Algebra

Long

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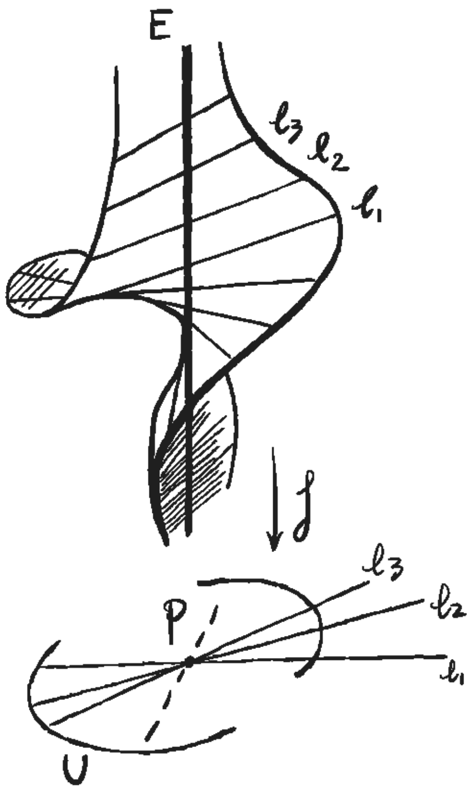

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0.0.1 Sheaves and Étalé Spaces

We now give a different description of sheaves via so-called étalé spaces.

Definition 0.0.1. *Let X be a topological space.*

(a) *A pair (E, π), where E is a topological space and $\pi : E \to X$ is a local homeomorphism, is called an* ***étalé space over X****.*

(b) *Let (E_1, π_1) and (E_2, π_2) be étalé spaces over X. A morphism $\varphi : (E_1, \pi_1) \to (E_2, \pi_2)$ of étalé spaces is a continuous map $\varphi : E_1 \to E_2$ such that $\pi_1 = \pi_2 \circ \varphi$.*

$$\begin{tikzcd} E_1 \arrow[rr, "\varphi"] \arrow[dr, "\pi_1"'] & & E_2 \arrow[dl, "\pi_2"] \\ & X & \end{tikzcd}$$

Denote by Ét/X the category of étalé spaces over X.

Definition 0.0.2. *Let (E, π) be an étalé space over X. For $U \subseteq X$ open, a* ***section*** *of E over U is a continuous map $s : U \to E$ with $\pi \circ s = \mathrm{id}_U$. The* ***fiber*** *of E over $x \in X$ is the set $E_x = \pi^{-1}(x)$. A morphism of étalé spaces $f : (E_1, \pi_1) \to (E_2, \pi_2)$ gives rise to maps $f_x : (E_1)_x \to (E_2)_x$ for $x \in X$.*

Proposition 0.0.3. *Let (E, π) be an étalé space over X. Then the following properties hold:*

(a) *The map π is an open map.*

(b) *For any open subset U of X and any continuous section $s \in \Gamma(U, E)$, the subset $s(U)$ is open in E, such open subsets form a basis for the topology of E.*

Now we justify our notation on the element of $\mathscr{F}(U)$: we show how to construct a sheaf from a étalé space using sections.

Definition 0.0.4. *Let (E, π) be an étalé space over X. Define a presheaf ΓE of E-valued functions by*

$$\Gamma E(U) = \{\textit{sections } s : U \to E\} = \{s : U \to E : s \textit{ is continuous and } \pi \circ s = \mathrm{id}_U\}.$$

It is a sheaf of E-valued functions by Example **??***.*

Let $\varphi : (E_1, \pi_1) \to (E_2, \pi_2)$ be a morphism of étalé spaces, we obtain a map of sheaves $\Gamma\varphi : \Gamma E_1 \to \Gamma E_2$ defined as follows: For every open subset U of X, the map $(\Gamma\varphi)_U$ is given by

$$(\Gamma\varphi)_U(s) = \varphi \circ s.$$

It is immediately checked that $\Gamma\varphi$ is a map of sheaves. Also, if $\varphi : E_1 \to E_2$ and $\psi : E_2 \to E_3$ are two maps of étalé spaces, then

$$\Gamma(\psi \circ \varphi) = \Gamma\psi \circ \Gamma\varphi \quad \text{and} \quad \Gamma\mathrm{id}_E = \mathrm{id}_{\Gamma E}.$$

This means that the construction $\Gamma : \text{Ét}/X \to \mathsf{Sh}(X)$ is functorial.

The next proposition tells us that the fibres of a étalé space are stalks of the sheaf ΓE.

Proposition 0.0.5. *Let (E, π) be a étalé space. For any $x \in X$, the stalk $(\Gamma E)_x$ is isomorphic to the fibre E_x at x. Furthermore, as a subspace of E, the fibre E_x has the discrete topology.*

Proof. For $x \in X$, the stalk $(\Gamma E)_x$ is the set of equivalence classes of pairs (U, s), where U is an open neighborhood of x and $s : U \to E$ is a section of π. Here (U, s) and (V, t) are equivalent if there exists $x \in W \subseteq U \cap V$ open such that $s|_W = t|_W$.

Fix a $x \in X$, for any open subset $x \in U \subseteq X$, we define a map $\tau_U : \Gamma E(U) \to E_x$ by

$$\tau_U(s) = s(x) \quad \text{for} \quad s \in \Gamma E(U).$$

Then it is obvious that τ commute with the restriction map, hence we get a induced map $\tau_x : (\Gamma E)_x \to E_x$.

For any $e \in E_x$, there exists an open neighborhood $e \in V \subseteq E$ such that $\pi|_V$ is a homeomorphism onto its open image. Then $(\pi|_V)^{-1}$ is obviously a section of π with $(\pi|_V)^{-1}(x) = e$. This shows that τ_x is surjective for every $x \in X$. Note that $E_x \cap V = \{e\}$, so E_x has the discrete topology.

Let $s : U \to E$ and $s' : U' \to E$ be sections with $s(x) = s'(x) = e$ for a point $x \in U \cap U'$. Then there exsits an open neighborhood $e \in V \subseteq E$ such that $\pi|_V : V \to \pi(V)$ is a homeomorphism. Let $W = \pi(V) \cap U \cap U'$, and replace V by $(\pi|_V)^{-1}(W)$. Now $\pi|_V : V \to W$ is a homeomorphism and $s|_W, s'|_W$ are both inverse of it, so $s|_W = s'|_W$. This implies $(s, U) \sim (s', U')$, so τ_x is injective. □

Now we attach conversely to every presheaf an étalé space as follows. Let $\mathscr{F}$ be a presheaf on X. Define $S\mathscr{F} := \coprod_{x \in X} \mathscr{F}_x$ to be the disjoint union of all the stalks. We have a projection map $\pi : S\mathscr{F} \to X$:

$$\pi(s) = x \quad \text{for} \quad s \in \mathscr{F}_x.$$

For every open subset U of X, we view each abstract section $s \in \mathscr{F}(U)$ as the actual function $\widetilde{s} : U \to S\mathscr{F}$ given by

$$\widetilde{s}(x) = s_x \quad \text{for} \quad s \in U.$$

By definition, $\widetilde{s}$ is a section of π. We give $S\mathscr{F}$ the finest topology which makes all the functions $\widetilde{s}$ continuous. Consequently, a subset Ω of $S\mathscr{F}$ is open if and only if for every open subset U of X and every $s \in \mathscr{F}(U)$, the subset

$$\widetilde{s}^{-1}(\Omega) = \{x \in U \mid \widetilde{s}(x) = s_x \in \Omega\}$$

is open in X. The space $S\mathscr{F}$ endowed with the above topology is called the **stalk space** of the presheaf $\mathscr{F}$.

Lemma 0.0.6. *For any $U \subseteq X$ open and $s \in \mathscr{F}(U)$, the set $\widetilde{s}(U)$ is open in $S\mathscr{F}$. Moreover,*

$$\{\widetilde{s}(U) : U \subseteq X \text{ open}, s \in \mathscr{F}(U)\}$$

consists of a basis for $S\mathscr{F}$.

Proof. It suffices to show that for any $x \in U$, any open subset V containing x and any $t \in \mathscr{F}(V)$, the subset

$$\widetilde{t}^{-1}\big(\widetilde{s}(U)\big) = \{x \in V : \widetilde{t}(x) = t_x \in \widetilde{s}(U)\} = \{x \in U \cap V : t_x = s_x\}$$

is open in X. However, $t_x = s_x$ means that there is some open subset $W \subseteq U \cap V$ containing x such that $s|_W = t|_W$, which means that $W \subseteq \widetilde{t}^{-1}\big(\widetilde{s}(U)\big)$. So $\widetilde{t}^{-1}(\widetilde{s}(U))$ is indeed open in X.

Now let U, V be open subsets of X and $s \in \mathscr{F}(U), t \in \mathscr{F}(V)$, respectively. We consider the set $\widetilde{s}(U) \cap \widetilde{t}(V)$. By definition, we have

$$\widetilde{s}(U) \cap \widetilde{t}(V) = \{u \in S\mathscr{F} : \pi(u) \in U \cap V, s_{\pi(u)} = t_{\pi(u)}\}.$$

Let $u \in \widetilde{s}(U) \cap \widetilde{t}(V)$, then $s_{\pi(u)} = t_{\pi(u)}$, so there is a neighborhood W of $\pi(u)$ such that $s|_W = t|_W$. This then means $\widetilde{s|_W}(W) \subseteq \widetilde{s}(U) \cap \widetilde{t}(V)$, so $\{\widetilde{s}(U)\}$ is a basis of $S\mathscr{F}$. □

Now by definition each $\widetilde{s}$ gives a local section of π. Since $\widetilde{s}$ is continuous by our topology, if we can show π is continuous, then π is a local homeomorphism. This is ture because

$$\pi^{-1}(U) = \bigcup_{\substack{s \in \mathscr{F}(V) \\ V \text{ is open}, V \subseteq U}} \widetilde{s}(V).$$

Proposition 0.0.7. *For a presheaf $\mathscr{F}$, the stalk space $S\mathscr{F}$, together with the projection $\pi : S\mathscr{F} \to X$, give a étalé space $(S\mathscr{F}, \pi)$.*

Moreover, let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism of presheaves, then the induced map $\varphi_x : \mathscr{F}_x \to \mathscr{G}_x$ gives a map between stalk spaces:

$$S\varphi : S\mathscr{F} \to \mathscr{G}, \quad S\varphi(e) = \varphi_x(e) \quad \text{for} \quad e \in \mathscr{F}_x.$$

It is clear that $\pi_{\mathscr{F}} = \pi_{\mathscr{G}} \circ S\varphi$, and for any open subset $U \subseteq X$ and $s \in \mathscr{F}(U)$, we have

$$S\varphi \circ \widetilde{s}(x) = S\varphi(s_x) = \varphi_x(s_x) = \big(\varphi_U(s)\big)_x = \widetilde{\varphi_U(s)}(x).$$

Therefore $S\varphi \circ \widetilde{s} = \widetilde{\varphi_U}(s)$. Since $\widetilde{s}$ and $\widetilde{\varphi_U}(s)$ are both local homeomorphisms, it follows that $S\varphi$ is continuous. We then obtain a functor $S : \mathsf{Psh}(X) \to \text{Ét}/X$.

Theorem 0.0.8. *There is a natural isomorphism from $\Gamma \circ S$ to the sheafification.*

Proof. Let $\mathscr{F}$ be a presheaf on X, $(S\mathscr{F}, \pi) = S(\mathscr{F})$ and $\mathscr{F}' = \Gamma S\mathscr{F}$. By construction $\widetilde{s} : U \to E$ is a section of $(S\mathscr{F}, \pi)$ and therefore an element of $\mathscr{F}'(U)$ for any open subset $U \subseteq X$ and $s \in \mathscr{F}(U)$. We define a morphism of presheaves $\kappa : \mathscr{F} \to \mathscr{F}'$ by $\mathscr{F}(U) \mapsto \mathscr{F}'(U), s \mapsto \widetilde{s}$.

Let $x \in X$ be an arbitrary point. By construction $(S\mathscr{F})_x = \mathscr{F}_x$ and due to Proposition 0.0.5 there is a bijective map

$$\tau_x : \mathscr{F}'_x \to (S\mathscr{F})_x = \mathscr{F}_x, \quad \widetilde{s} \mapsto \widetilde{s}(x).$$

For $U \subseteq X$ open, $s \in \mathscr{F}(U)$ we have

$$\tau_x\big(\kappa_x(s_x)\big) = \tau_x(\widetilde{s}_x) = \widetilde{s}(x) = s_x.$$

so that $\tau_x \circ \kappa_x = \mathrm{id}_{\mathscr{F}_x}$ and κ_x is bijective for every $x \in X$. It is straightforward to check that κ defines a natural transformation and due to Example **??** we attain an isomorphism from $\mathscr{F}'$ to the sheafification of $\mathscr{F}$ in a natural way making it a natural isomorphism. □

Theorem 0.0.9. *There is a natural isomorphism $S \circ \Gamma$ to the identity functor.*

Proof. Let (E, π) be an étalé space over X, $(E', \pi') = S \circ \Gamma E$. By Proposition 0.0.5 we have a bijection $\tau_x : (\Gamma E)_x \to E_x$ and by construction $E' = \coprod_{x \in X} (\Gamma E)_x$. This defines a bijective map $\tau : E' \to E$ with $\pi \circ \tau = \pi'$. For $U \subseteq X$ open and $s \in \Gamma E(U)$, we have

$$\tau(\widetilde{s}(x)) = \tau(s_x) = s(x).$$

for every $x \in X$. The topology of E' is the finest such that $\widetilde{s} : U \to E'$ is continuous for every s, U and the topology of E is the finest such that $s : U \to E$ is continuous for every s, U. This implies that τ is a homeomorphism:

$$\begin{aligned} W \subseteq E \text{ is open } &\iff s^{-1}(W) \subseteq X \text{ is open for every } s : U \to E \\ &\iff (\tau \circ \widetilde{s})^{-1}(W) \subseteq X \text{ is open for every } s : U \to E \\ &\iff \tau^{-1}(W) \text{ is open in } E' \end{aligned}$$

It is straightforward to check that this isomorphism is natural. □

As the sheafification of sheaf is the sheaf itself we deduce from Theorem0.0.8 and Theorem 0.0.9:

Proposition 0.0.10. *Let X be a topological space. The functors Γ and S yield an equivalence between the category Ét/X of étalé spaces over X and the category $\mathsf{Sh}(X)$ of sheaves on X.*

Example 0.0.11 (Étalé Spaces of constant sheaves). Let E be a set that we also consider as a discrete topological space. Let E_X be the constant sheaf with values in E on a topological space X. Then the corresponding étalé space is $(X \times E, \pi_1)$ because as a set, $SE_X = X \times E$, and for $U \subseteq X$ open the sections of π_1 over U are just the maps $x \mapsto (x, s(x))$, where $s : U \to E$ is locally constant.

Note that the map $\pi_1 : X \times E \to X$ is a trivial covering map. More generally, every covering map is an étalé space that is locally on X a trivial covering map. Hence we obtain the following result.

Proposition 0.0.12 (Locally constant sheaves). *A sheaf $\mathscr{F}$ on X is called* ***locally constant*** *if there exists an open covering $(U_i)_i$ of X such that $F|_{U_i}$ is a constant sheaf.*

The equivalence of $\mathsf{Sh}(X)$ and Ét/X yields an equivalence between the full subcategory of locally constant sheaves on X and the category of covering spaces of X.

Example 0.0.13. Let $\mathcal{L}$ be the sheaf of complex logarithms on $\mathbb{C} - \{0\}$,

$$\mathcal{L}(U) := \{L : U \to \mathbb{C} \text{ holomorphic} \mid \exp \circ L = \mathrm{id}_U\} \quad \text{for} \quad U \subseteq \mathbb{C} - \{0\}.$$

For every simply connected open subspace U of $\mathbb{C} - \{0\}$ the choice of a logarithm L_0 on U yields an isomorphism of sheaves of abelian groups $(2\pi i\mathbb{Z})_U \cong \mathcal{L}|_U$: one attaches to a locally constant function t with values in $2\pi i\mathbb{Z}$ on an open subset V of U the logarithm $L_0|_V + t$. Hence $\mathcal{L}$ is a locally constant sheaf of abelian groups. But it is not constant because $\mathcal{L}(\mathbb{C} - \{0\}) = \varnothing$. The associated étalé space to $\mathcal{L}$ is the covering map $\exp : \mathbb{C} \to \mathbb{C} - \{0\}$.

0.0.1.1 Inverse image and étalé space

We have already seen that there is a natural correspondence between sheaves and étalé space and that it is possible to describe the sheafification of a presheaf in terms of associated étalé spaces. We will now show that the formation of the inverse image of a presheaf has a simple description in terms of étalé spaces: The corresponding étalé space is given by the fiber product. Hence let us consider continuous maps $f : X' \to X$ and $\pi : E \to X$ of topological spaces. We form the fiber product and obtain the following commutative diagram, where $g : E \times_X X' \to E$ and $\pi' : E \times_X X' \to X'$ are the projections:

$$\begin{tikzcd} E \times_X X' \arrow[r, "\pi'"] \arrow[d, "g"] & X' \arrow[d, "f"] \\ E \arrow[r, "\pi"] & X \end{tikzcd}$$

Lemma 0.0.14. *Suppose that π has one of the following properties:*

(a) *homeomorphism.*

(b) *open topological embedding.*

(c) *local homeomorphism.*

Then π' has the same property.

Proof. Assertion (a) is clear because $\pi \mapsto \pi'$ is functorial: the fiber product is given by

$$E \times_X X' = \{(e, x') : \pi(e) = f(x')\},$$

If ω is a continuous inverse of π, then we have $e = \omega \circ \pi(e) = \omega \circ f(x')$. So the map

$$\omega : X' \to E \times_X X', \quad x' \mapsto (\omega \circ f(x'), x')$$

is a continuous inverse of π'.

If $\pi : E \to U$ is a homeomorphism for some $U \subseteq X$ open, then the restriction $\pi' : E' \to f^{-1}(U)$ is a homeomorphism by (a). This shows (b).

Finally, if there exists an open covering $(W_i)_i$ of E such that $\pi|_{W_i}$ is an open embedding for all i, then $(g^{-1}(W_i))_i$ is an open covering of E' and $\pi'|_{g^{-1}(W_i)}$ is an open embedding by (b). This proves (c). □

The fiber product construction above yields a functor $\text{Ét}/X \to \text{Ét}/X'$ by sending a morphism $f : E_1 \to E_2$ of étalé spaces over X to the map $E_1 \times_X X' \to E_2 \times_X X'$ induced by $f \times \mathrm{id}_{X'}$.

Proposition 0.0.15 (Inverse image via étalé spaces). *Let $f : X \to Y$ be a continuous map of topological spaces, $\mathscr{E}$ a presheaf on Y and (E, π) the étalé space over Y associated to $\mathscr{E}$. The functor that sends $\mathscr{E}$ to the sheaf associated to the étalé space $(E \times_Y X, \pi')$ is naturally isomorphic to the inverse image functor f^{-1}.*

Example 0.0.16 (Inverse image of constant sheaves). Let $f : X \to Y$ be a continuous map. Let E be a set, let E_Y be the sheaf of locally constant E-valued functions on Y. The corresponding étalé space is the projection $E \times Y \to Y$, where we consider E as a discrete topological space. Then the projection $(E \times Y) \times_Y X \to E \times X$ is a homeomorphism compatible with the projections to X. Hence $f^{-1}E_Y = E_X$.

Definition 0.0.17 (Pullback of sections). *Let $f : X \to Y$ be a continuous map, let $\mathscr{G}$ be a sheaf on Y, and let $t \in \mathscr{G}(V)$, $V \subseteq Y$ open. Let $\pi : G \to Y$ be the étalé space corresponding to $\mathscr{G}$ and consider t as a continuous section $t : V \to G$ of π. Then*

$$f^{-1}(t) : f^{-1}(V) \to G \times_Y X, \quad x \mapsto \big(t \circ f(x), x\big)$$

is a continuous section of $G \times_Y X \to X$. Hence we obtain a pullback map

$$f^{-1} : \mathscr{G}(V) \to (f^{-1}\mathscr{G})(f^{-1}(V)),$$

which is functorial in $\mathscr{G}$ and compatible with restrictions to smaller open subsets of Y.

Example 0.0.18. Let X be a topological space, let S be a subspace of X, and denote by $i : S \to X$ the inclusion. If $\mathscr{F}$ is a sheaf on X with corresponding étalé space $\pi : E \to X$, then the étalé space corresponding to $\mathscr{F}|_S$ is the usual restriction $\pi|_{\pi^{-1}(S)} : \pi^{-1}(S) \to S$. The pullback $i^{-1}(s)$ of a continuous section $s : U \to E$ of π for $U \subseteq X$ is simply the restriction $s|_{S \cap U}$.