Algebra

Xiaolong Pan

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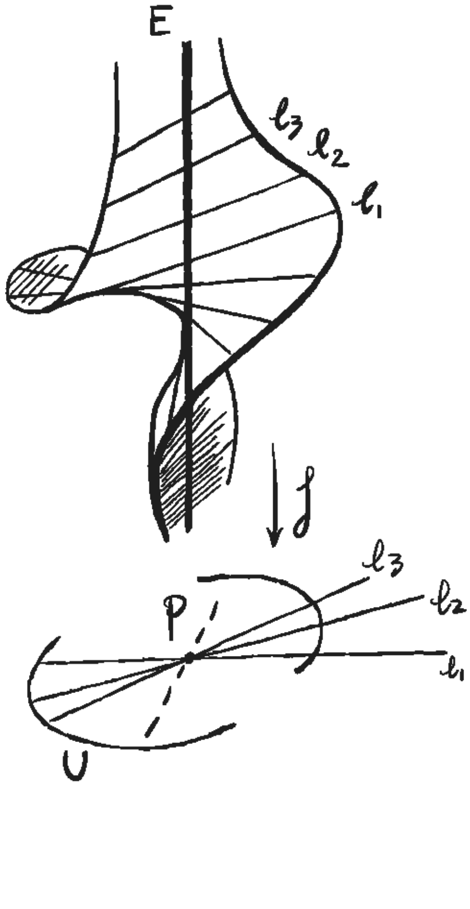

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Chapter 1

The theory of topos and étale cohomology of schemes

1.1 Fibre category

1.1.1 Categories over a fixed category

Let $\mathcal{E}$ be a category in a fixed universe, which is an object of **Cat**. We consider the category $\mathbf{Cat}_{/\mathcal{E}}$ of objects of **Cat** lying over $\mathcal{E}$; by definition, an object of this category is a functor $p : \mathcal{F} \to \mathcal{E}$. We say that the category $\mathcal{F}$, endowed with the functor p (called the **structural functor** of $\mathcal{F}$), is a category over $\mathcal{E}$, or an $\mathcal{E}$-category. By definition, a morphism of $\mathcal{E}$-categories $p : \mathcal{F} \to \mathcal{E}$, $q : \mathcal{G} \to \mathcal{E}$ is a functor $f : \mathcal{F} \to \mathcal{G}$ such that $qf = p$. We denote by $\mathrm{Hom}_{/\mathcal{E}}(\mathcal{F}, \mathcal{G})$ the set of morphism of $\mathcal{F}$ to $\mathcal{G}$ (also called **$\mathcal{E}$-functors**). The composition of $\mathcal{E}$-functors is given by that of the category **Cat**.

Now consider two $\mathcal{E}$-functors $f, g : \mathcal{F} \to \mathcal{G}$ and a morphism of functor $u : f \to g$. We say that u is an $\mathcal{E}$-homomorphism or a homomorphism of $\mathcal{E}$-functors, if for any object $\xi \in \mathrm{Ob}(\mathcal{F})$, we have

$$q(u(\xi)) = \mathrm{id}_{p(\xi)},$$

in other words, if $S = p(\xi) = q(f(\xi)) = q(g(\xi))$ is the corresponding object in $\mathcal{E}$, the morphism

$$u(\xi) : f(\xi) \to g(\xi)$$

in $\mathcal{G}$ is an id_S-morphism (in genera, for a morphism $\alpha : T \to S$ in $\mathcal{E}$ and a $\mathcal{E}$-category $q : \mathcal{G} \to \mathcal{E}$, a morphism v in $\mathcal{G}$ is called an α-morphism if $q(v) = \alpha$). If we have a third $\mathcal{E}$-functor $h : \mathcal{F} \to \mathcal{G}$ and an $\mathcal{E}$-homomorphism $v : g \to h$, then vu is also an $\mathcal{E}$-morphism. Therefore, the $\mathcal{E}$-functors from $\mathcal{F}$ to $\mathcal{G}$ and the $\mathcal{E}$-homomorphisms for a subcategory of the category $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ of functors from $\mathcal{F}$ to $\mathcal{G}$, which is called the **category of $\mathcal{E}$-functors** from $\mathcal{F}$ to $\mathcal{G}$, denoted by $\mathcal{H}om_{/\mathcal{E}}(\mathcal{F}, \mathcal{G})$.

We note that there is a natural composition law on the category $\mathcal{H}om_{/\mathcal{E}}(\mathcal{F}, \mathcal{G})$. In other words, we have a composition functor

$$\circ : \mathcal{H}om_{/\mathcal{E}}(\mathcal{F}, \mathcal{G}) \times \mathcal{H}om_{/\mathcal{E}}(\mathcal{G}, \mathcal{H}) \to \mathcal{H}om_{/\mathcal{E}}(\mathcal{F}, \mathcal{H}) \tag{1.1.1}$$

where $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$ are $\mathcal{E}$-categories. For this, we recall that there is a composition functor

$$\mathcal{H}om(\mathcal{F}, \mathcal{G}) \times \mathcal{H}om(\mathcal{G}, \mathcal{H}) \to \mathcal{H}om(\mathcal{F}, \mathcal{H}) \tag{1.1.2}$$

which on objects is the composition map $(f, g) \mapsto gf$ of $\mathcal{E}$-functors $f : \mathcal{F} \to \mathcal{G}$ and $g : \mathcal{G} \to \mathcal{H}$, and on morphisms, it transform (u, v), where $u : f \to f'$, $v : g \to g'$ are arrows of $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ and $\mathcal{H}om(\mathcal{G}, \mathcal{H})$, to the arrow

$$v \circ u : gf \to g'f'$$

defined by the relation

$$(v \circ u)(\xi) = v(f'(\xi)) \cdot g(u(\xi)) = g'(u(\xi)) \cdot v(f(\xi)).$$

(We use product to denote vertical composition, and $\circ$ by horizontal composition.) We thus obtain a homomorphism from gf to $g'f'$, which satisfies the following identities (hence we obtain the functor (1.1.2)):

$$\mathrm{id}_g \circ \mathrm{id}_f = \mathrm{id}_{gf}, \quad (v' \circ u') \circ (v \circ u) = (v' \circ v) \circ (u' \circ u).$$

Recall also that we have an associativity formula for the composition functor (1.1.2), which is expressed on the one hand by the associativity $(hg)f = h(gf)$ of the composition of functors, and on the other hand by the formula

$$(w \circ v) \circ u = w \circ (v \circ u)$$

where $u : f \to f'$, $v : g \to g'$, $v : h \to h'$ are morphisms of functors. It is now immediate that if $\mathcal{F}$ and $\mathcal{G}$ are $\mathcal{E}$-functors, the functor (1.1.2) induces the functor (1.1.1), as the horizontal composition of two $\mathcal{E}$-functors is clearly an $\mathcal{E}$-functor. Again, the induced functor (1.1.1) also satisfies the associativity, expressed by $(hg)f = h(gf)$ and $(w \circ v) \circ u = w \circ (v \circ u)$. We also note that the functor $\circ$ also satisfies the following formule:

$$v \circ \mathrm{id}_{\mathcal{F}} = v, \quad \mathrm{id}_{\mathcal{G}} \circ u = u,$$

where for simplicity we write $v \circ f$ or $u \circ g$ instead of $v \circ u$, if u (resp. v) is the identity automorphism of f (resp. g). We then conclude that the category $\mathcal{C}\dashv\sqcup_{/\mathcal{E}}$, with the Hom category $\mathcal{H}om_{/\mathcal{E}}$ and the composition functor $\circ$, is a strict 2-category.

It follows from our definition of $\circ$ that $\mathcal{H}om_{/\mathcal{E}}(\mathcal{F}, \mathcal{G})$ is a functor on $\mathcal{F}$, $\mathcal{G}$, on the product category $\mathbf{Cat}^{\mathrm{op}}_{/\mathcal{E}} \times \mathbf{Cat}_{/\mathcal{E}}$. In fact, if $g : \mathcal{G} \to \mathcal{G}'$ and $f : \mathcal{F}' \to \mathcal{F}$ are $\mathcal{E}$-functors, then we have the corresponding functors

$$\begin{aligned} g_* &: \mathcal{H}om_{/\mathcal{E}}(\mathcal{F}, \mathcal{G}) \to \mathcal{H}om_{/\mathcal{E}}(\mathcal{F}, \mathcal{G}') \\ f^* &: \mathcal{H}om_{/\mathcal{E}}(\mathcal{F}, \mathcal{G}) \to \mathcal{H}om_{/\mathcal{E}}(\mathcal{F}', \mathcal{G}). \end{aligned}$$

We now consider the base change of $\mathcal{E}$-categories. First, as **Cat** admits (small) projective limits, so does the category $\mathbf{Cat}_{/\mathcal{E}}$, and in particular the Cartesian products exist, which can be interpreted as fiber products in **Cat**. If $\mathcal{F}$ and $\mathcal{G}$ are two categories over $\mathcal{E}$, we denote by $\mathcal{F} \times_{\mathcal{E}} \mathcal{G}$ their product in $\mathbf{Cat}_{/\mathcal{E}}$, i.e. their fiber product over $\mathcal{E}$ in **Cat**. Therefore, $\mathcal{F} \times_{\mathcal{E}} \mathcal{G}$ is endowed with two projection $\mathcal{E}$-functors pr_1 and pr_2, which define, for any category $\mathcal{H}$ over $\mathcal{E}$, a bijection

$$\mathrm{Hom}_{/\mathcal{E}}(\mathcal{H}, \mathcal{F} \times_{\mathcal{G}} \mathcal{G}) \xrightarrow{\sim} \mathrm{Hom}_{/\mathcal{E}}(\mathcal{H}, \mathcal{F}) \times \mathrm{Hom}_{/\mathcal{E}}(\mathcal{H}, \mathcal{G}).$$

This bijection in fact provides an isomorphism of categories

$$\mathcal{H}om_{/\mathcal{E}}(\mathcal{H}, \mathcal{F} \times_{\mathcal{G}} \mathcal{G}) \xrightarrow{\sim} \mathcal{H}om_{/\mathcal{E}}(\mathcal{H}, \mathcal{F}) \times \mathcal{H}om_{/\mathcal{E}}(\mathcal{H}, \mathcal{G})$$

which has components $h \mapsto \mathrm{pr}_1 \circ h$ and $h \mapsto \mathrm{pr}_2 \circ h$ on 2-morphisms. We also remark that, as in the case of products of sets, we have

$$\mathrm{Ob}(\mathcal{F} \times_{\mathcal{E}} \mathcal{G}) = \mathrm{Ob}(\mathcal{F}) \times_{\mathrm{Ob}(\mathcal{E})} \mathrm{Ob}(\mathcal{G}), \quad \mathrm{Arr}(\mathcal{F} \times_{\mathcal{E}} \mathcal{G}) = \mathrm{Arr}(\mathcal{F}) \times_{\mathrm{Arr}(\mathcal{E})} \mathrm{Arr}(\mathcal{G}).$$

Now let $\lambda : \mathcal{E}' \to \mathcal{E}$ be a base change functor. For any category $\mathcal{F}$ over $\mathcal{E}$, we consider $\mathcal{F} \times_{\mathcal{E}} \mathcal{E}'$ as a category over $\mathcal{E}'$ via the canonical projection pr_2. Since this construction is clearly functorial over $\mathcal{F}$, we obtain a base change functor

$$\lambda^* : \mathbf{Cat}_{/\mathcal{E}} \to \mathbf{Cat}_{/\mathcal{E}'},$$

(which is adjoint to the restriction functor). As is the general case, the base change functor commutes with projective limits, and in particular transforms fiber products over $\mathcal{E}$ to fiber products over $\mathcal{E}'$.

Let $\mathcal{F}$ and $\mathcal{G}$ be two categories over $\mathcal{E}$, we want to define a canonical isomorphism

$$\mathcal{H}om_{/\mathcal{E}'}(\mathcal{F}',\mathcal{G}') \xrightarrow{\sim} \mathcal{H}om_{/\mathcal{E}}(\mathcal{F}',\mathcal{G}) \tag{1.1.3}$$

where $\mathcal{F}' = \mathcal{F} \times_{\mathcal{E}} \mathcal{E}'$, $\mathcal{G}' = \mathcal{G} \times_{\mathcal{E}} \mathcal{E}'$. For this, we consider the functor $\mathrm{pr}_1 : \mathcal{G}' \to \mathcal{G}$ and define (1.1.3) by

$$\mathcal{H}om(\mathcal{F}',\mathcal{G}') \xrightarrow{\sim} \mathcal{H}om(\mathcal{F}',\mathcal{G}), \quad F \mapsto \mathrm{pr}_1 \circ F.$$

It remains to verify that this induces a functor when restricted to $\mathcal{E}$-functors, and is bijective. But we note that if F, G are $\mathcal{E}'$-functors $\mathcal{F}' \to \mathcal{G}'$, then the map $u \mapsto \mathrm{pr}_1 \circ u$ induces an bijection

$$\mathrm{Hom}_{/\mathcal{E}'}(F,G) \xrightarrow{\sim} \mathrm{Hom}_{/\mathcal{E}}(F, \mathrm{pr}_1 \circ G)$$

whence our assertion.

From the isomorphism (1.1.3), we see that $\mathcal{H}om_{/\mathcal{E}'}(\mathcal{F} \times_{\mathcal{E}} \mathcal{E}', \mathcal{G} \times_{\mathcal{E}} \mathcal{E}')$ can be considered as a functor on $\mathcal{E}'$, $\mathcal{F}$ and $\mathcal{G}$, from the category $\mathbf{Cat}^{\mathrm{op}}_{/\mathcal{E}} \times \mathbf{Cat}^{\mathrm{op}}_{/\mathcal{E}} \times \mathbf{Cat}_{/\mathcal{E}}$ to $\mathbf{Cat}$, which is isomorphic to the functor $\mathcal{H}om_{/\mathcal{E}'}(\mathcal{F} \times_{\mathcal{E}} \mathcal{E}', \mathcal{G})$. In particular, for fixed $\mathcal{F}$ and $\mathcal{G}$, we obtain a functor

$$\mathcal{E}' \mapsto \mathcal{H}om_{/\mathcal{E}'}(\mathcal{F}',\mathcal{G}') = \mathcal{H}om_{/\mathcal{E}'}(\mathcal{F} \times_{\mathcal{E}} \mathcal{E}', \mathcal{G} \times_{\mathcal{E}} \mathcal{E}')$$

and in particular the functor $\lambda : \mathcal{E}' \to \mathcal{E}$ defines a morphism

$$\lambda^*_{\mathcal{F},\mathcal{G}} : \mathcal{H}om_{/\mathcal{E}}(\mathcal{F},\mathcal{G}) \to \mathcal{H}om_{/\mathcal{E}'}(\mathcal{F}',\mathcal{G}') \tag{1.1.4}$$

which we will now explain. For an $\mathcal{E}$-functor $f : \mathcal{F} \to \mathcal{G}$, the morphism $\lambda^*_{\mathcal{F},\mathcal{G}}$ sends f to its base change $f' = f \times_{\mathcal{E}} \mathcal{E}'$. On the other hand, if $f, g : \mathcal{F} \to \mathcal{G}$ are two $\mathcal{E}$-functors and $u : f \to g$ is a morphism of $\mathcal{E}$-functors, then u is associated with the morphism $u' : f' \to g'$, where for $\xi' = (\xi, S') \in \mathrm{Ob}(\mathcal{F}')$ (with $\xi \in \mathrm{Ob}(\mathcal{F})$, $S' \in \mathrm{Ob}(\mathcal{E}')$, $p(\xi) = \lambda(S') = S$), the morphism

$$u'(\xi') : f'(\xi') = (f(\xi), S') \to g'(\xi') = (g(\xi), S')$$

is defined by the formula

$$u'(\xi') = (u(\xi), \mathrm{id}_{S'}),$$

which is an S'-morphism in $\mathcal{G}'$ because $q(u(\xi)) = \lambda(\mathrm{id}_{S'}) = \mathrm{id}_S$.

Now consider an $\mathcal{E}$-functor $\lambda' : \mathcal{E}'' \to \mathcal{E}'$ and the corresponding functor

$$\mathcal{H}om_{/\mathcal{E}'}(\mathcal{F} \times_{\mathcal{E}} \mathcal{E}', \mathcal{G} \times_{\mathcal{E}} \mathcal{E}') \to \mathcal{H}om_{/\mathcal{E}''}(\mathcal{F} \times_{\mathcal{E}} \mathcal{E}'', \mathcal{G} \times_{\mathcal{E}} \mathcal{E}'').$$

This is none other than the functor (1.1.4) by taking $\mathcal{F}'$ and $\mathcal{G}'$ over $\mathcal{E}'$ and considering $\mathcal{E}''$ as an $\mathcal{E}'$-category, in view the transitivity of base change:

$$\mathcal{F}' \times_{\mathcal{E}'} \mathcal{E}'' \xrightarrow{\sim} \mathcal{F}'' = \mathcal{F} \times_{\mathcal{E}} \mathcal{E}'', \quad \mathcal{G}' \times_{\mathcal{E}'} \mathcal{E}'' \xrightarrow{\sim} \mathcal{G}'' = \mathcal{G} \times_{\mathcal{E}} \mathcal{E}''.$$

which implies a canonical isomorphism

$$\mathcal{H}om_{/\mathcal{E}''}(\mathcal{F}' \times'_{\mathcal{E}} \mathcal{E}'', \mathcal{G}' \times'_{\mathcal{E}} \mathcal{E}'') \xrightarrow{\sim} \mathcal{H}om_{/\mathcal{E}''}(\mathcal{F} \times_{\mathcal{E}} \mathcal{E}'', \mathcal{G} \times_{\mathcal{E}} \mathcal{E}'').$$

The functors we have just defined are compatible with the composition law $\circ$ on $\mathcal{H}om$. More precisely, if $\mathcal{F}, \mathcal{G}, \mathcal{H}$ are categories over $\mathcal{E}$ and we put

$$\mathcal{F}' = \mathcal{F} \times_{\mathcal{E}} \mathcal{E}', \quad \mathcal{G}' = \mathcal{G} \times_{\mathcal{E}} \mathcal{E}', \quad \mathcal{H}' = \mathcal{H} \times_{\mathcal{E}} \mathcal{E}',$$

then we have a commutative diagram

$$\begin{tikzcd}
\mathcal{H}om_{/\mathcal{E}}(\mathcal{F},\mathcal{G}) \times \mathcal{H}om_{/\mathcal{E}}(\mathcal{G},\mathcal{H}) \arrow[r, "\circ"] \arrow[d, "\lambda^*_{\mathcal{F},\mathcal{G}}\times\lambda^*_{\mathcal{G},\mathcal{H}}"'] & \mathcal{H}om_{/\mathcal{E}}(\mathcal{F},\mathcal{H}) \arrow[d, "\lambda^*_{\mathcal{F},\mathcal{H}}"] \\
\mathcal{H}om_{/\mathcal{E}}(\mathcal{F},\mathcal{G}) \times \mathcal{H}om_{/\mathcal{E}}(\mathcal{G},\mathcal{H}) \arrow[r, "\circ"] & \mathcal{H}om_{/\mathcal{E}}(\mathcal{F},\mathcal{H})
\end{tikzcd}$$

The commutativity of this diagram is often expressed by the following formule:

$$(gf)' = g'f', \quad (v \circ u)' = (v' \circ u')$$

where g, f and v, u are $\mathcal{E}$-functors or homomorphism of $\mathcal{E}$-functors.

In the following, we will be mainly interested in the functor $\mathcal{H}om_{/\mathcal{E}}(\mathcal{F},\mathcal{G})$ (and certain subcategories of it) when $\mathcal{F} = \mathcal{G}$, so we introduce the following notation:

$$\mathbf{\Gamma}(\mathcal{G}/\mathcal{E}) = \mathcal{H}om_{/\mathcal{E}}(\mathcal{E},\mathcal{G}), \quad \Gamma(\mathcal{G}/\mathcal{E}) = \mathrm{Ob}(\mathbf{\Gamma}(\mathcal{G}/\mathcal{E})) = \mathrm{Hom}_{/\mathcal{E}}(\mathcal{E},\mathcal{G}).$$

Remark 1.1.1. If the category $\mathcal{E}$ is a point, i.e., $\mathrm{Ob}(\mathcal{E})$ and $\mathrm{Arr}(\mathcal{E})$ are both reduced to a singleton (which signifies that $\mathcal{E}$ is a final object in **Cat**), then an $\mathcal{E}$-category is just an ordinary category, so $\mathbf{Cat}_{/\mathcal{E}}$ is isomorphic to **Cat**. Moreover, the category $\mathcal{H}om_{/\mathcal{E}}(\mathcal{F},\mathcal{G})$ are nothing but the usual category $\mathcal{H}om(\mathcal{F},\mathcal{G})$. Recall that we have the following adjoint formula

$$\mathrm{Hom}(\mathcal{H}, \mathcal{H}om(\mathcal{F},\mathcal{G})) \xrightarrow{\sim} \mathrm{Hom}(\mathcal{F}\times\mathcal{H},\mathcal{G}).$$

which allows us to axiomatically interpret $\mathcal{H}om(\mathcal{F},\mathcal{G})$ as an internal object of **Cat**, and any formula for $\mathcal{H}om$ can be extended to any category such that "Hom objects" exsits. There is also an analogous formula of $\mathcal{H}om_{/\mathcal{E}}(\mathcal{F},\mathcal{G})$, for arbitrary category $\mathcal{E}$:

$$\mathrm{Hom}(\mathcal{H}, \mathcal{H}om_{/\mathcal{E}}(\mathcal{F},\mathcal{G})) \xrightarrow{\sim} \mathrm{Hom}_{/\mathcal{E}}(\mathcal{F}\times\mathcal{H},\mathcal{G}).$$

In the way, the preceding properties we given for $\mathcal{H}om$ extends to any category such that the objects $\mathcal{H}om_{/\mathcal{E}}(\mathcal{F},\mathcal{E})$ exsits.

1.1.2 Categorical fibers and $\mathcal{E}$-equivalences

Let $\mathcal{F}$ be a category over $\mathcal{E}$, and $S \in \mathrm{Ob}(\mathcal{E})$. We define the **categorical fiber** of $\mathcal{F}$ at S to be the subcategory $\mathcal{F}_S$ of $\mathcal{F}$, obtained by taking the inverse image of the point subcategory $\{S\}$ of $\mathcal{E}$ defined by S. In other words, the objets of $\mathcal{F}_S$ are objects ξ of $\mathcal{F}$ such that $p(\xi) = S$, and morphisms of $\mathcal{F}_S$ are morphisms u of $\mathcal{F}$ such that $p(u) = \mathrm{id}_S$, i.e., the S-morphisms in $\mathcal{F}$. Alternatively, $\mathcal{F}_S$ is canonically isomorphic to the fiber product $\mathcal{F}\times_{\mathcal{E}}\{S\}$, so for any base change $\lambda : \mathcal{E}' \to \mathcal{E}$ and any $S' \in \mathrm{Ob}(\mathcal{E}')$, the projection $\mathrm{pr}_1 : \mathcal{F}' = \mathcal{F}\times_{\mathcal{E}}\mathcal{E}' \to \mathcal{F}$ induces an isomorphism $\mathcal{F}'_{S'} = \mathcal{F}_S$, where $S = \lambda(S')$.

Proposition 1.1.2. *Let $f : \mathcal{F} \to \mathcal{G}$ be an $\mathcal{E}$-functor. If f is fully faithful, then for any base change $\mathcal{E}' \to \mathcal{E}$, the corresponding functor $f' : \mathcal{F}' = \mathcal{F}\times_{\mathcal{E}}\mathcal{E}' \to \mathcal{G}' = \mathcal{G}\times_{\mathcal{E}}\mathcal{E}'$ is fully faithful.*

Proof. The varification is immediate from the definition of $\mathcal{F}'$, $\mathcal{G}'$ and f'. More generally, we can prove that any projective limit of fully faithful functors is fully faithful. □

We note that the analogous assertion of Proposition 1.1.2, with "fully faithful" replaced by "equivalence of categories", is flase. For example, if $f : \mathcal{F} \to \mathcal{E}$ is an equivalence of categories and $g : \mathcal{E} \to \mathcal{F}$ is its inverse, then for an object $S \in \mathrm{Ob}(\mathcal{E})$, the induced base change functor $f_S : \mathcal{F}_S \to \{S\}$ is essentially closed if and only if $g(S)$ is an object of $\mathcal{F}_S$ (which means g is a section of $\mathcal{F}$ over $\mathcal{E}$, or equivalently $f \circ g = \mathrm{id}_{\mathcal{E}}$). However, we have the following result.

Proposition 1.1.3. *Let $f : \mathcal{F} \to \mathcal{G}$ be an $\mathcal{E}$-functor. The following conditions are equivalent:*

(i) *There exists an $\mathcal{E}$-functor $g : \mathcal{G} \to \mathcal{F}$ and $\mathcal{E}$-isomorphisms*

$$gf \xrightarrow{\sim} \mathrm{id}_{\mathcal{F}}, \quad fg \xrightarrow{\sim} \mathrm{id}_{\mathcal{G}}.$$

(ii) *For any base change $\mathcal{E}' \to \mathcal{E}$, the functor $f' : \mathcal{F}' = \mathcal{F} \times_{\mathcal{E}} \mathcal{E}' \to \mathcal{G}' = \mathcal{G} \times_{\mathcal{E}} \mathcal{E}'$ is an equivalence of categories.*

(iii) *f is an equivalence of categories, and for any $S \in \mathrm{Ob}(\mathcal{E})$, the functor $f_S : \mathcal{F}_S \to \mathcal{G}_S$ induced by f is an equivalence of categories.*

(iii') *f is faithfully flat, and for any $S \in \mathrm{Ob}(\mathcal{E})$ and $\eta \in \mathrm{Ob}(\mathcal{G}_S)$, there exists $\xi \in \mathrm{Ob}(\mathcal{F}_S)$ and an S-isomorphism $u : f(\xi) \to \eta$.*

If these conditions are satisfied, we then say that f is an ***equivalence of categories over*** $\mathcal{E}$*, or an* $\mathcal{E}$***-equivalence****.*

Proof. Clearly (i) implies that f is an equivalence of categories (this is defined by the same condition, but the inverse is not required to be an $\mathcal{E}$-functor). On the other hand, it follows from the functoriality of base change that condition (i) is preserved under base change, so (i) implies (ii). Evidently (ii)⇒(iii), because $\mathcal{F}_S$ can be considered as a fiber product of f, and it is trivial that (iii) implies (iii'). It then remains to prove that (iii')⇒(i). For this, we choose for each $\eta \in \mathrm{Ob}(\mathcal{G})$ an object $g(\eta) \in \mathrm{Ob}(\mathcal{F})$ and an S-isomorphism $u(\eta) : f(g(\eta)) \to \eta$, where $S = q(\eta)$ (this is possible by condition (iii')). The fact that f is fully faithful implies that g can be considered as a functor from $\mathcal{G}$ to $\mathcal{F}$, and the $u(\eta)$ then define a homomorphism (hence an isomorphism) $u : fg \xrightarrow{\sim} \mathrm{id}_{\mathcal{G}}$. Moreover, by construction g is an $\mathcal{E}$-functor and u is an $\mathcal{E}$-homomorphism. By composition we see that $u \circ f : \mathcal{F} \to \mathcal{G}$ is an $\mathcal{E}$-homomorphism, isomorphic to f. Since f is fully faithful, we then obtain a functorial isomorphism $v : gf \to \mathrm{id}_{\mathcal{F}}$, defined by the condition that $f \circ v = u \circ f$, and it is clear that v is also an $\mathcal{E}$-homomorphism. □

Corollary 1.1.4. *Suppose that the structural functor $p : \mathcal{F} \to \mathcal{E}$ is transportable, i.e. for any isomorphism $\alpha : T \to S$ in $\mathcal{E}$ and any object ξ in $\mathcal{F}_T$, there exists an object η in $\mathcal{F}_S$ and an isomorphism $u : \xi \to \eta$ such that $p(u) = \alpha$. Then any $\mathcal{E}$-functor $f : \mathcal{F} \to \mathcal{G}$ which is an equivalence of categories, is an $\mathcal{E}$-equivalence.*

Proof. If f is an an equivalence of categories, then for any $S \in \mathrm{Ob}(\mathcal{E})$ and $\eta \in \mathrm{Ob}(\mathcal{G}_S)$, there exists $\xi \in \mathrm{Ob}(\mathcal{F})$ and an isomorphism $u : f(\xi) \xrightarrow{\sim} \eta$. Since f is an $\mathcal{E}$-functor, if $T = p(\xi)$, then $q(u)$ is an isomorphism from T to S (where $q : \mathcal{G} \to \mathcal{E}$ is the structural functor). By our hypothesis on p, there then exists an object ξ' in $\mathcal{F}_T$ and an isomorphism $v : \xi \to \xi'$ such that $p(v) = q(u)$:

$$\begin{tikzcd}
 & f(\xi) \arrow[rr, "u", "\sim"'] \arrow[dd] & & \eta \arrow[dd, "q"] \\
\xi \arrow[ur, "f"] \arrow[rr, "v", "\sim"'] \arrow[dr, "p"'] & & \xi' \arrow[ur, dashed] \arrow[dr, "p"] & \\
 & T \arrow[rr, "q(u)"] & & S
\end{tikzcd}$$

Then $f(\xi')$ is isomorphic to η and belongs to $\mathcal{G}_S$, so the corollary follows from the criterion (iii') of Proposition 1.1.3. □

Corollary 1.1.5. *Let $f : \mathcal{F} \to \mathcal{G}$ be an $\mathcal{E}$-equivalence. Then for any category $\mathcal{H}$ over $\mathcal{E}$, the corresponding functors*

$$f^* : \mathcal{H}om_{/\mathcal{E}}(\mathcal{G}, \mathcal{H}) \to \mathcal{H}om_{/\mathcal{E}}(\mathcal{F}, \mathcal{H})$$

$$f_* : \mathcal{H}om_{/\mathcal{E}}(\mathcal{H}, \mathcal{F}) \to \mathcal{H}om_{/\mathcal{E}}(\mathcal{H}, \mathcal{G})$$

are equivalences of categories.

Proof. This follows from criterion (i) of Proposition 1.1.3. In fact, if $g : \mathcal{G} \to \mathcal{F}$ is an $\mathcal{E}$-inverse of f, then g_* and g^* are inverses of f_* and f^*, respectively. □

1.1.3 Cartesian morphisms and Cartesian functors

Let $\mathcal{F}$ be a category over $\mathcal{E}$, with structural functor $p : \mathcal{F} \to \mathcal{E}$. We consider a morphism $\alpha : \eta \to \xi$ in $\mathcal{F}$, and let $S = p(\xi)$, $T = p(\eta)$, $f = p(\alpha)$:

$$\begin{tikzcd} \eta \arrow[r, "\alpha"] \arrow[d] & \xi \arrow[d] \\ T \arrow[r, "f"] & S \end{tikzcd} \tag{1.1.5}$$

We say that α is a **Cartesian morphism** if for any $\eta' \in \mathrm{Ob}(\mathcal{F}_T)$ and any f-morphism $u : \eta' \to \xi$, there exists a unique T-morphism $\tilde{u} : \eta' \to \eta$ such that $u = \alpha \circ \tilde{u}$.

$$\begin{tikzcd} \eta' \arrow[dr, dashed, "\tilde{u}"] \arrow[drr, bend left, "u"] \arrow[ddr, bend right] & & \\ & \eta \arrow[r, "\alpha"] \arrow[d] & \xi \arrow[d] \\ & T \arrow[r, "f"] & S \end{tikzcd} \tag{1.1.6}$$

This signifies that for any $\eta' \in \mathrm{Ob}(\mathcal{F}_T)$, the map

$$\mathrm{Hom}_T(\eta', \eta) \to \mathrm{Hom}_f(\eta', \xi), \quad v \mapsto \alpha \circ v \tag{1.1.7}$$

is bijective. In other words, the couple (η, α) represents the functor $\eta' \mapsto \mathrm{Hom}_f(\eta', \xi)$ from $\mathcal{F}_T^{\mathrm{op}} \to$ **Set**. If for any morphism $f : T \to S$ and any object $\xi \in \mathrm{Ob}(\mathcal{F}_S)$, there exists a such couple (η, α), i.e. a Cartesian morphism $\alpha : \eta \to \xi$ in $\mathcal{F}$ such that $p(\alpha) = f$, then η is uniquely determined in $\mathcal{F}_T$. In this case, we then say that the inverse image of ξ under f exists, and an object η together with a Cartesian f-morphism $\alpha : \eta \to \xi$ is called an **inverse image of ξ by f**. We often assume that such an inverse image is chosen whenever it exists ($\mathcal{F}$ being fixed), and denote it by $f^*_{\mathcal{F}}(\xi)$, or simply $f^*(\xi)$ or $\xi \times_S T$ if there is no risk of confusion. The canonical homomorphism $\alpha : \eta \to \xi$ is then denoted in this case by $\alpha_f(\xi)$. If for any $\xi \in \mathrm{Ob}(\mathcal{F}_S)$, the inverse image of ξ by f exists, we then say that the **inverse image functor of f in $\mathcal{F}$ exists**, and $\xi \mapsto f^*(\xi)$ defines a covariant functor from $\mathcal{F}_S$ to $\mathcal{F}_T$. This functorial dependence of $f^*(\xi)$ is explained as follows: consider Cartesian f-morphisms

$$\alpha : \eta \to \xi, \quad \alpha' : \eta' \to \xi'$$

and an S-morphism $\lambda : \xi \to \xi'$. Then there exits a unique T-morphism $\mu : \eta \to \eta'$ such that the diagram

$$\begin{tikzcd} \eta \arrow[r, "\alpha"] \arrow[d, "\mu"'] & \xi \arrow[d, "\lambda"] \\ \eta' \arrow[r, "\alpha'"] & \xi' \end{tikzcd}$$

is commutative (this follows from the fact that α' is Cartesian).

An $\mathcal{E}$-functor $F : \mathcal{F} \to \mathcal{G}$ is called a **Cartesian functor** is it transforms Cartesian morphisms to Cartesian morphisms. We denote by $\mathcal{H}om_{\mathrm{Cart}}(\mathcal{F},\mathcal{G})$ the full subcategory of $\mathcal{H}om_{/\mathcal{E}}(\mathcal{F},\mathcal{G})$ formed by Cartesian functors. For example, consider $\mathcal{E}$ as an $\mathcal{E}$-category via the identity functor. Then any morphism of $\mathcal{E}$ is Cartesian, so a Cartesian functor from $\mathcal{E}$ to $\mathcal{F}$ is a section $F : \mathcal{E} \to \mathcal{F}$ which transforms any morphism of $\mathcal{E}$ to a Cartesian morphism; such a functor is called a **Cartesian section** of $\mathcal{F}$ over $\mathcal{E}$.

Proposition 1.1.6. *Let $\mathcal{E}$ be a fixed category.*

(a) *A functor $\mathcal{F} : \mathcal{F} \to \mathcal{G}$ that is an $\mathcal{E}$-equivalence is a Cartesian functor.*

(b) *Let $F, G : \mathcal{F} \to \mathcal{G}$ be isomorphic $\mathcal{E}$-functors. If one of them is Cartesian, so is the other.*

(c) *The composition of two Cartesian functors is Cartesian.*

Proof. Assertion (c) is trivial by definition, and (b) is immediate from the definition of isomorphisms of $\mathcal{E}$-functors. Finally, (a) follows from criterion (iii) of Proposition 1.1.3. More precisely, in this case a morphism α in $\mathcal{F}$ is Cartesian if and only if $F(\alpha)$ is. □

Corollary 1.1.7. *Let $F : \mathcal{F} \to \mathcal{G}$ be an $\mathcal{E}$-equivalence. Then for any category $\mathcal{H}$ over $\mathcal{F}$, the corresponding functors*

$$F^* : \mathcal{H}om_{\mathrm{Cart}}(\mathcal{G},\mathcal{H}) \to \mathcal{H}om_{\mathrm{Cart}}(\mathcal{F},\mathcal{H}),$$
$$F_* : \mathcal{H}om_{\mathrm{Cart}}(\mathcal{H},\mathcal{F}) \to \mathcal{H}om_{\mathrm{Cart}}(\mathcal{H},\mathcal{G})$$

are equivalence of categories.

Proof. This follows from criterion (i) of Proposition 1.1.3 and Proposition 1.1.6. □

It follows from Proposition 1.1.6 that if we consider the subcategory $\mathbf{Cat}_{/\mathcal{E}}^{\mathrm{Cart}}$ of $\mathbf{Cat}_{/\mathcal{E}}$ whose objects are the same as $\mathbf{Cat}_{/\mathcal{E}}$ and whose morphisms are Cartesian functors, then we have a composition law

$$\circ : \mathcal{H}om_{\mathrm{Cart}}(\mathcal{F},\mathcal{G}) \times \mathcal{H}om_{\mathrm{Cart}}(\mathcal{G},\mathcal{H}) \to \mathcal{H}om_{\mathrm{Cart}}(\mathcal{F},\mathcal{H})$$

induced by that of $\mathcal{H}om_{/\mathcal{E}}(-,-)$. Therefore, we can consider $\mathcal{H}om_{\mathrm{Cart}}(\mathcal{F},\mathcal{G})$ as a functor from $(\mathbf{Cat}_{/\mathcal{E}}^{\mathrm{Cart}})^{\mathrm{op}} \times \mathbf{Cat}_{/\mathcal{E}}^{\mathrm{Cart}}$ to $\mathbf{Cat}$. In particular, if $\mathcal{F}$ is a category over $\mathcal{E}$, we define $\mathbf{\Gamma}_{\mathrm{Cart}}(\mathcal{F}/\mathcal{E})$ to be the category of Cartesian $\mathcal{E}$-functors $\mathcal{E} \to \mathcal{F}$, i.e. the Cartesian sections of $\mathcal{F}$ over $\mathcal{E}$, and write $\Gamma_{\mathrm{Cart}}(\mathcal{F}/\mathcal{E})$ for the objects of $\mathbf{\Gamma}_{\mathrm{Cart}}(\mathcal{F}/\mathcal{E})$. In view of the preceding remarks, $\Gamma_{\mathrm{Cart}}(\mathcal{F}/\mathcal{E})$ then defines a functor from $\mathbf{Cat}_{/\mathcal{E}}^{\mathrm{Cart}}$ to $\mathbf{Cat}$.

We now consider categories over $\mathcal{E}$ such that inverse images exists for any object of $\mathcal{E}$.

Definition 1.1.8. A category $\mathcal{F}$ over $\mathcal{E}$ is called a **fibre category** (and the structural functor $p : \mathcal{F} \to \mathcal{E}$ is called a **fibrant**) if it satisfies the following conditions:

(Fib1) For any morphism $f : T \to S$ in $\mathcal{E}$, the inverse image functor of f in $\mathcal{F}$ exists.

(Fib2) The composition of two Cartesian morphism is Cartesian.

A category $\mathcal{F}$ over $\mathcal{E}$ is said to be **prefibre** if it satisfies condition (Fib1).

If $\mathcal{F}$ is a (pre)fibre category over $\mathcal{E}$, a subcategory $\mathcal{G}$ of $\mathcal{F}$ is called a **(pre)fibre subcategory** if it is a fibre category (resp. prefibre category) over $\mathcal{E}$, and if the inclusion functor is Cartesian. If for example $\mathcal{G}$ is a full subcategory of $\mathcal{F}$, we see that this signifies that for any morphism $f : T \to S$ in $\mathcal{E}$ and any $\xi \in \mathrm{Ob}(\mathcal{G}_S)$, $f_{\mathcal{F}}^*(\xi)$ is T-isomorphisc to an object of $\mathcal{G}_T$. An interesting case is the following: $\mathcal{F}$ being a fibre category over $\mathcal{E}$, consider the subcategory $\mathcal{G}$ of $\mathcal{F}$ with the same objects, and whose morphisms are Cartesian morphisms of $\mathcal{F}$. In particular, the morphisms of $\mathcal{G}_S$ are the isomorphisms of $\mathcal{F}_S$. Then $\mathcal{G}$ is a fibre subcategory of $\mathcal{F}$, because the bijection

$$\mathrm{Hom}_T(\eta', \eta) \xrightarrow{\sim} \mathrm{Hom}_f(\eta', \xi)$$

of (1.1.7) sends T-isomorphisms to Cartesian morphisms, and conversely. By definition, the Cartesian sections $\mathcal{E} \to \mathcal{F}$ then correspond to $\mathcal{E}$-functors $\mathcal{E} \to \mathcal{G}$.

Example 1.1.9. Let $\mathcal{F}$ be a category over $\mathcal{E}$. Then the following conditions are equivalent:

(i) Every morphism of $\mathcal{F}$ is Cartesian.

(ii) $\mathcal{F}$ is a fibre category over $\mathcal{E}$ and for any $S \in \mathrm{Ob}(\mathcal{E})$, the category $\mathcal{F}_S$ is a groupoid.

In fact, it is clear that (i) implies (ii), and conversely, condition (ii) implies that for any morphism $\alpha : \eta \to \xi$ in $\mathcal{F}$, the object η is isomorphic to $f^*(\xi)$ in $\mathcal{F}_T$, where $f = p(\alpha)$.

A category $\mathcal{F}$ over $\mathcal{E}$ satisfying the above equivalent conditions is called a **fibre category in groupoids**. If $\mathcal{E}$ is a groupoid, then the above conditions is also equivalent to the following:

(iii) $\mathcal{F}$ is a groupoid and the structural functor $p : \mathcal{F} \to \mathcal{E}$ is transportable.

For example, if $\mathcal{E}$ and $\mathcal{F}$ are groupoids such that $\mathrm{Ob}(\mathcal{E})$ and $\mathrm{Ob}(\mathcal{F})$ are reduced to singletones, which means $\mathcal{E}$ and $\mathcal{F}$ are defined by groups E and F and the functor $p : \mathcal{F} \to \mathcal{E}$ is defined by a homomorphism $\phi : F \to E$. Then $\mathcal{F}$ is fibre over $\mathcal{E}$ if and only if p is surjective, i.e. if ϕ defines an extension of the group E by the group $G = \ker\phi$.

Proposition 1.1.10. *Let $\mathcal{F}$ be an $\mathcal{E}$-equivalence. For $\mathcal{F}$ to be a (pre)fibre category over $\mathcal{E}$, it is necessary and sufficient that $\mathcal{G}$ is.*

Proof. This follows from the definition and the fact that α is Cartesian if and only if $F(\alpha)$ is. □

Proposition 1.1.11. *Let $\mathcal{F}_1$, $\mathcal{F}_2$ be categories over $\mathcal{E}$, and $\alpha = (\alpha_1, \alpha_2)$ be a morphism in $\mathcal{F} = \mathcal{F}_1 \times_{\mathcal{E}} \mathcal{F}_2$. For α to be Cartesian, it is necessary and sufficient that α_1 and α_2 are Cartesian.*

Proof. Let $\alpha_i : \xi_i \to \eta_i$ be the given morphism, and $f : T \to S$ be the morphism such that $p_i(\alpha_i) = f$, where $p_i : \mathcal{F}_i \to \mathcal{E}$ is the structural functor. For any object $\eta' = (\eta'_1, \eta'_2)$ in $\mathcal{F}_T$, we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_T(\eta', \eta) \arrow[r] \arrow[d] & \mathrm{Hom}_f(\eta', \xi) \arrow[d] \\
\mathrm{Hom}_T(\eta'_1, \eta_1) \times \mathrm{Hom}_T(\eta'_2, \eta_2) \arrow[r] & \mathrm{Hom}_f(\eta'_1, \xi_1) \times \mathrm{Hom}_f(\eta'_2, \xi_2)
\end{tikzcd}$$

where the vertical arrows are bijections. Then the horizontal arrows are simutanously bijective, which shows that, if α_1, α_2 are Cartesian, then so is α. The converse of this can be obtained from the above diagram by taking $\eta'_i = \eta_i$ so $\mathrm{Hom}_T(\eta'_i, \eta_i) \neq \varnothing$ (for $i = 1$ or $i = 2$, respectively), which shows that α_1 and α_2 are Cartesian. □

Corollary 1.1.12. *Let $\mathcal{F} = \mathcal{F}_1 \times_{\mathcal{E}} \mathcal{F}_2$ and $F = (F_1, F_2)$ be an $\mathcal{E}$-functor $\mathcal{G} \to \mathcal{F}$. For F to be Cartesian, it is necessary and sufficient that F_1 and F_2 are Cartesian. Therefore, we have an isomorphism of categories*

$$\mathcal{H}om_{\mathrm{Cart}}(\mathcal{G}, \mathcal{F}_1 \times_{\mathcal{E}} \mathcal{F}_2) \xrightarrow{\sim} \mathcal{H}om_{\mathrm{Cart}}(\mathcal{G}, \mathcal{F}_1) \times \mathcal{H}om_{\mathrm{Cart}}(\mathcal{G}, \mathcal{F}_2)$$

and in particular an isomorphism

$$\mathbf{\Gamma}_{\mathrm{Cart}}(\mathcal{F}_1 \times_{\mathcal{E}} \mathcal{F}_2/\mathcal{E}) \xrightarrow{\sim} \mathbf{\Gamma}_{\mathrm{Cart}}(\mathcal{F}_1/\mathcal{E}) \times \mathbf{\Gamma}_{\mathrm{Cart}}(\mathcal{F}_2/\mathcal{E}).$$

Corollary 1.1.13. *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be (pre)fibre categories over $\mathcal{E}$, then the fiber product $\mathcal{F}_1 \times_{\mathcal{E}} \mathcal{F}_2$ is a (pre)fibre category.*

It is clear that the above results extends without difficulty to finite products of categories over $\mathcal{E}$. For example, a finite fiber product of (pre)fibre categories over $\mathcal{E}$ is again a (pre)fibre category over $\mathcal{E}$.

Proposition 1.1.14. *Let $\mathcal{F}$ be a category over $\mathcal{E}$, with structural functor $p : \mathcal{F} \to \mathcal{E}$, and $\lambda : \mathcal{E}' \to \mathcal{E}$ be a base change functor. Consider $\mathcal{F}' = \mathcal{F} \times_{\mathcal{E}} \mathcal{E}'$ as a category over $\mathcal{E}'$ via the projection $p' = p \times_{\mathcal{E}} \mathrm{id}_{\mathcal{E}'}$. Let α' be a morphism of $\mathcal{F}'$, for α' to be a Cartesian morphism, it is necessary and sufficient that its image α in $\mathcal{F}$ is a Cartesian morphism.*

Proof. If α is Cartesian, then it it immediate to see that α' is also Cartesian, using the universal property of base change. Conversely, assume that α is Cartesian, and let $\alpha' : \eta' \to \xi'$ be a morphism in $\mathcal{F}'$. Then we can write

$$\eta' = (\eta, T'), \quad \xi' = (\xi, S')$$

where η (resp. ξ) is the image of η' (resp. ξ') in $\mathcal{F}$, and we have

$$\lambda(T') = p(\eta) =: T, \quad \lambda(S') = p(\xi) =: S.$$

Now if γ is an object of $\mathcal{F}_T$, then it is the image of the object $\gamma' = (\gamma, T')$ in $\mathcal{F}'$, and we have

$$\mathrm{Hom}_T(\gamma, \eta) = \mathrm{Hom}_{T'}(\gamma', \eta'), \quad \mathrm{Hom}_f(\gamma, \xi) = \mathrm{Hom}_{f'}(\gamma', \eta')$$

where $f = p(\alpha)$ and $f' = p'(\alpha')$. In view of (1.1.7), it follows that α is Cartesian in $\mathcal{F}$. □

Corollary 1.1.15. *For any Cartesian functor $F : \mathcal{F} \to \mathcal{G}$ of categories over $\mathcal{E}$, the functor $F' = F \times_{\mathcal{E}} \mathcal{E}'$ from $\mathcal{F}' = \mathcal{F} \times_{\mathcal{E}} \mathcal{E}'$ to $\mathcal{G}' = \mathcal{G} \times_{\mathcal{E}} \mathcal{E}'$ is Cartesian.*

Corollary 1.1.16. *Let $\mathcal{F}$ be a (pre)fibre category over $\mathcal{E}$, then $\mathcal{F}' = \mathcal{F} \times_{\mathcal{E}} \mathcal{E}'$ is a (pre)fibre category over $\mathcal{E}'$.*

In particular, the functor $\mathcal{H}om_{/\mathcal{E}}(\mathcal{F}, \mathcal{G}) \to \mathcal{H}om_{/\mathcal{E}'}(\mathcal{F}', \mathcal{G}')$ considered in Section 1.1.1 induces a functor

$$\mathcal{H}om_{\mathrm{Cart}}(\mathcal{F}, \mathcal{G}) \to \mathcal{H}om_{\mathrm{Cart}}(\mathcal{F}', \mathcal{G}')$$

and in view of this, for fixed $\mathcal{F}, \mathcal{G}$, we can consider $\mathcal{H}om_{\mathrm{Cart}}(\mathcal{F}, \mathcal{G})$ as a functor on $\mathcal{E}'$, from the category $\mathbf{Cat}_{/\mathcal{E}}^{\mathrm{op}}$ to $\mathbf{Cat}$. If we take into account the isomorphism (1.1.3), then the Cartesian $\mathcal{E}'$-functors from $\mathcal{F}'$ to $\mathcal{G}'$ correspond to $\mathcal{E}$-functors $\mathcal{F} \times_{\mathcal{E}} \mathcal{E}' \to \mathcal{G}$ which sends any morphism whose projection to $\mathcal{F}$ is a Cartesian morphism, to a Cartesian morphism of $\mathcal{G}$. If $\mathcal{F} = \mathcal{E}$, we then obtain the following corollary:

Corollary 1.1.17. *The category $\mathbf{\Gamma}_{\mathrm{Cart}}(\mathcal{F}'/\mathcal{E}')$ is isomorphic to the full subcategory of $\mathcal{H}om_{/\mathcal{E}}(\mathcal{E}', \mathcal{F})$ formed by $\mathcal{E}$-functors $\mathcal{E}' \to \mathcal{F}$ which transforms morphisms of $\mathcal{E}'$ into Cartesian morphisms of $\mathcal{F}$. In particular, if $\mathcal{F}$ is a fibre category and $\widetilde{\mathcal{F}}$ is the subcategory of $\mathcal{F}$ whose morphisms are the Cartesian morphisms of $\mathcal{F}$, then we have a bijection*

$$\Gamma_{\mathrm{Cart}}(\mathcal{F}'/\mathcal{E}') \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{E}}(\mathcal{E}', \widetilde{\mathcal{F}}).$$

Proposition 1.1.18. *Let $\mathcal{F}$ and $\mathcal{G}$ be prefiber categories over $\mathcal{E}$, $F : \mathcal{F} \to \mathcal{G}$ be a Cartesian $\mathcal{E}$-functor. For F to be faithful (resp. fully faithful, resp. an $\mathcal{E}$-equivalence), it is necessary and sufficient that for any $S \in \mathrm{Ob}(\mathcal{E})$, the induced functor $F_S : \mathcal{F}_S \to \mathcal{G}_S$ is faithful (resp. fully faithful, resp. an $\mathcal{E}$-equivalence).*

Proof. One direction is already proved in Proposition 1.1.2 and Proposition 1.1.3, and the converse for $\mathcal{E}$-equivalences is given in Proposition 1.1.3 (withour the hypothesis that $\mathcal{F}$ and $\mathcal{G}$ are prefibre categories). Conversely, assume that (say) F_S is faithful for any $S \in \mathrm{Ob}(\mathcal{E})$. Let $\alpha, \beta : \eta \to \xi$ be two morphisms of $\mathcal{F}$ lying over a morphism $f : T \to S$ of $\mathcal{E}$ such that $F(\alpha) = F(\beta)$. Since $\mathcal{G}$ is Cartesian, α and β correspond to morphisms

$$\tilde{\alpha} : \eta \to f^*_{\mathcal{F}}(\xi), \quad \tilde{\beta} : \eta \to f^*_{\mathcal{F}}(\xi)$$

and since F is a Cartesian $\mathcal{E}$-functor, $F(\tilde{\alpha})$ and $F(\tilde{\beta})$ are the morphisms corresponding to $F(\alpha)$ and $F(\beta)$ under pullbacks in $\mathcal{G}$. Since $F(\tilde{\alpha})$ and $F(\tilde{\beta})$ are morphisms of objects of $\mathcal{F}$ with the same base, by hypothesis we must have $\tilde{\alpha} = \tilde{\beta}$, so $\alpha = \beta$ by the universal property of pullbacks. The same arguments can be applied to prove that F is full if F_S is full for any $S \in \mathrm{Ob}(\mathcal{E})$, and this completes the proof. □

We have defined Cartesian morphism via the pullback diagram (1.1.6). There is in fact a stronger notion for Cartesian morphisms, called *strongly Cartesian*. Briefly speaking, it allows the object η' in (1.1.6) to have a different base T', with an additional morphism $g : T' \to T$ being provided.

$$\begin{tikzcd}
\eta' \arrow[d] \arrow[dr, dashed, "\tilde{u}"] \arrow[drr, bend left, "u"] & & \\
T' \arrow[dr, "g"] & \eta \arrow[r, "\alpha"] \arrow[d] & \xi \arrow[d] \\
& T \arrow[r, "f"] & S
\end{tikzcd}$$

In other words, a strongly Cartesian morphism $\alpha : \eta \to \xi$ provides a bijection

$$\mathrm{Hom}_g(\eta', \eta) \xrightarrow{\sim} \mathrm{Hom}_{fg}(\eta', \xi). \tag{1.1.8}$$

It is clear that strongly Cartesian property is preserved under composition, so we could just define a fibre category to be an category $\mathcal{F}$ over $\mathcal{E}$ in which strong Cartesian pullbacks exist. However, this seemingly stronger condition in fact produces equivalent fibre categories, as the following proposition shows.

Proposition 1.1.19. *Let $\mathcal{F}$ be a prefibre category over $\mathcal{E}$. For $\mathcal{F}$ to be fibre, it is necessary and sufficent that any Cartesian morphism $\alpha : \eta \to \xi$ is strongly Cartesian.*

Proof. We have remarked that strongly Cartesian morphisms are closed under composition, so this condition is sufficient. Conversely, let $\mathcal{F}$ be a fibre category over $\mathcal{E}$. Let $\alpha : \eta \to \xi$ be a Cartesian morphism in $\mathcal{F}$ lying over a morphism $f : T \to S$, and $u : \eta' \to \xi$ be an fg-morphism, where $g : T' \to T$ is a morphism in $\mathcal{E}$. Let $\beta : g^*(\eta) \to \eta$ be the pullback of η under g; then the composition $\alpha\beta : g^*(\eta) \to \xi$ is Cartesian by hypothesis, and since $g^*(\eta)$ has base T', the morphism $u : \eta' \to \xi$ corresponds to a morphism $\tilde{v} : \eta' \to g^*(\eta)$, which in turn corresponds to a morphism $\tilde{u} : \eta' \to \eta$ such that $\alpha\tilde{u} = u$. This then proves that α is strongly Cartesian. □

Corollary 1.1.20. *Let $\mathcal{F}$ be a category over $\mathcal{E}$ and α be a morphism in $\mathcal{F}$. For α to be an isomorphism, it is necessary that $p(\alpha) = f$ is an isomorphism and α is Cartesian. The converse is true if $\mathcal{F}$ is Cartesian over $\mathcal{E}$.*

Proof. If α is an isomorphism, it is clear that so is $p(\alpha) = f : T \to S$, and for any $\eta' \in \mathrm{Ob}(\mathcal{F}_T)$, the map $u \mapsto \alpha \circ u$ is bijective and sends T-morphisms to f-morphisms, so α is Cartesian.

Conversely, suppose that f is an isomorphism and that α is Strongly Cartesian. Then from (1.1.8), we see that for any $\zeta \in \mathrm{Ob}(\mathcal{F})$, the map

$$\mathrm{Hom}(\zeta, \eta) \to \mathrm{Hom}(\zeta, \xi), \quad u \mapsto \alpha \circ u$$

is bijective (since f an isomorphism), so α is an isomorphism. □

Corollary 1.1.21. *Let $\alpha : \eta \to \xi$ and $\beta : \zeta \to \eta$ be morphisms in a fibre category $\mathcal{F}$ over $\mathcal{E}$. If α is Cartesian, then β is Cartesian if and only if $\alpha\beta$ is Cartesian.*

Proof. In fact, this property is true for strongly Cartesian morphisms, so the corollary follows from Proposition 1.1.19. □

1.1.4 Cleavages and pseudo-functors

Let $\mathcal{F}$ be a category over $\mathcal{E}$. We define a **cleavage** of $\mathcal{F}$ over $\mathcal{E}$ to be a function which attaches to each morphism $f : T \to S$ an inverse image *functor* of f in $\mathscr{F}$, denoted by $f^* : \mathscr{F}_S \to \mathscr{F}_T$ (the pullback functor of f). The cleavage is called **normalized** if $f = \mathrm{id}_S$ implies $f^* = \mathrm{id}_{\mathscr{F}_S}$. A category $\mathcal{F}$ over $\mathcal{E}$, together with a chosen (normalized) cleavage, is called a **(normalized) cloven category**.

It is evident that $\mathcal{F}$ admits a cleavage if and only if it is prefibre over $\mathcal{E}$, and in this case the cleavage can be chosen to be normalized. The set of cleavages over $\mathcal{F}$ corresponds to the subset K of $\mathrm{Arr}(\mathcal{F})$ satisfying the following conditions:

(a) the $\alpha \in K$ are Cartesian morphisms;

(b) for any morphism $f : T \to S$ in $\mathcal{E}$ and any $\xi \in \mathrm{Ob}(\mathcal{F}_S)$, there exists a unique f-morphism in K with target ξ.

For a cleavage defined by K to be normalized, it is necessary and sufficient that K satisfies the following additional condition:

(c) the identity morphisms of $\mathscr{F}$ belongs to K.

The morphisms of K are called the "transport morphismss" for this defined cleavage.

The notion of morphisms of cloven categories is clear: this is defined as an $\mathcal{E}$-functor $\mathcal{F} \to \mathcal{G}$ which sends transport morphisms to transport morphisms (and in particular is a Cartesian functor). In this way, cloven categories over $\mathcal{E}$ form a category, called the **category of cloven categories over** $\mathcal{E}$. It is clear that products exsits in this category, since if $\mathcal{F}$ is the product of cloven categories $\mathcal{F}_i$ over $\mathcal{E}$, then it is endowed with a natural cleavage. In particular, base changes exists for cloven categories.

We denote by $\alpha_f(\xi)$ the canonical morphism $\alpha_f(\xi) : f^*(\xi) \to \xi$, which is functorial on ξ, i.e. we have a functorial homomorphism

$$\alpha_f : i_T f^* \to i_S$$

where for $S \in \mathrm{Ob}(\mathcal{E})$, $i_S : \mathcal{F}_S \to \mathscr{F}$ denote the inclusion functor. Now consider morphisms $f : T \to S$ and $g : U \to T$ in $\mathcal{E}$, and let $\xi \in \mathrm{Ob}(\mathcal{F}_S)$. Then there is a unique U-morphism

$$c_{f,g}(\xi) : g^* f^*(\xi) \to (fg)^*(\xi)$$

such the the following diagram

$$\begin{tikzcd}
g^*(f^*(\xi)) \arrow[r, "\alpha_g(f^*(\xi))"] \arrow[d, "c_{f,g}(\xi)"'] & f^*(\xi) \arrow[d, "\alpha_f(\xi)"] \\
(fg)^*(\xi) \arrow[r, "\alpha_{fg}(\xi)"] & \xi
\end{tikzcd}$$

is commutative (in view of the definition of $(fg)^*(\xi)$). Moreover, this homomorphism is functorial on ξ, so we obtain a homomorphism

$$c_{f,g} : g^* f^* \to (fg)^*$$

of functors from $\mathcal{F}_S$ to $\mathcal{F}_U$.

Proposition 1.1.22. *For the cloven category $\mathcal{F}$ over $\mathcal{E}$ to be fibre, it is necessary and sufficient that $c_{f,g}$ are isomorphisms.*

Proof. Since $\mathcal{F}$ is already prefibre, it is fibre if and only if composition of Cartesian morphisms are Cartesian, which means exactly that the $c_{f,g}$ are isomorphisms. □

Corollary 1.1.23. *If $\mathcal{F}$ is a fibre cloven category, then for any isomorphism $f : T \to S$ in $\mathcal{E}$, f^* is an equivalence of categories.*

Proof. In fact, if g is the inverse of f, then f^* and g^* are inverses of each other. □

Proposition 1.1.24. *Let $\mathcal{F}$ be a cloven category over $\mathcal{E}$. Then for morphisms*

$$V \xrightarrow{h} U \xrightarrow{g} T \xrightarrow{f} S$$

and $\xi \in \mathrm{Ob}(\mathcal{F}_S)$, we have

$$c_{f,\mathrm{id}_T}(\xi) = \alpha_{\mathrm{id}_T}(f^*(\xi)), \quad c_{\mathrm{id}_S,f}(\xi) = f^*(\alpha_{\mathrm{id}_S}(\xi)), \tag{1.1.9}$$

$$c_{f,gh}(\xi) \cdot c_{g,h}(f^*(\xi)) = c_{fg,h}(\xi) \cdot h^*(c_{f,g}(\xi)) \tag{1.1.10}$$

Proof. □

The first two relations in Proposition 1.1.24, in the case of a normalized cleavage, can be written into the following simply form:

$$c_{f,\mathrm{id}_T} = \mathrm{id}_{f^*}, \quad c_{\mathrm{id}_S,f} = \mathrm{id}_{f^*}. \tag{1.1.11}$$

As for the third one, it is visualized by the commutativity of the diagram

$$\begin{tikzcd} h^*g^*f^*(\xi) \arrow[r, "c_{g,h}(f^*(\xi))"] \arrow[d, "h^*(c_{f,g}(\xi))"'] & (gh)^*(f^*(\xi)) \arrow[d, "c_{f,gh}(\xi)"] \\ h^*(fg)^*(\xi) \arrow[r, "c_{fg,h}"] & (fgh)^*(\xi) \end{tikzcd} \tag{1.1.12}$$

In the case of fibre categories (where the $c_{f,g}$ are isomorphisms), this commutativity can be expressed intuitively by the fact that the successive use of isomorphisms of the form $c_{f,g}$ does not lead to "contradictory identifications". We can also write this formula without the variable ξ, using the horizontal composition of homomorphisms of functors:

$$c_{fg,h} \cdot (h^* \circ c_{f,g}) = c_{f,gh} \cdot (c_{g,h} \circ f^*).$$

We now confine ourselves to normalized cloven categories. Any such category gives rise to the following objects:

(a) A map $S \mapsto \mathcal{F}_S$ from $\mathrm{Ob}(\mathcal{E})$ to **Cat**.

(b) A map $f \mapsto f^*$, associates each morphism $f : T \to S$ in $\mathcal{E}$ with a functor $f^* : \mathcal{F}_S \to \mathcal{F}_T$.

(c) A map $(f, g) \mapsto c_{f,g}$, associates each couple of morphisms f, g in $\mathcal{E}$ with a homomorphism of functors $c_{f,g} : g^* f^* \to (fg)^*$.

Moreover, these data satisfy the formule (1.1.11) and (1.1.12) given above. (N.B. If we had not limited ourselves to the case of a normalized cleavage, it would have been necessary to introduce an additional object, namely a function $S \mapsto \alpha_S$ which associates to any object S of $\mathcal{E}$ a functorial homomorphism $\alpha_S : (\mathrm{id}_S)^* \to \mathrm{id}_{\mathcal{F}_S}$, and condition (1.1.11) would then be replaced by condition (1.1.9)).

We are now going to show that how we can reconstruct (up to isomorphisms) the normalized cloven category $\mathcal{F}$ over $\mathcal{E}$ using the previous objects. For this, we introduce the notion of **pseudo-functors** from $\mathcal{E}^{\mathrm{op}}$ to **Cat**, which is a collection of objects satisfying conditions (a), (b), (c) (or more precisely, normalized pseudo-functors). From our preceding discussion, any normalized cloven category $\mathcal{F}$ over $\mathcal{E}$ defines a pseudo-functor $\mathcal{E}^{\mathrm{op}} \to \mathbf{Cat}$. Conversely, assume that we are given a pseudo-functor $\mathcal{E}^{\mathrm{op}} \to \mathbf{Cat}$, which associates an object $S \in \mathrm{Ob}(\mathcal{E})$ with a category $\mathcal{F}_S$. We put

$$\mathcal{F}_0 = \coprod_{S \in \mathrm{Ob}(\mathcal{E})} \mathrm{Ob}(\mathcal{F}_S),$$

which is the sum of the sets $\mathrm{Ob}(\mathcal{F}_S)$; there is an obvious map

$$p_0 : \mathcal{F}_0 \to \mathrm{Ob}(\mathcal{E}).$$

Let $\bar{\xi} = (S, \xi)$ and $\bar{\eta} = (T, \eta)$ be two elements of $\mathcal{F}_0$, where $\xi \in \mathrm{Ob}(\mathcal{F}_S)$, $\eta \in \mathrm{Ob}(\mathcal{F}_T)$, and let $f \in \mathrm{Hom}_{\mathcal{E}}(T, S)$; we set

$$h_f(\bar{\eta}, \bar{\xi}) = \mathrm{Hom}_{\mathcal{F}_T}(\eta, f^*(\xi)).$$

If $g : U \to T$ is another morphism in $\mathcal{E}$ and $\zeta \in \mathrm{Ob}(\mathcal{F}_U)$, then we define a composition law

$$\circ : h_f(\bar{\eta}, \bar{\xi}) \times h_g(\bar{\zeta}, \bar{\eta}) \to h_{fg}(\bar{\zeta}, \bar{\xi}),$$

i.e. a map

$$\mathrm{Hom}_{\mathcal{F}_T}(\eta, f^*(\xi)) \times \mathrm{Hom}_{\mathcal{F}_U}(\zeta, g^*(\eta)) \to \mathrm{Hom}_{\mathcal{F}_U}(\zeta, (fg)^*(\xi))$$

by the formula

$$u \circ v = c_{f,g}(\xi) \cdot g^*(u) \cdot v.$$

That is, $u \circ v$ is given by the composition

$$\zeta \xrightarrow{u} g^*(\eta) \xrightarrow{g^*(u)} g^* f^*(\xi) \xrightarrow{c_{f,g}(\xi)} (fg)^*(\xi) \ .$$

We then set $h(\bar{\eta}, \bar{\xi}) = \coprod_{f \in \mathrm{Hom}(T,S)} h_f(\bar{\eta}, \bar{\xi})$, so that the preceding composition law define a composition map

$$\circ : h(\bar{\eta}, \bar{\xi}) \times h(\bar{\zeta}, \bar{\eta}) \to h(\bar{\zeta}, \bar{\xi}).$$

The definition of $h(\bar{\eta}, \bar{\xi})$ gives a map

$$p_{\bar{\eta}, \bar{\xi}} : h(\bar{\eta}, \bar{\xi}) \to \mathrm{Hom}(T, S).$$

Proposition 1.1.25. *The set $\mathcal{F}_0$, togher with the $h(\bar{\eta}, \bar{\xi})$, give rise to a normalized cloven category $p : \mathcal{F}\mathcal{E}$.*

Proof. □

Example 1.1.26. Let $F : \mathcal{E}^{\mathrm{op}} \to \mathbf{Cat}$ be a functor. Then F can be considered as a pseudo-functor by setting

$$\mathcal{F}_S = F(S), \quad f^* = F(f), \quad c_{f,g} = \mathrm{id}_{(fg)^*},$$

so we can construct the fibre category $\mathcal{F}$ associated with F. For a cloven fibre category $\mathcal{F}$ over $\mathcal{E}$ to be isomorphic to that defined by a functor $F : \mathcal{E}^{\mathrm{op}} \to \mathbf{Cat}$, it is necessary and sufficient that the following condition is satisfied for any morphisms f, g of $\mathcal{E}$:

$$(fg)^* = g^* f^*, \quad c_{f,g} = \mathrm{id}_{(fg)^*}.$$

In terms of the set K of transport morhisms, this signifies that K is stable under composition. A cleavage of a category $\mathcal{F}$ over $\mathcal{E}$ satisfying this condition is said to be **splitting**, and a category $\mathcal{F}$ over $\mathcal{E}$ endowed with a splitting cleavage is called a **split fiber category over** $\mathcal{E}$. It is immediate that the category of split fiber categories over $\mathcal{E}$ is equivalent to $\mathcal{H}om(\mathcal{E}^{\mathrm{op}}, \mathbf{Cat})$.

If $\mathcal{F}$ is a fibre category over $\mathcal{E}$, there does not exists (in general) a split over $\mathcal{F}$. Suppose for example that $\mathrm{Ob}(\mathcal{E})$ and $\mathrm{Ob}(\mathcal{F})$ are both reduced to a singleton, and that the endomorphism sets are groups F and E, respectively. The functor p then corresponds to a group homomorphism $\phi : E \to F$, and is surjective if p is fibrant. We see that the cleavages of $\mathcal{F}$ over $\mathcal{E}$ corresponds to maps $s : E \to F$ such that $ps = \mathrm{id}_E$, and a cleavage over $\mathcal{F}$ is splitting if and only if s is a group homomorphism. Therefore, the fibre category $\mathcal{F}$ is split if and only if the extension F of the group E by $G = \ker \phi$ is trivial.

Suppose that $\mathcal{F}$ is a fibre category over $\mathcal{E}$ such that $\mathcal{F}_S$ are **rigid categories**, i.e. the automorphism groups of any object of $\mathcal{F}_S$ is trivial (in this case, $\mathcal{F}_S$ is also called a **setoid**). Then it is easy to see that there exists a splitting of $\mathcal{F}$ over $\mathcal{E}$. In fact, we first note that the question of the existence of a splitting is unchanged if we replaces $\mathcal{F}$ by an equivalent $\mathcal{E}$-category, which brings us to the case where $\mathcal{F}_S$ are *sets*. In this case, the pullback of an object $\xi \in \mathrm{Ob}(\mathcal{F}_S)$ under a morphism $f : T \to S$ in $\mathcal{E}$ is necessarily unique (if exists), so we conclude that there eixsts a unique cleavage of $\mathcal{F}$ over $\mathcal{E}$, which is necessarily splitting. In particular, we obtain an equivalence from the category of fibre categories in setoids over $\mathcal{E}$ to the category of functors $\mathcal{E}^{\mathrm{op}} \to \mathbf{Set}$.

Before we proceed further to give examples of fibre categories, we shall introduce here the notion of *cofibre* categories and bifibre categories. This is defined to be a category over $\mathcal{E}$ that has both pullbacks and pushouts. More precisely, consider a category $\mathscr{F}$ over $\mathcal{E}$, with structural functor $p : \mathcal{F} \to \mathcal{E}$. Then we also have a functor

$$p^{\mathrm{op}} : \mathcal{F}^{\mathrm{op}} \to \mathcal{E}^{\mathrm{op}}$$

on opposite categories. A morphism $\alpha : \eta \to \xi$ lying over $f : T \to S$ in $\mathcal{F}$ is said to be co-Cartesian if it is a Cartesian morphism in $\mathcal{F}^{\mathrm{op}}$ over $\mathcal{E}^{\mathrm{op}}$. Explicitly, this means for any object ξ' in $\mathcal{F}_S$, the map

$$\mathrm{Hom}_S(\xi, \xi') \to \mathrm{Hom}_f(\eta, \xi')$$

is bijective. In this case, we also say that (ξ, α) is a direct image of η under f, in the category $\mathcal{F}$. If such an image exists for any η in $\mathcal{F}_T$, we then obtain a direct image functor, denoted by $f_* : \mathcal{F}_T \to \mathcal{F}_S$ (as always, this depends on the choice of diect images); it is therefore defined by the isomorphism of bifunctors over $\mathcal{F}_T^{\mathrm{op}} \times \mathcal{F}_S$:

$$\mathrm{Hom}_S(f_*(\eta), \xi) \xrightarrow{\sim} \mathrm{Hom}_f(\eta, \xi).$$

If f_* exists, then for the inverse image functor f^* to exist, it is necessary and sufficient that f_* admits an adjoint functor, i.e. a functor $f^* : \mathcal{F}_S \to \mathcal{F}_T$ such that there is an isomorphism of bifunctors

$$\mathrm{Hom}_S(f_*(\eta), \xi) \xrightarrow{\sim} \mathrm{Hom}_T(\eta, f^*(\xi)).$$

Let $g : U \to T$ be another morphism in $\mathcal{E}$, and suppose that the inverse images and direct images exists for f, g and fg. Consider the functorial homomorphisms

$$c^{f,g} : (fg)_* \to f_*g_*, \quad c_{f,g} : g^*f^* \to (fg)^*.$$

We note that if we consider f_*g_* and g^*f^* as an adjoint pair, as well as $(fg)_*$ and $(fg)^*$, then the preceding homomorphisms are corresponded under adjunction. Therefore, one of them is an isomorphism if and only if the other is. In particular, we obtain the following result:

Proposition 1.1.27. *Suppose that the category $\mathcal{F}$ over $\mathcal{E}$ is prefibre and coprefibre. Then for it to be fibre (resp. cofibre), it is necessary and sufficient that it is cofibre.*

Of course, we say that $\mathcal{F}$ is coprefire (resp. cofibre) over $\mathcal{E}$ if $\mathcal{F}^{\mathrm{op}}$ is prefibre (resp. fibre) over $\mathcal{E}^{\mathrm{op}}$, and $\mathcal{F}$ is said to be **bifibre** if it is both fibre and cofibre over $\mathcal{E}$.

Example 1.1.28 (Arrow category over $\mathcal{E}$). Let $\mathcal{E}$ be a category. We define the **arrow category** over $\mathcal{E}$ as follows:

- Objects of $\mathbf{Arr}(\mathcal{E})$ are morphisms (arrows) in $\mathcal{E}$.
- If $f : T \to S$ and $g : Y \to X$ are objects of $\mathbf{Arr}(\mathcal{E})$, a morphism from f to g is defined to be a pair (u, v) of morphisms $u : Y \to T$, $v : X \to S$ such that the following diagram is commutative:

$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[d, "u"'] & X \arrow[d, "v"] \\ T \arrow[r, "f"] & S \end{tikzcd}$$

The functor which sends a morphism in $\mathcal{E}$ to its target then defines a canonical functor

$$p : \mathbf{Arr}(\mathcal{E}) \to \mathcal{E}.$$

For any object S of $\mathcal{E}$, the categorical fiber $\mathbf{Arr}(\mathcal{E})_S$ is canonically isomorphic to the category $\mathcal{E}_{/S}$ of objects lying over S.

Consider a morphism $f : T \to S$ in $\mathcal{E}$, which then correponds to a canonical functor

$$f_* : \mathcal{E}_{/T} = \mathbf{Arr}(\mathcal{E})_T \to \mathcal{E}_{/S} = \mathbf{Arr}(\mathcal{E})_S$$

and a functorial isomorphism

$$\mathrm{Hom}_S(f_*(\eta), \xi) \xrightarrow{\sim} \mathrm{Hom}_f(\eta, \xi)$$

which makes f_* a functor (called the **direct image functor** of f in $\mathbf{Arr}(\mathcal{E})$). Moreover, we have

$$(\mathrm{id}_S)_* = \mathrm{id}_{\mathbf{Arr}(\mathcal{E})_S}, \quad (fg)^* = f_*g_*, \quad c^{f,g} = \mathrm{id}_{fg},$$

so $\mathbf{Arr}(\mathcal{E})$ is endowed with a co-splitting over $\mathcal{E}$, and a fortiori is co-fibre over $\mathcal{E}$. We also note that by definition, a morphism

$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[d, "u"'] & X \arrow[d, "v"] \\ T \arrow[r, "f"] & S \end{tikzcd}$$

in $\mathbf{Arr}(\mathcal{E})$ is Cartesian if and only if this square is Cartesian in $\mathcal{E}$, i.e. Y is a fiber product of X and T over S. Therefore, the inverse image functor f^* exists if and only if fiber products exists in $\mathcal{E}$. It follows from Proposition 1.1.27 that if fiber products in $\mathcal{E}$, i.e. if $\mathcal{F}$ is prefibre over $\mathcal{E}$, then it is fibre over $\mathcal{E}$.

Example 1.1.29 (Category of presheaves over Top). Let $\mathcal{E} = \mathbf{Top}$ be the category of topological spaces. If T is a topological space, we denote by $\mathcal{U}(T)$ the category of open subsets of T, where the morphisms are induced by inclusions. If $\mathcal{C}$ is a category, a functor $F : \mathcal{U}(T)^{\mathrm{op}} \to \mathcal{C}$ is called a presheaf over T with values in $\mathcal{E}$, and a sheaf if it satisfies some additional exactness conditions. The category $\mathrm{PSh}(T)$ of presheaves over T with values in $\mathcal{C}$, is by definition the category $\mathcal{H}om(\mathcal{U}(T)^{\mathrm{op}}, \mathcal{C})$, and the category $\mathrm{Sh}(T)$ of sheaves over T with values in $\mathcal{T}$ is a full subcategory of $\mathrm{PSh}(T)$. If $f : T \to S$ is a morphism in $\mathcal{E}$, i.e. a continuous map of topological spaces, then it corresponds to a functor

$$\mathcal{U}(S) \to \mathcal{U}(T), \quad U \mapsto f^{-1}(U),$$

whence a functor

$$f_* : \mathrm{PSh}(T) \to \mathrm{PSh}(S)$$

called the direct image functor of presheaves under f. We also see that the direct image of a sheaf is a sheaf, so f_* induces a functor $f_* : \mathrm{Sh}(T) \to \mathrm{Sh}(S)$. Moreoever, by the associativity of the composition of the functors, we have, for a second map $g : U \to T$, the identity

$$(gf)_* = g_* f_*, \quad (\mathrm{id}_S)_* = \mathrm{id}_{\mathrm{PSh}(S)} .$$

Therefore we obtain a functor $S \mapsto \mathrm{PSh}(S)$ (resp. $S \mapsto \mathrm{Sh}(S)$) from the category $\mathcal{E}$ to **Cat**. The corresponding functor $S \mapsto \mathrm{PSh}(S)^{\mathrm{op}}$ (resp. $S \mapsto \mathrm{Sh}(S)^{\mathrm{op}}$) then defines a cofibre (and cosplitting) category over **Top**, called the **cofibre category of presheaves** (resp. **sheaves**) **with values in** $\mathcal{C}$, and denoted by PSh (resp. Sh). By the construction before Proposition 1.1.25, we see that a morphism of a presheaf G over T to a presheaf F over S is defined to be a couple (ϕ, f), where $f : T \to S$ is a continuous map and $\phi : F \to f_*(G)$ is a morphism in $\mathrm{PSh}(S)$. This description also applies to the morphisms of sheaves, since Sh is a full subcategory of PSh.

In most important cases, the categories PSh and Sh over $\mathcal{E}$ are also fibre categories, i.e., for any continuous map $f : T \to S$, the direct image functors $f_* : \mathrm{PSh}(T) \to \mathrm{PSh}(S)$ and $f_* : \mathrm{Sh}(T) \to \mathrm{Sh}(S)$ admit adjoints, which is then denoted by f^* and called the **inverse image functor** of presheaves (resp. sheaves) by the continuous f. The functor $f^* : \mathrm{PSh}(T) \to \mathrm{PSh}(S)$ exists as long as the category $\mathcal{C}$ admits inductive limits, so the problem is that for Sh. For example, if $\mathcal{C} = \mathbf{Set}$, then the inverse limit of a sheaf (considered as a presheaf) is in general not a sheaf, so we can not just hope that the inverse image functor of PSh restricts to Sh (despite their common notation f^*). Therefore, we see that Sh is a cofibre subcategory of PSh, but not a *fibre subcategory*, i.e. the inclusion functor $\mathrm{Sh} \to \mathrm{PSh}$ is not fibrant.

The cofibre category **PSh** can be deduced from a more general cofibre category. For this, we note that the association $\mathcal{U} \mapsto \mathcal{H}om(\mathcal{U}, \mathcal{C})$ is naturally a contravariant functor on $\mathcal{U}$, from the category **Cat** to **Cat**, so it defines a split fibre category over $\mathcal{E} = \mathbf{Cat}$, which we denote by $\mathbf{Cat}_{(-,\mathcal{C})}$. The objects of this category are the couples $(\mathcal{U}, p)$ of a category $\mathcal{U}$ and a functor $p : \mathcal{U} \to \mathcal{C}$, and a morphism of $(\mathcal{U}, p)$ to $(\mathcal{V}, q)$ is essentially a couple (f, u), where $f : \mathcal{U} \to \mathcal{V}$ is a functor and $u : p \to qf$ is a homomorphism of functors:

$$\begin{tikzcd} \mathcal{U} \arrow[rr, "f"] \arrow[dr, "p"'] \arrow[rr, phantom, shift right=4ex, "\overset{u}{\Rightarrow}"] & & \mathcal{V} \arrow[dl, "q"] \\ & \mathcal{C} & \end{tikzcd}$$

The composition of two morphisms $(f, u) : (\mathcal{U}, p) \to (\mathcal{V}, q)$ and $(g, v) : (\mathcal{V}, q) \to (\mathcal{W}, r)$ is given by the couple (fg, w), where $w : p \to rgf$ is given by the composition $w = (v \circ f) \cdot u$:

$$\begin{tikzcd} & \mathcal{V} \arrow[rr, "g"] \arrow[dr, "q"] \arrow[d, phantom, "\Uparrow u"] \arrow[rr, phantom, shift right=4ex, "\overset{v}{\Rightarrow}"] & & \mathcal{W} \arrow[dl, "r"] \\ \mathcal{U} \arrow[ur, "f"] \arrow[rr, "p"'] & & \mathcal{C} & \end{tikzcd}$$

The projection functor $\mathscr{F} = \mathbf{Cat}_{(-,\mathcal{C})} \to \mathcal{E}$ is then defined to associate a couple $(\mathcal{U}, p)$ with the object $\mathcal{U}$, and the categorical fiber $\mathcal{F}_{\mathcal{U}}$ over $\mathcal{U}$ is the category $\mathcal{H}om(\mathcal{U}, \mathcal{C})$. If $\mathcal{C}$ admits inductive limits, then the fibre category $\mathbf{Cat}_{(-,\mathcal{C})}$ over $\mathbf{Cat}$ is equally cofibre over $\mathbf{Cat}$, i.e. we can define the direct image of a functor $p : \mathcal{U} \to \mathcal{C}$ by a functor $f : \mathcal{U} \to \mathcal{V}$. The category PSh of presheaves is induced from $\mathcal{F}$ by the base change defined by the functor

$$\mathbf{Top}^{\mathrm{op}} \to \mathbf{Cat}, \quad S \mapsto \mathcal{U}(S)$$

which gives a fibre category on $\mathbf{Top}^{\mathrm{op}}$, and by passing to the opposite category, we obtain the cofibre category PSh of the presheaves over $\mathbf{Top}$. Note that under this identification, the inverse image (resp. direct image) of a functor corresponds to the direct image (resp. inverse image) of a presheaf.

Example 1.1.30. Let $\mathcal{F}$ be a category over $\mathcal{E}$ and S be an object of $\mathcal{E}$ acted by a group G. This object then corresponds to a functor $\lambda : \mathcal{E}' \to \mathcal{E}$ from category $\mathcal{E}'$ defined by G (with a single object and the endomorphism being G) to $\mathcal{E}$. By base change, we then obtain a category $\mathscr{F}'$ over $\mathcal{E}'$, which is fibre (resp. cofibre) if and only if $\mathcal{F}$ is fibre (resp. cofibre) over $\mathcal{E}$ (Proposition 1.1.14). A section of $\mathcal{E}'$ over $\mathcal{F}'$ (necessarily Cartesian, because $\mathcal{E}'$ is a gropoid and any isomorphism in $\mathcal{F}'$ is Cartesian) can also be interpreted as an $\mathcal{E}$-functor $\mathcal{E}' \to \mathcal{F}$ lying over λ, or also as an object ξ in $\mathcal{F}$ acted by G which is "lying over" the object S.

Example 1.1.31. If the base category $\mathcal{E}$ is reduced to two objects a, b and the only nontrivial morphisms are $f : a \to b$ and $g : b \to a$, which are inverses of each other, a normalized cloven category $\mathcal{F}$ over $\mathcal{E}$ is essentially the a system formed by two categories $\mathcal{F}_a$, $\mathcal{F}_b$ and an adjoint pair $G \dashv F : \mathcal{F}_a \to \mathcal{F}_b$ of functors, which are equivalences of categories. In fact, we can choose $\mathcal{F}_a$ and $\mathcal{F}_b$ to be the fibers of $\mathcal{F}$, the F, G are the functors f^* and g^*, and the two isomorphisms

$$u : FG \xrightarrow{\sim} \mathrm{id}_{\mathcal{F}_a}, \quad GF \xrightarrow{\sim} \mathrm{id}_{\mathcal{F}_b}$$

are $c_{g,f}$ and $c_{f,g}$. The usual compatibility conditions followws from (1.1.12) for the composition fgf and gfg.

An interesting case is the following: assume that we have

$$\mathcal{F}_b = \mathcal{F}_a^{\mathrm{op}}, \quad G = F^{\mathrm{op}}, \quad v = u^{\mathrm{op}}.$$

In general, a functor $D : \mathcal{C} \to \mathcal{C}^{\mathrm{op}}$ and an isomorphism $u : DD^{\mathrm{op}} \xrightarrow{\sim} \mathrm{id}_{\mathcal{C}}$ such that the isomorphism $u^{\mathrm{op}} : D^{\circ}D \xrightarrow{\sim} \mathrm{id}_{\mathcal{C}^{\mathrm{op}}}$ makes (D, D°) an adjoint pair is called an **autoduality** on $\mathcal{C}$.

Example 1.1.32. Let $\mathcal{E}$ be a discrete category, so that it is defined by a set $I = \mathrm{Ob}(\mathcal{E})$. Then a category $\mathcal{F}$ over $\mathcal{E}$ is given a family of categories $\mathcal{F}_i$ ($i \in I$), the categorical fibers, any category $\mathcal{F}$ over $\mathcal{E}$ is fibre, any $\mathcal{E}$-functor $\mathcal{F} \to \mathcal{G}$ is Cartesian, and we have a canonical isomorphism

$$\mathcal{H}om_{/\mathcal{E}}(\mathscr{F}, \mathcal{G}) \xrightarrow{\sim} \prod_i \mathcal{H}om(\mathscr{F}_i, \mathcal{G}_i).$$

In particular, the category $\mathbf{\Gamma}(\mathcal{F}/\mathcal{E}) = \mathbf{\Gamma}_{\mathrm{Cart}}(\mathcal{F}/\mathcal{E})$ is isomorphic to $\prod_i \mathcal{F}_i$.

Example 1.1.33. Suppose that $\mathcal{E}$ has exactly two objects S and T, and a unique nontrivial morphism $f : T \to S$. Then a category $\mathcal{F}$ over $\mathcal{E}$ is defined by two categories $\mathcal{F}_S$, $\mathcal{F}_T$ and a bifunctor $H(-,-)$ over $\mathcal{F}_T^{\mathrm{op}} \times \mathcal{F}_S$ with values in $\mathbf{Set}$. In fact, this bifunctor is given by $H(\eta, \xi) = \mathrm{Hom}_f(\eta, \xi)$, where $\eta \in \mathrm{Ob}(\mathcal{F}_T)$ and $\xi \in \mathrm{Ob}(\mathcal{F}_S)$. For the considered category to be fibre (or prefibre, which is the same), it is necessary and it is sufficient that the functor $H(\eta, -)$ be representable for any $\eta \in \mathrm{Ob}(\mathcal{F}_T)$, and for it to be cofibre, it is necessary and sufficient that the functor $H(-, \xi)$ be representable for any $\xi \in \mathrm{Ob}(\mathcal{F}_S)$.

1.1.5 Functors of fibre categories

Let $\mathcal{F}$ be a normalized cloven category over $\mathcal{E}$. For any object S of $\mathcal{E}$, we denote by $i_S : \mathcal{F}_S \to \mathcal{F}$ the inclusion functor. We then have a functorial homomorphism, for any morphism $f : T \to S$ in $\mathcal{E}$:

$$\alpha_f : i_T f^* \to i_S,$$

where f^* is the inverse image functor $\mathcal{F}_S \to \mathcal{F}_T$ for f defined by the cleavage. Now let $F : \mathcal{F} \to \mathcal{C}$ be a functor from $\mathcal{F}$ into a category $\mathcal{C}$. For $S \in \mathrm{Ob}(\mathcal{E})$, we set

$$F_S : F \circ i_S : \mathcal{F}_S \to \mathcal{C}$$

and for any morphism $f : T \to S$ in $\mathcal{E}$, we define

$$\varphi_f : F \circ \alpha_f : F_T f^* \to F_S.$$

We therefore obtain a family (F_S) of functors $\mathcal{F}_S \to \mathcal{C}$, and a family (φ_f) of homomorphisms of functors $F_T f^* \to F_S$. This family satisfies the following conditions:

(a) $\varphi_{\mathrm{id}_S} = \mathrm{id}_{F_S}$.

(b) For two morphisms $f : T \to S$ and $g : U \to T$ in $\mathcal{E}$, we have a commutative diagram

$$\begin{tikzcd}
F_U g^* f^* \arrow[r, "F_U \circ c_{f,g}"] \arrow[d, "\varphi_g \circ f^*"] & F_U(fg)^* \arrow[d, "\varphi_{fg}"] \\
F_T f^* \arrow[r, "\varphi_f"] & F_S
\end{tikzcd}$$

In fact, the fitst relation is trivial, and the second one is obtained by applying F to the commutative diagram

$$\begin{tikzcd}
g^* f^*(\xi) \arrow[r, "c_{f,g}(\xi)"] \arrow[d, "\alpha_g(f^*(\xi))"] & (fg)^*(\xi) \arrow[d, "\alpha_{fg}(\xi)"] \\
f^*(\xi) \arrow[r, "\varphi_f"] & \xi
\end{tikzcd}$$

for any object ξ in $\mathcal{F}_S$.

If $G : \mathscr{F} \to \mathcal{C}$ is another functor, given by functors $G_S : \mathcal{F}_S \to \mathcal{C}$ and the functorial homomorphisms $\psi_f : G_T f \to G_S$, and if $u : F \to G$ is a functorial homomorphism, then there is a corresponding homomorphism $u_S = u \circ i_S : F_S \to G_S$, and for any morphism $f : T \to S$ in $\mathcal{E}$, we have a commutative diagram

$$\begin{tikzcd}
F_T f^* \arrow[r, "\phi_f"] \arrow[d, "u_T \circ f^*"'] & F_S \arrow[d, "u_S"] \\
G_T f^* \arrow[r, "\psi_f"] & G_S
\end{tikzcd} \tag{1.1.13}$$

Proposition 1.1.34. *Let $\mathcal{H}(\mathcal{F}, \mathcal{C})$ be the category of couples of familys $(F_S)_{S \in \mathrm{Ob}(\mathcal{F})}$ of functors $\mathcal{F}_S \to \mathcal{C}$ and the families $(\varphi_f)_{f \in \mathrm{Arr}(\mathcal{F})}$ of functorial homomorphisms $F_T f^* \to F_S$, satisfying conditions (a) and (b), with morphisms being the families $(u_S)_{S \in \mathrm{Ob}(\mathcal{F})}$ of homomorphisms $F_S \to G_S$ such that the diagram (1.1.13) commutes. Then there is an isomorphism from $\mathcal{H}om(\mathcal{F}, \mathcal{C})$ to the category $\mathcal{H}(\mathcal{F}, \mathcal{C})$.*

Proof. □

1.2 Faithfully flat descent

The notion "descent" provides a general framework for "glueing" process of objects, and hence the "glueing" of categories. The most classical case of glueing is giving a topological space X and an open covering (X_i) of X; if for each i we are given a fiber space (say) E_i over X_i, and for each pair (i,j) an isomorphism f_{ji} from $E_i|_{X_{ij}}$ to $E_j|_{X_{ij}}$ (where $X_{ij} = X_i \cap X_j$), satisfying a transitive condition (that is, $f_{kj}f_{ji} = f_{ki}$), then we know that there exists a unique (up to isomorphism) space E over X, defined by the condition that we have isomorphisms $f_i : E|_{X_i} \to E_i$, satisfying the relations $f_{ji} = f_j f_i^{-1}$. Let X' be the sum of X_i, which is a fiber space over X (endowed with the continous map $X' \to X$). The data of the spaces E_i then can be interpreted as a fiber space E' over X', and the isomorphisms f_{ji} give an isomorphism of the inverse images E_1'' and E_2'' of E' (under the canonical projection) over $X'' = X \times_X X'$. The gluing condition can then be written as an identity between isomorphisms of the fiber spaces E_1''' and E_3''' over the triple fiber product $X''' = X' \times_X X' \times_X X'$ (where E_i''' is the inverse image of E' over X''' under the canonical projection of index i). The construction of E from E_i and f is a typical case of a "descent" process.

1.2.1 Universally effective morphisms

Let $\mathcal{C}$ be a fixed category. We recall that a morphism $u : T \to S$ is called an **epimorphism** if, for any object $T \in \mathcal{C}$, the corresponding map

$$X(S) = \mathrm{Hom}(S, X) \to X(T) = \mathrm{Hom}(T, X)$$

is injective. We say that u is a **universal epimorphism** if for any morphism $S' \to S$, the fiber product $T' = T \times_S S'$ exists and $u' : T' \to S'$ is an epimorphism.

Definition 1.2.1. A diagram

$$A \xrightarrow{u} B \underset{v_2}{\overset{v_1}{\rightrightarrows}} C$$

of maps of sets is called **exact** if u is injective and its image is formed by the elements $b \in B$ such that $v_1(b) = v_2(b)$. A diagram of the same type in $\mathcal{C}$ is called **exact** if for any object X of $\mathcal{C}$, the corresponding diagram of sets

$$A(X) \longrightarrow B(X) \rightrightarrows C(X)$$

is exact; we also say that (A, u) is a kernel of the couple (v_1, v_2). Dually, a diagram

$$C \underset{v_2}{\overset{v_1}{\rightrightarrows}} B \xrightarrow{u} A$$

in $\mathcal{C}$ is called exact if it is exact as a diagram in the opposite category $\mathcal{C}^{\mathrm{op}}$, i.e. if for any object X of $\mathcal{C}$, the corresponding diagram of sets

$$X(A) \longrightarrow X(B) \rightrightarrows X(C)$$

is exact. We also say that (A, u) is a cokernel of the couple (v_1, v_2).

Definition 1.2.2. A morphism $u : T \to S$ is called an effective morphism effective epimorphism if the fiber product $T \times_S T$ exist, and if the diagram

$$T \times_S T \underset{\mathrm{pr}_2}{\overset{\mathrm{pr}_1}{\rightrightarrows}} T \xrightarrow{u} S$$

is exact, i.e. if u is a cokernel of $(\mathrm{pr}_1, \mathrm{pr}_2)$. We say that u is a **universally effective epimorphism** if for any morphism $S' \to S$, the fiber product $T' = T \times_S S'$ exists and the morphism $u' : T' \to S'$ is an effective epimorphism.

A morphism $u : T \to S$ is called **squarable** if for any morphism $S' \to S$, the fiber product $T \times_S S'$ exists. The following lemma is trivial from the above definitions:

Lemma 1.2.3. *Consider morphisms $U \xrightarrow{v} T \xrightarrow{u} S$, then*

(a) *u, v epimorphism $\Rightarrow$ uv epimorphism $\Rightarrow$ u epimorphism.*

(b) *u, v universal epimorphism $\Rightarrow$ uv universal epimorphism $\Rightarrow$ u universal epimorphism.*

Corollary 1.2.4. *Let $u : X \to Y$ and $u' : X' \to Y'$ be universal epimorphisms such that $Y \times Y'$ exists. Then $X \times X'$ exists and $u \times u' : X \times X' \to Y \times Y'$ is a universal epimorphism.*

Definition 1.2.5. We say that an object S of $\mathcal{C}$ is squarable if its product by any object of $\mathcal{C}$ exists (if $\mathcal{C}$ has a final object e, this means $S \to e$ is squatable).

Lemma 1.2.6. *Let $u : X \to Y$ be a morphism in $\mathcal{C}_{/S}$; for this to be an epimorphism (resp. universal epimorphism, resp, effective epimorphism, resp. universally effective morphism), it is sufficient that the corresponding morphism in $\mathcal{C}$ is, and this is also necessary if S is a squarable object in $\mathcal{C}$.*

1.2.2 Descent morphisms

Let $f : S' \to S$ be a morphism such that $S'' = S' \times_S S'$ exists, and let $u' : X' \to S'$ be an object over S'. A **glueing datum** over X'/S', relative to f, is defined to be an S''-isomorphism

$$\phi : X''_1 \xrightarrow{\sim} X''_2$$

where X''_i $(i = 1, 2)$ denotes the inverse image (assumed to exist) of X'/S' under the projection $\mathrm{pr}_i : S'' \to S'$. We say the the glueing datum ϕ is a **descent datum** if it satisfies the cocycle condition:

$$\mathrm{pr}^*_{3,1}(\phi) = \mathrm{pr}^*_{3,2}(\phi)\mathrm{pr}^*_{2,1}(\phi)$$

where $\mathrm{pr}_{i,j}$ $(1 \leq j < i \leq 3)$ are the canonical projections of $S''' = S' \times_S S' \times_S S'$ to S'' (we assume that S''' exists), and $\mathrm{pr}^*_{i,j}(\phi)$ is the inverse image of ϕ, considered as an S'''-morphism from X'''_j to X'''_i, and for any integer $1 \leq k \leq 3$, X'''_k denotes the inverse image (assumed to exist) of X'/S' under the canonical projection $q_k : S''' \to S'$.

Let $f : S' \to S$ be a morphism such that $S'' = S' \times_S S'$ exists, and X be an object over X such that $X' = X \times_S S'$ and $X'' = X \times_S S''$ exist. Then the inverse images of X' by pr_i $(i = 1, 2)$ exist and are canonically isomorphic, so X'/S' is endowed with a canonical glueing datum relative to f. If S''' and $X''' = X \times_S S'''$ exist, then this is also a descent datum. If Y is another object over S satisfying the same conditions, then for any S-morphism $X \to Y$, the corresponding S'-morphism $X' \to Y'$ is *compatible* with the canonical glueing datum over X', Y'. If in particular $S' \to S$ is a squarable morphism, then

$$X \mapsto X' = X \times_S S'$$

is a functor from the category $\mathcal{C}_{/S}$ to the category of objects over S' endowed with a descent datum relative to f. We say that $f : S' \to S$ is a **descent morphism** (resp. an **effective descent morphism**) if f is squarable (i.e. for any $X \to S$, the fiber product $X' = X \times_S S'$ exists) and if the preceding functor $X \mapsto X' = X \times_S S'$ from $\mathcal{C}_{/S}$ to the category of objects over S' endowed with a descent datum relative to f, is fully faithfully (resp. an quivalence of categories).

We note that the first definition only concerns the notion of glueing (hence do not intervene the fiber product S'''): f is a descent morphism if f is squarable and $X \mapsto X'$ is a fully faithful functor from $\mathcal{C}_{/S}$ to the category of objects over S' endowed with a glueing datum relative to

f. To make this definition explicit, we note that it means that for two objects X, Y over S, the following diagram of sets

$$\mathrm{Hom}_S(X,Y) \longrightarrow \mathrm{Hom}_{S'}(X',Y') \underset{p_2}{\overset{p_1}{\rightrightarrows}} \mathrm{Hom}_{S''}(X'',Y'') \tag{1.2.1}$$

is exact, where $p_i(u')$ denotes the inverse image of $u' \in \mathrm{Hom}_{S'}(X',Y')$ by the projection $\mathrm{pr}_i : S'' \to S'$, for $i = 1, 2$. In fact, the kernel of the couple (p_1, p_2) is by definition the set of S'-morphisms $X' \to Y'$ compatible with the canonical glueing datum.

Note that, by the definition of inverse images Y', Y'', we have canonical bijections

$$\mathrm{Hom}_{S'}(X',Y') \cong \mathrm{Hom}_S(X',Y), \quad \mathrm{Hom}_{S''}(X'',Y'') \cong \mathrm{Hom}_S(X'',Y),$$

so the exactness of (1.2.1) is equivalent to that of

$$\mathrm{Hom}_S(X,Y) \longrightarrow \mathrm{Hom}_S(X',Y) \rightrightarrows \mathrm{Hom}_S(X'',Y) \tag{1.2.2}$$

obtained by applying $\mathrm{Hom}_S(-,Y)$ to the diagram in $\mathcal{C}_{/S}$:

$$X'' \rightrightarrows X' \longrightarrow X \tag{1.2.3}$$

which is deduced from

$$X'' \rightrightarrows X' \longrightarrow X$$

by base change $X \to S$. This proves, in view of Lemma 1.2.6, the first part of the following proposition:

Proposition 1.2.7. *If $f : S' \to S$ is a universally effective morphism, then it is a descent morphism, and the converse is also true if S is a squarable object in $\mathcal{C}$.*

Proof. It remains to prove that if f is a descent morphism, then is it a universally effective morphism, if S is squarable. This means for any $X \to S$, we muse show that the diagram (1.2.3) is exact, i.e. for any object Z of $\mathcal{C}$, the diagram of sets obtained by applying $\mathrm{Hom}(-,Z)$ is exact. Now by hypothesis $Z \times S$ exists; let Y be the corresponding object in $\mathcal{C}_{/S}$, then the transformed diagram of (1.2.3) by $\mathrm{Hom}(-,Z)$ is isomorphic to that by $\mathrm{Hom}_S(-,Y)$, and the latter is exact by our hypothesis on f. □

We can therefore apply the results of descent morphisms to universally effective epimorphisms, such as the following:

Proposition 1.2.8. *Let $f : S' \to S$ be a descent morphism (for example a universally effective epimorphism), then*

(a) *For any S-morphism $u : X \to Y$, u is an isomorphism (resp. a monomorphism) if and only if $u' : X' \to Y'$ is.*

(b) *Let X, Y be two sub-objects of S and X', Y' be the sub-objects of S' obtained by base change. For that X is dominated by Y (resp. equal to Y), it is necessary and sufficient that so is for X', Y'.*

Proof. For (a), it follows by definition that if u' is an isomorphism in the category of glueing objects, then u is an isomorphism. But we also note that any isomorphism of objects over S', compatible with glueing datum, is an isomorphism of glueing object, i.e. its inverse is also compatible with glueing datum. For (b), we are reduced to proving that if X' is dominated by Y', i.e. there exists an S'-morphism $X' \to Y'$, then the same is true for X, Y over S. Now as $Y' \to S'$, hence also $Y'' \to S''$, is a monomorphism, we see that $X' \to Y'$ is automatically compatible with the glueing datum, hence provides an S-morphism $X \to Y$. □

Corollary 1.2.9. *Let $f : S' \to S$ be a univerally effective epimorphism and $g : S \to T$ be a morphism such that $S \times_T S$ exists. Suppose that $S'' = S' \times_S S'$ is isomorphic to $S' \times_T S'$, then $g : S \to T$ is a monomorphism.*

Proof. In fact, consider the Cartesian diagram

$$\begin{tikzcd} S' \times_S S' \arrow[r, "\cong"] \arrow[d] & S' \times_T S' \arrow[d] \\ X \arrow[r] & S \times_T S \end{tikzcd}$$

In view of ([?], IV, 1.9), the morphism $S' \times_T S' \to S \times_T S$ is a universally effective epimorphism, so by Proposition 1.2.8 the diagonal morphism $S \to S \times_T S$ is also an isomorphism, which means $g : S \to T$ is a monomorphism. □

1.2.3 Descent of quasi-coherent modules

Let **Sch** be the category of schemes. We consider the category $\mathcal{F}$ of couples $(X, \mathscr{F})$, where X is a scheme and $\mathscr{F}$ is an $\mathscr{O}_X$-module over X. A morphism from $(X, \mathscr{F})$ to $(Y, \mathscr{G})$ is defined to be a couple (f, φ), where $f : X \to Y$ is a morphism of schemes and $\varphi : \mathscr{G} \to f_*(\mathscr{F})$ is a homomorphism. This category can be considered as a fibre category over **Sch**, where the inverse image functor of a morphism $f : X \to Y$ in **Sch** is given by the usual pullback functor of modules defined by f. As the inverse image of a quasi-coherent module is quasi-coherent, we see that the full subcategory formed by couples $(X, \mathscr{F})$, wherer $\mathscr{F}$ is a quasi-coherent $\mathscr{O}_X$-module, is a fibre subcategory. This category is called the **fibre category of quasi-coherent modules over schemes**, and denoted by **Qcoh**. By definition, the structural functor $p :$ **Qcoh** $\to$ **Sch** sends a couplt $(X, \mathscr{F})$ to its base scheme X, and the categorical fiber of **Qcoh** at an object X in **Sch** is the *opposite* category $\mathbf{Qcoh}(X)^{\mathrm{op}}$.

Recall that a morphism $f : X \to Y$ of ringed spaces is called **faithfully flat** if it is flat and surjective, and **quasi-compact** if the inverse image of any quasi-compact open is quasi-compact. By

Theorem 1.2.10. *Let* **Qcoh** *be the fibre category of quasi-coherent modules over schemes and $g : S' \to S$ be a faithfully flat and quasi-compact morphism of schemes. Then g is an effective descent morphism for* **Qcoh**.

1.3 Grothendieck topologies and sheaves

1.3.1 Topologies and basis

Let $\mathcal{C}$ be a category. A full subcategory $\mathcal{D}$ of $\mathcal{C}$ is called a **sieve** of $\mathcal{C}$ if it satisfies the following property: any object of $\mathcal{C}$ such that there exists a morphism from this object to an object of $\mathcal{D}$ is in $\mathcal{D}$. If X is an object of $\mathcal{C}$, then the sieves of the category $\mathcal{C}_{/X}$ are also called **sieves of X**.

Let $\mathscr{U}$ be a universe such that $\mathcal{C}$ is a $\mathscr{U}$-category, and $\mathrm{PSh}(\mathcal{C})$ be the corresponding category of presheaves. For any sieve of X, we can define a sub-object of X in $\mathrm{PSh}(\mathcal{C})$ by associating any object Y in $\mathcal{C}$ with the set of morphisms $f : Y \to X$ such that the object (Y, f) of $\mathcal{C}_{/X}$ belongs to the sieve. In this way, we obtain a correspondence between sieves and sub-objects:

Proposition 1.3.1. *The association defined above is a correspondence between the set of sieves of X and the set of sub-objects of X in* $\mathrm{PSh}(\mathcal{C})$.

Proof. It suffices to establish a converse process from sub-objects of X in $\mathrm{PSh}(\mathcal{C})$ to sieves of X. For this, we associate each sub-functor R of X with the category $\mathcal{C}_{/R}$ of objects lying over R. Since $R \subseteq X$, it is immediate that $\mathcal{C}_{/R}$ is a sieve of $\mathcal{C}_{/X}$, whose corresponding sub-object of X is R. □

Let $\mathcal{C}$ be a $\mathscr{U}$-category. In view of category sieve and sub-object correspondence, we shall identify sieves of X with sub-objects of X in the category $\mathrm{PSh}(\mathcal{C})$. In this way, for any presheaf F and any sieve R of X, we can then define $\mathrm{Hom}_{\mathrm{PSh}(\mathcal{C})}(R, F)$ to be the set of morphisms of functors $R \to F$, and we have a functorial isomorphism ([?], I, 3.5)

$$\mathrm{Hom}_{\mathrm{PSh}(\mathcal{C})}(R, F) \xrightarrow{\sim} \varprojlim_{Y \in \mathcal{C}_{/R}} F(Y). \tag{1.3.1}$$

Therefore, Proposition 1.3.1 allows us to transport the usual operations on functors to sieves; here are a few examples.

- **Base change**. Let R be a sieve of X and $f : Y \to X$ be a morphism in $\mathcal{C}$. The fiber product $R \times_X Y$ is then a sieve of Y, which is called the **base change** of R to Y. The corresponding subcategory of $\mathcal{C}_{/Y}$ is then the inverse image of the subcategories of $\mathcal{C}_{/X}$ defined by R under the morphism f.

- **Order relation, intersection, union**. The inclusion of sub-functors of X is an order relation, which then defines an order relation on sieves of X. We define the union and intersection of a family of sieves as the supremum and infimum of the corresponding family of sub-presheaves.

- **Image, generated sieve**. Let $(F_\alpha)_{\alpha \in I}$ be a family of presheaves and $f_\alpha : F_\alpha \to X$ be a morphism for each α, where X is an object of $\mathcal{C}$. We define the image of this family of morphisms to be the union of the images of f_α, which is then a sieve of X. In particular, if the F_α are objects of $\mathcal{C}$, the image sieve is called the **sieve generated by the morphisms $\boldsymbol{f_\alpha}$**. The corresponding subcategory of $\mathcal{C}_{/X}$ is by definition the full subcategory of $\mathcal{C}_{/X}$ formed by objects $Y \to X$ such that there exists an X-morphism of Y into one of the F_α.

The use of sieves allows us to define a topology on a category $\mathcal{C}$. Roughly speaking, this amounts to associate for each object X of $\mathcal{C}$ a class of sieves of X. A sieve of X in this class can then be considered as a *covering* of X, or a way to "localize" X.

Definition 1.3.2. Let $\mathcal{C}$ be a category. A **topology** $\mathcal{T}$ on $\mathcal{C}$ is the assignment to each object X of $\mathcal{C}$ of a set $\mathcal{T}(X)$ of sieves of X, so that the following conditions are satisfied:

(T1) (**Stable under base change**). For any object X of $\mathcal{C}$, any sieve $R \in \mathcal{T}(X)$ and any morphism $f : X \to Y$ in $\mathcal{C}$, the sieve $R \times_X Y$ of Y belongs to $\mathcal{T}(Y)$.

(T2) (**Local characterization**). Let R and R' be two sieves of X and assume that $R \in \mathcal{T}(X)$. If for any $Y \in \mathrm{Ob}(\mathcal{C})$ and any morphism $Y \to R$, the sieve $R' \times_X Y$ belongs to $\mathcal{T}(Y)$, then R' belongs to $\mathcal{T}(X)$.

(T3) For any object X of $\mathcal{C}$, X belongs to $\mathcal{T}(X)$.

The sieves in $\mathcal{T}(X)$ are called **covering sieves of $\boldsymbol{X}$**, or simply the **coverings of $\boldsymbol{X}$**. If $R \in \mathcal{T}(X)$, the inclusion $R \hookrightarrow X$ is also called a **refinement** of X. From the above axioms, we immediately deduce that the set of coverings of X is stable under finite intersection and that any sieve containing a covering sieve is a covering sieve. In fact, if R' contains a covering sieve R of X, then for any morphism $Y \to R$, we have the following Cartesian diagram

$$\begin{tikzcd}
R \times_X Y \arrow[r, hook] \arrow[d] & R' \times_X Y \arrow[r, hook] \arrow[d] & Y \arrow[d] \\
R \arrow[r, equal] \arrow[d, equal] & R \arrow[r, equal] \arrow[d, hook] & R \arrow[d, hook] \\
R \arrow[r, hook] & R' \arrow[r, hook] & X
\end{tikzcd}$$

from which we conclude that $R \times_X Y$ (and a fortiori $R' \times_X Y$) is equal to Y, so $R' \in \mathcal{T}(X)$ by axiom (T2). The set $\mathcal{T}(X)$ of covering sieves of X, ordered by the inclusion relation, is therefore cofiltered.

Let $\mathcal{C}$ be a category and $\mathcal{T}, \mathcal{T}'$ be topologies over $\mathcal{C}$. The topology $\mathcal{T}$ is called **finer than** $\mathcal{T}'$ (or equivalently, $\mathcal{T}'$ is **coarser than** $\mathcal{T}$) if for any object X of $\mathcal{C}$, any covering of X for the topology $\mathcal{T}'$ is also a covering for the topology $\mathcal{T}$. In this way, we define an order relation on the set of topologies over $\mathcal{C}$.

Example 1.3.3. Let $(\mathcal{T}_i)_{i\in I}$ be a family of topologies over $\mathcal{C}$. Then for any object X of $\mathcal{C}$, the intersection of the sets $\mathcal{T}_i(X)$ is easily verified to satisfy the axioms (T1), (T2) and (T3), so it defines a topology $\mathcal{T}$ over $\mathcal{C}$, called the **intersection** of the $\mathcal{T}_i$. This is the finest topology on $\mathcal{C}$ that coarser than any of the $\mathcal{T}_i$, and is clearly the infimum of the topologies $(\mathcal{T}_i)_{i\in I}$. On the other hand, the family $(\mathcal{T}_i)_{i\in I}$ also admits a supremum: the intersection of the topologies than are finer than each of the $\mathcal{T}_i$.

Example 1.3.4. The topology $\mathcal{T}$ such that $\mathcal{T}(X)$ is the set of sieves of X, is clearly the finest topology on $\mathcal{C}$, which is called the **discrete topology** on $\mathcal{C}$. On the other hand, the coarsest topology on $\mathcal{C}$ is given by $\mathcal{T}(X) = \{X\}$ for any object X of $\mathcal{C}$, which is called the **trivial topology** on $\mathcal{C}$.

A category $\mathcal{C}$, endowed with a topology, is called a **site**. The category $\mathcal{C}$ is called the underlying category of the site.

Definition 1.3.5. Let $\mathcal{C}$ be a site and X be an object of $\mathcal{C}$. A family $\{f_\alpha : X_\alpha \to X\}$ is called a **covering of X** if the sieve generated by the f_α is a covering sieve of X.

Let $\mathcal{C}$ be a category. If for each object X of $\mathcal{C}$ we are given a set of families of morphisms with target X, then there exists a coarsest topology $\mathcal{T}$ on $\mathcal{C}$ for which the given families of morphisms are coverings, namely the intersection of all these topologies. This topology is called the **topology generated by the set of families of morphisms**. In general, it is difficult to describe all the coverings in this topology, but the situation is this topology is generated by a *basis*:

Definition 1.3.6. Let $\mathcal{C}$ be a category. A **basis** (for a topology) on $\mathcal{C}$ is the assignment to each object X of $\mathcal{C}$ of a set $\mathrm{Cov}(X)$ of families of morphisms with target X (called **coverings** of X), which satisfies the following conditions:

(PT1) (**Stable under base change**). For any object X of C, any covering $\{X_\alpha \to X\}$ of X, and any morphism $Y \to X$ in $\mathcal{C}$, the fiber products $X_\alpha \times_X Y$ exists in $\mathcal{C}$ and the family $\{X_\alpha \times_X Y \to Y\}$ is a covering of X.

(PT2) (**Stable under composition**). If $\{X_\alpha \to X\}$ is a covering of X and for each α, $\{X_{\beta\alpha} \to X_\alpha\}$ is a covering of X_α, then the composite family $\{X_{\beta\alpha} \to X_\alpha \to X\}$ is a covering of X.

(PT3) The family $\{\mathrm{id}_X : X \to X\}$ is a covering of X.

For any given basis on $\mathcal{C}$, we can consider the topology over $\mathcal{C}$ generated by this basis. Note that if $\mathcal{C}$ is a category where fiber products exist, then any topology $\mathcal{T}$ of $\mathcal{C}$ can be defined by a basis, namely the one for which $\mathrm{Cov}(X)$ is formed by the covering families of X for the topology $\mathcal{T}$.

Proposition 1.3.7. *Let $\mathcal{C}$ be a category, $\mathcal{B}$ be a basis on $\mathcal{C}$, $\mathcal{T}$ the topology generated by $\mathcal{B}$, X an object of $\mathcal{C}$. Denote by $\mathcal{T}_{\mathcal{B}}(X)$ the set of sieves of X generated by the families of morphisms of the basis, and by $\mathcal{T}(X)$ the set of covering sieves of X for the topology $\mathcal{T}$. Then $\mathcal{T}_{\mathcal{B}}(X)$ is cofinal in $\mathcal{T}(X)$, in other words, for a sieve R of X to belong to $\mathcal{T}(X)$, it is necessary and sufficient that there exists a sieve R' of $\mathcal{T}_{\mathcal{B}}(X)$ such that $R' \subseteq R$.*

Proof. For any object X of $\mathcal{C}$, let $\mathcal{J}(X)$ be the set of sieves of X that contain a sieve of $\mathcal{T}_{\mathcal{B}}(X)$. We have evidently $\mathcal{J}(X) \subseteq \mathcal{T}(X)$, and to prove the converse inclusion, it suffices to show that the sets $\mathcal{J}(X)$ defines a topology over $\mathcal{C}$.

It is clear that the $\mathcal{J}(X)$ satisfy the axioms (T1) and (T3), and it remains to verify (T2). To this end, let R', R be sieves of X with $R \in \mathcal{T}_{\mathcal{B}}(X)$ such that for any morphism $Y \to R$, the sieve $R' \times_X Y$ of Y belongs to $\mathcal{J}(Y)$. By definition, R contains a sieve T which is generated by a covering $\{X_\alpha \to X\}$ of X. For each α, we have a canonical morphism $X_\alpha \to T$, defined by

$$\mathrm{Hom}(Y, X_\alpha) \to T(Y), \quad (Y \to X_\alpha) \mapsto (Y \to X_\alpha \to X),$$

so by our hypothesis the fiber product sieve $R' \times_X X_\alpha$ belongs to $\mathcal{J}(X_\alpha)$, hence contains a sieve generated by a covering $\{X_{\beta\alpha} \to X_\alpha\}$ of X_α. Unwinding the definitions, this means for any α and any morphism $f : Y \to X_{\beta\alpha} \to X_\alpha$, there exists a morphism $g : Y \to X$ in $R'(Y)$ such that g is equal to the composition of f with $X_\alpha \to X$. We therefore deduce that R' contains the sieve generated by the composite family $\{X_{\beta\alpha} \to X\}$, so by axiom (PT2), R' contains a sieve of $\mathcal{J}_{\mathcal{B}}(X)$ and hence belongs to $\mathcal{J}(X)$. □

In practice, we usually give basis to generate a topology on the category $\mathcal{C}$. In this case, by a topology, we in fact mean the topology generated by the basis defined by these arrows[1]. Here are some examples of topologies generated by basis, where for a family $\{f_\alpha : X_\alpha \to X\}$ of maps is called **jointly surjective** if the induced map $\coprod_\alpha X_\alpha \to X$ is surjective (in other words, the union of the images of f_α equal to X).

Example 1.3.8 (The site of a topological space). Let X be a topological space and let X_{cl} be the category of open subsets of X, with morphisms given by inclusions. Then we get a topology on X_{cl} by associating with each open subset $U \subseteq X$ the set of open coverings of U, whence a generated topology on X_{cl}. In this case, if $U_1 \to U$ and $U_2 \to U$ are arrows, the fiber product $U_1 \times_U U_2$ is the intersection $U_1 \cap U_2$.

Example 1.3.9 (The global classical topology). Let $\mathcal{C} = \mathbf{Top}$ be the category of topological spaces. If U is a topological space, then a covering of U will be a jointly surjective collection of open embeddings $U_i \to U$.

Example 1.3.10 (The small étale site for a scheme). Let X be a scheme and consider the full subcategory $X_{\text{ét}}$ of $\mathbf{Sch}_{/X}$, consisting of étale morphisms $U \to X$. Note that any morphism $U \to V$ of objects in $X_{\text{ét}}$ is necessarily étale, and a covering of $U \to X$ is a jointly surjective collection of morphisms $U_i \to U$.

Example 1.3.11 (The topologies on $\mathbf{Sch}_{/S}$). Let $\mathbf{Sch}_{/S}$ be the category of schemes over a fixed scheme S. We can define several topologies on $\mathbf{Sch}_{/S}$:

- The **Zariski topology** on $\mathbf{Sch}_{/S}$ is defined by collections of open coverings $\{U_i \to U\}$ of U.
- The **global étale topology** on $\mathbf{Sch}_{/S}$ is defined by jointly surjective collections of étale morphisms in $\mathbf{Sch}_{/S}$.
- The **fppf topology** on $\mathbf{Sch}_{/S}$ is defined by jointly surjective collections of flat morphisms that are locally of finite presentation (the abbreviation "fppf" stands for "fidèlement plat et de présentation finie").

[1] In fact, many authors define a Grothendieck topology to be a collection of families of morphisms satisfying axioms (PT1), (PT2) and (PT3).

1.3.2 Sheaves over a site

Let $\mathcal{C}$ be a site whose underlying category is a $\mathcal{U}$-category and $F : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ be a presheaf over $\mathcal{C}$. The functor F is called **separated** (resp. a **sheaf**) if for any object X of $\mathcal{C}$ and any covering sieve R of X, the map

$$\mathrm{Hom}_{\mathrm{PSh}(\mathcal{C})}(X, F) \to \mathrm{Hom}_{\mathrm{PSh}(\mathcal{C})}(R, F)$$

is injective (resp. bijective). The full subcategory of $\mathrm{PSh}(\mathcal{C})$ of sheaves over $\mathcal{C}$ is call the **category of sheaves of sets over** $\mathcal{C}$, and denoted by $\mathrm{Sh}(\mathcal{C})$. If there is no risk of ambiguity, this is simply called the category of sheaves over $\mathcal{C}$.

Proposition 1.3.12. *Let C be a $\mathcal{U}$-category and $\mathfrak{F} = (F_i)_{i\in I}$ be a family of presheaves over $\mathcal{C}$. For each object X of $\mathcal{C}$, let $\mathcal{T}_{\mathfrak{F}}(X)$ be the set of sieves R of X such that for any morphism $Y \to X$ in $\mathcal{C}$ and any $i \in I$, the map*

$$\mathrm{Hom}_{\mathrm{PSh}(\mathcal{C})}(Y, F_i) \to \mathrm{Hom}_{\mathrm{PSh}(\mathcal{C})}(R \times_X Y, F_i)$$

is injective (resp. bijective). Then the sets $\mathcal{T}_{\mathfrak{F}}(X)$ define a topology $\mathcal{T}$ over $\mathcal{C}$, which is the finest topology for which each F_i is a separated presheaf (resp. a sheaf).

Proof. □

Corollary 1.3.13. *Let $\mathcal{C}$ be a $\mathcal{U}$-category, and for any object X of $\mathcal{C}$, let $K(X)$ be a set of sieves of X satisfying axiom (T1). For a presheaf F over $\mathcal{C}$ to be a separated presheaf (resp. a sheaf) for the topology generated by the $K(X)$, it is necessary and sufficient that for any object X of C and any sieve $R \in K(X)$, the map*

$$\mathrm{Hom}_{\mathrm{PSh}(\mathcal{C})}(X, F) \to \mathrm{Hom}_{\mathrm{PSh}(\mathcal{C})}(R, F) \tag{1.3.2}$$

is injective (resp. bijective).

Proof. In fact, let $\mathcal{T}_F(X)$ be the set of sieves R of X such that for any morphism $Y \to X$ in $\mathcal{C}$ and any $i \in I$, the map

$$\mathrm{Hom}_{\mathrm{PSh}(\mathcal{C})}(Y, F) \to \mathrm{Hom}_{\mathrm{PSh}(\mathcal{C})}(R \times_X Y, F_i)$$

is bijective (resp. injective), and $\mathcal{T}_F$ be the topology generated by $\mathcal{T}_F(X)$. Since $K(X)$ satisfies (T1), the set $K(X)$ is contained in $\mathcal{T}_F(X)$ with the condition of the corollary, so the topology $\mathcal{T}$ generated by $K(X)$ is coarser than $\mathcal{T}_F$. By Proposition 1.3.12, it then follows that F is a separated presheaf (resp. a sheaf) for the topology $\mathcal{T}$. □

Corollary 1.3.14. *Let C be a $\mathcal{U}$-category endowed with a basis. For a presheaf F to be a separated presheaf (resp. a sheaf), it is necessary and sufficient that for any object X of $\mathcal{C}$ and for any covering $\{X_\alpha \to X\}$ of X, the following sequence*

$$F(X) \longrightarrow \textstyle\prod_\alpha F(X_\alpha) \rightrightarrows \prod_{\alpha,\beta} F(X_\alpha \times_X X_\beta)$$

is exact (resp. the map $F(X) \to \prod_\alpha F(X_\alpha)$ is injective.)

Proof. By ([?], I, 3.5), we have a functorial isomorphism

$$\mathrm{Hom}_{\mathrm{PSh}(\mathcal{C})}(F, R) = \varprojlim_{(X,u)\in\mathcal{C}_{/R}} F(X).$$

so the corollary follows from Corollary 1.3.13 and ([?], I, 2.12). □

Let $\mathcal{C}$ be a $\mathcal{U}$-category. We define the **canonical topology** of $\mathcal{C}$ to be the finest topology such that all representable functors are sheaves. A covering sieve of X for the canonical topology is called **universally effective epimorphic**. In other words, this means for any object Z of $\mathcal{C}$, the canonical map

$$\mathrm{Hom}_{\mathrm{PSh}(\mathcal{C})}(X, Z) \to \mathrm{Hom}_{\mathrm{PSh}(\mathcal{C})}(R, Z)$$

is bijective. More generally, a topology $\mathcal{T}$ over $\mathcal{C}$ such that all representable functors are sheaves is called **subcanonical**, so the canonical topology is the finest subcanonical topology on $\mathcal{C}$. In most cases, the topology considered over a category $\mathcal{C}$ is subcanonical, so the covering sieves of $\mathcal{C}$ are universally effective epimorphic. The only exception is the site $\mathrm{PSh}(\mathcal{C})$ obtained from a site $\mathcal{C}$, whose topology is finer, and often strictly finer, than the canonical topology.

Proposition 1.3.15. *For a sieve R of an object X of $\mathcal{C}$ to be universally effective epimorphic, it is necessary and sufficient that for any object $Y \to X$ of $\mathcal{C}_{/X}$ and any object Z of $\mathcal{C}$, the map*

$$\mathrm{Hom}_{\mathcal{C}}(Y, Z) \to \varprojlim_{U \in \mathcal{C}_{/(Y \times_X R)}} \mathrm{Hom}_{\mathcal{C}}(U, Z)$$

is a bijection.

Proof. By definition of canonical topology and Proposition 1.3.12, R is universally effective epimorphic if and only if for any for any object Z of $\mathcal{C}$ and any morphism $Y \to X$ in $\mathcal{C}$, the map

$$\mathrm{Hom}_{\mathrm{PSh}(\mathcal{C})}(Y, Z) \to \mathrm{Hom}_{\mathrm{PSh}(\mathcal{C})}(R \times_X Y, Z)$$

is bijective. On the other hand, by ([?], I, 3.5), we have a functorial isomorphism

$$\mathrm{Hom}_{\mathrm{PSh}(\mathcal{C})}(R \times_X Y, Z) = \varprojlim_{U \in \mathcal{C}_{/R \times_X Y}} h_Z(U) = \varprojlim_{U \in \mathcal{C}_{/R \times_X Y}} \mathrm{Hom}_{\mathcal{C}}(U, Z),$$

whence the proposition. □

Corollary 1.3.16. *Let $\mathcal{C}$ be a $\mathcal{U}$-category endowed with a basis. Then for a sieve R of X defined by a covering $\{X_\alpha \to X\}$ to be universally effective epimorphic, it is necessary and sufficient that for any object Z of $\mathcal{C}$, the sequence*

$$\mathrm{Hom}_{\mathcal{C}}(X, Z) \longrightarrow \prod_\alpha \mathrm{Hom}_{\mathcal{C}}(X_\alpha, Z) \rightrightarrows \prod_{\alpha,\beta} \mathrm{Hom}_{\mathcal{C}}(X_\alpha \times_X X_\beta, Z) \tag{1.3.3}$$

is exact. In particular, the covering families of the canonical topology of $\mathcal{C}$ are universally effective epimorphic families.

Proof. The first assertion follows from the proof of Corollary 1.3.14 and Proposition 1.3.15. As for the second one, it suffices to note that a family $\{X_\alpha \to X\}$ of morphisms in $\mathcal{C}$ is universally effective epimorphic if and only if the sequence (1.3.3) is exact. □

Remark 1.3.17. Proposition 1.3.15 provides a characterization of universally effective epimorphisms of a category $\mathcal{C}$, which is independent of the universe in which the presheaves take their values, with the sole condition that the sets of morphisms $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ of $\mathcal{C}$ belong to this universe. It therefore permits thus to define the canonical topology for any category $\mathcal{C}$, without specifying the universes we are considering.

1.3.3 Sheafifications of presheaves

Let $\mathcal{C}$ be a site. We define a **topological generating family** (or simply a **generating family** of $\mathcal{C}$ is there is no risk of confusion) of $\mathcal{C}$ to be a set G of objects of $\mathcal{C}$ such that any object of $\mathcal{C}$ is the target of a family of covering morphisms of $\mathcal{C}$ whose sources are elements of G. A site $\mathcal{C}$ is called a **$\mathscr{U}$-site** (where $\mathscr{U}$ is the fixed universe) if the underlying category is a $\mathscr{U}$-category and $\mathcal{C}$ has a $\mathscr{U}$-small topological generating family. If $\mathcal{C}$ is a category, we define a **$\mathscr{U}$-topology** over $\mathcal{C}$ to be a topology with which $\mathcal{C}$ is a $\mathscr{U}$-site. The site $\mathcal{C}$ is called **$\mathscr{U}$-small** (or by abusing of language, small) if the underlying category is small. It follows immediately from definition that any topology finer than a $\mathscr{U}$-topology is a $\mathscr{U}$-topology, and any small site is a $\mathscr{U}$-site.

Proposition 1.3.18. *Let $\mathcal{C}$ be a $\mathscr{U}$-site and G be a small topological generating family of $\mathcal{C}$. For any object X of $\mathcal{C}$, we denote by $\mathcal{T}_G(X)$ the set of covering sieves of X generated by the families of morphisms $\{u_\alpha : Y_\alpha \to X\}$, where $Y_\alpha \in G$. Then*

(a) *The set $\mathcal{T}_G(X)$ is $\mathscr{U}$-small and cofinal in the set $\mathcal{T}(X)$ of covering sieves of X, ordered by inclusion.*

(b) *For any $R \in \mathcal{T}_G(X)$, there exists a $\mathscr{U}$-small epimorphic family $\{u_\alpha : Y \to R\}$ with $Y \in G$.*

Proof. For a presheaf $F \in \mathrm{PSh}(\mathcal{C})$, we define

$$A(F) = \coprod_{Y \in G} \mathrm{Hom}(Y, F),$$

which is a $\mathscr{U}$-small set by our hypothesis. For any object X of $\mathcal{C}$, we have $|\mathcal{T}_G(X)| \leq 2^{|A(X)|}$, so the set $\mathcal{T}_G(X)$ is $\mathscr{U}$-small. Now for a sieve $R \in \mathcal{T}(X)$, let R' be the sieve of X generated by the family $\{u : Y \to R\}$, where $u \in A(R)$. We then have $R' \subseteq R$, and it suffices to prove that R' is a covering sieve. By axiom (T2), we only need to show that for any morphism $Z \to R$, where Z is an object of $\mathcal{C}$, the sieve $R' \times_X Z$ of Z is a covering sieve. But $R' \times_X Z$ contains the sieve generated by the family of morphisms $Y \to Z$, where $Y \in G$, which are coverings by hypothesis, so it is also a covering sieve by axiom (T2) again; this completes the proof of (a). Finally, we note that for any $R \in \mathcal{T}_G(X)$, the family $\{u : Y \to R\}_{u \in A(R)}$ is by hypothesis epimorphic, and is $\mathscr{U}$-small. □

Let $\mathcal{C}$ be a $\mathscr{U}$-site, $\mathscr{V}$ be a universe containing $\mathscr{U}$ such that $\mathcal{C}$ is $\mathscr{V}$-small, and G be a $\mathscr{V}$-small topological generating family of $\mathcal{C}$. The category $\mathrm{PSh}(\mathcal{C})$ of presheaves of $\mathscr{U}$-sets over $\mathcal{C}$ is then a $\mathscr{V}$-category. Let X be an object of $\mathcal{C}$; the set $\mathcal{T}(X)$ of covering sieves of X is $\mathscr{V}$-small, and directed under inclusion. For any $\mathscr{U}$-presheaf F, the inductive limit

$$\varinjlim_{R \in \mathcal{T}(X)} \mathrm{Hom}_{\mathrm{PSh}(\mathcal{C})}(R, F) \tag{1.3.4}$$

is then represented by an element of $\mathscr{V}$ ([?] I, 2.4.1). Moreover, it follows from Proposition 1.3.18 and (1.3.1) that, for any $R \in \mathcal{T}(X)$, $\mathrm{Hom}_{\mathrm{PSh}(\mathcal{C})}(R, F)$ is $\mathscr{U}$-small, and as $\mathcal{T}_G(X)$ is a $\mathscr{U}$-small cofinal set in $\mathcal{T}(X)$, it follows from ([?] I, 2.4.2) that the limit (1.3.4) is $\mathscr{U}$-small. We then choose, for any F and for any X, an element of $\mathscr{U}$ that represents this inductive limit and let

$$LF(X) = \varinjlim_{R \in \mathcal{T}(X)} \mathrm{Hom}_{\mathrm{PSh}(\mathcal{C})}(R, F). \tag{1.3.5}$$

If $g : Y \to X$ is a morphism in $\mathcal{C}$, the base change functor $g^* : \mathcal{T}(X) \to \mathcal{T}(Y)$ then defines a map

$$LF(g) : LF(X) \to LF(Y)$$

so that the construction $X \mapsto LF(X)$ is a presheaf on $\mathcal{C}$. We also note that since the family $\{\mathrm{id}_X : X \to X\}$ is an element of $\mathcal{T}(X)$, for any object X of $\mathcal{C}$, we have a map

$$\ell_F(X) : F(X) \to LF(X),$$

from which we obtain a morphism of functors $\ell_F : F \to LF$, and therefore a morphism

$$\ell : \mathrm{id} \to L.$$

Now let $R \hookrightarrow X$ be a refinement of X. The definition of $LF(X)$ and Yoneda's lemma then give a map

$$Z_R : \mathrm{Hom}_{\mathrm{PSh}(\mathcal{C})}(R, F) \to \mathrm{Hom}_{\mathrm{PSh}(\mathcal{C})}(X, LF),$$

and for any morphism $g : Y \to X$ in $\mathcal{C}$, we have the following commutative diagram:

$$\begin{tikzcd} \mathrm{Hom}_{\mathrm{PSh}(\mathcal{C})}(R, F) \arrow[r, "Z_R"] \arrow[d] & \mathrm{Hom}_{\mathrm{PSh}(\mathcal{C})}(X, LF) \arrow[d] \\ \mathrm{Hom}_{\mathrm{PSh}(\mathcal{C})}(R \times_X Y, F) \arrow[r, "Z_{R\times_X Y}"] & \mathrm{Hom}_{\mathrm{PSh}(\mathcal{C})}(Y, LF) \end{tikzcd} \tag{1.3.6}$$

where the vertical arrows are induced by the obvious morphisms.

Lemma 1.3.19. *Let LF be the presheaf defined above and $\ell : \mathrm{id} \to L$ be the induced morphism.*

(a) *For any refinement $i_R : R \hookrightarrow X$ and any morphism $u : R \to F$, the diagram*

$$\begin{tikzcd} R \arrow[r, hook, "i_R"] \arrow[d, "u"'] & X \arrow[d, "Z_R(u)"] \\ F \arrow[r, "\ell_F"] & LF \end{tikzcd} \tag{1.3.7}$$

is commutative.

(b) *For any morphism $v : X \to LF$, there exists a refinement R of X and a morphism $u : R \to F$ such that $Z_R(u) = v$.*

(c) *Let Y be an object of C and $u, v : Y \rightrightarrows F$ be two morphisms such that $\ell_F \circ u = \ell_F \circ v$. Then ther kernel of the couple (u, v) is a refinement of Y.*

(d) *Let R, R' be refinements of X and $u : R \to F$, $u' : R' \to F$ be morphisms. For that $Z_R(u) = Z_{R'}(u')$, it is necessary and sufficient that u and u' coincide on a refinement $R'' \hookrightarrow R \times_X R'$.*

Proof. In fact, any morphism $v : X \to LF$ corresponds to an element of the inductive limit $\varinjlim_{R\in\mathcal{T}(X)} \mathrm{Hom}_{\mathrm{PSh}(\mathcal{C})}(R, F)$, which is the canonical image of an element of $\mathrm{Hom}_{\mathrm{PSh}(\mathcal{C})}(R, F)$, and two such images coincide if and only if their canonical images to a further refinement are equal; this proves (b) and (d). Now if Y is an object of C and $u, v : Y \to F$ are two morphisms such that $\ell_F \circ u = \ell_F \circ v$, then by our preceding remarks, there exists a refinement R of Y such that u and v coincides on R. Since the kernel of (u, v) must contains R, it is therefore a refinement of Y.

Finally, in view of Yoneda's lemma, to prove (a), it suffices to show that the compositions of $Z_R(u) \circ i_R$ and $\ell_F \circ u$ with any morphism $g : Y \to R$ (Y being an object of $\mathcal{C}$) are equal. For this, we consider $f = i_R \circ g$ and the fiber product $R \times_X Y$:

$$\begin{tikzcd}
R \times_X Y \arrow[r, hook, "i'"] \arrow[d, "f'"'] & Y \arrow[d, "f"] \arrow[dl, "g"'] \\
R \arrow[r, hook, "i_R"] \arrow[d, "u"'] & X \arrow[d, "Z_R(u)"] \\
F \arrow[r, "\ell_F"] & LF
\end{tikzcd}$$

By the definition of ℓ_F, the morphism $Z_{R\times_X Y}(u \circ f')$ is equal to $\ell_F \circ u \circ g$. On the other hand, the commutative diagram (1.3.6) shows that $Z_{R\times_X Y}(u \circ f') = Z_R(u) \circ f$, so we conclude that

$$\ell_F \circ u \circ g = Z_R(u) \circ f = Z_R(u) \circ i_R \circ g. \qquad \square$$

Proposition 1.3.20. *Let L be the functor on* $\mathrm{PSh}(\mathcal{C})$ *defined by (1.3.5).*

(a) *The functor L is left exact.*

(b) *For any presheaf F, LF is a separated presheaf.*

(c) *The presheaf F is separated if and only if the morphism $\ell_F : F \to LF$ is a monomorphism, and in this case LF is a sheaf.*

(d) *The morphism $\ell_F : F \to LF$ is an isomorphism if and only if F is a sheaf.*

Proof. $\square$

Remark 1.3.21. If $\mathcal{J}(X)$ is a cofinal subset of $\mathcal{T}(X)$, we then have

$$LF(X) = \varinjlim_{R \in \mathcal{J}(X)} \mathrm{Hom}_{\mathrm{PSh}(\mathcal{C})}(R, F).$$

In particular, if the topology of $\mathcal{C}$ is defined by a basis $X \mapsto \mathrm{Cov}(X)$, the functor L can be then described by the covering families of $\mathrm{Cov}(X)$.

From Proposition 1.3.20, we then deduce the following theorem on the eixstence of sheafification functor on $\mathrm{PSh}(\mathcal{C})$:

Theorem 1.3.22. *For a $\mathcal{U}$-site $\mathcal{C}$, the inclusion functor $i : \mathrm{Sh}(\mathcal{C}) \to \mathrm{PSh}(\mathcal{C})$ admits a left adjoint #:*

$$\begin{tikzcd}
\mathrm{Sh}(\mathcal{C}) \arrow[r, shift left, "i"] & \mathrm{PSh}(\mathcal{C}) \arrow[l, shift left, "(-)^{\#}"]
\end{tikzcd}$$

The functor $i \circ \#$ is canonically isomorphic to $L \circ L$, and the sheaf $F^{\#}$ is called the ***sheafification*** *of a presheaf F. For a presheaf F, the adjunction morphism $F \to i(F^{\#})$ is induced by the morphism $\ell_{LF}\ell_F : F \to L \circ L(F)$.*

Remark 1.3.23. We note that, since L is left exact and the functor $\# = L \circ L$ is a left adjoint, the sheafification functor # is in fact exact on $\mathrm{PSh}(\mathcal{C})$. Recall that this result is obtained in the classical situation by the observation that sheafification does not change the stalks.

1.3.4 Exactness properties of the category of sheaves

The exactness properties of the category of sheaves is then induced from that of the category of presheaves via Theorem 1.3.22. In this paragraph, we explain this philosophy by single out some of the most useful standard statements.

1.3.5 The induced topology on $\mathrm{PSh}(\mathcal{C})$

Proposition 1.3.24. *Let $\mathcal{C}$ be a $\mathcal{U}$-site and $f : H \to K$ be a morphism in $\mathrm{PSh}(\mathcal{C})$. Then the following conditions are equivalent:*

(i) *For any morphism $X \to K$, with $X \in \mathrm{Ob}(\mathcal{C})$, the image of the corresponding morphism $H \times_K X \to X$ is a covering sieve of X.*

(ii) *The morphism $f^\# : H^\# \to K^\#$ is an epimorphism in $\mathrm{Sh}(\mathcal{C})$.*

(ii') *For any sheaf F over $\mathcal{C}$, the map $\mathrm{Hom}(K, F) \to \mathrm{Hom}(H, F)$ is injective.*

Proof. By the adjunction property of Theorem 1.3.22, for any sheaf F over $\mathcal{C}$ we have

$$\mathrm{Hom}(K, F) = \mathrm{Hom}(K^\#, F),$$

so it is clear that (ii) is equivalent to (ii'). On the other hand, in view of ([**?**], II, 4.4), condition (i) signifies that $(H \times_K X)^\# \to X^\#$ is an epimorphism. Since epimorphisms in $\mathrm{Sh}(\mathcal{C})$ are stable under base change and u is induced from $f^\#$ by base change, this proves (ii)⇒(i).

Conversely, since the family $\{X \to K : X \in \mathrm{Ob}(\mathcal{C})\}$ is epimorphic in $\mathrm{PSh}(\mathcal{C})$, so is the induced family $\{X^\# \to K^\# : X \in \mathrm{Ob}(\mathcal{C})\}$ in $\mathrm{Sh}(\mathcal{C})$ ([**?**] II, 4.1). Therefore, the morphism $f^\# : H^\# \to K^\#$ in $\mathrm{Sh}(\mathcal{C})$ is an epimorphism if and only if its base change $f^\# : H^\# \times_{K^\#} X^\# \to X^\#$ is an epimorphism for any $X \to K$, $X \in \mathrm{Ob}(\mathcal{C})$. This proves the implication (i)⇒(ii) in view of ([**?**], II, 4.1). □

A morphism $f : H \to K$ satisfying the equivalent conditions of Proposition 1.3.24 is called a **covering** morphism. A family $\{f_i : H_i \to K\}$ of morphims in $\mathrm{PSh}(\mathcal{C})$ is called **covering** if the induced morphism $f : \coprod_i H_i \to K$ is covering. A morphism $f : H \to K$ is called **bicovering** if it is covering and the diagonal morphism $H \to H \times_K H$ is covering. Similarly, a family $\{f_i : H_i \to K\}$ is called **bicovering** if the induced morphism $f : \coprod_i H_i \to K$ is bicovering[2].

By condition (i) of Proposition 1.3.24, to say that a family $\{f_i : H_i \to K\}$ is a covering signifies that for any morphism $X \to K$, where $X \in \mathrm{Ob}(\mathcal{C})$, the family $\{H_i \times_K X \to X\}$ generates a covering sieve of X. Or equivalently, by (ii), the family $\{f_i^\# : H_i^\# \to K^\#\}$ in $\mathrm{Sh}(\mathcal{C})$ is epimorphic (since the functor # commutes with direct sums).

Proposition 1.3.25. *Let $\mathcal{C}$ be a $\mathcal{U}$-site and $f : H \to K$ be a morphism in $\mathrm{PSh}(\mathcal{C})$. The following conditions are equivalent:*

(i) *The morphism f is bicovering.*

(i') *The morphism f is covering and for any couple of morphisms $u, v : X \rightrightarrows H$ with $X \in \mathrm{Ob}(\mathcal{C})$ such that $fu = fv$, the kernel (u, v) is a covering sieve of X.*

(ii) *The morphism $f^\# : H^\# \to K^\#$ is an isomorphism in $\mathrm{Sh}(\mathcal{C})$.*

(ii') *For any sheaf F over $\mathcal{C}$, the map $\mathrm{Hom}(K, F) \to \mathrm{Hom}(H, F)$ is a bijection.*

Proof. The equivalence (i)⇔(i') follows from condition (i) of Proposition 1.3.24, applied to the diagonal morphism $H \to H \times_K H$ (recall that $\ker(u, v)$ is given by the base change of the diagonal morphism $H \to H \times_K H$ along $(u, v)_K : X \to H \times_K H$), and that of (ii) and (ii') is trivial. We now prove that (i)⇒(ii), so let $f : H \to K$ be a bicovering morphism in $\mathrm{PSh}(\mathcal{C})$. The morphism $f^\# : H^\# \to K^\#$ is then an epimorphism by Proposition 1.3.24, and the diagonal morphism $H^\# \to H^\# \times_{K^\#} H^\#$ is also an epimorphism since the functor # commutes with fiber

[2] We note that, in view of condition (ii) of Proposition 1.3.24, the condition of being covering (resp. bicovering) for a family of morphisms in $\mathrm{PSh}(\mathcal{C})$ is stable under base change.

products. As the diagonal morphism is always a monomorphism, it is then an isomorphism ([**?**], II, 4.2), so $f^\#$ is a monomorphism, whence an isomorphism.

Conversely, if $f^\#$ is an isomorphism, then the diagonal morphism $H \to H^\# \times_{K^\#} H^\#$ is also an isomorphism. Since # commutes with fiber products, we conclude that the diagonal morphism $H \to H \times_K H$ is covering, so the morphism f is bicovering. □

Remark 1.3.26. It follows from Proposition 1.3.25 and the fact that # commutes with direct sums that a family $\{f_i : H_i \to K\}$ of morphisms in $\mathrm{PSh}(\mathcal{C})$ is bicovering if and only if the f_i induce an isomorphism $\coprod_i H_i^\# \to K^\#$, of equivalently, if and only if for any sheaf F, the map

$$\mathrm{Hom}(K, F) \to \prod_i \mathrm{Hom}(H_i, F)$$

is bijective.

Proposition 1.3.27. *Let $\mathcal{C}$ be a $\mathcal{U}$-site. Then there exists a (unique) topology on* $\mathrm{PSh}(\mathcal{C})$ *such that a family $H_i \to K$ of morphisms in* $\mathrm{PSh}(\mathcal{C})$ *is covering for this topology if and only if it is covering (in the sense of Proposition 1.3.24). This is also the coarsest topology $\mathcal{T}$ on* $\mathrm{PSh}(\mathcal{C})$ *satisfying the following conditions:*

(a) *$\mathcal{T}$ is finer than the canonical topology of* $\mathrm{PSh}(\mathcal{C})$*.*

(b) *Any covering family in $\mathcal{C}$ is covering in* $\mathrm{PSh}(\mathcal{C})$*.*

Proof. We first show that the covering families in the sense of Proposition 1.3.24 generates a topology $\mathcal{T}_\mathcal{C}$ on $\mathrm{PSh}(\mathcal{C})$.

The covering family of the canonical topology over $\mathrm{PSh}(\mathcal{C})$ are the universally effective epimorphic families of $\mathrm{PSh}(\mathcal{C})$ (Corollary 1.3.16). As # commutes with inductive limits, the topology $\mathcal{T}_\mathcal{C}$ is then finer than the canonical topology of $\mathrm{PSh}(\mathcal{C})$. Moreover, the covering families of $\mathcal{C}$ are also that of $\mathcal{T}_\mathcal{C}$ (Proposition 1.3.24(i)), so if $\mathcal{T}'$ denote the coarsest topology on $\mathrm{PSh}(\mathcal{C})$ satisfying conditions (a), (b), then $\mathcal{T}_\mathcal{C}$ is fiber than $\mathcal{T}'$. □

Remark 1.3.28. The proof of Proposition 1.3.27 in fact proves that any subcanonical topology $\mathcal{T}'$ over $\mathrm{PSh}(\mathcal{C})$, finer than the canonical topology, is the coarsest topology $\mathcal{T}$ of $\mathrm{PSh}(\mathcal{C})$ with the following properties:

(a) $\mathcal{T}$ is finer than the canonical topology of $\mathrm{PSh}(\mathcal{C})$.

(b) Any covering family for $\mathcal{T}'$ of the form $\{u_i : X_i \to X\}$, where X_i and X are objects of $\mathcal{C}$, is covering for $\mathcal{T}$.

In particular, any topology over $\mathrm{PSh}(\mathcal{C})$, finer than the canonical topology, is completely determined by its covering families of morphisms in $\mathcal{C}$.

Conversely, one can easily show that for any topology $\mathcal{T}'$ over $\mathrm{PSh}(\mathcal{C})$, finer than the canonical topology, the set of covering families of morphisms $\{X_i \to X\}$ in $\mathcal{C}$ is the set of covering families of a topology on $\mathcal{C}$. Therefore, we obtain a correspondence between topologies on $\mathcal{C}$ and topologies on $\mathrm{PSh}(\mathcal{C})$ finer than the canonical topology.

Let $\mathcal{C}$ be a small category. We denote by **Caf** the set of strictly full subcategories (any object isomorphic to an object of the subcategory is an object of it) of $\mathrm{PSh}(\mathcal{C})$ whose inclusion functor to $\mathrm{PSh}(\mathcal{C})$ admits a left adjoint which commutes with finite projective limits, and by $\mathscr{T}$ the set of topologies on $\mathcal{C}$. In view of Theorem 1.3.22, we have a map

$$\Phi : \mathscr{T} \to \mathbf{Caf}$$

which sends a topology over $\mathcal{C}$ to the sheaf category it defines. We now construct an inverse of the map Φ. For this, we need to associate for each element

$$\mathcal{C}' \underset{j'}{\overset{i'}{\rightleftarrows}} \mathrm{PSh}(\mathcal{C})$$

(where j' is the adjoint of i') of **Caf** a topology $\mathcal{T}_e$ over $\mathcal{C}$. For any object X of $\mathcal{C}$, we define $\mathcal{T}_e(X)$ to be the set of sub-objects of X whose canonical inclusion to X is transformed by j' into an isomorphism. It is immediately verified, using the assumptions made on j', that this defines a topology $\mathcal{T}_e$ over $\mathcal{C}$. We have therefore defined an application $\Psi : \mathbf{Caf} \to \mathscr{T}$.

Theorem 1.3.29 (J. Giraud). *The map Φ is an bijection with inverse map Ψ.*

Proof. The map $\Psi \circ \Phi$ is the identity in view of Proposition 1.3.25 and the axioms (T1), (T2). Conversely, let $i' : \mathcal{C}' \to \mathrm{PSh}(\mathcal{C})$ be an element of **Caf** with left adjoint j', $\mathcal{T}_e$ be the corresponding topology, and $\mathcal{C}_e$ be the sheaf category defined by $\mathcal{T}_e$. By using the definition of $\mathcal{T}_e$ and bicovering morphisms, we can then prove the equivalence of the following conditions for a morphism u of $\mathrm{PSh}(\mathcal{C})$:

(a) u is bicovering for the topology $\mathcal{T}_e$.

(b) u is transformed by j' to an isomorphism.

It is clear that $\mathcal{C}'$ is a full subcategory of $\mathcal{C}_e$, so it suffices to prove that any sheaf F for the topology $\mathcal{T}_e$ is an object of $\mathcal{C}'$. However, by the above equivalence, the morphism $F \to i' \circ j'(F)$ is bicovering ($j' \circ i'$ is isomorphic to the identity functor since j' is exact), whose source and target are sheaves. We deduce from Proposition 1.3.25(ii) that $F \to i' \circ j'(F)$ is an isomorphism, so $F \in \mathcal{C}'$ (recall that $\mathcal{C}'$ is strictly full). □

1.3.6 Topologies on the category of schemes

Let **Sch** be the category of schemes, to which we can assocate the Zariski topology, that is, the topology generated by the family of morphisms $\{S_i \to S\}$, where each $S_i \to S$ is an open immersion and the union of images of S_i is equal to S. A sheaf over the Zariski topology is also called of local nature: this is a contravariant functor $F : \mathbf{Sch}^{\mathrm{op}} \to \mathbf{Set}$ such that for any scheme S and any covering $\{S_i \to S\}$, we have an exact diagram

$$F(S) \longrightarrow \textstyle\prod_i F(S_i) \rightrightarrows \prod_{i,j} F(S_i \cap S_j)$$

In particular, a functor of local natura transforms direct sums to products. As any representable functor is a sheaf, this topology is coarser than the canonical topology.

To introduce (and handle) more topologies on **Sch**, we need a general criterion to identify the covering families of the topology generated by certain family of morphisms. This is contained in the following proposition.

Proposition 1.3.30. *Let $\mathcal{C}$ be a category and $\mathcal{C}'$ be a full subcategory. Let P be a set of families of morphisms of $\mathcal{C}$ with the same codomains, which is stable under composition and base change, and P' be a set of families of morphisms of $\mathcal{C}'$ containing the families of identity morphisms. We endow $\mathcal{C}$ with the topology generated by P and P' and suppose that the following conditions are satisfied:*

(a) *If $\{S_i \to S\} \in P'$ (hence $S_i, S \in \mathrm{Ob}(\mathcal{C}')$) and $T \to S$ is a morphism in $\mathcal{C}'$, then the fiber products $S_i \times_S T$ (in $\mathcal{C}$) exist and the family $\{S_i \times_S T \to T\}$ belongs to P'.*

(b) *For any $S \in \mathrm{Ob}(\mathcal{C})$, there exists $\{S_i \to S\} \in P$ with $S_i \in \mathrm{Ob}(\mathcal{C}')$ for each i.*

(c) *In the following situation*

$$S_{ijk} \xrightarrow{(P')} S_{ij} \xrightarrow{(P)} S_i \xrightarrow{(P')} S$$

where $S, S_i, S_{ij}, S_{ijk} \in \mathrm{Ob}(\mathcal{C}')$, $\{S_i \to S\} \in P'$, $\{S_{ij} \to S_i\} \in P$ *for each* i, $\{S_{ijk} \to S_{ij}\} \in P'$ *for any* i, j, *there exists a family* $\{T_n \to S\} \in P'$ *and for each* n *a multi-index* ijk *and a commutative diagram*

$$\begin{tikzcd} T_n \arrow[rr] \arrow[dr] & & S_{ijk} \\ & S \arrow[ur] & \end{tikzcd}$$

Then for a sieve R *of* $S \in \mathrm{Ob}(\mathcal{C})$ *to be covering, it is necessary and sufficient that there exists a composite family*

$$\begin{tikzcd} S_{ij} \arrow[r, dashed] \arrow[d, "(P')"'] & R \arrow[d] \\ S_i \arrow[r, "(P)"] & S \end{tikzcd} \tag{1.3.8}$$

where $S_i, S_{ij} \in \mathrm{Ob}(\mathcal{C}')$, $\{S_i \to S\} \in P$, $\{S_{ij} \to S_i\} \in P'$ *for each* i, *and the morphisms* $S_{ij} \to S$ *factors through* R.

Proof. Since the families in P and P' are covering, any family which is the composite of such families is again covering, so a sieve of the form indicated in the proposition is covering for $\mathcal{C}$, since it contains a covering sieve. Conversely, it suffices to prove that sieves of the form (1.3.8) form a topology, i,e, it suffices to verify the axioms (T1)–(T3).

To verify (T3), let $S \in \mathrm{Ob}(\mathcal{C})$. There exists by (b) a family $\{S_i \to S\} \in P$ with $S_i \in \mathrm{Ob}(\mathcal{C}')$. The families $\{\mathrm{id}_{S_i} : S_i \to S_i\}$ belong to P' by hypothesis, so the sieve S of S is of the following form:

$$\begin{tikzcd} S_i \arrow[r] \arrow[d, "(P')"'] & S \arrow[d, "\mathrm{id}_S"] \\ S_i \arrow[r, "(P)"] & S \end{tikzcd}$$

Now let R be a sieve of S with desired form (1.3.8) and R' be a sieve such that for any $T \to R$ in $\mathcal{C}$, the sieve $R' \times_T S$ is of the desired from. Then as the morphism $S_{ij} \to S$ factors through R, the sieve $R'_{ij} = R' \times_S S_{ij}$ of S_{ij} is of the desicred form by hypothesis:

$$\begin{tikzcd} R'_{ij} \arrow[r, hook] \arrow[dd] & S_{ij} \arrow[d, "(P')"'] \arrow[dr] & \\ & S_i \arrow[d, "(P)"'] & R \arrow[dl, hook] \\ R' \arrow[r, hook] & S & \end{tikzcd}$$

so for each ij, we have a diagram of the form

$$\begin{tikzcd} S_{ijkl} \arrow[d, "(P')"'] \arrow[dr] & \\ S_{ijk} \arrow[d, "(P)"'] & R'_{ij} \arrow[dl, hook] \\ S_{ij} & \end{tikzcd}$$

We have thus proved that there exists a composite family

$$S_{ijkl} \xrightarrow{(P')} S_{ijk} \xrightarrow{(P)} S_{ij} \xrightarrow{(P')} S_i \xrightarrow{(P)} S$$

belonging to $P \circ P' \circ P \circ P'$, which factors through R' and with all objects (except S) belong to $\mathcal{C}'$. Applying condition (c) to the family $\{S_{ijkl} \to S_i\}$, we then obtain for each i a family $\{T_{in} \to S_i\} \in P'$, such that $T_{in} \to S$ factors through one of the S_{ijkl}, hence through R':

$$\begin{tikzcd} T_{in} \arrow[r] \arrow[d, "(P')"'] & S_{ijkl} \arrow[dd] \\ S_i \arrow[d, "(P)"'] & \\ S & R' \arrow[l, hook'] \end{tikzcd}$$

The sieve R' of S is therefore of the desired form (1.3.8), which verifies axiom (T2).

Fianlly, as for axiom (T1), let R be a sieve of S of the desired form and $T \to S$ be a morphism in $\mathcal{C}$. Let $T_i = S_i \times_S T$; the family $\{T_i \to T\}$ then belongs to P, and applying condition (b), we obtain for each i a family $\{U_{ik} \to T_i\} \in P$, with $U_{ik} \in \mathrm{Ob}(\mathcal{C}')$. By the hypotesis on P, we have $\{U_{ik} \to T\} \in P$, so by condition (a), $U_{ik} \times_{S_i} S_{ij} = U_{ikj}$ is an object of $\mathcal{C}'$ and for each ik, $\{U_{ikj} \to U_{ik}\} \in P'$.

$$\begin{tikzcd} U_{ikj} \arrow[rr] \arrow[d, "(P')"'] & & S_{ij} \arrow[d, "(P')"] \arrow[ddr, bend left] & \\ U_{ik} \arrow[r, "(P)"] \arrow[dr, "(P)"'] & T_i \arrow[r] \arrow[d, "(P)"] & S_i \arrow[d, "(P)"] & \\ & T \arrow[r] & S & R \arrow[l, hook'] \end{tikzcd}$$

We therefore conclude that the family $\{U_{ikj} \to T\}$ factors through the sieve $T \times_S R$ of T, which is hence of the desired form. This proves axiom (T1) and completes the proof. $\square$

Corollary 1.3.31. *If $S \in \mathrm{Ob}(\mathcal{C}')$ and R is a sieve of S, then R is covering if and only if there exists a family $\{T_i \to S\} \in P'$ which factors through R.*

Proof. In fact, any such sieve is covering. Conversely, it suffices to apply (c) to the family $\{S_i \to S\}$ and the identity morphisms of S_i to deduce that any covering sieve is of the indicated form. $\square$

Corollary 1.3.32. *For a presheaf $F \in \mathrm{PSh}(\mathcal{C})$ to be separated (resp. a sheaf), it is necessary and sufficient that the morphisms*

$$F(S) \longrightarrow \textstyle\prod_i F(S_i)$$

is injective (resp. that the diagram

$$F(S) \longrightarrow \textstyle\prod_i F(S_i) \rightrightarrows \prod_{i,j} F(S_i \times_S S_j)$$

is exact) for $\{S_i \to S\} \in P$ and $\{S_i \to S\} \in P'$, respectively.

Proof. In fact, these conditions are necessary, because the families above are covering. Conversely, if R is the sieve of S of a family of morphisms $\{S_{ij} \overset{(P')}{\to} S_i \overset{(P)}{\to} S\}$, a diagram chasing shows that the above conditions imply that $\mathrm{Hom}(S, F) \to \mathrm{Hom}(R, F)$ is injective (resp. bijective). But any covering sieve R' of S contains a sieve generated by such a family and we have a commutative diagram

$$\begin{tikzcd} \mathrm{Hom}(S,F) \arrow[rr, "f"] \arrow[dr, "g"'] & & \mathrm{Hom}(R,F) \arrow[dl, "h"] \\ & \mathrm{Hom}(R',F) & \end{tikzcd}$$

If g is injective, then so is f, so in this case F is separated. In this case, since the morphism $R' \to R$ is covering, we see that h is also injective (cf. Proposition 1.3.24). Therefore, if g is bijective, so is f, hence F is a sheaf. □

Remark 1.3.33. The condition (c) of Proposition 1.3.30 is satisfies if P' is stable under composition and if any family $\{S_i \to S\}$ of morphisms in P with $S_i, S \in \mathrm{Ob}(\mathcal{C}')$ has a subfamily belonging to P'.

We now let $\mathcal{C} = \mathbf{Sch}$ be the category of schemes, and $\mathcal{C}'$ be the full subcategory formed by affine schemes. We shall consider the following sets P':

P_1': finite and surjective families, formed by flat morphisms;

P_2': finite and surjective families, formed by flat morphisms of finite presentation;

P_3': finite and surjective families, formed by étale morphisms;

P_4': finite and surjective families, formed by finite étale morphisms;

For each of these sets P_i' (except P_4'), the conditions of Proposition 1.3.30 are satisfied (as for (c), note that an affine scheme is quasi-compact, so any family of morphisms of $\mathcal{C}'$, belonging to P, contains a finite subfamily which is equally in P, hence in P_i' for $i = 1, 2, 3$). The corresponding topology $\mathcal{T}_i$ generated by P and P_i' is denoted and called by the following manner:

$\mathcal{T}_1$ is the faifufully flat and quasi-compact topology (fpqc);

$\mathcal{T}_2$ is the faifufully flat and finite presented topology (fppf);

$\mathcal{T}_3$ is the étale topology (ét);

$\mathcal{T}_4$ is the finite étale topology (étf).

As $P_1' \supseteq P_2' \supseteq P_3' \supseteq P_4'$, we have

$$\text{fpqc} \geq \text{fppf} \geq \text{ét} \geq \text{étf} \geq \text{Zar}.$$

Proposition 1.3.34. *Let $\mathcal{T}_i$ ($i = 1, 2, 3, 4$) be the topologies on* **Sch** *defined above.*

(a) *For a sieve R of S to be covering for $\mathcal{T}_i$ ($1 \leq i \leq 3$), it is necessary and sufficient that there exists a covering (S_α) of S by affine opens and for each α a family $\{S_{\alpha\beta} \to S_\alpha\} \in P_i'$, with $S_{\alpha\beta}$ affine, such that the the family $\{S_{\alpha\beta} \to S\}$ factors through R.*

(b) *For a presheaf F over* **Sch** *to be a sheaf for the fpqc topology (resp. fppf, étale, finite étale), it is necessary and sufficient that*

  (i) *F is a sheaf over the Zariski topology, i.e. a functor of local nature.*

(ii) *For any faithfully flat morphism (resp. faithfully flat morphism of finite presentation, resp. surjective étale, resp. finite surjective étale) $T \to S$ of affine schemes, we have an exact diagram*

$$F(S) \longrightarrow F(T) \rightrightarrows F(T \times_S T)$$

(c) *The topologies $\mathcal{T}_i$ $(1 \leq i \leq 4)$ are subcanonical.*

(d) *Any surjective family formed by open and flat morphisms (resp. flat and locally of finite presentation, resp. étale, resp. finite and étale) is covering for the fpqc topology (resp. fppf, resp. étale, resp. finite étale).*

(e) *Any finite and surjective family, formed by flat and quasi-compact morphisms, is covering for the fpqc topology.*

Proof. Assertion (a) follows from Proposition 1.3.30, and (b) follows from Corollary 1.3.32, since a sheaf for the Zariski topology transforms direct sums into products. Any representable functor is a sheaf for Zariski topology, and satisfies condition (ii) by ([**?**], VIII, 5.3), so $\mathcal{T}_1$ is subcanonical, which proves (c).

Let $\{S_i \to S\}$ be a family of morphisms as in (d). By considering a covering of S by affine opens, we are reduced to the case where S is affine. We first deal with the case where each $S_i \to S$ is flat and open (resp. étale). Let S_{ij} be a covering og S_i be affine opens. As the morphisms considered are open, the images T_{ij} of S_{ij} in S form an open covering of S. As S is affine, hence quasi-compact, there exists a finite subcover of T_{ij}, with i, j belongs to a finite set F. Then $S' = \coprod_F S_{ij}$ is affine, and the morphism $S' \to S$ belongs to P'_1 (resp. P'_3), hence is covering. As this factors through the given family $\{S_i \to S\}$, the latter is also covering.

In the case of the finite étale topology, each S_i is finite over S, hence is affine; in the preceding argument, we can then take for $\{S_{ij}\}$ the covering $\{S_i\}$ of S_i, and we obtain a morphism $S' \to S$ belonging to P'_4.

Now consider the case where $f_i : S_i \to S$ are flat and locally of finite presentation. For any $s \in S$, there exists (by the proof of ([**?**], IV$_4$, 17.16.2)) an affine subscheme $X(s)$ of one of the S_i such that $s \in f_i(X(s))$ and that the morphism $g_i : X(s) \to S$, restriction of f_i, is flat and quasi-finite. Then $g_i(X(s))$ is an open neighborhood $U(s)$ of s (([**?**], IV$_2$, 2.4.6)), and as S is affine, it is covered by a finite number of such opens $U(s_j)$, $j = 1, \ldots, n$. Therefore, $X' = \coprod_j X(s_j)$ is affine, and the morphism $X' \to S$ is surjective, flat, of finite presentation and quasi-finite, hence belongs to P'_2, which completes the proof of (d).

Finally, let $\{S_i \to S\}$ be a finite and surjective family of flat and quasi-compact morphisms. Let T_j be a covering of S be affine opens. Then $S_{ij} = T_j \times_S S_i$ is quasi-compact and hence has a finite affine covering T_{ijk}. Each morphism $T_{ijk} \to T_j$ is flat, and the family $\{T_{ijk} \to T_j\}$ is finite and surjective, hence covering for $\mathcal{T}_1$. The family $\{T_{ijk} \to S\}$ is hence also, by composition, covering, and it factors through the given family:

$$\begin{tikzcd} T_{ijk} \arrow[r] \arrow[dr] & S_{ij} \arrow[r] \arrow[d] & S_i \arrow[d] \\ & T_j \arrow[r] & S \end{tikzcd}$$

so the given family $\{S_i \to S\}$ is also covering for $\mathcal{T}_1$. □

1.4 Functoriality of categories of sheaves

In **??**, we studied the behavior of the categories of presheaves with respect to the functors between the underlying categories. In this section, we extend these results to sites and categories of sheaves.

1.4.1 Continuous functors

Let $\mathcal{C}$ and $\mathcal{D}$ be two $\mathcal{U}$-sites. A functor $u : \mathcal{C} \to \mathcal{D}$ on the underlying categories is called **continuous** if for any sheaf of sets F on $\mathcal{D}$, the presheaf $X \mapsto F(u(X))$ over $\mathcal{C}$ is a sheaf. Equivalenyly, this means we have an induced functor $u_* : \mathrm{Sh}(\mathcal{D}) \to \mathrm{Sh}(\mathcal{C})$ such that the diagram

$$\begin{tikzcd} \mathrm{Sh}(\mathcal{D}) \arrow[r, "u_*"] \arrow[d, "i_{\mathcal{D}}"'] & \mathrm{Sh}(\mathcal{C}) \arrow[d, "i_{\mathcal{C}}"] \\ \mathrm{PSh}(\mathcal{D}) \arrow[r, "u_\star"] & \mathrm{PSh}(\mathcal{C}) \end{tikzcd} \tag{1.4.1}$$

commutes, where if $i_{\mathcal{C}} : \mathrm{PSh}(\mathcal{C}) \to \mathrm{Sh}(\mathcal{C})$ and $i_{\mathcal{D}} : \mathrm{PSh}(\mathcal{D}) \to \mathrm{Sh}(\mathcal{D})$ are the canonical inclusion functors.

Remark 1.4.1. We use $(u^\star, u_\star, {}^\star u)$ to denote the adjoint triple $(u_!, u^*, u_*)$ of ([**?**] I).

Proposition 1.4.2. *Let $\mathcal{C}$ be a small site, $\mathcal{D}$ be a $\mathcal{U}$-site and $u : \mathcal{C} \to \mathcal{D}$ be a functor on the underlying categories. The following properties are equivalent:*

(i) *The functor u is continuous.*

(ii) *For any object X of $\mathcal{C}$ and any covering sieve R of X, the morphism $u^\star(R) \hookrightarrow u(X)$ is bicovering in* $\mathrm{PSh}(\mathcal{D})$.

(iii) *For any bicovering family $\{H_i \to K\}$ of* $\mathrm{PSh}(\mathcal{C})$*, the family $\{u^\star(H_i) \to u^\star(K)\}$ is bicovering in* $\mathrm{PSh}(\mathcal{D})$.

(iv) *There exists a functor $u^* : \mathrm{Sh}(\mathcal{C}) \to \mathrm{Sh}(\mathcal{D})$ which commutes with inductive limits and extends u, i.e. such that the following diagram is commutative up to isomorphisms:*

$$\begin{tikzcd} \mathcal{C} \arrow[r, "u"] \arrow[d, "\varepsilon_{\mathcal{D}}"'] & \mathcal{D} \arrow[d, "\varepsilon_{\mathcal{D}}"] \\ \mathrm{Sh}(\mathcal{C}) \arrow[r, "u^*"] & \mathrm{Sh}(\mathcal{D}) \end{tikzcd} \tag{1.4.2}$$

Moreover, if $\mathcal{C}$ is only assumed to be a $\mathcal{U}$-site, we still have the equivalence (i)⇔(iv), and the functor u^* of (iv) is necessarily a left adjoint of the functor $u_* : \mathrm{Sh}(\mathcal{D}) \to \mathrm{Sh}(\mathcal{C})$ (therefore uniquely determined up to isomorphisms).

Proof. The proof of (i)⇔(iv) when $\mathcal{C}$ is a $\mathcal{U}$-site will be done in **??**, so let us suppose that $\mathcal{C}$ is small (so that the functor $u_!$ is *defined*, cf. [**?**], I, 5.0). Let X be an object of $\mathcal{C}$ and $R \hookrightarrow X$ be a covering sieve. Then for any sheaf F over $\mathcal{D}$, by the adjunction property of $u^\star$ and $u_\star$, we have a commutative diagram

$$\begin{tikzcd} \mathrm{Hom}(u^\star(X), F) \arrow[r] \arrow[d] & \mathrm{Hom}(u^\star(R), F) \arrow[d] \\ \mathrm{Hom}(X, u_\star(F)) \arrow[r] & \mathrm{Hom}(R, u_\star(F)) \end{tikzcd}$$

where the vertical are isomorphisms. In view of Proposition 1.3.25, the equivalence of (i), (ii), (iii) is then easily deduced.

We now prove (i)⇒(iv), so assume that u is a continuous functor. For any sheaf G on $\mathcal{C}$, we define $u^*(G) = (u^\star(i_{\mathcal{C}}(G)))^\#$. Then for any sheaf F on $\mathcal{D}$, there are canoical isomorphisms

$$\mathrm{Hom}(G, u_*(F)) \cong \mathrm{Hom}(i_{\mathcal{C}}(G), i_{\mathcal{C}}(u_*(F))) \cong \mathrm{Hom}(i_{\mathcal{C}}(G), u_\star(i_{\mathcal{D}}(F)))$$

$$\cong \mathrm{Hom}(u^\star(i_{\mathcal{C}}(G)), i_{\mathcal{D}}(F)) \cong \mathrm{Hom}((u^\star(i_{\mathcal{C}}(G)))^\#, F)$$

(the first isomorphism follows from the fact that $i_{\mathcal{C}}$ is fully faithful, the second one is the commutative diagram (1.4.1), and the third one follows from adjunction). Therefore the functor u^* is left adjoint to u_*, and it clearly commutes with inductive limits. For any presheaf K over $\mathcal{C}$ and sheaf F over $\mathcal{D}$, we also have the following canonical isomorphisms

$$\begin{aligned}\mathrm{Hom}(u^*(K^\#), F) &\cong \mathrm{Hom}(K^\#, u_*(F)) \cong \mathrm{Hom}(K, i_{\mathcal{C}}(u_*(F))) \cong \mathrm{Hom}(K, u_\star(i_{\mathcal{D}}(F))) \\ &\cong \mathrm{Hom}(u^\star(K), i_{\mathcal{D}}(F)) \cong \mathrm{Hom}((u^\star(K))^\#, F).\end{aligned}$$

In particular, if K is representable, we then obtain a commutative diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, "u"] \arrow[d, "\varepsilon_{\mathcal{D}}"'] & \mathcal{D} \arrow[d, "\varepsilon_{\mathcal{D}}"] \\ \mathrm{Sh}(\mathcal{C}) \arrow[r, "u^*"] & \mathrm{Sh}(\mathcal{D}) \end{tikzcd}$$

where we use the commutative diagram ([**?**] I, 5.4 (3)).

Conversely, assume that we are given a functor $u^* : \mathrm{Sh}(\mathcal{D}) \to \mathrm{Sh}(\mathcal{D})$ satsisfying the conditions of (iv). Consider the diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, "u"] \arrow[d, "h"'] & \mathcal{D} \arrow[d, "h"] \\ \mathrm{PSh}(\mathcal{C}) \arrow[r, "u^\star"] \arrow[d, "(-)^\#"'] & \mathrm{PSh}(\mathcal{D}) \arrow[d, "(-)^\#"] \\ \mathrm{Sh}(\mathcal{C}) \arrow[r, "u^*"] & \mathrm{Sh}(\mathcal{D}) \end{tikzcd}$$

By the definition of $u^\star$, the upper square is commutative, and the vertical compositions are $\varepsilon_{\mathcal{C}}$ and $\varepsilon_{\mathcal{D}}$, respectively. For any object K in $\mathrm{PSh}(\mathcal{C})$, we have a canonical isomorphism ([**?**], I, 3.4)

$$\varinjlim_{(X\to K)\in\mathcal{C}_{/K}} h_X \xrightarrow{\sim} K.$$

As the functors $u^\star$, u^* and # commutes with inductive limits, we then deduce the following isomorphisms

$$\varinjlim_{(X\to K)\in\mathcal{C}_{/K}} u^*(X^\#) \xrightarrow{\sim} u^*(K^\#), \qquad \varinjlim_{(X\to K)\in\mathcal{C}_{/K}} (u^\star(X))^\# \xrightarrow{\sim} (u^\star(K))^\#.$$

Since $u^\star \circ h = h \circ u$, the commutativity of (1.4.2) then provides an isomorphism $u^*(K^\#) \cong (u^\star(K))^\#$, which is immediately verified to be funcotrial on K. The diagram

$$\begin{tikzcd} \mathrm{PSh}(\mathcal{C}) \arrow[r, "u^\star"] \arrow[d, "(-)^\#"'] & \mathrm{PSh}(\mathcal{D}) \arrow[d, "(-)^\#"] \\ \mathrm{Sh}(\mathcal{C}) \arrow[r, "u^*"] & \mathrm{Sh}(\mathcal{D}) \end{tikzcd}$$

is then commutative up to isomorphisms. As $\# \circ i$ is isomorphic to the identity functor, we obtain an isomorphism $u^* \cong (u^\star \circ i)^\#$, whence the uniqueness of u^*. Now let $f : H \to K$ be a bicovering of $\mathrm{PSh}(\mathcal{C})$. Then $f^\#$ is an isomorphism by Proposition 1.3.25, so $(u^\star(f))^\#$ (isomorphic to $u^*(f^\#)$) is also an isomorphism, which means $u^\star(f)$ is bicovering. This proves (iv)⇒(ii) and completes the proof. □

The functor $u^* : \mathrm{Sh}(\mathcal{C}) \to \mathrm{Sh}(\mathcal{D})$ of Proposition 1.4.2 can be intepreted as the "inverse image" functor induced by the functor u. Its properties are summarized in the following proposition:

Proposition 1.4.3. *Let $u : \mathcal{C} \to \mathcal{D}$ be a continuous functor from a small site $\mathcal{C}$ to a $\mathcal{U}$-site $\mathcal{D}$.*

(a) *The functor u^* is left adjoint to u_*.*

(b) *We have canonical isomorphisms $u^* \cong (u^\star \circ i)^\#$ and $u^* \circ \# \cong \# \circ u^\star$.*

(c) *The functor u^* commutes with inductive limits.*

(d) *If the functor $u^\star$ is left exact, then u^* is also left exact. More generally, the functor u^* commutes with finite projective limit if the functor $u^\star$ does.*

Proof. The first three assertions are proved in Proposition 1.4.2, and (d) follows from the isomorphism $u^* \cong (u^\star \circ i)^\#$ in (b), since i and # commute with finite projective limits. □

Remark 1.4.4. Combing ([?] I, 5.4) and Proposition 1.4.3, we obtain a diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, "u"] \arrow[d, "h"'] & \mathcal{D} \arrow[d, "h"] \\ \mathrm{PSh}(\mathcal{C}) \arrow[r, "u^\star"] \arrow[d, "(-)^\#"'] & \mathrm{PSh}(\mathcal{D}) \arrow[d, "(-)^\#"] \\ \mathrm{Sh}(\mathcal{C}) \arrow[r, "u^*"] & \mathrm{Sh}(\mathcal{D}) \end{tikzcd}$$

where the upper square is commutative and the lower square commutes up to isomorphisms. But the functors #, $u^\star$ and u^* are only defined up to isomorphism. One can check that we can choose them such that:

(a) The composition $h^\#$ is injective on the set of objects.

(b) The diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, "u"] \arrow[d, "h^\#"'] & \mathcal{D} \arrow[d, "h^\#"] \\ \mathrm{Sh}(\mathcal{C}) \arrow[r, "u^*"] & \mathrm{Sh}(\mathcal{D}) \end{tikzcd}$$

is commutative.

More precisely, we can choose the sheafification functors on $\mathrm{PSh}(\mathcal{C})$ and $\mathrm{PSh}(\mathcal{D})$ to satisfy condition (a). Then we can choose the functor u^* in such a way that condition (b) is fulfilled.

1.5 Grothendieck Topoi

We have seen in §§1.3.4 that various exactness properties for the category of sheaves over $\mathcal{C}$, where $\mathcal{C}$ is a small site, which can be expressed by saying that in many respects such a category inherit familiar properties from the category **Set** of (small) sets. In this section, we consider categories that can be realized as the category of sheaves of sets over a site. As we shall see, topoi behave much like the category of sets and possess a notion of localization.

1.5.1 Characterization of topoi

A category $\mathcal{X}$ is called a $\mathscr{U}$**-topos**, or simply a **topos**, if there exists a small site $\mathcal{C}$ such that $\mathcal{X}$ is equivalent the the category $\mathrm{Sh}(\mathcal{C})$ of sheaves of sets over $\mathcal{C}$. Let $\mathcal{X}$ be a topos. We will always endow $\mathcal{X}$ with the canonical topology, so that it is a site (and in fact a $\mathscr{U}$-site as we shall see). Unless futher explaination, we will not consider any other topology on $\mathcal{X}$.

We have seen in ([**?**] II, 4.8), ([**?**] II, 4.10) and ([**?**] II, 4.11) that a $\mathscr{U}$-topos $\mathcal{X}$ is a $\mathscr{U}$-category satisfying the following properties (these are called **Giraud's conditions**):

(a) Finite projective limits exist in $\mathcal{X}$.

(b) Small direct sums exist in $\mathcal{X}$ and are disjoint.

(c) All equivalence relations in $\mathcal{X}$ are effective.

(d) $\mathcal{X}$ admits a small set of generators.

In fact, we will see that these intrinsic properties characterize $\mathscr{U}$-topoi:

Theorem 1.5.1 (J. Giraud). *Let $\mathcal{X}$ be a $\mathscr{U}$-category. The following properties are equivalent:*

(i) *$\mathcal{X}$ is a $\mathscr{U}$-topos.*

(i′) *There exists a small site C endowed with a subcanonical topology such that projective limits exist in C and $\mathcal{X}$ is equivalent to the category of sheaves over C.*

(ii) *$\mathcal{X}$ satisfies Giraud's conditions.*

(iii) *The sheaves over $\mathcal{X}$ (endowed with the canonical topology) are representable, and $\mathcal{X}$ admits a small set of generators.*

(iv) *There exists a small category C and a fully faithful functor $i : \mathcal{X} \to \mathrm{PSh}(\mathcal{C})$ which admits an exact left adjoint.*

Proof. The equivalence of (i) and (iv) follows from Theorem 1.3.29, and we have remarked that (i)⇒(ii). As (i′) clearly implies (i), it remains to prove that (ii)⇒(iii) and (iii)⇒(i′).

We first deal with the implication (iii)⇒(i′), so let $\mathfrak{X} = (X_i)_{i\in I}$ be a small family of generators of $\mathcal{X}$ and assume that any sheaf over $\mathcal{X}$ is representable. As $\mathcal{X}$ is a $\mathscr{U}$-category, the set of isomorphism classes of finite diagrams in $\mathcal{X}$ with objects belonging to $\mathfrak{X}$ is $\mathscr{U}$-small. Therefore, the smallest set $\mathfrak{X}'$ of objects of $\mathcal{X}$ containing the finite projective limits of objects of $\mathfrak{X}^{(0)}$ is a countable union of small sets, hence is small. By redefining $\mathfrak{X}^{(n+1)} = (\mathfrak{X}^{(n)})'$ and $\mathfrak{X} = \bigcup_n \mathfrak{X}^{(n)}$, we see that, by adding generators if necessary, we can suppose that $\mathfrak{X}$ is stable under finite projective limits. □

Corollary 1.5.2. *Let $\mathcal{X}$ be a $\mathscr{U}$-topos and $\mathcal{C}$ be a full subcategory of $\mathcal{X}$ endowed with the induced topology ([**?**] III, 3.1). Consider the functor*

$$\mathcal{C} \to \mathrm{Sh}(\mathcal{C})$$

which associates an object X of $\mathcal{X}$ the restriction to $\mathcal{C}$ of the sheaf represented by X. This functor is an equivalence of categories if and only if $\mathrm{Ob}(\mathcal{C})$ is a topological generating family of $\mathcal{X}$.

Proof. It suffices to note that the considered functor factors into $\mathcal{X} \to \mathrm{Sh}(\mathcal{X}) \to \mathrm{Sh}(\mathcal{C})$, where the first one is an equivalence by Theorem 1.5.1(iii). The question is then reduced to determining whether $\mathrm{Sh}(\mathcal{X}) \to \mathrm{Sh}(\mathcal{C})$ is an equivalence, which can be deduced from the comparision lemma ([**?**] III, 4.1). □

1.6 Cohomology of topoi

1.6.1 Complements on abelian categories

In this paragraph, we recall some definitions and results concerning abelian categories which will be used later in the discussion of cohomology of topos. The material we give here can all be found on the famuous papar "Tohoku".

Proposition 1.6.1. *Let $\mathcal{A}$ be an abelian category with a generator. The following conditions are equivalent:*

(i) *The category $\mathcal{A}$ verifies the axiom (AB5): $\mathcal{A}$ possesses small direct sums and if $(X_i)_{i\in I}$ is a filtered small family of sub-objects of an object X of $\mathcal{A}$ and Y is a sub-object of X, then*

$$(\sup_i X_i) \cap Y = \sup_i (X_i \cap Y).$$

(ii) *Small filtered limits exist in $\mathcal{A}$ and commute with finite projective limits.*

An abelian category $\mathcal{A}$ possessing a generator and satisfying the axiom (AB5) is called a **Grothendieck category**. It can be proved that any Grothendieck category has enough injectives i.e. any object embeds itself in a injective object. Moreover, according to the result already cited, small products are representable in $\mathcal{A}$.

Proposition 1.6.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories and $F \dashv G : \mathcal{A} \to \mathcal{B}$ be an adjoint pair of additive functors. Consider the following properties:*

(i) *The functor F is exact.*

(ii) *The functor G transforms injective object in $\mathcal{B}$ to injective objects in $\mathcal{A}$.*

Then we always have (i)⇒(ii). If any nonzero object of $\mathcal{B}$ is the source of a nonzero morphism into an injective object (this is true for example if $\mathcal{B}$ has enough injectives), then (ii)⇒(i).

Proposition 1.6.3. *Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories and $F \dashv G : \mathcal{A} \to \mathcal{B}$ be an adjoint pair of additive functors. Suppose that:*

(a) *the category $\mathcal{B}$ has enough injectives;*

(b) *the equivalent conditions of Proposition 1.6.2 are satisfied;*

(c) *the functor F is faithful.*

Then the category $\mathcal{A}$ has enough injectives.

Remark 1.6.4. The category of abelian groups has enough injectives, so by applying Proposition 1.6.3, we deduce that any category of modules over a ring has enough injectives. Then applying the result of [5] (used in the proof of Proposition 1.6.1) and Proposition 1.6.2, we deduce that any Grothendieck category has enough injectives, which provides a new proof of this fact.

Proposition 1.6.5. *Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ be abelian categories and $F : \mathcal{A} \to \mathcal{B}$, $G : \mathcal{B} \to \mathcal{C}$ be two left exact additive functors. Suppose that $\mathcal{A}$ and $\mathcal{B}$ have enough injectives. Then the following conditions are equivalent:*

(i) *There exists a spectral functor*

$$E_2^{p,q} = R^p G \circ R^q F \Rightarrow R^{p+q}(G \circ F).$$

(ii) *The functor F transforms injective objects to G-acyclic objects.*

Proof. The implication (i)⇒(ii) is trivial because it suffices to apply this spectral sequence to an injective object of $\mathcal{A}$. The converse implication is the famous Grothendieck spectral sequence. □

Proposition 1.6.6. *Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact additive functor of abelian categories. Let $\mathfrak{M}$ be a collection of object of $\mathcal{A}$ possessing the following properties:*

(a) *Any object of $\mathcal{A}$ can be embedded into an element of $\mathfrak{M}$.*

(b) *If $X \oplus Y$ belongs to $\mathfrak{M}$, then X and Y belong to $\mathfrak{M}$.*

(c) *If we have an exact sequence $0 \to X' \to X \to X'' \to 0$ is where X' and X belong to $\mathfrak{M}$, then X'' belongs to $\mathfrak{M}$ and the sequence*

$$0 \longrightarrow F(X') \longrightarrow F(X) \longrightarrow F(X'') \longrightarrow 0$$

is exact. Moreover, the zero object belongs to $\mathfrak{M}$.

Then any injective object belongs to $\mathfrak{M}$, and the objects of $\mathfrak{M}$ are F-acyclic, i.e. for any $p > 0$ and any $X \in \mathfrak{M}$ we have $R^pF(X) = 0$. In particular, the resolutions by objects of $\mathfrak{M}$ computes the derived functors of F.

Proposition 1.6.7. *Let $\mathscr{U} \subseteq \mathscr{V}$ be universes, $\mathcal{A}$ (resp. $\mathcal{B}$) be an abelian $\mathscr{U}$-category (resp. $\mathscr{V}$-category) satisfying (AB5) and possesses a $\mathscr{U}$-small (resp. $\mathscr{V}$-small) topological generating family. Let $\varepsilon : \mathcal{A} \to \mathcal{B}$ be a fully faithful and exact functor. The following conditions are equivalent:*

(i) *There exists a generaring family $(X_i)_{i \in I}$ of $\mathcal{A}$ such that $(\varepsilon(X_i))_{i \in I}$ is a generating family of $\mathcal{B}$.*

(i′) *Any object of $\mathcal{B}$ is isomorphic to a quotient of an object of the form $\bigoplus_{\alpha \in A} \varepsilon(Y_\alpha)$, where A is $\mathscr{V}$-small.*

Under these equivalent conditions, we have:

(ii) *the functor ε transforms $\mathscr{U}$-small products to products (hence commutes with $\mathscr{U}$-small projective limits),*

and the following conditions are equivalent:

(a) *For any object Y of $\mathcal{A}$, any sub-object of $\varepsilon(Y)$ is isomorphic to the image under ε of a sub-object of Y.*

(a′) *There exists a generating family $(X_i)_{i \in I}$ of $\mathcal{A}$ such that the family $(\varepsilon(X_i))_{i \in I}$ is generating in $\mathcal{B}$ and that for any $i \in I$, any sub-object of $\varepsilon(X_i)$ is isomorphic to the image under ε of a sub-object of X_i.*

(b) *Any object of $\mathcal{B}$ is isomorphic to a sub-object of an object of the form $\prod_{\alpha \in A} \varepsilon(Y_\alpha)$, where A is $\mathscr{V}$-small.*

(c) *ε commutes with $\mathscr{U}$-small direct sums (hence commutes with $\mathscr{U}$-small inductive limits).*

Finally, if these conditions are satisfied, the functor ε transforms injective objects to injective objects.

Remark 1.6.8. If $\mathscr{U} = \mathscr{V}$, then the conditions (i) and (a′) imply that ε is an equivalence of categories, since we then have conditions (b), (ii) and (b).

1.6.2 Flat modules on ringed topoi

Let $(\mathcal{X}, A)$ be a ringed topos. A right (resp. left) A-module M is called **flat** if the functor $M \otimes_A (-)$ (resp. $(-) \otimes_A M$) from the category of left (resp. right) A-modules to the category of abelian sheaves on $\mathcal{X}$ is exact.

Proposition 1.6.9. *Let M be a (B, A)-bimodule.*

(a) *The following properties are equivalent:*

  (i) The right A-module M is flat.

  (ii) For any injective B-module I, the right A-module $\mathcal{H}om_B(M, I)$ is injective.

(b) *A filtered limit of flat modules is flat.*

(c) *If $M^\bullet$ is an exact complex of flat modules, then for any module F, the complex $M^\bullet \otimes_A F$ is exact.*

Proof. By the adjunction ([?] IV, 12.12), we have a canonical isomorphism

$$\mathrm{Hom}_B(M \otimes_A (-), -) \xrightarrow{\sim} \mathrm{Hom}_A(-, \mathcal{H}om_B(M, -)). \tag{1.6.1}$$

To establish the equivalence of (i) and (ii), it then suffices to apply Proposition 1.6.2. The isomorphism (1.6.1) also shows that tensor products commutes with inductive limits, so the fact the filtered inductive limits are exact (Proposition 1.6.1) implies the second assertion. To see that the complex $M^\bullet \otimes_A F$ is exact if $M^\bullet$ is an exact flat complex, it suffices to show that for any injective abelian sheaf I, the complex $\mathrm{Hom}^\bullet_{\mathbb{Z}}(M^\bullet \otimes_A F, I)$ is exact. This complex is isomorphic, in view of the adjunction formula, to the complex $\mathrm{Hom}^\bullet_A(F, \mathcal{H}om(M^\bullet, I))$, and by the equivalence (i)$\Rightarrow$(ii), the complex $\mathcal{H}om_{\mathbb{Z}}(M^\bullet, I)$ is an exact complex whose objects are injective, whence our conclusion. □

Proposition 1.6.10. *Let $(\mathcal{X}, A)$ be a ringed topos, X be an object of $\mathcal{X}$, $j : \mathcal{X}_{/X} \to \mathcal{X}$ be the localiztion functor, and M be a flat $A|_X$-module. Then $j_!(M)$ is a flat A-module. In particular, A_X is a flat A-module.*

Proof. Suppose that M is a right $A|_X$-module. For any left A-module N, we have a canonical isomorphism ([?] IV, 12)

$$N \otimes_A j_!(M) \xrightarrow{\sim} j_!(N \otimes_{A|_X} M).$$

The functors $j_!$ and j^* are exact by ([?] IV, 11.3.1) and ([?] IV, 11.12.2), and by hypothesis the functor $(-) \otimes_{A|_X} M$ is exact. We then conclude that the functor $(-) \otimes_A j_!(M)$ is exact, so $j_!(M)$ is exact. □

Proposition 1.6.11 (Projection formula for closed immersions). *Let $(\mathcal{X}, A)$ be a ringed topos, $i : \mathcal{Z} \to \mathcal{X}$ be a closed subtopos of $\mathcal{X}$, and put $A_{/\mathcal{Z}} = i^*(A)$. Then for any right $A_{/\mathcal{Z}}$-module M and any left A-module N, we have a canonical isomorphism*

$$i_*(M \otimes_{A_{/\mathcal{Z}}} i^*(N)) \xrightarrow{\sim} i_*(M) \otimes_A N. \tag{1.6.2}$$

Proof. Let $\mathcal{U}$ be the open complement of $\mathcal{Z}$ and $j : \mathcal{U} \to \mathcal{X}$ be the canonical open immersion. We have $j_*(i_*(M) \otimes_A N) \xrightarrow{\sim} 0 \otimes_{A_{/\mathcal{U}}} j^*(N)$ ([?] IV, 12), so $i_*(M) \otimes_A N$ is supported in $\mathcal{Z}$, and the corresponding unit morphism

$$i_*(M) \otimes_A N \to i_* i^*(i_*(M) \otimes_A N)$$

is an isomorphism ([?] IV, 14). We have $i^*(i_*(M) \otimes_A N) \cong i^* i_*(M) \otimes_{A_{/\mathcal{Z}}} i^*(N)$ by ([?] IV, 12), and $i^* i_*(M) \cong M$ since i is a closed immersion; whence the canonical isomorphism. □

Corollary 1.6.12. *Let $(\mathcal{X}, A)$ be a ringed topos, $i : \mathcal{Z} \to \mathcal{X}$ be a closed subtopos of $\mathcal{X}$, and put $A_{/\mathcal{Z}} = i^*(A)$. For any flat $A_{/\mathcal{Z}}$-module M, the A-module $i_*(M)$ is flat.*

Proof. It follows from Proposition 1.6.11 and ([?] IV 14) that the functor $N \mapsto i_*(M) \otimes_A N$ is exact, so $i_*(M)$ is flat. □

Let $(\mathcal{X}, A)$ be a ringed topos, $x : \mathcal{P} \to \mathcal{X}$ be a point of $\mathcal{X}$ (SGA IV, 6.1), and $\mathfrak{U}(x)$ be the category of neighborhoods of x (SGA IV, 6.8). For any object V of $\mathfrak{U}(x)$, we can associates an object of $\mathcal{X}$ (still denote by V), and a point $x_V : \mathcal{P} \to \mathcal{X}/V$ of $\mathcal{X}/V$. Moreover, any morphism $u : V \to W$ in $\mathfrak{U}(x)$ corresponds to a commutative diagram of topos (SGA IV, 6.7)

$$\begin{tikzcd} \mathcal{P} \arrow[r, "x_V"] \arrow[dr, "x_W"'] & \mathcal{X}/V \arrow[d, "j_u"] \\ & \mathcal{X}/W \end{tikzcd}$$

As an application of the materials given in this paragraph, we construct the Čech complex for a family of morphisms in $\mathcal{X}$ and prove its exactness under certain circumstances, that is, for epimorphic families. Now let $\mathcal{X}$ be a topos and $\mathfrak{U} = \{U_i \to X\}_{i \in I}$ be a small family of morphisms. For any ordered set $[n] = \{0, \dots, n\}$, we define

$$S_n(\mathfrak{U}) = \coprod_{f:[n]\to I} U_f$$

where the direct sum is taken over all maps $f : [n] \to I$, and for such a map f we define U_f to be

$$U_f := U_{f(1)} \times_X U_{f(2)} \times_X \cdots \times_X U_{f(n)}.$$

For any nondecreasing map $g : [m] \to [n]$, we have a morphism

$$s(g) : S_n(\mathfrak{U}) \to S_m(\mathfrak{U}) \tag{1.6.3}$$

defined in the following way: for any map $f : [n] \to I$, the restriction of $s(g)$ to the component U_f is the composition morphism

$$U_f \xrightarrow{s_f(g)} U_{fg} \hookrightarrow S_m(\mathfrak{U})$$

where $s_f(g) : U_f \to U_{fg}$ is the unique morphism such that for any $i \in [m]$, we have

$$\mathrm{pr}_{f(g(i))} s_f(g) = \mathrm{pr}_{f(g(i))}, \tag{1.6.4}$$

where pr_j is the j-th projection. We therefore obtain a contravariant functor $[n] \mapsto S_n$ from the category of finite sets to $\mathcal{X}$, or in other words, a semi-simplicial object $S_\bullet(\mathfrak{U})$ of $\mathcal{X}$. Note that this complex is canonically augmented by X. Any functor of $\mathcal{X}$ into a category $\mathcal{C}$ transforms $S_\bullet(\mathfrak{U})$ into a simplicial object of $\mathcal{C}$. In particular, if A is a ring object of $\mathcal{X}$, the "free A-module functor" transforms $S_\bullet(\mathfrak{U})$ into a simplicial complex of A-bimodules augmented by A_X, which is denoted by $A_\bullet(\mathfrak{U})$. We have

$$A_n(\mathfrak{U}) = \bigoplus_{f:[n]\to I} A_{U_{fg}}. \tag{1.6.5}$$

Let $\{d_i : S_n(\mathcal{C}) \to S_{n-1}(\mathcal{C})\}_{0 \le i \le n}$ be the face maps of $S_\bullet(\mathcal{C})$, then the complex $A_\bullet(\mathfrak{U})$ has the following form

$$\cdots \begin{smallmatrix} \xrightarrow{s_0} \\ \xrightarrow{s_1} \\ \xrightarrow{s_2} \end{smallmatrix} \bigoplus_{i,j} A_{U_i \times_X U_j} \begin{smallmatrix} \xrightarrow{s_0} \\ \xrightarrow{s_1} \end{smallmatrix} \bigoplus_i A_{U_i} \longrightarrow A_X \tag{1.6.6}$$

For such a complex, we can define a differential complex, augmented by A_X, by simply setting $d = \sum_i (-1)^i s_i$:

$$\cdots \xrightarrow{d} \bigoplus_{i,j} A_{U_i \times_X U_j} \xrightarrow{d} A_X \tag{1.6.7}$$

Proposition 1.6.13. *If the famly $\mathfrak{U}$ is epimorphic, the differential complex (1.6.7) is exact and hence a resolution of A_X.*

Proof. We denote by $\mathbb{Z}$ the constant sheaf with values $\mathbb{Z}$. By the definition of the "free A-module functor", we have, for any object Y of $\mathcal{X}$,

$$A_Y \cong \mathbb{Z}_Y \otimes_{\mathbb{Z}} A,$$

whence an isomorphism

$$A_\bullet(\mathfrak{U}) \cong \mathbb{Z}_\bullet(\mathfrak{U}) \otimes_{\mathbb{Z}} A.$$

As the components of $\mathbb{Z}_\bullet(\mathfrak{U})$ are all flat $\mathbb{Z}$-modules by Proposition 1.6.10, it suffices to prove the proposition for $A = \mathbb{Z}$.

Suppose first that $\mathcal{X}$ is the topos of sets. Then the augmented complex $S_\bullet(\mathfrak{U})$ is the direct sum of augmented complexed of the form

$$\cdots \; S \times S \times S \; S \times S \; S \longrightarrow *$$

where $*$ is the set with a single element. Since each of these complexes is homotopically trivial, we then conclude that $S_\bullet(\mathfrak{U})$ is a homotopically trivial augmented complex, whence the proposition in this case.

Now let $p : \mathbf{Set} \to \mathcal{X}$ be a point of $\mathcal{X}$. As the formation of the complex $\mathbb{Z}_\bullet(\mathfrak{U})$ commutes with inverse image functors between topos, $p^*(\mathbb{Z}_\bullet(\mathfrak{U})) \cong \mathbb{Z}_\bullet(p^*(\mathfrak{U}))$ is a resolution of $\mathbb{Z}_{p^*(X)} \cong p^*(\mathbb{Z}_X)$, which proves the proposition if $\mathcal{X}$ possesses enough fiber functors ([?] IV 4.6). This is the case in particular if $\mathcal{X}$ is the topos of presheaves over a small site $\mathcal{C}$, because for any object X of $\mathcal{C}$, $\Gamma(X, -)$ is a fiber functor. In the general case, $\mathcal{X}$ is equivalent to the topos of sheaves over a small site $\mathcal{C}$ (Theorem 1.5.1), and the epimorphic family $\mathfrak{U}$ is the image, under the sheafification functor, of an epimorphic family $\mathfrak{U}' = \{U_i' \to X'\}$. Therefore we have $\mathbb{Z}_\bullet(\mathfrak{U}) = (\mathbb{Z}_\bullet(\mathfrak{U}'))^\#$, which is a resolution of $(\mathbb{Z}_{X'})^\# \cong \mathbb{Z}_X$. □

1.6.3 Čech cohomology

1.6.3.1 The general notion of cohomology Let $(\mathcal{X}, A)$ be a ringed topos, M, N be two A-modules (say left modules). We denote by $\mathrm{Ext}^p_A(\mathcal{X}; M, N)$ (or simply $\mathrm{Ext}^p_A(M, N)$ if there is no risk of confusion) the value of the p-th right derived functor of the functor $\mathrm{Hom}_A(M, -)$ at N. In other words,

$$\mathrm{Ext}^p_A(\mathcal{X}; M, N) := R^p\mathrm{Hom}_A(M, -)(N). \tag{1.6.8}$$

The functors $\mathrm{Ext}^p_A(\mathcal{X}; M, N)$ then form a δ-functor on the variable N, and is also a contravariant functor on the variable M.

Let X be an object of $\mathcal{X}$. If $M = A_X$ is the free A-module generated by X ([?] IV, 12), we then write

$$\mathrm{Ext}^p_A(\mathcal{X}; A_X, N) = H^p(X, N). \tag{1.6.9}$$

Note that in this notation, the ring A no longer appears. This leads to no confusion because we will show that the formation of $H^p(X, -)$ commutes to the restriction of scalars, and the functor $H^p(X, -)$ is the p-th right derived functor of the functor $\mathrm{Hom}_A(A_X, -) = \mathrm{Hom}_{\mathcal{X}}(X, -)$, which is again denoted by $\Gamma(X, -)$. In particular, if X is the final object of $\mathcal{X}$, then $A_X = A$ and we write

$$\mathrm{Ext}^p_A(\mathcal{X}; A, N) = H^p(\mathcal{X}, N).$$

Let X be an object of $\mathcal{X}$ and $j : \mathcal{X}_{/X} \to \mathcal{X}$ be the localization morphism ([**?**] IV, 8). The functor j^* is exact on A-modules and admits a left adjoint functor $j_!$. Therefore j^* transforms injective modules to injective modules (Proposition 1.6.2), and for any A-module N and $A|_X$-module M, we have a canonical isomorphism

$$\mathrm{Ext}^p_{A|_X}(\mathcal{X}_{/X}; M, j^*(N)) \xrightarrow{\sim} \mathrm{Ext}^p_A(\mathcal{X}; j_!(M), N). \tag{1.6.10}$$

In particular, by setting $M = A$, we obtain canonical isomorphisms

$$H^p(\mathcal{X}_{/X}, j^*(N)) \xrightarrow{\sim} H^p(X, N). \tag{1.6.11}$$

For any object X of $\mathcal{X}$ and any couple M, N of A-modules, we put

$$\mathrm{Ext}^p_A(X; M, N) := \mathrm{Ext}^p_A(\mathcal{X}_{/X}; M|_X, N|_X) \tag{1.6.12}$$

From the above remarks, the functors $\mathrm{Ext}^p_A(X; M, -)$ are the derived functors of the functors $\mathrm{Hom}_{A|_X}(M|_X, (-)|_X)$, and the functors $(M, N) \mapsto \mathrm{Ext}^p_A(X; M, N)$ form an δ-functor with respect to each of the variables.

1.6.3.2 Cohomology for topos of presheaves Let $\mathcal{C}$ be a small category endowed with a presheaf of rings A, $\mathrm{PSh}(\mathcal{C})$ be the topos of presheaves over $\mathcal{C}$. We divide the computation of cohomology groups into two cases:

- Let X be a representable object of $\mathrm{PSh}(\mathcal{C})$. The functor which associates an A-module M with the group $\Gamma(X, M) = M(X)$ is then exact by ([**?**] I, 3), so we have $H^p(X, M) = 0$ for any $p > 0$ and any A-module M. In particular, since $M(X) = \mathrm{Hom}_A(A_X, M)$, we conclude that A_X is a projective A-module.

- Let S be a presheaf over $\mathcal{C}$. We have a canonical isomorphism for any A-module M ([**?**] I, 2):
$$\Gamma(S, M) = \mathrm{Hom}(S, M) = \varprojlim_{U \in \mathcal{C}_{/S}} M(U).$$
Moreover, for any injective A-module M, the $A|_S$-module $M|_S$ is injective ([**?**] 2.2). Therefore, the group $H^p(S, M)$ is the value at $M|_S$ of the p-th right derived functor of the functor $\varprojlim_{U \in \mathcal{C}_{/S}} \Gamma(U, -)$. Denote by $\varprojlim^p_{\mathcal{C}_{/S}}$ this derived functor, we then have canonical isomorphism
$$H^p(S, M) \xrightarrow{\sim} \varprojlim\nolimits^p_{\mathcal{C}_{/S}} M. \tag{1.6.13}$$
In particular, if S is the final object in $\mathcal{S}$, we then obtain a canonical isomorphism
$$H^p(\mathrm{PSh}(\mathcal{C}), M) \cong \varprojlim\nolimits^p_{\mathcal{C}} M. \tag{1.6.14}$$

We now turn to the computation of Čech cohomologies of $\mathrm{PSh}(\mathcal{C})$. Let X be an object of $\mathcal{C}$ and $\mathfrak{U} = \{U_i \to X\}_{i \in I}$ be a family of squarable morphisms in $\mathcal{C}$. We denote by $A_\bullet$ the simplicial complex (1.6.6):

$$A_\bullet : \quad \cdots \overset{s_0}{\underset{s_2}{\overset{s_1}{\rightrightarrows}}} \bigoplus_{i,j} A_{U_i \times_X U_j} \overset{s_0}{\underset{s_1}{\rightrightarrows}} \bigoplus_i A_{U_i} \longrightarrow A_X$$

For any A-module M, we denote by $C^\bullet(\mathfrak{U}, M)$ the complex obtained by applying the functor $\mathrm{Hom}_A(A_\bullet, M)$:

$$C^\bullet(\mathfrak{U}, M) : \ \prod_i M(U_i) \rightrightarrows \prod_{i,j} M(U_i \times_X U_j) \ \substack{\rightarrow\\\rightarrow\\\rightarrow} \ \cdots$$

The cohomology of this complex of abelian groups is denoted by $H^p(\mathfrak{U}, M) = H^p(C^\bullet(\mathfrak{U}, M))$.

Proposition 1.6.14. *With the above notions, let $R \hookrightarrow X$ be the sieve generated by $\mathfrak{U}$. Then we have a canonical isomorphism*

$$H^p(\mathfrak{U}, M) \xrightarrow{\sim} H^p(R, M) \tag{1.6.15}$$

Moreover, the functors $H^p(\mathfrak{U}, -)$ commutes with restrictions of scalars.

Proof. As R is a sub-object of X in $\mathrm{PSh}(\mathcal{C})$, the fiber products $U_{i_1} \times_R \cdots \times_R U_{i_p}$ and $U_{i_1} \times_X \cdots \times_X U_{i_p}$ are canonical isomorphic, so it follows from Proposition 1.6.13 that the complex $A_\bullet$ is a resolution of A_R. Now recall from our previous discussion that the components of $A_\bullet$ are projective A-modules, so by definition, the cohomology groups of $C^\bullet(\mathfrak{U}, M)$ are then canonically isomorphic to $\mathrm{Ext}^p_A(A_R, M)$, whence the isomorphism (1.6.15). The second assertion follows immediately from the description of the complex $C^\bullet(\mathfrak{U}, M)$. □

Corollary 1.6.15. *Let $\mathfrak{U} = \{U_i \to X\}$ and $\mathfrak{V} = \{V_i \to X\}$ be two families of morphisms with target X and*

$$\phi = (\phi : I \to J, f_i : U_i \to V_{\phi(i)}), \quad \phi = (\psi : I \to J, g_i : U_i \to V_{\phi(i)})$$

be morphisms (lying over X) from $\mathfrak{U}$ to $\mathfrak{V}$. Then ϕ and ψ induce equal morphisms $H^p(\mathfrak{U}, M) \to H^p(\mathfrak{U}, M)$. In particular, if the families $\mathfrak{U}$ and $\mathfrak{V}$ are equivalent (i.e. there exists a morphism from $\mathfrak{U}$ to $\mathfrak{V}$ and a morphism from $\mathfrak{V}$ to $\mathfrak{U}$), then the A-modules $H^p(\mathfrak{U}, M)$ and $H^p(\mathfrak{V}, M)$ are canonically isomorphic.

1.6.3.3 Cohomology for small sites Let $(\mathcal{C}, A)$ be a ringed $\mathscr{U}$-site, $\mathrm{Sh}(\mathcal{C})$ be the topos of sheaves over $\mathcal{C}$, and $\varepsilon : \mathcal{C} \to \mathrm{Sh}(\mathcal{C})$ be the canonical functor which associated an object of $\mathcal{C}$ with the associated sheaf. By abusing of languages, for any object X of $\mathcal{C}$ and any sheaf of A-modules M, we define $H^p(X, M)$ to be the p-th derived functor of the functor $\Gamma(X, -)$. Recall that if the topology on $\mathcal{C}$ is subcanonical, then the functor ε is fully faithful and we can identify $\mathcal{C}$ with a subcategory of $\mathrm{Sh}(\mathcal{C})$.

Now consider the inclusion functor $\mathcal{H}^0 : \mathrm{Sh}(\mathcal{C}_A) \to \mathrm{PSh}(\mathcal{C}_A)$ from the category of sheaves of A-modules to the category of presheaves of A-modules. For any sheaf of A-modules M and any object X of $\mathcal{C}$, we have by definition

$$\mathcal{H}^0(M)(X) = H^0(X, M) = M(X).$$

Since functor $\mathcal{H}^0$ is obviously left exact, we can define its right derived functors, which are denoted by $\mathcal{H}^p$. Note that as for any object X of $\mathcal{C}$, the functor $\Gamma(X, -)$ is exact on the category of presheaves, we have

$$\mathcal{H}^p(M)(X) = H^p(X, M) \tag{1.6.16}$$

for any sheaf of A-modules M, so the presheaf $\mathcal{H}^p(M)$ is defined by $X \mapsto H^p(X, M)$.

The definition of the cohomology group $H^p(X, M)$ is simple, but hard to compute. Because of this, it is necessary to introduce another cohomology group, the Čech cohomology group, which are much easily to handle. Suppose that $(\mathcal{C}, A)$ is a small ringed site, so that $\mathrm{PSh}(\mathcal{C})$ is a topos and we can apply the results of 1.6.3.2. Let X be an object of $\mathcal{C}$ and $R \hookrightarrow X$ be a covering sieve. For any presheaf of A-modules G, the groups $H^p(R, G)$ (which are computed in the topos $\mathrm{PSh}(\mathcal{C})$) are then called the **Čech cohomology groups of the presheaf G relative to the covering sieve R**. If $R \hookrightarrow X$ is generated by a covering family $\mathfrak{U} = \{U_i \to X\}$, these groups are then computed by the Čech complex $C^\bullet(\mathfrak{U}, G)$, and called the **Čech cohomology groups of the presheaf G relative to the covering family** $\mathfrak{U}$ (denoted by $H^p(\mathfrak{U}, G) = H^p(R, G)$). If M is a sheaf of A-modules over $\mathcal{C}$, the groups $H^p(\mathfrak{U}, \mathcal{H}^0(M))$ are then denoted by $H^p(\mathfrak{U}, M)$, and called the Čech cohomology groups of the sheaf M relative to the covering family $\mathfrak{U}$.

The cohomology group $H^p(R, M)$ thus defined is inadequate to reflect the cohomological natures of M, and in fact differs from the group $H^p(X, M)$. To fix this, we must apply a limit process as the case of classical Čech cohomologies. Now let $\check{\mathcal{H}}^0 : \mathrm{PSh}(\mathcal{C}_A) \to \mathrm{PSh}(\mathcal{C}_A)$ be the natrual extension of the functor $\mathcal{H}^0$ to the category of presheaves of A-modules (composed with the functor L). We then have, by (1.3.5), for any presheaf G and any object X of $\mathcal{C}$:

$$\check{\mathcal{H}}^0(G)(X) = \varinjlim_{R \hookrightarrow X} G(R) \tag{1.6.17}$$

where the inductive limit is taken over all covering sieves of X. From this, we see that the functor $\check{\mathcal{H}}^0$ is left exact, and we denote its right derived functors by $\check{\mathcal{H}}^p$. As the section functor $\Gamma(X, -)$ and taking filtered limits are both exact, it follows from (1.6.17) that

$$\check{\mathcal{H}}^p(G)(X) = \varinjlim_{R \hookrightarrow X} H^p(R, G), \tag{1.6.18}$$

The presheaves $\check{\mathcal{H}}^p(G)$ are then called the **presheaves of Čech chomologies of G**. For any object X of $\mathcal{C}$, the **Čech cohomology groups of** G are defined to be

$$\check{H}^p(X, G) := \check{\mathcal{H}}^p(G)(X). \tag{1.6.19}$$

If the topology of $\mathcal{C}$ is defined by a basis, which is most of the case in practice, we then have, in view of Proposition 1.6.14,

$$\check{H}^p(X, G) = \varinjlim_{\mathfrak{U}} H^p(\mathfrak{U}, G) \tag{1.6.20}$$

where the inductive limit is taken over all covering families $\mathfrak{U}$ of X, ordered by refinements. If M is a sheaf of A-modules, then by abusing of languages, we write

$$\check{H}^p(X, M) = \check{H}^p(X, \mathcal{H}^0(M)) = \check{\mathcal{H}}^p(\mathcal{H}^0(M))(X). \tag{1.6.21}$$

The groups $\check{H}^p(X, M)$ are called the **Čech cohomology groups of the sheaf M**. Note that although the functors $\check{H}^p$ are derived functors on the category of presheaves, they do not, in general, form a δ-functor on the category of sheaves.

1.6.3.4 Čech cohomology for $\mathscr{U}$-sites Let $(\mathcal{C}, A)$ be a ringed $\mathscr{U}$-site and $\mathscr{V}$ be a universe containing $\mathscr{U}$. Then the site $(\mathcal{C}, A)$ is also a $\mathscr{V}$-site, and we have a $\mathscr{U}$-topos $\mathrm{Sh}(\mathcal{C})_{\mathscr{U}}$, a $\mathscr{V}$-topos $\mathrm{Sh}(\mathcal{C})_{\mathscr{V}}$, and a canonical inclusion functor $\varepsilon : \mathrm{Sh}(\mathcal{C})_{\mathscr{U}} \to \mathrm{Sh}(\mathcal{C})_{\mathscr{V}}$. The functor ε is exact and fully faithful over the category of modules and transforms injective modules to injective modules (Proposition 1.6.7). For any couple of $\mathscr{U}$-sheaves of A-modules, we then have caonical isomorphisms

$$\mathrm{Ext}^p_A(\mathrm{Sh}(\mathcal{C})_{\mathscr{U}}; M, N) \xrightarrow{\sim} \mathrm{Ext}^p_{\varepsilon(A)}(\mathrm{Sh}(\mathcal{C})_{\mathscr{V}}; \varepsilon(M), \varepsilon(N)) \quad \text{for} \quad p \geq 0. \tag{1.6.22}$$

In particular, for any $\mathscr{U}$-sheaf of sets R over $\mathcal{C}$, we have canonical isomorphisms

$$H^p(R, M) \xrightarrow{\sim} H^p(\varepsilon(R), \varepsilon(M)); \tag{1.6.23}$$

and more particularly, for any object X of $\mathcal{C}$, we have canonical isomorphisms

$$H^p(X, M) \xrightarrow{\sim} H^p(X, \varepsilon(M)). \tag{1.6.24}$$

We can therefore say that the cohomology of sheaves does not depend on the choice of universes and one can always, for the need of a proof or a construction, augment the universe to calculate the cohomology of a sheaf.

Now consider the inclusion functor $\hat{\varepsilon} : \mathrm{PSh}(\mathcal{C})_{\mathscr{U}} \to \mathrm{PSh}(\mathcal{C})_{\mathscr{V}}$ from $\mathscr{U}$-presheaves to $\mathscr{V}$-presheaves. This functor is exact, so the derived functors of $\hat{\varepsilon}\mathcal{H}^0 : \mathrm{Sh}(\mathcal{C}_A) \to \mathrm{PSh}(\mathcal{C}_A)_{\mathscr{V}}$ are equal to $\hat{\varepsilon}\mathcal{H}^p$ for $p \geq 0$. By abusing of languages, we then denote them by $\mathcal{H}^p : \mathrm{Sh}(\mathcal{C}_A) \to \mathrm{PSh}(\mathcal{C}_A)_{\mathscr{V}}$. This enlargement of the universe has the following advantage if $\mathcal{C}$ is $\mathscr{V}$-small: The category $\mathrm{PSh}(\mathcal{C})_{\mathscr{U}}$ is in general not a $\mathscr{U}$-topos and the $\mathscr{U}$-presheaves of A-modules are not necessarily submodules of injective $\mathscr{U}$-presheaves, whereas the category of $\mathscr{V}$-presheaves is a $\mathscr{V}$-topos and therefore any $\mathscr{V}$-presheaf of A-modules is a sub-object of an injective $\mathscr{V}$-presheaf. Therefore for any $\mathscr{V}$-presheaf of sets R (and in particular when R is a $\mathscr{U}$-presheaf) and any $\mathscr{U}$-sheaf of A-modules M, the groups $H^p(R, \mathcal{H}^p(M))$ are defined by (1.6.3.1) and it follows from (1.6.23) that these groups do not depend on the considered universe. Similarly, for any pair of positive integer p and q, the presheaves $\check{\mathcal{H}}^p(\mathcal{H}^p(M))$ are defined by (1.6.21) and it follows from (1.6.23) and (1.6.20) that these presheaves do not depend on the universe $\mathscr{V}$ used to define them.

1.6.4 The Cartan-Laray spectral sequence

The classical Leray spectral sequence for a covering $\mathfrak{U}$ (also called the Čech-to-derive spectral sequence) of a topological space X relates the cohomology sheaf and Čech cohomology into a spectral sequence of the form

$$E_2^{p,q} = \check{H}^p(\mathfrak{U}, \mathcal{H}^q(X, \mathscr{F})) \Rightarrow H^{p+q}(X, \mathscr{F})$$

where $\mathscr{F}$ is a sheaf on X. This spectral sequence has many useful concequences. For example, if the cohomology vanishes for any finite intersections of the covering $\mathfrak{U}$, then the E_2-term degenerates and the edge morphisms yield an isomorphism of Čech cohomology for this covering to sheaf cohomology. This provides a method of computing sheaf cohomology using Čech cohomology: for instance, this happens if $\mathscr{F}$ is a quasi-coherent sheaf on a scheme and each element of $\mathfrak{U}$ is an open affine subscheme such that all finite intersections are again affine (e.g. if the scheme is separated).

In this paragraph we provide a direct genralization of the Leray spectral sequence for cohomology of topos. As we shall see, the language of derived functors and the Grothendieck spectral sequence can be used to give an easy proof of such generalizations. A relative version of this, which relates the sheaf cohomology with higher direct images of a morphism, will also be given after we introduce the notion of flasque sheaves.

Proposition 1.6.16. *Let $(\mathcal{C}, A)$ be a ringed $\mathscr{U}$-site and $\mathscr{V}$ be a universe containing $\mathscr{U}$. Then the functor $\mathcal{H}^0 : \mathrm{Sh}(\mathcal{C}_A) \to \mathrm{PSh}(\mathcal{C}_A)_{\mathscr{V}}$ transforms injective A-modules to injective presheaves. For any integer $p > 0$ and any A-module M, the sheaf associated with the presheaf $\mathcal{H}^p(M)$ is zero.*

Proof. We denote by $(-)^{\#}_{\mathscr{V}}$ the sheafification functor on $\mathscr{V}$-presheaves, and $\varepsilon : \mathrm{Sh}(\mathcal{C}_A) \to \mathrm{Sh}(\mathcal{C}_A)_{\mathscr{V}}$ the inclusion functor. Since we have $(\mathcal{H}^0)^{\#}_{\mathscr{V}} = \varepsilon$ by Proposition 1.3.20 and the functors $(-)^{\#}_{\mathscr{V}}$ and ε are exact, we conclude that $(\mathcal{H}^p)^{\#} = 0$ for any integer $p > 0$. Now for any $\mathscr{U}$-sheaf M and any $\mathscr{V}$-presheaf N, we have a functorial isomorphism

$$\mathrm{Hom}_{\mathrm{PSh}(\mathcal{C}_A)}(N, \mathcal{H}^0(M)) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Sh}(\mathcal{A})_{\mathscr{V}}}(N^{\#}_{\mathscr{V}}, \varepsilon(M)).$$

If M is injective, then $\varepsilon(M)$ is injective (Proposition 1.6.2) and as the functor $(-)^{\#}_{\mathscr{V}}$ is exact, the functor $\mathrm{Hom}_{\mathrm{PSh}(\mathcal{C}_A)}(-, \mathcal{H}^0(M))$ is exact, therefore $\mathcal{H}^0(M)$ is injective. □

Theorem 1.6.17. *Let $(\mathcal{C}, A)$ be a ringed $\mathscr{U}$-site, R be a $\mathscr{U}$-presheaf of sets over $\mathcal{C}$, M be a sheaf of A-modules. Then there exists a canonical spectral sequence*

$$E_2^{p,q} = H^p(R, \mathcal{H}^q(M)) \Rightarrow H^{p+q}(R^{\#}, M). \tag{1.6.25}$$

(If $\mathcal{C}$ is not $\mathscr{U}$-small, the term $H^p(R, \mathcal{H}^q(M))$ should be considered as the cohomology of the presheaf $\mathcal{H}^p(M)$ in the topos $\mathrm{PSh}(\mathcal{C}_A)_{\mathscr{V}}$, where $\mathscr{V}$ is a universe containing $\mathscr{U}$ such that $\mathcal{C}$ is $\mathscr{V}$-small.)

Proof. By the definition of the functor #, we have an isomorphism of functors
$$H^0(R^\#, M) \xrightarrow{\sim} H^0(R, \mathcal{H}^0(M)).$$
The functor $\mathcal{H}^0$ transforms injective objects to injective objects, so we conclude the spectral sequence from Proposition 1.6.5. $\square$

Corollary 1.6.18. *Let X be an object of $\mathcal{C}$ and $\mathfrak{U} = \{U_i \to X\}$ be a covering family of X. Then we have the following Cartan-Leray spectral sequence*
$$E_2^{p,q} = H^p(\mathfrak{U}, \mathcal{H}^p(M)) \Rightarrow H^{p+q}(X, M). \tag{1.6.26}$$

Proof. Let $R \hookrightarrow X$ be the sieve generated by $\mathfrak{U}$. As the sieve is covering, the sheaf associated with R is the sheaf associated with X (Proposition 1.3.25), so we have $H^{p+q}(R^\#, M) = H^{p+q}(X, M)$ in view of the definition of H^p. The corollary then follows from Proposition 1.6.13. $\square$

Corollary 1.6.19. *There exists a canonical spectral sequence on sheaves M and the objects X of $\mathcal{C}$:*
$$E_2^{p,q} = \check{H}^p(X, \mathcal{H}^p(M)) \Rightarrow H^{p+q}(X, M). \tag{1.6.27}$$
As X varies in $\mathcal{C}$, this spectral sequence gives a spectral sequence of presheaves
$$E_2^{p,q} = \check{\mathcal{H}}^p(\mathcal{H}^p(M)) \Rightarrow \mathcal{H}^{p+q}(M), \tag{1.6.28}$$
which gives canonical edge morphisms
$$\phi^p(M) : \check{\mathcal{H}}^p(M) \to \mathcal{H}^p(M), \tag{1.6.29}$$
$$\phi_X^p(M) : \check{H}^p(X, M) \to H^p(X, M). \tag{1.6.30}$$
The morphisms $\phi^p(M)$ and $\phi_X^p(M)$ are isomorphisms for $p = 0, 1$, and are monomorphisms for $p = 2$. In general, if the presheaf $\mathcal{H}^i(M)$ is zero for $0 < i < n$, then the morphisms $\phi^p(M)$ and $\phi_X^p(M)$ are isomorphic for $0 \le p \le n$ and monomorphic for $p = n + 1$.

Proof. The first spectral sequence are obtained by passing to inductive limits in the spectral sequence (1.6.26) over covering sieves $R \hookrightarrow X$, and the second one is induced in view of (1.6.19). By Theorem 1.6.17, the sheaf associated with the presheaf $\mathcal{H}^p(M)$ is zero if $p > 0$, which implies $\check{\mathcal{H}}^0\mathcal{H}^q(M) = 0$ for $p > 0$ (Proposition 1.3.20). The assertions on the induced morphisms ϕ^p and ϕ_X^p therefore follows. $\square$

Corollary 1.6.20. *Let $(\mathcal{X}, A)$ be a ringed topos and M be a (left) A-module. Denote by $\mathscr{M}$ the underlying abelian group of M. Then the functor $M \mapsto \mathscr{M}$ is exact and for any object X of $\mathcal{X}$, we have a canonical isomorphism*
$$H^0(X, M) \xrightarrow{\sim} H^0(X, \mathscr{M})$$
which extendes to isomorphisms
$$H^p(X, M) \xrightarrow{\sim} H^p(X, \mathscr{M}) \quad \text{for} \quad p \ge 0. \tag{1.6.31}$$

Proof. For any object Y of $\mathcal{X}$, we have
$$\check{H}^p(Y, M) = \varinjlim_{\mathfrak{U}} H^p(\mathfrak{U}, M), \quad \check{H}^p(Y, \mathscr{M}) = \varinjlim_{\mathfrak{U}} H^p(\mathfrak{U}, \mathscr{M})$$
where the limit is taken over covering families $\mathfrak{U}$. Since the cohomology $H^p(\mathfrak{U}, -)$ commutes with restriction of scalars (Proposition 1.6.13 (b)), we conclude that the canonical homomorphism $\check{H}^p(Y, M) \to \check{H}^p(Y, \mathscr{M})$ is an isomorphism. Now if M is an injective A-module, we have have $\check{\mathcal{H}}^p(\mathscr{M}) = 0$ for $p > 0$, whence $\mathcal{H}^p(\mathscr{M}) = 0$ by induction on p and use Corollary 1.6.19. It then follows that $H^p(X, \mathscr{M}) = 0$ for $p > 0$, so the functor $M \mapsto \mathscr{M}$ transforms injective objects to acyclic objects for the functor $H^0(X, -)$, and we can apply Proposition 1.6.5 to get the isomorphisms (1.6.31). $\square$

Example 1.6.21. Let G be a topological group and $\mathcal{B}G$ be the classifying topos. Let E_G be the left regular representation of G (given by left translations of G), which is an object of $\mathcal{B}G$. The canonical morphism from E_G to the final object e_G of B_G is easily seen to be an epimorphism, so it gives a covering $\mathfrak{U} = \{E_G \to e_G\}$ and, for any abelian sheaf F on $\mathcal{B}G$, a spectral sequence

$$E_2^{p,q} = H^p(\mathfrak{U}, \mathcal{H}^p(F)) \Rightarrow H^{p+q}(\mathcal{B}G, F). \tag{1.6.32}$$

By definition, the E_2 terms of this spectral sequence is computed as the cohomology of the following sequence

$$H^p(E_G, F) \longrightarrow H^p(E_G \times E_G, F) \longrightarrow H^p(E_G \times E_G \times E_G, F) \longrightarrow \cdots$$

Let $\mathrm{Topos}(G)$ denote the topos of sheaves over the big site $\mathbf{Top}_{/G}$ associated with G. We claim that for each integer $n > 0$, there is a canonical equivalence

$$\mathcal{B}G_{/(E_G)^{\times n}} \xrightarrow{\sim} \mathrm{Topos}(G^{\times(n-1)})$$

which associates each G-set X over $(E_G)^{\times n}$ with its orbit set X/G. To see this is an equivalence, it suffices to note that any G-set X over $(E_G)^{\times n}$ has a faithful action by G, hence isomorphic to a product of copies of G. By passing to quotient we then obtain a morphism.

1.6.5 Acyclic sheaves

Let $(\mathcal{X}, A)$ be a ringed topos, F be an A-module, S be a topological generating family of $\mathcal{X}$. The sheaf F is said to be ***S*-acyclic** if for any object X of S and any integer $p > 0$, we have $H^p(X, F) = 0$, and **$\mathcal{C}$-acyclic** if S is the family of sheaves assocated with the objects of $\mathcal{C}$. If S is equal to $\mathrm{Ob}(\mathcal{X})$, the S-acyclic sheaves are then called **flasque sheaves**.

Proposition 1.6.22. *Let $(\mathcal{C}, A)$ be a ringed $\mathcal{U}$-site, F be a sheaf of A-modules. Denote by $\mathcal{H}^0$: $\mathrm{Sh}(\mathcal{C}_A) \to \mathrm{PSh}(\mathcal{C}_A)$ the canonical inclusion functor. The following conditions are equivalent:*

(i) *F is $\mathcal{C}$-acyclic.*

(ii) *For any object X of $\mathcal{C}$ and any covering sieve $R \hookrightarrow X$, we have $H^p(R, \mathcal{H}^0(F)) = 0$ for $p > 0$.*

(iii) *For any object X of $\mathcal{C}$, we have $\check{H}^p(X, F) = 0$ for $p > 0$.*

Proof. If F is $\mathcal{C}$-acyclic, the presheaf $\mathcal{H}^p(F)$ is zero for $p > 0$, so the spectral sequence (1.6.25) gives an isomorphism $H^p(R, \mathcal{H}^0(F)) \cong H^p(X, F)$ for $p > 0$, which implies (ii). By passing to inductive limit on R, it is immediate that (ii)⇒(iii). Conversely, if $\check{H}^p(X, F) = 0$ for $p > 0$, then by induction and Corollary 1.6.19 we conclude that $\mathcal{H}^p(F) = 0$ for $p > 0$, whence (i). □

It follows from Proposition 1.6.22(ii) and Corollary 1.6.20 that the property of S-acyclicity only depends on the underlying abelian sheaf. In particular, a sheaf of A-modules is flasque if and only if the underlying abelian sheaf is flasque.

Corollary 1.6.23. *Let $(\mathcal{X}, A)$ be a ringed topos and F be an A-module. The following properties are equivalent:*

(i) *F is flasque;*

(ii) *for any epimorphic family $\mathfrak{U} = \{X_i \to X\}$, $H^p(\mathfrak{U}, F) = 0$ for $p > 0$.*

Proof. In the definition, we can take $\mathcal{C}$ to be the topos $\mathcal{X}$, so the corollary follows from the equivalence (i)⇔(ii) of Proposition 1.6.22. □

Any injective sheaf is by definition flasque, and flasque sheaves are S-acyclic for any topological generating family S. Note that a flasque sheaf is not necessarily injective (for example consider the topos of sets). An S-acyclic sheaf is also not necessarily flasque.

Proposition 1.6.24. *Let $(\mathcal{X}, A)$ be a topos, F be a flasque A-module, X be an object of $\mathcal{X}$. Then for any sub-object Y of X, the canonical homomorphism $H^0(X, F) \to H^0(Y, F)$ is surjective.*

Proof. Let Y be a sub-object of X such that the morphism $H^0(X, F) \to H^0(Y, F)$ is not surjective, and Z be the object obtained by glueing two copies of X along Y. The object Z is then covered by two sub-objects X_1 and X_2 isomorphic to X, and we have $X_1 \times_Z X_2 = Y$. If $\mathfrak{U} = \{X_1, X_2\}$ is the corresponding covering of Z, we then have $H^1(\mathfrak{U}, F) \neq 0$, which is a contradiction. □

The criterion of Proposition 1.6.24 is not sufficient to characterize, in the case of general topos, flasque sheaves. However, it characterizes them in the case of topos generated by their open sets and in particular in the case of topos associated with topological spaces. Therefore, our terminology adopted here coincides with the terminology for classical flasque sheaves over topological spaces.

Proposition 1.6.25. *Let $f : (\mathcal{X}, A) \to (\mathcal{Y}, B)$ be a morphism of ringed topoi.*

(a) *The functor f_* transforms flasque A-modules into flasque B-modules.*

(b) *Let S (resp. T) be a topological generating family of $\mathcal{X}$ (resp. $\mathcal{Y}$) such that $f^*(T) \subseteq S$. Then the functor f_* transforms S-acyclic A-modules to T-acyclic T-modules.*

(c) *If f is a flat morphism, the functor f_* transforms injective A-modules to injective B-modules.*

Proof. Let Y be an object of $\mathcal{Y}$, $\mathfrak{V} = \{Y_i \to Y\}$ be an epimorphic family, F be a flasque A-module, $C^\bullet(\mathfrak{U}, f_*(F))$ be the Čech complex of covering $\mathfrak{V}$. By using the adjunction of f_* and f^* and the fact that f^* commutes with fiber products, we then obtain a canonical isomorphism

$$C^\bullet(\mathfrak{V}, f_*(F)) \cong C^\bullet(f^*(\mathfrak{V}), F).$$

Since f^* also commutes with inductive limits, $f^*(\mathfrak{V})$ is an epimorphic family, and we then conclude that $H^p(f^*(\mathfrak{V}), F) = 0$ for $p > 0$ since F is flasque, whence

$$H^p(\mathfrak{V}, f_*(F)) = H^p(C^\bullet(\mathfrak{V}, f_*(F))) = H^p(C^\bullet(f^*(\mathfrak{V}), F)) = H^p(f^*(\mathfrak{V}), F) = 0 \quad \text{for} \quad p > 0.$$

We then conclude that $f_*(F)$ is flasque (Corollary 1.6.23), whence the first assertion. The second assertion can be done similarly if the family T is stable under fiber products. In the general case, we can use the spectral sequence (1.6.36) of the morphism f, since its proof only depends on assertion (a). Let F be S-acyclic sheaf. Then $R^p f_*(F)$ are the sheaves associated with the presheaves $Y \mapsto H^p(f^*(Y), F)$. As T is a topologically generating family and F is S-acyclic, we have $R^p f_*(F) = 0$ for $p > 0$. The spectral sequence (1.6.36) then provides a cannical isomorphism $H^p(Y, f_*(F)) \cong H^p(f^*(Y), F)$ for $Y \in \mathrm{Ob}(\mathcal{Y})$, so we conclude that $H^p(Y, f_*(F)) = 0$ for $p > 0$ and $Y \in T$, which means F is T-acyclic. Finally, if f is flat, then the functor f^* is exact on modules, so f_* transforms injective objects to injective objects (Proposition 1.6.5). □

Proposition 1.6.26. *Let F be an A-module over a ringed topos $(\mathcal{X}, A)$ and I be an injective A-module.*

(a) *The functor $M \mapsto \mathcal{H}om_A(M, G)$ is exact.*

(b) *The abelian sheaf $\mathcal{H}om_A(F, G)$ is flasque.*

Proof. To prove the first assertion, we consider an exact sequence

$$0 \longrightarrow F' \longrightarrow F \longrightarrow F'' \longrightarrow 0$$

which induces a sequence

$$0 \longrightarrow \mathcal{H}om_A(F'', G) \longrightarrow \mathcal{H}om_A(F, G) \longrightarrow \mathcal{H}om_A(F', G) \longrightarrow 0$$

To prove that this sequence is exact, it suffices to consider, for any object H of $\mathcal{X}$, the following sequence

$$0 \to \mathrm{Hom}_{\mathcal{X}}(H, \mathcal{H}om_A(F'', G)) \to \mathrm{Hom}_{\mathcal{X}}(H, \mathcal{H}om_A(F, G)) \to \mathrm{Hom}_{\mathcal{X}}(H, \mathcal{H}om_A(F', G)) \to 0$$

which is isomorphic to the sequence

$$0 \longrightarrow \mathrm{Hom}_A(A_H \otimes_A F'', G) \longrightarrow \mathrm{Hom}_A(A_H \otimes_A F, G) \longrightarrow \mathrm{Hom}_A(A_H \otimes_A F', G) \longrightarrow 0$$

Since the A-module A_H is flat (Proposition 1.6.10), this sequence is exact, whence the desired assertion. Now to prove that $\mathcal{H}om_A(F, G)$ is flasque, we consider an epimorphic family $\mathfrak{U} = \{X_i \to X\}$ and the complex $\mathbb{Z}_\bullet(\mathfrak{U})$ defined in (1.6.7). This complex is a flat resolution of the object $\mathbb{Z}_X$, which is also flat by (Proposition 1.6.10). From the definition of Čech cohomology, we then have

$$H^p(\mathfrak{U}, \mathcal{H}om_A(F, G)) \xrightarrow{\sim} H^p(\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_\bullet(\mathfrak{U}), \mathrm{Hom}_A(F, G))) \xrightarrow{\sim} H^p(\mathrm{Hom}_A(\mathbb{Z}_\bullet(\mathfrak{U}) \otimes_{\mathbb{Z}} F, G)).$$

The complex $Z_\bullet(\mathfrak{U}) \otimes_{\mathbb{Z}} F$ is exact at nonzero degrees because $\mathbb{Z}_\bullet(\mathfrak{U})$ is a flat resolution of a flat module, so $\mathrm{Hom}(\mathbb{Z}_\bullet(\mathfrak{U}) \otimes_{\mathbb{Z}} F, G)$ is acyclic at nonzero degrees, which proves our assertion. □

Proposition 1.6.27. *Let $(\mathcal{X}, A)$ be a ringed topos, F be a flasque (resp. injective) sheaf of A-modules.*

(a) *For any object X of $\mathcal{X}$, the $A|_X$-module $j_X^*(F)$ is flasque (resp. injective).*

(b) *For any closed $\mathcal{Z}$ of $\mathcal{X}$, the sheaf of sections of F supported in $\mathcal{Z}$ is flasque (resp. injective).*

Proof. The first assertion follows from Proposition 1.6.25 if F is flasque. The functor j_X^* admits a left adjoint $(j_X)_!$, so if F is injective, $j_X^*(F)$ is also injective (Proposition 1.6.2). As for (b), let $i : \mathcal{Z} \to \mathcal{X}$ be the inclusion morphism. The sheaf of sections of F with support in $\mathcal{Z}$ is then given by $i^*i^!(F)$ (**[?]** IV, 14). Since $i_*i^!$ is right adjoint to i_*i^*, which is exact by (**[?]** IV, 14), we conclude that it transforms injective sheaves to injective sheaves. Now let $\mathcal{U}$ be the open complement of $\mathcal{Z}$, $j : \mathscr{U} \to \mathcal{X}$ be the inclusion morphism, and F be a flasque sheaf. We then have an exact sequence (**[?]** IV, 14)

$$0 \longrightarrow i_*i^!(F) \longrightarrow F \longrightarrow j_*j^*(F) \tag{1.6.33}$$

For any object X of $\mathcal{X}$, we have $j_*j^*(F)(X) = F(X \times \mathcal{U})$ and the induced morphism $F(X) \to j_*j^*(F)(X)$ is given by the canonical inclusion $X \times \mathcal{U} \hookrightarrow X$. As F is flasque, this morphism is surjective by Proposition 1.6.24, so the last arrow of (1.6.33) is an epimorphism of presheaves. For any object X of $\mathcal{X}$, the long exact sequence induced by Proposition 1.6.24 shows that $H^p(X, i_*i^!(F)) = 0$ for $p > 0$, so $i_*i^!(F)$ is flasque. □

Example 1.6.28. Let X be a locally compact space and F be a c-soft sheaf on X.

Example 1.6.29. Let G be a discrete group and $\mathcal{B}G$ be the classifying topos. Prove that for any abelian sheaf F and any monomorphism $X \hookrightarrow Y$, the homomorphism $F(Y) \to F(X)$ is surjective. Let E_G be the group G considered as a G-set with left translations of G. Then the topos $\mathcal{B}G_{/E_G}$ can be identified with the pointed topos. The morphism $E_G \to e_G$ (e_G being the final object of $\mathcal{B}G$) is an epimorphism. For any abelian sheaf F of $\mathcal{B}G$, the sheaf $F|_{E_G}$ is flasque. Deduce that the property of being flasque (or injective) is not a local property.

Example 1.6.30. We say a topos $\mathcal{X}$ is **generated by its opens** if the opens of $\mathcal{X}$ (i.e. the subobjects of the final object e of E) form a generating family. Such a topos has the following property:

(P) Any epimorphic family $\{X_i \to X\}$ is dominated by an epimorphic family $\{U_j \to X\}$, where the $U_j \to X$ are monomorphisms.

As an application of flasque sheaves, we now consider a morphism $f : (\mathcal{X}, A) \to (\mathcal{Y}, B)$ of ringed topoi and the direct image functor $f_* : \mathcal{X} \to \mathcal{Y}$ induced by f. The functor f_* is left exact on the category of (left) modules, let $R^p f_*$ denote its right derived functors. The higher direct images $R^p f_*$ behaves much like the classical case, and we also have a Leray spectral sequence relating $R^p f_*$ and sheaf cohomology H^p.

Proposition 1.6.31. *Let $f : (\mathcal{X}, A) \to (\mathcal{Y}, B)$ be a morphism of ringed topoi and M be an A-module.*

(a) *The sheaf $R^p f_*(M)$ is the sheaf associated with the presheaf $Y \mapsto H^p(f^*(Y), M)$.*

(b) *The formation of $R^p f_*$ commutes with restriction of scalars.*

(c) *The formation $R^p f_*$ commutes with localization. More precisely, for any object Y of $\mathcal{X}$, if we denote by $f_{/X} : \mathcal{X}_{/X} \to \mathcal{Y}_{/Y}$ the induced morphism under localization, where $X = f^*(Y)$, we have, for any A-module M, a canonical isomorphism*

$$R^p(f_{/X})_*(M|_X) \xrightarrow{\sim} R^p f_*(M)|_Y \quad \textit{for} \quad p \geq 0. \tag{1.6.34}$$

Proof. We denote by $\hat{f}_* : \mathrm{PSh}(\mathcal{X}) \to \mathrm{PSh}(\mathcal{Y})$ the direct image functor for $\mathcal{U}$-presheaves (that is, $\hat{f}_*(M) = M \circ f^*$). As f^* and f_* are adjoints, we have an isomorphism $f_* = (\hat{f}_*)^\#$. But the functors $\#$ and $\hat{f}_*$ are exact, so this implies

$$R^p f_* \cong (\hat{f}_*)^\# \mathcal{H}^p,$$

which is equivalent to assertion (a). Assertion (b) then follows from (a) and Corollary 1.6.20. To prove (c), let Y be an object of $\mathcal{Y}$ and consider the induced morphism $f_{/X} : \mathcal{X}_{/X} \to \mathcal{Y}_{/Y}$, where $X = f^*(Y)$. By the definition of the morphism $f_{/X}$, we have the canonical isomorphism

$$(f_{/X})_*(M|_X) \xrightarrow{\sim} f_*(M)|_Y.$$

The case general for higher direct images is then deduced by noting that the localization functors are exact and transform injective objects into injective objects (Proposition 1.6.2). □

Proposition 1.6.32. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of topoi and T be a topological generating family of $\mathcal{Y}$. Then the sheaves M acyclic for the functors $H^0(f^*(Y), -)$, $Y \in T$ are acyclic for the functor f_*. In particular, the flasque sheaves are acyclic for f_*.*

Proof. This follows from the assertion of Proposition 1.6.31 (a). □

Proposition 1.6.33. *Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of topoi and M be an abelian sheaf on $\mathcal{X}$. Then we have a spectral sequence*

$$E_2^{p,q} = H^p(\mathcal{Y}, R^p f_*(M)) \Rightarrow H^{p+q}(\mathcal{X}, M). \tag{1.6.35}$$

More generally, for any object Y of $\mathcal{Y}$, we have a spectral sequence

$$E_2^{p,q} = H^p(Y, R^p f_*(M)) \Rightarrow H^{p+q}(f^*(Y), M). \tag{1.6.36}$$

Proof. By the definitions of the direct image and inverse image functors, we have a canonical isomorphism $H^0(Y, f_*(M)) \xrightarrow{\sim} H^0(f^*(Y), M)$. The functor f_* transforms injective objects to flasque objects (Proposition 1.6.25), so the spectral sequence follows from Proposition 1.6.5. □

Remark 1.6.34 (Flasque sheaves and changing the universe). Let $\mathcal{C}$ be a $\mathscr{U}$-site (for example a $\mathscr{U}$-topos) and $\mathscr{V}$ be a universe containing $\mathscr{U}$. Let $\varepsilon : \mathrm{Sh}(\mathcal{C})_{\mathscr{U}} \to \mathrm{Sh}(\mathcal{C})_{\mathscr{V}}$ be the canonical injection functor. Let F be an abelian $\mathscr{U}$-sheaf that is flasque over $\mathcal{C}$. Then for any object X of $\mathrm{Sh}(\mathcal{C})_{\mathscr{U}}$, we have $H^p(\varepsilon(X), \varepsilon(F)) = 0$ for $p > 0$. Since any object Y in $\mathrm{Sh}(\mathcal{C})_{\mathscr{V}}$ admits an epimorphic family $\{\varepsilon(X_i) \to Y\}$ where $\varepsilon(X_i) \to Y$ are monomorphisms, we then conclude that

$$H^p(Y, \varepsilon(F)) = \varprojlim\nolimits^p_{\varepsilon(Y) \hookrightarrow Y} F(X).$$

By reducing to the topos $\mathrm{Sh}(\mathrm{Ouv}(\mathcal{C}_{/Y}))_{\mathscr{V}}$ and use the fact that in this topos, a sheaf is flasque if it is locally flasque, we conclude that $\varepsilon(F)$ is flasque.

1.6.6 Local Ext and cohomology with closed support

1.6.6.1 The local and lobal Ext Let $(\mathcal{X}, A)$ be a ringed topos and M be a (left) A-module. The functor $N \mapsto \mathrm{Hom}_A(M, N)$ from the category of left A-modules to the category of abelian groups is left exact, and its derived functors are then denoted by $\mathcal{E}xt^p_A(M, N)$. In particular, we have

$$\mathcal{E}xt^0_A(M, N) = \mathcal{H}om_A(M, N).$$

By definition, for any object X of $\mathcal{X}$, we have the following canonical isomorphisms

$$H^0(X, \mathcal{E}xt^0_A(M, N)) = \mathrm{Ext}^0_A(X; M, N) = \mathrm{Hom}_{A|_X}(M|_X, N|_X).$$

Proposition 1.6.35. *Let $(\mathcal{X}, A)$ be a ringed topos and M, N be left A-modules.*

(a) *The formation of $\mathcal{E}xt^p_A$ commutes with localizations. More precisely, for any object X of $\mathcal{X}$, we have a functorial isomorphism*

$$\mathcal{E}xt^p_A(M, N)|_X \xrightarrow{\sim} \mathcal{E}xt^p_{A|_X}(M|_X, N|_X). \tag{1.6.37}$$

(b) *The sheaf $\mathcal{E}xt^p_A(M, N)$ is the sheaf associated with the presheaf $X \mapsto \mathrm{Ext}^p_A(X; M, N)$.*

(c) *There is a spectral sequence*

$$E_2^{p,q} = H^p(\mathcal{X}, \mathcal{E}xt^p_A(F, G)) \Rightarrow \mathrm{Ext}^p_A(\mathcal{X}; M, N). \tag{1.6.38}$$

More generally, for any object X of $\mathcal{X}$, we have a functorial spectral sequence

$$E_2^{p,q} = H^p(X, \mathcal{E}xt^p_A(F, G)) \Rightarrow \mathrm{Ext}^p_A(X; M, N). \tag{1.6.39}$$

Proof. By the definition of $\mathcal{H}om$, we have the isomorphism (1.6.37) for $p = 0$, so the general case follows from the fact that the localization functor is exact and transforms injective modules to injective modules (1.6.2). The functor $N \mapsto \mathcal{H}om_A(M, N) = \mathcal{E}xt^0_A(M, N)$ transforms injective modules to flasque sheaves (hence flasque for $H^0(X, -)$), so the spectral sequences (1.6.38) and (1.6.39) follow from Proposition 1.6.5. If X varies in $\mathcal{X}$, the spectral sequence (1.6.39) becomes a spectral sequence for presheaves, whence by passing to sheafification, a spectral sequence for presheaves. As the sheaf associated with the presheaf $X \mapsto H^p(X, -)$ is zero for $p > 0$ (Proposition 1.6.16), this spectral sequence degenerates and we obtain the isomorphism in (b). □

Proposition 1.6.36. *The functors $(M, N) \mapsto \mathcal{E}xt^p_A(M, N)$ form a δ-functor on the variable M or the variable N, and the same is true for the functors $(M, N) \mapsto \mathrm{Ext}^p_A(X; M, N)$ for any object X of $\mathcal{X}$.*

Proof. This follows from the general property of the functors Ext^p_A that for any object X of $\mathcal{X}$, the functors $(M, N) \mapsto \mathrm{Ext}^p_A(X; M, N)$ form a δ-functor on each of its variable, whence our assertion in view of Proposition 1.6.35 (b). □

1.6.6.2 Cohomology with support in a closed topos Let $(\mathcal{X}, A)$ be a ringed topos, M be an A-module, $\mathcal{Z}$ be a closed of $\mathcal{X}$ ([?] IV, 9), $\mathcal{U}$ be the complement of $\mathcal{Z}$. We denote by $H^0_{\mathcal{Z}}(\mathcal{X}, M)$ the group of sections of M whose support is contained in $\mathcal{Z}$ ([?] IV, 14) and $\mathcal{H}^0_{\mathcal{Z}}(M)$ the subsheaf of M defined by the section of M "with support in $\mathcal{Z}$" ([?] IV, 14). The functors $H^0_{\mathcal{Z}}(\mathcal{X}, -)$ and $\mathcal{H}^0_{\mathcal{Z}}(-)$ are both left exact by ([?] IV, 14). Their right derived functors are denoted by $H^p_{\mathcal{Z}}(\mathcal{X}, -)$ and $H^p_{\mathcal{Z}}(-)$, respectively, and called the cohomology group (resp. sheaf) of M with support in $\mathcal{Z}$.

We have the following canonical isomorphisms ([?] IV, 14):

$$H^0_{\mathcal{Z}}(\mathcal{X}, M) \xrightarrow{\sim} \mathrm{Hom}_A(A_{\mathcal{Z}}, M) \xrightarrow{\sim} \mathrm{Ext}^0_A(\mathcal{X}; A_{\mathcal{Z}}, M), \tag{1.6.40}$$

$$\mathcal{H}^0_{\mathcal{Z}}(M) \xrightarrow{\sim} \mathcal{H}om_A(A_{\mathcal{Z}}, M) \xrightarrow{\sim} \mathcal{E}xt^0_A(A_{\mathcal{Z}}, M), \tag{1.6.41}$$

whence the isomorphisms for $p \geq 0$:

$$H^p_{\mathcal{Z}}(\mathcal{X}, M) \xrightarrow{\sim} \mathrm{Ext}^p_A(\mathcal{X}; A_{\mathcal{Z}}, M), \tag{1.6.42}$$

$$\mathcal{H}^p_{\mathcal{Z}}(M) \xrightarrow{\sim} \mathcal{E}xt^p_A(A_{\mathcal{Z}}, M). \tag{1.6.43}$$

We remark that, since $A_{\mathcal{Z}}$ is a bimodule, the sheaves $\mathrm{Ext}^p_A(A_{\mathcal{Z}}, M) \cong \mathcal{H}^p_{\mathcal{Z}}(M)$ are canonically endowed with a structure of A-modules.

For any object X of $\mathcal{X}$, let $\mathcal{Z}_{/X}$ be the sub-topos of $\mathcal{X}_{/X}$ induced by localization (which is the complement of $\mathcal{U} \times X$). By definition, we have a canonical isomorphism

$$H^0(X, \mathcal{H}^0_{\mathcal{Z}}(M)) \xrightarrow{\sim} \mathcal{H}^0_{\mathcal{Z}_{/X}}(\mathcal{X}_{/X}, M|_X). \tag{1.6.44}$$

If we define

$$H^p_{\mathcal{Z}}(X, M) := H^p_{\mathcal{Z}_{/X}}(\mathcal{X}_{/X}, M|_X) \tag{1.6.45}$$

we then have, in view of (1.6.41), canonical isomorphisms

$$H^p_{\mathcal{Z}}(X, M) \xrightarrow{\sim} \mathrm{Ext}^p_A(X; A_{\mathcal{Z}}, M). \tag{1.6.46}$$

Proposition 1.6.37. *Let $(\mathcal{X}, A)$ be a ringed topos and $\mathcal{Z}$ be a closed of $\mathcal{X}$.*

(a) *The formation of $\mathcal{H}^p_{\mathcal{Z}}$ commutes with localizations. More precisely, for any object X of $\mathcal{X}$ and any A-module M, we have canonical isomorphisms*

$$\mathcal{H}^p_{\mathcal{Z}}(M)|_X \xrightarrow{\sim} \mathcal{H}^p_{\mathcal{Z}_{/X}}(M|_X). \tag{1.6.47}$$

(b) *The sheaf $\mathcal{H}^p_{\mathcal{Z}}(M)$ is the sheaf associated with the presheaf $X \mapsto H^p_{\mathcal{Z}}(X, M)$.*

(c) *There exists a spectral sequence*

$$E_2^{p,q} = H^p(\mathcal{X}, \mathcal{H}^p_{\mathcal{Z}}(M)) \Rightarrow H^{p+q}_{\mathcal{Z}}(\mathcal{X}, M). \tag{1.6.48}$$

More generally, for any object X of $\mathcal{X}$, there is a spectral sequence

$$E_2^{p,q} = H^p(\mathcal{X}, \mathcal{H}^p_{\mathcal{Z}}(M)) \Rightarrow H^{p+q}_{\mathcal{Z}}(\mathcal{X}, M). \tag{1.6.49}$$

Proof. From the isomorphisms (1.6.40) and (1.6.41), the proposition is a particular case of Proposition 1.6.35. □

Proposition 1.6.38. *With the notations of Proposition 1.6.37, let $j : \mathcal{U} \to \mathcal{X}$ be the canonical inclusion. For any A-module M, there exist an exact sequence of sheaves:*

$$0 \longrightarrow \mathcal{H}^0_{\mathcal{Z}}(M) \longrightarrow M \longrightarrow j_*(M|_U) \longrightarrow \mathcal{H}^1_{\mathcal{Z}}(M) \longrightarrow 0 \tag{1.6.50}$$

and isomorphisms for $p \geq 2$:

$$\mathcal{H}^p_{\mathcal{Z}} \xrightarrow{\sim} \mathcal{E}xt^{p-1}_A(A_{\mathcal{U}}, M) \xrightarrow{\sim} R^{p-1}j_*(M|_U). \tag{1.6.51}$$

Moreover, we have a long exact sequence

$$\cdots \longrightarrow H^p_{\mathcal{Z}}(\mathcal{X}, M) \longrightarrow H^p(\mathcal{X}, M) \longrightarrow H^p(\mathcal{U}, M) \longrightarrow H^{p+1}(\mathcal{X}, M) \longrightarrow \cdots \tag{1.6.52}$$

and more generally, for any object X of $\mathcal{X}$, we have a long exact sequence

$$\cdots \longrightarrow H^p_{\mathcal{Z}}(X, M) \longrightarrow H^p(X, M) \longrightarrow H^p(X \times \mathcal{U}, M) \longrightarrow H^{p+1}(\mathcal{X}, M) \longrightarrow \cdots \tag{1.6.53}$$

Proof. By the definition of $A_{\mathcal{Z}}$, we have an exact sequence ([?] iv, 14)

$$0 \longrightarrow A_{\mathcal{U}} \longrightarrow A \longrightarrow A_{\mathcal{Z}} \longrightarrow 0 \tag{1.6.54}$$

We have $\mathcal{H}om_A(A_{\mathcal{U}}, M) \cong j_*(M|_U)$ ([?] iv, 14), so $\mathcal{E}xt^p_A(A_{\mathcal{U}}, M) \cong R^p j_*(M|_U)$ by (1.6.10). On the other hand, the functor $\mathcal{E}xt^p_A(A, -)$ is zero for $p > 0$ and $\mathcal{E}xt^p_A(A_{\mathcal{Z}}, M) \cong \mathcal{H}^p_{\mathcal{Z}}(M)$, so the exact sequences (1.6.50) and (1.6.51) follow from induced the long exact sequence of $\mathcal{E}xt_A(-, M)$. The long exact sequences (1.6.51) and (1.6.52) follows from the long exact sequence of the δ-functors $\mathrm{Ext}^p_A(X; -, M)$ (note that $\mathrm{Ext}^p_A(X; A, -)$ is usually nontrivial). □

Proposition 1.6.39. *With the notations of Proposition 1.6.38:*

(a) *the flasque sheaves are acyclic for the functors $\mathcal{H}^0_{\mathcal{Z}}$ and $H^0_{\mathcal{Z}}(X, -)$;*

(b) *the functors $\mathcal{H}^p_{\mathcal{Z}}$ and $H^p_{\mathcal{Z}}(X, -)$ commutes with restriction of scalars.*

Proof. Let M be a flasque sheaf. Then Proposition 1.6.38 implies the equality $H^p_{\mathcal{Z}}(X, M) = 0$ for any X and any $p \geq 2$, and there is an exact sequence

$$0 \longrightarrow H^0_{\mathcal{Z}}(X, M) \longrightarrow H^0(X, M) \longrightarrow H^0(X \times \mathcal{U}, M) \longrightarrow H^1_{\mathcal{Z}}(X, M) \longrightarrow 0$$

Since the sheaf M is flasque, the morphism $H^0(X, M) \to H^0(X \times \mathcal{U}, M)$ is surjective (Proposition 1.6.24), so $H^1_{\mathcal{Z}}(X, M) = 0$ and M is acyclic for $H^0_{\mathcal{Z}}(X, M)$. By passing to associate sheaf, we conclude that M is acyclic for $\mathcal{H}^0_{\mathcal{Z}}$ (Proposition 1.6.37). It is clear that the functors $\mathcal{H}^0_{\mathcal{Z}}$ and $H^0_{\mathcal{Z}}(X, -)$ commutes with restriction of scalars, and as the flasque sheaves are acyclic for them, assertion (b) follows from Proposition 1.6.5. □

Proposition 1.6.40. *Let $(\mathcal{X}, A)$ be a ringed topos, $\mathcal{Z}$ be a closed of $\mathcal{X}$, $\mathcal{U}$ be the open complement of $\mathcal{Z}$, M, N be two A-modules. There exists a functorial isomorphism comtatible with closed base change:*

$$\mathcal{H}^0_{\mathcal{Z}}(\mathcal{H}om_A(M, N)) \xrightarrow{\sim} \mathcal{H}om_A(M, \mathcal{H}^0_{\mathcal{Z}}(N)) \xrightarrow{\sim} \mathcal{H}om_A(M \otimes_A A_{\mathcal{Z}}, N). \tag{1.6.55}$$

Proof. This is the result of ([?] iv, 14), in view of (1.6.41). □

In view of the results of Proposition 1.6.40, we put

$$\mathcal{H}om_{A,\mathcal{Z}}(M,N) = \mathcal{H}^0_{\mathcal{Z}}(\mathcal{H}om_A(M,N)). \tag{1.6.56}$$

It is easy to see that the functor $N \mapsto \mathcal{H}om_{A,\mathcal{Z}}(M,N)$ is left exact. Its right derived functors are denoted by $\mathcal{E}xt^p_{A,\mathcal{Z}}(M,N)$, and called the extension sheaf with support in $\mathcal{Z}$. With this definition, we then have

$$\mathcal{E}xt^0_{A,\mathcal{Z}}(M,N) = \mathcal{H}om_{A,\mathcal{Z}}(M,N). \tag{1.6.57}$$

If $M_{\mathcal{Z}} = M \otimes_A A_{\mathcal{Z}}$, then it follows from ([?] iv, 14) that $M_{\mathcal{Z}}$ is equal to the direct image of the inverse image of M to $\mathcal{Z}$. We then have, in view of Proposition 1.6.40, that

$$\mathcal{E}xt^p_{A,\mathcal{Z}}(M,N) = \mathcal{E}xt^p_A(M_{\mathcal{Z}},N). \tag{1.6.58}$$

We now pass from local to global invariants, so define

$$\mathrm{Hom}_{A,\mathcal{Z}}(M,N) := \mathrm{Ext}^0_{A,\mathcal{Z}}(\mathcal{X};M,N) = H^0_{\mathcal{Z}}(\mathcal{X},\mathcal{H}om_A(M,N)). \tag{1.6.59}$$

The group $\mathrm{Hom}_{A,\mathcal{Z}}(M,N)$ is the subgroup of $\mathrm{Hom}_A(M,N)$ with support contained in $\mathcal{Z}$ ([?] iv, 14), i.e. which is zero over $\mathcal{U}$. More generally, for any objet X of $\mathcal{X}$, we set

$$\mathrm{Ext}^0_{A,\mathcal{Z}}(X;M,N) := H^0_{\mathcal{Z}}(X,\mathcal{H}om_A(M,N)). \tag{1.6.60}$$

The functor $N \mapsto \mathrm{Ext}^0_{A,\mathcal{Z}}(X;M,N)$ is clearly left exact, its derived functors are then denoted by $\mathrm{Ext}^p_{A,\mathcal{Z}}(X;M,N)$ and called the extension group of M and N with support in $\mathcal{Z}$. The definition (1.6.45), (1.6.56) and (1.6.60) and the isomorphism of Proposition 1.6.40 then give the following isomorpihsms

$$\mathrm{Ext}^0_{A,\mathcal{Z}}(X;M,N) \xrightarrow{\sim} H^0(X,\mathcal{H}om_{A,\mathcal{Z}}(M,N)) \xrightarrow{\sim} \mathrm{Ext}^0_A(X;M,\mathcal{H}^0_{\mathcal{Z}}(N)) \xrightarrow{\sim} \mathrm{Ext}^0_A(X;M_{\mathcal{Z}},N). \tag{1.6.61}$$

whence the following isomorphisms

$$\mathrm{Ext}^p_{A,\mathcal{Z}}(X;M,N) \xrightarrow{\sim} \mathrm{Ext}^p_A(X;M_{\mathcal{Z}},N). \tag{1.6.62}$$

Proposition 1.6.41. *Let $(\mathcal{X},A)$ be a ringed topos and $\mathcal{Z}$ be a closed of $\mathcal{X}$.*

(a) *There exist two functorial spectral sequence*

$$\left.\begin{aligned} {}_IE_2^{p,q} &= \mathcal{H}^p_{\mathcal{Z}}(\mathcal{E}xt^p_A(M,N)) \\ {}_{II}E_2^{p,q} &= \mathcal{E}xt^p_A(M,\mathcal{H}^p_{\mathcal{Z}}(N)) \end{aligned}\right\} \Rightarrow \mathcal{E}xt^{p+q}_{A,\mathcal{Z}}(M,N) \tag{1.6.63}$$

which are compatible with closed base changes.

(b) *There exist three functorial spectral sequence*

$$\left.\begin{aligned} {}_IE_2^{p,q} &= H^p_{\mathcal{Z}}(X,\mathcal{E}xt^p_A(M,N)) \\ {}_{II}E_2^{p,q} &= H^p(X,\mathcal{E}xt^p_{A,\mathcal{Z}}(M,N)) \\ {}_{III}E_2^{p,q} &= \mathrm{Ext}^p_A(X;M,\mathcal{H}^p_{\mathcal{Z}}(N)) \end{aligned}\right\} \Rightarrow \mathrm{Ext}^{p+q}_{A,\mathcal{Z}}(X;M,N) \tag{1.6.64}$$

(c) *The sheaf $\mathcal{E}xt^p_{A,\mathcal{Z}}(M,N)$ is the sheaf associated with the presheaf $X \mapsto \mathrm{Ext}^p_{A,\mathcal{Z}}(X;M,N)$.*

Proof. If N is injective, the sheaf $\mathcal{H}om_A(M,N)$ is flasque by Proposition 1.6.26, hence acyclic for $\mathcal{H}^0_{\mathcal{Z}}$ (Proposition 1.6.39). The first spectral sequence of (1.6.63) then follows from (1.6.56) and Proposition 1.6.5. Similarly, if N is injective, $\mathcal{H}^0_{\mathcal{Z}}(N)$ is injective (Proposition 1.6.27), so the second spectral sequence of (1.6.63) follows from Proposition 1.6.5 and the second isomorphism of (1.6.55). The spectral sequences of (1.6.64) similarly follow from Proposition 1.6.5 and the first two isomorphisms of (1.6.61) and (1.6.60). Finally, by varying X in the second spectral sequence of (1.6.64), we obtain a spectral sequence

$$E_2^{p,q} = \mathcal{H}^p(\mathcal{E}xt^p_{A,\mathcal{Z}}(M,N)) \Rightarrow (X \mapsto \mathrm{Ext}^p_{A,\mathcal{Z}}(X;M,N)).$$

By passing to the associated sheaves and apply (1.6.16), we obtain a degenerate spectral sequence of sheaves, which provides the isomorphism of (c). □

Proposition 1.6.42. *The functors* $(M,N) \mapsto \mathcal{E}xt^p_{A,\mathcal{Z}}(M,N)$ *and* $(M,N) \mapsto \mathrm{Ext}^p_{A,\mathcal{Z}}(X;M,N)$ *are* δ*-functors on each of its variables. If* $M_{\mathcal{U}}$ *is the sheaf* $M \otimes_A A_{\mathcal{U}}$*, we have a long exact sequence*

$$\cdots \xrightarrow{\delta} \mathcal{E}xt^p_{A,\mathcal{Z}}(M,N) \longrightarrow \mathcal{E}xt^p_A(M,N) \longrightarrow \mathcal{E}xt^p_A(M_{\mathcal{U}},N) \xrightarrow{\delta} \mathcal{E}xt^{p+1}_{A,\mathcal{Z}}(M,N) \longrightarrow \cdots \tag{1.6.65}$$

and a similar long exact sequence by replacing $\mathcal{E}xt$ *with* Ext.

Proof. The functors $\mathcal{E}xt^p_A(-,-)$ and $\mathrm{Ext}^p_A(X;-,-)$ are δ-functors on each variables, and $M \mapsto M_{\mathcal{Z}}$ is exact because $A_{\mathcal{Z}}$ is flat (Corollary 1.6.12). The first assertion then follows from the isomorphisms (1.6.58) and (1.6.62). The long exact sequence (1.6.65) can be induced from the exact sequence

$$0 \longrightarrow M_{\mathcal{U}} \longrightarrow M \longrightarrow M_{\mathcal{Z}} \longrightarrow 0$$

in view of (1.6.58) and (1.6.62). □

1.7 The method of cohomological descent

Let X be a topological space and $\mathfrak{U} = (U_i)_{i\in I}$ be a covering of X, which we suppose to be open or locally finite and closd. Let $\mathscr{F}$ be an abelian sheaf over X, the Leray spectral sequence

$$\check{H}^p(\mathfrak{U}, \mathcal{H}^p(X,\mathscr{F})) \Rightarrow H^{p+q}(X,\mathscr{F}) \tag{1.7.1}$$

defined by $\mathfrak{U}$ can be described as follows: The covering $\mathfrak{U}$ defines a Čech resolution $\mathscr{C}^\bullet(\mathfrak{U},\mathscr{F})$, which is functorial on $\mathscr{F}$. On the other hand, for any sheaf $\mathscr{F}$ we have a canonical flasque resolution $C^\bullet(\mathscr{F})$, which is also functorial on $\mathscr{F}$. With these notations, the spectral sequence (1.7.1) is obtained, in the case where $\mathfrak{U}$ is an open covering, from the following double complex

$$\Gamma(X, \mathscr{C}^\bullet(\mathfrak{U}, C^\bullet(\mathscr{F}))),$$

and in the case where $\mathfrak{U}$ is locally finite and closed, from the following double complex

$$\Gamma(X, C^\bullet(\mathscr{C}^\bullet(\mathfrak{U},\mathscr{F}))).$$

We now seek a unified description of these double complexes. Let X_0 be the direct sum of the topological spces U_i and X_n be the $(n+1)$-fold fiber product of X_0 over X:

$$X_n = \coprod_{i_0,\dots,i_n\in I} U_{i_0}\cap\cdots\cap U_{i_n} = \coprod_{\sigma\in\mathrm{Hom}([n],I)} U_{\sigma(0)}\cap\cdots\cap U_{\sigma(n)}. \tag{1.7.2}$$

The X_n then form a simplicial system of topological spaces, and if $j_n : X_n \to X$ is the canonical projection, we have

$$\mathscr{C}^n(\mathfrak{U},\mathscr{F}) = (j_n)_* j_n^*(\mathscr{F}). \tag{1.7.3}$$

Note that the formation of the canonical flasque resolution commutes with restriction to open subsets and direct image under a closed immersions, so

(a) if $\mathfrak{U}$ is open,

$$\mathscr{C}^p(\mathfrak{U}, C^q(\mathscr{F})) = (j_q)_* j_q^*(C^p(\mathscr{F})) = (j_q)_*(C^p(j_q^*(\mathscr{F})));$$

(b) if $\mathfrak{U}$ is closed and locally finite,

$$C^p(\mathscr{C}^q(\mathfrak{U}, \mathscr{F})) = C^p((j_q)_* j_q^*(\mathscr{F})) = (j_q)_*(C^p(j_q^*(\mathscr{F}))).$$

Therefore, to obtain a unified description of (1.7.1), we see that it suffices to take the canonical flasque resolution of $j_q^*(\mathscr{F})$ over X_q for all q, and then apply the functor $(j_q)_*$ to this resolution.

We also note that this description is in fact meaningful for any simplicial system of spaces topological above X:

$$\Delta^0 \to \mathbf{Top}_{/X}, \quad [n] \mapsto X_n.$$

However, in this case the double complex

$$(j_q)_*(C^p(j_q^*(\mathscr{F}))) \tag{1.7.4}$$

is not in general a resolution of $\mathscr{F}$. We are then devoted to find sufficient conditions for which (1.7.4) is a resolution of $\mathscr{F}$. In this case, the spectral sequence (1.7.1) is generalized to a spectral sequence

$$\check{H}^p(H^q(X_p, j_p^*(\mathscr{F}))) \Rightarrow H^{p+q}(X, \mathscr{F}) \tag{1.7.5}$$

which is called the *descent spectral sequence*.

1.7.1 Topoi over a fixed category

In this section, we fix a universe $\mathscr{U}$ and only consider $\mathscr{U}$-topoi. We shall make constant use of the language of fibre categories, introduced in §1.1. Let's just fix some notations: if $\mathcal{F} \to \mathcal{E}$ is a fibrant (resp. cofibrant) functor, for a morphism $f : T \to S$, we denote by $f^* : \mathcal{F}_S \to \mathcal{F}_T$ (resp. $f_* : \mathcal{F}_T \to \mathcal{F}_S$) the inverse image (resp. direct image) functor associated with $\mathcal{F}$; each of these functors is defined up to a unique functorial isomorphism. If $F : \mathcal{F} \to \mathcal{G}$ is a $\mathcal{E}$-functor, for any object S of $\mathcal{E}$, we denote by $F_S : \mathcal{F}_S \to \mathcal{G}_S$ the restriction functor induced by F on categorical fibers.

1.7.1.1 $\mathcal{E}$-topoi Let $\mathcal{E}$ be a category and $p : \mathcal{X} \to \mathcal{E}$ be a bifibrant functor. We say that $\mathcal{X}$ is a **$\mathcal{E}$-topos** if the following categories are satisfied:

(F1) For any object S of $\mathcal{E}$, the categorical fiber $\mathcal{X}_S$ is a topos.

(F2) For any morphism $f : T \to S$ in $\mathcal{E}$, there exists a morphism of topoi $\alpha : \mathcal{X}_S \to \mathcal{X}_T$ such that $f_* = \alpha^*$ and $f^* = \alpha_*$.

We recall that if $\mathcal{X}$ and $\mathcal{Y}$ are topoi, a morphism from $\mathcal{X}$ to $\mathcal{Y}$ consists of an adjoint pair (Φ^*, Φ_*) of functors

$$\Phi_* : \mathcal{X} \to \mathcal{Y}, \quad \Phi^* : \mathcal{Y} \to \mathcal{X}$$

such that Φ^* is left exact (i.e. commutes with finite projective limits). Therefore, consition (F2) of $\mathcal{E}$-topos is just saying that the functor f_* is left exact, since we can then take the adjoint pair (f_*, f^*) to be the required morphism. If $\mathcal{E} = \Delta$ (resp. $\Delta \times \Delta$), a Δ-topos (resp. $\Delta \times \Delta$-topos) is then called a **simplicial topos** (resp. **bisimplicial topos**).

In practice, we often encounter $\mathcal{E}$-topoi induced by fibre categories fibered into dual topoi. Here is the precise definition of this terminology:

Definition 1.7.1. Let $\mathcal{F}$ be a bifibre category over $\mathcal{E}$. We say that $\mathcal{F}$ is **bifibered into dual topoi over $\mathcal{E}$** if $\mathcal{F}^{\mathrm{op}}$ is an $\mathcal{E}^{\mathrm{op}}$-topos.

Explicitly, a bifibre category $\mathcal{F}$ over $\mathcal{E}$ is bifibered into dual topoi over $\mathcal{E}$ if it satisfies the following conditions:

(FD1) For any object S of $\mathcal{E}$, the dual $\mathcal{F}_S^{\mathrm{op}}$ of the categorical fiber $\mathcal{F}_S$ is a topoi.

(FD2) For any morphism $f : T \to S$ in $\mathcal{E}$, there exists a morphism of topoi $\alpha : \mathcal{F}_S \to \mathcal{F}_T$ such that $\alpha^* = f^*$, $\alpha_* = f_*$ (or equivalently, the functor f^* is left exact).

Recall that a functor $X : \mathcal{D}^{\mathrm{op}} \to \mathcal{E}$ is often called an $\mathcal{D}$**-object** of $\mathcal{E}$, and we denote by X_i the image of an object i in $\mathcal{D}$ under X. The $\mathcal{D}$-objects of $\mathcal{E}$ then form a category, which is denoted by $\mathcal{E}_{\mathcal{D}}$. If S is an object of $\mathcal{E}$, an $\mathcal{D}$-object of $\mathcal{E}_{/S}$ is also called a $\mathcal{D}$**-object augmented by** $\boldsymbol{S}$. If $\mathcal{F} \to \mathcal{E}$ be a bifibre category into dual topoi over $\mathcal{E}$ and $X : \mathcal{D}^{\mathrm{op}} \to \mathcal{E}$ is a $\mathcal{D}$-object of $\mathcal{E}$, then the fiber product $(\mathcal{D}^{\mathrm{op}} \times_{\mathcal{E}} \mathcal{F})^{\mathrm{op}}$ is a $\mathcal{D}$-topos, which we denote by $X_{\mathcal{F}}$. Note that if i is an object of $\mathcal{D}$, the fiber of $X_{\mathcal{F}}$ is then given by

$$(X_{\mathcal{F}})_i = (\mathcal{D}^{\mathrm{op}} \times_{\mathcal{E}} \mathcal{F})^{\mathrm{op}} \times_{\mathcal{D}} i = \mathcal{F}^{\mathrm{op}} \times_{\mathcal{E}^{\mathrm{op}}} \mathcal{D} \times_{\mathcal{D}} i = \mathcal{F}^{\mathrm{op}} \times_{\mathcal{E}^{\mathrm{op}}} X_i = \mathcal{F}_{X_i}^{\mathrm{op}}. \tag{1.7.6}$$

The $\mathcal{D}$-topos $X_{\mathcal{F}}$ obtained from a bifibre category into dual topoi over $\mathcal{E}$ and any $\mathcal{D}$-object X is called the **topos induced by** $\boldsymbol{X}$.

Example 1.7.2. Suppose that the category $\mathcal{E}$ possesses finite fiber products. Let $f : T \to S$ be a morphism in $\mathcal{E}$. The bifunctor

$$\Delta^{\mathrm{op}} \times (\mathcal{E}_{/S})^{\mathrm{op}} \to \mathbf{Set}, \quad ([n], U) \mapsto \mathrm{Hom}_{\mathbf{Set}}([n], \mathrm{Hom}_S(U, T))$$

then defines a functor

$$\Delta^{\mathrm{op}} \to \mathcal{E}_{/S}, \quad [n] \mapsto T_n$$

where $T_n = T \times_S \cdots \times_S T$ is the $(n+1)$-fold fiber product representing the functor

$$Z \mapsto \mathrm{Hom}_{\mathbf{Set}}([n], \mathrm{Hom}_S(Z, T)).$$

In this way, we have defined a semi-simplicial object augmented by S, which is denoted by $[T|_f S]$ or $[T|S]$.

If X and Y are two semi-simplicial objects of $\mathcal{E}$ (or of $\mathcal{E}_{/S}$) and $u : X \to Y$ is a morphism of functors, we can similarly define a simplicial object $[X|_u Y]$ of $\mathcal{E}_\Delta$, which can be interpreted as a bisimplicial object of $\mathcal{E}$, using the isomorphism $(\mathcal{E}_\Delta)_\Delta \xrightarrow{\sim} \mathcal{E}_{\Delta\times\Delta}$.

Let $\mathcal{X}$ be a $\mathcal{E}$-topos and $\mathbf{\Gamma}(\mathcal{X}/\mathcal{E})$ be the category $\mathcal{H}om_{/\mathcal{E}}(\mathcal{X}, \mathcal{E})$. Let $\lambda : \mathcal{E}' \to \mathcal{E}$ be a functor; the category $\mathcal{E}' \times_{\mathcal{E}} \mathcal{X}$ is then a $\mathcal{E}'$-topos, and by composing with λ we obtain a functor

$$\lambda^* : \mathbf{\Gamma}(\mathcal{X}/\mathcal{E}) \to \mathbf{\Gamma}(\mathcal{E}' \times_{\mathcal{E}} \mathcal{X}/\mathcal{E}').$$

In the case where $\mathcal{E}'$ is reduced to a single object S of $\mathcal{E}$ (with identity morphism) and λ is the canonical inclusion, the category $\mathcal{E}' \times_{\mathcal{E}} \mathcal{X}$ is then the fiber $\mathcal{X}_S$, and the functor λ^* is identified with the evaluation functor

$$i_S^* : \mathbf{\Gamma}(\mathcal{X}/\mathcal{E}) \to \mathcal{X}_S, \quad u \mapsto u(S). \tag{1.7.7}$$

Proposition 1.7.3. *If $\mathcal{E}'$ is a $\mathcal{U}$-small category, the functor λ^* possesses a right adjont λ_* and a left adjoint $\lambda_!$.*

This in fact follows from a slight generalization of Kan's lemma, which we shall use frequently.

Lemma 1.7.4. *Let $\mathcal{I}$, $\mathcal{J}$ and $\mathcal{A}$ be cetegories over a category $\mathcal{E}$, and suppose that $\mathcal{I}$ is $\mathcal{U}$-small and that $\mathcal{A}$ is bifibre over $\mathcal{E}$. Let $\lambda : \mathcal{I} \to \mathcal{J}$ be a $\mathcal{E}$-functor and*

$$\lambda^* : \mathcal{H}om_{/\mathcal{E}}(\mathcal{J}, \mathcal{A}) \to \mathcal{H}om_{/\mathcal{E}}(\mathcal{I}, \mathcal{A})$$

be the functor induced by composition with λ. If in each fiber of $\mathcal{A}$ over $\mathcal{E}$, the $\mathcal{U}$-inductive limits (resp. projective limits) exist, then λ possesses a left adjoint (resp. a right adjoint).

Proof. □

Corollary 1.7.5. *Let $\mathcal{X}$ be a $\mathcal{E}$-topos and S be an object of $\mathcal{E}$. Then the evaluation functor $i_S^* : \mathbf{\Gamma}(\mathcal{X}/\mathcal{E}) \to \mathcal{X}_S$ admits a left adjoint $(i_S)_!$ and a right adjoint $(i_S)_*$:*

$$(i_S)_! : \mathcal{X}_S \to \mathbf{\Gamma}(\mathcal{X}/\mathcal{E}), \quad \xi \mapsto \big(T \mapsto \coprod_{f \in \mathrm{Hom}_{\mathcal{E}}(S,T)} f_*(\xi)\big), \tag{1.7.8}$$

$$(i_S)_* : \mathcal{X}_S \to \mathbf{\Gamma}(\mathcal{X}/\mathcal{E}), \quad \xi \mapsto \big(T \mapsto \prod_{f \in \mathrm{Hom}_{\mathcal{E}}(T,S)} f^*(\xi)\big). \tag{1.7.9}$$

Proposition 1.7.6. *Let $\mathcal{E}$ be a $\mathcal{U}$-small category and $\mathcal{X}$ be a $\mathcal{E}$-topos. Then the category $\mathbf{\Gamma}(\mathcal{X}/\mathcal{E})$ is a $\mathcal{U}$-topos.*

Proof. By considering the fibers of $\mathcal{X}$ and using Lemma 1.7.4, we verify that $\mathbf{\Gamma}(\mathcal{X}/\mathcal{E})$ satisfies the following conditions:

(a) Finite projective limits exist.

(b) Small direct sums exist, and are disjoint and universal.

(c) The equivalence relations are universally effective.

It then remains to prove that $\mathbf{\Gamma}(\mathcal{X}/\mathcal{E})$ possesses a $\mathcal{U}$-small generating family. For this, note that if for any object S of $\mathcal{E}$, $(\xi_\alpha^S)_{\alpha \in I_S}$ is a generating family of $\mathcal{X}_S$ (where I_S is a $\mathcal{U}$-small set), the family $((i_S)_!(\xi_\alpha^S))_{S,\alpha}$ is then a generating family of $\mathbf{\Gamma}(\mathcal{X}/\mathcal{E})$. □

We now consider the notion of morphisms between $\mathcal{E}$-topoi. For this, we need the following terminology: let $F : \mathcal{F} \to \mathcal{G}$ be an $\mathcal{E}$-functor, a **left $\mathcal{E}$-adjoint** of F is defined to be a functor $G : \mathcal{G} \to \mathcal{F}$ which is left adjoint to F and such that the canonical morphisms $1 \to GF$ and $FG \to 1$ are $\mathcal{E}$-morphisms. Under this conditions, it is easy to verify that if F is Cartesian, then G is also Cartesian.

Definition 1.7.7. Let $\mathcal{X}$ and $\mathcal{Y}$ be $\mathcal{E}$-topoi. A **morphism** from $\mathcal{X}$ to $\mathcal{Y}$ is a couple of $\mathcal{E}$-functors

$$\Phi_* : \mathcal{X} \to \mathcal{Y}, \quad \Phi^* : \mathcal{Y} \to \mathcal{X}$$

endowed with a $\mathcal{E}$-adjunction

$$\mu : \mathrm{Hom}_{\mathcal{X}}(\Phi^*(-), -) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{Y}}(-, \Phi_*(-)),$$

such that for any object S of $\mathcal{E}$, the couple $(\Phi_*(S), \Phi^*(S))$, endowed with the adjunction induced by μ, is a morphism of topoi from $\mathcal{X}_S$ to $\mathcal{Y}_S$.

Proposition 1.7.8. *Let $\mathcal{X}$ and $\mathcal{Y}$ be $\mathcal{E}$-topoi and $(\Phi_*, \Phi^*) : \mathcal{X} \to \mathcal{Y}$ be a $\mathcal{E}$-morphism. Suppose that $\mathcal{E}$ is a $\mathcal{U}$-small category, then the couple*

$$(\mathbf{\Gamma}(\Phi_*), \mathbf{\Gamma}(\Phi_*)) : \mathbf{\Gamma}(\mathcal{X}/\mathcal{E}) \to \mathbf{\Gamma}(\mathcal{Y}/\mathcal{E})$$

is a morphism of topoi.

Proof. This is trivial from the above definition of $\mathcal{E}$-morphisms. $\square$

A $\mathcal{E}$-morphism of $\mathcal{E}$-topoi is in fact completely determined by the restriction on each caegorical fibers. In order to construct such a morphism from the restrictions on fibers, we need the following lemma:

Lemma 1.7.9. *Let $\mathcal{X}$ and $\mathcal{Y}$ be bifibre categories over $\mathcal{E}$ and $\Phi : \mathcal{X} \to \mathcal{Y}$ be a Cartesian $\mathcal{E}$-functor such that for any object S of $\mathcal{E}$, $\Phi_S : \mathcal{X}_S \to \mathcal{Y}_S$ possesses a left adjoint. Then, by choosing for any object S a left adjoint of Φ_S, we obtain a $\mathcal{E}$-functor $\Psi : \mathcal{Y} \to \mathcal{X}$, which is $\mathcal{E}$-adjoint to Φ.*

Proof. $\square$

Remark 1.7.10. Under the conditions of Lemma 1.7.9, suppose that $\mathcal{X}$ and $\mathcal{Y}$ are two $\mathcal{E}$-topoi and that for any object S of $\mathcal{E}$, the left adjoint of Φ_S is left exact. Then, if $\Psi : \mathcal{Y} \to \mathcal{X}$ is the constructed functor, the couple $(\Phi, \Psi) : \mathcal{X} \to \mathcal{Y}$ is a morphism of $\mathcal{E}$-topos.

Example 1.7.11. Let $\mathcal{F} \to \mathcal{E}$ be a category bifibered into dual topoi over $\mathcal{E}$, X, Y be two $\mathcal{D}$-objects of $\mathcal{E}$, and $\alpha : X \to Y$ be a morphism of $\mathcal{D}$-objects. Then by choosing normalized cleveages for $\mathcal{X}$ and $\mathcal{X}^{\mathrm{op}}$, we obtain a morphism

$$(\alpha_*, \alpha^*) : X_{\mathcal{F}} \to Y_{\mathcal{F}}$$

of $\mathcal{D}$-topoi. On the other hand, for any object i of $\mathcal{D}$, the morphism $\alpha(i) : X_i \to Y_i$ in $\mathcal{E}$ induces functors

$$\alpha(i)_* : \mathcal{F}_{X_i} \to \mathcal{F}_{Y_i}, \quad \alpha(i)^* : \mathcal{F}_{Y_i} \to \mathcal{F}_{X_i}.$$

which, in view of (1.7.6), coincides with the opposite of the morphisms on fibers induced by (α_*, α^*).

1.7.1.2 Ringed $\mathcal{E}$-topoi

A **ringed $\mathcal{E}$-topos** is a couple $(\mathcal{X}, A)$ where $\mathcal{X}$ is a $\mathcal{E}$-topos and A is a ring of $\mathbf{\Gamma}(\mathcal{X}/\mathcal{E})$. If $(\mathcal{X}, A)$ is a ringed $\mathcal{E}$-topos, then for any object S of $\mathcal{E}$, A_S is a ring of the topos $\mathcal{X}_S$, and any morphism $f : T \to S$ induced a canonical morphism $A_T \to f^*(A_S)$. If $(\mathcal{X}, A)$ and $(\mathcal{Y}, B)$ are two ringed $\mathcal{E}$-topoi, a **morphism** from $(\mathcal{X}, A)$ to $(\mathcal{Y}, B)$ is defined to be a couple (Φ, θ), where $\Phi : \mathcal{X} \to \mathcal{Y}$ is a morphism of $\mathcal{E}$-topoi and $\theta : B \to \Gamma(\Phi_*)(A)$ is a ring homomorphism. By Proposition 1.7.8, if $\mathcal{E}$ is $\mathscr{U}$-small, such a morphism induces a morphism $(\mathbf{\Gamma}(\Phi), \theta) : (\mathbf{\Gamma}(\mathcal{X}/\mathcal{E}), A) \to (\mathbf{\Gamma}(\mathcal{Y}/\mathcal{E}), B)$ of ringed topoi.

Example 1.7.12. Let $\mathcal{F} \to \mathcal{E}$ be a category bifibered into dual topoi over $\mathcal{E}$ and $\mathcal{O}$ be a ring of $\mathbf{\Gamma}(\mathcal{F}^{\mathrm{op}}/\mathcal{E}) = \mathrm{Hom}_{/\mathcal{E}^{\mathrm{op}}}(\mathcal{E}^{\mathrm{op}}, \mathcal{F}^{\mathrm{op}})$. If $X : \mathcal{D}^{\mathrm{op}} \to \mathcal{E}$ is a $\mathcal{D}$-object of $\mathcal{E}$, then since $\mathbf{\Gamma}(X_{\mathcal{F}}/\mathcal{D}) = \mathcal{H}om(\mathcal{D}, \mathcal{F}^{\mathrm{op}})$, the $\mathcal{D}$-topos $X_{\mathcal{F}}$ is naturally a ringed via the composition

$$\mathcal{D} \xrightarrow{X^{\mathrm{op}}} \mathcal{E}^{\mathrm{op}} \xrightarrow{\mathcal{O}} \mathcal{F}^{\mathrm{op}}$$

and we denote by $(X_{\mathcal{F}}, \mathcal{O})$ the ringed $\mathcal{D}$-topos thus obtained. If $\alpha : X \to Y$ is a morphism of $\mathcal{D}$-objects, the induced morphism $(\alpha_*, \alpha^*) : X_{\mathcal{F}} \to Y_{\mathcal{F}}$ then induces canonically a morphism $(X_{\mathcal{F}}, \mathcal{O}) \to (Y_{\mathcal{F}}, \mathcal{O})$ of ringed $\mathcal{D}$-topoi.

A ringed $\mathcal{E}$-topos $(\mathcal{X}, A)$ defines a category $\mathbf{Mod}(\mathcal{X}, A)$ which is bifibered into abelian categories over $\mathcal{E}$, whose fiber over an object S of $\mathcal{E}$ is the category $\mathbf{Mod}(\mathcal{X}_S, A_S)$ of modules over the ringed topos $(\mathcal{X}_S, \mathcal{A}_S)$. With this notation, the category of A-modules of $\mathbf{\Gamma}(\mathcal{X}/\mathcal{E})$, denoted by $\mathbf{Mod}(\mathbf{\Gamma}(\mathcal{X}/\mathcal{E}), A)$, is then identified with the category $\mathrm{Hom}_{/\mathcal{E}}(\mathcal{E}, \mathbf{Mod}(\mathcal{X}, A))$.

Let $\varphi = (\Phi, \theta) : (\mathcal{X}, A) \to (\mathcal{Y}, A)$ be a morphism of ringed $\mathcal{E}$-topoi. We then have two functors

$$\varphi_* : \mathbf{Mod}(\mathcal{X}, A) \to \mathbf{Mod}(\mathcal{Y}, B), \quad \varphi^* : \mathbf{Mod}(\mathcal{Y}, B) \to \mathbf{Mod}(\mathcal{X}, A)$$

of the corresponding categories, defined in the following way:

- Let M be an object $\mathbf{Mod}(\mathcal{X}, A)$ lying over an object S of $\mathcal{E}$, then $\Phi_*(M)$ is a module over $\Phi_*(A_S)$ and, via the homomorphism $\theta_S : B_S \to \Phi_*(A_S)$, it can be considered a module over B_S, which is denoted by $\varphi_*(M)$. The functor φ_* is called the **direct image functor** induced by φ.
- Let N be an object of $\mathbf{Mod}(\mathcal{Y}, B)$ lying over an object S of $\mathcal{E}$. Then $\Phi^*(N)$ is a module over $\Phi^*(B_S)$ and
$$\phi^*(N) = \Phi^*(N) \otimes_{\Phi^*(B_S)} A_S$$
is canonically endowed with a module structure over A_S. In view of 1.7.9, we then obtain a functor φ^* which is left adjiont to φ_* and called the inverse image functor induced by φ.

The morphism φ is called **flat** if the functor $\mathbf{\Gamma}(\varphi^*)$ is exact.

Proposition 1.7.13. *Let $\lambda : \mathcal{E}' \to \mathcal{E}$ be a functor and $(\mathcal{X}, A)$ be a ringed topos. Then the canonical functor*
$$\lambda^* : \mathbf{Mod}(\mathbf{\Gamma}(\mathcal{X}/\mathcal{E}), A) \to \mathbf{Mod}(\mathbf{\Gamma}(\mathcal{X} \times_{\mathcal{E}} \mathcal{E}'/\mathcal{E}'), \lambda^*(A))$$
possesses a right adjoint f_* and left adjoint $f_!$ if $\mathcal{E}'$ is $\mathcal{U}$-small category. In particular, it is exact in this case.

Proof. This follows immediately from Lemma 1.7.4 and the identification
$$\mathrm{Hom}_{/\mathcal{E}}(\mathcal{E}, \mathbf{Mod}(\mathcal{X}, A)) \cong \mathbf{Mod}(\mathbf{\Gamma}(\mathcal{X}/\mathcal{E}), A).$$
□

1.8 Étale site and topos of schemes

In this section, we develop the we develop the elementary properties of étale topology and cohomology for schemes. Certain properties in this section is essentially valid for many other different topologies, such as the "fppf topology". However, there are also distinguished properties for the étale topology, due to the very particular nature of étale morphisms.

1.8.1 The étale topology

We denote by **Sch** the category of schemes (in a fixed universe $\mathcal{U}$). Recall that a morphism $f : X \to Y$ is étale if it is locally of finite presentation, formally smooth and formally unramified. Equivalently, this means f is locally of finite presentation and for any affine scheme Y' over Y and any subscheme Y'_0 defined by a nilpotent ideal, the map
$$\mathrm{Hom}(Y, X) \to \mathrm{Hom}_Y(Y'_0, X)$$
is bijective. We define the **étale topology** on **Sch** to be the topology generated by the basis given by $X \mapsto \mathrm{Cov}(X)$, where $\mathrm{Cov}(X)$ is formed by families of jointly surjective morphisms $\{u_i : X_i \to X\}$ such that each u_i is étale. For each object X of **Sch**, we consider the subcategory $\mathbf{Et}_{/X}$ of $\mathbf{Sch}_{/X}$ formed by étale schemes over X, and endow it with the induced étale topology of that on **Sch** (called the **étale topology**). The resulting site is denoted by $X_{\text{ét}}$, and called the **étale site** of X. We note that any morphism in $\mathbf{Et}_{/X}$ is étale (**??**), so a family $\{U_i \to U\}$ of morphisms in $X_{\text{ét}}$ is covering if and only if it is jointly surjective. The topos $\mathrm{Sh}(X_{\text{ét}})$ of sheaves over $X_{\text{ét}}$ is called the **étale topos** of X. We usually use capital letters F, G to denote a sheaf over the étale site $X_{\text{ét}}$, and the usual (Zariski) sheaves of modules over the scheme X will be still denoted by $\mathscr{F}, \mathscr{G}$.

Proposition 1.8.1. *Let X be a scheme, $\mathcal{C}$ be a full subcategory of $X_{\text{ét}}$ such that for any étale scheme Y over X which is affine, $\mathcal{C}$ contains an object which is isomorphic to Y. Then $\mathcal{C}$ is a topological generating family of the site $X_{\text{ét}}$, hence a set of generators for the topos $\mathrm{Sh}(X_{\text{ét}})$, and the restriction functor $\mathrm{Sh}(X_{\text{ét}}) \to \mathrm{Sh}(\mathcal{C})$ is an equivalence of categories (where $\mathcal{C}$ is endowed with the induced topology).*

Proof. This is trivial in view of the comparision lemma ([**?**] III). □

Corollary 1.8.2. *suppose that X is quasi-separated. Let $\mathcal{C}$ be the full subcategory of $X_{\text{ét}}$ formed by étale schemes over X that are of finite presentation, endowed with the induced topology of $X_{\text{ét}}$. Then*

(a) *$\mathcal{C}$ is stable under fiber products and is a site of finite type if X is quasi-compact.*

(b) *The restriction functor $\mathrm{Sh}(X_{\text{ét}}) \to \mathrm{Sh}(\mathcal{C})$ is an equivalence of categories.*

In particular, if X is quasi-compact and quasi-separated, the functors $H^p(X_{\text{ét}}, F)$ over abelian sheaves of $X_{\text{ét}}$ commute with inductive limits.

Proof. The first assertion is trivial in view of **??**, and the second one follows from Proposition 1.8.1 because $\mathcal{C}$ satisfies the condition of Proposition 1.8.1. In fact, since X is quasi-separated, if Y is quasi-compact, then it is quasi-compact over X, hence of finite presentation over X if Y is étale over X. Finally, if X is quasi-compact, then by replacing $X_{\text{ét}}$ with $\mathcal{C}$, the last assertion follows from (a) and ([**?**] VI 6.1.2(3)). □

As $X_{\text{ét}}$ is then a $\mathscr{U}$-site and $\mathrm{Sh}(X_{\text{ét}})$ is a $\mathscr{U}$-topos, the general constructions and results can be applied to them. In the sequel, unless specified, the topologies on $\mathbf{Et}_{/X}$ or **Sch** are understood to be the étale topologies, and we often write X instead of $X_{\text{ét}}$ or $\mathbf{Et}_{/X}$. For example, by sheaves of X, we mean sheaves for the étale topology on the site $X_{\text{ét}}$. We will denote by $\mathrm{Sh}(X_{\text{ét}})$ the category of these sheaves, which is a $\mathscr{U}$-topos. If F is an abelian sheaf on X, we will denote its cohomology group by $H^p(X, F)$, which should be understood to be given by $H^p(X_{\text{ét}}, F)$.

Let $f : X \to Y$ be a morphism of schemes. Then the base change operation induces a functor

$$f^* : Y_{\text{ét}} \to X_{\text{ét}}, \quad Y' \mapsto Y' \times_Y X \tag{1.8.1}$$

which commutes with finite projective limits and sends covering families to covering families, hence is continuous. By ([**?**] I), f^* therefore defines a functor

$$f_\star : \mathrm{PSh}(X_{\text{ét}}) \to \mathrm{PSh}(Y_{\text{ét}}), \quad f_\star(F)(Y') = F(Y' \times_Y X).$$

which admits a left adjoint

$$f^\star : \mathrm{PSh}(Y_{\text{ét}}) \to \mathrm{PSh}(X_{\text{ét}}), \quad f^\star(G)(U) = \varinjlim_{Y' \in \mathcal{I}_U^{\mathrm{op}}} G(Y')$$

We also note that in this case the category $\mathcal{I}_U$ can be identified with the category of commutative diagrams

$$\begin{tikzcd} & X \times_Y Y' \arrow[r] \arrow[d] & Y' \arrow[d] \\ U \arrow[ur] \arrow[r] & X \arrow[r, "f"] & Y \end{tikzcd}$$

where Y' is an étale Y-scheme. On the other hand, in view of Proposition 1.4.2, the functor f^* also induces a functor

$$f_*^{\text{ét}} : \mathrm{Sh}(X_{\text{ét}}) \to \mathrm{Sh}(Y_{\text{ét}}), \quad f_*^{\text{ét}}(F) = f_\star(F), \tag{1.8.2}$$

which admits a left adjoint

$$f_{\text{ét}}^* : \mathrm{Sh}(Y_{\text{ét}}) \to \mathrm{Sh}(X_{\text{ét}}), \quad f_{\text{ét}}^*(G) = (f^\star(G))^\# \tag{1.8.3}$$

so that $f_{\text{ét}}^*$ extends the functor (1.8.1) and commutes with inductive limits and finite projective limits. In other words, f defines a morphism of topos

$$f_{\text{ét}} : \mathrm{Sh}(X_{\text{ét}}) \to \mathrm{Sh}(Y_{\text{ét}}).$$

If $g : Y \to Z$ is another morphism of schemes, it is clear that we have canonical isomorphisms

$$(gf)_*^{\text{ét}} \cong g_*^{\text{ét}} f_*^{\text{ét}}, \quad (gf)_{\text{ét}}^* \cong f_{\text{ét}}^* g_{\text{ét}}^* \tag{1.8.4}$$

whence an isomorphism $(gf)_{\text{ét}} \cong g_{\text{ét}} f_{\text{ét}}$.

Since we only consider the étale topology in this section, we just write f^* and f_* instead of $f_{\text{ét}}^*$ and $f_*^{\text{ét}}$. If $f : X \to Y$ is a morphism of schemes, the functors $R^p f_*^{\text{ét}}$ are then simply denoted by $R^p f_*$. Recall that for a sheaf F over X, we have the following Leray spectral sequence:

$$E_2^{p,q} = H^p(Y, R^q f_*(F)) \Rightarrow H^{p+q}(X, F).$$

Similarly, if $f : X \to Y$ and $g : Y \to Z$ are morphisms, we have a Grothendieck spectral sequence

$$E_2^{p,q} = R^p g_*(R^q f_*(F)) \Rightarrow R^{p+q}(fg)_*(F).$$

If $f : X \to Y$ is an étale morphism, we can consider X as an object of $Y_{\text{ét}}$, and there is then a canonical isomorphism of sites

$$X_{\text{ét}} \xrightarrow{\sim} (Y_{\text{ét}})_{/X}.$$

In this case, the object $U \to Y$ is final in the category $\mathcal{I}_U$, so $f^\star(G)(U) = G(U)$ and we conclude that the functor

$$f^* : \mathrm{Sh}(Y_{\text{ét}}) \to \mathrm{Sh}(X_{\text{ét}})$$

is isomorphic to the restriction functor to Y:

$$f^*(G) = G \circ i_{X/Y} \tag{1.8.5}$$

where G is a sheaf over $Y_{\text{ét}}$ and $i_{X/Y} : X_{\text{ét}} \to Y_{\text{ét}}$ is the canonical inclusion.

Remark 1.8.3. Note that if X is nonempt, then $X_{\text{ét}}$ is not an element of the chosen universe $\mathcal{U}$. However, as we pointed out, it can be proved that we can find a full subcategory $\mathcal{C}$ of $X_{\text{ét}}$ satisfying the conditions of the "comparison lemma" ([**?**] III), so that the restriction functor induces an equivalence $\mathrm{Sh}(X_{\text{ét}}) \xrightarrow{\sim} \mathrm{Sh}(\mathcal{C})$. Therefore, the étale topos is equivalent to a topos defined by a small site, so it is a $\mathcal{U}$-site.

Example 1.8.4. Let Y be a scheme over X. Then Y defines a sheaf over $\mathbf{Sch}_{/X}$ for the étale topology, given by $Y(X') = \mathrm{Hom}_X(X', Y)$. In other words, the étale topology is subcanonical on $\mathbf{Sch}_{/X}$ (this fact is essentially due to the result of (SGA1 VIII 5.1)). A fortiori, the restriction of this sheaf on $X_{\text{ét}}$ is a sheaf, which is still denoted by Y if there is no risk of confusion. We also note that the functor thus obtained

$$\mathbf{Sch}_{/X} \to \mathrm{Sh}(X_{\text{ét}})$$

commutes with finite projective limits (this is trivial). This implies for example if Y is a group (resp. abelian group, ring, etc.) scheme over X, then the sheaf defined by Y is a sheaf of groups (resp. abelian group, ring, etc.). Note that the induced functor $X_{\text{ét}} \to \mathrm{Sh}(X_{\text{ét}})$ is nothing but the canonical functor of $X_{\text{ét}}$, which associates an object Y of $X_{\text{ét}}$ the functor it represents. It is therefore an isomorphism from the category $X_{\text{ét}}$ to a full subcategory of the étale topos $\mathrm{Sh}(X_{\text{ét}})$, by which we identify usually an object Y of $X_{\text{ét}}$ with the corresponding sheaf, which will be denoted h_Y or simply Y. It is clear that we have $H^0(X, Y) = \mathrm{Hom}_X(X, Y)$, and if F is a group scheme over X (abelian by our definition of cohomology groups, but note that this is also true for nonabelian group schemes), then $H^1(X, F)$ is canonically isomorphic to the classes of F-torsors over $X_{\text{ét}}$.

Remark 1.8.5. For *any* scheme Y over X, we denote by $\widetilde{Y}$ the associated sheaf over $X_{\text{ét}}$. If $f : X' \to X$ is a morphism of schemes, we have a evident homomorphism (functorial on Z)

$$f^*(\widetilde{Y}) \to \widetilde{Y_{(X')}}, \tag{1.8.6}$$

(where $Y_{(X')} = Y \times_X X'$), which is the adjoint homomorphism of the canonical homomorphism $\widetilde{Y} \to f_*(\widetilde{Y_{(X')}})$ obtained from the functorial homomorphism

$$\mathrm{Hom}_X(U, Y) \to \mathrm{Hom}_{X'}(U_{(X')}, Y_{(X')})$$

where $U \in \mathrm{Ob}(X_{\text{ét}})$. This homomorphism is in general not an isomorphism, i.e. the formation of associated étale sheaf does not commute with inverse image functors. However, if Y is a étale scheme over X (that is, $Y \in \mathrm{Ob}(X_{\text{ét}})$), then for any sheaf F' over $X'_{\text{ét}}$, we have

$$\mathrm{Hom}_{\mathrm{Sh}(X'_{\text{ét}})}(f^*(\widetilde{Y}), F') \cong \mathrm{Hom}_{\mathrm{Sh}(X_{\text{ét}})}(\widetilde{Y}, f_*(F')) \cong f_*(F')(Y) = F'(Y_{(X')}) \cong \mathrm{Hom}_{\mathrm{Sh}(X'_{\text{ét}})}(Y_{(X')}, F')$$

so (1.8.6) is an isomorphism in this case. Similarly, the functor

$$\varphi_X : \mathbf{Sch}_{/X} \to \mathrm{Sh}(X_{\text{ét}})$$

is not faithful if X is nonempty, but its restriction to $X_{\text{ét}}$, which is the canonical functor $X_{\text{ét}} \to \mathrm{Sh}(X_{\text{ét}})$, is fully faithful (since the topology on $X_{\text{ét}}$ is subcanonical).

Note that (1.8.6) commutes with (small) direct sums, as can be easily verified. In particular, if for any set I we denote by $I_X = \coprod_{i \in I} X$ the constant X-scheme over X, then the associated sheaf is none other than the constant sheaf $I_{X_{\text{ét}}}$ (direct sum of I copies of final sheaf over $X_{\text{ét}}$, which sends Y to $\coprod_{i \in I} Y$). As the functor $I \mapsto I_X$ commutes with finite projective limits, it transforms (abelian) groups to (abelian) groups, etc. If G is a ordinary abelian group, we write $H^p(X, G)$ for $H^p(X, G_X)$. Suppose, for example, that G is a finite group; then G_X is finite and hence affine over X, so using the remarks in Example 1.8.4, we obtain an interpretation of $H^1(X, G)$ as the classes of principal G-bundles over X (SGA1 V 2.7). If X is connected and endowed with a geometric point $\bar{x}$, then in terms of the étale fundamental group $\pi_1(X, \bar{x})$, we obtain an isomorphism

$$H^1(X, G) \xrightarrow{\sim} \mathrm{Hom}(\pi_1(X, \bar{x}), G) \tag{1.8.7}$$

(where we may suppose that G is commutative).

Example 1.8.6. Let $\mathscr{F}$ be an $\mathscr{O}_X$-module over X, in the sense of Zariski topology. Then we can define a presheaf over $\mathbf{Sch}_{/X}$ by the formula

$$\Gamma_{\mathscr{F}}(Y) = \Gamma(Y, \mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{O}_Y)$$

which is a sheaf for the étale cohomology in view of (SGA1 VIII 1.6) and (SGA1 IV 6.3.1). A fortiori, the restriction of $\Gamma_{\mathscr{F}}$ to $X_{\text{ét}}$ is a sheaf, which is also denoted by $\Gamma_{\mathscr{F}}$. By definition, we then have $H^0(X, \Gamma_{\mathscr{F}}) = \Gamma(\mathscr{F}) = H^0(X, \mathscr{F})$.

1.8.2 Comparision of cohomologies

We note that the examples considered in the previous paragraph are in fact in a natural way restrictions of sheaves defined on $\mathbf{Sch}_{/X}$ endowed with its étale topology (or even fpqc topology). In general, we have an inclusion functor $X_{\text{ét}} \hookrightarrow \mathbf{Sch}_{/X}$ which is continuous and commutes with finite projective limits, hence defines a morphism of sites

$$\iota : X_{\text{ét}} \to \mathbf{Sch}_{/X}$$

whence a functor $\iota_* : \mathrm{Sh}(\mathbf{Sch}_{/X}) \to \mathrm{Sh}(X_{\text{ét}})$ over the category of sheaves, given by $F \mapsto F \circ \iota$.

Proposition 1.8.7. *Let $\iota^* : \mathrm{Sh}(X_{\text{ét}}) \to \mathrm{Sh}(\mathbf{Sch}_{/X})$ be the functor defined by*

$$\iota^*(F)(Y) = (p_Y)^*_{\text{ét}}(F)(Y)$$

where $Y \in \mathrm{Ob}(\mathbf{Sch}_{/X})$ and p_Y is the structural morphism $Y \to X$. Then

(a) *The functor i^* is fully faithful and a left adjiont to ι_*, so for any sheaf G over $X_{\text{ét}}$, the canonical homomorphism $G \mapsto \iota_*\iota^*(G)$ is an isomorphism.*

(c) *Let G (resp. F) be an abelian sheaf over $X_{\text{ét}}$ (resp. $\mathbf{Sch}_{/X}$). Then the canonical homomorphism (defined as the edge-homomorphisms of the Leray spectral sequence) are isomorphisms*

$$H^*(X_{\text{ét}}, \iota_*(F)) \xrightarrow{\sim} H^*(\mathbf{Sch}_{/X}, F), \quad H^*(X_{\text{ét}}, G) \xrightarrow{\sim} H^*(\mathbf{Sch}_{/X}, \iota^*(G)).$$

Proof. Let Y be a scheme over X. Then in fact, the morphism ι is cocontinuous, so we can apply the result of ([**?**] III, 2.2). □

On the other hand, it is necessary to introduce on **Sch** (therefore on $\mathbf{Sch}_{/X}$) various topologies other than the étale topology. The coarsest of these is the *Zariski topology* (Zar), defined by the basis where the covering families are the surjective families of open immersions; it is less fine than the étale topology. The finest of these topologies is the "*faithfully flat and quasi-compact*" topology, for short (fpqc), which is the coarses topologies for which the covering families in the sense of Zariski, as well as the faithfully flat quasi-compact morphisms, are covering; the fpqc topology is finer as the topology spreads out. As we have already noted, the various examples of sheaves considered on $\mathbf{Sch}_{/X}$ are in fact already sheaves for the fpqc topology.

However, one should note that an abelian sheaf F over $\mathbf{Sch}_{/X}$ for the fpqc topology (or the fppf topology), the cohomology group of F for the fpqc topology is not always isomorphic to that for the étale topology (even if X is the spectrum of a field k and F is represented by an algebraic group over k). In general, we can show that the étale cohomology gives the "good" cohomological groups when the coefficient is taken in any étale group scheme (or more generally smooth scheme) over X, but this is no longer true for group schemes such as the radical group over X, in which case we need to replace the étale cohomology with the fpqc or fppf topology.

As an example of the relations between cohomologies relative to different topologies, we consider here the Zariski topologies and the étale topology. We denote by X_{Zar} the site of Zariski open sets of X, so that we have a canonical inclusion functor $i : X_{\text{Zar}} \to X_{\text{ét}}$, which defines an induced functor

$$i_* : \mathrm{Sh}(X_{\text{ét}}) \to \mathrm{Sh}(X_{\text{Zar}}), \quad F \mapsto F \circ i,$$

and the corresponding inverse image functor

$$i^* : \mathrm{Sh}(X_{\text{Zar}}) \to \mathrm{Sh}(X_{\text{ét}})$$

which is left adjoint to i_*. Geometrically, the couple (i^*, i_*) can be considered as a morphism of topos

$$i : \mathrm{Sh}(X_{\text{ét}}) \to \mathrm{Sh}(X_{\text{Zar}}).$$

We therefore deduce a homomorphism of cohomology functors

$$H^*(X_{\text{Zar}}, i_*(F)) \to H^*(X_{\text{ét}}, F)$$

and a Leray spectral sequence

$$E_2^{p,q} = H^p(X_{\text{Zar}}, R^q i_*(F)) \Rightarrow H^{p+q}(X_{\text{ét}}, F),$$

where F is an abelian sheaf over $X_{\text{ét}}$. This spectral sequence summarizes the general relations between étale cohomology and Zariski topology. Of course, for the constant coefficient sheafs F, this spectral sequence is in general far from being trivial, i.e. in general we have $R^q f_*(F) \neq 0$. However, the following result is true:

Proposition 1.8.8. *Let $\mathscr{F}$ be an $\mathscr{O}_X$-module on X (in the sense of Zariski topology) and $\mathscr{F}_{\text{ét}}$ be the associated sheaf over $X_{\text{ét}}$ (cf. Example 1.8.6). Then we have a canonical homomorphism of cohomological functors*

$$H^*(X, \mathscr{F}) \to H^*(X_{\text{ét}}, \mathscr{F}_{\text{ét}}). \tag{1.8.8}$$

If $\mathscr{F}$ is quasi-coherent, then the preceding homomorphism is an isomorphism.

Proof. In fact, the cohomology group $H^*(X, \mathscr{F})$ is equal to $H^*(X_{\text{Zar}}, i_*(\mathscr{F}_{\text{ét}}))$, and with the preceding notations, it suffices to prove that

$$R^q i_*(\mathscr{F}) = 0 \quad \text{for} \quad q > 0.$$

Since affine opens form a basis for the topology of X, it suffices to assume that X is affine and prove that

$$H^q(X_{\text{ét}}, \mathscr{F}_{\text{ét}}) = 0 \quad \text{for} \quad q > 0.$$

To this end, let $\mathcal{C}$ be the full subcategory of $X_{\text{ét}}$ formed by étale affine schemes over X. In view of Proposition 1.8.1, we have

$$H^q(X_{\text{ét}}, \mathscr{F}_{\text{ét}}) \cong H^q(\mathcal{C}, \mathscr{F}|_{\mathcal{C}}).$$

It then suffices to prove that $\mathscr{F}|_{\mathcal{C}}$ is $\mathcal{C}$ acyclic (**[?]** V, 4.1), or that it satisfies the condition of (**[?]** V, 4.3), i.e. that for all $U \in \mathrm{Ob}(\mathcal{C})$ and all covering family $\mathfrak{U} = \{U_i \to U\}$, we have $H^q(\mathfrak{U}, \mathscr{F}) = 0$ for $q > 0$. Since U is quasi-compact, we can assume that $\mathfrak{U}$ is finite, so by replacing the U_i with their direct sum, the covering family consists of a single morphism which is covering, i.e. (étale and) surjective. We are therefore reduced to proving that if $f : X \to Y$ is a surjective étale morphism of affine scheme and $\mathscr{F}$ a quasi-coherent module on Y, then $H^q(X/Y, \mathscr{F}) = 0$ for $q > 0$, which follows from the lemma below. □

Lemma 1.8.9. *Let $A \to B$ is a faithfully flat homomorphism of rings and M is an A-module. Then the sequence*

$$0 \longrightarrow M \longrightarrow M \otimes_A B \xrightarrow{d_0} M \otimes_A B \otimes_A B \xrightarrow{d_1} M \otimes_A B \otimes_A B \otimes_A B \longrightarrow \cdots$$

is exact, where

$$d_n(x \otimes b_0 \otimes \cdots \otimes b_n) = \sum_{i=0}^{n+1} (-1)^i x \otimes b_0 \otimes \cdots \otimes b_{i-1} \otimes 1 \otimes b_i \otimes \cdots \otimes b_n.$$

Proof. Since B is faithfully flat over A, it suffices to prove that the sequence is exact after tensoring with B, that is, the sequence

$$0 \longrightarrow M \otimes_A B \longrightarrow M \otimes_A B \otimes_A B \xrightarrow{d_0} M \otimes_A B \otimes_A B \otimes_A B \xrightarrow{d_1} \cdots$$

where

$$d_n(x \otimes b_0 \otimes \cdots \otimes b_n \otimes b) = \sum_{i=0}^{n+1} (-1)^i x \otimes b_0 \otimes \cdots \otimes b_{i-1} \otimes 1 \otimes b_i \otimes \cdots \otimes b_n \otimes b.$$

For this, we define a homotopy $h_n : M \otimes_A B^{\otimes(n+2)} \to M \otimes_A B^{\otimes(n+1)}$ by

$$h_n(x \otimes b_0 \otimes \cdots \otimes b_n \otimes b) = x \otimes b_1 \otimes \cdots \otimes b_n \otimes b_0 b.$$

It is then immediate to check that $d_{n-1}h_n + h_{n+1}d_n = \mathrm{id}$, whence our assertion. □

Corollary 1.8.10. *If X is an affine scheme, then $H^p(X_{\text{ét}}, \mathscr{F}_{\text{ét}}) = 0$ for $p > 0$.*

1.8.3 Cohomology of a projective limit of schemes

1.9 Fiber functor, supports, and cohomology of finite morphisms

In this section, we consider the fiber functors on étale sites and topos associated with a geometric point, and definte the support of a étale sheaf. As an application, we discuss the cohomology groups of finite morphisms of schemes.

1.9.1 Topological invariance of étale topos

Theorem 1.9.1. *Let $f : X' \to X$ be a surjective, radical and integral morphism (or equivalently, a universal homeomorphism, cf. [**?**] 18). Then the base change functor*

$$f^* : X_{\text{ét}} \to X'_{\text{ét}}$$

is an equivalence of sites (i.e an equivalence of sites whose quasi-inverse is continuous). Therefore, the functors

$$f_* : \mathrm{Sh}(X'_{\text{ét}}) \to \mathrm{Sh}(X_{\text{ét}}), \quad f^* : \mathrm{Sh}(X_{\text{ét}}) \to \mathrm{Sh}(X'_{\text{ét}})$$

are equivalences of categories and quasi-inverses of each other.

Proof. The first assertion is well-known if f is of finite presentation or if f is flat ([**?**] IX, 4.10 et 4.11), and the general case can be reduced to the case where f is of finite presentation. In fact, by ([**?**] IX 3.4) the functor f^* is fully faithful, so it suffices to prove that f^* is essentially surjective, i.e. that any étale scheme Z' over X' is obtained by an étale scheme Z over X. Using the fact that f is a universal homeomorphism, and the fully faithfullness of f^*, we can reduce to the case where $X = \mathrm{Spec}(A), X' = \mathrm{Spec}(A'), Z' = \mathrm{Spec}(B')$ are affine. Writting A' as an inductive limit of sub-A-algebras of finite type of A'_i, with B' coming from a étale algebra over an A'_i (cf. [**?**] 8), we may further assume that A' is integral and of finite type over A, hence finite over A. We then have an isomorphism $A' \cong A''/\mathfrak{I}$, where $A'' = A[T_1, \dots, T_n]$ and $\mathfrak{I}$ is an ideal of A''. Again, by writting $\mathfrak{I}$ as an inductive limit of its finitely generated subideals $\mathfrak{I}_i$ and putting $A'_i = A''/\mathfrak{I}_i$, we have $A' = \varinjlim_i A'_i$, so B' comes from an étale algebra B'_i over A'_i (cf. [**?**] 8). On the other hand, as $\mathrm{Spec}(A') \to \mathrm{Spec}(A)$ is surjective, so is the morphism $\mathrm{Spec}(A'_i) \to \mathrm{Spec}(A)$, and we can verify that if $\mathrm{Spec}(A') \to \mathrm{Spec}(A)$ is radical, so is $\mathrm{Spec}(A'_i) \to \mathrm{Spec}(A)$ for $i \gg 0$ (apply [**?**] 1.9.9 to $\mathrm{Spec}(A)$). This brings us back to the case where A' is one of A'_i, hence of finite presentation on A. This proves the first assertion, and the fact that f^* and f_* are quasi-inverse of each other is immediate. □

Corollary 1.9.2. *Let F be an abelian sheaf over X and F' be the inverse image over X'. Then the canonical homomorphisms*

$$H^p(X, F) \to H^p(X', F') \tag{1.9.1}$$

are isomorphisms. Similarly, if F' is an abelian sheaf over X' and F is its image over X, then the canonical homomorphisms (1.9.1) are isomorphisms.

Example 1.9.3. The following examples of Theorem 1.9.1 will be frequently used in this section.

(a) X' is a subscheme of X with the same underlying space, i.e. defined by a nilideal $\mathfrak{I}$.

(b) X is a scheme over a separably closed field k, k' is an algebraic closure of k, and $X' = X \otimes_k k'$.

(c) Let X be a geometrically unibrach scheme (for example an algebraic curve over a field k). Then, if X' is the nomarlization of X_{red}, by definition $X' \to X$ is radical, so Theorem 1.9.1 applies.

1.9.2 Étale sheaves over the spectrm of a field

Proposition 1.9.4. *Let k be a field, $\bar{k}$ be a separable closure of k, $G = \mathrm{Gal}(\bar{k}/k)$ the Galois group, $X = \mathrm{Spec}(k)$ and $\bar{X} = \mathrm{Spec}(\bar{k})$. Let $i : X_{\text{ét}} \to \mathrm{Sh}(X_{\text{ét}})$ be the canonical inclusion and consider the functor*

$$\Gamma : \mathrm{Sh}(X_{\text{ét}}) \to \mathcal{B}G, \quad \Gamma(F) = \varinjlim F(\mathrm{Spec}(k_\alpha))$$

where k_α runs through finite subextensions of $\bar{k}/k$ and $\mathcal{B}G$ is the classifying topos of G. Then i and Γ are equivalences of categories.

Proof. The composition functor

$$\Gamma \circ i : X_{\text{ét}} \to \mathrm{Sh}(X_{\text{ét}}) \to \mathcal{B}G$$

is an equivalence of categories, in view of the fundamental theorem of Galois theory ([**?**] V), so $X_{\text{ét}}$ is also a topos and a family $\{X_i \to X\}$ of morphisms in $X_{\text{ét}}$ is a covering if and only if it is jointly surjective, i.e. its image in $\mathcal{B}G$ is epimorphic, or equivalently, a covering family for the canonical topology of $\mathcal{B}G$. This proves that the topology on $X_{\text{ét}}$ is the canonical topology, so i is an equivalence, and the same holds for Γ. □

Corollary 1.9.5. *The functor Γ induces an equivalence from the category of abelian sheaves over $X = \mathrm{Spec}(k)$ to the category of Galois G-modules.*

Proof. In fact, Galois G-modules are exactly the "abelian groups" in the topos $\mathcal{B}G$. □

Corollary 1.9.6. *Let F be an abelian sheaf over $X = \mathrm{Spec}(k)$ and $M = \Gamma(F)$ be the associated Galois G-module. Then there is a canonical isomorphism of δ-functors*

$$H^\bullet(X, F) \xrightarrow{\sim} H^\bullet(G, M).$$

Corollary 1.9.7. *Suppose that k is separably closed. Then the functor*

$$\Gamma : \mathrm{Sh}(X_{\text{ét}}) \to \mathbf{Set}, \quad F \mapsto F(X)$$

is an equivalence of categories. If F is an abelian sheaf over $X = \mathrm{Spec}(k)$, we have $H^p(X, F) = 0$ for $p > 0$.

Proof. In this case the Galois group is trivial, so the functor Γ sends $\mathrm{Sh}(X_{\text{ét}})$ into **Set**. The rest of the corollary follows from Proposition 1.9.4 and Corollary 1.9.6. □

1.9.3 Fiber functor relative to a geometric point

Let X be a scheme. We recall that a *geometric point* of X is defined to be an X-morphism $\xi : \mathrm{Spec}(\Omega) \to X$, where Ω a separably closed field. Giving a geometric point of X is equivalent to giving an ordinary point $x \in X$ and a field extension $\kappa(x) \to \Omega$, and we also write ξ for the scheme $\mathrm{Spec}(\Omega)$. In practice we only take Ω to be a separable closure of $\kappa(x)$, and the corresponding geometric point is then denoted by $\bar{x}$. For a given point $x \in X$, the field $\overline{\kappa(x)}$ (resp. $\bar{x}$) is determined up to nonunique isomorphisms.

Remark 1.9.8. In most geometric questions, it is more natural to consider algebraically closed fields Ω. The convention made here is special to the study of étale topology, and is also the convention we adopted in the definition of the fundamental group. This difference of conventions is not essential because the key property of Ω used here is that any étale covering of the spectrum of Ω is trivial, i.e. that Ω is separately closed.

Definition 1.9.9. Let X be a scheme and $\xi : \mathrm{Spec}(\Omega) \to X$ be a geometric point X. The **(geometric) fiber functor** relative to ξ, denoted by $F \mapsto F_\xi$, is defined to be the composition

$$\mathrm{Sh}(X_{\text{ét}}) \xrightarrow{\xi^*} \mathrm{Sh}(\xi_{\text{ét}}) \xrightarrow{\Gamma_\xi} \mathbf{Set}$$

where Γ_ξ is the functor defined in Proposition 1.9.4. In view of Corollary 1.9.7, we can also say that a geometric point ξ of X defines a point of the topos $\mathrm{Sh}(X_{\text{ét}})$, by which we mean the associated fiber functor.

As the functor Γ_ξ (section functor over ξ) is an equivalence of categories, the fiber functor $F \mapsto F_\xi$ is equivalent essentially to the inverse image functor ξ^*. Therefore, it follows from **??** that if ξ is a geometric point of X such that there exists an X-morphism $\xi' \to \xi$ (i.e. ξ' corresponds to a separably closed extension Ω' of Ω), then the canonical homomorphism $F_\xi \to F_{\xi'}$ is an isomorphism. This suggests us that in the study of fiber functors, we can (by replacing Ω with a separably closed closure of k in Ω) assume that $\Omega = k(\xi)$ is a separable closure $\overline{\kappa(x)}$ of $\kappa(x)$. In this way, we usually denote the corresponding fiber functor by $F \mapsto F_{\bar{x}}$.

Let us point out an obvious property of transitivity (whose technical utility should not be confined exclusively to geometric points defined by separable closure of residual fields): Let $f : X' \to X$ be a morphism of schemes and $\xi' : \xi' \to X'$ be a geometric point of X. Then $\xi = f\xi' : \xi' \to X$ is a geometric point of X and we have a functorial isomorphism on $F \in \mathrm{Sh}(X_{\text{ét}})$:

$$f^*(F)_{\xi'} \cong F_\xi \tag{1.9.2}$$

which follows from the transitivity of inverse images $(f\xi)^* \cong \xi^* f^*$.

Our main purpose for introducing the fiber functor is of course to show that various properties a morphism of abelian sheaves over $X_{\text{ét}}$ can be extracted from the induced morphism on fibers (i.e. by "taking stalk"). Before proving this, we need an efficient way to compute the fibers. For this, we recall that in the classical case, the stalk of a sheaf $\mathscr{F}$ at a point $x \in X$ is comupted as the inductive limit:

$$\mathscr{F}_x = \varinjlim_{U \ni x} \mathscr{F}(U) \tag{1.9.3}$$

where U runs through neighborhoods of x in X. In the étale setting, we can also obtain an analogue of (1.9.3), provided that we replace topological neighborhoods by suitable étale neighborhoods over X: more precisely, an **étale neighborhoods of ξ in X** is defined to be a commutative diagram

$$\begin{tikzcd} & U \arrow[d] \\ \xi \arrow[ur] \arrow[r] & X \end{tikzcd}$$

where U is an étale over X. The etale neighborhoods of ξ in X clearly form a category, which will be denoted by $\mathcal{C}_\xi$ (in fact, this is nothing than the category $\mathcal{I}_\xi$). Since étale morphisms are stable under fiber products, it is not hard to see that the category $\mathcal{C}_\xi$ is filtered. It turns our thar the fiber F_ξ can be derived by taking inductive limits in this category, as the following proposition shows:

Proposition 1.9.10. *Let ξ be a geometric point of X and $\mathcal{C}_\xi$ be the category of etale neighborhoods of ξ in X. Then for any sheaf F (resp. presheaf P) over X, we have a canonical functorial isomorphism*

$$F_\xi \xrightarrow{\sim} \varinjlim_{U \in \mathcal{C}_\xi^{\mathrm{op}}} F(U), \quad (resp.(P^\#)_\xi \xrightarrow{\sim} \varinjlim_{U \in \mathcal{C}_\xi^{\mathrm{op}}} P(U)). \tag{1.9.4}$$

Proof. We first note that for any presheaf P over $\xi_{\text{ét}}$, if $P^\#$ is the associated sheaf, the natural homomorphism

$$P(\xi) \to P^\#(\xi)$$

is an isomorphism (this follows for example, by the explicit construction of $P^\#$ by (1.3.4)). Therefore, if F is a sheaf on X, we then have

$$F_\xi = \Gamma(\operatorname{Spec}(\Omega), \xi^*(F)) = \Gamma(\operatorname{Spec}(\Omega), \xi^\star(F)) = \varinjlim_{U \in \mathcal{C}_\xi^{\mathrm{op}}} F(U)$$

where we use the construction of ξ^* in Proposition 1.4.3. The assertion for presheaves can be proved similarly, using the observation that $\xi^*(P^\#) = (\xi^*(P))^\#$. □

Theorem 1.9.11. *Let X be a scheme.*

(a) *For any geometric point ξ of X, the fiber functor $F \mapsto F_\xi$ commutes with ($\mathcal{U}$-small) inductive limits and finite projective limits. In particular, it transforms sheaf of groups (resp. abelian sheaves, etc.) into groups (resp. abelian sheaves, etc.)*

(b) *As x turns therough the points of X, the fiber functors $F \mapsto F_{\bar{x}}$ form a conservative family of functors, i.e. if $u : F \to G$ is a homomorphism of sheaves over X, then u is an isomorphism if and only if for any $x \in X$, the corresponding homomorphism $u_{\bar{x}} : F_{\bar{x}} \to G_{\bar{x}}$ is an isomorphism.*

Proof. The first assertion folows from the exactness properties in 1.4.3. Now let $u : F \to G$ be a morphism of sheaves over X, such that for any $x \in X$, $u_{\bar{x}} : F_{\bar{x}} \to G_{\bar{x}}$ is a monomorphism. To show that u is a monomorphism, we consider $X' \in \operatorname{Ob}(X_{\text{ét}})$ and two elements $\varphi, \psi \in F(X')$ such that $u(\varphi) = u(\psi)$. By replacing X by X' and use the transitivity of fiber functor, we can assume that $X = X'$. For any $x \in X$, we have $u(\varphi)_{\bar{x}} = u(\psi)_{\bar{x}}$, i.e. $u_{\bar{x}}(\varphi_{\bar{x}}) = u_{\bar{x}}(\psi_{\bar{x}})$, whence $\varphi_{\bar{x}} = \psi_{\bar{x}}$ since $u_{\bar{x}}$ is a monomorphism. Using (1.9.4), we conclude that there exists an étale neighborhood U_x of ξ (where x is the locality of ξ) such that the inverse images of φ and ψ coincide on U_x. As x varies over X, the U form a covering family of X, so we conclude that $\varphi = \psi$.

Suppose now that $u_{\bar{x}}$ is also an epimorphism for each $x \in X$. To show that u is an epimorphism, it suffices to prove that for any $X' \in \operatorname{Ob}(X_{\text{ét}})$ and any element $\psi \in G(X')$, we can find an element $\varphi \in F(X')$ such that $u(X')(\varphi) = \psi$. We may assume that $X' = X$, and by (1.9.4), we see that there exist for each $x \in X$ an étale neighborhood U_x and an element $\varphi_x \in F(U_x)$ whose image in $G(U_x)$ is equal to the inverse image of ψ. Using the fact that u is a monomorphism, we see that the φ_x coincides on overlapings, so there exists a unique element $\varphi \in F(X)$ such that $u(\varphi) = \psi$. □

Corollary 1.9.12. *Let $u : F \to G$ be a homomorphism of sheaves. Then u is a monomorphism (resp. epimorphism) if and only if for any $x \in X$, $u_{\bar{x}} : F_{\bar{x}} \to G_{\bar{x}}$ is injective (resp. surjective).*

Corollary 1.9.13. *Let $u, v : F \to G$ be two morphism of sheaves over X. Then $u = v$ if and only if for any $x \in X$, we have $u_{\bar{x}} = v_{\bar{x}}$. In particular, if u, v are two sections of F, then $u = v$ if and only if $u_{\bar{x}} = v_{\bar{x}}$ for any $x \in X$.*

Corollary 1.9.14. *A sequence $F \to G \to H$ of sheaves over X is exact if and only if for any $x \in X$, the corresponding sequence $F_{\bar{x}} \to G_{\bar{x}} \to H_{\bar{x}}$ is exact.*

For any point $x \in X$, we consider the natural morphism $i_x : \operatorname{Spec}(\kappa(x)) \to X$. This gives rise to a canonical functor

$$i_x^* : \operatorname{Sh}(X_{\text{ét}}) \to \operatorname{Sh}(x_{\text{ét}})$$

(we also denote by x the scheme $\operatorname{Spec}(\kappa(x))$). If F is a sheaf over X, its inverse image under i_x is denoted by $F_x = i_x^*(F)$, which is a sheaf over $\operatorname{Spec}(\kappa(x))$ (hence identified with an étale scheme over $\kappa(x)$ in view of Proposition 1.9.4). This depends functorially on F, and the functor $i_x^* : F \mapsto F_x$ commutes with inductive limits and finite projective limits.

If $\bar{x} = \operatorname{Spec}(\overline{\kappa(x)})$, the fiber functor $F \mapsto F_{\bar{x}}$ is canonically isomorphic with the composition of the functor $F \mapsto F_x$ and the functor Γ of Proposition 1.9.4 (the latter is isomorphic to the fiber functor relative to the geometric point $\bar{x}$ of x). It then follows from Theorem 1.9.11 that the system of functors $(F \mapsto F_x)_{x \in X}$ is also conservative. On the other hand, if $\kappa(x)$ is spearably closed, i.e. $x = \bar{x}$, the notations of F_x and $F_{\bar{x}}$ (strictly speaking) contradictory (the first is an object of $\operatorname{Sh}(x_{\text{ét}})$, while the second is a set), but this is not essential, in view of Proposition 1.9.4.

From the preceding remark we see that if $\bar{x} = \operatorname{Spec}(\overline{\kappa(x)})$, then for any sheaf F over X, the group $G_x = \operatorname{Gal}(\overline{\kappa(x)}/\kappa(x))$ naturally acts on the fiber set $F_{\bar{x}}$, so that $F \mapsto F_{\bar{x}}$ can be viewed as a functor

$$\operatorname{Sh}(X_{\text{ét}}) \to \mathcal{B}G_x$$

which is essentially equivalent to the functor $F \mapsto F_x$.

Remark 1.9.15. Consider a subset E of X such that for any étale morphism $f : X' \to X$, $E' = f^{-1}(E)$ is very dense in X' ([**?**] 10.1.3), i.e. that for any f as above, any open subset of X' cointaing E' is equal to X'. Then the fiber functor $F_{\bar{x}}$ over $\operatorname{Sh}(X_{\text{ét}})$, for $x \in E$, form a conservative system (this can be seen from the proof of Theorem 1.9.11 (b)). This observation is applicable in the following cases:

(a) X is a Jacobson scheme, for example a locally algebraic scheme over a field, or over $\operatorname{Spec}(\mathbb{Z})$, and E is the set of closed points of X.

(b) X is locally of finite type over a scheme X and E is the set of points of X that is closed in the corresponding fiber.

(c) X is locally Noetherian and E is the set of $x \in X$ such that $\bar{x}$ is a finite set (i.e. an Artinian scheme) (c.f. [**?**] 10.5.3 et 10.5.5).

1.9.4 Applications to the calculation of $R^p f_*$

Let $f : X \to Y$ be a morphism of schemes and F be an abelian sheaf over X. In view of Theorem 1.9.11, to determine the sheaf $R^p f_*(F)$, it suffices to determine the geometric fibers $R^p f_*(F)_{\bar{y}}$ for $y \in Y$. But by Proposition 1.6.31, $R^p f_*(F)$ is the sheaf associated with the presheaf

$$\mathcal{H}^p : Y' \mapsto H^p(X \times_Y Y', F)$$

over Y, so by (1.9.4),

$$R^p f_*(F)_{\bar{y}} = \varinjlim_{Y'} \mathcal{H}^p(Y')$$

where Y' runs through the opposite category of étale neighborhoods of $\bar{y}$ in Y. By choosing an affine open neighborhood of y in Y, we can take the above limit in the cofinal subcategory of Y' consisting of affine schemes over U, and we obtain an isomorphism

$$R^p f_*(F)_{\bar{y}} \cong \varinjlim_{Y'} H^p(X \times_Y Y', F). \tag{1.9.5}$$

We now introduce $\bar{Y} = \operatorname{Spec}(\mathscr{O}_{Y,\bar{y}}) = \varinjlim Y'$ (cf. [**?**] VIII 4.5) and $\bar{X} = X \times_Y \bar{Y}$. Since the projective system $X \times_Y Y'$ (induced from that of the Y' by base change) has affine transition morphisms, we see that the general conditions of ([**?**] VII 5.1) are satisfied, so we have a caonical isomorphism

$$\bar{X} = \varprojlim X \times_Y Y'.$$

Denote by $\bar{F}$ the sheaf over $\bar{X}$ induced by F. We then obtain a canonical homomorphism

$$\varinjlim_{Y'} H^p(X \times_Y Y', F) \to H^p(\bar{X}, \bar{F})$$

whence by composing with (1.9.5), a canonical homomorphism

$$R^p f_*(F)_{\bar{y}} \to H^p(\bar{X}, \bar{F}) \tag{1.9.6}$$

which is clearly functorial on F.

Now suppose that $f : X \to Y$ is quasi-compact and quasi-separated, then so is the morphism $X' \times_Y Y' \to Y'$, and as Y' is chosen to be affine, the $X \times_Y Y'$ are quasi-compact and quasi-separated. Using (1.9.5) and the theorem of passing to limits ([?] VII 5.8), we then obtain the following theorem:

Theorem 1.9.16. *Let $f : X \to Y$ be a quasi-compact and quasi-separated morphism, F be an abelian sheaf over X, y be a point of $\bar{Y}$, and $\bar{y}$ be the geometric point lying over y. If $\bar{Y} = \mathrm{Spec}(\mathscr{O}_{Y,\bar{y}})$ is the corresponding strict localization scheme, $\bar{X} = X \times_Y \bar{Y}$, and $\bar{F}$ is the inverse image of F over $\bar{X}$, then the canonical homomorphism (1.9.6) is an isomorphism*

$$R^p f_*(F)_{\bar{y}} \xrightarrow{\sim} H^p(\bar{X}, \bar{F}).$$

This statement brings practically the determination of the fibers of $R^p f_*(F)$ to the determination of the cohomology of a scheme above a strictly local scheme, and will be used constantly thereafter. Technically, it's Theorem 1.9.16 that explains the important role of Henselian rings and strictly local rings in the study of étale cohomology.

Suppose that $f : X \to Y$ is a finite morphism. Then $\bar{f} : \bar{X} \to \bar{Y}$ is also a finite morphism, and as $\bar{Y}$ is strictly local, the scheme $\bar{X}$ is a finite sum of strictly local schemes $\bar{X}_i$. Therefore, using ([?] VIII 4.7), we have

$$H^p(\bar{X}, \bar{F}) = 0 \quad \text{for} \quad p > 0. \tag{1.9.7}$$

On the other hand, we note that the components $\bar{X}_i$ of $\bar{X}$ corresponds to points $\bar{x}_i$ of $\bar{X}$ lying over the closed point $\bar{y}$ of $\bar{Y}$, i.e. to points of $\bar{X}_{\bar{y}} = X_y \otimes_{\kappa(y)} \overline{\kappa(y)}$. Therefore, these poitns can be considered as geometric points of X, and $\bar{X}_i$ is then none other than the strict localization of X at $\bar{x}_i$. We then have $H^0(\bar{X}_i, \bar{F}) \cong F_{\bar{x}_i}$, whence

$$H^0(\bar{X}, \bar{F}) = \prod_i F_{\bar{x}_i}, \tag{1.9.8}$$

which is a functorial isomorphism on the sheaf F. The previous formule (1.9.5) and (1.9.6) then proves the following assertion:

Proposition 1.9.17. *Let $f : X \to Y$ be a finite morphism of schemes and y be a point of Y. Then for any sheaf F over X, we have a canonical isomorphism*

$$f_*(F)_{\bar{y}} \xrightarrow{\sim} \prod_{\bar{x} \in X_y \otimes_{\kappa(y)} \overline{\kappa(y)}} F_{\bar{x}}, \tag{1.9.9}$$

(therefore the formation of f_* commutes with base change) and if F is an abelian sheaf, we have

$$R^p f_*(F) = 0 \quad \textit{for} \quad p > 0. \tag{1.9.10}$$

We note that the first formula of Proposition 1.9.17 is in fact independent of the result of Theorem 1.9.16 and the theorem of passing to limits ([?] VII 5.7), and that it implies (thanks to Theorem 1.9.11) that f_* is an exact functor on the category of abelian sheaves, whence $R^p f_*(F) = 0$ for $p > 0$.

Corollary 1.9.18. *Let $f : X \to Y$ be an integral morphism. Then for any abelian sheaf F over X, we have $R^p f_*(F) = 0$ for $p > 0$. Moreover, the functor f_* over the sheaf of sets commutes with base change.*

Proof. In fact, we can reduce (thanks to Theorem 1.9.11) to the case where Y is strict local. But then $Y = \mathrm{Spec}(A)$, $X = \mathrm{Spec}(B)$, with B being an integral algebra over A, and as $B = \varinjlim B_i$, where B_i runs through subalgebras of finite type (hence finite over A) of B, we have $X = \varprojlim_i X_i$, where $X_i = \mathrm{Spec}(B_i)$. In view of ([?] VII 5.13), we can then assume that f is finite, and the claim follows from 1.9.17. □

Corollary 1.9.19. *Let $f : X \to Y$ be an immersion. Then for any sheaf F over X, the canonical morphism $f^* f_*(F) \to F$ is an isomorphism.*

Proof. As f factors into a closed immersion with an open immersion, and the corollary is trivial for open immersions ([?] IV), we may assume that f is closed. Then for any $x \in X$, by transitivity, the fiber $(f^* f_*(F))_{\bar{x}}$ is equal to $(f_*(F))_{\bar{x}}$, so it suffices to prove that the canonical homomorphism

$$(f_*(F))_{\bar{x}} \to F_{\bar{x}}$$

is bijective, which follows from Proposition 1.9.17. □

1.9.5 Localization functors and supports

Let U be a Zariski open of a scheme X. Then $U \in \mathrm{Ob}(X_{\text{ét}})$, and in fact that U is a sub-object of the final object X of $X_{\text{ét}}$, hence defines a sub-object $\widetilde{U}$ of the final object of $\mathrm{Sh}(X_{\text{ét}})$, i.e. an open of the étale topos $X_{\text{ét}}$ of X (cf. [?] 8.3).

Proposition 1.9.20. *The preceding map $U \mapsto \widetilde{U}$ is an isomorphism from the ordered set of (Zariski) open subsets of X to the set of opens of the étale topos $X_{\text{ét}}$.*

Proof. As $X_{\text{ét}} \to \mathrm{Sh}(X_{\text{ét}})$ is fully faithful, we see that the map $U \mapsto \widetilde{U}$ is order preserving, and in particular injective. To see that it is surjective, consider a subsheaf F of the final sheaf $\widetilde{X}$, and consider the objects of $(X_{\text{ét}})_{/F}$, i.e. the étale schemes X' over X such that $F(X') \neq \varnothing$ (note that, since F is a subobject of $\widetilde{X}$, $F(U)$ is either empty or a singleton). As $X' \to X$ is étale, it is an open map ([?] 2.4.6), and in particular its image is open in X. Let U be the union of the images of $X' \in \mathrm{Ob}((X_{\text{ét}})_{/F})$. As the family $\{X' \to U : X' \in \mathrm{Ob}((X_{\text{ét}})_{/F})\}$ is surjective, hence covering, we conclude by the sheaf condition that there is a unique element in $F(U)$ which maps to the unique element of $F(X')$ for each X'. In other words, U is an object of $(X_{\text{ét}})_{/F}$, so $(X_{\text{ét}})_{/F} = (X_{\text{ét}})_{/U}$, and $F = U$. □

By Proposition 1.9.20, we can therefore unambiguously speak of an "open" of X, without specifying whether we are considering the usual Zariski topology or the étale topology.

Corollary 1.9.21. *Let U be an open of X and $j : U \to X$ be the canonical inclusion. Then the functor*

$$j^* : \mathrm{Sh}(X_{\text{ét}}) \to \mathrm{Sh}(U_{\text{ét}})$$

induces an equivalence of categories

$$\mathrm{Sh}(X_{\text{ét}})_{/\widetilde{U}} \xrightarrow{\sim} \mathrm{Sh}(U_{\text{ét}}).$$

Proof. In general, for any site $\mathcal{C}$ where finite projective limit exists, and for any sub-object U of the final object e of $\mathcal{C}$, consider the functor $j : \mathcal{C} \to \mathcal{C}_{/U}$ defined by $j(S) = S \times U$. Then g is a morphism of sites and $j^* : \mathrm{Sh}(\mathcal{C}) \to \mathrm{Sh}(\mathcal{C}_{/U})$ induces an equivalence $\mathrm{Sh}(\mathcal{C})_{/\widetilde{U}} \xrightarrow{\sim} \mathrm{Sh}(\mathcal{C}/_U)$. The corollary then follows from the fact that $U_{\text{ét}}$ is canonically isomorphic to $(X_{\text{ét}})_{/U}$. □

In a pictorial way, we can express Corollary 1.9.21 by saying that the operations of "restriction to opens" in the usual sense of diagrams on the one hand, and in the sense of topos on the other, are compatible. Here is an analogous compatibility for "restriction to a closed":

Theorem 1.9.22. *Let X be a scheme, Z be a closed subscheme of X, $U = X - Z$ be the open complement, $i : Z \to X$ and $j : U \to X$ be the canonical immersions. Then the functor*

$$i_* : \mathrm{Sh}(Z_{\text{ét}}) \to \mathrm{Sh}(X_{\text{ét}})$$

is fully faithful, and for any sheaf F over X, F is isomorphic to a sheaf of the form $i_*(G)$ if and only if $j^*(F)$ is isomorphic to the final sheaf $\widetilde{U}$ over U.

Proof. As i_* and i^* are adjoints of each other, the fact that i_* is faithfully exact is equivalent to the fact that the functorial homomorphism

$$i^*(i_*(G)) \to G$$

is an isomorphism, which follows from Corollary 1.9.19. On the other hand, if $G \in \mathrm{Sh}(Z_{\text{ét}})$, then we verify that for any $V \in U_{\text{ét}}$,

$$i_*(G)(V) = G(V \times_X Z) = G(\varnothing)$$

so $j^*(i_*(G))$ is the final sheaf over U (Corollary 1.9.21). Conversely, if $F \in \mathrm{Sh}(X_{\text{ét}})$ is such that $j^*(F)$ is the final sheaf, to prove that F is of the form $i_*(G)$, it suffices to prove that the canonical morphism

$$F \to i_*(i^*(F))$$

is an isomorphism, or equivalently it induces isomorphism on fibers for any $x \in X$. If $x \in U$, this follows from the hypothesis on F, and if $x \in Z$, by (1.9.2), we are reduced to verify that the homomorphism induced by

$$i^*(F) \to i^*(i_*(i^*(G)))$$

is an isomorphism at $\bar{x}$. But this follows from Corollary 1.9.19 applied to $i^*(F)$ and i, whence the claim. $\square$

Corollary 1.9.23. *The functor i_* induces an equivalence from the category of abelian sheaves over Z to the category of abelian sheaves over X whose restriction to $U = X - Z$ is zero.*

In view of Proposition 1.9.20, we can identify opens of the topos $\mathrm{Sh}(X_{\text{ét}})$ with open subsets U of X, and Theorem 1.9.22 tells us that the complement topos $\mathcal{E}_{U^c}$ of ([?] IV 3) is canonically equivalent to the topos $\mathrm{Sh}(Z_{\text{ét}})$, where $Z = X - U$ is endowed with the induced reduced scheme structure. We can therefore apply the results of ([?] IV 3), which allows us to view a sheaf F over X as a triple (F, G, u), where F is a sheaf over U, F is a sheaf over U and $u : F \to i^*(j_*(G))$ is a homomorphism. We can also use the functors $i^!$ and $j_!$ defined in ([?] V 3):

$$j_! : \mathrm{Ab}(U_{\text{ét}}) \to \mathrm{Ab}(X_{\text{ét}}), \quad i^! : \mathrm{Ab}(X_{\text{ét}}) \to \mathrm{Ab}(Z_{\text{ét}})$$

where Ab means the category of abelian sheaves. In view of the preceding arguments, we introduce, for an abelian sheaf F on X (resp. a section s of a sheaf) the notion of the support of F (resp. of φ), which is defined to be the complement of the largets open subset of X on which the restriction of F (resp. φ) is zero. Also, we apply the notations $H^p_Y(X, F)$ and $\mathcal{H}^p_Y(F)$ to denote $H^p(\mathcal{Z}, F)$ and $\mathcal{H}^p_{\mathcal{Z}}(F)$, where $\mathcal{Z}$ is the complement of $\mathcal{E}_{/U}$. In particular, we have $\mathcal{H}^p_Y = R^p i_!$, etc. One may consult ([?] V 6) for the general properties of these functors.

1.9.6 Specialization functors of fiber functors

In **??**, we have associated for any geometric point ξ of a scheme X, a strictly local x-scheme

$$\bar{X}(\xi) = \operatorname{Spec}(\mathscr{O}_{X,\xi}),$$

which depends, in fact, only on the geometric point ξ defined by the separable closure $\overline{\kappa(x)}$ of $\kappa(x)$ in $\Omega = \kappa(\xi)$. We often restrict ourselves to geometric points $\xi = \operatorname{Spec}(\Omega)$ which are separable over X, i.e. such that Ω is a separable closure of $\kappa(x)$, or equivalently $\xi = \bar{x}$. An X-scheme Z is called a **strict localization** of X if it is X-isomorphic to a scheme of the form $\operatorname{Spec}(\mathscr{O}_{X,\bar{x}})$, and we then see that

$$\xi \mapsto \operatorname{Spec}(\mathscr{O}_{X,\xi}) = \bar{X}(\xi)$$

is a functor from the category of separable geometric points over X to the category of X-schemes that are strict localizations of X. This is not an equivalence of categories if we consider the X-morphisms in the second category, since there may be X-morphisms that are not isomorphisms. We therefore make the following definition:

Definition 1.9.24. Let ξ, ξ' be two geometric points of the scheme X. A **specialization** $\xi' \rightsquigarrow \xi$ is defined to be an X-morphism

$$\bar{X}(\xi') \to \bar{X}(\xi)$$

of the corresponding strict localizations. We say that ξ is a specialization of ξ', or that ξ' is a generalization of ξ, if there exists a specialization from ξ' to ξ.

It is clear that compositions of specilizations make sense, so the geometric points of X form a category, which is equivalent to the full subcategory of $\mathbf{Sch}_{/X}$ formed by strict localizations of X.

Lemma 1.9.25. *Let X be a scheme, Z be an X-scheme which is isomorphic to a filtered projective limit of étale X-schemes X_i with affine transition morphisms, ξ' be a geometric point of X, $Z' = \bar{X}(\xi')$ be the corresponding strict localization.*

(a) *The restriction map*

$$\operatorname{Hom}_X(Z', Z) \to \operatorname{Hom}_X(\xi', Z)$$

is bijective.

(b) *For the two members in (a) to be nonempty, it is necessary and sufficient that the image x' of ξ' in X is contained in that of Z.*

(c) *Let T be a Z-scheme, for that T is a strict localization of Z, it is necessary and sufficient that it is a strict localization of X.*

Proof. To prove (a), we can assume that Z is one of the X_i, i.e. where Z us étale over X. In this case, we have the following canonical bijections

$$\operatorname{Hom}_X(Z', Z) \cong \operatorname{Hom}_{Z'}(Z', Z' \times_X Z), \quad \operatorname{Hom}_X(\xi', Z) \cong \operatorname{Hom}_{\xi'}(\xi', \xi' \times_X Z).$$

It then suffices to show for any etale Z'-scheme W that the canonical map

$$\operatorname{Hom}_{Z'}(Z', W) \xrightarrow{\sim} \operatorname{Hom}_{\xi'}(\xi', \xi' \times_{Z'} W)$$

is bijective, which follows from **??** and (Fulei, 2.8.3(vii)).

As for (b), we note that if x' is the image of a point z of Z, then $\kappa(z)$ is a separable extension of $\kappa(x')$, so there exists a $\kappa(x')$-homomorphism from $\kappa(z)$ into $\kappa(\xi')$. The last assertion follows from (**??**). ∎

Proposition 1.9.26. *Let ξ, ξ' be two geometric points of a scheme X. Then the restriction map defines a bijection from the set $\mathrm{Hom}_X(\bar{X}(\xi'), \bar{X}(\xi))$ of specilization morphisms from ξ' to ξ to the set of X-morphisms $\mathrm{Hom}_X(\xi', \bar{X}(\xi))$.*

Proof. This follows from Lemma 1.9.25, as $\bar{X}(\xi')$ satisfies the conditions of Lemma 1.9.25. □

Corollary 1.9.27. *For that ξ is a specialization of ξ', it is necessary and sufficient that the same is true for the images x, x' of ξ, ξ' in X, i.e. that $\mathfrak{x}' \rightsquigarrow x$.*

Proof. As the morphism $\mathrm{Spec}(\mathscr{O}_{X,\xi}) \to \mathrm{Spec}(\mathscr{O}_{X,x})$ is faithfully flat, hence surjective, it suffices to apply Lemma 1.9.25 (b) (cf. **??**). □

Corollary 1.9.28. *For any scheme X, let $\mathrm{Pt}(X)$ be the category of geometric points of X, with morphisms given by specializations. Let ξ be a geometric point and $\bar{X}(\xi)$ be the corresponding strict localization. Then we have an equivalence of categories:*

$$\mathrm{Pt}(\bar{X}(\xi)) \xrightarrow{\sim} \mathrm{Pt}(X)_{/\xi}$$

obtained by associating any geometric point ξ' of $\bar{X}(\xi)$ with the corresponding geometric point of X and the specialization induced by the structural morphism $\xi' \to \bar{X}(\xi)$.

Proof. By Proposition 1.9.26 the described functor is fully faithful, and it is essentially surjective by definition. □

In other words, giving a generalization ξ' of a geometric point ξ is essentially equivalent to giving a geometric point of $\bar{X}(\xi)$. We also note that in view of Lemma 1.9.25 (c), the corresponding morphism $\bar{X}(\xi') \to \bar{X}(\xi)$ makes $\bar{X}(\xi')$ into a strict localization of $\bar{X}(\xi)$ relative to ξ'.

Now for any geometric point ξ of X and any sheaf F over X, the fiber F_ξ can be interprete as $\Gamma(\bar{X}(\xi), \bar{F}(\xi))$, where $\bar{F}(\xi)$ is the inverse image of F over $\bar{X}(\xi)$ (cf. [**?**] VIII). Then we see that any specialization

$$u : \xi' \to \xi$$

induces a canonical homomorphism

$$u^* : F_\xi \to F_{\xi'}$$

called the **specialization homomorphism** associated with the specialization u. It is evident, in view of the transitivity of inverse images, that the associated specialization homomorphism are transitive: $(wu)^* = u^* w^*$.

Let $\mathcal{X}$ be a topos, recall that a "*fiber functor*", of (by abusing of languages) a "*point*" of the topos $\mathcal{X}$, is defined to be a morphism from the topos **Set** (which is isomorphic to the sheaf category over a space reduced to a point) into $\mathcal{X}$. Equivalently, since **Set** is initial among topoi, this is given by a functor

$$\varphi^* : \mathcal{X} \to \mathbf{Set}$$

which commutes with inductive limits and finite projective limits. We can consider the set of fiber functors of $\mathcal{X}$ as objects of a full subcategory $\mathcal{H}om(\mathcal{X}, \mathbf{Set})$, called the **category of fiber functors of the topos** $\mathcal{X}$. Its opposite category is called the **category of points of** $\mathcal{X}$, and denoted by $\mathrm{Pt}(\mathcal{X})$ (cf. [**?**] IV 6.1).

If $\mathcal{X}$ is of the form $\mathrm{Sh}(X_{\text{ét}})$ where X is a scheme, we then have a functor

$$\mathrm{Pt}(X) \to \mathrm{Pt}(\mathrm{Sh}(X_{\text{ét}})) \tag{1.9.11}$$

sending a geometric point ξ of X to the fiber functor $F \mapsto F_\xi$ it corresponds.

Theorem 1.9.29. *Let X be a scheme. The functor (1.9.29) is an equivalence from the category of geometric points of X to the category of points of the étale topos $\mathrm{Sh}(X_{\text{ét}})$.*

Proof. □

1.9.7 Spectral sequences for integral morphisms

Proposition 1.9.30 (Descent Spectral Sequence). *Let $f : X \to Y$ be a surjective integral morphism and F be an abelian sheaf over Y. For any integer $p \geq 0$, let $\mathcal{H}^p(F)$ be the presheaf over $\mathbf{Sch}_{/Y}$ defined by*

$$\mathcal{H}^p(F)(Z) = H^i(Z, F_Z)$$

where F_Z is the inverse image of F on Z. Then there exists a functorial spectral sequence

$$E_2^{p,q} = H^p(X/Y, \mathcal{H}^p(F)) \Rightarrow H^{p+q}(X, F).$$

Here the symbol $H^p(X/Y, G)$, for a presheaf F over $\mathbf{Sch}_{/Y}$, denotes the *relative p-th Čech cohomology group* defined by the complex

$$C^n(X/Y, F) = F(X_{n+1})$$

where X_{n+1} is the $(n+1)$-fold fiber product of X over Y[3]. To establish Proposition 1.9.30, let $p_n : X_{n+1} \to Y$ be the canonical projection, and put

$$A^n = (p_n)_*(\mathbb{Z}_{X_{n+1}})$$

where $\mathbb{Z}_{X_{n+1}}$ denotes the constant sheaf $\mathbb{Z}$ on X_{n+1}. Then $(A^n)_{n\in\mathbb{N}}$ form a simplicial abelian sheaf over Y, whence a complex $A^\bullet$ of abelian sheaves over Y. It is evident that we have a canonical homomorphism

$$\varepsilon : \mathbb{Z}_Y \to A^0. \tag{1.9.12}$$

Lemma 1.9.31. *The complex $A^\bullet$, endowed with the homomorphism (1.9.12), is a resolution of $\mathbb{Z}_Y$. More generally, for any abelian sheaf F over Y, $A^\bullet \otimes_{\mathbb{Z}} F$ is a resolution of F.*

Proof. We can suppose that Y is affine, so $Y = \operatorname{Spec}(A)$, $X = \operatorname{Spec}(B)$, where B is an integral A-algebra. We then have $B = \varinjlim B_i$, where B_i runs through finite subalgebras of B, so $X = \varprojlim X_i$, where $X_i = \operatorname{Spec}(B_i)$. Using ([?] VII, 5.11), we see that the agumented complex $A^\bullet(X/Y)$ is the inductive limit of the augmented complex $A^\bullet(X_i/Y)$ (where $A^\bullet(X_i/Y)$ is the similar complex defined for X_i), so we may assume that f is finite. It suffices to prove that for any geometric point $\bar{y}$ of Y, the complex $A^\bullet_{\bar{y}}$ is a resolution of $\mathbb{Z}_{\bar{y}}$. But if $X_{\bar{y}}$ is the fiber of X at $\bar{y}$, it follows from Proposition 1.9.17 that the complex $A^\bullet_{\bar{y}}$ is none other than the analogue complex $A^\bullet(X_{\bar{y}}/\bar{y})$, so we may assume that Y is the spectrum of a separably closed field k. Using Theorem 1.9.1, we can also suppose that k is algebraically closed and X is reduced, so X is of the form I_Y, where I is a finite set. But then the complex $A^\bullet \otimes_{\mathbb{Z}} F$ is identified with the trivial cochain complex of the index set I with coefficients in $F_{\bar{y}}$, so it is a resolution of $F_{\bar{y}}$. □

Proof. As a concequence of Lemma 1.9.31, we obtain a hypercohomology spectral sequence

$$E_2^{p,q} = H^p(H^q(Y, A^\bullet \otimes_{\mathbb{Z}} F)) \Rightarrow H^{p+q}(Y, A^\bullet \otimes_{\mathbb{Z}} F) = H^{p+q}(Y, F).$$

On the other hand, by reducing to the case where f is finite and using Proposition 1.9.17, we have a canonical isomorphism $A^n \otimes_{\mathbb{Z}} F \xrightarrow{\sim} (p_n)_*(p_n^*(F))$, so

$$H^q(Y, A^\bullet \otimes_{\mathbb{Z}} F) \cong H^q(X_{n+1}, p_n^*(F)) = \mathcal{H}^q(F)(X_{n+1})$$

which gives the desired spectral sequence. □

[3]If $f : X \to Y$ is étale, this is equal to the Čech cohomology group $H^p(R, F)$, where R is the sieve generated by the covering $f : X \to Y$.

Remark 1.9.32. If we are given a locally finite closed covering (Y_i) of Y and $X = \coprod_i Y_i$, then the canonical morphism $f : X \to Y$ is clearly finite, and the E_2 page of spectral sequence of (1.9.30) is given by the cohomology of the complex

$$C^n(X/Y, \mathcal{H}^p(F)) = H^p(X_n, F|_{X_n}) = \bigoplus H^q(Y_{i_0\ldots i_p}, F|_{Y_{i_0\ldots i_p}}),$$

where $Y_{i_0\ldots i_p} = Y_{i_0} \cap \cdots \cap Y_{i_p}$. This is therefore the analogue of the Leray spectral sequence for a locally finite closed cover of an ordinary topological space.

Remark 1.9.33. One should note the analogy of the spectral sequence of Proposition 1.9.30 with the Leray spectral sequence for a covering (X_i) of Y (in this case X_i is étale over Y). The latter would be obtained formally by writing the spectral sequence Proposition 1.9.30 for $X = \coprod_i X_i$. It is possible in fact that these two spectral sequences admit a common generalization, which would be valid whenever we have a family of morphisms $\{X_i \to Y\}$, which is a "universally effective descent family" for the fibre category of étale sheaves.

Proposition 1.9.34 (Hochschild-Serre Spectral Sequence). *Let Y be a scheme, G be a profinite group, (G_i) be a projective system of finite quotient groups of G, and (X_i) be a projective system of principal G_i-bundles of Y over the system (G_i). Let $X = \varprojlim_i X_i$ (cf. [?] VII 5.1) and F be an abelian sheaf over Y. Then we have a spectral sequence*

$$E_2^{p,q} = H^p(G, \varprojlim H^q(X_i, F_i)) \Rightarrow H^{p+q}(Y, F),$$

where F_i is the inverse image of F over X_i, and $H^p(G, -)$ is the Galois cohomology.

Proof. Then $f : X \to Y$ is a covering in $Y_{\text{ét}}$, so we can apply the Leray spectral sequence for this morphism. In view of the canonical isomorphism $X_{n+1} \cong X \times G^n$, we see that for any presheaf F over $\mathbf{Sch}_{/Y}$, the complex $C^\bullet(X/Y, F)$ is equal to the complex of cochains on $F(X)$ with coefficients in G, whence the proposition. □

Corollary 1.9.35. *Suppose that Y is quasi-compact and quasi-separated, then the spectral sequence (1.9.34) becomes*

$$E_2^{p,q} = H^p(G, H^q(X, F_X)) \Rightarrow H^{p+q}(Y, F).$$

Proof. In fact, in view of ([?] VII, 5.8), we then have a canonical isomorphism

$$\varinjlim_i H^q(X_i, F_i) \cong H^q(X, F_X).$$ □

Corollary 1.9.36. *Let Y be a local Henselian scheme with closed point y, F be an abelian sheaf over Y, and F_0 the induced sheaf over $Y_0 = \mathrm{Spec}(\kappa(y))$. Then the canonical homomorphisms*

$$H^p(Y, F) \to H^p(Y_0, F_0)$$

are isomorphisms.

Proof. Let $\bar{y}$ be a geometric point over y, corresponding to a separably closure $\kappa(\bar{y})$ of $\kappa(y)$. As Y is Henselian, the strict localization X of Y at $\bar{y}$ is the projective limit of Galois étale neighborhoods X_i of $\bar{y}$, so we can apply Proposition 1.9.34. As $H^q(X, F_X) = 0$ for $q > 0$ in view of ([?] VIII 4.7), we then conclude the isomorphisms

$$H^p(G, F(X)) \xrightarrow{\sim} H^p(Y, F),$$

where G is the Galois group of X over Y, isomorphic to $\mathrm{Gal}(\kappa(\bar{y})/\kappa(y))$. Similarly, we have

$$H^p(G, F_0(X_0)) \xrightarrow{\sim} H^p(Y_0, F_0),$$

where $X_0 = X \times_{Y_0} Y \cong \mathrm{Spec}(\kappa(\bar{y}))$. Now the restriction homomorphism $F(X) \to F_0(X_0)$ is an isomorphism by ([?] VIII 4.8), whence the assertion of the corollary. □

1.9.8 Appendix: representability lemma

Lemma 1.9.37 (Representability Lemma). *A sheaf F over a scheme X is representable if and only if the following conditions are satisfied:*

(a) *The stalks of F are finite,*

(b) *For each étale scheme $U \to X$ over X and every two sections $s, t \in F(U)$, the set of points $x \in U$ such that the fibers $s_{\bar{x}}$, $t_{\bar{x}}$ are different in $F_{\bar{x}}$ is an open subset.*

1.10 Constructible sheaves and cohomology of curves

Let X be a topological space. A sheaf $\mathscr{F}$ over X is called **locally constant** if each point of X has an open neighbourhood U such that the restriction of $\mathscr{F}$ to U is isomorphic (in the category of sheaves on U) to a constant sheaf. It is a well-known fact that for a sufficiently good space, the category of locally constant sheaves of sets on X is equivalent to the category of coverings of X. Since étale morphisms are regarded as covering morphisms, we are therefore interested in the description of locally constant sheaves in the étale topology, together with their cohomology. However, the class of locally constant sheaves is well-bahaved (for example it is not closed under direct images with respect to proper maps, even closed immersions), so it become necessary to consider the smallest useful class of sheaves containing the finite constant sheaves (since we only consider finite étale coverings), which turns out to be *constructible sheaves*. The constructible sheaves over a scheme X form an ableian category, and is Noetherian if X is quasi-compact. Form another perspective, constructible sheaves are precisely those that can be represented by étale algebraic spaces of finite type.

1.10.1 Torsion sheaves

Let $\mathcal{X}$ be a topos and F be an abelian sheaf over $\mathcal{X}$. We can then define the multiplication by n (where n is an integer) on F, induced by the same operation on **Ab**, and we denote by $F[n]$ the kernel of this multiplication. In other words, for any object X of $\mathcal{X}$, $F[n](X)$ is the n-torsion subgroup of $F(X)$. If n and m are two integers such that $n \mid m$, then there is a natrual inclusion map

$$F[n] \to F[m]$$

which sends, for any object X of $\mathcal{X}$, the subgroup $F[n](X)$ into $F[m](X)$.

Definition 1.10.1. Let p be a prime number (or a set of prime numbers). A sheaf F is said to be **p-torsion**, or a **p-sheaf**, if the canonical morphism

$$\varinjlim_{n} F[n] \to F \tag{1.10.1}$$

is bijective, where n runs through the set of integers such that $p \dashv n$, i.e. such that every prime divisor of n belongs to p (if p is a prime number, this means n is a power of p). If p is equal to the set of all prime numbers, then F is simply called a **torsion-sheaf**.

The most important torsion sheaves within our consideration are constant sheaves induced by finite abelian groups: in fact, if G is a finite abelian group, then the constant sheaf defined by G is p-torsion, where p is the set of prime divisors of n. In practice, we often take $G = \mathbb{Z}/n\mathbb{Z}$, where n is coprime to the characterstic of the base field.

Proposition 1.10.2. *Let $\mathcal{X}$ be a topos, F be a sheaf on $\mathcal{X}$, and p be a set of prime numbers.*

(a) *The sheaf F is p-torsion if and only if F is the sheaf associated with a presheaf with values in p-torsion abelian groups*[4]. *In this case, this presheaf is isomorphic to the inductive limit $\varinjlim_n F[n]$, taken in the category of presheaves.*

(b) *If F is p-torsion and X is a quasi-compact*[5] *object of $\mathcal{X}$, then $F(X)$ is a p-torsion group.*

(c) *If $\mathcal{X}$ is locally of finite type (cf. [?] VI 1.1), then F is p-torsion if and only if for any quasi-compact object X of $\mathcal{X}$, $F(X)$ is a p-torsion group. In this case, for any quasi-compact object X of $\mathcal{X}$ and any integer $p \geq 0$, we have*

$$\varinjlim_{p \dashv n} H^p(X, F[n]) \xrightarrow{\sim} H^p(X, F).$$

(d) *If $u : \mathcal{X} \to \mathcal{Y}$ is a morphism of topos and G is a p-torsion sheaf over $\mathcal{Y}$, then the inverse image $u^*(G)$ is p-torsion over $\mathcal{X}$.*

(e) *If $u : \mathcal{X} \to \mathcal{Y}$ is a quasi-compact morphism of topos locally of finite type and F is a p-torsion sheaf over $\mathcal{X}$, then $R^p u_*(F)$ is a p-torsion sheaf over $\mathcal{Y}$ for any $p \geq 0$.*

Proof. It is clear that if a sheaf F is p-torsion, then F is the sheaf associated with the presheaf P whose section group $P(X)$ is the subgroup of p-torsion elements of $F(X)$. Conversely, let $F = P^\#$, where P is a presheaf with values in the category of p-torsion groups. From the exact sequence

$$0 \longrightarrow P[n] \longrightarrow P \xrightarrow{n} P$$

we then deduce an exact sequence

$$0 \longrightarrow (P[n])^\# \longrightarrow F \xrightarrow{n} F$$

whence $(P[n])^\# = F[n]$. As # commutes with inductive limits and $\varinjlim_{p \dashv n} P[n] \xrightarrow{\sim} P$, with limit taken in the category of presheaves, we then deduce that

$$\varinjlim_{p \dashv n} F[n] \xrightarrow{\sim} F$$

so F is p-torsion. The last assertion of (a) also follows easily from this isomorphism.

Now let F be p-torsion and X be a quasi-compact object of $\mathcal{X}$. Put $P = \varinjlim_n F[n]$, with limit taken in the category of presheaves. Then P can be considered as a sub-presheaf of F, so it is a separated and we have

$$P^\#(X) = \varinjlim \Big(\ker \big(\prod_i P(X_i) \rightrightarrows \prod_{i,j} P(X_i \times_X X_j) \big) \Big)$$

where the limit is taken for all covering families $\{X_i \to X\}$. But as X is quasi-compact, it suffices to take finite covering familes, and for such families $\prod_i P(X_i)$ is p-torsion, so the corresponding kernel is p-torsion. Therefore $P^\#(X) = F(X)$ is a p-torsion abelian group, and this proves assertion (b).

Now assume that $\mathcal{X}$ is locally of finite type. To verify that $\varinjlim_n F[n] \to F$ is bijective, it suffices to consider their values over a generating family of $\mathcal{X}$. We are then reduced to the situation of (b), and this proves the first assertion of (c); the second assertion follows from (a). Finally, (d) follows from the fact that $u^*(G^\#) = (u^\star(G))^\#$, and (e) is a concequence of (c). □

[4] An abelian group G is said to be p-torsion if the canonical homomorphism $\varinjlim_{p \dashv n} G[n] \to G$ is an isomorphism, where n runs through the set of integers such that all prime divisors of n belong to p.

[5] An object X of a topos is called **quasi-compact** if any covering family $\{X_i \to X\}$ can be dominated by a finite covering family. A morphism of topos if called quasi-compact if the inverse image of any quasi-compact object is still quasi-compact.

We also need a non-abelian variant of p-torsion sheaves, which turns our to be techniquely subtle. We therefore restrict ourselves to étale topos over a scheme X.

Definition 1.10.3. Let p be a prime number (or a set of prime numbers). A group G is called a ***p*-group** if it is finite with order n such that $p \dashv n$, and is called a **ind-*p*-group** if any finite subset of G generates a p-group. Equivalently, this means that the finite sub-p-groups G_α form a filtered set and that $G = \varinjlim G_\alpha$. If p is equal to the set of all prime numbers, then G is simply called a **ind-finite group**.

Proposition 1.10.4. *Let p be a prime number (or a set of prime numbers).*

(a) *A subgroup or a quotient group of a ind-p-group is also a ind-p-group.*

(b) *A filtered inductive limit of ind-p-groups is an ind-p-group.*

(c) *A finite projective limit of ind-p-groups is an ind-p-group.*

Proof. □

Definition 1.10.5. Let X be a scheme. A sheaf F of group over X is called a sheaf of ind-p-groups (resp. a sheaf of ind-finite groups) if for any étale morphism $U \to X$, where U is quasi-compact, $F(U)$ is an ind-p-group (resp. a ind-finite group).

Proposition 1.10.6. *Let F be a étale sheaf of groups over a scheme X and p be a prime number (or a set of prime numbers).*

(a) *F is a sheaf of ind-p-groups if and only if for any geometric point ξ of X, the fiber F_ξ is an ind-p-group.*

(b) *If $f : X \to Y$ is a morphism and G is a sheaf of ind-p-groups, then $f^*(G)$ is a sheaf of ind-p-groups.*

(c) *If $f : X \to Y$ is a quasi-compact morphism and F is a ind-p-group over X, then $f_*(F)$ is a sheaf of ind-p-groups.*

Proof. In view of (1.9.2), assertion (b) follows from (a), and (c) is trivial by (1.8.2) and **??**. Suppose that F is a sheaf of ind-p-groups, and let ξ be a geometric point of X. Then $F_\xi = \varinjlim F(X')$, where X' runs throught étale neighborhoods of ξ in X (Proposition 1.9.10). It is evident that the affine neighborhoods form a cofinal system, so F_ξ is an inductive limit of ind-p-groups, whence an ind-p-groups by Proposition 1.10.4. Conversely, suppose that for any geometric point ξ, the fiber F_ξ is an ind-p-group and let $U \to X$ be an étale morphism, where U is quasi-compact. Let $S \subseteq F(U)$ be a finite subset. For any given geometric point ξ of U, the image of S in F_ξ then generates a finite p-group. As a finite group is of finite presentation, we conclude that there exists an étale neighborhood U_ξ of ξ in U such that S generates a finite p-group in $F(U_\xi)$[6]. Since U is quasi-compact, finitely many U_ξ cover U, say $\{U_1, \dots, U_n\}$. The image of S in $\prod F(U_i)$ then genetates a finite group with order divisible by p, and as $F(U) \subseteq \prod F(U_i)$, we conclude that S generates a finite p-subgroup in $F(U)$, so F is a sheaf of ind-p-groups. □

1.10.2 Constructible sheaves

Let $\mathcal{X}$ be a topos and e be its final object. If S is a set, we can define a sheaf $S_\mathcal{X}$ associated with the presheaf $X \mapsto S$ for any object X in $\mathcal{X}$. The object of $\mathcal{X}$ representing $S_\mathcal{X}$ is then $\coprod_{s \in S} e$, and the sheaf $S_\mathcal{X}$ is called the **constant sheaf** with value S. Similarly, we can define the notion of constant groups, constant A-modules (A being a ring), and constant abelian sheaves. A

[6]Not clear for the reference ([**?**] VIII, 4).

morphism $f_{\mathcal{X}} : S_{\mathcal{X}} \to T_{\mathcal{X}}$ of constant sheaves is called **constant** if it is induced by a map of sets $f : S \to T$[7].

A sheaf F is called **locally constant** if there exists a covering $\{e_i \to e\}$ of e such that F is constant over each e_i. We define similarly the notion of locally constant sheaves of groups or A-modules, and locally constant morphism $f : F \to G$, if F, G are locally constant sheaves. Finally, a locally constant sheaf of sets is called **finite** if its local values are finite sets, and a locally constant sheaf of groups or A-modules is said to be **of finite type** (resp. **of finite presentation**) if its local values are finitely generated groups or A-modules (resp. finitely presented groups or A-modules).

Lemma 1.10.7. *Let $\mathcal{X}$ be a topos and F, G be sheaves over $\mathcal{X}$.*

(a) *Let $f : F \to G$ be a morphism of localy constant sheaves of sets over $\mathcal{X}$, and suppose that F is finite. Then f is locally constant.*

(b) *Let $f : F \to G$ be a morphism of locally constant sheaves of A-modules, where F is of finite type. Then f is locally constant, and the kernel and cokernel of f are locally constant.*

(c) *Let $0 \to F' \to F \to F'' \to 0$ be an exact sequence of sheaves of A-modules (resp. groups), where F' and F'' are locally constant and F'' is of finite presentation. Then F is locally constant.*

Proof. To prove (a) and (b), we may assume that F and G are constant, so let $F = M_{\mathcal{X}}$, $G = N_{\mathcal{X}}$ (M and N are A-modules), and let $S \subseteq M$ be a finite generating subset of M. Then f is determined by its restriction to $S_{\mathcal{X}}$ and f is locally constant if $f|_{S_{\mathcal{X}}}$. The first assertion of (b) then follows from (a), and the second assertion is a trivial concequence of the first one.

It now remains to prove (a). Suppose that $F = S_{\mathcal{X}}$, $G = T_{\mathcal{X}}$ are constant, and denote also by f the morphism of objects $S \times e \to T \times e$ induced by f (recall that $S_{\mathcal{X}}$ is represented by $S \times e$). Let $i_s : \{s\} \times e \to T \times e$ ($s \in S$) be the components of the morphism f. Since S is finite, it evidently suffcies to consider the case $f = i_s$, that is, where S is a singleton. Let $X_t \to e$ ($t \in T$) be the family of morphisms fitting the Cartesian diagram

$$\begin{tikzcd} X_t \arrow[r] \arrow[d] & e \arrow[d, "i_t"] \\ e \arrow[r, "f"] & T \times e \end{tikzcd}$$

where $i_t : e \to T \times e$ is the inclusion in the t-th component. The family $\{i_t\}$ is trivially covering, so $\{X_t \to e\}$ is also a covering. It is immediate to verify that the restriction of f to X_t is constant, whence the assertion of (a).

To prove (c), we may consider the case of A-modules. We can suppose that $F' = M'_{\mathcal{X}}$ and $F'' = M''_{\mathcal{X}}$ are constant, where M'' is an A-module of finite presentation. If M'' is free, then the morphism $F \to F''$ is locally constant by (a), so locally the extension F of F'' by F' splits, which proves that F is locally isomorphic to $F' \times F''$, hence is locally constant. In the general case, choosing a surjective homomorphism $\varphi : L'' \to M''$, where L'' is free of finite rank; the kernel R of φ is of finite type, and we consider the pull back diagram

$$\begin{tikzcd} 0 \arrow[r] & G \arrow[r] \arrow[d] & L \arrow[r] \arrow[d] & L''_{\mathcal{X}} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & M'_{\mathcal{X}} \arrow[r] & F \arrow[r] & M''_{\mathcal{X}} \arrow[r] & 0 \end{tikzcd}$$

where $L = L''_{\mathcal{X}} \times_{M''_{\mathcal{X}}} F$ and $G = L''_{\mathcal{X}} \times_{M''_{\mathcal{X}}} M'_{\mathcal{X}}$. Then L is an extension of $L''_{\mathcal{X}}$ by G, hence locally constant by the preceding arguments. □

[7]In general, the functor **Set** $\to \mathcal{X}$ sending S to $S \times e$ is not fully faithful.

We now return to the étale topos of a scheme X. As we shall see, locally constant sheaves over X corresponds to étale covering spaces of X.

Proposition 1.10.8. *Let X be a scheme and F be a sheaf of sets over X. If F is locally constant, then it is represented by a étale scheme U over X. If F has finite (resp. nonempty) fibers, then U is a finite (resp. surjective) étale covering of X. Conversely, if U is finite étale over X, then the sheaf represented by U is locally constant.*

Proof. Descent ([**?**] IX 4.1 if F has finite fibers, and [**?**] X 5.4 in the general case). □

Definition 1.10.9. A sheaf of sets (resp. groups, resp. A-modules) is called **constructible** if for any affine open subset $U \subseteq X$, there exists a finite stratification of U by locally closed constructible subschemes $U = \coprod_i U_i$ such that $F|_{U_i}$ is locally constant and finite (resp. finite, resp. finitely presentated[8]).

The finiteness hypothesis on local values amounts to saying that for each geometric point ξ of X, the fiber F_ξ is finite (resp. finite, resp. of finite presentation). It then follows from Lemma 1.10.7 (a) that a sheaf of groups F is constructible if and only if the underlying sheaf of sets is constructible. Note that if F is an abelian sheaf, then it is constructible as a sheaf of groups (or a sheaf of sets) if and only if it is constructible as a sheaf of $\mathbb{Z}$-modules, *and its fibers are finite*. Therefore, the sheaf $\mathbb{Z}_X$ is constructible as a $\mathbb{Z}$-module, but not as a sheaf of groups.

Proposition 1.10.10. *Let X be a scheme.*

(a) *If X is quasi-compact and quasi-separated, then a sheaf of sets (resp. groups, resp. A-modules) is constructible if and only if there exists a stratification of X into finitely many locally closed and constructible subsets $X = \coprod_i X_i$ such that $F|_{X_i}$ is locally constant and finite (resp. finite, resp. finitely presented).*

(b) *If (U_i) is an open covering of X, then for a sheaf F to be constructible, it suffices that $F|_{U_i}$ is constructible for each i.*

(c) *Let $f : Y \to X$ be a morphism and F be a constructible sheaf over Y, then $f^*(F)$ is constructible.*

(d) *Let $f : F \to G$ be a morphism of constructible sheaves of sets over X. Then the set of points $x \in X$ where $F_{\bar{x}} \to G_{\bar{x}}$ is surjective (resp. injective, resp. is isomorphic to a given map of sets) is a locally constructible subset of X.*

(e) *If X is locally Noetherian, then a sheaf of sets (resp. groups, resp. A-modules) F over X is constructible if and only if for any $x \in X$, there exists a nonempty open neighborhood of the closure $\overline{\{x\}}$ of x such that $F|_U$ is locally constant and finite (resp. finite, resp finitely presented).*

Proof. Evidently, if $f : Y \to X$ is a morphism and there exists a morphism and there exists a stratification of X into finitely many locally closed and constructible subsets $X = \coprod_i X_i$ such that F is locally constant over X_i, then the same holds for Y and $f^*(F)$ (cf. [**?**] 1.8.2). Conversely, suppose that X is quasi-compact and quasi-separated, and let $\{U_1, \dots, U_n\}$ be an affine open covering of X such that $F|_{U_i}$ is constructible for each i. To find a stratification of X as in (a), it suffices to consider such stratifications for U_n and $Y = X - U_n$ (with the reduced induced scheme structure), since they are constructible by the hypothesis that X is quasi-separated. Now such a stratification exists for U_n by hypothesis, and Y is a union of $n-1$ affine opens $V_i = Y \cap U_i$ with $F|_{V_i}$ being constructible, whence the assertion of (a) by induction on n. The same reasoning proves (b), in fact, the hypothesis in (b) implies that $F|_U$ is constructible for any sufficiently small affine opens, and we can cover an arbitrary open affine V by a finite number

[8] If A is not assumed to be Noetherian, then it is preferable, for the sake of Homological Algebra, to require that the A-module M has a resolution by free A-modules of finite rank.

of such opens. Since assertion (b) implies that the notion of constructibility is local, (c) and (d) then follows immediately from Lemma 1.10.7. Fianlly, the implication $\Rightarrow$ in (e) is immediate, and does not depend on the Noetherian hypothesis. On the other hand, if X is Noetherian, we can then prove the implication $\Leftarrow$ by Noetherian induction. $\square$

Proposition 1.10.11. *Let X be a quasi-compact and quasi-separated scheme and F be a constructible sheaf of groups (resp. A-modules) over X. Then there exists a finite filtration of F whose successive quotients are of the form $i_!(G)$, where $i : U \to X$ is the inclusion of a locally closed and constructible subset and G is a locally constant and constructible sheaf over U. If X is Noetherian, we can further assume that the U are irreducible.*

Proof. By Proposition 1.10.10 (a), there exists a stratification of X into finitely many locally closed and constructible subsets $X = \coprod_i X_i$ such that $F|_{X_i}$ is locally constant with finite (resp. finitely presented) fibers. We write $X_i = U_i \cap V_i^c$, where U_i and V_i are constructible open subsets, and let n be the number of open subsets of X in the subtopology $\mathcal{T}$ generated by $\{U_i, V_i\}$. We proceed by induction on n: if W is a minimal nonempty element of $\mathcal{T}$, it is evident that $F|_W$ is locally constant and of finite presentation. Let $i : W \to X$ be the inclusion morphism and $Y = X - W$, we then have an exact sequence

$$0 \longrightarrow i_!(i^*(F)) \longrightarrow F \longrightarrow F|_Y \longrightarrow 0$$

and we are th erefore reduced to the same assertion for Y and for the sheaf $F|_Y$ (Proposition 1.10.10 (c)), where we can apply the induction hypothesis. $\square$

Proposition 1.10.12. *Suppose that A is a Noetherian ring.*

(a) *A finite projective (resp. inductive) limit of constructible sheaves of sets or A-modules is constructible. In particular, the kernel, cokernel and image of a morphism of constructible sheaves of A-modules are constructible.*

(a') *A finite projective limit of constructible sheaves of groups is constructible, and if $f : F \to G$ is a morphisms of constructible sheaves of groups, the kernel, image and cokernel (if the image is normal) are constructible.*

(b) *Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of sheaves of A-modules (resp. groups), with M', M'' constructible. Then M is constructible.*

(c) *Let $M_1 \to M_2 \to M_3 \to M_4 \to M_5$ be an exact sequence of sheaves of A-modules. Then if M_i are constructible for $i = 1, 2, 4, 5$, so is M_3.*

Proof. We only consider the case for sheaves of A-modules, the corresponding assertions for sheaves of sets or groups can be done similarly. In this case, it suffices to prove that the kernel and cokernel of a morphism $f : F \to G$ of sheaves of A-module are constructible. Since the question is local over X, we can suppose that X is affine, hence quasi-compact and quasi-separated. Then by Proposition 1.10.10 (a), we are reduced to the case where F and G are locally constant, and we are done by using Lemma 1.10.7 (b) and the Noetherian hypothesis on A. Similarly, we can prove assertion (b) by using Lemma 1.10.7 (c). Finally, (c) follows from (a) and (b). $\square$

Proposition 1.10.13. *Let X be a quasi-compact and quasi-separated scheme and F be a sheaf of sets (resp. A-modules). For F to be constructible, it is necessary and sufficient that it is isomorphic to the cokernel of morphisms $H \rightrightarrows G$, where H and G are sheaves of sets represented by étale schemes of finite presentation over X (resp. that F is isomorphic to the cokernel of a homomorphism $A_V \to A_U$, where U, V are étale schemes of finite presentation over X).*

Proof. This condition is sufficient in view of Proposition 1.10.12 (a), so suppose that F is a constructible sheaf of sets. Using the existence of infinite direct sums in the étale site $X_{\text{ét}}$, we see that we can find an epimorphism $\widetilde{G} \to F$, where G is an étale scheme over X, which we can assume to be a sum of affine schemes G_i ($i \in I$), hence separated over X. For any finite subset J of I, let G_J be the sum of G_i for $i \in J$, and F_J be its image in F. As G_J and F are constructible, so is F_J (Proposition 1.10.8 and Proposition 1.10.12 (a)), so the set X_J of $x \in X$ such that the fibers of F and F_J at a geometric point of X over x is equal, is locally constructible (Proposition 1.10.10 (d)). As the family X_J is increasing with union X, and X is quasi-compact, one of the X_J is equal to X ([**?**] 1.9.9). Therefore, by replacing G with G_J, we can assume that G is affine, hence separated over X, and quasi-compact over X since X is quasi-separated (**??**). Let $H = \widetilde{G} \times_F \widetilde{G}$, which is a subsheaf of the representable sheaf $\widetilde{G \times_X G}$, hence representable by Lemma 1.9.37. As H are constructible by Proposition 1.10.12 (a), it follows from the previous arguments that it is quasi-compact, hence of finite presentation over X. This prove the first assertion of the proposition, and the second one can be proved by the same method. □

Corollary 1.10.14. *Let X be a quasi-compact and quasi-separated scheme scheme and U be an étale scheme over X. For the étale sheaf $\widetilde{U}$ to be constructible, it is necessary and sufficient that U is of finite presentation over X.*

Corollary 1.10.15. *Let X be a quasi-compact and quasi-separated scheme. Then any sheaf of sets (resp. ind-p-groups, resp. A-modules) over X is the inductive limit of a filtered system of constructible sheaves of sets (resp. ind-p-groups, resp. A-modules).*

Corollary 1.10.16. *Let X be a quasi-compact and quasi-separated scheme and F be a constructible sheaf of sets (resp. ind-p-groups, resp. A-modules) over X. Then the functor $\mathrm{Hom}(F, -)$ commutes with inductive limits. In the case where F is a sheaf of A-modules and A is Noetherian, the functors $\mathrm{Ext}^i(X; F, -)$ also commute with inductive limits.*

Corollary 1.10.17. *Let $\mathcal{I}$ be a filtered category and $i \mapsto X_i$ be a diagram of affine morphisms of quasi-compact and quasi-separated schemes indexed by $\mathcal{I}$. For any scheme Y, let $\mathrm{Sh}_c(Y)$ be the category of constructible sheaves of sets (resp. ind-p-groups, resp. A-modules) over Y. Then we have an equivalence of categories*

$$\varinjlim_i \mathrm{Sh}_c(X_i) \xrightarrow{\sim} \mathrm{Sh}_c(X)$$

where $X = \varprojlim_i X_i$. In particular, any constructible sheaf of sets (resp. ind-p-groups, resp. A-modules) over X is isomorphic to the inverse image of a constrictible sheaf of sets (resp. ind-p-groups, resp. A-modules) over X_i.

Proposition 1.10.18. *Let $f : X \to Y$ be a surjective morphism which is locally of finite presentation and F be a sheaf of sets (resp. groups, resp. A-modules). Then F is constructible if and only if $f^*(F)$ is constructible.*

Proposition 1.10.19. *Let X be a Noetherian scheme and A be a Notherian ring.*

(a) *The category of sheaves of sets (resp. ind-p-group, resp. A-modules) over X is locally Noetherian, that is, possesses a set of generators formed by Noetherian objects.*

(b) *A sheaf F of sets (resp. ind-p-group, resp. A-modules) over X is constructible if and only if it is Noetherian. If F is a sheaf of A-modules, then F is constructible if and only if it is the quotient of a finite sum of sheaves of the form A_U, where $U \to X$ is étale and of finite type.*

(c) *The constructible subsheaves of a sheaf F of sets (resp. ind-p-group, resp. A-modules) form an inductive system, and F is its inductive limit.*

Proposition 1.10.20. *Let X be a scheme, F be a constructible sheaf of sets (resp. ind-p-group, resp. A-modules) (if F is a sheaf of sets, suppose further that F has finite fibers). Let $x \in X$ and $\bar{x}$ be a geometric point over x. For F to be locally constant in a neighborhood of x, it is necessary and sufficient that for any geometric point $\bar{x}'$ and any specialization $\bar{x}' \rightsquigarrow \bar{x}$, the specialization homomorphism $F_{\bar{x}} \to F_{\bar{x}'}$ is an isomorphism.*

Definition 1.10.21. A scheme X is called **arc connected** if for any couple (ξ, η) of geometric points of X, there exist geometric points $\xi = \xi_0, \dots, \xi_n, \eta_1, \dots, \eta_n = \eta$ and specializations $\xi_i \rightsquigarrow \eta_i$ and $\xi_{i-1} \rightsquigarrow \eta_i$. We say that X is locally arc connected (for the étale topology) if for any étale morphism $U \to X$ there exists an étale covering $\{U_i \to U\}$ of U such that U_i is arc connected for each i.

We verity immediately that a locally Noetherian scheme is locally arc connected.

Proposition 1.10.22. *Let X be a scheme and F be a sheaf of sets (resp. groups, resp. A-modules) over X.*

(a) *Suppose that X is locally arc connected and the fibers of F are finite (resp. finite, resp. finitely presented). Then F is locally constant if and only if for any specialization $\xi \to \eta$ of geometric points, the specialization homomorphism $F_\eta \to F_\xi$ is bijective.*

(b) *Suppose that X is locally Noetherian and the fibers of F are finite (resp. finite, resp. finitely presented). Then, if for any specialization $\xi \to \eta$, the specialization homomorphism $F_\eta \to F_\xi$ is injective, then F is constructible.*

(c) *Suppose that X is locally Noetherian and F is a sheaf of sets with finite fibers. Let $d : X \to \mathbb{N}$ be the function $d(x) = |F_{\bar{x}}|$. Then F is constructible if and only if d is a constructible function, i.e. if and only if $d^{-1}(n)$ is constructible for any $n \in \mathbb{N}$.*

Proposition 1.10.23. *Let X be a scheme.*

(a) *For any finite and finitely presented morphism $f : X' \to X$ and any constructible sheaf F of sets (resp. ind-p-groups, resp. A-modules) over X', $f_*(F)$ is constructible.*

(b) *Suppose that X is quasi-compact and quasi-separated, and let F be a sheaf of sets (resp. ind-p-groups, resp. A-modules) over X. Then there exists a finite family $\{p_i : U_i \to X\}$ of finite morphisms and for each i, a constant sheaf F_i of sets (resp. ind-p-groups, resp. A-modules) over U_i, and a monomorphism*
$$F \hookrightarrow \prod_i (p_i)_*(C_i).$$
Moreover, if X is Noetherian, we can suppose that each U_i is integral.

1.10.3 The theories of Kummer and Artin-Schreier

We denote by $\mathbb{G}_{m,X}$ the sheaf represented by the multiplicative group $\operatorname{Spec}(\mathscr{O}_X[t, t^{-1}])$ over X. For any étale morphism $U \to X$, the section of $\mathbb{G}_{m,X}$ over U is then the invertible elements $\Gamma(U, \mathscr{O}_U^\times)$ of $\Gamma(U, \mathscr{O}_U)$. Let $n \in \mathbb{N}$, the kernel of the n-th power of $\mathbb{G}_{m,X}$ is called the "sheaf of n-th unit roots", and denoted by $\boldsymbol{\mu}_{n,X}$. We then have an exact sequence

$$0 \longrightarrow \boldsymbol{\mu}_{n,X} \longrightarrow \mathbb{G}_{m,X} \xrightarrow{n} \mathbb{G}_{m,X} \tag{1.10.2}$$

If $n \in \mathbb{N}$ is invertible over X (that is, if n is coprime to the characteristic of each residue field of X), then taking n-th power is a surjective morphism of sheaves:

Theorem 1.10.24 (Kummer). *If n is invertible over X, then we have an exact sequence*

$$0 \longrightarrow \boldsymbol{\mu}_{n,X} \longrightarrow \mathbb{G}_{m,X} \xrightarrow{n} \mathbb{G}_{m,X} \longrightarrow 0 \tag{1.10.3}$$

Proof. Let $u \in \mathbb{G}_{m,X}(U)$, where $U \to X$ is an étale morphism. Since n is invertible over U, the equation $T^n - u = 0$ is separable over U, i.e.

$$U' = \operatorname{Spec}(\mathscr{O}_U[T]/(T^n - u))$$

is étale over Y. As $U' \to U$ is surjective, it is covering and the restriction of u to U' therefore admits an n-th root, whence the assertion. □

Proposition 1.10.25. *Let X be a scheme and n be a positive integer invertible over X. Then the sheaf $\boldsymbol{\mu}_{n,X}$ is locally constant. If X is a scheme over a strictly local ring A such that n is invertible in A, then $\boldsymbol{\mu}_{n,X}$ is isomorphic to the constant sheaf $\mathbb{Z}/n\mathbb{Z}$.*

Proof. We note that for any étale scheme U over X, we have a bijection

$$\operatorname{Hom}_X(U, \operatorname{Spec}(\mathscr{O}_X[T]/(T^n - 1))) \xrightarrow{\sim} \{s \in \Gamma(U, \mathscr{O}_U^\times) : s^n = 1\},$$

so $\boldsymbol{\mu}_{n,X}$ is represented by the X-scheme $\operatorname{Spec}(\mathscr{O}_X[T]/(T^n - 1))$, which is finite and étale over X since n is invertible over X. By Proposition 1.10.8, we conclude that $\boldsymbol{\mu}_{n,X}$ is locally constant. If A is strictly Henselian and n is invertible in A, then $T^n - 1$ splits into a product of linear polynomials in $A[T]$ by (LeiFu 2.8.3 (v)). For any A-scheme X, $\operatorname{Spec}(\mathscr{O}_X[T]/(T^n - 1))$ is then a trivial étale covering of X of degree n. Hence $\boldsymbol{\mu}_{n,X}$ is isomorphic to the constant sheaf $\mathbb{Z}/n\mathbb{Z}$. □

In view of Proposition 1.10.25, we conclude that for any geometric point ξ of X, the fiber $(\boldsymbol{\mu}_{n,X})_\xi$ is isomorphic to $\mathbb{Z}/n\mathbb{Z}$, so the sheaf $\boldsymbol{\mu}_{n,X}$ is locally isomorphic to $(\mathbb{Z}/n\mathbb{Z})_X$. Kummer's theory thus gives information about the cohomology of X with values in certain constant sheaves. In fact, we have by definition

$$H^0(X, \mathbb{G}_{m,X}) = \Gamma(X, \mathscr{O}_{X_{\text{ét}}}^\times)$$

and for dimension 1, we have the following:

Theorem 1.10.26 (Hilbert's Theorem 90). *We have an isomorphism*

$$H^1(X, \mathbb{G}_{m,X}) \xrightarrow{\sim} \operatorname{Pic}(X)$$

where $\operatorname{Pic}(X)$ *is the group of invertible (Zariski) sheaves over X.*

Proof. Since as a Zariski sheaf $\mathbb{G}_{m,X}$ is isomorphic to $\mathscr{O}_X^\times$, it suffices to prove that the canonical homomorphism

$$H^1(X_{\text{Zar}}, \mathbb{G}_{m,X}) \to H^1(X_{\text{ét}}, \mathbb{G}_{m,X})$$

is bijective, which is equivalent to saying that $R^1 i_*(\mathbb{G}_{m,X}) = 0$, where $i : X_{\text{ét}} \to X_{\text{Zar}}$ is the canonical homomorphism. We are therefore reduced to the case where X is affine (Proposition 1.6.31). As we can calculte H^1 by Čech cohomology (Corollary 1.6.19) and as it suffices to take quasi-compact coverings of a quasi-compact X, this is an immediate consequence of the descent theory for sheaves (cf. FGA 190, or SGA VIII 1). □

Example 1.10.27. Let X be a scheme and consider the sheaf $\mathscr{O}_{X_{\text{ét}}}^\times$. By **??**, the étale cohomology group $H^1(X_{\text{ét}}, \mathscr{O}_{X_{\text{ét}}}^\times)$ is isomorphic to $H^1(X, \mathscr{O}_X^\times) = \operatorname{Pic}(X)$. On the other hand, if $X = \operatorname{Spec}(k)$ is the spectrum of a field and $G = \operatorname{Gal}(\bar{k}/k)$, then by Corollary 1.9.6 we have

$$H^1(X_{\text{ét}}, \mathscr{O}_{X_{\text{ét}}}^\times) = H^1(G, M)$$

where $M = \mathscr{O}_{X_{\text{ét}}}^\times(X) = k^\times$. Since $\operatorname{Pic}(X) = 0$ in this case, we conclude that $H^1(G, k^\times) = 0$, which is the famous theorem Hilbert's 90.

Now suppose that X is a Noetherian scheme without embedded components, and let x_j be the generic points of irreducible components of X. Let R_j be the local ring of X at x_j (which is Artinian) and $i_j : \mathrm{Spec}(R_j) \to X$ be the inclusion morphism. We have a canonical injection

$$\mathbb{G}_{m,X} \to \prod_j (i_j)_*(\mathbb{G}_{m,\mathrm{Spec}(R_j)})$$

of sheaves over $X_{\text{ét}}$ (resp. X_{Zar}). The cokernel of this morphism is called the sheaves of Cartier divisors over $X_{\text{ét}}$ (resp. X_{Zar}).

Corollary 1.10.28. *Let X be Noetherian without embedded components, $i : X_{\text{ét}} \to X_{\mathrm{Zar}}$ be the canonical morphism of sites, and D be the sheaf of Cartier divisors over $X_{\text{ét}}$. Then $i_*(D)$ is the sheaf D_{Zar} of Cartier divisors over X_{Zar}, and we have*

$$H^0(X_{\text{ét}}, D) = H^0(X_{\mathrm{Zar}}, D_{\mathrm{Zar}}).$$

Proof. If we apply i_* to the exact sequence

$$0 \longrightarrow \mathbb{G}_{m,X} \longrightarrow \textstyle\prod_j (i_j)_*(\mathbb{G}_{m,\mathrm{Spec}(R_j)}) \longrightarrow D \longrightarrow 0$$

we then obtain an exact sequence since $R^1 i_*(\mathbb{G}_{m,X}) = 0$, whence the assertion. □

In the case where X admits a characteristic $p > 0$, we have the following replacement of Theorem 1.10.24 for the study of p-torsions:

Theorem 1.10.29 (Artin-Schreier). *If X is of characteristic $p > 0$, we have an exact sequence*

$$0 \longrightarrow (\mathbb{Z}/p\mathbb{Z})_X \longrightarrow \mathscr{O}_X \xrightarrow{\wp} \mathscr{O}_X \longrightarrow 0$$

where $\wp$ is the morphism of abelian groups defined by $\wp(s) = s^p - s$.

Proof. Let $U = \mathrm{Spec}(A)$ be an affine étale scheme over X and $s \in \Gamma(U, \mathscr{O}_{X_{\text{ét}}}) = A$ be a section. Then the canonical morphism

$$V = \mathrm{Spec}(A[T]/(T^p - T - a)) \to U = \mathrm{Spec}(A)$$

is étale and surjective, hence an étale covering of U. The restriction of a to V is the image of T under the homomorphism $\wp$, so $\wp : \mathscr{O}_{X_{\text{ét}}} \to \mathscr{O}_{X_{\text{ét}}}$ is surjective. Note that for any étale scheme U over X, we have a bijection

$$\mathrm{Hom}_X(U, \mathrm{Spec}(\mathscr{O}_X[T]/(T^p - T))) \xrightarrow{\sim} \{s \in \Gamma(U, \mathscr{O}_U) : s^p - s = 0\}$$

so $\ker \wp$ is represented by the X-scheme $\mathrm{Spec}(\mathscr{O}_X[T]/(T^p - T))$. Since we have $T^p - T = \prod_{i \in \mathbb{Z}/p\mathbb{Z}}(T - i)$ over X, it follows that $\ker \wp \cong \mathbb{Z}/p\mathbb{Z}$. □

Proposition 1.10.30. *Let X be a scheme and G be a torsion-free abelian group.*

(a) *Let $i : x \to X$ be the inclusion of a point. Then*

$$H^1(X, i_*(G_x)) = 0$$

where G_x is the constant sheaf with value G.

(b) *If X is irreducible and for any geometric point $\bar{x}$ of X the strict localization $X_{\bar{x}}$ at $\bar{x}$ is irreducible (i.e. X is "geometrically unibranch"), then we have*

$$H^1(X, G_X) = 0.$$

Proof. As for (a), the Leray spectral sequence

$$E_2^{p,q} = H^p(X, R^q i_*(G_x)) \Rightarrow H^{p+q}(x, G_x)$$

provides an injection $H^1(X, i_*(G_x)) \to H^1(x, G_x)$. Now since $x = \mathrm{Spec}(\kappa(x))$ is the spectrum of a field, we have $H^1(x, G_x) = 0$[9], whence the assertion in (a). To prove (b), let $i : x \to X$ be the inclusion of the genetric point of X. We see immediately from the hypothesis of (b) that the canonical morphism $G_X \to i_*(G_x)$ is bijective, so (b) follows from (a). □

1.10.4 Cohomology of algebraic curves

Let X be a Noetherian scheme of dimension 1. Let $K(X)$ be the ring of rational functions of X, and $i : \mathrm{Spec}(K(X)) \to X$ be the inclusion morphism. Let F be an abelian sheaf over $\mathrm{Spec}(K(X))$ and consider the sheaves $R^p i_*(F)$ over X, where $p > 0$. If η is the generic point of an irreducible component of X, $\bar{\eta}$ is the geometric point lying over η, and $X_{\bar{\eta}} = \mathrm{Spec}(\mathscr{O}_{X,\bar{\eta}})$ is the corresponding strict localization, then Theorem 1.9.16, we have

$$(R^p i_*(F))_{\bar{\eta}} = H^p(\mathrm{Spec}(K(X)) \times_X X_\eta, \bar{F}) = H^p(\mathrm{Spec}(K(X) \otimes_{\mathscr{O}_X} \mathscr{O}_{X,\bar{\eta}}), \bar{F})$$

where $\bar{F}$ is the inverse image of F over $\mathrm{Spec}(K(X) \otimes_{\mathscr{O}_X} \mathscr{O}_{X,\bar{\eta}})$. Since we have

$$K(X) \otimes_{\mathscr{O}_X} \mathscr{O}_{X,\bar{\eta}} \cong K(X_{\bar{\eta}}) \cong \mathscr{O}_{X,\bar{\eta}},$$

it follows from ([**?**] VIII 4.7) that $(R^p i_*(F))_{\bar{\eta}}$ is zero. Since X is of dimension 1, such a sheaf is very special in nature:

Lemma 1.10.31. *Let X be a Noetherian scheme and F be a sheaf over X. The following conditions are equivalent:*

(i) *The fiber of F at any geometric point $\bar{x}$ of X over a nonclosed point $x \in X$ is zero.*

(ii) *Every section $s \in F(X')$ (where X' is étale of finite type over X) is zero except at a finite number of closed points of X.*

(iii) *The canonical morphism $F \to \prod_{x \in X} (i_x)_*(i_x)^*(F)$ induces an isomorphism*

$$F \xrightarrow{\sim} \bigoplus_{x \in X_0} (i_x)_*(i_x)^*(F)$$

where X_0 is the set of closed points of X and $i_x : x \to X$ is the inclusion.

*A sheaf over X satisfying the above equivalent conditions is called a **skyscraper sheaf**.*

Proof. Suppose that (i) is satisfied, and let $z \in F(X')$, where X' is étale of finite type over X. Then by hypothesis, the support of z is a closed subset of X' formed by closed points, hence finite (X' being Noetherian), whence (i)⇒(ii). The implication (ii)⇒(i) follows trivially from calculation on fibers (Eq. (1.9.4)), and (iii)⇒(i) since for any family (G_x) of sheaves over closed point x of X, the sheaf $F = \bigoplus_i (i_x)_*(G_x)$ over X satisfies (i), as it follows from the fact that the fiber functor commutes with direct sums and that the support of $(i_x)_*(G_x)$ is equal to $\{x\}$ (cf. Proposition 1.9.17).

It remains to prove that (ii)⇒(iii), so consider the canonical morphism

$$F \to \prod_{x \in X_0} (i_x)_*(i_x)^*(F)$$

[9] We have $H^1(x, G_x) = \varinjlim H^1(\mathrm{Gal}(k'/k), G)$, where $k = \kappa(x)$ and k' runs through finite Galois extensions of k. Now $H^1(\mathrm{Gal}(k'/k), G) = \mathrm{Hom}(\mathrm{Gal}(k'/k), G) = 0$ since $\mathrm{Gal}(k'/k)$ is a finite group, hence torsion.

By (ii), the image of F is contained in $\bigoplus_x (i_x)_*(i_x)^*(F)$, so we obtain a morphism

$$F \to \bigoplus_{x \in X_0} (i_x)_*(i_x)^*(F).$$

This is an isomorphism since we have the formula

$$((i_x)_*(i_x)^*(F))_{\bar{y}} = \begin{cases} F_{\bar{x}} & y = x, \\ 0 & y \neq x \end{cases}$$

for any closed point x and any $y \in X$. □

If X is a Noetherian scheme of dimension 1, then the non-closed points of X are exactly maximal points of X, so applying Lemma 1.10.31 we see that for an abelian sheaf F over $\mathrm{Spec}(K(X))$, the higher direct images $R^p i_*(F)$ is a skyscraper sheaf on X and we have

$$R^p i_*(F) \xrightarrow{\sim} \bigoplus_{x \in X_0} (i_x)_*(i_x)^*(R^p i_*(F))$$

On the other hand, as we have already remarked, the fiber of $R^p i_*(F)$ at a geometric point $\bar{x}$ of X is

$$(R^p i_*(F))_{\bar{x}} \cong H^p(\mathrm{Spec}(\mathscr{O}_{X,\bar{x}}), \bar{F}).$$

Suppose now that the residue field of any closed point x of X is separably closed, by Corollary 1.9.7, we then have

$$\begin{aligned} H^p(X, R^q i_*(F)) &\cong H^p(X, \bigoplus_{x \in X_0} (i_x)_*(i_x)^*(R^p i_*(F))) \cong \bigoplus_{x \in X_0} H^p(X, (i_x)_*(i_x)^*(R^q i_*(F))) \\ &\cong \bigoplus_{x \in X_0} H^p(x, (i_x)^*(R^q i_*(F))) \end{aligned}$$

which is equal to $\bigoplus_{x \in X_0} H^q(\mathrm{Spec}(\mathscr{O}_{X,x}), F)$ if $p = 0$, and zero if $p > 0$.

Corollary 1.10.32. *Let X be a Noetherian scheme of dimension 1 such that for any closed point of x of X, the residue field $\kappa(x)$ is separably closed. Let F be an abelian sheaf over $\mathrm{Spec}(K(X))$, and consider the Leray spectra lsequence*

$$E_2^{p,q} = H^p(X, R^q i_*(F)) \Rightarrow H^{p+q}(\mathrm{Spec}(K(X)), F).$$

Then we have

$$E_2^{p,q} = \begin{cases} \bigoplus_x H^q(\mathrm{Spec}(\mathscr{O}_{X,x}), F) & p = 0, \\ 0 & p > 0. \end{cases}$$

where X_x is the strict localization of X at x. Therefore, we obtain a long exact sequence

$$0 \to H^1(X, i_*(F)) \to H^1(\mathrm{Spec}(K(X)), F) \to \textstyle\bigoplus_x H^q(\mathrm{Spec}(\mathscr{O}_{X,x}), F) \to H^2(X, i_*(F)) \to \cdots$$

We now suppose that the global ℓ-dimension of $\mathrm{Spec}(K(X))$, for any prime number ℓ, is less than 1 (that is, $H^q(\mathrm{Spec}(K(X)), F) = 0$ for $q > 1$ and any ℓ-torsion sheaf over $\mathrm{Spec}(K(X))$). Let $K_1, \dots, K_n$ be the residue fields of $K(X)$, i.e. the residue fields of the maximal points of X. In view of Corollary 1.9.6, we have

$$\mathrm{gl.dim}_\ell(\mathrm{Spec}(K(X))) = \sup_i \{\mathrm{gl.dim}_\ell(\mathrm{Gal}(\bar{K}_i/K_i))\}$$

where $\mathrm{gl.dim}_\ell(G)$ is the global ℓ-dimension of the profinite group G. The residue field K_x of $K(X_{\bar{x}})$, for the strict localization $X_{\bar{x}}$ of X, is clearly identified with a (infinite) separable extension of K_i, so we have

$$\mathrm{gl.dim}_\ell(\mathrm{Gal}(\bar{K}_x/K_x)) \leq \mathrm{gl.dim}_\ell(\mathrm{Gal}(\bar{K}_i/K_i))$$

by (CG, II 4.1, Prop 10). Therefore, the global ℓ-dimension of $K(X_{\bar{x}})$ is also less or equal than 1. By applying Corollary 1.10.32 to an ℓ-adic sheaf F, we then obtain

$$H^q(X, i_*(F)) = 0 \quad \text{for} \quad q > 2, \tag{1.10.4}$$

and an exact sequence

$$0 \to H^1(X, i_*(F)) \to H^1(\mathrm{Spec}(K(X)), F) \to \bigoplus_x H^1(\mathrm{Spec}(K(X_{\bar{x}})), F) \to H^2(X, i_*(F)) \to 0 \tag{1.10.5}$$

Theorem 1.10.33. *Let X be a Noetherian scheme of dimension 1 and n be an integer that is invertible over X. Suppose that $\mathrm{gl.dim}_\ell(K(X)) \leq 1$ for any prime divisor ℓ of n and that for any closed point x of X, the residue field $\kappa(x)$ is separably closed. Then we have $H^q(X, \boldsymbol{\mu}_{n,X}) = 0$ for $q > 2$ and an exact sequence*

$$\begin{tikzcd} 0 \arrow[r] & H^0(X,\boldsymbol{\mu}_{n,X}) \arrow[r] & \Gamma(X,\mathscr{O}_X^\times) \arrow[r, "n"] & \Gamma(X,\mathscr{O}_X^\times) \arrow[dll, out=-10, in=170] & \\ & H^1(X,\boldsymbol{\mu}_{n,X}) \arrow[r] & \mathrm{Pic}(X) \arrow[r, "n"] & \mathrm{Pic}(X) \arrow[r] & H^2(X,\boldsymbol{\mu}_{n,X}) \arrow[r] & 0 \end{tikzcd} \tag{1.10.6}$$

Proof. Suppose that ℓ is a prime divisor of n, then by applying Theorem 1.10.26 and [Serre, J.P. Cohomologie Galoisienne, chap I, prop.12], we conclude that $H^1(\mathrm{Spec}(K(X)), \mathbb{G}_m) = 0$, and $H^q(\mathrm{Spec}(K(X)), \mathbb{G}_m)$ has no ℓ-torsion for $q \geq 2$. By replacing $K(X)$ by $K(X_{\bar{x}})$, we also have a similar result for $H^q(\mathrm{Spec}(K(X_{\bar{x}})), i_*(\mathbb{G}_m))$. Now in view of Lemma 1.10.31, for any sheaf F over X, the kernel P and cokernel Q of the canonical homomorphism $F \mapsto i_*(i^*(F))$ (where $i : \mathrm{Spec}(K(X)) \to X$ is the inclusion) is a skyscraper sheaf, so we have (since the residue fields of the closed points of X are supposed to be separably closed)

$$H^i(X, P) = H^i(X, Q) = 0 \quad \text{for} \quad i > 0.$$

This implies, in view of the long exact sequence on cohomology, that

$$H^i(X, F) \to H^i(X, i_*(i^*(F)))$$

is an isomorphism for $i \geq 2$, so $H^i(X, \mathbb{G}_m)$ has no ℓ-torsion for $i \geq 2$. Since $H^q(X, \boldsymbol{\mu}_{n,X})$ is n-torsion for any $q \geq 2$, we conclude the exact sequence (1.10.6), and $H^q(X, \boldsymbol{\mu}_{n,X}) = 0$ for $q \geq 3$[10]. □

Corollary 1.10.34. *Let X be a complete and irreducible algebraic curve over a separably closed field k, with characteristic $p > 0$. If n is coprime to p, then we have*

$$\begin{aligned} H^0(X, \boldsymbol{\mu}_{n,X}) &\cong \boldsymbol{\mu}_n(k), \\ H^1(X, \boldsymbol{\mu}_{n,X}) &\cong \mathrm{Pic}(X)_n = \{\mathscr{L} \in \mathrm{Pic}(X) : \mathscr{L}^{\otimes n} \cong \mathscr{O}_X\}, \\ H^2(X, \boldsymbol{\mu}_{n,X}) &\cong \mathrm{Pic}(X)/n\mathrm{Pic}(X) \cong \mathbb{Z}/n\mathbb{Z}, \end{aligned}$$

and $H^q(X, \boldsymbol{\mu}_{n,X}) = 0$ for $q \geq 3$.

[10]For this, one can consider the snake diagram induced by multiplication by n, and use skane lemma.

Proof. We have $\Gamma(X, \mathscr{O}_X^\times) = k^\times$, and the homomorphism $x \mapsto x^n$ is surjective, so it follows from the exact sequence (1.10.6) that

$$\begin{aligned} &H^0(X, \boldsymbol{\mu}_{n,X}) \cong \boldsymbol{\mu}_n(k), \\ &H^1(X, \boldsymbol{\mu}_{n,X}) \cong \mathrm{Pic}(X)_n = \{\mathscr{L} \in \mathrm{Pic}(X) : \mathscr{L}^{\otimes n} \cong \mathscr{O}_X\}, \\ &H^2(X, \boldsymbol{\mu}_{n,X}) \cong \mathrm{Pic}(X)/n\mathrm{Pic}(X). \end{aligned}$$

To see that $\mathrm{Pic}(X)/n\mathrm{Pic}(X) \cong \mathbb{Z}/n\mathbb{Z}$, let $\deg : \mathrm{Pic}(X) \to \mathbb{Z}$ be the degree homomorphism. Then the map deg is surjective, so let $\mathrm{Pic}^0(X)$ be its kernel. We have a commutative diagram of short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Pic}^0(X) \arrow[r] \arrow[d, "n"] & \mathrm{Pic}(X) \arrow[r, "\deg"] \arrow[d, "n"] & \mathbb{Z} \arrow[r] \arrow[d, "n"] & 0 \\ 0 \arrow[r] & \mathrm{Pic}^0(X) \arrow[r] & \mathrm{Pic}(X) \arrow[r, "\deg"] & \mathbb{Z} \arrow[r] & 0 \end{tikzcd}$$

By the snake lemma, we obtain an exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Pic}^0(X)_n \arrow[r] & \mathrm{Pic}(X)_n \arrow[r] & 0 \arrow[dll, rounded corners] \\ & \mathrm{Pic}^0(X)/n\mathrm{Pic}^0(X) \arrow[r] & \mathrm{Pic}(X)/n\mathrm{Pic}(X) \arrow[r] & \mathbb{Z}/n\mathbb{Z} \arrow[r] & 0 \end{tikzcd}$$

Now $\mathrm{Pic}^0(X)$ is isomorphic to the group of k-points $\mathrm{Jac}_X(k)$ of the Jacobian Jac_X of X, and the homomorphism $n : \mathrm{Jac}_X(k) \to \mathrm{Jac}_X(k)$ is surjective with kernel isomorphic to $(\mathbb{Z}/n\mathbb{Z})^{2g}$, where g is the genus of X, so we have

$$\begin{gathered} H^1(X, \boldsymbol{\mu}_{n,X}) \cong \mathrm{Pic}(X)_n \cong \mathrm{Pic}^0(X)_n \cong \mathrm{Jac}_X(k)_n \cong (\mathbb{Z}/n\mathbb{Z})^{2g}, \\ H^2(X, \boldsymbol{\mu}_{n,X}) \cong \mathrm{Pic}(X)/n\mathrm{Pic}(X) \cong \mathbb{Z}/n\mathbb{Z}. \end{gathered}$$

By Theorem 1.10.33, we have $H^q(X, \boldsymbol{\mu}_{n,X}) = 0$ for $q \geq 3$, so the claim follows. $\square$

1.10.5 Méthode de la trace

Let X be a scheme and $f : X' \to X$ be a étale morphism. Then we have an adjoint triple $(f_!, f^*, f_*)$ (the functor $f_!$ will be defined later). The adjunction map $\mathrm{id} \to f_*f^*$ is called the **restriction map**, and the adjunction map $f_!f^* \to \mathrm{id}$ is called the **trace map**. If f is finite, then the functor $f_!$ coincides with f_*, and in this case the trace morphism $f_*f^* \to \mathrm{id}$ can be characterized by the following properties:

(i) It commutes with étale localization on X.

(ii) If X' is the direct sum of d copies of X, so that $f_*f^*(F) \cong F^{\oplus d}$, then tr is the sum morphism $F^{\oplus d} \to F$.

Since any étale covering is locally given by a direct sum of X (Proposition 1.10.8), the uniquenss of such morphism is clear. The existence is ensured by the usual argument of descent.

From condition (ii), we see that for any sheaf F over X, the composition

$$F \xrightarrow{\text{res}} f_*f^*(F) \xrightarrow{\text{tr}} F$$

is the multiplication by the local degree of f. If this degree is a constant d, then this is given by the multiplication by d. The "méthode de la trace" is the following observation: if F is an abelian sheaf on X such that multiplication by d is an isomorphism on F, then the map

$$\mathrm{res}: H^q(X,F) \to H^q(X, f_*f^*(F)) = H^q(X', f^*(F))$$

is injective. In fact, we have $H^q(X, f_*f^*(F)) = H^q(X', f^*(F))$ by the vanishing of the higher direct images (Proposition 1.9.17) and the Leray spectral sequence (Proposition 1.6.33). In particular, if $H^q(Y, f^*(F)) = 0$, then $H^q(X,F) = 0$ as well.

As an example of this observation, we have the following corollary.

Corollary 1.10.35. *Let $f : X' \to X$ be an étale covering of constant degree d. If F is a p-torsion sheaf over X with $\gcd(p,d) = 1$, then $H^q(X', f^*(F)) = 0$ implies $H^q(X,F) = 0$.*

Proof. In fact, in this case multiplication by d is an isomorphism on F. □

Now let $i : U \to X$ be an immersion and $f : U' \to U$ be a finite étale morphism of degree d, which is assumed to be induced from a finite morphism $g : X' \to X$ by base change, i.e. we have a Cartesian diagram

$$\begin{tikzcd} U' \arrow[r, "i'"] \arrow[d, "f"'] & X' \arrow[d, "g"] \\ U \arrow[r, "i"] & X \end{tikzcd} \tag{1.10.7}$$

For an abelian sheaf F over U, we have

$$\begin{tikzcd} F \arrow[r, "\mathrm{res}"] \arrow[rr, bend right, "\cdot d"'] & f_*f^*(F) \arrow[r, "\mathrm{tr}"] & F \end{tikzcd}$$

Since $i_*f_* = g_*i'_*$ and $i_!f_* \cong g_*i'_!$, we have

$$\begin{tikzcd} i_*(F) \arrow[r, "\mathrm{res}"] \arrow[rr, bend right, "\cdot d"'] & g_*i'_*f^*(F) \arrow[r, "\mathrm{tr}"] & i_*(F) \end{tikzcd} \qquad \begin{tikzcd} i_!(F) \arrow[r, "\mathrm{res}"] \arrow[rr, bend right, "\cdot d"'] & g_*i'_!f^*(F) \arrow[r, "\mathrm{tr}"] & i_!(F) \end{tikzcd} \tag{1.10.8}$$

Applying this to the cohomology group, we then obtain

$$\begin{tikzcd} H^q(X, i_*(F)) \arrow[r, "\mathrm{res}"] \arrow[rr, bend right, "\cdot d"'] & H^q(X', i'_*f^*(F)) \arrow[r, "\mathrm{tr}"] & H^q(X, i_*(F)) \end{tikzcd} \tag{1.10.9}$$

$$\begin{tikzcd} H^q(X, i_!(F)) \arrow[r, "\mathrm{res}"] \arrow[rr, bend right, "\cdot d"'] & H^q(X', i'_!f^*(F)) \arrow[r, "\mathrm{tr}"] & H^q(X, i_!(F)) \end{tikzcd} \tag{1.10.10}$$

Proposition 1.10.36. *Let X be a quasi-compact and quasi-separated scheme, ℓ be a prime number, and H be a semi-exact functor on the category of ℓ-torsion sheaves over X with values in* **Ab**, *which is compatible with filtrant inductive limits. Then the following conditions are equivalent:*

(i) $H = 0$.

(ii) $H(f_*i_!(\mathbb{Z}/\ell\mathbb{Z})) = 0$ *for any finite morphism $f : Y \to X$ and any immersion $i : U \to Y$ such that U is of finite presentation over X. If X is Noetherian, we can restrict to the case where Y is integral.*

Proof. By Corollary 1.10.15 and Proposition 1.10.11, it suffices to verify that $H(F) = 0$ for $F = i_!(G)$, where $i : U \to X$ is the inclusion of a finitely presented subscheme of X and G is a locally constant ℓ-torsion sheaf over U. Let $U'' \to U$ be a principal étale covering with group g such that G is constant over U'', and let A be the ℓ-group such that $G|_{U''} \cong A_{U''}$. Note that the group g acts on A, and this action determines the structure of G. Let h be a Sylow ℓ-subgroup of g, and $U' = U''/h$ be the principal covering corresponding to h, so that the degree d of U' over U is coprime to ℓ. By Zariski's Main Theorem ([**?**], 8.12.6), we may choose a diagram

$$\begin{tikzcd} U' \arrow[r, "i'"] \arrow[d, "f"'] & X' \arrow[d, "g"] \\ U \arrow[r, "i"] & X \end{tikzcd}$$

where $g : X' \to X$ be a finite morphism and $i' : U' \to X'$ is an open immersion. Moreover, since the composition $U' \to X$ is finitely presented, we can take the scheme-theoretic closure of i' in X', and thus assume that i' has dense image. Since f is finite and U' is dense in X', we conclude that the induced morphism $U' \to U \times_X X'$ is a finite open immersion with dense image, so it is an isomorphism and the above diagram is Cartesian. From the diagram (1.10.8) and the fact that $\gcd(d, \ell) = 1$, we are reduced to showing that $H(g_* i'_! f^*(G)) = 0$.

Now it is well known that the only abelian group of ℓ-torsion which is simple under an operation of an ℓ-group h is $\mathbb{Z}/\ell\mathbb{Z}$ with the trivial operation, so it follows that there is a filtration of A as an h-module, whose successive quotients are isomorphic to $\mathbb{Z}/\ell\mathbb{Z}$ with trivial operation. This filtration gives a corresponding filtration of $f^*(G)$ by descent, and it is therefore sufficient to prove that $H(g_* i'_!(\mathbb{Z}/\ell\mathbb{Z})) = 0$; this proves the claim. □

Proposition 1.10.37. *Let X be a Noetherian scheme, ℓ be a prime number, and φ be a function defined on the set of finite integral schemes over X with values in $\mathbb{N}$. Suppose that $\varphi(Y) \leq \varphi(Y')$ if there exists an X-morphism $Y \to Y'$, and the inequality is strict if Y is a closed proper subscheme of Y'. For any X-scheme Y finite over X, we define $\varphi(Y) = \sup_i \varphi(Y_i)$, where Y_i are the irreducible components of Y, endowed with the reduced scheme structure. Then the following assertions are equivalent:*

(i) *For any scheme Y finite over X and any ℓ-torsion sheaf F over Y, we have $H^q(Y, F) = 0$ for $q > \varphi(Y)$.*

(ii) *For any scheme Y which is finite and integral over X, we have $H^q(Y, \mathbb{Z}/\ell\mathbb{Z}) = 0$ for $q > \varphi(Y)$.*

Proof. It suffices to prove that (ii)⇒(i). If (ii) is satisfied, we can apply Proposition 1.10.36 to the functor H^q, since it commutes with inductive limits and direct images under finite morphisms. We then reduce the verification of (i) to the case where Y is finite and integral over X and $F = i_!((\mathbb{Z}/\ell\mathbb{Z})_U)$, where $i : U \to Y$ is an immersion. Now we have an exact sequence

$$0 \longrightarrow i_!((\mathbb{Z}/\ell\mathbb{Z})_U) \longrightarrow (\mathbb{Z}/\ell\mathbb{Z})_Y \longrightarrow (\mathbb{Z}/\ell\mathbb{Z})_Z \longrightarrow 0$$

where $Z = Y \setminus U$. By induction on $\varphi(Y)$ and the assumption on the function φ, we may suppose that $H^q(Z, \mathbb{Z}/\ell\mathbb{Z}) = 0$ for $q > \varphi(Y) - 1$, whence $H^q(i_!(\mathbb{Z}/\ell\mathbb{Z})) = 0$ for $q > \varphi(Y)$ in view of the long exact sequence on cohomology. □

Corollary 1.10.38. *Let X be an algebraic curve over a separably closed field k with characteristic $p > 0$ and ℓ be a prime number which is coprime to p. Then for any ℓ-torsion sheaf F over X, we have $H^q(X, F) = 0$ for $p > 2$. If X is affine, then $H^q(X, F) = 0$ for $q > 1$.*

Proof. For Y finite and integral over X, we define $\varphi(Y) = 2\dim(Y)$ (resp. $\varphi(Y) = \dim(Y)$ if X, and hence Y, is affine). By Proposition 1.10.37, it suffices to show that $H^q(Y, \mathbb{Z}/\ell\mathbb{Z}) = 0$ for

$q > \varphi(Y)$. This is trivial for $\dim(Y) = 0$, so it suffices to show that $H^q(Y, \mathbb{Z}/\ell\mathbb{Z}) = 0$ if $q > 2$ for an integral curve Y. This follows from Theorem 1.10.33, since $\boldsymbol{\mu}_{\ell,Y} \cong (\mathbb{Z}/\ell\mathbb{Z})_Y$.

Suppose that X is affine. Since purely inseparable extensions do not affect the cohomology (Example 1.9.3 (b)), we may assume that k is algebraically closed. Let $f : X' \to X$ be the normalization of X, which is a finite morphism and an isomorphism except for a set of finitely many points of X, denoted by S. We have an exact sequence of sheaves over X:

$$0 \longrightarrow (\mathbb{Z}/\ell\mathbb{Z})_X \longrightarrow f_*((\mathbb{Z}/\ell\mathbb{Z})_{X'}) \longrightarrow E \longrightarrow 0$$

where E is concentrated over S, whence $H^q(X, E) = 0$ for $q > 0$. By Proposition 1.9.17, we are reduced to prove that

$$H^q(X, f_*((\mathbb{Z}/\ell\mathbb{Z})_{X'})) \cong H^q(X', \mathbb{Z}/\ell\mathbb{Z}) = 0 \quad \text{for} \quad q > 1$$

which means we can assume that X is normal from the begining. We may also suppose that X is connected. Now by Theorem 1.10.33, we have an exact sequence

$$\Gamma(X, \mathscr{O}_X^\times) \xrightarrow{\ell} \Gamma(X, \mathscr{O}_X^\times) \to H^1(X, \boldsymbol{\mu}_\ell) \to \mathrm{Pic}(X) \xrightarrow{\ell} \mathrm{Pic}(X) \to H^2(X, \boldsymbol{\mu}_\ell) \to 0$$

We now embed X into a complete smooth connected curve $\bar{X}$, so that can identify the elements of $\Gamma(X, \mathscr{O}_X^\times)$ with the rational functions on $\bar{X}$ whose divisors are contained in the finite set $\bar{X} - X$. Now we observe that the map $\ell : \mathrm{Pic}(X) \to \mathrm{Pic}(X)$ in the affine case is surjective, since every divisor on X can be extended to a divisor of degree 0 on $\bar{X}$ (as $\bar{X} - X \neq \varnothing$) and we have seen in the proof of Corollary 1.10.34 that the map $\ell : \mathrm{Pic}^0(\bar{X}) \to \mathrm{Pic}^0(\bar{X})$ is surjective. We therefore conclude that $H^2(X, \boldsymbol{\mu}_{\ell,X}) = 0$, whence the corollary. □

Remark 1.10.39. The reasoning of Proposition 1.10.36 in fact establishes the following result: Let X be a quasi-compact and quasi-separate scheme, ℓ a prime number, $\mathcal{C}$ a strictly full subcategory of the category of constructible ℓ-torsion abelian sheaves which is stable under direct factor and extensions. Suppose that for any finite morphism $f : Y \to X$ and any open immersion $i : U \to Y$ of Y that is finite presented over X, we have $f_*(i_!((\mathbb{Z}/\ell\mathbb{Z})_U)) \in \mathcal{C}$, then $\mathcal{C}$ contains any constructible ℓ-torsion sheaves.

Chapter 2

Group schemes

2.1 Algebraic structures

2.1.1 Algebraic structures on the category of presheaves

Given a kind of algebraic structure in the category of sets, we propose to extend it to the category $\mathcal{C}$. Let us first consider an example: the case of groups.

2.1.1.1 Group objects in $\widehat{\mathcal{C}}$ Let $G \in \widehat{\mathcal{C}}$, a **group structure on G** is defined to be the assignment of a group structure on the set $G(S)$ for any $S \in \mathrm{Ob}(\mathcal{C})$, so that for any morphism $f : S' \to S$ in $\mathcal{C}$, the map $G(f) : G(S) \to G(S')$ is a homomorphism of groups. If G and H are groups in $\widehat{\mathcal{C}}$, a **group homomorphism** from G to H is defined to be a morphism $\theta \in \mathrm{Hom}(G, H)$ such that for any object $S \in \mathrm{Ob}(\mathcal{C})$, the map $\theta(S) : G(S) \to H(S)$ is a homomorphism of groups. We denote by $\mathrm{Hom}_{\mathbf{Grp}}(G, H)$ the set of group homomorphisms from G to H, and by $\mathbf{Grp}_{\widehat{\mathcal{C}}}$ the category of groups in $\widehat{\mathcal{C}}$.

Example 2.1.1. Let $E \in \widehat{\mathcal{C}}$, then the object $\mathcal{A}ut(E)$ is endowed with a group structure. The final object e also possesses a unique group structure and is a final object in $\mathbf{Grp}_{\widehat{\mathcal{C}}}$.

Let G be a group in $\widehat{\mathcal{C}}$. For any $S \in \mathrm{Ob}(\mathcal{C})$, let $e_G(S)$ be the unit element in $G(S)$. The family $e_G(S)$ then defines an element $e_G \in \Gamma(G) = \mathrm{Hom}(e, G)$, which is a morphism of groups $e \to G$ and called the **unit section** of G. We also note that giving a group structure over G amounts to given a composition law over G, which is a morphism

$$\pi_G : G \times G \to G$$

such that for any $S \in \mathrm{Ob}(\mathcal{C})$, $\pi_G(S)$ is a group structure on $G(S)$. With the same manner, $f : G \to H$ is a group homomorphism is and only if the following diagram is commutative:

$$\begin{tikzcd} G \times G \arrow[r, "\pi_G"] \arrow[d, "(f,f)"'] & G \arrow[d, "f"] \\ H \times H \arrow[r, "\pi_H"] & H \end{tikzcd}$$

A sub-object H of G such that for any $S \in \mathrm{Ob}(\mathcal{C})$, $H(S)$ is a subgroup of $G(S)$ possessing evidently a group structure induced by that of G: that is, such that the monomorphism $H \to G$ is a morphism of groups. The group H endowed with this structure is called a **subgroup** of G.

If G and H are two groups in $\widehat{\mathcal{C}}$, the product $G \times H$ is endowed with a group structure such that for any $S \in \mathrm{Ob}(\mathcal{C})$, $G(S) \times H(S)$ is endowed with the product group structure. The group $G \times H$ endowed with this structure is called the product group of G and H (and this is also the product in the category $\mathbf{Grp}_{\widehat{\mathcal{C}}}$).

If G is a group in $\widehat{\mathcal{C}}$ then for any $S \in \mathrm{Ob}(\mathcal{C})$, G_S is also a group in $\widehat{\mathcal{C}_{/S}}$. If G and H are groups in $\widehat{\mathcal{C}}$, then we can define an object $\mathcal{H}om_{\mathbf{Grp}}(G,H)$ of $\widehat{\mathcal{C}}$ by

$$\mathcal{H}om_{\mathbf{Grp}}(G,H)(S) = \mathrm{Hom}_{\mathbf{Grp}}(G_S, H_S).$$

One should note that $\mathcal{H}om_{\mathbf{Grp}}(G,H)$ is in general not a group, nor a fortiori the object $\mathcal{H}om$ in the category $\mathbf{Grp}_{\widehat{\mathcal{C}}}$. We define similarly objects $\mathcal{I}so_{\mathbf{Grp}}(G,H)$, $\mathcal{E}nd_{\mathbf{Grp}}(G)$ and $\mathcal{A}ut_{\mathbf{Grp}}(G)$.

Definition 2.1.2. Let $G \in \mathrm{Ob}(\mathcal{C})$. A **group structure over *G*** is defined to be a group structure over $h_G \in \widehat{\mathcal{C}}$. If G and H are groups in $\mathcal{C}$, a group homomorphism from G to H is defined to be an element $f \in \mathrm{Hom}(G,H) \cong \mathrm{Hom}(h_G,h_H)$ which is a group homomorphism from h_G to h_H. We denote by $\mathbf{Grp}_{\mathcal{C}}$ the category of groups in $\mathcal{C}$. Note that there is a Cartesian square in **Cat**:

$$\begin{tikzcd}
\mathbf{Grp}_{\mathcal{C}} \arrow[r] \arrow[d] & \mathbf{Grp}_{\widehat{\mathcal{C}}} \arrow[d] \\
\mathcal{C} \arrow[r, "h"] & \widehat{\mathcal{C}}
\end{tikzcd}$$

The preceding definitions and constructions carries over to groups in $\mathcal{C}$, provided that the corresponding functors (products, $\mathcal{H}om$ objects, etc.) are representable in $\mathcal{C}$. They also applies to categories of the form $\mathcal{C}_{/S}$, and in this case, we denote by $\mathcal{H}om_{S\text{-}\mathbf{Grp}}$ for $\mathcal{H}om_{\mathbf{Grp}}$, etc.

More generally, if $\mathcal{T}$ is a kind of structure over n base sets defined by finite projective limits (for example, by the commutativity of some diagrams constructed from Cartesian products: monoid, group, action by group, module over a ring, Lie algebra over a ring, etc.), we can define the notion of $\mathcal{T}$ structure over n objects $F_1,\dots,F_n$ over $\widehat{\mathcal{C}}$: such a structure is the assignment of a $\mathcal{T}$ structure over the sets $F_1(S),\dots,F_n(S)$ for each $S \in \mathrm{Ob}(\mathcal{C})$, so that for any morphism $S' \to S$ in $\mathcal{C}$, the family of maps $(F_i(S) \to F_i(S'))$ is a poly-homomorphism for the $\mathcal{T}$ structure. We define in a similar way the morphisms of the $\mathcal{T}$ structure, whence a category of $\mathcal{T}$ objects in $\widehat{\mathcal{C}}$. The fully faithful functor h permits us to define the category of $\mathcal{T}$ objects in $\mathcal{C}$ as a fiber product in **Cat**.

Suppose now that in $\mathcal{C}$ the pullbacks exist, and let $\mathcal{T}$ be an algebraic structure defined by the data of certain morphisms between Cartesian products satisfying some axioms consisting of the commutativity of certain diagrams constructed by the previous arrows. A $\mathcal{T}$ structure on a family of objects of $\mathcal{C}$ will therefore be defined by certain morphisms between Cartesian products satisfying certain commutation conditions. It follows that if $\mathcal{C}$ and $\mathcal{C}'$ are two categories with products and $\lambda : \mathcal{C} \to \mathcal{C}'$ is a functor commuting with products, then for any family of objects (F_i) of $\mathcal{C}$ equipped with a $\mathcal{T}$ structure, the family $(f(F_i))$ of objects of $\mathcal{C}'$ will thereby be endowed with a $\mathcal{T}$ structure. For example, any group in $\mathcal{C}$ will be transformed into a group in $\mathcal{C}'$, any pair of a ring in $\mathcal{C}$ and a module over this ring will be transformed into an analogous pair in $\mathcal{C}'$, etc.

In particular, let $\mathcal{C}$ be a category, then the constant functor $E \mapsto E_S$ commutes with finite projective limits, and hence transforms groups into S-groups (i.e. groups in $\mathcal{C}_{/S}$), rings to S-rings, etc.

Remark 2.1.3. It is worth noting that the previous construction, applied to the category $\widehat{\mathcal{C}}$, restores the notions that have already been defined there. In others words, it amounts to the same thing to give oneself a $\mathcal{T}$ structure over an object of $\widehat{\mathcal{C}}$ when we consider this object as a functor on $\mathcal{C}$, or to give ourselves a $\mathcal{T}$ structure on the representable functor over $\mathcal{C}$ defined by this object. For example, let $G \in \widehat{\mathcal{C}}$; if the functor $F \mapsto \mathrm{Hom}_{\widehat{\mathcal{C}}}(F,G)$ is endowed with a group structure, then so is its restriction to $\mathcal{C}$. Conversely, if G is a group in $\widehat{\mathcal{C}}$, then the multiplication morphism $\pi_G : G \times G \to G$ induces for each $F \in \widehat{\mathcal{C}}$ a group structure over $\mathrm{Hom}_{\widehat{\mathcal{C}}}(F,G)$, which is functorial on F.

2.1.1.2 Group action in $\widehat{\mathcal{C}}$ Let $E \in \widehat{\mathcal{C}}$ and $G \in \mathbf{Grp}_{\widehat{\mathcal{C}}}$. A ***G*-object structure** over E is defined to be an assignment over $E(S)$, for each $S \in \mathrm{Ob}(\mathcal{C})$, a $G(S)$-set structure on $G(S)$, so that for any morphism $S' \to S$ in $\mathcal{C}$, the map $E(S) \to E(S')$ is compatible with the group homomorphism $G(S) \to G(S')$. As usual, this is equivalent to giving a morphism

$$\mu : G \times E \to E$$

which for each S endows $E(S)$ with a $G(S)$-set structure. On the other hand, since $\mathrm{Hom}(G \times E, E) \cong \mathrm{Hom}(G, \mathcal{E}nd(E))$, the morphism μ defines also a morphism $G \to \mathcal{E}nd(E)$ and it is immediate to see that this is a group homomorphism which sends G into $\mathcal{A}ut(E)$. Therefore, giving a G-object structure over E is equivalent to giving a group homomorphism

$$\rho : G \to \mathcal{A}ut(E).$$

In particular, any element $g \in G(S)$ defines an automorphism $\rho(g)$ of the functor E_S, that is, an automorphism of $E \times h_S$ which commutes with the projection $E \times h_S \to h_S$, and in particular an automorphism of $E(S')$ for any morphism $S' \to S$.

Definition 2.1.4. Let G be a group in $\widehat{\mathcal{C}}$ and E be a G-object. We denote by E^G the sub-object of E defined by

$$E^G(S) = \{x \in E(S) : x_{S'} \text{ is invariant under } G(S') \text{ for any morphism } S' \to S\}.$$

Here $x_{S'}$ is the image of x under $E(S) \to E(S')$. It is clear that E^G (called the **invariant sub-object** of E) is the largest sub-object of E on which G acts trivially. If F is a sub-object of E, we denote by $N_G(F)$ and $Z_G(F)$ the subgroups of G defined by

$$\begin{aligned} N_G(F)(S) &= \{g \in G(S) : \rho(g)F_S = F_S\} \\ &= \{g \in G(S) : \rho(S)F(S') = F(S') \text{ for any morphism } S' \to S\}, \\ Z_G(F)(S) &= \{g \in G(S) : \rho(g)|_{F_S} = \mathrm{id}\} \\ &= \{g \in G(S) : \rho(g)|_{F(S')} = \mathrm{id} \text{ for any morphism } S' \to S\}. \end{aligned}$$

In particular, let $x \in \Gamma(E)$, i.e. a collection of elements $x_S \in E(S)$, $S \in \mathrm{Ob}(\mathcal{C})$, such that for any morphism $f : S' \to S$, we have $E(f)(x_s) = x_{S'}$ (if $\mathcal{C}$ admits a final object S_0, then we have $\Gamma(E) = E(S_0)$). Then x can be considered as a sub-functor of E, also denoted by x, and we have $N_G(x) = Z_G(x)$. This common functor is also denoted by $\mathrm{Stab}_G(x)$ and called the **stablizer** of x. For any $S \in \mathrm{Ob}(\mathcal{C})$, we then have

$$\mathrm{Stab}_G(x)(S) = \{g \in G(S) : \rho(g)x_S = x_S\}.$$

Suppose that fiber products exist in $\mathcal{C}$. If $G = h_G$ (resp. $E = h_E$), where G is a group in $\mathcal{C}$ (resp. $E \in \mathrm{Ob}(\mathcal{C})$), and if $\mathcal{C}$ possesses a final object S_0, so that x is a morphism $S_0 \to E$, then the stablizer $\mathrm{Stab}_G(x)$ is represented by the fiber product $G \times_E S_0$, where $G \to E$ is the composition of $\mathrm{id}_G \times x : G = G \times S_0 \to G \times E$ and $\mu : G \times E \to E$.

Remark 2.1.5. The formation of E^G, $N_G(F)$ and $Z_G(F)$ commute with base changes, so for any $S \in \mathrm{Ob}(\mathcal{C})$, weh ave

$$(E^G)_S = (E_S)^{G_S}, \quad N_G(F)_S \cong N_{G_S}(F_S), \quad Z_G(F)_S \cong Z_{G_S}(F_S).$$

If G is a group in $\mathcal{C}$ and E is an object of $\widehat{\mathcal{C}}$ (resp. an object of $\mathcal{C}$), a G-object structure over E is defined to be an h_G-object structure over E (resp. h_E). With this definition, the above notations carries to $\mathcal{C}$, if the corresponding functors are representable. For example, if $N_{h_G}(h_F)$ is representable, then it is represented by a unique sub-object of G, which is then a subgroup of G and denoted by $N_G(F)$.

We say that the group G in $\widehat{\mathcal{C}}$ acts on a group H in $\widehat{\mathcal{C}}$ if H is endowed with a G-object structure such that, for any $g \in G(S)$, the automorphism of $H(S)$ defined by g is a group automorphism. This is the same to say that for any $g \in G(S)$, the automorphism $\rho(g)$ of H_S is an automorphism of groups in $\widehat{\mathcal{C}_{/S}}$, or that the morphism $G \to \mathcal{A}ut(H)$ sends G into $\mathcal{A}ut_{\mathbf{Grp}}(H)$.

In the above situation, there exists over $H \times G$ a unique group structure such that, for any $S \in \mathrm{Ob}(\mathcal{C})$, $(H \times G)(S)$ is the semi-direct product of the groups $H(S)$ and $G(S)$ relative to the given action of $G(S)$ on $H(S)$. This group is denoted by $H \rtimes G$ and called the semi-direct product of H by G. By definition, we then have

$$(H \rtimes G)(S) = H(S) \rtimes G(S).$$

Let G be a group in $\widehat{\mathcal{C}}$. For any morphism $S' \to S$ of $\mathcal{C}$ and any $g \in G(S)$, let $\mathrm{Inn}(g)$ be the automorphism of $G(S')$ defined by $\mathrm{Inn}(g)h = ghg^{-1}$. This definition extends to a morphism of groups in $\widehat{\mathcal{C}}$:

$$\mathrm{Inn} : G \to \mathcal{A}ut_{\mathbf{Grp}}(G) \subseteq \mathcal{A}ut(G).$$

The above definitions then apply to H and we have subgroups $N_G(E)$ and $Z_G(E)$ for any subobject E of G.

Definition 2.1.6. We define the **center** of G and denote by $Z(G)$ the subgroup $Z_G(G)$ of G. We say that G is **abelian** if $Z_G(G) = G$ or, equivalently, if $G(S)$ is abelian for any $S \in \mathrm{Ob}(\mathcal{C})$. A subgroup H of G is called **invariant** in G if $N_G(H) = G$, or equivalently, if $H(S)$ is invariant in $G(S)$ for any S. Moreover, we say that H is **cental** in G if $Z_G(H) = G$, or equivalently, if $H(S)$ is cental in $G(S)$ for any S.

Definition 2.1.7. Let $f : G \to G'$ be a group homomorphism. The kernel of f is the subgroup of G defined by

$$(\ker f)(S) = \{x \in G(S) : f(S)x = 1\} = \ker f(S)$$

for any $S \in \mathrm{Ob}(\mathcal{C})$. This is an invariant subgroup of G. Note that if G and G' belongs to $\mathcal{C}$, $\mathcal{C}$ possesses a final object S_0 and fiber products exist in $\mathcal{C}$, then $\ker(f)$ is represented by $S_0 \times_{G'} G$.

Definition 2.1.8. Let $E \in \widehat{\mathcal{C}}$ and G be a group acting on E. We say that the action of G on E is faithful if the kernel of the morphism $G \to \mathcal{A}ut(E)$ is trivial, that is, if for any $S \in \mathrm{Ob}(\mathcal{C})$ and $g \in G(S)$, the condition $g_{S'} \cdot x = x$ for any morphism $S' \to S$ and $x \in E(S')$ implies $g = 1$.

Many definitions and propositions of elementary group theory are easily transported to the setting of groups in $\widehat{\mathcal{C}}$. Let us simply point out the following which will be useful to us:

Proposition 2.1.9. *Let $f : W \to G$ be a group homomorphism and put $H(S) = \ker f(S)$ for $S \in \mathrm{Ob}(\mathcal{C})$. Let $u : G \to W$ be a group homomorphism which is a section of f. Then W is identified with a semi-direct product of H by G for the action of G over H defined by $(g, h) \mapsto \mathrm{Inn}(u(g))h$ for $g \in G(S)$, $h \in H(S)$ and $S \in \mathrm{Ob}(\mathcal{C})$.*

All the definitions and propositions are transported as usual to $\mathcal{C}$. We define in particular the semi-product of two groups H and G in $\mathcal{C}$, with G acting on H, when the Cartesian product $H \times G$ exists in $\mathcal{C}$. We have the following analogue of Proposition 2.1.9:

Proposition 2.1.10. *Let $f : W \to G$ and $i : H \to W$ be group homomorphisms in $\mathcal{C}$ such that for any $S \in \mathrm{Ob}(\mathcal{C})$, $(H(S), i(S))$ is a kernel of $f(S) : W(S) \to G(S)$. Let $u : G \to W$ be a homomorphism of groups in $\mathcal{C}$ which is a section of f. Then W is identified with the semi-direct product of H by G for the action of G over H such that if $S \in \mathrm{Ob}(\mathcal{C})$, $g \in G(S)$ and $h \in H(S)$, we have $\mathrm{Inn}(u(g))i(h) = i(ghg^{-1})$.*

To end this paragraph, we breifly introduce the concept of modules over a ring in $\widehat{\mathcal{C}}$. So let A and M be objects of $\widehat{\mathcal{C}}$, we say that F is a **module over the ring A**, of simply an A-module, if for each $S \in \mathrm{Ob}(\mathcal{C})$ the et $A(S)$ is endowed with a ring structure and $M(S)$ with a module structure over this ring, so that for any morphism $S' \to S$, the map $A(S) \to A(S')$ is a ring homomorphism and $M(S) \to M(S')$ is a bi-homomorphism. If the ring A is fixed, we define as usual morphisms of A-modules M, M', whence the abelian group $\mathrm{Hom}_A(M, M')$, and the category of A-modules, which we denote by $\mathbf{Mod}(A)$.

Proposition 2.1.11. *The category $\mathbf{Mod}(A)$ is endowed with an abelian category structure defined "argument by argument". Moreover, $\mathbf{Mod}(A)$ is an (AB5) category, that is, arbitrary direct sums exist in $\mathbf{Mod}(A)$ and if M is an A-module, N is a submodule, and $(M_i)_{i\in I}$ is a filtrant family of increasing submodules of M, then*

$$\bigcup_{i\in I}(M_i \cap N) = \Big(\bigcup_{i\in I} M_i\Big) \cap N.$$

Proof. In fact, let $f : M \to M'$ be a morphism of A-modules. We define the A-modules $\ker f$ (resp. $\mathrm{im}\, f$ and $\mathrm{coker}\, f$) so that for any $S \in \mathrm{Ob}(\mathcal{C})$, $(\ker f)(S) = \ker f(S)$ (resp. $\cdots$). Then $\ker f$ (resp. $\mathrm{coker}\, f$) is a kernel (resp. cokernel) of f, and we have an isomorphism of A-modules $M/\ker f \cong \mathrm{im}\, f$. This proves that $\mathbf{Mod}(A)$ is an abelian category.

Arbitrary direct sums exist in $\mathbf{Mod}(A)$ and are defined "argument by argument". Finally, if M is an A-module, N is a submodule, and $(M_i)_{i\in I}$ is a filtrant family of increasing submodules of M, then the inclusion

$$\bigcup_{i\in I}(M_i \cap N) \subseteq \Big(\bigcup_{i\in I} M_i\Big) \cap N$$

is an equality: in fact, if $S \in \mathrm{Ob}(\mathcal{C})$ and $x \in N(S) \cap \bigcup_i M_i(S)$, then there exists $i \in I$ such that $x \in N(S) \cap M_i(S)$. $\square$

Proposition 2.1.12. *If the category $\mathcal{C}$ is $\mathcal{U}$-small, then A is a generator for the category $\mathbf{Mod}(A)$. Concequently, $\mathbf{Mod}(A)$ is a Grothendieck category, hence possesses enough injectives.*

Proof. Let M be an A-module. For any $S \in \mathrm{Ob}(\mathcal{C})$, let $M_0(S)$ be a system of generators of the $A(S)$-module $M(S)$. Since, by hypothesis, $\mathcal{C}$ is small, we can consider the set $I = \coprod_{S\in\mathrm{Ob}(\mathcal{C})} M_0(S)$. We then have an epimorphism $A^{\oplus I} \to M$. This proves that A is a generator for $\mathbf{Mod}(A)$ (cf. [**?**] 1.9.1). As $\mathbf{Mod}(A)$ satisfies (AB5), it then follows from (cf. [**?**] 1.10.2) that $\mathbf{Mod}(A)$ has enough injectives. $\square$

Remark 2.1.13. If we consider $\mathbb{Z}$ as a constant functor on $\mathcal{C}$, then the category of $\mathbb{Z}$-modules is isomorphic to the category of abelian groups.

Definition 2.1.14. If M is an A-module, then for any $S \in \mathrm{Ob}(\mathcal{C})$, M_S is an A_S-module, so we can define an abelian group $\mathcal{H}om_A(M, N)$ by

$$\mathcal{H}om_A(M, N)(S) = \mathrm{Hom}_{A_S}(M_S, N_S).$$

We define similarly objects $\mathcal{I}so_A(M, N)$, $\mathcal{E}nd_A(M)$ and $\mathcal{A}ut_A(M)$, which are groups in $\widehat{\mathcal{C}}$ endowed with the structure of composition.

Definition 2.1.15. Let A be a ring in $\widehat{\mathcal{C}}$, M be an A-module and G be a group in $\widehat{\mathcal{C}}$. We denote by $A[G]$ the group algebra in $\widehat{\mathcal{C}}$ of G over A, so that for any $S \in \mathrm{Ob}(\mathcal{C})$, we have

$$(A[G])(S) = A(S)[G(S)].$$

An $\boldsymbol{A[G]}$**-module structure** on M is defined to be a G-object structure such that for any $S \in \mathrm{Ob}(\mathcal{C})$ and $g \in G(S)$, the automorphim of $F(S)$ defined by g is an automorphism of $A(S)$-module. Equivalently, this means the group homomorphism

$$\rho : G \to \mathcal{A}ut(M)$$

sends G to the subgroup $\mathcal{A}ut_A(M)$ of $\mathcal{A}ut(M)$. Therefore, given an $A[G]$-module structure on M, we have a group homomorphism

$$\rho : G \to \mathcal{A}ut_A(M).$$

We define similarly the abelian group $\mathrm{Hom}_{A[G]}(M, N)$ for $A[G]$-modules M, N, whence an additive category $\mathbf{Mod}(A[G])$.

The constructions above are immediately specialized in the case where G (or A, or both) are representable by objects of $\mathcal{C}$ which are thereby endowed with corresponding algebraic structures.

2.1.2 Algebraic structures on the category of schemes

We now apply the constructions of the previous paragraph to the category of schemes **Sch**, and more generally to categories $\mathbf{Sch}_{/S}$. We will simplify the notations in the following way: a group in **Sch** will also be called a **group scheme**, and a group scheme in $\mathbf{Sch}_{/S}$ will be called a **group scheme over** $\boldsymbol{S}$, or an $\boldsymbol{S}$**-group**, or A-group when S is the spectrum of a ring A.

2.1.2.1 Constant schemes The category of schemes admits direct sums and fiber products, while direct sums commute with base changes. We can then define the constant objects: for any set E, we have a scheme $E_{\mathbb{Z}}$ and for any scheme S, an S-scheme $E_S = (E_{\mathbb{Z}})_S$. Recall that for any S-scheme T, $\mathrm{Hom}_S(T, E_S)$ is the set of locally constant maps from the space T to E.

The functor $E \mapsto E_S$ commutes with finite projective limits. In particular, if G is a group, then G_S is a group scheme over S; if A is a ring, then A_S is a ring scheme over S, etc.

2.1.2.2 Affine S-groups Let T be an affine S-scheme, or an S-scheme that is affine over S. Then the $\mathscr{O}_S$-algebra $f_*(\mathscr{O}_T)$ (also denoted by $\mathscr{A}(T)$), where $f : T \to S$ is the structural morphism, is then quasi-coherent. Conversely, any quasi-coherent $\mathscr{O}_S$-algebra $\mathscr{A}$ corresponds to an affine S-scheme $\mathrm{Spec}(\mathscr{A})$, and the constructions $T \mapsto \mathscr{A}(T)$, $\mathscr{A} \mapsto \mathrm{Spec}(\mathscr{A})$ are quasi-inverses of each other. It follows that giving an algebraic structure on an affine S-scheme T is equivalent to giving the corresponding structure on $\mathscr{A}(T)$ in the opposite category to that of quasi-coherent $\mathscr{O}_S$-algebras. In particular, if G is an affine S-group over S, $\mathscr{A}(G)$ is endowed with an augmented $\mathscr{O}_S$-bialgebra structure, that is, we have the following homomorphisms of $\mathscr{O}_S$-algebras

$$\Delta : \mathscr{A}(G) \to \mathscr{A}(G) \otimes_{\mathscr{O}_S} \mathscr{A}(G), \quad \varepsilon : \mathscr{A}(G) \to \mathscr{O}_S, \quad \tau : \mathscr{A}(G) \to \mathscr{A}(G)$$

corresponding to the morphisms of S-schemes

$$\pi : G \times G \to G, \quad e_G : S \to G, \quad i : G \to G.$$

The maps Δ, ε and τ satisfy the following conditions (which express that G is an S-monoid):

(HA1) Δ is coassociative: the following diagram is commutative

$$\begin{tikzcd} \mathscr{A}(G) \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & \mathscr{A}(G) \otimes_{\mathscr{O}_S} \mathscr{A}(G) \arrow[d, "\mathrm{id} \otimes \Delta"] \\ \mathscr{A}(G) \otimes_{\mathscr{O}_S} \mathscr{A}(G) \arrow[r, "\Delta \otimes \mathrm{id}"] & \mathscr{A}(G) \otimes_{\mathscr{O}_S} \mathscr{A}(G) \otimes_{\mathscr{O}_S} \mathscr{A}(G) \end{tikzcd}$$

(HA2) Δ is compatible with ε: the following compositions are identities:

$$\mathscr{A}(G) \xrightarrow{\Delta} \mathscr{A}(G) \otimes_{\mathscr{O}_S} \mathscr{A}(G) \xrightarrow{\mathrm{id}\otimes\varepsilon} \mathscr{A}(G) \otimes_{\mathscr{O}_S} \mathscr{O}_S \xrightarrow{\sim} \mathscr{A}(G)$$

$$\mathscr{A}(G) \xrightarrow{\Delta} \mathscr{A}(G) \otimes_{\mathscr{O}_S} \mathscr{A}(G) \xrightarrow{\varepsilon\otimes\mathrm{id}} \mathscr{O}_S \otimes_{\mathscr{O}_S} \mathscr{A}(G) \xrightarrow{\sim} \mathscr{A}(G)$$

Also, in this case $(\mathscr{A}(G), \Delta, \varepsilon, \tau)$ is a Hopf algebra. Let us take advantage of the circumstance to notice once again that it follows from the definition of an S-group structure that in order to give such a structure on a S-scheme G affine over S, it is not necessary to verify anything on $\mathscr{A}(G)$, but simply endow each $G(S')$ for S' above S with a group structure functorial in S'. This remark applies mutatis mutandis to morphisms.

2.1.2.3 The groups $\mathbb{G}_a$ and $\mathbb{G}_m$ We consider the **additive group functor** $\mathbb{G}_a : \mathbf{Sch}^{\mathrm{op}} \to \mathbf{Set}$ defined by the formula

$$\mathbb{G}_a(S) = \Gamma(S, \mathscr{O}_S),$$

endowed with the group structure defined by the additive group structure of the ring $\Gamma(S, \mathscr{O}_S)$. This is represented by the affine scheme, which we denote also by $\mathbb{G}_a$, and which is then a group scheme

$$\mathbb{G}_a = \mathrm{Spec}(\mathbb{Z}[T]).$$

In fact, we have bijections

$$\mathrm{Hom}(S, \mathbb{G}_a) = \mathrm{Hom}_{\mathbf{Alg}}(\mathbb{Z}[T], \Gamma(S, \mathscr{O}_S)) \cong \Gamma(S, \mathscr{O}_S).$$

For any scheme S, we then have an affine S-group over S, which we denote by $\mathbb{G}_{a,S}$, and it corresponds to the $\mathscr{O}_S$-bigebra $\mathscr{O}_S[T]$ with the comultiplication and counit given by

$$\Delta(T) = T \otimes 1 + 1 \otimes T, \quad \varepsilon(T) = 0.$$

Let $\mathbb{G}_m : \mathbf{Sch}^{\mathrm{op}} \to \mathbf{Set}$ be the **multiplication group functor** defined by

$$\mathbb{G}_m(S) = \Gamma(S, \mathscr{O}_S)^\times,$$

where $\Gamma(S, \mathscr{O}_S)^\times$ denotes the multiplication group of invertible elements in the ring $\Gamma(S, \mathscr{O}_S)$, endowed with the canonical group structure. This is represented by an affine group, which is still denoted by $\mathbb{G}_m$:

$$\mathbb{G}_m = \mathrm{Spec}(\mathbb{Z}[T, T^{-1}]) = \mathrm{Spec}(\mathbb{Z}[\mathbb{Z}])$$

where $\mathbb{Z}[\mathbb{Z}]$ is the group algebra of the additive group $\mathbb{Z}$ over the ring $\mathbb{Z}$. In fact,

$$\mathrm{Hom}(S, \mathrm{Spec}(\mathbb{Z}[T, T^{-1}])) = \mathrm{Hom}_{\mathbf{Alg}}(\mathbb{Z}[T, T^{-1}], \Gamma(S, \mathscr{O}_S)) \cong \Gamma(S, \mathscr{O}_S)^\times.$$

For any scheme S, we then have an affine S-group $\mathbb{G}_{m,S}$ over S, which corresponds to the $\mathscr{O}_S$-bigebra $\mathscr{O}_S[\mathbb{Z}]$, with the comultiplication and counit given by

$$\Delta(x) = x \otimes x, \quad \varepsilon(x) = 1 \quad \text{for } x \in \mathbb{Z}.$$

We also note that the set $\Gamma(S, \mathscr{O}_S)$ is a ring for each scheme S, so we can endow the functor $\mathbb{G}_a$ with a natural ring structure, which we denote by $\mathbb{O}$. The ring $\mathbb{O}$ is represented by the scheme $\mathrm{Spec}(\mathbb{Z}[T])$, which is also denoted by $\mathbb{O}$, which is then a ring scheme in $\widehat{\mathbf{Sch}}$. For any scheme S, $\mathbb{O}_S = S \times_{\mathrm{Spec}(\mathbb{Z})} \mathrm{Spec}(\mathbb{Z}[T]) = \mathrm{Spec}(\mathscr{O}_S[T])$ is then an affine ring scheme over S. Note that this ring is also denoted by $S[T]$.

For any object F in $\widehat{\mathbf{Sch}}$, the set $\mathbb{O}(F) := \mathrm{Hom}(F, \mathbb{O})$ is then endowed with a ring structure and is functorial on F. In particular, if X is a scheme and we are given morphisms $x : X \to F$ and $f : F \to \mathbb{O}$ (that is, $x \in F(X)$ and $f \in \mathbb{O}(F)$), then $f(x) := f \circ x$ is an element in $\mathbb{O}(X) = \Gamma(X, \mathscr{O}_X)$.

Definition 2.1.16. Let $\pi : M \to X$ be a morphism in $\widehat{\mathbf{Sch}}$, and $\mathbb{O}_X = \mathbb{O} \times X$. We say that M is an $\mathbb{O}_X$**-module** if for each X-scheme X', we are given an $\mathbb{O}(X')$-module structure on $\mathrm{Hom}_X(X', M)$, which is functorial on X'. Equivalently, this amounts to giving oneself an X-abelian group structure $\mu : M \times_X M \to M$ on M and an "external law"

$$\mathbb{O} \times M = \mathbb{O}_X \times_X M \to M, \quad (f, m) \mapsto f \cdot m$$

which is an X-morphism and for any $x \in X(S)$, endows $M(x) = \{m \in M(S) : \pi(m) = x\}$ an $\mathbb{O}(S)$-module structure. In this case, for any $Y \in \widehat{\mathbf{Sch}}_{/X}$ (not necessarily representable), the set $\mathrm{Hom}_X(Y, M) = \Gamma(M_Y/Y)$ is an $\mathbb{O}(Y)$-module, which is functorial on Y.

2.1.2.4 Diagonalizable groups

The construction of $\mathbb{G}_m$ can be generalized in the following manner. Let M be an abelian group and $M_{\mathbb{Z}}$ be the constant group scheme associated with M. We then consider the functor $D(M) : \mathbf{Sch}^{\mathrm{op}} \to \mathbf{Set}$ defined by

$$D(M)(S) = \mathrm{Hom}_{\mathbf{Grp}}(M_{\mathbb{Z}}(S), \mathbb{G}_m(S)) \cong \mathrm{Hom}_{S\text{-}\mathbf{Grp}}(M_S, \mathbb{G}_{m,S}) \cong \mathcal{H}om_{\mathbf{Grp}}(M_{\mathbb{Z}}, \mathbb{G}_m)(S).$$

This is an abelian group in $\widehat{\mathbf{Sch}}$ and is represented by the group scheme $\mathrm{Spec}(\mathbb{Z}[M])$, which is still denoted by $D(M)$. In fact, for any scheme S, we have

$$\mathrm{Hom}(S, \mathrm{Spec}(\mathbb{Z}[M])) = \mathrm{Hom}_{\mathbf{Alg}}(\mathbb{Z}[M], \Gamma(S, \mathscr{O}_S)) \cong \mathrm{Hom}_{\mathbf{Grp}}(M, \Gamma(S, \mathscr{O}_S)^\times).$$

For any scheme S, we then obtain an affine group scheme over S:

$$D_S(M) = D(M)_S = \mathcal{H}om_{\mathbf{Grp}}(M_{\mathbb{Z}}, \mathbb{G}_m)_S = \mathcal{H}om_{\mathbf{Grp}}(M_S, \mathbb{G}_{m,S}).$$

This is associated with the $\mathscr{O}_S$-bigebra $\mathscr{O}_S[M]$, whose comultiplication and counit are defined by

$$\Delta(x) = x \otimes x, \quad \varepsilon(x) = 1 \quad \text{for} \quad x \in M.$$

If $f : M \to N$ is a homomorphism of abelian groups, we then have obtain a morphism of S-groups

$$D_S(f) : D_S(N) \to D_S(M),$$

whence a functor $D_S : M \mapsto D_S(M)$ from the category of abelian groups to the category of affine groups over S, which can also be described as the composition of the functor $M \mapsto M_S$ with the functor $M_S \mapsto \mathcal{H}om_{\mathbf{Grp}}(M_S, \mathbb{G}_{m,S})$. This functor clearlt commutes with base changes. An S-group isomorphic to a group of them form $D_S(M)$ is called **diagonalizable**. We note that the elements of M can be interpreted as some characters of $D_S(M)$, that is, certain elements of $\mathrm{Hom}_{\mathbf{Grp}}(D_S(M), \mathbb{G}_{m,S})$.

Example 2.1.17. It is clear that we have $D(\mathbb{Z}) = \mathbb{G}_m$ and $D(\mathbb{Z}^n) = (\mathbb{G}_m)^n$. We now consider the group scheme

$$\boldsymbol{\mu}_n = D(\mathbb{Z}/n\mathbb{Z})$$

which is called the **group of n-th roots of unity**. In fact, we have

$$\boldsymbol{\mu}_n(S) = \mathrm{Hom}_{\mathbf{Grp}}(\mathbb{Z}/n\mathbb{Z}, \Gamma(S, \mathscr{O}_S)^\times) = \{f \in \Gamma(S, \mathscr{O}_S) : f^n = 1\}.$$

The S-group $\boldsymbol{\mu}_{n,S}$ corresponds to the $\mathscr{O}_S$-algebra $\mathscr{O}_S[T]/(T^n - 1)$. Suppose in particular that S is the spectrum of a field k of characteristic p. Then by putting $T - 1 = s$, we have

$$k[T]/(T^p - 1) = k[s]/(s^p),$$

which shows that the underlying space of $\boldsymbol{\mu}_{p,S}$ is reduced to a single point, and the local ring of this point is the Artinian k-algebra $k[s]/(s^p)$. By the same ideas, we see that the S-schemes $\mathbb{G}_{a,S}$, $\mathbb{G}_{m,S}$, $\mathbb{O}_S$ are smooth on S, that $D_S(M)$ is flat on S and that it is formally smooth (resp. smooth) on S if and only if the residual characteristic of S does not divide the torsion of M (resp. and if moreover M is finite type).

Example 2.1.18. The above procedure applies to "classical groups" (linear groups GL_n, symplectic groups Sp_n, orthogonal groups O_n, etc.). We define for example GL_n as representing the functor such that

$$\mathrm{GL}_n(S) = \mathrm{GL}(n, \Gamma(S, \mathscr{O}_S)) = \mathrm{Aut}_{\mathscr{O}_S}(\mathscr{O}_S^n).$$

We can construct it for example as the open set of $\mathrm{Spec}(\mathbb{Z}[T_{ij}])$ ($1 \leq i, j \leq n$) defined by the function $f = \det(T_{ij})$, which is $\mathrm{Spec}(\mathbb{Z}[T_{ij}, f^{-1}])$.

2.1.2.5 Module functors in the category of schemes We now associate with any $\mathscr{O}_S$-module over the schema S, an $\mathbb{O}_S$-module (where $\mathbb{O}_S$ denotes the ring functor introduced in 2.1.2.3). This can be done in two different ways, as we shall now define.

Definition 2.1.19. Let S be a scheme. For any $\mathscr{O}_S$-module $\mathscr{F}$, we denote by $\Gamma_{\mathscr{F}}$ and $\check{\Gamma}_{\mathscr{F}}$ the contravariant functors over $\mathbf{Sch}_{/S}$ defined by

$$\Gamma_{\mathscr{F}}(S') = \Gamma(S', \mathscr{F} \otimes_{\mathscr{O}_S} \mathscr{O}_{S'}), \quad \check{\Gamma}_{\mathscr{F}}(S') = \mathrm{Hom}_{\mathscr{O}_{S'}}(\mathscr{F} \otimes_{\mathscr{O}_S} \mathscr{O}_{S'}, \mathscr{O}_{S'}).$$

Then $\Gamma_{\mathscr{F}}$ and $\check{\Gamma}_{\mathscr{F}}$ are endowed with natural structures of $\mathbb{O}_S$-modules (we note that $\mathbb{O}_S(S') = \Gamma(S', \mathscr{O}_{S'}) = \Gamma_{\mathscr{O}_S}(S')$), so that we obtain functors Γ and $\check{\Gamma}$ from the category of $\mathscr{O}_S$-modules to that of $\mathbb{O}_S$-modules, Γ being convariant and $\check{\Gamma}$ being contracovariant.

We often restrict ourselves to the category of quasi-coherent $\mathscr{O}_S$-modules, so that Γ and $\check{\Gamma}$ are considered as functors from $\mathbf{Qcoh}(\mathscr{O}_S)$ to the category of $\mathbb{O}_S$-modules:

$$\Gamma : \mathbf{Qcoh}(\mathscr{O}_S) \to \mathbf{Mod}(\mathbb{O}_S), \quad \check{\Gamma} : \mathbf{Qcoh}(\mathscr{O}_S)^{\mathrm{op}} \to \mathbf{Mod}(\mathbb{O}_S).$$

The reader should however note that most of the propositions in this paragraph do not rely on the quasi-coherence hypothesis.

Proposition 2.1.20. *Let S be a scheme.*

(a) *The functors Γ and $\check{\Gamma}$ commute with base changes: if $S' \to S$ is a morphism and $\mathscr{F}$ is a quasi-coherent $\mathscr{O}_S$-module, then $\Gamma_{\mathscr{F} \otimes \mathscr{O}_{S'}} \cong (\Gamma_{\mathscr{F}})_{S'}$ and $\check{\Gamma}_{\mathscr{F} \otimes \mathscr{O}_{S'}} \cong (\check{\Gamma}_{\mathscr{F}})_{S'}$.*

(b) *The functors Γ and $\check{\Gamma}$ are fully faithful: the canonical maps*

$$\mathrm{Hom}_{\mathscr{O}_S}(\mathscr{F}, \mathscr{F}') \to \mathrm{Hom}_{\mathbb{O}_S}(\Gamma_{\mathscr{F}}, \Gamma_{\mathscr{F}'}),$$
$$\mathrm{Hom}_{\mathscr{O}_S}(\mathscr{F}, \mathscr{F}') \to \mathrm{Hom}_{\mathbb{O}_S}(\check{\Gamma}_{\mathscr{F}'}, \check{\Gamma}_{\mathscr{F}})$$

are bijective.

(c) *The functors Γ and $\check{\Gamma}$ are additive: we have $\Gamma_{\mathscr{F} \oplus \mathscr{F}'} \cong \Gamma_{\mathscr{F}} \times_S \Gamma_{\mathscr{F}'}$ and $\check{\Gamma}_{\mathscr{F} \oplus \mathscr{F}'} \cong \check{\Gamma}_{\mathscr{F}} \times_S \check{\Gamma}_{\mathscr{F}'}$.*

Proof. Assertions (a) and (c) are clear from the definitions. As for (b), we note that by taking S' to be the open subsets of S, we can construct a homomorphism $u : \mathscr{F} \to \mathscr{F}'$ from an $\mathbb{O}_S$-homomorphism $f : \Gamma_{\mathscr{F}} \to \Gamma_{\mathscr{F}'}$, and it is immediate to verify that this gives an inverse of the canonical map $\mathrm{Hom}_{\mathscr{O}_S}(\mathscr{F}, \mathscr{F}') \to \mathrm{Hom}_{\mathbb{O}_S}(\Gamma_{\mathscr{F}}, \Gamma_{\mathscr{F}'})$. A similar argument shows that the canonical map $\mathrm{Hom}_{\mathscr{O}_S}(\mathscr{F}, \mathscr{F}') \to \mathrm{Hom}_{\mathbb{O}_S}(\check{\Gamma}_{\mathscr{F}'}, \check{\Gamma}_{\mathscr{F}})$ is also bijective. □

We recall that if F, F' are $\mathbb{O}_S$-modules, then $\mathcal{H}om_{\mathbb{O}_S}(F, F')$ denote that S-functor which associates any morphism $S' \to S$ with $\mathrm{Hom}_{\mathbb{O}_{S'}}(F_{S'}, F'_{S'})$.

Proposition 2.1.21. *We have the following canonical morphisms in $\mathbf{Mod}(\mathbb{O}_S)$:*

$$\begin{tikzcd}
\mathcal{H}om_{\mathbb{O}_S}(\Gamma_{\mathscr{F}}, \Gamma_{\mathscr{F}'}) \arrow[rr, "\sim"] & & \mathcal{H}om_{\mathbb{O}_S}(\check{\Gamma}_{\mathscr{F}'}, \check{\Gamma}_{\mathscr{F}}) \\
& \Gamma_{\mathcal{H}om_{\mathscr{O}_S}(\mathscr{F}, \mathscr{F}')} \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Proof. For each S-scheme S', we have a canonical homomorphism

$$\Gamma_{\mathcal{H}om_{\mathcal{O}_S}(\mathscr{F},\mathscr{F}')}(S') = \Gamma(\mathcal{H}om_{\mathcal{O}_S}(\mathscr{F},\mathscr{F}') \otimes \mathcal{O}_{S'}) \to \mathrm{Hom}_{\mathcal{O}_{S'}}(\mathscr{F} \otimes \mathcal{O}_{S'}, \mathscr{F}' \otimes \mathcal{O}_{S'}).$$

The proposition then follows from Proposition 2.1.20 (a) and (b). □

Remark 2.1.22. Let $\mathscr{F}$ be a quasi-coherent $\mathcal{O}_S$-module. Recall that the S-functor $\check{\Gamma}_{\mathscr{F}}$ is represented by an affine S-scheme which is denoted by $\mathbb{V}(\mathscr{F})$ and called the vector bundle defined by $\mathscr{F}$:

$$\mathbb{V}(\mathscr{F}) = \mathrm{Spec}(\boldsymbol{S}(\mathscr{F})),$$

where $\boldsymbol{S}(\mathscr{F})$ denotes the symmetric algebra over $\mathscr{F}$. On the other hand, the article ([]) shows that if S is Noetherian and $\mathscr{F}$ is a coherent $\mathcal{O}_S$-module, then $\Gamma_{\mathscr{F}}$ is representable if and only if $\mathscr{F}$ is locally free, and in this case we have an isomorphism $\Gamma_{\mathscr{F}} \cong \check{\Gamma}_{\mathscr{F}}$.

Proposition 2.1.23. *Let $\mathscr{F}$ and $\mathscr{F}'$ be quasi-coherent $\mathcal{O}_S$-modules and $\mathscr{A}$ be a quasi-coherent $\mathcal{O}_S$-algebra. Then we have a functorial isomorphism*

$$\mathrm{Hom}_S(\mathrm{Spec}(\mathscr{A}), \mathcal{H}om_{\mathbb{O}_S}(\Gamma_{\mathscr{F}'}, \Gamma_{\mathscr{F}})) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}_S}(\mathscr{F}', \mathscr{F} \otimes_{\mathcal{O}_S} \mathscr{A}).$$

Proof. If we put $X = \mathrm{Spec}(\mathscr{A})$, then the LHS is canonically isomorphic to $\mathcal{H}om_{\mathbb{O}_S}(\Gamma_{\mathscr{F}'}, \Gamma_{\mathscr{F}})(X)$, which by Proposition 2.1.20 is given by

$$\begin{aligned}\mathcal{H}om_{\mathbb{O}_S}(\Gamma_{\mathscr{F}'}, \Gamma_{\mathscr{F}})(X) &\cong \mathrm{Hom}_{\mathbb{O}_X}(\Gamma_{\mathscr{F}'\otimes\mathcal{O}_X}, \Gamma_{\mathscr{F}\mathcal{O}_X}) \cong \mathrm{Hom}_{\mathcal{O}_X}(\mathscr{F}' \otimes \mathcal{O}_X, \mathscr{F} \otimes \mathcal{O}_X)\\ &\cong \mathrm{Hom}_{\mathcal{O}_S}(\mathscr{F}', \varphi_*(\varphi^*(\mathscr{F})))\end{aligned}$$

where $\varphi : X \to S$ is the structural morphism. On the other hand, by **??** we have $\varphi_*(\varphi^*(\mathscr{F})) \cong \mathscr{F} \otimes \mathscr{A}$, so the assertion follows. □

Corollary 2.1.24. *We have a canonical isomorphism $\Gamma_{\mathscr{F}\otimes\mathscr{A}} \cong \mathcal{H}om_S(\mathrm{Spec}(\mathscr{A}), \Gamma_{\mathscr{F}})$.*

Proof. Let $f : S' \to S$ be an S-scheme and $X' = X \times_S S'$, we then have a Cartesian diagram

$$\begin{tikzcd} X' \arrow[r, "\varphi'"] \arrow[d, "f'"'] & S' \arrow[d, "f"] \\ X \arrow[r, "\varphi"] & S \end{tikzcd}$$

By **??** and **??**, X' is affine over S' and $\varphi'_*(\mathcal{O}_{X'}) = f^*(\mathscr{A})$, so

$$\mathcal{H}om_S(\mathrm{Spec}(\mathscr{A}), \Gamma_{\mathscr{F}})(S') = \mathrm{Hom}_{S'}(\mathrm{Spec}(f^*(\mathscr{A})), \Gamma_{f^*(\mathscr{F})})$$

and by Proposition 2.1.23 applied to $f^*(\mathscr{F})$, $\mathscr{F}' = \mathcal{O}_{S'}$ and $f^*(\mathscr{A})$, this is equal to

$$\Gamma(S', f^*(\mathscr{F}) \otimes f^*(\mathscr{A})) = \Gamma(S', f^*(\mathscr{F} \otimes \mathscr{A})) = \Gamma_{\mathscr{F}\otimes\mathscr{A}}(S'). \qquad \square$$

Proposition 2.1.25. *If $\mathscr{F}$ and $\mathscr{F}'$ are locally free of finite type, then the morphisms in Proposition 2.1.21 are isomorphisms.*

Proof. In fact, for any morphism $S' \to S$, we then have

$$\Gamma_{\mathcal{H}om_{\mathcal{O}_S}(\mathscr{F},\mathscr{F}')}(S') = \Gamma(S', \mathcal{H}om_{\mathcal{O}_S}(\mathscr{F},\mathscr{F}') \otimes \mathcal{O}_{S'}) = \mathrm{Hom}_{\mathcal{O}_S}(\mathscr{F},\mathscr{F}').$$

But this is also isomorphic to $\mathcal{H}om_{\mathbb{O}_S}(\Gamma_{\mathscr{F}}, \Gamma_{\mathscr{F}'})(S')$ and to $\mathcal{H}om_{\mathbb{O}_S}(\Gamma_{\mathscr{F}}, \Gamma_{\mathscr{F}'})(S')$, in view of Proposition 2.1.20 (b). □

Corollary 2.1.26. *Let $\mathscr{F}$ be a locally free $\mathscr{O}_S$-module of finite type and put $\check{\mathscr{F}} = \mathcal{H}om_{\mathscr{O}_S}(\mathscr{F}, \mathscr{O}_S)$. Then we have canonical isomorphisms*

$$\Gamma_{\check{\mathscr{F}}} \cong \mathcal{H}om_{\mathbb{O}_S}(\Gamma_{\mathscr{F}}, \mathbb{O}_S) \cong \check{\Gamma}_{\mathscr{F}}, \quad \check{\Gamma}_{\check{\mathscr{F}}} \cong \mathcal{H}om_{\mathbb{O}_S}(\check{\Gamma}_{\mathscr{F}}, \mathbb{O}_S) \cong \Gamma_{\mathscr{F}},$$

Proof. This follows from Proposition 2.1.25 by taking $\mathscr{F}' = \mathscr{O}_S$ and note that $\Gamma_{\mathscr{O}_S} = \mathbb{O}_S$. □

Proposition 2.1.27. *If $u : \mathscr{F} \to \mathscr{F}'$ is a morphism of locally free $\mathscr{O}_S$-modules of finite rank, then for $\Gamma_u : \Gamma_{\mathscr{F}} \to \Gamma_{\mathscr{F}'}$ to be a monomorphism, it is necessary and sufficient that f identifies $\mathscr{F}$ locally as a direct factor of $\mathscr{F}'$.*

Proof. One direction follows essentially from **??**. Conversely, if $\mathscr{F}$ is a direct factor of $\mathscr{F}'$, then for any $f : S' \to S$, $f^*(\mathscr{F})$ is a submodule of $f^*(\mathscr{F}')$, so $\Gamma_{\mathscr{F}}(S') = \Gamma(S', f^*(\mathscr{F}))$ is a submodule of $\Gamma_{\mathscr{F}'}(S') = \Gamma(S', f^*(\mathscr{F}'))$. □

2.1.2.6 The category of $\mathscr{O}_S[G]$-modules Let G be an S-group and $\mathscr{F}$ be an $\mathscr{O}_S$-module. Then an $\mathscr{O}_S[G]$**-module structure** on $\mathscr{F}$ is defined to be an $\mathbb{O}_S[h_G]$-module structure on $\Gamma_{\mathscr{F}}$. A morphism of $\mathscr{O}_S[G]$-modules is by definition a morphism of the associated $\mathbb{O}_S[h_G]$-modules. We thus obtain a category $\mathbf{Mod}(\mathscr{O}_S[G])$ of $\mathscr{O}_S[G]$-modules and the full subcategory $\mathbf{Qcoh}(\mathscr{O}_S[G])$ formed by quasi-coherent $\mathscr{O}_S$-modules. By definition, giving an $\mathscr{O}_S[G]$-module structure on $\mathscr{F}$ is equivalent to giving a morphism of groups

$$\rho : h_G \to \mathcal{A}ut_{\mathbb{O}_S}(\Gamma_{\mathscr{F}}).$$

Remark 2.1.28. Since by Proposition 2.1.20 we have an anti-isomorphism

$$\mathcal{A}ut_{\mathbb{O}_S}(\Gamma_{\mathscr{F}}) \cong \mathcal{A}ut_{\mathbb{O}_S}(\check{\Gamma}_{\mathscr{F}}),$$

we see that an $\mathbb{O}_S[h_G]$-module structure on $\Gamma_{\mathscr{F}}$ is equivalent to an $\mathbb{O}_S[h_G]$-module structure on $\check{\Gamma}_{\mathscr{F}}$, and these two structures are connected by the operation $\rho(g) \mapsto \rho^*(g^{-1})$, where ρ^* denotes the image of $\rho : h_G \to \mathcal{A}ut_{\mathbb{O}_S}(\Gamma_{\mathscr{F}})$ under the above isomorphism.

Remark 2.1.29. The categories we have just constructed can also be defined by the following Cartesian squares:

$$\begin{tikzcd}
\mathbf{Qcoh}(\mathscr{O}_S[G]) \arrow[r, hook] \arrow[d] & \mathbf{Mod}(\mathscr{O}_S[G]) \arrow[r] \arrow[d] & \mathbf{Mod}(\mathbb{O}_S[h_G]) \arrow[d, "\text{forget}"] \\
\mathbf{Qcoh}(\mathscr{O}_S) \arrow[r, hook] & \mathbf{Mod}(\mathscr{O}_S) \arrow[r, "\Gamma"] & \mathbf{Mod}(\mathbb{O}_S)
\end{tikzcd}$$

The categories $\mathbf{Mod}(\mathscr{O}_S)$ and $\mathbf{Mod}(\mathbb{O}_S)$ are abelian, but one should be careful that in general the functor Γ is not exact, neither left nor right.

Remark 2.1.30. Let $\mathscr{F}$ be an $\mathscr{O}_S[G]$-module. The **subsheaf of invariants** $\mathscr{F}^G$ is defined as follows: for any open subset U of S,

$$\mathscr{F}^G(U) = \Gamma_{\mathscr{F}}^G(U) = \{x \in \mathscr{F}(U) : g \cdot x_{S'} = x_{S'} \text{ for any morphism } f : S' \to U \text{ and } g \in G(S')\}$$

where $x_{S'}$ denotes the image of x in $\Gamma(S', f^*(\mathscr{F})) = \Gamma(U, f_*(f^*(\mathscr{F})))$.

Be careful that the natural morphism $\Gamma_{\mathscr{F}^G} \to \Gamma_{\mathscr{F}}^G$ is not an isomorphism in general. For example, if $S = \mathrm{Spec}(\mathbb{Z})$ and G is the constant group $\mathbb{Z}/2\mathbb{Z} = \{1, \tau\}$ acting on $\mathscr{F} = \mathscr{O}_S$ via $\tau \cdot 1 = -1$, then we have $\mathscr{F}^G = 0$ since the ring $\Gamma(U, \mathscr{F})$ has characteristic zero for any standard open U of S. However, it is clear that $\Gamma_{\mathscr{F}}^G(\mathrm{Spec}(R)) = R$ for any $\mathbb{F}_2$-algebra R.

From now on, we restrict ourselves to the case where the group scheme G is affine over S. Then, in view of Proposition 2.1.23, giving a morphism of S-functors

$$\rho : h_G \to \mathcal{A}ut_{\mathbb{O}_S}(\Gamma_{\mathscr{F}})$$

is equivalent to giving a morphism of $\mathscr{O}_S$-modules

$$\mu : \mathscr{F} \to \mathscr{F} \otimes_{\mathscr{O}_S} \mathscr{A}(G).$$

The condition that ρ is a group homomorphism is then translated into the folllowing conditions on μ:

(CM1) the following diagram is commutative:

$$\begin{tikzcd} \mathscr{F} \arrow[r, "\mu"] \arrow[d, "\mu"'] & \mathscr{F} \otimes_{\mathscr{O}_S} \mathscr{A}(G) \arrow[d, "\mathrm{id} \otimes \Delta"] \\ \mathscr{F} \otimes_{\mathscr{O}_S} \mathscr{A}(G) \arrow[r, "\mu \otimes \mathrm{id}"] & \mathscr{F} \otimes_{\mathscr{O}_S} \mathscr{A}(G) \otimes_{\mathscr{O}_S} \mathscr{A}(G) \end{tikzcd}$$

(CM2) the following composition is the identity:

$$\mathscr{F} \xrightarrow{\mu} \mathscr{F} \otimes_{\mathscr{O}_S} \mathscr{A}(G) \xrightarrow{\mathrm{id} \otimes \varepsilon} \mathscr{F} \otimes \mathscr{O}_S \xrightarrow{\sim} \mathscr{F}$$

These two axioms then endow a *comodule structure* on $\mathscr{F}$ over the bigebra $\mathscr{A}(G)$.

Put $\mathscr{A} = \mathscr{A}(G)$. If $\mathscr{F}$ and $\mathscr{F}'$ are $\mathscr{A}$-comodules, a morphism $f : \mathscr{F} \to \mathscr{F}'$ of comodules is then defined to be a morphism of $\mathscr{O}_S$-modules such that the following diagram is commutative:

$$\begin{tikzcd} \mathscr{F} \arrow[r, "f"] \arrow[d, "\mu_{\mathscr{F}}"'] & \mathscr{F}' \arrow[d, "\mu_{\mathscr{F}'}"] \\ \mathscr{F} \otimes \mathscr{A} \arrow[r, "f \otimes \mathrm{id}"] & \mathscr{F}' \otimes \mathscr{A} \end{tikzcd}$$

We thus obtain a category $\mathbf{CoMod}(\mathscr{A})$ of comodules over $\mathscr{A}$, and we denote by $\mathbf{CoQcoh}(\mathscr{A})$ the full subcategory formed by quasi-coherent $\mathscr{O}_S$-modules. From the above remarks, it is also clear that we have the following:

Proposition 2.1.31. *Let G be an affine S-group. Then we have equivalences of categories:*

$$\mathbf{Mod}(\mathscr{O}_S[G]) \cong \mathbf{CoMod}(\mathscr{A}(G)), \quad \mathbf{Qcoh}(\mathscr{O}_S[G]) \cong \mathbf{CoQcoh}(\mathscr{A}(G)).$$

If moreover $S = \mathrm{Spec}(A)$ is affine and we put $A[G] = \Gamma(S, \mathscr{A}(G))$, then we have an equivalence of categories

$$\mathbf{CoQcoh}(\mathscr{A}(G)) \cong \mathbf{CoMod}(A[G]).$$

Proposition 2.1.32. *Suppose that G is affine and flat over S. Then the category* $\mathbf{Mod}(\mathscr{O}_S[G])$ *(resp.* $\mathbf{Qcoh}(\mathscr{O}_S[G])$*), being equivalent to the category of $\mathscr{A}(G)$-comodules (resp. quasi-coherent over $\mathscr{O}_S$), is abelian.*

Proof. Suppose that $\mathscr{A} = \mathscr{A}(G)$ is a flat $\mathscr{O}_S$-module. Let $\mathscr{E}$ be an $\mathscr{A}$-comodule and $\mathscr{F}$ be a sub-$\mathscr{O}_S$-module of $\mathscr{E}$. As $\mathscr{A}$ is flat over $\mathscr{O}_S$, we can identify $\mathscr{F} \otimes \mathscr{A}$ (resp. $\mathscr{F} \otimes \mathscr{A} \otimes \mathscr{A}$) as a sub-$\mathscr{O}_S$-module of $\mathscr{E}$ (resp. $\mathscr{E} \otimes \mathscr{A} \otimes \mathscr{A}$). Assume that $\mu_{\mathscr{E}}$ sends $\mathscr{F}$ into $\mathscr{F} \otimes \mathscr{A}$, then the restriction $\mu_{\mathscr{F}} : \mathscr{F} \to \mathscr{F} \otimes \mathscr{A}$ induces a comodule structure on $\mathscr{F}$, and we say that $\mathscr{F}$ is a sub-comodule of $\mathscr{E}$. By passing to quotient, $\mu_{\mathscr{E}}$ then defies a morphism of $\mathscr{O}_S$-modules $\mathscr{E}/\mathscr{F} \to \mathscr{E}/\mathscr{F} \otimes \mathscr{A}$, which endows $\mathscr{E}/\mathscr{F}$ with an $\mathscr{A}$-comodule structure.

Now if $f : \mathscr{E} \to \mathscr{E}'$ is a morphism of $\mathscr{A}$-comodules, then $\ker f$ (resp. $\mathrm{im}\, f$) is a sub-$\mathscr{A}$-comodule of $\mathscr{E}$ (resp. $\mathscr{E}'$), and f induces an isomorphism $\mathscr{E}/\ker f \xrightarrow{\sim} \mathrm{im}\, f$ of $\mathscr{A}$-comodules. Moreover, if $\mathscr{E}$ and $\mathscr{E}'$ are quasi-coherent $\mathscr{O}_S$-modules, then so are $\ker f$ and $\mathrm{im}\, f$. Therefore, we conclude that $\mathbf{CoMod}(\mathscr{A})$ and $\mathbf{CoQcoh}(\mathscr{A})$ are abelian categories. $\square$

We now suppose further that G is a diagonalizable group, which means $\mathscr{A}(G)$ is the algebra of an abelian group M over the ring $\mathscr{O}_S$. If $\mathscr{F}$ is an $\mathscr{O}_S$-module, we then have

$$\mathscr{F} \otimes \mathscr{A}(G) = \coprod_{m \in M} \mathscr{F} \otimes m\mathscr{O}_S,$$

so giving a morphism $\mu : \mathscr{F} \to \mathscr{F} \otimes \mathscr{A}(G)$ is equivalent to giving a family of endomorphisms $(\mu_m)_{m \in M}$ of $\mathscr{F}$ such that for any section x of $\mathscr{F}$ over an open subset S, $(\mu_m(x))$ is a section of the direct sum $\coprod_{m \in M} \mathscr{F}$ (this means that over any sufficiently small open subset, there are only a finite number of restrictions of the $\mu_m(x)$ which are non-zero). For a morphism μ defined by

$$\mu(x) = \sum_{m \in M} \mu_m(x) \otimes m$$

to satisfy (CM1) and (CM2), it is necessary and sufficient that we have

$$\mu_m \circ \mu_n = \delta_{mn}\mu_m, \quad \sum_{m \in M} \mu_m = \mathrm{id}_{\mathscr{F}}$$

which signify that the μ_m are orthogonal projections adding up to the identity. We have therefore proved the following result:

Proposition 2.1.33. *If $G = D_S(M)$ is a diagonalizable group over S, then the category of $\mathscr{O}_S[G]$-modules (resp. quasi-coherent $\mathscr{O}_S[G]$-modules) is equivalent to the category of graded $\mathscr{O}_S$-modules (resp. quasi-coherent $\mathscr{O}_S[G]$-modules) of type M.*

Corollary 2.1.34. *The functor $\mathscr{A} \mapsto \mathrm{Spec}(\mathscr{A})$ induces an equivalence from the category of graded quasi-coherent $\mathscr{O}_S$-algebras of type M to the opposite category of that of affine S-schemes acted by the group $G = D_S(M)$.*

Proof. If X is an affine scheme over S acted by the affine S-group $D_S(M)$, then $\mathscr{A}(S)$ is a quasi-coherent $\mathscr{O}_S$-algebra which is acted by G, whence a graded $\mathscr{O}_S$-algebra of type M. The converse of this is immediate. □

Proposition 2.1.35. *Let G be a diagonalizable group over S. If*

$$0 \longrightarrow \mathscr{F}_1 \longrightarrow \mathscr{F}_2 \longrightarrow \mathscr{F}_3 \longrightarrow 0$$

is an exact sequence of quasi-coherent $\mathscr{O}_S[G]$-modules which split as a sequence of $\mathscr{O}_S$-modules, then it splits as a sequence of $\mathscr{O}_S[G]$-modules..

Proof. If $G = D_S(M)$, then each $\mathscr{F}_i$ is graded by the $(\mathscr{F}_i)_m$ and for each $m \in M$ the sequence

$$0 \longrightarrow (\mathscr{F}_1)_m \longrightarrow (\mathscr{F}_2)_m \longrightarrow (\mathscr{F}_3)_m \longrightarrow 0$$

of $\mathscr{O}_S$-modules is splitting. The proposition then follows from Proposition 2.1.33, since the corresponding result for graded modules is true. □

2.1.3 Cohomology of groups

2.1.3.1 The standard complex Let $\mathcal{C}$ be a category, G be a group in $\widehat{\mathcal{C}}$, A be a ring and M be a $A[G]$-module. For $n \geq 0$, we put

$$C^n(G,M) = \mathrm{Hom}(G^n, M), \quad \mathcal{C}^n(G,M) = \mathcal{H}om(G^n, M),$$

where G^0 is the final object e of $\widehat{\mathcal{C}}$. Then $\mathcal{C}^n(G,M)$ (resp. $C^n(G,M)$) is endowed evidently with a structure of $\mathbb{O}$-module (resp. $\Gamma(\mathbb{O})$-module), and we have

$$C^n(G,M) \cong \Gamma(\mathcal{C}^n(G,M)), \quad \mathcal{C}^n(G,M)(S) = C^n(G_S, M_S).$$

Giving an element of $C^n(G, M)$ is then equivalent to giving for each $S \in \mathrm{Ob}(\mathcal{C})$ an n-cochain of $G(S)$ in $M(S)$, which is functorial on S. The boundary operator

$$d : C^n(G(S), M(S)) \to C^{n+1}(G(S), M(S)),$$

which is defined by the formula

$$\begin{aligned}(df)(g_1, \ldots, g_{n+1}) = g_1 \cdot f(g_2, \ldots, g_{n+1}) + \sum_{i=1}^{n} (-1)^i f(g_1, \ldots, g_i g_{i+1}, \ldots, g_{n+1}) \\ + (-1)^{n+1} f(g_1, \ldots, g_n)\end{aligned}$$

is then functorial on S and hence defines a homomorphism

$$d : C^n(G, M) \to C^{n+1}(G, M)$$

such that $d \circ d = 0$. We then obtain a complex of abelian groups, which we denote by $C^\bullet(G, M)$. We define similarly a complex of A-modules $\mathcal{C}^n(G, M)$, and we have

$$C^\bullet(G, M) = \Gamma(\mathcal{C}^n(G, M)).$$

We denote by $H^n(G, M)$ (resp. $\mathcal{H}^n(G, M)$) the cohomology group of the complex $C^\bullet(G, M)$ (resp. $\mathcal{C}^\bullet(G, M)$). In particular, we have

$$\mathcal{H}^0(G, M) = M^G, \quad H^0(G, M) = \Gamma(M^G).$$

Remark 2.1.36. The set-theoretic definition of d is given to verify that $d \circ d = 0$. We can also define d in terms of the multiplication $m : G \times G \to G$ and the action $\mu : G \times M \to M$ as follows: for any $f \in C^n(G, M)$,

$$df = \mu \circ (\mathrm{id}_G \times f) + \sum_{i=1}^{n} (-1)^i f \circ (\mathrm{id}_{G^{i-1}} \times m \times \mathrm{id}_{G^{n-i}}) + (-1)^{n+1} f \circ \mathrm{pr}_{[1,n]},$$

where $\mathrm{pr}_{[1,n]}$ is the projection of $G^{n+1} = G^n \times G$ to G^n. Similarly, for any $S \in \mathrm{Ob}(\mathcal{C})$ and $f \in \mathrm{Ob}(\mathcal{C})^n(G, M)(S) = C^n(G_S, M_S)$, we have

$$df = \mu_S \circ (\mathrm{id}_G \times f) + \sum_{i=1}^{n} (-1)^i f \circ (\mathrm{id}_{G_S^{i-1}} \times m_S \times \mathrm{id}_{G_S^{n-i}}) + (-1)^{n+1} f \circ \mathrm{pr}_{[1,n]},$$

where m_S and μ_S are defined by base change.

We recall that $\mathbf{Mod}(A[G])$ is endowed with an abelian category structure, defined "argument by argument" (Proposition 2.1.11); therefore a sequence of $A[G]$-modules

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

is exact if and only the sequence of abelian groups

$$0 \longrightarrow M'(S) \longrightarrow M(S) \longrightarrow M''(S) \longrightarrow 0$$

is exact for any $S \in \mathrm{Ob}(\mathcal{C})$. If $\mathcal{C}$ is $\mathcal{U}$-small, then by Proposition 2.1.12, $\mathbf{Mod}(A[G])$ possesses enough injectives, so that the derived functors of the left exact functors $\mathcal{H}^0$ and H^0 can be defined. We now show that the functors $\mathcal{H}^n$ and H^n are isomorphic to the derived functors of $\mathcal{H}^0$ and H^0, respectively.

Definition 2.1.37. For any A-module P, we denote by $\mathrm{CoInd}(P)$ the object $\mathcal{H}om(G,P)$ of $\widehat{\mathcal{C}}$ endowed with the structure of an $A[G]$-module defined as follows: for any $S \in \mathrm{Ob}(\mathcal{C})$, we have $\mathcal{H}om(G,P)(S) = \mathrm{Hom}_S(G_S, P_S)$, and we act $g \in G(S)$ and $a \in A[S]$ on $\phi \in \mathrm{Hom}_S(G_S, P_S)$ by the formule

$$(g \cdot \phi)(h) = \phi(hg), \quad (a \cdot \phi)(h) = a\phi(h),$$

for any $h \in G(S')$ and $S' \to S$. Moreover, for any $\phi \in \mathrm{Hom}_S(G_S, P_S)$, we set

$$\varepsilon(\phi) = \phi(1) \in P(S)$$

where 1 denotes the unit element of $G(S)$. Then it is clear that the construction of $\mathrm{CoInd}(P)$ is functorial on P, and we have thus defined a functor $\mathrm{CoInd} : \mathbf{Mod}(A) \to \mathbf{Mod}(A[G])$ and a natural transform $\iota \circ \mathrm{CoInd} \to \mathrm{id}$, where ι denotes the forgetful functor.

Remark 2.1.38. Let G_1 and G_2 be two copies of G. Then the morphism

$$G_1 \times \mathrm{CoInd}(P) \to \mathrm{CoInd}(P), \quad (g_1, \phi) \mapsto (g_2 \mapsto \phi(g_2 g_1))$$

corresponds via the isomorphisms

$$\begin{aligned}\mathrm{Hom}(G_1 \times \mathrm{CoInd}(P), \mathrm{CoInd}(P)) &\cong \mathrm{Hom}(\mathrm{CoInd}(P), \mathcal{H}om(G_1, \mathcal{H}om(G_2, P)))\\ &\cong \mathrm{Hom}(\mathrm{CoInd}(P), \mathcal{H}om(G_2 \times G_1, P))\end{aligned}$$

to the morphism $\phi \mapsto ((g_2, g_1) \mapsto \phi(g_2 g_1))$, i.e. to the morphism

$$\mathcal{H}om(G,P) \to \mathcal{H}om(G_2 \times G_1, P)$$

induced by the multiplication $\mu_G : G \times G \to G$, $(g_2, g_1) \mapsto g_2 g_1$.

Lemma 2.1.39. *The functor* CoInd *is right adjoint to the forgetful functor* $\iota : \mathbf{Mod}(A[G]) \to \mathbf{Mod}(A)$. *More precisely,* $\varepsilon : \iota \circ \mathrm{CoInd} \to \mathrm{id}$ *induces for any* $M \in \mathbf{Mod}(A[G])$ *and* $P \in \mathbf{Mod}(A)$ *a bijection*

$$\mathrm{Hom}_{A[G]}(M, \mathrm{CoInd}(P)) \xrightarrow{\sim} \mathrm{Hom}_A(M, P).$$

Therefore, if I is an injective object of $\mathbf{Mod}(A)$, *then* $\mathrm{CoInd}(I)$ *is an injective object of* $\mathbf{Mod}(A[G])$.

Proof. To any A-morphism $f : M \to P$, we associate an element $\phi_f \in \mathrm{Hom}_A(M, \mathrm{CoInd}(P))$ defined as follows: for $S \in \mathrm{Ob}(\mathcal{C})$ and $m \in M(S)$, $\phi_f(m)$ is the element of $\mathrm{Hom}_S(G_S, P_S)$ such that for any $g \in G(S')$, $S' \to S$,

$$\phi_f(m)(g) = f(gm) \in P(S').$$

Then for any $h \in G(S)$, we have $\phi_f(hm) = h \cdot f(m)$, i.e. $\phi_f \in \mathrm{Hom}_{A[G]}(M, \mathrm{CoInd}(P))$. Now if $\phi \in \mathrm{Hom}_{A[G]}(M, \mathrm{CoInd}(P))$ and we denote, for $m \in M(S)$, $f(m) = \phi(m)(1)$, then

$$\phi_f(m)(g) = f(gm) = \phi(gm)(1) = (g \cdot \phi(m)) = \phi(m)(g),$$

so $\phi_f = \phi$. Conversely, it is clear that $\phi_f(m)(1) = f(m)$, whence the first claim. The second claim then follows since the forgetful functor ι is exact. □

Definition 2.1.40. Let M be an $A[G]$-module; the identity map on M (considered as an A-module) corresponds by adjunction to a morphism of $A[G]$-modules

$$\eta_M : M \to \mathrm{CoInd}(M)$$

such that for $S \in \mathrm{Ob}(\mathcal{C})$ and $m \in M(S)$, $\eta_M(m)$ is the morpism $G_S \to M_S$ such that for any $S' \to S$ and $g \in G(S')$, $\eta_M(m)(g) = g \cdot m_{S'} \in M(S')$. Note that η_M is a monomorphism: in fact, $\varepsilon_M : \mathrm{CoInd}(M) \to M$ is a morphism of A-modules such that $\varepsilon_M \circ \eta_M = \mathrm{id}_M$. Therefore, M is a direct factor of the A-module $\mathrm{CoInd}(M)$.

Lemma 2.1.41. *For any $P \in \mathbf{Mod}(A)$, we have*

$$H^n(G, \mathcal{H}om(G,P)) = 0, \quad \mathcal{H}^n(G, \mathcal{H}om(G,P)) = 0 \ \text{ for } \ n > 0.$$

Therefore, the functors $H^n(G,-)$ and $\mathcal{H}^n(G,-)$ are effacable for $n > 0$.

Proof. It suffices to prove that $\mathcal{C}^\bullet(G, \mathcal{H}om(G,P))$ and $C^\bullet(G, \mathcal{H}om(G,P))$ are null-homotopic at positive degrees. To this end, we only need to consider the second one, since the corresponding result can be derived via base changes. Now, we define for $n \geq 0$ a morphism

$$\sigma : C^{n+1}(G, \mathcal{H}om(G,P)) \to C^n(G, \mathcal{H}om(G,P)).$$

Let $f \in C^{n+1}(G, \mathcal{H}om(G,P))$; for any $S \in \mathrm{Ob}(\mathcal{C})$ and $g_1, \dots, g_n \in G(S)$, $\sigma(f)(g_1, \dots, g_n)$ is the element of $\mathrm{Hom}_S(G_S, P_S)$ such that for any $S' \to S$ and $x \in G(S')$,

$$\sigma(f)(g_1, \dots, g_n)(x) = f(x, g_1, \dots, g_n)(1) \in P(S'),$$

where 1 denotes the unit element of $G(S')$. Then σ is a null homotopy at positive degrees. In fact, for any $g_1, \dots, g_{n+1} \in G(S)$ and $x \in G(S')$, we have, on the one hand,

$$\begin{aligned} d\sigma(f)(g_1, \dots, g_{n+1})(x) = f(xg_1, g_2, \dots, g_{n+1})(1) + \sum_{i=1}^{n} (-1)^i f(x, g_1, \dots, g_i g_{i+1}, \dots, g_{n+1})(1) \\ + (-1)^{n+1} f(x, g_1, \dots, g_n)(1), \end{aligned}$$

and on the other hand,

$$\begin{aligned} \sigma(df)(g_1, \dots, g_{n+1})(x) = (xf(g_1, \dots, g_{n+1}))(1) - f(xg_1, g_2, \dots, g_{n+1})(1) \\ + \sum_{i=1}^{n} (-1)^{i+1} f(x, g_1, \dots, g_i g_{i+1}, g_{n+1}) + (-1)^{n+2} f(x, g_1, \dots, g_n)(1), \end{aligned}$$

whence

$$(d\sigma(f) + \sigma(df))(g_1, \dots, g_{n+1})(x) = (xf(g_1, \dots, g_{n+1}))(1) = f(g_1, \dots, g_{n+1})(x),$$

i.e. $d\sigma + \sigma d$ is the identity map on $C^{n+1}(G, \mathcal{H}om(G,P))$, for any $n \geq 0$. □

Proposition 2.1.42. *Suppose that $\mathcal{C}$ is $\mathcal{U}$-small, finite products exist in $\mathcal{C}$, and that G is representable. Then the functors $H^n(G,-)$ (resp. $\mathcal{H}^n(G,-)$) are the derived functors of $H^0(G,-)$ (resp. $\mathcal{H}^n(G,-)$) over the category of $A[G]$-modules.*

Proof. In view of (**[?]** 2.2.1 and 2.3), it suffices to show that the $H^n(G)$ (resp. $\mathcal{H}^n(G,-)$) form a cohomological functors, since they are effacable for $n > 0$ in view of Lemma 2.1.41. Let

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

be an exact sequence of $A[G]$-modules, and let $S \in \mathrm{Ob}(\mathcal{C})$. By hypothesis, G is represented by an object $G \in \mathrm{Ob}(\mathcal{C})$, and finite products exist in $\mathcal{C}$. In particular, $\mathcal{C}$ possesses a final object e. For each $n \geq 0$, the product $G^n \times h_S$ is then represented by $G^n \times S$ (where $G^0 = e$), and the sequence

$$0 \longrightarrow M'(G^n \times S) \longrightarrow M(G^n \times S) \longrightarrow M''(G^n \times S) \longrightarrow 0$$

is exact. Therefore, the sequence of A-modules

$$0 \longrightarrow \mathcal{C}^n(h_G, M') \longrightarrow \mathcal{C}^n(h_G, M) \longrightarrow \mathcal{C}^n(h_G, M'') \longrightarrow 0$$

is exact, which means $\mathcal{C}^\bullet(G,-)$, considered as a functor from $\mathbf{Mod}(A[G])$ to the category of complexes of $\mathbf{Mod}(A)$, is exact. It then follows from the induced long exact sequence that $\mathcal{H}^n(G,-)$ form a cohomological functor. As the functor Γ is exact, the same holds for the functors $H^n(G,-)$. □

2.1.3.2 Cohomology of $\mathscr{O}_S[G]$-modules Let S be a scheme, G be an S-group and $\mathscr{F}$ be a quasi-coherent $\mathscr{O}_S[G]$-module. We define the cohomology groups of G with values in $\mathscr{F}$ by

$$H^n(G,\mathscr{F}) = H^n(h_G,\Gamma_{\mathscr{F}}).$$

Suppose that G is affine over S, then by Corollary 2.1.24, this cohomology can be calculated in the following way: $H^n(G,\mathscr{F})$ is the n-th cohomology group of the complex $C^\bullet(G,\mathscr{F})$ whose n-th term is

$$C^n(G,\mathscr{F}) = \Gamma(S,\mathscr{F}\otimes\underbrace{\mathscr{A}(G)\otimes\cdots\otimes\mathscr{A}(G)}_{n\text{-fold}}).$$

If f (resp. a_i) is a section of $\mathscr{F}$ (resp. $\mathscr{A}(G)$) over an open subset of S, we then have

$$\begin{aligned} d(f\otimes a_1\otimes\cdots\otimes a_n) = \mu_{\mathscr{F}}(f)\otimes a_1\otimes\cdots\otimes a_n + \sum_{i=1}^n(-1)^i f\otimes a_1\cdots\otimes\Delta a_i\otimes\cdots\otimes a_n \\ +(-1)^{n+1}f\otimes a_1\otimes\cdots\otimes a_n\otimes 1 \end{aligned}$$

where $\Delta:\mathscr{A}(G)\to\mathscr{A}(G)\otimes\mathscr{A}(G)$ and $\mu_{\mathscr{F}}:\mathscr{F}\to\mathscr{F}\otimes\mathscr{A}(G)$ are induced from the cogebrea structure of $\mathscr{A}(G)$ and the comodule structure on $\mathscr{F}$. Note in passing that the cohomology of G with values in $\mathscr{F}$ therefore depends only on the comodule structure of $\mathscr{F}$ and the monoid structure of G. In particular, we obtain a functor

$$H^0(G,\mathscr{F}) = \Gamma(S,\mathscr{F}^G)$$

where $\mathscr{F}^G$ is the invariant sheaf of $\mathscr{F}$ defined in Remark 2.1.30.

Theorem 2.1.43. *Let S be an affine scheme and G be an affine and flat group over S. Then the functors $H^n(G,-)$ are the derived functors of $H^0(G,-)$ over the category of quasi-coherent $\mathscr{O}_S[G]$-modules.*

If G is affine and flat over S, then by Proposition 2.1.32, the category $\mathbf{Qcoh}(\mathscr{O}_S[G])$ is equivalent to the category $\mathbf{CoQcoh}(\mathscr{A}(G))$ of quasi-coherent $\mathscr{A}(G)$-comodules over $\mathscr{O}_S$ and is abelian. On the other hand, $\mathscr{A}(G)$ being a flat $\mathscr{O}_S$-module, the functor $\mathscr{F}\mapsto\mathscr{F}\otimes_{\mathscr{O}_S}\mathscr{A}(G)^{\otimes n}$ is exact; as S is also affine, we conclude that $C^\bullet(G,-)$ is an exact functor over $\mathbf{Qcoh}(\mathscr{O}_S[G])$.

We denote by Δ (resp. η) the coultiplication (resp. counit) of $\mathscr{A}(G)$. For any quasi-coherent $\mathscr{O}_S$-module $\mathscr{P}$, we denote by $\mathrm{Ind}(\mathscr{P}) = \mathscr{P}\otimes_{\mathscr{O}_S}\mathscr{A}(G)$ endowed with the $\mathscr{A}(G)$-comodule structure defined by

$$\mathrm{id}_{\mathscr{P}}\otimes\Delta:\mathscr{P}\otimes_{\mathscr{O}_S}\mathscr{A}(G)\to\mathscr{P}\otimes_{\mathscr{O}_S}\mathscr{A}(G)\otimes_{\mathscr{O}_S}\mathscr{A}(G);$$

this defines a functor $\mathrm{Ind}:\mathbf{Qcoh}(\mathscr{O}_S)\to\mathbf{Qcoh}(\mathscr{O}_S[G])$. It follows from Corollary 2.1.24 that we have an isomorphism of $\mathbb{O}_S[G]$-modules

$$\Gamma_{\mathrm{Ind}(\mathscr{P})}\cong\mathrm{CoInd}(\Gamma_{\mathscr{P}}) = \mathcal{H}om(G,\Gamma_{\mathscr{P}}). \tag{2.1.1}$$

Via this identification, the morphism $\varepsilon:\mathrm{CoInd}(\Gamma_{\mathscr{P}})\to\Gamma_{\mathscr{P}}$ then corresponds to the morphism $\mathrm{id}_{\mathscr{P}}\otimes\eta:\mathrm{Ind}(\mathscr{P})\to\mathscr{P}$ of $\mathscr{O}_S$-modules, where we use Proposition 2.1.20. From Lemma 2.1.39, we then conclude the following corolalry:

Corollary 2.1.44. *Let S be a scheme and G be an affine group over S. Then the functor* Ind *is right adjoint to the forgetful functor $\iota:\mathbf{Qcoh}(\mathscr{O}_S[G])\to\mathbf{Qcoh}(\mathscr{O}_S)$. More precisely, the map $\mathrm{id}_{\mathscr{P}}\otimes\eta:\mathrm{Ind}(\mathscr{P})\to\mathscr{P}$ induces for any object $\mathscr{M}$ of $\mathbf{Qcoh}(\mathscr{O}_S[G])$ a bijection*

$$\mathrm{Hom}_{\mathscr{O}_S[G]}(\mathscr{M},\mathrm{Ind}(\mathscr{P}))\xrightarrow{\sim}\mathrm{Hom}_{\mathscr{O}_S}(\mathscr{M},\mathscr{P}).$$

Therefore, if $\mathscr{I}$ is an injective object in $\mathbf{Qcoh}(\mathscr{O}_S)$, then $\mathrm{Ind}(\mathscr{I})$ is an injective object in $\mathbf{Qcoh}(\mathscr{O}_S)$.

Let $\mathscr{F}$ be an $\mathscr{O}_S[G]$-module and $\mu_{\mathscr{F}} : \mathscr{F} \to \mathrm{Ind}(\mathscr{F})$ be the map defining the $\mathscr{A}(G)$-comodule structure. It follows from the axioms (CM1) and (CM2) that $\mu_{\mathscr{F}}$ is a morphism of $\mathscr{O}_S[G]$-modules, and that $(\mathrm{id}_{\mathscr{F}} \otimes \eta) \circ \mu_{\mathscr{F}} = \mathrm{id}_{\mathscr{F}}$, so that $\mathscr{F}$ is a direct factor of $\mathrm{Ind}(\mathscr{F})$ considered as $\mathscr{O}_S$-modules. In particular, $\mu_{\mathscr{F}}$ is a monomorphism. As we have, by (2.1.1) and Lemma 2.1.41,

$$H^n(G, \Gamma_{\mathrm{Ind}(\mathscr{F})}) \cong H^n(G, \mathcal{H}om_S(G, \Gamma_{\mathscr{F}})) = 0 \quad \text{for} \quad n > 0$$

we conclude that $H^n(G, -)$ is effacable for $n > 0$.

Finally, as S is affine, $\mathbf{Qcoh}(\mathscr{O}_S)$ possesses enough injectives. Let $\mathscr{F} \rightarrowtail \mathscr{J}$ be a monomorphism of $\mathscr{O}_S$-modules where $\mathscr{J}$ is injective object of $\mathbf{Qcoh}(\mathscr{O}_S)$; then, $\mathscr{A}(G)$ being flat over $\mathscr{O}_S$, $\mathrm{Ind}(\mathscr{F})$ is a sub-$\mathscr{O}_S[G]$-module of $\mathrm{Ind}(\mathscr{J})$, so we conclude that

Corollary 2.1.45. *Under the hypothesis of Theorem 2.1.43, the abelian category $\mathbf{Qcoh}(\mathscr{O}_S[G])$ possesses enough injectives.*

In view of ([?] 2.2.1 and 2.3), we then conclude that proof of Theorem 2.1.43.

Remark 2.1.46. We can also prove Corollary 2.1.44by the following calculation. To any morphism of $\mathscr{O}_S[G]$-modules $\phi : \mathscr{M} \to \mathscr{P} \otimes_{\mathscr{O}_S} \mathscr{A}(G)$, we associate the $\mathscr{O}_S$-morphism $(\mathrm{id}_{\mathscr{P}} \otimes \eta) \circ \phi : \mathscr{M} \to \mathscr{P}$. Conversely, to any $\mathscr{O}_S$-morphism $f : \mathscr{M} \to \mathscr{P}$ we associate the $\mathscr{O}_S[G]$-morphism $(f \otimes \mathrm{id}_{\mathscr{A}(G)}) \circ \mu_{\mathscr{M}} : \mathscr{M} \to \mathrm{Ind}(\mathscr{P})$. On the one hand, from axiom (CM2) we see that

$$(\mathrm{id}_{\mathscr{P}} \otimes \eta) \circ (f \circ \mathrm{id}_{\mathscr{A}(G)}) \circ \mu_{\mathscr{M}} = (f \circ \mathrm{id}_{\mathscr{O}_S}) \circ (\mathrm{id}_{\mathscr{P}} \otimes \eta) \circ \mu_{\mathscr{M}} = f.$$

On the other hand, for any ϕ the following diagram is commutative:

$$\begin{tikzcd} \mathscr{M} \arrow[r, "\phi"] \arrow[d, "\mu_{\mathscr{M}}"'] & \mathscr{P} \otimes_{\mathscr{O}_S} \mathscr{A}(G) \arrow[d, "\mathrm{id}_{\mathscr{P}} \otimes \Delta"] \\ \mathscr{M} \otimes_{\mathscr{O}_S} \mathscr{A}(G) \arrow[r, "\phi \otimes \mathrm{id}_{\mathscr{A}(G)}"] & \mathscr{P} \otimes_{\mathscr{O}_S} \mathscr{A}(G) \otimes_{\mathscr{O}_S} \mathscr{A}(G) \end{tikzcd}$$

so it follows that

$$\begin{aligned} (((\mathrm{id}_{\mathscr{P}} \otimes \eta) \circ \phi) \otimes \mathrm{id}_{\mathscr{A}(G)}) \circ \mu_{\mathscr{M}} &= (\mathrm{id}_{\mathscr{P}} \otimes \eta \otimes \mathrm{id}_{\mathscr{A}(G)}) \circ (\phi \otimes \mathrm{id}_{\mathscr{A}(G)}) \circ \mu_{\mathscr{M}} \\ &= (\mathrm{id}_{\mathscr{P}} \otimes \eta \otimes \mathrm{id}_{\mathscr{A}(G)}) \circ (\mathrm{id}_{\mathscr{P}} \otimes \Delta) \circ \phi = \phi. \end{aligned}$$

This proves the first claim of Corollary 2.1.44, and the second one then follows.

Let $\mathscr{F}$ be an $\mathscr{O}_S[G]$-module. We have seen that the axiom (CM2) shows that considered as $\mathscr{O}_S$-modules, $\mathscr{F}$ is a direct factor of $\mathrm{CoInd}(\mathscr{F})$. This implies the following proposition:

Proposition 2.1.47. *Let S be an affine scheme and G be an affine and flat group scheme over S. Suppose that for any exact sequence*

$$0 \longrightarrow \mathscr{F}_1 \longrightarrow \mathscr{F}_2 \longrightarrow \mathscr{F}_3 \longrightarrow 0$$

of quasi-coherent $\mathscr{O}_S[G]$-modules, which splits as a sequence of $\mathscr{O}_S$-modules, also split as $\mathscr{O}_S[G]$-modules. Then the functors $H^n(G, -)$ are zero for $n > 0$.

Proof. In fact, by the hypothesis, the sequence of $\mathscr{O}_S[G]$-modules

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathrm{CoInd}(\mathscr{F}) \longrightarrow \mathrm{CoInd}(\mathscr{F})/\mathscr{F} \longrightarrow 0$$

is splitting, so $\mathscr{F}$ is a direct factor of $\mathrm{CoInd}(\mathscr{F})$ as an $\mathscr{O}_S[G]$-module. Since $\mathrm{CoInd}(\mathscr{F})$ has trivial higher cohomology, so does $\mathscr{F}$. □

Theorem 2.1.48. *Let S be an affine scheme and G be a diagonalizable S-group. Then for any quasi-coherent $\mathscr{O}_S[G]$-module $\mathscr{F}$, we have $H^n(G, \mathscr{F}) = 0$ for $n > 0$.*

Proof. This follows from Proposition 2.1.47 and Proposition 2.1.35. □

2.1.4 G-equivariant objects and modules

Let $\mathcal{C}$ be a category with a final object e and such that fiber products exist in $\mathcal{C}$. Let G be a group in $\widehat{\mathcal{C}}$, $\pi : M \to X$ be a morphism in $\widehat{\mathcal{C}}$, and $\lambda = \lambda_X : G \times X \to X$ be an action of G on X. In this paragraph, we denote by $Y \times_f M$ the fiber product of $\pi : M \to X$ and an X-functor $f : Y \to X$.

For any $U \in \mathrm{Ob}(\mathcal{C})$ and $x \in X(U)$, the **fiber** of M at x is defined by $M_x = U \times_x M$, i.e. for any $\phi : U' \to U$, we have

$$M_x(U') = \{m \in M(U') : \pi(m) = x_{U'} = \phi^*(x)\}.$$

Finally, if $g \in G(U)$, we denote by $g(x)$ the element $\lambda(g, x)$ in $X(U)$.

Definition 2.1.49. We say that M is a ***G*-equivariant object over *X***, or a ***G*-equivariant *X*-object**, if we are given an action $\Lambda : G \times M \to M$ of G on M compatible with λ, i.e. such that the following diagram is commutative:

$$\begin{array}{ccc} G \times M & \xrightarrow{\Lambda} & M \\ \big\downarrow{\scriptstyle \mathrm{id}_G \times \pi} & & \big\downarrow{\scriptstyle \pi} \\ G \times X & \xrightarrow{\lambda} & X \end{array}$$

This is equivalent to saying that we are given, for any morphism $(g, x) : U \to G \times X$, morphisms

$$\Lambda_x^U(g) : M_x(U) \to M_{g(x)}(U), \quad m \mapsto g \cdot m$$

satisfying $1 \cdot m = m$ and $g \cdot (h \cdot m) = (gh) \cdot m$ and functorial on the $(G \times X)$-object U. Alternatively, this means we are given morphisms of U-objects

$$\Lambda_x(g) : M_x \to M_{g(x)}$$

such that $\Lambda_x(1) = \mathrm{id}$ and $\Lambda_{h(x)}(g) \circ \Lambda_x(h) = \Lambda_x(gh)$.

Now let A be a ring in $\widehat{\mathcal{C}}$ and $A_X = A \times X$. Under the condition described above, we say that M is a ***G*-equivariant A_X-module** if it is an A_X-module and the action Λ is compatible with the A_X-module structure on M, that is, if for any morphism $(g, x) : U \to G \times X$, the map $\Lambda_x(g) : M_x \to M_{g(x)}$ is a morphism of A_U-modules.

Remark 2.1.50. In the above definition for G-equivariant objects, the conditions $\Lambda_x(1) = \mathrm{id}$ and $\Lambda_{h(x)}(g) \circ \Lambda_x(h) = \Lambda_x(gh)$ implies that $\Lambda_x(g)$ is an isomorphism, with inverse $\Lambda_{g(x)}(g^{-1})$. Conversely, if we suppose that each $\Lambda_x(g)$ is an isomorphism, the condition $\Lambda_{h(x)}(g) \circ \Lambda_x(h) = \Lambda_x(gh)$, applied to $h = 1$, then implies that $\Lambda_x(1) = \mathrm{id}$.

Remark 2.1.51. If M is an A_X-module, then in view of the universal property of fiber products, giving a morphism $\Lambda : G \times M \to M$ which is compatible with λ is equivalent to giving a homomorphism of $A_{G \times X}$-modules

$$\theta : G \times M = (G \times X) \times_{\mathrm{pr}_X} M \to (G \times X) \times_\lambda M, \quad (g, x, m) \mapsto (g, g(x), g \cdot m),$$

and the morphisms $\Lambda_x(g) : M_x \to M_{g(x)}, m \mapsto g \cdot m$ are isomorphisms of A_U-modules if and only if θ is an isomorphism. As we have supposed that each $\Lambda_x(h)$ is an isomorphism, the equality $\Lambda_x(1) = \mathrm{id}$ follows from the equality $\Lambda_{h(x)}(g) \circ \Lambda_x(h) = \Lambda_x(gh)$. Therefore, Λ is an action of G over M if and only the following diagram of $(G \times G \times X)$-isomorphisms is commutative (where we denote by m the multiplication of G and $f^*(\theta)$ is the isomorphism induced from θ under a base change $f : G \times G \times X \to G \times X$)

$$\begin{tikzcd}
(G \times G \times X) \times_{\mathrm{pr}_X \circ \mathrm{pr}_{23}} M \arrow[r, "\mathrm{pr}_{23}^*(\theta)", "\sim"'] \arrow[d, equal] & (G \times G \times X) \times_{\lambda \circ \mathrm{pr}_{23}} M \arrow[d, equal] \\
(G \times G \times X) \times_{\mathrm{pr}_X \circ (m \times \mathrm{id}_X)} M \arrow[d, "(m \times \mathrm{id}_X)^*(\theta)"', "\sim"] & (G \times G \times X) \times_{\mathrm{pr}_X \circ (\mathrm{id}_G \times \lambda)} M \arrow[d, "\sim"', "(\mathrm{id}_G \times \lambda)^*(\theta)"] \\
(G \times G \times X) \times_{\lambda \times (m \times \mathrm{id}_X)} M \arrow[r, equal] & (G \times G \times X) \times_{\lambda \circ (\mathrm{id}_G \times \lambda)} M
\end{tikzcd}$$

Remark 2.1.52. The above definitions extend to the case where G is only a monoid. In this case, giving an action $\Lambda : G \times M \to M$ that is compatible with λ and such that each $\Lambda_x(g) : M_x \to M_{g(x)}$ is a morphism of A_U-modules is equivalent to giving a morphism

$$\theta : G \times M = (G \times X) \times_{\mathrm{pr}_X} M \to (G \times X) \times_\lambda M, \quad (g, x, m) \mapsto (g, g(x), g \cdot m),$$

such as the diagram in Remark 2.1.51 (without the signs $\sim$ under the arrows) is commutative, and such that $\mathrm{pr}_M \circ \theta \circ (\varepsilon_G \times \mathrm{id}_M) = \mathrm{id}_M$, where ε_G denotes the unit section of G and pr_M the projection on M (this is added since in this case the equality $\Lambda_x(1) = \mathrm{id}$ can not be derived).

Let Y be another object of $\widehat{\mathcal{C}}$ which is endowed with an action $\lambda_Y : G \times Y \to Y$ by G and N be a G-equivariant A_X-module. A morphism $f : Y \to X$ in $\widehat{\mathcal{C}}$ (resp. a homomorphism of A_X-modules $\phi : M \to X$) is called G-equivariant if it commutes with the action of G, i.e. if we have $f(g \cdot y) = g \cdot f(y)$ (resp. $\phi(g \cdot m) = g \cdot \phi(m)$), which is equivalent to $f \circ \lambda_Y = \lambda_X \circ \mathrm{id}_G \times f$ (resp. $\phi \circ \Lambda_M = \Lambda_N \circ (\mathrm{id}_G \times \phi)$). We then obtain the following lemma:

Lemma 2.1.53. *Let $f : Y \to X$ be a G-equivariant morphism and M be a G-equivariant A-module. Then the inverse image $f^*(M) = Y \times_f M$ is a G-equivariant A_Y-module.*

Proof. □

2.2 Tangent spaces and Lie algebras

In this section, we construct the tangent spaces and Lie algebras in scheme theory. It will be useful not to restrict oneself to the diagrams themselves, but to also be intersted to certain functors on the category of schemes which are not necessarily representable. The exposition we give here easily generalize beyond the theory of schemes. For example, it is valid for the theory of complex analytic spaces, with suitable modifications.

2.2.1 The tangent bundle and tangent space

2.2.1.1 The functor $\mathcal{H}om_{Z/S}(X, Y)$ Let $\mathcal{C}$ be a category and S be an object of $\mathcal{C}$. We consider objects X, Y, Z in $\widehat{\mathcal{C}}$ with X, Y lying over Z and Z lying over S:

$$\begin{tikzcd}
X \arrow[dr, "p_X"'] & & Y \arrow[dl, "p_Y"] \\
& Z \arrow[d] & \\
& S &
\end{tikzcd}$$

Definition 2.2.1. We define an object $\mathcal{H}om_{Z/S}(X,Y)$ in $\widehat{\mathcal{C}_{/S}}$ by the formula

$$\mathcal{H}om_{Z/S}(X,Y)(S') = \mathrm{Hom}_{Z_{S'}}(X_{S'}, Y_{S'}) = \mathrm{Hom}_Z(X \times_S S', Y),$$

where S' is an object of $\mathcal{C}_{/S}$. We see that $\mathcal{H}om_{Z/S}(X,Y)$ is none other than the sub-object of $\mathcal{H}om_S(X,Y)$ formed by morphisms compatible with p_X and p_Y, that is, it is the kernel of the morphisms

$$\mathcal{H}om_S(X,Y) \rightrightarrows \mathcal{H}om_S(X,Z)$$

where the first map is defined by composing with p_Y and the seond one is the constant map of p_X.

On the other hand, we see as in (**??**) that, for any object T of $\widehat{\mathcal{C}}$ over S, we have a natural bijection

$$\mathrm{Hom}_S(T, \mathcal{H}om_{Z/S}(X,Y)) \cong \mathrm{Hom}_Z(X \times_S T, Y).$$

Moreover, by (**??**), if E, F are objects of $\widehat{\mathcal{C}}$ lying over Z, then

$$\mathrm{Hom}_Z(E, \mathcal{H}om_Z(F,Y)) \cong \mathrm{Hom}_Z(E \times_Z F, Y) \cong \mathrm{Hom}_Z(F, \mathcal{H}om_Z(E,Y)).$$

Apply this to $E = X$ and $F = Z \times_S T$, we then obtain the following bijections for any object T of $\widehat{\mathcal{C}_{/S}}$:

$$\mathrm{Hom}_S(T, \mathcal{H}om_{Z/S}(X,Y)) \cong \mathrm{Hom}_Z(X \times_S T, Y) \cong \begin{cases} \mathrm{Hom}_Z(Z \times_S T, \mathcal{H}om_Z(X,Y)), \\ \mathrm{Hom}_Z(X, \mathcal{H}om_Z(Z \times_S T, Y)). \end{cases} \tag{2.2.1}$$

Since these bijections are functorial over T, we then obtain isomorphisms of S-functors

$$\begin{tikzcd} \mathcal{H}om_S(T, \mathcal{H}om_{Z/S}(X,Y)) \arrow[rr, "\sim"] \arrow[dr, "\sim"'] & & \mathcal{H}om_{Z/S}(X, \mathcal{H}om_Z(Z \times_S T, Y)) \\ & \mathcal{H}om_{Z/S}(X \times_S T, Y) \arrow[ur, "\sim"'] & \end{tikzcd} \tag{2.2.2}$$

We also note that, by definition, for $Z = S$ we have $\mathcal{H}om_{S/S}(X,Y) = \mathcal{H}om_S(X,Y)$. On the other hand, if $X = Z$, we put

$$\mathrm{Res}_{Z/S} Y = \mathcal{H}om_{Z/S}(Z, Y),$$

by definition, we then have

$$\mathrm{Res}_{Z/S}(Y)(S') = \mathrm{Hom}_Z(Z \times_S S', Y) = \Gamma(Y_{S'}/Z_{S'}).$$

The functor $\mathrm{Res}_{Z/S} : \widehat{\mathcal{C}_{/Z}} \to \widehat{\mathcal{C}_{/S}}$ is a right adjoint of the base change functor from S to Z. In fact, for any S-functor U, by (2.2.1) we have

$$\mathrm{Hom}_S(U, \mathrm{Res}_{Z/S} Y) = \mathrm{Hom}_S(U, \mathcal{H}om_{Z/S}(Z,Y)) \cong \mathrm{Hom}_Z(U \times_S Z, Y).$$

(If $\mathcal{C} = \mathbf{Sch}$ and Z is an S-scheme, the functor $\mathrm{Res}_{Z/S}$ is called the **Weil restriction**.) We also ntoe that since for any $S' \in \mathrm{Ob}(\mathcal{C}_{/S})$ we have

$$\mathcal{H}om_{Z/S}(X,Y)(S') = \mathrm{Hom}_Z(X_{S'}, Y) \cong \mathrm{Hom}_X(X_{S'}, Y \times_Z X) = \mathcal{H}om_{X/S}(X, Y \times_Z X),$$

so we obtain an isomorphism

$$\mathcal{H}om_{Z/S}(X,Y) \cong \mathcal{H}om_{X/S}(X, Y \times_Z X) = \mathrm{Res}_{X/S}(Y \times_Z X),$$

which for $Z = S$ gives an isomorphism

$$\mathcal{H}om_S(X,Y) \cong \mathrm{Res}_{X/S} Y_X.$$

Remark 2.2.2. The functore $Y \mapsto \mathcal{H}om_{Z/S}(X, Y)$ commutes with products in the sense that we have a functorial isomorphism

$$\mathcal{H}om_{Z/S}(X, Y \times_Z Y') \cong \mathcal{H}om_{Z/S}(X, Y) \times_S \mathcal{H}om_{Z/S}(X, Y') \tag{2.2.3}$$

It follows that if Y is a Z-group (resp. Z-ring, etc.), then $\mathcal{H}om_{Z/S}(X, Y)$ is an S-group (resp. S-ring, etc.).

Moreover, let $\pi : M \to Y$ be an Y-functor in $\mathbb{O}_Y$-modules. Put $H = \mathcal{H}om_{Z/S}(X, Y)$, then $\mathcal{H}om_{Z/S}(X, M)$ is endowed with a natrual structure of $\mathbb{O}_H$-module. More precisely, for any $H' \to H$, $\mathrm{Hom}_H(H', \mathcal{H}om_{Z/S}(X, M))$ is endowed with a natural structure of $\mathbb{O}(H' \times_S X)$-module.

Remark 2.2.3. Moreover, let $\pi : M \to Y$ be a Y-functor in $\mathbb{O}_Y$-modules. Put $H = \mathcal{H}om_{Z/S}(X, Y)$, then $\mathcal{H}om_{Z/S}(X, M)$ is endowed with a natural $\mathbb{O}_H$-module structure; more precisely, for any $H' \to H$, $\mathrm{Hom}_H(H', \mathcal{H}om_{Z/S}(X, M))$ is endowed with a natural $\mathbb{O}(H' \times_S X)$-structure.

In fact, denote by $m : M \times_Y M \to M$ and $\lambda : \mathbb{O}_Y \times_Y M \to M$ the defining morphisms of abelian group structure and module structure of M. Let H' be an S-scheme over H, that is, we are given a Z-morphism $f : X \times_S H' \to Y$, which makes $X \times_S H'$ a Y-object. Then $\mathrm{Hom}_H(H', \mathcal{H}om_{Z/S}(X, M))$ is the set of Z-morphisms $\phi : X \times_S H' \to M$ such that $\pi \circ \phi = f$, that is, the Y-morphisms $X \times_S H' \to M$.

Let ϕ, ψ be two such morphisms, we define $\phi + \psi$ as the composition of Y-morphisms

$$X \times_S H' \xrightarrow{\phi \times \psi} M \times_Y M \xrightarrow{m} M$$

and we verify that this endows $\mathcal{H}om_{Z/S}(X, M)$ an abelian group structure over $H = \mathcal{H}om_{Z/S}(X, Y)$.

Similarly, if a is an element of $\mathbb{O}(X \times_S H')$, i.e. an S-morphism $a : X \times_S H' \to \mathbb{O}_S$, we define $a\phi$ as the composition $\lambda \circ (a \times \phi)$, where $a \times \phi$ denotes the Y-morphism from $X \times_S H$ to $\mathbb{O}_Y \times_Y M \cong \mathbb{O}_S \times_S M$ with components a and ϕ. We verify that this endows $\mathrm{Hom}_H(H', \mathcal{H}om_{Z/S}(X, M))$ with an $\mathbb{O}(X \times_S H')$-module structure, which is functorial on H'.

2.2.1.2 The scheme $I_S(\mathscr{M})$

Definition 2.2.4. Let S be a scheme and $\mathscr{M}$ be a quasi-coherent $\mathscr{O}_S$-module. We denote by $\mathscr{D}_{\mathscr{O}_S}(\mathscr{M})$ the quasi-coherent algebra $\mathscr{O}_S \oplus \mathscr{M}$ (where $\mathscr{M}$ is considered as a square zero ideal). We denote by $I_S(\mathscr{M})$ the S-scheme $\mathrm{Spec}(\mathscr{D}_{\mathscr{O}_S}(\mathscr{M}))$. In particular, we have $\mathscr{D}_{\mathscr{O}_S} = \mathscr{D}_{\mathscr{O}_S}(\mathscr{O}_S)$, $I_S = I_S(\mathscr{O}_S)$, which are called the **algebra of dual numbers over S** and the **dual number scheme over S**.

We then obtain a contravariant functor $\mathscr{M} \mapsto I_S(\mathscr{M})$ from the category of quasi-coherent $\mathscr{O}_S$-modules to the category of S-schemes. In particular, the morphisms $0 \to \mathscr{M}$ and $\mathscr{M} \to 0$ define respectively the structural morphism $\rho : I_S(\mathscr{M}) \to I_S(0) = S$ and a section $\varepsilon_{\mathscr{M}} : S \to I_S(\mathscr{M})$, which is called the **zero section** of $I_S(\mathscr{M})$.

As $\mathscr{M} \mapsto I_S(\mathscr{M})$ is a contravariant functor, for any endomorphism $a \in \mathrm{End}_{\mathscr{O}_S}(\mathscr{M})$, we have an S-endomorphism a^* of $I_S(\mathscr{M})$, and

$$1^* = \mathrm{id}, \quad (ab)^* = b^* \circ a^*, \quad 0^* = \varepsilon_{\mathscr{M}} \circ \rho, \quad a^* \circ \varepsilon_{\mathscr{M}} = \varepsilon_{\mathscr{M}}.$$

Therefore, the S-scheme $I_S(\mathscr{M})$ is endowed with a right action of the multiplicative monoid $\mathrm{End}_{\mathscr{O}_S}(\mathscr{M})$, which commutes with S-morphisms $I_S(\mathscr{M}) \to I_S(\mathscr{M}')$ induced by morphisms $\mathscr{M} \to \mathscr{M}'$. In particular, the operations a^* preserves the zero section of $I_S(\mathscr{M})$.

For any endomorphism $a \in \mathrm{End}_{\mathscr{O}_S}(\mathscr{M})$, $f : S' \to S$ and $m \in I_S(\mathscr{M})(S')$, we write $m \cdot a = a^*(m)$. Then we have

$$m \cdot 1 = m, \quad (m \cdot a) \cdot b = m \cdot (ab), \quad m \cdot 0 = \varepsilon_{\mathscr{M}}(\rho(m))$$

and, if $m = \varepsilon_{\mathscr{M}}(f)$, then $m \cdot a = m$.

Remark 2.2.5. The formation of $I_S(\mathscr{M})$ commutes with base changes: we have a canonical isomorphism
$$I_S(\mathscr{M})_{S'} \cong I_{S'}(\mathscr{M} \otimes_{\mathscr{O}_S} \mathscr{O}_{S'}).$$
For simplicity, we shall write $I_{S'}(\mathscr{M})$ for $I_S(\mathscr{M})_{S'}$. More generally, if X is an S-functor (not necessarily representable), then we define $I_X(\mathscr{M}) := I_S(\mathscr{M}) \times_S X$.

Remark 2.2.6. By consider the homotheties on $\mathscr{M}$, we see that the multiplicative monoid $\mathbb{O}(S')$ acts on the S'-scheme $I_{S'}(\mathscr{M})$, which is functorial on $\mathscr{M}$, i.e. the S-scheme $I_S(\mathscr{M})$ is endowed with a structure of an $\mathbb{O}_S$-object, which is functorial on $\mathscr{M}$. We then have a morphism of S-schemes
$$\lambda : I_S(\mathscr{M}) \times_S \mathbb{O}_S \to I_S(\mathscr{M}),$$
which satisfies the evident conditions. For any S-functor X, we then obtain by base change a morphism of X-functors
$$\lambda_X : I_X(\mathscr{M}) \times_S \mathbb{O}_S \to I_X(\mathscr{M})$$
which makes the S-functor $I_X(\mathscr{M})$ an object acted by the monoid $\mathbb{O}(X)$: any element a of $\mathbb{O}_X = \mathrm{Hom}_S(X, \mathbb{O}_S)$ defines an X-endomorphism a^* of $I_X(\mathscr{M})$. More precisely, if $x \in X(S')$ and $m \in I_S(\mathscr{M})(S') = I_{S'}(\mathscr{M})(S')$, then $a(x) = a \circ x$ belongs to $\mathbb{O}(S')$ and we have
$$(m, x) \cdot a = (m \cdot a(x), x).$$
This operation is functorial on $\mathscr{M}$ and preserves the zero section $\varepsilon_{\mathscr{M}} : X \to I_X(\mathscr{M})$, i.e. $a^* \circ \varepsilon_{\mathscr{M}} = \varepsilon_{\mathscr{M}}$ for any $a \in \mathbb{O}(X)$.

Even further, this operation is functorial on X in the following sense: if $\pi : Y \to X$ is a morphism of S-functors and $u : \mathbb{O}(X) \to \mathbb{O}(Y)$ is the corresponding ring homomorphism (i.e. $u(a) = a \circ \pi$ for $a \in \mathbb{O}(X)$), then the following diagram is commutative
$$\begin{tikzcd} I_Y(\mathscr{M}) \arrow[r, "u(a)^*"] \arrow[d, "\pi"'] & I_Y(\mathscr{M}) \arrow[d, "\pi"] \\ I_X(\mathscr{M}) \arrow[r, "a^*"] & I_X(\mathscr{M}) \end{tikzcd}$$

Let $\mathscr{M}$ and $\mathscr{N}$ be quasi-coherent $\mathscr{O}_S$-modules. The commutative diagram
$$\begin{tikzcd} & \mathscr{M} \oplus \mathscr{N} \arrow[dl] \arrow[dr] & \\ \mathscr{M} \arrow[dr] & & \mathscr{N} \arrow[dl] \\ & 0 & \end{tikzcd}$$
then defines a commutative diagram of S-schemes
$$\begin{tikzcd} & I_S(\mathscr{M} \oplus \mathscr{N}) & \\ I_S(\mathscr{M}) \arrow[ur] & & I_S(\mathscr{N}) \arrow[ul] \\ & S \arrow[ul, "\varepsilon_{\mathscr{M}}"] \arrow[ur, "\varepsilon_{\mathscr{N}}"'] \arrow[uu, "\varepsilon_{\mathscr{M}\oplus\mathscr{N}}"'] & \end{tikzcd} \tag{2.2.4}$$

Proposition 2.2.7. *For any S-scheme X, the diagram of functors over S obtained by applying the functor $\mathcal{H}om_S(-, X)$ to (2.2.4) is Cartesian:*
$$\begin{tikzcd} \mathcal{H}om_S(I_S(\mathscr{M} \oplus \mathscr{N}), X) \arrow[r] \arrow[d] & \mathcal{H}om_S(I_S(\mathscr{N}), X) \arrow[d] \\ \mathcal{H}om_S(I_S(\mathscr{M}), X) \arrow[r] & \mathcal{H}om_S(S, X) = X \end{tikzcd}$$

Proof. It suffices to verify that for any $S' \to S$, the diagram obtained by applying the functors on S' is Cartesian. As the formation of $I_S(\mathscr{P})$ commutes with base change, it then suffices to prove this for $S' = S$, hence to verify that the following diagram is Cartesian:

$$\begin{tikzcd} X(I_S(\mathscr{M} \oplus \mathscr{N})) \arrow[r] \arrow[d] \arrow[dr, "X(\varepsilon_{\mathscr{M}\oplus\mathscr{N}})"] & X(I_S(\mathscr{N})) \arrow[d, "X(\varepsilon_{\mathscr{N}})"] \\ X(I_S(\mathscr{M})) \arrow[r, "X(\varepsilon_{\mathscr{M}})"] & X(S) \end{tikzcd}$$

Now if $x \in X(S)$, it follows from ([?] III, 5.1) that the fiber $X(\varepsilon_{\mathscr{M}})^{-1}(x)$ is isomorphic to $\mathrm{Hom}_{\mathscr{O}_S}(x^*(\Omega^1_{X/S}), \mathscr{M})$. Since this latter functor clearly commutes with finite direct sums of $\mathscr{O}_S$-modules, our assertion follows. □

Corollary 2.2.8. *Let X be an S-scheme and $\mathscr{M}$ be a free $\mathscr{O}_X$-module of finite type. Then the S-functor $\mathcal{H}om_S(I_S(\mathscr{M}), X)$ is isomorphic to a finite product of copies of $\mathcal{H}om_S(I_S, X)$.*

Remark 2.2.9. It follows from the proof of Proposition 2.2.7 that $\mathcal{H}om_S(I_S, X)$ is isomorphic to the X-functor $\check{\Gamma}_{\Omega^1_{X/S}}$, and hence represented by the vector bundle $\mathbb{V}(\Omega^1_{X/S})$.

2.2.1.3 The tangent bundle and condition (E)

Definition 2.2.10. Let S be a scheme and $\mathscr{M}$ be a free $\mathscr{O}_S$-module of finite rank. Let X be a functor over S. The **tangent bundle of X over S relative to the $\mathscr{O}_S$-module $\mathscr{M}$** is defined to be the S-functor

$$T_{X/S}(\mathscr{M}) = \mathcal{H}om_S(I_S(\mathscr{M}), X).$$

In particular, the **tangent bundle of X over S** is the functor

$$T_{X/S} = T_{X/S}(\mathscr{O}_S) = \mathcal{H}om_S(I_S, X).$$

The construction $\mathscr{M} \mapsto T_{X/S}(\mathscr{M})$ is then a covariant functor from the category of free $\mathscr{O}_S$-modules of finite type to the category of S-functors. In particular, the morphisms $\mathscr{M} \to 0$ and $0 \to \mathscr{M}$ define respectively an S-morphism $\pi_{\mathscr{M}} : T_{X/S}(\mathscr{M}) \to T_{X/S}(0) \cong X$ and a section $\tau : X \to T_{X/S}(\mathscr{M})$, called the **zero section**. Moreover, it follows from the preceding remarks that $\mathbb{O}(S)$ is a monoid acting on the X-functor $T_{X/S}(\mathscr{M})$, which is functorial on $\mathscr{M}$.

Remark 2.2.11. We note that the projection $\pi_{\mathscr{M}} : T_{X/S}(\mathscr{M}) \to X$ is induced by the zero section $\varepsilon_{\mathscr{M}} : S \to I_S(\mathscr{M})$, while the zero section $\tau : X \to T_{X/S}(\mathscr{M})$ is induced by the structural morphism $\rho : I_S(\mathscr{M}) \to S$. For any point $t \in T_{X/S}(\mathscr{M})(S')$ (resp. $x \in X(S')$), which corresponds to an S-morphism $f : I_{S'}(\mathscr{M}) \to X$ (resp. $g : S' \to X$), we have

$$\pi(t) = f \circ (\mathrm{id}_{S'} \times \varepsilon_{\mathscr{M}}), \quad (\text{resp. } \tau(x) = g \circ (\mathrm{id}_{S'} \times \rho)).$$

It follows from the above definition that $\mathscr{M} \mapsto T_{X/S}(\mathscr{M})$ is a covariant functor from the category of free $\mathscr{O}_X$-modules of finite rank to that of functors over X. In particular, $\mathbb{O}(S)$ is a monoid operating on the X-functor $T_{X/S}(\mathscr{M})$, which *respects the functoriality of $\mathscr{M}$*.

Remark 2.2.12. In particular, the above arguments motivates the following construction. For any S-morphism $X' \to X$, we put

$$\Sigma(X', \mathscr{M}) = \mathrm{Hom}_X(X', T_{X/S}(\mathscr{M})).$$

We have an action of the multiplicative monoid $\mathrm{End}_{\mathscr{O}_S}(\mathscr{M})$ over $\Sigma(X', \mathscr{M})$, denoted by $(\lambda, x) \mapsto \lambda * x$, such that

$$\lambda * (\mu * x) = (\lambda\mu) * x, \quad 1 * x = x, \quad 0 * x = \tau_0 * \phi \tag{2.2.5}$$

where τ_0 is the zero section $X \to T_{X/S}(\mathscr{M})$. We have similarly an action of $\mathrm{End}_{\mathscr{O}_S}(\mathscr{M} \oplus \mathscr{M})$ over $\Sigma(X', \mathscr{M} \oplus \mathscr{M})$.

Moreover, let $m : \mathscr{M} \oplus \mathscr{M} \to \mathscr{M}$ (resp. $\delta : \mathscr{M} \to \mathscr{M} \oplus \mathscr{M}$) the addition (resp. diagonal map) of $\mathscr{M}$, and put $m_{X'} : \Sigma(X', \mathscr{M} \oplus \mathscr{M}) \to \Sigma(X', \mathscr{M})$ and $\delta_{X'} : \Sigma(X', \mathscr{M}) \to \Sigma(X', \mathscr{M} \oplus \mathscr{M})$ be the induced morphisms. For $\lambda, \mu \in \mathbb{O}(S)$, let h_λ (resp. $h_{\lambda,\mu}$) be the multiplication by λ on $\mathscr{M}$ (resp. by (λ, μ) on $\mathscr{M} \oplus \mathscr{M}$). Since $m \circ h_{\lambda,\lambda} = h_\lambda \circ m$ and $m \circ h_{\lambda,\mu} = h_{\lambda+\mu}$, we have, for $z \in \Sigma(X', \mathscr{M} \oplus \mathscr{M})$ and $x \in \Sigma(X', \mathscr{M})$:

$$\lambda * m(z) = m((\lambda, \lambda) * z), \quad m((\lambda, \mu) * \delta(x)) = (\lambda + \mu) * x. \tag{2.2.6}$$

Definition 2.2.13. Let $x \in X(S) = \mathrm{Hom}_S(S, X) = \Gamma(X/S)$. We then define the tangent space of X over S at the point x relative to $\mathscr{M}$ to be the S-functor obtained from $T_{X/S}(\mathscr{M})$ by base change via the morphism $x : S \to X$:

$$\begin{tikzcd} T_{X/S,x}(\mathscr{M}) \arrow[r] \arrow[d] & T_{X/S}(\mathscr{M}) \arrow[d, "\pi"] \\ S \arrow[r, "x"] & X \end{tikzcd}$$

In particular, $T_{X/S,x}(\mathscr{O}_X)$ is denoted by $T_{X/S,x}$, which is called the **tangent space of X over S at the point x**.

Remark 2.2.14. It follows from Remark 2.2.11 that, for any $t : S' \to S$, $T_{X/S,x}(\mathscr{M})(S')$ is the set of S-morphisms $f : I_{S'}(\mathscr{M}) \to X$ such that $f \circ (\mathrm{id}_{S'} \times \varepsilon_{\mathscr{M}}) = x \circ t$, where $\varepsilon_{\mathscr{M}} : S \to I_S(\mathscr{M})$ is the zero section.

Proposition 2.2.15. *If X is representable, then $T_{X/S}(\mathscr{M})$ and $T_{X/S,x}(\mathscr{M})$ are representable. In particular, $T_{X/S}$ and $T_{X/S,x}$ are represented by the vector bundles $\mathbb{V}(\Omega^1_{X/S})$ and $\mathbb{V}(x^*(\Omega^1_{X/S}))$.*

Proof. It suffices to prove for $T_{X/S}(\mathscr{M})$, since the analogous result follows from base change. By Corollary 2.2.8, it suffices to consider $T_{X/S}$, which follows from Remark 2.2.9. □

Remark 2.2.16. By Proposition 2.2.15, we can give a simple description of the vector bundle representing $T_{X/S,x}$: if $x : S \to X$ is an S-morphism, then the image of x is locally closed in S by **??**, hence defined by a quasi-coherent ideal $\mathscr{I}$ of an open subscheme of X. The quotient $\mathscr{I}/\mathscr{I}^2$ can then be considered as a quasi-coherent module over S, whose vector bundle $\mathbb{V}(\mathscr{I}/\mathscr{I}^2)$ is the desired representing scheme.

For example, let X be an algebraic scheme over a field X and x be a rational point of X over k. Let $\mathfrak{m}_x$ be the maximal ideal of the local ring $\mathscr{O}_{X,x}$, then we have $T_{X/k,x} = \mathbb{V}(\mathfrak{m}_x/\mathfrak{m}_x^2)$.

We now retun to the general situation. We first note that $T_{X/S,x}$ is a covariant functor from the category of free $\mathscr{O}_S$-modules of finite rank to that of functors over S. In particular, $\mathbb{O}_S$ is a set of perators of the functor $T_{X/S,x}(\mathscr{M})$, which respects the functoriality on $\mathscr{M}$.

Proposition 2.2.17. *The formulation of $T_{X/S}(\mathscr{M})$ and $T_{X/S,x}(\mathscr{M})$ commutes with base changes: for any S-scheme S', we have functorial isomorphisms*

$$T_{X_{S'}/S'}(\mathscr{M} \otimes \mathscr{O}_S) \xrightarrow{\sim} T_{X/S}(\mathscr{M})_{S'},$$
$$T_{X_{S'}/S',x'}(\mathscr{M} \otimes \mathscr{O}_S) \xrightarrow{\sim} T_{X/S,x}(\mathscr{M})_{S'}$$

where $x' = x_{S'}$.

Proof. This follows from the fact that $\mathcal{H}om$ commutes with base changes. □

Corollary 2.2.18. *The X-functor $T_{X/S}(\mathscr{M})$ (resp. the S-functor $T_{X/S,x}(\mathscr{M})$) is naturally endowed with an $\mathbb{O}_X$-object (resp. $\mathbb{O}_S$-object) structure, which is functorial on $\mathscr{M}$, and the isomorphism of Proposition 2.2.17 are isomorphism of $\mathbb{O}_{X_{S'}}$-objects (resp. $\mathbb{O}_{S'}$-objects).*

Proof. We first prove the case for $T_{X/S,x}(\mathscr{M})$. For any S' over S, $\mathbb{O}(S')$ acts on $\mathscr{M} \otimes \mathscr{O}_{S'}$, and hence on $T_{X_{S'}/S',x'}(\mathscr{M} \otimes \mathscr{O}_{S'}) = T_{X/S,x}(\mathscr{M})_{S'}$. It is easy to verify that this operation is functorail on S', so $T_{X/S,x}(\mathscr{M})$ is endowed with an $\mathbb{O}_S$-object structure.

For $T_{X/S}(\mathscr{M})$ this is more complicated. For each X' over X, put $T_{X/S}(\mathscr{M})_{X'} = T_{X/S}(\mathscr{M}) \times_X X'$; we need to endow $T_{X/S}(\mathscr{M})_{X'}(X') = \mathrm{Hom}_X(X', T_{X/S}(\mathscr{M}))$ with a structure of $\mathbb{O}(X')$-set which is functorial in X'. For this we construct the following diagram, where $X_{X'} = X \times_S X'$ and f' is the section of $X_{X'}$ over X' defined by $f : X' \to X$:

$$\begin{tikzcd}
 & & T_{X_{X'}/X'}(\mathscr{M}) \arrow[dll] \arrow[dd] & & \\
T_{X/S}(\mathscr{M}) \arrow[dd] & & & & T_{X/S}(\mathscr{M})_{X'} \arrow[ull] \arrow[llll] \arrow[dd] \\
 & & X_{X'} \arrow[dll] & & \\
X \arrow[drr] & & & & X' \arrow[llll, "f"'] \arrow[ull, "f'"'] \arrow[dll] \\
 & & S & &
\end{tikzcd}$$

This diagram, together with Remark 2.2.14, shows that $T_{X/S}(\mathscr{M})_{X'}(X')$ is identified with

$$T_{X_{X'}/X',f'}(\mathscr{M})(X') = \{X'\text{-morphisms } \psi : I_{X'}(\mathscr{M}) \to X_{X'} \text{ such that } \psi \circ \varepsilon_{\mathscr{M}} = f'\}, \qquad (2.2.7)$$

over which any $a \in \mathbb{O}(X')$ operates via the action over $I_{X'}(\mathscr{M})$, i.e. with the notations of 2.2.1.2, we have $a\psi = \psi \circ a^*$, so for any $X'' \to X'$ and $x \in I_{X'}(\mathscr{M})(X'')$, $(a\psi)(x) = \psi(x \cdot a)$. We then verify that this construction is functorial on X'. □

Remark 2.2.19. The operation of $\mathbb{O}_X$ over $T_{X/S}(\mathscr{M})$ can be simply defined as follows. For any $f : X' \to X$, by (2.2.7) we have[1]

$$\mathrm{Hom}_X(X', T_{X/S}(\mathscr{M})) = T_{X/S}(\mathscr{M})_{X'}(X') = \{\phi \in \mathrm{Hom}_S(I_{X'}(\mathscr{M}), X) \mid \phi \circ \varepsilon_{\mathscr{M}} = f\},$$

and we have seen in Remark 2.2.6 that $I_{X'}(\mathscr{M})$, considered as an S-functor, is endowed with an operation by the monoid $\mathbb{O}(X')$ which conserve the zero section $\varepsilon_{\mathscr{M}} : X' \to I_{X'}(\mathscr{M})$. Therefore, if we denote by a^* the endomorphism of $I_{X'}(\mathscr{M})$ defined by $a \in \mathbb{O}(X')$, then we have $a^*\phi = \phi \circ a$, which means for any $S' \to S$ and $(m, x') \in \mathrm{Hom}_S(S', I_S(\mathscr{M}) \times_S X')$,

$$(a\phi)(m, x') = \phi(m \cdot a(x'), x')$$

(note that $a^* \circ \varepsilon_{\mathscr{M}} = \varepsilon_{\mathscr{M}}$, whence $(a\phi) \circ \varepsilon_{\mathscr{M}} = f$). Similarly, the operation of $\mathbb{O}_S$ over $T_{X/S,x}(\mathscr{M})$ can be described as follows. For any $t : S' \to S$, $T_{X/S,x}(\mathscr{M})(S')$ is the set of S-morphisms $\phi : I_{S'}(\mathscr{M}) \to X$ such that $\phi \circ \varepsilon_{\mathscr{M}} = u \circ t$; for such a ϕ and $a \in \mathbb{O}(S')$, we have $a\phi = \phi \circ a^*$.

[1] If X' is representable, this equality can also be deduced from Remark 2.2.11 and the equivalence $\widehat{\mathbf{Sch}}_{/X} \xrightarrow{\sim} \widehat{\mathbf{Sch}_{/X}}$. In fact, the equivalence $\alpha : \widehat{\mathbf{Sch}}_{/X} \to \widehat{\mathbf{Sch}_{/X}}$ commutes with Yoneda embedding, so we have

$$\mathrm{Hom}_X(X', T_{X/S}(\mathscr{M})) \cong \mathrm{Hom}_X(X', \alpha(T_{X/S}(\mathscr{M}))) = \alpha(T_{X/S}(\mathscr{M}))(X') = \{\phi \in \mathrm{Hom}_S(I_{X'}(\mathscr{M}), X) : \pi_{\mathscr{M}}(\phi) = f\}.$$

and Remark 2.2.11 shows that $\pi_{\mathscr{M}}(\phi) = \phi \circ \varepsilon_{\mathscr{M}}$.

Let S be a scheme and X be an S-functor. We say that ***X* satisfies conditon (E) relative to *S*** if, for any $S' \to S$ and any free $\mathscr{O}_{S'}$-module $\mathscr{M}$ and $\mathscr{N}$ of finite rank, the diagram of sets

$$\begin{tikzcd} & X(I_{S'}(\mathscr{M}\oplus\mathscr{N})) \arrow[dl] \arrow[dr] & \\ X(I_{S'}(\mathscr{M})) \arrow[dr] & & X(I_{S'}(\mathscr{N})) \arrow[dl] \\ & X(S') & \end{tikzcd}$$

obtained by applying X to the diagram (2.2.4), is Cartesian. Equivalently, this means the functor $\mathscr{M} \mapsto T_{X/S}(\mathscr{M})$ transforms direct sums of free $\mathscr{O}_S$-modules of finite rank to products of X-functors. If this is the case, the same holds for the functor $\mathscr{M} \mapsto T_{X/S,x}(\mathscr{M}) = S \times_X T_{X/S}(\mathscr{M})$, for any $x \in \Gamma(X/S)$. By Proposition 2.2.7, we see that any representable functor satisfies condition (E).

We often say that "X/S satisfies condition (E)" to abbreviate that X satisfies condition (E) relative to S. In this case, the functor $\mathscr{M} \mapsto T_{X/S}(\mathscr{M})$ commutes with products, hence transforms groups to groups. In particular, $T_{X/S}(\mathscr{M})$ is an abelian X-group, and for the same reason $T_{X/S,x}(\mathscr{M})$ is an abelian S-group.

Proposition 2.2.20. *If X/S satisfies condition (E), the abelian group structure over $T_{X/S}(\mathscr{M})$ (resp. $T_{X/S,x}(\mathscr{M})$) and the operation of $\mathbb{O}_X$ (resp. $\mathbb{O}_S$) endow $T_{X/S}(\mathscr{M})$ (resp. $T_{X/S,x}(\mathscr{M})$) with the structure of an $\mathbb{O}_X$-module (resp. $\mathbb{O}_S$-module).*

Proof. The operation of $\mathbb{O}_X$ (resp. $\mathbb{O}_S$) is functorial on $\mathscr{M}$, so it respects the abelian group structure induced by the functoriality of $\mathscr{M}$. In fact, retain the notations of Remark 2.2.12. The structure of (abelian) X-group of $T_{X/S}(\mathscr{M})$ is deduced by the composition

$$T_{X/S}(\mathscr{M}) \times_X T_{X/S}(\mathscr{M}) \cong T_{X/S}(\mathscr{M}\oplus\mathscr{M}) \xrightarrow{m} T_{X/S}(\mathscr{M}),$$

and on the other hand the morphism

$$T_{X/S}(\mathscr{M}) \xrightarrow{\delta} T_{X/S}(\mathscr{M}\oplus\mathscr{M}) \cong T_{X/S}(\mathscr{M}) \times_X T_{X/S}(\mathscr{M})$$

is the diagonal morphism. We then deduce from the equality (2.2.6) and Remark 2.2.12 that

$$\lambda(x+y) = \lambda x + \lambda y, \quad (\lambda+\mu)x = \lambda x + \mu x,$$

for any $f : X' \to X$, $x, y \in \mathrm{Hom}_X(X', T_{X/S}(\mathscr{M}))$ and $\lambda, \mu \in \mathbb{O}(X')$. ☐

Remark 2.2.21. If X is representable, then it satisfies (E) and $T_{X/S}$ and $T_{X/S,x}$ are represented by vector bundles. The previous laws are the same as those which are deduced from the vector bundle structures.

Proposition 2.2.22. *If X/S satisfies condition (E), then $X_{S'}/S'$ satisfies condition (E) and the isomorphisms of Proposition 2.2.20 respects the $\mathbb{O}_{X_{S'}}$-module (resp. $\mathbb{O}_{S'}$-module) structure.*

Proof. The formulation of $I_S(\mathscr{M})$ commutes with base change, so the first assertion is immediate. The second one follows from the proof of Proposition 2.2.20. ☐

Proposition 2.2.23. *The functors $T_{X/S}(\mathscr{M})$ and $T_{X/S,x}(\mathscr{M})$ are functorial on X, which means if $f : X \to X'$ is an S-morphism, we have commutative diagrams*

$$\begin{tikzcd} T_{X/S}(\mathscr{M}) \arrow[r, "T(f)"] \arrow[d] & T_{X'/S}(\mathscr{M}) \arrow[d] \\ X \arrow[r] & X' \end{tikzcd} \qquad \begin{tikzcd} T_{X/S,x}(\mathscr{M}) \arrow[rr, "T_x(f)"] \arrow[dr] & & T_{X'/S,f\circ x}(\mathscr{M}) \arrow[dl] \\ & S & \end{tikzcd}$$

Moreover, if f is a monomorphism, so are $T(f)$ and $T_x(f)$.

Proof. The existence of $T(f)$ and $T_x(f)$, as well as the last assertion, follow immediately from definition. The commutativity of the diagrams then follows from the functoriality of these morphisms with respect to $\mathscr{M}$ and of the fact that $X = T_{X/S}(0)$. $\square$

Remark 2.2.24. In the situation of Proposition 2.2.23, suppose that X and X' are representable and r is the rank of the free $\mathscr{O}_S$-module $\mathscr{M}$. Then by Corollary 2.2.8, $T_{X/S}(\mathscr{M})$ is isomorphic to the product over X of r copies of $\mathbb{V}(\Omega^1_{X/S})$, and similarly for $T_{X'/S}(\mathscr{M})$. Therefore, the square in Proposition 2.2.23 are Cartesian if f is an open immersion, of more generally if $f^*(\Omega^1_{X'/S}) = \Omega^1_{X/S}$ (for example if f is étale). In this case, we have an isomorphism of S-functors

$$T_{X/S,x}(\mathscr{M}) \xrightarrow{\sim} T_{X'/S,f\circ x}(\mathscr{M}).$$

More generally, the Cartesian square of Proposition 2.2.23 defines a morphism of X-functors

$$\begin{tikzcd} T_{X/S}(\mathscr{M}) \arrow[rr] \arrow[dr] & & T_{X'/S}(\mathscr{M}) \times_{X'} X \arrow[dl] \\ & X & \end{tikzcd}$$

Proposition 2.2.25. *Let $f : X \to X'$ be an S-morphism. If X and X' satisfy condition (E) relative to S, then*

$$T_{X/S}(\mathscr{M}) \xrightarrow{T(f)} T_{X'/S}(\mathscr{M})_X \qquad (\textit{resp.}\quad T_{X/S,x}(\mathscr{M}) \xrightarrow{T_x(f)} T_{X'/S,f\circ x}(\mathscr{M}))$$

is a morphism of $\mathbb{O}_X$-modules (resp. $\mathbb{O}_S$-modules).

Proof. This follows from Proposition 2.2.23 by the functoriality on $\mathscr{M}$. $\square$

Proposition 2.2.26. *Let X and Y be functors over S. We have isomorphisms functorial on $\mathscr{M}$:*

$$T_{X/S}(\mathscr{M}) \times_S T_{Y/S}(\mathscr{M}) \xrightarrow{\sim} T_{(X\times_S Y)/S}(\mathscr{M}), \tag{2.2.8}$$
$$T_{X/S,x}(\mathscr{M}) \times_S T_{Y/S,y}(\mathscr{M}) \xrightarrow{\sim} T_{(X\times_S Y)/S,(x,y)}(\mathscr{M}), \tag{2.2.9}$$

Proof. The first isomorphism follows from (2.2.3), and the second one is deduced by base change via $(x, y) : S \to X \times_S Y$. $\square$

Corollary 2.2.27. *If X/S is endowed with an algebraic structure defined by finite Cartesian products, then $T_{X/S}(\mathscr{M})$ is endowed with the same structure and the projection $T_{X/S}(\mathscr{M}) \to X$ is a morphism of that structure.*

Proposition 2.2.28. *If X/S and Y/S satisfy condition (E), then $(X \times_S Y)/S$ satisfies condition (E) and (2.2.8) (resp. (2.2.9)) is an isomorphism of $\mathbb{O}_{X\times_S Y}$-modules (resp. $\mathbb{O}_S$-modules).*

Proof. Suppose that X/S and Y/S satisfy condition (E). Then by (2.2.8), so does $(X \times_S Y)/S$. Let $(x, y) : Z \to X \times_S Y$ be an S-morphism. To see that (2.2.8) is a morphism of $\mathbb{O}_{X\times_S Y}$-modules, in view of Remark 2.2.19, it suffices to show that the map

$$\begin{aligned}\{\phi \in \mathrm{Hom}_S(I_Z(\mathscr{M}), X) : \phi \circ \varepsilon_{\mathscr{M}} = x\} \times \{\psi \in \mathrm{Hom}_S(I_Z(\mathscr{M}), Y) : \psi \circ \varepsilon_{\mathscr{M}} = y\} \\ \to \{\theta \in \mathrm{Hom}_S(I_Z(\mathscr{M}), X \times_S Y) : \theta \circ \varepsilon_{\mathscr{M}} = (X, y)\}\end{aligned}$$

which to (ϕ, ψ) associated $\phi \times \psi$, is a morphism of $\mathbb{O}(Z)$-modules. But this is immediate, since for $a \in \mathbb{O}(Z)$ we have $a \cdot (\phi, \psi) = (\phi \circ a^*, \psi \circ a^*)$, and

$$(\phi \circ a^*) \times (\psi \circ a^*) = (\phi \times \psi) \circ a^* = a \cdot (\phi \times \psi).$$

Similarly, by using Remark 2.2.14, we can show that (2.2.9) is a morphism of O_S-modules. $\square$

If X is an S-group and $e : S \to X$ is the unit section, we define

$$\mathfrak{Lie}(X/S, \mathscr{M}) = T_{X/S,e}(\mathscr{M}),$$

that is, $\mathfrak{Lie}(X/S, \mathscr{M})$ is defined by the Cartesian square

$$\begin{array}{ccc} \mathfrak{Lie}(X/S,\mathscr{M}) & \xrightarrow{i} & T_{X/S}(\mathscr{M}) \\ \downarrow & & \downarrow{\scriptstyle \pi} \\ S & \xrightarrow{e} & X \end{array}$$

By Corollary 2.2.27, the projection $\pi : T_{X/S}(\mathscr{M}) \to X$ is a morphism of S-groups, and it then follows that $\mathfrak{Lie}(X/S, \mathscr{M})$ is endowed with an S-group structure, and is isomorphic via i to the kernel of π.

If, moreover, X/S satisfies condition (E), we shall see in Proposition 2.2.29 that the S-group structure of $\mathfrak{Lie}(X/S, \mathscr{M})$, induced by that of X, coincides with the abelian group structure induced by functoriality of $\mathscr{M}$. To this end we introduce the following terminology: an **H-set** is a set X endowed with a composition law with a two-sided unit, denoted by e_X or simply e. If $f : X \to Y$ is a morphism of H-sets, its kernel $\ker f$ is defined to be $f^{-1}(e_Y)$, which is a sub-H-set of X.

An H-object in a category $\mathcal{C}$ is defined by the usual manner: this is an object X of $\mathcal{C}$, endowed with a morphism $X \times X \to X$ such that there exists a section of X (over the final object) possessing the property of being a two-sided unit. Any $\mathcal{C}$-monoid, and in particular any $\mathcal{C}$-group is therefore an H-object. In particular, an H-object of the category of functors over a scheme S is called an ***S*-H-functor**. If X is an S-H-functor (for example, an S-group), and $e : S \to X$ is the unit section of X, we define

$$\mathfrak{Lie}(X/S, \mathscr{M}) = T_{X/S,e}(\mathscr{M}), \quad \mathfrak{Lie}(X/S) = \mathfrak{Lie}(X/S, \mathscr{O}_S).$$

By Corollary 2.2.27, we see that $T_{X/S}(\mathscr{M})$ and $\mathfrak{Lie}(X/S, \mathscr{M})$ are also S-H-functors, and we have morphisms of S-H-functors

$$\mathfrak{Lie}(X/S, \mathscr{M}) \xrightarrow{i} T_{X/S}(\mathscr{M}) \underset{\tau}{\overset{\pi}{\rightleftarrows}} X \tag{2.2.10}$$

where i is an isomorphism from $\mathfrak{Lie}(X/S, \mathscr{M})$ to $\ker \pi$ and τ is a section of π.

Proposition 2.2.29. *Let X be an S-H-object satisfying condition (E) relative to S. Then the S-H-object structure of $\mathfrak{Lie}(X/S, \mathscr{M})$ induced by that of X coincides with the S-group structure induced by functoriality on $\mathscr{M}$.*

Since X satisfies condition (E), we see that $\mathfrak{Lie}(X/S, \mathscr{M})$ is an H-object in the category of $\mathbb{O}_S$-modules. The proposition then follows from the following lemma:

Lemma 2.2.30. *Let $\mathcal{C}$ be a category. Let G be an H-object in the category of $\mathcal{C}$-H-objects (i.e. G is a $\mathcal{C}$-H-object endowed with a morphism of $\mathcal{C}$-H-objects $h : G \times G \to G$). Then h coincides with the composition law of G and is commutative.*

Proof. By taking the values of the functors on a variable argument, we are reduced to the case where $\mathcal{C}$ is the category of sets. We then have a set G and two maps $f, h : G \times G \to G$ such that

$$h(f(x,y), f(z,t)) = f(h(x,z), h(y,t)), \tag{2.2.11}$$

and we have two elements e, u of G such that $f(e,x) = f(X,e) = x$ and $h(u,x) = h(x,u) = x$. This is the famous Eckmann-Hilton argument[2], which we now provide a proof. We first note that by (2.2.11),

$$h(f(u,y), f(x,u)) = f(x,y) = h(f(x,u), f(u,y)). \tag{2.2.12}$$

[2]This argument is used to prove, for example, that higher homotopy groups are abelian.

In particular, for $y = e$ (resp. $x = e$), we obtain, respectively,

$$x = f(x,e) = h(f(u,e), f(x,u)) = h(u, f(x,u)) = f(x,u),$$
$$y = f(e,y) = h(f(e,u), f(u,y)) = h(u, f(u,y)) = f(u,y),$$

whence the equality $h(y,x) = f(x,y) = h(x,y)$ in view of (2.2.12). This proves the lemma, whence Proposition 2.2.29. □

Remark 2.2.31. The assertion of Proposition 2.2.29 can also be interpreted as follows: if we endow $\mathfrak{Lie}(X/S, \mathscr{M})$ with the abelian group structure induced by functoriality on $\mathscr{M}$, then the morphism $i : \mathfrak{Lie}(X/S, \mathscr{M}) \to T_{X/S}(\mathscr{M})$ is a morphism of S-H-objects.

Corollary 2.2.32. *If X is an S-H-functor satisfying condition (E) relative to S, any element of $X(I_S(\mathscr{M}))$, which projects to the unit element of $X(S)$, is invertible.*

Proof. This follows from the sequence (2.2.10) and Proposition 2.2.29, since $\mathfrak{Lie}(X/S, \mathscr{M})$ is a group hence any element has an inverse. □

Corollary 2.2.33. *If X is an S-monoid satisfying condition (E) relative to S, an element of $X(I_S(\mathscr{M}))$ is invertible if and only if its image in $X(S)$ is invertible.*

Proof. One direction is immediate, so assume that $x \in X(I_S(\mathscr{M}))$ is an element whose projection s to $X(S)$ is invertible in $X(S)$. Let s^{-1} be the inverse of s in $X(S)$, then $y = x\tau(s^{-1}) = x\tau(s)^{-1}$ is projective to the unit element of $X(S)$, and hence is invertible in $X(I_S(\mathscr{M}))$. If y^{-1} is this inverse, we then have

$$x \cdot \tau(s)^{-1} y^{-1} = (x\tau(s)^{-1}) \cdot (x\tau(s)^{-1})^{-1} = e,$$

so x is right invertible. Similarly, by considering $y' = \tau(s^{-1})x = \tau(s)^{-1}x$, we see that x is also left invertible, so it is invertible in $X(I_S(\mathscr{M}))$. □

Corollary 2.2.34. *If X is an S-group satisfying condition (E) relative to S, the two S-group laws on $\mathfrak{Lie}(X/S, \mathscr{M})$ coincide.*

Corollary 2.2.35. *Let G be an S-group satisfying condition (E) relative to S. For $n \in \mathbb{Z}$, let $n_G : G \to G$ be the morphism of S-functors defined by $g \mapsto g^n$. Then the induced morphism $\mathfrak{Lie}(n_G) : \mathfrak{Lie}(G/S) \to \mathfrak{Lie}(G/S)$ is the multiplication by n, i.e. the map which to any $x \in \mathfrak{Lie}(G/S)(S')$ associates nx.*

Proof. We first note that n_G is in general not a morphism of groups, but it perverses the unit section $e : S \to G$, hence the induced morphism $\mathfrak{Lie}(n_G) = T_e(n_G)$ sends $\mathfrak{Lie}(G/S)$ into itself. If we denote by $i : \mathfrak{Lie}(G/S) \to T_{G/S}$ the inclusion, then $\mathfrak{Lie}(n_G)$ is defined by the equality $i(\mathfrak{Lie}(n_G)(x)) = i(x)^n$, for any $S' \to S$ and $x \in \mathfrak{Lie}(G/S)(S')$. Now by Remark 2.2.31 we have $i(x)^n = i(nx)$, whence $\mathfrak{Lie}(n_G)(x) = nx$. □

Before deducing other consequences from Proposition 2.2.29, let us prove another result of functoriality:

Proposition 2.2.36. *In the situation of 2.2.1.1, we have a functorial isomorphism on $\mathscr{M}$:*

$$T_{\mathcal{H}om_{Z/S}(X,Y)/S}(\mathscr{M}) \xrightarrow{\sim} \mathcal{H}om_{Z/S}(X, T_{Y/Z}(\mathscr{M})).$$

Proof. In fact, by definition we have

$$T_{\mathcal{H}om_{Z/S}(X,Y)/S}(\mathscr{M}) = \mathcal{H}om_S(I_S(\mathscr{M}), \mathcal{H}om_{Z/S}(X,Y)) \cong \mathcal{H}om_{Z/S}(X, \mathcal{H}om_Z(Z \times_S I_S(\mathscr{M}), Y)),$$

where we have used the isomorphism (2.2.1) with $T = I_S(\mathscr{M})$. In view of the isomorphism $Z \times_S I_S(\mathscr{M}) \cong I_Z(\mathscr{M})$, we then obtain

$$T_{\mathcal{H}om_{Z/S}(X,Y)/S}(\mathscr{M}) \cong \mathcal{H}om_{Z/S}(X, \mathcal{H}om_Z(I_Z(\mathscr{M}), Y)) = \mathcal{H}om_{Z/S}(X, T_{Y/Z}(\mathscr{M})). \qquad \square$$

Corollary 2.2.37. *If Y/Z satisfies condition (E), then $\mathcal{H}om_{Z/S}(X,Y)/S$ satisfies condition (E) and the isomorphism of Proposition 2.2.36 respects the $\mathbb{O}$-module structure over $\mathcal{H}om_{Z/S}(X,Y)$.*

Proof. Let $\mathscr{M}, \mathscr{N}$ be two free $\mathscr{O}_S$-modules of finite rank. If Y/Z satisfies condition (E), then

$$T_{Y/Z}(\mathscr{M} \oplus \mathscr{N}) \cong T_{Y/Z}(\mathscr{M}) \times_Y T_{Y/Z}(\mathscr{N}).$$

The right side is a sub-functor of $T_{Y/Z}(\mathscr{M}) \times_S T_{Y/Z}(\mathscr{N})$ and via the isomorphism (2.2.3), we obtain an isomorphism

$$\mathcal{H}om_{Z/S}(X, T_{Y/Z}(\mathscr{M} \oplus \mathscr{N})) \cong \mathcal{H}om_{Z/S}(X, T_{Y/Z}(\mathscr{M})) \times_{\mathcal{H}om_{Z/S}(X,Y)} \mathcal{H}om_{Z/S}(X, T_{Y/Z}(\mathscr{N})).$$

Combined with Proposition 2.2.36, this implies

$$T_{\mathcal{H}om_{Z/S}(X,Y)/S}(\mathscr{M} \oplus \mathscr{N}) \cong T_{\mathcal{H}om_{Z/S}(X,Y)/S}(\mathscr{M}) \times_{\mathcal{H}om_{Z/S}(X,Y)} T_{\mathcal{H}om_{Z/S}(X,Y)/S}(\mathscr{N}),$$

so $\mathcal{H}om_{Z/S}(X,Y)$ satisfies condition (E).

For the second assertion, let $H = \mathcal{H}om_{Z/S}(X,Y)$ and consider an S-morphism $\Delta : H' \to \mathcal{H}om_{Z/S}(X,Y)$, that is, an Z-morphism $\delta : H' \times_S X \to Y$, which makes $H' \times_S X$ a Y-object. We then have a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_H(H', \mathcal{H}om_{Z/S}(X, T_{Y/Z}(\mathscr{M}))) \arrow[r, hook] \arrow[d, equal] & \mathrm{Hom}_S(H', \mathcal{H}om_{Z/S}(X, T_{Y/Z}(\mathscr{M}))) \arrow[d, equal] \\
\mathrm{Hom}_Y(H' \times_S X, T_{Y/Z}(\mathscr{M})) \arrow[r, hook] \arrow[d, equal] & \mathrm{Hom}_Z(H' \times_S X, T_{Y/Z}(\mathscr{M})) \arrow[d, equal] \\
\{\psi \in \mathrm{Hom}_Z(I_{H' \times_S X}(\mathscr{M}), Y) : \psi \circ \varepsilon_{\mathscr{M}} = \delta\} \arrow[r, hook] & \mathrm{Hom}_Z(I_{H' \times_S X}(\mathscr{M}), Y).
\end{tikzcd}$$

By Remark 2.2.3, the action of $\alpha \in \mathbb{O}(H' \times_S X)$ over $\Psi \in \mathrm{Hom}_Y(H' \times_S X, T_{Y/Z}(\mathscr{M}))$ is given as follows: for any $U \to S$ and $(h,x) \in \mathrm{Hom}_S(U, H' \times_S X)$ (U is then an Y-object via $\delta \circ (h,x)$), we have

$$(\alpha\Psi)(h,x) = \alpha(h,x)\Psi(h,x),$$

where $\alpha(h,x) \in \mathbb{O}(U)$ acts on $\Psi(h,x) \in T_{Y/Z}(\mathscr{M})(U)$ via the $\mathbb{O}_Y$-module structure of $T_{Y/Z}(\mathscr{M})$. By Remark 2.2.19, the latter is given, via the identification

$$\mathrm{Hom}_Y(H' \times_S X, T_{Y/Z}(\mathscr{M})) = \{\psi \in \mathrm{Hom}_Z(I_{H' \times_S X}(\mathscr{M}), Y) : \psi \circ \varepsilon_{\mathscr{M}} = \delta\},$$

by the following: for any $(m,h,x) \in \mathrm{Hom}_S(U, I_S(\mathscr{M}) \times_S H' \times_S X)$,

$$(\alpha\psi)(m,h,x) = \psi(m \cdot \alpha(h,x), h, x). \tag{2.2.13}$$

On the other hand, consider the tangent space $T_{H/S}(\mathscr{M}) = \mathcal{H}om_S(I_S(\mathscr{M}), H)$; we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_H(H', T_{H/S}(\mathscr{M})) \arrow[r, hook] \arrow[d, equal] & \mathrm{Hom}_S(H', T_{H/S}(\mathscr{M})) \arrow[d, equal] \\
\{\Phi \in \mathrm{Hom}_S(I_{H'}(\mathscr{M}), H) : \Phi \circ \varepsilon_{\mathscr{M}} = \Delta\} \arrow[r, hook] \arrow[d, equal, "(*)"] & \mathrm{Hom}_S(I_{H'}(\mathscr{M}), H) \arrow[d, equal] \\
\{\phi \in \mathrm{Hom}_Z(I_{H' \times_S X}(\mathscr{M}), Y) : \phi \circ \varepsilon_{\mathscr{M}} = \delta\} \arrow[r, hook] & \mathrm{Hom}_Z(I_{H' \times_S X}(\mathscr{M}), Y)
\end{tikzcd}$$

where the bijection $(*)$ is given as follows: for any $U \to S$ and $(m,h,x) \in \mathrm{Hom}(U, I_S(\mathscr{M}) \times_S H' \times_S X)$ (so that U is over Z via $U \xrightarrow{x} X \to Z$), we have $\Phi(m,h) \in \mathrm{Hom}_Z(X \times_S U, Y)$ and

$$\phi(m,h,x) = \Phi(m,h) \circ (x \times \mathrm{id}_U) \in \mathrm{Hom}_Z(U,Y). \tag{2.2.14}$$

By Remark 2.2.19 (where we replace X by $\mathcal{H}om_{Z/S}(X,Y)$ and X' by H'), the action of $a \in \mathbb{O}(H')$ over $\Phi \in \mathrm{Hom}_S(I_{H'}(\mathscr{M}), H)$ is given by

$$(a\Phi)(m,h) = \Phi(m \cdot a(h), h)$$

where $U \to S$ and $(m,h) \in \mathrm{Hom}_S(U, I_S(\mathscr{M}) \times_S H')$. Therefore, if ϕ (resp. $a\phi$) is the element of $\mathrm{Hom}_Z(I_{H' \times_S X}(\mathscr{M}), Y)$ associated with Φ (resp $a\Phi$), we have, by (2.2.14),

$$(a\Phi)(m,h,x) = \Phi(m \cdot a(h), h) \circ (x \times \mathrm{id}_U) = \phi(m \cdot a(h), h, x). \tag{2.2.15}$$

Together with (2.2.13), this shows that the isomorphism $T_{H/S}(\mathscr{M}) \xrightarrow{\sim} \mathcal{H}om_{Z/S}(X, T_{Y/Z}(\mathscr{M}))$ of Proposition 2.2.36 is an isomorphism of $\mathbb{O}(H)$-modules. Moreover, for any $H' \to H$, the $\mathbb{O}(H')$-module structure of $\mathrm{Hom}_H(H', T_{H/S}(\mathscr{M}))$ extends, in a functorial way on H', to an $\mathbb{O}(H' \times_S X)$-module structure. □

In particular, for $Z = S$, we obtain the following corollary:

Corollary 2.2.38. *We have a functorial isomorphism on $\mathscr{M}$:*

$$T_{\mathcal{H}om_S(X,Y)/S}(\mathscr{M}) \xrightarrow{\sim} \mathcal{H}om_S(X, T_{Y/S}(\mathscr{M})).$$

Moreover, if Y/S satisfies condition (E), then $\mathcal{H}om_S(X,Y)/S$ satisfies condition (E) and the preceding isomorphism respects the $\mathbb{O}$-module structure over $\mathcal{H}om_S(X,Y)$.

Let $u : X \to Y$ be an S-morphism, which can be identified with a constant morphism $\boldsymbol{u} : S \to \mathcal{H}om_S(X,Y)$ such that $\boldsymbol{u}(f) = u_{S'}$ for any $f : S' \to S$. The fiber product of $\boldsymbol{u}$ and $\mathcal{H}om_S(X, T_{Y/S}(\mathscr{M})) \to \mathcal{H}om_S(X,Y)$ is then identified with $\mathcal{H}om_{Y/S}(X, T_{Y/S}(\mathscr{M}))$, where X is over Y via u. Therefore, we deduce from the definition of $T_{\mathcal{H}om_S(X,Y)/S,\boldsymbol{u}}(\mathscr{M})$ and Corollary 2.2.38 the following:

Corollary 2.2.39. *Let $u : X \to Y$ be an S-morphism. We have a functorial isomorphism on $\mathscr{M}$ (where X is over Y via u):*

$$T_{\mathcal{H}om_S(X,Y)/S,\boldsymbol{u}}(\mathscr{M}) \xrightarrow{\sim} \mathcal{H}om_{Y/S}(X, T_{Y/S}(\mathscr{M})).$$

This is an isomorphism of $\mathbb{O}_S$-modules if Y/S satisfies condition (E).

In particular, for $Y = X$, $\mathcal{E}nd_S(X)$ is an S-functor in monoids, hence a fortiori an S-H-functor. Since $\mathfrak{Lie}(\mathcal{E}nd_S(X)/S, \mathscr{M})$ is by definition $T_{\mathcal{E}nd_S(X)/S,e}(\mathscr{M})$, where e is the unit section, we obtain (recall that $\mathcal{H}om_{X/S}(X, T_{X/S}(\mathscr{M})) \cong \mathrm{Res}_{X/S} T_{X/S}(\mathscr{M})$):

Corollary 2.2.40. *We have a functorial isomorphism on $\mathscr{M}$:*

$$\mathfrak{Lie}(\mathcal{E}nd_S(X)/S, \mathscr{M}) \xrightarrow{\sim} \mathrm{Res}_{X/S} T_{X/S}(\mathscr{M}).$$

This is an isomorphism of $\mathbb{O}_S$-modules if X/S satisfies condition (E).

Remark 2.2.41. Suppose that X/S satisfies condition (E). Then the functor $\mathrm{Res}_{X/S} T_{X/S}(\mathscr{M}) = \mathcal{H}om_{X/S}(X, T_{X/S}(\mathscr{M}))$ is endowed with a $\mathrm{Res}_{X/S}\mathbb{O}_X$-module structure, i.e. for any $S' \to S$,

$$\mathcal{H}om_{X/S}(X, T_{X/S}(\mathscr{M}))(S') = \{\psi \in \mathrm{Hom}_X(I_{S'}(\mathscr{M}) \times_S X, X) : \psi \circ (\varepsilon_{\mathscr{M}} \times \mathrm{id}_X) = \mathrm{pr}_X\}$$

is endowed with a $\mathbb{O}(X \times_S S')$-module structure, which is functorial on S'. This follows either from Proposition 2.2.20 and the properties of the functor $\mathrm{Res}_{X/S}$, or from the proof of Corollary 2.2.37.

We now give a geometric interpretation of the tangent bundle. Let U be an S-functor; by (??), we have isomorphism functorial on $\mathscr{M}$:

$$\begin{aligned} T_{X/S}(\mathscr{M})(U) &= \mathrm{Hom}_S(U, \mathcal{H}om_S(I_S(\mathscr{M}), X)) \cong \mathrm{Hom}_S(I_S(\mathscr{M}), \mathcal{H}om_S(U, X)) \\ &= \mathrm{Hom}_{I_S(\mathscr{M})}(U_{I_S(\mathscr{M})}, X_{I_S(\mathscr{M})}). \end{aligned}$$

In particular, the morphism $\mathscr{M} \to 0$ induces a commutative diagram

$$\begin{tikzcd} \mathrm{Hom}_S(U, T_{X/S}(\mathscr{M})) \arrow[r, "\sim"] \arrow[d, "\circ\pi_{\mathscr{M}}"] & \mathrm{Hom}_{I_S(\mathscr{M})}(U_{I_S(\mathscr{M})}, X_{I_S(\mathscr{M})}) \arrow[d] \\ \mathrm{Hom}_S(U, X) \arrow[r, equal] & \mathrm{Hom}_S(U, X) \end{tikzcd}$$

where the second vertical arrow is given by base change $\varepsilon_{\mathscr{M}} : S \to I_S(\mathscr{M})$. We therefore obtain the following proposition:

Proposition 2.2.42. *Let $h_0 : U \to X$ be an S-morphism. Then $\mathrm{Hom}_X(U, T_{X/S}(\mathscr{M}))$ is identified with the set of $I_S(\mathscr{M})$-morphisms $h : U_{I_S(\mathscr{M})} \to X_{I_S(\mathscr{M})}$ that extend h_0 (we view U (resp. X) as a sub-object of $U \times_S I_S(\mathscr{M})$ (resp. $X \times_S I_S(\mathscr{M})$) via $\mathrm{id}_U \times_S \varepsilon_{\mathscr{M}}$ (resp. $\mathrm{id}_X \times_S \varepsilon_{\mathscr{M}}$)).*

In particular, for $U = X$ and $h_0 = \mathrm{id}_X$, we obtain:

Corollary 2.2.43. *The set $\Gamma(T_{X/S}(\mathscr{M})/X)$ is identified with the set of $I_S(\mathscr{M})$-endomorphisms ϕ of $X_{I_S(\mathscr{M})}$ which induce identity on X, i.e. such that the following diagram is commutative:*

$$\begin{tikzcd} I_X(\mathscr{M}) \arrow[rr, "\phi"] & & I_X(\mathscr{M}) \\ & X \arrow[ul, "\varepsilon_{\mathscr{M}}"] \arrow[ur, "\varepsilon_{\mathscr{M}}"'] & \end{tikzcd}$$

On the other hand, by Corollary 2.2.39, $\Gamma(T_{X/S}(\mathscr{M})/X) \cong \mathfrak{Lie}(\mathcal{E}nd_S(X)/S, \mathscr{M})(S)$. If X/S satisfies condition (E), then $\mathcal{E}nd_S(X)/S$ satisfies condition (E) and $\mathfrak{Lie}(\mathcal{E}nd_S(X)/S, \mathscr{M})$ is then an $\mathbb{O}_S$-module (and in fact a $\mathrm{Res}_{X/S}\mathbb{O}_X$-module). Applying Proposition 2.2.29, we then deduce that

Proposition 2.2.44. *If X/S satisfies condition (E), the abelian group $\Gamma(T_{X/S}(\mathscr{M})/X)$ is identified with the set of $I_S(\mathscr{M})$-endomorphisms of $X_{I_S}(\mathscr{M})$ which induce identity on X. In particular, any $I_S(\mathscr{M})$-endomorphism of $X_{I_S}(\mathscr{M})$ which induces the identity on X is an automorphism.*

Corollary 2.2.45. *Let $u : X \to Y$ be an S-isomorphism with Y/S satisfying condition (E). Any $I_S(\mathscr{M})$-morphism of $X_{I_S(\mathscr{M})}$ to $Y_{I_S(\mathscr{M})}$ which extends u is an isomorphism.*

Proof. By Proposition 2.2.42 the considered set is identified with $\mathrm{Hom}_Y(X, T_{Y/S}(\mathscr{M}))$, which is isomorphic to $\Gamma(T_{Y/S}(\mathscr{M})/Y)$ by our hypothesis. □

Corollary 2.2.46. *If Y/S satisfies condition (E), the monomorphism $\mathcal{I}so_S(X,Y) \to \mathcal{H}om_S(X,Y)$ induces, for any $u \in \mathrm{Iso}_S(X,Y)$, an isomorphism*

$$T_{\mathcal{I}so_S(X,Y)/S,u}(\mathscr{M}) \xrightarrow{\sim} T_{\mathcal{H}om_S(X,Y)/S,u}(\mathscr{M}).$$

Proof. It suffices to see that $T_{\mathcal{I}so_S(X,Y)/S,u}(\mathscr{M}) \xrightarrow{\sim} T_{\mathcal{H}om_S(X,Y)/S,u}(\mathscr{M})$ is a bijection, for any $S' \to S$. By base change (cf. Proposition 2.2.26), it suffices to consider $S' = S$. In this case, we note that $T_{\mathcal{H}om_S(X,Y)/S,u}(\mathscr{M})(S)$ (resp. $T_{\mathcal{I}so_S(X,Y)/S,u}(\mathscr{M})(S)$) is the set of $I_S(\mathscr{M})$-morphisms (resp. automorphims) $X_{I_S(\mathscr{M})} \to Y_{I_S(\mathscr{M})}$ which extends u, and we can apply Corollary 2.2.45. □

Corollary 2.2.47. *If X/S satisfies (E), the monomorphism $\mathcal{A}ut_S(X) \to \mathcal{E}nd_S(X)$ induces, for any $u \in \mathcal{A}ut_S(X)$, an isomorphism $T_{\mathcal{A}ut_S(X)/S,u}(\mathscr{M}) \xrightarrow{\sim} T_{\mathcal{E}nd_S(X)/S,u}(\mathscr{M})$. In particular, we have*

$$\mathfrak{Lie}(\mathcal{A}ut_S(X)/S, \mathscr{M}) \xrightarrow{\sim} \mathfrak{Lie}(\mathcal{E}nd_S(X)/S, \mathscr{M}) \xrightarrow{\sim} \mathrm{Res}_{X/S} T_{X/S}(\mathscr{M})$$

so that $\mathfrak{Lie}(\mathcal{A}ut_S(X)/S, \mathscr{M})$ is endowed with a $\mathrm{Res}_{X/S}\mathbb{O}_X$-module structure.

Example 2.2.48. There exist functors possessing infinitesimal endomorphisms which are not automorphisms, and hence a fortiori do not satisfy condition (E). For any pointed set (E, x_0), let $M(E)$ be the free commutative monoid generated by E and $M_P(E, x_0)$ be the commutative monoid obtained by quotient $M(E)$ by the equivalence relation generated by $m \sim x_0 + m$. Then $(E, x_0) \to M_P(E, x_0)$ is the left adjoint of the forgetful functor from the category of commutative monoid to that of pointed sets. We say that $M_P(E, x_0)$ is the **free commutative monoid over the pointed set** (E, x_0).

Let X be the functor which associates any scheme S to the free commutative monoid over the set $\mathbb{O}(S)$, pointed by the zero element. A morphism $f : S \to I_{\mathbb{Z}} = \mathrm{Spec}(\mathbb{Z}[t])$ corresponds to a square zero element u_f of $\mathbb{O}(S)$, hence defines an endomorphism of $X(S)$ by $x \mapsto x + u_f$ (taken in $M_P(\mathbb{O}(S), 0)$). We thus obtain an endomorphism ϕ of $X_{I_{\mathbb{Z}}} = X \times_{\mathbb{Z}} I_{\mathbb{Z}}$, defined as follows. For any $f \in I_{\mathbb{Z}}(S)$ and $x \in X(S)$,

$$\phi(x, f) = (x + u_f, f).$$

If $f_0 : S \to I_{\mathbb{Z}}$ is the composition of the structural morphism $S \to \mathrm{Spec}(\mathbb{Z})$ and the zero section of $I_{\mathbb{Z}}$, the corresponding element $u_{f_0} = 0$, and hence $\phi(x, f_0) = (x, f_0)$ (since $x + 0 = x$ in $M_P(\mathbb{O}(S), 0)$). Since the map $X(S) \to X_{I_{\mathbb{Z}}}(S)$ is given by $x \mapsto (x, f_0)$, this shows that ϕ induces the identity on X, hence is an infinitesimal endomorphism of X which is evidently not an automorphism.

Suppose that X is representable. In this case, we have seen in Proposition 2.2.15 that the X-functor $T_{X/S}$ is represented by $\mathbb{V}(\Omega^1_{X/S})$, whence the bijections

$$\Gamma(T_{X/S}/X) \cong \mathrm{Hom}_X(\Omega^1_{X/S}, \mathscr{O}_S) \cong \mathrm{Der}_{\mathscr{O}_S}(\mathscr{O}_X). \tag{2.2.16}$$

This can also be deduced as follows. According to Proposition 2.2.44, $\Gamma(T_{X/S}/X)$ is identified with the set of **infinitesimal endomorphisms** of X (i.e. I_S-endomorphisms of X_{I_S} inducing the identity on X). Now X and X_{I_S} have the same underlying topological space, with structural sheaves being $\mathscr{O}_X$ and $\mathscr{D}_{\mathscr{O}_X} = \mathscr{O}_X \oplus \mathscr{M}$, where $\mathscr{M} = \mathscr{O}_X$ is considered as a square zero ideal. Let $\pi : \mathscr{D}_{\mathscr{O}_X} \to \mathscr{O}_X$ be the morphism of $\mathscr{O}_X$-algebras which is zero on $\mathscr{M}$, we then deduce that giving an infinitesimal endomorphism of X is equivalent to giving a morphism of $\mathscr{O}_S$-algebras $\phi : \mathscr{O}_X \to \mathscr{D}_{\mathscr{O}_X}$ such that $\pi \circ \phi = \mathrm{id}_{\mathscr{O}_X}$, which then amouts to giving an $\mathscr{O}_S$-derivation of the sheaf of rings $\mathscr{O}_X$.

Moreover, we see that if $D, D' \in \mathrm{Der}_{\mathscr{O}_S}(\mathscr{O}_X)$ and if we denote by ϕ_D the infinitesimal endomorphism correponding to D, then

$$\phi_{D+D'} = \phi_D \circ \phi_{D'}.$$

This shows that the identification

$$\{\text{infinitesimal endomorphisms of } X\} \cong \mathrm{Der}_{\mathscr{O}_S}(\mathscr{O}_X)$$

is an isomorphism of abelian groups. In view of Proposition 2.2.44 (and Remark 2.2.41), we have then isomorphism of abelian groups (as well as $\mathbb{O}(X)$-modules)

$$\Gamma(T_{X/S}/X) \xrightarrow{\sim} \mathrm{Der}_{\mathscr{O}_S}(\mathscr{O}_X)$$

which ressume the classical interpretation of tangent vectors in view of derivations of the structural sheaf. Recall also that $\Gamma(T_{X/S}/X)$ is equal to $H^0(X, \mathfrak{g}_{X/S})$, where $\mathfrak{g}_{X/S}$ is the dual of $\Omega^1_{X/S}$.

2.2.2 Tangent space of a group

Let G be a functor in groups over S. By Corollary 2.2.27, $T_{G/S}(\mathscr{M})$ and $\mathfrak{Lie}(G/S,\mathscr{M})$ are endowed with group structures over S and we have group morphisms

$$\mathfrak{Lie}(G/S,\mathscr{M}) \xrightarrow{i} T_{G/S}(\mathscr{M}) \underset{\tau}{\overset{\pi}{\rightleftarrows}} G \tag{2.2.17}$$

By definition i is an isomorphism from $\mathfrak{Lie}(G/S)(\mathscr{M})$ onto the kernel of π, and τ is a section of π. It then follows from Proposition 2.1.10 that we can identify $T_{G/S}(\mathscr{M})$ with a semi-direct product of G by $\mathfrak{Lie}(G/S,\mathscr{M})$.

Definition 2.2.49. The corresponding operation of G on $\mathfrak{Lie}(G/S,\mathscr{M})$ is denoted by

$$\mathrm{Ad} : G \to \mathcal{A}ut_{\mathbf{Grp}}(\mathfrak{Lie}(G/S,\mathscr{M}))$$

and called the adjoint representation (relative to $\mathscr{M}$) of G. For any $S' \to S$, we then have by definition, for $x \in G(S')$ and $X \in \mathfrak{Lie}(G/S,\mathscr{M})(S')$, that

$$\mathrm{Ad}(x)X = i^{-1}(\tau(x)i(x)\tau(x)^{-1}).$$

Definition 2.2.50. If G and H are two functors in groups over S and if $f : G \to H$ is a group morphism, then we have an induced morphism of exact sequences which is compatible with sections:

$$\begin{tikzcd} 1 \arrow[r] & \mathfrak{Lie}(G/S,\mathscr{M}) \arrow[r] \arrow[d, "\mathfrak{Lie}(f)"] & T_{G/S}(\mathscr{M}) \arrow[r] \arrow[d, "T(f)"] & G \arrow[r] \arrow[d, "f"] & 1 \\ 1 \arrow[r] & \mathfrak{Lie}(H/S,\mathscr{M}) \arrow[r] & T_{H/S}(\mathscr{M}) \arrow[r] & H \arrow[r] & 1 \end{tikzcd}$$

The morphism $\mathfrak{Lie}(f) = T_e(f)$ is the derived morphism of f. If G/S and H/S satisfy condition (E), then $\mathfrak{Lie}(f)$ respects the $\mathbb{O}_S$-module structure induced by functoriality on $\mathscr{M}$ (cf. Proposition 2.2.25).

Proposition 2.2.51. *Let* $g \in G(S)$*, then* $\mathrm{Ad}(g) : \mathfrak{Lie}(G/S,\mathscr{M}) \to \mathfrak{Lie}(G/S,\mathscr{M})$ *is the derived morphism of* $\mathrm{Inn}(g) : G \to G$.

Proof. In fact, $\mathrm{Ad}(g)X = i^{-1}(\mathrm{Inn}(g)i(X))$, which is none other than $T(\mathrm{Inn}(g))X$ by the definition of the derived morphism. □

Suppose that G/S satisfies condition (E). Then, by Proposition 2.2.29, the group structure of $\mathfrak{Lie}(G/S,\mathscr{M})$ defined from G coincides with that induced by the $\mathbb{O}_S$-module structure of $\mathscr{M}$. We then deduce from the preceding proposition and the functoriality of the operation of $\mathbb{O}_S$ (Proposition 2.2.25) that:

Corollary 2.2.52. *Suppose that* G/S *satisfies condition (E). Then* Ad *sends* G *into the subgroup* $\mathcal{A}ut_{\mathbb{O}_S}(\mathfrak{Lie}(G/S,\mathscr{M}))$ *of* $\mathcal{A}ut_{\mathbf{Grp}}(\mathfrak{Lie}(G/S),\mathscr{M})$*, that is, for any* $g \in G(S')$*,* $\mathrm{Ad}(g)$ *respects the* $\mathbb{O}(S')$*-module structure of* $\mathfrak{Lie}(G_{S'}/S',\mathscr{M})$*. In other words,* Ad *is a linear representation of* G *on the* $\mathbb{O}_S$*-module* $\mathfrak{Lie}(G/S,\mathscr{M})$.

Remark 2.2.53. Suppose that G/S satisfies condition (E). Then the derived morphism of the group law $m : G \times_S G \to G$ is none other than the addition law of $\mathfrak{Lie}(G/S,\mathscr{M})$ (m is not a morphism of groups, but $m(e,e) = e$, so the derived morphism $\mathfrak{Lie}(m)$ sends $T_{(G\times_S G)/S,(e,e)}(\mathscr{M}) = \mathfrak{Lie}(G/S,\mathscr{M}) \times_S \mathfrak{Lie}(G/S,\mathscr{M})$ into $\mathfrak{Lie}(G/S,\mathscr{M})$). For any $n \in \mathbb{Z}$, we show similarly that if $n_G : G \to G$ is the morphism of S-functors defined by $g \mapsto g^n$, then the derived morphism $\mathfrak{Lie}(n_G)$ is the multiplication by n on $\mathfrak{Lie}(G/S)$, cf. Corollary 2.2.35.

Now consider the S-functor $\mathcal{H}om_{G/S}(G, T_{G/S}(\mathscr{M}))$; for any $S' \to S$, we have $T_{G/S}(\mathscr{M})_{S'} \cong T_{G_{S'}/S'}(\mathscr{M})$ and hence

$$\mathcal{H}om_{G/S}(G, T_{G/S}(\mathscr{M}))(S') \cong \mathrm{Hom}_{G_{S'}}(G_{S'}, T_{G_{S'}/S'}(\mathscr{M})) = \Gamma(T_{G_{S'}/S'}(\mathscr{M})/G_{S'}).$$

Note that we have an isomorphism, functorial on S',

$$\mathrm{Hom}_{S'}(G_{S'}, \mathfrak{Lie}(G_{S'}/S', \mathscr{M})) \xrightarrow{\sim} \Gamma(T_{G_{S'}/S'}(\mathscr{M})/G_{S'}) \tag{2.2.18}$$

which to any $f : G_{S'} \to \mathfrak{Lie}(G_{S'}/S', \mathscr{M})$ associates the section $s_f : G_{S'} \to T_{G_{S'}/S'}(\mathscr{M})$ such that, for any $S'' \to S'$ and $g \in G(S'')$,

$$s_f(g) = i(f(g))\tau(g).$$

Let h be an automorphism of the functor $G_{S'}$ over S' (not necessarily respects the group structure). To any section s of $T_{G_{S'}/S'}(\mathscr{M})$, we can associate $h(s)$ defined by transport the structure: this for example the only section of $T_{G_{S'}/S'}(\mathscr{M})$ fitting into the commutative diagram

$$\begin{tikzcd} G_{S'} \arrow[r, "s"] \arrow[d, "h"'] & T_{G_{S'}/S'}(\mathscr{M}) \arrow[d, "T(h)"] \\ G_{S'} \arrow[r, "h(s)"] & T_{G_{S'}/S'}(\mathscr{M}) \end{tikzcd}$$

In particular, we can take h to be the right translation t_x by an element x of $G(S')$, that is, $h(g) = t_x(g) = g \cdot x$, for any $g \in G(S'')$, $S'' \to S'$. We have immediately

$$t_x(s_f) = s_{t_x(f)},$$

where $t_x(f) : G_{S'} \to \mathfrak{Lie}(G_{S'}/S', \mathscr{M})$ is defined by

$$t_x(f)(g) = f(g \cdot x^{-1})$$

for any $g \in G(S'')$, $S'' \to S'$. It follows that if we operate G on $\mathcal{H}om_{G/S}(G, T_{G/S}(\mathscr{M}))$ and $\mathcal{H}om_S(G, \mathfrak{Lie}(G/S, \mathscr{M}))$ by right translation in the following way: for any $S' \to S$, $x \in G(S')$, $\sigma \in \Gamma(T_{G_{S'}/S'}(\mathscr{M}/G_{S'}))$ and $f \in \mathrm{Hom}_{S'}(G_{S'}, \mathfrak{Lie}(G_{S'}/S', \mathscr{M}))$,

$$(\sigma \cdot x)(g) = \sigma(g \cdot x^{-1}) \cdot \tau(x), \quad (f \cdot x)(g) = f(g \cdot x^{-1}),$$

for any $g \in G(S'')$, $S'' \to S'$, then the isomorphism (2.2.18) respects the action of G.

In particular, by this isomorphism, the elements of $\mathcal{H}om_{G/S}(G, T_{G/S}(\mathscr{M}))^G(S')$ (called **right invariant sections** of $T_{G_{S'}/S'}(\mathscr{M})$) corresponds to constant morphisms of $G_{S'}$ into $\mathfrak{Lie}(G_{S'}/S', \mathscr{M})$ (i.e. which factors through the projection $G_{S'} \to S'$), or to elements of $\mathfrak{Lie}(G_{S'}/S', \mathscr{M})(S') = \mathfrak{Lie}(G/S, \mathscr{M})(S')$. We then have the following proposition:

Proposition 2.2.54. *The map $\mathfrak{Lie}(G/S, \mathscr{M})(S) \to \Gamma(T_{G/S}(\mathscr{M})/G)$ which associates an element $X \in \mathfrak{Lie}(G/S, \mathscr{M})(S)$ the section $x \mapsto X(\pi(x))$ is a bijection from $\mathfrak{Lie}(G/S, \mathscr{M})(S)$ onto the set of right invariant sections of $\Gamma(T_{G/S}(\mathscr{M})/G)$.*

Similarly, we can act G on $\mathcal{E}nd_{I_S(\mathscr{M})/S}(G_{I_S(\mathscr{M})})$ as follows: for any $S' \to S$, $x \in G(S')$ and $u \in \mathcal{E}nd_{I_S(\mathscr{M})/S}(G_{I_S(\mathscr{M})})(S') = \mathrm{End}_{I_{S'}}(G_{I_{S'}(\mathscr{M})})$,

$$(u \cdot x)(g) = u(g \cdot x^{-1}) \cdot x,$$

for any $g \in G(S'')$, $S'' \to I_{S'}(\mathscr{M})$. Then the morphism of Corollary 2.2.43

$$\mathcal{H}om_{G/S}(G, T_{G/S}(\mathscr{M})) \to \mathcal{E}nd_{I_S(\mathscr{M})/S}(G_{I_S(\mathscr{M})})$$

respects the operation of G and induces for any $S' \to S$ a bijection from $\Gamma(T_{G_{S'}/S'}(\mathscr{M})/G_{S'})$ and the set of $I_{S'}(\mathscr{M})$-endomorphisms u of $G_{I_{S'}(\mathscr{M})}$ inducing the identity on G and are invariant under right translations, i.e. satisfies $u_{S''} \cdot x = u_{S''}$ for any $S'' \to S'$ and $x \in G(S'')$. By Proposition 2.2.44, we then conclude the following theorem:

Proposition 2.2.55. *There exists a bijection (functorail on G) from the set $\mathfrak{Lie}(G/S, \mathscr{M})(S)$ to the set of $I_S(\mathscr{M})$-endomorphisms of $G_{I_S(\mathscr{M})}$ inducing the identity on G and commutes with right translations of G, and this is a group isomorphism if G/S satisfies condition (E).*

By considering the case $\mathscr{M} = \mathscr{O}_S$, we thus obtain the classical definitions of the Lie algebra of a group.

Before going further, let us establish some new corollaries of Proposition 2.2.36. Let X, Y be over Z and Z be over S, as in 2.2.1.1. As we have seen in Proposition 2.2.36, the isomorphisms (2.2.2):

$$\begin{tikzcd} \mathcal{H}om_S(I_S(\mathscr{M}), \mathcal{H}om_{Z/S}(X,Y)) \arrow[rr, "\cong"] \arrow[dr, "\cong"] & & \mathcal{H}om_{Z/S}(X, \mathcal{H}om_Z(I_Z(\mathscr{M}), Y)) \\ & \mathcal{H}om_{Z/S}(X \times_S I_S(\mathscr{M}), Y) \arrow[ur, "\cong"] & \end{tikzcd} \tag{2.2.19}$$

induces the isomorphism θ below

$$\begin{tikzcd} T_{\mathcal{H}om_{Z/S}(X,Y)}(\mathscr{M}) \arrow[rr, "\cong", "\theta"'] \arrow[dr, "\cong"] & & \mathcal{H}om_{Z/S}(X, T_{Y/Z}(\mathscr{M})) \\ & \mathcal{H}om_{Z/S}(X \times_S I_S(\mathscr{M}), Y) \arrow[ur, "\cong"] & \end{tikzcd} \tag{2.2.20}$$

By Remark 2.2.2, if Y is a Z-group, so is $\mathcal{H}om_Z(V, Y)$ for any $V \to Z$ (in particular for $V = I_Z(\mathscr{M})$); explicitly, if $Z'' \to Z' \to Z$ and $\phi, \psi \in \mathrm{Hom}_Z(V_{Z'}, Y)$, then $\phi \cdot \psi$ is defined by

$$(\phi \cdot \psi)(v) = \phi(v)\psi(v)$$

for any $v \in V_{Z'}(Z'')$.

Definition 2.2.56. Suppose that X and Y are Z-groups. Let $\mathcal{H}om_{(Z/S)\text{-}\mathbf{Grp}}(X, Y)$ be the subfunctor of $\mathcal{H}om_{Z/S}(X, Y)$ defined as follows: for any $S' \to S$,

$$\mathcal{H}om_{(Z/S)\text{-}\mathbf{Grp}}(X, Y)(S') = \mathrm{Hom}_{Z_{S'}\text{-}\mathbf{Grp}}(X_{S'}, Y_{S'}). \tag{2.2.21}$$

This definition applies equally if we replace Y by the Z-group $T_{Y/Z}(\mathscr{M})$.

We then easily see that $T_{\mathcal{H}om_{(Z/S)\text{-}\mathbf{Grp}}(X,Y)/S}(\mathscr{M})(S')$ corresponds, under the isomorphisms of (2.2.20), to $Z_{S'}$-morphisms $\phi : X_{S'} \times_{S'} I_{S'}(\mathscr{M}) \to Y_{S'}$ which is multiplicative on X, that is, which satisfies $\phi(x_1x_2, m) = \phi(x_1, m)\phi(x_2, m)$, and these correspond to $Z_{S'}$-group morphisms $X_{S'} \to T_{Y/Z}(\mathscr{M})_{S'}$. We then obtain the following:

Proposition 2.2.57. *Let X, Y be Z-groups and Z be over S. We have an isomorphism of S-functors, functorial on $\mathscr{M}$:*

$$T_{\mathcal{H}om_{(Z/S)\text{-}\mathbf{Grp}}(X,Y)}(\mathscr{M}) \xrightarrow{\sim} \mathcal{H}om_{(Z/S)\text{-}\mathbf{Grp}}(X, T_{Y/Z}(\mathscr{M})).$$

In particular, for $Z = S$, we obtain the following corollary. Before stating it, we note that if Y is an abelian S-group, then so is $T_{Y/S}(\mathscr{M})$, and hence $H = \mathrm{Hom}_{S\text{-}\mathbf{Grp}}(X, Y)$ and $\mathrm{Hom}_{S\text{-}\mathbf{Grp}}(X, T_{Y/S}(M))$, and finally is $T_{H/S}(\mathscr{M})$.

Corollary 2.2.58. *Let X, Y be S-groups. We have an isomorphism of S-functors, functorial on $\mathscr{M}$:*

$$T_{\mathcal{H}om_{S\text{-}\mathbf{Grp}}(X,Y)/S}(\mathscr{M}) \xrightarrow{\sim} \mathcal{H}om_{S\text{-}\mathbf{Grp}}(X, T_{Y/S}(\mathscr{M})).$$

If Y is commutative, then this is an isomorphism of abelian S-groups.

If Y is an $\mathbb{O}_S$-module, the functor $T_{Y/S}(\mathscr{M})$ (resp. $\mathfrak{Lie}(Y/S,\mathscr{M})$) is endowed with an $\mathbb{O}_S$-module structure deduced by that of Y, which we denote by $T'_{Y/S}(\mathscr{M})$ (resp. $\mathfrak{Lie}'(Y/S,\mathscr{M})$). Therefore, if X, Y are $\mathbb{O}_S$-modules, then $T'_{Y/S}(\mathscr{M}) = \mathcal{H}om_S(I_S(\mathscr{M}), Y)$ and $H = \mathcal{H}om_{\mathbb{O}_S}(X, Y)$, and hence $\mathcal{H}om_{\mathbb{O}_S}(X, T'_{Y/S}(\mathscr{M}))$ and $T'_{H/S}(\mathscr{M})$, are endowed with $\mathbb{O}_S$-module structures, and we have:

Corollary 2.2.59. *If X, Y are $\mathbb{O}_S$-modules, we have an isomorphism of $\mathbb{O}_S$-modules, functorial on $\mathscr{M}$:*

$$T'_{\mathcal{H}om_{\mathbb{O}_S}(X,Y)/S}(\mathscr{M}) \xrightarrow{\sim} \mathcal{H}om_{\mathbb{O}_S}(X, T_{Y/S}(\mathscr{M})).$$

Definition 2.2.60. Let X, L be S-groups and X acts on L by groups automorphisms. We define the sub-functor $\mathcal{Z}^1_S(X, L)$ of $\mathcal{H}om_S(X, L)$ as follows: for any $S' \to S$, $\mathcal{Z}^1_S(X, L)(S')$ is defined to be the set

$$\{\phi \in \mathrm{Hom}_{S'}(X_{S'}, L_{S'}) : \phi(x_1x_2) = \phi(x_1)(x_1 \cdot \phi(x_2)) \text{ for any } x_1, x_2 \in X(S''),\ S'' \to S'\}.$$

The functor $\mathcal{Z}^1_S(X, L)$ is called the **functor of cross homomorphisms** from X to L.

If L is an $\mathbb{O}_S[X]$-module, then $\mathcal{Z}^1_S(X, L)$ coincides with the kernel of the differential

$$d : \mathcal{H}om_S(X, L) \to \mathcal{H}om_S(X^2, L)$$

defined in 2.1.3.1. In particular, $\mathcal{Z}^1_S(X, L)$ is an $\mathbb{O}_S$-module in this case.

Let $u : X \to Y$ be a morphism of S-groups. We have seen in Corollary 2.2.39 that we have an isomorphism of S-functors, functorial on $\mathscr{M}$:

$$T_{\mathcal{H}om_S(X,Y)/S,u}(\mathscr{M}) \xrightarrow{\sim} \mathcal{H}om_{Y/S}(X, T_{Y/S}(\mathscr{M})). \tag{2.2.22}$$

On the other hand, as Y is an S-group, we have $T_{Y/S}(\mathscr{M}) = \mathfrak{Lie}(Y/S,\mathscr{M}) \rtimes Y$, whence an isomorphism

$$\begin{aligned}\mathcal{H}om_{Y/S}(X, T_{Y/S}(\mathscr{M})) &\xrightarrow{\sim} \mathcal{H}om_{Y/S}(X, \mathfrak{Lie}(Y/S,\mathscr{M}) \rtimes Y)\\ &\xrightarrow{\sim} \mathcal{H}om_{Y/S}(X, \mathfrak{Lie}(Y,S,\mathscr{M})_Y)\\ &\xrightarrow{\sim} \mathcal{H}om_S(X, \mathfrak{Lie}(Y,S,\mathscr{M})).\end{aligned} \tag{2.2.23}$$

For any $S' \to S$, denote by $u' : X' \to Y'$ the morphism induced by u from base change. Consider the S-functor defined as follows:

$$\begin{aligned}\mathcal{H}om_{(Y/S)\text{-}\mathbf{Grp}}(X, \mathfrak{Lie}(Y/S,\mathscr{M}) \rtimes Y)(S') &= \mathrm{Hom}_{Y'\text{-}\mathbf{Grp}}(X', (\mathfrak{Lie}(Y/S,\mathscr{M}) \rtimes Y)_{S'})\\ &= \mathrm{Hom}_{Y'\text{-}\mathbf{Grp}}(X', \mathfrak{Lie}(Y'/S',\mathscr{M}) \rtimes Y').\end{aligned}$$

The isomorphism (2.2.22) then induces an isomorphism

$$T_{\mathcal{H}om_{S\text{-}\mathbf{Grp}}(X,Y)/S,u}(\mathscr{M}) \xrightarrow{\sim} \mathcal{H}om_{(Y/S)\text{-}\mathbf{Grp}}(X, \mathfrak{Lie}(Y/S,\mathscr{M}) \rtimes Y). \tag{2.2.24}$$

The isomorphism (2.2.23) can be made explicit as follows. If $\Phi \in \mathcal{H}om_{Y/S}(X, \mathfrak{Lie}(Y/S,\mathscr{M}) \rtimes Y)$, then for any $S'' \to S' \to S$ and $x \in X(S'')$, we can write

$$\Phi(S')(x) = \phi(S')(x) \cdot u'(x) \quad \text{where} \quad \phi(S')(x) \in \mathfrak{Lie}(Y'/S',\mathscr{M})(S''),$$

which determines an element ϕ of $\mathcal{H}om_S(X, \mathfrak{Lie}(Y/S,\mathscr{M}))$. On the other hand, the composition of the morphisms

$$X \xrightarrow{u} Y \xrightarrow{\mathrm{Ad}} \mathrm{Aut}_{S\text{-}\mathbf{Grp}}(\mathfrak{Lie}(Y/S,\mathscr{M}))$$

defines an operation of X on $L = \mathfrak{Lie}(Y/S,\mathscr{M})$ by group automorphisms, and we note that $\Phi(S')$ is a group morphism if and only if for any $x_1, x_2 \in X(S'')$, we have

$$\phi(S')(x_1x_2) = \phi(S')(x_1)(u(x_1)\phi(S')(x_2)u(x_1)^{-1}) = \phi(S')(x_1)(x_1 \cdot \phi(S')(x_2)),$$

that is, if and only if $\phi \in \mathcal{Z}^1_S(X, \mathfrak{Lie}(Y/S,\mathscr{M}))$. We therefore obtain the following result:

Proposition 2.2.61. *Let $u : X \to Y$ be a morphism of S-groups. We have an isomorphism of S-functors, functorial on $\mathscr{M}$:*

$$T_{\mathcal{H}om_{S\text{-}\mathbf{Grp}}(X,Y)/S,u}(\mathscr{M}) \xrightarrow{\sim} \mathcal{Z}^1_S(X, \mathfrak{Lie}(Y/S, \mathscr{M})).$$

Suppose moreover that Y/S satisfies condition (E). Then it folows from Corollary 2.2.58, by the same proof of Corollary 2.2.37, that $\mathcal{H}om_{S\text{-}\mathbf{Grp}}(X,Y)/S$ satisfies condition (E). We then have (this also follows from Proposition 2.2.61)

$$T_{\mathcal{H}om_{S\text{-}\mathbf{Grp}}(X,Y)/S,u}(\mathscr{M} \oplus \mathscr{N}) \cong T_{\mathcal{H}om_{S\text{-}\mathbf{Grp}}(X,Y)/S,u}(\mathscr{M}) \times_S T_{\mathcal{H}om_{S\text{-}\mathbf{Grp}}(X,Y)/S,u}(\mathscr{N}).$$

Therefore, $T_{\mathcal{H}om_{S\text{-}\mathbf{Grp}}(X,Y)/S,u}(\mathscr{M})$ is endowed, as $\mathcal{Z}^1_S(X, \mathfrak{Lie}(Y/S, \mathscr{M}))$, with an $\mathbb{O}_S$-module structure, induced by functoriality on $\mathscr{M}$. We then deduce that the isomorphism Proposition 2.2.61 is an isomorphism of $\mathbb{O}_S$-modules in this case:

Proposition 2.2.62. *Let $u : X \to Y$ be a morphism of S-groups and suppose that Y/S satisfies condition (E). We have an isomorphism of $\mathbb{O}_S$-modules, functorial on $\mathscr{M}$:*

$$T_{\mathcal{H}om_{S\text{-}\mathbf{Grp}}(X,Y)/S,u}(\mathscr{M}) \xrightarrow{\sim} \mathcal{Z}^1_S(X, \mathfrak{Lie}(Y/S, \mathscr{M})).$$

Moreover, if Y/S satisfies condition (E), we deduce from Corollary 2.2.45, as the proof of Corollary 2.2.46, that for any $u \in \mathrm{Iso}_{S\text{-}\mathbf{Grp}}(X,Y)$ we have an isomorphism functorial on $\mathscr{M}$

$$T_{\mathcal{I}so_{S\text{-}\mathbf{Grp}}(X,Y)/S,u}(\mathscr{M}) \xrightarrow{\sim} T_{\mathcal{H}om_{S\text{-}\mathbf{Grp}}(X,Y)/S,u}(\mathscr{M}).$$

We then deduce the following corollaries:

Corollary 2.2.63. *Let $u : X \to Y$ be a morphism of S-groups. If Y/S satisfies condition (E), we have an isomorphism of $\mathbb{O}_S$-modules, functorial on $\mathscr{M}$:*

$$T_{\mathcal{I}so_{S\text{-}\mathbf{Grp}}(X,Y)/S,u}(\mathscr{M}) \xrightarrow{\sim} \mathcal{Z}^1_S(X, \mathfrak{Lie}(Y/S, \mathscr{M})).$$

Corollary 2.2.64. *Let X be an S-group. If X/S satisfies condition (E), we have an isomorphism of $\mathbb{O}_S$-modules, functorial on $\mathscr{M}$:*

$$\mathfrak{Lie}(\mathcal{A}ut_{S\text{-}\mathbf{Grp}}(X)/S, \mathscr{M}) \xrightarrow{\sim} \mathcal{Z}^1_S(X, \mathfrak{Lie}(X/S, \mathscr{M})).$$

If Y is abelian, then the adjoint representation of Y on $L = \mathfrak{Lie}(Y/S, \mathscr{M})$ is trivial, so we have $\mathcal{Z}^1_S(X, L) = \mathcal{H}om_{S\text{-}\mathbf{Grp}}(X, L)$. We thus have:

Corollary 2.2.65. *Let Y be an abelian S-group. We have an isomorphism of S-functors, functorial on $\mathscr{M}$:*

$$T_{\mathcal{H}om_{S\text{-}\mathbf{Grp}}(X,Y)/S,u}(\mathscr{M}) \xrightarrow{\sim} \mathcal{H}om_{S\text{-}\mathbf{Grp}}(X, \mathfrak{Lie}(Y/S, \mathscr{M})).$$

If Y/S satisfies condition (E), this is an isomorphism of $\mathbb{O}_S$-modules.

Consider now the case where X, Y are $\mathbb{O}_S$-modules. Recall that we denote by $T'_{Y/S}(\mathscr{M})$ (resp. $\mathfrak{Lie}'(Y/S, \mathscr{M})$) the functor $T_{Y/S}(\mathscr{M})$ (resp. $\mathfrak{Lie}(Y/S, \mathscr{M})$) endowed with the $\mathbb{O}_S$-module structure induced by that of Y. If Y/S satisfies condition (E), we always endow $\mathfrak{Lie}(Y/S, \mathscr{M})$ the $\mathbb{O}_S$-module structure defined by functoriality on $\mathscr{M}$. In this case, the abelian group structures of $\mathfrak{Lie}(Y/S, \mathscr{M})$ and $\mathfrak{Lie}'(Y/S, \mathscr{M})$ coincide (cf. Proposition 2.2.29), but this is in general not true for the module structures. For any $S' \to S$ and $a \in \mathbb{O}(S')$, we denote by $a \cdot' m$ (resp. $a \cdot m$) the action of a on $m \in \mathfrak{Lie}'(Y/S, \mathscr{M})(S')$ (resp. $m \in \mathfrak{Lie}(Y/S, \mathscr{M})(S')$), and similarly for the actions of a on $T'_{Y/S}(\mathscr{M})$ and $T_{Y/S}(\mathscr{M})$.

We have $T'_{Y/S}(\mathscr{M}) \cong \mathfrak{Lie}'(Y/S, \mathscr{M}) \oplus Y$ as $\mathbb{O}_S$-modules; therefore, we obtain, as in Corollary 2.2.65, that:

Proposition 2.2.66. *Let $u : X \to Y$ be a morphism of $\mathbb{O}_S$-modules. We have an isomorphism of S-functors, functorial on $\mathscr{M}$:*

$$T_{\mathcal{H}om_{\mathbb{O}_S}(X,Y)/S,u}(\mathscr{M}) \xrightarrow{\sim} \mathcal{H}om_{\mathbb{O}_S}(X, \mathfrak{Lie}'(Y/S,\mathscr{M})). \tag{2.2.25}$$

If Y/S satisfies condition (E), then $\mathcal{H}om_{\mathbb{O}_S}(X,Y)/S$ satisfies condition (E) and (2.2.25) is an isomorphism of $\mathbb{O}_S$-modules if we endow both sides the $\mathbb{O}_S$-module structure induced by functoriality on $\mathscr{M}$.

Remark 2.2.67. Let $u : X \to Y$ be a morphism of $\mathbb{O}_S$-modules. Denote by τ_u the map which to any morphism $\phi : X \to \mathfrak{Lie}'(Y/S,\mathscr{M})$ of $\mathbb{O}_S$-modules associates the morphism

$$\phi \oplus u : X \to T'_{Y/S}(\mathscr{M}) = \mathfrak{Lie}'(Y/S,\mathscr{M}) \oplus Y.$$

Then the isomorphism of Proposition 2.2.66 fits into the following diagram, functorial on $\mathscr{M}$:

$$\begin{tikzcd}
T_{\mathcal{H}om_{\mathbb{O}_S}(X,Y)/S,u} \arrow[r, "\cong"] \arrow[d, hook] & \mathcal{H}om_{\mathbb{O}_S}(X, \mathfrak{Lie}'(Y/S,\mathscr{M})) \arrow[d, hook, "\tau_u"] \\
T_{\mathcal{H}om_{\mathbb{O}_S}(X,Y)/S}(\mathscr{M}) \arrow[r, "\cong"] & \mathcal{H}om_{\mathbb{O}_S}(X, T'_{Y/S}(\mathscr{M}))
\end{tikzcd}$$

Moreover, if Y/S satisfies condition (E), we deduce from Corollary 2.2.45, as the proof of Corollary 2.2.46, that for any $u \in \mathrm{Iso}_{\mathbb{O}_S}(X,Y)$, we have

$$T_{\mathcal{I}so_{\mathbb{O}_S}(X,Y)/S}(\mathscr{M}) \cong T_{\mathcal{H}om_{\mathbb{O}_S}(X,Y)/S}(\mathscr{M}). \tag{2.2.26}$$

Corollary 2.2.68. *Let X be an $\mathbb{O}_S$-module satisfying condition (E) relative to S. We have an isomorphism, functorial on $\mathscr{M}$:*

$$\mathfrak{Lie}(\mathcal{A}ut_{\mathbb{O}_S}(X)/S,\mathscr{M}) \xrightarrow{\sim} \mathcal{H}om_{\mathbb{O}_S}(X, \mathfrak{Lie}'(X/S,\mathscr{M}))$$

which respects the $\mathbb{O}_S$-module structure induced by functoriality on $\mathscr{M}$. In particular, $\mathcal{A}ut_{\mathbb{O}_S}(X)/S$ satisfies condition (E).

Proof. The first assertion follows from (2.2.25) and (2.2.26). For the second one, as X/S satisfies condition (E), we have an isomorphism of $\mathbb{O}_S$-modules $\mathfrak{Lie}'(X/S,\mathscr{M}\oplus\mathscr{N}) \cong \mathfrak{Lie}'(X/S,\mathscr{M}) \times_S \mathfrak{Lie}'(X/S,\mathscr{N})$, and hence

$$\mathfrak{Lie}(\mathcal{A}ut_{\mathbb{O}_S}(X)/S,\mathscr{M}\oplus\mathscr{N}) \cong \mathfrak{Lie}(\mathcal{A}ut_{\mathbb{O}_S}(X)/S,\mathscr{M}) \times_S \mathfrak{Lie}(\mathcal{A}ut_{\mathbb{O}_S}(X)/S,\mathscr{N}).$$

In view of the sequence (2.2.17), this proves that $\mathcal{A}ut_{\mathbb{O}_S}(X)/S$ satisfies condition (E). □

Before going further towards this direction, let us take a closer look at the relations between Y, $\mathfrak{Lie}(Y/S)$ and $\mathfrak{Lie}'(Y/S)$. We first notice that (cf. Remark 2.2.14)

$$\mathfrak{Lie}(\mathbb{O}_S/S,\mathscr{M}) = \mathfrak{Lie}'(\mathbb{O}_S/S,\mathscr{M}) = \Gamma_{\mathscr{M}} \tag{2.2.27}$$

and that we have a canonical isomorphism

$$d : \mathbb{O}_S \xrightarrow{\sim} \mathfrak{Lie}(\mathbb{O}_S/S). \tag{2.2.28}$$

Now let F be an $\mathbb{O}_S$-module. For any $S_2 \to S_1 \to S$, we have a bihomomorphism

$$F(S_1) \to F(S_2), \quad \mathbb{O}(S_1) \to \mathbb{O}(S_2), \tag{2.2.29}$$

whence a morphism of $\mathbb{O}(S_2)$-modules

$$F(S_1) \otimes_{\mathbb{O}(S_1)} \mathbb{O}(S_2) \to F(S_2).$$

In particular, for $S_1 = S'$ and $S_2 = I_{S'}(\mathscr{M})$, we deduce a morphism of $\mathbb{O}(S')$-modules, functorial on $\mathscr{M}$

$$F(S') \otimes_{\mathbb{O}(S')} T_{\mathbb{O}_S/S}(\mathscr{M})(S') \to T'_{F/S}(\mathscr{M})(S').$$

With S' varies, we obtain morphisms of $\mathbb{O}_S$-modules, functorial on $\mathscr{M}$:

$$F \otimes_{\mathbb{O}_S} T_{\mathbb{O}_S/S}(\mathscr{M}) \to T'_{F/S}(\mathscr{M}). \tag{2.2.30}$$

These morphisms are functorial on $\mathscr{M}$, hence compatible with the projections of tangent bundles onto their bases; they then define morphisms of $\mathbb{O}_S$-modules

$$F \otimes_{\mathbb{O}_S} \mathfrak{Lie}(\mathbb{O}_S/S, \mathscr{M}) \to \mathfrak{Lie}'(F/S, \mathscr{M}) \tag{2.2.31}$$

such that the following diagram is commutative:

$$\begin{tikzcd}
0 \arrow[r] & F \otimes_{\mathbb{O}_S} \mathfrak{Lie}(\mathbb{O}_S/S, \mathscr{M}) \arrow[r] \arrow[d] & F \otimes_{\mathbb{O}_S} T_{\mathbb{O}_S/S}(\mathscr{M}) \arrow[r] \arrow[d] & F \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathfrak{Lie}'(F/S, \mathscr{M}) \arrow[r] & T'_{F/S}(\mathscr{M}) \arrow[r] & F \arrow[r] & 0
\end{tikzcd}$$

We can consider the morphisms (2.2.31) as morphisms of abelian S-groups:

$$F \otimes_{\mathbb{O}_S} \mathfrak{Lie}(\mathbb{O}_S/S, \mathscr{M}) \to \mathfrak{Lie}(F/S, \mathscr{M}). \tag{2.2.32}$$

By tensoring F with the isomorphism $d : \mathbb{O}_S \xrightarrow{\sim} \mathfrak{Lie}(\mathbb{O}_S/S)$, we then deduce (for $\mathscr{M} = \mathscr{O}_S$) a morphism of abelian S-groups

$$d : F \xrightarrow{\sim} F \otimes_{\mathbb{O}_S} \mathfrak{Lie}(\mathbb{O}_S/S) \to \mathfrak{Lie}(F/S). \tag{2.2.33}$$

Remark 2.2.69. If F/S satisfies condition (E), the morphisms (2.2.32) and (2.2.33) are not necessarily morphisms of $\mathbb{O}_S$-modules, if we endow both sides the module structure induced by functoriality on $\mathscr{M}$. For example, let k be a field with characteristic $p > 0$, $S = \operatorname{Spec}(k)$, and F be the $\mathbb{O}_S$-module which to any S-scheme T associates $F(T) = \Gamma(T, \mathscr{O}_T)$, endowed with the $\mathbb{O}(T)$-module structure obtained by acting a scalar via its p-th power, that is, $r \cdot f = r^p f$ for $r \in \mathbb{O}(T)$ and $f \in F(T)$. As an S-group, F is isomorphic to $\mathbb{G}_{a,S}$, so F satisfies condition (E) and $\mathfrak{Lie}(F/S)$ is identified with $\mathfrak{Lie}(\mathbb{G}_{a,S}/S) \cong \mathbb{O}_S$. Then, the morphism $d : F \to \mathfrak{Lie}(F/S)$ is, for any $T \to S$, the identity map $F(T) \to \mathbb{O}(T)$; it respects the abelian group structure, but not the $\mathbb{O}_S$-module structure.

Remark 2.2.70. We can explicit the morphisms (2.2.30) and (2.2.31) as follows. The morphism $\Theta : F \otimes_{\mathbb{O}_S} T_{\mathbb{O}_S/S}(\mathscr{M}) \to T'_{F/S}(\mathscr{M}) = \mathcal{H}om_S(I_S(\mathscr{M}), F)$ is defined so that for any $S' \to S$, $\alpha \in \mathbb{O}(I_{S'}(\mathscr{M}))$, and $f : S' \to F$,

$$\Theta(f \otimes \alpha) = \alpha(\tau_0 \circ f) = \alpha \cdot (f \circ \rho)$$

where $\tau_0 : F \to T'_{F/S}(\mathscr{M})$ is the zero section and $\rho : I_{S'}(\mathscr{M}) \to S'$ is the structural morphism.

Definition 2.2.71. We say that F is a **good $\mathbb{O}_S$-module** if the morphisms $F \otimes_{\mathbb{O}_S} T_{\mathbb{O}_S/S}(\mathscr{M}) \to T_{F/S}(\mathscr{M})$ (or equivalently, the morphisms $F \otimes_{\mathbb{O}_S} \mathfrak{Lie}(\mathbb{O}_S/S, \mathscr{M}) \to \mathfrak{Lie}(F/S, \mathscr{M})$) are isomorphisms of abelian S-groups (so that F/S satisfies condition (E)) and if moreover they respect the $\mathbb{O}_S$-module structures induced by functoriality on $\mathscr{M}$.

Proposition 2.2.72. *Let F be an $\mathbb{O}_S$-module. Consider the following conditions:*

(i) *F is a good $\mathbb{O}_S$-module.*

(ii) *F/S satisfies condition (E) and $d : F \to \mathfrak{Lie}(F/S)$ is an isomorphism of $\mathbb{O}_S$-modules.*

(iii) $\mathfrak{Lie}(F/S, \mathscr{M}) = \mathfrak{Lie}'(F/S, \mathscr{M})$.

Then we have (i)⇔(ii)⇒(iii).

Proof. The implication (i)⇒(ii) follows from definition. To see that (ii)⇒(ii), it suffices to show that the morphisms of abelian S-groups

$$F \otimes_{\mathbb{O}_S} \mathfrak{Lie}(\mathbb{O}_S/S, \mathscr{M}) \xrightarrow{\sim} \mathfrak{Lie}(F/S, \mathscr{M})$$

are isomorphisms of $\mathbb{O}_S$-modules. As F/S satisfies condition (E), the two members transform finite direct sums of copies of $\mathscr{O}_S$ into finite products of abelian S-groups. We are then reduced to the case where $\mathscr{M} = \mathscr{O}_S$, which follows by the hypothesis.

Finally, (i)⇒(iii) follows from the definition and the fact that the isomorphisms

$$F \otimes_{\mathbb{O}_S} \mathfrak{Lie}(\mathbb{O}_S/S, \mathscr{M}) \xrightarrow{\sim} \mathfrak{Lie}'(F/S, \mathscr{M})$$

of (2.2.31) is an isomorphism of $\mathbb{O}_S$-modules. □

Example 2.2.73. For any quasi-coherent $\mathscr{O}_S$-module $\mathscr{E}$, the $\mathbb{O}_S$-module $\Gamma_{\mathscr{E}}$ and $\check{\Gamma}_{\mathscr{E}}$ are good. In fact, for any $f : S' \to S$, the morphisms

$$\Gamma_{\mathscr{E}}(S') \otimes_{\mathbb{O}(S')} \mathbb{O}(I_{S'}(\mathscr{M})) \to T_{\Gamma_{\mathscr{E}}/S}(\mathscr{M})(S')$$
$$\check{\Gamma}_{\mathscr{E}}(S') \otimes_{\mathbb{O}(S')} \mathbb{O}(I_{S'}(\mathscr{M})) \to T_{\check{\Gamma}_{\mathscr{E}}/S}(\mathscr{M})(S')$$

correspond, respectively, to morphisms

$$\Gamma(S', f^*(\mathscr{E})) \otimes_{\mathbb{O}(S')} \Gamma(S', \mathscr{D}_{\mathscr{O}_{S'}}(\mathscr{M})) \to \Gamma(S', f^*(\mathscr{E}) \otimes_{\mathscr{O}_{S'}} \mathscr{D}_{\mathscr{O}_{S'}}(\mathscr{M})),$$
$$\mathrm{Hom}_{\mathscr{O}_{S'}}(f^*(\mathscr{E}), \mathscr{O}_{S'}) \otimes_{\mathbb{O}(S')} \Gamma(S', \mathscr{D}_{\mathscr{O}_{S'}}(\mathscr{M})) \to \mathrm{Hom}_{\mathscr{O}_{S'}}(f^*(\mathscr{E}), \mathscr{D}_{\mathscr{O}_{S'}}(\mathscr{M}));$$

which are both isomorphisms since $\mathscr{D}_{\mathscr{O}_{S'}}(\mathscr{M})$ is isomorphic, as $\mathscr{O}_{S'}$-module, to a finite direct sum of copies of $\mathscr{O}_{S'}$ (recall that $\mathscr{M}$ is assumed to be free).

Proposition 2.2.74. *Let F be a good $\mathbb{O}_S$-module. Then $\mathcal{A}ut_{\mathbb{O}_S}(F)/S$ satisfies condition (E) and we have a isomorphism (functorial on $\mathscr{M}$)*

$$\mathfrak{Lie}(\mathcal{A}ut_{\mathbb{O}_S}(F)/S, \mathscr{M}) \xrightarrow{\sim} \mathcal{H}om_{\mathbb{O}_S}(F, \mathfrak{Lie}(F/S, \mathscr{M}))$$

which respects the $\mathbb{O}_S$ induced by the functoriality on $\mathscr{M}$. In particular, we have an isomorphism of $\mathbb{O}_S$-modules

$$\mathfrak{Lie}(\mathcal{A}ut_{\mathbb{O}_S}(F)/S) \xrightarrow{\sim} \mathcal{E}nd_{\mathbb{O}_S}(F).$$

Morover, $\mathcal{E}nd_{\mathbb{O}_S}(F)$ is a good $\mathbb{O}_S$-module.

Proof. In fact, by Proposition 2.2.72, F/S satisfies condition (E) and

$$\mathfrak{Lie}(F/S, \mathscr{M}) = \mathfrak{Lie}'(F/S, \mathscr{M}) \cong F \otimes_{\mathbb{O}_S} \mathfrak{Lie}(\mathbb{O}_S/S, \mathscr{M}). \tag{2.2.34}$$

The first assertion then follows from Corollary 2.2.68. Put $E = \mathcal{E}nd_{\mathbb{O}_S}(F)$; by (2.2.34) and ([**?**] remarque 4.3.5), we have the following commutative diagram of abelian groups

$$\begin{tikzcd}
\mathcal{E}nd_{\mathbb{O}_S}(F) \otimes_{\mathbb{O}_S} \mathfrak{Lie}(\mathbb{O}_S/S, \mathscr{M}) \arrow[r, "d_E"] \arrow[d, equal] & \mathfrak{Lie}(\mathcal{E}nd_{\mathbb{O}_S}(F)/S, \mathscr{M}) \\
\mathcal{H}om_{\mathbb{O}_S}(F, F \otimes_{\mathbb{O}_S} \mathfrak{Lie}(\mathbb{O}_S/S, \mathscr{M})) \arrow[r, "d_F"] & \mathcal{H}om_{\mathbb{O}_S}(F, \mathfrak{Lie}(\mathcal{E}nd_{\mathbb{O}_S}(F)/S, \mathscr{M})) \arrow[u, "\cong"', "(*)"]
\end{tikzcd}$$

where d_F and $(*)$ are isomorphisms of $\mathbb{O}_S$-modules; therefore, so is d_E, and this proves the proposition. □

Remark 2.2.75. Put $\mathscr{O}_{I_S} = \mathscr{O}_S \oplus t\mathscr{O}_S$ (with $t^2 = 0$) and let F be a good $\mathbb{O}_S$-module. Then, for any $S' \to S$, the morphism

$$F(S') \oplus tF(S') = F(S') \otimes_{\mathbb{O}(S')} \mathbb{O}(I_{S'}) \to F(I_{S'}) = F(S') \oplus \mathfrak{Lie}(F/S)(S')$$

(which is the identity on $F(S')$) induces an isomorphism of $\mathbb{O}(S')$-modules $tF(S') \cong \mathfrak{Lie}(F/S)(S')$. By varying S', we then obtain an isomorphism $\mathfrak{Lie}(F/S) \cong tF$. For any $S' \to S$, we have, by Proposition 2.2.74, a commutative diagram

$$\begin{tikzcd}
\mathrm{End}_{\mathbb{O}_{S'}}(F_{S'}) \arrow[r, "\cong"] & \mathrm{Hom}_{\mathbb{O}_{S'}}(F_{S'}, tF_{S'}) \arrow[r, "\cong"] \arrow[d, hook] & \mathfrak{Lie}(\mathcal{A}ut_{\mathbb{O}_S}(F)/S)(S') \arrow[d, hook] \\
& \mathrm{Aut}_{\mathbb{O}(I_{S'})}(F_{I_{S'}}) \arrow[r, equal] & T_{\mathcal{A}ut_{\mathbb{O}_S}(F)/S}(S')
\end{tikzcd}$$

and we deduce from Remark 2.2.67 that any $X \in \mathrm{End}_{\mathbb{O}_{S'}}(F_{S'})$ corresponds to the element $\mathrm{id} + tX$ of $\mathrm{Aut}_{\mathbb{O}_{I_{S'}}}(F_{I_{S'}})$.

We say that the S-group G is **good** if G/S satisfies condition (E) and $\mathfrak{Lie}(G/S)$ is a good $\mathbb{O}_S$-module. Note that if F is a good $\mathbb{O}_S$-module, it is also a good S-groups: in fact, F/S satisfies condition (E) and $\mathfrak{Lie}(F/S) \cong F$ (cf. Proposition 2.2.72 (ii)) is a good $\mathbb{O}_S$-module.

Example 2.2.76. If G is representable, then it is good. In fact, G/S satisfies condition (E) and $\mathfrak{Lie}(G/S)$ is of the form $\mathbb{V}(\mathscr{E})$ by Proposition 2.2.15, hence good by Example 2.2.73.

Lemma 2.2.77. *Let G be an S-group such that G/S satisfies condition (E), and $F = \mathfrak{Lie}(G/S)$. Then F/S satisfies condition (E) and the abelian group morphism $d : F \to \mathfrak{Lie}(F/S)$ respects the $\mathbb{O}_S$-module structure. Therefore, G is good if and only $d : F \to \mathfrak{Lie}(F/S)$ is bijective.*

Proof. □

Theorem 2.2.78. *If F is a good $\mathbb{O}_S$-module, the S-group $\mathcal{A}ut_{\mathbb{O}_S}(F)$ is good.*

Proof. In fact, by Proposition 2.2.74, $\mathcal{A}ut_{\mathbb{O}_S}(F)/S$ satisfies condition (E) and $\mathfrak{Lie}(\mathcal{A}ut_{\mathbb{O}_S}(F)/S) \cong \mathcal{E}nd_{\mathbb{O}_S}(F)$ is a good $\mathbb{O}_S$-module. □

Example 2.2.79. Let F be the $\mathbb{O}_S$-module defined in Remark 2.2.69. Then, the canonical morphism $d : F \to \mathfrak{Lie}(F/S)$ is, for any $T \to S$, the identity morphism $F(T) \to \mathbb{O}(T)$. It respects the abelian group structure, but not the module structure, so F is not good.

Let G be an S-group and F be a good $\mathbb{O}_S$-module. Suppose that we are given a linear representation of G on F, that is, an S-group morphism

$$\rho : G \to \mathcal{A}ut_{\mathbb{O}_S}(F).$$

If G/S satisfies condition (E), we deduce from Proposition 2.2.74 and Proposition 2.2.25 a morphism of $\mathbb{O}_S$-modules

$$d\rho : \mathfrak{Lie}(G/S) \to \mathfrak{Lie}(\mathcal{A}ut_{\mathbb{O}_S}(F)/S) \cong \mathcal{E}nd_{\mathbb{O}_S}(F).$$

Moreover, put $\mathscr{O}_{I_S} = \mathscr{O}_S \oplus t\mathscr{O}_S$ (with $t^2 = 0$), we deduce from Remark 2.2.75 that, if $S' \to S$ and $X \in \mathfrak{Lie}(G/S)(S') \subseteq G(I_{S'})$, then we have the following equality in $\mathrm{Aut}_{\mathbb{O}_{I_{S'}}}(F_{I_{S'}})$:

$$\rho(X) = \mathrm{id} + td\rho(X), \tag{2.2.35}$$

i.e. for any $S'' \to I_{S'}$ and $f \in F(S')$, we have $\rho(X)(f) = f + td\rho(X)(f)$ in $F(S'')$.

Let G be a good S-group. Then $\mathfrak{Lie}(G/S)$ is a good $\mathbb{O}_S$-module, and we have a morphism of S-groups

$$\mathrm{Ad} : G \to \mathcal{A}ut_{\mathbb{O}_S}(\mathfrak{Lie}(G/S)).$$

We then deduce a morphism of $\mathbb{O}_S$-modules

$$\mathrm{ad} : \mathfrak{Lie}(G/S) \to \mathcal{E}nd_{\mathbb{O}_S}(\mathfrak{Lie}(G/S)),$$

or equivalently, an $\mathbb{O}_S$-bilinear morphism

$$\mathfrak{Lie}(G/S) \times_S \mathfrak{Lie}(G/S) \to \mathfrak{Lie}(G/S), \quad (x, y) \mapsto [x, y] := \mathrm{ad}(x) \cdot y$$

where x, y denote elements of $\mathfrak{Lie}(G/S)(S') = \mathfrak{Lie}(G_{S'}/S')(S')$. If G is commutative, then the action Ad is trivial, and we have $[x, y] = 0$.

Remark 2.2.80. We can give an equivalent definition of the bracket: note first that it suffices to do this for $x, y \in \mathfrak{Lie}(G/S)(S)$. We then note that there is a canonical isomorphism $I_S \times_S I_S \cong I_{I_S}$; to avoid confusions, we denote by I and I' the two copies of I_S and put $\mathcal{O}_I = \mathcal{O}_S[t]$, $\mathcal{O}_{I'} = \mathcal{O}_S[t']$, where $t^2 = t'^2 = 0$. We then have a commutative diagram

$$\begin{tikzcd}
I \times I' \arrow[r] \arrow[d] & I' \arrow[d] \\
I \arrow[r] & S
\end{tikzcd}$$

(the two arrows from $I \times I'$ identifying it as the dual number scheme over I or over I'), which gives a commutative diagram of abelian groups (where $L = \mathfrak{Lie}(G/S)$) by (2.2.17):

$$\begin{tikzcd}
 & & 1 \arrow[d] & 1 \arrow[d] & \\
 & & L(I) \arrow[r] \arrow[d] & L(S) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & L(I') \arrow[r] \arrow[d] & G(I \times I') \arrow[r] \arrow[d] & G(I') \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & L(S) \arrow[r] \arrow[d] & G(I) \arrow[r] \arrow[d] & G(S) \arrow[r] \arrow[d] & 1 \\
 & 1 & 1 & 1 &
\end{tikzcd} \tag{2.2.36}$$

The ninith vertex of this diagram is none other than $\mathfrak{Lie}(L/S)(S)$. If G is good, this is isomorphic to $L(S)$ and we then have the following diagram, where the rows and columns are exact sequences of groups and in view of the identification $L(I) = L(S) \oplus tL(S)$ (resp. $L(I') = L(S) \oplus t'L(S)$), the injection $L(S) \hookrightarrow L(I)$ (resp. $L(S) \hookrightarrow L(I')$) is given by $u \mapsto tu$ (resp. $u \mapsto t'u$):

$$\begin{tikzcd}
L(S) \arrow[r, "t"] \arrow[d, "t'"] & L(I) \arrow[r] \arrow[d] & L(S) \arrow[d] \\
L(I') \arrow[r] \arrow[d] & G(I \times I') \arrow[r] \arrow[d] & G(I') \arrow[d] \\
L(S) \arrow[r] & G(I) \arrow[r] & G(S)
\end{tikzcd} \tag{2.2.37}$$

Now in this diagram, if we take two elements x and y in $L(S)$ and choose arbitrarily element $\tilde{x} \in L(I)$ (resp. $\tilde{y} \in L(I')$) which maps to x (resp. to y), then the commutator $\tilde{x}\tilde{y}\tilde{x}^{-1}\tilde{y}^{-1}$ in $G(I \times I')$ does not depend on the choice of $\tilde{x}$ and $\tilde{y}$, and it is the image of an element $z \in L(S)$. In fact, if we identify x with its image under the canonical section $L(S) \to L(I)$ (and similarly for y), then $\tilde{x} = xu$ and $\tilde{y} = yv$, with $u, v \in L(S) = L(I) \cap L(I')$, and since $L(I)$, $L(I')$ are abelian, we have

$$\tilde{x}\tilde{y}\tilde{x}^{-1}\tilde{y}^{-1} = xuyvu^{-1}x^{-1}v^{-1}y^{-1} = xuyu^{-1}vx^{-1}v^{-1}y^{-1} = xyx^{-1}y^{-1}.$$

Moreover, this element is send to the unit element of $G(I)$ and of $G(I')$, hence comes from an element $z \in L(S)$. Finally, consider y (resp. x) as element of $L(I')$ (resp. $L(S) \subseteq G(I')$), by (2.2.35) we have

$$xyx^{-1} = \mathrm{Ad}(x)(y) = (\mathrm{id} + t'\mathrm{ad}(x))(y) = y + t'[x, y],$$

so the element $xyx^{-1}y^{-1}$ of $L(I')$ is the iamge of $z = [x, y] \in L(S)$.

From the above construction, we see that the bracket has the following properties:

(i) The bracket is functorial on G: more precisely, $G \mapsto \mathfrak{Lie}(G/S)$ is a functor from the category of good S-groups to the category of good $\mathbb{O}_S$-modules endowed with an $\mathbb{O}_S$-bilinear composition law.

(ii) We have $[x, y] + [y, x] = 0$. In fact, the diagram is symmetric, and by exchanging x and y we are considering the element $\tilde{y}\tilde{x}\tilde{y}^{-1}\tilde{x}^{-1}$, which is the inverse of $\tilde{x}\tilde{y}\tilde{x}^{-1}\tilde{y}^{-1}$.

Proposition 2.2.81. *Let F be a good $\mathbb{O}_S$-module. Via the identification $\mathfrak{Lie}(\mathcal{A}ut_{\mathbb{O}_S}(F)/S) = \mathcal{E}nd_{\mathbb{O}_S}(F)$, we have*

$$\mathrm{Ad}(g) \cdot Y = g \circ Y \circ g^{-1}, \quad [X, Y] = X \circ Y - Y \circ X,$$

for any $S' \to S$, $g \in \mathrm{Aut}_{\mathbb{O}_S}(F_{S'})$ and $X, Y \in \mathfrak{Lie}(\mathcal{A}ut_{\mathbb{O}_S}(F)/S)(S') = \mathrm{End}_{\mathbb{O}_S}(F_{S'})$.

Proof. By base change, we can assume that $S' = S$, which makes it possible to simplify the notations. Put $I = I_S$ and $\mathcal{O}_I = \mathcal{O}_S[t]$ (with $t^2 = 0$). Recall that the inclusion $i : \mathrm{End}_{\mathbb{O}_S}(F) \hookrightarrow \mathcal{A}ut_{\mathbb{O}_I}(F_I)$ sends Y to $\mathrm{id} + tY$, so by the definition of $\mathrm{Ad}(g)$, we have

$$\mathrm{id} + t\mathrm{Ad}(g)(Y) = g \circ (\mathrm{id} + tY) \circ g^{-1} = \mathrm{id} + t(g \circ Y \circ g^{-1}),$$

whence $\mathrm{Ad}(g)(Y) = g \circ Y \circ g^{-1}$.

Let I' be a second copy of I_S, and put $\mathcal{O}_{I'} = \mathcal{O}_S[t']$ (with $t'^2 = 0$). Apply the result of Remark 2.2.80 to $G = \mathcal{A}ut_{\mathbb{O}_S}(F)$ and $L = \mathfrak{Lie}(G/S) = \mathcal{A}ut_{\mathbb{O}_S}(F)$, where we identify X with its image under the canonical section $L(S) \hookrightarrow L(I)$; its image in $G(I \times I')$ is then $\mathrm{id} + t'X$, hence the inverse is $\mathrm{id} - t'X$. Similarly, Y is send to $\mathrm{id} + tY$, so the inverse is $\mathrm{id} - tY$. Then the commutator

$$(\mathrm{id} + t'X) \circ (\mathrm{id} + tY) \circ (\mathrm{id} - t'X) \circ (\mathrm{id} - tY) = \mathrm{id} + tt'(X \circ Y - Y \circ X)$$

is the image of $Z = X \circ Y - Y \circ X$ in $G(I \times I')$ (in fact, Z is send to $tZ \in L(I)$, hence to $\mathrm{id} + tt'Z \in G(I \times I')$). By Remark 2.2.80, we conclude that $[X, Y] = X \circ Y - Y \circ X$. □

Corollary 2.2.82. *Let G be a good S-group and $x, y, z \in \mathfrak{Lie}(G/S)(S')$. We have*

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0.$$

Proof. In fact, as G is good, $\mathfrak{Lie}(G/S)$ is a good $\mathbb{O}_S$-module and hence, by Theorem 2.2.78, $\mathcal{A}ut_{\mathbb{O}_S}(\mathfrak{Lie}(G/S))$ is a good S-group. Then, the morphism of S-groups

$$\mathrm{Ad} : G \to \mathcal{A}ut_{\mathbb{O}_S}(\mathfrak{Lie}(G/S))$$

gives by functoriality $\mathrm{ad}([x, y]) = [\mathrm{ad}(x), \mathrm{ad}(y)]$. Combined with Proposition 2.2.81, this shows that

$$\mathrm{ad}([x, y]) = [\mathrm{ad}(x), \mathrm{ad}(y)] = \mathrm{ad}(x) \circ \mathrm{ad}(y) - \mathrm{ad}(y) \circ \mathrm{ad}(x),$$

which implies the Jacobi identity by applied to an element z. □

Corollary 2.2.83. *Let G be a good S-group linearly acted on a good $\mathbb{O}_S$-module F (i.e. F is an $\mathbb{O}_S[G]$-module, G and F being good). Then the linear map $d\rho : \mathfrak{Lie}(G/S) \to \mathcal{E}nd_{\mathbb{O}_S}(F)$ is a representation, that is, we have*

$$d\rho([x,y]) = d\rho(x) \circ d\rho(y) - d\rho(y) \circ d\rho(x).$$

Proof. This follows from the functoriality of bracket and Proposition 2.2.81. □

To any good S-group (for example representable), we have associated a good $\mathbb{O}_S$-module $\mathfrak{Lie}(G/S)$ endowed functorially a bilinear map verifying

$$[x,y] + [y,x] = 0, \quad [x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0.$$

We therefore say that $\mathfrak{Lie}(G/S)$, endowed with this structure, is the **quasi-Lie algebra** of G over S. For any linear representation of G over a good $\mathbb{O}_S$-module F, we can associate a representation of the quasi-Lie algebra $\mathfrak{Lie}(G/S)$. In particular, the adjoint representation of G is associated to the adjoint representation of the quasi-Lie algebra.

Example 2.2.84. A group functor G over S is called very good if it is good and $\mathfrak{Lie}(G/S)$ is a Lie algebra over $\mathbb{O}_S$ (that is, if we have the identity $[x,x] = 0$). The following S-groups are very good: $\mathcal{A}ut_{\mathbb{O}_S}(F)$ for any good $\mathbb{O}_S$-module F (cf. Proposition 2.2.81 and Corollary 2.2.82), any representable group (see below), any good S-group admitting a monomorphism into a very good S-group (cf. Proposition 2.2.23), for example any good subgroup of a very good representable group, or any good S-group admitting a faithful representation over a good $\mathbb{O}_S$-module, for example any good S-group such that Ad is faithful.

Now suppose that G is a group scheme. By Proposition 2.2.55, $\mathfrak{Lie}(G/S)(S)$ is identified with right invariant infinitesimal automorphisms of G, hence by (2.2.16) with derivations of $\mathscr{O}_G$ over $\mathscr{O}_S$ invariant under right translations. Moreover, this identification respects the module structure and is an *anti-isomorphism* of Lie algebras: put $\mathscr{O}_I = \mathscr{O}_S[t]$ and $\mathscr{O}_{I'} = \mathscr{O}_S[t']$ and let $x \in L(I)$ and $y \in L(I')$. The left translation λ_x (resp. λ_y) is an S-automorphism of $G_{I\times I'}$ which induces the identity on $G_{I'}$ (resp. G_I) and which corresponds to an $\mathscr{O}_S$-automorphism

$$u = \mathrm{id} + td_x, \qquad (\text{resp.} \quad v = \mathrm{id} + t'd_y)$$

of $\mathscr{O}_{G_{I\times I'}} = \mathscr{O}_G[t,t']/(t^2,t'^2)$, where d_x, d_y are $\mathscr{O}_S$-derivations of $\mathscr{O}_G$ invariant under right translations. As the correspondence of S-automorphisms of $G_{I\times I'}$ and $\mathscr{O}_S$-automorphisms of $\mathscr{O}_{G_{I\times I'}}$ is contravariant, $\lambda_x\lambda_y\lambda_x^{-1}\lambda_y^{-1}$ corresponds to $v^{-1}u^{-1}vu = \mathrm{id} + tt'(d_yd_x - d_xd_y)$. We then deduce from Remark 2.2.80 that the map $x \mapsto -d_x$ is an isomorphism of Lie algebras. The preceding argument is valid for $\mathfrak{Lie}(G/S)(S') = \mathfrak{Lie}(G_{S'}/S')(S')$ for any $S' \to S$, so we recover the following classical definition:

Proposition 2.2.85. *Via the isomorphism $x \mapsto -d_x$, $\mathfrak{Lie}(G/S)$ is identified with the functor which associates any $S' \to S$ to the Lie algebra of derivations of $G_{S'}$ over S' invariant under right translations.*

As we have seen in Example 2.2.76 that any representable group is good, we conclude the following corollary:

Corollary 2.2.86. *Any representable grop is very good.*

Let $e : S \to G$ be the unit section of G. Put $\omega^1_{G/S} = e^*(\Omega^1_{G/S})$ and recall that (cf. Proposition 2.2.15) $\mathfrak{Lie}(G/S)$ is represented by the vector bundle $\mathbb{V}(\omega^1_{G/S})$. We then have assocaited functorially to any S-group scheme G a vector bundle $\mathrm{Lie}(G/S) = \mathbb{V}(\omega^1_{G/S})$ over S, which represents the functor $\mathfrak{Lie}(G/S)$, hence is endowed with the structure of a Lie algebra S-scheme over $\mathbb{O}_S$. Moreover, this construction commutes with base change and finite products.

Remark 2.2.87. Let $\pi : G \to S$ be the structural morphism. The $\mathscr{O}_G$-module $\Omega^1_{G/S}$ is evidently $(G \times_S G)$-equivariant and hence, by ([?] I, 6.8.1), we have $\Omega^1_{G/S} \cong \pi^*(\omega^1_{G/S})$. It follows for example that $\Omega^1_{G/S}$ is locally free (resp. locally free of finite rank) if $\omega^1_{G/S}$ is, which is in particular the case if S is the spectrum of a field (resp. if S is the spectrum of a field and G is locally of finite type over S). Moreover, by ([?] I, 6.8.2), $\omega^1_{G/S}$ is endowed with a canonical $\mathbb{O}_S[G]$-module structure, which induces over $\mathbb{V}(\omega^1_{G/S}) = \mathrm{Lie}(G/S)$ the adjoint representation.

On the other hand, e is an immersion, and is a closed immersion if G is separated over S (cf. **??**). Hence $\omega^1_{G/S}$ is identified with $\mathscr{I}/\mathscr{I}^2$, where $\mathscr{I}$ is the quasi-coherent ideal defining $e(S)$ in an open subset U of G in which $e(G)$ is closed (if G is separated over S, we can put $U = G$, and if $G = \mathrm{Spec}(\mathscr{A}(G))$ is affine over S, $\mathscr{I}$ is none other than the augmented ideal of $\mathscr{A}(G)$, i.e. the kernel of $e^\sharp : \mathscr{A}(G) \to \mathscr{O}_S$).

Remark 2.2.88. We deduce from the isomorphism $\Omega^1_{G/S} \cong \pi^*(\omega^1_{G/S})$ that the $\mathscr{O}_S$-module $\omega^1_{G/S}$ is identified with the sheaf $\pi^G_*(\Omega^1_{G/S})$ of right invariant differentials of G over S, that is, the sheaf whose sections over an open subset U of S are the sections of $\Omega^1_{G/S}$ over $\pi^{-1}(U)$ which are invariant under right translations (cf. ([?] I, 6.8.3)).

We denote by $\mathscr{L}ie(G/S)$ the sheaf of sections of the vector bundle $\mathrm{Lie}(G/S) \to S$, which is the $\mathscr{O}_S$-module $(\omega^1_{G/S})^\vee = \mathscr{H}om_{\mathscr{O}_S}(\omega^1_{G/S}, \mathscr{O}_S)$ dual to $\omega^1_{G/S}$ (cf. **??**). It is endowed with a Lie algebra structure over $\mathscr{O}_S$. As this construction does not commute with base change (in general), the Lie algebra structure on $\mathscr{L}ie(G/S)$ does not allow us to reconstruct the S-scheme structure on the $\mathbb{O}_S$-Lie algebra $\mathrm{Lie}(G/S)$. However, we have:

Proposition 2.2.89. *Suppose that $\omega^1_{G/S}$ is locally free of finite type. Then $\mathscr{L}ie(G/S)^\vee \cong (\omega_{G/S})^{\vee\vee} \cong \omega^1_{G/S}$ and hence*

$$\mathrm{Lie}(G/S) = \mathbb{V}(\omega^1_{G/S}) = \mathbb{V}(\mathscr{L}ie(G/S)^\vee) = \Gamma_{\mathscr{L}ie(G/S)}.$$

Proof. In fact, $\omega^1_{G/S}$ is reflexive if it is locally free of finite type, and the assertion follows from Corollary 2.1.26. □

Finally, let $G \to H$ be a monomorphism of group functors. Then $\mathfrak{Lie}(G/S) \to \mathfrak{Lie}(H/S)$ is also a monomorphism (cf. Proposition 2.2.23). As any monomorphism of vector bundles is a closed immersion[3], we obtain:

Corollary 2.2.90. *Let $G \to H$ be a monomorphism of S-groups.*

(i) $\mathrm{Lie}(G/S) \to \mathrm{Lie}(H/S)$ *is a closed immersion and hence $\omega^1_{H/S} \to \omega^1_{G/S}$ is an epimorphism.*

(ii) *If $\omega^1_{G/S}$ is locally free of finite type, then the corresponding morphism $\mathscr{L}ie(G/S) \to \mathscr{L}ie(H/S)$ is an isomorphism from $\mathscr{L}ie(G/S)$ to a submodule of $\mathscr{L}ie(H/S)$ which is locally a direct factor.*

Example 2.2.91. Let $S = \mathrm{Spec}(k)$ with k a field of characteristic p. Let $\boldsymbol{\alpha}_{p,S}$ be the S-functor which to any S-scheme T associates

$$\boldsymbol{\alpha}_{p,S}(T) = \{x \in \mathscr{O}(T) : x^p = 0\}.$$

Then $\boldsymbol{\alpha}_{p,S}$ is represented by $\mathrm{Spec}(\mathscr{O}_S[X]/(X^p))$, and hence is a very good S-group. It is also endowed with an $\mathbb{O}_S$-module structure, which is not very good, because the canonical morphism $\boldsymbol{\alpha}_{p,S} \to \mathfrak{Lie}(\boldsymbol{\alpha}_{p,S}/S) = \mathbb{G}_{a,S}$[4] is not bijective.

[3] Let $f : \mathscr{M} \to \mathscr{N}$ be a morphism of $\mathscr{O}_S$-modules and $\mathscr{P} = \operatorname{coker} f$. If $\mathbb{V}(\mathscr{N}) \to \mathbb{V}(\mathscr{M})$ is a monomorphism, the surjective morphism $\boldsymbol{S}(\mathscr{N}) \to \boldsymbol{S}(\mathscr{P})$ factors through $\mathscr{O}_S$, hence $\mathscr{P} = 0$.

[4] This can be deduced from the exact sequence (2.2.17), or we can also note that $\omega^1_{G/k} = k[X]$.

Example 2.2.92. Let Nil be the $\mathbb{Z}$-functor defined as follows: for any scheme S, $\mathrm{Nil}(S)$ is the nilideal of $\mathscr{O}_S$:

$$\mathrm{Nil}(S) = \{x \in \mathscr{O}(S) : \text{there exists } n \in \mathbb{N} \text{ such that } x^n = 0\}.$$

Let Nil^2, $\mathbb{O}_{\mathrm{red}}$ and F be the $\mathbb{Z}$-functors in groups which associate to any scheme S, respectively, the ideal $\mathrm{Nil}(S)^2$ and

$$\mathbb{O}_{\mathrm{red}}(S) = \mathscr{O}(S)/\mathrm{Nil}(S), \quad F(S) = \mathscr{O}(S)/\mathrm{Nil}(S)^2.$$

It is easily seen that $\mathfrak{Lie}(\mathbb{O}_{\mathrm{red}}/\mathbb{Z}) = 0$, hence the $\mathbb{O}_{\mathbb{Z}}$-module $\mathbb{O}_{\mathrm{red}}$ is not good (although it is a good $\mathbb{Z}$-group). If M, N are free $\mathbb{Z}$-modules of finite rank, we have

$$\mathrm{Nil}^2(I_S(M \oplus N)) = \mathrm{Nil}^2(S) \oplus \mathrm{Nil}^2(S) \otimes_{\mathbb{Z}} M \oplus \mathrm{Nil}(S) \otimes_{\mathbb{Z}} N$$

and hence

$$F(I_S(M \oplus N)) = F(S) \oplus \mathbb{O}_{\mathrm{red}}(S) \otimes_{\mathbb{Z}} M \oplus \mathbb{O}_{\mathrm{red}}(S) \otimes_{\mathbb{Z}} N.$$

We then deduce, on the one hand, that the $\mathbb{Z}$-functor F satisfies condition (E) and, on the other hand, that $\mathfrak{Lie}(F/\mathbb{Z}) = \mathbb{O}_{\mathrm{red}}$ (cf. (2.2.17)); as the latter is not a good $\mathbb{O}_{\mathbb{Z}}$-module, this shows that F is a $\mathbb{Z}$-group which satisfies condition (E) but is not good.

2.2.3 Calculation of some Lie algebras

2.2.3.1 Lie algebras of diagonalizable groups Let $G = D_S(M)$ be a diagonalizable group over S (cf. 2.1.2.4). The formation of $\mathfrak{Lie}(G/S)$ commutes with base change, so it suffices to consider this construction for $G = D(M)$. We then have

$$G(I_S) = \mathrm{Hom}_{\mathbf{Grp}}(M, \Gamma(I_S, \mathscr{O}_{I_S})^\times) = \mathrm{Hom}_{\mathbf{Grp}}(M, \Gamma(S, \mathscr{D}_{\mathscr{O}_S})^\times).$$

Now the section $S \to I_S$ induces a split exact sequence

$$1 \longrightarrow \Gamma(S, \mathscr{O}_S) \longrightarrow \Gamma(S, \mathscr{D}_{\mathscr{O}_S})^\times \longrightarrow \Gamma(S, \mathscr{O}_S)^\times \longrightarrow 0$$

which implies that $\mathfrak{Lie}(G)(S)$ is identified with $\mathrm{Hom}_{\mathbf{Grp}}(M, \mathbb{O}_S)$, endowed with the evident $\mathbb{O}(S)$-module structure. We then obtain by base change the following:

Proposition 2.2.93. *We have isomorphisms*

$$\mathcal{H}om_{S\text{-}\mathbf{Grp}}(M_S, \mathbb{O}_S) \xrightarrow{\sim} \mathfrak{Lie}(D_S(M)/S), \quad \mathcal{H}om_{\mathbf{Grp}}(\widetilde{M}_S, \mathscr{O}_S) \xrightarrow{\sim} \mathscr{L}ie(D_S(M)/S),$$

where, in the second isomorhism, $\widetilde{M}_S$ is the sheaf of constant group over S defined by M, and $\mathcal{H}om_{\mathbf{Grp}}$ is the sheaf of homomorphisms of groups.

Corollary 2.2.94. *If M is free of finite rank, then*

$$\Gamma_{\mathscr{L}ie(D_S(M)/S)} \xrightarrow{\sim} \mathfrak{Lie}(D_S(M)/S), \quad M^\vee \otimes_{\mathbb{Z}} \mathscr{O}_S \xrightarrow{\sim} \mathscr{L}ie(D_S(M)/S).$$

In particular, $\mathbb{O}_S \cong \mathfrak{Lie}(\mathbb{G}_{m,S}/S)$ and $\mathscr{O}_S \cong \mathscr{L}ie(\mathbb{G}_{m,S}/S)$.

Proof. The second isomorphism follows from Proposition 2.2.93 the isomorphism

$$M^\vee \otimes_{\mathbb{Z}} \mathscr{O}_S = \mathrm{Hom}_{\mathbb{Z}}(\widetilde{M}_S, \mathscr{O}_S) = \mathrm{Hom}_{\mathbf{Grp}}(\widetilde{M}_S, \mathscr{O}_S),$$

which it implies that $\Gamma_{\mathscr{L}ie(D_S(M)/S)} = \mathcal{H}om_{S\text{-}\mathbf{Grp}}(M_S, \mathbb{O}_S)$, whence the first isomorphism. □

2.2.3.2 Normalizers and centralizers Recall that a sequence $0 \to F' \to F \to F'' \to 0$ of $\mathbb{O}_S$-modules is called **exact** if for any $S' \to S$ the sequence $0 \to F'(S') \to F(S') \to F''(S') \to 0$ of $\mathbb{O}(S')$-modules is exact. Similarly, a sequence $1 \to G' \to G \to G'' \to 1$ of S-groups is exact if for any $S' \to S$ the sequence of groups $1 \to G'(S') \to G(S') \to G''(S') \to 1$ is exact.

Lemma 2.2.95. *Let $1 \to G' \to G \to G'' \to 1$ be an exact sequence of S-groups.*

(i) *The sequences $1 \to T_{G'/S}(\mathscr{M}) \to T_{G/S}(\mathscr{M}) \to T_{G''/S}(\mathscr{M}) \to 1$ and $1 \to \mathfrak{Lie}(G'/S, \mathscr{M}) \to \mathfrak{Lie}(G/S, \mathscr{M}) \to \mathfrak{Lie}(G''/S, \mathscr{M}) \to 1$ are exact.*

(ii) *Let $1 \to H' \to H \to H'' \to 1$ be a second exact sequence of groups; it is exact if and only if the following sequence is exact:*

$$1 \longrightarrow G' \times_S H' \longrightarrow G \times_S H \longrightarrow G'' \times_S H'' \longrightarrow 1$$

(iii) *If two of the S-groups G', G, G'' satisfy condition (E), then the third one satisfies condition (E).*

(iv) *If $0 \to F' \to F \to F' \to 0$ is an exact sequence of $\mathbb{O}_S$-modules and two of the modules F', F, F'' are good, the third one is good.*

(v) *If two of the S-groups are good, the third one is good.*

Lemma 2.2.96. *Let G be an S-group, E, H be G-objects, F be an $\mathbb{O}_S[G]$-module.*

(a) *The canonical homomorphism $E^G \times_S H^G \to (E \times_S H)^G$ is an isomorphism.*

(b) *If F is a good $\mathbb{O}_S$-module, so is F^G.*

If E is an S-group and F is a sub-S-group of E, we denote by E/F the S-functor which to any $S' \to S$ associates the set $E(S')/F(S')$ of classes $\bar{x} = xF(S')$, $x \in E(S')$. If E is an abelian group over S, then E/F is endowed with an abelian group structure.

Now let G be an S-group and K be a sub-S-group of G; put $E = \mathfrak{Lie}(G/S, \mathscr{M})$ and $F = \mathfrak{Lie}(K/S, \mathscr{M})$. The adjoint action of K on E stablize F, hence induces an action of K over the S-functor E/F. For any $S' \to S$, we then have

$$(E/F)^K(S') = \{\bar{x} \in E(S')/F(S') : f^*(x^{-1})\mathrm{Ad}(k)(f^*(x)) \in F(S'') \text{ for } f : S'' \to S', k \in K(S'')\}$$

where $f^*(x)$ denotes the image of x in $E(S'')$.

Theorem 2.2.97. *Let G be an S-group, K be a sub-S-group of G, $N = N_G(K)$ and $Z = Z_G(K)$.*

(i) *If the group law of $\mathfrak{Lie}(G/S, \mathscr{M})$ is abelian, then*

$$\mathfrak{Lie}(N/S, \mathscr{M})/\mathfrak{Lie}(K/S, \mathscr{M}) = \big(\mathfrak{Lie}(G/S, \mathscr{M})/\mathfrak{Lie}(K/S, \mathscr{M})\big)^K.$$

(ii) *If the group law of $\mathfrak{Lie}(G/S, \mathscr{M})$ is abelian, then $\mathfrak{Lie}(Z/S, \mathscr{M}) = \mathfrak{Lie}(G/S, \mathscr{M})^K$.*

(iii) *If G satisfies condition (E) (resp. if G and K satisfy condition (E)), then Z satisfies condition (E) (resp. N satisfies condition (E)).*

(iv) *If G is good (resp. very good), then Z is good (resp. very good).*

(v) *If G and K are good, then N is good. If moreover G is very good, then N is very good.*

Corollary 2.2.98. *We have $\mathfrak{Lie}(Z(G)/S) = \mathfrak{Lie}(G/S)^G$ if the group law of $\mathfrak{Lie}(G/S)$ is abelian.*

Corollary 2.2.99. *If the group law of $\mathfrak{Lie}(G/S)$ is abelian and K is a normal subgroup of G, then*

$$\mathfrak{Lie}(G/S, \mathscr{M})/\mathfrak{Lie}(K/S, \mathscr{M}) = \big(\mathfrak{Lie}(G/S, \mathscr{M})/\mathfrak{Lie}(K/S, \mathscr{M})\big)^K.$$

Let G be a good S-group acting linearly on a good $\mathbb{O}_S$-module F via

$$\rho : G \to \mathcal{A}ut_{\mathbb{O}_S}(F).$$

We have defined a corresponding linear representation

$$d\rho : \mathfrak{Lie}(G/S) \to \mathcal{E}nd_{\mathbb{O}_S}(F).$$

The subgroups $N_G(E)$ and $Z_G(E)$ are defined for any subset E of F. Similarly, for any $S' \to S$, we define

$$N_{\mathfrak{Lie}(G/S)}(E)(S') = \{X \in \mathfrak{Lie}(G/S) : d\rho(X)E_{S'} \subseteq E_{S'}\},$$
$$Z_{\mathfrak{Lie}(G/S)}(E)(S') = \{X \in \mathfrak{Lie}(G/S) : d\rho(X)E_{S'} = 0\}.$$

called the **normalizer** and **centralizer**, respectively, of E in F.

Theorem 2.2.100. *Let G be a good S-group acting linearly on a good $\mathbb{O}_S$-module F, and E be a sub-$\mathbb{O}_S$-module of F.*

(a) *We have $\mathfrak{Lie}(Z_G(E)/S) = Z_{\mathfrak{Lie}(G/S)}(E)$ and $Z_G(E)$ is a good S-group; it is very good if G is.*

(b) *Suppose that E is a good $\mathbb{O}_S$-module. Then $\mathfrak{Lie}(N_G(E)/S) = N_{\mathfrak{Lie}(G/S)}(E)$ and $N_G(E)$ is a good S-group; it is very good if G is.*

Example 2.2.101. Let G be a good S-group. Then Theorem 2.2.100 can be applied to the adjoint representation of G. Let E be a good submodule of $\mathfrak{Lie}(G/S)$, for which we can associate the normalizer and centralizer. By Theorem 2.2.100, their Lie algebras are respectively the normalizer and centralizer of E in $\mathfrak{Lie}(G/S)$, given by the usual definition:

$$N_{\mathfrak{Lie}(G/S)}(E)(S') = \{X \in \mathfrak{Lie}(G/S) : [X, E_{S'}] \subseteq E_{S'}\},$$
$$Z_{\mathfrak{Lie}(G/S)}(E)(S') = \{X \in \mathfrak{Lie}(G/S) : d\rho[X, E_{S'}] = 0\}.$$

Example 2.2.102. Let K be a sub-S-group of G, then $\mathfrak{Lie}(K/S)$ is a sub-$\mathbb{O}_S$-module of $\mathfrak{Lie}(G/S)$. Suppose that $\mathfrak{Lie}(K/S)$ is a good $\mathbb{O}_S$-module; we evidently have

$$N_G(K) \subseteq N_G(\mathfrak{Lie}(K/S)), \quad Z_G(K) \subseteq Z_G(\mathfrak{Lie}(K/S))$$

whence, by Theorem 2.2.100, we obtain

$$\mathfrak{Lie}(N_G(K)/S) \subseteq N_{\mathfrak{Lie}(G/S)}(\mathfrak{Lie}(K/S)), \quad \mathfrak{Lie}(Z_G(K)/S) \subseteq Z_{\mathfrak{Lie}(G/S)}(\mathfrak{Lie}(K/S)),$$

but none of these four inclusions is a priori an identity. In particular, if K is a normal subgroup of G, then $\mathfrak{Lie}(K/S)$ is an ideal of $\mathfrak{Lie}(G/S)$.

Example 2.2.103. Let S be a scheme, F be the good $\mathbb{O}_S$-module $\mathbb{O}_S^2$ endowed with the natural action of the good S-group $G = \mathrm{GL}_{2,S}$, and E be the sub-$\mathbb{O}_S$-module of F formed by couples (x_1, x_2) such that x_2 is nilpotent. Put $N = N_G(E)$, then $\mathfrak{Lie}(N/S) = \mathfrak{Lie}(G/S)$ while, for any $S' \to S$, we have

$$N_{\mathfrak{Lie}(G/S)}(E)(S') = \left\{ \begin{pmatrix} a & b \\ x & c \end{pmatrix} : a, b, c, x \in \mathscr{O}(S'), x \text{ nilpotent} \right\}$$

hence $\mathfrak{Lie}(N_G(E)/S) \neq N_{\mathfrak{Lie}(G/S)}(E)$.

By considering the semi-direct product $G' = F \rtimes G$, we obtain a similar conter-example where E is a sub-$\mathbb{O}_S$-module of $\mathfrak{Lie}(G'/S)$. We also note that with the notations above, $E = \mathfrak{Lie}(K/S)$ where K is the subgroup $\mathbb{O}_S \oplus \mathrm{Nil}^2$ of F (that is, for any $S' \to S$, $K(S')$ is formed by couples (x_1, x_2) where $x_2 \in \mathrm{Nil}(S')^2$).