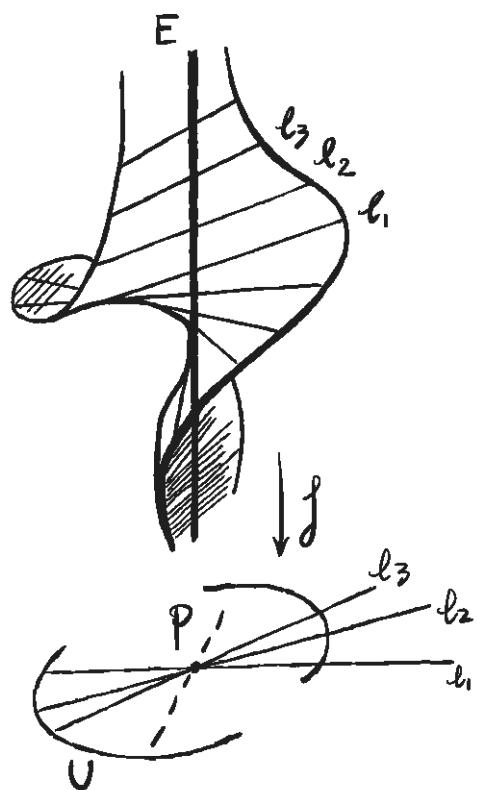


Algebra

Xiaolong Pan

April 17, 2023



Contents

Chapter 1

General topology

1.1 Topological structures

1.1.1 Topological spaces

Definition 1.1.1. A *topological structure* (or, more briefly, a *topology*) on a set X is a structure given by a set \mathcal{T} of subsets of X , having the following properties:

- (O1) $\emptyset, X \in \mathcal{T}$.
- (O2) Every union of sets of \mathcal{T} is a set of \mathcal{T} .
- (O3) Every finite intersection of sets of \mathcal{T} is a set of \mathcal{T} .

The sets of \mathcal{T} are called **open sets** of the topological structure defined by \mathcal{T} on X .

A **topo space** is a set endowed with a topological structure. The elements of a topological space are often called points. When a topology has been defined on a set X , this set is said to be the set underling the topological space X .

Example 1.1.2. If X is any set, the collection of all subsets of X is a topology on X ; it is called the **discrete topology**. The collection consisting of X and \emptyset only is also a topology on X ; we shall call it the **indiscrete topology**, or the **trivial topology**.

Example 1.1.3. Let X be a set; let \mathcal{T}_{fc} be the collection of all subsets U of X such that U^c either is finite or is all of X . Then \mathcal{T}_{fc} is a topology on X , called the **finite complement topology**. Both X and \emptyset are in \mathcal{T}_{fc} , since X^c is finite and \emptyset^c is all of X . If $\{U_\alpha\}$ is an indexed family of nonempty elements of \mathcal{T}_{fc} , to show that $\bigcap_\alpha U_\alpha$ is in \mathcal{T}_{fc} , we compute

$$\left(\bigcup_\alpha U_\alpha \right)^c = \bigcap_\alpha U_\alpha^c.$$

The latter set is finite because each set U_α^c is finite. If U_1, \dots, U_n are nonempty elements of \mathcal{T}_{fc} , to show that $\bigcap_{i=1}^n U_i$ is in \mathcal{T}_{fc} , we compute that

$$\left(\bigcap_{i=1}^n U_i \right)^c = \bigcup_{i=1}^n U_i^c.$$

The latter set is a finite union of finite sets and, therefore, finite.

Similarly, let \mathcal{T}_{cc} be the collection of all subsets U of X such that U^c either is countable or is all of X . Then \mathcal{T}_{cc} is a topology on X , called the **countable complement topology** on X .

Definition 1.1.4. Suppose that \mathcal{T}_1 and \mathcal{T}_2 are two topologies on a given set X . If $\mathcal{T}_1 \supseteq \mathcal{T}_2$, we say that \mathcal{T}_1 is **finer** than \mathcal{T}_2 ; if \mathcal{T}_1 properly contains \mathcal{T}_2 , we say that \mathcal{T}_1 is **strictly finer** than \mathcal{T}_2 . We also say that \mathcal{T}_2 is **coarser** than \mathcal{T}_1 , or **strictly coarser**, in these two respective situations. We say \mathcal{T}_1 is **comparable** with \mathcal{T}_2 if either $\mathcal{T}_1 \subseteq \mathcal{T}_2$ or $\mathcal{T}_2 \subseteq \mathcal{T}_1$.

1.1.2 Base of topologies

For each of the examples in the preceding part, we were able to specify the topology by describing the entire collection \mathcal{T} of open sets. Usually this is too difficult. In most cases, one specifies instead a smaller collection of subsets of X and defines the topology in terms of that.

Definition 1.1.5. If X is a set, a base for a topology on X is a collection \mathcal{B} of subsets of X (called **base elements**) such that

(B1) For each $x \in X$, there is at least one base element B containing x .

(B2) If x belongs to the intersection of two base elements B_1 and B_2 , then there is a base element B_3 containing x such that $B_3 \subseteq B_1 \cap B_2$.

If \mathcal{B} satisfies these two conditions, then we define the topology \mathcal{T} generated by \mathcal{B} as follows: A subset U of X is said to be open in X (that is, to be an element of \mathcal{T}) if for each $x \in U$, there is a base element $B \in \mathcal{B}$ such that $x \in B$ and $B \subseteq U$. Note that each base element is itself an element of \mathcal{T} . We now check that the collection \mathcal{T} is indeed a topology on X .

Proposition 1.1.6. The collection \mathcal{T} induced by a base \mathcal{B} is a topology on X .

Proof. If U is the empty set, it satisfies the defining condition of openness vacuously. Likewise, X is in \mathcal{T} , since for each $x \in X$ there is some base element B containing x and contained in X . Now let us take an indexed family $(U_\alpha)_{\alpha \in A}$, of elements of \mathcal{T} and consider $U = \bigcup_\alpha U_\alpha$. Given $x \in U$, there is an index α such that $x \in U_\alpha$. Since U_α is open, there is a base element B such that $x \in B \subseteq U_\alpha$. Then $x \in B \subseteq U$, so that U is open, by definition.

Now let us take two elements U_1 and U_2 of \mathcal{T} and show that $U_1 \cap U_2$ belongs to \mathcal{T} . Given $x \in U_1 \cap U_2$, choose a base element B_1 containing x such that $B_1 \subseteq U_1$; choose also a base element B_2 containing x such that $B_2 \subseteq U_2$. The second condition for a base enables us to choose a base element B_3 containing x such that $B_3 \subseteq B_1 \cap B_2$. Then $x \in B_3$ and $B_3 \subseteq U_1 \cap U_2$, so $U_1 \cap U_2$ belongs to \mathcal{T} , by definition. \square

Example 1.1.7. Let \mathcal{B} be the collection of all circular regions (interiors of circles) in the plane. Then \mathcal{B} satisfies both conditions for a base. In the topology generated by \mathcal{B} , a subset U of the plane is open if every x in U lies in some circular region contained in U .

Example 1.1.8. Let \mathcal{B} be the collection of all rectangular regions (interiors of rectangles) in the plane, where the rectangles have sides parallel to the coordinate axes. Then \mathcal{B} satisfies both conditions for a base. In this case, the condition (B2) is trivial, because the intersection of any two base elements is itself a base element (or empty). As we shall see later, the base \mathcal{B} generates the same topology on the plane as the base given in the preceding example.

Example 1.1.9. If X is any set, the collection of all one-point subsets of X is a base for the discrete topology on X .

Another way of describing the topology generated by a base is given in the following lemma:

Lemma 1.1.10. *Let X be a set and \mathcal{B} a base for a topology \mathcal{T} on X . Then \mathcal{T} equals the collection of all unions of elements of \mathcal{B} .*

Proof. Given a collection of elements of \mathcal{B} , they are also elements of \mathcal{T} . Because \mathcal{T} is a topology, their union is in \mathcal{T} . Conversely, given $U \in \mathcal{T}$, choose for each $x \in U$ an element B_x of \mathcal{B} such that $x \in B_x \subseteq U$. Then $U = \bigcup_{x \in U} B_x$, so U equals a union of elements of \mathcal{B} . \square

This lemma states that every open set U in X can be expressed as a union of base elements. This expression for U is not, however, unique. Thus the use of the term "base" in topology differs drastically from its use in linear algebra, where the equation expressing a given vector as a linear combination of base vectors is unique. We have described in two different ways how to go from a base to the topology it generates. Sometimes we need to go in the reverse direction, from a topology to a base generating it. Here is one way of obtaining a base for a given topology; we shall use it frequently.

Proposition 1.1.11. *Let X be a topological space. Suppose that \mathcal{B} is a collection of open sets of X such that for each open set U of X and each x in U , there is an element B of \mathcal{B} such that $x \in B \subseteq U$. Then \mathcal{B} is a base for the topology of X .*

Proof. We must show that \mathcal{B} is a base. The first condition for a base is easy: Given $x \in X$, since X is itself an open set, there is by hypothesis an element B of \mathcal{B} such that $x \in B \subseteq X$. To check the second condition, let x belong to $B_1 \cap B_2$, where B_1 and B_2 are elements of \mathcal{B} . Since B_1 and B_2 are open, so is $B_1 \cap B_2$. Therefore, there exists by hypothesis an element B_3 in \mathcal{B} such that $x \in B_3 \subseteq B_1 \cap B_2$.

Let \mathcal{T}' be the collection of open sets of X ; we must show that the topology \mathcal{T}' generated by \mathcal{B} equals the topology \mathcal{T} . First, note that if U belongs to \mathcal{T} and if $x \in U$, then there is by hypothesis an element B of \mathcal{B} such that $x \in B \subseteq U$. It follows that U belongs to the topology \mathcal{T}' , by definition. Conversely, if W belongs to the topology \mathcal{T}' , then W equals a union of elements of \mathcal{B} , by the preceding lemma. Since each element of \mathcal{B} belongs to \mathcal{T} and \mathcal{T} is a topology, W also belongs to \mathcal{T} . \square

When topologies are given by bases, it is useful to have a criterion in terms of the bases for determining whether one topology is finer than another. One such criterion is the following:

Proposition 1.1.12. *Let \mathcal{B}_1 and \mathcal{B}_2 be bases for the topologies \mathcal{T}_1 and \mathcal{T}_2 , respectively, on X . Then the following are equivalent:*

- (i) \mathcal{T}_1 is finer than \mathcal{T}_2 .
- (ii) For each $x \in X$ and each base element $B_2 \in \mathcal{B}_2$ containing x , there is a base element $B_1 \in \mathcal{B}_1$ such that $x \in B_1 \subseteq B_2$.

Proof. Assume (ii). Given an element U of \mathcal{T}_2 , we wish to show that $U \in \mathcal{T}_1$. Let $x \in U$. Since \mathcal{B}_2 generates \mathcal{T}_2 , there is an element $B_2 \in \mathcal{B}_2$ such that $x \in B_2 \subseteq U$. Condition (ii) tells us there exists an element $B_1 \in \mathcal{B}_1$ such that $x \in B_1 \subseteq B_2$. Then $x \in B_1 \subseteq U$, so $U \in \mathcal{T}_1$, by definition.

Conversely, assume (i). We are given $x \in X$ and $B_2 \in \mathcal{B}_2$, with $x \in B_2$. Now B_2 belongs to \mathcal{T}_2 by definition and $\mathcal{T}_2 \subseteq \mathcal{T}_1$ by condition (i); therefore, $B_2 \in \mathcal{T}_1$. Since \mathcal{T}_1 is generated by \mathcal{B}_1 , there is an element $B_1 \in \mathcal{B}_1$ such that $x \in B_1 \subseteq B_2$. \square

We now define three topologies on the real line \mathbb{R} , all of which are of interest.

Example 1.1.13. If \mathcal{B} is the collection of all open intervals in the real line,

$$(a, b) = \{x \in \mathbb{R} : a < x < b\}$$

the topology generated by \mathcal{B} is called the standard topology on the real line. Whenever we consider \mathbb{R} , we shall suppose it is given this topology unless we specifically state otherwise. If \mathcal{B}' is the collection of all half-open intervals of the form

$$[a, b) = \{x \in \mathbb{R} : a \leq x < b\},$$

where $a < b$, the topology generated by \mathcal{B}' is called the lower limit topology on \mathbb{R} . When \mathbb{R} is given the lower limit topology, we denote it by \mathbb{R}_l . Finally let K denote the set of all numbers of the form $1/n$, for $n \in \mathbb{Z}_+$, and let \mathcal{B}_K be the collection of all open intervals (a, b) , along with all sets of the form $(a, b) \setminus K$. The topology generated by \mathcal{B}_K will be called the K -topology on \mathbb{R} . When \mathbb{R} is given this topology, we denote it by \mathbb{R}_K .

It is easy to see that all three of these collections are bases; in each case, the intersection of two base elements is either another base element or is empty. The relation between these topologies is the following:

Proposition 1.1.14. *The topologies of \mathbb{R}_l and \mathbb{R}_K are strictly finer than the standard topology on \mathbb{R} , but are not comparable with one another.*

Proof. Let \mathcal{T} , \mathcal{T}_l , and \mathcal{T}_K be the topologies of \mathbb{R} , \mathbb{R}_l , and \mathbb{R}_K , respectively. Given a base element (a, b) for \mathcal{T} and a point x of (a, b) , the base element $[x, b)$ for \mathcal{T}_l contains x and lies in (a, b) . On the other hand, given the base element $[x, d)$ for \mathcal{T}_l , there is no open interval (a, b) that contains x and lies in $[x, d)$. Thus \mathcal{T}_l is strictly finer than \mathcal{T} .

A similar argument applies to \mathbb{R}_K . Given a base element (a, b) for \mathcal{T} and a point x of (a, b) , this same interval is a base element for \mathcal{T}_K that contains x . On the other hand, given the base element $B = (-1, 1) \setminus K$ for \mathcal{T}_K and the point 0 of B , there is no open interval that contains 0 and lies in B .

Finally, given the base element $[x, d)$ for \mathcal{T}_l , there is no base element of \mathcal{T}_K that contains x and lies in $[x, d)$, so \mathcal{T}_K is not finer than \mathcal{T}_l . Conversely, given the base element $B = (-1, 1) \setminus K$ for \mathcal{T}_K and the point 0 of B , there is no interval $[a, b)$ that contains 0 and lies in B . Thus \mathcal{T}_l is not finer than \mathcal{T}_K . \square

Example 1.1.15. Let X be a totally ordered set; assume X has more than one element. Let \mathcal{B} be the collection of all sets of the following types:

- (a) All open intervals (a, b) in X .

- (b) All intervals of the form $[a_0, b]$, where a_0 is the smallest element (if any) of X .
- (c) All intervals of the form $(a, b_0]$, where b_0 is the largest element (if any) of X .

The collection \mathcal{B} is a base for a topology on X , which is called the **order topology**.

A question may occur to us at this point. Since the topology generated by a base \mathcal{B} may be described as the collection of arbitrary unions of elements of \mathcal{B} , what happens if we start with a given collection of sets and take finite intersections of them as well as arbitrary unions? This question leads to the notion of a subbase for a topology.

Definition 1.1.16. A **subbase \mathcal{S}** for a topology on X is a collection of subsets of X whose union equals X . The topology generated by the subbase \mathcal{S} is defined to be the collection \mathcal{T} of all unions of finite intersections of elements of \mathcal{S} .

We must of course check that \mathcal{T} is a topology. For this purpose it will suffice to show that the collection \mathcal{B} of all finite intersections of elements of \mathcal{S} is a base. Given $x \in X$, it belongs to an element of \mathcal{S} and hence to an element of \mathcal{B} ; this is the first condition for a base. The second condition is trivial since the intersection of two elements of \mathcal{B} again belongs to \mathcal{B} .

1.1.3 Neighborhoods

Let X be a topological space and A any subset of X . A **neighbourhood of A** is any subset of X which contains an open set containing A . The neighbourhoods of a subset $\{x\}$ consisting of a single point are also called **neighbourhoods of the point x** .

Proposition 1.1.17. A set A is a neighbourhood of each of the points of a set B if and only if it is a neighborhood of B . In particular, a set is open if and only if it is a neighborhood of each of its points.

Proof. It is clear that every neighbourhood of a subset B of X is also a neighbourhood of each subset of B ; in particular, it is a neighbourhood of each point of B . Conversely, suppose A is a neighbourhood of each of the points of a set B , and let V be the union of the open sets contained in A ; then $V \subseteq A$, and since each point of B belongs to an open set contained in A , we have $B \subseteq V$; but V is open by virtue of (O2), hence A is a neighbourhood of B . \square

Remark 1.1.1. The everyday sense of the word "neighbourhood" is such that many of the properties which involve the mathematical idea of neighbourhood appear as the mathematical expression of intuitive properties; the choice of this term thus has the advantage of making the language more expressive. For this purpose it is also permissible to use the expressions "sufficiently near" and "as near as we please" in some statements. For example, ?? can be stated in the following form: a set A is open if and only if, for each $x \in A$, all the points sufficiently near x belong to A . More generally, we shall say that a property holds for all points sufficiently near a point x , if it holds at all points of some neighbourhood of x .

Let us denote by $\mathfrak{U}(x)$ the set of all neighbourhoods of x . The sets $\mathfrak{U}(x)$ have the following properties:

- (U1) Every subset of X which contains a set belonging to $\mathfrak{U}(x)$ itself belongs to $\mathfrak{U}(x)$.
- (U2) Every finite intersection of sets of $\mathfrak{U}(x)$ belongs to $\mathfrak{U}(x)$.

- (U3) Every element of $\mathfrak{U}(x)$ contains x .
- (U4) If V belongs to $\mathfrak{U}(x)$, then there is a set $W \in \mathfrak{U}(x)$ such that, for each $y \in W$, V belongs to $\mathfrak{U}(y)$.

These four properties of the sets $\mathfrak{U}(x)$ are characteristic. To be precise, we have:

Proposition 1.1.18. *If to each element $x \in X$ there corresponds a set $\mathfrak{U}(x)$ of subsets of X such that the properties (U1), (U2), (U3) and (U4) are satisfied, then there is a unique topological structure on X such that, for each $x \in X$, $\mathfrak{U}(x)$ is the set of neighbourhoods of x in this topology.*

Proof. By ??, if there is a topology on X satisfying these conditions, the set of open sets for this topology is necessarily the set \mathcal{T} of subsets A of X such that for each $x \in A$ we have $A \in \mathcal{B}(X)$; hence the uniqueness of this topology if it exists.

Now we prove that the set \mathcal{T} is a topology on X . It is clear that \mathcal{T} satisfies the axiom (O1). It satisfies axioms (O2) and (O3) because: for (O2), this follows immediately from (U1), and for (O3), from (U2). It remains to show that, in the topology defined by \mathcal{T} , $\mathcal{B}(x)$ is the set of neighbourhoods of x for each $x \in X$.

It follows from (U1) that every neighbourhood of x belongs to $\mathfrak{U}(x)$. Conversely, let V be a set belonging to $\mathcal{B}(x)$, and let U be the set of points $y \in X$ such that $V \in \mathfrak{U}(y)$; if we can show that $x \in U$, $U \subseteq V$ and $U \in \mathcal{T}$, then the proof will be complete.

We have $x \in U$ since $V \in \mathfrak{U}(x)$; also $U \subseteq V$, for every point $y \in U$ belongs to V by reason of (U3) and the hypothesis $V \in \mathfrak{U}(y)$. It remains to show that $U \in \mathcal{T}$, i.e. that $U \in \mathfrak{U}(y)$ for each $y \in U$; now if $y \in U$ then by (U4) there is a set $W \in \mathfrak{U}(y)$ such that for each $z \in W$ we have $V \in \mathfrak{U}(z)$; since $V \in \mathfrak{U}(z)$ means that $z \in U$, it follows that $W \subseteq U$, and therefore, by (U1) that $U \in \mathfrak{U}(y)$. \square

?? shows that a topology on X can be defined by means of the sets $\mathfrak{U}(x)$ of neighbourhoods of points of X , subject only to the axioms (U1), (U2), (U3) and (U4).

Definition 1.1.19. *In a topological space X , a fundamental system of neighbourhoods of a point x (resp. of a subset A of X) is any set \mathfrak{B} of neighbourhoods of x (resp. A) such that for each neighbourhood V of x (resp. of A) there is a neighbourhood $W \in \mathfrak{B}$ such that $W \subseteq V$.*

If \mathfrak{B} is a fundamental system of neighbourhoods of a subset A of X , then every finite intersection of sets of \mathfrak{B} contains a set of \mathfrak{B} . In fact, as we will see, a neighborhood base of x is just a filterbase for the neighborhood filter $\mathfrak{U}(x)$.

Proposition 1.1.20. *If X is a topological space, then for a set \mathcal{B} of open subsets of X to be a base of the topology of X it is necessary and sufficient that for each $x \in X$ the set of $V \in \mathcal{B}$ such that $x \in V$ is a fundamental system of neighbourhoods of x .*

Proof. It is clear that the condition is necessary. Conversely, if it is satisfied, then given any open set U and any $x \in U$ there is an open set $V_x \in \mathcal{B}$ such that $x \in V_x \subseteq U$. The union of the sets V_x for $x \in U$ is therefore equal to U . This completes the proof. \square

1.1.4 Closure, interior and boundary

Definition 1.1.21. *In a topological space X , a point x is said to be an **interior point** of a subset A of X if A is a neighbourhood of x . The set of interior points of A is called the **interior** of A and is denoted by $\text{Int } A$.*

According to the definition, a point x is an interior point of A if there is an open set contained in A which contains x ; it follows that $\text{Int } A$ is the union of all the open sets contained in A , and hence is the largest open set contained in A . Every point which is interior to the complement of a set A is said to be an **exterior point** of A , and the set of these points is called the **exterior** of A in X ; a point $x \in X$ which is an exterior point of A is therefore characterized by the property that x has a neighbourhood which does not meet A .

Now we define the closure of a subset $A \subseteq X$.

Definition 1.1.22. *The closure of a subset A of a topological space X is the set of all points $x \in X$ such that every neighbourhood of x meets A , and is denoted by \bar{A} .*

Every point which is not in the closure of A is exterior to A , and conversely; thus we have the formulae (which are duals of each other)

$$\bar{A}^c = \text{Int } A^c, \quad (\text{Int } A)^c = \overline{A^c}.$$

Hence, to any proposition on interiors of sets, there corresponds by duality a proposition on closures, and vice versa. In particular, the closure of a set A is the smallest closed set which contains A .

Proposition 1.1.23. *Let U be an open set in X ; then for every subset A of X , we have $U \cap \bar{A} \subseteq \overline{U \cap A}$.*

Proof. For suppose $x \in U \cap \bar{A}$; then if V is any neighbourhood of x , $V \cap U$ is a neighbourhood of x , since U is open; hence $V \cap U \cap A$ is not empty and therefore x lies in the closure of $U \cap A$. \square

If x lies in the closure of \bar{A} but not in A , then every neighbourhood of x contains a point of A other than x ; but if $x \in A$ it can happen that x has a neighbourhood which contains no point of A except x . We say then that x is an **isolated point** of A . In particular, x is isolated in the whole space X if and only if $\{x\}$ is an open set.

Example 1.1.24. Let X be the real line \mathbb{R} . If $A = (0, 1]$, then $\bar{A} = [0, 1]$, for every neighborhood of 0 intersects A , while every point outside $[0, 1]$ has a neighborhood disjoint from A . Similar arguments apply to the following subsets of X : If $B = \{1/n : n \in \mathbb{Z}_+\}$, then $\bar{B} = \{0\} \cup B$. If $C = \{0\} \cup (1, 2)$, then $\bar{C} = \{0\} \cup [1, 2]$. If \mathbb{Q} is the set of rational numbers, then $\overline{\mathbb{Q}} = \mathbb{R}$. If \mathbb{Z}_+ is the set of positive integers, then $\overline{\mathbb{Z}_+} = \mathbb{Z}_+$. If \mathbb{R}_+ is the set of positive reals, then the closure of \mathbb{R}_+ is the set $\mathbb{R}_+ \cup \{0\}$.

Example 1.1.25. Consider the subspace $A = (0, 1]$ of the real line \mathbb{R} . The set $B = (0, 1/2)$ is a subset of A ; its closure in \mathbb{R} is the set $[0, 1/2]$, and its closure in A is the set $[0, 1/2] \cap A = (0, 1/2]$.

Definition 1.1.26. *In a topological space X , a point x is said to be a **boundary point** of a set A if x lies in the closure of A and in the closure of A^c . The set of boundary points of A is called the **boundary** of A , denoted by ∂A .*

The boundary of A is therefore the set $\bar{A} \cap \overline{A^c}$, which is closed. A boundary point x of A is characterized by the property that every neighbourhood of x contains at least one point of A and at least one point of A^c ; x may or may not belong to A . The boundary of A is the same as

the boundary of A^c . The interior of A , the exterior of A and the boundary of A are mutually disjoint and their union is the whole space X .

Now we prove some basic properties of closures and interiors.

Proposition 1.1.27. *Let A, B be subsets of a topological space X . Then*

$$(a) \text{ Int}(A \cap B) = (\text{Int } A) \cap (\text{Int } B) \text{ and } \overline{A \cup B} = \overline{A} \cup \overline{B}.$$

$$(b) \text{ If } A \subseteq B \text{ then } \overline{A} \subseteq \overline{B} \text{ and } \text{Int } A \subseteq \text{Int } B.$$

$$(c) \text{ For a family } (A_i)_{i \in I} \text{ of subsets of } X, \text{ we have}$$

$$\overline{\bigcap_{i \in I} A_i} \subseteq \bigcap_{i \in I} \overline{A_i}, \quad \overline{\bigcup_{i \in I} A_i} \supseteq \bigcup_{i \in I} \overline{A_i}$$

and

$$\text{Int} \left(\bigcup_{i \in I} A_i \right) \supseteq \bigcup_{i \in I} \text{Int } A_i, \quad \text{Int} \left(\bigcap_{i \in I} A_i \right) \subseteq \bigcap_{i \in I} \text{Int } A_i.$$

Proof. Part (b) and (c) are clear by definition, so we only prove (a). Since $A \cap B$ is contained in A and B , it is clear that $\text{Int}(A \cap B) \subseteq \text{Int}(A) \cap \text{Int}(B)$. Conversely, if $x \in \text{Int}(A) \cap \text{Int}(B)$, then there are open sets U, V in X such that $x \in U \subseteq A$ and $x \in V \subseteq B$. Then $x \in U \cap V \subseteq A \cap B$, hence $x \in \text{Int}(A \cap B)$. This proves $\text{Int}(A \cap B) = (\text{Int } A) \cap (\text{Int } B)$, and the other part follows from duality. \square

There is yet another way of describing the closure of a set, a way that involves the important concept of limit point, which we consider now.

If A is a subset of the topological space X and if x is a point of X , we say that x is a limit point of A if every neighborhood of x intersects A in some point other than x itself. Said differently, x is a limit point of A if it belongs to the closure of $A \setminus \{x\}$. The point x may lie in A or not; for this definition it does not matter.

Example 1.1.28. Consider the real line \mathbb{R} . If $A = (0, 1]$, then the point 0 is a limit point of A and so is the point 1/2. In fact, every point of the interval $[0, 1]$ is a limit point of A , but no other point of \mathbb{R} is a limit point of A .

If $B = \{1/n : n \in \mathbb{Z}_+\}$, then 0 is the only limit point of B . Every other point x of \mathbb{R} has a neighborhood that either does not intersect B at all, or it intersects B only in the point x itself. If $C = \{0\} \cup (1, 2)$, then the limit points of C are the points of the interval $[1, 2]$. If \mathbb{Q} is the set of rational numbers, every point of \mathbb{R} is a limit point of \mathbb{Q} . If \mathbb{Z}_+ is the set of positive integers, no point of \mathbb{R} is a limit point of \mathbb{Z}_+ . If \mathbb{R}_+ is the set of positive reals, then every point of $\{0\} \cup \mathbb{R}_+$ is a limit point of \mathbb{R}_+ .

It follows from the definition of closure that the following result is valid.

Proposition 1.1.29. *Let A be a subset of the topological space X ; let A' be the set of all limit points of A . Then $\overline{A} = A \cup A'$.*

Corollary 1.1.30. *A subset of a topological space is closed if and only if it contains all its limit points.*

1.2 Continuous functions

1.2.1 Semicontinuity

Recall the usual topology of \mathbb{R} has a subbase $S = \{(a, +\infty) : a \in \mathbb{R}\} \cup \{(-\infty, a) : a \in \mathbb{R}\}$. Therefore a function $f : X \rightarrow \mathbb{R}$ is continuous if and only if $f^{-1}((a, +\infty))$ and $f^{-1}((-\infty, a))$ are open for every $a \in \mathbb{R}$. This motivates the following definition:

Definition 1.2.1. Let X be a topological space. A function $f : X \rightarrow \mathbb{R}$ is called **lower semicontinuous** at x if for any $\varepsilon > 0$ there exists a neighborhood U of x such that $f(U) \subseteq (f(x) - \varepsilon, +\infty)$, and **upper semicontinuous** if $f^{-1}((a, +\infty))$ is open for every $a \in \mathbb{R}$. A similar terminology is addressed for **upper semicontinuous**.

It is clear that f is continuous at x if and only if f is lower and upper semicontinuous at x , and f is continuous if and only if f is lower and upper semicontinuous. Now we collect some properties of semicontinuity.

Proposition 1.2.2. Let X be a topological space and $f : X \rightarrow \mathbb{R}$ be a function. Then

- (a) f is lower (upper) semicontinuous if and only if f is lower (upper) semicontinuous at every $x \in X$.
- (b) f is lower semicontinuous if and only if $-f$ is upper semicontinuous.
- (c) If \mathcal{F} is a family of lower semicontinuous functions, then $g(x) = \sup_{f \in \mathcal{F}} f(x)$ is lower semicontinuous.
- (d) A finite sum of lower semicontinuous functions is lower semicontinuous
- (e) If X is compact and $f : X \rightarrow \mathbb{R}$ is lower semicontinuous. Then f attains its minimum on X .

Proof. If f is lower semicontinuous, then for any $x \in X$ and $\varepsilon > 0$, the set $f^{-1}((f(x) - \varepsilon, +\infty))$ is open. Since x is contained in it, we can find a neighborhood U of x such that $U \subseteq f^{-1}((f(x) + \varepsilon, +\infty))$. Therefore f is lower semicontinuous at x . The converse is done similarly. This proved (a).

For (b), just note that $(-f)^{-1}((-\infty, a)) = f^{-1}((-a, +\infty))$, so $(-f)$ is upper semicontinuous if and only if f is lower semicontinuous.

Now assume that \mathcal{F} is a family of lower semicontinuous functions, then for any $a \in \mathbb{R}$ and $f \in \mathcal{F}$, the set $\{x : f(x) > a\}$ is open. Since we have

$$\{x : g(x) > a\} = \bigcup_{f \in \mathcal{F}} \{x : f(x) > a\},$$

we see the left side is also open, therefore g is lower semicontinuous.

Let f and g be lower semicontinuous functions, then for every $a \in \mathbb{R}$, we observe that

$$\begin{aligned} \{x : f(x) + g(x) > a\} &= \{x : f(x) > a - g(x)\} = \bigcup_{r \in \mathbb{Q}} \left(\{x : f(x) > r\} \cap \{x : a - g(x) < r\} \right) \\ &= \bigcup_{r \in \mathbb{Q}} \left(\{x : f(x) > r\} \cap \{x : g(x) > a - r\} \right). \end{aligned}$$

Therefore the left side is open, and so $f + g$ is also lower semicontinuous.

Finally, if f is lower semicontinuous and X is compact. Then an open cover of X is given by the collection $\{f^{-1}((a, +\infty)) : a \in \mathbb{R}\}$, hence has a finite subcover. This means $f(X)$ has a lower bound. Let α be the infimum of $f(X)$, and assume that f does not attain α . Then the collection $\{f^{-1}((\alpha + 1/n, +\infty)) : n \in \mathbb{N}\}$ is an open cover of X , which does not have a finite subcover. This is a contradiction. \square

Proposition 1.2.3. *Let X be a topological space and $A \subseteq X$ be a subset. Then*

- (a) *A is open if and only if χ_A is lower semicontinuous.*
- (b) *A is closed if and only if χ_A is upper semicontinuous.*

Proof. This follows from the observations

$$A = \chi_A^{-1}((0, +\infty)), \quad A^c = \chi_A^{-1}((-∞, 1)),$$

so A is open if and only if χ_A is lower semicontinuous, and A^c is open if and only if χ_A is upper semicontinuous. \square

Proposition 1.2.4. *Let X be a topological space and $f : X \rightarrow \mathbb{R}$. Let $x \in X$. Then*

- (a) *f is lower semicontinuous at x if and only if $f(x) \leq \underline{\lim} f(x_\alpha)$ for every net $\{x_\alpha\}$ converging to x .*
- (b) *f is upper semicontinuous at x if and only if $f(x) \geq \overline{\lim} f(x_\alpha)$ for every net $\{x_\alpha\}$ converging to x .*

Proof. If f is lower semicontinuous at x , then for any $\varepsilon > 0$, there exists a neighborhood U of x such that $f(U) \subseteq f^{-1}((f(x) - \varepsilon, +\infty))$. This then implies $\underline{\lim} f(x_\alpha) \geq f(x) - \varepsilon$, and since ε is arbitrary, that $\underline{\lim} f(x_\alpha) \geq f(x)$. Conversely, if $f(x) \leq \underline{\lim} f(x_\alpha)$, let $\{x_\alpha\}$ be the neighborhood net converging to x . Then for any $\varepsilon > 0$, there exist an index α_0 such that $f(x_\alpha) > f(x) - \varepsilon$ for any $\alpha \leq \alpha_0$. This then means f is lower semicontinuous at x . The claim for upper semicontinuous is proved similarly. \square

Example 1.2.5 (Rank function is lower continuous). Let $\text{rank} : \mathcal{M}_{mn}(\mathbb{R}) \rightarrow \mathbb{N} \subseteq \mathbb{R}$ be the rank function, then rank is lower continuous.

In fact, the set

$$M_r = \{A \in \mathcal{M}_{mn}(\mathbb{R}) : \text{rank}(A) > r\}$$

is open in $\mathcal{M}_{mn}(\mathbb{R})$ since if $\text{rank}(A) > r$, then A has a invertible square submatrix with order $> r$. By the continuity of determinant, there is a neighborhood U of A in $\mathcal{M}_{mn}(\mathbb{R})$ such that for any matrix $B \in U$, the square submatrix of B at the same place is invertible. This implies $\text{rank}(B) > r$, thus $U \subseteq M_r$.

1.2.2 Continuous functions

Definition 1.2.6. *A mapping f of a topological space X into a topological space Y is said to be **continuous at a point** $x_0 \in X$ if, given any neighbourhood V of $f(x_0)$ in Y , there is a neighbourhood U of x_0 in X such that the relation $x \in U$ implies $f(x) \in V$.*

The relation "for each $x \in U, f(x) \in V$ " is equivalent to $f(U) \subseteq V$ or again to $U \subseteq f^{-1}(V)$; in view of the neighbourhood axiom (U1) we see that the above definition is equivalent to the following: $f : X \rightarrow Y$ is said to be continuous at the point x_0 if, for each neighbourhood V of $f(x_0)$ in Y , $f^{-1}(V)$ is a neighbourhood of x_0 in X . Moreover, it is sufficient that $f^{-1}(V)$ is a neighbourhood of x_0 for each neighbourhood V belonging to a fundamental system of neighbourhoods of $f(x_0)$ in Y .

Proposition 1.2.7. *Let f be a mapping of a topological space X into a topological space Y . If f is continuous at x , and if x lies in the closure of a subset A of X , then $f(x)$ lies in the closure of $f(A)$.*

Proof. Let V be a neighbourhood of $f(x)$ in Y . Since f is continuous at x , $f^{-1}(V)$ is a neighbourhood of x in X . Hence $f^{-1}(V)$ meets A , from which it follows that V meets $f(A)$, and therefore $f(x)$ is in the closure of $f(A)$. \square

Proposition 1.2.8. *Let X, Y, Z be three topological spaces; let f be a mapping of X into Y , continuous at $x \in X$; let g be a mapping of Y into Z , continuous at $f(x)$. Then the composition $h = g \circ f : X \rightarrow Z$ is continuous at x .*

Proof. Let W be a neighbourhood of $h(x) = g(f(x))$ in Z . Since g is continuous at $f(x)$ it follows that $g^{-1}(W)$ is a neighbourhood of $f(x)$ in Y . But f is continuous at x , hence $f^{-1}(g^{-1}(W)) = h^{-1}(W)$ is a neighbourhood of x in X , and therefore h is continuous at x . \square

Definition 1.2.9. *A mapping of a topological space X into a topological space Y is said to be **continuous on X** (or just continuous) if it is continuous at each point of X .*

Example 1.2.10 (Example of continuous maps).

- (a) The identity mapping of a topological space X onto itself is continuous.
- (b) A constant map of a topological space into a topological space is continuous.
- (c) Every mapping of a discrete space into a topological space is continuous.

Theorem 1.2.11. *Suppose X and Y are topological spaces, and $f : X \rightarrow Y$ is any map. Then the following statements are equivalent:*

- (i) f is continuous on X .
- (ii) For every subset A of X , $f(\bar{A}) \subseteq \overline{f(A)}$.
- (iii) For every subset B of Y , $f^{-1}(\text{Int } B) \subseteq \text{Int } f^{-1}(B)$.
- (iv) The inverse image under f of every closed subset of Y is a closed subset of X .
- (v) The inverse image under f of every open subset of Y is an open subset of X .

Also, for f to be continuous, it is sufficient that (v) holds for bases of Y .

Proof. We have already seen (i) implies (ii). To see (ii) implies (iv), let F be a closed subset of Y and let $E = f^{-1}(F)$; then by hypothesis

$$f(\bar{E}) \subseteq \overline{f(E)} \subseteq F$$

hence $\bar{E} \subseteq f^{-1}(F) = E$, so that E is closed. By virtue of the relation $f^{-1}(Y \setminus U) = X \setminus f^{-1}(U)$ for every subset U of Y , (iv) implies (v).

Now suppose that (v) is satisfied. Then for any subset B of Y , $f^{-1}(\text{Int } B)$ is open in X and contained in $f^{-1}(B)$, hence $f^{-1}(\text{Int } B) \subseteq \text{Int } f^{-1}(B)$. Finally, assume (iii) and let x be any point of X and B be any neighbourhood of $f(x)$ in Y ; then there exists an open set W in Y such that $f(x) \in W \subseteq B$, and thus $f^{-1}(W) \subseteq \text{Int } f^{-1}(W) \subseteq \text{Int } f^{-1}(B)$. This means $f^{-1}(B)$ is a neighbourhood of x , which proves (i). The last remark is obvious, so the proof is completed. \square

One may also wonder when do the converse inclusions in ??(ii) and (iii) hold. In fact, we have the following result.

Proposition 1.2.12. *Suppose X and Y are topological spaces, and $f : X \rightarrow Y$ is any map.*

- (a) *f is closed if and only if $f(\bar{A}) \supseteq \overline{f(A)}$ for all $A \subseteq X$.*
- (b) *f is open if and only if $f^{-1}(\text{Int } B) \supseteq \text{Int } f^{-1}(B)$ for all $B \in Y$.*

Proof. If f is a closed map, then $f(\bar{A})$ is a closed subset in Y containing $f(A)$. This implies $\overline{f(A)} \subseteq f(\bar{A})$. Conversely, if $\overline{f(A)} \subseteq f(\bar{A})$ holds, so that for closed subset A ,

$$f(A) = f(\bar{A}) \supseteq \overline{f(A)},$$

then $f(A)$ is closed.

If f is open, let $x \in \text{Int } f^{-1}(B)$ and U be an open set such that $x \in U \subseteq f^{-1}(B)$. Then $f(U)$ is an open set contained in B , hence $f(x) \in \text{Int}(B)$, and $x \in f^{-1}(\text{Int } B)$. Conversely, let U be open in X ; if $f(U)$ is not open, then there is $x \in U$ such that $f(x) \notin \text{Int } f(U)$. That is, $x \notin f^{-1}(\text{Int } f(U))$. Since $f^{-1}(\text{Int } f(U)) \supseteq \text{Int } f^{-1}(f(U))$, we then get $x \notin \text{Int } f^{-1}(f(U))$. But $\text{Int } f^{-1}(f(U))$ contains U , this is a contradiction. \square

Proposition 1.2.13 (Properties of continuous maps).

- (a) *If $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are two continuous mappings, then $g \circ f : X \rightarrow Z$ is continuous.*
- (b) *For a bijection f of a topological space X onto a topological space Y to be a homeomorphism, it is necessary and sufficient that f and the inverse of f are continuous (or, as is also said, that f is bicontinuous).*

Proof. The first assertion is an immediate consequence of ??; the second follows from ?? and the definition of a homeomorphism. \square

?? shows that we may take the continuous mappings as morphisms of topological structures; from now on, we shall assume that we have made this choice of morphisms. In accordance with the general definitions, this allows us to define an ordering on the set of topologies on a given set X , which coincides with our definition of comparability of topologies. The criteria for a mapping to be continuous give the following criterion:

Proposition 1.2.14. *Given two topologies \mathcal{T}_1 and \mathcal{T}_2 on a set X , the following statements are equivalent:*

- (i) *\mathcal{T}_1 is finer than \mathcal{T}_2 .*
- (ii) *For each $x \in X$, each neighbourhood of x for \mathcal{T}_2 is a neighbourhood of x for \mathcal{T}_1 .*

- (iii) For each subset A of X , the closure of A in \mathcal{T}_1 is contained in the closure of A in \mathcal{T}_2 .
- (iv) Every subset of X which is closed in \mathcal{T}_2 is closed in \mathcal{T}_1 .
- (v) Every subset of X which is open in \mathcal{T}_2 is closed in \mathcal{T}_1 .

1.2.3 Initial topologies

Proposition 1.2.15. Let X be a set, let $(Y_i)_{i \in I}$ be a family of topological spaces, and for each $i \in I$ let f_i be a mapping of X into Y_i . Let \mathcal{S} be the set of subsets of X of the form $f_i^{-1}(U_i)$, where $i \in I$ and U_i is open in Y_i . Then \mathcal{S} is a subbase of a topology \mathcal{T} on X which is the initial topological structure on X for the family $(f_i)_{i \in I}$ and in particular is the coarsest topology on X for which the mappings f_i are continuous. More precisely, if g is a mapping of a topological space Z into X , then g is continuous at a point $z \in Z$ (with X carrying the topology \mathcal{T}) if and only if each of the functions $f_i \circ g$ is continuous at z .

Proof. The topology \mathcal{T} is just that induced by the subbase \mathcal{S} . We shall prove the last assertion of the proposition, which implies the others by reason of the general properties of initial structure. In the first place, the definition of \mathcal{S} shows that the f_i are continuous on X ; hence, if g is continuous at z , so are the mappings $f_i \circ g$. Conversely, suppose that all the mappings $f_i \circ g$ are continuous at z , and let U be a neighbourhood of $g(z)$ in X ; by definition, there is a finite subset J of I , and for each $i \in J$ an open subset V_i of Y_i such that U contains the set $\bigcap_{i=1}^n f_i^{-1}(V_i)$ and $g(z)$ belongs to this set. It follows that

$$g^{-1}(U) \supseteq g^{-1}\left(\bigcap_{i=1}^n f_i^{-1}(V_i)\right) = \bigcap_{i \in J} (f_i \circ g)^{-1}(V_i)$$

and the hypothesis implies that each of the sets $(f_i \circ g)^{-1}(V_i)$ is a neighbourhood of z in Z ; hence $g^{-1}(U)$ is also a neighbourhood of z in Z . This completes the proof. \square

The topology \mathcal{T} is called the **initial topology** induced by the family $(f_i)_{i \in I}$. It is clear that if \mathcal{B}_i is a base of the topology of Y_i ; then the set \mathcal{S}' of subsets of X of the form $f_i^{-1}(B_i)$ for $i \in I$ and $B_i \in \mathcal{B}_i$ for each $i \in I$ is also a subbase of the topology \mathcal{T} .

The general properties of initial structures imply in particular the following transitivity property:

Proposition 1.2.16. Let X be a set, $(Z_\beta)_{\beta \in J}$ a family of topological spaces, $(J_\alpha)_{\alpha \in I}$ a partition of B and $(Y_\alpha)_{\alpha \in I}$ a family of sets indexed by I . For each $\alpha \in I$, let g_α be a mapping of X into Y_α ; and for each $\beta \in J_\alpha$, let $h_{\beta\alpha}$ be a mapping of Y_α into Z_β , and put $f_\beta = h_{\beta\alpha} \circ g_\alpha$.

$$X \xrightarrow{g_\alpha} Y_\alpha \xrightarrow{h_{\beta\alpha}} Z_\beta$$

If each Y_α carries the initial topology induced by $(h_{\beta\alpha})_{\beta \in J_\alpha}$ then the initial topology on X induced by $(f_\beta)_{\beta \in J}$ is the same as the initial topology induced by $(g_\alpha)_{\alpha \in I}$.

Example 1.2.17 (Example of initial topologies).

- (a) Let X be a set, Y a topological space, f a mapping of X into Y ; the initial topology \mathcal{T} on X induced by f is called the **inverse image** under f of the topology of Y . It follows from

?? and the formula for the inverse image of a union and an intersection that the open (resp. closed) sets in the topology \mathcal{T} are the inverse images under f of the open (resp. closed) sets of Y ; consequently, for each $x \in X$, the sets $f^{-1}(W)$, where W runs through a neighbourhoods base of $f(x)$ in Y , form a fundamental system of neighbourhoods of x in the topology \mathcal{T} . Later we shall study, under the name of induced topology, the particular case in which X is a subset of Y and f is the canonical injection $X \rightarrow Y$; X , with the induced topology, is then called a **subspace** of Y .

- (b) Every family $(\mathcal{T}_i)_{i \in I}$ of topologies on a set X has a **least upper bound** \mathcal{T} in the ordered set of all topologies on X , i.e. there exists a topology on X which is coarsest among all the topologies on X which are finer than each of the \mathcal{T}_i . To see this we may apply ??, taking Y_i to be the set X with the topology \mathcal{T}_i and f_i to be the identity map $X \rightarrow X$. Then \mathcal{T} is the initial topology induced by $(f_i)_{i \in I}$.
- (c) Let \mathcal{S} be an arbitrary set of subsets of a set X , not necessarily covers X . Amongst the topologies on X for which the sets of \mathcal{S} are open, there is a topology \mathcal{T} which is coarser than all the others and which is called the topology generated by \mathcal{S} : For each set $U \in \mathcal{S}$ let \mathcal{T}_U be the topology whose open sets are \emptyset , U and X ; then \mathcal{T} is just the least upper bound of the topologies \mathcal{T}_U . By ??, if \mathcal{B} is the set of finite intersections of sets belonging to \mathcal{S} , then \mathcal{B} is a base of the topology \mathcal{T} . Note that, if we take $\mathcal{S}' = \mathcal{S} \cup \{X\}$, then \mathcal{T} is exactly the topology generated by \mathcal{S}' as a subbase. In view of this, we may also make abuse of language and say \mathcal{S} is a *subbase* of \mathcal{T} .
- (d) Let $(X_i)_{i \in I}$ be a family of topological spaces. The initial topology induced on the product space $X = \prod_{i \in I} X_i$ by the projections $\pi_i : X \rightarrow X_i$ is called the product of the topologies of the X_i 's; we shall study it in more detail later.

1.2.4 Final topologies

Proposition 1.2.18. *Let X be a set, let $(Y_i)_{i \in I}$ be a family of topological spaces, and for each $i \in I$ let f_i be a mapping of Y_i into X . Let \mathcal{T} be the set of subsets U of X such that $f_i^{-1}(U)$ is open in Y_i for each $i \in I$. Then \mathcal{T} is the final structure on X for the family (f_i) , and in particular \mathcal{T} is the finest topology on X for which the mappings f_i are continuous. In other words, if g is a mapping of X into a topological space Z , then g is continuous (with X carrying the topology \mathcal{T}) if and only if each of the mappings $g \circ f_i$ is continuous.*

Proof. It is immediately verified that \mathcal{T} satisfies the axioms (O1) and (O2). We shall prove the last assertion of the proposition, which implies the other assertions by reason of the general properties of final structures. It is clear that the f_i are continuous in the topology \mathcal{T} , by the definition of \mathcal{T} ; hence if g is continuous, so is each mapping $g \circ f_i$. Conversely, suppose that each $g \circ f_i$ is continuous, and let V be an open set in Z ; by hypothesis, $f_i^{-1}(g^{-1}(V))$ is open in Y_i for each $i \in I$; hence $g^{-1}(V) \in \mathcal{T}$, and the proof is complete. \square

The topology in ?? is called the **final topology** induced by (f_i) . It follows from definition that a subset F of X is closed in this topology if and only if $f_i^{-1}(F)$ is closed for all $i \in I$. The general properties of final structures imply the following transitivity property:

Proposition 1.2.19. Let X be a set, $(Z_\beta)_{\beta \in J}$ a family of topological spaces, $(J_\alpha)_{\alpha \in I}$ a partition of J and $(Y_\alpha)_{\alpha \in I}$ a family of sets indexed by I . For each $\alpha \in I$, let g_α be a mapping of Y_α into X ; and for each $\beta \in J_\alpha$, let $h_{\alpha\beta}$ be a mapping of Z_β into Y_α , and put $f_\beta = g_\alpha \circ h_{\alpha\beta}$.

$$Z_\beta \xrightarrow{h_{\alpha\beta}} Y_\alpha \xrightarrow{g_\alpha} X$$

If each Y_α carries the final topology induced by $(h_{\alpha\beta})_{\beta \in J_\alpha}$ then the final topology on X induced by $(f_\beta)_{\beta \in J}$ is the same as the final topology induced by $(g_\alpha)_{\alpha \in I}$.

Example 1.2.20 (Example of final topologies).

- (a) Let X be a topological space, R an equivalence relation on X , $Y = X/R$ the quotient set of X with respect to the relation R , $\pi : X \rightarrow Y$ the canonical mapping. The final topology on Y induced by π is called the **quotient** of the topology of X by the relation R ; we shall study it in more detail later.
- (b) Every family $(\mathcal{T}_i)_{i \in I}$ of topologies on a set X has a greatest lower bound \mathcal{T} in the set of all topologies on X , i.e. \mathcal{T} is the finest of all the topologies on X which are coarser than each of the \mathcal{T}_i . To see this we may apply ??, taking Y_i to be the set X with the topology \mathcal{T}_i and f_i to be the identity mapping $Y_i \rightarrow X$. Then the collection \mathcal{T} is given by the intersection of all \mathcal{T}_i . It is also called the **intersection** of the topologies \mathcal{T}_i .
- (c) Let $(X_i)_{i \in I}$ be a family of topological spaces and $X = \coprod_{i \in I} X_i$ be the coproduct of X_i 's; for each $i \in I$, let ι_i be the canonical (injective) mapping of X_i into X . The final topology \mathcal{T} on X induced by the mappings ι_i is called the **coproduct** of the topologies of the X_i and X with this topology is said to be the **coproduct** of the topological spaces X_i . Let us identify each of the X_i with a subset of X by means of ι_i ; then a set $A \subseteq X$ is open (resp. closed) in the topology \mathcal{T} if and only if each of the sets $A \cap X_i$ is open (resp. closed) in X_i . In particular, each of the X_i is both open and closed.

The following proposition generalizes the situation of coproducts of a family of topological spaces.

Proposition 1.2.21. Let X be a set, $(X_i)_{i \in I}$ a family of subsets of X . Suppose each X_i carries a topology \mathcal{T}_i such that for each pair of indices (i, j) :

- (a) $X_i \cap X_j$ is open (resp. closed) in each of the topologies \mathcal{T}_i and \mathcal{T}_j .
- (b) The topologies induced on $X_i \cap X_j$ by \mathcal{T}_i and \mathcal{T}_j coincide.

Let \mathcal{T} be the final topology on X induced by the inclusions $\iota_i : X_i \rightarrow X$, then for each $i \in I$ the subset X_i is open (resp. closed) in X in the topology \mathcal{T} and the topology induced by \mathcal{T} on X_i coincides with \mathcal{T}_i .

Proof. In view of ??, it is enough to show that for each i and each subset A_i of X_i , the following statements are equivalent:

- (i) A_i is open (resp. closed) in the topology \mathcal{T}_i .
- (ii) For each $j \in I$, $A_i \cap X_j$ is open (resp. closed) in the topology \mathcal{T}_j .

It is clear that (ii) implies (i) by taking $j = i$. Conversely, if (i) is satisfied, then $A_i \cap X_j$ is open (resp. closed) in $X_i \cap X_j$ for the topology \mathcal{T}_{ij} induced on $X_i \cap X_j$ by \mathcal{T}_i ; but \mathcal{T}_{ij} is also the topology induced on $X_i \cap X_j$ by \mathcal{T}_j ; hence $A_i \cap X_j$ is also the intersection of $X_i \cap X_j$ with a subset $B_j \subseteq X_j$ which is open (resp. closed) in the topology \mathcal{T}_j . Since $X_i \cap X_j$ is also open (resp. closed) in \mathcal{T}_j , so is $A_i \cap X_j$. This completes the proof. \square

1.2.5 Subspaces of a topological space

Let A be a subset of a topological space X . We have defined the topology induced on A by the topology of X as the inverse image of the latter under the canonical injection $A \rightarrow X$. An equivalent definition is as follows:

Definition 1.2.22. Let A be a subset of a topological space X . The topology induced on A by the topology of X is that in which the open sets are the intersections with A of open sets of X . The set A with this topology is called a **subspace** of X .

By the transitivity of initial topology (or from definition) we see that, if $B \subseteq A \subseteq X$, the subspace B of X is identical with the subspace B of the subspace A of X . If \mathcal{S} is a subbase (resp. base) of the topology of X , its trace \mathcal{S}_A on A is a subbase (resp. a base) of the topology induced on A .

In all questions which involve the elements or subsets of A , it is essential to distinguish carefully between their properties as points (resp. subsets) of X , and their properties as points (resp. subsets) of the subspace A . We shall make this distinction by using the phrases "in A ", "with respect to A ", or "relative to A " to refer to properties in the latter category (possibly contrasting them with the phrases "in X ", "with respect to X ", "relative to X ").

An open set of the subspace A need not be open in X , and a closed subset of A need not be closed. For these two conditions to be true, we have the following result.

Proposition 1.2.23. Let A be a subspace of a topological space X .

- (a) In order that every set which is open in A should be open in X it is necessary and sufficient that A is open in X .
- (b) In order that every set which is closed in A should be closed in X it is necessary and sufficient that A is closed in X .
- (c) Every neighbourhood of x relative to A is a neighbourhood of x relative to X if and only if A is a neighbourhood of x in X .

Proof. The condition in (a) is necessary, since A is open in A , and it is sufficient by virtue of (O2) and the definition of subspace topology. The other parts can be proved similarly. \square

Now let X be an ordered set in the order topology, and let A be a subset of X . The order relation on X , when restricted to A , makes A into an ordered set. However, the resulting order topology on A need not be the same as the topology that A inherits as a subspace of X . We give one example where the subspace and order topologies on A agree, and two examples where they do not.

Example 1.2.24. Consider the subset $A = [0, 1]$ of the real line \mathbb{R} , in the subspace topology. The subspace topology has as basis all sets of the form $(a, b) \cap A$, where (a, b) is an open interval in \mathbb{R} . Such a set is of one of the following types:

$$(a, b) \cap A = \begin{cases} (a, b) & \text{if } a \text{ and } b \text{ are in } A, \\ [0, b) & \text{if only } b \text{ is in } A, \\ (a, 1] & \text{if only } a \text{ is in } A, \\ A \text{ or } \emptyset & \text{if neither } a \text{ nor } b \text{ is in } A. \end{cases}$$

By definition, each of these sets is open in A . But sets of the second and third types are not open in the larger space \mathbb{R} .

Note that these sets form a basis for the order topology on A . Thus, we see that in the case of the set $A = [0, 1]$, its subspace topology (as a subspace of \mathbb{R}) and its order topology are the same.

Example 1.2.25. Let A be the subset $[0, 1] \cup \{2\}$ of \mathbb{R} . In the subspace topology on A the singleton $\{2\}$ is open, because it is the intersection of the open set $(3/2, 5/2)$ with A . But in the order topology on A , the set $\{2\}$ is not open. Any basis element for the order topology on A that contains 2 is of the form $(a, 2]$ for some $a \in A$; such a set necessarily contains points of A less than 2.

Example 1.2.26. Let $I = [0, 1]$. The dictionary order on $I \times I$ is just the restriction to $I \times I$ of the dictionary order on the plane $\mathbb{R} \times \mathbb{R}$. However, the dictionary order topology on $I \times I$ is not the same as the subspace topology on $I \times I$ obtained from the dictionary order topology on $\mathbb{R} \times \mathbb{R}$! For example, the set $\{1/2\} \times (1/2, 1]$ is open in $I \times I$ in the subspace topology, but not in the order topology since $(1/2, 1]$ is not the biggest point of $I \times I$ in dictionary order. The set $I \times I$ in the dictionary order topology will be called the **ordered square**.

Proposition 1.2.27. If A and B are two subsets of a topological space X , and $B \subseteq A$, then the closure of B in the subspace A is the intersection with A of the closure \bar{B} of B in X .

Proof. If $x \in A$, every neighbourhood of x in A is of the form $V \cap A$, where V is a neighbourhood of x in X . Since $V \cap B = (V \cap A) \cap B$, it follows that x lies in the closure of B with respect to A if and only if x lies in the closure of B with respect to X . \square

Corollary 1.2.28. A subset B of A is dense in A if and only if $\bar{B} \supseteq A$ in X , if and only if $\bar{B} = \bar{A}$.

Corollary 1.2.29. It follows that if A, B, C are three subsets of X such that $C \subseteq B \subseteq A$, and if B is dense in A , and C is dense in B , then C is dense in A .

Proposition 1.2.30. Let A be a dense subset of a topological space X ; then for each $x \in A$ and each neighbourhood V of x relative to A , the closure \bar{V} of V in X is a neighbourhood of x in X .

Proof. Since C is a neighbourhood of x relative to A , there exists an open subset U of X such that $x \subseteq U \cap A \subseteq V$. Then since U is open, by ?? we have

$$\bar{V} \supseteq \overline{U \cap A} \supseteq U \cap \bar{A} = U,$$

whence \bar{V} is a neighbourhood of x in X . \square

Now we consider continuity with respect to subspaces. Let X and Y be two topological spaces, f a mapping of X into Y , B a subset of Y which contains $f(X)$. The definition of the induced topology as an initial topology shows that f is continuous at $x \in X$ if and only if the mapping of X into the subspace B of Y , having the same graph as f , is continuous at x .

Now let A be a subset of X ; if f is continuous at $x \in A$ (resp. continuous on X), its restriction $f|_A$ is a mapping of the subspace A into Y , which is continuous at x (resp. continuous on A). We shall sometimes say that a mapping $f : X \rightarrow Y$ is continuous relative to A at $x \in A$ (resp. continuous relative to A) if its restriction $f|_A$ is continuous at x (resp. continuous on A).

It should be noted that $f|_A$ can be continuous without f being continuous at any point of X ; an example of this phenomenon is provided by the characteristic function χ_A of a subset A of X which is such that both A and its complement are dense in X . The function χ_A , being regarded as a mapping of X into the discrete space $\{0, 1\}$ is not continuous at any point of X , but its restriction to A is constant and therefore continuous.

On the other hand, if A is a neighbourhood in X of a point $x \in A$, and if $f : X \rightarrow Y$ is such that $f|_A$ is continuous at x , then f is continuous at x ; for each neighbourhood of x in A is a neighbourhood of x in X (local character of continuity).

To conclude, we give a condition in which the openness of a subset of X can be detected by an cover of X by subspaces.

Proposition 1.2.31. *Let $(A_i)_{i \in I}$ be a family of subsets of a topological space X , such that one of the following properties holds:*

- (a) *the interiors of A_i 's cover X ;*
- (b) *$(A_i)_{i \in I}$ is a locally finite closed covering of X .*

Then a subset of X is open (resp. closed) in X if and only if each of its intersection with A_i is open (resp. closed) in A_i .

Proof. Clearly if U is open (resp. closed) in X , then $U \cap A_i$ is open (resp. closed) in A_i . Conversely, suppose first that condition (a) is satisfied; since $U^c \cap A_i = A_i \setminus (U \cap A_i)$, it is enough, by duality, to consider the case in which each of the $U \cap A_i$ is open with respect to A_i . In this case $U \cap \text{Int } A_i$ is open in $\text{Int } A_i$ for each $i \in I$, and therefore open in X ; and since $U = \bigcup_{i \in I} (U \cap \text{Int } A_i)$ by hypothesis, it follows that U is open in X .

Now suppose that (b) is satisfied and let $F \subseteq X$; by duality again, we need only consider the case in which each of the $F \cap A_i$ is closed in A_i and therefore closed in X . Since the family $(F \cap A_i)$ is locally finite and $F = \bigcup_{i \in I} (F \cap A_i)$, it follows from ?? that F is closed in X . \square

1.3 Quotient spaces and quotient maps

1.3.1 Quotient spaces and maps

Let X be a topological space, R an equivalence relation on X . Recall that the quotient space of X by R is the quotient set X/R with the quotient topology induced by the canonical map $\pi : X \rightarrow X/R$. Unless the contrary is expressly stated, whenever we speak of X/R as a topological space, it is to be understood that we mean the quotient space of X by R . We shall often say that this topological space is the space obtained by identifying the points of X which belong to the same equivalence class mod R .

By the definition of quotient topology, the open sets in X/R are the sets $U \subseteq X/R$ such that $\pi^{-1}(U)$ is open in X . In other words, the open (resp. closed) sets in X/R are in one-to-one correspondence with the open (resp. closed) subsets of X which are *saturated* with respect to R and are the canonical images of these subsets. (Recall that a subset $U \subseteq X$ is **saturated** with respect to an equivalence relation if it is a union of equivalent classes, or equivalently $U = \pi^{-1}(\pi(U))$.)

From the characterization of final topology and transitivity, we get the following results for quotient topology.

Proposition 1.3.1. *Let X be a topological space, R an equivalence relation on X , π the canonical mapping of X onto X/R ; then a mapping f of X/R into a topological space Y is continuous if and only if $f \circ \pi$ is continuous on X .*

Proposition 1.3.2 (Transitivity of quotient spaces). *Let R and S be two equivalence relations on a topological space X such that $R \subseteq S$, and let S/R be the quotient equivalence relation on the quotient space X/R . Then the canonical bijection $(X/R)/(S/R) \rightarrow X/S$ is a homeomorphism.*

Note that ?? also shows that there is a one-to-one correspondence between the continuous mappings of X/R into Y and the continuous maps of X into Y which are constant on each equivalence class mod R .

Example 1.3.3. Consider the equivalence relation mod \mathbb{Z} on the real line \mathbb{R} ; the quotient space of \mathbb{R} by this relation is called the one-dimensional torus and is denoted by \mathbb{T} . The equivalence class of a point $x \in \mathbb{R}$ consists of all the points $x + n$, where n runs through the set \mathbb{Z} of integers. By ??, there is a one-to-one correspondence between the continuous functions on \mathbb{T} and the continuous functions on \mathbb{R} which are periodic with period 1.

Corollary 1.3.4. *Let X, Y be two topological spaces, R (resp. S) an equivalence relation on X (resp. Y), and let $f : X \rightarrow Y$ be a continuous mapping which is compatible with the equivalence relations R and S . Then the mapping $\tilde{f} : X/R \rightarrow Y/S$ induced by f is continuous.*

Proof. Let π_R and π_S be the canonical projection of R and S . To prove the claim, let V be a open subset of Y/S , so that $\pi_S^{-1}(V)$ is open in Y . Then from the following commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \pi_R \downarrow & & \downarrow \pi_S \\ X/R & \xrightarrow{\tilde{f}} & Y/S \end{array}$$

and the continuity of f , we conlude that $f^{-1}(\pi_S^{-1}(V)) = \pi_R^{-1}(\tilde{f}^{-1}(V))$ is open in X , whence $\tilde{f}^{-1}(V)$ is open in X/R . This proves the corollary. \square

Now using the notation of quotient topologies, we can establish a canonical decomposition of a continuous map. To be more precise, let X and Y be two topological spaces, $f : X \rightarrow Y$ a continuous map, R the equivalence relation $f(x) = f(y)$ on X . Consider the canonical decomposition

$$\begin{array}{ccccc} X & \xrightarrow{\pi} & X/R & \xrightarrow{\tilde{f}} & f(X) & \xrightarrow{\iota} & Y \\ & & \searrow f & & & & \end{array}$$

where π is the canonical (surjective) mapping of X onto the quotient space X/R , ι is the canonical injection of the subspace $f(X)$ into Y , and \tilde{f} is the bijection associated with f . It is immediately seen that \tilde{f} is continuous (by ??). But the bijection \tilde{f} is not necessarily a homeomorphism.

Proposition 1.3.5. *Let $f = \pi \circ \tilde{f} \circ \iota$ be the canonical decomposition of a continuous mapping $f : X \rightarrow Y$, and let R denote the equivalence relation $f(x) = f(y)$. Then the following three conditions are equivalent:*

- (i) \tilde{f} is a homeomorphism of X/R onto $f(X)$.
- (ii) The image under f of every open set which is saturated with respect to R is an open set in the subspace $f(X)$.
- (iii) The image under f of every closed set which is saturated with respect to R is a closed set in the subspace $f(X)$.

Proof. Since \tilde{f} is bijective, it is a homeomorphism if and only if \tilde{f} is open or closed. Now note that an open subset of X/R is the image under π of a saturated open subset of X , so the claim follows. \square

Example 1.3.6. Let X be a topological space and $(X_i)_{i \in I}$ a covering of X . Since for each $i \in I$ we have an inclusion $\iota_i : X_i \rightarrow X$, we can consider the coproduct $\tilde{X} = \coprod_{i \in I} X_i$ and get a map $\Phi : \tilde{X} \rightarrow X$. If we identify each X_i as a subspace of \tilde{X} , then Φ is a homeomorphism when restricted to each X_i . Let R be the equivalence relation $\Phi(x) = \Phi(y)$; the quotient space \tilde{X}/R is thus obtained by "pasting together" the overlaps of X_i 's.

Consider the bijection $\phi : \tilde{X}/R \rightarrow X$ associated with Φ , which is always continuous. In general ϕ is not a homeomorphism, as is shown by the example in which each X_i consists of a single point and X is not discrete.

However, if the interiors of the X_i cover X , or if $(X_i)_{i \in I}$ is a locally finite closed covering of X , then ϕ is a homeomorphism: for if U is any open subset of Y which is saturated with respect to R , then $\Phi(U)$ is open in X because for each $i \in I$ the set

$$\Phi(U) \cap X_i = \iota_i(U \cap Y_i)$$

is open in X_i since Φ_i is a homeomorphism on each X_i , hence $\Phi(U)$ is open by ??.

The following proposition gives a simple sufficient condition for \tilde{f} to be a homeomorphism:

Proposition 1.3.7. *Let $f : X \rightarrow Y$ be a continuous surjection, and let R denote the equivalence relation $f(x) = f(y)$. If there is a continuous section $s : Y \rightarrow X$ associated with f , then the mapping $\tilde{f} : X/R \rightarrow Y$ associated with f is a homeomorphism, and s is a homeomorphism of Y onto the subspace $s(Y)$ of X .*

Proof. If $\pi : X \rightarrow X/R$ is the canonical mapping, then \tilde{f} and $\pi \circ s$ are bijective, continuous and inverse to each other; likewise s and the restriction of f to $s(Y)$ are bijective, continuous and inverse to each other. These prove the claim. \square

If R is an equivalence relation on a topological space X and $\pi : X \rightarrow X/R$ is the canonical mapping, a continuous section $s : X/R \rightarrow X$ associated with f is also called a **continuous section** of X with respect to R ; the subspace $s(X/R)$ of X is then homeomorphic to X/R . If we

are given $s(X/R)$, then s is uniquely determined; $s(X/R)$ is often called, by abuse of language, a **(continuous) section** of X with respect to R .

Now let X and Y be topological spaces, and $q : X \rightarrow Y$ be a map. Then q is called a **quotient map** if it is surjective and Y has the quotient topology induced by q . Once q is known to be surjective, to say it is a quotient map is the same as saying that V is open in Y if and only if $q^{-1}(V)$ is open in X . It is immediate from the definition that every quotient map is continuous.

Proposition 1.3.8. *Let $f : X \rightarrow Y$ be a continuous surjection. If f is an open map, or a closed map, then f is an quotient map.*

Proof. Suppose that f is an open map. Let U be a subset of Y . By continuity, if U is open in Y then $f^{-1}(U)$ is open in X . On the other hand, since f is a surjection, we have $f(f^{-1}(U)) = U$. Hence $f^{-1}(U)$ is open if and only if U is open. A similar proof applies with open replaced by closed. \square

It is easy to see a composition of quotient maps is a quotient map; this fact follows from the equation

$$p^{-1}(q^{-1}(U)) = (q \circ p)^{-1}(U).$$

On the other hand, products of maps do not behave well; the cartesian product of two quotient maps need not be a quotient map. One needs further conditions on either the maps or the spaces in order for this statement to be true. One such, a condition on the spaces, is called local compactness. Another, a condition on the maps, is the condition that both the maps p and q be open maps. In that case, it is easy to see that $p \times q$ is also an open map, so it is a quotient map.

Example 1.3.9. A product of quotient maps need not to be a quotient map. Let $\pi : \mathbb{Q} \rightarrow \mathbb{Q}/\mathbb{Z}$ be the quotient map and let $h = \pi \times \text{id} : \mathbb{Q} \times \mathbb{Q} \rightarrow (\mathbb{Q}/\mathbb{Z}) \times \mathbb{Q}$. Let $r_0 = 1$ and for each non-zero $n \in \mathbb{N}$ let $r_n = \sqrt{2}/|n|$ —thus r_n is irrational and $r_n \rightarrow 0$ as $n \rightarrow \infty$. Let A_n be any open subset of $[n, n+1] \times \mathbb{R}$ such that the closure of A_n meets $\{n, n+1\} \times \mathbb{R}$ in the two points (n, r_n) and (n, r_{n+1}) . Let A be the union of these sets A_n , $n \in \mathbb{N}$, and let $B = \bar{A} \cap (\mathbb{Q} \times \mathbb{Q})$. Then B is closed in $\mathbb{Q} \times \mathbb{Q}$ and saturated with respect to h . By our construction of r_n , we see that in $(\mathbb{Q}/\mathbb{Z}) \times \mathbb{Q}$ the point $(\pi(0), 0)$ belongs to the closure of $h(B)$, so $h(B)$ is not closed. This then implies that h is not a quotient map.

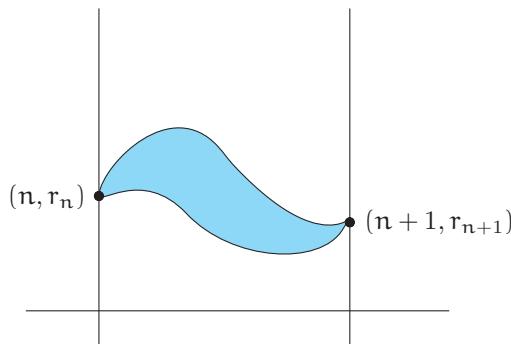


Figure 1.1: The open subset A_n .

On the other hand, we have the following result for coproducts.

Proposition 1.3.10. Let $(X_\alpha)_{\alpha \in A}$ and $(Y_\alpha)_{\alpha \in A}$ be topological spaces and for each $\alpha \in A$ let $p_\alpha : X_\alpha \rightarrow Y_\alpha$ be a quotient map. Then the map

$$p : \coprod_\alpha X_\alpha \rightarrow \coprod_\alpha Y_\alpha$$

which coincides with p_α on X_α is a quotient map.

Proof. Let V be a subset in $\coprod_\alpha Y_\alpha$, then by the definition of p , we have

$$p^{-1}(V) = \bigcup_\alpha p_\alpha^{-1}(V \cap Y_\alpha),$$

so $p^{-1}(V) \cap X_\alpha = p_\alpha^{-1}(V \cap Y_\alpha)$. The claim now follows from by the definition of coproduct topology and the fact that each p_α is a quotient map. \square

1.3.2 Quotient of a subspace

Let X be a topological space, A a subspace of X , and π the canonical map $X \rightarrow X/R$ of an equivalence relation R on X . Let R_A be the relation induced by R on A , and $\pi|_A = \iota \circ \tilde{\pi} \circ \pi_A$ be the canonical decomposition of $\pi|_A$, so that we have a commutative diagram

$$\begin{array}{ccccccc} A & \xrightarrow{\pi_A} & A/R_A & \xrightarrow{\tilde{\pi}} & \pi(A) & \xrightarrow{\iota} & X/R \\ & \searrow & \nearrow \pi|_A & & & & \end{array}$$

Proposition 1.3.11. The canonical bijection $\tilde{\pi} : A/R_A \rightarrow \pi(A)$ is continuous. Furthermore, the following three statements are equivalent:

- (i) $\tilde{\pi}$ is a homeomorphism.
- (ii) Every open subset of A which is saturated with respect to R_A is the intersection with A of an open subset of X which is saturated with respect to R .
- (iii) Every closed subset of A which is saturated with respect to R_A is the intersection with A of a closed subset of X which is saturated with respect to R .

Proof. Note that A/R_A carries the quotient topology induced by π_A , and $\pi(A)$ carries the subspace topology induced by X/R . Now the claim follows from ??: if U is an open subset of A which is saturated with respect to R_A , and $\tilde{\pi}(U) = \pi(U)$ is the intersection with $\pi(A)$ of an open subset V of X/R , then U is the intersection with A of the open subset $\pi^{-1}(V)$ of X , which is saturated with respect to R ; and conversely, if U is the intersection with A of an open subset W which is saturated with respect to R , then $\pi(U)$ is the intersection of $\pi(A)$ and $\pi(W)$, which is open in X/R . \square

Corollary 1.3.12. If A is an open (resp. closed) subset of X which is saturated with respect to R , then the canonical mapping $\tilde{\pi} : A/R_A \rightarrow \pi(A)$ is a homeomorphism.

Proof. If A is open (resp. closed) in X and saturated with respect to R , and if $B \subseteq A$ is open (resp. closed) in A and saturated with respect to R_A , then B is open (resp. closed) in X and saturated with respect to R . \square

Corollary 1.3.13. *If there is a continuous mapping $\varphi : X \rightarrow A$ such that $\varphi(x) \equiv x \bmod R$ for each $x \in X$, then $\pi(A) = X/R$ and the canonical mapping $\tilde{\pi} : A/R_A \rightarrow X/R$ is a homeomorphism.*

Proof. Since each equivalence class mod R meets A , the canonical image of $\pi(A)$ in X/R is the whole of X/R ; on the other hand, if U is open in A and is saturated with respect to R_A , it follows from the hypothesis that $\varphi^{-1}(U)$ is the set obtained by saturating U with respect to R ; since φ is continuous, $\varphi^{-1}(U)$ is open in X . The corollary follows from this fact by virtue of ??.

Example 1.3.14. Let R denote the equivalence relation $x \equiv y \bmod \mathbb{Z}$ on the real line \mathbb{R} and let I denote the closed interval $[0, 1]$; I contains at least one point of each equivalence class mod R . The canonical mapping of I/R_I onto the torus \mathbb{T} is a homeomorphism; for if F is closed in I (and hence in \mathbb{R}), in order to saturate F with respect to the relation R we have to take the union of the closed sets $F + n$ for all $n \in \mathbb{Z}$, which evidently form a locally finite family, so that their union is closed; the assertion follows from this. We remark that I/R_I is obtained by identifying the points 0 and 1 in I .

1.3.3 Hausdorff condition of quotient spaces

Typically, to prove that a given quotient space is Hausdorff, one has to resort to the definition. For open quotient maps, however, we have the following criterion.

Proposition 1.3.15. *Suppose $q : X \rightarrow Y$ is an open quotient map. Then Y is Hausdorff if and only if the set $\mathcal{R} := \{(x_1, x_2) : q(x_1) = q(x_2)\}$ is closed in $X \times X$.*

Proof. First assume Y is Hausdorff. If $(x_1, x_2) \notin \mathcal{R}$, then there are disjoint neighborhoods V_1 of $q(x_1)$ and V_2 of $q(x_2)$, and it follows that $q^{-1}(V_1) \times q^{-1}(V_2)$ is a neighborhood of (x_1, x_2) that is disjoint from \mathcal{R} . Thus \mathcal{R} is closed. (This implication does not require the assumption that q is open.)

Conversely, assume \mathcal{R} is closed. Given distinct points $y_1, y_2 \in Y$, choose $x_1, x_2 \in X$ such that $q(x_i) = y_i$. Because $(x_1, x_2) \notin \mathcal{R}$, there is a product neighborhood $U_1 \times U_2$ of (x_1, x_2) in $X \times X$ that is disjoint from \mathcal{R} . Since q is open, $q(U_1)$ and $q(U_2)$ are disjoint neighborhoods of y_1 and y_2 , respectively.

Corollary 1.3.16. *Suppose \sim is an equivalence relation on a space X . If the quotient map $X \rightarrow X/\sim$ is an open map, then X/\sim is Hausdorff if and only if \sim is a closed subset of $X \times X$.*

Theorem 1.3.17. *Suppose X is a compact Hausdorff space and $q : X \rightarrow Y$ is a quotient map. Then the following are equivalent.*

- (a) Y is Hausdorff.
- (b) q is a closed map.
- (c) The set $\mathcal{R} = \{(x_1, x_2) : q(x_1) = q(x_2)\}$ is closed in $X \times X$.

Proof. The implication (a) \Rightarrow (b) is immediate by closed map lemma, and (a) \Rightarrow (c) is proved just as in ?. Next we prove (b) \Rightarrow (a). Assuming q is a closed map, we begin by showing that its fibers are compact. Every point $y \in Y$ is the image of some $x \in X$, and since $\{x\}$ is a closed

subset of X , it follows that $\{y\} = q(\{x\})$ is a closed subset of Y . Thus by continuity, the fiber $q^{-1}(\{y\})$ is closed in X and hence compact.

To prove that Y is Hausdorff, suppose y_1 and y_2 are distinct points of Y . Then we can prove the disjoint compact subsets $q^{-1}(y_1)$ and $q^{-1}(y_2)$ of X have disjoint neighborhoods U_1 and U_2 , respectively. Define subsets $W_1, W_2 \subseteq Y$ by

$$W_i = Y - q(X - U_i) = \{y \in Y : q^{-1}(y) \subseteq U_i\}$$

Then $y_i \in W_i$ for each i by construction, and W_1 and W_2 are disjoint because U_1 and U_2 are. To complete the proof, the fact that q is closed implies that W_i is open.

Finally, we prove $(c) \Rightarrow (a)$. Assuming that \mathcal{R} is closed, we start again by showing that q has compact fibers. Given $y \in Y$ and $x \notin q^{-1}(y)$, let x_1 be any point in $q^{-1}(y)$. (Such a point exists because q is surjective.) Because \mathcal{R} is closed and $(x, x_1) \notin \mathcal{R}$, there is a product neighborhood $U_1 \times U_2$ of (x_1, x) that is disjoint from \mathcal{R} . It follows that U_2 is a neighborhood of x disjoint from $q^{-1}(y)$, for if x_2 were a point in $U_2 \cap q^{-1}(y)$, then (x_1, x_2) would lie in $\mathcal{R} \cap (U_1 \times U_2)$, which is empty. Thus $q^{-1}(y)$ is closed in X and hence compact.

Now let y_1 and y_2 be distinct points in Y . As before, there are disjoint open subsets $U_1 \supseteq q^{-1}(y_1)$ and $U_2 \supseteq q^{-1}(y_2)$, and we define W_1, W_2 as before. These are disjoint sets containing y_1 and y_2 , respectively, so we need only show they are open. Because q is a quotient map, W_i is open if and only if $q^{-1}(W_i)$ is open, which is the case if and only if $X - q^{-1}(W_i)$ is closed. From the definition of W_i , it follows that

$$\begin{aligned} X - q^{-1}(W_i) &= \{x \in X : q(x) \notin W_i\} = \{x \in X : q^{-1}(q(x)) \not\subseteq U_i\} \\ &= \{x \in X : \exists \tilde{x} \in X - U_i \text{ such that } q(x) = q(\tilde{x})\} \\ &= \pi_1(\mathcal{R} \cap (X \times X - U_i)) \end{aligned}$$

where $\pi_1 : X \times X \rightarrow X$ is the projection on the first factor. Observe that π_1 is a closed map by the closed map lemma. Our hypothesis on \mathcal{R} implies that $\mathcal{R} \cap (X \times X - U_i)$ is closed in $X \times X$, and therefore $X - q^{-1}(W_i)$ is closed in X . \square

1.3.4 Direct limit of topological spaces

Let $(X_\alpha)_{\alpha \in I}$ be a family of sets, and let X be the set which is the coproduct of the X_α . We shall identify each X_α with a subset of X by means of the canonical injection $\iota_\alpha : X_\alpha \rightarrow X$.

Let R be an equivalence relation on X such that each equivalence class of R has at most one element in each X_α ; for each pair of indices (α, β) let $A_{\alpha\beta}$ be the subset of X_α consisting of the elements x for which there is an element $y \in X_\beta$ which belongs to the equivalence class of x . Clearly to each $x \in A_{\alpha\beta}$ there is a unique $y \in X_\beta$ which is congruent to x mod R ; the mappings $\phi_{\beta\alpha} : A_{\beta\alpha} \rightarrow A_{\alpha\beta}$ so defined satisfy the following conditions:

- (a) For each $\alpha \in I$, $\phi_{\alpha\alpha}$ is the identity mapping of $A_{\alpha\alpha} = X_\alpha$.
- (b) For each triple of indices (α, β, γ) of I and each $x \in A_{\alpha\beta} \cap A_{\alpha\gamma}$, we have $\phi_{\beta\alpha}(x) \in A_{\beta\gamma}$ and $\phi_{\gamma\alpha}(x) = \phi_{\beta\gamma}(\phi_{\beta\alpha}(x))$.

Conversely, suppose that for each pair of indices (α, β) we are given a subset $A_{\alpha\beta}$ of X_α and a mapping $\phi_{\beta\alpha} : A_{\beta\alpha} \rightarrow A_{\alpha\beta}$ satisfying the conditions (a) and (b) above. It follows first

of all from (b) applied to the triples (α, β, α) and (β, α, β) that $\phi_{\alpha\beta} \circ \phi_{\beta\alpha}$ (resp. $\phi_{\beta\alpha} \circ \phi_{\alpha\beta}$) is the restriction of $\phi_{\alpha\alpha}$ (resp. $\phi_{\beta\beta}$) to $A_{\alpha\alpha}$ (resp. $A_{\beta\beta}$); hence we deduce from (a) that $\phi_{\beta\alpha}$ and $\phi_{\alpha\beta}$ are bijections which are inverses of each other. Now let R be the relation "there exist α, β such that $x \in A_{\alpha\beta}$, $y \in A_{\beta\alpha}$ and $y = \phi_{\beta\alpha}(x)$." It follows from (a) and what precedes that R is reflexive and symmetric; on the other hand, if

$$x \in A_{\alpha\beta}, \quad y = \phi_{\beta\alpha}(x) \in A_{\beta\alpha} \cap A_{\beta\gamma}, \quad z = \phi_{\gamma\beta}(y) \in A_{\gamma\beta},$$

then also $x = \phi_{\alpha\beta}(y)$ and by (b) we have $z = \phi_{\gamma\alpha}(x)$; thus R is transitive, and so is an equivalence relation on X . It follows also from (a) and from the definition of R that each equivalence class mod R has at most one element in each of the sets X_α , and that $A_{\alpha\beta}$ is the set of all $x \in X_\alpha$ for which there is an element $y \in X_\beta$ congruent to x mod R . We say that the quotient set X/R is obtained by pasting together the X_α along the $A_{\alpha\beta}$ by means of the bijections $\phi_{\beta\alpha}$. If $\pi : X \rightarrow X/R$ is the canonical mapping, the restriction of π to each X_α is a bijection of X_α onto $\pi(X_\alpha)$.

Now suppose that each X_α is a topological space, and let \mathcal{T}_α be its topology. Let \mathcal{T} be the finest topology on the set X/R for which the mappings $\pi \circ \iota_\alpha$ are continuous; then \mathcal{T} is the quotient by R of the topology on X which is the coproduct of the topologies \mathcal{T}_α . We say that the topological space X/R (with the topology \mathcal{T}) is obtained by pasting together the topological spaces X_α along the $A_{\alpha\beta}$ by means of the bijections $\phi_{\beta\alpha}$. The open (resp. closed) subsets of X/R are thus the canonical images of the subsets B of X which are saturated with respect to R and are such that $B \cap X_\alpha$ is open (resp. closed) in X_α for each $\alpha \in I$.

Since the restriction of π to each X_α is a bijection onto the subset $\pi(X_\alpha)$ of X/R , we can transport the topology \mathcal{T}_α to $\pi(X_\alpha)$ by means of this bijection, so that $\pi(X_\alpha)$ carries a topology $\tilde{\mathcal{T}}_\alpha$ and the topology \mathcal{T} on X/R is the finest for which the canonical injections $\pi(X_\alpha) \rightarrow X/R$ are continuous. In general, the topology induced by \mathcal{T} on $\pi(X_\alpha)$ is coarser than $\tilde{\mathcal{T}}_\alpha$, but not identical with the latter; even if the $\phi_{\beta\alpha}$ are homeomorphisms. However, it follows from ?? that, with the preceding notation:

Proposition 1.3.18. *Suppose that the $\phi_{\beta\alpha}$ are homeomorphisms and that each $A_{\alpha\beta}$ is open (resp. closed) in X_α ; then each $\pi(X_\alpha)$ is open (resp. closed) in X/R and the restriction of π to X_α is a homeomorphism of X_α onto the subspace $\pi(X_\alpha)$ of X/R .*

1.4 Product of topological spaces

1.4.1 Product spaces

Definition 1.4.1. *Given a family $(X_i)_{i \in I}$ of topological spaces, the **product space** of this family is the product set $X = \prod_{i \in I} X_i$ with the initial topology induced by the projections $\pi_i : X \rightarrow X_i$. The spaces X_i are called the factors of X .*

By virtue of ??, the product topology on X has as a base the set of finite intersections of sets of the form $\pi_i^{-1}(U_i)$, where U_i is open in X_i ; these sets are products $\prod_{i \in I} U_i$, where U_i is open in X_i for each i and $U_i = X_i$ for all but a finite number of indices. These sets will be called **elementary sets**.

If \mathcal{B}_i is a base of the topology of X_i for each $i \in I$, it is clear that the elementary sets $\prod_i B_i$ such that $B_i = X_i$ for all but finitely many indices form another base of the product topology.

The elementary sets of this type which contain a given point $x \in X$ thus form a fundamental system of neighbourhoods of x .

If I is a finite set, the construction of the product topology from the topologies of the factors X_i is simpler: the elementary sets are just products $\prod_{i \in I} U_i$ where U_i is any open subset of X_i for each $i \in I$.

Example 1.4.2. The product \mathbb{R}^n of n spaces identical with the real line \mathbb{R} is called real number space of n dimensions; \mathbb{R}^2 is also called the real plane. Likewise, starting from the rational line \mathbb{Q} , we define the rational number space of n dimensions \mathbb{Q}^n (rational plane for $n = 2$). The topology of the space \mathbb{R}^n has as a base the set of all products of n open intervals in \mathbb{R} , which are called open boxes of n dimensions. The open boxes which contain a point $x \in \mathbb{R}^n$ form a fundamental system of neighbourhoods of this point. Likewise the products of n closed intervals in \mathbb{R} are called closed boxes of n dimensions. The closed boxes to which x is interior also form a fundamental system of neighbourhoods of x . There are analogous results for \mathbb{Q}^n .

Proposition 1.4.3. Let $f = (f_i)$ be a mapping of a topological space Y into a product space $X = \prod_{i \in I} X_i$. Then f is continuous at a point $y \in Y$ if and only if f_i is continuous at y for each $i \in I$.

Proof. Since $f_i = f \circ \pi_i$, this is a special case of ??.

□

Corollary 1.4.4. Let $(X_i)_{i \in I}$ and $(Y_i)_{i \in I}$ be two families of topological spaces with the same set of indices. For each $i \in I$, let f_i be a mapping of X_i into Y_i . In order that the product mapping $f : (x_i) \rightarrow (f_i(x_i))$ should be continuous at a point $a = (a_i)$, it is necessary and sufficient that f_i is continuous at a_i for each $i \in I$.

Proof. The map f can be written as $x \mapsto (f_i(\pi_i(x)))$, so that by ?? the condition is sufficient. Conversely, for each $j \in I$ let g_j be the mapping of X_j into $\prod_{i \in I} X_i$ such that $\pi_j(g_j(x)) = x_j$ and $\pi_i(g_j(x)) = a_i$ for $i \neq j$. Then g_j is continuous at the point a_j , by ???. Since $f_j = \pi_j \circ f \circ g_j$, it follows that if f is continuous at a then f_j is continuous at a_j . □

Corollary 1.4.5. Let X, Y be two topological spaces. In order that a mapping $f : X \rightarrow Y$ should be continuous it is necessary and sufficient that the mapping $g : x \mapsto (x, f(x))$ is a homeomorphism of X onto the graph $\Gamma(f)$ (considered as a subspace of the product space $X \times Y$).

Proof. Since $f = \pi_2 \circ g$, the condition is sufficient. It is also necessary, for if f is continuous, then g is bijective and continuous (??) and the inverse of g is the restriction of π_1 to $\Gamma(f)$, which is continuous. □

Proposition 1.4.6 (Associativity of topological products). Let $(X_i)_{i \in I}$ be a family of topological spaces, $(I_\kappa)_{\kappa \in K}$ a partition of the set I , and for each $\kappa \in K$ let $X_\kappa = \prod_{i \in I_\kappa} X_i$. Then the canonical mapping of the product space $\prod_{i \in I} X_i$ onto the product space $\prod_{\kappa \in K} X_\kappa$ is a homeomorphism.

Proof. This is a special case of the transitivity of initial topologies.

□

Corollary 1.4.7. Let $\sigma \in S_I$ be a permutation of the set I . Then the mapping $(x_i) \mapsto (x_{\sigma(i)})$ is a homeomorphism of $\prod_{i \in I} X_i$ onto $\prod_{i \in I} X_{\sigma(i)}$.

Proof. Take $K = I$ and $I_i = \{\sigma(i)\}$ in the previous proposition.

□

Finally, we show that, the general concept of initial topologies can be defined using product spaces and product topology.

Proposition 1.4.8. Let X be a set, $(Y_i)_{i \in I}$ a family of topological spaces, and for each $i \in I$ let f_i be a mapping of X into Y_i . Let f be the mapping $x \mapsto (f_i(x))$ of X into $Y = \prod_{i \in I} Y_i$. Then the initial topology on X induced by (f_i) is the inverse image under f of the topology induced on $f(X)$ by the product topology on Y .

Proof. This is another application of the transitivity of initial topologies. \square

Corollary 1.4.9. For each $i \in I$ let A_i be a subspace of X_i . Then the topology induced on $A = \prod_{i \in I} A_i$ by the product topology on $\prod_{i \in I} X_i$ is the product of the topologies of the subspaces A_i .

Proof. Let ι_i be the canonical injection $A_i \rightarrow X_i$ and apply ?? to the mappings $f_i = \iota_i \circ \pi_i$. \square

There is another topology on the product space $X = \prod_{i \in I} X_i$ called the box topology, whose open sets are defined to be the product of open subsets in X_i . The box topology turns out to have too many open sets, for example, ?? does not hold for it.

Example 1.4.10. Consider \mathbb{R}^ω , the countably infinite product of \mathbb{R} with itself. That is, $\mathbb{R}^\omega = \prod_{n \in \mathbb{Z}_+} X_n$, where $X_n = \mathbb{R}$ for each n . Let us define a function $f : \mathbb{R} \rightarrow \mathbb{R}^\omega$ by the equation

$$f(x) = (x, x, \dots, x, \dots).$$

Each of the coordinate functions $f_n : \mathbb{R} \rightarrow \mathbb{R}$ is continuous; therefore, the function f is continuous if \mathbb{R}^ω is given the product topology. But f is not continuous if \mathbb{R}^ω is given the box topology. Consider, for example, the basis element

$$B = \prod_{n \in \mathbb{Z}_+} (-1/n, 1/n),$$

for the box topology. We assert that $f^{-1}(B)$ is not open in \mathbb{R} : If $f^{-1}(B)$ contains some interval $(-\delta, \delta)$ about the point 0, then we must have $(-\delta, \delta) \subseteq (-1/n, 1/n)$ for each n , whence $\delta = 0$, a contradiction.

1.4.2 Sections and projections

Proposition 1.4.11. Let X and Y be topological spaces. Then

- (a) for each $a \in X$, the map $y \mapsto (a, y)$ is a homeomorphism from Y to the subspace $\{a\} \times Y$ of $X \times Y$;
- (b) for each $b \in Y$, the map $x \mapsto (x, b)$ is a homeomorphism from X to the subspace $X \times \{b\}$ of $X \times Y$.

Proof. Since π_Y , restricted to $\{a\} \times Y$, is a continuous inverse of the given map in (a), this is a particular case of ?? applied to the constant function $x \mapsto a$. The same proof applies to (b). \square

Note that the mapping $y \mapsto (a, y)$ is a continuous section with respect to the equivalence relation $\pi_X(z) = \pi_X(z')$ in $X \times Y$; the quotient space of $X \times Y$ by this equivalence relation is therefore homeomorphic to Y .

Corollary 1.4.12. The section $A(x)$ of an open (resp. closed) set A of the product $X \times Y$ at an arbitrary point $x \in X$ is open (resp. closed) in Y .

Using ??, we also get the following useful fact.

Proposition 1.4.13. *The projection of $X \times Y$ onto either factor is an open map.*

Proof. Let U be open in $X \times Y$, then we have

$$\pi_X(U) = \bigcup_{y \in Y} U(y), \quad \pi_Y(U) = \bigcup_{x \in X} U(x),$$

and the proposition follows from the ??.

□

Note that the projection of a closed subset of $X \times Y$ onto a factor need not be closed. For example, in the real plane \mathbb{R}^2 , the hyperbola whose equation is $xy = 1$ is a closed set, but both its projections are equal to the complement of the point 0 in \mathbb{R} . and this is not a closed set.

Proposition 1.4.14. *Let X , Y and Z be three topological spaces, f a mapping of the product space $X \times Y$ into Z . If f is continuous at the point $(a, b) \in X \times Y$, then the partial mapping $y \mapsto f(a, y)$ of Y into Z is continuous at the point b .*

Proof. This mapping is the composition of f and the mapping $y \mapsto (a, y)$; hence the result follows from ??.

□

?? is often expressed by saying that a continuous function of two variables is continuous with respect to each of them separately.

Example 1.4.15. It is possible for all the partial mappings determined by a map $f : X \times Y \rightarrow Z$ to be continuous without f being continuous. For example if f is the mapping of the real plane \mathbb{R}^2 into \mathbb{R} defined by

$$f(x, y) = \begin{cases} \frac{xy}{x^2 + y^2} & (x, y) \neq (0, 0), \\ 0 & (x, y) = (0, 0). \end{cases}$$

Then all the partial mappings are continuous; but f is not continuous at $(0, 0)$.

Proposition 1.4.16. *In a product space $\prod_{i \in I} X_i$ the closure of a product of sets $\prod_{i \in I} A_i$ is the same as the product A_i of their closures.*

Proof. Suppose that $a = (a_i)$ lies in the closure of $\prod_{i \in I} A_i$; then for each $j \in I$, $a_j = \pi_j(a)$ is in the closure of A_j because of the continuity of π_j and therefore $a \in \prod_{i \in I} \bar{A}_i$. Conversely, let $b = (b_i) \in \prod_{i \in I} \bar{A}_i$ and let $\prod_{i \in I} U_i$ be any elementary set containing b ; for each $i \in I$, U_i contains a point $x_i \in A_i$; hence $\prod_{i \in I} U_i$ contains the point $(x_i) \in \prod_{i \in I} A_i$ and therefore b lies in the closure of $\prod_{i \in I} A_i$.

□

Corollary 1.4.17. *A product $\prod_{i \in I} A_i$ of non-empty sets is closed in the product space $\prod_{i \in I}$ if and only if A_i is closed in X_i for each $i \in I$.*

Remark 1.4.1. Note that if I is finite, a product $\prod_{i \in I} A_i$ is open provided that A_i is open in X_i for each $i \in I$; but this is no longer so if I is infinite.

As an application of ??, we give a certain family of dense subsets in a product space.

Proposition 1.4.18. *Let $a = (a_i)$ be any point of a product space $X = \prod_{i \in I} X_i$; then the set D of points $x \in X$ such that $\pi_i(x) = a_i$ except for a finite number of indices i is dense in X .*

Proof. For each $x \in X$ and each elementary set $U = \prod_{i \in I} U_i$ which contains x , we have $U_i = X_i$ except for indices i belonging to a finite subset of J ; if we take $y_i = x_i$ for $i \notin J$ and $y_i = a_i$ for $i \in J$, it is clear that $y = (y_i) \in D$ and $y \in U$; hence the result. \square

1.4.3 Inverse limit of topological spaces

Let I be a partially ordered (but not necessarily directed) set, in which the order relation is written as \preceq . For each $\alpha \in I$, let X_α be a topological space, and for each pair (α, β) such that $\alpha \preceq \beta$ let $f_{\alpha\beta}$ be a mapping of X_β into X_α . We say that $(X_\alpha, f_{\alpha\beta})$ is an inverse system of topological spaces if:

- (a) $(X_\alpha, f_{\alpha\beta})$ is an inverse system of sets;
- (b) Each $f_{\alpha\beta}$ is continuous.

Let X denote the set $\varprojlim X_\alpha$ and for each $\alpha \in I$ let f_α be the canonical mapping $X \rightarrow X_\alpha$; then the initial topology induced by the f_α 's is said to be the inverse limit (with respect to the $f_{\alpha\beta}$) of the topologies of the X_α and the set X with this topology is called the inverse limit of the inverse system of topological spaces $(X_\alpha, f_{\alpha\beta})$. Whenever we speak of $\varprojlim X_\alpha$ as a topological space, it is always to be understood that the topology of this space is the inverse limit of the topologies of the X_α unless the contrary is expressly stated.

The set X is the subset of the product $\prod_\alpha X_\alpha$ consisting of those points x such that

$$\pi_\alpha(x) = f_{\alpha\beta}(\pi_\beta(x)).$$

whenever $\alpha \preceq \beta$. It follows from ?? that the inverse limit of the topologies of the X_α is the same as the topology induced on X by the topology of the product space $\prod_\alpha X_\alpha$. If, for each $\alpha \in I$, A_α is a subspace of X_α such that the $(A_\alpha, f_{\alpha\beta}|_{A_\beta})$ form an inverse system of subsets of the X_α , then it is clear that the topological space $\varprojlim A_\alpha$ is a subspace of $\varprojlim X_\alpha$.

Let $(Y_\alpha, g_{\alpha\beta})$ be another inverse system of topological spaces indexed by the same set I , and for each $\alpha \in I$ let $\varphi_\alpha : X_\alpha \rightarrow Y_\alpha$ be a continuous mapping such that (φ_α) is an inverse system of mappings; then $\varphi = \varprojlim \varphi_\alpha$ is a continuous mapping of $X = \varprojlim X_\alpha$ into $Y = \varprojlim Y_\alpha$. For if g_α is the canonical mapping $Y \rightarrow Y_\alpha$, we have $g_\alpha \circ \varphi = \varphi_\alpha \circ f_\alpha$, so that $g_\alpha \circ \varphi$ is continuous for each $\alpha \in I$, and so φ is continuous.

Finally, suppose I is a directed set, and let J be a cofinal subset of I ; let \tilde{X} be the inverse limit of the inverse system of topological spaces $(X_\alpha, f_{\alpha\beta})_{\alpha, \beta \in J}$. Then the canonical bijection $\psi : X \rightarrow \tilde{X}$ is a homeomorphism. For we have $\pi_\alpha(\psi(x)) = \pi_\alpha(x)$ for each $\alpha \in J$, hence ψ is continuous; and if ϕ is the inverse of ψ , then for each $\alpha \in I$ there exists $\beta \in J$ such that $\alpha \preceq \beta$, and therefore $\pi_\alpha(\phi(z)) = f_{\alpha\beta}(\pi_\beta(z))$, which shows that ϕ is continuous, since the $f_{\alpha\beta}$'s are continuous.

Proposition 1.4.19. Let I be a directed set and J a cofinal subset of I . Let $(X_\alpha, f_{\alpha\beta})$ be an inverse system of topological spaces indexed by I ; let $X = \varprojlim X_\alpha$ and let $f_\alpha : X \rightarrow X_\alpha$ be the canonical mapping. Then the family of sets $f_\alpha^{-1}(U_\alpha)$, where α runs through J and U_α runs through a base \mathcal{B}_α of the topology of X_α for each $\alpha \in J$, is a base of the topology of X .

Proof. We know that the finite intersections of sets of the form $f_\alpha^{-1}(U_\alpha)$, $\alpha \in I$, U_α open in X_α form a base of the topology of X . If $\alpha_1, \dots, \alpha_n$ is a finite family of indices of I , then there exists

$\beta \in J$ such that $\alpha_i \preceq \beta$ for $1 \leq i \leq n$; hence $f_{\alpha_i} = f_{\alpha_i \beta} \circ f_\beta$; if we put $U_\beta = \bigcap_{i=1}^n f_{\alpha_i \beta}^{-1}(U_{\alpha_i})$, then

$$f_\beta^{-1}(U_\beta) = \bigcap_{i=1}^n f_{\alpha_i}^{-1}(U_{\alpha_i}).$$

But U_β is open and is therefore a union of sets belonging to \mathcal{B}_β . Hence the result. \square

Corollary 1.4.20. *Let A be a subset of X and let $A_\alpha = f_\alpha(A)$ for each $\alpha \in I$. Then:*

- (a) *The A_α (resp. the \bar{A}_α) form an inverse system of subsets of the X_α , and $\bar{A} = \bigcap_\alpha f_\alpha^{-1}(\bar{A}_\alpha) = \varprojlim \bar{A}_\alpha$.*
- (b) *If A is closed in X then $A = \varprojlim A_\alpha = \varprojlim \bar{A}_\alpha$.*

Proof. The first assertion of (a) follows from the relations $f_\alpha = f_{\alpha \beta} \circ f_\beta$ for $\alpha \preceq \beta$ and from the continuity of the f_α . Let A' denote $\bigcap_\alpha f_\alpha^{-1}(\bar{A}_\alpha)$, then it is clear that A' is closed and contains A , so that $\bar{A} \subseteq A'$. Conversely, let $x \in A'$; we have to show that x lies in the closure of A . By virtue of ??, it is enough to prove that every neighbourhood of x which is of the form $f_\alpha^{-1}(U_\alpha)$, with $\alpha \in I$ and U_α open in X_α meets A . Now, by hypothesis, $f_\alpha(x) \in U_\alpha$, and since $f_\alpha(x) \in \bar{A}_\alpha$ we have $U_\alpha \cap A_\alpha \neq \emptyset$, which means $A \cap f_\alpha^{-1}(A_\alpha) \neq \emptyset$.

To establish (b) it is enough to remark that, without any restriction on A , we have $A \subseteq \varprojlim A_\alpha \subseteq \varprojlim \bar{A}_\alpha$; now if A is closed, then from (a) we have $A = \varprojlim \bar{A}_\alpha$, so the claim follows. \square

Example 1.4.21. Let I be a directed set and $(X_\alpha)_{\alpha \in I}$ a family of subsets of a set Y , such that $X_\alpha \supseteq X_\beta$ whenever $\alpha \preceq \beta$. For each $\alpha \in I$ let \mathcal{T}_α be a topology on X_α such that \mathcal{T}_α is finer than the topology induced on X_α by \mathcal{T}_β whenever $\alpha \preceq \beta$. If we take $f_{\alpha \beta}$ to be the canonical injection $X_\beta \rightarrow X_\alpha$ for $\alpha \preceq \beta$, then $\varprojlim X_\alpha$ may be identified canonically with the intersection X of the X_α , with the topology which is the least upper bound of the topologies induced on X by the \mathcal{T}_α .

1.5 Connectedness

1.5.1 Connected spaces

The definition of connectedness for a topological space is a quite natural one. One says that a space can be "separated" if it can be broken up into two "globs"—disjoint open sets. Otherwise, one says that it is connected. From this simple idea much follows.

Definition 1.5.1. *Let X be a topological space. A **separation** of X is a pair U, V of disjoint nonempty open subsets of X whose union is X . The space X is said to be **connected** if there does not exist a separation of X .*

Connectedness is obviously a topological property, since it is formulated entirely in terms of the collection of open sets of X . Said differently, if X is connected, so is any space homeomorphic to X .

For a subspace A of a topological space X , there is another useful way of formulating the definition of connectedness:

Lemma 1.5.2. *If A is a subspace of X , a separation of A is a pair of disjoint nonempty sets U and V whose union is A , neither of which contains a limit point of the other. The space A is connected if there exists no separation of A .*

Proof. Suppose first that U and V form a separation of A . Then U is both open and closed in A . The closure of U in A is the set $\bar{U} \cap A$ (where \bar{U} as usual denotes the closure of U in X). Since U is closed in A , $U = \bar{U} \cap A$; or to say the same thing, $\bar{U} \cap V = \emptyset$. Since \bar{U} is the union of U and its limit points, V contains no limit points of U . A similar argument shows that U contains no limit points of V .

Conversely, suppose that U and V are disjoint nonempty sets whose union is A , neither of which contains a limit point of the other. Then $\bar{U} \cap V = \emptyset$ and $\bar{V} \cap U = \emptyset$; therefore, we conclude that $\bar{U} \cap A = U$ and $\bar{V} \cap A = V$. Thus both U and V are closed in A , and since $U \cup V = A$, they are open in A as well. \square

Example 1.5.3. Let X denote a two-point space in the indiscrete topology. Obviously there is no separation of X , so X is connected.

Example 1.5.4. Let A denote the subspace $[-1, 0] \cup (0, 1]$ of the real line \mathbb{R} . Each of the sets $[-1, 0]$ and $(0, 1]$ is nonempty and open in A (although not in \mathbb{R}); therefore, they form a separation of A . Alternatively, note that neither of these sets contains a limit point of the other. (They do have a limit point 0 in common, but that does not matter.)

Example 1.5.5. A subspace of \mathbb{R} is connected if and only if it is an interval: Clearly intervals are connected. Now suppose X is connected in \mathbb{R} and is not an interval. Then there are $x, y \in X$ and $z \notin X$ such that $x < z < y$. We set

$$O_1 = (-\infty, z) \cap X, \quad O_2 = X \cap (z, +\infty)$$

Then O_1 and O_2 is a separation of X , and since $x \in O_1, y \in O_2$ they are both nonempty. This is a contradiction since X is connected, hence X is an interval.

Example 1.5.6. Let X be the subspace $[-1, 1]$ of the real line. The sets $[-1, 0]$ and $(0, 1]$ are disjoint and nonempty, but they do not form a separation of X , because the first set is not open in X . Alternatively, note that the first set contains a limit point, 0, of the second. Indeed, there exists no separation of the space $[-1, 1]$.

Example 1.5.7. The rationals \mathbb{Q} are not connected. Indeed, the only connected subspaces of \mathbb{Q} are the one-point sets: If A is a subspace of \mathbb{Q} containing two points p and q , one can choose an irrational number α lying between p and q , and write A as the union of the open sets

$$A = (A \cap (-\infty, \alpha)) \cup (A \cap (\alpha, +\infty)).$$

Example 1.5.8. Consider the following subset of the plane \mathbb{R}^2 :

$$X = \{(x, y) \in \mathbb{R}^2 : y = 0\} \cup \{(x, y) : x > 0 \text{ and } xy = 1\}.$$

Then X is not connected; indeed, the two indicated sets form a separation of X because neither contains a limit point of the other.

We have given several examples of spaces that are not connected. How can one construct spaces that are connected? We shall now prove some useful properties of connectedness, and later use them to construct connected topological spaces. First, we give another criterion for connectedness which is quite simple but useful.

Proposition 1.5.9. *A topological space X is connected if and only if every continuous map $f : X \rightarrow \{0, 1\}$ is constant, with the space $\{0, 1\}$ given the discrete topology.*

Proof. Clearly if f is not constant then $f^{-1}(0) \cup f^{-1}(1)$ is a separation of X . Conversely, if U and V form a separation of X then the characteristic function χ_U is nonconstant and continuous. \square

Proposition 1.5.10 (Properties of Connected Spaces). *Let X be a topological space.*

- (a) *Suppose that U, V are disjoint open subsets of X . If A is a connected subset of X contained in $U \cup V$, then either $A \subseteq U$ or $A \subseteq V$.*
- (b) *Suppose X is any space and $A \subseteq X$ is connected. If B is a subset such that $A \subseteq B \subseteq \bar{A}$, then it is connected. In particular, \bar{A} is connected.*
- (c) *Let $\{C_\alpha\}_{\alpha \in I}$ be a collection of connected subspaces of X with a point in common. Then $\bigcup_{\alpha \in I} C_\alpha$ is connected.*
- (d) *Let $\{C_\alpha\}_{\alpha \in I}$ be a collection of connected subspaces of X with $C_\alpha \cap C_\beta \neq \emptyset$ for all $\alpha, \beta \in I$. Then $\bigcup_{\alpha \in I} C_\alpha$ is connected.*

Proof. For (a), if A contained points in both U and V , then $A \cap U$ and $A \cap V$ would form a separation of A . To prove (b), suppose A is connected and $A \subseteq B \subseteq \bar{A}$. Suppose that $B = U \cup V$ is a separation of B . By part (a), the set A must lie entirely in U or in V ; say $A \subseteq U$. Then $\bar{A} \subseteq \bar{U}$; since \bar{U} and V are disjoint, B cannot intersect V . This contradicts the fact that V is a nonempty subset of B .

For part (c), let p be a point that is contained in C_α for every α , and let $f : \bigcup_\alpha C_\alpha \rightarrow \{0, 1\}$ be a continuous function. Then f is constant when restricted to each C_α , since C_α is connected. Since C_α contains p for each α , it then follows that f is constant. This shows $\bigcup_\alpha C_\alpha$ is connected. Part (d) can be proved similarly. \square

Corollary 1.5.11. *Let X be a topological space and $(A_n)_{n \in \mathbb{N}}$ be an infinite sequence of connected subspaces such that $A_n \cap A_{n+1} \neq \emptyset$ for all $n \geq 0$. Then the union $\bigcup_n A_n$ is connected.*

Proof. By induction on n we see immediately that the set $B_n = \bigcup_{i=1}^n A_i$ is connected for all n , by ???. The sets B_n have a non-empty intersection; hence their union, equal to $\bigcup_n B_n$, is connected by ???. \square

Proposition 1.5.12. *The image of a connected space under a continuous map is connected.*

Proof. Let $f : X \rightarrow Y$ be a continuous map; let X be connected. We wish to prove the image space $f(X)$ is connected. Since the map obtained from f by restricting its range to the space is also continuous, it suffices to consider the case that f is surjective. Suppose that $Y = U \cup V$ is a separation of Y into two disjoint nonempty sets open in Y . Then $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint sets whose union is X ; they are open in X because f is continuous, and nonempty because f is surjective. Therefore, they form a separation of X , contradicting the assumption that X is connected. \square

Corollary 1.5.13. *Every quotient space of a connected space is connected.*

As for ??, we have the following converse.

Proposition 1.5.14. *Let X be a topological space and R an equivalence relation on X . If the quotient space X/R is connected, and if each equivalence class of R is connected, then X is connected.*

Proof. Suppose X is not connected. Then there is a partition of X into two open sets U, V . The sets U, V are saturated with respect to R in view of ??(a). The canonical images of U and V are therefore open sets in X/R and form a partition of X/R ; this contradicts the assumption that X/R is connected. \square

Proposition 1.5.15. *Every product of connected spaces is connected. Conversely, if a product of non-empty spaces is connected, then each of the factors is connected.*

Proof. Let $(X_i)_{i \in I}$ be a family of topological spaces and $X = \prod_{i \in I} X_i$. Since the projection $\pi_i : X \rightarrow X_i$ is continuous and surjective, if X is connected then so is each X_i . Conversely, assume that each X_i is connected and let $f : X \rightarrow \{0, 1\}$ be a continuous map, where $\{0, 1\}$ is given the discrete topology. Let $a = (a_i)$ be any point of X , and for each $j \in I$ let $f_j : X_j \rightarrow \{0, 1\}$ be the partial mapping defined by $f_j(x_j) = f((y_i))$, where $y_j = x_j$ and $y_i = a_i$ for $i \neq j$. Then f_j is continuous, so is constant since X_j is connected. It follows immediately by induction that $f(x) = f(a)$ for all points $x = (x_i)$ such that $x_i = a_i$ for all but finitely many indices of I . But these points form a dense subset of X by ???. Hence f is continuous on X and constant on a dense subset of X , and therefore constant on X . But this contradicts the definition of f . \square

Now that, however, the result of ?? depends on the topology we endow with the product space $\prod_{i \in I} X_i$. If we consider the box topology, it turns out that $\prod_{i \in I} X_i$ may be not connected even though each X_i is.

Example 1.5.16. Consider the cartesian product \mathbb{R}^ω in the box topology. We can write \mathbb{R}^ω as the union of the set A consisting of all bounded sequences of real numbers, and the set B of all unbounded sequences. These sets are disjoint, and each is open in the box topology. For if $x = (x_n)$ is a point of \mathbb{R}^ω , the open set

$$U = \bigcup_{n \in \mathbb{Z}_+} (x_n - 1/n, x_n + 1/n)$$

consists entirely of bounded sequences if x is bounded, and of unbounded sequences if x is unbounded. Thus, even though \mathbb{R} is connected, \mathbb{R}^ω is not connected in the box topology.

Example 1.5.17. The set $O(n)$ of all real orthogonal $n \times n$ matrices is not connected. The function $O(n) \rightarrow \{1, -1\}, A \mapsto \det A$ is continuous and surjective, since we have the formula

$$\det A = \sum_{\sigma \in \mathfrak{S}_n} (\text{sign } \sigma) a_{1\sigma(1)} \cdots a_{n\sigma(n)}$$

Since $\{1, -1\}$ has discrete topology, it follows that $O(n)$ is not connected.

1.5.2 Path connectedness

The theorems of the preceding part show us how to construct new connected spaces out of given ones. But where can we find some connected spaces to start with? The best place to begin is the real line. We shall prove that \mathbb{R} is connected, and so are the intervals and rays in \mathbb{R} . The fact that intervals and rays in \mathbb{R} are connected may be familiar to you from analysis. We prove it again here, in generalized form. It turns out that this fact does not depend on the algebraic properties of \mathbb{R} , but only on its order properties. To make this clear, we shall prove the theorem for an arbitrary ordered set that has the order properties of \mathbb{R} . Such a set is called a **linear continuum**.

Definition 1.5.18. A totally ordered set L having more than one element is called a **linear continuum** if the following hold:

- (a) L has the least upper bound property.
- (b) If $x < y$, there exists z such that $x < z < y$.

Theorem 1.5.19. If L is a linear continuum in the order topology, then L is connected, and so are intervals and rays in L .

Proof. Recall that a subspace A of L is said to be convex if for every pair of points a, b of A with $a < b$, the entire interval $[a, b]$ of points of L lies in A . We prove that if A is a convex subspace of L , then A is connected.

So suppose that A is the union of the disjoint nonempty sets U and V , each of which is open in A . Choose $a \in U$ and $b \in V$; suppose for convenience that $a < b$. The interval $[a, b]$ of points of L is contained in A . Hence $[a, b]$ is the union of the disjoint sets

$$U_0 = U \cap [a, b], \quad V_0 = V \cap [a, b]$$

each of which is open in $[a, b]$ in the subspace topology, which is the same as the order topology. The sets U_0 and V_0 are nonempty because $a \in U_0$ and $b \in V_0$. Thus, U_0 and V_0 constitute a separation of $[a, b]$. Let $c = \sup U_0$. We show that c belongs neither to U_0 nor to V_0 , which contradicts the fact that $[a, b]$ is the union of U_0 and V_0 .

Suppose that $c \in A_0$. Then $c \neq b$, so either $c = a$ or $a < c < b$. But U_0 is open in $[a, b]$, so there must be some interval of the form $[c, e)$ contained in U_0 . Because of order property (b) of the linear continuum L , we can choose a point z of L such that $c < z < e$. Then $z \in U_0$, contrary to the fact that c is an upper bound for U_0 .

Suppose that $c \in B_0$. Then $c \neq a$, so either $c = b$ or $a < c < b$. In either case, it follows from the fact that V_0 is open in $[a, b]$ that there is some interval of the form $(d, c]$ contained in V_0 . Since $(c, b]$ does not intersect U_0 , we see that $(d, b] = (d, c] \cup (c, b]$ does not intersect U_0 . Again, d is a smaller upper bound on U_0 than c , contrary to construction. \square

Corollary 1.5.20. The real line \mathbb{R} is connected and so are intervals and rays in \mathbb{R} .

As an application, we prove the intermediate value theorem of calculus, suitably generalized.

Theorem 1.5.21 (Intermediate value theorem). Let $f : X \rightarrow Y$ be a continuous map, where X is a connected space and Y is an ordered set in the order topology. If a and b are two points of X and if r is a point of Y lying between $f(a)$ and $f(b)$, then there exists a point c of X such that $f(c) = r$.

Proof. Assume the hypotheses of the theorem. The sets

$$U = f(X) \cap (-\infty, r) \quad \text{and} \quad V = f(X) \cap (r, +\infty)$$

are disjoint, and they are nonempty because one contains $f(a)$ and the other contains $f(b)$. Each is open in $f(X)$, being the intersection of an open ray in Y with $f(X)$. If there were no point c of X such that $f(c) = r$, then $f(X)$ would be the union of the sets U and V . Then U and V would constitute a separation of $f(X)$, contradicting the fact that the image of a connected space under a continuous map is connected. \square

Example 1.5.22. One example of a linear continuum different from \mathbb{R} is the ordered square. We check the least upper bound property. (The second property of a linear continuum is trivial to check.)

Let A be a subset of $I \times I$; let $\pi_1 : I \times I \rightarrow I$ be projection on the first coordinate; let $b = \sup \pi_1(A)$. If $b \in \pi_1(A)$, then A intersects the subset $\{b\} \times I$ of $I \times I$. Because $\{b\} \times I$ has the order type of I , the set $A \cap (\{b\} \times I)$ will have a least upper bound (b, c) , which will be the least upper bound of A .

If $b \notin \pi_1(A)$, then $(b, 0)$ is the least upper bound of A ; no element of the form (b', c) with $b' < b$ can be an upper bound for A , for then b' would be an upper bound for $\pi_1(A)$.

Example 1.5.23. If X is a well-ordered set, then $X \times [0, 1)$ is a linear continuum in the dictionary order. This set can be thought of as having been constructed by "fitting in" a set of the order type of $(0, 1)$ immediately following each element of X .

Connectedness of intervals in \mathbb{R} gives rise to an especially useful criterion for showing that a space X is connected; namely, the condition that every pair of points of X can be joined by a path in X :

Definition 1.5.24. Given points x and y of the space X , a **path** in X from x to y is a continuous map $p : [0, 1] \rightarrow X$ of the unit interval in the real line into X , such that $p(0) = x$ and $p(1) = y$. A space X is said to be **path-connected** if every pair of points of X can be joined by a path in X .

It is easy to see that a path-connected space X is connected. Suppose $X = U \cup V$ is a separation of X . Let $p : [0, 1] \rightarrow X$ be any path in X . Being the continuous image of a connected set, the set $p([0, 1])$ is connected, so that it lies entirely in either U or V . Therefore, there is no path in X joining a point of U to a point of V , contrary to the assumption that X is path-connected. The converse does not hold: a connected space need not be path-connected.

Example 1.5.25. Define punctured Euclidean space to be the space $\mathbb{R}^n \setminus \{0\}$, where 0 is the origin in \mathbb{R}^n . If $n > 1$, this space is path-connected: Given x and y different from 0 , we can join x and y by the straight-line path between them if that path does not go through the origin. Otherwise, we can choose a point z not on the line joining x and y , and take the broken-line path from x to z , and then from z to y .

Example 1.5.26. Being a linear continuum, the ordered square is connected. Let $x = (0, 0)$ and $y = (1, 1)$. We suppose there is a path $p : [0, 1] \rightarrow I_o^2$ joining x and y and derive a contradiction. The image set $p([0, 1])$ must contain every point (x, y) of I_o^2 , by the intermediate value theorem. Therefore, for each $x \in I$, the set

$$U_x = p^{-1}(\{x\} \times (0, 1))$$

is a nonempty subset of $[0, 1]$; by continuity, it is open in $[0, 1]$. Choose, for each $x \in I$, a rational number r_x belonging to U_x . Since the sets U_x are disjoint, the map $x \mapsto r_x$ is an injective mapping of I into \mathbb{Q} . This contradicts the fact that the interval I is uncountable.

Example 1.5.27. Let S denote the following subset of the plane

$$S = \{(x, y) : 0 < x \leq 1 \text{ and } y = \sin(1/x)\}.$$

Because S is the image of the connected set $(0, 1]$ under the continuous map $x \mapsto \sin(1/x)$, it is connected. Therefore, its closure \bar{S} in \mathbb{R}^2 is also connected. The set \bar{S} is a classical example in topology called the **topologist's sine curve**. It equals the union of S and the vertical interval $\{0\} \times [-1, 1]$. We show that \bar{S} is not path-connected.

Suppose there is a path $p : [a, b] \rightarrow \bar{S}$ beginning at the origin and ending at a point of S . The set of those t for which $p(t) \in \{0\} \times [-1, 1]$ is closed, so it has a largest element c . Then $p : [c, b] \rightarrow \bar{S}$ is a path that maps c into the vertical interval $\{0\} \times [-1, 1]$ and maps the other points of $[c, b]$ to points of S . Without loss of generality, we replace $[c, b]$ with $[0, 1]$.

Let $p(t) = (x(t), y(t))$. Then $x(0) = 0$, while $x(t) > 0$ and $y(t) = \sin(1/x(t))$ for $t > 0$. We show there is a sequence of points $t_n \rightarrow 0$ such that $y(t_n) = (-1)^n$. Then the sequence $y(t_n)$ does not converge, contradicting continuity of p . To find t_n , we proceed as follows: Given n , choose u with $0 < u < x(1/n)$ such that $\sin(1/u) = (-1)^n$. Then use the intermediate value theorem to find t_n with $0 < t_n < 1/n$ such that $x(t_n) = u$.

Proposition 1.5.28. Let X be a topological space. Let $\{C_\alpha\}_{\alpha \in I}$ be a collection of path-connected subspaces of X with a point in common. Then $\bigcup_\alpha C_\alpha$ is path-connected.

Proof. Let $p \in \bigcap_\alpha C_\alpha$, then for each $x \in \bigcup_\alpha C_\alpha$, there is a path from x to p . This shows $\bigcup_\alpha C_\alpha$ is path-connected. \square

Proposition 1.5.29. The image of a path-connected space under a continuous map is path-connected.

Proof. Let $f : X \rightarrow Y$ be a continuous map and X be path-connected. Let $a, b \in f(X)$, and choose $x, y \in X$ such that $f(x) = a, f(y) = b$. Then if $p : [0, 1] \rightarrow X$ be a path from x to y , then it follows that $f \circ p$ is a path from a to b . Therefore $f(X)$ is path-connected. \square

Proposition 1.5.30. Every product of path-connected spaces is path-connected. Conversely, if a product of non-empty spaces is path-connected, then each of the factors is path-connected.

Proof. Let $(X_i)_{i \in I}$ be a family of topological spaces and $X = \prod_{i \in I} X_i$. If X is path-connected, then since $X_i = \pi_i(X)$, we have X_i is path-connected for each $i \in I$. Conversely, assume that each X_i path-connected. Let $x = (x_i)_{i \in I}, y = (y_i)_{i \in I}$ be two points in X . By assumption, there exist continuous paths $p_i : [0, 1] \rightarrow X_i$ with $\gamma_i(0) = x_i$ and $\gamma_i(1) = y_i$. By definition of product, there exists a unique continuous $p : [0, 1] \rightarrow X$ such that $\pi_i \circ p = \gamma_i$ for all $i \in I$. That makes p a path from x to y , so X is path-connected. \square

Example 1.5.31. Let Ω be the first uncountable ordinal, and we use S_Ω to denote the set Ω together with the order topology. As a set, S_Ω consists of all countable ordinals. Let L denote the ordered set $S_\Omega \times [0, 1)$ in the dictionary order, with its smallest element deleted. The set L is a classical example in topology called the **long line**. We now show that L is path-connected.

Pick any two points $x = (\alpha, s)$ and $y = (\beta, t)$ on the long line, with $x < y$. If $\alpha = \beta$, then $s < t$ and the long line interval $[x, y]$ is readily homeomorphic to the real interval $[s, t]$, so x, y are connected by a path. Otherwise, since α and β are countable ordinals, the long line interval $[x, y]$ is the union of (in increasing order) $\alpha \times [s, 1)$, then (at most) countably many intervals of the form $\gamma \times [0, 1)$ with $\alpha < \gamma < \beta$, then $\beta \times [0, t]$, joined end to end. Again, this set is homeomorphic to a closed real interval, so that x, y are connected by a path.

1.5.3 Components and locally connectedness

Given a point x of a topological space X , the union of the connected subsets of X which contain x is connected; it is therefore the largest connected subset of X which contains x .

Definition 1.5.32. *The component (or connected component) of a point of a topological space X is the largest connected subset of X which contains this point. The components of a subset A of X are the components of the points of A , relative to the subspace A of X .*

If a space is connected, the component of each point is the whole space. A space X is said to be **totally disconnected** if the component of each point of X consists of the point alone. A subset A of X is totally disconnected if the subspace A of X is totally disconnected.

Proposition 1.5.33. *The component of any point in a topological space X is a closed set. The relation "y belongs to the component of x" is an equivalence relation R on X , and the equivalence classes are the components of X . The quotient space X/R is totally disconnected.*

Proof. If C is any component of X , it follows that \bar{C} is a connected set containing C . Since components are maximal connected sets, $C = \bar{C}$, so C is closed.

Since the union of connected sets which have a point in common is connected, the relation R is transitive, hence is an equivalence relation (since it is obviously reflexive and symmetric) and the equivalence class of x with respect to R is the component of x . It remains to show that X/R is totally disconnected.

Let $\pi : X \rightarrow X/R$ be the canonical mapping, and let F be a closed set in X/R containing at least two distinct points; the inverse image $\pi^{-1}(F)$ of F is closed in X , saturated with respect to R , and contains at least two distinct components of X and hence is not connected. Hence there exist two non-empty disjoint closed sets B, C in X such that $B \cup C = \pi^{-1}(F)$. The component of any point x of $\pi^{-1}(F)$ in $\pi(F)$ is the same as the component of x in X (by the definition of R) and therefore B and C , which are both open and closed in $\pi^{-1}(F)$, are saturated with respect to R . Hence $\pi(B)$ and $\pi(C)$ are closed in X/R , and $\pi(B) \cup \pi(C) = F$ and $\pi(B) \cap \pi(C) = \emptyset$; this shows that F is not connected and consequently that X/R is totally disconnected. \square

Proposition 1.5.34. *In a product space $X = \prod_{i \in I} X_i$ the component of $x = (x_i)$ in X is the product of the components of x_i in the factors X_i*

Proof. This product set is connected by ???. Conversely, if C is a connected subset of X which contains x , then $\pi_i(C)$ is a connected set which contains x_i ; since $C \subseteq \prod_i \pi_i(A)$, it follows that C is contained in the product of the components of the x_i 's. \square

Although components are always closed, they may not be open in general, so they do not necessarily disconnect the space. Consider the set \mathbb{Q}^2 of rational points in the plane, for

example: its components are single points, which are not open subsets. There is a condition, however, which ensure that all components are open.

Definition 1.5.35. A topological space X is said to be **locally connected** if each point of X has a fundamental system of connected neighbourhoods.

The existence, at each point x of a space X , of one connected neighbourhood of x by no means implies that X is locally connected. In particular, X can be connected but not locally connected. Conversely, a space can be locally connected but not connected (e.g. a discrete space which contains more than one point).

Example 1.5.36. The topologist's sine curve \bar{S} is connected but not locally connected. In fact, any neighbourhood of points in $\{0\} \times [-1, 1]$ should intersect the set S , so is not connected.

Example 1.5.37. Consider \mathbb{R}^2 with its standard topology and let K be the set $\{1/n : n \in \mathbb{Z}_+\}$. The set C defined by:

$$C = (\{0\} \times [0, 1]) \cup (K \times [0, 1]) \cup ([0, 1] \times \{0\})$$

considered as a subspace of \mathbb{R}^2 equipped with the subspace topology is known as the comb space. The space C is not locally connected, as every neighborhood of points in $\{0\} \times [0, 1]$ is not connected. But it is easily seen to be path-connected.

Proposition 1.5.38. A necessary and sufficient condition for a space X to be locally connected is that every component of an open set in X is open in X .

Proof. Suppose that X is locally connected; let U be an open set in X ; let C be a component of U . If x is a point of C , we can choose a connected neighborhood V of x such that $V \subseteq U$. Since V is connected, it must lie entirely in the component C of U . Therefore, C is open in X .

Conversely, suppose that components of open sets in X are open. Given a point x of X and a neighborhood U of x , let C be the component of U containing x . Now C is connected; since it is open in X by hypothesis, X is locally connected at x . \square

The components of a locally connected space X therefore form a partition of X into open sets, and hence X is the coproduct of its components.

Corollary 1.5.39. Let U be an open subset of a locally connected space X , and let V be a component of U . Then the boundary of V (relative to X) is contained in the boundary of U .

Proof. Since V is open and closed in U , a boundary point of V (relative to X) cannot belong to U , for it would also be a bounadry point of V relative to U , and there is none. \square

Proposition 1.5.40. Every quotient space of a locally connected space is locally connected.

Proof. Let X be a locally connected space, R an equivalence relation on X and $\pi : X \rightarrow X/R$ the canonical mapping of R . Let U be an open subset of X/R and C a component of U . Then $\pi^{-1}(C)$ is a union of components of $\pi^{-1}(U)$; for if $x \in \pi^{-1}(C)$ and if K is the component of x in $\pi^{-1}(U)$, then $\pi(K)$ is connected, is contained in U , and contains $\pi(x)$; hence $\pi(K) \subseteq C$ by the definition of C , and therefore $K \subseteq \pi^{-1}(C)$. Since X is locally connected and $\pi^{-1}(U)$ is open in X , it follows from ?? that $\pi^{-1}(C)$ is open in X ; consequently C is open in X/R and hence, by ?? again, X/R is locally connected. \square

Proposition 1.5.41. Let $(X_i)_{i \in I}$ be a family of topological spaces and $X = \prod_{i \in I} X_i$ be the product space.

- (a) If X_i is connected for all but a finite number of indices $i \in I$. Then the product space X is locally connected.
- (b) Conversely, if the product space X is locally connected, then each X_i is locally connected, and X_i is connected for all but a finite number of indices.

Proof. First assume (a), and let J be the finite subset of I such that X_i is not connected if and only if $i \in J$. Let $U = \prod_{i \in I} U_i$ be an elementary set containing a point $x = (x_i)$ of X and let K be the finite subset of I such that $U_i \neq X_i$ if and only if $i \in K$. Let $V_i = X_i$ for $i \in J \cup K$, and let V_i be a connected neighbourhood of x_i contained in V_i for $i \notin J \cup K$; then $V = \prod_{i \in I} V_i$ is connected (by ??) and is a neighbourhood of x contained in U . Hence X is locally connected.

Conversely, assume that X is locally connected. Let $a = (a_i)$ be a point of X and let V be a connected neighbourhood of a in X . Since we have $\pi_i(V) = X_i$ except for a finite number of indices, it follows that the X_i are connected, for all but a finite number of indices. On the other hand, for each $j \in I$, each $a_j \in X_j$ and each neighbourhood V_j of a_j in X_j , there is a point x of X such that $\pi_j(x) = a_j$, and

$$V = V_j \times \prod_{i \neq j} X_i$$

is a neighbourhood of x in X ; V therefore contains a connected neighbourhood W of x , whose projection $\pi_j(W)$ is a connected neighbourhood of a_j contained in V_j (the projection is open). Hence each X_j is locally connected. \square

1.5.4 Path components

We can also define an analogue of components with path-connectedness in place of connectedness. If X is any space, define a **path component** of X to be a maximal nonempty path-connected subset. The following properties of path components are immediate.

Proposition 1.5.42 (Properties of Path Components). Let X be any space.

- (a) The path components of X form a partition of X .
- (b) Each path component is contained in a single component, and each component is a disjoint union of path components.
- (c) Any nonempty path-connected subset of X is contained in a single path component.

Example 1.5.43. The topologist's sine curve \bar{S} is a space that has a single component (since it is connected) and two path components. One path component is the curve S and the other is the vertical interval $V = \{0\} \times [-1, 1]$. Note that S is open in \bar{S} but not closed, while V is closed but not open. If one forms a space from \bar{S} by deleting all points of V having rational second coordinate, one obtains a space that has only one component but uncountably many path components.

Similarly, we say that a space X is **locally path-connected** if each point of X has a fundamental system of path-connected neighbourhoods. Since path-connectedness implies connectedness, every locally path-connected space is locally connected. We have the following analogue of ??.

Proposition 1.5.44. *A space X is locally path connected if and only if for every open set U of X , each path component of U is open in X .*

Proof. The proof is identically the same as that of ??, replacing "connected" by "path-connected." \square

Proposition 1.5.45 (Properties of Locally Path-Connected Spaces). *Suppose X is a locally path-connected space, then the path components of X are equal to its components. In particular, X is connected if and only if it is path-connected.*

Proof. Let $x \in X$, and let C and P be the component and the path component containing x , respectively. By definition, we know that $P \subseteq C$ and C can be written as a disjoint union of path components, each of which is open in X . Suppose that $P \subsetneq C$, let Q denote the union of all the path components of X that are different from P and intersect C ; each of them necessarily lies in C , so that $C = P \cup Q$. Because X is locally path connected, each path component of X is open in X . Therefore, P (which is a path component) and Q (which is a union of path components) are open in X , so they constitute a separation of C . This contradicts the fact that C is connected. \square

1.6 Countability and separation axioms

There are actually several countability properties that are useful. We begin with the weakest one. Let X be a topological space, we say that X is **first countable** if there exists a countable neighborhood base at each point. A topological space is said to be **second countable** if it admits a countable base for its topology. A topological space X is said to be **separable** if it contains a countable dense subset, and to be a **Lindelöf** space if every open cover of X has a countable subcover.

Theorem 1.6.1 (Properties of Second Countable Spaces). *Suppose that X is a second countable space. Then:*

- (a) X is first countable.
- (b) X is Lindelöf.
- (c) X is separable.

Proof. Let \mathcal{B} be a countable base for X . To prove (a), just note that for any $p \in X$, the elements of \mathcal{B} that contain p form a countable neighborhood base at p .

Let $\{B_n\}$ be a countable base for X . For (b), let \mathcal{U} be an open over of X and set

$$J := \{n \in \mathbb{N} : \exists U \in \mathcal{U}, B_n \subseteq U\}.$$

Let $\mathcal{B}' = \{B_n\}_{n \in J}$. The collection \mathcal{B}' is countable, since it is indexed with a subset J of the positive integers. Furthermore, it covers X : Given a point $x \in X$, we can choose an element U of \mathcal{U} containing x . Since U is open, there is a base element B_n such that $x \in B_n \subseteq U$. Because B_n lies in an element of \mathcal{U} , the index n belongs to the set J , so $B_n \in \mathcal{B}'$. Thus \mathcal{B}' is a countable refinement of \mathcal{U} that covers X , and hence \mathcal{U} has a countable subcollection that covers X .

Finally, for (c), from each nonempty base element B_n , choose a point x_n . Let D be the set consisting of the points x_n . Then D is dense in X : Given any point x of X , every base element containing x intersects D , so x belongs to \overline{D} . \square

The above theorem has a partial converse when X is a metric space.

Theorem 1.6.2. *Let X be a metrizable space. Then the following are equivalent:*

- (a) X is second countable.
- (b) X is separable.
- (c) X is Lindelöf.

Proof. The implication (a) \Rightarrow (b) and (a) \Rightarrow (c) are done in ???. For (b) \Rightarrow (a), let B be a countable dense subset of X . Then the collection

$$\mathcal{B} = \{B_{1/n}(x) : x \in X, n \in \mathbb{N}\}$$

is a countable base for X . Finally, if X is Lindelöf, for each fixed $n \in \mathbb{N}$ consider the collection

$$\mathcal{U}_n = \{B_{1/n}(x) : x \in X\}$$

This is a cover of X , hence has a countable subcover \mathcal{U}'_n . Now vary n , and let $\mathcal{U} = \bigcup_{n=1}^{\infty} \mathcal{U}'_n$. Then \mathcal{U} is a base of X , and is countable. \square

1.7 Nets and filters

1.7.1 Nets

A **directed set** is a set A equipped with a binary relation \preceq such that

- $\alpha \preceq \alpha$ for all $\alpha \in A$;
- if $\alpha \preceq \beta$ and $\beta \preceq \gamma$, then $\alpha \preceq \gamma$;
- for any $\alpha, \beta \in A$ there exists $\gamma \in A$ such that $\alpha \preceq \gamma$ and $\beta \preceq \gamma$.

A **net** in a set X is a mapping $x : \alpha \mapsto x_\alpha$ from a directed set A into X . We shall usually denote such a mapping by $(x_\alpha)_{\alpha \in A}$, or just by (x_α) if A is understood, and we say that (x_α) is indexed by A .

Let X be a topological space and E a subset of X . A net $(x_\alpha)_{\alpha \in A}$ is **eventually in E** if there exists $\alpha_0 \in A$ such that $x_\alpha \in E$ for $\alpha \succeq \alpha_0$, and (x_α) is **frequently in E** if for every $\alpha \in A$ there exists $\beta \succeq \alpha$ such that $x_\beta \in E$. A point $x \in X$ is a **limit point** of (x_α) (or (x_α) converges to x) if for every neighborhood U of x , (x_α) is *eventually* in U , and x is a **cluster point** of (x_α) if for every neighborhood U of x , (x_α) is *frequently* in U .

The next propositions show that nets are a good substitute for sequences.

Proposition 1.7.1. *A topological space X is Hausdorff if and only if the limit of every convergent net is unique.*

Proof. If X is Hausdorff and (x_α) is a net converging to $x \in X$. Then for any $y \neq x$, there exists disjoint neighborhoods U, V such that $x \in U, y \in V$. If x_α also converges to y , then there exists β and β' such that $x_\alpha \in U$ for $\alpha \succeq \beta$ and $x_\alpha \in V$ for $\alpha \succeq \beta'$. Choose $\gamma \succeq \beta$ and $\gamma \succeq \beta'$, then $x_\gamma \in U \cap V = \emptyset$, which is a contradiction.

Conversely, if X is not Hausdorff, then there exists $x, y \in X, x \neq y$ such that $U \cap V \neq \emptyset$ for $U \in \mathfrak{U}(x)$ and $V \in \mathfrak{U}(y)$. Choose $x_{U,V} \in U \cap V$ for each $U \in \mathfrak{U}(x)$ and $V \in \mathfrak{U}(y)$, we then get a net $(x_{U,V})$ in X . It is clear that $(x_{U,V})$ converges to both x and y , thus the claim follows. \square

Proposition 1.7.2. *If X is a topological space, $E \subseteq X$, and $x \in X$, then x is a limit point of E if and only if there is a net in $E \setminus \{x\}$ that converges to x , and $x \in \bar{E}$ if and only if there is a net in E that converges to x .*

Proof. If x is a limit point of E , let \mathcal{N} be the set of neighborhoods of x , directed by reverse inclusion. For each $U \in \mathcal{N}$, pick $x \in (U \setminus \{x\}) \cap E$. Then x_U converges to x . Conversely, if $x_\alpha \in E \setminus \{x\}$ and $x_\alpha \rightarrow x$, then every punctured neighborhood of x contains some x_α , so x is an accumulation point of E . The second claim is then clear. \square

Proposition 1.7.3. *If X and Y are topological spaces and $f : X \rightarrow Y$, then f is continuous at $x \in X$ if and only if for every net (x_α) converging to x , $(f(x_\alpha))$ converges to $f(x)$.*

Proof. If f is continuous at x and V is a neighborhood of $f(x)$, then $f^{-1}(V)$ is a neighborhood of x . Hence, if $x_\alpha \rightarrow x$, then (x_α) is eventually in $f^{-1}(V)$, so $(f(x_\alpha))$ is eventually in V , and thus $f(x_\alpha) \rightarrow f(x)$. On the other hand, if f is not continuous at x , there is a neighborhood V of $f(x)$ such that $f^{-1}(V)$ is not a neighborhood of x , that is, $x \notin \text{Int } f^{-1}(V)$, or equivalently, $x \in \overline{f^{-1}(V^c)}$. By ??, there is a net (x_α) in $f^{-1}(V^c)$ that converges to x . But then $f(x_\alpha) \notin V$, so $f(x_\alpha)$ does not converge to $f(x)$. \square

Proposition 1.7.4. *A net (x_α) in a product $X = \prod_s X_s$ converges to x if and only if for each s , $\pi_s(x_\alpha) \rightarrow \pi_s(x)$ in X_s .*

Proof. If $x_\alpha \rightarrow x$, then since each π_s is continuous, $\pi_s(x_\alpha) \rightarrow \pi_s(x)$ by the previous theorem, for each s .

Suppose on the other hand that $\pi_s(x_\alpha) \rightarrow \pi_s(x)$ for each s . Let $U = \bigcap_{i=1}^n \pi_{s_i}^{-1}(U_{s_i})$ be a basic neighborhod of x in the product space. Then for each $i = 1, \dots, n$ there is a α_i such that whenever $\pi_{s_i}(x_\alpha) \in U_{s_i}$ for $\alpha \succeq \alpha_i$. Thus if β is picked greater than all of $\alpha_1, \dots, \alpha_n$, we have $\pi_{s_i}(x_\alpha) \in U$ for all $\alpha \succeq \beta$. It follows that $x_\alpha \rightarrow x$ in the product space. \square

A **subnet** of a net $(x_\alpha)_{\alpha \in A}$ is a net $(x_{\alpha_\beta})_{\beta \in B}$ together with a map $\beta \mapsto \alpha_\beta$ from B to A such that for every $\alpha_0 \in A$ there exists $\beta_0 \in B$ such that $\alpha_\beta \succeq \alpha_0$ whenever $\beta \succeq \beta_0$. Clearly if (x_α) converges to a point x , then so does any subnet (x_{α_β}) .

Proposition 1.7.5. *If $(x_\alpha)_{\alpha \in A}$ is a net in a topological space X , then $x \in X$ is a cluster point of (x_α) if and only if (x_α) has a subnet that converges to x .*

Proof. If (x_{α_β}) is a subnet converging to x and U is a neighborhood of x , choose $\beta_1 \in B$ such that $x_{\alpha_\beta} \in U$ whenever $\beta \succeq \beta_1$. Also, given $\alpha_0 \in A$, choose $\beta_2 \in B$ such that $\alpha_\beta \succeq \alpha_0$ for $\beta \succeq \beta_2$. Then there exists $\beta \in B$ with $\beta \succeq \beta_1$ and $\beta \succeq \beta_2$, and we have $\alpha_\beta \succeq \alpha_0$ and $x_{\alpha_\beta} \in U$. Thus (x_α) is frequently in U , so x is a cluster point of (x_α) .

Conversely, if x is a cluster point of (x_α) , let \mathcal{N} be the set of neighborhoods of x and make $\mathcal{N} \times A$ into a directed set by declaring that $(U, \alpha) \preceq (U', \alpha')$ iff $U \supseteq U'$ and $\alpha \preceq \alpha'$. For each $(U, \alpha) \in \mathcal{N} \times A$ we can choose $\alpha_{(U, \gamma)} \in A$ such that $\alpha_{(U, \gamma)} \succeq \gamma$ and $x_{\alpha_{(U, \gamma)}} \in U$. Then if $(U', \gamma') \succeq (U, \gamma)$ we have $\alpha_{(U', \gamma')} \succeq \gamma' \succeq \gamma$ and $x_{\alpha_{(U', \gamma')}} \in U' \subseteq U$, whence it follows that $(x_{\alpha_{(U, \gamma)}})$ is a subnet of (x_α) that converges to x . \square

Definition 1.7.6. A net (x_α) in a set X is an **ultranet** (universal net) if for each subset E of X , (x_α) is either eventually in E or eventually in E^c .

It follows from this definition that if an ultranet is frequently in E then it is eventually in E . In particular, an ultranet in a topological space must converge to each of its cluster points.

For any directed set A , the map $P : A \rightarrow X$, defined by $P(\alpha) = x$ for all $\alpha \in A$, gives an ultranet on X , called the **trivial ultranet**. Nontrivial ultranets can be proved to exist but none has ever been explicitly constructed. Most facts about ultranets are best developed using filters and ultrafilters as a vehicle.

Proposition 1.7.7. If (x_α) is an ultranet in X and $f : X \rightarrow Y$, then $(f(x_\alpha))$ is an ultranet in Y .

Proof. If $A \subseteq Y$, then $f^{-1}(A) = f^{-1}(A^c)^c$, so (x_α) is eventually in either $f^{-1}(A)$ or $f^{-1}(A^c)$, from which it follows that $(f(x_\alpha))$ is eventually in either A or A^c . Thus, $(f(x_\alpha))$ is an ultranet. \square

1.7.2 Filters

We have already seen that net provides an effective approach to the topology of a topological space X . In this part we introduce another tool to do this job, namely filters. Later we will see that these two concepts are equivalent.

Definition 1.7.8. A filter \mathfrak{F} on a set X is a nonempty collection of subsets of X with the properties:

- (F1) Every subset of X which contains a set of \mathfrak{F} belongs to \mathfrak{F} .
- (F2) Every finite intersection of sets of \mathfrak{F} belongs to \mathfrak{F} .
- (F3) The empty set is not in \mathfrak{F} .

A filter \mathfrak{F} on X defines a structure on X , the axioms of which are (F1), (F2) and (F3); this structure is called a **structure of a filtered set**, and the set X endowed with this structure is called a **set filtered by \mathfrak{F}** .

Example 1.7.9 (Example of Filters).

- (a) Let X be any set, $A \subseteq X$. Then $\{F \subseteq X : A \subseteq F\}$ is a filter on X with a particularly simple filter base, the collection consisting of the single set A .
- (b) Let X be any topological space, $A \subseteq X$. Then $\{U \subseteq X : A \subseteq \text{Int } U\}$ is a filter on X . In particular, the set $\mathfrak{U}(x)$ of all neighborhoods of $x \in X$ is a filter on X , and any neighborhood base at x is a filterbase for $\mathfrak{U}(x)$. This filter will sometimes be called the **neighborhood filter at x** .
- (c) If X is an infinite set, the complements of the finite subsets of X are the elements of a filter. This is called the **Fréchet filter on X** .

If \mathfrak{F}_1 and \mathfrak{F}_2 are filters on X , we say \mathfrak{F}_1 is **finer** than \mathfrak{F}_2 (or \mathfrak{F}_2 is **coarser** than \mathfrak{F}_1) if $\mathfrak{F}_1 \supseteq \mathfrak{F}_2$. Two filters are said to be comparable if one is finer than the other. The set of all filters on X is ordered by the relation " \mathfrak{F}_1 is coarser than \mathfrak{F}_2 "; this relation is induced by the inclusion relation. The filter formed by the single set X is the smallest element of the ordered set of all filters on X . We shall see later that, if X has more than one element, the set of all filters on X has no greatest element.

Let $(\mathfrak{F}_i)_{i \in I}$ be any nonempty family of filters on a set X , then the set

$$\mathfrak{F} = \bigcap_{i \in I} \mathfrak{F}_i$$

is a filter on X , called the **intersection** of the family of filters $(\mathfrak{F}_i)_{i \in I}$ and is obviously the greatest lower bound of the set of the \mathfrak{F}_i , in the ordered set of all filters on X .

Given a set \mathfrak{S} of subsets of a set X , let us consider whether there are any filters on X which contain \mathfrak{S} . If such a filter exists then by (F2) it contains also the set \mathfrak{S}' of finite intersections of sets of \mathfrak{S} ; hence a necessary condition for such a filter to exist is that the empty subset of X is not in \mathfrak{S}' . This condition is also sufficient, for by (F1) any filter which contains \mathfrak{S}' also contains the set \mathfrak{F} of subsets of X which contain a set of \mathfrak{S} . Now \mathfrak{F} clearly satisfies (F1); it satisfies (F2) by reason of the definition of \mathfrak{S}' ; and finally it satisfies (F3) because the empty set does not belong to \mathfrak{S} . Hence \mathfrak{F} is the coarsest filter which contains \mathfrak{S} , and we have proved:

Proposition 1.7.10. *A necessary and sufficient condition that there should exist a filter on X containing a set \mathfrak{S} of subsets of X is that no finite subset of \mathfrak{S} has an empty intersection.*

The filter \mathfrak{F} defined above is said to be **generated** by \mathfrak{S} , and \mathfrak{S} is said to be a **subbase** of \mathfrak{F} .

Corollary 1.7.11. *Let \mathfrak{F} be a filter on a set X , and A a subset of X . Then there is a filter \mathfrak{G} which is finer than \mathfrak{F} and such that $A \in \mathfrak{G}$, if and only if A meets all the sets of \mathfrak{F} .*

Corollary 1.7.12. *A set Φ of filters on a non-empty set X has a least upper bound in the set of all filters on X if and only if, for all finite sequences $(\mathfrak{F}_i)_{i=1}^n$ of elements of Φ and all $A_i \in \mathfrak{F}_i$, the intersection $A_1 \cap \dots \cap A_n$ is not empty.*

Corollary 1.7.13. *The ordered set of all filters on a non-empty set X is inductive (every linearly ordered subset has a upper bound).*

Proof. For every linearly ordered set Φ of filters satisfies the condition of ??, since the sets A_i all belong to the same \mathfrak{F}_j by hypothesis, and we can apply (F2). \square

If \mathfrak{S} is a subbase of a filter \mathfrak{F} on X , then \mathfrak{F} is not in general the set of subsets of X which contain a set of \mathfrak{S} ; for \mathfrak{S} to have this property it is necessary and sufficient that every finite intersection of sets of \mathfrak{S} should contain a set of \mathfrak{S} . Hence the following proposition:

Proposition 1.7.14. *Let \mathfrak{B} be a set of subsets of a set X . Then the set of subsets of X which contain a set of \mathfrak{B} is a filter if and only if \mathfrak{B} has the following two properties:*

(B1) *The intersection of two sets of \mathfrak{B} contains a set of \mathfrak{B} .*

(B2) *\mathfrak{B} is not empty, and the empty set is not in \mathfrak{B} .*

Definition 1.7.15. A set \mathfrak{B} of subsets of a set X which satisfies axioms (B1) and (B2) is said to be a **base** of the filter it generates. Two filter bases are said to be **equivalent** if they generate the same filter.

If \mathfrak{G} is a subbase of a filter \mathfrak{F} , then the set \mathfrak{B} of finite intersections of sets of \mathfrak{G} is a base of \mathfrak{F} .

Proposition 1.7.16. A subset \mathfrak{B} of a filter \mathfrak{F} on X is a base of \mathfrak{F} if and only if every set of \mathfrak{F} contains a set of \mathfrak{B} .

Proof. If \mathfrak{B} is a base of \mathfrak{F} , then clearly every set of \mathfrak{F} contains a set of \mathfrak{B} ; conversely, if every set of \mathfrak{F} contains a set of \mathfrak{B} , then the set of subsets of X containing a set of \mathfrak{B} coincides with \mathfrak{F} by reason of (F1). \square

Corollary 1.7.17. A filter \mathfrak{F}' with base \mathfrak{B}' is finer than a filter \mathfrak{F} with base \mathfrak{B} if and only if every set of \mathfrak{B} contains a set of \mathfrak{B}' .

Corollary 1.7.18. Two filter bases \mathfrak{B} and \mathfrak{B}' on a set X are equivalent if and only if every set of \mathfrak{B} contains a set of \mathfrak{B}' and every set of \mathfrak{B}' contains a set of \mathfrak{B} .

Example 1.7.19 (Example of Filterbases).

- (a) Let X be a topological space. ?? shows that the bases of the neighbourhood filter of a point $x \in X$ are precisely the fundamental systems of neighbourhoods of x .
- (b) Let (X, \preceq) be a non-empty directed set. For each $a \in X$, the set $S(a)$ of all $x \in X$ such that $a \preceq x$ will be called the section of X relative to the element a . Then the set \mathfrak{B} of sections of X is a filter base, for it clearly satisfies (B2), and if a, b are any two elements of X , then there is by hypothesis an element $c \in X$ such that $a \preceq c$ and $b \preceq c$, and therefore $S(a) \cap S(b) \supseteq S(c) \neq \emptyset$, so that (B1) is satisfied. The filter generated by \mathfrak{B} is called the **section filter** of the directed set X .

Finally, we introduce ultrafilters. An **ultrafilter** on a set X is a filter \mathfrak{F} such that there is no filter on X which is strictly finer than \mathfrak{F} (in other words, a maximal element in the ordered set of all filters on X). Since the ordered set of all filters on X is inductive, Zorn's lemma shows that:

Proposition 1.7.20. If \mathfrak{F} is any filter on a set X , there is an ultrafilter finer than \mathfrak{F} .

Theorem 1.7.21. Let \mathfrak{F} be a filter on a set X . Then the following are equivalent:

- (i) \mathfrak{F} is an ultrafilter.
- (ii) If $A \cup B \in \mathfrak{F}$, then either $A \in \mathfrak{F}$ or $B \in \mathfrak{F}$.
- (iii) For every subset $A \subseteq X$, either $A \in \mathfrak{F}$ or $A^c \in \mathfrak{F}$.

Proof. If there exist subsets A and B of X such that $A \notin \mathfrak{F}$ and $B \notin \mathfrak{F}$ but $A \cup B \in \mathfrak{F}$. Let \mathfrak{G} be the set of subsets F of X such that $A \cup F \in \mathfrak{F}$. It is straightforward to check that \mathfrak{G} is a filter on X , and \mathfrak{G} is strictly finer than \mathfrak{F} , since $B \in \mathfrak{G}$; but this contradicts the hypothesis that \mathfrak{F} is an ultrafilter. This proves (i) \Rightarrow (ii).

Also, (ii) implies (iii), since we have $A \cup A^c = X \in \mathfrak{F}$ for any $A \subseteq X$. Finally, assume (iii), and let $A \in \mathfrak{G}$ where \mathfrak{G} is a filter containing \mathfrak{F} . Then if $A \notin \mathfrak{F}$ we will have $A^c \in \mathfrak{F} \subseteq \mathfrak{G}$, which contradicts (F3). Thus \mathfrak{F} is an ultrafilter. \square

By induction, the following corollary is true.

Corollary 1.7.22. *If the union of a finite sequence $(A_i)_{i=1}^n$ of subsets of X belongs to an ultrafilter \mathfrak{F} , then at least one of the A_i belongs to \mathfrak{F} . In particular, if $(A_i)_{i=1}^n$ is a covering of X , then at least one of the A_i belongs to \mathfrak{F} .*

Example 1.7.23.

- (a) A filter \mathfrak{F} on X is a fixed ultrafilter if and only if $\mathfrak{F} = \mathfrak{U}(x)$ for some $x \in X$. By the criterion given in ??, each filter of this form is an ultrafilter. On the other hand, if \mathfrak{F} is a fixed ultrafilter, say $\bigcap \mathfrak{F} = A \neq \emptyset$, then \mathfrak{F} must be the filter of all sets containing A (since this is a filter containing \mathfrak{F}) and A must be a single point (since the filter of all sets containing $x \in A$ is finer than \mathfrak{F}). Such ultrafilters are called **trivial**.
- (b) The Frechet filter \mathfrak{F} on \mathbb{R} is contained in some ultrafilter \mathcal{G} . Since \mathfrak{F} is free, \mathcal{G} must be also be free.
- (c) The ultrafilter containing a given filter \mathfrak{F} need not be unique. For if \mathfrak{F} is the filter of all sets containing $A \subseteq X$, then for each $x \in A$, the filter of all sets containing x is an ultrafilter containing \mathfrak{F} . In fact, if a filter is contained in a unique ultrafilter, it is itself an ultrafilter.

Proposition 1.7.24. *Every filter \mathfrak{F} on a set X is the intersection of the ultrafilters finer than \mathfrak{F} .*

Proof. Clearly this intersection contains \mathfrak{F} . Conversely, let A be a subset of X which does not belong to \mathfrak{F} . Then A contains no set of \mathfrak{F} ; hence every $F \in \mathfrak{F}$ meets A^c and there is a filter \mathfrak{F}' which is finer than \mathfrak{F} and contains A^c . If \mathfrak{U} is an ultrafilter finer than \mathfrak{F}' it follows that $A \notin \mathfrak{F}$. This completes the proof. \square

1.7.3 Construction of filters

Proposition 1.7.25. *Let \mathfrak{F} be a filter on a set X and A a subset of X . Then the trace \mathfrak{F}_A of \mathfrak{F} on A is a filter if and only if each set of \mathfrak{F} meets A .*

Proof. Since $(F \cap G) \cap A = (F \cap A) \cap (G \cap A)$ we see that \mathfrak{F}_A satisfies (F2); again, if $F \cap A \subseteq G \subseteq A$ then $G = (F \cup G) \cap A \in \mathfrak{F}_A$, whence \mathfrak{F}_A satisfies (F1). Hence \mathfrak{F}_A is a filter if and only if it satisfies (F3), i.e. if and only if each set of \mathfrak{F} meets A . In particular, if $A \in \mathfrak{F}$ then \mathfrak{F}_A is a filter on A , by (F1) and (F3). \square

Definition 1.7.26. *Let A be a subset of a set X and \mathfrak{F} a filter on X . If the trace of \mathfrak{F} on A is a filter on A , this filter is said to be **induced** by \mathfrak{F} on A .*

If a filter \mathfrak{F} on X induces a filter on $A \subseteq X$, then the trace on A of a base of \mathfrak{F} is a base of \mathfrak{F}_A , by ??.

Example 1.7.27. Let X be a topological space, A a subset of X , x a point of X . In order that the trace on A of the **neighbourhood filter** \mathfrak{B} of x should be a filter on A , it is necessary and sufficient that every neighbourhood of x meets A , i.e. that x lies in the closure of A .

This example of an induced filter is of interest for two reasons: first because it plays an important role in the theory of limits and secondly because every filter can be defined in this way. Indeed, let \mathfrak{F} be a filter on a set X and let X' be the set obtained by adjoining a new

element ω to X , X being identified with the complement of $\{\omega\}$ in X' ; let \mathfrak{F}' be the filter on X' consisting of the sets $F \cup \{\omega\}$, where F runs through \mathfrak{F} . For each point $x \neq \omega$ of X' , let $\mathfrak{U}(x)$ be the set of all subsets of X' which contain x , and let $\mathfrak{U}(\omega)$ be \mathfrak{F}' ; then the $\mathfrak{U}(x)$ for $x \in X'$ obviously satisfy the axioms in ??, and therefore define a topology on X' for which they are the neighbourhood filters of points. Finally ω lies in the closure of X in this topology, and \mathfrak{F} is induced by $\mathfrak{F}' = \mathfrak{U}(\omega)$ on X . The topology thus defined on X' (resp. the set X' with this topology) is called the **topology** (resp. the topological space) **associated with \mathfrak{F}'** .

Proposition 1.7.28. *An ultrafilter \mathfrak{U} on a set X induces a filter on a subset A of X if and only if $A \in \mathfrak{U}$ and if this condition is satisfied then \mathfrak{U}_A is an ultrafilter on A .*

Proof. This follows from the characterization in ??.

Let \mathfrak{B} be a filter base on a set X , and let f be a mapping of X into a set Y ; then $f(\mathfrak{B})$ is a filter base on Y , for the relation $A \neq \emptyset$ implies $f(A) \neq \emptyset$, and we have $f(A \cap B) \subseteq f(A) \cap f(B)$. If \mathfrak{B}_1 is a base of a filter which is finer than the filter of base \mathfrak{B} , then $f(\mathfrak{B}_1)$ is a base of a filter finer than the filter of base $f(\mathfrak{B})$, by ??.

Proposition 1.7.29. *If \mathfrak{B} is an ultrafilter base on a set X and if f is a mapping of X into a set Y , then $f(\mathfrak{B})$ is an ultrafilter base on Y .*

Proof. Let N be a subset of Y . If $f^{-1}(N)$ contains a set M of \mathfrak{B} , then N contains $f(M)$; if not, then $f^{-1}(N)^c = f^{-1}(N^c)$ contains a set M of \mathfrak{B} and therefore N^c contains $f(M)$. Hence the result follows from ??.

Consider in particular the case where f is the canonical injection $A \hookrightarrow X$ of a subset A of a set X . If \mathfrak{B} is a filter base on A then $f(\mathfrak{B})$ is a filter base on X . The filter \mathfrak{F} on X generated by $f(\mathfrak{B})$ is called the **direct image** of \mathfrak{B} under f . If \mathfrak{B} is an ultrafilter base on A it is also an ultrafilter base on X by reason of ??.

Let us next examine whether the **inverse image** of a filter base is a filter base. Let \mathfrak{B} be a filter base on a set Y , and let f be a mapping of a set X into Y ; then $f^{-1}(\mathfrak{B})$ is a filter base on X if and only if $f^{-1}(N) \neq \emptyset$ for each $N \in \mathfrak{B}$. This condition can also be expressed by saying that every set of \mathfrak{B} meets $f(X)$. If this condition is satisfied, then $f(f^{-1}(\mathfrak{B}))$ is a base of a filter finer than the filter of base \mathfrak{B} (by $N \supseteq f(f^{-1}(N))$ and ??).

If \mathfrak{B} is a filter base on X it is clear that the above condition is satisfied by $\mathfrak{B}' = f(\mathfrak{B})$; $f^{-1}(f(\mathfrak{B}))$ is then a base of a filter coarser than the filter of base \mathfrak{B} (by $f(f^{-1}(M)) \supseteq M$ and ??).

Let $(X_i)_{i \in I}$ be a family of sets, and for each $i \in I$ let \mathfrak{B}_i be a filter base on X_i . Let \mathfrak{B} be the set of subsets of the product set $X = \prod_{i \in I} X_i$ which are of the form $\prod_{i \in I} M_i$ where $M_i = X_i$ except for a finite number of indices and where $M_i \in \mathfrak{B}_i$ for each $i \in I$ such that $M_i \neq X_i$. The formula $(\prod_{i \in I} M_i) \cap (\prod_{i \in I} N_i) = \prod_{i \in I} (M_i \cap N_i)$ shows that \mathfrak{B} is a filter base on X . Note that the filter of base \mathfrak{B} is also generated by the sets $\pi_i^{-1}(M_i)$, where $M_i \in \mathfrak{B}_i$ and i runs through I , since

$$\pi_i^{-1}(M_i) = M_i \times \prod_{j \neq i} X_j.$$

The filter of base \mathfrak{B} is the product filter of \mathfrak{F}_i , which we now define.

Definition 1.7.30. *Given a filter \mathfrak{F}_i on each set X_i of a family of sets $(X_i)_{i \in I}$, the product of the filters \mathfrak{F}_i is the filter on $X = \prod_{i \in I} X_i$, which has as a base the set of subsets of X of the form $\prod_{i \in I} M_i$, where*

$M_i \in \mathfrak{F}_i$ for each $i \in I$ and $M_i = X_i$ for all but a finite number of indices. The product filter is denoted $\prod_{i \in I} \mathfrak{F}_i$.

Proposition 1.7.31. *The product of the filters \mathfrak{F}_i is the coarsest filter \mathfrak{G} on X such that $\pi_i(\mathfrak{G}) = \mathfrak{F}_i$ for each $i \in I$.*

Proof. By definition, $\prod_{i \in I} \mathfrak{F}_i$ has a base consists of sets of the form $\prod_{i \in I} M_i$, where $M_i \in \mathfrak{F}_i$ for each $i \in I$ and $M_i = X_i$ for all but a finite number of indices. These sets are mapped under π_j to \mathfrak{F}_j , for each index $j \in I$. Thus $\pi_j(\prod_{i \in I} \mathfrak{F}_i) = \mathfrak{F}_j$ for each j .

Conversely, let \mathfrak{G} be a filter on X satisfying this condition. Then for each j and $M_j \in \mathfrak{F}_j$, $\pi_j(N) \subseteq M_j$ for some $N \subseteq X$. It then follows that $N \subseteq M_i \prod_{i \neq j} X_j$, whence $M_i \prod_{i \neq j} X_j \in \mathfrak{G}$. Since j is arbitrary, we see $\prod_{i \in I} \mathfrak{F}_i \subseteq \mathfrak{G}$. \square

Corollary 1.7.32. *If \mathfrak{B}_i is a base of \mathfrak{F}_i for each $i \in I$, then \mathfrak{B} is a base of the product filter $\prod_{i \in I} \mathfrak{F}_i$.*

Example 1.7.33. On a product $X = \prod_{i \in I} X_i$ of topological spaces, the neighbourhood filter of any point $x = (x_i)$ is the product of the neighbourhood filters of the x_i 's.

1.7.4 Limit of filters

Now we define the convergence of a filter.

Definition 1.7.34. *A filter \mathfrak{F} on a topological space X is said to converge to $x \in X$ (written $\mathfrak{F} \rightarrow x$) if $\mathfrak{U}(x) \subseteq \mathfrak{F}$. That is, if \mathfrak{F} is finer than the neighborhood filter at x . We say \mathfrak{F} has x as a **cluster point** (or, \mathfrak{F} clusters at x) if each $F \in \mathfrak{F}$ meets each $U \in \mathfrak{U}(x)$. It is clear that if $\mathfrak{F} \rightarrow x$, then \mathfrak{F} clusters at x .*

It will be convenient to have the notions of convergence and clustering available for filter bases; they generalize easily and obviously. A filter base \mathcal{B} **converges to x** if each $U \in \mathfrak{U}(x)$ contains some $B \in \mathcal{B}$ (iff the filter generated by \mathcal{B} converges to x); \mathcal{B} **clusters at x** if each $U \in \mathfrak{U}(x)$ meets each $B \in \mathcal{B}$ (iff the filter generated by \mathcal{B} clusters at x).

Lemma 1.7.35. *Let \mathfrak{F} be a filter on X . Then \mathfrak{F} clusters to $x \in X$ if and only if $x \in \bigcap_{M \in \mathfrak{F}} \bar{F}$.*

Proof. Since $x \in \bar{F}$ if and only if F meets every element of $\mathfrak{U}(x)$, the claim is clear by definition. \square

Example 1.7.36.

- (a) Let X be a topological space, $A \subseteq X$. The cluster points of the filter $\mathfrak{F} = \{U \subseteq X : A \subseteq U\}$ include each point of \bar{A} .
- (b) The Fréchet filter on \mathbb{R} has no cluster points.
- (c) Let \mathfrak{F} be the filter on \mathbb{R} generated by the filter base $\mathcal{B} = \{(0, \varepsilon) : \varepsilon > 0\}$. Then $\mathfrak{F} \rightarrow 0$ (although 0 does not belong to every element of \mathfrak{F}).

Proposition 1.7.37. *\mathfrak{F} has x as a cluster point if and only if there is a filter \mathfrak{G} finer than \mathfrak{F} which converges to x .*

Proof. If \mathfrak{F} has x as a cluster point, the collection $\mathcal{B} = \{U \cap F : U \in \mathfrak{U}(x), F \in \mathfrak{F}\}$ is a filterbase for a filter \mathcal{G} which is finer than \mathcal{G} and converges to x (note that $X \in \mathfrak{F}$, so $\mathfrak{U}(x)$ is contained in \mathcal{B} , and hence \mathcal{G}).

Conversely, if $\mathfrak{F} \subseteq \mathcal{G} \rightarrow x$, then each $F \in \mathfrak{F}$ and each neighborhood U of x belong to \mathcal{G} and hence meet, so \mathfrak{F} clusters at x . \square

According to the next propositions, filter convergence is adequate to the task of describing topological concepts.

Proposition 1.7.38. *A topological space X is Hausdorff if and only if the limit of every convergent filter is unique.*

Proof. If X is Hausdorff, let $x, y \in X$ be distinct and \mathfrak{F} be a filter converges to x and y . Then since x and y have disjoint neighborhoods, it follows that $\emptyset \in \mathfrak{F}$, which is impossible. Conversely, if X is not Hausdorff, then there exists $x, y \in X, x \neq y$ such that $U \cap V \neq \emptyset$ for any $U \in \mathfrak{U}(x), V \in \mathfrak{U}(y)$. Then $\mathcal{B} = \{U \cap V : U \in \mathfrak{U}(x), V \in \mathfrak{U}(y)\}$ is a filterbase, so it generates a filter \mathfrak{F} . It is clear that \mathfrak{F} converges to both x and y . \square

Proposition 1.7.39. *Let X be a topological space and $x \in X$. If $E \subseteq X$, then $x \in \bar{E}$ if and only if there is a filter \mathfrak{F} such that $E \in \mathfrak{F}$ and $\mathfrak{F} \rightarrow x$.*

Proof. If $x \in \bar{E}$, then $\mathcal{B} = \{U \cap E : U \in \mathfrak{U}(x)\}$ is a filter base. The resulting filter contains E and converges to x . Conversely, if $E \in \mathfrak{F} \rightarrow x$, then x is a cluster point of \mathfrak{F} and hence $x \in \bar{E}$. \square

If \mathfrak{F} is a filter on X and $f : X \rightarrow Y$, then $f(\mathfrak{F})$ is the filter on Y generated by $f(\mathfrak{F})$.

Proposition 1.7.40. *Let $f : X \rightarrow Y$ be a map between topological spaces. Then f is continuous at $x \in X$ if and only if whenever $\mathfrak{F} \rightarrow x$ in X then $f(\mathfrak{F}) \rightarrow f(x)$ in Y .*

Proof. Suppose f is continuous at x and $\mathfrak{F} \rightarrow x$. Let V be any neighborhood of $f(x)$ in Y . Then for some neighborhood U of x in X , $f(U) \subseteq V$. Then since $U \in \mathfrak{F}$, $V \in f(\mathfrak{F})$. Thus $f(\mathfrak{F}) \rightarrow f(x)$.

Conversely, suppose whenever $\mathfrak{F} \rightarrow x$ in X then $f(\mathfrak{F}) \rightarrow f(x)$ in Y . Let $\mathfrak{U}(x)$ be the filter of all neighborhoods of x in X . Then since $f(\mathfrak{U}(x)) \rightarrow f(x)$, each neighborhood V of $f(x)$ belongs to $f(\mathfrak{U}(x))$, so for some neighborhood U of x , $f(U) \subseteq V$. Thus f is continuous at x . \square

Proposition 1.7.41. *A filter \mathfrak{F} converges to x in $X = \prod_s X_s$ if and only if $\pi_s(\mathfrak{F}) \rightarrow \pi_s(x)$ in X_s for each s .*

Proof. If $\mathfrak{F} \rightarrow x$ then $\pi_s(\mathfrak{F}) \rightarrow \pi_s(x)$ for each s by continuity. Conversely, suppose that $\pi_s(\mathfrak{F}) \rightarrow \pi_s(x)$ for each s . Let $U = \bigcap_{i=1}^n \pi_{s_i}^{-1}(U_{s_i})$ be a basic neighborhood of x in X . Then U_{s_i} is a neighborhood of $\pi_{s_i}(x)$ for each i , so $U_{s_i} \in \pi_{s_i}(\mathfrak{F})$ for each i , and hence $\pi_{s_i}(F_i) \subseteq U_{s_i}$ for some $F_i \in \mathfrak{F}$. Then $\bigcap_{i=1}^n F_i \in \mathfrak{F}$ and $\bigcap_{i=1}^n F_i \subseteq \bigcap_{i=1}^n \pi_{s_i}^{-1}(U_{s_i})$, so $\bigcap_{i=1}^n \pi_{s_i}^{-1}(U_{s_i}) \in \mathfrak{F}$. Thus $\mathfrak{F} \rightarrow x$. \square

The similarities between net and filter convergence are manifest. Each describes the topology on a topological space with equal facility, "finer filters" provide a filter analog to "subnets." In addition, there is more than a casual relationship between the ideas behind the two approaches. Thus the fact that a formal bridge can be built between the two notions should come as no surprise.

Definition 1.7.42. If (x_α) is a net in X , the filter generated by the filter base \mathcal{B} consisting of the sets $B_{\alpha_0} = \{x_\alpha : \alpha \succeq \alpha_0\}, \alpha_0 \in A$, is called the **filter generated by** (x_α) .

Definition 1.7.43. If \mathfrak{F} is a filter on X , let $A_{\mathfrak{F}} = \{(x, F) : \in F, F \in \mathfrak{F}\}$. Then $A_{\mathfrak{F}}$ is directed by the relation $(x_1, F_1) \preceq (x_2, F_2)$ iff $F_2 \subseteq F_1$, so the map $A_{\mathfrak{F}} \rightarrow X, (x, F) \mapsto x$ is a net in X . It is called the **net based on** \mathfrak{F} .

Theorem 1.7.44. Let X be a topological space.

- (a) A filter \mathfrak{F} converges to x in X if and only if the net based on \mathfrak{F} converges to x .
- (b) A net (x_α) converges to x in X if and only if the filter generated by (x_α) converges to x .

Proof. Suppose $\mathfrak{F} \rightarrow x$. If U is a nhood of x , then $U \in \mathfrak{F}$. Pick $y_0 \in U$, then $(y_0, U) \in A_{\mathfrak{F}}$, and if $(y, F) \succeq (y_0, U)$ then $y \in F \subseteq U$. Thus the net based on \mathfrak{F} converges to x .

Conversely, suppose the net based on \mathfrak{F} converges to x . Let U be a nhood of x . Then for some $(y_0, F_0) \in A_{\mathfrak{F}}$ we have $(y, F) \succeq (y_0, F_0)$ implies $y \in U$. But then $F_0 \subseteq U$; otherwise, there is some $y \in F_0 \setminus U$, and then $(y, F_0) \succeq (y_0, F_0)$ but $y \notin U$. Hence $U \in \mathfrak{F}$, so $\mathfrak{F} \rightarrow x$.

The net (x_α) converges to x iff each nhood of x contains a tail of (x_α) . Since the tails of (x_α) are a base for the filter generated by (x_α) , the result follows. \square

1.8 Compactness

Recall that an **open cover** of a space X is a collection \mathcal{U} of open subsets of X whose union is X , and a **subcover** of \mathcal{U} is a subcollection of elements of \mathcal{U} that still covers X . A topological space X is said to be **compact** if every open cover of X has a finite subcover. Note that the empty set is compact.

As in the case of connectedness, to say that a subset of a topological space is compact is to say that it is a compact space when endowed with the subspace topology. In this situation, it is often useful to extend our terminology in the following way. If X is a topological space and $A \subseteq X$, a collection of subsets of X whose union contains A is also called a **cover of A** ; if the subsets are open in X we sometimes call it an **open cover of A** . We try to make clear in each specific situation which kind of open cover of A is meant: a collection of open subsets of A whose union is A , or a collection of open subsets of X whose union contains A . The following lemma shows that either kind of open cover can be used to detect compactness of a subspace; it is an immediate consequence of the definitions of compactness and the subspace topology.

Example 1.8.1. The real line \mathbb{R} is not compact, for the covering of \mathbb{R} by open intervals $\{(n, n+2) : n \in \mathbb{Z}\}$ contains no finite subcollection that covers \mathbb{R} .

In general, it takes some effort to decide whether a given space is compact or not. First we shall prove some general theorems that show us how to construct new compact spaces out of existing ones.

Lemma 1.8.2 (Compactness Criterion for Subspaces). If X is any topological space, a subset $A \subseteq X$ is compact (in the subspace topology) if and only if every cover of A by open subsets of X has a finite subcover.

Proof. Assume that A is compact and let \mathcal{U} be an open cover of A by open subsets of X . Then the intersection $A \cap \mathcal{U}$ is an open cover of A , hence there are finitely many sets U_1, \dots, U_k in \mathcal{U} such that $A = \bigcup_{i=1}^k (U_i \cap A)$. This then implies $A \subseteq \bigcup_{i=1}^k U_i$, so \mathcal{U} has a finite subcover.

Conversely, if every cover of A by open subsets of X has a finite subcover. Let \mathcal{U} be an open cover of A consisting of open subsets of A . Then by the definition of the subspace topology, each member of \mathcal{U} is the intersection with A of an open subset of X , and we then obtain an open cover of A by open subsets of X . Since this cover has a finite subcover, it is easy to see that \mathcal{U} also has a finite subcover, and hence A is compact. \square

Proposition 1.8.3. Suppose X is a topological space, and (x_i) is a sequence of points in X converging to $x \in X$. Then the set $A = \{x_i : i \in \mathbb{N}\} \cup \{x\}$ is compact in the subspace topology.

Proof. Suppose \mathcal{U} is a cover of A by open subsets of X . There is some set $U \in \mathcal{U}$ containing x , and U must contain x_i for all but finitely many i . Choosing one set in \mathcal{U} for each of those finitely many elements, we obtain a finite subcover of A . \square

The most important fact about compactness is that continuous images of compact spaces are compact.

Proposition 1.8.4. Let X and Y be topological spaces, and let $f : X \rightarrow Y$ be a continuous map. If X is compact, then $f(X)$ is compact.

Proof. Let \mathcal{U} be a cover of $f(X)$ by open subsets of Y . For each $U \in \mathcal{U}$, $f^{-1}(U)$ is an open subset of X . Since \mathcal{U} covers $f(X)$, every point of X is in some set $f^{-1}(U)$, so the collection $\{f^{-1}(U) : U \in \mathcal{U}\}$ is an open cover of X . By compactness of X , some finite number of these, say $f^{-1}(U_1), \dots, f^{-1}(U_k)$, cover X . Then it follows that U_1, \dots, U_k cover $f(X)$. \square

Proposition 1.8.5. Every closed subspace of a compact space is compact.

Proof. Let A be a closed subspace of the compact space X . Given a covering \mathcal{U} of A by sets open in X , let us form an open covering \mathcal{V} of X by adjoining to \mathcal{U} the single open set A^c . Some finite subcollection of \mathcal{V} covers X . If this subcollection contains the set A^c , discard A^c ; otherwise, leave the subcollection alone. The resulting collection is a finite subcollection of \mathcal{U} that covers A . \square

Proposition 1.8.6 (Separation of Compact Subsets). If X is a Hausdorff space and $A, B \subseteq X$ are disjoint compact subsets, there exist disjoint open subsets $U, V \subseteq X$ such that $A \subseteq U$ and $B \subseteq V$. In particular, compact subsets of X are closed.

Proof. First consider the case in which $B = \{y\}$ is a singleton. For each $x \in A$, there exist disjoint open subsets U_x containing x and V_x containing y by the Hausdorff property. The collection $\{U_x : x \in A\}$ is an open cover of A , so it has a finite subcover: call it $\{U_{x_1}, \dots, U_{x_k}\}$. Let $U = \bigcup_{i=1}^k U_{x_i}$ and $V = \bigcap_{i=1}^k V_{x_i}$. Then U and V are disjoint open subsets with $A \subseteq U$ and $\{y\} \subseteq V$, so this case is proved.

Next consider the case of a general compact subset B . The argument above shows that for each $y \in B$ there exist disjoint open subsets $U_y, V_y \subseteq X$ such that $A \subseteq U_y$ and $y \in V_y$. By compactness of B , finitely many of these, say V_{y_1}, \dots, V_{y_k} , cover B . Then setting $U = \bigcap_{i=1}^k U_{y_i}$ and $V = \bigcup_{i=1}^k V_{y_i}$ proves the result. \square

Example 1.8.7. One needs the Hausdorff condition in the hypothesis of ???. Consider, for example, the finite complement topology on the real line. The only proper subsets of \mathbb{R} that are closed in this topology are the finite sets. But every subset of \mathbb{R} is compact in this topology.

One important use of the preceding results is as a tool for verifying that a continuous map is closed:

Theorem 1.8.8. *Suppose $f : X \rightarrow Y$ is a continuous map from a compact space to a Hausdorff space, then it is closed.*

Proof. Let F be a closed subset of X , then it is compact, so the image $f(F)$ is also compact. By ??, $f(F)$ is closed. \square

Lemma 1.8.9 (The tube lemma). *Let X and Y be topological spaces and consider the product space $X \times Y$. Let A be a compact subset of X and B be a compact subset of Y . If N is an open set containing $A \times B$, then N contains a neighborhood of $A \times B$ of the form $U \times V$, where U is open in X and V is open in Y .*

Proof. By the definition of the product topology, for each $(a, b) \in A \times B$, there are open sets $U_{(a,b)} \subseteq X$ and $V_{(a,b)} \subseteq Y$ such that $(a, b) \in U_{(a,b)} \times V_{(a,b)} \subseteq N$. For any fixed $a \in A$, the collection $\{V_{(a,b)} : b \in B\}$ is an open cover of B , and thus has a finite subcover; namely, there exists a finite subset B_a of B such that $V_a := \bigcup_{b \in B_a} V_{(a,b)}$ contains B . For each $a \in A$, let $U_a = \bigcap_{b \in B_a} U_{(a,b)}$, which is a neighborhood of a in X since B_a is finite. Moreover, the construction of U_a and V_a implies that $\{a\} \times B \subseteq U_a \times V_a \subseteq N$. We now essentially repeat the argument to drop the dependence on a : By the compactness of A , there exists a finite subset $A_0 \subseteq A$ such that $U = \bigcup_{a \in A_0} U_a$ contains A . Set $V = \bigcap_{a \in A_0} V_a$, it then follows by the above reasoning that $A \times B \subseteq U \times V \subseteq N$, which completes the proof. \square

Example 1.8.10. The tube lemma is certainly not true if Y is not compact. For example, let Y be the y -axis in \mathbb{R}^2 , and let

$$N = \{(x, y) : |x| < (y^2 + 1)^{-1}\}.$$

Then N is an open set containing the set $\{0\} \times \mathbb{R}$, but it contains no tube about $\{0\} \times \mathbb{R}$

Theorem 1.8.11. *The product of finitely many compact spaces is compact.*

Proof. Let X and Y be compact spaces. Let \mathcal{A} be an open covering of $X \times Y$. Given $x_0 \in X$, the slice $x_0 \times Y$ is compact and may therefore be covered by finitely many elements A_1, \dots, A_m of \mathcal{A} . Their union $N = A_1 \cup \dots \cup A_m$ is an open set containing $x_0 \times Y$; by tube lemma, the open set N contains a tube $W \times Y$ about $x_0 \times Y$, where W is open in X . Then $W \times Y$ is covered by finitely many elements A_1, \dots, A_m of \mathcal{A} . Now since X is compact, $X \times Y$ is covered by finitely many such tubes, which implies $X \times Y$ is covered by finitely many elements of \mathcal{A} , thus is compact. \square

1.8.1 Limit point compactness and sequentially compactness

A space X is said to be **limit point compact** if every infinite subset of X has a limit point in X , and **sequentially compact** if every sequence of points in X has a subsequence that converges to a point in X .

Proposition 1.8.12. *Compactness implies limit point compactness, but not the converse.*

Proof. Suppose X is compact, and let $S \subseteq X$ be an infinite subset. If S has no limit point in X , then it is closed, hence compact, and every point $x \in X$ has a neighborhood U such that $U \cap S$ is either empty or $\{x\}$. Finitely many of these neighborhoods cover X . But since each such neighborhood contains at most one point of S , this implies that S is finite, which is a contradiction. \square

Example 1.8.13. Let Y consist of two points; give Y the topology consisting of Y and the empty set. Then the space $X = \mathbb{Z}_+ \times Y$ is limit point compact, for every nonempty subset of X has a limit point. It is not compact, for the covering of X by the open sets $U_n = \{n\} \times Y$ has no finite subcollection covering X .

Example 1.8.14. Consider the first uncountable ordinal S_Ω , in the order topology. The space S_Ω is not compact, since it has no largest element. However, it is limit point compact: Let A be an infinite subset of S_Ω . Choose a subset B of A that is countably infinite. Being countable, the set B has an upper bound b in S_Ω ; then B is a subset of the interval $[a_0, b]$ of S_Ω , where a_0 is the smallest element of S_Ω . Since S_Ω has the least upper bound property, the interval $[a_0, b]$ is compact. By the preceding theorem, B has a limit point x in $[a_0, b]$. The point x is also a limit point of A . Thus S_Ω is limit point compact.

Proposition 1.8.15. *Sequentially compactness implies limit point compactness, and for first countable Hausdorff spaces they are equivalent.*

Proof. Suppose that X is sequentially compact and let $A \subseteq X$ be an infinite subset. Let $\{x_n\} \subseteq A$ be a sequence such that x_n 's are distinct. Then $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ converging to a point $x \in X$. Then by ??, x is a limit point of A . Thus X is limit point compact.

Suppose X is first countable Hausdorff and limit point compact, and let (x_n) be any sequence of points in X . If the sequence takes on only finitely many values, then it has a constant subsequence, which is certainly convergent. So we may suppose it takes on infinitely many values. By hypothesis the set of values $\{x_n\}$ has a limit point $x \in X$. If x is actually equal to x_n for infinitely many values of n , again there is a constant subsequence and we are done; so by discarding finitely many terms at the beginning of the sequence if necessary we may assume $x_n \neq x$ for all n . Because X is first countable, there is a nested neighborhood base at x , say $\{V_n\}$. For such a neighborhood base, it is easy to see that any subsequence $\{x_{n_i}\}$ such that $x_{n_i} \subseteq V_i$ converges to x .

Since x is a limit point, we can choose n_1 such that $x_{n_1} \subseteq V_1$. Suppose by induction that we have chosen $n_1 < n_2 < \dots < n_k$ with $x_{n_i} \in V_i$. Since X is Hausdorff, the sequence takes on infinitely many values in V_{k+1} , so we can choose some $n_{k+1} > n_k$ such that $x_{n_{k+1}} \in V_{k+1}$. This completes the induction, and proves that there is a subsequence x_{n_k} converging to x . \square

We now show that these different versions of compactness coincide for certain spaces; for this purpose, we need the following lemma.

Lemma 1.8.16 (The Lebesgue number lemma). *Let \mathcal{U} be an open covering of the metric space (X, d) . If X is sequentially compact, there is a $\delta > 0$ such that for each subset of X having diameter less than δ , there exists an element of \mathcal{U} containing it. The number δ is called a **Lebesgue number** for the covering \mathcal{U} .*

Proof. Let \mathcal{U} be an open covering of X . We assume that there is no $\delta > 0$ such that each set of diameter less than δ has an element of \mathcal{U} containing it, and derive a contradiction.

Our assumption implies in particular that for each positive integer n , there exists a set C_n of diameter less than $1/n$ that is not contained in any element of \mathcal{U} . Choose a point $x_n \in C_n$ for each n . By hypothesis, some subsequence (x_{n_i}) of the sequence (x_n) converges, say to the point x . Now x belongs to some element U of the collection \mathcal{U} , and because U is open, there is an $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq U$. If we choose i so large that $1/n_i < \varepsilon/2$ and $d(x_{n_i}, x) < \varepsilon/2$, then C_{n_i} lies in the ε -neighborhood of x . But this means that $C_{n_i} \subseteq U$, contrary to the hypothesis. \square

Proposition 1.8.17. *For metric spaces and second countable topological spaces, sequentially compactness implies compactness.*

Proof. Suppose first that X is second countable and sequentially compact, and let \mathcal{U} be an open cover of X . Since X is also lindelöf, \mathcal{U} has a countable subcover $\{U_i\}$. Assume no finite subcollection of U_i 's covers X . Then for each i there exists $x_i \in X$ such that $x_i \notin U_1 \cup \dots \cup U_i$. By hypothesis, the sequence x_i has a convergent subsequence $x_{n_i} \rightarrow x$. Now, $x \in U_m$ for some m because the U_i 's cover X , and then convergence of the subsequence means that $x_{n_k} \in U_m$ for all but finitely many values of k . But by construction, $x_{n_k} \notin U_m$ as soon as $n_k \geq m$, which is a contradiction. This proves that second countable sequentially compact spaces are compact.

Second, let M be a sequentially compact metric space. We show that given $\varepsilon > 0$, there exists a finite covering of X by open ε -balls. Assume that there exists an $\varepsilon > 0$ such that X cannot be covered by finitely many ε -balls. Construct a sequence of points x_n of X as follows: First, choose x_1 to be any point of X . Noting that the ball $B_\varepsilon(x_1)$ is not all of X (otherwise X could be covered by a single ε -ball), choose x_2 to be a point of X not in $B_\varepsilon(x_1)$. In general, given x_1, \dots, x_n , choose x_{n+1} to be a point not in the union

$$B_\varepsilon(x_1) \cup \dots \cup B_\varepsilon(x_n)$$

using the fact that these balls do not cover X . Note that by construction $d(x_{n+1}, x_i) \geq \varepsilon$ for $1 \leq i \leq n$. Therefore, the sequence (x_n) can have no convergent subsequence; in fact, any ball of radius $\varepsilon/2$ can contain x_n for at most one value of n .

Let \mathcal{A} be an open covering of X . Because X is sequentially compact, the open covering \mathcal{A} has a Lebesgue number δ . Let $\varepsilon = \delta/3$; use sequentially compactness of X to find a finite covering of X by open ε -balls. Each of these balls has diameter at most $2\delta/3$, so it lies in an element of \mathcal{U} . Choosing one such element of \mathcal{U} for each of these ε -balls, we obtain a finite subcollection of \mathcal{U} that covers X . \square

Combining the results above, we have the following theorem.

Theorem 1.8.18. *Let X be a metrizable space. Then the following are equivalent:*

- (i) X is compact.
- (ii) X is limit point compact.
- (iii) X is sequentially compact.

Example 1.8.19. Recall that \bar{S}_Ω denotes the first uncountable ordinal S_Ω with the point Ω adjoined. It is easy to see that the space \bar{S}_Ω is not metrizable, for it does not satisfy the sequence

lemma: The point Ω is a limit point of S_Ω , but it is not the limit of a sequence of points of S_Ω , for any sequence of points of S_Ω has an upper bound in S_Ω . The space S_Ω , on the other hand, does satisfy the sequence lemma. Nevertheless, S_Ω is not metrizable, for it is limit point compact but not compact.

1.8.2 General characterization of compactness

We now show that a version of the Balzano-Weierstrass property holds for compact topological spaces. As one might suspect, it is merely necessary to replace sequences by nets.

Theorem 1.8.20. *If X is a topological space, the following are equivalent:*

- (a) *X is compact.*
- (b) *Every net in X has a cluster point.*
- (c) *Every net in X has a convergent subnet.*

Proof. The equivalence of (b) and (c) follows from ???. If X is compact and (x_α) is a net in X , let $E_\alpha = \{x_\beta : \beta \succeq \alpha\}$. Since for any $\alpha, \beta \in A$ there exists $\gamma \in A$ with $\alpha \preceq \gamma$ and $\beta \preceq \gamma$, the family $\{E_\alpha\}_{\alpha \in A}$ has the finite intersection property. If (x_α) has no cluster point, then E_α is closed for each α , so by compactness, $\bigcap_{\alpha \in A} E_\alpha \neq \emptyset$. If $x \in \bigcap_{\alpha \in A} E_\alpha$, and U is a neighborhood of x , then U intersects each E_α , which means that (x_α) is frequently in U , so x is a cluster point of (x_α) . This is a contradiction.

On the other hand, if X is not compact, let $\{U_\beta\}_{\beta \in B}$ be an open cover of X with no finite subcover. Let \mathcal{A} be the collection of finite subsets of B , directed by inclusion, and for each $A \in \mathcal{A}$ let x_A be a point in $(\bigcup_{\beta \in A} U_\beta)^c$. Then $(x_A)_{A \in \mathcal{A}}$ is a net with no cluster point. Indeed, if $x \in X$, choose $\beta \in B$ with $x \in U_\beta$. If $A \in \mathcal{A}$ and $A \succeq \{\beta\}$, then $x_A \notin U_\beta$, so (x_A) is not frequently in U_β , and thus x is not a cluster point of (x_A) . \square

Using the above characterization of compactness, we prove the following important theorem about compactness of product spaces. To prepare for it, we introduce some notation. Recall that an element x of $X = \prod_{\alpha \in A} X_\alpha$ is, strictly speaking, a mapping from A into $\bigcup_{\alpha \in A} X_\alpha$; namely, $x(\alpha) \in X_\alpha$ is the α -th coordinate of x , which we generally denote by $\pi_\alpha(x)$. If $B \subseteq A$, there is a natural map $\pi_B : X \rightarrow \prod_{\alpha \in B} X_\alpha$; namely, $\pi_B(x)$ is the restriction of the map x to B . (In particular, $\pi_{\{\alpha\}}$ is essentially identical to π_α , and we shall not distinguish between them.) If $p \in \prod_{\alpha \in B} X_\alpha$ and $q \in \prod_{\alpha \in B} X_\alpha$, we shall say that q is an **extension** of p if q extends p as a mapping, that is, if $B \subseteq C$ and $p(\alpha) = q(\alpha)$ for $\alpha \in B$.

Theorem 1.8.21 (Tychonoff's Theorem). *If $\{X_\alpha\}_{\alpha \in A}$ is any family of compact topological spaces, the $\prod_{\alpha \in A} X_\alpha$ is compact with the product topology.*

Proof. By ??, it is enough to show that any net $(x_i)_{i \in I}$ in X has a cluster point. We shall do this by examining cluster points of the nets $(\pi_B(x_i))$ in the subproducts of X . To wit, let

$$\mathcal{P} = \bigcup_{B \subseteq A} \{p \in \prod_{\alpha \in B} X_\alpha : p \text{ is a cluster point of the net } (\pi_B(x_i))\}.$$

It is clear that \mathcal{P} is nonempty, because each X_α is compact and so $(\pi_B(x_i))$ has cluster points when $B = \{\alpha\}$. Moreover, \mathcal{P} is partially ordered by extension; that is, $p \leq q$ if q is an extension of p as defined above.

Suppose that $\{p_\lambda : \lambda \in \Lambda\}$ is a linearly ordered subset of \mathcal{P} , where $p_\lambda \in \prod_{\alpha \in B_\lambda} X_\alpha$. Let $\tilde{B} = \bigcup_\lambda B_\lambda$ and \tilde{p} be the unique element of $\prod_{\alpha \in \tilde{B}} X_\alpha$ that extends every p_λ . We claim that $\tilde{p} \in \mathcal{P}$. Indeed, from the definition of the product topology, any neighborhood of \tilde{p} contains a set of the form $\prod_{\alpha \in \tilde{B}} U_\alpha$, where each U_α is open in X_α and $U_\alpha = X_\alpha$ for all but finitely many α , say $\alpha_1, \dots, \alpha_n$. Each of these α_j 's belongs to some B_λ , so by linearity of the ordering they all belong to a single B_λ . But then $\prod_{\alpha \in B_\lambda} U_\alpha$ is a neighborhood of p_λ , so $(\pi_{B_\lambda}(x_i))$ is frequently in $\prod_{\alpha \in B_\lambda} U_\alpha$; hence $(\pi_{\tilde{B}}(x_i))$ is frequently in $\prod_{\alpha \in \tilde{B}} U_\alpha$, and \tilde{p} is a cluster point of $(\pi_{\tilde{B}}(x_i))$. This implies $\{p_\lambda\}$ has an upper bound \tilde{p} in \mathcal{P} .

By Zorn's lemma, then, \mathcal{P} has a maximal element $p_0 \in \prod_{\alpha \in B_0} X_\alpha$. We claim that $B_0 = A$. If not, pick $\gamma \in A \setminus B_0$. By ?? there is a subnet $(\pi_{B_0}(x_{i_j}))_{j \in J}$ of $(\pi_{B_0}(x_i))$ that converges to p_0 , and since X_γ is compact, there is a subnet $(\pi_\gamma(x_{i_{j(k)}}))_{k \in K}$ of $(\pi_{B_0}(x_{i_j}))$ that converges to some $p_\gamma \in X_\gamma$. Let q be the unique element of $\prod_{\alpha \in B_0 \cup \{\gamma\}} X_\alpha$ that extends both p_0 and p_γ , then the net $(\pi_{B_0 \cup \{\gamma\}}(x_{i_{j(k)}}))_{k \in K}$ converges to q and hence q is a cluster point of $(\pi_{B_0 \cup \{\gamma\}}(x_i))$, contradicting the maximality of p_0 . Therefore p_0 is a cluster point of (x_i) , and we are done. \square

For interest, we give a proof about the equivalence of Tychonoff's theorem and the axiom of choice. We already know that the latter is equivalent to Zorn's lemma, so we only need to use Tychonoff's theorem to deduce the axiom of choice.

Proposition 1.8.22. *Tychonoff's theorem implies the axiom of choice.*

Proof. Suppose that $\{X_\alpha\}_{\alpha \in A}$ is a nonempty collection of nonempty sets. Pick a point ω that is not an element of any X_α , define $X_\alpha^* = X_\alpha \cup \{\omega\}$, and define a topology on X_α^* by declaring the open sets to be $\emptyset, X_\alpha, \{\omega\}$, and X_α^* . Evidently each X_α^* is compact, so Tychonoff's theorem implies that $X^* = \prod_{\alpha \in A} X_\alpha^*$ is compact. Let $F_\alpha = \pi_\alpha^{-1}(X_\alpha)$. The sets F_α are closed (X_α is closed and π_α is continuous), and by the axiom of choice for finite collections of sets which is provable from the other standard axioms of set theory, they have the finite intersection property. Indeed, given a finite set $B \subseteq A$, pick $x_\beta \in X_\beta$ for $\beta \in B$; then $\bigcap_{\beta \in B} F_\beta$ contains the point $x \in X$ such that $\pi_\beta(x) = x_\beta$ for $\beta \in B$ and $\pi_\alpha(x) = \omega$ for $\alpha \notin B$. By the compactness, $\bigcap_{\alpha \in A} F_\alpha$, which is precisely $\prod_{\alpha \in A} X_\alpha$, is nonempty. \square

1.8.3 Inverse limit of compact spaces

Proposition 1.8.23. *Let $(X_\alpha, f_{\alpha\beta})$ be an inverse system of compact Hausdorff spaces indexed by a directed set I such that $f_{\alpha\alpha}$ is the identity mapping for each $\alpha \in I$. Let $X = \varprojlim X_\alpha$ be the inverse limit and $f_\alpha : X \rightarrow X_\alpha$ the canonical mapping. Then*

(a) *X is compact Hausdorff and for each $\alpha \in I$ we have*

$$f_\alpha(X) = \bigcap_{\beta \succeq \alpha} f_{\alpha\beta}(X_\beta). \quad (1.8.1)$$

(b) *If the X_α are all non-empty then X is non-empty.*

Proof. The space X is a closed subspace of $\prod_\alpha X_\alpha$, which is compact and Hausdorff. For $\beta \succeq \alpha$ we define

$$L_\beta = \{y \in \prod_\gamma X_\gamma : f_{\gamma\beta}(y_\beta) = y_\gamma \text{ for } \gamma \preceq \beta\}.$$

Since each component of $\prod X_\gamma$ is Hausdorff, we can see that each L_β is closed in the product space by showing that every point in the complement is an interior point by using the continuity of the $f_{\alpha\beta}$'s. Thus each L_β is compact. Furthermore, by construction, we have $\bigcap_{\beta \in I} L_\beta \subseteq X$. Thus we only need to show $\bigcap_{\beta \in I} L_\beta \neq \emptyset$, and this follows from the observation that any finite intersection of the L_β 's is nonempty. This proves (b) and the first part of (a). Now, for $x_\alpha \in X_\alpha$, if we set

$$L_\beta(x_\alpha) = \{y \in \prod X_\gamma : \pi_\alpha(y) = x_\alpha \text{ and } f_{\gamma\beta}(y_\beta) = y_\gamma \text{ for } \gamma \preceq \beta\}$$

then we have

$$f_\alpha(X) = \{x_\alpha \in X_\alpha : \bigcap_{\beta \preceq \alpha} L_\beta(x_\alpha) \neq \emptyset\}$$

whence $f_\alpha(X) \subseteq \bigcap_{\beta \succeq \alpha} f_{\alpha\beta}(X_\beta)$. Conversely, if $x_\alpha \in \bigcap_{\beta \succeq \alpha} f_{\alpha\beta}(X_\beta)$ then $L_\beta(x_\alpha) \neq \emptyset$ for all $\beta \succeq \alpha$, whence $\bigcap_{\beta \succeq \alpha} L_\beta(x_\alpha) \neq \emptyset$ by compactness. This finishes the proof. \square

Corollary 1.8.24. *Let $(X_\alpha, f_{\alpha\beta})$ be an inverse system of topological spaces indexed by a directed set, such that for each pair of indices α, β for which $\alpha \preceq \beta$ and for each $x_\alpha \in X_\alpha$, $f_{\alpha\beta}^{-1}(x_\alpha)$ is compact. Then equation (??) is valid and $f_\alpha^{-1}(x_\alpha)$ is compact for each index α and each $x_\alpha \in X_\alpha$.*

Proof. For each $x_\alpha \in \bigcap_{\beta \succeq \alpha} f_{\alpha\beta}(X_\beta)$ and each $\alpha \in I$, let L_β denote $f_{\alpha\beta}^{-1}(x_\alpha)$. If $\alpha \preceq \beta \preceq \gamma$ then we have $f_{\beta\gamma}(L_\gamma) \subseteq L_\beta$ and the set of all indices $\beta \preceq \alpha$ is cofinal in the index set. It follows immediately that the $(L_\beta, f_{\beta\gamma})$ form an inverse system of topological spaces whose inverse limit L is homeomorphic to $f_\alpha^{-1}(x_\alpha)$. Since by hypothesis the L_β are compact and not empty, the corollary follows from ??.

Corollary 1.8.25. *Let $(X_\alpha, f_{\alpha\beta})$ and $(Y_\alpha, g_{\alpha\beta})$ be two inverse systems of topological spaces indexed by the same directed set I , and let (u_α) be an inverse system of mappings $u_\alpha : X_\alpha \rightarrow Y_\alpha$. Let $X = \varprojlim X_\alpha$, $Y = \varprojlim Y_\alpha$ and $u = \varprojlim u_\alpha$. Then:*

- (a) *If $y = (y_\alpha) \in Y$ is such that $u_\alpha^{-1}(y_\alpha)$ is compact and non-empty for each $\alpha \in I$, then $u^{-1}(y)$ is compact and non-empty.*
- (b) *If the X_α are compact Hausdorff, the Y_α are Hausdorff and the u_α surjective and continuous, then u is surjective.*

Proof. Let L_α denote $u^{-1}(y_\alpha)$; then clearly the L_α form an inverse system of topological spaces and $L = u^{-1}(y)$ is the inverse limit of the L_α . Assertion (a) therefore follows from ?? and (b) is an immediate consequence of (a). \square

1.8.4 Arzelá-Ascoli theorem

We now turn to the Arzelá-Ascoli theorem, which has to do with compactness in spaces of continuous mappings. There are several variants of this result; the theorems below are two of the most useful ones.

If X is a topological space and $\mathcal{F} \subseteq \mathbb{K}^X$ is a subset, \mathcal{F} is called **equicontinuous at $x \in X$** if for every $\varepsilon > 0$ there is a neighborhood U of x such that $|f(y) - f(x)| < \varepsilon$ for all $y \in U$ and all $f \in \mathcal{F}$; \mathcal{F} is called **equicontinuous** if it is equicontinuous at each $x \in X$. Also, \mathcal{F} is said to be **pointwise bounded** if $\mathcal{F}(x) = \{f(x) : f \in \mathcal{F}\}$ is bounded for each $x \in X$.

Lemma 1.8.26. *Let X be a topological space. The same topology is induced on equicontinuous subsets \mathcal{F} of $C(X)$ by the point-open topology \mathcal{T}_p and the compact-open topology \mathcal{T}_c .*

Proof. Let $\mathcal{F} \subseteq C(X)$ be equicontinuous. Generally, $\mathcal{T}_p \subseteq \mathcal{T}_c$, so we need only show that $\mathcal{T}_c \cap \mathcal{F} \subseteq \mathcal{T}_p \cap \mathcal{F}$. To this end, let $f_0 \in \mathcal{F}$. A $\mathcal{T}_c \cap \mathcal{F}$ -subbasic neighborhood of x_0 is of the form

$$U_c = \{f \in \mathcal{F} : \|f - f_0\|_K < r\}$$

where $r > 0$ and K is a compact subset of X . We now find $r' > 0$ and $x_1, \dots, x_n \in X$ such that the $\mathcal{T}_p \cap \mathcal{F}$ -neighborhood

$$U_p = \{f \in \mathcal{F} : \|f - f_0\|_{\{x_1, \dots, x_n\}} < r'\} \subseteq U_c.$$

To this end, let K and $r > 0$ be given. Since \mathcal{F} is equicontinuous, there exist neighborhoods V_x of x for each $x \in X$ such that $|f(x) - f(y)| < r/4$ for each $y \in V_x$ and each $f \in \mathcal{F}$. A finite number V_{x_1}, \dots, V_{x_n} cover the compact set K . Now consider $f \in \mathcal{F}$ such that $\|f - f_0\|_{\{x_1, \dots, x_n\}} < r/2$. For $x \in K$, there is an i such that $x \in V_{x_i}$. Thus

$$|f(x) - f_0(x)| \leq |f(x) - f(x_i)| + |f(x_i) - f_0(x_i)| + |f_0(x_i) - f_0(x)| < r/4 + r/2 + r/4 = r.$$

Since $x \in K$ is arbitrary, it follows that $f \in U_c$. □

Lemma 1.8.27. *Let X be a topological space and \mathcal{F} be an equicontinuous subset of \mathbb{K}^X . Then the pointwise closure $\text{cl}_p \mathcal{F}$ of \mathcal{F} in \mathbb{K}^X is an equicontinuous subset of $C(X)$.*

Proof. Let $\mathcal{F} \subseteq \mathbb{K}^X$ be equicontinuous. Given $r > 0$ and $x \in X$, there is an neighborhood V of x such that

$$|f(x) - f(y)| < r \tag{1.8.2}$$

for $y \in V$ and $f \in \mathcal{F}$. We show that (??) holds for each $g \in \text{cl}_p \mathcal{F}$ as well. To this end let $g \in \text{cl}_p \mathcal{F}$ and let $y \in V$. Since $g \in \text{cl}_p \mathcal{F}$, given $r' > 0$, there is an $f \in \mathcal{F}$ such that $\|f - g\|_{\{x, y\}} < r'$. Hence

$$|g(x) - g(y)| \leq |g(x) - f(x)| + |f(x) - f(y)| + |f(y) - g(y)| < r' + r + r'.$$

It follows that $\text{cl}_p \mathcal{F}$ is equicontinuous as well. □

As an immediate consequence of Lemmas ?? and ??, the \mathcal{T}_c -closure of an equicontinuous subset of \mathbb{K}^X is equicontinuous in $C(X, \mathbb{K}, c)$. In our next result we obtain some results connecting pointwise boundedness of equicontinuous sets and compactness.

Theorem 1.8.28 (Arzelá-Ascoli). *Let X be a topological space. If \mathcal{F} is an equicontinuous subset of $C(X, \mathbb{K}, c)$ and is pointwise bounded, then \mathcal{F} is precompact in $C(X, \mathbb{K}, c)$. Conversely, if X is locally compact and \mathcal{F} is precompact in $C(X, \mathbb{K}, c)$ then \mathcal{F} is equicontinuous and pointwise bounded.*

Proof. Suppose first that \mathcal{F} is equicontinuous and pointwise bounded. As such, each $\mathcal{F}(x)$ is precompact by the Heine-Borel theorem. Let $\text{cl}_p \mathcal{F}$ and $\text{cl}_c \mathcal{F}$ denote the \mathcal{T}_p - and \mathcal{T}_c -closures

of \mathcal{F} in $C(X)$, respectively. By ??, the pointwise closure $\text{cl}_p \mathcal{F}$ of \mathcal{F} computed in \mathbb{K}^X is an equicontinuous subset of $C(X)$. Consider the homeomorphism

$$\iota : C(X, \mathbb{K}, p) \rightarrow \mathbb{K}^X, \quad f \mapsto (f(x))_{x \in X}.$$

To see that $\text{cl}_p \iota(\mathcal{F})$ is compact in \mathbb{K}^X , note

$$\text{cl}_p \iota(\mathcal{F}) = \iota(\text{cl}_p \mathcal{F}) = \{(f(x))_{x \in X} : f \in \text{cl}_p \mathcal{F}\} = \prod_{x \in X} (\text{cl}_p \mathcal{F})(x) \subseteq \prod_{x \in X} \text{cl}_{\mathbb{K}} \mathcal{F}(x).$$

It follows from Tychonoff theorem that $\text{cl}_p \iota(\mathcal{F})$ is compact and therefore $\text{cl}_p \mathcal{F}$ is \mathcal{T}_p -compact. Since \mathcal{F} is equicontinuous, $\text{cl}_p \mathcal{F} = \text{cl}_c \mathcal{F}$ by ?? and $\text{cl}_c \mathcal{F}$ is \mathcal{T}_c -compact.

Conversely, suppose that X is locally compact and that $\mathcal{F} \subseteq C(X, \mathbb{K}, c)$ is relatively compact. For any $x \in X$, it is easy to see the evaluation map δ_x is continuous on $C(X, \mathbb{K}, c)$, hence each $\delta_x(\text{cl}_p \mathcal{F})$ is compact. It follows that $\delta_x(\mathcal{F})$ is precompact—hence bounded—for any $x \in X$.

To show equicontinuity of \mathcal{F} , consider $x \in X$ and $r > 0$. Since X is locally compact, we may choose a compact neighborhood K of x . Let $B_{p_K} = \{f \in C(X, \mathbb{K}, c) : \|f\|_K < 1\}$. Since \mathcal{F} is relatively \mathcal{T}_c -compact, there exists $n \in \mathbb{N}$ and $f_i \in \text{cl}_c \mathcal{F}$ with $i = 1, \dots, n$ such that $\{f_i + (r/3)B_{p_K}\}$ cover \mathcal{F} . Thus for each $y \in K$ and $f \in \mathcal{F}$, there is some f_i such that $|f(y) - f_i(y)| < r/3$. Since each f_i is continuous, there exist neighborhoods V_i of x such that

$$|f_i(x) - f_i(y)| < r/3 \quad \text{for } y \in V_i.$$

Now for any $f \in \mathcal{F}$, any $y \in K \cap V_1 \cap \dots \cap V_n$, we have, for some i whose choice depends on f ,

$$|f(x) - f(y)| \leq |f(x) - f_i(x)| + |f_i(x) - f_i(y)| + |f_i(y) - f(y)| < r.$$

Hence the equicontinuity of \mathcal{F} is established. \square

Theorem 1.8.29 (Arzelá-Ascoli). *Let X be a compact Hausdorff space and endow $C(X)$ with the uniform topology. Then a subspace \mathcal{F} in $C(X)$ is totally bounded if and only if \mathcal{F} is equicontinuous and pointwise bounded.*

Proof. If X is compact Hausdorff then $\mathcal{T}_c = \mathcal{T}_u$ and they can be metrized by the supremum norm. Moreover, $C(X, \|\cdot\|_\infty)$ is complete. Since totally boundedness is equivalent to precompactness in complete metric spaces. The claim follows. \square

Corollary 1.8.30. *Let X be a compact Hausdorff space and endow $C(X)$ with the uniform topology. Then a subspace \mathcal{F} in $C(X)$ is compact if and only if it is closed, pointwise bounded and equicontinuous.*

Proposition 1.8.31. *Let X be a σ -compact locally compact Hausdorff space. If $\{f_n\}$ is an equicontinuous, pointwise bounded sequence in $C(X)$, there exist $f \in C(X)$ and a subsequence of $\{f_n\}$ that converges to f uniformly on compact sets.*

Proof. Let $\{K_j\}$ be a exhaustion of X by compact sets (??). By ??, applied to $C(K_1)$, there is a subsequence $\{f_i^1\}$ of $\{f_n\}$ that is uniformly Cauchy on K_1 . Proceeding inductively, for $j \in \mathbb{N}$ we obtain a subsequence $\{f_i^j\}_{i=1}^\infty$ of $\{f_i^{j-1}\}_{i=1}^\infty$ that is uniformly Cauchy on K_j . Let $g_i = f_i^j$; then $\{g_i\}$ is a subsequence of $\{f_n\}$ which is (except for the first $j-1$ terms) a subsequence of

$\{f_i^j\}$ and hence is uniformly Cauchy on each K_j . Let $f = \lim g_i$. Then $f \in C(X)$ and $g_i \rightarrow f$ uniformly on compact sets by Propositions ?? and ??.

1.8.5 Compactification

A **compactification** of a topological space X is a pair (Y, ι) where Y is a compact Hausdorff space and ι is a homeomorphism from X onto a dense subset of Y . Since Hausdorff compact spaces satisfy many properties which are hereditary, compactification is not available for all topological spaces. In this part, we consider completely regular spaces and show that their compactifications correspond to subalgebras of the continuous functions.

Let X be a topological space and $BC(X)$ be the space of bounded continuous functions on X . Without further referring, we endow $BC(X)$ with the supremum norm, which makes it a Banach space. Recall that a subalgebra \mathcal{F} of $C(X, I)$ is said to **separate points and closed sets** if for every closed $E \subseteq X$ and every $x \in E^c$ there exists $f \in \mathcal{F}$ such that $f(x) = 0$ and $f(E) = \{1\}$. It follows that a T_1 space X admits a family \mathcal{F} that separates points and closed sets iff X is completely regular.

Let X be a completely regular space. A subalgebra \mathcal{A} of $BC(X)$ is called **completely regular** if

- (a) it is closed in $BC(X)$ and contains constant functions;
- (b) $\mathcal{A} \cap C(X, I)$ separates points and closed sets.

Due to the remark above, completely regular subalgebra always exists in $BC(X)$. An important example is provided by the following proposition.

Proposition 1.8.32. *Let (Y, ι) be a compactification of X , then $\mathcal{A}_Y = \{f \circ \iota : f \in C(Y)\}$ is a completely regular subalgebra of $BC(X)$.*

Proof. Clearly \mathcal{A}_Y contains constant functions, and it is easy to see $\mathcal{A}_Y \cap C(X, [0, 1])$ separates points and closed sets of X , since Y is completely regular. Now let f_α be a net in $C(Y)$ such that $f_\alpha \circ \iota \rightarrow f$ in $BC(X)$ for some $f \in BC(X)$. We now prove that f_α is Cauchy in $C(Y)$, so that it has a limit $\tilde{f} \in C(Y)$. With this, we immediate see $\tilde{f} \circ \iota = f$, so the proposition is proved.

Now, by definition, for any fixed $\varepsilon > 0$ there exists an index γ such that

$$|f_\alpha(\iota(x)) - f_\beta(\iota(x))| < \varepsilon/3, \quad \forall x \in X, \forall \alpha, \beta \preceq \gamma.$$

Now fix $\alpha, \beta \preceq \gamma$. Since $\iota(X)$ is dense in Y and f_α, f_β are continuous, for any $y \in Y$ there exists $x_0 \in X$ such that

$$|f_\alpha(\iota(x)) - f_\alpha(y)| < \varepsilon/3, \quad |f_\beta(\iota(x)) - f_\beta(y)| < \varepsilon/3,$$

whence

$$|f_\alpha(y) - f_\beta(y)| \leq |f_\alpha(y) - f_\alpha(\iota(x))| + |f_\alpha(\iota(x)) - f_\beta(\iota(x))| + |f_\beta(\iota(x)) - f_\beta(y)| < \varepsilon.$$

Since this is true for all $y \in Y$, we see (f_α) is Cauchy so the claim follows. \square

Note that for a compact Hausdorff Y , Y can be embedded into the dual space $C(Y)^*$ (with the weak* topology) by the evaluation map δ defined by

$$\delta : Y \rightarrow C(Y)^*, \quad y \mapsto \delta_y.$$

With this observation, we now consider two compactifications (Y_1, ι_1) and (Y_2, ι_2) of X . Via the map δ , the spaces Y_1 and Y_2 can be embedded into $C(Y_1)^*$ and $C(Y_2)^*$, respectively. One may wonder in what condition does there exist a continuous map from (Y_1, ι_1) to (Y_2, ι_2) . Instead of thinking Y_1 and Y_2 , let's try to define a map Ψ from $C(Y_1)^*$ to $C(Y_2)^*$. Let $p \in C(Y_1)^*$, we then need to determine how it acts on a function $g \in C(Y_2)$. For this, consider the restriction $g \circ \iota_2 \in \mathcal{A}_{Y_2}$. If we have $g \circ \iota_2 \in \mathcal{A}_1$, then there will be a unique function $\tilde{g} \in C(Y_1)$ such that $\tilde{g} \circ \iota_1 = g \circ \iota_2$, and we can then define

$$p(g) := p(\tilde{g}).$$

In this way, we see if $\mathcal{A}_{Y_2} \subseteq \mathcal{A}_{Y_1}$ then there is a map $\Psi : C(Y_1)^* \rightarrow C(Y_2)^*$. A crucial fact is that the map Ψ is continuous with respect to the weak* topology and this construction is functorial.

Proposition 1.8.33. *Let $(Y_1, \iota_1), (Y_2, \iota_2), (Y_3, \iota_3)$ be compactifications of X .*

- (a) *If $\mathcal{A}_{Y_2} \subseteq \mathcal{A}_{Y_1}$ then there is a continuous map $\Psi : C(Y_1)^* \rightarrow C(Y_2)^*$ which restricts to a continuous map $\psi : Y_1 \rightarrow Y_2$ such that the following diagram commutes:*

$$\begin{array}{ccccc} X & \xrightarrow{\iota_1} & Y_1 & \xhookrightarrow{\delta} & C(Y_1)^* \\ \parallel & & \downarrow \psi & & \downarrow \Psi \\ X & \xrightarrow{\iota_2} & Y_2 & \xhookrightarrow{\delta} & C(Y_2)^* \end{array}$$

If $\Psi' : C(Y_1)^ \rightarrow C(Y_2)^*$ is another map with this property, then $\Psi'|_{Y_1} = \psi$.*

- (b) *If $\mathcal{A}_{Y_3} \subseteq \mathcal{A}_{Y_2} \subseteq \mathcal{A}_{Y_1}$ then the following diagram commutes:*

$$\begin{array}{ccccc} X & \xrightarrow{\iota_1} & Y_1 & \xhookrightarrow{\delta} & C(Y_1)^* \\ \parallel & & \downarrow \psi & & \downarrow \Psi \\ X & \xrightarrow{\iota_2} & Y_2 & \xhookrightarrow{\delta} & C(Y_2)^* \\ \parallel & & \downarrow \phi & & \downarrow \Phi \\ X & \xrightarrow{\iota_3} & Y_3 & \xhookrightarrow{\delta} & C(Y_3)^* \end{array}$$

where Ψ and Φ are the maps in (a).

Proof. Let p_α be a net in $C(Y_1)^*$ which converges weakly to $p \in C(Y_1)^*$. Then for any $g \in C(Y_2)$, we see

$$\Psi(p_\alpha)(g) = p_\alpha(\tilde{g}) \rightarrow p(\tilde{g}) = \Psi(p)(g),$$

which shows $\Psi(p_\alpha) \rightarrow \Psi(p)$ weakly, so Ψ is continuous. Moreover, let $x \in X$, then

$$\Psi(\delta_{\iota_1(x)})(g) = \tilde{g}(\iota_1(x)) = g(\iota_2(x)) = \delta_{\iota_2(x)}(g).$$

Therefore $\Psi \circ \delta \circ \iota_1 = \delta \circ \iota_2$. In particular, since $\iota_1(X)$ is dense in Y_1 , we have

$$\Psi(\delta(Y_1)) = \Psi(\overline{\delta(\iota_1(X))}) \subseteq \overline{\delta(\iota_2(X))} = \delta(Y_2),$$

so Ψ restricts to a map $\psi : Y_1 \rightarrow Y_2$, which is continuous since δ is an embedding for both Y_1 and Y_2 . Finally, if Ψ' is another map making the diagram commutes, then it must coincide with Ψ on $\delta(\iota_1(X))$, hence we have $\Psi'|_{Y_1} = \psi$. This completes the proof of (a), and the validity of (b) follows from the definition of Φ and Ψ . \square

Corollary 1.8.34. *Let (Y_1, ι_1) and (Y_2, ι_2) be compactifications of X with $\mathcal{A}_{Y_1} = \mathcal{A}_{Y_2}$. Then the map $\Psi : C(Y_1)^* \rightarrow C(Y_2)^*$ induces a homeomorphism of (Y_1, ι_1) and (Y_2, ι_2) .*

Now we consider the converse process. With full generality, each nonempty subset $\mathcal{F} \subseteq C(X, I)$ canonically induces a map $\iota : X \rightarrow I^{\mathcal{F}}$ by the formula $\iota(x) = (f(x))_{f \in \mathcal{F}}$. We call ι the map from X into the cube $I^{\mathcal{F}}$ **associated to \mathcal{F}** .

Proposition 1.8.35. *Let X be a topological space, $\mathcal{F} \subseteq C(X, I)$, and $\iota : X \rightarrow I^{\mathcal{F}}$ be the map associated to \mathcal{F} . Then*

- (a) *The map ι is continuous.*
- (b) *If \mathcal{F} separates points, then ι is injective.*
- (c) *If X is T_1 and \mathcal{F} separates points and closed sets, then ι is an embedding.*

Proof. Part (a) and (b) are obvious. Next, observe that if \mathcal{F} separates points and closed sets and X is T_1 , then ι is injective. To prove the continuity of the inverse, suppose that U is open in X . If $x \in U$, choose $f \in \mathcal{F}$ with $f(x) = 0$ and $f(U^c) = \{1\}$, and let $V = \pi_f^{-1}([0, 1))$. Then V is open in $I^{\mathcal{F}}$ and $\iota(x) \in V \cap \iota(X) \subseteq \iota(U)$. Thus $\iota(U)$ is a neighborhood of $\iota(x)$ in $\iota(X)$, so $\iota(U)$ is open in $\iota(X)$. It follows that ι^{-1} is continuous. \square

Corollary 1.8.36. *Every compact Hausdorff space is homeomorphic to a closed subset of a cube.*

Now let \mathcal{A} be a completely regular subalgebra of $BC(X)$ and let $\mathcal{F} = \mathcal{A} \cap C(X, I)$. Then by ??, we have a compactification (Y, ι) associated with \mathcal{F} . We now consider the relation between \mathcal{A} and \mathcal{A}_Y .

Proposition 1.8.37. *Suppose that $\mathcal{F} \subseteq C(X, I)$ separates points and closed sets. Let (Y, ι) be the compactification of X associated to \mathcal{F} . Then \mathcal{A}_Y is the smallest closed subalgebra of $BC(X)$ that contains \mathcal{F} and constant functions.*

Proof. Let \mathcal{A} be the smallest closed subalgebra of $BC(X)$ that contains \mathcal{F} . Let $f \in \mathcal{F}$, then f can be extended to Y by

$$f(p) := \pi_f(p), \quad \forall p \in I^{\mathcal{F}},$$

If $p_\alpha \rightarrow p$ in $I^{\mathcal{F}}$, then we have $f(p_\alpha) = \pi_f(p_\alpha) \rightarrow \pi_f(p) = f(p)$, therefore this extension is continuous. This shows $\mathcal{F} \subseteq \mathcal{A}_Y$, and therefore $\mathcal{F} \subseteq \mathcal{A} \subseteq \mathcal{A}_Y$.

Let π_f be the coordinate map on $I^{\mathcal{F}}$, then by Stone-Weierstrass theorem, the algebra generated by the coordinate projections $\{\pi_f : f \in \mathcal{F}\}$ and the constant functions is dense in $C(I^{\mathcal{F}})$. Moreover, $\pi_f \circ \iota = f$ for each $f \in \mathcal{F}$. Since \mathcal{A} is closed and contains constant functions, it follows that $\mathcal{A} \supseteq \mathcal{A}_Y$, so the claim follows. \square

Theorem 1.8.38. Let X be a completely regular space. The compactifications of X are in one-to-one correspondence with the completely regular subalgebras of $BC(X)$.

Proof. If \mathcal{A} is a completely regular subalgebra, we denote by $(Y_{\mathcal{A}}, \iota)$ the compactification of X associated with $\mathcal{F} := \mathcal{A} \cap C(X, I)$. By the definition of completely regularity and ??, We have $\mathcal{A}_{Y_{\mathcal{A}}} = \mathcal{A}$.

Conversely, for each compactification (Y, ι) , we consider the map $(Y, \iota) \mapsto \mathcal{A}_Y$. We have already seen that \mathcal{A}_Y is completely regular. Moreover, if $Y_{\mathcal{A}_Y}$ is the compactification associated with \mathcal{A}_Y , then we have $\mathcal{A}_{Y_{\mathcal{A}_Y}} = \mathcal{A}_Y$ by ??, which implies Y are homeomorphic to $Y_{\mathcal{A}_Y}$, by ?. These together prove the theorem. \square

1.9 Local compactness

A topological space X is said to be **locally compact** if for every $p \in X$ there is a compact subset of X containing a neighborhood of p . In this generality, the definition is not particularly useful, and does not seem parallel to other definitions of what it means for a topological space to possess a property "locally", which usually entails the existence of a base of open subsets with a particular property. But when combined with the Hausdorff property, local compactness is much more useful. A subset A of a topological space X is said to be **precompact** (or sometimes **relatively compact**) in X if \bar{A} is compact.

Proposition 1.9.1. Let X be a Hausdorff space. The following are equivalent.

- (a) X is locally compact.
- (b) Each point of X has a precompact neighborhood.
- (c) X has a base of precompact open subsets.

Proof. Clearly $(c) \Rightarrow (b) \Rightarrow (a)$, so all we have to prove is $(a) \Rightarrow (c)$. It suffices to show that if X is a locally compact Hausdorff space, then each point $x \in X$ has a neighborhood base of precompact open subsets. Let $K \subseteq X$ be a compact set containing a neighborhood U of x . The collection \mathcal{V} of all neighborhoods of x contained in U is clearly a neighborhood base at x .

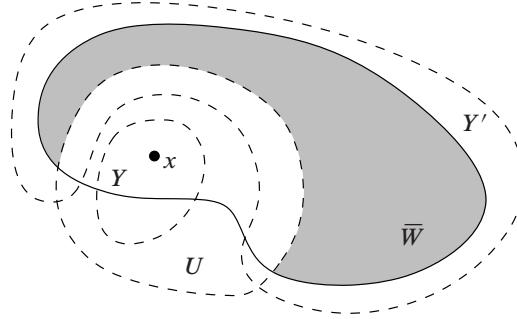
Because X is Hausdorff, K is closed in X . If $V \in \mathcal{V}$, then $\bar{V} \subseteq K$ (because $V \subseteq U \subseteq K$ and K is closed), and therefore \bar{V} is compact (because a closed subset of a compact set is compact). Thus \mathcal{V} is the required neighborhood base. \square

Lemma 1.9.2. Let X be a locally compact Hausdorff space. If $x \in X$ and U is any neighborhood of x , there exists a precompact neighborhood V of x such that $\bar{V} \subseteq U$.

Proof. Suppose U is a neighborhood of x . By ?? there is a precompact neighborhood W of x . Then $\bar{W} \setminus U$ is closed in \bar{W} and therefore compact. Because compact subsets can be separated by open subsets in a Hausdorff space, there are disjoint open subsets Y containing x and Y' containing $\bar{W} \setminus U$. Let $V = Y \cap W$. Because $\bar{V} \subseteq \bar{W}$, it is compact.

Because $V \subseteq Y \subseteq (Y')^c$, we have $\bar{V} \subseteq \overline{(Y')^c} = (Y')^c$. Since $\bar{W} \setminus U$ is contained in Y' , this then implies $\bar{V} \cap (\bar{W} \setminus U) = \emptyset$, and therefore $\bar{V} \subseteq \bar{W} \cap U \subseteq U$. \square

Proposition 1.9.3. Let X be a locally compact Hausdorff space, let K be a compact subset of X , and let U be an neighborhood of K . Then there is a precompact neighborhood V of K such that $\bar{V} \subseteq U$.



Proof. ?? implies that each point in K has a precompact neighborhood whose closure is included in U . Since K is compact, some finite collection of these neighborhoods covers K . Let V be the union of the sets in such a finite collection; then V is the required set. \square

Proposition 1.9.4. *Any open or closed subset of a locally compact Hausdorff space is a locally compact Hausdorff space.*

Proof. Let X be a locally compact Hausdorff space. Note that every subspace of X is Hausdorff, so only local compactness needs to be checked. If $Y \subseteq X$ is open, ?? says that any point in Y has a neighborhood whose closure is compact and contained in Y , so Y is locally compact. Suppose $Z \subseteq X$ is closed. Any $x \in Z$ has a precompact neighborhood U in X . Since $\overline{U \cap Z}$ is a closed subset of the compact set \overline{U} , it is compact. Since $\overline{U \cap Z} \subseteq \overline{Z} = Z$, it follows that $U \cap Z$ is a precompact neighborhood of x in Z . Thus Z is locally compact. \square

A subset of a topological space X is a G_δ if it is the intersection of a sequence of open subsets of X and is an F_σ if it is the union of a sequence of closed subsets of X .

Proposition 1.9.5. *Let X be a second countable locally compact Hausdorff space. Then each open subset of X is an F_σ and is in fact the union of a sequence of compact sets. Likewise, each closed subset of X is a G_δ .*

Proof. Suppose that \mathcal{U} is a countable base for the topology of X . Let U be an open subset of X , and let \mathcal{U}_U be the collection of those sets V in \mathcal{U} for which V is precompact and included in U . ?? implies that U is the union of the closures of the sets in \mathcal{U}_U . Thus U is the union of a countable collection of sets that are closed and, in fact, compact.

Now suppose that C is a closed subset of X . Then C^c is open and so is the union of a sequence $\{F_n\}$ of closed sets. Consequently C is the intersection of the open sets $\{F_n^c\}$. \square

Recall that a topological space (or a subset thereof) is σ -compact if it is the union of a countable collection of compact sets.

Corollary 1.9.6. *Every second countable locally compact Hausdorff space is σ -compact.*

Proof. Since a topological space is an open subset of itself, this is an immediate corollary of ?? \square

1.9.1 Continuous functions

If X is a topological space, the space \mathbb{R}^X of all real-valued functions on X can be topologized in various ways. One way, of course, is the product topology, that is, the topology of pointwise convergence. Another is the **topology of uniform convergence**, which is generated by the sets

$$\{g \in \mathbb{R}^X : \sup_{x \in X} |g(x) - f(x)| < 1/n\}$$

for $f \in \mathbb{R}^X$. Intermediate between these two topologies is the topology of uniform convergence on compact sets, which is generated by the sets

$$\{g \in \mathbb{R}^X : \sup_{x \in K} |g(x) - f(x)| < 1/n\}$$

where $f \in \mathbb{R}^X$ and $K \subseteq X$ is compact. We shall now examine this topology in the case where X is a locally compact Hausdorff space.

Proposition 1.9.7. *If X is a locally compact Hausdorff space, then $C(X)$ is a closed subspace of \mathbb{R}^X in the topology of uniform convergence on compact sets.*

Proof. If f is in the closure of $C(X)$, then f is a uniform limit of continuous functions on each compact $K \subseteq X$, so $f|_K$ is continuous. If $E \subseteq \mathbb{R}$ is closed, then $f^{-1}(E) \cap K = (f|_K)^{-1}(E)$ is closed for each compact K , so $f^{-1}(E)$ is closed (X is compactly generated), whence f is continuous. \square

Proposition 1.9.8. *Let X be a σ -compact locally compact Hausdorff space and $\{K_n\}$ be an exhaustion of X by compact subsets, then for each $f \in \mathbb{R}^X$ the sets*

$$\{g \in \mathbb{R}^X : \sup_{x \in K_n} |f(x) - g(x)| < 1/m\}$$

form a neighborhood base for f in the topology of uniform convergence on compact sets. Hence this topology is first countable, and $f_n \rightarrow f$ uniformly on compact sets if and only if $f_n \rightarrow f$ uniformly on each K_n .

Proof. These assertions follow easily from the observation that if $K \subseteq X$ is compact, then $\{\text{Int } K_n\}$ is an open cover of K and hence $K \subseteq K_n$ for some n . \square

One may wonder if there are enough continuous functions on a locally compact Hausdorff space. The following results answer this question.

Proposition 1.9.9 (Urysohn's Lemma, Locally Compact Version). *Let X be a locally compact Hausdorff space. If K is a compact subset of X and U is a neighborhood of K , then there is a continuous function $f : X \rightarrow [0, 1]$ such that $f \equiv 1$ on K and $f \equiv 0$ outside a compact subset of U .*

Proof. By ??, there exists a precompact neighborhood V of K such that $\overline{V} \subseteq U$. Then V is normal, so by Urysohn's lemma there exists a continuous function $f : \overline{V} \rightarrow [0, 1]$ such that $f \equiv 1$ on K and $f \equiv 0$ on ∂V . We extend f to X by setting $f = 0$ on V .

Now we show that f is a continuous function on X . Suppose that $E \subseteq [0, 1]$ is closed. If $0 \notin E$, then we have $f^{-1}(E) = (f|_{\overline{V}})^{-1}(E)$, and if $0 \in E$ we have $f^{-1}(E) = (f|_{\overline{V}})^{-1}(E) \cup (\overline{V})^c =$

$(f|_{\bar{V}})^{-1}(E) \cup (V)^c$ (note that ∂V is contained in $(f|_{\bar{V}})^{-1}(E)$). In either case, $f^{-1}(E)$ is closed, so f is continuous. \square

Proposition 1.9.10 (Tietze Extension Theorem, Locally Compact Version). *Let X be a locally compact Hausdorff space and K is a compact subset.*

- (a) *Any continuous map of K into a closed interval $[a, b]$ may be extended to a continuous map of all of X into $[a, b]$.*
- (b) *Any continuous map of K into \mathbb{R} may be extended to a continuous map of all of X into \mathbb{R} .*

Moreover, the extension may be taken to vanish outside a compact set.

Proof. We prove (a), and (b) follows as in tietze extension theorem. Let $f : K \rightarrow [a, b]$ be a continuous function. Let V be a precompact neighborhood of K . Then \bar{V} is normal, so by Tietze extension theorem there exists a continuous function $F : \bar{V} \rightarrow [a, b]$ such that $F|_K = f$. Since V is a neighborhood of K , by ?? there is a continuous function $\varphi : X \rightarrow [0, 1]$ such that $\varphi \equiv 1$ on K and $\varphi \equiv 0$ on a compact subset of V . Define $\tilde{F}(x) = F(x)\varphi(x)$. By the choice of φ , it is easy to see \tilde{F} is continuous and $\tilde{F}|_K = f$. Therefore \tilde{F} is the required function. \square

The preceding results show that locally compact Huasdorff spaces have a rich supply of continuous functions that vanish outside compact sets. Let us recall some terminology: If X is a topological space and $f \in C(X)$, the support of f , denoted by $\text{supp}(f)$, is the smallest closed set outside of which f vanishes, that is, the closure of $\{x : f(x) \neq 0\}$. If $\text{supp}(f)$ is compact, we say that f is **compactly supported**, and we define

$$C_c(X) = \{f \in C(X) : \text{supp}(f) \text{ is compact}\}$$

to be the set of continuous functions on X with compact support. Moreover, if $f \in C(X)$, we say that f **vanishes at infinity** if for every $\varepsilon > 0$ the set $\{x : |f(x)| \geq \varepsilon\}$ is compact, and we define

$$C_0(X) = \{f \in C(X) : f \text{ vanishes at infinity}\}.$$

Clearly $C_c(X) \subseteq C_0(X)$. Moreover, $C_0(X) \subseteq BC(X)$, because for $f \in C_0(X)$ the image of the set $\{x : |f(x)| \geq \varepsilon\}$ is compact, and $|f| < \varepsilon$ on its complement.

Proposition 1.9.11. *If X is a locally compact Hausdorff space, then $C_0(X)$ is the closure of $C_c(X)$ in the uniform metric.*

Proof. If $\{f_n\}$ is a sequence in $C_c(X)$ that converges uniformly to $f \in C(X)$, for each $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that $\|f - f_n\| < \varepsilon$. Then $|f(x)| < \varepsilon$ for $x \notin \text{supp}(f_n)$, so $f \in C_0(X)$.

Conversely, if $f \in C_0(X)$, for $n \in \mathbb{N}$ let $K_n = \{x : |f(x)| \geq 1/n\}$. Then K_n is compact, so by ?? there exists $\varphi_n \in C_c(X)$ with $0 \leq \varphi_n \leq 1$ and $\varphi_n \equiv 1$ on K_n . Let $f_n = \varphi_n f$. Then $f_n \in C_c(X)$ and $\|f_n - f\|_\infty \leq 1/n$, so f_n converges to f uniformly. \square

Now we derive some consequences of ?? which will be used later. Let us begin with the following lemma.

Lemma 1.9.12. *Let X be a Hausdorff space, let K be a compact subset of X , and let U_1 and U_2 be open subsets of X such that $K \subseteq U_1 \cup U_2$. Then there are compact sets K_1 and K_2 such that $K = K_1 \cup K_2$, $K_1 \subseteq U_1$ and $K_2 \subseteq U_2$.*

Proof. Let $L_1 = K \setminus U_1$ and $L_2 = K \setminus U_2$. Then L_1 and L_2 are disjoint and compact, and so according to ?? they can be separated by disjoint open sets, say by V_1 and V_2 . If we define K_1 and K_2 by $K_1 = K \setminus V_1$ and $K_2 = K \setminus V_2$, then K_1 and K_2 are compact, are included in U_1 and U_2 , respectively, and have K as their union. \square

Proposition 1.9.13. *Let X be a locally compact Hausdorff space, let f belong to $C_c(X)$, and let U_1, \dots, U_n be open subsets of X such that $\text{supp}(f) \subseteq U = \bigcup_{i=1}^n U_i$. Then there are functions f_1, \dots, f_n in $C_c(X)$ such that $f = f_1 + \dots + f_n$ and such that for each i the support of f_i is contained in U_i . Furthermore, if the function f is nonnegative, then the functions f_1, \dots, f_n can be chosen so that all are nonnegative.*

Proof. In case $n = 1$ we need only let f_1 be f . So we can begin by supposing that $n = 2$. Use ?? to construct compact sets K_1 and K_2 such that $K_1 \subseteq U_1$ and $K_2 \subseteq U_2$, and $\text{supp}(f) = K_1 \cup K_2$, and then use ?? to construct functions h_1 and h_2 that belong to $C_c(X)$ and satisfy $\chi_{K_i} \leq h_i \leq \chi_{U_i}$ and $\text{supp}(h_i) \subseteq U_i$ for $i = 1, 2$. Define functions g_1 and g_2 by $g_1 = h_1$ and $g_2 = h_2 - (h_1 \wedge h_2)$. Then g_1 and g_2 are non-negative, their supports are included in U_1 and U_2 , respectively, and they satisfy

$$g_1 + g_2 = h_1 + h_2 - (h_1 \wedge h_2) = h_1 \vee h_2 = 1.$$

at each x in $\text{supp}(f)$. We can complete the proof in the case where $n = 2$ by defining f_1 and f_2 to be fg_1 and fg_2 , and the general case follows by induction. \square

Proposition 1.9.14. *Let X be a locally compact Hausdorff space, let K_1, \dots, K_n be disjoint compact subsets of X , and let $\alpha_1, \dots, \alpha_n$ be real (or complex) numbers. Then there is a function f that belongs to $C_c(X)$ and satisfies*

- (a) $f(x) = \alpha_i$ if $x \in K_i$, for $i = 1, \dots, n$.
- (b) $\|f\|_\infty = \max\{|a_1|, \dots, |a_n|\}$.

Proof. We begin by constructing disjoint open sets U_1, \dots, U_n such that $K_i \subseteq U_i$ holds for each i . If $n = 2$, such sets are provided by ?.?. The general case follows by induction. Next we use ?? to choose functions f_1, \dots, f_n that belong to $C_c(X)$ and satisfy $\chi_{K_i} \leq f_i \leq \chi_{U_i}$ for $i = 1, \dots, n$. The required function f is now given by $f = \sum_{i=1}^n \alpha_i f_i$. \square

1.9.2 One-point compactification

If X is a noncompact locally compact Hausdorff space, it is possible to make X into a compact space by adding a single point "at infinity" in such a way that the functions in $C_0(X)$ are precisely those continuous functions f such that $f(x) \rightarrow 0$ as x approaches the point at infinity. Here is the precise definition of this space.

Proposition 1.9.15. *Let X be a topological space. Then X is locally compact Hausdorff if and only if there exists a space X^* satisfying the following conditions:*

- (a) X is a subspace of X^* .
- (b) $X^* \setminus X$ consists of a single point.
- (c) X^* is a compact Hausdorff space.

If X_1 and X_2 are two spaces satisfying these conditions, then there is a homeomorphism of X_1 with X_2 that equals the identity map on X .

Proof. We first verify uniqueness. Let X_1 and X_2 be two spaces satisfying these conditions. Define $h : X_1 \rightarrow X_2$ by letting h map the single point p of $X_1 \setminus X$ to the point q of $X_2 \setminus X$, and letting h equal the identity on X . We show that if U is open in X_2 , then $h(U)$ is open in X_1 . Symmetry then implies that h is a homeomorphism.

First, consider the case where U does not contain p . Then $h(U) = U$. Since U is open in X_1 and is contained in X , it is open in X . Because X is open in X_2 , the set U is also open in X_2 , as desired. Second, suppose that U contains p . Since $C = X_1 \setminus U$ is closed in X_1 , it is compact as a subspace of X_1 . Because C is contained in X , it is a compact subspace of X . Then because X is a subspace of X_2 , the space C is also a compact subspace of X_2 , and hence is closed in X_2 . So $h(U) = X_2 \setminus C$ is open, as desired.

Now we suppose X is locally compact Hausdorff and construct the space X^* . Let us take some object that is not a point of X , denote it by the symbol ∞ for convenience, and adjoin it to X , forming the set $X^* = X \cup \{\infty\}$. Topologize X^* by defining the collection of open sets of X^* to consist of (1) all sets U that are open in X , and (2) all sets of the form $X^* \setminus C$, where C is a compact subspace of X . We need to check that this collection is in fact a topology on X^* . The empty set is a set of type (1), and the space X^* is a set of type (2). Checking that the intersection of two open sets is open involves two cases:

$$(X^* \setminus C_1) \cap (X^* \setminus C_2) = X^* \setminus (C_1 \cup C_2), \quad U_1 \cap (X^* \setminus C_1) = U_1 \cap (X \setminus C_1).$$

Similarly, checking that arbitrary union open sets is open involves two cases:

$$\bigcup_{\alpha} (X^* \setminus C_{\alpha}) = X^* \setminus \bigcap_{\alpha} C_{\alpha}, \quad \left(\bigcup_{\alpha} U_{\alpha} \right) \cup \bigcup_{\beta} (X^* \setminus C_{\beta}) = U \cup (X^* \setminus C) = X^* \setminus (C \setminus U).$$

Therefore we have defined a topology on X^* .

Now we show that X is a subspace of X^* . Given any open set of X^* , we show its intersection with X is open in X . If U is of type (1), then $U \cap X = U$; if $U = X^* \setminus C$ is of type (2), then $U \cap X = X \setminus C$; both of these sets are open in X . Conversely, any set open in X is a set of type (1) and therefore open in X^* by definition.

To show that X^* is compact, let \mathcal{U} be an open covering of X^* . The collection \mathcal{U} must contain an open set of type (2), say $X^* \setminus C$, since none of the open sets of type (1) contain the point ∞ . Take all the members of \mathcal{U} different from $X^* \setminus C$ and intersect them with X ; they form a collection of open sets of X covering C . Because C is compact, finitely many of them cover C ; the corresponding finite collection of elements of \mathcal{U} will, along with the element $X^* \setminus C$, cover all of X^* . Therefore X^* is compact.

To show that X^* is Hausdorff, let x and y be two distinct points of X^* . If both of them lie in X , there are disjoint sets U and V open in X containing them, respectively, since X is Hausdorff. On the other hand, if $x \in X$ and $y = \infty$, we can choose a compact set C in X containing a neighborhood U of x . Then U and $X^* \setminus C$ are disjoint neighborhoods of x and ∞ , respectively, in X^* .

Finally, we prove the converse. Suppose a space X^* satisfying conditions (a)–(c) exists. Then X is Hausdorff because it is a subspace of the Hausdorff space X^* . Given $x \in X$, we

show X is locally compact at x . Choose disjoint open sets U and V of X^* containing x and the single point of $X^* \setminus X$, respectively. Then the set $C = X^* \setminus V$ is closed in X^* , so it is a compact subspace of X^* . Since C lies in X , it is also compact as a subspace of X ; it contains the neighborhood U of x . \square

If X itself should happen to be compact, then the space X^* of the preceding theorem is not very interesting, for it is obtained from X by adjoining a single isolated point. However, if X is not compact, then the point of $X^* \setminus X$ is a limit point of X , so that $\bar{X} = X^*$.

Definition 1.9.16. *If Y is a compact Hausdorff space and X is a proper subspace of Y whose closure equals Y , then Y is said to be a compactification of X . If $Y \setminus X$ equals a single point, then Y is called the one-point compactification of X .*

Example 1.9.17. The one-point compactification of the real line \mathbb{R} is homeomorphic with the circle, as you may readily check. Similarly, the one-point compactification of \mathbb{R}^2 is homeomorphic to the sphere S^2 . If \mathbb{R}^2 is looked at as the space \mathbb{C} of complex numbers, then $\mathbb{C} \cup \{\infty\}$ is called the **Riemann sphere**, or the **extended complex plane**.

Proposition 1.9.18. *Let X be a locally compact Hausdorff space and X^* be its one-point compactification. Let $f \in C(X)$, then f extends continuously to X^* if and only if f has a limit at the point infinity. That is, if and only if $f = g + a$ where $g \in C_0(X)$ and a is a constant, in which case the continuous extension is given by $f(\infty) = a$.*

Proof. Let f be the form stated above. We extend f to X^* by defining $f(\infty) = a$. We only need to check that f is continuous at ∞ , since f is in $C(X)$ and every open set of X is open in X^* . Let $\varepsilon > 0$ be arbitrary, then the set $C_\varepsilon = \{x : |g(x)| \geq \varepsilon\}$ is compact in X , and hence $X^* \setminus C_\varepsilon$ is a neighborhood of ∞ in X^* . By definition, this then implies $|f(x) - a| < \varepsilon$ for $x \in X^* \setminus C_\varepsilon$. Therefore f is continuous at ∞ , and so $f \in C(X^*)$.

Conversely, assume that $f \in C(X)$ has an extension \tilde{f} on X^* . Define $a = \tilde{f}(\infty)$ and $g(x) = f(x) - a$. We need to show that $g \in C_0(X)$. Let $\varepsilon > 0$, then there exists a neighborhood U of ∞ in X^* such that $\tilde{f}(U) \subseteq (a - \varepsilon, a + \varepsilon)$. By the topology of X^* , since $\infty \in U$, $X \setminus U$ is then compact in X . Since $\{x : |g(x)| \geq \varepsilon\} = \{x : |f(x) - a| \geq \varepsilon\}$ is contained in $X \setminus U$, it is also compact, as desired. \square

1.9.3 Stone-Weierstrass theorem

In this part we prove a far-reaching generalization of the well-known theorem of Weierstrass to the effect that any continuous function on a compact interval $[a, b]$ is the uniform limit of polynomials on $[a, b]$. In what follows, X will denote a compact Hausdorff space, and we equip the space $C(X)$ with the uniform metric.

A subset \mathcal{A} of $C(X, \mathbb{R})$ or $C(X, \mathbb{C})$ is said to be **separating** if for every $x, y \in X$ with $x \neq y$ there exists $f \in \mathcal{A}$ such that $f(x) \neq f(y)$. \mathcal{A} is called an **algebra** if it is a real (resp. complex) vector subspace of $C(X, \mathbb{R})$ (resp. $C(X, \mathbb{C})$) such that $fg \in \mathcal{A}$ whenever $f, g \in \mathcal{A}$. If $\mathcal{A} \subseteq C(X, \mathbb{R})$, \mathcal{A} is called a **lattice** if $\max(f, g)$ and $\min(f, g)$ are in \mathcal{A} whenever $f, g \in \mathcal{A}$. Since the algebra and lattice operations are continuous, one easily sees that if \mathcal{A} is an algebra or a lattice, so is its closure $\bar{\mathcal{A}}$ in the uniform metric.

Theorem 1.9.19 (Stone-Weierstrass Theorem). *Let X be a compact Hausdorff space. If \mathcal{A} is a separating closed subalgebra of $C(X, \mathbb{R})$, then either $A = C(X, \mathbb{R})$ or $A = \{f \in C(X, \mathbb{R}) : f(x_0) = 0\}$ for some $x_0 \in X$. The first alternative holds iff \mathcal{A} contains the constant functions.*

The proof will require several lemmas. The first one, in effect, proves the theorem when X consists of two points, and the second one is a special case of the classical Weierstrass theorem for $X = [-1, 1]$. After these two we return to the general case.

Lemma 1.9.20. *Consider \mathbb{R}^2 as an algebra under coordinatewise addition and multiplication. Then the only subalgebras of \mathbb{R}^2 are \mathbb{R}^2 , $\{(0, 0)\}$, and the linear span of $(1, 0)$, $(0, 1)$, or $(1, 1)$.*

Proof. The subspaces of \mathbb{R}^2 listed above are evidently subalgebras. If $\mathcal{A} \subseteq \mathbb{R}^2$ is a nonzero algebra and $(0, 0) \neq (a, b) \in \mathcal{A}$, then $(a^2, b^2) \in \mathcal{A}$. If a, b are nonzero and distinct then (a, b) and (a^2, b^2) are linearly independent, so $\mathcal{A} = \mathbb{R}^2$. The other possibilities give the other three subalgebras. \square

Lemma 1.9.21. *For any $\varepsilon > 0$ there is a polynomial P on \mathbb{R} such that $P(0) = 0$ and $||x| - P(x)| < \varepsilon$ for $x \in [-1, 1]$.*

Proof. Consider the Maclaurin series for $(1 - t)^{1/2}$:

$$(1 - t)^{1/2} = 1 - \sum_n c_n t^n.$$

By the ratio test, this series converges for $|t| < 1$. Moreover, by the monotone convergence theorem (applied to counting measure on \mathbb{N}),

$$\sum_n c_n = \lim_{t \rightarrow 1^-} \sum_n c_n t^n = 1 - \lim_{t \rightarrow 1^-} (1 - t)^{1/2} = 1.$$

It follows from the finiteness of $\sum_n c_n$ that the series $1 - \sum_n c_n t^n$ converges absolutely and uniformly on $[-1, 1]$, and its sum is $(1 - t)^{1/2}$ there. Therefore, given $\varepsilon > 0$, by taking a suitable partial sum of this series we obtain a polynomial Q such that $|(1 - t)^{1/2} - Q(t)| < \varepsilon/2$ for $t \in [-1, 1]$. Set $t = 1 - x^2$ and $R(x) = Q(1 - x^2)$, we obtain a polynomial R such that $||x| - R(x)| < \varepsilon/2$ for $x \in [-1, 1]$. In particular, $|R(0)| < \varepsilon/2$. If we set $P(x) = R(x) - R(0)$, we then get the claim. \square

Lemma 1.9.22. *If \mathcal{A} is a closed subalgebra of $C(X, \mathbb{R})$, then $|f| \in \mathcal{A}$ whenever $f \in \mathcal{A}$; hence \mathcal{A} is a lattice.*

Proof. If $f \in \mathcal{A}$ and $f \neq 0$, let $h = f / \|f\|_\infty$. Then h maps X into $[-1, 1]$, so if $\varepsilon > 0$ and P is as in ??, we have $||h| - P \circ h| < \varepsilon$. Since $P(0) = 0$, P has no constant term, so $P \circ h \in \mathcal{A}$ since \mathcal{A} is an algebra. Since \mathcal{A} is closed and ε is arbitrary, we have $|h| \in \mathcal{A}$ and hence $|f| \in \mathcal{A}$. This proves the first assertion, and the second one follows because $\max(f, g)$ and $\min(f, g)$ can be represented by f, g , and $|f - g|$. \square

Lemma 1.9.23. *Suppose \mathcal{A} is a closed lattice in $C(X, \mathbb{R})$ and $f \in C(X, \mathbb{R})$. If for every $x, y \in X$ there exists $h_{xy} \in \mathcal{A}$ such that $h_{xy}(x) = f(x)$ and $h_{xy}(y) = f(y)$, then $f \in \mathcal{A}$.*

Proof. Given $\varepsilon > 0$, for each $x, y \in X$ let

$$U_{xy} = \{z \in X : f(z) < h_{xy}(z) + \varepsilon\}, \quad V_{xy} = \{z \in X : f(z) > h_{xy}(z) - \varepsilon\}.$$

These sets are open and contain x and y . Fix y , then $\{U_{xy} : x \in X\}$ covers X , so there is a finite subcover $\{U_{x_i y}\}$. Let $h_y = \max(h_{x_1 y}, \dots, h_{x_n y})$, then $f < h_y + \varepsilon$ on X and $f > h_y - \varepsilon$ on $V_y = \bigcup_{i=1}^n V_{x_i y}$, which is open and contains y . Thus $\{V_y : y \in X\}$ is another open cover of X , so there is a finite subcover $\{V_{y_j}\}$. Let $h = \min(h_{y_1}, \dots, h_{y_m})$, then $\|f - h\| < \varepsilon$. Since \mathcal{A} is a lattice, $h \in \mathcal{A}$, and since \mathcal{A} is closed and ε is arbitrary, $f \in \mathcal{A}$. \square

Proof of ??. Given $x \neq y \in X$, let $\mathcal{A}_{xy} = \{(f(x), f(y)) : f \in \mathcal{A}\}$. Then \mathcal{A}_{xy} is a subalgebra of \mathbb{R}^2 as in ?? because $f \mapsto (f(x), f(y))$ is an algebra homomorphism. If $\mathcal{A}_{xy} = \mathbb{R}^2$ for all x, y , then Lemmas ?? and ?? imply that $\mathcal{A} = C(X, \mathbb{R})$. Otherwise, there exist x, y for which \mathcal{A}_{xy} is a proper subalgebra of \mathbb{R}^2 . It cannot be $\{(0, 0)\}$ or the linear span of $(1, 1)$ because \mathcal{A} separates points, so by ?? \mathcal{A}_{xy} is the linear span of $(1, 0)$ or $(0, 1)$. In either case there exists $x_0 \in X$ such that $f(x_0) = 0$ for all $f \in \mathcal{A}$. There is only one such x_0 since \mathcal{A} separates points, so if neither x nor y is x_0 , we have $\mathcal{A}_{xy} = \mathbb{R}^2$. Lemmas ?? and ?? now imply that $\mathcal{A} = \{f \in C(X, \mathbb{R}) : f(x_0) = 0\}$. Finally, if \mathcal{A} contains constant functions, there is no x_0 such that $f(x_0) = 0$ for all $f \in \mathcal{A}$, so \mathcal{A} must equal $C(X, \mathbb{R})$. \square

We have stated the Stone-Weierstrass theorem in the form that is most natural for the proof. However, in applications one is typically dealing with a subalgebra \mathcal{B} of $C(X, \mathbb{R})$ that is not closed, and one applies the theorem to $\mathcal{A} = \overline{\mathcal{B}}$. The resulting restatement of the theorem is as follows:

Corollary 1.9.24. *Suppose \mathcal{B} is a separating subalgebra of $C(X, \mathbb{R})$. If there exists $x_0 \in X$ such that $f(x_0) = 0$ for all $f \in \mathcal{B}$, then \mathcal{B} is dense in $\{f \in C(X, \mathbb{R}) : f(x_0) = 0\}$. Otherwise, \mathcal{B} is dense in $C(X, \mathbb{R})$.*

The classical Weierstrass approximation theorem is the special case where X is a compact subset of \mathbb{R}^n and \mathcal{B} is the algebra of polynomials on \mathbb{R}^n (restricted to X); here \mathcal{B} contains the constant functions, so the conclusion is that it is dense in $C(X, \mathbb{R})$.

The Stone-Weierstrass theorem, as it stands, is false for complex-valued functions. For example, the algebra of polynomials in one complex variable is not dense in $C(K)$ for most compact subsets K of \mathbb{C} . (In particular, if $\text{Int } K \neq \emptyset$, any uniform limit of polynomials on K must be holomorphic on $\text{Int } K$.) Here we shall give a simple proof that the function $f(z) = \bar{z}$ cannot be approximated uniformly by polynomials on the unit circle. If $P(z) = \sum_j a_j z^j$, then

$$\int_0^{2\pi} \overline{f(e^{it})} P(e^{it}) dt = \sum_{j=0}^n a_j \int_0^{2\pi} e^{i(j+1)t} dt = 0.$$

Since $|f| = 1$ on the unit circle we have

$$2\pi = \left| \int_0^{2\pi} \bar{f} f dt \right| \leq \left| \int_0^{2\pi} \bar{f}(f - P) dt \right| + \left| \int_0^{2\pi} \bar{f} P dt \right| \leq \int_0^{2\pi} |f - P| dt \leq 2\pi \|f - P\|_\infty.$$

Therefore, $\|f - P\|_\infty \geq 1$ for any polynomial P .

There is, however, a complex version of the Stone-Weierstrass theorem.

Theorem 1.9.25 (Complex Stone-Weierstrass Theorem). *Let X be a compact Hausdorff space. If \mathcal{A} is a separating closed complex subalgebra of $C(X, \mathbb{C})$ that separates points and is closed under complex conjugation, then either $A = C(X, \mathbb{C})$ or $A = \{f \in C(X, \mathbb{C}) : f(x_0) = 0\}$ for some $x_0 \in X$.*

Proof. Since $\operatorname{Re}(f) = (f + \bar{f})/2$ and $\operatorname{Im}(f) = (f - \bar{f})/2i$, the set $\mathcal{A}_{\mathbb{R}}$ of real and imaginary parts of functions in \mathcal{A} is a subalgebra of $C(X, \mathbb{R})$ to which the Stone-Weierstrass theorem applies. Since $\mathcal{A} = \mathcal{A}_{\mathbb{R}} + i\mathcal{A}_{\mathbb{R}}$, the desired result follows. \square

There is also a version of the Stone-Weierstrass theorem for locally compact Hausdorff spaces.

Theorem 1.9.26. *Let X be a locally compact Hausdorff space. If \mathcal{A} is a separating closed subalgebra of $C_0(X, \mathbb{R})$, then either $\mathcal{A} = C_0(X, \mathbb{R})$ or $\mathcal{A} = \{f \in C_0(X, \mathbb{R}) : f(x_0) = 0\}$ for some $x_0 \in X$.*

Proof. If there exists $x_0 \in X$ such that $f(x_0) = 0$ for all $f \in \mathcal{A}$, let Y be the one-point compactification of $X \setminus \{x_0\}$. Then by ?? we can write $\mathcal{A} \subseteq \{f \in C(Y, \mathbb{R}) : f(+\infty) = 0\}$. Since \mathcal{A} separates points on $X \setminus \{x_0\}$, there is no $y \in X \setminus \{x_0\}$ such that $f(y) = 0$ for all $f \in \mathcal{A}$; it follows that \mathcal{A} separates points in Y , and by the Stone-Weierstrass theorem we get $\mathcal{A} = \{f \in C(Y, \mathbb{R}) : f(+\infty) = 0\} = \{f \in C_0(X, \mathbb{R}) : f(x_0) = 0\}$.

If there is no $x_0 \in X$ such that $f(x_0) = 0$ for all $f \in \mathcal{A}$, let Y be the one-point compactification of X . By the same proof we see $\mathcal{A} = C_0(X, \mathbb{R})$. \square

By the same proof as ??, we also have the following complex version.

Theorem 1.9.27. *Let X be a locally compact Hausdorff space. If \mathcal{A} is a separating closed complex subalgebra of $C_0(X, \mathbb{C})$ that separates points and is closed under complex conjugation, then either $A = C(X, \mathbb{C})$ or $A = \{f \in C(X, \mathbb{C}) : f(x_0) = 0\}$ for some $x_0 \in X$.*

1.9.4 Metrizability

Now we consider metrizability of locally compact spaces.

Proposition 1.9.28. *A compact Hausdorff space is metrizable if and only if it is second countable.*

Proof. First suppose that X is compact and metrizable. Then X is separable and so is second countable.

Now suppose that X is a compact Hausdorff space that is second countable and let \mathcal{U} be a countable base. The space X is normal, and so for each pair of sets U, V that belong to \mathcal{U} and satisfy $U \cap V = \emptyset$ there is, by Urysohn's lemma, a continuous function $f : X \rightarrow [0, 1]$ that vanishes on \bar{U} and has value 1 everywhere on \bar{V} . Form a sequence $\{f_n\}$ by choosing one such function for each such pair of sets. Our next step is to check that this sequence of functions separates the points of X , and for this it is enough to show that for each pair x, y of distinct points in X there are sets U and V that belong to \mathcal{U} , have disjoint closures, and contain x and y , respectively. To construct such sets, choose disjoint open neighborhoods W_1 and W_2 of x and y , and use ?? to choose open sets U and V such that $x \in U \subseteq \bar{U} \subseteq W_1$ and $y \in V \subseteq \bar{V} \subseteq W_2$. By making U and V a bit smaller, if necessary, we can assume that they belong to \mathcal{U} . Thus the required sets U and V exist, and the sequence $\{f_n\}$ separates the points of X .

Define a function $d : X \times X \rightarrow \mathbb{R}$ by setting

$$d(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} |f_n(x) - f_n(y)|.$$

It is easy to use the fact that the functions $\{f_n\}$ separate the points of X to check that d is a metric on the set X and to use the fact that the functions $\{f_n\}$ are continuous (with respect to

the original topology on X) to check that the topology induced by d is weaker than the original topology. Since the original topology makes X a compact space, while the topology induced by d is weaker and Hausdorff, the two topologies must be the same (id is a continuous bijection from a compact space to a Hausdorff space). Thus the original topology on X is metrizable and in fact is metrized by d . \square

Our next task is to prove that each second countable locally compact Hausdorff space is metrizable. For this we need the following lemma.

Lemma 1.9.29. *Let X be a locally compact Hausdorff space. If X is second countable, then the one-point compactification of X is also second countable.*

Proof. Let \mathcal{U} be a countable base for the topology of X , and let \mathcal{U}_0 be the collection of those sets V in \mathcal{U} for which V is precompact. Arrange the sets in \mathcal{U}_0 in a sequence, say $\{V_k\}$. Then $X = \bigcup_k V_k$, since X is locally compact, and so for each compact subset K of X there is a positive integer n such that $K \subseteq \bigcup_{k=1}^n V_k$. Thus if U is an open neighborhood in X^* of the point at infinity and if $K = X^* \setminus U$, then there is a positive integer n such that $K \subseteq \bigcup_{k=1}^n V_k$ and hence such that $X^* \setminus \overline{\bigcup_{k=1}^n V_k} \subseteq U$. It follows that the sets in U , together with the sets $X^* \setminus \overline{\bigcup_{k=1}^n V_k}$, form a countable base for the topology of X . \square

Proposition 1.9.30. *Each second countable locally compact Hausdorff space is metrizable.*

Proof. Let X be a second countable locally compact Hausdorff space. ?? and ?? imply that the one-point compactification X^* of X is metrizable. Then X , as a subspace of the metrizable space X^* , is metrizable. \square

1.10 Proper maps

1.10.1 Proper maps

If f and g are closed continuous maps, it does not follow that $f \times g$ is closed, even if f is the identity map. An example is given below.

Example 1.10.1. Every constant mapping into a Hausdorff space is closed. But if f is the constant mapping $\mathbb{Q} \rightarrow \{0\}$, then $f \times \text{id}_{\mathbb{Q}}$ is the mapping $(x, y) \mapsto (0, y)$ of \mathbb{Q}^2 into \mathbb{Q}^2 , so it is the second projection and is not closed.

Definition 1.10.2. A map $f : X \rightarrow Y$ is said to be *proper* if f is continuous and if the mapping $f \times \text{id}_Z : X \times Z \rightarrow Y \times Z$ is closed, for every topological space Z .

If in the definition we take the space Z to consist of a single point, then we see that any proper map is closed. A partial converse holds when the map f is injective:

Proposition 1.10.3. *Let $f : X \rightarrow Y$ be a continuous injection. Then the following three statements are equivalent.*

- (i) f is proper.
- (ii) f is closed.
- (iii) f is a homeomorphism of X onto a closed subset of Y .

Proof. We have just seen that (i) implies (ii). Since the equivalence relation $f(x) = f(y)$ is the equality relation, the quotient space of X with respect to this relation can be identified with X ; hence (ii) implies (iii) by ?? . Finally, if (iii) is satisfied then $f \times \text{id}_Z$ is a homeomorphism of $X \times Z$ onto a closed subspace of $Y \times Z$ and is therefore a closed mapping; hence (iii) implies (i). \square

Proposition 1.10.4. *Let $f : X \rightarrow Y$ be a continuous map. If B is any subset of Y , let f_B denote the restriction of f on $f^{-1}(B)$.*

- (a) *If f is proper, so is f_B .*
- (b) *Let $(B_i)_{i \in I}$ be a family of subsets of Y whose interiors cover Y , or which is a locally finite closed covering of Y ; then if each of the mappings f_B is proper, so is f .*

Proof. Let Z be a topological space. If B is any subset of Y , we have

$$f_B \times \text{id}_Z = (f \times \text{id}_Z)_{B \times Z}.$$

If f is proper, then $f \times \text{id}_Z$ is closed, hence so is $(f \times \text{id}_Z)_{B \times Z}$ (??), whence (a) is proved. If now $(B_i)_{i \in I}$ satisfies one of the two conditions stated in (b), then the covering $(B_i \times Z)_{i \in I}$ of $Y \times Z$ has the same property; if the f_{B_i} are proper then the mappings $(f \times \text{id}_Z)_{B \times Z}$ are closed, whence $f \times \text{id}_Z$ is closed (??). This completes the proof. \square

Proposition 1.10.5. *Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be two continuous mappings.*

- (a) *If f and g are proper, then $g \circ f$ is proper.*
- (b) *If $g \circ f$ is proper and f is surjective, then g is proper.*
- (c) *If $g \circ f$ is proper and g is injective, then f is proper.*
- (d) *If $g \circ f$ is proper and Y is Hausdorff, then f is proper*

Proof. \square

1.10.2 Proper maps

If X and Y are topological spaces, a map $f : X \rightarrow Y$ (continuous or not) is said to be **proper** if the preimage of each compact subset of Y is compact. For example, every exhaustion function $f : X \rightarrow \mathbb{R}$ is a proper map.

In order to visualize the behavior of proper maps, it is useful to introduce the following definition: if X is a topological space, a sequence (x_i) in X is said to **diverge to infinity** if for every compact set $K \subseteq X$ there are at most finitely many values of i for which $x_i \in K$.

Lemma 1.10.6. *Suppose X is a first countable Hausdorff space. A sequence in X diverges to infinity if and only if it has no convergent subsequence.*

Proof. Assume first that (x_i) is a sequence in X that diverges to infinity. If there is a subsequence (x_{i_j}) that converges to $x \in X$, then the set $K = \{x\} \cup \{x_{i_j}\}$ is compact and contains infinitely many terms of the sequence, which is a contradiction. (This implication does not require the hypothesis that X is first countable and Hausdorff.)

Conversely, assume that (x_i) has no convergent subsequence. If $K \subseteq X$ is any compact set that contains x_i for infinitely many i , then there is a subsequence (x_{i_j}) such that $x_{i_j} \in K$ for all j . Because a compact, first countable Hausdorff space is sequentially compact, this subsequence in turn has a convergent subsequence, which is a contradiction. \square

Proposition 1.10.7. *Suppose X and Y are topological spaces and $f : X \rightarrow Y$ is a proper map. Then f takes every sequence diverging to infinity in X to a sequence diverging to infinity in Y .*

Proof. Suppose (x_i) is a sequence in X that diverges to infinity. If $(f(x_i))$ does not diverge to infinity, then there is a compact subset $K \subseteq Y$ that contains $f(x_i)$ for infinitely many values of i . It follows that x_i lies in the compact set $f^{-1}(K)$ for these values of i , which contradicts the assumption that (x_i) diverges to infinity. \square

Because the definition of properness is sometimes tricky to check directly, it is useful to have other sufficient conditions for a map to be proper. The next proposition gives several such conditions.

Proposition 1.10.8. *Suppose X and Y are topological spaces, and $f : X \rightarrow Y$ is a continuous map.*

- (a) *If X is compact and Y is Hausdorff, then f is proper.*
- (b) *If X is a second countable Hausdorff space and f takes sequences diverging to infinity in X to sequences diverging to infinity in Y , then f is proper.*
- (c) *If f is a closed map with compact fibers, then f is proper.*
- (d) *If f is a topological embedding with closed image, then f is proper.*
- (e) *If Y is Hausdorff and f has a continuous left inverse, then f is proper.*
- (f) *If f is proper and $A \subseteq X$ is any subset that is saturated with respect to f , then $f|_A : A \rightarrow f(A)$ is proper.*

Proof. Suppose X is compact and Y is Hausdorff. If $K \subseteq Y$ is compact, then it is closed in Y because Y is Hausdorff. By continuity, $f^{-1}(K)$ is closed in X and therefore compact.

Assume X is a second countable Hausdorff space, and suppose $f : X \rightarrow Y$ takes sequences diverging to infinity to sequences diverging to infinity. Let $K \subseteq Y$ be any compact set, and let $L = f^{-1}(K) \subseteq X$. Because of our hypothesis on X , to show that L is compact, it suffices to show that it is sequentially compact. Suppose on the contrary that (x_i) is a sequence in L with no convergent subsequence. It diverges infinity by ??, so our assumption about f implies that $(f(x_i))$ diverges to infinity. But this is impossible because $f(x_i)$ lies in the compact set K for all i .

Assume f is a closed map with compact fibers. Let $K \subseteq Y$ be compact, and let \mathcal{U} be a cover of $f^{-1}(K)$ by open subsets of X . If $y \in K$ is arbitrary, the fiber $f^{-1}(y)$ is covered by finitely many of the sets in \mathcal{U} , say U_1, \dots, U_k . The set $A_y = X - \bigcup_{i=1}^k U_i$ is closed in X and disjoint from $f^{-1}(y)$, so $V_y = Y - f(A_y)$ is open in Y and contains y . It follows from our construction that

$$f^{-1}(V_y) = f^{-1}(Y - f(A_y)) \subseteq X - A_y = U_1 \cap \dots \cap U_k.$$

Because K is compact, it is covered by finitely many of the sets V_y . Thus $f^{-1}(K)$ is covered by finitely many sets of the form $f^{-1}(V_y)$, each of which is covered by finitely many of the sets in \mathcal{U} , so it follows that $f^{-1}(K)$ is compact.

Now (d) follows from (c), because a topological embedding with closed image is a closed map, and its fibers are singletons, which are certainly compact.

Assume that Y is Hausdorff and $g : Y \rightarrow X$ is a continuous left inverse for f , and suppose K is a compact subset of Y . Then we observe that

$$x \in f^{-1}(K) \Rightarrow x = g(f(x)) \in g(K)$$

Since K is closed in Y (because Y is Hausdorff), it follows that $f^{-1}(K)$ is a closed subset of the compact set $G(K)$, so it is compact.

Finally, to prove (f), suppose $f : Y \rightarrow X$ is proper and $A \subseteq X$ is saturated. Let $K \subseteq f(A)$ be compact. The fact that A is saturated means that $(f|_A)^{-1}(K) = f(K)$, which is compact because f is proper. \square

A topological space X is said to be **compactly generated** if it has the following property:

$$A \text{ is closed in } X \iff A \cap K \text{ is closed in } X \text{ for any compact subset } K$$

It is easy to see that an equivalent definition is obtained by substituting open for closed.

Lemma 1.10.9. *First countable spaces and locally compact spaces are compactly generated.*

Proof. Let X be a space satisfying either of the two hypotheses, and let $A \subseteq X$ be a subset whose intersection with each compact set $K \subseteq X$ is closed in K .

First assume that X is first countable, and let $x \in \bar{A}$. By the sequence lemma, there is a sequence (a_i) of points in A converging to x . The set $K = \{x\} \cup \{a_i : i \in \mathbb{N}\}$ is compact, so $A \cap K$ is closed in K by hypothesis. Since x is the limit of a sequence of points in $A \cap K$, it must also be in $A \cap K \subseteq A$. Thus A is closed.

Now assume X is locally compact. Let $x \in A^c$, we want to find a neighborhood of x disjoint from A . Now let K be a compact subset of X containing a neighborhood U of x . If $U \cap A = \emptyset$, then we are done. Otherwise let $\tilde{U} = U \setminus (A \cap K)$. Since $A \cap K$ is closed, \tilde{U} is open in X . This is the desired neighborhood of x . \square

Theorem 1.10.10. *Suppose X is any topological space, Y is a compactly generated Hausdorff space and $F : X \rightarrow Y$ is a proper continuous map. Then F is a closed map.*

Proof. Let $A \subseteq X$ be a closed subset. We show that $F(A)$ is closed in Y by showing that its intersection with each compact subset is closed. If $K \subseteq Y$ is compact, then $F^{-1}(K)$ is compact since F is proper, and so is $A \cap F^{-1}(K)$ because it is a closed subset of a compact set. By the main theorem on compactness, $F(A \cap F^{-1}(K))$ is compact as well. This set is equal to $F(A) \cap K$. Because K is Hausdorff, $F(A) \cap K$ is closed in K . \square

Corollary 1.10.11. *If X is a topological space and Y is a compactly generated Hausdorff space, an embedding $F : X \rightarrow Y$ is proper if and only if it has closed image.*

Proof. If F is proper, then by ?? F is closed, so $F(X)$ is closed. Conversely, if $F(X)$ is closed, then by ??(d) we know F is proper. \square

Corollary 1.10.12. Suppose F is a proper continuous map from a topological space to a compactly generated Hausdorff space.

- (a) If F is surjective, it is a quotient map.
- (b) If F is injective, it is a topological embedding.
- (c) If F is bijective, it is a homeomorphism.

1.11 Baire Spaces

1.11.1 Nowhere dense sets and meager sets

We develop the elementary properties of nowhere dense sets here.

Definition 1.11.1. A subset A of a topological space X is called **nowhere dense** if its closure has empty interior.

Before giving some examples, we consider some elementary properties of nowhere dense sets.

Proposition 1.11.2. In a topological space X :

- (a) $A \subseteq X$ is nowhere dense iff $(\bar{A})^c$ is dense.
- (b) $A \subseteq X$ is nowhere dense iff every nonempty open subset U of X contains a nonempty open subset that is disjoint from A .

Proof. Assume that A is nowhere dense. If $A = \emptyset$, the result is clear so suppose that $A \neq \emptyset$. Then any nonempty open subset U of X meets the dense subset $(\bar{A})^c$ and the nonempty open set $U \cap (\bar{A})^c$ is disjoint from A . Conversely, the condition says that any nonempty open set meets $(\bar{A})^c$ which means that $(\bar{A})^c$ is dense and so A is nowhere dense by (a). \square

We point out that when we speak of a set $A \subseteq X$ as being nowhere dense, we mean with respect to X 's topology, not the topology it induces on X —a nonempty set X is never a nowhere dense subset of itself, because $\text{cl}_A A = A$.

Example 1.11.3.

- (a) Any subset of a nowhere dense set is nowhere dense. The empty set is nowhere dense; so are singletons in any Hausdorff space as long as the point is not isolated. The only nowhere dense subset of a discrete space is \emptyset .
- (b) The rationals \mathbb{Q} are not nowhere dense in \mathbb{R} , but the integers \mathbb{Z} are.
- (c) The set $\{1/n : n \in \mathbb{N}\} \cup \{0\}$ is nowhere dense in \mathbb{R} .
- (d) The boundary ∂A of a set A is the set $\bar{A} \cap \overline{(A^c)}$. If A is open, then $\partial A \subseteq A^c$ while a neighborhood of any point in ∂A must meet A and A^c . Thus ∂A has no interior. By the same reasoning, if A is closed, $\partial A \subseteq A$ and $\text{Int } \partial A = \emptyset$. Two things follow: a closed set is nowhere dense iff it coincides with its boundary; and boundaries of open or closed sets are nowhere dense.

- (e) A linear subspace M of a topological vector space X is nowhere dense or dense. Note that the closure of a subspace is again a subspace. If M is not nowhere dense then $\text{Int } \bar{M} \neq \emptyset$. By translation, it follows immediately that 0 is an interior point of \bar{M} . Thus \bar{M} is a neighborhood of 0 and is therefore equal to X .
- (f) In a Hausdorff topological vector space, finite-dimensional subspaces are closed. Thus, by (e), proper finite-dimensional subspaces are nowhere dense.

Now we present some elementary properties of nowhere dense sets.

Proposition 1.11.4. *Let X be a topological space.*

- (a) *Subsets of nowhere dense sets are nowhere dense.*
- (b) *Finite unions of nowhere dense sets are nowhere dense.*
- (c) *Suppose that $A \subseteq S \subseteq X$. If A is nowhere dense in S , then A is nowhere dense in X . Conversely, if S is open in X and A is nowhere dense in X , then A is nowhere dense in S .*

Proof. If A is nowhere dense then so is any $B \subseteq A$ since $\bar{B} \subseteq \bar{A}$. The key fact is that finite intersections of dense open sets are dense. To prove this, suppose that A and B are dense open sets. Thus, if U is a nonempty open set, $U \cap A \neq \emptyset$. As $U \cap A$ is a nonempty open set, it meets B nontrivially. Thus $A \cap B$ is dense.

Suppose that $A \subseteq S \subseteq X$ and that A is not nowhere dense in X . Then there is a nonempty open set U in X that does not meet $(\bar{A})^c$, i.e., such that $U \subseteq \bar{A}$ and $U \cap S \subseteq \bar{A} \cap S = \text{cl}_S A$. Since $U \cap A \subseteq U \cap S$ and $U \cap A \neq \emptyset$ because $U \subseteq \bar{A}$, this shows that $\text{cl}_S A$ has nonempty interior, i.e., that A is not nowhere dense in S .

For the converse, suppose that A is nowhere dense in X , S is open and U is a nonempty open subset of S . Since S is open in X , so is U . Since A is nowhere dense in X , U must contain a nonempty open subset V of X —hence also of S —that does not meet A . It follows from ?? that A is nowhere dense in S . \square

Corollary 1.11.5. *Let A be a nowhere dense subset in X and $U \subseteq X$ be open. Then $U \cap A$ is nowhere dense in U .*

Definition 1.11.6. *A subset of a topological space X is called **meager in X** if it can be written as a countable union of nowhere dense sets of X , or, equivalently, is a subset of such a union. Otherwise, E is called **nonmeager in X** . A topological space is called **meager** if it is a meager subset of itself.*

Example 1.11.7 (Examples of Meager and Nonmeager Sets).

- (a) As we can write the rationals \mathbb{Q} as a countable union of singletons, \mathbb{Q} is a meager subset of \mathbb{R} . Indeed, any countable Hausdorff space without isolated points is meager.
- (b) Any topological space which contains an isolated point x is nonmeager, as no set to which x belongs can be nowhere dense. Thus discrete spaces are never meager.
- (c) Singletons are always nonmeager subspaces. A singleton is a nonmeager subset of a topological space iff the point is isolated.

(d) View $\mathbb{Q} \times \mathbb{Q}$ together with the x -axis as a subspace of \mathbb{R}^2 . As the x -axis is a nowhere dense subset of \mathbb{R}^2 and we can enumerate $\mathbb{Q} \times \mathbb{Q}$, $(\mathbb{Q} \times \mathbb{Q}) \cup (\mathbb{R} \times \{0\})$ is a meager subset of \mathbb{R}^2 . However, as we will see, $\mathbb{R} \times \{0\}$ itself is a nonmeager subspace. The point is that the presence of a nonmeager subspace does not imply that the space itself is nonmeager.

An important property of meager sets is that any open subset of a meager set is again meager.

Proposition 1.11.8. *Let $U \subseteq X$ be an open subset. If X is meager, then so is U .*

Proof. If there are nowhere dense subsets $\{E_n\}$ of X such that $X = \bigcup_n E_n$, then $U = \bigcup_n (E_n \cap U)$. As U is open in X , each $E_n \cap U$ is nowhere dense in U by ?? and therefore U is meager. \square

It is then interesting that whether any open subsets of a nonmeager space is nonmeager. For this, we recall that empty set is meager, so we may concentrate on nonempty open subsets. It turns out that this property is equivalent to some other conditions, and is not true in general. The Baire spaces that we will define are exactly nonmeager spaces satisfying these conditions.

Theorem 1.11.9. *The following conditions on a topological space X are equivalent:*

- (i) *The union of countably many closed nowhere dense sets have empty interior.*
- (ii) *The intersection of countably many dense open sets is dense.*
- (iii) *Each nonempty open subset is nonmeager.*
- (iv) *Each meager subsets has dense complement.*

A **Baire space** is one which satisfies either of the conditions above.

Proof. It is clear that (i) \Leftrightarrow (ii). Assume (ii) and let $U \subseteq X$ be a meager open set so that there are nowhere dense subset $\{E_n\}$ such that $U = \bigcup_n E_n \subseteq \bigcup_n \bar{E}_n$. Then $U^c \supseteq \bigcap_n (\bar{E}_n)^c$ and since each $(\bar{E}_n)^c$ is dense and open, $\bigcap_n (\bar{E}_n)^c$ must be dense by hypothesis. Since U^c is closed, it follows that $U^c = X$ or $U = \emptyset$.

Suppose that nonempty open subsets are nonmeager in X . Let $A \subseteq X$ be meager, then $\text{Int } A$ must be meager, too. By hypothesis $\text{Int } A = \emptyset$ and therefore A^c is dense. This proves (iii) \Rightarrow (iv).

Finally, assume (iv). Let $\{E_n\}$ be a sequence of closed nowhere dense sets and let $E = \bigcup_n E_n$. Since E is meager, E^c is dense by hypothesis. Therefore $\text{Int } E = \emptyset$. \square

It is clear that every open subset of a Baire space is Baire. Thus a Baire space is **locally Baire**. It turns out that the converse is also true.

Proposition 1.11.10. *If each point in a topological space X has a neighborhood which is a Baire space, then X is a Baire space.*

Proof. Let $U \subseteq X$ be a nonempty open subset and let $x \in U$. By hypothesis, x has an open neighborhood V which is a Baire space. If U were meager in X , $U \cap V$ would be meager in V and open in V , contradicting the Baireness of V . \square

Example 1.11.11. Obviously, Baire spaces are nonmeager. To see that nonmeager spaces need not be Baire, consider the metric space $\mathbb{Q} \cup (0, 1)$. Since it contains the open nonmeager subspace $(0, 1)$, it is nonmeager. Since it has meager open sets such as $\{x \in \mathbb{Q} : x > 1\}$, it is not a Baire space. Note also that this is an incomplete metric space which is nonmeager.

Some interesting properties for Baire spaces are given below.

Proposition 1.11.12. *Let X be a Baire space and $\{f_s : s \in S\}$ a family of lower semicontinuous real-valued functions on X that is pointwise bounded from above on X . Then each nonempty open subset of X contains a nonempty open subset on which $\{f_s : s \in S\}$ is bounded from above.*

Proof. It suffices to prove the claim for X . The map $f = \sup_s f_s$ is lower semicontinuous. Decompose X into the union of the closed sets $F_n = f^{-1}([-n, n])$. Since X is nonmeager, some F_n has a nonempty interior. The theorem now follows. \square

Note that ?? remain valid for complex-valued functions, and even for functions $\{f_s : s \in S\}$ with values in a normed space by considering the functions $\|f_s\|$.

Proposition 1.11.13. *Let X be a space; let (Y, d) be a metric space. Let $f_n : X \rightarrow Y$ be a sequence of continuous functions such that $f_n(x) \rightarrow f(x)$ for all $x \in X$, where $f : X \rightarrow Y$. If X is a Baire space, the set of points at which f is continuous is dense in X .*

Proof. Given a positive integer N and given $\varepsilon > 0$, define

$$E_N(\varepsilon) = \{x \in X : d(f_n(x), f_m(x)) \leq \varepsilon \text{ for } n, m \geq N\}.$$

Note that $E_N(\varepsilon)$ is closed in X . For the set of those x for which $d(f_n(x), f_m(x)) \leq \varepsilon$ is closed in X , by continuity of f_n and f_m , and $E_N(\varepsilon)$ is the intersection of these sets for all $n, m \geq N$. For fixed ε , since $f_n \rightarrow f$, we have $\bigcup_N E_N(\varepsilon) = X$. Now let

$$U(\varepsilon) = \bigcup_{N=1}^{\infty} \text{Int } E_N(\varepsilon).$$

We claim that $U(\varepsilon)$ is open and dense. To show this, it suffices to show that for any nonempty open set V of X , there is an N such that the set $V \cap \text{Int } E_N(\varepsilon)$ is nonempty. For this purpose, we note first that for each N , the set $V \cap E_N(\varepsilon)$ is closed in V . Because V is a Baire space, at least one of these sets, say $V \cap E_M(\varepsilon)$, must contain a nonempty open set W of V . Because V is open in X , the set W is open in X ; therefore, it is contained in $\text{Int } E_M(\varepsilon)$.

Now we show that f_n converges to f uniformly on the set $C = \bigcap_n U_n(1/n)$, which is dense since X is a Baire space. To this end, let $\varepsilon > 0$ be fixed. Choose $N \in \mathbb{N}$ such that $1/N < \varepsilon$. Then since $C \subseteq U_N(1/N)$, we have $d(f_n(x), f_m(x)) < \varepsilon$ for $n, m \geq N$ and $x \in C$. Let $m \rightarrow +\infty$, this implies $d(f_n(x), f(x)) < \varepsilon$ for $n \geq N$, thus the claim follows. \square

1.11.2 Baire category theorem

Proposition 1.11.14 (Criterion for Baireness). *A topological space (X, \mathcal{T}) is a Baire space if there exists a relation \preceq among the collection \mathcal{T}^* of nonempty open subsets of X such that for $A, B, C, D \in \mathcal{T}^*$,*

- (a) *if $A \preceq B$, then $A \subseteq B$;*

- (b) for every $B \in \mathcal{T}^*$, there is an $A \in \mathcal{T}^*$ such that $A \preceq B$;
- (c) if $A \subseteq B \preceq C \subseteq D$, then $A \preceq D$;
- (d) if $\{A_n\}$ is a sequence of nonempty open sets such that $A_n \succeq A_{n+1}$ for each $n \in \mathbb{N}$, then $\bigcap_n A_n \neq \emptyset$.

Proof. Suppose that the relation \preceq satisfies (a)–(d) is defined on the nonempty open sets \mathcal{T}^* of the topological space (X, \mathcal{T}) . If X is not a Baire space, it must contain a nonempty meager open set U . Consequently, there must be a sequence $\{E_n\}$ of closed nowhere dense sets whose union contains U . We now construct a sequence $\{U_n\}$ of open subsets of U such that $U_n \succeq U_{n+1}$ and $U_n \cap (E_1 \cup \dots \cup E_n) = \emptyset$ for each $n \in \mathbb{N}$. With this, we see

$$\bigcap_{n=1}^{\infty} U_n = U \cap \bigcap_{n=1}^{\infty} U_n \subseteq \bigcap_{n=1}^{\infty} U_n \cap \bigcup_{n=1}^{\infty} E_n = \emptyset,$$

which contradicts condition (d).

Now we construct $\{U_n\}$. Since $\text{Int } E_1 = \emptyset$, certainly $U \not\subseteq E_1$. Hence $U \cap E_1^c$ is a nonempty open subset of U which does not meet E_1 . By (b), there is a nonempty open set U_1 such that $U_1 \preceq U \cap E_1^c$. By (c) we have $U_1 \preceq U$, so $U_1 \subseteq U$ and $U_1 \cap E_1 = \emptyset$. Since U_1 is a nonempty open set, $U_1 \not\subseteq (E_1 \cup E_2)$ and there must be a nonempty open set U_2 such that $U_2 \preceq U_1$ and $U_2 \cap (E_1 \cup E_2) = \emptyset$. By induction, the sequence $\{U_n\}$ mentioned above is now seen to exist. \square

We show next that a complete pseudometric space or locally compact Hausdorff space is a Baire space by showing that a relation such as the one of ?? may be defined on the nonempty open sets.

Theorem 1.11.15 (Baire Category Theorem). *Any locally compact Hausdorff space or complete pseudometric space is Baire.*

Proof. We first consider locally compact Hausdorff spaces. For nonempty open sets A and B of a locally compact Hausdorff space X , define $A \preceq B$ if A is precompact and $\bar{A} \subseteq B$. In view of ??, properties (a)–(c) of ?? are easy to verify. As for (d), suppose that $\{U_n\}$ is a sequence of nonempty open sets such that $U_n \succeq U_{n+1}$. The sets $\{\bar{U}_n : n > 2\}$ are nested in the compact space \bar{U}_2 , so they have nonempty intersection. It then follows that

$$\emptyset \neq \bigcap_{n=3}^{\infty} \bar{U}_n \subseteq \bigcap_{n=2}^{\infty} U_n \subseteq \bigcap_{n=1}^{\infty} U_n.$$

Thus X is Baire by ??.

Now we turn to complete pseudometric spaces. For any subset A of a complete pseudometric space (X, d) , let

$$d(A) = \text{diam}(A), \quad d^*(A) = \min\{1, d(A)\}$$

and note that $A \subseteq B$ implies $d^*(A) \leq d^*(B)$. For $A, B \in \mathcal{T}^*$, we define $A \preceq B$ iff $\bar{A} \subseteq B$ and $d(A) \leq (1/2)d^*(B)$. We show that \preceq satisfies conditions of ???. That $A \preceq B$ implies $A \subseteq B$ is clear. Moreover, for any $A \in \mathcal{T}^*$, let $x \in A \in \mathcal{T}^*$ and choose $0 < r < 1$ such that $B_r(x) \subseteq A$.

With $r' = d(B_r(x)) \leq r < 1$ and $U = B_{r'/4}(x)$, we have $U \preceq A$. As for (c), if $A \subseteq B \preceq C \subseteq D$, then $\bar{A} \subseteq C \subseteq D$; since $d(A) \leq (1/2)d^*(C) \leq (1/2)d^*(D)$, it follows that $A \preceq D$.

We now use the completeness of X to show that (d) is satisfied. For this purpose, let $\{U_n\}$ be a sequence of nonempty open sets, decreasing with respect to \preceq . Since $d^*(U_1) \leq 1$, we have $d(U_n) \leq 2^{-n+1}$ for each $n \in \mathbb{N}$. Hence, choosing $x_n \in U_n$ for each $n \in \mathbb{N}$ yields a d -Cauchy sequence. Since X is complete, $\{x_n\}$ has a limit $x \in X$. Since $x_m \in U_n$ for $m \geq n$, we have $x \in \bar{U}_n$ for each n , and thus $x \in \bigcap_n U_n$. \square

Corollary 1.11.16. *In a locally compact Hausdorff space or a complete metric space, every nonempty countable closed subset contains at least one isolated point.*

Proof. Assume X is such a space. Let $A \subseteq X$ be a nonempty countable closed subset, and assume that A has no isolated points. The fact that A is closed in X means that A itself is either a locally compact Hausdorff space or a complete metric space. For each $a \in A$, the singleton $\{a\}$ is nowhere dense in A : it is closed in A because A is Hausdorff, and it contains no nonempty open subset because A has no isolated points. Since A is a countable union of singletons, A is meager in A , which is a contradiction. \square

Proposition 1.11.17. *Let $\{U_n\}$ be a sequence of open dense subsets of a locally compact Hausdorff space or a complete metric space X with no isolated points and $U := \bigcap_n U_n$. Then U is dense in X and uncountable. In particular, no countable dense subset of X can be obtained as the intersection of a countable family of open sets, i.e., is not G_δ .*

Proof. Since X is Baire, we know U is dense. If $U = \{x_n\}$ then, since each $\{x_n\}$ is closed and nowhere dense in X , it follows that

$$U \cap \bigcap_n \{x_n\}^c = U \cap U^c = \emptyset$$

is open dense, which is a contradiction. \square

Corollary 1.11.18. *There is no function from \mathbb{R} to \mathbb{R} which is continuous at each rational point and discontinuous at each irrational point.*

Proof. The points of continuity of f form a G_δ set. But \mathbb{Q} is not G_δ by ??.

1.12 Topological manifolds

1.12.1 Definition of manifolds

A topological space M is said to be **locally Euclidean** of dimension n if every point of M has a neighborhood in M that is homeomorphic to an open subset of \mathbb{R}^n .

Lemma 1.12.1. *A topological space M is locally Euclidean of dimension n if and only if either of the following properties holds:*

- (a) *Every point of M has a neighborhood homeomorphic to an open ball in \mathbb{R}^n .*
- (b) *Every point of M has a neighborhood homeomorphic to \mathbb{R}^n .*

In addition, \mathbb{R}^n is homeomorphic to an open ball in \mathbb{R}^n .

Proof. It is immediate that any space with property (a) or (b) is locally Euclidean of dimension n . Conversely, suppose M is locally Euclidean of dimension n . Because any open ball in \mathbb{R}^n is homeomorphic to \mathbb{R}^n itself, properties (a) and (b) are equivalent, so we need only prove (a).

Given a point $p \in M$, let U be a neighborhood of p that admits a homeomorphism $\varphi : U \rightarrow V$, where V is an open subset of \mathbb{R}^n . The fact that V is open in \mathbb{R}^n means that there is some open ball B around $\varphi(p)$ that is contained in V , and then $\varphi^{-1}(B)$ is a neighborhood of p homeomorphic to B . \square

Suppose M is locally Euclidean of dimension n . If $U \subseteq M$ is an open subset that is homeomorphic to an open subset of \mathbb{R}^n , then U is called a **coordinate domain**, and any homeomorphism φ from U to an open subset of \mathbb{R}^n is called a **coordinate map**. The pair (U, φ) is called a **coordinate chart** (or just a chart) for M . A **coordinate domain** that is homeomorphic to a ball in \mathbb{R}^n is called a **coordinate ball**. The preceding lemma shows that every point in a locally Euclidean space is contained in a coordinate ball.

The definition of locally Euclidean spaces makes sense even when $n = 0$. Because \mathbb{R}^0 is a single point, ??(b) implies that a space is locally Euclidean of dimension 0 if and only if each point has a neighborhood homeomorphic to a one-point space, or in other words if and only if the space is discrete.

Definition 1.12.2 (topological manifold). An n -dimensional **topological manifold** is a second countable Hausdorff space that is locally Euclidean of dimension n .

The most obvious example of an n -manifold is \mathbb{R}^n itself. More generally, any open subset of \mathbb{R}^n —or in fact of any n -manifold—is again an n -manifold, as the next proposition shows.

Proposition 1.12.3. Every open subset of an n -manifold is an n -manifold.

Proof. Let M be an n -manifold, and let V be an open subset of M . Any $p \in V$ has a neighborhood (in M) that is homeomorphic to an open subset of \mathbb{R}^n ; the intersection of that neighborhood with V is still open, still homeomorphic to an open subset of \mathbb{R}^n , and is contained in V , so V is locally Euclidean of dimension n . Since any open subset of a Hausdorff space is Hausdorff and any open subset of a second countable space is second countable, we see that V is an n -manifold. \square

Theorem 1.12.4. A separable metric space that is locally Euclidean of dimension n is an n -manifold. And the converse is also true.

Proof. Every metric space is Hausdorff, and ?? shows that a separable metric space is second countable. For the converse, separability follows from ??, and metrizability follows from Urysohn metrization theorem. \square

Proposition 1.12.5. Suppose P is a second countable space and M is a quotient space of P . If M is locally Euclidean, then it is second countable. Thus if M is locally Euclidean and Hausdorff, it is a manifold.

Proof. Let $q : P \rightarrow M$ denote the quotient map, and let \mathcal{U} be a cover of M by coordinate balls. The collection $q^{-1}(\mathcal{U}) = \{q^{-1}(U) : U \in \mathcal{U}\}$ is an open cover of P , which has a countable subcover since P is Lindelöf. If we let $\mathcal{U}' \subseteq \mathcal{U}$ denote a countable subset of \mathcal{U} such that $q^{-1}(\mathcal{U}')$ covers P , then \mathcal{U}' is a countable cover of M by coordinate balls. Each such ball is second countable, so M is also second countable. \square

1.12.2 Manifolds with boundary

In many cases we will encounter spaces that would be manifolds except that they have a boundary. The following definition deals with this problem.

The closed n dimensional upper half-space $\mathbb{H}^n \subseteq \mathbb{R}^n$ is defined by

$$\mathbb{H}^n := \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_n \geq 0\}$$

Definition 1.12.6 (Manifolds with Boundary). An n -dimensional manifold with boundary is a second countable Hausdorff space in which every point has a neighborhood homeomorphic either to an open subset of \mathbb{R}^n , or to an open subset of \mathbb{H}^n .

Note that, despite the terminology, a manifold with boundary is not necessarily a manifold. When it is important to make the distinction, we say (U, φ) is an **interior chart** if (U, φ_0) is an open subset of \mathbb{R}^n , and a **boundary chart** if (U, φ) is an open subset of \mathbb{H}^n with $\varphi(U) \cap \partial\mathbb{H}^n \neq \emptyset$.

If M is an n -manifold with boundary, we define a coordinate chart for M to be a pair (U, φ) , where $U \subseteq M$ is open and φ is a homeomorphism from U to an open subset of either \mathbb{R}^n or \mathbb{H}^n . Just as in the case of manifolds, the set U is called a **coordinate domain**, and φ is called a coordinate map. We use the notation $\partial\mathbb{H}^n$ to denote the boundary of \mathbb{H}^n , and $\text{Int } \mathbb{H}^n$ to denote its interior, considering \mathbb{H}^n as a subset of \mathbb{R}^n . For $n > 0$, this means

$$\partial\mathbb{H}^n = \{(x_1, \dots, x_n) : x_n = 0\}, \quad \text{Int } \mathbb{H}^n = \{(x_1, \dots, x_n) : x_n > 0\}.$$

When $n = 0$, we have $\mathbb{H}^0 = \mathbb{R}^0 = \{0\}$, so $\text{Int } \mathbb{H}^0 = \mathbb{H}^0$ and $\partial\mathbb{H}^0 = \emptyset$. It follows that 0-dimensional manifolds with boundary are no different from 0-manifolds.

Example 1.12.7. The upper half-space \mathbb{H}^n itself is an n -manifold with boundary, with the identity map as a global coordinate map. Similarly, any closed or half-open interval in \mathbb{R} is a 1-manifold with boundary, for which charts are easy to construct. Another important example is the closed unit ball $\bar{\mathbb{B}}^n$ with the Euclidean topology. It is not hard to see intuitively that $\bar{\mathbb{B}}^n$ is an n -manifold with boundary

If M is an n -manifold with boundary, a point $p \in M$ is called an **interior point** of M if it is in the domain of an interior chart; and it is called a **boundary point** of M if it is in the domain of a boundary chart that takes p to $\partial\mathbb{H}^n$. The **boundary of M** , denoted by ∂M , is the set of all its boundary points, and its **interior**, denoted by $\text{Int } M$, is the set of all its interior points.

Note that these are new meanings for the terms boundary and interior, distinct from their use earlier in reference to subsets of topological spaces. If M is a manifold with boundary, it might have nonempty boundary in this new sense, irrespective of whether it has any boundary points as a subset of some other topological space. Usually the distinction is clear from the context, but if necessary we can always distinguish the two meanings by referring to the **topological boundary** (for the original meaning) or the **manifold boundary** (for this new meaning) as appropriate.

Theorem 1.12.8. If M is an n -dimensional manifold with boundary, then $\text{Int } M$ is an open subset of M , which is itself an n -dimensional manifold without boundary. And ∂M is an $(n - 1)$ -manifold without boundary.

There is a subtlety about these definitions that might not be immediately evident. Although the interior and boundary of M are well-defined subsets whose union is M , and it might seem intuitively rather obvious that they are disjoint from each other, we have no way of proving at this stage that a point $p \in M$ cannot be simultaneously a boundary point and an interior point, meaning that there is one interior chart whose domain contains p , and another boundary chart that sends p to $\partial\mathbb{H}^n$. After we have developed some more machinery, you will be able to prove the following theorem.

Theorem 1.12.9 (Invariance of the Boundary). *If M is a manifold with boundary, then a point of M cannot be both a boundary point and an interior point. Thus ∂M and $\text{Int } M$ are disjoint subsets whose union is M .*

Corollary 1.12.10. *If M is a nonempty n -dimensional manifold with boundary, then ∂M is closed in M , and M is an n -manifold if and only if $\partial M = \emptyset$.*

Proof. Because $\text{Int } M$ is an open subset of M by ??, it follows from ?? that $\partial M = M \setminus \text{Int } M$ is closed. If M is a manifold, then every point is in the domain of an interior chart, so $M = \text{Int } M$, and it follows from ?? that $\partial M = \emptyset$. Conversely, if $\partial M = \emptyset$, then $M = \text{Int } M$, which is a manifold by ??.

Corollary 1.12.11. *Let M be a smooth manifold with boundary, let (U, φ) be a boundary chart on M , and let p be a point in $\partial M \cap U$. Then $\varphi(p)$ is in $\partial\mathbb{H}^n$. Thus, a coordinate map sends boundary points to boundary points.*

Proof. If p is a boundary point, then $\varphi(p)$ is also a boundary point in \mathbb{H}^n . Thus by ?? we know $\varphi(p) \in \partial\mathbb{H}^n$. \square

More generally, we have the following result.

Theorem 1.12.12. *If M and N are smooth manifolds with boundary and $F : M \rightarrow N$ is a homeomorphism, then $F(\partial M) = \partial N$, hence $F(\text{Int } M) = F(\text{Int } N)$.*

Proof. Since F is a homeomorphism, every boundary point of M is mapped to boundary of N , and vice versa. Thus we have $F(\partial M) = \partial N$, and since M is the disjoint union of its interior and boundary, we also have $F(\text{Int } M) = \text{Int } N$. \square

Example 1.12.13. If $U \subseteq \mathbb{R}^n$ is an open subset and $f : U \rightarrow \mathbb{R}^k$ is any continuous map, the graph of f is the subset $\Gamma(f) \subseteq \mathbb{R}^{n+k}$ defined by

$$\Gamma(f) := \{(x, y) \in \mathbb{R}^{n+k} : x \in U \text{ and } y = f(x)\}$$

with the subspace topology inherited from \mathbb{R}^{n+k} . Let $\Phi_f : U \rightarrow \mathbb{R}^{n+k}$ be the continuous injective map

$$\Phi_f(x) = (x, f(x))$$

the the restriction to $\Gamma(f)$ of the projection $\pi : \mathbb{R}^{n+k} \rightarrow \mathbb{R}^n$ is a continuous inverse for Φ_f . Hence $\Gamma(f)$ is homeomorphic to U and is a manifold.

Example 1.12.14. Define \mathbb{RP}^n to be the set of 1-dimensional linear subspaces (lines through the origin) in \mathbb{R}^{n+1} , and \mathbb{CP}^n be the set of all 1-dimensional complex subspaces of \mathbb{C}^{n+1} , called n -dimensional complex projective space. Topologize them as the quotient $(\mathbb{R}^{n+1} \setminus \{0\})/\mathbb{R}^*$ and $(\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^*$. Then \mathbb{RP}^n is an n -manifold, \mathbb{CP}^n is a $2n$ -manifold.

Example 1.12.15. Let X be the subset $(\mathbb{R} \times \{0\}) \cup (\mathbb{R} \times \{1\}) \subseteq \mathbb{R}^2$. Define an equivalence relation on X by declaring $(x, 0) \sim (x, 1)$ if $x \neq 0$. Show that the quotient space X/\sim is locally Euclidean and second countable, but not Hausdorff. (This space is called the **line with two origins**)

Example 1.12.16. Consider the product $\mathbb{R}^d \times \mathbb{R}$, where \mathbb{R}^d means \mathbb{R} with discrete topology. This space is Hausdorff and Locally Euclidean, but not second countable.

1.12.3 Adjunction spaces

Definition 1.12.17. Suppose X and Y are topological spaces, A is a closed subspace of Y , and $f : A \rightarrow X$ is a continuous map. Let \sim be the equivalence relation on the disjoint union $X \amalg Y$ generated by $a \sim f(a)$ for all $a \in A$, and denote the resulting quotient space by

$$X \cup_f Y := (X \amalg Y)/\sim$$

Any such quotient space is called an **adjunction space**, and is said to be formed by attaching Y to X along f . The map f is called the **attaching map**. Note that the equivalence relation identifies each point $x \in X$ with all of the points (if any) in $f^{-1}(x) \in A$. If $A = \emptyset$, then $X \cup_f Y$ is just the disjoint union space $X \amalg Y$.

Theorem 1.12.18. Let $X \cup_f Y$ be an adjunction space, and let $q : X \amalg Y \rightarrow X \cup_f Y$ be the associated quotient map.

- (a) The restriction of q to X is a topological embedding, whose image set $q(X)$ is a closed subspace of $X \cup_f Y$.
- (b) The restriction of q to $Y \setminus A$ is a topological embedding, whose image set $q(Y \setminus A)$ is an open subspace of $X \cup_f Y$.
- (c) $X \cup_f Y$ is the disjoint union of $q(X)$ and $q(Y \setminus A)$.

Proof. We begin by showing that $q|_X$ is a closed map. Suppose that B is a closed subset of X . To show that $q(B)$ is closed in the quotient space, we need to show that $q^{-1}(q(B))$ is closed in $X \amalg Y$, which is equivalent to showing that its intersections with X and Y are closed in X and Y , respectively. From the form of the equivalence relation, it follows that $q^{-1}(q(B)) \cap X = B$, which is closed in X by assumption; and $q^{-1}(q(B)) = f^{-1}(B)$, which is closed in A by the continuity of f , and thus is closed in Y because A is closed in Y . It follows, in particular, that $q(X)$ is closed in $X \cup_f Y$.

Now $q|_X$ is clearly injective because the equivalence relation does not identify any points in X with each other. Because it is also closed, it is a topological embedding. This proves (a).

To prove (b), we just note that $Y \setminus A$ is a saturated open subset of $X \amalg Y$, so the restriction $q|_{Y \setminus A}$ is a quotient map, and since it is bijective it is a homeomorphism. Its image is open in $X \cup_f Y$ by definition of the quotient topology. Finally, part (c) is an easy consequence of the definition of the equivalence relation. \square

Because of the preceding proposition, one typically identifies X with $q(X)$ and $Y \setminus A$ with $q(Y \setminus A)$, considering each as a subspace of the adjunction space.

Example 1.12.19 (Adjunction Spaces).

- (a) Suppose X and Y are topological spaces with chosen base points $x \in X$ and $y \in Y$. Let $A = \{y\} \subseteq Y$, and define $f : A \rightarrow X$ by $f(y) = x$. Then the adjunction space $X \cup_f Y$ is just the wedge sum $X \wedge Y$.
- (b) Let $A = S^1 \subseteq \bar{B}^2$, and let $f : A \rightarrow \bar{B}^2$ be the inclusion map. Then the adjunction space is homeomorphic to S^2 .

Suppose M and N are n -dimensional manifolds with nonempty boundaries, such that ∂M and ∂N are homeomorphic. Let $h : \partial N \rightarrow \partial M$ be a homeomorphism. Assuming the theorem on the invariance of the boundary, we conclude that ∂N is a closed subset of N , so we can define the adjunction space $M \cup_h N$ (considering h as a map into M). This space is said to be formed by **attaching M and N together along their boundaries**.

Theorem 1.12.20. *With M , N , and h as above, $M \cup_h N$ is an n -manifold (without boundary). There are topological embeddings $e : M \rightarrow M \cup_h N$ and $f : N \rightarrow M \cup_h N$ whose images are closed subsets of $M \cup_h N$ satisfying*

$$e(M) \cup f(N) = M \cup_h N, \quad e(M) \cap f(N) = e(\partial M) = f(\partial N).$$

Proof. First we need to show that $M \cup_h N$ is locally Euclidean of dimension n . Let $q : M \amalg N \rightarrow M \cup_h N$ denote the quotient map, and write $S = q(\partial M \cup \partial N)$. Note that $\text{Int } M \amalg \text{Int } N$ is a saturated open subset of $M \amalg N$, and therefore q restricts to a quotient map from $\text{Int } M \amalg \text{Int } N$ onto $(M \cup_h N) \setminus S$. Because this restriction is injective, it is a homeomorphism, and thus $(M \cup_h N) \setminus S$ is locally Euclidean of dimension n . Thus we need only consider points in S .

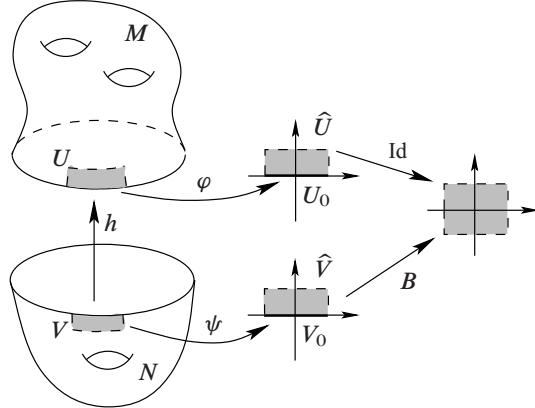


Figure 1.1: Attaching along boundaries.

Suppose $s \in S$, and let $y \in \partial N$ and $x = h(y) \in \partial M$ be the two points in the fiber $q^{-1}(s)$. We can choose coordinate charts (U, φ) for M and (V, ψ) for N such that $x \in U$ and $y \in V$, and let $\hat{U} = \varphi(U)$, $\hat{V} = \psi(V)$. It is useful in this proof to identify \mathbb{H}^n with $\mathbb{R}^{n-1} \times [0, +\infty)$ and \mathbb{R}^n with $\mathbb{R}^{n-1} \times \mathbb{R}$. By shrinking U and V if necessary, we may assume that $h(V \cap \partial N) = U \cap \partial M$, and that $\hat{U} = U_0 \times [0, \varepsilon)$ and $\hat{V} = V_0 \times [0, \varepsilon)$ for some $\varepsilon > 0$ and some open subsets $U_0, V_0 \subseteq \mathbb{R}^{n-1}$. Then we can write the coordinate maps as

$$\varphi(x) = (\varphi_0(x), \varphi_1(x)), \quad \psi(y) = (\psi_0(y), \psi_1(y))$$

for some continuous maps $\varphi_0 : U \rightarrow U_0$, $\varphi_1 : U \rightarrow [0, \varepsilon)$, $\psi_0 : V \rightarrow V_0$ and $\psi_1 : V \rightarrow [0, \varepsilon)$. Our assumption that x and y are boundary points means that $\varphi_1(x) = \psi_1(y) = 0$.

We wish to assemble these two charts into a map whose image is an open subset of \mathbb{R}^n , by matching them up along corresponding points in ∂M and ∂N . As they stand, however, the maps φ and ψ might not take corresponding boundary points to the same image point, so we need to adjust for that. Both of the restrictions

$$\varphi_0|_{U \cap \partial M} : U \cap \partial M \rightarrow U_0, \quad \psi_0|_{V \cap \partial N} : V \cap \partial N \rightarrow V_0$$

are homeomorphisms. Define a homeomorphism $\tau : V_0 \rightarrow U_0$ by

$$\tau = (\varphi_0|_{U \cap \partial M}) \circ h \circ (\psi_0|_{V \cap \partial N})^{-1},$$

and let $B : \widehat{V} \rightarrow \mathbb{R}^n$ be the map

$$B(x_1, \dots, x_n) = (\tau(x_1, \dots, x_{n-1}), -x_n).$$

Geometrically, B rearranges the boundary points according to the map τ , and then "flips" each vertical line segment above a boundary point to a line segment below the image point. Our construction ensures that for $y \in \partial N$,

$$B \circ \psi(y) = (\tau \circ \psi(y), 0) = (\varphi_0 \circ h(y), 0) = \varphi \circ h(y). \quad (1.12.1)$$

Now define $\tilde{\Phi} : U \amalg V \rightarrow \mathbb{R}^n$ by

$$\tilde{\Phi} = \begin{cases} \varphi(y), & y \in U; \\ B \circ \psi(y), & y \in V. \end{cases}$$

Because $U \amalg V$ is a saturated open subset of $M \amalg N$, the restriction of q to it is a quotient map onto the neighborhood $q(U \amalg V)$ of s , and (??) shows that $\tilde{\Phi}$ passes to the quotient and defines an injective continuous map $\Phi : q(U \amalg V) \rightarrow \mathbb{R}^n$. Since φ , ψ , and B are homeomorphisms onto their images, we can define an inverse for Φ as follows:

$$\Phi^{-1}(y) = \begin{cases} q \circ \varphi^{-1}(y), & y^n \geq 0; \\ q \circ \psi^{-1} \circ B^{-1}(y) & y^n \leq 0. \end{cases}$$

These two definitions agree where they overlap, so the resulting map is continuous by the gluing lemma. Thus Φ is a homeomorphism. Note that the image of Φ is exactly $U_0 \times (-\varepsilon, \varepsilon)$, which is open in \mathbb{R}^n . Therefore $M \cup_h N$ is locally Euclidean of dimension n .

The quotient space $M \cup_h N$ is second countable by ???. To prove that it is Hausdorff, we need to show that the fibers of q can be separated by saturated open subsets. It is straightforward to check on a case-by-case base that the preimages of sufficiently small coordinate balls will do.

It follows immediately from ??(a) that the quotient map q restricts to a topological embedding of M into $M \cup_h N$ with closed image. On the other hand, because h is a homeomorphism, it is easy to see that $M \cup_h N$ is also equal to $N \cup_{h^{-1}} M$, so q also restricts to a topological embedding of N with closed image. The union of the images of these embeddings is all of $M \cup_h N$,

and their intersection is the set S defined above, which is exactly the image of either boundary. \square

Here is an important example of the preceding construction.

Example 1.12.21 (The Double of a Manifold with Boundary). Suppose M is an n -dimensional manifold with boundary. If $h : \partial M \rightarrow \partial M$ is the identity map, the resulting quotient space $M \cup_h M$ is denoted by $D(M)$ and called the **double of M** . It can be visualized as the space obtained by attaching two copies of M to each other along their common boundary. (If $\partial M = \emptyset$, then $D(M)$ is just the disjoint union of two copies of M)

The following proposition is an immediate consequence of ??.

Proposition 1.12.22. *Every n -manifold with boundary is homeomorphic to a closed subset of an n -manifold without boundary.*

This construction can be used to extend many results about manifolds to manifolds with boundary. For example, any property that holds for all closed subsets of manifolds is also shared by manifolds with boundary.

1.13 Paracompactness

1.13.1 Paracompact spaces

Let X be a topological space. A collection \mathcal{A} of subsets of X is said to be **locally finite** if each point of X has a neighborhood that intersects at most finitely many of the sets in \mathcal{A} . Given a cover \mathcal{A} of X , another cover \mathcal{B} is called a **refinement** of \mathcal{A} if for each $B \in \mathcal{B}$ there exists some $A \in \mathcal{A}$ such that $B \subseteq A$. It is an **open refinement** if each $B \in \mathcal{B}$ is an open subset of X . (Note that every subcover of \mathcal{A} is a refinement of \mathcal{A} , but a refinement is not in general a subcover, because a refinement does not need to be composed of sets that are elements of \mathcal{A} .)

Here are some elementary properties of local finiteness. Given a collection \mathcal{A} of subsets of a topological space, let us use the notation $\bar{\mathcal{A}}$ to denote the collection of closures of sets in \mathcal{A} :

$$\bar{\mathcal{A}} := \{\bar{A} : A \in \mathcal{A}\}$$

Lemma 1.13.1. *Let X be a topological space and \mathcal{A} be a collection of subsets of X . Then \mathcal{A} is locally finite if and only $\bar{\mathcal{A}}$ is locally finite.*

Proof. If $\bar{\mathcal{A}}$ is locally finite, then it follows immediately that \mathcal{A} is locally finite. Conversely, suppose \mathcal{A} is locally finite. Given $x \in X$, let W be a neighborhood of x that intersects only finitely many of the sets in \mathcal{A} , say A_1, \dots, A_n . If W contains a point a in \bar{A} for some $A \in \mathcal{A}$, then $W \cap A \neq \emptyset$ since a is a limit point of A . It then follows that A is one of the sets A_1, \dots, A_n , and therefore W can only intersect finitely many sets in $\bar{\mathcal{A}}$. \square

Lemma 1.13.2. *If \mathcal{A} is a locally finite collection of subsets of X , then*

$$\overline{\bigcup_{A \in \mathcal{A}} A} = \bigcup_{A \in \mathcal{A}} \bar{A}$$

Proof. The right-hand side is contained in the lefthand side even without the assumption of local finiteness, so we only prove the reverse containment. We will prove the contrapositive: assuming $x \in X$ is not an element of $\bigcup_{A \in \mathcal{A}} \bar{A}$, we show it is not an element of $\overline{\bigcup_{A \in \mathcal{A}} A}$ either. By ??, x has a neighborhood U that intersects only finitely many sets in \bar{A} , say $\bar{A}_1, \dots, \bar{A}_k$. Then $U \setminus (\bar{A}_1 \cap \dots \cap \bar{A}_k)$ is a neighborhood of x that intersects none of the sets in \mathcal{A} , so $x \notin \overline{\bigcup_{A \in \mathcal{A}} A}$. \square

Definition 1.13.3. A space X is said to be **paracompact** if every open cover of X admits a locally finite open refinement. Every compact space is paracompact, because a finite subcover is a locally finite open refinement.

Proposition 1.13.4. A closed subspace of a paracompact space is paracompact.

Proof. If A is a closed subspace of a paracompact space X , every open cover of A is of the form $\{U_\alpha\} \cap A$, where U_α are open in X . Add in the open set $X - A$, we get an open cover of X , thus using paracompactness of X we get the claim. \square

A key topological fact about manifolds, as we show below, is that they are all paracompact. This is a consequence of second countability, and in fact is one of the most important reasons why second countability is included in the definition of manifolds. For spaces that are Hausdorff and locally Euclidean with countably many components, the two conditions are equivalent.

Before we prove that manifolds are paracompact, we need a preliminary result. If X is a topological space, a sequence $\{K_i\}_{i \in \mathbb{N}}$ of compact subsets of X is called an **exhaustion** of X by compact sets if $X = \bigcup_i K_i$ and $K_i \subseteq \text{Int } K_{i+1}$ for each i .

Proposition 1.13.5. A σ -compact locally compact Hausdorff space admits an exhaustion by compact sets.

Proof. Since X is a σ -compact, there exists a sequence $\{K_i\}$ of compact subsets such that $X = \bigcup_i K_i$. Every compact subset of X has a precompact open neighborhood by ???. Thus we may take U_1 to be a precompact open neighborhood of K_1 , and then, proceeding inductively, take U_n to be a precompact open neighborhood of $\bar{U}_{n-1} \cup K_n$. It follows that the sequence $\{U_i\}$ satisfies $K_i \subseteq U_i$ and $\bar{U}_i \subseteq U_{n+1}$. Let $\tilde{K}_i = \bar{U}_i$, then $\{\tilde{K}_i\}$ is a exhaustion of X by compact sets. \square

Theorem 1.13.6. Every σ -compact locally compact Hausdorff space is paracompact. In fact, given such a space X , an open cover \mathcal{U} of X , and any base \mathcal{B} for the topology of X , there exists a countable, locally finite open refinement of \mathcal{U} consisting of elements of \mathcal{B} .

Proof. Given X, \mathcal{U} and \mathcal{B} , by ??, let $(K_i)_{i \in \mathbb{N}}$ be an exhaustion of X by compact sets. For each i , let

$$V_i = K_{i+1} \setminus \text{Int } K_i, \quad W_i = \text{Int } K_{i+2} \setminus K_{i-1}$$

where we interpret $K_i = \emptyset$ if $i < 1$. Then V_i is a compact set contained in the open subset W_i . For each $x \in V_i$, there is some $U_x \in \mathcal{U}$ containing x , and because \mathcal{B} is a base, there exists $B_x \in \mathcal{B}$ such that $x \in B_x \subseteq U_x \cap W_i$. The collection of all such sets B_x as x ranges over V_i is an open cover of V_i , and thus has a finite subcover. The union of all such finite subcovers as i ranges over the positive integers is a countable open cover of M that refines \mathcal{U} . Because the

finite subcover of V_i consists of sets contained in W_i , and $W_i \cap W_j \neq \emptyset$ only for $i-2 < j < i+2$, the resulting cover is locally finite. \square

In view of the theorem, one may wonder the connection between paracompactness and σ -compactness for a locally Hausdorff space. The following theorem answers this question.

Theorem 1.13.7. *A locally compact Hausdorff space X is paracompact if and only if it is a disjoint union of open σ -compact subsets.*

Proof. Assume that X is paracompact. Using local compactness, cover the space with open precompact sets U_α . Using paracompactness we may assume this cover is locally finite. We shall inductively construct open precompact sets V_i . We start with $V_1 = U_\beta$ for some given β . If V_i has been defined then consider all the U_α which intersect \bar{V}_i : By compactness of \bar{V}_i and the local finiteness of the cover, this set of U_α 's is finite. Let V_{i+1} be the union of these U_α . Then \bar{V}_{i+1} is the union of the closures of this finite set of U_α 's and so it is compact. Also $\bar{V}_i \subseteq V_{i+1}$. Put $V = \bigcup_i V_i$, then V is open. By ?? \bar{V} is the union of the closures of these U_α 's. But by construction each of these closures is contained in some $\bar{V}_i \subseteq V_{i+1} \subseteq V$. Thus $V = \bar{V}$ is closed and by construction, is σ -compact. Now since V is closed and open, it is a connected component of X . Thus X is a disjoint union of open σ -compact subsets.

Conversely, assume that X is a union of σ -compact subsets. It is clear that a disjoint union of open paracompact spaces is paracompact, and so we may as well assume the space X to be σ -compact. Let $X = \bigcup_i C_i$ where the C_i are compact. We define an exhaustion of X by compact sets K_i .

- Let $K_1 = C_1$. To define K_2 , for each point $x \in C_1$ we find by local compactness a precompact neighborhood U_x of x . Then the collection $\{U_x\}$ covers C_1 , hence has a subcover $\{U_1^{(1)}, \dots, U_{i_1}^{(1)}\}$. We set

$$K_2 = C_2 \cup \bigcup_{j=1}^{i_1} \bar{U}_j^{(1)}$$

so K_2 is compact and $K_1 \subseteq \text{Int } K_2$.

- Assume the ascending compact sets K_1, \dots, K_n are defined. Now by the same argument we find precompact sets $\{U_1^{(n)}, \dots, U_{i_n}^{(n)}\}$ covers K_n . Then we set

$$K_{n+1} = C_n \cup \bigcup_{j=1}^{i_n} \bar{U}_j^{(n)}$$

Now K_{n+1} is compact and $K_n \subseteq \text{Int } K_{n+1}$. By induction we get the sequence $\{K_i\}$, and since each C_i is contained in K_i , we have $X = \bigcup_i K_i$.

Now with this exhaustion by compact sets, by arguing as in the proof of ??, we can show that X is paracompact. \square

One feature of paracompact Hausdorff spaces is that they satisfy a strengthened version of the Hausdorff property. A topological space X is said to be **normal** if it is Hausdorff and for every pair of disjoint closed subsets $A, B \subseteq X$, there exist disjoint open subsets $U, V \subseteq X$ such that $A \subseteq U$ and $B \subseteq V$. Similarly, X is said to be **regular** if it is Hausdorff and for every closed

subset $B \subseteq X$ and point $a \in B$, there exist disjoint open subsets $U, V \subseteq X$ such that $a \in U$ and $B \subseteq V$. Briefly, in normal spaces, closed subsets can be separated by open subsets; and in regular spaces, closed subsets can be separated from points by open subsets.

Lemma 1.13.8. *Let X be a Hausdorff space. Then X is normal if and only if it satisfies the following condition: whenever A is a closed subset of X and U is a neighborhood of A , there exists a neighborhood V of A such that $\bar{V} \subseteq U$.*

Proof. This is easily seen to be equivalent to the definition of normality by taking $B = U^c$. \square

Theorem 1.13.9. *Every paracompact Hausdorff space is normal.*

Proof. Suppose X is a paracompact Hausdorff space, and let A and B be disjoint closed subsets of X . We begin with the special case in which $B = \{q\}$ is a singleton; in other words, we prove that X is regular.

For each $p \in A$, because X is Hausdorff there exist disjoint open subsets U_p and V_p such that $p \in U_p$ and $q \in V_p$. The collection $\{U_p : p \in A\} \cup \{X \setminus A\}$ is an open cover of X . By paracompactness, it has a locally finite open refinement \mathcal{W} . Each of the open subsets in \mathcal{W} is contained either in U_p for some p , or in $X \setminus A$. Let \mathcal{U} be the collection of all the sets in \mathcal{W} that are contained in some U_p . Then \mathcal{U} is still locally finite, and it is an open cover of A . Moreover, if $U \in \mathcal{U}$, then there is a neighborhood V_p of q disjoint from U , so \bar{U} does not contain q .

Let $\mathfrak{U} = \bigcup_{U \in \mathcal{U}} U$, and $\mathfrak{V} = X \setminus \mathfrak{U}$. Because \mathcal{U} is locally finite, $\bar{\mathfrak{U}} = \bigcup_{U \in \mathcal{U}} \bar{U}$ by ???. Thus $q \notin \bar{\mathfrak{U}}$, so \mathfrak{U} and \mathfrak{V} are disjoint open subsets of X containing A and q , respectively. This completes the proof that X is regular. Next consider arbitrary disjoint closed subsets A and B . Exactly the same argument works in this case, using regularity in place of the Hausdorff condition. \square

The following result is a nontrivial property about normal spaces.

Theorem 1.13.10 (Urysohn's Lemma). *Suppose X is a normal topological space. If $A, B \subseteq X$ are disjoint closed subsets, then there exists a continuous function $f : X \rightarrow [0, 1]$ such that $f|_A \equiv 0$ and $f|_B \equiv 1$.*

Proof. We will construct for each rational number r an open subset $U_r \subseteq X$, such that the following properties hold

- (a) $U_r = \emptyset$ when $r < 0$, and $U_r = X$ when $r > 1$.
- (b) $A \subseteq U_0$.
- (c) $U_1 = X \setminus B$.
- (d) If $p < q$, then $\bar{U}_p \subseteq U_q$

we will do this by induction. We begin by defining $U_1 = X \setminus B$, and defining U_r for $r \notin [0, 1]$ by (a). By normality, there exists a neighborhood U_0 of A such that $\bar{U}_0 \subseteq U_1$. Let $\{r_i\}_{i \in \mathbb{N}}$ be a enumeration of the rationals in $[0, 1]$. By normality again, there exists an open subset $U_{r_1} \subseteq X$ such that $\bar{U}_0 \subseteq U_{r_1}$ and $\bar{U}_{r_1} \subseteq U_1$. Now use induction, suppose U_i has been defined for $i = 1, \dots, n$ such that $\bar{U}_0 \subseteq U_{r_i}$, $\bar{U}_{r_i} \subseteq U_1$ and $\bar{U}_{r_i} \subseteq U_{r_j}$ for $r_i < r_j$. Consider the next rational number r_{n+1} in the sequence, and let p be the largest number in the set $\{0, r_1, \dots, r_n, 1\}$ that is smaller than r_{n+1} , and q the smallest number in the same set that is larger than r_{n+1} . Then

$\bar{U}_p \subseteq U_q$. By normality, there exists an open subset $U_{r_{n+1}}$ such that $\bar{U}_p \subseteq U_{r_{n+1}}$ and $\bar{U}_{r_{n+1}} \subseteq U_q$. Continuing by induction, we obtain open subsets U_r for all rational r satisfying the properties above.

Now, define $f : X \rightarrow \mathbb{R}$ by

$$f(x) = \inf\{r \in \mathbb{Q} : x \subseteq U_r\}.$$

Because of (a), f is well defined and takes its values in $[0, 1]$. Properties (a) and (b) imply that $f(x) = 0$ for $x \in A$, and (a) and (c) imply that $f(x) = 1$ for $x \in B$. Thus it remains only to show that f is continuous.

Because sets of the form (a, ∞) and $(-\infty, a)$ form a subbase for the topology of \mathbb{R} , it suffices to show that the preimage under f of any such set is open. We begin with the following observations:

$$f(x) < a \iff x \in U_r \text{ for some rational } r < a \quad (1.13.1)$$

$$f(x) \leq a \iff x \in \bar{U}_r \text{ for any rational } r > a \quad (1.13.2)$$

The equivalence (??) is an easy consequence of the definition of f . To prove (??), suppose first that $f(x) \leq a$. If r is any rational number greater than a , then by definition of f , there is a rational number $s < r$ such that $x \in U_s \subseteq U_r \subseteq \bar{U}_r$. Conversely, suppose $x \in \bar{U}_r$ for every rational number greater than a . If $s > a$ is rational, choose a rational number r such that $s > r > a$; it follows from our hypothesis that $x \in \bar{U}_r \subseteq U_s$, which implies that $f(x) \leq s$. Since this is true for every rationals $s > a$, it follows that $f(x) \leq a$.

From (??) and (??) we conclude that

$$f^{-1}((-\infty, a)) = \bigcup_{\substack{r \in \mathbb{Q} \\ r < a}} U_r, \quad f^{-1}((a, \infty)) = X \setminus \bigcap_{\substack{r \in \mathbb{Q} \\ r > a}} \bar{U}_r$$

both of which are open. Thus f is continuous. \square

The next corollary of Urysohn's lemma is often the most useful. If X is a topological space and $f : X \rightarrow \mathbb{R}$ is a continuous function, the support of f , denoted by $\text{supp}(f)$, is the closure of the set $\{x \in X : f(x) \neq 0\}$:

$$\text{supp}(f) := \overline{\{x \in X : f(x) \neq 0\}}$$

If A is a closed subset of X and U is a neighborhood of A , a continuous function $f : X \rightarrow [0, 1]$ such that $f|_A \equiv 1$ and $\text{supp}(f) \subseteq U$ is called a **bump function** for A supported in U .

Theorem 1.13.11. *Let X be a normal space. If A is a closed subset of X and U is a neighborhood of A , there exists a bump function for A supported in U .*

Proof. Just apply Urysohn's Lemma with $B = X \setminus U$. \square

One immediate consequence of the Urysohn lemma is the useful theorem called the Tietze extension theorem. It deals with the problem of extending a continuous realvalued function that is defined on a subspace of a space X to a continuous function defined on all of X . This theorem is important in many of the applications of topology.

Theorem 1.13.12 (Tietze extension theorem). *Let X be a normal space and A be a closed subspace of X .*

- (a) *Any continuous map of A into a closed interval $[a, b]$ may be extended to a continuous map of all of X into $[a, b]$.*
- (b) *Any continuous map of A into \mathbb{R} may be extended to a continuous map of all of X into \mathbb{R} .*

Proof. We first prove (a). The idea of the proof is to construct a sequence of continuous functions g_n defined on the entire space X , such that the sequence g_n converges uniformly, and such that the restriction of g_n to A approximates f more and more closely as n becomes large. Then the limit function will be continuous, and its restriction to A will equal f .

Without loss of generality, we may assume that $[a, b] = [0, 1]$. We claim that there is a sequence $\{g_n : X \rightarrow [0, 2^{n-1}/3^n]\}$ of continuous functions such that $0 \leq f - \sum_{i=1}^n g_i \leq 2^{n-1}/3^n$ on A . To begin with, let $B = f^{-1}([0, 1/3])$ and $C = f^{-1}([2/3, 1])$. These are closed subsets of A , and since A itself is closed, they are closed in X . By Urysohn's lemma there is a continuous $g_1 : X \rightarrow [0, 1/3]$ with $g_1 = 0$ on B and $g_1 = 1/3$ on C ; it follows that $0 \leq f - g_1 \leq 2/3$ on A . In general, having found g_1, \dots, g_{n-1} , by the same reasoning we can find $g_n : X \rightarrow [0, 2^{n-1}/3^n]$ such that $g_n = 0$ on the set where $f - \sum_{i=1}^{n-1} g_i \leq 2^{n-1}/3^n$ and $g_n = 2^{n-1}/3^n$ on the set where $f - \sum_{i=1}^{n-1} g_i \geq (2/3)^n$. Let $F = \sum_{n=1}^{\infty} g_n$. Since $|g_n| \leq 2^{n-1}/3^n$, the partial sums of this series converge uniformly, so F is continuous. Moreover, on A we have $0 < f - F < (2/3)^n$ for all n , whence $F = f$ on A .

We now prove part (b) of the theorem, in which f maps A into \mathbb{R} . We can replace \mathbb{R} by the open interval $(0, 1)$, since this interval is homeomorphic to \mathbb{R} . So let f be a continuous map from A into $(0, 1)$. The half of the Tietze theorem already proved shows that we can extend f to a continuous map $F : X \rightarrow [0, 1]$ mapping X into the closed interval. Let us define a subset D of X by the equation

$$D = F^{-1}(0) \cup F^{-1}(1).$$

Since F is continuous, D is a closed subset of X . Because $F(A) = f(A)$, which is contained in $(0, 1)$, the set A is disjoint from D . By the Urysohn lemma, there is a continuous function $\phi : X \rightarrow [0, 1]$ such that $\phi(D) = \{0\}$ and $\phi(A) = \{1\}$. Define

$$\tilde{F} = F(x)\phi(x).$$

Then \tilde{F} is continuous, being the product of two continuous functions. Also, \tilde{F} is an extension of F with values in $(0, 1)$. This finishes the proof. \square

1.13.2 Partitions of unity

The applications of paracompactness are all based on a technical tool called a partition of unity, which can be used to blend together locally defined continuous maps into a global one.

For this purpose, we need to consider open covers indexed by some set. (Of course any open cover \mathcal{U} can be considered as an indexed family, just by taking the index set to be \mathcal{U} itself.) In this context, an indexed family $\mathcal{U} = (U_\alpha)_{\alpha \in A}$ of subsets of a topological space X is said to be a **locally finite** family if each point of X has a neighborhood that intersects U_α for at most finitely many values of α .

If $\mathcal{U} = (U_\alpha)_{\alpha \in A}$ is an indexed open cover of X , a **partition of unity** subordinate to \mathcal{U} is a family of continuous functions $\psi_\alpha : X \rightarrow \mathbb{R}$, indexed by the same set A , with the following properties:

- (i) $0 \leq \psi_\alpha(x) \leq 1$ for all $\alpha \in A$ and all $x \in X$.
- (ii) $\text{supp}(\psi_\alpha) \subseteq U_\alpha$.
- (iii) The family of supports $\{\text{supp}(\psi_\alpha)\}_{\alpha \in A}$ is locally finite.
- (iv) $\sum_{\alpha \in A} \psi_\alpha(x) = 1$ for all $x \in X$.

Because of the local finiteness condition (iii), the sum in (iv) has only finitely many nonzero terms in a neighborhood of each point, so there is no issue of convergence.

First we have a simple observation.

Proposition 1.13.13. *Suppose X is a topological space with the property that for every indexed open cover \mathcal{U} of X , there exists a partition of unity subordinate to \mathcal{U} . Then X is paracompact.*

Proof. Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ be any cover of X , and $\{\psi_\alpha\}_{\alpha \in A}$. Then since $\sum_{\alpha \in A} \psi_\alpha = 1$, the collection $\{\text{supp}(\psi_\alpha)\}_{\alpha \in A}$ covers X . Moreover, by definition this collection is a locally finite refinement of \mathcal{U} . Thus X is paracompact. \square

Now we want to prove the converse of ???. First we need some lemmas.

Lemma 1.13.14. *Suppose X is a paracompact Hausdorff space. If $\mathcal{U} = (U_\alpha)_{\alpha \in A}$ is an indexed open cover of X , then \mathcal{U} admits a locally finite open refinement $\mathcal{V} = (V_\alpha)_{\alpha \in A}$ indexed by the same set, such that $\bar{V}_\alpha \subseteq U_\alpha$ for each α .*

Proof. By ?? and ??, each $x \in X$ has a neighborhood Y_x such that $\bar{Y}_x \subseteq U_\alpha$ for some $\alpha \in A$. Since X is paracompact, the open cover $\{Y_x : x \in X\}$ has a locally finite open refinement. Let us index this refinement by some set B , and denote it by $\mathcal{Z} = (Z_\beta)_{\beta \in B}$. For each β , there is some $x \in X$ such that $Z_\beta \subseteq Y_x$, and therefore there is some $\alpha \in A$ such that $\bar{Z}_\beta \subseteq \bar{Y}_x \subseteq U_\alpha$. Define a function $a : B \rightarrow A$ by choosing some such index $\alpha \in A$ for each $\beta \in B$, and setting $a(\beta) = \alpha$.

For each $\alpha \in A$, define an open subset $V_\alpha \subseteq X$ by

$$V_\alpha = \bigcup_{a(\beta)=\alpha} Z_\beta$$

(If there are no indices β such that $a(\beta) = \alpha$, then $V_\beta = \emptyset$.) Because the family \mathcal{Z} is locally finite, the closure of V_α is equal to $\bigcup_{a(\beta)=\alpha} \bar{Z}_\beta$ by ??, which is contained in U_α as required. \square

Theorem 1.13.15 (Existence of Partitions of Unity). *Let X be a paracompact Hausdorff space. If \mathcal{U} is any indexed open cover of X , then there is a partition of unity subordinate to \mathcal{U} .*

Proof. Let $\mathcal{U} = (U_\alpha)_{\alpha \in A}$ be an indexed open cover of X . Apply ?? twice, we obtain locally finite open covers $\mathcal{V} = (V_\alpha)_{\alpha \in A}$ and $\mathcal{W} = (W_\alpha)_{\alpha \in A}$ such that $\bar{W}_\alpha \subseteq V_\alpha$ and $\bar{V}_\alpha \subseteq U_\alpha$.

Now, for each $\alpha \in A$, let $f_\alpha : X \rightarrow [0, 1]$ be a bump function for \bar{W}_α supported in V_α . Define $f : X \rightarrow \mathbb{R}$ by $f(x) = \sum_\alpha f_\alpha(x)$. Because $\text{supp}(f_\alpha) \subseteq V_\alpha$, the family of supports $\{\text{supp}(f_\alpha)\}_{\alpha \in A}$ is locally finite; thus each point of X has a neighborhood on which only finitely many terms of this sum are nonzero, so f is continuous. Because the sets $\{W_\alpha\}$ cover X , for each $x \in X$ there

is at least one α such that $x \in W_\alpha$ and thus $f_\alpha(x) = 1$, so f is everywhere positive. Therefore, we can define

$$\psi_\alpha(x) = \frac{f_\alpha(x)}{f(x)}$$

Then $\{\psi_\alpha\}_{\alpha \in A}$ is the desired partition of unity, as one may check. \square

Theorem 1.13.16 (Embeddability of Compact Manifolds). *Every compact manifold is homeomorphic to a subset of some Euclidean space.*

Proof. Suppose M is a compact n -manifold. By compactness we can obtain a cover of M by finitely many open subsets U_1, \dots, U_k , each of which is homeomorphic to \mathbb{R}^n . For each i , let $\varphi_i : U_i \rightarrow \mathbb{R}^n$ be a homeomorphism. Let $\{\psi_i\}$ be a partition of unity subordinate to this cover, and define functions $F_i : M \rightarrow \mathbb{R}^n$ by

$$F_i(x) = \begin{cases} \psi_i(x)\varphi_i(x), & x \in U_i \\ 0, & x \in M \setminus \text{supp}(\psi_i) \end{cases}$$

Each F_i is continuous by the pasting lemma.

Now define $F : M \rightarrow \mathbb{R}^{nk+k}$ by

$$F(x) = (F_1(x), \dots, F_k(x), \psi_1(x), \dots, \psi_k(x))$$

Then F is continuous, and if we can show it is injective, it is a topological embedding by the closed map lemma.

Suppose $F(x) = F(y)$ for some points $x, y \in M$. Since $\sum_i \psi_i(x) \equiv 1$, there is some i such that $\psi_i(x) > 0$ and therefore $x \in U_i$. Because $F(x) = F(y)$ implies $\psi_i(y) = \psi_i(x) > 0$, it follows that $y \in U_i$ as well. Then we see that $F_i(x) = F_i(y)$ implies $\varphi_i(x) = \varphi_i(y)$, which in turn implies that $x = y$ since φ_i is injective on U_i . \square

Theorem 1.13.17. *Suppose M is a topological manifold, and $B \subseteq M$ is any closed subset. Then there exists a continuous function $f : M \rightarrow [0, \infty)$ whose zero set is exactly B .*

Proof. First, consider the special case in which $M = \mathbb{R}^n$ and $B \subseteq \mathbb{R}^n$ is a closed subset. It is straightforward to check that

$$u(x) = \inf\{|x - y| : y \in B\}$$

does the trick.

Now, let M be an arbitrary n -manifold and let B be a closed subset of M . Let $\mathcal{U} = (U_\alpha)_{\alpha \in A}$ be a cover of M by open subsets homeomorphic to \mathbb{R}^n , and let $\{\psi_\alpha\}_{\alpha \in A}$ be a subordinate partition of unity. For each ψ_α the construction in the preceding paragraph yields a continuous function $u_\alpha : U_\alpha \rightarrow [0, \infty)$ such that $u_\alpha^{-1}(0) = B \cap U_\alpha$. Define $f : M \rightarrow \mathbb{R}$ by

$$f(x) = \sum_{\alpha \in A} \psi_\alpha(x) u_\alpha(x)$$

where each summand is to be interpreted as zero outside the support of ψ_α . Each term in this sum is continuous by the pasting lemma, and only finitely many terms are nonzero in a

neighborhood of each point, so this defines a continuous function on M . It is easy to check that it is zero exactly on B . \square

Corollary 1.13.18. *Suppose M is a topological manifold, and A, B are disjoint closed subsets of M . Then there exists a continuous function $f : M \rightarrow [0, 1]$ such that*

$$f^{-1}(0) = A, \quad f^{-1}(1) = B$$

Proof. Using the previous theorem, we can find $u, v : M \rightarrow [0, \infty)$ such that u vanishes exactly on A and v vanishes exactly on B , and then the function $f(x) = v(x)/(u(x) + v(x))$ satisfies the conclusion of the corollary. \square

For our last application of partitions of unity, we need the following definition. If M is a topological space, an **exhaustion function** for M is a continuous function $f : M \rightarrow \mathbb{R}$ such that for every $c \in \mathbb{R}$, the sublevel set $f^{-1}((-\infty, c])$ is compact. The name comes from the fact that as k ranges through the positive integers, the sublevel sets $f^{-1}((-\infty, k])$ form an exhaustion of M by compact sets.

Theorem 1.13.19. *Every manifold admits a positive exhaustion function.*

Proof. Let M be a manifold, let $\{U_i\}$ be a countable open cover of M by precompact open subsets, and let $\{\psi_i\}$ be a partition of unity subordinate to this cover. Define $f : M \rightarrow \mathbb{R}$ by

$$f(x) = \sum_{k=1}^{\infty} k\psi_k(x)$$

Then f is continuous since only finite terms are nonzero in a neighborhood of each point, and positive because $f(x) \geq \sum_k \psi_k(x) = 1$. For any positive integer m , if $x \notin \bigcup_{k=1}^m \bar{U}_k$, then $\psi_k(x) = 0$ for $1 \leq k \leq m$, so

$$f(x) = \sum_{k=m+1}^{\infty} k\psi_k(x) > \sum_{k=m+1}^{\infty} m\psi_k(x) = m \sum_{k=m+1}^{\infty} \psi_k(x) = m$$

The contrapositive of this last statement is that $f(x) \leq m$ implies $x \in \bigcup_{k=1}^m \bar{U}_k$. Let $c \in \mathbb{R}$ be arbitrary, and let m be any positive integer greater than c . It follows that $f^{-1}((-\infty, c])$ is a closed subset of the compact set $\bigcup_{k=1}^m \bar{U}_k$, and is therefore compact. \square

Chapter 2

Uniform structures

2.1 Uniform spaces

Definition 2.1.1. A *uniform structure* (or *uniformity*) on a set X is a structure given by a set \mathfrak{U} of subsets of $X \times X$ which satisfies the following axioms:

- (F1) If $U \in \mathfrak{U}$ and $U \subseteq V \subseteq X \times X$, then $V \in \mathfrak{U}$.
- (F2) If $U \in \mathfrak{U}$ and $V \in \mathfrak{U}$, then $U \cap V \in \mathfrak{U}$.
- (U1) Every set in \mathfrak{U} contains the diagonal Δ .
- (U2) If $U \in \mathfrak{U}$ then $U^{-1} \in \mathfrak{U}$, where $U^{-1} := \{(y, x) : (x, y) \in U\}$ is the *inverse* of U .
- (U3) If $U \in \mathfrak{U}$ then there is $V \in \mathfrak{U}$ such that $V \circ V \subseteq U$, where the composite of two sets $V, W \subseteq X \times X$ is defined by

$$V \circ W := \{(x, z) : \text{there exists } y \in X \text{ such that } (x, y) \in V \text{ and } (y, z) \in W\}.$$

The sets of \mathfrak{U} are called *entourages* of the uniformity defined on X by \mathfrak{U} . A set endowed with a uniformity is called a *uniform space*.

If U is an entourage of a uniformity on X , we may express the relation $(x, y) \in U$ by saying that " x and y are U -close." Note that if X is nonempty then every set in \mathfrak{U} is nonempty by (U1), and (F1) and (F2) imply that \mathfrak{U} is a filter on $X \times X$. There is only one uniformity on the empty set, namely $\{\emptyset\}$.

Throughout this part we shall also write V^2 instead of $V \circ V$, and in general $V^n = V \circ V^{n-1} = V^{n-1} \circ V$, for each integer $n > 1$ and each subset $V \subseteq X \times X$.

Lemma 2.1.2. The conjunction of axioms (U2) and (U3) is equivalent (assuming the other axioms of uniform structures) to the following axiom:

- (U2') For each $U \in \mathfrak{U}$ there exists $V \in \mathfrak{U}$ such that $V \circ V^{-1} \subseteq U$.

Proof. Clearly (U2) and (U3) imply (U2'). Conversely, if (U2') is satisfied, then for every $U \in \mathfrak{U}$, there exists $V \in \mathfrak{U}$ such that $V \circ V^{-1} \subseteq U$, and so

$$V^{-1} = \Delta \circ V^{-1} \subseteq V \circ V^{-1} \subseteq U,$$

thus $V \subseteq U^{-1}$ and by (F1) we have $U^{-1} \in \mathfrak{U}$. This proves (U2). Let $W = V \cap V^{-1}$; then $W \in \mathfrak{U}$ by what we have just proved and (F2), and we have $W \circ W \subseteq V \circ V^{-1} \subseteq U$. \square

Example 2.1.3. Let (X, d) be a metric space. Then a uniformity on X is given by subsets of the form

$$U_\varepsilon = \{(x, y) : d(x, y) < \varepsilon\}.$$

In this case, x and y are U_ε -close precisely when the distance between x and y is at most ε .

A **base** or **fundamental system** of entourages of a uniformity \mathfrak{U} is any set \mathfrak{B} of entourages such that every entourage contains a set belonging to \mathfrak{B} .

Example 2.1.4. Axiom (U3) shows that if n is any positive integer and V runs through a fundamental system of entourages, then the sets V^n (n times) again form a fundamental system of entourages.

Entourages V such that $V = V^{-1}$ are called **symmetric**. If V is any entourage then $V \cap V^{-1}$ and $V \cup V^{-1}$ are symmetric entourages, and axioms (F2) and (U2) show that the symmetric entourages form a fundamental system of entourages.

Proposition 2.1.5. A set \mathfrak{B} of subsets of $X \times X$ is a fundamental system of entourages of a uniformity \mathfrak{U} on X if and only if \mathfrak{B} is a filterbase and satisfies the following axioms:

(B1) Every set in \mathfrak{B} contains the diagonal.

(B2) For every $V \in \mathfrak{B}$ there exists $W \in \mathfrak{B}$ such that $W \subseteq V^{-1}$.

(B3) For every $V \in \mathfrak{B}$ there exists $W \in \mathfrak{B}$ such that $W \circ W \subseteq V$.

Proof. If \mathfrak{B} is a base for a uniformity \mathfrak{U} , then \mathfrak{B} clearly satisfies the claimed properties. Conversely, assume the properties for \mathfrak{B} , and let \mathfrak{U} be the filter in $X \times X$ generated by \mathfrak{B} . We prove that \mathfrak{U} is a uniformity.

First, since \mathfrak{U} is a filter, it satisfies (F1) and (F2). Also, by (B1) we see (U1) is satisfied. Now let $U \in \mathfrak{U}$, then $U \supseteq V$ for some $V \in \mathfrak{B}$. By (B2), there exists $W \in \mathfrak{B}$ such that $W \subseteq V^{-1} \subseteq U^{-1}$. Thus $U^{-1} \in \mathfrak{U}$ and (U2) holds. Also, by (B3) there also exists $W' \in \mathfrak{B}$ such that $W' \circ W' \subseteq V \subseteq U$. Since $W' \in \mathfrak{U}$, we see (U3) also holds. This completes the proof. \square

Example 2.1.6. Let X be a set and R be an equivalence relation on X . Let $\Gamma \subseteq X \times X$ be the graph of R . Then $\Delta \subseteq \Gamma$ and $\Gamma \circ \Gamma = \Gamma^{-1} = \Gamma$. The set of subsets of $X \times X$ which consists of the set C alone is therefore a fundamental system of entourages of a uniformity on X . In particular, if we take R to be the relation of equality, then $C = \Delta$ and the entourages of the corresponding uniformity are all the subsets of $X \times X$ which contain Δ ; this uniformity is called the **discrete uniformity** on X , and the set X endowed with this uniformity is called a **discrete uniform space**.

Example 2.1.7. On the set \mathbb{Z} of rational integers we can define a uniformity, important in the theory of numbers, as follows: given a prime number p , let U_n be the set of all pairs $(x, y) \in \mathbb{Z} \times \mathbb{Z}$ such that $x \equiv y \pmod{p^n}$, for each positive integer n . It is easily verified that the sets U_n (for a fixed p) form a fundamental system of entourages of a uniformity on \mathbb{Z} , called the **p -adic uniformity**.

Now let X and Y be two sets each endowed with uniformities whose sets of entourages are \mathfrak{U} and \mathfrak{V} respectively, then a bijection f of X onto Y is an isomorphism of the uniformity of X onto that of Y if $(f \times f)(\mathfrak{U}) = \mathfrak{V}$.

Now we define the topology induced by a uniformity.

Proposition 2.1.8. *Let X be a set endowed with a uniform structure \mathfrak{U} , and for each $x \in X$ let $\mathfrak{U}(x)$ be the set of subsets $U(x)$ of X , where*

$$U(x) = \{y \in X : (x, y) \in V\}$$

and U runs through the set of entourages of \mathfrak{U} . Then there is a unique topology on X such that, for each $x \in X$, $\mathfrak{U}(x)$ is the neighbourhood filter of x in this topology.

Proof. We have to show that the $\mathfrak{U}(x)$ satisfy conditions of ???. In view of the axioms of uniformity, we only need to prove (U4). To see this, let V be an entourage of \mathfrak{U} , W an entourage of \mathfrak{U} such that $W \circ W \subseteq V$; then if $(x, y) \in W$ and $(y, z) \in W$ we have $(x, z) \in V$, so that $W(y) \subseteq V(x)$ for all $y \in W(x)$, and therefore $V(x) \in \mathfrak{U}(y)$ for all $y \in W(x)$. This completes the proof. \square

The topology defined in ?? is called the **topology induced by the uniform structure \mathfrak{U}** .

Example 2.1.9. The topology induced by the additive uniformity on the set of real numbers is the topology of the real line. On any set X , the topology induced by the discrete uniformity is the discrete topology. The topology induced by the p -adic uniformity is the p -adic topology on \mathbb{Z} .

In the future, when we speak of the topology of a uniform space X , we shall always mean the topology induced by the uniform structure of the space, unless the contrary is expressly stated. The topological space obtained by putting this topology on the set X is sometimes called the topological space *underlying* the uniform space in question. For example, when we say that a uniform space is Hausdorff, or compact, or locally compact, etc., we mean that the underlying topological space has this property.

If X and Y are two uniform spaces, any isomorphism f of the uniform structure of X onto that of Y is also a homeomorphism of X onto Y ; we say that f is an **isomorphism** of the uniform space X onto the uniform space Y . It should be noted that a homeomorphism of X onto Y is not necessarily an isomorphism of the uniform structure of X onto that of Y . In other words, distinct uniformities on the same set X can induce the same topology. For example, on $(0, +\infty)$, the additive and the multiplicative uniformities (which are distinct) induce the same topology.

Proposition 2.1.10. *Let X be a uniform space. For every symmetric entourage V of X and every subset S of $X \times X$, $V \circ S \circ V$ is a neighbourhood of S in the product space $X \times X$, and the closure of S in this space is given by the formula*

$$\bar{S} = \bigcap_{V \in \mathfrak{S}} V \circ S \circ V \tag{2.1.1}$$

where \mathfrak{S} denotes the set of symmetric entourages of X .

Proof. Let V be a symmetric entourage of X . The relation $(x, y) \in V \circ S \circ V$ means that there is an element (p, q) of S such that $(x, p) \in V$ and $(q, y) \in V$: in other words (V being symmetric)

$x \in V(p)$ and $y \in V(q)$, that is $(x, y) \in V(p) \times V(q)$. Since $V(p) \times V(q)$ is a neighbourhood of (p, q) in $X \times X$, the first part of the proposition is proved. The relations $(x, p) \in V, (y, q) \in V$ can also be written $p \in V(x), q \in V(y)$, or $(p, q) \in V(x) \times V(y)$. As V runs through \mathfrak{S} , the sets $V(x) \times V(y)$ form a fundamental system of neighbourhoods of (x, y) in $X \times X$; for if U and V are any two entourages there is always a symmetric entourage $W \subseteq U \cap V$, so that $W(x) \times W(y) \subseteq U(x) \times V(y)$. Hence $V(x) \times V(y)$ meets S for each $V \in \mathfrak{S}$ if and only if $(x, y) \in \bar{S}$, and formula (??) follows. \square

Corollary 2.1.11. *If A is any subset of X and V is any symmetric entourage of X , then $V(A)$ is a neighbourhood of A in X , and*

$$\bar{A} = \bigcap_{V \in \mathfrak{S}} V(A) = \bigcap_{V \in \mathfrak{U}} V(A) \quad (2.1.2)$$

where \mathfrak{U} denotes the set of all entourages in X .

Proof. If $S = A \times A$, then $V \circ S \circ V = V(A) \times V(A)$ for any $V \in \mathfrak{S}$; for the relation "there exists $p \in A$ such that $(x, p) \in V$ " is by definition equivalent to $x \in V(A)$. The corollary now follows from $\bar{S} = \bar{A} \times \bar{A}$. \square

The set $V(A)$ in ?? is called the **V -neighborhood** of A . It should be remarked that as V runs through the set of entourages of X , the sets $V(A)$ do not necessarily form a fundamental system of neighbourhoods of A in X .

Example 2.1.12. On the real line \mathbb{R} , carrying the additive uniformity, as V runs through the filter of entourages of \mathbb{R} the sets $V(\mathbb{Z})$ do not form a fundamental system of neighbourhoods of the set \mathbb{Z} of rational integers: the set

$$\bigcup_{n \in \mathbb{Z}} \left(n - \frac{1}{2^{|n|+1}}, n + \frac{1}{2^{|n|+1}} \right)$$

is a neighborhood of \mathbb{Z} , but it can not contain any $V(\mathbb{Z})$ with V an entourage of \mathbb{R} .

Corollary 2.1.13. *The interiors (resp. the closures) of the entourages of X in $X \times X$ form a fundamental system of entourages of X .*

Proof. If V is any entourage of X , there is a symmetric entourage W such that $W \circ W \circ W \subseteq V$; since $W \circ W \circ W$ is a neighbourhood of W (Propositionuniformity on product space closure char), the interior of V in $X \times X$ contains W and is therefore an entourage. Furthermore, we have $W \subseteq \bar{W} \circ W \circ W \subseteq V$ by ??, and hence V contains the closure of an entourage. \square

Corollary 2.1.14. *Let X be a uniform space. Then the set of closed neighbourhoods of a point of X is a fundamental system of neighbourhoods of the point. Therefore, a Hausdorff uniform space is regular.*

Proof. If x is any point of X and V runs through the entourages of X which are closed in $X \times X$, then the sets $V(x)$ form a fundamental system of neighbourhoods of x in X by ??, and they are closed in X as sections of V . \square

Proposition 2.1.15. *A uniform space X is Hausdorff if and only if the intersection of all the entourages of its uniform structure is the diagonal Δ of $X \times X$. Every Hausdorff uniform space is regular.*

Proof. We have seen that the closed entourages form a fundamental system of entourages; if their intersection is Δ , then Δ is closed in $X \times X$ and consequently X is Hausdorff. Conversely, if X is Hausdorff then for every point $(x, y) \in \Delta$, there is an entourage V of X such that $y \notin V(x)$, or equivalently $(x, y) \notin V$; hence Δ is the intersection of all the entourages. \square

If a uniform space X is Hausdorff, we say that the uniform structure of X is Hausdorff. If \mathfrak{B} is a fundamental system of entourages for this structure, then X is Hausdorff if and only if the intersection of all the sets of \mathfrak{B} is Δ .

2.1.1 Uniformly continuous functions

Proposition 2.1.16. *Let X and Y be uniform spaces and $f : X \rightarrow Y$ be a map and $g = f \times f$. Then the following conditions are equivalent:*

- (i) *For each entourage V of Y , there exists an entourage U of X such that if $(x, y) \in U$ then $(f(x), f(y)) \in V$.*
- (ii) *For each entourage V of Y , $g^{-1}(V)$ is an entourage of X .*

In the admissible case, we say f is **uniformly continuous**.

Proof. Assume (i). Then we see $g^{-1}(V)$ contains the entourage U in (i), so it is an entourage of X . Conversely, if (ii) holds, then it follows from $g(g^{-1}(V)) \subseteq V$ that (i) holds. \square

It follows from definition that a uniformly continuous function must be continuous. On the other hand, a continuous mapping of a uniform space X into a uniform space Y need not be uniformly continuous, as is shown by the example $x \mapsto x^3$, a homeomorphism of \mathbb{R} onto itself, which is not uniformly continuous with respect to the additive uniformity

Example 2.1.17 (Examples of uniformly continuous maps).

- (a) The identity mapping of a uniform space onto itself is uniformly continuous.
- (b) A constant mapping of a uniform space into a uniform space is uniformly continuous.
- (c) Every mapping of a discrete uniform space into a uniform space is uniformly continuous.

The following properties of uniformly continuous maps are immediate.

Proposition 2.1.18.

- (a) *If $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are two uniformly continuous maps, then $g \circ f : X \rightarrow Z$ is uniformly continuous.*
- (b) *A bijection f of a uniform space X onto a uniform space Y is an isomorphism if and only if f and f^{-1} are uniformly continuous.*

Proof. The first claim follows from the interpretation of uniformly continuity in ??(i), and part (b) follows from ??(ii). \square

?? shows that we can take as morphisms of uniform structures the uniformly continuous mappings; we shall always assume in the future that the morphisms have been so chosen. In accordance with the general definitions, this allows us to define an order relation on the set of uniformities on a given set X :

Definition 2.1.19. If \mathfrak{U}_1 and \mathfrak{U}_2 are two uniform structures on the same set X , then \mathfrak{U}_1 is said to be *finer* than \mathfrak{U}_2 (and \mathfrak{U}_2 *coarser* than \mathfrak{U}_1) if the identity map $(X, \mathfrak{U}_1) \rightarrow (X, \mathfrak{U}_2)$ is uniformly continuous.

If \mathfrak{U}_1 is finer than \mathfrak{U}_2 and distinct from \mathfrak{U}_2 we say that \mathfrak{U}_1 is strictly finer than \mathfrak{U}_2 (and that \mathfrak{U}_2 is strictly coarser than \mathfrak{U}_1). Two uniformities are said to be comparable if one is finer than the other. By definition, the following results is clear.

Proposition 2.1.20.

- (a) If \mathfrak{U}_1 and \mathfrak{U}_2 are two uniformities on a set X , then \mathfrak{U}_1 is finer than \mathfrak{U}_2 if and only if every entourage of \mathfrak{U}_2 is an entourage of \mathfrak{U}_1 .
- (b) Let \mathfrak{U}_1 and \mathfrak{U}_2 be two uniformities on a set X and suppose that \mathfrak{U}_1 is finer than \mathfrak{U}_2 ; then the topology induced by \mathfrak{U}_1 is finer than the topology induced by \mathfrak{U}_2 .

It can happen that a uniformity \mathfrak{U} is strictly finer than a uniformity \mathfrak{U}' , but that the two induced topologies are identical. The following example shows this:

Example 2.1.21. Let X be a nonempty set. For each finite partition $\omega = (A_i)_{i=1}^n$ of X , let V_ω denote

$$\bigcup_{i=1}^n (A_i \times A_i)$$

The sets V_ω then form a fundamental system of entourages of a uniformity \mathfrak{U} on X . For if ω is any finite partition of X we have $\Delta \subseteq V_\omega$ and $V_\omega \circ V_\omega = V_\omega^{-1} = V_\omega$; and if $\beta = (B_j)$ and $\delta = (C_k)$ are two finite partitions of X , then those of the sets $B_j \cap C_k$ which are not empty form a finite partition ω of X , and we have $V_\omega \subseteq V_\beta \circ V_\delta$. The uniformity \mathfrak{U} is called the uniformity of finite partitions on X . The topology induced by \mathfrak{U} is the discrete topology, since for each $x \in X$ the sets $\{x\}$ and $\{x\}^c$ form a finite partition of X . Nevertheless, if X is infinite, it is clear that \mathfrak{U} is strictly coarser than the discrete uniformity.

2.1.2 Construction of uniform structures

Proposition 2.1.22. Let X be a set, $(Y_i)_{i \in I}$ be a family of uniform spaces. For each $i \in I$ let $f_i : X \rightarrow Y_i$ be a mapping of X into Y_i and put $g_i = f_i \times f_i$. Let \mathfrak{B} be the set of all finite intersections

$$U(V_{i_1}, \dots, V_{i_n}) = \bigcap_{k=1}^n g_{i_k}^{-1}(V_{i_k})$$

where V_{i_k} is an entourage of Y_{i_k} . Then \mathfrak{B} is a fundamental system of entourages of a uniformity \mathfrak{U} on X which is the initial uniform structure on X with respect to the family (Y_i) , and in particular \mathfrak{U} is the coarsest uniformity on X for which all the mappings f_i are uniformly continuous.

Proof. It is immediately seen that \mathfrak{B} is a filterbase satisfies axioms (B1). If $W_i = g_i^{-1}(V_i)$, then $W_i^{-1} = g_i^{-1}(V_i^{-1})$ and $W_i \circ W_i = g_i^{-1}(V_i \circ V_i)$; hence \mathfrak{B} also satisfies axioms (B2) and (B3) and is therefore a fundamental system of entourages of a uniformity \mathfrak{U} on X . Furthermore, it follows immediately from the definition of \mathfrak{U} that f_i is uniformly continuous for each index $i \in I$; hence $f_i \circ h$ is uniformly continuous for each $i \in I$ if h is. Conversely, suppose that $f_i \circ h$ is uniformly continuous for each $i \in I$, and consider a set $U(V_{i_1}, \dots, V_{i_n})$; by hypothesis, for each $1 \leq j \leq n$, there is an entourage W_k of Z such that the relation $(z, z') \in W_k$ implies

$(f_{i_k}(h(z)), f_{i_k}(h(z'))) \in V_k$; if $W = \bigcap_{j=1}^n W_j$, then these n relations are simultaneously satisfied whenever z and z' are W -close, so that we have then $(h(z), h(z')) \in U(V_{i_1}, \dots, V_{i_n})$ and the proof is complete. \square

Corollary 2.1.23. *The topology on X induced by the coarsest uniformity \mathfrak{U} , for which the f_i are uniformly continuous is also the coarsest topology for which the f_i are continuous.*

Proof. This is an immediate consequence of the definition of the neighbourhoods of a point in this latter topology. \square

The general properties of initial structures imply in particular the following transitivity property:

Proposition 2.1.24. *Let X be a set, $(Z_\beta)_{\beta \in J}$ a family of uniform spaces, $(J_\alpha)_{\alpha \in I}$ a partition of J and $(Y_\alpha)_{\alpha \in I}$ a family of sets indexed by I . For each $\alpha \in I$, let g_α be a mapping of X into Y_α ; and for each $\beta \in J_\alpha$, let $h_{\beta\alpha}$ be a mapping of Y_α into Z_β , and put $f_\beta = h_{\beta\alpha} \circ g_\alpha$.*

$$X \xrightarrow{g_\alpha} Y_\alpha \xrightarrow{h_{\beta\alpha}} Z_\beta$$

If each Y_α carries the initial uniform structure induced by $(h_{\beta\alpha})_{\beta \in J_\alpha}$ then the initial uniform structure on X induced by $(f_\beta)_{\beta \in J}$ is the same as the initial uniform structure induced by $(g_\alpha)_{\alpha \in I}$.

Let X be a set, Y a uniform space, f a mapping of X into Y . The coarsest uniformity \mathfrak{U} on X for which f is uniformly continuous is called the **inverse image** under f of the uniform structure of Y . It follows from ?? that the inverse images under $g = f \times f$ of the entourages of Y form a fundamental system of entourages for \mathfrak{U} . The topology induced by \mathfrak{U} is the inverse image under f of the topology of Y . Note that if $f : X \rightarrow Y$ is surjective, then the entourages of Y are the **direct images** under g of the entourages of X .

Let A be a subset of a uniform space X . The uniformity induced on A by the uniformity of X is the inverse image of the latter under the canonical injection $A \hookrightarrow X$. By ??, this is equivalent to the following definition:

Definition 2.1.25. *Let A be a subset of a uniform space X . The uniformity on A whose set of entourages is the trace on $A \times A$ of the set of entourages of X is called the uniformity induced on A by the uniformity of X .*

The topology induced by the uniformity induced on A is the same as the topology induced on A by the topology of X ; the set A , together with the uniformity and the topology induced by those of X , is called a **uniform subspace** of X .

If A is a subset of a uniform space X and if $f : X \rightarrow Y$ is a uniformly continuous mapping, then the restriction $f|_A$ is a uniformly continuous mapping of A into Y . If $B \subseteq Y$ is such that $f(X) \subseteq B$, then the mapping of X into the uniform subspace B of Y , having the same graph as f , is again uniformly continuous.

Proposition 2.1.26. *Let A be a dense subset of a uniform space X . Then the closures in $X \times X$ of the entourages of the uniform subspace A form a fundamental system of entourages of X .*

Proof. First we note that, if U is open in X then for any subset B of X we have $U \cap \bar{B} \subseteq \overline{U \cap B}$. In fact, if $x \in U \cap \bar{B}$, then for any neighborhood V of x , $U \cap V$ is also a neighborhood of x and thus $U \cap V \cap B \neq \emptyset$ (since $x \in \bar{B}$). This implies $x \in \overline{U \cap B}$.

Now let V be an open entourage of A ; it is the intersection of $A \times A$ with an open entourage U of X . By what we have proved, $\bar{V} \subseteq U \subseteq \bar{V}$. Thus the claim follows. \square

Note that every family $(\mathfrak{U}_i)_{i \in I}$ of uniformities on a set X has a **least upper bound** \mathfrak{U} in the ordered set of all uniformities on X : we have only to apply ??, taking Y_i to be the set X with the uniformity \mathfrak{U}_i and f_i to be the identity mapping $X \rightarrow Y_i$. The topology induced by \mathfrak{U} is just the least upper bound of the topologies induced by the \mathfrak{U}_i .

Example 2.1.27. If $\omega = (A_i)_{i=1}^n$ is any finite partition of a nonempty set X , the set $V_\omega = \bigcup_{i=1}^n A_i \times A_i$ by itself constitutes a fundamental system of entourages of a uniformity \mathfrak{U}_ω on X ; the uniformity of finite partitions on X (??) is then the least upper bound of the uniformities \mathfrak{U}_ω .

Using initial uniformity, we can consider products of uniform spaces. If $(X_i)_{i \in I}$ is a family of uniform spaces, the product uniform space of this family is the product set $X = \prod_{i \in I} X_i$ endowed with the coarsest uniformity for which the projections $\pi_i : X \rightarrow X_i$ are uniformly continuous. This uniformity is called the **product** of the uniformities of the X_i and the uniform spaces X_i are called the **factors** of X . By ??, the topology induced by the product uniformity on X is same as the product of the topologies of the X_i .

Proposition 2.1.28. Let $f = (f_i)$ be a mapping of a uniform space Y into a product uniform space $X = \prod_{i \in I} X_i$. Then f is uniformly continuous if and only if each f_i is uniformly continuous.

Proof. Since $f_i = \pi_i \circ f$, this is a particular case of ??.

Corollary 2.1.29. Let $(X_i)_{i \in I}$ and $(Y_i)_{i \in I}$ be two families of uniform spaces indexed by the same set I . For each $i \in I$, let f_i be a mapping of X_i into Y_i . If each of the f_i is uniformly continuous, then so is the product mapping

$$f : \prod_{i \in I} X_i \rightarrow \prod_{i \in I} Y_i, \quad (x_i) \mapsto (f_i(x_i)).$$

Conversely, if the X_i 's are nonempty and f is uniformly continuous, then each f_i is uniformly continuous.

Proof. The map f can be written as $x \mapsto (f_i(\pi_i(x)))$, and the first part of the corollary therefore follows from ???. Conversely, choose $a = (a_i) \in \prod_{i \in I} X_i$. For each $j \in I$ let g_j be the mapping of X_j into $\prod_{i \in I} X_i$ such that $\pi_j(g_j(x)) = x_j$ and $\pi_i(g_j(x)) = a_i$, whenever $i \neq j$. Then g_j is uniformly continuous at the point a_j . Since $f_j = \pi_j \circ f \circ g_j$, it follows that if f is uniformly continuous at a , then f_j is uniformly continuous at a_j . \square

The general criterion of transitivity of initial uniformities shows that, as for the product of topological spaces, the product of uniform spaces is associative and that the following is true:

Proposition 2.1.30. Let X be a set, let $(Y_i)_{i \in I}$ be a family of uniform spaces, and for each $i \in I$ let f_i be a mapping of X into Y_i . Let f be the mapping $x \mapsto (f_i(x))$ of X into $Y = \prod_{i \in I} Y_i$ and let \mathfrak{U} be the coarsest uniformity on X for which the f_i are uniformly continuous. Then \mathfrak{U} is the inverse image under f of the uniformity induced on $f(X)$ by the product uniformity on Y .

Corollary 2.1.31. *For each $i \in I$, let A_i be a subspace of X_i . Then the uniformity induced on $A = \prod_{i \in I} A_i$ by the product uniformity on $\prod_{i \in I} X_i$ is the same as the product of the uniformities of the subspaces A_i .*

To conclude this part, we introduce the inverse limit of uniform spaces. Let (I, \preceq) be a partially ordered set. For each $\alpha \in I$ let X_α be a uniform space, and for each pair of indices α, β such that $\alpha \preceq \beta$ there is a map $f_{\alpha\beta} : X_\beta \rightarrow X_\alpha$. We shall say that $(X_\alpha, f_{\alpha\beta})$ is an inverse system of uniform spaces if

- (a) $(X_\alpha, f_{\alpha\beta})$ is an inverse system sets.
- (b) Each map $f_{\alpha\beta}$ is uniformly continuous.

On $X = \varprojlim X_i$ the coarsest uniformity for which the canonical mappings $f_\alpha : X \rightarrow X_\alpha$ are uniformly continuous is called the inverse limit (with respect to the $f_{\alpha\beta}$) of the uniformities of the X_α , and the set X endowed with this coarsest uniformity is called the **inverse limit of the inverse system of uniform spaces** $(X_\alpha, f_{\alpha\beta})$.

Proposition 2.1.32. *Let I be a directed set, let $(X_\alpha, f_{\alpha\beta})$ be an inverse system of uniform spaces indexed by I , and let J be a cofinal subset of I . For each $\alpha \in I$ let f_α be the canonical mapping of $X = \varprojlim X_\alpha$ into X_α , and let g_α denote $f_\alpha \times f_\alpha$. Then the family of sets $g_\alpha^{-1}(V_\alpha)$, where α runs through J and where, for each $\alpha \in J$, V_α runs through a fundamental system of entourages of X_α , is a fundamental system of entourages of X .*

Finally, the topology on $X = \varprojlim X_\alpha$ induced by the inverse limit of the uniformities of the X_α is the inverse limit of the topologies of the X_α .

2.2 Complete spaces

2.2.1 Cauchy filters

Once a set X has been endowed with a uniform structure we can define what is meant by a "small" subset of X (relative to this structure): a "small" subset of X is one in which all the points are "very close" to each other. Precisely:

Definition 2.2.1. *If X is a uniform space and if V is an entourage of X , a subset A of X is said to be V -small if every pair of points of A are V -close (in other words, if $A \times A \subseteq V$).*

The following result can be viewed as a "triangle inequality" property for U -closeness.

Proposition 2.2.2. *In a uniform space X , if two sets A and B are V -small and intersect, then their union $A \cup B$ is $V \circ V$ -small.*

Proof. Let x and y be any two points of $A \cup B$, and let $z \in A \cap B$. Then $(x, z) \in V$ and $(z, y) \in V$, so that $(x, y) \in V \circ V$. \square

A filter \mathfrak{F} on a uniform space X is a Cauchy filter if for each entourage V of X there is a subset of X which is V -small and belongs to \mathfrak{F} .

Remark 2.2.1. An infinite sequence (x_n) of points of a uniform space X is said to be a **Cauchy sequence** if the elementary filter associated with the sequence is a Cauchy filter. It comes to the same thing to say that for each entourage V of X there is an integer N such that for all integers $n, m > N$, we have $(x_n, x_m) \in V$.

Example 2.2.3 (Example of Cauchy filters).

- (a) Let X be a discrete uniform space, then the Cauchy filters on X are trivial filters: since Δ is the unique entourage on X , any Cauchy filter must contain a singleton.
- (b) Let X be an infinite set and endow X with the finite partition uniformity. Then any Cauchy filter \mathfrak{F} on X is ultra: if $A \subseteq X$, then A and A^c form a partition of X , and with $V = (A \times A) \cup (A^c \times A^c)$, we see either A contains a set in \mathfrak{F} or A^c contains a set in \mathfrak{F} . Thus by ??, \mathfrak{F} is ultra.

Proposition 2.2.4. *On a uniform space X every convergent filter is a Cauchy filter.*

Proof. If x is any point of X and V is any symmetric entourage of X , then the neighbourhood $V(x)$ of x is $V \circ V$ -small. If \mathfrak{F} is a filter which converges to x , there is a set of \mathfrak{F} contained in $V(x)$, and therefore $V \circ V$ -small. Together with axiom (U3), this proves \mathfrak{F} is Cauchy. \square

Proposition 2.2.5. *Let $f : X \rightarrow Y$ be a uniformly continuous mapping. Then the image under f of a Cauchy filter base on X is a Cauchy filter base on Y .*

Proof. Let $g = f \times f$. If V is an entourage of Y , then $g^{-1}(V)$ is an entourage of X , and the image under f of a $g^{-1}(V)$ -small set is V -small; hence the result. \square

It follows in particular that if the uniformity of a uniform space X is replaced by a coarser uniformity, then every Cauchy filter with respect to the original uniformity remains a Cauchy filter with respect to the new uniformity. This fact can be easily remembered in the following form: the finer the uniformity, the fewer Cauchy filters there are.

Proposition 2.2.6. *Let X be a set, let $(Y_i)_{i \in I}$ be a family of uniform spaces, and for each $i \in I$ let f_i be a mapping of X into Y_i . Let X carry the coarsest uniformity \mathfrak{U} for which the f_i are uniformly continuous. Then in order that a filter base \mathfrak{B} on X should be a Cauchy filter base it is necessary and sufficient that $f_i(\mathfrak{B})$ should be a Cauchy filter base on Y_i for each $i \in I$.*

Proof. The condition is necessary by ???. Conversely, suppose that it is satisfied, and let $U(V_{i_1}, \dots, V_{i_n})$ be an entourage of the uniformity \mathfrak{U} . By hypothesis, for each index k there is a set $M_k \in \mathfrak{B}$ such that $f_{i_k}(M_k)$ is V_{i_k} -small. Let M be a set of \mathfrak{B} contained in M_k for $1 \leq k \leq n$; then for each pair of points x, y of M we have $(f_{i_k}(x), f_{i_k}(y)) \in V_{i_k}$, $1 \leq k \leq n$, so that

$$(x, y) \in U(V_{i_1}, \dots, V_{i_n}).$$

This completes the proof. \square

Corollary 2.2.7. *If a Cauchy filter on a uniform space X induces a filter on a subset A of X , then this filter is a Cauchy filter on the uniform subspace A .*

Corollary 2.2.8. *A filter base \mathfrak{B} on a product $\prod_{i \in I} X_i$ of uniform spaces is a Cauchy filter base if and only if, for each $i \in I$, $\pi_i(\mathfrak{B})$ is a Cauchy filter base on X_i .*

The minimal elements (with respect to inclusion) of the set of Cauchy filters on a uniform space X are called **minimal Cauchy filters** on X .

Proposition 2.2.9. *Let X be a uniform space. For each Cauchy filter \mathfrak{F} on X there is a unique minimal Cauchy filter \mathfrak{F}_0 coarser than \mathfrak{F} . If \mathfrak{B} is a base of \mathfrak{F} and \mathfrak{S} is a fundamental system of symmetric entourages of X , then the sets $\{V(M) : M \in \mathfrak{B}, V \in \mathfrak{S}\}$ form a base of \mathfrak{F}_0 .*

Proof. If M, M' are in \mathfrak{B} and V, V' are in \mathfrak{S} , then there is a set $N \in \mathfrak{B}$ and $W \in \mathfrak{S}$ such that $N \subseteq M \cap M'$ and $W \subseteq V \cap V'$. Hence

$$W(N) \subseteq V(M) \cap V'(M')$$

and therefore the sets $\{V(M) : M \in \mathfrak{B}, V \in \mathfrak{S}\}$ indeed form a base of a filter \mathfrak{F}_0 on X . Further, if M is V -small, then $V(M)$ is V^3 -small; hence \mathfrak{F}_0 is a Cauchy filter and is clearly coarser than \mathfrak{F} . To complete the proof it is enough to show that if \mathfrak{G} is a Cauchy filter coarser than \mathfrak{F} , then \mathfrak{G} is finer than \mathfrak{F}_0 . For each $M \in \mathfrak{B}$ and each $V \in \mathfrak{S}$ there is a set $N \in \mathfrak{G}$ which is V -small; since $N \in \mathfrak{F}$, N must meet M ; hence $N \subseteq V(M)$ and so $V(M) \in \mathfrak{G}$. \square

Corollary 2.2.10. *For each $x \in X$, the neighbourhood filter $\mathfrak{U}(x)$ of x in X is a minimal Cauchy filter.*

Proof. Take \mathfrak{F} in ?? to be the filter of all subsets of X which contain x , and take \mathfrak{B} to consist of the single element $\{x\}$. Then it is easy to see $\mathfrak{U}(x)$ is the minimal Cauchy filter contained in \mathfrak{F} . \square

Corollary 2.2.11. *Every cluster point x of a Cauchy filter \mathfrak{F} is a limit point of \mathfrak{F} .*

Proof. Since x is a cluster point of \mathfrak{F} , there is a filter \mathfrak{G} which is finer than both \mathfrak{F} and $\mathfrak{U}(x)$; since \mathfrak{F} is a Cauchy filter, so is \mathfrak{G} . If \mathfrak{F}_0 is the unique minimal Cauchy filter coarser than \mathfrak{F} , then both \mathfrak{F}_0 and $\mathfrak{U}(x)$ are minimal Cauchy filters coarser than \mathfrak{G} . Hence $\mathfrak{F}_0 = \mathfrak{U}(x)$, which shows that \mathfrak{F} converges to x . \square

Corollary 2.2.12. *Every Cauchy filter \mathfrak{F} which is coarser than a filter converging to a point x , also converges to x .*

Proof. This is a consequence of ??, since in this situation x is a cluster point of \mathfrak{F} . \square

Corollary 2.2.13. *If \mathfrak{F} is a minimal Cauchy filter, then every set of \mathfrak{F} has a nonempty interior which also belongs to \mathfrak{F} (in other words, \mathfrak{F} has a base consisting of open sets).*

Proof. Let V be any entourage of X ; then there is an open entourage $U \subseteq V$. For each subset M of X , $U(M)$ is open and contained in $V(M)$; hence the result, in view of ??.

2.2.2 Complete spaces

In a uniform space X , a Cauchy filter need not have a limit point.

Example 2.2.14. Let X be an infinite set, and consider the uniformity of finite partitions on X . Every ultrafilter \mathfrak{F} on X is a Cauchy filter with respect to this uniformity. For if $(A_i)_{i=1}^n$ is a finite partition of X and

$$V = \bigcup_{i=1}^n A_i \times A_i$$

is the corresponding entourage, then at least one of the A_i belongs to \mathfrak{F} , and A_i is V -small. On the other hand, X is an infinite discrete space, hence is not compact, and consequently there are ultrafilters on X which do not converge.

A **complete space** is a uniform space in which every Cauchy filter converges. For example, on a discrete uniform space X a Cauchy filter is a trivial ultrafilter, hence convergent; consequently, X is complete.

From definition and ??, we deduce the following proposition, known as Cauchy's criterion:

Proposition 2.2.15. *Let \mathfrak{F} be a filter on a set X , and let f be a mapping of X into a complete uniform space Y . Then f has a limit with respect to \mathfrak{F} if and only if the image of \mathfrak{F} under f is a Cauchy filter base.*

This criterion shows the importance of complete spaces in all questions involving the notion of limit: if a function takes its values in a complete space we can prove the existence of a limit without knowing in advance the value of the limit; this would be impossible if the definition of limit were the only criterion of convergence at our disposal.

A uniformity which is finer than the uniformity of a complete space need not be a uniformity of a complete space. However, we have the following proposition:

Proposition 2.2.16. *Let $\mathfrak{U}_1, \mathfrak{U}_2$ be two uniformities on a set X , and let $\mathcal{T}_1, \mathcal{T}_2$ be the topologies induced by these uniformities respectively. Suppose that \mathfrak{U}_1 is finer than \mathfrak{U}_2 , and that there is a fundamental system of entourages for \mathfrak{U}_1 which are closed in $X \times X$ in the topology $\mathcal{T}_2 \times \mathcal{T}_2$. Then a filter \mathfrak{F} on X converges in the topology \mathcal{T}_1 if and only if it is a Cauchy filter in the uniformity \mathfrak{U}_1 and converges in the topology \mathcal{T}_2 .*

Proof. The conditions are clearly necessary, because \mathcal{T}_2 is coarser than \mathcal{T}_1 . Conversely, suppose that the conditions are satisfied, and let x be a limit point of \mathfrak{F} with respect to \mathcal{T}_2 ; we shall show that x is a limit of \mathfrak{F} with respect to \mathcal{T}_1 . Let V be a symmetric entourage of \mathfrak{U}_1 which is closed in the topology $\mathcal{T}_2 \times \mathcal{T}_2$. By hypothesis, \mathfrak{F} contains a set M which is V -small; hence if $y \in M$ we have $M \subseteq V(y)$. But $V(y)$ is closed in the topology \mathcal{T}_2 ; hence x , which lies in the closure of M with respect to \mathcal{T}_2 (?), must belong to $V(y)$. It follows that $M \subseteq (V \circ V)(x)$, and the proposition is proved. \square

Corollary 2.2.17. *In the conditions of ??, if \mathfrak{U}_2 is a uniformity of a complete space, then so is \mathfrak{U}_1 .*

Proof. For every Cauchy filter with respect to \mathfrak{U}_1 is then a Cauchy filter with respect to \mathfrak{U}_2 and therefore converges in the topology \mathcal{T}_2 . \square

Now we consider completeness of subspaces. First, we have the following proposition.

Proposition 2.2.18. *Every closed subspace of a complete space is complete. Every complete subspace of a Hausdorff uniform space (complete or not) is closed.*

Proof. Let X be a complete space and let A be a closed subspace of X . If \mathfrak{F} is a Cauchy filter on A , then it is a Cauchy filter base on X and therefore converges to a point $x \in X$; but since A is closed we have $x \in A$, and therefore \mathfrak{F} converges in the subspace A .

Now let A be a non-closed subset of a Hausdorff uniform space X , and let $x \in \bar{A} \setminus A$. The trace \mathfrak{B}_A on A of the neighbourhood filter \mathfrak{B} of x in X is a Cauchy filter on A ; but it cannot converge to a point $y \in A$, otherwise y would be a limit point of \mathfrak{B} which is absurd since $x \neq y$ and X is Hausdorff. \square

The following proposition says, for the completeness of X , it suffices to consider Cauchy filters on a dense subset of X . This will be useful when we consider completion of uniform spaces.

Proposition 2.2.19. *Let X be a uniform space and let A be a dense subset of X such that every Cauchy filter base on A converges in X . Then X is complete.*

Proof. It is enough to show that every minimal Cauchy filter \mathfrak{F} on X is convergent. Since A is dense and since every set of \mathfrak{F} has a non-empty interior (??), the trace \mathfrak{F}_A of \mathfrak{F} on A is a Cauchy filter on A , hence converges to a point $x \in X$. Since \mathfrak{F} is coarser than the filter on X generated by \mathfrak{F}_A , it follows that \mathfrak{F} converges to x . \square

2.2.3 Product and inverse limit of complete spaces

Proposition 2.2.20. *Every product of complete uniform spaces is complete. Conversely, if a product of nonempty uniform spaces is complete, then each of the factors is a complete uniform space.*

Proof. The first assertion is a consequence of the characterization of Cauchy filters and convergent filters on a product space. Conversely, suppose $X = \prod_{i \in I} X_i$ is complete (the X_i being nonempty) and let \mathfrak{F}_j be a Cauchy filter on X_j . For each $i \neq j$ let \mathfrak{F}_i be a Cauchy filter on X_i and consider the product filter

$$\mathfrak{F} = \prod_{i \in I} \mathfrak{F}_i.$$

Then \mathfrak{F} is a Cauchy filter, hence is convergent, and therefore so is each $\mathfrak{F}_j = \pi_j(\mathfrak{F})$. \square

Corollary 2.2.21. *Let $(X_\alpha, f_{\alpha\beta})$ be an inverse system of uniform spaces. If the X_α are Hausdorff and complete, then so is $X = \varprojlim X_\alpha$.*

Proof. For X is Hausdorff and X can be identified with a closed subspace of $\prod_{i \in I} X_i$, the corollary therefore follows from ?? and ??.

An inverse limit of complete Hausdorff uniform spaces X_α can be empty, even if all the X_α are non-empty and all the $f_{\alpha\beta}$ are surjective, as is shown by the following example.

Example 2.2.22. For each finite subset α of the uncountable set \mathbb{R} , let X_α be the set of injections of α into the integers. Each X_α is countable, and for $\alpha \subseteq \beta$ the restriction map $X_\beta \rightarrow X_\alpha$ is surjective. But the inverse limit of the X_α is empty, since an element of it would give an injection of \mathbb{R} into the integers.

However, we have the following theorem:

Theorem 2.2.23 (Mittag-Leffler). *Let $(X_\alpha, f_{\alpha\beta})$ be an inverse system of complete Hausdorff uniform spaces, indexed by a directed set I which has a countable cofinal subset; suppose also that, for each $\alpha \in I$, X_α has a countable fundamental system of entourages. Finally, suppose that for each $\alpha \in I$ there is an index $\beta \succeq \alpha$ satisfying the following condition:*

$(ML_{\alpha\beta})$ *For each $\gamma \succeq \beta$, $f_{\alpha\gamma}(X_\gamma)$ is dense in $f_{\alpha\beta}(X_\beta)$.*

Let $X = \varprojlim X_\alpha$ and let $f_\alpha : X \rightarrow X_\alpha$ be the canonical map. Then for each $\alpha \in I$ and each $\beta \succeq \alpha$ satisfying $(ML_{\alpha\beta})$, $f_\alpha(X)$ is dense in $f_\beta(X)$ (and consequently X is nonempty if the X_α are all nonempty).

Proof. Let (λ_n) be a sequence of indices cofinal in I . Start with an index $\alpha_0 \in I$ and define recursively an increasing sequence (α_n) such that $\alpha_n \succeq \lambda_n$ and $(\text{ML}_{\alpha_n \alpha_{n+1}})$ is true. Clearly the sequence (α_n) is cofinal in I . We shall write f_{mn} in place of $f_{\alpha_m \alpha_n}$ for $m \leq n$ and put $Y_n = f_{n,n+1}(X_{\alpha_{n+1}})$. For $m \leq n$, we see $f_{mn}(Y_n)$ is dense in Y_m : by definition $f_{m,n+1}(X_{\alpha_{n+1}})$ is dense in $f_{m,m+1}(X_{\alpha_{m+1}}) = Y_m$, and we have

$$f_{m,n+1}(X_{\alpha_{n+1}}) = (f_{mn} \circ f_{n,n+1})(X_{\alpha_{n+1}}) = f_{mn}(Y_n).$$

Now we define a fundamental system $(V_{kn})_{k \in \mathbb{N}}$ of closed symmetric entourages of X_{α_n} for each n such that

$$(a) \quad V_{k+1,n} \circ V_{k+1,n} \subseteq V_{kn}.$$

$$(b) \quad (f_{n,n+1} \times f_{n,n+1})(V_{k,n+1}) \subseteq V_{kn}.$$

We proceed by induction. Let $(U_{kn})_{k \in \mathbb{N}}$ be a fundamental system of entourages of X_{α_n} . If we suppose that the V_{kn} have been defined for a given n and for all $k \in \mathbb{N}$, then since $f_{n,n+1}$ is uniformly continuous we can define the entourage $V_{k,n+1}$ by induction on k so that (b) is satisfied and

$$V_{k+1,n+1} \circ V_{k+1,n+1} \subseteq V_{k,n+1} \cap U_{k+1,n+1}.$$

This gives the desired system (V_{kn}) .

Now let $x_0 \in Y_0$. We shall show that for each positive integer k there is a point $z \in X$ such that $(x_0, f_{\alpha_0}(z)) \in V_{k-1,0}$; this will prove the theorem. Since $f_{n,n+1}(Y_{n+1})$ is dense in Y_n , we can define by induction a sequence of points $x_n \in Y_n$ such that

$$(x_n, f_{n,n+1}(x_{n+1})) \in V_{k+n,n}$$

By condition (b) it follows that if $m \leq n$ then

$$(f_{mn}(x_n), f_{m,n+1}(x_{n+1})) \in V_{k+n,m}. \quad (2.2.1)$$

From this we conclude that for fixed m the sequence $(f_{mn}(x_n))_{n \geq m}$ is a Cauchy sequence in X_{α_m} and therefore converges to a point z_m ; for by induction it follows from (??) that, for each pair of integers $p, q \geq m$, we have

$$(f_{mp}(x_p), f_{mq}(x_q)) \in V_{k+q-1,m} \circ V_{k+q-2,m} \circ \cdots \circ V_{k+p,m} \quad (2.2.2)$$

and by virtue of (a) it is clear that the right-hand side of (??) is contained in $V_{k+p-1,m}$. Let q increase indefinitely; we infer in particular that, for $p = m = 0$, we have $(x_0, z_0) \in V_{k-1,0}$ since $V_{k-1,0}$ is closed. On the other hand, from the relations $z_m = \lim_n f_{mn}(x_n)$ and from the continuity of $f_{m,m+1}$ we deduce that $f_{m,m+1}(z_{m+1}) = z_m$ for each $m \geq 0$. For each $\gamma \in I$ there is at least one integer n such that $\alpha_n \succeq \gamma$; putting $z_\gamma = f_{\gamma,\alpha_n}(z_n)$ we verify immediately that z_γ does not depend on the value of n such that $\alpha_n \succeq \gamma$, and that the family $(z_\gamma)_{\gamma \in I}$ so defined is a point z of $X = \varprojlim X_\alpha$. Since $f_{\alpha_0}(z) = z_0$, the proof is complete. \square

Corollary 2.2.24. *Let $(X_\alpha, f_{\alpha\beta})$ be an inverse system of sets indexed by a directed set I which has a countable cofinal subset, and suppose that the $f_{\alpha\beta}$ are surjective. Then if $X = \varprojlim X_\alpha$, the canonical mapping $f_\alpha : X \rightarrow X_\alpha$ is surjective for each $\alpha \in I$.*

Proof. Let each X_α carry the discrete uniformity, and apply ??.

□

Corollary 2.2.25. *Let I be a directed set which has a countable cofinal subset. Let $(X_\alpha, f_{\alpha\beta})$ and $(Y_\alpha, g_{\alpha\beta})$ be two inverse systems of sets indexed by I and (ϕ_α) be an inverse system of maps with $\phi = \varprojlim \phi_\alpha$. Let $y = (y_\alpha)$ be an element of $Y = \varprojlim Y_\alpha$ which satisfies the following condition:*

- *For each $\alpha \in I$ there is an index $\beta \succeq \alpha$ such that for all $\gamma \succeq \beta$ we have*

$$f_{\alpha\gamma}(\phi_\gamma^{-1}(y_\gamma)) = f_{\alpha\beta}(\phi_\beta^{-1}(y_\beta)).$$

Then there is an element $x \in X$ such that $\phi(x) = y$.

Proof. Apply ?? to the inverse system of sets $\phi_\alpha^{-1}(y_\alpha)$, each carrying the discrete uniformity. □

Example 2.2.26. Suppose we are given in \mathbb{C} :

- a sequence (a_n) of distinct points such that the sequence $(|a_n|)$ is increasing and tends to $+\infty$;
- for each n a rational function $R_n(z)$ defined in $\mathbb{C} \setminus \{a_n\}$ and having a pole at a_n ;
- an increasing sequence (B_n) of open discs centred at 0 whose union is \mathbb{C} , and such that none of the a_n is on the boundary of any of the discs B_m .

For each n , define $C_n = \bar{B}_n \setminus \{a_n\}$, and let X_n denote the set of all mappings from C_n to \mathbb{C} of the form

$$S(z) = P(z) + \sum_{a_k \in B_n} R_k(z)$$

where P is the restriction to C_n of a function which is continuous in \bar{B}_n and holomorphic in B_n . We define a metric in X_n by putting

$$d_n(S_1, S_2) = \sup_{z \in C_n} |S_1(z) - S_2(z)|.$$

It is easily verified that X_n is complete with respect to this metric. Finally, for $n \leq m$, we define a mapping $f_{mn} : X_m \rightarrow X_n$ such that if $S \in X_m$ then $f_{nm}(S)$ is the restriction of S to C_n . It is clear that the f_{mn} are uniformly continuous and that (X_m, f_{mn}) is an inverse system of uniform spaces. Now an element of the inverse limit $X = \varprojlim X_n$ can be canonically identified with a meromorphic function f in \mathbb{C} , whose only poles are the points a_n and which is such that for each n , $f(z) - R_n(z)$ is holomorphic at a_n . The classical theorem of Mittag-Leffler asserts that X is not empty; by virtue of ??, we have only to verify the condition (ML_{nn}) for all n . Let

$$S_n = P_n + \sum_{a_k \in B_n} R_k$$

be an element of X_n where P_n is continuous in \bar{B}_n and holomorphic in B_n ; for any $m \geq n$, let Q_{mn} be the restriction of $\sum_{a_i \in B_m \setminus B_n} R_i$ to C_n ; this sum is a holomorphic function in some neighbourhood of \bar{B}_n , hence (by Taylor's theorem) for each $\varepsilon > 0$ there is a polynomial P_{mn} such that $|Q_{mn} - P_{mn}| < \varepsilon$ in B_n ; if S_m is the restriction of $S_n + Q_{mn} - P_{mn}$ to C_m , we have $S_m \in X_m$ and $|S_m - S_n| < \varepsilon$ in B_n . This completes the proof.

2.2.4 Extension of uniformly continuous functions

The theorem of extension by continuity has important additions when the functions in question take their values in a complete Hausdorff uniform space.

Theorem 2.2.27. *Let A be a dense subset of a topological space X , and let f be a mapping of A into a complete Hausdorff uniform space Y . Then f can be extended by continuity to X if and only if, for each $x \in X$, the image under f of the trace on A of the neighbourhood filter of x in X is a Cauchy filter base in Y .*

Proof. This follows from the theorem of extension by continuity because Y is regular and because on Y convergent filters are the same as Cauchy filters. \square

When X is also a uniform space, there is the following theorem:

Theorem 2.2.28. *Let f be a function defined on a dense subspace A of a uniform space X , taking its values in a complete Hausdorff uniform space Y , and suppose that f is uniformly continuous on A . Then f can be extended to the whole of X by continuity, and the extended function \bar{f} is uniformly continuous.*

Proof. The existence of \bar{f} is an immediate consequence of ???. Let us show that \bar{f} is uniformly continuous. Let V be a closed symmetric entourage of Y , and let U be an entourage of X such that, when x and y are in A and are U -close, then $f(x)$ and $f(y)$ are V -close. We may assume that U is the closure $X \times X$ of an entourage W of A (??). We have $(\bar{f}(x), \bar{f}(y)) \in V$ when $(x, y) \in W$; since $\bar{f} \times \bar{f}$ is continuous in $X \times X$ and V is closed, we see $(\bar{f}(x), \bar{f}(y)) \in V$ when $(x, y) \in U = \overline{W}$. This proves the claim. \square

Corollary 2.2.29. *Let X_1, X_2 be two complete Hausdorff uniform spaces, and let A_1, A_2 be dense subspaces of X_1, X_2 respectively. Then every isomorphism $f : A_1 \rightarrow A_2$ extends to an isomorphism of X_1 onto X_2 .*

Proof. The map f is uniformly continuous in A_1 , hence extends to a uniformly continuous mapping $\bar{f} : X_1 \rightarrow X_2$. Likewise the inverse g of f extends to a uniformly continuous mapping $\bar{g} : X_2 \rightarrow X_1$. The function $\bar{g} \circ \bar{f}$ is therefore a continuous mapping of X_1 into itself whose restriction to A_1 is the identity mapping; by the density of A_1 it follows that $\bar{g} \circ \bar{f}$ is the identity mapping of X_1 ; similarly $\bar{f} \circ \bar{g}$ is the identity map of X_2 . Consequently \bar{f} and \bar{g} are bijections and are inverses of each other; they are also uniformly continuous and are therefore isomorphisms. \square

2.2.5 Completion of uniform spaces

Theorem 2.2.30. *Let X be a uniform space. Then there exists a complete Hausdorff uniform space \widehat{X} and a uniformly continuous mapping $\iota : X \rightarrow \widehat{X}$ having the following property:*

(P) *Given any uniformly continuous mapping f of X into a complete Hausdorff uniform space Y , there is a unique uniformly continuous mapping $g : X \rightarrow Y$ such that $f = g \circ \iota$.*

If (\widehat{X}', ι') is another pair consisting of a complete Hausdorff uniform space \widehat{X}' and a uniformly continuous mapping $\iota' : X \rightarrow \widehat{X}'$ having the property (P), then there is a unique isomorphism $\phi : \widehat{X} \rightarrow \widehat{X}'$ such that $\iota' = \phi \circ \iota$.

Proof. Let X be the set of minimal Cauchy filters on X . We shall define a uniform structure on X . For this purpose, if V is any symmetric entourage of X , let \widehat{V} denote the set of all pairs $(\mathfrak{X}, \mathfrak{Y})$ of minimal Cauchy filters which have in common a V -small set. We shall show that the sets \widehat{V} form a fundamental system of entourages of a uniform structure on \widehat{X} .

If V and V' are two symmetric entourages of X , then $W = V \cap V'$ is a symmetric entourage, and every set which is W -small is also V -small and V' -small; hence $\widehat{W} \subseteq \widehat{V} \cap \widehat{V}'$. Also, it is easy to see $\widehat{V} \neq \emptyset$ for any symmetric entourage V , therefore $\{\widehat{V}\}$ is a filter base. Now since each $\mathfrak{X} \in \widehat{X}$ is a Cauchy filter, we have by definition $(\mathfrak{X}, \mathfrak{X}) \in \widehat{V}$ for every symmetric entourage V of X ; hence axiom (B1) is satisfied. The sets V are symmetric by definition, hence (B2) is satisfied. Finally, given a symmetric entourage V of X , let W be a symmetric entourage such that $W \circ W \subseteq V$. Consider three minimal Cauchy filters $\mathfrak{X}, \mathfrak{Y}, \mathfrak{Z}$ such that $(\mathfrak{X}, \mathfrak{Y}) \in \widehat{W}$ and $(\mathfrak{Y}, \mathfrak{Z}) \in \widehat{W}$; then there are two W -small sets M, N such that $M \in \mathfrak{X} \cap \mathfrak{Y}$ and $N \in \mathfrak{Y} \cap \mathfrak{Z}$. Since M and N belong to \mathfrak{Y} , $M \cap N$ is not empty and therefore $M \cup N$ is $W \circ W$ -small (??) and hence V -small; since $M \cup N$ belongs to \mathfrak{X} and to \mathfrak{Z} , we have $\widehat{W} \circ \widehat{W} \subseteq \widehat{V}$, hence (B3) is satisfied.

We show next that the uniform space \widehat{X} is Hausdorff. Let $\mathfrak{X}, \mathfrak{Y}$ be two ininimal Cauchy filters on \widehat{X} such that $(\mathfrak{X}, \mathfrak{Y}) \in \widehat{V}$ for all symmetric entourages V of X . It follows immediately that the sets $M \cup N$, where $M \in \mathfrak{X}$ and $N \in \mathfrak{Y}$, form a base of a filter \mathfrak{Z} coarser than \mathfrak{X} and \mathfrak{Y} . Now \mathfrak{Z} is a Cauchy filter, since for every symmetric entourage V of X there is by hypothesis a V -small set P belonging to both \mathfrak{X} and \mathfrak{Y} and therefore belonging to \mathfrak{Z} . By the definition of minimal Cauchy filters, we have $\mathfrak{X} = \mathfrak{Y} = \mathfrak{Z}$, and this shows that \widehat{X} is Hausdorff.

Now we define the map ι . We know that for each $x \in X$ the neighbourhood filter $\mathfrak{U}(x)$ of x in X is a minimal Cauchy filter (??). So we define $\iota(x) = \mathfrak{U}(x)$. Let $j = \iota \times \iota$; we shall show that for each symmetric entourage V of X we have $j^{-1}(\widehat{V}) \subseteq V \subseteq j^{-1}[\widehat{V}^3]$, and this will implies that X carries the initial uniformity with respect to \widehat{X} and ι . Now, if $(\iota(x), \iota(y)) \in \widehat{V}$, there is a V -small set M which is a neighbourhood of each of x and y , hence $(x, y) \in V$. Conversely, if $(x, y) \in V$, it is immediately seen that the set $V(x) \cup V(y)$ is V^3 -small and is a neighbourhood of each of x and y .

The trace on $\iota(X)$ of a neighbourhood $\widehat{V}(\mathfrak{X})$ of a point of $\mathfrak{X} \in \widehat{X}$ is the set of all $\iota(x)$ such that $(\mathfrak{X}, \iota(x)) \in \widehat{V}$. This relation means that there is a V -small neighbourhood of x in X which belongs to \mathfrak{X} , i.e. that x is an interior point of a V -small set of \mathfrak{X} . Let M be the union of the interiors of all V -small sets of \mathfrak{X} ; then M is nonempty by ?? and from what has been said it follows that $\widehat{V}(\mathfrak{X}) \cap \iota(x) = \iota(M)$. We conclude that

- (i) $\widehat{V}(\mathfrak{X}) \cap \iota(X)$ is not empty, hence $\iota(X)$ is dense in \widehat{X} .
- (ii) The trace of $\widehat{V}(\mathfrak{X})$ on $\iota(X)$ belongs to the filter base $\iota(\mathfrak{X})$ on \widehat{X} ; hence this filter base converges in \widehat{X} to the point \mathfrak{X} .

Now let \mathfrak{F} be a Cauchy filter on $\iota(X)$; then from (ii) above and ??, $\iota^{-1}(\mathfrak{F})$ is a base of a Cauchy filter \mathfrak{G} on X . Let \mathfrak{X} be a minimal Cauchy filter coarser than \mathfrak{G} ; then $\iota(\mathfrak{X})$ is a Cauchy filter base on $\iota(X)$, and $\mathfrak{F} = \iota(\iota^{-1}(\mathfrak{F}))$ is finer than the filter whose base is $\iota(\mathfrak{X})$. Since the latter converges in \mathfrak{X} , so does \mathfrak{F} , and ?? therefore shows that \widehat{X} is complete.

We now verify the property (P). Let f be a uniformly continuous mapping of X into a complete Hausdorff uniform space Y . Let us first show that there is a unique uniformly continuous mapping $g_0 : \iota(X) \rightarrow Y$ such that $f = g_0 \circ \iota$. Since f is continuous, we have

$$f(x) = \lim f(\mathfrak{U}(x))$$

hence if we put $g_0(\iota(x)) = \lim f(\mathfrak{U}(x))$, we have $f = g_0 \circ \iota$; so it remains to show that g_0 is uniformly continuous in $\iota(X)$. Let U be an entourage of Y and let V be a symmetric entourage of X such that the relation $(x, y) \in V$ implies $(f(x), f(y)) \in U$; we have seen in (ii) that the relation $(\iota(x), \iota(y)) \in \widehat{V}$ implies $(x, y) \in V$, hence also implies $(g_0(\iota(x)), g_0(\iota(y))) \in U$, which proves our assertion.

Let g be the extension of g_0 by continuity to \widehat{X} (?); then $f = g_0 \circ \iota$, and it is clear that g is the unique continuous mapping of \widehat{X} into Y satisfying this relation, since $\iota(X)$ is dense in \widehat{X} . With this, the uniqueness part of the pair (\widehat{X}, ι) is clear now, \square

Definition 2.2.31. *The complete Hausdorff uniform space \widehat{X} defined in the proof of ?? is called the **Hausdorff completion** of X , and the mapping $\iota : X \rightarrow \widehat{X}$ is called the **canonical mapping** of X into its Hausdorff completion.*

Proposition 2.2.32. *Let X be a uniform space and \widehat{X} be its completion.*

- (a) *The subspace $\iota(X)$ is dense in \widehat{X} .*
- (b) *The graph of the equivalence relation $\iota(x) = \iota(y)$ is the intersection of the entourages of X .*
- (c) *The uniform structure of X is the inverse image under ι of that of \widehat{X} .*
- (d) *The entourages of $\iota(X)$ are the images under $\iota \times \iota$ of the entourages of X , and the closures in $\widehat{X} \times \widehat{X}$ of the entourages of $\iota(X)$ form a fundamental system of entourages of \widehat{X} .*
- (e) *The construction $X \mapsto \widehat{X}$ is functorial. That is, if X and Y be two uniform spaces and $f : X \rightarrow Y$ is a uniformly continuous map, then there is a unique uniformly continuous mapping $\widehat{f} : \widehat{X} \rightarrow \widehat{Y}$ such that the diagram*

$$\begin{array}{ccc} \widehat{X} & \xrightarrow{\widehat{f}} & \widehat{Y} \\ \uparrow & & \uparrow \\ X & \xrightarrow{f} & Y \end{array}$$

is commutative. Moreover, if $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are two uniformly continuous mappings, then $\widehat{g \circ f} = \widehat{g} \circ \widehat{f}$.

Proof. Part (a) and (c) have been proved in the course of the proof of ??; (e) follows from ??, and (d) is a consequence of (a) and (c) by virtue of general results proved earlier. The relation $\iota(x) = \iota(y)$ means by definition that x and y have the same neighbourhood filter. But this implies, by definition, that $(x, y) \in V$ for every entourage V of X , and the converse is obvious. \square

Corollary 2.2.33. *If X is a Hausdorff uniform space, then the canonical mapping $\iota : X \rightarrow \widehat{X}$ is an isomorphism of X onto the dense subspace $\iota(X)$ of \widehat{X} .*

When X is Hausdorff, \widehat{X} is said to be the **completion** of X , and we generally identify X with a dense subset of \widehat{X} by means of ι . ?? characterizes the completion of a Hausdorff uniform space:

Proposition 2.2.34. *If Y is a complete Hausdorff uniform space and X is a dense subspace of Y , then the canonical injection $X \rightarrow Y$ extends to an isomorphism of \widehat{X} onto Y .*

Proof. By by ??, every uniformly continuous mapping of X into a complete Hausdorff uniform space Z extends uniquely to a uniformly continuous mapping of Y into Z . \square

Proposition 2.2.35. *Let X be a complete Hausdorff uniform space, \mathfrak{U} its uniformity, and let Z be a dense subspace of X . If \mathfrak{U}' is a uniformity on X which is coarser than \mathfrak{U} , and which induces the same uniformity as \mathfrak{U} on Z , then $\mathfrak{U}' = \mathfrak{U}$.*

Proof. Let X' denote the set X with the uniformity \mathfrak{U}' . The composition of the identity mapping $X \rightarrow X'$ and the canonical mapping $X' \rightarrow \widehat{X}'$ is a uniformly continuous mapping $\varphi : X \rightarrow \widehat{X}'$. Since Z is Hausdorff for the uniform structure induced by \mathfrak{U}' , the restriction of φ to Z is by hypothesis an isomorphism of Z onto the dense subspace $\varphi(Z)$ of \widehat{X}' ; it follows from ?? that φ itself is an isomorphism of X onto \widehat{X}' , hence $X = \widehat{X}'$ and $\mathfrak{U}' = \mathfrak{U}$. \square

2.2.6 The Hausdorff space associated with a uniform space

We have seen that the completion of a uniform space is Hausdorff and complete. Since not all uniform spaces are Hausdorff, one may get from this a "Hausdorffification" functor on the category of uniform spaces.

Proposition 2.2.36. *Let X be a uniform space and ι the canonical mapping of X into its Hausdorff completion \widehat{X} . For each uniformly continuous mapping f of X into a Hausdorff uniform space Y , there is a unique uniformity continuous mapping $h : \iota(X) \rightarrow Y$ such that $f = h \circ \iota$.*

$$\begin{array}{ccc} & \iota(X) & \\ \iota \uparrow & \searrow h & \\ X & \xrightarrow{f} & Y \end{array}$$

Proof. We may identify Y with a subspace of its completion \widehat{Y} , and f can then be considered as a uniformly continuous mapping of X into \widehat{Y} . By virtue of ??, f is then of the form $f = g \circ \iota$, where g is a uniformly continuous mapping of \widehat{X} into \widehat{Y} . If h is the restriction of g to $\iota(X)$, then clearly $f = h \circ \iota$, and h maps $\iota(X)$ into Y . The uniqueness of h is trivial. \square

Definition 2.2.37. *The Hausdorff uniform space $\iota(X)$ defined in the proof of ?? is called the **Hausdorff uniform space associated with X** .*

The Hausdorff completion of X is thus the completion of the Hausdorff uniform space associated with X .

Corollary 2.2.38. *Let X, Y be two uniform spaces and X', Y' the associated Hausdorff spaces. For each uniformity continuous mapping $f : X \rightarrow Y$ there is a unique uniformity continuous mapping $f' : X' \rightarrow Y'$ for which the diagram*

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ \uparrow & & \uparrow \\ X & \xrightarrow{f} & Y \end{array}$$

is commutative.

The Hausdorff space associated with a uniform space may also be characterized by the following property:

Proposition 2.2.39. *Let X be a uniform space, $\iota(X)$ its associated Hausdorff space, and let f be a mapping of X onto a Hausdorff uniform space X' , such that the uniformity of X is the inverse image under f of the uniformity of X' . Then the mapping $g : \iota(X) \rightarrow X'$ such that $f = g \circ \iota$ is an isomorphism.*

Proof. By ??, g is uniformly continuous; also g is obviously surjective, and is also injective because the relation $f(x) = f(y)$ implies by definition that (x, y) belongs to all the entourages of X , and therefore that $\iota(x) = \iota(y)$ (??). Finally, the entourages of X' are the images under $f \times f$ of the entourages of X , hence they are also the images under $g \times g$ of the entourages of $\iota(X)$; hence the result. \square

2.2.7 Completion of subspaces and product spaces

Proposition 2.2.40. *Let X be a set, let $(Y_i)_{i \in I}$ be a family of uniform spaces, and for each $i \in I$ let f_i be a mapping of X into Y_i . Let X carry the initial uniformity \mathfrak{U} with respect to the maps $(f_i)_{i \in I}$. Then the Hausdorff completion \widehat{X} of X carries the initial uniformity with respect to the maps $(\widehat{f}_i)_{i \in I}$. Furthermore, if ι_i is the canonical mapping of Y_i into \widehat{Y}_i , and if $g_i = \iota_i \circ f_i$, then X may be identified with the closure in $\prod_{i \in I} \widehat{Y}_i$ of the image of X under the mapping $x \mapsto (g_i(x))$.*

Proof. Let X' (resp. Y'_i) be the Hausdorff uniform space associated with X (resp. Y'_i), and let $f'_i : X' \rightarrow Y'_i$ be the uniformly continuous mapping which makes the diagram

$$\begin{array}{ccc} X' & \xrightarrow{f'_i} & Y'_i \\ \iota \uparrow & & \uparrow \iota_i \\ X & \xrightarrow{f_i} & Y_i \end{array}$$

commutative. The transitivity of initial uniformities shows on the one hand that \mathfrak{U} is the coarsest uniformity for which the mappings $\iota_i \circ f_i : X \rightarrow \widehat{Y}_i$ are uniformly continuous and on the other hand that \mathfrak{U} is also the inverse image under ι of the coarsest uniformity \mathfrak{U}' on the set \widehat{X} for which the \widehat{f}_i are uniformly continuous. Now \mathfrak{U}' is Hausdorff, for if x_1, x_2 are two points of X such that $\iota_i(f_i(x_1)) = \iota_i(f_i(x_2))$ for each $i \in I$, then (x_1, x_2) belongs to all the entourages of \mathfrak{U} and hence $\iota(x_1) = \iota(x_2)$. It then follows from ?? that \mathfrak{U}' is the uniformity of the Hausdorff space X' associated with X .

The bijection $x' \mapsto (f'_i(x'))$ identifies X' with a uniform subspace of the product $\prod_{i \in I} Y'_i$. Since the Y'_i are Hausdorff, each Y'_i can be identified with a dense subspace of its completion \widehat{Y}_i , and hence $\prod_{i \in I} Y'_i$ can be identified with a dense subspace of $\prod_{i \in I} \widehat{Y}_i$. But $\prod_{i \in I} \widehat{Y}_i$ is Hausdorff and complete; the closure $\overline{X'}$ of X' in $\prod_{i \in I} \widehat{Y}_i$ is therefore a complete Hausdorff subspace which can be identified with the Hausdorff completion \widehat{X} of X ; under this identification the mappings \widehat{f}_i become the projections onto the factors \widehat{Y}_i , and the proposition is proved. \square

Corollary 2.2.41. *Let X be a uniform space and ι be the canonical mapping of X into its Hausdorff completion \widehat{X} ; let A be a subspace of X and $j : A \rightarrow X$ the canonical injection. Then $\widehat{j} : \widehat{A} \rightarrow \widehat{X}$ is an isomorphism of \widehat{A} onto the closure of $\iota(A)$ in X .*

Corollary 2.2.42. Let $(Y_i)_{i \in I}$ be a family of uniform spaces. Then the Hausdorff completion of the product space $\prod_{i \in I} Y_i$ is canonically isomorphic to the product $\prod_{i \in I} \widehat{Y}_i$.

2.3 Compact space and uniform spaces

2.3.1 Uniformity for compact Hausdorff spaces

A uniformity on a topological space X is said to be compatible with the topology of X if the latter coincides with the topology induced by the uniformity. A topological space is said to be **uniformizable**, and its topology is said to be **uniformizable**, if there exists a uniformity on the space which is compatible with its topology. There are topological spaces which are not uniformizable, for example any space which is not T_3 ; hence the question arises of determining under what conditions a topological space is uniformizable.

We shall not give a complete answer to this question here. In this part we shall examine only one important particular case, that in which X is compact. We have then the following theorem:

Theorem 2.3.1. On a Hausdorff compact space X there is exactly one uniformity compatible with the topology of X ; the entourages of this uniformity are all the neighbourhoods of the diagonal Δ in $X \times X$. Furthermore, X endowed with this uniformity is a complete uniform space.

Proof. The last part of the theorem is straightforward; for every Cauchy filter on X has a cluster point and is therefore convergent.

Let us show next that if there is a uniformity on X compatible with the topology of X , then the set \mathfrak{U} of entourages of this uniformity is the set of all neighbourhoods of Δ . We know already that every entourage is a neighbourhood of Δ , hence we have to show that conversely every neighbourhood of Δ belongs to \mathfrak{U} . Suppose that there is a neighbourhood U of Δ which is not in \mathfrak{U} ; then since no element in \mathfrak{U} is contained in U , the sets $V \setminus U$, as V runs through \mathfrak{U} , form a base of a filter \mathfrak{G} on the compact space $X \times X$; consequently \mathfrak{G} has a cluster point (a, b) not belonging to Δ . Since \mathfrak{U} is a filter coarser than \mathfrak{G} , (a, b) is also a cluster point of \mathfrak{U} . But the uniformity defined by \mathfrak{U} is Hausdorff by hypothesis; hence the intersection of the closures of the sets of \mathfrak{U} is Δ ; thus we arrive at a contradiction.

It remains therefore to show that the set \mathfrak{U} of neighbourhoods of Δ in $X \times X$ is the set of entourages of a uniformity compatible with the topology of X . For this it is enough to show that \mathfrak{U} is the set of entourages of a Hausdorff uniformity on X ; for then the topology induced by this uniformity will be coarser than the topology of X and must therefore coincide with the latter.

It is clear that \mathfrak{B} is a filter. Let us show that axioms (U2) and (U3) are also satisfied and that Δ is the intersection of the sets of \mathfrak{B} . Take the last point first: every set consisting of a single point $(x, y) \in X \times X$ is closed, since X is Hausdorff; hence if $x \neq y$, the complement of (x, y) in $X \times X$ is a neighbourhood of Δ . Since the symmetry $(x, y) \mapsto (y, x)$ is a homeomorphism of $X \times X$ onto itself, it follows that $V \in \mathfrak{B}$ implies $V^{-1} \in \mathfrak{B}$, whence (U2); Finally, suppose that (U3) is not satisfied; then there is a set $V \in \mathfrak{B}$ such that for all $W \in \mathfrak{B}$ the set $W^2 \setminus V$ is not empty; the sets $W^2 \setminus V$, as W runs through \mathfrak{B} , therefore form a filter base on $X \times X$, and this filter base has a cluster point (x, y) not in Δ . Now, since X is Hausdorff, it is regular, so there are disjoint open neighbourhoods U_1 of x and U_2 of y , and there are closed neighbourhoods

$V_1 \subseteq U_1$, $V_2 \subseteq U_2$ of x and y respectively. Let $U_3 = (V_1 \cup V_2)^c$, and consider the following neighbourhood of Δ :

$$W = \bigcup_{i=1}^3 (U_i \times U_i).$$

It follows immediately from these definitions that if $(u, v) \in W$ and $u \in V_1$ (resp. $u \in U_1$) then we must have $v \in U_1$ (resp. $v \in U_1 \cup U_3 = V_2^c$); hence the neighbourhood $V_1 \times V_2$ of (x, y) in $X \times X$ does not meet W , and we have a contradiction. This completes the proof. \square

Remark 2.3.1. For every finite open covering $\mathfrak{R} = (U_i)_{i=1}^n$ of X , the set

$$V_{\mathfrak{R}} = \bigcup_{i=1}^n U_i$$

is a neighbourhood of Δ in $X \times X$, and these sets $V_{\mathfrak{R}}$ form a fundamental system of neighbourhoods of Δ (and therefore a fundamental system of entourages of the unique uniformity on X): Let W be any neighbourhood of Δ in $X \times X$; then for each $x \in X$ there is an open neighbourhood U_x of x in X such that $U_x \times U_x \subseteq W$. Since the $\{U_x : x \in X\}$ form an open covering of X , there exist a finite number of points x_1, \dots, x_n such that the $\{U_{x_i}\}_{i=1}^n$ form a covering \mathfrak{R} of X . We have then $V_{\mathfrak{R}} \subseteq W$, which proves the assertion. For this reason the unique uniformity on X is often called the uniformity of finite open coverings.

Corollary 2.3.2. *Every subspace of a compact space is uniformizable, every locally compact space is uniformizable.*

Proof. Note that a locally compact space can be embedded into a compact space. \square

It can happen that there are several distinct uniformities compatible with the topology of a locally compact space. For example, we have seen that on an infinite discrete space there is more than one distinct uniformity compatible with the discrete topology.

Theorem 2.3.3. *Every continuous mapping f of a compact space X into a uniform space Y is uniformly continuous.*

Proof. Let $g = f \times f$, then g is continuous on $X \times X$; hence for each open entourage V of Y , $g^{-1}(V)$ is an open subset of $X \times X$, which evidently contains the diagonal. The theorem thus follows from ??, since the open entourages of Y form a fundamental system of entourages. \square

Corollary 2.3.4. *Let A be a dense subspace of a compact space X , and let f be a mapping of A into a complete Hausdorff uniform space Y . Then f can be extended by continuity to the whole of X if and only if f is uniformly continuous.*

2.3.2 Compactness of uniform spaces

Definition 2.3.5. A uniform space X is said to be precompact if its Hausdorff completion \widehat{X} is compact. A subset A of a uniform space X is said to be a precompact subset if the uniform subspace A of X is precompact.

Thus a subset A of a uniform space X is precompact if and only if the closure of $\iota(A)$ in \widehat{X} is compact.

Example 2.3.6. In any uniform space X , the set of points of Cauchy sequence (x_n) is precompact. Since the images of the x_n in \widehat{X} again form a Cauchy sequence, we can assume that X is Hausdorff; the closure in \widehat{X} of the set of points x_n then consists of the points x_n and $\lim_n x_n$ and is therefore compact.

Besides precompactness, we introduce another concept which is frequently used. A subset A of a uniform space X is called totally bounded if, for each entourage V of X , there is a finite covering of A by V -small sets. The following theorem for precompactness and totally boundedness is fundamental and important.

Theorem 2.3.7. *A subset A of a uniform space X is precompact if and only if it is totally bounded.*

Proof. Since the completion \widehat{A} is identified as a subspace of \widehat{X} , we may prove the result for X . Let $\iota : X \rightarrow \widehat{X}$ be the canonical mapping, then the entourages of X are the inverse images under $\iota \times \iota$ of the entourages of \widehat{X} . Suppose X is precompact, and let U be any entourage of \widehat{X} ; then there is a symmetric entourage W of \widehat{X} such that $W \circ W \subseteq U$. Since \widehat{X} is compact, there exist a finite number of points $x_1, \dots, x_n \in \widehat{X}$ such that the $W(x_i)$ (which are U -small) cover X . If V is the inverse image of U by $\iota \times \iota$, then the sets $\iota^{-1}(W(x_i))$ are V -small and cover X . This proves X is totally bounded.

Conversely, suppose that for each entourage V of X there is a finite covering of X by V -small sets. We have to show that every ultrafilter \mathfrak{F} on \widehat{X} is convergent; since \widehat{X} is complete, it is enough to show that \mathfrak{F} is a Cauchy filter, i.e. that for each closed entourage U of \widehat{X} there is a U -small set in \mathfrak{F} (??). Let V be the inverse image of U under $\iota \times \iota$, and let (B_j) be a finite covering of X by V -small sets; then the sets $C_j = \iota(B_j)$ are U -small and cover $\iota(X)$, so that

$$\widehat{X} = \bigcup_j \bar{C}_j.$$

On the other hand, since $C_j \times C_j \subseteq U$, and U closed in $X \times X$, we have $\bar{C}_j \times \bar{C}_j \subseteq U$, so that the \bar{C}_j are also U -small. Since \mathfrak{F} is an ultrafilter, one of the \bar{C}_j belongs to \mathfrak{F} . \square

Corollary 2.3.8. *A uniform space X is compact if and only if it is Hausdorff, complete and totally bounded.*

Proposition 2.3.9. *In a uniform space every subset of a precompact set, every finite union of precompact sets and the closure of every precompact set are precompact.*

Proof. The first two assertions are immediate consequences of ???. Let X be a uniform space, A a precompact subset of X , and let $\iota : X \rightarrow \widehat{X}$ be the canonical mapping. By continuity, $\iota(A)$ is contained in the closure of $\iota(A)$ in X , hence the closure of $\iota(A)$ in \widehat{X} is contained in a compact set and is therefore compact. \square

Remark 2.3.2. In a uniform space X a relatively compact set A is precompact, since A is contained in a compact set. On the other hand, even if X is Hausdorff, a precompact set need not be relatively compact in X , as is shown by the case where X itself is precompact but not compact.

Proposition 2.3.10. *Let $f : X \rightarrow Y$ be a uniformly continuous mapping. If A is any precompact subset of X , then $f(A)$ is a precompact subset of Y .*

Proof. For if $\iota_X : X \rightarrow \widehat{X}$ and $\iota_Y : Y \rightarrow \widehat{Y}$ are the canonical mappings, we have $\iota_Y(f(A)) = \widehat{f}(\iota_X(A))$ and hence $\iota_Y(f(A))$ is relatively compact in \widehat{Y} . \square

Proposition 2.3.11. *Let X be a set, let $(Y_i)_{i \in I}$ be a family of uniform spaces, and for each $i \in I$ let A be a mapping of X into Y_i . Let X carry the initial uniformity with respect to $(f_i)_{i \in I}$. Then a subset A of X is precompact if and only if $f_i(A)$ is a precompact subset of Y_i for each $i \in I$.*

Proof. The condition is necessary by virtue of ???. Sufficiency follows from the characterization of the Hausdorff completion of X given in ??, and Tychonoff's theorem. \square

To conclude, we prove the following two properties of compact subset of a uniform space.

Proposition 2.3.12. *In a uniform space X , let A be a compact set and B a closed set such that $A \cap B = \emptyset$. Then there is an entourage V of X such that $V(A)$ and $V(B)$ do not intersect.*

Proof. If the proposition were false, then none of the sets $A \cap V^2(B)$, where V runs through the set of symmetric entourages of X , would be empty; hence these sets would form a filter base on A , which would have a cluster a point $x_0 \in A$. Hence for each symmetric entourage V of X , $V(x_0)$ would meet B and therefore, as B is closed, we should have $x_0 \in B$, contrary to hypothesis. \square

Corollary 2.3.13. *Let A be a compact set in a uniform space X ; then as V runs through the set of entourages of X , the sets $V(A)$ form a fundamental system of neighbourhoods of A .*

Proof. Let U be any open neighbourhood of A , then $B = U^c$ is closed and does not meet A ; hence, by ??, there is an entourage V such that $V(A) \cap V(B) = \emptyset$, and therefore $V(A) \subseteq U$. \square

Chapter 3

Topological groups and rings

3.1 Topological groups

3.1.1 Topologies on groups

Definition 3.1.1. A topological group is a set G which comes a group structure and a topology and satisfies the following two axioms:

(GT1) The mapping $(x, y) \mapsto xy$ of $G \times G$ into G is continuous.

(GT2) The mapping $x \mapsto x^{-1}$ of G into G is continuous.

A group structure and a topology on a set G are said to be compatible if they satisfy (GT1) and (GT2). An immediate consequence of the axioms is that translation and inversion are homeomorphisms. We show next that translates of neighborhoods V of e constitute the neighborhoods of g .

Proposition 3.1.2. If \mathcal{B} is a neighborhood base at e in a topological group G , then for any $g \in G$, $g\mathcal{B}$ is a neighborhood base for $\mathcal{B}(g)$.

Proof. The map R_g is a homeomorphism taking e into g . Its inverse is the map $R_{g^{-1}}$. Thus if $V \in \mathcal{U}(g)$, then $g^{-1}V$ is a neighborhood of 0 . Since \mathcal{B} is a base, there exists $B \in \mathcal{B}$ such that $B \subseteq g^{-1}V$. Hence $gB \subseteq V$ and $g\mathcal{B}$ is a neighborhood base at g . \square

Recall that a topological space X is **homogeneous** if, given any two points $x, y \in X$ there is a homeomorphism of X onto X mapping x into y . If X is a topological group and $x, y \in X$, the translation map $R_{x^{-1}y}$ is a homeomorphism which takes x into y ; hence any topological group is homogeneous. A consequence of this fact is the following.

Proposition 3.1.3. A homomorphism mapping one topological group into another continuous at one point is continuous.

Proof. Suppose that G and H are topological groups and that $f : G \rightarrow H$ is a homomorphism. Suppose that f is continuous at $g_0 \in G$. To see that f is continuous everywhere, let $g \in G$ and suppose that the net $g_\alpha \rightarrow g$. Then $g_0g^{-1}g_\alpha \rightarrow g_0$. Since f is continuous at g_0 ,

$$f(g_0g^{-1}g_\alpha) = f(g_0)f(g^{-1})f(g_\alpha) \rightarrow f(g_0).$$

which implies that $f(g_\alpha) \rightarrow g$ and proves the continuity of f at g . \square

Finally, we consider the neighborhood base at e of a topological group G . Many important result depends on the property of $\mathfrak{B}(e)$, which we now establish.

Proposition 3.1.4. *Suppose G is a topological group and let \mathfrak{B} be the neighbourhood filter of the identity element of G . Then \mathfrak{B} satisfies the following conditions.*

(GV1) *Given any $U \in \mathfrak{B}$, there exists $V \in \mathfrak{B}$ such that $UV \subseteq U$.*

(GV2) *Given any $U \in \mathfrak{B}$, we have $U^{-1} \in \mathfrak{B}$.*

(GV3) *For all $g \in G$ and all $V \in \mathfrak{B}$, we have $gVg^{-1} \in \mathfrak{B}$.*

Conversely, if \mathfrak{B} is a filter on G satisfying the three conditions, then there is a unique topology on G , compatible with the group structure of G , for which \mathfrak{B} is the neighbourhood filter of the identity element e .

Proof. Since the map $(x, y) \mapsto xy$ and $x \mapsto x^{-1}$ are continuous at $x = y = e$, we obtain. Also, $x \mapsto gxg^{-1}$ is a homeomorphism which leaves e fixed, so (GV3) follows.

Conversely, assume that \mathfrak{B} is a filter on G satisfying these properties. If there is a topology with the required properties, then by what has been said above the neighbourhood filter of g coincides with each of the filters $g\mathfrak{B}$ and $\mathfrak{B}g$; hence the topology is unique, if it exists. Its existence will be established if we show that the filters $g\mathfrak{B}$ are the neighbourhood filters of a topology on G , and that this topology is compatible with the group structure of G .

The filter $g\mathfrak{B}$ satisfies axiom (V3) by reason of (GV1) and (GV2), as we have already seen; hence to show that $g\mathfrak{B}$ is the neighbourhood filter of g in a topology on G , we have to verify axiom (V4). Let then V be any set of \mathfrak{B} , and W a set of \mathfrak{B} such that $WW \subseteq V$; then for any $x \in gW$ we have $xW \subseteq gWW \subseteq gV$, so that gV belongs to the filter $g\mathfrak{B}$; hence (V4) is satisfied.

Let us now show that the topology defined by the neighbourhood filters $g\mathfrak{B}$ satisfies (GT2) and (GT3). Let a, b be any two points of G ; if we put $x = au$ and $y = bv$, then we have to show that xy^{-1} is as near as we please to ab^{-1} whenever u and v are close enough to e . Note that $(ab^{-1})^{-1}(xy^{-1}) = buv^{-1}b^{-1}$; let U be any neighbourhood of e , then we shall have $buv^{-1}b^{-1} \in U$ if $uv^{-1} \in b^{-1}Ub = V$. Now $V \in \mathfrak{B}$ by reason of (GV3), and by (GV1) and (GV2) there exists $W \in \mathfrak{B}$ such that $WW^{-1} \in V$; hence it is enough to take $u \in W$ and $v \in W$. This completes the proof. \square

A neighbourhood of e which coincides with its image under the symmetry $x \mapsto x^{-1}$ is said to be **symmetric**. If V is any neighbourhood of e , then UV^{-1} , $V \cap V^{-1}$ and $V \cup V^{-1}$ are symmetric neighbourhoods. By (GV2), the symmetric neighbourhoods form a fundamental system of neighbourhoods of e . Also it follows from (GV1) that when V runs through a fundamental system of neighbourhoods of e , the sets V^n (where n is a fixed nonzero integer) form a fundamental system of neighbourhoods of e .

Corollary 3.1.5. *Suppose G is a topological group and $\mathfrak{B}(e)$ be the neighborhood base of the identity and $\mathfrak{S}(e)$ be the symmetric neighborhood base of the identity. Then*

$$\overline{\{e\}} = \bigcap \mathfrak{B}(e) = \bigcap \mathfrak{S}(e).$$

Proof. Since $\mathfrak{B}(e)$ is a neighborhood base of e , it suffices to prove $\overline{\{e\}} = \bigcap \mathfrak{B}(e)$. If $g \in \overline{\{e\}}$, then for every $U \in \mathfrak{S}(e)$, the set gU meets $\{e\}$. That is, $e \in gU$. Since U is symmetric, this

implies $g \in U$ for all $U \in \mathfrak{S}(e)$. Conversely, if $g \in \bigcap \mathfrak{S}(e)$ then $g^{-1} \in \bigcap \mathfrak{S}(e)$. Hence $e \in gU$ for any $U \in \bigcap \mathfrak{S}(e)$, which implies $g \in \overline{\{e\}}$. \square

Corollary 3.1.6. *A topological group G is Hausdorff if and only if the intersection of the neighbourhoods of e consists only of the point e .*

Example 3.1.7. If \mathfrak{B} is a filter base on a group G consists of subgroups of G , then it is immediately seen that it satisfies axioms (GV1) and (GV2), since $HH^{-1} = H$ for any subgroup H of G . Hence the set \mathfrak{B} will be a fundamental system of neighbourhoods of e in a topology compatible with the group structure of G , provided that it satisfies (GV3); this will in particular be the case if all the subgroups in \mathfrak{B} are normal, hence always if G is commutative. The topology thus defined is Hausdorff, by ??, if and only if the intersection of all the subgroups in \mathfrak{B} consists only of e . The most interesting cases are those in which the subgroup $\{e\}$ is not in \mathfrak{B} , since otherwise the topology defined by \mathfrak{B} is the discrete topology: if $\{e\} \notin \mathfrak{B}$, the topology defined by \mathfrak{B} is Hausdorff only if \mathfrak{B} is an infinite set.

Since the intersection of two subgroups is a subgroup, we can define a topology on G , compatible with its group structure, starting from any set \mathfrak{F} of subgroups of G : let \mathfrak{S} be the set of all subgroups gHg^{-1} , where $H \in \mathfrak{F}$ and $g \in G$, and let \mathfrak{B} be the set of all finite intersections of subgroups belonging to \mathfrak{S} . Then \mathfrak{B} is a filter base and satisfies (GV3).

Consider in particular the additive group of a ring A . Every set \mathfrak{F} of ideals of A defines a topology compatible with this additive group structure. This topology is Hausdorff if the intersection of all the ideals of \mathfrak{F} is the zero ideal, and it is not discrete if no finite intersection of the ideals of \mathfrak{F} is the zero ideal. Topologies defined in this way play a large part in the theory of numbers.

Proposition 3.1.8. *Let G be a topological group. If U is open in G and B is a subset of G , then UB and BU are open.*

Proof. Note that $BU = \bigcup_{b \in B} bU$ and $UB = \bigcup_{b \in B} Ub$, so the claim follows from the fact that $g \mapsto bg$ and $g \mapsto gb$ are homeomorphisms. \square

Example 3.1.9. Let $G = (\mathbb{R}, +)$, the additive topological group of real numbers with the usual topology. The sets $A = \{n \in \mathbb{N} : n \geq 2\}$ and $B = \{-n - 1/n : n \geq 2\}$ are closed but $A + B$ contains $\{-1/n : n \geq 2\}$ which has 0 as a limit point so $A + B$ is not closed.

The product of a compact set and a closed set is generally closed, however. We prove this after establishing the following result.

Proposition 3.1.10. *Let G be a topological group, let K be a compact subset of G , and let U be an open neighborhood of K . Then there are open neighborhoods V_R and V_L of e such that $KV_R \subseteq U$ and $V_L K \subseteq U$.*

Proof. For each x in K choose open neighborhoods W_x and V_x of e such that $xW_x \subseteq U$ and $V_x W_x \subseteq W_x$. Then $\{xV_x\}_{x \in K}$ is an open cover of the compact set K , and so there is a finite collection x_1, \dots, x_n of points in K such that the sets $\{x_i V_{x_i} : i = 1, \dots, n\}$ cover K . Let $V_R = \bigcap_{i=1}^n V_{x_i}$. If $x \in K$, then there is an index i such that $x \in x_i V_{x_i}$, and so

$$xV_R \subseteq x_i V_{x_i} V_{x_i} \subseteq x_i W_{x_i} \subseteq U.$$

Since x was an arbitrary element of K , it follows that $KV_R \subseteq U$. The construction of V_L is similar. \square

Proposition 3.1.11. *Let G be a topological group. If K is compact in G and F is closed, then KF and FK are closed.*

Proof. Let $g \in \overline{KF}$. Then for any neighborhood U of e ,

$$(gU) \cap (KF) \neq \emptyset \iff (K^{-1}gU) \cap F \neq \emptyset. \quad (3.1.1)$$

To show that $g \in KF$, first observe that if $K^{-1}g \cap F = \emptyset$, then the compact set $K^{-1}g$ is contained in F^c , so by ?? there is some neighborhood V of e such that $(K^{-1}gV) \cap F = \emptyset$. This contradicts (??), so the claim follows. The result for FK can be proved similarly, using a modified version of ??.

As a final word on interior, closure, and multiplication, we prove the following result.

Proposition 3.1.12. *If A and B are subsets of a topological group, then*

$$\overline{AB} \subseteq \overline{A}\overline{B}, \quad (\text{Int } A)(\text{Int } B) \subseteq A(\text{Int } B) \subseteq \text{Int}(AB).$$

Proof. Let $g \in \overline{A}$ and $h \in \overline{B}$, and U be a neighborhood of e . By the continuity of product, there exists neighborhoods W_1, W_2 of e such that

$$(gW_1)(hW_2) \subseteq ghU.$$

Since $g \in \overline{A}$ and $h \in \overline{B}$, there exists $a \in A \cap (gW_1)$ and $b \in B \cap (hW_2)$. Then $ab \in AB \cap (ghU)$, and so $gh \in \overline{AB}$.

For the interiors, it is clear that $(\text{Int } A)(\text{Int } B) \subseteq A(\text{Int } B)$. Also, by ??, $A(\text{Int } B)$ is open. Since it is contained in AB , we get $A(\text{Int } B) \subseteq \text{Int}(AB)$. \square

Proposition 3.1.13. *Let G be a topological group.*

- (a) *Suppose that G is connected and that the identity element of G has a compact neighborhood C . Then G is σ -compact.*
- (b) *Suppose that G is σ -compact. Then there exists for every neighborhood U of the identity element e of G a sequence $\{g_n\}$ in G such that $G = \bigcup_n g_n U$.*
- (c) *If G is σ -compact and first countable, then G is second countable.*

Proof. First we prove (a). Since G is connected, G is generated by C . Replacing C by the compact subset $C \cup C^{-1}$, we see $G = \bigcup_n C^n$ with $C^n = \{g_1 g_2 \cdots g_n : g_i \in C\}$. But C^n is compact, as the image of the compact n -fold product $C \times \cdots \times C$ under the n -fold multiplication map $G \times \cdots \times G$. Therefore G is σ -compact.

Now for (b), suppose $G = \bigcup_n K_n$ with $K_n \subseteq G$ compact. For each fixed n , we have a cover of the compact set K_n :

$$\{gU : g \in K_n\}$$

Therefore there exist $g_{n,1}, \dots, g_{n,r_n} \in K_n$ with $K_n \subseteq \bigcup_{i=1}^{r_n} g_{n,i}U$, and hence

$$G \subseteq \bigcup_{n \in \mathbb{N}} \bigcup_{i=1}^{r_n} g_{n,i}U$$

Renumbering the countably many $g_{n,i}$ yields the desired result.

Finally, assume that G is σ -compact and first countable. Let $\{U_n\}_{n \in \mathbb{N}}$ be a fundamental system of neighborhoods of e . Then by ??, for all $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $U_m^{-1}U_m \subseteq U_n$.

By (b), there exist for all $n \in \mathbb{N}$ sequences $(g_{n,k})_k$ in G such that $G = \bigcup_{k \in \mathbb{N}} g_{n,k}U_n$. We show that

$$\mathcal{U} := \{g_{n,k}U_n : k, n \in \mathbb{N}\}$$

is a basis of the topology of G . In fact, let V be an open subset of G and $g \in V$. Then there exists $n \in \mathbb{N}$ such that $gU_n \subseteq V$, and we may choose $m \in \mathbb{N}$ such that $U_m^{-1}U_m \subseteq U_n$. Since $\{g_{m,k}U_m\}$ covers G , there is some $k \in \mathbb{N}$ such that $g \in g_{m,k}U_m$. Now

$$g_{m,k}U_m \subseteq gU_m^{-1}U_m \subseteq gU_n \subseteq V$$

Thus the claim follows. \square

Corollary 3.1.14. *A connected locally compact first countable topological group is second countable.*

Proof. This is a combination of ?? (a) and (c). \square

3.1.2 Topological subgroups

Proposition 3.1.15. *Let G be a topological group and let $H \subseteq G$ be a subgroup. Then its closure \bar{H} is also a subgroup. If H is normal, then \bar{H} is also normal.*

Proof. Consider the continuous map $\sigma(x, y) = xy^{-1}$. It suffices if we show $\sigma(\bar{H} \times \bar{H}) \subseteq \bar{H}$, or equivalently $\bar{H} \times \bar{H} \subseteq \sigma^{-1}(\bar{H})$. First note that, since H is a subgroup,

$$H \times H \subseteq \sigma^{-1}(H) \subseteq \sigma^{-1}(\bar{H}),$$

thus, by taking closure we get $\bar{H} \times \bar{H} \subseteq \sigma^{-1}(\bar{H})$. Now the claim follows from the inclusion $\bar{H} \times \bar{H} \subseteq \bar{H} \times \bar{H}$.

If H is normal, then the map $(g, x) \mapsto gxg^{-1}$ is continuous and maps $G \times H$ into H . Then it maps $G \times \bar{H}$ into \bar{H} , so \bar{H} is also continuous. \square

As a result, it is clear that $\overline{\{e\}}$ is a subgroup in any topological group. An interesting consequence of this proposition is the following.

Corollary 3.1.16. *A locally closed subgroup is closed.*

Proof. Let $H \subseteq G$ be a locally closed subgroup, then H is open and dense in \bar{H} , which is also a subgroup. Now let $h \in \bar{H}$, then hH and H are both open dense subsets of \bar{H} , so they intersect: $hH \cap H \neq \emptyset$. This means $h \in H$, so $H = \bar{H}$ is closed. \square

Proposition 3.1.17. *Suppose G is a topological group.*

- (a) Every open subgroup of G is also closed, hence a union of components.
- (b) For any neighborhood U of e , the subgroup $\langle U \rangle$ generated by U is open and closed in G .
- (c) For any connected subset $U \subseteq G$ containing e , the subgroup $\langle U \rangle$ is connected.
- (d) If G is connected, then every connected neighborhood of e generates G .

Proof. First, let H be an open subgroup of G , then every left coset $gH = \{gh : h \in H\}$ is also open in G because it is the image of H under the homeomorphism L_g . Because $G - H$ is the union of the cosets of H other than H itself, it is open, and therefore H is closed in G . Because H is both open and closed, it is a union of components. This proves (a).

For (b), let $U \subseteq G$ be any neighborhood of the identity, and let H be the subgroup generated by U . As a matter of notation, if A and B are subsets of G , let us write

$$AB = \{ab : a \in A, b \in B\}, \quad A^{-1} = \{a^{-1} : a \in A\}.$$

For each positive integer k , let W_k denote the set of all elements of G that can be expressed as products of k or fewer elements of $W \cup W^{-1}$. Then H is the union of all the sets W_k as k ranges over the positive integers. Now, W^{-1} is open because it is the image of W under the inversion map, which is a diffeomorphism. Thus, $W_1 = W \cup W^{-1}$ is open, and for each $k > 1$ we have

$$W_k = W_1 W_{k-1} = \bigcup_{g \in W_1} L_g(W_{k-1})$$

Because each L_g is a diffeomorphism, it follows by induction that each W_k is open, and thus H is open.

Now we prove (c). Suppose U is connected. Then W^{-1} is also connected because it is a homeomorphic image of W , and $W_1 = W \cup W^{-1}$ is connected because it is a union of connected sets with the identity in common. Therefore, $W_2 = m(W_1 \times W_1)$ is connected because it is the image of a connected space under the continuous multiplication map m , and it follows by induction that $W_k = m(W_1 \times W_{k-1})$ is connected for each k . Thus, $H = \bigcup_k W_k$ is connected because it is a union of connected subsets with the identity in common.

Finally, we deal with (d). Assume that G is connected. Since H is an open subgroup, it is also closed by (a), and it is not empty because it contains the identity. Thus $H = G$. \square

Proposition 3.1.18. In a topological group G , the component C of the identity element e is a closed normal subgroup. The component of a point $g \in G$ is the coset $gC = Cg$.

The component of the identity element e of G is called the **identity component** of G .

Proof. If $a \in C$, then $a^{-1}C$ is connected and contains a ; hence $C^{-1}C \subseteq C$, which shows that C is a subgroup of G . This subgroup is invariant under all automorphisms of G , and in particular under all inner automorphisms, therefore C is normal in G ; also C is closed since it is a maximal connected subset. Finally, the left translation $x \mapsto gx$ is a homeomorphism of G which sends e to g , and hence the component of g is gK . \square

Proposition 3.1.19. A subgroup H of a topological group G is:

- (a) open iff $\text{Int } H \neq \emptyset$.

- (b) closed if H is open.
- (c) discrete iff H has an isolated point

Proof. If H is open then $e \in \text{Int } H$. Conversely, suppose that H has an interior point g . For any h in H , the map $x \mapsto hg^{-1}x$ maps H homeomorphically onto H and maps g into h . Thus h is also an interior point of H . Thus H is open. Part (b) is established in ??, and (c) can be proved similarly. \square

3.1.3 Group actions

If G is a topological group and $g \in G$, left translation by g is the map $L_g : G \rightarrow G$ defined by $L_g(g') = gg'$. It is continuous, because it is equal to the composition

$$G \xrightarrow{i_g} G \times G \xrightarrow{m} G$$

where $i_g(h) = (g, h)$ and m is group multiplication. Because $L_g \circ L_{g^{-1}} = \text{id}_G$, left translation by any element of g is a homeomorphism of G . Similarly, right translation by g , $R_g(h) = hg$, is also a homeomorphism. A topological space X is said to be topologically homogeneous if for any $x, y \in X$, there is a homeomorphism $\varphi : X \rightarrow X$ taking x to y . Every topological group G is topologically homogeneous.

Definition 3.1.20. Suppose G is a group (not necessarily a topological group for now), and X is a set. A left action of G on X is a map $G \times X \rightarrow X$, written $(g, x) \mapsto g \cdot x$, with the following properties:

- (a) $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$ for all $x \in X$ and all $g_1, g_2 \in G$.
- (b) $1 \cdot x = x$ for all $x \in X$.

Similarly, a right action is a map $X \times G \rightarrow X$, written $(x, g) \mapsto x \cdot g$, with the same properties except that composition works in reverse: $(x \cdot g_1) \cdot g_2 = x \cdot (g_1 g_2)$.

Now, suppose X is a topological space and G is a group acting on X . The action is called an action by homeomorphisms if for each $g \in G$, the map $x \mapsto g \cdot x$ is a homeomorphism of X . If in addition G is a topological group, the action is said to be continuous if the map $G \times X \rightarrow X$ is continuous. The next proposition explains the relationship between the two concepts.

Proposition 3.1.21. Suppose G is a topological group acting on a topological space X .

- (a) If the action is continuous, then it is an action by homeomorphisms.
- (b) If G has the discrete topology, then the action is continuous if and only if it is an action by homeomorphisms.

Proof. First suppose the action is continuous. Then as the composition $x \mapsto (x, g) \mapsto g \cdot x$, the map $x \mapsto g \cdot x$ is continuous for each $g \in G$, and has a continuous inverse $x \mapsto g^{-1} \cdot x$. Thus G acts by homeomorphisms.

Now, suppose G has the discrete topology. If G acts by homeomorphisms, then the map $G \times X \rightarrow X$ defined by the action is continuous when restricted to each subset of the form $\{g\} \times X$. Since these subsets form an open cover of $G \times X$, this implies that the action is continuous. \square

For any $x \in X$, the set $G \cdot x = \{g \cdot x : g \in G\}$ is called the **orbit** of x . The action is said to be transitive if for every pair of points $x, y \in X$, there is a group element g such that $g \cdot x = y$, or equivalently if the orbit of each point is the entire space X . The action is said to be **free** if the only element of G that fixes any point in X is the identity, that is, if $g \cdot x = x$ for some x implies $g = 1$.

Example 3.1.22 (Continuous Group Actions).

- (a) The general linear group $\mathrm{GL}(n; \mathbb{R})$ acts on the left on \mathbb{R}^n by matrix multiplication, considering each vector in \mathbb{R}^n as a column matrix. The action is continuous, because the component functions of $g \cdot x$ are polynomial functions of the components of g and x . Given any nonzero vector $x \in \mathbb{R}^n$, we can find vectors (x_2, \dots, x_n) such that (x, x_2, \dots, x_n) is a basis for \mathbb{R}^n , and then the matrix G with columns (x, x_2, \dots, x_n) is invertible and takes the vector $(1, 0, \dots, 0)$ to x . If y is any other nonzero vector, the same argument shows that there is a matrix $h \in \mathrm{GL}(n; \mathbb{R})$ taking $(1, 0, \dots, 0)$ to y , and then hg^{-1} takes x to y . Thus there are only two orbits: $\mathbb{R}^n \setminus \{0\}$ and $\{0\}$.
- (b) The orthogonal group $\mathrm{O}(n)$ acts continuously on \mathbb{R}^n by matrix multiplication as well. This is just the restriction of the action in part (a) to $\mathrm{O}(n) \times \mathbb{R}^n \subseteq \mathrm{GL}(n; \mathbb{R}) \times \mathbb{R}^n$. Since any unit vector x can be completed to an orthonormal basis (x, x_2, \dots, x_n) the argument in the preceding paragraph shows that for any two unit vectors x and y there is an orthogonal matrix taking x to y . If x and y are any two nonzero vectors with the same length, there is an orthogonal matrix taking $x/|x|$ to $y/|y|$, and this matrix also takes x to y . Since multiplication by an orthogonal matrix preserves lengths of vectors, the orbits of the $\mathrm{O}(n)$ action on \mathbb{R}^n are $\{0\}$ and the spheres centered at 0.
- (c) The restriction of the action of $\mathrm{O}(n)$ to the unit sphere in \mathbb{R}^n yields a transitive action on \mathbb{S}^{n-1} .
- (d) The group \mathbb{R}^* acts on $\mathbb{R}^n \setminus \{0\}$ by scalar multiplication. The action is free, and the orbits are the lines through the origin (with the origin removed).
- (e) Any topological group G acts continuously, freely, and transitively on itself on the left by left translation: $g \cdot g' = L_g(g') = gg'$. Similarly, G acts on itself on the right by right translation.
- (f) If H is a subgroup of the topological group G (with the subspace topology), then group multiplication on the left or right defines a left or right action of H on G ; it is just the restriction of the action of G on itself to $H \times G$ or $G \times H$. This action is continuous and free, but in general not transitive.
- (g) The two-element discrete group $\{\pm 1\}$ or $\mathbb{Z}/2\mathbb{Z}$ acts freely on \mathbb{S}^n by multiplication: $\pm 1 \cdot x = \pm x$. This is an action by homeomorphisms, and because the group is discrete, it is continuous. Each orbit is a pair of antipodal points: $\{\pm x\}$.

Given an action of a group G on a space X (not necessarily continuous or even by homeomorphisms), we define a relation on X by saying $x_1 \sim x_2$ if there is an element $g \in G$ such that $g \cdot x_1 = x_2$. This is reflexive because $1 \cdot x = x$ for each x ; it is symmetric because $g \cdot x_1 = x_2$ implies $g^{-1} \cdot x_2 = x_1$; and it is transitive because $g \cdot x_1 = x_2$ and $g' \cdot x_2 = x_3$ imply $g'g \cdot x_1 = x_3$.

Thus it is an equivalence relation. The equivalence classes are precisely the orbits of the group action. The resulting quotient space is denoted by X/G , and is called the orbit space of the action. If the action is transitive, the orbit space is a single point, so only nontransitive actions yield interesting examples.

Let us examine the quotients determined by some of the group actions described in ??.

Example 3.1.23.

- (a) As we mentioned above, the action of $\mathrm{GL}(n; \mathbb{R})$ on \mathbb{R}^n has two orbits, so the quotient space has exactly two points: $a = q(\mathbb{R}^n \setminus \{0\})$ and $b = q(\{0\})$. The only saturated open subsets of \mathbb{R}^n are \mathbb{R}^n , $\mathbb{R}^n \setminus \{0\}$ and $\{0\}$, so the open subsets of the quotient space are $\{a, b\}, \{a\}, \emptyset$. This quotient space is not Hausdorff.
- (b) The quotient space of \mathbb{R}^n by $\mathrm{O}(n)$ is homeomorphic to $[0, \infty)$.
- (c) The real projective space \mathbb{RP}^n is exactly the orbit space of the action of \mathbb{R}^* on $\mathbb{R}^{n+1} \setminus \{0\}$ by scalar multiplication.

A particularly important special case arises when G is a topological group and we consider the action of a subgroup $H \subseteq G$ on G . An orbit of the right action of H on G is a set of the form $\{gh : h \in H\}$, which is precisely the left coset gH . Thus the orbit space of the right action of H on G is the set G/H of left cosets with the quotient topology. This quotient space is called the (left) coset space of G by H . (It is unfortunate but unavoidable that the right action produces a left coset space and vice versa. If G is commutative, the situation is simpler, because then the left action and right action of H are equal to each other.)

Proposition 3.1.24. *If a topological group G operates continuously on a topological space X , then the equivalence relation R defined by G is open.*

Proof. For the saturation with respect to R of an open subset U of X is the set $\bigcup_{g \in G} gU$, and each gU is open. \square

Let X (resp. Y) be a set with a group of operators G (resp. H) and let $\phi : G \rightarrow H$ be a homomorphism and $\psi : X \rightarrow Y$ a mapping. Then ϕ and ψ are said to be compatible if $\psi(g \cdot x) = \phi(g) \cdot \psi(x)$ for all $g \in G$ and all $x \in X$. If ϕ and ψ are continuous and compatible, passing to the quotients, ψ then induces a continuous mapping $X/G \rightarrow Y/H$.

Let G be a topological group acting continuously on a topological space X , and let π be the canonical mapping of X onto the orbit space X/G . Let A be any subset of X , and let $R(A)$ be the subspace of X which is the saturation of A with respect to the equivalence relation R defined by G (thus $R(A)$ is the union of the orbits of points of A , and is said to be the saturation of A with respect to G). Then G acts continuously on $R(A)$ by the restriction of $(g, x) \mapsto g \cdot x$ to $G \times R(A)$. Moreover, since R is open and $R(A)$ is saturated, it follows that

Proposition 3.1.25. *The canonical bijection of the subspace $\pi(A)$ onto the orbit space $R(A)/G$ is a homeomorphism.*

Now let S be an equivalence relation on X such that, for each $g \in G$, the mapping $x \mapsto g \cdot x$ is compatible with S [in other words, such that the relation xSy implies $(g \cdot x)S(g \cdot y)$; for the sake of brevity we shall say that the relation S is compatible with the group G . If π is the canonical mapping of X onto X/S , and if $g \cdot \bar{x}$ denotes the class of $g \cdot x$, then X/S has G as groups of operators with respect to the law $(g, \bar{x}) \mapsto g \cdot \bar{x}$. Moreover:

Proposition 3.1.26. *If the equivalence relation S on X is open and compatible with G , then G operates continuously on X/S .*

Proof. Since the relation of equality on G and the relation S on X are open, it is enough to show that the mapping $(g, x) \mapsto g \cdot \pi(x) = \pi(g \cdot x)$ of $G \times X$ into X/S is continuous; but this follows from the continuity of π and of the mapping $(g, x) \mapsto g \cdot x$. \square

Let G be a topological group and H a subgroup of G . Then H acts continuously on the right on G according to the multiplication $(g, h) \mapsto gh$ and the orbit of a point $g \in G$ is the left coset gH . The set of orbits is therefore what we have called in algebra the **homogeneous space** G/H .

Whenever we speak of G/H as a topological space, we shall always mean the orbit space of G (with respect to H) unless the contrary is expressly stated; i.e. the quotient space of G by the equivalence relation $x^{-1}y \in H$. In conformity with the general definitions, we say that the topology of this space is the **quotient by H of the topology of G** .

Proposition 3.1.27. *The group G operates continuously on every homogeneous space G/H .*

Proof. Since the equivalence relation $x^{-1}y \in H$ is open, by ?? we see the claim. \square

Proposition 3.1.28. *Let G be a topological group and let H be a subgroup of G . Then the homogeneous space G/H is Hausdorff if and only if H is closed in G .*

Proof. note that H is an equivalence class for the relation $x^{-1}y \in H$ and therefore, if G/H is Hausdorff, H is closed in G . Conversely, if H is closed, then the graph of this relation is closed in $G \times G$, since it is the inverse image of H under the continuous mapping $(x, y) \mapsto x^{-1}y$, so it follows from ?? that G/H is Hausdorff. \square

Proposition 3.1.29. *Let G be a topological group and let H be a subgroup of G . Then the homogeneous space G/H is discrete if and only if H is open in G .*

Proof. For the inverse images in G of the points of G/H under the canonical mapping are the cosets gH , $g \in H$; and these sets are open in G if and only if H is open in G . \square

Let X be a topological space on which a topological group G acts continuously and transitively. The space X is then (in the algebraic sense) a **homogeneous space** of G . Let x be a point of X , G_x its stabilizer. The continuous surjection $g \mapsto gx$ of G onto X factorizes canonically as

$$G \xrightarrow{\pi_x} G/G_x \xrightarrow{\phi_x} X$$

where π_x is the canonical mapping of G onto the homogeneous space G/G_x and ϕ_x is the bijection $gG_x \mapsto g \cdot x$ of G/G_x onto X . Since the composition $\phi_x \circ \pi_x$ is continuous, it follows from the property of the quotient topology that ϕ_x is continuous. However, ϕ_x is not necessarily a homeomorphism of G/G_x onto X . If ϕ_x is a homeomorphism for each $x \in X$, then we say that X is a topological homogeneous space (of the topological group G).

Proposition 3.1.30. *Let X be a topological space on which a topological group G acting continuously and transitively. For X to be a topological homogeneous space (relative to G) it is necessary and sufficient that for each $x \in X$, the orbit map $g \mapsto g \cdot x$ is open.*

Proof. For each $x \in X$, denote by θ_x the orbit map, which maps g to $g \cdot x$. Then for a subset $U \subseteq G/G_x$, we have

$$\phi_x(U) = \theta_x\pi_x^{-1}(U).$$

Therefore ϕ_x is an open map if and only if θ_x is open, and this proves the claim. \square

Proposition 3.1.31. *Let X be a topological space on which a topological group G acting continuously and transitively. For X to be a topological homogeneous space (relative to G) it is sufficient that for some point $x_0 \in X$ the map θ_{x_0} transforms each neighbourhood of e in G into a neighbourhood of x_0 in X .*

Proof. Every $x \in X$ can be written as $x = g \cdot x_0$ for some $g \in G$. If V is a neighbourhood of e , consider the set $V \cdot x$. We can write

$$V \cdot x = (Vg) \cdot x_0 = [g(g^{-1}Vg)] \cdot x_0.$$

Note that $g^{-1}Vg$ is a neighborhood of e , so $(g^{-1}Vg) \cdot x_0$ is a neighborhood of x_0 , by hypothesis. Since $y \mapsto g \cdot y$ is a homeomorphism of X onto itself, it follows that $[g(g^{-1}Vg)] \cdot x_0$ is a neighborhood of x , thus the map θ_x is open (by translation on G) and the claim follows. \square

3.1.4 Product and quotient of topological groups

If H is a normal subgroup of G then we know the homogeneous space G/H becomes a group. We now prove that this makes G/H into a topological group.

Proposition 3.1.32. *Let G be a topological group and let H be a normal subgroup of G . Then the quotient by H of the topology of G is compatible with the group structure of G/H .*

Proof. If $\pi : G \rightarrow G/H$ is the canonical mapping, then we have to show that $(\pi(x), \pi(y)) \mapsto \pi(x)\pi(y)^{-1}$ is a continuous mapping of $(G/H) \times (G/H)$ into G/H . Since the equivalence relation $x^{-1}y \in H$ is open, it is enough to show that $(x, y) \mapsto \pi(x)\pi(y)^{-1}$ is a continuous mapping of $G \times G$ into G/H . But this map is the composition of the continuous mappings π and $(x, y) \mapsto xy^{-1}$, hence is continuous. \square

Whenever in the sequel we consider a quotient group G/H of a topological group G as a topological group, it is always to be understood that the topology of G/H is the quotient by H of the topology of G , unless the contrary is expressly stated.

Proposition 3.1.33. *Let H be a subgroup of a topological group G . The canonical map $\pi : G \rightarrow G/H$ is open. If \mathfrak{B} is a fundamental system of neighbourhoods of e in G , then $\pi(\mathfrak{B})$ is a fundamental system of neighbourhoods of the identity element $\pi(e)$ of G/H .*

Proof. Let $U \subseteq G$ be open, then we have $\pi^{-1}(\pi(U)) = \bigcup_{h \in H} hU$, which is open since U is. Thus by the definition of the quotient topology, $\pi(U)$ is open in G/H . The second claim therefore follows. \square

In particular, ?? and ?? implies the following result.

Proposition 3.1.34. *Let G be a topological group and let H be a normal subgroup of G . Then G/H is Hausdorff if and only if H is closed, and is discrete if and only if H is open.*

If G is a topological group and N is the closure of $\{e\}$ in G , then N is a closed normal subgroup of G (??), hence G/N is Hausdorff; G/N is called the **Hausdorff group associated with G** .

Proposition 3.1.35. *If H is a discrete normal subgroup of a topological group G , then G/H is locally isomorphic to G .*

Proof. Let V be a neighbourhood of e in G which does not contain any point of H other than e , and let W be a symmetric open neighbourhood of e in G such that $W^2 \subseteq V$. Then the restriction to W of the canonical mapping π of G onto G/H is injective; for if $x, y \in W$ and $\pi(x) = \pi(y)$, then $x^{-1}y \in W^2 \subseteq V$ and $x^{-1}y \in H$, so that $x = y$. By ?? it follows that the restriction of π to W is a homeomorphism of W onto $\pi(W)$; since moreover π is a homomorphism, we conclude that G and G/H are locally isomorphic. \square

Proposition 3.1.36. *Let H be a normal subgroup of G and $f : G \rightarrow G$ a continuous homomorphism of G such that $f(H) \subseteq H$. Then the induced map $\tilde{f} : G/H \rightarrow G/H$ is continuous.*

Proof. We have the following commutative diagram:

$$\begin{array}{ccc} G & \xrightarrow{f} & G \\ \downarrow \pi & & \downarrow \pi \\ G/H & \xrightarrow{\tilde{f}} & G/H \end{array}$$

Let $V \subseteq G/H$ be an open set. Then to show $\tilde{f}^{-1}(V)$ is open, we note that

$$\pi^{-1}(\tilde{f}^{-1}(V)) = f^{-1}(\pi^{-1}(V)).$$

Since f and π are continuous, we see $\pi^{-1}(\tilde{f}^{-1}(V))$ is open, whence the claim. \square

Proposition 3.1.37. *Let $\phi : G \rightarrow H$ be a homomorphism between topological groups. Consider the following decomposition of ϕ :*

$$G \xrightarrow{\pi} G/\ker \phi \xrightarrow{\bar{\phi}} \text{im } \phi \xrightarrow{\iota} H$$

Then ι is an embedding and $\bar{\phi}$ is a group isomorphism, and ϕ is continuous iff $\bar{\phi}$ is. Also, $\bar{\phi}$ is open if and only if ϕ is open.

Proof. Note that $\phi = \iota \circ \bar{\phi} \circ \pi$ and π, ι are continuous. Thus if $\bar{\phi}$ is continuous, then so is ϕ ; if ϕ is continuous, then for any V open in H ,

$$\bar{\phi}^{-1}(V \cap \text{im } \phi) = \pi(\phi^{-1}(V))$$

is open in $G/\ker \phi$ since π is open. This shows the first claim, and the second can be proved similarly. \square

Since ϕ may not be open, the bijection $\bar{\phi}$ in ?? may not be a homeomorphism. For example, let G' be a non-discrete topological group, and G the topological group obtained by giving G' the discrete topology; then the identity mapping of G into G' is a continuous bijective homomorphism, but is not bicontinuous.

Definition 3.1.38. A continuous homomorphism ϕ of a topological group G into a topological group H is said to be a **strict morphism** of G into H if the bijective homomorphism $\bar{\phi}$, from $G / \ker \phi$ to $\text{im } \phi$, associated with ϕ , is an isomorphism of topological groups.

By ??, the following proposition is immediate.

Proposition 3.1.39. Let ϕ be a continuous homomorphism of a topological group G into a topological group H . Then the following statements are equivalent:

- (i) ϕ is a strict homomorphism.
- (ii) The image under ϕ of every open set in G is an open set in $f(G)$.
- (iii) The image under ϕ of every neighbourhood of the identity element in G is a neighbourhood of the identity element in H .

Example 3.1.40. From condition (ii) of ?? it follows that every continuous homomorphism of a topological group into a discrete group is a strict morphism. If G is compact and $\phi(G)$ is Hausdorff, then the bijective homomorphism $\bar{\phi}$ associated with ϕ is bicontinuous. Hence every continuous homomorphism of a compact group into a Hausdorff group is a strict morphism.

Example 3.1.41. Let ϕ be a strict morphism of G into G' and let ψ be a strict morphism of G' into G'' . If ϕ is surjective or if ψ is injective, it follows immediately from ?? that $\psi \circ \phi$ is a strict morphism of G into G'' . But this conclusion is no longer necessarily valid if neither of the two preceding conditions is satisfied, even if ϕ is injective and ψ surjective.

Example 3.1.42. Let ϕ be a continuous homomorphism of a topological group G into a topological group G' , and let H be a normal subgroup of G . Then ϕ induces a homomorphism $\bar{\phi}$ of the group G / H onto the quotient group $\phi(G) / \phi(H)$. This homomorphism $\bar{\phi}$ is continuous.

$$\begin{array}{ccc} G & \xrightarrow{\phi} & \phi(G) \\ \pi \downarrow & & \downarrow \pi' \\ G/H & \xrightarrow{\bar{\phi}} & \phi(G)/\phi(H) \end{array}$$

Moreover, if ϕ is a strict morphism of G into G' , then $\bar{\phi}$ is a strict morphism of G / H onto $\phi(G) / \phi(H)$: for if U is open in G / H , and if π (resp. π') denotes the canonical mapping of G onto G / H [resp. of $\phi(G)$ onto $\phi(G) / \phi(H)$], then we have $\bar{\phi}(U) = \pi'(\phi(\pi^{-1}(U)))$. Since $\pi^{-1}(U)$ is open in G , it follows that $\bar{\phi}(U)$ is open in $\phi(G) / \phi(H)$, which proves our assertion.

Now we consider subgroups of a quotient group. Let G be a topological group, let H be a normal subgroup of G , and let $\pi : G \rightarrow G / H$ be the canonical mapping. We know that if \bar{K} is a subgroup of G / H , then $K := \pi^{-1}(\bar{K})$ is a subgroup of G which contains H . Conversely, if K is a subgroup of G , then $\pi(K)$ is a subgroup of G / H ; moreover, there is a canonical bijection of the quotient group $K / (K \cap H)$ onto the subgroup $\pi(K)$ of G / H , and a canonical bijection of $\pi(K)$ onto the quotient group KH / H , and both these bijections are isomorphisms for the group structures.

Proposition 3.1.43. Let K be a subgroup of a topological group G , let H be a normal subgroup of G , and let π denote the canonical mapping of G onto G / H . Then the canonical bijection of $\pi(K)$ onto KH / H is an isomorphism of topological groups.

Proof. Note that HK is the saturation of K , so this follows from ?? \square

The canonical bijection of $K/(K \cap H)$ onto $\pi(K)$ is a continuous homomorphism, since it arises from the restriction of π to K by passing to the quotients; but in general the topological groups $K/(K \cap H)$ and KH/H are not isomorphic.

Example 3.1.44. For example, take G to be the additive group \mathbb{R} of real numbers, take H to be the group \mathbb{Z} of integers, and take K to be the group $\alpha\mathbb{Z}$ of integer multiples of an irrational number α . Then $K \cap H = \{0\}$, so that $K/(K \cap H)$ is a discrete group, isomorphic to \mathbb{Z} ; on the other hand, $K + H$ is dense in \mathbb{R} , hence $(K + H)/H$, which is locally isomorphic to $K + H$, is not a discrete group and consequently is not isomorphic to $K/(K \cap H)$.

Let G be a topological group operating continuously on a topological space X , and let K be a normal subgroup of G which is contained in the stabilizer of each point of X . For each $x \in X$, the relation $g^{-1}h \in K$ therefore implies $g \cdot x = h \cdot x$, and passing to the quotient we define a mapping $\bar{g} \mapsto \bar{g} \cdot x$ of G/K into X . It is immediately verified that with respect to this definition G/K acts on X . Moreover, this action is continuous: the relation of equality on X and the relation $g^{-1}h \in K$ on G are both open equivalence relations, and therefore the result follows from the continuity of the mapping

$$(g, x) \mapsto \bar{g} \cdot x = g \cdot x$$

of $G \times X$ into X .

Now let G be a topological group acting continuously on a topological space X , and let H be any normal subgroup of G ; then H acts continuously on X . Let R_H be the equivalence relation defined by H on X ; then R_H is open. The relation R_H is compatible with the group G , for if $(x, y) \in R_H$ then there exists $h \in H$ such that $y = h \cdot x$, hence for all $g \in G$ we have $g \cdot y = (ghg^{-1})g \cdot x$, and $ghg^{-1} \in H$ since H is normal in G ; thus $(g \cdot x, g \cdot y) \in R_H$. If \bar{x} denotes the canonical image of X onto X/H , the group G therefore acts continuously on X/H by

$$(g, \bar{x}) \mapsto \bar{g} \cdot \bar{x}.$$

Moreover, the group H is contained in the stabilizer of each of the points of X/H ; as we have seen above, G/H then acts continuously on X/H by

$$(\bar{g}, \bar{x}) \mapsto \bar{g} \cdot \bar{x}.$$

If R denotes the equivalence relation on X defined by G , then the relation R_H implies R , and the equivalence relation R/R_H on X/H is that defined by the group G/H . Hence we have the following result:

Proposition 3.1.45. *Let G be a topological group operating continuously on a topological space X , and let H be a normal subgroup of G . Then the canonical bijection of X/G onto $(X/H)/(G/H)$ is a homeomorphism.*

Corollary 3.1.46. *Let G be a topological group, H a normal subgroup of G , and K a normal subgroup of G which contains H . Then the canonical bijection of G/K onto $(G/H)/(K/H)$ is an isomorphism of topological groups.*

Proof. We know that this bijection is an isomorphism of groups, and ?? shows that it is a homeomorphism. \square

Proposition 3.1.47. *Let G be a group and let $(G_i)_{i \in I}$ be a family of topological groups such that for each $i \in I$ there is a homomorphism $f_i : G \rightarrow G_i$. Then the initial topology on G with respect to (f_i) is a group topology.*

Proof. For each $i \in I$ let $\mathfrak{B}_i(e)$ denote the filter of neighborhoods of e in G_i . Then a base $\mathfrak{U}_{\mathcal{T}}(e)$ at e for the initial topology \mathcal{T} for G determined by the family $(f_i)_{i \in I}$ is given by intersections of the form $\bigcap_{k \in K} f_k^{-1}(V_k)$, where K is a finite subset of I and $V_k \in \mathfrak{B}_k(e)$ for each $k \in K$. For i_1, \dots, i_n , let $V_{i_j} \in \mathfrak{B}_{i_j}(e)$, and choose symmetric $B_{i_j} \in \mathfrak{B}_{i_j}(e)$ such that $B_{i_j} B_{i_j}^{-1} \subseteq V_{i_j}$, then

$$(f_{i_j}^{-1}(B_{i_j})) (f_{i_j}^{-1}(B_{i_j}))^{-1} = f_{i_j}^{-1}(B_{i_j} B_{i_j}^{-1}) \subseteq f_{i_j}^{-1}(V_{i_j})$$

and therefore $(\bigcap_{j=1}^n f_{i_j}^{-1}(B_{i_j})) (\bigcap_{j=1}^n f_{i_j}^{-1}(B_{i_j}))^{-1} \subseteq (\bigcap_{j=1}^n f_{i_j}^{-1}(V_{i_j}))$. By ??, \mathcal{T} is a group topology on G . \square

An example is given by the product group topology. Let $\{G_i : i \in I\}$ be a family of topological groups and consider the product set $G = \prod_i G_i$. It is easy to verify that G is a group with respect to this pointwise operation and that each projection $\pi_i : G \rightarrow G_i$ is a homomorphism. The product topology \mathcal{T} , the coarsest topology with respect to which each projection is continuous, is the initial image topology determined by $\{\pi_i : i \in I\}$. Since projections are homomorphisms, it is a group topology.

Definition 3.1.48. *The topological group obtained by giving the product set*

$$G = \prod_{i \in I} G_i$$

the group structure which is the product of the group structures of the G_i and the topology which is the product of the topologies of the G_i is called the product of the topological groups G_i .

If H_i is a subgroup of G_i then the product of the topological groups H_i is isomorphic to the subgroup $\prod_i H_i$.

Proposition 3.1.49. *Let $(G_i)_{i \in I}$ be a family of topological groups, and let H be the normal subgroup of $G = \prod_i G_i$ consisting of all $x = (x_i)$ such that the x_i are equal to the identity element e_i of G_i except for a finite number of indices. Then the subgroup H is dense in G .*

Let $(X_i)_{i \in I}$ be a family of topological spaces, and for each $i \in I$ let G_i be a topological group acting continuously on X_i . It is clear that the product group $G = \prod_{i \in I} G_i$ then acts continuously on the product space $X = \prod_{i \in I} X_i$ by

$$((g_i), (x_i)) \mapsto (g_i \cdot x_i).$$

Moreover the orbit under G of a point $x = (x_i)$ of X is the product of the orbits of the X_i (with respect to the groups G_i). Let π_i be the canonical mapping of X_i onto X_i/G_i and let $\pi = (\pi_i)$ be the product mapping of X onto $\prod_{i \in I} (X_i/G_i)$; then the preceding remark shows that the bijection canonically associated with π maps the orbit space X/G onto $\prod_{i \in I} (X_i/G_i)$. Moreover:

Proposition 3.1.50. *The bijection of X/G onto $\prod_{i \in I}(X_i/G_i)$ canonically associated with (π_i) is a homeomorphism.*

Proof. Since the π_i are surjective and open, it follows that $\pi = (\pi_i)$ is an open mapping. \square

Corollary 3.1.51. *Let $(G_i)_{i \in I}$ be a family of topological groups, and for each $i \in I$ let H_i be a normal subgroup of G_i ; let π_i denote the canonical mapping of G_i onto G_i/H_i . Let $G = \prod_{i \in I} G_i$ and $H = \prod_{i \in I} H_i$. Then the bijective homeomorphism of G/H onto $\prod_{i \in I}(G_i/H_i)$ associated with the continuous homomorphism $(x_i) \mapsto (\pi_i(x_i))$ is an isomorphism of topological groups.*

3.1.5 \mathcal{S} -topologies

Let G be an commutative topological group, $\mathfrak{U}(0)$ the neighborhood filter at 0, and X be a set. Let G^X denote the set of all functions mapping X into G . For $f, g \in G^X$ and $x \in X$, define $(f + g)(x) = f(x) + g(x)$. With respect to these pointwise operations, G^X is an commutative group.

We topologize G^X in various ways that make it a topological group. Various different notions of convergence of functions-uniform, pointwise, etc.–all fit under this umbrella. This was very important, historically, for it shows that general topology is a broad enough tool to encompass these diverse types of convergence. Our general remarks below apply just as well to any subgroup of G^X . When X is a topological space, we can consider the subgroup $C(X, G)$ of continuous maps of X into G . The building blocks for the topologies for G^X to come are the sets

$$N(S, V) = \{f \in G^X : f(S) \subseteq V\}, \quad S \subseteq X, V \in \mathfrak{U}(0).$$

Their vital properties follow. The proofs of all parts are immediate.

Proposition 3.1.52. *For any $S_1, S_2 \subseteq X$ and $U, V \in \mathfrak{U}(0)$,*

- (a) $U \subseteq V \Rightarrow N(S, U) \subseteq N(S, V)$;
- (b) $N(S, -V) = -N(S, V)$; thus if V is symmetric, so is $N(S, V)$;
- (c) $N(S, U) - N(S, V) \subseteq N(S, U - V)$;
- (d) $S_1 \subseteq S_2 \Rightarrow N(S_1, V) \supseteq N(S_2, V)$;
- (e) for any family $\{S_i : i \in I\}$ of subsets of X , $N(\bigcup_i S_i, V) = \bigcap_i N(S_i, V)$.
- (f) for any family $\{V_i : i \in I\}$ of neighborhoods of 0 in G , $N(S, \bigcap_i V_i) = \bigcap_i N(S, V_i)$.

Let \mathcal{S} be a collection of subsets of X and consider $B_{\mathcal{S}}(0) = \{N(S, V) : S \in \mathcal{S}, V \in \mathfrak{U}(0)\}$. Since the 0 function belongs to each $N(S, V)$, finite intersections of sets from $B_{\mathcal{S}}(0)$ are nonempty. Let $S \in \mathcal{S}$ and $V \in \mathfrak{U}(0)$. Choose a symmetric neighborhood $U \subseteq V$ such that $U - U \subseteq V$. By ??, it follows that

$$N(S, U) - N(S, U) \subseteq N(S, U - U) \subseteq N(S, V)$$

thus showing that $B_{\mathcal{S}}(0)$ satisfies condition (FB) of ???. Thus $B_{\mathcal{S}}(0)$ forms a neighborhood subbase at 0 for a group topology $\mathcal{T}_{\mathcal{S}}$ for G^X , called the **topology of uniform convergence on \mathcal{S}** , or the **\mathcal{S} -topology**. We denote the neighborhood filter it determines at 0 by $\mathfrak{U}_{\mathcal{S}}(0)$.

Suppose the topological group (G, d) is metrizable and that \mathcal{B} consists of the sets $B_r(0)$, $r > 0$. Then, for any collection \mathcal{S} of subsets of X , the sets $N(S, B_r(0))$, $r > 0$, $S \in \mathcal{S}$, is a base at 0 for $\mathcal{T}_{\mathcal{S}}$.

Since $N(\bigcup_i S_i, V) = \bigcap_i N(S_i, V)$, if \mathcal{S} is closed under the formation of finite unions, then the $N(S, V)$ form a base at 0 rather than just a subbase. Actually, we can always assume that \mathcal{S} is closed with respect to the formation of finite unions in view of the following result.

Proposition 3.1.53. *Let \mathcal{S} be a collection of subsets of the set X . The \mathcal{S} -topology $\mathcal{T}_{\mathcal{S}}$ is unaffected by replacing \mathcal{S} by the collection \mathcal{S}^* of all subsets of finite unions of sets of \mathcal{S} .*

Proof. Since $\mathcal{S} \subseteq \mathcal{S}^*$, clearly $\mathcal{T}_{\mathcal{S}} \subseteq \mathcal{T}_{\mathcal{S}^*}$, so it only remains to show that $\mathfrak{U}_{\mathcal{S}^*}(0) \subseteq \mathfrak{U}_{\mathcal{S}}(0)$. To this end, suppose that $S_1, \dots, S_n \in \mathcal{S}$. By ??(d), for any neighborhood V of 0 in the topological group G ,

$$N\left(\bigcup_{i=1}^n S_i, V\right) = \bigcap_{i=1}^n N(S_i, V)$$

This implies $N(\bigcup_{i=1}^n S_i, V) \in \mathfrak{U}_{\mathcal{S}}(0)$ and the result follows. \square

Proposition 3.1.54. *Let \mathcal{S} be a collection of subsets of the set X . If the topological group G is Hausdorff and \mathcal{S} covers X then $(G^X, \mathcal{T}_{\mathcal{S}})$ is Hausdorff.*

Proof. It suffices to show that $\bigcap \mathfrak{U}_{\mathcal{S}}(0) = \{0\}$. To this end let $f \in \mathfrak{U}_{\mathcal{S}}(0)$ and choose any $x \in X$. Now select $S \in \mathcal{S}$ such that $x \in S$. Since $f \in \bigcap \mathfrak{U}_{\mathcal{S}}(0)$, $f(x) \in \bigcap \mathfrak{U}(0) = \{0\}$ (G is Hausdorff). It follows that $f(x) = 0$. Since x is arbitrary, $f = 0$. \square

?? is one instance of a somewhat more general phenomenon: namely, that the behavior of a function f on $\bigcup \mathcal{S}$ determines the position f will occupy in the \mathcal{S} -topology. For example, if f vanishes on $\bigcup \mathcal{S}$, then f is a limit point of the filter $\mathfrak{U}_{\mathcal{S}}(0)$.

We consider some particular \mathcal{S} -topologies next.

Example 3.1.55 (Topology of Uniform Convergence). Let $\mathcal{S} = \{X\}$. The sets $\mathfrak{U}_X(0) = \{N(X, V) : V \in \mathfrak{U}(0)\}$ form a filterbase in G^X by ???. The topology \mathcal{T}_u obtained this way is called the **topology of uniform convergence**. It is clear that a net (f_α) from (G^X, \mathcal{T}_u) converges to f iff for any $V \in \mathfrak{U}(0)$, $f_\alpha(x) - f(x) \in V$ eventually for all $x \in X$.

More generally, consider any subset S of X . The sets $\mathfrak{U}_S(0) = \{N(S, V) : V \in \mathfrak{U}(0)\}$ are a base at 0 for a group topology on G^X called the topology of uniform convergence on S .

Example 3.1.56. Let \mathcal{S} denote the class of all singletons of X . In view of ?? we get the same \mathcal{S} -topology by taking \mathcal{S} to be the set of all finite subsets of X . This \mathcal{S} -topology is called the **topology of pointwise convergence**; we denote \mathcal{T}_p by this topology and the neighborhood filter by $\mathfrak{U}_p(0)$. It is clear that a net (f_α) from (G^X, \mathcal{T}_p) converges to f iff the nets $(f_\alpha(x))$ in G converge to $f(x)$ for each $x \in X$. Another name for \mathcal{T}_p is the point-open topology. If we replace X by a subset S of X , we call the resulting topology the topology of pointwise convergence on S .

Since the collection of all finite subsets of X is generally a rather small collection, the topology of pointwise convergence is a fairly weak topology. A subbasic \mathcal{T}_p -neighborhood of 0 is a set of the form $\{f \in G^X : f(x) \in V\}$ for some $x \in X$ and neighborhood V of 0 in X which is $V \times \prod_{s \in X, s \neq x} G_s$, where $G_s = G$ for all $s \neq x$, i.e., a subbasic neighborhood of 0 for the product topology on G^X ; in other words the product and point-open topologies coincide on G^X . By

??, given any collection \mathcal{S} of subsets of X , the point-open topology on $\bigcup \mathcal{S}$ is coarser than the \mathcal{S} -topology.

Example 3.1.57 (Compact-Open Topology). Let \mathcal{S} be the collection of all compact subsets of a topological space X . We denote this \mathcal{S} -topology by \mathcal{T}_c . It is called variously the **compact-open topology**, **topology of uniform convergence on compact sets** and the **topology of compact convergence**. An open base at 0 for \mathcal{T}_c is given by the sets $N(K, V)$, where K is a compact subset of X and V an open neighborhood of 0 in X . Since there are generally more compact sets than finite sets, \mathcal{T}_c is generally finer than the topology of pointwise convergence.

3.1.6 Metrizability

In this part, G denotes a topological group. We show that a topological group is pseudometrizable iff it has a countable base at 0, in which case the topology is generated by an *invariant* pseudometric. After showing that quotients of pseudometrizable groups are pseudometrizable, we show that the topology of any topological group is generated by a family of invariant pseudometrics.

First, we exhibit a pseudometric that does not yield a group topology.

Example 3.1.58. We define a pseudometric d on the integers \mathbb{Z} as follows

$$d(j, k) = d(k, j) = \begin{cases} 0 & \text{if } j = k \text{ or } j < k \leq 0, \\ 1 & \text{if } j \leq 0 \leq k \text{ or } j > k \geq 0. \end{cases}$$

It is trivial to verify that d is a pseudometric and any neighborhood of 0 must contain the negative integers. Note that $B_{1/2}(1) = \{1\}$. Writing 1 as $1 + 0$, there can be no neighborhoods U of 0 and V of 1 such that $U + V \subseteq B_{1/2}(1)$. This, then, d induces an incompatible pseudometric topology.

If G is a group and d a pseudometric on G , d is called **left invariant** if $d(gx, gy) = d(x, y)$ whenever $g, x, y \in G$, and **right invariant** if $d(xg, yg) = d(x, y)$, and **invariant** if it is both right and left invariant. If G is a topological space and d a pseudometric on G , we shall say that G admits d if the d -topology of G agrees with its original topology.

Metrizability of a topology depends on its separation and countability properties. Since group topologies have a certain amount of inherent separation (namely, regularity) metrizability reduces to just a countability condition.

Theorem 3.1.59. *Let G be a topological group. Then G is pseudometrizable iff it has a countable base at e. If G is pseudometrizable then there exists a left invariant pseudometric d that generates its topology. Moreover, G is metrizable iff it is Hausdorff.*

Proof. That the condition is necessary is obvious. Therefore it is sufficient to prove that if G satisfies the first countability axiom, it is metrizable. To prove this, suppose that $\{V_n\}$ is a denumerable base of neighborhoods of 0. By ??(b), we can define a sequence $\{U_n\}$ of symmetric neighborhoods of e such that $U_1 = V_1$ and $U_{n+1}^3 \subseteq U_n \cap V_n$ for each $n \in \mathbb{N}$. Since $\{V_n\}$ is a base at e , so is $\{U_n\}$. As a first step toward getting a left invariant pseudometric that generates the

topology, consider the function f defined as follows:

$$f(x, y) = \begin{cases} 1 & \text{if } x^{-1}y \notin U_1, \\ 2^{-n} & \text{if } x^{-1}y \in U_n \setminus U_{n+1}, \\ 0 & \text{if } x^{-1}y \in U_n \text{ for each } n. \end{cases}$$

Obviously $f(x, x) = 0$. Also, the sets $U_r(g) = \{x \in G : f(g, x) < r\}$ are a complete system of neighborhoods for any point $g \in G$. Moreover, since U_n are symmetric, $x^{-1}y \in U_n$ iff $y^{-1}x \in U_n$, whence $f(x, y) = f(y, x)$. And finally, since $U_i U_j U_k \subseteq U_n$ if $n > i, j, k$, one sees that

$$\text{If } f(x, y) < r, f(y, \tilde{y}) < r, f(\tilde{y}, z) < r, \text{ then } f(x, z) < 2r. \quad (3.1.2)$$

The function d , defined next, is:

$$d(x, y) = \inf \sum_{i=0}^{p-1} f(z_i, z_{i+1})$$

where the infimum is taken over all finite sequences z_0, \dots, z_p for which $z_0 = x$ and $z_p = y$. In effect, we are seeking a path of shortest distance from x to y . We show that d is a left invariant pseudometric such that $f(x, y)/2 \leq d(x, y) \leq f(x, y)$. The second inequality is obvious; to prove the first, we show that

$$\sum_{i=0}^{p-1} f(z_i, z_{i+1}) \geq f(x, y)/2 \quad (3.1.3)$$

for any finite sequence $x = z_0, \dots, z_p = y$. We establish this by induction on p . Clearly, this holds for $p = 1$, so suppose that $p > 1$ and that (??) holds for all integers $q < p$ and x, y in G . Let A denote the sum on the left side of (??). Consider two possibilities: if $A \geq 1/2$, then (??) holds since $f(x, y) \leq 1$. If $A < 1/2$, we can find an integer h such that

$$\sum_{i=0}^{h-1} f(z_i, z_{i+1}) \leq A/2, \quad \sum_{i=h+1}^p f(z_i, z_{i+1}) < A/2.$$

By the induction hypothesis, $f(x, z_h)/2 \leq \sum_{i=0}^{h-1} f(z_i, z_{i+1})$, so $f(x, z_h) \leq A$. Similarly we have $f(z_{h+1}, y) \leq A$. Since clearly $f(z_h, z_{h+1}) \leq A$, by (??) we get $f(x, y) \leq 2A$.

If $A = 0$, then $f(z_i, z_{i+1}) = 0$ for $i = 0, \dots, p - 1$, which implies that $z_i^{-1}z_{i+1} \in U_n$ for every n . Thus $x^{-1}y \in U_n^p$ for every n . It follows that $x^{-1}y \in \bigcap_n U_n$ so $f(x, y) = 0$. Thus (??) holds when $A = 0$ as well and the proof of the pseudometrizability of G is complete. \square

If the topological group G is pseudometrizable then, for any normal subgroup H of G , so is the quotient group G/H . For this, it suffices to note that if G has a countable base $\{V_n\}$ at e , the canonical images $\pi(V_n)$ are a countable base at \bar{e} in G/H . In fact, even more is true.

Proposition 3.1.60. *Let G be a pseudometrizable topological group and H be a normal subgroup of G . Then the function $\bar{d} : G/H \rightarrow \mathbb{R}$ defined by*

$$\bar{d}(\bar{x}, \bar{y}) = d(xH, yH) = \inf_{h_1, h_2 \in H} d(xh_1, yh_2)$$

is a pseudometric on G/H inducing the quotient topology. If d is a metric and H is closed, then \bar{d} is a metric.

Proof. Let $\pi : G \rightarrow G/H$ denote the canonical map. We show that $\pi(B_r(e)) = B_r(\bar{e})$ for each $r > 0$. If $x \in B_r(0)$, then $d(\bar{x}, \bar{e}) \leq d(x, e) < r$; hence $\pi(B_r(e)) \subseteq B_r(\bar{e})$. Conversely, if $\bar{x} \in B_r(\bar{e})$, then there exists $h \in H$ such that $d(xh, e) < r$. Hence $xh \in B_r(0)$ and $\pi(xh) = \pi(x) = \bar{x}$. This proves the first part.

Now assume that d is a metric and H is closed. Let $\bar{x}, \bar{y} \in G/H$ such that $\bar{d}(\bar{x}, \bar{y}) = 0$. Then there exists $h_1, h_2 \in H$ such that $d(xh_1, yh_2) = 0$. Since d is a metric, this implies $xh_1 = yh_2$, and thus $\bar{x} = \bar{y}$. \square

Suppose that \mathcal{D} is a family of left invariant pseudometrics on a group G . For each $d \in \mathcal{D}$, the symmetric sets $\{x \in G : d(x, e) < r\}$ where $d \in D$ and $r > 0$ form a filter subbase satisfying the conditions of ??, and therefore are a filter subbase of open sets for a group topology \mathcal{T} on X . We call \mathcal{T} the topology determined by the family of pseudometrics \mathcal{D} . It is trivial to verify that \mathcal{T} is the weakest topology with respect to which each d is continuous. We show next that the topology of any topological group is generated by a family of left invariant pseudometrics.

Proposition 3.1.61. *The topology of any topological group (G, \mathcal{T}) is determined by a family of left invariant pseudometrics.*

Proof. Let $\mathfrak{U}(0)$ denote the collection of symmetric neighborhoods of 0 in the topological group (X, \mathcal{T}) . For each $U \in \mathfrak{U}(0)$ we may inductively define a sequence $\{U_n\}$ of symmetric neighborhoods of 0 such that $U_1 = U$ and $U_{n+1}^3 \subseteq U_n$ for $n \in \mathbb{N}$ as in the proof of ???. Also as in the proof of ???, there is a left invariant pseudometric d_U associated with each $U \in \mathfrak{U}(0)$ such that $x \in U$ iff $d(x, U) \leq 1$. In other words, the closed unit ball determined by d_U is just U so the topology generated by $\{d_U : U \in \mathfrak{U}(0)\}$ is finer than \mathcal{T} . By the proof of ??, each d_U is continuous; therefore each $\{x \in G : d_U(x, e) < r\}$, $r > 0$ is a neighborhood of 0 in G and the topology generated by $\{d_U : U \in \mathfrak{U}(0)\}$ is coarser than \mathcal{T} . \square

3.2 Complete topological groups

In this section we discuss the completeness of an commutative topological group G by means of Cauchy nets. Without further specification, all groups in this part will be commutative.

3.2.1 Uniformity on topological groups

In a topological group a we can perceive the possibility of defining a notion of "sufficiently near points" and hence a uniform structure, by operating as follows: if x and y are any two points of G , we apply to both points the translation which sends one of them, say x , to the identity element e ; the "proximity" of x and y is then evaluated in some sense by the neighbourhood V of e into which y is translated. This translation, which consists in multiplying both x and y by x^{-1} , can be carried out on the right or on the left, and we shall see that in either case we obtain effectively a uniformity on a which is compatible with the topology of G . Let us take the case in which the translations are performed on the left; then to each neighbourhood V of e there corresponds the set L_V of pairs $(x, y) \in G \times G$ such that $x^{-1}y \in V$. Let \mathfrak{L} be the family

of sets L_V , as V runs through the neighbourhood filter \mathfrak{B} of e . Then \mathfrak{L} is a fundamental system of entourages. For since $e \in V$, the diagonal Δ of $G \times G$ is contained in L_V for each $V \in \mathfrak{B}$, hence \mathfrak{L} is a filter base and satisfies axiom (B1); since the relations $x^{-1}y \in V$ and $y^{-1}x \in V^{-1}$ are equivalent, we have $L(V^{-1}) \in \mathfrak{L}$, so that (B2) is satisfied; and finally, the relations $x^{-1}y \in V$ and $y^{-1}z \in V$ imply $x^{-1}z \in V^2$; hence $L_V \circ L_V$ is contained in L_{V^2} , thus \mathfrak{L} satisfies (B3).

The uniformity defined by \mathfrak{L} is compatible with the topology of G , for the relations $y \in L_V(x)$ and $y \in xV$ are equivalent by definition; in other words $L_V(x) = xV$. We can also apply right translation to get another uniformity on G , which consists of set of the form

$$R_V = \{(x, y) \in G \times G : yx^{-1} \in V\}$$

with V a neighborhood of e in G . We may call this uniformity \mathfrak{R} . If $i : G \rightarrow G$ is the inverse map, then it is easy to see $i^{-1}(R_V) = L(V^{-1})$, so i is an isomorphism between the uniformities \mathfrak{L} and \mathfrak{R} .

To each proposition on the topology of a uniform space there corresponds a proposition on the topology of a group; the translation is made according to the definition and the formulae $L_V(x) = xV$, $L_V(A) = AV$, $R_V(x) = V$, $R_V(A) = VA$, which are immediate consequences of the definition. For example, if A is any non-empty subset of G then we have

$$\bar{A} = \bigcap_{V \in \mathfrak{B}} AV = \bigcap_{V \in \mathfrak{B}} VA.$$

The left and right uniformities on a topological group are in general distinct, but they both induce the group topology on G . Obviously they coincide if the group is commutative, for then $L_V = R_V$; also they coincide if the group is compact and Hausdorff (by ??). In general, we shall denote by G_l (resp. G_r) the uniform space obtained by giving the set G its left (resp. right) uniformity.

Proposition 3.2.1. *The left and right translations are isomorphisms of the left uniformity onto itself.*

Proof. As to the left translations, the result is clear, since the relation $x^{-1}y \in V$ is equivalent to $(ax)^{-1}(ay) \in V$ [in other words, the mapping $(x, y) \rightarrow (ax, ay)$ leaves L_V fixed]. For the right translations, the result follows from the group axioms: for $x^{-1}y \in V$ if and only if $(xa)^{-1}(ya) \in a^{-1}Va$ hence $x \mapsto xa$ is uniformly continuous on G_l . \square

Every inner automorphism $x \mapsto axa^{-1}$ of G is therefore an automorphism for the group structure of G , for the topology of G , and for both the uniformities of G .

Proposition 3.2.2. *Every continuous homomorphism f of a topological group G into a topological group H is uniformly continuous when considered as a mapping of G_l into H_l (or of G_r into H_r).*

Proof. For if V is a neighbourhood of the identity element in H , and $U = f^{-1}(V)$, then the relation $x^{-1}y \in U$ implies

$$f(x)^{-1}f(y) = f(x^{-1}y) \in V.$$

This proves the claim. \square

3.2.2 Complete groups

Definition 3.2.3. A topological group is said to be *complete* if its left and right uniformities are structures of complete spaces.

Since the left and right uniformities are isomorphic, for a group to be complete it is sufficient that one of its uniformities is a structure of a complete space. Every closed subgroup of a complete group is complete. Every product of complete groups is complete.

Example 3.2.4. Consider the additive group \mathbb{R} of real numbers. A metric for \mathbb{R} is given by $d(x, y) = |\arctan x - \arctan y|$. As a metric space (\mathbb{R}, d) is not complete for $\{n\}$ is a nonconvergent Cauchy sequence. Since, for $r > 0$, $B_r(0) = \{x \in \mathbb{R} : |\arctan x| < r\} = (-\tan r, \tan r)$, d induce the usual Euclidean topology on \mathbb{R} , so (\mathbb{R}, d) is a complete topological group. Note that $\{n\}$ is a Cauchy sequence in (\mathbb{R}, d) but not in the topological group \mathbb{R} with its usual topology.

The metric of ?? is not invariant, yet it induces a group topology on \mathbb{R} . Invariance, or rather the lack of it, is really the key to the situation of the example. If d is an invariant pseudometric, then $d(x_\alpha, x_\beta)$ goes to 0 iff $d(x_\alpha - x_\beta, 0)$ goes to 0, so what occurred in ?? cannot happen. In summary:

Proposition 3.2.5. If d is an invariant pseudometric on the group G then (G, d) is complete as a pseudometric space iff it is complete as a topological group.

We show next that the existence of a complete neighborhood of e implies that the group is complete.

Proposition 3.2.6. If the topological group G has a complete neighborhood V of e then G is complete.

Proof. Let V be a complete neighborhood of e . If (x_α) is a Cauchy net, there is some index β such that $x_\alpha \in x_\beta + V$ for $\alpha \succeq \beta$. As a translate of a complete set, $x_\beta + V$ is complete. Since $\{x_\alpha : \alpha \succeq \beta\}$ is also a Cauchy net, it converges to some $x \in x_\beta + V$; clearly, x is also a limit of (x_α) . \square

The most important fact about completeness is of course, that, compact sets are complete:

Proposition 3.2.7. A compact subset of a topological group G is complete. Hence any locally compact topological group G is complete.

Proof. If K is compact, then any filter in K has a cluster point, hence converges. As for the second statement, it suffices to note that a locally compact topological group has a compact neighborhood of 0, hence a complete neighborhood of 0. \square

Proposition 3.2.8. Let G_1 be a topological group, G_2 be a complete Hausdorff topological group, and let H_1 (resp. H_2) be a dense subgroup of G_1 (resp. G_2). Then every continuous homomorphism $f : H_1 \rightarrow H_2$ can be uniquely extended to a continuous homomorphism \tilde{f} of G_1 into G_2 . Furthermore, if G_1 is Hausdorff and complete, and if f is an isomorphism of H_1 onto H_2 , then \tilde{f} is an isomorphism of G_1 onto G_2 .

Proof. The map f is uniformly continuous with respect to the left uniformities of H_1 and H_2 by ??, hence admits a unique extension to a mapping \tilde{f} of G_1 into G_2 which is uniformly continuous with respect to the left uniformities of these groups (??). Moreover, by virtue of the

principle of extension of identities, \tilde{f} is a homomorphism of G_1 into G_2 , whence the first assertion of the Proposition. To prove the second assertion, it is enough to consider the isomorphism g of H_2 onto H_1 , which is the inverse of f , and its extension \tilde{g} to a continuous homomorphism of G_2 into G_1 ; by reason of the uniqueness of the extension, \tilde{f} and \tilde{g} are inverse of each other, hence are bijective. \square

Now let G be an commutative topological group. If X is any set, the set G^X of all maps of X into G forms a group with respect to pointwise addition of functions. We also discussed a class of group topologies for G^X called **S -topologies**: S is any family of subsets of X and the basic neighborhoods of 0 for the S -topology are the sets

$$N(S, V) = \{f \in G^X : F(S) \subseteq V\}$$

zwhere $S \in S$ and V is a neighborhood of 0 in G . By ??, we may suppose that S is closed with respect to the formation of subsets of finite unions. The result below provides a necessary and sufficient condition for S -convergence and also relates S -convergence and pointwise convergence. We denote the S -topology by \mathcal{T}_S .

Proposition 3.2.9. *With notation as above, a net (f_α) from G^X is \mathcal{T}_S -convergent to f iff (f_α) is a \mathcal{T}_S -Cauchy net and $f_\alpha(x) \rightarrow f(x)$ for each $x \in \bigcup S$.*

Proof. Suppose that $f_\alpha \rightarrow f$ with respect to \mathcal{T}_S . Then (f_α) is a \mathcal{T}_S -Cauchy net. Since S may be supposed to contain all subsets of finite unions of sets of S , given any neighborhood V of 0 in G and any $x \in \bigcup S$, $f_\alpha(x) - f(x) \in V$ eventually and the conditions are seen to be necessary.

Conversely, suppose that the \mathcal{T}_S -Cauchy net $f_\alpha \rightarrow f$ pointwise on $\bigcup S$. Since (f_α) is \mathcal{T}_S -Cauchy, given any neighborhood V of 0, there exists an index γ such that $f_\alpha - f_\beta \in N(S, V)$ for $\alpha, \beta \succeq \gamma$. Since $f_\alpha \rightarrow f$ pointwise on S , for any $x \in S$, there is an index $\alpha(x) \succeq \gamma$ such that $f_{\alpha(x)}(x) - f(x) \in V$. Hence, for any $x \in S$ and $\alpha \succeq \gamma$,

$$f_\alpha(x) - f(x) = f_\alpha(x) - f_{\alpha(x)}(x) + f_{\alpha(x)}(x) - f(x) \in V + V.$$

Since V is arbitrary, the desired result follows. \square

The next result provides a sufficient condition for the completeness of a function group (G^X, \mathcal{T}_S) .

Proposition 3.2.10. *If the topological group G is complete, then so is (G^X, \mathcal{T}_S) for any collection S of subsets of X .*

Proof. Let (f_α) be a Cauchy net in G^X and let $H = \bigcup S$. Since S may be assumed to contain singletons, given any $x \in H$ and neighborhood V of 0 in G , $f_\alpha - f_\beta \in N(\{x\}, V)$ eventually, i.e., $f_\alpha(x) - f_\beta(x) \in V$ eventually. For each $x \in H$, then $(f_\alpha(x))$ is a Cauchy net in G . Since G is complete, each $(f_\alpha(x))$ converges; we call the limit $f(x)$. For $x \notin H$, take $f(x) = 0$. Thus $f_\alpha \rightarrow f$ pointwise on H so, by ??, $f_\alpha \rightarrow f$ in the S -topology. \square

3.2.3 Completion of Hausdorff groups

For a Hausdorff topological group G , we know the uniform space G_l may be considered as a dense subspace of its completion \widehat{G}_l . We shall investigate whether G can be considered as

a dense subgroup of a complete Hausdorff group \widehat{G} . If so, then the uniform space G must be isomorphic to \widehat{G}_l , and we must therefore be able to define on \widehat{G}_l a topological group structure which induces the given topological group structure on G . Consequently we have to examine:

- (a) Whether we can extend by continuity the functions $(x, y) \mapsto xy$ and $x \mapsto x^{-1}$ to $\widehat{G}_l \times \widehat{G}_l$ and \widehat{G}_l respectively;
- (b) whether the functions thus extended do indeed define a group structure on \widehat{G}_l ;
- (c) when the preceding operations are possible, the topological group which they define is complete;
- (d) if there is a complete group satisfying the given conditions, then it is unique up to isomorphism.

Since the functions xy and x^{-1} are not in general uniformly continuous, we cannot apply the theorem of extension of uniformly continuous functions. Nevertheless, we can extend xy , by virtue of ?? and the following Proposition:

Proposition 3.2.11. *Let \mathfrak{F} and \mathfrak{G} be two Cauchy filters on G_l . Then the image of the filter $\mathfrak{F} \times \mathfrak{G}$ under the mapping $(x, y) \mapsto xy$ is a Cauchy filter base on G_l .*

Proof. Let us evaluate the "proximity" of xy and $\tilde{x}\tilde{y}$ in G_l ; in other words, let us form the product $(xy)^{-1}(\tilde{x}\tilde{y}) = y^{-1}x^{-1}\tilde{x}\tilde{y}$. For each $a \in G$, we can also write

$$(xy)^{-1}(\tilde{x}\tilde{y}) = (y^{-1}a)(a^{-1}x^{-1}\tilde{x}a)(a^{-1}\tilde{y}).$$

We shall see that by suitable choice of a , each of the three factors of this product is very small whenever the pairs (x, y) and (\tilde{x}, \tilde{y}) belong to a sufficiently small set of $\mathfrak{F} \times \mathfrak{G}$. Let V be any neighbourhood of e in G ; then there is a L_V -small set $B \in \mathfrak{G}$. Choose a in A , then if y and \tilde{y} are any two points of B , we have $y^{-1}a \in V$ and $a^{-1}\tilde{y} \in V$. On the other hand, the relation $a^{-1}x^{-1}\tilde{x}a \in V$ is equivalent to $x^{-1}\tilde{x} \in aVa^{-1} =: W$, and since W is a neighbourhood of e , there is a $L(W)$ -small set $A \in \mathfrak{F}$. Hence for all (x, y) and (\tilde{x}, \tilde{y}) in $A \times B$, we have $(xy)^{-1}(\tilde{x}\tilde{y}) \in V^3$, and this completes the proof. \square

In order that $x \mapsto x^{-1}$ can be extended by continuity to G_l , it is necessary and sufficient that the image, under the symmetry $x \mapsto x^{-1}$, of a Cauchy filter on G_l is a Cauchy filter on G_l (??). There are examples of topological groups in which this condition is not satisfied; we shall see this condition is sufficient for \widehat{G}_l being a group.

Theorem 3.2.12. *A Hausdorff topological group G is isomorphic to a dense subgroup of a complete group \widehat{G} if and only if the image, under the symmetry $x \mapsto x^{-1}$, of a Cauchy filter with respect to the left uniformity of G is a Cauchy filter with respect to this uniformity. The complete group \widehat{G} (which is called the completion of G) is then unique (up to isomorphism).*

Proof. We prove that the extended functions $(x, y) \mapsto xy$ and $x \mapsto x^{-1}$ define a group structure on \widehat{G}_l . First, the functions $x(yz)$ and $(xy)z$, defined on $\widehat{G}_l \times \widehat{G}_l \times \widehat{G}_l$ and equal on the dense subspace $G_l \times G_l \times G_l$, so the law of composition $(x, y) \mapsto xy$ is associative on \widehat{G}_l . For the same reason, the functions $x \mapsto x$, $x \mapsto ex$, $x \mapsto xe$ are identical on \widehat{G}_l , and the functions $x \mapsto e$, $x \mapsto xx^{-1}$, $x \mapsto x^{-1}x$ are identical on \widehat{G}_l .

We now prove the topological group \widehat{G}_l is complete. Let \mathfrak{L} be its left uniformity, and let \mathfrak{U} be the uniformity on \widehat{G}_l obtained by completing the left uniformity of G . Then \mathfrak{U} and \mathfrak{L} induce the same uniformity on G , and therefore every Cauchy filter base \mathfrak{B} on G with respect to \mathfrak{L} is also a Cauchy filter base with respect to \mathfrak{U} . Now \mathfrak{B} converges in \mathfrak{U} , since \mathfrak{U} is a complete uniformity; and since \mathfrak{U} and \mathfrak{L} induce the same topology on \widehat{G} it follows that \mathfrak{L} is a complete uniformity and \mathfrak{U} and \mathfrak{L} coincide. The uniqueness is now clear. \square

Example 3.2.13. Let \mathbb{R}_t denote \mathbb{R} endowed with the trivial topology. The nonempty basic open sets in $\mathbb{R} \times \mathbb{R}_t$ are of the form $(a, b) \times \mathbb{R}$ where (a, b) is an open interval in \mathbb{R} . Each $(a, b) \times \mathbb{R}$ intersects $\mathbb{R} \times \{0\}$ in $(a, b) \times \{0\}$ so $\mathbb{R} \times \{0\}$ is dense in $\mathbb{R} \times \mathbb{R}_t$. Since \mathbb{R}_t is obviously complete, so is $\mathbb{R} \times \mathbb{R}_t$. Thus, even though $\mathbb{R} \times \{0\}$ is complete, $\mathbb{R} \times \mathbb{R}_t$ is a completion of $\mathbb{R} \times \{0\}$ which is not topologically isomorphic to the Hausdorff space $\mathbb{R} \times \{0\}$ since $\mathbb{R} \times \mathbb{R}_t$ is not Hausdorff.

Proposition 3.2.14. *Let G be a Hausdorff topological group which has a completion \widehat{G} . Then the closures in \widehat{G} of the neighbourhoods of the identity element in G form a fundamental system of neighbourhoods of the identity element in \widehat{G} .*

Proof. Since \widehat{G} is regular, every neighbourhood of the identity element in \widehat{G} contains the closure V of an open neighbourhood U of e in G , and V is also the closure of the trace of U on G . \square

Let G be a group which is not necessarily Hausdorff; let $N = \overline{\{e\}}$ and let $G' = G/N$ be the Hausdorff group associated with G . If G' has a completion \widehat{G}' , this completion is called the **Hausdorff completion** of G and is denoted by \widehat{G} .

Proposition 3.2.15. *Let G be a topological group which has a Hausdorff completion \widehat{G} . Then every continuous homomorphism f of G into a complete Hausdorff group H can be uniquely factorized into $f = g \circ \iota$ where g is a continuous homomorphism of \widehat{G} into H and ι is the canonical mapping of G into \widehat{G} .*

$$\begin{array}{ccc} \widehat{G} & \xrightarrow{g} & H \\ \iota \uparrow & \nearrow f & \\ G & & \end{array}$$

Proof. Since the kernel of f is closed and contains e , it contains N , and hence f can be written as $f = g \circ \pi$, where g is a continuous homomorphism of G/N into H ; now apply ?? to g . \square

Proposition 3.2.16. *Let G and H be topological groups and $f : G \rightarrow H$ be a continuous homomorphism, then there a unique continuous homomorphism $\widehat{f} : \widehat{G} \rightarrow \widehat{H}$ such that the diagram*

$$\begin{array}{ccc} \widehat{G} & \xrightarrow{\widehat{f}} & \widehat{H} \\ \uparrow & & \uparrow \\ G & \xrightarrow{f} & H \end{array}$$

3.2.4 Completeness of commutative topological groups

We have already remarked that the left and right uniformities coincide on a commutative topological group G ; whenever we speak of the uniformity of G , it is this unique uniformity to which we refer.

Theorem 3.2.17. *Let G be a commutative topological group. Then the functions $x \mapsto x^{-1}$ and $(x, y) \mapsto xy$ are uniformly continuous on G and $G \times G$ respectively. Moreover G admits a Hausdorff completion \widehat{G} , and \widehat{G} is commutative.*

Proof. The uniform continuity of $x \mapsto x^{-1}$ follows from ??, and that of $(x, y) \mapsto xy$ from ??, since $(x, y) \mapsto xy$ is a continuous homomorphism of $G \times G$ into G . If G is Hausdorff, it satisfies the condition of ?? (as does every Hausdorff group whose left and right uniformities coincide); moreover the functions $(x, y) \mapsto xy$ and $(x, y) \mapsto yx$ are equal on $G \times G$, hence on $\widehat{G} \times \widehat{G}$; hence the second part of the theorem, by considering in the general case the Hausdorff group associated with G . \square

It follows in particular from this theorem that if f and g are two uniformly continuous mappings of a uniform space X into a commutative group G , written additively, then the functions $-f$ and $f + g$ are uniformly continuous.

Completeness of a commutative topological group G does not generally imply completeness of \widehat{G}/H even when H is closed subgroup of G . If G is metrizable, it does, however, as we show next.

Proposition 3.2.18. *If H is a closed subgroup of a complete commutative pseudometrizable topological group (G, d) where d is invariant, then G/H is complete.*

Proof. By ?? the quotient G/H can be pseudometrizable by the metric \bar{d} . Let (\bar{x}_n) be a Cauchy sequence from G/H . By choosing a subsequence if necessary, we may assume that $\bar{d}(\bar{x}_{n+1}, \bar{x}_n) < 2^{-n}$ for each $n \in \mathbb{N}$. Choose $h_1 \in H$ and let $w_1 = x_1 + h_1$. Then, for each n , choose $h_n \in H$ such that $w_n = x_n - x_{n-1} + h_n$ and $d(w_n, 0) < 2^{-(n-1)}$. Now form the sums $y_n = w_1 + \dots + w_n$ and note that

$$y_n = (x_1 + h_1) + (x_2 - x_1 + h_2) + \dots + (x_n - x_{n-1} + h_n) = x_n + (h_1 + \dots + h_n)$$

so that the canonical image $\pi(y_n) = \bar{x}_n$. Since, for each $n \in \mathbb{N}$,

$$d(y_n, y_{n+1}) = d(y_n - y_{n+1}, 0) = d(w_n, 0) \leq 2^{-(n-1)},$$

it follows that (y_n) is a Cauchy sequence in G and therefore has a limit y . By the continuity of π , $\bar{x}_n = \pi(y_n) \rightarrow \pi(y)$ and G/H is complete. \square

Proposition 3.2.19. *Let G be a commutative group, and let $\mathcal{T}_1, \mathcal{T}_2$ be two Hausdorff topologies compatible with the group structure of G . Suppose that \mathcal{T}_1 is finer than \mathcal{T}_2 and that there is a fundamental system of neighbourhoods of 0 for \mathcal{T}_1 which are closed for \mathcal{T}_2 . Let $\widehat{G}_1, \widehat{G}_2$ be the completions of G with respect to the topologies $\mathcal{T}_1, \mathcal{T}_2$, respectively, and let $f : \widehat{G}_1 \rightarrow \widehat{G}_2$ be the continuous homomorphism which extends the identity mapping of G . Then f is injective.*

Proof. Suppose that G is written additively. Let \mathcal{U}_1 be the uniformity on G corresponding to \mathcal{T}_1 : it will suffice to show that if \mathfrak{F} and \mathfrak{F}' are two minimal Cauchy filters with respect to \mathcal{U}_1 , which converges in G_2 to the same point a , then $\mathfrak{F} = \mathfrak{F}'$. For this, it is enough to show that $\mathfrak{F} \cap \mathfrak{F}'$ is a Cauchy filter with respect to \mathcal{U}_1 . Let V be a neighborhood of 0 in G with respect to \mathcal{T}_1 , such that V is closed in \mathcal{T}_2 , and let W be a symmetric neighborhood of 0 in G with respect to \mathcal{T}_1 , such that $W + W \subseteq V$. By hypothesis, there is a W -small set M (resp. M') in \mathfrak{F} (resp. \mathfrak{F}'); if

$x \in M$ and $y \in M$ we have $y - x \in W$, i.e., $y \in x + W$. If \bar{W} and \bar{V} are the closures of W and V in G_2 , it follows that $y \in x + \bar{W}$, and therefore, since a is in the closure of M , that $a \in x + \bar{M}$ for each $x \in M$. Similarly, $a \in x' + \bar{W}$ for each $x' \in M'$, and hence $x - x' \in \bar{W} + \bar{W}$; but since $(x, y) \mapsto x + y$ is a continuous map of $G_2 \times G_2$ into G_2 , we have $\bar{W} + \bar{W} \subseteq \bar{W} + \bar{W} \subseteq \bar{V}$. It follows that if $x \in M$ and $x' \in M'$, then $x - x' \in \bar{V} \cap G = V$, because V is closed in \mathcal{T}_2 ; and this completes the proof. \square

Corollary 3.2.20. *Under the hypotheses of ??, if A is a subset of G which is a complete subspace with respect to the uniformity corresponding to \mathcal{T}_2 , then A is also a complete subspace with respect to the uniformity corresponding to \mathcal{T}_1 .*

Proof. If A_1 is the closure of A in G_1 , then $f(A_1)$ is contained in the closure of A in G_2 , which by hypothesis is equal to A . Since $f(A) = A$ by definition and f is injective, we have $A_1 = A$. \square

3.3 Topological group with operators, topological rings, modules and fields

3.3.1 Topological group with operators

A **group with operators** (or **Ω -group**) (G, Ω) is defined as a group G together with an action of a set Ω on G :

$$\Omega \times G \rightarrow G, \quad (\omega, g) \mapsto g^\omega$$

that is distributive relative to the group law:

$$(g \cdot h)^\omega = g^\omega \cdot h^\omega$$

for all $g, h \in G$ and $\omega \in \Omega$. For each $\omega \in \Omega$, the application $g \mapsto g^\omega$ is then an endomorphism of G . From this, a Ω -group can also be viewed as a group G with an indexed family $(f_\omega)_{\omega \in \Omega}$ of endomorphisms of G . The set Ω is called the **operator domain** and f_ω are called the **homotheties** of G .

Given two groups G, H with same operator domain Ω , a **homomorphism of groups with operators** is a group homomorphism $\phi : G \rightarrow H$ satisfying

$$\phi(g^\omega) = \phi(g)^\omega$$

for all $g \in G$ and $\omega \in \Omega$. A subgroup S of G is called a **stable subgroup**, **Ω -subgroup** or **invariant subgroup** if it respects the homotheties, that is, $s^\omega \in S$ for all $s \in S$ and $\omega \in \Omega$,

On a set G , a structure of a group with operators and a topology are said to be **compatible** if the topology and the group structure of G are compatible and if in addition the endomorphisms of G produced by the operators are continuous. The set G , together with the given topology and structure of group with operators, is then said to be a **topological group with operators**.

If H is a stable subgroup of a topological group G with operators, then the topology induced on H by the topology of G is compatible with the structure of group with operators on H . Furthermore:

Proposition 3.3.1. *If H is a stable subgroup of a topological group G with operators, then the closure \bar{H} of H in G is a stable subgroup of G .*

Proof. We know already that \bar{H} is a subgroup of G . Also if f_ω is any operator on G , the image of \bar{H} under the continuous mapping f_ω is contained in \bar{H} , so the claim follows. \square

Let H be a stable normal subgroup of a topological group G with operators. Then for each operator f_ω on G , the mapping of G/H into itself induced by f_ω is continuous by ??, and the structure of group with operators on G/H is therefore compatible with the quotient topology on G/H . We therefore have the following result.

Proposition 3.3.2. *Let H be a stable normal subgroup of a topological group G with operators, then G/H is also a topological group with operators.*

Let $(G_i)_{i \in I}$ be a family of topological groups with operators, where each G_i is assumed to have the same operator domain Ω . For each $\omega \in \Omega$, the mapping $(x_i) \mapsto (x_i^\omega)$ of $G = \prod_{i \in I} G_i$ into itself is continuous, and the structure of groups with operators on G is therefore compatible with the product topology on G . This defines G into a topological group with operators.

If G is a Hausdorff topological group with operators and if G has a completion \widehat{G} , then every endomorphism f_ω of G defined by an operator on G can be extended by continuity to an endomorphism of \widehat{G} (??). Hence \widehat{G} has the structure of a topological group with operators, and the operator set is the same for \widehat{G} as for G .

3.3.2 Topological direct sum of stable subgroups

Since the study of commutative groups with operators is equivalent to the study of modules we shall sometimes allow ourselves to use the terminology proper to modules for arbitrary commutative groups with operators; thus we may speak of linear mappings instead of homomorphisms of commutative groups with operators, and we may use the word projector to denote an idempotent endomorphism of a commutative group with operators.

If a commutative topological group G with operators is the direct sum of a finite family $(G_i)_{i=1}^n$ of stable subgroups, then the canonical bijection $(x_i) \rightarrow \sum_{i=1}^n x_i$ of the product group $\prod_{i=1}^n G_i$ onto G is continuous, but is not necessarily a homeomorphism.

Definition 3.3.3. *Let G be a commutative topological group with operators, and let $(G_i)_{i=1}^n$ be a finite family of stable subgroups of G , such that G is the direct sum of the G_i . Then G is said to be the **topological direct sum** of the G_i if the canonical mapping $(x_i) \mapsto \sum_{i=1}^n x_i$ of the product group $\prod_{i=1}^n G_i$ onto G is a homeomorphism.*

Proposition 3.3.4. *Let G be a commutative topological group with operators which is the direct sum of stable subgroups $(G_i)_{i=1}^n$. Let $(P_i)_{i=1}^n$ be the family of projections associated with the decomposition $G = \sum_{i=1}^n G_i$. Then G is the topological direct sum of the G_i if and only if the P_i are continuous.*

Proof. Note that the mapping $x \mapsto (P_i(x))$ is the inverse of the mapping $(x_i) \mapsto \sum_{i=1}^n x_i$. \square

If G is the topological direct sum of two stable subgroups H, K , then K is said to be a topological complement of H in G ; this is the case if and only if the canonical mapping of G/H onto K an isomorphism of topological groups with operators.

Corollary 3.3.5. *Let G be a commutative topological group with operators, and let H be a stable subgroup of G . Then the following conditions are equivalent:*

- (i) *H has a topological complement in G .*
- (ii) *There is a continuous projection P on G such that $P(G) = H$.*
- (iii) *The identity mapping of H can be extended to a continuous linear mapping of G onto H .*

Proof. It follows from ?? that (i) implies (ii), and it is clear that (ii) implies (iii). Finally, if P is a continuous linear mapping of G onto H which extends the identity mapping of H , then P is a continuous projection, and the projections P and $I - P$ are associated with the direct sum decomposition $G = H + K$, where $K = P^{-1}(0)$. \square

Corollary 3.3.6. *If a Hausdorff commutative topological group with operators is the topological direct sum of a family $(G_i)_{i=1}^n$ of stable subgroups, then each of the subgroups G_i is closed in G .*

3.3.3 Topological rings and modules

Definition 3.3.7. *A topological ring is a set A which carries a ring structure and a topology satisfying the following axioms:*

- (RT1) *The mapping $(x, y) \mapsto x + y$ of $A \times A$ into A is continuous.*
- (RT2) *The mapping $x \mapsto -x$ of A into A is continuous.*
- (RT3) *The mapping $(x, y) \mapsto xy$ of $A \times A$ into A is continuous.*

The first two axioms express that the topology of A is compatible with its additive group structure. If a ring structure and a topology are given on a set A , they are said to be compatible if they satisfy these axioms.

In a topological ring every left homothety $x \mapsto ax$ (resp. every right homothety $x \mapsto xa$) is continuous (and is a homeomorphism if a is a unit of A). Since we can write identically

$$xy - ab = (x - a)(y - b) + (x - a)b + a(y - b)$$

the axiom (RT3) [in view of (RT1) and (RT2)] is equivalent to the conjunction of the following two axioms:

- (RT3a) Given any $a \in A$, the mappings $x \mapsto ax$ and $x \mapsto xa$ are continuous at the point 0.
- (RT3b) The mapping $(x, y) \mapsto xy$ of $A \times A$ into A is continuous at the point $(0, 0)$.

From this we can deduce a necessary and sufficient set of conditions which the filter \mathfrak{B} of neighbourhoods of 0 in a ring A must satisfy in order to define a topology on A compatible with its ring structure: \mathfrak{B} must satisfy axioms (GV1) and (GV2), and also the following two axioms: (note that (GV3) is automatically satisfied since we are considering abelian groups)

- (RV1) For all $a \in A$ and all $V \in \mathfrak{B}$, there exists $W \in \mathfrak{B}$ such that $aW \subseteq V$ and $Wa \subseteq V$.
- (RV2) For all $V \in \mathfrak{B}$, there exists $W \in \mathfrak{B}$ such that $WW \subseteq V$.

Example 3.3.8. Let \mathfrak{B} be a filter base on a ring A , consisting of two-sided ideals. Then \mathfrak{B} is a fundamental system of neighbourhoods of 0 for a topology compatible with the additive group structure of A , and it follows immediately from (RV1) and (RV2) that this topology is compatible with the ring structure of A . This case happens, for example, if \mathfrak{a} is an ideal of A and we set

$$\mathfrak{B} = \{\mathfrak{a}^n : n \in \mathbb{N} \text{ and } n > 0\}.$$

The topology therefore defined by \mathfrak{B} can be used to produce the \mathfrak{a} -adic completion of A .

If B is a subring of a topological ring A , then the topology induced on B by that of A is compatible with the ring structure of B . The topological ring structure thus defined on B is said to be induced by that of A . Similar to subgroups, we have the following result.

Proposition 3.3.9. *Let B be a subring (resp. left ideal, right ideal, two-sided ideal) of a topological ring A . Then the closure \bar{B} of B in A is a subring (resp. left ideal, right ideal, two-sided ideal) of A .*

Let I be a two-sided ideal in a topological ring A . By the same argument as for quotient groups, we see that the quotient of the topology of A by the relation $x - y \in I$ is compatible with the ring structure of A/I . In particular, if A is not Hausdorff, then the closure N of $\{0\}$ in A is a closed two-sided ideal, by ??; the quotient ring, which is Hausdorff, is called the Hausdorff ring associated with A .

Let $(A_i)_{i \in I}$ be a family of topological rings. On the set $A = \prod_{i \in I} A_i$ the product of the topologies of the A_i is compatible with the product of the ring structures of the A_i (same proof as for product groups); the topological ring A thus defined is called the **product** of the topological rings A_i .

When we speak of the uniformity of a topological ring, we always mean the uniformity of its additive group, unless the contrary is expressly stated; in particular, A is said to be a complete ring if the additive group of A is complete. Let A be a Hausdorff topological ring; as an additive group A can be considered as a dense subgroup of a complete Hausdorff commutative group A , which is determined up to isomorphism. In order that we should be able to consider A as a subring of a complete ring, it is necessary to be able to extend the function $(x, y) \mapsto xy$ by continuity to the space $\widehat{A} \times \widehat{A}$. The possibility of this extension will follow from the following more general theorem:

Theorem 3.3.10. *Let E, F, G be three complete Hausdorff commutative groups; let A be a dense subgroup of E and let B be a dense subgroup of F . If $f : A \times B \rightarrow G$ is a continuous \mathbb{Z} -bilinear mapping, then f can be extended by continuity to a continuous \mathbb{Z} -bilinear mapping of $E \times F$ into G .*

Proof. Let (a, b) be an arbitrary point of $E \times F$, and let $\mathfrak{A}, \mathfrak{B}$ be the traces on A, B respectively of the neighbourhood filters of (x_0, y_0) (\mathfrak{A} and \mathfrak{B} are filters, by hypothesis). To show that f can be extended by continuity, it is enough to show that $f(\mathfrak{A} \times \mathfrak{B})$ is a Cauchy filter-base on G (by ??). Consider the identity

$$\begin{aligned} f(\tilde{x}, \tilde{y}) - f(x, y) &= f(\tilde{x} - x, y) + f(x, \tilde{y} - y) + f(\tilde{x} - x, \tilde{y} - y) \\ &= f(\tilde{x} - x, b) + f(a, \tilde{y} - y) + f(\tilde{x} - x, \tilde{y} - b) + f(x - a, \tilde{y} - y) + f(\tilde{x} - x, \tilde{y} - y). \end{aligned}$$

We shall show that by taking (x, y) and (\tilde{x}, \tilde{y}) in a sufficiently small set of $\mathfrak{A} \times \mathfrak{B}$, and by choosing a and b suitably, we can make each of the terms on the right-hand side very small.

Let W be any neighbourhood of 0 in G ; since f is continuous at $(0, 0) \in A \times B$, there is a set $U \in \mathfrak{A}$ and a set $V \in \mathfrak{B}$ such that $f(\tilde{x} - x, \tilde{y} - y) \in W$ whenever $x, \tilde{x} \in U$ and $y, \tilde{y} \in V$. Take a point $a \in U$ and a point $b \in V$, then for all $x, \tilde{x} \in U$ and $y, \tilde{y} \in V$, we have

$$f(\tilde{x} - x, \tilde{y} - b) + f(x - a, \tilde{y} - y) \in W + W.$$

On the other hand, the partial mapping $x \mapsto f(x, b)$ is continuous on A ; hence there is a set $U' \subseteq U$ belonging to \mathfrak{A} and such that $f(\tilde{x} - x, b) \in W$ whenever $x, \tilde{x} \in U'$. Likewise, there exists $V' \subseteq V$ belonging to \mathfrak{B} such that $f(a, \tilde{y} - y) \in W$ whenever $y, \tilde{y} \in V'$. Consequently, if (x, y) and (\tilde{x}, \tilde{y}) are any two points of $U' \times V'$, then we have

$$f(\tilde{x}, \tilde{y}) - f(x, y) \in W + W + W + W.$$

This proves the existence of the extension \bar{f} of f . The fact that \bar{f} is \mathbb{Z} -bilinear is an immediate consequence of the principle of extension of identities. \square

In the application of this theorem to a Hausdorff topological ring A , we take E, F and G to be \widehat{A} , A and B to be the ring A , and f to be the \mathbb{Z} -bilinear mapping $(x, y) \mapsto xy$, which by hypothesis is continuous. We denote again by xy the value of the extended function on $\widehat{A} \times \widehat{A}$; this function is a law of composition on \widehat{A} , and to say that it is \mathbb{Z} -bilinear means that it is distributive on both sides with respect to addition; and it is also associative, by the principle of extension of identities. Consequently,

Proposition 3.3.11. *A Hausdorff topological ring A is isomorphic to a dense subring of a complete Hausdorff ring \widehat{A} , which is determined up to isomorphism (and is called the **completion** of A). If A is commutative (resp. has an identity element) then the same is true for \widehat{A} .*

Let A be a topological ring, not necessarily Hausdorff; let I be the closure of $\{0\}$ in A , and let $A' = A/I$ be the Hausdorff ring associated with A . Then \widehat{A}' , the completion of A' , is called the **Hausdorff completion** of A and is also denoted by \widehat{A} . One in the case of topological groups, the following results are valid.

Proposition 3.3.12. *Let A be a topological ring and C a complete Hausdorff topological ring. Then every continuous ring homomorphism f of A with A can be uniquely factorized as $f = g \circ \iota$, where g is a continuous homomorphism of \widehat{A} into C and ι is the canonical mapping of A into \widehat{A} .*

Proposition 3.3.13. *If A, B are two topological rings, and $f : A \rightarrow B$ is a continuous homomorphism, there is therefore a unique continuous homomorphism $\hat{f} : \widehat{A} \rightarrow \widehat{B}$ such that the diagram*

$$\begin{array}{ccc} \widehat{A} & \xrightarrow{\hat{f}} & \widehat{B} \\ \uparrow & & \uparrow \\ A & \xrightarrow{f} & B \end{array}$$

is commutative.

Definition 3.3.14. *Given a topological ring A with an identity element, a **topological left A -module** is a set M carrying a left A -module structure and a topology compatible with the additive group structure of M , and satisfying the following axiom:*

(MT) *The mapping $(a, x) \mapsto ax$ of $A \times M$ into M is continuous.*

We define similarly the notion of a topological right A -module; since every right A -module can be considered as a left A^{op} -module, where A^{op} is the opposite ring of A , and since the topology of A is compatible with the ring structure of A^{op} , there is no need to distinguish between topological right A -modules and topological left A^{op} -modules.

Example 3.3.15. Let A be a ring and let \mathfrak{B} be a filter base on A consisting of two-sided ideals of A ; let M be a left A -module. If we give A the topology (compatible with its ring structure) for which \mathfrak{B} is a fundamental system of neighbourhoods of 0 (??), and M the topology (compatible with its additive group structure) in which the sets IM , as I runs through \mathfrak{B} , form a fundamental system of neighbourhoods of 0, it is immediately verified that M is a topological A -module.

Example 3.3.16. Every commutative topological group is a topological \mathbb{Z} -module when the ring \mathbb{Z} is given the discrete topology.

Given a topological ring A , consider, on a left A -module M , a topology compatible with the additive group structure of M . By virtue of the identity

$$ax - a_0x_0 = (a - a_0)x_0 + a_0(x - x_0) + (a - a_0)(x - x_0)$$

the axiom (MT) is equivalent to the conjunction of the following three axioms:

- (MT1) For each $x_0 \in M$, the mapping $a \mapsto ax_0$ is continuous at the point 0.
- (MT2) For each $a_0 \in A$, the mapping $x \mapsto a_0x$ is continuous at the point 0.
- (MT3) The mapping $(a, x) \mapsto ax$ is continuous at the point $(0, 0)$.

We deduce from this a necessary and sufficient set of conditions that the filter \mathfrak{B} of neighbourhoods of 0 in an A -module M must satisfy in order to define a topology on M compatible with its module structure; \mathfrak{B} must satisfy the axioms (GV1) and (GV2), and in addition must satisfy the following three axioms:

- (MV1) For each $x_0 \in M$ and $V \in \mathfrak{B}$, there is a neighbourhood S of 0 in A such that $Sx_0 \subseteq V$.
- (MT2) For each $a_0 \in A$ and $V \in \mathfrak{B}$, there exists $W \in \mathfrak{B}$ such that $a_0W \subseteq V$.
- (MT3) For each $V \in \mathfrak{B}$ there exists $U \in \mathfrak{B}$ and a neighbourhood T of 0 in A such that $TU \subseteq V$.

If N is a submodule of a topological A -module M , it is clear that the topology induced on N by the topology of M is compatible with the module structure of N . As before, we have the following result on closure of a submodule.

Proposition 3.3.17. *Let M be a topological A -module, and N a submodule of M . Then \overline{N} is a submodule of M .*

On the quotient A -module M/N , the topology which is the quotient by N of the topology of M is compatible with the A -module structure. To see this it is enough to show that the mapping $(a, \bar{x}) \mapsto a\bar{x}$ onto M/N is continuous. Now, since we may identify the additive topological groups $A \times (M/N)$ and $(A \times M)/(\{0\} \times N)$, it is enough to show that the mapping

$(a, x) \mapsto a\bar{x}$ of $A \times M$ into M/N is continuous; and this is immediate, since the mapping in question is the composition of $x \mapsto \bar{x}$ and $(a, x) \mapsto ax$.

Let $(M_i)_{i \in I}$ be an arbitrary family of topological A -modules, and let $M = \prod_{i \in I} M_i$ be the A -module which is the product of the M_i . Then the product topology on M is compatible with the A -module structure of M . To prove this it is enough to show that the mapping $(a, x) \mapsto (a\pi_i(x))$ of $A \times M$ into M is continuous, or that for each index $i \in I$ the mapping $(a, x) \mapsto a\pi_i(x)$ is a continuous mapping of $A \times M \rightarrow M_i$, which is obvious.

Let A be a Hausdorff topological ring and M a Hausdorff topological A -module. Let \widehat{M} be the additive group which is the completion of the commutative topological group M . The \mathbb{Z} -bilinear mapping $(a, x) \mapsto ax$ of the product $A \times M$ of the additive groups A, M into the additive group M can be extended by continuity to a \mathbb{Z} -bilinear mapping of $A \times M$ into M by ??, and this mapping we continue to denote by $(a, x) \mapsto ax$. By virtue of the principle of extension of identities, this map then defines an \widehat{A} -module structure on \widehat{M} compatible with its topology. The topological \widehat{A} -module \widehat{M} thus defined is called the completion of the topological A -module M .

Let M be a topological module over a topological ring A , where neither A nor M is necessarily Hausdorff. Let I (resp. N) be the closure of $\{0\}$ in A (resp. M). Then I is a two-sided ideal of A and N is a sub- A -module of M ; furthermore, by continuity we have $ax \in N$ whenever $a \in I$ or $x \in N$. We can therefore define, by passing to the quotients, a mapping $(\bar{a}, \bar{x}) \mapsto a\bar{x}$ of $(A/I) \times (M/N)$ into M/N ; it is easily verified (by use of ??) that this mapping is continuous, and therefore defines a structure of a topological (A/I) -module on M/N . If we put $A' = A/I$ and $M' = M/N$, then the A' -module M' is called the **Hausdorff module associated with M** ; its completion is a topological module over the Hausdorff completion \widehat{A} of A , and this module is called the Hausdorff completion of M and is denoted by \widehat{M} .

Proposition 3.3.18. *every continuous homomorphism $f : M \rightarrow N$ of M into a complete Hausdorff A -module N factorizes uniquely into $f = g \circ \iota$, where g is a continuous homomorphism of \widehat{M} into N and ι is the canonical mapping of M into \widehat{M} .*

Proposition 3.3.19. *If M, N are two topological A -modules and $f : M \rightarrow N$ is a continuous homomorphism, then there is a unique continuous homomorphism $\widehat{f} : \widehat{M} \rightarrow \widehat{N}$ such that the diagram*

$$\begin{array}{ccc} \widehat{M} & \xrightarrow{\widehat{f}} & \widehat{N} \\ \uparrow & & \uparrow \\ M & \xrightarrow{f} & N \end{array}$$

is commutative.

3.3.4 Topological division ring and fields

Definition 3.3.20. *A topological division ring is a set K carrying a division ring structure and a topology compatible with the ring structure of K , and satisfying in addition the following axiom:*

(KT) *The mapping $x \mapsto x^{-1}$ of K^\times into K^\times is continuous.*

A commutative topological division ring is called a topological field.

A division ring structure and a topology on a set K are said to be compatible if the corresponding ring structure and the topology are compatible and if in addition axiom (KT) is satisfied.

Example 3.3.21. On any division ring K , the discrete topology is compatible with the division ring structure. A topological division ring whose topology is discrete is called a **discrete division ring**.

By definition, if K is a topological division ring, then the topology induced by that of K on the multiplicative group K^\times is compatible with the group structure of K^\times .

If $a \in K^\times$, the homotheties $x \mapsto ax$ and $x \mapsto xa$ are homeomorphisms of K onto itself; and so is the mapping $x \mapsto ax + y$ for all $y \in K$. Note that the homotheties $x \mapsto ax$ and $x \mapsto xa$ are automorphisms of the (topological) additive group of K . If V is any neighbourhood of 0 in K , it follows therefore that aV and Va are neighbourhoods of 0. If A is a division subring of a topological division ring K , then the topology induced on A by the topology of K is compatible with the division ring structure of A . The structure of a topological division ring thus defined on A is said to be induced by that of K . Furthermore, \bar{A} is also a division subring of K .

If K is a topological division ring it is necessary to distinguish between:

- (a) the uniformity of the additive group of K , which is defined on K and is called the **additive uniformity** of K ;
- (b) the left and right uniformities of the multiplicative group K^\times , which are defined on K^\times and are called (by abuse of language) the multiplicative uniformities of K

The uniformity induced on K^\times by the additive uniformity of K is in general distinct from the multiplicative uniformities of K . By ??, a Hausdorff topological division ring K can be considered as a dense subring of a complete Hausdorff ring \hat{K} . In order that \hat{K} should be a topological division ring it is necessary that the mapping $x \mapsto x^{-1}$ be extendable by continuity to \hat{K}^\times ; and this necessary condition is also sufficient, for the functions $x \mapsto xx^{-1}$, $x \mapsto x^{-1}x$ and $x \mapsto 1$ are then equal on \hat{K}^\times by reason of the principle of extension of identities, and therefore the value of the extended function is the inverse of x for each $x \in \hat{K}^\times$. In other words,

Proposition 3.3.22. *The completion \hat{K} of a Hausdorff topological division ring K is a topological division ring if and only if the image under the mapping $x \mapsto x^{-1}$ of every Cauchy filter (with respect to the additive structure) which does not have a cluster point at 0, is a Cauchy filter (with respect to the additive structure).*

There are topological division rings in which this condition is not satisfied and in which the ring \hat{K} has zero divisors. Moreover, even if the completion \hat{K} is a topological division ring, there is no a priori reason to assume that the multiplicative structures of \hat{K} are structures of a complete space. However, this will be the case for division rings K such that \hat{K} is locally compact, and for topological fields; for the latter, we have the following proposition:

Proposition 3.3.23. *If the additive uniform structure of a topological field K is a structure of a complete Hausdorff space, then the multiplicative structure on \hat{K}^\times is a structure of a complete space.*

Proof. We shall show that if \mathfrak{F} is a Cauchy filter with respect to the multiplicative structure on K^\times , then \mathfrak{F} is a Cauchy filter with respect to the additive structure on K , and does not converge

to 0; this will establish the result. Let U be any neighbourhood of 0 in K , let V be a closed neighbourhood of 0 such that $V \subseteq U$, $VV \subseteq U$ and $-1 \notin U$; then by hypothesis there is a set $A \in \mathfrak{F}$ such that, for all $x \in A$ and $y \in A$, we have $x^{-1}y \in 1 + A$. Let $a \in A$, then $A \subseteq a + aV$, and $a + aV$ is a closed set which does not contain 0 hence 0 is not in the closure of A and is therefore not a cluster point of \mathfrak{F} . Let W be a neighbourhood of 0 such that $aW \subseteq V$; then by hypothesis there is a set $B \in \mathfrak{F}$ such that $B \subseteq A$ and, for all $x \in B$ and $y \in B$, we have $x^{-1}y \in 1 + W$, hence

$$y - x \in xW \subseteq AW \subseteq aW + aVW.$$

But K is commutative, and therefore $aVW = aWV \subseteq VV \subseteq U$; consequently $y - x \in U + U$ and the proposition is proved. \square

3.3.5 Inverse limits of topological groups and rings

Throughout this part, I denotes a non-empty directed set and \preceq denotes the partial order relation in I . Unless the contrary is expressly stated, all the inverse systems considered are indexed by I .

Let $(X_\alpha, f_{\alpha\beta})$ be an inverse system of sets, and suppose that each X_α is endowed with an internal law of composition, everywhere defined, and written multiplicatively. Suppose also that the $f_{\alpha\beta}$ are homomorphisms with respect to these internal laws. Since

$$f_{\alpha\beta}(x_\beta y_\beta) = f_{\alpha\beta}(x_\beta) f_{\alpha\beta}(y_\beta)$$

whenever $\alpha \preceq \beta$ and x_α and y_β belong to X_β , it is clear that $X = \varprojlim X_\alpha$ is a stable subset of the product $\prod X_\alpha$, with respect to the internal law $(x_\alpha) \cdot (y_\alpha) = (x_\alpha y_\alpha)$. Let $(\Omega_\alpha, \varphi_{\alpha\beta})$ be another inverse system of sets, indexed by I , and suppose that each X_α carries an external law of composition, everywhere defined, written multiplicatively and having Ω_α as a set of operators, such that whenever $\alpha \preceq \beta$ we have

$$f_{\alpha\beta}(\lambda_\beta x_\beta) = \varphi_{\alpha\beta}(\lambda_\alpha) f_{\alpha\beta}(x_\beta)$$

for $\lambda_\alpha \in \Omega_\alpha$ and $x_\beta \in X_\beta$. Then we can define an external law on $\prod X_\alpha$ having $\prod \Omega_\alpha$ as a set of operators, by $(\lambda_\alpha) \cdot (x_\alpha) = (\lambda_\alpha \cdot x_\alpha)$. Restricting the set of operators to $\Omega = \varprojlim \Omega_\alpha$, we have an external law on $\prod X_\alpha$ with respect to which X is again a stable subset. The internal (resp. external) law thus defined on X is said to be the **inverse limit** of the internal (resp. external) laws of the X_α . In the case of external laws, it may happen that all the Ω_α are identical with the same set Ω_0 and that all the $\varphi_{\alpha\beta}$ are identity mappings; then, since I is a directed set, Ω can be identified with Ω_0 .

It is immediately verified that the usual properties of associativity, commutativity, existence of an identity element for an internal law (provided that the $f_{\alpha\beta}$'s map identity element to identity element), distributivity of an external law with respect to an internal law, etc., are preserved under passage to the inverse limit.

Let Σ be a species of algebraic structure and let Σ_0 be the impoverished structure corresponding to Σ . Whenever we speak of an inverse system of sets $(X_\alpha, f_{\alpha\beta})$ endowed with structures of species Σ , we shall always suppose that the $f_{\alpha\beta}$ are homomorphisms for these structures. If we endow $X = \varprojlim X_\alpha$ with the internal and external laws which are the re-

spective inverse limits of the internal and external laws of the X_α , then X carries an algebraic structure of species Σ_0 . Naturally it remains to be seen in each particular case whether or not this structure is of species Σ . For example, if $(G_\alpha, f_{\alpha\beta})$ is an inverse system of groups (resp. rings), then $\varprojlim G_\alpha$ is a subgroup (resp. subring) of $\prod X_\alpha$ and is called the inverse limit of the system $(G_\alpha, f_{\alpha\beta})$ of groups (resp. rings).

Let $(X_\alpha, f_{\alpha\beta})$ be an inverse system of sets and let $(G_\alpha, \varphi_{\alpha\beta})$ be an inverse system of groups; suppose that each X_α has G_α as a group of operators and that whenever $\alpha \preceq \beta$ have

$$f_{\alpha\beta}(g_\beta \cdot x_\beta) = \varphi_{\alpha\beta}(g_\beta) \cdot f_{\alpha\beta}(x_\beta) \quad (3.3.1)$$

for $x_\beta \in X_\beta$ and $g_\beta \in G_\beta$. Then $X = \varprojlim X_\alpha$ has $G = \varprojlim G_\alpha$ as group of operators. It follows from (??) that, if $\alpha \preceq \beta$, the mappings $f_{\alpha\beta}$ and $\varphi_{\alpha\beta}$ are compatible and hence define a mapping $\pi_{\alpha\beta} : X_\beta/G_\beta \rightarrow X_\alpha/G_\alpha$ of the quotient sets, such that $(X_\alpha/G_\alpha, \pi_{\alpha\beta})$ is an inverse system. Moreover, if $f_\alpha : X \rightarrow X_\alpha$ and $\varphi_\alpha : G \rightarrow G_\alpha$ are the canonical mappings, then f_α and φ_α are compatible and consequently define a mapping $\eta_\alpha : X/G \rightarrow X_\alpha/G_\alpha$ of the quotient sets; it is clear that the η_α form an inverse system of mappings, whose inverse limit is therefore a mapping $\eta : X/G \rightarrow \varprojlim X_\alpha/G_\alpha$, which is not necessarily either injective or surjective.

Again, let $(A_\alpha, f_{\alpha\beta})$ be an inverse system of rings and let $(M_\alpha, \varphi_{\alpha\beta})$ be an inverse system of commutative groups; and suppose that each M_α carries a left A_α -module structure in such a way that whenever $\alpha \preceq \beta$ we have

$$f_{\alpha\beta}(\lambda_\beta \cdot x_\beta) = \varphi_{\alpha\beta}(\lambda_\beta) \cdot f_{\alpha\beta}(x_\beta) \quad (3.3.2)$$

for $x_\beta \in M_\beta$ and $\lambda_\beta \in A_\beta$; then $\varprojlim M_\alpha$ has a structure of a left module over $\varprojlim A_\alpha$. If we suppose in addition that, for each α , A_α is commutative and that each M_α has an A_α -algebra structure, and finally that $(M_\alpha, \varphi_{\alpha\beta})$ is an inverse system of rings, then $\varprojlim M_\alpha$ has a structure of an algebra over $\varprojlim A_\alpha$.

Let $(G_\alpha, \varphi_{\alpha\beta})$ be an inverse system of groups, and for each α let H_α be a subgroup of G_α . If $\varphi_{\beta\alpha}(H_\beta) \subseteq H_\alpha$ whenever $\alpha \preceq \beta$, then the inverse system of subsets H_α of G_α is an inverse system of groups with respect to the restrictions of the $\varphi_{\alpha\beta}$, and $H = \varprojlim H_\alpha$ is a subgroup of $G = \varprojlim G_\alpha$. If each H_α is a normal subgroup of G_α , then H is a normal subgroup of G . If $(G'_\alpha, \psi_{\alpha\beta})$ is another inverse system of groups and, for each α , $u_\alpha : G_\alpha \rightarrow G'_\alpha$ is a homomorphism such that the u_α 's form an inverse system of mappings, then, if H_α is the kernel of u_α , we have $\psi_{\beta\alpha}(H_\beta) \subseteq H_\alpha$ whenever $\alpha \preceq \beta$. Thus $u = \varprojlim u_\alpha$ is a homomorphism of G into $G' = \varprojlim G'_\alpha$, and $H = \varprojlim H_\alpha$ is the kernel of u . If we put $K_\alpha = u_\alpha(G_\alpha)$, then we have $\psi_{\beta\alpha}(K_\beta) \subseteq K_\alpha$ whenever $\alpha \preceq \beta$, so that the K_α form an inverse system of subgroups of the G_α ; but $K = \varprojlim K_\alpha$ is not necessarily the image of G under u .

We shall say that an inverse system $(G_\alpha, \varphi_{\alpha\beta})$ is an **inverse system of topological groups** if the G_α are topological groups and the $\varphi_{\alpha\beta}$ are continuous homomorphisms. Then $G = \varprojlim G_\alpha$ is a subgroup of the product group $\prod G_\alpha$; if we endow G with the topological group structure induced by that of $\prod G_\alpha$, then the topological group so obtained is called the inverse limit of the inverse system of topological groups $(G_\alpha, \varphi_{\alpha\beta})$. If the G_α are Hausdorff (resp. Hausdorff and complete) then G is Hausdorff and closed in $\prod G_\alpha$ (resp. Hausdorff and complete). If $(G'_\alpha, \psi_{\alpha\beta})$ is another inverse system of topological groups and if, for each α , $u_\alpha : G_\alpha \rightarrow G'_\alpha$ is a continuous homomorphism such that the u_α form an inverse system of mappings, then

$u = \varprojlim u_\alpha$ is a continuous homomorphism of G into $G' = \varprojlim G'_\alpha$. The same results are valid when "topological group" is replaced by "topological ring."

Let $(X_\alpha, f_{\alpha\beta})$ be an inverse system of topological spaces, and let $(G_\alpha, \phi_{\alpha\beta})$ be an inverse system of topological groups. Suppose that each G_α operates continuously on X_α and that the relations ?? . We have seen that $X = \varprojlim X_\alpha$ has $G = \varprojlim G_\alpha$ as group of operators; furthermore, G operates continuously on X . For if f_α (resp. ϕ_α) is the canonical mapping $X \rightarrow X_\alpha$, (resp. $G \rightarrow G_\alpha$), then by definition

$$f_\alpha(g \cdot x) = \phi_\alpha(g) \cdot f_\alpha(x)$$

and therefore the mappings $(g, x) \mapsto f_\alpha(g \cdot x)$ are continuous on $X \times G$, which shows the continuity of $(g, x) \mapsto (g \cdot x)$. The mapping $\eta_\alpha : X/G \rightarrow X_\alpha/G_\alpha$, induced by f_α and ϕ_α is therefore continuous, and so is the mapping $\eta : X/G \rightarrow \varprojlim X_\alpha/G_\alpha$ defined by the η_α 's.

Proposition 3.3.24. *Suppose that the X_α and the G_α satisfy the above hypotheses.*

- (a) *If the stabilizer of each point of X_α is a compact subgroup of G_α for each $\alpha \in I$, then the stabilizer of each point $x = (x_\alpha)$ of X is a compact subgroup of G , then the orbit of x (with respect to G) is canonically homeomorphic to the inverse limit of the orbits of the x_α (with respect to G_α), and the canonical mapping $\eta : X/G \rightarrow \varprojlim X_\alpha/G_\alpha$ is injective.*
- (b) *If, for each $\alpha \in I$, every orbit of a point of X_α (with respect to G_α) is compact, then every orbit of a point of X (with respect to G) is relatively compact, and η is surjective. If also η is bijective, then every orbit of a point of X is compact.*

Proof. Let $x = (x_\alpha) \in X$ and for each $\alpha \in I$ let $Y_\alpha = G_\alpha \cdot x_\alpha$ be the orbit of x_α . If $\alpha \preceq \beta$, it follows from ?? and the relation $f_{\beta\alpha}(x_\beta) = x_\alpha$ that $f_{\beta\alpha}(Y_\beta) = Y_\alpha$, and hence that (Y_α) is an inverse system of subsets of the X_α . For each $\alpha \in I$ let $\theta_\alpha : G_\alpha \rightarrow Y_\alpha$ be the continuous mapping $g_\alpha \rightarrow g_\alpha \cdot x_\alpha$. Then the θ_α form an inverse system of mappings, and $\theta = \varprojlim \theta_\alpha$ is the continuous mapping $g \mapsto g \cdot x$ of G into the subspace $Y = \varprojlim Y_\alpha$ of X . The hypothesis of (a) implies that $\theta_\alpha^{-1}(y_\alpha)$ is compact for each $y_\alpha \in Y_\alpha$. Since also θ_α is surjective, the conditions of ?? are satisfied, and the first two assertions of (a) are therefore established. Hence, if $x = (x_\alpha)$ and $y = (y_\alpha)$ are such that x_α and y_α belong to the same orbit with respect to G_α for each $\alpha \in I$, then x and y belong to the same orbit with respect to G , and therefore η is injective.

Likewise, the hypothesis of (b) implies that the inverse system of canonical mappings $\pi_\alpha : X_\alpha \rightarrow X_\alpha/G_\alpha$ satisfies the conditions of ?? , and therefore its inverse limit $\pi = \varprojlim \pi_\alpha : X \rightarrow \varprojlim X_\alpha/G_\alpha$ is surjective, and the inverse image under π of every point of $\varprojlim X_\alpha/G_\alpha$ is compact. Since π factorizes into

$$X \rightarrow X/G \rightarrow \varprojlim X_\alpha/G_\alpha$$

where $X \rightarrow X/G$ is the canonical mapping, the assertions of (b) follow. \square

Corollary 3.3.25. *If the G_α are compact and the X_α are Hausdorff, then the conclusions of (a) and (b) in ?? are valid.*

Proof. The hypotheses of (a) and (b) are satisfied since every closed subgroup of G_α is compact and $u_\alpha : G_\alpha \rightarrow Y_\alpha$ is a continuous mapping of a compact space into a Hausdorff space. \square

Corollary 3.3.26. *If for each $\alpha \in I$, the group G_α operates transitively on the space X_α and if the stabilizer of each point of X_α is a compact subgroup of G_α , then G operates transitively on X and the stabilizer of each point of X is a compact subgroup of G .*

Proof. In this case we have $Y_\alpha = X_\alpha$ and the hypotheses of (a) is satisfied. \square

Corollary 3.3.27. *Suppose that the G_α are Hausdorff. For each $\alpha \in I$ let K_α be a compact subgroup of G_α such that $\phi_{\alpha\beta}(K_\beta) \subseteq K_\alpha$ whenever $\alpha \preceq \beta$. Then, if $K = \varprojlim K_\alpha$, the canonical mapping h of the homogeneous space G/K into $\varprojlim G_\alpha/K_\alpha$ is a homeomorphism.*

Proof. The fact that h is bijective follows from ??, applied by replacing X_α by G_α , and G_α by K_α operating by right translations. Let π be the canonical mapping $G \rightarrow G/K$, and for each α let ϕ_α be the canonical mapping $G \rightarrow G_\alpha$. If, for each α , V_α runs through a fundamental system of open neighbourhoods of the identity element e_α of G_α , then the sets $f_\alpha^{-1}(V_\alpha)$ form a fundamental system of neighbourhoods of the identity element e in G , and the sets $\pi(V \cdot K)$ form a fundamental system of neighbourhoods of $\pi(e)$ in G/K . We have to show that the image of $\pi(V \cdot K)$ under h contains a neighbourhood of $h(\pi(e))$, that is, there exists $\beta \succeq \alpha$ and a neighbourhood W_β of e_β in G_β such that $\phi_\beta^{-1}(W_\beta \cdot K_\beta) \subseteq V \cdot K$. Now, the relation $x \in V \cdot K$ is equivalent to the existence of y in K such that $\phi_\alpha(xy^{-1}) \in V_\alpha$, i.e., $\phi_\alpha(x) \in V_\alpha \cdot \phi_\alpha(K)$. Thus we get $V \cdot K = \phi_\alpha^{-1}(V_\alpha \cdot \phi_\alpha(K))$. Let $U_\alpha = V_\alpha \cdot \phi_\alpha(K)$, we shall see that there exists $\beta \succeq \alpha$ such that if we set $U_\beta = \phi_{\alpha\beta}^{-1}(U_\alpha)$, then $K_\beta \subseteq U_\beta$. It will then follow from ?? that there is a neighbourhood W_β of e_β in G_β such that $W_\beta K_\beta \subseteq V_\alpha$, and this will establish the relation

$$\phi_\beta^{-1}(W_\beta \cdot K_\beta) \subseteq \phi_\beta^{-1}(U_\beta) = V \cdot K.$$

We proceed by reductio ad absurdum: for each $\beta \succeq \alpha$, let M_β denote $K_\beta \setminus U_\beta$. Since $\phi_{\beta\gamma}^{-1}(U_\beta) = U_\gamma$ if $\gamma \succeq \beta$, the M_β form an inverse system of compact subsets of the G_β (for $\beta \succeq \alpha$). If they were all nonempty, then their inverse limit M would also be non-empty (??). It is clear that $M \subseteq K$ and $\phi_\alpha(M) \subseteq M_\alpha$; but this is absurd since $\phi_\alpha(K) \subseteq U_\alpha$, and the proof is therefore complete. \square

Let G be a group and let (H_α) be a family of normal subgroups of G such that $H_\alpha \supseteq H_\beta$ whenever $\alpha \preceq \beta$. For each α let $G_\alpha = G/H_\alpha$ and $\pi_{\alpha\beta}$ be the canonical homomorphism $G/H_\beta \rightarrow G/H_\alpha$, which therefore maps a coset of H_β in G to the coset of H_α in G . Clearly $(G_\alpha, \pi_{\alpha\beta})$ is an inverse system of groups; the elements of $\tilde{G} = \varprojlim G_\alpha$ are families $(T_\alpha)_{\alpha \in I}$ where T_α is a coset of H_α in G for each α , and $T_\alpha \supseteq T_\beta$ whenever $\alpha \preceq \beta$. The mapping $i : G \rightarrow \tilde{G}$ given by $g \mapsto (gH_\alpha)$ is the inverse limit of the canonical homomorphisms $G \rightarrow G/H_\alpha$; and is therefore a homomorphism of G into \tilde{G} , and the inverse image under i of an element $(T_\alpha)_{\alpha \in I}$ is equal to $\bigcap_\alpha T_\alpha$. The kernel of i is therefore $\bigcap_\alpha H_\alpha$, and the image of i consists of all families (T_α) in \tilde{G} whose intersection is non-empty.

Now suppose that G is a topological group; if we give each $G_\alpha = G/H_\alpha$ the quotient topology, it is clear that $(G_\alpha, \pi_{\alpha\beta})$ is an inverse system of topological groups, and that $i : G \rightarrow \tilde{G}$ is a continuous homomorphism.

Proposition 3.3.28. *Let G be a topological group and let $(H_\alpha)_{\alpha \in I}$ be a family of normal subgroups of G such that $H_\alpha \supseteq H_\beta$ whenever $\alpha \preceq \beta$, and which satisfy the following condition:*

(AP) *For each $\alpha \in I$, H_α is closed in G and every neighbourhood of the identity element e in G contains one of the H_α (in other words, the filter base formed by the H_α converges to e).*

Then the mapping $i : G \rightarrow \tilde{G}$ is a strict morphism of G onto $i(G)$, \tilde{G} is Hausdorff and $i(G)$ is dense in G . Finally, the kernel of i is the closure of $\{e\}$ in G . If in addition one of the H_α is complete, then i is surjective.

Proof. Since H_α is closed, the $G_\alpha = G/H_\alpha$ are Hausdorff, hence so is G (since it is a subspace of $\prod G_\alpha$). The kernel H of i is the intersection of the H_α and is therefore a closed subgroup of G . Since each neighbourhood of e contains some H_α it contains H and therefore H is the closure of $\{e\}$. Let us next show that $i(G)$ is dense in \tilde{G} . Let $\pi_\alpha : \tilde{G} \rightarrow G_\alpha$ be the canonical mapping, which is the restriction to \tilde{G} of the projection. Then $\varphi_\alpha = \pi_\alpha \circ i$ is the canonical mapping $G \rightarrow G/H_\alpha$.

$$\begin{array}{ccc} G & \xrightarrow{i} & \tilde{G} \\ & \searrow \varphi_\alpha & \downarrow \pi_\alpha \\ & & G/H_\alpha \end{array}$$

If U is any non-empty open set of \tilde{G} , then there is an index $\alpha \in I$ and a non-empty open set V_α in G_α such that $\pi_\alpha^{-1}(V_\alpha) \subseteq U$; therefore $\varphi_\alpha^{-1}(U_\alpha) \subseteq i^{-1}(U)$. But since φ_α is surjective, $\varphi_\alpha^{-1}(U_\alpha)$ is not empty, and therefore $i(G) \cap U \neq \emptyset$.

To see that i is a strict morphism of G onto $i(G)$, consider a neighbourhood V of e in G . There is a neighbourhood W of e in G such that $W^2 \subseteq V$, and an index $\alpha \in I$ such that $H_\alpha \subseteq W$. It then follows that V contains $WH_\alpha = \varphi_\alpha^{-1}(\varphi_\alpha(W)) = i^{-1}(\pi_\alpha^{-1}(\pi_\alpha(W)))$. Since $\pi_\alpha^{-1}(\pi_\alpha(W))$ is a neighbourhood of the identity element in \tilde{G} , the result follows.

Finally, suppose that there is an index $\gamma \in I$ such that H_γ is complete. To show that i is surjective it is enough to prove that every family $(T_\alpha) \in \tilde{G}$ has a non-empty intersection. Since T_γ is obtained by translation from H_γ , it is a complete subspace of G (with respect to both the right and the left uniformities). Moreover, since every neighbourhood U of e in G contains one of the H_α , the corresponding set T_α is U_l - (or U_r -) small, and hence the set of T_α contained in T_γ is a Cauchy filter base; therefore it converges in T_γ , and since the T_α are closed in G (since they are translations of the H_α), their intersection is not empty. \square

Corollary 3.3.29. *If the condition (AP) is satisfied and if in addition the groups G/H_α are complete, then the group G has a Hausdorff completion which can be identified with \tilde{G} and the mapping $i : G \rightarrow \tilde{G}$ is then identified with the canonical mapping.*

Proof. For \tilde{G} is then complete, and ?? shows that $i(G)$ is isomorphic with the Hausdorff group associated with G ; since it is dense in \tilde{G} , the corollary follows. \square

Corollary 3.3.30. *Let G be a group and let (H_α) be a family of normal subgroups of G directed with respect inclusion. If we endow G with the group topology for which the H_α form a fundamental system of neighbourhoods of the identity element e , then the Hausdorff group associated with G is isomorphic to $G' = G / (\bigcap_\alpha H_\alpha)$ and G' has a completion $\tilde{G}' = \varprojlim G/H_\alpha$. The canonical map $G \rightarrow G'$ extends to an isomorphism of \tilde{G} onto \tilde{G}' .*

Proof. The subgroup H_α of G is open and therefore also closed, and G/H_α is discrete; hence the conditions of ?? are satisfied. \square

For the remainder of this part we shall assume that G is Hausdorff and that (H_α) is a family of compact normal subgroups of G , which is directed with respect inclusion and which satisfies

the condition (AP); by virtue of ??, the mapping

$$i : G \rightarrow \tilde{G} = \varprojlim G/H_\alpha$$

is then an isomorphism of topological groups which permits us to identify G and \tilde{G} . We denote by π_α the canonical mapping $G \rightarrow G/H_\alpha$.

Lemma 3.3.31. *Under the hypotheses of ??, if E is any closed subset of G we have $E = \bigcap_\alpha EH_\alpha$.*

Proof. For E is the intersection of the sets EV where V runs through the neighbourhood filter of e , and every neighbourhood of e contains an H_α ; whence the result, since $E \subseteq EH_\alpha$. \square

Proposition 3.3.32. *Suppose that G is Hausdorff and that the H_α are compact and satisfy (AP).*

- (a) *Let L be a closed subgroup of G ; then, for each $\alpha \in I$, the subgroup $L_\alpha := \pi_\alpha(L)$ of $G_\alpha = G/H_\alpha$ is closed, and the isomorphism i of G onto $\varprojlim G_\alpha$ gives by restriction an isomorphism of L onto $\varprojlim L_\alpha$. If also L is normal in G , then L_α is normal in G_α for each $\alpha \in I$, and by passing to the quotients, i induces an isomorphism of G/L onto $\varprojlim G/L_\alpha$.*
- (b) *Conversely, for each $\alpha \in I$ let L_α be a closed subgroup of G_α such that $L_\alpha = \pi_{\alpha\beta}(L_\beta)$ whenever $\alpha \preceq \beta$. Then there is a unique closed subgroup L of G such that $L_\alpha = \pi_\alpha(L)$ for each $\alpha \in I$, and if in addition L_α is normal in G_α for each $\alpha \in I$, then L is normal in G .*

Proof. First assume (a). Since H_α is compact, LH_α is closed in G by ?? and therefore L_α is closed in G_α . Since i identifies the topological groups G and $\varprojlim G_\alpha$ and since $\varprojlim L_\alpha$ may be identified with a (topological) subgroup of $\varprojlim G_\alpha$, i identifies the subgroup $\bigcap_\alpha LH_\alpha$ of G with $\varprojlim L_\alpha$, and to prove the first assertion it is enough to remark that $L = \bigcap_\alpha LH_\alpha$ by ???. On the other hand, if L is normal, then for each $\alpha \in I$ the mapping $\pi'_\alpha : G/L \rightarrow G_\alpha/L_\alpha$ induced by π_α is a surjective strict morphism (??), whose kernel is the compact normal subgroup LH_α/L of G/L , the canonical image of the compact subgroup H_α of G . Since the subgroups LH_α/L of G/L satisfy condition (AP) (??), and since G/L is Hausdorff, the last assertion of (a) is a consequence of ??.

Now we consider (b). Let $\pi'_{\alpha\beta}$ be the restriction of $\pi_{\alpha\beta}$ to L_α . Then $(L_\alpha, f'_{\alpha\beta})$ is an inverse system of topological groups, whose inverse limit L can be identified with the subgroup $\bigcap_\alpha \pi_\alpha^{-1}(L_\alpha)$ of G . By hypothesis, $\pi'_{\alpha\beta}$ is surjective and its kernel is the compact subgroup $\pi_\beta(H_\beta) \cap L_\beta$ of L_α ; consequently we have $L_\alpha = \pi_\alpha(L)$ for all $\alpha \in I$ (??). If L' is another closed subgroup of G such that $\pi_\alpha(L') = L_\alpha$ for all $\alpha \in I$, then $L'H_\alpha = \pi_\alpha^{-1}(L_\alpha)$, whence

$$L' = \bigcap_\alpha L'H_\alpha = \bigcap_\alpha \pi_\alpha^{-1}(L_\alpha) = L.$$

Finally, the last assertion of (b) follows from the formula $L = \bigcap_\alpha \pi_\alpha^{-1}(L_\alpha)$, since the $\pi_\alpha^{-1}(L_\alpha)$ are now normal subgroups of G . \square

Now we give an application of the previous results.

Proposition 3.3.33. *Let $(G_\alpha, \phi_{\alpha\beta})$ be an inverse system of Hausdorff topological groups such that the $\phi_{\alpha\beta}$ are surjective strict morphisms with compact kernels. Then for each $\alpha \in I$ the canonical mapping ϕ_α of $G = \varprojlim G_\alpha$ into G_α is a surjective strict morphism whose kernel is compact.*

Proof. The facts that ϕ_α is surjective and that its kernel is compact are consequences of ???. It remains to show that ϕ_α is a strict morphism. Let e (resp. e_α) denote the identity element of G (resp. G_α). Each neighbourhood V of e in G contains a set of the form $\phi_\beta^{-1}(V_\beta)$, where V_β is a neighbourhood of e_β in G_β , and we may suppose that $\beta \succeq \alpha$. Since $\phi_{\beta\alpha}$ is a surjective strict morphism, $\phi_{\alpha\beta}(V_\beta)$ is a neighbourhood of e_α in G_α , and since ϕ_α is surjective, we have $V_\beta \subseteq \phi_\beta(V)$, whence

$$\phi_\alpha(V) = \phi_{\alpha\beta}(\phi_\beta(V)) \supseteq \phi_{\alpha\beta}(V_\beta)$$

This shows that $\phi_\alpha(V)$ is a neighbourhood of e_α in G_α . \square

If $H_\alpha = \ker \phi_\alpha$, then each H_α is a compact normal subgroup of G . The H_α clearly satisfy the condition (AP) and G_α can be identified with G/H_α . In particular ?? apply to G and the H_α .

Corollary 3.3.34. *Let $(G_\alpha, \phi_{\alpha\beta})$ be an inverse system of topological groups which satisfies the hypotheses of ???. Let $(G'_\alpha, \phi'_{\alpha\beta})$ be an inverse system of topological groups, and for each α let $u_\alpha : G_\alpha \rightarrow G'_\alpha$ be a surjective strict morphism with compact kernel, such that the u_α from an inverse system of mappings. Then $u = \varprojlim u_\alpha$ is a strict morphism of $G = \varprojlim G_\alpha$ onto $G' = \varprojlim G'_\alpha$, and its kernel is compact.*

Proof. Let N_α be the kernel of u_α and H_α the kernel of ϕ_α . Then $L_\alpha = \phi_\alpha^{-1}(N_\alpha)$ is then the kernel of the surjective strict morphism $v_\alpha = u_\alpha \circ \phi_\alpha : G \rightarrow G'_\alpha$. Since L_α / H_α is isomorphic to N_α , L_α is a compact normal subgroup of G . The kernel L of u is the intersection of the L_α . Let π denote the canonical mapping $G \rightarrow G/L$; then we can write $v_\alpha = w_\alpha \circ \pi$, where w_α is a strict morphism of G/L onto G'_α , whose kernel is L_α / L . Since the intersection of the L_α / L is the identity element of G/L , and since the L_α / L form a filter base and are compact, this filter base converges to the identity element of G/L . ?? then shows that $w = \varprojlim w_\alpha$ is an isomorphism of G/L onto G' . It follows that $w \circ \pi$ is a strict morphism of G onto G' , with kernel L . But it is clear that $u = w \circ \pi$, and thus the corollary is proved. \square

Chapter 4

Topologies encountered in algebraic geometry

4.1 Irreducibility

4.1.1 Irreducible topological spaces

A topological space X is called **irreducible** if every finite intersection of non-empty open sets of X is non-empty. By considering the empty family of open sets of X it is seen that an irreducible space is non-empty. For a topological space X to be irreducible, it is necessary and sufficient that it be non-empty and that the intersection of two non-empty open sets of X be always non-empty (or, what amounts to the same thing, that the union of two closed sets distinct from X be always distinct from X).

Proposition 4.1.1. *Let X be a non-empty topological space. The following assertions are equivalent.*

- (i) *X is irreducible.*
- (ii) *Any two non-empty open subsets of X have a non-empty intersection.*
- (iii) *Every non-empty open subset is dense in X .*
- (iv) *Every non-empty open subset is connected.*
- (v) *Every non-empty open subset is irreducible.*

Proof. It is clear that (i), (ii), and (iii) are equivalent. If there exist non-empty open subsets U_1 and U_2 that have an empty intersection, their union is a non-connected open subset. Conversely if U is a nonempty nonconnected subset we can write U as the disjoint union of two non-empty open subsets of U (and hence of X). This shows that (iv) and (i) are equivalent.

Obviously (v) implies (i). Let us show that (iii) implies (v). Let $U \subseteq X$ be open and non-empty, and V be an open subset of U . Then V is also open in X and therefore dense in X by (iii). But then V is certainly dense in U , so U is irreducible. \square

Example 4.1.2. A Hausdorff space X is irreducible if and only if it consists of a single point.

A subset E of a topological space X is called an **irreducible** set if the subspace E of X is irreducible. For this to be so, it is necessary and sufficient that, for every pair of sets U, V

which are open in X and meet E , $U \cap V$ also meet E , or (what amounts to the same thing) that, for every pair of sets F, G which are closed in X and satisfy $E \subseteq F \cup G$, $E \subseteq F$ or $E \subseteq G$.

Proposition 4.1.3. *Let E be an irreducible set of X . If $(F_i)_{i \in I}$ is a finite family of closed sets of X such that $E \subseteq \bigcup_i F_i$, then there exists $i \in I$ such that $E \subseteq F_i$.*

Proof. This follows from the definition of irreducibility and induction on the cardinality of I . \square

Proposition 4.1.4. *In a topological space X , for a set E to be irreducible, it is necessary and sufficient that its closure \bar{E} be so.*

Proof. For an open set of X to meet E , it is necessary and sufficient that it meet E . Then the proposition follows from the above remarks. \square

Proposition 4.1.5. *Let X be a topological space and let $(U_\alpha)_{\alpha \in A}$ be an open covering of X consists of irreducible open subsets such that $U_\alpha \cap U_\beta \neq \emptyset$ for $\alpha \neq \beta$. Then X is irreducible.*

Proof. Let us show that, for every non-empty open set V in X , $V \cap U_\alpha \neq \emptyset$ for all $\alpha \in A$; it then follows that $V \cap U_\alpha$ is dense in U_α by hypothesis, hence that V is dense in X and this proves that X is irreducible. Now there exists at least one index γ such that $V \cap U_\gamma \neq \emptyset$. As $U_\alpha \cap U_\gamma \neq \emptyset$ for all α and $V \cap U_\gamma$ is dense in U_γ , we see $V \cap U_\gamma \cap U_\alpha \neq \emptyset$ and a fortiori $V \cap U_\alpha \neq \emptyset$, which completes the proof of (b). \square

Proposition 4.1.6. *Let X and Y be two topological spaces and $f : X \rightarrow Y$ a continuous map. For every irreducible subset E of X , $f(E)$ is an irreducible subset of Y .*

Proof. If U, V are two open sets of Y which meet $f(E)$, then $f^{-1}(U)$ and $f^{-1}(V)$ are open sets of X which meet E . Consequently, $f^{-1}(U) \cap f^{-1}(V) = f^{-1}(U \cap V)$ meets E , which implies that $U \cap V$ meets $f(E)$ and proves the proposition. \square

Now let X be a topological space. Every maximal irreducible subset of a topological space X is called an **irreducible component** of X . It follows from ?? that every irreducible component of X is closed in X .

Proposition 4.1.7. *Let X be a topological space. Every irreducible subset of X is contained in an irreducible component of X and X is the union of its irreducible components.*

Proof. To prove the first assertion, it is sufficient, by virtue of Zorn's Lemma, to prove that the set \mathcal{I} of irreducible subsets of X is inductive. Let \mathcal{C} be a chain in \mathcal{I} . We show that the union E of the sets $F \in \mathcal{C}$ is irreducible. Let U, V be two open sets of X which meet E . As \mathcal{C} is totally ordered, there exists a set $F \in \mathcal{C}$ meeting U and V . As F is irreducible, $U \cap V$ meets F and hence also E , which proves that E is irreducible and hence that \mathcal{I} is inductive. The second assertion follows from the first, for every subset of X consisting of a single point is irreducible. \square

Corollary 4.1.8. *Every connected component of a topological space X is a union of irreducible components of X .*

Proof. Every irreducible subspace of X is connected by ?? and hence is contained in a connected component of X . \square

Proposition 4.1.9. *Let X be a topological space and $(Z_i)_{1 \leq i \leq n}$ a finite closed covering of X consisting of irreducible sets. Then the irreducible components of X are the maximal elements (with respect to inclusion) of the set of Z_i .*

Proof. We may restrict ourselves to the case where no two Z_i are comparable. If E is an irreducible subset of X , then $E \subseteq \bigcup_i Z_i$, and hence E is contained in one of the closed sets Z_i (??). This proves that the Z_i are the only maximal irreducible subsets of X . \square

Corollary 4.1.10. *Let X be a topological space and E a subspace with only a finite number of distinct irreducible components Q_1, \dots, Q_n . Then the irreducible components of the closure E in X are the closures \bar{Q}_i of the Q_i and $\bar{Q}_i \neq \bar{Q}_j$ when $i \neq j$.*

Proof. The space \bar{E} is the union of the \bar{Q}_i which are irreducible (??). As Q_i is closed in E , $\bar{Q}_i \cap E = Q_i$. As $Q_i \neq Q_j$, we see $\bar{Q}_i \neq \bar{Q}_j$ when $i \neq j$, whence the corollary by virtue of ??.

Example 4.1.11. Suppose that X has only a finite number of distinct irreducible components X_1, \dots, X_n . Then $U_i = (\bigcup_{j \neq i} X_j)^c$ is open in X and dense in X_i (since $X_i \not\subseteq \bigcup_{j \neq i} X_j$). The U_i are therefore non-empty open sets of X which are irreducible by ??, pairwise disjoint and with their union dense in X .

Proposition 4.1.12. *Let X be a topological space with finitely irreducible components $(X_i)_{1 \leq i \leq r}$. Then any open dense subset U of X intersects each X_i .*

Proof. If $U \cap X_i = \emptyset$, then the open subset $X - \bigcup_{j \neq i} X_j$ of X is disjoint from U , contradicting to the fact that U is dense in X . \square

Proposition 4.1.13. *Let U be an open subset of a topological space X . Then the map $C \mapsto \bar{C}$ (closure in X) gives a bijection between the set of irreducible closed subsets of U and the set of irreducible closed subsets of X meeting U , whose inverse is given by $Z \mapsto Z \cap U$. In particular, this bijection maps the set of irreducible components of U onto the set of irreducible components of X which meet U .*

Proof. If C is closed in U and irreducible, then \bar{C} is irreducible and $C = \bar{C} \cap U$. Conversely, if Z is irreducible and closed in X and meets U , then $Z \cap U$ is a non-empty open subset of Z , hence irreducible and dense in Z and, as Z is closed, $Z = \bar{Z} \cap U$. This proves the proposition. \square

4.1.2 Noetherian topological spaces

Definition 4.1.14. *A topological space is called **Noetherian** if the descending chain condition holds for closed subsets of X . A topological space is called **locally Noetherian** if every point has a neighbourhood which is Noetherian.*

Clearly, X is noetherian if and only if every non-empty set of closed subsets of X has a minimal element, if and only if every non-empty set of open subsets of X has a maximal element, or that every decreasing (resp. increasing) sequence of closed (resp. open) sets is stationary.

Proposition 4.1.15. *Any subspace of a Noetherian space is Noetherian. Let $(A_i)_{i \in I}$ be a finite covering of a topological space X . If the subspaces A_i of X are Noetherian, then X is Noetherian.*

Proof. Let X be a Noetherian space, A a subspace of X and (F_n) a decreasing sequence of subsets of A which are closed in A . Then $F_n = \bar{F}_n \cap A$ and the closures \bar{F}_n of the F_n in X form a decreasing sequence of closed subsets of X . As this sequence is stationary, so is the sequence (F_n) .

For (b), let (F_n) be a decreasing sequence of closed subsets of X . By hypothesis, each of the sequences $(E_n \cap A_i)_n$ is stationary. As I is finite, there is an integer n_0 such that, for $n \geq n_0$, $E_n \cap A_i = E_{n_0} \cap A_i$ for all $i \in I$. But

$$E_n = \bigcup_{i \in I} E_n \cap A_i$$

and hence the sequence (E_n) is stationary and X is Noetherian. \square

Proposition 4.1.16. *For a topological space X to be Noetherian, it is necessary and sufficient that every open set in X be quasi-compact.*

Proof. To show the necessity, it is sufficient, by virtue of ??, to prove that every Noetherian space X is quasi-compact. Let $(U_\alpha)_{\alpha \in A}$ be an open covering of X . The set of finite unions of sets U , is non-empty and hence admits a maximal element $V = \bigcup_{i \in H} U_i$ where H is a finite subset of I . By definition, $V \cup U_i = V$ for all $i \in I$ and hence $V = X$.

Conversely, suppose that every open set of X is quasi-compact and let (U_n) be an increasing sequence of open subsets of X . The union V of the U_n is open and hence quasi-compact. As (U_n) is an open covering of V , there is a finite subfamily of (U_n) which is a covering of V and hence $V = U_n$ for some index n , which proves that the sequence (U_n) is stationary. \square

Lemma 4.1.17 (Principle of Noetherian Induction). *Let E be an ordered set such that any nonempty subset of E contains a minimal element. Let F be a subset of E with the following property: if $a \in E$ is such that the relation $x < a$ implies $x \in F$, then $a \in F$. Then $E = F$.*

Proof. Suppose $F \neq E$, then $E \setminus F$ would have a minimal element b . By definition, $x \in F$ for all $x < b$, which implies that $b \in F$, a contradiction. \square

Proposition 4.1.18. *If X is a Noetherian space, the set of irreducible components of X (and a fortiori the set of connected components of X) is finite.*

Proof. It is sufficient to prove that X is a finite union of irreducible closed subsets (??). Let us show that the principle of Noetherian induction can be applied taking E to be the set of closed subsets of X , ordered by inclusion, and F to be the set of finite unions of irreducible closed subsets. Let Y be a closed subset of X such that every proper closed subset of Y belongs to F . If Y is irreducible, then $Y \in F$ by definition. Otherwise, Y is the union of two closed subsets Y_1, Y_2 which are distinct from Y . Then $Y_1 \in F$ and $Y_2 \in F$ by hypothesis, whence $Y \in F$ by definition of F . \square

Proposition 4.1.19. *Suppose X is a Noetherian topological space. Then every nonempty closed subset E can be expressed uniquely as a finite union $E = Z_1 \cup \dots \cup Z_n$ of irreducible closed subsets, none contained in any other.*

Proof. Just take the irreducible component of E . \square

Proposition 4.1.20. *Let $f : X \rightarrow Y$ be a continuous map of topological spaces.*

- (a) If X is Noetherian, then $f(X)$ is Noetherian
- (b) If X is locally Noetherian and f is open, then $f(X)$ is locally Noetherian.

Proof. For (a), suppose that $(Z_i)_i$ is a descending chain of closed subsets of $f(X)$. Then $(f^{-1}(Z_i))_i$ is a descending chain of closed subsets of X . Hence this chain stabilizes. Since $f(f^{-1}(Z_i)) = Z_i$ we conclude that $(Z_i)_i$ also stabilizes. In case (b), let $y \in f(X)$. Choose $x \in X$ with $f(x) = y$. By assumption there exists a neighbourhood $E \subseteq X$ of x which is Noetherian. Then $f(E) \subseteq f(X)$ is a neighbourhood which is Noetherian by part (a). \square

Proposition 4.1.21. *Let X be a locally Noetherian topological space. Then X is locally connected.*

Proof. Let $x \in X$ and V be a neighbourhood of x . We have to find a connected neighbourhood of x contained in V . By assumption there exists a neighbourhood U of x which is Noetherian. Then $U \cap V$ is Noetherian. Let $U \cap V = Z_1 \cup \dots \cup Z_n$ be the decomposition into irreducible components, then $W = \bigcup_{x \in Z_i} Z_i$ is a connected subset of $U \cap V$ containing x (since each Z_i is connected). \square

Proposition 4.1.22. *Let X be a locally Noetherian space, then the following are equivalent:*

- (i) *The irreducible components of X are open.*
- (ii) *The irreducible components coincide with the connected components.*
- (iii) *The connected components of X are irreducible.*
- (iv) *The irreducible components of X are disjoint.*

Proof. Since the irreducible components of X are closed and connected, clearly (i) implies (ii) and (ii) implies (iii). If (iii) holds, then the connected components are exactly the irreducible component since they are open and closed and disjoint. Thus (iii) implies (ii) and (iv).

Finally, we prove that (iv) implies (i). Let Z be an irreducible component of X and $x \in Z$ be arbitrary point. Choose a noetherian neighborhood U of x . It is easy to see that the irreducible components of U are exactly the intersection of that of X with U . Since U is noetherian, there are finitely many such irreducible components: say $Z = Z_1, \dots, Z_n$. Then since $Z_i \cap U$ are closed and disjoint, they are also open in U , hence in X . In particular, this means Z has an open neighborhood of x , which means it is open. \square

4.1.3 Generic points

4.2 Krull dimension

4.2.1 Krull dimension of a topological space

Definition 4.2.1. *Let I be an ordered set. A non-empty and totally ordered finite subset of I is called a **chain** of I . Let C be a chain of I , the smallest and the most large element of C are called the **ends** of C . The integer $|C| - 1$ is called the **length** of C . The relation of inclusion in the set of subsets of I induces a order relation in the set of chains of I . A chain C of I is said to be **saturated** if it is maximal among the chains of I having the same endpoints as C .*

Let X be a topological space. We endow the set of irreducible closed subsets of X of the order relation defined by the inclusion. When we talk about a chain of closed irreducible subsets of X , it will always be a chain in the sense of this order relation.

Definition 4.2.2. Let X be a topological space. The **dimension** or more precisely the **Krull dimension** of X , denoted by $\dim(X)$, is defined to be the supremum of chains of irreducible closed subsets of X . We set $\dim(X) = -\infty$ when X is the empty space. Let $x \in X$, the **Krull dimension of X at x** , denoted by $\dim_x(X)$, is defined to be the infimum of the dimensions of the open neighborhoods of x .

Note that if X is Hausdorff then any irreducible closed subset of X reduces to a point. The definition of the Krull dimension is therefore devoid of interest for Hausdorff spaces, but it is specially adapted to the topological spaces encountered in commutative algebra. In this part, no confusion is to be feared with other notions of dimension of topological spaces, and we will simply say "dimension" for "Krull dimension."

Proposition 4.2.3. Let X be a topological space.

- (a) Let Y be a subspace of X , then $\dim(Y) \leq \dim(X)$ and $\dim_y(Y) \leq \dim_y(X)$ for any point y of Y .
- (b) Let x be a point of X and U a neighborhood of x . Then $\dim_x(X) = \dim_x(U)$.

Proof. Let Y be a subspace of X . If Z is an irreducible closed subset of Y , its closure \bar{Z} in X is irreducible and $\bar{Z} \cap Y = Z$. Thus, any chain of irreducible closed subsets of Y defines, by passage to closure in X , a chain of irreducible closed subsets of X of the same length. This implies $\dim(Y) \leq \dim(X)$. If U is an open subset of X containing a point y of Y , we therefore have $\dim(U \cap Y) \leq \dim(U)$, whence $\dim_y(Y) \leq \dim_y(X)$. Now part (b) follows from (a). \square

Proposition 4.2.4. Let X be a topological space. Then $\dim(X) = \sup_{x \in X} \dim_x(X)$.

Proof. First we have $\dim_x(X) \leq \dim(X)$ by ??(a). On the other hand, if $Z_0 \subset \dots \subset Z_n$ is a chain of irreducible closed subsets and U a neighborhood of x_0 in X , where $x_0 \in Z_0$, then $Z_0 \cap U \subset \dots \subset Z_n \cap U$ is a chain of irreducible closed subsets of U (??). This shows $\dim_x(X) \geq n$, whence $\dim(X) \leq \sup_{x \in X} \dim_x(X)$. \square

Corollary 4.2.5. Let $(X_i)_{i \in I}$ be an open cover (or a locally finite closed cover) of a topological space X . Then

$$\dim(X) = \sup_{i \in I} \dim(X_i), \quad \dim_x(X) = \sup_{i \in I_x} \dim_x(X_i)$$

where I_x is the set $i \in I$ such that $x \in U_i$.

Proof. The claim is immediate when (X_i) is an open cover, and we now deal with the case of a locally finite closed cover. First suppose that I is finite. Let $Z_0 \subset \dots \subset Z_n$ be a chain of irreducible closed subsets of X . Then $Z_n = \bigcup_i Z_n \cap X_i$, and each $Z_n \cap X_i$ is closed in Z_n . Since I is finite, we see Z_n is contained in one of the X_i , whence $\dim(X) \leq \sup_i \dim(U_i)$, and the converse inequality follows from ??.

Now let $x \in X$ and for each $i \in I_x$ let U_i be a neighborhood of x such that $\dim(U_i \cap X_i) \leq n$, where $n = \sup_{i \in I_x} \dim_x(X_i)$, and let

$$U = \left(\bigcap_{i \in I_x} U_i \right) \cap \left(\bigcap_{i \in I - I_x} X_i^c \right).$$

Then U is open in X since I_x is finite, and since $(U \cap X_i)_{i \in I_x}$ is a finite closed cover of U , by the preceding argument we have

$$\dim(U) = \sup_{i \in I_x} \dim(U \cap X_i) \leq \sup_{i \in I_x} \dim(U_i \cap X_i) \leq n$$

whence $\dim_x(X) \leq n$ and the proof is finished in this case. The general case then follows from ??.

Proposition 4.2.6. *Let \mathcal{I} be the set of irreducible components of X . Then*

$$\dim(X) = \sup_{Y \in \mathcal{I}} \dim(Y).$$

Proof. It is clear that $\sup_{Y \in \mathcal{I}} \dim(Y) \leq \dim(X)$ according to ???. Now let $Z_0 \subset \dots \subset Z_n$ be a chain of irreducible closed subsets in X . Then Z_n is contained in a irreducible component Y of X , and therefore $Z_0 \subset \dots \subset Z_n$ is a chain of irreducible closed subsets in Y (since Y is closed), and we obtain $n \leq \sup_{Y \in \mathcal{I}} \dim(Y)$, whence the claim. \square

Proposition 4.2.7. *Let X be an irreducible topological space and Y a proper closed subset of X . Then $\dim(Y) < \dim(X)$.*

Proof. Since X is irreducible and closed, any chain $Z_0 \subset \dots \subset Z_n$ of irreducible closed subsets of Y can be extended by adding X at the end, whence $\dim(Y) \leq \dim(X) + 1$. \square

Corollary 4.2.8. *If Y is a closed subset of an irreducible finite-dimensional topological space X , and if $\dim Y = \dim X$, then $Y = X$.*

4.2.2 Codimension of a closed subset

Let X be a topological space. For an irreducible closed subset Z of X , the **codimension** of Z in X is defined to be the supremum of the lengths of the chains of closed irreducible subsets of X containing Z . If Y is a closed subset of X , the **codimension** of Y in X is defined to be the infimum of the codimensions in X of the irreducible components Y .

The codimension of a closed subset Y of X is therefore the infimum of the codimensions of the irreducible closed subsets of Y . We have $\text{codim}(\emptyset, X) = +\infty$ and, if X is not empty, $\text{codim}(X, X) = 0$. Since any nonempty closed subset of X contains an irreducible closed subset (for example the closure of a singleton), the codimension in X of a closed subset Y is always a positive integer or $+\infty$. It is zero if and only if Y contains an irreducible component of X . Note that by definition $\text{codim}(Y, X) \leq \dim(X)$, and $\dim(X) = \sup_Y \text{codim}(Y, X)$ where Y runs through all irreducible closed subsets of X . Also, if $Y' \subseteq Y$ are closed subsets of X , we have $\text{codim}(Y, X) \leq \text{codim}(Y', X)$.

Let Y be a closed subset of X and $(X_\alpha)_{\alpha \in A}$ (resp. $(Y_\beta)_{\beta \in B}$) be the family of irreducible components of X (resp. of Y). For each $\beta \in B$, let A_β be the subset $\alpha \in A$ such that $Y_\beta \subseteq X_\alpha$. Then since each irreducible closed subset of X is contained in one of X_α , it follows from definition that

$$\text{codim}(Y, X) = \inf_{\beta \in B} \sup_{\alpha \in A_\beta} \text{codim}(Y_\beta, X_\alpha). \quad (4.2.1)$$

In particular, if $(Y_i)_{i \in I}$ is a finite family of closed subsets of X and $Y = \bigcup_i Y_i$, then

$$\text{codim}(Y, X) = \inf_{i \in I} \text{codim}(Y_i, X)$$

since by ?? any irreducible component of Y is contained in one of the Y_i .

Proposition 4.2.9. *Let X be a topological space.*

(a) *For any closed subset Y of X , we have*

$$\dim(Y) + \text{codim}(Y, X) \leq \dim(X).$$

(b) *If $Y \subseteq Z \subseteq T$ are closed subsets of X , then*

$$\text{codim}(Y, Z) + \text{codim}(Z, T) \leq \text{codim}(Y, T).$$

Proof. To prove (a), we may assume that $\dim(X)$ is finite, whence $\dim(Y)$ and $\text{codim}(Y, X)$ are both finite. If $Z_0 \subset \dots \subset Z_n$ is a chain of irreducible closed subsets of Y such that $n = \dim(Y)$ and $Z_n \subset \dots \subset Z_{n+p}$ is a chain of that of X such that $p \geq \text{codim}(Y, X)$, then we get a chain $Z_0 \subset \dots \subset Z_{n+p}$ in X , whence $\dim(X) \geq n + p$ and part (a) follows.

To prove (b), we may assume that Y is irreducible. Since $\text{codim}(Y, Z) \leq \text{codim}(Y, T)$, the claim is true if $\text{codim}(Y, Z) = +\infty$. Otherwise, let Z_0 be an irreducible component of Z containing Y such that $\text{codim}(Y, Z) = \text{codim}(Y, Z_0)$. We have $\text{codim}(Z, T) \leq \text{codim}(Z_0, T)$, and we see, as the proof of (a), that $\text{codim}(Y, Z_0) + \text{codim}(Z_0, T) \leq \text{codim}(Y, T)$, whence (b). \square

Definition 4.2.10. *A topological space X is called **catenary** if for any pair (Y, Z) of irreducible closed subsets of X such that $Y \subseteq Z$, any saturated chain of subsets irreducible closed lines with ends Y and Z is of length $\text{codim}(Y, Z)$.*

It amounts to the same thing to say that, for any pair (Y, Z) of closed irreducible subsets of X such that $\text{codim}(Y, Z)$ is finite, all the saturated chains with ends Y and Z have the same length, and that, for any couple (Y, Z) such that $\text{codim}(Y, Z) = +\infty$, there is no saturated chain with Y and Z ends.

Any closed subspace of a catenary space is catenary. For a space is catenary, it is necessary and sufficient that its irreducible components are catenary.

Proposition 4.2.11. *Let X be a topological space. For X be catenary, it is necessary and sufficient that, for every triple (Y, Z, T) of irreducible closed subsets of X such that $Y \subseteq Z \subseteq T$, we have*

$$\text{codim}(Y, T) = \text{codim}(Y, Z) + \text{codim}(Z, T). \quad (4.2.2)$$

Proof. Suppose X is catenary. Concerning ??, it suffices to prove (??) when $\text{codim}(Y, Z)$ and $\text{codim}(Z, T)$ are finite. By putting end to end a saturated chain of irreducible closed subsets with endpoints Y and Z of length $\text{codim}(Y, Z)$ and a saturated chain of irreducible closed subsets with end points Z and T of length $\text{codim}(Z, T)$, we obtain a saturated chain with ends Y and T , of length $\text{codim}(Y, Z) + \text{codim}(Z, T)$. As X is catenary, this length is necessarily equal to $\text{codim}(Y, T)$.

Conversely, suppose that (??) holds for every triple (Y, Z, T) of irreducible closed subset of X such that $Y \subseteq Z \subseteq T$, and we show that X is catenary. For this, we prove by induction that for every saturated chain $Z_0 \subset \cdots \subset Z_n$ we have $\text{codim}(Z_0, Z_n) = n$. For $n = 0$ this is clear. Let $n > 0$, and suppose the property satisfied for chains of length smaller than $n - 1$. If $Z_0 \subset \cdots \subset Z_n$ is a maximal of length n , then $Z_0 \subseteq \cdots \subseteq Z_{n-1}$ is a saturated chain of length $n - 1$, so $\text{codim}(Z_0, Z_{n-1}) = n - 1$. Given the assumption made on X , we have $\text{codim}(Z_0, Z_n) = \text{codim}(Z_0, Z_{n-1}) + \text{codim}(Z_{n-1}, Z_n) = (n - 1) + 1 = n$. \square

Corollary 4.2.12. *Let X be an irreducible topological space of finite dimension. For X to be catenary, it is necessary and sufficient that, for every pair (Y, Z) of irreducible closed subsets of X such that $Y \subseteq Z$, we have*

$$\text{codim}(Y, Z) + \text{codim}(Z, X) = \text{codim}(Y, X).$$

Proof. The condition is necessary by ?? Conversely, suppose it is true, and for each irreducible closed subset Z of X define $c(Z) = \text{codim}(Z, X)$. Then for a triple triple (Y, Z, T) of irreducible closed subsets of X such that $Y \subseteq Z \subseteq T$,

$$\text{codim}(Y, Z) + \text{codim}(Z, T) = c(Y) - c(Z) + c(Z) - c(T) = c(Y) - c(T) = \text{codim}(Y, T)$$

whence X is catenary by ?? \square

Proposition 4.2.13. *Let X be a topological space of finite dimension. Suppose that all maximal chains of closed irreducible subsets of X have the same length. Then X is catenary, and for any irreducible closed subset Z of X , we have*

$$\text{codim}(Z, X) = \dim(X) - \dim(Z).$$

Moreover, for any pair (Y, Z) of irreducible closed subsets of X such that $Y \subseteq Z$, we have

$$\text{codim}(Y, Z) = \dim(Z) - \dim(Y).$$

Proof. Let Y and Z be irreducible closed subsets of X such that $Y \subseteq Z$. Let $Y_0 \subset \cdots \subset Y_p$ be a chain such that $Y_p = Y$ and $p = \dim(Y)$, $Z_0 \subset \cdots \subset Z_q$ a chain such that $Z_0 = Z$ and $q = \text{codim}(Z, X)$. Then for any saturated chain $T_0 \subset \cdots \subset T_r$ with $T_0 = Y$, $T_r = Z$, the chain

$$Y_0 \subset \cdots \subset Y_{p-1} \subseteq T_0 \subset \cdots \subset T_r \subseteq Z_1 \subset \cdots \subset Z_q$$

is maximal with length $p + q + r$. The hypotheses on X then implies $p + q + r = \dim(X)$, whence $r = \dim(X) - \dim(Y) - \text{codim}(Z, X)$. It follows that X is catenary and that, for Y and Z as above, we have

$$\dim(Y) + \text{codim}(Y, Z) = \dim(X) - \text{codim}(Z, X).$$

Put $Z = Y$, we see the second number is equal to $\dim(Z)$, whence the proposition. \square

Chapter 5

Integral domains

5.1 Prime and irreducible elements

Two elements $a, b \in R$ are associates if and only if $(a) = (b)$,

Lemma 5.1.1. *Let a, b be nonzero elements of an integral domain R . Then a and b are associates if and only if $a = ub$, for u a unit in R .*

Proof. Assume a and b are associates. Then there are $u, v \in R$ such that

$$a = ub, \quad b = va.$$

Therefore $a = uva$, which means $uv = 1$, u is a unit. The converse is clear. \square

Definition 5.1.2. *Let R an integral domain and $a \in R$.*

- *a is prime if the ideal (a) is prime. That is, a is not a unit and*

$$a \mid bc \Rightarrow a \mid b \text{ or } a \mid c.$$

- *a is irreducible if a is not a unit and*

$$a = bc \Rightarrow b \text{ is a unit or } c \text{ is a unit.}$$

Proposition 5.1.3. *A nonunit $a \in R$ is irreducible if and only if*

- (i) *If $a = bc$ for some $b, c \in R$ then a is an associate of b or c .*
- (ii) *If $a = bc$ for some $b, c \in R$ then $(a) = (b)$ or $(a) = (c)$.*
- (iii) *The ideal (a) is maximal among proper principal ideals.*

Proof. One only need to note that $b \mid a$ if and only if $(a) \subseteq (b)$. \square

Lemma 5.1.4. *Let R be an integral domain. Then every prime element is irreducible.*

Proof. Let a be a prime element in R and assume that $a = bc$ for some $b, c \in R$, then we have $bc \in (a)$. Since a is prime, we conclude $b \in (a)$ or $c \in (a)$, say $b \in (a)$. But then $(b) \subseteq (a) = (bc) \subseteq (b)$, whence $(a) = (b)$. This proves a is irreducible. \square

Example 5.1.5. The converse is not true in general. As an example, consider the ring $\mathbb{Z}[\sqrt{5}i]$. The element 2 is irreducible. However, $2 \mid 6 = (1 + \sqrt{5}i)(1 - \sqrt{5}i)$, but 2 does not divide $1 + \sqrt{5}i$ and 2 does not divide $1 - \sqrt{5}i$.

5.2 Domains with factorizations

Definition 5.2.1. Let R be an integral domain. An element $r \in R$ has a **factorization into irreducibles** if there exist irreducible elements q_1, \dots, q_n such that $r = q_1 \cdots q_n$. This factorization is **unique** if the elements q_i are determined by r up to order and associates.

Definition 5.2.2. An integral domain R is a **domain with factorizations** if every nonzero, nonunit element $r \in R$ has a factorization into irreducibles. It is a **unique factorization domain** if every nonzero, nonunit element $r \in R$ has a unique factorization into irreducibles.

Proposition 5.2.3. Let R be an integral domain, and r nonzero and nonunit. Assume that every ascending chain of principal ideals

$$(r) \subseteq (r_1) \subseteq (r_3) \subseteq \dots$$

stabilizes. Then r has a factorization into irreducibles.

Proof. Assume that r does not have a factorization into irreducible elements. In particular, r is itself not irreducible; thus $r_1, s_1 \in R$ such that $r = r_1 s_1$ and $(r) \subset (r_1), (r) \subset (s_1)$. If both r_1, s_1 have factorizations into irreducibles, the product of these factorizations gives a factorization of r ; thus we may assume that (e.g.) r_1 does not have a factorization into irreducibles. Therefore we have a chain

$$(r) \subset (r_1) \subset (r_3) \subset \dots$$

contradicting our hypothesis. □

Corollary 5.2.4. Factorizations exist in Noetherian domains.

Example 5.2.5. $\mathbb{Z}[X_1, X_2, \dots]$ is not a Noetherian domain, but a.c.c holds for principle ideals in it, so it has factorizations.

Now we give a sufficient and necessary condition for a domain to be a UFD. First we prove some preparations.

Lemma 5.2.6. Let R be a UFD, and let a, b, c be nonzero elements of R . Then

- (i) $(a) \subseteq (b)$ iff the multiset of irreducible factors of b is contained in the multiset of irreducible factors of a .
- (ii) a and b are associates iff the two multisets coincide.
- (iii) the irreducible factors of a product bc are the collection of all irreducible factors of b and of c .

Proposition 5.2.7. In a domain in which factorization into irreducibles is possible, factorization is unique if and only if every irreducible is prime.

Proof. Let's assume that we have two different decompositions

$$p_1 \cdots p_n = q_1 \cdots q_m$$

where all p_i and q_j are irreducible. Then $p_1 \cdots p_n \in (q_1)$, and (q_1) is a prime ideal (by hypothesis); thus $p_i \in (q_1)$ for some i , which we may assume to be 1 after changing the order of the factors. Therefore $p_1 = uq_1$ for some $u \in R$. Since p_1 is irreducible and q_1 is not a unit, necessarily u is a unit. Thus p_1 and q_1 are associates. Canceling p_1, q_1 and replacing p_2 by up_2 , we find

$$p_2 \cdots p_n = q_2 \cdots q_m.$$

Repeating this process matches all factors one by one. It is clear that $m = n$, because otherwise we will obtain $1 =$ a product of irreducibles, contradicting the fact that irreducibles are not units.

Conversely, assume that factorization in R is unique, so that R is a UFD. Let $a \in R$ be irreducible. Then a is not a unit, by definition of irreducible. Assume $bc \in (a)$: thus $(bc) \subseteq (a)$, and the irreducible factors of a , that is, a itself, must be among the factors of b or of c . We have $b \in (a)$ in the first case and $c \in (a)$ in the second. This shows that (a) is a prime ideal, as needed. \square

Theorem 5.2.8. *An integral domain R is a UFD if and only if*

- The a.c.c. for principal ideals holds in R .
- Every irreducible element of R is prime.

Proof. If these conditions are satisfied, then by ?? and ?? we know that R is a UFD. Conversely, if R is a UFD, then ?? shows that irreducible elements of R are prime. To prove that the a.c.c. for principal ideals holds, consider an ascending chain

$$(r_1) \subset (r_2) \subset (r_3) \subset \cdots$$

By lemma ?? this chain determines a corresponding descending chain of irreducible factors. A descending chain of finite multisets clearly stabilizes, and it follows that the original chain also stabilizes. \square

5.3 Greatest common divisors and Bézout domains

Our next step is going to be: how do we prove that prime and irreducible are the same thing in a particular domain? The answer will be that if have GCDs and they behave well, then we can prove it. We are going to need to start by defining what a gcd is in general.

Definition 5.3.1. *Let $a, b \in R$. We say that $d \in R$ is a greatest common divisor of a and b if*

- (1) $d | a$ and $d | b$,
- (2) If $c \in R$ is another element such that $c | a$ and $c | b$, then $c | d$.

The following result is clear.

Lemma 5.3.2. Let $a, b \in R$. Let $d \in R$.

- (1) d is a common divisor of a and b iff $(a, b) \subseteq (d)$.
- (2) d is a gcd of a and b iff (d) is a minimal element in the poset of principal ideals that contain (a, b) with respect to inclusion.

Notice that based on the definition, the two elements a and b may in principle have one gcd, or various gcds, or none. It can be easily showed that if d' is another gcd of a, b , then d associates with d' . Thus in fact we define the gcd to be an associate class.

Lemma 5.3.3. Let R be an integral domain. Let $a, b, c \in R - \{0\}$, then

- (1) $\gcd(a, b) = a$ iff $a \mid b$.
- (2) $c \gcd(a, b) = \gcd(ca, cb)$.
- (3) If $\gcd(ab, c) = 1$, then $\gcd(a, c) = \gcd(b, c) = 1$.
- (4) If $\gcd(a, b) = 1$ and $\gcd(a, c) = 1$, then $(a, bc) = 1$.
- (5) If $\gcd(a, b) = 1$ and $a \mid bc$, then $a \mid c$.

Proof. (1) One direction is obvious from the definition. So suppose $a \mid b$. Then since $a \mid a$, we have $a \mid \gcd(a, b)$. But $\gcd(a, b) \mid a$, thus $\gcd(a, b) = a$.

(2) Let $d_1 = \gcd(a, b)$ and $d_2 = \gcd(ca, cb)$. Then from $cd_1 \mid ca$ and $cd_1 \mid cb$ we conclude that $cd_1 \mid d_2$. Therefore

$$\begin{aligned} ca &= x \cdot d_2 = x \cdot ucd_1, \\ cb &= y \cdot d_2 = y \cdot ucd_1. \end{aligned}$$

These then implies $ud_1 \mid a$, $ud_1 \mid b$, and therefore $ud \mid d$, so u is a unit.

- (3) This follows from $(ab, c) \subseteq (a, c)$ and $(ab, c) \subseteq (b, c)$.
- (4) Suppose that $d \mid a$ and $d \mid bc$. Then $d \mid ab$ and $d \mid bc$ and therefore $d \mid \gcd(ab, bc) = b \gcd(a, c) = b$. Since $\gcd(a, b) = 1$, we conclude that d is a unit.
- (5) Now $a \mid bc$ and $a \mid ac$, thus $a \mid \gcd(ac, bc) = c \gcd(a, b) = c$. □

Definition 5.3.4. Let R be an integral domain. Consider the following two properties:

- (1) Given any non-zero elements $ab \in R$, there exists a gcd d of a and b .
- (2) Moreover, d can be written as $d = xa + yb$ for some $x, y \in R$.

We say that R is a **GCD-domain** when it satisfies the first property. We say that R is a **Bézout domain** when it satisfies both properties.

Proposition 5.3.5. Let R be an integral domain. The the following conditions are equivalent:

- (1) R is Bezout.
- (2) For $a, b \in R$, the ideal (a, b) is principal.
- (3) Every finitely generated ideal in R is principal.

Proof. Clearly, if R is Bezout, then (2) holds. Also, an inductive argument implies that (2) and (3) are equivalent. Now assume (2), and let a, b be elements in R . Then since (a, b) is principal, by ?? we know that $\gcd(a, b)$ exists, and is a linear combination of a, b . Thus R is Bezout. \square

Proposition 5.3.6. *Every UFD is a GCD-domain.*

Proof. Let $a, b \in R$, we can write

$$a = uq_1^{\alpha_1} \cdots q_r^{\alpha_r}, \quad b = vq_1^{\beta_1} \cdots q_r^{\beta_r}$$

where u and v are units, the elements q_i are irreducible, q_i is not an associate of q_j for $i \neq j$, and $\alpha_i, \beta_i \geq 0$. We claim that

$$d = q_1^{\min(\alpha_1, \beta_1)} \cdots q_r^{\min(\alpha_r, \beta_r)}$$

is a gcd of a and b . Indeed, d is clearly a divisor of a and b ; and if c also divides a and b , then the multiset of factors of c must be contained in both multisets of factors for a and b by ??; that is,

$$c = wq_1^{\gamma_1} \cdots q_r^{\gamma_r}.$$

with w a unit and $\gamma_i \leq \alpha_i, \beta_i$. This implies $c \mid d$ again by ??, as needed. \square

Proposition 5.3.7. *Let R be a GCD domain. Then being prime is the same as being irreducible in R .*

Proof. We already know that every prime is irreducible in every domain. Let $p \in R$. Assume that p is irreducible and that $p \mid ab$ with $a, b \in R$. Let $d = \gcd(p, a)$. Since $d \mid p$, d must be either a unit or associate to p .

If d is associate to p , we have that $p \mid d$ and $d \mid a$, so that $p \mid a$ and we are done. If d is a unit, then we have

$$\gcd(p, a) = 1 \text{ and } p \mid ab.$$

Thus by ?? we conclude $p \mid b$. \square

Theorem 5.3.8. *An integral domain R is a UFD if and only if*

- R is a GCD domain.
- The a.c.c. for principal ideals holds in R .

Proof. By ?? and ??, the given conditions imply that R is a UFD. Conversely, if R is a UFD, then by ?? and ??, the conditions are satisfied. \square

Theorem 5.3.9. *If R is a PID, then R is a Bézout domain and a UFD.*

Proof. Let $a, b \in R$. Then the ideal (a, b) is principal. Write $(a, b) = (d)$, then by ?? d is a gcd of a and b . Moreover, d is a linear combination of a and b . Thus R is a Bézout domain. Since R is Noetherian, R is a UFD by ??.

5.4 Unique factorization in polynomial rings

Lemma 5.4.1. *Let R be a ring, and let I be an ideal of R . Then*

$$\frac{R[X]}{IR[X]} = \frac{R}{I}[X].$$

Therefore, if I is a prime ideal of R , then $IR[X]$ is prime in $R[X]$.

Definition 5.4.2. *Let R be a commutative ring, and let $f = \sum_{i=1}^n a_i X^i \in R[X]$ be a polynomial.*

- f is **very primitive** if for all prime ideals \mathfrak{p} of R , $f \notin \mathfrak{p}R[X]$.
- f is **primitive** if for all principal prime ideals \mathfrak{p} of R , $f \notin \mathfrak{p}R[X]$.

Lemma 5.4.3. *Let R be a commutative ring and $f \in R[X]$ as above.*

- f is very primitive if and only if $(a_0, \dots, a_n) = 1$.
- If R is a UFD, then f is primitive if and only if $\gcd(a_0, \dots, a_n) = 1$.

Proof. That f is very primitive means the coefficients of f are not all contained in any one prime ideal, thus is equivalent to $(a_0, \dots, a_n) = 1$.

For the second point note that, in a UFD, $\gcd(a_0, \dots, a_n) \neq 1$ if and only if there exists an irreducible element $q \in R$ such that $(a_0, \dots, a_n) \subseteq (q)$. As (q) is then prime, the second point follows as well. \square

Lemma 5.4.4. *Let R be a commutative ring. Then for $f, g \in R[X]$,*

$$fg \text{ is primitive} \iff \text{both } f \text{ and } g \text{ are primitive.}$$

Proof. This is an easy consequence of ??:

$$\begin{aligned} fg \text{ is primitive} &\iff \forall \mathfrak{p} \text{ prime and principal in } R, fg \notin \mathfrak{p}R[X] \\ &\iff \forall \mathfrak{p} \text{ prime and principal in } R, f \notin \mathfrak{p}R[X] \text{ and } g \notin \mathfrak{p}R[X] \\ &\iff f \text{ is primitive and } g \text{ is primitive.} \end{aligned}$$

Thus the claim follows. \square

Definition 5.4.5. *Let R be a UFD. The **content** of a nonzero polynomial $f \in R[X]$, denoted cont_f , is the gcd of its coefficients.*

The proof of the following remarks is immediate from the definition.

Lemma 5.4.6. *Let R be a UFD, and let $f \in R[X]$. Then*

- $(f) = (\text{cont}_f)(f_1)$ where f_1 is primitive.
- if $(f) = (c)(g)$, with $c \in R$ and g primitive, then $(c) = (\text{cont}_f)$.

Now we prove a useful result about primitive polynomials.

Proposition 5.4.7 (Gauss's lemma). *Let R be a UFD, and let $f, g \in R[X]$. Then*

$$(\text{cont}_{fg}) = (\text{cont}_f)(\text{cont}_g).$$

Proof. This follows easily from our preparatory work. Write

$$(fg) = (\text{cont}_f)(f_1)(\text{cont}_g)(g_1) = (\text{cont}_f)(\text{cont}_g)(f_1g_1)$$

with f_1, g_1 primitive. By ?? f_1g_1 is primitive, thus $(\text{cont}_f)(\text{cont}_g)$ is the content of fg , as needed. \square

Now we consider the field of fractions of R . We first prove some consequences of Gauss's lemma.

Proposition 5.4.8. *Let R be a UFD and $f \in R[X]$. Suppose that $f = gh$ where $g, h \in K[X]$. Then there exists $\lambda \in K^\times$ such that $\lambda g, \lambda^{-1}h \in R[X]$.*

Proof. Let $c, d \in K$ be such that

$$g = cg_1, \quad h = dh_1,$$

where $g_1, h_1 \in R[X]$ are primitive. Then g_1h_1 is also primitive and from $cdg_1h_1 \in R[X]$ we conclude that $u := cd \in R$. Set $\lambda = d$, we then get $\lambda g = cdg_1 \in R[X]$ and $\lambda^{-1}h = h_1 \in R[X]$. \square

This result is also called Guass's lemma, and is used for example in algebraic number theory (the case $R = \mathbb{Z}$ and $K = \mathbb{Q}$).

Corollary 5.4.9. *Let R be an UFD with fraction field K . Let $f \in R[X]$ be a monic polynomial and suppose that f factors in $K[X]$ as gh with g and h monic. Then g and h have coefficients in R .*

Proof. By ?? there exists $\lambda \in K^\times$ such that $\lambda g, \lambda^{-1}h \in R[X]$. Since g, h are monic, it follows that $\lambda = 1$ and the claim holds. \square

Before we preceed to give our main theorem, we prove a lemma that will be used later.

Lemma 5.4.10. *Let R be a UFD, and let K be its field of fractions. For nonzero $f, g \in R[X]$, denote by $(f)_K, (g)_K$ the principal ideals $fK[X], gK[X]$ in $K[X]$. Then $(f) \subseteq (g)$ if and only if the following conditions are satisfied:*

- $(\text{cont}_f) \subseteq (\text{cont}_g)$.
- $(f)_K \subseteq (g)_K$.

The claim still holds if we replace all the inclusions by equality.

Proof. The condition $(f) \subseteq (g)$ clearly implies $(f)_K \subseteq (g)_K$ and $(\text{cont}_f) \subseteq (\text{cont}_g)$. Thus we may assume the converse.

Since $(f)_K \subseteq (g)_K$ means $f = gh$, where $h \in K[X]$. Write $h = a/b \cdot h_1$ where $a, b \in R$ and h_1 is primitive, we then have

$$bf = agh_1.$$

Since h_1 is primitive, by Gauss's lemma we conclude

$$(b \operatorname{cont}_f) = (a \operatorname{cont}_g).$$

By hypothesis we have $(b \operatorname{cont}_f) \subseteq (b \operatorname{cont}_g)$, thus

$$(a \operatorname{cont}_g) \subseteq (b \operatorname{cont}_g).$$

Since R is an integral domain and $(\operatorname{cont}_g) \neq (0)$, this implies $a = bc$ for some $c \in R$. But then $h = ch_1 \in R[X]$, so that $(f) \subseteq (g)$. \square

The first application is the following description of the irreducible elements of $R[X]$.

Proposition 5.4.11 (Irreducible criterion for polynomials of UFDs). *Let R be a UFD, and let K be its field of fractions. Let $f \in R[X]$ be a nonconstant polynomial. Then f is irreducible in $R[X]$ if and only if it is irreducible in $K[X]$ and primitive in $R[X]$.*

Proof. Assume that f is irreducible in $R[X]$, then f is primitive, otherwise we could factor out its content. Next, assume that $f = gh$, with $g, h \in K[X]$; we have to prove that either g or h is a unit in $K[X]$. Let $c, d \in K$ be such that

$$g = cg_1, \quad h = dh_1,$$

where $g_1, h_1 \in R[X]$ are primitive. Then g_1h_1 is also primitive and we have $(\operatorname{cont}_{g_1h_1}) = (1) = (\operatorname{cont}_f)$. Further, we have $(f)_K = (gh_K) = (g_1h_1)_K$, thus by ?? we obtain $(f) = (g_1h_1)$. As f is irreducible in $R[X]$, this implies that either g_1 or h_1 is a unit in $R[X]$. But then g or h were units in $K[X]$, verifying that f is irreducible in $K[X]$.

Conversely, assume that f is irreducible in $K[X]$ and primitive. If $f = gh$ for $g, h \in R[X]$, then view this equation as in $K[X]$, we find that g or h is a unit in $K[X]$. That is, $g \in K$ or $h \in K$. Assume that $g \in K$. Since $R[X] \cap K = R$ and f is primitive, g is then a unit in R . This implies that f is irreducible in $R[X]$. \square

Theorem 5.4.12 (Gauss). *If R is a UFD, then $R[X]$ is a UFD.*

Proof. We begin by verifying the a.c.c. for principal ideals in $R[X]$. Let

$$(f_1) \subseteq (f_2) \subseteq (f_3) \subseteq \cdots$$

be an ascending chain of principal ideals of $R[X]$. By ??, this induces ascending chains of principal ideals

$$(\operatorname{cont}_{f_1}) \subseteq (\operatorname{cont}_{f_2}) \subseteq (\operatorname{cont}_{f_3}) \subseteq \cdots \quad \text{and} \quad (f_1)_K \subseteq (f_2)_K \subseteq (f_3)_K \subseteq \cdots.$$

Since R and $K[X]$ are both UFDs, these chains stabilize. Thus by ?? again the original chain also stabilizes.

Next, we consider an irreducible element f of $R[X]$. By ?? f is primitive and irreducible in $K[X]$, thus is prime in $K[X]$. Consider the composition

$$\rho : R[X] \hookrightarrow K[X] \twoheadrightarrow \frac{K[X]}{(f)_K}.$$

We claim that $\ker \rho = (f)$. In fact, for $g \in R[X]$, $\rho(g) = 0$ is equivalent to $(g)_K \subseteq (f)_K$. Since we always have $(\text{cont}_g) \subseteq (\text{cont}_f) = 1$, by ?? this is exactly $(g) \subseteq (f)$. Since $\ker \rho = (f)$, we find that ρ induces an injective homomorphism

$$\tilde{\rho} : \frac{R[X]}{(f)} \hookrightarrow \frac{K[X]}{(f)_K}.$$

Since the ring on the right is an integral domain, this implies $R[X]/(f)$ is also an integral domain. Thus (f) is prime in $R[X]$. \square

Corollary 5.4.13. *If R is a UFD, then $R[X_1, \dots, X_n]$ is a UFD.*

Example 5.4.14. Let R be a UFD. Since every elements of $R[X_1, X_2, \dots]$ can be embedded to $R[X_1, \dots, X_n]$ for some n , we find that $R[X_1, X_2, \dots]$ is also a UFD. However, $R[X_1, X_2, \dots]$ is not Noetherian, thus we get an example of a non-Noetherian UFD.

Irreducibility of a polynomial in several variables is more difficult to determine than irreducibility of a polynomial of one variable, but here is one criterion.

Proposition 5.4.15. *Let k be a field, and view $f(X_1, \dots, X_n) \in k[X_1, \dots, X_n]$ as a polynomial in $R[X_n]$, where $R = k[X_1, \dots, X_n]$:*

$$f(X_n) = a_0(X_1, \dots, X_{n-1}) + \dots + a_m(X_1, \dots, X_{n-1})X_n^m.$$

If $f(X_n)$ is primitive and cannot be factored into two polynomials of lower degree in $R[X_n]$, then $f(X_1, \dots, X_n)$ is irreducible in $k[X_1, \dots, X_n]$.

Proof. Suppose that $f(X_n) = g(X_n)h(X_n)$ in $R[X_n]$; by hypothesis, $\deg g = 0$ or $\deg h = 0$. It follows, because f is primitive, that g or h is a unit in R . Therefore, $f(X_1, \dots, X_n)$ is irreducible in $k[X_1, \dots, X_n]$. \square

Corollary 5.4.16. *If $g(X_1, \dots, X_n)$ and $h(X_1, \dots, X_n)$ are relatively prime in $k[X_1, \dots, X_n]$, then $f(X_1, \dots, X_n, Y) = Yg(X_1, \dots, X_n) + h(X_1, \dots, X_n)$ is irreducible in $k[X_1, \dots, X_n, Y]$.*

Proof. Let $R = k[X_1, \dots, X_n]$. Note that f is primitive in $R[Y]$, because $(g, h) = 1$ forces any divisor of its coefficients g, h to be a unit. Since f is linear in Y , it is not the product of two polynomials in $R[Y]$ of smaller degree, and hence ?? shows that f is irreducible in $R[Y] = k[X_1, \dots, X_n, Y]$. \square

As an application we consider irreducibility in $\mathbb{Z}[X]$.

Proposition 5.4.17. *Let $f \in \mathbb{Z}[X]$ be a primitive polynomial, and let p be a prime integer. Assume $f \pmod p$ has the same degree as f and is irreducible in $\mathbb{Z}/p\mathbb{Z}[X]$. Then f is irreducible in $\mathbb{Z}[X]$.*

Proof. Argue contrapositively: if f is primitive and reducible in $\mathbb{Z}[X]$ and $\deg f = n$, then $f = gh$ with $\deg g = d, \deg h = e, d + e = n$. But then the same can be said of $f \pmod p$, so $f \pmod p$ is also reducible. \square

The hypothesis on degrees is necessary, precisely to take care of the special cases mentioned in the discussion preceding the statement. For example, $2X^3 + 3X^2 + 3X + 1$ equals $X^2 + X + 1 \pmod 2$, and the latter is irreducible in $\mathbb{Z}/2\mathbb{Z}[X]$; yet $2X + 1$ is a factor of the former. The point is of course that $2X + 1$ is a unit mod 2.

Remark 5.4.1. There are irreducible polynomials in $\mathbb{Z}[X]$ which are reducible modulo all primes! So ?? cannot be turned into a characterization of irreducibility in $\mathbb{Z}[X]$. It is however rather useful in producing examples.

Corollary 5.4.18. *There are irreducible polynomials in $\mathbb{Z}[X]$ and $\mathbb{Q}[X]$ of arbitrarily large degree.*

Proof. By ??, the statement for $\mathbb{Z}[X]$ implies the one for $\mathbb{Q}[X]$. By ??, it suffices to verify that there are irreducible polynomials in $\mathbb{Z}/p\mathbb{Z}[X]$ of arbitrarily large degree, for any prime integer p . This is a particular case of ??.

The following result is commonly established in $\mathbb{Z}[X]$, but it holds for general rings.

Theorem 5.4.19 (Eisenstein's criterion). *Let R be a commutative ring, and let \mathfrak{p} be a prime ideal of R . Let*

$$f = a_n X^n + \cdots + a_1 X + a_0 \in R[X]$$

be a polynomial, and assume that

- $a_n \notin \mathfrak{p}$.
- $a_i \in \mathfrak{p}$ for $0 \leq i \leq n-1$.
- $a_0 \notin \mathfrak{p}^2$.

Then f is not the product of polynomials of degree $< n$ in $R[X]$.

Proof. Argue by contradiction. Assume $f = gh$ in $R[X]$, with both $d = \deg g$ and $e = \deg h$ less than n ; write

$$g = b_d X^d + \cdots + b_1 X + b_0, \quad h = c_e X^e + \cdots + c_1 X + c_0.$$

and note that necessarily $d > 0$ and $e > 0$. Consider f modulo \mathfrak{p} : thus

$$\bar{f} = \overline{gh} \text{ in } (R/\mathfrak{p})[X].$$

By hypothesis, $\bar{f} = a_n x^n$ modulo \mathfrak{p} , where $a + n \neq 0$ in R/\mathfrak{p} . Since R/\mathfrak{p} is an integral domain, factors of \bar{f} must also be monomials: that is, necessarily

$$\bar{g} = \overline{b_d} X^d, \quad \bar{h} = \overline{c_e} X^e.$$

Since $d, e > 0$, this implies $b_0 \in \mathfrak{p}, c_0 \in \mathfrak{p}$. But then $a_0 = b_0 c_0 \in \mathfrak{p}^2$, contradiction. \square

Example 5.4.20. For all n and all primes p , the polynomial $x^n - p$ is irreducible in $\mathbb{Z}[X]$. This follows immediately from Eisenstein's criterion and gives an alternative proof of ??.

Proposition 5.4.21. *The polynomial*

$$\frac{X^p - 1}{X - 1} = X^{p-1} + X^{p-2} + \cdots + 1$$

is irreducible over \mathbb{Z} if and only if p is a prime.

Proof. If p is prime, plug into $X = X + 1$ yields

$$\frac{(X+1)^p - 1}{X} = X^{p-1} + \binom{p}{p-1} X^{p-2} + \dots + \binom{p}{1}$$

thus we only need to show p divides $\binom{p}{k}$ for all $k = 1, \dots, p-1$, and apply Eisenstein's criterion. But we have

$$\binom{p}{k} = \frac{p!}{k!(p-k)!}$$

and p divides the numerator and does not divide the denominator; the claim follows as p is prime.

For p not a prime, we write $p = mn$, and note that

$$\frac{X^p - 1}{X - 1} = \frac{X^{mn} - 1}{X - 1} = \frac{X^{mn} - 1}{X^m - 1} \cdot \frac{X^m - 1}{X - 1}$$

so the polynomial is reducible. \square

5.5 A prime ideal principle

We start with the two crucial definitions

Definition 5.5.1. An ideal family \mathcal{F} in a ring R with $R \in \mathcal{F}$ is said to be an **Oka family** if for $a \in R$ and $I \subseteq R$ an ideal,

$$(I, a) \in \mathcal{F}, (I : a) \in \mathcal{F} \Rightarrow I \in \mathcal{F}.$$

It is a **strongly Oka family** if for I, J ideals of R ,

$$(I, J) \in \mathcal{F}, (I : J) \in \mathcal{F} \Rightarrow I \in \mathcal{F}.$$

Definition 5.5.2. An ideal family \mathcal{F} in a ring R with $R \in \mathcal{F}$ is said to be an **Ako family** if for $a, b \in R$ and $I \subseteq R$ an ideal,

$$(I, a) \in \mathcal{F}, (I, b) \in \mathcal{F} \Rightarrow (I : ab) \in \mathcal{F}.$$

It is a **strongly Ako family** if for I, J ideals of R ,

$$(I, a) \in \mathcal{F}, (I, J) \in \mathcal{F} \Rightarrow (I, aJ) \in \mathcal{F}.$$

The following connections between the definitions are self-evident.

Proposition 5.5.3.

- (1) A strongly Oka family is Oka, and conversely if R is a principal ideal ring.
- (2) A strongly Ako family is Ako, and conversely if R is a principal ideal ring.

We will use \mathcal{F}^c to denote the set of all ideals of R not belonging to \mathcal{F} , and $\text{Max}(\mathcal{F}^c)$ for the set of maximal elements of \mathcal{F}^c with respect to the partial ordering given by inclusion. We will say \mathcal{F}^c is an **MP-family** if $\text{Max}(\mathcal{F}^c) \subseteq \text{Spec}(R)$.

With the notions of Oka and Ako families in place, we can now formulate the following general result.

Theorem 5.5.4. If \mathcal{F} is an Oka family or an Ako family, then \mathcal{F}^c is an MP-family, that is, $\text{Max}(\mathcal{F}^c) \subseteq \text{Spec}(R)$.

Proof. Suppose some $I \in \text{Max}(\mathcal{F}^c)$ is not prime. Since $I \neq R$, there exist $a, b \in R - I$ such that $ab \in I$. Then

$$(I, b), (I : a) \supseteq I, \quad (I, a), (I : b) \supseteq I.$$

Therefore, $(I, a), (I, b)$ and $(I : a), (I : b)$ all belong to \mathcal{F} . But $I = (I, ab) \notin \mathcal{F}$, so \mathcal{F} is neither Oka nor Ako. \square

Remark 5.5.1. The converse of ?? does not hold in general. For instance, let $(R, (t))$ be a DVR, and let $\mathcal{F} = \{(t)^i \mid i \neq 1, 4\}$. Then $\mathcal{F}^c = \{(0), (t), (t)^4\}$ is an MP-family since $\text{Max}(\mathcal{F}^c) = \{\mathfrak{m}\} \subseteq \text{Spec}(R)$. But \mathcal{F} is not Oka since $I := (t)^4 \notin \mathcal{F}$, but $(I, t^2) = (I : t^2) = (t)^2 \in \mathcal{F}$. From $(I, t^4) = (t^4) = I \notin \mathcal{F}$, we see that \mathcal{F} is also not Ako.

Proposition 5.5.5. For an ideal family \mathcal{F} in R with $R \in \mathcal{F}$, consider the following property, where I, J denote arbitrary ideals in R :

- (1) $I \subseteq J$ and $J, (I : J) \in \mathcal{F} \Rightarrow I \in \mathcal{F}$ if J/I is cyclic.
- (2) $I \subseteq J$ and $J, (I : J) \in \mathcal{F} \Rightarrow I \in \mathcal{F}$.

Then the condition Oka is equivalent to (1), and strongly Oka to (2).

Proof. If \mathcal{F} satisfies (2), then for $I, J \subseteq R$ with $(I, J) \in \mathcal{F}, (I : J) \in \mathcal{F}$, we have

$$I \subseteq (I, J) \quad \text{and} \quad (I : (I, J)) = (I : J)$$

Therefore by (2) we get $I \in \mathcal{F}$, hence \mathcal{F} is strongly Oka. Conversely, if \mathcal{F} is strongly Oka, since for ideals $I \subseteq J$ we have $(I, J) = J$, the condition (2) clearly holds. \square

Theorem 5.5.6. Let S be a multiplicative set in a ring R . An ideal $I \subseteq R$ that is maximal with respect to being disjoint from S is prime.

Proof. Let $\mathcal{F} = \{I \subseteq R \mid I \cap S \neq \emptyset\}$ be the set of ideals touching S . We claim that \mathcal{F} is an Ako family. In fact, if $(I, a) \in \mathcal{F}$ and $(I, b) \in \mathcal{F}$, then there are $k_1, k_2 \in R$ and $r_1, r_2 \in I$ such that

$$x = k_1a + r_1 \in S, \quad y = k_2b + r_2 \in S$$

Then since S is multiplicative, we have

$$S \ni xy = k_1k_2ab + k_1ar_2 + k_2br_1 + r_1r_2 \in (I, ab).$$

Therefore $(I, ab) \in \mathcal{F}$. \square

Theorem 5.5.7. Let I be an ideal in a ring R that is maximal with respect to not being finitely generated. Then I is prime.

Proof. Let \mathcal{F} be the set of all ideals of R that are finitely generated. If (I, x) and $(I : x)$ are finitely generated, we claim that I is also finitely generated. In fact, if

$$(I, x) = (p_1 + r_1x, p_2 + r_2x, \dots, p_n + r_nx), \quad p_i \in I, r_i \in R.$$

Let $I_0 = (p_1, \dots, p_n)$, for $p \in I$ we have

$$p = \sum_{i=1}^n c_i(p_i + r_i x) = \sum_{i=1}^n c_i p_i + x \sum_{i=1}^n c_i r_i$$

Note that $x \sum c_i r_i = p - \sum c_i p_i \in I$, so it is in $x(I : x)$. Therefore we have proved that

$$I = I_0 + x(I : x)$$

Now $(I : x)$ is finitely generated, thus I is also finitely generated.

This means \mathcal{F} is a Oka family, so the claim follows. \square

Theorem 5.5.8. *Let I be an ideal in a ring R that is maximal with respect to being nonprincipal. Then I is prime.*

Proof. Let \mathcal{F} be the set of all principal ideals of R . We show that \mathcal{F} satisfies the condition (2) in ??, thus is strongly Oka. For $I \subseteq J$ ideals such that $J, (I : J) \in \mathcal{F}$, write $J = (x)$ then we observe that

$$I = x \cdot (I : x) \in \mathcal{F}$$

since $I \subseteq J = (x)$. Therefore the claim follows. \square

5.6 Some results on UFDs and PIDs

5.6.0.0.1 On UFDs

Proposition 5.6.1 (Kaplansky's theorem). *An integral domain is a UFD iff every nonzero prime ideal contains a nonzero prime element.*

Proof. Let R be a UFD and \mathfrak{p} a nonzero prime ideal of R . Choose $0 \neq x \in \mathfrak{p}$ and write $x = p_1 \cdots p_r$ into irreducibles. Then since \mathfrak{p} is prime, one of the p_i 's is in \mathfrak{p} . Since R is a UFD, p_i 's are prime, so this gives one direction.

Conversely, assume the condition on prime ideals, if we can prove that every elements of R can be written into products of prime elements, then clearly R is a UFD. To show this, let

$$S = \{x \in R - \{0\} \mid x \text{ is a unit or } x \text{ can be written into product of prime elements}\}$$

Then clearly S is a multiplicative subset, so that a maximal element $I \subseteq R$ with respect to being disjoint from S is prime. If $I \neq 0$, then I contains a nonzero prime, which implies $I \cap S \neq \emptyset$, contradiction. Therefore $I = (0)$ and we get $S = R - \{0\}$, verifying our claim. \square

Corollary 5.6.2. *If R is a UFD, then every prime ideal of height 1 in R is principal.*

Proof. Since R is an integral domain, a prime \mathfrak{p} has height 1 means \mathfrak{p} does not properly contain any prime ideals. In particular $\mathfrak{p} \neq 0$, so by ?? \mathfrak{p} contains a nonzero prime element p . Since (p) is then a prime ideal contained in \mathfrak{p} , we must have $\mathfrak{p} = (p)$. \square

Proposition 5.6.3. *A Noetherian domain R is a UFD iff every prime ideal of height 1 in R is principal.*

Proof. One direction is showed in the previous corollary, so assume the later condition. Since the a.c.c. for principal ideals are satisfied, we only need to show that every irreducible element is prime. Let x be irreducible and \mathfrak{p} be a minimal prime of (x) , then since x is not a unit and not a zero-divisor, by Krull principal ideal theorem \mathfrak{p} has height 1. By our assumption \mathfrak{p} is principal, so write $\mathfrak{p} = (p)$. Clearly p is a prime element, and $(x) \subseteq (p)$. Since x is irreducible, we then conclude $(x) = (p) = \mathfrak{p}$, showing that x is prime. \square

5.6.0.0.2 On PIDs

Theorem 5.6.4. *Let R be an integral domain. Then R is a PID iff every prime ideal of R is principal.*

Proof. This is a consequence of ?? \square

Definition 5.6.5. *Let R be an integral domain, a function $v : R \rightarrow \mathbb{Z}^{\geq 0}$ is called a **Dedekind-Hasse norm** if it satisfies the following conditions:*

- (1) $v(a) = 0$ if and only if $a = 0$.
- (2) For any nonzero elements a and b in R either $b \mid a$ or there exist elements x and y in R such that

$$0 < v(xa - yb) < v(b).$$

Proposition 5.6.6. *An integral domain is a PID if and only if there is a Dedekind-Hasse norm on it.*

Proof. Assume the existence of the norm v , we prove that R is a PID. Let I be a nonzero ideal of R . We consider the set

$$S = \{x \mid x = v(a), 0 \neq a \in I\} \subseteq \mathbb{Z}^{\geq 0}$$

Then S has a minimal element, say x_0 . If $v(a_0) = x_0$ is the least of the v -values of nonzero elements of I , then $0 < v(a_0) \leq v(a)$ if a_0 does not divide a .

For $0 \neq a \in I$, there exists a pair $x, y \in R$ such that

$$0 < v(ax - a_0y) < v(a_0).$$

Since $ax - a_0y \in I$, this contradicts the choice of a_0 . Therefore, $a_0 \mid a$ for all $a \in I$, and I is principal.

Suppose that R is a PID. We define

$$v(a) = \begin{cases} 0 & \text{if } a = 0 \\ 2^n & \text{if } a \neq 0 \text{ and } n \text{ is the number of irreducible factors of } a \end{cases}$$

Then $v(a) = 0$ iff $a = 0$ by our definition. For nonzero elements $a, b \in R$ with $b \nmid a$, we write $I = (a, b)$. Since I is principal, $I = (d)$ for some $d \in R$. Then there exists a pair $x, y \in R$ such that $ax + by = d$, and from $d \mid b$ we get

$$0 < v(ax + by) = v(d) < v(b).$$

Thus v is a Dedekind-Hasse norm for R . \square

Proposition 5.6.7. *An integral domain is a PID iff it is Bezout and satisfies the a.c.c. for principal ideals.*

Proof. One direction is clear, assume the other. By ?? we know that R is a UFD, so we can define a norm as in ?? . The Bezout condition implies that this norm is a Dedekind-Hasse norm, thus R is a PID. \square

5.6.0.0.3 On UFD into PID

Proposition 5.6.8. *Let R be a UFD. Then R is a PID iff $\dim R = 1$.*

Proof. If R is a PID, then prime ideals are maximal in R , so $\dim R = 1$. Conversely, if $\dim R = 1$, then every nonzero prime ideal has height 1. By ?? they are principal. Now apply ?? . \square

Proposition 5.6.9. *Let R be a UFD. Then R is a PID iff maximal ideals of R are principal.*

Proof. Let \mathfrak{p} be a nonzero prime ideal of R . Then \mathfrak{p} is contained in a maximal ideal \mathfrak{m} , which by assumption is principal: $\mathfrak{m} = (x)$. Since R is a UFD, by ?? , \mathfrak{p} contains a nonzero prime element p . Then we have $(p) \subseteq (x) = \mathfrak{m}$. As p is irreducible, (p) is maximal among principal ideals, thus $(p) = (x) = \mathfrak{m}$. We then get $\mathfrak{p} = \mathfrak{m}$ is principal, and therefore R is a PID by ?? . \square

5.6.0.0.4 Fermat's Last Theorem for Polynomials

Theorem 5.6.10. *Let $f, g, h \in \mathbb{C}[X]$ be nonconstant polynomials such that no irreducible divides all of f, g, h . If*

$$f^n + g^n = h^n.$$

then $n \leq 2$.

Note that we cannot improve on this bound of $n \leq 2$. Indeed, if $f \in \mathbb{C}[X]$ then it is always true that

$$(1 - f^2)^2 + (2f)^2 = (1 + f^2)^2.$$

We present two proofs.

Proof 1: Induction and Roots of Unity. Because we are working over \mathbb{C} we have all roots of unity. Thus we can factor

$$h^n = f^n + g^n = (f + g)(f + \zeta g) \cdots (f + \zeta^{n-1}g) = \prod_{i=0}^{n-1} (f + \zeta^i g)$$

where $\zeta = e^{2\pi i/n}$. Note that for $i \neq j$, $f + \zeta^i g$ and $f + \zeta^j g$ are coprime. Indeed

$$f + \zeta^i g - (f + \zeta^j g) = (\zeta^i - \zeta^j)g, \quad f + \zeta^i g - (\zeta^i - \zeta^j) \frac{\zeta^i}{\zeta^i - \zeta^j} g = f.$$

so that $\gcd(f + \zeta^i g, f + \zeta^j g) = \gcd(f, g) = 1$. Now we are prepared for the first proof.

Let $\deg f + \deg g = d$ and $n \geq 3$. By above,

$$h^n = \prod_{i=0}^{n-1} (f + \zeta^i g)$$

These are pairwise coprime polynomials. Since h^n factors into n -th powers, each $f + \zeta^i g$ is also a n -th power.

We proceed by inducting on d . If $d = 2$, then f, g are linear and this is clearly impossible by degree considerations. Now suppose the theorem holds for all degrees less than d where $d > 2$. We have $f + g = a^n, f + \zeta g = b^n, f + \zeta^2 g = c^n$ (this is where the $n > 2$ hypothesis enters). Then

$$-\zeta a^n + (1 + \zeta) b^n = -\zeta f - \zeta g + (1 + \zeta)(f + \zeta g) = f + \zeta^2 g = c^n$$

Thus we obtain a relation $(\lambda a)^n + (\mu b)^n = (\nu c)^n$, where $\lambda, \mu, \nu \in \mathbb{C}$. Since a, b, c has degree strictly small than f, g , we get the claim. \square

Theorem 5.6.11 (Mason's theorem). *Let F be any field. Suppose $f, g, h \in F[X]$ are nonzero and that f, g, h are coprime. If f, g, h satisfy the equation $f + g = h$, then either $D(f) = D(g) = D(h) = 0$, or*

$$\max\{\deg f, \deg g, \deg h\} \leq \deg \sqrt{fgh} - 1.$$

Proof 2: Computations of Degrees. Suppose there exist $f, g, h \in \mathbb{C}[X]$ nonconstant, coprime such that

$$f^n + g^n = h^n.$$

Then by Mason's Theorem we have

$$\max\{\deg f^n, \deg g^n, \deg h^n\} \leq \deg \sqrt{fgh} - 1 \leq \deg f + \deg g + \deg h - 1.$$

The maximum of a finite set is at least the mean, so the left side is at least

$$\frac{\deg f^n + \deg g^n + \deg h^n}{3} = \frac{n}{3}(\deg f + \deg g + \deg h) \geq \deg f + \deg g + \deg h$$

contradiction. \square

5.7 Exercise

Exercise 5.7.1. Prove that the inclusion $i : \mathbb{Z} \rightarrow \mathbb{Q}$ is an epimorphism of rings, though it is not surjective.

Proof. If there are two ring homomorphism α_1, α_2 such that $\alpha_a|\mathbb{Z} = \alpha_2|\mathbb{Z} := \alpha$, then

$$\alpha_i\left(\frac{p}{q}\right) = \alpha_i(p)\alpha_i(q^{-1}) = \alpha(p)(\alpha(q))^{-1}$$

so they coincide on \mathbb{Q} , which show that i is an epimorphism of rings. \square

Exercise 5.7.2. Prove that the polynomial $(x - 1)(x - 2) \cdots (x - n) - 1$ is irreducible in $\mathbb{Q}[X]$ for all $n \geq 1$.

Proof. Suppose $h = fg$ is a nontrivial factorization of h where h is the polynomial above, then we see $f(i)g(i) = -1$ for $i = 1, \dots, n$. So $f(i), g(i) = \pm 1$ and $f(i) = -g(i)$. Now the polynomial $f(x) + g(x)$ has n roots with its degree strictly less than n , hence is identically zero.

So we conclude $f(x) = -g(x)$ and $h = -f^2$. But $h(x)$ also takes positive values, which is a contradiction. \square

Exercise 5.7.3. Prove that the polynomial $(x-1)(x-2)\cdots(x-n)+1$ is irreducible in $\mathbb{Q}[X]$ for all $n \geq 1, n \neq 4$.

Proof. Suppose $h = fg$ is a nontrivial factorization of h where h is the polynomial above, then we see $f(i)g(i) = 1$ for $i = 1, \dots, n$. Similar to the exercise above, we get $h = f^2$, so n must be even, let $n = 2k$.

The cases $n = 2$ and $n = 4$ are easy to verify, so we assume $n \geq 6$. Now $f(x)$ takes the value ± 1 on the set $\{1, \dots, 2k\}$, let denotes I, J the subset on which $f(x) = 1, f(x) = -1$ respectively, and assume $1 \in I$. Since $k \geq 3$, J has at least 3 distinct elements greater than 1, so there exists an element u in J such that $u-1 \geq 3$. But we must have $u-1 \mid f(u)-f(a) = -2$, contradiction. (We have $a-b \mid f(a)-f(b)$ since $a-b \mid a^n-b^n$ is true for every n .) \square

Exercise 5.7.4. Let R be a ring. The opposite ring R° is obtained from R by reversing the multiplication: that is, the product $a \bullet b$ in R° is defined to be $ba \in R$. Prove that the identity map $R \rightarrow R^\circ$ is an isomorphism if and only if R is commutative. Prove that $M_n(R)$ is isomorphic to its opposite (not via the identity map!). Explain how to turn right- R -modules into left- R -modules and conversely, if $R \simeq R^\circ$. (by this exercise, we can identify left and right module if we are considering commutative rings)

Proof. The first claim is clear, for the second, suppose $\rho : R \times M \rightarrow M$ is a left action on M . Now we define a new action σ on M by R° through:

$$\sigma : M \times R \rightarrow M, \quad (m, r) \mapsto \rho(r, m)$$

We verify that

$$\begin{aligned} \sigma(m, r+s) &= \rho(r+s, m) = \rho(r, m) + \rho(s, m) = \sigma(m, r) + \sigma(m, s) \\ \sigma(m+n, r) &= \rho(r, m+n) = \rho(r, m) + \rho(r, n) = \sigma(m, r) + \sigma(n, r) \\ \sigma(m, r \bullet s) &= \rho(r \bullet s, m) = \rho(sr, m) = \rho(s, \rho(r, m)) = \rho(s, \sigma(m, r)) = \sigma(\sigma(m, r), s) \\ \sigma(m, 1) &= \rho(1, m) = m \end{aligned}$$

this shows that σ is a right action. \square

Exercise 5.7.5. Let G be an abelian group. Prove that if G has a structure of \mathbb{Q} -vector space, then it has only one such structure.

Proof. For any $g \in G, g \neq 0$, we know g has infinite order since $ng = 0$ implies $g = 0$. Since the map ng and $\frac{1}{n}g$ are uniquely determined, G can only have one \mathbb{Q} -vector space structure. \square

Exercise 5.7.6. An abelian group A is a vector space over \mathbb{Q} if and only if it torsion free and is divisible.

Exercise 5.7.7. Let R be a commutative ring, and let I, J be ideals of R . Prove that $I \cdot (R/J) \simeq (I+J)/J$ as R -modules.

Proof. Since we have

$$\frac{I+J}{J} \simeq \frac{I}{I \cap J}$$

we only need to show $I/(I \cap J) \simeq I \cdot (R/J)$. Define a homomorphism:

$$\psi : I \rightarrow I \cdot \left(\frac{R}{J}\right), \quad i \mapsto i(1+J) = i+J$$

the kernel of ψ is exactly $I \cap J$. Every element in $I \cdot (R/J)$ has the form

$$ir + J \quad \text{for } i \in I, r \in R.$$

Since I is an ideal, $ir \in I$, so $\psi(ir) = ir + J$, which gives the surjectivity of ψ . So we find

$$\frac{I}{I \cap J} \simeq I \cdot \left(\frac{R}{J}\right)$$

as desired. \square

Exercise 5.7.8. Let I, J be ideals in a ring R , such that $I + J = (1)$. Prove that $IJ = I \cap J$.

Proof. Since $I + J = (1)$, there are elements $x \in I, y \in J$ such that $x + y = 1$. Then for $a \in I \cap J$, we have

$$a = a \cdot 1 = a(x + y) = ax + ay$$

Since $a \in I, a \in J$, we have $ax \in IJ$ and $ay \in IJ$. Thus $I \cap J \subseteq IJ$. Hence the claim follows. \square

Exercise 5.7.9. Let I, J be ideals in a ring R . Assume that $R/(IJ)$ is reduced (that is, it has no nonzero nilpotent elements). Prove that $IJ = I \cap J$.

Proof. For $a \in I \cap J$, we have $a^2 \in IJ$. Hence $(a + IJ)^2 = 0$ in $R/(IJ)$ and $a + IJ = 0$ since it is reduced. This means $a \in IJ$. Hence the claim follows. \square

Chapter 6

Chain conditions

6.1 General properties

Let Σ be a set partially ordered by a relation \leq .

Proposition 6.1.1. *The following conditions on Σ are equivalent:*

- (a) *Every increasing sequence $x_1 \leq x_2 \leq \dots$ in Σ is stationary.*
- (b) *Every non-empty subset of Σ has a maximal element.*

Proof. If (b) is false there is a non-empty subset T of Σ with no maximal element, and we can construct inductively a non-terminating strictly increasing sequence in T . The converse is easy to see. \square

If Σ is the set of submodules of a module M , ordered by the relation \subseteq , then (a) is called the **ascending chain condition** (a.c.c. for short) and (b) the **maximal condition**. A module M satisfying either of these equivalent conditions is said to be **Noetherian**.

If Σ is ordered by \supseteq , then (a) is the **descending chain condition** (d.c.c. for short) and (b) the **minimal condition**. A module M satisfying these is said to be **Artinian**.

Example 6.1.2 (Examples of Noetherian and Artinian rings).

- (a) A finite abelian group (as \mathbb{Z} -module) satisfies both a.c.c. and d.c.c.
- (b) The ring \mathbb{Z} satisfies a.c.c. but not d.c.c. For if $a \in \mathbb{Z}$ and $a \neq 0$ we have $(a) \supseteq (a^2) \supseteq (a^n) \supseteq \dots$ (strict inclusions).
- (c) Let G be the subgroup of \mathbb{Q}/\mathbb{Z} consisting of all elements whose order is a power of p , where p is a fixed prime. Then G has exactly one subgroup G_n of order p^n for each $n \geq 0$, and $G_0 \subseteq G_1 \subseteq \dots$ (strict inclusions) so that G does not satisfy the a.c.c. On the other hand the only proper subgroups of G are the G_m so that G does satisfy d.c.c.
- (d) The group H of all rational numbers of the form m/p^n ($m, n \in \mathbb{Z}, n \geq 0$) satisfies neither chain condition. For we have an exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow H \longrightarrow G \longrightarrow 0$$

so that H doesn't satisfy d.c.c. because \mathbb{Z} doesn't; and H doesn't satisfy a.c.c. because G doesn't.

- (e) The ring $k[X]$ where k is a field satisfies a.c.c. but not d.c.c. on ideals.
- (f) The polynomial ring $k[X_1, X_2, \dots]$ in an infinite number of indeterminates satisfies neither chain condition on ideals: for the sequence $(X_1) \subseteq (X_1, X_2) \subseteq \dots$ is strictly increasing, and the sequence $(X_1) \supseteq (X_1^2) \supseteq \dots$ is strictly decreasing.
- (g) We shall see later that a ring which satisfies d.c.c. **on ideals** must also satisfy a.c.c. **on ideals**. (This is not true in general for modules.)

Proposition 6.1.3. *A module M is a Noetherian A -module if and only if every submodule of M is finitely generated.*

Proof. Assume that M is Noetherian and let N be a submodule of M , and let Σ be the set of all finitely generated submodules of N . Then Σ is not empty (since $0 \in \Sigma$) and therefore has a maximal element, say N_0 . If $N_0 = N$, consider the submodule $N_0 + Ax$ where $x \in N$, $x \notin N_0$; this is finitely generated and strictly contains N_0 , so we have a contradiction. Hence $N = N_0$ and therefore N is finitely generated.

Conversely, assume that every submodule of M is finitely generated. Let $M_1 \subseteq M_2 \subseteq \dots$ be an ascending chain of submodules of M . Then $N = \bigcup_n M_n$ is a submodule of M , hence is finitely generated, say by x_1, \dots, x_r . Say $x_i \in M_{n_i}$, and let $n = \max_i n_i$, then each $x_i \in M_n$ hence $M_n = M$ and therefore the chain is stationary. \square

Because of ??, Noetherian modules are more important than Artinian modules: the Noetherian condition is just the right finiteness condition to make a lot of theorems work. However, many of the elementary formal properties apply equally to Noetherian and Artinian modules.

Proposition 6.1.4. *Let*

$$0 \longrightarrow L \xrightarrow{\alpha} M \xrightarrow{\beta} N \longrightarrow 0$$

be an exact sequence of A -modules. Then

- (a) M is Noetherian if and only if L and N are Noetherian.
- (b) M is Artinian if and only if L and N are Artinian.

Proof. An ascending chain of submodules of L (or N) gives rise to a chain in M , hence is stationary if M is Noetherian. Conversely, let (M_n) be an ascending chain of submodules of M ; then $(\alpha^{-1}(M_n))$ is a chain in L , and $(\beta(M_n))$ is a chain in N . If L and N are Noetherian, then for large enough n both these chains are stationary, and it follows that the chain (M_n) is stationary. \square

Proposition 6.1.5. *If M_1, \dots, M_n are Noetherian (resp. Artinian) A -modules, so is $\bigoplus_{i=1}^n M_i$.*

Proof. Apply induction and ?? to the exact sequence

$$0 \longrightarrow M_n \longrightarrow \bigoplus_{i=1}^n M_i \longrightarrow \bigoplus_{i=1}^{n-1} M_i \longrightarrow 0$$

we get the claim. \square

A ring A is said to be **Noetherian** (resp. **Artinian**) if it is so as an A -module, i.e., if it satisfies a.c.c. (resp. d.c.c.) on ideals.

Example 6.1.6.

- (a) Any field is both Artinian and Noetherian; so is the ring $\mathbb{Z}/n\mathbb{Z}$ ($n \neq 0$). The ring \mathbb{Z} is Noetherian, but not Artinian.
- (b) Any principal ideal domain is Noetherian.
- (c) The ring $k[X_1, X_2, \dots]$ is not Noetherian. But it is an integral domain, hence has a field of fractions. Thus a subring of a Noetherian ring need not be Noetherian.
- (d) Let X be a compact infinite Hausdorff space, $C(X)$ the ring of realvalued continuous functions on X . Take a strictly decreasing sequence $F_1 \supseteq F_2 \supseteq \dots$ of closed sets in X , and let $I_n = \{f \in C(X) : f(F_n) = 0\}$. Then the I_n form a strictly increasing sequence of ideals in $C(X)$: so $C(X)$ is not a Noetherian ring.

Proposition 6.1.7. *Let A be a Noetherian (resp. Artinian) ring, M a finitely generated A -module. Then M is Noetherian (resp. Artinian).*

Proof. Since M is a quotient of A^n for some n , the result follows by applying ?? and ??.

□

Proposition 6.1.8. *Let A be Noetherian (resp. Artinian), I an ideal of A . Then A/I is a Noetherian (resp. Artinian) ring.*

Proof. By ?? A/I is Noetherian (resp. Artinian) as an A -module, hence also as an A/I -module.

□

A **chain** of submodules of a module M is a sequence $(M_i)_{i=0}^n$ of submodules of M such that

$$M = M_0 \supset M_1 \supset \dots \supset M_n = 0$$

The **length** of the chain is defined to be n (the number of links). A **module composition series exist iff acc dcc** of M is a maximal chain, that is, each M_{i-1}/M_i is simple.

Proposition 6.1.9. *Suppose that M has a composition series of length n . Then every composition series of M has length n , and every chain in M can be extended to a composition series.*

Proof. Let $\ell(M)$ denote the least length of a composition series of a module M . We first note that if $N \subset M$ then $\ell(N) \leq \ell(M)$: Let (M_i) be a composition series of M of minimum length, and consider the submodules $N_i = N \cap M_i$ of N . Since $N_{i-1}/N_i \subseteq M_{i-1}/M_i$ and the latter is a simple module, we have either $N_{i-1}/N_i = M_{i-1}/M_i$ or else $N_{i-1} = N_i$; hence, removing repeated terms, we have a composition series of N , so that $\ell(N) \leq \ell(M)$. If $\ell(N) = \ell(M) = n$, then $N_{i-1}/N_i = M_{i-1}/M_i$ for each $1 \leq i \leq n$. This implies $M_{n-1} = N_{n-1}$, $M_{n-2} = N_{n-2}$, and finally $M = N$.

On the other hand, any chain in M has length smaller than $\ell(M)$. In fact, let

$$M = M_0 \supset M_1 \supset \dots \supset M_k = 0$$

be a chain of length k . Then by the preceding argument we have $\ell(M) > \ell(M_1) > \dots > \ell(M_k) = 0$, hence $\ell(M) \geq k$.

Now consider any composition series of M . If it has length k , then we must have $k \leq \ell(M)$, and therefore $k = \ell(M)$ by the definition of $\ell(M)$. This proves all composition series of M have the same length.

Finally, consider any chain of M . If its length is $\ell(M)$ it must be a composition series; if its length is strictly smaller than $\ell(M)$, then it is not a composition series, hence not maximal, and therefore new terms can be inserted until the length is $\ell(M)$. \square

Proposition 6.1.10. *M has a composition series if and only if M satisfies both chain conditions.*

Proof. One direction is clear: if M has a composition series, then every chain in M are of bounded length, hence both a.c.c. and d.c.c. hold. Conversely, if M satisfies both a.c.c. and d.c.c., we construct a composition series of M as follows. Since $M = M_0$ satisfies the maximum condition by ??, it has a maximal submodule $M_1 \subseteq M_0$. Similarly M_1 has a maximal submodule $M_2 \subseteq M_1$ and so on. Thus we have a strictly descending chain

$$M_0 \supset M_1 \supset M_2 \supset \dots$$

which by d.c.c. must be finite, and hence is a composition series of M . \square

A module satisfying both a.c.c. and d.c.c. is therefore called a **module of finite length**. By ?? all composition series of M have the same length $\ell(M)$, called the length of M . The **Jordan-Hölder theorem** applies to modules of finite length: if (M_i) and (M'_i) are any two composition series of M , there is a one-to-one correspondence between the set of quotients (M_{i-1}/M_i) and the set of quotients (M'_{i-1}/M'_i) , such that corresponding quotients are isomorphic. The proof is the same as for finite groups.

Proposition 6.1.11. *The length $\ell(M)$ is an additive function on the class of all A -modules of finite length.*

Proof. We have to show that if

$$0 \longrightarrow L \xrightarrow{\alpha} M \xrightarrow{\beta} N \longrightarrow 0$$

is an exact sequence, then $\ell(M) = \ell(L) + \ell(N)$. Take the image under α of any composition series of L and the inverse image under β of any composition series of N ; these fit together to give a composition series of M , hence the result. \square

Example 6.1.12. If A is a field, then A -modules are just vector spaces, and the notation of length coincides with dimension. In particular, a vector space M is of finite length if and only if it is finite-dimensional.

To conclude, we give a condition when Noetherian is equivalent to Artinian. This result will be used latter when we discuss Artinian rings.

Corollary 6.1.13. *Let A be a ring in which the zero ideal is a product $\mathfrak{m}_1 \cdots \mathfrak{m}_n$ of maximal ideals. Then A is Noetherian if and only if A is Artinian.*

Proof. Consider the chain of ideals

$$A \supset \mathfrak{m}_1 \supset \mathfrak{m}_1 \mathfrak{m}_2 \supset \mathfrak{m}_1 \cdots \mathfrak{m}_n = 0.$$

Each factor $\mathfrak{m}_1 \cdots \mathfrak{m}_{i-1} / \mathfrak{m}_1 \cdots \mathfrak{m}_i$ is a vector space over the field A/\mathfrak{m}_i . Hence a.c.c. if and only if d.c.c. for each factor. But a.c.c. (resp. d.c.c.) holds for each factor if and only if it holds for A , by repeated application of ???. Hence a.c.c. if and only if d.c.c. for A . \square

6.2 Noetherian rings

Proposition 6.2.1. *If A is Noetherian and ϕ is a homomorphism of A onto a ring B , then B is Noetherian.*

Proof. This follows from the exact sequence of ϕ . \square

Proposition 6.2.2. *Let A be a subring of B ; suppose that A is Noetherian and that B is finitely generated as an A -module. Then B is Noetherian.*

Proof. By ?? B is Noetherian as an A -module, hence also as a B -module. \square

Proposition 6.2.3. *In a noetherian ring A , every ideal \mathfrak{a} contains a power of its radical.*

Proof. If a_1, \dots, a_n generate $\sqrt{\mathfrak{a}}$, then there exists an integer k such that for $a_i^k \in \mathfrak{a}$ for all i . Let $m = n(k-1) + 1$, then $(\sqrt{\mathfrak{a}})^m$ is generated by the power products $\prod_i a_i^{\alpha_i}$, where $\sum_i \alpha_i = m$. Since $m > n(k-1)$, at least one of the exponents α_i is not less than k , and this proves that these products are in \mathfrak{a} , and therefore \mathfrak{a} contains $(\sqrt{\mathfrak{a}})^m$. \square

Corollary 6.2.4. *In a Noetherian ring A the nilradical $\mathfrak{n}(A)$ is nilpotent.*

Proposition 6.2.5. *If A is Noetherian and S is any multiplicatively closed subset of A , then $S^{-1}A$ is Noetherian.*

Proof. By ?? the ideals of $S^{-1}A$ are in one-to-one order-preserving correspondence with the contracted ideals of A , hence satisfy the maximal condition. \square

Corollary 6.2.6. *If A is Noetherian and \mathfrak{p} is a prime ideal of A , then $A_{\mathfrak{p}}$ is Noetherian.*

Now we prove the most important theorem concerning Noetherian rings.

Theorem 6.2.7 (Hilbert's Basis Theorem). *If A is Noetherian, then the polynomial ring $A[X]$ is Noetherian.*

Proof. We show that any ideal $I \subseteq A[X]$ is finitely generated. We inductively produce a set of generators f_i as follows.

For $n > 0$, if $I \neq (f_1, \dots, f_{n-1})$, let f_n be any nonzero element of $I - (f_1, \dots, f_{n-1})$ of lowest degree. Thus f_1 is any element of I of lowest degree, assuming $I \neq (0)$. If this procedure terminates, we are done. Otherwise, let $a_n \in A$ be the initial coefficient of f_n for $n > 0$. Then as A is Noetherian, there exists $N > 0$ such that $(a_1, \dots, a_n) = (a_1, \dots, a_{n+1})$ for all $n \geq N$. Say $a_{N+1} = \sum_{i=1}^N c_i a_i$. Then

$$f_{N+1} - \sum_{i=1}^N c_i f_i x^{\deg f_{N+1} - \deg f_i}$$

is an element of I that is nonzero (as $f_{N+1} \notin (f_1, \dots, f_N)$), and of lower degree than f_{N+1} , yielding a contradiction. \square

Corollary 6.2.8. *If A is Noetherian so is $A[X_1, \dots, X_n]$.*

Corollary 6.2.9. *Let B be a finitely-generated A -algebra. If A is Noetherian, then so is B .*

Proposition 6.2.10. *Let A be a Noetherian ring, then $A[[X]]$ is Noetherian.*

Proof. Set $B = A[[X]]$ and let \mathfrak{a} be an ideal of B , we will prove that \mathfrak{a} is finitely generated. Write $\mathfrak{a}(r)$ for the ideal of A formed by the leading coefficients a_r of $f = \sum_{i \geq r} a_i X^i$ as f runs through in $\mathfrak{a} \cap X^r B$, then we have

$$\mathfrak{a}(0) \subseteq \mathfrak{a}(1) \subseteq \mathfrak{a}(2) \subseteq \dots$$

Since A is Noetherian, there exists $s > 0$ such that $\mathfrak{a}(i+1) = \mathfrak{a}(i)$ for $i \geq s$; moreover, each $\mathfrak{a}(i)$ is finitely generated. For each $0 \leq i \leq s$ we take finitely many elements $a_{iv} \in A$ generating $\mathfrak{a}(i)$, and choose $g_{iv} \in \mathfrak{b} \cap X^i B$ having a_{iv} as the coefficient of X^i . These g_{iv} then generate \mathfrak{a} . Indeed, for $f \in \mathfrak{a}$ we can take a linear combination g_0 of the g_{0v} with coefficients in A such that $f - g_0 \in \mathfrak{a} \cap XB$, then proceed by induction to get

$$f - g_0 - g_1 - \dots - g_s \in \mathfrak{a} \cap X^{s+1}B$$

Now $\mathfrak{a}(i+1) = \mathfrak{a}(i)$ for $i \geq s$, so we can take linear combinations $g_{s+i} \in X^i g_{sv}$ with coefficients in A such that

$$f - g_0 - g_1 - \dots - g_s - g_{s+1} - \dots - g_{s+i} \in \mathfrak{a} \cap X^{s+i+1}B.$$

Letting i tends to infinity, we then see f can be generated by the g_{iv} , whence the claim. \square

6.3 Artin rings

Proposition 6.3.1. *An Artinian integral domain is a field.*

Proof. Let $x \neq 0$ in A , then we have a descending chain

$$(x) \supseteq (x^2) \supseteq \dots$$

By the d.c.c. there is $n > 0$ such that $(x^n) = (x^{n+1})$, in particular we have $x^n = ax^{n+1}$ for some $a \in A$. Since A is an integral domain, we get $ax = 1$, so x is a unit. Hence A is a field. \square

Proposition 6.3.2. *In an Artin ring A every prime ideal is maximal.*

Proof. Let \mathfrak{p} be a prime ideal of A . Then $B = A/\mathfrak{p}$ is an Artinian integral domain, hence a field. \square

Corollary 6.3.3. *In an Artin ring the nilradical is equal to the Jacobson radical.*

Proposition 6.3.4. *An Artin ring has only a finite number of maximal ideals.*

Proof. Consider the set of all finite intersections $\bigcap_{i=1}^n \mathfrak{m}_i$, where the \mathfrak{m}_i are maximal ideals. This set has a minimal element, say $\mathfrak{m}_1 \cap \dots \cap \mathfrak{m}_n$; hence for any maximal ideal \mathfrak{m} we have

$$\mathfrak{m} \cap \mathfrak{m}_1 \cap \dots \cap \mathfrak{m}_n = \mathfrak{m}_1 \cap \dots \cap \mathfrak{m}_n$$

, and therefore $\mathfrak{m} \supseteq \mathfrak{m}_1 \cap \dots \cap \mathfrak{m}_n$. By ?? $\mathfrak{m} \subseteq \mathfrak{m}_i$ for some i , hence $\mathfrak{m} = \mathfrak{m}_i$ since \mathfrak{m}_i is maximal. \square

Proposition 6.3.5. *In an Artin ring the nilradical is nilpotent.*

Proof. By d.c.c. we have $\mathfrak{n}(A)^k = I$ for k large enough. Suppose that $I \neq 0$, and let Σ denote the set of all ideals J such that $IJ \neq 0$. Then Σ is not empty, since $I \in \Sigma$. Let J_0 be a minimal element of Σ . Then we have $J_0 = (x)$, by the minimality of J_0 . But

$$(xI)I = xI^2 = xI \neq 0$$

and $xI \subseteq (x)$, hence $xI = (x)$ (again by minimality). Hence $x = xy$ for some $y \in I$, and therefore

$$x = xy = \cdots = xy^n = \cdots = 0$$

since $y \in I = \mathfrak{n}(A)^k \subseteq \mathfrak{n}(A)$ is nilpotent. This contradicts the choice of x , so $I = 0$. \square

By a **chain of prime ideals** of a ring A we mean a finite strictly increasing sequence

$$\mathfrak{p}_0 \supset \mathfrak{p}_1 \supset \cdots \supset \mathfrak{p}_n$$

the **length** of the chain is n . We define the dimension of A to be the supremum of the lengths of all chains of prime ideals in A : it is an integer ≥ 0 , or $+\infty$ (assuming $A \neq 0$). A field has dimension 0; the ring \mathbb{Z} has dimension 1.

Theorem 6.3.6. *A ring A is Artin if and only if A is Noetherian and $\dim A = 0$.*

Proof. Assume that A is Artinian. By ??, we have $\dim A = 0$. Let \mathfrak{m}_i ($1 \leq i \leq n$) be the distinct maximal ideals of A by ???. Then $\prod_{i=1}^n \mathfrak{m}_i^k \subseteq (\bigcap_{i=1}^n \mathfrak{m}_i)^k = \mathfrak{n}(A)^k = 0$. Hence by ?? A is Noetherian.

Conversely, assume that A is Noetherian and $\dim A = 0$. Then A has only a finite number of minimal prime ideals, and these are all maximal since $\dim A = 0$. Hence $\mathfrak{n}(A) = \bigcap_{i=1}^n \mathfrak{m}_i$ say; we have $\mathfrak{n}(A)^k = 0$ by ??, hence $\prod_{i=1}^n \mathfrak{m}_i^k = 0$ as in the previous part of the proof. Now by ??, A is an Artin ring. \square

If A is an Artin local ring with maximal ideal \mathfrak{m} , then \mathfrak{m} is the only prime ideal of A and therefore \mathfrak{m} is the nilradical of A . Hence every element of \mathfrak{m} is nilpotent, and \mathfrak{m} itself is nilpotent. Every element of A is either a unit or is nilpotent. An example of such a ring is $\mathbb{Z}/p^n\mathbb{Z}$, where p is prime and $n \geq 1$.

Proposition 6.3.7. *Let A be a Noetherian local ring, \mathfrak{m} its maximal ideal. Then exactly one of the following two statements is true:*

(a) $\mathfrak{m}^n \neq \mathfrak{m}^{n+1}$ for all n .

(b) $\mathfrak{m}^n = 0$ for some n , in which case A is an Artin local ring.

Proof. Suppose $\mathfrak{m}^n = \mathfrak{m}^{n+1}$ for some n . By Nakayama's lemma we have $\mathfrak{m}^n = 0$. Let \mathfrak{p} be any prime ideal of A . Then $\mathfrak{m}^n \subseteq \mathfrak{p}$, hence (taking radicals) $\mathfrak{m} = \mathfrak{p}$. Hence \mathfrak{m} is the only prime ideal of A and therefore A is Artinian. \square

Theorem 6.3.8 (Structure theorem for Artin rings). *An Artin ring A is uniquely a finite direct product of Artin local rings.*

Proof. Let $\mathfrak{m}_1, \dots, \mathfrak{m}_n$ be a complete list of distinct maximal ideals of A . From the proof of ??, we see $\prod_{i=1}^n \mathfrak{m}_i^k = \bigcap_{i=1}^n \mathfrak{m}_i^k = 0$ for some $k > 0$. By ?? the ideals \mathfrak{m}_i^k are coprime in pairs, hence $\bigcap_{i=1}^n \mathfrak{m}_i^k = \prod_{i=1}^n \mathfrak{m}_i^k$ by ???. Consequently by ?? again the natural mapping $A \rightarrow \prod_{i=1}^n (A/\mathfrak{m}_i^k)$ is an isomorphism. Each A/\mathfrak{m}_i^k is an Artin local ring, hence A is a direct product of Artin local rings.

Conversely, suppose $A \cong \prod_{i=1}^n A_i$, where the A_i are Artin local rings. Then for each i we have a natural surjective homomorphism (projection on the i -th factor) $\phi_i : A \rightarrow A_i$. Let $I_i = \ker \phi_i$. By ?? the I_i are pairwise coprime, and $\bigcap_{i=1}^n I_i = 0$. Let \mathfrak{q}_i be the unique prime ideal of A_i , and let \mathfrak{p}_i be its contraction $\mathfrak{p}_i = \phi_i^{-1}(\mathfrak{q}_i)$. The ideal \mathfrak{p}_i is prime and therefore maximal by ???. Since \mathfrak{q}_i is nilpotent it follows that I_i is \mathfrak{p}_i -primary, and hence $\bigcap_i I_i = (0)$ is a primary decomposition of the zero ideal in A . Since the I_i are pairwise coprime, so are the \mathfrak{p}_i , and they are therefore isolated prime ideals of (0) . Hence all the primary components I_i are isolated, and therefore uniquely determined by A , by the second uniqueness theorem. Hence the rings $A_i \cong A/I_i$ are uniquely determined by A . \square

Example 6.3.9. A ring with only one prime ideal need not be Noetherian (and hence not an Artin ring). Consider the ring

$$A = k[X_1, X_2, \dots] / (X_1, X_2^2, \dots)$$

it has only one prime ideal, namely the image of (X_1, X_2, \dots) , hence is a local ring of dimension 0. But A is not Noetherian, for it is not difficult to see that its prime ideal is not finitely generated.

If A is a local ring, \mathfrak{m} its maximal ideal, $k = A/\mathfrak{m}$ its residue field, the A -module $\mathfrak{m}/\mathfrak{m}^2$ is annihilated by \mathfrak{m} and therefore has the structure of a k -vector space. If \mathfrak{m} is finitely generated (e.g., if A is Noetherian), the images in $\mathfrak{m}/\mathfrak{m}^2$ of a set of generators of \mathfrak{m} will span $\mathfrak{m}/\mathfrak{m}^2$ as a vector space. and therefore $\dim_k(\mathfrak{m}/\mathfrak{m}^2)$ is finite.

Proposition 6.3.10. Let A be an Artin local ring. Then the following are equivalent:

- (a) every ideal in A is principal.
- (b) the maximal ideal \mathfrak{m} is principal.
- (c) $\dim_k(\mathfrak{m}/\mathfrak{m}^2) \leq 1$.

Proof. The implications $(a) \Rightarrow (b) \Rightarrow (c)$ are clear. Now we show $(c) \Rightarrow (a)$. If $\dim_k(\mathfrak{m}/\mathfrak{m}^2) = 0$, then $\mathfrak{m} = \mathfrak{m}^2$, hence $\mathfrak{m} = 0$ by Nakayama's lemma, and therefore A is a field and there is nothing to prove. If $\dim_k(\mathfrak{m}/\mathfrak{m}^2) = 1$, then \mathfrak{m} is a principal ideal by ?? (take $M = \mathfrak{m}$ there), say $\mathfrak{m} = (x)$. Let I be an ideal of A , other than (0) or (a) . We have $\mathfrak{m} = J(A)$ hence \mathfrak{m} is nilpotent by ?? and therefore there exists an integer r such that $I \subseteq \mathfrak{m}^r$, $I \notin \mathfrak{m}^{r+1}$; hence there exists $y \in I$ such that $y = ax^r$, $y \notin (x^{r+1})$; consequently $a \notin (x)$ and thus a is a unit in A . Hence $x^r \in I$, therefore $\mathfrak{m}^r = (x^r) \subseteq I$ and hence $I = \mathfrak{m}^r = (x^r)$. Hence I is principal. \square

Example 6.3.11. The rings $\mathbb{Z}/(p^n)$ (p prime), $k[X]/(f^n)$ (f irreducible) satisfy the conditions of ???. On the other hand, the Artin local ring $k[X^2, X^3]/(X^4)$ does not: here \mathfrak{m} is generated by X^2 and $X^3 \pmod{X^4}$, so that $\mathfrak{m}^2 = 0$ and $\dim_k(\mathfrak{m}/\mathfrak{m}^2) = 2$.

6.4 Exercise

Exercise 6.4.1. Let A be a Noetherian ring and let $f = \sum_{n=1}^{\infty} a_n X^n \in A[[X]]$. Prove that f is nilpotent if and only if each a_n is nilpotent.

Proof. One direction is verified in ???. For the other, assume each a_n is nilpotent. Then by ??, there is $m > 0$ such that $(J(A))^m = 0$. This means the product $a_I = \prod_{i=1}^m a_{n_i} = 0$. In f^m , the coefficient of each term is divisible by some a_I , so $f^m = 0$. \square

Exercise 6.4.2. Let A be a ring such that

- (a) for each maximal ideal \mathfrak{m} of A , the local ring $A_{\mathfrak{m}}$ is Noetherian.
- (b) for each $x \neq 0$ in A , the set of maximal ideals of A which contain x is finite.

Show that A is Noetherian.

Proof. Let $I \subseteq A$ be an ideal. By (b) the set M of maximal ideals containing I is finite, say $M = \{\mathfrak{m}_1, \dots, \mathfrak{m}_r\}$. For each $\mathfrak{m}_i \in M$, the extension $A_{\mathfrak{m}_i}I$ of I in $A_{\mathfrak{m}_i}$ is finitely generated since (a) says $A_{\mathfrak{m}}$ is Noetherian. We may take x_1, \dots, x_t such that their image in each $A_{\mathfrak{m}_i}$ generate $A_{\mathfrak{m}_i}I$. Let $z \neq 0$ in I , and $M_z = \{\mathfrak{m}_1, \dots, \mathfrak{m}_{r+s}\}$ be the maximal ideals containing z . Then for each \mathfrak{m}_{r+j} , there is $y_j \in I$ such that $y_j \notin \mathfrak{m}_{r+j}$. Now let

$$I_0 = (z, x_1, \dots, x_t, y_1, \dots, y_j)$$

Now for any maximal ideal \mathfrak{m} ,

- If $\mathfrak{m} \in M$, by construction we have $S^{-1}I_0 = S^{-1}I$.
- If $\mathfrak{m} \notin M_z$, we have $z \in A - \mathfrak{m}$, so $S^{-1}I = S^{-1}I_0 = (a)$.
- If $\mathfrak{n} \in M_z - M$, then there is $y_j \in A - \mathfrak{n}$, hence $S^{-1}I_0$. Also, $\mathfrak{m} \notin M$, so $I \cap A - \mathfrak{m} \neq \emptyset$ so $S^{-1}I = (a)$.

Hence I and I_0 have the same extension in $A_{\mathfrak{m}}$ for every maximal ideal \mathfrak{m} , by ?? the map $I_0 \rightarrow I$ is surjective, so I is finitely generated. \square

Exercise 6.4.3. Let A be a Noetherian local ring, \mathfrak{m} its maximal ideal and k its residue field, and let M be a finitely generated A -module. Then the following are equivalent:

- (a) M is free.
- (b) M is flat.
- (c) The mapping of $\mathfrak{m} \otimes_A M$ into $A \otimes_A M$ is injective.
- (d) $\text{Tor}_1^A(k, M) = 0$.

Proof. To see the implication $(b) \Rightarrow (c)$, consider the exact sequence

$$0 \longrightarrow \mathfrak{m} \longrightarrow A \longrightarrow k \longrightarrow 0$$

Tensoring with M gives a short exact sequence, if M is flat.

Now the equivalence $(c) \Leftrightarrow (d)$ comes from the exact sequence

$$0 \longrightarrow \text{Tor}_1^A(k, M) \longrightarrow \mathfrak{m} \otimes_A M \longrightarrow A \otimes_A M \longrightarrow k \otimes_A M \longrightarrow 0$$

Hence we only need to show $(d) \Rightarrow (a)$. Since M is finitely generated, there are $x_1, \dots, x_n \in M$ whose image in $M/\mathfrak{m}M$ forms a basis. By ??, they also generate M . Now consider the exact sequence

$$0 \longrightarrow K \longrightarrow A^n \xrightarrow{\phi} M \longrightarrow 0$$

where $K := \ker \phi$. Tensoring with k gives

$$0 \longrightarrow k \otimes_A K \longrightarrow k \otimes_A A^n \longrightarrow k \otimes M \longrightarrow 0$$

since $\text{Tor}_1^A(k, M) = 0$. Then since x_1, \dots, x_n is a basis, the map $1 \otimes \phi$ is an isomorphism, so $k \otimes_A K = 0$. Since A is Noetherian, K is finitely generated. Also, $k \otimes_A K = K/\mathfrak{m}K = 0$, and by Nakayama's lemma we conclude $K = 0$, so M is free. \square

Exercise 6.4.4. Let A be a Noetherian ring, M a finitely generated A -module. Then the following are equivalent:

- (a) M is a free A -module.
- (b) $M_{\mathfrak{p}}$ is a free $A_{\mathfrak{p}}$ -module, for all prime ideals \mathfrak{p} .
- (c) $M_{\mathfrak{m}}$ is a free $A_{\mathfrak{m}}$ -module, for all maximal ideals \mathfrak{m} .

In other words, flat = locally free.

Proof. By ??, and note that $A_{\mathfrak{p}}$ is a Noetherian local ring by ???. Hence the claim follows from ???. \square

Exercise 6.4.5. Let $(0) = \bigcap_{i=1}^n \mathfrak{q}_i$ be a reduced primary decomposition of the zero ideal in a Noetherian ring, and let \mathfrak{q}_i be \mathfrak{p}_i -primary. Let $\mathfrak{p}_i^{(r)}$ be the r -th **symbolic power** of \mathfrak{p}_i (that is, the saturation of \mathfrak{p}_i^r with respect to \mathfrak{p}_i). Show that for each $1 \leq i \leq n$ there exists an integer r_i such that $\mathfrak{p}_i^{(r_i)} \subseteq \mathfrak{q}_i$.

Suppose \mathfrak{q}_i is an isolated primary component. Then $A_{\mathfrak{p}_i}$ is an Artin local ring, hence if \mathfrak{m} is its maximal ideal we have $\mathfrak{m}^r = 0$ for all sufficiently large r , hence $\mathfrak{q}_i = \mathfrak{p}_i^{(r)}$ for all large r .

If \mathfrak{q}_i is an embedded primary component, then $A_{\mathfrak{p}_i}$ is not Artinian, hence the powers \mathfrak{m}^r are all distinct, and so the $\mathfrak{p}_i^{(r)}$ are all distinct. Hence in the given primary decomposition we can replace \mathfrak{q}_i by any of the infinite set of \mathfrak{p}_i -primary ideals $\mathfrak{p}_i^{(r_i)}$ where $r \geq r_i$ and so there are infinitely many reduced primary decompositions of 0 which differ only in the \mathfrak{p}_i -component.

Proof. By ?? for each \mathfrak{p}_i there is r_i such that $\mathfrak{p}_i^{r_i} \subseteq \mathfrak{q}_i \subseteq \mathfrak{p}_i$. Then

$$\mathfrak{p}_i^{(r_i)} = S_{\mathfrak{p}_i}(\mathfrak{p}_i^{r_i}) \subseteq S_{\mathfrak{p}_i}(\mathfrak{q}_i) = \mathfrak{q}_i$$

where we use $S_{\mathfrak{p}_i}(\mathfrak{q}_i) = \mathfrak{q}_i$ by ???. This proves the first claim.

If \mathfrak{p}_i is isolated, since the isolated ideal of (0) is minimal in A , $A_{\mathfrak{p}_i}$ contains exactly one prime ideal \mathfrak{m} , hence is an Artin local ring. Since $\mathfrak{m} = \mathfrak{n}(A_{\mathfrak{p}_i})$, by ?? $\mathfrak{m}^k = 0$ for some $k > 0$.

Also, for $r \geq r_i$ the intersection

$$\mathfrak{p}_i^{(r)} \cap \bigcap_{j \neq i} \mathfrak{q}_j \subseteq \mathfrak{p}_i^{(r_i)} \cap \bigcap_{j \neq i} \mathfrak{q}_j \subseteq \bigcap_{i=1}^n \mathfrak{q}_i = (0)$$

gives another primary decomposition of (0) , since by ?? the isolated primary component is unique, we conclude $\mathfrak{p}_i^{(r)} = \mathfrak{q}_i$ for $r \geq r_i$.

If \mathfrak{p}_i is embedded, we have $\mathfrak{p}_j \subset \mathfrak{p}_i$ for another $j \neq i$, hence we have the chain $\mathfrak{p}_j A_{\mathfrak{p}_i} \subseteq \mathfrak{p}_i A_{\mathfrak{p}_i}$ in $A_{\mathfrak{p}_i}$. It follows that $A_{\mathfrak{p}_i}$ is not an Artin ring, hence by ?? the powers \mathfrak{m}^r are all distinct. Now if $\mathfrak{p}_i^r = \mathfrak{p}_i^s$ for $r \neq s$, then $\mathfrak{p}_i^r A_{\mathfrak{p}_i} = \mathfrak{m}^r = \mathfrak{p}_i^s A_{\mathfrak{p}_i} = \mathfrak{m}^s$, contradiction. Hence the powers \mathfrak{p}_i^r are also distinct. Also, note that

$$(\mathfrak{p}_i^{(r)})^e = (\mathfrak{p}_i^r)^{ece} = (\mathfrak{p}_i^r)^e = \mathfrak{m}^r$$

so if $\mathfrak{p}_i^{(r)} = \mathfrak{p}_i^{(s)}$, then $\mathfrak{m}^r = \mathfrak{m}^s$ and hence $r = s$. This implies all $\mathfrak{p}_i^{(r)}$ are distinct. If $r \geq r_i$, then the intersection

$$\mathfrak{p}_i^{(r_i)} \cap \bigcap_{j \neq i} \mathfrak{q}_j \subseteq \bigcap_{i=1}^n \mathfrak{q}_i = (0)$$

gives another primary decomposition of (0) . Since $\mathfrak{p}_i^{(r)}$ are distinct, these are distinct primary decomposition of (0) , infinitely many, all minimal. \square

Exercise 6.4.6. Let A be a Noetherian ring. Prove that the following are equivalent:

- (a) A is Artinian.
- (b) $\text{Spec}(A)$ is discrete and finite.
- (c) $\text{Spec}(A)$ is discrete.

Proof. If A is Artinian, then every prime ideal is maximal, so $\text{Spec}(A)$ is discrete. If $\text{Spec}(A)$ is discrete, then $\dim A = 0$, hence A is Artinian. \square

Exercise 6.4.7. Let k be a field and A a finitely generated k -algebra. Prove that the following are equivalent:

- (a) A is Artinian.
- (b) A is a finite k -algebra.

Proof. Assume A is Artin, then A is a direct product of Artin local rings, hence we can assume A is an Artin local ring. Let \mathfrak{m} be the maximal ideal in A , then A/\mathfrak{m} is a field, hence is finite by Zariski's Lemma. Now A is Noetherian and Artinian, so A has a composition series

$$A = M_0 \supset M_1 \supset \cdots \supset M_n = 0$$

where each quotient $M_{i-1}/M_i \cong A/\mathfrak{p}$ for a prime ideal in A . Since A is local, $M_{i-1}/M_i \cong A/\mathfrak{m}$ is finite, hence by induction A is finite.

If A is finite, observe that the ideals of A are k -vector subspaces and therefore satisfy d.c.c. \square

Exercise 6.4.8. Let $\rho : A \rightarrow B$ be a ring homomorphism of finite type. Consider the following statements:

- (a) ρ is finite.
- (b) the fibres of ${}^a\rho$ are discrete subspaces of $\text{Spec}(B)$.
- (c) for each prime ideal \mathfrak{p} of A , the ring $B \otimes_A k(\mathfrak{p})$ is a finite $k(\mathfrak{p})$ -algebra.
- (d) the fibres of ${}^a\rho$ are finite.

Prove that (a) \Rightarrow (b) \Leftrightarrow (c) \Rightarrow (d).

Proof. We first observe that

$$C := B \otimes_A k(\mathfrak{p}) = B \otimes_A (A_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} k(\mathfrak{p})) = (B \otimes_A A_{\mathfrak{p}}) \otimes_{A_{\mathfrak{p}}} k(\mathfrak{p})$$

and that $\rho^{*-1}(\mathfrak{p}) = \text{Spec}(B \otimes_A k(\mathfrak{p}))$. Therefore, in view of ?? and ??, we see (b) is equivalent to (c).

Now if ρ is finite, then B is a finite algebra. Thus C is a finite dimensional $k(\mathfrak{p})$ -vector space. Finally, by ??, C is Artinian iff it is finite, and ?? says $\text{Spec}(C)$ is finite if this holds. \square

Exercise 6.4.9. Let A be a Noetherian ring and \mathfrak{q} a \mathfrak{p} -primary ideal in A . Consider chains of primary ideals from \mathfrak{q} to \mathfrak{p} . Show that all such chains are of finite bounded length, and that all maximal chains have the same length.

Proof. Since we are interested in ideals between \mathfrak{q} and \mathfrak{p} , let $B = (A/\mathfrak{q})_{\mathfrak{p}/\mathfrak{q}}$. Then ideals in B are those in between, while extension and contradiction preserves primarity. Also, note that since A is Noetherian, there is $n > 0$ such that $\mathfrak{p}^n \subseteq \mathfrak{q}$, hence $\mathfrak{m}^r = 0$ in B where \mathfrak{m} is the unique maximal ideal in B . Thus, by ?? B is an Artin local ring. Also, for any ideal \mathfrak{b} in B we have $\mathfrak{m}^n \subseteq \mathfrak{b} \subseteq \mathfrak{m}$, so \mathfrak{b} is \mathfrak{m} -primary.

Thus our question boils down to arbitrary chains of ideals in an Artinian ring B . By the a.c.c. and d.c.c., these must all have finite length. A maximal chain is a composition series, and these all have the same length. \square

Chapter 7

Linear algebras

In this section all rings are assumed to be commutative with unit. Let M be an A -module and let $(M_i)_{i \in I}$ be a family of submodules of M . Their sum $\sum M_i$ is the set of all (finite) sums $\sum_i x_i$ where $x_i \in M_i$ for all $i \in I$, and almost all the x . (that is, all but a finite number) are zero. $\sum_i M_i$ is the smallest submodule of M which contains all the M_i .

The intersection $\bigcap_i M_i$ is again a submodule of M . Thus the submodules of M form a complete lattice with respect to inclusion. We cannot in general define the **product** of two submodules, but we can define the product IM , where I is an ideal and M an A -module; it is the set of all finite sums $\sum_i a_i x_i$ with $a_i \in I$, $x_i \in M$, and is a submodule of M .

If N, P are submodules of M , we define $(N : P)$ to be the set of all $a \in A$ such that $aP \subseteq N$; it is an ideal of A . In particular, $(0 : M)$ is the set of all $a \in A$ such that $aM = 0$; this ideal is called the **annihilator** of M and is also denoted by $\text{Ann}(M)$.

If $I \subseteq \text{Ann}(M)$, we may regard M as an A/I -module, as follows: if $\bar{x} \in A/I$ is represented by $x \in A$, define $\bar{x} \cdot m$ to be xm ($m \in M$): this is independent of the choice of the representative x of \bar{x} , since $IM = 0$.

An A -module is **faithful** if $\text{Ann}(M) = 0$. If $\text{Ann}(M) = I$, then M is faithful as an A/I -module. If x is an element of M , the set of all multiples ax ($a \in A$) is a submodule of M , denoted by Ax or (x) . If $M = \sum_{i \in I} Ax_i$ then x_i are said to be a set of generators of M ; this means that every element of M can be expressed (not necessarily uniquely) as a finite linear combination of the x_i with coefficients in A . An A -module M is said to be finitely generated if it has a finite set of generators.

7.1 Modules and diagrams

7.1.1 Finitely generated modules

Proposition 7.1.1. *Let M be a finitely generated A -module, let I be an ideal of A , and let ϕ be an A -module endomorphism of M such that $\phi(M) \subseteq IM$. Then ϕ satisfies an equation of the form*

$$\phi^n + a_1\phi^{n-1} + \cdots + a_n = 0$$

where $a_i \in I$.

Proof. Let x_1, \dots, x_n be a set of generators of M . Then each $\phi(x_i) \in IM$, so that we have say

$$\phi(x_i) = \sum_{j=1}^n a_i^j x_j$$

where $a_i^j \in I$. Then let $T = [a_i^j]$, we see

$$\phi(x) = Tx.$$

Let $f(\lambda)$ be the characteristic polynomial of T , we see $f(\phi) = 0$. \square

Corollary 7.1.2. *Let M be a finitely generated A -module and let I be an ideal of A such that $IM = M$. Then there exists $x \equiv 1 \pmod{I}$ such that $xM = 0$.*

Proof. Take $\phi = \text{identity}$, $x = 1 + a_1 + \dots + a_n$ in ??.

\square

Proposition 7.1.3 (Nakayama's lemma). *Let M be a finitely generated A -module and I an ideal of A contained in the Jacobson radical of A . Then $IM = M$ if and only if $M = 0$.*

Proof. First Proof. By ?? we have $xM = 0$ for some $x \equiv 1 \pmod{\mathfrak{r}(A)}$. By ?? x is a unit in A , hence $M = x^{-1}xM = 0$.

Second Proof. Suppose $M \neq 0$, and let u_1, \dots, u_n be a minimal set of generators of M . Then $u_n \in IM$, hence we have an equation of the form

$$u_n = a_1 u_1 + \dots + a_n u_n$$

with the $a_i \in I$. Hence

$$(1 - a_n)u_n = a_1 u_1 + \dots + a_{n-1} u_{n-1}$$

since $a_n \in \mathfrak{r}(A)$, it follows that $1 - a_n$ is a unit in A . Hence u_n belongs to the submodule of M generated by u_1, \dots, u_{n-1} , contradiction. \square

Corollary 7.1.4. *Let M, N be finitely generated A -modules, $u : M \rightarrow N$ be a homomorphism, and I be an ideal of A contained in the Jacobson radical of A . If the induced homomorphism $\tilde{u} : M/IM \rightarrow N/IN$ is surjective, then u is surjective.*

Proof. By hypothesis we have $u(M) \otimes_A (A/I) = \text{im}(\tilde{u}) = N \otimes_A (A/I)$, so $(A/I) \otimes_A (N/u(M)) = 0$, and $N = u(M)$ by ??.

\square

Corollary 7.1.5. *Let M be a finitely generated A -module, N a submodule of M , I an ideal of A contained in the Jacobson radical of A . Then $M = IM + N$ if and only if $M = N$.*

Proof. Apply ?? to M/N , observing that $I(M/N) = M/N$.

\square

Let A be a local ring, \mathfrak{m} its maximal ideal, $k = A/\mathfrak{m}$ its residue field. Let M be a finitely generated A -module. $M/\mathfrak{m}M$ is annihilated by \mathfrak{m} , hence is naturally an A/\mathfrak{m} -module, i.e., a k -vector space, and as such is finite-dimensional.

Proposition 7.1.6. *Let x_1, \dots, x_n be elements of M whose images in $M/\mathfrak{m}M$ form a basis of this vector space. Then the x_i generate M .*

Proof. Let N be the submodule of M generated by the x_i . Then the composite map

$$N \rightarrow M \rightarrow M/\mathfrak{m}M$$

maps N onto $M/\mathfrak{m}M$, hence $N + \mathfrak{m}M = M$, hence $N = M$ by ??.

□

Proposition 7.1.7. *Let A be a local ring, and let M be a direct summand of a finitely generated free A -module: that is, there exists an A -module N such that $M \oplus N$ is a free A -module. Then M is free.*

Proof. Choose elements $m_1, \dots, m_r \in M$ whose cosets mod $\mathfrak{m}M$ are a basis of $M/\mathfrak{m}M$ as a vector space over the field A/\mathfrak{m} . By Nakayama's lemma, M is generated by m_1, \dots, m_r . Since M is a direct summand of a free module, it is projective. For the epimorphism $\pi : A^r \rightarrow M$ induced by the m_i , we get an lift $\psi : M \rightarrow A^r$ such that $\pi \circ \psi = \text{id}_M$. This means π splits, so $A^r \cong M \oplus K$, where $K := \ker \pi$. Tensoring this with A/\mathfrak{m} , we get $K/\mathfrak{m}K = 0$. On the other hand, K is a quotient of A^r , hence is finitely generated. Then by Nakayama's lemma, $K = 0$. Hence M is free. □

7.1.2 Exact sequence and the snake lemma

Proposition 7.1.8. *Let $L \xrightarrow{\alpha} M \xrightarrow{\beta} N \rightarrow 0$ be a sequence of A -modules. Then this sequence is exact if and only if for all A -modules Q , the sequence*

$$0 \longrightarrow \text{Hom}(N, Q) \xrightarrow{\beta^*} \text{Hom}(M, Q) \xrightarrow{\alpha^*} \text{Hom}(L, Q)$$

is exact.

Proof. The nontrivial case is the only if part. So suppose that the second sequence

$$0 \longrightarrow \text{Hom}(N, Q) \xrightarrow{\beta^*} \text{Hom}(M, Q) \xrightarrow{\alpha^*} \text{Hom}(L, Q)$$

is exact for all Q . We need to show the following is exact:

$$L \xrightarrow{\alpha} M \xrightarrow{\beta} N \longrightarrow 0$$

First of all, to show β is surjective. Let $Q = N/\text{im } \beta$, and consider the projection $\phi : N \rightarrow N/\text{im } \beta$. Note that

$$\beta^*(\phi) = \phi \circ \beta = 0.$$

Since β^* is injective, $\phi = 0$ must be the case, and hence $\text{im } \beta = N$, β is surjective.

For $\alpha^* \circ \beta^* = (\beta \circ \alpha)^* = 0$. Let $Q = N$ and choose $\text{id} : N \rightarrow N$. Then we can see

$$\alpha^* \circ \beta^*(\text{id}) = \text{id} \circ \beta \circ \alpha = 0.$$

so $\beta \circ \alpha = 0$, $\text{im } \alpha \subseteq \ker \beta$.

Now take $Q = M/\text{im } \alpha$, and let $\phi : M \rightarrow N$ be the projection. Then $\alpha^*(\phi) = \phi \circ \alpha = 0$, hence by the exactness, $\phi = \beta^*(\psi) = \psi \circ \beta$ for some $\psi : N \rightarrow M/\text{im } \alpha$. Consequently $\text{im } \alpha = \ker \phi \supseteq \ker \beta$. Hence we are done. □

Similarly, we can get the following result for the functor $\text{Hom}(Q, -)$.

Proposition 7.1.9. Let $0 \longrightarrow L \xrightarrow{\alpha} M \xrightarrow{\beta} N$ be a sequence of A -modules and homomorphisms. Then this sequence is exact if and only if for all A -modules M , the sequence

$$0 \longrightarrow \text{Hom}(Q, L) \xrightarrow{\alpha_*} \text{Hom}(Q, M) \xrightarrow{\beta_*} \text{Hom}(Q, N)$$

is exact.

A particular case of short exact sequence arises by considering the second projection from a direct sum: $M_1 \oplus M_2 \rightarrow M_2$; there is then an exact sequence

$$0 \longrightarrow M_1 \longrightarrow M_1 \oplus M_2 \longrightarrow M_2 \longrightarrow 0$$

These short exact sequences are said to **split**; more generally, a short exact sequence

$$0 \longrightarrow M_1 \longrightarrow N \longrightarrow M_2 \longrightarrow 0$$

is said to be exact if it is isomorphic to one of these sequences in the sense that there is a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & M_1 & \longrightarrow & N & \longrightarrow & M_2 & \longrightarrow & 0 \\ & & \downarrow \sim & & \downarrow \sim & & \downarrow \sim & & \\ 0 & \longrightarrow & M'_1 & \longrightarrow & M'_1 \oplus M'_2 & \longrightarrow & M'_2 & \longrightarrow & 0 \end{array}$$

Theorem 7.1.10. Let $\varphi : M \rightarrow N$ be an A -module homomorphism. Then

- φ has a left-inverse if and only if the sequence

$$0 \longrightarrow M \xrightarrow{\varphi} N \longrightarrow \text{coker } \varphi \longrightarrow 0$$

splits.

- φ has a right-inverse if and only if the sequence

$$0 \longrightarrow \ker \varphi \longrightarrow M \xrightarrow{\varphi} N \longrightarrow 0$$

splits.

Proof. If the sequence splits, then φ may be identified with the embedding of M into a direct sum $M \oplus M'$, and the projection $M \oplus M' \rightarrow M$ gives a left-inverse of φ . Conversely, assume that φ has a left-inverse ψ :

$$\begin{array}{ccccc} 0 & \longrightarrow & M & \xrightarrow{\varphi} & N \\ & & \searrow \text{id} & & \downarrow \psi \\ & & M & & \end{array}$$

Then we claim that N is isomorphic to $M \oplus \ker \psi$ and that φ corresponds to the identification of M with the first factor: $M \rightarrow M \oplus \ker \psi \simeq N$. The isomorphism $M \oplus \ker \psi \rightarrow N$ is given by

$$(m, k) \mapsto \varphi(m) + k$$

its inverse $N \rightarrow M \oplus \ker \psi$ is

$$n \mapsto (\psi(n), n - \varphi \circ \psi(n))$$

The element $n - \varphi \circ \psi(n)$ is in $\ker \psi$ as it should be, since

$$\psi(n - \varphi \circ \psi(n)) = \psi(n) - \psi \circ \varphi \circ \psi(n) = \psi(n) - \psi(n) = 0.$$

However, if φ has an right-inverse

$$\begin{array}{ccccc} M & \xrightarrow{\varphi} & N & \longrightarrow & 0 \\ \psi \uparrow & & \nearrow \text{id} & & \\ N & & & & \end{array}$$

Then the isomorphism $N \times \ker \varphi \rightarrow M$ becomes

$$(n, k) \mapsto \psi(n) + k$$

with the inverse $M \rightarrow N \times \ker \varphi$ being

$$m \mapsto (\varphi(m), m - \psi \circ \varphi(m))$$

This completes the proof. \square

Lemma 7.1.11 (snake lemma). *If given a snake diagram as follows*

$$\begin{array}{ccccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\ \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \\ 0 & \longrightarrow & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' \end{array}$$

where the two rows are exact. Then there is an homomorphism $\delta : \ker \gamma \rightarrow \text{coker } \alpha$ such that the

follows diagram commutes

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \ker f & \longrightarrow & \ker \alpha & \longrightarrow & \ker \beta & \longrightarrow & \ker \gamma & \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
 & & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow 0 \\
 & & \downarrow \alpha & & \downarrow \delta & \downarrow \beta & \downarrow \gamma & \\
 0 & \longrightarrow & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow & \\
 & & \text{coker } \alpha & \longrightarrow & \text{coker } \beta & \longrightarrow & \text{coker } \gamma & \longrightarrow & \text{coker } g' & \longrightarrow 0
 \end{array}$$

the homomorphism δ is called the **connection homomorphism**.

Proof. First, let $c \in \ker \gamma$. We can construct elements of A' as follows. Since g is surjective, there exists an element $b \in B$ such that $g(b) = c$. We now move vertically down by β , and take $\beta(b)$. The commutativity $\gamma \circ g = g' \circ \beta$ shows that

$$g'(\beta(b)) = \gamma(g(b)) = \gamma(c) = 0$$

whence $\beta(b)$ is in the kernel of g' in B' . By exactness, there exists an element $a' \in A'$ such that $f'(a') = \beta(b)$. We map a into $\text{coker } \alpha$ by the canonical projection, and still denote it by a .

In brief, we write

$$\delta(c) = a = f'^{-1} \circ \beta \circ g^{-1}(c)$$

$$\begin{array}{ccccc}
 & & c & & \\
 & & \downarrow & & \\
 & b & \xrightarrow{g} & c & \\
 & \downarrow \beta & & \downarrow \gamma & \\
 a & \xrightarrow{f'} & \beta(b) & \dashrightarrow & 0 \\
 & \downarrow & & & \\
 & a & & &
 \end{array}$$

We now verify that δ is well defined. First, f' is injective by exactness, a is uniquely determined by $\beta(b)$, hence by b . So the only verification is for b .

Suppose we choose a different b' mapping to the same c : $g(b') = c = g(b)$. Then $b' - b \in \ker g$, and by exactness, $\text{im } f = \ker g$, so there is $\ell \in A$ such that $\alpha(\ell) = b' - b$. It follows from the commutativity that

$$\beta(b' - b) = \beta(f(b' - b)) = f'(\alpha(\ell))$$

Following our construction, $f'^{-1}(\beta(b' - b)) = \alpha(\ell)$. But since the column is exact, the projec-

tion into $\text{coker } \alpha$ kills $\alpha(\ell)$, hence taking b' makes no difference to $\delta(c)$, that is, δ is well defined.

$$\begin{array}{ccccccc}
& & f & & g & & \\
\ell & \xrightarrow{\quad} & b' - b & \xrightarrow{\quad} & 0 & & \\
\downarrow \alpha & & \downarrow \beta & & & & \\
\alpha(\ell) & \xrightarrow{f'} & \beta(b' - b) & & & & \\
\downarrow & & & & & & \\
0 & & & & & &
\end{array}$$

Now we need to verify the exactness of the sequence. First we give the morphisms in the sequence

$$0 \rightarrow \ker f \xrightarrow{i} \ker \alpha \xrightarrow{f_*} \ker \beta \xrightarrow{g_*} \ker \gamma \xrightarrow{\delta} \text{coker } \alpha \xrightarrow{f'_*} \text{coker } \beta \xrightarrow{g'_*} \text{coker } \gamma \xrightarrow{j} \text{coker } g' \rightarrow 0$$

We just define $f_* := f|_{\ker \alpha}$, and similar for g_* , f'_* , g'_* . If $a \in \ker f$, then $f'(\alpha(a)) = 0$ by commutativity, and $\alpha(a) = 0$ by injectivity of f' . This gives an inclusion $\ker f \rightarrow \ker \alpha$, which is our i . Similarly, if $c \in \text{im } \gamma$, then there is an element $b \in B$ such that $\gamma(g(b)) = c$ by the surjectivity of g . Then we see $c = g'(\beta(b))$ by commutativity, hence $c \in \text{im } g'$. This gives $\text{im } \gamma \subseteq \text{im } g'$, and we have an induced quotient map $\text{coker } \gamma \rightarrow \text{coker } g'$. This is our j . By our definition, the exactness of two ends follows, now we begin to verify the rest.

Since $\ker f \subseteq \ker \alpha$, by our definition of f_* , the exactness at $\ker \alpha$ follows. Similar for the exactness of $\text{coker } \gamma$. Since the top row is exact, $\text{im}(f) = \ker(g)$. It follows that $g_* \circ f_* = 0$, so $\text{im } f_* \subseteq \ker g_*$. To verify the converse, assume $b \in \ker g_*$, that is, $b \in \ker \beta$ and $g(b) = 0$. By exactness of B , there is $a \in A$ such that $f(a) = b$. We show that $a \in \ker \alpha$: in fact,

$$f'(\alpha(a)) = \beta(f(a)) = \beta(b) = 0$$

Since f' is injective, this means $\alpha(a) = 0$, so $a \in \ker \alpha$. Thus $\ker g_* = \text{im } f_*$, giving the exactness at $\ker \beta$.

The exactness at $\text{coker } \beta$ arises similarly. Assume $b' + \text{im } \beta \in \ker g'_*$, then $g'(b') \in \text{im } \gamma$. By the surjectivity of g , $\exists b \in B$ such that $g'(b') = \gamma(g(b)) = g'(\beta(b))$. Then $b' = \beta(b) + x$ with $x \in \ker g'$. By exactness, $x = f'(a')$ for some $a' \in A'$. Then it follows

$$b' + \text{im } \gamma = \beta(b) + f'(a') + \text{im } \beta = f'(a')$$

this shows $\ker g'_* \subseteq \text{im } f'_*$, and the claim.

We verify the exactness about δ . First, let $c = g(b)$ for some $b \in \ker \beta$. Since $\beta(b) = 0$, by our definition of δ , $\delta(c) = f'^{-1}(\beta(b)) + \text{im } \alpha = \text{im } \alpha$. This gives $\text{im } g_* \subseteq \ker \delta$. Conversely, if $\delta(c + \text{im } \alpha) = \text{im } \alpha$ for some $c \in C$, we get $\delta(c) \in \text{im } \alpha$, hence $\delta(c) = \alpha(a)$ by some $a \in A$. By the construction of δ , choose some $b \in B$ such that $g(b) = c$, then $f'^{-1}(\beta(b)) = \delta(c) = \alpha(a)$. It follows that

$$\beta(b) = f'(\alpha(a)) = \beta(f(a))$$

and $b = f(a) + x$ for $x \in \ker \beta$. Applying g we get $c = g(b) = g(f(a)) + g(x) = g(x)$ since $g \circ f = 0$. Since $x \in \ker \beta$, $g_*(x) = g(x) = c$. This gives $\ker \delta \subseteq \text{im } g_*$, giving the exactness

at $\ker \gamma$. Finally, we deal with the exactness at $\text{coker } \alpha$. Choose $a' + \text{im } \alpha = \delta(c)$ for $c \in \ker \gamma$. Again the definition of δ : $f'(a') = f'(\delta(c)) = \beta(b)$ for some $g(b) = c, b \in B$. This already gives

$$f'_*(a' + \text{im } \alpha) = f'(\delta(c)) + \text{im } \beta = \beta(b) + \text{im } \beta = \text{im } \beta$$

For the other direction, let $a' + \text{im } \beta \in \ker f'_*$, that is, $f'(a') \in \text{im } \beta$. So let $f'(a') = \beta(b), b \in B$, we observe that $a' = f'^{-1} \circ \beta(b) = f'^{-1} \circ \beta \circ g^{-1}(g(b)) = \delta(g(b))$. This gives $\ker f'_* \subseteq \text{im } \delta$, finishing the proof. \square

7.1.3 Exercise

Exercise 7.1.1. Let A be a local ring, and $f_1, \dots, f_r \in A$ be elements in A generating a non-trivial ideal I of A . Suppose I is principal, show that $I = (f_i)$ for some f_i .

Proof. Let $I = (a)$ with $a \neq 0$ and \mathfrak{m} the unique maximal ideal of A . Then for every f_i we have $f_i = k_i a$ for $k_i \in A$. Now consider the module $\mathfrak{m}I$: we have $\mathfrak{m}I \subseteq I$. If one of k_i is a unit, then $I = (a_i)$ and we are done. So we assume this is not the case. Then, each k_i is contained in \mathfrak{m} , and hence $f_i = k_i a$ is in $\mathfrak{m}I$, hence $I \subseteq \mathfrak{m}I$. Now we have $\mathfrak{m}I = I$, and by Nakayame's lemma we have $I = 0$, which is contradiction. \square

Exercise 7.1.2. Let $A[X]$ be the ring of polynomials in one indeterminate over a ring A . Prove that $A[X]$ is a flat A -algebra.

Proof. We have

$$A[X] = \bigoplus_{i \in \mathbb{N}} (x^n)$$

Each (x^n) is free, hence flat, so $A[X]$ is flat. \square

Exercise 7.1.3. For any A -module, let $M[X]$ denote the set of all polynomials in x with coefficients in M , prove that

$$A[X] \otimes_A M \cong (A \otimes_A M)[X] \cong M[X].$$

Exercise 7.1.4. Let

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

be an exact sequence of A -modules. If L and N are finitely generated, then so is M .

Proof. We have $M \subseteq L + N$ if we identify this sequence as a projection. \square

Exercise 7.1.5. If M is a finitely generated A -module and $f : M \rightarrow M$ is a surjective homomorphism, then f is an isomorphism.

Proof. Define a $A[X]$ -module structure on M via $x \cdot m := f(m)$. Let $I := (x)$, we claim that $IM = M$: In fact, since f is surjective, for each $m \in M$ there is $n \in M$ such that $x \cdot n = f(n) = m$, so $m \in IM$. Now by ?? there exists $g \in I$ such that $(1 + g)M = 0$. By the definition of I , we can write $g(x) = xh(x)$.

Now we prove f is injective. Let $m \in M$ be such that $f(m) = 0$. Then

$$0 = (1 + xh(x)) \cdot m = m + h(x) \cdot f(m) = m + 0 = m.$$

This finishes the proof. \square

Exercise 7.1.6. Let $\mathcal{M} = (M_i, \mu_i)$ be a directed system directed by some set I . We shall construct an A -module M called the direct limit of the direct system M . Let C be the direct sum of the M_i and identify each module M_i with its canonical image in C . Let D be the submodule of C generated by all elements of the form $x_i - \mu_{ij}(x_i)$ where $i \leq j$ and $x_i \in M_i$. Let $M = C/D$, let $\mu : C \rightarrow M$ be the projection and let μ_i be the restriction of μ to M_i . Then clearly $\mu_j = \mu_{ij}\mu_i$. Show that every element of $M = \varinjlim M_i$ can be written in the form $\mu_i(x_i)$ for some $i \in I$ and some $x_i \in M_i$. Show also that if $\mu_i(x_i) = 0$ then there exists $j \geq i$ such that $\mu_{ij}(x_i) = 0$ in M_j .

Proof. For the first claim, $x \in C$ has a representative of the form $x = \sum_{j \in J} x_j$ for some finite subset J . Then we can get $i \in I$ such that $i \geq j$ for all $j \in J$. So

$$x = \mu\left(\sum_{j \in J} x_j\right) = \sum_{j \in J} \mu_j(x_j) = \sum_{j \in J} \mu_i \mu_{ji}(x_j) = \mu_i\left(\sum_{j \in J} \mu_{ji}(x_j)\right)$$

Let $x_i = \sum \mu_{ji}(x_j)$ we get the claim.

For the second claim, suppose $\mu_i(x_i) = 0$ for some $i \in I$. Then, $x_i \in M_i \cap D$ is a finite sum

$$x_i = \sum_{\ell} (y_{j_\ell} - \mu_{j_\ell k_\ell}(y_{j_\ell}))$$

for some finite subset J of I , and $y_{j_\ell} \in M_{j_\ell}$ for each $j \in J$ with $j_\ell \leq k_\ell$. Now, there exists $l \in I$ such that $l \geq k_\ell$ for all $k_\ell \in J \cup \{i\}$. Let $\pi_a : x \rightarrow M_a$ be the projectin, then we have

$$\pi_a = \sum_{j_\ell=a} y_{j_\ell} - \sum_{k_\ell=a} \mu_{j_\ell k_\ell}(y_{j_\ell}).$$

Hence

$$\mu_{al}(\pi_a(x_i)) = \sum_{j_\ell=a} \mu_{al}(y_{j_\ell}) - \sum_{k_\ell=a} \mu_{j_\ell l}(y_{j_\ell})$$

It follows that

$$\begin{aligned} \sum_a \mu_{al} \pi_a(x_i) &= \sum_a \left(\sum_{j_\ell=a} \mu_{al}(y_{j_\ell}) - \sum_{k_\ell=a} \mu_{j_\ell l}(y_{j_\ell}) \right) \\ &= \sum_{\ell} \sum_{a=j_\ell} \mu_{al}(y_{j_\ell}) - \sum_{\ell} \sum_{a=k_\ell} \mu_{j_\ell l}(y_{j_\ell}) \\ &= \sum_{\ell} \mu_{j_\ell l}(y_{j_\ell}) - \sum_{\ell} \mu_{j_\ell l}(y_{j_\ell}) = 0 \end{aligned}$$

Since $x_i \in M_i$, the left sum is $\mu_{il}(x_i)$, and we get $\mu_{il}(x_i) = 0$. □

Exercise 7.1.7. Let I be a directed set, and let (A_i, α_{ij}) , (B_i, β_{ij}) , and (C_i, γ_{ij}) be direct systems of A -modules over I . If $r_i : A_i \rightarrow B_i$ and $s_i : B_i \rightarrow C_i$ are morphisms of direct systems, and if

$$0 \longrightarrow A_i \xrightarrow{r_i} B_i \xrightarrow{s_i} C_i \longrightarrow 0$$

is exact for each $i \in I$, then there is an exact sequence

$$0 \longrightarrow \varinjlim A_i \xrightarrow{r} \varinjlim B_i \xrightarrow{s} \varinjlim C_i \longrightarrow 0$$

where $r = \vec{r}_i$, $s = \vec{s}_i$. Consequently, filtered colimits commute with homology in $A\text{-Mod}$.

Proof. Since direct limit is a left adjoint functor, it is right-exact. Hence we only need to prove r is injective. Suppose that $r(x) = 0$, where $x \in \varinjlim A_i$. Since the index set I is directed, allows us to write $x = \alpha_i(a_i)$, where $\alpha_i : A_i \rightarrow \bigoplus A_i$ is projection, similar for β_i . Now, by definition we have

$$r(x) = \beta_i r_i(a_i).$$

so $\beta_i r_i(a_i) = 0$. By ??, there is a $j \geq i$ such that $\beta_j r_i(a_i) = \beta_{ij} r_i(a_i) = 0$. Since r_i is a map of direct systems, we have

$$r_j \alpha_{ij} = \beta_{ij} r_i$$

Thus $r_j \alpha_{ij}(a_i) = 0$, and by the injectivity of r_j , we conclude $\alpha_{ij}(a_i) = 0$, since we have

$$x = \alpha_i(a_i) = \alpha_j \alpha_{ij}(a_i)$$

we conclude $x = 0$. \square

Exercise 7.1.8. Let $(M_i)_{i \in I}$ be a family of submodules of an A -module, such that for each pair of indices i, j in I there exists $k \in I$ such that $M_i + M_j \subseteq M_k$. Define $i \leq j$ to mean $M_i \subseteq M_j$ and let $\mu_{ij} : M_i \rightarrow M_j$ be the embedding of M_i in M_j . Show that

$$\varinjlim M_i = \sum_i M_i = \bigcup_i M_i.$$

In particular, any A -module is the direct limit of its finitely generated submodules.

Proof. We can easily check that $\bigcup_i M_i$ is a module and satisfies that universal property of $\varinjlim M_i$. And since it has the same description as $\sum M_i$, we get the equality. \square

Exercise 7.1.9. Let $\mathcal{M} = (M_i, \mu_{ij}), \mathcal{N} = (N_i, \nu_{ij})$ be direct systems of A -modules over the same directed set. Let M, N be the direct limits and $\mu_i : M \rightarrow M_i, \nu_i : N_i \rightarrow N$ the associated homomorphisms.

A homomorphism $\Phi : \mathcal{M} \rightarrow \mathcal{N}$ is by definition a family of A -module homomorphisms $\phi_i : M_i \rightarrow N_i$, such that $\phi_i \circ \mu_{ij} = \nu_{ij}$ whenever $i \leq j$. Show that Φ defines a unique homomorphism $\phi = \varinjlim \phi_i : M \rightarrow N$ such that $\phi \circ \mu_i = \nu_i$ for all $i \in I$.

Exercise 7.1.10. Consider a direct limit of rings $A = \varinjlim A_i$, prove that $\varinjlim A = 0$ if and only if $A_i = 0$ for some i .

Proof. If $A = 0$, then in particular $0 = 1$. By exercise ??, for fixed i , there exists $j \in I$ such that $\mu_{ij}(a) = 0$. Since μ_{ij} is also a ring homomorphism from A_i to A_j , this implies that $1 = 0$ in A_j and hence $A_j = 0$. \square

Exercise 7.1.11. Let (A_i, μ_i) be a direct system of rings and let $\mathfrak{n}(A_i)$ be the nilradical of A_i . Show that $\varinjlim \mathfrak{n}(A_i)$ is the nilradical of $\varinjlim A_i$. If each A_i is an integral domain, then $\varinjlim A_i$ is an integral domain.

Proof. Let a be a nilpotent element of A . Then $\exists n \in \mathbb{N}$ such that $a^n = 0$. By ??, $a = \mu_i(x)$ thus $0 = a^n = \mu_i(x^n)$. By ??, $\mu_i(x^n) = 0$ implies $\exists j \geq i$ such that $\mu_{ij}(x^n) = 0$. Hence, $\mu_{ij}(x)$ is in the nilradical of A_j , and

$$\mu_j(\mu_{ij}(x)) = \mu_i(x) = a$$

Hence $a \in \varinjlim \mathfrak{n}(A_i)$.

Conversely, if $a \in \varinjlim \mathfrak{n}(A_i)$, then by ??, there is $x_i \in M_i$ such that $\mu_i(x_i) = a$. Then

$$a^n = \mu_i(x_i^n) = 0$$

so there is $j \geq i$ such that $\mu_{ij}(x_i^n) = 0 = \mu_{ij}(x_i)^n$. This means $\mu_{ij}(x_i)$ is in $\mathfrak{n}(A_j)$. Since

$$\mu_j(\mu_{ij}(x_i)) = \mu_i(x_i) = a,$$

we see a is nilpotent in A . □

7.2 Flat modules

7.2.1 M -flat modules

Let A be a ring, E and M A -modules. Then E is said to be **flat for M** (or **M -flat**) if, for every A -module N and every injective homomorphism $\phi : M' \rightarrow M$, the homomorphism $1 \otimes v : E \otimes_A M' \rightarrow E \otimes_A M$ is injective.

Lemma 7.2.1. *For an A -module E to be M -flat, it is necessary and sufficient that for every finitely generated submodule N of M the canonical homomorphism $1 \otimes v : E \otimes_A N \rightarrow E \otimes_A M$ is injective.*

Proof. Suppose that this condition holds and let N be any submodule of M . Suppose that the canonical image in $E \otimes_A M$ of an element $z = \sum_i x_i \otimes y_i \in E \otimes M$ is zero and let M' be the finitely generated submodule of N generated by the y_i . As by hypothesis the composition map

$$E \otimes_A M' \rightarrow E \otimes_A N \rightarrow E \otimes_A M$$

is injective, the sum $z' = \sum_i x_i \otimes y_i$ considered as an element of $E \otimes_A M'$, is zero. Since z is the image of z' , we have $z = 0$, whence the lemma. □

Lemma 7.2.2. *Let E be an A -module and M an A -module such that E is M -flat. If N is either a submodule or a quotient module of M , then E is N -flat.*

Proof. The case in which N is a submodule is easy, as, if N' is a submodule of N , the composite homomorphism

$$E \otimes_A N' \rightarrow E \otimes_A N \rightarrow E \otimes_A M$$

is injective, hence so is $E \otimes_A N' \rightarrow E \otimes_A N$. Suppose then that N is a quotient module of M , that is there exists an exact sequence

$$0 \longrightarrow R \xrightarrow{i} M \xrightarrow{\pi} N \longrightarrow 0$$

Let N' be a submodule of N and $M' = \pi^{-1}(N')$. Let ι' denote the mapping of R to M' with the same graph as ι , π' the surjection $M' \rightarrow N'$ with the same graph as the restriction of π to M' ,

and ϕ, ψ, γ the canonical injections. The the followiwng diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & R & \xrightarrow{\iota'} & M' & \xrightarrow{\pi'} & N' \longrightarrow 0 \\ & & \downarrow \phi & & \downarrow \psi & & \downarrow \gamma \\ 0 & \longrightarrow & R & \xrightarrow{\iota} & M & \xrightarrow{\pi} & N \longrightarrow 0 \end{array}$$

is commutative and its rows are exact. Tensoring with E , since E is M -flat, we get the following diagram

$$\begin{array}{ccccccc} E \otimes_A R & \xrightarrow{1 \otimes \iota'} & E \otimes_A M' & \xrightarrow{1 \otimes \pi'} & E \otimes_A N' & \longrightarrow 0 \\ \downarrow 1 \otimes \phi & & \downarrow 1 \otimes \psi & & \downarrow 1 \otimes \gamma & & \\ 0 & \longrightarrow & E \otimes_A R & \xrightarrow{1 \otimes \iota} & E \otimes_A M & \xrightarrow{1 \otimes \pi} & E \otimes_A N \longrightarrow 0 \end{array}$$

with row exact. Now since $1 \otimes \phi$ is surjective and $1 \otimes \psi$ is injective, by snake lemma we see $1 \otimes \gamma$ is injective, whence the claim. \square

Lemma 7.2.3. *Let $(M_i)_{i \in I}$ be a family of A -modules, $M = \bigoplus_i M_i$ their direct sum and E an A -module. If, for all $i \in I$, E is flat for M_i , then E is flat for M .*

Proof. Suppose first that $I = \{1, 2\}$, and let M' be a submodule of $M = M_1 \oplus M_2$, M_1 and M_2 being canonically identified with submodules of M . Denote by M'_1 the intersection $M' \cap M_1$ and by M'_2 the image of M' in M_2 under the canonical projection π of M onto M_2 . We have a diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & M'_1 & \xrightarrow{\iota'} & M' & \xrightarrow{\pi'} & M'_2 \longrightarrow 0 \\ & & \downarrow \phi & & \downarrow \psi & & \downarrow \eta \\ 0 & \longrightarrow & M_1 & \xrightarrow{\iota} & M & \xrightarrow{\pi} & M_2 \longrightarrow 0 \end{array}$$

where ι, ι' are the canonical injections and π' is the mapping with the same graph as the restriction of π to M' , which is surjective. It is immediately verified that this diagram is commutative and that its rows are exact.

$$\begin{array}{ccccccc} E \otimes_A M'_1 & \xrightarrow{1 \otimes \iota'} & E \otimes_A M' & \xrightarrow{1 \otimes \pi'} & E \otimes_A M'_2 & \longrightarrow 0 \\ \downarrow 1 \otimes \phi & & \downarrow 1 \otimes \psi & & \downarrow 1 \otimes \eta & & \\ 0 & \longrightarrow & E \otimes_A M_1 & \xrightarrow{1 \otimes \iota} & E \otimes_A M & \xrightarrow{1 \otimes \pi} & E \otimes_A M_2 \longrightarrow 0 \end{array}$$

By hypotheses the two rows of this diagram are exact. As E is flat for M_1 and M_2 , $1 \otimes \phi$ and $1 \otimes \eta$ are injective. The snake lemma then shows that $1 \otimes \psi$ is injective and consequently E is M -flat.

If I is a finite set with n elements, the lemma follows by induction on n . In the general case, let M' be a finitely generated submodule of M . Then there exists a finite subset J of the indexing set I such that M' is contained in the direct sum $M^J = \bigoplus_{i \in J} M_i$. By the finite case E is flat for M^J . The canonical homomorphism $E \otimes_A M' \rightarrow E \otimes_A M^J$ is therefore injective. On the other hand, as M^J is a direct factor of M , the canonical homomorphism $E \otimes_A M^J \rightarrow E \otimes_A M$ is injective. Taking the composition, it follows that $E \otimes_A M' \rightarrow E \otimes_A M$ is injective and E is flat

for M by ??.

□

7.2.2 Flat modules

Proposition 7.2.4. *Let E be an A -module. The following three properties are equivalent:*

- (i) *E is flat for A (in other words, for every ideal \mathfrak{a} of A , the canonical homomorphism $E \otimes_A \mathfrak{a} \rightarrow E \otimes_A A$ is injective).*
- (ii) *E is M -flat for every A -module M .*
- (iii) *The functor $-\otimes_A E$ is exact, that is, E is flat.*

Proof. It is immediate that (ii) is equivalent to (iii) and (ii) implies (i). Conversely, if (i) holds, then by ??, E is fiat for every free left A -module. As every A -module is isomorphic to a quotient of a free module, it follows from ?? that E is flat for M . □

Remark 7.2.1. By ??, for an A -module E to be flat, it is necessary and sufficient that, for every finitely generated ideal \mathfrak{a} of A , the canonical mapping $E \otimes_A \mathfrak{a}$ with image $E\mathfrak{a}$ be injective.

Proposition 7.2.5. *Let $(E_i)_{i \in I}$ be a family of A -modules. For $E = \bigoplus_i E_i$ to be flat, it is necessary and sufficient that each of the E_i be flat.*

Proof. Let $M' \rightarrow M$ be an injective A -module homomorphism. For the direct sum homomorphism

$$\bigoplus_{i \in I} (E_i \otimes_A M) \rightarrow \bigoplus_{i \in I} (E_i \otimes_A M)$$

to be exact, it is necessary and sufficient that each of the homomorphisms $E_i \otimes_A M' \rightarrow E_i \otimes_A M$ be so, which proves the claim since $\bigoplus_{i \in I} (E_i \otimes_A M)$ is canonically identified with $E \otimes_A M$. □

Proposition 7.2.6. *Let I be an ordered set and $(E_\alpha, f_{\beta\alpha})$ a direct system of A -modules. If each of the E_α is flat, then $E = \varinjlim E_\alpha$ is flat.*

Proof. By hypothesis each of the sequences $0 \rightarrow E_\alpha \otimes_A M' \rightarrow E_\alpha \otimes_A M$ is exact; so then is the sequence $0 \rightarrow E \otimes_A M' \rightarrow E \otimes_A M$ since taking the direct limit commutes with tensor products and preserves exactness. □

Example 7.2.7.

- (a) For any ring A , A is clearly a flat A -module. Then it follows from ?? that every free A -module, and more generally every projective A -module, is a flat A -module.
- (b) If A is a semisimple ring, every A -module E is semisimple and hence a direct sum of simple modules. As each of these latter is isomorphic to a direct factor of A , E is projective and therefore flat.
- (c) Let $a \in A$ be such that the map $a_A : x \mapsto ax$ on A is injective (a is not a zero divisor). If E is a flat A -module, then the homothety a_E is injective. In particular, if A is integral, any flat A -module is torsion free. Conversely, if A is a PID, then any torsion free A -module is flat: in fact, if the A -module E is torsion-free, any finitely generated submodule of E is free, and E is the union of these flat submodules, hence flat (??).

Proposition 7.2.8. *Let A be a ring and E an A -module.*

- (a) *Suppose that E is flat. For every element a of A which is not a divisor of 0, the relations $x \in E$, $ax = 0$ imply $x = 0$.*
- (b) *Suppose that A is an integral domain in which every finitely generated ideal is principal. Then for E to be flat it is necessary and sufficient that E be torsion-free.*

Proof. We prove (a). Let $v : A \rightarrow A$ be the A -module homomorphism $t \mapsto at$. The hypothesis implies that v is injective. As E is flat, the homomorphism $1_E \otimes v : E \otimes_A A \rightarrow E \otimes_A A$ is also injective. When $E \otimes_A A$ is canonically identified with E , $1_E \otimes v$ becomes the endomorphism $x \mapsto ax$ of E . Thus the relation $ax = 0$ implies $x = 0$.

We prove (b). By (a), if E is flat, E is torsion-free. Conversely, let E be a torsion-free A -module and let \mathfrak{a} be a nonzero finitely generated ideal of A . By hypothesis $\mathfrak{a} = Aa$ for some nonzero $a \in A$ and $t \mapsto at$ is then an isomorphism v of A onto \mathfrak{a} . Using i to denote the canonical injection $\mathfrak{a} \rightarrow A$, $i \circ v$ is the homothety with ratio a on E and is injective since E is assumed to be torsion-free. Then $1_E \otimes (i \circ v) = (1_E \otimes i) \circ (1_E \otimes v)$. As $1_E \otimes v$ is an isomorphism, $1_E \otimes i$ is injective, which completes the proof. \square

Proposition 7.2.9. *Let E be an A -module. The three following properties are equivalent:*

- (i) *E is flat.*
- (ii) *For every exact sequence of A -modules of the form*

$$0 \longrightarrow G \xrightarrow{\phi} H \xrightarrow{\psi} E \longrightarrow 0 \quad (7.2.1)$$

and every A -module F , the sequence

$$0 \longrightarrow G \otimes_A F \xrightarrow{\phi \otimes 1} H \xrightarrow{\psi \otimes 1} E \otimes_A F \longrightarrow 0 \quad (7.2.2)$$

is exact.

- (iii) *There exists an exact sequence (??), where H is flat, such that the sequence (??) is exact for every A -module F of the form A/\mathfrak{a} , where \mathfrak{a} is a finitely generated ideal of A .*

Proof. The A -module F is isomorphic to a quotient of a free module. In other words, we have an exact sequence $0 \rightarrow R \xrightarrow{\iota} L \xrightarrow{\pi} F \rightarrow 0$ where L is free. Consider the diagram

$$\begin{array}{ccccccc} & G \otimes_A R & \xrightarrow{\phi \otimes 1_R} & H \otimes_A R & \xrightarrow{\psi \otimes 1_R} & E \otimes_A R & \longrightarrow 0 \\ & \downarrow 1_G \otimes \iota & & \downarrow 1_H \otimes \iota & & \downarrow 1_E \otimes \iota & \\ 0 & \longrightarrow & G \otimes_A L & \xrightarrow{\phi \otimes 1_L} & H \otimes_A L & \xrightarrow{\psi \otimes 1_L} & E \otimes_A L \longrightarrow 0 \\ & \downarrow 1_G \otimes \pi & & \downarrow 1_H \otimes \pi & & \downarrow 1_E \otimes \pi & \\ & G \otimes_A F & \xrightarrow{\phi \otimes 1_F} & H \otimes_A F & \xrightarrow{\psi \otimes 1_F} & E \otimes_A F & \longrightarrow 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ & 0 & & 0 & & 0 & \end{array} \quad (7.2.3)$$

It follows immediately that this diagram is commutative, and its rows and columns are exact. Apply the snake lemma to the first two rows, we get the following exact sequence

$$\ker(1_H \otimes \iota) \longrightarrow \ker(1_E \otimes \iota) \xrightarrow{\delta} G \otimes_A F \xrightarrow{\phi \otimes 1_F} H \otimes_A F \quad (7.2.4)$$

Now if E is flat, then $1_E \otimes \iota$ is injective, and the exact sequence shows that $\phi \otimes 1_F$ is injective, hence the sequence (??) is exact.

As (ii) obviously implies (iii), it remains to prove that (iii) implies (i). Consider the diagram (??) in the case $R = \mathfrak{a}$, $L = A$, $F = A/\mathfrak{a}$ and apply the exact sequence (??). By hypothesis $\phi \otimes 1_F$ is injective, so $\text{im } \delta = 0$. Moreover, as H is exact, we see $\ker(1_H \otimes \iota) = 0$, whence $\ker(1_E \otimes \iota) = 0$ and E is flat. \square

Proposition 7.2.10. *Let $0 \longrightarrow E_1 \xrightarrow{\phi} E_2 \xrightarrow{\psi} E_3 \longrightarrow 0$ be an exact sequence of A -modules. Suppose E_3 is flat. Then, for E_2 to be flat it is necessary and sufficient that E_1 be flat.*

Proof. Let $\eta : F' \rightarrow F$ be an injective homomorphism of A -modules. Consider the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & E_1 \otimes_A F' & \xrightarrow{\phi \otimes 1_{F'}} & E_2 \otimes_A F' & \xrightarrow{\psi \otimes 1_{F'}} & E_3 \otimes_A F' \longrightarrow 0 \\ & & \downarrow 1_{E_1} \otimes \eta & & \downarrow 1_{E_2} \otimes \eta & & \downarrow 1_{E_3} \otimes \eta \\ 0 & \longrightarrow & E_1 \otimes_A F & \xrightarrow{\phi \otimes 1_F} & E_2 \otimes_A F & \xrightarrow{\psi \otimes 1_F} & E_3 \otimes_A F \longrightarrow 0 \end{array}$$

It is commutative and its rows are exact by ???. Since E_3 is flat, we know $1_{E_3} \otimes \eta$ is injective. Now snake lemma gives the claim. \square

Let E and F be A -modules. Every element of $E \otimes_A F$ can be written in at least one way in the form $z = \sum_{i=1}^n e_i \otimes f_i$ where $e_i \in E$ and $f_i \in F$. The following lemma gives a condition under which this sum is zero:

Lemma 7.2.11. *Let $(f_i)_{i \in I}$ be a family of generators of F and $(e_i)_{i \in I}$ a family of elements of E of finite support. For $\sum_{i \in I} e_i \otimes f_i = 0$, it is necessary and sufficient that there exist a finite set J , a family $(x_j)_{j \in J}$ of elements of E and a family $(a_{ji})_{j \in J, i \in I}$ of elements of A with the following properties:*

- (a) the family (a_{ji}) has finite support;
- (b) $\sum_{i \in I} a_{ji} f_i = 0$ for all $j \in J$;
- (c) $e_i = \sum_{j \in J} a_{ji} x_j$ for all $i \in I$.

Proof. Consider the free A -module $A^{\oplus I}$, its canonical basis (u_i) and the homomorphism $g : A^{\oplus I} \rightarrow F$ such that $g(u_i) = f_i$ for all $i \in I$. Denoting by R the kernel of g , we have (since the f_i generate F) an exact sequence

$$R \xrightarrow{\iota} A^{\oplus I} \xrightarrow{g} F \longrightarrow 0$$

where ι denotes the canonical injection. We derive from this the exact sequence

$$E \otimes_A R \longrightarrow E \otimes_A A^{\oplus I} \xrightarrow{1 \otimes g} E \otimes_A F \longrightarrow 0 \quad (7.2.5)$$

Now, $E \otimes_A A^{\oplus I}$ is canonically identified with $E^{\oplus I}$ a family $e = (e_i) \in E^{\oplus I}$ being identified with $\sum_{i \in I} e_i \otimes u_i$. For such a family to belong to the kernel of $1_E \otimes g$, it is necessary and sufficient that $\sum_{i \in I} e_i \otimes f_i = 0$ in $E \otimes_A F$. Taking account of the exact sequence (??), this is equivalent to saying that e belongs to the image of $1_E \otimes \iota$, in other words there is a relation of the form

$$\sum_{i \in I} e_i \otimes u_i = \sum_{j \in J} x_j \otimes \iota(r_j) \quad (7.2.6)$$

where $x_i \in E$, $r_i \in R$ and J is finite. Writing $\iota(r_i) = \sum_{j \in I} a_{ji} u_j$, the hypothesis $r_j \in R$ implies the relation $\sum_{i \in I} a_{ji} f_i = 0$ for all $j \in J$. On the other hand the relation (??) implies $e_i = \sum_{j \in J} a_{ji} x_j$ for all $i \in I$ (??), which completes the proof. \square

Proposition 7.2.12. *For E to be F -flat, it is necessary and sufficient that the following condition be satisfied:*

(R) *If $(e_i)_{i \in I}$ and $(f_i)_{i \in I}$ are two finite families of elements of E and F respectively such that $\sum_{i \in I} e_i \otimes f_i = 0$ in $E \otimes_A F$, there exists a finite set J , a family $(x_j)_{j \in J}$ of elements of E and a family $(a_{ji})_{j \in J, i \in I}$ of elements of A , with the following properties:*

- (a) $\sum_{i \in I} a_{ji} f_i = 0$ for all $j \in J$.
- (b) $e_i = \sum_{j \in J} a_{ji} x_j$ for all $i \in I$.

Proof. Suppose that E is F -flat. Let (e_i) and (f_i) be finite families of elements such that $\sum_{i \in I} e_i \otimes f_i = 0$ in $E \otimes_A F$, and let F' be the submodule of F generated by the f_i . Since the canonical mapping $E \otimes_A F' \rightarrow E \otimes_A F$ is injective, we have also $\sum_{i \in I} e_i \otimes f_i = 0$ in $E \otimes_A F'$ and ?? can then be applied to E and F' . Thus families (x_j) and (a_{ji}) are obtained satisfying the conditions of (R).

Conversely, suppose that condition (R) holds. Let F' be a submodule of F and let $y = \sum_{i \in I} e_i \otimes f_i$ be an element of the kernel of the canonical mapping $E \otimes_A F' \rightarrow E \otimes_A F$. Since (R) holds, there exist families (x_j) and (a_{ji}) satisfying conditions (a) and (b). We conclude that, in $E \otimes_A F$,

$$y = \sum_{i,j} a_{ji} x_j \otimes f_i = \sum_{j \in J} (x_j \otimes \sum_{i \in I} a_{ji} f_i) = 0$$

hence the map $E \otimes F' \rightarrow E \otimes F$ is injective. \square

Corollary 7.2.13. *For an A -module E to be flat, it is necessary and sufficient that the following condition hold:*

(RP) *Every solution $\mathbf{y} \in M^n$ of a linear equation $P\mathbf{y} = 0$ with $P \in \mathcal{M}_{mn}(A)$ is a linear combination $\mathbf{y} = Q\mathbf{x}$, where $Q \in \mathcal{M}_{nk}(A)$, $\mathbf{x} \in M^k$ and $PQ = 0$.*

7.2.3 Intersection properties

Lemma 7.2.14. *Let E be an A -module, F an A -module and F_1, F_2 two submodule of F such that $F = F_1 + F_2$. Then the intersection of the canonical images of $E \otimes F_1$ and $E \otimes F_2$ in $E \otimes F$ is equal to the canonical image of $E \otimes (F_1 \cap F_2)$.*

Proof. Consider the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & F_1 \cap F_2 & \longrightarrow & F_1 & \longrightarrow & F_1/(F_1 \cap F_2) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \eta \\ 0 & \longrightarrow & F_2 & \longrightarrow & F_1 + F_2 & \longrightarrow & (F_1 + F_2)/F_2 \longrightarrow 0 \end{array}$$

where the unspecified arrows are the canonical injections and surjections and η is the canonical isomorphism. This diagram is commutative and its rows are exact. We derive (since $F = F_1 + F_2$) a commutative diagram

$$\begin{array}{ccccccc} E \otimes (F_1 \cap F_2) & \longrightarrow & E \otimes F_1 & \longrightarrow & E \otimes (F_1/(F_1 \cap F_2)) & \longrightarrow 0 \\ \downarrow & & \downarrow & & \downarrow 1_E \otimes \eta & & \\ E \otimes F_2 & \longrightarrow & E \otimes F & \longrightarrow & E \otimes ((F_1 + F_2)/F_2) & \longrightarrow 0 & \end{array}$$

The rows of this diagram are exact and $1_E \otimes \eta$ is an isomorphism. A simple diagram chasing shows the claim. \square

Proposition 7.2.15. *Let E be an A -module and F an A -module such that E is F -flat. Then, if F_1, F_2 are two submodules of F , we have*

$$E \otimes (F_1 \cap F_2) = (E \otimes F_1) \cap (E \otimes F_2).$$

Proof. As E is F -flat, $(E \otimes F_1) + (E \otimes F_2)$ is identified with $E \otimes (F_1 + F_2)$, which is a submodule of $E \otimes F$. The assertion then follows from ???. \square

Example 7.2.16. Let k be a field, and consider the subring $A = k[X^2, X^3]$ of the polynomial ring $B = k[X]$ in an indeterminate x . Then $X^2 A \cap X^3 A$ is the set of polynomials made up of terms of degree ≥ 5 in X , so that $(X^2 A \cap X^3 A)B = X^5 B$; but on the other hand $X^2 B \cap X^3 B = X^3 B$. Therefore by the above proposition B is not flat over A .

Proposition 7.2.17. *Let E be an A -module, E' a submodule of E , F an A -module and F' a submodule of F . Suppose that E/E' or F/F' is flat. Then we have*

$$E' \otimes F' = (E' \otimes F) \cap (E \otimes F').$$

Proof. Suppose for example that E/E' is flat and consider the diagram

$$\begin{array}{ccccccc} E' \otimes F' & \longrightarrow & E \otimes F' & \longrightarrow & (E/E') \otimes F & \longrightarrow 0 \\ \downarrow & & \downarrow & & \downarrow \eta & & \\ E' \otimes F & \longrightarrow & E \otimes F & \longrightarrow & (E/E') \otimes F & \longrightarrow 0 & \end{array}$$

where the arrows are the canonical homomorphisms. This diagram is commutative and its rows are exact. As E/E' is flat, η is injective. Then our assertion then follows from a diagram chasing as in ???. \square

Corollary 7.2.18. *Let E be an A -module and E' a submodule of E .*

- (a) Suppose that E/E' is flat. Then, for every ideal \mathfrak{a} of A , we have $\mathfrak{a}E' = \mathfrak{a}E \cap E'$.
- (b) Conversely, suppose that E is flat and, for every finitely generated ideal \mathfrak{a} of A we have $\mathfrak{a}E' = \mathfrak{a}E \cap E'$. Then E/E' is flat.

Proof. For (a), it is sufficient to apply ?? to the case $F = A$ and $F' = \mathfrak{a}$. Conversely, to prove that E/E' is flat, apply criterion (iii) of ?. It is then necessary to establish that the sequence

$$0 \longrightarrow E'/\mathfrak{a}E' \longrightarrow E/\mathfrak{a}E \longrightarrow E/(E' + \mathfrak{a}E) \longrightarrow 0$$

is exact for every finitely generated ideal \mathfrak{a} of A . Now this is precisely what relation $\mathfrak{a}E' = \mathfrak{a}E \cap E'$ expresses. \square

7.2.4 Extension of scalars and flatness

Proposition 7.2.19. *Let R, S be two rings, E an R -module and F an (R, S) -bimodule. Suppose that E is flat and that F is flat as an S -module. Then the S -module $E \otimes_R F$ is flat.*

Proof. Let M be a left S -module and M' a submodule of M . Since F is flat as a S -module, the canonical homomorphism $F \otimes_S M' \rightarrow F \otimes_S M$ is injective. Since E is flat, the canonical homomorphism

$$E \otimes_R (F \otimes_S M') \rightarrow E \otimes_R (F \otimes_S M)$$

is injective. As $E \otimes_R (F \otimes_S M')$ and $E \otimes_R (F \otimes_S M)$ are canonically identified with $(E \otimes_R F) \otimes_S M'$ and $(E \otimes_R F) \otimes_S M$ respectively, the canonical homomorphism

$$(E \otimes_R F) \otimes_S M' \rightarrow (E \otimes_R F) \otimes_S M$$

is injective, which proves that $E \otimes_R F$ is a flat S -module. \square

Corollary 7.2.20. *Let A be a ring, E, F two flat A -modules. Then $E \otimes_A F$ is a flat A -module.*

Proof. We see F is a (A, A) -bimodule and it is sufficient to apply ?? with $R = S = A$. \square

Corollary 7.2.21. *Let $\rho : A \rightarrow B$ be a ring homomorphism. If E is a flat A -module, the B -module $\rho_*(E) = E \otimes_A B$ is flat.*

Proof. The B -module B is flat, so the result follows from ?. \square

Corollary 7.2.22. *Let $\rho : A \rightarrow B$ be a ring homomorphism. If M is a flat A -module and B is a flat A -module, then $\rho^*(M)$ is a flat A -module.*

Proof. Apply ?? with $R = B, S = A, E = M$ and $F = B, B$ having a (B, A) -bimodule structure defined by f , the A -module $M \otimes_B B$ is then $\rho^*(M)$. \square

Example 7.2.23. Suppose that $A = K[X, Y]$ where K is a field. The maximal ideal \mathfrak{m} generated by X and Y is a torsion-free A -module, but not flat. In fact, consider the ring $B = A/(Y)$, which is isomorphic to $K[X]$, hence integral. The B -module $\mathfrak{m} \otimes_A B$ is isomorphic to $\mathfrak{m}/Y\mathfrak{m} = (X, Y)/(XY, Y^2)$, in which the class of Y is torsion (killed by X). Therefore, $\mathfrak{m} \otimes_A B$ is not a flat B -module, so \mathfrak{m} is not flat (??).

Proposition 7.2.24. *Let $\rho : R \rightarrow S$ be a ring homomorphism and F an R -module. The following two properties are equivalent:*

- (i) $\mathrm{Tor}_1^R(\rho^*(E), F) = 0$ for every S -module E .
- (ii) $\mathrm{Tor}_1^S(E, \rho_*(F)) = 0$ for every S -module E and $\mathrm{Tor}_1^R(\rho^*(S), F) = 0$.

Proof. Suppose that (i) holds. Taking $E = S$, we see that $\mathrm{Tor}_1^R(\rho^*(S), F) = 0$. We show also that $\rho_*(F)$ is a flat S -module. For that, we note that, if E is an S -module, the additive group $E \otimes_S \rho_*(F) = E \otimes_S (S \otimes_R F)$ is identified with $\rho_*(E) \otimes_R F$. Then if there is an exact sequence of S -modules

$$0 \longrightarrow E_1 \longrightarrow E_2 \longrightarrow E_3 \longrightarrow 0$$

we obtain, using (i), an exact sequence

$$0 \longrightarrow \rho^*(E_1) \otimes_R F \longrightarrow \rho^*(E_2) \otimes_R F \longrightarrow \rho^*(E_3) \otimes_R F \longrightarrow 0$$

or also

$$0 \longrightarrow E_1 \otimes_S \rho_*(F) \longrightarrow \rho_*(E_2) \otimes_S \rho_*(F) \longrightarrow \rho_*(E_3) \otimes_S \rho_*(F) \longrightarrow 0$$

which proves that $\rho_*(F)$ is flat. Conversely, if (ii) holds, we have first of all, for every free right S -module $L = S^{\oplus I}$, $\mathrm{Tor}_1^R(\rho^*(L), F) = \mathrm{Tor}_1^R(\rho^*(S), F)^{\oplus I} = 0$. Every right S -module E can be written in the form $E = L/H$ for a suitable free S -module L , then we have the exact sequence

$$0 = \mathrm{Tor}_1^R(\rho^*(L), F) \longrightarrow \mathrm{Tor}_1^R(\rho^*(E), F) \longrightarrow \rho^*(H) \otimes_R F \longrightarrow \rho^*(L) \otimes_R F \quad (7.2.7)$$

But as $\rho_*(F)$ is flat, the homomorphism $H \otimes_S \rho_*(F) \rightarrow L \otimes_S \rho_*(F)$ is injective and is identified with the homomorphism $\rho^*(H) \otimes_R F \rightarrow \rho^*(L) \otimes_R F$. Then it follows from (??) that $\mathrm{Tor}_1^R(\rho^*(E), F) = 0$. \square

7.2.5 Finitely presented modules

Let A be a ring. An exact sequence

$$L_1 \longrightarrow L_0 \longrightarrow E \longrightarrow 0$$

of A -modules, where L_0 and L_1 are free, is called a **presentation** of an A -module E . Every A -module E admits a presentation: we know there exists a surjective homomorphism $\phi : L_0 \rightarrow E$, where L_0 is free. If R is the kernel of ϕ , there exists similarly a surjective homomorphism $\psi : L_1 \rightarrow R$, where L_1 is free. If ψ is considered as a homomorphism of L_1 to L_0 , the sequence

$$L_1 \xrightarrow{\psi} L_0 \xrightarrow{\phi} E \longrightarrow 0$$

is exact by definition, whence our assertion. A presentation of a module E is called **finite** if the free modules L_0 and L_1 have finite bases. E is called a **finitely presented** A -module if it admits a finite presentation.

Proposition 7.2.25. *Let A be a ring.*

- (a) Every module admitting a finite presentation is finitely generated.
- (b) If A is a Noetherian ring, every finitely generated A -module admits a finite presentation.
- (c) Every finitely generated projective module admits a finite presentation.

Proof. Assertion (a) follows trivially from the definitions. If A is Noetherian and there exists a surjective homomorphism $\phi : L_0 \rightarrow E$, where L_0 is a free A -module with a finite basis, the kernel R of ϕ is finitely generated, hence there is a surjective homomorphism $\psi : L_1 \rightarrow R$, where L_1 is free and has a finite basis, and the exact sequence

$$L_1 \xrightarrow{\psi} L_0 \xrightarrow{\phi} E \longrightarrow 0$$

is a finite presentation of E ; whence (b).

Finally, suppose that E is a finitely generated projective module; then it is a direct factor of a finitely generated free module L_0 . The kernel R of the surjective homomorphism $L_0 \rightarrow E$ is then isomorphic to a quotient of L_0 and hence finitely generated and the proof is completed as for (b). \square

Proposition 7.2.26. Let A be a ring and E a finitely presented A -module. For every exact sequence

$$0 \longrightarrow F \xrightarrow{\iota} G \xrightarrow{\pi} E \longrightarrow 0$$

where G is finitely generated, the module F is finitely generated.

Proof. Let $L_1 \xrightarrow{\psi} L_0 \xrightarrow{\phi} E \rightarrow 0$ be a finite presentation. If (e_i) is a basis of L_0 , there exists for each i an element $g_i \in G$ such that $\pi(g_i) = \phi(e_i)$. the homomorphism $\eta : L_0 \rightarrow G$ such that $\eta(e_i) = g_i$ for all i then satisfies $\phi = \pi \circ \eta$. As $\phi \circ \psi = 0$, we have $\text{im}(\eta \circ \phi) \subseteq \ker \pi$, and as $\ker \pi$ is isomorphic to F , it can be seen that there is a homomorphism $\gamma : L_1 \rightarrow F$ such that the diagram

$$\begin{array}{ccccccc} L_1 & \xrightarrow{\psi} & L_0 & \xrightarrow{\phi} & E & \longrightarrow & 0 \\ \downarrow \gamma & & \downarrow \eta & & \downarrow 1_E & & \\ 0 & \longrightarrow & F & \xrightarrow{\iota} & G & \xrightarrow{\pi} & E \longrightarrow 0 \end{array}$$

is commutative. As ι is injective and ψ surjective, we can apply the snake lemma, in other words there is an exact sequence

$$0 = \ker 1_E \longrightarrow \text{coker } \gamma \longrightarrow \text{coker } \eta \longrightarrow \text{coker } 1_E = 0$$

This shows that $\text{coker } \gamma$ is isomorphic to $G/\eta(L_0)$, which is finitely generated by hypothesis. Moreover we have the exact sequence

$$0 \longrightarrow \gamma(L_1) \longrightarrow F \longrightarrow \text{coker } \gamma \longrightarrow 0$$

and as $\gamma(L_1)$ and $\text{coker } \gamma$ are finitely generated, so is F . \square

Proposition 7.2.27. Let M be an A -module. There exist a direct system $(M_\alpha, \varphi_{\beta\alpha})$ of finitely presented A -modules such that M is isomorphic to $\varinjlim M_\alpha$. If M is generated by n elements, then we may suppose that M_α is generated by n elements.

Proof. Consider a presentation

$$A^{\oplus K} \xrightarrow{\psi} A^{\oplus L} \xrightarrow{\phi} M \longrightarrow 0$$

Let I be the set of pairs $\alpha = (K', L')$, where K' (resp. L') is a finite subset of K (resp. L), such that ψ induces a map ψ_α from the submodule $A^{K'}$ of $A^{\oplus K}$ to the submodule $A^{L'}$ of $A^{\oplus L}$. For each $\alpha \in I$, let M_α be the cokernel of ψ_α and $\phi_\alpha : A^{L'} \rightarrow M_\alpha$ the canonical map, such that we have a commutative diagram with exact rows:

$$\begin{array}{ccccccc} A^{\oplus K} & \xrightarrow{\psi} & A^{\oplus L} & \xrightarrow{\phi} & M & \longrightarrow & 0 \\ \uparrow & & \uparrow & & \uparrow & & \\ A^{K'} & \xrightarrow{\psi_\alpha} & A^{L'} & \xrightarrow{\phi_\alpha} & M_\alpha & \longrightarrow & 0 \end{array}$$

We order the set I by the relation $(K', L') \preceq (K'', L'')$ if $K' \subseteq K''$ and $K' \subseteq K''$. For $\alpha \preceq \beta$, let $\varphi_{\beta\alpha} : M_\alpha \rightarrow M_\beta$ be the homomorphism induced by the inclusion of $A^{L'}$ to $A^{L''}$. It is immediate to verify that I is filtered and $(M_\alpha, \varphi_{\beta\alpha})$ is a direct system indexed by I . Passing to direct limit, we obtain a commutative diagram

$$\begin{array}{ccccccc} A^{\oplus K} & \xrightarrow{\psi} & A^{\oplus L} & \xrightarrow{\phi} & M & \longrightarrow & 0 \\ i \uparrow & & j \uparrow & & \uparrow & & \\ \varinjlim A^{K'} & \longrightarrow & \varinjlim A^{L'} & \longrightarrow & \varinjlim M_\alpha & \longrightarrow & 0 \end{array}$$

with exact rows. Since i and j are bijective, we conclude that φ is bijective by five lemma. In the case M can be generated by n elements, we have $|L| \leq n$, so each $|L'| \leq n$, which means M_α can be generated by n elements. \square

Let A and B be two rings, E an A -module, F an B -module and G a (B, A) -bimodule. Recall that we have a canonical homomorphism of \mathbb{Z} -modules

$$\nu : F \otimes_B \text{Hom}_A(E, G) \rightarrow \text{Hom}_A(E, F \otimes_B G)$$

such that, for all $y \in F$ and $\phi \in \text{Hom}_A(E, G)$, $\nu(y \otimes \phi)$ is the A -linear mapping $x \mapsto y \otimes \phi(x)$.

Proposition 7.2.28. Let A, B be two rings, E an A -module, F an B -module and G a (B, A) -bimodule. Suppose that F is flat. Then, if E is finitely generated (resp. finitely presented), the canonical homomorphism ν is injective (resp. bijective).

Proof. Consider A, B, F, G as fixed and for each A -module E set

$$T(E) = F \otimes_B \text{Hom}_A(E, G), \quad U(E) = \text{Hom}_A(E, F \otimes_B G)$$

and denote the homomorphism ν by ν_E . For every A -module homomorphism $v : E \rightarrow E'$, set $T(v) = 1_F \otimes \text{Hom}(v, 1_G)$ and $U(v) = \text{Hom}(v, 1_F \otimes 1_G)$. Let

$$L_1 \xrightarrow{\psi} L_0 \xrightarrow{\phi} E \longrightarrow 0$$

be a presentation (resp. finite presentation) of E . We suppose the free module L_0 (resp. the free modules L_0 and L_1) to be finitely generated. The diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & T(E) & \xrightarrow{T(\phi)} & T(L_0) & \xrightarrow{T(\psi)} & T(L_1) \\ & & \downarrow \nu_E & & \downarrow \nu_{L_0} & & \downarrow \nu_{L_1} \\ 0 & \longrightarrow & U(E) & \xrightarrow{U(\phi)} & U(L_0) & \xrightarrow{U(\psi)} & U(L_1) \end{array} \quad (7.2.8)$$

is commutative and the second row is exact. Moreover, the sequence

$$0 \longrightarrow \text{Hom}_A(E, G) \longrightarrow \text{Hom}_A(L_0, G) \longrightarrow \text{Hom}_A(L_1, G)$$

is exact, and as F is flat, the first row of (??) is also an exact sequence. Then we know that ν_{L_0} (resp. ν_{L_0} and ν_{L_1}) is bijective (resp. are bijective). If we assume only that ν_{L_0} is bijective, it follows from (??) that

$$\nu_{L_0} \circ T(\phi) = U(\phi) \circ \nu_E$$

is injective and hence so is ν_E . If we assume that both ν_{L_0} and ν_{L_1} are bijective, it follows from the four lemma that ν_E is surjective, and as we have just seen that ν_E is injective, it is bijective. \square

Now let $\rho : A \rightarrow B$ a ring homomorphism such that B is an A -algebra. Recall that for every ordered pair (E, F) of A -modules, we have a canonical B -homomorphism

$$\omega : \rho^*(\text{Hom}_A(E, F)) \rightarrow \text{Hom}_B(\rho^*(E), \rho^*(F))$$

such that for all $\phi \in \text{Hom}_A(E, F)$, we have $\omega(\phi \otimes 1) = \phi \otimes 1_B$.

Proposition 7.2.29. *Let $\rho : A \rightarrow B$ be a ring homomorphism and E, F two A -modules. Suppose that B is a flat A -module and E is finitely generated (resp. finitely presented). Then the canonical homomorphism ω is injective (resp. bijective).*

Proof. As ω is composed of the canonical homomorphisms

$$B \otimes_A \text{Hom}_A(E, F) \xrightarrow{\nu} \text{Hom}_A(E, B \otimes_A F) \xrightarrow{\cong} \text{Hom}_B(B \otimes_A E, B \otimes_A F)$$

the proposition is a consequence of ??.

Let E be an A -module and (M_i, ϕ_{ji}) be a direct limit of A -modules. Then the canonical map $M_i \rightarrow \varinjlim M_i$ induces a homomorphism $\text{Hom}_A(E, M_i) \rightarrow \text{Hom}_A(E, \varinjlim M_i)$, hence a canonical homomorphism

$$\alpha : \varinjlim \text{Hom}_A(E, M_i) \rightarrow \text{Hom}_A(E, \varinjlim M_i). \quad (7.2.9)$$

Proposition 7.2.30. *If the A -module E is finitely generated (resp. finitely presented), then the canonical homomorphism $(??)$ is injective (resp. bijective).*

Proof. The proof is similar to that of $??$, using the fact that taking direct limit is exact. \square

Corollary 7.2.31. *Any flat and finitely presented module is projective.*

Proof. Let E be such an A -module. By $??$, the canonical homomorphism

$$\mathrm{Hom}_A(E, A) \otimes_A E \rightarrow \mathrm{Hom}_A(E, E)$$

is bijective, which implies E is projective. \square

Proposition 7.2.32. *Let (A_α) be an inductive system of rings and $A = \varinjlim A_\alpha$. For any A -module M of finite presentation, there exists an index α and an A_α -module M_α of finite presentation such that M is isomorphic to $M_\alpha \otimes_{A_\alpha} A$.*

Proof. In fact, we can suppose that $M = A^r/P$ where P is a finitely generated sub- A -module of A^r . As $A^r = \varinjlim A_\alpha^r$ and P is finitely generated, there exists an index α and a finitely generated sub- A_α -module P_α such that P is the canonical image $P_\alpha \otimes_{A_\alpha} A$; the exactness of the tensor product shows that M is isomorphic to $(A_\alpha^r/P_\alpha) \otimes_{A_\alpha} A$. \square

We now consider A -algebras. We say an A -algebra B is of finite type if it is isomorphic to a polynomial algebra of the form $A[T_1, \dots, T_n]/\mathfrak{a}$. We say B is of **finite presentation** if moreover the ideal \mathfrak{a} is finitely generated. For any A -algebra A' , the ring $B' = B \otimes_A A'$ is of finite type if B is, since it is then isomorphic to a quotient of $A'[T_1, \dots, T_n]$ by the canonical image of $\mathfrak{a} \otimes_A A'$, which is finitely generated if \mathfrak{a} is.

If B is an A -algebra of finite presentation, and C is an B -algebra of finite presentation, then C is an A -algebra of finite presentation. In fact, if we suppose $B = A[S_1, \dots, S_m]/\mathfrak{a}$ and $C = B[T_1, \dots, T_n]/\mathfrak{b}$, where \mathfrak{a} and \mathfrak{b} are finitely generated, then the ring $B[T_1, \dots, T_n] = B \otimes_A A[T_1, \dots, T_n]$ is isomorphic to $A[S_1, \dots, S_m, T_1, \dots, T_n]/\mathfrak{a}'$, where \mathfrak{a}' is a finitely generated; the ideal \mathfrak{b} of $B[T_1, \dots, T_n]$ is then of the form $\mathfrak{b}'/\mathfrak{a}'$, where \mathfrak{b}' is an ideal of $A[S_1, \dots, S_m, T_1, \dots, T_n]$. As \mathfrak{a}' and $\mathfrak{b}'/\mathfrak{a}'$ is finitely generated in $A[S_1, \dots, S_m, T_1, \dots, T_n]$, so is \mathfrak{b}' .

If the ring A is Noetherian, then so is $A[T_1, \dots, T_n]$, so any A -algebra of finite type is of finite presentation. However, if A is not Noetherian and \mathfrak{a} is an ideal of A not finitely generated, and A/\mathfrak{a} is a finite algebra, then we can prove that it is not finitely presented.

Proposition 7.2.33. *If B is a finite A -algebra, there exist a surjective homomorphism of A -algebras $\varphi : B' \rightarrow B$, where B' is a finite A -algebra that is of finite presentation and a free A -module.*

Proof. In fact, there exists a finite system of generators $(a_i)_{1 \leq i \leq m}$ of the A -algebra B , such that each a_i satisfies the relation $F_i(a_i) = 0$, where $F_i \in A[T]$ is a monic polynomial of positive degree. The quotient A -algebra $B'_i = A[T]/(F_i)$ is then free with finite rank over A ; let c_i be the class of T in B'_i . There exists a homomorphism of A -algebras $\varphi_i : B'_i \rightarrow B$ such that $\varphi_i(c_i) = a_i$ and let $u : B' \rightarrow B$ be the tensor product of these homomorphisms, where $B' = B'_1 \otimes_A \cdots \otimes_A B'_m$. By construction φ is surjective, and if $C = A[T_1, \dots, T_m]$, C is identified with the tensor product $A[T_1] \otimes_A \cdots \otimes_A A[T_m]$ and B' with the tensor product $(A[T_1]/(F_1(T_1))) \otimes_A \cdots \otimes_A (A[T_m]/(F_m(T_m)))$, hence a quotient C/\mathfrak{c} , where \mathfrak{c} is the ideal generated by the $F_i(T_i)$. This shows that B' is an A -algebra of finite presentation. \square

Corollary 7.2.34. *Let B be a finite A -algebra that is an A -module of finite presentation. Then B is an A -algebra of finite presentation.*

Proof. With the notations of ??, the kernel \mathfrak{a} of $\varphi : B \rightarrow B$ is an A -module of finite type (??) and a fortiori a finite generated ideal of B' . Therefore B is an B' -algebra of finite presentation, and as B' is an A -algebra of finite presentation, we conclude that B is an A -algebra of finite presentation. \square

Proposition 7.2.35. *Any A -algebra B is a filtered limit of A -algebras of finite presentation.*

Proof. For any finite subset L of B , we consider the algebra $C_L = A[(T_m)_{m \in L}]$; if $L \subseteq N$ are two finite subsets of B , let $u_{N,L} : C_L \rightarrow C_N$ be the canonical injection such that any T_m for $m \in L$ are mapped to the same element. Let $u_L : C_L \rightarrow B$ be the homomorphism of A -algebras sending T_m to m . We consider the ideal $P_L = \ker u_L$ and the set of couples $\alpha = (L, R_L)$, where for a given L , R_L is a finite subset of P_L . It is clear that $u_{N,L}(P_L) \subseteq P_N$ for $L \subseteq N$. We order the set of these couples such that $(L, R_L) \leq (N, R_N)$ if $L \subseteq N$ and if $u_{N,L}(R_L) \subseteq R_N$. It is immediate that this defines an order relation and is filtered, since if (L, R_L) and (N, R_N) are two couples, we can form $M = L \cup N$ and let R_M be the union of the images of $u_{M,L}(R_L)$ and $u_{M,N}(R_N)$. The couple (M, R_M) clearly dominant (L, R_L) and (N, R_N) . For any couple $\alpha = (L, R_L)$, we then let $N(R_L) \subseteq P_L$ be the ideal generated by R_L in C_L ; the A -algebra $B_\alpha = C_L/N(R_L)$ is of finite presentation; for $\alpha \leq \beta = (N, R_N)$, we have $u_{N,L}(N(R_L)) \subseteq N(R_N)$, so we have a homomorphism $v_{\beta\alpha} : B_\alpha \rightarrow B_\beta$. It is immediate that $(B_\alpha, v_{\beta\alpha})$ is an inductive system of A -algebras. We now prove that its limit B' is isomorphic to B . Let $v_\alpha : B_\alpha \rightarrow B'$ be the canonical homomorphism and let $w_\alpha : B_\alpha \rightarrow B$ be the homomorphism induced by u_L by passing to quotient. The w_α form an inductive system, with limit $w : B' \rightarrow B$, and it suffices to prove that w is bijective. In view of the definition, it is clear that w is surjective. Suppose that $w(x') = 0$ for an element $x' \in B'$; then there exists a couple $\alpha = (L, R_L)$ and an element $x_\alpha \in B_\alpha$ such that $x' = v_\alpha(x_\alpha)$ and $w_\alpha(x_\alpha) = 0$. Then x_α is the canonical image of an element $y_L \in P_L$; if R'_L is the union of R_L with $\{y_L\}$ and $\beta = (L, R'_L)$, we then have $v_{\beta\alpha}(x_\alpha) = 0$, and therefore $x' = 0$. \square

Corollary 7.2.36. *Let A be a ring and B be an integral A -algebra. Then B is isomorphic to the filtered limit of finite A -algebras of finite presentation.*

Proof. We use the same reasoning, and choose for any element $m \in B$ a polynomial $F_m \in A[T]$ with positive degree and such that $F_m(m) = 0$. We consider for any finite subset L of B the finite A -algebra $C_L = \bigotimes_{m \in L} (A[T_m]/(F_m(T_m)))$, and define $u_{N,L}$ and u_L as in ???. The definition of P_L , R_L , $N(R_L)$ and the order relation are the same. As C_L is a finite A -algebra, $B_\alpha = C_L/N(R_L)$ is a finite A -algebra of finite presentation, and our assertion follows as in ???. \square

Proposition 7.2.37. *Let A be a ring and B be an A -algebra.*

(a) *If B is of finite type over A , for any filtered system (C_λ) of A -algebras, the canonical map*

$$\varinjlim \text{Hom}_A(B, C_\lambda) \rightarrow \text{Hom}_A(B, \varinjlim C_\lambda) \quad (7.2.10)$$

which, to any inductive system of homomorphisms, associates its corresponding inductive limit, is an injective map.

(b) In order that for any filtered system (C_λ) of A -algebras, the map (??) is bijective, it is necessary and sufficient that B is an A -algebra of finite presentation.

Proof. Let $(t_i)_{1 \leq i \leq n}$ be a system of generators of the A -algebra B . We need to prove that if (θ_λ) and (η_λ) are two inductive systems of homomorphisms $B \rightarrow C_\lambda$ such that $\varinjlim \theta_\lambda = \varinjlim \eta_\lambda$, there exists μ such that $\theta_\mu = \eta_\mu$. Now let $\varphi_{\mu\lambda} : C_\lambda \rightarrow C_\mu$ and $\varphi_\lambda : C_\lambda \rightarrow C = \varinjlim C_\lambda$ be the canonical homomorphisms of the inductive system (C_λ) . By hypothesis, for each index i , there exists an index λ_i such that $\varphi_{\lambda_i}(\theta_{\lambda_i}(t_i)) = \varphi_{\lambda_i}(\eta_{\lambda_i}(t_i))$, and if we choose $\lambda_i = \lambda$ for all i , there then exists an index $\mu \geq \lambda$ such that $\varphi_{\mu\lambda}(\theta_\lambda(t_i)) = \varphi_{\mu\lambda}(\eta_\lambda(t_i))$ for all i , which means $\theta_\mu(t_i) = \eta_\mu(t_i)$ for each i , hence $\theta_\mu = \eta_\mu$.

We now prove (b). Suppose that B is of finite presentation, so $B = A[T_1, \dots, T_n]/\mathfrak{a}$, where \mathfrak{a} is a finitely generated ideal. Denote by t_i the class of $T_i \bmod \mathfrak{a}$ and let $(P_j)_{1 \leq j \leq m}$ be a system of generators of \mathfrak{a} . Let $\theta : B \rightarrow C$ be a homomorphism of A -algebras, and put $c_i = \theta(t_i)$. We then have $P_j(c_1, \dots, c_n) = \theta(P_j(t_1, \dots, t_n)) = 0$ for each j . Now there exists an index λ and elements x_1, \dots, x_n in C_λ such that $c_i = \varphi_\lambda(x_i)$, so $\varphi_\lambda(P_j(x_1, \dots, x_n)) = P_j(c_1, \dots, c_n) = 0$, and there then exists another index $\mu \geq \lambda$ such that $\varphi_{\mu\lambda}(P_j(x_1, \dots, x_n)) = P_j(\varphi_{\mu\lambda}(x_1), \dots, \varphi_{\mu\lambda}(x_n)) = 0$ for all j . We then conclude that there exists a homomorphism of A -algebras $\theta_\mu : B \rightarrow C_\mu$ such that $\theta_\mu(t_i) = \varphi_{\mu\lambda}(x_i)$ for each i . If $\theta_\nu = \varphi_{\nu\mu} \circ \theta_\mu$ for any $\nu \geq \mu$, it is clear that $\theta = \varinjlim \theta_\nu$.

Finally we prove the necessity. Let (C_λ) be the filtered system of sub- A -algebras of finite type of B , so that $\varinjlim C_\lambda = B$. Since (??) is bijective, the identity map 1_B factors into $B \rightarrow C_\lambda \rightarrow B$ for an index λ sufficiently large, and this implies $C_\lambda = B$, so B is of finite type over A . Put then $B = C/\mathfrak{a}$, where $C = A[T_1, \dots, T_n]$ and \mathfrak{a} is an ideal of C . Then \mathfrak{a} is the filtered limit of the inductive system of finitely generated ideals $\mathfrak{a}_\lambda \subseteq \mathfrak{a}$ of C . Let $C_\lambda = C/\mathfrak{a}_\lambda$ and using the exactness of the filtered limit, we see B is isomorphic to the limit of the filtered system (C_λ) . Let $p_{\mu\lambda} : C_\lambda \rightarrow C_\mu$ and $p_\lambda : C_\lambda \rightarrow C$ be the canonical homomorphisms. There exists by hypothesis an index λ and an A -homomorphism $u : B \rightarrow C_\lambda$ such that the composition

$$B \xrightarrow{u} C_\lambda \xrightarrow{p_\lambda} B$$

is the identity. Let $q_\lambda : C \rightarrow C_\lambda$ be the canonical projection and put $t_i = p_\lambda(q_\lambda(T_i))$. We then have $p_\lambda(u(t_i)) = t_i = p_\lambda(q_\lambda(T_i))$, so $u(t_i) - q_\lambda(T_i) \in \mathfrak{a}/\mathfrak{a}_\lambda$ for each i . Therefore we can find $\mu \geq \lambda$ such that the n elements $u(t_i) - q_\lambda(T_i)$ belongs to $\mathfrak{a}_\mu/\mathfrak{a}_\lambda$, and $p_{\mu\lambda}(u(t_i)) = p_{\mu\lambda}(q_\lambda(T_i)) = q_\mu(T_i)$. By replacing λ by μ and u by $p_{\mu\lambda} \circ u$, we may then assume that $u(t_i) = q_\lambda(T_i)$ for each i . If $\gamma = p_\lambda \circ q_\lambda$ is the canonical homomorphism $C \rightarrow C/\mathfrak{a} = B$, we can then $u(\gamma(T_i)) = q_\lambda(T_i)$ for all i , whence $q_\lambda = u \circ \gamma$. But then necessarily $\mathfrak{a} = \mathfrak{a}_\lambda$, since if $z \in \mathfrak{a}$ we have $\gamma(z) = q_\lambda(z) = 0$, so $z \in \mathfrak{a}_\lambda$; hence $B = C_\lambda$. \square

Remark 7.2.2. One can wonder if the assumption that (??) is injective for any filtering inductive system (C_λ) imply that B be an A -algebra of finite type. We immediately see that this is not the case: take B equal to a localized $S^{-1}A$ of A , since $A \rightarrow S^{-1}A$ is a epimorphism in the category of rings.

Proposition 7.2.38. Let A_0 be a ring, (A_λ) be an inductive system of A_0 -algebras, and $A = \varinjlim A_\lambda$. If B is an A -algebra of finite presentation, there exists an index λ and an A_λ -algebra B_λ of finite presentation such that B is isomorphic to $B_\lambda \otimes_{A_\lambda} A$.

Proof. Let $\varphi_\lambda : A_\lambda \rightarrow A$ be the canonical homomorphism. By hypothesis, B is isomorphic to an algebra of the form $A[T_1, \dots, T_n]/\mathfrak{a}$, where \mathfrak{a} is a finitely generated ideal. Let $(F_j)_{1 \leq j \leq m}$

be a system of generators of \mathfrak{a} . There then exists an index λ such that the coefficients of the polynomials F_j belong to the image of A_λ under φ_λ . Because of this, there is a system of polynomials $(F_{j,\lambda})_{1 \leq j \leq m}$ of $A_\lambda[T_1, \dots, T_n]$ such that $F_j = \varphi_\lambda(F_{j,\lambda})$ for each j . If \mathfrak{a}_λ is the ideal of $A_\lambda[T_1, \dots, T_n]$ generated by the $F_{j,\lambda}$, \mathfrak{a} is then the image of $\mathfrak{a}_\lambda \otimes_{A_\lambda} A$ in $A[T_1, \dots, T_n] = A_\lambda[T_1, \dots, T_n] \otimes_{A_\lambda} A$; the ring $B_\lambda = A_\lambda[T_1, \dots, T_n]/\mathfrak{a}_\lambda$ then satisfies the requirement. \square

Remark 7.2.3. It is clear that even if B is a finite A -algebra, the localization $S^{-1}B$ of B are not in general A -algebras of finite type. We say that an A -algebra C is **essentially of finite type** if C is A -isomorphic to an A -algebra of the form $S^{-1}B$, where B is a A -algebra of finite type and S a multiplicative subset of B .

Proposition 7.2.39 (Properties of algebras essentially of finite type).

- (a) *If B is an A -algebra essentially of finite type and C is an B -algebra essentially of finite type, then C is an A -algebra essentially of finite type.*
- (b) *Let B be an A -algebra essentially of finite type and A' be an A -algebra. Then $B' = B \otimes_A A'$ is an A -algebra essentially of finite type.*

Proof. Let $B = S^{-1}B'$ and $C = T^{-1}C'$, where B' (resp. C') is an A -algebra (resp. B -algebra) of finite type and S (resp. T) is a multiplicative subset of B' (resp. C'). We can then suppose that $C' = B'[T_1, \dots, T_n]/\mathfrak{a}$ and as $B[T_1, \dots, T_n] = S^{-1}(B'[T_1, \dots, T_n])$, the ideal \mathfrak{a} is of the form $S^{-1}\mathfrak{a}'$, where \mathfrak{a}' is an ideal of $B'[T_1, \dots, T_n]$. This shows $C' = S^{-1}B''$, where $B'' = B'[T_1, \dots, T_n]/\mathfrak{a}'$ is a B' -algebra of finite type. Hence B'' is an A -algebra of finite type, and $C = T^{-1}(S^{-1}B'')$ is an A -algebra essentially of finite type. The assertion in (b) is clear. \square

If B is a local A -algebra essentially of finite type, B is of the form $C_{\mathfrak{q}}$, where C is an A -algebra of finite type and \mathfrak{q} an ideal of C (??). Let \mathfrak{p} be the image in A of the ideal \mathfrak{q} ; if we set $S = A - \mathfrak{p}$, $C_{\mathfrak{q}}$ is also the localization of $S^{-1}C$ at a prime ideal. As $S^{-1}C$ is an algebra of finite type over $A_{\mathfrak{p}} = S^{-1}A$, we see B is also an $A_{\mathfrak{p}}$ -algebra essentially of finite type, and the homomorphism $A_{\mathfrak{p}} \rightarrow B$ is local.

Proposition 7.2.40. *If B is a local A -algebra essentially of finite type, B is A -isomorphic to a quotient of the A -algebra of the form $C_{\mathfrak{q}}$, where C is a polynomial algebra $A[T_1, \dots, T_n]$ and \mathfrak{q} is a prime ideal of C .*

Proof. By definition, B is isomorphic to $C'_{\mathfrak{q}'}$ where C' is an A -algebra of finite type and \mathfrak{q}' is a prime ideal of C' . But $C' = C/\mathfrak{b}$ where $C = A[T_1, \dots, T_n]$ and \mathfrak{b} is an ideal of C . Hence $\mathfrak{q}' = \mathfrak{q}/\mathfrak{b}$, where \mathfrak{q} is a prime ideal of C , and $C'_{\mathfrak{q}'}$ is isomorphic to $C_{\mathfrak{q}}/\mathfrak{b}C_{\mathfrak{q}}$. \square

7.2.6 Structure of flat modules

Lemma 7.2.41. *Let I be a filtered set and $(E_\alpha, \varphi_{\beta\alpha})$ be a direct system of sets indexed by I . Let E be the filtered limit and $\varphi_\alpha : E_\alpha \rightarrow E$ the canonical maps. Let $f : I \rightarrow I$ be a map such that $f(\alpha) \succ \alpha$ for $\alpha \in I$, and suppose given, for each $\alpha \in I$, a set L_α and maps $u_\alpha : E_\alpha \rightarrow L_\alpha$, $v_\alpha : L_\alpha \rightarrow E_{f(\alpha)}$ such that*

the following diagram commutes

$$\begin{array}{ccc}
 E_\alpha & \xrightarrow{\varphi_{f(\alpha),\alpha}} & E_{f(\alpha)} \\
 u_\alpha \searrow & & \swarrow v_\alpha \\
 & L_\alpha &
 \end{array}$$

Let J be the ordered set obtained by endowing I with the relation $\alpha \preceq \beta$ if $\alpha = \beta$ or $f(\alpha) \preceq \beta$. If $\alpha, \beta \in J$ and $\alpha \preceq \beta$, let $\psi_{\beta\alpha} : L_\alpha \rightarrow L_\beta$ be the map such that

$$\psi_{\beta\alpha} = \begin{cases} 1 & \text{if } \alpha = \beta, \\ u_\beta \circ \varphi_{\beta,f(\alpha)} \circ v_\alpha & \text{if } f(\alpha) \preceq \beta. \end{cases}$$

If $\alpha \in J$, let $\psi_\alpha : L_\alpha \rightarrow E$ be the map $\varphi_{f(\alpha)} \circ v_\alpha$. Then the ordered set J is filtered, $(L_\alpha, \psi_{\beta\alpha})$ is a direct system indexed by J , (ψ_α) is a direct system of maps and the map $\psi : \varinjlim_{\alpha \in J} L_\alpha \rightarrow E$ induced by ψ_α is bijective.

Proof. It is clear J is filtered. If $\alpha, \beta \in J$ and $\alpha \prec \beta$, we have

$$\psi_\beta \circ \psi_{\beta\alpha} = \varphi_{f(\beta)} \circ v_\beta \circ u_\beta \circ \varphi_{\beta,f(\alpha)} \circ v_\alpha = \varphi_{f(\beta)} \circ \varphi_{f(\beta),\beta} \circ \varphi_{\beta,f(\alpha)} = \varphi_{f(\alpha)} \circ v_\alpha = \psi_\alpha;$$

similarly, if $\alpha, \beta, \gamma \in J$ and $\alpha \prec \beta \prec \gamma$,

$$\begin{aligned}
 \psi_{\gamma\beta} \circ \psi_{\beta\alpha} &= u_\gamma \circ \varphi_{\gamma,f(\beta)} \circ v_\beta \circ u_\beta \circ \varphi_{\beta,f(\alpha)} \circ v_\alpha \\
 &= u_\gamma \circ \varphi_{\gamma,f(\beta)} \circ \varphi_{f(\beta),\beta} \circ \varphi_{\beta,f(\alpha)} \circ v_\alpha = u_\gamma \circ \varphi_{\gamma,f(\alpha)} \circ v_\alpha = \psi_{\gamma\alpha}.
 \end{aligned}$$

We now prove the last assertion. For $\alpha \in J$, we have

$$\psi_\alpha \circ u_\alpha = \varphi_{f(\alpha)} \circ v_\alpha \circ u_\alpha = \varphi_{f(\alpha)} \circ \varphi_{f(\alpha),\alpha} = \varphi_\alpha,$$

hence $\varphi_\alpha(E_\alpha) = \psi_\alpha(u_\alpha(E_\alpha)) \subseteq \psi_\alpha(L_\alpha)$ and ψ is surjective. Let $\alpha \in J$ and $x, y \in L_\alpha$ be such that $\psi_\alpha(x) = \psi_\alpha(y)$, i.e., $\varphi_{f(\alpha)}(v_\alpha(x)) = \varphi_{f(\alpha)}(v_\alpha(y))$; there then exists $\beta \in I$, $\beta \succeq f(\alpha)$ such that

$$\varphi_{\beta,f(\alpha)}(v_\alpha(x)) = \varphi_{\beta,f(\alpha)}(v_\alpha(y)),$$

hence

$$\psi_{\beta,\alpha}(x) = u_\beta(\varphi_{\beta,f(\alpha)}(v_\alpha(x))) = u_\beta(\varphi_{\beta,f(\alpha)}(v_\alpha(y))) = \psi_{\beta,\alpha}(y).$$

This shows ψ is injective. □

Theorem 7.2.42 (D. Lazard). For an A -module E , the following conditions are equivalent:

- (i) E is flat;
- (ii) For any A -module P that is finitely presented, the canonical homomorphism

$$\mathrm{Hom}_A(P, A) \otimes_A E \rightarrow \mathrm{Hom}_A(P, E)$$

is surjective.

(iii) For any A -module P that is finitely presented and any homomorphism $\phi : P \rightarrow E$, there exist a finitely generated free A -module L and homomorphisms $\psi : P \rightarrow L$ and $\eta : L \rightarrow E$ such that $\phi = \eta \circ \psi$.

(iv) There exist a filtered ordered set I and a direct system $(L_i)_{i \in I}$ of free modules such that E is isomorphic to $\varinjlim L_i$.

Proof. We note that (iv) \Rightarrow (i) by ??, and (i) \Rightarrow (ii) by ?. We now show that (ii) \Rightarrow (iii): let P be a finitely presented A -module and $\phi : P \rightarrow E$ a homomorphism; by (ii), there exist $\psi_1, \dots, \psi_n \in \text{Hom}_A(P, A)$ and $w_1, \dots, w_n \in E$ such that $\phi(x) = \sum_i \psi_i(x)w_i$ for $x \in P$. If $\psi : P \rightarrow A^n$ is the homomorphism $x \mapsto (\psi_i(x))$ and $\eta : A^n \rightarrow E$ is the homomorphism $(a_i) \mapsto \sum_i a_i w_i$, we then have $\phi = \eta \circ \psi$.

Finally, assume that (iii) holds, and let $(E_\alpha, \varphi_{\beta, \alpha})$ be a filtered system of finitely presented A -modules (?). By replacing I with the product $I \times \mathbb{N}$ and set $E_{(\alpha, n)} = E_\alpha$ for each n , we may assume that I has no biggest element. For each $\alpha \in I$, by applying (iii) on the canonical map $\varphi_\alpha : E_\alpha \rightarrow E$, we can find a finitely generated A -module L_α and homomorphisms $u_\alpha : E_\alpha \rightarrow L_\alpha$, $w_\alpha : L_\alpha \rightarrow E$ such that $\varphi_\alpha = w_\alpha \circ u_\alpha$. Since L_α is finitely generated free and I has no biggest element, there exists $\beta \succ \alpha$ and a homomorphism $\tilde{v}_\alpha : L_\alpha \rightarrow E_\beta$ such that $w_\alpha = \varphi_\beta \circ \tilde{v}_\alpha$ (see the construction in ??):

$$\begin{array}{ccccc} E_\alpha & \xrightarrow{\varphi_\alpha} & E & & \\ \downarrow u_\alpha & \nearrow w_\alpha & \uparrow \varphi_\beta & \nearrow \varphi_\gamma & \\ L_\alpha & \xrightarrow{\tilde{v}_\alpha} & E_\beta & \xrightarrow{\varphi_{\gamma\alpha}} & E_\gamma \end{array}$$

But note that E_α is finitely presented and we have

$$\varphi_\beta \circ \tilde{v}_\alpha \circ u_\alpha = w_\alpha \circ u_\alpha = \varphi_\alpha = \varphi_\beta \circ \varphi_{\beta, \alpha}$$

so it follows from ?? that there exist $\gamma \succeq \beta$ such that

$$\varphi_{\gamma\beta} \circ \tilde{v}_\alpha \circ u_\alpha = \varphi_{\gamma\beta} \circ \varphi_{\beta, \alpha} = \varphi_{\gamma\alpha}.$$

Put $\gamma = f(\alpha)$ and let $v_\alpha = \varphi_{\gamma\beta} \circ \tilde{v}_\alpha$, we see $v_\alpha \circ u_\alpha = \varphi_{f(\alpha), \alpha}$, so ?? gives the claim. \square

Corollary 7.2.43. For any flat module E , the A -modules $T(E)$, $S(E)$, $\Lambda(E)$, $T^n(E)$, $S^n(E)$, $\Lambda^n(E)$ are flat.

Proof. In fact, E is a direct limit of free A -modules (L_i) , hence $T(E)$ (resp. $S(E)$, etc.) is a direct limit of free A -modules $(T(L_i))$ (resp. $S(L_i)$, etc.), hence flat. \square

7.3 Faithfully flat modules

7.3.1 Faithfully flatness

Proposition 7.3.1. Let E be an A -module. The following properties are equivalent:

- (i) For a sequence \mathcal{S} of A -modules to be exact, it is necessary and sufficient that the sequence $E \otimes_A \mathcal{S}$ be exact.

- (ii) E is flat and, for every A -module N , the relation $E \otimes_A N = 0$ implies $N = 0$.
- (iii) E is flat and, for every homomorphism $v : N' \rightarrow N$ of A -modules, the relation $1_E \otimes v = 0$ implies $v = 0$.
- (iv) E is flat and, for every maximal ideal \mathfrak{m} of A , $\mathfrak{m}E \neq E$.

Proof. Let \mathcal{S} be the sequence $0 \rightarrow N \rightarrow 0$. If $N \otimes_A M = 0$ then $\mathcal{S} \otimes_A M$ is exact, so \mathcal{S} is exact, and therefore $N = 0$. This shows (i) implies (ii)

We show that (ii) implies (iii). Suppose that (ii) holds and let $v : N' \rightarrow N$ be a homomorphism and P its image. As the image of $1_E \otimes v$ is identified with $E \otimes_A P$, the hypothesis $1_E \otimes v = 0$ implies $E \otimes_A P = 0$, hence $P = 0$ by (ii) and consequently $v = 0$.

Now suppose then that (iii) holds and consider a sequence

$$0 \longrightarrow M \xrightarrow{\phi} N \xrightarrow{\psi} P \longrightarrow 0 \quad (7.3.1)$$

of homomorphism of A -modules and the corresponding sequence

$$0 \longrightarrow E \otimes_A M \xrightarrow{1_E \otimes \phi} E \otimes_A N \xrightarrow{1_E \otimes \psi} E \otimes_A P \longrightarrow 0 \quad (7.3.2)$$

If the sequence (??) is exact, so is (??), since E is flat. Conversely, if (??) is exact, we have first $1_E \otimes (\psi \circ \phi) = (1_E \otimes \psi) \circ (1_E \otimes \phi) = 0$, hence $\psi \circ \phi = 0$ by hypothesis. Set $H = \text{im } \phi$ and $K = \ker \psi$, then H is contained in K by the above. Consider the exact sequence

$$0 \longrightarrow H \xrightarrow{\iota} K \xrightarrow{\pi} K/H \longrightarrow 0$$

ι and π being the canonical mappings. As E is flat, the sequence

$$0 \longrightarrow E \otimes_A H \xrightarrow{1_E \otimes \iota} E \otimes_A K \xrightarrow{1_E \otimes \pi} E \otimes_A (K/H) \longrightarrow 0$$

is exact, in other words, $E \otimes_A (K/H)$ is isomorphic to $(E \otimes_A K)/(E \otimes_A H)$, which is 0 by hypothesis, since $E \otimes_A H$ (resp. $E \otimes_A K$) is identified with the image of $1 \otimes \phi$ (resp. the kernel of $1 \otimes \psi$). But the relation $1_E \otimes \pi = 0$ implies $\pi = 0$ by hypothesis, hence $K = H$, which proves that the sequence (??) is exact at N . Similarly we can show that (??) is exact at M and P , so (iii) implies (i).

Finally we show the equivalence of (ii) and (iv). If (iv) holds, then $E/\mathfrak{m}E = E \otimes_A (A/\mathfrak{m}) \neq 0$ since $A/\mathfrak{m} \neq 0$, whence (iv). Conversely, suppose that (iv) holds. Then every proper ideal \mathfrak{a} of A is contained in a maximal ideal \mathfrak{m} , and the hypothesis $E \neq \mathfrak{m}E$ implies $E \neq \mathfrak{a}E$, in other words, $E \otimes_A (A/\mathfrak{a}) \neq 0$. That is to say, for every cyclic A -module $N \neq 0$, $E \otimes_A N \neq 0$. If now N is any nonzero A -module, it contains a cyclic submodule $N' \neq 0$. Since E is flat, $E \otimes_A N'$ is identified with a subgroup of $E \otimes_A N$ and we have just seen that $E \otimes_A N' \neq 0$, hence $E \otimes_A N \neq 0$. \square

An A -module E is called **faithfully flat** if it satisfies the equivalent conditions of ???. Note that if E is a faithfully flat A -module, E is a faithful A -module: for, if an element $a \in A$ is such that $ax = 0$ for all $x \in E$, the homothety $h_a : b \mapsto ab$ in A is such that $1_E \otimes h_a = 0$, whence $h_a = 0$ by property (iii) of ???, that is $a = 0$ since A has a unit element.

Example 7.3.2.

- (a) The direct sum of a flat module and faithfully flat module is a faithfully flat module by virtue of property (iv) of ?? and ??.
- (b) As A is faithfully flat by virtue of criterion (iv) of ??, it follows from (a) that every free module not reduced to 0 is faithfully flat. On the other hand, there exist non-zero direct factors of free modules (in other words, non-zero projective modules) which are faithful but not faithfully flat.
- (c) Let A be a PID. For an A -module E to be faithfully flat, it is necessary and sufficient that it be torsion-free and $E \neq pE$ for every irreducible element p of A . This follows immediately from ?? and criterion (iv) of ?. In particular, the \mathbb{Z} -module \mathbb{Q} is flat but not faithfully flat.

Proposition 7.3.3. *Let E be a faithfully flat A -module and $\phi : N' \rightarrow N$ an A -module homomorphism. For ϕ to be injective (resp. surjective, bijective), it is necessary and sufficient that $1_E \otimes \phi : E \otimes_A N' \rightarrow E \otimes_A N$ be so.*

Proof. This is an immediate consequence of criterion (i) of ??.

□

Proposition 7.3.4. *Let $0 \rightarrow E_1 \rightarrow E_2 \rightarrow E_3 \rightarrow 0$ be an exact sequence of A -modules. Suppose that E_1 and E_3 are flat and that one of them is faithfully flat. Then E_2 is faithfully flat.*

Proof. We know already that E is flat by ?. We verify that E has property (ii) of ?. Let N be an A -module. As E_3 is flat, there is an exact sequence

$$0 \longrightarrow E_1 \otimes_A N \longrightarrow E_2 \otimes_A N \longrightarrow E_3 \otimes_A N \longrightarrow 0$$

If $E \otimes_A N = 0$, it follows that $E_1 \otimes_A N$ and $E_2 \otimes_A N$ are zero. As one of the modules E_1, E_2 is faithfully flat, this implies that $N = 0$.

□

7.3.2 Extension of scalars

Proposition 7.3.5. *Let R, S be two rings, E an R -module and F an (R, S) -bimodule. Suppose that E is faithfully flat. Then for F to be a flat (resp. faithfully flat) S -module, it is necessary and sufficient that $E \otimes_R F$ be so.*

Proof. If F is flat, then $E \otimes_R F$ is flat by ?. Suppose that $E \otimes_R F$ is flat and let $v : N' \rightarrow N$ be an injective S -module homomorphism. The homomorphism

$$1_E \otimes 1_F \otimes v : E \otimes_R F \otimes_S N' \rightarrow E \otimes_R F \otimes_S N$$

is then injective. It follows from ?? that $1_F \otimes v : F \otimes_S N' \rightarrow F \otimes_S N$ is injective; then F is a flat S -module.

Suppose that F is faithfully flat and let N be a S -module such that $E \otimes_R F \otimes_S N = 0$. Since E is faithfully flat, this implies that $F \otimes_S N = 0$, whence $N = 0$ since F is faithfully flat. This proves that $E \otimes_R F$ is faithfully flat.

Suppose that $E \otimes_R F$ is faithfully flat and let N be a S -module such that $F \otimes_S N = 0$. Then $E \otimes_R F \otimes_S N = 0$, whence $N = 0$, which shows that F is faithfully flat.

□

Corollary 7.3.6. *Let A be a ring and E and F two faithfully flat A -modules. Then the A -module $E \otimes_A F$ is faithfully flat.*

Proposition 7.3.7. *Let $\rho : A \rightarrow B$ be a ring homomorphism. If E is a faithfully flat A -module, the B -module $\rho_*(E) = E \otimes_A B$ is faithfully flat.*

Proof. Note that the B -module B is faithfully flat, so apply ??.

□

Corollary 7.3.8. *If E is a faithfully flat A -module and \mathfrak{a} is an ideal of A , then the (A/\mathfrak{a}) -module $E/\mathfrak{a}E$ is faithfully flat.*

Proof. This follows from $E/\mathfrak{a}E = E \otimes_A (A/\mathfrak{a})$ and ??.

□

Proposition 7.3.9. *Let $\rho : A \rightarrow B$ be a ring homomorphism. Suppose that B is a faithfully flat A -module. Then, for an A -module E to be flat (resp. faithfully flat), it is necessary and sufficient that the B -module $\rho_*(E) = E \otimes_A B$ be flat (resp. faithfully flat).*

Proof. If E is flat (resp. faithfully flat), $\rho_*(E)$ is flat (resp. faithfully flat) by ?? (resp. by ??). Suppose that $\rho_*(E)$ is flat and let $v : N' \rightarrow N$ be an injective A -module homomorphism. By ??, the A -module $E \otimes_A B$ is flat, hence the homomorphism $1_E \otimes 1_B \otimes v$ is injective. As the right and left A -module structures on B coincide, this homomorphism is identified with

$$1_E \otimes v \otimes 1_B : E \otimes_A N' \otimes_A B \rightarrow E \otimes_A N' \otimes_A B$$

As B is a faithfully flat A -module, it follows that $1_E \otimes v$ is injective, which shows that E is flat.

Suppose finally that $\rho_*(E)$ is faithfully flat. First of all E is flat by the argument above. Also let N be an A -module such that $E \otimes_A N = 0$. Then $E \otimes_A N \otimes_A B = 0$, whence $E \otimes_A B \otimes_A B = 0$, which may also be written as $(E \otimes_A B) \otimes_B (B \otimes_A N) = 0$. As $\rho_*(E)$ is a faithfully flat B -module, this implies that $B \otimes_A N = 0$, whence $N = 0$ since B is a faithfully flat A -module. □

Proposition 7.3.10. *Let $\rho : A \rightarrow B$ be a ring homomorphism. Let E be a faithfully flat B -module. For $\rho^*(E)$ to be a flat (resp. faithfully flat) A -module, it is necessary and sufficient that B be a flat (resp. faithfully flat) A -module.*

Proof. Applying ?? with $R = B$, $S = A$, $E, F = B$, it is seen that B is a flat (resp. faithfully flat) A -module if and only if $E \otimes_B B = \rho^*(E)$ is a flat (resp. faithfully flat) A -module. □

Corollary 7.3.11. *Let $\rho : A \rightarrow B$ be a ring homomorphism. For B to be a faithfully flat A -module, it is sufficient that there exist one faithfully flat B -module which is also a faithfully flat A -module.*

Corollary 7.3.12. *Let $\rho : A \rightarrow B$ and $v : B \rightarrow C$ be a ring homomorphisms.*

- (a) *If C is a faithfully flat B -module and B is a faithfully flat A -module, then C is a faithfully flat A -module.*
- (b) *If C is a faithfully flat B -module and a faithfully flat A -module, then B is a faithfully flat A -module.*

Proof. Part (a) follows from ??, and (b) follows from ??.

□

7.3.3 Faithfully flat rings

Proposition 7.3.13. *Let $\rho : A \rightarrow B$ be a ring homomorphism. Suppose that there exists a B -module E such that $\rho^*(E)$ is a faithfully flat A -module. Then*

- (a) *For every A -module F , the canonical homomorphism $j : F \rightarrow B \otimes_A F$ is injective.*
- (b) *For every ideal \mathfrak{a} of A , $\mathfrak{a}^{ec} = \mathfrak{a}$.*
- (c) *The homomorphism ρ is injective.*
- (d) *For every maximal ideal \mathfrak{m} of A , there exists a maximal left ideal \mathfrak{M} of B such that $\mathfrak{M}^c = \mathfrak{m}$.*

Proof. We know from ?? that for every B -module M , the canonical A -homomorphism

$$i : \rho^*(M) \rightarrow \rho^*(M) \otimes_A B = \rho_*(\rho^*(M)), \quad y \mapsto y \otimes 1$$

is injective and that the A -module $i(\rho^*(M))$ is a direct factor of $\rho^*(M) \otimes_A B$. Since tensor commutes with direct sum, for every A -module F we have

$$i \otimes 1_F = 1_M \otimes j : \rho^*(M) \otimes_A F \rightarrow \rho^*(M) \otimes_A B \otimes_A F$$

is injective. Taking $M = E$, it follows that j is injective, and this proves assertion (a). Assertion (b) follows from (a) by taking $F = A/\mathfrak{a}$ and (c) from (b) by taking $\mathfrak{a} = \{0\}$.

Finally, if \mathfrak{m} is a maximal ideal of A , then $\mathfrak{m}^{ec} = \mathfrak{m}$ by (b), and consequently $\mathfrak{m}^e \neq B$. Then there exists a maximal ideal \mathfrak{M} of B containing \mathfrak{m}^e , and $\mathfrak{m} \subseteq \mathfrak{M}^c$ with $1 \notin \mathfrak{M}^c$. Consequently $\mathfrak{M}^c = \mathfrak{m}$. \square

Corollary 7.3.14. *Under the hypotheses of ??, if B is Noetherian (resp. Artinian), so is A .*

Proof. If (\mathfrak{a}_n) is a non-stationary increasing (resp. decreasing) sequence of ideals of A , the sequence (\mathfrak{a}_n^e) of ideals of B is non-stationary and increasing (resp. decreasing) since $\mathfrak{a}_n^{ec} = \mathfrak{a}_n$, contrary to the hypothesis. \square

An important application of ?? is when B is itself a faithfully flat A -module. But in this case we have the following more precise proposition:

Proposition 7.3.15. *Let $\rho : A \rightarrow B$ be a ring homomorphism. The following properties are equivalent:*

- (i) *The A -module B is faithfully flat.*
- (ii) *The homomorphism ρ is injective and the A -module $B/\rho(A)$ is flat.*
- (iii) *The A -module B is flat and, for every A -module F , the canonical homomorphism $F \rightarrow B \otimes_A F$ is injective.*
- (iv) *The A -module B is flat and, for every ideal \mathfrak{a} of A , $\mathfrak{a}^{ec} = \mathfrak{a}$.*
- (v) *The A -module B is flat and, for every maximal ideal \mathfrak{m} of A , there exists a maximal ideal \mathfrak{M} of B such that $\mathfrak{M}^c = \mathfrak{m}$.*

Proof. By ??, (i) implies each of the properties (iii), (iv), (v). On the other hand, if (v) holds, then $\mathfrak{m}^e \neq B$ for every maximal ideal \mathfrak{m} of A , and B is a faithfully flat A -module by criterion (iv) of ??, hence (v) implies (i).

We shall now prove that (iii) \Rightarrow (iv) \Rightarrow (ii) \Rightarrow (i), which will complete the proof. In the first place, (iii) implies (iv) by taking $F = A/\mathfrak{a}$ in (iii). If (iv) holds, by taking $\mathfrak{a} = \{0\}$ it follows that f is injective. Condition (iv) and ?? imply that $B/\rho(A)$ is a flat A -module, that is (iv) implies (ii). Finally, if (ii) holds, ?? applied to the exact sequence

$$0 \longrightarrow A \xrightarrow{\rho} B \longrightarrow B/\rho(A) \longrightarrow 0$$

shows that B is a faithfully flat A -module, since A is faithfully flat. \square

Under the conditions of ??, let us identify A with a subring of B by means of f . The relation $\mathfrak{a}^{ec} = \mathfrak{a}$ then reads $A \cap B\mathfrak{a} = \mathfrak{a}$. On the other hand, if F is an A -module, F is identified with its image in $B \otimes_A F$ under the canonical mapping $x \mapsto 1 \otimes x$. If F' is a submodule of F , then we denote by BF' the sub- B -module of $B \otimes_A F$ generated by F' . With this notation, we have:

Proposition 7.3.16. *Let B be a ring and A a subring of B such that B is a faithfully flat A -module. Let F be an A -module, F_1, F_2 two submodules of F . Then:*

- (a) *The canonical mapping $B \otimes_A F_1 \rightarrow B \otimes_A F$ is an isomorphism of $B \otimes_A F_1$ onto BF_1 .*
- (b) *$F \cap BF_1 = F_1$.*
- (c) *$B(F_1 \cap F_2) = BF_1 \cap BF_2$ and $B(F_1 + F_2) = BF_1 + BF_2$.*

Proof. As B is a flat A -module, the canonical mapping $B \otimes_A F_1 \rightarrow B \otimes_A F$ is injective. Taking account of the identifications made, its image is BF_1 , which proves (a). Assertion (b) follows from ??, applied with $E = B$, $E' = A$, and using the formulae. Assertion (c) follows from ??.

Proposition 7.3.17. *Let B be a ring and A a subring of B such that B is a faithfully flat A -module. For an A -module F to be finitely generated (resp. finitely presented), it is necessary and sufficient that the B -module $B \otimes_A F$ be finitely generated (resp. finitely presented).*

Proof. Without any hypothesis on B , clearly, if F is a finitely generated A -module, $B \otimes_A F$ is a finitely generated B -module. Conversely, if $B \otimes_A F$ is a finitely generated B -module, it is generated by a finite number of elements of the form $1 \otimes x_i$, with $x_i \in F$. If M is a sub- A -module of F generated by the x_i and j the canonical injection $M \rightarrow F$, then $1_B \otimes j$ is a surjective homomorphism, hence j is surjective, which proves that F is finitely generated.

If F admits a finite presentation, so does $B \otimes_A F$ without any hypothesis on B . It remains to prove that, if $B \otimes_A F$ admits a finite presentation, so does F . We already know that F is finitely generated, hence there exists a surjective homomorphism $\phi : L \rightarrow F$, where L is a finitely generated free A -module. Let R be the kernel of ϕ , so that $B \otimes_A R$ is identified with the kernel of the surjective homomorphism $1_B \otimes \phi : B \otimes_A L \rightarrow B \otimes_A F$. As $B \otimes_A F$ admits a finite presentation by hypothesis, we conclude that $B \otimes_A R$ is finitely generated (??). Then it follows from the argument above that R is a finitely generated A -module and consequently F admits a finite presentation. \square

Proposition 7.3.18. *Let B be a ring and A a subring of B such that B is a faithfully flat A -module. For an A -module F to be projective and finitely generated, it is necessary and sufficient that $B \otimes_A F$ be a finitely generated projective B -module.*

Proof. The condition is obviously necessary without any hypothesis on A or B . We prove that it is sufficient. If a finitely generated projective module admits a finite presentation, the hypothesis implies that F admits a finite presentation by virtue of ??, hence, by ?? for every A -module M , there is a canonical isomorphism

$$\omega : B \otimes_A \text{Hom}_A(F, M) \rightarrow \text{Hom}_B(B \otimes_A F, B \otimes_A M)$$

Then let $\phi : M \rightarrow M'$ be a surjective A -module homomorphism and consider the commutative diagram

$$\begin{array}{ccc} B \otimes_A \text{Hom}_A(F, M) & \xrightarrow{\omega} & \text{Hom}_B(B \otimes_A F, B \otimes_A M) \\ \downarrow 1_B \otimes \text{Hom}(1_F, \phi) & & \downarrow \text{Hom}(1_{B \otimes_A F}, 1_B \otimes \phi) \\ B \otimes_A \text{Hom}_A(F, M') & \xrightarrow{\omega} & \text{Hom}_B(B \otimes_A F, B \otimes_A M') \end{array}$$

As $1_B \otimes \phi$ is surjective and $B \otimes_A F$ is assumed projective, $\text{Hom}(1_{B \otimes_A F}, 1_B \otimes \phi)$ is surjective and so then is $1_B \otimes \text{Hom}(1_F, \phi)$. But as B is a faithfully flat A -module, $\text{Hom}(1_F, \phi)$ is itself surjective, hence F is a projective A -module. \square

Let $\rho : A \rightarrow B$ be a ring homomorphism. We consider the following condition

(PM) There exists a B -module E such that $\rho^*(E)$ is a faithfully flat A -module and for any prime ideal \mathfrak{P} of B , we have $E \otimes_B \kappa(\mathfrak{P}) \neq 0$.

The condition (PM) is satisfied when there exists a finitely generated B -module faithfully flat on A and with support equal to $\text{Spec}(B)$. This is the case, in particular, if the A -module B is faithfully flat. Note that the existence of a B -module faithfully flat over A implies ρ is injective by ?? and $\rho^* : \text{Spec}(B) \rightarrow \text{Spec}(A)$ is surjective by ??.

Example 7.3.19. Suppose that $\rho : A \rightarrow B$ is a local homomorphism of local rings and that there exists a B -module E flat over A and such that $E \otimes_B \kappa(\mathfrak{P}) \neq 0$ for any prime ideal \mathfrak{P} of B . Then E is faithfully flat on A and ρ therefore satisfies the property (PM). In fact we have $E/\mathfrak{m}_B E \neq 0$ and therefore $N \neq \mathfrak{m}_B N$ and a fortiori $E \neq \mathfrak{m}_A E$, so the conclusion follows from ??.

Proposition 7.3.20. *Let $\rho : A \rightarrow B$ be a ring homomorphism satisfying condition (PM).*

- (a) *Let $\nu : A \rightarrow A'$ be a ring homomorphism, then the homomorphism $\rho' : A' \rightarrow A' \otimes_A B$ satisfies condition (PM).*
- (b) *Let \mathfrak{P} be a prime ideal of A and $\mathfrak{p} = \mathfrak{P}^c$, then the canonical homomorphism $\rho_{\mathfrak{p}} : A_{\mathfrak{p}} \rightarrow B_{\mathfrak{p}}$ satisfies (PM).*
- (c) *The going-down property holds for ρ .*

Proof. Let E be a B -module such that $\rho^*(E)$ is a faithfully flat A -module and for any prime ideal \mathfrak{P} of B , we have $E \otimes_B \kappa(\mathfrak{P}) \neq 0$. For (a), the A' -module $E' = A' \otimes_A E$ is faithfully flat by

?? Let \mathfrak{P}' be a prime ideal of $B' = A' \otimes_A B$ and \mathfrak{P} its contraction in B . Then we have canonical isomorphisms

$$E' \otimes_{B'} \kappa(\mathfrak{P}') = E \otimes_B \kappa(\mathfrak{P}') = (E \otimes_B \kappa(\mathfrak{P})) \otimes_{\kappa(\mathfrak{P})} \kappa(\mathfrak{P}')$$

since $E \otimes_A \kappa(\mathfrak{P}) \neq 0$, we see $E' \otimes_{B'} \kappa(\mathfrak{P}') \neq 0$, whence ρ' satisfies condition (PM).

Let \mathfrak{p} and \mathfrak{P} be as in (b). By ??, the $A_{\mathfrak{p}}$ module $E_{\mathfrak{P}}$ is flat. On the other hand, every prime ideal of $B_{\mathfrak{P}}$ is of the form $\mathfrak{Q}B_{\mathfrak{P}}$ where \mathfrak{Q} is a prime ideal of B contained in \mathfrak{P} . We have $E \otimes_B \kappa(\mathfrak{Q}) \neq 0$ given the hypothesis made on E , and since $E_{\mathfrak{P}} \otimes_{B_{\mathfrak{P}}} \kappa(\mathfrak{Q}B_{\mathfrak{P}})$ is isomorphic to $E \otimes_B \kappa(\mathfrak{Q}B_{\mathfrak{P}})$, we have $E_{\mathfrak{P}} \otimes_{B_{\mathfrak{P}}} \kappa(\mathfrak{Q}B_{\mathfrak{P}}) \neq 0$, whence (c) by ?? Now (c) follows from (b) by ?? \square

Proposition 7.3.21. *Let A be a subring of a ring B . For B be a faithfully flat A -module, it is necessary and sufficient that the following condition holds:*

(LE) *Every solution $\mathbf{y} \in B^n$ of a linear equation*

$$P\mathbf{y} = \mathbf{b} \quad (7.3.3)$$

with $P \in \mathcal{M}_{mn}(A)$ and $\mathbf{b} \in A^m$, is of the form $\mathbf{y} = \mathbf{x} + Q\mathbf{z}$, where $\mathbf{x} \in A^n$ is a solution of ??, $\mathbf{z} \in B^k$ and $Q \in \mathcal{M}_{nk}(A)$ is such that $PQ = 0$.

Proof. First assume that B is a faithfully flat A -module. Then, as B is a flat A -module, every solution with elements in M of the homogeneous linear system associated with ?? is of the form $Q\mathbf{z}$, where $\mathbf{z} \in B^k$, $Q \in \mathcal{M}_{nk}(A)$ and $PQ = 0$. The problem then reduces to proving that the existence of a solution of ?? with elements in B implies the existence of one solution with elements in A .

Now if we write $\mathbf{p}_j \in A^m$ for the j -th column of P , then ?? is equivalent to the equation $\sum_{j=1}^n \mathbf{p}_j \otimes y_j = \mathbf{b}$ in $B \otimes_A A^m = B^m$. In other words, if M is the sub- A -module of A^m generated by the \mathbf{p}_j , the existence of the solution \mathbf{y} of ?? is equivalent to the relation $\mathbf{b} \in BM \cap A^m$. But as $BM \cap A^m \cap B^m = M$, it implies $\mathbf{d} \in M$, that is, the existence of a solution $\mathbf{x} \in A^m$ of ??.

Now suppose that the property is satisfied. We know already that B is a flat A -module by ?? we prove that, for every ideal \mathfrak{a} of A , $B\mathfrak{a} \cap A = \mathfrak{a}$, which shows that B is a faithfully flat A -module. Now, let $x \in B\mathfrak{a} \cap A$. There exists by hypothesis $y_i \in B$ and $a_i \in \mathfrak{a}$ such that $\sum_i a_i y_i = x$. The property (LE) applied to this linear equation with coefficients and right hand side in A shows that there exist $x_i \in A$ such that $x = \sum_i a_i x_i$, hence $x \in \mathfrak{a}$. \square

7.4 Injective modules

7.4.1 Injective modules

An A -module E is called injective if the functor $\text{Hom}_A(-, E)$ is exact. That is, for any exact sequence of A -modules:

$$0 \longrightarrow M' \xrightarrow{\phi} M \xrightarrow{\psi} M' \longrightarrow 0$$

the sequence obtained by applying $\text{Hom}(-, E)$ is exact:

$$0 \longrightarrow \text{Hom}_A(M'', E) \xrightarrow{\text{Hom}_A(\psi, 1)} \text{Hom}_A(M, E) \xrightarrow{\text{Hom}_A(\phi, 1)} \text{Hom}_A(M'', E) \longrightarrow 0$$

Since the contravariant functor $\text{Hom}_A(-, E)$ is left-exact, this is equivalent to say, for any injective homomorphism $\phi : M' \rightarrow M$, the map

$$\text{Hom}_A(\phi, 1) : \text{Hom}_A(M, E) \rightarrow \text{Hom}_A(M', E)$$

is surjective.

Proposition 7.4.1. *Let $(E_i)_{i \in I}$ be a family of A -module and $E = \prod_i E_i$. For the A -module E to be injective, it is necessary and sufficient that each E_i is injective.*

Proof. Let $\phi : M' \rightarrow M$ be an injective homomorphism of A -modules. For the product homomorphism $\prod_i \text{Hom}_A(M, E_i) \rightarrow \prod_i \text{Hom}_A(M', E_i)$ to be injective, it is necessary and sufficient that each homomorphism $\text{Hom}_A(M, E_i) \rightarrow \text{Hom}_A(M', E_i)$ is injective. This shows the claim since $\prod_i \text{Hom}_A(M, E_i)$ is identified with $\text{Hom}_A(M, \prod_i E_i)$. \square

Proposition 7.4.2 (Baer). *Suppose that E is an A -module. Then the following conditions are equivalent:*

- (i) E is injective;
- (ii) any homomorphism $f : \mathfrak{a} \rightarrow E$ from an ideal \mathfrak{a} of A to E can be extended to A .
- (iii) for any ideal \mathfrak{a} of A and any homomorphism $f : \mathfrak{a} \rightarrow E$, there exists $e \in E$ such that $f(a) = ae$ for all $a \in \mathfrak{a}$.

Proof. Suppose that E is injective. Let \mathfrak{a} be an ideal of A , $f : \mathfrak{a} \rightarrow E$ an A -homomorphism, and $i : \mathfrak{a} \rightarrow A$ the canonical injection. Then the map

$$\text{Hom}_A(i, 1) : \text{Hom}_A(A, E) \rightarrow \text{Hom}_A(\mathfrak{a}, E)$$

is surjective. If $g \in \text{Hom}_A(A, E)$ is such that $f = g \circ i$, we have

$$f(a) = g(a) = ag(1)$$

for each $a \in \mathfrak{a}$.

Conversely, suppose the condition in (ii) is satisfied. Let M be an A -module, N a submodule of M , and $\phi : N \rightarrow E$ a homomorphism. We must find an extension of ϕ to M . Let \mathcal{P} be the set of pairs (P, ψ) where P is a submodule of M containing N and $\psi : P \rightarrow E$ extends ϕ . This set is clearly inductive and nonempty, so by Zorn's lemma we can find a maximal element (P, ψ) . It suffices to prove that $P = M$. Let $x \in M$ and let \mathfrak{a} be the ideal of A such that $ax \in P$. Put $f(a) = \psi(ax)$ for $a \in \mathfrak{a}$, by assumption f can be extended to A , so there exists an element $e \in E$ such that $f(a) = ae$ for $a \in \mathfrak{a}$. Let $P' = P + Ax$ and $\psi' : P' \rightarrow E$ be the unique homomorphism such that $\psi'(p + ax) = \psi(p) + ae$ for $p \in P, a \in A$. Then (P', ψ') extends (P, ψ) , hence $P' = P$, which means $x \in P$. This completes the proof. \square

Corollary 7.4.3. *If the ring A is Noetherian, any direct sum of injective A -modules is injective.*

Proof. Let $(E_i)_{i \in I}$ be a family of injective A -modules, E the direct sum, \mathfrak{a} an ideal of A and $f : \mathfrak{a} \rightarrow E$ an A -homomorphism. Since A is Noetherian, \mathfrak{a} is finitely generated, and therefore the canonical map

$$\varphi : \bigoplus_{i \in I} \text{Hom}_A(\mathfrak{a}, E_i) \rightarrow \text{Hom}_A(\mathfrak{a}, E)$$

is bijective (if \mathfrak{a} is not finitely generated, φ is only injective). Let (f_i) be the preimage of f by φ . Since each E_i is injective, and the family (f_i) has finite support, there exist elements $(e_i)_{i \in I}$ of E such that $f_i(a) = ae_i$ for $a \in \mathfrak{a}$ and $i \in I$, hence $f(a) = a((e_i))$ for each $a \in \mathfrak{a}$, whence E is injective. \square

Suppose that A is integral. We say an A -module E is **divisible** if the homothety a_E is injective for every nonzero element a of A .

Corollary 7.4.4. *Suppose that A is an integral domain.*

- (a) *Any injective A -module is divisible.*
- (b) *Any torsion free A -module is divisible and injective.*
- (c) *If A is a PID, any divisible A -module is injective.*

Proof. If $a \in A$ is nonzero, then a_A is injective; for any A -module E , the homothety a_E is identified with

$$\text{Hom}(a_A, 1_E) = \text{Hom}_A(A, E) \rightarrow \text{Hom}_A(A, E)$$

hence E is divisible if $\text{Hom}(a_E, 1_E)$ is surjective for any $a \in A$ nonzero. Assertion (a) then follows.

Let E be a divisible A -module; suppose that A is principal (resp. E is torsion-free) and we show that E is injective by applying ???. Let \mathfrak{a} be an ideal of A and $f : \mathfrak{a} \rightarrow E$ a homomorphism. Let $x \in \mathfrak{a}$ be such that $\mathfrak{a} = Ax$ (resp. such that $x \neq 0$ if $\mathfrak{a} \neq 0$), and $e \in E$ such that $f(x) = xe$. Then for any $a \in \mathfrak{a}$, we have $f(a) = ae$: this is clear if $a \in Ax$, which proves our assertion if A is principal; if E is torsion free, then since $xa \in \mathfrak{a}$, we have $xf(a) = f(ax) = axe$, hence $f(a) = ae$ (since x is nonzero if $\mathfrak{a} \neq 0$). \square

7.4.2 Injective modules and cogenerators

Proposition 7.4.5. *Let B be a ring, F a B -module, and P a (B, A) -bimodule. If E is an injective B -module and P is a flat A -module, then $\text{Hom}_B(P, E)$ is an injective A -module.*

Proof. Let $\phi : M' \rightarrow M$ be an injective homomorphism of A -modules. We have a commutative diagram

$$\begin{array}{ccc} \text{Hom}_A(M, \text{Hom}_B(P, E)) & \xrightarrow{\text{Hom}_A(\phi, 1)} & \text{Hom}_A(M', \text{Hom}_B(P, E)) \\ \downarrow & & \downarrow \\ \text{Hom}_B(P \otimes_A M, E) & \xrightarrow{\text{Hom}_B(1_P \otimes \phi, 1_E)} & \text{Hom}_B(P \otimes M', E) \end{array}$$

where the vertical homomorphisms are canonical isomorphisms. Since P is flat over A , the homomorphism $1_P \otimes \phi : P \otimes_A M' \rightarrow P \otimes_A M$ is injective. Since E is injective, $\text{Hom}_B(1_P \otimes \phi, 1_E)$ is surjective, hence so is $\text{Hom}_A(\phi, 1)$, which shows $\text{Hom}_B(P, E)$ is injective. \square

We say a A -module E is a **cogenerator** if, for each A -module M and a nonzero element x of M , there exists a homomorphism $\phi : M \rightarrow E$ such that $\phi(x) \neq 0$. This concept is the dual of that of a generator: an A -module L is called a generator if, for each A -module M and a nonzero

element x of M , there exist a homomorphism $\phi : L \rightarrow M$ such that $x \in \phi(L)$. For example, the A -module A is a generator.

Proposition 7.4.6. *Let E be an injective A -module. For E to be a cogenerator, it is necessary and sufficient that $\text{Hom}_A(S, E) \neq 0$ for any simple A -module S .*

Proof. This condition is clearly necessary. Conversely, let M be an A -module and x a nonzero element of M ; the submodule Ax of M has a simple quotient S . If $\text{Hom}_A(S, E) \neq 0$, then $\text{Hom}_A(Ax, E) \neq 0$ and there exist a homomorphism $\phi : Ax \rightarrow E$ such that $\phi(x) \neq 0$. Since E is injective, ϕ can be then extended to a homomorphism from M to E , which maps x to a nonzero element. This shows E is a cogenerator. \square

Example 7.4.7. We know the \mathbb{Z} -module \mathbb{Q}/\mathbb{Z} is injective since it is divisible. We now show that it is a cogenerator. In fact, any simple \mathbb{Z} -module is isomorphic to $\mathbb{Z}/p\mathbb{Z}$ for some $p \neq 0$, and $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}/p\mathbb{Z}, \mathbb{Q}/\mathbb{Z})$ is nonzero: for example, it contains the map induced by the homomorphism $x \mapsto x/p$ from \mathbb{Z} to \mathbb{Q} .

Proposition 7.4.8. *Let B be a ring, E a injective B -module that is a cogenerator, P a (B, A) -bimodule. Suppose that P is flat over A and $P \otimes_A S \neq 0$ for any simple A -module S (i.e., faithfully flat over A). Then the A -module $\text{Hom}_B(P, E)$ is injective and a cogenerator.*

Proof. In fact, $\text{Hom}_B(P, E)$ is injective by ???. For any simple A -module S , $\text{Hom}_A(\text{Hom}_B(P, E))$ is isomorphic to $\text{Hom}_B(P \otimes_A S, E)$, hence is nonzero since $P \otimes_A S \neq 0$ and the B -module E is a cogenerator; the A -module $\text{Hom}_B(P, E)$ is then a cogenerator by ???. \square

Corollary 7.4.9. *The A -module $E_A = \text{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z})$ is injective and a cogenerator.*

Proof. Apply ?? to $B = \mathbb{Z}$, $E = \mathbb{Q}/\mathbb{Z}$ and $P = A$. \square

For any A -module M , put $I^0(M) = (E_A)^{\text{Hom}_A(M, E_A)}$ and let $e_M : M \rightarrow I^0(M)$ be the homomorphism such that, for $m \in M$,

$$e_M(m) = (\varphi(m))_{\varphi \in \text{Hom}_A(M, E_A)}.$$

Corollary 7.4.10. *The A -module $I^0(M)$ is injective and the homomorphism e_M is injective.*

Proof. The module $I^0(M)$ is injective since E_A is injective (??). The map e_M is injective since E_A is a cogenerator (by the definition of e_M). \square

Corollary 7.4.11. *Every A -module is isomorphic to a submodule of an injective A -module.*

Corollary 7.4.12. *For an A -module E to be injective, it is necessary and sufficient that for injective homomorphism $\phi : E \rightarrow F$ has an A -linear retraction.*

Proof. Suppose that E is injective and $\phi : E \rightarrow F$ is an injective homomorphism. Then

$$\text{Hom}_A(\phi, 1_E) : \text{Hom}_A(F, E) \rightarrow \text{Hom}_A(E, E)$$

is surjective; there then exists $\psi \in \text{Hom}_A(F, E)$ such that $\psi \circ \phi = 1_E$, so ψ is a retraction of ϕ . Conversely, there exists an injective A -module I and an injective homomorphism $\phi : E \rightarrow I$; if ϕ has a retraction, then E is injective by ???. \square

7.4.3 Injective hull of modules

Let M be an A -module and N be a submodule of M . The module M is said to be an essential extension of N (or N is said to be an **essential submodule** of M) if for every submodule P of M , $P \cap N = \{0\}$ implies $P = \{0\}$. An injective hull of M is defined to be a pair (I, i) , where I is an injective A -module and $i : M \rightarrow I$ is a homomorphism such that I is an essential extension of $i(M)$. We note that in particular i is injective, so we can identify M with the image $i(M)$ in I , and we then say that I is an injective hull of M .

Example 7.4.13. Suppose that A is integral and M is torsion-free. Let K be the fraction field of A and $i : M \rightarrow K \otimes_A M$ be the canonical homomorphism. Then $(K \otimes_A M, i)$ is an injective hull of M (??).

Theorem 7.4.14. *Let M be an A -module.*

- (a) *M has an injective hull.*
- (b) *If (I, i) and (J, j) are two injective hulls of M , there exist an isomorphism $\phi : I \rightarrow J$ such that $\phi \circ i = j$.*

Proof. By ??, we can suppose that M is a submodule of an injective A -module E . Consider the set \mathcal{F} of submodules I of E containing M and such that the canonical injection $i : M \rightarrow I$ makes I an essential extension of M , and order \mathcal{F} by inclusion. Since \mathcal{F} is inductive, it has a maximal element I . It then suffices to prove that I is a direct factor of E , which shows it is injective. Let N be a submodule of E such that $N \cap I = 0$, and maximal with this property. The homomorphisms

$$I \xrightarrow{\iota} E \xrightarrow{\pi} E/N$$

have injective composition. Since E is injective, there exist a homomorphism $\psi : E/N \rightarrow E$ such that $\iota = \psi \circ \pi \circ \iota$, which means $\psi(\pi(x)) = x$ for $x \in I$.

$$\begin{array}{ccccc} I & \xrightarrow{\iota} & E & \xrightarrow{\pi} & E/N \\ & \searrow \iota & & \swarrow \psi & \\ & & E & & \end{array}$$

Put $I' = \text{im } \psi$ and $i' : M \rightarrow I'$ the canonical injection. Then $I \subseteq I'$, and it suffices to show that I' is an essential extension of M , since then $I = I'$ and $\psi \circ \pi$ is a projection of E to I .

Now let P be a submodule of I' such that $P \cap M = \{0\}$. We have $P = \psi \circ \pi(Q)$, where Q is a submodule of E containing N ; moreover,

$$Q \cap M = \psi \circ \pi(Q \cap M) \subseteq P \cap M = \{0\}$$

hence $Q \cap I = \{0\}$ (the extension $M \rightarrow I$ is essential). By the maximality of N , this implies $Q = N$, which means $\pi(Q) = 0$ and hence $P = 0$. This proves the first claim.

Now let (I, i) and (J, j) be two injective hulls of M . Since J is injective, there exists a homomorphism $\phi : I \rightarrow J$ such that $\phi \circ i = j$; we have

$$i^{-1}(\ker \phi) = \ker j = \{0\}$$

hence $\ker \phi = 0$ and ϕ is injective. Then $\phi(I)$ is an injective submodule of J , hence a direct factor. Let P be a submodule of J such that $P \cap \phi(I) = \{0\}$. Then since $\phi(i(M)) = j(M) \subseteq \phi(I)$, we have $P \cap j(M) = \{0\}$, whence $P = 0$ since j is an essential extension. This implies $\phi(I) = J$ and ϕ is bijective. \square

Remark 7.4.1. Let (I, i) be an injective hull of M and $j : M \rightarrow J$ an injective homomorphism of M to an injective A -module J . By the above argument, there exist an injective homomorphism $\phi : I \rightarrow J$ such that $\phi \circ i = j$.

Proposition 7.4.15. *Let I be a nonzero injective A -module. The following conditions are equivalent:*

- (i) I is indecomposable;
- (ii) the intersection of any two nonzero submodules of I is nonzero;
- (iii) I is an injective hull of its nonzero submodules;
- (iv) the ring $\text{End}_A(I)$ is local.

Proof. Assume that I is indecomposable and let M be a nonzero module of I . Then there exists a submodule I' of I which is an injective hull of M (??). Since I' is nonzero and a direct factor of I , we have $I = I'$ if I is indecomposable. This proves (i) \Rightarrow (iii).

Now assume (iii), and let E, F be two submodules of M such that $E \cap F = \{0\}$. If $E \neq \{0\}$, then I is an injective hull of E , so $F = 0$ (since I is an essential extension of E). This shows (ii); also, we see (ii) implies (i).

The implication (iv) \Rightarrow (i) is given in ???. Conversely, suppose that I is indecomposable. Note first that any injective endomorphism ϕ of I is bijective (since $\phi(I)$ is then a nonzero direct factor submodule of I). Moreover, any endomorphism ϕ of I , whose restriction to a nonzero submodule E of I is injective, is injective (indeed, since (i) \Rightarrow (iii), I is an injective hull of E , so $E \cap \ker \phi = 0$ implies $\ker \phi = 0$). That being the case, let ϕ be a non-invertible element of $\text{End}_A(M)$; according to ??, we have to prove that $1 - \phi$ is invertible. Since ϕ is not injective, we have $\ker \phi \neq 0$; as the restriction of $1 - \phi$ to $\ker \phi$ is injective, $1 - \phi$ is injective, therefore bijective. \square

Corollary 7.4.16. *Let M be an A -module, I an injective hull of M . For I to be indecomposable, it is necessary and sufficient that the intersection of two nonzero submodules of M is nonzero.*

Proof. This condition is necessary by ???. Conversely, if I is a direct sum of nonzero submodules I_1, I_2 , then

$$I_1 \cap M \neq \{0\}, \quad I_2 \cap M \neq \{0\}, \quad (I_1 \cap M) \cap (I_2 \cap M) = \{0\}$$

which contradicts the assumption. \square

Lemma 7.4.17. *Let M be a Noetherian nonzero A -module, I an injective hull of M . Then I has an indecomposable injective submodule.*

Proof. We can clearly suppose that M is a submodule of I . Let N be a submodule of M , such that I is not an injective hull of N and maximal for this property. By ?? there exists a submodule

I_1 of I that is an injective hull of N ; then I_1 is a direct factor of I , let J be its complement. We have $J \neq 0$. If J' is a nonzero direct factor of J , we have $J' \cap M \neq \{0\}$ and

$$(J' \cap M) \cap N \subseteq J' \cap I_1 = \{0\}.$$

The submodule $N' = (J' \cap M) + N$ of M is then the direct sum of $J' \cap M$ and N , hence strictly contains N . Moreover N' is contained in the submodule $J' + I_1$ of J , which is a direct sum, therefore injective. By the maximality of N , we then have $J' + I_1 = I$, hence $J' = J$, and J is therefore indecomposable. \square

Let \mathcal{I} be the set of classes of injective indecomposable A -modules. We recall (??) that, if A is Noetherian, any direct sum of injective A -modules is injective.

Theorem 7.4.18. *Let I be an injective A -module.*

- (a) *If I is an injective hull of a Noetherian A -module M , I is the direct sum of a finite family of (injective) indecomposable submodules.*
- (b) *If A is Noetherian, I is a direct sum of a family of (injective) indecomposable submodules.*
- (c) *If I is a direct sum of a family of (injective) indecomposable submodules, there exist a unique family of cardinals $(\alpha_E)_{E \in \mathcal{I}}$ and a such that I is isomorphic to $\bigoplus_{E \in \mathcal{I}} E^{\oplus \alpha_E}$.*

Proof. We note that (c) follows from ?? and ?. Now we prove (a). Let N be a submodule of M whose injective hull is a direct sum of a finite family of indecomposable submodules, and maximal for this property (there are some since M is noetherian). According to ??, there exists a submodule I_1 of I which is an injective envelope of N . If $I_1 = I$, the claim is proved. Otherwise, let J be a complement of I_1 in I . Then J is an injective hull of the Noetherian module $J \cap M$, hence has a injective indecomposable submodule J' (?). Then $I_1 + J'$ is injective, a direct sum of injective indecomposable submodules, and an injective hull of the submodule $(I_1 + J') \cap M$ of M which strictly contains N . This contradicts the maximality of N , whence our claim.

Suppose now A is Noetherian. Let X be the union of the sets $\text{Hom}_A(E, I)$ for $E \in \mathcal{I}$. To each subset Y of X we associate an A -module E_Y and a homomorphism $f_Y : E_Y \rightarrow I$ in the following way: Y is the union of a family $(Y(E))_{E \in \mathcal{I}}$ where $Y(E) \subseteq \text{Hom}_A(E, I)$, we set

$$E_Y = \bigoplus_{E \in \mathcal{I}} E^{\oplus Y(E)}$$

and the component of f_Y on the direct factor of E_Y corresponding to the element y of $Y(E) \subseteq \text{Hom}_A(E, I)$ is $y : E \rightarrow I$ (i.e., the action of f_Y on the y -th component is determined by y). Let Y be a subset of X such that f_Y is injective and Y is maximal with respect to this property. It suffices to prove that f_Y is bijective. If not, let J be a complement of the injective submodule $\text{im } f_Y$ of I . Since J is nonzero, it possesses a Noetherian nonzero submodule (since A is Noetherian), hence a nonzero injective submodule J' which is an injective hull of a Noetherian module. By (a), J' is a nonempty direct sum of a finite family of (injective) indecomposable submodules. There then exists a finite nonempty subset Y' of X such that $f_{Y'}$ maps bijectively from $E_{Y'}$ to J' . Since $\text{im } f_Y \cap J' = \{0\}$, we have $Y \cap Y' = \emptyset$ and $f_{Y \cup Y'}$ is injective; this contradicts the maximality of Y , whence our assertion. \square

Chapter 8

Homological algebras for modules

8.1 Complexes of modules

In this paragraph, we let A be a commutative ring with unit. Without further specifying, all A -modules refers to left A -modules, and all graded modules are of type \mathbb{Z} . If M is a graded A -module with graduation $(M_n)_{n \in \mathbb{Z}}$, we set $M^n = M_{-n}$ and say M_n (resp. M^n) is the homogeneous component of descending degree n (resp. of ascending degree n) of M . If $\phi : M \rightarrow N$ is a graded homomorphism of degree p of graded A -modules, we note $\phi_n : M_n \rightarrow N_{n+p}$ (resp. $\phi^n : M^n \rightarrow N^{n-p}$) the homomorphism deduced from ϕ ; it is called the homogeneous component of descending degree (resp. ascending) n of ϕ ; we also say that ϕ is of descending degree p or of ascending degree $-p$.

8.1.1 Complexes of A -modules

A **differential complex** (or simple complex) of A -modules is a pair (M, d) formed by a graded A -module M and a graded endomorphism $d : M \rightarrow M$ of descending degree -1 such that $d \circ d = 0$.

If M_n (resp. M^n) is the homogeneous component of descending (resp. ascending) degree n of M , the datum of d is equivalent to that of the sequence of homomorphisms

$$\cdots \longrightarrow M_{n+1} \xrightarrow{d_{n+1}} M_n \xrightarrow{d_n} M_{n-1} \longrightarrow \cdots$$

resp.

$$\cdots \longrightarrow M^{n-1} \xrightarrow{d^{n-1}} M^n \xrightarrow{d^n} M^{n+1} \longrightarrow \cdots$$

such that $d_n \circ d_{n+1} = 0$ (resp. $d^n \circ d^{n-1} = 0$) for each $n \in \mathbb{Z}$. By abuse of language, we will also call complex the data of such a sequence of A -modules and homomorphisms.

Any graded A -module will be tacitly considered as a complex in providing zero differential; the complexes thus obtained will be called complexes with zero differential. In particular, any A -module M will be equipped with the unique structure of A -complex such that $M_0 = M^0 = M$. The complex (M, d) is said to be zero if M is reduced to 0. In the following, we denote by 0 a zero complex.

Adjoining the ordered set \mathbb{Z} with elements $-\infty$ and $+\infty$, we denote by $\bar{\mathbb{Z}}$ the obtained set, and extend the order relation on \mathbb{Z} to $\bar{\mathbb{Z}}$ by defining $-\infty < n < +\infty$ for any $n \in \mathbb{Z}$. The set

$\bar{\mathbb{Z}}$ then has maximal and minimal elements. If M is a complex, we define the left and right bounds of M to be the elements $b_l(M)$ and $b_r(M)$ of $\bar{\mathbb{Z}}$ by

$$b_l(M) = \sup\{n \in \mathbb{Z} : M_n \neq 0\}, \quad b_r(M) = \inf\{n \in \mathbb{Z} : M_n \neq 0\}.$$

The complex M is said to be **positive** if $b_r(M) \geq 0$, and **left bounded** if $b_l(M) \neq +\infty$. Similarly, M is **negative** if $b_l(M) \leq 0$, and **right bounded** if $b_r(M) \neq -\infty$. Finally, we say M is **bounded** if it is left bounded and right bounded. The length of M , denote by $l(M)$, is the element $\bar{\mathbb{Z}}$ defined by the following: if M is zero then $l(M) = -\infty$; if M is bounded and nonzero, $l(M) = b_r(M) - b_l(M)$; if M is not bounded, $l(M) = +\infty$.

We say that the complex (M, d) is free, projective, flat, injective, if each of its component M_n is. We note that the complex (M, d) is projective or flat if and only if the module M is (??), but M can be free without the complex (M, d) being free (since a direct factor of a module free is not always free), just as (M, d) can be injective without M being injective.

Let (M, d) be a complex. We set $Z(M, d) = \ker d$ and $B(M, d) = \text{im } d$. These are graded submodules of M , which are called **cycles** and **boundaries** of (M, d) . The homogeneous components of $Z(M, d)$ and $B(M, d)$ are denoted by $Z_n(M, d) = Z^{-n}(M, d)$ and $B_n(M, d) = B^{-n}(M, d)$. We have

$$Z_n(M, d) = \ker d_n, \quad B_n(M, d) = \text{im } d_{n+1}, \quad Z^n(M, d) = \ker d^n, \quad B^n(M, d) = \text{im } d^{n-1}.$$

Since $d \circ d = 0$, we have $B(M, d) \subseteq Z(M, d)$; two cycles are then said to be **homologous** if their difference is a boundary. The quotient graded module $H(M, d) = Z(M, d)/B(M, d)$ and called the **homology module** of (M, d) . Its elements are called the homology classes, and the homogeneous components of $H(M, d)$ are denoted by $H_n(M, d) = H^{-n}(M, d)$.

Example 8.1.1. If M has zero differentials, then $Z(M) = M$ and $B(M) = 0$, and $H(M)$ is identified with M .

Let (M, d) and (N, d) be complexes (note that here we do not distinguish differentials of M and N and we denote them by d). A **morphism** of (M, d) to (N, d) is a graded A -homomorphism α of degree 0 from M to N such that

$$d \circ \alpha = \alpha \circ d.$$

We use $\text{Hom}_{\mathcal{C}(A)}(M, N)$ to denote the set of morphisms from M to N (here $\mathcal{C}(A)$ means the category of complexes of A -modules). If α is such a morphism, then for each n , we have $d_n \circ \alpha_n = \alpha_{n-1} \circ d_n$ and $d^n \circ \alpha^n = \alpha^{n+1} \circ d^n$. Note that

$$\alpha(Z(M)) \subseteq Z(N), \quad \alpha(B(M)) \subseteq B(N)$$

so we get induced maps

$$Z(\alpha) : Z(M) \rightarrow Z(N), \quad B(\alpha) : B(M) \rightarrow B(N), \quad H(\alpha) : H(M) \rightarrow H(N).$$

The homogeneous components of them are denoted by $Z_n(\alpha)$, $Z^n(\alpha)$, ect. It is easy to check that these definitions behave well for addition and composition of morphisms, and therefore are functorial. If the homomorphism $H(\alpha)$ is bijective, we say α is a **quasi-isomorphism**.

We say a complex (M, d) is **null-homologous** if $H(M) = 0$, which means the canonical morphism $0 \rightarrow M$ (resp. $M \rightarrow 0$) is a quasi-isomorphism. We say (M, d) is acyclic at descending degree n (resp. ascending degree n) if $H_n(M) = 0$ (resp. $H^n(M) = 0$).

Let (M, d) be a complex and $r \in \mathbb{Z}$. The **r -fold translation** of (M, d) is the complex $(M[r], d[r])$ defined by the following: $M[r]$ is the A -module obtained by descending the graduation of M by r :

$$M[r]_n = M_{n+r}, \quad M[r]^n = M^{n-r}$$

and in particular $M[r]_0 = M_r$; we also note that M is the direct sum of its graded submodules $M[-r]$, for $r \in \mathbb{Z}$ (resp. $M[r]^r$ for $r \in \mathbb{Z}$). We put $d[r] = (-1)^r d$, so that $Z(M[r]) = Z(M)[r]$, $B(M[r]) = B(M)[r]$ and $H(M[r]) = H(M)[r]$. For example, we see d is a morphism of M to $M[-1]$ and

$$H(d) : H(M) \rightarrow H(M)[-1]$$

is zero. For a morphism $\alpha : M \rightarrow N$ and $r \in \mathbb{Z}$, we then have a morphism $\alpha[r] : M[r] \rightarrow N[r]$ and

$$\alpha[r]_n = \alpha_{n+r}, \quad \alpha[r]^n = \alpha^{n-r}.$$

On the set $A \times A$, the following definition

$$(a, b) + (a', b') = (a + a', b + b')$$

$$(a, b)(a', b') = (aa', ab' + ba')$$

give a ring structure on $A \times A$, denoted by $A(\varepsilon)$, whose unit is $1 = (1, 0)$. The injection $a \mapsto (a, 0) = a1$ then identify A with a subring of $A(\varepsilon)$. The A -module $A(\varepsilon)$ is free with basis $\{1, \varepsilon\}$ where $\varepsilon = (0, 1)$; we have $\varepsilon^2 = 0$ and ε is central in $A(\varepsilon)$. If A is commutative, $A(\varepsilon)$ is an algebra of dual numbers over A .

Endow $A(\varepsilon)$ with the graduation such that $A(\varepsilon)_0 = A \cdot 1$, $A(\varepsilon)_{-1} = A \cdot \varepsilon$, and $A(\varepsilon)_n = 0$ for $n \neq 0$. It is clear that giving a structure of A -complex on a set M amounts to giving on M a structure of graded $A(\varepsilon)$ -modulus, where the differential d corresponding to the homothety ε_C . Similarly, morphisms of complexes correspond to graded homomorphism of graded $A(\varepsilon)$ -modules of degree 0. The category of graded $A(\varepsilon)$ -modules is then equivalent to the category of complexes of A -modules. We can then apply the theory of graded modules to complexes.

The notion of graded sub- $A(\varepsilon)$ -module then corresponds that of sub-complexes: a subcomplex of the complex (M, d) is therefore a graded submodule N of M such that, for all $n \in \mathbb{Z}$, $d_n(N_n) \subseteq N_{n-1}$; if we denote restrict the differential d on N , then (N, d) is a complex structure, called induced by (M, d) . Without further specifying, any sub-complex will be provided with of the induced structure. Similarly, we can explain the notions of quotient complexes, exact sequence of complexes, kernel, cokernel, image of a morphism of complex. Also, we have the notations of direct sum of complexes, inductive system and inductive limit of complexes.

Let (M_i, d_i) be a family of complexes. We define the product $\prod_i M_i$ to be the complex M defined by the following:

- (a) For each $n \in \mathbb{Z}$, M_n is the product A -module $\prod_i (M_i)_n$.
- (b) For each $n \in \mathbb{Z}$, the differential $d_n : M_n \rightarrow M_{n-1}$ is the product of the components $(d_i)_n$.

If I is finite, $\prod_i M_i$ is equal to $\bigoplus_i M_i$. We note that in general, the A -module $\prod_i M_i$ is not the

product of the A -modules M_i .

Consider a family (resp. an direct family) $(M_i)_{i \in I}$ of complexes. Let M be the direct sum (resp. direct limit) of M_i , and $\alpha_i : M_i \rightarrow M$ the canonical morphism. Then the homomorphism $H(\alpha_i) : H(M_i) \rightarrow H(M)$ then defines a canonical homomorphism $\bigoplus_i H(M_i)$ (resp. $\varinjlim_i H(M_i)$) to $H(M)$. Similarly, the projections $\prod_i M_i \rightarrow M_i$ defines a graded homomorphism of degree 0, called canonical, from $H(\prod_i M_i)$ to $\prod_i (H(M_i))$.

Proposition 8.1.2. *For any family $(M_i)_{i \in I}$ of complexes, the canonical homomorphisms*

$$\bigoplus_i H(M_i) \rightarrow H\left(\bigoplus_i M_i\right), \quad H\left(\prod_i M_i\right) \rightarrow \prod_i H(M_i)$$

are bijective. Moreover, for any direct system (M_i) of complexes, the canonical homomorphism

$$\varinjlim_i H(M_i) \rightarrow H\left(\varinjlim_i M_i\right)$$

is bijective.

Proof. This result follows from ??, ?? and ??.

□

8.1.2 Long exact homological sequence

In this part, we consider a short exact sequence of complexes:

$$0 \longrightarrow L \xrightarrow{\alpha} M \xrightarrow{\beta} N \longrightarrow 0$$

Let Γ be the set of elements $x \in M$ such that $dx \in \text{im } \alpha$. For $x \in \Gamma$, we have

$$d(\alpha^{-1}(dx)) = \alpha^{-1}(dd(x)) = 0, \quad d\beta(x) = \beta(dx) \in \text{im}(\beta \circ \alpha) = 0$$

so $\alpha^{-1}(dx) \in Z(L)$ and $\beta(x) \in Z(N)$. Consider then the homomorphism $\varphi : \Gamma \rightarrow H(N) \times H(L)$ defined by

$$\varphi(x) = (\beta(x), \alpha^{-1}(dx)).$$

Lemma 8.1.3. *The image $\varphi(\Gamma)$ of Γ in $H(N) \times H(L)$ is the graph of a homomorphism of degree -1 from $H(N)$ to $H(L)$.*

Proof. If $x \in \Gamma$ and if $\beta(x) \in B(N)$, then $\alpha^{-1}(dx) \in B(L)$: in fact, there exist $y \in N$ such that $\beta(x) = dy$, also $z \in N$ such that $y = \beta(z)$. Then $\beta(x) = d\beta(y) = \beta(dy)$, so $x - dy = \alpha(t)$ for some $t \in L$. This being so, we have $dx = d\alpha(t) = \alpha(dt)$, so $\alpha^{-1}(dx) = dt \in B(L)$. Also, since β is surjective, any element of $Z(N)$ is the image under β of some x in M such that $\beta(dx) = 0$, i.e., $dx \in \text{im } \alpha$, which means $x \in \Gamma$. We then conclude that $\varphi(\Gamma)$ is the graph of a function. Since φ is bihomogeneous of bidegree $(0, -1)$, the claim follows. □

The graded homomorphism of degree -1 from $H(N)$ to $H(L)$ is called the connecting homomorphism relative to the exact sequence (α, β) . We denote it by $\partial(\alpha, \beta)$ or $\partial_{\alpha, \beta}$, or simply ∂ . Its homogeneous components are denoted by

$$\partial_n(\alpha, \beta) : H_n(N) \rightarrow H_{n-1}(L), \quad \partial^n(\alpha, \beta) : H^n(N) \rightarrow H^{n+1}(L).$$

By definition, to construct the image of a class $\gamma \in H_n(N)$, we choose a cycle $y \in Z_n(N)$ whose class is γ , and an element x in M_n such that $\beta(x) = y$. Then dx is of the form $\alpha(t)$, $t \in L_{n-1}$, and $\partial(\gamma)$ is defined to be the class of t . In other words, $\partial_n(\alpha, \beta)$ is obtained from the correspondence $\alpha_{n-1}^{-1} \circ d_n \circ \beta_n^{-1}$ from N_n to L_{n-1} , by passing to subsets $Z_n(N)$ and $Z_{n-1}(L)$, then to their quotients $H_n(N)$ and $H_{n-1}(L)$. This then shows that, if we replace L, M, N and α, β by $L[r], M[r], N[r]$ and $\alpha[r], \beta[r]$, then

$$\partial(\alpha[r], \beta[r]) = (-1)^r \partial(\alpha, \beta).$$

Similarly, if λ and μ are invertible elements in the center of A , then

$$\partial(\lambda\alpha, \mu\beta) = \lambda^{-1}\mu^{-1}\partial(\alpha, \beta).$$

The homomorphism $\partial(\alpha, \beta)$ can also be deduced from the snake lemma. For this, just note that the diagram

$$\begin{array}{ccccccc} L_n/B_n(L) & \xrightarrow{\alpha_n} & M_n/B_n(M) & \xrightarrow{\beta_n} & N_n/B_n(N) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & Z_{n-1}(L) & \xrightarrow{Z_{n-1}(\alpha)} & Z_{n-1}(M) & \xrightarrow{Z_{n-1}(\beta)} & Z_{n-1}(N) \end{array}$$

commutes with two row exact. The homomorphism $\partial(\alpha, \beta)$ is then the connecting homomorphism in the snake lemma.

Theorem 8.1.4. *The short exact sequence induces a long exact sequence*

$$\dots \longrightarrow H_{n+1}(N) \xrightarrow{\partial_{n+1}} H_n(L) \xrightarrow{H_n(\alpha)} H_n(M) \xrightarrow{H_n(\beta)} H_n(N) \longrightarrow \dots$$

Corollary 8.1.5. *If two of the complexes L, M, N are null-homologous, then the third one is null-homologous. For the morphism α (resp. β) to be a quasi-isomorphism, it is necessary and sufficient that N (resp. L) is null-homologous. For $\partial(\alpha, \beta)$ to be bijective, it is necessary and sufficient that M is null-homologous.*

Corollary 8.1.6. *Let α be a morphism of complexes. If $\ker \alpha$ and $\text{coker } \alpha$ are null-homologous, then α is a quasi-isomorphism.*

Proposition 8.1.7. *Consider a commutative diagram of complexes with exact rows:*

$$\begin{array}{ccccccc} 0 & \longrightarrow & L & \xrightarrow{\alpha} & M & \xrightarrow{\beta} & N \longrightarrow 0 \\ & & \downarrow u & & \downarrow v & & \downarrow w \\ 0 & \longrightarrow & L' & \xrightarrow{\alpha'} & M' & \xrightarrow{\beta'} & N' \longrightarrow 0 \end{array}$$

Then $H(u) \circ \partial(\alpha, \beta) = \partial(\alpha', \beta') \circ H(w)$.

Proof. Let z be a cycle in $H(N)$ and x an element of M such that $\beta(x) = z$. Then we have

$$(\partial_{\alpha', \beta'} \circ H(w))(z) = \partial_{\alpha', \beta'}(w(z)) = (\alpha')^{-1}(dv(x)) = u(\alpha^{-1}(dx))$$

$$= H(u)(\alpha^{-1}(dx)) = H(u) \circ \partial_{\alpha,\beta}(z),$$

so the claim follows. \square

Example 8.1.8. Let M be a complex. Consider the canonical exact sequence

$$0 \longrightarrow Z(M) \longrightarrow M \longrightarrow B(M)[-1] \longrightarrow 0$$

and let $i : B(M) \rightarrow Z(M)$ be the canonical injection. Then the connecting homomorphism $\partial : H(B(M))[-1] \rightarrow H(Z(M))[-1]$ is identified with $H(i)[-1]$.

8.1.3 Homotopies

Let (M, d) and (N, d) be two complexes, α and β be morphisms from M to N . A **homotopy** from α to β is defined to be a graded A -homomorphism h of degree 1 from M to N such that

$$\beta - \alpha = d \circ h + h \circ d. \quad (8.1.1)$$

We say f and g are homotopic if there exists such a homotopy.

If γ is another morphism from M to N and h (resp. s) is a homotopy from α to β (resp. β to γ), then $h + s$ is a homotopy from α to γ . Therefore, the relation " α is homotopic to β " is an equivalence relation on $\text{Hom}_{C(A)}(M, N)$, whose classes are called homotopy of morphisms from M to N .

Proposition 8.1.9. If α and β are two homotopic morphisms from M to N , then $H(\alpha) = H(\beta)$.

Proof. Let h be a homotopy from α to β , then

$$(\beta - \alpha)(Z(M)) = (d \circ h + h \circ d)(Z(M)) = (d \circ h)(Z(M)) \subseteq B(M)$$

hence $H(\beta - \alpha) = 0$ and $H(\alpha) = H(\beta)$. \square

Corollary 8.1.10. A morphism homotopic to a quasi-isomorphism is a quasi-isomorphism.

Proposition 8.1.11. Let $\alpha : M \rightarrow N$, $\beta : M \rightarrow N$, $\gamma : L \rightarrow M$, $\eta : N \rightarrow P$ be morphisms of complexes. If h is a homotopy from α to β , then $\eta \circ h \circ \gamma$ is a homotopy from $\eta \circ \alpha \circ \gamma$ to $\eta \circ \beta \circ \gamma$.

Proof. This is clear from the definition, by composing (??) with η and γ . \square

Corollary 8.1.12. Let M, N, P be complexes, α, β two morphisms from M to N , and γ, η two morphisms from N to P . If h (resp. s) is a homotopy from α to β (resp. from γ to η), then $s \circ \alpha + \eta \circ h$ is a homotopy from $\gamma \circ \alpha$ to $\eta \circ \beta$.

Proof. In fact, $s \circ \alpha$ is a homotopy from $\gamma \circ \alpha$ to $\eta \circ \alpha$, and $\eta \circ h$ is a homotopy from $\eta \circ \alpha$ to $\eta \circ \beta$. \square

Let $\alpha : M \rightarrow N$ be a morphism of complexes. We say α is a **homotopy equivalence** if there exist a morphism $\beta : N \rightarrow M$ such that $\alpha \circ \beta$ and $\beta \circ \alpha$ are homotopic to identity. In this case, we say β is a **homotopy inverse** of α . If β and β' are homotopy inverses of α , then β and β' are homotopic (in fact, $\beta' = \beta' \circ 1_N$ is homotopic to $\beta' \circ \alpha \circ \beta$, which is homotopic to $1_M \circ \beta = \beta$).

Proposition 8.1.13. *A homotopy equivalence is a quasi-isomorphism; a composition of homotopy equivalences is a homotopy equivalence. A morphism homotopic to a homotopy equivalence is a homotopy equivalence.*

Proof. Let $\alpha : M \rightarrow N$ and $\gamma : N \rightarrow P$ be homotopy equivalence of complexes, with inverses $\beta : N \rightarrow M$ and $\eta : P \rightarrow N$. Then

$$H(\beta) \circ H(\alpha) = H(\beta \circ \alpha) = H(1_M) = 1_{H(M)}$$

so $H(\alpha)$ is an isomorphism. Now $(\beta \circ \eta) \circ (\gamma \circ \alpha)$ is homotopic to $\beta \circ 1_N \circ \alpha$ (??), hence to 1_M ; similarly, $(\gamma \circ \alpha) \circ (\beta \circ \eta)$ is homotopic to 1_P , so $\gamma \circ \alpha$ is a homotopy equivalence. Finally, if $\mu : M \rightarrow N$ is homotopic to α , then $\beta \circ \mu$ is homotopic to $\beta \circ \alpha$, hence to identity. This shows β is a homotopy inverse of μ , so μ is a homotopy equivalence. \square

Corollary 8.1.14. *Let $\alpha : M \rightarrow N$ be a morphism and $\eta : L \rightarrow M$, $\gamma : N \rightarrow P$ be homotopy equivalences. Then for $\gamma \circ \alpha \circ \eta$ is a homotopy equivalence (resp. a quasi-isomorphism), it is necessary and sufficient that α is a homotopy equivalence (resp. a quasi-isomorphism).*

Proof. If α is a homotopy equivalence (resp. a quasi-isomorphism), then so is $\gamma \circ \alpha \circ \eta$ by ???. Conversely, let $\bar{\gamma}$ and $\bar{\eta}$ be the homotopy inverses of γ and η , respectively. Then $\bar{\gamma} \circ \gamma \circ \alpha \circ \eta \circ \bar{\eta}$ is homotopic to α , whence the conclusion. \square

We say a complex M is null-homotopic if the identity map 1_M is homotopic to zero, which means there exists a graded A -homomorphism h of degree 1 on M such that $1_M = d \circ h + h \circ d$. This also amounts to saying that the unique morphism $0 \rightarrow M$ (resp. $M \rightarrow 0$) is a homotopy equivalence. A null-homotopic complex is null-homologous by ??.

Example 8.1.15. Let $\phi : M \rightarrow N$ and $\psi : N \rightarrow P$ be homomorphisms of A -modules such that $\psi \circ \phi = 0$. Let C be the complex

$$0 \longrightarrow M \xrightarrow{\phi} N \xrightarrow{\psi} P \longrightarrow 0$$

Then C is null-homologous if and only if it is exact, and is null-homotopic if and only if it is splitting. In fact, if C is homotopic to zero, there exist A -homomorphisms $h : P \rightarrow N$ and $s : N \rightarrow M$ such that

$$\psi \circ h = 1_P, \quad h \circ \psi + \phi \circ s = 1_N, \quad s \circ \phi = 1_M.$$

$$\begin{array}{ccccccc} 0 & \longrightarrow & M & \xrightarrow{\phi} & N & \xrightarrow{\psi} & P \longrightarrow 0 \\ & & \swarrow s & & \nearrow h & & \\ 0 & \longrightarrow & M & \xrightarrow{\phi} & N & \xrightarrow{\psi} & P \longrightarrow 0 \end{array}$$

Then the homomorphism h gives a splitting for C . Conversely, if C is splitting, then let $h : P \rightarrow N$ be a section of P in N . Then we can define s by $\phi \circ s = 1_N - h \circ \psi$, this is possible since ϕ is injective and $\phi \circ (s \circ \phi - 1_M) = (1_N - h \circ \psi) \circ \phi - \phi = 0$.

8.1.4 Splitting complexes

Proposition 8.1.16. *Let (M, d) be a complex. The following conditions are equivalent:*

- (i) there exists a homotopy equivalence from (M, d) to $(H(M), 0)$;
- (ii) there exists an A -endomorphism s of degree 1 on M such that $d = d \circ s \circ d$;
- (iii) $B(M)$ and $Z(M)$ are direct factors of M ;
- (iv) (M, d) is a direct sum of sub-complexes which are either of length 0 or of length 1 and null-homologous.

The complex M is called **splitting** if the above conditions hold, and s is called a **split** for M .

Proof. Let $\varphi : M \rightarrow H(M)$ be a homotopy equivalence; there exists then a homomorphism $\psi : H(M) \rightarrow M$ and an endomorphism s on C of degree 1 such that $\psi \circ \varphi = 1_M - s \circ d - d \circ s$. We have $d \circ \psi = \psi \circ 0 = 0$, so

$$0 = d \circ \psi \circ \varphi = d - d \circ s \circ d - d \circ d \circ s = d - d \circ s \circ d$$

so s is the endomorphism on M satisfying conditions in (ii).

We also note that if s is an endomorphism as in (ii), then $d \circ (1_M - s \circ d) = 0$ and $1_M - s \circ d$ is the identity on $Z(M)$, so $1_M - s \circ d$ is a projection from M to $Z(M)$. Similarly, $(d \circ s) \circ d = d$, so $d \circ s$ is a projection from M to $B(M)$. This shows $Z(M)$ and $B(M)$ are direct factors of M , so (ii) implies (iii).

We now show the implication (iii) \Rightarrow (iv). For $n \in \mathbb{Z}$, put $Z_n = Z_n(M)$, $B_n = B_n(M)$ and choose submodules K_n and L_n such that

$$M_n = L_n \oplus Z_n, \quad Z_n = K_n \oplus B_n.$$

Let $E_{(n)} = K_n[-n]$ and $F_{(n)}$ be the following complex: $(F_{(n)})_n = L_n$, $(F_{(n)})_{n-1} = B_{n-1}$, and the differential is the restriction of d_n on L_n :

$$\cdots \longrightarrow 0 \longrightarrow L_n \xrightarrow{d_n} B_{n-1} \longrightarrow 0 \longrightarrow \cdots$$

Then $E_{(n)}$ and $F_{(n)}$ are sub-complexes of (M, d) and we have $(M, d) = \bigoplus_{n \in \mathbb{Z}} (E_{(n)} \oplus F_{(n)})$. Each $E_{(n)}$ is, either zero, of length 0; and each $F_{(n)}$ is either null or of length 1 and null-homologous, hence (iv).

Finally, to see (iv) \Rightarrow (i), it suffices to note that (i) is satisfied for any complex M of length 0, or of length 1 and null-homologous. \square

Example 8.1.17 (Example of splitting complexes).

- (a) A complex with zero differentials is splitting.
- (b) The null-homotopic complexes are the splitting complexes that are null-homologous, i.e., the complexes M such that $H(M) = 0$ and $Z(M)$ is a direct factor of M .
- (c) Let $\phi : M \rightarrow N$ be a homomorphism of A -modules and C the complex

$$0 \longrightarrow M \xrightarrow{\phi} N \longrightarrow 0$$

Then C is splitting if and only if $\ker \phi$ is a direct factor of M and $\text{im } \phi$ is a direct factor of N .

(d) We have the following exact sequences

$$0 \longrightarrow Z_n(M) \longrightarrow M_n \longrightarrow B_{n-1}(M) \longrightarrow 0$$

$$0 \longrightarrow H_n(M) \longrightarrow M_n / B_n(M) \longrightarrow B_{n-1}(M) \longrightarrow 0$$

So the complex M is splitting if $B(M)$ and $H(M)$ are projective (resp. if $B_n(M)$ and $H_n(M)$ are injective for each n). In particular, if A is principal, a free complex M is splitting if and only if $H(M)$ is free.

Remark 8.1.1. Suppose that the canonical exact sequence of graded A -modules

$$0 \longrightarrow B(M) \longrightarrow Z(M) \xrightarrow{\pi} H(M) \longrightarrow 0$$

is splitting (this is for example if $H(M)$ is projective or $B(M)$ is injective); let $\sigma : H(M) \rightarrow Z(M)$ be a graded A -linear section of π , and ψ the homomorphism $x \mapsto \sigma(x)$ from $H(M)$ to M . Then ψ is a quasi-isomorphism from $(H(M), 0)$ to M such that $H(\psi) = 1_{H(M)}$.

Remark 8.1.2. Suppose that the canonical exact sequence of graded A -modules

$$0 \longrightarrow H(M) \xrightarrow{i} M / B(M) \longrightarrow B(M)[-1] \longrightarrow 0$$

is splitting (this is for example if $B(M)$ is projective or $H(M)$ is injective); let $\tau : M / B(M) \rightarrow H(M)$ be a graded A -linear section of i , and ϕ the homomorphism from M to $H(M)$ which associates to an element of M the image under τ of its class modulo $B(M)$. Then ϕ is a quasi-isomorphism from M to $(H(M), 0)$ such that $H(\phi) = 1_{H(M)}$.

8.1.5 Cone and cylinder of a morphism of complexes

Let $\alpha : (M, d) \rightarrow (N, d)$ be a morphism of complexes. We define graded A -modules $\text{Cyl}(\alpha)$ and $\text{Cone}(\alpha)$ by

$$\text{Cyl}(\alpha) = M \oplus M[-1] \oplus N, \quad \text{Cone}(\alpha) = M[-1] \oplus N$$

and define graded A -linear maps of degree -1 :

$$\begin{aligned} D : \text{Cyl}(\alpha) &\rightarrow \text{Cyl}(\alpha), & D(x_1, x_2, y) &= (dx_1 + x_2, -dx_2, dy + \alpha(x_2)), \\ D : \text{Cone}(\alpha) &\rightarrow \text{Cone}(\alpha), & D(x, y) &= (-dx, dy + \alpha(x)) \end{aligned}$$

It is easy to see that $(\text{Cyl}(\alpha), D)$ and $(\text{Cone}(\alpha), D)$ are complexes. Equivalently, the differentials of $\text{Cyl}(\alpha)$ and $\text{Cone}(\alpha)$ are given by the following diagrams:

$$\begin{array}{ccccccc} \cdots & \longrightarrow & M_{n+1} & \xrightarrow{d} & M_n & \xrightarrow{d} & M_{n-1} \longrightarrow \cdots \\ & & \oplus & \nearrow 1_{M_n} & \oplus & \nearrow 1_{M_{n-1}} & \oplus \\ \cdots & \longrightarrow & M_n & \xrightarrow{-d} & M_{n-1} & \xrightarrow{-d} & M_{n-2} \longrightarrow \cdots \\ & & \oplus & \searrow \alpha_n & \oplus & \searrow \alpha_{n-1} & \oplus \\ \cdots & \longrightarrow & N_{n+1} & \xrightarrow{d} & N_n & \xrightarrow{d} & N_{n-1} \longrightarrow \cdots \end{array}$$

$$\begin{array}{ccccccc} \cdots & \longrightarrow & M_n & \xrightarrow{-d} & M_{n-1} & \xrightarrow{-d} & M_{n-2} \longrightarrow \cdots \\ & & \oplus & \searrow \alpha_n & \oplus & \searrow \alpha_{n-1} & \oplus \\ & & & d & & d & \\ \cdots & \longrightarrow & N_{n+1} & \xrightarrow{d} & N_n & \xrightarrow{d} & N_{n-1} \longrightarrow \cdots \end{array}$$

In particular, it is easy to see that M and N can be viewed as sub-complexes of $\text{Cyl}(\alpha)$ and $\text{Cone}(\alpha)$, and we have the following split exact sequences

$$0 \longrightarrow M \xrightarrow{\iota_M} \text{Cyl}(\alpha) \xrightarrow{\tilde{\pi}} \text{Cone}(\alpha) \longrightarrow 0 \quad (8.1.2)$$

$$0 \longrightarrow N \xrightarrow{\tilde{\iota}_N} \text{Cone}(\alpha) \xrightarrow{\tilde{\pi}_M} M[-1] \longrightarrow 0 \quad (8.1.3)$$

Example 8.1.18. Let $\phi : M \rightarrow N$ be a homomorphism of A -modules; then $\text{Cone}(\phi)$ is of the form

$$0 \longrightarrow M \xrightarrow{\phi} N \longrightarrow 0$$

In particular, we see $H_1(\text{Cone}(\phi)) = \ker \phi$ and $H_0(\text{Cone}(\phi)) = \text{coker } \phi$.

We define two maps ι_N and π_N between N and $\text{Cyl}(\alpha)$:

$$\begin{aligned} \iota_N : N &\rightarrow \text{Cyl}(\alpha), & y &\mapsto (0, 0, y) \\ \pi_N : \text{Cyl}(\alpha) &\rightarrow N, & (x_1, x_2, y) &\mapsto \alpha(x_1) + y. \end{aligned}$$

Proposition 8.1.19. *The morphisms $\iota_N : N \rightarrow \text{Cyl}(\alpha)$ and $\pi_N : \text{Cyl}(\alpha) \rightarrow N$ are homotopy inverses of each other.*

Proof. It is clear that $\pi_N \circ \iota_N = 1_N$; on the other hand, if $\sigma : \text{Cyl}(\alpha) \rightarrow \text{Cyl}(\alpha)$ is the A -linear map of degree 1 such that $\sigma(x_1, x_2, y) = (0, x_1, 0)$, then

$$D \circ \sigma + \sigma \circ D + \iota_N \circ \pi_N = 1_{\text{Cyl}(\alpha)}$$

so σ is a homotopy equivalence and the claim follows. \square

We can summarize ?? by the following commutative diagram where rows are exact and columns are homotopy equivalent:

$$\begin{array}{ccccccc} 0 & \longrightarrow & N & \xrightarrow{\tilde{\iota}_N} & \text{Cone}(\alpha) & \xrightarrow{\tilde{\pi}_M} & M[-1] \longrightarrow 0 \\ & & \downarrow \iota_N & & \downarrow 1 & & \\ 0 & \longrightarrow & M & \xrightarrow{\iota_M} & \text{Cyl}(\alpha) & \xrightarrow{\tilde{\pi}} & \text{Cone}(\alpha) \longrightarrow 0 \\ & & \downarrow 1 & & \downarrow \pi_N & & \\ & & M & \xrightarrow{\alpha} & N & & \end{array}$$

Corollary 8.1.20. *For any morphism $\alpha : M \rightarrow N$ of complexes, there exists an injective morphism $\tilde{\alpha} : M \rightarrow \tilde{N}$ and a homotopy equivalence $\beta : \tilde{N} \rightarrow N$ such that $\alpha = \beta \circ \tilde{\alpha}$.*

Proof. This follows by taking $\tilde{N} = \text{Cone}(\alpha)$, and using ??.

Lemma 8.1.21. *Let $\alpha : M \rightarrow N$ be a morphism.*

- (a) The connecting homomorphism $\partial_{n+1}(\tilde{\iota}_N, \tilde{\pi}_M) : H_n(M) \rightarrow H_n(N)$ is given by $H_n(\alpha)$.
- (b) The connecting homomorphism $\partial_{n+1}(\iota_M, \tilde{\pi}) : H_n(\text{Cone}(\alpha)) \rightarrow H_{n-1}(M)$ is given by $H_n(\tilde{\pi}_M)$.

Proof. Let $x \in Z_n(M)$; since $x = \tilde{\pi}_M(x, 0)$ and

$$D(x, 0) = (-dx, \alpha(x)) = (0, \alpha(x)) = \tilde{\iota}_N(\alpha(x))$$

we see $(\tilde{\iota}_N, \tilde{\pi}_M)$ sends x to the class of $\alpha(x)$ in $H_n(N)$. This proves (a).

Let $(x, y) \in \text{Cone}_n(\alpha)$ such that $D(x, y) = 0$; we then have $(-dx, \alpha(x) + dy) = 0$. Since $(x, y) = \tilde{\pi}(0, x, y)$ and

$$D(0, x, y) = (x, -dx, dy + \alpha(x)) = (x, 0, 0) = \iota_M(x)$$

we see $\partial(\iota_M, \tilde{\pi})$ sends (x, y) to the class of x in $H_{n-1}(M)$, whence (b). \square

Proposition 8.1.22. *We have a long exact sequence*

$$\cdots \longrightarrow H_n(M) \xrightarrow{H_n(\alpha)} H_n(N) \xrightarrow{H_n(\tilde{\iota}_N)} H_n(\text{Cone}(\alpha)) \xrightarrow{H_n(\tilde{\pi}_M)} H_{n-1}(M) \longrightarrow \cdots \quad (8.1.4)$$

Proof. This follows from the long exact sequence induced from (??) and noting that the connecting homomorphism $H_n(M) \rightarrow H_n(N)$ is given by $H_n(\alpha)$. \square

Corollary 8.1.23. *For α to be a quasi-isomorphism, it is necessary and sufficient that $\text{Cone}(\alpha)$ is null-homologous.*

Remark 8.1.3. Consider the following diagram with two exact rows induced from (??) and (??):

$$\begin{array}{ccccccc} \cdots & \longrightarrow & H_n(M) & \xrightarrow{H_n(\iota_M)} & H_n(\text{Cyl}(\alpha)) & \xrightarrow{H_n(\tilde{\pi})} & H_n(\text{Cone}(\alpha)) \xrightarrow{\partial_n(\iota_M, \tilde{\pi})} H_{n-1}(M) \longrightarrow \cdots \\ & & \downarrow 1 & & \downarrow H_n(\pi_N) & & \downarrow 1 \\ \cdots & \longrightarrow & H_n(M) & \xrightarrow{H_n(\alpha)} & H_n(N) & \xrightarrow{H_n(\tilde{\iota}_N)} & H_n(\text{Cone}(\alpha)) \xrightarrow{H(\tilde{\pi}_M)} H_{n-1}(M) \longrightarrow \cdots \end{array} \quad (8.1.5)$$

This diagram is commutative since

- (a) $\alpha = \pi_N \circ \iota_M$ so $H_n(\alpha) = H_n(\pi_N) \circ H_n(\iota_M)$.
- (b) $H_n(\pi_N) = H_n(\iota_N)^{-1}$ and $\tilde{\iota}_N = \tilde{\pi} \circ \iota_N$, so $H_n(\tilde{\pi}) = H_n(\tilde{\iota}_N) \circ H_n(\pi_N)$.
- (c) $H_n(\tilde{\pi}_M) = \partial_n(\iota_M, \tilde{\pi})$.

Now consider a short exact sequence of complexes:

$$0 \longrightarrow M \xrightarrow{\alpha} N \xrightarrow{\beta} L \longrightarrow 0 \quad (8.1.6)$$

Define a graded A -linear map $\varphi : \text{Cone}(\alpha) \rightarrow L$ of degree 0 by $\varphi(x, y) = \beta(y)$. We then have a

commutative diagram of A -complexes with exact rows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & M & \xrightarrow{\iota_M} & \text{Cyl}(\alpha) & \xrightarrow{\tilde{\pi}} & \text{Cone}(\alpha) \longrightarrow 0 \\ & & \downarrow 1 & & \downarrow \pi_N & & \downarrow \varphi \\ 0 & \longrightarrow & M & \xrightarrow{\alpha} & N & \xrightarrow{\beta} & L \longrightarrow 0 \end{array} \quad (8.1.7)$$

Proposition 8.1.24. *The map φ is a quasi-isomorphism.*

Proof. From the definition of $\text{Cone}(\alpha)$, we have

$$\varphi \circ D(x, y) = \varphi(-dx, dy + \alpha(x)) = \beta(dy) + \beta(\alpha(x)) = \beta(d(y)) = d(\beta(y)) = d(\varphi(x, y)),$$

so φ is a morphism of complexes. Moreover, φ is surjective with kernel identified with the complex (K, d_K) such that $K = M[-1] \oplus \alpha(M)$ and d_K is the restriction of the differential of $\text{Cone}(\alpha)$. If $d_K(x, \alpha(x')) = 0$, then we have $-dx = 0$ and $\alpha(x) + d\alpha(x') = 0$. Since α is injective, this implies $x + dx' = 0$, so

$$(x, \alpha(x')) = (-dx', \alpha(x')) = d_K(x', 0).$$

This shows $H(K) = 0$, whence φ is a quasi-isomorphism. \square

Corollary 8.1.25. *The following diagram*

$$\begin{array}{ccccc} & & H(\text{Cone}(\alpha)) & & \\ & \nearrow H(\tilde{\iota}_N) & & \searrow H(\tilde{\pi}_M) & \\ H(N) & & H(\varphi) & & H(M)[-1] \\ \searrow H(\beta) & & \downarrow & & \nearrow \partial(\alpha, \beta) \\ & & H(L) & & \end{array}$$

is commutative and $H(\varphi)$ is bijective.

Proof. From the commutative diagram (??), we have $H(1_M) \circ \partial(\tilde{\alpha}, \tilde{\pi}) = \partial(\alpha, \beta) \circ H(\varphi)$ and $\partial(\iota_M, \tilde{\pi}) = H(\tilde{\pi}_M)$ by ??, so

$$\partial(\alpha, \beta) \circ H(\varphi) = H(\tilde{\pi}_M).$$

Moreover, from the diagram (??) and (??), $H(\beta) \circ H(\pi_N) = H(\varphi) \circ H(\tilde{\pi}) = H(\varphi) \circ H(\tilde{\iota}_N) \circ H(\pi_N)$. Since $H(\pi_N)$ is bijective, we then have $H(\beta) = H(\varphi) \circ H(\pi_N)$. \square

We then have $\partial(\alpha, \beta) = H(\varphi)^{-1} \circ H(\tilde{\pi}_M)$, which provides a new definition of the connecting homomorphism $\partial(\alpha, \beta)$. Note also that if we identify $H(\text{Cone}(\alpha))$ to $H(L)$ by $H(\varphi)$, the previous corollary means that the exact sequence (??) is then identified with the exact sequence of homology relative to (??).

8.1.6 Euler-Poincaré characteristic

In this part, we consider a set \mathcal{C} of classes of A -modules which is additive and left exact, i.e. which satisfies the following two conditions:

- (A) If M and N are two A -modules of type \mathcal{C} , $M \oplus N$ is of type \mathcal{C} .
- (E) If $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ is an exact sequence of A -modules and if M and N are of type \mathcal{C} , then L is of type \mathcal{C} .

We say \mathcal{C} is stable if \mathcal{C} satisfies the following two conditions:

- (S1) If $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ is an exact sequence of A -modules and L and M are of type \mathcal{C} , then M is of type \mathcal{C} .
- (S2) For any homomorphism f of A -modules of type \mathcal{C} , $\ker f$ and $\text{coker } f$ are of type \mathcal{C} .

We denote by $K(\mathcal{C})$ the Grothendieck group of \mathcal{C} and $[M]$ the element in $K(\mathcal{C})$ defined by an A -module M . Let G be an abelian group and $\varphi : K(\mathcal{C}) \rightarrow G$ be a homomorphism.

Example 8.1.26.

- (a) If A is a field, we can let \mathcal{C} to be the classes of finite dimensional vector spaces and $\varphi : K(\mathcal{C}) \rightarrow \mathbb{Z}$ be the map that associates M with its dimension $\dim(M)$.
- (b) We can let \mathcal{C} be the classes of A -modules of finite length and set $\varphi([M]) = \ell_A(M)$.

We say a graded A -module M is of type \mathcal{C} if each M_n is of type \mathcal{C} . Let M be a graded A -module bounded and of type \mathcal{C} , with (M_n) the graduation. The φ -characteristic of M , denoted by $\chi_\varphi(M)$ or simple $\chi(M)$, is the element $\sum_n (-1)^n \varphi([M_n])$ of G . This definition applies in particular if M is a bounded complex of A -modules.

Example 8.1.27.

- (a) If M is bounded of type \mathcal{C} , so is $M[r]$ for any $r \in \mathbb{Z}$, and we have $\chi(M[r]) = (-1)^r \chi(M)$.
- (b) Let $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ be an exact sequence of graded A -modules of type \mathcal{C} . If L, M, N are bounded, then we have

$$\chi(M) = \chi(L) + \chi(N).$$

- (c) Let $\alpha : M \rightarrow N$ be a morphism of bounded complexes of type \mathcal{C} . Then $\text{Cone}(\alpha)$ is bounded of type \mathcal{C} , and we have

$$\chi(\text{Cone}(\alpha)) = \chi(N) - \chi(M).$$

- (d) Let M be a graded A -module bounded and of type \mathcal{C} . The Poincaré polynomial of M relative to φ is the element $P_M(t) = \sum \varphi([M_n]) t^n \in G \otimes \mathbb{Z}[t, t^{-1}]$. We have $P_M(1) = \varphi([M])$ and $P_M(-1) = \chi(M)$.

Lemma 8.1.28. *Let M be a bounded complex of type \mathcal{C} . If $H(M) = 0$ then $\chi(M) = 0$,*

Proof. This follows from ??.

□

Proposition 8.1.29. *Let M, N be bounded complexes of type \mathcal{C} . If M and N are quasi-isomorphic, then $\chi(M) = \chi(N)$.*

Proof. In fact, $\alpha : M \rightarrow N$ is a quasi-isomorphism if and only if the mapping cone $\text{Cone}(\alpha)$ is null-homologous. Then since $\chi(\text{Cone}(\alpha)) = \chi(N) - \chi(M)$, we conclude the claim. \square

Proposition 8.1.30. *Let M be a bounded complex of type \mathcal{C} .*

(a) *If \mathcal{C} is stable, $H(M)$ is of type \mathcal{C} .*

(b) *If $H(M)$ is of type \mathcal{C} , so are $B(M)$ and $Z(M)$, and we have $\chi(H(M)) = \chi(M)$.*

Proof. If \mathcal{C} is stable, for each n the module $Z_n(M)$ is of type \mathcal{C} as the kernel of $d_n : M_n \rightarrow M_{n-1}$, and $H_n(M)$ is of type \mathcal{C} as the cokernel of $M_{n+1} \rightarrow Z_n$.

Suppose that $H(M)$ is of type \mathcal{C} . The canonical exact sequences

$$0 \longrightarrow Z_n(M) \longrightarrow M_n \longrightarrow B_{n-1}(M) \longrightarrow 0$$

$$0 \longrightarrow B_n(M) \longrightarrow Z_n(M) \longrightarrow H_n(M) \longrightarrow 0$$

shows by induction on n that $Z_n(M)$ and $B_n(M)$ are of type \mathcal{C} for each n . We then have

$$\chi(M) = \chi(Z(M)) + \chi(B(M))[-1] = \chi(Z(M)) - \chi(B(M)) = \chi(H(M)).$$

This completes the proof. \square

Corollary 8.1.31. *If \mathcal{C} is stable and M is bounded of type \mathcal{C} , the graded module $H(M)$ is bounded of type \mathcal{C} and $\chi(H(M)) = \chi(M)$.*

Proposition 8.1.32. *Let $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ be an exact sequence of complexes.*

(a) *If $H(L), H(M), H(N)$ are bounded of type \mathcal{C} , we have*

$$\chi(H(M)) = \chi(H(L)) + \chi(H(N)).$$

(b) *If \mathcal{C} is stable and two of the graded modules of $H(L), H(M), H(N)$ are bounded of type \mathcal{C} , then so is the third one.*

Proof. Part (a) follows from ?? applied to the long exact sequence induced. Part (b) follows similarly. \square

Corollary 8.1.33. *Suppose that \mathcal{C} is stable, and let $\alpha : M \rightarrow N$ be a morphism of complexes such that $H(M)$ and $H(N)$ are bounded of type \mathcal{C} . Then $H(\text{Cone}(\alpha))$ is bounded of type \mathcal{C} , and we have*

$$\chi(H(\text{Cone}(\alpha))) = \chi(H(N)) - \chi(H(M)).$$

Proof. This follows from ?? applied to the exact sequence

$$0 \longrightarrow N \longrightarrow \text{Cone}(\alpha) \longrightarrow M[-1] \longrightarrow 0$$

\square

8.1.7 The de Rham complex

In this part we let A be a commutative k -algebra over some commutative ring k . Let $d^0 : A \rightarrow \Omega_{A/k}^1$ be the canonical k -derivation of A into the A -module $\Omega_{A/k}^1$ of k -differentials of A . Let $\Omega_{A/k}$ be the graded k -algebra $\bigwedge_A(\Omega_{A/k}^1)$.

Proposition 8.1.34. *There exists a unique k -antiderivation $d : \Omega_{A/k} \rightarrow \Omega_{A/k}$ of degree 1 such that $d \circ d = 0$ which extends $d^0 : A \rightarrow \Omega_{A/k}^1$.*

Proof. We first show the uniqueness of d . Since $d \circ d = 0$, for $y, x_1, \dots, x_p \in A$, we have

$$d(ydx_1 \wedge \cdots \wedge dx_p) = dy \wedge dx_1 \wedge \cdots \wedge dx_p.$$

The A -module $\Omega_{A/k}^p$ is generated by the elements $dx_1 \wedge \cdots \wedge dx_p$, which proves the uniqueness.

For the existence, it suffices to construct a k -homomorphism $d^1 : \Omega_{A/k}^1 \rightarrow \Omega_{A/k}^2$ such that $d^1 \circ d^0 = 0$ and

$$d^1(x\omega) = d^0(x) \wedge \omega + xd^1(\omega) \quad \text{for } x \in A, \omega \in \Omega_{A/k}^1. \quad (8.1.8)$$

In fact, it then follows from ?? and ?? that there exists an antiderivation $d : \Omega_{A/k} \rightarrow \Omega_{A/k}$ which coincides with d^0 at degree 0 and with d^1 at degree 1. As d^0 is zero on k , the antiderivation d is k -linear; as $d^1 \circ d^0 = 0$, we have $d \circ d = 0$ since $\Omega_{A/k}$ is generated as A -algebra by the elements $d^0(a)$ for $a \in A$.

To define d^1 , recall that $\Omega_{A/k}^1$ is equal to the A -module $\mathfrak{I}/\mathfrak{I}^2$, where \mathfrak{I} is the kernel of the multiplication map $m : A \otimes_k A \rightarrow A$. Consider the k -linear map $u : A \otimes_k A \rightarrow \Omega_{A/k}^2$ defined by $u(x \otimes y) = d^0(y) \wedge d^0(x)$. We have

$$u(ax \otimes y - x \otimes ay) = d^0(y) \wedge d^0(ax) - d^0(ay) \wedge d^0(x) = d^0(xy) \wedge d^0(a)$$

for x, y, a in A , whence

$$u((a \otimes 1 - 1 \otimes a)\xi) = d^0(m(\xi)) \wedge d^0(a), \quad \xi \in A \otimes_k A, a \in A. \quad (8.1.9)$$

Since \mathfrak{I} is generated as an A -module by the elements $(a \otimes 1 - 1 \otimes a)$ for $a \in A$, we then deduce that $u(\mathfrak{I}^2) = 0$; thus u defines a k -linear map $d^1 : \mathfrak{I}/\mathfrak{I}^2 \rightarrow \Omega_{A/k}^2$.

Let $\xi = b \otimes 1$ in ??, where $b \in A$, we obtain $d^1(bd^0(a)) = d^0(b) \wedge d^0(a)$; it then follows that $d^1 \circ d^0 = 0$ and $d^1(a\omega) = d^0(a) \wedge \omega + ad^1(\omega)$ for $a \in A$ and $\omega = bd^0(a)$. Since $\Omega_{A/k}^1$ is generated as an A -module by the elements $bd^0(a)$, formula ?? is then satisfied for $\omega \in \Omega_{A/k}$, which completes the proof. \square

We say the elements $\Omega_{A/k}^p$ are **exterior differential forms** of degree p of A over k , and the antiderivation d is the **exterior differential** of $\Omega_{A/k}$. The complex $(\Omega_{A/k}, d)$ is called the de Rham complex of A over k , and its cohomology is called the de Rham cohomology of A over k .

Example 8.1.35. Let A be the polynomial ring $k[X_1, \dots, X_n]$. Then $\Omega_{A/k}^1$ is the free A -module with basis dX_1, \dots, dX_n . Consequently, if for a subset $I = \{i_1, \dots, i_p\}$ of $\{1, \dots, n\}$ we set $dX_I = dX_{i_1} \wedge \cdots \wedge dX_{i_p}$ (here $i_1 < \cdots < i_p$), the A -module $\Omega_{A/k}^p$ has a basis dX_i , where I runs

through the set of subsets of $\{1, \dots, n\}$ with cardinal p . We have

$$d(PdX_I) = dP \wedge dX_I = \sum_{i \notin I} (-1)^{n(I,i)} \frac{\partial P}{\partial X_i} dX_{I \cup \{i\}},$$

where $n(I,i)$ denotes the number of elements in I that are strictly smaller than i .

The cycles of $Z^p(\Omega_{A/k})$ are then elements $\omega = \sum_{|I|=p} P_I dX_I$ such that, for any subset J of $\{1, \dots, n\}$ with $p+1$ elements, we have

$$\sum_{i \in J} (-1)^{n(J,i)} \frac{\partial P_{J \setminus \{i\}}}{\partial X_i} = 0.$$

The element ω is a boundary if we can choose, for each subset J with $(p-1)$ elements and a polynomial $Q_J \in A$ such that

$$P_I = \sum_{j \in I} (-1)^{n(I,j)} \frac{\partial Q_{I \setminus \{j\}}}{\partial X_j}.$$

We shall see that the de Rham complex of A over k is acyclic at positive degrees if k is a \mathbb{Q} -algebra.

Now let M be an A -module. We say a k -linear map $\nabla^0 : M \rightarrow M \otimes_A \Omega_{A/k}^1$ is a **connection** on M if it satisfies

$$\nabla^0(am) = a\nabla^0(m) + m \otimes da \quad \text{for } a \in A, m \in M. \quad (8.1.10)$$

Proposition 8.1.36. *Let ∇^0 be a connection on M .*

- (a) *There exists a unique k -linear map ∇ of degree 1 on the $\Omega_{A/k}$ -module $M \otimes_A \Omega_{A/k}$ which extends ∇^0 and satisfies the identity*

$$\nabla(x\omega) = \nabla(x)\omega + (-1)^p x(d\omega) \quad \text{for } x \in M \otimes_A \Omega_{A/k}^p, \omega \in \Omega_{A/k}. \quad (8.1.11)$$

- (b) *The composition ∇^2 is $\Omega_{A/k}$ -linear; in particular the map $R = \nabla^1 \circ \nabla^0$ is A -linear, and we have*

$$\nabla^2(m \otimes \omega) = R(m)\omega \quad \text{for } m \in M, \omega \in \Omega_{A/k}$$

Proof. The uniqueness of ∇ is clear. We now define a k -homomorphism $\bar{\nabla} : M \otimes_k \Omega_{A/k} \rightarrow M \otimes_A \Omega_{A/k}$ by

$$\bar{\nabla}(m \otimes \omega) = \nabla^0(m)\omega + m \otimes d\omega \quad \text{for } m \in M, \omega \in \Omega_{A/k}.$$

It then follows from (??) that $\bar{\nabla}(am \otimes \omega) = \bar{\nabla}(m \otimes a\omega)$, so we get a k -homomorphism ∇ on $M \otimes_A \Omega_{A/k}$ which extends ∇^0 at degree 0. We now verify (??): for $m \in M, \alpha \in \Omega_{A/k}^p, \omega \in \Omega_{A/k}$,

$$\begin{aligned} \nabla((m \otimes \alpha)\omega) &= \nabla(m \otimes (\alpha \wedge \omega)) = \nabla^0(m)(\alpha \wedge \omega) + m \otimes d(\alpha \wedge \omega) \\ &= \nabla^0(m)\alpha\omega + (m \otimes d\alpha)\omega + (-1)^p(m \otimes \alpha)d\omega \\ &= (\nabla(m \otimes \alpha))\omega + (-1)^p(m \otimes \alpha)d\omega; \end{aligned}$$

this proves (??) for $x = m \otimes \alpha$, and the general case follows by linearity.

Let $x \in M \otimes_A \Omega_{A/k}^p$ and $\omega \in \Omega_{A/k}$; by applying (??), we obtain

$$\begin{aligned}\nabla^2(x\omega) &= \nabla(\nabla(x)\omega) + (-1)^p \nabla(xd\omega) \\ &= (\nabla \circ \nabla(x))\omega + (-1)^{p+1} \nabla(x)(d\omega) + (-1)^p \nabla(x)(d\omega) = (\nabla \circ \nabla(x))\omega.\end{aligned}$$

this shows the first assertion in (b), and the second one is immediate. \square

The map $R : M \rightarrow M \otimes_A \Omega_{A/k}^2$ is called the **curvature homomorphism** of ∇ , and ∇ is called **integrable** if $R = 0$, or equivalently $\nabla \circ \nabla = 0$. In this case, the couple $(M \otimes_A \Omega_{A/k}, \nabla)$ is a complex, called the **de Rham complex** of (M, ∇^0) over k .

8.2 Resolutions

8.2.1 Resolutions of modules

In this part, we identify a module M with the complex $M[0]$. A **left resolution** of M is a pair (P, p) where P is a positive complex and $p : P \rightarrow M$ is a quasi-isomorphism. Similarly, a **right resolution** is a pair (e, E) where E is a negative complex and $e : M \rightarrow E$ is a quasi-isomorphism. The **length** of a resolution is defined to be the length of the complex. If (P, p) and (P', p') (resp. (e, E) and (e', E')) are two left (resp. right) resolutions of M , a morphism $f : P \rightarrow P'$ such that $p' \circ f = p$ (resp. $g : E \rightarrow E'$ such that $g \circ e = e'$) is called a morphism of resolutions.

A **projective** resolution (resp. **free**, **flat**) of an A -module is defined to be a left resolution (P, p) of M such that the complex P is projective (resp. free, flat). An **injective** resolution of M is a right resolution (e, E) of M such that E is injective.

Example 8.2.1.

- (a) Suppose that A is a PID; let M be an A -module and $(x_i)_{i \in I}$ a family of generators of M . Let L_0 be the free module $A^{\oplus I}$ and $p : L_0 \rightarrow M$ be the canonical homomorphism such that $p(e_i) = x_i$. The homomorphism p is surjective and its kernel L_1 is a free module, so we have a free resolution for M of length 1:

$$0 \longrightarrow L_1 \longrightarrow L_0 \xrightarrow{p} M \longrightarrow 0$$

If I is finite, L_0 and L_1 are finitely generated.

- (b) Suppose that A is commutative; let E be an A -module and u an endomorphism of E . Let E_u be the $A[X]$ -module obtained by acting X by u on E . Then we have an exact sequence

$$0 \longrightarrow A[X] \otimes_A E \xrightarrow{\psi} A[X] \otimes_A E \xrightarrow{\varphi} E_u \longrightarrow 0$$

where $\varphi(p \otimes x) = px$ and $\psi(p \otimes x) = Xp \otimes x - p \otimes u(x)$ for $p \in A[X]$ and $x \in E$. This is then a projective resolution (resp. free, finitely generated) for E of length 1 if E is a projective (resp. free, finitely generated) A -module.

Proposition 8.2.2. *Let $f : M \rightarrow N$ be a homomorphism of A -modules. Let (P_M, p_M) and (P_N, p_N) be projective resolutions of M and N , respectively. Then there exists a morphism $\tilde{f} : P_M \rightarrow P_N$ which is unique up to homotopy equivalence, such that $p_N \circ \tilde{f} = f \circ p_M$.*

Proposition 8.2.3. Let $f : M \rightarrow N$ be a homomorphism of A -modules. Let (e_M, E_M) and (e_N, E_N) be injective resolutions of M and N , respectively. Then there exists a morphism $\tilde{f} : E_M \rightarrow E_N$ which is unique up to homotopy equivalence, such that $\tilde{f} \circ p_M = P_M \circ f$.

8.2.2 The canonical free resolution

For any A -module M , denote by $L_0(M)$ the free A -module $A^{\oplus M}$ with basis M , $(e_m)_{m \in M}$ the canonical basis and $p_M : L_0(M) \rightarrow M$ the homomorphism such that $p_M(e_m) = m$ for $m \in M$. Put $Z_0(M) = \ker p_M$ and let $i_M : Z_0(M) \rightarrow L_0(M)$ be the canonical injection. We then have an exact sequence

$$0 \longrightarrow Z_0(M) \xrightarrow{i_M} L_0(M) \xrightarrow{p_M} M \longrightarrow 0$$

We define a graded module $L(M)$ by $L_n(M) = 0$ for $n < 0$ and by induction on $n > 0$, that

$$L_n(M) = L_0(Z_{n-1}(M)), \quad Z_n(M) = Z_0(Z_{n-1}(M)).$$

Also, there is a homomorphism $d_n : L_n(M) \rightarrow L_{n-1}(M)$ defined by

$$\begin{cases} d_n = 0 & n \leq 0, \\ d_1 = i_M \circ p_{Z_0(M)} \\ d_m = i_{Z_{n-2}(M)} \circ p_{Z_{n-1}(M)} & n > 1 \end{cases}$$

We then have an exact sequence

$$\dots \longrightarrow L_n(M) \xrightarrow{d_n} L_{n-1}(M) \longrightarrow \dots \longrightarrow L_0(M) \xrightarrow{p_M} M \longrightarrow 0$$

which gives a morphism $p_M : (L(M), d) \rightarrow M$. Therfore, $(L(M), p_M)$ is a free resolution of M , called the **canonical free resolution** of M .

Let $f : M \rightarrow N$ be a homomorphism of A -modules. Then there is a unique homomorphism $L_0(f) : L_0(M) \rightarrow L_0(N)$ such that $L_0(f)(e_m) = e_{f(m)}$ for $m \in M$. We have

$$p_N \circ L_0(f) = f \circ p_M$$

so $L_0(f)$ induces an A -homomorphism $Z_0(f) : Z_0(M) \rightarrow Z_0(N)$, and

$$i_N \circ Z_0(f) = L_0(f) \circ i_M.$$

Put $L_n(f) = 0$ for $n < 0$ and we define by induction on $n > 0$ homomorphisms $L_n(f) : L_n(M) \rightarrow L_n(N)$ and $Z_n(f) : Z_n(M) \rightarrow Z_n(N)$ by

$$\begin{cases} L_n(f) = L_0(Z_{n-1}(f)) \\ Z_n(f) = Z_0(Z_{n-1}(f)). \end{cases}$$

Proposition 8.2.4. The morphism $L(f) : L(M) \rightarrow L(N)$ is a morphism of complexes, and we have $p_M \circ L(f) = f \circ p_N$.

Proof. We must show that, for $n > 0$, the formula

$$d_n^N \circ L_n(f) = L_{n-1}(f) \circ d_n^M.$$

Now we have

$$\begin{aligned} d_1^N \circ L_1(f) &= i_N \circ p_{Z_0(N)} \circ L_0(Z_0(f)) = i_N \circ Z_0(f) \circ p_{Z_0(M)} \\ &= L_0(f) \circ i_M \circ p_{Z_0(M)} = L_0(f) \circ d_1^M, \end{aligned}$$

and for $n > 1$,

$$\begin{aligned} d_n^N \circ L_n(f) &= i_{Z_{n-2}(N)} \circ p_{Z_{n-1}(N)} \circ L_0(Z_{n-1}(f)) = i_{Z_{n-2}(N)} \circ Z_{n-2}(f) \circ p_{Z_{n-1}(M)} \\ &= i_{Z_{n-2}(N)} \circ Z_0(Z_{n-2}(f)) \circ p_{Z_{n-1}(M)} = L_0(Z_{n-2}(f)) \circ i_{Z_{n-2}(M)} \circ p_{Z_{n-1}(M)} \\ &= L_{n-1}(f) \circ d_n^M. \end{aligned}$$

This proves the claim. \square

Let $g : N \rightarrow P$ be another homomorphism. Then from the definition of L , it is clear that

$$L(g \circ f) = L(g) \circ L(f).$$

Therefore, the operation L defines a functor from the category of A -modules to the category of complexes of free modules.

8.2.3 The canonical injective resolution

Let F the the A -module $\text{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z})$. For any A -module M , we set $I^0(M) = F^{\text{Hom}_A(M,F)}$ and $e_M : M \rightarrow I^0(M)$ the homomorphism that associates a element $m \in M$ the family $(\varphi(m))_{\varphi \in \text{Hom}_A(M,F)}$. By ??, the A -module $I^0(M)$ is injective and e_M is injective. Let $K^0(M) = \text{coker } e_M$ and $q_M : I^0(M) \rightarrow K^0(M)$ the canonical projection. We then have an exact sequence

$$0 \longrightarrow M \xrightarrow{e_M} I^0(M) \xrightarrow{q_M} K^0(M) \longrightarrow 0$$

We define a graded A -module $I(M)$ by setting $I^n(M) = 0$ for $n < 0$ and define by induction on $n > 0$ that

$$I^n(M) = I^0(K^{n-1}(M)), \quad K^n = K^0(K^{n+1}(M)).$$

Also, there is an A -homomorphism $\delta^n : I^n(M) \rightarrow I^{n+1}(M)$ defined by

$$\begin{cases} \delta^n = 0 & n < 0 \\ \delta^0 = e_{K^0(M)} \circ q_M \\ \delta^n = e_{K^n(M)} \circ q_{K^{n-1}(M)} & n > 0. \end{cases}$$

We then get an exact sequence

$$0 \longrightarrow M \xrightarrow{e_M} I^0(M) \xrightarrow{\delta^0} \cdots \longrightarrow I^n(M) \xrightarrow{\delta^n} I^{n+1}(M) \longrightarrow \cdots$$

so that, if $e_M : M \rightarrow (I(M), \delta)$ is the morphism of complexes, we obtain an injective resolution for M , called the **canonical injective resolution** of M .

Let $f : M \rightarrow N$ be a homomorphism of A -modules. Let $I^0(f)$ be the homomorphism from $I^0(M) = F^{\text{Hom}_A(M, F)}$ to $I^0(N) = F^{\text{Hom}_A(N, F)}$ sending $(x_\varphi)_{\varphi \in \text{Hom}_A(M, F)}$ to $(x_{\psi \circ f})_{\psi \in \text{Hom}_A(N, F)}$. We then have

$$I^0(f) \circ e_M = e_N \circ f.$$

Therefore, $I^0(f)$ induces a homomorphism $K^0(f) : K^0(M) \rightarrow K^0(N)$ and we have

$$K^0(f) \circ q_M = q_N \circ K^0(f).$$

Again, if we define $I^n(f) = 0$ for $n < 0$ and for $n > 0$,

$$\begin{cases} I^n(f) = I^0(K^{n-1}(f)) \\ K^n(f) = K^0(K^{n-1}(f)) \end{cases}$$

Then we can show as in ?? that $I(f) : I(M) \rightarrow I(N)$ is a morphism of complexes and $I(f) \circ e_M = e_N \circ f$. If $g : N \rightarrow P$ is another homomorphism, then

$$I(g \circ f) = I(g) \circ I(f)$$

so I defines a functor from the category of A -modules to that of complexes of injective A -modules.

8.2.4 Finitely generated resolutions

We have seen that any A -module possesses free resolutions and injective resolutions. Now suppose that A is left Noetherian and let M be a finitely generated A -module. We construct by induction sequences $(L_n)_{n \geq 0}$, $(Z_n)_{n \geq 0}$, $(d_n)_{n \geq 0}$ such that for $n \geq 0$, L_n is a finitely generated free A -module, Z_n is a submodule of L_n and $d_{n+1} : L_{n+1} \rightarrow L_n$ a homomorphism.

For this, we first choose a finite family of generators $(m_i)_{i \in I_0}$ of M , put $L_0 = A^{\oplus I_0}$, define $p : L_0 \rightarrow M$ by $p(e_i) = m_i$ and set $Z_0 = \ker p$. Now for $n \geq 0$, assume that the modules L_n and Z_n are defined. Then Z_n is a submodule of L_n which is finitely generated (since A is left Noetherian); choose a finite generating family $(x_{n,i})_{i \in I_{n+1}}$ of Z_n , put $L_{n+1} = Z_n^{\oplus I_{n+1}}$, define d_{n+1} by $d_{n+1}(e_i) = x_{n,i}$ and set $Z_{n+1} = \ker d_{n+1}$. By this process, we get an exact sequence

$$\dots \longrightarrow L_{n+1} \xrightarrow{d_{n+1}} L_n \longrightarrow \dots \longrightarrow L_0 \xrightarrow{p} M \longrightarrow 0$$

with each L_n free and finitely generated. In particular, this shows that any finitely generated A -modules possesses a free resolution with each term finitely generated. In fact, we can prove a more general result:

Proposition 8.2.5. *Let M be an A -complex and let $n_0 \in \mathbb{Z}$ suc that $H_n(M) = 0$ for $n < n_0$.*

- (a) *There exists a free A -complex L such that $L_n = 0$ for $n < n_0$ and a quasi-isomorphism $f : L \rightarrow M$.*
- (b) *Suppose that A is left Noetherian and the A -modules $H_n(M)$ are finitely generated. Then there*

exists a free A -complex L such that $L_n = 0$ for $n < n_0$ and L_n is finitely generated for each n , and a quasi-isomorphism $f : L \rightarrow M$.

Proof. Let M' be the sub-complex of M such that $M'_n = M_n$ for $n > n_0$, $M'_{n_0} = Z_{n_0}(M)$, and $M'_n = 0$ for $n < n_0$. Then the canonical injection from M' to M is a quasi-isomorphism, so by replacing M with M' , we may assume that $M_n = 0$ for $n < n_0$. The proposition then follows from applying ?? for $r \geq n_0$. \square

Lemma 8.2.6. *Let M be a complex and $r \in \mathbb{Z}$. There exists a complex N and a quasi-isomorphism $\alpha : N \rightarrow M$ such that $\alpha_n : N_n \rightarrow M_n$ is an isomorphism for $n < r$ and N_r is a free A -module. If A is left Noetherian and the A -modules $H_r(M)$ and M_{r-1} are finitely generated, N_r can be chosen to be finitely generated.*

8.2.5 Minimal projective resolutions

Let M be an A -module and

$$\cdots \longrightarrow P_n \xrightarrow{d_n} P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \xrightarrow{d_0} M \longrightarrow 0 \quad (8.2.1)$$

be a resolution of M . We say (??) is a minimal projective resolution for M if for $n \geq 0$, the homomorphism $\delta_n : P_n \rightarrow \text{im } d_n$ induced by d_n is a projective cover.

Proposition 8.2.7. *Let M be an A -module, P and P' two minimal projective resolutions of M and $f : P \rightarrow P'$ a morphism of resolutions. Then f is an isomorphism. In particular, minimal projective resolutions of M are all isomorphic.*

Proof. Let's set $\tilde{P}_n = P_n$ and $\tilde{P}'_n = P'_n$ for $n \neq -1$ and $\tilde{P}_{-1} = \tilde{P}' = M$; let $f_{-1} = 1_M$. We prove by induction that $f_n : \tilde{P}_n \rightarrow \tilde{P}'_n$ is an isomorphism. This is clear for $n = -1$; suppose then f_n and f_{n-1} are isomorphisms. Then it follows from the diagram

$$\begin{array}{ccc} \tilde{P}_n & \xrightarrow{d_n} & \tilde{P}_{n-1} \\ \downarrow f_n & & \downarrow f_{n-1} \\ \tilde{P}'_n & \xrightarrow{d'_n} & \tilde{P}'_{n-1} \end{array}$$

that f_n induces an isomorphism g_n from $\ker d_n$ to $\ker d'_n$. Now from the uniqueness of projective covers, we then conclude that f_{n+1} is an isomorphism. \square

Corollary 8.2.8. *Let M be an A -module, P and P' be projective resolutions of M . Suppose that P is minimal. Let $f : P \rightarrow P'$ and $g : P' \rightarrow P$ be two morphisms of resolutions. Then f is injective, g is surjective, and P' is a direct sum of $\text{im } f$ and $\ker g$. Moreover $\ker g$ is null-homologous.*

Proof. In fact, $\alpha = g \circ f$ is an automorphism of P by ?? . If $\tilde{f} = f \circ \alpha^{-1}$, then $\text{im } \tilde{f} = \text{im } f$ and $g \circ \tilde{f} = 1_P$, which implies $P' = \text{im } \tilde{f} \oplus \ker g$. Now the sequence

$$0 \longrightarrow \ker g \longrightarrow P' \xrightarrow{g} P \longrightarrow 0$$

is exact, so $\ker g$ is null-homologous. \square

Proposition 8.2.9. *Let M be an A -module and (P, p) a projective resolution of M . Let \mathfrak{r} be the Jacobson radical of A . Suppose that, either P_n is finitely generated for all n , or \mathfrak{r} is nilpotent. Then for (P, p) to be minimal, it is necessary and sufficient that the complex $(A/\mathfrak{r}) \otimes_A P$ has zero differentials, or equivalently $d(P_{n+1}) \subseteq \mathfrak{r}P_n$ for all n .*

Proof. If (P, p) is minimal, then by ?? the homomorphism

$$1 \otimes \delta_n : (A/\mathfrak{r}) \otimes_A P_n \rightarrow (A/\mathfrak{r}) \otimes_A \text{im } d_n$$

is an isomorphism. It then follows from the exact sequence

$$0 \longrightarrow \text{im } d_{n+1} \xrightarrow{j_n} P_n \xrightarrow{\delta_n} \text{im } d_n \longrightarrow 0$$

that the homomorphism $1 \otimes j_n : (A/\mathfrak{r}) \otimes_A \text{im } d_{n+1} \rightarrow (A/\mathfrak{r}) \otimes_A P_n$ is zero. Since $d_{n+1} = j_n \circ \delta_{n+1}$, this shows $1_{A/\mathfrak{r}} \otimes d_{n+1} = 0$ for all n .

Conversely, assume that $1 \otimes d_n$ is zero for all n , which means $\text{im } d_n = \ker d_{n-1}$ is contained in $\mathfrak{r}P_{n-1}$. Since δ_{n-1} is surjective, it then follows from ?? that δ_{n-1} is a projective cover for all $n \geq 1$, so (P, p) is minimal. \square

Proposition 8.2.10. *Suppose that A is a local left Noetherian ring, and let M be a finitely generated A -module. Then M possesses a minimal projective resolution (P, p) where P_n is free and finitely generated for all $n \geq 0$.*

Proof. In fact, in this case any finitely generated module has a projective cover, so we may construct the complex (P, d) just as $(L(M), d)$, which means P_0 is the projective cover of M , and P_{n+1} is the projective cover of $\ker d_n$. The resulting complex is clearly minimal. \square

8.2.6 Graded resolutions

In this part we let A be a graded ring with graduation $(A_n)_{n \in \mathbb{Z}}$ such that $A_n = 0$ for $n < 0$. We say a graded A -module M is bounded from below if $M_n = 0$ for n small enough. It is clear that every finitely generated graded A -module is bounded from below.

Proposition 8.2.11. *If A be a bounded below A -module (resp. if M is a finitely generated A -module and A is left Noetherian), there exists an exact sequence of graded A -modules*

$$\cdots \longrightarrow L_n \xrightarrow{d_n} L_{n-1} \longrightarrow \cdots \longrightarrow L_1 \xrightarrow{d_1} L_0 \xrightarrow{d_0} M \longrightarrow 0$$

where L_i are free graded modules that are bounded from below (resp. free graded modules that are finitely generated), and d_i are graded homomorphisms of degree 0.

Proof. If N is a graded A -module that is bounded from below (resp. finitely generated and A is left Noetherian), there exist a free graded A -module L that is bounded from below (resp. finitely generated) and a surjective graded homomorphism $L \rightarrow N$. Consider the kernel of the map $L \rightarrow N$ and repeat this process, we see the claim follows. \square

8.2.7 The standard resolution

Suppose that A is an algebra over a commutative ring k . For $n \geq 0$, let B_n be the tensor product of k of $(n+2)$ modules equal to A . Then B_n is an (A, A) -bimodule by acting A on the first and last factors of the tensor product. For $n \geq 1$, we define homomorphisms of bimodules d_n^i and d_n from B_n to B_{n-1} by the formulae

$$d_n^i(x_0 \otimes \cdots \otimes x_{n+1}) = x_0 \otimes \cdots \otimes x_i x_{i+1} \otimes \cdots \otimes x_{n+1} \quad \text{for } 0 \leq i \leq n$$

and $d_n = \sum_{i=0}^n (-1)^i d_n^i$. It is clear that, if $i < j$, then

$$d_{n-1}^i \circ d_n^j = d_{n-1}^{j-1} \circ d_n^i$$

so we have

$$d_{n-1} \circ d_n = \sum_{0 \leq i < j \leq n} (-1)^{i+j} d_{n-1}^i \circ d_n^j + \sum_{0 \leq j \leq i \leq n-1} (-1)^{i+j} d_{n-1}^i \circ d_n^j = 0$$

Therefore, if we set $B_n = 0$ for $n < 0$ and $d_n = 0$ for $n \leq 0$, the sequence (B_n, d_n) is a complex of (A, A) -bimodules, which is then denoted by $B(A)$. For any left A -module M , we denote by $B(A, M)$ the complex formed by $B_n \otimes M$ and $d_n \otimes 1_M$. This is a complex of left A -modules.

We define an A -linear map ε_M from $B_0(A, M) = (A \otimes_k A) \otimes_A M$ to M by the formula

$$\varepsilon_M(a \otimes b \otimes m) = abm \quad \text{for } a, b \in A, m \in M.$$

We have $\varepsilon_M \circ d_1 = 0$, so the graded homomorphism $\bar{\varepsilon}_M : B(A, M) \rightarrow M$, which coincides with ε_M at degree M , is a morphism of complexes.

Proposition 8.2.12. *The map $\bar{\varepsilon}_M : B(A, M) \rightarrow M$ is a homotopy equivalence of complexes of A -modules. In particular, $(B(A, M), \bar{\varepsilon}_M)$ is a left resolution for the A -module M .*

Proof. For $n \geq 0$, we define a k -linear map

$$s_n(x_0 \otimes \cdots \otimes x_{n+1}) = 1 \otimes x_0 \otimes \cdots \otimes x_{n+1} \quad \text{for } x_0, \dots, x_{n+1} \in A.$$

This is a homomorphism of left A -modules, and one can verify that

$$d_{n+1}^i \circ s_n = s_{n-1} \circ d_n^{i-1}, \quad d_{n+1}^0 \circ s_n = 1_{B_n}$$

for $n \geq 1$ and $1 \leq i \leq n+1$. This then implies

$$d_{n+1} \circ s_n + s_{n-1} \circ d_n = 1_{B_n} \quad \text{for } n \geq 1. \tag{8.2.2}$$

Moreover, we have

$$d_1 \circ s_0(x_0 \otimes x_1) = x_0 \otimes x_1 - 1 \otimes x_0 x_1 \quad \text{for } x_0, x_1 \in A. \tag{8.2.3}$$

Let $\eta : A \rightarrow A \otimes_k A$ be the map defined by $\eta(a) = 1 \otimes a$, and $\bar{\eta} : A \rightarrow B(A)$ the morphism of complexes which coincides with η at degree 0. It is clear that $\bar{\varepsilon}_A \circ \bar{\eta} = 1_A$, and the formulae (??), (??) show that $d \circ s + s \circ d = 1_{B(A)} - \bar{\eta} \circ \bar{\varepsilon}_A$. Put $\bar{\eta}_M = \bar{\eta} \otimes 1_M$, $d_M = d \otimes 1_M$ and $s_M = s \otimes 1_M$,

we then deduce that

$$\bar{\varepsilon}_M \circ \bar{\eta}_M = 1_M, \quad d_M \circ s_M + s_M \circ d_M = 1_{B(A,M)} - \bar{\eta}_M \circ \bar{\varepsilon}_M.$$

Therefore $\bar{\varepsilon}_M$ is a homotopy equivalence of complexes of A -modules. \square

The left resolution $(B(A,M), \bar{\varepsilon}_M)$ is called the **standard resolution** of the A -module M . If A and M are projective k -modules (resp. free, flat), the resolution $B(A,M)$ is a projective (resp. free, flat) resolution of M .

8.2.8 Resolutions and Grothendieck groups

If \mathcal{C} is a set of classes of A -modules, we say a left resolution (P, p) is **bounded of type \mathcal{C}** if the complex P is bounded of type \mathcal{C} .

Theorem 8.2.13. *Let \mathcal{C}_0 and \mathcal{C} be two additive and left exact sets of classes of A -modules such that $\mathcal{C}_0 \subseteq \mathcal{C}$ and any A -module of type \mathcal{C} has a left resolution bounded of type \mathcal{C} . Then the induced homomorphism $\alpha : K(\mathcal{C}_0) \rightarrow K(\mathcal{C})$ is bijective; if M is an A -module of type \mathcal{C} and (P, p) is a left resolution of M bounded of type \mathcal{C}_0 , we have $\alpha^{-1}([M]_{\mathcal{C}}) = \chi_{\mathcal{C}_0}(P)$.*

8.3 Tensor product

From now on, we will assume that A is a k -algebra where k is a commutative ring. The role of k is of an auxiliary nature; we have in mind mainly the following three special cases:

- (a) we consider a general ring A , and put $k = \mathbb{Z}$.
- (b) we consider a general ring A , and let k to be the center of A .
- (c) we consider a commutative ring A and set $k = A$.

8.3.1 Tensot product of complexes

Let (M, d) be a complex of right A -modules and (N, d) be a complex of left A -modules. Then we can endow the tensor product $M \otimes_A N$ a graduation by setting

$$(M \otimes_A N)_n = \sum_{p+q=n} (M_p \otimes N_q).$$

Let D be the k -linear endomorphism of degree -1 on $M \otimes_A N$ such that

$$D(x \otimes y) = dx \otimes y + (-1)^p x \otimes dy, \quad x \in M_p, y \in N_q. \quad (8.3.1)$$

We have $D \circ D = 0$ since, by (??)

$$D^2(x \otimes y) = ddx \otimes y + (-1)^{p-1} dx \otimes dy + (-1)^p dx \otimes dy - x \otimes ddy = 0.$$

The resulting complex $(M \otimes_A N, D)$ is called the **tensor product** of the complexes (M, d) and (N, d) .

Example 8.3.1 (Example of tensor products of complexes).

- (a) If N is reduced to a single term N_0 , then $(M \otimes N)_n = M_n \otimes_A N_0$ and $D = d \otimes 1_{N_0}$. For example $M \otimes_A A$ is identified with M .
- (b) For any integer r , we have $(M \otimes_A N)[r] = M[r] \otimes_A N$. However, $(M \otimes_A N)[r]$ and $M \otimes_A N[r]$ are in general different (this is a sign issue).

Let p, q be integers, $x \in Z_p(M)$, $y \in Z_q(N)$; then the element $x \otimes y$ of $M_p \otimes N_q$ belongs to $Z_{p+q}(M \otimes N)$ by (??). Moreover, if $x' \in M_{p+1}$, $y' \in N_{q+1}$, we have

$$(x + dx') \otimes (y + dy') = x \otimes y + D(x' \otimes y + (-1)^p(x + dx') \otimes y').$$

By passing to quotients, we then deduce a canonical k -linear map

$$\gamma_{p,q}(M, N) : H_p(M) \otimes_A H_q(N) \rightarrow H_{p+q}(M \otimes_A N).$$

If we give $H(M) \otimes_A H(N)$ the graduation such that

$$(H(M) \otimes H(N))_n = \sum_{p+q=n} H_p(M) \otimes_A H_q(N)$$

then $\gamma_{p,q}$ together define a graded k -linear map of degree 0

$$\gamma(M, N) : H(M) \otimes_A H(N) \rightarrow H(M \otimes_A N).$$

Proposition 8.3.2. *If the complexes M and N are positive, then $M \otimes_A N$ is positive and the canonical k -linear map*

$$\gamma_{0,0}(M, N) : H_0(M) \otimes_A H_0(N) \rightarrow H_0(M \otimes_A N)$$

is bijective.

Proof. Consider two exact sequences of the form

$$M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \longrightarrow 0 \quad N' \xrightarrow{\gamma} N \xrightarrow{\eta} N'' \longrightarrow 0$$

We will show that the canonical homomorphism $\beta \otimes \eta : M \otimes_A N \rightarrow M'' \otimes N''$ is surjective with kernel equals to $\text{im}(\alpha \otimes 1_N) + \text{im}(1_M \otimes \gamma)$. For this, consider the commutative diagram with exact rows are columns

$$\begin{array}{ccccccc}
& 0 & & 0 & & 0 & \\
& \uparrow & & \uparrow & & \uparrow & \\
M' \otimes_A N'' & \xrightarrow{\alpha \otimes 1_{N''}} & M \otimes_A N'' & \xrightarrow{\beta \otimes 1_{N''}} & M'' \otimes_A N'' & \longrightarrow & 0 \\
& 1_{M'} \otimes \eta \uparrow & 1_M \otimes \eta \uparrow & 1_{M''} \otimes \eta \uparrow & & & \\
M' \otimes_A N & \xrightarrow{\alpha \otimes 1_N} & M \otimes_A N & \xrightarrow{\beta \otimes 1_N} & M'' \otimes_A N & \longrightarrow & 0 \\
& 1_{M'} \otimes \gamma \uparrow & 1_M \otimes \gamma \uparrow & 1_{M''} \otimes \gamma \uparrow & & & \\
M' \otimes_A N' & \xrightarrow{\alpha \otimes 1_{N'}} & M \otimes_A N' & \xrightarrow{\beta \otimes 1_{N'}} & M'' \otimes_A N' & \longrightarrow & 0
\end{array} \tag{8.3.2}$$

The first assertion is clear since we have $\beta \otimes \eta = (\beta \otimes 1_{N''}) \circ (1_M \otimes \eta)$, where both maps are surjective. Now, for an element $z \in M \otimes N$ to be in the kernel of $\beta \otimes \eta$, it is necessary and sufficient that $(1_M \otimes \eta)(z)$ to be contained in the kernel of $\beta \otimes 1_{N''}$, which, in view of the exactness of (??), is equal to the image of $\alpha \otimes 1_{N''}$. Since the homomorphism $1_{M'} \otimes \eta : M' \otimes_A N \rightarrow M' \otimes_A N''$, we then conclude that $(\beta \otimes \eta)(z) = 0$ if and only if there exists $a \in M' \otimes_A N$ such that

$$(1_M \otimes \eta)(z) = (\alpha \otimes \eta)(a) = (1_M \otimes \eta)(\alpha \otimes 1_N(a)).$$

If $b = z - \alpha \otimes 1_N(a)$, then we have $(1_M \otimes \eta)(b) = 0$, which means $b \in \text{im}(1_M \otimes \gamma)$. This in turn implies $z \in \text{im}(\alpha \otimes 1_N) + \text{im}(1_M \otimes \gamma)$, which is our claim. Finally, by applying our claim to the following exact sequences

$$M_1 \longrightarrow M_0 \longrightarrow H_0(M) \longrightarrow 0 \quad N_1 \longrightarrow N_0 \longrightarrow H_0(N) \longrightarrow 0$$

and noting that the image of $M_1 \otimes N_0$ and $M_0 \otimes N_1$ generate $B_0(M \otimes_A N)$, we conclude that $H_0(M \otimes_A N)$ is identified with $H_0(M) \otimes_A H_0(N)$, which completes the proof. \square

Let $\alpha : M \rightarrow M'$ and $\beta : N \rightarrow N'$ be morphism of complexes of right A -module and left A -modules, respectively. Then $\alpha \otimes \beta : M \otimes_A N \rightarrow M' \otimes_A N'$ is a morphism of complex of k -modules. In fact, it is graded of degree 0, and for $x \in M_p, y \in N_q$,

$$\begin{aligned} (\alpha \otimes \beta)(D(x \otimes y)) &= \alpha(dx) \otimes \beta(y) + (-1)^p \alpha(x) + \beta(dy) \\ &= d\alpha(x) \otimes \beta(y) + (-1)^p \alpha(x) \otimes d(\beta(y)) = D(\alpha(x) \otimes \beta(y)). \end{aligned}$$

Moreover, we have the following commutative diagram

$$\begin{array}{ccc} H(M) \otimes_A H(N) & \xrightarrow{\gamma(M,N)} & H(M \otimes_A N) \\ H(\alpha) \otimes H(\beta) \downarrow & & \downarrow H(\alpha \otimes \beta) \\ H(M') \otimes_A H(N') & \xrightarrow{\gamma(N',N')} & H(M' \otimes_A N') \end{array}$$

Let A° be the opposite k -algebra of A , M° (resp. N°) the complex M (resp. N) considered as a complex of left A° -modules (resp. right A° -modules). Let

$$\sigma(M, N) : M \otimes_A N \rightarrow N^\circ \otimes_{A^\circ} M^\circ$$

be the unique graded k -linear map of degree 0 such that, for $x \in M_p, y \in N_q$, we have

$$\sigma(M, N)(x \otimes y) = (-1)^{pq} x \otimes y.$$

Proposition 8.3.3. *The map $\sigma(M, N) : M \otimes_A N \rightarrow N^\circ \otimes_{A^\circ} M^\circ$ is an isomorphism of complexes of k -modules, whose inverse is $\sigma(N^\circ, M^\circ)$.*

Proof. Since the map $\sigma(M, N)$ and $\sigma(N^\circ, M^\circ)$ are inverses of each other, it amounts to show that $\sigma(M, N)$ is a morphism of complexes. Now for $x \in M_p = M_p^\circ, y \in N_q = N_q^\circ$, we have (we

denote by D° the differential of $N^\circ \otimes_{A^\circ} M^\circ$

$$\begin{aligned}\sigma(M, N)(D(x \otimes y)) &= \sigma(M, N)(dx \otimes y + (-1)^p x \otimes dy) \\ &= (-1)^{(p+1)q} y \otimes dx + (-1)^{p+p(q+1)} dy \otimes x \\ &= (-1)^{pq} dy \otimes x + (-1)^{pq+q} y \otimes dx \\ &= (-1)^{pq} D^\circ(y \otimes x) = D^\circ \circ \sigma(M, N)(x \otimes y).\end{aligned}$$

This proves the claim. \square

The isomorphism $\sigma(M, N) : M \otimes_A N \rightarrow N^\circ \otimes_{A^\circ} M^\circ$ is called the **commutation isomorphism** of the tensor product $M \otimes_A N$.

Let $\alpha : M \rightarrow M'$ and $\beta : N \rightarrow N'$ be morphism of complexes of right A -module and left A -modules, respectively. Then we have a commutative diagram

$$\begin{array}{ccc}M \otimes_A N & \xrightarrow{\sigma(M, N)} & N^\circ \otimes_{A^\circ} M^\circ \\ \alpha \otimes \beta \downarrow & & \downarrow \beta \otimes \alpha \\ M' \otimes_A N' & \xrightarrow{\sigma(M', N')} & N'^\circ \otimes_{A^\circ} M'^\circ\end{array}$$

Now assume that A is commutative. We consider three complexes M, N, P of A -modules. The canonical homomorphism of A -modules

$$\varphi : (M \otimes_A N) \otimes_A P \rightarrow M \otimes_A (N \otimes_A P)$$

is an isomorphism of complexes, which can be verified from the definition of differentials. Moreover generally, let $(M^{(i)}, d^{(i)})_{1 \leq i \leq r}$ be a family of complexes of A -modules. Define a graduation on the A -module $M = \bigotimes_{i=1}^r M^{(i)}$ such that

$$M_n = \sum_{p_1 + p_2 + \dots + p_r = n} (M^{(1)})_{p_1} \otimes \dots \otimes (M^{(r)})_{p_r}$$

and define a graded A -homomorphism of degree -1 by

$$D(x_1 \otimes \dots \otimes x_r) = \sum_{j=1}^r (-1)^{p_1 + \dots + p_{j-1}} x_1 \otimes \dots \otimes x_{j-1} \otimes dx_j \otimes x_{j+1} \otimes \dots \otimes x_r$$

where $x_i \in (M^{(i)})_{p_i}$ for each $i = 1, \dots, r$. Then (M, D) is a complex of A -modules, called that tensor product of the family $(M^{(i)}, d^{(i)})$. We define similarly a graded homomorphism of degree 0

$$\gamma(M^{(i)}) : \bigotimes_{i=1}^r H(M^{(i)}) \rightarrow H(\bigotimes_{i=1}^r M^{(i)}).$$

Remark 8.3.1. Suppose that each $M^{(i)}$ has a structure of graded A -algebra and $d^{(i)}$ is an antiderivation. Then with the graded algebra structure on the tensot product $\bigotimes_{i=1}^r M^{(i)}$, the differential D is an antiderivation.

8.3.2 Tensor products and homotopies

Proposition 8.3.4. *Let M, M' be complexes of right A -modules and N, N' be complexes of left A -modules. Let $\alpha_1 : M \rightarrow M'$, $\alpha_2 : M \rightarrow M'$ and $\beta_1 : N \rightarrow N'$, $\beta_2 : N \rightarrow N'$ be morphism of complexes.*

- (a) *If α_1 is homotopic to α_2 and β_1 is homotopic to β_2 , then the two morphisms $\alpha_1 \otimes \beta_1$ and $\alpha_2 \otimes \beta_2$ from $M \otimes N$ to $M' \otimes N'$ are homotopic.*
- (b) *If α_1 and β_1 are homotopy equivalences, then $\alpha_1 \otimes \beta_1$ is a homotopy equivalence.*
- (c) *If M and N are null-homotopic, then $M \otimes_A N$ is null-homotopic.*

Proof. If α_1 is homotopic to α_2 and β_1 is homotopic to β_2 , there exist graded homomorphisms $s : M \rightarrow M'$ and $h : N \rightarrow N'$ of degree 1 such that

$$\alpha_1 - \alpha_2 = ds + sd, \quad \beta_1 - \beta_2 = dh + hd.$$

Let $S : M \otimes_A N \rightarrow M' \otimes_A N'$ be the unique graded homomorphism of degree 1 such that, for $x \in M_p, y \in N_q$, we have

$$S(x \otimes y) = s(x) \otimes \beta_1(y) + (-1)^p \alpha_2(x) \otimes h(y).$$

We then have

$$\begin{aligned} (DS + SD)(x \otimes y) &= D(s(x) \otimes \beta_1(y) + (-1)^p \alpha_2(x) \otimes h(y)) + S(dx \otimes y) + (-1)^p S(x \otimes dy) \\ &= ds(x) \otimes \beta_1(y) + (-1)^{p+1} s(x) \otimes d\beta_1 y + (-1)^p d\alpha_2(x) \otimes h(y) + \alpha_2(x) \otimes dh(y) \\ &\quad + s(dx) \otimes \beta_1(y) + (-1)^{p-1} \alpha_2(dx) \otimes h(y) + (-1)^p s(x) \otimes \beta_1(dy) + \alpha_2(x) \otimes h(dy) \\ &= (ds + sd)(x) \otimes \beta_1(y) + \alpha_2(x) \otimes (dh + hd)(y) \\ &= (\alpha_1 - \alpha_2)(x) \otimes \beta_1(y) + \alpha_2(x) \otimes (\beta_1 - \beta_2)(y) \\ &= \alpha_1(x) \otimes \beta_1(y) - \alpha_2(x) \otimes \beta_2(y). \end{aligned}$$

This shows $DS + SD = \alpha_1 \otimes \beta_1 - \alpha_2 \otimes \beta_2$, whence (a). The rest of the assertions follow from (a) immediately. \square

Corollary 8.3.5. *Let N be a splitting complex of left A -modules such that $H(N)$ is flat. Then for any complex M of right A -modules, the canonical map*

$$\gamma(M, N) : H(M) \otimes_A H(N) \rightarrow H(M \otimes_A N)$$

is bijective. In particular, this happens if $B(M)$ and $H(M)$ are projective.

Proof. By ??, there exists a homotopy equivalence $\beta : N \rightarrow H(N)$, so from ?? we conclude that $1_M \otimes \beta$ is a homotopy equivalence from $M \otimes_A N$ to $M \otimes_A H(N)$. Since

$$H(1_M \otimes \beta) \circ \gamma(M, N) = \gamma(M, H(N)) \circ (1_M \otimes H(\beta))$$

and $H(1_M \otimes \beta)$ and $H(\beta)$ are bijective, it suffices to show that $\gamma(M, H(N))$ is bijective, and we are then reduced to the case where N is flat and have zero differential. In this case the

canonical exact sequences

$$\begin{aligned} 0 \longrightarrow Z(M) &\xrightarrow{i} M \xrightarrow{\delta} B(M)[-1] \longrightarrow 0 \\ 0 \longrightarrow B(M) &\xrightarrow{j} Z(M) \xrightarrow{\pi} H(M) \longrightarrow 0 \end{aligned}$$

induces the following exact sequences:

$$\begin{aligned} 0 \longrightarrow Z(M) \otimes_A N &\xrightarrow{i \otimes 1} M \otimes_A N \xrightarrow{\delta \otimes 1} B(M)[-1] \otimes_A N \longrightarrow 0 \\ 0 \longrightarrow B(M) \otimes_A N &\xrightarrow{j \otimes 1} Z(M) \otimes_A N \xrightarrow{\pi \otimes 1} H(M) \otimes_A N \longrightarrow 0 \end{aligned}$$

Since $d = i \circ j \circ \delta$, we have $D = d \otimes 1_N = (i \otimes 1) \circ (j \otimes 1) \circ (\delta \otimes 1)$. This shows the canonical map $Z(M) \otimes_A N \rightarrow Z(M \otimes_A N)$ and $B(M) \otimes_A N \rightarrow B(M \otimes_A N)$ are bijective, hence so is $\gamma(M, N)$ by passing to quotient. \square

We now consider a positive complex E which is flat at each degree. We shall show that the operation $- \otimes E$ preserves quasi-isomorphisms. For this, we first establish some lemmas.

Lemma 8.3.6. *Let M be a complex of right A -modules and E a complex of left A -modules. Suppose that $H(M) = 0$ and E is flat and right bounded. Then $H(M \otimes_A E) = 0$.*

Proof. For $k \in \mathbb{Z}$, let $T^{(k)}$ be the sub-complex of $M \otimes_A E$ such that

$$T_n^{(k)} = \sum_{p+q=n, q \leq k} M_p \otimes_A E_q.$$

Then $T^{(k-1)} \subseteq T^{(k)}$ and we have an exact sequence of complexes:

$$0 \longrightarrow T^{(k-1)} \xrightarrow{i_k} T^{(k)} \xrightarrow{\pi} M \otimes_A E_k[-k] \longrightarrow 0$$

where i_k is the canonical inclusion and π the canonical projection. By ??, we have $H(M \otimes_A E_k[-k]) = 0$, so i_k is a quasi-isomorphism. We have $T^{(k)} = 0$ for k small enough, since E is right bounded, hence $H(T^{(k)}) = 0$ for any k by induction on k . Finally, the canonical morphism $\varinjlim T^{(k)} \rightarrow M \otimes_A E$ is evidently an isomorphism, so $H(M \otimes_A E) = 0$ (??). \square

Lemma 8.3.7. *If $\alpha : M \rightarrow N$ is a morphism of complexes of right bounded A -modules and E is a complex of right A -modules, then the complexes $\text{Cone}(\alpha) \otimes_A E$ and $\text{Cone}(\alpha \otimes 1_E)$ are isomorphic.*

Proof. By definition, $\text{Cone}(\alpha) \otimes_A E$ is the graded module $(M[-1] \oplus N) \otimes_A E$ with the differential D defined by

$$D((x, y) \otimes z) = (-dx, dy + \alpha(x)) \otimes z + (-1)^p (x, y) \otimes dz$$

for $x \in M_{p-1}, y \in N_p, z \in E_q$. On the other hand, the complex $\text{Cone}(\alpha \otimes 1_E)$ is the graded module $(M \otimes_A N)[-1] \oplus (N \otimes_A E)$ with the differential D defined by

$$\begin{aligned} D(x \otimes z, y \otimes z) &= (-dx \otimes z - (-1)^{p-1} x \otimes dz, dy \otimes z + (-1)^p y \otimes dz + \alpha(x) \otimes z) \\ &= (-dx \otimes z, dy \otimes z + \alpha(x) \otimes z) + (-1)^p (x \otimes dz, y \otimes dz) \end{aligned}$$

where $x \in M_{p-1}, y \in N_p, z \in E_q$. Our claim is then immediate. \square

Proposition 8.3.8. *Let $\alpha : M \rightarrow N$ be a quasi-isomorphism of complexes of right A -modules and E be a complex of right A -modules which is flat and right bounded. Then*

$$\alpha \otimes 1_E : M \otimes_A E \rightarrow N \otimes_A E$$

is a quasi-isomorphism of complexes of k -modules.

Proof. In fact, by ??, α (resp. $\alpha \otimes 1_E$) is a quasi-isomorphism if and only if $\text{Cone}(\alpha)$ (resp. $\text{Cone}(\alpha \otimes 1_E)$) is exact. We then conclude the assertion from ?? and ??.

8.3.3 The twisted product and its properties

For an A -module E , we denote by $p_E : L(E) \rightarrow E$ the canonical free resolution of E . Now let M be a right A -module and N a left A -module. The **twisted product** of M and N are defined to be the graded k -module

$$\text{Tor}^A(M, N) = H(L(M) \otimes_A L(N)).$$

The homogeneous components of $\text{Tor}^A(M, N)$ are then

$$\text{Tor}_n^A(M, N) = H_n(L(M) \otimes L(N)).$$

Since $L(M)$ and $L(N)$ are positive, we have $\text{Tor}_n^A(M, N) = 0$ for $n < 0$.

Let $\alpha : M \rightarrow M'$ be a homomorphism of right A -modules and $\beta : N \rightarrow N'$ a homomorphism of left A -modules. We set $\text{Tor}^A(\alpha, \beta) = H(L(f) \otimes_A L(\beta))$; this is a graded homomorphism of k -modules

$$\text{Tor}^A(\alpha, \beta) : \text{Tor}^A(M, N) \rightarrow \text{Tor}^A(M', N')$$

whose homogeneous components are then

$$\text{Tor}_n^A(\alpha, \beta) : \text{Tor}_n^A(M, N) \rightarrow \text{Tor}_n^A(M', N').$$

By ??, the canonical homomorphism

$$\gamma_{0,0} : H_0(L(M) \otimes_A L_0(N)) \rightarrow H_0(L(M) \otimes_A L(N))$$

is bijective; using the isomorphisms $M \rightarrow H_0(L(M))$ and $N \rightarrow H_0(L(N))$, we then get a canonical isomorphism

$$\gamma_{M,N} : M \otimes_A N \rightarrow \text{Tor}_0^A(M, N).$$

Recall that we have $L(1_M) = 1_{L(M)}$ and $L(1_N) = 1_{L(N)}$, so by passing to homology,

$$\text{Tor}^A(1_M, 1_N) = 1_{\text{Tor}^A(M, N)}.$$

If $\alpha' : M' \rightarrow M''$ (resp. $\beta' : N' \rightarrow N''$) is a homomorphism of right A -modules (resp. left A -modules), then $L(\beta' \circ \beta) = L(\beta') \circ L(\beta)$ and $L(\alpha' \circ \alpha) = L(\alpha') \circ L(\alpha)$, so

$$\text{Tor}^A(\alpha' \circ \alpha, \beta' \circ \beta) = \text{Tor}^A(\alpha', \beta') \circ \text{Tor}^A(\alpha, \beta).$$

Therefore, the operation Tor^A is a bifunctor on its two variables.

Consider the following morphisms of k -complexes:

$$L(M) \otimes_A N \xleftarrow{1_{L(M)} \otimes p_N} L(M) \otimes_A L(N) \xrightarrow{p_M \otimes 1_{L(N)}} M \otimes_A L(N)$$

and the k -homomorphisms induced on homology:

$$H(L(M) \otimes_A N) \xleftarrow{\psi_M(N)} \text{Tor}^A(M, N) \xrightarrow{\psi_N(M)} H(M \otimes_A L(N))$$

By ??, $1_{L(M)} \otimes p_N$ and $p_M \otimes 1_{L(N)}$ are quasi-isomorphisms, so we conclude that:

Proposition 8.3.9. *The k -homomorphisms*

$$\begin{aligned} \psi_M(N) : \text{Tor}^A(M, N) &\rightarrow H(L(M) \otimes_A N) \\ \psi_N(M) : \text{Tor}^A(M, N) &\rightarrow H(M \otimes_A L(N)) \end{aligned}$$

are bijective.

Corollary 8.3.10. *If M or N is flat, then $\text{Tor}_i^A(M, N) = 0$ for $i > 0$.*

Proof. Suppose that N (resp. M) is flat; then $p_M \otimes 1_N : L(M) \otimes_A N \rightarrow M \otimes_A N$ (resp. $1_M \otimes p_N : M \otimes_A L(N) \rightarrow M \otimes_A N$) is a quasi-isomorphism by ??, hence $H_i(L(M) \otimes_A N)$ (resp. $H_i(M \otimes_A L(N))$) is zero for $i > 0$. \square

Remark 8.3.2. If $\beta : N \rightarrow N'$ is a homomorphism of left A -modules, we have

$$(1_{L(M)} \otimes \beta) \circ (1_{L(M)} \otimes 1_N) = (1_{L(M)} \otimes 1_N) \circ (1_{L(M)} \circ L(\beta)),$$

so the diagram

$$\begin{array}{ccc} \text{Tor}^A(M, N) & \xrightarrow{\psi_M(N)} & H(L(M) \times_A N) \\ \downarrow \text{Tor}^A(1, \beta) & & \downarrow H(1 \otimes \beta) \\ \text{Tor}^A(M, N') & \xrightarrow{\psi_M(N')} & H(L(M) \otimes_A N') \end{array}$$

is commutative. Simialrly, if $\alpha : M \rightarrow M'$ is a homomorphism of right A -modules, we have a commutative diagram:

$$\begin{array}{ccc} \text{Tor}^A(M, N) & \xrightarrow{\psi_N(M)} & H(M \times_A L(N)) \\ \downarrow \text{Tor}^A(\alpha, 1) & & \downarrow H(\alpha \otimes 1) \\ \text{Tor}^A(M', N) & \xrightarrow{\psi_N(M')} & H(M' \otimes_A L(N)) \end{array}$$

Proposition 8.3.11. *The map $(\alpha, \beta) \mapsto \text{Tor}^A(\alpha, \beta)$:*

$$\text{Hom}_A(M, M') \times \text{Hom}_A(N, N') \rightarrow \text{Hom}_k(\text{Tor}^A(M, N), \text{Tor}^A(M', N'))$$

is k -linear.

Proof. Let $\alpha \in \text{Hom}_A(M, M')$, $\beta_1, \beta_2 \in \text{Hom}_A(N, N')$, and $\lambda_1, \lambda_2 \in k$. Then the morphisms $\lambda_1(L(\alpha) \otimes \beta_1) + \lambda_2(L(\alpha) \otimes \beta_2)$ and $L(\alpha) \otimes (\lambda_1\beta_1 + \lambda_2\beta_2)$ from $L(M) \otimes_A N$ to $L(M) \otimes_A N'$ coincide; by ?? and ??, we then have

$$\text{Tor}^A(\alpha, \lambda_1\beta_1 + \lambda_2\beta_2) = \lambda_1\text{Tor}^A(\alpha, \beta_1) + \lambda_2\text{Tor}^A(\alpha, \beta_2).$$

Similarly, the map $\alpha \mapsto \text{Tor}^A(\alpha, \beta)$ is k -linear. \square

Corollary 8.3.12. *Let $\lambda \in k$. If λ annihilates M or N , it annihilates $\text{Tor}^A(M, N)$.*

Proof. In fact, we have $\lambda 1_{\text{Tor}(M, N)} = \text{Tor}^A(\lambda 1_M, 1_N) = \text{Tor}^A(1_M, \lambda 1_N)$, so the claim follows. \square

Proposition 8.3.13. *Let $(M_i)_{i \in I}$ be a family of right A -modules and $(N_j)_{j \in J}$ be a family of left A -modules. The following canonical homomorphism is bijective.*

$$\bigoplus_{i \in I, j \in J} \text{Tor}^A(M_i, N_j) \rightarrow \text{Tor}^A\left(\bigoplus_{i \in I} M_i, \bigoplus_{j \in J} N_j\right)$$

Proof. It suffices to show that for a right module M (resp. for a left module N), the canonical homomorphism $\bigoplus_{j \in J} \text{Tor}^A \rightarrow \text{Tor}^A(M, \bigoplus_{j \in J} N_j)$ (resp. the canonical homomorphism $\bigoplus_{i \in I} \text{Tor}^A(M_i, N) \rightarrow \text{Tor}^A(\bigoplus_{i \in I} M_i, N)$) is bijective. This follows from ?? and the isomorphisms $\bigoplus_j (L(M) \otimes_A N_j) \rightarrow L(M) \otimes_A (\bigoplus_j N_j)$ and $\bigoplus_i (M_i \otimes_A L(N)) \rightarrow (\bigoplus_i M_i) \otimes_A L(N)$. \square

Simialrly, since tensor product commutes with direct limits, we have the following:

Proposition 8.3.14. *Let $(M_\alpha, \phi_{\alpha'\alpha})_{\alpha \in I}$ be a inductive system of right A -modules and $(N_\beta, \psi_{\beta'\beta})_{\beta \in J}$ be a inductive system of left A -modules. The following graded k -homomorphism is bijective:*

$$\varinjlim_{(\alpha, \beta) \in I \times J} \text{Tor}^A(M_\alpha, N_\beta) \rightarrow \text{Tor}^A\left(\varinjlim_{\alpha \in I} M_\alpha, \varinjlim_{\beta \in J} N_\beta\right)$$

Corollary 8.3.15. *Let $(M_i, \phi_{ji})_{i \in I}$ be a inductive system of right A -modules and $(N_i, \psi_{ji})_{i \in I}$ be a inductive system of left A -modules. The following canonical graded k -homomorphism is bijective:*

$$\varinjlim_{i \in I} \text{Tor}^A(M_i, N_i) \rightarrow \text{Tor}^A\left(\varinjlim_{i \in I} M_i, \varinjlim_{i \in I} N_i\right)$$

8.3.4 The long exact sequence of twisted products

Let M be a right A -module. Recall that for a left A -module N , we have an isomorphism $\psi_M(N) : \text{Tor}^A(M, N) \rightarrow H(L(M) \otimes_A N)$. Now consider an exact sequence

$$\mathcal{E} : 0 \longrightarrow N' \xrightarrow{\alpha} N \xrightarrow{\beta} N'' \longrightarrow 0$$

of left A -modules. The induced sequence k -complexes

$$\mathcal{E}^M : 0 \longrightarrow L(M) \otimes_A N' \xrightarrow{1 \otimes \alpha} L(M) \otimes_A N \xrightarrow{1 \otimes \beta} L(M) \otimes_A N'' \longrightarrow 0$$

is then exact by ???. Let $\partial(\mathcal{E}^M) : H(L(M) \otimes_A N'') \rightarrow H(L(M) \otimes_A N')$ be the connecting homomorphism. The corresponding homomorphism:

$$\partial(M, \mathcal{E}) : \text{Tor}^A(M, N'') \rightarrow \text{Tor}^A(M, N')$$

is called the connecting homomorphism of the twisted product relative to the module M and the exact sequence \mathcal{E} . This is a graded k -homomorphism of degree -1 , whose homogeneous components are $\partial_n(M, \mathcal{E}) : \text{Tor}_n^A(M, N'') \rightarrow \text{Tor}_{n-1}^A(M, N')$.

Theorem 8.3.16. *We have a long exact sequence*

$$\dots \longrightarrow \text{Tor}_1^A(M, N'') \xrightarrow{\partial_1(M, \mathcal{E})} M \otimes_A N' \xrightarrow{1 \otimes \alpha} M \otimes_A N \xrightarrow{1 \otimes \beta} M \otimes_A N'' \longrightarrow 0$$

Proof. In fact, the diagram

$$\begin{array}{ccccccc} \text{Tor}(M, N') & \xrightarrow{\text{Tor}(1, \alpha)} & \text{Tor}(M, N) & \xrightarrow{\text{Tor}(1, \beta)} & \text{Tor}(M, N'') & \xrightarrow{\partial(M, \mathcal{E})} & \text{Tor}(M, N') \\ \downarrow \psi_M(N') & & \downarrow \psi_M(N) & & \downarrow \psi_M(N'') & & \downarrow \psi_M(N') \\ H(L(M) \otimes N') & \xrightarrow{H(1 \otimes \alpha)} & H(L(M) \otimes N) & \xrightarrow{H(1 \otimes \beta)} & H(L(M) \otimes N'') & \xrightarrow{\partial(\mathcal{E}^M)} & H(L(M) \otimes N') \\ & & & & & & \xrightarrow{H(1 \otimes \alpha)} \\ & & & & & & H(L(M) \otimes N) \end{array}$$

is commutative. Now the first row is exact and the vertical maps are isomorphisms, so our claim follows. \square

Corollary 8.3.17. *If we have $\text{Tor}_1^A(M, N'') = 0$ (for example, if N'' is flat), then the sequence*

$$0 \longrightarrow M \otimes_A N' \xrightarrow{1 \otimes \alpha} M \otimes_A N \xrightarrow{1 \otimes \beta} M \otimes_A N'' \longrightarrow 0$$

is exact.

Example 8.3.18. Let \mathfrak{a} be an ideal of A and consider the exact sequence

$$0 \longrightarrow \mathfrak{a} \longrightarrow A \longrightarrow A/\mathfrak{a} \longrightarrow 0$$

of left A -modules. Since A is a flat module, we have $\text{Tor}^A(M, A) = 0$, so from the long exact sequence in ?? we deduce isomorphisms

$$\text{Tor}_{i+1}^A(M, A/\mathfrak{a}) \rightarrow \text{Tor}_i^A(M, \mathfrak{a}) \quad \text{for } i > 0$$

and an exact sequence

$$0 \longrightarrow \text{Tor}_1^A(M, A/\mathfrak{a}) \longrightarrow M \otimes_A \mathfrak{a} \longrightarrow M \otimes_A A \longrightarrow M \otimes_A (A/\mathfrak{a}) \longrightarrow 0$$

It then follows that $\text{Tor}_1^A(M, A/\mathfrak{a})$ is identified with the kernel of the canonical map $M \otimes_A \mathfrak{a} \rightarrow M$. For example, if M is a module of the form A/\mathfrak{b} where \mathfrak{b} is a right ideal of A , we obtain an isomorphism $\text{Tor}_1^A(A/\mathfrak{b}, A/\mathfrak{a}) \cong (\mathfrak{a} \cap \mathfrak{b})/\mathfrak{ab}$.

Proposition 8.3.19. Let $M \rightarrow M'$ be a homomorphism of right A -modules and

$$\mathcal{E} : 0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0$$

$$\mathcal{F} : 0 \longrightarrow F' \longrightarrow F \longrightarrow F'' \longrightarrow 0$$

be a commutative diagram of left A -modules with exact rows. Then the diagram of k -modules

$$\begin{array}{ccc} \mathrm{Tor}^A(M, E'') & \xrightarrow{\partial(M, \mathcal{E})} & \mathrm{Tor}^A(M, E') \\ \downarrow & & \downarrow \\ \mathrm{Tor}^A(M', F'') & \xrightarrow{\partial(M', \mathcal{F})} & \mathrm{Tor}^A(M', F') \end{array}$$

is commutative.

Proof. Note that we have the following commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & L(M) \otimes_A E' & \longrightarrow & L(M) \otimes_A E & \longrightarrow & L(M) \otimes_A E'' \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & L(M') \otimes_A F' & \longrightarrow & L(M') \otimes_A F & \longrightarrow & L(M') \otimes_A F'' \longrightarrow 0 \end{array}$$

so the result follows from the naturality of the connecting homomorphism. \square

8.3.5 Künneth formula

Now consider a complex (M, d) of right A -modules and (N, d) a complex of left A -modules. We have the following canonical exact sequences

$$0 \longrightarrow Z(M) \xrightarrow{j} M \xrightarrow{\delta} B(M)[-1] \longrightarrow 0 \tag{8.3.3}$$

$$0 \longrightarrow B(M) \xrightarrow{i} Z(M) \xrightarrow{\pi} H(M) \longrightarrow 0 \tag{8.3.4}$$

We then obtain a k -homomorphism

$$H(\delta \otimes 1) : H(M \otimes_A N) \rightarrow H(B(M) \otimes_A N)[-1],$$

and a connecting homomorphism

$$\partial(B(M), H(N)) : \mathrm{Tor}_1^A(H(M), H(N)) \rightarrow B(M) \otimes_A H(N).$$

If we give $\mathrm{Tor}_1^A(H(M), H(N))$ the graduation whose degree n homogeneous component is $\sum_{p+q=n} \mathrm{Tor}_1^A(H_p(M), H_q(N))$, this homomorphism is then graded of degree 0. Finally, we call that we have a canonical homomorphism

$$\gamma(B(M), N) : B(M) \otimes_A H(N) \rightarrow H(B(M) \otimes_A N).$$

Theorem 8.3.20. Suppose that the A -modules $B(M)$ and $Z(M)$ are flat. There exists a unique graded homomorphism of k -modules of degree -1 ,

$$\alpha : H(M \otimes_A N) \rightarrow \text{Tor}_1^A(H(M), H(N))$$

completing the following diagram

$$\begin{array}{ccc} H(M \otimes_A N) & \xrightarrow{\alpha} & \text{Tor}_1^A(H(M), H(N))[-1] \\ H(\delta \otimes 1) \downarrow & & \downarrow \partial(B(M), H(N)) \\ H(B(M) \otimes_A N)[-1] & \xleftarrow{\gamma(B(M), N)} & (B(M) \otimes_A H(N))[-1] \end{array}$$

and such that the following sequence of k -modules is exact:

$$0 \longrightarrow H(M) \otimes_A H(N) \xrightarrow{\gamma(M, N)} H(M \otimes_A N) \xrightarrow{\alpha} \text{Tor}_1^A(H(M), H(N))[-1] \longrightarrow 0 \quad (8.3.5)$$

Proof. Since $B(M)$ is flat, we have the following exact sequence (??) induced from (??):

$$0 \longrightarrow Z(M) \otimes_A N \xrightarrow{j \otimes 1} M \otimes_A N \xrightarrow{\delta \otimes 1} (B(M) \otimes_A N)[-1] \longrightarrow 0 \quad (8.3.6)$$

We claim that the connecting homomorphism from $H(B(M) \otimes_A N)$ to $H(Z(M) \otimes_A N)$ is given by $H(i \otimes 1)$. To see this, let $z \in Z(B(M) \otimes_A N)$; since $B(M)$ is flat, we have $Z(B(M) \otimes_A N) = B(M) \otimes_A Z(N)$, so z is of the form $z = \sum_\lambda da_\lambda \otimes b_\lambda$, where $a_\lambda \in M$, $b_\lambda \in N$, and $db_\lambda = 0$. The image of the class of z under the connecting homomorphism is by definition given by the class of $D(\sum a_\lambda \otimes b_\lambda) = \sum da_\lambda \otimes b_\lambda = (i \otimes 1)(z)$, so our claim is justified. Now, the long exact sequence associated with (??) is then

$$\cdots \longrightarrow H(B(M) \otimes_A N) \xrightarrow{H(i \otimes 1)} H(Z(M) \otimes_A N) \longrightarrow \text{Tor}_1^A(H(M), H(N))[-1] \longrightarrow \cdots$$

$$\text{Tor}_1^A(H(M), H(N))[-1] \xrightarrow{H(j \otimes 1)} H(M \otimes_A N) \xrightarrow{H(\delta \otimes 1)} H(B(M) \otimes_A N)[-1] \longrightarrow \cdots$$

On the other hand, since $Z(M)$ is flat, the following sequence deduced from (??) is exact:

$$0 \longrightarrow \text{Tor}_1^A(H(M), H(N)) \xrightarrow{\partial} B(M) \otimes_A H(N) \longrightarrow \text{Tor}_1^A(H(M), H(N))[-1]$$

$$\text{Tor}_1^A(H(M), H(N))[-1] \xrightarrow{i \otimes 1} Z(M) \otimes_A H(N) \xrightarrow{\pi \otimes 1} H(M) \otimes_A H(N) \longrightarrow 0$$

Recall that we have the following canonical homomorphisms:

$$\gamma_B = \gamma(B(M), N) : B(M) \otimes_A H(N) \rightarrow H(B(M) \otimes_A N)$$

$$\gamma_Z = \gamma(Z(M), N) : Z(M) \otimes_A H(N) \rightarrow H(Z(M) \otimes_A N)$$

$$\gamma_H = \gamma(H(M), N) : H(M) \otimes_A H(N) \rightarrow H(M \otimes_A N)$$

whence a commutative diagram of graded k -modules, with exact rows:

$$\begin{array}{ccccccc}
 B(M) \otimes_A H(N) & \xrightarrow{i \otimes 1} & Z(M) \otimes_A H(N) & \xrightarrow{\pi \otimes 1} & H(M) \otimes_A H(N) & \longrightarrow & 0 \\
 \downarrow \gamma_B & & \downarrow \gamma_Z & & \downarrow \gamma_H & & \\
 H(B(M) \otimes_A N) & \xrightarrow{H(i \otimes 1)} & H(Z(M) \otimes_A N) & \xrightarrow{H(j \otimes 1)} & H(M \otimes_A N) & \xrightarrow{H(\delta \otimes 1)} & H(B(M) \otimes_A N)[-1] \\
 & & & \downarrow \alpha & & & \uparrow \gamma_B \\
 0 & \longrightarrow & \text{Tor}_1^A(H(M), H(N))[-1] & \xrightarrow{\partial} & (B(M) \otimes_A H(N))[-1] & &
 \end{array}$$

Since the complexes $B(M)$ and $Z(M)$ have zero differentials and are flat, the homomorphisms γ_B and γ_Z are bijective by [??](#). This implies, by the four lemma, on the one hand that the γ_H is injective with image equal to $\ker H(\delta \otimes 1)$ (since $\pi \otimes 1$ is surjective and γ_Z is bijective, we have $\text{im } \gamma_H = \text{im } H(j \otimes 1)$), on the other hand that $\gamma_B \circ \partial(B(M), H(N))$ is injective with image equal to $\text{im } H(\delta \otimes 1)$. These together proves the theorem. \square

Corollary 8.3.21. *Suppose that $B(M)$ and $B(N)$ are projective and $Z(M)$ is flat. Then the exact sequence [\(??\)](#) splits.*

Proof. It suffices to show that, if $B(M)$ and $B(N)$ are projective, then the canonical homomorphism

$$\gamma(M, N) : H(M) \otimes_A H(N) \rightarrow H(M \otimes_A N)$$

possesses a k -linear retraction. For this, we recall that by [??](#), there exists quasi-isomorphisms $\varphi : M \rightarrow H(M)$ and $\psi : N \rightarrow H(N)$ such that $H(\varphi) = 1_{H(M)}$ and $H(\psi) = 1_{H(N)}$. In the commutative diagram

$$\begin{array}{ccc}
 H(M) \otimes_A H(N) & \xrightarrow{\gamma(M, N)} & H(M \otimes_A N) \\
 \downarrow H(\varphi) \otimes H(\psi) & & \downarrow H(\varphi \otimes \psi) \\
 H(M) \otimes_A H(N) & \xrightarrow{\gamma(H(M), H(N))} & H(M) \otimes_A H(N)
 \end{array}$$

the maps $H(\varphi) \otimes H(\psi)$ and $\gamma(H(M), H(N))$ are identities, whence our claim. \square

Corollary 8.3.22 (Universal Coefficients Formula). *Suppose that the ring A is a PID. If the complexes M and N are free, the sequence [\(??\)](#) of A -modules is exact and split.*

Proof. In this case, $B(M)$, $Z(M)$, and $B(N)$ are submodules of the free modules M and N , hence are free, and we can apply [??](#). \square

Corollary 8.3.23 (Künneth Formula). *Suppose that M and $H(M)$ are flat and M is right bounded. Then the canonical homomorphism*

$$\gamma(M, N) : H(M) \otimes_A H(N) \rightarrow H(M \otimes_A N)$$

is bijective.

Proof. By [??](#), it suffices to prove that $B(M)$ and $Z(M)$ are flat. Now from the exact sequences [\(??\)](#) and [\(??\)](#), we see the flatness of $B_{n-1}(M)$ implies that of $Z_n(M)$, which in turn implies that

$B_n(M)$ is flat. Since $B_n(M) = 0$ for n small enough, the assertion then follows by induction on n . \square

Corollary 8.3.24. *Let $\alpha : M \rightarrow M'$ be a quasi-isomorphism of complexes of right A -modules that are flat and right bounded. Then for any complex N of left A -modules, the morphism $\alpha \otimes 1_E : M \otimes_A N \rightarrow M' \otimes_A N$ is a quasi-isomorphism.*

8.4 Modules of extensions

We retain the assumption that A is a k -algebra over a commutative ring k . Without further specification, by A -modules we mean left A -modules.

8.4.1 The Hom complex of complexes

Let (M, d) and (N, d) be A -complexes. Consider the graded k -module $\text{Homgr}_A(M, N)$: for each $n \in \mathbb{Z}$, the homogeneous component $\text{Homgr}_A(M, N)_n$ is the k -module of A -linear graded homomorphisms of degree n from M to N . In other words, $\text{Homgr}_A(M, N)$ is identified with the A -module

$$\bigoplus_{n \in \mathbb{Z}} \prod_{p \in \mathbb{Z}} \text{Hom}_A(M_p, N_{p+n}) = \bigoplus_{n \in \mathbb{Z}} \prod_{p+q=n} \text{Hom}_A(M^p, N_q).$$

We define a k -linear map $D_n : \text{Homgr}_A(M, N)_n \rightarrow \text{Homgr}_A(M, N)_{n-1}$ by the formula

$$D_n(f) = d \circ f - (-1)^n f \circ d. \quad (8.4.1)$$

It is clear that we have

$$\begin{aligned} D_{n-1} \circ D_n(f) &= D_{n-1}(d \circ f - (-1)^n f \circ d) \\ &= d^2 \circ f - (-1)^n d \circ f \circ d - (-1)^{n-1} d \circ f \circ d - f \circ d^2 = 0 \end{aligned}$$

The couple $(\text{Homgr}_A(M, N), D)$ is then a complex of k -modules, called the **Hom complex** of M and N . For simplicity, we just denote this complex by $(\text{Hom}(M, N), D)$. For example, $\text{Hom}(A, N)$ is canonically identified with N , and for $r \in \mathbb{Z}$ we have $\text{Hom}(M, N)[r] = \text{Hom}(M, N[r])$.

The elements of $Z_n(\text{Hom}(M, N))$ are the graded homomorphisms f of degree n from M to N such that $d \circ f = (-1)^n f \circ d$, which are the morphisms from the complex M to $N[n]$. We call these homomorphisms as **morphisms of complexes of degree n** from M to N . If $f, g \in Z_n(\text{Hom}(M, N))$ and $s \in \text{Hom}(M, N)_{n+1}$, then the condition $f - g = Ds$ amounts to that s is a homotopy of the morphisms f and g from M to $N[n]$, so $H_n(\text{Hom}(M, N))$ is the k -module of homotopy classes of morphisms of degree n from M to N .

Let $\gamma \in Z_n(\text{Hom}(M, N))$ be represented by $f \in Z_n(\text{Hom}(M, N))$ and $p \in \mathbb{Z}$. Then f is a morphism from M to $N[n]$, so $H_p(f)$ is a homomorphism from $H_p(M)$ to $H_{p+n}(N)$. Since $H_p(f)$ does not depend on the homotopy class of f , we get a canonical induced homomorphism of k -modules:

$$H_n(\text{Hom}(M, N)) \rightarrow \text{Hom}_A(H_p(M), H_{p+n}(N)),$$

whence a canonical graded k -linear homomorphism of degree 0:

$$\lambda(M, N) : H(\text{Hom}(M, N)) \rightarrow \text{Hom}(H(M), H(N)).$$

The homogeneous component of $\lambda(M, N)$ are denoted by

$$\lambda^n(M, N) : H^n(\text{Hom}(M, N)) \rightarrow \prod_{p+q=n} \text{Hom}_A(H_p(M), H^q(N)).$$

Proposition 8.4.1. *If M is positive and N is negative, the complex $\text{Hom}(M, N)$ is negative, and the canonical k -linear map*

$$\lambda^0(M, N) : H^0(\text{Hom}(M, N)) \rightarrow \text{Hom}_A(H_0(M), H^0(N))$$

is bijective.

Proof. We have the following exact sequences:

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^0(N) & \xrightarrow{i} & N^0 & \longrightarrow & N^1 \\ & & M_1 & \xrightarrow{d_1} & M_0 & \xrightarrow{\pi} & H_0(M) \longrightarrow 0 \end{array} \quad (8.4.2)$$

Since $\text{Hom}^0(M, N)$ is identified with $\text{Hom}_A(M_0, N^0)$, the k -modules $Z^0(\text{Hom}(M, N))$ is identified with the set of homomorphisms $f : M_0 \rightarrow N^0$ such that $d^0 \circ f = 0$ and $f \circ d = 0$. Since $B^0(\text{Hom}(M, N))$ is zero, the map λ^0 then associates the class of f modulo $\{0\}$ to the homomorphism $\varphi : H_0(M) \rightarrow H^0(N)$ such that $f = i \circ \varphi \circ \pi$, hence our proposition. \square

Let $\alpha : M' \rightarrow M$ and $\beta : N \rightarrow N'$ be morphisms of complexes. Then the canonical homomorphism $\text{Hom}(\alpha, \beta) : \text{Hom}(M, N) \rightarrow \text{Hom}(M', N')$, defined by $f \mapsto \beta \circ f \circ \alpha$, is a morphisms of complexes, which follows immediately from (??). Moreover, the following diagram is commutative:

$$\begin{array}{ccc} \text{H}(\text{Hom}(M, N)) & \xrightarrow{\lambda(M, N)} & \text{Hom}(H(M), H(N)) \\ \text{H}(\text{Hom}(\alpha, \beta)) \downarrow & & \downarrow \text{Hom}(H(\alpha), H(\beta)) \\ \text{H}(\text{Hom}(M', N')) & \xrightarrow{\lambda(M', N')} & \text{Hom}(H(M'), H(N')) \end{array}$$

Proposition 8.4.2. *Let $0 \rightarrow M' \xrightarrow{\alpha} M \xrightarrow{\beta} N \rightarrow 0$ be an exact sequence of complexes.*

(a) *If F is a projective complex and E is an injective complex, the following sequences are exact:*

$$\begin{aligned} 0 &\longrightarrow \text{Hom}(F, M') \longrightarrow \text{Hom}(F, M) \longrightarrow \text{Hom}(F, M'') \longrightarrow 0 \\ 0 &\longrightarrow \text{Hom}(M', E) \longrightarrow \text{Hom}(M, E) \longrightarrow \text{Hom}(M'', E) \longrightarrow 0 \end{aligned}$$

(b) *If the sequence $0 \rightarrow M' \xrightarrow{\alpha} M \xrightarrow{\beta} N \rightarrow 0$ splits, then for any complex E , the sequences*

$$\begin{aligned} 0 &\longrightarrow \text{Hom}(E, M') \longrightarrow \text{Hom}(E, M) \longrightarrow \text{Hom}(E, M'') \longrightarrow 0 \\ 0 &\longrightarrow \text{Hom}(M', E) \longrightarrow \text{Hom}(M, E) \longrightarrow \text{Hom}(M'', E) \longrightarrow 0 \end{aligned}$$

are exact as complexes of k -modules.

Proof. In case (a), we have the exact sequences

$$0 \longrightarrow \text{Hom}_A(F_p, M'_q) \longrightarrow \text{Hom}_A(F_p, M_q) \longrightarrow \text{Hom}_A(F_p, M''_q) \longrightarrow 0$$

$$0 \longrightarrow \text{Hom}_A(M'_p, E_q) \longrightarrow \text{Hom}_A(M_p, E_q) \longrightarrow \text{Hom}_A(M''_p, E_q) \longrightarrow 0$$

for any $p, q \in \mathbb{Z}$. By taking product and direct sum, we then conclude (a). The proof of (b) is similar. \square

8.4.2 Hom complex and homotopies

Proposition 8.4.3. *Let $\alpha_1 : M' \rightarrow M$, $\alpha_2 : M' \rightarrow M$, $\beta_1 : N \rightarrow N'$, $\beta_2 : N \rightarrow N'$ be morphisms of complexes.*

- (a) *If α_1 is homotopic to α_2 and β_1 is homotopic to β_2 , then $\text{Hom}(\alpha_1, \beta_1)$ is homotopic to $\text{Hom}(\alpha_2, \beta_2)$.*
- (b) *If α_1 and β_1 are homotopy equivalences, so is $\text{Hom}(\alpha_1, \beta_1)$.*
- (c) *If M and N are null-homotopic, so is $\text{Hom}(M, N)$.*

Proof. If $\alpha_1 \sim \alpha_2$ and $\beta_1 \sim \beta_2$, there exist homomorphisms $s : M' \rightarrow N$ and $h : N \rightarrow N'$ of degree 1 such that

$$\alpha_1 - \alpha_2 = ds + sd, \quad \beta_1 - \beta_2 = dh + hd.$$

Let $W : \text{Hom}(M, N) \rightarrow \text{Hom}(M', N')$ be the graded homomorphism of degree 1 such that

$$W(f) = hf\alpha_1 + (-1)^n\beta_2 fs$$

for $f \in \text{Hom}(M, N)_n$. We then have

$$\begin{aligned} (DW + WD)(f) &= D[hf\alpha_1 + (-1)^n\beta_2 fs] + W[df - (-1)^nfd] \\ &= dhf\alpha_1 - (-1)^{n+1}hf\alpha_1 d + (-1)^n d\beta_2 fs + \beta_2 fsd \\ &\quad + hdf\alpha_1 + (-1)^{n-1}\beta_2 df fs - (-1)^n hfd\alpha_1 + \beta_2 f ds \\ &= [dh + hd]f\alpha_1 + \beta_2 f[sd + ds] \\ &= (\beta_1 - \beta_2)f\alpha_1 + \beta_2 f(\alpha_1 - \alpha_2) \\ &= \beta_1 f\alpha_1 - \beta_2 f\alpha_2. \end{aligned}$$

This shows $DW + WD = \text{Hom}(\alpha_1, \beta_1) - \text{Hom}(\alpha_2, \beta_2)$, so assertion (a) follows. The rest of the claims follow from (a). \square

Corollary 8.4.4. *If M is splitting and $H(M)$ is projective (resp. if N is splitting and $H(N)$ is injective), the canonical homomorphism*

$$\lambda(M, N) : H(\text{Hom}(M, N)) \rightarrow \text{Hom}(H(M), H(N))$$

is bijective.

8.5 Koszul complex

8.5.1 The Koszul complex associated with a linear form

Let A be a ring, E an A -module, $\alpha : E \rightarrow A$ a linear form, and $\wedge(E)$ the exterior algebra of the A -module E . For $x \in \wedge(E)$, we denote by $d_\alpha(x)$ the interior product $x \lrcorner \alpha$. In other words, d_α is given by

$$d_\alpha(e_1 \wedge \cdots \wedge e_n) = \sum_{i=1}^n (-1)^{i+1} \alpha(e_i) e_1 \wedge \cdots \wedge \hat{e}_i \wedge \cdots \wedge e_n \quad (8.5.1)$$

for e_1, \dots, e_n in E . The map $d_\alpha : \wedge(E) \rightarrow \wedge(E)$ is an antiderivation of degree -1 with square zero, which is the unique antiderivation on the A -algebra $\wedge(E)$ extending $\alpha : \wedge^1(E) \rightarrow \wedge^0(E)$. The resulting complex $(\wedge(E), d_\alpha)$ is denoted by $K^A(\alpha)$ or $K(\alpha)$, called the **Koszul complex associated with α** . The homogeneous components of $K(\alpha)$ is then $K_n(\alpha) = \wedge^n(E) = K^{-n}(\alpha)$. It is clear that $K(\alpha)$ is positive and $H_0(K(\alpha)) = A/\mathfrak{a}$ where \mathfrak{a} is the ideal $\alpha(E)$ of A .

Now for any complex of A -modules M , we put

$$\begin{aligned} K_\bullet^A(\alpha, M) &= M \otimes_A K^A(\alpha) & K_A^\bullet(\alpha, M) &= \text{Hom}(K^A(\alpha), M), \\ H_\bullet^A(\alpha, M) &= H(M \otimes_A K^A(\alpha)) & H_A^\bullet(\alpha, M) &= H(\text{Hom}(K^A(\alpha), M)). \end{aligned}$$

There are then canonical homomorphisms

$$\gamma_0 : H_0(M) \otimes_A A/\mathfrak{a} \rightarrow H_0^A(\alpha, M), \quad \lambda^0 : H_A^0(\alpha, M) \rightarrow \text{Hom}_A(A/\mathfrak{a}, H^0(M)).$$

By ?? and ??, if the complex M is positive (resp. negative), then $K_\bullet^A(\alpha, M)$ (resp. $K_A^\bullet(\alpha, M)$) is positive (resp. negative), and γ_0 (resp. λ_0) is bijective.

Let $f : M \rightarrow N$ be a morphism of complexes. Then we have the following morphisms:

$$K_\bullet^A(\alpha, f) : K_\bullet^A(\alpha, M) \rightarrow K_\bullet^A(\alpha, N), \quad K_A^\bullet(\alpha, f) : K_A^\bullet(\alpha, M) \rightarrow K_A^\bullet(\alpha, N),$$

which are nothing but $f \otimes 1_{K(\alpha)}$ and $\text{Hom}(1_{K(\alpha)}, f)$. The induced maps on homology modules are denoted by $H_\bullet^A(\alpha, f)$ and $H_A^\bullet(\alpha, f)$. It is clear that these constructions are functorial.

Proposition 8.5.1. *For $x \in E$, let $L_x : y \mapsto x \wedge y$ be the left multiplication on $\wedge(E)$; then*

$$d_\alpha \circ L_x + L_x \circ d_\alpha = \alpha(x) \cdot 1_{\wedge(E)}.$$

Proof. In fact, we have

$$(d_\alpha \circ L_x + L_x \circ d_\alpha)(y) = d_\alpha(x \wedge y) + x \wedge d_\alpha(y);$$

since d_α is an antiderivation, this is equal to $d_\alpha(x) \wedge y = \alpha(x)y$. \square

Corollary 8.5.2. *If α is surjective then $K(\alpha)$ is null-homotopic, so is $K_\bullet^A(\alpha, M)$ and $K_A^\bullet(\alpha, M)$ for any complex M .*

Proof. If α is surjective, then in particular there exist $x \in E$ such that $\alpha(x) = 1$. Then $K(\alpha)$ is null-homotopic by ??, with homotopy given by L_x . The complexes $K_\bullet^A(\alpha, M)$ and $K_A^\bullet(\alpha, M)$ are then null-homotopic by ??.

Corollary 8.5.3. *Let M be a complex and $\text{Ann}(M)$ its annihilator. Then $\mathfrak{a} + \text{Ann}(M)$ annihilates $H_\bullet^A(\alpha, M)$ and $H_A^\bullet(\alpha, M)$.*

Proof. For any $\lambda \in \mathfrak{a}$, the homothety $\lambda_{K(\alpha)}$ is homotopic to zero by ??, hence so is $1_M \otimes \lambda_{K(\alpha)}$ and $\text{Hom}(\lambda_{K(\alpha)}, 1_M)$ (?? and ??); it follows that λ annihilates $H_\bullet^A(\alpha, M)$ and $H_A^\bullet(\alpha, M)$. If $\lambda \in \text{Ann}(M)$, then $1_M \otimes \lambda_{K(\alpha)}$ and $\text{Hom}(\lambda_{K(\alpha)}, 1_M)$ are already zero maps. \square

Suppose that E is projective (resp. $K(\alpha)$ is acyclic at positive degrees). Then the complex $\Lambda(E)$ is projective (resp. is a resolution of A/\mathfrak{a}). By ?? (resp. ??), we then have, for any A -module M , the homomorphisms

$$\begin{aligned} H_\bullet^A(\alpha, M) &\rightarrow \text{Tor}_\bullet^A(A/\mathfrak{a}, M) & \text{Ext}_A^\bullet(A/\mathfrak{a}, M) &\rightarrow H_A^\bullet(\alpha, M), \\ \text{Tor}_\bullet^A(A/\mathfrak{a}, M) &\rightarrow H_\bullet^A(\alpha, M) & H_A^\bullet(\alpha, M) &\rightarrow \text{Ext}_A^\bullet(A/\mathfrak{a}, M). \end{aligned}$$

If E is projective and $K(\alpha)$ is acyclic at positive degrees, the homomorphisms above are bijective and inverses of each other ??).

Proposition 8.5.4. *Let $(E_i)_{i \in I}$ be a family of A -modules, where I is finite and totally ordered. Let α be a linear form on $\bigoplus_{i \in I} E_i$ and α_i its restriction on E_i . Then the canonical isomorphism of A -algebras:*

$$\eta : \bigotimes_{i \in I}^g \Lambda(E_i) \rightarrow \Lambda\left(\bigoplus_{i \in I} E_i\right)$$

is an isomorphism from the complex $\bigotimes_{i \in I} K(\alpha_i)$ to $K(\alpha)$.

Proof. In fact, by ??, the differential D of the complex $\bigotimes_{i \in I} K(\alpha_i)$ is an antiderivation; the antiderivations d_α and $\eta \circ D \circ \eta^{-1}$ on $\Lambda(\bigoplus E_i)$ both coincide on $\bigoplus E_i$ with the map $x \mapsto \alpha(x) \cdot 1$, hence are equal ??). \square

Proposition 8.5.5. *Let M and N be two complexes of A -modules. Then we have the following canonical isomorphisms of complexes:*

$$M \otimes_A K_\bullet^A(\alpha, N) \rightarrow K_\bullet^A(\alpha, M \otimes_A N), \tag{8.5.2}$$

$$\text{Hom}(N, K_A^\bullet(\alpha, M)) \rightarrow \text{Hom}(K_\bullet^A(\alpha, N), M). \tag{8.5.3}$$

Proof. Recall that we have the following isomorphisms:

$$\begin{aligned} M \otimes_A (N \otimes_A K(\alpha)) &\rightarrow (M \otimes_A N) \otimes_A K(\alpha), \\ \text{Hom}(N, \text{Hom}(K(\alpha), M)) &\rightarrow \text{Hom}(N \otimes_A K(\alpha), M); \end{aligned}$$

The claim thus follows in view of the definition of K_\bullet^A and K_A^\bullet . \square

Corollary 8.5.6. *Let $\alpha : E \rightarrow A$ and $\beta : F \rightarrow A$ be two linear forms. Then we have the following isomorphisms of complexes:*

$$K_\bullet^A(\alpha \oplus \beta, M) \rightarrow K_\bullet^A(\alpha, K_\bullet^A(\beta, M)) \tag{8.5.4}$$

$$K_A^\bullet(\beta, K_A^\bullet(\alpha, M)) \rightarrow K_A^\bullet(\alpha \oplus \beta, M); \tag{8.5.5}$$

which by passing to homology induce the following isomorphisms of A -modules:

$$H_\bullet^A(\alpha \oplus \beta, M) \rightarrow H_\bullet^A(\alpha, K_\bullet^A(\beta, M)), \quad (8.5.6)$$

$$H_A^\bullet(\beta, K_A^\bullet(\alpha, M)) \rightarrow H_A^\bullet(\alpha \oplus \beta, M). \quad (8.5.7)$$

Proof. This follows by setting $N = K(\beta)$ in (??) and (??). Note that $K_\bullet^A(\alpha, K(\beta))$ is equal by definition to $K(\alpha) \otimes_A K(\beta)$, which is identified with $K(\alpha \oplus \beta)$, where $\alpha \oplus \beta$ is the linear form $(x, y) \mapsto \alpha(x) + \beta(y)$. \square

Note that the homomorphism deduced from the product of the algebra $\Lambda(E)$

$$m : K^A(\alpha) \otimes_A K^A(\alpha) \rightarrow K^A(\alpha)$$

is a morphism of complexes (since d_α is an antiderivation). Suppose that E is free with rank n and composing with the morphism $K^A(\alpha) \rightarrow \Lambda^n(E)[-n]$ (which is just the identity in degree n), we deduce a morphism of complexes

$$\chi : K^A(\alpha) \otimes_A K^A(\alpha) \rightarrow \Lambda^n(E)[-n];$$

by ??, this morphism corresponds to a canonical morphism of complexes:

$$\phi : K^A(\alpha) \rightarrow \text{Hom}(K^A(\alpha), \Lambda^n(E)[-n])$$

which is bijective. For any complex M , we then deduce an isomorphism of complexes

$$\begin{array}{ccc} K_\bullet^A(\alpha, M) & = & M \otimes_A K^A(\alpha) \xrightarrow{1_M \otimes \phi} M \otimes \text{Hom}(K^A(\alpha), \Lambda^n(E)[-n]) \\ & \downarrow & \downarrow \\ K_A^\bullet(\alpha, M \otimes_A \Lambda^n(E)[-n]) & = & \text{Hom}(K^A(\alpha), M \otimes_A \Lambda^n(E)[-n]) \end{array}$$

and hence a canonical isomorphism on homology modules:

$$H_\bullet^A(\alpha, M) \rightarrow H_A^{n-\bullet}(\alpha, M \otimes_A \Lambda^n(E)). \quad (8.5.8)$$

Since E is free of rank n , we know the module $\Lambda^n(E)$ is isomorphic to A , so in particular, we have an isomorphism $H_\bullet^A(\alpha, M) \rightarrow H_A^{n-\bullet}(\alpha, M)$. Note however that this isomorphism is not canonical.

Proposition 8.5.7. *Consider an exact sequence of complexes $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$.*

(a) *If E is flat then there is a long exact sequence of $H_\bullet^A(\alpha, -)$:*

$$\cdots \longrightarrow H_n^A(\alpha, L) \longrightarrow H_n^A(\alpha, M) \longrightarrow H_n^A(\alpha, N) \longrightarrow H_{n-1}^A(\alpha, L) \longrightarrow \cdots$$

(b) *If E is projective then there is also a long exact sequence of $H_A^\bullet(\alpha, -)$:*

$$\cdots \longrightarrow H_A^n(\alpha, L) \longrightarrow H_A^n(\alpha, M) \longrightarrow H_A^n(\alpha, N) \longrightarrow H_A^{n+1}(\alpha, L) \longrightarrow \cdots$$

Proof. If E is flat (resp. projective) then $\Lambda(E)$ is flat (resp. projective), so the sequence

$$0 \longrightarrow K_\bullet^A(\alpha, L) \longrightarrow K_\bullet^A(\alpha, M) \longrightarrow K_\bullet^A(\alpha, N) \longrightarrow 0$$

resp. the sequence

$$0 \longrightarrow K_A^\bullet(\alpha, L) \longrightarrow K_A^\bullet(\alpha, M) \longrightarrow K_A^\bullet(\alpha, N) \longrightarrow 0$$

The rest then follows from the long exact sequence theorem for homology modules. \square

Proposition 8.5.8. *Let $\rho : A \rightarrow B$ be a ring homomorphism, $\rho_*(E)$ the B -module $E \otimes_A B$, and $\rho_*(\alpha) : \rho^*(E) \rightarrow B$ the linear form $1 \otimes \alpha$. Then the canonical homomorphism*

$$\psi : \Lambda_B(B \otimes_A E) \rightarrow B \otimes_A \Lambda_A(E)$$

is an isomorphism of complexes of B -modules. In particular:

- (a) *for any complex N of B -modules, we have an isomorphism $\rho^*(K_\bullet^B(\rho_*(\alpha), N)) \rightarrow K_\bullet^A(\alpha, \rho^*(N))$ of complexes of A -modules, which is given by the following composition:*

$$N \otimes_B (\Lambda_B(B \otimes_A E)) \xrightarrow{1_N \otimes \psi} N \otimes_B B \otimes_A \Lambda_A(E) \longrightarrow B \otimes_A \Lambda_A(E)$$

- (b) *for any complex M of A -modules, we have an isomorphism $K_\bullet^A(\alpha, \rho_*(M)) \rightarrow \rho_*(K_\bullet^A(\alpha, M))$ of complexes of B -modules, whence an isomorphism*

$$H_\bullet^A(\alpha, \rho_*(M)) \rightarrow \rho_*(H_\bullet^A(\alpha, M))$$

if B is flat over A .

Proof. The homomorphism ψ is an isomorphism by (A, III, p.83, prop.8 ??), so assertion (a) follows from (A, III, p.85, prop.14 ??), and (b) follows from (A, X, p.66, cor.2 ??). \square

Let F be an A -module and $\beta : F \rightarrow A$ a linear form. Let $f : E \rightarrow F$ be a homomorphism such that $\beta \circ f = \alpha$. Then it follows from formula (??) that the homomorphism $\Lambda(f) : \Lambda(E) \rightarrow \Lambda(F)$ satisfies

$$d_\alpha \circ \Lambda(f) = \Lambda(f) \circ d_\beta$$

hence defines a morphism $\Lambda(f) : K^A(\alpha) \rightarrow K^A(\beta)$ of complexes. If M is an A -complex, we then get morphisms:

$$1_M \otimes \Lambda(f) : K_\bullet^A(\alpha, M) \rightarrow K_\bullet^A(\beta, M), \quad \text{Hom}(\Lambda(f), 1_M) : K_A^\bullet(\alpha, M) \rightarrow K_A^\bullet(\beta, M).$$

8.5.2 The canonical Koszul complex of $S(E)$

Let A be a ring, E an A -module, $S(E)$ the symmetric algebra over E , and $S(E) \otimes_A E$ the $S(E)$ -module induced by extending scalars. Let $\alpha : S(E) \otimes_A E \rightarrow S(E)$ be the linear form defined by $\alpha(s \otimes x) = sx$ for $s \in S(E)$ and $x \in E$. By the canonical isomorphism of $S(E)$ -modules (??)

$$\Lambda_{S(E)}(S(E) \otimes_A E) \rightarrow S(E) \otimes_A \Lambda(E)$$

the differential of the complex $K^{S(E)}(\alpha)$ then corresponds to a map

$$d : S(E) \otimes_A \Lambda(E) \rightarrow S(E) \otimes_A \Lambda(E)$$

such that for $x_1, \dots, x_p, y_1, \dots, y_q$ in E , we have

$$d((x_1 \cdots x_p) \otimes (y_1 \wedge \cdots \wedge y_q)) = \sum_{i=1}^q (-1)^{i+1} y_i x_1 \cdots x_p \otimes (y_1 \wedge \cdots \wedge \hat{y}_i \wedge \cdots \wedge y_q). \quad (8.5.9)$$

Note that d maps $S^p(E) \otimes \Lambda^q(E)$ to $S^{p+1}(E) \otimes \Lambda^{q-1}(E)$, so the complex $S(E) \otimes \Lambda(E)$ breaks down into the direct sum of complexes described by the following diagrams

$$0 \longrightarrow S^0(E) \otimes_A \Lambda^n(E) \longrightarrow S^1(E) \otimes_A \Lambda^{n-1}(E) \longrightarrow \cdots \longrightarrow S^n(E) \otimes_A \Lambda^0(E) \longrightarrow 0$$

which we denote by \mathcal{E}_n . If the A -module E is a direct sum of a finite family $(E_i)_{i \in I}$ where I is totally ordered, the canonical bijection

$$\bigotimes_{i \in I}^g (S(E_i) \otimes_A \Lambda(E_i)) \rightarrow S(E) \otimes_A \Lambda(E)$$

is an isomorphism of A -complexes (this follows from ?? and formula (??)).

Proposition 8.5.9. *If E is a flat A -module, then the complex \mathcal{E}_n is exact for $n > 0$.*

Proof.

□

8.5.3 Koszul complexes and alternating maps

Let k be a ring, M an k -module, I a finite set and $p \geq 0$ an integer. We say a map $m : I^p \rightarrow M$ is **alternating** if the following conditions are satisfied:

- (a) for any permutation $\sigma \in \mathfrak{S}_p$ and any sequence $(\alpha_1, \dots, \alpha_p) \in I^p$, we have

$$m(\alpha_{\sigma(1)}, \dots, \alpha_{\sigma(p)}) = \text{sgn}(\sigma)m(\alpha_1, \dots, \alpha_p);$$

- (b) for any sequence $(\alpha_1, \dots, \alpha_p) \in I^p$ such that two of the indices in $(\alpha_1, \dots, \alpha_p)$ are equal, we have $m(\alpha_1, \dots, \alpha_p) = 0$.

We denote by $C^p(I, M)$ the k -module of alternating maps from I^p to M .

If E_0 is a k_0 -module and $(e_i)_{i \in I}$ is a family of elements in E_0 , we define a map

$$\tau : \text{Hom}_k(\Lambda^p(E_0), M) \rightarrow C^p(I, M)$$

by setting $\tau(f)(\alpha_1, \dots, \alpha_p) = f(e_{\alpha_1} \wedge \cdots \wedge e_{\alpha_p})$ for $f \in \text{Hom}_k(\Lambda^p(E_0), M)$. We also have a map

$$\eta : C^p(I, M) \rightarrow M \otimes_k \Lambda^p(E_0).$$

To define η , consider an element $m \in C^p(I, M)$; for any element $(\alpha_1, \dots, \alpha_p)$ of I^p , the element $m(\alpha_1, \dots, \alpha_p) \otimes (e_{\alpha_1} \wedge \cdots \wedge e_{\alpha_p})$ of $\Lambda^p(E_0) \otimes_k M$ only depends on the subset $J = \{\alpha_1, \dots, \alpha_p\}$;

we denote it by $\eta_J(m)$. The map η is then given by

$$\eta(m) = \sum_J \eta_J(m).$$

where J runs through subset of I with exactly p elements. By ??, if the k -module E_0 is free with basis $(e_i)_{i \in I}$, the k -linear maps τ and η are bijective.

Let M be a k -module and $\mathbf{x} = (x_i)_{i \in I}$ be a finite permutable family of k -endomorphisms of M . Consider the polynomial $A = k[(X_i)_{i \in I}]$ and endow M with the A -module structure such that $P(X_i)m = P(x_i)m$ for $P \in A$ and $m \in M$. Also, let $\alpha : A^I \rightarrow A$ be the linear form that sends e_i to X_i for each $i \in I$, where $(e_i)_{i \in I}$ is the canonical basis for A^I . Consider the Koszul complex of k -modules $K_A^\bullet(\alpha, M)$ and $K_A^\bullet(\alpha, M)$; we have the canonical isomorphisms

$$\begin{aligned} K_A^\bullet(\alpha, M) &= \text{Hom}_A(\Lambda_A(A^I), M) \rightarrow \text{Hom}_k(\Lambda_k(k^I), M), \\ M \otimes_k \Lambda_k(k^I) &\rightarrow M \otimes_A \Lambda_A(A^I) = K_A^\bullet(\alpha, M); \end{aligned}$$

whence by composition with the isomorphisms τ and η of the isomorphisms of k -modules

$$\theta^p : K_A^p(\alpha, M) \rightarrow C^p(I, M), \quad \theta_p : C^p(I, M) \rightarrow K_A^p(\alpha, M).$$

By transporting or structure, we then get differentials ∂^p and ∂_p on $C^p(I, M)$:

$$\partial^p : C^p(I, M) \rightarrow C^{p+1}(I, M), \quad \partial_p : C^p(I, M) \rightarrow C^{p-1}(I, M).$$

For example, the differential ∂^p is given by

$$(\partial^p m)(\alpha_1, \dots, \alpha_{p+1}) = \sum_{j=1}^{p+1} (-1)^{j+1} x_{\alpha_j} m(\alpha_1, \dots, \widehat{\alpha_j}, \dots, \alpha_{p+1}). \quad (8.5.10)$$

The complex $(C^p(I, M), \partial^p)$ (resp. the complex $(C^p(I, M), \partial_p)$) is denoted by $K^\bullet(\mathbf{x}, M)$ (resp. $K_\bullet(\mathbf{x}, M)$) and called the ascending (resp. descending) **Koszul complex** associated with the module M and the sequence $(x_i)_{i \in I}$. We then have isomorphisms of complexes of k -modules:

$$\theta^\bullet : K_A^\bullet(\alpha, M) \rightarrow K^\bullet(\mathbf{x}, M)$$

Remark 8.5.1. Conversely, let B be a k -algebra, E a free B -module with basis $(e_i)_{i \in I}$, and M a B -module. The giving a linear form $\alpha : E \rightarrow B$ is equivalent to giving a family $\mathbf{x} = (x_i)_{i \in I}$ of elements of B , by the relation $x_i = \alpha(e_i)$. The complex of k -modules underlying $K_B^\bullet(\alpha, M)$ (resp. $K_\bullet^B(\alpha, M)$) is then identified, by the isomorphisms θ^\bullet (resp. θ_\bullet), with the Koszul complex $K^\bullet(\mathbf{x}, M)$ (resp. $K_\bullet(\mathbf{x}, M)$). For example, $K^B(\alpha)$ is identified with $K^\bullet(\mathbf{x}, B)$.

We introduce the notations $H_\bullet(\mathbf{x}, M)$, $H^\bullet(\mathbf{x}, M)$, etc. The previous results apply mutatis mutandis, with the module A^I being free. For example, we have isomorphisms

$$H_0(\mathbf{x}, M) \rightarrow M/\mathfrak{x}M, \quad H^0(\mathbf{x}, M) \rightarrow \text{Hom}_A(A/\mathfrak{x}, M),$$

where \mathfrak{x} denotes the ideal $\sum Ax_i$ generated by x_i ; and if $K_\bullet(\mathbf{x}, A)$ is acyclic at positive degrees,

we have isomorphisms

$$H_\bullet(x, M) \rightarrow \text{Tor}_\bullet^A(k, M), \quad \text{Ext}_A^\bullet(k, M) \rightarrow H^\bullet(x, M).$$

Finally, suppose that I is totally ordered, for example $I = \{1, \dots, n\}$; then we can identify $\Lambda^n(A^I)$ to A by choosing the basis element $e_1 \wedge \dots \wedge e_n$. Similarly, we can identify $K_p^A(\alpha, M)$ with $K_A^{n-p}(\alpha, M)$. By transporting structure by the isomorphisms θ_p and θ^{n-p} , we get an isomorphism

$$C^p(I, M) \rightarrow C^{n-p}(I, M)$$

where an element $m \in C^p(I, M)$ corresponds to an element \check{m} of $C^{n-p}(I, M)$ such that

$$m(\alpha_1, \dots, \alpha_p) = \check{m}(\beta_1, \dots, \beta_{n-p})$$

if $(\alpha_1, \dots, \alpha_p, \beta_1, \dots, \beta_{n-p})$ is an even permutation of $\{1, \dots, n\}$. Also note that when $I = \{1, \dots, n\}$, we can identify $C^p(I, M)$ with the set of families $m(\alpha_1, \dots, \alpha_p)$ of elements of M where $\alpha_1 < \dots < \alpha_p$.

Example 8.5.10. Let $M = k[T_1, \dots, T_n]$; the Koszul complex $K(\partial/\partial T, M)$ associated with the sequence $(\partial/\partial T_1, \dots, \partial/\partial T_n)$ is then identified with the de Rham complex of $k[X_1, \dots, X_n]$ over k . In fact, for an element $m \in C^p(I, M)$, where $I = \{1, \dots, n\}$, we can associate a differential form

$$\omega(m) = \sum_{\alpha_1 < \dots < \alpha_p} m(\alpha_1, \dots, \alpha_p) dx_{\alpha_1} \wedge \dots \wedge d_{x_{\alpha_p}}.$$

The assertion follows from the following computation

$$\begin{aligned} d\omega(m) &= \sum_{\alpha_1 < \dots < \alpha_p} \sum_{i=1}^n (\partial/\partial T_i)(m(\alpha_1, \dots, \alpha_p)) dx_i \wedge dx_{\alpha_1} \wedge \dots \wedge d_{x_{\alpha_p}} \\ &= \sum_{\alpha_1 < \dots < \alpha_{p+1}} \sum_{i=1}^{p+1} (-1)^{i+1} (\partial/\partial T_{\alpha_i})(m(\alpha_1, \dots, \widehat{\alpha_i}, \dots, \alpha_{p+1})) dx_{\alpha_1} \wedge \dots \wedge d_{x_{\alpha_{p+1}}} \\ &= \omega(\partial^p m). \end{aligned}$$

which shows ω is a complex isomorphism.

8.5.4 Completely secant sequences

We now turn to the most important case, where E is just the free module A . Let $\alpha : A \rightarrow A$ be a linear form on A and $\alpha(1) = x \in A$. The complex $K(\alpha)$ is then of length 1 and given by

$$0 \longrightarrow A \xrightarrow{x_A} A \longrightarrow 0$$

Hence for any A -module M , the complexes $K_\bullet^A(\alpha, M)$ and $K_A^\bullet(\alpha, M)$ are just

$$0 \longrightarrow M \xrightarrow{x_M} M \longrightarrow 0$$

We then have isomorphisms

$$H_0(x, M) \xrightarrow{\sim} A/xA \otimes_A M \xleftarrow{\sim} H^1(x, M), \quad H_1(x, M) \xrightarrow{\sim} \text{Hom}_A(A/xA, M) \xleftarrow{\sim} H^0(x, M).$$

Lemma 8.5.11. *Let K be an A -complex such that $K_i = 0$ for $i \neq 0, 1$, and M an A -complex.*

(a) *If K is flat, for any integer $p \in \mathbb{N}$ we have an exact sequence*

$$0 \longrightarrow H_0(K \otimes_A H_p(M)) \longrightarrow H_p(K \otimes_A M) \longrightarrow H_1(K \otimes_A H_{p-1}(M)) \longrightarrow 0$$

(b) *If K is projective, for any integer $p \in \mathbb{N}$ we have an exact sequence*

$$0 \rightarrow H^1(\text{Hom}(K, H^{p-1}(M))) \rightarrow H_p(\text{Hom}(K, M)) \rightarrow H^0(\text{Hom}(K, H_{p-1}(M))) \rightarrow 0$$

Proof. We only prove (a), since (b) follows similarly. For each i , let $K_{(i)}$ be the complex $K_i[-i]$. We then have an exact sequence of complexes, which splits as sequences of A -modules:

$$0 \longrightarrow K_{(0)} \xrightarrow{\alpha} K \xrightarrow{\beta} K_{(1)} \longrightarrow 0$$

The sequence of complexes

$$0 \longrightarrow K_{(0)} \otimes_A M \xrightarrow{\alpha \otimes 1} K \otimes_A M \xrightarrow{\beta \otimes 1} K_{(1)} \otimes_A M \longrightarrow 0 \tag{8.5.11}$$

is then exact, with K being flat. By ??, the homomorphisms

$$\gamma(K_{(0)}, M) : K_{(0)} \otimes_A H_p(M) \rightarrow H_p(K_{(0)} \otimes_A M) \tag{8.5.12}$$

$$\gamma(K_{(1)}, M) : K_{(1)} \otimes_A H_{p-1}(M) \rightarrow H_p(K_{(1)} \otimes_A M) \tag{8.5.13}$$

are bijective. We now calculate the connecting homomorphism $\partial(\alpha \otimes 1, \beta \otimes 1)$: by definition, it maps the class of a cycle $\sum a_i \otimes b_i$ to the class of $\sum da_i \otimes b_i$, which means

$$\partial(\alpha \otimes 1, \beta \otimes 1) \circ \gamma(K_{(1)}, M) = \gamma(K_{(0)}, M) \circ (d_K \otimes 1).$$

Combining with the isomorphisms ?? and ??, the long exact sequence associated with ?? is then of the form

$$K_1 \otimes H_p(M) \xrightarrow{d_K \otimes 1} K_0 \otimes H_p(M) \longrightarrow H_p(K \otimes M) \longrightarrow K_1 \otimes H_{p-1}(M) \xrightarrow{d_K \otimes 1} K_0 \otimes H_{p-1}(M)$$

which proves (a) by taking homology modules on both ends. \square

Proposition 8.5.12. *For any complex M , we have an exact sequence*

$$0 \longrightarrow A/xA \otimes_A H_p(M) \longrightarrow H_p(x, M) \longrightarrow \text{Hom}_A(A/xA, H_{p-1}(M)) \longrightarrow 0$$

Proof. This follows by applying ??(a) to the complex $K(\alpha)$, which is flat. \square

Corollary 8.5.13. For $H_p(x, M) = 0$, it is necessary and sufficient that the homothety with ratio x is surjective on $H_p(M)$ and injective on $H_{p-1}(M)$.

Proof. By the exact sequence in ??, for $H_p(x, M) = 0$ it is necessary and sufficient that $A/xA \otimes_A H_p(M)$ and $\text{Hom}_A(A/xA, H_{p-1}(M))$ are both zero. Since $A/xA \otimes_A H_p(M)$ is identified with $H_p(M)/xH_p(M)$ and $\text{Hom}_A(A/xA, H_{p-1}(M))$ is identified with the elements of $H_{p-1}(M)$ annihilated by x , we see the corollary follows. \square

Corollary 8.5.14. Let $\mathbf{x} = (x_1, \dots, x_n)$ be a sequence of elements of A , M an A -module, and $\tilde{\mathbf{x}} = (x_1, \dots, x_{n-1})$. Then we have an exact sequence

$$0 \longrightarrow A/x_n A \otimes_A H_p(\tilde{\mathbf{x}}, M) \longrightarrow H_p(\mathbf{x}, M) \longrightarrow \text{Hom}_A(A/x_n, H_{p-1}(\tilde{\mathbf{x}}, M)) \longrightarrow 0$$

Proof. By noting that $K_\bullet^A(\mathbf{x}, M)$ is the tensor product of $K(x)$ and $K_\bullet^A(\tilde{\mathbf{x}}, M)$, this is just an application of ?? \square

Corollary 8.5.15. Let $\mathbf{x} = (x_1, \dots, x_n)$ be a sequence of elements of A and M an A -module. For $H_i(\mathbf{x}, M) = 0$ for any $i > 0$, it is necessary and sufficient that the homothety with ratio x is bijective on $H_i(\tilde{\mathbf{x}}, M)$ for $i > 0$, and injective on $M/\tilde{\mathfrak{x}}M$, where $\tilde{\mathfrak{x}}$ is the ideal generated by $\tilde{\mathbf{x}}$.

Let $\mathbf{x} = (x_i)_{i \in I}$ be a sequence of elements of A . We say \mathbf{x} is **completely secant** for M if $H_i(\mathbf{x}, M) = 0$ for each $i > 0$. If the family \mathbf{x} is finite, this also means $H^i(\mathbf{x}, M) = 0$ for $i < |I|$. Now ?? just gives a equivalent condition for a sequence $\mathbf{x} = (x_1, \dots, x_n)$ to be completely secant, and in particular, we get a special example:

Proposition 8.5.16. Let $\mathbf{x} = (x_1, \dots, x_n)$ be a sequence of elements of A . If for each i the homothety with ratio x_i is injective on the A -module $M_i = M/(x_1M + \dots + x_{i-1}M)$, then the sequence \mathbf{x} is completely secant for M .

Proof. We say a sequence $\mathbf{x} = (x_1, \dots, x_n)$ is M -regular if it satisfies the conditions given in the proposition. It is clear that if \mathbf{x} is M -regular then so is $\tilde{\mathbf{x}} = (x_1, \dots, x_{n-1})$. So by induction using ??, we reduce to the case where $n = 1$, and this is trivial by ?? \square

We note that the regular property of $\mathbf{x} = (x_1, \dots, x_n)$ depends on the order of the x_i : for example the sequence $(1, 0)$ is clearly M -regular, but $(0, 1)$ is not regular unless M is zero. On the other hand, the fact that a sequence is completely secant does not depend on the order of its terms.

Example 8.5.17. Let k be a ring, $A = k[X_1, \dots, X_n]$, and $\mathbf{x} = (X_1, \dots, X_n)$. Then \mathbf{x} is A -regular and ?? restores the acyclicity at positive degrees of the complex $S_k(k^n) \otimes_k \Lambda_k(k^n)$ (??). Similarly, the sequence $\mathbf{x} = (X_1, \dots, X_n)$ is regular in the ring $k[\![X_1, \dots, X_n]\!]$.

Proposition 8.5.18. Let A be a ring and $\mathbf{x} = (x_1, \dots, x_n)$ a sequence of elements of A . Let \mathfrak{x} be the ideal generated by \mathbf{x} in A .

- (a) If $\mathfrak{x} = A$ then the family \mathbf{x} is completely secant for M .
- (b) Let $(a_{ij}) \in \text{GL}_n(A)$ and put $y_i = \sum_j a_{ij}x_j$. If the sequence \mathbf{x} is completely secant for M , then $\mathbf{y} = (y_1, \dots, y_n)$ is also completely secant.

- (c) If k_1, \dots, k_n are positive integers, the sequence $(x_1^{k_1}, \dots, x_n^{k_n})$ is completely secant for M if and only if \mathbf{x} is completely secant for M .
- (d) Let N be a flat A -module. If the family \mathbf{x} is completely secant for M , so it is for $M \otimes_A N$.
- (e) Let $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$ be an exact sequence of A -modules. If the family \mathbf{x} is completely secant for L and N , so it is for M .

Proof. The first assertion is trivial. Let $(a_{ij}) \in \mathrm{GL}_n(A)$ be as in (b) and $f : A^n \rightarrow A^n$ be the isomorphism defined by $(a_{ij})^t$. Then the map $1_M \otimes \Lambda(f)$ is an isomorphism from $K_\bullet(\mathbf{y}, M)$ to $K_\bullet(\mathbf{x}, M)$, whence (b) follows.

To prove (c), it suffices to show that if $k \geq 1$ is an integer, the sequence $(x_1, \dots, x_{n-1}, x_n^k)$ is completely secant for M if and only if \mathbf{x} is. Let $\tilde{\mathbf{x}} = (x_1, \dots, x_{n-1})$, then by ??, the first condition (resp. the second) means the homothety with ratio x_n^k (resp. x_n) is bijective on $H_i(\tilde{\mathbf{x}}, M)$ for $i > 0$ and injective on $M/\tilde{\mathbf{x}}M$. These two conditions are clearly equivalent, so (c) follows.

Let N be a flat A -module. Then the complex $K_\bullet(\mathbf{x}, M \otimes_A N)$ is isomorphic to $K_\bullet(\mathbf{x}, M) \otimes_A N$, hence $H_\bullet(\mathbf{x}, M \otimes_A N) = H_\bullet(\mathbf{x}, M)$. This proves (d), and assertion (e) follows from the long exact sequence induced from the given short exact sequence. \square

Remark 8.5.2. If the family \mathbf{x} is completely secant for A , the complex $K_\bullet(\mathbf{x}, A)$ then defines a free resolution for the A -module A/\mathfrak{x} . Then for $i \geq 0$ we have the following isomorphisms

$$\mathrm{Ext}_A^{n-i}(A/\mathfrak{x}, M) \xrightarrow{\sim} H^{n-i}(\mathbf{x}, M) \xrightarrow{\sim} H_i(\mathbf{x}, M) \xrightarrow{\sim} \mathrm{Tor}_i^A(A/\mathfrak{x}, M). \quad (8.5.14)$$

Example 8.5.19. We say the sequence $\mathbf{x} = (x_1, \dots, x_n)$ is **M -coregular** if the homothety $(x_i)_M$ is surjective on the module

$$\ker(x_1)_M \cap \dots \cap \ker(x_{i-1})_M$$

for each i . Similar to ??, we can show that $H_i(\mathbf{x}, M) = 0$ for $i < n$ if \mathbf{x} is M -coregular.

Take for example $A = k[D_1, \dots, D_n]$, where k is a \mathbb{Q} -algebra, and $M = k[T_1, \dots, T_n]$, where D_i acts on M by $D_i P = \partial P / \partial T_i$. We then verify that the sequence (D_1, \dots, D_n) is M -coregular. Combine this with ??, we deduce that the de Rham complex of $k[T_1, \dots, T_n]$ over k is acyclic at positive degrees.

Let A be a ring, M an A -module, \mathfrak{x} an ideal of A . We recall that the \mathfrak{x} -adic topology on M is the topology compatible with the module structure on M and such that $\mathfrak{x}^r M$ form a fundamental neighborhood of 0 in M . Suppose now \mathfrak{x} is generated by a sequence $\mathbf{x} = (x_1, \dots, x_n)$ of elements of A . Consider the graded A -module $\bigoplus_r \mathfrak{x}^r M$ and the graded A -homomorphism of degree 0:

$$a_M^{\mathfrak{x}} : A[X_1, \dots, X_n] \otimes_A M \rightarrow \bigoplus_{r \in \mathbb{N}} \mathfrak{x}^r M$$

such that $a_M^{\mathfrak{x}}(P \otimes m) = P(x_1, \dots, x_n)m$ if P is a homogeneous polynomial and $m \in M$. Let \mathfrak{d} be the ideal of $A[X_1, \dots, X_n]$ generated by the elements $(x_i X_j - x_j X_i)$ for $1 \leq i < j \leq n$. Then $P(x_1, \dots, x_n) = 0$ if $P \in \mathfrak{d}$, so $a_M^{\mathfrak{x}}$ factors into a graded A -homomorphism of degree 0:

$$\alpha_M^{\mathfrak{x}} : (A[X_1, \dots, X_n]/\mathfrak{d}) \otimes_A M \rightarrow \bigoplus_{r \in \mathbb{N}} \mathfrak{x}^r M.$$

By tensoring with A/\mathfrak{x} , we deduce from $\alpha_M^{\mathfrak{x}}$ another graded A -homomorphism of degree 0:

$$\beta_M^{\mathfrak{x}} : (A/\mathfrak{x})[X_1, \dots, X_n] \otimes_A M \rightarrow \text{gr}_{\mathfrak{x}}(M).$$

For each i , let M_i be the quotient A -module $M/(x_1M + \dots + x_{i-1}M)$.

Lemma 8.5.20. *Let $A_0 = \mathbb{Z}[T_1, \dots, T_n]$ and $\rho : A_0[X_1, \dots, X_n] \rightarrow A_0[U]$ be the homomorphism of A_0 -algebras given by $\rho(X_i) = T_iU$. Then the kernel of ρ is the ideal \mathfrak{d}_0 of $A_0[X_1, \dots, X_n]$ generated by the elements $(T_iX_j - T_jX_i)$ for $1 \leq i < j \leq n$. If \mathfrak{t} is the ideal generated by T_1, \dots, T_n then ρ induces an isomorphism*

$$\bar{\rho} : A_0[X_1, \dots, X_n]/\mathfrak{d}_0 \rightarrow \bigoplus_{r \in \mathbb{N}} \mathfrak{t}^r.$$

Proof. □

Theorem 8.5.21. *Consider the following conditions:*

- (i) *The sequence \mathbf{x} is M -regular.*
- (ii) *The sequence \mathbf{x} is completely secant for M .*
- (iii) $H_1(\mathbf{x}, M) = 0$.
- (iv) *The homomorphism $\alpha_M^{\mathfrak{x}} : (A[X_1, \dots, X_n]/\mathfrak{d}) \otimes_A M \rightarrow \bigoplus_{r \in \mathbb{N}} \mathfrak{x}^r M$ is bijective.*
- (v) *The homomorphism $\beta_M^{\mathfrak{x}} : (A/\mathfrak{x})[X_1, \dots, X_n] \otimes_A M \rightarrow \text{gr}_{\mathfrak{x}}(M)$ is bijective.*

We have (i) \Rightarrow (ii) \Rightarrow (iii) \Leftrightarrow (iv) \Rightarrow (v). If for each i the A -module M_i is separated for the \mathfrak{x} -adic topology, the above conditions are equivalent.

Proof. The implication (i) \Rightarrow (ii) is already proved, and (ii) \Rightarrow (iii) is clear; so is (iv) \Rightarrow (v), since $\beta_M^{\mathfrak{x}}$ is identified with $\alpha_M^{\mathfrak{x}} \otimes 1_{A/\mathfrak{x}}$. Also, we note that the homomorphisms $\alpha_M^{\mathfrak{x}}$ and $\beta_M^{\mathfrak{x}}$ are surjective.

To prove (iv) \Rightarrow (iii), consider the homomorphism $(\alpha_M^{\mathfrak{x}})_1$ induced on the degree one component by $\alpha_M^{\mathfrak{x}}$. Let E be the graded A -module $A[X_1, \dots, X_n]$; the A -module E_1 is free with basis X_1, \dots, X_n , and \mathfrak{d}_1 is the sub- A -module of E_1 generated by the elements $x_iX_j - x_jX_i$ for $1 \leq i < j \leq n$. The A -module $((E/\mathfrak{d}) \otimes_A M)_1$ is then identified with $K_1(\mathbf{x}, M)/B_1(K_{\bullet}(\mathbf{x}, M))$, and the homomorphism $(\alpha_M^{\mathfrak{x}})_1$ is identified with the map from $K_1(\mathbf{x}, M)/B_1(K_{\bullet}(\mathbf{x}, M))$ to $B_0(K_{\bullet}(\mathbf{x}, M))$ induced by d_1 . The condition that $H_1(\mathbf{x}, M) = 0$ is then equivalent to the injectivity of $(\alpha_M^{\mathfrak{x}})_1$, which proves the implication (iv) \Rightarrow (iii).

To prove that (iii) implies (iv), we consider the ring $A_0 = \mathbb{Z}[T_1, \dots, T_n]$ and let \mathfrak{t} be the ideal of A_0 generated by T_1, \dots, T_n . Endow M the A_0 -module structure such that $T_i m = x_i m$ for $m \in M$. Then the homology group $H_i(\mathbf{x}, M)$ is identified with $H_i(\mathbf{T}, M)$, where $\mathbf{T} = (T_1, \dots, T_n)$. Since \mathbf{T} is regular for A_0 , it follows from the implication (i) \Rightarrow (ii) that the complex $K_{\bullet}(\mathbf{T}, A_0)$ is a free resolution for the A_0 -module A_0/\mathfrak{t} , which is identified with \mathbb{Z} with the A_0 -structure defined by $T_i \mathbb{Z} = 0$ for each i . Therefore, the condition (iii) is equivalent to $\text{Tor}_1^{A_0}(M, \mathbb{Z}) = 0$.

We first show that (iii) implies $\text{Tor}_1^{A_0}(M, A_0/\mathfrak{t}^r) = 0$ for $r \geq 1$. This is true for $r = 1$ as we have seen. In the general case, consider the exact sequence

$$0 \longrightarrow \mathfrak{t}^r/\mathfrak{t}^{r+1} \xrightarrow{\iota_r} A_0/\mathfrak{t}^{r+1} \xrightarrow{\pi_r} A_0/\mathfrak{t}^r \longrightarrow 0 \tag{8.5.15}$$

The A_0 -module $\mathfrak{t}^r/\mathfrak{t}^{r+1}$ is isomorphic to a finite product of modules isomorphic to \mathbb{Z} , so $\mathrm{Tor}_1^{A_0}(M, \mathfrak{t}^r/\mathfrak{t}^{r+1}) = 0$. We then deduce, by induction on r , that $\mathrm{Tor}_1^{A_0}(M, A_0/\mathfrak{t}^r) = 0$ for each $r \geq 1$. The sequence (??) then provides, by tensoring with M , an exact sequence

$$0 \longrightarrow (\mathfrak{t}^r/\mathfrak{t}^{r+1}) \otimes_{A_0} M \longrightarrow M/\mathfrak{t}^{r+1}M \longrightarrow M/\mathfrak{t}^rM \longrightarrow 0 \quad (8.5.16)$$

hence an isomorphism $(\mathfrak{t}^r/\mathfrak{t}^{r+1}) \otimes_{A_0} M$ to $\mathfrak{t}^rM/\mathfrak{t}^{r+1}M$. By considering the exact sequence

$$0 \longrightarrow \mathfrak{t}^{r+1} \longrightarrow \mathfrak{t}^r \longrightarrow \mathfrak{t}^r/\mathfrak{t}^{r+1} \longrightarrow 0$$

and induction on r , we then conclude that the map $m_r : \mathfrak{t}^r \otimes_{A_0} M \rightarrow \mathfrak{t}^rM$, induced by the operations of A_0 on M , is an isomorphism. Now consider the following commutative diagram:

$$\begin{array}{ccc} (A_0[X_1, \dots, X_n]/\mathfrak{d}_0) \otimes_{A_0} M & \xrightarrow{\bar{\rho} \otimes 1_M} & \bigoplus_r (\mathfrak{t}^r \otimes_{A_0} M) \\ \downarrow \eta & & \downarrow \oplus m_r \\ (A[X_1, \dots, X_n]/\mathfrak{d}) \otimes_A M & \xrightarrow{\alpha_M^r} & \bigoplus_r \mathfrak{t}^r M \end{array}$$

where η is the canonical isomorphism of extension of scalars:

$$(A_0[X_1, \dots, X_n]/\mathfrak{d}_0) \otimes_{A_0} M \cong (A_0[X_1, \dots, X_n]/\mathfrak{d}_0) \otimes_{A_0} A \otimes_A M \cong (A[X_1, \dots, X_n]/\mathfrak{d}) \otimes_A M.$$

By ??, we then see condition (iv) follows.

Finally, consider the following exact sequence induced from (??):

$$\begin{array}{ccccccc} \mathrm{Tor}_1^{A_0}(A_0/\mathfrak{t}^{r+1}, M) & \xrightarrow{\mathrm{Tor}(\iota_r, 1_M)} & \mathrm{Tor}_1^{A_0}(A_0/\mathfrak{t}^r, M) & \xrightarrow{\mathrm{Tor}(\pi_r, 1_M)} & (\mathfrak{t}^r/\mathfrak{t}^{r+1}) \otimes_{A_0} M & \longrightarrow & \\ & & & & \downarrow \iota_r \otimes 1_M & & \\ & & & & (A_0/\mathfrak{t}^r) \otimes_{A_0} M & \xrightarrow{\pi_r \otimes 1_M} & (A_0/\mathfrak{t}^{r+1}) \otimes_{A_0} M \longrightarrow 0 \end{array} \quad (8.5.17)$$

The kernel of $\pi_r \otimes 1_M$ is identified with $\mathfrak{t}^rM/\mathfrak{t}^{r+1}M$ and the A_0 -module $\mathfrak{t}^r/\mathfrak{t}^{r+1}$ is identified with the homogeneous component of degree r of A_0 . Therefore, the homomorphism $(\mathfrak{t}^r/\mathfrak{t}^{r+1}) \otimes_{A_0} M \rightarrow \mathfrak{t}^rM/\mathfrak{t}^{r+1}M$ induced by $\iota_r \otimes 1_M$ is identified with the homogeneous component of degree r of β_M^r and we get the following exact sequence from (??):

$$\mathrm{Tor}_1^{A_0}(A_0/\mathfrak{t}^{r+1}, M) \xrightarrow{\mathrm{Tor}_1^{A_0}(\iota_r, 1_M)} \mathrm{Tor}_1^{A_0}(A_0/\mathfrak{t}^r, M) \xrightarrow{\mathrm{Tor}_1^{A_0}(\pi_r, 1_M)} (\mathfrak{t}^r/\mathfrak{t}^{r+1}) \otimes_{A_0} M \xrightarrow{(\beta_M^r)_r} \mathfrak{t}^rM/\mathfrak{t}^{r+1}M$$

The condition (v) is then equivalent to the following condition:

(v') the homomorphism $\mathrm{Tor}_1^{A_0}(\pi_r, 1_M) : \mathrm{Tor}_1^{A_0}(A_0/\mathfrak{t}^{r+1}, M) \rightarrow \mathrm{Tor}_1^{A_0}(A_0/\mathfrak{t}^r, M)$ is surjective for each $r \geq 1$.

We are now left to prove that (v) \Rightarrow (i) if the modules M_i are separated for the \mathfrak{x} -adic topology. Let $\bar{M} = M/x_1M$. By definition, the sequence \mathbf{x} is regular for M if and only if $(x_1)_M$ is injective and the sequence $\tilde{\mathbf{x}} = (x_2, \dots, x_n)$ is regular for \bar{M} . Reasoning by induction on n , it suffices to prove that, if M is separated for the \mathfrak{x} -adic topology and if β_M^r is bijective, then $(x_1)_M$

is injective and $\beta_M^{\tilde{x}}$ is bijective. Now the bijectivity of β_M^x implies in particular that the homothety with ratio x_1 is injective on $\text{gr}_{\mathfrak{x}}(M)$ (since it is mapped by β_M^x to the homothety of X_1 on $(A/\mathfrak{x})[X_1, \dots, X_n] \otimes_A M$), hence $(x_1)_M$ is injective since the \mathfrak{x} -topology on M is separated (??). It therefore suffices to prove that if $(x_1)_M$ is injective and M verifies the condition (v'), then \bar{M} verifies the condition (v') for \tilde{x} . By hypothesis we have an exact sequence

$$0 \longrightarrow M \xrightarrow{(x_1)_M} M \longrightarrow \bar{M} \longrightarrow 0$$

Put $\bar{A}_0 = A_0/T_1 A_0$, $\bar{\mathfrak{t}} = \mathfrak{t}\bar{A}_0$. Let $E \rightarrow M$ be a free resolution of the A_0 -module M ; since the homothety with ratio T_1 is injective on A_0 , it is injective on E , and we have an exact sequence of complexes:

$$0 \longrightarrow E \xrightarrow{(T_1)_E} E \longrightarrow \bar{E} \longrightarrow 0$$

where $\bar{E} = E/T_1 E$, and a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & E & \xrightarrow{(T_1)_E} & E & \longrightarrow & \bar{E} & \longrightarrow 0 \\ & & \downarrow \gamma & & \downarrow \gamma & & \downarrow \bar{\gamma} & \\ 0 & \longrightarrow & M & \xrightarrow{(x_1)_M} & M & \longrightarrow & \bar{M} & \longrightarrow 0 \end{array}$$

Since γ is an quasi-isomorphism, we conclude from the induced long exact sequence that $\bar{\gamma} : \bar{E} \rightarrow \bar{M}$ is a free resolution for the \bar{A}_0 -module \bar{M} . For any \bar{A}_0 -module P , we have a canonical isomorphism $P \otimes_{A_0} E \cong P \otimes_{\bar{A}_0} \bar{E}$, whence by passing to homology an isomorphism

$$\varphi_P : \text{Tor}_1^{A_0}(P, M) \rightarrow \text{Tor}_1^{\bar{A}_0}(P, \bar{M}).$$

which is functorial in the sense that if $f : P \rightarrow Q$ be an \bar{A}_0 -homomorphism, then

$$\varphi_Q \circ \text{Tor}_1^{A_0}(f, 1_M) = \text{Tor}_1^{\bar{A}_0}(f, 1_M) \circ \varphi_P.$$

This being so, suppose the condition (v') holds for M . Let $r \geq 1$ be an integer; we have a commutative diagram with exact rows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & A_0/\mathfrak{t}^r & \xrightarrow{T_1} & A_0/\mathfrak{t}^{r+1} & \longrightarrow & \bar{A}_0/\bar{\mathfrak{t}}^{r+1} & \longrightarrow 0 \\ & & \downarrow \pi_{r-1} & & \downarrow \pi_r & & \downarrow \bar{\pi}_r & \\ 0 & \longrightarrow & A_0/\mathfrak{t}^r & \xrightarrow{T_1} & A_0/\mathfrak{t}^{r+1} & \longrightarrow & \bar{A}_0/\bar{\mathfrak{t}}^{r+1} & \longrightarrow 0 \end{array}$$

which induces the following commutative diagram

$$\begin{array}{ccccccc} \text{Tor}_1^{A_0}(A_0/\mathfrak{t}^{r+1}, M) & \longrightarrow & \text{Tor}_1^{A_0}(\bar{A}_0/\bar{\mathfrak{t}}^{r+1}, M) & \longrightarrow & M/\mathfrak{x}^r M & \xrightarrow{x_1} & M/\mathfrak{x}^{r+1} M \\ \downarrow \text{Tor}_1(\pi_r, 1) & & \downarrow \text{Tor}_1(\bar{\pi}_r, 1) & & \downarrow & & \downarrow \\ \text{Tor}_1^{A_0}(A_0/\mathfrak{t}^r, M) & \longrightarrow & \text{Tor}_1^{A_0}(\bar{A}_0/\bar{\mathfrak{t}}^r, M) & \longrightarrow & M/\mathfrak{x}^{r-1} M & \xrightarrow{x_1} & M/\mathfrak{x}^r M \end{array} \quad (8.5.18)$$

with exact rows. Now the multiplication by x_1 defines an injection from $M/\mathfrak{x}^r M$ to $M/\mathfrak{x}^{r+1} M$: this follows immediately, by induction on r , from the exact sequence

$$0 \longrightarrow \mathfrak{x}^{r-1}M/\mathfrak{x}^r M \longrightarrow M/\mathfrak{x}^r M \longrightarrow M/\mathfrak{x}^{r-1}M \longrightarrow 0$$

and the injectivity of the homothety with ratio x_1 on $\text{gr}_{\mathfrak{x}}(M)$. Therefore, the first horizontal maps in both rows of (??) are surjective, and condition (v') then implies that the homomorphism $\text{Tor}_1^{\bar{A}_0}(\bar{\pi}_r, 1_M)$ is surjective for $r \geq 1$. By composing with the isomorphisms $(\varphi_{\bar{A}_0/\bar{\mathfrak{x}}^{r+1}})^{-1}$ and $\varphi_{\bar{A}_0/\bar{\mathfrak{x}}^r}$, we then deduce that the homomorphism

$$\text{Tor}_1^{\bar{A}_0}(\bar{\pi}_r, 1_M) : \text{Tor}_1^{\bar{A}_0}(\bar{A}_0/\bar{\mathfrak{x}}^{r+1}, M) \rightarrow \text{Tor}_1^{\bar{A}_0}(\bar{A}_0/\bar{\mathfrak{x}}^r, M)$$

is surjective for $r \geq 1$, whence the condition (v') for \bar{M} . This completes the proof. \square

Corollary 8.5.22. *If A is a Zariski ring and \mathfrak{x} is a defining ideal of A . Then for any finitely generated A -module M , the conditions in ?? are equivalent.*

Proof. By ?? the modules M_i are separated for the \mathfrak{x} -adic topology. \square

Corollary 8.5.23. *Suppose that A is a graded ring of positive degrees, M is a graded A -module that is bounded below, and the x_i are homogeneous elements of positive degrees. Then the conditions in ?? are equivalent.*

Proof. The \mathfrak{x} -adic topology is in fact separated for any graded A -module N bounded below, since the condition $N_n = 0$ for $n < n_0$ implies that $\mathfrak{x}^\nu N \subseteq \sum_{i \geq n_0 + \nu} N_i$ for any $\nu \geq 0$. \square

Corollary 8.5.24. *Suppose that the modules M_i are separated for the \mathfrak{x} -adic topology and let $1 \leq p \leq n$ be an integer. For the sequence \mathbf{x} to be completely secant for M , it is necessary and sufficient that the sequence (x_1, \dots, x_p) is completely secant for M and the sequence (x_{p+1}, \dots, x_n) is completely secant for $M/(x_1 M + \dots + x_p M)$.*

Remark 8.5.3. Let $\mathbf{x} = (x_1, \dots, x_n)$ be a sequence of elements of A such that $H_1(\mathbf{x}, A) = 0$. Then from the construction of $K_\bullet(\alpha)$, the kernel of the surjective homomorphism $\alpha : A^n \rightarrow \mathfrak{x}$ defined by $\alpha(\sum_i a_i e_i) = \sum_i a_i x_i$ is the image of $\wedge^2(A^n)$ under d_α , which by ?? is generated by the elements $x_j e_i - x_i e_j$. Therefore, the A -algebra $A[X_1, \dots, X_n]/\mathfrak{d}$ is isomorphic to the symmetric algebra $S_A(\mathfrak{x})$ (A, III, p.69, prop.4 ??). It then follows from ?? that the algebra homomorphism $S_A(\mathfrak{x}) \rightarrow \bigoplus_n \mathfrak{x}^n$ induced from the canonical injection of \mathfrak{x} to $\bigoplus_n \mathfrak{x}^n$ is an isomorphism. The same holds for the homomorphism $S_A(\mathfrak{x}/\mathfrak{x}^2) \rightarrow \text{gr}_{\mathfrak{x}}(A)$.

8.5.5 Extension class associated with a regular sequence

Let A be a ring, M an A -module and $\mathbf{x} = (x_1, \dots, x_n)$ a sequence of elements of A . Let $M_i = M/(x_1 M + \dots + x_{i-1} M)$ for each $i = 0, 1, \dots, n+1$ (we set $M_0 = M$), and let

$$\bar{x}_i : M_{i-1} \rightarrow M_i, \quad i = 1, \dots, n$$

be the A -homomorphism composing the homothety with ratio x_i on M_{i-1} with the canonical projection from M_{i-1} to M_i . Finally, let $\pi : M_n \rightarrow M/\mathfrak{x} M$ be the canonical projection. Suppose

that x is M -regular, then the sequence

$$0 \longrightarrow M \xrightarrow{\bar{x}_1} M_1 \xrightarrow{\bar{x}_2} \cdots \xrightarrow{\bar{x}_n} M_n \xrightarrow{\pi} M/\mathfrak{x}M \longrightarrow 0 \quad (8.5.19)$$

is exact. The element $\theta_x \in \text{Ext}_A^n(M/\mathfrak{x}M, M)$ associated with this sequence is then called the **extension class associated with the M -regular sequence x** .

For each integer $1 \leq i \leq n$, the sequence (??) can be decomposed into the following two exact sequences:

$$\begin{aligned} 0 &\longrightarrow M \xrightarrow{\bar{x}_1} M_1 \xrightarrow{\bar{x}_2} \cdots \xrightarrow{\bar{x}_i} M_i \longrightarrow M_{i+1} \longrightarrow 0 \\ 0 &\longrightarrow M_{i+1} \xrightarrow{\bar{x}_{i+1}} M_{i+1} \xrightarrow{\bar{x}_{i+2}} \cdots \xrightarrow{\bar{x}_n} M_n \xrightarrow{\pi} M/\mathfrak{x}M \longrightarrow 0 \end{aligned}$$

which are none other than the exact sequences associated with the sequence (x_1, \dots, x_i) which is regular for M and that with (x_{i+1}, \dots, x_n) which is regular for M_{i+1} . Note that the extension classes associated with these sequences are

$$\theta_{(x_1, \dots, x_i)} \in \text{Ext}_A^i(M_{i+1}, M), \quad \theta_{(x_{i+1}, \dots, x_n)} \in \text{Ext}_A^{n-i}(M/\mathfrak{x}M, M_{i+1})$$

and we have by ?? that

$$\theta_{(x_1, \dots, x_n)} = \theta_{(x_1, \dots, x_i)} \circ \theta_{(x_{i+1}, \dots, x_n)}.$$

Moreover, by ?? the Koszul complex $K^\bullet(x, M)$ is acyclic at degrees $\neq n$, so we have an exact sequence

$$0 \longrightarrow M \xrightarrow{\partial^0} K^1(x, M) \xrightarrow{\partial^1} \cdots \longrightarrow K^n(x, M) \xrightarrow{\varepsilon} M/\mathfrak{x}M \longrightarrow 0 \quad (8.5.20)$$

where we identify $K^0(x, M)$ with M and the map ε sends each element $m \in K^n(x, M)$ to the class in $M/\mathfrak{x}M$ of $m(1, 2, \dots, n) \in M$.

Proposition 8.5.25. *Let x be a M -regular sequence. Then the element of $\text{Ext}_A^n(M/\mathfrak{x}M, M)$ associated with the sequence (??) is $(-1)^{n(n+1)/2}\theta_x$.*

Proof. Recall that $K^i(x, M)$ can be viewed as the set $C^i(I, M)$ where $I = \{1, \dots, n\}$ and the differential ∂^i is given by (??). For each $i = 0, \dots, n$, define an A -linear map

$$\alpha^i : K^i(x, M) \rightarrow M_i, \quad m \mapsto m(1, 2, \dots, i).$$

It is then clear that α^0 is the identity map on M and $\pi \circ \alpha^n = \varepsilon$. Moreover, $\alpha^{i+1} \circ \partial^i(m)$ is the image in M_{i+1} of the element

$$\sum_{j=1}^{i+1} (-1)^{j+1} x_j m(1, 2, \dots, \hat{j}, \dots, i+1).$$

Since x_j annihilates M_{i+1} for $j = 1, \dots, i$, the element $\alpha^{i+1} \circ \partial^i(m)$ then equals to the image of $(-1)^i x_{i+1} m(1, 2, \dots, i)$ in M_{i+1} and we therefore have

$$\alpha^{i+1} \circ \partial^i = (-1)^i \bar{x}_{i+1} \circ \alpha^i.$$

By ?? and ??, the element in $\text{Ext}_A^n(M/\mathfrak{x}M, M)$ is equal to $\prod_{i=1}^n (-1)^i \theta_x$, whence our claim. \square

Corollary 8.5.26. suppose further that the modules M_i are separated for the \mathfrak{x} -adic topology, and let $(a_{ij}) \in \text{GL}_n(A)$. Put

$$y_i = \sum_j a_{ij} x_j$$

and $\mathbf{y} = (y_1, \dots, y_n)$. Then the sequence \mathbf{y} is regular for M and we have $\theta_{\mathbf{y}} = \det(a_{ij})^{-1} \theta_x$.

Proof. In fact, the sequence \mathbf{y} is regular for M by ?? and ??; the second assertion then follows from ?? and ??.

Proposition 8.5.27. Let $\mathbf{x} = (x_1, \dots, x_n)$ be a regular sequence for M and N is an A -module annihilated by \mathfrak{x} . Then $\text{Ext}_A^i(N, M) = 0$ for $i < n$ and the map $\alpha \mapsto \theta_{\mathbf{x}} \circ \alpha$ from $\text{Hom}_A(N, M/\mathfrak{x}M)$ to $\text{Ext}_A^n(N, M)$ is bijective.

Proof. We prove by induction that the homomorphism

$$\psi^i : \text{Ext}_A^{i-n}(N, M/\mathfrak{x}M) \rightarrow \text{Ext}_A^i(N, M), \quad \alpha \mapsto \theta_{\mathbf{x}} \circ \alpha$$

is bijective for $i \leq n$. The assertion is trivial for $n = 0$, so suppose that $n \geq 1$ and put $M_1 = M/x_1 M$, $\tilde{\mathbf{x}} = (x_2, \dots, x_n)$, so that $\tilde{\mathbf{x}}$ is M_1 -regular. By the induction hypothesis, the homomorphism

$$\bar{\psi}^{i-1} : \text{Ext}_A^{i-n}(N, M/\mathfrak{x}M) \rightarrow \text{Ext}_A^{i-1}(N, M_1), \quad \alpha \mapsto \theta_{\tilde{\mathbf{x}}} \circ \alpha$$

is bijective for $i \leq n$. Now consider the exact sequence

$$0 \longrightarrow M \xrightarrow{(x_1)_M} M \longrightarrow M_1 \longrightarrow 0$$

where the associated connecting homomorphism is

$$\varphi^i : \text{Ext}_A^{i-1}(N, M_1) \rightarrow \text{Ext}_A^i(N, A), \quad \beta \mapsto \theta_{x_1} \circ \beta.$$

Since $\text{Ext}^i(1_N, (x_1)_M) = \text{Ext}^i((x_1)_N, 1_M) = 0$, we deduce the exact sequences

$$0 \longrightarrow \text{Ext}_A^{i-1}(N, M) \longrightarrow \text{Ext}_A^{i-1}(N, M_1) \xrightarrow{\varphi^i} \text{Ext}_A^i(N, M) \longrightarrow 0$$

Now by induction hypothesis, $\text{Ext}_A^{i-1}(N, M_1) = 0$ for $i < n$, so $\text{Ext}_A^i(N, M) = 0$ for $i < n$, which implies φ^i is bijective for $i \leq n$. Since for $\alpha \in \text{Ext}_A^{i-n}(N, M/\mathfrak{x}M)$ we have

$$\psi^i(\alpha) = \theta_{\mathbf{x}} \circ \alpha = \theta_{x_1} \circ \theta_{\tilde{\mathbf{x}}} \circ \alpha = \varphi^i \circ \bar{\psi}^{i-1}(\alpha) \tag{8.5.21}$$

the map ψ^i is therefore bijective for $i \leq n$. \square

Chapter 9

Localizations

9.1 Prime ideals and maximal ideals

9.1.1 Prime ideals

Definition 9.1.1. An ideal \mathfrak{p} of a ring A is called **prime** if the ring A/\mathfrak{p} is an integral domain.

A maximal ideal \mathfrak{m} of A is prime since A/\mathfrak{m} is a field; then it follows from Krull's theorem that every proper ideal of A is contained in at least one prime ideal. In particular, for prime ideals to exist in a ring A , it is necessary and sufficient that A be not reduced to 0.

Let $\rho : A \rightarrow B$ be a ring homomorphism and \mathfrak{q} an ideal of B . Set $\mathfrak{p} = \rho^{-1}(\mathfrak{q})$. Then the homomorphism $\bar{\rho} : A/\mathfrak{p} \rightarrow B/\mathfrak{q}$ derived from ρ by taking quotients is injective. Suppose that \mathfrak{q} is prime; as the ring B/\mathfrak{q} is an integral domain, so is A/\mathfrak{p} , being isomorphic to a subring of B/\mathfrak{q} . Consequently the ideal $\mathfrak{p} = \rho^{-1}(\mathfrak{q})$ is prime. In particular, let A be a subring of B . For every ideal \mathfrak{q} of B , $\mathfrak{q} \cap A$ is a prime ideal of A .

If ρ is surjective, then $\bar{\rho}$ is an isomorphism, so the conditions " \mathfrak{p} is prime" and " \mathfrak{q} is prime" are then equivalent. Hence, if \mathfrak{p} and \mathfrak{a} are ideals of A such that $\mathfrak{a} \subseteq \mathfrak{p}$, a necessary and sufficient condition for \mathfrak{p} to be prime is that $\mathfrak{p}/\mathfrak{a}$ be prime in A/\mathfrak{a} .

Proposition 9.1.2. Let A be a ring, $\mathfrak{a}_1, \dots, \mathfrak{a}_n$ ideals of A and \mathfrak{p} a prime ideal of A . If \mathfrak{p} contains the product $\prod_{i=1}^n \mathfrak{a}_i$, it contains at least one of the \mathfrak{a}_i .

Proof. Suppose in fact that \mathfrak{p} contains none of the \mathfrak{a}_i . For $1 \leq i \leq n$ there exists then an element $s_i \in \mathfrak{a}_i - \mathfrak{p}$. Then $s = \prod_{i=1}^n s_i$ is contained in $\mathfrak{a}_1 \cdots \mathfrak{a}_n$ and is not contained in \mathfrak{p} , which is absurd. \square

Corollary 9.1.3. Let \mathfrak{m} be a maximal ideal of A . Then for every integer $n > 0$, the only prime ideal containing \mathfrak{m}^n is \mathfrak{m} .

Proposition 9.1.4. Let A be a ring, \mathfrak{a} a non-empty set of A which is closed under addition and multiplication and $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ ideals of A . Suppose that \mathfrak{a} is contained in the union of the \mathfrak{p}_i and that at most two of the \mathfrak{p}_i are not prime. Then \mathfrak{a} is contained in one of the \mathfrak{p}_i .

Proof. We argue by induction on n . The proposition is trivial if $n = 1$. Suppose that $n \geq 2$; if there exists an index j such that $\mathfrak{a} \cap \mathfrak{p}_j \subseteq \bigcup_{i \neq j} \mathfrak{p}_i$, then the set \mathfrak{a} , which is the union of the $\mathfrak{a} \cap \mathfrak{p}_i$ where $1 \leq i \leq n$, is contained in $\bigcup_{i \neq j} \mathfrak{p}_i$, and hence in one of the \mathfrak{p}_i , by the induction hypothesis. Suppose then that such an index does not exist.

For every $1 \leq j \leq n$ let y_j be an element of $\mathfrak{a} \cap \mathfrak{p}_j$ not belonging to any \mathfrak{p}_i for $i \neq j$. Let k be chosen such that \mathfrak{p}_k is prime if $n > 2$ and chosen arbitrarily if $n = 2$. Let $z = y_k + \prod_{j \neq k} y_j \in \mathfrak{a}$. If $i \neq k$ then $\prod_{j \neq k} y_j$ belongs to \mathfrak{p}_i , but $y_k \notin \mathfrak{p}_i$, whence $z \notin \mathfrak{p}_i$. On the other hand, $\prod_{j \neq k} y_j$ does not belong to \mathfrak{p}_k , as none of the factors y_j belongs to it and \mathfrak{p}_k is prime if $n > 2$; as $y_k \in \mathfrak{p}_k$, z does not belong to \mathfrak{p}_k , and the proposition is established. \square

Proposition 9.1.5. *Let A be a ring. Let $\mathfrak{a}_1, \dots, \mathfrak{a}_n$ be ideals and let \mathfrak{p} be a prime ideal containing $\bigcap_{i=1}^n \mathfrak{a}_i$. Then $\mathfrak{p} \supseteq \mathfrak{a}_i$ for some i . If $\mathfrak{p} = \bigcap_{i=1}^n \mathfrak{a}_i$, then $\mathfrak{p} = \mathfrak{a}_i$ for some i .*

Proof. Suppose that $\mathfrak{p} \not\supseteq \mathfrak{a}_i$ for all i . Then for each $1 \leq i \leq n$ there exist $x_i \in \mathfrak{a}_i$ such that $x_i \notin \mathfrak{p}$, and therefore $\prod x_i \in \prod \mathfrak{a}_i \subseteq \bigcap_i \mathfrak{a}_i$; but $\prod x_i \notin \mathfrak{p}$ (since \mathfrak{p} is prime). Hence $\mathfrak{p} \not\supseteq \bigcap_i \mathfrak{a}_i$, contradiction. Finally, if $\mathfrak{p} = \bigcap_{i=1}^n \mathfrak{a}_i$, then $\mathfrak{p} \subseteq \mathfrak{a}_i$ for all i and hence $\mathfrak{p} = \mathfrak{a}_i$ for some i . \square

9.1.2 Nilradical and Jacobson radical

An element $x \in A$ is **nilpotent** if $x^n = 0$ for some $n > 0$.

Proposition 9.1.6. *The set \mathfrak{n} of all nilpotent elements in a ring A is an ideal, and A/\mathfrak{n} has no nilpotent element except 0. This is called the **nilradical** of A .*

Proof. If $x \in \mathfrak{n}$, clearly $ax \in \mathfrak{n}$ for all $a \in A$. Let $x, y \in \mathfrak{n}$: say $x^m = 0, y^n = 0$. By the binomial theorem (which is valid in any commutative ring), $(x+y)^{m+n-1}$ is a sum of integer multiples of products $x^r y^s$. Since $r+s = m+n-1$, we can not have $r < m, s < n$ simultaneously. Hence $(x+y)^{m+n-1} = 0, x+y \in \mathfrak{n}$. \square

The following proposition gives an alternative definition of the nilradical.

Proposition 9.1.7. *The nilradical of A is the intersection of all the prime ideals of A .*

Proof. We divide the proof into two parts. We first prove that \mathfrak{n} is contained in every prime ideal of A . Suppose a is an element of the nilradical of A , so $a^n = 0$. And let \mathfrak{p} be a prime ideal of A . If a is not contained in \mathfrak{p} , then $a + \mathfrak{p}$ is nonzero in A/\mathfrak{p} . But

$$(a + \mathfrak{p})^n = a^n + \mathfrak{p} = 0 + \mathfrak{p}$$

since \mathfrak{p} is prime, A/\mathfrak{p} is an integral domain, which is a contradiction.

Now consider the family \mathcal{F} of ideals of A that intersect the set $\{r^n\}_{n \in \mathbb{N}}$ only at 0. Clearly \mathcal{F} is nonempty and ordered by inclusion. So by Zorn's lemma there is a maximal element \mathfrak{p} in \mathcal{F} .

Now we prove that \mathfrak{p} is prime: Suppose $a, b \notin \mathfrak{p}$ but $ab \in \mathfrak{p}$. Then the ideals $\mathfrak{p} + (a)$ and $\mathfrak{p} + (b)$ properly contain \mathfrak{p} . Since \mathfrak{p} is maximal, these two ideals are not in \mathcal{F} . So there are $m, n > 0$ such that

$$r^m \in \mathfrak{p} + (a), \quad r^n \in \mathfrak{p} + (b)$$

Now we find $r^{m+n} \in \mathfrak{p} + (ab) = \mathfrak{p}$, which is a contradiction. This shows at least one of a, b belongs to \mathfrak{p} , hence \mathfrak{p} is prime.

Since we find a prime ideal \mathfrak{p} such that $r \notin \mathfrak{p}$, we conclude that r is not in the intersection of all prime ideals. The contrapositive gives another inclusion, which finishes the proof. \square

Proposition 9.1.8. *Let A be a commutative ring, and let \mathfrak{n} be its nilradical. Then A/\mathfrak{n} contains no nonzero nilpotent elements. (Such a ring is said to be **reduced**.)*

Definition 9.1.9. *Define the **Jacobson radical** \mathfrak{r} of A is defined to be the intersection of all the maximal ideals of A .*

Proposition 9.1.10. *An element r of A is in the Jacobson radical \mathfrak{r} if and only if $1 + rs$ is invertible for every $s \in A$.*

Proof. Consider the ideal $\mathfrak{a} := (1 + rs)$, if $1 + rs$ is not invertible, this is a proper ideal, hence is contained in some maximal ideal \mathfrak{m} . Since r is in the Jacobson radical, it is contained in \mathfrak{m} , and then we find $1 \in \mathfrak{m}$. But \mathfrak{m} is proper, this is not possible.

Assume $1 + rs$ is invertible for every $s \in A$. Consider a maximal ideal \mathfrak{m} : If $r \notin \mathfrak{m}$, then $\mathfrak{m} + rA$ is an ideal properly contains \mathfrak{m} , then $A = \mathfrak{m} + rA$ since \mathfrak{m} is maximal. In particular, $1 = m + rs$ for some $m \in \mathfrak{m}$ and $s \in A$. Then $m = 1 - rs$ is invertible by assumption, which contradicts that \mathfrak{m} is a maximal ideal. \square

9.1.3 Coprime ideals

Two ideals $\mathfrak{a}, \mathfrak{b}$ are said to be **coprime** (or **comaximal**) if $\mathfrak{a} + \mathfrak{b} = (1)$. Clearly two ideals $\mathfrak{a}, \mathfrak{b}$ are coprime if and only if there exist $x \in \mathfrak{a}$ and $y \in \mathfrak{b}$ such that $x + y = 1$.

Proposition 9.1.11. *Let \mathfrak{a} and \mathfrak{b} be two relatively prime ideals of a ring A . Let \mathfrak{a}' and \mathfrak{b}' be two ideals of A such that every element of \mathfrak{a} (resp. \mathfrak{b}) has a power in \mathfrak{a}' (resp. \mathfrak{b}'). Then \mathfrak{a}' and \mathfrak{b}' are relatively prime.*

Proof. Under the given hypothesis, every prime ideal which contains \mathfrak{a}' contains \mathfrak{a} and every prime ideal which contains \mathfrak{b}' contains \mathfrak{b} . If a prime ideal contains \mathfrak{a}' and \mathfrak{b}' , then it contains \mathfrak{a} and \mathfrak{b} , which is absurd, since \mathfrak{a} and \mathfrak{b} are relatively prime; hence \mathfrak{a}' and \mathfrak{b}' are relatively prime. \square

Proposition 9.1.12. *Let $\mathfrak{a}, \mathfrak{b}_1, \dots, \mathfrak{b}_n$ be ideals of a ring A . If \mathfrak{a} is relatively prime to each of the \mathfrak{b}_i , it is relatively prime to $\prod_{i=1}^n \mathfrak{b}_i$.*

Proof. Let \mathfrak{p} be a prime ideal of A . If \mathfrak{p} contains \mathfrak{a} and $\prod_{i=1}^n \mathfrak{b}_i$ then it contains one of the \mathfrak{b}_i , which is absurd since \mathfrak{a} and \mathfrak{b}_i are relatively prime. \square

Proposition 9.1.13. *Let A be a ring and $\mathfrak{a}_1, \dots, \mathfrak{a}_n$ ideals of A . Define a homomorphism*

$$\phi : A \rightarrow \prod_{i=1}^n (A/\mathfrak{a}_i), \quad \phi(x) = (x + \mathfrak{a}_1, \dots, x + \mathfrak{a}_n).$$

(a) *If $\mathfrak{a}_i, \mathfrak{a}_j$ are coprime whenever $i \neq j$, then $\prod_i \mathfrak{a}_i = \bigcap_i \mathfrak{a}_i$.*

(b) *ϕ is surjective if and only if $\mathfrak{a}_i, \mathfrak{a}_j$ are coprime whenever $i \neq j$.*

(c) *ϕ is injective if and only if $\bigcap_i \mathfrak{a}_i = (0)$.*

Proof. First we prove (a). The case $n = 2$ is dealt with above. Suppose $n > 2$ and the result true for $\mathfrak{a}_1, \dots, \mathfrak{a}_{n-1}$, and let $\mathfrak{b} = \mathfrak{a}_1 \cap \dots \cap \mathfrak{a}_{n-1}$. Since $\mathfrak{a}_i + \mathfrak{a}_j = (1)$ we have equations $x_i + y_i = 1$ ($x_i \in \mathfrak{a}_i, y_i \in \mathfrak{a}_n$) and therefore

$$\prod_{i=1}^{n-1} x_i = \prod_{i=1}^{n-1} (1 - y_i) \equiv 1 \pmod{\mathfrak{a}_n}.$$

This means $\mathfrak{a}_n + \mathfrak{b} = (1)$. And so

$$\prod_{i=1}^n \mathfrak{a}_i = \mathfrak{b}\mathfrak{a}_n = \mathfrak{b} \cap \mathfrak{a}_n = \bigcap_{i=1}^n \mathfrak{a}_i.$$

Now we turn to part (b). First assume that ϕ is surjective. Let us show for example that $\mathfrak{a}_1, \mathfrak{a}_2$ are coprime. There exists $x \in A$ such that $\phi(x) = (1, 0, \dots, 0)$; hence

$$x \equiv 1 \pmod{\mathfrak{a}_1} \quad \text{and} \quad x \equiv 0 \pmod{\mathfrak{a}_2},$$

so that $1 = (1 - x) + x \in \mathfrak{a}_1 + \mathfrak{a}_2$. The other case is done similarly.

Now suppose that \mathfrak{a}_i and \mathfrak{a}_j are coprime. It is enough to show, for example, that there is an element $x \in A$ such that $\phi(x) = (1, 0, \dots, 0)$. Since $\mathfrak{a}_1 + \mathfrak{b}_i = (1)$ for $i > 1$, we have equations

$$u_i + v_i = 1, \quad u_i \in \mathfrak{a}_1, v_i \in \mathfrak{a}_i, i = 2, \dots, n.$$

Take $x = \prod_{i=2}^n v_i$. then

$$x = \prod_{i=2}^n v_i = \prod_{i=2}^n (1 - u_i) = 1 \pmod{\mathfrak{a}_1}.$$

Hence $\phi(x) = (1, 0, \dots, 0)$. The other case is done similarly. \square

Corollary 9.1.14 (Chinese remainder theorem). *Let $\mathfrak{a}_1, \dots, \mathfrak{a}_n$ be ideals of A such that $\mathfrak{a}_i + \mathfrak{b}_j = (1)$ for all $i \neq j$. Then the natural homomorphism*

$$\phi : A \rightarrow \prod_{i=1}^n (A/\mathfrak{a}_i)$$

is surjective and induces an isomorphism $\tilde{\phi} : A / \prod_{i=1}^n \mathfrak{a}_i \rightarrow \prod_{i=1}^n (A/\mathfrak{a}_i)$.

9.1.4 Local rings and semilocal rings

A ring A with exactly one maximal ideal \mathfrak{m} is called a **local ring**. The field $k = A/\mathfrak{m}$ is called the residue field of A .

Proposition 9.1.15. *Let $A \neq 0$ be a ring. The following are equivalent:*

- (a) *A has a proper ideal \mathfrak{m} and every element of $A - \mathfrak{m}$ is a unit.*
- (b) *A is not the zero ring and for every $x \in A$ either x or $1 - x$ is invertible or both.*

Proof. Let $\mathfrak{m} \neq (1)$ be an ideal. If every element of $A - \mathfrak{m}$ is a unit then \mathfrak{m} is clearly maximal and every ideal of A is contained in \mathfrak{m} . Thus A is local. The converse is clear.

If A is local and x is not a unit. Then $x \in \mathfrak{m}$, and $1 - x$ must be a unit since otherwise $1 = 1 - x + x \in \mathfrak{m}$. Conversely, the condition in (b) holds, and \mathfrak{m}_1 and \mathfrak{m}_2 are distinct maximal ideal of A . Then by the Chinese remainder theorem we can choose $x \in A$ such that $x \equiv 0 \pmod{\mathfrak{m}_1}$ and $x \equiv 1 \pmod{\mathfrak{m}_2}$. Then x is not invertible and neither is $1 - x$ which is a contradiction. \square

Lemma 9.1.16. *Let $\varphi : A \rightarrow B$ be a ring map. Assume A and B are local rings. The following are equivalent:*

- (a) $\varphi(\mathfrak{m}_A) \subseteq \mathfrak{m}_B$.
- (b) $\varphi^{-1}(\mathfrak{m}_B) = \mathfrak{m}_A$.
- (c) For any $x \in A$, if $\varphi(x)$ is invertible in B , then x is invertible in A .

If φ satisfies these conditions, we say φ is a **local homomorphism**.

Proof. For (a) \Rightarrow (b), the condition $\varphi(\mathfrak{m}_A) \subseteq \mathfrak{m}_B$ clearly implies $\mathfrak{m}_A \subseteq \varphi^{-1}(\varphi(\mathfrak{m}_A)) \subseteq \varphi^{-1}(\mathfrak{m}_B)$, hence $\mathfrak{m}_A = \varphi^{-1}(\mathfrak{m}_B)$.

For (b) \Rightarrow (c), assume that $\varphi^{-1}(\mathfrak{m}_B) = \mathfrak{m}_A$. If $\varphi(x)$ is invertible, then $\varphi(x) \notin \mathfrak{m}_B$, so $x \notin \mathfrak{m}_A$. This implies that x is invertible. The implication (c) \Rightarrow (a) is clear. \square

Now we consider semilocal rings. They are characterized by the following proposition.

Proposition 9.1.17. *Let A be a ring. The following properties are equivalent:*

- (a) The set of maximal ideals of A is finite.
- (b) The quotient of A by its Jacobson radical is the direct product of a finite number of fields.

Proof. Suppose that the quotient of A by its Jacobson radical $\mathfrak{r}(A)$ is a direct product of a finite number of fields. Then $A/\mathfrak{r}(A)$ possesses only a finite number of ideals and a fortiori only a finite number of maximal ideals. As every maximal ideal contains $\mathfrak{r}(A)$, the maximal ideals of A are the inverse images of the maximal ideals of $A/\mathfrak{r}(A)$ under the canonical homomorphism $A \rightarrow A/\mathfrak{r}(A)$; hence they are finite in number.

Conversely, suppose that A has a finite number of distinct maximal ideals $\mathfrak{m}_1, \dots, \mathfrak{m}_n$. The A/\mathfrak{m}_i are fields and it follows from ?? that the canonical map $A \rightarrow \prod_{i=1}^n A/\mathfrak{m}_i$ is surjective; as its kernel $\bigcap_{i=1}^n \mathfrak{m}_i$ is the Jacobson radical A . Thus $A/\mathfrak{r}(A)$ is isomorphic to $\prod_{i=1}^n A/\mathfrak{m}_i$. \square

Every local ring is semilocal. Every quotient of a semilocal ring is semilocal. Every finite product of semi-local rings is semi-local. Another example, generalizing the construction of the local rings $A_{\mathfrak{p}}$, is provided by the following proposition:

Proposition 9.1.18. *Let A be a ring and $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ prime ideals of A . We write $S = \bigcap_{i=1}^n (A - \mathfrak{p}_i) = A - \bigcup_{i=1}^n \mathfrak{p}_i$.*

- (a) The ring $S^{-1}A$ is semilocal; if $\mathfrak{q}_1, \dots, \mathfrak{q}_r$ are the distinct maximal elements (with respect to inclusion) of the set of \mathfrak{p}_i , the maximal ideals of $S^{-1}A$ are the $S^{-1}\mathfrak{q}_j$ and these ideals are distinct.
- (b) The ring $A_{\mathfrak{p}_i}$ is canonically isomorphic to $(S^{-1}A)_{S^{-1}\mathfrak{p}_i}$.
- (c) If A is an integral domain, then $S^{-1}A = \bigcap_{i=1}^n A_{\mathfrak{p}_i}$ in the field of fractions of A .

Proof. The ideals of A not meeting S are the ideals contained in the union of the \mathfrak{p}_i and hence in at least one of the \mathfrak{p}_i (by ??); the \mathfrak{q}_i are therefore the maximal elements of the set of ideals not meeting S ; consequently, the $S^{-1}\mathfrak{q}_i$ are the maximal ideals of $S^{-1}A$ by ?. Also, (b) follows from ??.

Suppose that A is an integral domain. If $\mathfrak{p}_i \subseteq \mathfrak{p}_j$ then $A_{\mathfrak{p}_i} \supseteq A_{\mathfrak{p}_j}$; to prove (c), we may therefore suppose that no two \mathfrak{p}_i are comparable. Then it follows from (a) and ?? that $S^{-1}A = \bigcap_{i=1}^n (S^{-1}A)_{S^{-1}\mathfrak{p}_i}$; whence (c) in view of (b). \square

Corollary 9.1.19. *Let A be an integral domain and $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ prime ideals of A , no two of which are comparable with respect to inclusion. If $A = \bigcap_{i=1}^n A_{\mathfrak{p}_i}$ in the field of fractions of A , then the maximal ideals of A are $\mathfrak{p}_1, \dots, \mathfrak{p}_n$.*

Proof. Setting $S = \bigcap_{i=1}^n (A - \mathfrak{p}_i)$. Then by ?? $S^{-1}A = A$; hence the elements of S are invertible in A and $S^{-1}\mathfrak{p}_i = \mathfrak{p}_i$ for all i . Our assertion then follows by virtue of ??(a). \square

9.1.5 Operations on ideals

If $\mathfrak{a}, \mathfrak{b}$ are ideals in a ring A , their **ideal quotient** is

$$(\mathfrak{a} : \mathfrak{b}) = \{x \in A : x\mathfrak{b} \subseteq \mathfrak{a}\}$$

which is an ideal. In particular, $(0 : \mathfrak{b})$ is called the **annihilator** of \mathfrak{b} and is also denoted by $\text{Ann}(\mathfrak{b})$: it is the set of all $x \in A$ such that $x\mathfrak{b} = 0$. In this notation the set of all zero-divisors in A is

$$D = \bigcup_{x \neq 0} \text{Ann}(x)$$

If \mathfrak{b} is a principal ideal (x) , we shall write $(\mathfrak{a} : x)$ in place of $(\mathfrak{a} : (x))$.

Example 9.1.20. If $A = \mathbb{Z}$, $\mathfrak{a} = (m)$, $\mathfrak{b} = (n)$, where $m = \prod_p p^{\mu_p}$, $n = \prod_p p^{\nu_p}$, then $(\mathfrak{a} : \mathfrak{b}) = (q)$ where $q = \prod_p p^{\gamma_p}$ and

$$\gamma_p = \max(\mu_p - \nu_p, 0) = \mu_p - \min(\mu_p, \nu_p)$$

Hence $q = m/(m, n)$.

Proposition 9.1.21. *For ideals $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ of A , we have the following properties.*

- (a) $\mathfrak{a} \subseteq (\mathfrak{a} : \mathfrak{b})$.
- (b) $(\mathfrak{a} : \mathfrak{b})\mathfrak{b} \subseteq \mathfrak{a}$.
- (c) $((\mathfrak{a} : \mathfrak{b}) : \mathfrak{c}) = (\mathfrak{a} : \mathfrak{b}\mathfrak{c}) = ((\mathfrak{a} : \mathfrak{c}) : \mathfrak{b})$.
- (d) $(\bigcap_i \mathfrak{a}_i : \mathfrak{b}) = \bigcap_i (\mathfrak{a}_i : \mathfrak{b})$.
- (e) $(\mathfrak{a} : \sum_i \mathfrak{b}_i) = \bigcap_i (\mathfrak{a} : \mathfrak{b}_i)$.

Proof. Part (a), (b) and (e) follow from the definition. As for (c), we have

$$((\mathfrak{a} : \mathfrak{b}) : \mathfrak{c}) = \{x \in A : xc \subseteq (\mathfrak{a} : \mathfrak{b})\} = \{x \in A : x\mathfrak{c}\mathfrak{b} \subseteq \mathfrak{a}\} = (\mathfrak{a} : \mathfrak{c}\mathfrak{b}),$$

and the second equality holds since $\mathfrak{b}\mathfrak{c} = \mathfrak{c}\mathfrak{b}$.

Now we prove (d). If $x\mathfrak{b} \subseteq \bigcap_i \mathfrak{a}_i$, then it is contained in $(\mathfrak{a}_i, \mathfrak{b})$ for all i , hence $(\bigcap_i \mathfrak{a}_i : \mathfrak{b}) \subseteq \bigcap_i (\mathfrak{a}_i : \mathfrak{b})$. Conversely, if $x \in (\mathfrak{a}_i, \mathfrak{b})$ for all i , then $X\mathfrak{b} \subseteq \mathfrak{a}_i$, hence in $\bigcap_i \mathfrak{a}_i$. \square

Now if \mathfrak{a} is any ideal of A , the **radical** of \mathfrak{a} is defined to be

$$\sqrt{\mathfrak{a}} = \{x \in A : x^n \in \mathfrak{a} \text{ for some } n > 0\}.$$

If $\phi : A \rightarrow A/\mathfrak{a}$ is the standard homomorphism, then $\sqrt{\mathfrak{a}} = \phi^{-1}(\mathfrak{n}(A/\mathfrak{a}))$ and hence $\sqrt{\mathfrak{a}}$ is an ideal. Moreover, according to ??, we see the radical of an idea \mathfrak{a} is the intersection of the prime ideals which contain \mathfrak{a} :

$$\sqrt{\mathfrak{a}} = \bigcap_{\mathfrak{p} \supseteq \mathfrak{a}} \mathfrak{p}.$$

Example 9.1.22. If $A = \mathbb{Z}$, $\mathfrak{a} = (m)$, let p_i ($1 \leq i \leq r$) be the distinct prime divisors of m . Then $\sqrt{\mathfrak{a}} = (p_1 \cdots p_r) = \bigcap_i (p_i)$.

Proposition 9.1.23. Let $\mathfrak{a}, \mathfrak{b}$ be ideals of A . Prove the following

$$(a) \quad \mathfrak{a} \subseteq \sqrt{\mathfrak{a}}.$$

$$(b) \quad \sqrt{\sqrt{\mathfrak{a}}} = \sqrt{\mathfrak{a}}.$$

$$(c) \quad \sqrt{\mathfrak{ab}} = \sqrt{\mathfrak{a} \cap \mathfrak{b}} = \sqrt{\mathfrak{a}} \cap \sqrt{\mathfrak{b}}.$$

$$(d) \quad \sqrt{\mathfrak{a}} = (1) \text{ if and only if } \mathfrak{a} = (1).$$

$$(e) \quad \sqrt{\mathfrak{a} + \mathfrak{b}} = \sqrt{\sqrt{\mathfrak{a}} + \sqrt{\mathfrak{b}}}.$$

$$(f) \quad \text{If } \mathfrak{p} \text{ is prime, } \sqrt{\mathfrak{p}^n} = \mathfrak{p} \text{ for all } n > 0.$$

Proof. Part (a), (d) are clear, and (b) holds since

$$x^n \in \sqrt{\mathfrak{a}} \Rightarrow (x^n)^m \in \mathfrak{a} \Rightarrow x^{nm} \in \mathfrak{a}.$$

For (c), since $\mathfrak{ab} \subseteq \mathfrak{a} \cap \mathfrak{b}$, we have $\sqrt{\mathfrak{ab}} \subseteq \sqrt{\mathfrak{a} \cap \mathfrak{b}}$. Conversely, if $x \in \sqrt{\mathfrak{a} \cap \mathfrak{b}}$, then $x^n \in \mathfrak{a} \cap \mathfrak{b}$ for some $n > 0$. This means $x^n \in \mathfrak{a}$ and $x^n \in \mathfrak{b}$, hence $x^{2n} = x^n \cdot x^n \subseteq \mathfrak{ab}$, which means $x \in \sqrt{\mathfrak{ab}}$.

The argument above also gives $x \in \sqrt{\mathfrak{a}}$ and $x \in \sqrt{\mathfrak{b}}$, hence $\sqrt{\mathfrak{a} \cap \mathfrak{b}} \subseteq \sqrt{\mathfrak{a}} \cap \sqrt{\mathfrak{b}}$. The other direction is also easy to verify.

Now we prove (e). Since $\mathfrak{a} + \mathfrak{b} \subseteq \sqrt{\mathfrak{a}} + \sqrt{\mathfrak{b}}$, we have $\sqrt{\mathfrak{a} + \mathfrak{b}} \subseteq \sqrt{\sqrt{\mathfrak{a}} + \sqrt{\mathfrak{b}}}$. Now assume that $x^n \in \sqrt{\mathfrak{a} + \mathfrak{b}}$, then $x^n = m + n$ for $m^a \in \mathfrak{a}$ and $n^b \in \mathfrak{b}$. Then

$$(x^n)^{a+b} = (m+n)^{a+b} = \sum_{i=0}^{a+b} \binom{a+b}{i} m^i n^{a+b-i}$$

It is clear that the terms of the sum either belongs to \mathfrak{a} or \mathfrak{b} . Hence this sum belongs to $\mathfrak{a} + \mathfrak{b}$. So $x \in \sqrt{\mathfrak{a} + \mathfrak{b}}$.

For (f), since $\mathfrak{p}^n \subseteq \mathfrak{p}$, we have $\sqrt{\mathfrak{p}^n} \subseteq \sqrt{\mathfrak{p}}$. But since \mathfrak{p} is prime, we have $\sqrt{\mathfrak{p}} = \mathfrak{p}$, and therefore $\sqrt{\mathfrak{p}^n} \subseteq \mathfrak{p}$. Conversely, if $x \in \mathfrak{p}$, then $x^n \in \mathfrak{p}^n$, and therefore $x \in \sqrt{\mathfrak{p}^n}$. These together prove (f). \square

Proposition 9.1.24. Let $\mathfrak{a}, \mathfrak{b}$ be ideals in a ring A such that $\sqrt{\mathfrak{a}}, \sqrt{\mathfrak{b}}$ are coprime. Then $\mathfrak{a}, \mathfrak{b}$ are coprime.

Proof. we have

$$\sqrt{\mathfrak{a} + \mathfrak{b}} = \sqrt{\sqrt{\mathfrak{a}} + \sqrt{\mathfrak{b}}} = \sqrt{(1)} = (1)$$

hence $\mathfrak{a} + \mathfrak{b} = (1)$, by ??.

Let $\rho : A \rightarrow B$ be a ring homomorphism. If \mathfrak{a} is an ideal in A , the set $\rho(\mathfrak{a})$ is not necessarily an ideal in B (e.g., let ρ be the embedding of \mathbb{Z} in \mathbb{Q} , the field of rationals, and take \mathfrak{a} to be any non-zero ideal in \mathbb{Z}). We define the **extension** \mathfrak{a}^e of \mathfrak{a} to be the ideal generated by $\rho(\mathfrak{a})$ in B : explicitly,

$$\mathfrak{a}^e = \left\{ \sum y_i \rho(x_i) : x_i \in \mathfrak{a}, y_i \in B \right\}$$

If \mathfrak{b} is an ideal of B , then $\rho^{-1}(\mathfrak{b})$ is always an ideal of A , called the **contraction** \mathfrak{b}^c of \mathfrak{b} . If \mathfrak{b} is prime, \mathfrak{b}^c is prime. If \mathfrak{a} is prime, \mathfrak{a}^e need not to be prime: For example, if $\rho : \mathbb{Z} \rightarrow \mathbb{Q}$ is the inclusion and \mathfrak{p} is any nonzero prime ideal of \mathbb{Z} , then $\mathfrak{p}^e = \mathbb{Q}$ is not prime.

We can factorize ρ as follows:

$$A \xrightarrow{\tilde{\rho}} \rho(B) \xrightarrow{j} B$$

where $\tilde{\rho}$ is surjective and j is injective. For ρ the situation is very simple: there is a one-to-one correspondence between ideals of $\rho(A)$ and ideals of A which contain $\ker f$, and prime ideals correspond to prime ideals. For j , on the other hand, the general situation is very complicated. The classical example is from algebraic number theory.

Example 9.1.25. Consider $\mathbb{Z} \rightarrow \mathbb{Z}[i]$, where $i = \sqrt{-1}$. A prime ideal (p) of \mathbb{Z} may or may not stay prime when extended to $\mathbb{Z}[i]$. In fact $\mathbb{Z}[i]$ is a principal ideal domain (because it has a Euclidean algorithm) and the situation is as follows:

- (a) $(2)^e = ((1+i)^2)$, the square of a prime ideal in $\mathbb{Z}[i]$.
- (b) If $p \equiv 1 \pmod{4}$ then $(p)^e$ is the product of two distinct prime ideals.
- (c) If $p \equiv 3 \pmod{4}$ then $(p)^e$ is prime in $\mathbb{Z}[i]$.

In fact the behavior of prime ideals under extensions of this sort is one of the central problems of algebraic number theory.

Proposition 9.1.26. Let $\rho : A \rightarrow B$, \mathfrak{a} and \mathfrak{b} be as before. Then

- (a) $\mathfrak{a} \subseteq \mathfrak{a}^{ec}$, $\mathfrak{b} \supseteq \mathfrak{b}^{ce}$.
- (b) $\mathfrak{b}^c = \mathfrak{b}^{cec}$, $\mathfrak{a}^e = \mathfrak{a}^{ece}$.
- (c) If C is the set of contracted ideals in A and if E is the set of extended ideals in B , then

$$C = \{ \mathfrak{a} \mid \mathfrak{a}^{ec} = \mathfrak{a} \}, \quad E = \{ \mathfrak{b} \mid \mathfrak{b}^{ce} = \mathfrak{b} \},$$

and $\mathfrak{a} \mapsto \mathfrak{a}^e$ is a bijective map of C onto E , whose inverse is $\mathfrak{b} \mapsto \mathfrak{b}^c$.

Proof. The first part is clear, and (b) follows from (a):

$$\mathfrak{b}^c \subseteq (\mathfrak{b}^c)^{ec} = \mathfrak{b}^{cec} \quad \text{and} \quad \mathfrak{b} \supseteq \mathfrak{b}^{ce} \Rightarrow \mathfrak{b}^c \supseteq \mathfrak{b}^{cec}$$

Hence $\mathfrak{b}^c = \mathfrak{b}^{cec}$. Also, part (c) is a consequence of part (b). \square

Proposition 9.1.27. *If $\mathfrak{a}_1, \mathfrak{a}_2$ are ideals of A and if $\mathfrak{b}_1, \mathfrak{b}_2$ are ideals of S , then we have*

- (a) $(\mathfrak{a}_1 + \mathfrak{a}_2)^e = \mathfrak{a}_1^e + \mathfrak{a}_2^e$ and $(\mathfrak{b}_1 + \mathfrak{b}_2)^c \supseteq \mathfrak{b}_1^c + \mathfrak{b}_2^c$.
- (b) $(\mathfrak{a}_1 \cap \mathfrak{a}_2)^e \subseteq \mathfrak{a}_1^e \cap \mathfrak{a}_2^e$ and $(\mathfrak{b}_1 \cap \mathfrak{b}_2)^c = \mathfrak{b}_1^c \cap \mathfrak{b}_2^c$.
- (c) $(\mathfrak{a}_1 \mathfrak{a}_2)^e = \mathfrak{a}_1^e \mathfrak{a}_2^e$ and $(\mathfrak{b}_1 \mathfrak{b}_2)^c \supseteq \mathfrak{b}_1^c \mathfrak{b}_2^c$.
- (d) $(\mathfrak{a}_1 : \mathfrak{a}_2)^e \subseteq (\mathfrak{a}_1^e : \mathfrak{a}_2^e)$ and $(\mathfrak{b}_1 : \mathfrak{b}_2)^c \subseteq (\mathfrak{b}_1^c : \mathfrak{b}_2^c)$.
- (e) $(\sqrt{\mathfrak{a}})^e \subseteq \sqrt{\mathfrak{a}^e}$ and $(\sqrt{\mathfrak{b}})^c = \sqrt{\mathfrak{b}^c}$.

Therefore, the set of ideals E is closed under sum and product, and C is closed under intersection and radical.

Proof. Note that the extension and contraction preserve inclusions. Since f is a ring homomorphism, part (a), (b), and (c) are easy to see. We first show part (d). If $x \in (\mathfrak{a}_1 : \mathfrak{a}_2)^e$, then $x = \sum y_i f(x_i)$ for $x_i \in (\mathfrak{a}_1 : \mathfrak{a}_2)$. Pick one term $y_i f(x_i)$, since $x_i \mathfrak{a}_2 \subseteq \mathfrak{a}_1$, we have $f(x_i) f(\mathfrak{a}_2) \subseteq f(\mathfrak{a}_1)$. So we have $y_i f(x_i) \in (\mathfrak{a}_1^e : \mathfrak{a}_2^e)$ and then $x \in (\mathfrak{a}_1^e : \mathfrak{a}_2^e)$. This gives $(\mathfrak{a}_1 : \mathfrak{a}_2)^e \subseteq (\mathfrak{a}_1^e : \mathfrak{a}_2^e)$. Similarly, if $x \in (\mathfrak{b}_1 : \mathfrak{b}_2)^c$, then $f(x) \mathfrak{b}_2 \subseteq \mathfrak{b}_1$. So $f(x \mathfrak{b}_2^c) \subseteq f(x) \mathfrak{b}_2 \subseteq \mathfrak{b}_1$. This shows $x \mathfrak{b}_2^c \subseteq \mathfrak{b}_1^c$, hence $(\mathfrak{b}_1 : \mathfrak{b}_2)^c \subseteq (\mathfrak{b}_1^c : \mathfrak{b}_2^c)$.

We turn to part (e). If $x = \sum y_i f(x_i)$ for $x_i^{n_i} \in \mathfrak{a}$. Then

$$(y_i f(x_i))^{n_i} = y_i^n f(x_i^{n_i}) \in \mathfrak{a}^e.$$

Since x is a finite sum, we get $x^N \in \mathfrak{a}^e$ for some N . Hence $x \in \sqrt{\mathfrak{a}^e}$, which shows $(\sqrt{\mathfrak{a}})^e \subseteq \sqrt{\mathfrak{a}^e}$. Finally, if $f(x) \in \sqrt{\mathfrak{b}}$, then $(f(x))^n = f(x^n) \in \mathfrak{b}$, so $x \in \sqrt{\mathfrak{b}^c}$. So $(\sqrt{\mathfrak{b}})^c \subseteq \sqrt{\mathfrak{b}^c}$. Conversely, if $x^n \in \mathfrak{b}^c$, then $f(x^n) = (f(x))^n \in \mathfrak{b}$. Hence $f(x) \in \sqrt{\mathfrak{b}}$, $x \in (\sqrt{\mathfrak{b}})^c$. \square

9.1.6 Exercise

Exercise 9.1.1. Let x be a nilpotent element of a ring A . Show that $1 + x$ is a unit of A . Deduce that the sum of a nilpotent element and a unit is a unit.

Proof. Assume $x^n = 0$ for $n > 0$, then consider the polynomials

$$1 + x^n, \quad 1 - x^n$$

Either of them has a root -1 : $1 + x^n$ for n odd, $1 - x^n$ for n even. Then we can insert x to get a decomposition of 1 since $x^n = 0$. Hence $1 + x$ is a unit.

For $u + x$ with u a unit, since there is $t \in A$ such that $ur = 1$, we have $r(u + x) = ur + xr = 1 + xr$. Since $(xr)^n = 0$, $1 + xr$ is a unit. Then $u + x$ is also a unit. \square

Exercise 9.1.2. Let A be a ring and let $A[X]$ be the ring of polynomials in an indeterminate X , with coefficients in A . Let $f = a_0 + a_1 X + \cdots + a_n X^n \in A[X]$. Prove that

- (a) f is a unit in $A[X]$ if and only if a_0 is a unit in A and a_1, \dots, a_n are nilpotent.
- (b) f is nilpotent if and only if a_0, a_1, \dots, a_n are nilpotent.
- (c) f is a zero-divisor if and only if there exists $a \neq 0$ in A such that $af = 0$.
- (d) f is said to be primitive if $(a_0, a_1, \dots, a_n) = (1)$. Prove that if $f, g \in A[X]$, then fg is primitive if and only if f and g are primitive.

These results can be generalized to a polynomial ring $A[X_1, \dots, X_r]$ in several indeterminates.

Proof. Let f be a unit in $A[X]$. Then for any prime ideal \mathfrak{p} of A , the image \bar{f} in $(A/\mathfrak{p})[X]$ is also a unit. Since A/\mathfrak{p} is an integral domain, this implies $\deg(\bar{f}) = 0$, hence $a_1, \dots, a_n \in \mathfrak{p}$ and $a_0 \notin \mathfrak{p}$. Since this holds for any prime ideal of A , it follows that a_0 is a unit and $a_1, \dots, a_n \in \bigcap \mathfrak{p} = \mathfrak{n}(A)$. That is, a_1, \dots, a_n are nilpotents. Conversely, if a_0 is a unit, then f has an inverse g in $A[[X]]$. If in addition a_1, \dots, a_n are nilpotent, this inverse g is in $A[X]$, hence f is a unit in $A[X]$.

Similarly, if f is nilpotent then its image in $(A/\mathfrak{p})[X]$ is nilpotent for any prime ideal \mathfrak{p} , hence is zero. This implies $a_0, \dots, a_n \in \mathfrak{n}(A)$, so they are nilpotent. The converse can be proved similarly.

Let $g(X) = b_0 + b_1X + \dots + b_mX^m$ be such that $fg = 0$ and $g \neq 0$. If $f = 0$ then the claim follows, so we may assume that f is nonzero and in particular that $a_n \neq 0$. Then $a_n b_m = 0$ follows immediately. Note that $f \cdot (a_n g)$ is also zero, so by replacing g with $a_n g$ we eliminate the degree of g by 1. This argument always holds except $\deg g = 0$, so there is $a \in A, a \neq 0$ such that $af = 0$. The converse is trivial.

Note that a polynomial is primitive just if no maximal ideal contains all its coefficients. Let $\mathfrak{m} \subseteq A$ be maximal. Since A/\mathfrak{m} is a field, $A[X]/\mathfrak{m}[X] = (A/\mathfrak{m})[X]$ is an integral domain. Thus

$$f, g \notin \mathfrak{m}[X] \iff \bar{f}, \bar{g} \neq 0 \text{ in } (A/\mathfrak{m})[X] \iff \bar{f}\bar{g} \neq 0 \text{ in } (A/\mathfrak{m})[X] \iff fg \notin \mathfrak{m}[X]$$

Therefore no maximal ideal contains all the coefficients of fg just if the same holds for f and g . \square

Exercise 9.1.3. In the ring $A[X]$, the Jacobson radical is equal to the nilradical.

Proof. Since maximal ideal is prime, $\mathfrak{n}(A) \subseteq \mathfrak{r}(A)$ holds for all ring A . Now assume $f(x) = a_0 + a_1x + \dots + a_nx^n$ is in $J(A[X])$, then $1 + f(x)g(x)$ is a unit for any $g(x) \in A[X]$. In particular, $1 + f(x)$ is a unit, then by ??, a_i are nilpotent for $i \geq 1$. Let $g(x) = 1 + X$, we obtain $a_0 + a_1$ is nilpotent (because it is the coefficient of X in $1 + fg$), hence a_0 is nilpotent. Again by ??, $f(x)$ is nilpotent. \square

Exercise 9.1.4. Let A be a ring and let $A[[X]]$ be the ring of formal power series with coefficients in A . Show that

- (a) f is a unit in $A[[X]]$ if and only if a_0 is a unit in A .
- (b) If f is nilpotent, then a_n is nilpotent for all $n \geq 0$. Is the converse true?
- (c) f belongs to the Jacobson radical of $A[[X]]$ if and only if a_0 belongs to the Jacobson radical of A .

- (d) The contraction of a maximal ideal \mathfrak{m} of $A[[X]]$ is a maximal ideal of A , and \mathfrak{m} is generated by \mathfrak{m}^c and X .
- (e) Every prime ideal of A is the contraction of a prime ideal of $A[[X]]$.

Proof. If f is a unit, then a_0 is a unit. If a_0 is a unit, consider the expansion

$$\frac{1}{1 + a_0^{-1}X} \sum_{n=0}^{\infty} (-a_0^{-1}X)^n$$

Plug into $X = f - a_0$, we get an inverse of $1 + a_0^{-1}(f - a_0) = a_0^{-1}f$, hence $a_0^{-1}f$ is a unit, so is f .

If f is nilpotent, then a_0 is nilpotent, and $f - a_0$ is nilpotent. By repeating this we can show a_n are all nilpotent.

To see that the converse fails, let Y_n be a collection of indeterminates over \mathbb{Z} , indexed by the natural numbers. Let

$$A = \mathbb{Z}[Y_1, Y_2, \dots] / (Y_1, Y_2^2, Y_3^3, \dots)$$

Let

$$f(X) = \sum_n Y_n X^n \in A[[X]]$$

Then $a_n = Y_n$ is nilpotent for all n , but there is no N such that $f^N = 0$ (this would establish some non-trivial relation amongst the Y_n for $n > N$ other than $Y_n^n = 0$).

$f \in J(A[[X]])$ if and only if $1 + fg$ is a unit for all $g \in A[[X]]$. Since for $g = \sum b_n X^n$, the constant term of $1 + fg$ is $1 + a_0 b_0$, the condition is equivalent to that $1 + a_0 b_0$ is a unit for all b_0 , and hence to $a_0 \in \mathfrak{r}(A)$.

Since x has constant term $0 \in \mathfrak{r}(A)$ in $A[[X]]$, by (c) above we get $X \in J(A[[X]])$, and hence $(X) \subseteq J(A[[X]])$. As $\mathfrak{m} - (X) = \mathfrak{m}^c$, we get $\mathfrak{m} = \mathfrak{m}^c + (X)$. Now $A/\mathfrak{m}^c \cong A[[X]]/\mathfrak{m}$ is a field, so \mathfrak{m}^c is maximal.

To see $I = I^{ec}$, we only need to verify I^e is prime when I is prime. Let $f, g \in A[[X]]$ with constant term a_0, b_0 . If $f, g \notin \mathfrak{p}^e$, then $a_0, b_0 \notin \mathfrak{p}$. Since fg has constant term $a_0 b_0$, which is not in \mathfrak{p} , $fg \notin \mathfrak{p}^e$. So \mathfrak{p}^e is prime. \square

Exercise 9.1.5. Let A be a ring, $\mathfrak{n}(A)$ its nilradical. Show that the following are equivalent:

- (a) A has exactly one prime ideal.
- (b) Every element of A is either a unit or nilpotent.
- (c) $A/\mathfrak{n}(A)$ is a field.

Proof. Assume A has exactly one prime ideal \mathfrak{p} . Suppose $x \in A$ is not a unit, then $x \in \mathfrak{m}$ for some maximal ideal \mathfrak{m} . Since \mathfrak{m} is also prime, we must have $\mathfrak{m} = \mathfrak{p}$. Hence $x \in \mathfrak{p}$. Since we also have $\mathfrak{n}(A) = \mathfrak{p}$, x is nilpotent.

Since $\mathfrak{n}(A)$ contains all nilpotent elements, $A - \mathfrak{n}(A)$ contains only units. Then every elements in $A/\mathfrak{n}(A)$ is a unit, so $A/\mathfrak{n}(A)$ is a field.

If $A/\mathfrak{n}(A)$ is a field, then $A - \mathfrak{n}(A)$ contains only units. Let \mathfrak{p} be a prime ideal in A , then $\mathfrak{p} \in$

$\mathfrak{n}(A)$ since it contains no units, and then $\mathfrak{n}(A) = \mathfrak{p}$ since $\mathfrak{n}(A) \subseteq \mathfrak{p}$. So A contains only one prime ideal $\mathfrak{n}(A)$. \square

Proof. Let $x \in A$ be idempotent, and let \mathfrak{m} be the unique maximal ideal. Then in A/\mathfrak{m} we see

$$\bar{x}(1 - \bar{x}) = 0$$

so $\bar{x} = 0$ or $\bar{x} = 1$, which means $x \in \mathfrak{m}$ or $x \in 1 + \mathfrak{m}$. Since $\mathfrak{m} = \mathfrak{r}(A)$, if $x \in \mathfrak{m}$ then $1 - x$ is a unit, and if $x \in 1 + \mathfrak{m}$ then x is a unit. Since $x(1 - x) = 0$, in the former case $x = 0$, and in the latter case $x = 1$. \square

Exercise 9.1.6. In a ring A , let Σ be the set of all ideals in which every element is a zero-divisor. Show that the set Σ has maximal elements and that every maximal element of Σ is a prime ideal. Hence the set of zero-divisors in A is a union of prime ideals.

Proof. Clearly $(0) \in \Sigma$, so Σ is non-empty. Let $(I_i)_{i \in I}$ be a chain in Σ . The union \mathfrak{a} of the chain is still an ideal, and consists only of zero-divisors. Thus Σ , so by Zorn's Lemma Σ has maximal elements.

Let I be a maximal element of Σ , and suppose that $xy \in I$ with $y \notin I$. Then $I \subsetneq (I, a)$ so y is not a zero-divisor by the maximality of I . But xy is a zero-divisor, so there exists some $z \in A, z \neq 0$ such that $zxy = 0$. As y is not a zero-divisor we must have that $xz = 0$, so that x is a zero-divisor and thus is contained in I . \square

9.2 Rings and modules of fractions

9.2.1 Rings of fractions

Definition 9.2.1. A subset S of a commutative ring A is a **multiplicative subset** (or **multiplicatively closed**) if

- $1 \in S$.
- if $s, t \in S$ then $st \in S$.

In other words, S is a subsemigroup of the multiplicative semigroup of A .

Example 9.2.2 (Example of multiplicative subsets).

- (a) For every $a \in A$, the set of a^n , where $n \in \mathbb{N}$, is a multiplicative subset of A .
- (b) Let A be a ring and \mathfrak{a} an ideal of A , then $1 + \mathfrak{a}$ is multiplicative closed.
- (c) Let \mathfrak{p} be an ideal of A . For $A - \mathfrak{p}$ to be a multiplicative subset of A , it is necessary and sufficient that \mathfrak{p} be prime.
- (d) The set of elements of A which are not divisors of zero is a multiplicative subset of A .
- (e) If S and T are multiplicative subsets of A , the set ST of products st , where $s \in S$ and $t \in T$, is a multiplicative subset.
- (f) Let \mathcal{S} be a directed set (with respect to inclusion) of multiplicative subsets of A . Then $T = \bigcup_{S \in \mathcal{S}} S$ is a multiplicative subset of A , as any two elements of T belong to some subset $S \in \mathcal{S}$, hence their product belongs to T .

Proposition 9.2.3. Let A be a ring, S a multiplicative subset, and \mathfrak{a} an ideal with $\mathfrak{a} \cap S = \emptyset$. Set $\mathcal{S} := \{\text{ideals } \mathfrak{b} \mid \mathfrak{b} \supseteq \mathfrak{a} \text{ and } \mathfrak{b} \cap S = \emptyset\}$. Then \mathcal{S} has a maximal element \mathfrak{p} , and every such \mathfrak{p} is prime.

Proof. By Zorn's lemma, similar to ??.

For every subset S of a ring A , there exist multiplicative subsets of A containing S , for example A itself. The intersection of all these subsets is the smallest multiplicative subset of A containing S ; it is said to be **generated** by S . It follows immediately that it is the set consisting of all the finite products of elements of S .

Proposition 9.2.4. Let A be a ring and S a subset of A . There exists a ring \tilde{A} and a homomorphism $i : A \rightarrow \tilde{A}$ with the following properties:

- (a) the elements of $i(S)$ are invertible in \tilde{A} ;
- (b) for every homomorphism f of A to a ring B such that the elements of $f(S)$ are invertible in B , there exists a unique homomorphism $\tilde{f} : \tilde{A} \rightarrow B$ such that $f = \tilde{f} \circ i$.

Proof. Define a relation on the set of pairs $A \times S$ as follows:

$$(a, s) \sim (a', s') \iff (\exists t \in S) t(s'a - sa') = 0 \quad (9.2.1)$$

Now we denote by a/s the equivalence class of (a, s) , and define the operations $+, \cdot$ on such fractions as

$$\frac{a}{s} + \frac{a'}{s'} = \frac{as' + a's}{ss'}, \quad \frac{a}{s} \cdot \frac{a'}{s'} = \frac{aa'}{ss'}.$$

The set $S^{-1}A$ of fractions, endowed with the operations $+, \cdot$, is a commutative ring and the function $a \mapsto a/1$ defines a ring homomorphism $i_A^S : A \rightarrow S^{-1}A$. It is now clear that $i(s)$ is invertible for any $s \in S$. If $\rho : A \rightarrow B$ is a homomorphism such that $\rho(S)$ are invertible, then the map $\tilde{\rho} : S^{-1}A \rightarrow B$ defined by $\tilde{\rho}(a/s) = \rho(a)\rho(s)^{-1}$ is well-defined and clearly satisfies the requirement. \square

Let A be a ring, S a subset of A and \bar{S} the multiplicative subset generated by S . The ring of fractions of A defined by \bar{S} and denoted by $S^{-1}A$ is the quotient set of $A \times \bar{S}$ under the equivalence relation (??) with the ring structure defined by

$$a/s + b/t = (ta + bs)/st, \quad (a/s)(b/t) = ab/st.$$

for $a, b \in A$ and $s, t \in \bar{S}$. The canonical map i_A^S of A to $S^{-1}A$ is the homomorphism $a \mapsto a/1$, which makes $S^{-1}A$ into an A -algebra.

Corollary 9.2.5. If $\rho : A \rightarrow B$ is a ring homomorphism such that

- (a) $\rho(S)$ is invertible in B .
- (b) $\rho(a) = 0$ if and only if $as = 0$ for some $s \in S$.
- (c) Every element of B is of the form $\rho(a)\rho(s)^{-1}$.

Then there is a unique isomorphism $h : S^{-1}A \rightarrow B$ such that $\rho = h \circ i$.

Proof. We only have to show that $h : S^{-1}A \rightarrow B$, defined by

$$h(a/s) = \rho(a)\rho(s)^{-1}$$

is an isomorphism. By (c), h is surjective. To show h is injective, look at the kernel of h : if $h(a/s) = 0$, then $\rho(a) = 0$, hence by (b) we have $as = 0$ for some $s \in S$, hence $(a, s) = (0, 1)$, i.e., $a/s = 0$ in $S^{-1}A$. \square

Example 9.2.6 (Example of localizations).

- (a) Let A be a commutative ring, and let \mathfrak{p} be a prime ideal of A . Then the set $S = A - \mathfrak{p}$ is multiplicatively closed. The localizations $S^{-1}A$, $S^{-1}M$ are then denoted $A_{\mathfrak{p}}$, $M_{\mathfrak{p}}$. The process of passing from A to $A_{\mathfrak{p}}$ is called **localization at \mathfrak{p}** .
- (b) Let $f \in A$ and let $S = \{f^n\}$, we write A_f for $S^{-1}A$ in this case.
- (c) Let $A = \mathbb{Z}$, $\mathfrak{p} = (p)$ where p a prime number. Then $\mathbb{Z}_{\mathfrak{p}}$ is set of all rational numbers m/n where n is prime to p ; if $f \in \mathbb{Z}$ and $f \neq 0$, then \mathbb{Z}_f is the set of all rational numbers whose denominator is a power of f .
- (d) Let $A = k[X_1, \dots, X_n]$, where k is a field and the X_i are independent indeterminates, \mathfrak{p} a prime ideal in A . Then $A_{\mathfrak{p}}$ is the ring of all rational functions f/g , where $g \notin \mathfrak{p}$. If V is the variety defined by the ideal \mathfrak{p} , that is to say the set of all $x = (x_1, \dots, x_n) \in k^n$ such that $f(x) = 0$ whenever $f \in \mathfrak{p}$, then (provided k is infinite) $A_{\mathfrak{p}}$ can be identified with the ring of all rational functions on k^n which are defined at almost all points of V ; it is the local ring of k^n **along the variety V** . This is the prototype of the local rings which arise in algebraic geometry.

Proposition 9.2.7. *Let A, B be two rings, S a subset of A , T a subset of B and $\rho : A \rightarrow B$ a homomorphism such that $\rho(S) \subseteq T$. There exists a unique homomorphism $\tilde{\rho}$ from $S^{-1}A$ to $T^{-1}B$ such that $\tilde{\rho}(a/1) = \rho(a)/1$ for all $a \in A$.*

Suppose further that T is contained in the multiplicative subset of B generated by $\rho(S)$. Then, if ρ is surjective (resp. injective) so is $\tilde{\rho}$.

Proof. The first assertion amounts to saying that there exists a unique homomorphism $\tilde{\rho} : S^{-1}A \rightarrow T^{-1}B$ giving a commutative diagram:

$$\begin{array}{ccc} A & \xrightarrow{\rho} & B \\ i_A^S \downarrow & & \downarrow i_B^T \\ S^{-1}A & \xrightarrow{\tilde{\rho}} & T^{-1}B \end{array}$$

Now the relation $\rho(S) \subseteq T$ implies that $i_B^T(\rho(S))$ is invertible in $T^{-1}B$ for all $s \in S$ and it is sufficient to apply ?? to $i_B^T \circ \rho$. It follows easily from the definition that, for all $a \in A$ and $s \in \bar{S}$ (multiplicative subset of A generated by S),

$$\tilde{\rho}(a/s) = \rho(a)/\rho(s)$$

Suppose that T is contained in the multiplicative subset generated by $\rho(S)$, which is precisely $\rho(\bar{S})$. Then it follows from the formula above that, if ρ is surjective, so is $\tilde{\rho}$. Suppose now that

ρ is injective. Let a/s be an element of the kernel of $\tilde{\rho}$. As the multiplicative subset generated by T is $\rho(\bar{S})$, there is an element $s_1 \in S$ such that $\rho(s_1)\rho(a) = 0$, whence $\rho(s_1a) = 0$ and consequently $s_1a = 0$ since ρ is injective; then $a/s = 0$, which proves that $\tilde{\rho}$ is injective. \square

Corollary 9.2.8. *Let A be a ring, S a subset of A and ρ an injective homomorphism from A to a ring B such that the elements of $\rho(S)$ are invertible in B . The unique homomorphism $\tilde{\rho} : S^{-1}A \rightarrow B$ such that $\rho = \tilde{\rho} \circ i_A^S$ is then injective.*

Corollary 9.2.9. *Let A be a ring and S, T two subsets of A such that $S \subseteq T$. Then there exists a unique homomorphism $i_A^{T,S} : S^{-1}A \rightarrow T^{-1}A$ such that $i_A^T = i_A^{T,S} \circ i_A^S$.*

Proof. For all $a \in A$, $i_A^{T,S}$ then maps the element a/s in $S^{-1}A$ to the element a/s in $T^{-1}A$. \square

Corollary 9.2.10. *Let A, B, C be three rings, S (resp. T, U) a multiplicative subset of A (resp. B, C), $\rho : A \rightarrow B, \nu : B \rightarrow C$ two homomorphisms and $\eta = \nu \circ \rho$ the composite homomorphism. Suppose that $\rho(S) \subseteq T$ and $\nu(T) \subseteq U$. Let $\tilde{\rho} : S^{-1}A \rightarrow T^{-1}B, \tilde{\nu} : T^{-1}B \rightarrow U^{-1}C$, and $\tilde{\eta} : S^{-1}A \rightarrow U^{-1}C$ be the induced homomorphisms. Then $\tilde{\eta} = \tilde{\nu} \circ \tilde{\rho}$.*

In particular, if S, T, U are three multiplicative subsets of A such that $S \subseteq T \subseteq U$, then $i_A^{U,S} = i_A^{U,T} \circ i_A^{T,S}$.

Corollary 9.2.11. *Let S be a subset of a ring A , B a subring of $S^{-1}A$ containing the set $T := i_A^S(A)$. Let $j : B \rightarrow S^{-1}A$ be the canonical injection, then the unique homomorphism $\eta : T^{-1}B \rightarrow S^{-1}A$ such that $\eta \circ i_A^S = j$ is an isomorphism.*

Proof. The map η is injective by ???. The ring $\eta(T^{-1}B)$ contains T and the inverse of the elements of T . Hence it is equal to $S^{-1}A$. \square

9.2.2 Modules of fractions

The canonical homomorphism $i_A^S : A \rightarrow S^{-1}A$ allows us to consider every $S^{-1}A$ -module as an A -module.

Proposition 9.2.12. *Let A be a ring, S a subset of A , M an A -module and \tilde{M} the A -module $M \otimes_A S^{-1}A$. If i_M^S is the canonical A -homomorphism $x \mapsto x \otimes 1$ of M to \tilde{M} , then:*

- (a) *For all $s \in S$, the homothety $z \mapsto sz$ of M' is bijective.*
- (b) *For every A -module N such that, for all $s \in S$, the homothety $z \mapsto sz$ of N is bijective, and every homomorphism $\phi : M \rightarrow N$, there exists a unique homomorphism $\tilde{\phi} : \tilde{M} \rightarrow N$ such that $\phi = \tilde{\phi} \circ i_M^S$.*

Proof. For every A -module N and all $a \in A$, denote by h_a the homothety $y \mapsto ay$ in N . Then $a \mapsto h_a$ is a ring homomorphism from A to $\text{End}_A(N)$. To say that h_a is bijective means that h_a is an invertible element of $\text{End}_A(N)$. Suppose that, for all $s \in S$, h_s is invertible in $\text{End}_A(N)$. The elements h_a where $a \in A$ and the inverses of the elements h_s where $s \in S$ then generate in $\text{End}_A(N)$ a commutative subring B and the homomorphism $a \mapsto h_a$ from A to B is such that the images of the elements of S are invertible. Then it follows that there exists a unique homomorphism \tilde{h} of $S^{-1}A$ to B such that

$$\tilde{h}(a/s) = h_a h_s^{-1}.$$

We know that such a homomorphism defines on N an $S^{-1}A$ -module structure such that $(a/s) \cdot y = h_s^{-1}(a \cdot y)$. The A -module structure derived from this $S^{-1}A$ -module structure by means of the homomorphism i_A^S is precisely the structure given initially.

Conversely, if N is an $S^{-1}A$ -module and it is considered as an A -module by means of i_A^S , the homotheties $y \mapsto sy$, for $s \in S$, are bijective, for $y \mapsto (1/s)y$ is its inverse; and the $S^{-1}A$ -module structure on N derived from its A -module structure by the process described above is the $S^{-1}A$ -module structure given initially. Thus there is a canonical one-to-one correspondence between $S^{-1}A$ -modules and A -modules in which the homotheties induced by the elements of S are bijective. Moreover, if N, N' are two A -modules with this property every A -module homomorphism $\phi : N \rightarrow N'$ is also a homomorphism of the $S^{-1}A$ -module structures of N and N' , as, for all $y \in N$ and all $s \in S$, we may write $\phi(y) = \phi(s \cdot ((1/s)y)) = s \cdot \phi((1/s)y)$, whence $\phi((1/s)y) = (1/s)\phi(y)$; the converse is obvious.

This being so, the statement is just the characterization of the module obtained from M by extending the scalars to $S^{-1}A$, taking account of the above interpretation. \square

Let A be a ring, S a subset of A , \bar{S} the multiplicative subset of A generated by S and M an A -module. Then the $S^{-1}A$ -module $M \otimes_A S^{-1}A$ is called the **module of fractions of M defined by S** and denoted by $S^{-1}M$.

Proposition 9.2.13. *Let A be a ring. Let $S \subseteq A$ be a multiplicative subset. Let M be an A -module. Then*

$$S^{-1}M = \varinjlim_{s \in \bar{S}} M_s$$

where the partial ordering on S is given by $s \geq s'$ if and only if $s = as'$ for some $a \in A$ in which case the map $M_{s'} \rightarrow M_s$ is given by $m/s^e = ma^e/(s')^e$.

Proposition 9.2.14. *Let S be a multiplicative subset of A and M an A -module. For $m/s = 0$ where $m \in M$ and $s \in S$, it is necessary and sufficient that there exist $s' \in S$ such that $s'm = 0$.*

Proof. If $s' \in S$ is such that $s'm = 0$, clearly $m/s = (s'm)/(ss') = 0$. Conversely, suppose that $m/s = 0$. As $1/s$ is invertible in $S^{-1}A$, $m/1 = 0$. For every sub- A -module P of $S^{-1}A$ containing 1, we denote by $\beta(P, m)$ the image of $(m, 1)$ under the canonical map of $M \times P$ to $M \otimes_A P$. Then $(MS^{-1}A, m) = 0$, so there exists a finitely generated submodule P of $S^{-1}A$ containing 1 and such that $\beta(P, m) = 0$. For all $t \in S$ we denote by A_t the set of a/t , where $a \in A$. As P is finitely generated, there exists $t \in S$ such that $P \subseteq A_t$, whence $\beta(A_t, m) = 0$. The map $a \mapsto a/t$ from A to A_t is surjective; let B be its kernel. Then we have a surjective map $h : M \otimes_A A \rightarrow M \otimes_A A_t$ whose kernel is $B \otimes_A M$. By definition $\beta(A_t, m) = h(tm)$ and consequently tm can be expressed in the form $\sum_i b_i m_i$, where $b_i \in B$ and $m_i \in M$. Since $b_i/t = 0$ for all i , there exists $t' \in S$ such that $t'tm = 0$, whence the claim. \square

Corollary 9.2.15. *The module $S^{-1}M$ can be identified with the set $M \times S$ modulo the equivalence relation*

$$(m, s) \sim (m', s') \iff \exists t \in S \text{ such that } t(ms' - m's) = 0$$

and the $S^{-1}A$ -module structure defined by $(a/s)(m, t) = (am, st)$.

Corollary 9.2.16. *Let M be a finitely generated A -module. For $S^{-1}M = 0$, it is necessary and sufficient that there exists $s \in S$ such that $sM = 0$.*

Proof. Without any conditions on M , clearly the relation $sM = 0$ for some $s \in S$ implies $S^{-1}M = 0$. Conversely, suppose that $S^{-1}M = 0$ and let x_1, \dots, x_n be a system of generators of M . The $m_i/1$ generate the $S^{-1}A$ -module $S^{-1}M$, hence to say that $S^{-1}M = 0$ amounts to saying that $m_i/1 = 0$ for all i . By ?? there exist $s_i \in S$ such that $s_i m_i = 0$ and, taking $s = s_1 \cdots s_n$, we see $s m_i = 0$ for all i and hence $sM = 0$. \square

Corollary 9.2.17. *Let M be a finitely generated A -module. For an ideal \mathfrak{a} of A to be such that $\mathfrak{a}M = M$, it is necessary and sufficient that there exist $a \in \mathfrak{a}$ such that $(1 + a)M = 0$.*

Proof. Clearly the relation $(1 + a)M = 0$ implies $M = aM$. To prove the converse, we use the following result: For every ideal \mathfrak{a} of A , the set $S = 1 + \mathfrak{a}$ is a multiplicative subset of A and the set $S^{-1}\mathfrak{a}$ is an ideal contained in the Jacobson radical of $S^{-1}A$. The first assertion is obvious, as well as the fact that $S^{-1}\mathfrak{a}$ is an ideal of $S^{-1}A$. On the other hand, $(1/1) + (a/s) = (s + a)/s$ and $s + a \in S$ for all $s \in S$ and $a \in \mathfrak{a}$ by definition of S . Hence $(1/1) + (a/s)$ is invertible in $S^{-1}A$ for all $a/s \in S^{-1}\mathfrak{a}$, which completes the proof of this result. This being done, if we set $N = S^{-1}M$, clearly N is a finitely generated $S^{-1}A$ -module. If $\mathfrak{a}M = M$, then $S^{-1}\mathfrak{a}N = N$ and it follows that $N = 0$ by Nakayama's Lemma. The corollary then follows from ?? \square

Remark 9.2.1. Let M be an A -module, f be an element of A . Consider a sequence (M_n) of A -modules, where $M_n = M$ and for any integers $m \leq n$, let $\varphi_{nm} : M_m \rightarrow M_n$ be the multiplication by f^{n-m} . Then it is immediate that (M_n, φ_{nm}) form an inductive system of A -modules, so let $N = \varinjlim M_n$ and $\varphi_n : N \rightarrow M_n$ be the canonical homomorphisms. We now define a functorial isomorphism from N to M_f : for each n , let $\theta_n : z \mapsto z/f^n$ be an A -homomorphism from $M_n = M$ to M_f , and since we clearly have $\theta_n \circ \varphi_{nm} = \theta_m$, let $\theta : N \rightarrow M_f$ be the induced homomorphism. Since any element of M_f is of the form z/f^n for some n , so θ is surjective. On the other hand, if $\theta(\varphi_n(z)) = 0$, which means $z/f^n = 0$, then there exists an integer $k > 0$ such that $f^k z = 0$, so $\varphi_{n+k,n}(z) = 0$, which means $\varphi_n(z) = 0$. We can then identify M_f with $\varinjlim M_n$ by the isomorphism θ .

Proposition 9.2.18. *Let A, B be two rings, S a multiplicative subset of A , T a multiplicative subset of B and f a homomorphism from A to B such that $f(S) \subseteq T$. Let M be an A -module, N a B -module and $\phi : M \rightarrow N$ an A -linear homomorphism. Then there exists a unique $S^{-1}A$ -linear map $\tilde{\phi} : S^{-1}M \rightarrow T^{-1}N$ such that $\tilde{\phi}(m/1) = \phi(m)/f(s)$ for all $m \in M$.*

Proof. The map $i_M^T \circ \phi$ from M to $T^{-1}N$ is A -linear. Moreover, if $s \in S$, then $f(s) \in T$, hence the homothety induced by s on $T^{-1}N$ is bijective. The existence and uniqueness of $\tilde{\phi}$ then follow from ???. Then, for $m \in M$ and $s \in S$,

$$\tilde{\phi}(m/s) = \phi(m)/f(s)$$

With the same notation, let C be a third ring, U a multiplicative subset of C , g a homomorphism from B to C such that $g(T) \subseteq U$, P a C -module, ψ a B -linear map from N to P and $\tilde{\phi}$ the $T^{-1}B$ -linear map from B to $U^{-1}P$ associated with ψ . Then

$$\widetilde{(\psi \circ \phi)} = \tilde{\psi} \circ \tilde{\phi}$$

where the left hand side is the A -linear map $S^{-1}M \rightarrow U^{-1}P$ associated with $\psi \circ \phi$. Similarly, if

ϕ' is a second A -linear map from M to N , then

$$\widetilde{(\phi + \phi')} = \tilde{\phi}' + \tilde{\phi}$$

the left-hand side being the A -linear map $S^{-1}M \rightarrow T^{-1}N$ associated with $\phi + \phi'$. \square

If, in ??, we take $B = A$, $T = S$ and $\phi = \text{id}_A$, it is easily seen that $\tilde{\phi}$ is just the map $\phi \otimes 1 : M \otimes_A S^{-1}A \rightarrow N \otimes_A S^{-1}A$. We shall henceforth denote it by $S^{-1}\phi$. If S is the complement of a prime ideal \mathfrak{p} of A , we write $\phi_{\mathfrak{p}}$ instead of $S^{-1}\phi$.

Proposition 9.2.19. *Let f be a homomorphism from a ring A to a ring B and S a multiplicative subset of A . Then there exists a unique map $j : f(S)^{-1}B \rightarrow S^{-1}B$ (where B is considered as an A -module by means of f) such that $j(b/f(s)) = b/s$ for all $b \in B$, $s \in S$. If $\tilde{f} : S^{-1}A \rightarrow f(S)^{-1}B$ is the ring homomorphism associated with f , then $j \circ \tilde{f} = S^{-1}f$. The map j is an isomorphism of the $S^{-1}A$ -module structure on $f(S)^{-1}B$ defined by \tilde{f} onto that on $S^{-1}B$ and also of the B -module structure on $f(S)^{-1}B$ onto that on $S^{-1}B$ (resulting from the definition $S^{-1}B = S^{-1}A \otimes_A B$).*

Proof. If $b, b' \in B$ and $s, s' \in S$, the conditions $b/s = b'/s'$ and $b/f(s) = b'/f(s')$ are equivalent, as follows from ??, which establishes the existence of j and shows that j is bijective. The uniqueness of j is obvious. Clearly j is an additive group isomorphism. If $a \in A$, $b \in B$, $s, t \in S$ then

$$(a/s) \cdot (b/f(t)) = \tilde{f}(a/s)(b/f(t)) = f(a)/f(s)(b/f(t)) = (f(a)b)/f(st)$$

from which it follows that j is $S^{-1}A$ -linear. Clearly $j \circ \tilde{f} = S^{-1}f$. Finally, if $b, b' \in B$, $s \in S$, then

$$j(b \cdot (b'/f(s))) = j(bb'/f(s)) = bb'/s = b \cdot (b'/s),$$

which proves the last assertion. \square

The map j of ?? is called the canonical isomorphism of $f(S)^{-1}B$ onto $S^{-1}B$. These two sets are in general identified by means of f . Then $\tilde{f} = S^{-1}f$ and $i_B^S = i_B^{f(S)}$.

9.2.3 Change of multiplicative subset

Let A be a ring, S a multiplicative subset of A and M an A -module. If T is a multiplicative subset of A containing S , it follows from ?? that there exists a unique $S^{-1}A$ -linear map $i_M^{T,S} : S^{-1}M \rightarrow T^{-1}M$ such that $i_M^T = i_M^{T,S} \circ i_M^S$. The map i_M^S maps the element m/s of $S^{-1}M$ to the element m/s of $T^{-1}M$. It is easily verified that $i_M^{T,S} = i_A^{T,S} \otimes 1_M$. If U is a third multiplicative subset of A such that $T \subseteq U$, then $i_M^{U,S} = i_M^{U,T} \circ i_M^{T,S}$. Moreover, if $\phi : M \rightarrow N$ is an A -module homomorphism, the diagram

$$\begin{array}{ccc} S^{-1}M & \xrightarrow{s^{-1}\phi} & S^{-1}N \\ \downarrow i_M^{T,S} & & \downarrow i_N^{T,S} \\ T^{-1}M & \xrightarrow{T^{-1}\phi} & T^{-1}N \end{array}$$

is commutative.

Proposition 9.2.20. *Let A be a ring and S, T two multiplicative subsets of A .*

- (a) There exists a unique isomorphism i_A from the ring $(ST)^{-1}A$ onto the ring $(S^{-1}T)^{-1}(S^{-1}A)$ such that the diagram

$$\begin{array}{ccc} A & \xrightarrow{i_A^S} & S^{-1}A \\ i_A^{ST} \downarrow & & \downarrow i_{S^{-1}A}^{S^{-1}T} \\ (ST)^{-1}A & \xrightarrow{i} & (S^{-1}T)^{-1}(S^{-1}A) \end{array}$$

- (b) Let M be an A -module. There exists an $(ST)^{-1}A$ -isomorphism i_M from the $(ST)^{-1}A$ -module $(ST)^{-1}M$ onto the $(S^{-1}T)^{-1}(S^{-1}A)$ -module $(S^{-1}T)^{-1}(S^{-1}M)$ such that the diagram

$$\begin{array}{ccc} M & \xrightarrow{i_M^S} & S^{-1}M \\ i_M^{ST} \downarrow & & \downarrow i_{S^{-1}M}^{S^{-1}T} \\ (ST)^{-1}M & \xrightarrow{i_M} & (S^{-1}T)^{-1}(S^{-1}M) \end{array}$$

Proof. It suffices to prove (a). We use the definition of $(ST)^{-1}A$ as the solution of a universal map problem. Let B be a ring and f a homomorphism from A to B such that $f(ST)$ consists of invertible elements. As $f(S)$ consequently consists of invertible elements, there exists a unique homomorphism $f_1 : S^{-1}A \rightarrow B$ such that $f = f_1 \circ i_A^S$. For all $t \in T$, $f_1(i_A^S(t)) = f(t)$ is invertible in B by hypothesis, hence $f_1(S^{-1}T)$ consists of invertible elements; then there exists a unique homomorphism f_2 from $(S^{-1}T)^{-1}(S^{-1}A)$ to B such that $f_1 = f_2 \circ i_{S^{-1}A}^{S^{-1}T}$, whence $f = f_2 \circ i$, where $i = i_{S^{-1}A}^{S^{-1}T} \circ i_A^S$.

Moreover, if $f'_2 : (S^{-1}T)^{-1}(S^{-1}A) \rightarrow B$ is a second homomorphism such that $f'_2 \circ i = f$, then $(f'_2 \circ i_{S^{-1}A}^{S^{-1}T}) \circ i_A^S = (f_2 \circ i_{S^{-1}A}^{S^{-1}T}) \circ i_A^S$, whence $f'_2 \circ i_{S^{-1}A}^{S^{-1}T} = f_2 \circ i_{S^{-1}A}^{S^{-1}T}$ and consequently $f'_2 = f_2$. As the images under i of the elements of ST are invertible, the ordered pair $((S^{-1}T)^{-1}(S^{-1}A), i)$ is a solution of the universal map problem (relative to A and ST). This shows the existence and uniqueness of i_A . \square

Corollary 9.2.21. If S and T are two multiplicative subsets of A with $S \subseteq T$, then writing $S^{-1}T$ for the image of T in $S^{-1}A$, we have $(S^{-1}T)^{-1}(S^{-1}A) = T^{-1}A$.

Corollary 9.2.22. If $S \subseteq A$ is a multiplicative set and \mathfrak{p} is a prime ideal of A disjoint from S then $(S^{-1}A)_{S^{-1}\mathfrak{p}} = A_{\mathfrak{p}}$. In particular if $\mathfrak{p}_1 \subseteq \mathfrak{p}_2$ are prime ideals of A , then $(A_{\mathfrak{p}_2})_{\mathfrak{p}_1} = A_{\mathfrak{p}_1}$.

Proposition 9.2.23. Let S, T be multiplicatively closed subsets of A , such that $S \subseteq T$. Then the following statements are equivalent:

- (a) The homomorphism $i_A^{T,S}$ is bijective.
- (b) For every A -module M , the homomorphism $i_M^{T,S} : S^{-1}M \rightarrow T^{-1}M$ is bijective.
- (c) For each $t \in T$, $t/1$ is a unit in $S^{-1}A$.
- (d) For each $t \in T$ there exists $x \in A$ such that $xt \in S$.
- (e) Every prime ideal which meets T also meets S .

Proof. It has been seen above that $i_M^{T,S} = i_A^{T,S} \otimes 1_M$, which immediately proves the equivalence of (a) and (b). Also, the equivalence of (c) and (d) is easy to see. By ??, since $S \subseteq T$, $T^{-1}A$ is identified with $(S^{-1}T)^{-1}(S^{-1}A)$ and (a) is equivalent to saying that the elements of $S^{-1}T$ are invertible in $S^{-1}A$, so (a) is equivalent to (c). Now by definition, it is clear that (d) implies (e).

So assume (e). To show the injectivity of $i_A^{T,S}$, let $i_A^{T,S}(x/s) = 0$ in $T^{-1}A$, so that $xt = 0$ for some $t \in T$. Now assume that $x/s \neq 0$, which is to say $\text{Ann}(x) \cap S = \emptyset$. Then by ??, there is a prime ideal $\mathfrak{p} \supseteq \text{Ann}(x)$ such that $\mathfrak{p} \cap S = \emptyset$. By assumption, we also have $\mathfrak{p} \cap T = \emptyset$, which is a contradiction. This proves $x/s = 0$, so $i_A^{T,S}$ is injective.

For the surjectivity, let $t \in T$. We claim that there exists $x \in A$ such that $xt \in S$. Otherwise, $(t) \cap S = \emptyset$, and by ??, there is a prime ideal $\mathfrak{p} \supseteq (t)$ such that $\mathfrak{p} \cap S = \emptyset$. Then we have $\mathfrak{p} \cap T = \emptyset$, which is a contradiction. \square

By ??, we see that amongst the multiplicative subsets T of A containing S and satisfying the equivalent conditions of ??, there exists a greatest, consisting of all the elements of A which divide an element of S . This set can be characterized by another property, which we now define.

A multiplicative subset S of a ring A is called **saturated** if the relation $xy \in S$ implies $x \in S$ and $y \in S$. If this holds, then for any $x \in A \setminus S$ and any $s \in S$ we have $ax \notin S$, which shows $x/1$ is not a unit in $S^{-1}A$, hence there is a maximal ideal \mathfrak{m} contains $x/1$. Now consider the contraction \mathfrak{m}^c in A . It contains x , and is prime since \mathfrak{m} is prime. Hence we see x is contained in a prime ideal \mathfrak{p} disjoint from A , thus $A \setminus S$ is a union of prime ideals of A . In fact this condition is also sufficient: if $A \setminus S = \bigcup \mathfrak{p}_i$ for prime ideals \mathfrak{p}_i , then assume $xy \in S$. If $x \in A \setminus S$, then $x \in \mathfrak{p}_i$ for some \mathfrak{p}_i . Since \mathfrak{p}_i is an ideal, $xy \in \mathfrak{p}_i$ for any $y \in A$, which means $xy \in A \setminus S$, a contradiction. So $x, y \in S$.

Proposition 9.2.24. *Let S be a multiplicative subset of A and define*

$$\bar{S} = \{x \in A : \text{there exists } y \in S \text{ such that } xy \in S\}.$$

*Then S is the smallest saturated multiplicative subset containing S , called the **saturation** of S , and $A \setminus \bar{S}$ is the union of the prime ideals of A not meeting S .*

Proof. It is easy to see \bar{S} is a multiplicative subset and saturated. In particular, $A \setminus \bar{S} = \bigcup_i \mathfrak{p}_i$, where \mathfrak{p}_i are prime ideals. Now if \mathfrak{p} is a prime ideal such that $\mathfrak{p} \cap S = \emptyset$, then for any $x \in \mathfrak{p}$, if there exists $y \in S$ such that $xy \in S$, then $xy \in \mathfrak{p} \cap S$, which is a contradiction. This shows $\mathfrak{p} \subseteq A \setminus S$, which finishes the proof. \square

Corollary 9.2.25. *Let S, T be multiplicatively closed subsets of A , such that $S \subseteq T$. Then $i_M^{T,S}$ is bijective if and only if T is contained in the saturation of S .*

Proposition 9.2.26. *Let A be a ring, S be a multiplicative subset of A . If B is a subring of $S^{-1}A$ containing $i_A^S(A)$, then we have*

$$S^{-1}A = S^{-1}B = T^{-1}B \quad \text{where} \quad T = \{b \in B : b \text{ is a unit of } S^{-1}A\}.$$

Proof. By definition of T we have $S \subseteq T$ and $T^{-1}B$ can be identified with a subring of $S^{-1}A$.

We can write

$$\begin{array}{ccc}
 A & \xrightarrow{i_A^S} & S^{-1}A \\
 \downarrow & & \downarrow i \\
 B & \xrightarrow{i_B^T} & T^{-1}B \\
 \downarrow & \nearrow j & \\
 S^{-1}A & &
 \end{array}$$

where i and j the natural maps. Also, by definition we have $i \circ j = 1_{S^{-1}A}$. Also, by the definition of T and ?? we see $S^{-1}B = T^{-1}B$. Now, for any $b/t \in T^{-1}B$, we have $t^{-1} = a/s$ where $a \in A$, $s \in S$. Since $b \in B \subseteq S^{-1}A$, we then see

$$b/t = i(b) \cdot i(a/s) = i(b \cdot a/s)$$

whence i is surjective. Thus i and j are mutually inverse, giving an isomorphism $S^{-1}A \cong T^{-1}B$. \square

Proposition 9.2.27. *Let M be an A -module. For every submodule N' of the $S^{-1}A$ -module $S^{-1}M$, let $\phi(N')$ be the inverse image of N' under i_M^S . Then:*

- (a) $S^{-1}\phi(N') = N'$.
 - (b) *For every submodule N of M , the submodule $\phi(S^{-1}N)$ of M consists of those $m \in M$ for which there exists $s \in S$ such that $sm \in N$.*
 - (c) *ϕ is an isomorphism of the set of sub- $S^{-1}A$ -modules of $S^{-1}M$ onto the set of submodules Q of M which satisfy the following condition:*
- (MS) *If $sm \in Q$, where $s \in S$, $m \in M$, then $m \in Q$.*

Proof. Obviously $S^{-1}\phi(N') \subseteq N'$. Conversely, if $n' = m/s \in N'$, then $m/1 \in N'$, hence in $m \in \phi(N')$ and consequently $n' \in S^{-1}(\phi(N'))$, whence (a). For an element $m \in M$ to be such that $m \in \phi(S^{-1}N)$, it is necessary and sufficient that $m/1 \in S^{-1}N$, that is, there exist $s \in S$ and $n \in N$ such that $m/1 = n/s$, which means there exists $s' \in S$ such that $s'sm = s'n \in N$, whence (b). Finally, the relation $sm \in \phi(N')$ is equivalent by definition to $sm/1 \in N'$ and as $s/1$ is invertible in $S^{-1}A$, this implies $m/1 \in N'$, or $m \in \phi(N')$, hence $\phi(N')$ satisfies condition (MS). On the other hand, it follows from (b) that, if N satisfies (MS), then $\phi(S^{-1}N) = N$, which completes the proof of (c). \square

The submodule $\phi(S^{-1}N)$ is called the **saturation** of N in M with respect to S , and the submodules satifying condition (MS) (and hence equal to their saturations) are said to be **saturated** with respect to S . The submodule $\phi(S^{-1}N)$ is the kernel of the composite homomorphism

$$M \xrightarrow{\pi} M/N \xrightarrow{i_{M/N}^S} S^{-1}M/S^{-1}N$$

where π is the canonical homomorphism, as follows from the commutativity of the diagram

$$\begin{array}{ccc} M & \xrightarrow{\pi} & M/N \\ \downarrow i_M^S & & \downarrow i_{M/N}^S \\ S^{-1}M & \longrightarrow & S^{-1}M/S^{-1}N \end{array}$$

If S is the complement in A of a prime ideal \mathfrak{p} , $\phi(S^{-1}N)$ is also called the saturation of N in M with respect to \mathfrak{p} .

Corollary 9.2.28. *Let N_1, N_2 be two submodules of an A -module M . For $S^{-1}N_1 \subseteq S^{-1}N_2$, it is necessary and sufficient that the saturation of N_1 with respect to S be contained in that of N_2 .*

Corollary 9.2.29. *If M is a Noetherian (resp. Artinian) A -module, then $S^{-1}M$ is a Noetherian (resp. Artinian) $S^{-1}A$ -module. In particular, if the ring A is Noetherian (resp. Artinian), so is the ring $S^{-1}A$.*

9.2.4 Properties of the localization operation

Proposition 9.2.30. *Let A be a ring and S a multiplicative subset of A . Then $S^{-1}A$ is A -flat.*

Proof. If $\phi : M \rightarrow N$ is an injective homomorphism of A -modules, it is necessary to establish that $S^{-1}\phi : S^{-1}M \rightarrow S^{-1}N$ is injective. Now, if m/s is such that $\phi(m)/s = 0$, this implies the existence of a $t \in S$ such that $t\phi(m) = 0$ or $\phi(tm) = 0$. As ϕ is injective, it follows that $tm = 0$, whence $m/s = 0$ in $S^{-1}M$. \square

Corollary 9.2.31. *Formation of fractions commutes with formation of finite sums, finite intersections and quotients. Precisely, if N, P are submodules of an A -module M , then*

- (a) $S^{-1}(N + P) = S^{-1}N + S^{-1}P$.
- (b) $S^{-1}(N \cap P) = S^{-1}N \cap S^{-1}P$.
- (c) *The $S^{-1}A$ -modules $S^{-1}(M/N)$ and $(S^{-1}M)/(S^{-1}N)$ are isomorphic.*

Proof. Part (a) follows readily from the definitions and (b) is easy to verify: if $y/s = z/t$ ($y \in N$, $z \in P$, $s, t \in S$) then $u(ty - sz) = 0$ for some $u \in S$, hence $w = uty = usz \in N \cap P$ and therefore $y/s = w/stu \in S^{-1}(N \cap P)$. Consequently $S^{-1}(N \cap P) \subseteq S^{-1}(N) \cap S^{-1}P$, and the reverse inclusion is obvious.

Now to prove (c), we apply S^{-1} to the exact sequence

$$0 \longrightarrow N \longrightarrow M \longrightarrow M/N \longrightarrow 0$$

Since localization is exact, we get the claim. \square

Proposition 9.2.32. *Let M be a finitely generated A -module, S a multiplicatively closed subset of A . Then $S^{-1}\text{Ann}(M) = \text{Ann}(S^{-1}M)$.*

Proof. If this is true for two A -modules, M, N , it is true for $M + N$:

$$S^{-1}(\text{Ann}(M + N)) = S^{-1}(\text{Ann}(M) \cap \text{Ann}(N)) = \text{Ann}(S^{-1}M) \cap \text{Ann}(S^{-1}N)$$

$$= \text{Ann}(S^{-1}M + S^{-1}N) = \text{Ann}(S^{-1}(M + N))$$

where we use $\text{Ann}(M)$. Hence it is enough to prove for M generated by a single element: then $M = A/I$ as A -module, where $\mathfrak{a} = \text{Ann}(M)$. We have

$$S^{-1}(M) = S^{-1}(A/\mathfrak{a}) \cong (S^{-1}A)/(S^{-1}\mathfrak{a})$$

by ??, which proves the claim.

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Corollary 9.2.33. If N, P are submodules of an A -module M and if P is finitely generated, then $S^{-1}(N : P) = (S^{-1}N : S^{-1}P)$.

Proof. We have $(N : P) = \text{Ann}((N + P)/P)$, so the claim follows by ??.

□

Proposition 9.2.34. Let I be a directed set, $(S_i)_{i \in I}$ an increasing family of multiplicative subsets of a ring A , and $S = \bigcup_i S_i$. We write $\rho_{\beta\alpha} = i_A^{S_\beta, S_\alpha}$. Then $(S_\alpha^{-1}A, \rho_{\beta,\alpha})$ is a direct system of rings and, if for $\alpha \in I$, ρ_α is the canonical map of $S_\alpha^{-1}A$ to $\varinjlim S_\alpha^{-1}A$, there exists a unique isomorphism $i : \varinjlim S_\alpha^{-1}A \rightarrow S^{-1}A$ such that $i \circ \rho_\alpha = i_A^{S_\alpha, S}$ for all $\alpha \in I$.

$$\begin{array}{ccccccc}
 S_\alpha^{-1}A & \xrightarrow{\rho_{\beta\alpha}} & S_\beta^{-1}A & \xrightarrow{\rho_{\gamma\beta}} & S_\gamma^{-1}A & \longrightarrow & \dots \\
 & \searrow & \swarrow & \searrow & \downarrow & & \\
 & & \rho_\alpha & \rho_\beta & \rho_\gamma & & \\
 & & & & & \nearrow & \\
 & & & & & & \lim_{\longrightarrow} S_\alpha^{-1}A \xrightarrow{i} S^{-1}A
 \end{array}$$

Proof. For $\alpha \preceq \beta \preceq \gamma$, $\rho_{\gamma\alpha} = \rho_{\gamma\beta} \circ \rho_{\beta\alpha}$, hence $(S_\alpha^{-1}A, \rho_{\beta\alpha})$ is a direct system of rings, and we write $B = \varinjlim S_\alpha^{-1}A$. As

$$i_A^{S,S_\alpha} = i_A^{S,S_\beta} \circ \rho_{\beta\alpha}$$

for $\alpha \preceq \beta$, (i_A^{S, S_α}) is a direct system of homomorphisms and $i = \varinjlim i_A^{S, S_\alpha}$ is the unique map such that $i \circ \rho_\alpha = i_A^{S, S_\alpha}$ for all $\alpha \in I$.

The homomorphisms $\rho_\alpha \circ i_A^{S_\alpha}$ are all equal since $\rho_{\alpha\beta} \circ i_A^{S_\alpha} = i_A^{S_\beta}$ for $\alpha \preceq \beta$, so let $\phi : A \rightarrow B$ be their common value. Clearly elements of $\phi(S)$ are invertible in B , which shows there exists a homomorphism $\tilde{\phi} : S^{-1}A \rightarrow B$ such that $\tilde{\phi} \circ i_A^S = \phi$. Then

$$i \circ \tilde{\phi} \circ i_A^S = i \circ \phi = i \circ \rho_\alpha \circ i_A^{S_\alpha} = i_A^{S, S_\alpha} \circ i_A^{S_\alpha} = i_A^S$$

for all $\alpha \in I$ and consequently $i \circ \tilde{\phi}$ is the identity map of $S^{-1}A$. On the other hand, for all $\alpha \in I$,

$$\tilde{\phi} \circ i \circ \rho_\alpha \circ i_A^{S_\alpha} = \tilde{\phi} \circ i_A^{S, S_\alpha} \circ i_A^{S_\alpha} = \tilde{\phi} \circ i_A^S = \phi = \rho_\alpha \circ i_A^{S_\alpha}$$

whence $\tilde{\phi} \circ i \circ \rho_\alpha = \rho_\alpha$ for all $\alpha \in I$; it follows that $\tilde{\phi} \circ i$ is the identity automorphism of B and consequently i is an isomorphism. \square

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Corollary 9.2.35. Under the hypotheses of ??, let M be an A -module. We write $\tau_{\beta\alpha} = i_M^{S_\beta, S_\alpha}$ for $\alpha \preceq \beta$. Then $(S_\alpha^{-1}M, \tau_{\beta,\alpha})$ is a direct system of rings and, if for $\alpha \in I$, τ_α is the canonical map of $S_\alpha^{-1}M$ to $\varinjlim S_\alpha^{-1}M$, there exists a unique isomorphism $i : \varinjlim S_\alpha^{-1}M \rightarrow S^{-1}M$ such that $i \circ \tau_\alpha = i_M^{S_\alpha, S_\alpha}$ for all $\alpha \in I$.

Proof. The corollary follows immediately from the definitions $S_\alpha^{-1}M = M \otimes_A S_\alpha^{-1}A$ and $S^{-1}M = M \otimes_A S^{-1}A$ and the fact that taking direct limits commutes with tensor products. \square

9.2.5 Ideals under localization

Proposition 9.2.36. *Let S be a multiplicative subset of a commutative ring A , and consider the localization operation. Let \mathfrak{a} be an ideal in A , and \mathfrak{b} an ideal in $S^{-1}A$.*

- (a) $(\mathfrak{b}^c)^e = \mathfrak{b}$, hence every ideal in $S^{-1}A$ is an extended ideal.
- (b) $(\mathfrak{a}^e)^c = \bigcup_{s \in S} (\mathfrak{a} : s)$. Hence $S^{-1}\mathfrak{a}$ is proper if and only if $\mathfrak{a} \cap S \neq \emptyset$ and \mathfrak{a} is a contracted ideal if and only if no element of S is a zero-divisor in A/\mathfrak{a} .
- (c) The assignment $\mathfrak{p} \rightarrow S^{-1}\mathfrak{p}$ gives an inclusion-preserving bijection between the set of prime ideals of A disjoint from S and the set of ideals of $S^{-1}A$.
- (d) The operation S^{-1} commutes with formation of finite sums, products, intersections and radicals.

Proof. Recall that we always have $\mathfrak{b} \subseteq (\mathfrak{b}^c)^e$. Conversely, if $a/s \in \mathfrak{b}$ then $a/1 \in \mathfrak{b}$, so $a \in \mathfrak{b}^c$. It follows that $a/s \in S^{-1}(\mathfrak{b}^c) = (\mathfrak{b}^c)^e$, so $\mathfrak{b} = (\mathfrak{b}^c)^e$. For (b), by definition we have

$$\begin{aligned} (\mathfrak{a}^e)^c &= \{a \in A : a/1 \in \mathfrak{a}^e\} = \{a \in A : (sa - r)u = 0 \text{ for some } r \in I \text{ and } u, s \in S\} \\ &= \{a \in A : (\exists s \in S) sx \in \mathfrak{a}\} = \bigcup_{s \in S} (\mathfrak{a} : s). \end{aligned}$$

From this, we see $\mathfrak{a}^e = (\bigcup_{s \in S} (\mathfrak{a} : s))^e$, and therefore $1 \in \mathfrak{a}^e$ if and only if $\mathfrak{a} \cap S \neq \emptyset$. From the description of \mathfrak{a}^{ec} , we also find that $\mathfrak{a}^{ec} = \mathfrak{a}$ if and only if $\bigcup_{s \in S} (\mathfrak{a} : s) \subseteq \mathfrak{a}$, if and only if no element of S is a zero-divisor in A/\mathfrak{a} . This proves (b).

If \mathfrak{p} is prime in A such that $\mathfrak{p} \cap S = \emptyset$, then we see that $(\mathfrak{p} : s) = \mathfrak{p}$ for all $s \in S$, whence $(\mathfrak{p}^e)^c = \mathfrak{p}$. Thus $(\mathfrak{p}^e)^c$ is a bijection on prime ideals of A . If \mathfrak{p} is prime in $S^{-1}A$, then \mathfrak{p}^c is prime in A . Further, by part (b) we have $\mathfrak{p}^c \cap S = \emptyset$.

If \mathfrak{p} is prime in A such that $\mathfrak{p} \cap S = \emptyset$, then $S^{-1}\mathfrak{p}$ is a proper ideal, as we see in (b). Now we prove the extension $\mathfrak{p}^e = S^{-1}\mathfrak{p}$ is prime. Let $x/s, y/t \notin S^{-1}\mathfrak{p}$ but $xy/st \in S^{-1}\mathfrak{p}$. Then by definition for some $p \in \mathfrak{p}, u \in S$ we have

$$u(xy - pst) = 0.$$

That is, $uxy = ustp \in \mathfrak{p}$. Since $\mathfrak{p} \cap S = \emptyset$, we must have $xy \in \mathfrak{p}$. But since $x/s, y/t \notin S^{-1}\mathfrak{p}$, $x \notin \mathfrak{p}$ and $y \notin \mathfrak{p}$. This is a contradiction. Thus the extension and contraction corresponds the prime ideals in $S^{-1}A$ and prime ideals in A disjoint from S .

Finally, we deal with (e). For sums and products, this follows from ??; for intersections, from Corollary ???. As to radicals, we have $S^{-1}\sqrt{\mathfrak{a}} \subseteq \sqrt{S^{-1}\mathfrak{a}}$ from ?? since $S^{-1}\mathfrak{a} = \mathfrak{a}^e$. For the converse, let $x/s \in \sqrt{S^{-1}\mathfrak{a}}$, then $x^n/s^n \in S^{-1}\mathfrak{a}$ for some $n > 0$. This means $ux^n \in \mathfrak{a}$ for some $u \in S$. Then $u^n x^n$ also belongs to \mathfrak{a} , so $ux \in \sqrt{\mathfrak{a}}$. Moreover, since $x/s = (ux)/(us)$, we get $x/s \in S^{-1}\sqrt{\mathfrak{a}}$. \square

Corollary 9.2.37. *There is an inclusion-preserving bijection between the prime ideals of $A_{\mathfrak{p}}$ and the prime ideals of A contained in \mathfrak{p} . Hence $A_{\mathfrak{p}}$ is a local ring.*

Proof. If $a, b \notin \mathfrak{p}$, then $ab \notin \mathfrak{p}$. So $A - \mathfrak{p}$ is a multiplicative set. The claim now comes from ??.

The elements a/s with $a \in \mathfrak{p}$ form an ideal \mathfrak{m} in $A_{\mathfrak{p}}$. If $b/t \notin \mathfrak{m}$, then $b \notin \mathfrak{p}$, so $b \in A - \mathfrak{p}$ and is a unit in $A_{\mathfrak{p}}$. By ??, $A_{\mathfrak{p}}$ is a local ring.

Or we can use the correspondence to prove $S^{-1}\mathfrak{p}$ is the unique maximal ideal. \square

Corollary 9.2.38. *If $\mathfrak{n}(A)$ is the nilradical of A , the nilradical of $S^{-1}A$ is $S^{-1}\mathfrak{n}(A)$.*

Remark 9.2.2. Thus the passage from A to $A_{\mathfrak{p}}$ cuts out all prime ideals except those contained in \mathfrak{p} . In the other direction, the passage from A to A/\mathfrak{p} cuts out all prime ideals except those containing \mathfrak{p} . Hence if $\mathfrak{p}, \mathfrak{q}$ are prime ideals such that $\mathfrak{p} \supseteq \mathfrak{q}$, then by localizing with respect to \mathfrak{p} and taking the quotient mod \mathfrak{q} (in either order: these two operations commute, by Corollary ??), we restrict our attention to those prime ideals which lie between \mathfrak{p} and \mathfrak{q} . In particular, if $\mathfrak{p} = \mathfrak{q}$ we end up with a field, called the **residue field at \mathfrak{p}** , which can be obtained either as the field of fractions of the integral domain A/\mathfrak{p} or as the residue field of the local ring $A_{\mathfrak{p}}$.

Proposition 9.2.39. *Let A be a ring and S a multiplicative subset of A . For every ideal \mathfrak{b} of $S^{-1}A$, let $\mathfrak{a} = (i_A^S)^{-1}(\mathfrak{b})$ be such that $S^{-1}\mathfrak{a} = \mathfrak{b}$.*

- (a) *Let π be the canonical homomorphism $A \rightarrow A/\mathfrak{a}$. Then the homomorphism from $S^{-1}A$ to $\pi(S)^{-1}(A/\mathfrak{a})$ canonically associated with π is surjective and its kernel is \mathfrak{b} , which defines, by taking quotients, a canonical isomorphism of $(S^{-1}A)/\mathfrak{b}$ onto $\pi(S)^{-1}(A/\mathfrak{a})$. Moreover, the canonical homomorphism from A/\mathfrak{a} to $\pi(S)^{-1}(A/\mathfrak{a})$ is injective.*
- (b) *If \mathfrak{q} is a prime ideal of $S^{-1}A$ and $\mathfrak{p} = (i_A^S)^{-1}(\mathfrak{q})$, then there exists an isomorphism of the ring of fractions $A_{\mathfrak{p}}$ onto the ring $(S^{-1}A)_{\mathfrak{q}}$, which maps a/b to $(a/1)/(b/1)$, where $a \in A$, $b \in A - \mathfrak{p}$.*

Proof. The ring $\pi(S)^{-1}(A/\mathfrak{a})$ can be identified with $S^{-1}(A/\mathfrak{a})$ by means of the canonical isomorphism between these two modules. The exact sequence $0 \rightarrow \mathfrak{a} \rightarrow A \rightarrow A/\mathfrak{a} \rightarrow 0$ then induces an exact sequence

$$0 \longrightarrow S^{-1}\mathfrak{a} \longrightarrow S^{-1}A \longrightarrow S^{-1}(A/\mathfrak{a}) \longrightarrow 0$$

whose existence proves the first assertion of (a), taking account of the fact that $\mathfrak{b} = S^{-1}\mathfrak{a}$. Since \mathfrak{a} is saturated with respect S , the conditions $a \in A$, $s \in S$, $as \in \mathfrak{a}$ imply $a \in \mathfrak{a}$; the homothety of ratio s on A/\mathfrak{a} is then injective, which proves the second assertion of (i).

Suppose that \mathfrak{q} is prime and such that \mathfrak{p} is also prime. The set $T = A - \mathfrak{p}$ is a multiplicative subset of A which contains S , whence $ST = T$. Then it follows from ?? that there exists a unique isomorphism i of $T^{-1}A = A_{\mathfrak{p}}$ onto $(S^{-1}T)^{-1}(S^{-1}A)$ such that

$$i(a/b) = (a/1)/(b/1)$$

where $a \in A$ and $b \in T$. On the other hand $S^{-1}T$ obviously does not meet \mathfrak{q} . Conversely, let $a/s \in S^{-1}A$; since $1/s$ is invertible in $S^{-1}A$, the condition $a/s \notin \mathfrak{q}$ is equivalent to $i_A^S(a) \notin \mathfrak{q}$ and hence to $a \notin \mathfrak{p}$. It follows that $S^{-1}A - \mathfrak{q} = S^{-1}T$ and hence $(S^{-1}T)^{-1}(S^{-1}A) = (S^{-1}A)_{\mathfrak{q}}$. \square

Proposition 9.2.40. *Let $\rho : A \rightarrow B$ be a ring homomorphism and let \mathfrak{p} be a prime ideal of A . Then \mathfrak{p} is the contraction of a prime ideal of B if and only if $\mathfrak{p}^{ec} = \mathfrak{p}$.*

Proof. One direction is trivial. Conversely, suppose that $\mathfrak{p}^{ec} = \mathfrak{p}$ and consider the multiplicative subset $S = \rho(A - \mathfrak{p})$ of B . The hypothesis shows that $S \cap \mathfrak{p}^e = \emptyset$, so there exists a prime ideal \mathfrak{q} of B disjoint with S such that $\mathfrak{p}^e \subseteq \mathfrak{q}$. Then \mathfrak{q}^c contains \mathfrak{p} and is disjoint with $A - \mathfrak{p}$, whence $\mathfrak{q}^c = \mathfrak{p}$. \square

Corollary 9.2.41. *Let $\rho : A \rightarrow B$ be a ring homomorphism.*

- (a) *Suppose that there exists a B -module E such that $\rho^*(E)$ is a faithfully flat A -module. Then, for every prime ideal \mathfrak{p} of A , there exists a prime ideal \mathfrak{P} of B such that $\mathfrak{P}^c = \mathfrak{p}$.*
- (b) *Conversely, suppose that B is a flat A -module. If for every prime ideal \mathfrak{p} of A , there exists an ideal \mathfrak{P} of B such that $\mathfrak{P}^c = \mathfrak{p}$, then B is a faithfully flat A -module.*

Proof. The hypothesis in (a) implies that, for every ideal \mathfrak{a} of A , we have $\mathfrak{a}^{ec} = \mathfrak{a}$ (??), and it is sufficient to apply ?? . Now assume the conditions in (b). It is sufficient to show that, for every maximal ideal \mathfrak{m} of A , there exists a maximal ideal \mathfrak{M} of B such that $\mathfrak{M}^c = \mathfrak{m}$ (??). Now there exists by hypothesis an ideal \mathfrak{Q} of B such that $\mathfrak{Q}^c = \mathfrak{m}$. As $\mathfrak{Q} \neq B$, there exists a maximal ideal \mathfrak{M} of B containing \mathfrak{Q} and consequently $\mathfrak{M}^c = \mathfrak{m}$ since \mathfrak{M}^c cannot contain 1. \square

Corollary 9.2.42. *Let $\rho : A \rightarrow B$ be a ring homomorphism such that B is a flat A -module. Let \mathfrak{P} be a prime ideal of B and let $\mathfrak{p} = \mathfrak{P}^c$. Then ${}^a\rho_{\mathfrak{p}} : \text{Spec}(B_{\mathfrak{P}}) \rightarrow \text{Spec}(A_{\mathfrak{p}})$ is surjective.*

Proof. The map $\rho_{\mathfrak{p}} : A_{\mathfrak{p}} \rightarrow B_{\mathfrak{P}}$ is a local homomorphism since we have

$$(\mathfrak{p}A_{\mathfrak{p}})^e = \left(\frac{\mathfrak{p}}{A - \mathfrak{p}} \right)^e = \frac{\mathfrak{p}^e}{f(A - \mathfrak{p})} = \frac{\mathfrak{P}^{ce}}{f(A - \mathfrak{p})} \subseteq \mathfrak{P}B_{\mathfrak{P}} \neq B_{\mathfrak{P}},$$

so $B_{\mathfrak{P}}$ is a faithfully flat $A_{\mathfrak{p}}$ -module by ?? , and the claim follows from ?? . \square

Proposition 9.2.43. *Let B be a ring and A a subring of B . For every minimal prime ideal \mathfrak{p} of A there exists a minimal prime ideal \mathfrak{P} of B such that $\mathfrak{P} \cap A = \mathfrak{p}$.*

Proof. Let \mathfrak{p} be a minimal prime of A and $S = A - \mathfrak{p}$. Then the ring $A_{\mathfrak{p}}$ is identified with a subring of $S^{-1}B$ and it has a single prime ideal $\mathfrak{p}A_{\mathfrak{p}}$ since \mathfrak{p} is minimal. As $S^{-1}B$ is not reduced to 0 (since it contains $A_{\mathfrak{p}}$), it has at least one prime ideal \mathcal{P} and therefore $\mathcal{P} \cap A_{\mathfrak{p}} = \mathfrak{p}A_{\mathfrak{p}}$. If $\mathfrak{P}' = (i_B^S)^{-1}(\mathcal{P})$, we have

$$i_A^S(\mathcal{P} \cap A) \subseteq \mathcal{P} \cap A_{\mathfrak{p}} = \mathfrak{p}A_{\mathfrak{p}},$$

hence $\mathfrak{P}' \cap A \subseteq \mathfrak{p}$, and as \mathfrak{p} is minimal, $\mathfrak{P}' \cap A = \mathfrak{p}$. If \mathfrak{P} is a minimal prime ideal of B contained in \mathfrak{P}' , then we also have $\mathfrak{P} \cap A = \mathfrak{p}$, whence the claim. \square

9.2.6 Localization of tensor products and Hom sets

Proposition 9.2.44. *Let A be a ring and S a multiplicative subset of A .*

- (a) *If M and N are two A -modules, then there is a canonical isomorphism*

$$S^{-1}M \otimes_{S^{-1}A} S^{-1}N \cong S^{-1}(M \otimes_A N).$$

- (b) *If M and N are two $S^{-1}A$ -modules, the canonical homomorphism $M \otimes_A N \rightarrow M \otimes_{S^{-1}A} N$ derived from the A -bilinear map $(x, y) \mapsto x \otimes y$ is bijective.*

Proof. Assertion (a) is an immediate consequence of the definition $S^{-1}M = M \otimes_A S^{-1}A$ and the associativity of tensor products. To prove (b), we note first that in M and N , considered as A -modules, the homotheties induced by the elements $s \in S$ are bijective, hence $M = S^{-1}M$ and $N = S^{-1}N$ and similarly $S^{-1}(M \otimes_A N) = M \otimes_A N$. Then (b) is then a special case of (a). \square

Corollary 9.2.45. *Let M be an A -module and \mathfrak{a} an ideal of A . The sub- $S^{-1}A$ -modules*

$$(S^{-1}\mathfrak{a})(S^{-1}M), \quad \mathfrak{a}(S^{-1}M), \quad (S^{-1}\mathfrak{a})i_M^S(M), \quad S^{-1}(\mathfrak{a}M)$$

of $S^{-1}M$ are identical. In particular, if \mathfrak{a} and \mathfrak{b} are two ideals of A , then

$$(S^{-1}\mathfrak{a})(S^{-1}\mathfrak{b}) = \mathfrak{a}(S^{-1}\mathfrak{b}) = (S^{-1}\mathfrak{a})\mathfrak{b} = S^{-1}(\mathfrak{a}\mathfrak{b}).$$

Proposition 9.2.46. *Let A be a ring and S a multiplicative subset of A .*

- (a) *If M and N are two A -modules and M is finitely generated (resp. finitely presented), the canonical homomorphism*

$$S^{-1}\text{Hom}_A(M, N) \rightarrow \text{Hom}_{S^{-1}A}(S^{-1}M, S^{-1}N)$$

is injective (resp. bijective).

- (b) *If M and N are two $S^{-1}A$ -modules, the canonical bijection*

$$\text{Hom}_{S^{-1}A}(M, N) \rightarrow \text{Hom}_A(M, N)$$

is bijective.

Proof. As $S^{-1}A$ is a flat A -module, (a) is a particular case of ???. On the other hand, we note that every A -homomorphism of $S^{-1}A$ -modules is necessarily $S^{-1}A$ -linear, whence (b). \square

Proposition 9.2.47. *Let A, B be two rings, $\rho : A \rightarrow B$ a homomorphism, S a multiplicative subset of A , and $\tilde{\rho} : S^{-1}A \rightarrow S^{-1}B$ the homomorphism corresponding to ρ .*

- (a) *For every B -module M there exists a unique $S^{-1}A$ -isomorphism*

$$i : S^{-1}\rho_*(M) \rightarrow \tilde{\rho}_*(S^{-1}M)$$

- (b) *For every A -module M , there exists a unique isomorphism*

$$j : (S^{-1}M) \otimes_{S^{-1}A} (S^{-1}B) \rightarrow S^{-1}(M \otimes_A B).$$

Proof. If we consider $S^{-1}M$ as an A -module by means of the composite homomorphism $i_M^S \circ \rho$, the homotheties induced by the elements of S are bijective, hence there exists a unique homomorphism i . It is clear that i is surjective; moreover, if $m \in M, s \in S$ and $m/\rho(s) = 0$, then there exists $t \in S$ such that $\rho(t)m = 0$, and so $tm = 0$ in $\rho_*(M)$, hence $m/s = 0$ in $S^{-1}\rho_*(M)$.

For (b), as $(S^{-1}M) \otimes_{S^{-1}A} (S^{-1}B) = (M \otimes_A S^{-1}A) \otimes_{S^{-1}A} S^{-1}B$, we see

$$S^{-1}(M \otimes_A B) = (M \otimes_A B) \otimes_B (S^{-1}B)$$

the existence of j follows from the associativity of tensor products. \square

9.3 Local rings and passing from local to global

9.3.1 Local rings

Proposition 9.3.1. *Let A be a ring and I the set of non-invertible elements of A . The set I is the union of the ideals of A which are distinct from A . Moreover, the following conditions are equivalent:*

- (i) I is an ideal.
- (ii) The set of ideals of A distinct from A has a greatest element.
- (iii) A has a unique maximal ideal.

Proof. The relation $x \in I$ is equivalent to $1 \notin xA$ and hence $xA \neq A$. If \mathfrak{a} is an ideal of A distinct from A and $x \in \mathfrak{a}$, then $xA \subseteq \mathfrak{a}$, hence $xA \neq A$ and $x \in I$. Hence every ideal of A distinct from A is contained in I and every element $x \in I$ belongs to a principal ideal $xA \neq A$. This proves the first assertion, which immediately implies the equivalence of (i), (ii) and (iii). \square

A ring A is called a **local ring** if it satisfies the equivalent conditions of ???. The quotient of A by its Jacobson radical (which is then the unique maximal ideal of A) is called the **residue field** of A . Let A, B be two local rings and $\mathfrak{m}, \mathfrak{n}$ their respective maximal ideals. A homomorphism $\rho : A \rightarrow B$ is called local if $\rho(\mathfrak{m}) \subseteq \mathfrak{n}$. Note that this amounts to saying that $\rho^{-1}(\mathfrak{n}) = \mathfrak{m}$, for $\rho^{-1}(\mathfrak{n})$ is then an ideal containing \mathfrak{m} and not containing 1 and hence equal to \mathfrak{m} . Taking quotients, we then derive canonically from ρ an injective homomorphism $A/\mathfrak{m} \rightarrow B/\mathfrak{n}$ from the residue field of A to that of B .

Example 9.3.2 (Example of local rings).

- (a) A field is a local ring. A ring which is reduced to 0 is not a local ring.
- (b) Let A be a local ring and κ its residue field. The ring of formal power series $B = A[[X_1, \dots, X_n]]$ is a local ring, for the non-invertible elements of B are the formal power series whose constant terms are not invertible in A . The canonical injection of A into B is a local homomorphism and the corresponding injection of residue fields is an isomorphism.
- (c) Let \mathfrak{a} be an ideal of a ring A which is only contained in a single maximal ideal \mathfrak{m} . Then A/\mathfrak{a} is a local ring with maximal ideal $\mathfrak{m}/\mathfrak{a}$ and residue field canonically isomorphic to A/\mathfrak{m} . This applies in particular to the case $\mathfrak{a} = \mathfrak{m}^n$, where \mathfrak{m} is any maximal ideal of A (??). If A itself is a local ring with maximal ideal \mathfrak{m} , then for every proper ideal \mathfrak{a} of A , A/\mathfrak{a} is a local ring, the canonical homomorphism $A \rightarrow A/\mathfrak{a}$ a local homomorphism and the corresponding homomorphism of residue fields an isomorphism.
- (d) Let X be a topological space, x_0 a point of X and A the ring of germs at the point x_0 of real-valued functions continuous in a neighbourhood of x_0 . Clearly, for the germ at x_0 of a continuous function f to be invertible in A , it is necessary and sufficient that $f(x_0) \neq 0$, since this implies that $f(x) \neq 0$ in a neighbourhood of x_0 . The ring A is therefore a local

ring whose maximal ideal \mathfrak{m} is the set of germs of functions which are zero at x_0 . Taking quotients, the map $f \mapsto f(x_0)$ of A to \mathbb{R} gives an isomorphism of the residue field A/\mathfrak{m} onto \mathbb{R} .

Proposition 9.3.3. *Let A be a ring and \mathfrak{p} a prime ideal of A . The ring $A_{\mathfrak{p}}$ is local, and its maximal ideal is the ideal $\mathfrak{p}A_{\mathfrak{p}} = \mathfrak{p}_{\mathfrak{p}}$, generated by the canonical image of \mathfrak{p} in $A_{\mathfrak{p}}$. Its residue field is canonically isomorphic to the field of fractions of A/\mathfrak{p} .*

Proof. Let $S = A - \mathfrak{p}$ and $i_A^S : A \rightarrow A_{\mathfrak{p}}$ be the canonical homomorphism; the hypothesis that \mathfrak{p} is prime implies that \mathfrak{p} is saturated with respect to S , hence $(i_A^S)^{-1}(\mathfrak{p}A_{\mathfrak{p}}) = \mathfrak{p}$ and, as the ideals of A not meeting S are those contained in \mathfrak{p} , the first two assertions are special cases of ???. Moreover, if π is the canonical homomorphism $A \rightarrow A/\mathfrak{p}$, $\pi(S)$ is the set of nonzero elements of the integral domain A/\mathfrak{p} and hence the last assertion is a special case of ???. \square

Let A be a ring and \mathfrak{p} a prime ideal of A . The ring $A_{\mathfrak{p}}$ is called the local ring of A at \mathfrak{p} , or the local ring of \mathfrak{p} , when there is no ambiguity. If A is a local ring and \mathfrak{m} its maximal ideal, the elements of $A - \mathfrak{m}$ are invertible and hence $A_{\mathfrak{m}}$ is canonically identified with A .

Example 9.3.4.

- (a) Let p be a prime number. The local ring $\mathbb{Z}_{\mathfrak{p}}$ is the set of rational numbers a/b , where a, b are rational integers with b prime to p . The residue field of $\mathbb{Z}_{\mathfrak{p}}$ is isomorphic to the prime field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$.
- (b) Let V be an affine algebraic variety, A the ring of regular functions on V , W an irreducible subvariety of V and \mathfrak{p} the (necessarily prime) ideal of A consisting of the functions which are zero at every point of W . The ring $A_{\mathfrak{p}}$ is called the local ring of W on V .

Proposition 9.3.5. *Let A, B be two rings, $\rho : A \rightarrow B$ a homomorphism, \mathfrak{q} a prime ideal of B and $\mathfrak{p} = \mathfrak{q}^c$. Then there is a canonical homomorphism $\rho_{\mathfrak{q}} : A_{\mathfrak{p}} \rightarrow B_{\mathfrak{q}}$ which is local.*

Proof. As $\rho(A - \mathfrak{p}) \subseteq B - \mathfrak{q}$, a canonical homomorphism $\rho_{\mathfrak{q}} : A_{\mathfrak{p}} \rightarrow B_{\mathfrak{q}}$ is derived from ρ , and is immediate that $\rho_{\mathfrak{q}}(\mathfrak{p}A_{\mathfrak{p}}) \subseteq \mathfrak{q}B_{\mathfrak{q}}$, hence $\rho_{\mathfrak{q}}$ is local. \square

9.3.2 Modules over a local ring

Proposition 9.3.6. *Let A be a ring, \mathfrak{m} an ideal of A contained in the Jacobson radical of A and M an A -module. Suppose that one of the following conditions holds:*

- (a) M is finitely generated;
- (b) \mathfrak{m} is nilpotent.

Then the relation $(A/\mathfrak{m}) \otimes_A M = 0$ implies $M = 0$.

Proof. The assertion with respect to hypothesis (a) is precisely Nakayama's Lemma. On the other hand, the relation $(A/\mathfrak{m}) \otimes_A M = 0$ is equivalent to $M = \mathfrak{m}M$ and hence implies $M = \mathfrak{m}^nM$ for every integer $n > 0$; whence the assertion with respect to hypothesis (b). \square

Corollary 9.3.7. Let A be a ring, \mathfrak{m} an ideal of A contained in the Jacobson radical of A , M and N two A -modules and $\phi : M \rightarrow N$ an A -linear map. If N is finitely generated or \mathfrak{m} is nilpotent and

$$1 \otimes \phi : (A/\mathfrak{m}) \otimes_A M \rightarrow (A/\mathfrak{m}) \otimes_A N$$

is surjective, then ϕ is surjective.

Proof. Since $(A/\mathfrak{m}) \otimes_A (N/\phi(M))$ is canonically isomorphic to $((A/\mathfrak{m}) \otimes_A M) / \text{im}(1 \otimes \phi)$, the hypothesis implies $(A/\mathfrak{m}) \otimes_A (N/\phi(M)) = 0$, hence $N/\phi(M) = 0$ by ?? \square

Corollary 9.3.8. Let A be a ring, \mathfrak{m} an ideal of A contained in the Jacobson radical of A , M an A -module and $(x_i)_{i \in I}$ a family of elements of M . If M is finitely generated or \mathfrak{m} is nilpotent and the elements $1 \otimes x_i$ generate the (A/\mathfrak{m}) -module $M/\mathfrak{m}M$, then the x_i generate M .

Proposition 9.3.9. Let A be a ring, \mathfrak{m} an ideal of A contained in the Jacobson radical of A and M an A -module. Suppose that one of the following conditions holds:

- (a) M is finitely presented;
- (b) \mathfrak{m} is nilpotent.

Then, if $M/\mathfrak{m}M$ is a free (A/\mathfrak{m}) -module and the canonical homomorphism $\mathfrak{m} \otimes_A M \rightarrow M$ is injective, then M is a free A -module. More precisely, if $(x_i)_{i \in I}$ is a family of elements of M such that $(1 \otimes x_i)$ is a basis of the (A/\mathfrak{m}) -module $M/\mathfrak{m}M$, then (x_i) is a basis of M .

Proof. We already know that the x_i generate M (??). We shall see that they are linearly independent over A . To this end, let us consider the free A -module $L = A^I$. Let (e_i) be its canonical basis and $\phi : A^I \rightarrow M$ the A -linear map such that $\phi(e_i) = x_i$ for all $i \in I$. If R is the kernel of ϕ , we shall prove that that $\mathfrak{m}R = R$. Let j be the canonical injection $R \rightarrow L$; then there is a commutative diagram

$$\begin{array}{ccccccc} \mathfrak{m} \otimes R & \xrightarrow{1 \otimes j} & \mathfrak{m} \otimes L & \xrightarrow{1 \otimes \phi} & \mathfrak{m} \otimes M & \longrightarrow & 0 \\ \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \\ 0 & \longrightarrow & R & \xrightarrow{j} & L & \xrightarrow{\phi} & M \longrightarrow 0 \end{array}$$

in which the two rows are exact. By hypothesis we have $\ker \gamma = 0$, hence by snake lemma, there is an exact sequence

$$0 \longrightarrow \text{coker } \alpha \longrightarrow \text{coker } \beta \longrightarrow \text{coker } \gamma \longrightarrow 0$$

Now $\text{coker } \beta = (A/\mathfrak{m}) \otimes L$ and $\text{coker } \gamma = (A/\mathfrak{m}) \otimes M$. Since $(1 \otimes e_i)$ is a basis for $(A/\mathfrak{m}) \otimes L$ and its image in $(A/\mathfrak{m}) \otimes M$ is a basis for $(A/\mathfrak{m}) \otimes M$, we see $\text{coker } \alpha = 0$, whence $\mathfrak{m}R = R$.

Under hypothesis (a), $(A/\mathfrak{m}) \otimes_A M$ is a finitely generated module, hence I is necessarily finite and R is a finitely generated A -module. Then by ?? we have $R = 0$. Similarly, under hypothesis (b) we also have $R = 0$, so the proof is completed. \square

Corollary 9.3.10. Let A be a local ring and \mathfrak{m} its maximal ideal. For a family (y_i) of elements of M to be a basis of a direct factor of M , it is necessary and sufficient that the family $(1 \otimes y_i)$ be free in $M/\mathfrak{m}M$.

Proof. If this condition holds, it can be assumed that (x_i) is a subfamily of a family (x_i) of elements of M such that $(1 \otimes x_i)$ is a basis of $M/\mathfrak{m}M$ and ?? then proves that (x_i) is a basis of M . \square

Corollary 9.3.11. *Let A be a local ring, \mathfrak{m} its maximal ideal and M an A -module. Suppose that one of the following conditions holds:*

- (a) M is finitely presented;
- (b) \mathfrak{m} is nilpotent.

Then the following properties are equivalent:

- (i) M is free;
- (ii) M is projective;
- (iii) M is flat;
- (iv) the canonical homomorphism $\mathfrak{m} \otimes_A M \rightarrow M$ is injective;
- (v) $\text{Tor}_A^1(A/\mathfrak{m}, M) = 0$.

Proof. The implications (i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv) are immediate, and we know that (iv) is equivalent to (v). As A/\mathfrak{m} is a field, $(A/\mathfrak{m}) \otimes_A M$ is a free (A/\mathfrak{m}) -module and ?? shows that (iv) implies (i). \square

Proposition 9.3.12. *Let A be a local ring and \mathfrak{m} its maximal radical. Let M and N be two finitely generated free A -modules and $\phi : M \rightarrow N$ a homomorphism. The following properties are equivalent:*

- (i) ϕ is an isomorphism of M onto a direct factor of N ;
- (ii) $1 \otimes \phi : (A/\mathfrak{m}) \otimes_A M \rightarrow (A/\mathfrak{m}) \otimes_A N$ is injective;
- (iii) ϕ is injective and $\text{coker } \phi$ is a free A -module;
- (iv) the transpose homomorphism $\phi^t : N^* \rightarrow M^*$ is surjective.

Proof. We know that, if $N/\phi(M)$ is free, then $\phi(M)$ is a direct factor of N , hence (iii) implies (i). Conversely, (i) implies that $\text{coker } \phi$, isomorphic to a complement of $\phi(M)$ in N , is a finitely generated projective A -module and a fortiori finitely presented, hence this module is free by ?? and thus (i) implies (iii).

On the other hand, (i) obviously implies (ii). For simplicity we write $\tilde{M} = (A/\mathfrak{m}) \otimes_A M$ and $\tilde{N} = (A/\mathfrak{m}) \otimes_A N$. As M and N are finitely generated, the duals \tilde{M}^* and \tilde{N}^* of the (A/\mathfrak{m}) -modules \tilde{M} and \tilde{N} are canonically identified with $M^* \otimes_A (A/\mathfrak{m})$ and $N^* \otimes_A (A/\mathfrak{m})$ and $(1 \otimes \phi)^t$ with $1 \otimes \phi^t$. As \tilde{M} and \tilde{N} are vector spaces over the field A/\mathfrak{m} , the hypothesis that $1 \otimes \phi$ is injective implies that $(1 \otimes \phi)^t$ is surjective. ?? then shows that ϕ^t is surjective and we have thus proved that (ii) implies (iv).

Finally we show that (iv) implies (i). Suppose that ϕ^t is surjective. As M^* is free, there exists a homomorphism $\psi^t : M^* \rightarrow N^*$ such that $\phi^t \circ \psi^t = \text{id}$. As M and N are free and finitely generated, there exists a homomorphism $\psi : N \rightarrow M$ such that the dual of ψ is ψ^t , whence

$$\text{id}_M^t = \text{id}_{M^*} = \phi^t \circ \psi^t = (\psi \circ \phi)^t$$

whence $\psi \circ \phi = \text{id}$. This proves that ϕ is an isomorphism of M onto a submodule which is a direct factor of N . \square

Corollary 9.3.13. *Under the hypotheses of ?? the following properties are equivalent:*

- (i) ϕ is an isomorphism of M onto N ;
- (ii) M and N have the same rank;
- (iii) $1 \otimes \phi : (A/\mathfrak{m}) \otimes_A M \rightarrow (A/\mathfrak{m}) \otimes_A N$ is bijective.

Proof. Clearly (i) implies (ii); (ii) implies that $1 \otimes \phi$ is surjective; moreover the hypothesis that M and N have the same rank implies that so do the vector spaces $(A/\mathfrak{m}) \otimes_A M$ and $(A/\mathfrak{m}) \otimes_A N$ over A/\mathfrak{m} , hence $1 \otimes \phi$ is bijective and (ii) implies (iii). Finally, condition (iii) implies, by ??, that N is the direct sum of $\phi(M)$ and a free submodule P and ϕ is an isomorphism of M onto $\phi(M)$. If $P \neq 0$, then $(A/\mathfrak{m}) \otimes_A P \neq 0$ and $1 \otimes \phi$ would not be surjective; hence (iii) implies (i). \square

Proposition 9.3.14. *Let A be a reduced local ring, \mathfrak{m} its maximal ideal, $(\mathfrak{p}_i)_{i \in I}$ the family of minimal prime ideals of A , K_i the field of fractions of A/\mathfrak{p}_i and M a finitely generated A -module. For M to be free it is necessary and sufficient that*

$$[(A/\mathfrak{m}) \otimes_A : (A/\mathfrak{m})] = [K_i \otimes_A M : K] \quad (9.3.1)$$

for all $i \in I$.

Proof. If M is free, clearly the two sides of (??) are equal to the rank of M for all $i \in I$. Suppose now that the condition is satisfied and denote by n the common value of the two sides of (??). By ?? M has a system of n generators x_1, \dots, x_n . Suppose first that A is an integral domain, in which case $\mathfrak{p}_i = 0$ for all $i \in I$. The elements $(1 \otimes x_i)$ generate the vector space $K \otimes M$ over the field of fractions K of A . But as by hypothesis this space is of dimension n over K , the elements $1 \otimes x_i$ are linearly independent over K . It follows that the x_i are linearly independent over A and hence form a basis of M .

Passing to the general case, there exists a surjective homomorphism η from $L = A^n$ onto M . Consider the commutative diagram

$$\begin{array}{ccc} L & \xrightarrow{\eta} & M \\ \downarrow \pi_L & & \downarrow \pi_M \\ \prod_i ((A/\mathfrak{p}_i) \otimes L) & \xrightarrow{\tilde{\eta}} & \prod_i ((A/\mathfrak{p}_i) \otimes M) \end{array}$$

where ϕ_L (resp. π_M) is the map $x \mapsto (\phi_i(x))$ (resp. $y \mapsto (\psi_i(y))$), with $\phi_i : L \rightarrow (A/\mathfrak{p}_i) \otimes L$ (resp. $\psi_i : M \rightarrow (A/\mathfrak{p}_i) \otimes M$) being the canonical map, and $\tilde{\eta}$ is the product of the $1_{A/\mathfrak{p}_i} \otimes \eta$. Then as

$$((A/\mathfrak{p}_i)/(\mathfrak{m}/\mathfrak{p}_i)) \otimes_{A/\mathfrak{p}_i} ((A/\mathfrak{p}_i) \otimes_A M) = (A/\mathfrak{m}) \otimes_A M$$

and A/\mathfrak{p}_i is a local integral domain, it follows from the first part of the argument that each of the $M, 1_{A/\mathfrak{p}_i} \otimes \eta$ is an isomorphism; then so is $\tilde{\eta}$. On the other hand, as A is reduced, $\bigcap_i \mathfrak{p}_i = (0)$ whence $\bigcap_i \mathfrak{p}_i L = (0)$ since L is free. As this shows that π_L is injective, it follows that $\tilde{\eta} \circ \pi_L$ is injective, hence η is injective and, as η is surjective by definition, this shows that M is free. \square

9.3.3 Passing from local to global

Proposition 9.3.15. *Let A be a ring, \mathfrak{m} a maximal ideal of A and M an A -module. If there exists an ideal \mathfrak{a} of A such that \mathfrak{m} is the only maximal ideal of A containing \mathfrak{a} and $\mathfrak{a}M = 0$, then the canonical homomorphism $M \rightarrow M_{\mathfrak{m}}$ is bijective.*

Proof. By hypothesis A/\mathfrak{a} is then a local ring with maximal ideal $\mathfrak{m}/\mathfrak{a}$; M can be considered as an (A/\mathfrak{a}) -module. For all $s \in A - \mathfrak{m}$ the canonical image of s in A/\mathfrak{a} is invertible, hence the homothety $x \mapsto sx$ of M is bijective from the definition of $M_{\mathfrak{m}}$ as the solution of a universal problem, whence the proposition. \square

In particular, if there exists $k > 0$ such that $\mathfrak{m}^k M = 0$, the homomorphism $M \rightarrow M_{\mathfrak{m}}$ is bijective.

Proposition 9.3.16. *Let A be a ring, \mathfrak{m} a maximal ideal of A , M an A -module and $k > 0$ an integer. The canonical homomorphism $M \rightarrow M_{\mathfrak{m}}/\mathfrak{m}^k M_{\mathfrak{m}}$ is surjective, has kernel $\mathfrak{m}^k M$ and defines an isomorphism of $M/\mathfrak{m}^k M$ onto $M_{\mathfrak{m}}/\mathfrak{m}^k M_{\mathfrak{m}}$.*

Proof. It follows from ?? the homomorphism $M/\mathfrak{m}^k M \rightarrow (M/\mathfrak{m}^k M)_{\mathfrak{m}}$ is bijective. On the other hand $(M/\mathfrak{m}^k M)_{\mathfrak{m}}$ is canonically identified with $M_{\mathfrak{m}}/(\mathfrak{m}^k M)_{\mathfrak{m}}$ and hence $(\mathfrak{m}^k M)_{\mathfrak{m}} = \mathfrak{m}^k M_{\mathfrak{m}}$, whence there is an isomorphism of $M/\mathfrak{m}^k M$ onto $M_{\mathfrak{m}}/\mathfrak{m}^k M_{\mathfrak{m}}$ which maps the class of an element $x \in M$ to the class of $x/1$. \square

Corollary 9.3.17. *Let A be a ring, $\mathfrak{m}_1, \dots, \mathfrak{m}_n$ distinct maximal ideals of A , M an A -module and k_1, \dots, k_n positive integers. The canonical homomorphism from M to $\bigoplus_{i=1}^n M_{\mathfrak{m}_i}/\mathfrak{m}_i^{k_i} M_{\mathfrak{m}_i}$ is surjective and its kernel is $(\bigcap_{i=1}^n \mathfrak{m}_i^{k_i})M$.*

Proposition 9.3.18. *Let A be a commutative ring, and let M be an A -module. The following are equivalent*

- (a) $M = 0$.
- (b) $M_{\mathfrak{p}} = 0$ for every prime ideal \mathfrak{p} .
- (c) $M_{\mathfrak{m}} = 0$ for every maximal ideal \mathfrak{m} .

Proof. Clearly (a) \Rightarrow (b) \Rightarrow (c). So we only need to show (c) \Rightarrow (a). Assume $M_{\mathfrak{m}} = 0$ for every maximal ideal \mathfrak{m} but $M \neq 0$. Let x be a non-zero element of M , consider the ideal $\text{Ann}(x)$: It is proper hence contained in a maximal ideal \mathfrak{m} . Consider the module $M_{\mathfrak{m}}$. We must have $x/1 = 0$ since $M_{\mathfrak{m}} = 0$, hence x is killed by some element in $A - \mathfrak{m}$. But this is impossible since $\text{Ann}(x) \subseteq \mathfrak{m}$. \square

Proposition 9.3.19. *Let $\phi : M \rightarrow N$ be an A -module homomorphism. Then the following are equivalent:*

- (a) ϕ is injective (resp. surjective).
- (b) $\phi_{\mathfrak{p}} : M_{\mathfrak{p}} \rightarrow N_{\mathfrak{p}}$ is injective (resp. surjective) for each prime ideal \mathfrak{p} of A .
- (c) $\phi_{\mathfrak{m}} : M_{\mathfrak{m}} \rightarrow N_{\mathfrak{m}}$ is injective (resp. surjective) for each maximal ideal \mathfrak{m} of A .

Proof. Since localization is exact, we see (a) \Rightarrow (b) \Rightarrow (c). Now assume (c) and let $M' = \ker \phi$, then the sequence $0 \rightarrow M' \rightarrow M \rightarrow N$ is exact, hence $0 \rightarrow M'_{\mathfrak{m}} \rightarrow M_{\mathfrak{m}} \rightarrow N_{\mathfrak{m}}$ is exact and therefore $M'_{\mathfrak{m}} \cong \ker \phi_{\mathfrak{m}} = 0$ by assumption. Hence $M' = 0$ by ??, so ϕ is injective. For the other part of the proposition, just reverse all the arrows. \square

Corollary 9.3.20. *The localization functor is faithfully exact in the following sense: let A be a commutative ring, and let*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0 \quad (9.3.2)$$

be a sequence of A -modules. Then ?? is exact if and only if the induced sequence of $A_{\mathfrak{p}}$ -modules

$$0 \longrightarrow A_{\mathfrak{p}} \longrightarrow B_{\mathfrak{p}} \longrightarrow C_{\mathfrak{p}} \longrightarrow 0$$

is exact for every prime ideal \mathfrak{p} of A , if and only if it is exact for every maximal ideal \mathfrak{m} .

Corollary 9.3.21. *Let M be an A -module, N a submodule of M and x an element of M . For $x \in N$, it is necessary and sufficient that, for any maximal ideal \mathfrak{m} , the canonical image of x in $M_{\mathfrak{m}}$ belong to $N_{\mathfrak{m}}$.*

Corollary 9.3.22. *Let M be an A -module and, for all maximal ideal \mathfrak{m} , let $\phi_{\mathfrak{m}} : M \rightarrow M_{\mathfrak{m}}$ be the canonical map. The homomorphism $x \mapsto (\phi_{\mathfrak{m}}(x))$ of M to $\bigoplus_{\mathfrak{m}} M_{\mathfrak{m}}$ is injective.*

Corollary 9.3.23. *Let A be an integral domain, K its field of fractions and M a torsion free A -module such that M is canonically identified with a sub- A -module of $K \otimes_A M$. Then, for any maximal ideal \mathfrak{m} , $M_{\mathfrak{m}}$ is canonically identified with a sub- A -module of $K \otimes_A M$ and $M = \bigcap_{\mathfrak{m}} M_{\mathfrak{m}}$.*

Proof. As M is identified with a submodule of $K \otimes_A M$, $M_{\mathfrak{m}}$ is identified with a sub- $A_{\mathfrak{m}}$ -module of $(K \otimes_A M)_{\mathfrak{m}} = K_{\mathfrak{m}} \otimes_A M = K \otimes_A M$, so $M_{\mathfrak{m}}$ is torsion-free. Moreover, the commutativity of the diagram

$$\begin{array}{ccc} M & \longrightarrow & K \otimes_A M \\ \downarrow & & \downarrow \\ M_{\mathfrak{m}} & \longrightarrow & (K \otimes_A M) \end{array}$$

proves that the canonical map $M \mapsto M_{\mathfrak{m}}$ is injective. The corollary then follows from ?? applied to the A -module $K \otimes_A M$ and its submodule M . \square

Proposition 9.3.24. *Let S be a multiplicative subset of a ring A and M be an A -module. If M is flat (resp. faithfully/flat), $S^{-1}M$ is a flat (resp. faithfully flat) $S^{-1}A$ -module and a flat A -module.*

Proof. As $S^{-1}M = M \otimes_A S^{-1}A$, the first assertion follows from ?? (resp. ??); moreover, $S^{-1}A$ is a flat A -module, hence if M is a flat A -module, so is $S^{-1}M$ by virtue of ?. \square

Proposition 9.3.25. *Let A be a ring, B an A -algebra and T a multiplicative subset of B . If N is a B -module which is flat as an A -module, $T^{-1}N$ is a flat A -module.*

Proof. We have $T^{-1}N = T^{-1}B \otimes_B N$, so the proposition follows from ?. \square

Proposition 9.3.26. *Let $\rho : A \rightarrow B$ be a homomorphism and N be a B -module. The following properties are equivalent:*

- (a) N is a flat A -module;

- (b) for every maximal ideal \mathfrak{N} of B , $N_{\mathfrak{N}}$ is a flat A -module;
- (c) for every maximal ideal \mathfrak{N} of B , if $\mathfrak{m} = \mathfrak{N}^c$, $N_{\mathfrak{N}}$ is a flat $A_{\mathfrak{m}}$ -module.

Proof. For all $a \notin \mathfrak{m}$, the homothety of $N_{\mathfrak{N}}$ induced by a is bijective, hence $N_{\mathfrak{N}}$ is canonically identified with $(N_{\mathfrak{N}})_{\mathfrak{m}}$ and the equivalence of (b) and (c) then follows. The fact that (a) implies (b) is a special case of [??](#). It remains to prove that (b) implies (a), that is, that, if (b) holds, for every injective A -module homomorphism $u : M \rightarrow M'$, the homomorphism $v = 1 \otimes u : N \otimes_A M \rightarrow N \otimes_A M'$ is injective. Now, v is also a B -module homomorphism and, for it to be injective, it is necessary and sufficient that $v_{\mathfrak{N}} : (N \otimes_A M)_{\mathfrak{N}} \rightarrow (N \otimes_A M')_{\mathfrak{N}}$ be so for every maximal ideal \mathfrak{N} of B ([??](#)). As we have

$$(N \otimes_A M)_{\mathfrak{N}} = B_{\mathfrak{N}} \otimes_B (N \otimes_A M) = N_{\mathfrak{N}} \otimes_A M$$

the homomorphism $v_{\mathfrak{N}}$ is identified with $1 \otimes u : N_{\mathfrak{N}} \otimes_A M \rightarrow N_{\mathfrak{N}} \otimes_A M'$, which is injective since $N_{\mathfrak{N}}$ is a flat A -module by hypothesis. \square

Proposition 9.3.27. *For an A -module M to be flat (resp. faithfully flat), it is necessary and sufficient that, for every maximal ideal \mathfrak{m} of A , $M_{\mathfrak{m}}$ be a flat (resp. faithfully flat) $A_{\mathfrak{m}}$ -module.*

Proof. By [??](#), the sequence

$$0 \longrightarrow A \otimes M \longrightarrow B \otimes M \longrightarrow C \otimes M \longrightarrow 0$$

is exact if and only if

$$0 \longrightarrow (A \otimes M)_{\mathfrak{p}} \longrightarrow (B \otimes M)_{\mathfrak{p}} \longrightarrow (C \otimes M)_{\mathfrak{p}} \longrightarrow 0$$

is exact for any prime ideal \mathfrak{p} , iff for any maximal ideal \mathfrak{m} . From [??](#), we have $(A \otimes_A M)_{\mathfrak{p}} \simeq A_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} M_{\mathfrak{p}}$, so the claim follows. \square

Proposition 9.3.28. *Let A be an integral domain with field of fractions K . We consider any ring of fractions of A as a subring of K . Then in this sense we have*

$$A = \bigcap_{\mathfrak{p} \in \text{Spec}(A)} A_{\mathfrak{p}} = \bigcap_{\mathfrak{m} \in \text{Max}(A)} A_{\mathfrak{m}}$$

Proof. For $x \in K$ the set $I = \{a \in A \mid ax \in A\}$ is an ideal of A . Now it is easy to see that $x \in A_{\mathfrak{p}}$ if and only if $I \cap (A - \mathfrak{p}) \neq \emptyset$, which is equivalent to that $I \not\subseteq \mathfrak{p}$, so if $x \in A_{\mathfrak{m}}$ for every maximal ideal \mathfrak{m} then I is not contained in any maximal ideal of A , whence $1 \in I$, that is, $x \in A$. \square

Corollary 9.3.29. *Let A be an integral domain and $f \in A$, then $A_f = \bigcap_{f \notin \mathfrak{p}} A_{\mathfrak{p}}$.*

Proof. If A is an integral domain, then A_f is also a domain. Now $\text{Spec}(A_f) = \{\mathfrak{p} \in \text{Spec}(A) \mid f \notin \mathfrak{p}\}$, so by [??](#) we get the claim. \square

9.4 The spectrum of a ring

9.4.1 The space $\text{Spec}(A)$

Let A be a ring. The **spectrum of A** is the set of prime ideals of A . It is usually denoted $\text{Spec}(A)$. To avoid confusion, we will sometimes write $[\mathfrak{p}]$ for the point of $\text{Spec}(A)$ corresponding to the prime \mathfrak{p} of A . Of course, the zero ideal (0) is a prime if A is a domain.

Example 9.4.1. Here are several examples.

- (a) Prime ideals in \mathbb{Z} is the form (p) with p prime, and every set $\{(p)\} = V(p)$ is closed. Hence $\text{Spec}(\mathbb{Z})$ is a line, with one closed point for each prime number in \mathbb{Z} , and one non-closed point corresponding to the ideal (0) .
- (b) As \mathbb{R} is a field, $\text{Spec}(\mathbb{R})$ consists of a single point, corresponding to the ideal (0) .
- (c) The maximal ideals of the ring $\mathbb{C}[X]$ are all of the form $(x - a)$ for $a \in \mathbb{C}$. The only non-maximal prime ideal of $\mathbb{C}[X]$ is (0) . Thus $\text{Spec}(\mathbb{C}[X])$ is the complex plane together with a single non-closed point, corresponding to (0) .
- (d) There are two types of maximal ideals in $\mathbb{R}[X]$, those generated by a polynomial of degree one, and those generated by a polynomial of degree two. The degree one maximal ideals are all of the form $(x - a)$ for $a \in \mathbb{R}$. The maximal ideals of degree two correspond to complex conjugate pairs of elements of $\mathbb{C} - \mathbb{R}$. There is only one non-maximal prime ideal, (0) . Therefore $\text{Spec}(\mathbb{R}[X])$ is the upper-half plane, together with a single non-closed point.

Each element $f \in A$ defines a function, which we also write as f , on the space $\text{Spec}(A)$: if $x = \mathfrak{p} \in X = \text{Spec}(A)$, we denote by $\kappa(x)$ or $\kappa(\mathfrak{p})$ the quotient field of the integral domain A/\mathfrak{p} , called the **residue field** of X at x , and we define $f(x) \in \kappa(x)$ to be the image of f via the canonical maps

$$A \rightarrow A/\mathfrak{p} \rightarrow \kappa(\mathfrak{p}).$$

The functions induced by elements of A are called the **regular functions** on $\text{Spec}(A)$.

Example 9.4.2. Consider the ring of polynomials $\mathbb{C}[X]$, and let $p(x)$ be a polynomial. If $\alpha \in \mathbb{C}$ is a number, then $(x - \alpha)$ is a prime of $\mathbb{C}[X]$, and in fact is maximal. So $\kappa(x - \alpha)$ is just the field \mathbb{C} , and the value of $p(x)$ at the point $(x - \alpha) \in \text{Spec}(\mathbb{C}[X])$ is the number $p(\alpha)$.

More generally, if A is the coordinate ring of an affine variety V over an algebraically closed field K and \mathfrak{m} is the maximal ideal corresponding to a point $x \in V$ in the usual sense, then $\kappa(x) = K$ and $f(x)$ is the value of f at x in the usual sense.

In general, the function f has values in fields that vary from point to point. Moreover, f is not necessarily determined by the values of this function. For example, if K is a field, the ring $A = K[X]/(x^2)$ has only one prime ideal, which is (x) ; and thus the element $x \in A$, albeit nonzero, induces a function whose value is 0 at every point of $\text{Spec}(A)$.

By using regular functions, we make $\text{Spec}(A)$ into a topological space. For each subset $S \subseteq A$, let

$$V(S) = \{x \in \text{Spec}(A) : f(x) = 0 \text{ for all } f \in S\} = \{\mathfrak{p} \in \text{Spec}(A) : \mathfrak{p} \supseteq S\}.$$

The impulse behind this definition is to make each $f \in A$ behave as much like a continuous function as possible. Of course the fields $\kappa(x)$ have no topology, and since they vary with x the usual notion of continuity makes no sense. But at least they all contain an element called zero, so one can speak of the locus of points in $\text{Spec}(A)$ on which f is zero; and if f is to be like a continuous function, this locus should be closed. Since intersections of closed sets must be closed, we are led immediately to the definition above: $V(S)$ is just the intersection of the loci where the elements of S vanish.

Proposition 9.4.3. *Let A be a ring.*

- (a) *If \mathfrak{a} is the ideal generated by E , then $V(E) = V(\mathfrak{a}) = V(\sqrt{\mathfrak{a}})$.*
- (b) *Let $\mathfrak{a}, \mathfrak{b}$ be ideals of A , then $V(\mathfrak{a}) \subseteq V(\mathfrak{b})$ iff $\mathfrak{b} \subseteq \sqrt{\mathfrak{a}}$.*
- (c) *If $(E_i)_{i \in I}$ is any family of subsets of A , then*

$$V\left(\bigcup_{i \in I} E_i\right) = \bigcap_{i \in I} V(E_i).$$

- (d) *For any ideals $\mathfrak{a}, \mathfrak{b}$ of A , we have $V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b})$.*

In particular, the sets $V(E)$ satisfy the axioms for closed sets in a topological space. The resulting topology is called the **Zariski topology** on $\text{Spec}(A)$.

Proof. We only prove (d). By ??, we have

$$V(\mathfrak{a}\mathfrak{b}) = V(\sqrt{\mathfrak{a}\mathfrak{b}}) = V(\sqrt{\mathfrak{a} \cap \mathfrak{b}}) = V(\mathfrak{a} \cap \mathfrak{b}).$$

Now we only need to prove $V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b})$. Since $\mathfrak{a} \cap \mathfrak{b}$ is contained in \mathfrak{a} and \mathfrak{b} , for either $\mathfrak{a} \subseteq \mathfrak{p}$ or $\mathfrak{b} \subseteq \mathfrak{p}$ we have $\mathfrak{a} \cap \mathfrak{b} \subseteq \mathfrak{p}$, whence $V(\mathfrak{a}) \cup V(\mathfrak{b}) \subseteq V(\mathfrak{a} \cap \mathfrak{b})$. Conversely, let $\mathfrak{p} \in V(\mathfrak{a} \cap \mathfrak{b})$ so that $\mathfrak{a} \cap \mathfrak{b} \subseteq \mathfrak{p}$. By ??, $\mathfrak{a} \subseteq \mathfrak{p}$ or $\mathfrak{b} \subseteq \mathfrak{p}$, so $\mathfrak{p} \in V(\mathfrak{a}) \cup V(\mathfrak{b})$. \square

Let X be the prime spectrum of a ring A ; for all $f \in A$, let us denote by X_f the set of prime ideals of A not containing f . Then $X_f = X - V(f)$ and X_f is therefore an open set, called the **distinguished (or standard) open subsets** of X . By ??, every closed subset of X is an intersection of closed sets of the form $V(f)$ and hence the X_f form a base of the spectral topology on X .

Proposition 9.4.4. *Ler A be a ring, $f, g \in A$ and $X = \text{Spec}(A)$.*

- (a) $D(f) = \emptyset$ iff f is nilpotent and $D(f) = X$ iff f is a unit.
- (b) $D(f) = D(g)$ iff $\sqrt{(f)} = \sqrt{(g)}$.
- (c) $D(f) \cap D(g) = D(fg)$.
- (d) *If $\mathfrak{a} \subseteq A$ is an ideal and \mathfrak{p} is a prime of A with $\mathfrak{p} \notin V(\mathfrak{a})$, then there exists an $f \in A$ such that $\mathfrak{p} \in D(f)$ and $D(f) \cap V(I) = \emptyset$.*
- (e) *If $(f_i)_{i \in A}$ is a family of elements in A , then $\bigcup_i D(f_i) = X - V(\{f_i\})$.*

Proof. Part (a), (b) and (c) are immediate from the definition of $D(f)$ and ???. Now if $\mathfrak{p} \notin V(I)$, then there exists $f \in A$ such that $f \in \mathfrak{a} - \mathfrak{p}$. Then $f \notin \mathfrak{p}$ so that $\mathfrak{p} \in D(f)$. Also, if $\mathfrak{q} \in D(f)$ then $f \notin \mathfrak{q}$ and thus \mathfrak{a} is not contained in \mathfrak{q} , which means $D(f) \cap V(\mathfrak{a}) = \emptyset$. This proves (d).

Finally, for (e) we note that

$$\bigcup_{i \in I} D(f_i) = \bigcup_{i \in I} (X - V(f_i)) = X - \bigcap_{i \in I} V(f_i) = X - V\left(\bigcup_{i \in I} \{f_i\}\right).$$

This completes the proof. \square

For every subset Y of X , let us denote by $I(Y)$ the intersection of the prime ideals of A which belong to Y . Clearly $I(Y)$ is an ideal of A and the map $Y \mapsto I(Y)$ is decreasing with respect to inclusion in X and A , and we have

$$I\left(\bigcup_{i \in I} Y_i\right) = \bigcap_{i \in I} I(Y_i).$$

Proposition 9.4.5. *Let A be a ring, \mathfrak{a} an ideal of A and Y a subset of $X = \text{Spec}(A)$.*

- (a) *$V(\mathfrak{a})$ is closed in X and $I(Y)$ is a radical ideal of A .*
- (b) *$I(V(\mathfrak{a}))$ is the radical of \mathfrak{a} and $V(I(Y))$ is the closure of Y in X .*
- (c) *If E is any subset of A and Y is any subset of X , then*

$$V(E) = V(I(V(E))), \quad I(Y) = I(V(I(Y))).$$

- (d) *The maps I and V define decreasing bijections, one of which is the inverse of the other, between the set of closed subsets of X and the set of radical ideals of A .*

Proof. Assertion (a) and the first assertion of (b) follow from the definitions. If a closed set $V(E)$ (for some $E \subseteq A$) contains Y , then $E \subseteq \mathfrak{p}$ for every prime ideal $\mathfrak{p} \in Y$, whence $E \subseteq I(Y)$ and consequently $V(E) \supseteq V(I(Y))$. As $Y \subseteq V(I(Y))$, $V(I(Y))$ is then the smallest closed set of X containing Y , which completes the proof of (b). Finally, it follows from (b) that, if \mathfrak{a} is a prime ideal equal to its radical, then $I(V(\mathfrak{a})) = \mathfrak{a}$ and that, if Y is closed in X , then $V(I(Y)) = Y$. This proves (d). \square

Corollary 9.4.6. *For every family $(Y_i)_{i \in I}$ of closed subsets of X , $I(\bigcap_{i \in I} Y_i)$ is the radical of the sum of the ideals $I(Y_i)$.*

Proof. It follows from ?? that $I(\bigcap_{i \in I} Y_i)$ is the smallest ideal which is equal to its radical and contains all the $I(Y_i)$; this ideal then contains $\sum_{i \in I} I(Y_i)$ and therefore also the radical of it, whence the corollary. \square

Corollary 9.4.7. *The space $X = \text{Spec}(A)$ is Noetherian if A is a Noetherian ring.*

Proof. If A is Noetherian, every ascending chain of prime ideals is stable, whence every descending chain of irreducible closed sets is stable. In other words, $\text{Spec}(A)$ is Noetherian. \square

Remark 9.4.1. Note that a ring A can be non-Noetherian even though its spectrum is Noetherian. For example, consider the ring

$$A = k[x_1, x_2, \dots]/(x_1, x_2^2, x_3^3, \dots)$$

where k is a field. Then A is not Noetherian since the ideal

$$\mathfrak{m} = (x_1, x_2, x_3, \dots)/(x_1, x_2^2, x_3^3, \dots)$$

is not finitely generated. However, this is the unique prime ideal of A (in fact maximal), so $\text{Spec}(A)$ is a singleton and hence Noetherian.

Corollary 9.4.8. Let A be a ring. For a subset Y of $X = \text{Spec}(A)$ to be irreducible, it is necessary and sufficient that the ideal $I(Y)$ be prime.

Proof. Writing $\mathfrak{p} = I(Y)$, we note that, for an element $f \in A$, the relation $f \in \mathfrak{p}$ is equivalent to $Y \subseteq V(f)$. Suppose that Y is irreducible and let f, g be elements of A such that $fg \in \mathfrak{p}$. Then

$$Y \subseteq V(fg) = V(f) \cup V(g).$$

as Y is irreducible and $V(f)$ and $V(g)$ are closed, $Y \subseteq V(f)$ or $Y \subseteq V(g)$, hence $f \in \mathfrak{p}$ or $g \in \mathfrak{p}$, which proves that \mathfrak{p} is prime.

Suppose now that \mathfrak{p} is prime; then $Y = V(\mathfrak{p})$ by ?? and, as \mathfrak{p} is prime, $\mathfrak{p} = I(\{\mathfrak{p}\})$, whence $\bar{Y} = V(I(\{\mathfrak{p}\})) = \overline{\{\mathfrak{p}\}}$. As a set consisting of a single point is irreducible, Y is irreducible. \square

Corollary 9.4.9. For a ring A to be such that $X = \text{Spec}(A)$ is irreducible, it is necessary and sufficient that the quotient of A by its nilradical \mathfrak{n} be an integral domain.

Proof. By ??, $I(X)$ is the radical of the ideal (0) , that is \mathfrak{n} . \square

Remark 9.4.2. Note that a ring A can be non-Noetherian even though $\text{Spec}(A)$ is Noetherian. For example, the ring $A = k[x_1, x_2, \dots]/(x_1^2, x_2^2, \dots)$ has a unique maximal ideal $\mathfrak{m} = (x_1, x_2, \dots)$, whence $\text{Spec}(A)$ is a singleton and is Noetherian. But \mathfrak{m} is not finitely generated, so A is not Noetherian.

Corollary 9.4.10. The space $X = \text{Spec}(A)$ is a Kolmogoroff space.

Proof. If \mathfrak{p} and \mathfrak{q} are two different points of X , we have, either $\mathfrak{p} \not\subseteq \mathfrak{q}$ or $\mathfrak{q} \not\subseteq \mathfrak{p}$, hence one of the points $\mathfrak{p}, \mathfrak{q}$ does not belong to the closure of the other. \square

Proposition 9.4.11. Let A be a ring. The topology on $X = \text{Spec}(A)$ has the following properties:

- (a) X is quasi-compact.
- (b) X has a basis for the topology consisting of quasi-compact opens.
- (c) The intersection of any two quasi-compact opens is quasi-compact.

Proof. It suffices to prove that any covering of $\text{Spec}(A)$ by standard opens can be refined by a finite covering. Thus suppose that $\text{Spec}(A) = \bigcup_i D(f_i)$ for a set of elements f_i of A . This means

that $\bigcap_i V(f_i) = \emptyset$. According to ?? this means that $V(\{f_i\}_{i \in I}) = \emptyset$. Then the ideal generated by the f_i is the unit ideal of A . This means that we can write

$$1 = \sum_{i=1}^n a_i f_i.$$

It follows that $\text{Spec}(A) = \bigcup_{i=1}^n D(f_i)$. Since $D(f) \cong \text{Spec}(A_f)$, each standard open is quasi-compact, and so their intersection $D(f) \cap D(g) = D(fg)$ is also quasi-compact. \square

Proposition 9.4.12. *Let A be a ring and $X = \text{Spec}(A)$.*

- (a) *The irreducible closed subsets of X are exactly the subsets $V(\mathfrak{p})$, with $\mathfrak{p} \subseteq A$ a prime.*
- (b) *The irreducible components of X are exactly the subsets $V(\mathfrak{p})$, with $\mathfrak{p} \subseteq A$ a minimal prime.*

Proof. By ?? $V(\mathfrak{p}) = \overline{\{\mathfrak{p}\}}$ is the closure of a singleton and hence irreducible. conversely, let $V(\mathfrak{a}) \subseteq X$ with \mathfrak{a} a radical ideal. If \mathfrak{a} is not prime, then choose $a, b \in A$, $a, b \notin \mathfrak{a}$ with $ab \in \mathfrak{a}$. In this case $V(\mathfrak{a}, a) \cup V(\mathfrak{a}, b) = V(\mathfrak{a})$, but neither $V(\mathfrak{a}, b) = V(\mathfrak{a})$ nor $V(\mathfrak{a}, a) = V(\mathfrak{a})$ since $\sqrt{(\mathfrak{a}, a)} \neq \sqrt{\mathfrak{a}} = \mathfrak{a}$, $\sqrt{(\mathfrak{a}, b)} \neq \mathfrak{a}$. Hence $V(\mathfrak{a})$ is not irreducible. \square

Corollary 9.4.13. *Let A be a ring, $X = \text{Spec}(A)$, and $\mathfrak{p} \in X$.*

- (a) *The set of irreducible closed subsets of X containing \mathfrak{p} is in one-to-one correspondence with primes $\mathfrak{q} \supseteq \mathfrak{p}$.*
- (b) *The set of irreducible component of X containing \mathfrak{p} is in one-to-one correspondence with minimal primes $\mathfrak{q} \supseteq \mathfrak{p}$.*

Corollary 9.4.14. *The set of minimal prime ideals of a Noetherian ring A is finite.*

Proof. The spectrum of a Noetherian is a Noetherian topological space, hence has finitely many irreducible components. \square

Proposition 9.4.15. *Let A be a ring and \mathfrak{p} be a minimal prime of A . Let $V \subseteq \text{Spec}(A)$ be a quasi-compact open not containing the point \mathfrak{p} . Then there exists an $f \in A$, $f \notin \mathfrak{p}$ such that $D(f) \cap V = \emptyset$.*

Proof. Since V is quasi-compact we may write $V = \bigcup_{i=1}^n D(g_i)$. Since $\mathfrak{p} \notin V$ we have $g_i \in \mathfrak{p}$ for all i . Since \mathfrak{p} is minimal, we see $\mathfrak{n}(A_{\mathfrak{p}}) = \mathfrak{p}A_{\mathfrak{p}}$, so each g_i is nilpotent in $A_{\mathfrak{p}}$. Hence we can find an $f \in A$, $f \notin \mathfrak{p}$ such that $f g_i^{n_i} = 0$ for some $n_i > 0$. Then $D(f) \cap D(g_i) = D(f g_i^{n_i}) = \emptyset$, so this $D(f)$ works. \square

Proposition 9.4.16. *Let A be a ring and $X = \text{Spec}(A)$. Let U be an open subset of X , then U is dense in X if and only if U meets every irreducible components of X . In particular, for an element $f \in A$ to be such that $D(f)$ is dense in X , it is necessary and sufficient that f does not belong to any minimal prime ideal of A . Moreover, every open dense subset of X contains an open subset of the form $D(f)$ with f not a zero divisor.*

Proof. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ be the minimal prime ideals of A and set $X_i = V(\mathfrak{p}_i)$. If $\mathfrak{p}_i \in U$ for all i , then $U \cap X_i \neq \emptyset$ so U is dense (in that for this direction we do not need to assume that X has finitely many irreducible components). Conversely, assume that U is dense and let $Z = \bigcup_{X_i \cap U \neq \emptyset} X_i$. Then Z is closed and contains U . But then $Z = X$ since U is dense, so there is no X_i such that

$X_i \cap U = \emptyset$. In particular, since for an element $f \in A$, $D(f) \cap V(\mathfrak{p}_i) \neq \emptyset$ if and only if $f \notin \mathfrak{p}_i$, we see (f) is dense if and only if $f \notin \mathfrak{p}_i$ for all i .

On the other hand, if U is an open dense subset of X , the complement of U is of the form $V(\mathfrak{a})$, where \mathfrak{a} is an ideal which is not contained in any of the \mathfrak{p}_i ; it is therefore not contained in their union, and there then exists $f \in \mathfrak{a}$ contained in S ; hence $D(f) \subseteq U$. \square

Proposition 9.4.17. *Let A be a ring. Let $X = \text{Spec}(A)$ as a topological space. The following are equivalent*

- (a) X is Hausdorff.
- (b) X is totally disconnected.
- (c) Every prime ideal of A is maximal.
- (d) Every quasi-compact open of X is closed.
- (e) Every standard open $D(f) \subseteq X$ is closed.

Proof. It is clear that (d) and (e) are equivalent as every quasi-compact open is a finite union of standard opens. The implication (c) \Rightarrow (d) follows from ???. Assume (d) holds. Let $\mathfrak{p}, \mathfrak{p}'$ be distinct primes of A . Choose an $f \in \mathfrak{p}'$, $f \notin \mathfrak{p}$. Then $\mathfrak{p}' \notin D(f)$ and $\mathfrak{p} \in D(f)$. By (d) the open $D(f)$ is also closed. Hence \mathfrak{p} and \mathfrak{p}' are in disjoint open neighbourhoods whose union is X . Thus X is Hausdorff and totally disconnected. Thus (d) implies (a) and (b). Finally, if (b) holds, then the closure of $\{\mathfrak{p}\}$ must be itself, so we see (c) holds. If X is Hausdorff then every point is closed, so (a) implies (c). These together finish the proof. \square

9.4.2 Functoriality of the spectrum

We will now show that $A \mapsto \text{Spec}(A)$ defines a contravariant functor from the category of rings to the category of topological spaces. Let $\rho : A \rightarrow B$ be a homomorphism of rings. If \mathfrak{q} is a prime ideal of B , $\rho^{-1}(\mathfrak{q})$ is a prime ideal of A . Therefore we obtain a map

$${}^a\rho = \text{Spec}(\rho) : \text{Spec}(B) \rightarrow \text{Spec}(A), \quad \mathfrak{q} \mapsto \rho^{-1}(\mathfrak{q}).$$

Proposition 9.4.18. *Let $\rho : A \rightarrow B$ be a ring homomorphism.*

- (a) *For every subset $S \subseteq A$, we have $({}^a\rho)^{-1}(V(S)) = V(\rho(S))$, so ${}^a\rho$ is continuous.*
- (b) *If \mathfrak{b} is an ideal of B , then $\overline{{}^a\rho(V(\mathfrak{b}))} = V(\rho^{-1}(\mathfrak{b}))$.*

Proof. A prime ideal \mathfrak{q} of B contains $\rho(S)$ if and only if $\rho^{-1}(\mathfrak{q})$ contains S , so (a) holds. For part (b), we can rewrite the left hand side as $VI({}^a\rho(V(\mathfrak{b})))$. But

$$I({}^a\rho(V(\mathfrak{b}))) = \bigcap_{\mathfrak{q} \in V(\mathfrak{b})} \rho^{-1}(\mathfrak{q}) = \rho^{-1}\left(\bigcap_{\mathfrak{q} \in V(\mathfrak{b})} \mathfrak{q}\right) = \rho^{-1}(\sqrt{\mathfrak{b}}) = \sqrt{\rho^{-1}(\mathfrak{b})}.$$

and the claim follows by applying V . \square

Corollary 9.4.19. *The map ${}^a\rho$ has dense image if and only if every element of $\ker(\rho)$ is nilpotent.*

Proof. By letting $\mathfrak{b} = 0$ in ?? we get $\overline{^a\rho(Y)} = V(\ker \rho)$, so the claim follows from the fact that $V(\mathfrak{a}) = \text{Spec}(A)$ if and only if $\mathfrak{a} \subseteq \mathfrak{n}(A)$. \square

Proposition 9.4.20. *Suppose that for any $g \in B$ there exists $f \in A$ such that $g = u\rho(f)$, where u is invertible in B (this is the case if ρ is surjective). Then ${}^a\rho$ is a homeomorphism from X onto ${}^a\rho(X)$. In particular, $\text{Spec}(A)$ and $\text{Spec}(A/\mathfrak{n}(A))$ are naturally homeomorphic.*

Proof. We show that for any subset $F \subseteq B$, there exist $E \subseteq A$ such that $V(F) = V(\rho(E))$; by virtue of the T_0 -axiom and the formula of ??(a), this will first imply that ${}^a\rho(X)$ is injective, then, still by virtue of ??(a), that ${}^a\rho(X)$ is a homeomorphism. Now, it suffices for each $g \in F$ to take $f \in A$ such that $u\rho(f) = g$ with u invertible in B ; the set E of these elements then satisfies the requirement. \square

Corollary 9.4.21. *Let $\rho : A \rightarrow B$ be a homomorphism of rings. Let \mathfrak{a} be an ideal of A and let \mathfrak{b} be its extension in B . Let $\bar{\rho} : A/\mathfrak{a} \rightarrow B/\mathfrak{b}$ be the induced homomorphism. If $\text{Spec}(A/\mathfrak{a})$ is identified with its canonical image $V(\mathfrak{a})$ in $\text{Spec}(A)$, and $\text{Spec}(B/\mathfrak{b})$ with its image $V(\mathfrak{b})$ in $\text{Spec}(B)$, then ${}^a\bar{\rho}$ is the restriction of ${}^a\rho$ to $V(\mathfrak{b})$.*

Proof. Let $\mathfrak{q} \in V(\mathfrak{b})$, then $\mathfrak{a} \subseteq \mathfrak{a}^{ec} = \mathfrak{b}^c \subseteq \mathfrak{q}^c$, so ${}^a\rho(V(\mathfrak{b})) \subseteq V(\mathfrak{a})$. Let $\pi_A : A \rightarrow A/\mathfrak{a}$, $\pi_B : B \rightarrow B/\mathfrak{b}$ be the quotient maps. Then

$$\bar{\rho} \circ \pi_A(a) = \bar{\rho}(a + \mathfrak{a}) = \rho(a) + \mathfrak{b} = \pi_B \circ \rho(a).$$

Hence $\bar{\rho} \circ \pi_A = \pi_B \circ \rho$, which implies ${}^a(\pi_A) \circ {}^a\bar{\rho} = {}^a\rho \circ {}^a(\pi_B)$, so we have the following commutative diagram:

$$\begin{array}{ccc} \text{Spec}(B/\mathfrak{b}) & \xrightarrow{{}^a\bar{\rho}} & \text{Spec}(A/\mathfrak{a}) \\ {}^a(\pi_B) \downarrow & & \downarrow {}^a(\pi_A) \\ V(\mathfrak{b}) & \xrightarrow{{}^a\rho} & V(\mathfrak{a}) \end{array}$$

which is exactly the claim. \square

Proposition 9.4.22. *Let S be a multiplicative subset of A and let $i_A^S : A \rightarrow S^{-1}A$ be the canonical homomorphism. Then ${}^a(i_A^S)$ is a homeomorphism of $\text{Spec}(S^{-1}A)$ onto the subspace of $\text{Spec}(A)$ consisting of prime ideals \mathfrak{p} with $S \cap \mathfrak{p} = \emptyset$.*

Proof. We already know that ${}^a(i_A^S)$ is injective by ??, and similar to ??, we see ${}^a(i_A^S)$ is closed, so the claim follows. \square

Due to ??, we may therefore identify $\text{Spec}(S^{-1}A)$ with a subspace of $\text{Spec}(A)$ and write $S^{-1}X$ for $\text{Spec}(S^{-1}A)$, if $X = \text{Spec}(A)$.

Corollary 9.4.23. *Let A be a ring and $f \in A$. Then the canonical map $i_A^{S_f} : A \rightarrow A_f$ induces a homeomorphism $\text{Spec}(A_f) \cong D(f) \subseteq \text{Spec}(A)$, with the inverse map given by $\mathfrak{p} \mapsto \mathfrak{p}A_f$.*

Corollary 9.4.24. *Let A be a ring and \mathfrak{p} a prime ideal of A . Then the canonical image of $\text{Spec}(A_{\mathfrak{p}})$ in $\text{Spec}(A)$ is equal to the intersection of all the open neighborhoods of \mathfrak{p} in $\text{Spec}(A)$.*

Proof. We have that $\text{Spec}(A_f) = \{\mathfrak{q} \in \text{Spec}(A) : f \notin \mathfrak{q}\}$ and $\text{Spec}(A_{\mathfrak{p}}) = \{\mathfrak{q} \in \text{Spec}(A) : \mathfrak{q} \subseteq \mathfrak{p}\}$, therefore

$$\text{Spec}(A_{\mathfrak{p}}) = \bigcap_{f \in A - \mathfrak{p}} \text{Spec}(A_f) = \bigcap_{\mathfrak{p} \in D(f)} D(f).$$

That is, $\text{Spec}(A_{\mathfrak{p}})$ is the intersection of all basic open neighborhoods of the point \mathfrak{p} in $\text{Spec}(A)$. Since every open set is a union of principal open sets, we get the claim. \square

Proposition 9.4.25. *Let $\rho : A \rightarrow B$ be a homomorphism and S a multiplicative subset of A . Then ${}^a(S^{-1}\rho) : \text{Spec}(S^{-1}B) \rightarrow \text{Spec}(S^{-1}A)$ is the restriction of ${}^a\rho$ on $\text{Spec}(S^{-1}B)$ and*

$$\text{Spec}(S^{-1}B) = ({}^a\rho)^{-1}(\text{Spec}(S^{-1}A)).$$

Proof. Since $S^{-1}B = \rho(S)^{-1}B$, we have the following commutative diagram:

$$\begin{array}{ccc} A & \xrightarrow{\rho} & B \\ i_A^S \downarrow & & \downarrow i_B^S \\ S^{-1}A & \xrightarrow{S^{-1}\rho} & S^{-1}B \end{array}$$

whence the first claim. Now, by definition, for $\mathfrak{q} \in \text{Spec}(B)$ we see $\mathfrak{q} \cap \rho(S) = \emptyset$ if and only if $\mathfrak{q}^c \cap S = \emptyset$, so the last claim follows. \square

Corollary 9.4.26. *Let $\rho : A \rightarrow B$ be a homomorphism and \mathfrak{p} be a prime ideal of A . Then the continuous map ${}^a\nu : \text{Spec}(B \otimes_A \kappa(\mathfrak{p})) \rightarrow \text{Spec}(B)$, induced by the homomorphism $\nu : B \rightarrow B \otimes_A \kappa(\mathfrak{p})$, induces a homeomorphism from $\text{Spec}(B \otimes_A \kappa(\mathfrak{p}))$ onto the fiber of \mathfrak{p} under the map ${}^a\rho$.*

Proof. The homomorphism ν is the composition of the quotient homomorphism $B \rightarrow \mathfrak{p}^e$ and of the canonical homomorphism B/\mathfrak{p}^e in its ring of fractions $(B/\mathfrak{p}^e)_{\mathfrak{p}}$. According to ?? and ??, ν^* thus induces a homeomorphism of $\text{Spec}(B \otimes_A \kappa(\mathfrak{p}))$ on the subspace of $\text{Spec}(B)$ consists of the prime ideals \mathfrak{P} of B which contain \mathfrak{p}^e and are disjoint of $\rho(A - \mathfrak{p})$. That is to say, which are lying above \mathfrak{p} . \square

Example 9.4.27. Let A be an integral domain with just one non-zero prime ideal \mathfrak{p} . and let K be the field of fractions of A . Let $B = (A/\mathfrak{p}) \times K$. Define $\rho : A \rightarrow B$ by $\rho(x) = (\bar{x}, x)$, where x is the image of \bar{x} in A/\mathfrak{p} . Then ${}^a\rho$ is bijective but not a homeomorphism.

In fact, the idals in A/\mathfrak{p} are $(0), A/\mathfrak{p}$, that in K is $(0), K$. So by the claim below the ideals in $(A/\mathfrak{p}) \times K$ are

$$(0) \times (0), (A/\mathfrak{p}) \times (0), (0) \times K, (A/\mathfrak{p}) \times K.$$

Note that $(0) \times (0)$ is not prime: $(a, 0) \cdot (0, b) = 0$ but neither of them is zero. So

$$\text{Spec}(B) = \{(A/\mathfrak{p}) \times (0), (0) \times K\}$$

Since $\text{Spec}(A) = \{0, \mathfrak{p}\}$, ${}^a\rho$ is bijective.

Now $\text{Spec}(B)$ is a two point Hausdorff space, since the prime ideals in B are maximal. But

$\text{Spec}(A)$ is non-Hausdorff because it contains 0 and

$$\overline{\{0\}} = \{\mathfrak{p} \subseteq A : \mathfrak{p} \supseteq (0)\} = \{0, \mathfrak{p}\}$$

Therefore ${}^a\rho$ cannot be a homeomorphism.

Example 9.4.28. Let A be a principal ideal domain. In this case, the maximal ideals are of the form (p) for a prime element p of A , and all prime ideals are maximal or the zero ideal. Therefore the closed points of $\text{Spec}(A)$ correspond to equivalence classes of prime elements. Let $\eta \in \text{Spec}(A)$ be the point corresponding to the zero ideal. Then the closure of $\{\eta\}$ is $\text{Spec}(A)$.

As A is a principal ideal domain, every closed subset of $\text{Spec}(A)$ is of the form $V(f)$ for some $f \in A$. Assume $f \neq 0$ and let $f = p_1^{e_1} \cdots p_r^{e_r}$ with pairwise non-equivalent prime elements p_i and integers e_i . Then $V(f)$ consists of those closed points which correspond to the prime divisors of f , that is, $V(f) = \{(p_1), \dots, (p_r)\}$. Therefore the proper closed subsets are the finite sets consisting of closed points.

If A is a local principal ideal domain, but not a field (i.e., A is a discrete valuation ring), $\text{Spec}(A)$ consists only of two points η and x , where \mathfrak{p}_x is the maximal ideal and $\mathfrak{p} = \{0\}$. The only nontrivial open subset of $\text{Spec}(A)$ is then $\{\eta\}$.

Example 9.4.29. Let $A = R[X]$, where R is a PID. We assume that R is not a field and let K be its field of fractions. Let $X = \text{Spec}(R[X])$. Then the prime elements of $R[X]$ are either of the form p , where p is a prime element of R , or of the form f , where $f \in R[X]$ is a primitive polynomial which is irreducible in $K[X]$

If $p \in R$ is a prime element, then the closure $V(p)$ is homeomorphic to $\text{Spec}((R/p)[X])$. Since $(R/p)[X]$ is a PID but not be a field, we see that (p) is not a maximal ideal, and the prime ideals in $V(p)$ different from (p) are the maximal ideals generated by p and f where $f \in R[X]$ is a polynomial such that its image in $(R/p)[X]$ is irreducible.

The situation is more complicated for prime ideals of the form (f) , where f is a primitive irreducible polynomial. If the leading coefficient of f is a unit in R , it is possible to divide in $R[X]$ by f with unique remainder, and therefore $R[X]/(f)$ is finitely generated as R -module. This implies that (f) is not a maximal ideal by ??, as otherwise R would be a field.

For other primitive irreducible polynomials f , (f) might be a maximal ideal, namely if R contains only finitely many prime elements (up to equivalence): If $0 \neq a \in R$ is an element which is divisible by all prime elements of R we have, with $f = ax - 1$,

$$R[X]/(f) \cong R[a^{-1}] = K$$

which shows that (f) is a maximal ideal.

9.4.3 The support of a module

Let A be a ring and M an A -module. The set of prime ideals \mathfrak{p} of A such that $M_{\mathfrak{p}} \neq 0$ is called the **support** of M and is denoted by $\text{supp}(M)$. As every maximal ideal of A is prime, it follows immediately from ?? that for A -module M to be equal to 0, it is necessary and sufficient that $\text{supp}(M) = \emptyset$.

Example 9.4.30. Let \mathfrak{a} be an ideal of A , then we have $\text{supp}(A/\mathfrak{a}) = V(\mathfrak{a})$. In fact, if \mathfrak{p} is a prime of A such that $\mathfrak{a} \not\subseteq \mathfrak{p}$, then $(A/\mathfrak{a})_{\mathfrak{p}} = 0$; if on the other hand $\mathfrak{a} \subseteq \mathfrak{p}$, then $\mathfrak{a}A_{\mathfrak{p}}$ is contained in the maximal ideal $\mathfrak{p}A_{\mathfrak{p}}$ of $A_{\mathfrak{p}}$ and $(A/\mathfrak{a})_{\mathfrak{p}}$ is isomorphic to $A_{\mathfrak{p}}/\mathfrak{a}A_{\mathfrak{p}}$ and hence is non-zero; whence our assertion.

Proposition 9.4.31. Let A be a ring and M an A -module.

(a) If N is a submodule of M , then

$$\text{supp}(M) = \text{supp}(N) \cup \text{supp}(M/N).$$

(b) If M is the sum of a family $(M_i)_{i \in I}$ of submodules, then

$$\text{supp}(M) = \bigcup_{i \in I} \text{supp}(M_i).$$

Proof. From the exact sequence $0 \rightarrow N \rightarrow M \rightarrow M/N \rightarrow 0$, we derive, for every prime ideal \mathfrak{p} of A , the exact sequence

$$0 \longrightarrow N_{\mathfrak{p}} \longrightarrow M_{\mathfrak{p}} \longrightarrow (M/N)_{\mathfrak{p}} \longrightarrow 0$$

For $M_{\mathfrak{p}}$ to be reduced to 0, it is necessary and sufficient that $N_{\mathfrak{p}}$ and $(M/N)_{\mathfrak{p}}$ be so. In other words, the relation $\mathfrak{p} \notin \text{supp}(M)$ is equivalent to $\mathfrak{p} \notin \text{supp}(N)$ and $\mathfrak{p} \notin \text{supp}(M/N)$, which proves (a). Part (b) can be proved similarly. \square

Corollary 9.4.32. Let A be a ring, M an A -module, $(m_i)_{i \in I}$ a system of generators of M and \mathfrak{a}_i the annihilator of m_i . Then $\text{supp}(M) = \bigcup_{i \in I} V(\mathfrak{a}_i)$.

Proof. We have $\text{supp}(M) = \bigcup_{i \in I} \text{supp}(Am_i)$ by ???. On the other hand, Am_i is isomorphic to the A -module A/\mathfrak{a}_i and we have seen that $\text{supp}(A/\mathfrak{a}_i) = V(\mathfrak{a}_i)$, hence the claim. \square

Proposition 9.4.33. Let A be a ring, M an A -module and \mathfrak{a} its annihilator. If M is finitely generated, then $\text{supp}(M) = V(\mathfrak{a})$ and $\text{supp}(M)$ is therefore closed in $\text{Spec}(A)$.

Proof. Let m_1, \dots, m_n be a system of generators of M and let \mathfrak{a}_i be the annihilator of m_i for each i . Then $\mathfrak{a} = \bigcap_{i=1}^n \mathfrak{a}_i$, hence $V(\mathfrak{a}) = \bigcup_{i=1}^n V(\mathfrak{a}_i)$ and the proposition follows from the ???. \square

Corollary 9.4.34. Let A be a ring, M a finitely generated A -module and a an element of A . For a to belong to every prime ideal of the support of M , it is necessary and sufficient that the homothety of M with ratio a be nilpotent.

Proof. It follows from ?? that the intersection of the prime ideals belonging to $\text{supp}(M)$ is the radical of the annihilator \mathfrak{a} of M . To say that a belongs to this radical is equivalent to say that there exist a power a^n such that $a^n M = 0$. \square

Corollary 9.4.35. Let A be a Noetherian ring, M a finitely generated A -module and \mathfrak{a} an ideal of A . For $\text{supp}(M) \subseteq V(\mathfrak{a})$, it is necessary and sufficient that there exist an integer n such that $\mathfrak{a}^n M = 0$.

Proof. If \mathfrak{b} is the annihilator of M , the relation $\text{supp}(M) \subseteq V(\mathfrak{a})$ is equivalent to $V(\mathfrak{b}) \subseteq V(\mathfrak{a})$ by ?? and hence to $\mathfrak{a} \subseteq \sqrt{\mathfrak{b}}$. Since A is Noetherian, this condition is also equivalent to the existence of an integer $n > 0$ such that $\mathfrak{a}^n \subseteq \mathfrak{b}$. \square

Lemma 9.4.36. *Let A be a local ring and M and N two finitely generated A -modules. If $M \neq 0$ and $N \neq 0$, then $M \otimes N \neq 0$.*

Proof. Let κ be the residue field of A . By virtue of ??, $\kappa \otimes_A M \neq 0$, and $\kappa \otimes_A N \neq 0$, then we deduce that

$$(\kappa \otimes_A M) \otimes_\kappa (\kappa \otimes_A N) \neq 0$$

But, since the tensor product is associative, this tensor product is isomorphic to

$$M \otimes_A (\kappa \otimes_\kappa \kappa) \otimes_A N = M \otimes_A \kappa \otimes_A N$$

and therefore to $\kappa \otimes_A (M \otimes_A N)$, whence the lemma. \square

Proposition 9.4.37. *Let M, N be two finitely generated modules over a ring A , then*

$$\text{supp}(M \otimes_A N) = \text{supp}(M) \cap \text{supp}(N).$$

Proof. We need to prove that, if \mathfrak{p} is a prime ideal of A , the relations " $(M \otimes_A N)_{\mathfrak{p}} \neq 0$ " and " $M_{\mathfrak{p}} = 0$ and $N_{\mathfrak{p}} \neq 0$ " are equivalent. As the $A_{\mathfrak{p}}$ -modules $M_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} N_{\mathfrak{p}}$ and $(M \otimes_A N)_{\mathfrak{p}}$ are isomorphic, our assertion follows from ??.

Corollary 9.4.38. *Let M be a finitely generated A -module and \mathfrak{n} its annihilator. For every ideal \mathfrak{a} of A , $\text{supp}(M/\mathfrak{a}M) = V(\mathfrak{a}) \cap V(\mathfrak{n}) = V(\mathfrak{a} + \mathfrak{n})$.*

Proof. We have $M/\mathfrak{a}M = M \otimes_A (A/\mathfrak{a})$ and A/\mathfrak{a} is finitely generated, so the claim follows from ??.

Lemma 9.4.39. *Let A, B be two local rings, $\rho : A \rightarrow B$ a local homomorphism and M a finitely generated A -module. If $M \neq 0$, then $\rho_*(M) \neq 0$.*

Proof. Let \mathfrak{m} be the maximal ideal of A and $\kappa = A/\mathfrak{m}$ the residue field. The hypothesis implies that $B \otimes_A \kappa = B/\mathfrak{m}B \neq 0$ and $M \otimes_A \kappa = M/\mathfrak{m}M \neq 0$ by ?. Since the tensor product is associative, we have

$$(M \otimes_A B) \otimes_A \kappa = M \otimes_A (B \otimes_A \kappa) = M \otimes_A (\kappa \otimes_\kappa (B \otimes_A \kappa)) = (M \otimes_A \kappa) \otimes_\kappa (B \otimes_A \kappa)$$

therefore $M \otimes_A B \neq 0$. \square

Proposition 9.4.40. *Let A, B be two rings, $\phi : A \rightarrow B$ a homomorphism and $\phi^* : \text{Spec}(B) \rightarrow \text{Spec}(A)$ the induced map. Then for every A -module M ,*

$$\text{supp}(M \otimes_A B) \subseteq \phi^{*-1}(\text{supp}(M)).$$

If also M is finitely generated, then the equality holds.

Proof. Let \mathfrak{q} be a prime ideal of B and $\mathfrak{p} = \phi^{-1}(\mathfrak{q})$. Suppose that \mathfrak{q} belongs to $\text{supp}(M \otimes_A B)$, then

$$(M \otimes_A B) \otimes_B B_{\mathfrak{q}} = M \otimes_A B_{\mathfrak{q}} = (M \otimes_A A_{\mathfrak{p}}) \otimes_A B_{\mathfrak{q}}.$$

since the homomorphism $A \rightarrow B \rightarrow B_{\mathfrak{q}}$ factors into $A \rightarrow A_{\mathfrak{p}} \rightarrow B_{\mathfrak{q}}$. The hypothesis $M \otimes_A B \otimes_B B_{\mathfrak{q}} \neq 0$ implies therefore $M \otimes_A A_{\mathfrak{p}} \neq 0$, whence the first assertion. As the homomorphism $\phi_{\mathfrak{q}} : A_{\mathfrak{p}} \rightarrow B_{\mathfrak{q}}$ is local, the second assertion follows from ??.

Proposition 9.4.41. *Let A be a ring and M a finitely generated A -module. For every prime ideal $\mathfrak{p} \in \text{supp}(M)$, there exists a non-zero A -homomorphism $\eta : M \rightarrow A/\mathfrak{p}$.*

Proof. Let $\mathfrak{p} \in \text{supp}(M)$. As M is finitely generated and $M \neq 0$,

$$M_{\mathfrak{p}}/\mathfrak{p}M_{\mathfrak{p}} = M_{\mathfrak{p}} \otimes_A (A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}) \neq 0$$

Let $\kappa = A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$ be the field of fractions of the integral domain A/\mathfrak{p} . Since $M_{\mathfrak{p}}/\mathfrak{p}M_{\mathfrak{p}}$ is a vector space over κ which is not reduced to 0, there exists a non-zero linear form $u : M_{\mathfrak{p}}/\mathfrak{p}M_{\mathfrak{p}} \rightarrow \kappa$. If x_1, \dots, x_n is a system of generators of M , and \bar{x}_i is the image of x_i in the (A/\mathfrak{p}) -module $M_{\mathfrak{p}}/\mathfrak{p}M_{\mathfrak{p}}$. Let $\alpha \in A/\mathfrak{p}$ be such that $\alpha u(\bar{x}_i) \in A/\mathfrak{p}$ for $1 \leq i \leq n$, then $v = \alpha u$ is a non-zero (A/\mathfrak{p}) -linear map from $M_{\mathfrak{p}}/\mathfrak{p}M_{\mathfrak{p}}$ to A/\mathfrak{p} . The composition

$$M \longrightarrow M_{\mathfrak{p}} \longrightarrow M_{\mathfrak{p}}/\mathfrak{p}M_{\mathfrak{p}} \xrightarrow{v} A/\mathfrak{p}$$

is therefore the required homomorphism. \square

9.4.4 Clopen subsets of the spectrum

Lemma 9.4.42. *Let A be a ring. Let $e \in A$ be an idempotent. In this case*

$$\text{Spec}(A) = D(e) \amalg D(1 - e).$$

Proof. Note that an idempotent e of a domain is either 1 or 0. Hence we see that

$$D(e) = \{\mathfrak{p} \in \text{Spec}(A) : e \neq 0 \text{ in } A/\mathfrak{p}\} = \{\mathfrak{p} \in \text{Spec}(A) : e = 1 \text{ in } A/\mathfrak{p}\}.$$

Similarly we have

$$D(1 - e) = \{\mathfrak{p} \in \text{Spec}(A) : e \neq 1 \text{ in } A/\mathfrak{p}\} = \{\mathfrak{p} \in \text{Spec}(A) : e = 0 \text{ in } A/\mathfrak{p}\}.$$

Since the image of e is either 1 or 0 we deduce that $D(e)$ and $D(1 - e)$ cover all of $\text{Spec}(A)$. \square

Lemma 9.4.43. *Let A_1 and A_2 be rings. Let $A = A_1 \times A_2$. The maps $A \rightarrow A_1, (x, y) \mapsto x$ and $A \rightarrow A_2, (x, y) \mapsto y$ induce continuous maps $\text{Spec}(A_1) \rightarrow \text{Spec}(A)$ and $\text{Spec}(A_2) \rightarrow \text{Spec}(A)$. The induced map*

$$\text{Spec}(A_1) \amalg \text{Spec}(A_2) \rightarrow \text{Spec}(A)$$

is a homeomorphism. In other words, the spectrum of $A = A_1 \times A_2$ is the disjoint union of the spectrum of A_1 and the spectrum of A_2 .

Proof. Write $1 = e_1 + e_2$ with $e_1 = (1, 0)$ and $e_2 = (0, 1)$. Note that e_1 and $e_2 = 1 - e_1$ are idempotents. Thus $\text{Spec}(A) = D(e_1) \amalg D(e_2)$ by the previous lemma. Now consider the localization A_{e_1} : $\{0\} \times A_2$ is mapped to zero in it, so $A_1 = A_{e_1}$. Similarly we have $A_{e_2} = A_2$. Thus by ?? $D(e_1) = \text{Spec}(A_1), D(e_2) = \text{Spec}(A_2)$. \square

We already know that an idempotent element gives an open and closed subset of $\text{Spec}(A)$. We now prove a converse.

Proposition 9.4.44. Let A be a ring. For each $U \subseteq \text{Spec}(A)$ which is open and closed there exists a unique idempotent $e \in A$ such that $U = D(e)$. This induces a one-to-one correspondence between open and closed subsets $U \subseteq \text{Spec}(A)$ and idempotents $e \in A$.

Proof. Let $U \subseteq \text{Spec}(A)$ be open and closed, and V be its complement. We can write U and V as unions of standard opens such that

$$U = \bigcup_{i=1}^n D(f_i) \quad \text{and} \quad W = \bigcup_{j=1}^m D(g_j).$$

Since $\text{Spec}(A) = U \amalg V$, we observe that the collection $\{f_i, g_j\}$ must generate the unit ideal in A by ?? So the following sequence is exact

$$0 \longrightarrow A \xrightarrow{\alpha} \bigoplus_{i=1}^n A_{f_i} \oplus \bigoplus_{j=1}^m A_{g_j} \xrightarrow{\beta} \bigoplus_{i_1, i_2} A_{f_{i_1} f_{i_2}} \oplus \bigoplus_{i, j} A_{f_i g_j} \oplus \bigoplus_{j_1, j_2} A_{g_{j_1} g_{j_2}}$$

However, notice that for any pair i, j , $D(f_i g_j) = D(f_i) \cap D(g_j) = \emptyset$ since $D(f_i) \subseteq U$ and $D(g_j) \subseteq V$. Therefore by ?? we see $\text{Spec}(A_{f_i g_j}) = D(f_i g_j) = \emptyset$, which implies $A_{f_i g_j}$ is the zero ring for each pair i, j . Consider the element $(1, \dots, 1, 0, \dots, 0) \in \bigoplus_{i=1}^n A_{f_i} \oplus \bigoplus_{j=1}^m A_{g_j}$ whose coordinates are 1 in each A_{f_i} and 0 in each A_{g_j} . It is sent to 0 under the map

$$\beta : \bigoplus_{i=1}^n A_{f_i} \oplus \bigoplus_{j=1}^m A_{g_j} \rightarrow \bigoplus_{i_1, i_2} A_{f_{i_1} f_{i_2}} \oplus \bigoplus_{j_1, j_2} A_{g_{j_1} g_{j_2}}$$

so by the exactness of the sequence, there must be some element of A whose image under α is $(1, \dots, 1, 0, \dots, 0)$, which we denote by e . We see that $\alpha(e^2) = \alpha^2(e) = (1, \dots, 1, 0, \dots, 0) = \alpha(e)$. Since α is injective, $e = e^2$ in A and so e is an idempotent of A . We claim that $U = D(e)$. Notice that for any j , the map $A \mapsto A_{g_j}$ maps e to 0. Therefore there must be some positive integer k_j such that $g_j^{k_j} e = 0$ in A . Multiplying by e as necessary, we see that $(g_j e)^{k_j} = 0$, so $g_j e$ is nilpotent in A . By ?? $D(e) \cap D(g_j) = D(eg_j) = \emptyset$. So since $V = \bigcup_{j=1}^m D(g_j)$, we get $D(e) \cap V = \emptyset$; and then $D(e) \subseteq U$. Furthermore, for any i , the map $A \mapsto A_{f_i}$ maps e to 1, so there must be some l_i such that $f_i^{l_i}(e - 1) = 0$ in A . Hence $f_i^{l_i} e = f_i^{l_i}$. Suppose $\mathfrak{p} \in \text{Spec}(A)$ contains e , then \mathfrak{p} contains $f_i^{l_i} e = f_i^{l_i}$, and since \mathfrak{p} is prime, $f_i \in \mathfrak{p}$. So $V(e) \subseteq V(f_i)$, implying that $D(f_i) \subseteq D(e)$. Therefore $U = \bigcup_{i=1}^n D(f_i) \subseteq D(e)$, and $U = D(e)$. Therefore any open and closed subset of $\text{Spec}(A)$ is the standard open of an idempotent as desired. \square

Corollary 9.4.45. Let A be a nonzero ring. Then $\text{Spec}(A)$ is connected if and only if A has no non-trivial idempotents.

Proposition 9.4.46. Let A be a ring. Let \mathfrak{a} be a finitely generated ideal. Assume that $\mathfrak{a} = \mathfrak{a}^2$. Then $V(\mathfrak{a})$ is open and closed in $\text{Spec}(A)$, and $A/\mathfrak{a} \cong A_e$ for some idempotent $e \in A$.

Proof. By Nakayama's Lemma there exists an element $f = 1 + a$, $a \in \mathfrak{a}$ in A such that $f\mathfrak{a} = 0$. It follows that $V(\mathfrak{a}) = D(f)$ by a simple argument. Also, $0 = fa = a + a^2$, and hence

$$f^2 = (1 + a)^2 = 1 + 2a + a^2 = 1 + a = f,$$

so f is an idempotent. Consider the canonical map $A \rightarrow A_f$. It is surjective since $x/f^n =$

$x/f = xf/f^2 = x/1$. Any element of \mathfrak{a} is in the kernel since $f\mathfrak{a} = 0$, and $x/1 = 0$ in A_f implies $f^n x = fx = (1 + \mathfrak{a})x = 0$, which means $x \in \mathfrak{a}$. \square

9.4.5 Glueing properties of localization

In this part we show that given an open covering $\text{Spec}(A) = \bigcup_{i=1}^n D(f_i)$ by standard opens, and given an element $h_i \in A_{f_i}$ for each i such that $h_i = h_j$ as elements of $A_{f_i f_j}$, then there exists a unique $h \in A$ such that the image of h in A_{f_i} is h_i . This result can be viewed as a sheaf property for $\text{Spec}(A)$, and will play an important rule when we consider schemes.

Proposition 9.4.47. *Let A be a ring, and let $f_1, \dots, f_n \in A$ generate the unit ideal of A . Then the following sequence is exact:*

$$0 \longrightarrow A \xrightarrow{\alpha} \bigoplus_i A_{f_i} \xrightarrow{\beta} \bigoplus_{i,j} A_{f_i f_j} \quad (9.4.1)$$

where the maps $\alpha : A \rightarrow \bigoplus_i A_{f_i}$ and $\beta : \bigoplus_i A_{f_i} \rightarrow \bigoplus_{i,j} A_{f_i f_j}$ are defined as

$$\alpha(x) = (x/1, \dots, x/1), \quad \beta(x_1/f_1^{r_1}, \dots, x_n/f_n^{r_n}) = x_i/f_i^{r_i} - x_j/f_j^{r_j}.$$

Proof. First let $x \in A$ such that $\alpha(x) = 0$. This means $x/1 = 0$ in A_{f_i} and so there exists n_i for each i such that

$$f_i^{n_i} x = 0.$$

Since f_1, \dots, f_n generate A , we can find $a_i \in A$ such that $1 = \sum_{i=1}^n a_i f_i$. Then for all $N \geq \sum_i n_i$, we have

$$1^N = \left(\sum_{i=1}^n a_i f_i \right)^N = \sum_{u_1, \dots, u_n} \binom{N}{u_1, \dots, u_n} a_i^{u_i} \cdots a_n^{u_n} f_1^{u_1} \cdots f_n^{u_n}.$$

where each term has a factor of at least $f_i^{n_i}$ for some i . Therefore,

$$x = 1^N \cdot x = \sum_{u_1, \dots, u_n} \binom{N}{u_1, \dots, u_n} a_i^{u_i} \cdots a_n^{u_n} f_1^{u_1} \cdots f_n^{u_n} x = 0.$$

Thus, if $\alpha(x) = 0$, then $x = 0$ and so α is injective.

Now we check that the image of α equals the kernel of β . First, note that for $x \in A$ we have $\beta(\alpha(x)) = 0$, therefore the image of α is in the kernel of β . Now assume we have $x_1, \dots, x_n \in A$ such that

$$\beta\left(\frac{x_1}{f_1^{r_1}}, \dots, \frac{x_n}{f_n^{r_n}}\right) = 0.$$

Then, for all pairs i, j , there exists an n^{ij} such that

$$f_i^{n_{ij}} f_j^{n_{ij}} (x_i f_j^{r_j} - x_j f_i^{r_i}) = 0.$$

Choosing N so $N \geq n_{ij}$ for all i, j , we see that

$$f_i^N f_j^N (x_i f_j^{r_j} - x_j f_i^{r_i}) = 0.$$

Define elements \tilde{x}_i and \tilde{f}_i of A as follows:

$$\tilde{f}_i = f_i^{N+r_i}, \quad \tilde{x}_i = f_i^N x_i$$

so that $x_i/f_i^{r_i} = \tilde{x}_i/\tilde{f}_i$. Then we can use this to rewrite the above equation to get the following equality, for all i, j ,

$$\tilde{f}_j \tilde{x}_i = \tilde{f}_i \tilde{x}_j.$$

Since f_1, \dots, f_n generate A , we clearly have that $\tilde{f}_1, \dots, \tilde{f}_n$ also generate A . Therefore, there exist a_1, \dots, a_n in A so that

$$1 = \sum_{i=1}^n a_i \tilde{f}_i.$$

Therefore, we finally conclude that for all i ,

$$\frac{x_i}{f_i^{r_i}} = \frac{\tilde{x}_i}{\tilde{f}_i} = \sum_{j=1}^n \frac{a_j \tilde{f}_j \tilde{x}_i}{\tilde{f}_i} = \sum_{j=1}^n \frac{a_j \tilde{f}_i \tilde{x}_j}{\tilde{f}_i} = \frac{\sum_{j=1}^n a_j \tilde{x}_j}{1}.$$

which means

$$\alpha \left(\sum_{j=1}^n a_j \tilde{x}_j \right) = \left(\frac{x_1}{f_1^{r_1}}, \dots, \frac{x_n}{f_n^{r_n}} \right).$$

as required. \square

Corollary 9.4.48. Let A be a ring. Let f_1, \dots, f_n be elements of A generating the unit ideal. Let M be an A -module. Then the following sequence is exact:

$$0 \longrightarrow M \xrightarrow{\alpha} \bigoplus_i M_{f_i} \xrightarrow{\beta} \bigoplus_{i,j} M_{f_i f_j} \tag{9.4.2}$$

Proof. This can be proved as ??, using the definition of M_f . \square

Proposition 9.4.49. Let A be a ring. Let $f_1, \dots, f_n \in A$ and M be an A -module. Then $M \rightarrow \bigoplus M_{f_i}$ is injective if and only if the homomorphism

$$\psi : M \rightarrow \bigoplus_{i=1}^n M, \quad m \mapsto (f_1 m, \dots, f_n m)$$

is injective.

Proof. The map $M \rightarrow \bigoplus_{i=1}^n M_{f_i}$ is injective if and only if for all $m \in M$ and $e_1, \dots, e_n \geq 1$ such that $f_i^{e_i} m = 0$, $1 \leq i \leq n$ we have $m = 0$. This clearly implies that ψ is injective. Conversely, suppose ψ is injective and $m \in M$ and $e_1, \dots, e_n \geq 1$ are such that $f_i^{e_i} m = 0$. Let $N \geq e_i$ for all i , then $f_i^N m = 0$ for all i , which is $\psi^N(m) = 0$. Since ψ is injective, this implies $m = 0$. \square

Proposition 9.4.50. Let $(f_i)_{i \in I}$ be a finite family of elements of a ring A generating the unit ideal of A . Then the ring $B = \prod_{i \in I} A_{f_i}$ is a faithfully flat A -module.

Proof. We know each of the A_{f_i} is a flat A -module, hence so is B . On the other hand, if \mathfrak{p} is a prime ideal of A , there exists an index i such that $f_i \notin \mathfrak{p}$ and $\mathfrak{p} f_i = \mathfrak{p} A_{f_i}$ is therefore a prime ideal of A_{f_i} . Then $\mathfrak{p} B \subseteq \mathfrak{p} A_{f_i} \times \prod_{j \neq i} A_{f_j} \neq B$ since $\mathfrak{p} A_{f_i} \neq A_{f_i}$. This suffices to imply that B is a faithfully flat A -module. \square

Proposition 9.4.51. Let $(f_i)_{i \in I}$ be a finite family of elements of a ring A generating the unit ideal of A .

- (a) For an A -module M to be zero, it is necessary and sufficient that, for every index i , the A_{f_i} -module M_{f_i} is zero.
- (b) For an A -module M to be finitely generated (resp. finitely presented), it is necessary and sufficient that, for every index i , the A_{f_i} -module M_{f_i} be finitely generated (resp. finitely presented).
- (c) Let $\phi : M \rightarrow N$ be a homomorphism of A -modules. Then ϕ is injective (resp. surjective) if and only if for every index i , the homomorphism $\phi_{f_i} : M_{f_i} \rightarrow N_{f_i}$ is injective (resp. surjective).

Proof. If $M = 0$ then clearly $M_{f_i} = 0$ for all i . Conversely, if each M_{f_i} is zero, then the A -module $P = \bigoplus_{i \in I} M_{f_i}$ is zero. But $P = M \otimes_A \prod_{i \in I} A_{f_i}$, so the claim in (a) follows from ???. Also, (c) follows immediately from (a).

Now if M is finitely generated (resp. finitely presented) then clearly every M_{f_i} is finitely generated (resp. finitely presented). Conversely, if all the M_{f_i} are finitely generated (resp. finitely presented), then P is a finitely generated (resp. finitely presented) B -module (for we can obviously suppose that for each i there is an exact sequence $A_f^m \rightarrow A_f^n \rightarrow M_{f_i} \rightarrow 0$, where m and n are independent of i). Again by ??, we see M is finitely generated (resp. finitely presented). \square

Corollary 9.4.52. Let $(f_i)_{i \in I}$ be a finite family of elements of a ring A generating the unit ideal of A . Then A is Noetherian if and only if for every index $i \in I$, then ring A_{f_i} is Noetherian.

Proof. This follows from ?? by applying to ideals of A . \square

Corollary 9.4.53. Let $(f_i)_{i \in I}$ be a finite family of elements of a ring A generating the unit ideal of A . For an A -algebra B to be of finite type (resp. finitely presented) over A , it is necessary and sufficient that, for every index i , the A_{f_i} -algebra B_{f_i} be of finite type (resp. finitely presented) over A_{f_i} .

Proof. One direction is clear. Conversely, assume that each B_{f_i} is of finite type (finite presented) over A_{f_i} . Since A_{f_i} is finitely presented, we then see each B_{f_i} is of finite type (finite presented) over A . For each $i \in I$ take a finite generating set S_i of B_{f_i} . Without loss of generality, we may assume that the elements of S_i are in the image of the localization map $B \rightarrow B_{f_i}$, so we take a finite set T_i of preimages of the elements of S_i in B . Let T be the union of these sets. This is still a finite set. Consider the algebra homomorphism $A[X_t]_{t \in T} \rightarrow B$ induced by T . Since it is an algebra homomorphism, the image B' is an A -submodule of the A -module B , so we can consider the quotient module B/B' . Now it is clear that ?? implies both of the claims (note that $\{f_i\}$ also generates the unit ideal in $A[X_t]$). \square

Corollary 9.4.54. Let $\rho : A \rightarrow B$ be a ring map and $\{g_i\}_{i \in I}$ elements of B generating the unit ideal of B . Then for B to be of finite type (finite presented) over A , it is necessary and sufficient that for every index $i \in I$, the algebra B_{g_i} is of finite type (finite presented) over A .

Proof. Choose h_i in B such that $1 = \sum_i g_i h_i$. We first prove that, if B_{g_i} is of finite type, then B is of finite type. For this, for each $i \in I$ choose a finite subset S_i which generate B_{g_i} as an A -algebra and let T_i be its preimage in B (as in the proof of ??). Consider the A -subalgebra $B' \subseteq B$ generated by $\{g_i\}$, $\{h_i\}$ and T_i for all $i \in I$. Since localization is exact, we see that $B'_{g_i} \rightarrow B_{g_i}$ is injective. On the other hand, it is surjective by our choice of T_i . The elements $\{g_i\}$

generate the unit ideal in B' as $h_i \in B'$, so $B' \rightarrow B$ viewed as an B' -module homomorphism is an isomorphism by ??.

Now we consider the case for finite presentation. Assume that each B_{g_i} is finitely presented. We already know that B is of finite type, so we may identify B with $A[x_1, \dots, x_n]/\mathfrak{a}$ for some ideal \mathfrak{a} . Moreover, we may choose elements $\tilde{g}_i, \tilde{h}_i \in A[x_1, \dots, x_n]$ whose image in B is g_i and h_i . Then we see

$$B_{g_i} = A[x_1, \dots, x_n, y_i]/(\mathfrak{a}_i + (1 - y_i \tilde{g}_i)),$$

where \mathfrak{a}_i is the ideal of $A[x_1, \dots, x_n, y_i]$ generated by \mathfrak{a} . By ??, we may choose a finite list of elements $f_{ij} \in \mathfrak{a}$ such that the image of f_{ij} in $\mathfrak{a}_i + (1 - y_i \tilde{g}_i)$ generate the ideal $\mathfrak{a}_i + (1 - y_i \tilde{g}_i)$. If we set

$$B' = A[x_1, \dots, x_n]/(\sum_i \tilde{g}_i \tilde{h}_i - 1, f_{ij}),$$

there is then a surjective A -algebra map $B' \rightarrow B$. The classes of the elements \tilde{g}_i in B' generate the unit ideal and by construction the maps $B'_{g_i} \rightarrow B_{g_i}$ are injective, so we conclude that $B' \rightarrow B$ is injective, hence an isomorphism. \square

Proposition 9.4.55. *Let \mathfrak{a} be an ideal of A , $f \in A$ such that A_f/\mathfrak{a}_f is an A -algebra of finite presentation. Then \mathfrak{a}_f is finitely generated in A_f .*

Proof. By hypothesis, there exists a polynomial algebra $B = A[T_1, \dots, T_n]$ and a surjective A -homomorphism $\rho : B \rightarrow A_f/\mathfrak{a}_f$, whose kernel \mathfrak{b} is finitely generated. Let $\pi : A_f \rightarrow A_f/\mathfrak{a}_f$ be the canonical homomorphism; for each $1 \leq i \leq n$, there exists $t_i \in A$ such that $\pi(t_i/s^k) = \rho(T_i)$ (we can choose an integer k for all index i). Consider then the A_f -algebra $C = A_f[T_1, \dots, T_n]$ and the homomorphism $\varphi : C \rightarrow A_f$ such that $\varphi(T_i) = t_i/s^k$. The homomorphism φ is clearly surjective, and so is the composition $\rho' = \pi \circ \varphi : C \rightarrow A_f/\mathfrak{a}_f$.

$$\begin{array}{ccc} B = A[T_1, \dots, T_n] & & \\ \downarrow \rho & & \\ A_f & \xrightarrow{\pi} & A_f/\mathfrak{a}_f \\ \varphi \swarrow & & \uparrow \rho' \\ C = A_f[T_1, \dots, T_n] & & \end{array}$$

On the other hand, any polynomial in C is of the form P/s^m , where $P \in B$, and we have $\rho'(P/s^m) = (1/s^m)\rho(P)$. As $1/s$ is invertible in A_f , the relation $\rho'(P/s^m) = 0$ in A_f/\mathfrak{a}_f is equivalent to $\rho(P) = 0$, and therefore the kernel \mathfrak{b}' of ρ' is generated by the image of \mathfrak{b} in C , and a fortiori is finitely generated in C . As $\mathfrak{b}' = \varphi^{-1}(\pi^{-1}(0)) = \varphi^{-1}(\mathfrak{a}_f)$ and φ is surjective, we see $\mathfrak{a}_f = \varphi(\mathfrak{b}')$, so \mathfrak{a}_f is finitely generated in A_f . \square

Remark 9.4.3. If B is an A -algebra of finite presentation, B' is an A -algebra of finite type, and $\rho : B' \rightarrow B$ is a surjective homomorphism, then the kernel \mathfrak{b} of ρ is finitely generated in B' . In fact, there exists a surjective homomorphism $\psi : C \rightarrow B'$, where C is a polynomial algebra over A ; as $\rho \circ \psi$ is surjective, $\ker(\rho \circ \psi)$ is a finitely generated ideal in C by ?? . As we have $\mathfrak{b} = \psi(\ker(\rho \circ \psi))$, \mathfrak{b} is then finitely generated in B' .

Conversely, if B' is an A -algebra of finite presentation, $\rho : B' \rightarrow B$ is a surjective homomorphism, and if $\mathfrak{b} = \ker \rho$ is finitely generated in B' , then B is an A -algebra of finite presentation. In fact, we have a surjective homomorphism $\psi : C \rightarrow B'$ where C is a polynomial algebra and $\ker \psi$ is finitely generated in C . As $\ker(\rho \circ \psi)$ is generated by $\ker \psi$ and a system of elements whose image in B' generate \mathfrak{b} , it is finitely generated and therefore B is an A -algebra of finite presentation.

9.5 Finitely generated projective modules

9.5.1 Local characterization

Proposition 9.5.1. *Let A be a ring, $\phi : M \rightarrow N$ an A -module homomorphism and \mathfrak{p} a prime ideal of A .*

- (a) *Suppose that $\phi_{\mathfrak{p}} : M_{\mathfrak{p}} \rightarrow N_{\mathfrak{p}}$ is surjective and that N is finitely generated. Then there exists $f \in A - \mathfrak{p}$ such that $\phi_f : M_f \rightarrow N_f$ is surjective.*
- (b) *Suppose that $\phi_{\mathfrak{p}} : M_{\mathfrak{p}} \rightarrow N_{\mathfrak{p}}$ is bijective, that M is finitely generated and that N is finitely presented. Then there exists $f \in A - \mathfrak{p}$ such that $\phi_f : M_f \rightarrow N_f$ is bijective.*

Proof. Let P and Q be the kernel and cokernel of ϕ . If $f \in A$, the kernel and cokernel of ϕ_f (resp. $\phi_{\mathfrak{p}}$) are P_f and Q_f (resp. $P_{\mathfrak{p}}$ and $Q_{\mathfrak{p}}$). In the case (a), $\phi_{\mathfrak{p}}$ is surjective so $Q_{\mathfrak{p}} = 0$; as N is finitely generated, so is Q and therefore there exists $f \in A - \mathfrak{p}$ such that $fQ = 0$, whence $Q_f = 0$.

Under the hypotheses of (b), the sequence $0 \rightarrow P_f \rightarrow M_f \rightarrow N_f \rightarrow 0$ is exact, hence P_f is finitely generated. Now, $(P_f)_{\mathfrak{p}P_f} = P_{\mathfrak{p}} = 0$, hence there exists $g/f^d \in A_f - \mathfrak{p}A_f$ (whence $g \in A - \mathfrak{p}$) such that $(g/f^d)P_f = 0$. Then as $1/f$ is invertible, we have $(g/1)P_f = 0$, whence $P_{fg} = (P_f)_{g/1} = 0$. Since $fg \in A - \mathfrak{p}$ and $Q_{fg} = 0$, we get the claim. \square

Corollary 9.5.2. *If N is finitely presented and $N_{\mathfrak{p}}$ is a free $A_{\mathfrak{p}}$ -module of rank p , there exists $f \in A - \mathfrak{p}$ such that N_f is a free A_f -module of rank p .*

Proof. There exist by hypothesis elements x_1, \dots, x_p such that the $x_i/1$ form a basis of the free $A_{\mathfrak{p}}$ -modulc $N_{\mathfrak{p}}$. Consider the homomorphism $\phi : A^p \rightarrow N$ such that $\phi(e_i) = x_i$, (e_i) being the canonical basis of A^p . As $\phi_{\mathfrak{p}}$ is bijective by hypothesis, there exists $f \in A - \mathfrak{p}$ such that ϕ_f is bijective, by virtue of ??.

Remark 9.5.1. In the language of algebraic geometry, Corallary ?? shows that if \mathcal{F} is a coherent sheaf over a scheme X such that $\mathcal{F}|_x$ is free of rank n for some point $x \in X$, then there exist a neighborhood U of x such that $\mathcal{F}|_U$ is free of rank n .

Corollary 9.5.3. *Let N be a finitely generated A -module and consider the function ϕ on $X = \text{Spec}(A)$ defined by*

$$\phi(\mathfrak{p}) = \dim_{\kappa(\mathfrak{p})}(N_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} \kappa(\mathfrak{p})).$$

Then ϕ is upper semi-continuous, i.e., for any $n \in \mathbb{Z}$, the set $\{\mathfrak{p} \in X : \phi(\mathfrak{p}) < n\}$ is open.

Proof. Note that $\phi(\mathfrak{p}) = [N_{\mathfrak{p}}/\mathfrak{p}N_{\mathfrak{p}} : \kappa(\mathfrak{p})]$. By Nakayama's Lemma, this number is equal to the minimal number of generators of $N_{\mathfrak{p}}$ as an $A_{\mathfrak{p}}$ module. Let m_1, \dots, m_r be such a minimal

generating set with $r < n$. Write $m_i = x_i/f$ for $x_i \in N$ and $f \in A - \mathfrak{p}$, then we get a map $\phi : A_f^r \rightarrow N_f$ defined by $\phi(e_i/1) = x_i/f$, such that $\phi_{\mathfrak{p}}$ is surjective. By ?? there exist some $g \in A - \mathfrak{p}$ such that $\phi_g : A_{fg}^r \rightarrow N_{fg}$ is surjective. Then for any $\mathfrak{q} \in D(fg)$, by localizing we see $\phi_{\mathfrak{q}} : A_{\mathfrak{q}}^r \rightarrow N_{\mathfrak{q}}$ is also surjective, whence $\phi(\mathfrak{q}) \leq r < n$. Therefore the set $\{\mathfrak{p} \in X : \phi(\mathfrak{p}) < n\}$ is open. \square

Theorem 9.5.4. *Let A be a ring and P an A -module. Then the following properties are equivalent:*

- (i) *P is a finitely generated projective module.*
- (ii) *P is a finitely presented module and, for every maximal ideal \mathfrak{m} of A , $P_{\mathfrak{m}}$ is a free $A_{\mathfrak{m}}$ -module.*
- (iii) *P is a finitely generated module, for all $\mathfrak{p} \in \text{Spec}(A)$, the $A_{\mathfrak{p}}$ -module $P_{\mathfrak{p}}$ is free and, if we denote its rank by $\text{rank}_{\mathfrak{p}}$, the function $\mathfrak{p} \mapsto \text{rank}_{\mathfrak{p}}$ is locally constant in the topological space $\text{Spec}(A)$.*
- (iv) *There exists a finite family $(f_i)_{i \in I}$ of elements of A , generating the ideal A , such that for all $i \in I$, the A_{f_i} -module P_{f_i} is free of finite rank.*
- (v) *For every maximal ideal \mathfrak{m} of A , there exists $f \in A - \mathfrak{m}$ such that $P_{\mathfrak{m}}$ is free of finite rank.*

Proof. By ??, we know that a finitely generated projective module is finitely presented. If P is a projective A -module, $P_{\mathfrak{m}} = P \otimes_A A_{\mathfrak{m}}$ is a projective $A_{\mathfrak{m}}$ -module. Finally, as $A_{\mathfrak{m}}$ is a local ring, every finitely presented projective $A_{\mathfrak{m}}$ -module is free (??). This shows (i) implies (ii). Also, (ii) implies (v) by ??.

Now assume (v), and let E be the set of $f \in A$ such that P_f is a finitely generated free A_f -module. The hypothesis implies that E is contained in no maximal ideal of A , hence E generates the ideal A and there therefore exist a finite family $(f_i)_{i \in I}$ of elements of E that generates A , whence (iv).

Assume (iv) and consider the ring $B = \prod_{i \in I} A_{f_i}$ and the B -module $M = \bigoplus_{i \in I} P_{f_i} = P \otimes_A B$. For every index i , there exists a free A_{f_i} -module L_i such that P_i is a direct factor of L_i and it may be assumed that all the L_i have the same rank; then $L = \bigoplus_{i \in I} L_i$ is a free B -module of which M is a direct factor, in other words M is a finitely generated projective B -module. As B is a faithfully flat A -module, we conclude that P is a finitely generated projective A -module by ??.

Finally, we show that (iv) implies (iii) and (iii) implies (v). If (iv) holds, then it follows from ?? that P is finitely generated. On the other hand, for every prime ideal \mathfrak{p} of A , there exists an index i such that $f_i \notin \mathfrak{p}$; then $P_{\mathfrak{p}} = (P_{f_i})_{\mathfrak{p} f_i}$ and hence by hypothesis $P_{\mathfrak{p}}$ is free and of the same rank as P_{f_i} , which proves (iii).

Assume (iii), let \mathfrak{m} be a maximal ideal of A ; write $\text{rank}_{\mathfrak{m}} = n$ and let x_1, \dots, x_n be a basis of $P_{\mathfrak{m}}$. We can assume that the x_i are canonical images of elements $p_i \in P$ to within multiplication by an invertible element of $A_{\mathfrak{m}}$. Let (e_i) be the canonical basis of A^n and $\phi : A^n \rightarrow P$ the homomorphism such that $\phi(e_i) = p_i$. As P is finitely generated, it follows from ?? that there exists $f \in A - \mathfrak{m}$ such that ϕ_f is surjective. We conclude that ϕ_{fg} is also surjective for all $g \in A - \mathfrak{m}$ and by hypothesis there exists $g \in A - \mathfrak{m}$ such that $\text{rank}_{\mathfrak{p}} = n$ for $\mathfrak{p} \in D(g)$. Then, replacing f by fg , we may assume that $\text{rank}_{\mathfrak{p}} = n$ for all $\mathfrak{p} \in D(f)$. Then $\phi_{\mathfrak{p}} : A_{\mathfrak{p}}^n \rightarrow P_{\mathfrak{p}}$ is a surjective homomorphism and $P_{\mathfrak{p}}$ and $A_{\mathfrak{p}}^n$ are both free $A_{\mathfrak{p}}$ -modules of the same rank; hence by ?? $\phi_{\mathfrak{p}}$ is bijective for all $\mathfrak{p} \in D(f)$. Let \mathfrak{p}' be a prime ideal of A_f and let \mathfrak{p} be its inverse image in A under the canonical map; if $(A_f^n)_{\mathfrak{p}'}$ and $(P_f)_{\mathfrak{p}'}$ are identified with $A_{\mathfrak{p}}^n$ and $P_{\mathfrak{p}}$ under the canonical isomorphisms, $(\phi_f)_{\mathfrak{p}'}$ is then identified with $\phi_{\mathfrak{p}}$ and is consequently bijective. We conclude that ϕ is bijective, which establishes (v). \square

Corollary 9.5.5. Suppose that the equivalent properties of the statement of ?? hold. Let n be a positive integer such that, for every family (x_1, \dots, x_n) of elements of P , there exists a family (a_1, \dots, a_n) of elements of A , which are not all divisors of zero and for which $\sum_{i=1}^n a_i x_i = 0$. Then, for all $\mathfrak{p} \in \text{Spec}(A)$, $\text{rank}_{\mathfrak{p}} \leq n$.

Proof. Let \mathfrak{p} be a prime ideal of A , set $\text{rank}_{\mathfrak{p}} = r$ and let (y_1, \dots, y_r) be a basis of the free $A_{\mathfrak{p}}$ -module $P_{\mathfrak{p}}$. There exist elements (x_1, \dots, x_r) of P and $s \in A - \mathfrak{p}$ such that $y_i = x_i/s$ for all i . If $r > n$, then there exist a family (a_1, \dots, a_r) of elements of A such that $\sum_{i=1}^r a_i x_i = 0$, which are not all divisors of zero (for example take (a_1, \dots, a_n) as in the hypothesis and let $a_i = 0$ for $i > n$). But then we have $\sum_{i=1}^r (a_i/1)y_i = 0$ and therefore $a_i = 0$ in $A_{\mathfrak{p}}$, which implies a_i is a divisor of zero, contradiction. \square

Corollary 9.5.6. Every finitely presented flat module is projective.

Proof. If P is a finitely presented flat A -module and \mathfrak{m} a maximal ideal of A , the $A_{\mathfrak{m}}$ -module $P_{\mathfrak{m}}$ is flat and finitely presented and hence free (??). Condition (ii) of ?? therefore holds. \square

9.5.2 The rank function

Definition 9.5.7. Let P be a finitely generated projective A -module. For every prime ideal \mathfrak{p} of A , the rank of the free $A_{\mathfrak{p}}$ -module $P_{\mathfrak{p}}$ is called the **rank** of P at \mathfrak{p} and is denoted by $\text{rank}_{\mathfrak{p}}(P)$.

By ?? the integer-valued function $\mathfrak{p} \mapsto \text{rank}_{\mathfrak{p}}(P)$ is locally constant on $X = \text{Spec}(A)$. It is therefore constant if X is connected and in particular if the ring A is an integral domain.

Definition 9.5.8. Let n be a positive integer. A projective A -module P is said to be of rank n if it is finitely generated and $\text{rank}_{\mathfrak{p}}(P) = n$ for every prime ideal \mathfrak{p} of A .

Clearly every finitely generated free A -module L is of rank n in the sense of the above definition, n being equal to the dimension (or rank) of L . A projective module of rank 0 is zero. If A is not reduced to 0 and a projective A -module P is of rank n , the integer n is determined uniquely; it is then denoted by $\text{rank}(P)$.

Theorem 9.5.9. Let P be an A -module and n a positive integer. The following properties are equivalent:

- (i) P is projective of rank n .
- (ii) P is finitely generated and, for every maximal ideal \mathfrak{m} of A , the $A_{\mathfrak{m}}$ -module $P_{\mathfrak{m}}$ is free of rank n .
- (iii) P is finitely generated and, for every prime ideal \mathfrak{p} of A , the $A_{\mathfrak{p}}$ -module $P_{\mathfrak{p}}$ is free of rank n .
- (iv) For every maximal ideal \mathfrak{m} of A , there exists $f \in A - \mathfrak{m}$ such that the $A_{\mathfrak{m}}$ -module $P_{\mathfrak{m}}$ is free of rank n .

Proof. By definition and ??, (i) and (iii) are equivalent; (ii) implies (iii), as, for every prime ideal \mathfrak{p} of A , there exists a maximal ideal \mathfrak{m} containing \mathfrak{p} and, writing $\mathfrak{p}' = \mathfrak{p}_{\mathfrak{m}}$, $P_{\mathfrak{p}}$ is isomorphic to $(P_{\mathfrak{m}})_{\mathfrak{p}'}$. If $P_{\mathfrak{m}}$ is free of rank n , so then is $P_{\mathfrak{p}}$. Property (iii) implies (i) by virtue of ?? and the fact that, iff $f \in A - \mathfrak{m}$ and $\mathfrak{m}' = \mathfrak{m}_f$, $P_{\mathfrak{m}}$ is isomorphic to $(P_f)_{\mathfrak{m}'}$ and therefore the ranks of P_f and $P_{\mathfrak{m}}$ are equal. Finally, this last argument and ?? show that (iv) implies (ii). \square

Corollary 9.5.10. If A is an integral domain and P is a projective A -module, then $\text{rank}_{\mathfrak{p}}(P) = \text{rank}(P)$ for all prime ideal \mathfrak{p} of A .

Proof. It is sufficient to apply ??(iii) with $\mathfrak{p} = (0)$. \square

Let E and F be two finitely generated projective A -modules. We know that $E \oplus F$, $E \otimes_A F$, $\text{Hom}_A(E, F)$ and the dual E^* of E are projective and finitely generated, so is the exterior power $\bigwedge^p E$ for every integer $p > 0$. Also, it follows immediately from definition and the exactness of localization that, for every prime ideal \mathfrak{p} of A ,

$$\begin{aligned}\text{rank}_{\mathfrak{p}}(E \oplus F) &= \text{rank}_{\mathfrak{p}}(E) + \text{rank}_{\mathfrak{p}}(F), & \text{rank}_{\mathfrak{p}}(E \times F) &= \text{rank}_{\mathfrak{p}}(E)\text{rank}_{\mathfrak{p}}(F), \\ \text{rank}_{\mathfrak{p}}(\text{Hom}_A(E, F)) &= \text{rank}_{\mathfrak{p}}(E)\text{rank}_{\mathfrak{p}}(F), & \text{rank}_{\mathfrak{p}}(E^*) &= \text{rank}_{\mathfrak{p}}(E), & \text{rank}_{\mathfrak{p}}(\bigwedge^p E) &= \binom{\text{rank}_{\mathfrak{p}}(E)}{p}.\end{aligned}$$

If the ranks of E and F are defined, so are those of $E \oplus F$, $E \otimes_A F$, $\text{Hom}_A(E, F)$, E^* and $\bigwedge^p E$ and the above equations also hold with the index \mathfrak{p} omitted. Moreover:

Corollary 9.5.11. *For a finitely generated free projective A -module P to be of rank n , it is necessary and sufficient that $\bigwedge^n P$ be of rank 1.*

Proposition 9.5.12. *Let B be a A -algebra and P a projective A -module of rank n . The B -module $B \otimes_A P$ is then projective of rank n .*

Proof. We know that $B \otimes_A P$ is projective and finitely generated. If \mathfrak{q} is a prime ideal of B and \mathfrak{p} its inverse image in A , then

$$(B \otimes_A P)_{\mathfrak{q}} = (P \otimes_A B) \otimes_B B_{\mathfrak{q}} = P \otimes_A B_{\mathfrak{q}} = (P \otimes_A A_{\mathfrak{p}}) \otimes_A B_{\mathfrak{q}}$$

and, as, by hypothesis, $P \otimes_A A_{\mathfrak{p}}$ is a free $A_{\mathfrak{p}}$ -module of rank n , $(B \otimes_A P)_{\mathfrak{q}}$ is a free $B_{\mathfrak{q}}$ -module of rank n . \square

Proposition 9.5.13. *Let A be a semi-local ring and P a finitely generated projective A -module. If the rank of P is defined, P is a free A -module.*

Proof. Suppose first that $A = \prod_{i=1}^n K_i$ where K_i are fields. Then the K_i are then identified with the minimal ideals of A and, for all i , the sum $\mathfrak{p}_i = \sum_{j \neq i} K_j$ is a maximal ideal of A , and the \mathfrak{p}_i 's are the only prime ideals of A . Every finitely generated A -module P is therefore the direct sum of its isotypical components P_i , P_i being isomorphic to a direct sum of a finite number r_i of A -modules isomorphic to K_i (??); the ring $A_{\mathfrak{p}_i}$ is identified with K_i and annihilates the P_j of index $j \neq i$, hence $n = \text{rank}_{\mathfrak{p}_i}(P)$; if all the r_i are equal to the same number r , P is isomorphic to A^r , whence the proposition in this case. In the general case, let \mathfrak{r} be the Jacobson radical of A and $B = A/\mathfrak{r}$; as B is a product of fields, the projective B -module $P \otimes_A B$ is free by the remarks preceding ???. Also P is a flat A -module and the proposition then follows from ???. \square

9.5.3 Projective modules of rank 1

Theorem 9.5.14. *Let A be a ring and M a finitely generated A -module.*

- (a) *If there exists an A -module N such that $M \otimes_A N$ is isomorphic to A , the module M is projective of rank 1.*
- (b) *Conversely, if M is projective of rank 1 and M^* is the dual of M , the canonical homomorphism $v : M^* \otimes_A M \rightarrow A$ corresponding to the canonical bilinear form $(x^*, x) \mapsto \langle x^*, x \rangle$ on $M^* \times M$ is bijective.*

Proof. First assume that $M \otimes_A N \cong A$ for some A -module N . It is required to prove that, for every maximal ideal \mathfrak{m} of A , the $A_{\mathfrak{m}}$ -module $M_{\mathfrak{m}}$ is free of rank 1. We are free to replace A by $A_{\mathfrak{m}}$ and hence may assume that A is a local ring. Let $k = A/\mathfrak{m}$. The isomorphism $\eta : M \otimes_A N \rightarrow A$ defines an isomorphism

$$\eta \otimes 1_k : (M/\mathfrak{m}M) \otimes_k (N/\mathfrak{m}N) \rightarrow k$$

as the rank over k of $(M/\mathfrak{m}M) \otimes_k (N/\mathfrak{m}N)$ is the product of the ranks of $M/\mathfrak{m}M$ and $N/\mathfrak{m}N$, these latter are necessarily equal to 1, in other words $M/\mathfrak{m}M$ is monogenous. It follows that M is monogenous by [??](#). On the other hand, the annihilator of M also annihilates $M \otimes_A N$ and hence is zero, which proves that M is isomorphic to A .

For (b), assume that M is projective of rank 1. It is sufficient to prove that, for every maximal ideal \mathfrak{m} of A , $v_{\mathfrak{m}}$ is an isomorphism. As M is finitely presented, $(M^*)_{\mathfrak{m}}$ is canonically identified with the dual $(M_{\mathfrak{m}})^*$ [\(??\)](#) and, as $M_{\mathfrak{m}}$ is free of rank 1 like its dual $(M_{\mathfrak{m}})^*$, clearly the canonical homomorphism $v_{\mathfrak{m}} : (M_{\mathfrak{m}})^* \otimes_{A_{\mathfrak{m}}} (M_{\mathfrak{m}}) \rightarrow A_{\mathfrak{m}}$ is bijective, which completes the proof. \square

Remark 9.5.2. If M is projective of rank 1 and N is such that $M \otimes_A N$ is isomorphic to A , then N is isomorphic to M^* , for there are isomorphisms

$$N \rightarrow N \otimes_A A \rightarrow N \otimes M \otimes M^* \rightarrow A \otimes M^* \rightarrow M^*.$$

Now let $A^n \rightarrow M$ be a surjective homomorphism. Then by applying the dual functor $\text{Hom}(-, A)$ we get an injection $N \cong M^* \rightarrow (A^n)^* \cong A^n$. If we further assume that A is Noetherian, then N is a finitely generated A -module.

In the language of algebraic geometry, [??](#) and the argument above show that a coherent sheaf \mathcal{F} on a Noetherian scheme X is invertible (locally of rank 1) if and only if there exist a coherent sheaf \mathcal{G} such that $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} \cong \mathcal{O}_X$. This justifies the terminology invertible: it means that \mathcal{F} is an invertible element of the monoid of coherent sheaves under the operation \otimes .

Proposition 9.5.15. *Let M and N be projective A -modules of rank 1. Then $M \otimes_A N$, $\text{Hom}_A(M, N)$ and the dual M^* of M are projective of rank 1.*

Let us now note that every finitely generated A -module is isomorphic to a quotient module of $L = A^n$. We may therefore speak of the set $F(A)$ of classes of finitely generated A -modules with respect to isomorphism. We denote by $P(A)$ the subset of $F(A)$ consisting of the classes of projective A -modules of rank 1 and by $\text{cl}(M)$ the image in $P(A)$ of a projective A -module M of rank 1. It is immediate that, for two projective A -modules M, N of rank 1, $\text{cl}(M \otimes_A N)$ depends only on $\text{cl}(M)$ and $\text{cl}(N)$. As definition we set

$$\text{cl}(M) + \text{cl}(N) = \text{cl}(M \otimes_A N) \tag{9.5.1}$$

and an internal law of composition is thus defined on $P(A)$.

Proposition 9.5.16. *The set $P(A)$ of classes of projective A -modules of rank 1, with the law of composition [??](#), is a commutative group. If M is a projective A -module of rank 1 and M^* is its dual, then*

$$\text{cl}(M^*) = -\text{cl}(M), \quad \text{cl}(A) = 0.$$

Proof. The associativity and commutativity of the tensor product show that the law of composition (6) is associative and commutative. The isomorphism between $A \otimes_A M$ and M prove

that $\text{cl}(A)$ is the identity element under this law and, by virtue of ??, $\text{cl}(M) + \text{cl}(M^*) = \text{cl}(A)$, whence the proposition. \square

Let B be an A -algebra and M a projective A -module of rank 1. Then $B \otimes_A M$ is a projective B -module of rank 1 by ?. Then there exists a map called canonical $\phi : P(A) \rightarrow P(B)$ such that

$$\phi(\text{cl}(M)) = \text{cl}(B \otimes_A M).$$

The equation $(M \otimes_A B) \otimes_B (N \otimes_A B) = (M \otimes_A N) \otimes_A B$ for two A -modules M, N proves that the map ϕ is a group homomorphism.

Remark 9.5.3. Condition (v) of ?? (equivalent to the fact that P is projective and finitely generated) may also be expressed by saying that the sheaf of modules P over $X = \text{Spec}(A)$ associated with P is locally free and of finite type and may therefore be interpreted as the sheaf of sections of a vector bundle over X . Conversely, every vector bundle over X arises from a finitely generated projective module, which is determined up to within a unique isomorphism. The projective modules of rank n thus correspond to the vector bundles all of whose fibres have dimension n . In particular, the vector bundles of rank 1 correspond to the projective modules of rank 1. If we denote by \mathcal{O}_X the structure sheaf of X and by \mathcal{O}_X^\times the sheaf of units of \mathcal{O}_X (whose sections over an open set U of X are the invertible elements of the ring of sections of \mathcal{O}_X over U), it follows that the group $P(A)$ is isomorphic to the first cohomology group $H^1(X, \mathcal{O}_X^\times)$.

9.5.4 Non-degenerate submodules and invertible submodules

In this part and the two following, A denotes a ring, S a multiplicative subset of A consisting of elements which are not divisors of zero in A , and B the ring $S^{-1}A$. The ring A is then canonically identified with a subring of B . The elements of S are therefore invertible in B . One of the most important special cases for applications is that where A is an integral domain and S is the set of nonzero elements of A ; in this case B is the field of fractions of A .

Let M be a sub- A -module of B . Then M is called **non-degenerate** if $B \cdot M = B$. Note that if B is a field, this condition simply means that M is not reduced to 0.

Proposition 9.5.17. *Let M be a sub- A -module of B . The following conditions are equivalent:*

- (i) M is non-degenerate;
- (ii) M meets S ;
- (iii) If $i_M : M \rightarrow B$ is the canonical injection, then $S^{-1}i_M : S^{-1}M \rightarrow B$ is bijective.

Proof. Part (i) implies (ii), for if $B \cdot M = B$, there exists $a \in A$, $s \in S$ and $x \in M$ such that $(a/s)x = 1$, hence $ax = s$ belongs to $S \cap M$. To see that (ii) implies (iii), note that $S^{-1}i$ is already injective, since localization is an exact functor. Moreover, if $x \in M \cap S$, the image under $S^{-1}i$ of $x/x \in S^{-1}M$ in B is equal to 1 and $S^{-1}i$ is therefore surjective. Finally, (iii) clearly implies (i). \square

Corollary 9.5.18. *If M and N are two non-degenerate sub- A -modules of B , the A -modules $M + N$, $M \cdot N$ and $M \cap N$ are non-degenerate.*

Proof. The assertion is trivial for $M + N$. On the other hand if $s \in S \cap M$ and $t \in S \cap N$, then $st \in S \cap (M \cdot N)$ and $st \in S \cap (M \cap N)$. \square

Given two sub- A -modules M and N of B , let us denote by $(N : M)$ the sub- A -module of B consisting of those $b \in B$ such that $bM \subseteq N$. If every $b \in (N : M)$ is mapped to the homomorphism $h_b : x \mapsto bx$ of M to N , a canonical homomorphism $b \mapsto h_b$, is obtained from $(N : M)$ to $\text{Hom}_A(M, N)$.

Proposition 9.5.19. *Let M, N be two sub- A -modules of B . If M is non-degenerate, the canonical homomorphism from $(N : M)$ to $\text{Hom}_A(M, N)$ is bijective.*

Proof. Let $s \in S \cap M$. If $b \in (N : M)$ is such that $bx = 0$ for all $x \in M$, then $bs = 0$ whence $b = 0$ since s is not a divisor of 0 in B . On the other hand, let $f \in \text{Hom}_A(M, N)$ and set $b = f(s)/s$; for all $x \in M$, there exists $t \in S$ such that $tx \in A$. Then

$$f(x) = f(stx)/(st) = (txf(s))/(st) = bx$$

whence $b \in (N : M)$ and $f = h_b$, which proves the proposition. \square

In particular, $(A : M)$ is canonically identified with the dual M^* of M , the canonical bilinear form on $M^* \times M$ being identified with the restriction to $(A : M) \times M$ of the multiplication $B \times B \rightarrow B$.

Definition 9.5.20. *A sub- A -module M of B is called **invertible** if there exists a sub- A -module N such that $M \cdot N = A$.*

Example 9.5.21. If b is invertible element of B , the A -module Ab is invertible, as is seen by taking $N = Ab^{-1}$.

Proposition 9.5.22. *Let M be an invertible sub- A -module of B . Then:*

- (a) *There exists $s \in S$ such that $As \subseteq M \subseteq As^{-1}$ (and in particular M is nondegenerate).*
- (b) *$(A : M)$ is the only sub- A -module N of B such that $M \cdot N = A$.*
- (c) *M is finitely generated and $(A : M) = A$ if and only if $M = A$.*

Proof. If there exists a sub- A -module N such that $M \cdot N = A$, then

$$B \cdot M = B \cdot (B \cdot M) \supseteq B \cdot (M \cdot N) = B \cdot A = B$$

hence M is non-degenerate. Similarly N is non-degenerate. If $t_1 \in S \cap M$ and $t_2 \in S \cap N$, the element $s = t_1t_2$ belongs to $S \cap M \cap N$, whence $Ms \subseteq M \cdot N = A$ and therefore $As \subseteq M \subseteq As^{-1}$.

On the other hand, obviously $N \subseteq (A : M)$, whence

$$A = M \cdot N \subseteq M \cdot (A : M) \subseteq A$$

whence $A = M \cdot (A : M)$. Multiplying the two sides by N , we deduce then $N = (A : M)$, this prove (b). Now since $M \cdot N = A$, then there is an equation

$$\sum_i x_i y_i, \quad x_i \in M, y_i \in N,$$

so M is generated by (x_i) since $x = \sum_i (xy_i)x_i$ for each $x \in M$. Moreover, if $(A : M) = A$ then since M is invertible we have $M \cdot A = A$, which implies $M = A$. \square

Theorem 9.5.23. *Let M be a non-degenerate sub- A -module of B . The following properties are equivalent:*

- (i) M is invertible.
- (ii) M is projective.
- (iii) M is projective of rank 1.
- (iv) M is a finitely generated A -module and, for every maximal ideal \mathfrak{m} of A , the $A_{\mathfrak{m}}$ -module $M_{\mathfrak{m}}$ is principal.

Proof. Let us show first the equivalence of properties (i), (ii) and (iii). If (i) holds and N is sub- A -module of B such that $M \cdot N = A$, then there is a relation

$$\sum_i m_i n_i = 1$$

For all $x \in M$, set $v_i(x) = n_i x$, the v_i are linear forms on M and by $x = \sum_i v_i(x)m_i$ for all $x \in M$. This proves that M is projective by ?? and generated by the x_i ; hence M is a finitely generated projective module.

Let \mathfrak{m} be a maximal ideal of A . We show that the integer $r = \text{rank}_{\mathfrak{m}}(M)$ is equal to 1. Let $S_{\mathfrak{m}}$ be the image of S in $A_{\mathfrak{m}}$. As the elements of S are not divisors of 0 in A , those of $S_{\mathfrak{m}}$ are not divisors of 0 in $A_{\mathfrak{m}}$, since $A_{\mathfrak{m}}$ is a flat A -module. Then $(S_{\mathfrak{m}})^{-1}A_{\mathfrak{m}} \neq 0$ and, as $M_{\mathfrak{m}}$ is a free $A_{\mathfrak{m}}$ -module of rank r , $(S_{\mathfrak{m}})^{-1}M_{\mathfrak{m}}$ is a free $(S_{\mathfrak{m}})^{-1}A_{\mathfrak{m}}$ -module of rank r . Also note that $(S_{\mathfrak{m}})^{-1}A_{\mathfrak{m}}$ (resp. $(S_{\mathfrak{m}})^{-1}M_{\mathfrak{m}}$) is canonically identified with $(S^{-1}A)_{\mathfrak{m}}$ (resp. $(S^{-1}M)_{\mathfrak{m}}$). Now $S^{-1}M = B$ by ??(iii) and hence $(S^{-1}M)_{\mathfrak{m}}$ is a free $(S^{-1}A)_{\mathfrak{m}}$ of rank 1, which proves that $r = 1$ and shows the implication (i) \Rightarrow (iii).

The implication (iii) \Rightarrow (ii) is trivial. Let us show that (ii) \Rightarrow (i). There exists by hypothesis a family (not necessarily finite) $(f_i)_{i \in I}$ of linear forms on M and a family $(m_i)_{i \in I}$ of elements of M such that, for all $x \in M$, the family $(f_i(x))$ has finite support and $x = \sum_i m_i f_i(x)$ (??). Since M is non-degenerate, $f_i(x) = n_i x$ for some $n_i \in (A : N)$ by virtue of ???. Taking x as an element of $M \cap S$, it is seen that of necessity $n_i = 0$ except for a finite number of indices and $\sum_i m_i n_i = 1$. This obviously implies $M \cdot (A : M) = A$, whence (i).

By definition we have (iii) implies (iv). Let us show the converse. Since M is non-degenerate, its annihilator is zero by ??, then so is the annihilator of $M_{\mathfrak{m}}$. As $M_{\mathfrak{m}}$ is assumed to be a monogenous $A_{\mathfrak{m}}$ -module, it is therefore free of rank 1 and it then follows from ?? that M is projective of rank 1. \square

Corollary 9.5.24. *Every invertible sub- A -module of B is flat and finitely presented.*

Proof. This follows from ??(iii). \square

Proposition 9.5.25. *Let M, N be two sub- A -modules of B . Suppose that M is invertible.*

- (a) *The canonical homomorphism $M \otimes_A N \rightarrow M \cdot N$ is bijective.*
- (b) *$(N : M) = N \cdot (A : M)$ and $N = (N : M) \cdot M$.*

Proof. Let i_N be the canonical injection from N to B . Since M is a flat A -module by ??, $1 \otimes i_N : M \otimes_A N \rightarrow M \otimes_A B$ is injective. But, as $B = S^{-1}A$, the B -module $M \otimes_A B$ is equal to $S^{-1}M$ and hence is identified with B since M is non-degenerate. If this identification is made, the image of $1 \otimes i_N$ is $M \cdot N$, whence (a).

Now write $M^{-1} := (A : M)$, then we have

$$M^{-1} \cdot N \subseteq (N : M) = M^{-1} \cdot M \cdot (N : M) \subseteq M^{-1} \cdot N$$

which proves $M^{-1} \cdot N = (N : M)$, whence $N = M \cdot (N : M)$. \square

9.5.5 The class group of invertible submodules

Under multiplication, the sub- A -modules of B form a commutative monoid, with A as identity element. Then the invertible modules are the invertible elements and therefore form a commutative group \mathfrak{I} . We have seen that the inverse of $M \in \mathfrak{I}$ is $M^{-1} := (A : M)$.

Let A^\times (resp. B^\times) be the multiplicative group of invertible elements of A (resp. B) and let i_A^S denote the canonical injection $A \rightarrow B$. For all $b \in B^\times$, $\theta(b) = bA$ is an invertible sub- A -module. The map $\theta : B^\times \rightarrow \mathfrak{I}$ is a homomorphism whose kernel is $i_A^S(A^\times)$. Its cokernel will be denoted by \mathfrak{C} or $\mathfrak{C}(A)$. The group \mathfrak{C} is called the **group of classes of invertible sub- A -modules** of B . The following exact sequence has been constructed

$$1 \longrightarrow A^\times \xrightarrow{i_A^S} B^\times \xrightarrow{\theta} \mathfrak{I} \xrightarrow{\pi} \mathfrak{C} \longrightarrow 1$$

where 1 denotes the group consisting only of the identity element and π is the canonical map $\mathfrak{I} \rightarrow \mathfrak{C}$.

As every invertible sub- A -module M of B is projective of rank 1, the element $\text{cl}(M) \in P(A)$ is defined.

Proposition 9.5.26. *The map $\text{cl} : \mathfrak{I} \rightarrow P(A)$ defines, by taking quotients, an isomorphism from \mathfrak{C} onto the kernel of the canonical homomorphism*

$$\phi : P(A) \rightarrow P(B).$$

In other words, there is an exact sequence

$$1 \longrightarrow A^\times \xrightarrow{i_A^S} B^\times \xrightarrow{\theta} \mathfrak{I} \xrightarrow{\text{cl}} P(A) \xrightarrow{\phi} P(B)$$

Proof. It follows from ?? and the definition of addition in $P(A)$ that $\text{cl}(M \cdot N) = \text{cl}(M) + \text{cl}(N)$ for M, N in \mathfrak{I} , which shows that cl is a homomorphism. If $M \in \mathfrak{I}$ is isomorphic to A , there exists $b \in B$ such that $M = Ab$ and, as M is invertible, there exists $b' \in B$ such that $bb' = 1$, in other words $b \in B^\times$. The converse is immediate. Hence the kernel of cl is contained in $\theta(B^\times)$.

Let us now determine the image of cl . If $M \in \mathfrak{I}$, then $M \otimes_A B = S^{-1}M = B$, whence $\text{cl}(M) \in \ker \phi$. Conversely, let P be a projective A -module of rank 1 such that $P(B) = P \otimes_A B$ is B -isomorphic to B . As P is a flat A -module, the injection $i : A \rightarrow B$ defines an injection $i \otimes i_B : P \rightarrow P \otimes_A B = B$ and P is thus identified with a sub- A -module of B . By virtue of ??(iii) P is non-degenerate and ?? shows that P is invertible. The kernel of ϕ is therefore equal to the

image of cl. □

Corollary 9.5.27. *For two invertible sub- A -modules of B to have the same image in \mathfrak{C} , it is necessary and sufficient that they be isomorphic.*

Corollary 9.5.28. *If the ring B is semi-local, the group \mathfrak{C} of classes of invertible sub- A -modules of B is canonically identified with the group $P(A)$ of classes of projective A -modules of rank 1.*

Proof. In this case $P(B) = 0$ by ??.

□

Example 9.5.29. Let A be an integral domain and S is the set of nonzero elements of A , so that B is the field of fractions of A . The invertible sub- A -modules of B are also called in this case **invertible fractional ideals**. Those which are monogenous free A -modules are just the fractional principal ideals.

9.5.6 Exercise

Exercise 9.5.1. let A be a commutative ring. The following are equivalent:

- (a) For every prime ideal \mathfrak{p} , the localization $A_{\mathfrak{p}}$ is an integral domain.
- (b) For every maximal ideal \mathfrak{m} , the localization $A_{\mathfrak{m}}$ is an integral domain.
- (c) For $x, y \in A$ such that $xy = 0$ we have $\text{Ann}(x) + \text{Ann}(y) = A$.

Proof. It is clear that (a) implies (b); for (b) \Rightarrow (c), let $xy = 0$, assume $\text{Ann}(x) + \text{Ann}(y) \subseteq \mathfrak{m}$ for some maximal ideal \mathfrak{m} , then $xy/1 = 0$ in $A_{\mathfrak{m}}$. Then, since $A_{\mathfrak{m}}$ is an integral domain, we have $x/1 = 0$ or $y/1 = 0$. Assume $x/1 = 0$, then $xs = 0$ for some $s \in A - \mathfrak{m}$. But $\text{Ann}(x) \subseteq \mathfrak{m}$, this is a contradiction. Similar for $y/1 = 0$.

We show that (c) \Rightarrow (a). First we have the following observation: If \mathfrak{p} is a prime ideal, then for any $x \in A$ such that $x/1 \neq 0 \in A_{\mathfrak{p}}$ then $\text{Ann}(x) \subseteq \mathfrak{p}$. In fact, assume $tx = 0$ for $t \in A$. Then $t \notin A - \mathfrak{p}$, hence $t \in \mathfrak{p}$. Now, suppose that $x/s \cdot y/t = 0 \in A_{\mathfrak{p}}$. Then $xyz = 0$ for some $z \in A - \mathfrak{p}$. It will suffice to prove that $x/1 = 0$ or $y/1 = 0$. Suppose otherwise, namely that $x/1 \neq 0$ and $y/1 \neq 0$. Then, $xz/1 \neq 0$ since $z/1$ is invertible. By the observation, we know that $\text{Ann}(xz) \subseteq \mathfrak{p}$ and $\text{Ann}(y) \subseteq \mathfrak{p}$, thus $\text{Ann}(x) + \text{Ann}(y) \subseteq \mathfrak{p}$, which is a contradiction. □

Exercise 9.5.2. Let A be a ring. Suppose that, for each prime ideal \mathfrak{p} , the local ring $A_{\mathfrak{p}}$ has no nilpotent element $\neq 0$. Show that A has no nilpotent element $\neq 0$. If each $A_{\mathfrak{p}}$ is an integral domain, is A necessarily an integral domain?

Proof. By corollary ??, we have $\mathfrak{n}(A_{\mathfrak{p}}) = \mathfrak{n}(A)_{\mathfrak{p}}$ so $\mathfrak{n}(A)_{\mathfrak{p}} = 0$ for any prime ideal \mathfrak{p} . By ??, $\mathfrak{n}(A) = 0$.

The answer for next question is no. Suppose that $A = \prod_{i=1}^n k_i$, where $k_i = k$ an algebraic closed field. This is not integral domain, for example $(1, \dots, 0) \cdot (0, \dots, 1) = 0$. Then $\text{Spec}(A) = \{\mathfrak{p}_i := 0 \times \prod_{i \neq j} k_j\}$. Consider the localization $A_{\mathfrak{p}_i}$:

$$A - \mathfrak{p}_i = \{(x_1, \dots, x_n) \in A : x_i \neq 0\}$$

We define a map $\varphi_i : A_{\mathfrak{p}_i} \rightarrow k_i$ by

$$\frac{x}{s} \mapsto \frac{x_i}{s_i}$$

This is surjective of course, since we can choose $s_i = 1$. To show the injectivity, let $x_i/s_i = y_i/t_i$, then since k_i is a field, we have

$$x_i t_i = y_i s_i.$$

Let $e_i = (0, \dots, 1, \dots, 0)$, then $e_i \in A - \mathfrak{p}_i$, and

$$e_i(xt - ys) = x_i t_i - y_i s_i = 0$$

so $x/s = y/t$. Hence φ_j is an isomorphism. In particular, $A_{\mathfrak{p}_i}$ is an integral domain. \square

Exercise 9.5.3. Let A be a ring and let Σ be the set of all multiplicatively closed subsets S of A such that $0 \notin S$. Show that Σ has maximal elements, and that $S \in \Sigma$ is maximal if and only if $A - S$ is a minimal prime ideal of A .

Moreover, by replace 0 with any ideal I , we can show $S \in \Sigma$ is maximal if and only if $A - \mathfrak{p}$ is a minimal prime ideal among those containing I .

Proof. By Zorn's lemma, Σ has a maximal element. For any set $S \in \Sigma$, it is clear that $A - S$ has the prime property. When S is maximal in Σ , we claim that $A - S$ is an ideal. Let $a \in A$, consider the smallest multiplicatively closed subsets containing a and S : When $a \in S$, it is S , and when $a \notin S$, it is

$$S_a := \{sa^n : s \in S, n \in \mathbb{N}\}$$

Now assume $a, b \in A - S$, then S_a and S_b both contains S properly. By the maximality, $0 \in S_a, S_b$, so there are $s_a, s_b \in S, m, n \in \mathbb{N}$ such that

$$s_a a^n = s_b b^m = 0$$

Now assume $a + b \in S$, then $s_a s_b (a + b) \in S$. Consider the power

$$(s_a s_b (a + b))^{m+n} = (s_a s_b)^{m+n} (a + b)^{m+n} = 0$$

since every term of the sum is zero. This is a contradiction since $(s_a s_b (a + b))^{m+n} \in S$. So $a + b \in A - S$.

Let $r \in A$ and $a \notin S$. If $ra \in S$, then $rs_a a \in S$, and $(rs_a a)^n = 0 \in S$, contradiction. Hence $A - S$ is a prime ideal. Since for any prime ideal \mathfrak{p} , $A - \mathfrak{p}$ is multiplicatively closed and does not contain (0) , it follows that $A - S$ is minimal.

Now assume \mathfrak{p} is a minimal prime ideal, then $A - \mathfrak{p} \in \Sigma$. By Zorn's lemma, there is a maximal elements $T \in \Sigma$ such that $A - \mathfrak{p} \subseteq T$. Then $A - T$ is prime and contained in \mathfrak{p} , by assumption we have $T = A - \mathfrak{p}$. So $A - \mathfrak{p}$ is maximal. \square

Exercise 9.5.4. Let A be an integral domain and M an A -module. An element $x \in M$ is a torsion element of M if $\text{Ann}(x) \neq 0$, that is if x is killed by some non-zero element of A .

Show that the torsion elements of M form a submodule of M . This submodule is called the **torsion submodule** of M and is denoted by $T(M)$. If $T(M) = 0$, the module M is said to be torsion-free. Show that

- (a) If M is any A -module, then $M/T(M)$ is torsion-free.
- (b) If $f : M \rightarrow N$ is a module homomorphism, then $f(T(M)) \subseteq T(N)$.

- (c) The functor T is left-exact.
- (d) If M is any A -module, then $T(M)$ is the kernel of the map $x \mapsto 1 \otimes x$ of M into $K \otimes_A M$, where K is the field of fractions of A .

Proof. Let

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N$$

be an exact sequence. Consider the sequence

$$0 \longrightarrow T(L) \xrightarrow{T(f)} T(M) \xrightarrow{T(g)} T(N)$$

where $T(f) = f|_{T(M)}$. Clearly $T(g) \circ T(f) = 0$. Since we have $\ker T(f) \subseteq \ker f$, we see $T(f)$ is injective.

Let $m \in \ker T(g)$, then m is also in $\ker g$, so there is $\ell \in L$ such that $f(\ell) = m$. m is torsion, so there is $r \in A$ such that $rm = 0$. Then $f(r\ell) = rf(\ell) = rm = 0$, and $r\ell = 0$ since f is injective. This implies $\ell \in T(L)$, so $T(f)(\ell) = m$.

Let $S = A - \{0\}$, then we have

$$(K \otimes_A M = S^{-1}A \otimes_A M = A \otimes_{S^{-1}A} S^{-1}M \cong S^{-1}M).$$

Now $1 \otimes m$ is $m/1$ in $S^{-1}M$, so $1 \otimes m = 0$ if and only if there is $a \in A$ such that $am = 0$. \square

Exercise 9.5.5. Let S be a multiplicatively closed subset of an integral domain A . Show that $T(S^{-1}M) = S^{-1}T(M)$. Deduce that the following are equivalent:

- (a) M is torsion-free.
- (b) $M_{\mathfrak{p}}$ is torsion-free for all prime ideals \mathfrak{p} .
- (c) $M_{\mathfrak{m}}$ is torsion-free for all maximal ideals \mathfrak{m} .

Proof. Let $m/s \in T(S^{-1}M)$, then there is $y/t \in S^{-1}A$ such that $ym/st = 0$. Hence $\exists u \in S$ such that

$$uym = 0$$

This implies $m \in T(m)$, so $m/s \in S^{-1}T(M)$.

Let $m/s \in S^{-1}T(M)$. Then there is $r \in A : rm = 0$. Then $r/1 \cdot m/s = 0$ so $m/s \in T(S^{-1}M)$. M is torsion free if and only if $T(M) = 0$. Use ??.

\square

Chapter 10

Graded rings and filtrations

10.1 Graduation on rings and modules

10.1.1 Graded ring and graded module

Definition 10.1.1. Given an abelian group G and a set Δ , a **graduation** of type Δ on G is a family $(G_\lambda)_{\lambda \in \Delta}$, of which G is the direct sum. The set G with the structure defined by its group law and the graduation is called a **graded group** of type Δ .

The set Δ is called the **set of degrees** of G . An element $x \in G$ is called **homogeneous** if it belongs to one of the G_λ , **homogeneous of degree λ** if $x \in G_\lambda$. The element 0 is therefore homogeneous of all degrees; but if $x \neq 0$ is homogeneous, it belongs to only one of the G_λ ; the index λ such that $x \in G_\lambda$ is then called the **degree** of x (or sometimes the **weight** of x) and is sometimes denoted by $\deg(x)$. Every $x \in G$ may be written uniquely as a sum $\sum_\lambda x_\lambda$ of homogeneous elements with $x_\lambda \in G_\lambda$; y_λ is called the **homogeneous component** of degree λ , or simply the component of degree λ of x .

Example 10.1.2. Given any commutative monoid Δ with identity element 0 and abelian group G , a graduation $(G_\lambda)_{\lambda \in \Delta}$ is defined on G by taking $G_0 = G$ and $G_\lambda = \{0\}$ for $\lambda \neq 0$; this graduation is called **trivial**.

Example 10.1.3. Let Δ and Δ' be two sets and ρ a map of Δ into Δ' . Let $(G_\lambda)_{\lambda \in \Delta}$ be a graduation of type Δ on a commutative group G ; for $\mu \in \Delta'$, let G_μ be the sum of the G_λ 's such that $\rho(\lambda) = \mu$; clearly $(G_\mu)_{\mu \in \Delta'}$ is a graduation of type Δ' on G , said to be **derived** from (G_λ) by means of the map ρ .

When Δ is a commutative group written additively and ρ the map $\lambda \mapsto -\lambda$ of Δ onto itself, (G_μ) is called the **opposite graduation** of (G_λ) .

Example 10.1.4. If $\Delta = \Delta_1 \times \Delta_2$ is a product of two sets, a graduation of type Δ is called a **bigraduation** of types Δ_1, Δ_2 . For all $\lambda \in \Delta_1$, let $G_\lambda = \bigoplus_{\mu \in \Delta_2} G_{\lambda\mu}$; for all $\mu \in \Delta_2$, let $G_\mu = \bigoplus_{\lambda \in \Delta_1} G_{\lambda\mu}$; clearly $(G_\lambda)_{\lambda \in \Delta_1}$ is a graduation of type Δ_1 and $(G_\mu)_{\mu \in \Delta_2}$ is a graduation of type Δ_2 on G ; these graduations are called the **partial graduations** derived from the bigraduation $(G_{\lambda\mu})$. Note that $G_{\lambda\mu} = G_\lambda \cap G_\mu$. Conversely, if $(G_\lambda)_{\lambda \in \Delta_1}$ and $(G_\mu)_{\mu \in \Delta_2}$ are two graduations on G such that G is the direct sum of the $G_{\lambda\mu} = G_\lambda \cap G_\mu$, these subgroups form a bigraduation of types Δ_1, Δ_2 on G , of which $(G_\lambda)_{\lambda \in \Delta_1}$ and $(G_\mu)_{\mu \in \Delta_2}$ are the partial graduations.

Example 10.1.5. Let Δ_0 be a commutative monoid written additively, with identity element denoted by 0; let I be any set and $\Delta_0^{\oplus I}$ denote the submonoid of the product set Δ_0^I consisting of the families $(\lambda_i)_{i \in I}$ of finite support. Let $\rho : \Delta \rightarrow \Delta_0$ be the surjective (codiagonal) homomorphism of Δ into Δ_0 defined by $\rho((\lambda_i)) = \sum_i \lambda_i$. From every graduation of type Δ a graduation of type Δ_0 is derived by means of ρ ; it is called the **total graduation** associated with the given "multigraduation" of type Δ .

Often the set Δ may also endow some algebraic structure and one want to have operations between different components of the graduation (G_λ) . In the following, we may always assume that Δ is a commutative monoid, and consider graduations of type Δ on rings and modules.

Definition 10.1.6. Let Δ be a commutative monoid. Given a ring A and a graduation (A_λ) of type Δ on the additive group A , this graduation is said to be **compatible** with the ring structure on A if

$$A_\lambda A_\mu \subseteq A_{\mu+\lambda} \quad \text{for all } \lambda, \mu \in \Delta.$$

The ring A with this graduation is called a **graded ring** of type Δ .

Proposition 10.1.7. If the elements of Δ are cancellable and (A_λ) is a graduation of type Δ compatible with the structure of a ring A , then A_0 is a subring of A .

Proof. As $A_0 A_0 \subseteq A_0$, by definition, it suffices to prove that $1 \in A_0$. Let $1 = \sum_{\lambda \in \Delta} e_\lambda$ be the decomposition of 1 into its homogeneous components. If $x \in A_\mu$ then $x = x \cdot 1 = \sum_\lambda x e_\lambda$; comparing the components of degree μ , (since every element in Δ is cancellable) we get $x = x e_0$. Since this relation is true for every homogeneous element of A , it is true for all $x \in A$; in particular $1 = 1 \cdot e_0 = e_0 \in A_0$. \square

Definition 10.1.8. Let A be a graded ring of type Δ , (A_λ) its graduation and M a left A -module; a graduation (M_λ) of type Δ on the addition group M is compatible with the A -module structure on M if

$$A_\lambda M_\mu \subseteq M_{\lambda+\mu} \quad \text{for all } \lambda, \mu \in \Delta.$$

The module M with this graduation is then called a **graded module** of type Δ over the graded ring A .

When the elements of Δ are cancellable, it follows from ?? that the M_λ are A_0 -modules. Clearly if A is a graded ring of type Δ , the A is itself a left graded A -module of type Δ .

Let A be a graded ring and M be a graded A -module. An element $x \in M$ is **homogeneous** if $x \in M_\lambda$ for some λ , and λ is then called the **degree** of x . A general element $x \in M$ can be written uniquely in the form $x = \sum_\lambda x_\lambda$ with $x_\lambda \in M_\lambda$; x_λ is called the **homogeneous term of x of degree λ** .

Example 10.1.9. On any ring A the trivial graduation of type Δ is compatible with the ring structure. If A is graded by the trivial graduation, for a graduation (M_λ) of type Δ on an A -module M to be compatible with the A -module structure, it is necessary and sufficient that the M_λ be submodules of M .

Example 10.1.10. Let A be a graded ring of type Δ , M a graded A -module of type Δ and ρ a homomorphism of Δ into a commutative monoid Δ' whose identity element is denoted by 0.

Then A is a graded ring of type Δ' and M a graded module of type Δ' for the graduations of type Δ' derived from ρ and the graduations of type Δ on A and M : this follows immediately from the relation $\rho(\lambda + \mu) = \rho(\lambda) + \rho(\mu)$.

In particular, if $\Delta = A_1 \times A_2$ is a product of two commutative monoids, the projections π_1 and π_2 are homomorphisms and the corresponding graduations are just the partial graduations derived from the graduations of type Δ ; these partial graduations are thus compatible with the ring structure of A and the module structure of M . Similarly, if $\Delta = \Delta_0^{\oplus I}$ (where Δ_0 is a commutative monoid with identity element denoted by 0), the total graduation of type Δ_0 derived from the graduation of type Δ on A (resp. M) by means of the codiagonal homomorphism is compatible with the ring structure on A (resp. with the module structure on M).

Example 10.1.11. Let A be a graded ring of type Δ , M a graded A -module of type Δ and A_0 an element of Δ ; for $\lambda \in \Delta$, let $M'_\lambda = M_{\lambda + \lambda_0}$, and let M' be the submodule $M' = \bigoplus_{\lambda \in \Delta} M'_\lambda$ is an A -module and the M'_λ form on M' a graduation of type A compatible with the A -module structure of M' ; the graded A -module M' of type Δ thus defined is said to be obtained by shifting by λ_0 the graduation of M and it is denoted by $M(\lambda_0)$. When Δ is a group, the underlying A -module of the graded A -module M' is identified with M since in this case $\lambda \mapsto \lambda + \lambda_0$ is an isomorphism of Δ onto itself.

Example 10.1.12. Let A be a ring. The polynomial ring $A[X]$ in one indeterminate is graded of type \mathbb{N} by the subgroups AX^n . Similarly, the polynomial ring $A[X_1, \dots, X_n]$ is graded of type \mathbb{N} by the subgroups of homogeneous polynomials.

The graduations most often used are of type \mathbb{Z} or of type \mathbb{Z}^n ; when we speak of graded (resp. bigraded, trigraded, etc.) modules and rings without mentioning the type, it is understood that we mean graduations of type \mathbb{Z} (resp. $\mathbb{Z}^2, \mathbb{Z}^3$ etc.); a graded ring (resp. module) of type \mathbb{N} is also called a **graded ring (resp. module) with positive degrees**.

Now we introduce the morphisms between graded rings and modules.

Definition 10.1.13. Let A, B be two graded rings with the same type Δ . and $(A_\lambda), (B_\lambda)$ their respective graduations. A ring homomorphism $\rho : A \rightarrow B$ is called **graded** if $\rho(A_\lambda) \subseteq B_\lambda$ for all $\lambda \in \Delta$.

Let M, N be two graded modules of type Δ over a graded ring A of type Δ . Let $\phi : M \rightarrow N$ be an A -homomorphism and δ an element of Δ ; ϕ is called **graded of degree δ** if $\phi(M_\lambda) \subseteq N_{\lambda + \delta}$ for all $\lambda \in \Delta$.

An A -homomorphism $\phi : M \rightarrow N$ is called **graded** if there exists $\delta \in \Delta$ such that ϕ is graded of degree δ . If $\phi \neq 0$ and every element of Δ is cancellable, then the degree δ of ϕ is uniquely determined.

If $\rho : A \rightarrow B$ and $\nu : B \rightarrow C$ are two graded homomorphisms of graded rings of type Δ , so is $\nu \circ \rho : A \rightarrow C$; for a map $\rho : A \rightarrow B$ to be a graded ring isomorphism, it is necessary and sufficient that ρ be bijective and that ρ and the inverse map ρ^{-1} be graded homomorphisms. It also suffices for this that f be a bijective graded homomorphism. Thus it is seen that graded homomorphisms can be taken as the morphisms of the category of graded ring structure of type Δ .

Similarly, if $\phi : M \rightarrow N$ and $\psi : N \rightarrow P$ are two graded homomorphisms of graded A -modules of type Δ of respective degrees δ and γ , $\psi \circ \phi : M \rightarrow P$ is a graded homomorphism of degree $\delta + \gamma$. If δ admits an inverse $-\delta$ in Δ and $\phi : M \rightarrow N$ is a bijective graded homomorphism of degree δ , the inverse map $\phi^{-1} : N \rightarrow M$ is a bijective graded homomorphism of

degree $-\delta$. It follows as above that the graded homomorphisms of degree 0 can be taken as the morphisms of the category of graded A -module of type Δ .

Example 10.1.14. If M is a graded A -module and $M(\lambda_0)$ is a graded A -module obtained by shifting, the \mathbb{Z} -linear map of $M(\lambda_0)$ into M which coincides with the canonical injection on each $M_{\lambda+\lambda_0}$ is a graded homomorphism of degree λ_0 (which is bijective when Δ is a group).

Example 10.1.15. If a is a homogeneous element of degree δ belonging to the centre of A , the homothety $x \mapsto ax$ of any graded A -module M is a graded homomorphism of degree δ .

Remark 10.1.1. A graded A -module M is called a graded free A -module if there exists a basis $(x_i)_{i \in I}$ of M consisting of homogeneous elements. Suppose it is and Δ is a commutative group; let λ_i be the degree of x_i and consider for each i the shifted A -module $A(-\lambda_i)$; if e_i denotes the element 1 of A considered as an element of degree λ_i in $A(-\lambda_i)$, the A -linear map $\phi : \bigoplus_{i \in I} A(-\lambda_i) \rightarrow M$ such that $\phi(e_i) = x_i$ for all i is a graded A -module isomorphism.

Example 10.1.16. A direct system $(A_\alpha, f_{\beta\alpha})$ of graded rings of type Δ is a direct system of rings such that each A_α is graded of type Δ and each $f_{\beta\alpha}$ is a homomorphism of graded ring. If (A_α^λ) is the graduation of A_α and we write

$$A = \varinjlim A_\alpha, \quad A^\lambda = \varinjlim A_\alpha^\lambda,$$

it follows that (A^λ) is a graduation of A and this graduation is compatible with the ring structure on A . The graded ring A is called the **direct limit** of the direct system of graded rings $(A_\alpha, f_{\beta\alpha})$. If $f_\alpha : A_\alpha \rightarrow A$ is the canonical map, then f_α is a homomorphism of graded rings.

Similarly, a direct system $(M_\alpha, \phi_{\beta\alpha})$ of graded A -modules of type Δ is a direct system of A -modules such that each M_α is graded of type Δ and each $\phi_{\beta\alpha}$ is a homomorphism of graded A -modules of degree 0. If (M_α^λ) is the graduation of M_α and

$$M = \varinjlim M_\alpha, \quad M^\lambda = \varinjlim M_\alpha^\lambda,$$

then (M^λ) is a graduation of M and this graduation is compatible with the module structure on M . The graded module M is called the **direct limit** of the direct system of graded A -modules $(M_\alpha, \phi_{\beta\alpha})$. If $\phi_\alpha : M_\alpha \rightarrow M$ is the canonical map, then ϕ_α is a homomorphism of graded modules of degree 0.

Proposition 10.1.17. Let A be a graded ring of type Δ , M a graded A -module of type Δ , (M_λ) its graduation and N a sub- A -module of M . The following properties are equivalent:

- (i) $N = \bigoplus_\lambda (N \cap M_\lambda)$.
- (ii) The homogeneous components of every element of N belongs to N
- (iii) N can be generated by homogeneous elements.

Proof. Every element of N can be written uniquely as a sum of elements of the M_λ , and hence it is immediate that (i) and (ii) are equivalent and that (i) implies (iii). We show that (iii) implies (ii). Then let $(x_i)_{i \in I}$ be a family of homogeneous generators of N and let δ_i be the degree of x_i .

Every element of N can be written as $\sum_i a_i x_i$ with $a_i \in A$; if $a_{i\mu}$ is the component of a_i of degree μ , the conclusion follows from the relation

$$\sum_{i \in I} \left(\sum_{\mu \in \Delta} a_{i\mu} x_i \right) = \sum_{\mu \in \Delta} \left(\sum_{\mu + \delta_i = \lambda} a_{i\mu} x_i \right).$$

This completes the proof. \square

When a submodule N of M has the equivalent properties stated in ??, clearly the $N \cap M_\lambda$ form a graduation compatible with the A -module structure of N , called the graduation induced by that on M ; N with this graduation is called a graded submodule of M .

Corollary 10.1.18. *If N is a graded submodule of M and (x_i) is a generating system of N , then the homogeneous components of the (x_i) form a generating system of N .*

Corollary 10.1.19. *If N is a finitely generated submodule of M , then N admits a finite generating system consists of homogeneous elements.*

A graded submodule of A is called a **graded ideal** of the graded ring A . For every subring B of A , if we set $B_\lambda = B \cap A_\lambda$ then $B_\lambda B_\mu \subseteq B_{\lambda+\mu}$, thus the graduation induced on B by that on A is compatible with the ring structure on B ; B is then called a **graded subring** of A .

If N is a graded submodule of a graded A -module M and $(M_\lambda)_{\lambda \in \Delta}$ is the graduation of M , the submodules $(M_\lambda + N)/N$ of M/N form a graduation compatible with the structure of this quotient module. For, if $N_\lambda = M_\lambda \cap N$, then $(M_\lambda + N)/N$ is identified with M_λ/N_λ and it follows from ?? that M/N is their direct sum. Moreover,

$$A_\lambda(M_\mu + N) \subseteq A_\lambda M_\mu + N \subseteq M_{\lambda+\mu} + N$$

which establishes our assertion. The graduation $((M_\lambda + N)/N)_{\lambda \in \Delta}$ is called the **quotient graduation** of that on M by N and the quotient module M/N with this graduation is called the **graded quotient module** of M by the graded submodule N ; the canonical homomorphism $\pi : M \rightarrow M/N$ is a graded homomorphism of degree 0 for this graduation.

If \mathfrak{a} is a graded ideal of A , the quotient graduation on A/\mathfrak{a} is compatible with the ring structure on A/\mathfrak{a} . The ring A/\mathfrak{a} with this graduation is called the **quotient graded ring** of A by \mathfrak{a} . The canonical homomorphism $\pi : A \rightarrow A/\mathfrak{a}$ is a homomorphism of graded rings for this graduation.

Proposition 10.1.20. *Let A be a graded ring of type Δ , M and N two graded A -modules of type Δ and $\phi : M \rightarrow N$ a graded A -homomorphism of degree δ . Then:*

- (a) *$\text{im } \phi$ is a graded submodule of N .*
- (b) *If δ is a regular element of Δ , then $\ker \phi$ is a graded submodule of M .*
- (c) *If $\delta = 0$, the bijection $M/\ker \phi \rightarrow \text{im } \phi$ canonically associated with ϕ is an isomorphism of graded modules.*

Proof. Assertion (a) follows immediately from the definitions and ??(iii). If x is an element of M such that $\phi(x) = 0$ and $x = \sum x_\lambda$ is its decomposition into homogeneous components (where x_λ is of degree λ), then

$$\sum_{\lambda} \phi(x_\lambda) = \phi(x) = 0$$

and $\phi(x_\lambda)$ is of degree $\lambda + \delta$; if δ is regular the relation $\lambda + \delta = \mu + \delta$ implies $\lambda = \mu$, hence the $\phi(x_\lambda)$ are the homogeneous components of $\phi(x)$ and necessarily $\phi(x_\lambda) = 0$ for all $\lambda \in \Delta$, which proves (b). The bijection $\pi : M / \ker \phi \rightarrow \text{im } \phi$ canonically associated with ϕ is then a graded homomorphism of degree δ , as follows from the definition of the quotient graduation; whence (c) when $\delta = 0$. \square

Corollary 10.1.21. *Let A, B be two graded rings of type Δ and $\rho : A \rightarrow B$ a graded homomorphism of graded rings. Then $\text{im } \rho$ is a graded subring of B , $\ker \rho$ a graded ideal of type Δ and the bijection $A / \ker \rho \rightarrow \text{im } \rho$ canonically associated with ρ is an isomorphism of graded rings.*

Proof. It suffices to apply ?? to ρ considered as a homomorphism of degree 0 of graded \mathbb{Z} -modules. \square

Proposition 10.1.22. *Let A be a graded ring of type Δ and M a graded A -module of type Δ .*

- (a) *Every sum and every intersection of graded submodules of M is a graded submodule.*
- (b) *If x is a homogeneous element of M of degree μ which is cancellable in Δ , then the annihilator of x is a graded ideal of A .*
- (c) *If the elements of Δ are cancellable, the annihilator of a graded submodule of M is a graded ideal of A .*

Proof. If (N_i) is a family of graded submodules of M , property (iii) of ?? shows that the sum of the N_i is generated by homogeneous elements and property (ii) of ?? proves that the homogeneous components of every element of $\bigcap_i N_i$ belongs to $\bigcap_i N_i$; whence (a).

To prove (b), it suffices to note that $\text{Ann}(x)$ is the kernel of the homomorphism $a \mapsto ax$ of the A -module A into M and that this homomorphism is graded of degree μ ; the conclusion follows from ??(b). Finally (c) is a consequence of (a) and (b) for the annihilator of a graded submodule N of M is the intersection of the annihilators of the homogeneous elements of N , by virtue of ??.

Example 10.1.23. Let M be a graded A -module and E a submodule of M ; it follows from ??(a) that there exists a largest graded submodule N_1 of M contained in E and a smallest graded submodule N_2 of M containing E ; N_1 is the set of $x \in E$ all of whose homogeneous components belong to E and N_2 is the submodule of M generated by the homogeneous components of a generating system of E .

Let Δ be a commutative monoid, A a graded ring of type Δ , $(A_\lambda)_{\lambda \in \Delta}$ its graduation and E an A -algebra. A graduation $(E_\lambda)_{\lambda \in \Delta}$ of type Δ on the additive group E is said to be **compatible** with the A -algebra structure on E if it is compatible both with the A -module and with the ring structure on E , in other words, if, for all λ, μ in Δ ,

$$A_\lambda E_\mu \subseteq E_{\lambda+\mu}, \quad E_\lambda E_\mu \subseteq E_{\lambda+\mu}. \quad (10.1.1)$$

The A -algebra E , with this graduation, is then called a **graded algebra** of type Δ over the graded ring A .

When the graduation on A is trivial (that is, $A_0 = A$, $A_\lambda = \{0\}$ for $\lambda \neq 0$), condition ?? means that the E_λ are sub- A -modules of E . This leads to the definition of the notion of graded

algebra of type Δ over a non-graded commutative ring A : A is given the trivial graduation of type Δ and the above definition is applied.

When we consider graded A -algebras E with a unit element 1, it will always be understood that 1 is of degree 0. It follows that if an invertible element $x \in E$ is homogeneous and of degree p , its inverse x^{-1} is homogeneous and of degree $-p$: it suffice to decompose x^{-1} as a sum of homogeneous elements in the relation $x^{-1}x = xx^{-1} = 1$.

Let E and F be two graded algebras of type Δ over a graded ring A of type Δ . An A -algebra homomorphism $u : E \rightarrow F$ is called a **graded algebra homomorphism** is $u(E_\lambda) \subseteq F_\lambda$ for all $\lambda \in \Delta$ (where (E_λ) and (F_λ) denote the respective graduations of E and F); where E and F are associative and unital and u is unital, this condition means that u is a graded ring homomorphism.

Remark 10.1.2. Let E be a graded A -algebra of type \mathbb{N} . Then E often is identified with a graded A -algebra of type \mathbb{Z} by writing $E_n = \{0\}$ for $n < 0$.

Remark 10.1.3. The definition can also be interpreted by saying that E is a graded A -module and that A -linear map

$$m : E \otimes_A E \rightarrow E$$

defining the multiplication on E , is homogeneous of degree 0 when $E \otimes_A E$ is given its graduation of type Δ .

To define a graded A -algebra structure of type Δ on the graded ring A , with E as underlying graded A -module, therefore amounts to defining for each ordered pair (λ, μ) of elements of Δ a \mathbb{Z} -bilinear map

$$m_{\lambda\mu} : E_\lambda \times E_\mu \rightarrow E_{\lambda+\mu}$$

such that for every triple of indices (λ, μ, ν) and for $\alpha \in A_\lambda$, $x \in E_\mu$, $y \in E_\nu$, we have $\alpha m_{\mu\nu}(x, y) = m_{\lambda+\mu, \nu}(\alpha x, y) = m_{\mu, \lambda+\nu}(x, \alpha y)$.

Example 10.1.24.

- (a) Let B be a graded ring of type Δ ; if B is given its canonical \mathbb{Z} -algebra structure, B is a graded A -algebra (\mathbb{Z} being given the trivial graduation).
- (b) Let M be a magma and $\phi : M \rightarrow \Delta$ a homomorphism. For all $\lambda \in \Delta$, we write $M_\lambda = \phi^{-1}(\lambda)$; then $M_\lambda M_\nu \subseteq M_{\lambda+\mu}$. Let A be a graded commutative ring of type Δ and $(A_\lambda)_{\lambda \in \Delta}$ its graduation; we shall define a graded A -algebra structure on the algebra $E = A^{\oplus M}$ of the magma M . To this end, let E_λ denote the additive subgroup of E generated by the elements of the form αm such that $\alpha \in A_\mu$, $m \in M_\nu$, and $\mu + \nu = \lambda$. As the M_λ are pairwise disjoint, E is the direct sum of the $A_\mu M_\nu$ and hence is the direct sum of E_λ and it is immediate that E_λ satisfy the condition (??). Therefore it defines on E the desired graded A -algebra structure. If M admits an identity element e , it may also be supposed that $\phi(e) = 0$. A particular case is the one where the graduation of the ring A is trivial; then E_λ is the sub- A -module of E generated by M_λ . More particularly, if we take $M = \mathbb{N}^{\oplus I}$, $\Delta = \mathbb{N}$ and ϕ the map such that $\phi((n_i)) = \sum_i n_i$, the ring A having the trivial graduation, a graduation is thus obtained on the polynomial algebra $A[X_i]_{i \in I}$, for which the degree of a homogeneous polynomial is the total degree.

We now take M to be the free monoid $\Delta(X)$ of a set X and ϕ the homomorphism $\Delta(X) \rightarrow \mathbb{N}$ which associates with each word its length. Thus a graded A -algebra structure is obtained on the free associated algebra of the set X .

Let E be a graded algebra of type Δ over a graded ring A of type Δ . If F is a sub- A -algebra of E which is a graded sub- A -module, then the graduation (F_λ) on F is compatible with its A -algebra structure, since $F_\lambda = F \cap E_\lambda$; in this case F is called a graded subalgebra of E and the canonical injection $F \rightarrow E$ is a graded algebra homomorphism.

Similarly, if \mathfrak{a} is a left (resp. right) ideal of E which is a graded sub- A -module, then $E_\lambda \mathfrak{a}_\mu \subseteq \mathfrak{a}_{\lambda+\mu}$ (resp. $\mathfrak{a}_\lambda E_\mu \subseteq \mathfrak{a}_{\lambda+\mu}$), since $\mathfrak{a}_\lambda = \mathfrak{a} \cap E_\lambda$; then \mathfrak{a} is called a graded ideal of the algebra E . If \mathfrak{b} is a graded two-sided ideal of E the quotient graduation on the module E/\mathfrak{b} is compatible with the algebra structure on E/\mathfrak{b} and the canonical homomorphism $E \rightarrow E/\mathfrak{b}$ is a graded algebra homomorphism.

If $u : E \rightarrow F$ is a graded algebra homomorphism, then $\text{im}(u)$ is a graded sub-algebra of F and $\ker u$ is a graded two-sided ideal of E and the bijection $E/\ker u \rightarrow \text{im } u$ canonically associated with u is a graded algebra isomorphism.

Proposition 10.1.25. *Let A be a graded commutative ring of type Δ , E a graded A -algebra of type Δ and S a set of homogeneous elements of E . Then the sub- A -algebra (resp. left ideal, right ideal, two-sided ideal) generated by S is a graded subalgebra (resp. graded ideal).*

Proof. The subalgebra of E generated by S is the sub- A -module generated by the finite products of elements of S , which are homogeneous. Simialrly, the left (resp. right) ideal generated by S is the sub- A -module generated by the elements of the form $u_1(u_2(\cdots(u_n s))\cdots)$ (resp. $(\cdots((su_n)u_{n-1})\cdots)u_2)u_1$ where $s \in S$ and the $u_j \in E$ are homogeneous and these products are homogeneous, whence in the case the conclusion by **??**. Finally, the two-sided ideal generated by S is the union of the sequence $(\mathfrak{J}_n)_{n \geq 1}$, where \mathfrak{J}_1 is the left ideal generated by S and \mathfrak{J}_{2n} (resp. \mathfrak{J}_{2n+1}) is the right (resp. left) ideal generated by \mathfrak{J}_{2n-1} (resp. \mathfrak{J}_{2n}), which completes the proof. \square

Let $(A_\alpha, \phi_{\beta\alpha})$ is a directed direct system of graded commutative rings of type Δ and for each α let E_α be a graded A_α -algebra of type Δ ; for $\alpha \leq \beta$ let $f_{\beta\alpha} : E_\alpha \rightarrow E_\beta$ be an A_α -homomorphism of graded algebras and suppose that $f_{\gamma\alpha} = f_{\gamma\beta} \circ f_{\beta\alpha}$ for $\alpha \leq \beta \leq \gamma$; then we shall write $(E_\alpha, f_{\beta\alpha})$ a directed direct system of graded algebras of type Δ over the directed direct system $(A_\alpha, \phi_{\beta\alpha})$ of graded commutative rings of type Δ . Then we know that $E = \varinjlim E_\alpha$ has canonically a graded module structure of type Δ over the graded ring $A = \varinjlim A_\alpha$ and a multiplication such that $E^\lambda E^\mu \subseteq E^{\lambda+\mu}$ (where (E^λ) denotes the graduation on E); then this multiplication and the graded A -module structure on E define on E a graded A -algebra structure of type Δ . The set E with this structure is called a direct limit of the direct system $(E_\alpha, f_{\beta\alpha})$ of graded algebras. Moreover, if F is a graded A -algebra of type Δ and (u_α) a direct system of A_α -homomorphisms $u_\alpha : E_\alpha \rightarrow F$, $u = \varinjlim u_\alpha$ is an A -homomorphism of graded algebras.

10.1.2 Tensor products and Hom sets

Let Δ be a commutative monoid with its identity element denoted by 0 , A a graded ring of type Δ and M, N graded A -modules of type Δ . Let (A_λ) (resp. $(M_\lambda), (N_\lambda)$) be the graduation of A (resp. M, N); the tensor product $M \otimes_{\mathbb{Z}} N$ of the \mathbb{Z} -modules M and N is the direct sum of

the $M_\lambda \otimes_{\mathbb{Z}} N_\mu$ and hence the latter form a bigraduation of types Δ, Δ on this \mathbb{Z} -module. Consider on $M \otimes_{\mathbb{Z}} N$ the total graduation of type Δ associated with this bigraduation; it consists of the sub- \mathbb{Z} -modules $P_\lambda = \sum_{\mu+\nu=\lambda} (M_\mu \otimes_{\mathbb{Z}} N_\nu)$. It is known that the \mathbb{Z} -module $M \otimes_A N$ is the quotient of $M \otimes_{\mathbb{Z}} N$ by the sub- \mathbb{Z} -module Q generated by the elements $(ax) \otimes y - x \otimes (ay)$, where $x \in M, y \in N$ and $a \in A$; if, for all $\lambda \in \Delta$, $x_\lambda, y_\lambda, a_\lambda$ are the homogeneous components of degree λ of x, y, a respectively, clearly $(ax) \otimes y - x \otimes (ay)$ is the sum of the homogeneous elements $(a_\nu x_\mu) \otimes y_\lambda - x_\mu \otimes (a_\nu y_\lambda)$, in other words Q is a graded sub- \mathbb{Z} -module of $M \otimes_{\mathbb{Z}} N$ and the quotient

$$M \otimes_A N = (M \otimes_{\mathbb{Z}} N)/Q$$

therefore has canonically a graded \mathbb{Z} -module structure of type Δ . Moreover, the graduation which we have just defined on $M \otimes_A N$ is compatible with its A -module structure. For if $x \in M_\lambda, y \in N_\mu, a \in A_\nu$, the element $a(x \otimes y)$ belongs to $(M \otimes_{\mathbb{Z}} N)_{\lambda+\mu+\nu}$ and hence its image in $M \otimes_A N$ belongs to $(M \otimes_A N)_{\lambda+\mu+\nu}$, which establishes our assertion. When we speak of $M \otimes_A N$ as a graded A -module, we always mean with the structure thus defined, unless otherwise mentioned. Note that $(M \otimes_A N)$ can be defined as the additive group of $M \otimes_A N$ generated by the $x_\mu \otimes y_\nu$, where $x_\mu \in M_\mu, y_\nu \in N_\nu$ and $\mu + \nu = \lambda$.

Let M' (resp. N') be another graded A -module and $\phi : M \rightarrow M', \psi : N \rightarrow N'$ graded homomorphisms of respective degrees α and β . Then it follows immediately from the above remark that $\phi \otimes \psi$ is a graded A -module homomorphism of degree $\alpha + \beta$.

Similarly, a graduation (compatible with the A -module structure) is similarly defined on the tensor product of any finite number of graded A -modules; it is moreover immediate that the associativity isomorphisms such as $(M \otimes N) \otimes P \rightarrow M \otimes (N \otimes P)$ are isomorphisms of graded modules.

Proposition 10.1.26. *Let M and N be graded A -modules of type Δ , P a graded A -module of type Δ and let f be a A -bilinear map of $M \times N$ to P such that*

$$f(x_\lambda, y_\mu) \in P_{\lambda+\mu} \quad \text{for } x_\lambda \in M_\lambda, y_\nu \in N_\nu, \lambda, \mu \in \Delta.$$

Then $f(x, y) = \tilde{f}(x \otimes y)$ where $\tilde{f} : M \otimes_A N \rightarrow P$ is a graded A -module homomorphism of degree 0.

Proof. It is clear that \tilde{f} is A -linear, and the given condition implies that it is a homomorphism of graded modules. \square

Let B be another graded ring of type Δ and $\rho : A \rightarrow B$ a homomorphism of graded rings; then B is a graded A -module of type Δ via the map ρ . If E is a graded A -module of type Δ and $B \otimes_A E$ is given the graded A -module structure of type Δ defined above, the canonical B -module structure is compatible with the graduation of

$$\rho_!(E) = B \otimes_A E.$$

The graded B -module thus obtained is said to be obtained by extending the ring of scalars to B by means of f and when we speak of $\rho_!(E)$ as a graded B -module, we always mean this structure, unless otherwise mentioned.

Now we define graduations on the Hom sets. For simplicity we suppose that Δ is a group. Let A be a graded ring of type Δ and M, N two graded A -modules of type Δ . Let H_λ denote the additive group of graded homomorphisms of degree λ of M into N ; in the additive group

$\text{Hom}_A(M, N)$ of all homomorphisms of M into N (with the un-graded A -module structures) the sum (for $\lambda \in \Delta$) of the H_λ is direct. For, if there is a relation $\sum_\lambda \phi_\lambda = 0$ with $\phi_\lambda \in H_\lambda$ for all λ , it follows that $\sum_\lambda \phi_\lambda(x_\mu) = 0$ for all μ and all $x_\mu \in M_\mu$. As the elements of Δ are cancellable, $\phi_\lambda(x_\mu)$ is the homogeneous component of $\sum_\lambda \phi_\lambda(x_\mu)$ of degree $\lambda + \mu$; hence $\phi_\lambda(x_\mu) = 0$ for every ordered pair (μ, λ) and every $x_\mu \in M_\mu$, which implies $\phi_\lambda = 0$ for all $\lambda \in \Delta$. We shall denote (in this paragraph) by $\text{HomHomgr}_A(M, N)$ the additive subgroup of $\text{Hom}_A(M, N)$ the sum of the H_λ and we shall call it the additive group of graded A -module homomorphisms of M into N . For the canonical A -module structure on $\text{Hom}_A(M, N)$, $\text{Homgr}_A(M, N)$ is a submodule and the graduation (H_λ) is compatible with the A -module structure: for if $a_\nu \in A_\nu$, $x_\mu \in N_\nu$ and $\phi_\lambda \in H_\lambda$, then by definition $(a_\nu \phi_\lambda)(x_\mu) = a_\nu \phi_\lambda(x_\mu) \in N_{\lambda+\mu+\nu}$ and hence $a_\nu \phi_\lambda \in H_{\lambda+\nu}$.

Let M' and N' be two graded A -modules of type Δ . and $\phi : M' \rightarrow M$, $\psi : N \rightarrow N'$ graded homomorphisms of respective degrees α and β . Then it is immediate that $\text{Hom}(\phi, \psi) : \eta \mapsto \psi \circ \eta \circ \phi$ maps $\text{Homgr}_A(M, N)$ into $\text{Homgr}_A(M', N')$ and that its restriction to $\text{Homgr}_A(M, N)$ is a graded homomorphism into $\text{Homgr}_A(M', N')$ of degree $\alpha + \beta$. In particular, $\text{Homgr}_A(M, M)$ is a graded subring of $\text{End}_A(M)$, which is denoted by $\text{Homgr}_A(M)$.

If M and N are graded A -modules, the set $\text{Homgr}_A(M, N)$ is in general distinct from $\text{Hom}_A(M, N)$. However there is a special case where these tow sets equal.

Proposition 10.1.27. *Let A be a graded ring of type Δ and M a finitely generated graded A -module of type Δ . Then for any graded A -module N of type Δ , we have $\text{Homgr}_A(M, N) = \text{Hom}_A(M, N)$.*

Proof. Assume that M is generated by homogeneous elements x_1, \dots, x_n ; let d_i be the degree of x_i . Let $\phi \in \text{Hom}_A(M, N)$ and for all $\lambda \in \Delta$ let $z_{i,\lambda}$ denote the homogeneous component of $\phi(x_i)$ of degree $\lambda + d_i$. We show that there exists a homomorphism $\phi_\lambda : M \rightarrow N$ such that $\phi_\lambda(x_i) = z_{i,\lambda}$. for all i . For this, it suffices to prove that if $\sum_i a_i x_i = 0$ with $a_i \in A$ for $1 \leq i \leq n$, then $\sum_i a_i z_{i,\lambda} = 0$ for all $\lambda \in \Delta$, for then the formula $\sum_i a_i x_i \mapsto \sum_i a_i z_{i,\lambda}$ defines a map and satisfies the condition. To this end, it can be assumed that each a_i is homogeneous of degree δ_i such that $d_i + \delta_i = \mu$ for all i ; then $\sum_i a_i \phi(x_i) = 0$; taking the homogeneous component of degree $\lambda + \mu$ on the left-hand sider, we obtain $\sum_i a_i z_{i,\lambda} = 0$, whence the existence of the homomorphism ϕ_λ ; clearly ϕ_λ is graded of degree λ . Finally, $\phi_\lambda = 0$ except for a finite number of values of λ , and $\phi = \sum_\lambda \phi_\lambda$. by definition, which proves our assertion. \square

In particular, $\text{Homgr}_A(A, M) = \text{Hom}_A(A, M)$ for every graded A -module M ; moreover $\text{Hom}_A(A, M)$ has a graded A -module structure, and it is immediate that with this structure the canonical map of M into $\text{Hom}_A(A, M)$ is a graded A -module isomorphism.

Similarly, $\text{Homgr}_A(M, A)$ has a graded A -module structure; it is called the **graded dual** of the graded A -module M and is denoted by $M^{*\text{gr}}$, or simply M^* when no confusion results. If $\phi : M \rightarrow N$ is a graded homomorphism of degree δ , it follows from the above that the restriction to $N^{*\text{gr}}$ of ϕ^* is a graded homomorphism of the graded dual $N^{*\text{gr}}$ into the graded dual $M^{*\text{gr}}$, of degree δ , called the graded dual of ϕ .

Let M, N, P, Q be graded A -modules of type Δ . Then there are canonical graded homomorphisms of degree 0:

$$\text{Homgr}_A(M, \text{Homgr}_A(N, P)) \rightarrow \text{Homgr}_A(M \otimes_A N, P)$$

$$\text{Homgr}_A(M, N) \otimes_A P \rightarrow \text{Homgr}_A(M, N \otimes_A P)$$

$$\text{Homgr}_A(M, P) \otimes \text{Homgr}_A(N, Q) \rightarrow \text{Homgr}_A(M \otimes_A N, P \otimes_A Q)$$

(the tensor products being given the graduations defined above) obtained by restricting the canonical homomorphisms for Hom sets; for, if $\phi : M \rightarrow \text{Homgr}_A(N, P)$ is graded of degree δ , then, for all $x \in M_\lambda$, $\phi(x)$ is a graded homomorphism $N \rightarrow P$ of degree $\delta + \lambda$ and hence, for $y \in N_\mu$, $\phi(x)(y) \in P_{\delta+\lambda+\mu}$; if $\psi : M \otimes_A N \rightarrow P$ corresponds canonically to ϕ , it is then seen that ψ is a graded homomorphisms of degree δ , whence our assertion concerning the first map; moreover it is seen that this homomorphism is bijective. The argument is similar for the other maps.

10.1.3 Graduation by an ordered group

An order structure (denoted by \leq) on a commutative group Δ written additively is said to be compatible with the group structure if, for all $\lambda, \mu, \rho \in \Delta$, the relation $\lambda \leq \mu$ implies $\lambda + \rho \leq \mu + \rho$. The group Δ with this order structure is then called an ordered group.

Let Δ be an ordered commutative group, A a graded ring of type Δ and (A_λ) its graduation. We say A is a **graded ring with positive degrees** if $A_\lambda = \{0\}$ for $\lambda < 0$. In this case, it follows from definition that $A_+ = \bigoplus_{\lambda > 0} A_\lambda$ is a graded ideal of A .

Proposition 10.1.28. *Let Δ be an ordered group, A a graded ring of type Δ with positive degrees, (A_λ) its graduation, M a graded A -module of type Δ and (M_λ) its graduation. Suppose that there exists λ_0 such that $M_{\lambda_0} \neq \{0\}$ and $M_\lambda = \{0\}$ for $\lambda < \lambda_0$. Then $A_+ M \neq M$.*

Proof. Let x be a non-zero element of M_{λ_0} ; suppose that $x \in A_+ M$. Then $x = \sum_i a_i x_i$ where the a_i are nonzero homogeneous elements of A_+ and the x_i nonzero homogeneous elements of M with $\deg(x) = \deg(a_i) + \deg(x_i)$ for all i . But, as $\deg(a_i) > 0$, $\lambda_0 = \deg(a_i) + \deg(x_i) > \deg(x_i)$, which contradicts the hypothesis. \square

Corollary 10.1.29. *Let Δ be an ordered group, A a graded ring of type Δ with positive degrees. If M is a finitely generated graded A -module such that $A_+ M = M$, then $M = \{0\}$.*

Proof. Suppose $M \neq \{0\}$. Let λ_0 be a minimal element of the set of degrees of a finite generating system of M consisting of nonzero homogeneous elements; then the hypotheses of ?? would be fulfilled, which implies a contradiction. \square

Corollary 10.1.30. *Let Δ be an ordered group, A a graded ring of type Δ with positive degrees. If M is a finitely generated graded A -module and N is a graded submodule of M such that $N + A_+ M = M$, then $N = M$.*

Proof. The quotient module M/N is a finitely generated graded A -module and the hypothesis implies that $A_+(M/N) = M/N$, hence $M/N = 0$. \square

Corollary 10.1.31. *Let Δ be an ordered group, A a graded ring of type Δ with positive degrees. Let $\phi : M \rightarrow N$ be a graded homomorphism of graded A -modules, where N is assumed to be finitely generated. If the homomorphism*

$$\phi \otimes \text{id} : M \otimes_A (A/A_+) \rightarrow N \otimes_A (A/A_+)$$

is surjective, then ϕ is surjective.

Proof. Note that $\phi(M)$ is a graded submodule of N and we have

$$(M/\phi(M)) \otimes_A (A/A_+) \cong (N \otimes_A (A/A_+))/\text{im}(\phi \otimes \text{id})$$

as (A/A_+) -modules. The hypothesis therefore implies $(N/\phi(M)) \otimes_A (A/A_+) = \{0\}$ and hence $N = \phi(M)$ by ?? \square

Remark 10.1.4. It follows from the proof of ?? that Corollaries ?? and ?? (resp. ??) are still valid when, instead of assuming that M (resp. N) is finitely generated, the following hypothesis is made: there exists a subset Δ^+ of Δ satisfying the following conditions:

- (a) $M_\lambda = \{0\}$ for all $\lambda \notin \Delta^+$.
- (b) every non-empty subset of Δ^+ has a least element.

This will be the case if $\Delta = \mathbb{Z}$ and M (resp. N) is a graded module with positive degrees.

Proposition 10.1.32. Suppose that $\Delta = \mathbb{Z}$. With the hypothesis of ??, consider the graded A_0 -module $N = M/A_+M$ and suppose the following conditions hold:

- (a) each of the N_λ considered as an A_0 -module admits a basis $(y_{i\lambda})_{i \in I_\lambda}$.
- (b) the canonical homomorphism $A_+ \otimes_A M \rightarrow M$ is injective.

Then M is a graded free A -module and, to be precise, if $x_{i\lambda}$ is an element of M_λ whose image in N_λ is $y_{i\lambda}$, the family $(x_{i\lambda})_{(i,\lambda) \in I}$ (where I is the disjoint union of the I_λ) is a basis of M .

Proof. We know that there is a graded free A -module L (of graduation (L_λ)) and a surjective homomorphism $p : L \rightarrow M$ of degree 0 such that $p(e_{i\lambda}) = x_{i\lambda}$ for all $(i, \lambda) \in I$, where $(e_{i\lambda})_{(i,\lambda) \in I}$ is a basis of L consisting of homogeneous elements $e_{i\lambda} \in L_\lambda$. It follows from the above Remark that p is surjective. Consider the graded A -module $R = \ker p$ and note that $R_\lambda = \{0\}$ for $\lambda < \lambda_0$ by definition; we need to prove that $R = \{0\}$ and by ?? it suffices to show that $A_+R = R$. Consider the following commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & A_+ \otimes R & \longrightarrow & A_+ \otimes L & \xrightarrow{1 \otimes p} & A_+ \otimes M \longrightarrow 0 \\ & & \downarrow \phi & & \downarrow \psi & & \downarrow \eta \\ 0 & \longrightarrow & R & \longrightarrow & L & \xrightarrow{p} & M \longrightarrow 0 \end{array}$$

where the vertical maps are induced from the canonical injection $A_+ \rightarrow A$. We need to show that ϕ is surjective. Note that, as L is free (hence flat), ψ is injective. Also, η is injective by hypothesis. Then by the snake lemma, we have an exact sequence

$$R/A_+R \longrightarrow L/A_+L \xrightarrow{\bar{p}} M/A_+M \longrightarrow 0$$

By hypothesis we see \bar{p} is a bijection, hence the claim follows. \square

10.1.4 Graded rings of type \mathbb{Z}

In this paragraph, all the graduations considered are assumed to be of type \mathbb{Z} . If A (resp. M) is a graded ring (resp. graded module), A_i (resp. M_i) will denote the set of homogeneous elements of degree i in A (resp. M). Recall that if $A_i = \{0\}$ (resp. $M_i = \{0\}$) for $i < 0$, A (resp. M) will, to abbreviate, be called a graded ring (resp. module) with positive degrees.

Proposition 10.1.33. *Let $A = \bigoplus_{i \geq 0} A_i$ be a graded ring with positive degrees and (x_λ) a family of homogeneous elements of A_+ . The following conditions are equivalent:*

- (i) *The ideal of A generated by the family (x_λ) is equal to A_+ .*
- (ii) *The family (x_λ) is a system of generators of the A_0 -algebra A .*
- (iii) *For all $i \geq 0$, the A_0 -module A_i is generated by the elements of the form $\prod_\lambda x_\lambda^{n_\lambda}$ which are of degree i in A .*

Proof. Clearly conditions (ii) and (iii) are equivalent. If they hold, every element of A_+ is of the form $f(x_\lambda)$ where f is a polynomial of $A_0[X_\lambda]$ with no constant term; then $A_+ = \sum_\lambda A x_\lambda$, which proves that (iii) implies (i). Conversely, suppose that condition (i) holds. Let $A' = A_0[x_\lambda]$ be the sub- A_0 -algebra of A generated by the family (x_λ) and let us show that $A' = A$. For this, it is sufficient to show that $A_i \subseteq B$ for all $i \geq 0$. We proceed by induction on i , the property being obvious for $i = 0$. Then let $y \in A_i$ with $i \geq 1$. Since $y \in A_+$, which is the ideal generated by (x_λ) , there exists a family (a_λ) of elements of A of finite support such that $y = \sum a_\lambda x_\lambda$ and we may assume that each of the a_λ is homogeneous of degree $i - \deg(x_\lambda)$ (by replacing it if need be by its homogeneous component of that degree); as $\deg(x_\lambda) > 0$, the induction hypothesis shows that $a_\lambda \in A'$ for all λ , whence $y \in A'$ and $A_i \subseteq A'$, which proves that (i) implies (ii). \square

Proof. Part (a) follows from ???. As for (b), the condition in (b) is sufficient by Hilbert's basis theorem. If A is Noetherian, $A_0 = A/A_+$ is Noetherian and A_+ is finitely generated as an ideal. Thus A is an A_0 -algebra of finite type by (a). \square

Corollary 10.1.34. *Let $A = \bigoplus_{i \geq 0} A_i$ be a graded ring with positive degrees.*

- (a) *The ideal A_+ is finitely generated if and only if A is an A_0 -algebra of finite type.*
- (b) *The ring A is Noetherian if and only if A_0 is Noetherian and A is an A_0 -algebra of finite type.*
- (c) *Suppose that the conditions in (a) hold and let $M = \bigoplus_{i \in \mathbb{Z}} M_i$ be a finitely generated graded A -module. Then, for all $i \in \mathbb{Z}$, M_i is a finitely generated A_0 -module and there exists i_0 such that $M_i = \{0\}$ for $i < i_0$.*

Proof. Part (a) follows from ???. As for (b), the condition in (b) is sufficient by Hilbert's basis theorem. If A is Noetherian, $A_0 = A/A_+$ is Noetherian and A_+ is finitely generated as an ideal. Thus A is an A_0 -algebra of finite type by (a).

We prove (c). We may suppose that A is generated (as an A_0 -algebra) by homogeneous elements a_1, \dots, a_r of degree $i \geq 1$ and M is generated (as an A -module) by homogeneous elements x_1, \dots, x_n ; let $h_i = \deg(a_i)$ and $k_j = \deg(x_j)$. Clearly M_n consists of the linear combinations with coefficients in A_0 of the elements $a_1^{\alpha_1} \cdots a_r^{\alpha_r} x_j$ such that the α_i are positive integers

satisfying the relation $k_j + \sum_{i=1}^r h_i \alpha_i = n$; for each n there is only a finite number of families $(\alpha_i)_{i=1}^r$ satisfying these conditions, since $h_i \geq 0$ for all i ; we conclude that M_n is a finitely generated A_0 -module and moreover clearly $M_n = \{0\}$ when $n < \inf(k_i)$. \square

Let $A = \bigoplus_{i \in \mathbb{Z}} A_i$ be a graded ring and $M = \bigoplus_{i \in \mathbb{Z}} M_i$ a graded A -module; for each ordered pair (d, r) with $d \geq 1$ and $0 \leq r \leq d - 1$, set

$$A^{(d)} = \bigoplus_{i \in \mathbb{Z}} A_{id}, \quad M^{(d,r)} = \bigoplus_{i \in \mathbb{Z}} M_{id+r}.$$

Clearly $A^{(d)}$ is a graded subring of A and $M^{(d,r)}$ a graded $A^{(d)}$ -module; moreover, if N is a graded submodule of M , $N^{(d,r)}$ is a graded sub- $A^{(d)}$ -module of $M^{(d,r)}$. We shall write $M^{(d)}$ instead of $M^{(d,0)}$; for each $d \geq 1$, M is the direct sum of the $A^{(d)}$ -modules $M^{(d,r)}$ for $0 \leq r \leq d - 1$.

Proposition 10.1.35. *Let A be a graded ring with positive degrees and $M = \bigoplus_{i \in \mathbb{Z}} M_i$ be a graded A -module. Suppose that A is an A_0 -algebra of finite type and M a finitely generated A -module. Then*

- (a) *For every ordered pair (d, r) of integers such that $d \geq 1$ and $0 \leq r \leq d - 1$, $M^{(d,r)}$ is a finitely generated $A^{(d)}$ -module.*
- (b) *$A_+^{(d)}$ is a finitely generated $A^{(d)}$ -module and therefore $A^{(d)}$ is an A_0 -algebra of finite type.*

Proof. Let us show that A is a finitely generated $A^{(d)}$ -module. Let a_1, \dots, a_s be a system of generators of the A_0 -algebra A consisting of homogeneous elements. The elements of A (finite in number) of the form $a_1^{\alpha_1} \cdots a_s^{\alpha_s}$ such that $0 \leq \alpha_i \leq d$ for $1 \leq i \leq s$ constitute a system of generators of the $A^{(d)}$ -module A ; for every system of integers $n_i \geq 0$, $1 \leq i \leq s$, there are positive integers q_i and r_i such that $n_i = q_i d + r_i$, where $0 \leq r_i < d$ for $1 \leq i \leq s$; then

$$a_1^{n_1} \cdots a_s^{n_s} = (a_1^{q_1} \cdots a_s^{q_s})(a_1^{r_1} \cdots a_s^{r_s}).$$

which proves our assertion. Then, if M is a finitely generated A -module, it is also a finitely generated $A^{(d)}$ -module; as M is the direct sum of the $M^{(d,r)}$ for $0 \leq r \leq d - 1$, each of the $M^{(d,r)}$ is a finitely generated $A^{(d)}$ -module, which proves (a).

Now by ??, A_+ is a finitely generated A -module, so (a) implies $A_+^{(d)}$ is a finitely-generated A_0 -module, which again by ?? implies $A^{(d)}$ is a finite A_0 -algebra. \square

Proposition 10.1.36. *Let $A = \bigoplus_{i \geq 0} A_i$ be a graded ring with positive degrees which is a finitely generated A_0 -algebra and $M = \bigoplus_{i \in \mathbb{Z}} M_i$ a graded A -module. Let $(x_\lambda)_{\lambda \in I}$ be a system of homogeneous generators of M such that $\sup_\lambda \deg(x_\lambda) < +\infty$. Then there exists $n_0 > 0$ and $d > 0$ such that, for any $n \geq n_0$ we have $M_{n+d} = A_d M_n$.*

Proof. Let A be generated as an A_0 -algebra by homogeneous elements a_1, \dots, a_s , and set $d_i = \deg(a_i)$. Let d be a common multiple of all the d_i and set $b_i = a_i^{d/d_i}$ for $1 \leq i \leq s$, so that $\deg(b_i) = d$ for all i . Consider the set Z of elements of the form $a_1^{\alpha_1} \cdots a_s^{\alpha_s} x_\lambda$ with $0 \leq \alpha_i \leq d/d_i$ for $1 \leq i \leq s$ and $\lambda \in I$. Then by the hypothesis on (x_λ) , we can choose $n_0 > 0$ be larger than the degree of every element of Z . If $n \geq n_0$ then any element of M_{n+d} can be written as an A_0 -linear combination of elements $a_1^{n_1} \cdots a_s^{n_s} x_j$ as above. If $n_i = (d/d_i)\beta_i + \gamma_i$ with $\gamma_i < d/d_i$,

then we can write

$$a_1^{n_1} \cdots a_s^{n_s} x_j = b_1^{\beta_1} \cdots b_s^{\beta_s} (a_1^{\gamma_1} \cdots a_s^{\gamma_s} x_j) = b_1^{\beta_1} \cdots b_s^{\beta_s} z$$

where every b_i is homogenous of degree d and $z \in Z$ is homogeneous of some degree t . Note that $n + d > n_0 \geq t$, so the last expression makes sense. Now it is clear that $b_1^{\beta_1} \cdots b_s^{\beta_s} z$ belongs to $A_d M_n$, so $M_{n+d} = A_d M_n$. \square

Corollary 10.1.37. *Let A be a graded ring such that $A = A_0[A_1]$, M a graded A -module and $(x_\lambda)_{\lambda \in I}$ a system of homogeneous generators of M such that $\deg(x_\lambda) \leq n_0$ for all $\lambda \in I$. Then for all $n \geq n_0$ and all $d > 0$, $M_{n+d} = A_d M_n$.*

Proof. In this case the integer d in ?? can be chosen to be 1, and the set Z has elements of degree smaller than n_0 . Thus we have $M_{n+d} = A_1 M_n$ for $n \geq n_0$, and by induction $M_{n+d} = A_d M_n$ for $n \geq n_0$ and $d > 0$. \square

Corollary 10.1.38. *Let A be a graded ring such that $A = A_0[A_1]$ and let $S = \bigoplus_{i \geq 0} S_i$ be a graded A -algebra with positive degrees which is a finitely generated A -module. Then there exists an integer $n_0 > 0$ such that:*

- (a) *For $n \geq n_0$ and $d > 0$, $S_{n+d} = S_d S_n$.*
- (b) *For $d \geq n_0$, $S^{(d)} = S_0[S_d]$.*

Proof. By ?? there exists an integer $n_0 \geq 0$ such that, for $n \geq n_0$ and $d > 0$, $S_{n+d} = A_d S_n$, whence a fortiori $S_{n+d} = S_d S_n$, which establishes (a). Then, for $d \geq n_0$ and $i > 0$, $S_{id} = (S_d)^i$ as follows by induction on i applying (a); this establishes (b). \square

Corollary 10.1.39. *Let $A = \bigoplus_{i \geq 0} A_i$ be a graded ring with positive degrees which is a finitely generated A_0 -algebra. There exists an integer $d > 0$ such that $A^{(md)} = A_0[A_{md}]$ for all $m \geq 1$.*

Proof. In ?? we can assume that $d \geq n_0$ (since the proof works with d any common multiple of the d_i). And therefore with $A = M$ we have $A_{d+d} = (A_d)^2$ and inductively $A_{id} = (A_d)^i$ for all $i > 0$. This shows the claim. \square

Proposition 10.1.40. *Let Δ be an ordered group and A a graded ring of type Δ . If the product in A of two homogeneous nonzero elements is nonzero, then the ring A is integral.*

Proof. Let $x = \sum_\lambda x_\lambda$ and $y = \sum_\lambda y_\lambda$ be two non-zero elements of A , with x_λ, y_λ being homogeneous of degree λ for all $\lambda \in \Delta$. Let α (resp. β) be the greatest of the elements $\lambda \in \Delta$ such that $x_\lambda \neq 0$ (resp. $y_\lambda \neq 0$); it is immediate that if $\lambda \neq \alpha$ or $\mu \neq \beta$, either $x_\lambda y_\mu = 0$ or $\deg(x_\lambda y_\mu) < \lambda + \alpha$; the homogeneous component of xy of degree $\alpha + \beta$ is therefore $x_\alpha y_\beta$, which is non-zero by hypothesis; whence $xy \neq 0$. \square

Proposition 10.1.41. *Let $\mathfrak{p} = \bigoplus_{i \geq 0} \mathfrak{p}_i$ be a graded ideal of A ; for \mathfrak{p} to be prime, it is necessary and sufficient that for homogeneous elements $x, y \in A$, $xy \notin \mathfrak{p}$ if and only if $x \notin \mathfrak{p}$ and $y \notin \mathfrak{p}$.*

Proof. The condition is obviously necessary. Conversely, if it is fulfilled, then in the graded ring $A/\mathfrak{p} = \bigoplus_{i \geq 0} A_i/\mathfrak{p}_i$ the product of two homogeneous nonzero elements is nonzero and hence A/\mathfrak{p} is an integral domain. \square

Let \mathfrak{a} be an arbitrary ideal of A . We can associate to it a homogeneous ideal $\mathfrak{a}^h = \bigoplus_{i \geq 0} (\mathfrak{a} \cap A_i)$. It follows from the definition that $\mathfrak{a}^h \subseteq \mathfrak{a}$ and \mathfrak{a} is homogeneous if and only if $\mathfrak{a} = \mathfrak{a}^h$.

Proposition 10.1.42. *Let A be a graded ring with positive degrees and $\mathfrak{p} \subseteq A$ be a prime ideal. Let \mathfrak{p}^h be the homogeneous ideal generated by the homogeneous elements of \mathfrak{p} . Then \mathfrak{p}^h is a prime ideal of A .*

Proof. For any homogeneous element $f, g \in A$ such that $fg \in \mathfrak{p}^h \subseteq \mathfrak{p}$, we have $f \in \mathfrak{p}$ or $g \in \mathfrak{p}$. Then by the definition of \mathfrak{p}^h , this implies $f \in \mathfrak{p}^h$ or $g \in \mathfrak{p}^h$. Thus \mathfrak{p}^h is prime by ??.

Corollary 10.1.43. *Let $A = \bigoplus_{i \geq 0} A_i$ be a graded ring with positive degrees.*

(a) *Any minimal prime of A is a homogeneous ideal.*

(b) *Given a homogeneous ideal $\mathfrak{a} \subseteq A$ any minimal prime over \mathfrak{a} is homogeneous.*

Proof. The first claim is a direct application of ??, since the ideal \mathfrak{p}^h is contained in \mathfrak{p} . The second follows by applying the result on A/\mathfrak{a} . \square

Let A be a graded ring with positive degrees. Two graded ideals $\mathfrak{a} = \bigoplus_{i \geq 0} \mathfrak{a}_i$ and $\mathfrak{b} = \bigoplus_{i \geq 0} \mathfrak{b}_i$ of A are said to be **equivalent** if there exists an integer n_0 such that $\mathfrak{a}_n = \mathfrak{b}_n$ for $n \geq n_0$ (clearly it is an equivalence relation). A graded ideal is called **essential** if it is not equivalent to A_+ . As we will see, this notation plays an important role when we study graded prime ideals.

Proposition 10.1.44. *Let A be a graded ring with positive degrees, \mathfrak{a} a graded ideal, and \mathfrak{p} a graded prime ideal of A with $\mathfrak{p} \not\supseteq A_+$. Then $\mathfrak{a} \subseteq \mathfrak{p}$ if and only if there exists an integer $n_0 \geq 0$ such that $\mathfrak{a}_n \subseteq \mathfrak{p}_n$ for all $n \geq n_0$.*

Proof. One implication is clear. Conversely, assume that $\mathfrak{a}_n \subseteq \mathfrak{p}_n$ for $n \geq n_0$. We now show that $\mathfrak{a}_{n_0-1} \subseteq \mathfrak{p}_{n_0-1}$. Since $\mathfrak{p} \not\supseteq A_+$, there exist $e > 0$ and $a \in A_e$ with $a \notin \mathfrak{p}$. If $b \in \mathfrak{a}_{n_0-1}$ then $ab \in \mathfrak{a}_{n_0-1+e} \subseteq \mathfrak{p}_{n_0-1+e}$, so $b \in \mathfrak{p}$ by primeness. This proves the claim. \square

Corollary 10.1.45. *Let A be a graded ring with positive degrees and $\mathfrak{p}, \mathfrak{q}$ be graded prime ideals and \mathfrak{p} with $\mathfrak{p} \not\supseteq A_+$ and $\mathfrak{q} \not\supseteq A_+$. Then $\mathfrak{p} = \mathfrak{q}$ if and only if they are equivalent.*

Proposition 10.1.46. *Let \mathfrak{a} be a graded ideal of A and $n_0 > 0$ an integer. For there to exist a graded prime ideal \mathfrak{p} such that $\mathfrak{p}_n = \mathfrak{a}_n$ for $n \geq n_0$, it is necessary and sufficient that, for any homogeneous elements x, y of degrees $\geq n_0$, the relation $xy \in \mathfrak{a}$ implies $x \in \mathfrak{a}$ or $y \in \mathfrak{a}$. If there exists $n \geq n_0$ such that $\mathfrak{a}_n \neq A_n$, then the prime ideal satisfying the above condition is unique.*

Proof. The condition of the statement is obviously necessary. If $\mathfrak{a}_n = A_n$ for all $n \geq n_0$, clearly every prime ideal containing A_+ is a solution to the problem; there may therefore be several prime ideals which solve the problem; however, any two of these ideals are obviously equivalent.

Now suppose that there exists a homogeneous element $a \in A_d$ with $d \geq n_0$ not belonging to \mathfrak{a}_d . Now we define

$$\mathfrak{p} = \{x \in A : ax \in \mathfrak{a}\}.$$

Clearly \mathfrak{p} is an ideal of A ; as the homogeneous components of ax are the products by a of those of x and a is a graded ideal, \mathfrak{p} is a graded ideal; moreover, $1 \notin \mathfrak{p}$ and hence $\mathfrak{p} \neq A$. To prove that \mathfrak{p} is prime, let $x \in A_m$ and $y \in A_n$ satisfy $x \notin \mathfrak{p}$ and $y \notin \mathfrak{p}$. Then $ax \notin \mathfrak{a}_{m+d}$ and $ay \notin \mathfrak{a}_{n+d}$, hence $a^2xy \notin \mathfrak{a}_{m+n+2d}$ by hypothesis. Since \mathfrak{a} is an ideal, we then get $axy \notin \mathfrak{a}_{m+n+d}$, so $xy \in \mathfrak{p}$.

Now if $n \geq n_0$ and $a \in A$, the conditions $x \in \mathfrak{a}_n$ and $ax \in \mathfrak{a}_{n+d}$ are equivalent by hypothesis and hence $\mathfrak{p} \cap A_n = \mathfrak{a}_n$, which completes the proof of the existence of the graded prime ideal \mathfrak{p} which solves the problem. The uniqueness part now follows from ??.

Proposition 10.1.47. *Let $A = \bigoplus_{i \geq 0} A_i$ be a graded ring with positive degrees and $d > 0$ an integer.*

- (a) *For every essential graded prime ideal \mathfrak{p} of A , $\mathfrak{p} \cap A^{(d)}$ is an essential graded prime ideal of $A^{(d)}$.*
- (b) *Conversely, for every essential graded prime ideal \mathfrak{p}' of $A^{(d)}$, there exists a unique (necessarily essential) graded prime ideal \mathfrak{p} of A such that $\mathfrak{p} \cap A^{(d)} = \mathfrak{p}'$.*

Proof. If $a \in A_i$ does not belong to \mathfrak{p}_i , then a^d does not belong to \mathfrak{p}_{id} , and hence $\mathfrak{p} \cap A^{(d)}$ is essential. This proves (a). Now let \mathfrak{p}' be an essential graded prime ideal of $A^{(d)}$. If \mathfrak{p} is a graded prime ideal of A such that $\mathfrak{p} \cap A^{(d)} = \mathfrak{p}'$, then for all $n \geq 0$, the set $\mathfrak{p} \cap A_n$ must be equal to the set \mathfrak{a}_n of $x \in A_n$ such that $x^d \in \mathfrak{p}'$. Let us show that $\mathfrak{a} = \bigoplus_{n \in \mathbb{N}} \mathfrak{a}_n$ is a graded prime ideal, which will prove (b). As \mathfrak{p}' is prime, $\mathfrak{a}_n = \mathfrak{p}_n$ when n is a multiple of d . Now if $x \in \mathfrak{a}_n$, $y \in \mathfrak{a}_m$, then $(x - y)^{2d}$ is the sum of terms each of which is a product of x^d or y^d by a homogeneous element of degree nd and hence $(x - y)^{2d} \in \mathfrak{p}'$ and, since \mathfrak{p}' is prime, $(x - y)^d \in \mathfrak{p}'$ and therefore \mathfrak{a}_n is a subgroup of A . As \mathfrak{p}' is an ideal of $A^{(d)}$, \mathfrak{a} is a graded ideal of A ; finally, the relation $(xy)^d \in \mathfrak{p}'$ implies $x^d \in \mathfrak{p}'$ or $y^d \in \mathfrak{p}'$, which completes the proof. \square

Let A be a graded ring with positive degrees and \mathfrak{p} an essential graded prime ideal of A . The set S of homogeneous elements of A not belonging to \mathfrak{p} is multiplicative and the ring of fractions $S^{-1}A$ is therefore graded canonically (note that there will in general be homogeneous nonzero elements of negative degree in this graduation). We shall denote by $A_{\mathfrak{p}}$ the subring of $S^{-1}A$ consisting of the homogeneous elements of degree 0, in other words the set of fractions x/s , where x and s are homogeneous of the same degree in A and $s \notin \mathfrak{p}$. Similarly, for every graded A -module M , $S^{-1}M$ is graded canonically (loc. cit.) and we shall denote by $M_{\mathfrak{p}}$ the subgroup of homogeneous elements of degree 0, which is obviously an $A_{\mathfrak{p}}$ -module.

Proposition 10.1.48. *Let \mathfrak{p} be a graded prime ideal of A , $d > 0$ an integer, and \mathfrak{p}' be the graded prime ideal $\mathfrak{p} \cap A^{(d)}$ of $A^{(d)}$. For every graded A -module M , the homomorphism $(M^{(d)})_{\mathfrak{p}'} \rightarrow M_{\mathfrak{p}}$ induced from the canonical injection $M^{(d)} \rightarrow M$ is bijective.*

Proof. If S is the set of homogeneous elements of A not belonging to \mathfrak{p} and $S^{(d)} = S \cap A^{(d)}$, the canonical homomorphism $\phi : (S^{(d)})^{-1}M^{(d)} \rightarrow S^{-1}M$ is a homogeneous homomorphism of degree 0 and it is injective, for, if $x \in M_{nd}$ satisfies $sx = 0$ for $s \in A_m$, $s \notin \mathfrak{p}$, then also $s^d x = 0$ and $s^d \in A_{md}$, $s^d \notin \mathfrak{p}'$. It remains to show that the image under ϕ of $(M^{(d)})_{\mathfrak{p}'}$ is the whole of $M_{\mathfrak{p}}$; but if $x \in M_n$, $s \in A_n$ and $s \notin \mathfrak{p}$, then also $x/s = (xs^{d-1})/s^d$ where $xs^{d-1} \in A_{nd}$, $s^d \in A_{md}$ and $s^d \notin \mathfrak{p}'$, whence our assertion. \square

Proposition 10.1.49. *Suppose A is a graded ring with positive degrees, $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ homogeneous prime ideals and \mathfrak{a} a homogeneous ideal. Assume that $\mathfrak{a} \cap A_+ \not\subseteq \mathfrak{p}_i$ for all i , then there exists a homogeneous element $x \in \mathfrak{a} \cap A_+$ of positive degree such that $x \notin \mathfrak{p}_i$ for $1 \leq i \leq r$.*

Proof. We may assume that $\mathfrak{a} \subseteq A_+$ and there are no inclusions among the \mathfrak{p}_i . The result is true for $r = 1$. Suppose the result holds for $r - 1$. Pick $x \in \mathfrak{a}$ homogeneous of positive degree such that $x \notin \mathfrak{p}_i$ for all $i = 1, \dots, r - 1$. If $x \notin \mathfrak{p}_r$ we are done, so assume that $x \in \mathfrak{p}_r$. Since $\mathfrak{a}\mathfrak{p}_1 \cdots \mathfrak{p}_{r-1}$ is a homogeneous ideal that is not contained in \mathfrak{p}_r , there is a homogeneous element $y \in \mathfrak{a}\mathfrak{p}_1 \cdots \mathfrak{p}_{r-1}$ of positive degree such that $y \notin \mathfrak{p}_r$. Then $x^{\deg(y)} + y^{\deg(x)}$ works. \square

10.2 Filtration and topologies

10.2.1 Filtrated rings and modules

Definition 10.2.1. An increasing (resp. decreasing) sequence $(G_n)_{n \in \mathbb{Z}}$ of subgroups of a commutative group G is called an **increasing** (resp. **decreasing**) **filtration** on G . A commutative group with a filtration is called a **filtered group**.

If $(G_n)_{n \in \mathbb{Z}}$ is an increasing (resp. decreasing) filtration on a commutative group G and we write $G'_n = G_{-n}$, clearly $(G'_n)_{n \in \mathbb{Z}}$ is a decreasing (resp. increasing) filtration on G . We may therefore restrict our study to decreasing filtrations and hence forth when we speak of a filtration, we shall mean a decreasing filtration, unless otherwise stated.

Given a decreasing filtration $(G_n)_{n \in \mathbb{Z}}$ on a commutative group G , clearly $\bigcap_{n \in \mathbb{Z}} G_n$ and $\bigcup_{n \in \mathbb{Z}} G_n$ are subgroups of G ; the filtration is called **separated** if $\bigcap_{n \in \mathbb{Z}} G_n$ is reduced to the identity element and **exhaustive** if $\bigcup_{n \in \mathbb{Z}} G_n = G$.

Definition 10.2.2. Given a ring A , a filtration $(A_n)_{n \in \mathbb{Z}}$ over the additive group A is called compatible with the ring structure on A if

$$(a) \quad A_m A_n \subseteq A_{m+n} \text{ for all } m, n \in \mathbb{Z}.$$

$$(b) \quad 1 \in A_0.$$

The ring A with this filtration is then called a **filtered ring**.

Conditions (a) and (b) show that A_0 is a subring of A and the A_n 's are A_0 -modules. The set $B = \bigcup_{n \in \mathbb{Z}} A_n$ is a subring of A and the set $\mathfrak{n} = \bigcap_{n \in \mathbb{Z}} A_n$ is an ideal of B ; for if $x \in \mathfrak{n}$ and $a \in A_p$, then for all $k \in \mathbb{Z}$ we have $x \in A_{k-p}$, whence $ax \in A_k$ by (a); therefore $ax \in \mathfrak{n}$. An important particular case is that $A_0 = A$; then $A_n = A$ for all $n \leq 0$ and all the A_n are ideals of A .

Definition 10.2.3. Let A be a filtered ring, $(A_n)_{n \in \mathbb{Z}}$ its filtration and M an A -module. A filtration $(M_n)_{n \in \mathbb{Z}}$ on M is called compatible with its module structure over the filtered ring A if

$$A_m M_n \subseteq M_{m+n} \quad \text{for all } m, n \in \mathbb{Z}. \tag{10.2.1}$$

The A -module M with this filtration is called a **filtered module**.

If M is a filtered module with filtration $(M_n)_{n \in \mathbb{Z}}$, then the M_n are all A_0 -modules; if $B = \bigcup_{n \in \mathbb{Z}} A_n$, clearly $\bigcup_{n \in \mathbb{Z}} M_n$ is a B -module and so is $\bigcap_{n \in \mathbb{Z}} M_n$ by the same argument as above for \mathfrak{n} . If $A_0 = A$, all the M_n 's are submodules of M .

Example 10.2.4. On a ring A the sets A_n such that $A_n = 0$ for $n > 0$, $A_n = A$ for $n \leq 0$ form what is called a **trivial filtration** associated with the trivial graduation on A ; on an A -module M , every filtration (M_n) consisting of sub- A -modules is then compatible with the module structure on M over the filtered ring A . Then it is possible to say that every filtered commutative group G is a filtered \mathbb{Z} -module, if \mathbb{Z} is given the trivial filtration.

Example 10.2.5. Let A be a graded ring of type \mathbb{Z} ; for all $i \in \mathbb{Z}$, let A^n be the subgroup of homogeneous elements of degree n in A and set $A_n = \bigoplus_{i \geq n} A^i$. Then it is immediate that (A_n) is an exhaustive and separated decreasing filtration which is compatible with the ring

structure on A ; this filtration is said to be associated with the graduation (A^n) and the filtered ring A is said to be associated with the given graded ring A .

Now let M be a graded module of type \mathbb{Z} over the graded ring A and for all $i \in \mathbb{Z}$ let M^n be the subgroup of homogeneous elements of degree n of M . If $M_n = \bigoplus_{i \geq n} M^i$, then (M_n) is an exhaustive and separated decreasing filtration which is compatible with the module structure on M over the filtered ring A ; this filtration is said to be **associated** with the graduation (M^n) and the filtered module M is said to be **associated** with the given graded module M .

Example 10.2.6. Let A be a filtered ring, $(A_n)_{n \in \mathbb{Z}}$ its filtration and M an A -module. Let us write $M_n = A_n M$; it follows that

$$A_m M_n = A_m A_n M \subseteq A_{m+n} M = M_{m+n}$$

and that $M_0 = M$; therefore (M_n) is an exhaustive filtration which is compatible with the A -module structure on M . This filtration is said to be **derived** from the given filtration (A_n) on A ; note that it is not necessarily separated, even if (A_n) is separated.

Example 10.2.7 (The \mathfrak{a} -adic filtration). Let A be a ring and \mathfrak{a} an ideal of A . Let us write $A_n = \mathfrak{a}^n$ for $n > 0$ and $A_n = A$ for $n \leq 0$. It is immediate that (A_n) is an exhaustive filtration on A , called the **\mathfrak{a} -adic filtration**. Let M be an A -module; the filtration (M_n) derived from the \mathfrak{a} -adic filtration on A is called the \mathfrak{a} -adic filtration on M . If B is an A -algebra, then $\mathfrak{b} = \mathfrak{a}^\epsilon$ is an ideal of B and for every B -module N and $n \in \mathbb{Z}$ we have $\mathfrak{b}^n N = \mathfrak{a}^n N$, therefore the \mathfrak{b} -adic filtration on N coincides with the \mathfrak{a} -adic filtration (if N is considered as an A -module).

Let G be a filtered group and $(G_n)_{n \in \mathbb{Z}}$ its filtration; clearly, for every subgroup H of G , $(H \cap G_n)_{n \in \mathbb{Z}}$ is a filtration said to be **induced** by that on G . Similarly, if H is a normal subgroup of G , the family $((H + G_n)/H)_{n \in \mathbb{Z}}$ is a filtration on the group G/H , called the **quotient** under H of the filtration on G .

If G and G' are filtered groups and (G_n) , (G'_n) are their filtrations. Then $(G_n \times G'_n)_{n \in \mathbb{Z}}$ is a filtration on $G \times G'$ called the **product** of the filtrations on G and G' , which is exhaustive (resp. separated) if (G_n) and (G'_n) are.

Now let A be a filtered ring and (A_n) its filtration; on every subring B of A , clearly the filtration induced by that on A is compatible with the ring structure on B . If \mathfrak{a} is an ideal of A , the quotient filtration on A/\mathfrak{a} of that on A is compatible with the structure of this ring, for

$$(A_m + \mathfrak{a})(A_n + \mathfrak{a}) \subseteq A_m A_n + \mathfrak{a} \subseteq A_{m+n} + \mathfrak{a}.$$

If A' is another filtered ring, the product filtration on $A \times A'$ is compatible with the structure of this ring.

Simialrly, let M be a filtered A -module and (M_n) its filtration; on every submodule N of M , the filtration induced by that on M is compatible with the A -module structure on N and, on the quotient module M/N , the quotient filtration of that on M is compatible with the A -module structure, as

$$A_m(N + M_n) \subseteq N + A_m M_n \subseteq N + M_{m+n}.$$

Note that if the filtration on M is derived from that on A , so is the quotient filtration on M/N because we have $(N + A_n M)/N = A_n(M/N)$. However, this is not true in general for the filtration induced on N .

Let A be a filtered ring, M a filtered A -module and (M_n) the filtration of M . For all $x \in M$ let $v(x)$ denote the least upper bound in $\bar{\mathbb{R}}$ of the set of integers $n \in \mathbb{Z}$ such that $x \in M_n$. Then the following equivalences hold:

$$\begin{cases} v(x) = +\infty & x \in \bigcap_{n \in \mathbb{Z}} M_n \\ v(x) = p & x \in M_p \setminus M_{p+1} \\ v(x) = -\infty & x \notin \bigcup_{n \in \mathbb{Z}} M_n \end{cases}$$

The map $v : M \rightarrow \bar{\mathbb{R}}$ is called the **order function** of the filtered module M . If v is known then so are the M_n , for M_n is the set of $x \in M$ such that $v(x) \geq n$; the fact that the M_n are additive subgroups of M implies the relation

$$v(x+y) \geq \min\{v(x), v(y)\} \quad (10.2.2)$$

The above definition applies in particular to the filtered A -module A . If v_A is the order function of A and v_M the order function on M , it follows from (??) that for $a \in A$, and $x \in M$,

$$v_M(ax) \geq v_A(a) + v_M(x)$$

whenever the right-hand side is defined. In particular, for $a \in A$ and $b \in A$, we have $v_A(ab) \geq v_A(a) + v_A(b)$.

10.2.2 The associated graded module

Let G be a commutative group (written additively) and (G_n) a filtration on G . Let us write

$$\text{gr}_n(G) = G_n/G_{n+1}, \quad \text{gr}(G) = \bigoplus_{n \in \mathbb{Z}} \text{gr}_n(G).$$

The commutative group $\text{gr}(G)$ is then a graded group of type \mathbb{Z} , called the graded group associated with the filtered group G , the homogeneous elements of degree n of $\text{gr}(G)$ being those of $\text{gr}_n(G)$.

Now let A be a filtered ring, (A_n) its filtration, M a filtered A -module and (M_n) its filtration. For all $p \in \mathbb{Z}, q \in \mathbb{Z}$, we can define a map

$$\text{gr}_p(A) \times \text{gr}_q(M) \rightarrow \text{gr}_{p+q}(M), \quad (\bar{a}, \bar{x}) \mapsto \bar{a}\bar{x}. \quad (10.2.3)$$

This is well-defined in view of the equality $ax - by = (a - b)x + b(x - y)$ and the relations $A_{p+1}M_q \subseteq M_{p+q+1}$, $A_pM_{q+1} \subseteq M_{p+q+1}$. It is immediate that the map defined is \mathbb{Z} -bilinear; by linearity, we derive a \mathbb{Z} -bilinear map

$$\text{gr}(A) \times \text{gr}(M) \rightarrow \text{gr}(M).$$

If this definition is first applied to the case $M = A$, the map (??) is an internal law of composition on $\text{gr}(A)$, which it is immediately verified is associative and has an identity element which is the canonical image in $\text{gr}_0(A)$ of the unit element of A ; it therefore defines on $\text{gr}(A)$ a ring structure and the graduation $(\text{gr}_n(A))_{n \in \mathbb{Z}}$ is by definition compatible with this structure.

The graded ring $\text{gr}(A)$ (of type \mathbb{Z}) thus defined is called the **graded ring associated with the filtered ring A** . The map (??) is on the other hand a $\text{gr}(A)$ -module external law on $\text{gr}(M)$, the module axioms being trivially satisfied, and the graduation $(\text{gr}_n(M))_{n \in \mathbb{Z}}$ on $\text{gr}(M)$ is obviously compatible with this module structure. The graded $\text{gr}(A)$ -module $\text{gr}(M)$ (of type \mathbb{Z}) thus defined is called the graded module associated with the filtered A -module M .

Example 10.2.8. Let A be a ring and t an element of A which is not a divisor of 0. Let us give A the (t) -adic filtration. Then the associated graded ring $\text{gr}(A)$ is canonically isomorphic to the polynomial ring $(A/(t))[X]$. For $\text{gr}_n(A) = 0$ for $n < 0$ and by definition the ring $\text{gr}_0(A)$ is the ring $A/(t)$. We now note that by virtue of the hypothesis on t the relation $at^n \equiv 0 \pmod{(t^{n+1})}$ is equivalent to $a \equiv 0 \pmod{t}$; if τ is the canonical image of t in $\text{gr}_1(A)$, every element of $\text{gr}_n(A)$ may then be written uniquely in the form $\alpha\tau^n$, where $\alpha \in \text{gr}_0(A)$; whence our assertion.

Example 10.2.9. Let K be a ring and A the ring of formal power series

$$A = K[[X_1, \dots, X_r]].$$

Let \mathfrak{m} be the ideal of A whose elements are the formal power series with no constant term. Let us give A the \mathfrak{m} -adic filtration; if M_1, \dots, M_s are the distinct monomials in X_1, \dots, X_r of total degree $n - 1$, clearly every formal power series u of total order $\deg(u) \geq n$ may be written as $\sum_{k=1}^s u_k M_k$, where the u_k belong to \mathfrak{m} ; it is seen that \mathfrak{m}^n is the set of formal power series u such that $\deg(u) \geq n$, which shows that \deg is the order function for the \mathfrak{m} -adic filtration. Then clearly, for every formal power series $u \in \mathfrak{m}^n$, there exists a unique homogeneous polynomial of degree n in the X_i 's which is congruent to $u \pmod{\mathfrak{m}^{n+1}}$, namely the sum of terms of degree n of u ; we conclude that $\text{gr}(A)$ is canonically isomorphic to the polynomial ring $K[X_1, \dots, X_r]$.

Example 10.2.10. More generally, let A be a ring and \mathfrak{a} an ideal of A and A be given the \mathfrak{a} -adic filtration. Then the identity map of the A/\mathfrak{a} -module $\mathfrak{a}/\mathfrak{a}^2$ onto itself can be extended uniquely to a homomorphism ϕ from the symmetric algebra $S(\mathfrak{a}/\mathfrak{a}^2)$ to the A/\mathfrak{a} -algebra $\text{gr}(A)$; it follows from the definition of $\text{gr}(A)$ that ϕ is a surjective homomorphism of graded algebras; for $n \geq 1$, every element of $\text{gr}_n(A)$ is a sum of classes mod \mathfrak{a}^{n+1} of elements of the form $y = x_1 \cdots x_n$, where $x_i \in \mathfrak{a}$ for $1 \leq i \leq n$; if ξ_i is the class of $x_i \pmod{\mathfrak{a}^2}$, clearly the class of $y \pmod{\mathfrak{a}^{n+1}}$ is the element $\phi(\xi_1) \cdots \phi(\xi_n)$, whence our assertion. In particular, every system of generators of the A/\mathfrak{a} -module $\mathfrak{a}/\mathfrak{a}^2$ is a system of generators of the A/\mathfrak{a} -algebra $\text{gr}(A)$.

If now M is an A -module and M is given the \mathfrak{a} -adic filtration, it is seen similarly that the graded $\text{gr}(A)$ -module $\text{gr}(M)$ is generated by $\text{gr}_0(M) = M/\mathfrak{a}M$. To be precise, the restriction Γ to $\text{gr}(A) \times \text{gr}_0(M)$ of the external law on the $\text{gr}(A)$ -module $\text{gr}(M)$ is a \mathbb{Z} -bilinear map of $\text{gr}(A) \times \text{gr}_0(M)$ to $\text{gr}(M)$. Moreover, it is immediately verified that, for $\alpha \in \text{gr}(A)$, $\beta \in \text{gr}_0(A)$, $\xi \in \text{gr}_0(M)$, we have $\Gamma(\alpha\beta, \xi) = \Gamma(\alpha, \beta\xi)$ and hence Γ defines a canonical surjective $\text{gr}_0(A)$ -linear map

$$\gamma_M : \text{gr}(A) \otimes_{\text{gr}_0(A)} \text{gr}_0(M) \rightarrow \text{gr}(M) \tag{10.2.4}$$

Example 10.2.11. Let A be a graded ring of type \mathbb{Z} and M a graded A -module of type \mathbb{Z} ; let A^i , (resp. M^i) be the subgroup of homogeneous elements of degree i A (resp. M). Let A and M be given the filtrations associated with their graduations. Then it is immediate that the \mathbb{Z} -linear map

map $A \rightarrow \text{gr}(A)$ which maps an element of A^n to its canonical image in

$$\text{gr}_n(A) = \bigoplus_{i \geq n} A^i / \bigoplus_{i \geq n+1} A^i$$

is a graded ring isomorphism. A canonical graded A -module isomorphism $E \rightarrow \text{gr}(E)$ is defined similarly.

Proposition 10.2.12. *Let A be a filtered ring, $(A_n)_{n \in \mathbb{Z}}$ its filtration and v its order function. Suppose that $\text{gr}(A)$ is an integral domain. Then, for every elements a, b of the ring $B = \bigcup_n A_n$, we have $v(ab) = v(a) + v(b)$.*

Proof. As $\mathfrak{n} = \bigcap_{n \in \mathbb{Z}} A_n$ is an ideal of the ring B , the formula holds if $v(a)$ or $v(b)$ is equal to $+\infty$. If not, $v(a) = r$ and $v(b) = s$ are integers. The classes \bar{a} of $a \bmod A_{r+1}$ and \bar{b} of $b \bmod A^{s+1}$ are nonzero by definition, whence by hypothesis $\bar{a}\bar{b} \neq 0$ in $\text{gr}(A)$ and therefore $ab \notin A_{r+s+1}$; as $ab \in A_{r+s}$, we see $v(ab) = r + s$. \square

Corollary 10.2.13. *Let A be a filtered ring and $(A_n)_{n \in \mathbb{Z}}$ its filtration. Define $B = \bigcup_{n \in \mathbb{Z}} A_n$ and $\mathfrak{n} = \bigcap_{n \in \mathbb{Z}} A_n$. If the ring $\text{gr}(A)$ is integral, so is B/\mathfrak{n} .*

Proof. If a and b are elements of B not belonging to \mathfrak{n} , then $v(a) \neq +\infty$ and $v(b) \neq +\infty$, whence $v(ab) \neq +\infty$ and therefore $ab \notin \mathfrak{n}$. \square

10.2.3 Homomorphism compatible with filtrations

Let G, G' be two commutative groups (written additively), (G_n) a filtration on G and (G'_n) a filtration on G' . A homomorphism $\phi : G \rightarrow G'$ is called **compatible** with the filtrations on G and G' if $\phi(G_n) \subseteq G'_n$ for all $n \in \mathbb{Z}$. If this holds, then by taking quotients we get a homomorphism $\text{gr}_n(\phi) : G_n/G_{n+1} \rightarrow G'_n/G'_{n+1}$ there is therefore a unique additive group homomorphism $\text{gr}(\phi) : \text{gr}(G) \rightarrow \text{gr}(G')$ such that, for all $n \in \mathbb{Z}$, $\text{gr}(\phi)$ coincides with $\text{gr}_n(\phi)$ on $\text{gr}_n(G)$. The map $\text{gr}(\phi)$ is called the **graded group homomorphism associated with ϕ** . If G'' is a third filtered group and $\psi : G' \rightarrow G''$ is a homomorphism which is compatible with the filtrations, then $\psi \circ \phi$ is a homomorphism which is compatible with the filtrations and we have

$$\text{gr}(\psi \circ \phi) = \text{gr}(\psi) \circ \text{gr}(\phi)$$

Proposition 10.2.14. *Let G be a filtered group and H a subgroup of G ; let H be given the induced filtration and G/H the quotient filtration. If $\iota : H \rightarrow G$ is the canonical injection and $\pi : G \rightarrow G/H$ the canonical surjection, then ι and π are compatible with the filtrations and the sequence*

$$0 \longrightarrow \text{gr}(H) \xrightarrow{\text{gr}(\iota)} \text{gr}(G) \xrightarrow{\text{gr}(\pi)} \text{gr}(G/H) \longrightarrow 0$$

is exact.

Proof. The first assertion is obvious: if (G_n) is the filtration on G , then

$$(H \cap G_n) \cap G_{n+1} = H \cap G_{n+1}.$$

and hence $\text{gr}(\iota)$ is injective; moreover the canonical map $G_n \rightarrow (H + G_n)/H$ is surjective, hence so is $\text{gr}(\pi)$ and $\text{gr}(\pi) \circ \text{gr}(\iota) = \text{gr}(\pi \circ \iota) = 0$. Finally, let $\xi \in \text{gr}_n(G)$ belong to the kernel of

$\text{gr}(\pi)$; then there exists $x \in \xi$ such that $x \in H + G_{n+1}$; but as $G_{n+1} \subseteq G_n$,

$$G_n \cap (H + G_{n+1}) = (H \cap G_n) + G_{n+1}$$

and hence $x = y + z$ where $y \in H \cap G_n$ and $z \in G_{n+1}$; this proves that ξ is the class mod G_{n+1} of $\iota(y)$, in other words it belongs to the image of $\text{gr}(H)$ under $\text{gr}(\iota)$. \square

Example 10.2.15. Note that the functor $\text{gr}(\cdot)$ is not exact. Let $A = A' = k[X]$ where k is a field and give A the trivial filtration while A' the (X) -adic filtration. Then the identity map η from A to A' is compatible with the filtrations and we get a sequence

$$0 \longrightarrow \text{gr}(A) \xrightarrow{\text{gr}(\eta)} \text{gr}(A') \longrightarrow 0$$

It is easy to see the map $\text{gr}(\eta)$ is neither injective nor surjective. The point is that the filtration on A differs from the filtration induced by A' via the map η .

If now A and B are two filtered rings and $\rho : A \rightarrow B$ a ring homomorphism which is compatible with the filtrations, it is immediately verified that the graded group homomorphism $\text{gr}(\rho) : \text{gr}(A) \rightarrow \text{gr}(B)$ is also a ring homomorphism. In particular, if A' is a subring of A with the induced filtration, $\text{gr}(A')$ is canonically identified with a graded subring of $\text{gr}(A)$ (??). If \mathfrak{a} is an ideal of A and A/\mathfrak{a} is given the quotient filtration, $\text{gr}(A/\mathfrak{a})$ is canonically identified with the quotient graded ring $\text{gr}(A)/\text{gr}(\mathfrak{a})$ (??).

Finally, let A be a filtered ring, M, N two filtered A -modules and $\phi : M \rightarrow N$ an homomorphism compatible with the filtrations. Then it is immediate that $\text{gr}(\phi) : \text{gr}(M) \rightarrow \text{gr}(N)$ is a $\text{gr}(A)$ -linear map and hence a graded homomorphism of degree 0 of graded $\text{gr}(A)$ -modules. Moreover, if $\psi : M \rightarrow N$ is another A -homomorphism compatible with the filtrations, so is $\phi + \psi$ and we have $\text{gr}(\phi + \psi) = \text{gr}(\phi) + \text{gr}(\psi)$.

10.3 Filtrations and completions

Let G be a commutative group filtered by a family $(G_n)_{n \in \mathbb{Z}}$ of subgroups of G . There exists a unique topology on G which is compatible with the group structure and for which the G_n constitute a fundamental system of neighbourhoods of the identity element 0 of G (??); it is called the topology on G defined by the filtration (G_n) . When we use topological notions concerning a filtered group, we shall mean, unless otherwise stated, with the topology defined by the filtration. Note that the G_n , being subgroups of G , are both open and closed (??). As G is commutative, we deduce that G admits a Hausdorff completion group \widehat{G} (??). For every subset E of G , the closure of E in G is equal to

$$\bar{E} = \bigcap_{n \in \mathbb{Z}} (E + G_n) = \bigcap_{n \in \mathbb{Z}} (G_n + E).$$

In particular $\bigcap_{n \in \mathbb{Z}} G_n$ is the closure of $\{0\}$; thus it is seen that for the topology on G to be Hausdorff it is necessary and sufficient that the filtration (G_n) be separated. On the other hand, for the topology on G to be discrete, it is necessary and sufficient that there exist $n \in \mathbb{Z}$ such that $G_n = \{0\}$; in this case the filtration (G_n) is called **discrete**.

Not let G' be another filtered group and $\phi : G \rightarrow G'$ a homomorphism compatible with the filtrations; the definition of the topologies on G and G' shows immediately that ϕ is continuous. If H is a subgroup of G , the topology induced on H by that on G (resp. the quotient topology with respect to H of that on G) is the topology on H (resp. G/H) defined by the filtration induced by that on G (resp. quotient topology of that on G). The product topology of those on G and G' is the topology defined by the product of filtrations on G and G' .

Proposition 10.3.1. *Let A be a filtered ring, (A_n) its filtration and $B = \bigcup_{n \in \mathbb{Z}} A_n$. If M is a filtered B -module, (M_n) its filtration and $N = \bigcup_{n \in \mathbb{Z}} M_n$ of M , then the map $(a, x) \mapsto ax$ from $B \times N$ to N is continuous.*

Proof. Let $a_0 \in B$, $x_0 \in N$; there exists by hypothesis integers r, s such that $a_0 \in A_r$ and $x_0 \in M_s$. The equality

$$ax - a_0x_0 = (a - a_0)x_0 + a_0(x - x_0) + (a - a_0)(x - x_0)$$

shows that if $a - a_0 \in A_i$ and $x - x_0 \in M_j$, then $ax - a_0x_0$ belongs to $M_{i+s} + M_{j+r} + M_{i+j}$. Then, given an integer n , $ax - a_0x_0 \in M_n$ provided $i \geq n - s$, $j \geq n - r$ and $i + j \geq n$; that is so long as i and j are sufficiently large. \square

Corollary 10.3.2. *The ring B is a topological ring and the B -module N is a topological B -module.*

It is seen in particular that a filtered ring A whose filtration is exhaustive is a topological ring; if this is so every filtered A -module whose filtration is exhaustive is a topological A -module.

Example 10.3.3. Let A be a ring and \mathfrak{a} an ideal of A ; the topology defined on A by the \mathfrak{a} -adic filtration is called the \mathfrak{a} -adic topology; as the \mathfrak{a} -adic filtration is exhaustive, A is a topological ring with this topology. Similarly, for every A -module M , the topology defined by the \mathfrak{a} -adic filtration is called the \mathfrak{a} -adic topology on M ; M is a topological A -module under this topology.

Proposition 10.3.4. *Let A be a ring filtered by an exhaustive filtration (A_n) and \mathfrak{p} an ideal of A . Suppose that the ideal $\text{gr}(\mathfrak{p})$ of the ring $\text{gr}(A)$ is prime. Then the closure of \mathfrak{p} in A is a prime ideal.*

Proof. We know that $\text{gr}(A/\mathfrak{p})$ is isomorphic to $\text{gr}(A)/\text{gr}(\mathfrak{p})$ (??) and hence an integral domain; we conclude that $A/\bigcap_{n \in \mathbb{Z}} (\mathfrak{p} + A_n)$ is an integral domain by ???. Thus the closure $\bar{\mathfrak{p}} = \bigcap_{n \in \mathbb{Z}} (\mathfrak{p} + A_n)$ is a prime ideal. \square

10.3.1 Complete filtered groups

Proposition 10.3.5. *Let G be a filtered group with filtration (G_n) . The following conditions are equivalent:*

- (i) *G is a complete topological group.*
- (ii) *The associated Hausdorff group $G' = G / (\bigcap_n G_n)$ is complete.*
- (iii) *Every Cauchy sequence in G is convergent.*
- (iv) *Every family $(x_\lambda)_{\lambda \in I}$ of elements of G' which converges to 0 with respect to the filter \mathfrak{F} complements of finite subsets of I is summable in G' .*

Proof. For a filter on G to be a Cauchy filter (resp. a convergent filter), it is necessary and sufficient that its image under the canonical map $G \rightarrow G'$ be a Cauchy (resp. convergent) filter (??); whence first of all the equivalence of (i) and (ii); on the other hand, as G' is first countable, the equivalence of (i) and (iii) follows.

Suppose that G' is complete and let $(x_\lambda)_{\lambda \in I}$ be a family of elements of G' which converge to 0 with respect to G' . For every neighbourhood V' of 0 in G' which is a subgroup of G' , there exists a finite subset J of I such that $x_\lambda \in V'$ whenever $\lambda \in I \setminus J$; then $\sum_{\lambda \in K} x_\lambda \in V'$ for every finite subset K of I not meeting J , which shows that the family $(x_\lambda)_{\lambda \in I}$ is summable. Conversely, suppose that condition (iv) holds and let (x_n) be a Cauchy sequence on G' , the family $(x_{n+1} - x_n)$ is then summable and in particular the series with general term $x_{n+1} - x_n$ is convergent and hence the sequence (x_n) is convergent. \square

Let G be a filtered group whose filtration (G_n) ; the quotient groups G/G_n are discrete and hence complete, since the G_n 's are open in G . For each $n \in \mathbb{Z}$, we have a canonical map $\pi_n : G \rightarrow G/G_n$, and for $m < n$ there is a canonical map $\tau_{mn} : G/G_n \rightarrow G/G_m$ such that $(G/G_n, \pi_{mn})$ is an inverse system of discrete groups which is compatible with the maps (π_n) . By the universal property of inductive limits, we then get a map $\pi : G \rightarrow \varprojlim G/G_n$. If $\iota : G \rightarrow \widehat{G}$ is the canonical map of G to its Hausdorff completion \widehat{G} , then as the G/G_n are complete, there exists a unique topological group isomorphism $\psi : \widehat{G} \rightarrow \varprojlim G/G_n$ such that $\pi = \psi \circ \iota$ (??), we shall call it the **canonical isomorphism** of \widehat{G} onto $\varprojlim G/G_n$.

$$\begin{array}{ccc}
 G/G_n & \xrightarrow{\tau_{mn}} & G/G_m \\
 \pi_n \swarrow & & \searrow \tau_m \\
 \varprojlim G/G_n & & \\
 \pi \downarrow & & \uparrow \iota \\
 G & & \widehat{G} \xrightarrow{\psi} \varprojlim G/G_n
 \end{array}$$

With the result above, we can apply the properties of inverse limit to completion. First let us observe that the inverse system $\{G/G_n\}$ has the special property that the transition maps $\tau_{mn} : G/G_n \rightarrow G/G_m$ are always surjective. Any inverse system with this property we shall call a **surjective system**. For such a system, we have the following result:

Proposition 10.3.6. *If $0 \rightarrow \{A_n\} \rightarrow \{B_n\} \rightarrow \{C_n\} \rightarrow 0$ is an exact sequence of inverse systems then we have an exact sequence*

$$0 \rightarrow \varprojlim A_n \rightarrow \varprojlim B_n \rightarrow \varprojlim C_n \rightarrow \varprojlim^1 A_n \rightarrow \varprojlim^1 B_n \rightarrow \varprojlim^1 C_n \rightarrow 0$$

Moreover, $\varprojlim^1 A_n = 0$ if $\{A_n\}$ is a surjective system.

Proof. For an inverse system $\{A_n\}$ let $A = \prod_n A_n$ and define $d_A : A \rightarrow A$ by

$$d_A((a_n)) = (a_n - \tau_{n,n+1}(a_{n+1}))$$

so that $\ker d_A = \varprojlim A_n$ and we set $\varprojlim^1 A_n := \text{coker } d_A$. The exact sequence of inverse systems

then defines a commutative diagram of exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow 0 \\ & & \downarrow d_A & & \downarrow d_B & & \downarrow d_C & \\ 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow 0 \end{array}$$

and hence by snake lemma an exact sequence

$$0 \rightarrow \ker d_A \rightarrow \ker d_B \rightarrow \ker d_C \rightarrow \text{coker } d_A \rightarrow \text{coker } d_B \rightarrow \text{coker } d_C \rightarrow 0$$

To complete the proof we have only to prove that d_A is surjective if $\{A_n\}$ is a surjective system. But this is clear because to show d_A surjective we have only to solve inductively the equations $x_n - \tau_{n,n+1}(x_{n+1}) = a_n$ for $x_n \in A_n$, given $a_n \in A_n$. \square

Corollary 10.3.7. *Let G be a filtered group and H a subgroup of G ; let H be given the induced filtration and G/H the quotient filtration. If $\iota : H \rightarrow G$ is the canonical injection and $\pi : G \rightarrow G/H$ the canonical surjection, then the sequence*

$$0 \longrightarrow \widehat{H} \longrightarrow \widehat{G} \longrightarrow \widehat{G/H} \longrightarrow 0$$

is exact.

Proof. We have an exact sequence of the correspond inverse systems so ?? gives the claim immediately. \square

Corollary 10.3.8. *Let G be a filtered group with filtration (G_n) . Then \widehat{G}_n is a subgroup of \widehat{G} and $\widehat{G}/\widehat{G}_n \cong G/G_n$. Therefore, we have $\text{gr}(G) \cong \text{gr}(\widehat{G})$.*

Proof. Just apply ?? with $H = G_n$, in which case G/G_n has the discrete topology so that $\widehat{G/G_n} = G/G_n$. \square

Example 10.3.9. Let G be a complete filtered group. Every closed subgroup of G with the induced filtration is complete. Every quotient group of G with the quotient filtration is complete.

Example 10.3.10. Let A be a ring and \mathfrak{a} an ideal. The completion \widehat{A} of A under the \mathfrak{a} -adic topology is called the \mathfrak{a} -adic completion of A . The canonical map $\iota : A \rightarrow \widehat{A}$ is a continuous ring homomorphism, whose kernel is $\bigcap_n \mathfrak{a}^n$.

Let M be an A -module, the completion \widehat{M} of M under the \mathfrak{a} -adic topology is a topological \widehat{A} -module. If $\phi : M \rightarrow N$ is any A -module homomorphism, then $\phi(\mathfrak{a}^n M) = \mathfrak{a}^n \phi(M) \subseteq \mathfrak{a}^n N$, and therefore ϕ is continuous (with respect to the \mathfrak{a} -topologies on M and N) and so defines $\widehat{\phi} : \widehat{M} \rightarrow \widehat{N}$.

Example 10.3.11. Let A be a filtered ring whose filtration we denote by $(A_n)_{n \in \mathbb{Z}}$; let $B = A[[X_1, \dots, X_r]]$. For all $\alpha = (\alpha_1, \dots, \alpha_s) \in \mathbb{N}^s$, we write $|\alpha| = \sum_{i=1}^s e_i$ and $X^\alpha = X_1^{\alpha_1} \cdots X_s^{\alpha_s}$ so that every element $P \in B$ can be written uniquely $P = \sum_{\alpha \in \mathbb{N}^s} c_{\alpha,P} X^\alpha$, where $c_{\alpha,P} \in A$. For all $n \in \mathbb{Z}$, define

$$B_n = \{P = \sum_{\alpha} c_{\alpha,P} X^\alpha : c_{\alpha,P} \in A_{n-|\alpha|} \text{ for all } \alpha \in \mathbb{N}^s\}.$$

Clearly B_n is an additive subgroup of B ; on the other hand, if $P \in B_n$ and $Q \in B$, then, for all $\alpha \in \mathbb{N}^s$, $c_{\gamma,PQ} = \sum_{\alpha+\beta=\gamma} c_{\alpha,P}c_{\beta,Q}$; as the relation $\alpha + \beta = \gamma$ implies $|\alpha| \leq |\gamma|$, we have $PQ \in B_n$, so B_n is an ideal of B . Moreover, if $Q \in B_m$, then for $\alpha + \beta = \gamma$,

$$c_{\alpha,P}c_{\beta,Q} \in A_{n-|\alpha|}A_{m-|\beta|} \subseteq A_{m+n-|\gamma|},$$

which proves that $(B_n)_{n \in \mathbb{Z}}$ is a filtration compatible with the ring structure on B (for obviously $1 \in B_0$). When in future we speak of B as a filtered ring, we shall mean, unless otherwise stated, with the filtration (B_n) . Clearly $\bigcap_{n \in \mathbb{Z}} B_n$ is the set of formal power series all of whose coefficients belong to $\bigcap_{n \in \mathbb{Z}} A_n$; then, if A is Hausdorff, so is B . If $A_0 = A$, then $B_0 = B$.

Proposition 10.3.12. *With the above notation, suppose that $A_0 = A$ and let h denote the map $P \mapsto (c_{\alpha,P})_{\alpha \in \mathbb{N}^s}$. Then h is an isomorphism of the additive topological group B onto the additive topological group $A^{\mathbb{N}^s}$. The polynomial ring $A[X_1, \dots, X_s]$ is dense in B ; if A is complete, so is B .*

Proof. Clearly h is bijective; $V_n = h(B_n)$ is the set of $(c_\alpha)_{\alpha \in \mathbb{N}^s}$ such that $c_\alpha \in A_{n-|\alpha|}$ for all $\alpha \in \mathbb{N}^s$ such that $|\alpha| \leq n$; as these elements α are finite in number, V_n is a neighbourhood of 0 in $A^{\mathbb{N}^s}$. Conversely, if V is a neighbourhood of 0 in $A^{\mathbb{N}^s}$, there is a finite subset E of \mathbb{N}^s and an integer n such that the conditions $c_\alpha \in A_n$ for all $\alpha \in E$ imply $(c_\alpha) \in V$; if then n is the greatest of the integers $n + |\alpha|$ for $\alpha \in E$, then $h(A_n) \subseteq V$, which proves the first assertion. Moreover, with n and E defined as above, $h(P - \sum_{\alpha \in E} c_{\alpha,P}X^\alpha) \in V$ for all $P \in B$, which shows that $A[X_1, \dots, X_s]$ is dense in B . The last assertion follows from the first and the fact that a product of complete spaces is complete. \square

Let \mathfrak{a} be an ideal of A and suppose that (A_n) is the \mathfrak{a} -adic filtration; then, if \mathfrak{b} is the ideal of B generated by \mathfrak{a} and X_1, \dots, X_s , the filtration (B_n) is the \mathfrak{b} -adic filtration. For clearly, for all $n \geq 0$, \mathfrak{b}^n is generated by the elements cX^α such that $c \in \mathfrak{b}^{n-|\alpha|}$ for all $\alpha \in \mathbb{N}^s$ such that $|\alpha| \leq n$, whence $\mathfrak{b}^n \subseteq B_n$. Conversely, for all $P \in B_n$, $P = P_1 + P_2$, where

$$P_1 = \sum_{|\alpha| < n} c_{\alpha,P}X^\alpha, \quad P_2 = \sum_{|\alpha| \geq n} c_{\alpha,P}X^\alpha$$

Clearly it is possible to write $P_2 = \sum_{|\alpha|=n} Q_\alpha X^\alpha$, where the Q_α are elements of B , whence $P_2 \in \mathfrak{b}^n$; on the other hand, since $P \in B_n$, we have $c_{\alpha,P}X^\alpha \in \mathfrak{b}^n$ for all $|\alpha| \leq n$, whence $P_1 \in \mathfrak{b}^n$.

Corollary 10.3.13. *Let A be a ring and $B = A[[X_1, \dots, X_r]]$ the ring of formal power series in s indeterminates over A and \mathfrak{n} the ideal of B consisting of the formal power series with no constant term. The ring B is Hausdorff and complete with the \mathfrak{n} -adic topology and the polynomial ring $A[X_1, \dots, X_r]$ is everywhere dense in B .*

Proof. It is sufficient to apply what has just been said to the case $\mathfrak{m} = \{0\}$, so that the \mathfrak{m} -adic topology on A is trivial and A is thus complete. \square

10.3.2 Completion of filtered rings and modules

Let A be a filtered ring, M a filtered A -module and (A_n) and (M_n) the respective filtrations of A and M ; we shall assume that these filtrations are exhaustive so that for the corresponding topologies A is a topological ring and M a topological A -module (??). Then we have defined

\widehat{A} as a topological ring and \widehat{M} as a topological \widehat{A} -module. If $\iota : A \rightarrow \widehat{A}$ is the canonical homomorphism, then $\iota(A_m)\iota(A_n) \subseteq \iota(A_{m+n})$, whence by the continuity of multiplication in \widehat{A} ,

$$\widehat{A}_n\widehat{A}_m \subseteq \widehat{A}_{m+n}$$

where \widehat{A}_n is the closure of $\iota(A_n)$ in A . It can be similarly shown that $\widehat{A}_n\widehat{M}_m \subseteq \widehat{M}_{m+n}$, in other words,

Proposition 10.3.14. *Let A be a filtered ring and M a filtered A -module, the respective filtrations (A_n) , (E_n) of A and M being exhaustive. Then (\widehat{A}_n) is a filtration compatible with the ring structure on \widehat{A} and (\widehat{M}_n) is a filtration compatible with the module structure on \widehat{A} over the filtered ring \widehat{A} . Moreover, these filtrations are exhaustive and define respectively the topologies on \widehat{A} and \widehat{M} . Finally, the canonical maps $\text{gr}(A) \rightarrow \text{gr}(\widehat{A})$ and $\text{gr}(M) \rightarrow \text{gr}(\widehat{M})$ of graded \mathbb{Z} -modules are respectively a graded ring isomorphism and a graded $\text{gr}(A)$ -module isomorphism.*

In what follows, for every uniform space X , ι_X will denote the canonical map from X to its Hausdorff completion \widehat{X} and $X_0 = \iota_X(X)$ the uniform subspace of \widehat{X} , which is the Hausdorff space associated with X . Recall that the topology on X is the inverse image under ι_X of that on X_0 . Recall also that, for every uniformly continuous map $f : X \rightarrow Y$, \widehat{f} denotes the uniformly continuous map from \widehat{X} to \widehat{Y} such that $\widehat{f} \circ \iota_X = \iota_Y \circ f$. If X is a uniform subspace of Y and i is the canonical injection, then \widehat{X} is identified with a uniform subspace of \widehat{Y} and \widehat{i} is the canonical injection of \widehat{X} into \widehat{Y} (??).

Lemma 10.3.15. *Let $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$ be an exact sequence of strict morphism of topological groups. Suppose that X, Y, Z admit Hausdorff completion groups and that the identity elements of X, Y, Z admit countable fundamental systems of neighbourhoods. Then*

$$0 \longrightarrow \widehat{X} \longrightarrow \widehat{Y} \longrightarrow \widehat{Z} \longrightarrow 0$$

is an exact sequence of strict morphisms.

Proof. Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be the given maps. We have already seen that the completion of a strict injection is strict injective, so it remains to show the following: if $Y = X/N$ (where N is a subgroup of X) and $f : X \rightarrow Y$ is the canonical map, then \widehat{f} is a surjective strict morphism with kernel \widehat{N} .

Let $X_0 = \iota_X(X)$ and $Y_0 = \iota_Y(Y)$; let $f_0 : X_0 \rightarrow Y_0$ be the map which coincides with \widehat{f} on X_0 ; as ι_X (resp. ι_Y) is a surjective strict morphism of X onto X_0 (resp. Y onto Y_0), f_0 is a surjective strict morphism by ???. Now X_0 and Y_0 are metrizable, so it follows from ?? that $\widehat{f}_0 = \widehat{f}$ is a surjective strict morphism and has kernel the closure \widehat{N}'_0 in \widehat{X} of the kernel N'_0 of f_0 . Then it will be sufficient for us to prove that $\widehat{N}'_0 = \widehat{N}$. Now N'_0 obviously contains $N_0 = \iota_X(N)$, so $\widehat{N}'_0 \supseteq \widehat{N}$; it will be sufficient to show that N'_0 is contained in the closure \overline{N}_0 of N_0 in X_0 . Now,

$$U = \iota_X^{-1}(X_0 \setminus \overline{N}_0) = X \setminus \iota_X^{-1}(\overline{N}_0)$$

is an open set in X which does not meet N ; as f is a surjective strict morphism, $V = f(U)$ is an open set in Y not containing the identity element 0 of Y and hence not meeting the closure of $\{0\}$; then $\iota_Y(V)$ does not contain the identity element of Y_0 . But $\iota_Y(V) = f_0(X_0 \setminus N_0)$ and hence $N'_0 \subseteq \overline{N}_0$, which complete the proof. \square

Proposition 10.3.16. *Let A be a filtered ring, (A_n) its filtration, M an A -module and (M_n) the filtration on M derived from that on A . Suppose that the filtration (A_n) is exhaustive and the module M is finitely generated. If $\iota_M : M \rightarrow \widehat{M}$ is the canonical map, then, for all $n \in \mathbb{Z}$,*

$$\widehat{M}_n = \widehat{A}_n \widehat{M} = \widehat{A}_n \iota_M(M), \quad \widehat{M} = \widehat{A} \widehat{M} = \widehat{A} \iota_M(M).$$

In particular \widehat{M} is a finitely generated \widehat{A} -module.

Proof. By hypothesis there exists a surjective homomorphism $\phi : L \rightarrow M$, where $L = A^I$, I being a finite set; let L be given the product filtration, consisting of the $L_n = A_n^I$; then $\widehat{L} = \widehat{A}^I$ and $\widehat{L}_n = \widehat{A}_n^I$. Let $\iota_L : L \rightarrow \widehat{L}$ be the canonical map and $(e_i)_{i \in I}$ the canonical basis of L ; for an element $\sum_i a_i \iota_L(e_i)$ of \widehat{L}_n to belong to \widehat{L}_n , it is necessary and sufficient that $a_i \in \widehat{A}_n$ for all i , thus we see $\widehat{L}_n = \widehat{A}_n \widehat{L} = \widehat{A}_n \iota_L(L)$.

Now note that by definition $\phi(L_n) = M_n$ and hence ϕ is a strict morphism of L onto M . ?? then shows that $\widehat{\phi} : \widehat{L} \rightarrow \widehat{M}$ is a surjective strict morphism. As L_n is an open subgroup of \widehat{L} , $\widehat{\phi}(\widehat{L}_n)$ is an open (and therefore closed) subgroup of M ; but by what we have just proved,

$$\widehat{\phi}(\widehat{L}_n) = \widehat{\phi}(\widehat{A}_n \iota_L(L)) = \widehat{A}_n \iota_M(M).$$

As $\iota_M(M_n) \subseteq \widehat{A}_n \iota_M(M)$, this then implies

$$\widehat{M}_n \subseteq \widehat{A}_n \iota_M(M) \subseteq \widehat{A}_n \widehat{M} \subseteq \widehat{M}_n$$

and therefore $\widehat{M}_n = \widehat{A}_n \widehat{M} = \widehat{A}_n \iota_M(M)$. Take the union of all $n \in \mathbb{Z}$, we obtain the second formula. \square

Corollary 10.3.17. *Under the conditions of ??, if A is complete, so is M .*

Proof. As the canonical map $A \rightarrow \widehat{A}$ is then surjective, $M = \iota_M(M)$ by ?? and the conclusion follows from ??.

Corollary 10.3.18. *Let A be a ring, \mathfrak{a} a finitely generated ideal of A and \widehat{A} the Hausdorff completion of A with respect to the \mathfrak{a} -adic topology. Then $\widehat{\mathfrak{a}}^n = (\widehat{\mathfrak{a}})^n = \widehat{A}\mathfrak{a}^n$ for every integer $n > 0$ and the topology on \widehat{A} is the \mathfrak{a} -adic topology.*

Proof. Let us write $A_n = \mathfrak{a}^n$, which is a finitely generated ideal of A . The formula $\mathfrak{a}^p A_n = \mathfrak{a}^{n+p}$ shows that the topology induced on A_n by the \mathfrak{a} -adic topology coincides with the \mathfrak{a} -adic topology on the A -module A_n . By ?? applied to $M = A_n$, $\widehat{A}_n = \widehat{A} A_n$, in other words $\widehat{\mathfrak{a}}^n = \widehat{A}\mathfrak{a}^n$. In particular $\widehat{\mathfrak{a}} = \widehat{A}\mathfrak{a}$, whence $(\widehat{\mathfrak{a}})^n = \widehat{A}\mathfrak{a}^n$. \square

Example 10.3.19. Let A be a graded ring of type \mathbb{N} and let $(A^n)_{n \in \mathbb{N}}$ be its graduation; let it be given the associated filtration which is separated and exhaustive. The additive group A is canonically identified with the subspace $\bigoplus_{n \in \mathbb{N}} A^n$ in $B = \prod_{n \in \mathbb{N}} A^n$. If B is given the topology the product of the discrete topologies, the topology induced on A is the topology defined by the filtration on A ; also B is a complete topological group and A is dense in B . The additive topological group B is then identified with the completion \widehat{A} of the Hausdorff additive group A and it has a unique ring structure which makes it the completion of the topological ring A . To define multiplication in this ring, note that, if we write $A_n = \bigoplus_{i \geq n} A^i$, the closure in B of the ideal A_n is the set B_n of $x = (x_n)$ such that $x_n = 0$ for $i < n$. Let $x = (x_n)$, $y = (y_n)$ be two

elements of B and $z = (z_n)$ their product. Then, for all $n > 0$, $x \equiv x^{(n)} \pmod{B_{n+1}}$ and $y \equiv y^{(n)} \pmod{B_{n+1}}$. But $x^{(n)}$ and $y^{(n)}$ belong to A and it is therefore seen that, for all $n \in \mathbb{N}$,

$$z_n = \sum_{i+j=n} x_i y_j.$$

In particular, we again obtain that, if A is a ring, the completion of $A[X_1, \dots, X_r]$ with the filtration associated with its usual graduation (by total degree) is canonically identified with the ring of formal power series $A[\![X_1, \dots, X_r]\!]$.

Example 10.3.20. Let α be a non-zero non-invertible element of a principal ideal domain; the (α) -adic topology on A is also called the α -adic topology; it is Hausdorff for the intersection of the ideals (α^n) reduces to 0. Note that the completion of A with respect to this topology is not necessarily an integral domain. The associated graded ring $\text{gr}(A) = \text{gr}(\widehat{A})$ is canonically isomorphic to $(A/\alpha)[X]$. If $A = \mathbb{Z}$, the completion of \mathbb{Z} with respect to the α -adic topology ($n > 1$) is denoted by \mathbb{Z}_α , and its elements are called α -adic integers.

Example 10.3.21. Let k be a field, $A = k[(X_n)_{n \in \mathbb{N}}]$, and $\mathfrak{m} = (X_1, X_2, \dots)$. We will think of an element $f \in \widehat{A}$ as a (possibly) infinite sum

$$f(X) = \sum_I a_I X^I$$

(using multi-index notation) such that for each $d \geq 0$ there are only finitely many nonzero a_I for $|I| = d$. The maximal ideal $\widehat{\mathfrak{m}}$ in \widehat{A} is the collection of f with zero constant term. In particular, the element

$$f = X_1 + X_2^2 + X_3^2 + \dots$$

is in $\widehat{\mathfrak{m}}$ but not in $\mathfrak{m}\widehat{A}$. This shows the inclusion $\mathfrak{m}\widehat{A} \subseteq \widehat{\mathfrak{m}}$ is proper.

Proposition 10.3.22. *Let A be a ring and $(\mathfrak{a}_\lambda)_{\lambda \in I}$ a family of ideals of A , distinct from A , such that \mathfrak{a}_λ and \mathfrak{a}_μ are relatively prime for $\lambda, \mu \in I$. For every family $s = (s_\lambda)_{\lambda \in I}$ of positive integers of finite support, set $\mathfrak{a}_s = \bigcap_{\lambda \in I} \mathfrak{a}_\lambda^{s_\lambda}$, and let \mathcal{T} be the topology compatible with the ring structure on A such that the \mathfrak{a}_s 's form a fundamental system of neighbourhoods of 0. Let \widehat{A} be the Hausdorff completion of A with respect to this topology. On the other hand, for all $\lambda \in I$, let A_λ be the ring A with the \mathfrak{a}_λ -adic topology and let \widehat{A}_λ be its Hausdorff completion. If $\phi : A \rightarrow \prod_{\lambda \in I} A_\lambda$ denotes the diagonal homomorphism, then ϕ is continuous and the corresponding homomorphism*

$$\widehat{\phi} : \widehat{A} \rightarrow \prod_{\lambda \in I} \widehat{A}_\lambda$$

is a topological ring isomorphism.

Proof. Note that \mathcal{T} is the least upper bound of all \mathfrak{a}_λ -adic topologies. The first assertion follows from ???. Let us set $B = \prod_{\lambda \in I} A_\lambda$; as the topology on A is finer than each of the \mathfrak{a}_λ -adic topologies, the maps $\pi_\lambda \circ \phi$ are continuous and hence ϕ is continuous. Also, $\phi(\mathfrak{a}_s)$ is the intersection of the diagonal Δ of B and the open set $\bigcap_{\lambda \in I} \pi_\lambda^{-1}(\mathfrak{a}_\lambda^{s_\lambda})$, it follows that ϕ is a strict morphism from the additive group A to B with image A . Now A is dense in B . For let $b = (a_\lambda)$ be an element of B ; every neighbourhood of b in B contains a set of the form $b + V$, where $V = \bigcap_{\lambda \in I} \pi_\lambda^{-1}(\mathfrak{a}_\lambda^{s_\lambda})$ for a family $s = (s_\lambda)$ of positive integers with finite support. As the \mathfrak{a}_λ 's

are relatively prime in pairs, there exists $x \in A$ such that $x \equiv a_\lambda \pmod{\mathfrak{a}_\lambda^{s_\lambda}}$ for all $\lambda \in I$ (by CRT, note that (s_λ) has finite support) and hence $(b + V) \cap \Delta \neq \emptyset$. The Hausdorff completion of the group B/Δ is then $\{0\}$; applying ?? to the exact sequences $0 \rightarrow A \rightarrow B \rightarrow B/\Delta \rightarrow 0$, we see $\hat{\phi}$ is an isomorphism of \widehat{A} onto \widehat{B} . \square

Corollary 10.3.23. *Let A be a PID and P a representative system of elements of A such that the ideals πA with $\pi \in P$ run through all the maximal ideals of A . The topology on A with respect to which the nonzero ideals of A form a fundamental system of neighbourhoods of 0, which is compatible with the ring structure on A , is Hausdorff and the completion of A with this topology is canonically isomorphic to the product of the completions of A with respect to the π -adic topologies, where π runs through P .*

Proof. The principal ideals πA where $\pi \in P$ are maximal and distinct and hence relatively prime, we have already seen that the π -adic topologies are Hausdorff and hence so is the topology defined in the statement of ??, which is finer than each of the π -adic topologies. \square

If ?? is applied when $A = \mathbb{Z}$, we denote by $\widehat{\mathbb{Z}}$ the completion of \mathbb{Z} with respect to the topology for which all the nonzero ideals of \mathbb{Z} form a fundamental system of neighbourhoods of 0, the ring isomorphic to the product $\prod_{p \in P} \mathbb{Z}_p$ of the rings of p -adic integers (P being the set of prime numbers), and is usually denoted by $\widehat{\mathbb{Z}}$.

Proposition 10.3.24. *Let A be a ring, $(\mathfrak{m}_i)_{1 \leq i \leq r}$ a finite family of distinct maximal ideals of A and set*

$$\mathfrak{t} = \prod_{i=1}^r \mathfrak{m}_i = \bigcap_{i=1}^n \mathfrak{m}_i, \quad S = \bigcap_{i=1}^r (A - \mathfrak{m}_i).$$

Endow A with the \mathfrak{t} -adic topology, $S^{-1}A$ the $S^{-1}\mathfrak{t}$ -adic topology and each of the local ring $A_{\mathfrak{m}_i}$ the $(\mathfrak{m}_i A_{\mathfrak{m}_i})$ -adic topology. Let $\phi : A \rightarrow S^{-1}A$ and $\psi : S^{-1}A \rightarrow \prod_{i=1}^r A_{\mathfrak{m}_i}$ be the canonical homomorphisms. Then ϕ and ψ are continuous and the corresponding homomorphisms $\hat{\phi}$ and $\hat{\psi}$ are topological ring isomorphism.

Proof. Since $\mathfrak{m}_i \cap S = \emptyset$ for each $i \in I$, the ideal $\mathfrak{m}'_i = \mathfrak{m}_i B$ of B is maximal and we have $S^{-1}\mathfrak{t} = \bigcap_{i=1}^r \mathfrak{m}'_i$. Also, $B_{\mathfrak{m}'_i} = A_{\mathfrak{m}_i}$ up to a canonical isomorphism. As $\phi^{-1}(\mathfrak{t}B) = \mathfrak{t}$ and $\psi_i^{-1}(\mathfrak{m}_i A_{\mathfrak{m}_i}) \supseteq \mathfrak{t}B$, ϕ and ψ are continuous. Then it is sufficient to prove that, if

$$\omega = \psi \circ \phi : A \rightarrow \prod_{i=1}^r A_{\mathfrak{m}_i}$$

then $\hat{\omega}$ is an isomorphism of \widehat{A} onto $\prod_{i=1}^r \widehat{A}_{\mathfrak{m}_i}$, for this result applied to B and the \mathfrak{m}'_i will show that $\hat{\psi}$ is an isomorphism and therefore also $\hat{\phi}$. Note that every product of powers of the \mathfrak{m}_i contains a power of \mathfrak{t} and hence the \mathfrak{t} -adic topology is the least upper bound of the \mathfrak{m}_i -adic topologies; moreover, if A_i denotes the ring A with the \mathfrak{m}_i -adic topology and $\gamma : A \rightarrow \prod_{i=1}^r A_i$ is the diagonal map, then $\hat{\gamma} : \widehat{A} \rightarrow \prod_{i=1}^r \widehat{A}_i$ is an isomorphism (??). Then it all amounts to proving that, if $v_i : A_i \rightarrow A_{\mathfrak{m}_i}$ is the canonical map, then $\hat{v}_i : \widehat{A}_i \rightarrow \widehat{A}_{\mathfrak{m}_i}$ is an isomorphism. Now, for all n , the map

$$v_{i,n} : A / \mathfrak{m}_i^n \rightarrow A_{\mathfrak{m}_i} / \mathfrak{m}_i^n A_{\mathfrak{m}_i}$$

derived from v_i by taking quotients is an isomorphism; our assertion follows from the fact that \widehat{A}_i (resp. $\widehat{A}_{\mathfrak{m}_i}$) is the inverse limit of the discrete rings A / \mathfrak{m}_i^n (resp. $A_{\mathfrak{m}_i} / \mathfrak{m}_i^n A_{\mathfrak{m}_i}$). \square

As a particular example, we want to consider the \mathfrak{m} -adic completion of a maximal ideal \mathfrak{m} of A . For this, we need the following useful lemma.

Lemma 10.3.25. *Let A be a complete Hausdorff topological ring, in which there exists a fundamental system of neighbourhoods of 0 consisting of additive subgroups of A .*

- (a) *For all $x \in A$ such that $\lim_n x^n = 0$, the element $1 - x$ is invertible in A and its inverse is equal to $\sum_n x^n$.*
- (b) *Let \mathfrak{a} be an ideal of A such that $\lim_n x^n = 0$ for any $x \in \mathfrak{a}$. For an element y of A to be invertible, it is necessary and sufficient that its class mod \mathfrak{a} be invertible in A/\mathfrak{a} . In particular, \mathfrak{a} is contained in the Jacobson radical of A .*

Proof. Note that

$$(1 - x)(1 + x + x^2 + \cdots + x^n) = 1 - x^{n+1}$$

so for (a) it all amounts to proving that the series with general term x^n is convergent in A when $\lim_n x^n = 0$; now, by hypothesis, for every symmetric neighbourhood V of 0 in A , there exists an integer $p > 0$ such that $x^n \in V$ for all $n \geq p$. We conclude that

$$x^p + \cdots + x^q \in V$$

for all $q \geq p$ and our assertion then follows from Cauchy's criterion, since A is complete.

Suppose that there exists $y \in A$ such that $yz \equiv 1 \pmod{\mathfrak{a}}$. The hypothesis on \mathfrak{a} implies, by (a), that yz is invertible in A and hence y is invertible in A . In particular, every $x \in \mathfrak{a}$ is such that $1 - x$ is invertible in A and, as \mathfrak{a} is an ideal of A , it is contained in the Jacobson radical of A . \square

Proposition 10.3.26. *Let A be a ring and \mathfrak{m} a maximal ideal of A . The Hausdorff completion \widehat{A} of A with respect to the \mathfrak{m} -adic topology is a local ring whose maximal ideal is $\widehat{\mathfrak{m}}$.*

Proof. If $\mathfrak{n} = \bigcap_{n \geq 1} \mathfrak{m}^n$, then \widehat{A} is the completion of the Hausdorff ring A/\mathfrak{n} associated with A and, as $\mathfrak{m}/\mathfrak{n}$ is maximal in A/\mathfrak{n} , we may assume that A is Hausdorff with respect to the \mathfrak{m} -adic topology. As A/\mathfrak{m} and $\widehat{A}/\widehat{\mathfrak{m}}$ are isomorphic rings, $\widehat{\mathfrak{m}}$ is maximal in \widehat{A} . As the topology on A is defined by the filtration (\mathfrak{m}^n) , the proposition will be a consequence of ??: apply it to the topological ring \widehat{A} and the ideal $\widehat{\mathfrak{m}}$, as, for all $x \in \widehat{\mathfrak{m}}$, $x^n \in (\widehat{\mathfrak{m}})^n \subseteq \widehat{\mathfrak{m}^n}$ and the sequence (x^n) therefore tends to 0. \square

Example 10.3.27. If we take $A = \mathbb{Z}$, every maximal ideal of \mathbb{Z} is of the form $p\mathbb{Z}$ where p is prime. The ring of p -adic numbers \mathbb{Z}_p is then a local ring of which $\mathfrak{p}\mathbb{Z}_p$ is the maximal ideal and whose residue field is isomorphic to $\mathbb{Z}/p\mathbb{Z} = \mathbb{F}_p$, and $\mathbb{Z}_{(p)}$ with the $p\mathbb{Z}_{(p)}$ -adic topology is identified with a topological subring of \mathbb{Z}_p containing \mathbb{Z} .

Corollary 10.3.28. *Let A be a semi-local ring, $\mathfrak{m}_1, \dots, \mathfrak{m}_r$ its distinct maximal ideals and*

$$\mathfrak{t} = \bigcap_{i=1}^r \mathfrak{m}_i = \prod_{i=1}^r \mathfrak{m}_i$$

its Jacobson radical. Then the Hausdorff completion \widehat{A} of A with respect to the \mathfrak{t} -adic topology is a semi-local ring, canonically isomorphic to the product $\prod_{i=1}^r \widehat{A}_{\mathfrak{m}_i}$, where $\widehat{A}_{\mathfrak{m}_i}$ is the Hausdorff completion ring of the local ring $A_{\mathfrak{m}_i}$ with respect to the $(\mathfrak{m}_i A_{\mathfrak{m}_i})$ -adic topology.

10.3.3 Lifting properties of the associated graduation

Theorem 10.3.29. Let X, Y be two filtered groups with filtrations $(X_n), (Y_n)$; let $\phi : X \rightarrow Y$ be a homomorphism compatible with the filtrations.

- (a) Suppose that the filtration (X_n) is exhaustive. For $\text{gr}(\phi)$ to be injective, it is necessary and sufficient that $\phi^{-1}(Y_n) = X_n$ for all $n \in \mathbb{Z}$.
- (b) Suppose that one of the following hypotheses holds:
 - (α) X is complete and Y Hausdorff.
 - (β) Y is discrete.

Then, for $\text{gr}(\phi)$ to be surjective, it is necessary and sufficient that $Y_n = \phi(X_n)$ for all $n \in \mathbb{Z}$.

Proof. To say that the map $\text{gr}_n(\phi)$ is injective means that

$$X_n \cap \phi^{-1}(Y_{n+1}) \subseteq X_{n+1}.$$

This is obviously the case if $\phi^{-1}(Y_{n+1}) = X_{n+1}$. Conversely, if this holds for all n , we deduce by induction on k that

$$X_m \cap \phi^{-1}(Y_{n+1}) \subseteq X_{n+1} \quad \text{for all } m, n \in \mathbb{Z} \text{ with } m < n + 1.$$

As the filtration (X_n) is exhaustive, we see $\bigcup_{m < n} X_m = X$, hence $\phi^{-1}(Y_{n+1}) \subseteq X_{n+1}$ for all $n \in \mathbb{Z}$ and therefore $\phi^{-1}(Y_{n+1}) = X_{n+1}$, which completes the proof of (a).

To say that the map $\text{gr}_n(\phi)$ is surjective means that

$$Y_n = \phi(X_n) + Y_{n+1}.$$

This is obviously the case if $Y_n = \phi(X_n)$. Conversely, suppose that $Y_n = \phi(X_n) + Y_{n+1}$ for all $n \in \mathbb{Z}$. Let n be an integer and y an element of Y_n ; we now define a sequence (x_k) of elements of X_n such that for all $k \geq 0$,

$$x_{k+1} \equiv x_k \pmod{X_{n+k}} \quad \text{and} \quad \phi(x_k) \equiv y \pmod{Y_{n+k}}.$$

First we take x_0 equal to the identity element of X , which certainly gives $\phi(x_0) \equiv y \pmod{Y_n}$. Suppose that $x_k \in X_n$ has been constructed such that $\phi(x_k) \equiv y \pmod{Y_{n+k}}$; then $y - \phi(x_k) \in Y_{n+k} = \phi(X_{n+k}) + Y_{n+k+1}$ so there exists $t \in X_{n+k}$ such that $y - \phi(x_k) \equiv \phi(t) \pmod{Y_{n+k+1}}$ and hence $\phi(x_k + t) \equiv y \pmod{Y_{n+k+1}}$; it then suffices to take $x_{k+1} = x_k + t$ to carry out the induction.

This being done, if Y is discrete, there exists $k \geq 0$ such that $Y_{n+k} = \{0\}$, whence $\phi(x_k) = y$ and hence in this case it has been proved that $\phi(X_n) = Y_n$ for all n . Suppose now that X is complete and Y Hausdorff. As $x_k - x_j \in X_{n+k}$ for $j \geq k$, (x_k) is a Cauchy sequence in X ; as X_n is closed in X and hence complete, this sequence has at least one limit x in X_n . By virtue of the continuity of ϕ , $\phi(x)$ is the unique limit of the sequence $(\phi(x_k))$ in Y , Y being Hausdorff. But the relations $\phi(x_k) \equiv y \pmod{Y_{n+k+1}}$ show that y is also a limit of this sequence, whence $\phi(x) = y$ and it has also been proved that $\phi(X_n) = Y_n$. \square

Corollary 10.3.30. Suppose that X is Hausdorff and its filtration exhaustive. Then, if $\text{gr}(\phi)$ is injective, ϕ is injective.

Proof. Since $\phi^{-1}(Y_n) = X_n$, we have $\ker \phi \subseteq \bigcap_n \phi^{-1}(Y_n) = \bigcap_n X_n$, whence the corollary. \square

Corollary 10.3.31. Suppose that one of the following hypotheses holds:

- (α) X is complete, Y is Hausdorff and its filtration is exhaustive;
- (β) Y is discrete and its filtration is exhaustive.

Then, if $\text{gr}(\phi)$ is surjective, ϕ is surjective.

Proof. In this case $Y = \bigcup_n Y_n = \bigcup_n \phi(X_n) \subseteq \phi(X)$ by ??.

\square

Corollary 10.3.32. Suppose that X and Y are Hausdorff, the filtrations of X and Y exhaustive and X complete. Then, if $\text{gr}(\phi)$ is bijective, ϕ is bijective.

Proposition 10.3.33. Let A be a filtered ring, M a filtered A -module and (A_n) and (M_n) the respective filtrations on A and M . Suppose that A is complete and the filtration (M_n) is exhaustive and separated. Let $(x_i)_{i \in I}$ be a finite family of elements of M and for $i \in I$ let n_i be an integer such that $x_i \in M_{n_i}$. Finally let ξ_i be the class of x_i in $\text{gr}_{n_i}(M)$. Then, if (ξ_i) is a system of generators of the $\text{gr}(A)$ -module $\text{gr}(M)$, then (x_i) is a system of generators of the A -module M .

Proof. In the A -module $L = A^I$ let L_n denote the set (a_i) such that $a_i \in A_{n-n_i}$ for all $i \in I$; if p and q are the least and greatest of the n_i , then $A_{n-p}^I \subseteq L_n \subseteq A_{n-q}^I$, so the topology defined on L by the definition (L_n) is the same as the product topology; hence L is a complete filtered A -module. As L is free, there exists an A -linear map $\phi : L \rightarrow M$ such that $\phi((a_i)) = \sum_i a_i x_i$ and it is obviously compatible with the filtrations; we must prove that ϕ is surjective and for this it is sufficient, by virtue of ??, to show that $\text{gr}(\phi)$ is surjective or also that, for all $x \in M_n$, there exist a family (a_i) such that $a_i \in A_{n-n_i}$ for all $i \in I$ and $x \equiv \sum_i a_i x_i \pmod{M_{n+1}}$. Let ξ be the class of x in $\text{gr}_n(M)$; since the ξ_i generate the $\text{gr}(A)$ -module $\text{gr}(M)$, there exist $\alpha_i \in \text{gr}(A)$ such that $\xi = \sum_i a_i \xi_i$, and we may assume that $\alpha_i \in \text{gr}_{n-n_i}(A)$ by replacing if need be α_i by its homogeneous component of degree $n - n_i$. Then α_i the image of an element $a_i \in A_{n-n_i}$ and the family (a_i) has the required property. \square

Corollary 10.3.34. Let A be a complete filtered ring and M a filtered A -module whose filtration is exhaustive and separated. If $\text{gr}(M)$ is a finitely generated (resp. Noetherian) $\text{gr}(A)$ -module, then M is a finitely generated (resp. Noetherian) A -module.

Proof. If $\text{gr}(M)$ is finitely generated, there is a finite system of homogeneous generators and ?? shows that M is finitely generated. Suppose now that $\text{gr}(M)$ is Noetherian and let N be a submodule of M ; the filtration induced on N by that on M is exhaustive and separated and $\text{gr}(N)$ is identified with a sub- $\text{gr}(A)$ -module of $\text{gr}(M)$ by ?? and hence is finitely generated by hypothesis; we conclude that N is a finitely generated A -module and hence M is Noetherian. \square

Corollary 10.3.35. Let A be a complete Hausdorff filtered ring whose filtration is exhaustive. If $\text{gr}(A)$ is Noetherian, so is A .

Corollary 10.3.36. *Let A be a complete filtered ring, (A_n) its filtration, M a Hausdorff filtered A -module, (E_n) its filtration and N a finitely generated submodule of M ; suppose that $A_0 = A$ and $M_0 = M$.*

- (a) *If for all $n \in \mathbb{Z}$ we have $M_n = M_{n+1} + A_n N$, then $M = N$.*
- (b) *If the filtration on M is derived from that on A , the relation $M = M_1 + N$ implies $M = N$.*

Proof. Let ξ_1, \dots, ξ_s be the classes mod M_1 of a finite system of generators of N . It follows from the given hypothesis that for all $n \geq 0$ every element of $\text{gr}_n(M)$ can be expressed in the form $\sum_{i=1}^s \alpha_i \xi_i$, where $\alpha_i \in \text{gr}(A)$; the ξ_i therefore generate the $\text{gr}(A)$ -module $\text{gr}(M)$, which proves (a) by virtue of ?? If the filtration on M is derived from that on A , the relation $M = M_1 + N$ implies

$$M_n = A_n M = A_n M_1 + A_n N = A_n A_1 M + A_n N \subseteq A_{n+1} M + A_n N = M_{n+1} + A_n N \subseteq M_n$$

and assertion (b) then follows from this. \square

Proposition 10.3.37. *Let A be a filtered ring and (A_n) its filtration. Suppose that there exist a subring C of A_0 such that $C \cap A_1 = \{0\}$ and C is identified with a subring of $\text{gr}_0(A)$. Let (x_1, \dots, x_r) be a family of elements of A with $x_i \in A_{n_i}$ and let ξ_i be the class of x_i in $\text{gr}_{n_i}(A)$.*

- (a) *If the family (ξ_i) of elements of $\text{gr}(A)$ is algebraically independent over C , then the family (x_i) is algebraically independent over C .*
- (b) *If the filtration on A is exhaustive and discrete and (ξ_i) is a system of generators of the C -algebra $\text{gr}(A)$, then (x_i) is a system of generators of the C -algebra A .*

Proof. Let B be the polynomial algebra $C[X_1, \dots, X_r]$ over C and give B the graduation (B_n) of type \mathbb{Z} where B_n is the set of C -linear combinations of the monomials $X_1^{s_1}, \dots, X_r^{s_r}$ such that $\sum_i n_i s_i = n$. Let ϕ be the homomorphism $f \mapsto f(x_1, \dots, x_r)$ from the C -algebra B to the C -algebra A . By definition, $\phi(B_n) \subseteq A_n$ for all $n \in \mathbb{Z}$ and hence ϕ is compatible with the filtrations. The hypothesis of (a) means that $\text{gr}(\phi)$ is injective. As the filtration on B is exhaustive and separated, ?? may be applied and ϕ is injective, which proves the conclusion of (a). Similarly, the hypothesis (b) on the (ξ_i) means that $\text{gr}(\phi)$ is surjective. As A is discrete and its filtration is exhaustive, ?? may be applied and ϕ is surjective, which proves the conclusion of (b). \square

Proposition 10.3.38. *Let A be a complete Hausdorff filtered ring and (A_n) its filtration. Suppose that there exist a subring C of A_0 such that $C \cap A_1 = \{0\}$ and C is identified with a subring of $\text{gr}_0(A)$. Let (x_1, \dots, x_r) be a family of elements of A with $x_i \in A_{n_i}$ and let ξ_i be the class of x_i in $\text{gr}_{n_i}(A)$.*

- (a) *There exists a unique C -homomorphism ρ from $B = C[\![X_1, \dots, X_r]\!]$ to A such that $\rho(X_i) = x_i$.*
- (b) *If the family (x_i) is algebraically independent over C , the homomorphism ρ is injective.*
- (c) *If the filtration on A is exhaustive and (ξ_i) is a system of generators of the C -algebra $\text{gr}(A)$, then ρ is surjective.*

Proof. As $n_i \geq 1$ for all i , we have $\sum_i n_i s_i \geq \sum_i s_i$ for every monomial $X_1^{s_1}, \dots, X_r^{s_r}$ and on the other hand $\sum_i n_i s_i \leq N$ if N is the greatest of the n_i . If B_n denotes the set of formal power series whose non-zero terms $a_s X^s$ satisfy $\sum_i n_i s_i \leq n$, it follows from ?? that B is Hausdorff and complete with the exhaustive filtration (B_n) and that $B' = C[X_1, \dots, X_r]$ is dense in B . Moreover the homomorphism ϕ defined in the proof of ?? is continuum on B' and can be extended uniquely to a continuous homomorphism $\rho : B \rightarrow A$, since A is Hausdorff and complete, which proves (a). Also, $\text{gr}(B) = \text{gr}(B')$ and $\text{gr}(\phi) = \text{gr}(\rho)$, so (b) and (c) follow respectively from ?? and ?? in view Of the hypotheses on A . \square

Let A be a local ring, \mathfrak{m} its maximal ideal and M an A -module; let A and M be given the \mathfrak{m} -adic filtrations and let $\text{gr}(A)$ and $\text{gr}(M)$ be the graded ring and the graded $\text{gr}(A)$ -module associated with A and M . We have seen (??) that the canonical map (??) is always surjective; we are going to consider the following property of M :

(GR) The canonical map $\gamma_M : \text{gr}(A) \otimes_{\text{gr}_0(A)} \text{gr}_0(M) \rightarrow \text{gr}(M)$ is bijective.

Proposition 10.3.39. *Let A be a local ring, \mathfrak{m} its maximal ideal, M, N two A -modules and $\phi : N \rightarrow M$ an A -homomorphism. Suppose that M and N are given the \mathfrak{m} -adic filtrations and*

- (a) M satisfies property (GR);
- (b) $\text{gr}_0(\phi) : \text{gr}_0(N) \rightarrow \text{gr}_0(M)$ is injective.

Then $\text{gr}(\phi) : \text{gr}(N) \rightarrow \text{gr}(M)$ is injective and N and $P = \text{coker } \phi$ satisfy property (GR).

Proof. It is immediately verified that the diagram

$$\begin{array}{ccc} \text{gr}(A) \otimes_{\text{gr}_0(A)} \text{gr}_0(N) & \xrightarrow{1 \otimes \text{gr}_0(\phi)} & \text{gr}(A) \otimes_{\text{gr}_0(A)} \text{gr}_0(M) \\ \downarrow \gamma_N & & \downarrow \gamma_M \\ \text{gr}(N) & \xrightarrow{\text{gr}(\phi)} & \text{gr}(M) \end{array}$$

is commutative. As $\text{gr}_0(A) = A/\mathfrak{m}$ is a field, the hypothesis implies that $1 \otimes \text{gr}_0(\phi)$ is injective; as by hypothesis γ_M is injective, so is $\gamma_M \circ (1 \otimes \text{gr}_0(\phi))$. This implies first that γ_N is injective and hence bijective and therefore that $\text{gr}(\phi)$ is injective. The formula $\phi^{-1}(\mathfrak{m}^n M) = \mathfrak{m}^n N$ is then a consequence of ??(a).

Also, let us write $N' = \phi(N)$ and let $\iota : N' \rightarrow M$ be the canonical injection. If $\pi : M \rightarrow P = M/N'$ is the canonical homomorphism, then in the commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{gr}(A) \otimes_{\text{gr}_0(A)} \text{gr}_0(N') & \longrightarrow & \text{gr}(A) \otimes_{\text{gr}_0(A)} \text{gr}_0(M) & \longrightarrow & \text{gr}(A) \otimes_{\text{gr}_0(A)} \text{gr}(P) \longrightarrow 0 \\ & & \downarrow \gamma_{N'} & & \downarrow \gamma_M & & \downarrow \gamma_P \\ 0 & \longrightarrow & \text{gr}(N) & \xrightarrow{\text{gr}(\iota)} & \text{gr}(M) & \xrightarrow{\text{gr}(\pi)} & \text{gr}(P) \longrightarrow 0 \end{array}$$

the lower row is exact by ?? and so is the upper row by virtue of ?? and the fact that $\text{gr}_0(A)$ is a field. The first part of the argument applied to ι shows that $\gamma_{N'}$ is bijective; as γ_M is also bijective by hypothesis, we conclude that γ_P is bijective by the snake lemma. \square

Corollary 10.3.40. *Under the hypotheses of ??, if we assume also that N is Hausdorff with the \mathfrak{m} -adic filtration, then ϕ is injective.*

Proposition 10.3.41. *Let A be a ring, \mathfrak{m} an ideal of A contained in the Jacobson radical of A and M an A -module. Let A and M be given the \mathfrak{m} -adic filtrations. Suppose that one of the following conditions holds:*

- (a) *M is a finitely generated A -module and A is Hausdorff.*
- (b) *\mathfrak{m} is nilpotent.*

Then for M to be a free A -module, it is necessary and sufficient that $M/\mathfrak{m}M$ be a free (A/\mathfrak{m}) -module and that M satisfies property (GR).

Proof. If M is a free A -module and (x_λ) a basis of M , $\mathfrak{m}^n M$ is the direct sum of the submodules $\mathfrak{m}^n x_\lambda$ of M for all $n \geq 0$; then $\mathfrak{m}^n M/\mathfrak{m}^{n+1} M$ is identified with the direct sum of the $\mathfrak{m}^n x_\lambda/\mathfrak{m}^{n+1} x_\lambda$. We deduce first (for $n = 0$) that the classes $1 \otimes x_\lambda$ of the x_λ in $M/\mathfrak{m}M = (A/\mathfrak{m}) \otimes_A M$ form a basis of the (A/\mathfrak{m}) -module $M/\mathfrak{m}M$, since the canonical map

$$(\mathfrak{m}^n/\mathfrak{m}^{n+1}) \otimes_A (M/\mathfrak{m}M) \rightarrow \mathfrak{m}^n M/\mathfrak{m}^{n+1} M$$

is bijective for all $n \geq 0$; hence γ_M is bijective. Note that this part of the proof uses neither condition (a) nor condition (b).

Suppose conversely that the conditions of the statement hold and let $(x_\lambda)_{\lambda \in I}$ be a family of elements of M whose classes mod $\mathfrak{m}M$ form a basis of the (A/\mathfrak{m}) -module $M/\mathfrak{m}M$; let L be the free A -module $A^{\oplus I}$ and (e_λ) a basis of L ; let $\phi : L \rightarrow M$ the A -linear map such that $\phi(e_\lambda) = x_\lambda$ for all $\lambda \in I$. Now, each of the hypotheses (a) and (b) implies that A is Hausdorff and hence so is L with the \mathfrak{m} -adic filtration, since $\mathfrak{m}^n L = (\mathfrak{m}^n)^{\oplus I}$. Also, since $\text{gr}(L)$ is identified with $\text{gr}(A) \otimes_{A/\mathfrak{m}} (L/\mathfrak{m}L)$ from the first part of the proof, it is possible to write $\text{gr}(\phi)$ as

$$\text{gr}(L) \xrightarrow{\cong} \text{gr}(A) \otimes_{A/\mathfrak{m}} (L/\mathfrak{m}L) \xrightarrow{\eta} \text{gr}(A) \otimes_{A/\mathfrak{m}} (M/\mathfrak{m}M) \xrightarrow{\gamma_M} \text{gr}(M)$$

where η is the bijection map the class $1 \otimes \bar{e}_\lambda$ onto $1 \otimes \bar{x}_\lambda$. The hypothesis then implies that $\text{gr}(\phi)$ is bijective and the conclusion follows with the aid of ??, ?? and ?? \square

Recall that we have already seen that if A is a ring and \mathfrak{a} is an ideal of A such that A is Hausdorff and complete with the \mathfrak{a} -adic topology, then the topological ring A is canonically identified with the inverse limit of the discrete rings $A_i = A/\mathfrak{a}^{i+1}$ ($i \in \mathbb{N}$) with respect to the canonical homomorphisms

$$\pi_{ij} : A/\mathfrak{a}^{j+1} \rightarrow A/\mathfrak{a}^{i+1}$$

for $i \leq j$; note that π_{ij} is surjective and if \mathfrak{n}_{ij} is its kernel, then

$$\mathfrak{n}_{ij} = \mathfrak{a}^{i+1}/\mathfrak{a}^{j+1} = (\mathfrak{a}/\mathfrak{a}^{j+1})^{i+1} = (\mathfrak{n}_{0,j})^{i+1};$$

in particular $(\mathfrak{n}_{0,j})^{j+1} = 0$. Conversely, we have the following result:

Proposition 10.3.42. *Let (A_i, π_{ij}) be an inverse system of discrete rings indexed by \mathbb{N} and let (M_i, u_{ij}) be an inverse system of modules over the inverse system of rings (A_i, π_{ij}) . Let \mathfrak{n}_j be the kernel of $\pi_{0,j} : A_j \rightarrow A_0$ and set $A = \varprojlim A_i$, $M = \varprojlim M_i$. Suppose that*

- (a) for $i \leq j$, π_{ij} and u_{ij} are surjective;
- (b) for $i \leq j$, the kernel of π_{ij} and u_{ij} are \mathfrak{n}_j^{i+1} and $\mathfrak{n}_j^{i+1}M_j$, respectively.

Then the following assertions hold:

- (i) A is a complete Hausdorff topological ring, M is a complete topological A -module and the canonical homomorphisms $\pi_i : A \rightarrow A_i$, $u_i : M \rightarrow M_i$ are surjective.
- (ii) If M_0 is a finitely generated A_0 -module, then M is a finitely generated A -module. In fact, every finite subset E of M such that $u_0(E)$ generates M_0 is a system of generators of M .

Proof. The first assertion (i) follows from ?? and ?. For each $i \in \mathbb{N}$, let $\mathfrak{m}_{i+1} = \ker \pi_i$, $N_{i+1} = \ker u_i$; then by hypothesis (b),

$$\mathfrak{m}_{i+1} = \varprojlim_k \pi_{i,i+k}^{-1}(0) = \varprojlim_k \mathfrak{n}_{i+1}^{i+k}, \quad N_{i+1} = \varprojlim_k u_{i,i+k}^{-1}(0) = \varprojlim_k \mathfrak{n}_{i+k}^{i+1}M_{i+k};$$

and as π_{i+k} and u_{i+k} are surjective, we also have

$$\pi_{i+k}(\mathfrak{m}_{i+1}) = \mathfrak{n}_{i+k}^{i+1}, \quad u_{i+k}(N_{i+1}) = \mathfrak{n}_{i+k}^{i+1}M_{i+k}. \quad (10.3.1)$$

We prove that $\mathfrak{m}_i N_j \subseteq N_{i+j}$ for $i, j \geq 1$, which amounts to showing that $u_{i+j-1}(\mathfrak{m}_i N_j) = 0$. For this, note that $\mathfrak{n}_k^{k+1} = 0$ for $k \geq 0$ as it is the kernel of $\pi_{kk} = \text{id}$, so

$$u_{i+j-1}(\mathfrak{m}_i N_j) = \pi_{i+j-1}(\mathfrak{m}_i) u_{i+j-1}(N_j) = \mathfrak{n}_{i+j-1}^i (\mathfrak{n}_{i+j-1}^j M_{i+j-1}) = 0;$$

we see similarly that $\mathfrak{m}_i \mathfrak{m}_j \subseteq \mathfrak{m}_{i+j}$. If for $i \leq 0$ we set $\mathfrak{m}_i = A$ and $N_i = M$, then $(\mathfrak{m}_i)_{i \in \mathbb{Z}}$ is a filtration of A and $(N_i)_{i \in \mathbb{Z}}$ is a filtration of M compatible with that of A . The topology on A and M are obviously those defined by these filtrations. Now let \mathfrak{a} be an ideal of A such that $\pi_1(\mathfrak{a}) = \mathfrak{n}_1$ and M' be the submodule of M generated by a subset E of M such that $u_0(E)$ generates M_0 ; we are going to show that

$$N_i = \mathfrak{a}^i M' + N_{i+1} \quad \text{for } i \geq 0. \quad (10.3.2)$$

Let us write $\mathfrak{a}_i = \pi_i(\mathfrak{a})$ and $M'_i = u_i(M')$; from the definition of N_{i+1} , it then suffices to show that

$$u_i(N_i) = \mathfrak{a}_i^i M'_i.$$

This is true if $i = 0$, since $N_0 = M$ and $M'_0 = M_0$ by hypothesis. If $i \geq 1$, then $u_i(N_i) = \mathfrak{n}_i^i M_i$ by ??). As $\pi_{1,i}$ is surjective and $\pi_{0,i} = \pi_{0,1} \circ \pi_{1,i}$, $\pi_{1,i}$ maps the kernel \mathfrak{n}_i of $h_{0,i}$ onto the kernel of $h_{0,1}$ and $\mathfrak{n}_i = \pi_{1,i}^{-1}(\mathfrak{n}_1)$, so

$$\pi_{1,i}(\mathfrak{a}_i) = \pi_1(\mathfrak{a}) = \mathfrak{n}_1 = \pi_{1,i}(\mathfrak{n}_i);$$

as the kernel of $\pi_{1,i}$ is \mathfrak{n}_i^2 , we conclude that $\mathfrak{a}_i \subseteq \mathfrak{n}_i \subseteq \mathfrak{a}_i + \mathfrak{n}_i^2$, whence $\mathfrak{n}_i = \mathfrak{a}_i + \mathfrak{n}_i^2$. Moreover $u_{0,i}(M'_i) = u_0(M') = M_0 = u_{0,i}(M_i)$ and as $\ker u_{0,i} = \mathfrak{n}_i M_i$, $M_i = M'_i + \mathfrak{n}_i M_i$, so

$$\mathfrak{n}_i M_i = (\mathfrak{a}_i + \mathfrak{n}_i^2)^i (M'_i + \mathfrak{n}_i M_i).$$

Now $\mathfrak{a}_i^k \mathfrak{n}_i^{i+1-k} \subseteq \mathfrak{n}_i^{i+1} = 0$ for $0 \leq k \leq i$, so it follows that $u_i(N_i) = \mathfrak{n}_i M_i = \mathfrak{a}_i^i M'_i$, which proves ?. Moreover, $\mathfrak{m}_1 = \pi_1^{-1}(\mathfrak{n}_1)$, so $\mathfrak{a} \subseteq \mathfrak{m}_1$ and therefore $\mathfrak{a}^i \subseteq \mathfrak{m}_1^i \subseteq \mathfrak{m}_i$, whence $N_i \subseteq$

$\mathfrak{m}_i M' + N_{i+1}$. On the other hand, obviously $\mathfrak{m}_i M \subseteq N_i$ and so $N_i = \mathfrak{m}_i M' + N_{i+1}$ for $i \geq 0$; it then follows from ?? that $M' = M$, which completes the proof. \square

10.4 The adic topology on Noetherian rings

Let A be a filtered ring, M a filtered A -module and (A_n) and (M_n) be the corresponding filtrations. Since we only consider the \mathfrak{a} -adic filtration, we may assume that all filtrations are exhaustive and consist of submodules. The **Rees ring** of A is then defined by

$$\text{Rees}(A) = \bigoplus_{n \in \mathbb{N}} A_n X^n \subseteq A[X]$$

and the **Rees module** of M is defined by

$$\text{Rees}(M) = \bigoplus_{n \in \mathbb{N}} M_n \otimes_A AX^n \subseteq M \otimes_A A[X].$$

Note that $\text{Rees}(M)$ is a graded $\text{Rees}(A)$ -module of type \mathbb{N} , since we have

$$A_m X^m (M_n \otimes_A AX^n) \subseteq (A_m M_n \otimes_A AX^{m+n}) \subseteq M_{m+n} \otimes_A AX^{m+n}.$$

The most important case is that \mathfrak{a} is an ideal of A and A, M are given the \mathfrak{a} -adic filtration. In this case we write $\text{Rees}_{\mathfrak{a}}(A)$ for the Rees ring $\bigoplus_{n \in \mathbb{N}} \mathfrak{a}^n X^n$ and $\text{Rees}_{\mathfrak{a}}(M)$ for $\bigoplus_{n \in \mathbb{N}} \mathfrak{a}^n M \otimes_A AX^n$.

10.4.1 Good filtrations

Let A be a ring, \mathfrak{a} an ideal of A , M an A -module and (M_n) a filtration on the additive group M consisting of submodules of M . The filtrations (M_n) is called **\mathfrak{a} -good** if:

- (a) $\mathfrak{a}M_n \subseteq M_{n+1}$ for all $n \in \mathbb{Z}$.
- (b) There exists an integer n_0 such that $\mathfrak{a}M_n = M_{n+1}$ for all $n \geq n_0$.

If (M_n) is \mathfrak{a} -good, then by induction on we see $\mathfrak{a}^p M_n = M_{n+p}$ for all $n \geq n_0$ and $p > 0$. Note that condition (a) means that the filtration (M_n) is compatible with the A -module structure on M if A is given the \mathfrak{a} -adic filtration. Clearly, on every A -module M , the \mathfrak{a} -adic filtration is \mathfrak{a} -good.

Theorem 10.4.1. *Let A be a ring, \mathfrak{a} an ideal of A , and give A the \mathfrak{a} -adic filtration. Let M be an A -module and (M_n) a filtration of M consisting of finitely generated submodules. Suppose that $\mathfrak{a}M_n \subseteq M_{n+1}$ for all n . Then the two following conditions are equivalent:*

- (a) *The filtration (M_n) is \mathfrak{a} -good.*
- (b) *$\text{Rees}_{\mathfrak{a}}(M)$ is a finitely generated $\text{Rees}_{\mathfrak{a}}(A)$ -module.*

Proof. Suppose that $\mathfrak{a}M_n = M_{n+1}$ for $n \geq n_0 \geq 0$. For $i \leq n_0$, let $(e_{ij})_{1 \leq j \leq r_i}$ be a finite system of generators of the A -module M_i . As the A -module $M_n \otimes_A AX^n$ is generated by the elements $e_{nj} \otimes X^n$ for $0 \leq n \leq n_0$ and is equal to

$$\mathfrak{a}^{n-n_0} M_{n_0} \otimes_A AX^n$$

for $n > n_0$, the $\text{Rees}(A)$ -module $\text{Rees}(M)$ is generated by the elements $e_{nj} \otimes X^n$ for $0 \leq n \leq n_0$ and $1 \leq j \leq r_n$; then it is certainly finitely generated.

Conversely, if $\text{Rees}(M)$ is a finitely generated $\text{Rees}(A)$ -module, it is generated by a finite family of elements of the form $e_k \otimes X^{n_k}$, where $e_k \in M_{n_k}$. Let n_0 be the greatest of the integers n_k . Then for $n \geq n_0$ and $x \in M_n$,

$$x \otimes X^n = \sum_k t_k(e_k \otimes X^{n_k})$$

where $t_k \in \text{Rees}(A)$; replacing if need be to by its homogeneous component of degree $n - n_k$, we may assume that $t_k = a_k X^{n-n_k}$ where $a_k \in \mathfrak{a}^{n-n_k}$. As the unique element X^n forms a basis of the A -module AX^n , the equation $x \otimes X^n = (\sum_k a_k e_k) \otimes X^n$ implies $x = \sum_k a_k e_k$. Then $M_n \subseteq \mathfrak{a}^{n-n_0} M_{n_0}$; since the opposite inclusion is obvious by hypothesis, we see $M_n = \mathfrak{a}^{n-n_0} M_{n_0}$, whence $\mathfrak{a}M_n = M_{n+1}$ for $n \geq n_0$. \square

Lemma 10.4.2. *Let A be a Noetherian ring and \mathfrak{a} an ideal of A . Then the subring $\text{Rees}_{\mathfrak{a}}(A)$ of $A[X]$ is Noetherian.*

Proof. Note that $\text{Rees}_{\mathfrak{a}}(A)$ is an A -algebra generated by $\mathfrak{a}X$; as A is Noetherian, $\mathfrak{a}X$ is a finitely generated A -module, so the claim follows from Hilbert basis theorem. \square

Proposition 10.4.3. *Let A be a filtered ring, \mathfrak{a} an ideal of A , and M is an A -module with a \mathfrak{a} -good filtration. If N is a submodule of M then the quotient filtration on M/N is \mathfrak{a} -good. If further A is Noetherian, then the submodule filtration on N is \mathfrak{a} -good.*

Proof. If N and M/N are endowed with the submodule (resp. quotient) filtrations respectively, then $\text{Rees}_{\mathfrak{a}}(N)$ is a $\text{Rees}_{\mathfrak{a}}(A)$ -submodule of $\text{Rees}_{\mathfrak{a}}(M)$ and we have $\text{Rees}_{\mathfrak{a}}(M/N) \cong \text{Rees}_{\mathfrak{a}}(M)/\text{Rees}_{\mathfrak{a}}(N)$ as $\text{Rees}_{\mathfrak{a}}(A)$ -modules. Since the filtration on M is \mathfrak{a} -good, $\text{Rees}_{\mathfrak{a}}(M)$ is a finitely generated $\text{Rees}_{\mathfrak{a}}(A)$ -module and so $\text{Rees}_{\mathfrak{a}}(M/N)$ is finitely generated, which implies the filtration on M/N is \mathfrak{a} -good. If further A is Noetherian then $\text{Rees}_{\mathfrak{a}}(A)$ is Noetherian, and so $\text{Rees}_{\mathfrak{a}}(N)$ is finitely generated. It then follows that the filtration on N is \mathfrak{a} -good. \square

Corollary 10.4.4 (Artin-Rees Lemma). *Let A be a Noetherian ring, \mathfrak{a} an ideal of A , M a finitely generated A -module and N a submodule of M . The filtration induced on N by the \mathfrak{a} -adic filtration on M is \mathfrak{a} -good. In other words, there exists an integer n_0 such that for $n \geq n_0$ we have*

$$\mathfrak{a}(\mathfrak{a}^n M \cap N) = \mathfrak{a}^{n+1} M \cap N.$$

Corollary 10.4.5. *Let A be a Noetherian ring and $\mathfrak{a}, \mathfrak{b}$ two ideals of A . Then there exists an integer $n > 0$ such that $\mathfrak{a}^n \cap \mathfrak{b} \subseteq \mathfrak{a}\mathfrak{b}$.*

Proof. By the Artin-Rees Lemma, there exist $n > 0$ such that $\mathfrak{a}^{n+1} \cap \mathfrak{b} = \mathfrak{a}(\mathfrak{a}^n \cap \mathfrak{b}) \subseteq \mathfrak{a}\mathfrak{b}$. \square

Corollary 10.4.6. *Let A be a Noetherian ring, \mathfrak{a} an ideal of A and x an element of A which is not a divisor of 0. There exists an integer $n_0 > 0$ such that, for all $n \geq n_0$, we have $(\mathfrak{a}^n : x) \subseteq \mathfrak{a}^{n-n_0}$.*

Proof. ?? applied to $M = A$, $N = Ax$ shows that there exists n_0 such that, for all $n \geq n_0$, $\mathfrak{a}^n \cap Ax = \mathfrak{a}^{n-n_0}(\mathfrak{a}^{n_0} \cap Ax)$. Then, if $xy \in \mathfrak{a}^n$,

$$xy \in \mathfrak{a}^n \cap Ax \subseteq \mathfrak{a}^{n-n_0}x$$

and, as x is not a divisor of 0, we deduce that $y \in \mathfrak{a}^{n-n_0}$. \square

Corollary 10.4.7. *Let A be a Noetherian ring, \mathfrak{a} an ideal of A , M a finitely generated A -module and (M_n) and (M'_n) two filtrations consisting of submodules of M . Suppose that the filtrations (M_n) and (M'_n) are compatible with the A -module structure on M when A is given the \mathfrak{a} -adic filtration. If the filtration (M_n) is \mathfrak{a} -good and $M'_n \subseteq M_n$ for all $n \in \mathbb{Z}$, then the filtration (M'_n) is \mathfrak{a} -good.*

Proof. This is a special case of ??.

Lemma 10.4.8. *Let A, B be two Noetherian rings, $\rho : A \rightarrow B$ a ring homomorphism, M a finitely generated A -module and N a finitely generated B -module. Then $\text{Hom}_A(M, N)$ is a finitely generated B -module.*

Proof. There exists by hypothesis a surjective A -homomorphism $\eta : A^n \rightarrow M$; the map $\phi \mapsto \phi \circ \eta$ of $\text{Hom}_A(M, N)$ to $\text{Hom}_A(A^n, N)$ is therefore injective and, as B is Noetherian, it is sufficient to prove that $\text{Hom}_A(A^n, N)$ is a finitely generated B -module; which is immediate since it is isomorphic to N^n . \square

Proposition 10.4.9. *Let A be a Noetherian ring, \mathfrak{a} an ideal of A and M, N two finitely generated A -modules. If (N_n) is an \mathfrak{a} -good filtration on N , the submodules $\text{Hom}_A(E, N_n)$ form an \mathfrak{a} -good filtration on the A -module $\text{Hom}_A(M, N)$.*

Proof. As $\mathfrak{a}^k N_n \subseteq N_{n+k}$ for $n \in \mathbb{Z}$ and $k \geq 0$, it is also true that

$$\mathfrak{a}^k \text{Hom}_A(M, N_n) \subseteq \text{Hom}_A(M, N_{n+k})$$

the family $(\text{Hom}_A(M, N_n))$ is then a filtration on $\text{Hom}_A(M, N)$ compatible with its module structure over the ring A filtered by the \mathfrak{a} -adic filtration. Since M is finitely generated, there exists an integer $r > 0$ and a surjective A -homomorphism $\eta : A^r \rightarrow M$ which defines an injective A -homomorphism

$$\eta^* : \text{Hom}_A(M, N) \rightarrow \text{Hom}_A(A^r, N)$$

clearly η^* is compatible with the filtrations $(\text{Hom}_A(M, N))$ and $(\text{Hom}_A(A^r, N))$. As the A -modules $\text{Hom}_A(M, N)$ and $\text{Hom}_A(A^r, N)$ are finitely generated ??, it is sufficient by virtue of ?? to show that the filtration $(\text{Hom}_A(A^r, N))$ is \mathfrak{a} -good; but this is immediate by virtue of the existence of the canonical isomorphism $\text{Hom}_A(A^r, N) \rightarrow N^r$, and the fact that the relation $\mathfrak{a}N = N_{n+1}$ implies $\mathfrak{a}(N_n^r) = (\mathfrak{a}N_n)^r = N_{n+1}^r$. \square

Proposition 10.4.10. *Let A be a Noetherian ring and \mathfrak{a} an ideal of A such that A is Hausdorff and complete with respect to the \mathfrak{a} -adic topology. Let M be a filtered A -module over the filtered ring A , the filtration (M_n) of A being such that $M_0 = M$ and M is Hausdorff with respect to the topology defined by (M_n) . Then the following conditions are equivalent:*

- (i) M is a finitely generated A -module and (M_n) is an \mathfrak{a} -good filtration.
- (ii) $\text{gr}(M)$ is a finitely generated $\text{gr}(A)$ -module.
- (iii) $\text{gr}_n(M)$ is a finitely generated A -module for all n and there exists n_0 such that for $n \geq n_0$ the canonical homomorphism $\text{gr}_1(A) \otimes_A \text{gr}_n(M) \rightarrow \text{gr}_{n+1}(M)$ is surjective.

Proof. It follows immediately from the definitions that (i) implies (iii). Also, (ii) is equivalent to (iii): the fact that (ii) implies (iii) is a consequence of ??; conversely, if (iii) holds, clearly $\text{gr}(M)$ is generated as a $\text{gr}(A)$ -module by the sum of the $\text{gr}_p(M)$ for $p \leq n_0$ and hence by hypothesis admits a finite system of generators. It remains to prove that (iii) implies (i).

Assume (iii), as the $\text{gr}_n(M)$ are finitely generated and $M_0 = M$, clearly first, by induction on n , M/M_n is a finitely generated A -module for all n ; it will therefore be sufficient to prove that, for $n > n_0$, M_n is a finitely generated A -module and that $\mathfrak{a}M_n = M_{n+1}$. Now, consider the A -module M_{n+1} with the exhaustive and separated filtration $(M_{n+k})_{k \geq 1}$. Now since $\text{gr}_1(A) = \mathfrak{a}/\mathfrak{a}^2$, hypothesis (iii) implies that the image of $\mathfrak{a}M_n$ in $\text{gr}_{n+1}(M)$ is equal to $\text{gr}_{n+1}(M)$ and generates the graded $\text{gr}(A)$ -module $\text{gr}(M_{n+1})$. As $\text{gr}_{n+1}(M)$ is by hypothesis a finitely generated A -module, it follows from ?? that $\mathfrak{a}M_n = M_{n+1}$ and that M_{n+1} is a finitely generated A -module. \square

Remark 10.4.1. Note that in the proof of ??, only the implication (iii) \Rightarrow (i) uses the hypothesis A is \mathfrak{a} -adic complete and Hausdorff. In particular, we see if A is Noetherian and the filtration (M_n) is \mathfrak{a} -good then $\text{gr}(M)$ is a finitely generated $\text{gr}(A)$ -module.

10.4.2 The adic topology on Noetherian rings

Proposition 10.4.11. *Let A be a Noetherian ring, \mathfrak{a} an ideal of A and M a finitely generated A -module. All the \mathfrak{a} -good filtrations on M define the same topology on M , namely the \mathfrak{a} -adic topology.*

Proof. Let (M_n) be an \mathfrak{a} -good filtration on M . As this filtration is exhaustive, every element of M belongs to one of the M_n and, as M is finitely generated (hence Noetherian) and the M_n are A -modules, there exists an integer n_1 such that $M_{n_1} = M$. On the other hand let n_0 be such that $\mathfrak{a}M_n = M_{n+1}$ for $n \geq n_0$; then, for $n > n_0 - n_1$,

$$\mathfrak{a}^n M \subseteq M_{n+n_1} = \mathfrak{a}^{n+n_1-n_0} M_{n_0} \subseteq \mathfrak{a}^{n+n_1-n_0} M,$$

which proves the proposition. \square

Theorem 10.4.12 (Krull). *Let A be a Noetherian ring, \mathfrak{a} an ideal of A , M a finitely generated A -module and N a submodule of M . Then the \mathfrak{a} -adic topology on N is induced by the \mathfrak{a} -adic topology on M .*

Proof. It follows from Artin-Rees Lemma that the filtration induced on N by the \mathfrak{a} -adic filtration on M is \mathfrak{a} -good and the conclusion then follows from ?. \square

Corollary 10.4.13. *Let A be a Noetherian ring, \mathfrak{a} an ideal of A , M an A -module and N a finitely generated A -module. Then every homomorphism $\phi : M \rightarrow N$ is a strict morphism for the \mathfrak{a} -adic topologies.*

Proof. As $\phi(\mathfrak{a}^n M) = \mathfrak{a}^n \phi(M)$, ϕ is a strict morphism of M onto $\phi(M)$ for the \mathfrak{a} -adic topologies on these two modules and the \mathfrak{a} -adic topology on $\phi(M)$ is induced by the \mathfrak{a} -adic topology on N by ??, hence the claim. \square

Corollary 10.4.14. Let $0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0$ be an exact sequence of finitely-generated modules over a Noetherian ring A . Let \mathfrak{a} be an ideal of A , then the sequence of \mathfrak{a} -adic completions

$$0 \longrightarrow \widehat{M}_1 \longrightarrow \widehat{M}_2 \longrightarrow \widehat{M}_3 \longrightarrow 0$$

is exact.

Proposition 10.4.15. Let A be a Noetherian ring, \mathfrak{a} an ideal of A and M a finitely generated A -module. The closure $\bigcap_{n=0}^{\infty} \mathfrak{a}^n M$ of $\{0\}$ in M with respect to the \mathfrak{a} -adic topology is the set of $x \in M$ for which there exists an element $a \in \mathfrak{a}$ such that $(1-a)x = 0$.

Proof. If $x = ax \bmod \mathfrak{a}$ where $a \in \mathfrak{a}$, then $x = a^n x$ for every integer $n > 0$ and hence $x \in \bigcap_n \mathfrak{a}^n M$. Conversely, if $x \in \bigcap_n \mathfrak{a}^n M$, Ax is contained in the intersection of the neighbourhoods of 0 in M ; it then follows from ?? that the \mathfrak{a} -adic topology on Ax , which is induced by that on M , is the trivial topology; as $\mathfrak{a}x$ is by definition a neighbourhood of 0 with this topology, $\mathfrak{a}x = Ax$ and hence there exists $a \in \mathfrak{a}$ such that $x = ax$. \square

Corollary 10.4.16 (Krull's Intersection Theorem). Let A be a Noetherian ring and \mathfrak{a} an ideal of A . Then the ideal $\bigcap_n \mathfrak{a}^n$ is the set of elements $x \in A$ for which there exists $a \in \mathfrak{a}$ such that $(1-a)x = 0$. In particular, for $\bigcap_n \mathfrak{a}^n = \{0\}$, it is necessary and sufficient that no element of $1 + \mathfrak{a}$ is a divisor of 0 in A .

Remark 10.4.2. The hypothesis that A is Noetherian is essential in this corollary. For example, let A be the ring of infinitely differentiable maps from \mathbb{R} to itself and let \mathfrak{m} be the (maximal) ideal of A consisting of the functions f such that $f(0) = 0$. It is immediate that $\bigcap_n \mathfrak{m}^n$ is the set of functions f such that $f^{(n)}(0) = 0$ for all $n \geq 0$ and there exist such functions with $f(x) \neq 0$ for all $x \neq 0$, for example the function f defined by $f(x) = e^{-1/x^2}$ for $x \neq 0$ and $f(0) = 0$. Also, note that for such an f , the element $1 + f$ is even invertible in A .

10.4.3 The adic completion of a Noetherian ring

Let A be a ring, \mathfrak{a} an ideal of A and M an A -module; let \widehat{A} and \widehat{M} denote the respective Hausdorff completions of A and M with respect to the \mathfrak{a} -adic topology and $\iota_M : M \rightarrow \widehat{M}$ the canonical map. The A -bilinear map $(a, x) \mapsto a\iota_M(x)$ of $\widehat{A} \times M$ to \widehat{M} defines a canonical A -linear map

$$\alpha_M : \widehat{A} \otimes_A M \rightarrow \widehat{M}.$$

Let $\phi : M \rightarrow N$ be an A -module homomorphism and let $\hat{\phi} : \widehat{M} \rightarrow \widehat{N}$ be the map obtained by passing to the Hausdorff completions; for $a \in \widehat{A}$ and $x \in M$,

$$\alpha_N(a \otimes \phi(x)) = a\iota_M(\phi(x)) = a\hat{\phi}(\iota_M(x)) = \hat{\phi}(\alpha_M(a \otimes x)).$$

In other words, the following diagram is commutative:

$$\begin{array}{ccc} \widehat{A} \otimes_A M & \xrightarrow{1 \otimes \phi} & \widehat{A} \otimes N \\ \downarrow \alpha_M & & \downarrow \alpha_N \\ \widehat{M} & \xrightarrow{\hat{\phi}} & \widehat{N} \end{array}$$

Finally, recall that by ??, if M is finitely generated then the homomorphism α_M is surjective.

Theorem 10.4.17. *Let A be a Noetherian ring, \mathfrak{a} an ideal of A and M, N, L be finitely generated A -modules. Then*

- (a) *If $0 \rightarrow M \rightarrow N \rightarrow L \rightarrow 0$ is an exact sequence of A -modules, then the sequence*

$$0 \longrightarrow \widehat{M} \longrightarrow \widehat{N} \longrightarrow \widehat{L} \longrightarrow 0$$

obtained by passing to the Hausdorff completions (with respect to the \mathfrak{a} -adic topologies) is exact.

- (b) *The canonical \widehat{A} -linear map $\alpha_M : \widehat{A} \otimes M \rightarrow \widehat{M}$ is bijective.*

- (c) *The A -module \widehat{A} is flat.*

Proof. We have seen that M and L carry the submodule filtration and quotient filtration, respectively, so claim (a) follows from ?. Assertion (b) is obvious when $M = A$ and the case where M is a free finitely generated A -module can be immediately reduced to that. In the general case, M admits a finite presentation

$$0 \longrightarrow N \longrightarrow A^n \longrightarrow M \longrightarrow 0$$

so we derive a commutative diagram

$$\begin{array}{ccccccc} \widehat{A} \otimes_A N & \longrightarrow & \widehat{A} \otimes_A A^n & \longrightarrow & \widehat{A} \otimes_A M & \longrightarrow & 0 \\ \downarrow \alpha_N & & \downarrow \alpha_{A^n} & & \downarrow \alpha_M & & \\ 0 & \longrightarrow & \widehat{N} & \longrightarrow & \widehat{A}^n & \longrightarrow & \widehat{M} \longrightarrow 0 \end{array}$$

The first row is exact and so is the second by (a). Since A is Noetherian we know that N is also finitely generated, so α_M and α_N are both surjective. Since α_{A^n} is bijective, we then deduce that α_M is injective by the snake lemma.

Then it follows from (a) and (b) that, if \mathfrak{a} is an ideal of A (necessarily finitely generated), the canonical map $\widehat{A} \otimes_A \mathfrak{a} \rightarrow \widehat{A}$ is injective, being the composition of $\mathfrak{a} \rightarrow \widehat{A}$ and $\alpha_{\mathfrak{a}}$, which proves that A is a flat A -module. \square

Under the conditions of Theorem, $\widehat{A} \otimes_A M$ is often identified with M by means of the canonical map α_M . If $\phi : M \rightarrow N$ is a homomorphism of finitely generated A -modules, $\widehat{\phi} : \widehat{M} \rightarrow \widehat{N}$ is then identified with $1 \otimes \phi$ by virtue of the commutativity of diagram.

Corollary 10.4.18. *Let A be a Noetherian ring, \mathfrak{a} an ideal of A and \widehat{A} the Hausdorff completion of A with respect to the \mathfrak{a} -adic topology. If an element $a \in A$ is not a divisor of zero in A , its canonical image \widehat{a} in \widehat{A} is not a divisor of zero in \widehat{A} .*

Proof. This follows from the fact that \widehat{A} is a flat A -module. \square

Corollary 10.4.19. *Let A be a Noetherian ring, \mathfrak{a} an ideal of A , M a finitely generated A -module and N and P two submodules of M . With the \mathfrak{a} -adic topologies and let ι_M be the canonical map from M to \widehat{M} , then*

$$\widehat{N} = \widehat{A}\iota_M(N), \quad (\widehat{N+P}) = \widehat{N} + \widehat{P}, \quad \widehat{N \cap P} = \widehat{N} \cap \widehat{P}, \quad (\widehat{N:P}) = (\widehat{N}:\widehat{P}).$$

Moreover, if \mathfrak{a} and \mathfrak{b} are two ideals of A , then $\widehat{\mathfrak{ab}} = \widehat{\mathfrak{a}}\widehat{\mathfrak{b}}$.

Proof. By ??, \widehat{M} , \widehat{N} , \widehat{P} are canonically identified with $\widehat{A} \otimes_A M$, $\widehat{A} \otimes_A N$, and $\widehat{A} \otimes_A P$, which establishes the formulas. Finally, as $\widehat{\mathfrak{a}} = \widehat{A}\mathfrak{a}$, $\widehat{\mathfrak{b}} = \widehat{A}\mathfrak{b}$, we see that

$$\widehat{\mathfrak{ab}} = \widehat{A}\mathfrak{ab} = \widehat{A}\mathfrak{a}\widehat{A}\mathfrak{b} = \widehat{\mathfrak{a}}\widehat{\mathfrak{b}},$$

and this finishes the proof. \square

Corollary 10.4.20. *Let A be a Noetherian ring, \mathfrak{a} an ideal of A and \widehat{A} the Hausdorff completion of A with respect to the \mathfrak{a} -adic topology. Then the topology on \widehat{A} is the $\widehat{\mathfrak{a}}$ -adic topology.*

Proof. By ?? we have $\widehat{(\mathfrak{a}^n)} = (\widehat{\mathfrak{a}})^n$. Since $(\widehat{\mathfrak{a}^n})$ is a fundamental system of neighbourhoods in \widehat{A} by ??, the claim follows. \square

Corollary 10.4.21. *If A is a Noetherian ring, the ring of normal power series $A[[X_1, \dots, X_n]]$ is a flat A -module.*

Proof. The ring $A[[X_1, \dots, X_n]]$ is the completion of $B = A[X_1, \dots, X_n]$ with respect to the \mathfrak{m} -adic topology, where \mathfrak{m} is the set of polynomials with no constant terms. As B is Noetherian, $A[[X_1, \dots, X_n]]$ is a flat B -module by ?? and, as B is free A -module, we deduce that $A[[X_1, \dots, X_n]]$ is a flat A -module by ?. \square

Proposition 10.4.22. *Let A be a Noetherian ring, \mathfrak{a} an ideal of A , \widehat{A} the Hausdorff completion of A with respect to the \mathfrak{a} -adic topology and ι the canonical map from A to \widehat{A} . Then*

- (a) *The map $\mathfrak{m} \mapsto \widehat{\mathfrak{m}}$ is a bijection of the set of maximal ideals of A containing \mathfrak{a} onto the set of maximal ideals of \widehat{A} and $\mathfrak{n} \mapsto \iota^{-1}(\mathfrak{n})$ is the inverse bijection.*
- (b) *Let \mathfrak{m} be a maximal ideal of A containing \mathfrak{a} . Then the homomorphism $\iota_{\mathfrak{m}} : A_{\mathfrak{m}} \rightarrow \widehat{A}_{\widehat{\mathfrak{m}}}$ derived from ι is an embedding with dense image if $A_{\mathfrak{m}}$ is given the $\mathfrak{m}A_{\mathfrak{m}}$ -adic topology and $\widehat{A}_{\widehat{\mathfrak{m}}}$ the $\widehat{\mathfrak{m}}\widehat{A}_{\widehat{\mathfrak{m}}}$ -adic topology.*

Proof. The assertion of (a) follows immediately from the fact that the canonical homomorphism $A/\mathfrak{a} \rightarrow \widehat{A}/\widehat{\mathfrak{a}}$ derived from ι is bijective and the fact that every maximal ideal of \widehat{A} contains $\widehat{\mathfrak{a}}$, since $\widehat{\mathfrak{a}}$ is contained in the Jacobson radical of \widehat{A} (?).

Finally let us prove (b). As $\mathfrak{m} = \iota^{-1}(\widehat{\mathfrak{m}})$, $\iota(A - \mathfrak{m}) \subseteq \widehat{A} - \widehat{\mathfrak{m}}$ and ι certainly defines a homomorphism $\iota_{\mathfrak{m}} : A_{\mathfrak{m}} \rightarrow \widehat{A}_{\widehat{\mathfrak{m}}}$. Let us show that $\iota_{\mathfrak{m}}$ is injective; let $a \in A$ and $s \in A - \mathfrak{m}$ be such that

$$\iota_{\mathfrak{m}}(a/s) = \iota(a)/\iota(s) = 0.$$

then there exists $t \in \widehat{A} - \widehat{\mathfrak{m}}$ such that $t\iota(a) = 0$ and the annihilator of $\iota(a)$ in A is therefore not contained in \mathfrak{m} . Now, if \mathfrak{b} is the annihilator of a in A , then the annihilator of $\iota(a)$ in A is $\widehat{\mathfrak{b}}$ (?); hence $\mathfrak{b} \not\subseteq \mathfrak{m}$, which shows that $a/s = 0$.

Moreover, there is a commutative diagram

$$\begin{array}{ccc} A/\mathfrak{m}^k & \longrightarrow & A_{\mathfrak{m}}/(\mathfrak{m}A_{\mathfrak{m}})^k \\ \downarrow \tau & & \downarrow \tau_{\mathfrak{m}} \\ \widehat{A}/\widehat{\mathfrak{m}}^k & \longrightarrow & \widehat{A}_{\widehat{\mathfrak{m}}}/(\widehat{\mathfrak{m}}\widehat{A}_{\widehat{\mathfrak{m}}})^k \end{array}$$

where τ and τ_m are derived from ι and ι_m respectively and the horizontal arrows are the canonical isomorphisms. As m^k is an open ideal of A (since it contains a^k), τ is bijective and hence so is τ_m . This shows first that $(\hat{m}\hat{A}_{\hat{m}})^k = \iota_m((mA_m)^k)$ and hence the topology on A_m is induced by that on $\hat{A}_{\hat{m}}$. Moreover, $\hat{A}_{\hat{m}} = A_m + (\hat{m}\hat{A}_{\hat{m}})^k$ for all $k > 0$ and hence A_m is dense in $\hat{A}_{\hat{m}}$. \square

Corollary 10.4.23. *Let A be a Noetherian local (resp. semi-local) ring and \mathfrak{r} its Jacobson radical. Then \hat{A} is a Noetherian local (resp. semi-local) ring whose Jacobson radical is $\hat{\mathfrak{r}}$.*

Proof. As \hat{A}/\hat{m} is isomorphic to A/m , it is a Noetherian ring and $\hat{m} = \iota(\mathfrak{m})$ is a finitely generated A -module and therefore \hat{A} is Noetherian. The rest part follows from ?? and the third formula in ??.

Corollary 10.4.24. *Let A be a Noetherian ring and \mathfrak{m} an maximal ideal of A . Then the \mathfrak{m} -adic completion \hat{A} of A is a Noetherian local ring and is isomomorphic to the $\mathfrak{m}A_m$ -adic completion \widehat{A}_m of A_m .*

Proof. Taking \mathfrak{a} to be the maximal ideal \mathfrak{m} in ??(a), we see the \mathfrak{m} -adic completion \hat{A} is Noetherian local with maximal ideal \hat{m} . Then by ??(b), since $\hat{A}_{\hat{m}}$ is identified with \hat{A} , we have an dense embedding $\iota_m : A_m \rightarrow \hat{A}$, where A_m is given the $\mathfrak{m}A_m$ -adic topology and \hat{A} the \hat{m} -adic topology. Since \hat{A} is complete, this proves $\hat{A} \cong \widehat{A}_m$. \square

10.4.4 Zariski rings

For a topological ring A , if the given topology on A is the \mathfrak{a} -adic topology for an ideal \mathfrak{a} of A , then \mathfrak{a} is called a **defining ideal** of the topology on A .

Remark 10.4.3. Let A be a Noetherian ring and \mathfrak{a} be an ideal of A . Note that if \mathfrak{b} is a defining ideal of the \mathfrak{a} -adic topology on A , then there exists an integer $n > 0$ such that $\mathfrak{b}^n \subseteq \mathfrak{a}$ and hence $\mathfrak{b} \subseteq \sqrt{\mathfrak{a}}$. Conversely, since A is Noetherian, there exists an integer $k > 0$ such that $(\sqrt{\mathfrak{a}})^k \subseteq \mathfrak{a}$ (??) and hence $\sqrt{\mathfrak{a}}$ is the largest defining ideal of the \mathfrak{a} -adic topology.

Proposition 10.4.25. *Let A be a Noetherian ring and \mathfrak{a} an ideal of A . The following properties are equivalent:*

- (i) \mathfrak{a} is contained in the Jacobson radical of A .
- (ii) Every finitely generated A -module is Hausdorff with the \mathfrak{a} -adic topology.
- (iii) For every finitely generated A -module M , every submodule of M is closed with respect to the \mathfrak{a} -adic topology on M .
- (iv) Every maximal ideal of A is closed with respect to the \mathfrak{a} -adic topology.
- (v) The Hausdorff completion \hat{A} is a faithfully flat A -module.

Proof. Let us show that (i) implies (ii). Suppose that \mathfrak{a} is contained in the Jacobson radical of A and let M be a finitely generated A -module. If $x \in M$ and $a \in \mathfrak{a}$ are such that $(1 - a)x = 0$, then $x = 0$, for $1 - a$ is invertible in A . Then M is Hausdorff with the \mathfrak{a} -adic topology.

Next we show that (ii) implies (iii). Suppose (ii) holds. Let M be a finitely generated A -module and N a submodule of M . Then M/N is Hausdorff with the \mathfrak{a} -adic topology, which is the quotient topology of the \mathfrak{a} -adic topology on M ; thus N is closed in M .

Clearly (iii) implies (iv). Now assume (iv), then for every maximal ideal \mathfrak{m} of A , the A -module A/\mathfrak{m} is Hausdorff with the \mathfrak{a} -adic topology. This implies $\mathfrak{a}(A/\mathfrak{m}) \neq A/\mathfrak{m}$, unless the \mathfrak{a} -adic topology on A/\mathfrak{m} were the trivial topology and A/\mathfrak{m} were reduced to 0, which is absurd since A/\mathfrak{m} is a field. The canonical image of \mathfrak{a} in A/\mathfrak{m} is therefore an ideal of A/\mathfrak{m} distinct from A/\mathfrak{m} and hence reduced to 0; then $\mathfrak{a} \subseteq \mathfrak{m}$, which proves that \mathfrak{a} is contained in the Jacobson radical of A .

Finally, for every finitely generated A -module M , the canonical map $M \hookrightarrow M \otimes_A \widehat{A}$ is identified with the canonical map $M \hookrightarrow \widehat{M}$ from M to its Hausdorff completion with respect to the \mathfrak{a} -adic topology (by ??) and the kernel of this map is then the closure of $\{0\}$ in M with respect to this topology. As we already know that \widehat{A} is a flat A -module, the equivalence of (v) and (ii) follows from the characterization of faithfully flat modules (??). This finishes the proof. \square

A topological ring A is called a **Zariski ring** if it is Noetherian and there exists a defining ideal \mathfrak{a} for the topology on A satisfying the equivalent conditions of ???. A Zariski ring A is necessarily Hausdorff (??) and every defining ideal of its topology is contained in the Jacobson radical of A .

Example 10.4.26 (Examples of Zariski rings).

- (a) Let A be a Noetherian ring and \mathfrak{a} an ideal of A . If A is Hausdorff and complete with the \mathfrak{a} -adic topology, then A is a Zariski ring with this topology. In particular, \widehat{A} is a Zariski ring.
- (b) Every quotient ring A/\mathfrak{b} of a Zariski ring A is a Zariski ring, for it is Noetherian and, if \mathfrak{a} is a defining ideal of A , then $\mathfrak{a}(A/\mathfrak{b}) = (\mathfrak{a} + \mathfrak{b})/\mathfrak{b}$ is contained in the Jacobson radical of A/\mathfrak{b} .
- (c) Let A be a Noetherian semi-local ring and \mathfrak{t} its Jacobson radical. Then A with the \mathfrak{t} -adic topology is a Zariski ring. This will always be the topology in question (unless otherwise stated) when we consider a Noetherian semi-local ring as a topological ring.

If A is a Zariski ring and M is a finitely generated A -module, then since M is Hausdorff, we may (by virtue of ??) identify M with a subset of \widehat{M} by means of the canonical map ι_M . With this identification, by ?? we have

Corollary 10.4.27. *Let A be a Zariski ring, M a finitely generated A -module and N a submodule of M . Then $N = \widehat{N} \cap M$.*

Corollary 10.4.28. *Let A be a Zariski ring and M a finitely generated A -module. If \widehat{M} is a free \widehat{A} -module, then M is a free A -module.*

Proof. Let \mathfrak{a} be a defining ideal of A , which is therefore contained in the Jacobson radical of A . We apply the criterion of ???: the canonical map $\iota_M : M \rightarrow \widehat{M}$ defines a bijection $i_M : M/\mathfrak{a}M \rightarrow \widehat{M}/\widehat{\mathfrak{a}M}$. Similarly the canonical map $\iota_A : A \rightarrow \widehat{A}$ defines a bijection $i_A : A/\mathfrak{a} \rightarrow \widehat{A}/\widehat{\mathfrak{a}}$, which is a ring isomorphism. Then $\widehat{M}/\widehat{\mathfrak{a}M}$ is given an $(\widehat{A}/\widehat{\mathfrak{a}})$ -module structure and hence (by means of i_A) an (A/\mathfrak{a}) -module structure. It is immediate that i_M is (A/\mathfrak{a}) -linear, so that it is an (A/\mathfrak{a}) -module isomorphism. As $\widehat{M}/\widehat{\mathfrak{a}M}$ is a free $(\widehat{A}/\widehat{\mathfrak{a}})$ -module, $M/\mathfrak{a}M$ is a free (A/\mathfrak{a}) -module.

On the other hand, let $\eta : \mathfrak{a} \otimes_A M \rightarrow M$ be the canonical homomorphism; as $(\mathfrak{a} \otimes_A M) \otimes_A \widehat{A}$ is canonically identified with $\widehat{\mathfrak{a}} \otimes_A \widehat{M}$ and $M \otimes_A \widehat{A}$ with \widehat{M} , the hypothesis that M is a free A -module implies that the homomorphism $\eta \otimes 1 : \widehat{\mathfrak{a}} \otimes_A \widehat{M} \rightarrow \widehat{M}$ is injective. As \widehat{A} is a faithfully flat A -module, we conclude that η is injective and the conditions for applying the above mentioned criterion are indeed fulfilled. \square

Corollary 10.4.29. *Let A be a Zariski ring such that A is an integral domain and let \mathfrak{a} be an ideal of A . If the ideal $\widehat{\mathfrak{a}}$ of A is principal, then \mathfrak{a} is principal.*

Corollary 10.4.30. *Let A be a Zariski ring such that \widehat{A} is an integral domain, L the field of fractions of \widehat{A} and $K \subseteq L$ the field of fractions of A . Then $\widehat{A} \cap K = A$.*

Proof. Clearly $A \subseteq \widehat{A} \cap K$; on the other hand, if $x \in \widehat{A} \cap K$, then $\widehat{A}x \subseteq \widehat{A}$ and hence, as $\widehat{A}x = \widehat{A} \otimes_A Ax$, we have

$$\widehat{A} \otimes_A ((Ax + A)/A) = (\widehat{A} \otimes_A Ax)/(\widehat{A} \otimes_A A) = 0.$$

As \widehat{A} is a faithfully flat A -module, we deduce that $Ax \subseteq A$, whence $x \in A$. \square

Corollary 10.4.31. *Let A be a Noetherian ring, M, N be finitely generated A -modules and $\phi : M \rightarrow N$ a homomorphism. For every maximal ideal \mathfrak{m} of A , let $A(\mathfrak{m})$ (resp. $M(\mathfrak{m}), N(\mathfrak{m})$) denote the Hausdorff completion of A (resp. M, N) with respect to the \mathfrak{m} -adic topology and $\phi(\mathfrak{m}) : M(\mathfrak{m}) \rightarrow N(\mathfrak{m})$ the corresponding homomorphism to ϕ . For ϕ to be injective (resp. surjective, bijective, zero), it is necessary and sufficient that $\phi(\mathfrak{m})$ be so for every maximal ideal \mathfrak{a} of A .*

Proof. We know that for ϕ to be injective (resp. surjective, bijective, zero), it is necessary and sufficient that $\phi_{\mathfrak{m}} : M_{\mathfrak{m}} \rightarrow N_{\mathfrak{m}}$ be so for every maximal ideal \mathfrak{m} of A . We now note that $A_{\mathfrak{m}}$ is a Noetherian local ring and hence a Zariski ring and there is a canonical A -algebra isomorphism $(A_{\mathfrak{m}})(\mathfrak{m}A_{\mathfrak{m}}) \rightarrow A(\mathfrak{m})$ (??). On the other hand, there is a commutative diagram

$$\begin{array}{ccccc} M_{\mathfrak{m}} \otimes_{A_{\mathfrak{m}}} A(\mathfrak{m}) & \longrightarrow & M \otimes_A A(\mathfrak{m}) & \longrightarrow & M(\mathfrak{m}) \\ \downarrow \phi_{\mathfrak{m}} \otimes 1 & & \downarrow \phi \otimes 1 & & \downarrow \phi(\mathfrak{m}) \\ N_{\mathfrak{m}} \otimes_{A_{\mathfrak{m}}} A(\mathfrak{m}) & \longrightarrow & N \otimes_A A(\mathfrak{m}) & \longrightarrow & N(\mathfrak{m}) \end{array}$$

where the horizontal arrows on the left arise from the associativity of the tensor product and the isomorphisms $M_{\mathfrak{m}} \rightarrow M \otimes_A A_{\mathfrak{m}}, N_{\mathfrak{m}} \rightarrow N \otimes_A A_{\mathfrak{m}}$. As M and N are finitely generated A -modules, it follows from ?? that the rows of this diagram consist of isomorphisms; thus we are reduced to proving that $\phi_{\mathfrak{m}}$ being injective (resp. surjective, bijective, zero) is equivalent to $\phi_{\mathfrak{m}} \otimes 1$ being so. But this follows from the fact that $(A_{\mathfrak{m}})(\mathfrak{m}A_{\mathfrak{m}})$ (and hence also $A(\mathfrak{m})$) is a faithfully flat $A_{\mathfrak{m}}$ -module. \square

Proposition 10.4.32. *Let A, B be two rings and $\rho : A \rightarrow B$ be a ring homomorphism. Suppose that A is Noetherian and B is a finitely generated A -module (with the structure defined by f). Let \mathfrak{a} be an ideal of A , then*

- (a) *For the \mathfrak{a}^e -adic topology on B to be Hausdorff, it is necessary and sufficient that the elements of $1 + \rho(\mathfrak{a})$ be not divisors of 0 in B .*

- (b) If A with the \mathfrak{a} -adic topology is a Zariski ring, then B with the \mathfrak{a}^e -adic topology is a Zariski ring.
- (c) If ρ is injective (thus identifying A with a subring of B), the \mathfrak{a}^e -adic topology on B induces on A the \mathfrak{a} -adic topology.

Proof. Recall that the \mathfrak{a}^e -adic filtration on B coincides with the \mathfrak{a} -adic filtration on the A -module B . Assertion (a) is thus a special case of ?? and assertion (c) a special case of Artin-Rees Lemma. Finally let us show (b). Suppose that A is a Zariski ring with the \mathfrak{a} -adic topology and let N be a finitely generated B -module; it is also a finitely generated A -module and the \mathfrak{a} -adic and \mathfrak{a}^e -adic filtrations on N coincide; then N is Hausdorff with the \mathfrak{a}^e -adic topology. Finally the A -module B is Noetherian and hence the ring B is Noetherian, which completes the proof that B is a Zariski ring. \square

Proposition 10.4.33. *Let A, B be two Zariski rings, \widehat{A}, \widehat{B} their completions, $\rho : A \rightarrow B$ a continuous ring homomorphism and $\widehat{\rho} : \widehat{A} \rightarrow \widehat{B}$ the homomorphism obtained from ρ by passing to the completions. If $\widehat{\rho}$ is bijective, then the A -module B is faithfully flat.*

Proof. As A and B are Hausdorff, the hypothesis that $\widehat{\rho}$ is bijective implies first that ρ is injective. Identifying (algebraically) A with $\rho(A)$ by means of ρ and \widehat{A} with \widehat{B} by means of $\widehat{\rho}$, we then obtain the inclusions $A \subseteq B \subseteq \widehat{A} = \widehat{B}$. Then we know that \widehat{A} is a faithfully flat A -module and a faithfully flat B -module, hence B is a faithfully flat A -module. \square

Proposition 10.4.34. *Let A be a Noetherian local ring, \mathfrak{m} its maximal ideal, \widehat{A} its \mathfrak{m} -adic completion and B a ring such that $A \subseteq B \subseteq \widehat{A}$. Suppose that B is a Noetherian local ring whose maximal ideal \mathfrak{n} satisfies the relation $\mathfrak{n} = \mathfrak{m}B$. Then for $k \geq 0$,*

$$\mathfrak{n}^k = \mathfrak{m}^k B = \widehat{\mathfrak{m}}^k \cap B,$$

the \mathfrak{n} -adic topology on B is induced by the $\widehat{\mathfrak{m}}$ -adic topology on \widehat{A} , B is a faithfully flat A -module and there is an isomorphism of \widehat{A} onto the \mathfrak{n} -adic completion \widehat{B} of B , which extends the canonical injection $A \rightarrow B$.

Proof. It is sufficient to verify the relation $\mathfrak{n}^k = \widehat{\mathfrak{m}}^k \cap B$, for, as B is dense in A and the \mathfrak{n} -adic topology is induced by the $\widehat{\mathfrak{m}}$ -adic topology, the last assertion will follow from ?? and the last but one from ?. The injections $A \rightarrow B \rightarrow \widehat{A} = \widehat{B}$ induce injective homomorphisms

$$A/(\widehat{\mathfrak{m}} \cap A) \longrightarrow B/(\widehat{\mathfrak{m}} \cap B) \longrightarrow \widehat{A}/\widehat{\mathfrak{m}}$$

We know that $\widehat{\mathfrak{m}} \cap A = \mathfrak{m}$ and that $A/\mathfrak{m} \cong \widehat{A}/\widehat{\mathfrak{m}}$, hence $B/(\widehat{\mathfrak{m}} \cap B) \cong \widehat{A}/\widehat{\mathfrak{m}}$, which shows that $B/(\widehat{\mathfrak{m}} \cap B)$ is a field, hence that $\widehat{\mathfrak{m}} \cap B$ is a maximal ideal of B and therefore $\widehat{\mathfrak{m}} \cap B = \mathfrak{n}$. As $A/\mathfrak{m} \cong \widehat{A}/\widehat{\mathfrak{m}}$ we have $\widehat{A} = A + \widehat{\mathfrak{m}}$, thus $B = A + \mathfrak{n} = A + \mathfrak{m}B$. By induction on k we deduce that

$$B = A + \mathfrak{m}^k B = A + \mathfrak{n}^k$$

for all $k > 1$. As $\mathfrak{n}^k \subseteq \widehat{\mathfrak{m}}^k + B$, it is sufficient to show that $\widehat{\mathfrak{m}}^k \cap B \subseteq \mathfrak{n}^k$. If $b \in \widehat{\mathfrak{m}}^k \cap B$, we may write $b = a + z$ where $a \in A, z \in \mathfrak{n}^k$; whence

$$a = b - z \in \widehat{\mathfrak{m}}^k \cap A = \mathfrak{m}^k \subseteq \mathfrak{n}^k$$

and $b \in \mathfrak{n}^k$. □

Example 10.4.35. An important case where this applies is the following: B is the ring of integral series in n variables over a complete valued field K , which converge in the neighbourhood of 0, A is the local ring $K[X_1, \dots, X_n]_{\mathfrak{m}}$ where \mathfrak{m} is the maximal ideal consisting of the polynomials with no constant term and \widehat{A} is the ring of formal power series $K[[X_1, \dots, X_n]]$.

Proposition 10.4.36. Let A be a Noetherian ring, \mathfrak{a} an ideal of A , S the multiplicative subset $1 + \mathfrak{a}$ of A and M a finitely generated A -module.

- (a) $S^{-1}A$ is a Zariski ring with the $(S^{-1}\mathfrak{a})$ -adic topology.
- (b) The canonical map $i : M \rightarrow S^{-1}M$ is continuous if M is given the \mathfrak{a} -adic topology and $S^{-1}M$ the $(S^{-1}\mathfrak{a})$ -adic topology and $\widehat{i} : \widehat{M} \rightarrow \widehat{S^{-1}M}$ is an isomorphism.

Proof. Every element of $1 + S^{-1}\mathfrak{a}$ is of the form

$$1 + (a/(1+b)) = (1+a+b)/(1+b)$$

where $a, b \in \mathfrak{a}$; it is therefore invertible in $S^{-1}A$, which proves that $S^{-1}\mathfrak{a}$ is contained in the Jacobson radical of $S^{-1}A$. As $S^{-1}A$ is Noetherian, it is then a Zariski ring with the $(S^{-1}\mathfrak{a})$ -adic topology, which proves (a). Let us show (b). For all $n > 0$, we have

$$i^{-1}((S^{-1}\mathfrak{a})^n M) = i^{-1}(S^{-1}\mathfrak{a}^n M) = \mathfrak{a}^n M$$

since by ??, if $x \in i^{-1}(S^{-1}\mathfrak{a}^n M)$ then $(1-a)x = y$, where $a \in \mathfrak{a}, y \in \mathfrak{a}^n M$, whence

$$x = (1+a+a^2+\dots+a^{n-1})y + a^n x \in \mathfrak{a}^n M.$$

This proves that i is continuous and a strict morphism (??). Moreover, the kernel of i , which is the set of $x \in M$ for which there exists some $s \in S$ such that $sx = 0$, is identical with the kernel of the canonical map $\iota : M \rightarrow \widehat{M}$. Then there exists a topological isomorphism $\iota_0 : i(M) \rightarrow \iota(M)$ such that the following diagram is commutative:

$$\begin{array}{ccccc} & & \iota(M) & \hookrightarrow & \widehat{M} \\ M & \begin{array}{c} \nearrow \iota \\ \searrow i \end{array} & \downarrow \iota_0 & & \downarrow \widehat{i} \\ & & i(M) & \hookrightarrow & \widehat{S^{-1}M} \end{array}$$

Therefore the problem reduces to verify that $i(M)$ is dense in $S^{-1}M$. Now every element of $S^{-1}M$ may be written as $x/(1-a)$, where $a \in \mathfrak{a}$, and it is immediately verified that

$$x/(1-a) \equiv (1+a+\dots+a^{n-1})x \pmod{S^{-1}\mathfrak{a}^n M}$$

which completes the proof. □

10.5 Lifting in complete rings

10.5.1 Strongly relatively prime polynomials

Let A be a ring. Two elements x, y of A are called **strongly relatively prime** if the principal ideals (x) and (y) are relatively prime, in other words if $Ax + Ay = A$; it amounts to the same to say that there exist two elements a, b of A such that $ax + by = 1$.

Lemma 10.5.1 (Euclid Lemma). *Let x, y be two strongly relatively prime elements of A ; if $z \in A$ is such that x divides yz , then x divides z .*

Proof. If $1 = ax + by$, then $z = x(az) + (yz)b$, so x divides z . \square

If x and y are strongly relatively prime in A , then $(xy) = (x) \cap (y)$. If A is an integral domain, two strongly relatively prime elements then have an lcm equal to their product and are therefore relatively prime in the sense that $\gcd(x, y) = 1$. Conversely, if A is a principal ideal domain, two relatively prime elements are also strongly relatively prime, as follows from Bezout's identity.

For polynomial rings there is the following result:

Proposition 10.5.2. *Let A be a ring and F, G two strongly relatively prime polynomials in $A[X]$. Suppose that F is monic and of degree s . Then every polynomial T in $A[X]$ may be written uniquely in the form*

$$T = PF + QG \quad (10.5.1)$$

where $P, Q \in A[X]$ and $\deg(Q) < s$. If further $\deg(T) \leq n$ and $\deg(G) \leq n - s$, then $\deg(P) \leq n - s$.

Proof. As F is monic, $FP \neq 0$ for every nonzero polynomial P of $A[X]$ and in this case $\deg(FP) = s + \deg(P)$. Let T be any polynomial in $A[X]$. As the ideal generated by F and G is the whole of $A[X]$, there exist polynomials P_1 and Q_1 such that

$$T = P_1F + Q_1G.$$

As F is monic of degree s , Euclidean division shows that there exist two polynomials Q' and Q'' such that $Q_1 = Q''F + Q'$ where $\deg(Q') < s$. Then we deduce that

$$T = P_1F + Q_1G = P_1F + Q''FG + Q'G = PF + Q'G$$

where $P = P_1 + Q''G$. To show the uniqueness of formula (??), it is sufficient to prove that the relations

$$0 = PF + QG, \quad \deg(Q) < s.$$

imply $P = Q = 0$. Now, if this holds, F divides QG and, as F and G are strongly relatively prime, F divides Q by ??, which is impossible unless $Q = 0$. This then implies $PF = 0$, and so $P = 0$ by the remark at the beginning.

Finally, suppose that $\deg(T) \leq n$ and $\deg(G) \leq n - s$. With the polynomial T in the form (??),

$$\deg(QG) \leq \deg(Q) + \deg(G) < s + \deg(G) \leq n$$

and therefore

$$s + \deg(P) = \deg(PF) = \deg(T - QG) \leq n$$

whence $\deg(P) \leq n - s$. □

Example 10.5.3. For a polynomial $P \in A[X]$ to be strongly relatively prime to $X - a$ (where $a \in A$), it is necessary and sufficient that $P(a)$ be invertible in A . For if P and $X - a$ are strongly relatively prime, it follows from ?? that there exist $c \in A$ and a polynomial $Q \in A[X]$ such that $1 = (X - a)Q + cP$, whence $cP(a) = 1$ and $P(a)$ is invertible. Conversely, by Euclidean division

$$P = (X - a)G + P(a)$$

and, if $P(a) = b^{-1}$ where $b \in A$, we deduce that $1 = bP - b(X - a)G$, which shows that P and $X - a$ are strongly relatively prime.

Let A and B be two rings and $\rho : A \rightarrow B$ a ring homomorphism. If $P = \sum_{n=0}^{\infty} a_n X^n$ is a formal power series in $A[[X]]$, let $f(P)$ denote the formal power series $\sum_{n=0}^{\infty} \rho(a_n) X^n$ in $B[[X]]$. If P is a polynomial, so is $\rho(P)$ and, if further P is monic, then $\rho(P)$ is monic of the same degree as P . Finally, $P \mapsto f(P)$ is clearly a homomorphism of $A[[X]]$ to $B[[X]]$ which extends f and maps X to X .

Proposition 10.5.4. *Let A and B be two rings, $\rho : A \rightarrow B$ a homomorphism and P, Q two polynomials in $A[X]$. If P and Q are strongly relatively prime in $A[X]$, then $\rho(P)$ and $\rho(Q)$ are strongly relatively prime in $B[X]$. The converse is true if ρ is surjective, if its kernel is contained in the Jacobson radical of A and if P is monic.*

Proof. Suppose that P and Q are strongly relatively prime; then there exist polynomials U, V in $A[X]$ such that $PU + QV = 1$; we deduce that

$$\rho(P)\rho(U) + \rho(Q)\rho(V) = 1$$

whence the first assertion. To show the second, let \mathfrak{a} denote the kernel of ρ . Let $M = A[X]$ and N be the ideal of M generated by P and Q . As ρ is surjective and $\rho(P)$ is monic, ?? shows that for every polynomial $T \in A[X]$ there exist two polynomials U, V in $A[X]$ such that

$$\rho(T) = \rho(P)\rho(U) + \rho(Q)\rho(V)$$

whence the relation $M = N + \mathfrak{a}M$. Now, M/N is a finitely generated A -module, for every polynomial is congruent mod P to a polynomial of degree strictly smaller than $\deg(P)$, P being monic. As $\mathfrak{a}(M/N) = M/N$ and \mathfrak{a} is contained in the Jacobson radical of A , Nakayama's Lemma shows that $M/N = 0$, which means that P and Q are strongly relatively prime. □

10.5.2 Restricted power series and Hensel's lemma

Definition 10.5.5. *A commutative topological ring A is said to be linearly topologized (and its topology is said to be **linear**) if there exists a fundamental system \mathcal{B} of neighbourhoods of 0 consisting of ideals of A .*

Note that in such a ring, the ideals $\mathfrak{a} \in \mathcal{B}$ are open and closed. For all $\mathfrak{a} \in \mathcal{B}$, the quotient topological ring A/\mathfrak{a} is then discrete. For $\mathfrak{a}, \mathfrak{b} \in \mathcal{B}$ and $\mathfrak{b} \subseteq \mathfrak{a}$, let

$$\pi_{\mathfrak{ab}} : A/\mathfrak{b} \rightarrow A/\mathfrak{a}$$

be the canonical map. We know that $(A/\mathfrak{a}, \pi_{\mathfrak{ab}})$ is an inverse system of discrete rings (relative to the indexing set \mathcal{B} which is ordered by inclusion and directed), whose inverse limit is a complete Hausdorff linearly topologized ring \tilde{A} . Further, a strict morphism $\iota : A \rightarrow \tilde{A}$ is defined, whose kernel is the closure of $\{0\}$ in A and whose image is dense in \tilde{A} , so that \tilde{A} is canonically identified with the Hausdorff completion of A .

Definition 10.5.6. *Given a commutative topological ring A , a formal power series*

$$P = \sum_{\alpha \in \mathbb{N}^p} c_\alpha X^\alpha$$

*in the ring $A[[X_1, \dots, X_p]]$ is called **restricted** if, for every neighbourhood V of 0 in A , there is only a finite number of coefficients c_α not belonging to V (in other words, the family (c_α) tends to 0 in A with respect to the filter of complements of finite subsets of \mathbb{N}^p .*

If A is linearly topologized, the restricted formal power series in $A[[X_1, \dots, X_p]]$ form a subring of $A[[X_1, \dots, X_p]]$, denoted by $A\{X_1, \dots, X_p\}$, for if $P = \sum a_\alpha X^\alpha$ and $Q = \sum b_\alpha X^\alpha$ are two restricted formal power series and \mathfrak{a} a neighbourhood of 0 in A which is an ideal of A , there exists an integer m such that $a_\alpha \in \mathfrak{a}$ and $b_\alpha \in \mathfrak{a}$ for every $\alpha \in \mathbb{N}^p$ such that $|\alpha| \geq m$. Now, if

$$PQ = \sum c_\alpha X^\alpha \quad \text{where} \quad c_\alpha = \sum_{\beta+\gamma=\alpha} a_\beta b_\gamma.$$

We conclude that if $|\alpha| \geq 2m$, then $|\beta| \geq m$ or $|\gamma| \geq m$ and hence, since \mathfrak{a} is an ideal, $c_\alpha \in \mathfrak{a}$ so long as $|\alpha| \geq 2m$, which establishes our assertion. Moreover, every derivative $\partial(PQ)/\partial X_i$ ($1 \leq i \leq p$) of a restricted formal power series is restricted, as follows immediately from the definition and the fact that the neighbourhoods $\mathfrak{a} \in \mathcal{B}$ are additive subgroups of A .

Let us always assume that A is linearly topologized and let \mathcal{B} be a fundamental system of neighbourhoods of 0 in A consisting of ideals of A ; for all $\mathfrak{a} \in \mathcal{B}$, let $\pi_{\mathfrak{a}} : A \rightarrow A/\mathfrak{a}$ be the canonical homomorphism. By definition, for every restricted formal power series $T \in A\{X_1, \dots, X_n\}$,

$$\pi_{\mathfrak{a}}(T) \in (A/\mathfrak{a})[X_1, \dots, X_p]$$

Clearly $((A/\mathfrak{a})[X_1, \dots, X_p], \pi_{\mathfrak{ab}})$ is an inverse system of rings and $(\pi_{\mathfrak{a}})$ is an inverse system of homomorphisms $A\{X_1, \dots, X_p\} \rightarrow (A/\mathfrak{a})[X_1, \dots, X_p]$; as every polynomial is a restricted formal power series, $\pi_{\mathfrak{a}}$ is surjective; its kernel $N_{\mathfrak{a}}$ is the ideal of $A\{X_1, \dots, X_p\}$ consisting of the restricted formal power series all of whose coefficients belong to \mathfrak{a} ; we shall give $A\{X_1, \dots, X_p\}$ the (linear) topology for which the $N_{\mathfrak{a}}$ (for $\mathfrak{a} \in \mathcal{B}$) form a fundamental system of neighbourhoods of 0 (a topology which obviously depends only on that on A). Then it follows from ?? that

$$\pi = \varprojlim \pi_{\mathfrak{a}} : A\{X_1, \dots, X_p\} \rightarrow \varprojlim_{\mathfrak{a}} (A/\mathfrak{a})[X_1, \dots, X_p]$$

is a strict morphism whose kernel is the closure of $\{0\}$ in $A\{X_1, \dots, X_p\}$ and whose image is dense in $B = \varprojlim_{\mathfrak{a}} (A/\mathfrak{a})[X_1, \dots, X_p]$.

Proposition 10.5.7. *If the linearly topologized ring A is Hausdorff and complete, the canonical homomorphism π is a topological ring isomorphism.*

Proof. For all $\alpha \in \mathbb{N}^p$ and all $\mathfrak{a} \in \mathcal{B}$, let $\phi_\alpha^\mathfrak{a}$ be the map $(A/\mathfrak{a})[X_1, \dots, X_p] \rightarrow A/\mathfrak{a}$ which maps every polynomial to the coefficient of X^α in this polynomial; clearly the $(\phi_\alpha^\mathfrak{a})_\mathfrak{a}$ form an inverse system of (A/\mathfrak{a}) -module homomorphisms (relative to the ordered set \mathcal{B}) and, as A is canonically identified with $\varprojlim_{\mathfrak{a}} (A/\mathfrak{a})$ by hypothesis $\phi_\alpha = \varprojlim_{\mathfrak{a}} \phi_\alpha^\mathfrak{a}$ is a continuous A -homomorphism from B to A . For every element $S = (S_\mathfrak{a})_{\mathfrak{a} \in \mathcal{B}}$ of B , we consider the formal power series $T = \sum_{\alpha} \phi_\alpha(S) X^\alpha$. For each $\mathfrak{a} \in \mathcal{B}$, since $S_\mathfrak{a}$ is a polynomial, there exist $n > 0$ such that $\phi_\alpha^\mathfrak{a}(S_\mathfrak{a}) = 0$ for $|\alpha| \geq n$, which implies $\phi_\alpha^\mathfrak{b}(S_\mathfrak{b}) \in \mathfrak{a}/\mathfrak{b}$ for all $\mathfrak{b} \in \mathcal{B}$, $\mathfrak{b} \subseteq \mathfrak{a}$. Taking limits for these \mathfrak{b} 's, we then see $\phi_\alpha(S) \in \mathfrak{a}$ for $|\alpha| \geq n$, which proves T is restricted. Also, it is immediate that $\pi(T) = S$. As A is Hausdorff, the intersection of the $N_\mathfrak{a}$ reduces to 0 and hence π is bijective, which completes the proof, since π is a strict morphism. \square

Proposition 10.5.8. *Let A, B be two linearly topologized rings, B being Hausdorff and complete, and $\rho : A \rightarrow B$ a continuous homomorphism. For every family $(b_i)_{1 \leq i \leq p}$ of elements of B , there exists a unique continuous homomorphism*

$$\tilde{\rho} : A\{X_1, \dots, X_p\} \rightarrow B$$

such that $\tilde{\rho}(a) = f(a)$ for all $a \in A$ and $\tilde{\rho}(X_i) = b_i$ for $1 \leq i \leq p$.

Proof. There exists a unique homomorphism $v : A[X_1, \dots, X_p] \rightarrow B$ such that $v(a) = \rho(a)$ for $a \in A$ and $v(X_i) = b_i$, for $1 \leq i \leq p$. Moreover, if \mathfrak{b} is a neighbourhood of 0 in B which is an ideal, $\rho^{-1}(\mathfrak{b})$ is an ideal of A which is a neighbourhood of 0 and, for every polynomial $P \in N_\mathfrak{a}$, clearly $v(P) \in \mathfrak{b}$ and hence v is continuous. As $A[X_1, \dots, X_p]$ is dense in $A\{X_1, \dots, X_p\}$, the existence and uniqueness of $\tilde{\rho}$ follow from ?? and the principle of extension of identities. \square

Example 10.5.9. In the special case where $A = B$ and f is the identity map we shall write $P(b_1, \dots, b_p)$ for the value of $\tilde{f}(P)$ for every restricted formal power series $P \in A\{X_1, \dots, X_p\}$.

Now we turn to the most important application of restricted power series in this part. In a topological ring A , an element x is called **topologically nilpotent** if 0 is a limit of the sequence (x^n) . If A is a linearly topologized ring, to say that x is topologically nilpotent means that for every open ideal \mathfrak{a} of A the canonical image of x in A/\mathfrak{a} is a nilpotent element of that ring. If $\mathfrak{r}_\mathfrak{a}$ is the nilradical of A/\mathfrak{a} , clearly $(\mathfrak{r}_\mathfrak{a})$ is an inverse system of subsets and the set \mathfrak{t} of topological nilpotent elements of A is the inverse image of $\varprojlim_{\mathfrak{a}} \mathfrak{r}_\mathfrak{a}$ under the canonical homomorphism $A \rightarrow \varprojlim A/\mathfrak{a}$; it is therefore a closed ideal of A . If also A is Hausdorff and complete, this ideal is contained in the Jacobson radical of A and, for an element $x \in A$ to be invertible, it is necessary and sufficient that its class mod \mathfrak{t} be invertible in A/\mathfrak{t} (??). Note that if A is a ring and \mathfrak{a} an ideal of A , the elements of \mathfrak{a} are topologically nilpotent with respect to the \mathfrak{a} -adic topology.

Theorem 10.5.10 (Hensel's Lemma). *Let A be a complete Hausdorff linearly topologized ring. Let \mathfrak{m} be a closed ideal of A whose elements are topologically nilpotent. Let A/\mathfrak{m} be the quotient topological ring and $\pi : A \rightarrow A/\mathfrak{m}$ the canonical map. Let T be a restricted formal power series in $A\{X\}$, \bar{P} a monic polynomial in $(A/\mathfrak{m})[X]$ and \bar{Q} a restricted formal power series in $(A/\mathfrak{m})\{X\}$. Suppose that $\pi(T) = \bar{P}\bar{Q}$ and that \bar{P} and \bar{Q} are strongly relatively prime in $(A/\mathfrak{m})\{X\}$. Then there exists a unique*

ordered pair (P, Q) consisting of a monic polynomial $P \in A[X]$ and a restricted formal power series $Q \in A\{X\}$ such that

$$T = PQ, \quad \pi(P) = \bar{P}, \quad \pi(Q) = \bar{Q}. \quad (10.5.2)$$

Moreover, P and Q are strongly relatively prime in $A\{X\}$ and, if T is a polynomial, so is Q .

Proof. The proof is divided into several steps. In the first three we assume that A is discrete, in which case T and Q are *polynomials*. First assume that $\mathfrak{m}^2 = 0$. Let P_1, Q_1 be two polynomials of $A[X]$ such that P_1 is monic and $\pi(P_1) = \bar{P}$, $\pi(Q_1) = \bar{Q}$. ?? shows that P_1 and Q_1 are strongly relatively prime; hence ?? there exists a unique ordered pair of polynomials (U, V) of $A[X]$ such that

$$T - P_1 Q_1 = P_1 U + Q_1 V, \quad \deg(V) < \deg(P_1) = \deg(\bar{P}).$$

Applying π on this equation, we then see

$$\bar{P}\pi(U) + \bar{Q}\pi(V) = \pi(T) - \bar{P}\bar{Q} = 0.$$

As \bar{P} is monic, \bar{P} and \bar{Q} strongly relatively prime and $\deg(\pi(V)) < \deg(\bar{P})$, the uniqueness part of ?? implies that $\pi(U) = \pi(V) = 0$, in other words the coefficients of P_1 and Q_1 belong to \mathfrak{m} and the relation $\mathfrak{m}^2 = 0$ gives

$$T = P_1 Q_1 + P_1 U + Q_1 V = P_1 Q_1 + P_1 U + Q_1 V + UV = (P_1 + V)(Q_1 + U)$$

so the polynomials $P = P_1 + V$ and $Q = Q_1 + U$ are the solutions to the problem.

Now suppose that \mathfrak{m} is nilpotent and let n be the smallest integer such that $\mathfrak{m}^n = 0$ and let us argue by induction on $n > 2$, the theorem having been shown for $n = 2$. Let

$$A' = A/\mathfrak{m}^{n-1}, \quad \mathfrak{m}' = \mathfrak{m}/\mathfrak{m}^{n-1}$$

As $(\mathfrak{m}')^{n-1} = 0$, by induction hypothesis there exists a unique ordered pair (P', Q') of polynomials in $A'[X]$ such that P' is monic and

$$T' = P' Q', \quad \pi'(P') = \bar{P}, \quad \pi'(Q') = \bar{Q},$$

where $\pi' : A' \rightarrow A'/\mathfrak{m}' = A/\mathfrak{m}$ and $\theta : A \rightarrow A'$ denote the canonical homomorphisms, and $T' = \theta(T)$. On the other hand, as $(\mathfrak{m}')^2 = 0$, there exists a unique ordered pair (P, Q) of polynomials in $A[X]$ such that P is monic and

$$T = PQ, \quad \theta(P) = P', \quad \theta(Q) = Q'.$$

As $\pi = \pi' \circ \theta$, this shows the existence and uniqueness of P and Q satisfying ???. Moreover P' and Q' are strongly relatively prime by the induction hypothesis and hence so are P and Q .

Next we turn to the case where A is discrete. In this case \mathfrak{m} is no longer necessarily nilpotent, but it is always a nilideal (that is, every element in \mathfrak{m} is nilpotent), since \mathfrak{m} is topological nilpotent and $\{0\}$ is open in A . Let P_0, Q_0 be two polynomials of $A[X]$ such that $(P_0) = \bar{P}$, $\pi(Q_0) = \bar{Q}$ and P_0 is monic. Let us consider the ideal \mathfrak{n} of A generated by the coefficients of $T - P_0 Q_0$; it is finitely generated and contained in \mathfrak{m} , hence it is nilpotent and by definition,

if $\psi : A \rightarrow A/\mathfrak{n}$ is the canonical map, then $\psi(T) = \psi(P_0)\psi(Q_0)$. Moreover, $\psi(P_0)$ and $\psi(Q_0)$ are strongly relatively prime, as follows from the hypothesis on P and Q and ?? applied to the canonical homomorphism $A/\mathfrak{n} \rightarrow A/\mathfrak{m}$. By virtue of the nilpotent case, there therefore exists an ordered pair (P, Q) of polynomials in $A[X]$ such that P is monic and relations ?? hold. The fact that P and Q are strongly relatively prime implies also here that P and Q are strongly relatively prime in $A[X]$ by virtue of ??, for \mathfrak{m} is contained in the Jacobson radical of A . Suppose finally that P_1, Q_1 are two polynomials in $A[X]$ satisfying ?? and such that P_1 is monic and let \mathfrak{n}_1 be the finitely generated ideal of A generated by the coefficients of $P - P_1$ and the coefficients of $Q - Q_1$; as \mathfrak{n}_1 is contained in \mathfrak{m} , it is nilpotent and, if $\psi_1 : A \rightarrow A/\mathfrak{n}_1$ is the canonical map, then $\psi_1(P) = \psi_1(P_1)$ and $\psi_1(Q) = \psi_1(Q_1)$. The uniqueness property for nilpotent case therefore implies $P = P_1$ and $Q = Q_1$.

Finally, we deal with the general case. Let \mathcal{B} be a fundamental system of neighbourhoods of 0 in A consisting of ideals of A . For all $\mathfrak{a} \in \mathcal{B}$, let $\pi_{\mathfrak{a}}$ be the canonical map $A \rightarrow A/\mathfrak{a}$, $\phi_{\mathfrak{a}}$ the canonical map

$$A/\mathfrak{a} \rightarrow (A/\mathfrak{a})/((\mathfrak{m} + \mathfrak{a})/\mathfrak{a}) = A/(\mathfrak{m} + \mathfrak{a}),$$

and $\psi_{\mathfrak{a}}$ the canonical map $A/\mathfrak{m} \rightarrow A/(\mathfrak{m} + \mathfrak{a})$ and write $T_{\mathfrak{a}} = \pi_{\mathfrak{a}}(T)$, $\bar{P}_{\mathfrak{a}} = \psi_{\mathfrak{a}}(\bar{P})$, $\bar{Q}_{\mathfrak{a}} = \psi_{\mathfrak{a}}(\bar{Q})$. As each ring A/\mathfrak{a} is discrete, the preceding argument can be applied to it and we see that there exists a unique ordered pair $(P_{\mathfrak{a}}, Q_{\mathfrak{a}})$ of polynomials in $(A/\mathfrak{a})[X]$ such that $P_{\mathfrak{a}}$ is monic and $T_{\mathfrak{a}} = P_{\mathfrak{a}}Q_{\mathfrak{a}}$, $\phi_{\mathfrak{a}}(P_{\mathfrak{a}}) = \bar{P}_{\mathfrak{a}}$, $\phi_{\mathfrak{a}}(Q_{\mathfrak{a}}) = \bar{Q}_{\mathfrak{a}}$. The uniqueness of this ordered pair implies that, if $\mathfrak{a}, \mathfrak{b} \in \mathcal{B}$, $\mathfrak{b} \subseteq \mathfrak{a}$ and $\pi_{\mathfrak{ab}} : A/\mathfrak{b} \rightarrow A/\mathfrak{a}$ is the canonical map, then $P_{\mathfrak{a}} = \pi_{\mathfrak{ab}}(P_{\mathfrak{b}})$ and $Q_{\mathfrak{a}} = \pi_{\mathfrak{ab}}(Q_{\mathfrak{b}})$. Then it follows from the canonical identification of $A\{X\}$ with $\varprojlim_{\mathfrak{a}} (A/\mathfrak{a})[X]$ that there exists $P \in A\{X\}$ and $Q \in A\{X\}$ such that $T = PQ$ and $\pi_{\mathfrak{a}}(P) = P_{\mathfrak{a}}, \pi_{\mathfrak{a}}(Q) = Q_{\mathfrak{a}}$ for all $\mathfrak{a} \in \mathcal{B}$. Moreover

$$\psi_{\mathfrak{a}}(\bar{P} - \pi(P)) = \psi_{\mathfrak{a}}(\bar{Q} - \pi(Q)) = 0,$$

for all $\mathfrak{a} \in \mathcal{B}$, which means that for all $\mathfrak{a} \in \mathcal{B}$ the coefficients of $\bar{P} - \pi(P)$ and $\bar{Q} - \pi(Q)$ all belong to $(\mathfrak{m} + \mathfrak{a})/\mathfrak{m}$. But, as \mathfrak{m} is closed in A , $\cap_{\mathfrak{a}} (\mathfrak{m} + \mathfrak{a}) = \mathfrak{m}$, whence $\bar{P} = \pi(P)$, $\bar{Q} = \pi(Q)$ and P and Q certainly satisfy ?. Moreover, as the $P_{\mathfrak{a}}$ are monic and of the same degree, the restricted formal power series P is a monic polynomial. If (P', Q') were another ordered pair satisfying ?? and such that P' is a monic polynomial, we would deduce that

$$T_{\mathfrak{a}} = \pi_{\mathfrak{a}}(P')\pi_{\mathfrak{a}}(Q'), \quad \phi_{\mathfrak{a}}(\pi_{\mathfrak{a}}(P')) = \bar{P}_{\mathfrak{a}}, \quad \phi_{\mathfrak{a}}(\pi_{\mathfrak{a}}(Q')) = \bar{Q}_{\mathfrak{a}}.$$

By the uniqueness in the discrete case we have $\pi_{\mathfrak{a}}(P') = P_{\mathfrak{a}}$ and $\pi_{\mathfrak{a}}(Q') = Q_{\mathfrak{a}}$ for all $\mathfrak{a} \in \mathcal{B}$, which implies that $P = P'$ and $Q = Q'$. Let us show finally that P and Q are strongly relatively prime; by virtue of the discrete case and ??, for all $\mathfrak{a} \in \mathcal{B}$, there exists a unique ordered pair $(U_{\mathfrak{a}}, V_{\mathfrak{a}})$ of polynomials in $(A/\mathfrak{a})[X]$ such that

$$1 = P_{\mathfrak{a}}U_{\mathfrak{a}} + Q_{\mathfrak{a}}V_{\mathfrak{a}}, \quad \deg(V_{\mathfrak{a}}) < \deg(P_{\mathfrak{a}}) = \deg(\bar{P}).$$

The uniqueness of this ordered pair shows immediately that, for $\mathfrak{a}, \mathfrak{b} \in \mathcal{B}$ with $\mathfrak{b} \subseteq \mathfrak{a}$, $U_{\mathfrak{a}} = \pi_{\mathfrak{ab}}(U_{\mathfrak{b}})$ and $V_{\mathfrak{a}} = \pi_{\mathfrak{ab}}(V_{\mathfrak{b}})$. Taking account of ??, we conclude that there exist two restricted formal power series U, V of $A\{X\}$ such that $U_{\mathfrak{a}} = \pi_{\mathfrak{a}}(U)$, $V_{\mathfrak{a}} = \pi_{\mathfrak{a}}(V)$ and $1 = PU + QV$.

It remains to verify that, if T is a polynomial, so is Q . Now, the $Q_{\mathfrak{a}}$ are polynomials by

construction and, as P_{α} is monic, the relation $T_{\alpha} = P_{\alpha}Q_{\alpha}$ implies

$$\deg(Q_{\alpha}) \leq \deg(T_{\alpha}) \leq \deg(T)$$

for all $\alpha \in \mathcal{B}$, whence immediately the required result by definition of Q . \square

10.5.3 System of equations in complete rings

Let A be a ring; we shall say that a system

$$\mathbf{f} = (f_1, \dots, f_p) \in (A[[X_1, \dots, X_q]])^p$$

of formal power series in the X_1, \dots, X_q with coefficients in A , is **without constant term** if this is true of all the f_j . For every systems of formal power series

$$\mathbf{f} = (f_1, \dots, f_p) \in (A[[X_1, \dots, X_q]])^p, \quad \mathbf{g} = (g_1, \dots, g_q) \in (A[[X_1, \dots, X_r]])^q$$

such that \mathbf{g} is without constant term, we shall denote by $\mathbf{f} \circ \mathbf{g}$ (or $\mathbf{f}(\mathbf{g})$) the system of formal power series $f_j(g_1, \dots, g_q)$ in $(A[[X_1, \dots, X_r]])^p$. If

$$\mathbf{h} = (h_1, \dots, h_r) \in (A[[X_1, \dots, X_s]])^r$$

is a third system without constant term, then we have

$$(\mathbf{f} \circ \mathbf{g}) \circ \mathbf{h} = \mathbf{f} \circ (\mathbf{g} \circ \mathbf{h}).$$

For every system \mathbf{f} , we shall denote by $M_{\mathbf{f}}$ or $M_{\mathbf{f}}(X)$ the Jacobian matrix $(\partial f_i / \partial X_j)$, where i is the index of the rows and j that of the columns. For two systems \mathbf{f} and \mathbf{g} , where \mathbf{g} is without constant term, we have

$$M_{\mathbf{f} \circ \mathbf{g}} = M_{\mathbf{f}}(\mathbf{g}) \cdot M_{\mathbf{g}}. \tag{10.5.3}$$

where $M_{\mathbf{f}}(\mathbf{g})$ is the matrix whose elements are obtained by substituting g_j for X_j in each series element of $M_{\mathbf{f}}$. We shall denote by $M_{\mathbf{f}}(0)$ the matrix of constant terms of the elements of $M_{\mathbf{f}}$. Then we deduce from (??) that

$$M_{\mathbf{f} \circ \mathbf{g}}(0) = M_{\mathbf{f}}(0) \cdot M_{\mathbf{g}}(0).$$

Given an integer $n > 0$, we shall write

$$\mathbf{1}_n = \mathbf{X} = (X_1, \dots, X_n) \in (A[[X_1, \dots, X_n]])^n$$

which will be considered as a matrix with a single column.

For every system $\mathbf{f} = (f_1, \dots, f_n) \in (A[[X_1, \dots, X_n]])^n$, $M_{\mathbf{f}}$ is a square matrix of order n ; we shall denote by $J_{\mathbf{f}}$ or $J_{\mathbf{f}}(X)$ its determinant and by $J_{\mathbf{f}}(0)$ the constant term of $J_{\mathbf{f}}$, equal to $\det M_{\mathbf{f}}(0)$. If $\mathbf{g} = (g_1, \dots, g_n)$ is a system without constant term in $(A[[X_1, \dots, X_n]])^n$, then

$$J_{\mathbf{f} \circ \mathbf{g}} = J_{\mathbf{f}}(\mathbf{g}) J_{\mathbf{g}}, \quad J_{\mathbf{f} \circ \mathbf{g}}(0) = J_{\mathbf{f}}(0) J_{\mathbf{g}}(0).$$

Proposition 10.5.11. *Let A be a ring and $\mathbf{f} = (f_1, \dots, f_n)$ a system without constant term of n series in $A[[X_1, \dots, X_n]]$. Suppose that $J_{\mathbf{f}}(0)$ is invertible in A . Then there exists a system without constant term $\mathbf{g} = (g_1, \dots, g_n)$ of n series in $A[[X_1, \dots, X_n]]$ such that $\mathbf{f} \circ \mathbf{g} = \mathbf{1}_n$. This system is unique and $\mathbf{g} \circ \mathbf{f} = \mathbf{1}_n$.*

Proof. The existence and uniqueness of \mathbf{g} are clear. It then follows that $J_{\mathbf{f}}(0)J_{\mathbf{g}}(0) = 1$ and hence $J_{\mathbf{g}}(0)$ is also invertible. We conclude that there exists a system $\mathbf{h} = (h_1, \dots, h_n)$ of n series without constant term in $A[[X_1, \dots, X_n]]$ such that $\mathbf{g} \circ \mathbf{h} = \mathbf{1}_n$; from this relation it then follows that

$$\mathbf{h} = \mathbf{1}_n \circ \mathbf{h} = (\mathbf{f} \circ \mathbf{g}) \circ \mathbf{h} = \mathbf{f} \circ (\mathbf{g} \circ \mathbf{h}) = \mathbf{f} \circ \mathbf{1}_n = \mathbf{f}$$

so the second claim follows. \square

To abbreviate, we shall say in what follows that a ring satisfies **Hensel's conditions** if it is linearly topologized, Hausdorff and complete; given an ideal \mathfrak{m} in such a ring, \mathfrak{a} (or the ordered pair (A, \mathfrak{m})) will be said to satisfy Hensel's conditions if \mathfrak{m} is closed in A and its elements are topologically nilpotent. The ideal \mathfrak{t} of A consisting of all the topologically nilpotent elements satisfies Hensel's conditions.

In particular, if A is a ring and \mathfrak{m} an ideal of A and A is Hausdorff and complete with respect to the \mathfrak{m} -adic topology, the ordered pair (A, \mathfrak{m}) satisfies Hensel's conditions.

Proposition 10.5.12. *Let A be a commutative ring, B a ring satisfying Hensel's conditions and $\phi : A \rightarrow B$ a homomorphism. For every family $\mathbf{x} = (x_1, \dots, x_n)$ of topologically nilpotent elements of B , there exists a unique homomorphism $\tilde{\phi}$ from $A[[X_1, \dots, X_n]]$ to B such that $\tilde{\phi}(a) = \phi(a)$ for all $a \in A$ and $\tilde{\phi}(X_i) = x_i$ for $1 \leq i \leq n$. Moreover, if \mathfrak{m} denotes the ideal of series without constant term in $A[[X_1, \dots, X_n]]$, then $\tilde{\phi}$ is continuous for the \mathfrak{m} -adic topology.*

Proof. Let \mathfrak{a} be the finitely generated ideal generated in B by the x_i 's; for every open ideal \mathfrak{b} of B , the images of the x_i in B/\mathfrak{b} are nilpotent, hence the ideal $(\mathfrak{a} + \mathfrak{b})/\mathfrak{b}$ is nilpotent in B/\mathfrak{b} and there exists an integer k such that, for $\alpha \in \mathbb{N}^n$ with $|\alpha| \geq k$ we have $\mathbf{x}^\alpha \in \mathfrak{b}$. As every element of \mathfrak{m}^k is a finite sum of formal power series of the form $X^\alpha g(X)$, where $|\alpha| \geq k$, it is seen that, if $\tilde{\phi}$ solves the problem, then $\tilde{\phi}(\mathfrak{m}^k) \subseteq \mathfrak{b}$, which proves the continuity of $\tilde{\phi}$. There obviously exists a unique homomorphism

$$v : A[X_1, \dots, X_n] \rightarrow B$$

such that $v(a) = \phi(a)$ for $a \in A$ and $v(X_i) = x_i$, for $1 \leq i \leq n$ and the above argument shows that v is continuous with respect to the topology induced on $A[X_1, \dots, X_n]$ by the \mathfrak{m} -adic topology. As $A[X_1, \dots, X_n]$ is dense in $A[[X_1, \dots, X_n]]$ with the \mathfrak{m} -adic topology and B is Hausdorff and complete, this completes the proof of the existence and uniqueness of $\tilde{\phi}$. \square

If $B = A$ and ϕ is the identity map, we shall write $f(x_1, \dots, x_n)$ or $f(\mathbf{x})$ for the element $\tilde{\phi}(f)$ for every formal power series $f \in A[[X_1, \dots, X_n]]$. For every system $\mathbf{f} = (f_1, \dots, f_r)$ of formal power series of $A[[X_1, \dots, X_n]]$, let $\mathbf{f}(\mathbf{x})$ denote the element $(f_1(\mathbf{x}), \dots, f_r(\mathbf{x}))$ of A^r , then it is said to be obtained by substituting the x_i for the X_i in \mathbf{f} . If $n \leq m$ and F is a formal power series of $A[[X_1, \dots, X_m]]$, it is possible to consider F as a formal power series in X_{n+1}, \dots, X_m with coefficients in $A[[X_1, \dots, X_n]]$; let

$$F(x_1, \dots, x_n, X_{n+1}, \dots, X_m)$$

denote the formal power series in $A[\![X_{n+1}, \dots, X_m]\!]$ obtained by substituting the x_i for the X_i in the coefficients of F , for $1 \leq i \leq n$.

Corollary 10.5.13. *Let A be a ring satisfying Hensel's condition and $\mathbf{x} = (x_1, \dots, x_n)$ a family of topologically nilpotent elements of A . Let $\mathbf{g} = (g_1, \dots, g_q)$ be a system without constant term of series in $A[\![X_1, \dots, X_n]\!]$ and $\mathbf{f} = (f_1, \dots, f_p)$ a system of formal power series in $A[\![X_1, \dots, X_q]\!]$. Then $\mathbf{g}(\mathbf{x}) = (g_1(\mathbf{x}), \dots, g_q(\mathbf{x}))$ is a family of topologically nilpotent elements of A and*

$$(\mathbf{f} \circ \mathbf{g})(\mathbf{x}) = \mathbf{f}(\mathbf{g}(\mathbf{x})). \quad (10.5.4)$$

Proof. The fact that the $g_i(\mathbf{x})$ are topologically nilpotent follows immediately from ?? and the fact that in A the ideal of topologically nilpotent elements is closed. Relation (??) is obvious when the f, g are polynomials; on the other hand, if \mathfrak{m} and \mathfrak{m}' are the ideals of series without constant term in $A[\![X_1, \dots, X_q]\!]$ and $A[\![X_1, \dots, X_n]\!]$, respectively, clearly the relation $f \in \mathfrak{m}^k$ implies $f(g_1, \dots, g_q) \in (\mathfrak{m}')^k$. The two sides of (??) are therefore continuous functions of \mathbf{f} to $(A[\![X_1, \dots, X_q]\!])^p$ if $A[\![X_1, \dots, X_q]\!]$ is given the \mathfrak{m} -adic topology, by virtue of the above remark and ??, whence the claim. \square

In what follows, for a ring A and an ideal \mathfrak{m} of A we shall denote by $\mathfrak{m}^{\times n}$ the product set $\prod_{i=1}^n \mathfrak{m}_i$ in A^n , where $\mathfrak{m}_i = \mathfrak{m}$ for all i , to avoid ambiguity.

Proposition 10.5.14. *Let A be a ring and \mathfrak{m} an ideal of A such that the ordered pair (A, \mathfrak{m}) satisfies Hensel's conditions. Let $\mathbf{f} = (f_1, \dots, f_n)$ be a system without constant term of series in $A[\![X_1, \dots, X_n]\!]$ such that $J_{\mathbf{f}}(0)$ is invertible in A . Then, for all $\mathbf{x} \in \mathfrak{m}^{\times n}$, $\mathbf{f}(\mathbf{x}) \in \mathfrak{m}^{\times n}$ and $\mathbf{x} \mapsto \mathbf{f}(\mathbf{x})$ is a bijection of $\mathfrak{m}^{\times n}$ onto itself, the inverse bijection being $\mathbf{x} \mapsto \mathbf{g}(\mathbf{x})$, where \mathbf{g} is given by relation $\mathbf{f} \circ \mathbf{g} = \mathbf{1}_n$.*

Proof. The fact that $\mathbf{f}(\mathbf{x}) \in \mathfrak{m}^{\times n}$ is obvious when the f_i are polynomials and follows in the general case from ?? and the fact that \mathfrak{m} is closed in A . The other assertions of the proposition are then immediate consequences of ??). \square

Corollary 10.5.15. *With the assumption of ??, let \mathfrak{a} be a closed ideal of A contained in \mathfrak{m} . Then the relation $\mathbf{x} \equiv \mathbf{y} \pmod{\mathfrak{a}^{\times n}}$ is equivalent to $\mathbf{f}(\mathbf{x}) \equiv \mathbf{f}(\mathbf{y}) \pmod{\mathfrak{a}^{\times n}}$.*

Proof. For every formal power series $f \in A[\![X_1, \dots, X_n]\!]$,

$$f(X_1, \dots, X_n) - f(Y_1, \dots, Y_n) = \sum_{i=1}^n (X_i - Y_i) h_i(X_1, \dots, X_n, Y_1, \dots, Y_n)$$

where the h_i belong to $A[\![X_1, \dots, X_n, Y_1, \dots, Y_n]\!]$. It follows immediately that the relation $\mathbf{x} \equiv \mathbf{y} \pmod{\mathfrak{a}^{\times n}}$ implies $\mathbf{f}(\mathbf{x}) \equiv \mathbf{f}(\mathbf{y}) \pmod{\mathfrak{a}^{\times n}}$. The converse is obtained by replacing \mathbf{f} by its "inverse" \mathbf{g} . \square

Theorem 10.5.16. *Let A be a ring and \mathfrak{m} an ideal of A such that the ordered pair (A, \mathfrak{m}) satisfies Hensel's conditions. Let $\mathbf{f} = (f_1, \dots, f_n)$ be a system of n elements of $A\{X_1, \dots, X_n\}$, let $\mathbf{a} \in A^n$ and write $e = J_{\mathbf{f}}(\mathbf{a})$. There exists a system $\mathbf{g} = (g_1, \dots, g_n)$ of restricted formal power series without constant term in $A\{X_1, \dots, X_n\}$ such that*

$$(a) \quad M_{\mathbf{g}}(0) = \mathfrak{a}_n.$$

(b) For all $\mathbf{x} \in A^n$,

$$\mathbf{f}(\mathbf{a} + e\mathbf{x}) = \mathbf{f}(\mathbf{a}) + M_{\mathbf{f}}(\mathbf{a}) \cdot e\mathbf{g}(\mathbf{x}). \quad (10.5.5)$$

(c) Let $\mathbf{h} = (h_1, \dots, h_n)$ be the system of formal power series without constant term such that $\mathbf{g} \circ \mathbf{h} = \mathbf{a}_n$. For all $\mathbf{y} \in \mathfrak{m}^{\times n}$,

$$\mathbf{f}(\mathbf{a} + e\mathbf{h}(\mathbf{y})) = \mathbf{f}(\mathbf{a}) + M_{\mathbf{f}}(\mathbf{a}) \cdot e\mathbf{y}. \quad (10.5.6)$$

Proof. For every formal power series $f \in A[\![X_1, \dots, X_n]\!]$,

$$f(\mathbf{X} + \mathbf{Y}) = f(\mathbf{X}) + M_f(\mathbf{X}) \cdot \mathbf{Y} + \sum_{1 \leq i \leq j \leq n} G_{ij}(\mathbf{X}, \mathbf{Y}) Y_i Y_j \quad (10.5.7)$$

where the G_{ij} are well determined formal power series in $A[\![X_1, \dots, X_n, Y_1, \dots, Y_n]\!]$. If f is restricted, so are the elements of M_f , and the G_{ij} for these formal power series are polynomials if f is a polynomial and it follows from their uniqueness that for every open ideal \mathfrak{a} of A , denoting by $\pi_{\mathfrak{a}} : A \rightarrow A/\mathfrak{a}$ the canonical map, the image of G_{ij} under $\pi_{\mathfrak{a}}$ is the coefficient of $Y_i Y_j$ in $\pi_{\mathfrak{a}}(F)$ where F is the formal power series $f(\mathbf{X} + \mathbf{Y})$ in $A[\![X_1, \dots, X_n, Y_1, \dots, Y_n]\!]$; whence our assertion.

This being so, writing formula (??) for each series f_i , we obtain for all $x \in A^n$,

$$\mathbf{f}(\mathbf{a} + e\mathbf{x}) = \mathbf{f}(\mathbf{a}) + M_{\mathbf{f}}(\mathbf{a}) \cdot e\mathbf{x} + e^2 \mathbf{r}(\mathbf{x}) \quad (10.5.8)$$

where $\mathbf{r} = (r_1, \dots, r_n)$ is a system of restricted formal power series each of which is of total order ≥ 2 . Now there exists a square matrix $N \in \mathcal{M}_n(A)$ such that $M_{\mathbf{f}}(\mathbf{a})N = e\mathbf{a}_n$, whence using this in (??),

$$\mathbf{f}(\mathbf{a} + e\mathbf{x}) = \mathbf{f}(\mathbf{a}) + M_{\mathbf{f}}(\mathbf{a}) \cdot e\mathbf{x} + M_{\mathbf{f}}(\mathbf{a})N \cdot e\mathbf{r}(\mathbf{x})$$

Writing $\mathbf{g} = \mathbf{1}_n + N \cdot \mathbf{r}$, we see that \mathbf{g} satisfies conditions (a) and (b); then it is sufficient to replace \mathbf{x} by $\mathbf{h}(\mathbf{y})$ to obtain (c). \square

Corollary 10.5.17. *Let A be a ring and \mathfrak{m} an ideal of A such that the ordered pair (A, \mathfrak{m}) satisfies Hensel's conditions. Let $f \in A\{X\}$, $a \in A$ and write $e = f'(a)$. If $f(a) \equiv 0 \pmod{e^2\mathfrak{m}}$, then there exists $b \in A$ such that $f(b) = 0$ and $b \equiv a \pmod{e\mathfrak{m}}$. If b' is another element such that $f(b') = 0$ and $b' \equiv a \pmod{e\mathfrak{m}}$, then $e(b - b') = 0$. In particular, b is unique if e is not a divisor of zero in A .*

Proof. Let $f(a) = e^2 c$ where $c \in \mathfrak{m}$; formula (??) for $n = 1$ gives

$$f(a + eh(y)) = e^2(c + y)$$

and it is therefore sufficient to take $y = -c$, whence $b = a + eh(-y)$. Moreover if $b = a + ex$, $b' = a + ex'$ are such that $x, x' \in \mathfrak{m}$ and $f(b) = f(b') = 0$, then we deduce from (??) that $e^2(g(x) - g(x')) = 0$. As $g(X) - g(Y) = (X - Y)p(X, Y)$, where p is restricted and $p(0, 0) = 1$, we see $g(x) - g(x') = (x - x')v$ where $v \in A$. Note that since \mathfrak{m} is closed, p is restricted and 1 is the constant term of p , we have $p(x, x') - 1 \in \mathfrak{m}$, whence $v \in 1 + \mathfrak{m}$ and so v is invertible. This proves the relation $e(b - b') = 0$. \square

Example 10.5.18. Let p be a prime number $\neq 2$ and n an integer whose class mod p is a nonzero square in the prime field \mathbb{F}_p . If \mathbb{Z}_p is the ring of p -adic integers, the application of ?? to the polynomial $X^2 - n$ shows that n is a square in \mathbb{Z}_p ; for example 7 is a square in \mathbb{Z}_3 .

Example 10.5.19. Let $A = K[[Y]]$ be the ring of formal power series in one indeterminate with coefficients in a field K ; with the (Y) -adic topology, the ring A is Hausdorff and complete and the map $f(Y) \mapsto f(0)$ defines by passing to the quotient ring an isomorphism of κ_A onto the field K . By ??, if $F(Y, X)$ is a polynomial in X with coefficients in A and a is a simple root of $F(0, X)$ in K , there exists a unique formal power series $f(Y)$ such that $f(0) = a$ and $F(Y, f(Y)) = 0$.

Corollary 10.5.20. Let A be a ring and \mathfrak{m} an ideal of A such that the ordered pair (A, \mathfrak{m}) satisfies Hensel's conditions. Let r, n be integers such that $0 \leq r < n$ and $\mathbf{f} = (f_{r+1}, \dots, f_n)$ is a system of $n - r$ elements of $A\{X_1, \dots, X_n\}$. Let $J_{\mathbf{f}}^{(n-r)}$ denote the minor of $M_{\mathbf{f}}(X)$ consisting of the columns of index j such that $r + 1 \leq j \leq n$. Let $\mathbf{a} \in A^n$ be such that $J_{\mathbf{f}}(\mathbf{a})$ is invertible in A and $\mathbf{f}(\mathbf{a}) \equiv 0 \pmod{\mathfrak{m}^{\times(n-r)}}$. Then there exists a unique $\mathbf{x} = (x_1, \dots, x_n) \in A^n$ such that $x_j = a_j$ for $1 \leq j \leq r$ and $\mathbf{x} \equiv \mathbf{a} \pmod{\mathfrak{m}^{\times n}}$ and $\mathbf{f}(\mathbf{x}) = 0$.

Proof. Substituting a_j for X_j for $1 \leq j \leq r$ in the f_i , we see immediately that we may restrict our attention to the case where $r = 0$ to prove the corollary. ?? shows then that \mathbf{f} defines a bijection on $\mathbf{a} + \mathfrak{m}^{\times n}$ onto $\mathbf{f}(\mathbf{a}) + \mathfrak{m}^{\times n} = \mathfrak{m}^{\times n}$; the corollary follows from the fact that $0 \in \mathfrak{m}^{\times n}$. \square

10.6 Complements on complete rings

10.6.1 Admissible rings

Let A be a linearly topologized ring and \mathfrak{I} be an open ideal of A . We say that \mathfrak{I} is a nilideal if for any open neighborhood V of 0 in A , there exists an integer $n > 0$ such that $\mathfrak{I}^n \subseteq V$ (which means, by abusing language, that (\mathfrak{I}^n) tends to 0). The linearly topologized A is called **preadmissible** if there exists a nilideal of A , and A is **admissible** if it is preadmissible and is complete and separated.

It is clear that if \mathfrak{I} is a nilideal and \mathfrak{K} is an open ideal of A , then $\mathfrak{I} \cap \mathfrak{K}$ is also a nilideal. The nilideals of a preadmissible ring A then form a fundamental system of open ideals (but note that the power \mathfrak{I}^n of a nilideal is not necessarily open).

Lemma 10.6.1. Let A be a linearly topologized ring.

- (a) For an element $x \in A$ to be topologically nilpotent, it is necessary and sufficient that for any open ideal \mathfrak{I} of A , the canonical image of x in A/\mathfrak{I} is nilpotent. The set \mathfrak{T} of topological nilpotent elements of A is therefore an ideal.
- (b) Suppose that A is preadmissible and let \mathfrak{I} be a nilideal of A . For an element $x \in A$ to be topologically nilpotent, it is necessary and sufficient that the canonical image of x in A/\mathfrak{I} is nilpotent. The ideal \mathfrak{T} is then the inverse image of the nilradical of A/\mathfrak{I} in A , and hence open.

Proof. By definition, x is topologically nilpotent if and only if x^n is contained in \mathfrak{I} for sufficiently large n , whence the assertion of (a). As for assertion (b), it suffices to note that for any neighborhood V of 0 in A , there exists $n > 0$ such that $\mathfrak{I}^n \subseteq V$. If $x \in A$ is such that $x^m \in \mathfrak{I}$, then $x^{mq} \in V$ for $q \geq n$, so x is topologically nilpotent. \square

Proposition 10.6.2. *Let A be a preadmissible ring and \mathfrak{J} be a nilideal of A .*

- (a) *For an ideal \mathfrak{J} of A to be contained in a nilideal, it is necessary and sufficient that there exists an integer $n > 0$ such that $\mathfrak{J}^n \subseteq \mathfrak{J}$.*
- (b) *For an element $x \in A$ to be contained in a nilideal, it is necessary and sufficient that it is topologically nilpotent.*

Proof. If $\mathfrak{J}^n \subseteq \mathfrak{J}$, then $\mathfrak{J} + \mathfrak{J}$ is a nilideal, because it is open and $(\mathfrak{J} + \mathfrak{J})^n \subseteq \mathfrak{J}$; this proves assertion (a). For (b), the condition is evidently necessary, and it is sufficient because if this is satisfied, then there exists an integer $n > 0$ such that $x^n \in \mathfrak{J}$, so $\mathfrak{J} = \mathfrak{J} + Ax$ is a nilideal (it is open and $\mathfrak{J}^n \subseteq \mathfrak{J}$). \square

Corollary 10.6.3. *In a preadmissible ring A , an open prime ideal contains any nilideal.*

Proof. If $x \in \mathfrak{J}$ is an element of a nilideal and \mathfrak{p} is an open prime ideal, then x is topologically nilpotent by ??(b), so $x^n \in \mathfrak{p}$ for some integer $n > 0$, and hence $x \in \mathfrak{p}$. \square

Corollary 10.6.4. *Let A be a preadmissible ring. Then the following properties for an ideal \mathfrak{J}_0 of A are equivalent:*

- (i) \mathfrak{J}_0 is the largest nilideal of A ;
- (ii) \mathfrak{J}_0 is a maximal nilideal of A ;
- (iii) \mathfrak{J}_0 is a nilideal such that the ring A/\mathfrak{J}_0 is reduced.

For there exists an ideal \mathfrak{J}_0 satisfying these properties, it is necessary and sufficient that there exists a nilideal \mathfrak{J} such that the nilradical A/\mathfrak{J} is nilpotent. In this case \mathfrak{J}_0 is then equal to the ideal \mathfrak{T} of topologically nilpotent elements of A , and we denote by A_{red} the reduced quotient ring A/\mathfrak{T} .

Proof. It is clear that (i) implies (ii), and (iii) implies (i) in view of ??(b) and ??(b). On the other hand, if \mathfrak{J}_0 is maximal among nilideals, then for any nilideal \mathfrak{J} , there exists an integer $n > 0$ such that $\mathfrak{J}_0^n \subseteq \mathfrak{J}$, so $(\mathfrak{J}_0 + \mathfrak{J})$ is a nilideal of A . By the maximality, we then conclude that $\mathfrak{J}_0 + \mathfrak{J} = \mathfrak{J}_0$, so $\mathfrak{J} \subseteq \mathfrak{J}_0$ and \mathfrak{J}_0 is the largest nilideal. The last assertion follows from ??(a) and ??(b). \square

Corollary 10.6.5. *A Noetherian preadmissible ring admits a largest nilideal.*

Proof. If A is a Noetherian preadmissible ring, then A/\mathfrak{J} is also Noetherian, so its nilradical is nilpotent. \square

We note that any defining ideal \mathfrak{J} of A is necessarily a nilideal, but the converse is not true. In view of this, we say that a ring A is **preadic** if there exists a nilideal \mathfrak{J} of A such that \mathfrak{J}^n form a fundamental system of open neighborhoods of 0 in A (which means each \mathfrak{J}^n is open, or that \mathfrak{J} is a defining ideal of A). The ring A is **adic** if it is preadic and separated and complete. If \mathfrak{J} is a nilideal of a preadic ring A with defining ideal \mathfrak{J} , then it is easy to see that each \mathfrak{J}^n is open in A , hence also a defining ideal of A . In other words, if A is a preadic ring, then any nilideal defines the topology of A , hence is a defining ideal of A .

Proposition 10.6.6. *Let A be a admissible ring and \mathfrak{J} be a nilideal of A . Then \mathfrak{J} is contained in the Jacobson radical of A . In particular, if A is adic, then it is a Zariski ring.*

Proof. In fact, since A is separated and complete, the conclusion follows from ??(b) as any element of \mathfrak{I} is topologically nilpotent. \square

We shall now provide a characterization for admissible rings via projective limits of discrete rings. For this, let us recall that if A is a linearly topologized ring and (\mathfrak{I}_λ) is a fundamental system of open ideals of A , then the canonical homomorphisms $\varphi_\lambda : A \rightarrow A/\mathfrak{I}_\lambda$ form a projective system of discrete rings, hence define a continuous homomorphism $\varphi : A \rightarrow \varprojlim A/\mathfrak{I}_\lambda$ (the latter can be identified with the completion of A). This homomorphism is injective if and only if A is separated (and hence identified A with a dense subring of $\varprojlim A/\mathfrak{I}_\lambda$), and is an isomorphism if A is separated and complete.

Proposition 10.6.7. *For a linearly topologized ring A to be admissible, it is necessary and sufficient that it is isomorphic to a projective limit $A = \varprojlim A_\lambda$, where $(A_\lambda, u_{\lambda\mu})$ is a projective system of discrete rings and the index set I has a smallest element 0 and satisfies the following conditions:*

- (a) *the homomorphisms $u_\lambda : A \rightarrow A_\lambda$ are surjective;*
- (b) *the kernel \mathfrak{I}_λ of $u_{0,\lambda} : A_\lambda \rightarrow A_0$ is nilpotent.*

If these are satisfied, the kernel \mathfrak{I} of $u_0 : A \rightarrow A_0$ is equal to $\varprojlim \mathfrak{I}_\lambda$.

Proof. This condition is necessary in view of the above remarks, since we can choose a fundamental system (\mathfrak{I}_λ) of neighborhoods of 0 formed by nilideals contained in a fixed nilideal \mathfrak{I}_0 , and note that $\mathfrak{I}_0^n \subseteq \mathfrak{I}_\lambda$ for sufficiently large n . The converse of this follows from the definition of projective limits, and the last assertion is immediate. \square

Let A be an admissible ring and \mathfrak{I} be an ideal of A contained in a nilideal (which means (\mathfrak{I}^n) tends to 0). We can then consider the \mathfrak{I} -adic topology on A , and the hypothesis on A implies that $\bigcap_n \mathfrak{I}^n = 0$, so the \mathfrak{I} -adic topology on A is separated.

Proposition 10.6.8. *If A is an admissible ring and \mathfrak{I} be an ideal of A contained in a nilideal such that the \mathfrak{I}^n are closed in A , then A is separated and complete for the \mathfrak{I} -adic topology.*

Proof. This follows from ??, since the \mathfrak{I} -adic topology is finer than the original topology on A . \square

Corollary 10.6.9. *Let A be an admissible ring, \mathfrak{I} be a nilideal of A such that the \mathfrak{I}^n are closed in A . For A to be Noetherian, it is necessary and sufficient that A/\mathfrak{I} is Noetherian and that $\mathfrak{I}/\mathfrak{I}^2$ is a finitely generated (A/\mathfrak{I}) -module.*

Proof. These conditions are clearly necessary. Conversely, if they are satisfied, then since $\text{gr}(A)$ is generated by $\text{gr}_1(A) = \mathfrak{I}/\mathfrak{I}^2$ over $\text{gr}_0(A) = A/\mathfrak{I}$, it is Noetherian by Hilbert basis theorem, and the conclusion follows from ??.

If A is an adic ring and \mathfrak{I} is a defining ideal of A , then A is the projective limit of the discrete rings $A_i = A/\mathfrak{I}^{i+1}$, which is identified with the \mathfrak{I} -adic completion of A . Conversely, we can in fact use this to characterize adic rings such that there exists a defining ideal \mathfrak{I} such that $\mathfrak{I}/\mathfrak{I}^2$ is finitely generated, as the following proposition shows:

Proposition 10.6.10. *Let A be a ring, \mathfrak{I} be an ideal of A such that $\mathfrak{I}/\mathfrak{I}^2$ is a finitely generated A/\mathfrak{I} -module.*

- (a) The ring $\widehat{A} = \varprojlim(A/\mathfrak{J}^{n+1})$ is an adic ring. If $\widehat{\mathfrak{J}}$ is the closure of the canonical image of \mathfrak{J} in A , then $\widehat{\mathfrak{J}}$ is a finitely generated nilideal of \widehat{A} , and $\widehat{\mathfrak{J}}^n$ is the closure of the canonical image of \mathfrak{J}^n in A . Moreover, $\widehat{A}/\widehat{\mathfrak{J}}^n$ is isomorphic to A/\mathfrak{J}^n and $\widehat{\mathfrak{J}}/\widehat{\mathfrak{J}}^2$ to $\mathfrak{J}/\mathfrak{J}^2$, and $\widehat{\mathfrak{J}}$ is isomorphic to $\varprojlim(\mathfrak{J}/\mathfrak{J}^{n+1})$.
- (b) Let M be an A -module such that $M/\mathfrak{J}M$ is a finitely generated A/\mathfrak{J} -module. Then $\widehat{M} = \varprojlim(M/\mathfrak{J}^{n+1}M)$ is a finitely generated A -module, $\widehat{\mathfrak{J}}^n \widehat{M}$ is the closure of the canonical image of $\mathfrak{J}^n M$, and $\widehat{M}/\widehat{\mathfrak{J}}^n \widehat{M}$ is isomorphic to $M/\mathfrak{J}^n M$.

Proof. It is clear that the conditions of ?? are satisfied. \square

10.6.2 Complete localizations

Let A be a linearly topologized ring, (\mathfrak{J}_λ) be a fundamental system of neighborhoods of 0 formed by ideals of A , and S be a multiplicative subset of A . Let $u_\lambda : A \rightarrow A_\lambda = A/\mathfrak{J}_\lambda$ be the canonical homomorphism, and for $\mathfrak{J}_\mu \subseteq \mathfrak{J}_\lambda$, let $u_{\lambda\mu} : A_\mu \rightarrow A_\lambda$ be the canonical homomorphism. Put $S_\lambda = u_\lambda(S)$, so that $u_{\lambda\mu}(S_\mu) = S_\lambda$. The homomorphisms $u_{\lambda\mu}$ then induce canonical surjections $S_\mu^{-1}A_\mu \rightarrow S_\lambda^{-1}A_\lambda$, which form a projective system and we denote by $A\{S^{-1}\}$ its projective limit. It is clear that this does not depend on the choice of the system (\mathfrak{J}_λ) .

Proposition 10.6.11. *The ring $A\{S^{-1}\}$ is homeomorphic to the completion of the localization $S^{-1}A$ for the topology induced by the closed ideals $S^{-1}\mathfrak{J}_\lambda$.*

Proof. In fact, if $v_\lambda : S^{-1}A \rightarrow S_\lambda^{-1}A_\lambda$ is the canonical homomorphism induced by u_λ , the kernel of v_λ is $S^{-1}\mathfrak{J}_\lambda$ and v_λ is surjective, and the completion of $S^{-1}A$ is identified with the limit $A\{S^{-1}\}$. \square

Corollary 10.6.12. *If \widehat{S} is the image of S in the completion \widehat{A} of A , then $A\{S^{-1}\}$ is canonically identified with $\widehat{A}\{\widehat{S}^{-1}\}$.*

Remark 10.6.1. Note that even if A is separated and complete, the localization $S^{-1}A$ many be neither separated nor complete. For example, let S be the subset f^n for $n \geq 0$, where f is a topologically nilpotent element in A but not nilpotent. Then $S^{-1}A$ is nonzero, but for any λ there exists an integer n such that $f^n \in \mathfrak{J}_\lambda$, so $1 = f^n/f^n \in S^{-1}\mathfrak{J}_\lambda$ and $S^{-1}\mathfrak{J}_\lambda = S^{-1}A$.

Corollary 10.6.13. *If 0 is not contained in the closure of S , then the ring $A\{S^{-1}\}$ is nonzero.*

Proof. In fact, 0 is not contained in the closure of $\{1\}$ in the ring $S^{-1}A$, since otherwise $1 \in S^{-1}\mathfrak{J}_\lambda$ for any ideal \mathfrak{J}_λ , whence $\mathfrak{J}_\lambda \cap S \neq \emptyset$, a contradiction. \square

The ring $A\{S^{-1}\}$ is called the **complete localization** of the ring A . It is clear that the inverse image of $S^{-1}\mathfrak{J}_\lambda$ in A contains \mathfrak{J}_λ , so the canonical homomorphism $A \rightarrow S^{-1}A$ is continuous. By composing with the canonical homomorphism $A \rightarrow A\{S^{-1}\}$, we then obtain a canonical continuous homomorphism $A \rightarrow A\{S^{-1}\}$, which is the projective limit of the homomorphisms $A \rightarrow S_\lambda^{-1}A_\lambda$. The ring $A\{S^{-1}\}$ satisfies a similar universal property as the localization rings:

Proposition 10.6.14. *Any continuous homomorphism u from A into a separated and complete linearly topologized ring B such that $u(S)$ is invertible in B factors uniquely through $A\{S^{-1}\}$:*

$$A \longrightarrow A\{S^{-1}\} \xrightarrow{\tilde{u}} B$$

where \tilde{u} is continuous.

Proof. In fact, u factors through $S^{-1}A$:

$$A \longrightarrow S^{-1}A \xrightarrow{\tilde{v}} B$$

Now since for any open ideal \mathfrak{K} of B , $u^{-1}(\mathfrak{K})$ contains an ideal \mathfrak{I}_λ , we see that $\tilde{v}^{-1}(\mathfrak{K})$ contains $S^{-1}\mathfrak{I}_\lambda$, so \tilde{v} is continuous. Since B is complete and separated, the homomorphism \tilde{v} then factors into the following

$$S^{-1}A \longrightarrow A\{S^{-1}\} \xrightarrow{\tilde{u}} B$$

where \tilde{u} is continuous; this proves the proposition. \square

Let B be a second linearly topologized ring, T be a multiplicative subset of B , and $\varphi : A \rightarrow B$ be a continuous homomorphism such that $\varphi(S) \subseteq T$. Then by ??, the continuous homomorphism

$$A \xrightarrow{\varphi} B \longrightarrow B\{T^{-1}\}$$

then factors into

$$A \longrightarrow A\{S^{-1}\} \xrightarrow{\tilde{\varphi}} B\{T^{-1}\}$$

where $\tilde{\varphi}$ is continuous. In particular, if $A = B$ and φ is the identity, we see that for any multiplicative subsets $S \subseteq T$ we have a continuous homomorphism $\rho^{T,S} : A\{S^{-1}\} \rightarrow A\{T^{-1}\}$, obtained by passing to completions from the canonical homomorphism $S^{-1}A \rightarrow T^{-1}A$. If U is a third multiplicative subset of A such that $S \subseteq T \subseteq U$, we have $\rho^{U,S} = \rho^{U,T} \circ \rho^{T,S}$.

Let S_1 and S_2 be multiplicative subsets of A , and \tilde{S}_2 be the canonical image of S_2 in $A\{S_1^{-1}\}$. We then have a canonical isomorphism $A\{(S_1S_2)^{-1}\} \cong A\{S_1^{-1}\}\{\tilde{S}_2^{-1}\}$, which comes from the canonical isomorphism $(S_1S_2)^{-1}A \cong \tilde{S}_2^{-1}(S_1^{-1}A)$, where \tilde{S}_2 is the canonical image of S_2 in $S_1^{-1}A$. Let \mathfrak{a} be an open ideal of A . We may suppose that $\mathfrak{I}_\lambda \subseteq \mathfrak{a}$ for any λ , and therefore $S^{-1}\mathfrak{I}_\lambda \subseteq S^{-1}\mathfrak{a}$ in the ring $S^{-1}A$, which means $S^{-1}\mathfrak{a}$ is an open ideal of $S^{-1}A$. We denote by $\mathfrak{a}\{S^{-1}\}$ the completion of $S^{-1}\mathfrak{a}$, which is identified with $\varprojlim(S^{-1}\mathfrak{a}/S^{-1}\mathfrak{I}_\lambda)$ and is an open ideal of $A\{S^{-1}\}$. Moreover, the discrete ring $A\{S^{-1}\}/\mathfrak{a}\{S^{-1}\}$ is canonically isomorphic to $S^{-1}A/\mathfrak{a} = S^{-1}(A/\mathfrak{a})$.

Proposition 10.6.15. *Let A be a linearly topologized ring and S be a multiplicative subset.*

- (a) *Any open ideal of $A\{S^{-1}\}$ is of the form $\mathfrak{a}\{S^{-1}\}$, where \mathfrak{a} is an open ideal of A .*
- (b) *The map $\mathfrak{p} \mapsto \mathfrak{p}\{S^{-1}\}$ is an increasing bijection from the set of open prime ideals \mathfrak{p} in A such that $\mathfrak{p} \cap S = \emptyset$ to the set of open prime ideals of $A\{S^{-1}\}$. Moreover, the fraction field of $A\{S^{-1}\}/\mathfrak{p}\{S^{-1}\}$ is canonically isomorphic to A/\mathfrak{p} .*

Proof. If \mathfrak{a}' is an open ideal of $A\{S^{-1}\}$, then \mathfrak{a}' contains an ideal of the form $\mathfrak{I}_\lambda\{S^{-1}\}$, hence is the inverse image of an ideal of $S^{-1}A/S^{-1}\mathfrak{I}_\lambda$, which is necessarily of the form $S^{-1}\mathfrak{a}$, where $\mathfrak{a} \supseteq \mathfrak{I}_\lambda$. The second assertion follows from ?? and the fact that taking completion does not change residue fields. \square

Proposition 10.6.16. *Let A be a linearly topologized ring and S be a multiplicative subset of A .*

- (a) *If A is admissible, so is $A' = A\{S^{-1}\}$, and for any nilideal \mathfrak{J} of A , $\mathfrak{J}' = \mathfrak{J}\{S^{-1}\}$ is a nilideal of $A\{S^{-1}\}$.*

(b) Suppose that A is adic and \mathfrak{I} is a defining ideal of A such that $\mathfrak{I}/\mathfrak{I}^2$ is finitely generated over A/\mathfrak{I} . Then A' is an \mathfrak{I}' -adic ring and $\mathfrak{I}'/\mathfrak{I}'^2$ is finitely generated over A'/\mathfrak{I}' .

Proof. If \mathfrak{I} is a nilideal of A , then it is clear that $S^{-1}\mathfrak{I}$ is nilideal of $S^{-1}A$, because $(S^{-1}\mathfrak{I})^n = S^{-1}\mathfrak{I}^n$ for each n . Now let \bar{A} be the separated ring associated with $S^{-1}A$, and $\bar{\mathfrak{I}}$ be the image of $S^{-1}\mathfrak{I}$ in \bar{A} . Then the image of $S^{-1}\mathfrak{I}^n$ is $\bar{\mathfrak{I}}^n$, so $\bar{\mathfrak{I}}^n$ tends to 0 in \bar{A} . As \mathfrak{I}' is the closure of $\bar{\mathfrak{I}}$ in A' , \mathfrak{I}'^n is contained in the closure of $\bar{\mathfrak{I}}^n$, hence tends to 0 in A' .

For (b), put $A_i = A/\mathfrak{I}^{i+1}$, and let $u_{ij} : A_j \rightarrow A_i$ be the canonical homomorphism for $i \leq j$. Let S_i be the image of S in A_i , put $A'_i = S_i^{-1}A_i$, and let $u'_{ij} : A'_j \rightarrow A'_i$ be the induced homomorphism. We show that the projective system (A'_i, u'_{ij}) satisfies the conditions of ???. It is clear that each u'_{ij} is surjective, and by the flatness of localization, the kernel of u'_{ij} is equal to $S_j^{-1}(\mathfrak{I}^{i+1}/\mathfrak{I}^{j+1}) = \mathfrak{I}_j'^{i+1}$, where $\mathfrak{I}_j' = S_j^{-1}(\mathfrak{I}/\mathfrak{I}^{i+1})$. Finally, $\mathfrak{I}_1'/\mathfrak{I}_1'^2 = S_1^{-1}(\mathfrak{I}/\mathfrak{I}^2)$ is finitely generated over A'_1 since $\mathfrak{I}/\mathfrak{I}^2$ is finitely generated over A/\mathfrak{I} , and $A'_0 = S_0^{-1}(A/\mathfrak{I})$ is Noetherian if A is Noetherian. \square

Corollary 10.6.17. Under the hypothesis of ??(b), we have $(\mathfrak{I}\{S^{-1}\})^n = \mathfrak{I}^n\{S^{-1}\}$.

Proof. This is already proved in the proof of ??, in view of ??.

Proposition 10.6.18. Let A be an adic Noetherian ring and S be a multiplicative subset of A . Then $A\{S^{-1}\}$ is a flat A -module.

Proof. If \mathfrak{I} is a defining ideal of A , $A\{S^{-1}\}$ is the completion of the Noetherian ring $S^{-1}A$ for the $S^{-1}\mathfrak{I}$ -adic topology, so $A\{S^{-1}\}$ is a flat $S^{-1}A$ -module (??). As $S^{-1}A$ is flat over A , we conclude the proposition. \square

Corollary 10.6.19. Let A be an adic Noetherian ring and $T \subseteq S$ be multiplicative subsets of A . Then $A\{S^{-1}\}$ is a flat $A\{T^{-1}\}$ -module.

Proof. We have remarked that $A\{S^{-1}\}$ is canonically identified with $A\{T^{-1}\}\{S_0^{-1}\}$, where S_0 is the canonical image of S in $A\{T^{-1}\}$, and $A\{T^{-1}\}$ is Noetherian by ??.

For any element $f \in A$, we denote by $A_{\{f\}}$ the complete localization $A\{S_f^{-1}\}$, where $S_f = \{f^n\}$. For any open ideal \mathfrak{a} of A , we denote by $\mathfrak{a}_{\{f\}}$ the ideal $\mathfrak{a}\{S_f^{-1}\}$. If g is another element of A , we then have a continuous homomorphism $A_{\{f\}} \rightarrow A_{\{fg\}}$. As f runs through a multiplicative subset S of A , the rings $A_{\{f\}}$ then form an inductive system, and we set $A_{\{S\}} = \varinjlim_{f \in S} A_{\{f\}}$. For any $f \in S$, we have a canonical homomorphism $A_{\{f\}} \rightarrow A\{S^{-1}\}$, and they form an inductive system which induces a canonical homomorphism $A_{\{S\}} \rightarrow A\{S^{-1}\}$.

Proposition 10.6.20. If A is a Noetherian ring, $A\{S^{-1}\}$ is a flat module over $A_{\{S\}}$.

Proof. By ?? the ring $A\{S^{-1}\}$ is flat over each $A_{\{f\}}$ for $f \in S$, so the conclusion follows from ??.

Proposition 10.6.21. Let \mathfrak{p} be an open prime ideal of an admissible ring A , and let $S = A - \mathfrak{p}$. Then the ring $A\{S^{-1}\}$ and $A_{\{S\}}$ are local, the canonical homomorphism $A_{\{S\}} \rightarrow A\{S^{-1}\}$ is local, and the residue fields of $A_{\{S\}}$ and $A\{S^{-1}\}$ are isomorphic to $\kappa(\mathfrak{p})$.

Proof. Let $\mathfrak{I} \subseteq \mathfrak{p}$ be a nilideal of A . We then have $S^{-1}\mathfrak{I} \subseteq S^{-1}\mathfrak{p} = \mathfrak{p}A_{\mathfrak{p}}$, so $A_{\mathfrak{p}}/S^{-1}\mathfrak{I}$ is a local ring, and we conclude from ??(a) that $A\{S^{-1}\}$ is a local ring. Put $\mathfrak{m} = \varinjlim_{f \in S} \mathfrak{p}_{\{f\}}$, which is an ideal of $A_{\{S\}}$, we then see that any element of $A_{\{S\}} - \mathfrak{m}$ is invertible, so $A_{\{S\}}$ is a local ring with maximal ideal \mathfrak{m} . In fact, any such element is the image of an element $z \in A_{\{f\}} = \mathfrak{p}_{\{f\}}$ in $A_{\{S\}}$, for an element $f \in S$. Its canonical image z_0 in $A_{\{f\}}/\mathfrak{I}_{\{f\}} = S_f^{-1}(A/\mathfrak{I})$ is then not contained in $S_f^{-1}(\mathfrak{p}/\mathfrak{I})$, which means $z_0 = \bar{x}/\bar{f}^k$, where $x \notin \mathfrak{p}$ and \bar{x}, \bar{f} are the classes of $x, f \bmod \mathfrak{I}$. As $x \in S$, we have $g = xf \in S$, so $y_0 = x^{k+1}/g^k$, the canonical image of $x/f^k \in S_f^{-1}A$ in the ring $S_g^{-1}A$, then admits an inverse $x^{k-1}f^{2k}/g^k$. This implies a fortiori that the image of y_0 in $S_g^{-1}A/S_g^{-1}\mathfrak{I}$ is invertible, so (??) the canonical image y of z in $A_{\{g\}}$ is invertible, and so is its image in $A_{\{S\}}$. Now, the image of $\mathfrak{p}_{\{f\}}$ in $A\{S^{-1}\}$ is contained in the maximal ideal $\mathfrak{p}\{S^{-1}\}$ of this ring, so the image of \mathfrak{m} in $A\{S^{-1}\}$ is contained in $\mathfrak{p}\{S^{-1}\}$, which means the homomorphism $A_{\{S\}} \rightarrow A\{S^{-1}\}$ is local. Finally, as any element $A\{S^{-1}\}/\mathfrak{p}\{S^{-1}\}$ is the image of an element of $S_f^{-1}A$ for some $f \in S$, the homomorphism $A_{\{S\}} \rightarrow A\{S^{-1}\}/\mathfrak{p}\{S^{-1}\}$ is surjective, and hence induces an isomorphism on residue fields. This residue field is isomorphic to $\kappa(\mathfrak{p})$ since taking completion does not change residue fields. \square

Corollary 10.6.22. *Under the hypotheses of ??, if we suppose that A is an adic Noetherian ring, the local rings $A\{S^{-1}\}$ and $A_{\{S\}}$ are Noetherian, and $A\{S^{-1}\}$ is a faithfully flat $A_{\{S\}}$ -module.*

Proof. By ??(b) and ??, the ring $A\{S^{-1}\}$ is Noetherian and flat over $A_{\{S\}}$. As the homomorphism $A_{\{S\}} \rightarrow A\{S^{-1}\}$ is local, we then conclude that $A\{S^{-1}\}$ is faithfully flat over $A_{\{S\}}$, and hence Noetherian (??). \square

10.6.3 Complete tensor products

Let A be a linearly topologized ring and M, N be two linearly topologized A -modules. Let \mathfrak{I}, V, W be neighborhoods of 0 in A, M, N , respectively, which are A -modules and such that $\mathfrak{I} \cdot M \subseteq V, \mathfrak{I} \cdot N \subseteq W$. Then the quotient modules M/V and N/W can be considered as (A/\mathfrak{I}) -modules. If \mathfrak{I}, V, W runs through the systems of open neighborhoods satisfying the preceding conditions, it is immediate that the modules $(M/V) \otimes_{A/\mathfrak{I}} (N/W)$ form a projective system of modules over the projective system A/\mathfrak{I} , so by passing to limit we obtain a module over the completion \widehat{A} of A , which is called the complete tensor product of M and N , and denoted by $M \hat{\otimes}_A N$. Note that since M/V is canonically identified with \widehat{M}/\widehat{V} , where \widehat{V} is the closure of V in \widehat{M} , the complete tensor product is canonically identified with $\widehat{M} \hat{\otimes}_{\widehat{A}} \widehat{N}$.

We also note that the tensor products $(M/V) \otimes_A (N/W)$ and $(M/V) \otimes_{A/\mathfrak{I}} (N/W)$ are both canonically identified with the quotient $(M \otimes_A N)/(\text{im}(V \otimes_A N) + \text{im}(M \otimes_A W))$, so the complete tensor product $M \hat{\otimes}_A N$ is the completion of the A -module $M \otimes_A N$ endowed with the topology induced by the submodules $\text{im}(V \otimes_A N) + \text{im}(M \otimes_A W)$. For simplicity, we say that this topology is the **tensor product** of the given topologies on M and N .

Let $u : M \rightarrow M'$ and $v : N \rightarrow N'$ be continuous homomorphisms of linearly topologized A -modules. It is immediate that $u \otimes v$ is continuous for the tensor product topologies on $M \otimes_A N$ and $M' \otimes_A N'$, respectively, so by passing to completion, we deduce a continuous homomorphism $M' \hat{\otimes}_A N' \rightarrow M \hat{\otimes}_A N$, which we denote by $u \hat{\otimes} v$. In this way, the construction $(M, N) \mapsto M \hat{\otimes}_A N$ is then a bifunctor on the category of linearly topologized A -modules. Moreover, in the same way we can define the complete tensor product for finitely many lin-

early topologized A -modules, and it is clear that this construction satisfies the commutativity and associativity.

Let B and C be linearly topologized A -algebras. Then the tensor product of the topologies on B and C is a linear topology on $B \otimes_A C$, so $B \hat{\otimes}_A C$ is endowed with a topological \widehat{A} -algebra structure. This algebra is called the complete tensor product of the algebras B and C .

The homomorphisms $b \mapsto b \otimes 1$ and $c \mapsto 1 \otimes c$ of B and C into $B \otimes_A C$ are continuous for the tensor product topology, so by composing with the canonical homomorphism from $B \otimes_A C$ to its completion, we obtain canonical homomorphisms $\rho : B \rightarrow B \hat{\otimes}_A C$ and $\sigma : C \rightarrow B \hat{\otimes}_A C$. The algebra $B \hat{\otimes}_A C$ and the homomorphisms ρ and σ then satisfies the following universal property:

Proposition 10.6.23. *For any complete and separated linearly topologized A -algebra D and any couple of continuous A -homomorphisms $u : B \rightarrow D$, $v : C \rightarrow D$, there exists a continuous A -homomorphism $w : B \hat{\otimes}_A C \rightarrow D$ such that the following diagram is commutative*

$$\begin{array}{ccccc} A & \longrightarrow & B & & \\ \downarrow & & \downarrow \rho & & \\ C & \xrightarrow{\sigma} & B \hat{\otimes}_A C & \xrightarrow{u} & D \\ & \searrow v & \swarrow w & & \\ & & D & & \end{array}$$

Proof. In fact, there exists a unique homomorphism $w_0 : B \otimes_A C \rightarrow D$ such that $u(b) = w_0(b \otimes 1)$ and $v(c) = w_0(1 \otimes c)$, and it suffices to prove that w_0 is continuous, since it then induces a continuous homomorphism w by passing to completion. Now if \mathfrak{M} is an open ideal of D , there exists by hypothesis open ideals $\mathfrak{K} \subseteq B$, $\mathfrak{R} \subseteq C$ such that $u(\mathfrak{K}) \subseteq \mathfrak{M}$, $v(\mathfrak{R}) \subseteq \mathfrak{M}$, so the image of $\text{im}(\mathfrak{K} \otimes C) + \text{im}(B \otimes \mathfrak{R})$ under w_0 is contained in \mathfrak{M} , whence the assertion. \square

Proposition 10.6.24. *If B and C are two preadmissible A -algebras, $B \hat{\otimes}_A C$ is admissible, and if \mathfrak{K} (resp. \mathfrak{R}) is a nilideal of B (resp. C), then the closure of $\mathfrak{L} = \text{im}(\mathfrak{K} \otimes C) + \text{im}(B \otimes \mathfrak{R})$ in $B \hat{\otimes}_A C$ is a nilideal.*

Proof. It suffices to show that \mathfrak{L}^n tends to 0 in the tensor product topology, and this follows immediately from the inclusion $\mathfrak{L}^{2n} \subseteq \text{im}(\mathfrak{K}^n \otimes C) + \text{im}(B \otimes \mathfrak{R}^n)$. \square

Proposition 10.6.25. *Let B, C be linearly topologized A -algebras, M be a linearly topologized B -module, whose topology is finer than the topology induced by B . Then the canonical isomorphism $M \otimes_B (B \otimes_A C) \cong M \otimes_A C$ is a topological isomorphism for the tensor product topologies. In particular, the complete tensor product $M \hat{\otimes}_B (B \hat{\otimes}_A C)$ is topologically isomorphic to $M \hat{\otimes}_A C$.*

Proof. An open neighborhood of 0 in the tensor product $M \otimes_B (B \otimes_A C)$ contains a neighborhood of the form

$$W = \text{im}(V \otimes_B (B \otimes_A C)) + \text{im}(M \otimes_B (\text{im}(\mathfrak{K} \otimes_A C)) + \text{im}(B \otimes_A \mathfrak{R}))$$

where V is an open submodule of M , \mathfrak{K} (resp. \mathfrak{R}) is an open ideal of B (resp. C). This submodule can also be written as

$$\text{im}(V \otimes_A C) + \text{im}(M \otimes_A \mathfrak{R}) + \text{im}(\mathfrak{K}M \otimes_A C)$$

and hence contains $W' = \text{im}(V \otimes_A C) + \text{im}(M \otimes_A \mathfrak{R})$, which is an open neighborhood of 0 in $M \otimes_A C$. Conversely, if \mathfrak{K} is taken so that $\mathfrak{K}M \subseteq V$, which is possible by hypothesis, then $W = W'$ and the first assertion follows. The second assertion follows from the equality

$$M \hat{\otimes}_B (B \hat{\otimes}_A C) = \widehat{M \hat{\otimes}_{\widehat{B}} (B \otimes_A C)} = (M \otimes_B \widehat{(B \otimes_A C)}) = \widehat{M \otimes_A C} = M \hat{\otimes}_A C. \quad \square$$

Proposition 10.6.26. *Let A be a preadic ring, \mathfrak{I} be a defining ideal of A , and M be a finitely generated A -module. Then for any adic and Noetherian topological A -algebra B , $B \otimes_A M$ is identified with the complete tensor product $B \hat{\otimes}_A M$.*

Proof. If \mathfrak{K} is a defining ideal of B , then there exists by hypothesis an integer m such that $\mathfrak{I}^m B \subseteq \mathfrak{K}$, so

$$\text{im}(B \otimes_A \mathfrak{I}^m M) = \text{im}(\mathfrak{I}^m B \otimes_A M) \subseteq \text{im}(\mathfrak{K}^n B \otimes_A M) = \mathfrak{K}^n (B \otimes_A M).$$

We then conclude that over $B \otimes_A M$, the tensor product topology is equivalent to the \mathfrak{K} -adic topology. As $B \otimes_A M$ is a finitely generated B -module, it is equal to the \mathfrak{K} -adic completion (??), since B is a Zariski ring (??). \square

Proposition 10.6.27. *Let A be a topological ring, B, C be topological Noetherian local A -algebras with maximal ideals $\mathfrak{m}, \mathfrak{n}$, respectively, endowed with the adic topologies. Suppose that C is complete and that the residue field B/\mathfrak{m} is a finitely generated A -module. Let $E = B \hat{\otimes}_A C$ be the complete tensor product.*

- (a) *E is a complete Noetherian semi-local ring.*
- (b) *The ideal $\mathfrak{m}E$ is contained in the radical of E , and for any $k > 0$, $E/\mathfrak{m}^k E$ is isomorphic to $(B/\mathfrak{m}^k) \otimes_A C$.*
- (c) *If C is a flat A -module, then E is a flat B -module.*

Proof. \square

Corollary 10.6.28. *Under the hypotheses of ??, let M be a finitely generated B -module, endowed with the \mathfrak{m} -adic topology. Then $M \hat{\otimes}_A C$ is isomorphic to $M \otimes_B E$.*

Proof. In fact, as $\mathfrak{m}E \subseteq \mathfrak{r}E$, the tensor product topology on $M \otimes_B E$ is the \mathfrak{r} -adic topology, and as E is complete for this topology, so is $M \otimes_B E$. It then suffices to apply ?? \square

Corollary 10.6.29. *Let k be a field, A, B be Noetherian local rings containing k , with maximal ideals $\mathfrak{m}, \mathfrak{n}$, respectively. Suppose that B is complete and that the residue field A/\mathfrak{m} is a finite extension of k . Let C be the complete tensor product $A \hat{\otimes}_k B$.*

- (a) *C is a complete Noetherian semi-local ring.*
- (b) *C is a flat A -module and a flat B -module.*
- (c) *$\mathfrak{m}C$ is contained in the Jacobson radical of C and $C/\mathfrak{m}C$ is isomorphic to $(A/\mathfrak{m}) \otimes_k B$.*
- (d) *The residue fields of C are finite extensions of that of B .*

Moreover, if A' is a finite local A -algebra containing A , the canonical homomorphism $A \hat{\otimes}_k B \rightarrow A' \hat{\otimes}_k B$ is injective.

Proof. \square

10.6.4 Flatness of graded modules

Let A be a ring and \mathfrak{I} an ideal of A . An A -module M is called **ideally Hausdorff with respect to \mathfrak{I}** (or simply ideally Hausdorff if there is no ambiguity) if, for every finitely generated ideal \mathfrak{a} of A , the A -module $\mathfrak{a} \otimes_A M$ is Hausdorff with the \mathfrak{I} -adic topology.

Example 10.6.30 (Examples of ideally Hausdorff modules).

- (a) If A is Noetherian and \mathfrak{I} is contained in the Jacobson radical of A (in other words if A is a Zariski ring with the \mathfrak{I} -adic topology), every finitely generated A -module is ideally Hausdorff by ??.
- (b) Every direct sum of ideally Hausdorff modules is an ideally Hausdorff module, by virtue of the observation

$$\mathfrak{I}^n(\mathfrak{a} \otimes_A \bigoplus_{i \in I} M_i) = \mathfrak{I}^n \bigoplus_{i \in I} (\mathfrak{a} \otimes_A M_i) = \bigoplus_{i \in I} \mathfrak{I}^n(\mathfrak{a} \otimes_A M_i).$$

- (c) If an A -module M is flat and Hausdorff with the \mathfrak{I} -adic topology, it is ideally Hausdorff, for $\mathfrak{a} \otimes_A M$ is then identified with a submodule of M and the \mathfrak{I} -adic topology on $\mathfrak{a} \otimes_A M$ is finer than the topology induced on $\mathfrak{a} \otimes_A M$ by the \mathfrak{I} -adic topology on M , which is Hausdorff by hypothesis.

Let A be a ring, \mathfrak{I} an ideal of A , M an A -module and $\text{gr}(A)$ and $\text{gr}(M)$ the graded ring and graded $\text{gr}(A)$ -module associated respectively with the ring A and with the module M with the \mathfrak{I} -adic filtrations. We have seen that for every nonnegative integer n there is a surjective \mathbb{Z} -module homomorphism (see ??))

$$\gamma_n : (\mathfrak{I}^n / \mathfrak{I}^{n+1}) \otimes_{A/\mathfrak{I}} (M / \mathfrak{I}M) \rightarrow \mathfrak{I}^n M / \mathfrak{I}^{n+1} M$$

and a graded homomorphism of degree 0 of graded $\text{gr}(A)$ -modules

$$\gamma_M : \text{gr}(A) \otimes_{\text{gr}_0(A)} \text{gr}_0(M) \rightarrow \text{gr}(M)$$

whose restriction to $\text{gr}_n(A) \otimes \text{gr}_0(M)$ is γ_n for all n and which is therefore surjective.

Proposition 10.6.31. *Let A be a ring, \mathfrak{I} an ideal of A and M an A -module. The following conditions are equivalent:*

- (i) *For all $n > 0$, $\text{Tor}_1^A(A/\mathfrak{I}^n, M) = 0$.*
- (ii) *For all $n > 0$, the canonical homomorphism $\theta_n : \mathfrak{I}^n \otimes_A M \rightarrow \mathfrak{I}^n M$ is bijective.*

Moreover these conditions imply:

- (iii) *The canonical homomorphism $\gamma_M : \text{gr}(A) \otimes_{\text{gr}_0(A)} \text{gr}_0(M) \rightarrow \text{gr}(M)$ bijective.*

Conversely, if \mathfrak{I} is nilpotent, then (iii) implies (i) and (ii).

Proof. The equivalence of (i) and (ii) follows from the exact sequence

$$\text{Tor}_1^A(A, M) \longrightarrow \text{Tor}_1^A(A/\mathfrak{I}^n, M) \longrightarrow \mathfrak{I}^n \otimes_A M \longrightarrow A \otimes_A M$$

Consider next the diagram

$$\begin{array}{ccccccc}
 \mathfrak{J}^{n+1} \otimes_A M & \longrightarrow & \mathfrak{J}^n \otimes_A M & \longrightarrow & (\mathfrak{J}^n / \mathfrak{J}^{n+1}) \otimes_A (M / \mathfrak{J}M) & \longrightarrow & 0 \\
 \downarrow \theta_{n+1} & & \downarrow \theta_n & & \downarrow \gamma_n & & \\
 0 & \longrightarrow & \mathfrak{J}^{n+1}M & \longrightarrow & \mathfrak{J}^nM & \longrightarrow & \text{gr}_n(M) \longrightarrow 0
 \end{array} \tag{10.6.1}$$

where we note that $(\mathfrak{J}^n / \mathfrak{J}^{n+1}) \otimes_A (M / \mathfrak{J}M)$ is canonically identified with $(\mathfrak{J}^n / \mathfrak{J}^{n+1}) \otimes_{A/\mathfrak{J}} (M / \mathfrak{J}M)$. This diagram is commutative by definition of γ_n , and its rows are exact. If (ii) holds, θ_n and θ_{n+1} are bijective and so therefore is γ_n , by definition of cokernel, hence (ii) implies (iii).

Conversely, assuming that \mathfrak{J} is nilpotent, let us show that (iii) implies (ii). We shall argue by descending induction on n , since $\mathfrak{J}^n \otimes_A M = 0$ for n sufficiently large. Suppose then that in diagram (??), γ_n and θ_{n+1} are bijective. Then so is θ_n by the snake lemma, and the induction proves θ_n is bijective for all n . \square

Theorem 10.6.32. *Let A be a ring, \mathfrak{J} an ideal of A and M an A -module. Consider the following properties:*

- (i) M is a flat A -module.
- (ii) $\text{Tor}_1(N, M) = 0$ for every A -module N annihilated by \mathfrak{J} .
- (iii) $M / \mathfrak{J}M$ is a flat A / \mathfrak{J} -module and the canonical map $\mathfrak{J} \otimes_A M \rightarrow \mathfrak{J}M$ is bijective.
- (iv) $M / \mathfrak{J}M$ is a flat A / \mathfrak{J} -module and the canonical map $\gamma_M : \text{gr}(A) \otimes_{\text{gr}_0(A)} \text{gr}_0(M) \rightarrow \text{gr}(M)$ is bijective.
- (v) For all $n > 0$, $M / \mathfrak{J}^n M$ is a flat (A / \mathfrak{J}^n) -module.

Then (i) \Rightarrow (ii) \Leftrightarrow (iii) \Rightarrow (iv) \Leftrightarrow (v). If further \mathfrak{J} is nilpotent or if A is Noetherian and M is ideally Hausdorff, then these properties are equivalent.

Proof. The implication (i) \Rightarrow (ii) is immediate. Also, condition (ii) is equivalent to the following:

$$(ii') \quad \text{Tor}_1^A(N, M) = 0 \text{ for every } A\text{-module } N \text{ annihilated by a power of } \mathfrak{J}.$$

In fact, if (ii) holds, then in particular $\text{Tor}_1^A(\mathfrak{J}^n / \mathfrak{J}^{n+1}, M) = 0$ for all n . From the exact sequence

$$0 \rightarrow \mathfrak{J}^{n+1}N \longrightarrow \mathfrak{J}^nN \longrightarrow \mathfrak{J}^nN / \mathfrak{J}^{n+1}N \longrightarrow 0$$

we derive the exact sequence

$$\text{Tor}_1^A(\mathfrak{J}^{n+1}N, M) \longrightarrow \text{Tor}_1^A(\mathfrak{J}^nN, M) \longrightarrow \text{Tor}_1^A(\mathfrak{J}^nN / \mathfrak{J}^{n+1}N, M)$$

and, as there exists an integer m such that $\mathfrak{J}^m N = 0$, we deduce by descending induction on n that $\text{Tor}_1^A(\mathfrak{J}^nN, M) = 0$ for all $n < m$ and in particular for $n = 0$. It follows from this that if \mathfrak{J} is nilpotent, (ii) implies (i), for (ii') then means that $\text{Tor}_1^A(N, M) = 0$ for every A -module N and hence that M is flat. Note that the equivalence (ii) \Leftrightarrow (iii) is a special case of ?? applied to $R = A$, $S = A / \mathfrak{J}$, $F = M$, $E = N$, taking account of the fact that an (A / \mathfrak{J}) -module N is the same as an A -module N annihilated by \mathfrak{J} .

Now if (ii) holds, so does (ii') and ?? shows that γ_M is an isomorphism. On the other hand, we already know that (ii) implies (iii) and hence $M/\mathfrak{J}M$ is a flat (A/\mathfrak{J}) -module, which completes the proof that (ii) implies (iv). Also, ?? shows that, if \mathfrak{J} is nilpotent, (iv) implies (iii). Taking account of the equivalence of (ii) and (iii), we have therefore proved in this case that (i), (ii), (iii) and (iv) are equivalent.

We prove the equivalence of (iv) and (v). For all $n > 0$, $M/\mathfrak{J}^n M$ has a canonical (A/\mathfrak{J}^n) -module structure. If it is filtered by the $(\mathfrak{J}/\mathfrak{J}^n)$ -adic filtration, it is immediate that

$$\text{gr}_m(M/\mathfrak{J}^n M) = \begin{cases} \text{gr}_m(M) & m < n \\ 0 & m \geq n. \end{cases}$$

For all $k > 0$, let $A_k = A/\mathfrak{J}^k$, $\mathfrak{J}_k = \mathfrak{J}/\mathfrak{J}^k$, and $M_k = M/\mathfrak{J}^k M$. Let $(iv)_k$ (resp. $(v)_k$) denote the assertion derived from (iv) (resp. (v)) by replacing A, \mathfrak{J}, M by A_k, \mathfrak{J}_k, M_k . It follows from what has just been said that (iv) is equivalent to "for all $k > 0$, $(iv)_k$ " and obviously (v) is equivalent to "for all $k > 0$, $(v)_k$ ". Then it will suffice to establish the equivalence $(iv)_k$ and $(v)_k$ for all k or also to show that $(iv) \Leftrightarrow (v)$ when \mathfrak{J} is nilpotent. Now we have seen that in that case (iv) is equivalent to (i), and (i) is clearly equivalent to (v). We have therefore shown the equivalence $(iv) \Leftrightarrow (v)$ in all cases and also that of all the properties of the theorem in the case where \mathfrak{J} is nilpotent.

Finally we show the implication $(v) \Rightarrow (i)$ when A is Noetherian and M ideally Hausdorff. It is sufficient to prove that for every ideal \mathfrak{a} of A the canonical map $j : \mathfrak{a} \otimes_A M \rightarrow M$ is injective. As $\mathfrak{a} \otimes_A M$ is Hausdorff with the \mathfrak{J} -adic topology, it suffices to verify that $\ker j \subseteq \mathfrak{J}^n(\mathfrak{a} \otimes_A M)$ for every integer $n > 0$. By ??, there exists an integer k such that $\mathfrak{a}_k := \mathfrak{a} \cap \mathfrak{J}^k \subseteq \mathfrak{J}^n \mathfrak{a}$. Now, denoting by $\iota : \mathfrak{a}_k \rightarrow \mathfrak{a}$, $\pi : \mathfrak{a} \rightarrow \mathfrak{a}/\mathfrak{a}_k$ and $\psi : \mathfrak{a}/\mathfrak{a}_k \rightarrow A/\mathfrak{J}^k$ the canonical maps, there is a commutative diagram

$$\begin{array}{ccccccc} \mathfrak{a}_k \otimes_A M & \xrightarrow{\iota \otimes 1_M} & \mathfrak{a} \otimes_A M & \xrightarrow{\pi \otimes 1_M} & (\mathfrak{a}/\mathfrak{a}_k) \otimes_A M & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow \psi \otimes 1_M & & \\ M & \longrightarrow & & & (A/\mathfrak{J}^k) \otimes_A M & & \end{array}$$

in which the first row is exact. The map $\psi \otimes 1_M$ is injective since it can also be written into

$$\psi \otimes 1_{M/\mathfrak{J}^k M} : (\mathfrak{a}/\mathfrak{a}_k) \otimes_{A/\mathfrak{J}^k} (M/\mathfrak{J}^k M) \rightarrow M/\mathfrak{J}^k M$$

and, as ψ is injective and by (v) $M/\mathfrak{J}^k M$ is a flat (A/\mathfrak{J}^k) -module. By a simple diagram chasing we see $\ker j \subseteq \text{im}(\iota \otimes 1_M)$, which shows the claim and completes the proof. \square

Proposition 10.6.33. *Let A be a ring, \mathfrak{J} an ideal of A and B a Noetherian A -algebra such that \mathfrak{J} is contained in the Jacobson radical of B . Then every finitely generated B -module M is an ideally Hausdorff A -module with respect to \mathfrak{J} .*

Proof. We shall see more generally that for every finitely generated A -module N , $N \otimes_A M$ is Hausdorff with the \mathfrak{J} -adic topology. Let \mathfrak{r} be the Jacobson radical of B . As $\mathfrak{J}B$ is contained in \mathfrak{r} and the B -module $N \otimes_A M$ is canonically identified with $N_{(B)} \otimes_B M$ by virtue of the associativity of the tensor product, the \mathfrak{J} -adic topology on $N \otimes_A M$ is therefore identified with a finer

topology than the \mathfrak{r} -adic topology on $N_{(B)} \otimes_B M$. But this latter topology is Hausdorff since $N_{(B)} \otimes_B M$ is a finitely generated B -module, whence the conclusion. \square

Proposition 10.6.34. *Let A be a ring, B an A -algebra, \mathfrak{J} an ideal of A and M a B -module. Suppose that B is a Noetherian ring and a flat A -module and that M is ideally Hausdorff with respect to $\mathfrak{J}B$. The following conditions are equivalent:*

- (i) M is a flat B -module.
- (ii) M is a flat A -module and $M/\mathfrak{J}M$ is a flat $(B/\mathfrak{J}B)$ -module.

If further the canonical homomorphism $A/\mathfrak{J} \rightarrow B/\mathfrak{J}B$ is bijective, then these are also equivalent to:

- (iii) M is a flat A -module.

Proof. Condition (i) implies (ii) by ?? and ?? and the fact that $M/\mathfrak{J}M$ is isomorphic to $M \otimes_B B/\mathfrak{J}B$. Suppose condition (ii) holds. To show that M is a flat B -module, we shall apply ?? with A replaced by B and \mathfrak{J} by $\mathfrak{J}B$. It will therefore be sufficient to show that the canonical map $\phi : \mathfrak{J}B \otimes_B M \rightarrow \mathfrak{J}M$ is injective. Let ϕ_1 be the canonical map $\mathfrak{J} \otimes_A B \rightarrow \mathfrak{J}B$ and ϕ_2 the canonical isomorphism $\mathfrak{J} \otimes_A M \rightarrow (\mathfrak{J} \otimes_A B) \otimes_B M$. Then

$$\mathfrak{J} \otimes_A M \xrightarrow{\phi_2} (\mathfrak{J} \otimes_A B) \otimes_B M \xrightarrow{\phi_1 \otimes 1_M} \mathfrak{J}B \otimes_B M \xrightarrow{\phi} \mathfrak{J}M$$

Now $\psi : \mathfrak{J} \otimes_A M \rightarrow \mathfrak{J}M$ is an isomorphism since M is a flat A -module, whilst ϕ_1 is an isomorphism because B is flat over A . Therefore ϕ is then an isomorphism.

Let $\rho : A/\mathfrak{J} \rightarrow B/\mathfrak{J}B$ be the canonical homomorphism. The (A/\mathfrak{J}) -module structure on $M/\mathfrak{J}M$ derived by means of ρ is isomorphic to that on $M \otimes_A (A/\mathfrak{J})$. Then it follows that, if M is a flat A -module, $M/\mathfrak{J}M$ is a flat (A/\mathfrak{J}) -module and hence also a flat $(B/\mathfrak{J}B)$ -module if ρ is an isomorphism. We have thus proved that (iii) \Rightarrow (ii) in that case. \square

Corollary 10.6.35. *Let A be a Noetherian ring, \mathfrak{J} an ideal of A , \widehat{A} the Hausdorff completion of A with respect to the \mathfrak{J} -adic topology and M an ideally Hausdorff \widehat{A} -module with respect to $\widehat{\mathfrak{J}}$. For M to be a flat A -module, it is necessary and sufficient that M be a flat \widehat{A} -module.*

Proof. We know in fact that A is a Noetherian ring and a flat A -module, that $\widehat{\mathfrak{J}} = \mathfrak{J}\widehat{A}$ and that the canonical homomorphism $A/\mathfrak{J} \rightarrow \widehat{A}/\widehat{\mathfrak{J}}$ is bijective. ?? can therefore be applied. \square

Proposition 10.6.36. *Let A and B be two Noetherian rings, $\rho : A \rightarrow B$ a ring homomorphism, \mathfrak{J} an ideal of A and \mathfrak{K} an ideal of B containing $\mathfrak{J}B$ and contained in the Jacobson radical of B . Let \widehat{A} be the Hausdorff completion of A with respect to the \mathfrak{J} -adic topology and \widehat{B} the Hausdorff completion of B with respect to the \mathfrak{K} -adic topology. Let M be a finitely generated B -module and \widehat{M} its Hausdorff completion with respect to the \mathfrak{K} -adic topology. Then the following properties are equivalent:*

- (i) M is a flat A -module.
- (ii) \widehat{M} is a flat A -module.
- (iii) \widehat{M} is a flat \widehat{A} -module.

Proof. As B with the \mathfrak{K} -adic topology is a Zariski ring, \widehat{B} is a faithfully flat B -module and \widehat{M} is canonically isomorphic to $M \otimes_B \widehat{B}$. It is immediately verified that this canonical isomorphism is an isomorphism of the A -module structure on \widehat{M} onto the A -module structure on $M \otimes_B \widehat{B}$ derived from that on M . Applying ??, we see that for M to be a flat A -module, it is necessary and sufficient that \widehat{M} be a flat A -module. Moreover, \widehat{M} is a finitely generated B -module and $\mathfrak{J}\widehat{B}$ is contained in $\widehat{\mathfrak{K}} = \mathfrak{K}\widehat{B}$ and hence in the Jacobson radical of \widehat{B} (??). Therefore \widehat{M} is an ideally Hausdorff \widehat{A} -module with respect to $\widehat{\mathfrak{J}}$ by ?? . Conditions (ii) and (iii) are therefore equivalent by the Corollary to ?? . \square

10.7 Exercise

Exercise 10.7.1. Show that p -adic completion is not a right-exact functor on the category of all \mathbb{Z} -modules.

Proof. Let $A = (\mathbb{Z}/p\mathbb{Z})^{\oplus \mathbb{N}}$ and $B = \bigoplus_n \mathbb{Z}/p^n\mathbb{Z}$. Consider the short exact sequence

$$0 \longrightarrow A \xrightarrow{\iota} B \xrightarrow{\pi} B/A \longrightarrow 0$$

If we give them the topology induced by B , namely $\{\iota^{-1}(p^n B)\}$ and $\{\pi(p^n B)\}$, then by ?? the induced completion sequence is exact. Note that on $B/\alpha(A)$, the induced topology is just the same as the p -adic topology. So if we instead use the p -adic topology, the sequence is not exact. \square

Remark 10.7.1. Though it does preserve surjectivity, because if $\rho : B \rightarrow C$ is a surjection, we have

$$\rho(p^n B) = p^n \rho(B) = p^n C,$$

so we have a surjective map of surjective inverse systems, and ?? tells us the map $\widehat{\rho} : \widehat{B} \rightarrow \widehat{C}$ is surjective.

The p -adic completion is not left-exact, by the same example; the essential reason is that given a general homomorphism $\alpha : A \rightarrow B$, we needn't have $\alpha(p^n A) = \alpha(A) \cap p^n B$, so the p -adic topology on A is not that induced from B and the hypotheses of ?? are not met.

Exercise 10.7.2. In ??, let $A_n = \iota^{-1}(p^n B)$, and consider the exact sequence

$$0 \longrightarrow A_n \longrightarrow A \longrightarrow A/A_n \longrightarrow 0$$

Show that \varprojlim is not right exact, and compute $\varprojlim^1 A_n$.

Proof. From ?? we already have

$$A_n = \bigoplus_{j > n} G_j, \quad A/A_n = \bigoplus_{j=1}^n G_j$$

where $G_i = \mathbb{Z}/p\mathbb{Z}$ for all i . The maps

$$A_{n+1} \hookrightarrow A_n, \quad A \xrightarrow{\text{id}} A, \quad A/A_{n+1} \hookrightarrow A/A_n$$

give rise to an inverse system, hence we can form the inverse limits, and by ?? we get an exact sequence

$$0 \rightarrow \varprojlim A_n \rightarrow \varprojlim A \rightarrow \varprojlim A/A_n \rightarrow \varprojlim^1 A_n \rightarrow \varprojlim^1 A \rightarrow \varprojlim^1 A/A_n \rightarrow 0$$

Since $\{A\}$ and $\{A/A_n\}$ are surjective inverse systems, $\varprojlim^1 A = \varprojlim^1 A/A_n = 0$. Also, the inverse limits can be calculate to be

$$\varprojlim A_n = 0, \quad \varprojlim A = A = \bigoplus_{j=1}^n G_j, \quad \varprojlim A/A_n = \prod_{i=1}^{\infty} G_i$$

Hence we get a short exact sequence

$$0 \longrightarrow \bigoplus_{j=1}^n G_j \xrightarrow{\psi} \prod_{j=1}^{\infty} G_j \longrightarrow \varprojlim^1 A_n \longrightarrow 0$$

Then the map ψ is simply the inclusion, and we see $\varprojlim^1 A_n = \prod_{j=1}^{\infty} G_j / \bigoplus_{j=1}^n G_j$. \square

Exercise 10.7.3. Let A be a Noetherian ring and let $\mathfrak{a}, \mathfrak{b}$ be ideals in A . If M is any A -module, let $M^\mathfrak{a}, M^\mathfrak{b}$ denote its \mathfrak{a} -adic and \mathfrak{b} -adic completions respectively. If M is finitely generated, prove that $(M^\mathfrak{a})^\mathfrak{b} \cong M^{\mathfrak{a}+\mathfrak{b}}$.

Proof. Take the \mathfrak{a} -adic completion of $0 \rightarrow \mathfrak{b}^m M \rightarrow M \rightarrow M/\mathfrak{b}^m M \rightarrow 0$, we get

$$0 \longrightarrow (\mathfrak{b}^m M)^\mathfrak{a} \longrightarrow M^\mathfrak{a} \longrightarrow (M/\mathfrak{b}^m M)^\mathfrak{a} \longrightarrow 0$$

By ?? we have

$$\begin{aligned} (\mathfrak{b}^m M)^\mathfrak{a} &\cong A^\mathfrak{a} \otimes_A \mathfrak{b}^m M = A^\mathfrak{a} \otimes_A \mathfrak{b}^m \otimes_A M = A^\mathfrak{a} \otimes_A M \otimes_A \mathfrak{b}^m \\ &\cong \mathfrak{b}^m \otimes M^\mathfrak{a} = \mathfrak{b}^m M^\mathfrak{a} \end{aligned}$$

so from the sequence we get $(M/\mathfrak{b}^m M)^\mathfrak{a} \cong M^\mathfrak{a}/\mathfrak{b}^m M^\mathfrak{a}$. Now we have

$$(M^\mathfrak{a})^\mathfrak{b} = \varprojlim_m M^\mathfrak{a}/\mathfrak{b}^m M^\mathfrak{a} = \varprojlim_m (M/\mathfrak{b}^m M)^\mathfrak{a} = \varprojlim_m \varprojlim_n \frac{M/\mathfrak{b}^m M}{\mathfrak{a}^n(M/\mathfrak{b}^m M)}.$$

Since we have

$$\mathfrak{a}^n(M/\mathfrak{b}^m M) \cong \frac{\mathfrak{a}^n M + \mathfrak{b}^m M}{\mathfrak{b}^m M}.$$

we get

$$(M^\mathfrak{a})^\mathfrak{b} = \varprojlim_m \varprojlim_n M/(\mathfrak{a}^n M + \mathfrak{b}^m M) \cong \varprojlim_n M/(\mathfrak{a}^n M + \mathfrak{b}^n M)$$

Now note that

$$(\mathfrak{a} + \mathfrak{b})^{2n} \subseteq \mathfrak{a}^n + \mathfrak{b}^n \subseteq (\mathfrak{a} + \mathfrak{b})^n$$

so the topology induced by $\mathfrak{a}^n M + \mathfrak{b}^n M$ is the same as that by $(\mathfrak{a} + \mathfrak{b})^n$, hence we get the desired result. \square

Exercise 10.7.4. Let A be the local ring of the origin in \mathbb{C}^n (i.e., the ring of all rational functions $f/g \in \mathbb{C}(z_1, \dots, z_n)$ with $g(0) \neq 0$), let B be the ring of power series in z_1, \dots, z_n which

converge in some neighborhood of the origin, and let C be the ring of formal power series in z_1, \dots, z_n , so that $A \subseteq B \subseteq C$. Show that B is a local ring and that its completion for the maximal ideal topology is C . Assuming that B is Noetherian, prove that B is A -flat.

Proof. The surjective ring homomorphism

$$\phi : B \rightarrow \mathbb{C}, \quad f \mapsto f(0)$$

shows the ideal $\mathfrak{m} = (z_1, \dots, z_n)$ of power series with zero constant term is maximal. To show B is local, note that if $f \notin \mathfrak{m}$, then $f(0) \neq 0$ and f is a unit in C , and thus a unit in B . So \mathfrak{m} is maximal.

We first compute the completion, note that

$$\mathbb{C}[z_1, \dots, z_n] \subseteq A \subseteq B \subseteq C$$

and that C is the completion of $\mathbb{C}[z_1, \dots, z_n]$ by the (z_1, \dots, z_n) -adic topology. Thus we have

$$C = \widehat{\mathbb{C}[z_1, \dots, z_n]} \subseteq \widehat{A} \subseteq \widehat{B} \subseteq \widehat{C} = C$$

So $\widehat{A} = \widehat{B} = C$. Since we know C is a faithfully flat B -algebra and A -algebra, we conclude B is a flat A -algebra. \square

Chapter 11

Associated prime ideals and primary decomposition

11.1 Associated prime ideals of a module

11.1.1 Associated prime ideals

Let M be a module over a ring A . A prime ideal \mathfrak{p} is said to be **associated with M** if there exists $x \in M$ such that \mathfrak{p} is equal to the annihilator of x . The set of prime ideals associated with M is denoted by $\text{Ass}_A(M)$, or simply $\text{Ass}(M)$.

Example 11.1.1. Let \mathfrak{a} be an ideal in the polynomial ring $A = \mathbb{C}[X_1, \dots, X_n]$, V the corresponding affine algebraic variety and V_1, \dots, V_p the irreducible components of V . If M is taken to be the ring A/\mathfrak{a} of functions which are regular on V , the set of prime ideals associated with M consists of the ideals of V_1, \dots, V_p and in general other prime ideals each of which contains one of the ideals of the V_i .

As the annihilator of 0 is A , an element $x \in M$ whose annihilator is a prime ideal is necessarily nonzero. To say that a prime idea \mathfrak{p} is associated with M amounts to saying that M contains a submodule isomorphic to A/\mathfrak{p} (namely Ax , for all $x \in M$ whose annihilator is \mathfrak{p}). If an A -module M is the union of a family $(M_i)_{i \in I}$ of submodules, then clearly

$$\text{Ass}(M) = \bigcup_{i \in I} \text{Ass}(M_i). \quad (11.1.1)$$

Proposition 11.1.2. *For every prime ideal \mathfrak{p} of a ring A and every nonzero submodule M of A/\mathfrak{p} , $\text{Ass}(M) = \{\mathfrak{p}\}$.*

Proof. As the ring A/\mathfrak{p} is an integral domain, the annihilator of an nonzero element of A/\mathfrak{p} is \mathfrak{p} . \square

Proposition 11.1.3. *Let M be a module over a ring A . Every maximal element of the set of ideals $\text{Ann}(x)$ of A , where x runs through the set of nonzero elements of M , belongs to $\text{Ass}(M)$.*

Proof. Let $\mathfrak{p} = \text{Ann}(x)$ (where $x \in M$ is nonzero) be such a maximal element; it is sufficient to show that \mathfrak{p} is prime. As $x \neq 0$, \mathfrak{p} is proper. Let a, b be elements of A such that $ab \in \mathfrak{p}$ but $a \notin \mathfrak{p}$. Then $ax \neq 0$ and $\mathfrak{p} \subseteq \text{Ann}(ax)$. As \mathfrak{p} is maximal, $\text{Ann}(ax) = \mathfrak{p}$, whence $b \in \mathfrak{p}$, so that \mathfrak{p} is prime. \square

Corollary 11.1.4. *Let M be a module over a Noetherian ring A . Then the condition $M \neq \{0\}$ is equivalent to $\text{Ass}(M) \neq \emptyset$.*

Proof. If $M = \{0\}$, clearly $\text{Ass}(M)$ is empty (without any hypothesis on A). If $M \neq \{0\}$, the set of ideals of the form $\text{Ann}(x)$, where $x \in M$ and $x \neq 0$, is non-empty and consists of ideals of A . As A is Noetherian, this set has a maximal element; then it suffices to apply ??.

Corollary 11.1.5. *Let A be a Noetherian ring, M an A -module and $a \in A$. For the homothety on M with ratio a to be injective, it is necessary and sufficient that a belong to no prime ideal associated with M .*

Proof. If a belongs to a prime ideal $\mathfrak{p} \in \text{Ass}(M)$, then $\mathfrak{p} = \text{Ann}(x)$ where $x \in M$ and $x \neq 0$, whence $ax = 0$ and the homothety with ratio a is not injective. Conversely, if $ax = 0$ for some $x \in M$ such that $x \neq 0$, then $Ax \neq \{0\}$, whence $\text{Ass}(Ax) \neq \emptyset$. Let $\mathfrak{p} \in \text{Ass}(Ax)$, then obviously $\mathfrak{p} \in \text{Ass}(M)$ and $\mathfrak{p} = \text{Ann}(bx)$ for some $b \in A$; then $a \in \mathfrak{p}$, since $abx = 0$.

Corollary 11.1.6. *The set of divisors of zero in a Noetherian ring A is the union of the ideals in $\text{Ass}(A)$.*

Proposition 11.1.7. *If we have a short exact sequence of A -modules*

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

then $\text{Ass}(L) \subseteq \text{Ass}(M) \subseteq \text{Ass}(L) \cup \text{Ass}(N)$.

Proof. The inclusion $\text{Ass}(L) \subseteq \text{Ass}(M)$ is obvious. Let $\mathfrak{p} \in \text{Ass}(M)$, E be a submodule of M isomorphic to A/\mathfrak{p} and $F = E \cap L$. If $F = \{0\}$, E is isomorphic to a submodule of N , whence $\mathfrak{p} \in \text{Ass}(N)$. If $F \neq \{0\}$, the annihilator of every nonzero element of F is \mathfrak{p} (??) and hence $\mathfrak{p} \in \text{Ass}(F) \subseteq \text{Ass}(L)$.

Corollary 11.1.8. *If $M = \bigoplus_{i \in I} M_i$ then $\text{Ass}(M) = \bigcup_{i \in I} \text{Ass}(M_i)$.*

Proof. It may be reduced to the case where I is finite by means of (??), then to the case where $|I| = 2$ by induction. Then let $I = \{i, j\}$, where $i \neq j$. As M/M_i is isomorphic to M_j , we have $\text{Ass}(M) \subseteq \text{Ass}(M_i) \cup \text{Ass}(M_j)$. Moreover, $\text{Ass}(M_i)$ and $\text{Ass}(M_j)$ are contained in $\text{Ass}(M)$ by ??, whence the result.

Corollary 11.1.9. *Let M be an A -module and $(Q_i)_{i \in I}$ a finite family of submodules of M . If $\bigcap_{i \in I} Q_i = \{0\}$ then*

$$\text{Ass}(M) \subseteq \bigcup_{i \in I} \text{Ass}(M/Q_i).$$

Proof. The canonical map $M \rightarrow \bigoplus_i (M/Q_i)$ is injective; then it suffices to apply ?? and ??.

Proposition 11.1.10. *Let M be an A -module and Φ a subset of $\text{Ass}(M)$. Then there exists a submodule N of M such that $\text{Ass}(N) = \text{Ass}(M) - \Phi$ and $\text{Ass}(M/N) = \Phi$.*

Proof. Let \mathcal{M} be the set of submodules P of M such that $\text{Ass}(P) \subseteq \text{Ass}(M) - \Phi$. Formula (??) shows that the set \mathcal{M} , ordered by inclusion, is inductive. Moreover, $\{0\} \in \mathcal{M}$ and hence $\mathcal{M} \neq \emptyset$. Let N be a maximal element of \mathcal{M} . Then $\text{Ass}(N) \subseteq \text{Ass}(M) - \Phi$. We shall see that $\text{Ass}(M/N) \subseteq \Phi$, which, by ??, will complete the proof. Let $\mathfrak{p} \in \text{Ass}(M/N)$; then M/N contains a submodule F/N isomorphic to A/\mathfrak{p} . By ?? and ??, $\text{Ass}(F) \subseteq \text{Ass}(N) \cup \{\mathfrak{p}\}$. Since N is maximal in \mathcal{M} , $F \notin \mathcal{M}$ and hence $\mathfrak{p} \in \Phi$.

Proposition 11.1.11. *Let A be a ring, S a multiplicative subset of A , Φ the set of ideals of A which do not meet S and M an A -module. Then:*

- (a) *The map $\mathfrak{p} \mapsto S^{-1}\mathfrak{p}$ is a bijection of $\text{Ass}_A(M) \cap \Phi$ onto a subset of $\text{Ass}_{S^{-1}A}(S^{-1}M)$.*
- (b) *If $\mathfrak{p} \in \Phi$ is a finitely generated ideal and $S^{-1}\mathfrak{p} \in \text{Ass}_{S^{-1}A}(S^{-1}M)$, then $\mathfrak{p} \in \text{Ass}_A(M)$.*

Proof. Recall that the map $\mathfrak{p} \mapsto S^{-1}\mathfrak{p}$ is a bijection of Φ onto the set of prime ideals of $S^{-1}A$. If $\mathfrak{p} \in \text{Ass}_A(M) \cap \Phi$, \mathfrak{p} is the annihilator of a monogenous submodule N of M ; then $S^{-1}\mathfrak{p}$ is the annihilator of the monogenous submodule $S^{-1}N$ of $S^{-1}M$ (??) and hence $S^{-1}\mathfrak{p} \in \text{Ass}_{S^{-1}A}(S^{-1}M)$. Conversely, suppose that $\mathfrak{p} \in \Phi$ is finitely generated and such that $S^{-1}\mathfrak{p}$ is associated with $S^{-1}M$, then there exists $x \in M$ and $t \in S$ such that $S^{-1}\mathfrak{p}$ is the annihilator of x/t . Let a_1, \dots, a_n be a system of generators of \mathfrak{p} ; then $(a_i)(x/t) = 0$ and hence there exists $s_i \in S$ such that $s_i a_i x = 0$. Let us write $s = s_1 \cdots s_n$, then for all $a \in \mathfrak{p}$ we have $sax = 0$, whence $\mathfrak{p} \subseteq \text{Ann}(sx)$. On the other hand, if $b \in A$ satisfies $bsx = 0$, then $b/1 \in S^{-1}\mathfrak{p}$ by definition, whence $b \in \mathfrak{p}$. Then $\mathfrak{p} = \text{Ann}(sx)$ and $\mathfrak{p} \in \text{Ass}_A(M)$. \square

Corollary 11.1.12. *If the ring A is Noetherian, the map $\mathfrak{p} \mapsto S^{-1}\mathfrak{p}$ is a bijection of $\text{Ass}_A(M) \cap \Phi$ onto $\text{Ass}_{S^{-1}A}(S^{-1}M)$.*

Proposition 11.1.13. *Let A be a Noetherian ring, M an A -module, S a multiplicative subset of A and Φ the set of elements of $\text{Ass}_A(M)$ which do not meet S . Then the kernel N of the canonical map $M \mapsto S^{-1}M$ is the unique submodule of M which satisfies the relations*

$$\text{Ass}(N) = \text{Ass}(M) - \Phi, \quad \text{Ass}(M/N) = \Phi.$$

Proof. By ??, there exists a submodule N' of M which satisfies the relations $\text{Ass}(N') = \text{Ass}(M) - \Phi$ and $\text{Ass}(M/N') = \Phi$. We need to prove $N' = N$. Consider the commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{\pi} & M/N' \\ i_M^S \downarrow & & \downarrow i_{M/N'}^S \\ S^{-1}M & \xrightarrow{S^{-1}\pi} & S^{-1}(M/N') \end{array}$$

where π , i_M^S , and $i_{M/N'}^S$ are the canonical homomorphisms. We shall show that $S^{-1}\pi$ and $i_{M/N'}^S$ are injective, which will prove that i_M^S and π have the same kernel and hence $N = N'$.

As $\text{Ass}(N') \cap \Phi = \emptyset$, every element of $\text{Ass}(N')$ meets S . Then by ?? we have $\text{Ass}_{S^{-1}A}(S^{-1}N') = \emptyset$, whence $S^{-1}N' = \{0\}$ by ?? . This proves that $S^{-1}\pi$ is injective. On the other hand, if x belongs to the kernel K of $i_{M/N'}^S$, then $\text{Ann}(x) \cap S \neq \emptyset$, so $\text{Ass}(K) \cap \Phi = \emptyset$. But $\text{Ass}(K) \subseteq \text{Ass}(M/N') = \Phi$, which means $\text{Ass}(K) = \emptyset$ and $K = \{0\}$ by ?? . This shows $i_{M/N'}^S$ is injective. \square

Let M be a module over a ring A . Recall that the set of prime ideals \mathfrak{p} of A such that $M_{\mathfrak{p}} \neq 0$ is called the *support* of M and is denoted by $\text{supp}(M)$.

Proposition 11.1.14. *Let A be a ring and M an A -module.*

- (a) *Every prime ideal \mathfrak{p} of A containing an element of $\text{Ass}(M)$ belongs to $\text{supp}(M)$.*
- (b) *Conversely, if A is Noetherian, every ideal $\mathfrak{p} \in \text{supp}(M)$ contains an element of $\text{Ass}(M)$.*

Proof. If \mathfrak{p} contains an element \mathfrak{q} of $\text{Ass}(M)$, then $\mathfrak{q} \cap (A - \mathfrak{p}) = \emptyset$ and hence, if we write $S = A - \mathfrak{p}$, $S^{-1}\mathfrak{p}$ is a prime ideal associated with $S^{-1}M = M_{\mathfrak{p}}$, and a fortiori $M_{\mathfrak{p}} \neq \emptyset$, hence $\mathfrak{p} \in \text{supp}(M)$. Conversely, if A is Noetherian, so is $A_{\mathfrak{p}}$. If $M_{\mathfrak{p}} \neq \emptyset$, then $\text{Ass}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}) \neq \emptyset$ and hence there exists $\mathfrak{q} \in \text{Ass}_A(M)$ such that $\mathfrak{q} \cap (A - \mathfrak{p}) = \emptyset$ by ??.

Corollary 11.1.15. *If M is a module over a Noetherian ring, then $\text{Ass}(M) \subseteq \text{supp}(M)$ and these two sets have the same minimal elements.*

Corollary 11.1.16. *The nilradical of a Noetherian ring A is the intersection of the ideals $\mathfrak{p} \in \text{Ass}(A)$.*

11.1.2 Associated prime ideals of Noetherian rings

Proposition 11.1.17. *Let A be a Noetherian ring and M a finitely generated A -module. Then there exists a chain $(M_i)_{0 \leq i \leq n}$ of submodules of M such that M_i/M_{i+1} is isomorphic to A/\mathfrak{p}_i where \mathfrak{p}_i is a prime ideal of A .*

Proof. Let \mathcal{M} be the set of submodules of M which have a chain with the property of the statement. As \mathcal{M} is non-empty (for $\{0\}$ belongs to \mathcal{M}) and M is Noetherian, \mathcal{M} has a maximal element N . If $M \subseteq N$, then $M/N \neq \emptyset$ and hence $\text{Ass}(M/N) \neq \emptyset$ by ?? . The module M/N therefore contains a submodule N'/N isomorphic to an A -module of the form A/\mathfrak{p} , where \mathfrak{p} is prime; then by definition $N' \in \mathcal{M}$, which contradicts the maximal character of N . Then necessarily $N = M$. \square

Theorem 11.1.18. *Let M be a finitely generated module over a Noetherian ring A and $(M_i)_{0 \leq i \leq n}$ a chain of submodules of M such that M_i/M_{i+1} is isomorphic to A/\mathfrak{p}_i where \mathfrak{p}_i is a prime ideal of A . Then*

$$\text{Ass}(M) \subseteq \{\mathfrak{p}_0, \dots, \mathfrak{p}_{n-1}\} \subseteq \text{supp}(M). \quad (11.1.2)$$

The minimal elements of these three sets are the same and coincide with the minimal elements of the set of prime ideals containing $\text{Ann}(M)$.

Proof. The inclusion $\text{Ass}(M) \subseteq \{\mathfrak{p}_0, \dots, \mathfrak{p}_{n-1}\}$ follows immediately from ?? . For each i , we have

$$\mathfrak{p}_i \in \text{supp}(A/\mathfrak{p}_i) = \text{supp}(M_i/M_{i+1})$$

so $\mathfrak{p}_i \in \text{supp}(M_i) \subseteq \text{supp}(M)$?? , which shows $\{\mathfrak{p}_0, \dots, \mathfrak{p}_{n-1}\} \subseteq \text{supp}(M)$. ?? shows that $\text{Ass}(M)$ and $\text{supp}(M)$ have the same minimal elements and ?? shows that these are just the minimal elements of $\{\mathfrak{p}_0, \dots, \mathfrak{p}_{n-1}\}$. The last assertion then follows from ?? . \square

Corollary 11.1.19. *If M is a finitely generated module over a Noetherian ring A , $\text{Ass}(M)$ is finite.*

Proposition 11.1.20. *Let A be a Noetherian ring, \mathfrak{a} an ideal of A and M a finitely generated A -module. The following conditions are equivalent:*

- (i) *there exists a nonzero element $x \in M$ such that $\mathfrak{a}x = 0$;*
- (ii) *for all $a \in \mathfrak{a}$, there exists a nonzero element $x \in M$ such that $ax = 0$;*
- (iii) *there exists $\mathfrak{p} \in \text{Ass}(M)$ such that $\mathfrak{a} \subseteq \mathfrak{p}$.*

Proof. Clearly (i) implies (ii). By virtue of ??, condition (ii) means that the ideal \mathfrak{a} is contained in the union of the prime ideals associated with M and hence in one of them since $\text{Ass}(M)$ is finite ??, thus (ii) implies (iii). Finally, if there exists $\mathfrak{p} \in \text{Ass}(M)$ such that $\mathfrak{a} \subseteq \mathfrak{p}$, then \mathfrak{p} is the annihilator of a nonzero element $x \in M$ such that $\mathfrak{a}x = 0$, thus (iii) implies (i). \square

Proposition 11.1.21. *Let A be a Noetherian ring, \mathfrak{a} an ideal of A and M a finitely generated A -module. For there to exist an integer $n > 0$ such that $\mathfrak{a}^n M = 0$, it is necessary and sufficient that \mathfrak{a} be contained in the intersection of the prime ideals associated with M .*

Proof. This intersection is also that of the minimal elements of $\text{supp}(M)$ and to say that \mathfrak{a} is contained in this intersection is equivalent to saying that $V(\mathfrak{a}) \supseteq \text{supp}(M)$. The conclusion then follows from ??.

Given an A -module M , an endomorphism ϕ of M is called **locally nilpotent** if, for all $x \in M$, there exists an integer $n(x) > 0$ such that $\phi(x)^{n(x)} = 0$. It is clear that, if M is finitely generated, every locally nilpotent endomorphism is nilpotent.

Corollary 11.1.22. *Let A be a Noetherian ring, M an A -module and a an element of A . For the homomorphism $h_a : x \mapsto ax$ of M to be locally nilpotent, it is necessary and sufficient that a belong to every ideal of $\text{Ass}(M)$.*

Proof. The condition of the statement is equivalent to saying that for all $x \in M$ there exists $n(x) > 0$ such that $(Aa)^{n(x)}(Ax) = 0$. By ?? this means also that a belongs to all the prime ideals associated with the submodule Ax of M . The corollary then follows from the fact that $\text{Ass}(M)$ is the union of the $\text{Ass}(Ax)$ where x runs through M . \square

Proposition 11.1.23. *Let A be a Noetherian ring, M a finitely generated A -module and N an A -module. Then*

$$\text{Ass}(\text{Hom}_A(M, N)) = \text{Ass}(N) \cap \text{supp}(M).$$

Proof. By hypothesis, M is isomorphic to an A -module of the form A^n/R , hence $\text{Hom}_A(M, N)$ is isomorphic to a submodule of $\text{Hom}_A(A^n, N)$ and the latter is isomorphic to N^n . Now, $\text{Ass}(N^n) = \text{Ass}(N)$ by ??, thus $\text{Ass}(\text{Hom}_A(M, N)) \subseteq \text{Ass}(N)$. On the other hand, for every prime ideal \mathfrak{p} of A , $\text{Hom}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}})$ is isomorphic to $(\text{Hom}_A(M, N))_{\mathfrak{p}}$ by ??

$$\text{supp}(\text{Hom}_A(M, N)) \subseteq \text{supp}(M).$$

Then we conclude from ?? that $\text{Ass}(\text{Hom}_A(M, N)) \subseteq \text{supp}(M)$.

Conversely, let \mathfrak{p} be a prime ideal of A belonging to $\text{Ass}(N) \cap \text{supp}(M)$. By definition, N contains a submodule isomorphic to A/\mathfrak{p} . On the other hand, since M is finitely generated and $M_{\mathfrak{p}} \neq 0$, we know that there exists a nonzero homomorphism $\eta : M \rightarrow A/\mathfrak{p}$??). As there exists an injective homomorphism $\iota : A/\mathfrak{p} \rightarrow N$, we have $\iota \circ \eta \in \text{Hom}(M, N)$ and it is nonzero. On the other hand, since the annihilator of every nonzero element of A/\mathfrak{p} is \mathfrak{p} , we see $\text{Ann}(\iota \circ \eta) = \mathfrak{p}$, whence $\mathfrak{p} \in \text{Ass}(\text{Hom}_A(M, N))$. \square

11.2 Primary decomposition

11.2.1 Primary submodules

Proposition 11.2.1. *Let A be a Noetherian ring and M an A -module. Then the following conditions are equivalent:*

- (i) $\text{Ass}(M)$ is reduced to a single element.
- (ii) $M \neq 0$ and every homothety of M is either injective or locally nilpotent.

If these conditions are fulfilled and \mathfrak{p} is the set of $a \in A$ such that the homothety of ratio a is locally nilpotent, then $\text{Ass}(M) = \{\mathfrak{p}\}$.

Proof. This follows from ?? and ??.

□

Let A be a Noetherian ring, M an A -module and N a submodule of M . If the module M/N satisfies the conditions of ??, N is called **\mathfrak{p} -primary** with respect to M (or in M).

In particular consider the case $M = A$, where the submodules of M are then the ideals of A and hence an ideal \mathfrak{q} of A is called primary if $\text{Ass}(A/\mathfrak{q})$ has a single element or, what amounts to the same, if $A \neq \mathfrak{q}$ and every divisor of zero in the ring A/\mathfrak{q} is nilpotent. If \mathfrak{q} is \mathfrak{p} -primary, then since $\text{supp}(A/\mathfrak{q}) = V(\text{Ann}(A/\mathfrak{q})) = V(\mathfrak{q})$, by ?? \mathfrak{p} is the radical of the ideal \mathfrak{q} .

Example 11.2.2 (Example of primary submodules).

- (a) If \mathfrak{p} is a prime ideal of A , then \mathfrak{p} is \mathfrak{p} -primary.
- (b) Let \mathfrak{q} be an ideal of A such that there exists a single prime ideal \mathfrak{m} (necessarily maximal) containing \mathfrak{q} . Then, if M is an A -module such that $\mathfrak{q}M \neq M$, $\mathfrak{q}M$ is \mathfrak{m} -primary with respect to M , for every element of $\text{Ass}(M/\mathfrak{q}M)$ contains \mathfrak{q} , hence is equal to \mathfrak{m} and $\text{Ass}(M/\mathfrak{q}M) \neq \emptyset$. In particular we see \mathfrak{q} is an \mathfrak{m} -primary ideal in A .
- (c) Let \mathfrak{m} be a maximal ideal of a Noetherian ring A . Then the \mathfrak{m} -primary ideals are the ideals \mathfrak{q} of A for which there exists an integer $n > 0$ such that $\mathfrak{m}^n \subseteq \mathfrak{q} \subseteq \mathfrak{m}$, for if $\mathfrak{m}^n \subseteq \mathfrak{q} \subseteq \mathfrak{m}$, then \mathfrak{m} is the only prime ideal containing \mathfrak{q} and the conclusion follows from (b). Conversely, if \mathfrak{q} is \mathfrak{m} -primary, then \mathfrak{m} is the radical of \mathfrak{q} and there therefore exists $n > 1$ such that $\mathfrak{m}^n \subseteq \mathfrak{q}$ by ?. In particular, if A is a PID then the primary ideals are (0) and the ideals of the form (p^n) , where p is a prime element and $n > 0$.

Remark 11.2.1. The powers of any prime ideal are not necessarily primary ideals. For an example, let K be a field and A the quotient ring of $K[X, Y, Z]/(Z^2 - XY)$. Let x, y, z be the canonical images of X, Y, Z in A . Then the ideal $\mathfrak{p} = (x, z)$ is prime, since $A/\mathfrak{p} \cong K[X]$. Note that $\mathfrak{p}^2 = (x^2, z^2, xz) = (x^2, xy, xz)$ is not primary, since $x \notin \mathfrak{p}^2$, $y \notin \mathfrak{p}$ but $xy \in \mathfrak{p}^2$. Also, we have the following decomposition

$$\mathfrak{p}^2 = (x) \cap (x, y, z^2)$$

On the other hand, there exist primary ideals which are not powers of prime ideals. For example, consider the ring $A = \mathbb{Z}[X]$ and the maximal ideal $\mathfrak{m} = (2, X)$. The ideal $\mathfrak{q} = (4, X)$ is \mathfrak{m} -primary since $\sqrt{\mathfrak{q}} = \mathfrak{m}$, but it is not a power of \mathfrak{m} .

Proposition 11.2.3. *Let M be a module over a Noetherian ring A , \mathfrak{p} a prime ideal of A and $(Q_i)_{i \in I}$ a non-empty finite family of submodules of M which are \mathfrak{p} -primary with respect to M . Then $\bigcap_{i \in I} Q_i$ is \mathfrak{p} -primary with respect to M .*

Proof. The quotient $M/(\bigcap_i Q_i)$ is isomorphic to a submodule of the direct sum $\bigoplus_i (M_i/Q_i)$. Now $\text{Ass}(\bigoplus_i (M_i/Q_i)) = \bigcup_i \text{Ass}(M_i/Q_i) = \{\mathfrak{p}\}$, whence $\text{Ass}(M/(\bigcap_i Q_i)) = \{\mathfrak{p}\}$ and $\bigcap_i Q_i$ is \mathfrak{p} -primary. \square

Proposition 11.2.4. *Let A be a Noetherian ring, S a multiplicative subset of A , \mathfrak{p} a prime ideal of A , M an A -module, N a submodule of M and i_M^S the canonical map of M to $S^{-1}M$.*

- (a) *Suppose that $\mathfrak{p} \cap S \neq \emptyset$. If N is \mathfrak{p} -primary with respect to M , then $S^{-1}N = S^{-1}M$.*
- (b) *Suppose that $\mathfrak{p} \cap S = \emptyset$. Then for N to be \mathfrak{p} -primary with respect to M , it is necessary and sufficient that N be of the form $(i_M^S)^{-1}(N')$, where N' is a sub- $S^{-1}A$ -module of $S^{-1}M$ which is $(S^{-1}\mathfrak{p})$ -primary with respect to $S^{-1}M$. In this case we have $N' = S^{-1}N$.*

Proof. If $\mathfrak{p} \cap S \neq \emptyset$ and N is \mathfrak{p} -primary with respect to M , then

$$\text{Ass}_{S^{-1}A}(S^{-1}(M/N)) = \emptyset$$

and hence $S^{-1}(M/N) = 0$, whence $S^{-1}N = S^{-1}M$.

Suppose that $\mathfrak{p} \cap S = \emptyset$. If N is \mathfrak{p} -primary with respect to M , then $\text{Ass}_{S^{-1}A}(S^{-1}(M/N)) = \{S^{-1}\mathfrak{p}\}$ and hence the submodule $N' = S^{-1}N$ of $S^{-1}M$ is $(S^{-1}\mathfrak{p})$ -primary. Moreover, if $s \in S$ and $m \in M$ are such that $sm \in N$, then $m \in N$, for the homothety with ratio s in M/N is injective, whence $N = (i_M^S)^{-1}(N')$.

Conversely, let N' be a submodule of $S^{-1}M$ which is $(S^{-1}\mathfrak{p})$ -primary with respect to $S^{-1}M$. Let us write $N = (i_M^S)^{-1}(N')$, then $N' = S^{-1}N$ and

$$\text{Ass}_{S^{-1}A}(S^{-1}(M/N)) = \text{Ass}_{S^{-1}A}((S^{-1}M)/N') = \{S^{-1}\mathfrak{p}\}.$$

Since N is saturated, the canonical map $M/N \rightarrow S^{-1}(M/N)$ is injective, whence no prime ideal of A associated with M/N meets S (??). It then follows that $\text{Ass}(M/N) = \{\mathfrak{p}\}$, so that N is \mathfrak{p} -primary with respect to M . \square

11.2.2 Primary decompositions

Let A be a Noetherian ring, M an A -module and N a submodule of M . A finite family $(Q_i)_{i \in I}$ of submodule of M which are primary with respect to M and such that $N = \bigcap_i Q_i$ is called a **primary decomposition** of N in M .

Example 11.2.5. Let us take $A = \mathbb{Z}$, $M = \mathbb{Z}$, $N = (n)$ for some integer $n > 0$. If $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ is the decomposition of n into prime factors, then

$$(n) = (p_1^{\alpha_1}) \cap \cdots \cap (p_k^{\alpha_k})$$

is a primary decomposition of (n) in \mathbb{Z} .

By an abuse of language, the relation $N = \bigcap_{i \in I} Q_i$ is called a primary decomposition of N in M . It amounts to the same to say that $\{0\} = \bigcap_{i \in I} (Q_i/N)$ is a primary decomposition of $\{0\}$

in M/N . If $(Q_i)_{i \in I}$ is a primary decomposition of N in M , the canonical map from M/N to $\bigcap_{i \in I} (M/Q_i)$ is injective. Conversely let N be a submodule of M and ϕ an injective homomorphism from M/N to a finite direct sum $P = \bigoplus_i P_i$, where each set $\text{Ass}(P_i)$ is reduced to a single element \mathfrak{p}_i . Let ϕ_i be the homomorphism $M/N \rightarrow P_i$ obtained by taking the composition off with the projection $P \rightarrow P_i$, and let Q_i/N be the kernel of ϕ_i ; then the Q_i distinct from M are primary with respect to M and $N = \bigcap_{i \in I} Q_i$. Moreover, $\text{Ass}(M/N) \subseteq \bigcup_{i \in I} \{\mathfrak{p}_i\}$ by virtue of ??.

Theorem 11.2.6. *Let M be a finitely generated module over a Noetherian ring and let N be a submodule of M . Then there exists a primary decomposition of N of the form*

$$N = \bigcap_{\mathfrak{p} \in \text{Ass}(M/N)} Q(\mathfrak{p}) \quad (11.2.1)$$

where for each $\mathfrak{p} \in \text{Ass}(M/N)$, $Q(\mathfrak{p})$ is \mathfrak{p} -primary with respect to M .

Proof. We may replace M by M/N and therefore suppose that $N = 0$. By ??, $\text{Ass}(M)$ is finite. By ??, there exists, for each $\mathfrak{p} \in \text{Ass}(M)$, a submodule $Q(\mathfrak{p})$ of M such that $\text{Ass}(M/Q(\mathfrak{p})) = \{\mathfrak{p}\}$ and $\text{Ass}(Q(\mathfrak{p})) = \text{Ass}(M) - \{\mathfrak{p}\}$. Let us write $P = \bigcap_{\mathfrak{p} \in \text{Ass}(M)} Q(\mathfrak{p})$. For all $\mathfrak{p} \in \text{Ass}(M)$, $\text{Ass}(P) \subseteq \text{Ass}(Q(\mathfrak{p}))$ and hence $\text{Ass}(P) = \{\mathfrak{p}\}$, which implies $P = 0$ and therefore proves the theorem. \square

Let M be a module over a Noetherian ring and N a submodule of M . A primary decomposition $N = \bigcap_{i \in I} Q_i$ of N in M is called **reduced** if the following conditions are fulfilled:

- (a) there exists no index $i \in I$ such that $\bigcap_{j \neq i} Q_j \subseteq Q_i$;
- (b) if $\text{Ass}(M/Q_i) = \{\mathfrak{p}_i\}$, the \mathfrak{p}_i 's are distinct.

From every primary decomposition $N = \bigcap_{i \in I} Q_i$ of N in M a reduced primary decomposition of M in N can be deduced as follows: let J be a minimal element of the set of subsets I' of I such that $N = \bigcap_{i \in I'} Q_i$. Clearly $(Q_i)_{i \in J}$ satisfies condition (a). Then let Φ be the set of \mathfrak{p}_i for $i \in J$. For all $\mathfrak{p} \in \Phi$, let $H(\mathfrak{p})$ be the set of $i \in J$ such that $\mathfrak{p}_i = \mathfrak{p}$ and let $Q(\mathfrak{p}) = \bigcap_{i \in H(\mathfrak{p})} Q_i$. It follows from ?? that $Q(\mathfrak{p})$ is \mathfrak{p} -primary with respect to M ; further $N = \bigcap_{\mathfrak{p} \in \Phi} Q(\mathfrak{p})$ and the family $(Q(\mathfrak{p}))_{\mathfrak{p} \in \Phi}$ is therefore a reduced primary decomposition of N in M .

Proposition 11.2.7. *Let M be a module over a Noetherian ring, N a submodule of M , $N = \bigcap_{i \in I} Q_i$ a primary decomposition of N in M and, for all $i \in I$, let $\{\mathfrak{p}_i\} = \text{Ass}(M/Q_i)$. For this decomposition to be reduced, it is necessary and sufficient that the \mathfrak{p}_i be distinct and belong to $\text{Ass}(M/N)$. In this case, we have*

$$\text{Ass}(M/N) = \bigcup_{i \in I} \mathfrak{p}_i, \quad \text{Ass}(Q_i/N) = \bigcup_{j \neq i} \{\mathfrak{p}_j\} \quad \text{for all } i \in I. \quad (11.2.2)$$

Proof. If the condition of the statement is fulfilled, $N = \bigcap_{j \neq i} Q_j$ cannot hold, for we would deduce that $\text{Ass}(M/N) \subseteq \bigcup_{j \neq i} \{\mathfrak{p}_j\}$ by ??, contrary to the hypothesis. The primary decomposition $(Q_i)_{i \in I}$ of N is then certainly reduced.

Conversely, assume that $(Q_i)_{i \in I}$ is reduced. Note that $\text{Ass}(M/N) \subseteq \bigcup_{i \in I} \{\mathfrak{p}_i\}$ always holds by ?? . On the other hand, for all $i \in I$, let us write $P_i = \bigcap_{j \neq i} Q_j$. Then $P_i \cap Q_i = N$ and $P_i \neq N$ if $(Q_i)_{i \in I}$ is reduced, hence P_i/N is non-zero and is isomorphic to the submodule $(P_i + Q_i)/Q_i$ of M/Q_i , whence $\{\mathfrak{p}_i\} = \text{Ass}(P_i/N)$ by ?? . As $P_i/N \subseteq M/N$, we get $\mathfrak{p}_i \in \text{Ass}(M/N)$, which

completes the proof of the necessity of the condition in the statement and the first formula of (??). Finally, for each $i \in I$, as $N = \bigcap_{j \neq i} (Q_i \cap Q_j)$, by ?? we have

$$\text{Ass}(Q_i/N) \subseteq \bigcup_{j \neq i} \text{Ass}(Q_i/(Q_i \cap Q_j)) = \bigcup_{j \neq i} \text{Ass}((Q_i + Q_j)/Q_j) \subseteq \bigcup_{j \neq i} \{\mathfrak{p}_j\}.$$

Taking account of the first formula of (??) and ??, the second formula of (??) follows easily. \square

Corollary 11.2.8. *Let A be a Noetherian ring, M an A -module, N a submodule of M and $(Q_i)_{i \in I}$ a primary decomposition of N in M . Then $|I| \geq |\text{Ass}(M/N)|$, and the equality holds if and only if $(Q_i)_{i \in I}$ is reduced.*

Proof. It follows from the remarks preceding ?? that there exists a reduced primary decomposition $(R_i)_{i \in J}$ of N in M such that $\text{Card}(J) \leq \text{Card}(I)$. The first assertion then follows from the second and the latter is a consequence of ??.

Proposition 11.2.9. *Let A be a Noetherian ring, M an A -module and N a submodule of M . Let $N = \bigcap_{i \in I} Q_i$ be a reduced primary decomposition of N where Q_i is \mathfrak{p}_i -primary. If \mathfrak{p}_i is a minimal element of $\text{Ass}(M/N)$, then Q_i is equal to the saturation on N with respect to \mathfrak{p}_i .*

Proof. We can obviously restrict our attention to the case where $N = 0$, replacing if need be M by M/N . If \mathfrak{p}_i is minimal in $\text{Ass}(M)$, then the set of elements of $\text{Ass}(M)$ which do not meet $A - \mathfrak{p}_i$ reduces to \mathfrak{p}_i . The proposition then follows from the second formula of (??) above and ??, since the kernel of the canonical map $M \mapsto M_{\mathfrak{p}_i}$ equals to the saturation of 0 with respect to \mathfrak{p}_i . \square

The minimal elements of $\text{Ass}(M/N)$ are called the **minimal or isolated prime ideals** associated with M/N , and the prime ideals $\mathfrak{p} \in \text{Ass}(M/N)$ which are not minimal are called the **embedded prime ideals** associated with M/N . If M/N is finitely generated, then by ?? we know the isolated ideals of M/N are just the minimal prime ideals containing $\text{supp}(M/N) = V(\text{Ann}(M/N))$, so for $\mathfrak{p} \in \text{Ass}(M/N)$ to be embedded, it is necessary and sufficient that $V(\mathfrak{p})$ be not an irreducible component of $\text{supp}(M/N)$.

If $(Q(\mathfrak{p}))$ and $(R(\mathfrak{p}))$ are two reduced primary decompositions of N in M , then by ?? we see $Q(\mathfrak{p}) = R(\mathfrak{p})$ if \mathfrak{p} is an isolated prime associated with M/N , being the saturation of N with respect to \mathfrak{p} . However, it may happen that $Q(\mathfrak{p}) \neq R(\mathfrak{p})$ for \mathfrak{p} an embedded prime, as the following example shows.

Example 11.2.10. Consider the ring $A = K[X, Y]$ where K is a field. Then for every positive integer n , a primary decomposition of the ideal $\mathfrak{a} = (X^2, XY)$ is

$$(X^2, XY) = (X) \cap (X^2, XY, Y^n)$$

where (X) is prime and (X^2, XY, Y^n) is (X, Y) -primary. Note that (X) is isolated and (X, Y) is embedded.

Given a submodule N of a module M over a Noetherian ring A , to simplify we shall denote by $D_I(M/N)$, in this part, the set of reduced primary decompositions of N in M whose indexing set is I .

Proposition 11.2.11. Let A be a Noetherian ring, M an A -module, N a submodule of M and $I = \text{Ass}(M/N)$. Let S be a multiplicative subset of A and I_S the subset I consisting of the indices i such that $S \cap \mathfrak{p}_i = \emptyset$. Let $\phi(N)$ be the saturation of N with respect to S in M . Then:

- (a) If $(Q_i)_{i \in I}$ is an element of $D_I(M/N)$, the family $(Q_i)_{i \in I_S}$ is an element of $D_{I_S}(M/\phi(N))$ and the family $(S^{-1}Q_i)_{i \in I_S}$ is an element of $D_{I_S}(S^{-1}M/S^{-1}N)$.
- (b) The map $(Q_i)_{i \in I_S} \mapsto (S^{-1}Q_i)_{i \in I_S}$ is a bijection from $D_{I_S}(M/\phi(N))$ to $D_{I_S}(S^{-1}M/S^{-1}N)$.
- (c) If $(Q_i)_{i \in I_S}$ is an element of $D_{I_S}(M/\phi(N))$ and $(R_i)_{i \in I}$ is an element of $D_I(M/N)$, then the family $(T_i)_{i \in I}$ defined by $T_i = Q_i$ when $i \in I_S$ and $T_i = R_i$ for $i \in I - I_S$ is an element of $D_I(M/N)$.

Proof. We know that for $i \in I_S$, $S^{-1}Q_i$ is $S^{-1}\mathfrak{p}_i$ -primary and that for $i \in I - I_S$, $S^{-1}Q_i = S^{-1}M$. As $S^{-1}N = \bigcap_{i \in I} S^{-1}Q_i$, we see $S^{-1}N = \bigcap_{i \in I_S} S^{-1}Q_i$. The ideals $S^{-1}\mathfrak{p}_i$ for $i \in I_S$ are distinct and their set is $\text{Ass}(S^{-1}M/S^{-1}N)$ by ???. Then by ?? $(S^{-1}Q_i)_{i \in I_S}$ is a reduced primary decomposition of $S^{-1}N$. Moreover, Q_i is saturated for $i \in I_S$, hence $\phi(N) = (i_M^S)^{-1}(S^{-1}N) = \bigcap_{i \in I_S} Q_i$, and $(Q_i)_{i \in I_S}$ is obviously a reduced primary decomposition of $\phi(N)$ in M .

As $S^{-1}\phi(N) = S^{-1}N$, we may replace N by $\phi(N)$ and suppose that $I = I_S$. Let $(P_i)_{i \in I}$ be a reduced primary decomposition of $S^{-1}N$ in $S^{-1}M$ and let us write $Q_i = (i_M^S)^{-1}(P_i)$. It follows from ?? that Q_i is \mathfrak{p}_i -primary and $(Q_i)_{i \in I}$ is a reduced primary decomposition of N of M by virtue of Corollaries ???. Finally, since $I = I_S$, by ?? we see every \mathfrak{p}_i -primary submodule of M is saturated, so the two maps

$$D_I(M/N) \rightarrow D_I(S^{-1}M/S^{-1}N) \quad \text{and} \quad D_I(S^{-1}M/S^{-1}N) \rightarrow D_I(M/N)$$

are inverse of each other, which proves (b).

Finally, we prove (c). From (a) we have $\phi(N) = \bigcap_{i \in I_S} R_i$, whence

$$N = \phi(N) \cap \bigcap_{i \in I - I_S} R_i = \left(\bigcap_{i \in I_S} Q_i \right) \cap \left(\bigcap_{i \in I - I_S} R_i \right)$$

and it follows immediately from ?? that this primary decomposition is reduced. \square

Corollary 11.2.12. The maps

$$D_I(M/N) \rightarrow D_{I_S}(M/\phi(N)) \quad \text{and} \quad D_I(M/N) \rightarrow D_{I_S}(S^{-1}M/S^{-1}N)$$

defined in ?? are surjective.

Proof. ??(c) shows that the map $D_I(M/N) \rightarrow D_{I_S}(M/\phi(N))$ is surjective and ??(b) then shows that $D_I(M/N) \rightarrow D_{I_S}(M/N) \rightarrow D_{I_S}(S^{-1}M/S^{-1}N)$ is surjective. \square

11.2.3 Finite length modules

If an A -module M is of finite length, we shall denote this length by $\ell_A(M)$ or $\ell(M)$. Recall that every Artinian ring is Noetherian and that every finitely generated module over an Artinian ring is of finite length.

Proposition 11.2.13. *Let M be a finitely generated module over a Noetherian ring A . The following properties are equivalent:*

- (i) M is of finite length.
- (ii) Every ideal $\mathfrak{p} \in \text{Ass}(M)$ is a maximal ideal of A .
- (iii) Every ideal $\mathfrak{p} \in \text{supp}(M)$ is a maximal ideal of A .

Proof. Let (M_i) be a chain of submodules of M such that M_i/M_{i+1} is isomorphic to A/\mathfrak{p}_i where \mathfrak{p}_i is prime. If M is of finite length, so is each of the A -modules A/\mathfrak{p}_i , which implies that each of the rings A/\mathfrak{p}_i is Artinian. But as A/\mathfrak{p}_i is an integral domain, it is therefore a field, in other words \mathfrak{p}_i is maximal; we conclude that (i) implies (ii). Condition (ii) implies (iii) by ???. Finally, if all the ideals of $\text{supp}(M)$ are maximal, so are the \mathfrak{p}_i , hence the A/\mathfrak{p}_i are simple A -modules and M is of finite length, which completes the proof. \square

Corollary 11.2.14. *For every module of finite length M over a Noetherian ring A , we have $\text{Ass}(M) = \text{supp}(M)$.*

Proof. Every element of $\text{supp}(M)$ is then minimal in $\text{supp}(M)$ and the conclusion follows from ???. \square

Corollary 11.2.15. *Let M be a finitely generated module over a Noetherian ring A and \mathfrak{p} a prime ideal of A . For $M_\mathfrak{p}$ to be a non-zero $A_\mathfrak{p}$ -module of finite length, it is necessary and sufficient that \mathfrak{p} be a minimal element of $\text{Ass}(M)$.*

Proof. By ??, $\text{Ass}_{A_\mathfrak{p}}(M_\mathfrak{p})$ is the set of ideals $\mathfrak{q}A_\mathfrak{p}$, where \mathfrak{q} runs through the set of ideals of $\text{Ass}(M)$ which are contained in \mathfrak{p} . Thus by ??, for $M_\mathfrak{p}$ to be an $A_\mathfrak{p}$ -module of finite length, it is necessary and sufficient that no element of $\text{Ass}(M)$ be strictly contained in \mathfrak{p} . On the other hand, for $M_\mathfrak{p} \neq 0$, it is necessary and sufficient by definition that $\mathfrak{p} \in \text{supp}(M)$, that is that \mathfrak{p} contain an element of $\text{Ass}(M)$. This proves the corollary. \square

Remark 11.2.2. Note that ?? is not equivalent to the condition that each prime ideal $\mathfrak{p} \in \text{Ass}(M)$ is *maximal in $\text{Ass}(M)$* . In fact, this condition signifies that each element of $\text{Ass}(M)$ is minimal, so is a minimal prime ideal of $\text{supp}(M)$ (in other words, M has no embedded prime ideals).

Remark 11.2.3. Let M be a finitely generated module over a Noetherian ring A and let (M_i) be a chain of M such that M_i/M_{i+1} is isomorphic to A/\mathfrak{p}_i where \mathfrak{p}_i is a prime ideal of A . If \mathfrak{p} is a minimal element of $\text{Ass}(M)$, the modules $(M_i)_\mathfrak{p}$ then form a chain of $M_\mathfrak{p}$ and $(M_i)_\mathfrak{p}/(M_{i+1})_\mathfrak{p}$ is isomorphic to $(A/\mathfrak{p}_i)_\mathfrak{p}$ and hence to $\{0\}$ if $\mathfrak{p}_i \neq \mathfrak{p}$, and to $(A/\mathfrak{p})_\mathfrak{p}$ which is a field, if $\mathfrak{p}_i = \mathfrak{p}$.

Proposition 11.2.16. *Let M be a module of finite length over a Noetherian ring A .*

- (a) *There exists a unique primary decomposition $\{0\} = \bigcap_{\mathfrak{p} \in \text{Ass}(M)} Q(\mathfrak{p})$ of $\{0\}$ with respect to M indexed by $\text{Ass}(M)$, where $Q(\mathfrak{p})$ is \mathfrak{p} -primary with respect to M .*
- (b) *There exists an integer n_0 such that, for all $n \geq n_0$ and all $\mathfrak{p} \in \text{Ass}(M)$, $Q(\mathfrak{p}) = \mathfrak{p}^n M$.*
- (c) *For all $\mathfrak{p} \in \text{Ass}(M)$, the canonical map of M to $M_\mathfrak{p}$ is surjective and its kernel is $Q(\mathfrak{p})$. Therefore $M_\mathfrak{p}$ can be identified with $M/Q(\mathfrak{p})$.*
- (d) *The canonical injection of M into $\bigoplus_{\mathfrak{p} \in \text{Ass}(M)} (M/Q(\mathfrak{p}))$ is bijective.*

Proof. Since each ideal of M is maximal, part (a) follows from ???. As $M/Q(\mathfrak{p})$ is finitely generated, by ?? there exists $n_0 > 0$ such that $\mathfrak{p}^n M \subseteq Q(\mathfrak{p})$ for all $n \geq n_0$. But as \mathfrak{p} is a maximal ideal, $\mathfrak{p}^n M$ is \mathfrak{p} -primary with respect to M and, as $\bigcap_{\mathfrak{p} \in \text{Ass}(M)} \mathfrak{p}^n M = \{0\}$, it follows from (a) that necessarily $\mathfrak{p}^n M = Q(\mathfrak{p})$ for all $\mathfrak{p} \in \text{Ass}(M)$, whence (b) follows. As the \mathfrak{p}^n , for $\mathfrak{p} \in \text{Ass}(M)$, are relatively prime in pairs, the canonical map $M \mapsto \bigoplus_{\mathfrak{p} \in \text{Ass}(M)} (M/\mathfrak{p}^n M)$ is surjective, whence (d).

Now $\text{Ass}(Q(\mathfrak{p})) = \text{Ass}(M) - \{\mathfrak{p}\}$ and $\text{Ass}(M/Q(\mathfrak{p})) = \{\mathfrak{p}\}$. As the elements of $\text{Ass}(M)$ are maximal ideals, \mathfrak{p} is the only element of $\text{Ass}(M)$ which does not meet $A - \mathfrak{p}$. Then $Q(\mathfrak{p})$ is the kernel of the canonical map $i_{\mathfrak{p}} : M \rightarrow M_{\mathfrak{p}}$ by ???. If $s \in A - \mathfrak{p}$, the homothety of $M/Q(\mathfrak{p})$ with ratio s is injective by virtue of the relation $\text{Ass}(M/Q(\mathfrak{p})) = \{\mathfrak{p}\}$. Since $M/Q(\mathfrak{p})$ is Artinian, this homothety is bijective. Therefore, if $m \in M$ and $s \in A - \mathfrak{p}$, then there exists $n \in M$ such that $\bar{m} = s\bar{n}$ in $M/Q(\mathfrak{p})$, whence $m - ns \in Q(\mathfrak{p})$. By the characterization of $M_{\mathfrak{p}}$, we can then find $t \in A - \mathfrak{p}$ such that $t(m - ns) = 0$, that is, $m/s = n/t$ in $M_{\mathfrak{p}}$. This proves the canonical map $M \rightarrow M_{\mathfrak{p}}$ is surjective and finishes the proof. \square

Corollary 11.2.17. *If M is a module of finite length over a Noetherian ring A , then*

$$\ell_A(M) = \sum_{\mathfrak{p} \in \text{Ass}(M)} \ell_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}).$$

Proof. This will follow from ??(d) if we prove that

$$\ell_A(M/Q(\mathfrak{p})) = \ell_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}).$$

Now, since $\text{Ass}(M/Q(\mathfrak{p})) = \{\mathfrak{p}\}$, for all $s \in A - \mathfrak{p}$ the homothety with ratio s on $M/Q(\mathfrak{p})$ is injective; the homothety with ratio s on every submodule N of $M/Q(\mathfrak{p})$ is therefore injective and, as N is Artinian, it is bijective. We conclude that the sub- A -modules of $M/Q(\mathfrak{p})$ are the images under the bijection $M_{\mathfrak{p}} \rightarrow M/Q(\mathfrak{p})$ of the sub- $A_{\mathfrak{p}}$ -modules of $M_{\mathfrak{p}}$, whence our assertion. \square

Proposition 11.2.18. *Let A be a Noetherian ring. The following conditions are equivalent:*

- (i) A is Artinian.
- (ii) All the prime ideals of A are maximal ideals.
- (iii) All the prime ideals of $\text{Ass}(A)$ are maximal ideals.

If these conditions are fulfilled, A has only a finite number prime ideals, which are all maximal and associated with the A -module A . Further, A is a semi-local ring and its Jacobson radical is nilpotent.

Proof. To say that A is Artinian is equivalent to saying that A is an A -module of finite length, hence (i) and (ii) are equivalent by ???. We already know that (i) and (ii) are equivalent by ??.

Suppose they hold. As every prime ideal of A belongs to $\text{supp}(A)$ and every element of $\text{supp}(A)$ contains an element of $\text{Ass}(A)$, it follows from (iii) that $\text{Ass}(A)$ is the set of all prime ideals of A . Then A has only a finite number of prime ideals, all of them maximal and associated with the A -module A . Finally, we know that the Jacobson radical of an Artinian ring is nilpotent. \square

Corollary 11.2.19. *Every Artinian ring A is isomorphic to the direct composition of a finite family of Artinian local rings.*

Proof. It follows from ?? and ?? (c) and (d) that, if (\mathfrak{m}_i) is the family of maximal ideals of A , the canonical map $A \rightarrow \prod_i A_{\mathfrak{m}_i}$ is bijective. \square

Corollary 11.2.20. *Let A be a Noetherian ring and \mathfrak{a} an ideal of A . The following conditions are equivalent:*

- (i) A is a semi-local ring and \mathfrak{a} is a defining ideal of A .
- (ii) A is a Zariski ring with the \mathfrak{a} -adic topology and A/\mathfrak{a} is Artinian.

Proof. If (a) holds then by ?? we know A is a Zariski ring with the \mathfrak{r} -adic topology, where \mathfrak{r} is the Jacobson radical of A . Moreover, as by hypothesis \mathfrak{a} contains a power of \mathfrak{r} , every prime ideal of A which contains \mathfrak{a} also contains \mathfrak{r} and is therefore maximal, since \mathfrak{r} is a finite intersection of maximal ideals. This shows that A/\mathfrak{a} is Artinian.

Conversely, if (b) holds, then $\mathfrak{a} \subseteq \mathfrak{r}$, so every maximal ideal \mathfrak{p} of A containing \mathfrak{r} must contain \mathfrak{a} . As A/\mathfrak{a} is Artinian, the ideals $\mathfrak{p}/\mathfrak{a}$ are finite in number (??) and hence A has only a finite number of maximal ideals, which implies that it is semi-local. \square

Corollary 11.2.21. *Let $\rho : A \rightarrow B$ be a ring homomorphism. Suppose that A is semi-local and Noetherian and that B is a finitely generated A -module. Then the ring B is semi-local and Noetherian. If \mathfrak{a} is a defining ideal of A , then $\mathfrak{b} = \mathfrak{a}^e$ is a defining ideal of B .*

Proof. We know that B is a Zariski ring with the \mathfrak{b} -adic topology (??). As A/\mathfrak{a} is Artinian by ?? and B/\mathfrak{b} is a finitely generated (A/\mathfrak{a}) -module, it is an Artinian ring, hence B is semilocal and \mathfrak{b} is a defining ideal of B by ??.

Corollary 11.2.22. *Let A be a complete semi-local Noetherian ring, \mathfrak{a} a defining ideal of A , M a finitely generated A -module and (N_n) a decreasing sequence of submodules of M such that $\bigcap_n N_n = 0$. Then, for all $p > 0$, there exists $n > 0$ such that $N_n \subseteq \mathfrak{m}^p M$.*

Proposition 11.2.23. *Let A be a Noetherian ring and $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ the prime ideals associated with the A -module A .*

- (a) *The set $S = \bigcap_{i=1}^n (A - \mathfrak{p}_i)$ is the set of elements which are not divisors of 0 in A .*
- (b) *If all the \mathfrak{p}_i are minimal elements of $\text{Ass}(A)$, then the total ring of fractions $S^{-1}A$ of A is Artinian.*
- (c) *If the ring A is reduced, then all the \mathfrak{p}_i are minimal elements of $\text{Ass}(A)$ (and therefore are the minimal prime ideals of A) and each of the $A_{\mathfrak{p}_i}$ is a field. For each index i , the canonical homomorphism $S^{-1}A \rightarrow A_{\mathfrak{p}_i}$ is surjective and its kernel is $S^{-1}\mathfrak{p}_i$. Finally, the canonical homomorphism $S^{-1}A \rightarrow \prod_{i=1}^n (S^{-1}A / S^{-1}\mathfrak{p}_i) \cong \prod_{i=1}^n A_{\mathfrak{p}_i}$ is bijective.*

Proof. The fact that S is the set of elements which are not divisors of 0 in A has already been seen in ?. The prime ideals of $S^{-1}A$ are of the form $S^{-1}\mathfrak{p}$, where \mathfrak{p} is a prime ideal of A contained in $\bigcup_i \mathfrak{p}_i$, and hence is contained in one of the \mathfrak{p}_i . If \mathfrak{p}_i is a minimal element of $\text{Ass}(A)$, it is then a minimal element of $\text{Spec}(A) = \text{supp}(A)$. If each of the \mathfrak{p}_i is a minimal element of $\text{Ass}(A)$, we then see that the prime ideals of $S^{-1}A$ are the $S^{-1}\mathfrak{p}_i$ and they are therefore all maximal, which proves that $S^{-1}A$ is Artinian.

Suppose finally that the ring A is reduced. Then $\bigcap_i \mathfrak{p}_i = \{0\}$, and we deduce that $\{0\} = \bigcap_i \mathfrak{p}_i$ is a reduced primary decomposition of the ideal $\{0\}$ by ???. In particular, none of the \mathfrak{p}_i can contain a \mathfrak{p}_j of index $j \neq i$ and therefore the \mathfrak{p}_i are all minimal elements of $\text{Ass}(A)$. The ring $S^{-1}A$ is then Artinian by (b). The $S^{-1}\mathfrak{p}$ are prime ideals associated with the $S^{-1}A$ -module $S^{-1}A$ and $\{0\} = S^{-1}(\bigcap_{i=1}^n \mathfrak{p}_i) = \bigcap_{i=1}^n S^{-1}\mathfrak{p}_i$. As the $S^{-1}\mathfrak{p}_i$ are distinct, $(S^{-1}\mathfrak{p}_i)$ is a reduced primary decomposition of $\{0\}$ in $S^{-1}A$ by ???. ?? then shows that the canonical homomorphism $\phi_i : S^{-1}A \rightarrow (S^{-1}A)_{\mathfrak{p}_i}$ is surjective and has kernel $S^{-1}\mathfrak{p}_i$, and the canonical homomorphism

$$S^{-1}A \rightarrow \prod_{i=1}^n (S^{-1}A / S^{-1}\mathfrak{p}_i)$$

is bijective. We know moreover that the canonical homomorphism $S^{-1}A \rightarrow A_{\mathfrak{p}_i}$ is given by

$$S^{-1}A \xrightarrow{\phi_i} (S^{-1}A)_{\mathfrak{p}_i} \xrightarrow{\cong} (S^{-1}A)_{S^{-1}\mathfrak{p}_i} \xrightarrow{\cong} A_{\mathfrak{p}_i}$$

Finally, it follows from ?? that $(S^{-1}A)_{S^{-1}\mathfrak{p}_i}$ is isomorphic to $S^{-1}A / S^{-1}\mathfrak{p}_i$, and hence is a field since $S^{-1}\mathfrak{p}_i$ is a maximal ideal. \square

11.2.4 Extension of scalars

In this part, A and B will denote two rings and we shall consider a ring homomorphism $\rho : A \rightarrow B$ which makes B into an A -algebra. Recall that, for every B -module P , $\rho^*(P)$ is the A -module whose structure is defined by $a \cdot y = \rho(a)y$ for all $a \in A, y \in P$.

Lemma 11.2.24. *Let A be a Noetherian ring, \mathfrak{p} a prime ideal of A , M an A -module whose annihilator contains a power of \mathfrak{p} and such that $\text{Ass}_A(M) = \{\mathfrak{p}\}$. Let P be a B -module such that $\rho^*(P)$ is a flat A -module. The condition $\mathfrak{P} \in \text{Ass}_B(M \otimes_A P)$ then implies $\mathfrak{P}^c = \mathfrak{p}$.*

Proof. If n is such that $\mathfrak{p}^n M = 0$, then $\mathfrak{p}^n B \subseteq \text{Ann}(M \otimes_A P)$, whence $\mathfrak{p}^n B \subseteq \mathfrak{P}$, which implies $\mathfrak{p}^n \subseteq \mathfrak{P}^c$ and therefore $\mathfrak{p} \subseteq \mathfrak{P}$ since \mathfrak{P}^c is prime. Moreover, if $a \in A - \mathfrak{p}$, the homothety h_a with ratio a on M is injective. As $h_a \otimes 1_P$ is the homothety $h_{\rho(a)}$ with ratio $\rho(a)$ on $M \otimes_A P$ and $\rho^*(P)$ is flat, we see $h_{\rho(a)}$ is injective. This proves that $\rho(a) \notin \mathfrak{P}$, whence $\mathfrak{P}^c = \mathfrak{p}$. \square

Proposition 11.2.25. *Let $\rho : A \rightarrow B$ be a ring homomorphism, M an A -module and P a B -module such that $\rho^*(P)$ is a flat A -module. Then*

$$\text{Ass}_B(M \otimes_A P) \supseteq \bigcup_{\mathfrak{p} \in \text{Ass}_A(M)} \text{Ass}_B(P / \mathfrak{p}P) \tag{11.2.3}$$

and the equality holds if A is Noetherian.

Proof. Let $\mathfrak{p} \in \text{Ass}_A(M)$. By definition there exists an injection $A / \mathfrak{p} \rightarrow M$, and by tensoring with P we get an injection $P / \mathfrak{p}P \rightarrow M \otimes_A P$, whence $\text{Ass}_B(P / \mathfrak{p}P) \subseteq \text{Ass}_B(M \otimes_A P)$ and (??) follows. Suppose now that A is Noetherian and let us prove the opposite inclusion.

Suppose first that M is a finitely generated A -module and that $\text{Ass}_A(M)$ is reduced to a single element \mathfrak{p} . By ?? there exists a chain (M_i) of M such that M_i / M_{i+1} is isomorphic to A / \mathfrak{p}_i where \mathfrak{p}_i is a prime ideal of A containing \mathfrak{p} . As P is a flat A -module, $(M_i \otimes_A P)$ is then a chain

of $M \otimes_A P$ such that

$$(M_i \otimes_A P) / (M_{i+1} \otimes_A P) = (A/\mathfrak{p}_i) \otimes_A P = P/\mathfrak{p}_i P.$$

By ??, we get

$$\text{Ass}_B(M \otimes_A P) \subseteq \bigcup_{i=0}^{n-1} \text{Ass}_B(P/\mathfrak{p}_i P).$$

We know that M is annihilated by a power of \mathfrak{p} by ??, so ?? shows that $\mathfrak{P}^c = \mathfrak{p}$ for all $\mathfrak{P} \in \text{Ass}_B(M \otimes_A P)$. Also, since $P/\mathfrak{p}_i P$ is annihilated by \mathfrak{p}_i , by ?? we have $(\mathfrak{P}')^c = \mathfrak{p}_i$ for all $\mathfrak{P}' \in \text{Ass}_B(P/\mathfrak{p}_i P)$. This then shows $\text{Ass}_B(M \otimes_A P) \cap \text{Ass}_B(P/\mathfrak{p}_i P) = \emptyset$ if $\mathfrak{p}_i \neq \mathfrak{p}$, which proves the claim in this case.

Next we drop the assumption $\text{Ass}(M) = \{\mathfrak{p}\}$ and assume only that M is finitely generated. Let $\text{Ass}_A(M) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$ and let $\{0\} = \bigcap_i Q_i$ be a corresponding reduced primary decomposition of $\{0\}$. Then M is isomorphic to a submodule of the direct sum of the submodules $M_i = M/Q_i$ and, as $f^*(P)$ is a flat A -module, $M \otimes_A P$ is isomorphic to a submodule of the direct sum of the submodules $M_i \otimes_A P$. We deduce that

$$\text{Ass}_B(M \otimes_A P) \subseteq \bigcup_{i=1}^n \text{Ass}_B(M_i \otimes_A P).$$

But M_i is a finitely generated A -module such that $\text{Ass}_A(M_i)$ is reduced to a single element \mathfrak{p}_i . By what we have proved, $\text{Ass}_B(M_i \otimes_A P) = \text{Ass}_B(P/\mathfrak{p}_i P)$, whence the claim in this case.

Now we turn to the general case. The B -module $M \otimes_A P$ is the union of the submodules $N \otimes_A P$, where N runs through the set of finitely generated submodules of the A -module M . If \mathfrak{P} belongs to $\text{Ass}_B(M \otimes_A P)$, then there exists a finitely generated submodule N of M such that $\mathfrak{P} \in \text{Ass}_B(N \otimes_A P)$. By the preceding argument, there exists $\mathfrak{p} \in \text{Ass}_A(N)$ such that $\mathfrak{P} \in \text{Ass}_B(P/\mathfrak{p}P)$. As $\text{Ass}_A(N) \subseteq \text{Ass}_A(M)$, this completes the proof. \square

Corollary 11.2.26. *In the situation of ??, if A is Noetherian and $\mathfrak{P} \in \text{Ass}_B(M \otimes_A P)$, then $\mathfrak{P}^c \in \text{Ass}_A(M)$ and \mathfrak{P}^c is the only prime ideal \mathfrak{p} of A such that $\mathfrak{P} \in \text{Ass}_B(P/\mathfrak{p}P)$.*

Proof. By ?? we see $\mathfrak{P} \in \text{Ass}_B(P/\mathfrak{p}P) = \text{Ass}_B((A/\mathfrak{p}) \otimes_A P)$ where $\mathfrak{p} \in \text{Ass}_A(M)$. Then ?? says $\mathfrak{P}^c = \mathfrak{p}$. \square

Corollary 11.2.27. *Suppose that A and B are Noetherian and that B is a flat A -module. Let \mathfrak{p} be a prime ideal of A , Q a \mathfrak{p} -primary submodule of an A -module M and \mathfrak{P} a prime ideal of B . For $Q \otimes_A B$ to be a \mathfrak{P} -primary submodule of $M \otimes_A B$, it is necessary and sufficient that $\mathfrak{p}B$ be a \mathfrak{P} -primary ideal of B .*

Proof. Let us apply ?? to the A -module M/Q and the B -module B . Then $\text{Ass}_A(M/Q) = \{\mathfrak{p}\}$ and $(M/Q) \otimes_A B$ is isomorphic to $(M \otimes_A B)/(Q \otimes_A B)$ and hence $\text{Ass}_B((E \otimes_A B)/(Q \otimes_A B)) = \text{Ass}_B(B/\mathfrak{p}B)$. To say that $Q \otimes_A B$ is \mathfrak{P} -primary in $M \otimes_A B$ therefore means that $\text{Ass}_B(B/\mathfrak{p}B)$ is reduced to $\{\mathfrak{P}\}$, whence the corollary. \square

Proposition 11.2.28. *Suppose that A and B are Noetherian and that B is a flat A -module. Let M be an A -module and N a submodule of M such that, for every prime ideal $\mathfrak{p} \in \text{Ass}_A(M/N)$, $\mathfrak{p}B$ is a prime ideal of B or equal to B . Let $N = \bigcap_{\mathfrak{p} \in \text{Ass}_A(M/N)} Q(\mathfrak{p})$ be a reduced primary decomposition of N in M , $Q(\mathfrak{p})$ being \mathfrak{p} -primary for all $\mathfrak{p} \in \text{Ass}(M/N)$.*

- (a) If $\mathfrak{p} \in \text{Ass}(M/N)$ and $\mathfrak{p}B = B$, then $Q(\mathfrak{p}) \otimes_A B = M \otimes_A B$.
- (b) If $\mathfrak{p} \in \text{Ass}(M/N)$ and $\mathfrak{p}B$ is prime, then $Q(\mathfrak{p}) \otimes_A B$ is $\mathfrak{p}B$ -primary in $M \otimes_A B$.
- (c) If Φ is the set of $\mathfrak{p} \in \text{Ass}(M/N)$ such that $\mathfrak{p}B$ is prime, then

$$N \otimes_A B = \bigcap_{\mathfrak{p} \in \Phi} (Q(\mathfrak{p}) \otimes_A B)$$

and this relation is a reduced primary decomposition of $N \otimes_A B$ in $M \otimes_A B$.

Proof. If $\mathfrak{p}B = B$, ?? applied to $M/Q(\mathfrak{p})$ and B shows that

$$\text{Ass}_B((M/Q(\mathfrak{p})) \otimes_A B) = \emptyset$$

and, as B is Noetherian and is a flat A -module, we conclude that $Q(\mathfrak{p}) \otimes_A B = M \otimes_A B$. Assertion (b) follows from ??, taking $\mathfrak{P} = \mathfrak{p}B$. Finally the relation $N \otimes_A B = \bigcap_{\mathfrak{p} \in \Phi} (Q(\mathfrak{p}) \otimes_A B)$ follows from the fact that B is a flat A -module. Also, for each $\mathfrak{p} \in \Phi$ we have $\text{Ass}_B((M/Q(\mathfrak{p})) \otimes B) = \{\mathfrak{p}B\}$, whence $\mathfrak{p} = (\mathfrak{p}B)^c$ by ??, and $\mathfrak{p}B \neq \mathfrak{q}B$ for two distinct ideals $\mathfrak{p}, \mathfrak{q}$ of Φ . On the other hand, by ??,

$$\text{Ass}((M \otimes_A B)/(N \otimes_A B)) = \Phi$$

we conclude from ?? that

$$N \otimes_A B = \bigcap_{\mathfrak{p} \in \Phi} (Q(\mathfrak{p}) \otimes_A B)$$

is a reduced primary decomposition. \square

Corollary 11.2.29. *In the situation of ??, suppose that $\mathfrak{p}B$ is prime for all $\mathfrak{p} \in \text{Ass}_A(M/N)$. Then, if $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ are the minimal elements of $\text{Ass}_A(M/N)$, the \mathfrak{p}_iB are minimal elements of $\text{Ass}_A((M \otimes_A B)/(N \otimes_A B))$.*

Proof. It is easy to see $\mathfrak{p}B$ is a minimal prime if \mathfrak{p} is, and by ?? we see $\mathfrak{p}_iB \neq \mathfrak{p}_jB$ when $i \neq j$. \square

Example 11.2.30.

- (a) Let us take $B = S^{-1}A$, where S is a multiplicative subset of A . If A is Noetherian, the hypotheses of ?? are satisfied and we recover a part of ??.
- (b) Let A be a Noetherian ring, \mathfrak{a} an ideal of A and B the Hausdorff completion of A with respect to the \mathfrak{a} -adic topology. Then B is a flat A -module and ?? may be applied with $P = B$. But in general the hypotheses of ?? are not satisfied for the prime ideals of A .
- (c) Let A be a Noetherian ring and B the polynomial algebra $A[X_1, \dots, X_n]$. Then B is Noetherian and is a free A -module and therefore flat. Also, if \mathfrak{p} is a prime ideal of A , $B/\mathfrak{p}B$ is isomorphic to $(A/\mathfrak{p})[X_1, \dots, X_n]$, which is an integral domain, and hence $\mathfrak{p}B$ is prime. The hypotheses of ?? are therefore satisfied for every A -module M and every submodule N of M .

11.2.5 Primary decompositions of graded modules

Proposition 11.2.31. *Let Δ be a torsion free commutative group, A a graded ring of type Δ and M a graded A -module of type Δ . Every prime ideal associated with M is graded and is the annihilator of a homogeneous element of M .*

Proof. We know that Δ can be given a total order structure compatible with its group structure. Let \mathfrak{p} be a prime ideal associated with M , the annihilator of an element $x \in M$, and let $(x_i)_{i \in \Delta}$ be the family of homogeneous components of x ; let $i(1) < i(2) < \dots < i(r)$ be the values of i for which $x_i \neq 0$. Consider an element $a \in \mathfrak{p}$ and let $(a_i)_{i \in \Delta}$ be the family of its homogeneous components; we shall prove that $a_i \in \mathfrak{p}$ for all $i \in \Delta$, which will show that \mathfrak{p} is a homogeneous ideal.

We argue by induction on the number of indices i such that $a_i \neq 0$. Our assertion is obvious if this number is 0; if not, let m be the greatest of the indices i for which $a_i \neq 0$; if we prove that $a_m \in \mathfrak{p}$, the induction hypothesis applied to $a - a_m$ will give the conclusion. Now, $ax = 0$; for all $j \in \Delta$, using the fact that the homogeneous component of degree $m + j$ of ax is 0, we obtain $\sum_{i \in \Delta} a_{m-i} x_{j+i} = 0$; we conclude that $a_m x_j$ is a linear combination of the x_i of indices $i > j$. In particular, $a_m x_{i(r)} = 0$, whence, by descending induction on $k < r$, $a_m^{r-k+1} x_{i(k)} = 0$. Then $a_m^r x = 0$, whence $a_m^r \in \mathfrak{p}$ and, as \mathfrak{p} is prime, $a_m \in \mathfrak{p}$.

We now show that \mathfrak{p} is the annihilator of a homogeneous element of M . Let us write $\mathfrak{b}_n = \text{Ann}(x_{i(n)})$ for $1 \leq n \leq r$. For every homogeneous element b of \mathfrak{p} and all n the homogeneous component of bx of degree $i(n) + \deg(b)$ is $bx_{i(n)}$, hence $bx_{i(n)} = 0$ and therefore $b \in \mathfrak{b}_n$; as \mathfrak{p} is generated by its homogeneous elements, $\mathfrak{p} \subseteq \mathfrak{b}_n$. On the other hand, clearly $\bigcap_{n=1}^r \mathfrak{b}_n = \mathfrak{p}$; as \mathfrak{p} is prime, there exists an n such that $\mathfrak{b}_n \subseteq \mathfrak{p}$ (??), whence $\mathfrak{b}_n = \mathfrak{p} = \text{Ann}(x_{i(n)})$, which completes the proof. \square

Corollary 11.2.32. *For every (necessarily homogeneous) prime ideal \mathfrak{p} associated with a graded A -module M , there exists an index $k \in \Delta$ such that the shifted graded A -module $(A/\mathfrak{p})(k)$ is isomorphic to a graded submodule of M .*

Proof. With the notation of the proof of ??, consider the homomorphism obtained, by taking quotients, from the homomorphism $a \mapsto ax_{i(n)}$ of A to M ; the latter is a graded homomorphism of degree $i(n)$ and hence it gives on taking quotients a graded bijective homomorphism of degree $i(n)$ of A/\mathfrak{p} onto a graded submodule of M . \square

Proposition 11.2.33. *Let Δ be a torsion-free commutative group, A a graded Noetherian ring of type Δ and M a graded finitely generated A -module of type Δ . There exists a composition series $(M_i)_{0 \leq i \leq n}$ consisting of graded submodules of M such that the graded module M_i/M_{i+1} is isomorphic to a shifted graded module $(A/\mathfrak{p}_i)(k_i)$, where \mathfrak{p}_i is a homogeneous prime ideal of A and $k_i \in \Delta$.*

Proof. It is sufficient to retrace the argument of ?? taking on this occasion \mathcal{M} to be the set of graded submodules of M with a composition series with the properties of the statement; we conclude using ??.

Proposition 11.2.34. *Let Δ be a torsion-free commutative group, A a graded Noetherian ring of type Δ , \mathfrak{p} a homogeneous ideal of A and M a graded A -module of type Δ not reduced to 0. Suppose that for every homogeneous element a of \mathfrak{p} the homothety of ratio a on M is almost nilpotent and that for every homogeneous element b of $A - \mathfrak{p}$ the homothety of ratio b on M is injective. Then \mathfrak{p} is prime and the submodule $\{0\}$ of M is \mathfrak{p} -primary.*

Proof. It suffices to show that $\text{Ass}(M) = \{p\}$ (??). Let q be a prime ideal associated with M ; it is a homogeneous ideal and it is the annihilator of a homogeneous element $x \neq 0$ of M by ???. For every homogeneous element a of q , $ax = 0$ and hence the homothety of ratio a on M is not injective, whence $a \in p$. Conversely, let b be a homogeneous element of p ; there exists an integer $n > 0$ such that $b^n x = 0$, whence $b^n \in \text{Ann}(x) = q$ and, as q is prime, $b \in q$. As p and q are generated by their respective homogeneous element, $p = q$, which proves that $\text{Ass}(M) \subseteq \{p\}$. As $M \neq \{0\}$, $\text{Ass}(M) \neq \emptyset$ (??), whence $\text{Ass}(M) = \{p\}$. \square

Proposition 11.2.35. *Let Δ be a torsion-free commutative group, A a graded Noetherian ring of type Δ and M a graded A -module of type Δ . Let p be a prime ideal of A and N a submodule of M which is p -primary with respect to M .*

- (a) *The homogeneous ideal p^h of A is prime.*
- (b) *The graded submodule N^h of N is p^h -primary with respect to M .*

Proof. We know that the homogeneous elements of p^h (resp. N^h) are just the homogeneous elements of p (resp. N). Let a be a homogeneous element of p ; if x is a homogeneous element of M , there exists an integer $n > 0$ such that $a^n x \in N$; as $a^n x$ is homogeneous, $a^n x \in N^h$; as every $y \in M$ is the direct sum of a finite number of homogeneous elements, we conclude that there exists an integer $m > 0$ such that $a^m y \in N^h$, so that the homothety with ratio a in M/N^h is almost nilpotent.

Consider now a homogeneous element b of $A - p^h$; then $b \notin p$ since b is homogeneous. Let x be an element of M such that $bx \in N^h$ and let $(x_i)_{i \in \Delta}$ be the family of homogeneous components of x . As N^h is graded, $bx_i \in N^h$ for all i , hence $bx_i \in N$ and, as $b \notin p$, we conclude that $x_i \in N$; as x_i is homogeneous, $x_i \in N^h$, whence $x \in N^h$ and the homothety with ratio b on M/N^h is injective. The proposition then follows from ?? applied to p^h and M/N^h . \square

Proposition 11.2.36. *Let Δ be a torsion-free commutative group, A a graded Noetherian ring of type Δ , M a graded A -module of type M , N a graded submodule of M and $N = \bigcap_{i \in I} Q_i$ a primary decomposition of N in M .*

- (a) *Let Q_i^h be the graded submodule of Q_i , then the Q_i^h are primary and $N = \bigcap_{i \in I} Q_i^h$.*
- (b) *If the primary decomposition $N = \bigcap_{i \in I} Q_i$ is reduced, so is the primary decomposition $N = \bigcap_{i \in I} Q_i^h$, and for all $i \in I$ the prime ideals corresponding to Q_i and Q_i^h are equal.*
- (c) *If Q_i corresponds to a prime ideal p_i which is a minimal element $\text{Ass}(M/N)$, Q_i is a graded submodule of M .*

Proof. We have seen (??) that the Q_i^h are primary with respect to M and $N \subseteq Q_i^h \subseteq Q_i$, which proves (a). ?? also shows that the prime ideal $p_i i^h$ corresponding to Q_i^h is the largest homogeneous ideal contained in the prime ideal p_i corresponding to Q_i . If the decomposition $N = \bigcap_{i \in I} Q_i$ is reduced, $p_i \in \text{Ass}(M/N)$ for all i (??), hence p_i is a graded ideal (??) and therefore $p_i = p_i^h$; then $\text{Ass}(M/N) = \bigcup_{i \in I} \{p_i\}$ (??), which proves that the decomposition $N = \bigcap_{i \in I} Q_i^h$ is reduced (??). Finally, if p_i is a minimal element of $\text{Ass}(M/N)$, then $p_i^h = p_i$ since p_i is homogeneous (??), whence $Q_i^h = Q_i$, by virtue of ??.

Chapter 12

Integral dependence

12.1 Integral elements

Proposition 12.1.1. Let A be a ring, R an algebra over A and x an element of R . The following properties are equivalent:

- (i) x is a root of a monic polynomial in the polynomial ring $A[X]$.
- (ii) The subalgebra $A[x]$ of R is a finitely generated A -module.
- (iii) There exists a faithful $A[x]$ -module which is a finitely generated A -module.

Proof. The implications $(i) \Rightarrow (ii) \Rightarrow (iii)$ are clear. Now assume that there exists a faithful $A[x]$ -module M which is finitely generated as an A -module. Let ϕ to be multiplication by x and $I = A$ (we have $xM \subseteq M$ since M is an $A[x]$ -module). Since M is faithful, we have $x^n + a_{n-1}x^{n-1} + \dots + a_0 = 0$ for suitable $a_i \in A$ in view of ??, whence (i) holds. \square

An element $x \in R$ is called integral over A if it satisfies the equivalent properties of ???. A relation of the form $P(x) = 0$, where P is a monic polynomial in $A[X]$ is also called an equation of integral dependence with coefficients in A .

Example 12.1.2.

- (a) Let K be a field and R a K -algebra; to say that an element $x \in R$ is integral over K is equivalent to saying that x is a root of a non-constant polynomial in the ring $K[X]$. Generalizing the terminology introduced when R is an extension of K , the elements $x \in R$ which are integral over K are also called the **algebraic elements** of R over K .
- (b) The elements of $\mathbb{Q}(i)$ which are integral over \mathbb{Z} are the elements of the form $a + ib$ where $a \in \mathbb{Z}$ and $b \in \mathbb{Z}$ ("Gaussian integers"); the elements of $\mathbb{Q}(\sqrt{5})$ which are integral over \mathbb{Z} are the elements of the form $(a + b\sqrt{5})/2$ where a and b belong to \mathbb{Z} .

Proposition 12.1.3. Let A be a ring, R an algebra over A and x an element of R . For x to be integral over A , it is necessary and sufficient that $A[x]$ be contained in a subalgebra R' of R which is a finitely generated A -module.

Proof. The condition is obviously necessary by virtue of ???. It is also sufficient by virtue of ??, for R' is a faithful $A[x]$ -module (since it contains the unit element of R). \square

Corollary 12.1.4. Let A be a Noetherian ring, R an A -algebra and x an element of R . For x to be integral over A , it is necessary and sufficient that there exist a finitely generated submodule of R containing $A[x]$.

Proof. The condition is necessary by virtue of ???. It is sufficient for if $A[x]$ is a sub- A -module of a finitely generated A -module, it is itself a finitely generated A -module. \square

Let A be a ring. An A -algebra R is called **integral over A** if every element of R is integral over A . Recall that R is called finite over A if R is a finitely generated A -module, and finite type if a finitely generated A -algebra. It follows from ?? that every finite A -algebra is integral.

Example 12.1.5. If M is a finitely generated A -module, the algebra $\text{End}_A(M)$ of endomorphisms of M is integral over A by virtue of ???. In particular, for every integer n , the matrix algebra $\mathcal{M}_n(A) = \text{End}_A(A^n)$ is integral (and even finite) over A .

Proposition 12.1.6. Let A, B be two rings, R an A -algebra, S an B -algebra, and $\rho : A \rightarrow B$ and $\tau : R \rightarrow S$ two ring homomorphisms such that the diagram

$$\begin{array}{ccc} A & \xrightarrow{\rho} & B \\ \downarrow & & \downarrow \\ R & \xrightarrow{\tau} & S \end{array}$$

is commutative. If an element $x \in R$ is integral over A , then $\tau(x)$ is integral over B .

Proof. If $P(x) = 0$ where $P \in A[X]$, then $\tau(x)$ is a zero of $\rho(P)$, hence $\tau(x)$ is integral over B . \square

Corollary 12.1.7. Let A be a ring, B an A -algebra and C a B -algebra. Then every element $x \in C$ which is integral over A is integral over B .

Corollary 12.1.8. Let K be a field, L an extension of K and x, y two elements of L which are conjugate over K . If A is a subring of K and x is integral over A , then y is also integral over A .

Proof. There exists a K -isomorphism σ of $K(x)$ onto $K(y)$ such that $\sigma(x) = y$ and the elements of A are invariant under σ . \square

Proposition 12.1.9. Let $(R_i)_{1 \leq i \leq n}$ be a finite family of A -algebras and let $R = \prod_{i=1}^n R_i$ be their product. For an element $x = (x_i)$ of R to be integral over A , it is necessary and sufficient that each of the x_i be integral over A . For R to be integral over A , it is necessary and sufficient that each of the R_i be integral over A .

Proof. It is obviously sufficient to prove the first assertion. The condition is necessary by ???. Conversely, if each of the x_i is integral over A , the subalgebra $A[x_i]$ of R_i is a finitely generated A -module and hence so is the subalgebra $\prod_{i=1}^n A[x_i]$ of R . As $A[x]$ is contained in this subalgebra, x is integral over A . \square

Proposition 12.1.10. Let A be a ring, B an A -algebra and let x_1, \dots, x_n be integral over A . Then the ring $A[x_1, \dots, x_n]$ is a finitely generated A -module.

Proof. Proof by induction. For $n = 1$, this is a part of ???. Assume $n > 1$, let $A_{n-1} = A[x_1, \dots, x_{n-1}]$; then by the induction hypothesis A_{n-1} is a finitely generated A -module. Moreover, $A_n = A_{n-1}[x_n]$ is a finitely generated A_{n-1} -module by the case $n = 1$, since x_n is integral over A_{n-1} . Hence A_n is finitely generated as an A -module. \square

Corollary 12.1.11. *Let A be a ring, R an A -algebra and E a set of elements of R which are integral over A . Then the sub- A -algebra of R generated by E is integral over A .*

Proof. Let B be the A -algebra generated by E in R . Then every element of B belongs to a sub- A -algebra of B generated by a finite subset of E , whence is integral over A by ???. \square

Corollary 12.1.12. *The set \bar{A} of elements of B which are integral over A is a subring of B containing A .*

Proof. If $x, y \in \bar{A}$ then $A[x, y]$ is a finitely generated A -module by ???. Hence $x \pm y$ and xy are integral over A , by (c) of ???. \square

Proposition 12.1.13. *Let A be a ring and B and R two A -algebras. If R is integral over A , then $R \otimes_A B$ is integral over B .*

Proof. Consider any element $y = \sum_i \otimes b_i$ of $R \otimes_A B$, where the x_i belong to R and the b_i to B . As $x_i \otimes a_i = (x_i \otimes 1)a_i$ and the $x_i \otimes 1$ are integral over B by ??, so is x . \square

Corollary 12.1.14. *Let R be a ring and A, B, C subrings of R such that $A \subseteq B$. If B is integral over A , then $C[B]$ is integral over $C[A]$.*

Proof. The rng $B \otimes_A C[A]$ is integral over $C[A]$ by ?? and hence so is the canonical image $C[B]$ of $B \otimes_A C[A]$ in R (considered as an A -algebra) by ???. \square

Proposition 12.1.15. *Let A be a ring and B an A -algebra. If B integral over A , \mathfrak{b} is an ideal of B and $\mathfrak{a} = \mathfrak{b}^c$, then B/\mathfrak{b} is integral over A/\mathfrak{a} .*

Proof. The follows by reducing an integral equation of an element of B mod \mathfrak{b} . \square

Proposition 12.1.16. *Let A be a ring, B an A -algebra and C a B -algebra. If B is integral over A and C is integral over B , then C is integral over A .*

Proof. It is sufficient to verify that every $x \in C$ is integral over A . Then we have an equation

$$x^n + b_{n-1}x^{n-1} + \cdots + b_0, \quad b_i \in B.$$

The ring $B' = A[b_1, \dots, b_n]$ is a finitely generated A -module by ??, and $B'[X]$ is a finitely generated B' -module (since x is integral over B'). Hence $B'[X]$ is a finitely generated A -module and therefore x is integral over A by (c) of ???. \square

Corollary 12.1.17. *Let A be a ring and R, S two A -algebras integral over A . Then $R \otimes_A S$ is integral over A .*

Proof. The ring $R \otimes_A S$ is integral over S by ?? and hence the conclusion follows from ???. \square

12.1.1 The integral closure of a ring

Let A be a ring and R a A -algebra. The sub- A -algebra \bar{A} of R consisting of the elements of R integral over A is called the integral closure of A in R . If \bar{A} is equal to the canonical image of A in R , A is called integrally closed in R .

Remark 12.1.1. If A is a field, the integral closure \bar{A} of A in R consists of the elements of R which are algebraic over A . Generalizing the terminology used for field extensions, \bar{A} is then also called the algebraic closure of the field A in the algebra R and A is called algebraically closed in R if $\bar{A} = A$.

If A is an integral domain, the integral closure of A in its field of fractions is called the **integral closure of A** . An integral domain is called **integrally closed** if it is equal to its integral closure.

Proposition 12.1.18. *Let A be a ring and R an A -algebra. The integral closure \bar{A} of A in R is a subring integrally closed in R .*

Proof. The integral closure of \bar{A} in R is integral over A . It is therefore equal to \bar{A} . \square

Corollary 12.1.19. *The integral closure of an integral domain A is an integrally closed domain.*

Proof. Let K be the field of fractions of A and B the integral closure of A . Clearly K is the field of fractions of B and it is sufficient to apply ?? to $R = K$. \square

Proposition 12.1.20. *Let R be a ring, $(B_i)_{i \in I}$ a family of subrings of R and for each $i \in I$ let A_i be a subring of B_i . If each A_i is integrally closed in B_i , then $A = \bigcap_i A_i$ is integrally closed in $B = \bigcap_i B_i$.*

Proof. This follows from ??.

Corollary 12.1.21. *Every intersection of a non-empty family of integrally closed subdomains of an integral domain A is an integrally closed domain.*

Proof. It is sufficient to apply ??: taking R and the B_i equal to the field of fractions K of A and noting that a subring of K integrally closed in K is a fortiori an integrally closed domain since its field of fractions is contained in K . \square

Proposition 12.1.22. *Let A be a ring, $(R_i)_{1 \leq i \leq n}$ a finite family of A -algebras and \bar{A}_i the integral closure of A in R_i . Then the integral closure of A in $R = \prod_{i=1}^n R_i$ is $\prod_{i=1}^n \bar{A}_i$.*

Proof. This is an immediate consequence of ??.

Corollary 12.1.23. *Let A be a reduced Noetherian ring, $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ its distinct minimal prime ideals, K_i the field of fractions of the integral domain A/\mathfrak{p}_i and \bar{A}_i the integral closure of A in K_i . Then the canonical isomorphism of the total ring of fractions $Q(A)$ of A onto $\prod_{i=1}^n K_i$ maps the integral closure of A in $Q(A)$ onto the product ring $\prod_{i=1}^n \bar{A}_i$.*

Proof. This follows from ?? and ??.

Corollary 12.1.24. *For a reduced Noetherian ring to be integrally closed in its total ring of fractions, it is necessary and sufficient that it be a direct product of integrally closed (Noetherian) domains.*

12.1.2 Examples of integrally closed domains

Proposition 12.1.25. *A UFD is integrally closed.*

Proof. Let A be a UFD and K be its fraction field. Then an element $x = a/b \in K$ (where a, b are coprime) is integral over A if and only if $x \in A$. In fact, if there is an equation

$$a^n/b^n + c_{n-1}a^{n-1}/b^{n-1} + \cdots + c_1a/b + c_0 = 0$$

then multiplying b^n on both sides we see $b \mid a$, which is a contradiction. \square

Example 12.1.26. Let k be a field and X an indeterminate over k ; set $A = k[X^2, X^3] \subseteq B = k[X]$. Then A and B both have the same field of fractions $K = k(X)$. Since B is a UFD, it is integrally closed; but X is integral over A , so that B is the integral closure of A in K .

Note that in this example $A \cong k[X, Y]/(Y^2 - X^3)$. Thus A is the coordinate ring of the plane curve $Y^2 = X^3$, which has a singularity at the origin. The fact that A is not integrally closed is related to the existence of this singularity.

Proposition 12.1.27. *Let A be a ring, R an A -algebra and P and Q monic polynomials in $R[X]$. If the coefficients of PQ are integral over A , then the coefficients of P and Q are integral over A .*

Proof. There exists a ring R' containing R and families of elements $(a_i), (b_j)$ of R' such that in $R'[X]$ we have

$$P(X) = \prod_{i=1}^n (X - a_i), \quad Q(X) = \prod_{j=1}^m (X - b_j)$$

The coefficients of PQ are integral over A and so belong to the integral closure A' of A in R' . Thus the elements a_i and b_j are integral over A' and hence belong to A' . It follows that the coefficients of P and Q are integral over A . \square

Let A be an integral domain, K its field of fractions and L a K -algebra. Given an element $x \in L$ algebraic over K , the polynomials $P \in K[X]$ such that $P(x) = 0$ form a nonzero ideal a of $K[X]$, necessarily principal. There exists a unique monic polynomial which generates a . Generalizing the terminology, this monic polynomial will be called the **minimal polynomial** of x over K .

Corollary 12.1.28. *Let A be an integral domain, K its field of fractions and x an element of a K -algebra L . Then x is integral over A if and only if the coefficients of the minimal polynomial P of x over K are integral over A .*

Proof. If all coefficients of the minimal polynomial of x are integral over A , then by ?? we see x is integral over A . Conversely, assume that x is integral over A . Then there exists by hypothesis a monic polynomial $Q \in A[X]$ such that $Q(x) = 0$. As P divides Q in $K[X]$, it follows from ?? that the coefficients of P are integral over A . \square

Example 12.1.29. Let A be a UFD in which 2 is a unit and $a \in A$. Then $A[\sqrt{a}]$ is an integrally closed domain if and only if a is square-free. In fact, let α be a square root of a . Let K be the field of fractions of A ; then A is integrally closed in K , so that if $\alpha \in K$ we have $\alpha \in A$ and $A[\alpha] = A$, and the assertion is trivial. If $\alpha \notin K$ then the field of fractions of $A[\alpha]$ is $K(\alpha)$, and

every element $\xi \in K(\alpha)$ can be written in a unique way as $\xi = x + y\alpha$ with $x, y \in K$. The minimal polynomial of ξ over K is

$$X^2 - 2xX + (x^2 - y^2a)$$

Since α is integral over A , $\xi \in K(\alpha)$ is integral over $A[\alpha]$ if and only if it is integral over A , and by ??, if and only if $2x \in A$ and $x^2 - y^2a \in A$.

If so, then by assumption, $2x \in A$ implies $x \in A$. Hence $y^2a \in A$. From this, if some prime p of A divides the denominator of y , we must have $p^2 \mid a$, which is impossible if a is square-free. Thus $y \in A$ and $A[\alpha]$ is integrally closed in this case.

Note that if a is not square-free then we can choose y such that $y^2a \in A$ and $y \notin A$. Thus, in this case $A[\alpha]$ is not integrally closed.

Example 12.1.30. If A is an integrally closed domain and \mathfrak{p} is a prime ideal in A , then the residue ring A/\mathfrak{p} is not integrally closed in general. In fact, any finite integral domain $k[x_1, \dots, x_n]$ over a field k is of the form A/\mathfrak{p} , where A is the polynomial ring $k[X_1, \dots, X_n]$. But finite integral domains are not in general integrally closed. In the case $n = 2$ the simplest example is the one in which \mathfrak{p} is the principal ideal $(X_1^2 - X_2^3)$. In that case, x_1/x_2 does not belong to the ring $k[x_1, x_2]$, but x_1/x_2 is integral over that ring since $(x_1/x_2)^2 = x_2$.

Let A be a ring and R an A -algebra. The homomorphism $A \rightarrow R$ defining the A -algebra structure on R can be extended uniquely to a homomorphism $A[X] \rightarrow R[X]$ of polynomial rings over A and R , leaving X invariant and hence $R[X]$ is given a canonical $A[X]$ -algebra structure.

Proposition 12.1.31. Let A be a ring, R an A -algebra and P a polynomial in $R[X_1, \dots, X_n]$. For P to be integral over $A[X_1, \dots, X_n]$, it is necessary and sufficient that the coefficients of P be integral over A .

Proof. By considering the polynomials of $R[X_1, \dots, X_n]$ as polynomials in X_n with coefficients in $R[X_1, \dots, X_{n-1}]$, we see immediately that it is reduced to proving the proposition for $n = 1$. Then let P be a polynomial in $R[X]$. It follows immediately from ?? that, if the coefficients of P are in the integral closure B of A in R , the element P , which belongs to $B[X] = B \otimes_A A[X]$, is integral over $A[X]$. Conversely, suppose that P is integral over $A[X]$ and let

$$Q(Y) = Y^m + F_{m-1}Y^{m-1} + \cdots + F_0$$

be a monic polynomial with coefficients $F_i \in A[X]$ with P as a root. Let r be an integer strictly greater than all the degrees of the polynomials P and F_i and let us write $P_1(X) = P(X) - X^r$. Then P_1 is a root of the polynomial

$$Q_1(Y) = Q(Y + X^r) = Y^m + G_{m-1}Y^{m-1} + \cdots + G_0$$

with coefficients in $A[X]$. We may therefore write

$$G_0 = -P_1(P_1^{m-1} + G_{m-1}P_1^{m-2} + \cdots + G_1).$$

Now the choice of r implies that $-P_1$ is a monic polynomial of $R[X]$ and so is $G_0(X) = Q(X^r)$, the degrees of the polynomials $F_i(X)X^{r(m-i)}$ being all smaller than rm for $i \geq 1$. We conclude

first of all that the polynomial

$$P_1^{m-1} + G_{m-1}P_1^{m-2} + \cdots + G_1$$

of $R[X]$ is also monic. Moreover, as the coefficients of G_0 belong to A , ?? shows that P_1 has coefficients integral over A and the coefficients of P are therefore certainly integral over A . \square

Corollary 12.1.32. *Let A be a ring, R an A -algebra and \bar{A} the integral closure of A in R . Then the integral closure of $A[X_1, \dots, X_n]$ in $R[X_1, \dots, X_n]$ is equal to $\bar{A}[X_1, \dots, X_n]$.*

Corollary 12.1.33. *Let A be an integral domain and \bar{A} its integral closure. Then the integral closure of the polynomial ring $A[X_1, \dots, X_n]$ is $\bar{A}[X_1, \dots, X_n]$.*

Proof. Arguing by induction on n , the problem is immediately reduced to the case $n = 1$. Let K be the field of fractions of A , which is also that of \bar{A} . If an element P of the field of fractions $K(X)$ of $A[X]$ is integral over $A[X]$, it belongs to the polynomial ring $K[X]$, for the latter is a principal ideal domain and hence integrally closed. The corollary then follows from ?? applied to $R = K$. \square

Corollary 12.1.34. *Let A be an integral domain. For the polynomial ring $A[X_1, \dots, X_n]$ to be integrally closed, it is necessary and sufficient that A be integrally closed.*

12.1.3 Completely integrally closed domains

Proposition 12.1.35. *Let A be a ring, K the field of fraction of A and x an element of K . The following properties are equivalent:*

- (i) *There exists nonzero $a \in A$ such that $ax^n \in A$.*
- (ii) *All the powers x^n (with $n \geq 0$) are contained in a finitely generated sub- A -module of K .*
- (iii) *The subalgebra $A[X]$ of K is a fractional ideal of K .*

Proof. It is clear that (i) and (iii) are equivalent and (iii) implies (ii). Also, if all powers x^n is contained in a finitely generated sub- A -module M of K , then it can be easily seen that there exists $s \in A$ such that $sM \subseteq A$, whence (i) holds. \square

An element x of K is called almost integral over A if it satisfies the conditions of ???. An integral domain A is called **completely integrally closed** if every almost integral element of K belongs to A . Clearly a completely integrally closed domain is integrally closed. Conversely, ?? shows that an integrally closed Noetherian domain is completely integrally closed. If (A_i) is a family of completely integrally closed domains with the same field of fractions K , then $A = \bigcap_i A_i$ is completely integrally closed. For if $x \in K$ is such that for some non-zero d in A , $dA[x]$ belongs to A , the hypothesis implies that $x \in A_i$ for all i and hence $x \in A$.

Proposition 12.1.36. *Let A be a completely integrally closed domain. Then every polynomial ring $A[X_1, \dots, X_n]$ (resp. every ring of formal power series $A[[X_1, \dots, X_n]]$) is completely integrally closed.*

Proof. By induction on n , it is sufficient to prove that $A[X]$ (resp. $A[[X]]$) is completely integrally closed. Then let P be an element of the field of fractions of $A[X]$ (resp. $A[[X]]$) and suppose that

there exists a non-zero, element $Q \in A[X]$ (resp. $Q \in A[\![X]\!]$) such that, $QP^m \in A[X]$ (resp. $QP^m \in A[\![X]\!]$) for every integer $m \geq 0$. If K is the field of fractions of A , then $A[X]$ (resp. $A[\![X]\!]$) is a subring of $K[X]$ (resp. $K[\![X]\!]$) and $K[X]$ (resp. $K[\![X]\!]$) is a principal ideal domain and hence integrally closed and Noetherian and therefore completely integrally closed, therefore we have already seen that $P \in K[X]$ (resp. $P \in K[\![X]\!]$). Write

$$P = \sum_{k=0}^{\infty} a_k X^k, \quad Q = \sum_{k=0}^{\infty} b_k X^k$$

where $a_k \in K$ and $b_k \in A$, and we argue by reductio ad absurdum by supposing that the a_k do not all belong to A . Then there is a least index i such that $a_i \notin A$. If we write $P_1 = \sum_{k=0}^{i-1} a_k X^k \in A[X]$, it follows immediately from the hypothesis that also $Q(P - P_1)^m \in A[X]$ (resp. $Q(P - P_1)^m \in A[\![X]\!]$) for all $m \geq 0$. Let j be the least integer such that $b_j \neq 0$. Then in $Q(P - P_1)^m$ the term of least degree with a nonzero coefficient is $b_j a_i^m X^{j+mi}$ and hence $b_j a_i^m \in A$ for all $m \geq 0$. But as A is completely integrally closed this implies $a_i \in A$, contrary to the hypothesis. \square

Proposition 12.1.37. *Let A be a filtered ring whose filtration is exhaustive and such that every principal ideal of A is closed under the topology defined by the filtration. If the associated graded ring $\text{gr}(A)$ is a completely integrally closed domain, then A is a completely integrally closed domain.*

Proof. Let $(A_n)_{n \in \mathbb{Z}}$ be the filtration defined on A . As $\bigcap_n A_n$ is the closure of the ideal (0) , the hypothesis implies first that the filtration (A_n) is separated and, as $\text{gr}(A)$ is an integral domain, so then is A (??). Let $x = a/b$ be an element of the field of fractions K of A for which there exists a nonzero element d of A such that $dx^n \in A$ for all n . We must prove that $a \in Ab$ and, as by hypothesis the ideal Ab is closed, it is sufficient to show that, for all $n \in \mathbb{Z}$, $a \in A_b + A_n$. As the filtration of A is exhaustive, there exists an integer $p \in \mathbb{Z}$ such that $a \in Ab + A_p$. It will therefore suffice to prove that the relation $a \in Ab + A_m$ implies $a \in Ab + A_{m+1}$ for any $m \in \mathbb{N}$.

Suppose then $a = by + z$ where $y \in A$ and $z \in A_m$. Since by hypothesis $dx^n \in A$ for all n , we have $d(x - y)^n \in A$ for all n . In other words, $dz^n = b^n t_n$ where $t_n \in A$ for all n . We may assume that $z \neq 0$. Let v denote the order function on A and let us write $v(z) = n_0$. If \bar{x} denote the canonical image of an element $x \in A$ in $\text{gr}(A)$, then since $b \neq 0$, we deduce that $\bar{d}(\bar{z}/\bar{b})^n \in \text{gr}(A)$ for all $n \in \mathbb{N}$. Since $\text{gr}(A)$ is completely integrally closed, there exists an element $\bar{s} \in \text{gr}(A)$ such that $\bar{z} = \bar{b}\bar{s}$. Decomposing \bar{s} into a sum of homogeneous elements, it is further seen (since \bar{z} and \bar{b} are homogeneous) that \bar{s} may be assumed to be homogeneous and that is the image of an element $s \in A$. Then $v(bs) = v(z) = n_0$ and $z \equiv bs \pmod{A_{n_0+1}}$. As we have $n_0 \geq m$, a fortiori $z \equiv bs \pmod{A_{m+1}}$, hence $a \equiv b(y + s) \pmod{A_{m+1}}$ and the claim follows. \square

12.1.4 The integral closure of a ring of fractions

Proposition 12.1.38. *Let A be a ring, R an A -algebra, \bar{A} the integral closure of A in R and S a multiplicative subset of A . Then $S^{-1}\bar{A}$ is the integral closure of $S^{-1}A$ in $S^{-1}B$.*

Proof. Let $b/s \in S^{-1}\bar{A}$, where b is integral over A , say

$$b^n + a_{n-1}b^{n-1} + \cdots + a_0 = 0, \quad a_i \in A.$$

Then the equation of b gives an equation

$$(b/s)^n + (a_{n-1}/s)(b/s)^{n-1} + \cdots + a_0/s^n = 0$$

Hence $S^{-1}\bar{A}$ is integral over $S^{-1}A$. Conversely, if $b/s \in S^{-1}B$ is integral over $S^{-1}A$, then we have an equation of the form

$$(b/s)^n + (a_{n-1}/s_{n-1})(b/s)^{n-1} + \cdots + a_0/s_0 = 0$$

where $a_i \in A$, $s_i \in S$. Let $t = s_1 \cdots s_n$, and multiply this equation by $(st)^n$ throughout. Then it becomes an equation of integral dependence for bt over A . Hence $bt \in \bar{A}$ and therefore $b/s = bt/st \in S^{-1}\bar{A}$. \square

Corollary 12.1.39. *Let A be an integral domain, \bar{A} its integral closure and S a multiplicative subset such that $0 \notin S$. Then the integral closure of $S^{-1}A$ is $S^{-1}\bar{A}$.*

Proof. The field of fractions K of A is also the field of fractions of $S^{-1}A$ since $0 \notin S$, so ?? is then applied to K . \square

Corollary 12.1.40. *Let A be an integral domain, K its field of fractions, R an algebra over K and B the integral closure of A in R . The elements of R which are algebraic over K are the elements of the form $a^{-1}b$ where $b \in B$ and $a \in A$. If L is the algebraic closure of K in R , then there exists a basis of L over K contained in B .*

Proof. The first assertion follows from ?? applied in the case $S = A - \{0\}$. If $(x_i)_{i \in I}$ is a basis of L over K , then there exists for all $i \in I$ an element a_i of A such that $a_i x_i \in B$. Then $(a_i x_i)_{i \in I}$ is also a basis of L over K . \square

Proposition 12.1.41. *Let A be an integral domain. Then the following are equivalent:*

- (i) A is integrally closed.
- (ii) $A_{\mathfrak{p}}$ is integrally closed for each prime ideal \mathfrak{p} .
- (iii) $A_{\mathfrak{m}}$ is integrally closed for each maximal ideal \mathfrak{m} .

Proof. It follows from ?? that the condition (iii) is necessary. The condition is sufficient, for $A = \bigcap_{\mathfrak{m}} A_{\mathfrak{m}}$ by ?? and it is sufficient to apply the ??.

Corollary 12.1.42. *Let A be an integral domain, K its field of fractions and S a multiplicative subset of A such that $0 \notin S$.*

- (a) *Let B be a subring of K which is integral over A and let \mathfrak{f} be the annihilator of the A -module B/A (called the **conductor** of A in B). Then $S^{-1}\mathfrak{f}$ is contained in the conductor of $S^{-1}A$ in $S^{-1}B$ and is equal to it if B is a finitely generated A -module.*
- (b) *Let \bar{A} be the integral closure of A . For $S^{-1}A$ to be integrally closed, it is sufficient that the conductor \mathfrak{f} of A in \bar{A} meet S . This condition is also necessary if \bar{A} is a finitely generated A -module.*

Proof. As $\mathfrak{f}B \subseteq A$, $(S^{-1}\mathfrak{f})(S^{-1}B) \subseteq S^{-1}A$ and hence $S^{-1}\mathfrak{f}$ is contained in $\text{Ann}(S^{-1}B/S^{-1}A)$. If B is a finitely generated A -module, the equation $S^{-1}\mathfrak{f} = \text{Ann}(S^{-1}B/S^{-1}A)$ is a special case of ??, $S^{-1}B/S^{-1}A$ being canonically identified with $S^{-1}(B/A)$. This proves (a).

By ??, $S^{-1}\bar{A}$ is the integral closure of $S^{-1}A$. As the relations $\mathfrak{f} \cap S \neq \emptyset$ and $S^{-1}\mathfrak{f} = S^{-1}A$ are equivalent, (b) is an immediate consequence of (a). \square

Corollary 12.1.43. *Let A be an integral domain, \bar{A} its integral closure and \mathfrak{f} the conductor of A in \bar{A} . Suppose that \bar{A} is a finitely generated A -module. Then the prime ideals \mathfrak{p} of A such that $A_{\mathfrak{p}}$ is not integrally closed are those which contain \mathfrak{f} .*

12.1.5 Norms and traces of integral elements

Proposition 12.1.44. *Let A be a ring, B be an A -algebra, and T be a square matrix of order n over B . The following properties are equivalent:*

- (i) T is integral over A ;
- (ii) There exists a finitely generated sub- A -module M of B^n such that $T(M) \subseteq M$ and M is a system of generators of the B -module B^n .
- (iii) The coefficients of the characteristic polynomial of T are integral over A .

Proof. If $\chi(X)$ is the characteristic polynomial of X , the Cayley-Hamilton Theorem shows that $\chi(T) = 0$ and, as $\chi(X)$ is a monic polynomial, (iii) implies (i) by the transitivity of integrality. Suppose in the second place that (i) holds. If e_1, \dots, e_n is the canonical basis of B^n , the sub- A -module M of B generated by the $T^k e_i$ is a finitely generated A -module, since the A -algebra $A[T]$ is a finitely generated A -module. As M contains the e_i , it is seen that (i) implies (ii). The converse is a consequence of ??.

Finally let us prove that (i) implies (iii). As T is integral over A and a fortiori over the polynomial ring $A[X]$, the polynomial $X - T$ is integral over $A[X]$ by ?. By ??, the problem is seen to reduce (by replacing T by $X - T$ and A by $A[X]$) to proving that, if T is integral over A , then $\det(T)$ is an element of B which is integral over A . Now, we have seen above that the endomorphism ϕ of B^n defined by the matrix T preserves a finitely generated sub- A -module M containing the e_i . Then $\wedge^n M$ is a finitely generated sub- A -module in $\wedge^n B^n$ containing $e_1 \wedge \dots \wedge e_n$ and which is stable under $\wedge^n \phi$, in other words under the homothety of ratio $\det(T)$. As $e_1 \wedge \dots \wedge e_n$ generates $\wedge^n B^n$, ?? then proves that $\det(T)$ is integral over A . \square

Corollary 12.1.45. *Let A be an integral domain, K its field of fractions and L a finite dimensional K -algebra. If $x \in L$ is integral over A , then the coefficients of the characteristic polynomial of x over K are integral over A .*

Proof. Let h_x be the homothety of ratio x on L . Then by definition h_x is integral over A , and the characteristic polynomial of h_x is that of x over K . \square

Corollary 12.1.46. *Let A be an integral domain, K its field of fractions and L a finite dimensional K -algebra. If $x \in L$ is integral over A , then $N_{L/K}(x)$ and $\text{tr}_{L/K}(x)$ are integral over A .*

Proposition 12.1.47. *If L/K is a finite separable extension, then $(x, y) = \text{tr}_{L/K}(xy)$ is a nondegenerate bilinear form on L .*

Proof. Let θ be a primitive element for L , so that $L = K(\theta)$. Then the bilinear form $(x, y) \mapsto \text{tr}_{L/K}(xy)$ has matrix $G = (\text{tr}_{L/K}(\theta^{i-1}\theta^{j-1}))$ under the basis $1, \theta, \dots, \theta^{n-1}$. It is nondegenerate because, if $\theta_1 = \theta, \dots, \theta_n$ are the K -conjugates of θ , then

$$\det(G) = \prod_{i < j} (\theta_i - \theta_j) \neq 0.$$

This proves the claim. \square

Theorem 12.1.48. *Let A be an integrally closed domain, K its quotient field, L a finite-dimensional separable K -algebra, and B the integral closure of A in L . There exists a basis $\{x_1, \dots, x_n\}$ of L over K such that B is contained in the A -module $\sum_i Ax_i$.*

The proposition will follow from the following more precise lemma:

Lemma 12.1.49. *Under the hypotheses of ??, let (w_1, \dots, w_n) be a basis of L over K contained in B (??); then there is a unique dual basis (w_1^*, \dots, w_n^*) of L over K with respect to the trace form $\text{tr}_{L/K}$; if $d = D_{L/K}(w_1, \dots, w_n)$ is the discriminant of the basis (w_1, \dots, w_n) , then $d \neq 0$ and*

$$\sum_{i=1}^n Aw_i \subseteq B \subseteq \sum_{i=1}^n Aw_i^* \subseteq d^{-1} \left(\sum_{i=1}^n Aw_i \right). \quad (12.1.1)$$

In particular, if d is invertible in A , B is a free A -module with basis (w_1, \dots, w_n) .

Proof. As L is a separable K -algebra, $d \neq 0$ by ?? and the K -bilinear form $\text{tr}_{L/K}$ is non-degenerate. This shows the existence and uniqueness of the dual basis (w_i^*) . This being so, the first inclusion of ?? is obvious. Let x be an element of B ; let us write $x = \sum_{i=1}^n \xi_i w_i^*$ where $\xi_i \in K$; for all i , we have $\xi_i = \text{tr}_{L/K}(xw_i)$, so ξ_i is integral over A ?? and, as A is integrally closed, $\xi_i \in A$ for all i ; this shows the second inclusion in ???. Finally, let us write $w_j^* = \sum_{i=1}^n \alpha_{ji} w_i$ where $\alpha_{ji} \in K$; then $\sum_{i=1}^n \alpha_{ji} \text{tr}_{L/K}(w_i w_k) = \delta_{jk}$ for all j and k ; Cramer's formulae show that the α_{ji} belong to $d^{-1}A$, whence the third inclusion ???. The last assertion follows immediate from ???, which in this case gives $B = \sum_{i=1}^n Aw_i$. \square

Proof. We first notice that $L = S^{-1}B$, where $S = A - \{0\}$; that is, given any element x of L there exists a non-zero element s of A such that $sx \in B$: in fact, if $p(X) = X^n + a_{n-1}X^{n-1} + \dots + a_0$ is the minimal polynomial of x over K , and if we take a common denominator s in A such that $sc_i = a_i \in A$, then we have

$$(sx)^n + a_{n-1}(sx)^{n-1} + \dots + s^{n-1}a_0 = 0$$

and sx is integral over A . It follows from this observation that there exists a basis $\{u_1, \dots, u_n\}$ of L over K such that $u_i \in B$ for every i . Let $\{v_1, \dots, v_n\}$ be the dual basis of $\{u_1, \dots, u_n\}$ under the bilinear form $\text{tr}(xy)$. If an element $x = \sum_j a_j x_j$ is in B , we have $xu_i \in B$ for every i , whence $\text{tr}(xu_i) \in A$ (A is integrally closed). Since $\text{tr}(xu_i) = \sum_j a_j \text{tr}(u_i v_j) = a_i$, we then have $B \subseteq \sum_j Av_j$. \square

Corollary 12.1.50. *The assumptions being the same as in ??, let us furthermore assume that the ring A is Noetherian. Then B is a finite A -module and is a Noetherian ring.*

Proof. In fact, B is a submodule of the finite A -module $\sum_i Ax_i$, and is therefore a finite A -module. Thus B satisfies the a.c.c. as an A -module, and a fortiori satisfies the a.c.c. as an B -module. That is, B is Noetherian. \square

Corollary 12.1.51. *The assumptions being the same as in ??, let us furthermore assume that A is a PID. Then there exists a basis $\{y_i\}$ of L over K such that $B = \sum_i Ay_i$.*

Proof. It was just shown that B is contained in an A -module generated by n elements x . Hence B has a basis consisting of n elements $\{y_i\}$. Since $L = S^{-1}B$, the set $\{y_i\}$ is necessarily also a basis of L over K . \square

?? is of particular importance for the case in which A is either the ring \mathbb{Z} of rational integers, L being then an algebraic number field, or a polynomial ring $k[X]$ in one variable over a field k , L being then a field of algebraic functions of one variable. In the first case, the elements of L which are integral over \mathbb{Z} are called the **algebraic integers** of the number field L ; in the second case, the elements of L which are integral over $k[X]$ are called the **integral functions** of the function field L (with respect to the element X). ?? shows that these algebraic integers (or integral functions) are the linear combinations, with ordinary integral coefficients (or with coefficients in $k[X]$), of $n = [L : K]$ linearly independent algebraic integers y_i . Such a basis $\{y_i\}$ of L over the rational field (or over the rational function field $k(X)$) is called an **integral basis** of L .

Example 12.1.52. Let X be an indeterminate over a field k of characteristic $\neq 2$, and p an irreducible polynomial over k . Let $A = k[X]$ and $K = k(X)$, then the function field $L = k(X, Y)$ admits $\{1, Y\}$ as an integral basis (with respect to A). In fact, in the first place 1 and Y are integral over A . Furthermore, let f be an element of L which is integral over A . We write $f = a(X) + b(X)Y$ with $a, b \in k(X)$. Then the trace $2a(X)$ and the norm $a(X)^2 - b(X)^2 p(X)$ of f over $k(X)$ belong to $k[X]$, whence both $a(X)$ and $b(X)$ are polynomials, since otherwise $p(X)$ would be divisible by the square of the denominator of $b(X)$. Consequently, the integral closure of $A = k[X]$ in $L = k(X, Y)$ is the ring $k[X, Y]$.

Proposition 12.1.53. *Let k be a field, L a separable extension of k and A an integrally closed k -algebra. If the ring $L \otimes_k A$ is an integral domain, it is integrally closed.*

Proof. Let K be the field of fractions of A ; as k is a field, $L \otimes_k A$ is canonically identified with a sub- k -algebra of $L \otimes_k K$ and L and A with sub- k -algebras of $L \otimes_k A$. Moreover, since an element $s \neq 0$ of A is not a divisor of 0 in A , $1 \otimes s$ is not a divisor of zero in $L \otimes_k A$ since L is flat over k ; identifying s with $1 \otimes s$, it is therefore seen that, if $S = A \setminus \{0\}$, $L \otimes_k K$ is identified with $S^{-1}(L \otimes_k K)$; as $L \otimes_k A$ is assumed to be an integral domain, $L \otimes_k K$ is thus identified with a subring of the field of fractions Ω of $L \otimes_k A$.

First suppose that L is a *finite* extension of k ; then $L \otimes_k K$ is an algebra of finite rank over K and by hypothesis has no divisor of 0; hence it is a field (it is sandwiched by K and Ω) and it is in this case the field of fraction Ω of $L \otimes_k A$. Let (w_1, \dots, w_n) be a basis of L over k , which is therefore also a basis of $L \otimes_k K$ over K . There exists a basis (w_1^*, \dots, w_n^*) of L such that $\text{tr}_{L/k}(w_i w_j^*) = \delta_{ij}$ (?); every $z \in L \otimes_k K$ may be written uniquely $z = \sum_{i=1}^n a_i w_i$ where $a_i \in K$; then

$$\text{tr}_{(L \otimes K)/K}(zw_j^*) = \sum_{i=1}^n a_i \text{tr}_{(L \otimes K)/K}(w_i w_j^*)$$

and as in L the traces $\text{tr}_{(L \otimes K)/K}$ and $\text{tr}_{L/K}$ coincide, finally $\text{tr}_{(L \otimes K)/K}(zw_j^*) = a_j$ for each j . Note on the other hand that the elements of L are integral over k and hence also over A ; therefore $L \otimes_k A$ is integral over A (??). This being so, suppose that $z \in L \otimes_k K$ is integral over $L \otimes_k A$; then z is integral over A , hence so is zw_j^* and therefore also $a_j = \text{tr}_{L/K}(zw_j^*)$ for all j (??). As A is integrally closed, $a_j \in A$ for all j and hence $z \in L \otimes_k A$, which proves the proposition in this case.

Suppose now that L is a finitely generated separable extension of k ; then there exists a separating transcendence basis (x_1, \dots, x_d) of L over k ; as L and K are algebraically disjoint over k in the field Ω , the x_i are algebraically independent over K ; hence $A[x_1, \dots, x_d]$ is integrally closed (??). Let T be the set of nonzero elements of the ring $R = k[x_1, \dots, x_d] \subseteq L$, so that the field $k_1 = k(x_1, \dots, x_d) \subseteq L$ is equal to $T^{-1}R$; then

$$k_1 \otimes_k A = (T^{-1}R) \otimes_k A = T^{-1}R \otimes_R (R \otimes_k A) = T^{-1}(R \otimes_k A) = T^{-1}A[x_1, \dots, x_d]$$

by the associativity of the tensor product, hence this domain is integrally closed (??). But $L \otimes_k A$ is identified with $L \otimes_{k_1} (k_1 \otimes_k A)$ and by definition L is a finite separable extension of k_1 ; it follows therefore from the preceding argument that $L \otimes_k A$ is integrally closed.

In the general case, if z is an element of Ω which is integral over $L \otimes_k A$, it satisfies a relation of the form $z^n + b_{n-1}z^{n-1} + \dots + b_0 = 0$, where the b_i belong to $L \otimes_k A$; then there exists a finitely generated sub-extension L' of L over k such that the b_i belong to $L' \otimes_k A$ for all i and z to $L' \otimes_k K$. Then it follows from the previous argument that $z \in L' \otimes_k A$ and hence $L \otimes_k A$ is integrally closed. \square

Remark 12.1.2. Let V be an affine irreducible algebraic variety over a field k and A the ring of functions regular on V defined over k ; if A is integrally closed, V is called normal over k ; ?? shows that, if V is normal over k , it remains normal over every separable extension L of k .

Corollary 12.1.54. Let k be a field and A and B two integrally closed k -algebras. Suppose that the ring $A \otimes_k B$ is an integral domain and that the fields of fractions K and L of A and B respectively are separable over k . Then the ring $A \otimes_k B$ is an integrally closed domain.

Proof. As A and B are identified with subalgebras of $A \otimes_k B$, K and L are identified with subfields of the field of fractions Ω of $A \otimes_k B$ which are linearly disjoint over k (??). It then follows from ?? that $A \otimes_k L$ and $K \otimes_k B$ are integrally closed domains; as their intersection is $A \otimes_k B$ (??), $A \otimes_k B$ is an integrally closed domain (??). \square

12.1.6 Integers over a graded ring

All the graduations considered in this part are of type \mathbb{Z} ; if A is a graded ring and $i \in \mathbb{Z}$, A_i denotes the set of homogeneous elements of degree i of the ring A . Let A be a graded ring and B a graded A -algebra. Let x be a homogeneous element of B which is integral over A ; then there is a relation

$$x^n + a_{n-1}x^{n-1} + \dots + a_0 = 0 \quad a_i \in A.$$

Let $m = \deg(x)$ and let a'_i be the homogeneous component of degree $m(n-i)$ of a_i for each i ; then obviously

$$x^n + a'_{n-1}x^{n-1} + \dots + a'_0 = 0 \tag{12.1.2}$$

in other words x satisfies an equation of integral dependence with homogeneous coefficients.

Let $A[X, X^{-1}]$ denote the ring of fractions $S^{-1}A[X]$ of the polynomial ring $A[X]$ in one indeterminate, S being the multiplicative subset of $A[X]$ consisting of the powers X^n of X ; as X is not a divisor of 0 in $A[X]$ it is immediate that the X^i 's form a basis over A of the A -module $A[X, X^{-1}]$. For every element $a \in A$ with homogeneous components $(a_i)_{i \in \mathbb{Z}}$, we write

$$j_A(a) = \sum_{i \in \mathbb{Z}} a_i X^i \in A[X, X^{-1}]$$

it is immediate that $j_A : A \rightarrow A[X, X^{-1}]$ is an injective ring homomorphism.

Proposition 12.1.55. *Let $A = \bigoplus_{i \in \mathbb{Z}} A_i$ be a graded ring and B a graded A -algebra. Then the set R of elements of B integral over A is a graded subalgebra of B . Moreover, if $A_i = 0$ for $i < 0$ and R is a reduced ring, then $R_i = 0$ for $i < 0$.*

Proof. The diagram

$$\begin{array}{ccc} A & \xrightarrow{\rho} & B \\ \downarrow j_A & & \downarrow j_B \\ A[X, X^{-1}] & \xrightarrow{\tilde{\rho}} & B[X, X^{-1}] \end{array}$$

(where ρ is the homomorphism defining the A -algebra structure on B and $\tilde{\rho}$ the homomorphism canonically derived from it) is commutative, as is immediately verified from the definition. Let x be an element of B integral over A ; then $j_B(x)$ is integral over $A[X, X^{-1}]$ (??) and it therefore follows from ?? that there exists an integer $m > 0$ such that $X^m j_B(x)$ is an element of $B[X]$ integral over $A[X]$. We then deduce from ?? that the coefficients of the polynomial $X^m j_B(x)$ are integral over A ; as these coefficients are by definition the homogeneous components of x , it is seen that these are integral over A , which proves that R is a graded subalgebra of B .

Suppose now that $x \in R_i$ where $i < 0$; the remark at the beginning of this part shows that x satisfies an equation of the form (??) where $a'_k \in A_{ki}$ for $1 \leq k \leq n$. If $A_j = 0$ for $j < 0$, then $x^n = 0$ and if B is a reduced ring we conclude that $x = 0$ and hence $R_i = 0$ for all $i < 0$ in this case. \square

Recall that, if $A = \bigoplus_{i \in \mathbb{Z}} A_i$ is a graded ring and S is a multiplicative subset of A consisting of homogeneous elements, a graded ring structure is defined on $S^{-1}A$ by taking the set $(S^{-1}A)_i$ of homogeneous elements of degree i to be the set of elements of the form a/s , where $a \in A$ and $s \in S$ are homogeneous and such that $\deg(a) - \deg(s) = i$.

Lemma 12.1.56. *Let $A = \bigoplus_{i \in \mathbb{Z}} A_i$ be a graded integral domain and S the set of nonzero homogeneous elements of A .*

- (a) *Every nonzero homogeneous element of $S^{-1}A$ is invertible, the ring $K = (S^{-1}A)_0$ is a field and the set of $i \in \mathbb{Z}$ such that $(S^{-1}A)_i \neq 0$ is a subgroup $q\mathbb{Z}$ of \mathbb{Z} (where $q \neq 0$).*
- (b) *Suppose that $q \geq 1$ and let t be a non-zero element of $(S^{-1}A)_q$. Then the K -homomorphism f of the polynomial ring $K[X]$ to $S^{-1}A$ which maps X to t extends to an isomorphism of $K[X, X^{-1}]$ onto $S^{-1}A$ and $S^{-1}A$ is integrally closed.*

Proof. The assertions in (a) follow immediately from the definitions and the hypothesis that A is an integral domain, for if a/s and b/t are two nonzero homogeneous elements of $S^{-1}A$ of degrees i and j , ab/st is a nonzero homogeneous element and of degree $i + j$. To show (b), we note that since t is invertible in $S^{-1}A$ the homomorphism f extends in a unique way to a homomorphism $\tilde{f} : K[X, X^{-1}] \rightarrow S^{-1}A$ and necessarily $\tilde{f}(X^{-1}) = t^{-1}$. On the other hand, by definition of q , every nonzero homogeneous element of $S^{-1}A$ is of degree qn ($n \in \mathbb{Z}$) and hence can be written uniquely in the form λt^n where $\lambda \in K$ (since $S^{-1}A$ is an integral domain); hence \tilde{f} is bijective. Finally, we know that $K[X]$ is integrally closed and hence so is $K[X, X^{-1}]$ (??), which completes the proof of the Lemma. \square

Proposition 12.1.57. *Let $A = \bigoplus_{i \in \mathbb{Z}} A_i$ be a graded integral domain and S the set of nonzero homogeneous elements of A . The integral closure B of A is then a graded subring of $S^{-1}A$. If further $A_i = 0$ for $i < 0$, then $B_i = 0$ for $i < 0$.*

Proof. If $A = A_0$, the proposition is trivial. Otherwise we may apply ??; the ring $S^{-1}A$ is an integrally closed domain and therefore $B \subseteq S^{-1}A$; as $S^{-1}A$ is graded, so is B by ??; the latter assertion also follows from ??.

Corollary 12.1.58. *With the hypotheses and notation of ??, if every homogeneous element of $S^{-1}A$ which is integral over A belongs to A , then A is integrally closed.*

Proof. Then $B_i \subseteq A$ for all $i \in \mathbb{Z}$, hence $B = A$. \square

Corollary 12.1.59. *If $A = \bigoplus_{i \in \mathbb{Z}} A_i$ is an integrally closed graded domain, the domain A_0 is integrally closed.*

Proof. The field of fractions K_0 of A_0 is identified (in the notation of ??) with a subring of the ring of homogeneous elements of degree 0 of $S^{-1}A$; every element of K_0 integral over A_0 (and a fortiori over A) belongs therefore by hypothesis to A_0 . \square

Corollary 12.1.60. *Let $A = \bigoplus_{i \in \mathbb{Z}} A_i$ be an integrally closed graded domain. Then, for every integer $d > 0$, the ring $A^{(d)}$ is an integrally closed domain.*

Proof. Let U be the set of nonzero homogeneous elements of $A^{(d)}$ and let x be a homogeneous element of $U^{-1}A^{(d)}$ integral over $A^{(d)}$ and hence over A ; as $x \in S^{-1}A$, x belongs to A by hypothesis; as its degree is divisible by d , it belongs to $A^{(d)}$ and it then follows from ?? that $A^{(d)}$ is integrally closed. \square

12.1.7 The lift of prime ideals

Let A, B be two rings and $\rho : A \rightarrow B$ a ring homomorphism. An ideal \mathfrak{b} of B is said to lie over an ideal \mathfrak{a} of A if $\mathfrak{a} = \rho^{-1}(\mathfrak{b})$.

To say that a prime ideal \mathfrak{P} of B lies over an ideal \mathfrak{p} of A therefore means that \mathfrak{p} is the image of \mathfrak{P} under the continuous map $f^* : \text{Spec}(B) \rightarrow \text{Spec}(A)$ associated with f . Note that for there to exist an ideal of B lying over the ideal (0) of A , it is necessary and sufficient that $\rho : A \rightarrow B$ be injective.

Let \mathfrak{a} be an ideal of A . By taking quotients the homomorphism ρ gives a homomorphism $\bar{\rho} : A/\mathfrak{a} \rightarrow B/\mathfrak{b}$. To say that \mathfrak{b} is an ideal of B lying over \mathfrak{a} is equivalent to saying that $\mathfrak{a}B \subseteq \mathfrak{b}$ and that $\mathfrak{b}/\mathfrak{a}B$ is an ideal of $B/\mathfrak{a}B$ lying over (0) .

Lemma 12.1.61. Let $\rho : A \rightarrow B$ be a ring homomorphism, S a multiplicative subset of A , i_A^S and i_B^S the canonical homomorphisms, so that there is a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\rho} & B \\ \downarrow i_A^S & & \downarrow i_B^S \\ S^{-1}A & \xrightarrow{S^{-1}\rho} & S^{-1}B \end{array}$$

Let \mathfrak{p} be a prime ideal of A such that $\mathfrak{p} \cap S = \emptyset$. Then $\mathfrak{b} \mapsto S^{-1}\mathfrak{b}$ is a bijective map of the set of ideals of B lying over \mathfrak{p} and saturated with S onto the set of ideals of $S^{-1}B$ lying over $S^{-1}\mathfrak{p}$. In particular, $\mathfrak{P} \mapsto S^{-1}\mathfrak{P}$ is a bijection on the set of prime ideals of B lying over \mathfrak{p} onto the set of prime ideals of $S^{-1}B$ lying over $S^{-1}\mathfrak{p}$.

Proof. This follows from ?? and ??.

□

Proposition 12.1.62. Let $A \subseteq B$ be integral domains, B integral over A . Then B is a field if and only if A is a field.

Proof. If A is a field, then, for all nonzero y in B , $A[y]$ is by hypothesis a finitely generated A -module. As $A[y]$ is an integral domain, it is a field and a fortiori y is invertible in B and hence B is a field. Conversely, suppose that B is a field and let x be nonzero in A . As $x^{-1} \in B$ and B is integral over A , there is an equation of integral dependence

$$x^{-n} + a_{n-1}z^{-(n-1)} + \cdots + a_0 = 0$$

where the $a_i \in A$; now this relation shows that

$$-z^{-1} = a_{n-1} + a_{n-2}z + \cdots + a_0z^{n-1} \in A,$$

hence A is a field.

□

Corollary 12.1.63. Let $\rho : A \rightarrow B$ be a ring homomorphism such that B is integral over A , \mathfrak{P} a prime ideal of B and $\mathfrak{p} = f^{-1}(\mathfrak{P})$. For \mathfrak{p} to be maximal, it is necessary and sufficient that \mathfrak{P} is maximal.

Proof. Let $\bar{\rho} : A/\mathfrak{p} \rightarrow B/\mathfrak{P}$ be the homomorphism derived from f by taking quotients. Then A/\mathfrak{p} and B/\mathfrak{P} are integral domains and B/\mathfrak{P} is integral over A/\mathfrak{p} . To say that \mathfrak{p} (resp. \mathfrak{P}) is maximal means that A/\mathfrak{p} (resp. B/\mathfrak{P}) is a field. The corollary then follows from ??.

□

Corollary 12.1.64. Let $\rho : A \rightarrow B$ a ring homomorphism such that B is integral over A , \mathfrak{p} a prime ideal of A and \mathfrak{P}_1 and \mathfrak{P}_2 two ideals of B lying over \mathfrak{p} such that $\mathfrak{P}_1 \subseteq \mathfrak{P}_2$. If \mathfrak{P}_1 and \mathfrak{P}_2 are prime, then $\mathfrak{P}_1 = \mathfrak{P}_2$.

Proof. Let us write $S = A - \mathfrak{p}$. Then $S^{-1}B$ is integral over $S^{-1}A$ by ??, $S^{-1}\mathfrak{p}$ is a maximal ideal of $S^{-1}A$, $S^{-1}\mathfrak{P}_1$ and $S^{-1}\mathfrak{P}_2$ are ideals of $S^{-1}B$ lying over $S^{-1}\mathfrak{p}$ and $S^{-1}\mathfrak{P}_1 \subseteq S^{-1}\mathfrak{P}_2$. As $S^{-1}\mathfrak{P}_1$ and $S^{-1}\mathfrak{P}_2$ are prime, they are maximal by ?? and hence $S^{-1}\mathfrak{P}_1 = S^{-1}\mathfrak{P}_2$. Therefore $\mathfrak{P}_1 = \mathfrak{P}_2$.

□

Corollary 12.1.65. Let B be an integral domain, A a subring of B such that B is integral over A and f a homomorphism from B to a ring C . If the restriction of f to A is injective, then f is injective.

Proof. If \mathfrak{b} is the kernel of f , the hypothesis means that $\mathfrak{b} \cap A = (0)$. As B is an integral domain, ?? may be applied taking \mathfrak{p} and \mathfrak{P}_1 to be the ideal (0) of A and the ideal (0) of B respectively, whence $\mathfrak{b} = (0)$. \square

Corollary 12.1.66. *Let $A \subseteq B$ be rings, B integral over A and \mathfrak{m} a maximal ideal of A . Suppose that there are only finitely many distinct maximal ideals $\mathfrak{M}_1, \dots, \mathfrak{M}_s$ in B lying over \mathfrak{m} . Let \mathfrak{Q}_i be the saturation of $\mathfrak{m}B$ with respect to \mathfrak{M}_i . Then:*

- (a) *In the ring B/\mathfrak{Q}_i , the zero divisors are the elements of $\mathfrak{M}_i/\mathfrak{Q}_i$, and they are nilpotent.*
- (b) $\mathfrak{m}B = \bigcap_{i=1}^s \mathfrak{Q}_i = \prod_{i=1}^s \mathfrak{Q}_i$.
- (c) *The canonical homomorphism $B/\mathfrak{m}B \rightarrow \prod_{i=1}^s B/\mathfrak{Q}_i$ is an isomorphism.*

Proof. For a prime ideal of B to contain $\mathfrak{m}B$, it is necessary and sufficient that its intersection with A contains \mathfrak{m} , and hence that it lies over \mathfrak{m} , since \mathfrak{m} is maximal in A . Therefore \mathfrak{M}_i are therefore the only prime ideals of B containing $\mathfrak{m}B$ and therefore $\sqrt{\mathfrak{m}B} = \bigcap_{i=1}^s \mathfrak{M}_i$. By definition of \mathfrak{Q}_i , the image of an element of $B - \mathfrak{M}_i$ in B/\mathfrak{Q}_i is not a divisor of 0. On the other hand, as the \mathfrak{M}_i are distinct maximal ideals, for every index i there exists an element a_i belonging to $\bigcap_{j \neq i} \mathfrak{M}_j$ and not to \mathfrak{M}_i . Then for all $x \in \mathfrak{M}_i$ we have $a_i x \in \sqrt{\mathfrak{m}B}$, hence the image of $a_i x$ in B/\mathfrak{Q}_i is nilpotent, and, as that of a_i is not a divisor of 0, we conclude that the image of x is nilpotent. In other words \mathfrak{M}_i is the radical of \mathfrak{Q}_i and proves (a).

It follows that \mathfrak{Q}_i are relatively coprime in pair, and (c) will be a consequence of (b). To establish (b), we note that in the ring $B/\mathfrak{m}B$ the $\mathfrak{M}_j/\mathfrak{m}B$ are the only maximal ideals and $\mathfrak{Q}_j/\mathfrak{m}B$ is the saturation of (0) with respect to $\mathfrak{M}_j/\mathfrak{m}B$. We may therefore restrict our attention to the case $\mathfrak{m}B = (0)$. The assertion of (b) then follows from ??, by the definition of a saturation of (0) . \square

Theorem 12.1.67. *Let $\rho : A \rightarrow B$ be a ring homomorphism such that B is integral over A and let \mathfrak{p} be a prime ideal of A containing $\ker \rho$. Then there exists a prime ideal \mathfrak{P} of B such that $\mathfrak{P} \cap A = \mathfrak{p}$.*

Proof. By consider the image $\rho(A)$ in B , we may assume that ρ is injective. Suppose first that A is a local ring and \mathfrak{p} the maximal ideal of A . Then, for every maximal ideal \mathfrak{M} of B , \mathfrak{M}^c is a maximal ideal of A and hence equal to \mathfrak{p} , which proves the theorem in this case (since B contains A by hypothesis and is therefore not reduced to 0). In the general case, let us write $S = A - \mathfrak{p}$, then $S^{-1}A$ is a local ring whose maximal ideal is $S^{-1}\mathfrak{p}$, $S^{-1}f : S^{-1}A \rightarrow S^{-1}A$ is injective and $S^{-1}B$ is integral over $S^{-1}A$. Then there exists a prime ideal \mathfrak{P}' of $S^{-1}B$ lying over $S^{-1}\mathfrak{p}$ and we know that $\mathfrak{P}' = S^{-1}\mathfrak{P}$ where \mathfrak{P} is a prime ideal of B lying over \mathfrak{p} by ??.

Corollary 12.1.68. *Let $\rho : A \rightarrow B$ be a ring homomorphism such that B is integral over A , \mathfrak{a} and \mathfrak{p} two ideals of A such that $\mathfrak{a} \subseteq \mathfrak{P}$ and \mathfrak{b} an ideal of B lying over \mathfrak{a} . Suppose that \mathfrak{p} is prime, then there exists a prime ideal \mathfrak{P} of B lying over \mathfrak{p} and containing \mathfrak{b} .*

Proof. If $\bar{\rho} : A/\mathfrak{a} \rightarrow B/\mathfrak{b}$ is the homomorphism derived from f by taking quotients, then $\bar{\rho}$ is injective by hypothesis and B/\mathfrak{b} is integral over A/\mathfrak{a} , so the claim follows from ??.

Corollary 12.1.69. *Let A be a ring and B a ring containing A and integral over A . Then there exists one-to-one correspondence between maximal ideals of A and maximal ideals of B .*

Proof. This follows from ?? and ??.

Corollary 12.1.70. Let A be a ring and B a ring containing A and integral over A . If \mathfrak{R} is the Jacobson radical of B , then $\mathfrak{R} \cap A$ is the Jacobson radical of A .

Corollary 12.1.71 (Going-up theorem). Let A be a ring and B a ring containing A and integral over A . Let $\mathfrak{p}_1 \subset \dots \subset \mathfrak{p}_n$ be a chain of prime ideals of A and $\mathfrak{P}_1 \subset \dots \subset \mathfrak{P}_m$ a chain of prime ideals of B such that $\mathfrak{P}_i \cap A = \mathfrak{p}_i$ for $1 \leq i \leq m$ and $m < n$. Then the chain $\mathfrak{P}_1 \subset \dots \subset \mathfrak{P}_m$ can be extended to a chain $\mathfrak{P}_1 \subset \dots \subset \mathfrak{P}_n$ such that $\mathfrak{P}_i \cap A = \mathfrak{p}_i$ for $1 \leq i \leq n$.

Proof. By induction we reduce immediately to the case $m = 1$, $n = 2$, and this follows from ??.

Corollary 12.1.72. Let $\rho : A \rightarrow B$ be a ring homomorphism such that B is integral over A . Then the associated continuous map $\rho^* : \text{Spec}(B) \rightarrow \text{Spec}(A)$ is closed.

Proof. For every ideal \mathfrak{b} of B , B/\mathfrak{b} is integral over B , hence also over A and $\text{Spec}(B/\mathfrak{b})$ is identified with the closed subspace $V(\mathfrak{b})$ of $\text{Spec}(B)$. To show that ρ^* is closed, we see then (replacing B by B/\mathfrak{b}) that it is sufficient to prove that the image of $\text{Spec}(B)$ under ρ^* is a closed subset of $\text{Spec}(A)$. Now it follows from ?? that this image is just the set of prime ideals of A containing the ideal $\ker \rho$ and this set is closed by definition of the topology on $\text{Spec}(A)$.

Corollary 12.1.73. Let A be a ring, B a ring containing A and integral over A and ϕ a homomorphism from A to an algebraically closed field Ω . Then ϕ can be extended to a homomorphism from B to Ω .

Proof. Let \mathfrak{p} be the kernel of ϕ , which is a prime ideal since $\phi(A) \subseteq \Omega$ is an integral domain. Let \mathfrak{P} be a prime ideal of B lying over \mathfrak{p} . Then A/\mathfrak{p} is canonically identified with a subring of B/\mathfrak{P} and B/\mathfrak{P} is integral over A/\mathfrak{p} . The homomorphism ϕ defines, by taking the quotient, an isomorphism of A/\mathfrak{p} onto the subring $\phi(A)$ of Ω , which can be extended to an isomorphism ψ of the field of fractions K of A/\mathfrak{p} onto a subfield of Ω . As the field of fractions L of B/\mathfrak{P} is algebraic over K , ψ can be extended to an isomorphism ψ' of L onto a subfield of Ω . If $\pi : B \rightarrow B/\mathfrak{P}$ is the canonical homomorphism, $\psi' \circ \pi$ is a homomorphism from B to Ω extending ϕ .

Proposition 12.1.74. Let $\rho : A \rightarrow B$ be a ring homomorphism such that B is integral over A , \mathfrak{p} a prime ideal of A , and $S = A - \mathfrak{p}$. If $(\mathfrak{P}_i)_{i \in I}$ is the family of all the prime ideals of B lying over \mathfrak{p} and $T = \bigcap_{i \in I} (B - \mathfrak{P}_i)$, then $S^{-1}B = T^{-1}B$.

Proof. By definition $\rho(S) \subseteq T$ and, as $\rho(S)^{-1}B = S^{-1}B$, it suffices to prove, by virtue of ??, that, if a prime ideal \mathfrak{Q} of B does not meet $\rho(S)$, it does not meet T either. Now, suppose that $\mathfrak{Q} \cap \rho(S) = \emptyset$ and let $\mathfrak{q} = \rho^{-1}(\mathfrak{Q})$. Then $\mathfrak{q} \cap S = \emptyset$, in other words $\mathfrak{q} \subseteq \mathfrak{p}$. As \mathfrak{Q} lies over \mathfrak{q} by definition, it follows from ?? that there is an index i such that $\mathfrak{Q}_i \subseteq \mathfrak{p}_i$ and hence $\mathfrak{Q} \cap T = \emptyset$, which completes the proof.

Corollary 12.1.75. Let A be an integral domain, \bar{A} its integral closure, and \mathfrak{p} a prime ideal of A . If $(\mathfrak{P}_i)_{i \in I}$ is the family of all prime ideals of \bar{A} lying over \mathfrak{p} and $T = \bigcap_{i \in I} (\bar{A} - \mathfrak{P}_i)$, then $T^{-1}\bar{A}$ is the integral closure of $A_{\mathfrak{p}}$.

Proof. By ??, the integral closure of $A_{\mathfrak{p}}$ is $\bar{A}_{\mathfrak{p}}$, which equals to $T^{-1}\bar{A}$ by ??.

Proposition 12.1.76. Let $\rho : A \rightarrow B$ be a ring homomorphism such that B is a finitely generated A -module. Then, for every prime ideal \mathfrak{p} of A , the set of prime ideals of B lying over \mathfrak{p} is finite.

Proof. By taking localization and quotient, we may reduce the problem to $\mathfrak{p} = (0)$ and A is a field. The ring B is then an A -algebra of finite rank and therefore Artinian and we know that in such an algebra there is only a finite number of prime ideals. \square

Proposition 12.1.77. *Let A be an integrally closed domain and B a ring containing A and integral over A . Suppose that B is a torsion-free A -module. Let $\mathfrak{p} \subseteq \mathfrak{q}$ be two prime ideals of A and \mathfrak{Q} a prime ideal of A lying over \mathfrak{q} . Then there exists a prime ideal \mathfrak{P} of B lying over \mathfrak{p} and such that $\mathfrak{P} \subseteq \mathfrak{Q}$.*

Proof. Suppose first that B is an integral domain. Let K, L be the fields of fractions of A and B respectively. Let Ω be the algebraic closure of L and \bar{A} the integral closure of A in Ω , so that $A \subseteq B \subseteq \bar{A}$. Let \mathcal{P} be a prime ideal of \bar{A} lying over \mathfrak{p} , then by ?? there is a prime ideal \mathcal{Q} of \bar{A} lying over \mathfrak{q} and such that $\mathcal{P} \subseteq \mathcal{Q}$. Finally let \mathcal{Q}_1 be a prime ideal of \bar{A} lying over \mathfrak{Q} . By ??, there exists a K -automorphism σ of Ω such that $\sigma(\mathcal{Q}) = \mathcal{Q}_1$. Then $\sigma(\mathcal{P})$ is a prime ideal of \bar{A} lying over \mathfrak{p} such that $\sigma(\mathcal{P}) \subseteq \mathcal{Q}_1$ and hence $\mathfrak{P} = B \cap \sigma(\mathcal{P})$ is a prime ideal of B lying over \mathfrak{p} and contained in \mathfrak{Q} .

$$\begin{array}{ccccc} \sigma(\mathcal{P}) & \xhookrightarrow{\quad} & \mathcal{Q}_1 & & \\ \downarrow \sigma & & \downarrow & & \downarrow \sigma \\ \mathcal{P} & \xhookrightarrow{\quad} & \mathcal{Q} & \xhookrightarrow{\quad} & \\ \downarrow \mathfrak{P} & & \downarrow & & \downarrow \\ \mathfrak{Q} & \xhookrightarrow{\quad} & \mathfrak{q} & & \\ \downarrow \mathfrak{q} & & \downarrow & & \downarrow \\ \mathfrak{p} & \xhookrightarrow{\quad} & \mathfrak{q} & \xhookrightarrow{\quad} & \end{array}$$

We pass to the general case. As A is an integral domain and \mathfrak{Q} is prime, the subsets $A \setminus \{0\}$ and $B \setminus \mathfrak{Q}$ of B are multiplicative; then their product $S = (A \setminus \{0\})(B \setminus \mathfrak{Q})$ is a multiplicative subset of B which does not contain 0 since the non-zero elements of A are not divisors of 0 in B . Then there exists a prime ideal \mathfrak{M} of B disjoint from S , in other words such that $\mathfrak{M} \subseteq \mathfrak{Q}$ and $\mathfrak{M} \cap A = 0$. Let π be the canonical homomorphism $B \rightarrow B/\mathfrak{M}$. Then the restriction of π to A is injective and hence $\pi(A)$ is integrally closed. Since B/\mathfrak{M} is an integral domain, the first part of the proof proves that there exists a prime ideal \mathfrak{N} of B/\mathfrak{M} such that $\mathfrak{N} \cap \pi(A) = \pi(\mathfrak{p})$ and $\mathfrak{N} \subseteq \pi(\mathfrak{Q})$. The ideal $\mathfrak{P} = \pi^{-1}(\mathfrak{N})$ is a prime ideal of B and $\mathfrak{P} \subseteq \mathfrak{Q}$, since \mathfrak{Q} contains the kernel of π . As π is injective on A , we see $\mathfrak{P} \cap A = \mathfrak{p}$. \square

Corollary 12.1.78 (Going-down theorem). *Let A be an integrally closed domain and B a ring containing A and integral over A . Suppose that B is a torsion-free A -module. Let $\mathfrak{p}_1 \supseteq \dots \supseteq \mathfrak{p}_n$ be a chain of prime ideals of A and $\mathfrak{P}_1 \supseteq \dots \supseteq \mathfrak{P}_m$ a chain of prime ideals of B such that $\mathfrak{P}_i \cap A = \mathfrak{p}_i$ for $1 \leq i \leq m$ and $m < n$. Then the chain $\mathfrak{P}_1 \supseteq \dots \supseteq \mathfrak{P}_m$ can be extended to a chain $\mathfrak{P}_1 \supseteq \dots \supseteq \mathfrak{P}_n$ such that $\mathfrak{P}_i \cap A = \mathfrak{p}_i$ for $1 \leq i \leq n$.*

Corollary 12.1.79. *let \mathfrak{p} be a prime ideal of A . The prime ideals of B lying over \mathfrak{p} are the minimal elements of the set \mathcal{E} of prime ideals of B containing $\mathfrak{p}B$.*

Proof. A prime ideal of B lying over \mathfrak{p} is minimal in \mathcal{E} by virtue of ???. Conversely, let \mathfrak{Q} be a minimal element of \mathcal{E} . As $\mathfrak{Q} \cap A \supseteq \mathfrak{p}$, the going down theorem shows that there exists a prime ideal \mathfrak{P} of B lying over \mathfrak{p} such that $\mathfrak{P} \subseteq \mathfrak{Q}$. As \mathfrak{P} contains $\mathfrak{p}B$, the hypothesis made on \mathfrak{Q} implies that $\mathfrak{P} = \mathfrak{Q}$ and hence \mathfrak{Q} lies over \mathfrak{p} . \square

Proposition 12.1.80. *Let $\rho : A \rightarrow B$ be a flat homomorphism of rings. Then ρ has the going-down property.*

Proof. Let $\mathfrak{p}_1 \subseteq \mathfrak{p}$ be prime ideals in A , \mathfrak{P} a prime ideal in B such that $\mathfrak{P} \cap A = \mathfrak{p}$. Consider the localization $A_{\mathfrak{p}}$ and $B_{\mathfrak{P}}$. From ?? we know the map $\rho^* : \text{Spec}(B_{\mathfrak{P}}) \rightarrow \text{Spec}(A_{\mathfrak{p}})$ is surjective, hence there is a prime ideal \mathfrak{P}'_1 lying over \mathfrak{p}_1 . Contracting \mathfrak{P}'_1 to B gives the desired prime ideal. \square

Theorem 12.1.81. *Let $A \subseteq B$ be integral domains such that A is integrally closed and B is integral over A . Let $\phi : \text{Spec}(B) \rightarrow \text{Spec}(A)$ be the canonical map. For $b \in B$, let*

$$f(X) = X^n + a_{n-1}X^{n-1} + \cdots + a_0$$

be a monic polynomial with coefficients in A having b as a root and of minimal degree. Then

$$\phi(D(b)) = \bigcup_{i=1}^n D(a_i).$$

In particular, ϕ is an open map.

Proof. By hypothesis, f is the minimal polynomial of b over the field of fraction of A . If we set $C = A[b]$ then $C \cong A[X]/(f(X))$ is a free A -module with basis $1, b, b^2, \dots, b^{n-1}$ and is hence faithfully flat over A . Suppose that $\mathfrak{P} \in D(b)$, so that $\mathfrak{P} \in \text{Spec}(B)$ with $b \notin \mathfrak{P}$, and set $\mathfrak{p} = \mathfrak{P} \cap A$. Then $\mathfrak{p} \in \bigcup_i D(a_i)$, since otherwise $a_i \in \mathfrak{p}$ for all i , and so $b^n \in \mathfrak{p}$, hence $b \in \mathfrak{p}$, which is a contradiction. So we have $\phi(D(b)) \subseteq \bigcup_{i=1}^n D(a_i)$.

Conversely, let $\mathfrak{p} \in \bigcup_i D(a_i)$ and let \mathcal{P} be a prime of C lying over \mathfrak{p} and \mathfrak{P} a prime of B lying over \mathcal{P} . Suppose that $b \in \sqrt{\mathfrak{p}C}$, then for sufficiently large m we have

$$b^m = \sum_{i=1}^n p_i b^i, \quad p_i \in \mathfrak{p}.$$

By the assumption on $f(X)$ we must have $m \geq n$. Then $X^m - \sum_{i=1}^n p_i X^i$ is divisible by $f(X)$ in $A[X]$, which implies that X^m is divisible by $\bar{f}(X) = X^n + \sum_{i=1}^{n-1} \bar{a}_i X^i$ in $(A/\mathfrak{p})[X]$; since at least one of the \bar{a}_i is nonzero, this is a contradiction.

Thus $b \notin \sqrt{\mathfrak{p}C}$ and there exists some $\mathcal{P}_1 \in \text{Spec}(C)$ with $b \notin \mathcal{P}_1$ and $\mathfrak{p}C \subset \mathcal{P}_1$. By ?? we have $\mathcal{P} \subseteq \mathcal{P}_1$, so $\mathfrak{P} \in D(b)$ since otherwise $b \in \mathfrak{P} \cap C = \mathcal{P} \subseteq \mathcal{P}_1$, which is a contradiction. This proves $\phi(D(b)) = \bigcup_{i=1}^n D(a_i)$. Since any open set of $\text{Spec}(B)$ is a union of open sets of the form $D(a)$, it follows that ϕ is open. \square

Proposition 12.1.82. *A ring homomorphism $\rho : A \rightarrow B$ is said to have the **going-up property** (resp. the **going-down property**) if the conclusion of the going-up theorem (resp. the going-down theorem) holds for B and its subring $\rho(A)$. Let $\rho^* : \text{Spec}(B) \rightarrow \text{Spec}(A)$ be the map associated with ρ .*

(i) *Consider the following three statements:*

- (a) ρ^* is a closed map.
- (b) ρ has the going-up property.
- (c) Let \mathfrak{P} be any prime ideal of B and let $\mathfrak{p} = \mathfrak{P}^c$. Then $\rho^* : \text{Spec}(B/\mathfrak{P}) \rightarrow \text{Spec}(A/\mathfrak{p})$ is surjective.

Then we have (a) \Leftrightarrow (b) \Leftrightarrow (c).

(ii) Consider the following three statements:

(a') ρ^* is an open map.

(b') ρ has the going-down property.

(c') Let \mathfrak{P} be any prime ideal of B and let $\mathfrak{p} = \mathfrak{P}^c$. Then $\rho^* : \text{Spec}(B_{\mathfrak{P}}) \rightarrow \text{Spec}(A_{\mathfrak{p}})$ is surjective.

Then we have (a') \Rightarrow (b') \Leftrightarrow (c').

Proof. The equivalences (b) \Leftrightarrow (c) and (b') \Leftrightarrow (c') are clear. For (a) \Rightarrow (b), assume that ρ^* is closed, we claim that $\rho^*(V(\mathfrak{P})) = V(\mathfrak{p})$ if $\mathfrak{p} = \mathfrak{P}^c$. To see this, we note that since ρ^* is closed, $\rho^*(V(\mathfrak{P}))$ is closed and contains \mathfrak{p} , so $V(\mathfrak{p}) = \overline{\{\mathfrak{p}\}} \subseteq \rho^*(V(\mathfrak{P}))$. But $\mathfrak{P}' \supseteq \mathfrak{P}$ implies $(\mathfrak{P}')^c \supseteq \mathfrak{P}^c = \mathfrak{p}$, hence we also have $\rho^*(V(\mathfrak{P})) \subseteq V(\mathfrak{p}_1)$, and therefore $\rho^*(V(\mathfrak{P})) = V(\mathfrak{p})$. In particular, the map ρ^* is surjective on $V(\mathfrak{p})$, so if $\mathfrak{p} \subseteq \mathfrak{p}'$ then $\mathfrak{p}' \in V(\mathfrak{p})$ and there is a $\mathfrak{P}' \in V(\mathfrak{P})$ such that $(\mathfrak{P}')^c = \mathfrak{p}'$.

For (a') \Rightarrow (c'), we first prove the following *downward property*: If $\mathfrak{p}' \subseteq \mathfrak{p}$ and $\mathfrak{p} \in U$ for open set U , then $\mathfrak{p}' \in U$. In fact, write $U = X \setminus C$ where C is closed, then $\mathfrak{p}' \in C$ would imply $V(\mathfrak{p}') = \overline{\{\mathfrak{p}'\}} \subseteq C$ since C is closed. But $\mathfrak{p} \in V(\mathfrak{p}')$ is not in C , contradiction. By this property, if $\rho^*(U)$ is open, then for any $\mathfrak{p}' \supseteq \mathfrak{p}$ in A and \mathfrak{P} in B such that $\mathfrak{P}^c = \mathfrak{p}$. Then let U be a neighborhood of \mathfrak{P} in B , so $\rho^*(U)$ is a neighborhood of \mathfrak{p}_1 in A . Now since $\mathfrak{p}' \subseteq \mathfrak{p}$, we conclude $\mathfrak{p}' \in f(U)$, so \mathfrak{p}' has a preimage \mathfrak{P}' . This shows the map $\rho^* : \text{Spec}(B_{\mathfrak{P}}) \rightarrow \text{Spec}(A_{\mathfrak{p}})$ is surjective.

Now assume the going up property, we show that ρ^* is closed. To this end, let $V(\mathfrak{b})$ be a closed subset of $\text{Spec}(B)$. Since $V(\mathfrak{b})$ is identified with $\text{Spec}(B/\mathfrak{b})$ and the induced map $A/\mathfrak{a} \rightarrow B/\mathfrak{b}$ satisfies the going up property (where $\mathfrak{a} = \mathfrak{b}^c$), we only need to show the image of $\text{Spec}(B)$ under ρ^* is closed. For this, let T be the image of ρ^* , and let $\mathfrak{p} \in \text{Spec}(A)$ be in the closure of T . This means for every $f \in A - \mathfrak{p}$ we have $D(f) \cap T \neq \emptyset$. Note that $D(f) \cap T$ is the image of $\text{Spec}(B_f)$ in $\text{Spec}(A)$. Hence we conclude that $B_f \neq 0$. Since $B_{\mathfrak{p}}$ is the directed limit of the rings B_f with $f \in A - \mathfrak{p}$, we conclude that $B_{\mathfrak{p}} \neq 0$ by ???. If \mathcal{P}' is a prime ideal of $B_{\mathfrak{p}}$, then $\mathfrak{p}' = \mathcal{P}' \cap A$ is a prime ideal of A such that $\mathfrak{p}' \subseteq \mathfrak{p}$, and it is in the image of ρ^* since $\mathfrak{P}' \cap A = \mathfrak{p}'$, where $\mathfrak{P}' = (i_B^{\mathfrak{p}})^{-1}(\mathcal{P}')$. As we assumed that ρ satisfies the going up property, there exist a prime ideal \mathfrak{P} of B lying over \mathfrak{p} , whence $\mathfrak{p} \in T$ and T is closed. \square

12.2 Group action on algebras

12.2.1 Group acting on an algebra

Given a ring R , a R -algebra A and a group G acting on A , we shall say the action of G is **R -linear** or G is a R -action if for each $\sigma \in G$, the map $x \mapsto \sigma(x)$ is an endomorphism of the R -algebra A . We shall denote by A^G the set of elements of A which are invariant under G . Clearly it is a sub- R -algebra of A . We shall say that the **action is locally finite** if every orbit of G in A is finite.

Proposition 12.2.1. *Let A be a R -algebra and G a locally finite R -action on A . Then A is integral over the subalgebra A^G .*

Proof. For all $x \in A$, let x_1, \dots, x_n be the distinct elements of the orbit of x under G . For all $\sigma \in G$, there exists a permutation $\pi_\sigma \in \mathfrak{S}_n$ such that $\sigma(x_i) = x_{\pi_\sigma(i)}$ for $1 \leq i \leq n$. Therefore the elementary symmetric functions of the x_i are elements of A which are invariant under G , in other words elements of A^G . As x is a root of the monic polynomial $\prod_i (X - x_i)$ and the coefficients of this polynomial belong to A^G , x is integral over A^G . \square

Theorem 12.2.2. *Let A be a finitely generated R -algebra and G a locally finite R -action on A . Then A is a finitely generated A^G -module. If further R is Noetherian, A^G is a finitely generated R -algebra.*

Proof. Let a_1, \dots, a_n generate the R -algebra A . Then we see $A = A^G[a_1, \dots, a_n]$ and the a_i are integral over A^G by ??, the first assertion follows. The second is a consequence of the following lemma:

Let A be a Noetherian ring, B a finitely generated A -algebra and C a sub- A -algebra of B such that B is integral over C . Then C is a finitely generated A -algebra. (12.2.1)

To prove the lemma, let x_1, \dots, x_n be a finite system of generators of the A -algebra B . For all i , there exists by hypothesis a monic polynomial $P_i \in C[X]$ such that $P_i(x_i) = 0$. Let C' be the sub- A -algebra of C generated by the coefficients of the P_i . Clearly the x_i are integral over C' and $B = C'[x_1, \dots, x_n]$, hence B is a finitely generated C' -module. On the other hand C' is a Noetherian ring, hence C is a finitely generated C' -module, which proves that C is a finitely generated A -algebra. \square

Let S be a multiplicative subset of a ring A and G a group acting on A and for which S is stable. Then, for all $\sigma \in G$, there exists a unique endomorphism $z \mapsto \sigma(z)$ of the ring $S^{-1}A$ such that $\sigma(a/1) = \sigma(a)/1$ for all $a \in A$, which is given by $\sigma(a/s) = \sigma(a)/\sigma(s)$. If τ is another element of G , then clearly $\sigma(\tau(z)) = \sigma\tau(z)$ for all $z \in S^{-1}A$, whence the group G operates on the ring $S^{-1}A$.

Proposition 12.2.3. *Let A be a R -algebra, G a locally finite R -action on A , S a multiplicative subset of A stable under G and S^G the set $S \cap A^G$. Then the canonical map $i_A^{S^G, S} : (S^G)^{-1}A \rightarrow S^{-1}A$ is an isomorphism which maps $(S^G)^{-1}A^G$ to $(S^{-1}A)^G$.*

Proof. For all $s \in S$, let $s_1 = s, s_2, \dots, s_n$ be the distinct elements of the orbit of s under G . As $\prod_{i=1}^n s_i \in S^G$, the first assertion follows from ???. Identifying canonically $(S^G)^{-1}A$ with $S^{-1}A$, clearly every element of $(S^G)^{-1}A^G$ is invariant under G . Conversely, let a/s be an element of $(S^G)^{-1}A$ which is invariant under G . If a_1, \dots, a_m are the distinct elements of the orbit of a under G , then $a_j/s = a/s$ for $1 \leq j \leq m$ and therefore there exists $t \in S^G$ such that $t(a_j - a) = 0$ for $1 \leq j \leq m$. In other words, ta is invariant under G and, as $a/s = (ta)/(ts)$, certainly $a/s \in (S^G)^{-1}A^G$. \square

Corollary 12.2.4. *Let A be an integral domain, K its field of fractions and G a locally finite R -action on A . Then G acts on K and K^G is the field of fractions of A^G .*

12.2.2 Decomposition group and inertial group

Let B be a ring and G a group acting on B . Given a prime ideal \mathfrak{P} of B the subgroup of elements $\sigma \in G$ such that $\sigma(\mathfrak{P}) = \mathfrak{P}$ is called the **decomposition group** of \mathfrak{P} (with respect to

G) and is denoted by $G^Z(\mathfrak{P})$. The ring of elements of B invariant under $G^Z(\mathfrak{P})$ is called the **decomposition ring** of \mathfrak{P} (with respect to G) and is denoted by $A^Z(\mathfrak{P})$.

We often write G^Z and A^Z instead of $G^Z(\mathfrak{P})$ and $A^Z(\mathfrak{P})$ respectively, when there is no ambiguity. For all $\sigma \in G^Z(\mathfrak{P})$, we also denote by $z \mapsto \sigma(z)$ the endomorphism of the ring B/\mathfrak{P} derived from the endomorphism $x \mapsto \sigma(x)$ of B by taking quotients; clearly the group $G^Z(\mathfrak{P})$ operates in this way on the ring B/\mathfrak{P} . The subgroup of $G^Z(\mathfrak{P})$ consisting of those σ such that the endomorphism $z \mapsto \sigma(z)$ of B/\mathfrak{P} is the identity is called the **inertia group** of \mathfrak{P} (with respect to G) and denoted by $G^T(\mathfrak{P})$. Similarly, we write $A^T(\mathfrak{P})$ for the subring of B invariant under $G^T(\mathfrak{P})$.

If A is the subring of B consisting of the invariants of G (i.e., $A = B^G$), clearly we have

$$A \subseteq A^Z(\mathfrak{P}) \subseteq A^T(\mathfrak{P}) \subseteq B.$$

It follows from definition that, for $\rho \in G$,

$$G^Z(\rho(\mathfrak{P})) = \rho G^Z(\mathfrak{P})\rho^{-1}, \quad G^T(\rho(\mathfrak{P})) = \rho G^T(\mathfrak{P})\rho^{-1}. \quad (12.2.2)$$

We may also write $\mathfrak{P}^Z := \mathfrak{P} \cap A^Z$ and $\mathfrak{P}^T = \mathfrak{P} \cap A^T$.

If, for all $\sigma \in G^Z(\mathfrak{P})$, $\bar{\sigma}$ denotes the automorphism $z \mapsto \overline{\sigma(z)}$ of B/\mathfrak{P} , then the map $\sigma \mapsto \bar{\sigma}$ is a homomorphism (called canonical) of $G^Z(\mathfrak{P})$ to the group of automorphisms of B/\mathfrak{P} leaving invariant the elements of $A^Z/(\mathfrak{P}^Z)$, and by definition $G^T(\mathfrak{P})$ is the kernel of this canonical homomorphism, so it is a normal subgroup of $G^Z(\mathfrak{P})$. If $\kappa(\mathfrak{P})$ is the field of fractions of B/\mathfrak{P} , every automorphism of B/\mathfrak{P} can be extended uniquely to an automorphism of $\kappa(\mathfrak{P})$, so that $\sigma \mapsto \bar{\sigma}$ can be considered as a homomorphism from $G^Z(\mathfrak{P})$ to the group of automorphisms of $\kappa(\mathfrak{P})$. Note that, since $G^T(\mathfrak{P})$ is normal in $G^Z(\mathfrak{P})$, the ring $A^T(\mathfrak{P})$ is stable under G , in view of ??.

Proposition 12.2.5. *Let B be a ring, G a group acting on B , A the ring of invariants of G , \mathfrak{P} a prime ideal of B and S a multiplicative subset of A not meeting \mathfrak{P} . Then*

$$G^Z(S^{-1}\mathfrak{P}) = G^Z(\mathfrak{P}), \quad G^T(S^{-1}\mathfrak{P}) = G^T(\mathfrak{P})$$

and, if the action of G is locally finite, then

$$S^{-1}A^Z(\mathfrak{P}) = A^Z(S^{-1}\mathfrak{P}), \quad S^{-1}A^T(\mathfrak{P}) = A^T(S^{-1}\mathfrak{P}).$$

Proof. As the elements of S are invariant under G , clearly, if $\sigma(\mathfrak{P}) = \mathfrak{P}$, also $\sigma(S^{-1}\mathfrak{P}) = S^{-1}\mathfrak{P}$. Conversely, suppose that $\sigma \in G$ is such that $\sigma(S^{-1}\mathfrak{P}) = S^{-1}\mathfrak{P}$, then, if $x \in \mathfrak{P}$, we have $\sigma(x/1) \in S^{-1}\mathfrak{P}$ and therefore there exists $s \in S$ such that $s\sigma(x) \in \mathfrak{P}$, whence $\sigma(x) \in \mathfrak{P}$ since \mathfrak{P} is prime and $s \notin \mathfrak{P}$. This proves that $\sigma(\mathfrak{P}) \subseteq \mathfrak{P}$ and it can be similarly shown that $\sigma^{-1}(\mathfrak{P}) \subseteq \mathfrak{P}$, hence $\sigma(\mathfrak{P}) = \mathfrak{P}$ and $\sigma \in G^Z(\mathfrak{P})$. If $\sigma \in G^T(\mathfrak{P})$, then $\sigma(x) - x \in \mathfrak{P}$ for all $x \in B$, hence also, for all $s \in S$,

$$\sigma(x/s) - (x/s) = (\sigma(x) - x)/s \in S^{-1}\mathfrak{P}$$

and therefore $\sigma \in G^T(S^{-1}\mathfrak{P})$. Conversely, suppose that $\sigma \in G^T(S^{-1}\mathfrak{P})$. Then, for all $x \in B$, $\sigma(x/1) - (x/1) \in S^{-1}\mathfrak{P}$ and therefore there exists $s \in S$ such that $s(\sigma(x) - x) \in \mathfrak{P}$, whence as above $\sigma(x) - x \in \mathfrak{P}$, which proves that $\sigma \in G^T(\mathfrak{P})$. The last two assertions follow from ??.

Proposition 12.2.6. *Let B be a ring, G a finite group acting on B and A the ring of invariants of G so that B is integral over A .*

- (a) *Given two prime ideals $\mathfrak{P}, \mathfrak{Q}$ of B lying over the same prime ideal \mathfrak{p} of A , there exists $\sigma \in G$ such that $\sigma(\mathfrak{P}) = \mathfrak{Q}$. In other words, G operates transitively on the set of prime ideals of B lying over \mathfrak{p} .*
- (b) *Let \mathfrak{P} be a prime ideal of B and $\mathfrak{p} = \mathfrak{P} \cap A$. Then $\kappa(\mathfrak{P})$ is a normal extension of $\kappa(\mathfrak{p})$ and the canonical homomorphism $\sigma \mapsto \bar{\sigma}$ defines by taking the quotient an isomorphism of $G^Z(\mathfrak{P})/G^T(\mathfrak{P})$ onto $\text{Gal}(\kappa(\mathfrak{P})/\kappa(\mathfrak{p}))$. In other words, we get the following exact sequence*

$$1 \longrightarrow G^T(\mathfrak{P}) \longrightarrow G^Z(\mathfrak{P}) \longrightarrow \text{Gal}(\kappa(\mathfrak{P})/\kappa(\mathfrak{p})) \longrightarrow 1$$

Proof. If $x \in \mathfrak{Q}$, then $\prod_{\sigma \in G} \sigma(x) \in \mathfrak{Q} \cap A = \mathfrak{p} \subseteq \mathfrak{P}$. Hence there exists $\sigma \in G$ such that $\sigma(x) \in \mathfrak{P}$, that is $x \in \sigma^{-1}(\mathfrak{P})$. We then conclude that $\mathfrak{Q} \subseteq \bigcup_{\sigma \in H} \sigma(\mathfrak{P})$, and hence (G is finite) $\mathfrak{Q} \subseteq \sigma(\mathfrak{P})$ for some $\sigma \in G$. Since \mathfrak{Q} and $\sigma(\mathfrak{P})$ both lie over \mathfrak{p} , we see $\mathfrak{Q} = \sigma(\mathfrak{P})$.

To see that $\kappa(\mathfrak{P})$ is a normal extension of $\kappa(\mathfrak{p})$, it suffices to prove that every element $\bar{x} \in B/\mathfrak{P}$ is a root of a polynomial P in $\kappa(\mathfrak{p})[X]$ all of whose roots are in B/\mathfrak{P} . Now, let $x \in B$ be a representative of the class \bar{x} . Then the polynomial $P(X) = \prod_{\sigma \in G} (X - \sigma(x))$ has all its coefficients in A . Let $\bar{P}(X)$ be the polynomial in $(A/\mathfrak{p})[X]$ which is the image of $P(X)$, then we see $\bar{P}(X) = \prod_{\sigma \in G} (X - \overline{\sigma(x)})$ and therefore solves the problem.

Clearly, for all $\sigma \in G^Z$, $\bar{\sigma}$ is a $\kappa(\mathfrak{p})$ -automorphism of $\kappa(\mathfrak{P})$. It remains to verify that $\sigma \mapsto \bar{\sigma}$ maps G^Z onto the group of all $\kappa(\mathfrak{p})$ -automorphisms of $\kappa(\mathfrak{P})$. Since $\kappa(\mathfrak{p})$ and $\kappa(\mathfrak{P})$ are not changed under localization and by ?? neither G^Z nor its operation on $\kappa(\mathfrak{P})$ is changed, we may therefore restrict our attention to the case where \mathfrak{p} is maximal, in which case we know that so is \mathfrak{P} and every element of $\kappa(\mathfrak{P})$ is therefore of the form \bar{x} for some x in B . It has been seen above that such an element is a root of a polynomial in $\kappa(\mathfrak{p})[X]$ of degree smaller than $|G|$. As every finite separable extension of $\kappa(\mathfrak{p})$ admits a primitive element, it is seen that every finite separable extension of $\kappa(\mathfrak{p})$ contained in $\kappa(\mathfrak{P})$ is of degree smaller than $|G|$, whence the separable closure $\kappa(\mathfrak{p})^{\text{sep}}$ in $\kappa(\mathfrak{P})$ is of degree smaller than $|G|$. Let $y \in B$ be an element such that \bar{y} is a primitive element of $\kappa(\mathfrak{p})^{\text{sep}}$. The ideals $\sigma(\mathfrak{P})$ for $\sigma \in G - G^Z$ are maximal and distinct from \mathfrak{P} by definition. By Chinese remainder theorem there then exists $x \in B$ such that $x \equiv y \pmod{\mathfrak{P}}$ and $x \in \sigma(\mathfrak{P})$ for all $\sigma \in G - G^Z$. Now let θ be a $\kappa(\mathfrak{p})$ -automorphism of $\kappa(\mathfrak{P})$ and let $P(X) = \prod_{\sigma \in G} (X - \overline{\sigma(x)})$. As \bar{x} is a root of P and $P \in \kappa(\mathfrak{p})[X]$, $\theta(\bar{x})$ is also a root of P in $\kappa(\mathfrak{P})$ and hence there exists $\tau \in G$ such that

$$\theta(\bar{x}) = \tau(\bar{x}).$$

Since $\sigma(\bar{x}) = 0$ for all $\sigma \in G - G^Z$ and $\theta(\bar{x}) \neq 0$, we conclude that necessarily $\tau \in G^Z$. As θ and τ have the same value for the primitive element $\bar{x} = \bar{y}$ of $\kappa(\mathfrak{p})^{\text{sep}}$, they coincide on $\kappa(\mathfrak{p})^{\text{sep}}$ and, as $\kappa(\mathfrak{P})$ is an inseparable extension of $\kappa(\mathfrak{p})^{\text{sep}}$, they coincide on $\kappa(\mathfrak{P})$. \square

Corollary 12.2.7. *With the hypotheses and notation of ??, let f_1 and f_2 be two homomorphisms of B to a field Ω with the same restriction to A . Then there exists a $\sigma \in G$ such that $f_2 = f_1 \circ \sigma$.*

Proof. Let \mathfrak{P}_i be the kernel of f_i , which is a prime ideal of B . By hypothesis $\mathfrak{P}_1 \cap A = \mathfrak{P}_2 \cap A$ and this intersection is a prime ideal \mathfrak{p} of A . There therefore exists $\tau \in G$ such that $\tau(\mathfrak{P}_2) = \mathfrak{P}_1$.

Replacing f_1 by the homomorphism $f_1 \circ \tau$, we may then assume that $\mathfrak{P}_1 = \mathfrak{P}_2$ (an ideal which we shall denote by \mathfrak{P}). By taking the quotient and localization we then derive from f_1 and f_2 two injective homomorphisms \tilde{f}_1, \tilde{f}_2 from $\kappa(\mathfrak{P})$ to Ω . As $\kappa(\mathfrak{P})$ is a normal extension of $\kappa(\mathfrak{p})$, we see $\tilde{f}_2 \circ \tilde{f}_1^{-1}$ is a $\kappa(\mathfrak{p})$ -automorphism of $\kappa(\mathfrak{P})$ (??) and it follows from ?? that $\tilde{f}_2 \circ \tilde{f}_1^{-1} = \bar{\sigma}$ for some $\sigma \in G$. In particular, for all $x \in B$ the elements $f_2(x)$ and $f_1(\sigma(x))$ are equal. \square

Proposition 12.2.8. *Let B be a ring, G a finite group acting on B and H a subgroup of G . Let A^G and A^H be the rings of invariants of G and H , respectively, so that $A^G \subseteq A^H$. For a prime ideal \mathfrak{P} of B , set $\mathfrak{p}^G = \mathfrak{P} \cap A^G$ and $\mathfrak{p}^H = \mathfrak{P} \cap A^H$.*

- (a) *For H to be contained in the decomposition group $G^Z(\mathfrak{P})$, it is necessary and sufficient that \mathfrak{P} be the only prime ideal of A lying over \mathfrak{p}^H .*
- (b) *If H contains $G^Z(\mathfrak{P})$, then*
 - (α) *The rings A^G/\mathfrak{p}^G and A^H/\mathfrak{p}^H have the same field of fractions;*
 - (β) *The maximal ideal of the local ring $(A^H)_{\mathfrak{p}^H}$ is generated by the image of \mathfrak{p}^G .*
- (c) *Suppose further that B is an integral domain and that $\bigcap_{n=1}^{\infty} (\mathfrak{p}^G)^n B_{\mathfrak{P}} = 0$, then the conditions of (b) imply that $G^Z(\mathfrak{P})$ preserves the elements of A^H .*

Proof. It follows from ??(a) that the prime ideals of B lying over \mathfrak{p}^H are the ideals of the form $\sigma(\mathfrak{P})$ where $\sigma \in H$; whence immediately (a). We now prove (b); by taking localization, we may assume that \mathfrak{p}^G is maximal. To establish (α) it will be sufficient to prove that

$$A^H = A^G + \mathfrak{p}^H, \quad (12.2.3)$$

for this will show that the fields (A^G/\mathfrak{p}^G) and $(A^H)/(\mathfrak{p}^H)$ are canonically isomorphic. By ?? there is only a finite number of primes of B lying over \mathfrak{p}^G , therefore there is only a finite number of prime ideals of A^H lying over \mathfrak{p}^G ; let $\mathfrak{n}_1, \dots, \mathfrak{n}_r$ denote those of these ideals that are different from \mathfrak{p}^H . Let x be an element of A^H , as the ideals \mathfrak{p}^H and \mathfrak{n}_i are maximal, there exists $y \in A^H$ such that $y \equiv x \pmod{\mathfrak{p}^H}$ and $y \in \mathfrak{n}_i$ for $1 \leq i \leq r$. Let $y_1 = y, y_2, \dots, y_s$ be distinct elements of the orbit of y under G . Clearly

$$z = y_1 + y_2 + \cdots + y_s \in A^G$$

and to establish (??) it will be sufficient to show that $y_i \in \mathfrak{P}$ for $i \geq 2$, for then we shall deduce that $z - y \in \mathfrak{P} \cap A^H = \mathfrak{p}^H$, whence $x \in A^G + \mathfrak{p}^H$ since $x \equiv y \pmod{\mathfrak{p}^H}$. Now let $i \geq 2$ and $\sigma \in G$ be such that $\sigma(y) = y_i$. Then $\sigma^{-1}(\mathfrak{P})$ does not lie over \mathfrak{p}^H , for otherwise there would exist $\tau \in H$ such that $\sigma^{-1}(\mathfrak{P}) = \tau(\mathfrak{P})$, whence $\sigma\tau \in G^Z \subseteq H$ (??(a)). But as $y \in A^H$ and $\sigma(\tau(y)) = \sigma(y) \neq y$, this is a contradiction. We conclude that $\sigma^{-1}(\mathfrak{P})$ lies over one of the ideals \mathfrak{n}_j and as $y \in \mathfrak{n}_j$ by construction, we see $y \in \sigma^{-1}(\mathfrak{P})$ or $y_i = \sigma(y) \in \mathfrak{P}$.

To prove (β) we show that \mathfrak{p}^H is contained in the saturation \mathfrak{q} of the ideal $(\mathfrak{p}^G)A^H$ with respect to \mathfrak{p}^H . As \mathfrak{p}^H is contained in none of the \mathfrak{n}_i , by ?? it suffices to prove that

$$\mathfrak{p}^H \subseteq \mathfrak{q} \cup \mathfrak{n}_1 \cup \cdots \cup \mathfrak{n}_r. \quad (12.2.4)$$

For this, we consider an element $x \in \mathfrak{p}^H$ belonging to none of the \mathfrak{n}_i . Let $x_1 = x, x_2, \dots, x_m$ be the distinct elements of the orbit of x under G . Write $u = x_1 x_2 \cdots x_m$ and $v = x_2 \cdots x_m$, then

$u \in A^G$. On the other hand, if $\tau \in H$ then $\tau(x) = x$ and hence necessarily $\tau(x_i) \neq x$ for $i \geq 2$, which shows that $\tau(v) = v$ and hence $v \in A^H$. It can be shown as in the proof of (α) that, if $\sigma \in G$ is such that $\sigma(x) = x_i$ where $i \geq 2$, then $\sigma^{-1}(\mathfrak{P})$ lies over one of the \mathfrak{n}_j and, as $x \notin \mathfrak{n}_j$, we have $x \notin \sigma^{-1}(\mathfrak{P})$ whence $x_i \notin \mathfrak{P}$. We conclude that $v \notin \mathfrak{P}$ and therefore $v \notin \mathfrak{p}^H$. On the other hand, clearly $u \in \mathfrak{P} \cap A^G = \mathfrak{p}^G$ and the relation $u = xv$ then shows that x is in the saturation of $\mathfrak{p}^G(A^H)$ with respect to \mathfrak{p}^H and hence establishes (??).

Suppose that B is an integral domain, that $\bigcap_{n=0}^{\infty} (\mathfrak{p}^G)^n B_{\mathfrak{P}} = 0$ and that conditions (α) and (β) of (b) hold. With the same notation as in (b), by localization we may assume also that the ideal \mathfrak{p}^G is maximal. The hypotheses (α) and (β) then imply that for all $n > 0$ we have (by induction)

$$(A^H)_{\mathfrak{p}^H} = A^G + (\mathfrak{p}^G)^n (A^H)_{\mathfrak{p}^H}.$$

Then let σ be an element of G^Z and x be an element of A^H . For all $n > 0$, there exists $a_n \in A^G$ such that $x - a_n \in (\mathfrak{p}^G)^n (A^H)_{\mathfrak{p}^H} \subseteq (\mathfrak{p}^G)^n B_{\mathfrak{P}}$. As $\sigma(a_n) = a_n$ and $\sigma(\mathfrak{P}) = \mathfrak{P}$, we deduce that $\sigma(x) - x \in (\mathfrak{p}^G)^n B_{\mathfrak{P}}$. Since this relation holds for all n , we conclude from the hypothesis that $\sigma(x) = x$. \square

Corollary 12.2.9. *Under the hypotheses of ?? the rings $(A^G)/(\mathfrak{p}^G)$ and $A^Z/(\mathfrak{P}^Z)$ have the same field of fractions and the maximal ideal of the local ring $(A^Z)_{\mathfrak{P}^Z}$ is generated by \mathfrak{p}^G .*

Proof. This follows from ?? by taking $H = G^Z$. \square

Corollary 12.2.10. *Let B be an integral domain, G a finite group acting on B , A the ring of invariants of G and \mathfrak{P} a prime ideal of B . Let K , K^Z and L be the fields of fractions of A , A^Z and L respectively. Then B is a Galois extension of K with Galois group G and the subfields E of L containing K and such that \mathfrak{P} is the only prime ideal of B lying over the ideal $\mathfrak{P} \cap E$ of $B \cap E$ are just those which contain K^Z .*

Proof. By ?? applied to $S = A - \{0\}$, we see K is the field of invariants of G in L and similarly K^Z is the field of invariants of G^Z . By ?? we see L is a Galois extension of K with Galois group G . By Galois correspondence, we see the condition " E contains K^Z " is equivalent to " H is contained in G^Z ", where H is the fixed group of E . As E is then the field of invariants of H in L , $B \cap E$ is the ring of invariants of H in B . The second assertion then follows from ?? \square

Proposition 12.2.11. *Let B be an integral domain, G a finite group acting on B , A the ring of invariants of G , \mathfrak{p} a prime ideal of A and \mathfrak{P} a prime of B lying over A . Then*

- (a) *The residue field of \mathfrak{P}^T is the separable closure of $\kappa(\mathfrak{p})$ in $\kappa(\mathfrak{P})$.*
- (b) *The extension K^T/K is Galois and $\text{Gal}(K^T/K)$ is isomorphic to $\text{Gal}(\kappa(\mathfrak{P}^T)/\kappa(\mathfrak{p}))$.*
- (c) *There exists a correspondence between the set of all intermediate extensions of $K^T/E/K$ and the set of all separable extensions $\kappa(\mathfrak{P}^T)/\kappa/\kappa(\mathfrak{p})$. The extension $\kappa/\kappa(\mathfrak{p})$ is a Galois extension if and only if E/K is Galois. Moreover, κ is the residue field of $\mathfrak{P} \cap E$.*

Proof. By localization this may be reduced to the case where \mathfrak{p} is a maximal ideal of A , which implies that \mathfrak{P} , \mathfrak{P}^Z and \mathfrak{P}^T are maximal in B , A^Z and A^T respectively. Since G^T is a normal subgroup of G , we know K^T/K is Galois with Galois group G/G^T . Since K^T/K is normal, by ?? we see $\kappa(\mathfrak{P}^T)$ is a normal extension of $\kappa(\mathfrak{p})$, and ?? shows that $\text{Gal}(K^T/K)$ is isomorphic to $\text{Gal}(\kappa(\mathfrak{P}^T)/\kappa(\mathfrak{p}))$, hence the correspondence between the sets of intermediate fields $K \subseteq$

$E \subseteq K^T$ and $\kappa(\mathfrak{p}) \subseteq \kappa \subseteq \kappa(\mathfrak{P}^T)$ follows from Galois correspondence as well as the statement concerning normality. To see the last statement of (c), observe that K^T is also the inertial field of $\mathfrak{P} \cap E$ for any intermediate extension $K \subseteq E \subseteq K^T$, therefore if $\kappa(E)$ is the residue field of $\mathfrak{P} \cap E$, then $\text{Gal}(K^T/E) \cong \text{Gal}(\kappa(\mathfrak{P}^T)/\kappa(E))$, by what we have just seen.

For all $x \in B$, the polynomial $P(X) = \prod_{\sigma \in G^T} (X - \sigma(x))$ has its coefficients in the inertia ring A^T and, by definition of G^T , all its roots in B are congruent mod \mathfrak{P} . Let $\pi : B \rightarrow B/\mathfrak{P}$ be the canonical homomorphism; the polynomial $\pi(P)(X)$ over A^T/\mathfrak{P}^T therefore has all its roots in B/\mathfrak{P} equal to $\pi(x)$, which shows that $\kappa(\mathfrak{P})$ is purely inseparable over $\kappa(\mathfrak{P}^T)$, so $\kappa(\mathfrak{p})^{\text{sep}} \subseteq \kappa(\mathfrak{P}^T)$.

We know that $\kappa(\mathfrak{p})^{\text{sep}}/\kappa(\mathfrak{p})$ is a Galois extension and it follows from ?? that its Galois group is isomorphic to $G' = G^Z/G^T$. As $\kappa(\mathfrak{P}^T)$ is a purely inseparable extension of $\kappa(\mathfrak{p})^{\text{sep}}$, $\kappa(\mathfrak{P}^T)$ is a normal extension of $\kappa(\mathfrak{p})$ and the separable degree of $\kappa(\mathfrak{P}^T)$ over $\kappa(\mathfrak{p})$ is

$$[\kappa(\mathfrak{P}^T) : \kappa(\mathfrak{p})]_s = [\kappa(\mathfrak{p})^{\text{sep}} : \kappa(\mathfrak{p})] = [G^Z : G^T] =: q.$$

It remains to see that $\kappa(\mathfrak{P}^T)$ is a separable extension of $\kappa(\mathfrak{p})$. We have seen above that G' is identified with an automorphism group of A^T and that A^Z is the ring of invariants of G' . If $x \in A^T$, the polynomial $Q(X) = \prod_{\sigma' \in G'} (X - \sigma'(x))$ therefore has its coefficients in A^Z . The polynomial $\pi(Q)$ over A^Z/\mathfrak{p}^Z whose coefficients are the images of those of Q under the homomorphism π is of degree q and has a root $\pi(x) \in A^T/\mathfrak{p}^T$. As $A^Z/\mathfrak{p}^Z = \kappa(\mathfrak{p})$ by ??(b), we see that every element of $\kappa(\mathfrak{P}^T)$ is of degree $\leq q$ over $\kappa(\mathfrak{p})$.

This being so, let κ be the field of invariants of the Galois group $\text{Gal}(\kappa(\mathfrak{P}^T)/\kappa(\mathfrak{p}))$; then $[\kappa(\mathfrak{P}^T) : \kappa] = q$ by ???. Let ξ be a primitive element of $\kappa(\mathfrak{P}^T)$ over κ ; as it is of degree q over κ and of degree $\leq q$ over $\kappa(\mathfrak{p})$, it is of degree q over $\kappa(\mathfrak{p})$ and its minimal polynomial over κ has coefficients in $\kappa(\mathfrak{p})$; this shows that ξ is separable over $\kappa(\mathfrak{p})$. On the other hand, for all $\zeta \in \kappa$, there exists a power p^f of the characteristic p such that $\zeta^{p^f} \in \kappa(\mathfrak{p})$. We conclude that $\kappa(\mathfrak{p})(\xi - \zeta)$, which contains

$$(\xi - \zeta)^{p^f} = \xi^{p^f} - \zeta^{p^f}$$

contains ξ^{p^f} and consequently $\kappa(\mathfrak{p})(\xi^{p^f})$. But as ξ is separable over $\kappa(\mathfrak{p})$, $\kappa(\mathfrak{p})(\xi) = \kappa(\mathfrak{p})(\xi^{p^f})$ by ??, whence $\kappa(\mathfrak{p})(\xi) \subseteq \kappa(\mathfrak{p})(\xi - \zeta)$. As ξ is of degree q over $\kappa(\mathfrak{p})$ and $\xi - \zeta$ of degree $\leq q$, it follows that $\kappa(\mathfrak{p})(\xi) = \kappa(\mathfrak{p})(\xi - \zeta)$, whence $\zeta \in \kappa(\mathfrak{p})(\xi)$. This shows that ζ is separable over $\kappa(\mathfrak{p})$, hence $\kappa = \kappa(\mathfrak{p})$ and $\kappa(\mathfrak{P}^T)$ is separable over $\kappa(\mathfrak{p})$. \square

Corollary 12.2.12. *If the order of the inertia group $G^T(\mathfrak{P})$ is relatively prime to the characteristic p of $\kappa(\mathfrak{p})$, then $\kappa(\mathfrak{P})/\kappa(\mathfrak{p})$ is a Galois extension.*

Proof. With the notation of the proof of ??, the polynomial $\pi(P)$ has coefficients in $\kappa(\mathfrak{P}^T) = \kappa(\mathfrak{p})^{\text{sep}}$; and all its roots equal to the image \bar{x} of x in B/\mathfrak{P} . We immediately deduce that $\pi(P)$ is a power of a minimal polynomial of \bar{x} over $\kappa(\mathfrak{p})^{\text{sep}}$. But the latter has degree equal to a power of p and hence, as the degree of $\pi(P)$ is equal to the order of G^T , the hypothesis implies that $\pi(P)$ has degree zero, in other words $\bar{x} \in \kappa(\mathfrak{p})^{\text{sep}}$. Thus $\kappa(\mathfrak{P}^T) = \kappa(\mathfrak{p})^{\text{sep}}$ and $\kappa(\mathfrak{P})/\kappa(\mathfrak{p})$ is a Galois extension. \square

12.2.3 Applications for integrally closed domains

Lemma 12.2.13. *Let A be an integrally closed domain, K its field of fractions, p the characteristic of K , L a radicial extension of K and B a subring of L containing A and integral over A . For every prime ideal \mathfrak{p} of A , there exists a unique prime ideal \mathfrak{P} of B lying over \mathfrak{p} and \mathfrak{P} is the set of $x \in B$ such that there exists an integer $n > 0$ for which $x^{p^n} \in \mathfrak{p}$.*

Proof. The existence of \mathfrak{P} follows from ???. If $x \in \mathfrak{P}$, there exists $n > 0$ such that $x^{p^n} \in K$, whence $x^{p^n} \in A$ since A is integrally closed, hence $x^{p^n} \in A \cap \mathfrak{P} = \mathfrak{p}$. Conversely, if $x \in B$ is such that $x^{p^n} \in \mathfrak{p} \subseteq \mathfrak{P}$ then $x \in \mathfrak{p}$ since \mathfrak{P} is prime. \square

Proposition 12.2.14. *Let A be an integrally closed domain, K its field of fractions, L a normal extension of K and B the integral closure of A in L . Then*

- (a) *For every prime ideal \mathfrak{p} of A , the group G of K -automorphisms of L acts transitively on the set of prime ideals of B lying over \mathfrak{p} .*
- (b) *For every prime ideal \mathfrak{P} of B and $\mathfrak{p} = \mathfrak{P} \cap A$, the residue field $\kappa(\mathfrak{P})$ is a normal extension of $\kappa(\mathfrak{p})$ and the canonical homomorphism $\sigma \mapsto \bar{\sigma}$ defines by taking the quotient an isomorphism of $G^Z(\mathfrak{P})/G^T(\mathfrak{P})$ onto $\text{Gal}(\kappa(\mathfrak{P})/\kappa(\mathfrak{p}))$.*

Proof. Suppose first that L is a finite Galois extension of K . Then $A = B \cap K$ since A is integrally closed and A is therefore the ring of invariants of G in B . As G is finite, the proposition follows in this case from ??.

Suppose secondly that L is any Galois extension of K . Then L is the union of a right directed family $(K_\alpha)_{\alpha \in I}$ of finite Galois extensions of K . To show (a), consider two prime ideals $\mathfrak{P}, \mathfrak{Q}$ of B lying over \mathfrak{p} . For all $\alpha \in I$, $\mathfrak{P} \cap K_\alpha$ and $\mathfrak{Q} \cap K_\alpha$ are two prime ideals of $B \cap K_\alpha$ lying over \mathfrak{p} . Since $B \cap K_\alpha$ is the integral closure of A in K_α and the restrictions to K_α of the elements of G form the group of K -automorphisms of K_α , it follows from the finite case that there exists $\sigma \in G$ such that $\sigma(\mathfrak{P} \cap K_\alpha) = \mathfrak{Q} \cap K_\alpha$. Let \mathcal{E}_α be the set of $\sigma \in G$ which have the above property. Let $\sigma \in G - \mathcal{E}_\alpha$, then for all $\tau \in G$ leaving invariant the elements of K_α , we have

$$(\sigma\tau)(\mathfrak{P} \cap K_\alpha) = \sigma(\mathfrak{P} \cap K_\alpha) \neq \mathfrak{Q} \cap K_\alpha$$

and hence $\sigma\tau \in G - \mathcal{E}_\alpha$. It follows that \mathcal{E}_α is closed in the topological Galois group G and clearly the family \mathcal{E}_α is directed. As G is compact and the \mathcal{E}_α are non-empty, the intersection \mathcal{E} of the family (\mathcal{E}_α) is non-empty and $\sigma(\mathfrak{P}) = \mathfrak{Q}$ for all $\sigma \in \mathcal{E}$, whence (a).

To show (b), note that $\kappa(\mathfrak{P})$ is the union of the right directed family $(\kappa_\alpha)_{\alpha \in I}$ where κ_α is the residue field of $\mathfrak{P} \cap K_\alpha$. As each κ_α is a normal extension of $\kappa(\mathfrak{p})$ by ??, so is $\kappa(\mathfrak{P})$. On the other hand, let θ be a $\kappa(\mathfrak{p})$ -automorphism of $\kappa(\mathfrak{P})$. By virtue of ?? applied to $B \cap K_\alpha$, there exists for all α a non-empty set \mathcal{F}_α of elements $\sigma \in G$ such that $\sigma(\mathfrak{P} \cap K_\alpha) = \mathfrak{P} \cap K_\alpha$ and $\theta(\bar{x}) = \overline{\sigma(x)}$ for all $x \in B \cap K_\alpha$. As above it is seen that \mathcal{F}_α is closed in G and, as (\mathcal{F}_α) is a left directed set, its intersection \mathcal{F} is non-empty. Clearly for $\sigma \in G$ we have $\sigma \in G^Z$ and $\bar{\sigma} = \theta$, which completes the proof of (b) in this case.

Finally we deal with the general case. The field of invariants K_1 of G is a purely inseparable extension of K . There therefore exists a single prime ideal \mathfrak{p}_1 of $A_1 = B \cap K_1$ lying over \mathfrak{p} (??). If \mathfrak{P} and \mathfrak{Q} are two prime ideals of B lying over \mathfrak{p} , then they lie over \mathfrak{p}_1 . As L is a Galois extension of K_1 and $B \cap K_1$ is integrally closed by ??, it follows from the Galois case that there exists a

$\sigma \in G$ such that $\sigma(\mathfrak{P}) = \mathfrak{Q}$, whence (a). On the other hand, clearly the residue field of \mathfrak{p}_1 is a purely inseparable extension of $\kappa(\mathfrak{p})$. As $\kappa(\mathfrak{P})$ is a normal extension of κ_1 by the Galois case, we then see $\kappa(\mathfrak{P})$ is a normal extension of $\kappa(\mathfrak{p})$ (since every $\kappa(\mathfrak{p})$ -isomorphism of $\kappa(\mathfrak{P})$ is a κ_1 -isomorphism). This last remark shows also, taking account of the Galois case, that every $\kappa(\mathfrak{p})$ -automorphism of $\kappa(\mathfrak{P})$ is of the form $\bar{\sigma}$ where $\sigma \in G^Z(\mathfrak{P})$, which completes the proof of (b). \square

Corollary 12.2.15. *Let A be an integrally closed domain, K its field of fractions, L a normal extension of K and B the integral closure of A in L . Let θ_1 and θ_2 be two homomorphisms of B to a field Ω with the same restriction on A . Then there exists a K -automorphism σ of L such that $\theta_1 = \theta_2 \circ \sigma$.*

Proof. This can be proved as ??.

Proposition 12.2.16. *Let A be an integrally closed domain, K its field of fractions, L a finite algebraic extension of K and B a subring of L containing A and integral over A .*

- (a) *For every prime ideal \mathfrak{p} of A the set of prime ideals of B lying over \mathfrak{p} is at most $[L : K]_s$.*
- (b) *If \mathfrak{P} is a prime ideal of B lying over \mathfrak{p} , every element of $\kappa(\mathfrak{P})$ is of degree smaller than $[L : K]$ over $\kappa(\mathfrak{p})$.*

Proof. We may first restrict our attention to the case where L is a separable extension of K , for in general L is a purely inseparable extension of the separable closure K_0 of K contained in L , and $[L : K]_s = [K_0 : K]$ by definition and, if $A_0 = B \cap K_0$, the prime ideals of A_0 and B are in one-to-one correspondence ??).

Suppose therefore that L is separable over K and let N be the Galois extension of K generated by L in an algebraic closure of K , G its Galois group, \bar{A} the integral closure of A in N and \mathcal{P} a prime ideal of \bar{A} lying over \mathfrak{p} . Let H be the Galois group of N over L and G^Z the decomposition group of \mathcal{P} . The prime ideals of \bar{A} lying over \mathfrak{p} are the $\sigma(\mathcal{P})$ where $\sigma \in G$?? and the relation $\sigma(\mathcal{P}) = \sigma'(\mathcal{P})$ means that $\sigma' = \sigma\tau$ where $\tau \in G^Z$. On the other hand, in order that $\sigma(\mathcal{P}) \cap L = \sigma'(\mathcal{P}) \cap L$, it is necessary and sufficient that $\sigma(\mathcal{P}) = \theta\sigma(\mathcal{P})$, where $\theta \in H$??, whence finally $\sigma' = \theta\sigma\tau$ where $\theta \in H$ and $\tau \in G^Z$. The number of prime ideals of B lying over \mathfrak{p} is therefore equal to the cardinality of the double coset $H \backslash G / G^Z$, and this number is smaller than $[G : H] = [L : K]$ by Galois theory.

The coefficients of the minimal polynomial (over K) of any element $x \in B$ belong to A ??). Applying the canonical homomorphism $B \rightarrow B/\mathfrak{P}$ to the coefficients of this polynomial, an equation of integral dependence with coefficients in A/\mathfrak{p} and of degree $\leq [L : K]$ is obtained for the class mod \mathfrak{P} of x , whence the conclusion. \square

Proposition 12.2.17. *Let A be an integrally closed domain, K its field of fractions, L a Galois extension of K , B the integral closure of A in L , \mathfrak{P} a prime ideal of B , $\mathfrak{p} = A \cap \mathfrak{P}$. For a subfield E of L containing K , set $A(E) = B \cap E$ and $\mathfrak{p}(E) = \mathfrak{P} \cap E$.*

- (a) *If E is contained in K^Z , then \mathfrak{p} and $\mathfrak{p}(E)$ have the same residue field and the maximal ideal of $(A(E))_{\mathfrak{p}(E)}$ is generated by \mathfrak{p} .*
- (b) *Conversely, if these two conditions in (a) hold and $\bigcap_{n=0}^{\infty} \mathfrak{p}^n B_{\mathfrak{P}} = 0$, then E is contained in K^Z .*

Proof. Let G be the Galois group of L over K and H the Galois group of L over E . Then to say that $E \subseteq K^Z$ means that $G^Z \subseteq H$ and the assertions are therefore special cases of ?? when $[L : K]$ is finite. In the general case the argument is as in the proof of ??.

Proposition 12.2.18. *Let A be an integrally closed domain, K its field of fractions, L a Galois extension of K , B the integral closure of A in L , \mathfrak{P} a prime ideal of B , $\mathfrak{p} = A \cap \mathfrak{P}$.*

- (a) *The residue field $\kappa(\mathfrak{P}^T)$ is the separable closure of $\kappa(\mathfrak{p})$ in $\kappa(\mathfrak{P})$.*
- (b) *The extension K^T/K is Galois and $\text{Gal}(K^T/K)$ is isomorphic to $\text{Gal}(\kappa(\mathfrak{P}^T)/\kappa(\mathfrak{p}))$.*
- (c) *There exists an inclusion-preserving bijective correspondence between the set of all intermediate extensions $K \subseteq E \subseteq K^T$ and the set of all separable extensions $\kappa(\mathfrak{p}) \subseteq \kappa \subseteq \kappa(\mathfrak{P}^T)$. The extension $\kappa/\kappa(\mathfrak{p})$ is a Galois extension if and only if E/K is Galois. Moreover, in this case κ is the residue field of $\mathfrak{P} \cap E$.*

Proof. The ring A is the ring of invariants in B of the Galois group of L over K . If L is of finite degree over K , the corollary then follows from ?? and ?. Consider now the general case, L therefore being the union of a right directed set (K_α) of finite Galois extensions of K .

Suppose now that $x \in A^T$. There exists α such that $x \in A^T(\mathfrak{P} \cap K_\alpha)$ and ?? shows that the class \bar{x} of x mod $\mathfrak{P}^T \cap K_\alpha$ is algebraic and separable over $\kappa(\mathfrak{p})$. A fortiori the class mod \mathfrak{P}^T of x is separable over $\kappa(\mathfrak{p})$. To complete the proof of the corollary, it is sufficient to show that $\kappa(\mathfrak{P})$ is a purely inseparable extension of $\kappa(\mathfrak{P}^T)$. Now, $\kappa(\mathfrak{P})$ is the union of the right directed family of residue fields κ_α of $\mathfrak{P} \cap K_\alpha$. It follows therefore from ?? that, if an element of $\kappa(\mathfrak{P})$ belongs to κ_α , it is purely inseparable over the residue field of $\mathfrak{P}^T \cap K_\alpha$, and a fortiori over $\kappa(\mathfrak{P}^T)$. □

Let A be an integrally closed domain, K its field of fractions, L a normal extension of K and B the integral closure of A in L . The field of invariants $K^Z(\mathfrak{P})$ (resp. $K^T(\mathfrak{P})$) of the group $G^Z(\mathfrak{P})$ (resp. $G^T(\mathfrak{P})$) in the field L is called the **decomposition field** (resp. inertia field) of \mathfrak{P} with respect to K . We also write K^Z (resp. K^T) in place of $K^Z(\mathfrak{P})$ (resp. $K^T(\mathfrak{P})$). It follows from ?? that K^Z (resp. K^T) is the field of fractions of the ring A^Z (resp. A^T), and by ?? A^Z (resp. A^T) is the integral closure of A in K^Z (resp. K^T).

Proposition 12.2.19. *Let A be an integrally closed domain, K its field of fractions, L a Galois extension of K , B the integral closure of A in L , \mathfrak{P} a prime ideal of B . For a subfield E of L containing K , set $A(E) = B \cap E$ and $\mathfrak{p}(E) = \mathfrak{P} \cap E$. Then the decomposition field (resp. inertia field) of \mathfrak{P} with respect to E is EK^Z (resp. EK^T). If further E is a Galois extension of K , the decomposition field of $\mathfrak{p}(E)$ with respect to K is $E \cap K^Z$.*

Proof. If H is the Galois group of L over E , clearly the decomposition group (resp. inertia group) of \mathfrak{P} with respect to E is $H \cap G^Z$ (resp. $H \cap G^T$) and the first assertion follows from Galois theory if L is a finite Galois extension of K . In the general case it follows from the fact that A^Z (resp. A^T) is the union of the $A^Z(\mathfrak{P} \cap K_\alpha)$ (resp. $A^T(\mathfrak{P} \cap K_\alpha)$): every element $x \in L$ belongs to some K_α and if it is invariant under $G^Z(\mathfrak{P}) \cap H$ (resp. $G^T(\mathfrak{P}) \cap H$) it is also invariant under $G^Z(\mathfrak{P} \cap K_\beta) \cap H$ (resp. $G^T(\mathfrak{P} \cap K_\beta) \cap H$) for some suitable β ; hence it belongs by the beginning of the argument to $EK^Z(\mathfrak{P} \cap K_\alpha) \subseteq EK^Z$ (resp. $EK^T(\mathfrak{P} \cap K_\alpha) \subseteq EK^T$).

Suppose now that E is a Galois extension of K . The restriction to E of every $\sigma \in G^Z$ then preserves $\mathfrak{p}(E)$ and hence belongs to the decomposition group of $\mathfrak{p}(E)$ with respect to K . Conversely, let τ be an automorphism of E belonging to this group and let σ be an extension of τ

to a K -automorphism of L . We write $\mathfrak{Q} = \sigma(\mathfrak{P})$. As \mathfrak{P} and \mathfrak{Q} both lie over $\mathfrak{p}(E)$, there exists an automorphism $\rho \in H$ such that $\mathfrak{Q} = \rho(\mathfrak{P})$, whence $\rho^{-1}\sigma \in G^Z$ and τ is the restriction of $\rho^{-1}\sigma$ to E . In other words, the decomposition group of $\mathfrak{p}(E)$ with respect to K is identical with the group of restrictions to E of the automorphisms $\sigma \in G^Z$ which proves the second assertion. \square

12.3 Finite generated algebras over a field

In this part, k will be a field. Recall that, if A is a ring, by a finite generated A -algebra we mean an A -algebra B that is isomorphic to a quotient of some polynomial algebra $A[X_1, \dots, X_n]$.

Theorem 12.3.1 (Normalization Lemma). *Let A be a finitely generated k -algebra and let $\mathfrak{a}_1 \subseteq \mathfrak{a}_2 \subset \dots \subset \mathfrak{a}_p$ be an increasing finite sequence of proper ideals of A . Then there exists a sequence $(x_i)_{1 \leq i \leq n}$ of elements of A which are algebraically independent over k and such that:*

- (a) *A is integral over the ring $B = k[x_1, \dots, x_n]$.*
- (b) *There exists an increasing sequence $(h_j)_{1 \leq j \leq n}$ of integers such that for all j the ideal $\mathfrak{a}_j \cap B$ of B is generated x_1, \dots, x_{h_j} .*

Proof. It is sufficient to prove the theorem when A is a polynomial algebra $k[Y_1, \dots, Y_m]$. In fact, assume the theorem for this case. In the general case, A is isomorphic to a quotient of such an algebra \tilde{A} by an ideal $\tilde{\mathfrak{a}}_0$. Let $\tilde{\mathfrak{a}}_j$ denote the inverse image of \mathfrak{a}_j in \tilde{A} and let $(\tilde{x}_i)_{1 \leq i \leq r}$ be elements of \tilde{A} satisfying the conditions of the statement for the ring \tilde{A} and the increasing sequence of ideals of $\tilde{\mathfrak{a}}_0 \subset \dots \subset \tilde{\mathfrak{a}}_p$. Then the images x_i of the \tilde{x}_i in A for $i > h_0$ satisfy the desired conditions. In fact, condition (b) is obvious and condition (a) follows from ???. Finally, if the x_i for $i > h_0$ were not algebraically independent over k , there would be a non-zero polynomial $Q \in k[X_{h_0+1}, \dots, X_r]$ such that $Q(x_{h_0+1}, \dots, x_r) \in \tilde{\mathfrak{a}}_0 \cap \tilde{B}$, where $\tilde{B} = k[\tilde{x}_1, \dots, \tilde{x}_r]$. But by hypothesis, the ideal $\mathfrak{a}_0 \cap \tilde{B}$ is generated by x_1, \dots, x_{h_0} , a contradiction which proves our assertion.

We shall therefore suppose in the rest of the proof that $A = k[Y_1, \dots, Y_m]$ and we shall argue by induction on p . First assume $p = 1$, and we may further assume that \mathfrak{a}_i is a principal ideal generated by an element $x_1 \notin k$. Then $x_1 = P(Y_1, \dots, Y_m)$, where P is a non-constant polynomial. We shall see that, for a suitable choice of integers $r_i > 0$, the ring A is integral over $B = k[x_1, x_2, \dots, x_m]$, where

$$x_i = Y_i - Y_1^{r_1} \quad \text{for } 2 \leq i \leq m. \quad (12.3.1)$$

For this it is sufficient to choose the r_i such that Y_1 is integral over B (??). Now, there is the relation

$$P(Y_1, x_2 + Y_1^{r_2}, \dots, x_m + Y_1^{r_m}) - x_1 = 0 \quad (12.3.2)$$

Write $P = \sum_{\alpha} a_{\alpha} Y^{\alpha}$, where $\alpha \in \mathbb{N}^m$ and $a_{\alpha} \in k$ are nonzero elements of k . Then (??) becomes

$$\sum_{\alpha} a_{\alpha} Y_1^{\alpha_1} (x_2 + Y_1^{r_2})^{\alpha_2} \cdots (x_m + Y_1^{r_m})^{\alpha_m} - x_1 = 0. \quad (12.3.3)$$

Let us write $f(\alpha) = \alpha_1 + r_2\alpha_2 + \cdots + r_m\alpha_m$ and suppose that the r_i are chosen so that the $f(\alpha)$ are distinct (it suffices for example to take $r_i = h^i$, where h is an integer strictly greater than

all the α_i . Then there will be a unique system $\alpha = (\alpha_1, \dots, \alpha_m)$ such that $f(\alpha)$ is maximal and relation (??) may be written

$$a_\alpha Y_1^{f(\alpha)} + \sum_{j < f(\alpha)} Q_j(x_1, \dots, x_m) Y_1^j = 0. \quad (12.3.4)$$

where the Q_i are polynomials in $k[Y_1, \dots, Y_m]$. As $a_\alpha \neq 0$ is invertible in k , (??) is certainly an equation of integral dependence with coefficients in B , whence our assertion.

The field of fractions $k(Y_1, \dots, Y_m)$ of A is therefore algebraic over the field of fractions $k(x_1, \dots, x_m)$ of B , which proves that the x_i are algebraically independent. Moreover, every element $z \in \mathfrak{a}_1 \cap B$ may be written $z = x_1 \tilde{z}$, where $\tilde{z} \in A \cap k(x_1, \dots, x_m)$. But $A \cap k(x_1, \dots, x_m) = k[x_1, \dots, x_m] = B$ since B is integrally closed by ??, whence $\mathfrak{a}_1 \cap B = Bx_1$.

We now do not assume that \mathfrak{a}_1 is principal and prove the theorem for $p = 1$ by induction on m . We may obviously suppose that $\mathfrak{a}_1 \neq 0$. Let x_1 be a non-zero element of \mathfrak{a}_1 . By the argument above there exist t_2, \dots, t_m such that x_1, t_2, \dots, t_m are algebraically independent over k , A is integral over $C = k[x_1, t_2, \dots, t_m]$ and $x_1 A \cap C = x_1 C$. By the induction hypothesis there exist algebraically independent elements x_2, \dots, x_m of $k[t_2, \dots, t_m]$ and an integer h such that $k[t_2, \dots, t_m]$ is integral over $\tilde{B} = k[x_2, \dots, x_m]$, and the ideal $\mathfrak{a}_1 \cap \tilde{B}$ is generated by x_2, \dots, x_h . Then C is integral over $B = k[x_1, x_2, \dots, x_m]$ and hence so is A , therefore x_1, \dots, x_m are algebraically independent over k . Finally, as $x_1 \in \mathfrak{a}_1$ and $B = \tilde{B}[x_1]$, $\mathfrak{a}_1 \cap B = Bx_1 + (\mathfrak{a}_1 \cap \tilde{B})$ and, as $\mathfrak{a}_1 \cap \tilde{B}$ is generated (in \tilde{B}) by x_2, \dots, x_h , we see $\mathfrak{a}_1 \cap B$ is generated (in B) by x_1, x_2, \dots, x_h .

Now assume we have proved the theorem for $p - 1$ ideals and let t_1, \dots, t_m be elements of A satisfying the conditions of the theorem for the increasing sequence of ideals $\mathfrak{a}_1 \subset \dots \subset \mathfrak{a}_{p-1}$ and let us write $r = h_{p-1}$. Apply the preceding proof for \mathfrak{a}_p , there exist algebraically independent elements x_{r+1}, \dots, x_m of $C = k[t_{r+1}, \dots, t_m]$ and an integer s such that C is integral over $\tilde{B} = k[x_{r+1}, \dots, x_m]$ and the ideal $\mathfrak{a}_p \cap \tilde{B}$ is generated by x_{r+1}, \dots, x_s . Now write $x_i = t_i$ for $1 \leq i \leq r$ and $h_p = s$, then A is integral over $C[t_1, \dots, t_r]$ and hence also over $B = k[x_1, \dots, x_m] = \tilde{B}[t_1, \dots, t_r]$ since C is integral over \tilde{B} (??). In particular, we see the x_i are algebraically independent over k . On the other hand, for $j \leq p - 1$, the ideal

$$\mathfrak{a}_j \cap k[x_1, \dots, x_r, t_{r+1}, \dots, t_m]$$

is by hypothesis the set of polynomials in $x_1, \dots, x_r, t_{r+1}, \dots, t_m$ all of whose monomials contain one of the elements x_1, \dots, x_{h_j} . As x_{r+1}, \dots, x_m are polynomials in t_{r+1}, \dots, t_m with coefficients in k , it is seen immediately that a polynomial in x_1, \dots, x_m (with coefficients in k) can belong to \mathfrak{a}_j only if all its monomials contain one of the elements x_1, \dots, x_{h_j} . Finally, as x_1, \dots, x_r belong to \mathfrak{a}_{p-1} and hence also to \mathfrak{a}_p , $\mathfrak{a}_p \cap \tilde{B}[x_1, \dots, x_r]$ consists of the polynomials in x_1, \dots, x_r with coefficients in \tilde{B} whose constant term belongs to $\mathfrak{a}_p \cap \tilde{B}$. This ideal is therefore generated by $x_1, \dots, x_r, x_{r+1}, \dots, x_s$. Therefore we see $(x_i)_{1 \leq i \leq m}$ satisfies the required conditions, which proves the theorem. \square

Corollary 12.3.2. *Let A be an integral domain and B a finitely generated A -algebra containing A as a subring. Then there exist a nonzero element s of A and a subalgebra C of B isomorphic to a polynomial algebra $A[Y_1, \dots, Y_n]$ such that B_s is integral over C_s .*

Proof. We write $S = A - \{0\}$ and let $k = S^{-1}A$ the field of fractions of A . Clearly $S^{-1}B$ is a finitely generated k -algebra and, as it contains k by hypothesis, it is not reduced to 0. By

?? applied to $a_1 = (0)$, there exists therefore a finite sequence $(x_i)_{1 \leq i \leq n}$ of elements of $S^{-1}B$ which are algebraically independent over k and such that $S^{-1}B$ is integral over $k[x_1, \dots, x_n]$. Let $(z_j)_{1 \leq j \leq m}$ be a system of generators of the A -algebra B . Then in $S^{-1}B$ each of the $z_j/1$ satisfies an equation of integral dependence

$$(z_j/1)^{q_j} + \sum_{k < q_j} P_{kj}(x_1, \dots, x_n)(z_j/1)^k = 0.$$

where the P_{kj} are polynomials in the x_i with coefficients in k . There exists an element $s \neq 0$ of A such that we may write $x_i = y_i/s$ where $y_i \in B$ and all the coefficients of the P_{kj} are of the form c/s where $c \in A$. Finally, replacing if need be, s by a product of elements of S , we may assume that in B ,

$$sz_j^{q_j} + \sum_{k < q_j} Q_{kj}z_j^k = 0. \quad (12.3.5)$$

where the Q_{kj} are polynomials in y_1, \dots, y_n with coefficients in A . If we write $\tilde{z}_j = sz_j$, it is seen from ?? by multiplying s^{q_j-1} that \tilde{z}_j is integral over $\tilde{B} = A[y_1, \dots, y_n]$. We show that the y_i are algebraically independent over A . If there is a relation of the form $\sum_{\alpha} a_{\alpha} y_1^{\alpha_1} \cdots y_n^{\alpha_n} = 0$ where $a_{\alpha} \in A$ for all α , we deduce that $\sum_{\alpha} b_{\alpha} x_1^{\alpha_1} \cdots x_n^{\alpha_n} = 0$ in $S^{-1}B$, where $b_{\alpha} = a_{\alpha}s^{|\alpha|}$. By hypothesis therefore $b_{\alpha} = 0$ for all α , whence $a_{\alpha} = 0$ for all α . Moreover, by ?? in the ring B_s each of the $\tilde{z}_j/1$ is integral over \tilde{B}_s and, as $z_j/1 = (\tilde{z}_j/1)(1/s)$ in B_s , it is seen that the $z_j/1$ are integral over \tilde{B}_s , which completes the proof. \square

Corollary 12.3.3. *Let L be a field, A a subring of L and K the field of fractions of A . If L is a finitely generated A -algebra, then $[L : K]$ is finite and there exists a nonzero element a in A such that $K = A_a$.*

Proof. It follows from ?? that there exist elements x_1, \dots, x_n of L and an element $a \neq 0$ of A such that x_1, \dots, x_n are algebraically independent over A (and therefore over K) and that L is integral over the subring $A[x_1, \dots, x_n]_a$. Then it follows from ?? that $A[x_1, \dots, x_n]_a$ is a field. But the only invertible elements of a polynomial ring $C[Y_1, \dots, Y_n]$ over an integral domain C are the invertible elements of C . Applying this remark to $C = A_a$, it is seen that necessarily $n = 0$ and that A_a is a field equal to K by definition of the latter. As L is integral over K and is a finitely generated K -algebra, the degree $[L : K]$ is finite. \square

Corollary 12.3.4. *Let A be an integral domain, B a finitely generated A -algebra and b a nonzero element of B such that $z^n b \neq 0$ for all $z \neq 0$ in A and every integer $n > 0$. Let $\rho : A \rightarrow B$ be the canonical homomorphism. Then there exists a nonzero a in A such that, for every homomorphism ϕ from A to an algebraically closed field Ω such that $\phi(a) \neq 0$, there exists a homomorphism $\tilde{\phi}$ from B to Ω such that $\tilde{\phi}(b) \neq 0$ and $\phi = \tilde{\phi} \circ \rho$.*

Proof. The hypothesis on b implies that, if i is the canonical homomorphism $x \mapsto x/1$ of B to B_b , the homomorphism $i \circ \rho$ of A to B_b is injective. By ?? there therefore exist an element $a \neq 0$ of A and a subring \tilde{B} of B_b such that $(B_b)_a$ is integral over \tilde{B}_a and \tilde{B} is isomorphic to a polynomial algebra $A[Y_1, \dots, Y_n]$. Let ϕ be a homomorphism from A to an algebraically closed field Ω such that $\phi(a) \neq 0$. There exists a homomorphism from $A[Y_1, \dots, Y_n]$ to Ω extending ϕ and hence there exists a homomorphism from \tilde{B} to Ω extending ϕ , which we still denote by ϕ . As $\phi(a) \neq 0$ in Ω , there exists a homomorphism ψ from \tilde{B}_a to Ω such that

$$\psi(x/a^n) = \phi(x)(\phi(a))^{-n}$$

for all $x \in \tilde{B}$ and all $n > 0$. Finally, as $(B_b)_a$ is integral over \tilde{B}_a , there exists a homomorphism $\tilde{\psi}$ from $(B_b)_a$ to Ω extending ψ (??). If $j : x \mapsto x/1$ is the canonical homomorphism from B to $(B_b)_a$, then $\tilde{\phi} = \tilde{\psi} \circ j$ solves the problem for $j(b)$ is invertible in $(B_b)_a$ and hence $\tilde{\psi}(j(b)) \neq 0$ in Ω . \square

Theorem 12.3.5. *Let A be a finitely generated integral k -algebra, K its field of fractions, L a finite algebraic extension of K , and \bar{A} the integral closure of A in L . Then \bar{A} is a finitely generated A -module and a finitely generated k -algebra.*

Proof. By ?? there exists a subalgebra C of A isomorphic to a polynomial algebra $k[X_1, \dots, X_n]$ and such that A is integral over C . Then \bar{A} is obviously the integral closure of C in L (??). We may therefore confine our attention to the case where $A = k[X_1, \dots, X_n]$.

Let N be the normal extension of L (in an algebraic closure of L) generated by L , which is a finite algebraic extension of K . It will suffice to prove that the integral closure of A in N is a finitely generated A -module, for \bar{A} is a sub- A -module of that ring and A is a Noetherian ring. We may therefore confine our attention to the case where L is a finite normal extension of K . Then we know that L is a (finite) Galois extension of a (finite) purely inseparable extension K' of K . If A' is the integral closure of A in K' , \bar{A} is the integral closure of A' in L and it will suffice to prove that A' is a finitely generated A -module and \bar{A} is a finitely generated A' -module. Now, if it has been proved that A' is a finitely generated A -module, it is a Noetherian domain, integrally closed by definition. The fact that \bar{A} is a finitely generated A' -module will follow from ??.

We see that we may confine our attention to the case where $A = k[X_1, \dots, X_n]$ and where L is a finite purely inseparable extension of $K = k(X_1, \dots, X_n)$. Then L is generated by a finite family of elements $(y_i)_{1 \leq i \leq m}$ and there exists a power q of the characteristic of k such that $y_i^q \in k(X_1, \dots, X_n)$ for all i . Let $(c_i)_{1 \leq i \leq r}$ be the coefficients of the numerators and denominators of the rational functions in X_1, \dots, X_n that equal to y_i^q . Then L is contained in the extension $L' = k'(X_1^{1/q}, \dots, X_n^{1/q})$, where $k' = k(c_1^{1/q}, \dots, c_r^{1/q})$ (we are in an algebraic closure of L) and \bar{A} is contained in the algebraic closure A' of A in L' . Now, k' is algebraic over k and hence $C' = k'[X_1, \dots, X_n]$ is integral over A . As $k'[X_1^{1/q}, \dots, X_n^{1/q}]$ is integrally closed by ??, it is seen that this ring is the integral closure of C' in L' and hence also that of A , in other words, $A' = k'[X_1^{1/q}, \dots, X_n^{1/q}]$. Now clearly A' is a finitely generated C' -module and, as k' is a finite extension of k , C' is a finitely generated A -module and hence A' is a finitely generated A -module. Since A is Noetherian and $\bar{A} \subseteq A'$, we see \bar{A} is a finitely generated A -module. \square

12.3.1 The Nullstellensatz

Proposition 12.3.6. *Let A be a finitely generated algebra over a field k and Ω the algebraic closure of k .*

- (a) *If $A \neq \{0\}$, there exists a nonzero k -homomorphism from A to Ω .*
- (b) *Let ϕ_1, ϕ_2 be two k -homomorphisms from A to Ω . For ϕ_1 and ϕ_2 to have the same kernel, it is necessary and sufficient that there exist a k -automorphism σ of Ω such that $\phi_2 = \sigma \circ \phi_1$.*
- (c) *Let \mathfrak{a} be an ideal of A . For \mathfrak{a} to be maximal, it is necessary and sufficient that it be the kernel of a k -homomorphism from A to Ω .*

- (d) For an element x of A to be such that $\phi(x) = 0$ for every k -homomorphism from A to L , it is necessary and sufficient that x be nilpotent.

Proof. Assertion (a) follows ?? applied replacing A by k , B by A , b by the unit element of B and ϕ by the canonical injection of k into Ω .

If ϕ is a k -homomorphism from A to Ω , $\phi(A)$ is a subring of Ω containing k . As Ω is an algebraic extension of k , $\phi(A)$ is a field and, if \mathfrak{a} is the kernel of ϕ , A/\mathfrak{a} is isomorphic to $\phi(A)$, therefore is a field, which proves that \mathfrak{a} is maximal. Conversely, if \mathfrak{a} is a maximal ideal of A it follows from (a) that there exists a k -homomorphism from A/\mathfrak{a} to Ω and hence a k -homomorphism of A to Ω whose kernel \mathfrak{b} contains \mathfrak{a} . But as \mathfrak{a} is maximal, $\mathfrak{b} = \mathfrak{a}$, this proves (c).

We now prove (b). If σ is a k -automorphism of Ω such that $\phi_2 = \sigma \circ \phi_1$, clearly ϕ_1 and ϕ_2 have the same kernel. Conversely, suppose that ϕ_1 and ϕ_2 have the same kernel. Then there exists a k -isomorphism σ_0 of the field $\phi_1(A)$ onto the field $\phi_2(A)$ such that $\phi_2 = \sigma_0 \circ \phi_1$. But by ??, σ_0 extends to a k -automorphism σ of Ω and hence $\phi_2 = \sigma \circ \phi_1$.

Finally, if $x \in A$ is such that $x^n = 0$, for every k -homomorphism ϕ from A to Ω , $(\phi(x))^n = \phi(x^n) = 0$ and hence $\phi(x) = 0$ since Ω is a field. Conversely, suppose that $x \in A$ is not nilpotent. Then A_x is a finitely generated A -algebra (and therefore a finitely generated k -algebra) not reduced to 0 and hence there exists a k -homomorphism ψ from A_x to Ω by (a). If $j : A \rightarrow A_x$ is the canonical homomorphism, $\phi = \psi \circ j$ a k -homomorphism from A to Ω and $\phi(x)\psi(1/x) = \psi(x/1)\psi(1/x) = \psi(1) = 1$, whence $\phi(x) \neq 0$. \square

Lemma 12.3.7. *Let A be a finitely generated algebra over a field k , $(a_i)_{1 \leq i \leq n}$ a system of generators of this algebra and \mathfrak{r} the ideal of algebraic relations between the a_i with coefficients in k . For every extension field L of k , the map $\phi \mapsto (\phi(a_i))$ is a bijection of the set of k -homomorphisms from A to L onto the set of zeros of \mathfrak{r} in L^n .*

Proof. There exists a unique k -algebra homomorphism η of $k[X_1, \dots, X_n]$ onto A such that $\eta(X_i) = a_i$ for $1 \leq i \leq n$ and by definition \mathfrak{r} is the kernel of η . The map $\phi \mapsto \phi \circ \eta$ is a bijection of the set of k -homomorphisms from A to Ω onto the set of k -homomorphisms from $k[X_1, \dots, X_n]$ to Ω which are zero on \mathfrak{r} . For every polynomial $P \in k[X_1, \dots, X_n]$ and every element $\mathbf{x} = (x_1, \dots, x_n) \in L^n$ we write $\eta_{\mathbf{x}}(P) = P(\mathbf{x})$. Then the map $\mathbf{x} \mapsto \eta_{\mathbf{x}}$ is a bijection of L^n onto the set of k -homomorphisms from $k[X_1, \dots, X_n]$ to L (such a homomorphism being determined by its values at the X_i). To say that $\eta_{\mathbf{x}}$ is zero on \mathfrak{r} means that \mathbf{x} is a zero of \mathfrak{r} in L^n , whence the lemma. \square

Proposition 12.3.8 (Hilbert's Nullstellensatz). *Let k be a field and Ω an algebraic closure of k .*

- (a) *Every proper ideal \mathfrak{r} of $k[X_1, \dots, X_n]$ admits at least one zero in Ω^n .*
- (b) *Let \mathbf{x}, \mathbf{y} be two elements of Ω^n . For the set of polynomials of $k[X_1, \dots, X_n]$ zero at \mathbf{x} to be identical with the set of polynomials of $k[X_1, \dots, X_n]$ zero at \mathbf{y} , it is necessary and sufficient that there exists a k -automorphism σ such that $\mathbf{y} = \sigma(\mathbf{x})$.*
- (c) *For an ideal \mathfrak{a} of $k[X_1, \dots, X_n]$ to be maximal, it is necessary and sufficient that there exist an \mathbf{x} in Ω^n such that \mathfrak{a} is the set of polynomials of $k[X_1, \dots, X_n]$ zero at \mathbf{x} .*
- (d) *For a polynomial P of $k[X_1, \dots, X_n]$ to be zero on the set of zeros in Ω^n of an ideal \mathfrak{r} of $k[X_1, \dots, X_n]$, it is necessary and sufficient that there exist an integer $n > 0$ such that $P^n \in \mathfrak{r}$.*

Proof. Apply ?? to the algebra $A = k[X_1, \dots, X_n]/\mathfrak{r}$ and use ??, we obtain part (a) and (d). Also, part (b) and (c) follows by the same argument applied to $A = k[X_1, \dots, X_n]$. \square

12.3.2 Jacobson rings

Proposition 12.3.9. *Let A be a ring. Then the following conditions are equivalent.*

- (i) *Every prime ideal of A is the intersection of a family of maximal ideals.*
- (ii) *For every ideal \mathfrak{a} of A , the Jacobson radical of A/\mathfrak{a} be equal to its nilradical.*
- (iii) *For every ideal \mathfrak{a} of A , the radical of \mathfrak{a} is the intersection of the maximal ideals of A containing \mathfrak{a} .*
- (iv) *Every prime ideal in A which is not maximal is equal to the intersection of the prime ideals which contain it strictly.*

Proof. The Jacobson radical (resp. nilradical) of A/\mathfrak{a} is the intersection of the maximal (resp. prime) ideals of A/\mathfrak{a} . Thus (ii) means for every ideal \mathfrak{a} of A the intersection of the prime ideals containing \mathfrak{a} is equal to the intersection of the maximal ideals containing \mathfrak{a} . This condition obviously holds for every ideal \mathfrak{a} of A if (i) holds, and it implies (i) by taking \mathfrak{a} to be prime ideals of A . This proves the equivalence of (i) and (ii). Also, it is clear that (iii) is equivalent to (ii)

Now assume (i) and let \mathfrak{p} be a prime ideal of A that is not maximal. Then by (i), since \mathfrak{p} is non maximal, we have

$$\mathfrak{p} = \bigcap_{\mathfrak{m} \supseteq \mathfrak{p}, \mathfrak{m} \in \text{Max}(A)} \mathfrak{m} \supseteq \bigcap_{\mathfrak{q} \supsetneq \mathfrak{p}, \mathfrak{q} \in \text{Spec}(A)} \mathfrak{q} \supseteq \mathfrak{p}$$

whence (iv) holds. Conversely, suppose there is a prime ideal which is not an intersection of maximal ideals. Passing to the quotient ring, we may then assume that A is an integral domain whose Jacobson radical \mathfrak{r} is not zero. Let a be a non-zero element of \mathfrak{r} . Then $A_a \neq 0$ since a is not nilpotent, hence has a maximal ideal, whose contraction in A is a prime ideal \mathfrak{p} such that $a \notin \mathfrak{p}$. Since $a \in \mathfrak{r}$ we see \mathfrak{p} is not maximal. Moreover, by our choice of \mathfrak{p} , every prime ideal of A that strictly contains \mathfrak{p} has nonempty with the multiplicative subset $S = \{a^n : n > 0\}$ and therefore contains a . Hence a is contained in the intersection of prime ideals containing \mathfrak{p} strictly, which therefore is not equal to \mathfrak{p} . This shows the equivalence of (i) and (iv) and completes the proof. \square

A ring A is called a **Jacobson ring** if it satisfies the equivalent conditions ??.

Example 12.3.10 (Examples of Jacobson rings).

- (a) Every field is a Jacobson ring.
- (b) The ring \mathbb{Z} is a Jacobson ring, the unique prime ideal which is not maximal (0) being the intersection of the maximal ideals (p) of \mathbb{Z} , where p runs through the set of prime numbers.
- (c) Let A be a Jacobson ring and let \mathfrak{a} be an ideal of A . Then A/\mathfrak{a} is a Jacobson ring, for the ideals of A/\mathfrak{a} are of the form $\mathfrak{b}/\mathfrak{a}$, where \mathfrak{b} is an ideal of A containing \mathfrak{a} and $\mathfrak{b}/\mathfrak{a}$ is prime (resp. maximal) if and only if \mathfrak{b} is.

Proposition 12.3.11. *Let A be a PID and $(p_i)_{i \in I}$ a representative system of irreducible elements of A . For A to be a Jacobson ring, it is necessary and sufficient that I be infinite.*

Proof. The maximal ideals of A are the (p_i) . If I is finite, their intersection is the ideal (x) where $x = \prod_i p_i$ and hence different from (0) . On the other hand, if I is infinite, the intersection of the (p_i) is (0) , since every nonzero element of A can be divisible by only a finite number of irreducible elements. The proposition then follows from the fact that (0) is the only prime ideal which is not maximal in A . \square

Proposition 12.3.12. *Let A be a ring and B an A -algebra integral over A . If A is a Jacobson ring, so is B .*

Proof. Replacing A by its canonical image in B , we may assume that $A \subseteq B$. Let \mathfrak{P} be a prime ideal of B and let $\mathfrak{p} = A \cap \mathfrak{P}$. There exists by hypothesis a family $(\mathfrak{m}_i)_{i \in I}$ of maximal ideals of A whose intersection is equal to \mathfrak{p} . For all $i \in I$ there exists a maximal ideal \mathfrak{M}_i of B lying over \mathfrak{m}_i and containing \mathfrak{P} by the going up theorem. If we write $\mathfrak{Q} = \bigcap_i \mathfrak{M}_i$, then $\mathfrak{Q} \cap A = \bigcap_i \mathfrak{m}_i = \mathfrak{p}$ and $\mathfrak{Q} \supseteq \mathfrak{P}$, whence $\mathfrak{Q} = \mathfrak{P}$. \square

Proposition 12.3.13. *Let A be a Jacobson ring, B a finitely generated A -algebra and $\rho : A \rightarrow B$ the canonical homomorphism. Then B is a Jacobson ring and for every maximal ideal \mathfrak{M} of B , $\mathfrak{m} = \mathfrak{M}^c$ is a maximal ideal of A and B/\mathfrak{M} is a finite algebraic extension of A/\mathfrak{m} .*

Proof. Let \mathfrak{P} be a prime ideal of B and $\mathfrak{p} = \mathfrak{P}^c$. Let v be a nonzero element of B/\mathfrak{P} . As B/\mathfrak{P} is a finitely generated integral (A/\mathfrak{p}) -algebra and the canonical homomorphism $\eta : A/\mathfrak{p} \rightarrow B/\mathfrak{P}$ is injective, there exists a nonzero element u of A/\mathfrak{p} such that, for every homomorphism ϕ from A/\mathfrak{p} to an algebraically closed field Ω such that $\phi(u) \neq 0$, there exists a homomorphism ψ from B/\mathfrak{P} to Ω such that $\psi(v) \neq 0$ and for which $\phi = \psi \circ \eta$ (??). Since A is a Jacobson ring, there exists a maximal ideal \mathfrak{m} of A containing \mathfrak{p} and such that $u \notin \mathfrak{m}/\mathfrak{p}$. We take Ω to be an algebraic closure of A/\mathfrak{m} and ϕ to be the canonical homomorphism $A/\mathfrak{p} \rightarrow \Omega$. Let $\psi : B/\mathfrak{P} \rightarrow \Omega$ be a homomorphism such that $\phi = \psi \circ \eta$ and $\psi(v) \neq 0$. Then

$$A/\mathfrak{m} \subseteq \phi(B/\mathfrak{P}) \subseteq \Omega$$

hence $\psi(B/\mathfrak{P})$ is a subfield of Ω and the kernel of ψ is therefore a maximal ideal of B/\mathfrak{P} not containing v . Thus it is seen that the intersection of the maximal ideals of B/\mathfrak{P} is reduced to 0 , which proves that B is a Jacobson ring. Moreover, if \mathfrak{P} is maximal, ψ is necessarily injective and hence $\mathfrak{p} = \mathfrak{m}$ is maximal. Finally B/\mathfrak{P} is then a finitely generated algebra over the field A/\mathfrak{m} and hence is a finite extension of A/\mathfrak{m} . \square

Corollary 12.3.14. *Let A be a Jacobson ring. If there exists a finitely generated A -algebra B containing A and which is a field, then A is a field and B is an algebraic extension of A .*

Proof. It suffices to apply ?? with $\mathfrak{M} = (0)$. \square

Corollary 12.3.15. *Every finitely generated algebra A over \mathbb{Z} is a Jacobson ring. For a prime ideal \mathfrak{p} of A to be maximal, it is necessary and sufficient that the ring A/\mathfrak{p} be finite.*

Proof. If the integral domain A/\mathfrak{p} is finite, it is a field, as, for every $u \neq 0$ in A/\mathfrak{p} , the map $v \mapsto uv$ of A/\mathfrak{p} to itself is injective and hence bijective since A/\mathfrak{p} is finite. Conversely, for every

maximal ideal \mathfrak{m} of A , the inverse image of \mathfrak{m} in \mathbb{Z} is a maximal ideal (p) and A/\mathfrak{m} is finite over the prime field $\mathbb{Z}/(p) = \mathbb{F}_p$ by ??.

Remark 12.3.1. Let A be a Jacobson ring. Then the ring $A[\![X]\!]$ is not Jacobson: in fact, let \mathfrak{p} be a prime ideal of A , then $\mathfrak{p}A[\![X]\!]$ is not an intersection of maximal ideals of $A[\![X]\!]$ since any such ideal contains $X \cdot A[\![X]\!]$.

Theorem 12.3.16. *Let A be a ring. Then the following are equivalent.*

- (i) *A is a Jacobson ring.*
- (ii) *Every finitely generated A -algebra B that is a field is finite over A .*

Proof. First assume (ii). Let \mathfrak{p} be a prime ideal of A which is not maximal, and let $B = A/\mathfrak{p}$. Let v be a non-zero element of B . Then $B_v = B[v^{-1}]$ is a finitely generated A -algebra. If it is a field then by condition (ii) it is finite over A , hence integral over B and therefore B is a field by ???. Since \mathfrak{p} is not maximal, we then see B_v is not a field and therefore has a non-zero maximal ideal, whose contraction in B is a non-zero prime ideal \mathfrak{q} such that $v \notin \mathfrak{q}$. By ??(iv), this shows A is a Jacobson ring.

Conversely, let A be a Jacobson ring and B be a finitely generated A -algebra that is a field. Since a quotient ring of a Jacobson ring is Jacobson, we can assume B contains A as a subring. Let $s \in A$ be such that for every homomorphism ϕ from A to an algebraically closed field Ω such that $\phi(s) \neq 0$, there exists a homomorphism $\tilde{\phi}$ from B to Ω extending ϕ (??). Since A is then a integral domain and Jacobson, there exists a maximal ideal \mathfrak{m} of A not containing s . We take Ω to be the algebraic closure of A/\mathfrak{m} and ϕ the canonical homomorphism $A \rightarrow A/\mathfrak{m} \rightarrow \Omega$. Then ϕ extends to a homomorphism ψ of B into Ω . Since B is a field, ψ is injective, and so B is algebraic (thus finite algebraic) over A/\mathfrak{m} , whence finite over A . \square

Since a field is Jacobson, we get immediately the following statement.

Corollary 12.3.17 (Zariski's Lemma). *Let k ba a field and K a finitely generated k -algebra. If K is a field then it is a finite field extension of k .*

12.4 Exercise

Exercise 12.4.1. Let A be a Noetherian ring and $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ be all the minimal prime ideals of A . Suppose that $A_{\mathfrak{p}}$ is an integral domain for all $\mathfrak{p} \in \text{Spec}(A)$. Then

- (a) $\text{Ass}(A) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$.
- (b) $\mathfrak{n}(A) = \bigcap_{i=1}^n \mathfrak{p}_i = (0)$.
- (c) $\mathfrak{p}_i + \bigcap_{j \neq i} \mathfrak{p}_j = A$ for all i .

Deduce that $A \cong A/\mathfrak{p}_1 \times \dots \times A/\mathfrak{p}_n$.

Proof. Recall $\mathfrak{n}(A_{\mathfrak{p}}) = (\mathfrak{n}(A))_{\mathfrak{p}}$ from ??, but $(\mathfrak{n}(A))_{\mathfrak{p}} = (0)$ for all $\mathfrak{p} \in \text{Spec}(A)$, so by ?? $\mathfrak{n}(A) = \bigcap_{i=1}^n \mathfrak{p}_i = (0)$. By ??, we then have

$$\text{Ass}(A) \subseteq \bigcup_{i=1}^n \text{Ass}(A/\mathfrak{p}_i) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}.$$

Since $\text{supp}(A) = \text{Spec}(A)$ and $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ are the minimal elements of $\text{Spec}(A)$, we then get $\text{Ass}(A) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$.

We first prove that $\mathfrak{p}_i + \mathfrak{p}_j = A$ for any $i \neq j$: Assume the converse, then $\mathfrak{p}_i + \mathfrak{p}_j \subseteq \mathfrak{m}$ for some maximal ideal \mathfrak{m} . Then, in the localization $A_{\mathfrak{m}}$ there are two distinct minimal ideals $\mathfrak{p}_i A_{\mathfrak{m}}$ and $\mathfrak{p}_j A_{\mathfrak{m}}$. Thus

$$\mathfrak{n}(A_{\mathfrak{m}}) = \mathfrak{p}_i A_{\mathfrak{m}} \cap \mathfrak{p}_j A_{\mathfrak{m}}$$

Since $A_{\mathfrak{m}}$ is an integral domain, and $\mathfrak{p}_i \cap \mathfrak{p}_j \neq 0$, this is a contradiction.

Now since $\mathfrak{p}_i + \mathfrak{p}_j = A$, for $j \neq i$ we have

$$x_j + y_j = 1, \quad x_j \in \mathfrak{q}_i, y_j \in \mathfrak{q}_j$$

Now

$$\prod_{j \neq i} (1 - x_i) = \prod_{j \neq i} y_j \in \bigcap_{j \neq i} \mathfrak{q}_j$$

and expand the term $\prod_{j \neq i} (1 - x_i)$ we get $1 + x$ for $x \in \mathfrak{q}_i$. Hence $\mathfrak{q}_i + \bigcap_{j \neq i} \mathfrak{p}_j = A$.

Finally, \mathfrak{p}_i and \mathfrak{p}_j are coprime, by the Chinese remainder theorem we have the claimed isomorphism. \square

Chapter 13

Valuation rings

13.1 Valuation rings of fields

Let A be an integral domain, K its field of fractions. A is a **valuation ring** if, for each $x \in K$ and $x \neq 0$, either $x \in A$ or $x^{-1} \in A$ (or both). The case $A = K$ is the trivial valuation ring.

Proposition 13.1.1. *Let A be an integral domain, K its field of fractions. Then A is a valuation ring of K if and only if ideals of A is totally ordered by inclusion.*

Proof. Assume that A is a valuation ring of K and let I, J be two ideals such that there is $b \in J$, $b \notin I$. Now for $a \in I$, either $a/b \in A$ or $b/a \in A$. If $b/a \in A$, then $b = (b/a) \cdot a \in I$, contradiction. So $a/b \in A$, and $a = b(a/b) \in J$. Hence $I \subseteq J$. Conversely, assume that ideals of A are totally ordered. Let $a, b \in A$. Then $(a) \subseteq (b)$ or $(b) \subseteq (a)$. Hence it is clear $a/b \in A$ or $b/a \in A$. This shows A is a valuation ring. \square

Corollary 13.1.2. *If A is a valuation ring and \mathfrak{p} is a prime ideal of A , then A/\mathfrak{p} and $A_{\mathfrak{p}}$ are valuation rings of their fields of fractions.*

Proposition 13.1.3. *If A is a valuation ring of K , then*

(a) *A is a local ring and if we write \mathfrak{m} for the maximal ideal, then*

$$K - A = \{x \in K^\times : x^{-1} \in \mathfrak{m}\}$$

Thus A is determined by K and \mathfrak{m} .

(b) *A is integrally closed.*

Proof. By ??, A has a unique maximal ideal, so is a local ring. Moreover, if $x \notin A$, then $x^{-1} \in A$ and x^{-1} is not a unit in A , so $x^{-1} \in \mathfrak{m}$. conversely, if $x^{-1} \in \mathfrak{m}$, then $x \notin A$ since x^{-1} is not a unit. Thus

$$K - A = \{x \in K^\times : x^{-1} \in \mathfrak{m}\}$$

Now given K and \mathfrak{m} , the set $K - A$ is then determined, so $A = K - (K - A)$ is uniquely determined.

Let $x \in K$ be integral over A . Then we have, say,

$$x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0$$

with the $b_i \in A$. If $x \in A$ there is nothing to prove. If not, then $x^{-1} \in A$, hence $x = -(a_{n-1} + a_{n-2}x^{-1} + \cdots + a_0x^{-n}) \in A$. \square

Proposition 13.1.4. *Let A be an integral domain, then the followings are equivalent:*

- (i) A is a valuation ring.
- (ii) A is a local domain and any finitely generated ideal is principal.

Proof. Let A be a valuation ring. By ?? A is a local ring. Let $I = (x_1, \dots, x_n)$, since by ?? the set $\{(x_1), \dots, (x_n)\}$ is totally ordered. Let's assume that (x_1) is the biggest element in it, then $(x_i) \subseteq (x_1)$ for all i , that is, $x_i = a_i x_1$. This implies I is generated by x_1 .

Conversely, assume (b), and let \mathfrak{m} be the unique maximal ideal of A . Write \mathfrak{m} as the unique maximal ideal of A , so $J(A) = \mathfrak{m}$. Let $a, b \in A$, and write $(a, b) = (h)$. Then we have

$$a = uh, \quad b = vh, \quad h = xa + yb, \quad u, v, x, y \in A.$$

Then $h = xuh + yvh$ are since A is a domain we get $1 = xu + yv$. If u is not a unit, then $u \in \mathfrak{m}$ and hence $1 - xu = yv$ is a unit. This implies v is a unit, and vice versa. Hence either $a/b \in A$ or $b/a \in A$, so A is a valuation ring. \square

Proposition 13.1.5. *Let A be a valuation ring. Every finitely generated torsion-free A -module is free. Every torsion-free A -module is flat.*

Proof. Let M be a finitely generated torsion-free A -module and let x_1, \dots, x_n be generators of E which are minimal in number; we show that they are linearly independent. If $\sum_{i=1}^n a_i x_i = 0$ is a non-trivial relation between the x_i , one of the a_i 's, say a_1 , divides all the others since the set of principal ideals of A is totally ordered by inclusion; then $a_1 \neq 0$ since the relation is non-trivial. As M is torsion-free, we can divide by a_1 , which amounts to assuming that $a_1 = 1$. But then x_1 is a linear combination of x_2, \dots, x_n , contrary to the minimal character of n . Hence M is free. The second claim follows since every finitely generated ideal is principal. \square

Quite generally, we write \mathfrak{m}_A for the maximal ideal of a local ring A . If A and B are local rings with $B \subseteq A$ and $\mathfrak{m}_B \cap A = \mathfrak{m}_A$, we say that B **dominates** A and write $B \geq A$. If $B \geq A$ and $B \neq A$, we write $B > A$.

Proposition 13.1.6. *Let A and B be local rings such that $A \subseteq B$. Then the following conditions are equivalent:*

- (i) $\mathfrak{m}_A \subseteq \mathfrak{m}_B$;
- (ii) B dominates A ;
- (iii) the ideal generated by \mathfrak{m}_A in B is proper.

Proof. If $\mathfrak{m}_A \subseteq \mathfrak{m}_B$, then $\mathfrak{m}_B \cap A$ is a proper prime ideal containing the maximal ideal of A , so $\mathfrak{m}_A = \mathfrak{m}_B \cap A$ and B dominates A . If B dominates A , then the ideal $\mathfrak{m}_A B$ is contained in \mathfrak{m}_B and so is proper, this shows (ii) \Rightarrow (iii). Finally, if (iii) holds, $\mathfrak{m}_A B$ is contained in the unique maximal ideal of \mathfrak{m}_B of B , hence (i). \square

Theorem 13.1.7. *Let K be a field, $A \subseteq K$ a subring, and \mathfrak{p} a prime ideal of A . Then there exists a valuation ring B of K satisfying*

$$A \subseteq B, \quad \mathfrak{m}_B \cap A = \mathfrak{p}.$$

Moreover, valuation rings are maximal with respect to the domination relation.

Proof. Replacing A by $A_{\mathfrak{p}}$ we can assume that A is a local ring with $\mathfrak{p} = \mathfrak{m}_A$. Now write \mathcal{F} for the set of all subrings of K containing A and to which \mathfrak{p} extended to a nontrivial ideal. Now $A \in \mathcal{F}$, and if $\mathcal{L} \subseteq \mathcal{F}$ is a subset totally ordered by inclusion then the union of all the elements of \mathcal{L} is again an element of \mathcal{F} , so that, by Zorn's lemma \mathcal{F} has a maximal element B . Since $\mathfrak{p}^e \neq B$, there is a maximal ideal \mathfrak{m} of B containing \mathfrak{p}^e . Then $B \subseteq B_{\mathfrak{m}} \in \mathcal{F}$, so that $B = B_{\mathfrak{m}}$, and B is local. Also $\mathfrak{p} \subseteq \mathfrak{m} \cap A$ and \mathfrak{p} is a maximal ideal of A , so that $\mathfrak{m} \cap A = \mathfrak{p}$. Thus it only remains to prove that B is a valuation ring of K .

Let $x \in K$. If $x \notin B$ then since $B[x] \notin \mathcal{F}$ and $B[x] \supseteq A$, we must have $1 \in \mathfrak{p}B[x]$, and there exists a relation of the form

$$1 = a_0 + a_1x + \cdots + a_nx^n, \quad a_i \in \mathfrak{p}B$$

Since $a_0 \in \mathfrak{p}B \subseteq \mathfrak{m} = J(B)$, $1 - a_0$ is unit of B and we can modify this to get a relation

$$1 = b_1x + \cdots + b_nx^n, \quad b_i \in \mathfrak{m} \tag{13.1.1}$$

Among all such relations, choose one for which n is as small as possible. If we also have $x^{-1} \notin B$, then by the same argument, there is a relation

$$1 = c_1x^{-1} + \cdots + c_mx^{-m}, \quad c_i \in \mathfrak{m} \tag{13.1.2}$$

for which m is as small as possible. If $n \geq m$ we can multiply (??) by b_nx^n and subtract from (??) to obtain a relation of the form (??) but with a strictly smaller degree n , which is a contradiction; if $n < m$ then we get the same contradiction on interchanging the roles of x and x^{-1} . Thus if $x \notin B$ we must have $x^{-1} \in B$.

Finally, if $A \subseteq B$ are valuation rings of K , then by ?? we have $\mathfrak{m}_B \subseteq \mathfrak{m}_A$, and $\mathfrak{m}_A = \mathfrak{m}_B$ iff $A = B$. It follows that B dominates A iff $A = B$. \square

Theorem 13.1.8. *Let K be a field, $A \subseteq K$ a subring, and let \bar{A} be the integral closure of A in K . Then \bar{A} is the intersection of all the valuation rings of K containing A . If A is local, \bar{A} is the intersection of the valuation rings of K which dominate A .*

Proof. Every element $x \in \bar{A}$ is integral over A , so over any valuation ring $B \supseteq A$. But since B is integer closed, we must have $x \in B$, thus $\bar{A} \subseteq B$ for every valuation ring containing A . Conversely, let $x \notin \bar{A}$. Then $x \notin A[x^{-1}]$, so x^{-1} is not a unit of $A[x^{-1}]$. It follows that x^{-1} is contained in a maximal ideal \mathfrak{M} of $A[x^{-1}]$, and by ?? there exists a valuation ring B of K such that $B \supseteq A[x^{-1}]$ and $\mathfrak{m}_B \cap A[x^{-1}] = \mathfrak{M}$. Now since $x^{-1} \in \mathfrak{M} \subseteq \mathfrak{m}_B$, we have $x \notin B$. Further $\mathfrak{M} \cap A$ is a maximal ideal of A , hence, if A is local, $\mathfrak{M} \cap A = \mathfrak{m}_A$ and B dominates A . \square

Corollary 13.1.9. *For an integral domain to be integrally closed, it is necessary and sufficient that it be the intersection of a family of valuation rings of its field of fractions.*

Corollary 13.1.10. *Let K be a field, L an extension of K and A a valuation ring of K . The integral closure of A in L is the intersection of the valuation rings B of L such that $B \cap K = A$.*

Proof. If B is a valuation ring of L , then $B \cap K$ is a valuation ring of K and B dominates $B \cap K$. For B to dominate A , it is necessary and sufficient that $B \cap K$ dominate A and therefore be equal to it. \square

13.2 Ordered groups and valuations

Let Γ be a commutative group. A order \leq on Γ is said to be compatible with the group structure of Γ if for $\alpha, \beta, \gamma, \delta \in \Gamma$,

$$\alpha \geq \beta, \gamma \geq \delta \Rightarrow \alpha + \gamma \geq \beta + \delta.$$

In this case, the group Γ is called an **ordered group**, and a **totally ordered group** if its order is a total order. In this section we only consider totally ordered groups and if Γ is an ordered group, we usually make an ordered set $\Gamma \cup \{\infty\}$ by adding to Γ an element ∞ bigger than all the elements of Γ , and fix the conventions $\infty + \alpha = \infty$ for $\alpha \in \Gamma$ and $\infty + \alpha = \infty$.

Example 13.2.1 (Examples of totally ordered groups).

- (a) The additive group of real numbers \mathbb{R} , or any subgroup of this.
- (b) The direct product \mathbb{Z}^n of n copies of \mathbb{Z} , with lexicographical order.
- (c) The group \mathbb{Q} of rational integers.

Theorem 13.2.2. *For a commutative group Γ to be such that there exists on Γ a total ordering compatible with the group structure of Γ , it is necessary and sufficient that Γ be torsion free.*

Proof. If there exists such an order structure on Γ and if $\alpha > 0$, then $\alpha + \mu > 0$ for all $\mu > 0$ and in particular, by induction we see $n\alpha \geq 0$ which proves that Γ is torsion free. Conversely, if Γ is torsion-free, it is a sub- \mathbb{Z} -module of a vector \mathbb{Q} -space, which may be assumed of the form $\mathbb{Q}^{\oplus I}$. If I is given a well ordering and $\mathbb{Q}^{\oplus I}$ its usual ordering, the set $\mathbb{Q}^{\oplus I}$ with the lexicographical ordering is totally ordered. It is immediate that this ordering is compatible with the additive group structure of $\mathbb{Q}^{\oplus I}$. \square

Let K be a field and Γ a totally ordered group. A map $v : K \rightarrow \Gamma \cup \{\infty\}$ is called an **additive valuation** or just a **valuation** of K if it satisfies the conditions

- (a) $v(xy) = v(x) + v(y)$.
- (b) $v(x+y) \geq \min\{v(x), v(y)\}$.
- (c) $v(x) = \infty$ if and only if $x = 0$.

If we write K^\times for the multiplicative group of K then v defines a homomorphism $K^\times \rightarrow \Gamma$ whose image is a subgroup of Γ , called the **value group** of v . The valuation v defined by $(x) = 0$ for any $x \in K^\times$ is called the **trivial valuation**.

Proposition 13.2.3. *Let v be a valuation on a field K . For any elements $x_1, \dots, x_n \in K$, we have*

$$v\left(\sum_{i=1}^n x_i\right) \geq \min\{v(x_1), \dots, v(x_n)\}.$$

Moreover, if there exists a single index k such that $v(x_k) = \min_i v(x_i)$, then the equality holds. In particular, if $v(x) \neq v(y)$, then $v(x+y) = \min\{v(x), v(y)\}$.

Proof. The first inequality follows by induction. Now if there exists a single index k such that $v(x_k) = \min_i v(x_i)$, then, writing $y = \sum_{i \neq k} x_i$ and $z = \sum_{i=1}^n x_i$, we have $v(y) > v(x_k)$ and $v(z) \geq v(x_k)$. If $v(z) > v(x_k)$, then the relation $x_k = z - y$ implies $v(x_k) \geq \min\{v(y), v(z)\} > v(x_k)$, which is absurd. Thus $v(z) = v(x_k)$ and the claim is proved. \square

Valuations are connected to valuation rings in the following way. If v is a valuation on K , we set

$$A_v = \{x \in K \mid v(x) \geq 0\}, \quad \mathfrak{m}_v = \{x \in K \mid v(x) > 0\}, \quad \kappa_v = A_v/\mathfrak{m}_v.$$

Then by the properties of v we see that A_v is a valuation ring and \mathfrak{m}_v is a maximal ideal. We call A_v the **valuation ring** of v , \mathfrak{m}_v the **valuation ideal** of v , and κ_v the **residue field** of v . Conversely, if A is a valuation ring of K , then the set $\Gamma = \{xA : x \in K^\times\}$ is a group with operation defined by $xA \cdot yA = xyA$. Moreover, if we define an order \leq on Γ by

$$xA \leq yA \iff yA \subseteq xA \iff y/x \in A$$

then Γ becomes a totally ordered group with \leq (since A is a valuation ring), and we obtain an additive valuation of K with value group Γ by setting $v(x) = xA$ and $v(0) = \infty$. It is easy to see the valuation ring of v is A .

The additive valuation corresponding to a valuation ring A is not quite unique, so we introduce the following notation. Two valuations v and w on a field K with value groups Γ_v and Γ_w are called equivalent if and only if there exists an order-isomorphism $\varphi : \Gamma_v \rightarrow \Gamma_w$ such that $w = \varphi \circ v$.

Theorem 13.2.4. *Let K be a field. Then valuation rings of K are in one-to-one correspondence to equivalent class of valuations on K .*

Proof. One direction is clear: If two valuations are equivalent, then they determine the same valuation ring. For the converse, assume that v and w are valuations on K with value groups Γ_v and Γ_w such that $A_v = A_w$. We define the homomorphism $\varphi : \Gamma_v \rightarrow \Gamma_w$ by

$$\varphi(v(x)) = w(x) \text{ for } x \in K^\times.$$

Since $U(A_v) = U(A_w)$, it is easy to see φ is well defined and injective. Clearly φ satisfies the condition $w = \varphi \circ v$ and is surjective. Now, for $x, y \in K^\times$ we have

$$\varphi(v(x) + v(y)) = \varphi(v(xy)) = w(xy) = w(x) + w(y),$$

and

$$v(x) > 0 \iff x \in A_v = A_w \iff w(x) = \varphi(v(x)) > 0.$$

Thus φ is an order-isomorphism from Γ_v to Γ_w and v and w are equivalent. \square

In view of this proposition, we think of valuation rings and additive valuations as being two aspects of the same thing, and we may assume that all valuations are *surjective*. In this part, we use this point of view to establish some results for valuation rings.

The first non-trivial examples of valuations are the **p -adic valuation** v_p on \mathbb{Q} where p is a prime number, and similarly the p -adic valuation v_p on the rational function field $k(X)$ where p is any irreducible polynomial from $k[X]$, k being an arbitrary field. In the second case, v_p restricted to k is trivial.

There is one more interesting valuation on $k[X]$, trivial on k . This is the degree valuation v_∞ defined for non-zero polynomials $f, g \in k[X]$ by

$$v_\infty(f/g) = \deg g - \deg f.$$

One easily checks the axioms of a valuation. Moreover, f/g is a unit in the valuation ring A_{v_∞} if and only if $\deg f = \deg g$. Thus if

$$f = \sum_{i=0}^n a_i X^i$$

with $a_i \in k$ and $a_n \neq 0$, then we see that $f(X)/X^n$ is a unit that maps to a_n in the residue class field. Hence it follows that the residue class field is exactly k .

Proposition 13.2.5. *Every non-trivial valuation on \mathbb{Q} is a p -adic valuation for some rational prime p . Every non-trivial valuation on $k(X)$, trivial on k , is either the degree valuation v_∞ or a p -adic valuation for some irreducible polynomial $p \in k[X]$.*

Proof. Let K be either \mathbb{Q} or $k(X)$ and v some non-trivial valuation on K . Then the valuation ring A is different from K . In the second case v is assumed to be trivial on k . This means that $k \subseteq A$.

First consider the case $K = \mathbb{Q}$. Since $1 \in A$, we have $\mathbb{Z} \subseteq A$. As $A \neq \mathbb{Q}$, at least one prime p must lie in \mathfrak{m}_A . If q is a prime different from p , we have $ap + bq = 1$ for some $a, b \in \mathbb{Z}$, whence $q \notin \mathfrak{m}_A$. Hence all primes $q \neq p$ are units in A . Using the factorization of integers we therefore see that for $a, b \in \mathbb{Z}$, relatively prime, we have $a/b \in A$ iff $p \nmid b$, whence $A = \mathbb{Z}_{(p)}$, i.e., v is equivalent to the p -adic valuation v_p .

Let $K = k(X)$ and $\mathbb{Z} \subseteq A \neq K$. If $X \in A$, then $k[X] \subseteq O$ and we can argue as in the case of \mathbb{Q} replacing \mathbb{Z} by $k[X]$. If $X \notin A$, then $X^{-1} \in \mathfrak{m}_A$. In this case $v(X) < 0$ and $v(X^m) < v(X^n)$ whenever $0 < n < m$. Since $v(a) = 0$ for every $a \in k^\times$, we get

$$v(a_n X^n + \cdots + a_1 X + a_0) = v(a_n X^n) = nv(X)$$

in the case $a_n \neq 0$. Hence the value group of v is $v(X)\mathbb{Z}$. By sending $v(X)$ to -1 , we therefore get an order-preserving isomorphism with \mathbb{Z} , showing that v is equivalent to the degree valuation. \square

Definition 13.2.6. *A subset Δ of a totally ordered group Γ is called a **segment** if Δ is non-empty and if for any element α of Γ which belongs to Δ , all the elements β of Γ which lie between α and $-\alpha$ also belong to Δ . A subgroup of Γ is called an **isolated subgroup** if it is a segment of Γ .*

Isolated subgroups occurs naturally in the theory of totally ordered groups due to the following proposition.

Proposition 13.2.7. *A subgroup Δ of Γ is isolated if and only if it is the kernel of an order homomorphism from Γ to a totally ordered group.*

Proof. Let $\phi : \Gamma \rightarrow \Gamma'$ be an order-homomorphism between tally ordered groups and $\Delta = \ker \phi$. we show that Δ is isolated. In fact, if $\alpha \in \Delta$ and β lie between α and $-\alpha$, then since $\phi(\alpha) = \varphi(-\alpha)$, we must have $0 \leq \phi(\beta) \leq 0$, whence $\beta \in \Delta$.

Conversely, let Δ be a isolated subgroup of Γ and Γ/Δ be the quotient group. We define an order on Γ/Δ by

$$\alpha + \Delta \leq \beta + \Delta \iff \alpha \leq \beta.$$

This definition makes sense since for $\alpha > 0$ and $\alpha \notin \Delta$ we have $\alpha > \beta$ for all $\beta \in \Delta$. It is easy to see Γ/Δ becomes a tally ordered group and the quotient map $\pi : \Gamma \rightarrow \Gamma/\Delta$ is an order homomorphism with kernel Δ . \square

Note that the set of segements of a tally ordered group Γ is totally ordered by inclusion. Therefore, we defined the **rank** of Γ , denoted by $\text{rank}(\Gamma)$, to be the maximal length of ascending chains in this set (just like the definition of the Krull dimension of a ring). The **rank of a valuation** is defined to be the rank of the corresponding value group.

Using the valuation map, there is a connection between idelas of a valuation ring and isolated subgroups of its value group, which we now explain.

Theorem 13.2.8. *Let $v : K^\times \rightarrow \Gamma$ be a valuation and A be its valuation ring. For any subset $S \subseteq A$, we define*

$$\Delta_S = \{\alpha \in \Gamma : \alpha \notin v(S) \text{ and } \alpha \notin -v(S)\}.$$

- (a) *If I is an ideal of A then Δ_I is a segment in Γ , and is an isolated subgroup if and only if I is prime.*
- (b) *The map $I \mapsto \Delta_I$ is an order-reversing bijection from the set of all ideals of A onto the set of all segments of Γ .*

Proof. Let I be an ideal of A . To show Δ_I is a segment, we only need to verify that, if $\alpha \notin \Delta_I$ and $\beta > \alpha > 0$, then $\beta \notin \Delta_I$. Now by definition, if $\alpha \notin \Delta_I$ and $\alpha > 0$, then $\alpha \in v(I)$. Since $\beta > \alpha$, we have $\beta - \alpha \in v(A)$, and hence $\beta = \alpha + \beta - \alpha \in v(A)v(I) \subseteq v(I)$. Furthermore, from the definition of Δ_I , it is easy to see Δ_I is a subgroup if and only if I is prime. For (b), we construct an inverse of the map $I \mapsto \Delta_I$. Let Δ be a segment in Γ , and define

$$I_\Delta = \{a \in A : v(a) > \alpha \text{ for all } \alpha \in \Delta\}.$$

It is easy to see I_Δ is an ideal of A , and we have $\Delta_{I_\Delta} = \Delta$, $I_{\Delta_I} = I$, so the claim is proved. \square

Corollary 13.2.9. *The rank of a valuation equals the Krull dimension of its valuation ring.*

13.2.1 Valuations of rank one

As an important example, we consider the condition under which a valuation has rank 1. By ??, we know this happens if and only if $\dim A_v = 1$. Here we provides another characterization.

Proposition 13.2.10. *Let v be a nontrivial valuation, Γ be the value group of v , and A the valuation ring. Then the following are equivalent.*

- (i) $\text{rank}(v) = 1$, or equivalently $\dim A = 1$.
- (ii) Γ is order isomorphic to a subgroup of \mathbb{R} .
- (iii) Γ is Archimedean, that is, for any $\alpha, \beta \in \Gamma$ with $\alpha > 0$, there exist $n \in \mathbb{N}$ such that $nx > y$.

Proof. We will show that (iii) \Rightarrow (ii) \Rightarrow (i) \Rightarrow (iii). First, assume that Γ is Archimedean, and we may suppose that $\Gamma \neq \{0\}$. Fix some $0 < \alpha \in \Gamma$. For any $\beta \in \Gamma$, there is a well-defined largest integer n such that $n\alpha \leq \beta$ (if $\beta \geq 0$ this is clear by assumption; if $\beta < 0$, let m be the smallest integer such that $-\beta \leq m\alpha$, and set $n = -m$), which we will denote by n_0 . Now set $\beta_1 = \beta - n_0\alpha$ and let n_1 be the largest integer n such that $n\alpha < 10\beta_1$; we have $0 \leq n_1 < 10$. Set $\beta_2 = 10\beta_1 - n_1\alpha$ and let n_2 be the largest integer n such that $n\alpha < 10\beta_2$. Continuing in the same way, we find a sequence of integers $\{n_i\}$ and set $\phi(\beta)$ to be the real number given by the decimal expression $n_0.n_1n_2\dots$. Then it can easily be checked that $\phi : \Gamma \rightarrow \mathbb{R}$ is order-preserving and injective. Now let $\beta, \beta' \in \Gamma$. For $r \in \mathbb{N}$ arbitrary, let n and n' be the interger determined by the inequalities

$$n\alpha \leq 10^r\beta < (n+1)\alpha, \quad n'\alpha \leq 10^r\beta' \leq (n'+1)\alpha.$$

Then we see $(n+n')\alpha \leq 10^r(\beta + \beta') < (n+n'+2)\alpha$. If we use $\phi_r(\beta)$ to denote the number obtained by taking the first r decimal places of $\phi(\beta)$, then $n = \phi_r(\beta)$, $n' = \phi_r(\beta')$, and hence

$$\begin{aligned} |\phi(\beta + \beta') - \phi(\beta) - \phi(\beta')| &= |\phi(\beta + \beta') - \phi_r(\beta + \beta')| + |\phi_r(\beta + \beta') - \phi_r(\beta) - \phi_r(\beta')| \\ &\quad + |\phi_r(\beta) - \phi(\beta)| + |\phi(\beta') - \phi_r(\beta')| \leq 4 \cdot 10^{-r}. \end{aligned}$$

Since r is arbitrary, we see ϕ is a homomorphism, hence Γ is order isomorphic to a subgroup of \mathbb{R} .

Now let Γ be a nonzero subgroup of \mathbb{R} . Since $\Gamma \neq 0$, A is not a field. Suppose that \mathfrak{p} is a prime ideal of A distinct from \mathfrak{m}_A . Let $a \in \mathfrak{m}_A - \mathfrak{p}$ and set $v(a) = \alpha$. Suppose that $0 \neq b \in \mathfrak{p}$, and set $\beta = v(b)$; then $\beta \in \Gamma$ and $\alpha > 0$, so that $n\alpha > y$ for some sufficiently large natural number n . This means $a^n/b \in A$, so that $a^n \in bA \subseteq \mathfrak{p}$; then since \mathfrak{p} is prime we have $a \in \mathfrak{p}$, which is a contradiction. Thus $\mathfrak{p} = (0)$ and the only prime ideals in A are (0) and \mathfrak{m}_A . That is, $\dim A = 1$.

Finally, assume that $\dim A = 1$. If $0 \neq b \in \mathfrak{m}_A$ then \mathfrak{m}_A is the unique prime ideal containing b , and hence $\sqrt{bA} = \mathfrak{m}_A$. Thus for any $a \in \mathfrak{m}_A$ there exists a natural number n such that $a^n \in bA$. From this one sees easily that Γ satisfies the Archimedean axiom. \square

Proposition 13.2.11. *Let K be a field, v a nontrivial valuation on K and A the ring of v . For A to be completely integrally closed, it is necessary and sufficient that v be of rank 1.*

Proof. Suppose v is of rank 1. Let $x \in K$ be such that the $A[x]$ are all contained in a finitely generated sub- A -module of K . There exists $d \in A - \{0\}$ such that $dx^n \in A$ for all $n \geq 0$. Then $v(d) + nv(x) \geq 0$, that is $n(-v(x)) \leq v(d)$ for all $n \geq 0$, whence $-v(x) \leq 0$ (since Γ_v is Archimedean) and $x \in A$. Thus A is completely integrally closed.

Suppose now that v is not of rank 1. Then there exist $y \in \mathfrak{m}_v$ and $t \in A$ such that $nv(y) < v(t)$ for all $n \geq 0$. Then $ty^{-n} \in A$ for all $n \geq 0$, but $y^{-1} \notin A$. Hence A is not completely integrally closed. \square

Corollary 13.2.12. *Let K be a field, $(v_i)_{i \in I}$ a family of valuations of rank 1 on K and A the intersection of the rings of the v_i . Then A is a completely integrally closed domain.*

Proof. An intersection of completely integrally closed rings is completely integrally closed, hence the claim. \square

13.2.2 Discrete valuation rings

A valuation ring whose value group is isomorphic to \mathbb{Z} is called a **discrete valuation ring** (DVR). Discrete refers to the fact that the value group is a discrete subgroup of \mathbb{R} .

Example 13.2.13. The two standard examples are:

- (a) $K = \mathbb{Q}$. Take a fixed prime p , then any non zero $x \in \mathbb{Q}$ can be written uniquely in the form $p^a y$, where $a \in \mathbb{Z}$ and both numerator and denominator of y are prime to p . Define $v_p(x)$ to be a . The valuation ring of v_p is the local ring $\mathbb{Z}_{(p)}$.
- (b) $K = k(X)$, where k is a field and X an indeterminate. Take a fixed irreducible polynomial $f \in k[X]$ and define v_f just as in (a). The valuation ring of v_f is then the local ring $k[X]_{(f)}$.
- (c) Let k be a field, and let $k((X))$ be the field of formal Laurent series in one variable over k . For every non-zero series

$$f(X) = \sum_{n \geq n_0} a_n X^n, \quad a_{n_0} \neq 0$$

one defines the order $v(f)$ of f to be the integer n_0 . One obtains thereby a discrete valuation of $k((X))$, whose valuation ring is $k[[X]]$, the set of formal series; its residue field is k .

Proposition 13.2.14. *Let A be a valuation ring. Then the following conditions are equivalent.*

- (a) A is a DVR.
- (b) A is a PID.
- (c) A is Noetherian.
- (d) The value group Γ of A is discrete under order topology and has rank 1.

Proof. Let K be the field of fractions of A and \mathfrak{m} its maximal ideal. Since A is a valuation ring, we know finitely generated idels of A are principal, so A is a PID if and only if it is Noetherian. Now, we first show that if A_v is Noetherian then v must have rank 1. For suppose that $\text{rank}(v) > 1$, there must exist a nontrivial isolated subgroup Δ . Fix a positive element $x \in \Delta$, then we get a chain of elements

$$x < 2x < \cdots < nx < \cdots .$$

Since Δ is proper, we can find $y \in G$ which does not belong to Δ . Then since Δ is isolated, y is bigger than any elements in Δ , and in particular bigger than any nx , $n \in \mathbb{N}$. Equivalently, we get a decreasing sequence of positive elements in G :

$$y - x > y - 2x > \dots > y - nx > \dots > 0$$

which gives an infinite strictly descending segments in G , hence an infinite ascending sequence of ideals in A_v (by ??). Hence R_v is not Noetherian.

Now since v has rank 1, G can be identified as a subgroup of \mathbb{R} . Moreover, the maximal ideal \mathfrak{m} is principal, which means G has a smallest positive element. Then we see G is discrete and hence isomorphic to \mathbb{Z} . The converse is obvious, since the group \mathbb{Z} has no infinite descending sequence of positive elements.

Finally, it is clear that \mathbb{Z} is discrete and has rank 1. Conversely, if the value group G of A has rank 1 then it can be embedded into \mathbb{R} , and is isomorphic to \mathbb{Z} if it is discrete. This completes the proof. \square

If A is a DVR with maximal ideal \mathfrak{m} then an element $t \in R$ such that $\mathfrak{m} = (t)$ is called a **uniformising element** (or **prime element**) of A .

Example 13.2.15. A valuation ring B whose maximal ideal \mathfrak{m}_B is principal does not have to be a DVR. In fact, the maximal ideal of a valuation ring is principal if and only if its value group G has a smallest positive element, if and only if G is discrete under order topology.

To obtain a counter-example, let K be a field, and A a DVR of K . If \bar{B} is a DVR for the residue field $k = A/\mathfrak{m}_A$, then their composite B will be a valuation ring of K with rank bigger than 2, hence not a DVR. However, let $s \in B$ be a preimage of a uniformising element \bar{s} of \bar{B} . Then $s \notin \mathfrak{m}_A$ and we have

$$\mathfrak{m}_A \subseteq \mathfrak{m}_B \subseteq B \subseteq A, \quad \mathfrak{m}_B/\mathfrak{m}_A = \bar{s}\bar{B},$$

so $\mathfrak{m}_B = \mathfrak{m}_A + sB$. On the other hand, since $s \in A - \mathfrak{m}_A$, we have $s^{-1} \in A$ and thus $s^{-1}\mathfrak{m}_A \subseteq A \subseteq B$. This implies $\mathfrak{m}_A \subseteq sB$, so $\mathfrak{m}_B = sB$.

The previous theorem gives a characterisation of DVRs among valuation rings; now we consider characterisations among all rings.

Proposition 13.2.16. Let A be a ring with fraction field K . Then the following conditions are equivalent:

- (a) A is a DVR.
- (b) A is a local PID, and not a field.
- (c) A is a Noetherian local ring with positive dimension and the maximal ideal \mathfrak{m}_A is principal.
- (d) A is a one-dimensional normal Noetherian local ring.

Proof. If A is a DVR, conditions (b), (c), (d) hold by ??, ?? and ???. Conversely, we now show that either one of these three conditions implies A is a DVR.

If A is a local PID and not a field, then A is Noetherian and the maximal ideal \mathfrak{m} is nonzero and principal, so (b) implies (c). Now assume (c), and let $\mathfrak{m} = (t)$ be the maximal ideal of A .

Then t can not be nilpotent since otherwise \mathfrak{m} is the unique prime ideal of A , whence $\dim A = 0$. By the Krull intersection theorem we have $\bigcap_{n=1}^{\infty} (t^n) = (0)$, so that for $0 \neq x \in A$ there is a well-determined n such that $x \in (t^n)$ but $x \notin (t^{n+1})$. If $x = t^n u$, then since $u \notin (t)$ it must be a unit. Similarly, for $0 \neq y \in A$ we have $y = t^m v$, with v a unit. Therefore $yz = t^{m+n} uv \neq 0$, and so A is an integral domain. Finally, any element α of K can be written $\alpha = x^\nu u$, with u a unit of A and $\nu \in \mathbb{Z}$, and it is easy to see that setting $v(t) = \nu$ defines an additive valuation of K whose valuation ring is A .

Finally, assume that A is a one-dimensional normal Noetherian local ring. Let x be a nonzero element of \mathfrak{m} . By hypothesis, \mathfrak{m} is the only nonzero prime ideal, so $\sqrt{(x)} = \mathfrak{m}$. Then by ?? (x) contains a power of \mathfrak{m} , and we may assume that $(x) \neq \mathfrak{m}$. Then there exists $n > 1$ such that $\mathfrak{m}^n \subseteq (x)$ but $\mathfrak{m}^{n-1} \not\subseteq (x)$. If $y \in \mathfrak{m}^{n-1} - (x)$ and $\beta = x/y \in K$, then we have $\beta^{-1} = y/x \notin A$ since $y \notin (x)$. Since A is integrally closed, β^{-1} is not integral over A . But then $\beta^{-1}\mathfrak{m} \not\subseteq \mathfrak{m}$, in view of ?. Now use $y \in \mathfrak{m}^{n-1}$ we have

$$\beta^{-1}\mathfrak{m} = (y/x)\mathfrak{m} \subseteq (1/x)\mathfrak{m}^n \subseteq (1/x)(x) = A.$$

Thus $\beta^{-1}\mathfrak{m}$ is an ideal of A , and it must be A since $\beta^{-1}\mathfrak{m} \not\subseteq \mathfrak{m}$. Hence \mathfrak{m} is the principal ideal (β) , and by the preceding argument we can show that A is a DVR. \square

To conclude this part, we introduce the concept of rational rank of a totally ordered group Γ , denoted by $\text{rank}_{\mathbb{Q}}(\Gamma)$, which is defined to be the maximal number of rationally independent elements of Γ . In other words,

$$\text{rank}_{\mathbb{Q}}(\Gamma) := \dim_{\mathbb{Q}}(\mathbb{Q} \otimes_{\mathbb{Z}} \Gamma).$$

Note that the rational rank of a group Γ is zero if and only if Γ is a torsion group. If Γ is a value group of a valuation, Γ is totally ordered, then its rational rank is zero if and only if $\Gamma = \{0\}$, i.e. if and only if the valuation is the trivial valuation.

Proposition 13.2.17. *Let Γ be an abelian group and Δ a subgroup of Γ . Then*

- (a) $\text{rank}_{\mathbb{Q}}(\Gamma) = \text{rank}_{\mathbb{Q}}(\Gamma/\Delta) + \text{rank}_{\mathbb{Q}}(\Delta)$.
- (b) *If Γ is ordered then $\text{rank}(\Gamma) = \text{rank}(\Gamma/\Delta) + \text{rank}(\Delta)$.*
- (c) *If Γ is ordered then $\text{rank}(\Gamma) \leq \text{rank}_{\mathbb{Q}}(\Gamma)$.*

Proof. Since \mathbb{Q} is \mathbb{Z} -flat (it is torsion free), we see that

$$(\mathbb{Q} \otimes_{\mathbb{Z}} \Gamma) / (\mathbb{Q} \otimes_{\mathbb{Z}} \Delta) = \mathbb{Q} \otimes_{\mathbb{Z}} (\Gamma/\Delta)$$

whence the first equality follows. Now it is easy to see the isolated groups in Γ/Δ are in one-to-one correspondence to isolated subgroups in Γ containing Δ , so the second equality follows. Finally, assume that Γ is ordered and let $\Delta_0 \leq \Delta_1 \leq \dots \leq \Delta_{s-1}$ be a strict chain of isolated subgroups of Γ . For each i , choose an element $\alpha_i \in \Delta_i \setminus \Delta_{i-1}$ (with $\Delta_s = \Gamma$), then we see $\alpha_1, \dots, \alpha_r$ are rationally independent: let $\sum_{i=1}^t r_i \alpha_i = 0$ with $r_t \neq 0$ and $t \leq s$, then $r_t \alpha_t \in \Delta_{t-1}$, and since Δ_{t-1} is a segment and $r_t \neq 0$, we find $\alpha_t \in \Gamma_{t-1}$, contradiction. Thus $\text{rank}(\Gamma) \leq \text{rank}_{\mathbb{Q}}(\Gamma)$. \square

13.3 Compare valuation rings of a field

As an immediate observation, if \mathfrak{p} is a prime ideal of A , then the localized ring $A_{\mathfrak{p}}$ is a local ring containing A . Therefore, $A_{\mathfrak{p}}$ is also a valuation of K . We have seen in ?? that such a prime ideal \mathfrak{p} corresponds to a proper isolated subgroup Δ of G . Now we determine the value group of $A_{\mathfrak{p}}$.

Proposition 13.3.1. *The value group of $A_{\mathfrak{p}}$ is isomorphic to the quotient group Γ/Δ , where Δ is the isolated subgroup of Γ corresponds to \mathfrak{p} . In this way the valuation $v_{\mathfrak{p}}$ associated to $A_{\mathfrak{p}}$ is given by the composition of $v : K^{\times} \rightarrow \Gamma$ and $\pi : \Gamma \rightarrow \Gamma/\Delta$.*

Proof. We define $\tilde{v} = \pi \circ v : K^{\times} \rightarrow \Gamma/\Delta$, which is easily seen to be a valuation of K . If we embed $A_{\mathfrak{p}}$ in K , then we have $v(A_{\mathfrak{p}}) = \Delta \cup \Gamma_+$. On the other hand, we observe that

$$A_{\tilde{v}} = \{x \in K : \pi(v(x)) \geq 0\} = \{x \in K : v(x) > \Delta \text{ or } v(x) \in \Delta\} = A_{\mathfrak{p}},$$

so the claim follows from ??.

□

With ??, we can now classify all valuation rings of K containing A : they are in one-to-one correspondence to prime ideals of A .

Proposition 13.3.2. *Let A be a valuation ring and \mathfrak{m}_A be the maximal ideal of A .*

- (a) *Any localization $A_{\mathfrak{p}}$ of A at a prime ideal \mathfrak{p} of A is a valuation ring of K containing A , and \mathfrak{p} is the maximal ideal of $A_{\mathfrak{p}}$.*
- (b) *The map $\mathfrak{p} \mapsto A_{\mathfrak{p}}$ is a bijection from the set of prime ideals of A onto the set of rings B with $A \subseteq B \subseteq K$, which is orderreversing for the relation of inclusion. The inverse map is defined by $B \mapsto \mathfrak{m}_B$.*

Proof. We have already seen that $A_{\mathfrak{p}}$ is a valuation ring containing A . Now, from the definition of the valuation ideal and ??, we have

$$\mathfrak{m}_{A_{\mathfrak{p}}} = \{x \in K : v_{\mathfrak{p}}(x) > 0\} = \{x \in K : \pi(v(x)) > 0\} = \{x \in K : v(x) > \Delta\} = \mathfrak{p}.$$

Therefore \mathfrak{p} is the maximal ideal of $A_{\mathfrak{p}}$.

Now let B be a ring such that $A \subseteq B \subseteq K$. Note that if $x \in \mathfrak{m}_B$ then $x^{-1} \notin B \supseteq A$, and hence $x \in \mathfrak{m}_A$. This implies $\mathfrak{m}_B \subseteq \mathfrak{m}_A \subseteq A \subseteq B$. Also, note that by (a), \mathfrak{m}_B is the maximal ideal of both $A_{\mathfrak{m}_B}$ and B . Since a valuation ring is uniquely determined by its maximal ideal, we get $B = A_{\mathfrak{m}_B}$, hence the claim. □

Since $A_{\mathfrak{p}}$ is a valuation ring and \mathfrak{p} is its maximal ideal, we can consider its residue field $\kappa_{\mathfrak{p}} = A_{\mathfrak{p}}/\mathfrak{p}$. Then since A is a valuation ring, it is easy to see $\bar{A} = A/\mathfrak{p}$ is a valuation ring of $\kappa_{\mathfrak{p}}$. Moreover, we can determine the value group of \bar{A} .

Proposition 13.3.3. *Let \mathfrak{p} be a prime ideal of A with corresponding isolated subgroup Δ . Then \bar{A} is a valuation ring of $\kappa_{\mathfrak{p}}$ with value group Δ .*

Proof. Let $\pi : A_{\mathfrak{p}} \rightarrow A_{\mathfrak{p}}/\mathfrak{p}$ be the quotient map. Then since A/\mathfrak{p} is local with maximal ideal $\mathfrak{m}_A/\mathfrak{p}$, we see $U(A/\mathfrak{p}) = \pi(U(A))$, hence there is an isomorphism ψ given by

$$\psi : U(A_{\mathfrak{p}})/U(A) \rightarrow U(A_{\mathfrak{p}}/\mathfrak{p})/U(A/\mathfrak{p}), \quad x + U(A) \mapsto \pi(x) + U(A/\mathfrak{p}).$$

Also, we note that $U(A_{\mathfrak{p}})/U(A) \cong \Delta$, with the isomorphism given by

$$\psi : U(A_{\mathfrak{p}})/U(A) \rightarrow \Delta, \quad x + U(A) \mapsto v(x).$$

With these observations, we now define a valuation on $\kappa_v^\times = U(A_{\mathfrak{p}}/\mathfrak{p})$ by

$$\bar{v} : U(A_{\mathfrak{p}}/\mathfrak{p}) \rightarrow \Delta, \quad \bar{v}(\pi(x)) = \psi(\phi^{-1}(\pi(x) + U(A/\mathfrak{p}))) = v(x).$$

With this, it is easy to check that

$$A_{\bar{v}} = \{\pi(x) : \bar{v}(\pi(x)) \geq 0\} = \{\pi(x) : v(x) \geq 0\} = A/\mathfrak{p}.$$

Thus the claim follows. \square

It turns out that all valuation rings contained in a given valuation ring arise in this way: we have the following proposition.

Proposition 13.3.4. *Let A be a valuation ring of K and $\pi : A \rightarrow A/\mathfrak{m}_A$ be the canonical map.*

- (a) *Let $B \subseteq A$ be a valuation ring of K , then the image \bar{B} under π_A is a valuation ring of κ_v .*
- (b) *Let \bar{B} be a valuation ring of κ_v , then its preimage $B \subseteq A$ under π_A is a valuation ring of K .*

Proof. Let $B \subseteq A$ be a valuation ring of K and $\pi(x) \in \kappa_v - \bar{B}$. Then $x \notin B$, and hence $x^{-1} \in B$. Since $\pi(x^{-1}) = \pi(x)^{-1}$, we see $\pi(x)^{-1} \in \bar{B}$, hence \bar{B} is a valuation ring of κ_v .

Conversely, let \bar{B} be a valuation ring of K and B be its preimage under π . Let $x \in K - B$, then $\pi(x) \in \kappa_v - \bar{B}$ and hence $\pi(x^{-1}) \in \bar{B}$. Then it follows that $x^{-1} \in B$ so B is a valuation of K . \square

The formulation of ?? can be defined on valuations as follows. Let $v : K^\times \rightarrow \Gamma$ be a valuation of K with residue field κ_v , and let $\bar{v} : \kappa_v \rightarrow \bar{\Gamma}$ be a valuation of κ_v . Then we can define a new valuation w of K by letting its valuation ring to be

$$A_w = \{a \in A : \bar{v}(\bar{a}) \geq 0\}.$$

The valuation w is called the composite of v and \bar{v} , and denoted by $w = v \circ \bar{v}$.

Let v and w be two valuations of K . We say w dominates v , or $w \geq v$, if $A_w \subseteq A_v$ or equivalently $\mathfrak{m}_w \supseteq \mathfrak{m}_v$. Then ?? just says all valuations $w \geq v$ are composite of v . By ?? and ??, the following result is immediate.

Proposition 13.3.5. *If w is the composite valuation $v \circ \bar{v}$ we have the equalities*

$$\text{rank}(w) = \text{rank}(v) + \text{rank}(\bar{v}), \quad \text{rank}_{\mathbb{Q}}(w) = \text{rank}_{\mathbb{Q}}(v) + \text{rank}_{\mathbb{Q}}(\bar{v}).$$

In fact, by ?? and ??, if we have the valuations v of K and \bar{v} of κ_v , then the composite valuation $w = v \circ \bar{v}$ defines an extension of the value group Γ of v by the value group $\bar{\Gamma}$ of \bar{v} , i.e. an exact sequence of totally ordered groups:

$$0 \longrightarrow \bar{\Gamma} \longrightarrow \widetilde{\Gamma} \longrightarrow \Gamma \longrightarrow 0$$

If this exact sequence splits, the value group $\tilde{\Gamma}$ is isomorphic to the group $\Gamma \times \bar{\Gamma}$ with the lexicographic order. If the valuation v is a discrete valuation of rank one, i.e. for $\Gamma = \mathbb{Z}$, the exact sequence always splits (this follows from the observation that \mathbb{Z} can be embedded into any nonzero totally ordered group) and we can describe the composite valuation $w = v \circ \bar{v}$ in the following way. The maximal ideal of the valuation ring A_v associated to v is generated by an element t and we can associate to any non zero element x in K the nonzero element $\pi(xt^{-v(x)})$ in the residue field, where $\pi : A_v \rightarrow \kappa_v$ is the canonical map. The composite valuation w is then defined by $w(x) = (v(x), \bar{v}(\pi(xt^{-v(x)})))$.

13.4 Dependent valuations and induced topology

With the notation of composite, we say two valuations v, w of K are **dependent** if they are the composite of a common nontrivial valuation. If these do not hold, we say v and w are **independent**. Equivalently, by ?? v and w are independent if and only if the valuation rings A_v and A_w are not contained in a proper valuation ring of K , if and only if AB , the smallest ring containing A and B , is a proper subring of K .

We see from ?? that the dependence relation is an equivalence relation. Indeed, if A and B are dependent and also B and C are dependent, then AB and BC are both overrings of B . Thus there is some inclusion, say $AB \subseteq BC$. But then A and B are both contained in BC . Hence they are dependent. We shall now study the topology induced by a valuation on a field K . Let $v : K^\times \rightarrow \Gamma$ be a valuation of K . For each $\gamma \in \Gamma$ and each $x \in K$ we define the set

$$U_\gamma(x) = \{y \in K : v(y - x) > \gamma\}.$$

These sets form a basis of open neighborhoods of x , due to the following observations:

- $x \in U_\gamma(x)$,
- If $y \in U_\gamma(x) \cap U_{\gamma'}(x')$ and $\delta > \max\{\gamma, \gamma'\}$, then for $z \in U_\delta(y)$, we have

$$v(z - x) = v(z - y + y - x) > \gamma, \quad v(z - x') = v(z - y + y - x') > \gamma'$$

whence $U_\delta(y) \subseteq U_\gamma(x) \cap U_{\gamma'}(x')$.

Therefore, the sets $\{U_\gamma(x) : x \in K, \gamma \in \Gamma\}$ is a basis for a topology \mathcal{T}_v on K , called the **topology induced by the valuation** v . Some properties of the topology \mathcal{T}_v are gathered below.

Proposition 13.4.1. *Let v be a valuation on K and \mathcal{T}_v the topology induced by v .*

- (a) *The topology \mathcal{T}_v is Hausdorff and totally disconnected, and it is discrete if and only if v is trivial.*
- (b) *The field operations are continuous with respect to \mathcal{T}_v .*
- (c) *The valuation $v : K^\times \rightarrow \Gamma$ is continuous if Γ is given the discrete topology.*
- (d) *The quotient topology induced on the residue field κ_v is discrete.*

Proof. Let A be the valuation ring of v . Let $x, y \in K$ and $x \neq y$. If $\gamma = v(x - y)$, then $\gamma \neq \infty$ and we see $U_\gamma(x) \cap U_\delta(y) = \emptyset$ if $\gamma, \delta > v(x - y)$. Therefore \mathcal{T}_v is Hausdorff. Also, it is easy to see \mathcal{T}_v

is discrete if and only if v is trivial. We next claim that the set $U_\gamma^c(x) = \{y \in K : v(y - x) \leq \gamma\}$ is also open in K . To prove this, we first observe that, if $v(z - y) > v(y - x)$ then we have $(z - y)/(y - x) \in \mathfrak{m}_A$, whence $1 + (z - y)/(y - x) = (z - x)/(y - x)$ is a unit in A . That is, we have

$$v(z - y) > v(y - x) \Rightarrow v(z - x) = v(y - x).$$

From this, it follows that $U_{v(y-x)} \subseteq U_\gamma(x)^c$ if $y \in U_\gamma(x)^c$, so $U_\gamma(x)^c$ is open. Since $U_\gamma(x)$ is a basis for \mathcal{T}_v , it follows that \mathcal{T}_v is totally disconnected.

Since $v(x + y) \geq \min\{v(x), v(y)\}$, we have $U_\gamma(x) + U_\gamma(y) \subseteq U_\gamma(x + y)$. Moreover, from

$$ab - xy = (a - x)(b - y) + (a - x)y + (b - y)x$$

it follows that $U_\gamma(x)U_\gamma(y) \subseteq U_\delta(xy)$, where $\delta = \min\{2\gamma, \gamma + v(x), \gamma + v(y)\}$. These shows addition and multiplication are continuous under \mathcal{T}_v .

Let $x_0 \in K^\times$, if $x \in K^\times$ satisfies $v(x - x_0) > \max\{\gamma + 2v(x_0), v(x_0)\}$, then we have $v(x) = v(x_0)$ and

$$\begin{aligned} v(x^{-1} - x_0^{-1}) &= v(x^{-1}(x_0 - x)x_0^{-1}) = v(x_0 - x) - v(x) - v(x_0) \\ &> \gamma + 2v(x_0) - 2v(x_0) = \gamma \end{aligned} \tag{13.4.1}$$

hence the inverse map is also continuous. These proves (b). Also, the single condition $v(x - x_0) > v(x_0)$ implies $v(x) = v(x_0)$, hence the map $v : K^\times \rightarrow G$ is continuous if G is given the discrete topology. Finally, note that $\mathfrak{m}_v = U_0(0)$ is clopen in A and so the quotient topology is discrete. \square

Theorem 13.4.2. *Two nontrivial valuations v and w of K are dependent if and only if they induce the same topology on K .*

Proof. Since two dependent valuation rings have a common non-trivial coarsening, to show that they induce the same topology it is enough to consider the particular case $A_v \subseteq A_w$.

Let $v : K^\times \rightarrow \Gamma$ be the valuation. Since $A_v \subseteq A_w$, by ?? there exists a isolated subgroup Δ of Γ such that w is given by $K^\times \rightarrow \Gamma \rightarrow \Gamma/\Delta$. Since w is nontrivial, $A_w \neq K$ and $\Delta \neq \Gamma$. Write

$$U_\gamma(0) = \{x \in K : v(x) > \gamma\}, \quad U_{\gamma+\Delta}(0) = \{x \in K : w(x) > \gamma + \Delta\},$$

then we see $U_{\gamma+\Delta}(0) \subseteq U_\gamma(0)$. On the other hand, if $v(x) > 2\gamma$ with $0 < \gamma \notin \Delta$ then $2\gamma > \gamma + \Delta$, hence $U_{2\gamma}(0) \subseteq U_{\gamma+\Delta}(0)$, whence v and w induces the same topology on K .

Conversely, let \mathfrak{m}_v and \mathfrak{m}_w be the valuation ideals of v and w . If v and w induce the same topology on K , then w is an open neighbourhood of 0 in \mathcal{T}_v , so there exists $a \in K^\times$ such that $a\mathfrak{m}_v \subseteq \mathfrak{m}_w$. As \mathfrak{m}_w is the maximal ideal of the valuation ring A_w , the set $K - \mathfrak{m}_w$ is multiplicatively closed. Thus we can form the ring

$$A = \{x/y \in K : x \in A_v, y \in A_v - \mathfrak{m}_w\} = \{x/y \in K : v(x) \geq 0, v(y) \geq 0, w(y) \leq 0\}.$$

Since $A_v \subseteq A$, A is also a valuation ring. Moreover, A contains A_w since if $x \in A_w - A_v$ then $w(x) \geq 0$ and we have $x = 1/x^{-1} \in A$. Finally, $A \neq K$: let $z \in A_v \setminus \{0\}$, then $\frac{1}{az} \notin A$ since otherwise $1/(az) = x/y$ with $x \in A_v$ and $y \in A_v - \mathfrak{m}_w$, and we have $y = azx \in a\mathfrak{m}_v \subseteq \mathfrak{m}_w$, a contradiction. Hence v and w are dependent. \square

?? is another way to see that the dependence relation among the valuation rings of a field K is an equivalence relation, as we already saw above. Let A be a non-trivial valuation ring of K and take the dependence class $[A]$ of all non-trivial valuation rings of K dependent on A . Clearly $[A]$ is an upwardly directed set with respect to the partial order of inclusion.

Proposition 13.4.3. *Let A be any non-trivial valuation ring of K . Then we have the following case distinction:*

- (a) *$[A]$ has a maximal valuation ring B_0 which is a maximal non-trivial overring of A ; moreover B_0 has dimension 1 and its maximal ideal is the intersection of all non-zero prime ideals of A , or*
- (b) *there is no maximal non-trivial overring of A . Then the maximal ideals \mathfrak{m}_B of valuation rings $B \in [A]$ form a neighborhood system of 0 for the topology induced by A . In this case the set of all non-zero prime ideals of A is also a neighborhood system of 0.*

Proof. If $[A]$ has a maximal element B_0 , then it is clear that B_0 contains A and is the maximal overring of A . Thus the first part follows from ???. Now assume that $[A]$ has no maximal element. Suppose we are given a positive $\delta \in \Gamma$. We seek a valuation ring $B \supseteq A$ such that $B \neq K$ and whose maximal ideal \mathfrak{m}_B satisfies $\mathfrak{m}_B \subseteq U_\delta(0)$. Let Δ be the convex hull of the subgroup generated by δ in Γ , i.e.,

$$\Delta = \{\gamma \in \Gamma : \gamma, -\gamma < n\delta \text{ for some } n \in \mathbb{N}\}.$$

Then Δ defines a valuation ring $B \supseteq A$ with $\mathfrak{m}_B \subseteq U_\delta(0)$. It remains to show that $B \neq K$.

If $B = K$, then $\Delta = \Gamma$ and thus δ is an element of Γ such that Γ is the convex hull of $\delta\mathbb{Z}$. Let Δ^* be the largest isolated subgroup of Γ not containing δ . Then $\Gamma^* = \Gamma/\Delta^*$ is archimedean ordered and thus the overring A^* of A corresponding to Δ^* is maximal according to ???. This contradicts our assumption. \square

We are now in a position to prove an approximation theorem for independent valuations. For this we need some preparations.

Lemma 13.4.4. *Let v_1, \dots, v_n be valuations on the field K and $x \in K^\times$. Then there exists a polynomial $f(X)$ of the form*

$$f(X) = 1 + n_1X + \dots + n_{k-1}X^{k-1} + X^k, \quad n_i \in \mathbb{Z}, k \geq 2 \quad (13.4.2)$$

such that $f(x) \neq 0$ and $z = f(x)^{-1}$ satisfies

$$\begin{aligned} v_i(z) &= 0 && \text{if } v_i(x) \geq 0, \\ v_i(z) + v_i(x) &> 0 && \text{if } v_i(x) < 0. \end{aligned}$$

Proof. Let I be the set of indices i such that $v_i(x) \geq 0$. For all $i \in I$, let x_i denote the canonical image of x in κ_{A_i} . For all $i \in I$ we construct a polynomial f_i as follows: if there exists a polynomial $g(X)$ of the form ?? such that $g(\bar{x}_i) = 0$ in κ_{A_i} , we take f_i to be such a polynomial; otherwise we take $f_i = 1$. Then we write $f(X) = 1 + X^2 \prod_{i \in I} f_i(X)$. It is obviously a polynomial of the form ???. If $i \in I$, then $f(x) \in A_i$ and also $f(\bar{x}_i) \neq 0$ by construction; hence $f(x) \notin \mathfrak{m}_i$, $v_i(f(x)) = 0$ and $v_i(z) = 0$. If $i \notin I$, then $v_i(x) < 0$, whence $v_i(f(x)) = kv_i(x)$ (by

??) and

$$v_i(x) + v(z) = (1 - k)v_i(x) > 0,$$

so the lemma is proved. \square

Proposition 13.4.5. *Let A_1, \dots, A_n be valuation rings of a field K , $B = \bigcap_{i=1}^n A_i$ and $\mathfrak{p}_i = \mathfrak{m}_{A_i} \cap B$. Then $A_i = B_{\mathfrak{p}_i}$ and the fraction field of B is K .*

Proof. Clearly $B_{\mathfrak{p}_i} \subseteq A_i$ since $B \subseteq A_i$ and $B - \mathfrak{p}_i \subseteq A - \mathfrak{m}_i$ is a unit in A_i . Now let x be a nonzero element in A_i . We apply the lemma to x and valuations v_i associated with the A_i . Then $v_i(z) \geq 0$ and $v_i(zx) \geq 0$ for all i , hence $z, zx \in B$. As $v_i(x) \geq 0$ and $v_i(z) = 0$, we see $z \notin \mathfrak{p}_i$. Hence $x = zx/z \in B_{\mathfrak{p}_i}$. The field of fractions of B then contains A_i and hence is K . \square

Proposition 13.4.6. *With the hypotheses of ??, suppose further that $A_i \not\subseteq A_j$ for $i \neq j$. Then the \mathfrak{p}_i 's are distinct maximal ideals of B and every maximal ideal of B is equal to one of the \mathfrak{p}_i .*

Proof. If $\mathfrak{p}_i \subseteq \mathfrak{p}_j$ for $i \neq j$ then $A_i = B_{\mathfrak{p}_i} \supseteq B_{\mathfrak{p}_j} = A_j$, which is a contradiction. Then the claim follows from ??.

Corollary 13.4.7. *Suppose that A_1, \dots, A_n are incomparable valuation rings. For every family of elements $a_1 \in A_1, \dots, a_n \in A_n$, there exists $x \in B$ such that $x \equiv a_i \pmod{\mathfrak{m}_i}$ for each i .*

Proof. Since the \mathfrak{p}_i are maximal ideals of B , $A/\mathfrak{m}_A = B_{\mathfrak{p}_i}/\mathfrak{p}_i B_{\mathfrak{p}_i} = B/\mathfrak{p}_i$ and it may therefore be assumed that $a_i \in B$ for all i . The corollary then follows from the fact that the canonical map from B to $\prod_{i=1}^n (B/\mathfrak{p}_i)$ is surjective. \square

Corollary 13.4.8. *Suppose that A_1, \dots, A_n are incomparable valuation rings. There exist elements x_1, \dots, x_n of K such that $v_i(x_i) = 0$ and $v_j(x_i) > 0$ for $j \neq i$.*

Proof. For each index i apply ?? to the family (a_i) such that $a_i = 1$ and $a_j = 0$ for $j \neq i$. \square

Corollary 13.4.9. *Every valuation ring of K containing B contains one of the A_i .*

Proof. We may confine our attention to the case where A_i 's are incomparable. Let A be a valuation ring of K containing B . We write $\mathfrak{p} = \mathfrak{m}_A \cap B$. There exists a maximal ideal \mathfrak{p}_i of B containing \mathfrak{p} , whence $A_i = B_{\mathfrak{p}_i} \subseteq B_{\mathfrak{p}} \subseteq A$. \square

Now for independent valuations, we have the following approximation theorem.

Theorem 13.4.10 (Approximation Theorem). *Suppose A_1, \dots, A_n are pairwise independent valuation rings of K . For every i , let $v_i : K^\times \rightarrow \Gamma_i$ be the valuation corresponding to A_i . Then for any $a_1, \dots, a_n \in K$ and $\gamma_1 \in \Gamma_1, \dots, \gamma_n \in \Gamma_n$, there exists an $x \in K$ with*

$$v_i(x - a_i) > \gamma_i \quad \text{for all } i \in \{1, \dots, n\}.$$

Proof. Let A_i be the valuation ring of v_i and set $B = \bigcap_{i=1}^n A_i$, $\mathfrak{p}_i = \mathfrak{m}_{A_i} \cap B$. By ?? we have $A_i = B_{\mathfrak{p}_i}$ and K is the fraction field of B , so the a_i may be written as $a_i = b_i/s$ with $b_i \in B$ and $s \in B \setminus \{0\}$. If we write $x = y/s$ and $\gamma'_i = \gamma_i + v(s)$, then

$$v_i(x - a_i) = v_i(y - b_i) - v(s) > \gamma_i \iff v_i(y - b_i) > \gamma'_i.$$

This shows that we may assume that $a_i \in B$ for all i ; we may also assume that $\gamma_i > 0$ for all i . Define ideals of A_i and B by

$$I_i = \{z \in K : v_i(z) \geq \gamma_i\} \subseteq \mathfrak{m}_i, \quad J_i = I_i \cap B \subseteq \mathfrak{p}_i.$$

For $x \in B$, $v_i(x - a_i) \geq \gamma_i$ is equivalent to $x \equiv a_i \pmod{I_i}$. We therefore need to show that the canonical homomorphism $B \rightarrow \prod_{i=1}^n (B/J_i)$ is surjective, that is, J_i 's are pairwise coprime. As the maximal ideals of B are the \mathfrak{p}_i , it will suffice for this to show that $J_i \not\subseteq \mathfrak{p}_j$ for $i \neq j$.

Suppose that there exists i, j such that $J_i \subseteq \mathfrak{p}_j$ and $i \neq j$. We shall see shortly that the radical of J_i is a prime ideal \mathfrak{p} of B , so that $\mathfrak{p} \subseteq \mathfrak{p}_j$. Since $\gamma_i > 0$ we also have $J_i \subseteq \mathfrak{p}_i$, so $A_j = B_{\mathfrak{p}_j} \subseteq B_{\mathfrak{p}}$ and similarly $A_i \subseteq B_{\mathfrak{p}}$. Now, as $I_i \neq (0)$ and $I_i = B_{\mathfrak{p}_i}J_i$ (??), $J_i \neq (0)$ whence $\mathfrak{p} \neq (0)$ and $B_{\mathfrak{p}} \neq K$. This contradicts the hypothesis that A_i and A_j are independent.

It remains to show that $\mathfrak{p} = \sqrt{J_i}$ is prime for each i . Suppose that $xy \in \mathfrak{p}$, so that $x^n y^n \in J_i$ for some i . Then, if for example $v(x) \geq v(y)$, we have

$$v(x^{2n}) = 2nv(x) \geq nv(x) + nv(y) = v(x^n y^n) \geq \gamma_i$$

whence $x^{2n} \in J_i$ and $x \in \mathfrak{p}$. This finishes the proof. \square

Corollary 13.4.11. *For every family of elements $\gamma_1 \in \Gamma_1, \dots, \gamma_n \in \Gamma_n$, there exists $x \in K$ such that $v_i(x) = \gamma_i$.*

Proof. We may assume that $A_i \neq K$ for all i . Then, there exists for all i an $a_i \in K$ such that $v_i(a_i) = \gamma_i$, and an $\alpha_i \in \Gamma_i$ such that $\gamma_i < \alpha_i$. We apply ?? to these elements a_i there exists a $x \in K$ such that $v_i(x - a_i) > \gamma_i = v_i(a_i)$, whence $v_i(x) = v_i(a_i) = \gamma_i$. \square

Here we introduce the concept of absolute values over a field, which has a deep connection with valuations of rank 1. Let K be a field. An **absolute value** on K is a map $|\cdot| : K \rightarrow \mathbb{R}$ satisfying the following axioms for all $x, y \in K$,

- (a) $|x| > 0$ for all $x \neq 0$ and $|0| = 0$.
- (b) $|xy| = |x||y|$.
- (c) $|x + y| \leq |x| + |y|$.

The absolute value sending all $x \neq 0$ to 1 is called the **trivial absolute value** on K . Note that by (b) we have $|1| = 1$, $|-x| = |x|$, and $|x^{-1}| = |x|^{-1}$, thus $|\cdot|$ is a group homomorphism from K^\times to \mathbb{R}^\times .

Proposition 13.4.12. *The set $\{|n \cdot 1| : n \in \mathbb{Z}\}$ is bounded if and only if $|\cdot|$ satisfies the **ultrametric inequality***

$$|x + y| \leq \max\{|x|, |y|\}$$

for all $x, y \in K$.

Proof. If $|\cdot|$ satisfies the inequality, then by induction, the set $\{|n \cdot 1| : n \in \mathbb{Z}\}$ is bounded by 1. Conversely, let $|n \cdot 1| \leq C$. Then

$$|x + y|^n = |(x + y)^n| \leq \sum_v \left| \binom{n}{v} x^v y^{n-v} \right| \leq (n+1)C \max\{|x|, |y|\}^n.$$

Taking n -th roots and letting n go to infinity proves the assertion of the proposition. \square

If an absolute value satisfies the ultrametric inequality, it is called **non-archimedean**; otherwise it is called **archimedean**. Clearly, if $\text{char } K \neq 0$, K cannot carry any archimedean absolute value. A typical example of an archimedean absolute value is the ordinary absolute value on \mathbb{R} , which is given by $|x|_0 = x$ if $x \geq 0$ and $-x$ if $x < 0$. Note that if $v : K^\times \rightarrow \mathbb{R}$ is a rank 1 valuation on K then the absolute value defined by

$$|x| = e^{-v(x)}, \quad x \in K$$

is non-archimedean. Conversely, if $|\cdot|$ is a non-archimedean absolute value then $v(x) := -\ln|x|$ is a valuation on K . Thus we see non-archimedean absolute values corresponds to rank 1 valuations.

An absolute value $|\cdot|$ on K defines a metric by taking $|x - y|$ as distance, for $x, y \in K$. In particular, $|\cdot|$ induces a topology on K . If two absolute values induce the same topology on K , they are called dependent (otherwise independent).

Proposition 13.4.13. *Let $|\cdot|_1$ and $|\cdot|_2$ be two non-trivial absolute values on K . They are dependent if and only if for all $x \in K$,*

$$|x|_1 < 1 \Rightarrow |x|_2 < 1.$$

If they are dependent, then there exists a real number $\lambda > 0$ such that $|x|_1 = |x|_2^\lambda$ for all $x \in K$.

Proof. For $|\cdot|_1$ and $|\cdot|_2$ non-trivial and dependent absolute values on K there exists $\varepsilon > 0$ such that $\{x \in K : |x|_1 < \varepsilon\} \subseteq \{x \in K : |x|_2 < 1\}$. If $|x|_1 < 1$, there is $m \geq 1$ such that $(|x|_1)^m = |x^m|_1 < \varepsilon$. Hence $(|x|_2)^m = |x^m|_2 < 1$ and consequently $|x|_2 < 1$, as required.

Conversely, by the non-triviality of $|\cdot|_1$, there exists $z \in K$ with $|z|_2 > 1$. Thus $|z^{-1}|_1 < 1$ and so $|z^{-1}|_1 < 1$, by assumption. Hence $|z|_2 > 1$, too. Now we claim that, for every $x \in K$, $x \neq 0$, we have

$$\frac{\log|x|_1}{\log|x|_2} = \frac{\log|z|_1}{\log|z|_2}.$$

In fact, for $m, n \in \mathbb{Z}$, $n > 0$, such that

$$\frac{m}{n} > \frac{\log|x|_1}{\log|z|_1},$$

it follows that $(|z|_1)^m > (|x|_1)^n$. Consequently, $|x^n z^{-m}|_1 < 1$ and then, by assumption, $|x^n z^{-m}|_2 < 1$. Walking back the steps of the last argument one gets

$$\frac{m}{n} > \frac{\log|x|_2}{\log|z|_2}.$$

It then follows that

$$\frac{\log|x|_1}{\log|z|_1} \geq \frac{\log|x|_2}{\log|z|_2}.$$

Similarly one proves the reverse inequality. So the claim follows.

Now set $\lambda = \log|z|_1 / \log|z|_2$, it follows that $|x|_1 = |x|_2^\lambda$, for every $x \in K$, as required. Finally, the last equation implies that $|\cdot|_1$ and $|\cdot|_2$ are dependent. \square

Now we prove an approximation theorem for absolute values.

Theorem 13.4.14. Let K be a field and $|\cdot|_1, \dots, |\cdot|_n$ be non-trivial pairwise-independent absolute values on K . Moreover let $x_1, \dots, x_n \in K$, and $0 < \varepsilon \in \mathbb{R}$. Then there exists $x \in K$ such that $|x - x_i| < \varepsilon$ for all i .

Proof. We shall first prove that for every $1 \leq i \leq n$ there exists $a_i \in K$ such that $|a_i|_i > 1$ and $|a_i|_j < 1$, for all $j \neq i$. We may fix $i = 1$ without loss of generality, and write $a = a_1$. We proceed by induction on n . For $n = 2$, ?? implies the existence of $b, c \in K$ such that

$$|b|_1 < 1, |b|_2 \geq 1, \quad |c|_1 \geq 1, |c|_2 < 1.$$

Thus $a = b^{-1}c$ has the desired properties.

Assume next that there is $y \in K$ such that $|y|_1 > 1$ and $|y|_j < 1$ for all $j = 2, \dots, n-1$. Applying the first case to $|\cdot|_1$ and $|\cdot|_n$, one has $|z|_1 > 1$ and $|z|_n < 1$, for some $z \in K$. Therefore, if $|y|_n \leq 1$, then $|zy^\nu|_1 > 1$ and $|zy^\nu|_n < 1$ for every integer $\nu \geq 1$. On the other hand, for a sufficiently large integer $\nu \geq 1$, $|zy^\nu|_j < 1$ for every $j = 2, \dots, n-1$. For such a ν , $a = zy^\nu$ satisfies the requirements.

Consider now the case $|y|_n > 1$, and form the sequence

$$w_\nu = \frac{y^\nu}{1 + y^\nu}$$

The usual properties of sequences of ordinary real numbers imply that $\lim_\nu |w_\nu|_j = 0$ for $j = 2, \dots, n-1$ and $\lim_\nu |w_\nu - 1|_j = 0$ for $j = 1$ and n . Consequently, $\lim_\nu |zw_\nu|_j = 0$ for $j = 2, \dots, n-1$ and $\lim_\nu |zw_\nu|_j = |z|_j$ for $j = 1$ and n . Hence, for sufficiently large ν , $a = zw_\nu$ has the required properties.

Now we prove that for any real number $\varepsilon > 0$ and every i such that $1 \leq i \leq n$, there exists $c_i \in K$ such that $|c_i - 1|_i < \varepsilon$ and $|c_i|_j < \varepsilon$, for all $j \neq i$. As before, it is enough to consider the case $i = 1$. Let $a \in K$ satisfy $|a|_1 > 1$ and $|a|_j < 1$ for $j > 1$. Then the sequence $|a^\nu/(1 + a^\nu)|_j$ converges to 1 for $j = 1$, and converges to 0 if $j > 1$. Thus for sufficiently large ν ,

$$c_1 = \frac{a^\nu}{1 + a^\nu}$$

has the required property.

According to the argument above, there exist elements c_1, \dots, c_n in K such that c_i is close to 1 at $|\cdot|_i$, and for every $j \neq i$, c_i is close to 0 at $|\cdot|_j$. The element $x = c_1x_1 + \dots + c_nx_n$ is then arbitrarily close to x_i at $|\cdot|_i$, for every $i = 1, \dots, n$, and therefore satisfies the requirements of the theorem. \square

We now prove some result about topological vector spaces over a valued field, which will be used later.

Proposition 13.4.15. Let (K, v) be a valued field with valued group Γ . Let X be a topological vector space over K which is Hausdorff and of dimension 1. Suppose that v is not trivial. For all $x_0 \in X$, the map $a \mapsto ax_0$ of K onto X is a topological isomorphism.

Proof. This map is a continuous algebraic isomorphism. It is sufficient to show that it is bicontinuous. Let $\alpha \in \Gamma$. We need to show that there exists a neighbourhood V of 0 in X such that the relation $ax_0 \in V$ implies $v(a) > \alpha$. Let $a_0 \in K^\times$ be such that $v(a_0) = \alpha$. As X is Hausdorff,

there exists a neighbourhood W of 0 in X such that $a_0x_0 \notin W$. Since v is not improper, there exist a neighbourhood W' of 0 in X and an element β of Γ such that the relations $y \in W'$ and $v(a) \geq \beta$ imply $ay \in W'$. Let $a_1 \in K^\times$ be such that $v(a_1) = -\beta$. The relations $ax_0 \in a_1^{-1}W'$ and $v(a) \leq \alpha$ imply $a_1ax_0 \in W'$ and

$$v(a_0a^{-1}a_1^{-1}) = v(a_0) - v(a) - v(a_1) = \alpha + \beta - v(a) \geq \beta$$

hence $a_0x_0 = (a_0a^{-1}a_1^{-1})a_1ax_0 \in W$, which is a contradiction. In other words, the relation $ax_0 \in a_1^{-1}W'$ implies $v(a) > \alpha$, so we are done. \square

Proposition 13.4.16. *Suppose that v is not trivial. If M is a closed maximal subspace of a topological vector space X over K then any algebraic complement N of M is a topological complement.*

Proof. Let M be a closed maximal subspace of a topological vector space X . Since M is maximal, $\dim X/M = 1$; hence if N is an algebraic complement of M , $\dim N = 1$. Therefore there must be some $x \neq 0$ such that $N = Kx$. Consequently $X = M \oplus Kx$ and each vector y has a unique representation in the form $y = m + tx$, $m \in M$, $t \in K$. To show that Kx is a topological complement of M , we use the criterion of ??: We show that the projection P on Kx along M , $tx + m \mapsto tx$, is continuous.

To this end note that $N(P) = M$. As M is closed and $x \notin M$, there exists a neighborhood U of 0 in X such that $(x + U) \cap M = \emptyset$ and $xU \subseteq U$ whenever $v(x) \geq -\gamma_0$, for some fixed $\gamma_0 \in \Gamma$. Now if $m + tx \in U$ then we must have $v(t) > \gamma_0$: otherwise we have $v(t) \leq \gamma_0$ and so $v(t^{-1}) \geq -\gamma_0$, thus $t^{-1}(m + tx) = m/t + x \in U$, which contradicts $(x + U) \cap M = \emptyset$. Hence if $\gamma > \gamma_0$ and $m + tx \in U_\gamma(0)U$, then $t > \gamma + \gamma_0$.

To establish the continuity of P at 0, suppose that the net $m_\alpha + t_\alpha x \rightarrow 0$. As such, for any $\gamma > 0$, $m_\alpha + t_\alpha x \in U_\gamma(0)U$ eventually. Therefore $t_\alpha > \gamma + \gamma_0$ eventually. In other words, $t_\alpha \rightarrow 0$ in K , which implies that $t_\alpha x = P(m_\alpha + t_\alpha x) \rightarrow 0$, and proves the continuity of P . \square

Proposition 13.4.17. *Suppose that v is nontrivial and K is complete. Let X be a topological vector space over K , which is Hausdorff and of finite dimension n . For every basis $(e_i)_{i=1}^n$ of X over K , the map $(a_i) \mapsto \sum_{i=1}^n a_i e_i$ from K^n onto X is a topological vector space isomorphism.*

Proof. This follows from induction, using ?? and ??.

\square

Corollary 13.4.18. *Suppose that v is nontrivial and K is complete. Let X be a Hausdorff topological vector space over K and F a finite-dimensional vector subspace of X . Then F is closed.*

13.5 Completion of valued fields

Since a non-trivial absolute value v makes K into a topological field, we may consider the completion of K with respect to the additive uniform structure. The next theorem will show that every field K with a non-trivial valuation can be densely embedded into a field complete with respect to a valuation extending the given one on K .

Theorem 13.5.1. *Let K be a field, v a valuation on K and Γ the value group of v with the discrete topology.*

(a) *The completion ring \widehat{K} of K (with respect to \mathcal{T}_v) is a topological field.*

- (b) The map $v : K^\times \rightarrow \Gamma$ can be extended uniquely to a continuous map $\hat{v} : \widehat{K}^\times \rightarrow \Gamma$. The map \hat{v} is a valuation on \widehat{K} with value group Γ .
- (c) The topology on \widehat{K} is the topology defined by the valuation \hat{v} .
- (d) If $U_\gamma(x), V_\gamma(x)$ and $\widehat{U}_\gamma(x), \widehat{V}_\gamma(x)$ are the basic neighborhood of x in (K, v) and (\widehat{K}, \hat{v}) , then $\widehat{U}_\gamma(x)$ is the closure of $U_\gamma(x)$ in \widehat{K} and $\widehat{V}_\gamma(x)$ is the closure of $V_\gamma(x)$ in \widehat{K} .
- (e) The valuation ring of \hat{v} is the completion \widehat{A} of the valuation ring A of v ; the maximal ideal $\widehat{\mathfrak{m}}$ is the completion of the maximal ideal \mathfrak{m} of v .
- (f) $\widehat{A} = A + \widehat{\mathfrak{m}}$ and the residue field of \hat{v} is canonically identified with that of v .

Proof. By ??, to prove (a), it suffices to show the following: let \mathfrak{U} be a Cauchy filter (with respect to the additive uniform structure) on K^\times for which 0 is not a cluster point; then the image of \mathfrak{U} under the bijection $x \mapsto x^{-1}$ is a Cauchy filter (with respect to the additive uniform structure). For since 0 is not a cluster point of \mathfrak{U} , there exists $V \in \mathfrak{U}$ and $\gamma \in \Gamma$ such that γ is an upper bound of $v(M)$. Let $\alpha \in \Gamma$. If V' is an element of \mathfrak{U} contained in M and such that $v(x - y) > \max\{\alpha + 2\gamma, \gamma\}$ for $x, y \in V'$, then $v(x^{-1} - y^{-1}) > \gamma$ for $x, y \in V$ (see ??)). Whence (a) follows.

By ??, $v : K^\times \rightarrow \Gamma$ is a continuous homomorphism from K^\times to Γ and hence can be extended uniquely to a continuous homomorphism \hat{v} from \widehat{K}^\times to Γ . The relation $\hat{v}(x + y) \geq \min\{\hat{v}(x), \hat{v}(y)\}$ holds in K^\times and hence also holds in \widehat{K}^\times by continuity. Thus \hat{v} (extended by \widehat{K} by $\hat{v}(0) = \infty$) is a valuation on \widehat{K} and (b) is proved.

We now show (d). Let $\gamma \in \Gamma$ and $x \in \overline{U}_\alpha(0)$. For y in $U_\alpha(0)$ sufficiently close to x , $v(y) = \hat{v}(y) = \hat{v}(x)$ and hence $\hat{v}(x) > \gamma$. Conversely, let $x \in \widehat{K}^\times$ be such that $\hat{v}(x) > \alpha$; for y in K^\times sufficiently close to x , $v(y) = \hat{v}(y) = \hat{v}(x)$ and therefore $y \in U_\gamma(0)$, whence $x \in \overline{U}_\gamma(0)$. Thus $\overline{U}_\gamma(0)$ is the set of $x \in \widehat{K}$ such that $\hat{v}(x) > \gamma$. The general case for $U_\gamma(x)$ now follows from homogeneity, and the result for $V_\gamma(x)$ can be proved similarly.

Taking account of ??, assertion (c) is a consequence of (d). Assertion (e) is a special case of (d). Finally let $x \in \widehat{A}$; since \widehat{A} is the closure of A in \widehat{K} , there exists $y \in A$ such that $\hat{v}(x - y) > 0$; then $z = x - y \in \widehat{\mathfrak{m}}$ and hence $x = y + z \in A + \widehat{\mathfrak{m}}$; thus $\widehat{A} = A + \widehat{\mathfrak{m}}$ which shows (f). \square

13.5.1 Archimedean complete fields

Let K be a field complete with respect to an archimedean absolute value $|\cdot|$. Since the set $\{|n \cdot 1| : n \in \mathbb{Z}\}$ is not bounded, $\text{char } K = 0$. Thus K contains the field \mathbb{Q} of rationals. We shall first show that $|\cdot|$ restricted to \mathbb{Q} is dependent on the usual absolute value of \mathbb{Q} . Thus the complete field K contains the completion of \mathbb{Q} with respect to the ordinary absolute value, i.e., K contains \mathbb{R} as a closed subfield. We shall then show that K must be equal to \mathbb{R} or to \mathbb{C} . Consequently, every field K admitting an archimedean absolute value may be considered as a subfield of \mathbb{C} or even \mathbb{R} with the absolute value dependent on the induced one from \mathbb{C} (or from \mathbb{R}).

Proposition 13.5.2. *Every archimedean absolute value on \mathbb{Q} is dependent on the usual one.*

Proof. Let $|\cdot|$ be an archimedean absolute value on \mathbb{Q} . Denote by $|\cdot|_0$ the usual absolute value on \mathbb{Q} . Next, for integers $m, n \geq 2$ and $t \geq 1$, expand m^t in powers of n :

$$m^t = c_0 + c_1 n + \cdots + c_s n^s, \quad 0 \leq c_0, \dots, c_s < n, c_s \neq 0.$$

Since each c_i is integer, we have $|c_i| \leq c_i < n$. It follows that

$$|m|^t \leq \sum_{i=0}^s |c_i| |n|^i \leq n \sum_{i=0}^s |n|^i \leq n(s+1) \max\{1, |n|^s\}.$$

As $n^s \leq m^t$, $s \leq t(\log m)/\log n$. Thus

$$|m|^t \leq n \left(\frac{t \log m}{\log n} + 1 \right) \max\{1, |n|\}^{t(\log m/\log n)}$$

or equivalently,

$$|m| \leq n^{1/t} \left(\frac{t \log m}{\log n} + 1 \right)^{1/t} \max\{1, |n|\}^{\log m/\log n}.$$

Letting t go to infinity and taking limits, one gets

$$|m| \leq \max\{1, |n|\}^{\log m/\log n}.$$

Now, if $|n| < 1$ for some $n \in \mathbb{N}$, the above inequality implies $|m| < 1$ for every integer $m \geq 2$, contradicting the archimedeaness of $|\cdot|$. Therefore, $|n| > 1$ for all integers $n \geq 2$, and thus

$$|m| \leq |n|^{\log m/\log n}.$$

Interchanging the roles of m and n in the above inequality gives the reverse inequality. Hence

$$|m| = |m|^{\log m/\log n}$$

Therefore, if $m > n \geq 2$, then $\log m/\log n > 1$ and so $|m| > |n|$. Since $|-m| = |m|$ for all $m \in \mathbb{Z}$, it follows that $|m|_0 > |n|_0$ implies $|m| > |n|$, for non-zero $m, n \in \mathbb{Z}$. Consequently, if $m/n \in \mathbb{Q}$ satisfies $|m/n|_0 < 1$, then $|m/n| < 1$. By ??, $|\cdot|_0$ and $|\cdot|$ are dependent. \square

In case E is a field extension of K , every absolute value $|\cdot|'$ of E that restricts to $|\cdot|$ on K is a norm of E compatible with $|\cdot|$. If E has finite degree over K and K is complete with respect to $|\cdot|$, the next proposition will imply that up to equivalence of norms (and hence up to dependence as absolute values), E admits only one absolute value $|\cdot|'$ restricting to $|\cdot|$ on K . Moreover, E is complete with respect to $|\cdot|'$.

Proposition 13.5.3. *Let K be a field complete with respect to a non-trivial absolute value $|\cdot|$. Then every two norms (compatible with $|\cdot|$) of a finite dimensional K -vector space E are equivalent.*

Proof. We shall prove that any such norm on E is equivalent to the max-norm of E . This will be done by induction on the dimension n of the K -vector space E . For $n = 1$ the statement is obvious. Assume the proposition is proved for $n - 1$, $n \geq 2$. One inequality is very simple to prove. Fix a basis $\omega_1, \dots, \omega_n$ of E over K , and for $\xi = x_1\omega_1 + \dots + x_n\omega_n$, denote $\|\xi\|_{\max} = \max_i |x_i|$. Then

$$\|\xi\| \leq \sum_{i=1}^n |x_i| \|\omega_i\| \leq C \|\xi\|_{\max} \quad \text{where } C = n \max_i \|\omega_i\|.$$

We must now prove that there exists a number $C' > 0$ such that for all $\xi \in E$,

$$\|\xi\|_{\max} \leq C' \|\xi\|.$$

Suppose no such number exists. Then, for every positive integer m there exists $\xi \in E$ such that $\|\xi\|_{\max} > m \|\xi\|$. Let j be such that $|x_j| = \max_{1 \leq i \leq n} |x_i|$. Letting $\xi_m = x_j^{-1} \xi$ we get $\|\xi_m\|_{\max} = 1$ and thus $\|\xi_m\| < 1/m$. For every $m \geq 1$, one of the components of ξ_m equals 1. Thus there must be an infinite subset T of \mathbb{N} and a fixed number j such that the j -th component of ξ_m equals 1 for all $m \in T$. We fix this number j from now until the end of the proof.

Consider the subspace E_1 of E consisting of all vectors whose j -th coordinate is equal to 0, equipped with the norm induced by $\|\cdot\|$. By induction, the restrictions of $\|\cdot\|$ and max-norm $\|\cdot\|_{\max}$ to E_1 are equivalent. In particular, a sequence of elements of E_1 converges to $\zeta \in E_1$ with respect to $\|\cdot\|_{\max}$ if and only if it converges to ζ with respect to $\|\cdot\|$.

For each $m \in T$ we can write $\xi_m = \omega_j + \zeta_m$, for some $\zeta_m \in E_1$. Now, for every $\varepsilon > 0$, take $N \in \mathbb{N}$ such that $2/N < \varepsilon$. If $m, n \geq N, m, n \in T$, then

$$\|\zeta_m - \zeta_n\| = \|\zeta_m + \omega_j - \omega_j - \zeta_n\| = \|\xi_m - \xi_n\| \leq \|\xi_m\| + \|\xi_n\| < \frac{1}{m} + \frac{1}{n} \leq \frac{2}{N} < \varepsilon.$$

Consequently, (ζ_m) is a Cauchy sequence with respect to the restriction of $\|\cdot\|$ to E_1 . From the induction hypothesis, it follows that this sequence is also a Cauchy sequence with respect to the max-norm. Since E_1 is complete with respect to the max-norm, this sequence converges to some $\zeta \in E_1$ (with respect to the max-norm).

The choice of T implies $\|\omega_j + \zeta_m\| < 1/m$ for each $m \in T$. Since the restrictions of $\|\cdot\|$ and the max-norm to E_1 are equivalent,

$$\|\zeta - \zeta\| \leq C \|\zeta_m - \zeta\|_{\max}$$

for some number $C > 0$. Therefore,

$$\|\omega_j + \zeta\| \leq \|\omega_j + \zeta_m\| + \|\zeta - \zeta_m\| \leq \frac{1}{m} + C \|\zeta - \zeta_m\|_{\max}.$$

Letting $m \in T$ go to infinity, the right-hand side of the preceding inequality tends to 0. Hence $\omega_j + \zeta = 0$. But, this cannot occur, because $\zeta \in E_1$ has the j -th coordinate equal to 0 and $\omega_1, \dots, \omega_n$ is a basis of E over K . This contradiction finishes the proof of the proposition. \square

Theorem 13.5.4. *Let K be a field containing \mathbb{R} and having an absolute value that induces the ordinary one on \mathbb{R} . Then $K = \mathbb{R}$ or $K = \mathbb{C}$.*

In particular, the only fields complete with respect to an archimedean absolute value $|\cdot|$ are (up to isomorphism) \mathbb{R} and \mathbb{C} with $|\cdot|$ dependent on the ordinary absolute value.

This theorem is a consequence of the following proposition:

Proposition 13.5.5 (Gelfand-Mazur). *Let A be a commutative normed algebra with identity and assume that A contains an element j such that $j^2 = -1$, and let $\mathbb{C} = \mathbb{R} + j\mathbb{R} \subseteq A$ (identify $\mathbb{R} \cdot 1$ with \mathbb{R}). Then for every nonzero element $x_0 \in A$, there exists an element $c \in \mathbb{C}$ such that $x_0 - c$ is not invertible in A .*

Proof. Suppose that $x_0 - z$ is invertible for all $z \in \mathbb{C}$. The map $f : \mathbb{C} \rightarrow A$ defined by $f(z) = (x_0 - z)^{-1}$, is then well defined. Moreover, we shall see that taking inverses is a continuous operation on the group of units of A , from which it will follow that f is continuous. In order to show that $x \mapsto x^{-1}$ is continuous on the group of units of A , note that for any units a and x in A ,

$$\|x^{-1} - a^{-1}\| = \|(a - x)a^{-1}x^{-1}\| \leq \|x - a\| \|a^{-1}\| \|x^{-1}\|.$$

Thus it remains to show that $\|x^{-1}\|$ is bounded as x varies through units near a . Let $\|a^{-1}\| \|x - a\| \leq 1/2$, and set $w = a^{-1}(x - a)$. Then clearly $\|w\| \leq 1/2$. Hence we get

$$\left\| \frac{1}{1+w} \right\| = \left\| 1 - \frac{w}{1+w} \right\| \leq 1 + \frac{\|w\|}{1+\|w\|} \leq 1 + \frac{1}{2} \left\| \frac{1}{1+w} \right\|.$$

This implies $\|(1+w)^{-1}\| \leq 1/2$. Thus finally we get

$$\|x^{-1}\| = \|a^{-1}(1+w)^{-1}\| \leq 2\|a^{-1}\|.$$

Back to the proof of the proposition, observe that for $0 \neq z \in \mathbb{C}$,

$$f(z) = \frac{1}{x_0 - z} = \frac{1}{z} \left(\frac{1}{x_0 z^{-1} - 1} \right).$$

Since z^{-1} and $x_0 z^{-1}$ approach 0 when z goes to infinity in \mathbb{C} , it follows that $f(z) \rightarrow 0$ when $z \rightarrow \infty$. On the other hand, the map $z \mapsto \|f(z)\|$ is continuous, being the composition of two continuous maps. Consequently, $\|f(z)\| \rightarrow 0$ when $z \rightarrow \infty$. Hence this map may be considered as a real valued continuous map on the one-point compactification $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ of \mathbb{C} . Hence $\|f\|$ has a maximum $M \in \mathbb{R}$, $M > 0$. Let D be the set of elements $z \in \mathbb{C}$ such that $\|f(z)\| = M$. Then D is a non-empty, bounded, and closed subset of \mathbb{C} . We shall prove that it is also open, a contradiction.

For $z_0 \in D$, let us write $y = x_0 - z_0$ and let t be a nonzero complex number to be determined later. Consider the polynomial $h(X) = X^n - t^n = \prod_{j=1}^n (X - \zeta_n^j t)$, whence

$$\frac{h'(X)}{h(X)} = \frac{nX^{n-1}}{X^n - t^n} = \sum_{j=1}^{n-1} \frac{1}{X - \zeta_n^j t} + \frac{1}{X - t}.$$

Dividing by n and replacing every occurrence of X by y , taking account of the definitions of f and y , we obtain

$$f(z_0 + t) + \sum_{j=1}^{n-1} f(z_0 + \zeta_n^j t) - nf(z_0) = \frac{ny^{n-1}}{y^n - t^n} - \frac{n}{y} = \frac{n}{y} \cdot \frac{t^n y^{-n}}{1 - t^n y^{-n}}. \quad (13.5.1)$$

If we choose t such that $\|ty^{-1}\| < 1$, then the last expression in (??) tends to 0 as n tends to ∞ ; hence

$$\|f(z_0 + t)\| = \lim_{n \rightarrow \infty} \|nf(z_0) - \sum_{j=1}^{n-1} f(z_0 + \omega_n^j t)\|. \quad (13.5.2)$$

Now, $\|f(z_0)\| = M$ and $\|f(z_0 + \omega_n^j t)\| \leq M$ by definition of M , whence

$$\|nf(z_0) - \sum_{j=1}^{n-1} f(z_0 + \omega_n^j t)\| \geq n\|f(z_0)\| - \sum_{j=1}^{n-1} \|f(z_0 + \omega_n^j t)\| \geq nM - (n-1)M = M.$$

Therefore by (??), letting n tend to ∞ , $\|f(z_0 + t)\| \geq M$ and by definition of M this implies $\|f(z_0 + t)\| = M$; in other words $z_0 + t \in D$. This proves that the set D is open in \mathbb{C} ; as it is also closed and non-empty and \mathbb{C} is connected, $D = \mathbb{C}$ and f is therefore constant on \mathbb{C} ; as this function tends to 0 at the point at infinity, $\|f(z)\| = 0$ in \mathbb{C} and in particular $\|f(0)\| = \|x^{-1}\| = 0$, which is absurd. \square

Proof of ??. We apply ?? to the \mathbb{R} -algebra K , and let the given absolute value serve as norm. If K contains \mathbb{C} , then $K = \mathbb{C}$, because every element of K is invertible. If K does not contain \mathbb{C} , let $L = K(j)$, where $j^2 = -1$. Define a norm on L by putting $\|x + yj\| = \|x\| + \|y\|$ for $x, y \in K$. This clearly makes L a normed \mathbb{R} -algebra. Moreover, by standard calculations one proves that $\|1\| = 1$ and $\|zw\| \leq \|z\|\|w\|$. Now, applying ?? to $A = L$, we obtain $L = \mathbb{C}$ as before. Thus K must be \mathbb{R} . \square

13.5.2 Non-archimedean complete fields

As we have already noticed, non-archimedean absolute values corresponds to rank 1 valuations. Therefore, for non-archimedean absolute values, we will use this additive notation and refer non-archimedean absolute values to valuations of rank 1.

Now we prove an important property of a field K complete with respect to a (non-trivial) valuation v . This theorem is widely known as Hensel's Lemma.

Theorem 13.5.6 (Hensel's Lemma). *Let K be a field complete with respect to a non-archimedean absolute value v . Let $f \in A_v[X]$ be a polynomial, and let $a_0 \in A_v$ be such that $v(f(a_0)) > 2v(f'(a_0))$. Then there exists some $a \in A_v$ with $f(a) = 0$ and $v(a_0 - a) > v(f'(a_0))$.*

Proof. The natural way to reach the conclusions of the theorem is to construct a suitable Cauchy sequence which must converge to a root a of f (recall that every polynomial f is a continuous map).

We use the Newton approximation method: define a sequence (a_n) by

$$a_{n+1} = a_n - \frac{f(a_0)}{f'(a_0)}.$$

We will show that (a_n) has a limit a and this element a satisfies the requirements. In fact, set $r = v(f(a_0)) - 2v(f'(a_0)) > 0$, we will prove by induction that

- (a) $v(a_n) \geq 0$.
- (b) $v(a_{n+1} - a_n) = v(f(a_n)/f'(a_n)) \geq v(f(a_n)/f'(a_n)^2) \geq 2^n r$.
- (c) $v(a_n - a_0) \geq v(f(a_0)/f'(a_0))$.
- (d) $v(f'(a_n)) = v(f'(a_0))$.

By hypothesis, (a), (b), and (c) holds for $n = 0$. Now assume these conditions for n , then we can see (a) holds for $n + 1$. Since a_n and $f(a_n)/f'(a_n)$ are in A_v , by Taylor expansion we have

$$\begin{aligned} v(f(a_{n+1})) &= v\left(f\left(a_n - \frac{f(a_n)}{f'(a_n)}\right)\right) = v\left(f(a_n) - f'(a_n) \cdot \frac{f(a_n)}{f'(a_n)} + M\left(\frac{f(a_n)}{f'(a_n)}\right)^2\right) \\ &= v\left(M\left(\frac{f(a_n)}{f'(a_n)}\right)^2\right) \geq v\left(\left(\frac{f(a_n)}{f'(a_n)}\right)^2\right) \end{aligned} \quad (13.5.3)$$

where $M \in A_v$. Similarly,

$$\frac{f'(a_{n+1})}{f'(a_n)} = \frac{1}{f'(a_n)} \left(f'(a_n) + M' \frac{f(a_n)}{f'(a_n)} \right) = 1 + M' \frac{f(a_n)}{f'(a_n)^2}$$

for some $M' \in A_v$, whence

$$v\left(\frac{f'(a_{n+1})}{f'(a_n)}\right) = 0 \quad (13.5.4)$$

and $v(f'(a_{n+1})) = v(f'(a_n)) \neq \infty$. Now by (??) and (??),

$$\begin{aligned} v\left(\frac{f(a_{n+1})}{f'(a_{n+1})^2}\right) &\geq v\left(\left(\frac{f(a_n)}{f'(a_n)}\right)^2\right) - v(f'(a_{n+1})^2) = v\left(\left(\frac{f(a_n)}{f'(a_n)}\right)^2\right) - v(f'(a_n)^2) \\ &= 2v\left(\frac{f(a_n)}{f'(a_n)^2}\right) \geq 2^{n+1}r. \end{aligned} \quad (13.5.5)$$

This proves condition (b) for $n + 1$. Moreover, by (??),

$$v\left(\frac{f(a_{n+1})}{f'(a_{n+1})^2}\right) \geq v\left(\frac{f(a_n)}{f'(a_n)^2}\right).$$

From (??), this implies

$$v\left(\frac{f(a_{n+1})}{f'(a_{n+1})}\right) \geq v\left(\frac{f(a_n)}{f'(a_n)}\right) \geq \dots \geq v\left(\frac{f(a_0)}{f'(a_0)}\right)$$

which is to say

$$v(a_{n+1} - a_n) \geq v(a_n - a_{n-1}) \geq \dots \geq v(a_1 - a_0),$$

and therefore

$$v(a_n - a_0) = v\left(\sum_{i=1}^n (a_i - a_{i-1})\right) = v(a_1 - a_0) = v\left(\frac{f(a_0)}{f'(a_0)}\right).$$

This proves (c), and completes the induction process.

From (a) and (b) we see (a_n) is a Cauchy sequence in K , so admits a limit $a \in K$. Also, since the valuation is continuous, we have $v(a) \geq 0$, whence $a \in A_v$. Taking limit in the recursion formula, we see $f(a) = 0$. Also, by (c), we have $v(a - a_0) \geq v(f(a_0)/f'(a_0)) > v(f'(a_0))$. \square

Corollary 13.5.7. *Let K, v be as in ???. If $f \in A_v[X]$ has a simple zero a_0 in the residue field κ_v , i.e., $\bar{f}(\bar{a}_0) = 0$ and $\bar{f}'(\bar{a}_0) \neq 0$, then f has a zero $a \in A_v$ such that $\bar{a} = \bar{a}_0$.*

The proof of ?? clearly shows that under the assumption of ??, the sequence $(f(a_n))$ converges to 0. At this point, essential use is made of the fact that $v(K^\times)$ is a subgroup of the additive reals. Thus even without the completeness of K , we could still notice:

Proposition 13.5.8. *Let v be a non-archimedean absolute value on K . Then for every $f \in A_v[X]$, if \bar{f} has a simple zero in κ_v , then $f(K)$ approximates 0.*

Now consider the completion (\widehat{K}, \hat{v}) of the field with respect to a non-archimedean absolute value v , but now using the additive notation for absolute values. The density of K in \widehat{K} has the following important consequence.

Theorem 13.5.9. *Denote by \widehat{A}_v , $\kappa_{\hat{v}}$ and A_v , κ_v the valuation ring and the residue field of \hat{v} and v , respectively. Then the residue fields κ_v and $\kappa_{\hat{v}}$, as well as the groups Γ_v and $\Gamma_{\hat{v}}$, are canonically isomorphic.*

Proof. It follows from the constructions that $A_{\hat{v}} \cap K = A_v$ and $\mathfrak{m}_{\hat{v}} \cap K = \mathfrak{m}_v$, where $\mathfrak{m}_{\hat{v}}$ and \mathfrak{m}_v are the respective maximal ideals. Thus the map that sends the residue class of $a \in A_v$ to the residue class $\bar{a} \in A_{\hat{v}}/\mathfrak{m}_{\hat{v}}$ is well defined; and it is clearly a ring homomorphism. It remains to be seen that it is surjective. For every $x \in A_{\hat{v}}$, the set $x + \mathfrak{m}_{\hat{v}}$ is an open neighbourhood of x . It consists of all elements z such that $\hat{v}(z - x) > 0$, or $|z - x| < 1$ in terms of the absolute value. Thus the set $(x + \mathfrak{m}_v) \cap K$ is non-empty, by the density property. Hence the residue class of $y \in (x + \mathfrak{m}_v) \cap K$ is sent by the map above to \bar{x} , as required.

Similarly, the map $\Gamma_v \rightarrow \Gamma_{\hat{v}}$ sending $v(x)$ to $\hat{v}(x)$ for every $x \in K^\times$ is an order-preserving group monomorphism. In order to show surjectivity, let $x \in \widehat{K}^\times$ be given. By the density of K in \widehat{K} there exists $z \in K$ with $\hat{v}(z - x) > v(x)$, or $|z - x|_{\widehat{K}} < |x|$ in terms of the absolute value. But then $\hat{v}(z) = v(x)$. \square

Example 13.5.10. Now consider the p -adic valuation on \mathbb{Q} ; the completion of (\mathbb{Q}, v_p) is denoted by \mathbb{Q}_p , and is called the field of **p -adic numbers**. The valuation ring of \mathbb{Q}_p , denoted by \mathbb{Z}_p , is the ring of p -adic integers; it is the topological closure of \mathbb{Z} in \mathbb{Q}_p , as we shall see below. According to our previous discussion of this example, ?? implies that the residue class field of \mathbb{Z}_p is \mathbb{F}_p . Observe also that p is a local parameter for v_p in both fields, \mathbb{Q} and \mathbb{Q}_p .

Example 13.5.11. If we fix X as the irreducible polynomial p , then the completion of $(k(X), v_X)$ is the field $k((X))$ of formal Laurent series over k , and the valuation ring is $k[[X]]$, the ring of formal power series.

The last two examples above are special cases of the following more general result:

Proposition 13.5.12. *Let v be a discrete absolute value on the field K , with uniformizer π . Then every element $x \in K^\times$ can be written uniquely as a convergent series*

$$x = \sum_{i=v}^{\infty} r_i \pi^i$$

where $v = v(x)$, $r_v \neq 0$, and the coefficients r_i are taken from a set $R \subseteq A_v$ of representatives of the residue classes in the field κ_v (i.e., the canonical map $A_v \rightarrow \kappa_v$ induces a bijection of R onto κ_v). In particular, the valuation ring of \widehat{K} is a complete discrete valuation ring with uniformizer π , and we have an isomorphism of topological rings $\widehat{A} \cong \varprojlim A / \pi^n A$.

Proof. We proceed by induction. As observed above, $u = x\pi^{-v}$ is a unit in A_v . Choose $r_v \in R$ such that $\bar{r}_v = \bar{u}$. Then clearly $v(x\pi^{-v} - r_v) > 0$ or, equivalently,

$$v(x - r_v \pi^v) > v(\pi^v) = v.$$

Let $x_1 = x - r_\nu \pi^\nu$ and $\nu_1 = v(x_1) > \nu$. Then by the same argument we get $r_{\nu_1} \in R$ such that

$$v(x - (r_\nu \pi^\nu + r_{\nu_1} \pi^{\mu_1})) = v(x_1 - r_{\mu_1} \pi^{\mu_1}) > \nu_1.$$

Repeating this argument and adding "zero coefficients" (i.e. a representative for zero in κ_v) if necessary, we obtain the existence of the "series"

$$r_\nu \pi^\nu + r_{\nu+1} \pi^{\nu+1} + \dots$$

At the same time we see that it converges to x .

The uniqueness of the coefficients is clear. Indeed, otherwise 0 would have a representation

$$0 = (r_\nu - r'_\nu) \pi^\nu + (r_{\nu+1} - r'_{\nu+1}) \pi^{\nu+1} + \dots$$

with $r_\nu \neq r'_\nu \in R$, and hence $r_\nu - r'_\nu \neq 0$ in κ_v . But $v(0) = \infty$, a contradiction.

Let \widehat{A} be the valuation ring of $\widehat{\mathcal{O}}$. Then \widehat{A} is complete (it is closed in \widehat{K}) and contains A . For each $n \geq 1$ we define a ring homomorphism $\phi_n : \widehat{A} \rightarrow A/\pi^n A$ as follows: for each $x = \sum_{i=0}^{\infty} r_i \pi^i$ let $\phi_n(x)$ be the n -th truncation $\sum_{i=0}^{n-1} r_i \pi^i$. We thus obtain an infinite sequence of surjective maps $\phi_n : \widehat{A} \rightarrow A/\pi^n A$ that are compatible in that for all $n \geq m > 0$ and all $x \in \widehat{A}$ the image of $\phi_n(x)$ in $A/\pi^m A$ is $\phi_m(x)$. This defines a surjective ring homomorphism $\phi : \widehat{A} \rightarrow \varprojlim A/\pi^n A$. Now note that $\ker \phi = \bigcap_n \pi^n \widehat{A} = \{0\}$ so ϕ is injective and therefore an isomorphism.

To show that ϕ is also a homeomorphism, it suffices to note that if $x + \pi^m A$ is a coset of $\pi^m A$ in A and U is the corresponding open set in $\varprojlim A/\pi^n A$, then $\phi^{-1}(U)$ is the closure of $x + \pi^m A$ in \widehat{A} , which is the coset $x + \pi^m \widehat{A}$, an open subset in \widehat{A} (as explained in the discussion above, every open set in the inverse limit corresponds to a finite union of cosets $x + \pi^m A$ for some m). Conversely ϕ maps open sets $x + \pi^m \widehat{A}$ to open sets in $\varprojlim A/\pi^n A$. \square

Example 13.5.13. Returning once more to our typical examples, any p -adic number $z \in \mathbb{Q}_p^\times$ has a unique representation in the form

$$z = \sum_{i=\nu}^{\infty} a_i p^i$$

where $\nu = v_p(z)$, $0 \leq a_i < p$ for every i , and $a_\nu \neq 0$. The set of representatives chosen here is then $R = \{0, \dots, p-1\}$. If $z \in \mathbb{Z}_p$, i.e., if $v(z) \geq 0$, then $z = \sum_{i=0}^{\infty} a_i p^i$. This shows, in particular, that \mathbb{Z} is dense in \mathbb{Z}_p . The reader should be aware of the fact that addition of two "series" of the form $\sum_{i=\nu}^{\infty} a_i p^i$ is not coefficient-wise, as the set R is not closed under addition. As a simple example observe that (choosing $p = 7$)

$$5p^i + 4p^i = p^{i+1} + 2p^i$$

Example 13.5.14. For the X -adic valuation of $k(X)$, we can take the elements of k itself as representatives of the residue field. In this case every $z \in k(X)^\times$ has a unique representation in the form

$$z = \sum_{i=\nu}^{\infty} a_i X^i$$

where $v_X(z) = \nu \in \mathbb{Z}$ and $a_i \in k$ for every i . This time, addition of two such series is coefficient-wise. These series are called formal Laurent series. They form a field $k((X))$, the field of formal Laurent series. The canonical discrete absolute value on $k((X))$ is just given by

$$v\left(\sum_{i=\nu}^{\infty} a_i X^i\right) = \nu \quad \text{if } \nu \neq 0.$$

Clearly its valuation ring consists of the ring $k[[X]]$ of formal power series, i.e., series of the type $\sum_{i=0}^{\infty} a_i X^i$.

13.6 Extensions of valuation rings

In this part we consider valuation rings in different fields. In particular, we will see how different valuation rings relate in a field extension, and we will prove a fundamental inequality in this situation.

13.6.1 Ramification index and inertial degree

Let $K \subseteq L$ be a field extension and w a valuation of L . Then it is easy to see $v = w|_K$ is a valuation of K and for maximal ideals we have

$$\mathfrak{m}_v = \mathfrak{m}_w \cap K.$$

That is, A_v is dominated by A_w . Conversely, if v is a valuation of K , then by ?? we can find a valuation ring of L dominating A_v . If w is the valuation of this ring, then the valuation v is given by the restriction of w on K and for value groups we have $\Gamma_v \subseteq \Gamma_w$. Also, since A_w dominates A_v , we also have an extension of residue fields : $\kappa_v \subseteq \kappa_w$. Now we make the following definition.

Definition 13.6.1. *The index $[\Gamma_w : \Gamma_v]$ is called the **ramification index** of w over v and denoted by $e(w/v)$. If this index equals 1 then w is called **unramified** over v .*

Definition 13.6.2. *The index $[\kappa_w : \kappa_v]$ is called the **inertial degree** of w over v and denoted by $f(w/v)$.*

In particular, if $e(w/v) = 1$ and $f(w/v) = 1$, the extension w/v is called **immediate**. We follow the usual convention that both e and f can be either finite or infinite, without distinguishing between different infinite cardinalities. With this convention, we have the following transitivity due to that of group index and field degree.

Proposition 13.6.3. *Let $K \subseteq L \subseteq L'$ be field extension, w' an extension of L' and w, v the restriction of w' to L' and L . Then*

$$e(w'/v) = e(w'/w)e(w/v), \quad f(w'/v) = f(w'/w)f(w/v).$$

Now, before we prove anything about the quantities e and f , we first provide some examples.

Example 13.6.4. Let (K, v) be a valued field with value group Γ_v and Γ a totally ordered group with $\Gamma_v \subseteq \Gamma$. Consider some t transcendental over K and any $\gamma \in \Gamma$. Define a valuation w on

$K(t)$ with values in Γ via

$$w\left(\sum_{i=0}^n a_i t^i\right) = \min\{v(a_i) + i\gamma : 0 \leq i \leq n\}$$

for $f = \sum_{i=0}^n a_i t^i \in K[t]$ and extend w to all of $K(t)$. Then w is a well-defined valuation extending v with value group $\Gamma_w = \Gamma_v + \gamma\mathbb{Z}$.

The following inequality is fundamental for extension of valuations.

Proposition 13.6.5. *Let K be a field, L an extension of K of degree n , w a valuation on L and v its restriction to K . Choose $x_1, \dots, x_e \in L^\times$ and $y_1, \dots, y_f \in A_w$ such that*

- (a) *the values $w(x_1), \dots, w(x_e)$ are representatives of the distinct cosets of Γ_w/Γ_v .*
- (b) *the residues $\bar{y}_1, \dots, \bar{y}_f \in \kappa_w$ are linearly independent over κ_v ;*

Then for all $a_{ij} \in K$, we have

$$w\left(\sum_{ij} a_{ij} x_i y_j\right) = \min_{ij} \{w(a_{ij} x_i y_j)\}$$

In particular, the products $x_i y_j$ are linearly independent over K .

Proof. Let $a_{ij} \in K$, not all zero, and $I \in \{1, \dots, e\}$ and $J \in \{1, \dots, f\}$ such that

$$w(a_{IJ} x_I) = \min_{ij} \{w(a_{ij} x_i)\}.$$

and observe first that $w(a_{IJ} x_I) < w(a_{ij} x_i)$ for all $i \neq I$ since otherwise

$$w(x_I) - w(x_i) = w(a_{ij}) - w(a_{IJ}) = v(a_{ij}/a_{IJ}) \in \Gamma_v,$$

which contradicts our assumption on x .

Next, write $z = \sum_{ij} a_{ij} x_i y_j$. Since $w(y_j) \geq 0$ for each j , for the sake of obtaining a contradiction, we assume that $w(z) > w(a_{IJ} x_I)$. Then $z(a_{IJ} x_I)^{-1} \in \mathfrak{m}_w$. Also, according to the previous paragraph we have $a_{ij} x_i (a_{IJ} x_I)^{-1} \in \mathfrak{m}_w$ for all $i \neq I$. Dividing everything by $a_{IJ} x_I$ one gets

$$\sum_j \frac{a_{Ij}}{a_{IJ}} y_j = \frac{z}{a_{IJ} x_I} - \sum_{j=1}^f \sum_{i \neq I} \frac{a_{ij} x_i}{a_{IJ} x_I} y_j \in \mathfrak{m}_w$$

which gives a relation $\sum_j \bar{a}_{Ij} (\bar{a}_{IJ})^{-1} \bar{y}_j = 0$, this contradicts the hypothesis made on y_j . \square

Corollary 13.6.6. *Let K be a field, L a finite extension of K of degree n , w a valuation on L and v its restriction to K . Then the $e(w/v)f(w/v) \leq n$. In particular, $e(w/v)$ and $f(w/v)$ are finite.*

Theorem 13.6.7. *Let K be a field, L an algebraic extension of K , w a valuation on L , v its restriction to K . Then Γ_w/Γ_v is torsion and κ_w/κ_v is algebraic.*

Proof. Pick $x \in L$ such that $w(x) = \gamma \in \Gamma_w$. If $L' = K(x)$ and v' is the restriction of w on L' then by ??, since L'/K is finite degree, the group $\Gamma_{v'}/\Gamma_v$ is finite, whence torsion. It then follows that Γ_w/Γ_v is torsion.

Similarly, for $x \in A_w$ we take L' and v' as above. It follows from ?? that the residue class field $\kappa_{v'}$ is a finite extension of κ_v . Therefore $\bar{x} \in \kappa_{v'} \subseteq \kappa_w$ is algebraic over κ_v . \square

Corollary 13.6.8. *Let K be a field, L an algebraic extension of K , w a valuation on L , v its restriction to K . Then the map $\Delta \mapsto \Delta \cap \Gamma_v$ is a one-to-one correspondence between isolated subgroups of Γ_w and Γ_v . In particular, $\text{rank}(w) = \text{rank}(v)$.*

Proof. It is clear that $\Delta \cap \Gamma_v$ is an isolated subgroup, if Δ is. Now let $\Delta \subseteq \Gamma_v$ be an isolated subgroup and Δ' denote the set of $\alpha \in \Gamma_w$ such that there exists $\beta \in \Delta$ satisfying $-\beta \leq \alpha \leq \beta$; it is immediately verified that Δ' is an isolated subgroup of Γ_v with $\Delta' \cap \Delta = \Delta$ (since Δ is isolated); hence the map $\Delta \mapsto \Delta \cap \Gamma_v$ is surjective. Finally, let Δ_1 and Δ_2 be two isolated subgroups of Γ_w such that $\Delta := \Delta_1 \cap \Gamma_v = \Delta_2 \cap \Gamma_v$ and assume, for example, $\Delta_1 \subseteq \Delta_2$; then Δ_2/Δ_1 is a totally ordered group and is isomorphic to a quotient group of Δ_2/Δ which itself is identified with a subgroup of Γ_w/Γ_v . Hence Δ_2/Δ_1 is a torsion group and therefore reduces to 0. \square

Corollary 13.6.9. *Suppose that L is a finite extension of K . For w to be discrete, it is necessary and sufficient that v be discrete.*

Proof. If w is discrete, then Γ_v is isomorphic to a non-zero subgroup of \mathbb{Z} and hence to \mathbb{Z} . Conversely, if v is discrete, Γ_v is isomorphic to \mathbb{Z} and Γ_w/Γ_v is a finite group. Hence Γ_w is a finitely generated commutative group of rank 1 and torsion-free. Consequently it is isomorphic to \mathbb{Z} . \square

13.6.2 Prolongations of a valuation

Given a valuation ring $A \subseteq K$, and an extension L/K , any valuation ring B of L dominating A is called a **prolongation** of A to L . Equivalently, if v is the valuation corresponding to K , then prolongations of A corresponds to extensions of v to L . In this part, we consider the connection bewteen these prolongations.

The following lemma will be used throughout this part, and it states that two distinct prolongations of A to an algebraic extension L/K are incomparable.

Lemma 13.6.10. *Suppose L/K is an algebraic extension of fields, A is a valuation ring of K , and B_1, B_2 are two prolongations of A to L . If $B_1 \subseteq B_2$, then $B_1 = B_2$.*

Proof. The valuation ring B_1 maps to a valuation ring $\bar{B}_1 = B_1/\mathfrak{m}_{B_2}$ of the residue field $\kappa_{B_2} = B_2/\mathfrak{m}_{B_2}$. Since B_1 is an extension of A , it follows that \bar{B}_1 is also an extension of κ_v . According to ??, κ_{B_2} is an algebraic extension of κ_v , so it follows that \bar{B}_1 is also a field. But \bar{B}_1 is a valuation ring of κ_{B_2} , so it follows that $\bar{B}_1 = \kappa_{B_2}$ whence $B_1 = B_2$. \square

There may exist infinitely many valuation rings of L lying over A , of course. But sometimes, their number has a natural bound. This is the content of the next result.

Theorem 13.6.11. *Let L be algebraic over K and $[L : K]_s < \infty$. Let A be a valuation ring of K . Then the number n of all prolongations of A to L is finite, and $n \leq [L : K]_s$.*

Proof. Let B_1, \dots, B_n be the distinct prolongations of A to L , with maximal ideals $\mathfrak{m}_1, \dots, \mathfrak{m}_n$, respectively. Also, by ?? these prolongations are pairwise incomparable. Therefore, by ?? there exist $x_1, \dots, x_n \in L$ such that for all $i, j \in \{1, \dots, n\}$,

$$x_i - 1 \in \mathfrak{m}_i, \quad x_i \in \mathfrak{m}_j \text{ for } i \neq j.$$

If K has characteristic $p > 0$, pick k large enough to guarantee that $x_i^{p^k} \in K^s$, the separable closure of K in L . Then we claim that these n elements $x_i^{p^k}$ are linearly independent over K . Consequently, $n \leq [K^s : K] = [L : K]_s$, and the result follows.

We shall prove the claim by contradiction. For $a_1, \dots, a_n \in K$, not all zero, such that $\sum_{i=1}^n a_i x_i^{p^k} = 0$, pick j such that

$$v(a_j) = \min\{v(a_1), \dots, v(a_n)\}.$$

Then $a_j \neq 0$ and we have

$$x_j^{p^k} = -\sum_{i \neq j} a_i x_i^{p^k} \in \mathfrak{m}_j.$$

Since this would imply $x_j \in \mathfrak{m}_j$ and hence $1 \in \mathfrak{m}_j$, we get the desired contradiction. \square

Corollary 13.6.12. *If the extension L/K is purely inseparable, then every valuation ring of K has a unique prolongation to L .*

With ?? in hand, we now classifies all prolongations of a given valuation ring to L : they corresponds to maximal ideals of the integral closure of A in L .

Theorem 13.6.13. *Let L be an algebraic extension of a field K , A be a valuation ring of K , and R the integral closure of A in L .*

- (a) *If B is a prolongation of A to L then $\mathfrak{m} = \mathfrak{m}_B \cap R$ is a maximal ideal of R .*
- (b) *If \mathfrak{m} is a maximal ideal of R then $R_\mathfrak{m}$ is a prolongation of A to L .*
- (c) *The map $B \mapsto \mathfrak{m}_B \cap R$ gives a one-to-one correspondence between maximal ideals of R and prolongations of A to L , and its inverse is given by $\mathfrak{m} \mapsto R_\mathfrak{m}$.*

Proof. Note that any prolongation B of A to L must contains R , by ???. Since \mathfrak{m}_B is maximal in B , \mathfrak{m} is also maximal in R . Conversely, since a contraction of maximal ideal is maximal, we see $R_\mathfrak{m}$ is a prolongation of A for every maximal ideal \mathfrak{m} in R .

It remains to prove that, for a prolongation B of A to L we have $B = R_\mathfrak{m}$, where $\mathfrak{m} = \mathfrak{m}_B \cap R$. The inclusion $R_\mathfrak{m} \subseteq B$ is clear. We first consider the case where the extension L/K is finite. In this case, there are finitely many prolongations of A to L by ??, say B_1, \dots, B_n with maximal ideals $\mathfrak{m}_1, \dots, \mathfrak{m}_n$. Then by ?? we have $B_i = R_{\mathfrak{m}_i}$, where $\mathfrak{m}_i = \mathfrak{m}_{B_i} \cap R$, whence the claim holds in this case.

Now consider the general case. Let L' be a subfield of L such that L'/K is finite. Then we see $R' = R \cap L'$ is the integral closure of A in L' and $B' = B \cap L'$ is a prolongation of A to L' with maximal ideal $\mathfrak{m}_{B'} = \mathfrak{m}_B \cap L'$. By the argument above, we get $B' = R'_{\mathfrak{m}'}$, where $\mathfrak{m}' = \mathfrak{m} \cap L' = \mathfrak{m}_{B'} \cap R'$. Since the subfield L' is arbitrary, it follows that $B = R_\mathfrak{m}$. \square

Next we shall consider the set of all prolongations of a fixed valuation ring of a field K to normal extensions of K . It turns out that the automorphism group $\text{Aut}(L/K)$ plays an important role.

Theorem 13.6.14. *Suppose L/K is a normal extension of fields, with $G = \text{Aut}(L/K)$. Suppose A is a valuation ring of K , and B_1 and B_2 are valuation rings in L extending A . Then B_1 and B_2 are conjugate over K , i.e., there exists $\sigma \in G$ with $\sigma(B_1) = B_2$.*

Proof. First, split the extension L/K into the steps $K \subseteq K^s \subseteq L$. ?? implies that every extension of A to K^s has just one prolongation to L . Furthermore, $\text{Aut}(K^s/K)$ and G can be canonically identified. Therefore, we see that it is enough to consider the case where L is separable over K .

We may assume $[L : K] < \infty$, and the general case follows by considering the compactness of Galois groups. In this case let

$$H_1 = \{\sigma \in G : \sigma(B_1) = B_1\}, \quad H_2 = \{\tau \in G : \tau(B_2) = B_2\}.$$

Then H_1 and H_2 are subgroups of G . Moreover, for every $\sigma \in H_1$, it follows that $\sigma(\mathfrak{m}_1) = \mathfrak{m}_1$, for the maximal ideal of B_1 . Indeed, it is enough to observe that $\sigma(\mathfrak{m}_1)$ must be the maximal ideal of $\sigma(B_1)$. Analogously for the maximal ideal \mathfrak{m}_2 of B_2 , it follows that $\tau(\mathfrak{m}_2) = \mathfrak{m}_2$ for all $\tau \in H_2$. Next write G as disjoint unions of cosets of H_1 and H_2 , respectively:

$$G = \bigcup_{i=1}^n H_1 \sigma_i^{-1}, \quad G = \bigcup_{k=1}^m H_2 \tau_k^{-1}.$$

Suppose now, for the sake of contradiction, that $\sigma_i(B_1) \not\subseteq \tau_k(B_2)$ and $\tau_k(B_2) \not\subseteq B_1$ for all i, k . Since $\sigma_1^{-1}, \dots, \sigma_n^{-1}$ is a complete set of representatives of cosets of H_1 , for all $i \neq j$, $\sigma_i(B_1) \not\subseteq \sigma_j(B_1)$. Similarly, $\tau_k(H_2) \not\subseteq \tau_l(H_2)$ for every $k \neq l$. Now take

$$C = \bigcap_{i=1}^n \sigma_i(H_1) \cap \bigcap_{k=1}^m \tau_k(H_2).$$

According to ??, there exists an $x \in C$ such that, for each i, k ,

$$x - 1 \in \sigma_i(\mathfrak{m}_1), \quad x \in \tau_k(\mathfrak{m}_2)$$

As a consequence, for $\sigma \in G$, writing $\sigma = \rho \sigma_i^{-1}$ and $\rho \in H_1$, it follows that

$$\sigma(x - 1) \in \rho \sigma_i^{-1}(\sigma_i(\mathfrak{m}_1)) = \rho(\mathfrak{m}_1) = \mathfrak{m}_1.$$

Analogously, $\sigma(x) \in \mathfrak{m}_2$ for every $\sigma \in G$. Taking norms, it then follows

$$N_{L/K}(a) = \prod_{\sigma \in G} \sigma(a) \in (\mathfrak{m}_1 + 1) \cap K = \mathfrak{m} + 1$$

and

$$N_{L/K}(a) = \prod_{\sigma \in G} \sigma(a) \in \mathfrak{m}_2 \cap K = \mathfrak{m}.$$

This contradiction implies $\sigma_i(\mathfrak{m}_1) \subseteq \tau_j(\mathfrak{m}_2)$ or $\tau_j(\mathfrak{m}_2) \subseteq \sigma_i(\mathfrak{m}_1)$. Thus the claim holds in this case by ??.

□

Now, since we now all prolongations of A are conjugate under $\text{Aut}(L/K)$, we are ready to prove the following theorem.

Theorem 13.6.15. *Let L be a normal extension of a field K , A a valuation ring of K , and B a prolongation of A to L . Let v and w be valuations corresponding to A and B .*

- (a) *For $\sigma \in \text{Aut}(L/K)$, the map $w \circ \sigma$ is the unique valuation of L that corresponds to $\sigma^{-1}(B)$. In particular, if $\sigma(B) = B$, then $w \circ \sigma = w$.*
- (b) *κ_B is a normal extension of κ_v .*
- (c) *The map $x \mapsto \overline{\sigma(x)}$ induces a K -isomorphism from $\sigma^{-1}(B)/\sigma^{-1}(\mathfrak{m}_B)$ onto κ_B . In particular, if $\sigma(B) = B$ then $\bar{\sigma} \in \text{Aut}(\kappa_B/\kappa_v)$.*
- (d) *For every $\sigma \in \text{Aut}(L/K)$ we have $e(\sigma^{-1}(B)/A) = e(B/A)$ and $f(\sigma^{-1}(B)/A) = f(B/A)$.*

Proof. Part (a), (b), (d) are easily verified, so we concentrate on (b). Let $\bar{f} \in \kappa_v[X]$ be an irreducible polynomial with a root $\bar{x} \in \kappa_v$. Let R be the integral closure of A in L and write $\mathfrak{m} = \mathfrak{m}_B \cap R$. By ??, we have $B = R_{\mathfrak{m}}$. Observe next that R is fixed by $\text{Aut}(L/K)$. Now let $x \in R$ be a preimage of \bar{x} , and let $g \in A[X]$ be the minimal polynomial of x over K . Since L/K is normal, g splits completely as $g(X) = \prod_{i=1}^n (X - x_i)$ with $x_i \in R$. From $\bar{g}(\bar{x}) = 0$, it follows that \bar{f} divides \bar{g} . But $\bar{g} = \prod_{i=1}^n (X - \bar{x}_i)$ in κ_B , hence \bar{f} has all its roots in κ_B , proving (b). \square

13.6.3 Henselian fields

In this part we introduce the basic concepts of a Henselian field. These will be used when we establish the structure theorem about extension of valuations.

We have seen that every rank-one valuation v of a complete field K has a unique prolongation to each algebraic extension of K . By ??, also every valuation of a separably closed field K has this property. Valuation rings, or valuations, with this property are very important. They are the so-called "Henselian" valuation rings. We shall see that they are the suitable substitute for rank-one valuation rings of complete fields. In particular, they are the valuation rings for which Hensel's Lemma holds.

A valued field (K, v) is called **Henselian** if v has a unique extension to every algebraic extension L of K . We see from the definition that the property of being Henselian is hereditary. Let \bar{K} be an algebraic closure and K^s be the separable closure of K in \bar{K} . We then have the following apparently easier characterization of Henselian valuations.

Proposition 13.6.16. *A valued field (K, v) is Henselian if and only if it extends uniquely to K^s .*

Proof. By definition, if (K, v) is Henselian then v extends uniquely to K^s . Conversely, take an algebraic extension L of K . By ??, every extension of v to $L \cap K^s$ has an extension to K^s . By assumption, v extends uniquely to $L \cap K^s$, so v has a unique prolongation to L . \square

The next theorem will highlight the importance of Henselian valuation rings. Comparing with completions we see that rank-one complete valuation rings are a particular case of Henselian valuation rings. Before stating the theorem we need a little preparation.

Let $v : K^\times \rightarrow \Gamma$ be a valuation of the field K . The **Gauss extension** w of v to the rational function field $K(X)$ is given by

$$w\left(\sum_{i=0}^n a_i X^i\right) = \min_i \{v(a_i)\}.$$

We say a polynomial is primitive if $w(f) = 0$. The following properties of primitive polynomials hold, which can be seen as a generalization of the well known Gauss's lemma.

Lemma 13.6.17. *Let $v : K^\times \rightarrow \Gamma$ be a valuation with valuations ring A .*

- (a) *If two polynomials f and g are primitive, then so is their product fg .*
- (b) *Every $f \in K[X]$ admits a decomposition $f = af$ with $a \in K$ and $\tilde{f} \in K[X]$ primitive.*
- (c) *If $f \in A[X]$ decomposes as $f = \prod_{i=1}^m g_i$ with irreducible factors $g_1, \dots, g_m \in K[X]$, then there are $h_1, \dots, h_m \in A[X]$, irreducible in $K[X]$, such that $f = \prod_{i=1}^m h_i$.*

Proof. Part (a) and (b) are easy to prove. To see (c), write $f = af$ and $g_i = b_i \tilde{g}_i$, where $a, b_1, \dots, b_m \in K$ and $\tilde{f}, \tilde{g}_1, \dots, \tilde{g}_m$ are primitive. Then

$$v(a) = w(f) = \sum_{i=1}^n w(g_i) = w(b_1 \cdots b_m).$$

Since $f \in A[X]$, we have $v(a) \geq 0$, whence $b_1 \cdots b_m \in A$. Define $h_1 = b_1 \cdots b_m \tilde{g}_1$ and $h_i = \tilde{g}_i$, we see the desired decomposition. \square

Theorem 13.6.18. *Let (K, v) be a valued field with valuation ring A and \mathfrak{m} its maximal ideal. The following statements are equivalent:*

(i) (K, v) is Henselian.

(ii) For each irreducible polynomial $f \in A[X]$ with $\bar{f} \notin (\kappa_v)[X]$, there exists $g \in A[X]$ such that \bar{g} is irreducible in $(\kappa_v)[X]$ and $\bar{f} = \bar{g}^s$, for some $s \geq 1$.

(iii) Let $f, g, h \in A[X]$ satisfy $\bar{f} = \bar{g}\bar{h}$, with \bar{g}, \bar{h} relatively prime in $(\kappa_v)[X]$. Then there exist $g_1, h_1 \in A[X]$ such that

$$f = g_1 h_1, \quad \bar{g}_1 = \bar{g}, \bar{h}_1 = \bar{h}, \quad \deg g_1 = \deg g, \deg h_1 = \deg h.$$

(iv) For each $f \in A[X]$ and $a \in A$ with $\bar{f}(\bar{a}) = 0$ and $\bar{f}'(\bar{a}) \neq 0$, there exists an $\alpha \in A$ with $f(\alpha) = 0$ and $\bar{\alpha} = \bar{a}$.

(v) For each $f \in A[X]$ and $a \in A$ with $v(f(a)) > 2v(f'(a))$, there exists an $\alpha \in A$ with $f(\alpha) = 0$ and $v(a - \alpha) > v(f'(\alpha))$.

(vi) Every polynomial $f(X) = X^n + a_{n-1}X^{n-1} + \cdots + a_1X + a_0 \in A[X]$ with $a_{n-1} \notin \mathfrak{m}$ and $a_{n-2}, \dots, a_0 \in \mathfrak{m}$ has a zero in K .

(vii) Every polynomial $f(X) = X^n + X^{n-1} + \cdots + a_1X + a_0 \in A[X]$ with $a_{n-2}, \dots, a_0 \in \mathfrak{m}$ has a zero in K .

Proof. Let (K, v) be Henselian, denote by w the unique extension of v to \bar{K} , the algebraic closure of K . Write B the valuation ring of w and \mathfrak{m}_B for the maximal ideal, and κ_B for the residue class field of B . Since w is the unique extension of v , for every K -automorphism σ of \bar{K} we have $\sigma(B) = B$ and $\sigma(\mathfrak{m}_B) = \mathfrak{m}_B$. Let f an be irreducible polynomial $f \in A[X]$ with $\bar{f} \notin (\kappa_v)[X]$. Write

$$f(X) = \prod_{i=1}^n (aX - x_i)$$

where $a, x_1, \dots, x_n \in K$ and a is an n -th root of the leading coefficient of f . Then $a \in B$ since $w(a) = v(a)/n \geq 0$. Also, $(-1)^n x_1 \dots x_n = f(0) \in A$.

The roots $x_1/a, \dots, x_n/a$ of f are all K -conjugate, and w is invariant under $\text{Aut}(\bar{K}/K)$. Consequently, there exists $\gamma \in \Gamma_w$ such that $w(x_j/a) = \gamma$ for all j . So, for $\delta = \gamma + w(a)$, we have that $w(x_j) = \delta$ for each j . Therefore, $x_1, \dots, x_n \in A$ implies $\delta \geq 0$, and thus $x_1, \dots, x_n \in \mathfrak{m}_B$ or $x_1, \dots, x_n \in B - \mathfrak{m}_B$.

In the first case, $\bar{f} = (\bar{a}X)^n$ and the claim (b) follows. In the second case, we obtain

$$\bar{f} = \prod_{i=1}^n (\bar{a}X - \bar{x}_i)$$

with $\bar{x}_i \neq 0$. Since $\bar{f} \notin (\kappa_v)[X]$, we also have $\bar{a} \neq 0$. For the sake of seeking a contradiction, let us assume that $\bar{f} = \bar{g}\bar{h}$ for some relatively prime polynomials \bar{g} and \bar{h} with $g, h \in A[X]$. Let x_i/a be a root of \bar{g} and take some $x_j \neq x_i$ such that x_j/a is a root of \bar{h} . Hence $g(x_i/a) \in \mathfrak{m}_A$ and $h(x_j/a) \in \mathfrak{m}_A$. Take $\sigma \in \text{Aut}(\bar{K}/K)$ such that $\sigma(x_i/a) = x_j/a$. Then

$$g(x_j/a) = g(\sigma(x_i/a)) = \sigma(g(x_i/a)) \in \sigma(\mathfrak{m}_B) = \mathfrak{m}_B$$

which means x_j/a is also a root of \bar{g} , contradiction. This proves (i) \Rightarrow (ii).

Now assume (ii), and let $f = gh$ with $f, g, h \in A[X]$ and g, h coprime. By ??, let $f = \prod_{i=1}^m g_i$ be a factorization of f with irreducible factors $g_1, \dots, g_m \in A[X]$. By condition (ii), for every i , either $g_i \in K$ or there exists $p_i \in A[X]$ such that p_i is irreducible and $g_i = p_i^{t_i}$ for some $t_i \geq 1$. Clearly, we may assume that p_i has no non-zero coefficient in \mathfrak{m}_A . Renumbering the polynomials g_1, \dots, g_m , we may assume that

$$\bar{g} = \bar{a} \prod_{i=1}^k \bar{p}_i^{t_i}, \quad \bar{h} = \bar{h} \prod_{i=k+1}^{\ell} \bar{p}_i^{t_i}, \quad \bar{c} = \prod_{i=\ell+1}^m \bar{p}_i^{t_i}$$

for some $a, b, c \in A - \mathfrak{m}$, since \bar{g} and \bar{h} are coprime. Define now

$$g_1 = a \prod_{i=1}^k p_i^{t_i}, \quad h_1 = \left(b \prod_{i=k+1}^{\ell} p_i^{t_i} \right) \left(c^{-1} \prod_{i=\ell+1}^m p_i^{t_i} \right)$$

then the polynomials g_1 and h_1 satisfies the requirements in (iii), so we have shown (ii) \Rightarrow (iii).

Next, we will deduce (iv) from (iii). If $f \in A[X]$ and $a \in A$ with $\bar{f}(\bar{a}) = 0$ and $\bar{f}'(\bar{a}) \neq 0$, set $g(X) = X - a$ and $\bar{h} = \bar{f}/\bar{g}$, then $\bar{f} = \bar{g}\bar{h}$. Since $\bar{f}'(\bar{a}) \neq 0$, \bar{g} and \bar{h} are coprime, so by condition (iii) there exist polynomials g_1 and h_2 such that

$$f = g_1 h_1, \quad \bar{g}_1 = \bar{g} = X - \bar{a}, \quad \bar{h}_1 = \bar{h}, \quad \deg g_1 = \deg g = 1, \quad \deg h_1 = \deg h.$$

It then follows that $g_1 = u(X - b)$ for some $u \in A^\times$ and $b \in A$, with $\bar{u} = 1$ and $\bar{b} = \bar{a}$. Clearly $f(b) = 0$, so this proves (iv).

For each $f \in A[X]$ and $a \in A$ with $v(f(a)) > 2v(f'(a))$, we have $f'(a) \neq 0$. Now by a simple computation we know $f(a - X) = f(a) - f'(a)X + X^2g(X)$ for some $g \in A[X]$, and by a change of variable $X = f'(a)Y$ we can define

$$h(Y) := \frac{f(a - f'(a)Y)}{f'(a)^2} = \frac{f(a)}{f'(a)^2} - Y + Y^2g(f'(a)Y).$$

Since $v(f(a)) > 2v(f'(a))$ we have $h \in A[Y]$. Note that $f(a)/f'(a)^2 \in \mathfrak{m}_A$, so \bar{h} has a simple root $\bar{0}$ in κ_v . If condition (iv) holds, this will give a root $b \in \mathfrak{m}$ of h . Then by the construction of h , $\alpha = a - f'(a)b$ will a root of f . Moreover, since $v(b) > 0$, we have $v(a - \alpha) > v(f'(a))$. This shows (iv) \Rightarrow (v).

The implication (v) \Rightarrow (vi) is immediate. In fact, let $f(X) = X^n + a_{n-1}X^{n-1} + \cdots + a_1X + a_0$ be as in (vi). Then

$$\bar{f}(X) = X^n + \bar{a}_{n-1}X^{n-1} = X^{n-1}(X + \bar{a}_{n-1}).$$

Hence $-\bar{a}_{n-1}$ is a nonzero simple zero of \bar{f} . In particular, $f(-a_{n-1}) > 0$ and $f'(-a_{n-1}) \in A - \mathfrak{m}$. This means

$$v(f(-a_{n-1})) > 0 = 2v(f'(-a_{n-1}))$$

thus f has a root in A , by (v).

It is clear that (vi) \Rightarrow (vii). Conversely, if (vii) holds and $f(X) = X^n + a_{n-1}X^{n-1} + \cdots + a_1X + a_0$ is a polynomial as in (vi), then the changing $X = a_{n-1}Y$ and dividing by a_{n-1}^n , we obtain

$$g(Y) = Y^n + Y^{n-1} + \frac{a_{n-2}}{a_{n-1}^2}Y^{n-2} + \cdots + \frac{a_1}{a_{n-1}^n}Y + \frac{a_0}{a_{n-1}^n}.$$

It is clear that a root of $g(Y)$ gives a root of f , so (vi) and (vii) are equivalent.

Finally, we prove (vii) \Rightarrow (i). Suppose (K, v) were not henselian. Then there would be a finite Galois extension L/K with Galois group $\text{Gal}(L/K)$ in which v has more than one extension. Let B be a prolongation of A to L and set $H = \{\sigma \in \text{Gal}(L/K) : \sigma(B) = B\}$. As B is not the only prolongation of A to L , the group H is a proper subgroup of $\text{Gal}(L/K)$, and thus the fixed field $L' := L^H$ of H is a proper extension of K . Let B_1, \dots, B_n be all conjugates of B in L with maximal ideals $\mathfrak{m}_1, \dots, \mathfrak{m}_n$ and consider the subring

$$R = \bigcap_{i=1}^n (B_i \cap L')$$

of L' . By ??, we find $\alpha \in R$ such that $\alpha - 1 \in \mathfrak{m}_1$ and $\alpha \in \mathfrak{m}_i$ for $i = 2, \dots, n$. Since $n > 1$, α cannot lie in K , so its minimal polynomial

$$f = \min(\alpha, K) = X^n + a_{n-1}X^{n-1} + \cdots + a_1X + a_0$$

cannot have a zero in K . However, if $\alpha_1 = \alpha, \alpha_2, \dots, \alpha_n$ are conjugates of α in L , we have

$$a_{n-i} = (-1)^i e_i(\alpha_1, \dots, \alpha_n), \quad (13.6.1)$$

where e_i is the i -th symmetric function. For each $i \geq 2$, we have $\alpha_i \neq \alpha$, so $\alpha_i = \tau(\alpha)$ for

some $\tau \in \text{Gal}(L/K) \setminus H$ (note that $\alpha \in L'$). Hence $\tau^{-1}(B) = B_j$ for some $j \geq 2$. Since by the approximation condition we have $\alpha \in \mathfrak{m}_j = \tau^{-1}(\mathfrak{m}_1)$, we then get $\alpha_i = \tau(\alpha) \in \mathfrak{m}_1$. By (??), this implies $1 + a_{n-1} \in \mathfrak{m}_1$ and $a_{n-2}, \dots, a_1, a_0 \in \mathfrak{m}_1$. Thus by (vi) the polynomial f has a root in K , contradiction. \square

Corollary 13.6.19. *Let (K, w) be a composition of valuations (K, v) and (κ_v, \bar{v}) . Then (K, w) is Henselian if and only if both (K, v) and (κ_v, \bar{v}) are Henselian.*

Proof. Suppose (K, w) is Henselian. Then (K, v) is also Henselian, using $\mathfrak{m}_v \subseteq \mathfrak{m}_w \subseteq A_w \subseteq A_v$ and ???. To show that (κ_v, \bar{v}) is Henselian, let $\bar{f} = X^n + X^{n-1} + \dots + \bar{a}_0$, with $a_i \in \mathfrak{m}_{\bar{v}}$ we must show that \bar{f} has a zero in κ_v (again using ??). The polynomial

$$f = X^n + X^{n-1} + \dots + a_0 \in A_w[X]$$

has a zero $x \in A_w$ (yet again by ??), since $a_i \in \mathfrak{m}_v \subseteq \mathfrak{m}_w$; therefore $\bar{x} \in A_{\bar{v}}$ is a zero of \bar{f} .

Now we prove the converse. Let $f = X_n + a_{n-1}X^{n-1} + \dots + a_0 \in A_w[X]$, and suppose that \bar{f} has a simple zero in the residue field κ_w . As κ_w is also the residue field of \bar{v} , and \bar{v} is Henselian, this simple zero of \bar{f} lifts to κ_v . As the field (K, v) is also Henselian, the zero (which is again simple) can be lifted further to K . \square

Let (K, v) be a valued field. Assume L/K is a Galois extension. We then say (K, v) is **L -Henselian** if v has exactly one extension to L . By taking $L = K^s$, the separable closure of K , we return to our usual definition of Henselian field. Just as ??, we can also obtain many equivalent characterizations of L -Henselian fields. One need only go through the proof of ?? and restrict consideration to those polynomials f that split in L . The corresponding condition for the Galois extension K^s/K is always satisfied, hence, of course, was not mentioned in ??.

13.6.4 The fundamental inequality

Definition 13.6.20. *Let K be a field, v a valuation on K and L an extension of K . A family $(w_i)_{i \in I}$ of valuations on L which extend v and such that every valuation on L extending v is equivalent to a unique w_i is called a **complete system of extensions of v to L** .*

We have already proved that the completion is an immediate extension of a given valued field. Now we go deeper to this result and consider the case of a extension of valued fields.

Proposition 13.6.21. *Let (K, v) be a valued field and $(\widehat{K}, \widehat{v})$ its completion of K . Let L be a finite extension of K of degree n .*

- (a) *Let w be a valuation on L extending v and $(\widehat{L}_w, \widehat{w})$ denote the completion of (L, w) . Identity \widehat{K} with the closure of K in \widehat{L}_w , then*

$$e(\widehat{w}/\widehat{v}) = e(w/v), \quad f(\widehat{w}/\widehat{v}) = f(w/v), \quad e(w/v)f(w/v) \leq [\widehat{L}_w : \widehat{K}] \leq n.$$

- (b) *Every set of pairwise independent valuations on L extending a nontrivial valuation v is finite. Let $(w_i)_{1 \leq i \leq s}$ denote pairwise independent valuations on L extending v such that every valuation on L extending v is dependent on one of the w_i ; let L_i be the field L with the topology defined by w_i*

and \widehat{L}_i its completion. Then the canonical map

$$\phi : \widehat{K} \otimes_K L \rightarrow \prod_{i=1}^s \widehat{L}_i$$

(extending by continuity the diagonal map $L \rightarrow \prod_{i=1}^s L_i$) is surjective, its kernel is the Jacobson radical of $\widehat{K} \otimes_K L$. In particular, $\sum_{i=1}^s [\widehat{L}_i : \widehat{K}] \leq n$.

Proof. Let us first prove (a). Suppose that v is nontrivial. As v and \widehat{v} (resp. w and \widehat{w}) have the same order group and the same residue field, we see $e(\widehat{w}/\widehat{v}) = e(w/v)$ and $e(\widehat{w}/\widehat{v}) = e(w/v)$. In particular, $e(w/v)f(w/v) \leq [\widehat{L}_w : \widehat{K}]$ by ???. Finally the vector sub- \widehat{K} -space of \widehat{L}_w generated by L is closed in \widehat{L}_w (by ???) and also dense in \widehat{L}_w so it equals to \widehat{L}_w . This shows $[\widehat{L}_w : \widehat{K}] \leq n$ and completes the proof of (a).

We now pass to (b). We may still assume that v is not trivial. Let (w_1, \dots, w_r) be any finite family of pairwise independent valuations on L extending v . The image of L in $\prod_{i=1}^r L_i$ under the diagonal map is dense by the approximation theorem, and $\prod_i L_i$ is dense in $\prod_{i=1}^r \widehat{L}_i$. Hence the canonical Image of $\widehat{K} \otimes_K L$ in $\prod_{i=1}^r \widehat{L}_i$ is dense. On the other hand this image is a vector sub- \widehat{K} -space of $\prod_{i=1}^r \widehat{L}_i$; as $\prod_{i=1}^r \widehat{L}_i$ is of finite dimension over \widehat{K} by (a), every subspace of $\prod_{i=1}^r \widehat{L}_i$ is closed. Thus the image of $\widehat{K} \otimes_K L$ is closed and equal to $\prod_{i=1}^r \widehat{L}_i$. As the dimension of $K \otimes_K L$ over K is n , we see $\sum_{i=1}^r [\widehat{L}_i : \widehat{K}] \leq n$. This shows in particular that the integer r is bounded above by n and shows the first assertion of (b).

We now take (w_1, \dots, w_s) as in the statement. The fact that

$$\phi : \widehat{K} \otimes_K L \rightarrow \prod_{i=1}^s \widehat{L}_i$$

is surjective have already been shown. It remains to verify that the kernel of ϕ is the Jacobson radical J of $K \otimes_K L$. As $\prod_{i=1}^s \widehat{L}_i$ is semi-simple, we see $J \subseteq \ker \phi$. On the other hand, for every maximal ideal \mathfrak{m} of $\widehat{K} \otimes_K L$, the quotient field $L(\mathfrak{m}) = (\widehat{K} \otimes_K L)/\mathfrak{m}$ is a composite extension of \widehat{K} and L over K (??). There exists a valuation μ on $L(\mathfrak{m})$ extending \widehat{v} ; the restriction w of μ to L then extends v . As $[L(\mathfrak{m}) : \widehat{K}]$ is finite, $L(\mathfrak{m})$ is complete with respect to μ (??). Now the closure of L in $L(\mathfrak{m})$ is a field containing \widehat{K} and L and hence is equal to $L(\mathfrak{m})$. Consequently $L(\mathfrak{m})$ is identified with the completion \widehat{L}_w and \mathfrak{m} is the kernel of the canonical map of $\widehat{K} \otimes_K L$ onto \widehat{L}_w . Now, by hypothesis, there exists an index i such that w and w_i are dependent; whence $\widehat{L}_w = \widehat{L}_i$. Thus $\ker \phi \subseteq \mathfrak{m}$, which proves that $\ker \phi \subseteq J$ and completes the proof. \square

Corollary 13.6.22. *If K is complete with respect to v and v is nontrivial, two valuations on L extending v are dependent.*

Proof. In this case we have $\widehat{K} \otimes_K L = K \otimes_K L = L$, so the claim follows by ??(b). \square

Corollary 13.6.23. *If the extension L/K or \widehat{K}/K is separable then the map ϕ is an isomorphism.*

Proof. In this case the Jacobson radical of $\widehat{K} \otimes_K L$ is zero. \square

Theorem 13.6.24 (The Fundamental Inequality). *Let (K, v) be a valued field and L a finite extension of K of degree n . Then every complete system $(w_i)_{i \in I}$ of extensions of v to L is finite and we have*

$$\sum_{i \in I} e(w_i/v)f(w_i/v) \leq n.$$

Proof. Since the theorem is trivial if v is trivial, we shall assume that v is nontrivial. Let (w_1, \dots, w_s) be any finite family of valuations on L extending v , no two of which are equivalent. We shall show that $\sum_{i=1}^s e(w_i/v)f(w_i/v) \leq n$. This will prove the theorem.

We argue by induction on s and suppose therefore that the inequality has been established for the case of valuations of number smaller than s . We distinguish two cases.

Suppose that there exist at least two independent valuations of w_i . Then there exists a partition $[1, s] = I_1 \cup \dots \cup I_t$ of $[1, s]$ such that:

- (i) for w_i and w_j to be dependent, it is necessary and sufficient that i and j belong to the same I_k ;
- (ii) $|I_k| < s$ for all k .

We choose in each I_k an index i_k . Let \widehat{L}_k denote the completion of L with respect to v_{i_k} and $n_k = [\widehat{L}_k : \widehat{K}]$. For all $i \in I_k$, w_i defines on L the same topology as w_{i_k} and hence may be extended to a valuation \widehat{w}_i on \widehat{L}_k whose restriction to \widehat{K} is \widehat{v} . Since no two of the w_i for $i \in I_k$ are equivalent, the same is true of the \widehat{w}_i . The induction hypothesis applied to the ordered pair $(\widehat{K}, \widehat{L}_k)$ shows

$$\sum_{i \in I_k} e(w_i/v)f(w_i/v) \leq n_k.$$

As $\sum_k n_k \leq n$ by ??, we see the claim follows in this case.

We now pass to the case where any two of the w_i are dependent. In this case there is a nontrivial valuation w' of L such that $w_i = w \circ \bar{w}_i$, where \bar{w}_i is a valuation of the residue field κ_w . Denote by v' the restriction of w' to K , then v is also a composition of v' , say $v = v' \circ \bar{v}$, where \bar{v} is a valuation of $\kappa_{v'}$. If we choose w such that the valuation rings A_{w_i} generates $A_{w'}$, then the valuation rings of \bar{w}_i generates $\kappa_{w'}$, so that they are not all dependent. From the first part of the proof, we then have

$$\sum_{i=1}^s e(\bar{w}_i/\bar{v})f(\bar{w}_i/\bar{v}) \leq [\kappa_{w'} : \kappa_{v'}] = f(w'/v')$$

and hence

$$\sum_{i=1}^s e(w'/v')e(\bar{w}_i/\bar{v})f(\bar{w}_i/\bar{v}) \leq e(w'/v')f(w'/v') \leq n.$$

Now it suffices to prove that

$$f(w_i/v) = f(\bar{w}_i/\bar{v}), \quad e(w'/v')e(\bar{w}_i/\bar{v}) = e(w_i/v). \quad (13.6.2)$$

For this, we note that v and \bar{v} (resp. w_i and \bar{w}_i) have the same residue field, so the first equality holds. For the second there is, by ?? and ??, the following commutative diagram, where the

rows are exact sequences and the vertical arrows represent canonical injections:

$$\begin{array}{ccccccc}
 & 0 & & 0 & & 0 & \\
 & \downarrow & & \downarrow & & \downarrow & \\
 0 & \longrightarrow & \Gamma_{\bar{v}} & \longrightarrow & \Gamma_v & \longrightarrow & \Gamma_{v'} \longrightarrow 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
 0 & \longrightarrow & \Gamma_{\bar{w}_i} & \longrightarrow & \Gamma_{w_i} & \longrightarrow & \Gamma_{w'} \longrightarrow 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
 0 & \rightarrow & \Gamma_{\bar{w}_i}/\Gamma_{\bar{v}} & \rightarrow & \Gamma_{w_i}/\Gamma_v & \rightarrow & \Gamma_{w'}/\Gamma_{v'} \rightarrow 0 \\
 & \downarrow & & \downarrow & & \downarrow & \\
 0 & & 0 & & 0 & & 0
 \end{array}$$

From this we see the second equality is valid, therefore the proof is completed. \square

Now we proceed a step further to investigate when the equality in ?? holds. For this, we need the following definition.

Definition 13.6.25. Let Γ be a totally ordered group. A subset of Γ is called **major** if the relations $\alpha \in M, \beta \in \Gamma$ and $\beta \geq \alpha$ imply $\beta \in M$.

Example 13.6.26. Let Δ be an isolated subgroup of a totally ordered group Γ . Then the set

$$M = \{\alpha \in \Gamma : \alpha \geq \gamma \text{ for all } \gamma \in \Delta\}$$

is a major subset of Γ .

Let K be a field, v a valuation on K , and A the ring of v and Γ the order group of v . For every major subset $M \subseteq G$, let $\mathfrak{a}(M)$ be the set of $x \in K$ such that $v(x) \in M \cup \{\infty\}$. Clearly $\mathfrak{a}(M)$ is a sub- A -module of K .

Proposition 13.6.27. The map $M \mapsto \mathfrak{a}(M)$ is an increasing bijection of the set of major subsets of Γ onto the set of sub- A -modules of K .

Proof. Let \mathfrak{b} be a sub- A -module of K . The set of $v(x)$ for $x \in \mathfrak{b} \setminus \{0\}$ is a major subset $M(\mathfrak{b})$ of Γ . We now show the following equations are satisfied:

- (a) $M(\mathfrak{a}(N)) = N$ for every major subset N of Γ ;
- (b) $\mathfrak{a}(M(\mathfrak{b})) = \mathfrak{b}$ for every sub- A -module \mathfrak{b} of K .

Part (a) is easy, since, for all $\alpha \in N$, there exists $x \in K$ such that $v(x) = \alpha$. Then obviously $\mathfrak{b} \subseteq \mathfrak{a}(M(\mathfrak{b}))$; conversely, let $x \in \mathfrak{a}(M(\mathfrak{b}))$ and suppose $x \neq 0$, then $v(x) \in M(\mathfrak{b})$ and therefore there exists $y \in \mathfrak{b}$ such that $v(x) = v(y)$; whence $x = uy$ where $v(u) = 0$, which proves that $x \in \mathfrak{b}$ and completes the proof. \square

Corollary 13.6.28. Let Γ_+ be the set of positive elements in Γ . The map $M \mapsto \mathfrak{a}(M)$ is a bijection of the set of major subsets of Γ_+ onto the set of ideals of A .

Proof. As $A = \mathfrak{a}(\Gamma_+)$, $\mathfrak{a}(M) \subseteq A$ is equivalent to $M \subseteq \Gamma_+$. \square

Definition 13.6.29. Let Γ be a totally ordered group and Δ a subgroup of Γ of finite index. The number of major subsets of Γ consisting of strictly positive elements and containing all strict positive elements of Δ is called the **initial index** of Δ in Γ and denoted by $\varepsilon(\Gamma, \Delta)$.

This initial index is a natural number by virtue of the following proposition:

Proposition 13.6.30. Let Γ be a totally ordered group and Δ a subgroup of Γ of finite index. If Γ has no least positive element, then $\varepsilon(\Gamma, \Delta) = 1$ for all Δ . If there exists a least positive element of Γ and Γ_0 denotes the subgroup it generates, then

$$\varepsilon(\Gamma, \Delta) = [\Gamma_0 : \Gamma_0 \cap \Delta].$$

Proof. In the first case, let α be a strict positive element in Γ . The set of $\beta \in \Gamma$ such that $0 < \beta < \alpha$ is infinite and hence there exist two elements of this set which are distinct and congruent modulo Δ ; their difference is an element γ of Γ such that $0 < \gamma < \alpha$. Hence every major subset which contains all the strictly positive elements of Δ contains α and hence all strict positive elements of Γ .

In the second case, let δ be the least strict positive element of Γ and let n be the least positive integer such that $m\delta \in \Delta$. Clearly $n = [\Gamma_0 : \Gamma_0 \cap \Delta]$. On the other hand, writing $M(\alpha)$ for the set of $\beta \in \Gamma$ such that $\beta > \alpha$, it is immediately seen that the admissible major sets are just $M(\delta), M(2\delta), \dots, M(n\delta)$. \square

Corollary 13.6.31. The initial index $\varepsilon(\Gamma, \Delta)$ divides the index $[\Gamma : \Delta]$ and is equal to it if Γ is isomorphic to \mathbb{Z} .

Definition 13.6.32. Let K be a field, L a finite extension of K , w a valuation on L , v its restriction to K and Γ_v and Γ_w their order groups. The initial index of Γ_v in Γ_w is called the **initial ramification index** of w with respect to v (or with respect to K) and denoted by $\varepsilon(w/v)$.

From the above corollary, $\varepsilon(w/v)$ divides $e(w/v)$ with equality in the case of a discrete valuation.

Proposition 13.6.33. Let K be a field, L a finite extension of K , w a valuation on L , v its restriction to K . Then

$$\dim_{\kappa_v}(A_w/\mathfrak{m}_v A_w) = \varepsilon(w/v)f(w/v).$$

Proof. The ideals of A_w containing \mathfrak{m}_v and distinct from A_w correspond to the major subsets of Γ_w , consisting of strict positive elements and containing the elements strict positive of Γ_v . They are therefore equal in number to $\varepsilon(w/v)$ and, as they form a totally ordered set under inclusion, this number is equal to the length of the quotient ring $A_w/\mathfrak{m}_v A_w$. Now a module of length 1 over A_w is a 1-dimensional vector space over κ_w and hence a module of length $f(w/v)$ over A ; hence, as $A_w/\mathfrak{m}_v A_w$ is of length $\varepsilon(w/v)$ over A_w , it is of length $\varepsilon(w/v)f(w/v)$ over A_v , that is over κ_v . \square

Theorem 13.6.34. Let K be a field, v a valuation on K , A its ring, \mathfrak{m} its maximal ideal, L a finite extension of K of degree n , B the integral closure of A in L and $(w_i)_{1 \leq i \leq s}$ a complete system of extensions of v to L . Then

$$\dim_{\kappa_v}(B/\mathfrak{m} B) = \sum_{i=1}^s \varepsilon(w_i/v)f(w_i/v) \leq \sum_{i=1}^s e(w_i/v)f(w_i/v) \leq n.$$

Proof. Let B_i be the ring of w_i ; then $B_i = B_{\mathfrak{m}_i}$, where \mathfrak{m}_i runs through the family of maximal ideals of B . Let \mathfrak{q}_i be the saturation of $\mathfrak{m}B$ with respect to \mathfrak{m}_i . By ??, the canonical homomorphism $B/\mathfrak{m}B \rightarrow \prod_{i=1}^s B/\mathfrak{q}_i$ is an isomorphism and \mathfrak{m}_i is the only maximal ideal containing \mathfrak{q}_i . Hence B/\mathfrak{q}_i is canonically isomorphic to $(B/\mathfrak{q}_i)_{\mathfrak{m}_i}$. On the other hand, we note that

$$(B/\mathfrak{q}_i)_{\mathfrak{m}_i} = B_{\mathfrak{m}_i}/\mathfrak{m}B_{\mathfrak{m}_i} = B_i/\mathfrak{m}B_i,$$

therefore there is a canonical isomorphism $B/\mathfrak{m}B \rightarrow \prod_{i=1}^s B_i/\mathfrak{m}B_i$, whence the result in view of ??.

□

Theorem 13.6.35. *With the hypotheses and notation of ??, the following conditions are equivalent:*

- (a) B is a finitely generated A -module;
- (b) B is a free A -module;
- (c) $\dim_{\kappa_v}(B/\mathfrak{m}B) = n$;
- (d) the equality in ?? holds and $\varepsilon(w_i/v) = e(w_i/v)$ for all i .

Proof. The equivalence of (a) and (b) follows from ?. Also, the equivalence of (c) and (d) follows from ?. It remains to show that (c) and (b) are equivalent.

First, assume that B is a free A -module. From the proof of ??, we see for any $x \in L$ there exists $s \in A$ such that sx is integral over K , hence in B . Since K is the fraction field of A , any independent set of B over A is an independent subset of L over K . These together show that any basis of B over A is also a basis of L over K , hence $\text{rank}_A(B) = n$. Next, we prove that $\dim_{\kappa_v}(B/\mathfrak{m}B)$ also equals to n . Let $\{x_1, \dots, x_n\}$ be a basis of B over A . If \bar{x}_i denotes the image of x_i in $B/\mathfrak{m}B$ then $\bar{x}_1, \dots, \bar{x}_n$ span the vector space $B/\mathfrak{m}B$ over κ_v . We assert that the n vectors are independent, hence the claim. For this, it suffices to show that if we have a relation of the form $\sum_{i=1}^n a_i x_i \in \mathfrak{m}B$, $a_i \in A$, then the a_i 's necessarily belong to A . But this follows at once from the linear independence of the x_i over A , for we have, by assumption: $\sum_{i=1}^n a_i x_i = \sum_{i=1}^n b_i x_i$, where the b_i are suitable elements of \mathfrak{m} , and this relation implies $a_i = b_i$ for each i .

Conversely, assume that $\dim_{\kappa_v}(B/\mathfrak{m}B) = n$. Let x_1, \dots, x_n be elements of B whose canonical images in $B/\mathfrak{m}B$ form a basis of $B/\mathfrak{m}B$ and let $M \subseteq B$ be the sub- A -module which they generate. As M is torsion-free ($M \subseteq B$ and B is an integral domain) and finitely generated, it is free by ?. We shall see that $B = M$. Let $y \in B$; we write $N = M + Ay$; this is also a free A -module. The canonical injections $M \rightarrow N \rightarrow B$ give canonical homomorphisms

$$M/\mathfrak{m}M \rightarrow N/\mathfrak{m}N \rightarrow B/\mathfrak{m}B.$$

Now, by hypothesis, $M/\mathfrak{m}M \rightarrow B/\mathfrak{m}B$ is surjective and $B/\mathfrak{m}B$ is n -dimensional, hence $M/\mathfrak{m}M$ and $N/\mathfrak{m}N$ are n -dimensional and $M/\mathfrak{m}M \rightarrow N/\mathfrak{m}N$ is surjective. As N is finitely generated, $M \rightarrow N$ is then surjective by ??, whence $M = N$ and $B = M$. Hence B is free.

□

Corollary 13.6.36. *With the same hypotheses and notation, suppose further that v is discrete and L separable. Then*

$$\sum_{i=1}^s e(w_i/v)f(w_i/v) = n.$$

Proof. In this case B is a free A -module of rank n by ??, since A is a PID.

□

Corollary 13.6.37. Let (K, v) be a discrete valued complete field and L a finite extension of K of degree n . Then v admits a unique (up to equivalence) extension w to L , the ring B of w is a finitely generated free module over the ring A of v and $e(w/v)f(w/v) = n$.

Proof. All the extensions of v to L are dependent by ??, they are therefore equivalent. This shows the uniqueness of w . The integral closure of A in L is therefore B . As v is discrete, the topology induced on A by that on K is the \mathfrak{m} -adic topology (where $\mathfrak{m} = \mathfrak{m}_A$); the ring A is complete, for it is closed in K . We conclude that, since $B/\mathfrak{m}B$ is a finite-dimensional vector (A/\mathfrak{m}) -space ??, B is a finitely generated A -module ??). It is therefore free and $e(w'/v)f(w'/v) = n$ by ??.

Corollary 13.6.38. Suppose that v is of rank 1 and that the equivalent conditions of ?? hold. If \widehat{L}_i is the completion of L with respect to w_i , then the degree $n_i = [\widehat{L}_i : \widehat{K}]$ is equal to $e(w_i/v)f(w_i/v)$ for all i and the canonical homomorphism

$$\phi : \widehat{K} \otimes_K L \rightarrow \prod_{i=1}^s \widehat{L}_i$$

is bijective. For all $x \in L$, the characteristic polynomial $\min_{L/K}(x)$ is equal to the product of the characteristic polynomials $\min_{\widehat{L}_i/\widehat{K}}(x)$. In particular, we have

$$\text{tr}_{L/K}(x) = \sum_{i=1}^s \text{tr}_{\widehat{L}_i/\widehat{K}}(x), \quad N_{L/K}(x) = \prod_{i=1}^s N_{\widehat{L}_i/\widehat{K}}(x), \quad v(N_{L/K}(x)) = \sum_{i=1}^s n_i w_i(x). \quad (13.6.3)$$

Proof. As no two of the w_i are equivalent and they are of rank 1 by ??, they are independent and ?? therefore shows that $e(w_i/v)f(w_i/v) \leq n_i$ for all i and $\sum_i n_i \leq n$. The first assertion therefore follows from these inequalities and the relation $\sum_i e(w_i/v)f(w_i/v) = n$. Under the isomorphism ϕ the endomorphism $z \mapsto z(1 \otimes x)$ of $\widehat{K} \otimes_K L$ (for $x \in L$) is transformed into the endomorphism of $\prod_{i=1}^s \widehat{L}_i$, leaving invariant each of the factors and reducing on each factor to multiplication by x (L_i being canonically imbedded in its completion \widehat{L}_i); whence the assertion relating to the characteristic polynomial of x and the first two formulae of ??). Finally, let E be a finite quasi-Galois extension of \widehat{K} containing \widehat{L}_i . As \widehat{K} is complete and v has rank 1, there exists only one valuation (up to equivalence) w on E extending \widehat{v} ??). Then for every \widehat{K} -automorphism σ of E we have $w(\sigma(x)) = w(x)$ and therefore

$$\widehat{v}(N_{\widehat{L}_i/\widehat{K}}(x)) = n_i w_i(x)$$

which proves the formula. □

13.7 Exercise

Exercise 13.7.1. If A is a valuation ring of dimension ≥ 2 then the formal power series ring $A[[X]]$ is not integrally closed.

Proof. Let $0 \subset \mathfrak{p}_1 \subset \mathfrak{p}_2$ be a strict chain. Let $b \in \mathfrak{p}_1, a \in \mathfrak{p}_2 - \mathfrak{p}_1$, then $a^n/b \notin A$ for all n :

$$a^n/b \in A \Rightarrow a^n = (a^n/b) \cdot b \in \mathfrak{p}_1 \Rightarrow a \in \mathfrak{p}_1 (\mathfrak{p}_1 \text{ is prime})$$

Now A is a valuation ring so $ba^{-n} \in A$ for all n .

Let $f = \sum_{n=0}^{\infty} u_n x^n$ be a solution of

$$f^2 + af + x = 0$$

We shall take $u_0 = -a$, then from the equation

$$f^2 + af = \sum_{n=0}^{\infty} (u_0 u_n + \cdots + u_n u_0 + a u_n) x^n$$

we claim that $u_n \in a^{-2n+1} A$ for all $n \geq 1$. In fact, we can compute $u_1 = a^{-1}$. And assume this holds for u_1, \dots, u_{n-1} , for u_n we have an equation

$$u_0 u_n + u_1 u_{n-1} + \cdots + u_{n-1} u_1 + u_n u_0 + a u_n = 0$$

For $1 \leq i \leq n-1$, since $u_i \in a^{-2i+1} A$ we have

$$u_i u_{n-i} = a^{-2i+1} r_1 \cdot a^{-2(n-i)+1} r_2 = a^{-2n+2} r_1 r_2 \in a^{-2n+2} A$$

Hence

$$u_n = -(a + 2u_0)^{-1} (u_1 u_{n-1} + \cdots + u_{n-1} u_1) \in a^{-1} \cdot a^{-2n+2} A = a^{-2n+1} A.$$

Now we only need to prove that f is in the field of fractions of $A[\![X]\!]$. By our observation we have $bf \in A[\![X]\!]$, so $f \in b^{-1} A[\![X]\!] \subseteq \text{Frac}(A[\![X]\!])$. But since $a \in \mathfrak{p}_2$, $a^{-1} \notin A$, which means $f \notin A[\![X]\!]$. So $A[\![X]\!]$ is not integrally closed. \square

Exercise 13.7.2. If v is an additive valuation of a field K and $\alpha_1, \dots, \alpha_n \in K$ are such that $\alpha_1 + \cdots + \alpha_n = 0$ then there exist two indices i, j such that $i \neq j$ and $v(\alpha_i) = v(\alpha_j)$.

Proof. If all $\alpha_i = 0$ then $v(\alpha_i) = \infty$ for all i , and the claim holds. Hence we may assume that $\alpha_n \neq 0$, and from $\alpha_1 + \cdots + \alpha_n = 0$ we get

$$1 = -\left(\frac{\alpha_1}{\alpha_n} + \cdots + \frac{\alpha_{n-1}}{\alpha_n}\right).$$

Now, since $1 \notin \mathfrak{m}$, there exist an index $1 \leq i \leq n-1$ such that α_i/α_n is not in \mathfrak{m} , hence is a unit. Then by our definition $v(\alpha_i/\alpha_n) = 0$, so $v(\alpha_i) = v(\alpha_n)$. \square

Exercise 13.7.3. Let k be a field, x and y indeterminates, and supposest α is a positive irrational number. Then the map $v : k[x, y] \rightarrow \mathbb{R} \cup \{\infty\}$ defined by

$$\sum_{n,m} c_{n,m} x^n y^m \mapsto \min\{n + m\alpha \mid c_{n,m} \neq 0\}$$

determines a valuation of $k(x, y)$ with value group $\mathbb{Z} + \alpha\mathbb{Z}$.

Proof. For $f = \sum_{n,m} a_{n,m} x^n y^m$, $g = \sum_{n,m} b_{n,m} x^n y^m$, we have

$$v(fg) = \min\{n_1 + n_2 + (m_1 + m_2)\alpha \mid a_{n_1, m_1} \neq 0, b_{n_2, m_2} \neq 0\}$$

which is easily seen to be equal to $v(f) + v(g)$.

For $v(f + g)$, it is also easy to verify $v(f + g) \geq \min\{v(f), v(g)\}$. Also, $v(0) = \infty$. Thus by the previous exercise, v extends to a valuation $k(x, y) \rightarrow \mathbb{R} \cup \{\infty\}$. It is clear that $\Gamma_v = \mathbb{Z} + \alpha\mathbb{Z}$. \square

Exercise 13.7.4. Let A be a DVR and \mathfrak{m} its maximal ideal; then the \mathfrak{m} -adic completion \widehat{A} of A is again a DVR.

Proof. Let $\mathfrak{m} = (x)$, then every element in A has a form rx^n for r a unit. Hence we can think the completion of A as the power series ring $(A - \mathfrak{m})[[t]]$. Thus, we can define a valuation for $f = \sum_{n=0}^{\infty} u_n t^n$:

$$v : \widehat{A} \rightarrow \mathbb{Z}, \quad v(f) = \{n \mid u_n \neq 0\}$$

Since $f = \sum_{n=0}^{\infty} u_n t^n$ is a unit if and only if u_0 is a unit, this is well defined, and extending it to the field of fractions of \widehat{A} gives makes \widehat{A} a valuation ring. \square

Chapter 14

Krull domains and divisors

14.1 Krull domains

14.1.1 Fractional ideals

Definition 14.1.1. Let A be an integral domain and K its field of fractions. A sub- A -module \mathfrak{a} of K such that there exists a nonzero element $d \in A$ for which $d\mathfrak{a} \subseteq A$ is called a **fractional ideal** of A (or of K , by an abuse of language).

Every finitely generated sub- A -module \mathfrak{a} of K is a fractional ideal: for if a_1, \dots, a_n is a system of generators of \mathfrak{a} , we may write $a_i = x_i/y_i$, where $x_i, y_i \in A$ and $y_i \neq 0$. If $d = y_1 \cdots y_n$ then clearly $d\mathfrak{a} \subseteq A$. In particular the principal sub- A -modules of K are fractional ideals. It is clear that every ideal of A is a fractional ideal. To avoid confusion, these will also be called the **integral ideals** of A . We denote by $\mathfrak{F}(A)$ the set of non-zero fractional ideals of A . Given two elements $\mathfrak{a}, \mathfrak{b}$ of $\mathfrak{F}(A)$, we shall write $\mathfrak{a} \preceq \mathfrak{b}$ for the relation "every fractional principal ideal containing \mathfrak{a} also contains \mathfrak{b} ". Clearly this relation is a **preordering** on $\mathfrak{F}(A)$ (it is reflexive and transitive). Consider the associated equivalence relation " $\mathfrak{a} \preceq \mathfrak{b}$ and $\mathfrak{b} \preceq \mathfrak{a}$ " and let $\mathfrak{D}(A)$ be the equivalent class of this relation. We shall say that the elements of $\mathfrak{D}(A)$ are the **divisors** of A and, for every fractional ideal $\mathfrak{a} \in \mathfrak{F}(A)$, we shall denote by $\text{div}(\mathfrak{a})$ (or $\text{div}_A(\mathfrak{a})$) the canonical image of \mathfrak{a} in $\mathfrak{D}(A)$ and we shall say that $\text{div}(\mathfrak{a})$ is the **divisor of \mathfrak{a}** . If $a = Ax$ is a fractional principal ideal, we write $\text{div}(x)$ instead of $\text{div}(Ax)$ and $\text{div}(x)$ is called the **divisor of x** . The elements of $\mathfrak{D}(A)$ of the form $\text{div}(x)$ are called **principal divisors**. By taking the quotient, the preordering \preceq on $\mathfrak{F}(A)$ defines on $\mathfrak{D}(A)$ an ordering which we shall denote by \leq .

For any $\mathfrak{a} \in \mathfrak{F}(A)$ there exists by hypothesis some $d \in A$ such that $\mathfrak{a} \subseteq Ad^{-1}$. The intersection $\tilde{\mathfrak{a}}$ of the fractional principal ideals containing \mathfrak{a} is therefore an element of $\mathfrak{F}(A)$. Clearly the relation $\mathfrak{a} \preceq \mathfrak{b}$ is equivalent to the relation $\tilde{\mathfrak{a}} \supseteq \tilde{\mathfrak{b}}$, which is the case if $\mathfrak{a} \supseteq \mathfrak{b}$. For two elements $\mathfrak{a}, \mathfrak{b}$ of $\mathfrak{F}(A)$ to be equivalent modulo R , it is necessary and sufficient that $\tilde{\mathfrak{a}} = \tilde{\mathfrak{b}}$.

Definition 14.1.2. Every element \mathfrak{a} of $\mathfrak{F}(A)$ such that $\mathfrak{a} = \tilde{\mathfrak{a}}$ is called a **divisorial fractional ideal** of A .

In other words a divisorial ideal is just a non-zero intersection of a non-empty family of fractional principal ideals. Every non-zero intersection of divisorial ideals is a divisorial ideal. If \mathfrak{a} is divisorial, so is $\mathfrak{a}x$ for all $x \in K$, the map $\mathfrak{b} \mapsto \mathfrak{b}x$ being a bijection of the set of fractional principal ideals onto itself. By definition, for all $\mathfrak{a} \in \mathfrak{F}(A)$, $\tilde{\mathfrak{a}}$ is then the least divisorial ideal containing \mathfrak{a} and is equivalent to \mathfrak{a} modulo R . Moreover, if \mathfrak{b} is a divisorial ideal equivalent to

\mathfrak{a} modulo R , then $\tilde{\mathfrak{a}} = \tilde{\mathfrak{b}} = \mathfrak{b}$. Hence $\tilde{\mathfrak{a}}$ is the unique divisorial ideal \mathfrak{b} such that $\text{div}(\mathfrak{a}) = \text{div}(\mathfrak{b})$ (in other words, the restriction of the map $\mathfrak{a} \mapsto \text{div}(\mathfrak{a})$ to the set of divisorial ideals is bijective).

Let \mathfrak{a} and \mathfrak{b} be two fractional ideals of K . Recall that $(\mathfrak{b} : \mathfrak{a})$ denotes the set of $x \in K$ such that $x\mathfrak{a} \subseteq \mathfrak{b}$. This is obviously an A -module. Note that $(\mathfrak{b} : \mathfrak{a})$ is also a fractional ideal, for if d is a non-zero element of A such that $d\mathfrak{b} \subseteq A$ and $d\mathfrak{a} \subseteq A$ and a is a non-zero element of $A \cap \mathfrak{a}$, then $da(\mathfrak{b} : \mathfrak{a}) \subseteq A$. On the other hand, if $b \neq 0$ belongs to \mathfrak{b} , then $bd\mathfrak{a} \subseteq \mathfrak{b}$, hence $bd \in (\mathfrak{b} : \mathfrak{a})$ and $(\mathfrak{b} : \mathfrak{a}) \neq 0$. Note that the definition of $(\mathfrak{b} : \mathfrak{a})$ can also be written as

$$(\mathfrak{b} : \mathfrak{a}) = \bigcap_{x \in \mathfrak{a}, x \neq 0} \mathfrak{b}x^{-1}. \quad (14.1.1)$$

Proposition 14.1.3. *Let $\mathfrak{a}, \mathfrak{b}$ be fractional ideals of A .*

- (a) *If \mathfrak{b} is a divisorial ideal then $(\mathfrak{b} : \mathfrak{a})$ is divisorial.*
- (b) *In order that $\text{div}(\mathfrak{a}) = \text{div}(\mathfrak{b})$, it is necessary and sufficient that $(A : \mathfrak{a}) = (A : \mathfrak{b})$.*
- (c) *We have $\tilde{\mathfrak{a}} = (A : (A : \mathfrak{a}))$.*

Proof. Assertion (a) follows immediately from equation (??) since, if \mathfrak{b} is divisorial, so is $\mathfrak{b}x^{-1}$ for all $x \neq 0$. To show (b), let $P(\mathfrak{a})$ denote the set of fractional principal ideals containing \mathfrak{a} . The relation $Ax \in P(\mathfrak{a})$ is equivalent to $x^{-1}\mathfrak{a} \subseteq A$ and hence to $x^{-1} \in (A : \mathfrak{a})$. As the relation $\text{div}(\mathfrak{a}) = \text{div}(\mathfrak{b})$ is by definition equivalent to $P(\mathfrak{a}) = P(\mathfrak{b})$, it is also equivalent to $(A : \mathfrak{a}) = (A : \mathfrak{b})$.

Finally, as $\mathfrak{a}(A : \mathfrak{a}) \subseteq A$, we see $\mathfrak{a} \subseteq (A : (A : \mathfrak{a}))$. Replacing \mathfrak{a} by $(A : \mathfrak{a})$ in this formula, it is seen that $(A : \mathfrak{a}) \subseteq (A : (A : (A : \mathfrak{a})))$. On the other hand, the relation $\mathfrak{a} \subseteq (A : (A : \mathfrak{a}))$ implies

$$(A : \mathfrak{a}) \supseteq (A : (A : (A : \mathfrak{a})))$$

Therefore $(A : \mathfrak{a}) = (A : (A : (A : \mathfrak{a})))$ and it follows from (b) that $\text{div}(\mathfrak{a}) = \text{div}(A : (A : \mathfrak{a}))$. As $(A : (A : \mathfrak{a}))$ is divisorial by (a), certainly $\tilde{\mathfrak{a}} = (A : (A : \mathfrak{a}))$, which proves (c). \square

Proposition 14.1.4 (Lattice Structure in $\mathfrak{D}(A)$).

- (a) *In $\mathfrak{D}(A)$ every non-empty set bounded above admits a least upper bound. More precisely, if (\mathfrak{a}_i) is a non-empty family of elements of $\mathfrak{F}(A)$ which is bounded above, then*

$$\sup_i (\text{div}(\mathfrak{a}_i)) = \text{div}(\bigcap_i \tilde{\mathfrak{a}}_i).$$

- (b) *In $\mathfrak{D}(A)$ every non-empty set bounded below admits a greatest lower bound. More precisely, if (\mathfrak{a}_i) is a non-empty family of elements of $\mathfrak{F}(A)$ which is bounded below, then*

$$\inf_i (\text{div}(\mathfrak{a}_i)) = \text{div}(\sum_i \tilde{\mathfrak{a}}_i).$$

- (c) *$\mathfrak{D}(A)$ is a lattice.*

Proof. Let (\mathfrak{a}_i) be a non-empty family of elements of $\mathfrak{F}(A)$ which is bounded above. To say that a divisorial ideal \mathfrak{b} bounds this family above amounts to saying that it is contained in all the $\tilde{\mathfrak{a}}_i$. Hence $\bigcap_i \tilde{\mathfrak{a}}_i \neq 0$ and $\sup_i (\text{div}(\mathfrak{a}_i)) = \text{div}(\bigcap_i \tilde{\mathfrak{a}}_i)$.

Now let (\mathfrak{a}_i) be a non-empty family of elements of $\mathfrak{F}(A)$ which is bounded below. To say that a divisorial ideal \mathfrak{b} bounds this family below means that it contains all the $\tilde{\mathfrak{a}}_i$, so we see $\inf_i(\text{div}(\mathfrak{a}_i)) = \text{div}(\sum_i \tilde{\mathfrak{a}}_i)$.

Finally, it is sufficient to note that, if $\mathfrak{a}, \mathfrak{b}$ are in $\mathfrak{F}(A)$, then $\mathfrak{a} \cap \mathfrak{b}$ and $\mathfrak{a} + \mathfrak{b}$ are both nonzero fractional ideals, so the claim follows. \square

Corollary 14.1.5. *If $x, y, x + y \in K^\times$ then $\text{div}(x + y) \geq \inf\{\text{div}(x), \text{div}(y)\}$.*

Proof. We have $A(x + y) \subseteq Ax + Ay$ and hence $\text{div}(x + y) \geq \text{div}(Ax + Ay)$. \square

14.1.2 The monoid structure on $\mathfrak{D}(A)$

Proposition 14.1.6. *Let $\mathfrak{a}, \mathfrak{a}', \mathfrak{b}, \mathfrak{b}'$ be elements of $\mathfrak{F}(A)$. Then the relations $\mathfrak{a} \succeq \mathfrak{a}'$ and $\mathfrak{b} \succeq \mathfrak{b}'$ imply $\mathfrak{ab} \succeq \mathfrak{a}'\mathfrak{b}'$.*

Proof. We may restrict our attention to the case where $\mathfrak{b} = \mathfrak{b}'$, by transitivity. Then let Ax be a fractional principal ideal containing $\mathfrak{a}'\mathfrak{b}$. For every non-zero element y of \mathfrak{b} , $Ax \supseteq \mathfrak{a}'y$ and hence $Axy^{-1} \supseteq \mathfrak{a}'$, whence $Axy^{-1} \supseteq \mathfrak{a}$ and $Ax \supseteq \mathfrak{a}y$. Varying y , it is seen that $Ax \supseteq \mathfrak{ab}$, whence $\mathfrak{ab} \succeq \mathfrak{a}'\mathfrak{b}$. \square

It follows from ?? that multiplication on $\mathfrak{F}(A)$ defines, by passing to the quotient, a law of composition on $\mathfrak{D}(A)$ which is obviously associative and commutative. It is written additively so that we may write:

$$\text{div}(\mathfrak{ab}) = \text{div}(\mathfrak{a}) + \text{div}(\mathfrak{b})$$

for $\mathfrak{a}, \mathfrak{b}$ in $\mathfrak{F}(A)$. Clearly $\text{div}(1)$ is an identity element for this addition. This element is denoted by 0. ?? proves further that the order structure on $\mathfrak{D}(A)$ is compatible with this addition:

$$\begin{aligned} \inf(\text{div}(\mathfrak{a}) + \text{div}(\mathfrak{b}), \text{div}(\mathfrak{a}) + \text{div}(\mathfrak{c})) &= \inf(\text{div}(\mathfrak{ab}), \text{div}(\mathfrak{ac})) = \text{div}(\mathfrak{ab} + \mathfrak{ac}) \\ &= \text{div}(\mathfrak{a}(\mathfrak{b} + \mathfrak{c})) = \text{div}(\mathfrak{a}) + \text{div}(\mathfrak{b} + \mathfrak{c}) \\ &= \text{div}(\mathfrak{a}) + \inf(\text{div}(\mathfrak{b}), \text{div}(\mathfrak{c})). \end{aligned}$$

For a fractional ideal \mathfrak{a} to be such that $\text{div}(\mathfrak{a}) \geq 0$ in $\mathfrak{D}(A)$, it is necessary and sufficient that $\mathfrak{a} \subseteq A$ (in other words, that \mathfrak{a} be an integral ideal of A).

For two elements x, y of K^\times , the relation $\text{div}(x) = \text{div}(y)$ is equivalent to $Ax = Ay$. Therefore the set of principal divisors of A with the order relation and the monoid law induced by that on $\mathfrak{D}(A)$ is an ordered group canonically isomorphic to the multiplicative group of fractional principal ideals ordered by the opposite order relation to inclusion. The relation S between two elements P, Q of $\mathfrak{D}(A)$:

$$\text{there exists } x \in K^\times \text{ such that } P = Q + \text{div}(x)$$

is therefore an equivalence relation since the relation $P = Q + \text{div}(x)$ is equivalent to $Q = P + \text{div}(x^{-1})$. If P and Q are congruent modulo S , they are called **equivalent divisors** of A . Clearly moreover the relation S is compatible with the law of the monoid $\mathfrak{D}(A)$ and the latter therefore defines, by taking quotients, a monoid structure on $\mathfrak{D}(A)/S$. This monoid is called the **divisor class monoid** of A .

Proposition 14.1.7. *Let $\mathfrak{a}, \mathfrak{b}$ be two divisorial fractional ideals of A . Then $\text{div}(\mathfrak{a})$ and $\text{div}(\mathfrak{b})$ are equivalent divisors if and only if there exists $x \in K^\times$ such that $\mathfrak{b} = x\mathfrak{a}$.*

Proof. If $\text{div}(\mathfrak{b}) = \text{div}(\mathfrak{a}) + \text{div}(x)$ for some $x \in K^\times$, then $\text{div}(\mathfrak{b}) = \text{div}(x\mathfrak{a})$ and, as \mathfrak{b} and $x\mathfrak{a}$ are divisorial, $\mathfrak{b} = x\mathfrak{a}$, which proves the proposition. \square

Let \mathfrak{a} be an invertible fractional ideal; then $\mathfrak{a} = (A : (A : \mathfrak{a}))$ by ??(b) and hence \mathfrak{a} is divisorial by ???. The group $\mathfrak{I}(A)$ of invertible fractional ideals is therefore identified with a subgroup of the monoid $\mathfrak{D}(A)$ and the canonical image of $\mathfrak{I}(A)$ in $\mathfrak{D}(A)/S$ with the group of classes of projective A -modules of rank 1 ??).

Proposition 14.1.8. *Let A be an integral domain. For the monoid $\mathfrak{D}(A)$ of divisors of A to be a group, it is necessary and sufficient that A be completely integrally closed.*

Proof. Suppose that $\mathfrak{D}(A)$ is a group. Let $x \in K$ and suppose that $A[x]$ is contained in a finitely generated sub- A -module of K . Then we have seen that $\mathfrak{a} = A[x]$ is an element of $\mathfrak{F}(A)$. Then $x\mathfrak{a} \subseteq \mathfrak{a}$ and hence $\text{div}(x) + \text{div}(\mathfrak{a}) \geq \text{div}\mathfrak{a}$. Since $\mathfrak{D}(A)$ is an ordered group, we conclude that $\text{div}(x) \geq 0$, whence $x \in A$. Thus A is completely integrally closed.

Conversely, suppose that A is completely integrally closed. Let \mathfrak{a} be a divisorial ideal. We shall show that $\text{div}(\mathfrak{a}(A : \mathfrak{a})) = 0$, which will prove that $\mathfrak{D}(A)$ is a group. As $\mathfrak{a}(A : \mathfrak{a}) \subseteq A$, it suffices to verify that every fractional principal ideal Ax^{-1} which contains $\mathfrak{a}(A : \mathfrak{a})$ also contains A . Now, for $y \in K^\times$, the relation $Ay \supseteq \mathfrak{a}$ implies $y^{-1} \in (A : \mathfrak{a})$, whence $y^{-1}\mathfrak{a} \subseteq \mathfrak{a}(A : \mathfrak{a}) \subseteq Ax^{-1}$ and hence $x\mathfrak{a} \subseteq Ay$. As \mathfrak{a} is divisorial, we then deduce that $x\mathfrak{a} \subseteq \mathfrak{a}$, whence $x^n\mathfrak{a} \subseteq \mathfrak{a}$ for all $n \in \mathbb{N}$. Since \mathfrak{a} is divisorial, there exist elements x_0, x_1 of K^\times such that $Ax_0 \subseteq \mathfrak{a} \subseteq Ax_1$; therefore $x^n x_0 \in Ax_1$, whence $x^n \in Ax_1 x_0^{-1}$. As A is completely integrally closed, $x \in A$, that is $Ax^{-1} \supseteq A$, which completes the proof. \square

Remark 14.1.1. Note that, if A is completely integrally closed (and even Noetherian), a divisorial ideal of A is not necessarily invertible, in other words, in general $\mathfrak{I}(A) \neq \mathfrak{D}(A)$.

Corollary 14.1.9. *Let A be a completely integrally closed domain and \mathfrak{a} a divisorial fractional ideal of A . Then, for every nonzero fractional ideal \mathfrak{b} of A , $\text{div}(\mathfrak{a} : \mathfrak{b}) = \text{div}(\mathfrak{a}) - \text{div}(\mathfrak{b})$.*

Proof. By the formula ??,

$$\text{div}(\mathfrak{a} : \mathfrak{b}) = \text{div}\left(\bigcap_{y \in \mathfrak{b}, y \neq 0} y^{-1}\mathfrak{a}\right) = \sup_{y \in \mathfrak{b}, y \neq 0} \text{div}(y^{-1}\mathfrak{a})$$

taking account of ?? and the fact that the fractional ideals $y^{-1}\mathfrak{a}$ are divisorial. But since $D(A)$ is an ordered group,

$$\sup_{y \in \mathfrak{b}, y \neq 0} \text{div}(y^{-1}\mathfrak{a}) = \sup_{y \in \mathfrak{b}, y \neq 0} (\text{div}(\mathfrak{a}) - \text{div}(y)) = \text{div}(\mathfrak{a}) - \inf_{y \in \mathfrak{b}, y \neq 0} \text{div}(y) = \text{div}(\mathfrak{a}) - \text{div}(\mathfrak{b})$$

so the claim follows. \square

14.1.3 Krull domains

Definition 14.1.10. *An integral domain A is called a **Krull domain** if there exists a family $(v_i)_{i \in I}$ valuations on the field of fractions K of A with the following properties:*

- (K1) the valuations v_i are discrete;
- (K2) the intersection of the rings of the v_i is A ;
- (K3) for all $x \in K^\times$, the set of indices $i \in I$ such that $v_i(x) \neq 0$ is finite.

Such a family $(v_i)_{i \in I}$ is called a **defining family of valuations**.

Example 14.1.11 (Examples of Krull domains).

- (a) Every discrete valuation ring is a Krull domain.
- (b) More generally, every PID A is a Krull domain. For let $(p_i)_{i \in I}$ be a representative system of irreducible elements of A and let v_i be the valuation on the field of fractions of A defined by p_i . It is immediately seen that the family $(v_i)_{i \in I}$ satisfies properties (K1), (K2) and (K3).

Let A be a Krull domain and let $(v_i)_{i \in I}$ be a family of valuations on the field of fractions K of A satisfying (K1), (K2) and (K3). The v_i may be assumed to be normed. For all $\mathfrak{a} \in I(A)$, we shall write:

$$v_i(\mathfrak{a}) = \sup_{\mathfrak{a} \subseteq Ax} v_i(x).$$

Then $v_i(\mathfrak{a}) \in \mathbb{Z}$, for, if a is a non-zero element of \mathfrak{a} , the relation $Ax \supseteq Aa$ implies that $v_i(x) \leq v_i(a)$ (by (K2)), which shows that the family $\{v_i(x) : \mathfrak{a} \subseteq Ax\}$ is bounded above. We establish the following properties.

Proposition 14.1.12. Let A be a Krull domain, $(v_i)_{i \in I}$ a family of normed valuations on K satisfying (K1), (K2) and (K3), and $v_i(\mathfrak{a})$ be defined above.

- (a) If $x \in K^\times$ then $v_i(Ax) = v_i(x)$.
- (b) Let \mathfrak{a} and \mathfrak{b} be two divisorial fractional ideals of A . In order that $\mathfrak{a} \subseteq \mathfrak{b}$, it is necessary and sufficient that $v_i(\mathfrak{a}) \geq v_i(\mathfrak{b})$ for all $i \in I$.
- (c) For all $\mathfrak{a} \in I(A)$, the indices $i \in I$ such that $v_i(\mathfrak{a}) \neq 0$ are finite in number.

Proof. If $Ay \supseteq Ax$, then $v_i(y) \leq v_i(x)$ by (K2) and the minimum value of $v_i(y)$ is taken at $y = x$. This proves (a). For a divisorial ideal \mathfrak{a} be, the relation $y \in \mathfrak{a}$ is equivalent to the relation " $\mathfrak{a} \subseteq Ax$ implies $y \in Ax$ ". Now, by (K2), the relation $y \in Ax$ is equivalent to $v_i(y) \geq v_i(x)$ for all $i \in I$, whence we see $y \in \mathfrak{a}$ if and only if $v_i(y) \geq v_i(\mathfrak{a})$ for all i , and (b) follows.

Finally, for $\mathfrak{a} \in I(A)$ there exist x, y in K^* such that $Ax \subseteq \mathfrak{a} \subseteq Ay$. By properties (a) and (b), $v_i(x) \geq v_i(\mathfrak{a}) \geq v_i(y)$ for all $i \in I$. It then follows from (K3) that (c) holds. \square

Corollary 14.1.13. If A is a Krull domain and $(v_i)_{i \in I}$ is a defining family of normed valuations on K , the map $\mathfrak{a} \mapsto (v_i(\mathfrak{a}))_{i \in I}$ is a decreasing injective map of the set of divisorial integer ideals of A (ordered by inclusion) to the set of positive elements of the ordered group the direct sum $\mathbb{Z}^{\oplus I}$. In particular every non-empty family of divisorial integral ideals of A admits a maximal element.

The condition in ?? is also sufficient. In fact, we have the following characterization for Krull domains.

Theorem 14.1.14. *Let A be an integral domain. For A to be a Krull domain, it is necessary and sufficient that the two following conditions be satisfied:*

- (a) *A is completely integrally closed.*
- (b) *every non-empty family of divisorial integral ideals of A admits a maximal element (with respect to inclusion).*

Moreover, if $P(A)$ is the set of irreducible elements of $\mathfrak{D}(A)$, then $P(A)$ is a basis of the \mathbb{Z} -module $\mathfrak{D}(A)$ and the positive elements of $\mathfrak{D}(A)$ are the linear combinations of the elements of $P(A)$ with positive coefficients.

Proof. A Krull domain is completely integrally closed by ??, and it satisfies (b) by ?? . Conversely, let A be an integral domain satisfying properties (a) and (b) of the statement. Since A is completely integrally closed, $\mathfrak{D}(A)$ is an ordered group and a lattice. By condition (b) of the statement, every non-empty family of positive elements of $\mathfrak{D}(A)$ has a minimal element. Let $P(A)$ be the set of irreducible elements of $\mathfrak{D}(A)$. Then by ?? $P(A)$ is a basis of the \mathbb{Z} -module $\mathfrak{D}(A)$ and the positive elements of $\mathfrak{D}(A)$ are the linear combinations with positive integer coefficients of the elements of $P(A)$. Thus, for $x \in K^\times$, integers $v_P(x)$ are defined (for $P \in P(A)$) by writing:

$$\text{div}(x) = \sum_{P \in P(A)} v_P(x)P. \quad (14.1.2)$$

From the relations $\text{div}(xy) = \text{div}(x) + \text{div}(y)$ and $\text{div}(x+y) \geq \inf\{\text{div}(x), \text{div}(y)\}$ for $x, y, x+y \in K^\times$, we deduce that the v_P are discrete valuations on K . In order that $x \in A$, it is necessary and sufficient that $\text{div}(x) \geq 0$, that is that $v_P(x) \geq 0$ for all $P \in P(A)$. Thus the v_P satisfy conditions (K1) and (K2) and obviously also (K3). \square

Corollary 14.1.15. *For a Noetherian ring to be a Krull domain, it is necessary and sufficient that it be an integrally closed domain.*

Proof. An integrally closed Noetherian domain is completely integrally closed. \square

Note that there are non-Noetherian Krull domains, for example the polynominal ring $K[X_n]_{n \in \mathbb{N}}$ over a field K in an infinity of indeterminates.

14.1.4 Essential valuations for a Krull domain

Let A be a Krull domain and K its field of fractions. The valuations defined by formula (??) (for $x \in K^\times$) are called the **essential valuations** of K (or A). We have remarked in the course of the proof of ?? that the valuations (v_P) satisfy properties (K1), (K2) and (K3). Moreover, these discrete valuations v_P are normed: for every irreducible divisor $P \in P(A)$, $P < 2P$ and hence, if \mathfrak{a} and \mathfrak{b} are the divisorial ideals corresponding to P and $2P$, then $\mathfrak{a} \supsetneq \mathfrak{b}$. For $x \in \mathfrak{a} - \mathfrak{b}$, $\text{div}(x) \geq P$ and $\text{div}(x) \not\geq 2P$, whence $v_P(x) = 1$, which proves our assertion.

Proposition 14.1.16. *Let A be a Krull domain, K its field of fractions and $(v_P)_{P \in P(A)}$ the family of its essential valuations. Let $(n_P)_{P \in P(A)}$ be a family of integers with finite support. Then the set of $x \in K$ such that $v_P(x) \geq n_P$ for all $P \in P(A)$ is the divisorial ideal \mathfrak{a} of A such that $\text{div}(\mathfrak{a}) = \sum_P n_P P$.*

Proof. Let $x \in K^\times$ and \mathfrak{a} the divisorial ideal such that $\text{div}(\mathfrak{a}) = \sum_P n_P P$. In order that $x \in \mathfrak{a}$, it is necessary and sufficient that $Ax \subseteq \mathfrak{a}$, hence that $\text{div}(x) \geq \text{div}(\mathfrak{a})$ and hence, by (??), that $v_P(x) \geq n_P$ for all $P \in P(A)$. \square

Proposition 14.1.17. *Let A be a Krull domain, K its field of fractions, $(v_i)_{i \in I}$ a defining family of valuations on K and A_i the ring of v_i . Let S be a multiplicative subset of A not containing 0, then*

$$S^{-1}A = \bigcap_{S^{-1}A \subseteq A_i} A_i.$$

In particular $S^{-1}A$ is a Krull domain.

Proof. Let $J = \{i \in I : S^{-1}A \subseteq A_i\}$ and write $B = \bigcap_{i \in J} A_i$, then we see $i \in J$ if and only if v_i is zero on S . Let $x \in B$ and $J(x)$ denote the finite set of indices $i \in I$ such that $v_i(x) < 0$. For each $i \in J(x)$ we have $x \notin A_i$, hence $i \notin J$ and so there exists $s_i \in S$ such that $v_i(s_i) > 0$. Let n_i be a positive integer such that $v_i(s_i^{n_i}x) \geq 0$ and we write $s = \prod_{i \in J} s_i^{n_i}$. Then $v_i(s_i x) \geq 0$ for all $i \in I$ and hence $s x \in A$ and $x \in s^{-1}A$. Thus $B = S^{-1}A$. \square

Corollary 14.1.18. *Let P be an irreducible divisor of A and \mathfrak{p} the corresponding divisorial ideal. Then \mathfrak{p} is a prime ideal of height 1, the ring of v_P is $A_{\mathfrak{p}}$ and the residue field of v_P is identified with the field of fractions of A/\mathfrak{p} .*

Proof. By ?? we see v_P is zero on $S = A - \mathfrak{p}$ and positive on \mathfrak{p} , hence $\mathfrak{p} = \mathfrak{m}_{v_P} \cap A$ and \mathfrak{p} is prime. On the other hand, since P is irreducible, for every divisor $Q \neq P$, $Q \not\geq P$ and hence the divisorial ideal \mathfrak{q} corresponding to Q is not contained in \mathfrak{p} . This proves $\text{ht}\mathfrak{p} = 1$ and the corollary then follows from ?? and ?? \square

Corollary 14.1.19. *Let A be a Krull domain, K its field of fractions and $(v_i)_{i \in I}$ a defining family of valuations. Then every essential valuation of A is equivalent to one of the v_i .*

Proof. Let P be an irreducible divisor of A and \mathfrak{p} the corresponding divisorial ideal. By ?? we have

$$A_{\mathfrak{p}} = \bigcap_{A_{\mathfrak{p}} \subseteq A_i} A_i = \bigcap_{\mathfrak{m}_i \cap A \subseteq \mathfrak{p}} A_i.$$

If $\mathfrak{m}_i \cap A = \{0\}$ then $A_i = K$ which is a contradiction, so $\mathfrak{m}_i \cap A \neq \{0\}$, and since \mathfrak{p} is of height 1 we must have $\mathfrak{m}_i \cap A = \mathfrak{p}$. Since \mathfrak{p} is nonzero, we then see there are only finitely many $i \in I$ such that $A_{\mathfrak{p}} \subseteq A_i$ by (K3). Now $A_{\mathfrak{p}}$ is an intersection of finitely many valuation rings and we can apply ?? ?? to claim that there is a unique $i \in I$ such that $A_{\mathfrak{p}} = A_i$, so v_P is equivalent to v_i . \square

Proposition 14.1.20. *Let A be a Krull domain, $(v_P)_{P \in P(A)}$ the family of its essential valuations and $\mathfrak{a} \in \mathfrak{F}(A)$. Then the coefficient of P in $\text{div}(\mathfrak{a})$ is $\inf_{x \in \mathfrak{a}} (v_P(x))$. If \mathfrak{p} is the divisorial prime ideal corresponding to the irreducible divisor P , then $\mathfrak{a}A_{\mathfrak{p}} = \tilde{\mathfrak{a}}A_{\mathfrak{p}}$.*

Proof. As $\mathfrak{a} = \sum_{x \in \mathfrak{a}} Ax$, ?? shows that $\text{div}(\mathfrak{a}) = \inf_{x \in \mathfrak{a}} \text{div}(Ax)$, whence our first assertion. The second follows immediately, since $\text{div}(\tilde{\mathfrak{a}}) = \text{div}(\mathfrak{a})$ and $A_{\mathfrak{p}}$ is the ring of the discrete valuation v_P . \square

Proposition 14.1.21. *Let A be an integrally closed Noetherian domain.*

(a) Let P be an irreducible divisor of A and \mathfrak{p} the corresponding divisorial prime ideal. Then the n -th symbolic power $\mathfrak{p}^{(n)}$ (that is, the saturation of \mathfrak{p}^n with respect to $S = A - \mathfrak{p}$) is the set of $x \in A$ such that $v_P(x) \geq n$ and is a \mathfrak{p} -primary ideal.

(b) Let \mathfrak{a} be a divisorial integral ideal and $\text{div}(\mathfrak{a}) = \sum_{i=1}^r n_i P_i$, with \mathfrak{p}_i the corresponding prime ideal of P_i . Then

$$\mathfrak{a} = \bigcap_{i=1}^r \mathfrak{p}_i^{(n_i)}$$

is the unique reduced primary decomposition of \mathfrak{a} and the \mathfrak{p}_i are isolated primes of \mathfrak{a} .

Proof. By ??, the relation $x \in \mathfrak{p}^n A_{\mathfrak{p}} = (\mathfrak{p} A_{\mathfrak{p}})^n$ is equivalent to $v_P(x) \geq n$. On the other hand, as $A_{\mathfrak{p}}$ is a discrete valuation ring, $(\mathfrak{p} A_{\mathfrak{p}})^n$ is $(\mathfrak{p} A_{\mathfrak{p}})$ -primary and hence $\mathfrak{p}^{(n)}$ is \mathfrak{p} -primary; this shows (a). ?? certainly shows that $\mathfrak{a} = \bigcap_{i=1}^r \mathfrak{p}_i^{(n_i)}$. As $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ are primes of height 1, this primary decomposition is reduced. The uniqueness follows from ?? \square

14.1.5 Approximation theorem

As the essential valuations of a Krull domain are discrete and normed, no two of them are equivalent and hence they are independent. ?? to the approximation theorem may therefore be applied to them: given some $n_i \in \mathbb{Z}$ and some essential valuations v_i finite in number and distinct, there exists $x \in K$ such that $v_i(x) = n_i$ for all i . But here there is a more precise result:

Proposition 14.1.22. *Let v_1, \dots, v_r be essential valuations of a Krull domain A and n_1, \dots, n_r integers. There exists an element x of the field of fractions K of A such $v_i(x) = n_i$ for all i and $v(x) \geq 0$ for every essential valuation v of A distinct from v_1, \dots, v_r .*

Proof. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ be the divisorial ideals of A corresponding to the valuations v_1, \dots, v_r . There exists $y \in K$ such that $v_i(y) = n_i$ for all i by ?. The essential valuations w_1, \dots, w_s of A distinct from the v_i for which the integer $w_i(y) = -m_j < 0$ are finite in number. Let $\mathfrak{q}_1, \dots, \mathfrak{q}_s$ be the corresponding ideals. There exists no inclusion relation between $\mathfrak{p}_1, \dots, \mathfrak{p}_r, \mathfrak{q}_1, \dots, \mathfrak{q}_s$ since these ideals correspond to irreducible divisors and these ideals are prime. Hence the integral ideal $\mathfrak{a} = \mathfrak{q}_1^{m_1} \cdots \mathfrak{q}_s^{m_s}$ is contained in none of the \mathfrak{p}_i and is therefore not contained in their union. Therefore there exists $z \in \mathfrak{a}$ such that $z \notin \mathfrak{p}_i$ for all i . Then $v_i(z) = 0$ for all i and $w_j(z) \geq m_j$ for all j ; hence the element $x = yz$ solves the problem. \square

Corollary 14.1.23. *Let A be a Krull domain, K its field of fractions and $\mathfrak{a}, \mathfrak{b}$ and \mathfrak{c} three divisorial fractional ideals of A such that $\mathfrak{a} \subseteq \mathfrak{b}$. Then there exists $x \in K$ such that $\mathfrak{a} = \mathfrak{b} \cap x\mathfrak{c}$.*

Proof. Let $(v_i)_{i \in I}$ be the family of essential valuations of A and let (m_i) (resp. $(n_i), (p_i)$) be the family of integers (zero except for a finite number of indices) such that \mathfrak{a} (resp. $\mathfrak{b}, \mathfrak{c}$) is the set of $x \in K$ for which $v_i(x_i) \geq m_i$ (resp. n_i, p_i) for all $i \in I$ (?). The set J of $i \in I$ such that $m_i > n_i$ is finite. As $p_i = m_i = 0$ except for a finite number of indices, ?? shows that there exists $x \in K^\times$ such that $v_i(x^{-1}) + m_i = p_i$ for $i \in J$ and

$$v_i(x^{-1}) + m_i \geq p_i$$

for $i \in I - J$. Then, for all $i \in I$, $m_i = \sup(n_i, v_i(x) + p_i)$. Whence $\mathfrak{a} = \mathfrak{b} \cap x\mathfrak{c}$. \square

Corollary 14.1.24. *Let A be a Krull domain. For a fractional ideal \mathfrak{a} of A to be divisorial, it is necessary and sufficient that it be the intersection of two fractional principal ideals.*

Proof. The sufficiency is obvious. The necessity follows from ?? by taking \mathfrak{b} and \mathfrak{c} to be principal and such that $\mathfrak{b} \supseteq \mathfrak{a}$. \square

14.1.6 Prime ideals of height 1

We have seen in ?? that irreducible divisors corresponds to prime ideals of height 1. Now we show that the converse also holds.

Theorem 14.1.25. *Let A be a Krull domain and \mathfrak{p} an integral ideal of A . For \mathfrak{p} to be the divisorial ideal corresponding to an irreducible divisor, it is necessary and sufficient that \mathfrak{p} be a prime ideal of height 1.*

Proof. If \mathfrak{p} is the divisorial ideal corresponding to an irreducible divisor, we know that \mathfrak{p} is prime of height 1. Conversely, let \mathfrak{p} be a nonzero prime ideal. As $A_{\mathfrak{p}} \neq K$, by ?? $A_{\mathfrak{p}}$ is the intersection of a non-empty family (A_i) of essential valuation rings, each A_i being of the form $A_{\mathfrak{q}_i}$?? and from $A_{\mathfrak{p}} \subseteq A_{\mathfrak{q}_i}$ we deduce that $\mathfrak{q}_i \subseteq \mathfrak{p}$. Thus, if \mathfrak{p} is of height 1, then $\mathfrak{p} = \mathfrak{q}$, which shows that \mathfrak{p} is the divisorial ideal corresponding to an irreducible divisor. \square

Corollary 14.1.26. *In a Krull domain every non-zero prime ideal \mathfrak{p} contains a prime ideal of height 1. If \mathfrak{p} is not of height 1, then $\text{div}(\mathfrak{p}) = 0$ and $(A : \mathfrak{p}) = A$.*

Proof. The first assertion is clear. If \mathfrak{p} is not of height 1 and \mathfrak{q} is a prime ideal of height 1 contained in \mathfrak{p} , then $\mathfrak{q} \subseteq \tilde{\mathfrak{p}}$ and $\mathfrak{q} \neq \tilde{\mathfrak{p}}$. As $\text{div}(\mathfrak{q})$ is irreducible, necessarily $\text{div}(\mathfrak{p}) = \text{div}(\tilde{\mathfrak{p}}) = 0$; hence $\text{div}(A : \mathfrak{p}) = 0$ and, as $(A : \mathfrak{p})$ is divisorial, we get $(A : \mathfrak{p}) = A$. \square

Corollary 14.1.27. *Let A be a Krull domain, K its field of fractions, v a valuation on K which is positive on A and \mathfrak{p} the set of $x \in A$ such that $v(x) > 0$. If the prime ideal \mathfrak{p} is of height 1, v is equivalent to an essential valuation of A .*

Proof. Let B be the ring of v and \mathfrak{m} its ideal. Then $\mathfrak{m} \cap A = \mathfrak{p}$ and hence $A_{\mathfrak{p}} \subseteq B$. Now $A_{\mathfrak{p}}$ is a DVR by ?? and $B \neq K$, thus $B = A_{\mathfrak{p}}$. \square

Theorem 14.1.28. *Let A be an integral domain. For A to be a Krull domain, it is necessary and sufficient that the following properties are satisfied:*

- (a) *For all prime ideal \mathfrak{p} of A of height 1, $A_{\mathfrak{p}}$ is a DVR.*
- (b) *$A = \bigcap_{\text{ht } \mathfrak{p}=1} A_{\mathfrak{p}}$.*
- (c) *For all $x \neq 0$ in A , there exists only a finite number of prime ideals \mathfrak{p} of height 1 such that $x \in \mathfrak{p}$.*

Moreover, in the admissible case, the valuations corresponding to the A for prime ideals of height 1 are the essential valuations of A .

Proof. The conditions are trivially sufficient. Their necessity follows immediately from ??, ?? and the fact that the essential valuations of A is a defining family. \square

Proposition 14.1.29. *Let A be an integrally closed Noetherian domain and \mathfrak{a} an integral ideal of A . Then \mathfrak{a} is divisorial if and only if the prime ideals associated with A/\mathfrak{a} are of height 1.*

Proof. Recall that, if $\mathfrak{a} = \bigcap_{i=1}^n \mathfrak{q}_i$ is a reduced primary decomposition of \mathfrak{a} and \mathfrak{p}_i denotes the prime ideal corresponding to \mathfrak{q}_i , the prime ideals associated with A/\mathfrak{a} are just the \mathfrak{p}_i . The necessity then follows from [??](#). Conversely, if, in the above notation, the \mathfrak{p}_i are of height 1, then $A_{\mathfrak{p}_i}$ is a DVR and each \mathfrak{p}_i is isolated, so $\mathfrak{q}_i = \mathfrak{q}_i A_{\mathfrak{p}_i} \cap A$ ([??](#)). Denoting by v_i the essential valuation corresponding to \mathfrak{p}_i , there therefore exists an integer n_i such that \mathfrak{q}_i is the set of $x \in A$ such that $v_i(x) \geq n_i$ (n_i is just the integer such that $\mathfrak{q}_i A_{\mathfrak{p}_i} = (\mathfrak{p}_i A_{\mathfrak{p}_i})^{n_i}$ in $A_{\mathfrak{p}_i}$). This shows the \mathfrak{q}_i are divisorial by [??](#), hence also is \mathfrak{a} . \square

Proposition 14.1.30. *Let A be a local Krull domain (in particular an integrally closed local Noetherian domain) and \mathfrak{m} its maximal ideal. The following conditions are equivalent:*

- (i) A is a DVR;
- (ii) \mathfrak{m} is invertible;
- (iii) $(A : \mathfrak{m}) \neq A$;
- (iv) \mathfrak{m} is divisorial;
- (v) \mathfrak{m} is the only non-zero prime ideal of A .

Proof. As every non-zero ideal of a DVR is principal, it is invertible and hence (i) implies (ii). If \mathfrak{m} is invertible, its inverse is $(A : \mathfrak{m})$ and hence $(A : \mathfrak{m}) \neq A$, hence (ii) implies (iii). If $(A : \mathfrak{m}) \neq A$, then $(A : (A : \mathfrak{m})) \neq A$. Now $\mathfrak{m} \subseteq (A : (A : \mathfrak{m}))$, hence $\mathfrak{m} = (A : (A : \mathfrak{m}))$ since \mathfrak{m} is maximal, so that \mathfrak{m} is divisorial. Thus (iii) implies (iv). The fact that (iv) implies (v) follows from [??](#). Finally, if \mathfrak{m} is the only non-zero prime ideal of A , it is of height 1 and hence $A_{\mathfrak{m}}$ is a DVR by [??](#). As A is local, $A_{\mathfrak{m}} = A$, which shows that (v) implies (i). \square

14.1.7 Permanence properties

Proposition 14.1.31. *Let A be a Krull domain, K its field of fractions, L a finite extension of K and R the integral closure of A in L . Then R is a Krull domain. The essential valuations of R are the normed discrete valuations on L which are equivalent to the extensions of the essential valuations of A .*

Proof. Let $(v_i)_{i \in I}$ be a complete family of extensions to L of the essential valuations of A . Since the degree $n = [L : K]$ is finite, the v_i are discrete valuations on L by [??](#). Let B_i be the ring of v_i . Then $R \subseteq \bigcap_{i \in I} B_i$ by [??](#). Conversely, every element of $x \in \bigcap_i B_i$ is integral over each of the essential valuation rings of A by [??](#), hence the coefficients of the minimal polynomial of x over K belong to A ([??](#)), so that $x \in R$; thus $R = \bigcap_i B_i$. Now let x be a non-zero element of R . It satisfies an equation of the form

$$x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 = 0$$

where $a_i \in A$ and $a_0 \neq 0$. If $v_i(x) > 0$, then $v_i(a_0) > 0$. Now the essential valuations v of A such that $v(a_0) > 0$ are finite in number and the valuations on L extending a given valuation on K are also finite in number, hence $v_i(x) = 0$ except for a finite number of indices $i \in I$. Thus it has been proved that R is a Krull domain.

It remains to show that the v_i are equivalent to essential valuations of R , that is, that the prime ideal \mathfrak{p}_i consisting of the $x \in R$ such that $v_i(x) > 0$ is of height 1. If this were not so,

there would exist a prime ideal \mathfrak{q} of R such that $(0) \subseteq \mathfrak{q} \subseteq \mathfrak{p}_i$ distinct from (0) and \mathfrak{p}_i . Then $(0) \subseteq \mathfrak{q} \cap A \subseteq \mathfrak{p}_i \cap A$ and $\mathfrak{q} \cap A$ would be distinct from (0) and $\mathfrak{p}_i \cap A$ by ?? . The prime ideal $\mathfrak{p}_i \cap A$ would therefore not be of height 1, which contradicts the fact that it corresponds to an essential valuation of A . \square

Corollary 14.1.32. *Let \mathfrak{p} (resp. \mathfrak{P}) be a prime ideal of A (resp. R) of height 1 and v (resp. w) the essential valuation of A (resp. R) corresponding to it. For \mathfrak{P} to lie over \mathfrak{p} , it is necessary and sufficient that the restriction of w to K be equivalent to v .*

Proof. The valuation w is equivalent to the extension of an essential valuation v' of A . Let $\mathfrak{q} = \mathfrak{P} \cap A$, which is a prime ideal of A of height 1. For the restriction of w to K to be equivalent to v , it is necessary and sufficient that $v' = v$ and hence that $\mathfrak{q} = \mathfrak{p}$. \square

Lemma 14.1.33. *Let A be an integral domain, K its field of fraction and L an extension of K . If $(A_i)_{i \in I}$ is a family of Krull domains contained in L satisfying the following two conditions:*

- (a) $A = \bigcap_i A_i$.
- (b) *Each nonzero element of A is a nonunit in only finitely many A_i .*

Then A is a Krull doamin.

Proof. For each $i \in I$ let $(v_{ij})_{j \in J_i}$ be a defining family of valuations of A_i . Let $x \in A$ be nonzero, then x is not a unit for only finitely many A_i 's. Also, for each i , since A_i is a Krull domain, $v_{ij}(x) \neq 0$ for only finitely many $j \in J_i$, whence we see the family $(v_{ij})_{i \in I, j \in J_i}$ satisfies the condition (K3) and hence A is a Krull domain. \square

Proposition 14.1.34. *Let A be a Krull domain, then the ring $A[X_1, \dots, X_n]$ is a Krull domain.*

Proof. Arguing by induction on n , it is sufficient to show that, if X is an indeterminate, $A[X]$ is a Krull domain. Let K be the field of fractions of A . The field of fractions of $A[X]$ is $K(X)$. The ring $K[X]$ is a PID and therefore a Krull ring. Moreover, for every prime ideal \mathfrak{p} of A of height 1, let $\bar{v}_{\mathfrak{p}}$ be the extension of v to $K(X)$ defined by

$$\bar{v}_{\mathfrak{p}}\left(\sum_j a_j X^j\right) = \inf_j v_{\mathfrak{p}}(a_j)$$

for $\sum_j a_j X^j \in K[X]$. Then the valuation ring of $\bar{v}_{\mathfrak{p}}$ in $K(X)$ is $A[X]_{\mathfrak{p}A[X]}$. We also note that $K \cap A[X]_{\mathfrak{p}A[X]} = A_{\mathfrak{p}}[X]$, in fact, for $\sum_j a_j X^j \in K[X]$, the relation $\bar{v}_{\mathfrak{p}}(\sum_j a_j X^j) \geq 0$ is equivalent to $v_{\mathfrak{p}}(a_j) \geq 0$ for all j , hence equivalent to $\sum_j a_j X^j \in A_{\mathfrak{p}}[X]$. With this, we have

$$A[X] = K[X] \cap \bigcap_{\text{ht } \mathfrak{p}=1} A[X]_{\mathfrak{p}A[X]}$$

Since A is Krull we see every nonzero element in $A[X]$ is a nonunit in only finitely many $A_{\mathfrak{p}}[X]$, so the condition of ?? is satisfied and we see $A[X]$ is a Krull domain. \square

Corollary 14.1.35. *Let A be a Krull domain, then the prime ideals of $A[X]$ of height 1 are:*

- (a) *the prime ideals of the form $\mathfrak{p}A[X]$, where \mathfrak{p} is a prime ideal of A of height 1;*
- (b) *the prime ideals of the form $\mathfrak{m} \cap A[X]$, where \mathfrak{m} is a (necessarily principal) prime ideal of $K[X]$.*

Proof. This follows from the observation that that the prime ideals of height 1 in $K[X]$ are exactly maximal ideals. \square

Proposition 14.1.36. *Let A be a Krull domain, then the ring $A[\![X_1, \dots, X_n]\!]$ is a Krull domain.*

Proof. By induction we only need to show $A[\![X]\!]$ is Krull. Since A is Krull we have $A = \bigcap_{\text{ht}\mathfrak{p}=1} A_{\mathfrak{p}}$, and therefore $A[\![X]\!] = \bigcap_{\text{ht}\mathfrak{p}=1} A_{\mathfrak{p}}[\![X]\!]$. Also, each $A_{\mathfrak{p}}$ is a DVR and thus $A_{\mathfrak{p}}[\![X]\!]$ is integrally closed and Noetherian, hence a Krull domain. However, as X is a non-unit of all the rings $A_{\mathfrak{p}}[\![X]\!]$, we can not apply ?? directly. On the other hand, note that

$$A[\![X]\!] = K[\![X]\!] \cap \bigcap_{\text{ht}\mathfrak{p}=1} (A_{\mathfrak{p}}[\![X]\!][X^{-1}]).$$

Now the hypothesis in ?? is easily verified for the rings $A_{\mathfrak{p}}[\![X]\!][X^{-1}]$. Indeed, an element $f(X) = \sum_{i=r}^{\infty} a_i X^i$ (with $a_r \neq 0$) in $A[\![X]\!]$ is a nonunit in $A_{\mathfrak{p}}[\![X]\!][X^{-1}]$ if and only if a_r is a nonunit of $A_{\mathfrak{p}}$, and there finitely many such \mathfrak{p} . Therefore $A[\![X]\!]$ is a Krull domain. \square

14.1.8 Divisor classes in a Krull domain

Let A be a Krull domain. Recall that the group $\mathfrak{D}(A)$ of divisors of A is the free commutative group generated by the set $P(A)$ of its irreducible elements and that $P(A)$ is identified with the set of prime ideals of A of height 1. For $\mathfrak{p} \in P(A)$ we shall denote by $v_{\mathfrak{p}}$ the normed essential valuation corresponding to \mathfrak{p} . Recall that the ring of $v_{\mathfrak{p}}$ is $A_{\mathfrak{p}}$. We shall denote by $\mathfrak{P}(A)$ the subgroup of $\mathfrak{D}(A)$ consisting of the principal divisors and by $\mathfrak{C}(A) = \mathfrak{D}(A)/\mathfrak{P}(A)$ the divisor class group of A .

Proposition 14.1.37. *Let A be a Krull domain and B a Krull domain containing A . Suppose that the following condition holds*

(PD) *For every prime ideal \mathfrak{P} of B of height 1, the prime ideal $\mathfrak{P} \cap A$ is zero or of height 1.*

For $\mathfrak{p} \in P(A)$ the $\mathfrak{P} \in P(B)$ such that $\mathfrak{P} \cap A = \mathfrak{p}$ are finite in number. We write

$$i(\mathfrak{p}) = \sum_{\mathfrak{P} \in P(B), \mathfrak{P} \cap A = \mathfrak{p}} e(\mathfrak{P}/\mathfrak{p}) \mathfrak{P},$$

where $e(\mathfrak{P}/\mathfrak{p})$ denotes the ramification index of $v_{\mathfrak{P}}$ over $v_{\mathfrak{p}}$. Then i defines, by linearity, an increasing homomorphism of $\mathfrak{D}(A)$ to $\mathfrak{D}(B)$, which enjoys the following properties:

(a) for every non-zero element x of the field of fractions of A ,

$$i(\text{div}_A(x)) = \text{div}_B(x),$$

(b) for all D_1, D_2 in $\mathfrak{D}(A)$,

$$i(\sup\{D_1, D_2\}) = \sup\{i(D_1), i(D_2)\}.$$

Proof. Let $\mathfrak{p} \in P(A)$, consider a non-zero element a of \mathfrak{p} ; the $\mathfrak{P} \in P(B)$ which contain a are finite in number by ?. A fortiori the $\mathfrak{P} \in P(B)$ such that $\mathfrak{P} \cap A = \mathfrak{p}$ are finite in number. We now show (a). By additivity, it may be assumed that $x \in A - \{0\}$. By definition, $\text{div}_B(x) =$

$\sum_{\mathfrak{P} \in P(B)} v_{\mathfrak{P}}(x) \mathfrak{P}$. For all $\mathfrak{P} \in P(B)$ such that $v_{\mathfrak{P}}(x) > 0$, $\mathfrak{P} \cap A$ is non-zero (for x is in it) and is therefore of height 1 by (PD). Setting $\mathfrak{p} = \mathfrak{P} \cap A$, by definition of the ramification index, $v_{\mathfrak{P}}(x) = e(\mathfrak{P}/\mathfrak{p})v_{\mathfrak{p}}(x)$ (since $v_{\mathfrak{P}}$ and $v_{\mathfrak{p}}$ are normed). As $\text{div}_A(x) = \sum_{\mathfrak{p} \in P(A)} v_{\mathfrak{p}}(x) \mathfrak{p}$, and $i(\mathfrak{q}) = 0$ for all $\mathfrak{q} \in P(A)$ which is not of the form $\mathfrak{Q} \cap A$ for some $\mathfrak{Q} \in P(B)$, we deduce (a).

To prove (b) we write

$$D_1 = \sum_{\mathfrak{p} \in P(A)} n_1(\mathfrak{p}) \mathfrak{p}, \quad D_2 = \sum_{\mathfrak{p} \in P(A)} n_2(\mathfrak{p}) \mathfrak{p}$$

the coefficient of \mathfrak{p} in $\sup\{D_1, D_2\}$ is $\sup\{n_1(\mathfrak{p}), n_2(\mathfrak{p})\}$. Let \mathfrak{P} be an element of $P(B)$. If $\mathfrak{P} \cap A = (0)$, the coefficients of \mathfrak{P} in $i(D_1), i(D_2)$, and hence also in $\sup\{i(D_1), i(D_2)\}$ are zero. Also the coefficient of \mathfrak{P} in $i(\sup\{D_1, D_2\})$ is zero. If $\mathfrak{P} \cap A \neq (0)$, it is a prime ideal \mathfrak{p} of height 1 (by (PD)); writing $e = e(\mathfrak{P}/\mathfrak{p})$, the coefficients of \mathfrak{P} in $i(D_1), i(D_2)$ and $i(\sup\{D_1, D_2\})$ are respectively $en_1(\mathfrak{p}), en_2(\mathfrak{p})$ and $e \sup\{n_1(\mathfrak{p}), n_2(\mathfrak{p})\}$. That of $\sup\{i(D_1), i(D_2)\}$ is

$$\sup\{en_1(\mathfrak{p}), en_2(\mathfrak{p})\} = e \sup\{n_1(\mathfrak{p}), n_2(\mathfrak{p})\}.$$

This proves (b). □

Under the hypotheses of ??, it follows from (a) that i defines, by taking quotients, a canonical homomorphism \bar{i} of $\mathfrak{C}(A)$ to $\mathfrak{C}(B)$.

Example 14.1.38. Let A, B be Krull domains and B be integral over A . In this case, for the prime ideal \mathfrak{P} of B to be of height 1, it is necessary and sufficient that $\mathfrak{p} = \mathfrak{P} \cap A$ be of height 1 (??), so the condition (PD) is satisfied.

Proposition 14.1.39. Let A and B be Krull domains such that B contains A and is a flat A -module. Then:

- (a) the condition (PD) is satisfied;
- (b) for every divisorial ideal \mathfrak{a} of A , $B\mathfrak{a}$ is the divisorial ideal of B which corresponds to the divisor $i(\text{div}_A(\mathfrak{a}))$.

Proof. To show (a), suppose that there exists a prime ideal \mathfrak{P} of B of height 1 such that $\mathfrak{P} \cap A$ is neither 0 nor of height 1. Take an element $x \neq 0$ in $\mathfrak{P} \cap A$. The ideals \mathfrak{p}_i of A of height 1 which contain x are finite in number and none contains $\mathfrak{P} \cap A$. There therefore exists an element y of $\mathfrak{P} \cap A$ such that $y \notin \mathfrak{p}_i$ for all i . Thus $\text{div}_A(x)$ and $\text{div}_A(y)$ are relatively prime elements of the ordered group $\mathfrak{D}(A)$, so that

$$\sup\{\text{div}_A(x), \text{div}_A(y)\} = \text{div}_A(x) + \text{div}_A(y) = \text{div}_A(xy).$$

As $\sup\{\text{div}_A(x), \text{div}_A(y)\} = \text{div}(Ax \cap Ay)$ and the ideals $Ax \cap Ay$ and Axy are divisorial, we deduce that $Ax \cap Ay = Axy$. Since B is a flat A -module, this implies $Bx \cap By = Bxy$ (??), and therefore

$$\sup\{v_{\mathfrak{P}}(x), v_{\mathfrak{P}}(y)\} = v_{\mathfrak{P}}(Bx \cap By) = v_{\mathfrak{P}}(xy) = v_{\mathfrak{P}}(x) + v_{\mathfrak{P}}(y).$$

Since $v_{\mathfrak{P}}(x)$ and $v_{\mathfrak{P}}(y)$ are positive integers (which hold since x and y are in \mathfrak{P}), this is a contradiction. Thus (a) has been proved by reductio ad absurdum.

We now show (b). If \mathfrak{a} is a divisorial ideal of A , it is the intersection of two fractional principal ideals (??), say

$$\mathfrak{a} = d^{-1}(Aa \cap Ab)$$

where a, b, d are in A and nonzero. As B is flat over A , $Ba = d^{-1}(Ba \cap Bb)$, which shows that $B\mathfrak{a}$ is divisorial. This shows also that $\text{div}_B(B\mathfrak{a}) = \sup\{\text{div}_B(a), \text{div}_B(b)\} - \text{div}_B(d)$, using ?? (a) and (b), it is seen that

$$\begin{aligned}\text{div}_B(B\mathfrak{a}) &= \sup\{i(\text{div}_A(a)), i(\text{div}_A(b))\} - i(\text{div}_A(d)) \\ &= i(\sup\{\text{div}_A(a), \text{div}_A(b)\}) - i(\text{div}_A(d)) \\ &= i(\text{div}_A(Aa \cap Ab)) - i(\text{div}_A(d)) \\ &= i(\text{div}_A(d^{-1}(Aa \cap Ab))) = i(\text{div}_A(a))\end{aligned}$$

which completes the proof. \square

Corollary 14.1.40. *Let A be a local Krull domain and B a DVR such that B dominates A and is a flat A -module. Then A is a field or a DVR.*

Proof. Let \mathfrak{M} be the maximal ideal of B . By (PD), $\mathfrak{M} \cap A$ is zero or of height 1. As it is, by hypothesis the maximal ideal of A , our assertion follows from ?? \square

Remark 14.1.2. As the elements of $P(B)$ form a basis of $\mathfrak{D}(B)$ and two distinct ideals of $P(A)$ cannot be the traces on A of the same ideal of $P(B)$, for the injectivity of i it amounts to verifying that $i(\mathfrak{p}) \neq 0$ for all $\mathfrak{p} \in P(A)$. Now this is equivalent to say every $\mathfrak{p} \in P(A)$ is the contraction of an element in $P(B)$, and in particular this holds if B is a faithfully flat A -algebra or B is a superring of A and integral over A .

In what follows, we propose to study the canonical homomorphism i from $\mathfrak{C}(A)$ to $\mathfrak{C}(B)$ for certain ordered pairs of Krull domains A, B .

Proposition 14.1.41. *Let A be a Zariski ring such that its completion \widehat{A} is a Krull domain. Then A is a Krull domain and the canonical homomorphism \bar{i} from $\mathfrak{C}(A)$ to $\mathfrak{C}(\widehat{A})$ (which is defined since \widehat{A} is a flat A -module) is injective.*

Proof. As \widehat{A} is an integral domain and $A \subseteq \widehat{A}$, A is an integral domain. Let L be the field of fractions of \widehat{A} and $K \subseteq L$ that of A . As $A = \widehat{A} \cap K$ (??), A is a Krull domain. The fact that $\bar{i} : \mathfrak{C}(A) \rightarrow \mathfrak{C}(\widehat{A})$ is injective follows from ?? and the fact that, if $\mathfrak{b}\widehat{A}$ is principal, \mathfrak{b} is principal (??) \square

Now let A be a Krull domain and S a multiplicative subset of A not containing 0. The group $\mathfrak{D}(A)$ (resp. $\mathfrak{D}(S^{-1}A)$) is the free commutative group with basis the set of $\text{div}(\mathfrak{p})$ (resp. $\text{div}(S^{-1}\mathfrak{p})$), where \mathfrak{p} runs through the set of prime ideals of A of height 1 (resp. the set of prime ideals of A of height 1 such that $\mathfrak{p} \cap S = \emptyset$) and, if $\mathfrak{p} \cap S = \emptyset$, then $i(\text{div}(\mathfrak{p})) = \text{div}(S^{-1}\mathfrak{p})$. Thus $\mathfrak{D}(S^{-1}A)$ is identified with the direct factor of $\mathfrak{D}(A)$ generated by the elements $\text{div}(\mathfrak{p})$ such that $\mathfrak{p} \cap S = \emptyset$ and admits as complement the free commutative subgroup of $\mathfrak{D}(A)$ with basis the set of $\text{div}(\mathfrak{p})$ such that $\mathfrak{p} \cap S \neq \emptyset$. We shall denote this complement by $\mathfrak{G}(S)$. As the map $i : \mathfrak{D}(A) \rightarrow \mathfrak{D}(S^{-1}A)$ is surjective, so is the induced map $\bar{i} : \mathfrak{C}(A) \rightarrow \mathfrak{C}(S^{-1}A)$ and

$$\ker \bar{i} = (\mathfrak{G}(S) + \mathfrak{P}(A)) / \mathfrak{P}(A) = \mathfrak{G}(S) / (\mathfrak{G}(S) \cap \mathfrak{P}(A)). \quad (14.1.3)$$

In fact, if an element of $\mathfrak{D}(S^{-1}A)$ is equal to $\text{div}_{S^{-1}A}(x/s)$, where $x \in A$ and $s \in S$, it is then the image under i of the principal divisor $\text{div}_A(x/s)$ (??).

Proposition 14.1.42. *Let A be a Krull domain and S a multiplicative subset of A not containing 0. Then the canonical homomorphism \bar{i} from $\mathfrak{C}(A)$ to $\mathfrak{C}(S^{-1}A)$ is surjective. If further S is generated by a family of elements p_i such that the principal ideals Ap_i are all prime, then \bar{i} is bijective.*

Proof. Suppose now that S is generated by a family of elements $(p_i)_{i \in I}$ of A such that the principal ideals Ap_i are all prime. Then, if \mathfrak{p} is a prime ideal of A of height 1 such that $\mathfrak{p} \cap S \neq \emptyset$, \mathfrak{p} contains a product of powers of the p_i and therefore one of the p_i , say p_0 . As Ap_0 is non-zero and prime and \mathfrak{p} is of height 1, it follows that $\mathfrak{p} = Ap_0$. In the above notation, we then have $\mathfrak{G}(S) \subseteq \mathfrak{P}(A)$ and the kernel of \bar{i} is zero. \square

Proposition 14.1.43. *Let R be a Krull domain and consider the polynomial ring $R[X]$. The canonical homomorphism of $\mathfrak{C}(R)$ to $\mathfrak{C}(R[X])$ is bijective.*

Proof. Let R be a Krull domain; take A to be the polynomial ring $A = R[X]$ and S to be the set $R - (0)$ of non-zero constant polynomials of A . The prime ideals \mathfrak{p} of A of height 1 such that $\mathfrak{p} \cap S \neq \emptyset$ are those of the form \mathfrak{p}_0A , where \mathfrak{p}_0 is a prime ideal of R of height 1. Hence, in the notation introduced above, $\mathfrak{G}(S)$ is identified with $\mathfrak{D}(R)$ by identifying $\text{div}_A(\mathfrak{p}_0A)$ with $\text{div}_R(\mathfrak{p}_0)$. On the other hand $\mathfrak{G}(S) \cap \mathfrak{P}(A)$ is identified with $\mathfrak{P}(R)$: for if an ideal \mathfrak{a}_0 of R generates a principal ideal $f(X)A$ in $A = R[X]$, then $f(0) \in \mathfrak{a}_0A$ since \mathfrak{a}_0A is a graded ideal of the ring A (graded by the usual degree of polynomials) and hence $f(0) \in \mathfrak{a}_0$. Further, for any $a \in \mathfrak{a}_0$ we have $a = f(X)g(X)$ where $g(X) \in R$, whence $a = f(0)g(0)$. It follows that \mathfrak{a}_0 is the principal ideal of R generated by $f(0)$. Finally, denoting by K the field of fractions of R , $S^{-1}A$ is identified with the polynomials ring $K[X]$, which is a principal ideal domain; hence $\mathfrak{C}(S^{-1}A) = (0)$. Thus $\mathfrak{C}(A) = \ker \bar{i}$ is identified with $\mathfrak{C}(R)$ and we have proved the proposition. \square

14.1.9 Dedekind domains

Let A be an integral domain. Clearly the following conditions are equivalent:

- (i) no two of the non-zero prime ideals of A are comparable with respect to inclusion;
- (ii) the non-zero prime ideals of A are maximal;
- (iii) the non-zero prime ideals of A are of height 1.

A Krull domain which satisfies the above equivalent conditions is called a **Dedekind domain**.

Example 14.1.44 (Examples of Dedekind domains).

- (a) Every principal ideal domain is a Dedekind domain.
- (b) Let K be a finite extension of \mathbb{Q} and A the integral closure of \mathbb{Z} over K . The ring A is a Krull domain by ???. Let \mathfrak{p} be a non-zero prime ideal of A . Then by the going down theorem $\mathfrak{p} \cap \mathbb{Z}$ is non-zero and hence is a maximal ideal of \mathbb{Z} . Hence \mathfrak{p} is a maximal ideal of A . Therefore, A is a Dedekind domain. In general, A is not a principal ideal domain.

- (c) Let V be an affine algebraic variety and A the ring of functions regular on V . Suppose that A is not a field (i.e. that V is not reduced to a point). For A to be a Dedekind domain, it is necessary and sufficient that V be an irreducible curve with no singular point: for to say that A is an integral domain amounts to saying that V is irreducible; to say that every non-zero prime ideal of A is maximal amounts to saying that $\dim A = 1$, so A is a curve; finally, as A is Noetherian, to say that it is a Krull domain amounts to saying that it is integrally closed, that is, V is a normal curve, or also that it has no singular point.
- (d) A ring of fractions $S^{-1}A$ of a Dedekind domain A is a Dedekind domain if $0 \notin S$. For $S^{-1}A$ is a Krull domain and every nonzero prime ideal of $S^{-1}A$ is maximal.

Theorem 14.1.45. *Let A be an integral domain and K its field of fractions. Then the following conditions are equivalent:*

- (i) *A is a Dedekind domain;*
- (ii) *A is a Krull domain and every nontrivial valuation on K which is positive on A is equivalent to an essential valuation of A ;*
- (iii) *A is a Krull domain and every nonzero fractional ideal \mathfrak{a} of A is divisorial;*
- (iv) *every nonzero fractional ideal \mathfrak{a} of A is invertible;*
- (v) *A is a Noetherian integrally closed domain and every non-zero prime ideal of A is maximal;*
- (vi) *A is Noetherian and, for every maximal ideal \mathfrak{m} of A , $A_{\mathfrak{m}}$ is either a field or a DVR;*
- (vii) *A is Noetherian and, for every maximal ideal \mathfrak{m} of A , $A_{\mathfrak{m}}$ is a PID.*

Proof. We show first the equivalence of (i) and (ii). ?? shows immediately that (i) implies (ii). Conversely, (ii) implies (i), since, for every prime ideal \mathfrak{p} of A , there exists a valuation ring of K which dominates $A_{\mathfrak{p}}$, and (ii) this implies that \mathfrak{p} has height 1, so A is a Dedekind domain.

If A is a Dedekind domain and \mathfrak{a} is a non-zero fractional ideal, then $\mathfrak{a}A_{\mathfrak{m}} = \tilde{\mathfrak{a}}A_{\mathfrak{m}}$ for every maximal ideal \mathfrak{m} (by ??) and hence $\mathfrak{a} = \tilde{\mathfrak{a}}$; thus (i) implies (iii).

We now show that (iii) implies (iv). If (iii) holds, the map $\mathfrak{a} \mapsto \text{div}(\mathfrak{a})$ is a bijection of $\mathfrak{F}(A)$ onto $\mathfrak{D}(A)$, as it is a homomorphism and $\mathfrak{D}(A)$ is a group, every element of $\mathfrak{F}(A)$ is invertible.

We show that (iv) implies (v). If (iv) holds, every integral ideal of A is finitely generated and hence A is Noetherian. As $\mathfrak{F}(A)$ is a group, $\mathfrak{D}(A)$ is a group and A is therefore completely integrally closed. Finally, if \mathfrak{p} is a non-zero prime ideal of A and \mathfrak{m} is a maximal ideal of A containing \mathfrak{p} , the ring $A_{\mathfrak{m}}$ is a PID by ???. As $\mathfrak{p}A_{\mathfrak{m}}$ is prime and non-zero, necessarily $\mathfrak{p}A_{\mathfrak{m}} = \mathfrak{m}A_{\mathfrak{m}}$, whence $\mathfrak{p} = \mathfrak{m}$ and \mathfrak{p} is maximal.

We now show that (v) implies (vi). If \mathfrak{m} is a maximal ideal of A and (v) holds, $A_{\mathfrak{m}}$ is an integrally closed Noetherian domain and its maximal ideal $\mathfrak{m}A_{\mathfrak{m}}$ is, either (0), or the only non-zero prime ideal of $A_{\mathfrak{m}}$. Hence $A_{\mathfrak{m}}$ is a field or a DVR by ??.

The fact (vi) implies (vii) is obvious. We show finally that (vii) implies (i). As A is the intersection of the $A_{\mathfrak{m}}$ where \mathfrak{m} runs through the set of maximal ideals (??), (vii) implies that A is integrally closed and Noetherian and hence that A is a Krull domain. On the other hand, it can be shown that every non-zero prime ideal of A is maximal as in the proof that (iv) implies (v). \square

Proposition 14.1.46. *A semi-local Dedekind domain is a PID.*

Proof. Let A be a semi-local Dedekind domain, K its field of fractions, $\mathfrak{m}_1, \dots, \mathfrak{m}_n$ its maximal ideals and v_1, \dots, v_n the corresponding essential valuations. These are the only essential valuations of A . Let \mathfrak{a} be a non-zero integral ideal of A . Since it is divisorial, there exists integers q_1, \dots, q_r such that \mathfrak{a} is the set of $x \in K$ such that $v_i(x) \geq q_i$ for all i (??). Let x_0 be an element of K such that $v_i(x_0) = q_i$ for all i (??). Then \mathfrak{a} is the set of $x \in K$ such that $v_i(xx_0^{-1}) \geq 0$ for all i and thus $\mathfrak{a} = Ax_0$. \square

If A is a Dedekind domain, it has been seen, in ??, that the group $\mathfrak{D}(A)$ of divisors of A is identified with the group $\mathfrak{F}(A)$ of nonzero fractional ideals \mathfrak{a} (as A is Noetherian, every non-zero fractional ideal is finitely generated). The divisor class group $\mathfrak{C}(A)$ of A is then identified with the group of class of nonzero ideals of A .

Let A be a Dedekind domain, $\mathfrak{F}(A)$ the ordered multiplicative group of nonzero fractional ideals of A and $\mathfrak{D}(A)$ the group of divisors of A . The isomorphism $\mathfrak{a} \mapsto \text{div}(\mathfrak{a})$ of $\mathfrak{F}(A)$ onto $\mathfrak{D}(A)$ maps the non-zero prime ideals of A to irreducible divisors and hence the multiplicative group $\mathfrak{F}(A)$ admits as basis the set of non-zero prime ideals of A . In other words, every non-zero fractional ideal \mathfrak{a} of A admits a unique decomposition of the form:

$$\mathfrak{a} = \prod \mathfrak{p}^{n_{\mathfrak{p}}} \quad (14.1.4)$$

where the product extends to the non-zero prime ideals of A , and the exponents $n_{\mathfrak{p}}$ equals $v_{\mathfrak{p}}(\mathfrak{a})$, where $v_{\mathfrak{p}}$ denotes the essential valuation corresponding to \mathfrak{p} . Further \mathfrak{a} is integral if and only if the $n_{\mathfrak{p}}$ are all positive. The relation (??) is called the **decomposition of \mathfrak{a} into prime factors**. For $\mathfrak{a}, \mathfrak{b} \in \mathfrak{F}(A)$, write

$$\mathfrak{a} = \prod \mathfrak{p}^{n_{\mathfrak{p}}}, \quad \mathfrak{b} = \prod \mathfrak{p}^{m_{\mathfrak{p}}}$$

then we have

$$\begin{aligned} \mathfrak{a}\mathfrak{b} &= \prod \mathfrak{p}^{n_{\mathfrak{p}}+m_{\mathfrak{p}}}, & (\mathfrak{a} : \mathfrak{b}) &= \mathfrak{a}\mathfrak{b}^{-1} = \prod \mathfrak{p}^{n_{\mathfrak{p}}-m_{\mathfrak{p}}}, \\ \mathfrak{a} + \mathfrak{b} &= \prod \mathfrak{p}^{\inf\{n_{\mathfrak{p}}, m_{\mathfrak{p}}\}}, & \mathfrak{a} \cap \mathfrak{b} &= \prod \mathfrak{p}^{\sup\{n_{\mathfrak{p}}, m_{\mathfrak{p}}\}}. \end{aligned}$$

Proposition 14.1.47. *Let A be a Dedekind domain, K its field of fractions and \mathcal{P} the set of non-zero prime ideals of A . For $\mathfrak{p} \in \mathcal{P}$ let $v_{\mathfrak{p}}$ denote the corresponding essential valuation of A . Let $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ be distinct elements of \mathcal{P} and n_1, \dots, n_r integers and x_1, \dots, x_r elements of K . Then there exists $x \in K$ such that $v_{\mathfrak{p}_i}(x - x_i) \geq n_i$ for all i and $v_{\mathfrak{p}}(x) \geq 0$ for all $\mathfrak{p} \in \mathcal{P}$ distinct from the \mathfrak{p}_i .*

Proof. Replacing if need be the n_i by greater integers, they may be assumed all to be positive. We examine first the case where the x_i are in A . It obviously amounts to finding an $x \in A$ satisfying the congruences

$$x \equiv x_i \pmod{\mathfrak{p}_i^{n_i}}$$

and the existence of x then follows from Chinese remainder theorem.

We pass now to the general case. We may write $x_i = y_i/s$, where y_i, s are in A . Writing $x = y/s$, it amounts to finding a $y \in A$ such that, on the one hand, $v_{\mathfrak{p}_i}(y - y_i) \geq n_i + v_{\mathfrak{p}_i}(s)$ and, on the other, $v_{\mathfrak{p}}(y) \geq v_{\mathfrak{p}}(s)$ for all $\mathfrak{p} \in \mathcal{P}$ distinct from the \mathfrak{p}_i . As $v_{\mathfrak{p}}(s) = 0$ except for a finite number of indices \mathfrak{p} , it is thus reduced to the above case, whence the proposition. \square

Proposition 14.1.48. *Let A be a Dedekind domain and \mathfrak{a} a fractional ideal of A . Then \mathfrak{a} is generated by at most two elements.*

Proof. By multiplying by an element of A , we can reduce to the case that \mathfrak{a} is an ideal of A . Write $\mathfrak{a} = \prod \mathfrak{p}_i^{n_i}$ as a finite product and choose $a \in \mathfrak{a}$ with $aA = \prod \mathfrak{q}_j^{m_j} \prod \mathfrak{p}_i^{f_i}$ and the \mathfrak{q}_j different from \mathfrak{p}_i . Then since $aA \subseteq \mathfrak{a}$ we have $f_i \geq n_i$ for each i . Choose $b \in A$ such that $v_{\mathfrak{p}_i}(b) = n_i$ and $v_{\mathfrak{q}_j}(b) = 0$, then we see

$$aA + bA = \prod \mathfrak{p}_i^{\inf\{f_i, f_i+1\}} \mathfrak{q}_j^{\inf\{0, m_j\}} = \prod \mathfrak{p}_i^{n_i} = \mathfrak{a}.$$

This proves the claim. \square

14.1.10 The Krull-Akizuki theorem

Lemma 14.1.49. *Let A be a one-dimensional Noetherian domain and M a finitely generated torsion A -module. Then the length $\ell_A(M)$ of M is finite.*

Proof. As M is a torsion module, every prime ideal associated with M is nonzero and therefore maximal. The lemma then follows from [??](#). \square

Lemma 14.1.50. *Let A be a ring, M an A -module and $(M_i)_{i \in I}$ a directed family of submodules of M with union M . Then $\ell_A(M) = \sup_i \ell_A(M_i)$.*

Proof. We have $\ell_A(M_i) \leq \ell_A(M)$ for all i . The lemma is obvious if no integer exceeds the $\ell_A(M_i)$, both sides then being infinite. Otherwise, let i_0 be an index for which $\ell_A(M_{i_0})$ takes its greatest value. Then $M = M_{i_0}$ since the family (M_i) is directed, whence our assertion in this case. \square

Lemma 14.1.51. *Let A be a one-dimensional Noetherian domain and M a torsion-free A -module of rank $r < \infty$. Then for every nonzero $a \in A$, we have*

$$\ell_A(M/aM) \leq r \cdot \ell_A(A/aA). \quad (14.1.5)$$

Proof. [??](#) shows that $\ell_A(A/Aa)$ is finite. We show [??](#) first in the case where M is finitely generated. As M is torsion-free and of rank r , there exists a submodule L of M which is isomorphic to A^r and such that $Q = M/L$ is a finitely generated torsion A -module and hence of finite length [??](#). For every integer $n \geq 1$, we have an exact sequence

$$0 \longrightarrow L/(a^n M \cap L) \longrightarrow M/a^n M \longrightarrow Q/a^n Q \longrightarrow 0$$

as $a^n L \subseteq a^n M \cap L$, this implies

$$\ell_A(M/a^n M) \leq \ell_A(L/a^n L) + \ell_A(Q/a^n Q) \leq \ell_A(L/a^n L) + \ell_A(Q). \quad (14.1.6)$$

Now, since M is torsion-free, multiplication by a defines an isomorphism of M/aM onto aM/a^2M and similarly for L ; whence, by induction on n , the formulae:

$$\ell_A(M/a^n M) = n \cdot \ell_A(M/aM), \quad \ell_A(L/a^n L) = n \cdot \ell_A(L/aL) \quad (14.1.7)$$

Taking account of [??](#), we deduce that

$$\ell_A(M/aM) \leq \ell_A(L/aL) + n^{-1} \ell_A(Q) \quad (14.1.8)$$

for all $n > 0$. As L is isomorphic to A^r , we see $\ell_A(L/aL) = r\ell_A(A/aA)$, so we get (??) by letting n tend to infinity in (??).

We now pass to the general case. Let (M_i) be the family of finitely generated submodules of M . Then the module $T = M/aM$ is the union of the submodules $T_i = M_i/(M_i \cap aM)$. Now, T_i is isomorphic to a quotient of M_i/aM_i and hence

$$\ell_A(T_i) \leq r \cdot \ell_A(A/aA)$$

by what we have just proved. Whence $\ell_A(T) \leq r \cdot \ell_A(A/aA)$ by ??.

□

Proposition 14.1.52 (Krull-Akizuki). *Let A be a one-dimensional Noetherian domain, K its field of fractions, L a finite extension of K and B a subring of L containing A . Then B is a one-dimensional Noetherian domain. Moreover, for every nonzero ideal \mathfrak{b} of B , B/\mathfrak{b} is an A -module of finite length.*

Proof. Let \mathfrak{b} be a non-zero ideal of B . We shall show that B/\mathfrak{b} is an A -module of finite length (hence, a fortiori, a B -module of finite length) and that \mathfrak{b} is a finitely generated B -module. A non-zero element y of \mathfrak{b} satisfies an equation of the form:

$$a_n y^n + a_{n-1} y^{n-1} + \cdots + a_1 y + a_0 = 0, \quad a_i \in A, a_0 \neq 0.$$

This equation shows that $a_0 \in By \subseteq \mathfrak{b}$. Applying ?? to $M = B$, it is seen that B/a_0B is an A -module of finite length, and so is B/\mathfrak{b} which is a quotient module of it. Further the B -module \mathfrak{b} contains, as a submodule, a_0B which is finitely generated. As \mathfrak{b}/a_0B is of finite length (as a submodule of B/a_0B) and hence finitely generated, \mathfrak{b} is certainly a finitely generated B -module.

The above shows first that B is Noetherian. On the other hand, if \mathfrak{P} is a nonzero prime ideal of B , the ring B/\mathfrak{P} is an integral domain and of finite length and hence is a field, so that \mathfrak{P} is maximal.

□

Corollary 14.1.53. *Let A be a one-dimensional Noetherian domain, K its field of fractions, L a finite extension of K and B a subring of L containing A . Then for every prime ideal \mathfrak{p} of A , the set of prime ideals of B lying over \mathfrak{p} is finite.*

Proof. Suppose first that $\mathfrak{p} = (0)$. Then the only prime ideal \mathfrak{P} of B such that $\mathfrak{P} \cap A = (0)$ is (0) : otherwise, writing $S = A - \{0\}$, then $S^{-1}\mathfrak{P}$ would be a non-zero prime ideal of $S^{-1}B$ and $S^{-1}B$ is just the field of fractions of B , for it is a subring of L containing K , whence an absurd conclusion. If \mathfrak{p} is nonzero, it follows from ?? that $B/\mathfrak{p}B$ is a finite-dimensional vector space over the field A/\mathfrak{p} , hence an Artinian ring and therefore has only a finite number of prime ideals, which proves that there is only a finite number of prime ideals of B containing \mathfrak{p} .

□

Corollary 14.1.54. *Let A be a one-dimensional Noetherian domain, K its field of fractions, L a finite extension of K and B a subring of L containing A . Then the integral closure of A in L is a Dedekind domain. In particular, the integral closure of a Dedekind domain in a finite extension of its field of fractions is a Dedekind domain.*

Proof. This integral closure is a one-dimensional integrally closed Noetherian domain, hence Dedekind.

□

Proposition 14.1.55. Let A be a Dedekind domain, K its field of fractions, L a finite extension of K and B the integral closure of A in L . Let \mathfrak{p} be a non-zero prime ideal of A , $v_{\mathfrak{p}}$ the corresponding essential valuation of K and

$$B\mathfrak{p} = \prod_i \mathfrak{P}_i^{e_i}$$

the decomposition of the ideal $B\mathfrak{p}$ as a product of prime ideals. Then:

- (a) the prime ideals of B lying over \mathfrak{p} are the \mathfrak{P}_i such that $e_i > 0$.
- (b) the valuations v_i on L corresponding to these ideals \mathfrak{P}_i are, up to equivalence, the valuations on L extending $v_{\mathfrak{p}}$.
- (c) $[B/\mathfrak{P}_i : A/\mathfrak{p}] = f(v_i/v)$ and $e_i = e(v_i/v)$.

Proof. To say that a prime ideal \mathfrak{P} of B lies over \mathfrak{p} amounts to saying that $\mathfrak{P} \supseteq \mathfrak{p}$, hence that $\mathfrak{P} \supseteq B\mathfrak{p}$ and that \mathfrak{P} contains one of the \mathfrak{P}_i , such that $e_i > 0$. Now part (b) follows from ???. Finally, the residue field of v is identified with A/\mathfrak{p} and that of v_i with B/\mathfrak{P}_i , hence the first claim of (c). Let t (resp. t_i) be a uniformizer for v (resp. v_i). Then

$$tB\mathfrak{p}_i = tA_{\mathfrak{p}}B\mathfrak{p}_i = \mathfrak{p}B\mathfrak{p}_i = \left(\prod_i \mathfrak{P}_j^{e_j} \right) B\mathfrak{p}_i = \prod_j (\mathfrak{p}_j B\mathfrak{p}_i)^{e_j} = (\mathfrak{p}_i B\mathfrak{p}_i)^{e_i} = t_i^{e_i} B\mathfrak{p}_i$$

since $\mathfrak{p}_j B\mathfrak{p}_i = B\mathfrak{p}_i$ for $j \neq i$, whence the second claim of (c), since $e(v_i/v) = v_i(t)$. \square

14.1.11 Unique factorization domains

A Krull domain all of whose divisorial ideals are principal is called a **factorial** or **unique factorization domain**. In other words, A is factorial if and only if $\mathfrak{C}(A)$ is reduced to 0.

Example 14.1.56.

- (a) Every PID is factorial (and, recall, is a Dedekind domain). Conversely, every factorial Dedekind domain is a PID by ??.
- (b) In particular, if K is a field, the rings $K[X]$ and $K[\![X]\!]$ are factorial domains.
- (c) The local ring of a simple point of an algebraic variety is a factorial domain. The ring of germs of functions analytic at the origin of \mathbb{C}^n is a factorial domain.

Theorem 14.1.57. Let A be an integral domain. The following conditions are equivalent

- (i) A is factorial;
- (ii) the ordered group of non-zero fractional principal ideal of A is a direct sum of groups isomorphic to \mathbb{Z} (ordered by the product order).
- (iii) Every non-empty family of integral principal ideals of A has a maximal element and the intersection of two principal ideals of A is a principal ideal;
- (iv) Every non-empty family of integral principal ideals of A has a maximal element and every irreducible element of A is prime;
- (v) A is a Krull domain and every prime ideal of height 1 is principal.

Proof. We shall denote by K the field of fractions of A and by \mathcal{P} (or $\mathcal{P}(A)$) the ordered group of non-zero fractional principal ideals of A . We show that (i) implies (ii). If A is factorial, \mathcal{P} is isomorphic to the group of divisors of A and hence to a direct sum of groups \mathbb{Z} .

Note now that the relation "the intersection of two integral principal ideals of A is a principal ideal" means that every ordered pair of elements of A admits a lcm, that is, \mathcal{P} is a lattice-ordered group. The fact that (ii) implies (iii) (and even is equivalent to it) therefore follows from ???. The fact that (iii) implies (iv) follows from ???. The fact that (iv) implies (ii) follows from ?? applied to the group \mathcal{P} .

We show that (ii) implies (v). If (ii) holds, there is an isomorphism of \mathcal{P} onto $\mathbb{Z}^{\oplus I}$. Let $(v_i(x))_{i \in I}$ denote the element of $\mathbb{Z}^{\oplus I}$ corresponding to the ideal Ax . It is seen immediately that each v_i is a discrete valuation on K , that A is the intersection of the rings of the v_i and that, for $x \neq 0$ in A , $v_i(x) \neq 0$ for a finite number of indices i . Hence A is a Krull domain. On the other hand, let \mathfrak{p} be a prime ideal of A of height 1; it contains a non-zero element a which is necessarily not invertible and hence also (by definition of a prime ideal) one of the irreducible elements p of A . As p is prime and non zero, $\mathfrak{p} = p$, which proves that \mathfrak{p} is principal.

Finally we show that (v) implies (i). Let \mathfrak{a} be a divisorial ideal of A . There exist prime ideals \mathfrak{p}_i of A of height 1 such that $\text{div}(\mathfrak{a}) = \sum_i n_i \text{div}(\mathfrak{p}_i)$ where $n_i \in \mathbb{Z}$. If (v) holds, \mathfrak{p}_i is of the form Ap_i whence $\text{div}(\mathfrak{a}) = \text{div}(Ap_i^{n_i})$ and hence $\mathfrak{a} = \prod_i Ap_i^{n_i}$ since \mathfrak{a} is divisorial. \square

Proposition 14.1.58. *Let A be a Krull domain. If every divisorial ideal of A is invertible, then, for every maximal ideal \mathfrak{m} of A , $A_{\mathfrak{m}}$ is factorial. The converse is true if it is also assumed that every divisorial ideal of A is finitely generated (in particular if A is Noetherian).*

Proof. Suppose that every divisorial ideal of A is invertible. As $A_{\mathfrak{m}}$ is a Krull domain, every divisorial ideal \mathfrak{a} of $A_{\mathfrak{m}}$ is the intersection of two principal fractional ideals (??), hence $\mathfrak{a} = \mathfrak{b}A_{\mathfrak{m}}$, where \mathfrak{b} is a divisorial ideal of A (??). As \mathfrak{b} is invertible by hypothesis, we deduce from ?? that \mathfrak{a} is principal and hence $A_{\mathfrak{m}}$ is a factorial domain.

Conversely, if all the $A_{\mathfrak{m}}$ are factorial and \mathfrak{a} is a finitely generated divisorial ideal of A , then $\mathfrak{a}A_{\mathfrak{m}}$ is a divisorial ideal of $A_{\mathfrak{m}}$, as follows from ?? and ???. By hypothesis $\mathfrak{a}A_{\mathfrak{m}}$ is principal and hence it follows from ?? that \mathfrak{a} is invertible. \square

Let A be an integral domain, K its field of fractions and $U(A)$ the multiplicative group of invertible elements of A . Recall that there is a canonical isomorphism of K^{\times}/U onto the group \mathcal{P} of non-zero fractional principal ideals of A . Condition (ii) of ?? may then be translated as follows:

Proposition 14.1.59. *Let A be an integral domain. For A to be factorial, it is necessary and sufficient that there exist a subset P of A such that every $a \in A$ may be written uniquely in the form $a = u \prod_{p \in P} p^{n_p}$, where $u \in U(A)$ and the n_p are positive integers which are zero except for a finite number of them.*

If P satisfies this condition, clearly all its elements are irreducible and every irreducible element of A is associated with a unique element of P . Recall that P is then called a representative system of irreducible elements of A .

Suppose always that A is factorial. It has been seen that the group \mathcal{P} is a lattice. In particular, every element of K^{\times} may be written in an essentially unique way in the form of an

irreducible fraction. Any two elements a, b of K^\times have a gcd and a lcm. If $a = u \prod p^{n_p}$ and $b = v \prod p^{m_p}$ are decompositions of a and b as products of irreducible elements, then, then

$$\gcd(a, b) = w \prod p^{\inf\{n_p, m_p\}}, \quad \text{lcm}(a, b) = w \prod p^{\sup\{n_p, m_p\}}.$$

For all $p \in P$, the map $a \mapsto n_p$ is a discrete valuation v_p on K whose ring is obviously A_{Ap} . It follows from ?? that the v_p are just the essential valuations of A and that the ideals Ap are just the prime ideals of A of height 1.

Proposition 14.1.60. *Let A be a Krull domain and S a multiplicative subset of A not containing 0.*

- (a) *If A is factorial, $S^{-1}A$ is factorial;*
- (b) *If S is generated by a family of elements p_i such that the principal ideals Ap_i are prime and $S^{-1}A$ is factorial, then A is factorial.*

Proof. This follows from the definition of factorial domains and ??.

□

Let A be a factorial domain, K its field of fractions and f a non-zero element of $K[X]$. The **content** of f is defined to be the gcd of the coefficients of f (up to a unit of A) and denote by $\text{cont}(f)$. Let v be a valuation on K which is essential for A and \bar{v} its canonical extension to $K[X]$, then

$$\bar{v}(f) = v(\text{cont}(f)).$$

Moreover, it is easy to see $\text{cont}(f)$ is uniquely determined by this property.

Lemma 14.1.61 (Guass's Lemma). *Let f, g be non-zero elements of $K[X]$, then*

$$\text{cont}(fg) = \text{cont}(f)\text{cont}(g).$$

Proof. For every valuation v on K which is essential for A , let \bar{v} denote its canonical extension to $K[X]$. Then

$$v(\text{cont}(fg)) = \bar{v}(fg) = \bar{v}(f) + \bar{v}(g) = v(\text{cont}(f)) + v(\text{cont}(g)) = v(\text{cont}(fg))$$

which proves the claim.

□

Note that for every $f \in K[X]$, $f/\text{cont}(f)$ is a polynomial with content 1 (such a polynomial is called **primitive**) and we can always write

$$f = \text{cont}(f)\tilde{f}$$

where \tilde{f} is a primitive polynomial.

Theorem 14.1.62. *Let A be a factorial domain, K its field of fractions, (p_i) a representative system of irreducible elements of A and (P_α) a representative system of primitive irreducible polynomials of $K[X]$. Then:*

- (a) *$A[X]$ is a factorial domain;*
- (b) *the set p_i and P_α is a representative system of irreducible elements of $A[X]$.*

Proof. Let f be a non-zero element of $A[X]$. In the ring $K[X]$, f can be decomposed uniquely in the form:

$$f = a \prod_{\alpha} P_{\alpha}^{n_{\alpha}}$$

where $a \in K^{\times}$ and $n_{\alpha} \geq 0$. ?? proves that a is the content of f . Hence $a \in A$. As A is factorial, a can be decomposed uniquely in the form:

$$a = u \prod_i p_i^{m_i}$$

whence the existence and uniqueness of the decomposition

$$f = u \prod_i p_i^{m_i} \prod_{\alpha} P_{\alpha}^{n_{\alpha}}.$$

Note that this theorem proves that every element of A admits the same decomposition into irreducible elements in A and $A[X]$. The gcd of a family of elements of A is therefore the same in A and in $A[X]$. \square

Corollary 14.1.63. *If A is a factorial domain, the domain $A[X_1, \dots, X_n]$ is factorial. Also the domain $A[X_n]_{n \in \mathbb{N}}$ is factorial.*

Proposition 14.1.64. *Let A be a Zariski ring and \widehat{A} its completion. If \widehat{A} is a factorial domain, A is a factorial domain.*

Proof. This follows from definition of factorial domain and ?? \square

Proposition 14.1.65. *Let C be a ring which is either a field or a DVR. Then the domain of formal power series $C[[X_1, \dots, X_n]]$ is factorial.*

Proof. Let \mathfrak{m} be the maximal ideal of C and π a generator of \mathfrak{m} (if C is a field, then $\pi = 0$). Let C be given the \mathfrak{m} -adic topology, which is Hausdorff. As C is a Noetherian local ring, $B = C[[X_1, \dots, X_n]]$ is a Noetherian local ring and its completion is $\widehat{C}[[X_1, \dots, X_n]]$. By the Corollary to ??, it suffices to prove that $\widehat{C}[[X_1, \dots, X_n]]$ is factorial. Now, if C is a field, then $\widehat{C} = C$; if C is a DVR, the same is true of \widehat{C} (?). We shall therefore assume in the remainder of the proof that C is complete.

Arguing by induction starting with the trivial case $n = 0$, we shall assume that it has been proved that $A = C[[X_1, \dots, X_{n-1}]]$ is factorial. We shall identify B with $A[[X_n]]$ and denote by \mathfrak{M} the maximal ideal of A (generated by π, X_1, \dots, X_{n-1}). We shall prove that every non-zero element g of B is, in an essentially unique way, a product of irreducible elements.

Let K be the field $C/C\pi$. As $B/B\pi$ is identified with $K[[X_1, \dots, X_n]]$, the ideal $B\pi$ is prime and π is irreducible. If $\pi \neq 0$, $B_{B\pi}$ is therefore the ring of a normed discrete valuation w ; every nonzero element g of B may therefore be written as $g = \pi^{w(g)} f$, where $f \in B$ and f is not a multiple of π . It will therefore suffice to show that f is an essentially unique product of extremal elements. Now the canonical image of f in $K[[X_1, \dots, X_n]]$ is not zero; ?? therefore shows that there exists an automorphism of B which maps f to an element \tilde{f} such that the coefficients of $\tilde{f}(0, \dots, 0, X_n)$ are not all in $C\pi$; this means that the coefficients of the series \tilde{f} , considered as a formal power series in X_n , are not all in \mathfrak{M} . It will suffice to prove our assertion for \tilde{f} .

In what follows, all the elements of B will be considered as formal power series in X_n with coefficients in A . By ?? (applicable since C and therefore A are separable and complete and the reduced series of \tilde{f} is nonzero), \tilde{f} is associated, in B , with a unique distinguished polynomial F . By ??, every series which divides \tilde{f} (or, what amounts to the same, which divides F) is associated with a distinguished polynomial which divides F and every decomposition of \tilde{f} is, to within invertible factors, of the form $\tilde{f} = uF_1 \cdots F_r$, where u is invertible and the F_i are irreducible distinguished polynomials (in B) such that $F = F_1 \cdots F_r$. By ??, the F_i are also irreducible in $A[X_n]$. Now, as A is factorial by the induction hypothesis, so is $A[X_n]$; hence, since they are monic, the F_i are uniquely determined by F (up to a permutation). This shows the uniqueness of the decomposition $\tilde{f} = uF_1 \cdots F_r$; its existence follows from the fact that B is Noetherian, which completes the proof. \square

14.2 Modules over integrally closed Noetherian domains

Throughout this section, A will be a integral domain with field of fractions K . Then $P(A)$, $\mathfrak{D}(A)$ and $\mathfrak{C}(A)$ will respectively denote the set of prime ideals of A of height 1, the divisor group of A and the divisor class group of A , these latter being written additively.

The general method of studying finite modules over an integrally closed Noetherian domain A consists of "localizing" the modules with respect to all the prime ideals $\mathfrak{p} \in P(A)$ of height 1 in A . As $A_{\mathfrak{p}}$ is then a DVR, the structure of finitely generated $A_{\mathfrak{p}}$ -modules is well known and therefore gives information about the structure of finitely generated A -modules. In the particular case where A is a Dedekind domain, we can arrive at as complete a theory as when A is a principal ideal domain.

14.2.1 Lattices of a vector space

Let V be a finite-dimensional vector space over the field K . A **lattice of V with respect to A** (or simply a lattice of V) is defined to be any sub- A -module M of V such that there exist two free sub- A -modules L_1, L_2 of V such that $L_1 \subseteq M \subseteq L_2$ and $\text{rank}_A(L_1) = \dim_K V$.

Example 14.2.1.

- (a) If we take $V = K$, the lattices of K are just the nonzero fractional ideals of K .
- (b) If $\dim_K V = n$, every free sub- A -module L of V has a basis containing at most n elements, every subset of V which is free over A being free over K . For L to be a lattice of V , it is necessary and sufficient that L have a basis of n elements (in other words, that $\text{rank}_A(L) = n$).
- (c) If A is a PID, every lattice M of V is a finitely generated A -module (since A is Noetherian) which is torsion-free and hence a free A -module.

Proposition 14.2.2. *For a sub- A -module M of V to be a lattice of V , it is necessary and sufficient that $KM = V$ and that M be contained in a finitely generated sub- A -module of V .*

Proof. The conditions are obviously necessary, for a free sub- A -module of V with the same rank as V generates V . Conversely, if $KM = V$, M contains a basis a_1, \dots, a_n of V over K and hence it contains the free sub- A -modulc L_1 generated by the a_i . On the other hand, if $M \subseteq M_1$,

where M_1 is a sub- A -module of V generated by a finite number of elements b_i and e_1, \dots, e_n is a basis of V over K , there exists a nonzero element s of A such that each of the b_i is a linear combination of the $s^{-1}e_i$ with coefficients in A . If L_2 is the free sub- A -modules of V generated by the $s^{-1}e_i$, then $M \subseteq L_2$. \square

Corollary 14.2.3. Suppose that A is Noetherian. For a sub- A -module M of V to be a lattice of V , it is necessary and sufficient that $KM = V$ and M be finitely generated.

Proposition 14.2.4. Let M be a lattice of V and M_1 a sub- A -module of V . If there exist two elements $x, y \in K^\times$ such that $xM \subseteq M_1 \subseteq yM$, then M_1 is a lattice of V . Conversely, if M_1 is a lattice of V , there exist two non-zero elements a, b of A such that $aM \subseteq M_1 \subseteq b^{-1}M$.

Proof. If L_1, L_2 are two free lattices of V such that $L_1 \subseteq M \subseteq L_2$, the relations $xM \subseteq M_1 \subseteq yM$ imply $xL_1 \subseteq M_1 \subseteq yL_2$ and xL_1 and yL_2 are free lattices. Conversely, if M_1 is a lattice and e_1, \dots, e_n is a basis of L_2 over A , the relation $KM_1 = V$ implies the existence of $x = a/s \in K^\times$ (where a and s are non-zero elements of A) such that $xe_i \in M_1$ for all i , whence $xM \subseteq xL_2 \subseteq M_1$ and a fortiori $aM \subseteq M_1$. Exchanging the roles of M and M_1 it can be similarly shown that there exists $b \neq 0$ in A such that $bM_1 \subseteq M$. \square

Proposition 14.2.5 (Properties of Lattices).

- (a) If M_1 and M_2 are lattices of V , so are $M_1 \cap M_2$ and $M_1 + M_2$.
- (b) If W is a vector subspace of V and M is a lattice of V , then $M \cap W$ is a lattice of W .
- (c) Let V, V_1, \dots, V_k be vector spaces of finite dimension over K and let $f : V_1 \times \dots \times V_k \rightarrow V$ be a multilinear map whose image generates V . If M_i is a lattice of V_i for each i , then the sub- A -module of V generated by $f(M_1 \times \dots \times M_k)$ is a lattice of V .
- (d) Let V and W be two vector spaces of finite dimension over K , M a lattice of V and N a lattice of W . Then the sub- A -module $(N : M)$ of $\text{Hom}_K(V, W)$, consisting of the K -linear maps f such that $f(M) \subseteq N$, is a lattice of $\text{Hom}_K(V, W)$.

Proof. For (a), by virtue of ??, there exist non-zero a and b in A such that $aM_1 \subseteq M_2 \subseteq b^{-1}M_1$. We conclude that $M_1 \cap M_2$ and $M_1 + M_2$ lie between aM_1 and $b^{-1}M_1$ and are therefore lattices by virtue of ??.

As for (b), let S be a complement of W in V , L_W a free lattice of W and L_S a free lattice of S , so that $L = L_S + L_W$ is a free lattice of V . Then there exist x, y in K^\times such that $xL \subseteq M \subseteq yL$. We deduce that $xL_W \subseteq M \cap W \subseteq yL_W$, which shows that $M \cap W$ is a lattice of W .

Now we prove (c). As $KM_i = V_i$ clearly by linearity $f(M_1 \times \dots \times M_k)$ generates the vector K -space V . On the other hand, for all i , there exists a finitely generated sub- A -module N_i of V_i such that $M_i \subseteq N_i$. The sub- A -module N of V generated by $f(N_1 \times \dots \times N_k)$ is finitely generated and contains M and hence M is a lattice of V (??).

Finally, consider (d). Let P (resp. Q) be a free lattice of V (resp. W) containing M (resp. contained in N). Obviously $(N : M) \subseteq (Q : P)$. Now it is immediate that $(Q : P)$ is isomorphic to $\text{Hom}_A(P, Q)$, hence is a free A -module of rank $(\text{rank}_A P)(\text{rank}_A Q)$ and therefore a lattice of $\text{Hom}_K(V, W)$. Similarly, if P' (resp. Q') is a free lattice of V (resp. W) contained in M , (resp. containing N), then $(Q' : P') \subseteq (N : M)$ and $(Q' : P')$ is a lattice of $\text{Hom}_K(V, W)$, whence the conclusion. \square

In the notation of ??(d), the canonical map $(N : M) \rightarrow \text{Hom}_A(M, N)$ which maps every K -linear map $f \in (N : M)$ to the A -linear map from M to N which has the same graph as $f|_M$, is bijective: for every A -linear map $g : M \rightarrow N$ can be imbedded in a K -linear map

$$g \otimes 1 : M \otimes_A K \rightarrow N \otimes_A K$$

and by ?? $M \otimes_A K$ and $N \otimes_A K$ are respectively identified with V and W .

In particular, if we take $W = K$, $N = A$, $\text{Hom}_K(V, W)$ is just the dual vector K -space V^* of V and $(A : M)$ is identified with the dual A -module M^* of M . We shall henceforth make this identification and we shall say that M^* is the dual lattice of M : it is therefore the set of $x^* \in V^*$ such that $\langle x^*, x \rangle \in A$ for all $x \in M$.

Corollary 14.2.6. *Let U, V, W be three vector spaces of finite dimension over K and $f : U \times V \rightarrow W$ a left non-degenerate K -bilinear map. If M is a lattice of V and N a lattice of W , then the set $(N :_f M)$ of $x \in U$ such that $f(x, y) \in N$ for all $y \in M$ is a lattice of U .*

Proof. Let $\phi_f : U \rightarrow \text{Hom}_K(V, W)$ be the K -linear map left associated with f such that $\phi_f(x)$ is the linear map $y \mapsto f(x, y)$. Recall that to say that f is left non-degenerate means that ϕ_f is injective. By ??(d) $(N : M)$ is a lattice of $\text{Hom}_K(V, W)$, as $(N :_f M) = \phi_f^{-1}(N : M)$ and ϕ_f is injective, the corollary follows from ??(b). \square

Example 14.2.7. Let S be a (not necessarily associative) K -algebra of finite dimension with a unit element. Then the bilinear map $(x, y) \mapsto xy$ of $S \times S$ to S is (left and right) non-degenerate. If M and N are lattices of S with respect to A , so are $M \cdot N$ and the set of $x \in S$ such that $xM \subseteq N$. Note that there exists a sub- A -algebra of S containing the unit element of S which is a lattice of S : for consider a basis e_1, \dots, e_n of S such that e_1 is the unit element of S and let $e_i e_j = \sum_k c_{ij}^k e_k$ be the multiplication table of S , so that $c_{1j}^k = \delta_j^k$ and $c_{i1}^k = \delta_i^k$. Let $s \in A$ be a non-zero element such that $\tilde{c}_{ij}^k := s \cdot c_{ij}^k \in A$ for all triplets of indices (i, j, k) . If we write $w_i = s^{-1}e_i$ for $i \geq 2$, then

$$w_i w_j = s \tilde{c}_{ij}^1 e_1 + \sum_{k \geq 2} \tilde{c}_{ij}^k w_k$$

for $i, j \geq 2$. The lattice of S with basis e_1 and the w_2, \dots, w_n is a sub- A -algebra of S with unit element e_1 .

Example 14.2.8. Let V be a finite-dimensional vector space over K and f a non-degenerate bilinear form on V . If M is a lattice of V , it follows from the ?? that the set M_f^* of $x \in V$ such that $f(x, y) \in A$ for all $y \in M$ is also a lattice of V . If $\phi_f : V \rightarrow V^*$ is the linear map left associated with f (which is bijective), $\phi_f(M_f^*)$ is just the dual lattice M^* of M .

Proposition 14.2.9. *Let B be an integral domain, A a subring of B and K and L the respective field of fractions of A and B . Let V be a finite-dimensional vector space over K .*

- (a) *For every lattice M of V with respect to A , the image BM of $M_{(B)} = M \otimes_A B$ in $V_{(L)} = V \otimes_K L$ is a lattice of $V_{(L)}$ with respect to B .*
- (b) *Suppose further that B is a flat A -module. Then the canonical map $M_{(B)} \rightarrow BM$ is bijective. If further B is faithfully flat, the map which maps every lattice M of V with respect to A to the lattice BM of $V_{(L)}$ with respect to B is injective.*

Proof. As $KM = V$, clearly $L(BM) = V_{(L)}$. On the other hand M is contained in a finitely generated sub- A -module M_1 of V and hence BM is contained in BM_1 which is a finitely generated B -module; whence assertion (a).

We see that $V_{(L)} = V \otimes_K L = V \otimes_A L$ and, as L is a flat B -module, it is also a flat A -module. If B is a flat A -module, then the canonical map $M \otimes_A B \rightarrow V \otimes_A B$ is injective. On the other hand, since V is a free K -module and K a flat A -module, V is a flat A -module and hence the canonical map $V \otimes_A B \rightarrow V \otimes_A L$ is injective, which establishes the first assertion. To see also that the relation $BM_1 = BM_2$ implies $M_1 = M_2$ for two lattices M_1, M_2 of V with respect to A when B is a faithfully flat A -module, note first that $BM_1 \cap BM_2 = B(M_1 \cap M_2)$. We may therefore confine our attention to the case where $M_1 \subseteq M_2$ and our assertion then follows from ?? applied to the canonical injection $M_1 \rightarrow M_2$. \square

Corollary 14.2.10. *Suppose that A is a DVR. Let \widehat{A} be its completion and let \widehat{K} be the field of fractions of \widehat{A} . The map η which maps every lattice M of V to the lattice $\widehat{A}M$ of $\widehat{V} = V \otimes_K \widehat{K}$ with respect to A is bijective and its inverse maps every lattice \widehat{M} of \widehat{V} with respect to \widehat{A} to its intersection $\widehat{M} \cap V$ (V being canonically identified with a vector sub- K -space of \widehat{V}).*

Proof. If L is a free lattice of V , the lattices aL (for $a \in A$ nonzero) form a fundamental system of neighbourhoods of 0 for a topology \mathcal{T} on V (compatible with its A -module structure), which (when a basis of L over A is taken) is identified with the product topology on K^n . By virtue of ??, a fundamental system of neighbourhoods of 0 for \mathcal{T} also consists of all the lattices of V with respect to A . Clearly \widehat{V} is the completion of V with respect to \mathcal{T} . Moreover, if \mathfrak{m} is the maximal ideal of A , the topology \mathcal{T} induces on every lattice M of V with respect to A the \mathfrak{m} -adic topology since M is a finitely generated A -module (??) and $\widehat{A}M$ is the completion of M with respect to this topology (??). Moreover, as M is open (and therefore closed) in V , $\widehat{A}M \cap V = M$, which proves again the fact that η is injective (which follows directly from ??(b), since \widehat{A} is a faithfully flat A -module).

Finally, if \widehat{M} is a lattice of \widehat{V} with respect to A and $M = \widehat{M} \cap V$, then since $\widehat{A}L$ is a free lattice of \widehat{V} with respect to \widehat{A} and every element of \widehat{A} is the product of an element of A and an invertible element of \widehat{A} , there exists $a, b \in \widehat{A}$ such that $a\widehat{A}L \subseteq \widehat{M} \subseteq b^{-1}\widehat{A}L$, whence $aL \subseteq \widehat{M} \cap V \subseteq b^{-1}L$, so $\widehat{M} \cap V$ is a lattice of V with respect to A . Moreover \widehat{M} is open in V and, as V is dense in \widehat{V} , \widehat{M} is the completion of $\widehat{M} \cap V = M$. This proves that η is surjective, whence the corollary. \square

Let S be a multiplicative subset of A not containing 0. Then we can apply ?? to $B = S^{-1}A$ with $L = K, BM = S^{-1}M$. Hence $S^{-1}M$ is a lattice of V with respect to $S^{-1}A$. Moreover:

Proposition 14.2.11. *Let V, W be vector spaces of finite dimension over K , M a lattice of V and N a lattice of W . If M is finitely generated, then*

$$S^{-1}(N : M) = (S^{-1}N : S^{-1}M)$$

in $\text{Hom}_K(V, W)$.

Proof. Clearly the left hand side is contained in the right hand side. Conversely, let $f \in (S^{-1}N : S^{-1}M)$ and let x_1, \dots, x_n be a system of generators of M . There exists $s \in S$ such that $f(x_i) \in s^{-1}N$ for all i and hence $sf \in (N : M)$, which proves the proposition. \square

14.2.2 Duality and reflexive modules

From now on the domain A is assumed to be Noetherian and integrally closed and that $P(A)$ denotes the set of prime ideals of A of height 1. Every lattice with respect to A is a finitely generated A -module by ??.

Let V be a vector space of finite dimension over K , V^* its dual and V^{**} its bidual. We shall identify V and V^{**} by means of the canonical map J_M . Let M be a lattice of V , recall that the dual A -module M^* of M is canonically identified with the dual lattice of M , which is the set of $x^* \in V^*$ such that $\langle x^*, x \rangle \in A$ for all $x \in M$. The bidual A -module M^{**} of M is therefore a lattice of V which contains M . Moreover $M^{***} = M^*$, for the relation $M \subseteq M^{**}$ implies $(M^{**})^* \subseteq M^*$ and on the other hand $M^* \subseteq (M^*)^{**}$ by the above.

If \mathfrak{p} is a prime ideal, ?? applied with $N = A$ gives the relation $(M^*)_{\mathfrak{p}} = (M_{\mathfrak{p}})^*$, which justifies the notation $M_{\mathfrak{p}}^*$ for both terms.

Theorem 14.2.12. *If M is a lattice of V , then $M^* = \bigcap_{\mathfrak{p} \in P(A)} M_{\mathfrak{p}}^*$.*

Proof. Clearly M^* is contained in each of the $M_{\mathfrak{p}}^*$. Conversely, suppose that $x^* \in \bigcap_{\mathfrak{p} \in P(A)} M_{\mathfrak{p}}^*$. If $x \in M$, then $\langle x^*, x \rangle \in \bigcap_{\mathfrak{p} \in P(A)} A_{\mathfrak{p}} = A$, and thus $x^* \in M^*$. \square

Corollary 14.2.13. *If M is a lattice of V , then $M^{**} = \bigcap_{\mathfrak{p} \in P(A)} M_{\mathfrak{p}}$.*

Proof. Theorem ?? applied to M^* shows that $M^{**} = \bigcap_{\mathfrak{p} \in P(A)} M_{\mathfrak{p}}^{**}$. But as $A_{\mathfrak{p}}$ is a principal ideal domain, $M_{\mathfrak{p}}^*$ is a finitely generated free $A_{\mathfrak{p}}$ -module and hence $M_{\mathfrak{p}}^{**}$ is canonically identified with $M_{\mathfrak{p}}$, whence the corollary. \square

For any lattice M with respect to A , the canonical map $J_M : M \rightarrow M^{**}$ identifies an element $x \in M$ with itself, for x is the unique element y of $V = V^{**}$ such that $\langle x^*, x \rangle = \langle x^*, y \rangle$ for all $x^* \in M^*$, since M^* generates V^* . We shall say that M is **reflexive** if $M^{**} = M$. As we have above $M^* = (M^*)^{**}$, it is seen that the dual of any lattice M is always reflexive.

Remark 14.2.1. Let M be a finitely generated A -module; it is immediate that the dual M^* of M , identified with a sub- A -module of $\text{Hom}_A(M, K)$, is a lattice of the vector K -space $\text{Hom}_A(M, K)$; in particular, every finitely generated reflexive A -module is isomorphic to a lattice of a suitable vector K -space.

Theorem 14.2.14. *If M is a lattice of V , the following conditions are equivalent:*

- (i) M is reflexive;
- (ii) $M = \bigcap_{\mathfrak{p} \in P(A)} M_{\mathfrak{p}}$;
- (iii) $\text{Ass}(V/M) \subseteq P(A)$.

Proof. The equivalence of (i) and (ii) follows from the ???. If (ii) holds, then $\bigcap_{\mathfrak{p} \in P(A)} (V/M_{\mathfrak{p}}) = \{0\}$ in V/M , hence by ?? we have

$$\text{Ass}(V/M) \subseteq \bigcup_{\mathfrak{p} \in P(A)} \text{Ass}(V/M_{\mathfrak{p}}).$$

As $V/M_{\mathfrak{p}}$ is an $A_{\mathfrak{p}}$ -module, an element of $A - \mathfrak{p}$ cannot annihilate a nonzero element of $V/M_{\mathfrak{p}}$, since the elements of $A - \mathfrak{p}$ are invertible in $A_{\mathfrak{p}}$. The elements of $\text{Ass}(V/M_{\mathfrak{p}})$ are therefore

contained in \mathfrak{p} and are nonempty, since $V/M_{\mathfrak{p}}$ is a torsion $A_{\mathfrak{p}}$ -module. As \mathfrak{p} is of height 1, necessarily $\text{Ass}(V/M_{\mathfrak{p}}) = \{p\}$ if $V/M_{\mathfrak{p}} \neq \{0\}$ and $\text{Ass}(V/M_{\mathfrak{p}}) = \emptyset$ if $V/M_{\mathfrak{p}} = \{0\}$; hence $\text{Ass}(V/M) \subseteq P(A)$.

Finally, if condition (iii) holds, then

$$\text{Ass}(M^{**}/M) \subseteq \text{Ass}(V/M) \subseteq P(A)$$

On the other hand, if $\mathfrak{p} \in P(A)$, then it has been seen in the proof of the ?? that ${}_{\mathfrak{p}}^{**} = M_{\mathfrak{p}}$, whence $\mathfrak{p} \notin \text{Ass}(M^{**}/M)$ by ???. We conclude that $\text{Ass}(M^{**}/M) = \emptyset$, whence $M^{**} = M$. \square

Corollary 14.2.15. *Let M, N be two lattices of V with respect to A such that N is reflexive. In order that $M \subseteq N$, it is necessary and sufficient that, for all $\mathfrak{p} \in P(A)$, $M_{\mathfrak{p}} \subseteq N_{\mathfrak{p}}$.*

Proof. The condition is obviously necessary and, if it is fulfilled, then

$$\bigcap_{\mathfrak{p} \in P(A)} M_{\mathfrak{p}} \subseteq \bigcap_{\mathfrak{p} \in P(A)} N_{\mathfrak{p}} = N$$

As $M \subseteq M^{**} = \bigcap_{\mathfrak{p} \in P(A)} M_{\mathfrak{p}}$, certainly $M \subseteq N$. \square

Example 14.2.16 (Example of reflexive submodules).

- (a) Every free lattice is reflexive.
- (b) Take $V = K$. For a fractional ideal \mathfrak{a} of K to be a reflexive lattice, it is necessary and sufficient that it be a divisorial ideal by the observation $\mathfrak{a}^* = (A : \mathfrak{a})$ and ??.
- (c) Let M be a lattice with respect to A . If S is a multiplicative subset of A not containing 0, ?? shows that $S^{-1}(M^*) = (S^{-1}M)^*$. If M is reflexive, $S^{-1}M$ is therefore a reflexive lattice with respect to $S^{-1}A$.
- (d) If M is a finitely generated A -module and T its torsion submodule, the dual M^* of M is the same as the dual of M/T , since, for every linear form f on M , the image $f(T)$ is a torsion submodule of A and hence zero. As M/T is isomorphic to a lattice of a vector space over K , it is seen that the dual of every finitely generated A -module is reflexive.

Proposition 14.2.17. *Let V be a vector space of finite dimension over K .*

- (a) *If M_1 and M_2 are reflexive lattices of V , so is $M_1 \cap M_2$.*
- (b) *If W is a vector subspace of V and M is a reflexive lattice of V , then $M \cap W$ is a reflexive lattice of W .*
- (c) *Let V, W be two vector spaces of finite dimension over K and M (resp. N) a lattice of V (resp. W). If N is reflexive, the lattice $(N : M)$ is reflexive.*

Proof. For (a), note that $(M_1 \cap M_2)_{\mathfrak{p}} = (M_1)_{\mathfrak{p}} \cap (M_2)_{\mathfrak{p}}$ for all $\mathfrak{p} \in P(A)$. If $M_1 = \bigcap_{\mathfrak{p} \in P(A)} (M_1)_{\mathfrak{p}}$ and $M_2 = \bigcap_{\mathfrak{p} \in P(A)} (M_2)_{\mathfrak{p}}$, then

$$M_1 \cap M_2 = \bigcap_{\mathfrak{p} \in P(A)} (M_1 \cap M_2)_{\mathfrak{p}}$$

whence the conclusion by virtue of ?? . Similarly $(M \cap W)_{\mathfrak{p}} = M_{\mathfrak{p}} \cap W$, whence (b). Now for (c), as M is finitely generated, it follows from ?? that $(N : M)_{\mathfrak{p}} = (N_{\mathfrak{p}} : M_{\mathfrak{p}})$. Moreover, the relation $N = \bigcap_{\mathfrak{p} \in P(A)} N_{\mathfrak{p}}$ implies

$$(N : M) = \bigcap_{\mathfrak{p} \in P(A)} (N_{\mathfrak{p}} : M_{\mathfrak{p}})$$

For if $f \in \bigcap_{\mathfrak{p} \in P(A)} (N_{\mathfrak{p}} : M_{\mathfrak{p}})$ and $x \in M$, then $f(x) \in N_{\mathfrak{p}}$ for all $\mathfrak{p} \in P(A)$, whence $f \in (N : M)$. This shows $(N : M)$ is reflexive. \square

Proposition 14.2.18. *Let $0 \rightarrow M \rightarrow N \rightarrow Q \rightarrow 0$ be an exact sequence of A -modules. Suppose that N is finitely generated and torsion-free.*

- (a) *If M is reflexive, then $\text{Ass}(Q) \subseteq P(A) \cup \{(0)\}$ (in other words, every ideal associated with Q is either zero or of height 1).*
- (b) *Conversely, if N is reflexive and $\text{Ass}(Q) \subseteq P(A) \cup \{(0)\}$, then M is reflexive.*

Proof. As A is Noetherian, M is finitely generated. If we write $V = M_{(K)}$ and $W = N_{(K)}$, then M (resp. N) is canonically identified with a lattice of V (resp. W) (??). Consider the two exact sequences:

$$0 \rightarrow V/M \rightarrow W/M \rightarrow W/V \rightarrow 0, \quad 0 \rightarrow Q \rightarrow W/M \rightarrow W/N \rightarrow 0.$$

from which we deduce that

$$\text{Ass}(Q) \subseteq \text{Ass}(W/M) \subseteq \text{Ass}(V/M) \cup \text{Ass}(W/V)$$

If M is reflexive, then $\text{Ass}(V/M) \subseteq P(A)$. On the other hand, clearly $\text{Ass}(W/V)$ is, either empty, or reduced to $\{0\}$, whence (a). Similarly,

$$\text{Ass}(V/M) \subseteq \text{Ass}(W/M) \subseteq \text{Ass}(Q) \cup \text{Ass}(W/N)$$

The hypotheses N is reflexive then imply that $\text{Ass}(V/M) \subseteq P(A) \cup \{(0)\}$. But V/M is a torsion A -module and hence $(0) \notin \text{Ass}(V/M)$; ?? then shows that M is reflexive. \square

Proposition 14.2.19. *Let A and B be two rings, $\rho : A \rightarrow B$ a ring homomorphism and M a finitely generated A -module. Suppose that A is Noetherian and that B is a flat A -module. Then, if M is reflexive, so is the B -module $M_{(B)} = M \otimes_A B$.*

Proof. We know that there exists a canonical isomorphism $\omega_M : (M^*)_{(B)} \rightarrow (M_{(B)})^*$, such that

$$\langle \omega_M(x^* \otimes 1), x \otimes 1 \rangle = f(\langle x^*, x \rangle)$$

for $x \in M$, $x^* \in M^*$. As M is a quotient of a finitely generated free A -module L , M^* is isomorphic to a sub- A -module of the dual L^* and L^* is free and finitely generated; since A is Noetherian, M^* is therefore also a finitely generated A -module, whence an isomorphism $\omega_{M^*} : (M^{**})_{(B)} \rightarrow ((M^*)_{(B)})^*$ such that

$$\langle \omega_{M^*}(x^{**} \otimes 1), x^* \otimes 1 \rangle = f(\langle x^{**}, x^* \rangle)$$

for $x^* \in M$ and $x^{**} \in M^{**}$. On the other hand, there is an isomorphism $\omega_M^t : (M_{(B)})^{**} \rightarrow ((M^*)_{(B)})^*$, whence by composition a canonical isomorphism

$$\phi = (\omega_M^t)^{-1} \circ \omega_{M^*} : (M^{**})_{(B)} \rightarrow (M_{(B)})^{**}$$

such that, in the above notation:

$$\langle \phi(x^{**} \otimes 1), \omega_M(x^* \otimes 1) \rangle = f(\langle x^{**}, x^* \rangle). \quad (14.2.1)$$

We consider now the canonical homomorphism $J_M : M \rightarrow M^{**}$ and show that the composite homomorphism:

$$\psi = (J_M \otimes 1) \circ \phi : M_{(B)} \rightarrow (M^{**})_{(B)} \rightarrow (M_{(B)})^{**}$$

is just the canonical homomorphism $J_{M_{(B)}}$. This follows immediately from (??) which gives the relations:

$$\langle \psi(x \otimes 1), \omega_M(x^* \otimes 1) \rangle = f(\langle J_M(x), x^* \rangle) = f(\langle x^*, x \rangle) = \langle \omega_M(x^* \otimes 1), x \otimes 1 \rangle.$$

and from the fact that the elements $\omega_M(x^* \otimes 1)$ generate $(M_{(B)})^*$. This being so, the hypothesis that M is reflexive means that J_M is bijective, hence so is $J_M \otimes 1$ and therefore $\psi = J_{M_{(B)}}$ is bijective, which shows the proposition. \square

14.2.3 Local construction of reflexive modules

We keep the notation and hypotheses of the last paragraph. We shall say that a property holds "for almost all $\mathfrak{p} \in P(A)$ " if the set of $\mathfrak{p} \in P(A)$ for which it is not true is finite.

Lemma 14.2.20. *Let \mathfrak{p} and \mathfrak{q} be two prime ideals of A such that (0) is the only prime ideal of A contained in $\mathfrak{p} \cap \mathfrak{q}$. Then, for every sub- A -module E of V , $(E_{\mathfrak{p}})_{\mathfrak{q}} = KE$.*

Proof. Let S be the multiplicative subset $(A - \mathfrak{p})(A - \mathfrak{q})$ of A ; by ??, $(E_{\mathfrak{p}})_{\mathfrak{q}} = S^{-1}E$. The prime ideals of $S^{-1}A$ correspond to the prime ideals \mathfrak{q}' of A such that $\mathfrak{p}' \cap S = \emptyset$ and by hypothesis (0) is the only prime ideal of A not meeting S ; hence $S^{-1}A = K$ and $S^{-1}E = KE$. \square

Theorem 14.2.21. *Let V be a vector space of finite dimension over K and M a lattice of V with respect to A .*

- (a) *Let N be a lattice of V with respect to A . Then for every prime ideal \mathfrak{p} of A , $N_{\mathfrak{p}}$ is a lattice of V with respect to $A_{\mathfrak{p}}$ and, for almost all $\mathfrak{p} \in P(A)$, $N_{\mathfrak{p}} = M_{\mathfrak{p}}$.*
- (b) *Conversely, suppose given for all $\mathfrak{p} \in P(A)$ a lattice $N(\mathfrak{p})$ of V with respect to $A_{\mathfrak{p}}$ such that $N(\mathfrak{p}) = M_{\mathfrak{p}}$ for almost all $\mathfrak{p} \in P(A)$. Then $N = \bigcap_{\mathfrak{p} \in P(A)} N(\mathfrak{p})$ is a reflexive lattice of V with respect to A and it is the only reflexive lattice of V with respect to A whose localization at \mathfrak{p} is $N(\mathfrak{p})$ for all $\mathfrak{p} \in P(A)$.*

Proof. The first assertion follows from ???. Moreover, there exist x, y in K^\times such that $xN \subseteq M \subseteq yN$. We know that, for almost all $\mathfrak{p} \in P(A)$, $v_{\mathfrak{p}}(x) = v_{\mathfrak{p}}(y) = 0$, which shows that x and y are invertible in A and hence $M_{\mathfrak{p}} = N_{\mathfrak{p}}$.

Now we prove (b). We may replace M by $x^{-1}M$ where $x \neq 0$ in A and assume that $N(\mathfrak{p}) \subseteq M_{\mathfrak{p}}$ for all $\mathfrak{p} \in P(A)$. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ be the elements of $P(A)$ such that $N(\mathfrak{p}) =$ for \mathfrak{p} distinct from the \mathfrak{p}_i , we write:

$$Q = M \cap N(\mathfrak{p}_1) \cap \cdots \cap N(\mathfrak{p}_n).$$

As each of the $N(\mathfrak{p}_i)$ contain a free lattice with respect to $A_{\mathfrak{p}_i}$, it contains a fortiori a lattice of V with respect to A , hence Q contains a lattice of V with respect to A (??) and, as Q is contained in M , Q is a lattice with respect to A . If $\mathfrak{p} \in P$ is distinct from the \mathfrak{p}_i , ?? applied to $N(\mathfrak{p}_i)$ gives

$$(N(\mathfrak{p}_i))_{\mathfrak{p}} = ((N(\mathfrak{p}_i))_{\mathfrak{p}_i})_{\mathfrak{p}} = KN(\mathfrak{p}_i) = V$$

since the \mathfrak{p}_i and \mathfrak{p} are of height 1. Then

$$Q_{\mathfrak{p}} = M_{\mathfrak{p}} \cap (N(\mathfrak{p}_i))_{\mathfrak{p}} \cap \cdots \cap (N(\mathfrak{p}_n))_{\mathfrak{p}} = M_{\mathfrak{p}} = N(\mathfrak{p}).$$

On the other hand, if \mathfrak{p} is equal to \mathfrak{p}_i , then $(N(\mathfrak{p}_i))_{\mathfrak{p}_j} = V$ for $i \neq j$ by the argument as above and $(N(\mathfrak{p}_i))_{\mathfrak{p}_i} = N(\mathfrak{p}_i)$, whence

$$Q_{\mathfrak{p}_i} = M_{\mathfrak{p}_i} \cap N(\mathfrak{p}_i) = N(\mathfrak{p}_i).$$

We have therefore proved that $Q_{\mathfrak{p}} = N(\mathfrak{p})$ for all $\mathfrak{p} \in P(A)$. Then $N = Q^{**} = \bigcap_{\mathfrak{p} \in P(A)} Q_{\mathfrak{p}}$ is reflexive and satisfies the relations $N_{\mathfrak{p}} = Q_{\mathfrak{p}} = N(\mathfrak{p})$ for all $\mathfrak{p} \in P(A)$. The uniqueness property follows immediately from ??.

Proposition 14.2.22. *Let M be a finitely generated A -module. The following conditions are equivalent:*

- (i) $M_{\mathfrak{p}} = 0$ for every prime ideal \mathfrak{p} of height ≤ 1 .
- (ii) The annihilator \mathfrak{a} of M is nonzero and $(A : \mathfrak{a}) = A$.

An A -module M is called **pseudo-zero** if it is finitely generated and it satisfies the equivalent conditions above.

Proof. We know (by ??) that the condition $M_{\mathfrak{p}} = 0$ is equivalent to $\mathfrak{a} \not\subseteq \mathfrak{p}$ and hence to $\mathfrak{a}A_{\mathfrak{p}} = A_{\mathfrak{p}}$. On the other hand, for every integral ideal $\mathfrak{b} \neq 0$ of A , the relation " $\mathfrak{b}A_{\mathfrak{p}} = A_{\mathfrak{p}}$ for all $\mathfrak{p} \in P(A)$ " is equivalent to $\text{div}(\mathfrak{b}) = \text{div}(A) = 0$ in $\mathfrak{D}(A)$ (??), or also to $\text{div}(A : \mathfrak{b}) = 0$ and, as $(A : \mathfrak{b})$ is divisorial, this relation is also equivalent to $(A : \mathfrak{b}) = A$. The proposition then follows by noting that to say that $\mathfrak{a} \not\subseteq (0)$ means that $\mathfrak{a} \neq (0)$. \square

Example 14.2.23 (Example of pseudo-zero modules).

- (a) If A is a Dedekind domain, every prime ideal of A is of height 1, so to say that M is pseudo-zero means then that $\text{supp}(M) = \emptyset$ and hence that $M = 0$.
- (b) Let k be a field and $A = k[X, Y]$ the polynomial ring over k in two indeterminates. If \mathfrak{m} is the maximal ideal $AX + AY$ of A , the A -module A/\mathfrak{m} is pseudo-zero, for its annihilator \mathfrak{m} is not of height ≤ 1 since it contains the principal prime ideals AX and AY and is distinct from them; therefore $(A : \mathfrak{m}) = A$.

Let $\phi : M \rightarrow N$ a homomorphism of A -modules. Then ϕ is called **pseudo-injective** (resp. **pseudo-surjective**, **pseudo-zero**) if $\ker \phi$ (resp. $\text{coker } \phi$, $\text{im } \phi$) is pseudo-zero. Also ϕ is called

pseudo-bijective if it is both pseudo-injective and pseudo-surjective. A pseudo-bijective homomorphism is also called a **pseudo-isomorphism**.

Suppose that M and N are finitely generated. Then, for $\phi : M \rightarrow N$ to be pseudo-injective (resp. pseudo-surjective, pseudo-zero), it is necessary and sufficient that, for all $\mathfrak{p} \in P(A) \cup \{(0)\}$, $\phi_{\mathfrak{p}} : M_{\mathfrak{p}} \rightarrow N_{\mathfrak{p}}$ be injective (resp. surjective, zero) (this follows from the exactness of the A -module $A_{\mathfrak{p}}$).

Example 14.2.24. Let M be a torsion-free finitely generated A -module. Then M is identified with a lattice of $V = M \otimes_A K$. We have seen that $(M_{\mathfrak{p}})_{\mathfrak{q}} = V$ for $\mathfrak{p} \neq \mathfrak{q}$ in $P(A)$, therefore $M_{\mathfrak{p}} = M_{\mathfrak{p}}^{**}$ for all $\mathfrak{p} \in P(A)$. Also, for $\mathfrak{p} = 0$, $M_{\mathfrak{p}}$ and $M_{\mathfrak{p}}^{**}$ are both equal to V . Therefore the canonical map $J_M : M \rightarrow M^{**}$ of M to its bidual is a pseudo-isomorphism.

Example 14.2.25. For all $\mathfrak{p} \in P(A)$, the canonical map $\phi : A/\mathfrak{p}^n \rightarrow A/\mathfrak{p}^{(n)} = A/(A \cap \mathfrak{p}^n A_{\mathfrak{p}})$ is a pseudo-isomorphism, as, for all $\mathfrak{q} \in P(A)$ distinct from \mathfrak{p} , $A_{\mathfrak{q}}/\mathfrak{p}^n A_{\mathfrak{q}} = A_{\mathfrak{q}}/\mathfrak{p}^{(n)} A_{\mathfrak{q}} = 0$ and $A_{\mathfrak{p}}/\mathfrak{p}^n A_{\mathfrak{p}} = A_{\mathfrak{p}}/\mathfrak{p}^{(n)} A_{\mathfrak{p}}$.

Theorem 14.2.26. Let E be a finitely generated A -module, T the torsion submodule of E and $M = E/T$. There exists a pseudo-isomorphism $\phi : E \rightarrow T \times M$.

For this theorem, we shall first prove two lemmas.

Lemma 14.2.27. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_k$ be a family of prime ideals of A of height 1 and let $S = \bigcap_{i=1}^k (A - \mathfrak{p}_i)$. Then the ring $S^{-1}A$ is a principal ideal domain.

Proof. The ring $S^{-1}A$ is a semi-local ring whose maximal ideals are the $\mathfrak{m}_i = S^{-1}\mathfrak{p}_i$ for $1 \leq i \leq k$, the local ring $(S^{-1}A)_{\mathfrak{m}_i}$ being isomorphic to $A_{\mathfrak{p}_i}$ and hence a discrete valuation ring. The ring $S^{-1}A$ is therefore a Dedekind domain and, as it is semi-local it is a principal ideal domain. \square

Lemma 14.2.28. There exists a homomorphism $\eta : E \rightarrow T$ whose restriction to T is both a homothety and a pseudo-isomorphism.

Proof. Let \mathfrak{a} be the annihilator of T . As T is a finitely generated torsion A -module, $\mathfrak{a} \neq 0$. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_k$ be the prime ideals of height 1 containing \mathfrak{a} (which are finite in number). If this number is 0, T is pseudo-zero (??) and we may take $\eta = 0$. Otherwise, let $S = \bigcap_{i=1}^k (A - \mathfrak{p}_i)$; by ??, $S^{-1}A$ is a PID and hence $S^{-1}M$, which is a torsion-free finitely generated $S^{-1}A$ -module, is free and, as $S^{-1}M = (S^{-1}E)/(S^{-1}T)$, $S^{-1}T$ is a direct factor of $S^{-1}E$. Now,

$$\text{Hom}_{S^{-1}A}(S^{-1}E, S^{-1}T) = S^{-1}\text{Hom}_A(E, T)$$

hence there exist $s_0 \in S$ and $\eta_0 \in \text{Hom}_A(E, T)$ such that $s_0^{-1}\eta_0$ is a projector of $S^{-1}E$ onto $S^{-1}T$. If $\psi_0 \in \text{Hom}_A(T, T)$ denotes the restriction of η_0 to T , there therefore exists $s_1 \in S$ such that $s_1\psi_0(x) = s_1s_0x$ for all $x \in T$. Writing $s = s_1s_0$, $\eta = s_1\eta_0$, $\psi = s_1\psi_0$, then ψ is the homothety of ratio s on T and is the restriction of η to T . It remains to verify that ψ is a pseudo-isomorphism.

Now, if $\mathfrak{p} = 0$ or if $\mathfrak{p} \in P(A)$ is distinct from the \mathfrak{p}_i , then $T_{\mathfrak{p}} = 0$ and $\psi_{\mathfrak{p}} : T_{\mathfrak{p}} \rightarrow T_{\mathfrak{p}}$ is an isomorphism. If on the contrary \mathfrak{p} is equal to one of the \mathfrak{p}_i , then s is invertible in $A_{\mathfrak{p}_i}$ and $h_{\mathfrak{p}_i}$ is the homothety of ratio s on $T_{\mathfrak{p}_i}$ is also an isomorphism, which completes the proof of ?? \square

Proof of ??. Let $\eta : E \rightarrow T$ be a homomorphism satisfying the properties of ???. Let ψ be the restriction of η to T and let π be the canonical projection of E onto M . We show that the

homomorphism $\phi = (\eta, \pi) : E \rightarrow T \times M$ solves the problem. There is the commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & T & \longrightarrow & E & \longrightarrow & M & \longrightarrow 0 \\ & & \downarrow \psi & & \downarrow \phi & & \downarrow 1_M \\ 0 & \longrightarrow & T & \longrightarrow & T \times M & \longrightarrow & M & \longrightarrow 0 \end{array}$$

where the rows are exact. The snake diagram gives the exact sequence:

$$0 \longrightarrow \ker \psi \longrightarrow \ker \phi \longrightarrow 0 \longrightarrow \text{coker } \psi \longrightarrow \text{coker } \phi \longrightarrow 0$$

and hence $\ker \phi$ is isomorphic to $\ker \psi$ and $\text{coker } \phi$ to $\text{coker } \psi$. As ψ is a pseudo-isomorphism, so is ϕ . \square

We can say that "to within a pseudo-isomorphism" ?? reduces the study of finitely generated A -modules to that of torsion-free modules on the one hand and to that of torsion modules on the other. Moreover, we have seen above that a torsion-free module is pseudo-isomorphic to its bidual and hence to a reflexive module. As for torsion modules, there is the following result, which determines them to within a pseudo-isomorphism:

Theorem 14.2.29. *Let T be a finitely generated torsion A -module. There exist two finite families $(n_i)_{i \in I}$ and $(\mathfrak{p}_i)_{i \in I}$ where the n_i are positive integers and the \mathfrak{p}_i are prime ideals of A of height 1 such that there exists a pseudo-isomorphism of T to $\bigoplus_{i \in I} A/\mathfrak{p}_i^{n_i}$. Moreover, the families $(n_i)_{i \in I}$ and $(\mathfrak{p}_i)_{i \in I}$ with this property are unique to within a bijection of the indexing set and the \mathfrak{p}_i containing the annihilator of T .*

Proof. Let $T' = \bigoplus_{i \in I} A/\mathfrak{p}_i^{n_i}$. If $\phi : T \rightarrow T'$ is a pseudo-isomorphism and $\mathfrak{p} \in P(A)$, then $\phi_{\mathfrak{p}} : T_{\mathfrak{p}} \rightarrow T'_{\mathfrak{p}}$ is an isomorphism. Now, $T'_{\mathfrak{p}}$ is the direct sum of the $A_{\mathfrak{p}}/\mathfrak{p}^{n_i}A_{\mathfrak{p}}$ the sum being over the indices i such that $\mathfrak{p}_i = \mathfrak{p}$. The $\mathfrak{p}^{n_i}A_{\mathfrak{p}}$ are therefore the elementary divisors of the torsion $A_{\mathfrak{p}}$ -module $T_{\mathfrak{p}}$ and are therefore unique.

We may confine our attention to the case where $T \neq 0$. Let \mathfrak{a} be the annihilator (non-zero and distinct from A) of T and $\mathfrak{p}_1, \dots, \mathfrak{p}_k$ the prime ideals of A of height 1 containing \mathfrak{a} (which are finite in number and $S = \bigcap_{i=1}^k (A - \mathfrak{p}_i)$). The semi-local ring $B = S^{-1}A$ is a principal ideal domain and has maximal ideals the $\mathfrak{m}_i = S^{-1}\mathfrak{p}_i$; as $S^{-1}T$ is a finitely generated torsion B -module, it is isomorphic to a finite direct sum $\bigoplus_{i=1}^k B/\mathfrak{m}_i^{n_i}$. As $B/\mathfrak{m}_i^{n_i}$ is isomorphic to $S^{-1}(A/\mathfrak{p}_i^{n_i})$, we have obtained a torsion A -module T' of the desired type and an isomorphism ϕ_0 of $S^{-1}T$ onto $S^{-1}T'$. As $\text{Hom}_{S^{-1}A}(S^{-1}T, S^{-1}T')$ is equal to $S^{-1}\text{Hom}_A(T, T')$, there exist $s \in S$ and a homomorphism $\phi : T \rightarrow T'$ such that $\phi_0 = s^{-1}\phi$. It remains to show that ϕ is a pseudo-isomorphism. Now if $\mathfrak{p} = 0$ or $\mathfrak{p} \in P(A)$ is distinct from the \mathfrak{p}_i , then $T_{\mathfrak{p}} = T'_{\mathfrak{p}} = 0$. If on the contrary \mathfrak{p} is one of the \mathfrak{p}_i , s is invertible in $A_{\mathfrak{p}_i}$ and, as $\phi_{\mathfrak{p}_i} = s(\phi_0)$ and $(\phi_0)_{\mathfrak{p}_i}$ is an isomorphism of $T_{\mathfrak{p}_i} = (S^{-1}T)_{\mathfrak{m}_i}$ onto $T'_{\mathfrak{p}_i} = (S^{-1}T')_{\mathfrak{m}_i}$, so is $\phi_{\mathfrak{p}_i}$. \square

Given an exact sequence of A -modules $0 \rightarrow E \rightarrow F \rightarrow G \rightarrow 0$, if E and G are pseudo-zero, so is F , as follows from definition and the exactness of localization. In the language of categories, we may then say that, in the category \mathcal{C} of A -modules, the sub-category \mathcal{C}' of pseudo-zero modules is full and we may then define the quotient category \mathcal{C}/\mathcal{C}' . The objects in this category are also A -modules but the set of morphisms from E to F (for E, F in \mathcal{C}) is the direct limit of the set of commutative groups $\text{Hom}_A(E', F')$, where E' (resp. F') runs through the set of submodules of E (resp. the set of quotient modules F/F'' of F) such that

E/E' (resp. F'') is pseudo-zero. Of course, for every ordered pair of A -modules E, F , there is a canonical homomorphism $\text{Hom}_{\mathcal{C}}(E, F) \rightarrow \text{Hom}_{\mathcal{C}/\mathcal{C}'}(E, F)$. To say that a homomorphism $\phi \in \text{Hom}_A(E, F)$ is pseudo-zero (resp. pseudo-injective, pseudo-surjective, pseudo-bijective) means that its canonical image in $\text{Hom}_{\mathcal{C}/\mathcal{C}'}(E, F)$ is zero (resp. a monomorphism, an epimorphism, an isomorphism).

14.2.4 Divisors of finite generated torsion modules

We keep to assume that A is Noetherian and integrally closed. Recall that $\mathfrak{D}(A)$ denotes the divisor group of A , written additively. We know that $\mathfrak{D}(A)$ is the free \mathbb{Z} -module generated by the elements of $P(A)$.

Let T be a finitely generated torsion A -module. For all $\mathfrak{p} \in P(A)$, $T_{\mathfrak{p}}$ is a finitely generated torsion $A_{\mathfrak{p}}$ -module and hence a module of finite length (??). We shall denote this length by $\ell_{\mathfrak{p}}(T)$. Now $T_{\mathfrak{p}} = 0$ for all \mathfrak{p} not containing the annihilator of T and hence for almost all \mathfrak{p} , which justifies the following definition:

Definition 14.2.30. *If T is a finitely generated torsion A -module, the divisor:*

$$\chi(T) = \sum_{\mathfrak{p} \in P(A)} \ell_{\mathfrak{p}}(T) \cdot \mathfrak{p}$$

is called the content of T .

Proposition 14.2.31. *Let χ be the content function defined above.*

- (a) *The function χ is additive.*
- (b) *If T_1 and T_2 are pseudo-isomorphic, then $\chi(T_1) = \chi(T_2)$.*
- (c) *In order that $\chi(T) = 0$, it is necessary and sufficient that T be pseudo-zero.*

Proof. In view of the definition, it suffices to consider for each $\mathfrak{p} \in P(A)$ the values of $\ell_{\mathfrak{p}}$ for the torsion modules considered. The claimed properties then follows from that of length function. \square

Corollary 14.2.32. *If there is a long exact sequence*

$$0 \longrightarrow T_n \longrightarrow T_{n-1} \longrightarrow \dots \longrightarrow T_0 \longrightarrow 0$$

of finitely generated torsion A -modules, then $\sum_{i=0}^n (-1)^i \chi(T_i) = 0$.

Recall that we may speak of the set $F(A)$ of classes of finitely generated A -modules with respect to the relation of isomorphism; for every finitely generated A -module M , let $\text{cl}(M)$ denote the corresponding element of $F(A)$. We shall denote by $T(A)$ the subset of $F(A)$ consisting of the classes of finitely generated torsion A -modules. Clearly χ defines a map of $T(A)$ to $\mathfrak{D}(A)$, also denoted by χ , such that $\chi(\text{cl}(T)) = \chi(T)$.

Proposition 14.2.33. *Let G be a commutative group and $\delta : T(A) \rightarrow G$ a function. For every finitely generated torsion A -module T , we also write, by an abuse of language, $\delta(T) = \delta(\text{cl}(T))$. Suppose that the following conditions are satisfied:*

(a) The function δ is additive.

(b) If T is pseudo-zero, then $\delta(T) = 0$.

Then there exists a unique homomorphism $\theta : \mathfrak{D}(A) \rightarrow G$ such that $\delta = \theta \circ \chi$.

Proof. As $\chi(A/\mathfrak{p}) = \mathfrak{p}$ for all \mathfrak{p} , necessarily $\theta(\mathfrak{p}) = \delta(A/\mathfrak{p})$ for all $\mathfrak{p} \in P(A)$, which proves the uniqueness of θ , since the elements of $P(A)$ form a basis of $\mathfrak{D}(A)$. Conversely, let θ be the homomorphism from $\mathfrak{D}(A)$ to G such that $\theta(\mathfrak{p}) = \delta(A/\mathfrak{p})$ for all $\mathfrak{p} \in P(A)$ and let us show that it solves the problem. For this, we write

$$\psi(T) = \delta(T) - \theta(\chi(T))$$

for every finitely generated torsion A -module T . Clearly conditions (a) and (b) are also satisfied if δ is replaced by ψ . On the other hand, $\psi(A/\mathfrak{p}) = 0$ if $\mathfrak{p} \in P(A)$. If \mathfrak{p} is a nonzero prime ideal and not in P , the annihilator of A/\mathfrak{p} is contained in no ideal of $P(A)$, hence A/\mathfrak{p} is pseudo-zero and therefore $\psi(A/\mathfrak{p}) = 0$. This being so, every finitely generated torsion A -module T admits a chain of submodules whose factors are isomorphic to A -modules of the form A/\mathfrak{p} , where $\mathfrak{p} \in \text{supp}(T)$ (??), and hence $\mathfrak{p} \neq 0$ since T is a torsion module. By induction on the length of this decomposition series, we deduce (in view of property (a) for ψ) that $\psi(T) = 0$. \square

Remark 14.2.2. We may consider the quotient category \mathcal{A}/\mathcal{A}' of the category \mathcal{A} of finitely generated torsion A -modules by the full sub-category \mathcal{A}' of pseudo-zero finitely generated torsion A -modules. In the language of Abelian categories, ?? then expresses the fact that the Grothendieck group of the Abelian category \mathcal{A}/\mathcal{A}' is canonically isomorphic to $\mathfrak{D}(A)$.

Proposition 14.2.34. If \mathfrak{a} is a nonzero ideal of A , then

$$\chi(A/\mathfrak{a}) = \chi((A : \mathfrak{a})/A) = \text{div}(\mathfrak{a}).$$

Proof. Let $\mathfrak{p} \in P(A)$. Then since $A_{\mathfrak{p}}$ is a DVR we have $\mathfrak{a}A_{\mathfrak{p}} = \mathfrak{p}^{n_{\mathfrak{p}}}A_{\mathfrak{p}}$ where some $n_{\mathfrak{p}} \neq 0$. As $(A/\mathfrak{a})_{\mathfrak{p}} = A_{\mathfrak{p}}/\mathfrak{a}A_{\mathfrak{p}} = (A/a)$, we then see $\ell_{\mathfrak{p}}(A/\mathfrak{a}) = n_{\mathfrak{p}}$, whence $\chi(A/\mathfrak{a}) = \sum_{\mathfrak{p} \in P(A)} n_{\mathfrak{p}}\mathfrak{p} = \text{div}(\mathfrak{a})$ by ?? . On the other hand, $(A : a)_{\mathfrak{p}} = (A_{\mathfrak{p}} : \mathfrak{a}A_{\mathfrak{p}}) = \mathfrak{p}^{-n_{\mathfrak{p}}}A_{\mathfrak{p}}$, hence $\ell_{\mathfrak{p}}((A : \mathfrak{a})/A) = n_{\mathfrak{p}}$ and we conclude in the same way. \square

Corollary 14.2.35. Let M be a finitely generated torsion A -module. If $\bigoplus_{i \in I} A/\mathfrak{p}_i^{n_i}$ is the canonical A -module to which M is pseudo-isomorphic to, then

$$\chi(M) = \sum_{i \in I} n_i\mathfrak{p}_i.$$

Proof. By ?? , we see $\chi(A/\mathfrak{p}_i^{n_i}) = \text{div}(\mathfrak{p}_i^{n_i}) = n_i\mathfrak{p}_i$, whence the claim. \square

Now let V be a vector space of dimension n over K and M a lattice of V with respect to A . Let W be the exterior power $\wedge^n V$, which is a one-dimensional vector space over K and let M_W denote the lattice of W generated by the image of M under the canonical map $V^n \rightarrow \wedge^n V$ (??). If e is a basis of W over K , we may write $M_W = \mathfrak{a}e$, where \mathfrak{a} is a nonzero fractional ideal of A .

Let M' be another lattice of V and let us write $M'_W = \mathfrak{a}'e$, where \mathfrak{a}' is a nonzero fractional ideal of A . The divisor $\text{div}(\mathfrak{a}) - \text{div}(\mathfrak{a}')$ does not depend on the choice of basis e of W , \mathfrak{a} and \mathfrak{a}' being multiplied by the same element of K^{\times} when the basis is changed. We shall write

$\chi(M, M') = \text{div}(\mathfrak{a}) - \text{div}(\mathfrak{a}')$ and say that this divisor is the **relative invariant** of M' with respect to M . Clearly, if M, M', M'' are three lattices of V , then:

$$\chi(M, M') + \chi(M', M'') + \chi(M, M'') = 0, \quad \chi(M, M') + \chi(M', M) = 0.$$

For all $\mathfrak{p} \in P(A)$, it follows immediately from the definitions that $(M_W)_{\mathfrak{p}} = (M_{\mathfrak{p}})_W$. Moreover, since $M_{\mathfrak{p}}$ is then a free A -module since A is a principal ideal domain, a basis of M over A is a basis of V over K , hence $(M_{\mathfrak{p}})_W = \bigwedge^n (M_{\mathfrak{p}})$ and the fractional ideal $\mathfrak{a}_{\mathfrak{p}}$ equals $\mathfrak{a}A_{\mathfrak{p}}$. If we set $\mathfrak{a}_{\mathfrak{p}} = \mathfrak{p}^{n_{\mathfrak{p}}} A_{\mathfrak{p}}$ and $\mathfrak{a}'_{\mathfrak{p}} = \mathfrak{p}^{n'_{\mathfrak{p}}} A_{\mathfrak{p}}$, then

$$\chi(M, M') = \sum_{\mathfrak{p} \in P(A)} (n_{\mathfrak{p}} - n'_{\mathfrak{p}}) \mathfrak{p},$$

which may also be written as:

$$\chi(M, M') = \sum_{\mathfrak{p} \in P(A)} \chi(M_{\mathfrak{p}}, M'_{\mathfrak{p}}).$$

identifying $\mathfrak{D}(A_{\mathfrak{p}})$ with the sub- \mathbb{Z} -module of $\mathfrak{D}(A)$ generated by \mathfrak{p} .

Proposition 14.2.36. *Let M be a lattice of V and ϕ a K -automorphism of V . Then:*

$$\chi(M, \phi(M)) = -\text{div}(\det(\phi)).$$

Proof. For all $\mathfrak{p} \in P(A)$, we have $\bigwedge^n \phi(M_{\mathfrak{p}}) = \bigwedge^n \phi(M)_{\mathfrak{p}}$. If e_1, \dots, e_n is a basis of $M_{\mathfrak{p}}$, then

$$\bigwedge^n (M_{\mathfrak{p}}) = A_{\mathfrak{p}} \cdot e_1 \wedge \cdots \wedge e_n, \quad \bigwedge^n \phi(M_{\mathfrak{p}}) = A_{\mathfrak{p}}(\det(\phi)) \cdot e_1 \wedge \cdots \wedge e_n$$

whence the claim. \square

Proposition 14.2.37. *If M, M' are two lattices of V such that $M' \subseteq M$, then M/M' is a finitely generated torsion A -module and $\chi(M, M') = -\chi(M/M')$.*

Proof. Clearly $M/M' \subseteq V/M'$ is a finitely generated torsion module. On the other hand, for all $\mathfrak{p} \in P(A)$, since $A_{\mathfrak{p}}$ is a PID, we know that there exist bases e_1, \dots, e_n of $M_{\mathfrak{p}}$ and e'_1, \dots, e'_n of $M'_{\mathfrak{p}}$ such that $e'_i = \pi^{\nu_i} e_i$ for all i and integers ν_i , π being a uniformizer of $A_{\mathfrak{p}}$. Therefore (in the notation introduced above) $n_i - n'_i = \sum_{i=1}^n \nu_i$. Also, $(M/M')_{\mathfrak{p}} = M_{\mathfrak{p}}/M'_{\mathfrak{p}}$ is isomorphic to the torsion $A_{\mathfrak{p}}$ -module $\bigoplus_{i=1}^n A_{\mathfrak{p}}/\mathfrak{p}_i^{\nu_i} A_{\mathfrak{p}}$, and hence its length is $\sum_{i=1}^n \nu_i$, which proves the proposition. \square

Corollary 14.2.38. *Let L_1, L_2 be two free A -modules of the same rank n and let $\phi : L_1 \rightarrow L_2$ be a homomorphism. For $\text{coker } \phi$ to be a torsion A -module, it is necessary and sufficient that $\det(\phi) \neq 0$ and in this case,*

$$\chi(\text{coker } \phi) = \text{div}(\det(\phi)).$$

Proof. The modules L_1 and L_2 can be considered as lattices in $V_1 = L_1 \otimes_A K$ and $V_2 = L_2 \otimes_A K$ respectively, ϕ extending to a K -homomorphism $\phi_{(K)}$ from V_1 to V_2 . Then $(\text{coker } \phi)_{(K)} = \text{coker } \phi_{(K)}$, and to say that $\text{coker } \phi$ is a torsion A -module means that $\text{coker } \phi_{(K)} = 0$. Now, it amounts to the same to say that $\phi_{(K)}$ is surjective or that $\det(\phi_{(K)}) = \det(\phi) \neq 0$, whence the

first assertion. On the other hand, we may write $\phi(L_1) = \psi(L_2)$, where ϕ is an endomorphism of L_2 of determinant $\det(\phi)$. As $\text{coker } \phi = L_2 / \phi(L_1)$, the formula follows from ??.

Example 14.2.39. If $A = \mathbb{Z}$, the divisor group of A is identified with the multiplicative group \mathbb{Q}_+^\times of rational numbers > 0 . For every finite commutative group T , $\chi(T)$ is the order of T ; the above corollary shows that the order of the group $\text{coker } \phi$ is equal to the absolute value of $\det(\phi)$.

14.2.5 Divisor classes of finite generated modules

Recall that $\mathfrak{C}(A)$ denotes the divisor class group of A , the quotient of $\mathfrak{D}(A)$ by the subgroup of principal divisors. For every divisor $D \in \mathfrak{D}(A)$, we shall denote by $c(D)$ its class in $\mathfrak{C}(A)$.

Proposition 14.2.40. *Let M be a finitely generated A -module. There exists a free submodule L of M such that M/L is a torsion module and the element $c(\chi(M/L))$ of \mathfrak{C} does not depend on the free submodule L . The element $c(M) := -c(\chi(M/L))$ will be called the divisor class attached to M .*

Proof. We write $S = A - \{0\}$ and let $V = S^{-1}M = M \otimes_A K$. If n is the dimension of V over K , there exist n elements e_1, \dots, e_n of M whose canonical images in V form a basis of V . These elements are obviously linearly independent in M and hence generate a free submodule L of M such that $S^{-1}(M/L) = S^{-1}M/S^{-1}L = 0$, so that M/L is a torsion module.

Now let L_1 be another free submodule of M of rank n . Since $S^{-1}L = S^{-1}L_1$, there exists $s \in S$ such that $sL_1 \subseteq L$. We may therefore limit ourselves to proving that, if $L_1 \subseteq L_2$ are two free submodules of M of rank n , then

$$c(\chi(M/L_1)) = c(\chi(M/L_2)).$$

Now, $\chi(M/L_1) = \chi(M/L_2) + \chi(L_2/L_1)$ and it follows from ?? that $\chi(L_2/L_1)$ is a principal divisor and therefore the claim follows. \square

Proposition 14.2.41 (Properties of the divisor class of modules).

- (a) *The function c is additive.*
- (b) *If M_1 and M_2 are pseudo-isomorphic, then $c(M_1) = c(M_2)$.*
- (c) *If T is a torsion A -module, then $c(T) = -c(\chi(T))$.*
- (d) *If \mathfrak{a} is a nonzero fractional ideal of A , then $c(\mathfrak{a}) = -c(\text{div}(\mathfrak{a}))$.*
- (e) *If L is a free A -module then $c(L) = 0$.*

Proof. To prove (a), consider a exact sequence

$$0 \longrightarrow M_1 \xrightarrow{\phi} M_2 \xrightarrow{\psi} M_3 \longrightarrow 0$$

Then there are free sub-module L_1 (resp. L_3) of M_1 (resp. M_3) such that M_1/L_1 (resp. M_3/L_3) is a torsion module. Since L_3 is free and ψ is surjective, there exists in $\psi^{-1}(L_3)$ a free complement L_{23} of $\ker \psi$ which is isomorphic to L_3 . But $\ker \psi = \text{im } \phi$ contains $\phi(L_1) = L_{12}$ which is free

since ϕ is injective. The sum $L_2 = L_{12} + L_{23}$ is direct and L_2 is therefore a free submodule of M_2 . There is moreover the commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & L_1 & \longrightarrow & L_2 & \longrightarrow & L_3 & \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow & \\ 0 & \longrightarrow & M_1 & \xrightarrow{\phi} & M_2 & \xrightarrow{\psi} & M_3 & \longrightarrow 0 \end{array}$$

where the rows are exact and the vertical arrows are injections. We therefore obtain from the snake diagram the exact sequence:

$$0 \longrightarrow M_1/L_1 \longrightarrow M_2/L_2 \longrightarrow M_3/L_3$$

As M_1/L_1 and M_3/L_3 are torsion modules, this exact sequence shows first that so is M_2/L_2 and then

$$\chi(M_2/L_2) = \chi(M_1/L_1) + \chi(M_3/L_3)$$

which proves (a).

Assertions (c) and (e) are obvious from the definition. We prove (b). Therefore let $\phi : M_1 \rightarrow M_2$ be a pseudo-isomorphism and let L_1 be a free submodule of M_1 such that M_1/L_1 is a torsion module. We set $L_2 = \phi(L_1)$; as $\ker \phi$ is pseudo-zero, it is a torsion module, hence $\ker \phi \cap L_1 = 0$ and therefore L_2 is free. Let $\bar{\phi} : M_1/L_1 \rightarrow M_2/L_2$ be the homomorphism derived from ϕ by taking quotients; $\ker \phi$ is isomorphic to $\ker \bar{\phi}$ and $\text{coker } \phi$ to $\text{coker } \bar{\phi}$ and hence $\bar{\phi}$ is a pseudo-isomorphism. Moreover $\text{coker } \bar{\phi} = M_2/\bar{\phi}(M_1)$ is a torsion module and so is $\phi(M_1)/L_2 = \phi(M_1/L_1)$, hence M_2/L_2 is a torsion module and it follows that $\chi(M_1/L_1) = \chi(M_2/L_2)$.

Finally it remains to prove (d). Let $x \in K^\times$ be such that $\mathfrak{a} \subseteq xA$. By considering the exact sequence $0 \rightarrow \mathfrak{a} \rightarrow xA \rightarrow xA/\mathfrak{a} \rightarrow 0$, we obtain

$$c(\mathfrak{a}) = c(xA) - c(xA/\mathfrak{a}) = -c(xA/\mathfrak{a})$$

by (a) and (e). But xA/\mathfrak{a} is isomorphic to $A/x^{-1}\mathfrak{a}$, whence, by virtue of (c),

$$c(xA/\mathfrak{a}) = -c(\chi(A/x^{-1}\mathfrak{a})) = -c(\text{div}(x^{-1}\mathfrak{a})) = -c(\text{div}(\mathfrak{a}))$$

This completes the proof. □

Corollary 14.2.42. *If there is a long exact sequence*

$$0 \longrightarrow M_n \longrightarrow M_{n-1} \longrightarrow \cdots \longrightarrow M_0 \longrightarrow 0$$

of finitely generated A -modules, then $\sum_{i=0}^n (-1)^i c(M_i) = 0$.

Proof. We argue by induction on n , the case $n = 2$ being ??(a). If $N_{n-1} = \text{coker}(M_n \rightarrow M_{n-1})$, there are the two exact sequences:

$$0 \longrightarrow M_n \longrightarrow M_{n-1} \longrightarrow N_{n-1} \longrightarrow 0$$

$$0 \longrightarrow N_{n-1} \longrightarrow M_{n-2} \longrightarrow \cdots \longrightarrow M_0 \longrightarrow 0$$

The first shows that M_{n-1} is finitely generated and the induction hypothesis gives

$$(-1)^{n-1}c(N_{n-1}) + \sum_{i=0}^{n-2} c(M_i) = 0$$

and

$$c(N_{n-1}) = c(M_{n-1}) - c(M_n),$$

whence the corollary. \square

Corollary 14.2.43. *If a nonzero divisorial fractional ideal \mathfrak{a} of A admits a finite free resolution, it is principal.*

Proof. In fact we apply ?? to a finite free resolution of \mathfrak{a} :

$$0 \longrightarrow L_n \longrightarrow L_{n-1} \longrightarrow \cdots \longrightarrow L_0 \longrightarrow \mathfrak{a} \longrightarrow 0$$

By ??(e), we see $c(\mathfrak{a}) = 0$, so $\text{div}(\mathfrak{a})$ is principal (??(d)). As \mathfrak{a} is assumed to be divisorial, it is principal. \square

Corollary 14.2.44. *If every nonzero divisorial ideal of A admits a finite free resolution, A is factorial.*

If M is a finitely generated A -module, we shall denote its rank by $\text{rank}(M)$ (recall that it is the dimension over K of $M \otimes_A K$). We know that $\text{rank}(M)$ is an additive function. We write

$$\gamma(M) = (\text{rank}(M), c(M)) \in \mathbb{Z} \times \mathfrak{C}(A).$$

Then γ is also additive and, if M is pseudo-zero, $\gamma(M) = 0$ (since M is a torsion module). There exists a unique map from $F(A)$ to $\mathbb{Z} \times \mathfrak{C}(A)$, also denoted by γ , such that $\gamma(M) = \gamma(\text{cl}(M))$ for every finitely generated A -module M . We shall see that the above properties essentially characterize γ :

Proposition 14.2.45. *Let G be a commutative group and δ a map from the set $F(A)$ of classes of finitely generated A -modules to G . For every finitely generated A -module M we also write, by an abuse of language, $\delta(M) = \delta(\text{cl}(M))$. Suppose the following conditions are satisfied:*

(a) δ is additive.

(b) If T is pseudo-zero, then $\delta(T) = 0$.

Then there exists a unique homomorphism $\delta : \mathbb{Z} \times \mathfrak{C}(A) \rightarrow G$ such that $\delta = \theta \circ \gamma$.

Proof. By ??, every element of $\mathbb{Z} \times \mathfrak{C}(A)$ is of the form $(\text{rank}(M), c(M))$ for some suitable finitely generated A -module M ; whence the uniqueness of θ . We apply ?? to the restriction of $-\delta$ to $T(A)$: then there exists a homomorphism $\theta_0 : \mathfrak{D}(A) \rightarrow G$ such that

$$-\delta(T) = \theta_0(\chi(T)).$$

for every finitely generated torsion A -module T . Let x be a non-zero element of A ; applying property (a) to the exact sequence:

$$0 \longrightarrow A \xrightarrow{h_x} A \longrightarrow A/xA \longrightarrow 0$$

where h_x is multiplication by x , we obtain $\delta(A/xA) = 0$, whence $\theta_0(\text{div}(x)) = 0$. Taking quotients, θ_0 therefore defines a homomorphism $\theta_1 : \mathfrak{C}(A) \rightarrow G$ and $\delta(T) = \theta_1(c(T))$ for every torsion A -module T . We show now that the homomorphism θ defined by $\theta(n, z) = n\delta(A) + \theta_0(z)$ solves the problem. For this, we write $\delta'(M) = \delta(M) - \theta(\gamma(M))$ for every finitely generated A -module M . Clearly condition (a) is still satisfied if δ is replaced by δ' . Moreover, $\delta'(M) = 0$ when M is a torsion module or a free module. But as for every finitely generated A -module M , there exists a free sub-module L of M such that M/L is a torsion module, property (a) shows that $\delta'(M) = 0$ for every finitely generated A -module M . \square

Remark 14.2.3. In the language of Abelian categories, ?? shows that $\mathbb{Z} \times \mathfrak{C}(A)$ is canonically isomorphic to the Grothendieck group of the quotient category \mathcal{F}/\mathcal{F}' , where \mathcal{F} is the category of finitely generated A -modules and \mathcal{F}' the full sub-category of \mathcal{F} consisting of the pseudo-zero modules.

14.2.6 Finite field extensions

In this part A and B denote two integrally closed Naetherian domains such that $A \subseteq B$ and B is a finitely generated A -module. Let K and L the fields of fractions of A and B respectively. We shall write instead of $\text{div}_A, \chi_A, c_A, \chi_A$ respectively where A -modules are concerned and use analogous notation for B -modules.

We know that B is integral over A by ??, so for a prime ideal \mathfrak{P} of B to be of height 1, it is necessary and sufficient that $\mathfrak{p} = \mathfrak{P} \cap A$ be of height 1. Moreover by ??, for $\mathfrak{p} \in P(A)$, there is only a finite number of prime ideals $\mathfrak{P} \in P(B)$ lying over \mathfrak{p} . To abbreviate, we shall denote by $\mathfrak{P}|\mathfrak{p}$ the relation " \mathfrak{P} lies over \mathfrak{p} ". We shall then denote by $e_{\mathfrak{P}/\mathfrak{p}}$ or $e(\mathfrak{P}/\mathfrak{p})$ the ramification index of the valuation $v_{\mathfrak{P}}$ over the valuation $v_{\mathfrak{p}}$ and by $f_{\mathfrak{P}/\mathfrak{p}}$ or $f(\mathfrak{P}/\mathfrak{p})$ the residue degree $f(v_{\mathfrak{P}}/v_{\mathfrak{p}})$. Recall that the discrete valuations $v_{\mathfrak{p}}$ and $v_{\mathfrak{P}}$ are normed and that $f_{\mathfrak{P}/\mathfrak{p}}$ is the degree of the field of fractions of B/\mathfrak{P} over the field of fractions of A/\mathfrak{p} . We set $n = \text{rank}_A(B)$, where B is considered as an A -module; hence by definition $n = [L : K]$ and, for all $\mathfrak{p} \in P(A)$, n is also the rank of the free $A_{\mathfrak{p}}$ -module $B_{\mathfrak{P}}$ for all $\mathfrak{P}|\mathfrak{p}$. Then it follows from ?? that for all $\mathfrak{p} \in P(A)$:

$$\sum_{\mathfrak{P}|\mathfrak{p}} e_{\mathfrak{P}/\mathfrak{p}} f_{\mathfrak{P}/\mathfrak{p}} = n. \quad (14.2.2)$$

This being so, as $\mathfrak{D}(A)$ and $\mathfrak{D}(B)$ are free \mathbb{Z} -modules, we define an increasing homomorphism of ordered groups $N : \mathfrak{D}(B) \rightarrow \mathfrak{D}(A)$ (also denoted by $N_{B/A}$) by the condition:

$$N(\mathfrak{P}) = f_{\mathfrak{P}/\mathfrak{p}} \mathfrak{p} \quad \text{for } \mathfrak{P} \in P(B) \text{ and } \mathfrak{P}|\mathfrak{p}.$$

On the other hand we have defined an increasing homomorphism of ordered groups $i : \mathfrak{D}(A) \rightarrow$

$\mathfrak{D}(B)$ (also denoted by $i_{B/A}$) by the condition:

$$i(\mathfrak{p}) = \sum_{\mathfrak{P}|\mathfrak{p}} e_{\mathfrak{P}/\mathfrak{p}} \mathfrak{P}$$

for $\mathfrak{p} \in P(A)$. Clearly for every family (D_i) (resp. (E_i)) of divisors of A (resp. B):

$$i(\sup_i D_i) = \sup_i i(D_i), \quad i(\inf_i D_i) = \inf_i i(D_i), \quad (14.2.3)$$

$$N(\sup_i E_i) = \sup_i N(E_i), \quad N(\inf_i E_i) = \inf_i N(E_i). \quad (14.2.4)$$

Also, the formula (??) implies $N \circ i = n \cdot 1_{\mathfrak{D}(A)}$.

Recall that in ?? we have proved that $i(\text{div}_A(a)) = \text{div}_B(a)$ for any $a \in A$. Now let $b \in B$ and consider $N(\text{div}_B(b))$. We recall that the norm $N_{L/K}(b)$ of b is in A and, as $v_{\mathfrak{P}}$ and $v_{\mathfrak{p}}$ are normed, we have

$$(v_{\mathfrak{P}})|_K = e_{\mathfrak{P}/\mathfrak{p}} \cdot v_{\mathfrak{p}} \quad \text{for } \mathfrak{P}|\mathfrak{p}.$$

The formula in (??) then becomes

$$v_{\mathfrak{p}}(N_{L/K}(b)) = \sum_{\mathfrak{P}|\mathfrak{p}} f_{\mathfrak{P}/\mathfrak{p}}(b)$$

which implies $N(\text{div}_B(b)) = \text{div}_A(N_{L/K}(b))$. Therefore, by taking quotients, the homomorphisms N and i define homomorphisms which will also be denoted, by an abuse of language, by:

$$i : \mathfrak{C}(A) \rightarrow \mathfrak{C}(B), \quad N : \mathfrak{C}(B) \rightarrow \mathfrak{C}(A).$$

Proposition 14.2.46. *Let $A \subseteq B$ be Noetherian integrally closed domains and B a finitely generated A -module.*

- (a) *For E to be a pseudo-zero B -module, it is necessary and sufficient that the A -module $\text{Res}_A^B E$ be pseudo-zero.*
- (b) *For E to be finitely generated torsion B -module, it is necessary and sufficient that $\text{Res}_A^B E$ is a finitely generated torsion A -module and in this case we have*

$$\chi_A(\text{Res}_A^B E) = N(\chi_B(E)).$$

- (c) *For E to be finitely generated B -module, it is necessary and sufficient that $\text{Res}_A^B E$ be a finitely generated A -module and in this case we have*

$$c_A(\text{Res}_A^B E) = N(c_B(E)) + \text{rank}_B(E)c_A(B), \quad \text{rank}_A(\text{Res}_A^B E) = n \cdot \text{rank}_B(E).$$

Proof. As B is a finitely generated A -module, for E to be a finitely generated B -module, it is necessary and sufficient that $\text{Res}_A^B E$ be a finitely generated A -module. Moreover, if \mathfrak{b} is the annihilator of E , $\mathfrak{b} \cap A = \mathfrak{a}$ is the annihilator of $\text{Res}_A^B E$. As B is integral over A , there is no ideal other than 0 lying over the ideal 0 of A and hence it amounts to the same to say that $\mathfrak{a} \neq 0$ or that $\mathfrak{b} \neq 0$.

By virtue of this last remark, we may confine our attention to the case where E is a torsion B -module. If \mathfrak{b} is contained in a prime ideal $\mathfrak{P} \in P(B)$, \mathfrak{a} is contained in $\mathfrak{P} \cap A = \mathfrak{p}$, which is of height 1. Conversely, if \mathfrak{a} is contained in a prime ideal $\mathfrak{p} \in P(A)$, there exists a prime ideal \mathfrak{P} of B which contains \mathfrak{b} and lies over \mathfrak{p} . Assertion (a) follows from these remarks and ??.

For every finitely generated torsion B -module E , we write

$$\delta(E) = \chi_A(\text{Res}_A^B E).$$

clearly δ satisfies that conditions ?? (taking account of part (a)). There therefore exists a homomorphism $\theta : \mathfrak{D}(B) \rightarrow \mathfrak{D}(A)$ such that $\delta(E) = \theta(\chi_B(E))$ for every finitely generated torsion B -module E . The homomorphism θ is determined by its value for every B -module of the form B/\mathfrak{P} where $\mathfrak{P} \in P(B)$, since $\chi_B(B/\mathfrak{P}) = \mathfrak{P}$. Now, for every prime ideal $\mathfrak{q} \neq \mathfrak{p} = \mathfrak{P} \cap A$ in $P(A)$, $\mathfrak{p} \not\subseteq \mathfrak{q}$ and hence $(B/\mathfrak{P})_{\mathfrak{q}} = 0$. On the other hand, $\mathfrak{P}B_{\mathfrak{P}}$ is a maximal ideal of $B_{\mathfrak{P}}$ and $(B/\mathfrak{P})_{\mathfrak{p}} = B_{\mathfrak{P}}/\mathfrak{P}B_{\mathfrak{P}}$ is isomorphic to the field of fractions of B/\mathfrak{P} , that is to the residue field of $v_{\mathfrak{P}}$. Its length as an $A_{\mathfrak{p}}$ -module is therefore $f_{\mathfrak{P}/\mathfrak{p}}$, which proves that $\theta = N$, by the definition of χ_A .

If T is the torsion submodule of E , then $\text{Res}_A^B T$ is the torsion submodule of $\text{Res}_A^B E$ and $\text{Res}_A^B(E/T) = \text{Res}_A^B E / \text{Res}_A^B T$. To prove (c) we may therefore confine our attention to the case where E is torsion-free. Then E is identified with a sub- B -module of $E_{(L)}$ and contains a basis e_1, \dots, e_m over L . If b_1, \dots, b_s is a basis of L over K consisting of elements of B , the $b_j e_i$ form a basis of $E_{(L)}$ over K consisting of elements of E , whence the second equality (c). On the other hand, let F be a free sub- B -module of E such that E/F is a torsion B -module; as $\text{Res}_A^B F$ is a direct sum of $\text{rank}_B(E)$ A -modules isomorphic to B , by ??,

$$c_A(\text{Res}_A^B F) = \text{rank}_B(E) \cdot c_A(B).$$

Moreover, by definiton of N and part (b), we have

$$c_A(\text{Res}_A^B(E/F)) = -c_A(N(\chi_B(E/F))) = -N(c_B(\chi_B(E/F))) = N(c_B(E))$$

Then it suffices to apply ?? to finish the proof. □

Proposition 14.2.47. *Let E be a finitely generated B -module. For E to be reflexive, it is necessary and sufficient that $\text{Res}_A^B E$ be a reflexive A -module.*

Proof. We have remarked in the proof of ?? that for E to be a torsion-free B -module, it is necessary and sufficient that $\text{Res}_A^B E$ be a torsion-free A -module. We may therefore assume that E is a lattice of $W = E \otimes_B L$ with respect to B . We shall use the following lemma:

Lemma 14.2.48. *Let W be a vector space of finite dimension over L and let E be a lattice of W with respect to B . Then, for all $\mathfrak{p} \in P(A)$, we have $(\text{Res}_A^B E)_{\mathfrak{p}} = \bigcap_{\mathfrak{P} \mid \mathfrak{p}} E_{\mathfrak{P}}$.*

If $S = A - \mathfrak{p}$, the prime ideals of the ring $S^{-1}B$ are generated by the prime ideals of B not meeting S , in other words the ideals \mathfrak{P}_i lying over \mathfrak{p} and the ideal (0) . This shows that $S^{-1}B$ is a semi-local ring whose maximal ideals are the $\mathfrak{m}_i = S^{-1}\mathfrak{P}_i$ for $1 \leq i \leq m$. Moreover the local ring $(S^{-1}B)_{\mathfrak{m}_i}$ is isomorphic to $B_{\mathfrak{P}_i}$ and hence is a discrete valuation ring. The ring $S^{-1}B$ is therefore a Dedekind domain and, as it is semi-local, it is a principal ideal domain. This being so, $(\text{Res}_A^B E)_{\mathfrak{p}}$ is equal to $S^{-1}E$ considered as an $A_{\mathfrak{p}}$ -module; by the above, $S^{-1}E$ is a free

lattice of W with respect to $S^{-1}B$ (hence reflexive) and ?? may therefore be applied to it, giving $S^{-1}E = \bigcap_i (S^{-1}E)_{\mathfrak{m}_i}$. But $(S^{-1}E)_{\mathfrak{m}_i} = E_{\mathfrak{P}_i}$, which proves the lemma.

Returning to the proof, by the above lemma we have

$$\bigcap_{\mathfrak{P} \in P(B)} E_{\mathfrak{P}} = \bigcap_{\mathfrak{p} \in P(A)} (\text{Res}_A^B E)_{\mathfrak{p}}.$$

and the conclusion follows from ??.

□

Corollary 14.2.49. *The ring B is a reflexive A -module.*

Proposition 14.2.50. *Let $A \subseteq B$ be Noetherian integrally closed domains and B a finitely generated A -module.*

- (a) *For a finitely generated A -module M to be pseudo-Zero, it is necessary and sufficient that $M \otimes_A B$ be a pseudo-zero B -module.*
- (b) *If M is a finitely generated torsion A -module, then $M \otimes_A B$ is a finitely generated B -module and*

$$\chi_B(M \otimes_A B) = i(\chi_A(M)).$$

- (c) *If M is a finitely generated A -module, $M \otimes_A B$ is a finitely generated B -module*

$$c_B(M \otimes_A B) = i(c_A(M)), \quad \text{rank}_B(M \otimes_A B) = \text{rank}_A(M).$$

Proof. Let \mathfrak{P} be a prime ideal of B and $\mathfrak{p} = \mathfrak{P} \cap A$, then $(M \otimes_A B)_{\mathfrak{P}} = M \otimes_A B_{\mathfrak{P}}$, and on the other hand

$$M \otimes_A B_{\mathfrak{P}} = (M \otimes_A A_{\mathfrak{p}}) \otimes_{A_{\mathfrak{p}}} B_{\mathfrak{P}} = M_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} B_{\mathfrak{P}}.$$

the relation $M_{\mathfrak{p}} = 0$ is therefore equivalent to $(M \otimes_A B)_{\mathfrak{P}} = 0$. It suffices to apply this remark to the ideal $\mathfrak{P} = (0)$ and the ideals $\mathfrak{P} \in P(B)$ to prove (a), taking account of the definition. To prove (b), we shall use the following lemma:

Lemma 14.2.51. *Let M_1, M_2 be two finitely generated A -modules, and $\phi : M_1 \rightarrow M_2$ an injective homomorphism. Then the kernel of $\phi \otimes 1 : M_1 \otimes_A B \rightarrow M_2 \otimes_A B$ is pseudo-zero.*

Let \mathfrak{p} be a prime ideal of A of height ≤ 1 . Then $(M_i \otimes_A B)_{\mathfrak{p}} = (M_i)_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} B_{\mathfrak{p}}$ and $(\phi \otimes 1_B)_{\mathfrak{p}} = \phi_{\mathfrak{p}} \otimes 1_{B_{\mathfrak{p}}}$ the hypothesis that ϕ is injective implies that so is $\phi_{\mathfrak{p}}$. On the other hand, in view of the choice of \mathfrak{p} , $A_{\mathfrak{p}}$ is a principal ideal domain and $B_{\mathfrak{p}}$ a finitely generated torsion-free $A_{\mathfrak{p}}$ -module and hence free; we conclude that $\phi_{\mathfrak{p}} \otimes 1_{B_{\mathfrak{p}}}$ is itself injective. If $I = \ker(\phi \otimes 1)$, then $I_{\mathfrak{p}} = \ker((\phi \otimes 1)_{\mathfrak{p}})$, therefore $I_{\mathfrak{p}} = 0$, whence a fortiori $I_{\mathfrak{P}} = (I_{\mathfrak{p}})_{\mathfrak{P}} = 0$ for $\mathfrak{P} \mid \mathfrak{p}$, which proves the lemma.

We return now to the proof of (b). For every finitely generated torsion A -module M , write $\delta(M) = \chi_B(M \otimes_A B)$. It follows from (a) that, if M is pseudo-zero, then $\delta(M) = 0$. On the other hand, consider an exact sequence of finitely generated torsion A -modules:

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$$

It follows from the above lemma that there is an exact sequence of B -modules:

$$0 \longrightarrow I \longrightarrow M_1 \otimes_A B \longrightarrow M_2 \otimes_A B \longrightarrow M_3 \otimes_A B \longrightarrow 0$$

where I is pseudo-zero. Using the additivity of χ we therefore see δ is additive. We therefore conclude from ?? that there exists a homomorphism $\delta : \mathfrak{D}(A) \rightarrow \mathfrak{D}(B)$ such that $\delta(M) = \theta(\chi_A(M))$ for every finitely generated torsion A -module M . To prove that $\theta = i$, it suffices to show that $\delta(A/\mathfrak{p}) = i(\mathfrak{p})$ for all $\mathfrak{p} \in P(A)$. Now $(A/\mathfrak{p}) \otimes_A B = B/\mathfrak{p}B$ and, for all $\mathfrak{P} \in P(B)$, $(B/\mathfrak{p}B)_{\mathfrak{P}} = B_{\mathfrak{P}}/\mathfrak{p}B_{\mathfrak{P}}$. The last module is 0 if \mathfrak{P} does not lie over \mathfrak{p} . If on the contrary $\mathfrak{P}|\mathfrak{p}$, then $B_{\mathfrak{P}}/\mathfrak{p}B_{\mathfrak{P}}$ is a $B_{\mathfrak{P}}$ -module of length $e(\mathfrak{P}/\mathfrak{p})$ by definition of the ramification index. Therefore $\chi_B(B/\mathfrak{p}B) = \sum_{\mathfrak{P}|\mathfrak{p}} e_{\mathfrak{P}/\mathfrak{p}} \mathfrak{P} = i(\mathfrak{p})$, which proves (b).

The second formula in (c) is immediate, for

$$(M \otimes_A B) \otimes_B L = M \otimes_A L = M \otimes_A K \otimes_K L$$

and the rank of $(M \otimes_A K) \otimes_K L$ over L is equal to the rank of $M \otimes_A K$ over K . To show the first, consider a free submodule H of M such that $Q = M/H$ is a torsion A -module. Applying the lemma as above, we obtain an exact sequence of B -modules:

$$0 \longrightarrow I \longrightarrow H \otimes_A B \longrightarrow M \otimes_A B \longrightarrow Q \otimes_A B \longrightarrow 0$$

where I is pseudo-zero. It therefore follows from ??(a), (b) and (e) that

$$c_B(M \otimes_A B) = c_B(Q \otimes_A B) - c_B(\chi_B(Q \otimes_A B)) = -c_B(i(\chi_A(Q))) = -i(c_A(\chi_A(Q))) = i(c_A(M))$$

by virtue of (b), which completes the proof. \square

14.2.7 Modules over Dedekind domains

We now assume that A is a Dedekind domain. Then we know that the ideals $\mathfrak{p} \in P(A)$ are maximal and that they are the only nonzero prime ideals of A . The group $\mathfrak{D}(A)$ is identified with the group $\mathfrak{F}(A)$ of fractional nonzero ideals of A .

Proposition 14.2.52. *Let A be a Dedekind domain. Every pseudo-zero A -module is zero. Every pseudo-injective (resp. pseudo-surjective, pseudo-bijective, pseudo-zero) A -module homomorphism is injective (resp. smjeetive, bijective, zero).*

Proof. The first assertion has already been shown; the others follow from it immediately. \square

Proposition 14.2.53. *Let A be a Dedekind domain and M a finitely generated A -module. The following properties are equivalent:*

- (a) M is torsion-free;
- (b) M is reflexive;
- (c) M is projective.

Proof. We already know (with no hypothesis on the integral domain A) that (ii) implies (i) and that (iii) implies (ii). If M is torsion-free, it is identified with a lattice of $V = M \otimes_A K$ with

respect to A . $M_{\mathfrak{p}}$ is therefore a free $A_{\mathfrak{p}}$ -module for every maximal ideal $\mathfrak{p} \in P(A)$, since $A_{\mathfrak{p}}$ is a principal ideal domain. The conclusion then follows from ??.

Corollary 14.2.54. *Let M be a finitely generated A -module and let T be its torsion submodule. Then T is a direct factor of M .*

Proof. As M/T is torsion-free and finitely generated, it is projective by ?? and the corollary therefore follows. \square

Proposition 14.2.55. *Let A be a Dedekind domain and T a finitely generated torsion A -module. There exist two finite families $(n_i)_{i \in I}$ and $(\mathfrak{p}_i)_{i \in I}$ where the n_i are positive integers and the \mathfrak{p}_i are elements of $P(A)$, such that T is isomorphic to the direct sum $\bigoplus_{i \in I} (A/\mathfrak{p}_i^{n_i})$. Further, the families $(n_i)_{i \in I}$ and $(\mathfrak{p}_i)_{i \in I}$ are unique up to within a bijection of the indexing set.*

Proof. This follows from ?? since a pseudo-isomorphism is here an isomorphism. \square

Proposition 14.2.56. *Let A be a Dedekind domain and M a finitely generated torsion-free A -module of rank $n \geq 1$. Then there exists a nonzero ideal \mathfrak{a} of A such that M is isomorphic to the direct sum of the modules A^{n-1} and \mathfrak{a} . Moreover, the class of the ideal \mathfrak{a} is determined uniquely by this condition.*

Proof. ?? shows that there exists a free submodule L of M such that M/L is isomorphic to an ideal \mathfrak{b} of A . If $\mathfrak{b} = 0$, we take $\mathfrak{a} = A$. Otherwise, \mathfrak{b} is of rank 1, hence $L = A^{n-1}$ and \mathfrak{b} is a projective module by ?? . M is therefore isomorphic to the direct sum of L and \mathfrak{b} , which proves the first part of the proposition. Moreover, it follows from ?? that $c(M) = c(\mathfrak{a})$ whence the uniqueness of the class of \mathfrak{a} . \square

Chapter 15

Dimension and Hilbert function

15.1 Dimension of rings

15.1.1 Dimension of rings and modules

Definition 15.1.1. Let A be a ring. Then **dimension** of A , denoted by $\dim(A)$, is defined to be the Krull dimension of the topological space $\text{Spec}(A)$. For a prime ideal \mathfrak{p} of A , the dimension of A at \mathfrak{p} , denoted by $\dim_{\mathfrak{p}}(A)$, is defined to be $\dim_{\mathfrak{p}}(\text{Spec}(A))$.

Recall that the map $\mathfrak{p} \mapsto V(\mathfrak{p})$ is a bijection from prime ideals of A to irreducible closed subsets of $\text{Spec}(A)$. The dimension of A is then the supremum of the length of chains of prime ideals of A . Since we do not consider the zero ring, $\dim(A)$ will always be a positive integer.

Let $\mathfrak{p} \in \text{Spec}(A)$, the sets $D(f)$, where f runs through A , form a basis of the topology of $\text{Spec}(A)$, and \mathfrak{p} belongs to the open set $D(f)$ if and only if f does not belong to \mathfrak{p} . Therefore, $\dim_{\mathfrak{p}}(A)$ is the infimum of the numbers $\dim(A_f)$, where f runs through $A - \mathfrak{p}$ (??).

Example 15.1.2 (Examples of dimension of rings).

- (a) For $\dim(A) = 0$, it is necessary and sufficient that every prime ideal of A is maximal. Thus a zero-dimensional integral domain is a field, and a zero-dimensional Noetherian ring is Artinian.
- (b) A Dedekind domain is a Noetherian integrally closed domain of dimension one. More generally, a ring is a finite product of Dedekind's rings if and only if it is Noetherian, reduced, integrally closed in its total ring of fractions, and of dimension one.
- (c) Let k be a field and A an k -algebra that is integral over k . Then since $\dim(k) = 0$ we have $\dim(A) = 0$.
- (d) For a ring A we have

$$\dim(A[X]) \geq \dim(A) + 1.$$

In fact, if $\mathfrak{p}_0 \subset \dots \subset \mathfrak{p}_n$ is a chain of prime ideals of A , then we have a chain $\mathfrak{P}_0 \subset \dots \subset \mathfrak{P}_n \subset \mathfrak{P}_{n+1}$ of prime ideals of $A[X]$, where $\mathfrak{P}_i = \mathfrak{p}_i A[X]$ for $1 \leq i \leq n$ and $\mathfrak{P}_{n+1} = \mathfrak{p}_n A[X] + XA[X]$. By the same reasoning, we prove the inequality $\dim(A[X]) \geq \dim(A) + 1$. We deduces by induction the inequalities

$$\dim(A[X_1, \dots, X_n]) \geq \dim(A) + n, \quad \dim(A[[X_1, \dots, X_n]]) \geq \dim(A) + n.$$

Later we will see that the equality holds if A is a Noetherian ring.

Proposition 15.1.3. *Let A be a ring.*

- (a) *For any ideal \mathfrak{a} of A , we have $\dim(A/\mathfrak{a}) \leq \dim(A)$.*
- (b) *Let S be a multiplicative subset of A , then $\dim(S^{-1}A) \leq \dim(A)$.*
- (c) *We have $\dim(A) = \sup_{\mathfrak{p}} \dim(A/\mathfrak{p}) = \sup_{\mathfrak{m}} \dim(A_{\mathfrak{m}})$, where \mathfrak{p} runs through minimal prime ideals of A and \mathfrak{m} runs through maximal ideals of A .*
- (d) *Assume that A has finitely many minimal prime ideals. Then for any prime ideal \mathfrak{p} of A ,*

$$\dim_{\mathfrak{p}}(A) = \sup_{\mathfrak{q}} \dim_{\mathfrak{p}/\mathfrak{q}}(A/\mathfrak{q})$$

where \mathfrak{q} runs through minimal prime ideals of A contained in \mathfrak{p} .

- (e) *Let \mathfrak{a} be an ideal of A which is not contained in any minimal prime ideal of A . Then we have $\dim(A) \geq \dim(A/\mathfrak{a}) + 1$. In particular, if A is an integral domain, then $\dim(A) \geq \dim(A/\mathfrak{a}) + 1$ for any nonzero ideal \mathfrak{a} of A .*

Proof. By ??, $\text{Spec}(A/\mathfrak{a})$ is homeomorphic to the subspace $V(\mathfrak{a})$ of $\text{Spec}(A)$, and we get $\dim(A/\mathfrak{a}) \leq \dim(A)$ therefore. Similarly, part (b) follows from ?. Also, by ??, the irreducible components of $\text{Spec}(A)$ are homeomorphic to $\text{Spec}(A/\mathfrak{p})$, where \mathfrak{p} are minimal prime ideals of A . This proves (c), and (d) follows from ??.

Finally, let \mathfrak{a} be an ideal of A which is not contained in any minimal prime ideal of A . We prove that, for any chain $\mathfrak{p}_0 \subset \cdots \subset \mathfrak{p}_n$ of prime ideals of A such that $\mathfrak{a} \subseteq \mathfrak{p}_0$ we have $\dim(A) \geq n + 1$. By the hypothesis on \mathfrak{a} we see \mathfrak{p}_0 is not minimal, so there exists a prime ideal \mathfrak{p}_{-1} of A contained in \mathfrak{p}_0 , distinct from \mathfrak{p}_0 , and $\mathfrak{p}_{-1} \subset \mathfrak{p}_0 \subset \cdots \subset \mathfrak{p}_n$ is a chain of prime ideals of A , whence $\dim(A) \geq n + 1$. \square

Corollary 15.1.4. *Let $\rho : A \rightarrow B$ be a ring homomorphism. Then $\dim(B)$ is the supremum of the numbers $\dim(B/\mathfrak{p}^e)$, where \mathfrak{p} covers the set of minimal prime ideals of A .*

Proof. In fact, for each minimal prime ideal \mathfrak{P} of B , there exists a minimal prime ideal \mathfrak{p} of A contained in \mathfrak{P}^c , and therefore

$$\dim(B/\mathfrak{P}) \leq \dim(B/\mathfrak{p}^e) \leq \dim(B)$$

whence the claim by ??.

\square

For an ideal \mathfrak{a} of A , we define the **height** (resp. **coheight**) $\text{ht}(\mathfrak{a})$ (resp. $\text{coht}(\mathfrak{a})$) of \mathfrak{a} to be the codimension (resp. dimension) of $V(\mathfrak{a})$ in $\text{Spec}(A)$. Also, the ring is called **catenary** if $\text{Spec}(A)$ is catenary.

Proposition 15.1.5. *Let A be a ring.*

- (a) *The height of a prime ideal \mathfrak{p} is the supremum of the length of chains $\mathfrak{p}_0 \subseteq \cdots \subseteq \mathfrak{p}_n$ of prime ideals of A such that $\mathfrak{p}_n = \mathfrak{p}$.*

(b) Let \mathfrak{p} be a prime ideal of A and \mathfrak{a} an ideal of A . Then $\dim(A_{\mathfrak{p}}/\mathfrak{a}A_{\mathfrak{p}}) = -\infty$ if \mathfrak{a} is not contained in \mathfrak{p} , and $\dim(A_{\mathfrak{p}}/\mathfrak{a}A_{\mathfrak{p}}) = \text{codim}(V(\mathfrak{p}), V(\mathfrak{a}))$ if \mathfrak{a} is contained in \mathfrak{p} . In particular, for a prime ideal \mathfrak{p} of A , we have $\dim(A_{\mathfrak{p}}) = \text{ht}(\mathfrak{p})$.

(c) For any ideal \mathfrak{a} of A , we have $\text{ht}(\mathfrak{a}) = \inf_{\mathfrak{p} \supseteq \mathfrak{a}} \text{ht}(\mathfrak{p}) = \inf_{\mathfrak{p} \supseteq \mathfrak{a}} \dim(A_{\mathfrak{p}})$.

Proof. Assertion (a) follows from definition, and (b) follows from the fact that the map $\mathfrak{q} \mapsto \mathfrak{q}(A_{\mathfrak{p}}/\mathfrak{a}A_{\mathfrak{p}})$ is an increasing isomorphism of the set of prime ideals \mathfrak{q} of A such that $\mathfrak{a} \subseteq \mathfrak{q} \subseteq \mathfrak{p}$ on the set of prime ideals of the local ring $A_{\mathfrak{p}}/\mathfrak{a}A_{\mathfrak{p}}$. Now let \mathfrak{a} be an ideal of A . Then the irreducible closed subset of $V(\mathfrak{a})$ are the sets $V(\mathfrak{p})$, where \mathfrak{p} is a prime ideal of A containing \mathfrak{a} . Part (c) then follows from the definition of codimension. \square

Corollary 15.1.6. Let \mathfrak{p} be a prime ideal of A and S a multiplicative subset of A not meeting \mathfrak{p} . Then $\text{ht}(\mathfrak{p}) = \text{ht}(S^{-1}\mathfrak{p})$.

Proof. This follows from ??.

Proposition 15.1.7. Let A be a ring. Let $\mathfrak{a}, \mathfrak{b}$ be proper ideals of A such that $\mathfrak{a} \subseteq \mathfrak{b}$. Then

$$\text{codim}(V(\mathfrak{b}), V(\mathfrak{a})) \leq \dim(A/\mathfrak{a}) - \dim(A/\mathfrak{b})$$

In particular, for any ideal \mathfrak{a} of A , we have $\text{ht}(\mathfrak{a}) + \dim(A/\mathfrak{a}) \leq \dim(A)$.

Proof. This from ?? and of the relations $\dim(A/\mathfrak{a}) = \dim(V(\mathfrak{a}))$, $\dim(A/\mathfrak{b}) = \dim(V(\mathfrak{b}))$. \square

Proposition 15.1.8. Let A be a ring.

(a) For A to be catenary, it is necessary and sufficient that, for every triple $(\mathfrak{p}, \mathfrak{q}, \mathfrak{r})$ of prime ideals of A such that $\mathfrak{r} \subseteq \mathfrak{q} \subseteq \mathfrak{p}$, we have

$$\dim(A_{\mathfrak{p}}/\mathfrak{q}A_{\mathfrak{p}}) + \dim(A_{\mathfrak{q}}/\mathfrak{r}A_{\mathfrak{q}}) = \dim(A_{\mathfrak{p}}/\mathfrak{r}A_{\mathfrak{p}}).$$

(b) If A is an integral domain with finite dimension, then for A to be catenary, it is necessary and sufficient that, for every pair $(\mathfrak{p}, \mathfrak{q})$ of prime ideals of A such that $\mathfrak{q} \subseteq \mathfrak{p}$, we have

$$\text{ht}(\mathfrak{q}) + \dim(A_{\mathfrak{p}}/\mathfrak{q}A_{\mathfrak{p}}) = \text{ht}(\mathfrak{p}).$$

Proof. This follows directly from ?? and ??.

Proposition 15.1.9. Let A be a ring of finite dimension such that all maximal chains of prime ideals have the same length. Then A is catenary, and for any prime ideal \mathfrak{p} of A we have

$$\text{ht}(\mathfrak{p}) = \dim(A) - \text{coht}(\mathfrak{p}).$$

Moreover, for any pair $(\mathfrak{p}, \mathfrak{q})$ of prime ideals such that $\mathfrak{q} \subseteq \mathfrak{p}$, we have

$$\dim(A_{\mathfrak{p}}/\mathfrak{q}A_{\mathfrak{p}}) = \text{coht}(\mathfrak{q}) - \text{coht}(\mathfrak{p}).$$

Proof. This is a reformulation of ??.

Now let M be a finitely generated A -module. The **dimension** (resp. **codimension**) of M , denoted by $\dim_A(M)$ (resp. $\text{codim}(M)$), is defined to be the dimension (resp. codimension) of the support of M . The support of the A -module A is $\text{Spec}(A)$, so the dimension of the A -module A equals the dimension of the ring A . Also, since M is finitely generated, we have

$$\text{supp}(M) = V(\mathfrak{a}) = \text{supp}(A/\mathfrak{a}),$$

where \mathfrak{a} is the annihilator of M . Consequently, the dimension of the A -module M , the dimension of the A -module A/\mathfrak{a} , the dimension of the ring A/\mathfrak{a} and the dimension of the (A/\mathfrak{a}) -module M are all the supremum of the set of lengths of the chains $\mathfrak{p}_0 \subset \cdots \subset \mathfrak{p}_n$ of prime ideals of A such that $\mathfrak{a} \subseteq \mathfrak{p}_0$. By ??(c), the dimension of M is also the supremum of the dimensions of the rings (or of the A -modules) A/\mathfrak{p} , where \mathfrak{p} runs through the set of prime ideals of A , minimal among those which contain \mathfrak{a} .

Example 15.1.10. If A is Noetherian, then by ??, we see $\dim_A(M) = 0$ if and only $\text{supp}(M)$ consists of maximal ideals, or equivalently M is of finite length.

Example 15.1.11. If M is a finitely generated module over a Noetherian ring A , $\dim_A(M)$ is then the supremum of the numbers $\dim(A/\mathfrak{p})$, where \mathfrak{p} runs through the set $\text{Ass}_A(M)$ of prime ideals of A associated with M ??.

Proposition 15.1.12. Let A be a ring and M a finitely generated A -module.

- (a) For each $\mathfrak{p} \in \text{supp}(M)$, we have $\dim_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}) = \text{codim}(V(\mathfrak{p}), \text{supp}(M))$.
- (b) We have $\dim_A(M) = \sup_{\mathfrak{p}} \dim_{A_{\mathfrak{p}}}(M_{\mathfrak{p}})$, where \mathfrak{p} runs through prime ideals in $\text{supp}(M)$ (resp. maximal ideal in $\text{supp}(M)$).
- (c) Let $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ be an exact sequence of finitely generated A -modules. Then $\dim_A(M) = \sup\{\dim_A(M'), \dim_A(M'')\}$.

Proof. Let \mathfrak{a} be the annihilator of M , then the annihilator of $M_{\mathfrak{p}}$ is $\mathfrak{a}A_{\mathfrak{p}}$, whence $\dim_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}) = \dim(A_{\mathfrak{p}}/\mathfrak{a}A_{\mathfrak{p}})$, and the claim follows from ?? and $\text{supp}(M) = V(\mathfrak{a})$. Also, part (b) follows from (a), and (c) follows from ?? \square

Now let A be a Noetherian ring. Let $Z(A)$ be the free \mathbb{Z} -module generated by the set of closed irreducible subsets of $\text{Spec}(A)$. For any irreducible closed subset Y of $\text{Spec}(A)$, we denote by $[Y]$ the element correspondent of $Z(A)$. The elements of $Z(A)$ are sometimes called **cycles**. Let M be a finitely generated A -module. For any prime ideal \mathfrak{p} of A which is a minimal element of $\text{supp}(M)$, we have $0 < \ell_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}) < +\infty$ by ?. We set

$$z(M) = \sum_{\mathfrak{p}} \ell_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}) \cdot [V(\mathfrak{p})]$$

where \mathfrak{p} runs through the finite set of minimal element of $\text{supp}(M)$.

Example 15.1.13. For any $\mathfrak{p} \in \text{Spec}(A)$, we have $z(A/\mathfrak{p}) = [V(\mathfrak{p})]$. More generally, let M be a finitely generated A -module, and let $(M_i)_{1 \leq i \leq n}$ be a chain of M such that for each i , the modulus M_i/M_{i+1} is isomorphic to A/\mathfrak{p}_i , where \mathfrak{p}_i is a prime ideal of A . Then we have $z(M) = \sum_{i \in J} \ell_i[V(\mathfrak{p}_i)]$, where J is the subset of I consists of i such that \mathfrak{p}_i be a minimal element of $\{\mathfrak{p}_0, \dots, \mathfrak{p}_{n-1}\}$, and ℓ_i is the number of indices j such that $\mathfrak{p}_j = \mathfrak{p}_i$.

For any integer d , let $Z_{\leq d}$ (resp. Z_d , resp. $Z^{\geq d}$, resp. Z^d) be the \mathbb{Z} -submodule of $Z(A)$ generated by the elements $[V(\mathfrak{p})]$ where \mathfrak{p} is a prime ideal of A such that $\text{coht}(\mathfrak{p}) \leq d$ (resp. $\text{coht}(\mathfrak{p}) = d$, resp. $\text{ht}(\mathfrak{p}) \geq d$, resp. $\text{ht}(\mathfrak{p}) = d$). An element of Z_d (resp. Z^d) is called a **cycle** of dimension d (resp. codimension d). We have

$$Z_{\leq d} = Z_{\leq d-1} \oplus Z_d, \quad Z^{\geq d} = Z^{\geq d-1} \oplus Z^d.$$

Let \mathcal{C} be the set of classes of finitely generated A -modules, and for each integer d , let $C_{\leq d}$ (resp. $C^{\geq d}$) be the subgroup generated by finitely generated A -modules of dimension $\leq d$ (resp. of codimension $\geq d$).

Lemma 15.1.14. *Let M be a finitely generated A -module.*

- (a) *For M be in $C_{\leq d}$, it is necessary and sufficient that $z(M) \in Z_{\leq d}$. In this case, the projection $z_d(M)$ of $z(M)$ on Z_d parallel to $Z_{\leq d-1}$ is given by*

$$z_d(M) = \sum_{\text{coht}(\mathfrak{p})=d} \ell_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}) \cdot [V(\mathfrak{p})].$$

- (b) *For M be in $C^{\geq d}$, it is necessary and sufficient that $z(M) \in Z^{\geq d}$. In this case, the projection $z^d(M)$ of $z(M)$ on Z^d parallel to $Z^{\geq d-1}$ is given by*

$$z^d(M) = \sum_{\text{ht}(\mathfrak{p})=d} \ell_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}) \cdot [V(\mathfrak{p})].$$

Proof. For M to be dimension $\leq d$, it is necessary and sufficient that for any minimal prime \mathfrak{p} of $\text{supp}(M)$, we have $\text{coht}(\mathfrak{p}) \leq d$, which means $z(M) \in Z_{\leq d}$. Suppose now $\dim(M) \leq d$, and let $\mathfrak{p} \in \text{Spec}(A)$ be such that $\text{coht}(\mathfrak{p}) = d$. Then, either $\mathfrak{p} \notin \text{supp}(M)$ and $M_{\mathfrak{p}} = 0$, or $\mathfrak{p} \in \text{supp}(M)$ and \mathfrak{p} is a minimal element of $\text{supp}(M)$. The coefficient of $[V(\mathfrak{p})]$ in $z(M)$ is in both cases $\ell_{A_{\mathfrak{p}}}(M_{\mathfrak{p}})$, whence (a). Part (b) is proved analogously, noting that a finitely generated module M is in $C^{\geq d}$ if and only if we have $M_{\mathfrak{p}} = 0$ for any prime ideal \mathfrak{p} of height $< d$. \square

By ??, the sets $C_{\leq d}$ and $C^{\geq d}$ are hereditary, and we can consider the corresponding Grothendieck groups $K(C_{\leq d})$ and $K(C^{\geq d})$, which are subgroups of $K(\mathcal{C})$, the Grothendieck group of finitely generated A -modules. By ??, the functions z_d and z^d are additive, whence extends to homomorphisms

$$\zeta_d : K(C_{\leq d}) \rightarrow Z_d, \quad \zeta^d : K(C^{\geq d}) \rightarrow Z^d$$

Moreover, since $C_{\leq d-1} \subseteq C_{\leq d}$ and $C^{\geq d+1} \subseteq C^{\geq d}$, we have canonical homomorphisms

$$i_d : K(C_{\leq d-1}) \rightarrow K(C_{\leq d}), \quad i^d : K(C^{\geq d+1}) \rightarrow K(C^{\geq d}).$$

Proposition 15.1.15. *The following sequence of \mathbb{Z} -modules*

$$K(C_{\leq d-1}) \xrightarrow{i_d} K(C_{\leq d}) \xrightarrow{\zeta_d} Z_d \longrightarrow 0$$

$$K(C^{\geq d+1}) \xrightarrow{i^d} K(C^{\geq d}) \xrightarrow{\zeta^d} Z^d \longrightarrow 0$$

are exact.

Proof. We have $\zeta_d \circ i_d = 0$ by ???. For every $\mathfrak{p} \in \text{Spec}(A)$ such that $\text{coht}(\mathfrak{p}) = d$, we see $\zeta_d([A/\mathfrak{p}]) = z_d(Z/\mathfrak{p}) = [V(\mathfrak{p})]$, so the homomorphism ζ_d is surjective. Now by ???, $K(\mathcal{C}_{\leq d})$ is generated by the elements $[A/\mathfrak{p}]$, where $\mathfrak{p} \in \text{Spec}(A)$ and $\text{coht}(\mathfrak{p}) \leq d$. Therefore, every element ξ can be written into $\xi = i_d(\eta) + \sum_{i=1}^k n_i [A/\mathfrak{p}_i]$, where $\eta \in K(\mathcal{C}_{\leq d-1})$, $n_i \in \mathbb{Z}$ and $\text{coht}(\mathfrak{p}_i) = d$ for each i . From this, we see $\zeta_d(\xi) = \sum_{i=1}^k n_i [V(\mathfrak{p}_i)]$, and $\zeta_d(\xi) = 0$ if and only if $\xi \in \text{im } i_d$, which proves the claim. The proof for i^d and ζ^d can be done similarly. \square

Example 15.1.16. Let A be a Noetherian integral domain. Then $Z^0 = \mathbb{Z}\text{Spec}(A)$ and $z^0(M) = \text{rank}(M) \cdot \text{Spec}(A)$. Therefore the modules in $\mathcal{C}^{\geq 1}$ are the torsion modules.

Example 15.1.17. Assume that A is Noetherian and integrally closed. Then Z^1 is identified with the group $\mathfrak{D}(A)$ of divisors of A . The modules in $\mathcal{C}^{\geq 2}$ are therefore pseudo-zero modules ???. If M is a finitely generated torsion module, then $z^1(M) \in Z^1\mathfrak{D}(A)$ is the content $\chi(M)$ of M . ?? and ?? is then equivalent to the exactness of the sequence $K(\mathcal{C}^{\geq 2}) \rightarrow K(\mathcal{C}^{\geq 1}) \rightarrow Z^1 \rightarrow 0$.

Example 15.1.18. The elements in \mathcal{C}_0 are finitely generated A -modules of dimension 0, that is, of finite length. Let $\epsilon : Z_0 \rightarrow \mathbb{Z}$ be the evaluation map, then $\ell_A(M) = \epsilon(z_0(M))$, in view of ??.

Example 15.1.19. Let A be a integral domain of finite dimension. Let $d = \dim(A)$, then $\mathcal{C}_{\leq d} = \mathcal{C}$ and $Z_d = \mathbb{Z}\text{Spec}(A)$. Also, we have $z_d(M) = \text{rank}(M)\text{Spec}(A) = z^0(M)$, whence the modules in $\mathcal{C}_{\leq d-1}$ are precisely the torsion A -modules.

15.1.2 Dimension and going-down property

Let $\rho : A \rightarrow B$ be a ring homomorphism. Recall that we say ρ has the going-down property if it satisfies the going down theorem.

Proposition 15.1.20. Let $\rho : A \rightarrow B$ be a ring homomorphism with the going-down property. Let F be a closed subset of $\text{Spec}(A)$. If Y is an irreducible component of the inverse image of F under $\rho^* : \text{Spec}(B) \rightarrow \text{Spec}(A)$, then the closure of $\rho^*(Y)$ is an irreducible component of F .

Proof. Let \mathfrak{a} be an ideal of A such that $F = V(\mathfrak{a})$. By ??, the inverse image by ρ^* of F is the subset $V(\mathfrak{a}^e)$ of $\text{Spec}(B)$. Let $Y = V(\mathfrak{P})$ be an irreducible component of $V(\mathfrak{a}^e)$, then the closure of $\rho^*(Y)$ is the irreducible closed subset $V(\mathfrak{P}^c)$. So we have to prove that, if \mathfrak{P} is a prime ideal minimal among the prime ideals of B containing \mathfrak{a}^e , then \mathfrak{P}^c is minimal among the prime ideals of A containing \mathfrak{a} .

Suppose the contrary, then there would exist a prime ideal \mathfrak{q} of A with $\mathfrak{a} \subseteq \mathfrak{q} \subset \mathfrak{P}^c$. According to the going-down property, there exist a prime ideal \mathfrak{Q} of B lying over \mathfrak{q} such that $\mathfrak{Q} \subset \mathfrak{P}$, whence $\mathfrak{a}^e \subseteq \mathfrak{Q} \subset \mathfrak{P}$, contrary to the hypothesis made on \mathfrak{P} . \square

Proposition 15.1.21. Let $\rho : A \rightarrow B$ be a ring homomorphism with the going-down property. Then we have the inequality

$$\dim(B) \geq \dim(A) + \inf_{\mathfrak{m} \in \text{Max}(A)} \dim(B/\mathfrak{m}B) \geq \dim(A) + \inf_{\mathfrak{p} \in \text{Spec}(A)} \dim(B \otimes_A \kappa(\mathfrak{p})). \quad (15.1.1)$$

Proof. We have $\dim(A) = \sup_{\mathfrak{m}} \dim(A_{\mathfrak{m}})$, so it suffices to show the inequality

$$\dim(B) \geq \dim(A_{\mathfrak{m}}) + \dim(B/\mathfrak{m}B) \quad (15.1.2)$$

for each maximal ideal \mathfrak{m} of A . In other words, we have to prove the inequality $\dim(B) \geq n + r$ if $\mathfrak{p}_0 \subset \dots \subset \mathfrak{p}_n$ is a chain of prime ideals of A contained in \mathfrak{m} and $\bar{\mathfrak{Q}}_0 \subset \dots \subset \bar{\mathfrak{Q}}_r$ is a chain of prime ideals of $B/\mathfrak{m}B$. For $1 \leq i \leq r$, there exists a prime ideal \mathfrak{Q}_{n+i} of B containing $\mathfrak{m}B$ such that $\bar{\mathfrak{Q}}_i = \mathfrak{Q}_{n+i}/\mathfrak{m}B$, and $\mathfrak{Q}_n \subset \dots \subset \mathfrak{Q}_{n+r}$ is a chain of prime ideals of B . Let \mathfrak{Q} be a prime ideal lying over \mathfrak{m} , then by the going down property, there exists prime ideals $\mathfrak{Q}_0 \subset \dots \subset \mathfrak{Q}_{n-1} \subset \mathfrak{Q}$ such that \mathfrak{Q}_i is lying over \mathfrak{p}_i for $0 \leq i \leq n - 1$. Then we see $\mathfrak{Q}_0 \subset \dots \subset \mathfrak{Q}_{n-1} \subset \mathfrak{Q}_n \subset \dots \subset \mathfrak{Q}_{n+r}$ is a chain of prime ideals of B , whence $\dim(B) \geq n + r$. \square

Corollary 15.1.22. *Let $\rho : A \rightarrow B$ be a ring homomorphism with the going-down property. Then for any ideal \mathfrak{a} of A we have $\text{ht}(\mathfrak{a}) \leq \text{ht}(\mathfrak{a}^e)$.*

Proof. Let \mathfrak{P} be a prime ideal of B containing \mathfrak{a}^e , and $\mathfrak{p} = \mathfrak{P}^c$. By ??, the homomorphism $\rho_{\mathfrak{p}} : A_{\mathfrak{p}} \rightarrow B_{\mathfrak{P}}$ satisfies condition (PM), whence $\dim(A_{\mathfrak{p}}) \leq \dim(B_{\mathfrak{P}})$ by ???. But $\text{ht}(\mathfrak{a}) \leq \dim(A_{\mathfrak{p}})$ since \mathfrak{p} contains \mathfrak{a} , so $\text{ht}(\mathfrak{a}) \leq \dim(B_{\mathfrak{P}})$ for any prime ideal \mathfrak{P} containing \mathfrak{a}^e , whence $\text{ht}(\mathfrak{a}) \leq \text{ht}(\mathfrak{a}^e)$. \square

Remark 15.1.1. Let $\rho : A \rightarrow B$ be a local homomorphism of Noetherian local rings satisfying the going down property. Then (??) the equality holds in (??). On the other hand, the inequality in (??) can be strict (c.f. ??).

15.1.3 Dimension of finite type algebras

Proposition 15.1.23. *Let $\rho : A \rightarrow B$ be a ring homomorphism. If $n = \sup_{\mathfrak{p}} \dim(B \otimes_A \kappa(\mathfrak{p}))$ with \mathfrak{p} taking over prime ideals of A , then we have the inequality*

$$\dim(B) + 1 \leq (n + 1)(\dim(A) + 1).$$

In particular, if $\dim(A)$ is finite and $\dim(B \otimes_A \kappa(\mathfrak{p}))$ is bounded from above, then $\dim(B)$ is finite.

Proof. We may suppose that $\dim(A) < +\infty$ and $n < +\infty$. Let $\mathfrak{P}_0 \subset \dots \subset \mathfrak{P}_m$ be a chain of prime ideals of B and set $\mathfrak{p}_i = \mathfrak{P}_i^c$. Then the sequence \mathfrak{p}_i is increasing, so the set of its values is of cardinal $\leq \dim(A) + 1$. For each $\mathfrak{p} \in \text{Spec}(A)$, the set of \mathfrak{P}_i such that $\mathfrak{P}_i = \mathfrak{p}$ is a chain of the subset $\rho^{*-1}(\mathfrak{p})$ of $\text{Spec}(B)$, therefore has cardinality less than $\dim(B \otimes_A \kappa(\mathfrak{p})) + 1$, and consequent to $(n + 1)$. It follows that $m + 1 \leq (n + 1)(\dim(A) + 1)$, hence the proposition. \square

Corollary 15.1.24. *Let A be a ring, then*

$$\dim(A) + 1 \leq \dim(A[X]) \leq 2 \dim(A) + 1.$$

Proof. One direction is established. For the other, note that for any prime ideal $\mathfrak{p} \in \text{Spec}(A)$, $A[X] \otimes_A \kappa(\mathfrak{p})$ is isomorphic to $\kappa(\mathfrak{p})[X]$, which is a PID hence of dimension 1. Thus the other side comes from ??.

Corollary 15.1.25. *Let A be a ring and B a finitely generated A -algebra. If $\dim(A) < +\infty$, then we also have $\dim(B) < +\infty$.*

Proof. It follows from ?? that the polynomial ring $A[X_1, \dots, X_n]$ has finite dimension for all n , whence $\dim(B) < +\infty$. \square

Lemma 15.1.26. *Let $\rho : A \rightarrow B$ be a ring homomorphism such that, for each prime ideal \mathfrak{p} of A , the algebra $B \otimes_A \kappa(\mathfrak{p})$ is integral over $\kappa(\mathfrak{p})$. If \mathfrak{P} and \mathfrak{Q} are prime ideals of B such that $\mathfrak{P} \subset \mathfrak{Q}$, then $\mathfrak{P}^c \neq \mathfrak{Q}^c$.*

Proof. If $\mathfrak{P}^c = \mathfrak{Q}^c = \mathfrak{p}$, then $\mathfrak{P} \subset \mathfrak{Q}$ is a chain in $B \otimes_A \kappa(\mathfrak{p})$, whence $\dim(B \otimes_A \kappa(\mathfrak{p})) \geq 1$. But since $\kappa(\mathfrak{p})$ is a field and $B \otimes_A \kappa(\mathfrak{p})$ is integral over $\kappa(\mathfrak{p})$, we have $\dim(B \otimes_A \kappa(\mathfrak{p})) = 0$, a contradiction. \square

Theorem 15.1.27. *Let $\rho : A \rightarrow B$ be a ring homomorphism such that B is integral over A , and $\rho^* : \text{Spec}(B) \rightarrow \text{Spec}(A)$ the induced map.*

- (a) *Let M be a finitely generated A -module. Then $\dim_B(M \otimes_A B) \leq \dim_A(M)$. If ρ^* is surjective, then $\dim_B(M \otimes_A B) = \dim_A(M)$.*
- (b) *Let \mathfrak{b} be an ideal of B . Then $\text{ht}(\mathfrak{b}) \leq \text{ht}(\mathfrak{b}^c)$ and $\dim(B/\mathfrak{b}) = \dim(A/\mathfrak{b}^c)$. If ρ^* is surjective, then $\text{ht}(\mathfrak{a}^e) \leq \text{ht}(\mathfrak{a})$ for any ideal \mathfrak{a} of A .*
- (c) *Suppose that B is a finitely generated A -module and let N be a finitely generated B -module. Then we have $\dim_B(N) = \dim_A(N)$. In particular, we have $\dim(B) = \dim_A(B)$.*

Proof. By ??, the algebra $B \otimes_A \kappa(\mathfrak{p})$ is integral over $\kappa(\mathfrak{p})$. Let $\mathfrak{P}_0 \subset \dots \subset \mathfrak{P}_m$ be a chain of prime ideals of B . Then by ??, the ideals $\mathfrak{p}_i = \mathfrak{P}_i^c$ are pairwise distinct, whence $\mathfrak{p}_0 \subset \dots \subset \mathfrak{p}_m$ is a chain of prime ideals of A , and $m \leq \dim(A)$. This proves $\dim(B) \leq \dim(A)$.

Suppose further that ρ^* is surjective. Let $\mathfrak{p}_0 \subset \dots \subset \mathfrak{p}_n$ be a chain of prime ideals of A . Then there exists a prime ideal \mathfrak{P}_0 of B lying over \mathfrak{p}_0 . According to ??, we can construct, by induction on n , a chain $\mathfrak{P}_0 \subset \dots \subset \mathfrak{P}_n$ of prime ideals of B such that \mathfrak{P}_i is lying over \mathfrak{p}_i for each i . We therefore have $n \leq \dim(B)$ and consequently $\dim(A) = \dim(B)$.

This proves (a) in the case $M = A$. In the general case, let \mathfrak{a} be the annihilator of M , so that the support of M is identified with $\text{Spec}(A/\mathfrak{a})$, and we have $\dim_A(M) = \dim(A/\mathfrak{a})$. According to ??, the support of $M \otimes_A B$ is the inverse image by ρ^* of the support of M , therefore identifies with $\text{Spec}(B/\mathfrak{a}^e)$, and we have $\dim_B(M \otimes_A B) = \dim(B/\mathfrak{a}^e)$. It remains to notice that the homomorphism $\bar{\rho} : A/\mathfrak{a} \rightarrow B/\mathfrak{a}^e$ deduced from ρ makes B/\mathfrak{a}^e an integral (A/\mathfrak{a}) -algebra, and that $\bar{\rho}^*$ is surjective when ρ^* is.

Now we prove (b). By the definition of height, it suffices to prove that $\text{ht}(\mathfrak{b}) \leq \dim(A_{\mathfrak{p}})$ for any prime ideal \mathfrak{p} of A containing \mathfrak{b}^c . Let \mathfrak{p} be such an ideal, according to ??, there exists a prime ideal \mathfrak{P} of B above \mathfrak{p} and containing \mathfrak{b} , and we have $\text{ht}(\mathfrak{b}) \leq \dim(B_{\mathfrak{P}})$. Now $B_{\mathfrak{P}}$ is identified with a ring of fractions of the integral $A_{\mathfrak{p}}$ -algebra $B \otimes_A A_{\mathfrak{p}}$, from which we get

$$\dim(B_{\mathfrak{P}}) \leq \dim(B \otimes_A A_{\mathfrak{p}}) \leq \dim(A_{\mathfrak{p}})$$

by (a) and ?? . We have thus proved the inequality $\text{ht}(\mathfrak{b}) \leq \text{ht}(\mathfrak{b}^c)$. Moreover, the homomorphism of A/\mathfrak{a} in B/\mathfrak{b} deduced from ρ is injective and makes B/\mathfrak{b} an integral (A/\mathfrak{a}) -algebra. We therefore have $\dim(B/\mathfrak{b}) = \dim(A/\mathfrak{a})$ according to (a). Assume now ρ^* surjective and let \mathfrak{a} be an ideal of A and \mathfrak{p} a prime ideal of A containing \mathfrak{a} . There exists by hypothesis a prime ideal \mathfrak{P} of B above \mathfrak{p} . We have $\mathfrak{a}^e \subseteq \mathfrak{P}$, hence $\text{ht}(\mathfrak{a}^e) \leq \text{ht}(\mathfrak{P}) \leq \text{ht}(\mathfrak{p})$ according to the above argument. Passing to the infimum, we obtain $\text{ht}(\mathfrak{a}^e) \leq \text{ht}(\mathfrak{a})$. Now part (c) follows from (b) by letting \mathfrak{b} be the annihilator of N and noting that the annihilator of $\rho^*(N)$ is then \mathfrak{b}^c . \square

Theorem 15.1.28. Let A be an integrally closed ring, B a ring containing A and integral over A . Suppose that B is a torsion-free A -module, then

- (a) For any ideal \mathfrak{a} of A , we have $\text{ht}(\mathfrak{a}) = \text{ht}(\mathfrak{a}^e)$.
- (b) For any ideal \mathfrak{b} of B , we have $\text{ht}(\mathfrak{b}) = \text{ht}(\mathfrak{b}^c)$.

Proof. Let $\rho : A \rightarrow B$ be the canonical homomorphism. Let \mathfrak{a} be an ideal of A . If $\mathfrak{a} = A$, then (a) is clear by ???. Suppose $\mathfrak{a} \neq A$, so that $\mathfrak{a}^e \neq B$. Since ρ is injective, ρ^* is surjective by ???. Let \mathfrak{P} be a prime ideal of B containing \mathfrak{a}^e and $\mathfrak{p} = \mathfrak{P} \cap A$. We have $\mathfrak{a} \subseteq \mathfrak{p}$, whence $\text{ht}(\mathfrak{a}) \leq \text{ht}(\mathfrak{p})$. Let $\mathfrak{p}_0 \subset \dots \subset \mathfrak{p}_n$ be a chain of prime ideals of A with $\mathfrak{p}_n = \mathfrak{p}$. According to the going down theorem (??), we have a chain $\mathfrak{P}_0 \subset \dots \subset \mathfrak{P}_n$ of prime ideals of B such that $\mathfrak{P}_n = \mathfrak{P}$ and \mathfrak{P}_i is above \mathfrak{p}_i . We then have $n \leq \text{ht}(\mathfrak{P})$, whence $\text{ht}(\mathfrak{a}) \leq \text{ht}(\mathfrak{P})$. By taking infimum we get $\text{ht}(\mathfrak{a}) \leq \text{ht}(\mathfrak{a}^e)$. The inequality $\text{ht}(\mathfrak{a}^e) \leq \text{ht}(\mathfrak{a})$ follows from ??, whence the first equality.

Let \mathfrak{b} be an ideal of B and $\mathfrak{a} = \mathfrak{b}^c$. We have $\mathfrak{a}^e \subseteq \mathfrak{b}$, hence $\text{ht}(\mathfrak{a}) = \text{ht}(\mathfrak{a}^e) \leq \text{ht}(\mathfrak{b})$. The inequality $\text{ht}(\mathfrak{b}) \leq \text{ht}(\mathfrak{a})$ follows from ??, hence the theorem. \square

15.1.4 Dimension of finite type algebras over a field

As a special case of the materials discussed in the last paragraph, we now consider finite type algebras over a field k . As we shall see, the dimension theory for such algebras is extremely simple and beautiful as it connects with the transcendental degree of such algebras.

Lemma 15.1.29 (Noether's Normalization Theorem). Let A be a finitely generated k -algebra and $\mathfrak{p}_0 \subset \dots \subset \mathfrak{p}_m$ a maximal chain of prime ideals of A . Then there exists an integer $n \geq m$ and a sequence $(x_i)_{1 \leq i \leq n}$ of elements of A algebraically independent over k such that

- (a) A is integral over $B = k[x_1, \dots, x_n]$.
- (b) The ideal $\mathfrak{p}_j \cap B$ is generated by x_1, \dots, x_{n-m+j} .

Proof. By ??, there exists an integer $n \geq 0$, a sequence $(x_i)_{1 \leq i \leq n}$ of elements of A algebraically independent over k and an increasing sequence $(h_j)_{1 \leq j \leq m}$ such that A is integral over $B = k[x_1, \dots, x_n]$ and $\mathfrak{q}_j = \mathfrak{p}_j \cap B$ is generated by x_1, \dots, x_{h_j} . By passing to the quotients, we deduce from the injective canonical homomorphism of B into A an injective homomorphism from B/\mathfrak{q}_j into A/\mathfrak{p}_j which makes A/\mathfrak{p}_j an integral (B/\mathfrak{q}_j) -algebra. As the ring B/\mathfrak{q}_j is isomorphic to a polynomial algebra in $n - h_j$ indeterminate over k , it is integrally closed. According to ??, we therefore have

$$1 = \text{ht}(\mathfrak{p}_{j+1}/\mathfrak{p}_j) = \text{ht}(\mathfrak{q}_{j+1}/\mathfrak{q}_j) \geq h_{j+1} - h_j.$$

whence $h_{j+1} \leq h_j + 1$. Since h_j is increasing and the ideals \mathfrak{q}_j are pairwise distinct, we get $h_{j+1} = h_j + 1$. Also note that $h_m = n$ since \mathfrak{q}_m is maximal (\mathfrak{p}_m is maximal), so $h_j = n - m + j$ and the claim follows. \square

Theorem 15.1.30. Let A be a finitely generated k -algebra.

- (a) For every minimal prime ideal \mathfrak{p} of A , the maximal chains of prime ideals of A starting from \mathfrak{p} have length $\text{tr.deg}_k(\kappa(\mathfrak{p}))$.
- (b) The ring A is catenary and its dimension is the supremum of the integers $\text{tr.deg}_k(\kappa(\mathfrak{p}))$, where \mathfrak{p} runs through minimal prime ideals of A .

(c) Suppose that A is an integral domain. Then the maximal chains of prime ideals of A have the same length, and the dimension of A equals to $\text{tr. deg}_k(K)$, where K is the field of fraction of A .

Proof. Suppose that A is an integral domain and consider a maximal chain $\mathfrak{p}_0 \subset \dots \subset \mathfrak{p}_m$ of prime ideals of A . We have $\mathfrak{p}_0 = 0$. We then deduce from ?? the existence of a injective homomorphism $\varphi : k[X_1, \dots, X_m] \rightarrow A$ of k -algebras which makes A an integral $k[X_1, \dots, X_m]$ -algebra. Consequently, the transcendence degree over k of the field of fractions of A is equal to m , hence (c). Assertion (a) follows from (c) applied to the ring A/\mathfrak{p} and assertion (b) is an immediate consequence of (a). \square

Corollary 15.1.31. Let n be a positive integer, then $\dim(k[X_1, \dots, X_n]) = n$. Moreover, for a finitely generated k -algebra A to be of dimension n , it is necessary and sufficient that there exists a injective k -homomorphism $\varphi : k[X_1, \dots, X_n] \rightarrow A$ such that A is finite over $k[X_1, \dots, X_n]$.

Proof. This follows from ??, ??, and ??(a). \square

Corollary 15.1.32. Let A be a finitely generated k -algebra that is an integral domain. Then for any prime ideal \mathfrak{p} of A , we have

$$\text{ht}(\mathfrak{p}) = \dim(A_{\mathfrak{p}}) = \dim(A) - \dim(A/\mathfrak{p}) = \dim(A) - \text{tr. deg}_k(\kappa(\mathfrak{p})).$$

Proof. This is a consequence of ?? and ??.

Corollary 15.1.33. Let A be a finitely generated k -algebra and let f be an element of A which does not belong to any minimal prime ideal of A . Then $\dim(A) = \dim(A_f)$.

Proof. The map $\mathfrak{p} \mapsto \mathfrak{p}A_f$ is a bijection between the set of minimal prime ideals of A and that of A_f . Since A/\mathfrak{p} and $A_f/\mathfrak{p}A_f$ have the same field of fraction, it suffices to apply ??.

Corollary 15.1.34. Let A be a finitely generated k -algebra and \mathfrak{p} a prime ideal of A .

- (a) For \mathfrak{p} to be maximal, it is necessary and sufficient that $\kappa(\mathfrak{p})$ is a finite extension of k .
- (b) Let $f \in A - \mathfrak{p}$. Then the ideal \mathfrak{p} is a maximal ideal of A if and only if $\mathfrak{p}A_f$ is a maximal ideal of A_f .

Proof. If \mathfrak{p} is a maximal ideal of A , then A/\mathfrak{p} is a field, whence a finitely generated extension of k of transcendence degree 0. It is therefore a finite extension of k (??). Conversely, if the field $\kappa(\mathfrak{p})$ is a finite extension of k , we have $\dim(A/\mathfrak{p}) = 0$, so \mathfrak{p} is maximal. The assertion (b) now follows from (a), taking into account that A/\mathfrak{p} and $A_f/\mathfrak{p}A_f$ have the same field of fractions. \square

Corollary 15.1.35. Let A be a finitely generated k -algebra, \mathfrak{p} a prime ideal of A and $(\mathfrak{p}_i)_{i \in I}$ the family of minimal prime ideals of A . Then we have

$$\dim_{\mathfrak{p}}(A) = \sup_{i \in I_{\mathfrak{p}}} \dim(A/\mathfrak{p}_i) = \dim(A_{\mathfrak{p}}) + \dim(A/\mathfrak{p}) = \dim(A_{\mathfrak{p}}) + \text{tr. deg}_k(\kappa(\mathfrak{p}))$$

where $I_{\mathfrak{p}}$ is the subset consists of $i \in I$ such that $\mathfrak{p}_i \subseteq \mathfrak{p}$.

Proof. We have $\dim_{\mathfrak{p}}(A) = \sup_{i \in I_{\mathfrak{p}}} \dim_{\mathfrak{p}/\mathfrak{p}_i}(A/\mathfrak{p}_i)$ by ??, since A is Noetherian. Also, from ?? we conclude that

$$\dim_{\mathfrak{p}/\mathfrak{p}_i}(A/\mathfrak{p}_i) = \inf_{f \notin \mathfrak{p}/\mathfrak{p}_i} \dim((A/\mathfrak{p}_i)_f) = \dim(A/\mathfrak{p}_i)$$

so the first equality follows. Now since A/\mathfrak{p}_i is an integral domain, ?? shows $\dim(A/\mathfrak{p}_i) = \dim((A/\mathfrak{p}_i)_{\mathfrak{p}}) + \dim(A/\mathfrak{p})$. The other equalities then follows from the observation that $\dim(A_{\mathfrak{p}}) = \sup_{i \in I_{\mathfrak{p}}} \dim((A/\mathfrak{p}_i)_{\mathfrak{p}})$ and ??.

Corollary 15.1.36. *Let A be a finitely generated k -algebra, n a positive integer. Then the following conditions are equivalent:*

- (i) *For every $\mathfrak{p} \in \text{Ass}(A)$, we have $\dim(A/\mathfrak{p}) = n$.*
- (ii) *Every prime ideal associated with A is minimal and all irreducible components of $\text{Spec}(A)$ are of dimension n .*
- (iii) *There exists an injective homomorphism $\varphi : k[X_1, \dots, X_n] \rightarrow A$ such that A is a finitely generated torsion-free $k[X_1, \dots, X_n]$ -module.*

Proof. The equivalence of (i) and (ii) is immediate. Now suppose that (ii) holds. Then $\dim(A) = n$ and by ?? there is an injective k -homomorphism

$$\varphi : k[X_1, \dots, X_n] \rightarrow A$$

such that A is finite over $k[X_1, \dots, X_n]$. For any prime ideal $\mathfrak{p} \in \text{Ass}(A)$, the ring A/\mathfrak{p} is then integral over $k[X_1, \dots, X_n]$, and we have so $n = \dim(A/\mathfrak{p}) = \dim(k[X_1, \dots, X_n]/\varphi^{-1}(\mathfrak{p}))$ according to ??(a), which proves $\varphi^{-1}(\mathfrak{p}) = 0$. By ??, this shows the image by the injective homomorphism φ of a nonzero element of $k[X_1, \dots, X_n]$ is not a divisor of 0 in A , whence (c).

Conversely, assume that (iii) holds. For each prime ideal $\mathfrak{p} \in \text{Ass}(A)$, the homomorphism $\varphi : k[X_1, \dots, X_n] \rightarrow A/\mathfrak{p}$ induced by φ is injective by ??, whence $\dim(A/\mathfrak{p}) = n$ by ??.

Corollary 15.1.37. *Under the condition of ??, $\dim_{\mathfrak{p}}(A) = \dim(A)$ for every prime ideal \mathfrak{p} of A .*

Proof. This follows from ?? and ??.

Proposition 15.1.38. *Let A and B be finitely generated k -algebras and $\rho : A \rightarrow B$ a k -algebra homomorphism. Suppose that A is an integral domain and B is a torsion-free A -module, and let K be the field of fraction of A . Then we have*

$$\dim(B) = \dim(A) + \dim(B \otimes_A K).$$

Proof. Suppose that B is an integral domain. The algebra $B \otimes_A K$ is then a ring of fractions of B defined by a multiplicative part not containing 0. It therefore has field of fractions the field of fractions L of B . According to ??, we have

$$\dim(B) = \text{tr.deg}_k(L), \quad \dim(A) = \text{tr.deg}_k(K), \quad \dim(B \otimes_A K) = \text{tr.deg}_K(L).$$

By the transitivity of transcendental degree, we see the claim is true in this case.

Let's consider to the general case. Every minimal prime ideal \mathfrak{P} of B is formed by divisors of zero in B , so is lying over the ideal 0 of A . By ?? it follows that the map $\mathfrak{P} \mapsto \mathfrak{P}(B \otimes_A K)$ is a bijection of the set of minimal prime ideals of B on the set of minimal prime ideals of $B \otimes_A K$. The proposition follows therefore from the first part of the proof and from ??.

Corollary 15.1.39. *Let $\rho : A \rightarrow B$ be an injective homomorphism of finitely generated k -algebras. Then $\dim(A) \leq \dim(B)$.*

Proof. In fact, let \mathfrak{p} be a minimal prime of A such that $\dim(A) = \dim(A/\mathfrak{p})$. Then there exists a prime ideal \mathfrak{P} of A lying over \mathfrak{p} by ?? . According to ?? applied to A/\mathfrak{p} and B/\mathfrak{P} , we have $\dim(A) = \dim(A/\mathfrak{p}) \leq \dim(B/\mathfrak{P}) \leq \dim(B)$, whence the corollary. \square

Lemma 15.1.40. *Let A and B be k -algebras that are integral domains, M a torsion-free A -module, N a torsion-free B -module. Then the ring $A \otimes_k B$ is an integral domain, and $M \otimes_k N$ is a torsion-free $A \otimes_k B$ module.*

Proof. Let K (resp. L) be the field of fraction of A (resp. B). Then there exists a set I (resp. J) such that M (resp. N) is isomorphic to a submodule of $K^{\oplus I}$ (resp. $L^{\oplus J}$). The $(A \otimes_k B)$ -module $M \otimes_k N$ is then isomorphic to a submodule of $K^{\oplus I} \otimes_k L^{\oplus J}$, which is isomorphic to $(K \otimes_k L)^{\oplus(I \times J)}$. As $K \otimes_k L$ is a ring of fractions of the ring $A \otimes_k B$, it is a torsion-free $A \otimes_k B$ -module, hence the lemma. \square

Proposition 15.1.41. *Let k' be an extension field of k , A a finitely generated k -algebra and B a finitely generated k' -algebra.*

(a) *The k' -algebra $A \otimes_k B$ is finitely generated and we have*

$$\dim(A \otimes_k B) = \dim(A) + \dim(B).$$

(b) *Let \mathfrak{r} be a prime ideal of $A \otimes_k B$. If \mathfrak{p} (resp. \mathfrak{q}) is the contraction of \mathfrak{r} to A (resp. B), then*

$$\dim_{\mathfrak{r}}(A \otimes_k B) = \dim_{\mathfrak{p}}(A) + \dim_{\mathfrak{q}}(B).$$

Proof. Put $n = \dim(A)$ and $m = \dim(B)$. There exists according to ?? injective homomorphisms of algebras $\varphi : k[X_1, \dots, X_n] \rightarrow A$ and $\psi : k'[Y_1, \dots, Y_m] \rightarrow B$ making respectively of A and B finite algebras over $k[X_1, \dots, X_n]$ and $k'[Y_1, \dots, Y_m]$. The homomorphism $\varphi \otimes \psi$ is then injective and makes $A \otimes_k B$ a finite algebra over the k' -algebra $k[X_1, \dots, X_n] \otimes_k k'[Y_1, \dots, Y_m]$, which identifies with $k'[X_1, \dots, X_n, Y_1, \dots, Y_m]$ (to see that $\varphi \otimes \psi$ is injective, we may consider the following commutative diagram

$$\begin{array}{ccc} k[X] \otimes k'[Y] & \xrightarrow{1 \otimes \psi} & k[X] \otimes B \\ \varphi \otimes 1 \downarrow & & \downarrow \varphi \otimes 1 \\ A \otimes k'[Y] & \xrightarrow{1 \otimes \psi} & A \otimes B \end{array}$$

where $\varphi \otimes \psi$ is the composition homomorphism, and note that all modules are flat since we are tensoring over k). We then have $\dim(A \otimes_k B) = n + m$ by ??, which proves (a).

Note that when A and B are integral domains, the $k'[X_1, \dots, X_n, Y_1, \dots, Y_m]$ -module $A \otimes_k B$ is torsion free by ?? and therefore we have

$$\dim_{\mathfrak{r}}(A \otimes_k B) = \dim(A \otimes_k B) = \dim(A) + \dim(B)$$

for any prime ideal \mathfrak{r} of $A \otimes_k B$ by ??.

Now let us show (b). Let \mathfrak{r}_0 be a minimal prime ideal of $A \otimes_k B$ contained in \mathfrak{r} , and let \mathfrak{p}_0 (resp. \mathfrak{q}_0) be the contraction of \mathfrak{r}_0 to A (resp. B). The ring $(A \otimes_k B)/\mathfrak{r}_0$ is isomorphic to a quotient of the ring $(A/\mathfrak{p}_0) \otimes_k (B/\mathfrak{q}_0)$. Thus we have, by (a),

$$\dim((A \otimes_k B)/\mathfrak{r}_0) \leq \dim((A/\mathfrak{p}_0) \otimes_k (B/\mathfrak{q}_0)) = \dim(A/\mathfrak{p}_0) + \dim(B/\mathfrak{q}_0).$$

According to ??, this shows

$$\dim_{\mathfrak{r}}(A \otimes_k B) \leq \dim_{\mathfrak{p}}(A) + \dim_{\mathfrak{q}}(B).$$

Conversely, let \mathfrak{p}_0 (resp. \mathfrak{q}_0) be a minimal prime ideal of A (resp. B) contained in \mathfrak{p} (resp. \mathfrak{q}). By the observation above, we have

$$\dim(A/\mathfrak{p}_0) + \dim(B/\mathfrak{q}_0) = \dim_{\bar{\mathfrak{r}}}((A/\mathfrak{p}_0) \otimes_k (B/\mathfrak{q}_0)) \leq \dim_{\mathfrak{r}}(A \otimes_k B)$$

where $\bar{\mathfrak{r}}$ is the image of \mathfrak{r} by the canonical surjection $A \otimes_k B \rightarrow (A/\mathfrak{p}_0) \otimes_k (B/\mathfrak{q}_0)$. Applying ??, we deduce the inequality

$$\dim_{\mathfrak{p}}(A) + \dim_{\mathfrak{q}}(B) \leq \dim_{\mathfrak{r}}(A \otimes_k B)$$

whence (b) follows. □

Corollary 15.1.42. *Let A be a finitely generated k -algebra, k' an extension field of k , and A' the k' -algebra $A \otimes_k k'$.*

- (a) *We have $\dim(A') = \dim(A)$.*
- (b) *Let \mathfrak{p}' be a prime ideal of A and \mathfrak{p} its contraction to A . Then $\dim_{\mathfrak{p}'}(A') = \dim_{\mathfrak{p}}(A)$.*
- (c) *Let \mathfrak{p}' be a minimal prime ideal of A and \mathfrak{p} be the contraction of \mathfrak{p}' . Then \mathfrak{p} is minimal and $\dim(A'/\mathfrak{p}') = \dim(A/\mathfrak{p})$.*
- (d) *If k' is a purely inseparable extension of k , then the canonical map ${}^a\rho : \text{Spec}(A') \rightarrow \text{Spec}(A)$ is a homeomorphism.*

Proof. Part (a) and (b) follows from ?? by setting $B = k'$. For (c), note that the homomorphism $\nu : k \rightarrow k'$ satisfies condition (PM), so the ideal \mathfrak{p}' is minimal by ??(a) and (c). By corollary ??, we then have

$$\dim(A'/\mathfrak{p}') = \dim_{\mathfrak{p}'}(A') = \dim_{\mathfrak{p}}(A) = \dim(A/\mathfrak{p})$$

which completes the proof of (c).

Finally, assume that k'/k is purely inseparable, and let $\rho : A \rightarrow A'$ be the canonical homomorphism. Let $\mathfrak{p} \in \text{Spec}(A)$, recall that the fiber of \mathfrak{p} under ${}^a\rho$ is homeomorphic to $\text{Spec}(\kappa(\mathfrak{p}) \otimes_k k')$. Now the set \mathfrak{n} of nilpotent elements of $\kappa(\mathfrak{p}) \otimes_k k'$ is a prime ideal by ?? and

the quotient ring $(\kappa(\mathfrak{p}) \otimes_k k')/\mathfrak{n}$ is a field by (??). Therefore $(^a\rho)^{-1}(\mathfrak{p})$ is reduced to a singleton. It follows that the map ${}^a\rho$ is a bijection of $\text{Spec}(A')$ on $\text{Spec}(A)$. Since we have observed that ρ has the going up property (??), it follows from ?? that ${}^a\rho$ is a closed map, whence a homeomorphism. \square

Theorem 15.1.43 (Grothendieck's Generic Freeness Lemma). *Let A be a Noetherian domain, B be an A -algebra of finite type, and M be a finitely generated B -module. Then there exists a nonzero element $f \in A$ such that M_f is a free A_f -module.*

Proof. Let K be the fraction field of A ; then $B \otimes_A K$ is a finite type algebra over K and $M \otimes_A K$ is a finitely generated $(B \otimes_A K)$ -module. We proceed by induction on the dimension $d = \dim(M \otimes_A K)$. If $d = -\infty$, we have $M \otimes_A K = 0$, which means M is annihilated by a nonzero element of $f \in A$, so $M_f = 0$ and the theorem is trivially satisfied.

Now assume that $d \geq 0$. By ??, there exists a filtration $(M_i)_{0 \leq i \leq n}$ of the B -module M such that $N_i = M_i/M_{i+1} \cong B/\mathfrak{P}_i$, where \mathfrak{P}_i is a prime ideal of B . If the theorem is proved for each N_i , then there exists for each i an element $f_i \in A$ such that $(N_i)_{f_i}$ is free over A_{f_i} . If we set $f = f_1 \cdots f_{n-1}$, then $(N_i)_f$ is a free A_f -module for $0 \leq i \leq n-1$, and since $(N_i)_f = (M_i)_f/(M_{i+1})_f$, the A_f -module M_f then possesses a filtration by free modules, hence is free. By replacing B with B/\mathfrak{P} (where \mathfrak{P} is a prime ideal of B), which is of finite type over A , we see that we are reduced to the case where $M = B$ and B is an integral domain. By ??, there then exists a nonzero element $g \in A$ and algebraically independent elements x_1, \dots, x_m of B such that B_g is integral over $A_g[x_1, \dots, x_m]$. By replacing A by A_g and B by B_g , we may assume that B is integral over $C = A[x_1, \dots, x_m]$, hence a finitely generated torsion-free C -module. By ??, we also note that the dimension of $B \otimes_A K$ is equal to m , so we have $m = d$.

Now if r is the rank of the torsion-free C -module B , there exists an exact sequence of C -modules

$$0 \longrightarrow C^r \longrightarrow B \longrightarrow M' \longrightarrow 0$$

where M' is a finitely generated torsion C -module. The support of M' then does not contain the generic point of $\text{Spec}(C)$, and therefore the support of $M' \otimes_A K$ does not contain the generic point of $\text{Spec}(C \otimes_A K)$ (??). We then conclude that $\dim_C(M') < d$, so by the induction hypotheses, there exists a nonzero element $f \in A$ such that M'_f is free over A_f . Since C_f is also free over A_f (a basis of C_f is the image of the monomials in the x_i), we then conclude that B_f is a free A_f -module. \square

15.1.5 Exercise

Exercise 15.1.1. In this exercise, we develop a way to measure the complexity of a partially ordered set.

- (a) Prove that, for any partially ordered set E we can associate an element of $\mathbb{N} \cup \{\pm\infty\}$, denoted by $\text{dev}(E)$, and called the **deviation** of E , which satisfies the following conditions:
 - (α) A trivial poset (one in which no two elements are comparable) has deviation $-\infty$.
 - (β) A poset E is said to have deviation at most n (for a positive integer n) if for every descending chain of elements $(a_k)_{k \in \mathbb{N}}$, all but a finite number of the posets (a_k, a_{k+1}) have deviation less than n . The deviation of E (if it exists) is the minimum value of n for which this is true.

- (b) Show that for $\text{dev}(E) \leq 0$, it is necessary and sufficient that any decreasing sequence of E is stationary. We have $\text{dev}(\mathbb{N}) = 0$, $\text{dev}(\mathbb{Z}) = 1$, $\text{dev}(\mathbb{Q}) = +\infty$.
- (c) Let E and F be partially ordered sets. Show that if there exists an increasing map from E to F , then $\text{dev}(E) \leq \text{dev}(F)$. Show that $\text{dev}(E \times F) = \sup\{\text{dev}(E), \text{dev}(F)\}$.

Exercise 15.1.2. Let A be a ring (not necessarily commutative), M a left A -module, we denote by $\text{dev}(M)$ the deviation of the set of submodules of M , ordered by inclusion. We put $\text{dev}(A)$ to be the deviation of the left A -module A .

- (a) If N is a sub- A -module of M , then

$$\text{dev}(M) = \sup\{\text{dev}(N), \text{dev}(M/N)\}.$$

- (b) Suppose that A is commutative. If \mathfrak{p} and \mathfrak{q} are distinct prime ideals of A and $\mathfrak{p} \subseteq \mathfrak{q}$, then $\text{dev}(A/\mathfrak{p}) > \text{dev}(A/\mathfrak{q})$. In particular, $\dim(A) \leq \text{dev}(A)$.
- (c) Suppose that A is commutative and Noetherian. Let \mathcal{P}_A be the set of prime ideals of A such that $\dim(A/\mathfrak{p}) = \dim(A)$ and S be complement of the union of prime ideals in \mathcal{P}_A . Let M be a finitely generated A -module. If $S^{-1}M = 0$ then

$$\text{dev}(M) \leq \sup_{\mathfrak{p} \notin \mathcal{P}_A} \text{dev}(A/\mathfrak{p}).$$

- (d) Suppose that A is commutative and Noetherian. Prove that in this case we have $\dim(A) = \text{dev}(A)$ and $\dim(M) = \text{dev}(M)$ for any finitely generated A -module M .

Proof. For a module E , we let $\mathcal{L}(E)$ denote the lattice of submodules of E . Then if N is a submodule of M , there are ordered homomorphisms $\mathcal{L}(N) \rightarrow \mathcal{L}(M)$ and $\mathcal{L}(M/N) \rightarrow \mathcal{L}(M)$, whence

$$\text{dev}(M) \geq \sup\{\text{dev}(N), \text{dev}(M/N)\}.$$

On the other hand, we have an ordered homomorphism

$$\mathcal{L}(M) \rightarrow \mathcal{L}(N) \times \mathcal{L}(M/N), \quad M' \mapsto (M' \cap N, M' + N/N).$$

This proves the equality in (a).

Suppose that A is commutative and let $\mathfrak{p} \subseteq \mathfrak{q}$ be distinct prime ideals. By replacing A with A/\mathfrak{p} , we may assume that $\mathfrak{p} = 0$. For any nonzero element $x \in \mathfrak{q}$, consider the sequence $(x^n A)$ in A . We have

$$\text{dev}(A) > \text{dev}(x^n A / x^{n+1} A) = \text{dev}(A/xA) \geq \text{dev}(A/\mathfrak{q})$$

so the claim in (b) follows. Now let $\mathfrak{p}_0 \subset \cdots \subset \mathfrak{p}_n$ be a saturated chain in A . Then $\text{dev}(A/\mathfrak{p}_0) > \cdots > \text{dev}(A/\mathfrak{p}_n)$, which shows $\text{dev}(A) \geq n = \dim(A)$.

Suppose that A is Noetherian and let M be a finitely generated A -module. Then there is a composition series $(M_i)_{0 \leq i \leq n}$ of M such that M/M_{i+1} is isomorphic to A/\mathfrak{p}_i . If $S^{-1}M = 0$, we have $\mathfrak{p}_i \notin \mathcal{P}_A$ for all i . Since $\text{dev}(M) = \sup_i \text{dev}(M_i/M_{i+1})$, we conclude that

$$\text{dev}(M) \leq \sup_{\mathfrak{p} \notin \mathcal{P}_A} \text{dev}(A/\mathfrak{p}).$$

Finally, we show that $\text{dev}(A) = \dim(A)$ by induction on $\dim(A)$. So assume that $\text{dev}(B) = \dim(B)$ for every Noetherian commutative B such that $\dim(B) < \dim(A)$. Then in the notation of (c), since $\dim(A/\mathfrak{p}) < \dim(A)$, we have $\text{dev}(A/\mathfrak{p}) = \dim(A/\mathfrak{p})$ for $\mathfrak{p} \notin \mathcal{P}_A$, whence

$$\text{dev}(M) \leq \sup_{\mathfrak{p} \in \mathcal{P}_A} \dim(A/\mathfrak{p}) < \dim(A) \quad (15.1.3)$$

if M is finitely generated and $S^{-1}M = 0$. Now let (\mathfrak{a}_k) be a sequence of decreasing ideals in A . Since $S^{-1}A$ is Artinian (it has dimension zero), there exists an integer N such that

$$S^{-1}\mathfrak{a}_n = S^{-1}\mathfrak{a}_{n+1} = \dots$$

for $n \geq N$. In other words, $S^{-1}(\mathfrak{a}_n/\mathfrak{a}_{n+1}) = 0$ for $n \geq N$, so by (??) we have $\text{dev}(\mathfrak{a}_n/\mathfrak{a}_{n+1}) < \dim(A)$ for $n \geq N$. From the definition of $\text{dev}(A)$, this shows $\text{dev}(A) \leq \dim(A)$, which gives $\text{dev}(A) = \dim(A)$.

Now let M be a finitely generated A -module and let $(M_i)_{0 \leq i \leq n}$ be the composition series of M such that $M_i/M_{i+1} \cong A/\mathfrak{p}_i$, where \mathfrak{p}_i is a prime ideal of M . Then

$$\text{dev}(M) = \sup_i \{\text{dev}(M_i/M_{i+1})\} = \sup_i \{\text{dev}(A/\mathfrak{p}_i)\} = \sup_i \{\dim(A/\mathfrak{p}_i)\}.$$

We also note that, since the minimal elements in $\{\mathfrak{p}_0, \dots, \mathfrak{p}_{n-1}\}$ are the embedded primes for M , which correspond to irreducible components of $\mathfrak{a} = \text{Ann}(M)$. Since $\dim(M) = \dim(V(\mathfrak{a}))$ is the supremum of the dimension of these components, we conclude that $\text{dev}(M) = \dim(M)$. \square

Exercise 15.1.3. Let A be a Noetherian commutative ring, \mathfrak{a} an ideal of A contained in the Jacobson radical of A , and $\text{gr}(A)$ the graded ring associated with the \mathfrak{a} -adic filtration. Associate to each ideal of A an ideal of $\text{gr}(A)$, and deduce that $\text{dev}(A) \leq \text{dev}(\text{gr}(A))$. Consequently, deduce that $\dim(A) \leq \dim(\text{gr}(A))$.

Exercise 15.1.4. Let A be a ring such that

- (a) For every maximal ideal \mathfrak{m} of A , $A_{\mathfrak{m}}$ is Noetherian.
- (b) Every nonzero element $x \in A$ is contained in a finite number of maximal ideals of A .

Show that A is Noetherian.

Proof. Let (\mathfrak{a}_n) be an increasing sequence of ideals in A , and assume that $\mathfrak{a}_1 \neq 0$. Then by condition (b), the ideal \mathfrak{a}_1 is contained in finitely many maximal ideals, say $\mathfrak{m}_1, \dots, \mathfrak{m}_r$. Since each ring $A_{\mathfrak{m}_i}$ is Noetherian, there exists an integer N such that

$$\mathfrak{a}_n A_{\mathfrak{m}_i} = \mathfrak{a}_{n+1} A_{\mathfrak{m}_i} = \dots$$

for $n \geq N$ and $i = 1, \dots, r$. But if \mathfrak{m} is a maximal ideal other than \mathfrak{m}_i , then $\mathfrak{a}_n \not\subset \mathfrak{m}$ for all n , so

$$\mathfrak{a}_n A_{\mathfrak{m}} = \mathfrak{a}_{n+1} A_{\mathfrak{m}} = \dots = A_{\mathfrak{m}}$$

trivially holds. By ??, we conclude that $\mathfrak{a}_n = \mathfrak{a}_{n+1} = \dots$ for $n \geq N$, so A is Noetherian. \square

Exercise 15.1.5. Let K be a field, $(n_i)_{i \geq 1}$ a strict increasing sequence of positive integers; for each $i \geq 0$, put \mathfrak{p}_i to be the ideal of the ring $R = K[(X_j)_{j \in \mathbb{N}}]$ generated by the elements X_j where $n_1 + \dots + n_i \leq j < n_1 + \dots + n_{i+1}$. Let S be the complement of the union of the \mathfrak{p}_i 's, and consider the ring $A = S^{-1}R$. Prove that $S^{-1}R$ is Noetherian but has infinite dimension.

Proof. We first note that a nonzero element in $S^{-1}R$ can only be contained in finitely many $S^{-1}\mathfrak{p}_i$ (since this is true in R for \mathfrak{p}_i). Since the ideals of $S^{-1}A$ correspond to ideals of R contained in $\bigcup_i \mathfrak{p}_i$, and any such ideal can only be contained in finitely many \mathfrak{p}_i , by prime avoidance we conclude that the $S^{-1}\mathfrak{p}_i$ are the only maximal ideals in $S^{-1}R$. Moreover, for each $S^{-1}\mathfrak{p}_i$, $(S^{-1}R)_{S^{-1}\mathfrak{p}_i}$ is isomorphic to $R_{\mathfrak{p}_i}$, which is Noetherian since it is just $K(x_I)[x_I]_{(x_I)}$ where I is indices appears in the generators of \mathfrak{p}_i and J its complement. By ??, we conclude that $S^{-1}R$ is Noetherian. Now it is easy to see each maximal ideal $S^{-1}\mathfrak{p}_i$ has height $\geq n_i$ in $S^{-1}R$, so $S^{-1}R$ has infinite dimension. \square

Exercise 15.1.6. Let A be the ring of germs at 0 of smooth functions on \mathbb{R} . We propose to show that the ring A is of infinite dimension. We denote by C the algebra of smooth functions on \mathbb{R} , F the ideal of functions zero on a neighborhood of 0, so that $A = C/F$.

- (a) Let $(x_n)_{n \in \mathbb{N}}$ be a strict decreasing sequence of positive numbers tending to 0, and let \mathcal{U} be an ultrafilter on \mathbb{N} finer than the filter of complements of finite subsets. Show that the set J_1 of $f \in C$ such that the set of $n \in \mathbb{N}$ such that $f(x_n) = 0$ belongs to \mathcal{U} , is a prime ideal of C .
- (b) We denote by $\tau_n(f)$ the function $x \mapsto f(x + x_n)$. Show that if J is a prime ideal of C , the set $\varphi(J)$ of $f \in C$ such that the set of $n \in \mathbb{N}$ where $\tau_n(f) \in J$ belongs to \mathcal{U} is a prime ideal of C .
- (c) We define a sequence I_n of ideals of C by $I_0 = J_1$ and $I_{n+1} = \varphi(I_n)$. Show that $(I_n)_{n \in \mathbb{N}}$ is a strict decreasing sequence of prime ideals of C containing F .

Proof. Let (x_n) and \mathcal{U} be as in (a), and define J_1 by

$$J_1 = \{f \in C : \{n \in \mathbb{N} : f(x_n) = 0\} \in \mathcal{U}\}.$$

Now let $f, g \in C$ and assume that $fg \in J_1$; put

$$E_f = \{n \in \mathbb{N} : f(x_n) = 0\}, \quad E_g = \{n \in \mathbb{N} : g(x_n) = 0\}.$$

Then by definition, the set $E_{fg} = \{n \in \mathbb{N} : f(x_n)g(x_n) = 0\}$ belongs to \mathcal{U} . Assume that $E_f, E_g \notin \mathcal{U}$, then since \mathcal{U} is a ultrafilter, we have $E_f^c, E_g^c \in \mathcal{U}$. But then

$$E_f^c \cap E_g^c = \{n \in \mathbb{N} : f(x_n) \neq 0, g(x_n) \neq 0\} = \{n \in \mathbb{N} : f(x_n)g(x_n) \neq 0\} = E_{fg}^c \in \mathcal{U}$$

which is a contradiction since $E_{fg} \in \mathcal{U}$. This proves that either of E_f and E_g is in \mathcal{U} , so J_1 is a prime ideal.

Now for a prime ideal J of C , define $\varphi(J)$ by

$$\varphi(J) = \{f \in C : \{n \in \mathbb{N} : \tau_n(f) \in J\} \in \mathcal{U}\}.$$

Again, for an element $f \in C$ we put $E_f(J) = \{n \in \mathbb{N} : \tau_n(f) \in J\}$. By noting that, since J is a prime ideal,

$$E_f(J)^c \cap E_g(J)^c = \{n \in \mathbb{N} : \tau_n(f) \notin J, \tau_n(g) \notin J\} = \{n \in \mathbb{N} : \tau_n(f)\tau_n(g) \notin J\} = E_{fg}(J)^c,$$

it is easy to show that $\varphi(J)$ is a prime ideal, just as in (a).

Define the sequence (I_n) by $I_0 = J_1$ and $I_{n+1} = \varphi(I_n)$. To prove the first assertion in (c), we first note that, since \mathcal{U} is finer than the finite complement filter on \mathbb{N} , every element in \mathcal{U} is infinite (otherwise its complement belongs to \mathcal{U}). Let $f \in F$, then there exists a neighborhood U of 0 such that $f|_U = 0$. Then since $x_n \rightarrow 0$, there exists an integer N such that $x_n \in U$ when $n \geq N$, whence $f(x_n) = 0$ for $n \geq N$. Since \mathcal{U} is finer than the finite complement filter on \mathbb{N} , this means $E_f = \{n \in \mathbb{N} : f(x_n) = 0\} \in \mathcal{U}$, so $f \in J_1$. This shows $F \subseteq J_1 = I_0$. Now let J be a prime ideal of C containing F , and consider $\varphi(J)$. Let $f \in F$, and let U be a neighborhood of 0 such that $f|_U \equiv 0$. Then since x_n is strict decreasing and tend to 0, there exists $N > 0$ such that $(x_n + U) \cap U$ contains a neighborhood of 0 for $n \geq N$, whence $\tau_n(f) \in F \subseteq J$ for $n \geq N$. This shows $f \in \varphi(J)$, so $F \subseteq \varphi(J)$. Thus we have shown that I_n contains F for each n .

It remains to prove that (I_n) is strict decreasing. We prove a more general fact: let J be a prime ideal of C such that, if f_n, f are elements in C and $f_n \rightarrow f$ pointwisely and $f_n \in J$ for all n , then $f \in J$ (we say J is *pointwise closed* if this is true). Then $\varphi(J) \subseteq J$ and $\varphi(J)$ is pointwise closed. For this, let $f \in \varphi(J)$, then $E_f(J)$ is infinite, hence contains a sequence (y_n) tends to 0; we have $\tau_{y_n}(f) \in J$ for each n , and letting n tends to infinite shows that $f \in J$, since $y_n \rightarrow 0$ by hypothesis and J is pointwise closed; this shows $\varphi(J) \subseteq J$. \square

Exercise 15.1.7. Let R be a DVR and π a uniformizer for R . In the ring $R[T]$, the ideal $\mathfrak{m}_1 = (\pi T - 1)$ is maximal with height 1, the ideal $\mathfrak{m}_2 = (\pi) + (T)$ is maximal with height 2. The field $R[T]/\mathfrak{m}_1$ and $R[T]/\mathfrak{m}_2$ are isomomorphic to the field of fractions of R and to the residual field of R respectively. We have

$$\dim(R[T]) = 2, \quad \dim(R[T]/\mathfrak{m}_1) + \dim(R[T]_{\mathfrak{m}_1}) = 1, \quad \dim(\text{gr}_{\mathfrak{m}_1}(R[T])) = 1. \quad (15.1.4)$$

Proof. Let K be the fraction field of R . Then it is easy to see $R[T]/\mathfrak{m}_1 \cong R[\pi^{-1}] = K$, so \mathfrak{m}_1 is maximal. Moreover, since \mathfrak{m}_1 is principal, it has height 1 by ???. Also, the quotient $R[T]/\mathfrak{m}_2$ is clearly isomomorphic to κ_R , so \mathfrak{m}_2 is maximal, and has height 2 by ??.

Clearly $\dim(R[T]) = \dim(R) + 1 = 2$ since R is Noetherian. The ring $R[T]_{\mathfrak{m}_1}$ has dimension equal to $\text{ht}(\mathfrak{m}_1)$, which is 1. These prove the first two equalities in (??). To see $\dim(\text{gr}_{\mathfrak{m}_1}(R[T])) = 1$, note that $\text{gr}_{\mathfrak{m}_1}(R[T])$ is isomomorphic to a polynomial ring over K . \square

Exercise 15.1.8. With the notations of the previous exercise, suppose that the residue field of R and fraction field of R are isomomorphic (this is the case for example when $R = k[[X]]$, the field k being the field of fractions of a ring of formal series with an infinite number of indeterminates with coefficients in a field). Let σ be an isomorphism of $R[T]/\mathfrak{m}_1$ to $R[T]/\mathfrak{m}_2$. Let C be the subring of $R[T] = E$ formed of the elements of E whose classes modulo \mathfrak{m}_1 and \mathfrak{m}_2 are associated by σ : in other words, if $\pi_i : R[T] \rightarrow R[T]/\mathfrak{m}_i$ is the natural map, then

$$C = \{x \in R[T] : \sigma(\pi_1(x)) = \pi_2(x)\}.$$

Show that C is noetherian and that $\mathfrak{m}_1\mathfrak{m}_2 = \mathfrak{m}_1 \cap \mathfrak{m}_2$ is a maximal ideal of C . Show that E is

the integral closure of C (note that we have $E = C + (\pi T - 1)C$ and that $(\pi T - 1)^2 + (\pi T - 1)$ belongs to C).

Let A be the local ring $C_{m_1 m_2}$. Then A is integral with dimension 2, hence catenary; the integral closure $B = E \otimes_C A$ of A is a semi-local Noetherian ring with exactly two maximal ideals \mathfrak{n}_1 and \mathfrak{n}_2 , and we have $\dim(B_{\mathfrak{n}_1}) = 1$, $\dim(B_{\mathfrak{n}_2}) = 2$.

Proof.

□

Exercise 15.1.9. Retain the notations of the previous exercise, and identify B with a quotient ring of the polynomial ring $A[U]$ by the prime ideal generated by $U^2 + U - \pi T(\pi T - 1)$. Let \mathfrak{q}_i be the ideal of $A[U]$ such that $\mathfrak{n}_i = \mathfrak{q}_i/\mathfrak{p}$. Then $\text{ht}(\mathfrak{q}_1) = 3$, $\text{ht}(\mathfrak{p}) = 1$ and the chain $\mathfrak{p} \subseteq \mathfrak{q}_1$ is saturated. In particular, $A[U]$ and $A[U]_{\mathfrak{q}_1}$ are not catenary.

Exercise 15.1.10. Let A be a ring.

- (a) Assume that A is an integral domain and let $f \in A[T]$, $f \neq 0$. Prove that there exists a maximal ideal \mathfrak{m} of $A[T]$ such that $f \notin \mathfrak{m}$.
- (b) Prove that $\dim(\text{Max}(A[T])) \geq \dim(A) + 1$, so $\dim(\text{Max}(A[T])) = \dim(A[T]) = \dim(A) + 1$ when A is Noetherian.
- (c) Let k be a field, S the subset $1 + (X)$ of $k[X, Y]$, and $A = S^{-1}k[X, Y]$. Show that we have $\dim(\text{Max}(A)) = 1$ and $\dim(\text{Spec}(A)) = 2$. Generalize this to several indeterminates.
- (d) Show that for any couple of integers $0 \leq n < m$, there exists a Noetherian ring A such that $\dim(\text{Max}(A)) = n$ and $\dim(\text{Max}(A[T])) = m$.
- (e) Let A be the Noetherian ring defined in ???. Show that $\dim(\text{Max}(A)) = 1$ and $\dim(\text{Max}(A[T])) = \infty$.

Proof. Since the Jacobson radical of $A[T]$ is zero if A is integral, we have (a). Now let $\mathfrak{p}_0 \subset \dots \subset \mathfrak{p}_n$ be a chain of prime ideals in A . Consider the chain

$$\mathfrak{p}_0[T] \subset \dots \subset \mathfrak{p}_1[T] \subset \dots \subset \mathfrak{p}_n[T] \subset (\mathfrak{p}_n, T)$$

of prime ideals of $A[T]$ and put

$$\begin{cases} U_i = V(\mathfrak{p}_i[T]) \cap \text{Max}(A[T]), & 0 \leq i \leq n \\ U_{n+1} = V(\mathfrak{p}_n, T) \cap \text{Max}(A[T]). \end{cases}$$

It is easy to see the chain $(U_i)_{0 \leq i \leq n+1}$ is decreasing and each U_i is closed; the fact it is strict follows immediately from (a). It remains to show that each U_i is irreducible. This shows the equality $\dim(\text{Max}(A[T])) \geq \dim(A) + 1$. Recall that $\dim(\text{Max}(A)) \leq \dim(A)$ for any ring and $\dim(A[T]) = \dim(A) + 1$ if A is Noetherian. So if A is Noetherian, we have

$$\dim(A) + 1 \leq \dim(\text{Max}(A[T])) \leq \dim(A[T]) = \dim(A) + 1.$$

Let A be the ring in (c). Then the ring $A = S^{-1}k[X, Y]$ is just the ring $S^{-1}k[Y][X]$, and it is easy to see the chain $0 \subseteq (X) \subseteq (X, Y)$ corresponds to a chain of length 2 in A , whence $\dim(A) = 2$. For the space $\text{Max}(A)$, note that a maximal ideal $(X - a, Y - b)$ is disjoint from

S if and only if $a = 0$, which indicates $\dim(\text{Max}(A)) = 1$ at least when k is algebraically closed. In general, let $0 \leq n \leq m$ and consider the ring $k[X_1, \dots, X_m]$. Let S be the subset $1 + (X_1, \dots, X_{m-n})$, and $A = S^{-1}k[X_1, \dots, X_m]$. Then $\dim(A) = m$ and $\dim(\text{Max}(A)) = n$. Note that in this case, since A is Noetherian, we have $\dim(\text{Max}(A[T])) = \dim(A) + 1 = m + 1$, so (d) is proved.

Finally, consider the example of ???. Since A is Noetherian, we have

$$\dim(\text{Max}(A[T])) = \dim(A[T]) = \infty.$$

But the maximal ideals of A are the $S^{-1}\mathfrak{p}_i$'s, and if $S^{-1}\mathfrak{p}$ is a prime ideal of A , it is contained in finitely many $S^{-1}\mathfrak{p}_i$'s. However, note that each singleton in $\text{Max}(A)$ is closed, so the set $V(S^{-1}\mathfrak{p}) \cap \text{Max}(A)$ is not irreducible unless it is a singleton or the whole space $\text{Max}(A)$. This shows $\dim(\text{Max}(A)) = 1$ (in fact, the topology on $\text{Max}(A)$ is the cofinite topology, hence has Krull dimension 1). \square

Exercise 15.1.11. Let R be a DVR with fraction field K and n an integer. Let A be the subring of the polynomial ring $K[X_1, \dots, X_n]$ consists of polynomials whose constant terms are in R . Then the canonical homomorphism $\rho : R \rightarrow A$ satisfies the going down property and the canonical homomorphism $R/\mathfrak{m}_R \rightarrow A/\mathfrak{m}_R A$ is an isomorphism. On the other hand, we have $\dim(A) \geq n + 1$ and $\dim(R) = 1$. As long as $n \geq 1$, we have $\dim(A) > \dim(R) + \dim(A/\mathfrak{m}_R A)$. Show that A is not Noetherian. By using formal series, construct an similar example where A is local and not Noetherian.

Proof. It is clear that A is a subring of $K[X_1, \dots, X_n]$ and $A/\mathfrak{m}_R A$ is isomorphic to R/\mathfrak{m}_R ; A is flat over R since it is torsion-free (??). Since $\mathfrak{a}^{ec} = \mathfrak{a}$ for every ideal of R , by ?? we conclude that A is faithfully flat over R , whence the going down property is satisfied (??). Let $\mathfrak{m}_R[X_1, \dots, X_n]$ be the ideal consists of polynomials with coefficients in \mathfrak{m}_R . Then it is clear that $\mathfrak{m}_R[X_1, \dots, X_n]$ is prime in A and we have a chain

$$0 \subset \mathfrak{m}_R[X_1, \dots, X_n] \subset \mathfrak{m}_R[X_1, \dots, X_n] + (X_1) \subset \cdots \subset \mathfrak{m}_R[X_1, \dots, X_n] + (X_1, \dots, X_n)$$

in the ring $K[X_1, \dots, X_n]$. Contracting this to A , we get a chain of length $n + 1$ in A , so $\dim(A) \geq n + 1$. Since $A/\mathfrak{m}_R A \cong R/\mathfrak{m}_R$ is a field, this shows

$$\dim(A) \geq n + 1 > \dim(R) + \dim(A/\mathfrak{m}_R A)$$

as long as $n \geq 1$.

To see A is not Noetherian, we claim that the ideal $\mathfrak{p} = (X_1, \dots, X_n) \cap A$ in A is not finitely generated. In fact, if $f_1, \dots, f_n \in A$ generate \mathfrak{p} , then $\sum_i f_i(0)g_i(0)$ can take any element in K as g_i varies in A . But this is not possible since $f_i(0), g_i(0) \in R$.

Finally, replacing $K[X_1, \dots, X_n]$ with $K[\![X_1, \dots, X_n]\!]$, we see the resulting ring A is then local with maximal ideal $\mathfrak{m}_R[\![X_1, \dots, X_n]\!] + (X_1, \dots, X_n)$. \square

Exercise 15.1.12. Let R be a DVR with fraction field K and residue field k . Put $A = K \times k$, and let $\rho : R \rightarrow A$ be the homomorphism deduced from the canonical homomorphisms $R \rightarrow K$ and $R \rightarrow k$. Then the homomorphism ρ is injective, the induced map $\rho^* : \text{Spec}(A) \rightarrow \text{Spec}(R)$ is surjective, A is a finitely generated R -algebra, and we have $\dim(A) < \dim(R)$.

Proof. Since $R \rightarrow K$ is injective, it is clear that ρ is injective. Also, we have $\text{Spec}(A) = \text{Spec}(K) \amalg \text{Spec}(k)$, which consists of two elements $(0)_K$ and $(0)_k$. The image of $(0)_K$ under ρ is $(0)_R$, and that of $(0)_k$ is \mathfrak{m}_R . Therefore ρ^* is surjective. Since K and k are finitely generated R -algebra, so is A , and we have $\dim(A) = 0 < \dim(R) = 1$. \square

Exercise 15.1.13. Let A be an integral domain and $\mathfrak{P}_0 \subset \mathfrak{P}_1 \subset \mathfrak{P}_2$ be a chain of prime ideals in $A[X]$. Show that $\mathfrak{P}_0, \mathfrak{P}_1, \mathfrak{P}_2$ can not have the same contraction in A . In particular, if A is one-dimensional and \mathfrak{P}_i are nonzero, then $\mathfrak{P}_0 \cap A = \{0\}$, $\mathfrak{P}_2 = (\mathfrak{P}_1 \cap A) \cdot A[X]$, and \mathfrak{P}_2 is maximal.

Proof. Let $\mathfrak{P}_0 \cap A = \mathfrak{P}_1 \cap A = \mathfrak{P}_2 \cap A = \mathfrak{p}$. Then $\mathfrak{P}_i = \mathfrak{p}A[X] + (X)$ for $i = 0, 1, 2$, which contradicts that \mathfrak{P}_i are distinct. Now assume that $\dim(A) = 1$ and \mathfrak{P}_i are nonzero. Recall that we have $2 \leq \dim(A[X]) \leq 3$ (??), and since 0 is a prime ideal of $A[X]$, the chain $\mathfrak{P}_0 \subset \mathfrak{P}_1 \subset \mathfrak{P}_2$ shows that $\dim(A[X]) = 3$. Therefore \mathfrak{P}_2 is a maximal ideal in $A[X]$ and it contains (X) . Now consider the contracted chain

$$0 \subseteq \mathfrak{P}_0^c \subseteq \mathfrak{P}_1^c \subseteq \mathfrak{P}_2^c.$$

Since A has dimension 1 and \mathfrak{P}_2 is maximal in $A[X]$, \mathfrak{P}_2^c must be a maximal ideal in A . The inclusion $0 \subseteq \mathfrak{P}_0^c$ can not be strict since otherwise we have $\mathfrak{P}_0^c = \mathfrak{P}_1^c = \mathfrak{P}_2^c$. This proves $\mathfrak{P}_0^c = \{0\}$. We also note that $\mathfrak{P}_1^c \neq \{0\}$ since otherwise $\mathfrak{P}_0 = \mathfrak{P}_1 = (X)$ (again by ??). Therefore $\mathfrak{P}_1^c = \mathfrak{P}_2^c$ is a maximal ideal of A , and since $\mathfrak{P}_1 \neq \mathfrak{P}_2$, we have $X \notin \mathfrak{P}_2$ and $\mathfrak{P}_2 = \mathfrak{P}_2^c \cdot A[X]$. \square

Exercise 15.1.14. Let A be an integrally closed domain with K the fraction field. For any element $x \in K$, we denote by $A[x]$ the sub- A -algebra of K generated by x . Let x be nonzero in K and $x^{-1} \notin A$; let $A[X]$ be the polynomial ring over A , and $\varphi : A[X] \rightarrow A[x]$ the canonical map $P \mapsto P(x)$.

- (a) The kernel of φ is the ideal \mathfrak{Q} generated by the polynomials of the form $aX + b$, where $a, b \in A$ and $ax + b = 0$.
- (b) Let A be local with maximal ideal \mathfrak{m}_A . Then the ideal $\mathfrak{m}_A A[x]$ of $A[x]$ is prime, and is maximal if and only if $x \in A$.

Proof. Let φ be the canonical map. Assume that $P(X) = \sum_{i=0}^n a_i X^i$ and $P(x) = 0$. Then by multiplying a_n^{n-1} , we get

$$(a_n x)^n + a_{n-1} (a_n x)^{n-1} + \cdots + a_n^{n-2} a_1 (a_n x) + a_n^{n-1} a_0 = 0,$$

so $-b := a_n x$ is integral over A , hence belongs to A . Consider the polynomial $a_n X + b$: it has x as a root and the same leading coefficient with P . Hence $a_n X + b \mid P(X)$ and assertion (a) is proved. We also note that, since $x^{-1} \notin A$, the element b can not be invertible, hence $b \in \mathfrak{m}_A$.

Now by (a), we have

$$A[x]/\mathfrak{m}_A A[x] \cong A[X]/(aX + b, \mathfrak{m}_A) = \kappa_A[X]/(\bar{a}X + \bar{b}) = \kappa_A[X]/(\bar{a}X)$$

where a, b are the generators of $\ker \varphi$, and we have remarked that $b \in \mathfrak{m}_A$. Since these are irreducible elements, we conclude that $\mathfrak{m}_A A[x]$ is prime, and maximal iff there exists $a \in A$ in these generators such that $\bar{a} \neq 0$. But this just means $a \notin \mathfrak{m}_A$, hence invertible, and we then have $x = a^{-1}b \in A$. \square

Exercise 15.1.15. Let A be an integrally closed domain with fraction field K . Given a prime ideal \mathfrak{p} of A , an element x of K such that $x \notin A_{\mathfrak{p}}$, and $x^{-1} \notin A_{\mathfrak{p}}$, the ideal $\mathfrak{p}A[x]$ of $A[x]$ is prime; we have $\mathfrak{p}A[x] \cap A = \mathfrak{p}$ and the canonical homomorphism $(A/\mathfrak{p})[X] \rightarrow A[x]/\mathfrak{p}A[x]$ is an isomorphism.

Proof. Since $x \notin A_{\mathfrak{p}}$, and $x^{-1} \notin A_{\mathfrak{p}}$, we have $x \notin A$, so we can prove that $\mathfrak{p}A[x]$ is prime. Since A is integrally closed, it is not hard to see $\mathfrak{p}A[x] \cap A = \mathfrak{p}$. Since $x \notin A_{\mathfrak{p}}$ and $x^{-1} \notin A_{\mathfrak{p}}$, by Exercise ??, the kernel of the canonical map $\varphi : A[X] \rightarrow A[x]$ the ideal \mathfrak{Q} generated by the polynomials of the form $aX + b$, where $a, b \in \mathfrak{p}$ and $ax + b = 0$. Since we have

$$A[x]/\mathfrak{p}A[x] = A[X]/(\mathfrak{p}, \mathfrak{Q}) = (A/\mathfrak{p})[X]/\bar{\mathfrak{Q}}$$

and the image of \mathfrak{Q} in A/\mathfrak{p} is zero, we see $(A/\mathfrak{p})[X]$ is isomorphic to $A[x]/\mathfrak{p}A[x]$. \square

Exercise 15.1.16. We say an integral domain A is an **F -ring** if $\dim(A) = 1$ but $\dim(A[X]) > 2$.

- (a) Using the previous exercise, show that if A is a one-dimensional integral domain then $\dim(A[X]) = 2$ if and only if the localization at any prime ideal of the integral closure of A is a valuation ring.
- (b) Show that if we have $\dim(A) = n$ and $\dim(A[X]) > n + 1$, there exists a prime ideal \mathfrak{p} of A such that $\text{ht}(\mathfrak{p}) = 1$ and, either $\dim(A_{\mathfrak{p}}[X]) > 2$, or $\dim((A/\mathfrak{p})[X]) > \dim(A/\mathfrak{p}) + 1$.

Proof. Let A be as in (a) and by taking integral closure and note that if R is the integral closure of A , then $R[X]$ is the integral closure of $A[X]$, we may assume that A is integrally closed. Assume that there exists $0 \neq x \in K$ such that $x, x^{-1} \notin A_{\mathfrak{p}}$. Then by ?? applied to $A_{\mathfrak{p}}$, the ideal $\mathfrak{p}A_{\mathfrak{p}}[x]$ is not maximal in $A_{\mathfrak{p}}[x]$, whence $A_{\mathfrak{p}}[x]$ has dimension ≥ 2 . But since $A_{\mathfrak{p}}[x]$ is a quotient of $A_{\mathfrak{p}}[X]$, this implies $\dim(A[X]) \geq \dim(A_{\mathfrak{p}}[X]) \geq 3$, so $\dim(A[X]) = 2$ iff there is not such x , or in other words $A_{\mathfrak{p}}$ is a valuation ring.

Now let A be n -dimensional and $\dim(A[X]) \neq n + 1$. Suppose that for some minimal prime ideal \mathfrak{p} of A , $\mathfrak{p}A[X]$ is not minimal in $A[X]$; that is, there exists a prime ideal \mathfrak{P} of $A[X]$ such that

$$0 \subset \mathfrak{P} \subset \mathfrak{p}A[X].$$

Then $0 \subset \mathfrak{P}A_{\mathfrak{p}}[X] \subset \mathfrak{p} \cdot A_{\mathfrak{p}}[X]$ is also a chain of prime ideals in $A_{\mathfrak{p}}[X]$, as one easily verifies. Since $\mathfrak{p} \cdot A_{\mathfrak{p}}[X]$ is not maximal (for example $\mathfrak{p} \cdot A_{\mathfrak{p}}[X] + (X)$ contains it), this shows that $A_{\mathfrak{p}}$ is an F -ring. We pass then to the case that $\mathfrak{p}A[X]$ is minimal for every minimal prime ideal \mathfrak{p} of A . Let

$$0 \subset \mathfrak{P}_1 \subset \cdots \subset \mathfrak{P}_{n+2}$$

be a chain of prime ideals in $A[X]$ (recall that $\dim(A[X]) \geq n + 2$). If $\mathfrak{P}_1 \cap A = \mathfrak{p}$ is nonzero, then A/\mathfrak{p} is at most $(n - 1)$ -dimensional, and $A[X]/\mathfrak{p}A[X]$ is a polynomial ring in one variable over A/\mathfrak{p} and is at least $(n + 1)$ -dimensional, which proves the claim. So we may suppose $\mathfrak{P}_1 \cap A = 0$. But then $\mathfrak{P}_2 \cap A = \mathfrak{p}_2 \neq 0$ (by ??); let \mathfrak{p} be a minimal prime ideal contained in \mathfrak{p}_2 -such exists since A is finite dimensional; then $\mathfrak{p}A[X] \subseteq \mathfrak{P}_2$, properly, since $\mathfrak{p}A[X]$ is minimal but \mathfrak{P}_2 is not. Replacing \mathfrak{P}_1 by $\mathfrak{p}A[X]$, we come back to a previous case, and the proof is complete. \square

Exercise 15.1.17. Let O be an integrally closed ring, with fraction field k . We put $d = \dim(O)$ and $t = \dim(O[X])$. Let k' be a non-trivial extension of k in which k is algebraically closed and let R be a discrete valuation ring (which is not a field) with residual field k' . We denote K the field of fractions of R and A the set of elements of R whose image in k' belongs to O .

- (a) Show that the ring A is integrally closed with fraction field K .
- (b) Let \mathfrak{p} be the kernel of the surjective homomorphism $A \rightarrow O$. Show that every nonzero prime ideal of A contains \mathfrak{p} and hence $\dim(A) = d + 1$.
- (c) Considering an element x of R whose image in k' does not belong to k , show that the local ring $A_{\mathfrak{p}}$ is not a valuation ring, and that $\mathfrak{p}A_{\mathfrak{p}}[x]$ is not minimal among non-zero prime ideals of $A_{\mathfrak{p}}[x]$. By using ??, deduce that we have $\dim(A[x]) \geq t + 2$. Conclude that we have $\dim(A[x]) = t + 2$ by a direct argument.
- (d) Finally, show that for any pair (d, t) of integers with $d + 1 \leq t \leq 2d + 1$, there exists an integrally closed domain A of dimension d such that $\dim(A[X]) = t$.

Proof. Let v be the valuation of K associated with R . Note that any element in K can be written as a/b where $a, b \in R$. Moreover, we can assume that a, b has positive valuations, whence are in $\mathfrak{m}_R \subseteq A$. This shows the fraction field of A is K . Now let $x \in K$ be integral over A , and let

$$x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0$$

be an equation of integral dependence. Dividing this equation by x^s and supposing x^{-1} to have residue 0 in k' , we get the contradiction $1 = 0$, so $x^{-1} \notin \mathfrak{m}_R$ and thus $x \in R$; we then have

$$\bar{x}^n + \bar{a}_{n-1}\bar{x}^{n-1} + \cdots + \bar{a}_0 = 0$$

where the bars indicate residues. Since each \bar{a}_i is in O and k is algebraically closed in k' we deduce that $\bar{x} \in K$; whence $\bar{x} \in O$, since O is integrally closed. Hence A is integrally closed.

Let \mathfrak{p} be the kernel of the canonical map $A \rightarrow O$. Then \mathfrak{p} is a prime ideal. From the definitions one obtains $A/\mathfrak{p} \cong O$, whence A is at least $(n + 1)$ -dimensional. If \mathfrak{q} is a nonzero prime ideal in A , then \mathfrak{q} contains \mathfrak{p} : In fact, let $x \in \mathfrak{q}$; since $x \in A \subseteq R$, we have $v(x) = e \geq 0$. Since any element in \mathfrak{p} has valuation ≥ 1 , its $(e + 1)$ -th power is divisible by x , whence $\mathfrak{p}^{e+1} \subseteq \mathfrak{q}$ and so $\mathfrak{p} \subseteq \mathfrak{q}$. From this it follows that A is at most $(d + 1)$ -dimensional, so $\dim(A) = d + 1$.

The quotient ring $A_{\mathfrak{p}}$ is integrally closed and has one nonzero prime ideal. Moreover it is not a valuation ring if $k \neq k'$. In fact, let $x \in R$ be an element having residue in k' but not in k (as in (c)). Note that x is in the quotient field of $A_{\mathfrak{p}}$, which is K ; but neither x nor x^{-1} has residue in k , so neither x nor x^{-1} is in $A_{\mathfrak{p}}$. Thus $A_{\mathfrak{p}}$ is not a valuation ring, and hence is an F -ring (??(a) applied to $A_{\mathfrak{p}}$). It follows that $\mathfrak{p}A[X]$ is not minimal among nonzero prime ideals in $A[X]$. Now

$$A[X]/\mathfrak{p}A[X] \cong (A/\mathfrak{p})[X] \cong O[X]$$

so $A[X]$ is at least $(t + 2)$ -dimensional (recall that is integral).

Finally, let $0 = \mathfrak{P}_0 \subset \mathfrak{P}_1 \subset \cdots \subset \mathfrak{P}_n$ be a chain of prime ideals in $A[X]$. Assume that the chain $0 \subset \mathfrak{P}_1$ is saturated; then $\mathfrak{P}_1 \cap A = 0$, as otherwise

$$\mathfrak{p} \subset \mathfrak{P}_1 \cap A \quad \text{and} \quad \mathfrak{p} \cdot A[X] \subset \mathfrak{P}_1.$$

Similarly one concludes that if the chain $0 \subset \mathfrak{P}_1 \subset \mathfrak{P}_2$ is saturated, then

$$\mathfrak{P}_2 \cap A = \mathfrak{p} \text{ and } \mathfrak{P}_2 = \mathfrak{p}A[X]$$

(by ??, $\mathfrak{P}_2 \cap A$ can not be 0). From this it follows at once that $A[X]$ is at most $(t+2)$ -dimensional.

We say an integral domain A is of type (d, t) if $\dim(A) = d$ and $\dim(A[X]) = t$. Now any field is of type $(0, 1)$, and from the above construction we get an integrally closed domain of type $(1, 3)$. Note that a Noetherian integrally closed domain is of type $(1, 2)$. We now prove (d) by induction on d . So suppose by induction that for some d and each t (where $d+1 \leq t \leq 2d+1$), we have an integrally closed ring of type (d, t) . Note that if $d+3 \leq t \leq 2d+3$, then $d+1 \leq t-2 \leq 2d+1$ and from an integrally closed ring of type $(d, t-2)$ we get an integrally closed ring of type (d, t) . If on the other hand $t = d+2$, then a Noetherian integrally closed domain of dimension $d+1$ (for example the ring $k[X_1, \dots, X_{d+1}]$ where k is a field) is of type $(d+1, d+2)$, so the claim is proved. \square

Exercise 15.1.18.

- (a) Let A be an integral domain, K its fraction field and n an integer. Let t_1, \dots, t_n be elements of K and $\varphi : A[X_1, \dots, X_n] \rightarrow A[t_1, \dots, t_n]$ the homomorphism such that $\varphi(X_i) = t_i$. Show that the height of the kernel of φ equals to n .
- (b) Deduce that for any n -tuple (t_1, \dots, t_n) of elements of K , we have

$$\dim(A[t_1, \dots, t_n]) \leq \dim(A[X_1, \dots, X_n]) - n.$$

Exercise 15.1.19. Let A be an integral domain with fraction field K . Let $0 \subset \mathfrak{p}_1 \subset \dots \subset \mathfrak{p}_n$ be a chain of prime ideals of A . Show that there exists a valuation ring R of K and a chain of prime ideals $0 \subset \mathfrak{q}_1 \subset \dots \subset \mathfrak{q}_n$ of R such that $\mathfrak{q}_i \cap A = \mathfrak{p}_i$.

Exercise 15.1.20. Let A be an integral domain and K its fraction field. For any integer $k \geq 0$, the following properties are equivalent:

- (i) Any subring B of K containing A has dimension $\leq k$.
- (ii) Any valuation ring R of K containing A has dimension $\leq k$.
- (iii) For any k elements (t_1, \dots, t_k) of K , we have $\dim(A[t_1, \dots, t_k]) \leq k$.
- (iv) We have $\dim(A[X_1, \dots, X_m]) \leq k+m$ for any integer $m \geq 0$.

15.2 Dimension of Noetherian rings

15.2.1 Dimension of a quotient ring

Proposition 15.2.1. Let A be a Noetherian integral domain, x a nonzero element of A and \mathfrak{p} a minimal prime belonging to x . Then \mathfrak{p} has height 1.

Proof. Let $\mathfrak{q} \subset \mathfrak{p}$ be a prime ideal distinct from \mathfrak{p} . Then $x \notin \mathfrak{q}$ by the hypothesis on \mathfrak{p} . Since A is integral, $A_{\mathfrak{p}}$ is identified with a subring of $A_{\mathfrak{q}}$. For any positive integer n , we denote by

\mathfrak{q}_n the ideal $\mathfrak{q}^n A_{\mathfrak{q}} \cap A_{\mathfrak{p}}$ of $A_{\mathfrak{p}}$. The minimal property of \mathfrak{p} means that the local ring $A_{\mathfrak{p}}/xA_{\mathfrak{p}}$ is 0-dimensional. It is therefore Artinian and by ?? there is a positive integer n_0 such that

$$\mathfrak{q}_n + xA_{\mathfrak{p}} = \mathfrak{q}_{n+1} + xA_{\mathfrak{p}} \quad \text{for } n \geq n_0.$$

Let's now fix an integer $n \geq n_0$. Given $y \in \mathfrak{q}_n$, there exists $a \in A_{\mathfrak{p}}$ such that $y - ax \in \mathfrak{q}_{n+1}$. We then have $ax \in \mathfrak{q}_n$, whence $a \in \mathfrak{q}_n$ since $x \notin \mathfrak{q}$, and finally we have $y \in \mathfrak{q}_{n+1} + x\mathfrak{q}_n$. So we have proved that

$$\mathfrak{q}_n = \mathfrak{q}_{n+1} + x\mathfrak{q}_n.$$

As x belongs to the maximal ideal of the Noetherian local ring $A_{\mathfrak{p}}$, the Nakayama lemma shows that we have $\mathfrak{q}_n = \mathfrak{q}_{n+1}$. As $\mathfrak{q}_n A_{\mathfrak{q}} = (\mathfrak{q} A_{\mathfrak{q}})^n$, we conclude that

$$(\mathfrak{q} A_{\mathfrak{q}})^n = (\mathfrak{q} A_{\mathfrak{q}})^{n+1} \quad \text{for } n \geq n_0.$$

Since $A_{\mathfrak{q}}$ is a local Noetherian ring, we have $\bigcap_n (\mathfrak{q} A_{\mathfrak{q}})^n = \{0\}$ by ??, whence $(\mathfrak{q} A_{\mathfrak{q}})^{n_0} = 0$ and the prime ideal $\mathfrak{q} A_{\mathfrak{q}}$ of $A_{\mathfrak{q}}$ reduces to 0. Then we must have $\mathfrak{q} = 0$ and therefore \mathfrak{p} has height 1. \square

Lemma 15.2.2. *Let A be a Noetherian ring, $\mathfrak{p}_0 \subset \cdots \subset \mathfrak{p}_n$ a chain of prime ideals of A and x an element of \mathfrak{p}_n . Then there exists a chain $\mathfrak{p}'_0 \subset \cdots \subset \mathfrak{p}'_n$ of prime ideals of A such that $\mathfrak{p}'_0 = \mathfrak{p}_0$, $\mathfrak{p}'_n = \mathfrak{p}_n$, and $x \in \mathfrak{p}'_1$.*

Proof. We prove by induction. The case $n = 1$ is trivial, so suppose $n \geq 2$ and that x does not belong to \mathfrak{p}_{n-1} (otherwise we can apply to $\mathfrak{p}_0 \subset \cdots \subset \mathfrak{p}_{n-1}$ the induction hypothesis). Let \mathfrak{p}'_{n-1} be a minimal element of the set of prime ideals of A contained in $\mathfrak{p}'_n = \mathfrak{p}_n$ and containing $\mathfrak{p}_{n-2} + Ax$. According to ??, the ideal $\mathfrak{p}'_{n-1}/\mathfrak{p}_{n-2}$ of the ring A/\mathfrak{p}_{n-2} is of height 1, and since $\mathfrak{p}_{n-2} \subset \mathfrak{p}_{n-1} \subset \mathfrak{p}_n$ is a chain of length 2, so is $\mathfrak{p}_{n-2} \subset \mathfrak{p}'_{n-1} \subset \mathfrak{p}'_n$. Since now $x \in \mathfrak{p}'_{n-1}$, the induction hypothesis applied to the chain $\mathfrak{p}_0 \subset \cdots \subset \mathfrak{p}'_{n-1}$ shows that there is a chain $\mathfrak{p}'_0 \subset \cdots \subset \mathfrak{p}'_{n-1}$ with $x \in \mathfrak{p}'_1$ and $\mathfrak{p}'_0 = \mathfrak{p}_0$. The chain

$$\mathfrak{p}'_0 \subset \cdots \subset \mathfrak{p}'_{n-1} \subset \mathfrak{p}'_n$$

then satisfies the requirements. \square

Proposition 15.2.3. *Let A be a Noetherian ring and \mathfrak{a} an ideal contained in the Jacobson radical of A and generated by m elements. Then we have*

$$\dim(A/\mathfrak{a}) \leq \dim(A) \leq \dim(A/\mathfrak{a}) + m.$$

Proof. The inequality $\dim(A/\mathfrak{a}) \leq \dim(A)$ is clear. By induction it suffices to prove the inequality

$$\dim(A) \leq \dim(A/xA) + 1$$

for any element x contained in the Jacobson radical of A . That is, to show that $\dim(A/xA) \geq n - 1$ for any chain $\mathfrak{p}_0 \subset \cdots \subset \mathfrak{p}_n$ of prime ideals of A . Now since x is in the Jacobson radical of A , we may assume that $x \in \mathfrak{p}_n$. Then it suffices to construct a chain $\mathfrak{q}_1 \subset \cdots \subset \mathfrak{q}_n$ of prime ideals of A , with $x \in \mathfrak{q}_1$ and this follows from ??.

Corollary 15.2.4.

- (a) Every Noetherian semilocal ring is of finite dimension.
- (b) Let A be a Noetherian ring. Then every proper ideal of A has finite height.
- (c) Any decreasing sequence of prime ideals of a Noetherian ring A is stationary.

Proof. Let A be a Noetherian semilocal ring and \mathfrak{r} its Jacobson radical. The quotient ring A/\mathfrak{r} is then of dimension 0 by ???. If the ideal \mathfrak{r} is generated by m elements, then we have $\dim(A) \leq m$ by ??.

Let A be Noetherian and \mathfrak{a} a proper ideal of A . Let \mathfrak{m} be a maximal ideal of A containing A , then $0 \leq \text{ht}(\mathfrak{a}) \leq \dim(A_{\mathfrak{m}})$. Since $A_{\mathfrak{m}}$ is a Noetherian local ring, we have $\dim(A_{\mathfrak{m}}) < +\infty$ by (a), whence (b) follows.

Finally, any finite strictly decreasing sequence $(\mathfrak{p}_i)_{1 \leq i \leq n}$ of prime ideals of a Noetherian ring A defines a chain $\mathfrak{p}_n \subset \dots \subset \mathfrak{p}_0$, hence $n < \dim(A_{\mathfrak{p}_0}) < +\infty$ and (c) follows. \square

Corollary 15.2.5. *Let A be a Noetherian local ring.*

- (a) *Let $x \in \mathfrak{m}_A$. Then $\dim(A) - 1 \leq \dim(A/xA) \leq \dim(A)$, and $\dim(A/xA) = \dim(A) - 1$ if and only if x does not belong to any of the minimal prime ideals \mathfrak{p} of A such that $\dim(A/\mathfrak{p}) = \dim(A)$, and it suffices that x is not a zero divisor in A .*
- (b) *Let \mathfrak{a} be a proper ideal of A such that $\dim(A/\mathfrak{a}) < \dim(A)$. Then there exists $x \in \mathfrak{a}$ such that $\dim(A/xA) = \dim(A) - 1$.*
- (c) *If $\dim(A) \geq 1$, then there exists $x \in \mathfrak{m}_A$ such that $\dim(A/xA) = \dim(A) - 1$.*

Proof. By ??, we see $\dim(A) - 1 \leq \dim(A/xA) \leq \dim(A)$. For $\dim(A/xA) = \dim(A) = n$, it is necessary and sufficient that there exists a chain $\mathfrak{p}_0 \subset \dots \subset \mathfrak{p}_n$ such that $x \in \mathfrak{p}_0$, and this is equivalent to say there exists a prime ideal \mathfrak{p}_0 containing x such that $\dim(A/\mathfrak{p}_0) = \dim(A)$. But such an ideal prime \mathfrak{p}_0 is necessarily minimal, and any element of \mathfrak{p}_0 is therefore a divisor of 0 in A . This proves (a).

Let Φ be the set of minimal prime ideals of A , and Φ' the subset $\mathfrak{p} \in \Phi$ such that $\dim(A) = \dim(A/\mathfrak{p})$. Since A is Noetherian, Φ is finite. Let \mathfrak{a} be a proper ideal such that $\dim(A/\mathfrak{a}) < \dim(A)$. Then for every $\mathfrak{p} \in \Phi'$ we have $\dim(A/\mathfrak{a}) < \dim(A/\mathfrak{p})$, whence $\mathfrak{a} \not\subseteq \mathfrak{p}$. Then by ??, there exists $x \in \mathfrak{a}$ such that $x \notin \mathfrak{p}$ for all $\mathfrak{p} \in \Phi'$, and therefore $\dim(A/xA) = \dim(A) - 1$ by (a), whence (b). Now (c) follows from (b) by taking $\mathfrak{a} = \mathfrak{m}_A$. \square

Example 15.2.6. The claim in ?? fails if A is not Noetherian. For example, let A be a valuation ring of dimension $d \geq 2$. Then the prime ideals of A form a chain of length d :

$$0 = \mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \dots \subset \mathfrak{p}_d = \mathfrak{m}_A.$$

Therefore, for each i , there exists $x_i \in \mathfrak{p}_i \setminus \mathfrak{p}_{i-1}$, and it is easy to see \mathfrak{p}_i is the only minimal prime belonging to x_i .

15.2.2 Dimension and secant sequences

Let A be a Noetherian, M a finitely generated A -module, S a subset of A contained in the Jacobson radical of A . Let \mathfrak{S} be the ideal generated by S and \mathfrak{a} the annihilator of M . Then we have

$$\dim_A(M) = \dim(A/\mathfrak{a}) \leq \dim(A). \quad (15.2.1)$$

Moreover, the support of the A -module M/SM is equal to $V(\mathfrak{a} + \mathfrak{S})$ by ??, hence

$$\dim_A(M/SM) = \dim(A/(\mathfrak{a} + I)). \quad (15.2.2)$$

By ??, we then have the following inequality

$$\dim_A(M/SM) \leq \dim_A(M) \leq |S| + \dim_A(M/SM). \quad (15.2.3)$$

Definition 15.2.7. Let A be a Noetherian local ring and S a subset of \mathfrak{m}_A . Then S is said to be **secant for M** if

$$\dim_A(M) = \dim_A(M/SM) + |S|. \quad (15.2.4)$$

A family $(x_i)_{i \in I}$ of elements of \mathfrak{m}_A is said to be **secant for M** if

$$\dim_A(M) = |I| + \dim(M / \sum x_i M)$$

and an element $x \in \mathfrak{m}_A$ is called **secant for M** if $\{x\}$ is secant for M .

By definition, if \mathfrak{a} is the annihilator of M , then a subset S of \mathfrak{m}_A is secant for M if and only if it is secant for the A -module A/\mathfrak{a} . Also note that, since A is Noetherian and local, we have $\dim(A) < +\infty$, so if S is M -secant then S is finite.

Proposition 15.2.8. Let A be a Noetherian local ring, M a finitely generated A -module, and S, S' two disjoint subsets of \mathfrak{m}_A . Then the set $S \cup S'$ is secant for M if and only if S is secant for M and S' is secant for $M' = M/SM$.

Proof. For this, we first observe that

$$M/(SM + S'M) = (M/SM)/((SM + S'M)/SM) = M'/S'M'$$

and therefore

$$\dim_A(M/(SM + S'M)) = \dim_A(M'/S'M') \geq \dim_A(M') - |S'| \geq \dim_A(M) - |S| - |S'|.$$

This proves the claim by the definition of secanteness. \square

Corollary 15.2.9. Let A be a Noetherian local ring and x_1, \dots, x_r be distinct elements of \mathfrak{m}_A . Define $M_1 = M$ and $M_{i+1} = M_i/x_i M_i$. Then the following conditions are equivalent.

(i) The sequence (x_1, \dots, x_r) is M -secant.

(ii) The element x_i is M_i -secant for each i .

(iii) The element x_i is $M/(x_1M + \cdots + x_{i-1}M)$ -secant for each i .

Proof. The equivalence of (i) and (ii) follows by applying ?? on the subsets $\{x_1\}, \dots, \{x_n\}$, and that of (i) and (iii) follows similarly. \square

Proposition 15.2.10. Let A be a Noetherian local ring, M a finitely generated A -module, x_1, \dots, x_r elements of \mathfrak{m}_A and n_1, \dots, n_r integers. Then the sequence (x_1, \dots, x_r) is secant for M if and only if the sequence $(x_1^{n_1}, \dots, x_r^{n_r})$ is secant for M .

Proof. We may reduce the case $r = 1$, so let $x \in \mathfrak{m}_A$ and n be a positive integer. If \mathfrak{a} is the annihilator of M , then by ??,

$$\text{supp}(M/xM) = V(\mathfrak{a}) \cap V(x), \quad \text{supp}(M/x^nM) = V(\mathfrak{a}) \cap V(x^n).$$

By ?? we have $V(x^n) = V(x)$, whence the proposition. \square

Proposition 15.2.11. Let A be a Noetherian local ring and M a finitely generated A -module. For an element x in \mathfrak{m}_A to be secant for M , it is necessary and sufficient that it does not belong to any minimal elements \mathfrak{p} of $\text{supp}(M)$ such that $\dim(A/\mathfrak{p}) = \dim_A(M)$, and it suffices that the homothety h_x with ratio x on M be injective.

Proof. Let \mathfrak{a} be the annihilator of M . Then x is secant for M means it is secant for A/\mathfrak{a} , and if h_x is injective on M , then x is not a zero divisor of A/\mathfrak{a} . The claim now follows from ?? applying to the ring A/\mathfrak{a} . \square

Corollary 15.2.12. Any sequence of elements of \mathfrak{m}_A that is complete secant for M is secant for M .

Proof. Let (x_1, \dots, x_r) be a sequence of elements of \mathfrak{m}_A that is complete secant for M . Put $M_0 = M$ and $M_i = M_{i-1}/x_i M_{i-1}$ for $1 \leq i \leq r$. Then by ??, the homothety of ratio x_i is injective on M_{i-1} , so $\dim_A(M_i) = \dim_A(M_{i-1}) - 1$ by ?. We then conclude that

$$\dim_A(M) = r + \dim_A(M/(x_1M + \cdots + x_rM)),$$

so the sequence (x_1, \dots, x_r) is secant for M . \square

Remark 15.2.1. A sequence (x_1, \dots, x_r) is said to be **regular for M** (or an **M -regular sequence**) if for each i , the element x_i is not a zerodivisor on $M/(x_1M + \cdots + x_{i-1}M)$. By ?? and ??, it follows that every regular sequence for M is secant for M .

Theorem 15.2.13. Let A be a Noetherian local ring, M a finitely generated A -module, and S a subset of \mathfrak{m}_A .

- (a) If M/SM has finite length, then $|S| \geq \dim_A(M)$.
- (b) If S is secant for M , then $|S| \leq \dim_A(M)$.
- (c) Any secant subset for M is contained in a maximal secant subset for M .
- (d) The following conditions are equivalent:
 - (i) S is a maximal secant sequence for M ;
 - (ii) S is a secant sequence for M and $|S| = \dim_A(M)$;

- (iii) M/SM has finite length and $|S| = \dim_A(M)$;
- (iv) S is a secant sequence for M and M/SM has finite length.

Proof. Since $S \subseteq \mathfrak{m}_A$, by Nakayama lemma we have $M/SM \neq 0$, whence $\dim_A(M/SM) \geq 0$ with equality if and only if M/SM is of finite length. Assertions (a) and (b) then follows from (??) and (??), as well as the equivalences of (ii), (iii), and (iv) in (d).

By (a), any secant subset for M with cardinality $\dim_A(M)$ is maximal. It remains to prove that, if S is secant for M and if $|S| < \dim_A(M)$, then S is not maximum. Let \mathfrak{a} be the annihilator of M , and B the local Noetherian ring $A/(\mathfrak{a} + I)$, where I is the ideal generated by S . According to ??(c), there exists an element x of \mathfrak{m}_A such that $\dim(B/xB) = \dim(B) - 1$, whence $x \notin S$. According to ??, the subst $S \cup \{x\}$ of \mathfrak{m}_A is secant for A/\mathfrak{a} , hence for M . This proves assertion (c), and the equivalence of (i) and (ii) in (d). \square

Corollary 15.2.14. *Let A be a Noetherian local ring, M a finitely generated A -module. Then the dimension of M is the smallest of the positive integers d for which it d there exists a sequence (x_1, \dots, x_d) of elements of \mathfrak{m}_A such that the A -module $M / \sum_i x_i M$ is of finite length.*

Proof. As \emptyset is a secant subset for M , ??(c) shows the existence of a secant sequence for M , say (x_1, \dots, x_d) . But then $d = \dim_A(M)$ and the A -module $M / \sum_i x_i M$ is of finite length by property (iii) of ??(d). Conversely if $(x_1, \dots, x_{d'})$ is a sequence of elements of \mathfrak{m}_A such that the A -module of $M / \sum_i x_i M$ is of finite length, we have $d' \geq \dim_A(M)$ according to ??(a). \square

If $\dim_A(M) = d$, then a sequence (x_1, \dots, x_d) of elements of \mathfrak{m}_A such that $M / \sum_i x_i M$ is of finite length is called a **system of parameters** for M . Note that such a sequence must be secant for M , by (iii) of ??(d).

Let A be a Noetherian local ring and \mathfrak{m}_A be its maximal ideal. Endow A with its \mathfrak{m}_A -adic topology, recall that an ideal \mathfrak{a} of A is called a **defining ideal** of A if the \mathfrak{a} -adic topology coincides with its \mathfrak{m}_A -adic topology.

Proposition 15.2.15. *Let A be a Noetherian local ring and \mathfrak{a} an ideal of A . Then the following conditions are equivalent.*

- (i) \mathfrak{a} is a defining ideal of A .
- (ii) \mathfrak{a} is \mathfrak{m}_A -primary.
- (iii) There exist a positive integer n such that $\mathfrak{m}_A^n \subseteq \mathfrak{a} \subseteq \mathfrak{m}_A$.
- (iv) \mathfrak{a} is a proper ideal and A/\mathfrak{a} is an A -module of finite length.

Proof. By ??, we see (ii) and (iii) are equivalent, also (i) is equivalent to (iii) by the definition of adic topologies and the fact that \mathfrak{m}_A is a prime ideal. Finally, the equivalence of (i) and (iv) follows from ?? and $\text{supp}(A/\mathfrak{a}) = V(\mathfrak{a})$. \square

Corollary 15.2.16. *The dimension of a local Noetherian ring A is the smallest of the positive integers d for which there exists a defining ideal of A generated by d elements.*

Proof. This follows from ?? and ??.

Let A be a ring and $X = \text{Spec}(A)$ be its spectrum. A closed subset of the form $V(x)$ with $x \in A$ of X is called a **hypersurface** of X .

Proposition 15.2.17. *Let A be a Noetherian ring, Y be a closed subset of $X = \text{Spec}(A)$, H_1, \dots, H_m be hypersurfaces of X , and $\tilde{Y} = Y \cap H_1 \cap \dots \cap H_m$.*

- (a) *For any closed subset V of Y contained in \tilde{Y} , we have $\text{codim}(V, \tilde{Y}) \geq \text{codim}(V, Y) - m$.*
- (b) *For any irreducible component Z subset of \tilde{Y} , we have $\text{codim}(Z, Y) \leq m$.*
- (c) *If Z is an irreducible closed subset of X contained in Y such that $\text{codim}(Z, Y) \leq m$, there are hypersurfaces $\tilde{H}_1, \dots, \tilde{H}_m$ such that Z is an irreducible component of $Y \cap \tilde{H}_1 \cap \dots \cap \tilde{H}_m$.*

Proof. Let \mathfrak{a} be an ideal of A and x_1, \dots, x_m be elements of A such that $Y = V(\mathfrak{a})$ and $H_i = V(x_i)$ for each i . Let $V(\mathfrak{p})$ be an irreducible closed subset contained in Y , where \mathfrak{p} is a prime ideal of A containing \mathfrak{a} . Suppose first that Z is contained in \tilde{Y} and denote by ξ_i the image of x_i in the Noetherian local ring $B = A_{\mathfrak{p}}/\mathfrak{a}A_{\mathfrak{p}}$. According to ??, we have

$$\text{codim}(Z, Y) = \dim(B), \quad \text{codim}(Z, \tilde{Y}) = \dim(B/(\xi_1B + \dots + \xi_mB))$$

so by ?? we have

$$\text{codim}(Z, \tilde{Y}) \geq \text{codim}(Z, Y) - m. \quad (15.2.5)$$

If Z is an irreducible component of \tilde{Y} , we have $\text{codim}(Z, \tilde{Y}) = 0$, whence $\text{codim}(Z, Y) \leq m$. This proves (b), and (a) follows by taking in both sides of (??) the lower bound on the set of irreducible components Z of V .

Conversely, suppose that we have $\text{codim}(Z, Y) \leq m$, ie, $\dim(B) \leq m$. As any element of $A_{\mathfrak{p}}$ is the product of an invertible element of $A_{\mathfrak{p}}$ by the image of an element of A , ?? demonstrates the existence of elements $\tilde{x}_1, \dots, \tilde{x}_m$ of A whose images in B generate a defining ideal of B . Set $\tilde{H}_i = V(\tilde{x}_i)$, it is then clear that Z is an irreducible component of $Y \cap \tilde{H}_1 \cap \dots \cap \tilde{H}_m$. \square

Corollary 15.2.18 (Krull's Hauptidealsatz). *Let A be a Noetherian ring and x_1, \dots, x_m be elements of A .*

- (a) *For any prime ideal \mathfrak{p} belonging to $Ax_1 + \dots + Ax_m$, we have $\text{ht}(\mathfrak{p}) \leq m$.*
- (b) *If \mathfrak{p} is a prime ideal of A such that $\text{ht}(\mathfrak{p}) \leq m$, then there exist elements $\tilde{x}_1, \dots, \tilde{x}_m$ of A such that \mathfrak{p} is a minimal prime belonging to $A\tilde{x}_1 + \dots + A\tilde{x}_m$.*

Proof. This is a reformulation of ?? in the case $Y = X$. \square

Corollary 15.2.19. *Let A be a Noetherian ring and H be a hypersurface of $X = \text{Spec}(A)$. Then the codimension of H in X is equal to 0 or 1. We have $\text{codim}(H, X) = 1$ if and only if H does not contain any irreducible component of X . If so, all the irreducible components of H are of codimension 1 in X .*

Proof. By ?? we have $\text{codim}(H, X) \leq 1$, and we have $\text{codim}(H, X) = 0$ if and only if H contains an irreducible component of X . By definition,

$$\text{codim}(H, X) = \inf_Z \text{codim}(Z, X)$$

where Z runs through the set of irreducible components of H . The claim then follows from these remarks. \square

Corollary 15.2.20. *Let A be a Noetherian ring, Y be an irreducible closed subset of $X = \text{Spec}(A)$, and H a hypersurface in X . Then only three cases are possible:*

- (a) $Y \subseteq H$;
- (b) $Y \cap H$ is nonempty and each of its irreducible components satisfies $\text{codim}(Z, X) = 1$;
- (c) $Y \cap H$ is empty.

Proof. Suppose that $Y \cap H$ is nonempty and is not equal to Y . Then any irreducible component Z of $Y \cap H$ satisfies $\text{codim}(Z, Y) \leq 1$ by ???. Since Z and Y are both irreducible closed subsets, $\text{codim}(Z, Y) = 0$ iff $Y = Z$, which is impossible. \square

Corollary 15.2.21. *If A is a Noetherian UFD, then the prime ideals of height 1 of A are the principal ideals generated by the irreducible elements of A . If moreover A is local, we have $\dim(A/\mathfrak{p}) = \dim(A) - 1$ for any prime ideal \mathfrak{p} of height 1 of A .*

Proof. Let x be an irreducible element of A . Then Ax is a prime ideal because x is prime, of height 1 because A is integral. Let \mathfrak{p} be a prime ideal of height 1 of A . Then $V(\mathfrak{p})$ is an irreducible component of a hypersurface $V(x)$ for some x (??). Let $x = \prod_i p_i^{n_i}$ be a decomposition of x into products of irreducible elements. Then the irreducible components of $V(x)$ is the $V(p_i)$, so $\mathfrak{p} = Ap_i$ for some i . The last statement follows from ???. \square

Proposition 15.2.22. *Let A be a Noetherian ring and $\mathfrak{p} \subset \mathfrak{q}$ be an unsaturated chain of prime ideals of A . Then the set E of prime ideals \mathfrak{r} of A such that $\mathfrak{p} \subset \mathfrak{r} \subset \mathfrak{q}$ is infinite and we have*

$$\bigcap_{\mathfrak{r} \in E} \mathfrak{r} = \mathfrak{p}, \quad \bigcup_{\mathfrak{r} \in E} \mathfrak{r} = \mathfrak{q}.$$

Proof. By passing to the quotient ring, we may assume that $\mathfrak{p} = \{0\}$. Now by ?? we have $\bigcup_{\mathfrak{r} \in E} \mathfrak{r} = \mathfrak{q}$, and it follows from ?? that E is infinite.

Now let y be a nonzero element in $\bigcap_{\mathfrak{r} \in E} \mathfrak{r}$. Since the height of \mathfrak{q} is finite by ??, by localization at a prime ideal of height 2, we may assume that $\text{ht}(\mathfrak{q}) = 2$. Then the ring A/yA has dimension 1, hence each prime ideal $\mathfrak{r} \in E$ is minimal. But a Noetherian ring can have only finitely many minimal prime ideals, a contradiction. Therefore $\bigcap_{\mathfrak{r} \in E} \mathfrak{r} = \{0\}$, and the proof is finished. \square

Proposition 15.2.23. *Let A be a Noetherian ring with dimension ≥ 2 , and h an integer such that $0 < h < \dim(A)$.*

- (a) *The ring A has infinitely many prime ideals of A with height h .*
- (b) *If A is of finite dimension, then there are infinitely many prime ideals \mathfrak{p} such that $\text{ht}(\mathfrak{p}) = h$, $\text{coht}(\mathfrak{p}) = \dim(A) - h$.*

Proof. Since a Noetherian local ring has finite dimension, each prime ideal of A has finite height. Since $h < \dim(A)$, there exist an integer $n > h$, a prime ideal \mathfrak{p} with $\text{ht}(\mathfrak{p}) = n$, and a chain $\mathfrak{p}_0 \subset \cdots \subset \mathfrak{p}_n = \mathfrak{p}$ of prime ideals of A . Then $\text{ht}(\mathfrak{p}_i) = i$ for each i , so $\text{ht}(\mathfrak{r}) = h$ for every prime ideal \mathfrak{r} such that $\mathfrak{p}_{h-1} \subset \mathfrak{r} \subset \mathfrak{p}_{h+1}$. By ?? the set E of such prime ideals \mathfrak{r} is infinite, whence (a).

If A is of finite dimension, then we may take $n = \dim(A)$ in the above proof. Then for each $\mathfrak{r} \in E$, we have $\text{ht}(\mathfrak{r}) = h$ and $\text{coht}(\mathfrak{r}) \leq n - h$. Since $\mathfrak{r} \subset \mathfrak{p}_{h+1} \subset \cdots \subset \mathfrak{p}_n$ is a chain of length $n - h$, we then get $\text{coht}(\mathfrak{r}) = \dim(A) - h$. \square

Example 15.2.24. There exist non-Noetherian integral domains of dimension 2 with only one prime ideal of height 1, for example the ring of a valuation of rank 2.

15.2.3 Extension of scalars

Proposition 15.2.25. Let $\rho : A \rightarrow B$ be a local homomorphism of Noetherian local rings, M a finitely generated A -module and N a finitely generated B -module. If $\bar{B} = B \otimes_A \kappa_A$ and $\bar{N} = N \otimes_B \bar{B}$, then

$$\dim_B(M \otimes_A N) \leq \dim_A(M) + \dim_{\bar{B}}(\bar{N})$$

and the equality holds if N is flat over A .

Proof. Let $S \subseteq \mathfrak{m}_A$ be a system of parameters for the A -module M and $T \subseteq \mathfrak{m}_B$ a system of parameters for the \bar{B} -module \bar{N} . Let E be the B -module $M \otimes_A N$. Since ρ is local, we have $\rho(S) \subseteq \mathfrak{m}_B$, and

$$\begin{aligned} E/(\rho(S)E + TE) &= E \otimes_B (B/(\rho(S)B + TB)) = M \otimes_A N \otimes_B (B/TB) \otimes_A (A/SA) \\ &= M \otimes_A (A/SA) \otimes_A N \otimes_B (B/TB) = (M/SM) \otimes_A (N/TN) \end{aligned}$$

Moreover, by ??, we have

$$\text{supp}(M/SM) = \{\mathfrak{m}_A\}, \quad \text{supp}(\bar{N}/T\bar{N}) = \{\mathfrak{m}_B\}$$

which implies, according to ?? and ??, that

$$\begin{aligned} \text{supp}((M/SM) \otimes_A (N/TN)) &= \text{supp}(N/TN) \cap \rho^{*-1}(\{\mathfrak{m}_A\}) \\ &= \text{supp}((N/TN) \otimes_A \kappa_A) = \text{supp}(\bar{N}/T\bar{N}) = \{\mathfrak{m}_B\}. \end{aligned}$$

Therefore the B -module $E/(\rho(S)E + TE)$ has finite length, which implies by ?? that

$$\dim_B(E) \leq |S| + |T| = \dim_A(B) + \dim_{\bar{B}}(\bar{N}).$$

Suppose now that N is flat over A . Let \mathfrak{a} (resp. \mathfrak{b}) be the annihilator of M (resp. N). Then we have

$$\text{supp}(E) = \text{supp}(N) \cap \rho^{*-1}(\text{supp}(N)) = V(\mathfrak{b}) \cap \rho^{*-1}(V(\mathfrak{a})) = V(\mathfrak{b} + \mathfrak{a}B),$$

and therefore $\dim_B(M \otimes_A N) = \dim(B/(\mathfrak{b} + \mathfrak{a}B))$. On the other hand,

$$\dim_A(M) + \dim_{\bar{B}}(\bar{N}) = \dim(A/\mathfrak{a}) + \dim(B/(\mathfrak{b} + \mathfrak{m}_A B)).$$

Let $A' = A/\mathfrak{a}$ and $B' = B/(\mathfrak{b} + \mathfrak{a}B)$ and let $\rho' : A' \rightarrow B'$ be the local homomorphism deduced from ρ by passing to quotients. Since the annihilator of N is \mathfrak{b} , N is a finitely generated B/\mathfrak{b} -module with support $\text{Spec}(B/\mathfrak{b})$, and flat on A . By ?? and ??, the local homomorphism $A \rightarrow B/\mathfrak{b}$ deduced from ρ therefore has the property (PM). By extension of the scalars, we deduce that ρ' has the property (PM). According to ??, we have

$$\dim(B') \geq \dim(A') + \dim(B'/\mathfrak{m}_{A'} B')$$

and as the ring $B'/\mathfrak{m}_{A'}B'$ is isomorphic to $B/(\mathfrak{b} + \mathfrak{m}_A B)$, our assertion follows. \square

Corollary 15.2.26. *Let $\rho : A \rightarrow B$ be a local homomorphism of Noetherian local rings.*

- (a) *We have $\dim(B) \leq \dim(A) + \dim(B \otimes_A \kappa_A)$, and the equality holds if B is flat over A .*
- (b) *Suppose that B is flat over A , then a subset S of \mathfrak{m}_A is secant for A if and only if $\rho(S)$ is secant for B .*

Proof. The assertion in (a) follows from ??, by setting $M = A$, $N = B$. For (b), if B is flat over A , then

$$\dim(B) = \dim(A) + \dim(\bar{B})$$

where $\bar{B} = B \otimes_A \kappa_A$. Since ρ is injective by ??, we have $|\rho(S)| = |S|$. Finally $B' = B/\rho(S)B$ is a flat module on $A' = A/SA$, hence

$$\dim(B') = \dim(A') + \dim(\bar{B})$$

since $B'/\mathfrak{m}_{A'}B'$ is isomorphic to \bar{B} . The claim in (b) then follows from these equalities. \square

Corollary 15.2.27. *Let $\rho : A \rightarrow B$ be a ring homomorphism of Noetherian rings. Then*

$$\dim(B) \leq \dim(A) + \sup_{\mathfrak{p} \in \text{Spec}(A)} \dim(B \otimes_A \kappa(\mathfrak{p})).$$

Proof. Let \mathfrak{P} be a prime ideal of B and $\mathfrak{p} = \mathfrak{P}^c$. By ??, we have

$$\dim(B_{\mathfrak{P}}) \leq \dim(A_{\mathfrak{p}}) + \dim(B_{\mathfrak{P}} \otimes_A \kappa(\mathfrak{p})) \leq \dim(A) + \dim(B \otimes_A \kappa(\mathfrak{p}))$$

taking supremum on \mathfrak{P} , we get the claim. \square

Corollary 15.2.28. *Let $\rho : A \rightarrow B$ be a ring homomorphism of Noetherian rings and $\mathfrak{p} \subset \mathfrak{q}$ be prime ideals of A .*

- (a) *If the going-up property holds for ρ , then*

$$\dim(B \otimes_A \kappa(\mathfrak{p})) \leq \dim(B \otimes_A \kappa(\mathfrak{q})).$$

- (b) *If the going-down property holds for ρ , then*

$$\dim(B \otimes_A \kappa(\mathfrak{p})) \geq \dim(B \otimes_A \kappa(\mathfrak{q})).$$

Proof. First assume the going-up property for ρ . Let $\mathfrak{P}_0 \subset \dots \subset \mathfrak{P}_r$ be a chain of prime ideals of B lying over \mathfrak{p} and $\mathfrak{p} = \mathfrak{p}_0 \subset \dots \subset \mathfrak{p}_s = \mathfrak{q}$ be a chain of prime ideals of A . Then by the going-up property, there is a chain $\mathfrak{P}_r \subset \dots \subset \mathfrak{P}_{r+s}$ of prime ideals of B such that $\mathfrak{P}_{r+i}^c = \mathfrak{p}_i$. If $\mathfrak{Q} := \mathfrak{P}_{r+s}$, then \mathfrak{Q} is lying over \mathfrak{q} and $\text{ht}(\mathfrak{Q}/\mathfrak{p}^e) \geq r+s$. Applying ?? to the homomorphism $(A/\mathfrak{p})_{\mathfrak{q}} \rightarrow (B/\mathfrak{p}^e)_{\mathfrak{Q}}$, we get

$$r+s \leq \text{ht}(\mathfrak{Q}/\mathfrak{p}^e) \leq \text{ht}(\mathfrak{q}/\mathfrak{p}) + \dim(B_{\mathfrak{Q}}/\mathfrak{q}B_{\mathfrak{Q}}) \leq s + \dim(B \otimes_A \kappa(\mathfrak{q}))$$

whence the claim in (a).

Now we prove (b), so assume the going down property for ρ . We may assume that $\text{ht}(\mathfrak{q}/\mathfrak{p}) = 1$, and it is enough to prove that, given a chain $\mathfrak{Q}_0 \subset \dots \subset \mathfrak{Q}_t$ of prime ideals of B lying over \mathfrak{q} such that $\text{ht}(\mathfrak{Q}_i/\mathfrak{Q}_{i-1}) = 1$, we can construct a chain of prime ideals $\mathfrak{P}_0 \subset \dots \subset \mathfrak{P}_t$ of B lying over \mathfrak{p} such that $\mathfrak{P}_i \subset \mathfrak{Q}_i$ and $\text{ht}(\mathfrak{P}_i/\mathfrak{P}_{i-1}) = 1$ for all i .

The existence of \mathfrak{P}_0 is guaranteed by the going-down property, and if $\mathfrak{P}_0, \dots, \mathfrak{P}_i$ is constructed, take $x \in \mathfrak{q} \setminus \mathfrak{p}$ and let $\mathfrak{T}_1, \dots, \mathfrak{T}_s$ be the minimal prime ideals belonging to $\mathfrak{P}_i + xB$. Then by ?? we have $\text{ht}(\mathfrak{T}_j/\mathfrak{P}_i) = 1$, while $\text{ht}(\mathfrak{Q}_{i+1}/\mathfrak{P}_i) \geq 2$, so there exists an element $y \in \mathfrak{Q}_{i+1} \setminus \bigcup_{j=1}^s \mathfrak{T}_j$. Let \mathfrak{P}_{i+1} be a minimal prime ideal of $\mathfrak{P}_i + yB$, then $\mathfrak{P}_{i+1} \neq \mathfrak{T}_j$ for all j and ?? shows that $\text{ht}(\mathfrak{P}_{i+1}/\mathfrak{P}_i) = 1$, hence $\rho(x) \notin \mathfrak{P}_{i+1}$. Since $x \in \mathfrak{q}$ and $\text{ht}(\mathfrak{q}/\mathfrak{p}) = 1$, we must have $\mathfrak{P}_{i+1}^c = \mathfrak{p}$, which finishes the induction process. \square

Corollary 15.2.29. *Let A be a Noetherian ring and n a positive integer. Then*

$$\dim(A[X_1, \dots, X_n]) = \dim(A) + n.$$

Proof. Let $B = A[X_1, \dots, X_n]$, then for any prime ideal \mathfrak{p} of A , $B \otimes_A \kappa(\mathfrak{p})$ is identified with the polynomial ring on $\kappa(\mathfrak{p})$ with indeterminates X_1, \dots, X_n , whence of dimension n . By ?? we then have $\dim(B) \leq \dim(A) + n$, and the reverse inequality is already established. \square

Corollary 15.2.30. *Let $\rho : A \rightarrow B$ be a homomorphism of Noetherian rings and $\rho^* : \text{Spec}(B) \rightarrow \text{Spec}(A)$ be the induced map.*

(a) *If ρ^* is surjective, then $\text{ht}(\mathfrak{a}^e) \leq \text{ht}(\mathfrak{a})$ for any ideal \mathfrak{a} of A .*

(b) *If B is a faithfully flat A -module, then $\text{ht}(\mathfrak{a}^e) = \text{ht}(\mathfrak{a})$ for any ideal \mathfrak{a} of A .*

Proof. If B is faithfully flat, then ρ^* is surjective by ?? and we have $\text{ht}(\mathfrak{a}) \leq \text{ht}(\mathfrak{a}^e)$ by ?. Therefore it suffices to prove assertion (a). Assume that ρ^* is surjective and let \mathfrak{p} be a prime ideal of A containing \mathfrak{a} such that $\text{ht}(\mathfrak{p}) = \text{ht}(\mathfrak{a})$. Let \mathfrak{P} be a prime ideal of B lying over \mathfrak{p} whose image in $B \otimes \kappa(\mathfrak{p})$ is minimal. Then $\dim(B_{\mathfrak{P}} \otimes \kappa(\mathfrak{p})) = 0$ and ?? implies $\dim(B_{\mathfrak{P}}) \leq \dim(A_{\mathfrak{p}})$, whence

$$\text{ht}(\mathfrak{a}^e) \leq \text{ht}(\mathfrak{P}) = \dim(B_{\mathfrak{P}}) \leq \dim(A_{\mathfrak{p}}) = \text{ht}(\mathfrak{p}) = \text{ht}(\mathfrak{a}).$$

This proves the claim. \square

Proposition 15.2.31. *Let A be a Noetherian ring, \mathfrak{a} an ideal of A , and M a finitely generated A -module. Let \widehat{A} (resp. \widehat{M}) be the Hausdorff completions of A (resp. M) under the \mathfrak{a} -adic topology.*

(a) *Let \mathfrak{p} be a prime ideal of A containing \mathfrak{a} . Then $\widehat{\mathfrak{p}}$ is a prime ideal of \widehat{A} and $\dim_{\widehat{A}_{\widehat{\mathfrak{p}}}}(\widehat{M}_{\widehat{\mathfrak{p}}}) = \dim_{A_{\mathfrak{p}}}(M_{\mathfrak{p}})$.*

(b) *We have $\dim_{\widehat{A}}(\widehat{M}) = \sup_{\mathfrak{m}} \dim_{\widehat{A}_{\widehat{\mathfrak{p}}}}(\widehat{M}_{\widehat{\mathfrak{p}}}) = \sup_{\mathfrak{m}} \dim_{A_{\mathfrak{m}}}(M_{\mathfrak{m}})$, where \mathfrak{m} runs through prime (resp. maximal) ideals of A containing \mathfrak{a} . In particular, $\dim_{\widehat{A}}(\widehat{M}) \leq \dim_A(M)$.*

Proof. Since the \mathfrak{a} -adic topology on A/\mathfrak{p} is trivial, by ?? $\widehat{A}/\widehat{\mathfrak{p}}$ is identified with A/\mathfrak{p} , so $\widehat{\mathfrak{p}}$ is a prime ideal of \widehat{A} . Since \widehat{A} is flat over A , $\widehat{A}_{\widehat{\mathfrak{p}}}$ is flat over $A_{\mathfrak{p}}$. Moreover the canonical homomorphism of A in \widehat{A} induces an isomorphism of A/\mathfrak{a} on $\widehat{A}/\widehat{\mathfrak{a}}$, therefore also an isomorphism of $A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$ on $\widehat{A}_{\widehat{\mathfrak{p}}}/\mathfrak{p}\widehat{A}_{\widehat{\mathfrak{p}}}$. We conclude assertion (a) by applying ?? to the homomorphism $A_{\mathfrak{p}} \rightarrow \widehat{A}_{\widehat{\mathfrak{p}}}$ and to the modules $M_{\mathfrak{p}}$ and $\widehat{A}_{\widehat{\mathfrak{p}}}$, noting that $M_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} \widehat{A}_{\widehat{\mathfrak{p}}}$ is isomorphic to $\widehat{M}_{\widehat{\mathfrak{p}}}$.

By ??, the map $\mathfrak{m} \mapsto \hat{\mathfrak{m}}$ is a bijection from the set of maximal ideals of A containing \mathfrak{a} to the set of maximal ideals of \widehat{A} . Thus assertion (b) follows from ??.

Corollary 15.2.32. *Let A be a Noetherian Zariski ring. Then for any finitely generated A -module M , we have $\dim_A(M) = \dim_{\widehat{A}}(\widehat{M})$.*

Proof. The topology on A is \mathfrak{a} -adic, where \mathfrak{a} is an ideal contained in the Jacobson radical of A . Thus the claim follows from ??.

Corollary 15.2.33. *Let A be a Noetherian ring, \mathfrak{a} an ideal of A , and \widehat{A} the Hausdorff completion with respect to the \mathfrak{a} -adic topology. Then $\dim(\widehat{A}) \leq \dim(A)$, and the equality holds if A is local and \mathfrak{a} is proper.*

Corollary 15.2.34. *Let A be a Noetherian ring and n a positive integer. Then*

$$\dim(A[\![X_1, \dots, X_n]\!]) = \dim(A) + n.$$

Proof. The ring $A[\![X_1, \dots, X_n]\!]$ is the Hausdorff completion of $A[X_1, \dots, X_n]$ with respect to the \mathfrak{m} -adic topology, where \mathfrak{m} is generated by X_1, \dots, X_n . We then have

$$\dim(A[\![X_1, \dots, X_n]\!]) \leq \dim(A[X_1, \dots, X_n]) = \dim(A) + n$$

and the reverse inequality is already established. \square

Corollary 15.2.35. *Let A be a Noetherian ring and \mathfrak{a} an ideal of A . Suppose that A is Hausdorff and complete under the \mathfrak{a} -adic topology. Then for any positive integer n , we have*

$$\dim(A\{X_1, \dots, X_n\}) = \dim(A) + n.$$

Proof. The ring $A\{X_1, \dots, X_n\}$ is the \mathfrak{a} -adic completion of $A[X_1, \dots, X_n]$, so

$$\dim(A\{X_1, \dots, X_n\}) \leq \dim(A) + n.$$

The reverse inequality can be established just like the power series ring. \square

Corollary 15.2.36. *Let A be a Noetherian local ring, identified with a subring of its completion \widehat{A} , and B a subring of \widehat{A} containing A . Suppose that B is a Noetherian local ring and that we have $\mathfrak{m}_A B = \mathfrak{m}_B$. Then $\dim(A) = \dim(B)$.*

Proof. By ?? the canonical injection of A into B extends to an isomorphism of \widehat{A} on the completion \widehat{B} of B , whence $\dim(B) = \dim(A)$. \square

15.2.4 Exersise

Exercise 15.2.1. Let k be a field, and set

$$A = k[x, y], \quad B = [x, y, x/y], \quad \mathfrak{p} = (x, y)A, \quad \mathfrak{q} = (y, x/y)B$$

check that

$$\mathfrak{p} = \mathfrak{q} \cap A, \quad \text{ht}\mathfrak{p} = \text{ht}\mathfrak{q} = 2, \quad \dim B_{\mathfrak{q}}/\mathfrak{p}B_{\mathfrak{q}} = 1$$

so that

$$\text{ht}\mathfrak{q} < \text{ht}\mathfrak{p} + \dim B_{\mathfrak{q}}/\mathfrak{p}B_{\mathfrak{q}}$$

Show also by a concrete example that the going-down property does not hold between A and B .

Proof. Clearly we can see $(x, y)A \subseteq (y, x/y)B$ so that $\mathfrak{p} \subseteq \mathfrak{q} \cap A$. Also, \mathfrak{p} is a maximal ideal in A , hence $\mathfrak{p} = \mathfrak{q} \cap A$.

We have $(y) = (y, x/y)$ in B , so

$$(y, x/y) \supset (y, x) \supset (0)$$

is a maximal chain of \mathfrak{q} . For \mathfrak{p} , it is easy to verify $\text{height } \mathfrak{p} = 2$.

For $\dim B_{\mathfrak{q}}/\mathfrak{p}B_{\mathfrak{q}}$, note that

$$(x, y)B \subset (y, x/y)B$$

Since (x, y) is maximal in A , if there is an ideal Q sandwiched between them, then Q contains a polynomial in x/y , hence contains x/y since Q is prime. This implies $\dim B_{\mathfrak{q}}/\mathfrak{p}B_{\mathfrak{q}} = 1$.

Consider the following chain in A :

$$P_1 = (y)A \subset (x, y)A = P_2$$

And choose a prime $Q_2 = (x, y)B$ in B such that $Q_2^c = P_2$. If Q_1 is a prime ideal in B such that $Q_1^c = P_1$, then $(y)A \subseteq Q_1$, in particular $y \in Q_1$. Note that since $x/y \in B$, this implies $x = (x/y) \cdot y \in Q_1$. But this means $Q_2 \subseteq Q_1$, hence there is no prime ideal contained in Q_2 satisfying the condition, therefore the going down property does not hold. \square

Exercise 15.2.2. Let A be a ring, U an open subset of $\text{Spec}(A)$.

- (a) Show that the map $\mathfrak{p} \mapsto V(\mathfrak{p}) \cap U$ induce a bijection of U onto the set of irreducible closed subsets of U .
- (b) Show that $\dim(U)$ is the supremum of the length of chains of prime ideals of A contained in U .
- (c) Let Φ be the set of minimal prime ideals of A contained in U . Show that

$$\dim(U) = \sup_{\mathfrak{p} \in \Phi} \dim(V(\mathfrak{p}) \cap U)$$

15.3 Hilbert-Samuel series

Let A be an Artinian ring and consider the power series ring $A[[T]]$ with one indeterminate on A . In this section, we denote by $A((T))$ the fraction ring $A[[T]]_T$ of $A[[T]]$ at the element T , which is identified with the subring of the Laurent series ring $A[[T, T^{-1}]]$ consisting of elements $\sum_{n \in \mathbb{Z}} a_n T^n$ that is **bounded below**, that is, there exist $n_0 \in \mathbb{N}$ such that $a_n = 0$ for all $n < n_0$. The ring $\mathbb{Z}((T))$ will be our main focus in this section.

For any $n, p \in \mathbb{Z}$, we extend the definition of the binomial coefficient $\binom{n}{p}$ by defining $\binom{n}{p} = 0$ if $p < 0$ or $p > n$.

Lemma 15.3.1. *The element $(1 - T)$ is invertible in $\mathbb{Z}((T))$, and for any $r > 0$ we have*

$$(1 - T)^{-r} = \sum_{n \in \mathbb{Z}} \binom{n+r-1}{r-1} T^n = \sum_{n \in \mathbb{N}} \binom{n+r-1}{r-1} T^n.$$

Proof. The element $1 - T$ is invertible with inverse $\sum_{n \in \mathbb{N}} T^n$, so

$$(1 - T)^{-r} = \left(\sum_{n \in \mathbb{N}} T^n \right)^r = \sum_{n_1, \dots, n_r \in \mathbb{N}} T^{n_1 + \dots + n_r} = \sum_{n \in \mathbb{N}} \binom{n+r-1}{r-1} T^n$$

so the claim follows. \square

Let $Q(T) \in \mathbb{Z}[T, T^{-1}]$ and r be a positive integer. If $P(T) = (1 - T)^{-r}Q(T)$, then it is easy to see $P(T) \in \mathbb{Z}((T))$. In fact, if

$$Q(T) = \sum_{n \in \mathbb{Z}} a_n T^n, \quad P = \sum_{n \in \mathbb{Z}} b_n T^n$$

then

$$b_n = \sum_{i \in \mathbb{Z}} a_i \binom{n-i+r-1}{r-1} = \sum_{i \leq n} a_i \binom{n-i+r-1}{r-1}$$

where the summation is finite since $Q(T)$ is bounded below. If n_0 is the supremum of the integers $i \in \mathbb{Z}$ such that $a_i \neq 0$, then for $n \geq n_0$, we can write

$$b_n = \sum_{i \in \mathbb{Z}} a_i \binom{n-i+r-1}{r-1} = \frac{1}{(r-1)!} \sum_{i \in \mathbb{Z}} a_i \prod_{j=1}^{r-1} (n-i+j).$$

If $c = Q(1) = \sum_i a_i$, then we get

$$b_n = c \frac{n^{r-1}}{(r-1)!} + \rho_n n^{r-2} \tag{15.3.1}$$

where the rational number ρ_n tends to a limit as n increases indefinitely. Therefore we deduce the relationship

$$Q(1) = (r-1)! \lim_{n \rightarrow \infty} n^{1-r} b_n. \tag{15.3.2}$$

For two elements $F = \sum_n a_n T^n$ and $G = \sum_n b_n T^n$ in $\mathbb{Z}((T))$, we denote $F \leq G$ by the relation $a_n \leq b_n$ for all n , which is an order relation compatible with the ring structure of $\mathbb{Z}((T))$. We have $(1 - T)^{-r} \geq 1$, and if an element Q is bigger than 0, then the integer $Q(1)$ is positive.

Lemma 15.3.2. *Let P be a nonzero element in $\mathbb{Z}((T))$ such that $(1 - T)^r P \in \mathbb{Z}[T, T^{-1}]$ for some $r \in \mathbb{Z}$. Then there exists a unique $Q \in \mathbb{Z}[T, T^{-1}]$ such that $P = (1 - T)^{-d} Q$ and $Q(1) \neq 0$, for some $d \in \mathbb{Z}$. If $P \geq 0$, then $Q(1) \geq 0$.*

Proof. Since $(1 - T)^r P \in \mathbb{Z}[T, T^{-1}]$, we can write $P = (1 - T)^{-1} T^n R(T)$ with $R(T) \in \mathbb{Z}[T]$, and by Euclidean division we may assume that $R(1) \neq 0$, which proves the existence of d and Q . On the other hand, if we have

$$(1 - T)^r Q(T) = (1 - T)^s R(T)$$

with $r > s$ and $Q, R \in \mathbb{Z}[T, T^{-1}]$, then $R(T) = (1 - T)^{r-s}Q(T)$, whence $R(1) = 0$. This proves the uniqueness part.

Finally, assume that $P \geq 0$. If $P = (1 - T)^{-d}Q$ with $d < 0$ then $P(1) = 0$ and we must have $P = 0$, which is a contradiction. Therefore $d \geq 0$, and if $d = 0$ then $P(1) = Q(1) \geq 0$. If $d > 0$, then $Q(1)$ is given by the formula (??), so $Q(1) \geq 0$. \square

Lemma 15.3.3. *Let P, Q be elements of $\mathbb{Z}[T, T^{-1}]$, $p, q \in \mathbb{Z}$ and $P(1) > 0$. If*

$$(1 - T)^{-p}P \leq (1 - T)^{-q}Q$$

then, either $p < q$, or $p = q$ and $P(1) \leq Q(1)$.

Proof. Suppose $p \geq q$, then we have $(1 - T)^{-p}[(1 - T)^{p-q}Q - P] \geq 0$. Since $R = (1 - T)^{p-q}Q - P \in \mathbb{Z}[T, T^{-1}]$, by ?? we have $R(1) \geq 0$. If $p > q$, then $R(1) = -P(1) \geq 0$, which is a contradiction. If $p = q$, then $R(1) = Q(1) - P(1)$, whence $P(1) \leq Q(1)$. \square

15.3.1 Poincaré series of graded modules over polynomial rings

Let A_0 be an Artinian ring, I a finite set and consider the ring $A = A_0[(X_i)_{i \in I}]$. For each $i \in I$, let d_i be a positive integer. Endow A with the graded ring structure of type \mathbb{Z} such that the elements of A_0 are homogeneous of degree 0 and each X_i is homogeneous of degree d_i . If $d_i = 1$ for each i , we then get the usual graduation on the polynomial ring A .

Let M be a finitely generated graded A -module such that all the homogeneous components of M are A_0 -module of finite length. The **Poincaré series** of M , denoted by P_M , is the element in $\mathbb{Z}((T))$ defined by

$$P_M = \sum_{n \in \mathbb{Z}} \ell_{A_0}(M_n) \cdot T^n.$$

Theorem 15.3.4. *The element $Q_M = P_M \prod_{i \in I} (1 - T^{d_i})$ is in $\mathbb{Z}[T, T^{-1}]$.*

Proof. If $I = \emptyset$, then $A = A_0$, and the family $(\ell_{A_0}(M_n))_{n \in \mathbb{Z}}$ has finite support since M is a finitely generated A_0 -module, whence has finite length. The theorem is therefore proved in this case.

Now we prove by induction on the cardinality of I . Fix $j \in I$, $J = I \setminus \{j\}$, and denote by A' the subring of A generated by A_0 and the X_i with $i \in J$. Consider the homothety h_{X_j} with ratio X_j with kernel R and cokernel S . Then for each $n \in \mathbb{Z}$, we have an exact sequence

$$0 \longrightarrow R_{n-d_j} \longrightarrow M_{n-d_j} \xrightarrow{h_{X_j}} M_n \longrightarrow S_n \longrightarrow 0$$

whence R_n and S_n have finite length, and we have

$$\ell_{A_0}(M_n) - \ell_{A_0}(M_{n-d_j}) = \ell_{A_0}(S_n) - \ell_{A_0}(R_{n-d_j}). \quad (15.3.3)$$

Since M is a finitely generated module over the Noetherian A , it is Noetherian so the A -modules R and S are finitely generated. Since they are both annihilated by X_j , they are finitely generated as A' -modules. According to the induction hypothesis, the elements Q_R and Q_S are in $\mathbb{Z}[T, T^{-1}]$. Moreover, due to (??), we have

$$(1 - T^{d_j})P_M = P_M - T^{d_j}P_M = P_S - T^{d_j}P_R \quad (15.3.4)$$

which implies

$$Q_M = P_M \prod_{i \in I} (1 - T^{d_i}) = P_S \prod_{i \in J} (1 - T^{d_i}) - T^{d_j} P_R \prod_{i \in J} (1 - T^{d_i}).$$

This shows Q_M is in $\mathbb{Z}[T, T^{-1}]$, and our conclusion is then proved. \square

Example 15.3.5. Let M_0 be an A_0 -module and $M = A \otimes_{A_0} M_0$. Then with the notations above, we have $R = 0$ and $S = A' \otimes_{A_0} M_0$, so $Q_M = Q_S$ by (??). Since $Q_{M_0} = \ell_{A_0}(M_0)$, by induction $Q_M = \ell_{A_0}(M_0)$, whence

$$P_M = \ell_{A_0}(M_0) \prod_{i \in I} (1 - T^{d_i})^{-1}.$$

Corollary 15.3.6. Assume that $d_i = 1$ for all $i \in I$ and set $r = |I|$, $c_M = Q_M(1)$.

(a) If $r = 0$, then $\ell_{A_0}(M) = c_M$.

(b) If $r = 1$, then $\ell_{A_0}(M_n) = c_M$ for n large enough.

(c) If $r > 1$, then $\ell_{A_0}(M_n) = \frac{c_M}{(r-1)!} n^{r-1} + \rho_n n^{r-2}$, where ρ_n has a limit when $n \rightarrow +\infty$.

Proof. In this case we have $P_M = Q_M(1 - T)^{-r}$, so this follows from the formula (??). \square

For simplicity, we may endow A with the usual graduation henceforth, so we have

$$P_M = Q_M(1 - T)^{-r}.$$

Example 15.3.7 (Example of Poincaré series for graded modules).

(a) Let $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ be an exact sequence of graded A -modules and of homomorphisms of degree 0 such that M is finitely generated over A and M_n of finite length over A_0 for each n . Then by the additivity of length, we have

$$P_M = P_{M'} + P_{M''}, \quad Q_M = Q_{M'} + Q_{M''}, \quad c_M = c_{M'} + c_{M''}.$$

(b) Let $M(p)$ be the modulu deduced from M by shifting p from the graduation of M . Since $M(p)_n = M_{p+n}$, we have

$$P_{M(p)} = T^{-p} P_M, \quad Q_{M(p)} = T^{-p} Q_M, \quad c_{M(p)} = c_M.$$

(c) Let M be a free graded A -module generated by linearly independent homogeneous elements of degrees $\delta_1, \dots, \delta_s$. As M is isomorphic to $A(-\delta_1) \oplus \dots \oplus A(-\delta_s)$, we have

$$P_M = \ell(A_0) \left(\sum_{i=1}^s T^{\delta_i} \right) (1 - T)^{-r}, \quad Q_M = \ell(A_0) \left(\sum_{i=1}^s T^{\delta_i} \right), \quad c_M = s \cdot \ell(A_0).$$

(d) Let M be a finitely generated A -module and suppose that there exists a long exact sequence of graded A -modules and of homomorphisms of degree 0

$$0 \longrightarrow L_n \longrightarrow L_{n-1} \longrightarrow \dots \longrightarrow L_0 \longrightarrow M \longrightarrow 0$$

such that for each k , the module L_k is free and generated by linearly independent homogeneous elements of degrees $\delta_{k,1}, \dots, \delta_{k,m_k}$. Then by (c),

$$Q_M = \ell(A_0) \sum_{k=0}^n \sum_{j=1}^{m_k} (-1)^k T^{\delta_{k,j}}, \quad c_M = \ell(A_0) \sum_{k=0}^n (-1)^k m_k.$$

Proposition 15.3.8. *Let M be a graded A -module. If M is generated by M_0 and M_0 is an A_0 -module with finite length. Then we have*

$$P_M \leq (1 - T)^{-r} \ell_{A_0}(M_0), \quad c_M \leq \ell_{A_0}(M_0).$$

Moreover, the following conditions are equivalent:

- (i) $c_M = \ell_{A_0}(M_0)$.
- (ii) $P_M = \ell_{A_0}(M_0)(1 - T)^{-r}$.
- (iii) The canonical homomorphism $\varphi : A \otimes_{A_0} M_0 \rightarrow M$ is bijective.

Proof. Let R be the kernel of φ . Then since φ is surjective, we have

$$P_M = P_{A \otimes_{A_0} M_0} - P_R = \ell_{A_0}(M_0)(1 - T)^{-r} - P_R, \quad c_M = \ell_{A_0}(M_0) - c_R$$

which proves the first assertion. Now the conditions (i), (ii), and (iii) are equivalent to $c_R = 0$, $P_R = 0$, and $R = 0$, respectively, so we have (iii) \Rightarrow (ii) \Rightarrow (i). It remains to show that $c_R = 0$ implies $R = 0$. Now assume that $R \neq 0$ and let

$$0 = M^r \subset M^{r-1} \subset \cdots \subset M^0 = M_0$$

be a Jordan-Hölder series of the A_0 -module M_0 . Let R^k be the intersection of R and the image of $A \otimes_{A_0} M^k$ in $A \otimes_{A_0} M_0$. Then there exists an integer k such that $R^k \neq R^{k-1}$. Define $L = R^{k-1}/R^k$, we have $0 \leq c_L \leq c_R$, and it suffices to prove that $c_L \neq 0$. Now, if K is the quotient field of A_0 by the maximal ideal annihilating M^{k-1}/M^k , then L is identified with a nonzero graded submodule of $K[(X_i)_{i \in I}]$. So L contains a submodule isomorphic to a shifted module of $K[(X_i)_{i \in I}]$. As $c_{K[(X_i)_{i \in I}]} = 1$, we have $c_L \geq 1$, which proves the claim. \square

Proposition 15.3.9. *Suppose that A_0 is a field and M is a finitely generated graded A -module. If K is the field of fraction of A , then c_M is equal to the rank of the A -module M , which is also the dimension of the K -vector space $M \otimes_A K$.*

Proof. This is clear if $M = A$, since $c_A = 1$. On the other hand, let $x \in A$, homogeneous of degree d , and not zero. We have $(A/xA) \otimes_A K = 0$. From the exact sequence

$$0 \longrightarrow A(-d) \xrightarrow{h_x} A \longrightarrow A/xA \longrightarrow 0$$

we get $c_{A/xA} = 0$. The proposition is therefore verified when M is generated by a homogeneous element. The general case follows, since any finite type graded A -module has a composition sequence whose quotients are of the previous form. \square

Example 15.3.10. Let A_0 be a field, P a homogeneous polynomial in A of degree s , and M the module A/PA . Since the rank of M equals to the degree of P , by ?? we have $c_M = s$. On the other hand, it is not hard to compute that

$$\ell_{A_0}(M_n) = \binom{n+r-1}{r-1} - \binom{n-s+r-1}{r-1}$$

and therefore

$$c_M = (r-1)! \lim_{n \rightarrow \infty} n^{1-r} \left(\binom{n+r-1}{r-1} - \binom{n-s+r-1}{r-1} \right) = s$$

verifying our claim.

15.3.2 Hilbert-Samuel series of a good-filtered module

Let A be a Noetherian ring, \mathfrak{I} an ideal of A , and M a finitely generated A -module. Recall that an exhaustive filtration $\mathcal{F} = (M_n)_{n \in \mathbb{Z}}$ on M consisting of submodules is called **\mathfrak{I} -good** if

- (a) $\mathfrak{I}M_n \subseteq M_{n+1}$ for all $n \in \mathbb{Z}$.
- (b) There exists an integer n_0 such that $\mathfrak{I}M_n = M_{n+1}$ for all $n \geq n_0$.

Similar to the case of series, we say the filtration \mathcal{F} is **bounded below** if there exist $n_1 \in \mathbb{Z}$ such that $M_n = M$ for all $n \leq n_1$. Note that if the filtration \mathcal{F} is exhaustive and M is finitely generated, then \mathcal{F} is bounded below since M is then Noetherian.

Lemma 15.3.11. *Let (M_n) be an \mathfrak{I} -good filtration on M . If $M/\mathfrak{I}M$ has finite length, then M/M_{n+1} and M_n/M_{n+1} have finite length for all $n \in \mathbb{Z}$.*

Proof. Since \mathcal{F} is bounded below, there exist n_1 such that $M_{n_1} = M$. Also, as (M_n) is \mathfrak{I} -good, there exist n_0 such that $\mathfrak{I}M_n = M_{n+1}$ for all $n \geq n_0$. Then for all $n \in \mathbb{Z}$,

$$\mathfrak{I}^{n-n_1}M \subseteq M_n \subseteq \mathfrak{I}^{n-n_0}M \tag{15.3.5}$$

where we define $\mathfrak{I}^n = A$ for $n \leq 0$. Then $\ell(M/M_n) \leq \ell(M/\mathfrak{I}^{n-n_1}M)$ and it suffices to prove that $\mathfrak{I}^nM/\mathfrak{I}^{n+1}M$ has finite length for each n . So we are reduced to the case of \mathfrak{I} -adic filtration. Let (x_1, \dots, x_r) be a finite generating system of the A -module \mathfrak{I} , and let I be the finite set of monomials of total degree n in r variables X_1, \dots, X_r . The homomorphism from $(M/\mathfrak{I}M)^I$ to $\mathfrak{I}^nM/\mathfrak{I}^{n+1}M$ which maps $(u_m)_{m \in I}$ to $\sum_m m(x_1, \dots, x_r)u_m$ is surjective. Since $M/\mathfrak{I}M$ has finite length, so does $\mathfrak{I}^nM/\mathfrak{I}^{n+1}M$. \square

Suppose henceforth that $M/\mathfrak{I}M$ has finite length. Let $\mathcal{F} = (M_n)$ be an \mathfrak{I} -good filtration on M . We define therefore the **Hilbert-Samuel series** H_M of M with respect to the \mathfrak{I} -good filtration (M_n) by

$$H_{M,\mathcal{F}} = \sum_{n \in \mathbb{Z}} \ell_{A/\mathfrak{I}}(M_n/M_{n+1}) \cdot T^n$$

and the map $n \mapsto \ell_{A/\mathfrak{I}}(M_n/M_{n+1})$ is called the **Hilbert-Samuel function** of M with respect to (M_n) .

Since there exists an integer n_1 such that $M_n = M$ for $n \leq n_1$, we see H_M is an element in $\mathbb{Z}((T))$. We denote by $H_{M,\mathfrak{I}}$ by the Hilbert-Samuel series of M with respect to the \mathfrak{I} -adic filtration, that is,

$$H_{M,\mathfrak{I}} = \sum_{n \in \mathbb{Z}} \ell_{A/\mathfrak{I}}(\mathfrak{I}^n M / \mathfrak{I}^{n+1} M) \cdot T^n.$$

If $P \in \mathbb{Z}((T))$ and $r \in \mathbb{N}$, we may use $P^{(1)}$ to denote the series $(1 - T)^{-r}P$. In other words, if $P = \sum_{n \in \mathbb{Z}} a_n T^n$, then

$$P^{(1)} = \sum_{n \in \mathbb{Z}} \left(\sum_{i \leq n} a_i \right) T^n.$$

Proposition 15.3.12. *Let \mathcal{F} be an \mathfrak{I} -good filtration on M . Then*

$$H_{M,\mathcal{F}}^{(1)} = \sum_{n \in \mathbb{Z}} \ell_A(M/M_{n+1}) \cdot T^n.$$

If \mathcal{F}' is another \mathfrak{I} -good filtration on M , then there exists an integer m such that $H_{M,\mathcal{F}'}^{(1)} \geq T^m H_{M,\mathcal{F}}^{(1)}$.

Proof. Since $\ell_{A/\mathfrak{I}}(M_n/M_{n+1}) = \ell_A(M_n/M_{n+1})$, the first claim follows from the definition of $H_{M,\mathcal{F}}$ and the observation

$$\sum_{i \leq n} \ell_{A/\mathfrak{I}}(M_i/M_{i+1}) = \sum_{i \leq n} \ell_A(M_i/M_{i+1}) = \ell_A(M/M_{n+1}).$$

If $\mathcal{F}' = (M'_n)$ is another filtration, then there exist an integer n_2 such that $M'_n \subseteq \mathfrak{I}^{n-n_2}M$ for all n , whence $M'_n \subseteq M_{n-(n_2-n_1)}$, and the second claim therefore follows. \square

Theorem 15.3.13. *Let A be a Noetherian ring, \mathfrak{I} an ideal of A , M a finitely generated A -module such that $M/\mathfrak{I}M$ is nonzero and has finite length. Let \mathcal{F} be an \mathfrak{I} -good filtration on M , then there exists a positive integer d and a uniquely determined element R in $\mathbb{Z}[T, T^{-1}]$ such that*

$$H_{M,\mathcal{F}} = (1 - T)^{-d}R \text{ and } R(1) > 0.$$

Moreover, the integers d and $R(1)$ are independent of the filtration \mathcal{F} .

Proof. Consider the graded ring $\text{gr}(A)$ and the graded module $\text{gr}(M)$. Since $M_{n_1} = M$ and $\mathfrak{I}M_n = M_{n+1}$ for $n \geq n_0$, $\text{gr}(M)$ is generated by $\bigoplus_{n_1 \leq n \leq n_0} \text{gr}_n(M)$, so is finitely generated. Furthermore, if (x_1, \dots, x_r) is a finite generating system of the A -module \mathfrak{I} , then $\text{gr}(A)$ is generated by $\text{gr}_0(A)$ and the classes of x_i modulo \mathfrak{I}^2 , so is isomorphic to a graded ring quotient of $B = (A/\mathfrak{I})[X_1, \dots, X_r]$. According to ??, we have

$$(1 - T)^r H_{M,\mathcal{F}} \in \mathbb{Z}[T, T^{-1}].$$

Since $H_{M,\mathcal{F}} \neq 0$, by ?? there exist $d \in \mathbb{N}$ and $R \in \mathbb{Z}[T, T^{-1}]$ uniquely determined such that $R(1) > 0$ and $H_{M,\mathcal{F}} = (1 - T)^{-d}R$.

If \mathcal{F}' is another \mathfrak{I} -good filtration on M , then similarly

$$H_{M,\mathcal{F}'} = (1 - T)^{-d'}R'.$$

By ??, there exists an integer m such that

$$(1 - T)^{-d'-1}R' \geq T^m (1 - T)^{-d-1}R$$

and due to ??, this implies $d' \geq d$, or $d' = d$ and $R'(1) \geq R(1)$. Exchange the role of \mathcal{F} and \mathcal{F}' , we then get $d = d'$ and $R'(1) = R(1)$. \square

Remark 15.3.1. In the notations of ??, if $R = \sum_{i \in \mathbb{Z}} a_i T^i$ and suppose that $d > 0$, then the relation $H_{M,\mathcal{F}} = (1 - T)^{-d} R$ implies

$$\ell_{A/\mathfrak{J}}(M_n/M_{n+1}) = \sum_{i \in \mathbb{Z}} a_i \binom{n-i+d-1}{d-1} = \sum_{i \leq n} a_i \binom{n-i+d-1}{d-1}$$

Similarly, the relation $H_{M,\mathcal{F}}^{(1)} = (1 - T)^{-d-1} R$ implies

$$\ell_A(M/M_{n+1}) = \sum_{i \in \mathbb{Z}} a_i \binom{n-i+d}{d} = \sum_{i \leq n} a_i \binom{n-i+d}{d}$$

Let A be a Noetherian ring, \mathfrak{J} an ideal of A , M a finitely generated A -module such that $M/\mathfrak{J}M$ has finite length. If $M \neq \mathfrak{J}M$, then by ?? there exists integers $d_{\mathfrak{J}}(M) \geq 0$ and $e_{\mathfrak{J}}(M) > 0$ such that, for any \mathfrak{J} -good filtration \mathcal{F} on M ,

$$H_{M,\mathcal{F}} = (1 - T)^{-d_{\mathfrak{J}}(M)} R, \quad R(1) = e_{\mathfrak{J}}(M).$$

If $M = \mathfrak{J}M$, then we may set $d_{\mathfrak{J}}(M) = -\infty$ and $e_{\mathfrak{J}}(M) = 0$.

Corollary 15.3.14. Let A be a Noetherian ring, \mathfrak{J} an ideal of A , M a finitely generated A -module such that $M/\mathfrak{J}M$ is nonzero and has finite length. Let \mathcal{F} be an \mathfrak{J} -good filtration on M .

- (a) For $d_{\mathfrak{J}}(M) \leq 0$, it is necessary and sufficient that the sequence $(\mathfrak{J}^n M)$ be stationary, or that the sequence (M_n) is stationary. In this case, we have, for all n large enough,

$$\ell_A(M/M_{n+1}) = \ell_A(M/\mathfrak{J}^{n+1} M) = e_{\mathfrak{J}}(M).$$

- (b) Suppose that $d_{\mathfrak{J}}(M) > 0$, then

$$\ell_{A/\mathfrak{J}}(M_n/M_{n+1}) = e_{\mathfrak{J}}(M) \frac{n^{d_{\mathfrak{J}}(M)-1}}{(d_{\mathfrak{J}}(M)-1)!} + \rho_n n^{d_{\mathfrak{J}}(M)-2} \quad (15.3.6)$$

and

$$\ell_A(M/M_{n+1}) = e_{\mathfrak{J}}(M) \frac{n^{d_{\mathfrak{J}}(M)}}{d_{\mathfrak{J}}(M)!} + \sigma_n n^{d_{\mathfrak{J}}(M)-1}. \quad (15.3.7)$$

where ρ_n and σ_n have limits when $n \rightarrow \infty$.

Proof. This follows from formulas ?? and ??, and the definitions above. \square

Example 15.3.15. Let k be an algebraically closed field and $F \in k[X_1, \dots, X_r]$ be an irreducible polynomial F in A contained in the maximal ideal $\mathfrak{m} = (X_1, \dots, X_r)$, and let m be the largest integer such that $F \in \mathfrak{m}^m$. Set $A = (k[X_1, \dots, X_r]/(F))_{\mathfrak{m}}$ and let $\mathfrak{J} = (\mathfrak{m}/(F))_{\mathfrak{m}}$ be the maximal ideal of A ; we want to determine $d_{\mathfrak{J}}(A)$ and $e_{\mathfrak{J}}(A)$. For this, we first note that $A/\mathfrak{J} = k$ by the Nullstellensatz and

$$0 \longrightarrow \mathfrak{J}^n/\mathfrak{J}^{n+1} \longrightarrow A/\mathfrak{J}^{n+1} \longrightarrow A/\mathfrak{J}^n \longrightarrow 0$$

Therefore we are reduced to compute $\dim_k(A/\mathfrak{J}^n)$. To this end, we first note that

$$A/\mathfrak{J}^n \cong (k[X_1, \dots, X_r]/(F))/(\mathfrak{m}^n/(F)) = k[X_1, \dots, X_n]/(\mathfrak{m}^n + (F)).$$

and we can use the following exact sequence to compute $\dim_k(k[X_1, \dots, X_r]/(\mathfrak{m}^n + (F)))$:

$$0 \rightarrow k[X_1, \dots, X_r]/\mathfrak{m}^{n-m} \xrightarrow{\psi} k[X_1, \dots, X_r]/\mathfrak{m}^n \rightarrow k[X_1, \dots, X_r]/(\mathfrak{m}^n + (F)) \rightarrow 0$$

where ψ is the homomorphism given by $\psi(\bar{G}) = \bar{F}\bar{G}$. Since

$$\dim_k(k[X_1, \dots, X_r]/\mathfrak{m}^n) = \sum_{k=0}^{n-1} \binom{k+r-1}{r-1} \quad (15.3.8)$$

it follows that

$$\dim_k(k[X_1, \dots, X_r]/(\mathfrak{m}^n + (F))) = \sum_{k=n-m}^{n-1} \binom{k+r-1}{r-1}. \quad (15.3.9)$$

Since $\binom{n+r-1}{r-1} = \frac{n^{r-1}}{(r-1)!} + \dots$, from ?? we deduce that

$$\dim_k(k[X_1, \dots, X_r]/(\mathfrak{m}^n + (F))) = m \frac{n^{r-1}}{(r-1)!} + \dots$$

This shows $d_{\mathfrak{J}}(A) = r - 1$ and $e_{\mathfrak{J}}(A) = m$. In the language of algebraic geometry, the ring A is the coordinate ring of the variety $V = V(F)$, and the integer m is called the **multiplicity** of the point $p = (0, \dots, 0)$ in V .

Example 15.3.16. Suppose that \mathfrak{J} is contained in the Jacobson radical of A . Then from the Nakayama lemma, the sequence $(\mathfrak{J}^n M)$ is stationary if and only if we have $\mathfrak{J}^n M = 0$ for n large enough. It then follows from part (a) of ?? that we have $d_{\mathfrak{J}}(M) \leq 0$ if and only if M is of finite length and in this case $e_{\mathfrak{J}}(M) = \ell_A(M)$.

Consider now the special case A is local, $\mathfrak{J} = \mathfrak{m}$ is the maximal ideal of A , and $M = A$. Then by ??, we see $\mathfrak{m}^n = 0$ if and only if A is Artinian. By ??, in this case we have $d_{\mathfrak{m}}(A) = 0$ and $e_{\mathfrak{m}}(A) = \ell_A(A)$.

Furthermore, to provide a concrete example, let consider $A = k[X]/(X^n)$ where k is an algebraically closed field. Then $\mathfrak{m} = (X)/(X^n)$ is the maximal ideal of A and $\mathfrak{m}^n = 0$, so $e_{\mathfrak{m}}(A) = n$.

Proposition 15.3.17. Let A be a Noetherian ring, x_1, \dots, x_r elements of A , \mathfrak{J} the ideal they generate and M a finitely generated A -module such that $M/\mathfrak{J}M$ is non-zero and of finite length.

- (a) We have $d_{\mathfrak{J}}(M) \leq r$, and if $d_{\mathfrak{J}}(M) = r$ then $e_{\mathfrak{J}}(M) \leq \ell_A(M/\mathfrak{J}M)$.
- (c) If (x_1, \dots, x_r) is completely secant for M , then $d_{\mathfrak{J}}(M) = r$ and $e_{\mathfrak{J}}(M) = \ell_A(M/\mathfrak{J}M)$. The converse is also true if \mathfrak{J} belongs to the Jacobson radical of A .

Proof. Let R be the ring $(A/\mathfrak{J})[X_1, \dots, X_r]$, and give $N = \bigoplus_n \mathfrak{J}^n M / \mathfrak{J}^{n+1} M$ be the graded R -module structure for which h_{X_i} is the multiplication by the class $x_i \bmod \mathfrak{J}^2$. Then we have

$$P_N = H_{M,\mathfrak{J}} = (1-T)^{-d_{\mathfrak{J}}(M)} R = (1-T)^{-r} Q_N$$

where $R(1) = e_{\mathfrak{J}}(M) > 0$ and $Q_N(1) = c_N$. By ??, we therefore have either $d_{\mathfrak{J}}(M) < r$ and $c_N = 0$, or $d_{\mathfrak{J}}(M) = r$ and $c_N = e_{\mathfrak{J}}(M)$. Furthermore, according to ??, we have $c_N \leq \ell_A(M/\mathfrak{J}M)$, and the equality holds if and only if canonical homomorphism

$$\varphi : (A/\mathfrak{J})[X_1, \dots, X_r] \otimes_{A/\mathfrak{J}} (M/\mathfrak{J}M) \rightarrow \bigoplus_n \mathfrak{J}^n M / I^{n+1} M$$

is bijective. This then proves the claim, in view of ??.

Proposition 15.3.18. *Let $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ be an exact sequence of finitely generated modules over an Noetherian ring A and \mathfrak{J} be an ideal of A .*

- (a) *For $M/\mathfrak{J}M$ to have finite length, it is necessary and sufficient that $M'/\mathfrak{J}M'$ and $M''/\mathfrak{J}M''$ both have finite length.*
- (b) *Suppose that $M/\mathfrak{J}M$ has finite length. Then there are exactly the following three cases:*
 - (i) $d_{\mathfrak{J}}(M) = d_{\mathfrak{J}}(M') > d_{\mathfrak{J}}(M'')$ and $e_{\mathfrak{J}}(M) = e_{\mathfrak{J}}(M')$.
 - (ii) $d_{\mathfrak{J}}(M) = d_{\mathfrak{J}}(M'') > d_{\mathfrak{J}}(M')$ and $e_{\mathfrak{J}}(M) = e_{\mathfrak{J}}(M'')$.
 - (iii) $d_{\mathfrak{J}}(M) = d_{\mathfrak{J}}(M') = d_{\mathfrak{J}}(M'')$ and $e_{\mathfrak{J}}(M) = e_{\mathfrak{J}}(M') + e_{\mathfrak{J}}(M'')$.

Proof. By ?? and ??, for the module $M/\mathfrak{J}M$ to have finite length, it is necessary and sufficient that the element in

$$\text{supp}(M/\mathfrak{J}M) = \text{supp}(M) \cap V(\mathfrak{J})$$

is maximal. Since $\text{supp}(M) = \text{supp}(M') \cap \text{supp}(M'')$, the assertion in (a) follows.

Now endow M with an \mathfrak{J} -good filtration \mathcal{F} (for example the \mathfrak{J} -adic filtration), M' the submodule filtration \mathcal{F}' , and M'' the quotient filtration \mathcal{F}'' . By ??, the filtrations \mathcal{F}' and \mathcal{F}'' are \mathfrak{J} -good. Then we have for each n an exact sequence of A -modules

$$0 \longrightarrow M'_n / M'_{n+1} \longrightarrow M_n / M_{n+1} \longrightarrow M''_n / M''_{n+1} \longrightarrow 0$$

which implies $H_{M,\mathcal{F}} = H_{M',\mathcal{F}'} + H_{M'',\mathcal{F}''}$, and

$$(1-T)^{-d_{\mathfrak{q}}(M)} R = (1-T)^{-d_{\mathfrak{q}}(M')} R' + (1-T)^{-d_{\mathfrak{q}}(M'')} R''$$

with $R, R', R'' \in \mathbb{Z}[T, T^{-1}]$. Assertion (b) therefore follows. \square

Theorem 15.3.19. *Let A be a Noetherian local ring, \mathfrak{J} a proper ideal of A and M a finitely generated A -module such that $M/\mathfrak{J}M$ is non-zero and of finite length. Then the integer $d_{\mathfrak{J}}(M)$ equals to the dimension of the A -module M .*

Proof. We may suppose that $M \neq 0$. Let (x_1, \dots, x_r) be a system of parameters for M and \mathfrak{x} the ideal they generate. Then by ?? we have $d_{\mathfrak{x}}(M) \leq r$. Since $\mathfrak{x} \subseteq \mathfrak{J}$, we have $H_{M,\mathfrak{J}}^{(1)} \leq H_{M,\mathfrak{x}}^{(1)}$ and therefore ??

$$d_{\mathfrak{J}}(M) \leq d_{\mathfrak{x}}(M) \leq r = \dim_A(M).$$

We now prove the reverse inequality $\dim_A(M) \leq d_{\mathfrak{J}}(M)$ by induction on $\dim_A(M)$. The claim is clear when $\dim_A(M) = 0$. Suppose then we have $\dim_A(M) > 0$, and $\dim_A(N) \leq$

$d_{\mathfrak{J}}(N)$ for any finitely generated A -module N such that $\dim_A(N) < \dim_A(M)$. If

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$$

is a composition sequence of M , we have

$$\dim_A(M) = \sup(\dim_A(M_i/M_{i-1})), \quad d_{\mathfrak{J}}(M) = \sup(d_{\mathfrak{J}}(M_i/M_{i-1}))$$

by ?? and ?. Due to ??, we are now reduced to the case that M is of the form A/\mathfrak{p} , where \mathfrak{p} is a prime ideal of A such that $\mathfrak{p} \neq \mathfrak{m}_A$ (otherwise $\dim(A/\mathfrak{p}) = 0$). Let $x \in \mathfrak{m}_A - \mathfrak{p}$, then the homothety h_x with ratio x is injective on M , and we have an exact sequence

$$0 \longrightarrow M \xrightarrow{h_x} M \longrightarrow M/xM \longrightarrow 0$$

By ??, the element x is secant for M , so we have $\dim_A(M/xM) = \dim_A(M) - 1$. On the other hand, since $e_{\mathfrak{J}}(M/xM) > 0$, from ?? we conclude $d_{\mathfrak{J}}(M/xM) \leq d_{\mathfrak{J}}(M) - 1$. Now by the induction hypothesis,

$$\dim_A(M) = \dim_A(M/xM) + 1 \leq d_{\mathfrak{J}}(M/xM) + 1 \leq d_{\mathfrak{J}}(M)$$

which finishes the induction process. \square

Corollary 15.3.20. *Let A be a Noetherian ring, \mathfrak{J} an ideal of A , M a finitely generated A -module such that $M/\mathfrak{J}M$ is nonzero and has finite length. Then*

- (a) $d_{\mathfrak{J}}(M)$ is the supremum of the integers $\dim_{A_{\mathfrak{m}}}(M_{\mathfrak{m}})$, where \mathfrak{m} runs through the set $\text{supp}(M) \cap V(\mathfrak{J})$.
- (b) $e_{\mathfrak{J}}(M)$ is the sum of the integers $e_{\mathfrak{J}_{\mathfrak{m}}}(M_{\mathfrak{m}})$, where \mathfrak{m} runs through the elements of $\text{supp}(M) \cap V(\mathfrak{J})$ such that $\dim_{A_{\mathfrak{m}}}(M_{\mathfrak{m}}) = d_{\mathfrak{J}}(M)$.

Proof. By ??, the length of $M/\mathfrak{J}^n M$ is the sum of the $\ell_{A_{\mathfrak{m}}}(M_{\mathfrak{m}}/\mathfrak{J}_{\mathfrak{m}}^n M_{\mathfrak{m}})$ where \mathfrak{m} runs through $\text{Ass}(M/\mathfrak{J}^n M)$. Since $M/\mathfrak{J}^n M$ has finite length, the elements of $\text{Ass}(M/\mathfrak{J}^n M)$ are maximal ideals and we have

$$\text{Ass}(M/\mathfrak{J}^n M) = \text{supp}(M/\mathfrak{J}^n M) = \text{supp}(M) \cap V(\mathfrak{J}^n) = \text{supp}(M) \cap V(\mathfrak{J})$$

(?? and ??), so we have

$$H_{M,\mathfrak{J}} = \sum_{\mathfrak{m}} H_{M_{\mathfrak{m}},\mathfrak{J}_{\mathfrak{m}}}$$

where \mathfrak{m} runs through prime ideals in $\text{supp}(M) \cap V(\mathfrak{J})$. The claims then follow from this. \square

Corollary 15.3.21. *Let A be a Noetherian ring, \mathfrak{J} an ideal of A , and M a finitely generated A -module such that $M/\mathfrak{J}M$ has finite length. Let \widehat{A} and \widehat{M} be the Hausdorff completions of A and M under the \mathfrak{J} -adic topology. Then $d_{\mathfrak{J}}(M) = \dim_{\widehat{A}}(\widehat{M})$. If \mathfrak{J} is contained in the Jacobson radical of A then $d_{\mathfrak{J}}(M) = \dim_A(M)$.*

Proof. By ?? and ??, we have

$$\dim_{\widehat{A}}(\widehat{M}) = \sup_{\mathfrak{m} \in V(\mathfrak{J})} \dim_{A_{\mathfrak{m}}}(M_{\mathfrak{m}}) = \sup_{\mathfrak{m} \in \text{supp}(M) \cap V(\mathfrak{J})} \dim_{A_{\mathfrak{m}}}(M_{\mathfrak{m}}) = d_{\mathfrak{J}}(M).$$

If \mathfrak{J} is contained in the Jacobson radical of A then we have $\dim_A(M) = \dim_{\widehat{A}}(\widehat{M})$, whence $d_{\mathfrak{J}}(M) = \dim_A(M)$. \square

Lemma 15.3.22. *Let A be a ring, M an A -module, and $(P_n), (Q_n)$ two filtrations on M of submodules with (Q_n) bounded below. Suppose that $Q_n \subseteq P_n$ and $\ell_A(P_n/Q_n) < +\infty$ for each $n \in \mathbb{Z}$. Then*

$$\begin{aligned} \sum_{n \in \mathbb{Z}} \ell_A((P_{n+1} \cap Q_n)/Q_{n+1})T^n &\leq \sum_{n \in \mathbb{Z}} \ell_A(P_{n+1}/Q_{n+1})T^n \\ &\leq (1-T)^{-1} \sum_{n \in \mathbb{Z}} \ell_A((P_{n+1} \cap Q_n)/Q_{n+1})T^n. \end{aligned} \quad (15.3.10)$$

Proof. It suffices to prove the following inequalities:

$$\ell_A((P_{n+1} \cap Q_n)/Q_{n+1}) \leq \ell_A(P_{n+1}/Q_{n+1}) \leq \sum_{i \leq n} \ell_A((P_{n+1} \cap Q_i)/Q_{i+1}). \quad (15.3.11)$$

The first part is immediate. For the second part, since (Q_n) is bounded below, for i small enough we have $P_{n+1} \cap Q_i = P_{n+1}$, whence

$$\ell_A(P_{n+1}/Q_{n+1}) \leq \sum_{i \leq n} \ell_A((P_{n+1} \cap Q_i)/(P_{n+1} \cap Q_{i+1})).$$

On the other hand, the module $(P_{n+1} \cap Q_i)/(P_{n+1} \cap Q_{i+1})$ is isomorphic to $(P_{n+1} \cap Q_i + Q_{i+1})/Q_{i+1}$, which is a submodule of $(P_{i+1} \cap Q_i)/Q_{i+1}$ when $i \leq n$. From these remarks we see (??) follows. \square

Proposition 15.3.23. *Let A be a ring, M an A -module and \mathcal{F} a bounded below filtration on M consisting of submodules such that M_n/M_{n+1} has finite length for all $n \in \mathbb{Z}$. Let ϕ be an endomorphism of M with kernel M' and cokernel M'' . Endow M' with the submodule filtration \mathcal{F}' and M'' the quotient filtration \mathcal{F}'' . If there exist an integer δ such that $\phi(M_n) \subseteq M_{n+\delta}$ and $\text{gr}(\phi)$ is the associated homomorphism on $\text{gr}(M)$, then we have*

$$H_{M',\mathcal{F}'} \leq P_{\ker \text{gr}(\phi)}, \quad (15.3.12)$$

$$(1-T^\delta)H_{M,\mathcal{F}}^{(1)} + T^\delta P_{\ker \text{gr}(\phi)} \leq H_{M'',\mathcal{F}''}^{(1)} \leq (1-T^\delta)H_{M,\mathcal{F}}^{(1)} + T^\delta P_{\ker \text{gr}(\phi)}^{(1)}. \quad (15.3.13)$$

Proof. The sequence $N_n = \phi^{-1}(M_{n+\delta})$ forms a filtration on M and $M_n \subseteq N_n$ for all $n \in \mathbb{Z}$. By definition, we have $\ker(\text{gr}(\phi)_n) = (N_{n+1} \cap M_n)/M_{n+1}$, and

$$P_{\ker \text{gr}(\phi)} = \sum_{n \in \mathbb{Z}} \ell_A((N_{n+1} \cap M_n)/M_{n+1})T^n. \quad (15.3.14)$$

For each n , the module $(M' \cap M_n)/(M' \cap M_{n+1})$ is identified with the submodule $(M' \cap M_n + M_{n+1})/M_{n+1}$ of $(N_{n+1} \cap M_n)/M_{n+1}$, so follows from (??) that $H_{M',\mathcal{F}'} \leq P_{\ker \text{gr}(\phi)}$. According to ??, we also have

$$P_{\ker \text{gr}(\phi)} \leq \sum_{n \in \mathbb{Z}} \ell_A(N_{n+1}/M_{n+1})T^n \leq P_{\ker \text{gr}(\phi)}^{(1)}. \quad (15.3.15)$$

For each $n \in \mathbb{Z}$, we have an exact sequence

$$0 \longrightarrow N_{n+1}/M_{n+1} \longrightarrow M/M_{n+1} \longrightarrow M/M_{n+\delta+1} \longrightarrow M''/M''_{n+\delta+1} \longrightarrow 0$$

This then implies, by the additivity of length, that

$$\ell_A(M''/M''_{n+\delta+1}) = \ell_A(M/M_{n+\delta+1}) - \ell_A(M/M_{n+1}) + \ell_A(N_{n+1}/M_{n+1}).$$

Multiply by $T^{n+\delta}$ on both sides and take summation on n , we then get

$$H_{M'',\mathcal{F}''}^{(1)} = (1 - T^\delta) H_{M,\mathcal{F}}^{(1)} + T^\delta \sum_{n \in \mathbb{Z}} \ell_A(N_{n+1}/M_{n+1}) T^n. \quad (15.3.16)$$

The inequality (??) then follows from (??) and (??). \square

Corollary 15.3.24. *Under the conditions of ??, we have $H_{M'',\mathcal{F}''} \geq (1 - T^\delta) H_{M,\mathcal{F}}^{(1)}$, and the equality holds if and only if $\text{gr}(\phi)$ is injective. In this case, we have $M' \subseteq \bigcap_n M_n$, and the following sequence is exact:*

$$0 \longrightarrow \text{gr}(M) \xrightarrow{\text{gr}(\phi)} \text{gr}(M) \longrightarrow \text{gr}(M'') \longrightarrow 0$$

Proof. The first assertion follows from (??). Suppose now $\text{gr}(\phi)$ is injective. By ?? we have $\ker \phi \subseteq \phi^{-1}(M_{n+\delta}) = M_n$ for all n , whence $M' \subseteq \bigcap_n M_n$. Now consider the following exact sequence

$$0 \longrightarrow M/M' \xrightarrow{\bar{\phi}} M \longrightarrow M/\phi(M) \longrightarrow 0$$

Since $\text{gr}(\phi)$ is injective, we have $\bar{\phi}^{-1}(M_n) = M_{n-\delta}/M'$. Hence the filtration on M/M' deduced by the filtration \mathcal{F} on M is the filtration $(M_{n-\delta}/M')$. The associated graded module is $\text{gr}(M)(-\delta)$ and we have an exact sequence of graded modules

$$0 \longrightarrow \text{gr}(M)(-\delta) \longrightarrow \text{gr}(M) \longrightarrow \text{gr}(M'') \longrightarrow 0$$

which proves our claim. \square

Proposition 15.3.25. *Let A be a Noetherian ring, M a finitely generated A -module, \mathfrak{I} an ideal of A such that $M/\mathfrak{I}M$ has finite length, and \mathcal{F} an \mathfrak{I} -good filtration on M . Let (x_1, \dots, x_s) be a sequence of elements of A , $(\delta_1, \dots, \delta_s)$ a sequence of positive integers such that $x_i \in \mathfrak{I}^{\delta_i}$ for each i , and let ξ_i be the image of x_i in $\text{gr}_{\delta_i}(A) = \mathfrak{I}^{\delta_i}/\mathfrak{I}^{\delta_i+1}$.*

(a) *Endow the A -module $\bar{M} = M/\sum_i x_i M$ with the \mathfrak{I} -good filtration $\bar{\mathcal{F}}$. Then we have*

$$H_{\bar{M},\bar{\mathcal{F}}}^{(s)} \geq \prod_{i=1}^s (1 - T^{\delta_i}) H_{M,\mathcal{F}}^{(s)}. \quad (15.3.17)$$

(b) *For the equality in (a) holds, it is necessary and sufficient that the sequence (ξ_1, \dots, ξ_s) of elements of $\text{gr}(A)$ is completely secant for $\text{gr}(M)$. In this case, the canonical homomorphism*

$$\theta : \text{gr}(M)/\sum_i \xi_i \cdot \text{gr}(M) \rightarrow \text{gr}(\bar{M})$$

is bijective.

(c) Suppose the condition in (b) hold, and the modules $M_i = M/(x_1M + \dots + x_iM)$ is Hausdorff in the \mathfrak{I} -adic topology for each i . Then the sequence (x_1, \dots, x_s) is completely secant for M .

Proof. When $s = 1$, we have $\bigcap_n M_n = \bigcap_n \mathfrak{I}^n M$ and the sequence $\{\xi_1\}$ is completely secant for $\text{gr}(M)$ if and only if the homothety with ratio ξ_1 in $\text{gr}(M)$ is injective. The claim then results immediately from ?? applied to the homothety $\phi = h_{x_1}$ in M .

Suppose then $s \geq 2$ and we prove by induction on s . By the induction hypothesis applied on $M_1 = M/x_1M$ with the quotient filtration \mathcal{F}_1 and the sequence (x_2, \dots, x_s) , we have

$$H_{\overline{M}, \overline{\mathcal{F}}}^{(s-1)} \geq \prod_{i=2}^s (1 - T^{\delta_i}) H_{M_1, \mathcal{F}_1}^{(s-1)} \quad (15.3.18)$$

and the equality holds if and only if the sequence (ξ_2, \dots, ξ_s) is completely secant for the $\text{gr}(A)$ -module $\text{gr}(M_1)$. Since the element $(1 - T^{\delta_1})/(1 - T)$ is positive, the case $s = 1$ already treated and the formula (??) provide the inequalities

$$H_{\overline{M}, \overline{\mathcal{F}}}^{(s)} \geq \prod_{i=2}^s (1 - T^{\delta_i}) H_{M_1, \mathcal{F}_1}^{(s)} \geq \prod_{i=1}^s (1 - T^{\delta_i}) H_{M_1, \mathcal{F}_1}^{(s)} \quad (15.3.19)$$

which proves (a).

We can have equality in (??) only if we have simultaneously equality in (??) and equality

$$H_{M_1, \mathcal{F}_1}^{(1)} = (1 - T^{\delta_1}) H_{M, \mathcal{F}}^{(1)}. \quad (15.3.20)$$

This last relation means that $\{\xi_1\}$ is completely secant for $\text{gr}(M)$ and implies that the canonical homomorphism from $\text{gr}(M)/\xi_1\text{gr}(M)$ to $\text{gr}(M_1)$ is a isomorphism. In other words, we have equality in (??) if and only if $\{\xi_1\}$ is completely secant for $\text{gr}(M)$ and $\{\xi_2, \dots, \xi_s\}$ is completely secant for $\text{gr}(M)$. This means $\{\xi_1, \dots, \xi_s\}$ is completely secant for $\text{gr}(M)$. We have thus demonstrated the equivalence of the two conditions of (b).

Now suppose that $\{\xi_1, \dots, \xi_s\}$ is completely secant for $\text{gr}(M)$ and M_i is Hausdorff for the \mathfrak{I} -adic topology for all i . According to the above argument and the induction hypothesis, the sequence (x_2, \dots, x_s) is completely secant for M_1 . As we have $M_1 = M/x_1M$ and that $\{x_1\}$ is completely secant for M , the sequence (x_1, x_2, \dots, x_s) is completely secant for M . \square

15.4 Regular local rings

15.4.1 Regularity of local rings

Let (A, \mathfrak{m}) be a Noetherian local ring and $(x_i)_{i \in I}$ a family of elements of \mathfrak{m} . In view of ??, it amounts to the same thing to suppose that the family $(x_i)_{i \in I}$ generates the ideal \mathfrak{m} of A , or that the classes of x_i modulo \mathfrak{m}^2 generate the κ_A -vector space $\mathfrak{m}/\mathfrak{m}^2$. In this case, we have $\dim(A) \leq |I|$ by ??, and therefore

$$\dim(A) \leq [\mathfrak{m}/\mathfrak{m}^2 : \kappa_A] \leq |I|$$

for every generating family $(x_i)_{i \in I}$ of the ideal \mathfrak{m} .

Definition 15.4.1. A Noetherian local ring (A, \mathfrak{m}) is called **regular** if $\dim(A) = [\mathfrak{m}/\mathfrak{m}^2 : \kappa_A]$. A

family of elements of \mathfrak{m} whose classes modulo \mathfrak{m}^2 form a basis of the κ_A -space $\mathfrak{m}/\mathfrak{m}^2$ is called a **system of parameters** for A .

A system of parameters in a regular local ring A is therefore a finite family $(x_i)_{i \in I}$ generating the ideal \mathfrak{m} of A and such that $|I| = \dim(A)$. Conversely, if the maximal ideal \mathfrak{m} of a Noetherian local ring A is generated by d elements with $d \leq \dim(A)$, then the ring A is regular.

Example 15.4.2 (Examples of regular local rings).

- (a) The regular local rings of dimension 0 (resp. 1) are the fields (resp. the discrete valuation rings). In fact, the claim is clear for dimension 0, and a discrete valuation ring is clearly regular (since it is a PID). Conversely, assume that (A, \mathfrak{m}) is a regular local ring of dimension one, with t its generator of \mathfrak{m} . For all $n \geq 0$ we have $\dim_{\kappa_A}(\mathfrak{m}^n/\mathfrak{m}^{n+1}) = 1$ because it is generated by t^n (and it cannot be zero since otherwise \mathfrak{m} will be the nilradical of A , which implies $\dim(A) = 0$, contradiction). In particular $\mathfrak{m}^n = (t^n)$ and the graded ring $\text{gr}_{\mathfrak{m}}(A)$ is isomorphic to the polynomial ring $\kappa_A[T]$. For $x \in A \setminus \{0\}$, define $v(x)$ to be the largest integer n such that $x \in \mathfrak{m}^n$. Then we have $v(xy) = v(x) + v(y)$ for all $x, y \in A \setminus \{0\}$. We extend this to the field of fractions K of A . Then it is clear that A is the set of elements of K which have positive valuation. Hence we see that A is a discrete valuation ring. Note that if (A, \mathfrak{m}) is a discrete valuation ring, then an element t of \mathfrak{m} is a uniformizer if and only if $\{t\}$ is a system of parameters of A .
- (b) Let k be a field and n a positive integer. The ring $A = k[[X_1, \dots, X_n]]$ is Noetherian and local with dimension n . Since its maximal ideal \mathfrak{m} is generated by X_1, \dots, X_n , we see A is regular. Moreover, a sequence (F_1, \dots, F_n) is a system of parameters if and only if their image in $\mathfrak{m}/\mathfrak{m}^2$ is linearly independent, or equivalently the Jacobi matrix $(\partial F_i / \partial X_j)$ is nonsingular at the point 0.
- More generally, let A be a regular local ring of dimension r . The ring $A[[X_1, \dots, X_n]]$ is then a regular local ring of dimension $r + n$. If (a_1, \dots, a_r) is a system of parameters for A , then $(a_1, \dots, a_r, X_1, \dots, X_n)$ is a system of parameters for $A[[X_1, \dots, X_n]]$.
- (c) Let A be a regular local ring of dimension r . The ring $A\{X_1, \dots, X_n\}$ of restricted formal power series is a regular local ring of dimension $n + r$. If (a_1, \dots, a_r) is a system of parameters for A , then $(a_1, \dots, a_r, X_1, \dots, X_n)$ is a system of parameters for $A\{X_1, \dots, X_n\}$.
- (d) Let k be a field, A a finitely generated k -algebra that is an integral domain and \mathfrak{m} a maximal ideal of A . Then the Noetherian local ring $A_{\mathfrak{m}}$ is regular if and only if we have $\dim(A) = [\mathfrak{m}/\mathfrak{m}^2 : A/\mathfrak{m}]$. In fact, we have $\dim(A_{\mathfrak{m}}) = \dim(A)$ by ?? and the A/\mathfrak{m} -vector space $\mathfrak{m}A_{\mathfrak{m}}/(\mathfrak{m}A_{\mathfrak{m}})^2$ is isomorphic to $\mathfrak{m}/\mathfrak{m}^2$. In particular, if k is algebraically closed, this condition is equivalent to $\dim(A) = [\mathfrak{m}/\mathfrak{m}^2 : k]$ by the Nullstellensatz.
- (e) Let A be a regular local ring. We will see later that the local ring $A_{\mathfrak{p}}$ is regular for any prime ideal \mathfrak{p} of A .

Proposition 15.4.3. *Let $\rho : A \rightarrow B$ be a local homomorphism of Noetherian local rings such that B is flat over A . Suppose that $\mathfrak{m}_B = \mathfrak{m}_A^e$, then $\dim(B) = \dim(A)$ and B is regular if and only if A is regular.*

Proof. The first assertion follows from ?? . Since B is flat over A , we can identify $\mathfrak{m}_B/\mathfrak{m}_B^2$ with $B \otimes_A (\mathfrak{m}_A/\mathfrak{m}_A^2)$ or with $\kappa_B \otimes_{\kappa_A} (\mathfrak{m}_A/\mathfrak{m}_A^2)$. We then have

$$[\mathfrak{m}_B/\mathfrak{m}_B^2 : \kappa_A] = [\mathfrak{m}_A/\mathfrak{m}_A^2 : \kappa_A]$$

so the second claim follows. \square

Corollary 15.4.4. *A Noetherian local ring A is regular if and only if its completion \widehat{A} is.*

Proof. The completion \widehat{A} is flat over A and $\widehat{\mathfrak{m}}$ is generated by \mathfrak{m} in \widehat{A} , so we can apply ?? . \square

Theorem 15.4.5. *Let A be a Noetherian local ring. The following conditions are equivalent:*

- (i) A is regular.
- (ii) The ideal \mathfrak{m}_A is generated by a subset of \mathfrak{m}_A secant for A .
- (iii) The ideal \mathfrak{m}_A is generated by a subset of \mathfrak{m}_A completely secant for A .
- (iv) The homomorphism $\gamma : S_A(\mathfrak{m}_A/\mathfrak{m}_A^2) \rightarrow \text{gr}(A)$ is bijective.
- (v) There exists a positive integer r such that $H_{A,\mathfrak{m}_A} = (1 - T)^{-r}$.
- (vi) We have $H_{A,\mathfrak{m}_A} = (1 - T)^{-d}$ where $d = \dim(A)$.

If these conditions are fulfilled, every system of parameters for A is a completely secant sequence for A .

Proof. Since every secant subset has cardinality smaller than $\dim(A)$, we see (iii) \Rightarrow (ii) \Rightarrow (i). To see (iv) \Rightarrow (iii), let (x_1, \dots, x_r) be a sequence of elements of \mathfrak{m}_A whose classes modulo \mathfrak{m}_A^2 form a basis (ξ_1, \dots, ξ_r) for $\mathfrak{m}_A/\mathfrak{m}_A^2$. If property (iv) is satisfied, $\text{gr}(A)$ is then the polynomial algebra $\kappa_A[\xi_1, \dots, \xi_r]$ and the sequence (x_1, \dots, x_r) is completely secant by ?? . This also proves the last assertion of the theorem.

Let $r = [\mathfrak{m}_A/\mathfrak{m}_A^2 : \kappa_A]$. Then the Poincaré series of the graded module $S = S_A(\mathfrak{m}_A/\mathfrak{m}_A^2)$ over κ_A is equal to

$$P_S = \sum_{n \in \mathbb{N}} [S_n : \kappa_A] T^n = (1 - T)^{-r}.$$

Suppose that the homomorphism γ is not bijective. As γ is surjective, there exists a homogeneous element u of S of degree $d > 0$, canceled by γ . Then

$$H_{A,\mathfrak{m}_A} = P_S - P_{\ker \gamma} \leq P_S - P_{uS} = (1 - T^d)(1 - T)^{-r} = (1 + T + \dots + T^{d-1})(1 - T)^{-(r-1)}.$$

By ?? and ?? , we get $\dim(A) < r$, so A is not regular.

Finally, let us prove the equivalence of conditions (iv) and (vi). If we have $H_{A,\mathfrak{m}_A} = (1 - T)^{-r}$ for some r , then by ?? we must have $r = \dim(A)$. Also, $[\mathfrak{m}_A/\mathfrak{m}_A^2 : \kappa_A] = r$ because $(1 - T)^{-r} = 1 + rT + \dots$. Therefore our claim follows. \square

Corollary 15.4.6. *A regular local ring is integrally closed, and in particular an integral domain.*

Proof. If A is a regular local ring, then $\text{gr}(A)$ is isomorphic to a polynomial algebra over a field, so it is integrally closed. Since A is a Zariski ring, by ?? we see A is integrally closed. \square

Corollary 15.4.7. *Let $\rho : A \rightarrow B$ be a local homomorphism of Noetherian local rings. Suppose that A is complete and B is regular. Then for ρ to be bijective, it is necessary and sufficient that it induces bijections of κ_A on κ_B and of $\mathfrak{m}_A/\mathfrak{m}_A^2$ to $\mathfrak{m}_B/\mathfrak{m}_B^2$.*

Proof. These conditions are clearly necessary. Suppose now ρ induces bijections of κ_A on κ_B and of $\mathfrak{m}_A/\mathfrak{m}_A^2$ to $\mathfrak{m}_B/\mathfrak{m}_B^2$. Since the ring $\text{gr}(A)$ is generated by $\text{gr}_0(A)$ and $\text{gr}_1(A)$, it follows that $\text{gr}(\rho)$ is bijective. Therefore ρ is bijective by ??.

Corollary 15.4.8. *Let k be a ring and A a Noetherian local k -algebra with residue field equal to k . Then A is regular if and only if the completion \widehat{A} is isomorphic to a formal series ring $k[[X_1, \dots, X_n]]$.*

Proof. This follows from ?? and ??.

15.4.2 Quotients of regular local rings

Proposition 15.4.9. *Let A be a Noetherian local ring, $\mathbf{x} = (x_1, \dots, x_r)$ a sequence of elements of \mathfrak{m}_A and \mathfrak{x} the ideal it generates. The following conditions are equivalent.*

- (i) A is regular and \mathbf{x} can be extended to a system of parameters for A .
- (ii) A/\mathfrak{x} is regular and \mathbf{x} is secant for A .
- (iii) A/\mathfrak{x} is regular and \mathbf{x} is completely secant for A .

Moreover, if these conditions are satisfied, then \mathfrak{x} is a prime ideal of A .

Proof. Clearly (iii) implies (ii). Suppose now that \mathbf{x} is secant for A and the Noetherian local ring A/\mathfrak{x} is regular. Let (x_{r+1}, \dots, x_d) be a sequence of elements of A whose class modulo I form a system of parameters for A/\mathfrak{x} . Then the sequence (x_1, \dots, x_d) generates the maximal ideal \mathfrak{m}_A of A , and we have

$$\dim(A) = r + \dim(A/\mathfrak{x}) = r + (d - r) = d.$$

Therefore, A is regular and (x_1, \dots, x_d) is a system of parameters for A .

Finally, assume that (i) is satisfied. Then A is regular and \mathbf{x} is completely secant for A , hence secant for A . Then we have

$$\dim(A/\mathfrak{x}) = \dim(A) - r$$

and the classes x_1, \dots, x_r modulo \mathfrak{m}_A^2 is linearly independent over κ_A , so

$$[\mathfrak{m}_A/(\mathfrak{m}_A^2 + \mathfrak{x}) : \kappa_A] = [\mathfrak{m}_A/\mathfrak{m}_A^2 : \kappa_A] - r.$$

Therefore we see A/\mathfrak{x} is regular, and in particular \mathfrak{x} is prime since A/\mathfrak{x} is integral. □

Corollary 15.4.10. *Let A be a Noetherian local ring and x an element of \mathfrak{m}_A . The following conditions are equivalent.*

- (i) A is regular, and x does not belong to \mathfrak{m}_A .
- (ii) A/xA is regular and $\dim(A/xA) = \dim(A) - 1$.

(iii) A/xA is regular and x is not a divisor of 0 in A .

Proof. This follows by applying ?? on $\mathfrak{x} = xA$. \square

Corollary 15.4.11. *Let A be a regular local ring and \mathfrak{I} an ideal of A . Then A/\mathfrak{I} is regular if and only if \mathfrak{I} is generated by a subset of a system of parameters for A .*

Proof. The condition is sufficient by ?. Suppose now A/\mathfrak{I} is regular, and $\mathbf{x} = (x_1, \dots, x_r)$ a sequence of elements of \mathfrak{I} whose classes modulo \mathfrak{m}_A^2 form a basis for $(\mathfrak{I} + \mathfrak{m}_A^2)/\mathfrak{m}_A^2$ over κ_A . Let \mathfrak{x} be the ideal generated by \mathbf{x} , so we have $\mathfrak{x} \subseteq \mathfrak{I}$ and \mathbf{x} is part of a system of parameters for A , so the local Noetherian ring A/\mathfrak{x} is regular. Moreover, the vector spaces $\mathfrak{m}_A/(\mathfrak{I} + \mathfrak{m}_A^2)$ and $\mathfrak{m}_A/(\mathfrak{x} + \mathfrak{m}_A^2)$ have the same dimension over κ_A . Consequently, the regular local rings A/\mathfrak{I} and A/\mathfrak{x} have the same dimension. As the ideals \mathfrak{I} and \mathfrak{x} are prime and we have $\mathfrak{x} \subseteq \mathfrak{I}$, we finally have $\mathfrak{I} = \mathfrak{x}$. \square

Example 15.4.12. Let k be a field. Then the ring $A = k[[X_1, \dots, X_n]]$ of power series is local. Let \mathfrak{I} be a proper ideal of A . By ??, for A/\mathfrak{I} to be regular, it is necessary and sufficient that we can find a positive integer r and elements F_1, \dots, F_r of A , generating \mathfrak{I} , and such that the matrix $(\partial F_i / \partial X_j(0))$ is of rank r (Jacobian criterion). We then have $\dim(A/\mathfrak{I}) = n - r$.

15.4.3 Eisenstein polynomials

Let A be a ring, \mathfrak{p} a prime ideal of A , and P a polynomial in $A[T]$. We say P is an **Eisenstein polynomial** for \mathfrak{p} if it satisfies the following conditions:

- (a) P is monic with degree $d \geq 2$;
- (b) $P(T) \equiv T^d \pmod{\mathfrak{p}A[T]}$;
- (c) $P(0) \notin \mathfrak{p}^2$.

In other words, an Eisenstein polynomial for \mathfrak{p} is of the form $P(T) = T^d + \sum_{i=0}^{d-1} a_i T^i$, with $d \geq 1$ and a_1, \dots, a_{d-1} belong to \mathfrak{p} with $a_0 \in \mathfrak{p} \setminus \mathfrak{p}^2$.

We say P is an Eisenstein polynomial for $\mathfrak{p}A_{\mathfrak{p}}$ if the canonical image of P in the polynomial ring $A_{\mathfrak{p}}[T]$ is an Eisenstein polynomial for the ideal $\mathfrak{p}A_{\mathfrak{p}}$. This means P is an Eisenstein polynomial for \mathfrak{p} and that it also satisfies the following condition, stronger than (c):

- (c') any element a in A such that $aP(0) \in \mathfrak{p}^2$ belongs to \mathfrak{p} .

Proposition 15.4.13. *Let A be a ring, \mathfrak{p} a prime ideal of A and $P \in A[T]$ an Eisenstein polynomial for \mathfrak{p} .*

- (a) P has no decomposition of the form $P = P_1 P_2$, where P_1, P_2 are two monic polynomials in $A[T]$ unequal to 1.
- (b) Suppose that A is integrally closed with K its fraction field. Then P is irreducible in $K[T]$.

Proof. Let π be the canonical map from A to the residue field $\kappa(\mathfrak{p})$ of \mathfrak{p} and $\tilde{\pi} : A[T] \rightarrow \kappa(\mathfrak{p})[T]$ be the extension of π such that $\tilde{\pi}(T) = T$. Suppose that $P = P_1 P_2$ where P_1, P_2 are two monic polynomials unequal to 1. Then we have $\tilde{\pi}(P) = T^d = \tilde{\pi}(P_1)\tilde{\pi}(P_2)$ in $\kappa(\mathfrak{p})[T]$, with d the degree of P . If d_i is the degree of P_i , we get $\tilde{\pi}(P_i) = T^{d_i}$, which means $P_i(T) \equiv T^{d_i} \pmod{\mathfrak{p}A[T]}$, in particular $P_i(0) \in \mathfrak{p}$. But then $P(0) = P_1(0)P_2(0) \in \mathfrak{p}^2$, which is a contradiction. The assertion (b) now follows from (a). \square

Proposition 15.4.14. *Let A be a Noetherian local ring and P_1, \dots, P_r be monic polynomials in $A[T]$ such that $P_i(T) \equiv T^{d_i} \pmod{\mathfrak{m}_A A[T]}$ and $\deg(P_i) \geq 2$ for each i . Let \mathfrak{q} be the ideal in $A[T_1, \dots, T_r]$ generated by $P_1(T_1), \dots, P_r(T_r)$ and B the A -algebra $A[T_1, \dots, T_r]/\mathfrak{q}$. For each i , we denote by d_i the degree of P_i , ξ_i the class of $T_i \pmod{\mathfrak{q}}$, and γ_i the class of $c_i = P_i(0) \pmod{\mathfrak{m}_A^2}$.*

(a) *B is a Noetherian local ring with maximal ideal*

$$\mathfrak{m}_B = B\mathfrak{m}_A + \sum_{i=1}^r B\xi_i.$$

We have $\dim(A) = \dim(B)$ and $[\kappa_B : \kappa_A] = 1$. Moreover, the monomials $\xi_1^{\alpha_1} \cdots \xi_r^{\alpha_r}$, with $0 \leq \alpha_i < d$ for $1 \leq i \leq r$, form a basis for the A -module B .

- (b) *Let λ be the canonical homomorphism from $\mathfrak{m}_A/\mathfrak{m}_A^2$ to $\mathfrak{m}_B/\mathfrak{m}_B^2$. Then the kernel of λ is the κ_A -vector space generated by $\gamma_1, \dots, \gamma_r$ and the classes ξ_1, \dots, ξ_r form a basis over κ_A for the cokernel of λ .*
- (c) *For B to be regular, it is necessary and sufficient that A is regular and the $\gamma_1, \dots, \gamma_r$ is linearly independent in the κ_A -vector space $\mathfrak{m}_A/\mathfrak{m}_A^2$.*

Proof. The A -algebra B is isomorphic to the tensor product $B_1 \otimes_A \cdots \otimes_A B_r$ where $B_i = A[T]/(P_i)$ for each i . It then follows that the monomials $\xi_1^{\alpha_1} \cdots \xi_r^{\alpha_r}$, with $0 \leq \alpha_i < d$ for $1 \leq i \leq r$, form a basis for the A -module B . In particular, B is integral over A , so A and B has the same dimension.

By ??, B is Noetherian, and any maximal ideal of B contains $B\mathfrak{m}_A$. Conversely, from the hypothesis on P_1, \dots, P_r and the relations $P_i(\xi_i) = 0$, we get $\xi_i^{d_i} \in B\mathfrak{m}_A$ for each i . Therefore any maximal ideal of B containing ξ_1, \dots, ξ_r must contain the ideal $\mathfrak{n} = B\mathfrak{m}_A + B\xi_1 + \cdots + B\xi_r$. But we have $\mathfrak{m}_A = A \cap \mathfrak{n}$ and $B = A + \mathfrak{n}$, so B/\mathfrak{n} is isomorphic to A/\mathfrak{m}_A and therefore is maximal in B . This shows B is local with maximal ideal $B\mathfrak{m}_A + \sum_{i=1}^r B\xi_i$, and we have $[\kappa_B : \kappa_A] = 1$. This proves (a).

Let φ be the canonical homomorphism of $(A/\mathfrak{m}_A^2)[T_1, \dots, T_r]$ to B/\mathfrak{m}_B^2 . By part (a), the kernel of φ is the ideal generated by the classes $\bar{P}_i(T_i)$ of $P_i(T_i)$ modulo $\mathfrak{m}_A^2 A[T_1, \dots, T_r]$, the monomials $T_i T_j$, and $x T_i$ with $1 \leq i, j \leq r$ and $x \in \mathfrak{m}_A/\mathfrak{m}_A^2$. By hypothesis, we have $P_i(T) \equiv T^{d_i} \pmod{\mathfrak{m}_A A[T]}$, so we can replace $\bar{P}_i(T_i)$ with γ_i in the description above. As a result, if we set $\mathfrak{r} = \mathfrak{m}_A^2 + \sum_{i=1}^r A c_i$, the ring B/\mathfrak{m}_B^2 is then isomorphic to $(A/\mathfrak{r})[T_1, \dots, T_r]$ quotient the ideal generated by the monomials $T_i T_j$ and $x T_i$, with $1 \leq i, j \leq r$ and $x \in \mathfrak{m}_A/\mathfrak{r}$. If we denote by τ_i the class of ξ_i modulo \mathfrak{m}_B^2 , this means

$$B/\mathfrak{m}_B^2 = (A/\mathfrak{r}) \oplus \kappa_A \tau_1 \oplus \cdots \oplus \kappa_A \tau_r. \quad (15.4.1)$$

and

$$\mathfrak{m}_B/\mathfrak{m}_B^2 = (\mathfrak{m}_A/\mathfrak{r}) \oplus \kappa_A \tau_1 \oplus \cdots \oplus \kappa_A \tau_r. \quad (15.4.2)$$

Assertion (b) now follows from these.

According to formula ?? and the relation $[\kappa_B : \kappa_A] = 1$, we get

$$[\mathfrak{m}_B/\mathfrak{m}_B^2 : \kappa_B] = [\mathfrak{m}_A/\mathfrak{m}_A^2 : \kappa_A] + (r - [\mathfrak{r}/\mathfrak{m}_A^2 : \kappa_A]). \quad (15.4.3)$$

But the κ_A -vector space $\mathfrak{r}/\mathfrak{m}_A^2$ is generated by $\gamma_1, \dots, \gamma_r$, and we have

$$\dim(B) = \dim(A) \leq [\mathfrak{m}_A/\mathfrak{m}_A^2 : \kappa_A]. \quad (15.4.4)$$

Combine formula (??) and formula (??), we see assertion (c) follows. \square

Corollary 15.4.15. *Let A be a regular Noetherian local ring and $P \in A[T]$ an Eisenstein polynomial for \mathfrak{m}_A . Then $B = A[T]/(P)$ is a Noetherian local regular ring, with the same dimension as A , and $[\kappa_B : \kappa_A] = 1$. In fact, we have $\mathfrak{m}_B = B\mathfrak{m}_A + B\xi$, where ξ is the class of T modulo (P) .*

Proof. By definition we have $\deg(P) \geq 2$ and $P(T) \equiv T^d \pmod{\mathfrak{m}_A}$, so this is a special case of ?? with $r = 1$. \square

Proposition 15.4.16. *Let A be an integral domain, with fraction field K , and L an algebraic extension of K . Let B be the integral closure of A in L and \mathfrak{p} a prime ideal of A . Suppose that the local ring $A_{\mathfrak{p}}$ is regular and Noetherian; let ξ be a primitive element for L over K (i.e., $L = K(\xi)$) and suppose that the minimal polynomial of ξ over K is an Eisenstein polynomial $P \in A[T]$ for $\mathfrak{p}A_{\mathfrak{p}}$.*

- (a) *There exists in B a unique prime ideal \mathfrak{P} lying over \mathfrak{p} .*
- (b) *The local ring $B_{\mathfrak{P}}$ is Noetherian and regular, with the same dimension as $A_{\mathfrak{p}}$. Moreover, $B_{\mathfrak{P}} = A_{\mathfrak{p}}[\xi]$.*
- (d) *The canonical homomorphism from A/\mathfrak{p} to B/\mathfrak{P} induces an isomorphism on the fraction fields.*

Proof. Set $C = A_{\mathfrak{p}}[\xi]$ and let d be the degree of P . By ?? applied to the ring $A_{\mathfrak{p}}$, the Eisenstein polynomial P is irreducible in $K[T]$ and $(1, \xi, \dots, \xi^{d-1})$ is a basis for L over K , and for C over $A_{\mathfrak{p}}$. Since P is monic, the kernel of the canonical homomorphism from $A_{\mathfrak{p}}[T]$ to C is equal to (P) ; in other words, $C = A_{\mathfrak{p}}[T]/(P)$, so we can apply ??: C is a regular Noetherian local with dimension equal to that of $A_{\mathfrak{p}}$, the maximal ideal \mathfrak{m}_C of C is generated by $\mathfrak{p} \cup \{\xi\}$ and the residue field κ_C is a trivial extension of $\kappa(\mathfrak{p})$. To prove the proposition, it suffices to find a unique prime ideal \mathfrak{P} of B lying over \mathfrak{p} , and such that $C = B_{\mathfrak{P}}$.

Let $S = A - \mathfrak{p}$, then the integral closure of $A_{\mathfrak{p}}$ in L is equal to $S^{-1}B$. But t is integral over $A_{\mathfrak{p}}$, and $C = A_{\mathfrak{p}}[\xi]$ is a regular local ring, hence integrally closed (??). It then follows that $C = S^{-1}B$, so $S^{-1}B$ is local. By ?? and ??, there then exists a unique prime ideal of $S^{-1}B$ lying over $\mathfrak{p}A_{\mathfrak{p}}$, which corresponds to a unique prime ideal \mathfrak{P} of B lying over \mathfrak{p} , and we have $B_{\mathfrak{P}} = S^{-1}B = C$ (??). \square

Corollary 15.4.17. *With the hypothesis in ??, suppose that $A_{\mathfrak{p}}$ is a DVR. Then $B_{\mathfrak{P}}$ is a DVR, ξ is a uniformizer for $B_{\mathfrak{P}}$, and we have*

$$f(B_{\mathfrak{P}}/A_{\mathfrak{p}}) = 1, \quad e(B_{\mathfrak{P}}/A_{\mathfrak{p}}) = [L : K]. \quad (15.4.5)$$

Proof. In fact, DVRs are regular Noetherian local rings with dimension 1; set $d = [L : K]$ and $P(T) = T^d + \sum_{i=0}^{d-1} a_i T^i$. Let $v_{\mathfrak{P}}$ be the valuation on $B_{\mathfrak{P}}$. Then since P is an Eisenstein polynomial, $a_0 \in \mathfrak{m}_{A_{\mathfrak{p}}} \setminus \mathfrak{m}_{A_{\mathfrak{p}}}^2$ and $a_i \in \mathfrak{m}_{A_{\mathfrak{p}}}$ for each i , whence

$$d = v_{\mathfrak{P}}(\xi^d) = v_{\mathfrak{P}}(-\sum_{i=0}^{d-1} a_i \xi^i) = \min_i \{v_{\mathfrak{P}}(a_i) + i\} = v_{\mathfrak{P}}(a_0).$$

This shows $d = e(B_{\mathfrak{P}}/A_{\mathfrak{P}})$. Since $[\kappa_B : \kappa_A] = 1$, we also have $f(B_{\mathfrak{P}}/A_{\mathfrak{P}}) = 1$. \square

Example 15.4.18 (Examples of extensions by Eisenstein polynomials).

- (a) Let $A = \mathbb{Z}$ and $L = \mathbb{Q}(p^{1/d})$, where p is a prime number and $d \geq 2$. Let B be the integral closure of \mathbb{Z} in L . Then $P(T) = T^d - p$ is an Eisenstein polynomial in $\mathbb{Z}[T]$ for $p\mathbb{Z}_{(p)}$, and there exists a unique prime ideal \mathfrak{P} of B lying over $p\mathbb{Z}$. Since $\mathbb{Z}_{(p)}$ is a DVR, there exists on $\mathbb{Q}(p^{1/d})$ a unique discrete valuation $v_{\mathfrak{P}}$ such that $v(p) > 0$. We have $[L : K] = v_{\mathfrak{P}}(p) = d$, and B/\mathfrak{P} is a field with p elements. The valuation ring $B_{\mathfrak{P}}$ of $v_{\mathfrak{P}}$ is equal to $\mathbb{Z}_{(p)}[p^{1/d}]$.
- (b) Let $A = \mathbb{Z}$ and $L = \mathbb{Q}(\zeta)$ where $\zeta = e^{2\pi i/p^f}$, with p a prime and $f \geq 1$. Let B be the integral closure of \mathbb{Z} in L and P the polynomial in $\mathbb{Z}[T]$ defined by

$$P(T-1) = \frac{T^{p^f} - 1}{T^{p^{f-1}} - 1}.$$

Set $d = p^f - p^{f-1}$. We have $P(\zeta - 1) = 0$, $P(0) = p$, and

$$P(T-1) \equiv (T-1)^d \pmod{p\mathbb{Z}[T]},$$

so $P(T) \equiv T^d$. Therefore P is an Eisenstein polynomial for $p\mathbb{Z}_{(p)}$. There exists a unique prime ideal \mathfrak{P} of B lying over $p\mathbb{Z}$, and we have $B_{\mathfrak{P}} = \mathbb{Z}_{(p)}[\zeta]$. Moreover, $\zeta - 1$ is a uniformizer for $B_{\mathfrak{P}}$ and

$$[L : K] = d = p^f - p^{f-1}.$$

If $v_{\mathfrak{P}}$ is the unique normalized valuation on $\mathbb{Q}(\zeta)$ such that $v_{\mathfrak{P}}(p) > 0$, we have $v_{\mathfrak{P}}(p) = d$. Moreover, the field B/\mathfrak{P} has p elements. We can prove that B is equal to $\mathbb{Z}[\zeta]$.

15.5 Dimension and graded rings

Let R be a graded ring of type \mathbb{Z} and $(R_n)_{n \in \mathbb{Z}}$ its graduation; we assume that $R_n = 0$ for $n < 0$. For each $n \in \mathbb{Z}$, let $R_{\geq n} = \bigoplus_{i \geq n} R_i$. We have $R = R_{\geq 0}$ and each $R_{\geq n}$ is an homogeneous ideal of R . Denote by S the multiplicative subset $1 + R_{\geq n}$ consists of elements of R with degree zero term equals to 1, and consider the localization $S^{-1}R$. By identifying R with a subring of $\widehat{R} = \prod_n R_n$, we see S is invertible in the completion \widehat{R} , whence $S^{-1}R$ can be identified with a subring of \widehat{R} containing R . For $s \in S$ and $a \in R_{\geq n}$, the element $s^{-1}a - a$ of \widehat{R} belongs to $\prod_{i \geq n+1} R_i$; therefore we have $S^{-1}R_{\geq n} = (S^{-1}R) \cap \prod_{i \geq n} R_i$. From these we deduce the following results:

Proposition 15.5.1. *Let $R = \bigoplus_{n \in \mathbb{N}} R_n$ be a graded ring and S be defined as above.*

- (a) *The ideals $S^{-1}R_{\geq n}$ form an exhaustive and separated filtration of $S^{-1}R$.*
- (b) *The homomorphism $i_S : R \rightarrow S^{-1}R$ induces an isomorphism u_n of R_n to $S^{-1}R_{\geq n}/S^{-1}R_{\geq n+1}$ for each n ; the u_n are homogeneous components of an isomorphism of R onto the graded ring associated with $S^{-1}R$, filtered by $S^{-1}R_{\geq n}$.*

Remark 15.5.1. An element a/s of $S^{-1}R$ with $a \in R$, $s \in S$, is invertible if and only if the degree 0 component of a is invertible in R_0 . Therefore, if R_0 is a local ring, so is $S^{-1}R$ and the canonical injection $R_0 \rightarrow S^{-1}R$ induces an isomorphism on residue fields.

Remark 15.5.2. Suppose that R is generated by R_0 and R_1 ; then for each n we have $R_{n+1} = R_1 \cdot R_n$, so $R_{\geq n+1} = R_1 \cdot R_{\geq n}$ and $S^{-1}R_{\geq n+1} = R_1 \cdot S^{-1}R_{\geq n}$. As a result, the filtration $(S^{-1}R_{\geq n})$ of $S^{-1}R$ is $S^{-1}R_{\geq 1}$ -adic.

Example 15.5.2.

- (a) Let \mathfrak{p} be a homogeneous prime ideal of $\mathbb{C}[X_0, \dots, X_n]$ different to the ideal generated by the X_i 's (denoted by \mathfrak{m}). Let $X \subseteq \mathbb{CP}^n$ be the projective variety defined by \mathfrak{p} and C the algebraic variety in \mathbb{C}^{n+1} defined by \mathfrak{p} (so that C is the **cone** of X). Let $R = \mathbb{C}[X_0, \dots, X_n]/\mathfrak{p}$ be the coordinate ring of C and consider the localization $S^{-1}R$. We claim that $S^{-1}R$ is the local ring at the origin, that is, $S^{-1}R = R_{\mathfrak{m}_0}$. This follows from the fact that a prime ideal \mathfrak{q} is disjoint from S if and only if every element of \mathfrak{q} has zero constant term, or equivalently is contained in the maximal ideal \mathfrak{m} . (Therefore S and $R \setminus \mathfrak{m}/\mathfrak{p}$ have the same saturation.)
- (b) Let A be a local ring and \mathfrak{a} a proper ideal of A . Then $R = \bigoplus_n \mathfrak{a}^n/\mathfrak{a}^{n+1}$ is a graded ring and $R_0 = A/\mathfrak{a}$ is local; it is generated by R_0 and R_1 . The ring $S^{-1}R$ is then local and the filtration $(S^{-1}R_{\geq 1})$ is the $S^{-1}R_{\geq 1}$ -adic filtration. Note that in general the rings A and $S^{-1}R$ are not isomorphic.

15.5.1 Chains of homogeneous ideals

In this part, we will use $\text{dimgr}(R)$ to denote the supremum of the length of chains homogeneous prime ideals in R ; similarly, if \mathfrak{p} is a homogeneous ideal of R , we use $\text{htgr}(\mathfrak{p})$ to denote the supremum of the length of chains homogeneous prime ideals in R for which \mathfrak{p} is the largest element. Note that if \mathfrak{p} is a homogeneous prime ideal of R , we have $\mathfrak{p} \cap S = \emptyset$; in fact, if \mathfrak{p} contains an element with degree zero component 1, then $1 \in \mathfrak{p}$. The map $\mathfrak{p} \mapsto S^{-1}\mathfrak{p}$ from the set of homogeneous prime ideals of R to the set of prime ideals of $S^{-1}R$ is then injective and increasing. Consequently, we have

$$\text{dimgr}(R) \leq \dim(S^{-1}R) \leq \dim(R), \quad \text{htgr}(\mathfrak{p}) \leq \text{ht}(S^{-1}\mathfrak{p}) = \text{ht}(\mathfrak{p}). \quad (15.5.1)$$

Example 15.5.3. Let A be an arbitrary ring and consider the polynomial ring $A[X]$. For a homogeneous prime ideal \mathfrak{p} of $A[X]$, we set $\mathfrak{p}_0 = \mathfrak{p} \cap A$ to be the constant part of elements in \mathfrak{p} . We distinguish two cases:

- (a) If $X \in \mathfrak{p}$, then it is easy to see $\mathfrak{p} = \mathfrak{p}_0 + X \cdot A[X]$ because any polynomial $f \in \mathfrak{p}$ can be written as $f = f(0) + X \cdot g$.
- (b) If $X \notin \mathfrak{p}$, then $\mathfrak{p} = \mathfrak{p}_0 \cdot A[X]$. In fact, let $f \in \mathfrak{p}$ and write

$$f(X) = a_0 + a_1X + \cdots + a_nX^n$$

with $a_i \in A$. Then $a_0 \in \mathfrak{p}_0$ by definition, and we have $g = a_1X + \cdots + a_nX^n \in \mathfrak{p}$. Note that g is divisible by X , so $g/X \in \mathfrak{p}$ since \mathfrak{p} is prime and $X \notin \mathfrak{p}$. Replacing this process we eventually conclude that $a_i \in \mathfrak{p}_0$ for all i , whence $f \in \mathfrak{p}_0 \cdot A[X]$.

We now compute $\text{htgr}(\mathfrak{p})$. Let $\mathfrak{q}_0 \subset \mathfrak{q}_1 \subset \cdots \subset \mathfrak{q}_n = \mathfrak{p}_0$ be a saturated chain in A , so that $n = \text{ht}(\mathfrak{p}_0)$. Then by extending them in $A[X]$, we get a chain

$$\mathfrak{q}_0 \cdot A[X] \subset \mathfrak{q}_1 \cdot A[X] \subset \cdots \subset \mathfrak{q}_n \cdot A[X] = \mathfrak{p}_0 \cdot A[X].$$

In case (b), we have $\mathfrak{p} = \mathfrak{p}_0 \cdot A[X]$, and any homogeneous prime ideal $\mathfrak{p}' \subseteq \mathfrak{p}$ of $A[X]$ satisfies $\mathfrak{p}' = \mathfrak{p}'_0 \cdot A[X]$ (using the claim in (b)). From this, it is easy to conclude that $\text{htgr}(\mathfrak{p}) = \text{ht}(\mathfrak{p})$ in this case.

On the other hand, in case (a) we claim that the chain $\mathfrak{p}_0 \cdot A[X] \subseteq \mathfrak{p} = \mathfrak{p}_0 + X \cdot A[X]$ is saturated for homogeneous prime ideals of A , so that $\text{htgr}(\mathfrak{p}) = \text{ht}(\mathfrak{p}) + 1$. For this, let $\mathfrak{p}' \subseteq \mathfrak{p}$ be a homogeneous prime ideal containing $\mathfrak{p}_0 \cdot A[X]$ and choose $f \in \mathfrak{p}' \setminus (\mathfrak{p}_0 \cdot A[X])$; write $f = a_0 + a_1X + \cdots + a_nX^n$ as usual, and recall that $a_0 \in \mathfrak{p}_0$ with $a_i \notin \mathfrak{p}_0$. Since $f \notin \mathfrak{p}_0 \cdot A[X]$, we may assume that $a_d \notin \mathfrak{p}_0$ with $1 \leq d \leq n$. As \mathfrak{p}' is homogeneous, each homogeneous component of f belongs to \mathfrak{p}' , so $a_dX^d \in \mathfrak{p}' \subseteq \mathfrak{p}$. But every element of \mathfrak{p} has constant term in \mathfrak{p}_0 by definition and \mathfrak{p}' is prime, so $X^d \in \mathfrak{p}'$, and therefore $X \in \mathfrak{p}'$. This shows $\mathfrak{p}' = \mathfrak{p}$ and justifies our claim.

In summary, we have proved the following equality

$$\text{htgr}(\mathfrak{p}) = \begin{cases} \text{ht}(\mathfrak{p}_0) + 1 & \text{if } X \in \mathfrak{p}, \\ \text{ht}(\mathfrak{p}_0) & \text{if } X \notin \mathfrak{p}. \end{cases}$$

Now by taking \mathfrak{p} to be the maximal ideals of $A[X]$, we deduce that $\dimgr(A[X]) \leq \dim(A) + 1$. However, it is easy to see this equality actually holds, for example we can take a saturated chain $\mathfrak{q}_0 \subset \mathfrak{q}_1 \subset \cdots \subset \mathfrak{q}_n$ in A , extending it to a chain

$$\mathfrak{q}_0 \cdot A[X] \subset \mathfrak{q}_1 \cdot A[X] \subset \cdots \subset \mathfrak{q}_n \cdot A[X] \subset \mathfrak{q}_n \cdot A[X] + X \cdot A[X].$$

It is easy to see the ideals above are homogeneous and prime, which shows $\dimgr(A[X]) \geq \dim(A) + 1$.

Finally, recall that in ?? we have shown that for any positive integers n and d with $n + 1 \leq d \leq 2n + 1$, there exists a ring A such that $\dim(A) = n \dim(A[X]) = d$. In view of our previous result, we see the inequality in (??) can be strict. But also recall that for a Noetherian ring A , we always have $\dim(A[X]) = \dim(A) + 1$ (??).

We now turn to our main theorem of this part, which asserts that if the graded ring R is Noetherian then $\dimgr(R) = \dim(R)$. First, we recall that for an arbitrary ideal \mathfrak{a} of R , we denote by \mathfrak{a}^h the ideal generated by homogeneous elements in \mathfrak{a} : in other words $\mathfrak{a}^h = \sum_n (\mathfrak{a} \cap R_n)$. We note that \mathfrak{a}^h is a homogeneous and contained in \mathfrak{a} . Moreover, by ??, \mathfrak{p}^h is prime if \mathfrak{p} is prime.

Lemma 15.5.4. *Any maximal element of the set of homogeneous prime ideals of R is a maximal ideal of R containing $R_{\geq 1}$.*

Proof. Let \mathfrak{m} be a proper homogeneous ideal of R . Then we have

$$\mathfrak{m} \subseteq (\mathfrak{m} \cap R_0) + R_{\geq 1} \neq R.$$

If \mathfrak{m} is maximal, then $\mathfrak{m} = \mathfrak{m}_0 + R_{\geq 1}$, where \mathfrak{m}_0 is a maximal ideal of R_0 . □

Lemma 15.5.5. *Let \mathfrak{p} and \mathfrak{q} be distinct prime ideals of R such that $\mathfrak{q} \subset \mathfrak{p}$. If $\mathfrak{q}^h = \mathfrak{p}^h$, then \mathfrak{q} is homogeneous, \mathfrak{p} is not homogeneous, and $\text{ht}(\mathfrak{p}/\mathfrak{q}) = 1$.*

Proof. By replacing R with R/\mathfrak{q}^h , we can assume that $\mathfrak{q}^h = 0$, so that R is integral (since \mathfrak{q}^h is prime), $\mathfrak{p}^h = 0$. It suffices to prove that $\text{ht}(\mathfrak{p}) \leq 1$, because $\text{ht}(\mathfrak{p}) = 0$ implies $\mathfrak{p} = \{0\}$ when

R is integral. Now since $\mathfrak{p}^h = \{0\}$, we have $\mathfrak{p} \cap R_n = \{0\}$ for each n , so \mathfrak{p} is disjoint from the multiplicative subset $T = \bigcup_n (R_n \setminus \{0\})$. The ring $R_{\mathfrak{p}}$ is then isomorphic to a subring of $T^{-1}R$, and we get

$$\text{ht}(\mathfrak{p}) = \dim(R_{\mathfrak{p}}) \leq \dim(T^{-1}R).$$

However, by ?? the ring $T^{-1}R$ is isomorphic to $K[X, X^{-1}]$, where K is a field. Therefore $\dim(T^{-1}R) \leq 1$ and $\text{ht}(\mathfrak{p}) \leq 1$, which proves the claim. \square

Proposition 15.5.6. *Let \mathfrak{p} be a prime ideal of R . If $\mathfrak{p} \neq \mathfrak{p}^h$ then $\text{ht}(\mathfrak{p}^h) = \text{ht}(\mathfrak{p}) - 1$.*

Proof. By ??, \mathfrak{p}^h is a prime ideal contained in \mathfrak{p} , so $\text{ht}(\mathfrak{p}^h) \leq \text{ht}(\mathfrak{p}) - 1$. The proposition is trivial if $\text{ht}(\mathfrak{p}^h) = +\infty$, so we may assume that $\text{ht}(\mathfrak{p}^h) < +\infty$. We now demonstrate the inequality $\text{ht}(\mathfrak{p}) \leq \text{ht}(\mathfrak{p}^h) + 1$ by proving that, for any prime ideal \mathfrak{q} contained in \mathfrak{p} and distinct from \mathfrak{p} , we have $\text{ht}(\mathfrak{q}) \leq \text{ht}(\mathfrak{p}^h)$.

We now proceed by induction on $\text{ht}(\mathfrak{p})$. We distinguish two cases: if $\mathfrak{q}^h \neq \mathfrak{p}^h$, then we have $\text{ht}(\mathfrak{q}^h) < \text{ht}(\mathfrak{p}^h)$ since $\mathfrak{q}^h \subset \mathfrak{p}^h$; also, the equality

$$\text{ht}(\mathfrak{q}) \leq \text{ht}(\mathfrak{q}^h) + 1$$

holds by the induction hypothesis if $\mathfrak{q} \neq \mathfrak{q}^h$ and is trivial if $\mathfrak{q} = \mathfrak{q}^h$. Consequently, we get $\text{ht}(\mathfrak{q}) \leq \text{ht}(\mathfrak{q}^h) + 1 \leq \text{ht}(\mathfrak{p}^h)$, which proves our claim. On the other hand, if $\mathfrak{q}^h = \mathfrak{p}^h$, then $\mathfrak{q} = \mathfrak{q}^h$ by ??, so $\text{ht}(\mathfrak{q}) = \text{ht}(\mathfrak{q}^h) = \text{ht}(\mathfrak{p}^h)$. \square

Theorem 15.5.7. *Suppose that the ring R is Noetherian.*

- (a) *Any chain of homogeneous prime ideals of R , saturated as a chain of homogeneous prime ideals, is saturated as a chain of prime ideals.*
- (b) *For any homogeneous prime ideal \mathfrak{p} of R , we have $\text{htgr}(\mathfrak{p}) = \text{ht}(S^{-1}\mathfrak{p}) = \text{ht}(\mathfrak{p})$.*
- (c) *We have $\dimgr(R) = \dim(S^{-1}R) = \dim(R)$.*

Proof. To show (a), it suffices to prove that, if \mathfrak{p} and \mathfrak{q} are distinct homogeneous prime ideals of R such that $\mathfrak{q} \subset \mathfrak{p}$ and there exists no homogeneous prime ideals between \mathfrak{q} and \mathfrak{p} , then $\text{ht}(\mathfrak{p}/\mathfrak{q}) = 1$. By quotienting by \mathfrak{q} , we may assume that $\mathfrak{q} = \{0\}$ (the quotient ring is still Noetherian). In other words, it suffices to show that, if R is integral and Noetherian, and if \mathfrak{p} is an homogeneous prime ideal of R minimal among nonzero homogeneous prime ideals, then $\text{ht}(\mathfrak{p}) = 1$. Now let a be a nonzero homogeneous element of \mathfrak{p} and \mathfrak{r} a prime ideal of R such that $a \in \mathfrak{r} \subset \mathfrak{p}$ and is minimal with this property. Then \mathfrak{r}^h is a nonzero prime ideal ($a \in \mathfrak{r}^h$) contained in \mathfrak{r} , so $\mathfrak{r}^h = \mathfrak{r}$, which means \mathfrak{r} is homogeneous. By the hypothesis on \mathfrak{p} , this means $\mathfrak{p} = \mathfrak{r}$. Since R is Noetherian and integral, \mathfrak{p} has height 1 (??), which proves (a).

Let \mathfrak{p} be a homogeneous prime ideal of R . We already have

$$\text{htgr}(\mathfrak{p}) \leq \text{ht}(S^{-1}\mathfrak{p}) \leq \text{ht}(\mathfrak{p}).$$

We now prove the converse inequality by induction on $\text{htgr}(\mathfrak{p})$. If $\text{htgr}(\mathfrak{p}) = 0$, \mathfrak{p} is minimal among the homogeneous prime ideals, so it is a minimal prime ideal of R (??) and $\text{ht}(\mathfrak{p}) = 0$. Suppose now $\text{ht}(\mathfrak{p}) > 0$, we show the inequality $\text{ht}(\mathfrak{q}) \leq \text{htgr}(\mathfrak{p}) - 1$ for any prime ideal \mathfrak{q}

contained in \mathfrak{p} and distinct from \mathfrak{p} . We distinguish two cases: If \mathfrak{q} is homogeneous, this is true from the induction hypothesis:

$$\text{ht}(\mathfrak{q}) \leq \text{htgr}(\mathfrak{q}) \leq \text{htgr}(\mathfrak{p}) - 1.$$

If \mathfrak{q} is not homogeneous, we have $\mathfrak{q}^h \subset \mathfrak{q} \subset \mathfrak{p}$, so $\text{ht}(\mathfrak{q}^h) \leq \text{htgr}(\mathfrak{q}^h)$ by the induction hypothesis. Recall that $\text{ht}(\mathfrak{q}) = \text{ht}(\mathfrak{q}^h) + 1$ by ??, so $\text{ht}(\mathfrak{q}) \leq \text{htgr}(\mathfrak{q}^h) + 1$. The proof will now be completed if we can prove the inequality $\text{htgr}(\mathfrak{q}^h) \leq \text{htgr}(\mathfrak{p}) - 2$. Assume on the contrary that we have $\text{htgr}(\mathfrak{q}^h) = \text{htgr}(\mathfrak{p}) - 1$; then the chain $\mathfrak{q}^h \subset \mathfrak{p}$ is saturated by part (a), which contradicts the fact $\mathfrak{q}^h \subset \mathfrak{q} \subset \mathfrak{p}$.

Finally, we show that inequality $\dim(R) \leq \text{dimgr}(R)$, or equivalently $\text{ht}(\mathfrak{p}) \leq \text{dimgr}(R)$ for any prime ideal \mathfrak{p} of R . If \mathfrak{p} is homogeneous, we have $\text{ht}(\mathfrak{p}) = \text{htgr}(\mathfrak{p}) \leq \text{dimgr}(R)$ by (b). If \mathfrak{p} is not homogeneous, we have $\text{ht}(\mathfrak{p}) = \text{htgr}(\mathfrak{p}^h) + 1$ by ??; let \mathfrak{m} be a homogeneous maximal ideal of R containing \mathfrak{p}^h (note that the homogeneous ideal is not maximal among homogeneous prime ideals by ??, since we have $\mathfrak{p}^h \subset \mathfrak{p}$). Then by ??, \mathfrak{m} is maximal, so distinct from \mathfrak{p}^h , and we have

$$\text{htgr}(\mathfrak{p}^h) + 1 \leq \text{htgr}(\mathfrak{m}) \leq \text{dimgr}(R).$$

This shows $\text{ht}(\mathfrak{p}) \leq \text{dimgr}(R)$, which finishes the proof. \square

15.5.2 Dimension of graded modules

Again R denote a graded ring with positive degrees and $S = 1 + R_{\geq 1}$. We now let M be a graded R -module of type \mathbb{Z} . Then $S^{-1}M$ is an $S^{-1}R$ -module, and if we write $M_{\geq n} = \bigoplus_{i \geq n} M_i$, the submodules $S^{-1}M_{\geq n}$ form an exhaustive and separated filtration of $S^{-1}M$ and the canonical map $M \mapsto S^{-1}M$ induces an isomorphism of M onto the graded module of $S^{-1}M$ associated to this filtration.

Lemma 15.5.8. *Suppose that R is generated by $R_0 \cup R_1$ and that M is generated by $\bigoplus_{i \leq n_0} M_i$ for an integer n_0 . Then the filtration $(S^{-1}M_{\geq n})$ of $S^{-1}M$ is good for the ideal $S^{-1}R_{\geq 1}$ of $S^{-1}R$.*

Proof. For $n \geq n_0$, we have $M_{\geq n+1} = R_1 \cdot M_{\geq n}$, so

$$S^{-1}M_{\geq n+1} = R_1 \cdot S^{-1}M_{\geq n} = S^{-1}R_{\geq 1} \cdot S^{-1}M_{\geq n},$$

which shows the filtration $(S^{-1}M_{\geq n})$ is $(S^{-1}R_{\geq 1})$ -good. \square

Proposition 15.5.9. *Suppose that R is Noetherian and M is finitely generated. Then $\dim_R(M) = \dim_{S^{-1}R}(S^{-1}M)$.*

Proof. Let \mathfrak{a} be the annihilator of the R -module M , which is a homogeneous ideal of R . Since M is finitely generated, the annihilator of $S^{-1}M$ is the ideal $S^{-1}\mathfrak{a}$. We have $\dim_R(M) = \dim(R/\mathfrak{a})$ and $\dim_{S^{-1}R}(S^{-1}M) = \dim(S^{-1}R/S^{-1}\mathfrak{a})$ by definition. Since R/\mathfrak{a} is Noetherian, an application of ?? proves the claim. \square

Proposition 15.5.10. *Suppose that R_0 is an Artinian local ring, R is generated by $R_0 \cup R_1$, and R_1 is a finitely generated R_0 -module. If M is nonzero and finitely generated as an R -module, then M_n is an*

R_0 -module with finite length for each n , and there exists $Q(T) \in \mathbb{Z}[T, T^{-1}]$ such that $Q(1) > 0$ and the following equality holds in the ring $\mathbb{Z}((T))$:

$$\sum_{n \in \mathbb{Z}} \ell_{R_0}(M_n) \cdot T^n = (1 - T^{-d})Q(T),$$

where $d = \dim_R(M)$.

Proof. The ring $S^{-1}R$ is local and Noetherian by ??, the $S^{-1}R$ -module $S^{-1}M$ is finitely generated and nonzero, and $d = \dim_R(M)$ is nonzero. Also, by ?? the filtration $(S^{-1}M_{\geq n})$ is $S^{-1}R_{\geq 1}$ -good and $S^{-1}R_{\geq 1}$ is a defining ideal of $S^{-1}R$ (??). Finally, we have $\ell_{S^{-1}R}(S^{-1}M_{\geq n}/S^{-1}M_{\geq n+1}) = \ell_{R_0}(M_n)$ for each n , so it suffices to apply ??.

Corollary 15.5.11. *If R is a Noetherian local ring and \mathfrak{a} is a defining ideal of R , then $\dim(R) = \dim(\text{gr}_{\mathfrak{a}}(R))$.*

Proof. Apply ?? on the module $M = B = \text{gr}_{\mathfrak{a}}(R)$, we get

$$\sum_{n=0}^{\infty} \ell_{R/\mathfrak{a}}(\mathfrak{a}^n/\mathfrak{a}^{n+1}) \cdot T^n = (1 - T)^{-d}Q(T)$$

where $d = \dim(\text{gr}_{\mathfrak{a}}(R))$ and $Q(1) \neq 0$. We have $d = \dim(R)$ by ??, whence the corollary. \square

Corollary 15.5.12. *Suppose that M is a finitely generated R -module. Let a be a homogeneous element of R with positive degree, and not belonging to any minimal element \mathfrak{p} in $\text{supp}(M)$ such that $\dim(R/\mathfrak{p}) = \dim_R(M)$. Then $\dim_R(M/aM) = \dim_R(M) - 1$.*

Proof. Put $d = \dim_R(M)$; by the hypothesis on a , we have $\dim_R(M/aM) < d$. Also, by ??, we have

$$\dim_R(M/aM) = \dim_{S^{-1}M}(S^{-1}M/(b/1) \cdot S^{-1}M)$$

and the formula (??) implies the following inequality:

$$\dim_{S^{-1}R}(S^{-1}M/(b/1) \cdot S^{-1}M) \geq \dim_{S^{-1}R}(S^{-1}M) - 1.$$

Finally, $\dim_{S^{-1}R}(S^{-1}M) = \dim_R(M) = d$ by ?? . We then conclude that $\dim_R(M/bM) \geq d - 1$, which proves the claim. \square

Proposition 15.5.13. *Suppose that R_0 is an Artinian local ring, R is an R_0 -algebra of finite type and M is a finitely generated R -module.*

- (a) *Let a_1, \dots, a_n be homogeneous elements of R with positive degrees and φ the homomorphism from $R_0[X_1, \dots, X_n]$ to R map X_i to a_i for each i . Then $S^{-1}M/\sum_{i=1}^n (a_i/1) \cdot S^{-1}M$ has finite length over $S^{-1}R$ if and only if $\varphi^*(M)$ is a finitely generated module over $R_0[X_1, \dots, X_n]$.*
- (b) *There exists a family (a_1, \dots, a_d) of elements of R , homogeneous with the same positive degree, where $d = \dim_R(M)$, such that $(a_1/1, \dots, a_d/1)$ is a maximal secant sequence for the $S^{-1}R$ -module $S^{-1}M$. If moreover R is generated by R_1 as an R_0 -algebra and the residue field of R_0 is infinite, each a_i can be taken to have degree 1.*

Proof. Put $N = M / \sum_{i=1}^n a_i M$, we have $\dim_R(N) = \dim_{S^{-1}R}(S^{-1}N)$ by ???. Also, the $S^{-1}R$ -module $S^{-1}N$ is of finite length if and only if the R -module N has finite length, that is, N is a finitely generated R_0 -module. If $\varphi^*(M)$ is the pullback module over $R_0[X_1, \dots, X_n]$, we have $\varphi^*(N) = \varphi^*(M) / \sum_{i=1}^n X_i \cdot \varphi^*(M)$. Therefore $\varphi^*(M)$ is a finitely generated $R_0[X_1, \dots, X_n]$ -module if and only if N is a finitely generated R_0 -module, which proves (a).

Let's assume the hypothesis in ??(b). We may assume that $\dim_R(M) > 0$. Note that any minimal element of $\text{supp}(M)$ is homogeneous (??). According to ??, there exists a homogeneous element a of R with positive degree, not belonging to any minimal element \mathfrak{p} in $\text{supp}(M)$ such that $\dim(R/\mathfrak{p}) = \dim_R(M)$. By ??, we have $\dim_R(M/aM) = \dim_R(M) - 1$. Suppose further that R is generated by R_1 as an R_0 -algebra and the residue field k of R_0 is infinite. For any minimal element \mathfrak{p} in $\text{supp}(M)$ such that $\dim(R/\mathfrak{p}) = \dim_R(M)$, consider the subspace $V_{\mathfrak{p}} = (\mathfrak{p} \cap R_1) \otimes_{R_0} k$ of the k -vector space $V = R_1 \otimes_{R_0} k$. These subspaces are proper: if $V_{\mathfrak{p}} = V$, then $\mathfrak{p} \cap H_1 = H_1$ by ??, so $H_1 \subseteq \mathfrak{p}$ and $\dim_R(M) = \dim(R/\mathfrak{p}) \leq \dim(R/R_1) = 0$, which contradicts to our assumption. Since k is infinite, the union of $V_{\mathfrak{p}}$'s is distinct from V . If $a \in R_1$ is such that $a \otimes 1$ is not contained in any of the $V_{\mathfrak{p}}$, then a is not contained in the minimal element \mathfrak{p} such that $\dim(R/\mathfrak{p}) = \dim_R(M)$, whence $\dim_R(M/aM) = \dim_R(M) - 1$.

By induction on the dimension $d = \dim_R(M)$, we can construct a sequence (b_1, \dots, b_d) of elements in R , with $n_i = \deg(b_i) > 0$, such that $M / \sum_{i=1}^d b_i M$ is a R -module of finite length. Moreover, if R is generated by R_1 as an R_0 -algebra and the residue field of R_0 is infinite, each b_i can be taken to have degree 1. Now by ??, $\dim_{S^{-1}R}(S^{-1}M) = d$ and

$$\dim_{S^{-1}R}(S^{-1}M / \sum_{i=1}^d (b_i/1) \cdot S^{-1}M) = 0.$$

Therefore $(b_1/1, \dots, b_d/1)$ is a maximal secant sequence for $S^{-1}M$. Put $a_i = b_i^{(n_1 \cdots n_d)/n_i}$, we see the a_i 's have the same degree, and $(a_1/1, \dots, a_d/1)$ is a maximal secant sequence for $S^{-1}M$ by ???. \square

Corollary 15.5.14. Suppose that R_0 is a field and R is an R_0 -algebra of finite type. Put $n = \dim(R)$. Then there exists homogeneous elements a_1, \dots, a_n of R with the same positive degree such that the R_0 -homomorphism $\varphi : R_0[X_1, \dots, X_n] \rightarrow R$, defined by $\varphi(X_i) = a_i$ for each i , is injective and makes R a finitely generated $R_0[X_1, \dots, X_n]$ -module. If R is generated by R_1 as an R_0 -algebra and R_0 is infinite, each a_i can be taken to have degree 1.

Proof. There exists by ?? an R_0 -homomorphism φ of the indicated form making R a finitely generated $R_0[X_1, \dots, X_n]$ -module. By ??, we also have

$$\dim(R) = n = \dim(R_0[X_1, \dots, X_n]).$$

Since the ring $R_0[X_1, \dots, X_n]$ is integral and $R \cong R_0[X_1, \dots, X_n] / \ker \varphi$, we conclude that $\ker \varphi = 0$. \square

Remark 15.5.3. Assume the hypothesis in ???. Let (h_1, \dots, h_r) be a basis for the R_0 -vector space R_1 . For $\lambda = (\lambda_1, \dots, \lambda_r) \in R_0^r$, put $h_{\lambda} = \lambda_1 h_1 + \cdots + \lambda_r h_r$. Then ?? and ?? imply the following result: the set of elements $(\lambda^{(1)}, \dots, \lambda^{(n)})$ of $(R_0^r)^n$ such that the elements $a_i = h_{\lambda_i} \in R_1$ satisfies the conclusion of ?? contains the complement of the union of a finite number of vector subspaces of $(R_0^r)^n$ distinct from the entire space.

Corollary 15.5.15. *Let A be a Noetherian local ring and d a non-negative integer. For $\dim(A) \geq n$, it is necessary and sufficient that for any integer $r \geq 0$ we have*

$$[\mathfrak{m}_A^r / \mathfrak{m}_A^{r+1} : \kappa_A] \geq \binom{n+r-1}{n-1}.$$

Moreover, the equality holds if and only if A is regular of dimension n .

Proof. By ?? and ??, this condition is sufficient. We now demonstrate the necessity, so assume that $\dim(A) \geq n$. Consider the graded ring $\text{gr}(A) = \text{gr}_{\mathfrak{m}_A}(A)$; let k be an extension of the residue field κ_A such that k is infinite, and put $R = k \otimes_{\kappa_A} \text{gr}(A)$. The ring R has dimension $\geq n$ by ???. We deduce by ?? the existence of an injective k -algebra homomorphism $\varphi : R_0[X_1, \dots, X_n] \rightarrow R$. Therefore for any integer $r \geq 0$,

$$[\mathfrak{m}_A^r / \mathfrak{m}_A^{r+1} : \kappa_A] = [R_r : R_0] \geq \binom{n+r-1}{n-1},$$

and the equality holds if and only if φ is bijective, whence A is regular (??). \square

Remark 15.5.4. Note that we have the equality

$$\binom{n+r}{n} = \sum_{i=0}^r \binom{n+i-1}{n-1}, \quad \ell_A(A/\mathfrak{m}_A^{r+1}) = \sum_{i=0}^r [\text{gr}_i(A) : \kappa_A].$$

Therefore the condition in ?? is equivalent to $\ell_A(A/\mathfrak{m}_A^{r+1}) \geq \binom{n+r}{n}$.

15.5.3 Dimension and graded algebras

Lemma 15.5.16. *Let A be a ring, \mathfrak{r} its Jacobson radical, $R = \bigoplus_{i \in \mathbb{Z}} R_i$ a graded A -algebra, and $M = \bigoplus_{i \in \mathbb{Z}} M_i$ a graded R -module. Suppose that each M_i is a finitely generated A -module and $M/\mathfrak{r}M$ is a finitely generated $R/\mathfrak{r}R$ module. Then M is a finitely generated R -module.*

Proof. Let m_1, \dots, m_n be homogeneous elements of M whose images generate $M/\mathfrak{r}M$. Let N be the graded submodule of M generated by m_1, \dots, m_n . We have $M_i = N_i + \mathfrak{r}M_i$ for each i , whence $M_i = N_i$ (??); this shows $M = N$. \square

Lemma 15.5.17. *Let $\rho : A \rightarrow B$ be a ring homomorphism and S be a multiplicative subset of A . Suppose that B is an A -algebra of finite type and $S^{-1}B$ is a finitely generated $S^{-1}A$ -module. Then there exists $f \in S$ such that B_f is a finitely generated A_f -module.*

Proof. Let X be a generating set of the A -algebra B . For each $x \in X$, the image of x in $S^{-1}B$ is integral over $S^{-1}A$, so there exist an integer $n(x) \geq 0$, elements $b_1(x), \dots, b_{n(x)}(x)$ of A , and an element $f(x) \in S$, such that

$$f(x)x^{n(x)} + b_1(x)x^{n(x)-1} + \cdots + b_{n(x)}(x) = 0.$$

Let $f = \prod_{x \in X} f(x)$; the image of $x \in X$ in B_f is then integral over A_f , so B_f is a finitely generated A_f -module. \square

Proposition 15.5.18. *Suppose that R is an R_0 -algebra of finite type. Then the function $\mathfrak{p} \mapsto \dim(R \otimes_{R_0} \kappa(\mathfrak{p}))$ is upper semi-continuous on $\text{Spec}(A_0)$.*

Proof. Since R is an R_0 -algebra of finite type, each R_i is finitely generated as an R_0 -module (??). Let $\mathfrak{p} \in \text{Spec}(R_0)$ and suppose $\dim(R \otimes_{R_0} \kappa(\mathfrak{p})) = n \geq 0$. By ??, there exists a_1, \dots, a_n in R , homogeneous of degree $d > 0$, such that the $\kappa(\mathfrak{p})$ -homomorphism $\bar{\varphi} : \kappa(\mathfrak{p})[X_1, \dots, X_n] \rightarrow R \otimes_{R_0} \kappa(\mathfrak{p})$ map X_i to $a_i \otimes 1$ for each i , makes $R \otimes_{R_0} \kappa(\mathfrak{p})$ a finitely generated $\kappa(\mathfrak{p})[X_1, \dots, X_n]$ -module. Let φ be the R_0 -homomorphism from $R_0[X_1, \dots, X_n] = H$ to R map X_i to a_i for each i .

For $m \in \mathbb{Z}$, define

$$\tilde{R}_m = \bigoplus_{(m-1)d < i \leq md} R_i.$$

Then (\tilde{R}_m) is a graduation of type \mathbb{Z} on R which is compatible with the $R_0[X_1, \dots, X_n]$ -module structure induced by φ ; moreover, each \tilde{R}_m is finitely generated over R_0 . Now note that the Jacobson radical of $(R_0)_{\mathfrak{p}}$ is $\mathfrak{p}(R_0)_{\mathfrak{p}}$ and

$$R_{\mathfrak{p}}/\mathfrak{p}(R_0)_{\mathfrak{p}}R_{\mathfrak{p}} = R \otimes_{R_0} \kappa(\mathfrak{p}), \quad H_{\mathfrak{p}}/\mathfrak{p}(R_0)_{\mathfrak{p}}H_{\mathfrak{p}} = \kappa(\mathfrak{p})[X_1, \dots, X_n].$$

By using ??, we conclude that $R_{\mathfrak{p}}$ is a finitely generated $H_{\mathfrak{p}}$ -module. Now apply ??, there then exists $f \in R_0 \setminus \mathfrak{p}$ such that R_f is finitely generated H_f -module. For any $\mathfrak{q} \in \text{Spec}((R_0)_f)$, $R \otimes_{R_0} \kappa(\mathfrak{q})$ is then a finitely generated $\kappa(\mathfrak{q})[X_1, \dots, X_n]$ -module, whence $\dim(R \otimes_{R_0} \kappa(\mathfrak{q})) \leq n$ by ?? This completes the proof. \square

From now on, we assume that R_0 is a field and R is an R_0 -algebra of finite type. In this case, we set $R_+ = R_{\geq 1}$, which is a maximal ideal of R . The ring $S^{-1}R$ can be then identified with the local ring R_{R_+} , whose maximal ideal is $(R_+)_{R_+} = S^{-1}R_+$ with residue field A_0 .

Proposition 15.5.19. Suppose that R_0 a field and R is a finite type R_0 -algebra.

- (a) We have $\dim(R) \leq [R_+/R_+^2 : R_0]$.
- (b) The following conditions are equivalent:
 - (i) $\dim(R) = [R_+/R_+^2 : R_0]$.
 - (ii) The Noetherian local ring $S^{-1}R$ is regular.
 - (iii) R is generated as an R_0 -algebra by a family of homogeneous elements with positive degrees, algebraically independent over R_0 .
- (c) Suppose that the conditions in (b) holds, and let $a_1, \dots, a_n \in R$ be homogeneous elements with positive degrees. Then the following conditions are equivalent:
 - (i') The images of a_i 's form a basis for the R_0 -vector space R_+/R_+^2 .
 - (ii') The images of a_i 's form a system of parameters in the Noetherian local ring $S^{-1}R$.
 - (iii') The a_i 's are algebraically independent over R_0 and generate R as an R_0 -algebra.

Proof. Since R is Noetherian, we have $\dim(R) = \dim(S^{-1}R)$ by ??; also,

$$[R_+/R_+^2 : R_0] = [(S^{-1}R_+)/((S^{-1}R_+)^2 : R_0)]$$

by ?? These together prove (a) and the equivalences (i) \Leftrightarrow (ii), (i') \Leftrightarrow (ii'). The implications (iii) \Rightarrow (i) and (iii') \Rightarrow (i') are trivial. To establish (i) \Rightarrow (iii), suppose that $\dim(R) = [R_+/R_+^2 : R_0]$

and let a_1, \dots, a_n be homogeneous elements of R with positive degrees whose classes in R_+ / R_+^2 form a basis. Consider the homomorphism $\varphi : R_0[X_1, \dots, X_n]$ to R map X_i to a_i . The ideal R_+ of R is generated by a_i , so the homomorphism φ is surjective. Thus we have

$$\dim(R_0[X_1, \dots, X_n]) = n = \dim(R) = \dim(R_0[X_1, \dots, X_n] / \ker \varphi)$$

which implies $\ker \varphi = 0$ (recall that $R_0[X_1, \dots, X_n]$ is an integral domain). This implies (iii), and the implication (i') \Rightarrow (iii') can be done similarly. \square

If R satisfies the hypothesis in (b), we say R is a regular graded R_0 -algebra, or a polynomial graded R_0 -algebra. In this case, any family (a_1, \dots, a_n) satisfying the hypothesis in (c) will be called a system of parameters of R .

Remark 15.5.5. In the notations of (c), let d_i be the degree of a_i for each i . Then the Poincaré series of H is given by

$$P_R = \sum_{n \in \mathbb{Z}} [R_n : R_0] \cdot T^n = \prod_{i=1}^n (1 - T^{d_i})^{-1}.$$

Conversely, if there exist integers $d_i > 0$ such that $P_R = \prod_i (1 - T^{d_i})^{-1}$, we may conclude that R is a polynomial graded algebra. For example, if R is generated by X with degree 1 and Y with degree 2 with the sole relation $X^2 = 0$, we have $P_R = (1 - T)^{-1}$, so R is regular.

15.6 Multiplicity

15.6.1 Multiplicity relative to an ideal

Let A be a Noetherian ring and M be a finitely generated A -module. Let \mathfrak{a} be an ideal of A contained in the Jacobson radical of A and such that $M/\mathfrak{a}M$ is of finite length. Suppose that M is nonzero and put $d = \dim_A(M)$. Recall that (??) there exists a unique integer $e_{\mathfrak{a}}(M) > 0$ such that, for $n \geq 1$,

$$\ell_A(M/\mathfrak{a}^{n+1}M) = e_{\mathfrak{a}}(M) \frac{n^d}{d!} + \beta_n n^{d-1} \quad (15.6.1)$$

where β_n has a limit when n goes to infinity. The integer $e_{\mathfrak{a}}(M)$ (or $e_{\mathfrak{a}}^A(M)$ if we want to emphasize the ring A) is called the **multiplicity** of M relative to \mathfrak{a} . If A is a local ring and \mathfrak{m} is the maximal ideal of A , we write $e(M)$ or $e^A(M)$ for $e_{\mathfrak{m}}^A(M)$. Recall that equivalently, we have

$$\ell_A(\mathfrak{a}^n M / \mathfrak{a}^{n+1} M) = e_{\mathfrak{a}}(M) \frac{n^{d-1}}{(d-1)!} + \alpha_n n^{d-1}$$

where α_n has a limit when n goes to infinity.

Remark 15.6.1. Let \mathfrak{b} be an ideal of A contained in the Jacobson radical of A and contains \mathfrak{a} . Then $M/\mathfrak{b}^n M$ is a quotient of $M/\mathfrak{a}^n M$, whence $e_{\mathfrak{b}}(M) \leq e_{\mathfrak{a}}(M)$ by the formula (??). Moreover, if the \mathfrak{b} -adic filtration on M is \mathfrak{a} -good, then $e_{\mathfrak{a}}(M) = e_{\mathfrak{b}}(M)$ by ??.

Recall that we can reduce the calculus of the multiplicities to the case where A is local since, according to ??, we have

$$e_{\mathfrak{a}}(M) = \sum_{\mathfrak{m}} e_{\mathfrak{a}_{\mathfrak{m}}}(M_{\mathfrak{m}})$$

where \mathfrak{m} runs through the elements of $\text{supp}(M) \cap V(\mathfrak{a})$ such that $\dim_{A_{\mathfrak{m}}}(M_{\mathfrak{m}}) = d_{\mathfrak{a}}(M)$.

Proposition 15.6.1. *Let \widehat{A} (resp. \widehat{M}) be the \mathfrak{a} -adic completion of A (resp. M), then $e_{\mathfrak{a}}^A(M) = e_{\mathfrak{a}}^{\widehat{A}}(\widehat{M})$.*

Proof. This follows from the fact that $\text{gr}_{\mathfrak{a}}(M) \cong \text{gr}_{\mathfrak{a}}(\widehat{M})$. \square

Proposition 15.6.2. *If A is regular, then $e(A) = 1$.*

Proof. If A is regular then $H_{A,\mathfrak{m}_A} = (1 - T)^{-r}$ (??), whence $e(A) = 1$. \square

Example 15.6.3. By definition $e_{\mathfrak{a}^r}(M) = r^d e_{\mathfrak{a}}(M)$ where $d = \dim_A(M)$. Consequently, if A is a regular local ring, we have $e_{\mathfrak{m}_A^r}(A) = r^d$. For example, if A is a discrete valuation ring, then $e_{\mathfrak{m}_A^r}(A) = r$, whence $e_{\mathfrak{a}}(A) = \ell(A/\mathfrak{a})$ for any ideal \mathfrak{a} .

For an ideal \mathfrak{a} of A contained in the Jacobson radical of A , we let $\mathcal{C}(\mathfrak{a})$ denote the set of classes of finitely generated A -modules M such that $M/\mathfrak{a}M$ is of finite length. For each $d \in \mathbb{N}$, let $\mathcal{C}(\mathfrak{a})_{\leq d}$ be the subset of $\mathcal{C}(\mathfrak{a})$ consists of classes of A -modules of dimension $\leq d$. We have a map $e_{\mathfrak{a},d} : \mathcal{C}(\mathfrak{a})_{\leq d} \rightarrow \mathbb{Z}$ such that $e_{\mathfrak{a},d}(M) = e_{\mathfrak{a}}(M)$ if $\dim_A(M) = d$ and $e_{\mathfrak{a},d}(M) = 0$ otherwise. This map is additive by ??, so we get an induced homomorphism, still denoted by $e_{\mathfrak{a},d}$, from the Grothendieck group $K(\mathcal{C}(\mathfrak{a})_{\leq d})$ to \mathbb{Z} , which is zero on $K(\mathcal{C}(\mathfrak{a})_{\leq d-1})$.

Proposition 15.6.4. *Let M be a finitely generated A -module with dimension $d \geq 0$. Let \mathfrak{a} be an ideal of A contained in the Jacobson radical of A such that $M/\mathfrak{a}M$ has finite length. Then*

$$e_{\mathfrak{a}}(M) = \sum_{\text{coht}(\mathfrak{p})=d} \ell_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}) \cdot e_{\mathfrak{a}}(A/\mathfrak{p})$$

Proof. By ??, M has a composition series $(M_i)_{0 \leq i \leq n}$ such that M_i/M_{i+1} is isomorphic to A/\mathfrak{p}_i , where \mathfrak{p}_i is a prime ideal of M . Also, by ??, the number $\ell_{A_{\mathfrak{p}}}(M_{\mathfrak{p}})$ is the times of i such that $M_i/M_{i+1} \cong A/\mathfrak{p}$. The proposition then follows from ?? \square

Corollary 15.6.5. *Suppose that A is Noetherian semi-local and let \mathfrak{a} be a defining ideal for A .*

- (a) *We have $e_{\mathfrak{a}}(A) = \sum_{\mathfrak{p}} \ell_{A_{\mathfrak{p}}}(A_{\mathfrak{p}}) \cdot e_{\mathfrak{a}}(A/\mathfrak{p})$, where \mathfrak{p} turns through minimal primes of A such that $\text{coht}(\mathfrak{p}) = \dim(A)$.*
- (b) *Suppose that A is integral and M is a finitely generated A -module such that $\dim_A(M) = \dim(A)$. Then $e_{\mathfrak{a}}(M) = \text{rank}(M) \cdot e_{\mathfrak{a}}(A)$.*

Proof. Part (a) follows directly from ?? by applying on $M = A$. Under the hypothesis of (b), the minimal prime ideal of A is (0) , and $\ell_{A_{(0)}}(M_{(0)}) = \text{rank}(M)$, so the claim follows. \square

15.6.2 Extension of scalars

Proposition 15.6.6. *Let $\rho : A \rightarrow B$ be a local homomorphism of Noetherian local rings. Let N be a finitely generated B -module, flat over A , and such that $N \otimes_A \kappa_A$ is a B -module of finite length. If M is a nonzero finitely generated A -module and \mathfrak{a} is a proper ideal of A such that $M/\mathfrak{a}M$ is of finite length, then $(M \otimes_A N)/\mathfrak{a}^e(M \otimes_A N)$ is a B -module of finite length, and we have*

$$e_{\mathfrak{a}^e}^B(M \otimes_A N) = \ell_B(N \otimes_A \kappa_A) \cdot e_{\mathfrak{a}}^A(M).$$

Proof. Let L be an A -module of finite length r . Then by ?? and ??, L has a composition series of length r , with quotients κ_A . Since N is flat over A , the B -module $L \otimes_A N$ possesses a composition series of length r , with quotients $N \otimes_A \kappa_A$. Therefore the length of $L \otimes_A N$ is $r \cdot \ell_B(N \otimes_A \kappa_A)$. The B -module $(M \otimes_A N)/(\mathfrak{a}^e)^n(M \otimes_A N)$ is isomorphic to $(M/\mathfrak{a}^n M) \otimes_A N$ for each $n \in \mathbb{N}$, the proposition then follows from the definition of multiplicity. \square

Corollary 15.6.7. Suppose that $\rho : A \rightarrow B$ be a local homomorphism of Noetherian local rings, B is flat over A and $\mathfrak{m}_A^e = \mathfrak{m}_B$. Then

$$e_{\mathfrak{a}^e}^B(M \otimes_A B) = e_{\mathfrak{a}}^A(M).$$

Proof. In this case we have $B \otimes_A \kappa_A = B/\mathfrak{m}_A^e$, which has length 1 over B , so the claim follows from ??.

Example 15.6.8. Let X be a complex algebraic variety, $\mathcal{O}_{X,x}$ the local ring of X at a rational point x , and X^{an} the analytic space associated with X . Let $\mathcal{O}_{X^{an},x}$ be the local ring of X^{an} at x . Then $e(\mathcal{O}_{X^{an},x}) = e(\mathcal{O}_{X,x})$.

Proposition 15.6.9. Suppose that A is Noetherian semi-local and let $\rho : A \rightarrow B$ be a ring homomorphism making B a finitely generated A -module. Let N be a nonzero finitely generated B -module and \mathfrak{a} an ideal of A contained in the Jacobson radical of A such that $N/\mathfrak{a}N$ has finite length. Let $\mathfrak{m}_1, \dots, \mathfrak{m}_r$ be the maximal ideals of B such that $\dim_{B_{\mathfrak{m}_i}}(N_{\mathfrak{m}_i}) = \dim_B(N)$ (by ?? B is semi-local). Put $B_i = B_{\mathfrak{m}_i}$ and $\mathfrak{a}_i = \mathfrak{a}B_i$ for each i . Then we have the equalities

$$\dim_A(N) = \dim_B(N), \quad e_{\mathfrak{a}^e}^B(N) = \sum_{i=1}^r e_{\mathfrak{a}_i}^{B_i}(N_{\mathfrak{m}_i}), \quad e_{\mathfrak{a}}^A(N) = \sum_{i=1}^r [B/\mathfrak{m}_i : A/\mathfrak{m}_i^c] \cdot e_{\mathfrak{a}_i}^{B_i}(N_{\mathfrak{m}_i}).$$

Proof. Since B is integral over A , we have $\dim_A(N) = \dim_B(N)$. The second equality follows from ?? (note that \mathfrak{m}_i is in $V(\mathfrak{a}^e)$ for each i since $\mathfrak{m}_i^c \supset \mathfrak{a}$ by ??). Now let E be a B -module of finite length; we have

$$\ell_A(E) = \sum_{\mathfrak{m}} [B/\mathfrak{m} : A/\mathfrak{m}^c] \cdot \ell_{B_{\mathfrak{m}}}(E_{\mathfrak{m}}),$$

where \mathfrak{m} runs through maximal ideals of B : this is evident when E is one of the B/\mathfrak{m} , and the general case follows since E has a composition sequence with quotients isomorphic to B/\mathfrak{m} , with \mathfrak{m} a maximal ideal of B . Apply this formula to the B -module $N/\mathfrak{a}^{n+1}N$, we deduce the equality from the definition of multiplicity. \square

Corollary 15.6.10. If $[B/\mathfrak{m}_i : A/\mathfrak{m}_i^c] = 1$ for each i , then $e_{\mathfrak{a}}^A(N) = e_{\mathfrak{a}^e}^B(N)$.

Lemma 15.6.11. Let $\rho : A \rightarrow B$ be a ring homomorphism and \mathfrak{p} a prime ideal of A . Consider the following conditions:

- (i) The canonical homomorphism $\tilde{\rho} : A_{\mathfrak{p}} \rightarrow A_{\mathfrak{p}} \otimes_A B$ is bijective.
- (ii) There is only one prime ideal \mathfrak{P} of B lying over \mathfrak{p} and the canonical homomorphism $\rho_{\mathfrak{p}} : A_{\mathfrak{p}} \rightarrow B_{\mathfrak{P}}$ is bijective.

We have (i) \Rightarrow (ii). If \mathfrak{p} is minimal, or if B is integral over A , then (i) \Leftrightarrow (ii).

Proof. The ring $A_{\mathfrak{p}} \otimes_A B$ can be identified with the localization $S^{-1}B$ by the multiplicative subset $S = \rho(A - \mathfrak{p})$. The prime ideals of $S^{-1}B$ are of the form $S^{-1}\mathfrak{P}$, where \mathfrak{P} is a prime ideal of B such that $\mathfrak{P}^c \subseteq \mathfrak{p}$. If \mathfrak{P} is such an ideal, $(S^{-1}B)_{S^{-1}\mathfrak{P}}$ is identified with $B_{\mathfrak{P}}$.

If condition (i) is satisfied, there exists (??) a unique prime ideal \mathfrak{P} of B such that $\mathfrak{P}^c = \mathfrak{p}$. Moreover, $B_{\mathfrak{P}}$ is identified with the ring $(S^{-1}B)_{S^{-1}\mathfrak{P}}$, hence with $(A_{\mathfrak{p}})_{\mathfrak{q}}$, where \mathfrak{q} is the inverse image of $S^{-1}\mathfrak{P}$ under the isomorphism $\tilde{\rho} : A_{\mathfrak{p}} \rightarrow S^{-1}B$; then $\tilde{\rho}^{-1}(S^{-1}\mathfrak{P}) = (A - \mathfrak{p})^{-1}\mathfrak{p} = \mathfrak{p}A_{\mathfrak{p}}$, hence (ii).

Conversely, assume (ii) and let \mathfrak{P} be the unique prime ideal of B lying over \mathfrak{p} . Since $(S^{-1}B)_{S^{-1}\mathfrak{P}}$ is identified with $B_{\mathfrak{P}}$, it suffices to prove that $S^{-1}B$ is a local ring with maximal ideal $S^{-1}\mathfrak{P}$, which means any prime ideal \mathfrak{Q} of B such that $\mathfrak{Q}^c \subseteq \mathfrak{p}$ is contained in \mathfrak{P} . If \mathfrak{p} is minimal, we have $\mathfrak{Q}^c = \mathfrak{p}$, whence $\mathfrak{Q} = \mathfrak{P}$. If B is integral over A , then by the going up theorem, there exists a prime ideal \mathfrak{P}' of B such that $\mathfrak{Q} \subseteq \mathfrak{P}'$ and $(\mathfrak{P}')^c = \mathfrak{p}$. Then $\mathfrak{P}' = \mathfrak{P}$, whence $\mathfrak{Q} \subseteq \mathfrak{P}$. \square

Lemma 15.6.12. *Suppose that A is Noetherian and semi-local. Let \mathfrak{a} be a defining ideal of A , and $\rho : A \rightarrow B$ a ring homomorphism making B a finitely generated A -module. Suppose that, for any prime ideal \mathfrak{p} (necessarily minimal) of A such that $\dim(A/\mathfrak{p}) = \dim(A)$, there exists a unique prime ideal \mathfrak{P} of B lying over \mathfrak{p} such that the canonical homomorphism $\rho_{\mathfrak{p}} : A_{\mathfrak{p}} \rightarrow B_{\mathfrak{P}}$ is bijective. Then $\dim_A(B) = \dim(A)$ and $e_{\mathfrak{a}}^A(B) = e_{\mathfrak{a}}^A(A)$.*

Proof. Let \mathcal{P}_A (resp. \mathcal{P}_B) be the set of prime ideals of A such that $\dim(A/\mathfrak{p}) = \dim(A)$ (resp. $\dim_A(B/\mathfrak{p}^e) = \dim_A(B)$); we have $\mathcal{P}_A \neq \emptyset$. Let $\mathfrak{p} \in \mathcal{P}_A$; there exists by hypothesis a unique ideal of B lying over \mathfrak{p} . By ?? we have $\mathfrak{p}^{ec} = \mathfrak{p}$, and ?? then shows that $\dim(A/\mathfrak{p}) = \dim(B/\mathfrak{p}^e) = \dim_A(B/\mathfrak{p}^e)$, whence

$$\dim_A(B) \geq \dim_A(B/\mathfrak{p}^e) = \dim(A/\mathfrak{p}) = \dim(A) \geq \dim_A(B).$$

This implies $\mathcal{P}_A \subseteq \mathcal{P}_B$ and $\dim(A) = \dim_A(B)$. Conversely, if $\mathfrak{p} \in \mathcal{P}_B$, we have the inequalities

$$\dim_A(B/\mathfrak{p}^e) = \dim_A(B) = \dim(A) \geq \dim(A/\mathfrak{p}) \geq \dim_A(B/\mathfrak{p}^e)$$

so $\mathfrak{p} \in \mathcal{P}_A$ and therefore $\mathcal{P}_A = \mathcal{P}_B$. Now by ?? and ??, we have

$$e_{\mathfrak{a}}^A(A) = \sum_{\mathfrak{p} \in \mathcal{P}_A} \ell_{A_{\mathfrak{p}}}(A_{\mathfrak{p}}) \cdot e_{\mathfrak{a}}^A(A/\mathfrak{p}), \quad e_{\mathfrak{a}}^A(B) = \sum_{\mathfrak{p} \in \mathcal{P}_B} \ell_{A_{\mathfrak{p}}}(A_{\mathfrak{p}} \otimes_A B) \cdot e_{\mathfrak{a}}^A(A/\mathfrak{p}).$$

Also, by ??, under the hypothesis on ρ we have $\ell_{A_{\mathfrak{p}}}(A_{\mathfrak{p}} \otimes_A B) = \ell_{A_{\mathfrak{p}}}(A_{\mathfrak{p}}) = 1$ for every $\mathfrak{p} \in \mathcal{P}_A$, so $e_{\mathfrak{a}}^A(A) = e_{\mathfrak{a}}^A(B)$. \square

Proposition 15.6.13. *Suppose that A is a Noetherian semi-local ring and reduced; let \mathfrak{a} be a defining ideal of A ; let $Q(A)$ be the total ring of fraction of A , and B a sub A -algebra of $Q(A)$ that is finitely generated as an A -module. Then B is semi-local and \mathfrak{a}^e is a defining ideal of B . Moreover, if for any maximal ideal \mathfrak{m} of B such that $\dim(B_{\mathfrak{m}}) = \dim(B)$ we have $[B/\mathfrak{m} : A/\mathfrak{m}^c] = 1$, then $e_{\mathfrak{a}}^A(A) = e_{\mathfrak{a}^e}^B(B)$.*

Proof. By Corollary ??, B is semi-local with defining ideal \mathfrak{a}^e . Then $e_{\mathfrak{a}^e}^B(B) = e_{\mathfrak{a}}^A(A)$ by ???. Recall that $Q(A)$ can be identified with the ring $\prod_{\mathfrak{p}} A_{\mathfrak{p}}$, where \mathfrak{p} runs through minimal prime ideals of A (??), so the canonical map $A_{\mathfrak{p}} \rightarrow A_{\mathfrak{p}} \otimes_A B$ is bijective for any minimal prime \mathfrak{p} of A . The proposition them follows from ?? and ???. \square

Example 15.6.14. Let k be a field with $\text{char}(k) \neq 2$ and let A be the local ring $k[\![X, Y]\!]/(X^2 + Y^2)$, with residue field k . It is easy to compute that $e_{\mathfrak{m}_A}^A(A) = 2$. Put $B = k[\![X, T]\!]/(T^2 + 1)$ where

$T = Y/X$, we distinguish two cases:

- (a) If -1 is a square of an element i in k , then B has two maximal ideals generated by $\{X, T+i\}$ and $\{X, T-i\}$, respectively, with residue field k . We have $e_{m_A B}^B(B) = 2$.
- (b) If -1 is not a square in k , then B has a unique maximal ideal (X) with residue field $k[T]/(T^2 + 1)$, and we have $e_{m_A B}^B(B) = 1$.

15.6.3 Multiplicity and secant sequences

Proposition 15.6.15. Suppose that A is a Noetherian local ring and $s \geq 1$ be an integer. For $1 \leq i \leq s$, let δ_i be a positive integer, x_i an element of $m_A^{\delta_i}$, and ξ_i the its class in $m_A^{\delta_i}/m_A^{\delta_i+1}$. Suppose that (x_1, \dots, x_s) is a secant sequence for A and let \mathfrak{x} be the ideal of A generated by (x_1, \dots, x_s) . Then $e(A/\mathfrak{x}) \geq \delta_1 \cdots \delta_s e(A)$ and the equality holds if (ξ_1, \dots, ξ_s) is a completely secant sequence for $\text{gr}(A)$.

Proof. Put $B = A/\mathfrak{x}$, and consider the formal series

$$H_A = \sum_{n=0}^{\infty} \ell_{\kappa_A}(m_A^n/m_A^{n+1}) \cdot T^n, \quad H_B = \sum_{n=0}^{\infty} \ell_{\kappa_B}(m_B^n/m_B^{n+1}) \cdot T^n.$$

Recall that, by ??, we have the following inequality in $\mathbb{Z}[[T]]$:

$$H_B^{(r)} \geq \prod_{i=1}^s (1 - T^{\delta_i}) H_A^{(s)} \tag{15.6.2}$$

and the equality holds if and only if (ξ_1, \dots, ξ_s) is completely secant for $\text{gr}(A)$. Put $d = \dim(A)$, so that $\dim(B) = d - s$ since (x_1, \dots, x_s) is secant for A . By ??, there then exists elements R_A and R_B in $\mathbb{Z}[T, T^{-1}]$ such that

$$H_A = (1 - T)^{-d} R_A, \quad H_B = (1 - T)^{-d+s} R_B$$

and we have $R_A(1) = e(A)$, $R_B(1) = e(A/\mathfrak{x})$. Plug this into ??, we find that

$$(1 - T)^{-d} R_B(T) = H_B^{(s)} \geq \prod_{i=1}^s (1 - T^{\delta_i}) H_A^{(s)} = (1 - T)^{-d} R(T) R_A(T)$$

where $R(T)$ is the polynomial in $\mathbb{Z}[T]$ given by

$$R(T) = \prod_{i=1}^s \frac{1 - T^{\delta_i}}{1 - T}.$$

Since $R(1) = \delta_1 \cdots \delta_s$, we conclude that $e(B) \geq \delta_1 \cdots \delta_s e(A)$, and the equality holds if and only if (ξ_1, \dots, ξ_s) is completely secant for $\text{gr}(A)$. \square

Remark 15.6.2. We can show conversely that, if A is a regular local ring and (x_1, \dots, x_s) is an arbitrary sequence of elements in \mathfrak{m} , then $e(A/\mathfrak{x}) = \delta_1 \cdots \delta_s$ if and only if the sequence (ξ_1, \dots, ξ_s) is completely secant for $\text{gr}(A)$. The point is that in this case we do not need to assume that (x_1, \dots, x_s) is secant for A .

Example 15.6.16. Let A be the formal series ring $k[[X_1, \dots, X_n]]$ over a field k ; let F_1, \dots, F_s be elements of A , \mathfrak{a} the ideal they generate, and $B = A/\mathfrak{a}$. Let $P_1, \dots, P_s \in k[X_1, \dots, X_n]$ be the

initial forms of the series F_1, \dots, F_s and $\delta_1, \dots, \delta_s$ be their degrees, respectively. By ??, if the sequence (F_1, \dots, F_s) is secant for A , we have $e(B) \geq \delta_1, \dots, \delta_s$, and the equality holds if and only if the sequence (P_1, \dots, P_s) is completely secant for the ring $k[X_1, \dots, X_n]$.

Consider for example the ring $B = k[[X, Y]]/\mathfrak{a}$, where \mathfrak{a} is generated by $X^2 + Y^3$ and $X^2 + Y^4$; the preceding inequality is $e(B) \geq 4$, and this inequality is strict because (X^2, X^2) is not completely secant. On the other hand, note that \mathfrak{a} is also generated by $X^2 + Y^3$ and $Y^4 - Y^3$, for which the sequence of initial forms, namely (X^2, Y^3) , is completely secant, so we obtain that $e(B) = 6$.

From now on, let A be a Noetherian ring, \mathfrak{a} an ideal of A contained in the Jacobson radical of A , and M a finitely generated A -module such that $M/\mathfrak{a}M$ has finite length. Recall that in this case we can talk about the multiplicity $e_{\mathfrak{a}}(M)$ of M relative to \mathfrak{a} .

Proposition 15.6.17. *Let δ be a positive integer, x an element of \mathfrak{a}^δ , ξ its class in $\mathfrak{a}^\delta/\mathfrak{a}^{\delta+1}$, and φ the multiplication by ξ on $\text{gr}(M)$.*

- (a) *The dimension of the A -module M/xM is equal to $\dim_A(M)$ or $\dim_A(M) - 1$. In the second case, we have $e_{\mathfrak{a}}(M/xM) \geq \delta e_{\mathfrak{a}}(M)$.*
- (b) *Suppose that $\dim_A(M) \geq 1$ and the kernel of φ has finite length over A/\mathfrak{a} . Then we have $\dim_A(M/xM) = \dim_A(M) - 1$. Moreover:*
 - (i) *If $\dim_A(M) > 1$, then $e_{\mathfrak{a}}(M/xM) = \delta e_{\mathfrak{a}}(M)$.*
 - (ii) *If $\dim_A(M) = 1$, then for any $n \geq 0$ we have*

$$n\delta e_{\mathfrak{a}}(M) \leq \ell_A(M/x^n M) \leq n\delta e_{\mathfrak{a}}(M) + \ell_{A/\mathfrak{a}}(\ker \varphi^n)$$

$$\text{and } \delta e_{\mathfrak{a}}(M) = e_{x A}(M) \leq \ell_A(M/xM).$$

Proof. Put $M' = M/xM$; consider the Hilbert-Samuel series $H_M = H_{M, \mathfrak{a}}$ and $H_{M'} = H_{M', \mathfrak{a}}$, as well as the Poincaré series $P(T) = \sum_{n=0}^{\infty} \ell_{A/\mathfrak{a}}(\ker \varphi_n) \cdot T^n$ (where φ_n is the restriction of φ on $\text{gr}_n(M)$). By ??, there then exists elements R_M and $R_{M'}$ in $\mathbb{Z}[T, T^{-1}]$ such that

$$H_M = (1-T)^{-d} R_M, \quad H_{M'} = (1-T)^{-d'} R_{M'}$$

where $d = \dim_A(M)$, $d' = \dim_A(M')$, $R_M(1) = e_{\mathfrak{a}}(M)$, and $R_{M'} = e_{\mathfrak{a}}(M')$. By ??, we have the inequality

$$(1-T^\delta) H_M^{(1)} \leq H_{M'}^{(1)} \leq (1-T^\delta) H_M^{(1)} + T^\delta P^{(1)}$$

in the ring $\mathbb{Z}((T))$. Posing $R(T) = (1-T^\delta)/(1-T)$, this is equivalent to

$$(1-T)^{-d} R(T) R_M(T) \leq (1-T)^{-d'-1} R_{M'}(T) \leq (1-T)^{-d} R(T) R_M(T) + (1-T)^{-1} T^\delta P(T). \quad (15.6.3)$$

By ??, the first inequality implies that, either $d < d' + 1$, or $d = d' + 1$ and $R(1)R_M(1) \leq R_{M'}(1)$, which means $e_{\mathfrak{a}}(M') \geq \delta e_{\mathfrak{a}}(M)$. Since we always have $d' \leq d$, this proves (a).

Under the hypothesis of (b), we have $P(T) \in \mathbb{Z}[T]$ and $P(1) = \ell_A(\ker \varphi)$. The second inequality of ?? shows

$$(1-T)^{-d'-1} R_{M'}(T) \leq (1-T)^{-d} [R(T) R_M(T) + T^\delta (1-T)^{-d-1} P(T)].$$

Suppose that $d > 1$, again by ??, either $d' + 1 < d$, or $d' + 1 = d$ and $R_{M'}(1) = R(1)R(1)$, which means $e_{\mathfrak{a}}(M') = \delta e_{\mathfrak{a}}(M)$. This demonstrate the case (i) in (b), in view of part (a).

Suppose now $d = 1$. Then by the same reasoning, we have $d' = 0$ and

$$R_{M'}(1) \leq R(1)R_M(1) + P(1).$$

Recall that if $d' = 0$ then $R_{M'}(1) = \ell_A(M')$; therefore we conclude that

$$\delta e_{\mathfrak{a}}(M) \leq \ell_A(M/xM) \leq \delta e_{\mathfrak{a}}(M) + \ell_A(\ker \varphi). \quad (15.6.4)$$

Since $\ker \varphi$ has finite length, φ_m is injective for m large enough, so $(\varphi^n)_m$ is also injective for m large enough for each integer $n \geq 1$, which means $\ker \varphi^n$ has finite length. By replacing x with x^n in ??, we then get

$$n\delta e_{\mathfrak{a}}(M) \leq \ell_A(M/x^n M) \leq n\delta e_{\mathfrak{a}}(M) + \ell_A(\ker \varphi^n). \quad (15.6.5)$$

Now the submodules $\ker \varphi^n$ of the $\text{gr}(A)$ -Noetherian module $\text{gr}(M)$ form an increasing sequence therefore stationary and that each of them is of finite length over A/\mathfrak{a} . Dividing by n in inequality ?? and letting n tend to $+\infty$, we find that $e_{xA}(M) = \delta e_{\mathfrak{a}}(M)$ by definition of $e_{xA}(M)$??.

Lemma 15.6.18. *Let R be a Noetherian graded ring with positive degrees, E a finitely generated graded R -module such that E_n has finite length over R_0 for each $n \in \mathbb{Z}$. The following conditions are equivalent:*

- (i) *E is an R -module of finite length;*
- (ii) *there exists an integer n_0 such that $E_n = 0$ for $n \geq n_0$;*
- (iii) *every associated prime of E contains R_+ .*

Proof. The equivalence of (i) and (ii) is clear. Let \mathfrak{p} be an associated prime of E . If (iii) is satisfied, we have $\mathfrak{p} = \mathfrak{p}_0 + R_+$, where \mathfrak{p}_0 is a prime ideal of R_0 , and the R -module R/\mathfrak{p} is isomorphic to R_0/\mathfrak{p}_0 . By ??, the R_0 -module R_0/\mathfrak{p}_0 is isomorphic to a submodule of one of the E_k , hence has finite length. Consequently, \mathfrak{p}_0 is a maximal ideal of R_0 , and \mathfrak{p} is then maximal. Since \mathfrak{p} is arbitrary, it follows from ?? that E has finite length.

Conversely, assume that E has finite length and let \mathfrak{p} be an associated prime of E . Then \mathfrak{p} is homogeneous by ?? and maximal by ??, hence contains R_+ ??.

Proposition 15.6.19. *Let $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ be the prime ideals of $\text{gr}(A)$ associated with the graded module $\text{gr}(M)$ and not containing $\text{gr}_1(A)$. Let $\delta > 0$ be an integer, ξ an element of $\text{gr}_{\delta}(A)$, and φ the homothety of $\text{gr}(M)$ with ratio ξ . Then for φ_n to be injective for all n large enough, it is necessary and sufficient that ξ does not belong to any of the \mathfrak{p}_i .*

Proof. In fact, the prime ideals associated with the $\text{gr}(A)$ -module $\ker \varphi$ is the primes associated with $\text{gr}(M)$ and contain ξ . By ??, $(\ker \varphi)_n$ is zero for n large enough, if and only if these prime ideals contain $\text{gr}_+(A)$ (or equivalently they contain $\text{gr}_1(A)$), whence the proposition. □

An element x of A is said to be **superficial** for M relative to \mathfrak{a} if it is contained in \mathfrak{a} and if, for n large enough, the map $\mathfrak{a}^n M / \mathfrak{a}^{n+1} M \rightarrow \mathfrak{a}^{n+1} M / \mathfrak{a}^{n+2} M$ induced by the multiplication by x is

injective. If moreover $x \in \mathfrak{a}^\delta$, then x is said to be superficial of order δ for M . In the notation of ??, we see x is superficial of order δ for M if its class ξ in $\text{gr}(A)$ belongs to $\text{gr}_\delta(A)$ and is not contained in the \mathfrak{p}_i .

By ??, there exists a homogeneous element of $\text{gr}(A)$ with positive degree that does not belong to any \mathfrak{p}_i . Consequently, there exists an integer $\delta > 0$ and an element of A that is superficial of order δ for M .

Remark 15.6.3. We note that, by ??, an element x of A satisfies the hypothesis in ?? if and only if it is superficial for M relative to \mathfrak{a} .

Example 15.6.20. Suppose that A is local with maximal ideal \mathfrak{m} and residue field k , and consider the canonical surjection $\pi : \mathfrak{a} \rightarrow \mathfrak{a} \otimes_A k$. Since $\mathfrak{a} \otimes_A k = \mathfrak{a}/\mathfrak{a}\mathfrak{m}$ and $\mathfrak{a} \subseteq \mathfrak{m}$, we see π factors through the canonical map $\mathfrak{a} \rightarrow \mathfrak{a}/\mathfrak{a}^2$:

$$\mathfrak{a} \longrightarrow \mathfrak{a}/\mathfrak{a}^2 \xrightarrow{\bar{\pi}} \mathfrak{a} \otimes_A k$$

By Nakayama lemma, the subspace $V_i = \bar{\pi}(\mathfrak{p}_i \cap (\mathfrak{a}/\mathfrak{a}^2))$ of $\mathfrak{a} \otimes_A k$ is proper. If $\alpha \in \mathfrak{a} \otimes_A k$ is not contained in any of the V_i , then $\pi^{-1}(\alpha)$ consists of superficial elements for M by ???. In particular, if k is infinite, the union of V_i is distinct from $\mathfrak{a} \otimes_A k$ so there exists a superficial element for M .

Theorem 15.6.21. Let A be a Noetherian ring, \mathfrak{a} an ideal of A contained in the Jacobson radical of A and M a finitely generated A -module such that $M/\mathfrak{a}M$ is of finite length. Let x_1, \dots, x_r be elements of \mathfrak{a} and \mathfrak{x} the ideal they generate.

- (a) We have $\dim_A(M/\mathfrak{x}M) \geq \dim_A(M) - r$.
- (b) If $\dim_A(M/\mathfrak{x}M) = \dim_A(M) - r$, then $e_{\mathfrak{a}}(M/\mathfrak{x}M) \geq e_{\mathfrak{a}}(M)$.
- (c) If $r < \dim_A(M)$ and for each i the element x_i is superficial for $M/(x_1M + \dots + x_{i-1}M)$ relative to \mathfrak{a} , then

$$\dim_A(M/\mathfrak{x}M) = \dim_A(M) - 1, \quad e_{\mathfrak{a}}(M/\mathfrak{x}M) = e_{\mathfrak{a}}(M).$$

- (d) If $r = \dim_A(M)$ and for each i the element x_i is superficial for $M/(x_1M + \dots + x_{i-1}M)$ relative to \mathfrak{a} , then

$$e_{\mathfrak{a}}(M) = e_{\mathfrak{x}}(M) \leq \ell_A(M/\mathfrak{x}M) < +\infty.$$

Proof. The assertions in (a), (b), and (c) for $r = 1$ follows from ??, and the general case follows by induction. Now assume the hypothesis in (d) and put $\mathfrak{x}' = Ax_1 + \dots + Ax_{r-1}$, $M' = M/\mathfrak{x}'M$; then $M/\mathfrak{x}M$ is identified with M'/x_rM' . By part (c), we have $\dim_A(M') = 1$ and $e_{\mathfrak{a}}(M) = e_{\mathfrak{a}}(M')$; ?? also implies that $M/\mathfrak{x}M$ has finite length and

$$e_{\mathfrak{a}}(M') = e_{x_r A}(M') \leq \ell_A(M'/\mathfrak{x}_r M') = \ell_A(M/\mathfrak{x}M).$$

On the other hand, since $x_r^n M' = \mathfrak{x}^n M'$ for each n , we have $e_{x_r A}(M') = e_{\mathfrak{x}}(M')$. By applying (b) to x_1, \dots, x_{r-1} , we also obtain that $e_{\mathfrak{x}}(M') \geq e_{\mathfrak{x}}(M)$, so

$$e_{\mathfrak{x}}(M) \leq e_{\mathfrak{x}}(M') = e_{x_r A}(M') = e_{\mathfrak{a}}(M') = e_{\mathfrak{a}}(M).$$

Since \mathfrak{x} is contained in \mathfrak{a} , this implies $e_{\mathfrak{x}}(M) = e_{\mathfrak{a}}(M)$ and completes the proof. \square

Corollary 15.6.22. Suppose that A is local, with residue field infinite, and put $d = \dim_A(M)$. Then there exists a sequence (x_1, \dots, x_d) of elements in \mathfrak{a} such that, if \mathfrak{x} is the ideal they generated, then

$$e_{\mathfrak{a}}(M) = e_{\mathfrak{x}}(M) \leq \ell_A(M/\mathfrak{x}M) < +\infty$$

Proof. This follows from ?? and ??.

□

Remark 15.6.4. In the situation of ??, we have

$$e_{\mathfrak{a}}(M) = e_{\mathfrak{x}}(M) \leq \ell_A(M/\mathfrak{x}M), \quad \ell_A(M/\mathfrak{a}M) \leq \ell_A(M/\mathfrak{x}M).$$

The three cases

$$e_{\mathfrak{a}}(M) < \ell_A(M/\mathfrak{a}M), \quad e_{\mathfrak{a}}(M) = \ell_A(M/\mathfrak{a}M), \quad e_{\mathfrak{a}}(M) > \ell_A(M/\mathfrak{a}M)$$

are all possible.

Chapter 16

Complete Noetherian local rings

In this section, all rings are assumed to be commutative with unit. We denote by 1_A for the unit of a ring A . If \mathfrak{p} a prime ideal of A , we denote by $\kappa(\mathfrak{p})$ the residue field of $A_{\mathfrak{p}}$. If A is local, \mathfrak{m}_A will denote its maximal ideal and κ_A its residue field.

We say a ring homomorphism $\rho : A \rightarrow B$ is flat (resp. faithfully flat) if B is a flat A -module (resp. faithfully flat A -module). Recall that by ??, if A and B are local rings, ρ is faithfully flat if and only if it is local and flat.

16.1 Witt vectors and Witt rings

16.1.1 Witt polynomials

In this part, we fix a prime number p . For a positive integer n , we define the **n -th Witt polynomial** Φ_n to be

$$\Phi_n(X_0, \dots, X_n) = \sum_{i=0}^n p^i X_i^{p^{n-i}} = X_0^{p^n} + pX_1^{p^{n-1}} + \dots + p^n X_n. \quad (16.1.1)$$

Clearly we have $\Phi_0 = X_0$ and the following recursive formulae

$$\Phi_{n+1}(X_0, \dots, X_{n+1}) = \Phi_n(X_0^p, \dots, X_n^p) + p^{n+1} X_{n+1}, \quad (16.1.2)$$

$$\Phi_{n+1}(X_0, \dots, X_{n+1}) = X_0^{p^{n+1}} + p\Phi_n(X_1, \dots, X_{n+1}). \quad (16.1.3)$$

If we assign X_i with weight p^i , the polynomial Φ_n is isobare with weight p^n .

Lemma 16.1.1. *Let A be a filtered ring with $(A_n)_{n \in \mathbb{Z}}$ its filtration. Let x, y be elements of A congruent mod A_m , then*

$$x^{p^n} \equiv y^{p^n} \pmod{A_{m+n}}. \quad (16.1.4)$$

Proof. Let $P = \sum_{i=0}^{p-1} X^i Y^{p-1-i}$ in $\mathbb{Z}[X, Y]$. From the hypothesis on x and y , we have

$$P(x, y) \equiv P(x, x) \equiv px^{p-1} \pmod{A_m}.$$

But we have $A_m + pA \subseteq A_1$, from which $P(x, y) \in A_1$. Finally, $x^p - y^p = (x - y)P(x, y)$ is contained in $A_m A_1 \subseteq A_{m+1}$, which complete the induction process. \square

Proposition 16.1.2. Let A be a filtered ring with $(A_n)_{n \in \mathbb{Z}}$ its filtration. Suppose that $A_0 = A$ and $p \cdot 1 \in A_1$. Let a_0, \dots, a_n and b_0, \dots, b_n be elements of A , and $m \geq 1$ be an integer. If $a_i \equiv b_i \pmod{A_m}$ for each i , then

$$\Phi_i(a_0, \dots, a_i) \equiv \Phi_i(b_0, \dots, b_i) \pmod{A_{m+i}} \text{ for } 0 \leq i \leq n. \quad (16.1.5)$$

Moreover, the converse holds if for each integer $k \geq 1$ and $x \in A$, the relation $px \in A_{k+1}$ implies $x \in A_k$.

Proof. We prove (??) by induction on n . The case $n = 0$ is immediate, so suppose $n \geq 1$. Apply (??) on a_i and b_i , we get

$$a_i^p \equiv b_i^p \pmod{A_{m+1}} \text{ for } 0 \leq i \leq n-1, \quad (16.1.6)$$

also, by the induction hypothesis on (a_0, \dots, a_{n-1}) and (b_0, \dots, b_{n-1}) ,

$$\Phi_{n-1}(a_0^p, \dots, a_{n-1}^p) \equiv \Phi_{n-1}(b_0^p, \dots, b_{n-1}^p) \pmod{A_{m+n}}. \quad (16.1.7)$$

Therefore, from (??) we conclude that

$$\Phi_n(a_0, \dots, a_n) - p^n a_n \equiv \Phi_n(b_0, \dots, b_n) - p^n b_n \pmod{A_{m+n}}. \quad (16.1.8)$$

Now $a_n - b_n$ is contained in A_m , the element $p^n a_n - p^n b_n$ then belongs to A_{m+n} and this implies the congruence

$$\Phi_n(a_0, \dots, a_n) \equiv \Phi_n(b_0, \dots, b_n) \pmod{A_{m+n}}.$$

Conversely, assume that for each integer $k \geq 1$ and $x \in A$, the relation $px \in A_{k+1}$ implies $x \in A_k$. We prove the inverse direction by induction on n . The case $n = 0$ is clear, and suppose that $n \geq 1$. Since $a_i \equiv b_i \pmod{A_m}$ for $0 \leq i \leq n-1$, by the induction hypothesis, we deduce the congruence (??), (??), and (??). But by hypothesis $\Phi_n(a_0, \dots, a_n) \equiv \Phi_n(b_0, \dots, b_n) \pmod{A_{m+n}}$, so $p^n(a_n - b_n) \in A_{m+n}$. Since $px \in A_{k+1}$ implies $x \in A_k$, we conclude that $a_n \equiv b_n \pmod{A_m}$, which completes the proof. \square

Let A be a ring, and give $A^{\mathbb{N}}$ the structure of product ring. We define two maps f_A and v_A by the following formulae

$$f_A(\mathbf{a}) = (a_1, a_2, \dots), \quad v_A(\mathbf{a}) = (0, pa_0, pa_1, \dots).$$

where $\mathbf{a} = (a_n)_{n \in \mathbb{N}}$. For any positive integer m , let Φ_m be the map on $A^{\mathbb{N}}$ defined by $\mathbf{a} = (a_n)_{n \in \mathbb{N}} \mapsto \Phi_m(a_0, \dots, a_m)$. We denote by Φ_A the map $\mathbf{a} \mapsto (\Phi_n(\mathbf{a}))_{n \in \mathbb{N}}$ on $A^{\mathbb{N}}$.

Lemma 16.1.3. Suppose that A is endowed with an endomorphism σ satisfying $\sigma(a) = a^p \pmod{pA}$ for any $a \in A$. Let a_0, \dots, a_{n-1} be elements of A . Put $u_i = \Phi_i(a_0, \dots, a_i)$ for each i , and let u_n be an element of A . Then the following conditions are equivalent:

- (i) There exists $a_n \in A$ such that $u_n = \Phi_n(a_0, \dots, a_n)$.
- (ii) We have $\sigma(u_{n-1}) \equiv u_n \pmod{p^n A}$.

Proof. For $0 \leq i \leq n - 1$, we have $\sigma(a_i) \equiv a_i^p \pmod{pA}$. Applying ?? on the filtration $(p^n A)_{n \in \mathbb{N}}$ and $m = 1$, we have a congruence

$$\sigma(u_{n-1}) = \Phi_{n-1}(\sigma(a_0), \dots, \sigma(a_{n-1})) \equiv \Phi_{n-1}(a_0^p, \dots, a_0^p) \pmod{p^n A}.$$

But, from formula ??, the relation $u_n = \Phi_n(a_0, \dots, a_n)$ is equivalent to

$$u_n = \Phi_{n-1}(a_0^p, \dots, a_{n-1}^p) + p^n a_n.$$

Thus the lemma is proved. \square

Proposition 16.1.4. *Let A be a ring and f_A, v_A , and Φ_A be the maps defined above.*

- (a) *If p is not a divisor of zero in A , the map Φ_A is injective.*
- (b) *If p is invertible in A , the map Φ_A is bijective.*
- (c) *If σ is an endomorphism of A satisfying $\sigma(a) = a^p \pmod{pA}$ for any $a \in A$, the image of Φ_A is a subring of $A^\mathbb{N}$ invariant under f_A and v_A , which consists of elements $(u_n)_{n \in \mathbb{N}}$ such that $\sigma(u_n) \equiv u_{n+1} \pmod{p^{n+1} A}$ for all $n \in \mathbb{N}$.*

Proof. If $\mathbf{a} = (a_n)_{n \in \mathbb{N}}$ and $\mathbf{u} = (u_n)_{n \in \mathbb{N}}$ are elements of $A^\mathbb{N}$, the relation $\Phi_A(\mathbf{a}) = \mathbf{u}$ is equivalent, by formula ??, to

$$\begin{cases} u_0 = a_0, \\ u_n = \Phi_{n-1}(a_0^p, \dots, a_{n-1}^p) + p^n a_n \text{ for } n \geq 1. \end{cases} \quad (16.1.9)$$

Let $\mathbf{u} = (u_n)_{n \in \mathbb{N}}$ be in $A^\mathbb{N}$. If p is not a zero divisor of A (resp. if p is invertible in A), there then exists at most (resp. exactly) one sequence $(a_n)_{n \in \mathbb{N}}$ satisfying ??, whence (a) and (b).

Now by ??, the image of Φ_A consists of elements $\mathbf{u} = (u_n)_{n \in \mathbb{N}}$ of $A^\mathbb{N}$ such that $\sigma(u_n) = u_{n+1} \pmod{p^{n+1} A}$ for all $n \in \mathbb{N}$. Since σ is an endomorphism, it follows that this is a subring of $A^\mathbb{N}$ and invariant under f_A and v_A . \square

Remark 16.1.1. Let $\mathbf{a} = (a_n)_{n \in \mathbb{N}}$ and $\mathbf{u} = (u_n)_{n \in \mathbb{N}}$ be elements of $A^\mathbb{N}$ such that $\Phi_A(\mathbf{a}) = \mathbf{u}$. Then by ??, we deduce the following assertions:

- (a) If the u_n , for $0 \leq n \leq m$, is contained in a subring B of A and if, for any $x \in A$, the relation $px \in B$ implies $x \in B$, then a_n , for $0 \leq n \leq m$, are contained in B .
- (b) If A is graded of type \mathbb{N} , p is not a zero divisor of A , and u_n is homogeneous of degree dp^n for $0 \leq n \leq m$ (where $d \in \mathbb{N}$), then a_n is homogeneous of degree dp^n for $0 \leq n \leq m$.

Now let A be the polynomial ring $\mathbb{Z}[\mathbf{X}, \mathbf{Y}]$ with indeterminates $\mathbf{X} = (X_n)_{n \in \mathbb{N}}$ and $\mathbf{Y} = (Y_n)_{n \in \mathbb{N}}$. Let θ be the endomorphism of A defined by $\theta(X_n) = X_n^p$ and $\theta(Y_n) = Y_n^p$ for $n \in \mathbb{N}$. Then p is not a divisor of zero in A and the set of $a \in A$ such that $\theta(a) = a^p \pmod{pA}$ is a subring of A containing X_n and Y_n , hence equal to A . By ??(a) and (c), there exist elements $\mathbf{S} = (S_n)_{n \in \mathbb{N}}$,

$\mathbf{P} = (P_n)_{n \in \mathbb{N}}$, $\mathbf{I} = (I_n)_{n \in \mathbb{N}}$, and $\mathbf{F} = (F_n)_{n \in \mathbb{N}}$ in $A^{\mathbb{N}}$ characterized by the equalities

$$\begin{cases} \Phi_A(\mathbf{S}) = \Phi_A(\mathbf{X}) + \Phi_A(\mathbf{Y}) \\ \Phi_A(\mathbf{P}) = \Phi_A(\mathbf{X})\Phi_A(\mathbf{Y}) \\ \Phi_A(\mathbf{I}) = -\Phi_A(\mathbf{X}) \\ \Phi_A(\mathbf{F}) = f_A(\Phi_A(\mathbf{X})). \end{cases} \quad (16.1.10)$$

The elements S_n , P_n , I_n , and F_n of A are then characterized by the following formulae (where $n \in \mathbb{N}$):

$$\Phi_n(S_0, \dots, S_n) = \Phi_n(X_0, \dots, X_n) + \Phi_n(Y_0, \dots, Y_n), \quad (16.1.11)$$

$$\Phi_n(P_0, \dots, P_n) = \Phi_n(X_0, \dots, X_n)\Phi_n(Y_0, \dots, Y_n), \quad (16.1.12)$$

$$\Phi_n(I_0, \dots, I_n) = -\Phi_n(X_0, \dots, X_n), \quad (16.1.13)$$

$$\Phi_n(F_0, \dots, F_n) = \Phi_{n+1}(X_0, \dots, X_{n+1}). \quad (16.1.14)$$

Formula (??) makes it possible in practice to determine the polynomials S_n , P_n , I_n and F_n step by step.

Example 16.1.5 (First terms of the polynomials S_n , P_n , I_n , and F_n).

(a) We have

$$S_0 = X_0 + Y_0, \quad S_1 = X_1 + Y_1 - \sum_{i=1}^{p-1} \frac{1}{p} \binom{p}{i} X_0 Y_0^{p-i}.$$

Furthermore, $S_n - X_n - Y_n$ is in the ring $\mathbb{Z}[X_0, \dots, X_{n-1}, Y_0, \dots, Y_{n-1}]$.

(b) We have

$$P_0 = X_0 Y_0, \quad P_1 = p X_1 Y_1 + X_0^p Y_1 + X_1 Y_0^p.$$

(c) If $p \neq 2$, we have $I_n = -X_n$. If $p = 2$, then

$$I_0 = -X_0, \quad I_1 = -(X_0^2 + X_1), \quad I_2 = -X_0^4 - X_0^2 X_1 - X_1^2 - X_2.$$

(d) We have

$$F_0 = X_0^p + p X_1, \quad F_1 = X_1^p + p X_2 - \sum_{i=0}^{p-1} \binom{p}{i} p^{p-i-1} X_0^{pi} X_1^{p-i}.$$

Note that $\Phi_n(F_0, \dots, F_n) \equiv \Phi_n(X_0^p, \dots, X_n^p) \pmod{p^{n+1}A}$ for each $n \in \mathbb{N}$ since $F_n \equiv X_n^p \pmod{pA}$ for each $n \in \mathbb{N}$.

Remark 16.1.2. A subset $J \subseteq \mathbb{N}_+$ is called **divisor-stable** provided that $J \neq \emptyset$, and if $n \in J$, then all proper divisors of n are also in J . If J is a divisor-stable set, we let $\mathfrak{p}(J)$ denote the set of prime numbers contained in J . For any element j in J , define the polynomial φ_j of $\mathbb{Z}[(X_j)_{j \in J}]$ by the formula

$$\varphi_j = \sum_{d|j} d X_d^{j/d}.$$

Note that we have, for each integer $n \geq 0$, that

$$\varphi_{p^n} = \Phi_n(X_{p^0}, \dots, X_{p^n}).$$

For any ring A and any $n \in J$, we denote by φ_n the map from A^J to A given by $(a_j)_{j \in J} \mapsto \varphi_n((a_j)_{j \in J})$; we denote by φ_A , or simply φ , the map A^J to A defined by $\mathbf{a} = (a_j)_{j \in J} \mapsto (\varphi_n(\mathbf{a}))_{n \in J}$. Let $\mathcal{A} = \mathbb{Z}[(X_j)_{j \in J}, (Y_j)_{j \in J}]$ be the polynomial ring of indeterminates $\mathbf{X} = (X_j)_{j \in J}$ and $\mathbf{Y} = (Y_j)_{j \in J}$. We can show that there exists elements $\mathbf{s} = (s_j)_{j \in J}$, $\mathbf{p} = (p_j)_{j \in J}$ and $\mathbf{i} = (i_j)_{j \in J}$ characterized by the following equalities:

$$\begin{aligned}\varphi_{\mathcal{A}}(\mathbf{s}) &= \varphi_{\mathcal{A}}(\mathbf{X}) + \varphi_{\mathcal{A}}(\mathbf{Y}) \\ \varphi_{\mathcal{A}}(\mathbf{p}) &= \varphi_{\mathcal{A}}(\mathbf{X})\varphi_{\mathcal{A}}(\mathbf{Y}) \\ \varphi_{\mathcal{A}}(\mathbf{i}) &= -\varphi_{\mathcal{A}}(\mathbf{X}).\end{aligned}$$

In particular, this definition works, for example, if J is the set of integers $j \geq 1$.

16.1.2 The ring of Witt vectors

Let A be a ring. If $\mathbf{a} = (a_n)_{n \in \mathbb{N}}$ and $\mathbf{b} = (b_n)_{n \in \mathbb{N}}$ are elements of $A^{\mathbb{N}}$, we denote by $S_A(\mathbf{a}, \mathbf{b})$, or simply $S(\mathbf{a}, \mathbf{b})$, the sequence $(S_n(a_0, \dots, a_n, b_0, \dots, b_n))_{n \in \mathbb{N}}$. Similarly, we denote by $P_A(\mathbf{a}, \mathbf{b})$ and $I_A(\mathbf{a})$ the resulting substitutions. By substituting X_n into a_n , Y_n into b_n , for each $n \in \mathbb{N}$, in the formulae (??), (??), and (??), we obtain the equalities

$$\Phi_A(S_A(\mathbf{a}, \mathbf{b})) = \Phi_A(\mathbf{a}) + \Phi_A(\mathbf{b}) \quad (16.1.15)$$

$$\Phi_A(P_A(\mathbf{a}, \mathbf{b})) = \Phi_A(\mathbf{a})\Phi_A(\mathbf{b}) \quad (16.1.16)$$

$$\Phi_A(I_A(\mathbf{a})) = -\Phi_A(\mathbf{a}). \quad (16.1.17)$$

We will denote by $W(A)$ the set $A^{\mathbb{N}}$ endowed with the laws of composition S_A and P_A .

Let $\rho : A \rightarrow B$ be a ring homomorphism. We denote $W(\rho)$ the map $\rho^{\mathbb{N}} : A^{\mathbb{N}} \rightarrow B^{\mathbb{N}}$ which sends an element $\mathbf{a} = (a_n)_{n \in \mathbb{N}}$ to $(\rho(a_n))_{n \in \mathbb{N}}$. From the definition, we have

$$W(\rho) \circ S_A = S_B \circ (W(\rho) \times W(\rho)) \quad (16.1.18)$$

$$W(\rho) \circ P_A = P_B \circ (W(\rho) \times W(\rho)) \quad (16.1.19)$$

$$W(\rho) \circ I_A = I_B \circ W(\rho) \quad (16.1.20)$$

$$W(\rho) \circ \Phi_A = \Phi_B \circ W(\rho) \quad (16.1.21)$$

Lemma 16.1.6. *Let A be a ring. Then there exists a surjective homomorphism $\rho : B \rightarrow A$, where B is a ring satisfying the following condition: p is not a divisor of zero in B , and there exists an endomorphism σ of B such that $\sigma(b) \equiv b^p \pmod{pB}$ for any $b \in B$.*

Proof. It suffices to take $B = \mathbb{Z}[(X_a)_{a \in A}]$, and let σ be the endomorphism on B defined by $\sigma(X_a) = X_a^p$ for any $a \in A$. The homomorphism $\rho : B \rightarrow A$ is defined to send X_a to a for any $a \in A$. \square

Theorem 16.1.7. *Let A be a ring.*

- (a) *With the map S_A and P_A , the set $W(A)$ is a commutative ring. The addition identity is the sequence $\mathbf{0}_A$ with zero terms, and the multiplication identity is the sequence $\mathbf{1}_A$ of which all the terms are zero except that of index 0 which is equal to 1. The additive inverse of an element \mathbf{a} in $W(A)$ is $I_A(\mathbf{a})$.*

- (b) Let $\rho : A \rightarrow B$ be a homomorphism of rings. Then the induced map $W(\rho) : W(A) \rightarrow W(B)$ is a homomorphism of rings.
- (c) The map Φ_A is a homomorphism from $W(A)$ to the product ring $A^{\mathbb{N}}$. In particular, for each $n \in \mathbb{N}$, the map $\Phi_n : \mathbf{a} \mapsto \Phi_n(a_0, \dots, a_n)$ is a homomorphism from $W(A)$ to A .

Proof. Concerning the formulae (??), (??), (??), and (??), it suffices to prove (a). Let $\rho : B \rightarrow A$ be a ring homomorphism satisfying the conditions in ???. Let B' be the image of Φ_B in $B^{\mathbb{N}}$, which by ?? consists of elements $(b_n)_{n \in \mathbb{N}}$ such that $\sigma(b_n) \equiv b_{n+1} \pmod{p^{n+1}B}$ for all $n \in \mathbb{N}$. By ??, Φ_B induces a bijection between $W(B)$ and B' . In view of the formulas (??) to (??) and the relations $\Phi_n(\mathbf{0}_B) = 0$ and $\Phi_n(\mathbf{1}_B) = 1$, we get a ring structure on $W(B)$ by transporting structure, with addition identity $\mathbf{0}_B$, multiplication identity $\mathbf{1}_B$, and additive inverse $I_B(b)$.

The map $W(\rho) : W(B) \rightarrow W(A)$ is surjective (since ρ is surjective). By the formulae (??) and (??), the equivalence relation R on $W(B)$ associated with the map $W(\rho)$ is compatible with the ring structure on $W(B)$. Thus $W(\rho)$ is a bijection from the quotient ring $W(B)/R$ to $W(A)$, compatible with addition and multiplication. Assertion (a) then follows by transporting structure. \square

For a ring A , the ring $W(A)$ is called the **Witt ring** with coefficients in A . For \mathbf{a} in $W(A)$ and $n \in \mathbb{N}$, the element $\Phi_n(\mathbf{a}) = \Phi_n(a_0, \dots, a_n)$ is called the **phantom components** of order n of \mathbf{a} .

Remark 16.1.3. Retain the notations in ???. Let A be a ring. If \mathbf{a} and \mathbf{b} are elements of A^J and $\mathbf{r} = (r_j)_{j \in J}$ belongs to A^J , we denote by $r_A(\mathbf{a}, \mathbf{b})$ the element $(r_j(\mathbf{a}, \mathbf{b}))_{j \in J}$ of A^J . Let $U(A)$ be the set A^J endowed with the maps s_A and p_A . We can show that, with the addition s_A and multiplication p_A , $U(A)$ is a commutative ring; we call it the **universal Witt ring** of A . The addition identity of $U(A)$ is the element with components all zero, and the multiplication identity is the element with components zero except 1_A at index 1; the additive inverse of \mathbf{a} in $U(A)$ is $i_A(\mathbf{a})$. The map φ_A is a homomorphism of $U(A)$ to the product ring A^J .

From now on, we use $+$ and \times to denote the addition and multiplication in the ring $W(A)$; for simplicity, we write $\mathbf{0}$ for $\mathbf{0}_A$ and $\mathbf{1}$ for $\mathbf{1}_A$. Let us introduce¹ two maps F_A and V_A on $W(A)$ by the formulae

$$F_A(\mathbf{a}) = (F_n(a_0, \dots, a_{n+1}))_{n \in \mathbb{N}}, \quad (16.1.22)$$

$$V_A(\mathbf{a}) = (0, a_0, a_1, \dots). \quad (16.1.23)$$

where $\mathbf{a} = (a_n)_{n \in \mathbb{N}}$. The map V_A is often called the **shift operator**. Note that by (??) we have the formula

$$\Phi_n(F_0(\mathbf{a}), \dots, F_n(\mathbf{a})) = \Phi_{n+1}(a_0, \dots, a_{n+1}).$$

We can also write it in the form

$$\Phi_A \circ F_A = f_A \circ \Phi_A. \quad (16.1.24)$$

¹The letter F is the initial of the name of Frobenius, and the latter V is that of the German word *Verschiebung*, which means shift.

From the recursive formula (??), we can also prove that

$$\Phi_A \circ V_A = v_A \circ \Phi_A. \quad (16.1.25)$$

Let $\rho : A \rightarrow B$ be a ring homomorphism. It is clear that we have

$$W(\rho) \circ F_A = F_B \circ W(\rho), \quad (16.1.26)$$

$$W(\rho) \circ V_A = V_B \circ W(\rho). \quad (16.1.27)$$

Proposition 16.1.8. *Let A be a ring and $W(A)$ be the Witt ring with coefficients A .*

- (a) *The map F_A is an endomorphism of the ring $W(A)$, and V_A is an endomorphism of the additive group of $W(A)$.*
- (b) *For $a \in W(A)$, we have $F_A(V_A(a)) = p \cdot a$.*
- (c) *Let a and b be elements in $W(A)$. Then we have*

$$V_A(a \times F_A(b)) = V_A(a) \times b, \quad (16.1.28)$$

$$V_A(a) \times V_A(b) = p \cdot V_A(a \times b). \quad (16.1.29)$$

- (d) *Put $\mu = V_A(\mathbf{1}) = (0, 1, 0, \dots)$. Then for b in $W(A)$ we have*

$$V_A(F_A(b)) = \mu \times b. \quad (16.1.30)$$

- (e) *For any element a in $W(A)$, denote by a^p the p -th power of a in $W(A)$. Then*

$$F_A(a) \equiv a^p \pmod{pW(A)}. \quad (16.1.31)$$

Proof. Let $\rho : B \rightarrow A$ be a ring homomorphism satisfying the conditions of ???. Then $W(\rho) : W(B) \rightarrow W(A)$ is a surjective ring homomorphism, and $\Phi_B : W(B) \rightarrow B^\mathbb{N}$ is an injective homomorphism of rings. Furthermore, $f_B : B^\mathbb{N} \rightarrow B^\mathbb{N}$ is a ring homomorphism. By the formulae (??) and (??), we have

$$\Phi_B \circ F_B = f_B \circ \Phi_B, \quad W(\rho) \circ F_B = F_A \circ W(\rho),$$

which proves the first assertion of (a). The second one can be proved similarly, using (??) and (??).

Let a be an element of $W(A)$, and choose a element x in $W(B)$ whose image under $W(\rho)$ is a . Put $\xi = \Phi_B(x)$. By definition of f_B and v_B , we have $f_B(v_B(\xi)) = p \cdot \xi$. By formulae (??) and (??), the elements $F_B(V_B(x))$ and $p \cdot x$ of $W(B)$ then have the same image under the injective map Φ_B , and therefore are equal. The formula $F_A(V_A(a)) = p \cdot a$ follows then from (??) and (??). This proves (b).

Reasoning analogously, we reduce the demonstration of formula (??) to the relation

$$v_B(\xi f_B(\eta)) = v_B(\xi)\eta$$

where ξ, η are in $B^{\mathbb{N}}$. This follows from the equalities

$$\xi f_B(\eta) = (\xi_0\eta_1, \xi_1\eta_2, \dots), \quad v_B(\xi)\eta = (0, p\xi_0\eta_1, p\xi_1\eta_2, \dots).$$

Taking into account (a) and (b), the formula (??) results from the formula (??), where we replace b by $V_A(b)$. Formula (??) is the particular case $a = 1$ of formula (??).

Simialrly, we deduce the formula (??) from the relation

$$f_B(\xi) \equiv \xi^p \pmod{p\Phi_B(B^{\mathbb{N}})}$$

where ξ^p denote the product of $B^{\mathbb{N}}$. By ??(c), this is equivalent to the fact that for all $n \geq 0$, we have

$$\sigma(\xi_{n+1} - \xi_n^p) \equiv \xi_{n+2} - \xi_{n+1}^p \pmod{p^{n+2}B}.$$

But, for $n \geq 0$, we have $\sigma(\xi_n) \equiv \xi_{n+1} \pmod{p^{n+1}B}$, since $\xi = \Phi_B(x)$ (again by ??(c)); we then deduce from ?? that

$$\sigma(\xi_n)^p \equiv \xi_{n+1}^p \pmod{p^{n+2}B},$$

which completes the proof. \square

16.1.3 Filtration and topology of the Witt ring

Lemma 16.1.9. *Let A be a ring and $m \geq 1$ an integer. Then*

$$a = (a_0, \dots, a_{m-1}, 0, \dots) + (\underbrace{0, \dots, 0}_{m\text{-terms}}, a_m, a_{m+1}, \dots)$$

for any a in $W(A)$.

Proof. Let $\rho : B \rightarrow A$ be a ring homomorphism satisfying the conditions of ???. Then $W(\rho) : W(B) \rightarrow W(A)$ is a surjective ring homomorphism, and $\Phi_B : W(B) \rightarrow B^{\mathbb{N}}$ is an injective homomorphism of rings. It suffices to prove that

$$\Phi_n(b) = \Phi_n(b_0, \dots, b_{m-1}, 0, \dots) + \Phi_n(0, \dots, 0, b_m, b_{m+1}, \dots) \quad (16.1.32)$$

for any b in $W(B)$ and $m \geq 1$. But we have

$$\Phi_n(b_0, \dots, b_{m-1}, 0, \dots) = \begin{cases} \Phi_n(b_0, \dots, b_n) & \text{if } 0 \leq n < m \\ \sum_{i=0}^{m-1} p^i b_i^{p^{n-i}} & \text{if } m \leq n \end{cases}$$

and

$$\Phi_n(0, b_m, b_{m+1}, \dots) = \begin{cases} 0 & \text{if } 0 \leq n < m \\ \sum_{i=m}^n p^i b_i^{p^{n-i}} & \text{if } m \leq n, \end{cases}$$

whence the formula (??). \square

Let A be a ring. For each integer $m \geq 0$, we denote by $V_m(A)$ the set of Witt vectors $a = (a_n)_{n \in \mathbb{N}}$ such that $a_n = 0$ for $0 \leq n < m$, which is also the m -fold image of V_A . From ??

and induction on m , we have the formulae

$$V^m(\mathbf{a} + \mathbf{b}) = V^m(\mathbf{a}) + V^m(\mathbf{b}), \quad (16.1.33)$$

$$V^m(\mathbf{a}) \times \mathbf{b} = V^m(\mathbf{a} \times F^m(\mathbf{b})). \quad (16.1.34)$$

We set $V_m(A) = W(A)$ if $m < 0$. The sequence $(V_m(A))_{m \in \mathbb{Z}}$ is then a decreasing filtration of additive subgroups of $W(A)$, and is compatible with the ring structure of $W(A)$ if and only if the ring A has characteristic p .

In the following, we denote by \mathcal{T} the topology on $W(A)$ associated with the filtration $(V_m(A))_{m \in \mathbb{Z}}$. As $V_m(A)$ is an ideal of $W(A)$ for all $m \in \mathbb{Z}$, the topology \mathcal{T} is compatible with the ring structure of $W(A)$ (??). Let $\mathbf{a} \in W(A)$; the sets $\mathbf{a} + V_m(A)$, for $m \in \mathbb{N}$, form a fundamental system of neighborhoods of \mathbf{a} in \mathcal{T} . By ??, $\mathbf{a} + V_m(A)$ consists of Witt vectors \mathbf{b} such that $a_i = b_i$ for $0 \leq i < m$. Consequently, \mathcal{T} is none other than the topology product on $A^{\mathbb{N}}$ of the discrete topology on each of the factors, and $W(A)$ is therefore a separated and complete topological ring.

Let τ_A be the map of A to $W(A)$ that sends an element a of A to $(a, 0, 0, \dots)$. We have $\Phi_n(\tau(a)) = a^{p^n}$ for each $n \in \mathbb{N}$. For any ring homomorphism $\rho : B \rightarrow A$, it is clear that $W(\rho) \circ \tau_B = \tau_A \circ \rho$.

Proposition 16.1.10. *Let a, b be elements of A and $\mathbf{x} = (x_n)_{n \in \mathbb{N}}$ an element of $W(A)$.*

(a) *We have the formulas*

$$\tau(ab) = \tau(b) \times \tau(b), \quad (16.1.35)$$

$$\tau(a) \times \mathbf{x} = (a^{p^n} x_n)_{n \in \mathbb{N}}. \quad (16.1.36)$$

(b) *The series $\sum_n V^n(\tau(x_n))$ converges in $W(A)$ to \mathbf{x} .*

Proof. Let n be a positive integer. The polynomial $P_n(X_0, \dots, X_n, Y_0, \dots, Y_n)$ is isobare with weight p^n if we assign X_i with weight p^i . We then have

$$P_n(X_0, 0, \dots, 0, Y_0, \dots, Y_n) = X_0^{p^n} P_n(1, 0, \dots, 0, Y_0, \dots, Y_n).$$

Since $\mathbf{1} = (1, 0, 0, \dots)$ is a unit in the Witt ring with coefficients in $\mathbb{Z}[(X_n)_{n \in \mathbb{N}}, (Y_n)_{n \in \mathbb{N}}]$, we have

$$P_n(1, 0, \dots, 0, Y_0, \dots, Y_n) = Y_n.$$

By Substituting X_0 with a and Y_i with x_i , we deduce that

$$P_n(a, 0, \dots, 0, x_0, \dots, x_n) = a^{p^n} x_n.$$

By the definition of product in $W(A)$, this proves (??), and (??) is a special case of (??).

We now prove (b). By definition, $V^n(\tau(x_n))$ is the sequence of which all the components are zero, except that of index n which is equal to x_n . It follows from ??, by induction on m , which we have

$$\sum_{n=0}^m V^n(\tau(x_n)) = (x_0, \dots, x_m, 0, 0, \dots)$$

for each $m \geq 0$; we then deduce (b) by passing to limit in the topology \mathcal{T} of $W(A)$, which is the product of the discrete topology on A . \square

Now for each integer $n \geq 1$, let $W_n(A)$ be the quotient ring $W(A)/V_n(A)$. For any elements a_0, \dots, a_{n-1} of A , we denote by $[a_0, \dots, a_{n-1}]$ or $[a_i]_{0 \leq i < n}$ the class modulo $V_n(A)$ of the element $(a_0, \dots, a_{n-1}, 0, 0, \dots)$ of $W(A)$. By ??, the map $(a_0, \dots, a_{n-1}) \mapsto [a_0, \dots, a_{n-1}]$ from A^n to $W_n(A)$ is bijective. For this reason, we say that the elements of $W_n(A)$ are the Witt vectors of length n ; similarly, one sometimes qualifies as Witt vectors of infinite length the elements of $W(A)$.

We denote by π_n the canonical homomorphism from $W(A)$ to $W_n(A)$. According to ??, for $\mathbf{a} = (a_n)_{n \in \mathbb{N}}$ in $W(A)$, we have

$$\pi_n(\mathbf{a}) = [a_0, \dots, a_{n-1}].$$

By the definition of the ring structure of $W(A)$, we have the following description of operations in $W_n(A)$:

$$\begin{aligned} [a_0, \dots, a_{n-1}] + [b_0, \dots, b_{n-1}] &= [S_i(a_0, \dots, a_i; b_0, \dots, b_i)]_{0 \leq i < n} \\ [a_0, \dots, a_{n-1}] \times [b_0, \dots, b_{n-1}] &= [S_i(a_0, \dots, a_i; b_0, \dots, b_i)]_{0 \leq i < n} \\ -[a_0, \dots, a_{n-1}] &= [I_i(a_0, \dots, a_i)]_{0 \leq i < n}. \end{aligned}$$

Moreover, the addition identity of $W_n(A)$ is $[0, \dots, 0]$ and the multiplication identity of $W(A)$ is $[1, 0, \dots, 0]$.

Let i be an integer such that $0 \leq i \leq n$. By passing to quotient, the homomorphism Φ_i from $W(A)$ to A defines a homomorphism from $W_n(A)$ to A , still denoted by Φ_i . It associates an Witt vector the element $\Phi_i(a_0, \dots, a_i)$ (called the phantom component of index i of $[a_0, \dots, a_{n-1}]$).

Let $\rho : A \rightarrow B$ be a ring homomorphism. By passing to quotient, the ring homomorphism $W(\rho)$ induces a homomorphism $W_n(\rho) : W_n(B) \rightarrow W_n(A)$. We have the following formula

$$W_n(\rho)[b_0, \dots, b_{n-1}] = [\rho(b_0), \dots, \rho(b_{n-1})] \quad (16.1.37)$$

for $[b_0, \dots, b_{n-1}]$ in $W_n(B)$.

Let m and n be integers such that $1 \leq n \leq m$. We have $V_n(A) \supset V_m(A)$, whence a canonical homomorphism $W_m(A) = W(A)/V_m(A)$ to $W_n(A) = W(A)/V_n(A)$, denoted by $\pi_{n,m}$. More precisely,

$$\pi_{n,m}[a_0, \dots, a_{m-1}] = [a_0, \dots, a_{n-1}] \quad (16.1.38)$$

for $[a_0, \dots, a_{m-1}]$ in $W_m(A)$. The family $(W_n(A), \pi_{n,m})$ is a inverse system of rings and the map $\pi : \mathbf{a} \mapsto (\pi_n(\mathbf{a}))_{n \geq 1}$ is a ring homomorphism from $W(A)$ to $\varprojlim W_n(A)$. Since $W(A)$ is separated and complete for the filtration $(V_n(A))_{n \in \mathbb{Z}}$, the canonical homomorphism π is an isomorphism of topological rings, where $W_n(A)$ has the discrete topology for each $n \geq 1$.

Henceforth, the homomorphisms π_n and $\pi_{n,m}$ will be called the projection from $W(A)$ to $W_n(A)$ and $W_m(A)$ to $W_n(A)$, respectively.

Example 16.1.11.

- (a) The homomorphism $\Phi_0 : W_1(A) \rightarrow A$ is an isomorphism.

(b) Explicitly, the operations on $W_2(A)$ are given by

$$\begin{aligned}[a_0, a_1] + [b_0, b_1] &= [a_0 + b_0, a_1 + b_1 - \sum_{i=1}^{p-1} \frac{1}{p} \binom{p}{i} a_0^i b_0^{p-i}], \\ [a_0, a_1] \times [b_0, b_1] &= [a_0 b_0, a_0^p b_1 + a_1 b_0^p + p a_1 b_1].\end{aligned}$$

where $[a_0, a_1]$ and $[b_0, b_1]$ are in $W_2(A)$. The phantom component of $[a_0, a_1]$ is a_0 and $a_0^p + p a_1$.

(c) Let $n \geq 1$ be an integer. If a_0, \dots, a_{n-1} and b_0, \dots, b_{n-1} are integers such that $a_i \equiv b_i \pmod{p}$ for each i , then (??)

$$\Phi_{n-1}(a_0, \dots, a_{n-1}) \equiv \Phi_{n-1}(b_0, \dots, b_{n-1}) \pmod{p^n}.$$

Therefore, by passing to quotient, Φ_{n-1} defines a homomorphism $\varphi_n : W_n(\mathbb{Z}/p\mathbb{Z}) \rightarrow \mathbb{Z}/p^n\mathbb{Z}$. The image of φ_n is a subgroup of $\mathbb{Z}/p^n\mathbb{Z}$ containing 1, hence the whole $\mathbb{Z}/p^n\mathbb{Z}$. Since $\mathbb{Z}/p^n\mathbb{Z}$ and $W_n(\mathbb{Z}/p\mathbb{Z})$ are both finite sets, φ_n is an isomorphism.

Let m and n be integers such that $1 \leq n \leq m$. There exists a ring homomorphism $\alpha_{n,m} : \mathbb{Z}/p^m\mathbb{Z} \rightarrow \mathbb{Z}/p^n\mathbb{Z}$; moreover, the diagram

$$\begin{array}{ccc} \mathbb{Z}/p^m\mathbb{Z} & \xrightarrow{\alpha_{n,m}} & \mathbb{Z}/p^n\mathbb{Z} \\ \downarrow & & \downarrow \\ W_m(\mathbb{Z}/p\mathbb{Z}) & \xrightarrow{\pi_{n,m}} & W_n(\mathbb{Z}/p\mathbb{Z}) \end{array}$$

is commutative. This then gives a map $\varphi = \varprojlim \varphi_n$ and an isomorphism of topological rings $W(\mathbb{Z}/p\mathbb{Z}) = \varprojlim W_n(\mathbb{Z}/p\mathbb{Z})$ and $Z_p = \varprojlim \mathbb{Z}/p^n\mathbb{Z}$.

Let $m, n \geq 1$ be integers. By construction, we have an exact sequence of additive groups

$$0 \longrightarrow W(A) \xrightarrow{V^m} W(A) \xrightarrow{\pi_m} W_m(A) \longrightarrow 0 \tag{16.1.39}$$

By passing to quotient, the endomorphism V^n of the additive group of $W(A)$ defines a homomorphism V_m^n on the additive group $W_m(A)$ to $W_{m+n}(A)$. In other words, we have a commutative diagram

$$\begin{array}{ccc} W(A) & \xrightarrow{V^n} & W(A) \\ \downarrow \pi_n & & \downarrow \pi_{n+m} \\ W_m(A) & \xrightarrow{V_m^n} & W_{n+m}(A) \end{array}$$

Moreover, from (??) we deduce the following exact sequence

$$0 \longrightarrow W_m(A) \xrightarrow{V_m^n} W_{m+n}(A) \xrightarrow{\pi_{n,n+m}} W_n(A) \longrightarrow 0 \tag{16.1.40}$$

and for $[a_0, \dots, a_{m-1}]$ in $W_m(A)$,

$$V_m^n[a_0, \dots, a_{m-1}] = [\underbrace{0, \dots, 0}_{n \text{ terms}}, a_0, \dots, a_{m-1}] \quad (16.1.41)$$

From ??(c), we have $FV^{m+1}(\mathbf{a}) = p \cdot V^m(\mathbf{a})$ for \mathbf{a} in $W(A)$ and therefore $F(V_{m+1}(A)) \subseteq V_m(A)$. By induction on n , we then deduce that F^n maps $V_{n+m}(A)$ to $V_m(A)$, and therefore defines, by passing to quotient, a homomorphism $F_m^n : W_{n+m}(A) \rightarrow W_m(A)$. By construction, we have the following exact sequence

$$\begin{array}{ccc} W(A) & \xrightarrow{F^n} & W(A) \\ \downarrow \pi_{n+m} & & \downarrow \pi_m \\ W_{n+m}(A) & \xrightarrow{F_m^n} & W_m(A) \end{array}$$

Recall the polynomial F_i in $\mathbb{Z}[X_0, \dots, X_{i+1}]$ for each $i \geq 0$. The homomorphism F_m^1 of $W_{m+1}(A)$ to $W_m(A)$ has the following explicit expression

$$F_m^1[a_0, \dots, a_m] = [F_i(a_0, \dots, a_{i+1})]_{0 \leq i < m}. \quad (16.1.42)$$

Let $\mathbf{a}, \mathbf{a}' \in W_m(A)$ and $\mathbf{b} \in W_{m+1}(A)$. The following formulae then follows from ??:

$$\begin{aligned} F_m^1(V_m^1(\mathbf{a})) &= p \cdot \mathbf{a} \\ V_m^1(\mathbf{a} \times F_m^1(\mathbf{b})) &= V_m^1(\mathbf{a}) \times \mathbf{b} \\ V_m^1(\mathbf{a}) \times V_m^1(\mathbf{a}') &= p \cdot V_m^1(\mathbf{a} \times \mathbf{a}') \\ V_m^1(F_m^1(\mathbf{b})) &= \boldsymbol{\mu}_{m+1} \times \mathbf{b} \end{aligned}$$

where $\boldsymbol{\mu}_{m+1} = [0, 1, 0, \dots, 0]$.

16.1.4 The Witt ring with characteristic p

Proposition 16.1.12. *Let A be a ring with characteristic p . Then for \mathbf{a}, \mathbf{b} in $W(A)$ and positive integers m, n , we have*

$$F(\mathbf{a}) = (a_n^p)_{n \in \mathbb{N}} \quad (16.1.43)$$

$$p \cdot \mathbf{a} = VF(\mathbf{a}) = FV(\mathbf{a}) = (0, a_0^p, a_1^p, \dots) \quad (16.1.44)$$

$$V^m(\mathbf{a}) \times V^n(\mathbf{b}) = V^{m+n}(F^n(\mathbf{a}) \times F^m(\mathbf{b})). \quad (16.1.45)$$

Proof. The formula ?? follows from ??(d). We immediately deduce from this the equality

$$VF(\mathbf{a}) = FV(\mathbf{a}) = (0, a_0^p, a_1^p, \dots)$$

and the equality $p \cdot \mathbf{a} = FV(\mathbf{a})$ is already proved in ??(b), whence ???. Now recall that by ??, we have

$$V^m(\mathbf{a}) \times V^n(\mathbf{b}) = V^m(\mathbf{a} \times F^m(V^n(\mathbf{b}))).$$

By (??), we also deduce that

$$V^n(F^m(\mathbf{b})) \times \mathbf{a} = V^n(F^m(\mathbf{b}) \times F^n(\mathbf{a})).$$

Formula (??) then follows from these two equalities and the relation $F^m \circ V^n = V^n \circ F^m$, which comes from (??). \square

Corollary 16.1.13. *Let m and n be positive integers, then*

$$V_m(A) \times V_n(A) \subseteq V_{m+n}(A).$$

Proof. This follows from (??), since $V_m(A)$ is the image of V^m . \square

Remark 16.1.4. Let A be a ring with characteristic p . By ??, we have the formulae

$$F_m^n[a_0, \dots, a_{n+m-1}] = [a_0^{p^n}, \dots, a_{m-1}^{p^n}], \quad p^n \cdot [a_0, \dots, a_{n+m-1}] = [\underbrace{0, \dots, 0}_{n \text{ terms}}, a_0^{p^n}, \dots, a_{m-1}^{p^n}]$$

for any Witt vector $[a_0, \dots, a_{n+m-1}]$ of length $n + m$.

Proposition 16.1.14. *Let A be a ring.*

- (a) *For each integer $k \geq 1$, we have $V_1(A)^k = p^{k-1} \cdot V_1(A)$.*
- (b) *Suppose that A is a ring with characteristic p . Then the $V_1(A)$ -adic topology and the p -adic topology coincide on $W(A)$, and is finer than the topology \mathcal{T} . The ring $W(A)$ is separated and complete for the p -adic topology.*

Proof. We prove (a) by induction on k . The case $k = 1$ is evident, so suppose $k \geq 2$. By the induction hypothesis, we have $V_1(A)^{k-1} = p^{k-2} \cdot V_1(A)$ and Consequently $V_1(A)^k = p^{k-2} \cdot V_1(A)^2$. But by ??(d), $V_1(A)^2 = p \cdot V_1(A)$, so (a) follows.

Suppose now that A has characteristic p . Then by (??),

$$pW(A) = VF(W(A)) \subseteq V_1(A)$$

combine this with (a), we deduce by induction that $p^k \cdot W(A) \subseteq V_1(A)^k \subseteq p^{k-1} \cdot W(A)$, and by ?? the inclusion $V_1(A)^k \subseteq V_k(A)$, for each $k \geq 1$. The first assertion in (b) then follow.

Let $k \geq 1$ be an integer. By (??), the ideal $p^k \cdot W(A)$ of $W(A)$ is the set of elements $\mathbf{a} = (a_n)_{n \in \mathbb{N}}$ of $W(A)$ such that $a_n = 0$ for $n < k$ and $a_n \in A^{p^k}$ for $n \geq k$. It is therefore closed in the topology \mathcal{T} . Since $W(A)$ is separated and complete for the topology \mathcal{T} and that the ideals $p^k \cdot W(A)$ of $W(A)$, for $k \geq 1$, form a base of neighborhoods of $\mathbf{0}$ in $W(A)$ for the p -adic topology, the ring $W(A)$ is separated and complete for the p -adic topology (??). \square

Proposition 16.1.15. *Let A be a perfect ring of characteristic p .*

- (a) *For any element $\mathbf{a} = (a_n)_{n \in \mathbb{N}}$ in $W(A)$, the series $\sum_n p^n \tau(a_n^{p^{-n}})$ converges in $W(A)$ to \mathbf{a} .*
- (b) *In $W(A)$, the $V_1(A)$ -adic topology, the p -adic topology, and the topology \mathcal{T} coincide. Moreover precisely, we have $V_n(A) = p^n W(A) = V_1(A)^n$ for each integer $n \geq 0$. In particular, Φ_0 defines an isomorphism of $W(A)/pW(A)$ to A .*

Proof. By definition, the map $a \mapsto a^p$ is an automorphism of A . ?? then show that F is an automorphism of $W(A)$, and for $n \in \mathbb{N}$, we have

$$p^n \cdot W(A) = V^n F^n(W(A)) = V^n(W(A)) = V_n(A).$$

In particular, $V_1(A)^n = (pW(A))^n = p^n \cdot W(A)$. Assertion (b) then follows. By ??, we have

$$p^n \cdot \tau(a_n^{p^{-n}}) = V^n F^n \tau(a_n^{p^{-n}}) = V^n \tau(a_n)$$

so (a) follows from ??.

Proposition 16.1.16. *Let A be a field with characteristic p . Then $W(A)$ is a local integral domain, separated and complete, with maximal ideal $V_1(A)$ and residue field isomorphic to A . If A is perfect, the ring $W(A)$ is a DVR, and its maximal ideal is $pW(A)$.*

Proof. The homomorphism Φ_0 defines an isomomorphic of $W(A)/V_1(A)$ to A . The ideal $V_1(A)$ of $W(A)$ is therefore maximal. The ring $W(A)$ is separated and complete for the $V_1(A)$ -topology ??, so is local with maximal ideal $V_1(A)$ by ??.

Let \mathbf{a} and \mathbf{b} be two nonzero elements in $W(A)$. There exists integers $n, m \geq 0$ and elements $\tilde{\mathbf{a}} = (\tilde{a}_n)_{n \in \mathbb{N}}$ and $\tilde{\mathbf{b}} = (\tilde{b}_n)_{n \in \mathbb{N}}$ of $W(A)$ such that $\mathbf{a} = V^m(\tilde{\mathbf{a}})$, $\mathbf{b} = V^n(\tilde{\mathbf{b}})$ and the elements \tilde{a}_0, \tilde{b}_0 are nonzero. Then the $m + n$ -th component of $\mathbf{a} \times \mathbf{b}$ is equal to the 0-th component of $F^n(\tilde{\mathbf{a}}) \times F^m(\tilde{\mathbf{b}})$ (formula ??), which is $\tilde{a}^{p^n} \tilde{b}^{p^m}$ (formula ?? and ??). Therefore $\mathbf{a} \times \mathbf{b}$ is nonzero and $W(A)$ is integral.

If the field A is perfect, the maximal ideal $V_1(A)$ of $W(A)$ is equal to $pW(A)$ by ??(b) and therefore $W(A)$ is a DVR by ??(c). \square

16.2 Cohen rings

In this part, we fix a prime integer p .

16.2.1 p -rings

Let C be a ring. We say C is a **p -ring** if the ideal pC of C is maximal and if C is separated and complete for the pC -adic topology.

If $p1_C$ is nilpotent in C and the ideal pC is maximal, then C is a p -ring and the pC -adic topology is discrete. In particular, every field with characteristic p is a p -ring.

Proposition 16.2.1. *Let C be a p -ring.*

- (a) *The ring C is local with maximal ideal pC .*
- (b) *Suppose that $p1_C$ is nilpotent and let d be the smallest positive integer such that $p^d 1_C = 0$. Then the ideals of C are of the form $p^i C$ and $p^i C \neq p^j C$ if i, j are distinct integers such that $0 \leq i, j \leq d$. The C -module C has length d .*
- (c) *Suppose that $p1_C$ is not nilpotent. Then C is a DVR whose residue field is of characteristic p , and the fraction field of characteristic 0. The ideals $p^n C$, where $n \in \mathbb{N}$, are distinct; they are the nonzero ideals of C . The C -module C has infinite length.*

Proof. Assertion (a) follows from ?? . By hypothesis we have $\bigcap_{n \in \mathbb{N}} p^n C = \{0\}$. Let $x \neq 0$ be in C ; there exists an integer $n \geq 0$ such that $x \in p^n C$, $x \notin p^{n+1} C$. Write $x = p^n y$ where $y \in C$, then $y \notin pC$, so it is invertible.

Suppose that $p1_C$ is not nilpotent. If x_1 and x_2 are two nonzero elements in C , there exist integers $n_1, n_2 \geq 0$ and invertible elements y_1, y_2 of C such that $x_1 = p^{n_1} y_1$, $x_2 = p^{n_2} y_2$. We then have $x_1 x_2 = p^{n_1+n_2} y_1 y_2 \neq 0$, so C is integral. Since C is a local ring, not a field, and the maximal ideal $\mathfrak{m}_C = pC$ is principal, we conclude that C is a DVR (??). The nonzero ideals of C are then of the form $p^n C$, and are all distinct. In particular, the ring C is not Artinian, so the C -module C has infinite length. The residue field C/pC of C clearly has characteristic p . Let q be the characteristic of the fraction field of C . We have $p1_C \neq 0$, so $q \neq p$. But if $q \neq 0$, we have $q1_C \neq 0$ and C/pC then has characteristic $q \neq p$, which is absurd. This proves (c).

Suppose that $p1_C$ is nilpotent and let d be the smallest positive integer such that $p^d 1_C = 0$. We have a sequence of ideals

$$C \supseteq pC \supseteq \cdots \supseteq p^{d-1}C \supseteq p^d C = \{0\}. \quad (16.2.1)$$

If i is an integer such that $0 \leq i < d$ and $p^i C = p^{i+1} C$, then

$$p^{d-i-1} p^i C = p^{d-i-1} p^{i+1} C = p^d C = \{0\}$$

contradicting the hypothesis $p^{d-1} 1_C \neq 0$. Therefore the ideals in the sequence above are all distinct. Let \mathfrak{a} be an ideal of C and i the smallest integer such that $\mathfrak{a} \supseteq p^i C$. Let $x \neq 0$ be in \mathfrak{a} ; we can write $x = p^m u$ where $m \geq 0$ and u is invertible. Then $p^m 1_C \in \mathfrak{a}$, whence $p^m C \subseteq \mathfrak{a}$ and $m \geq i$. This shows $x \in p^i C$, so $\mathfrak{a} = p^i C$. We now conclude that (??) is a Jordan-Hölder sequence for the C -module C , so C has length d . \square

Corollary 16.2.2. *If the p -ring C is integral, it is a DVR, or a field with characteristic p .*

Proof. Suppose that C is integral. If $p1_C$ is nilpotent, then $p1_C = 0$, and $\{0\}$ is a maximal ideal of C , whence C is a field with characteristic p . Otherwise, $p1_C$ is not nilpotent and C is a DVR by ?? . \square

Corollary 16.2.3. *Let C be a p -ring and \mathfrak{a} a proper ideal of C . Then C/\mathfrak{a} is a p -ring.*

Proof. We may suppose that $\mathfrak{a} \neq \{0\}$. Then there exists an integer $i \geq 1$ such that $\mathfrak{a} = p^i C$. The ideal pC/\mathfrak{a} of C/\mathfrak{a} is maximal and we have $p^i 1_{C/\mathfrak{a}} = 0$, so C/\mathfrak{a} is a p -ring. \square

Let C be a p -ring. The length of C , denoted by $l(C)$, is the supremum in $\overline{\mathbb{R}}$, of the set of integers $n \geq 1$ such that $p^{n-1} 1_C \neq 0$. By ?? , $l(C)$ is finite and equal to the length of the C -module C , or is $+\infty$, in which case the C -module C has infinite length.

Example 16.2.4 (Example of p -rings).

- (a) For each integer $n \geq 1$, the ring $\mathbb{Z}/p^n \mathbb{Z}$ is a p -ring of length n . The ring \mathbb{Z}_p of p -adic integers is a p -ring of length infinite.
- (b) Let K be a perfect field of characteristic p . By ?? , the Witt ring $W(K)$ is a p -ring of infinite length. The map $(a_n)_{n \in \mathbb{N}} \mapsto a_0$ induces by passing to quotient an isomorphism of

$W(K)/pW(K)$ to the field K . For each integer $n \geq 1$, the ring

$$W_n(K) = W(K)/p^nW(K)$$

is a p -ring of length n . Note that $W(\mathbb{F}_p) = \mathbb{Z}_p$, so this generalizes the example in (a).

Proposition 16.2.5. *Let C_1 and C_2 be p -rings and $\rho : C_1 \rightarrow C_2$ a ring homomorphism. Let $\bar{\rho}$ be the homomorphism of $\kappa_{C_1} = C_1/pC_1$ to $\kappa_{C_2} = C_2/pC_2$ induced by ρ on the residue field.*

- (a) *We have $l(C_1) \geq l(C_2)$, and ρ is injective if and only if $l(C_1) = l(C_2)$.*
- (b) *For ρ to be surjective, it is necessary and sufficient that $\bar{\rho}$ being an isomorphism.*
- (c) *For ρ to be an isomorphism, it is necessary and sufficient that $\bar{\rho}$ being an isomorphism and we have $l(C_1) = l(C_2)$.*

Proof. Let $n \geq 1$ be an integer. We have $\rho(p^{n-1}1_{C_1}) = p^{n-1}1_{C_2}$, so the relation $p^{n-1}1_{C_2} \neq 0$ implies $p^{n-1}1_{C_1} \neq 0$ and they are equivalent if ρ is injective. We then have $l(C_2) \leq l(C_1)$ where the equality holds if ρ is injective. If ρ is not injective, there exists an integer $i < l(C_1)$ such that the kernel of ρ is the ideal p^iC_1 of C_1 ; we have $p^i1_{C_2} = 0$, so $l(C_2) \leq i$. This proves (a). Since κ_{C_1} and κ_{C_2} are fields, the homomorphism $\bar{\rho}$ is injective. If ρ is surjective, so is $\bar{\rho}$ and hence it is an isomorphism. Conversely, assume that $\bar{\rho}$ is surjective. Then for each integer $n \geq 0$, the map $\bar{\rho}_n : p^nC_1/p^{n+1}C_1 \rightarrow p^nC_2/p^{n+1}C_2$ induced by ρ is surjective. Since C is complete for the pC_1 -adic topology, ρ is then surjective by ???. This proves (b). Finally, (a) and (b) imply (c). \square

Proposition 16.2.6. *Let $(C_n, \pi_{n,m})$ be a inverse system of rings indexed by \mathbb{N} . Suppose that C_n is a p -ring for each $n \in \mathbb{N}$ and the homomorphism $\pi_{n,m}$ are surjective. Then $C = \varprojlim C_n$ is a p -ring, and for each $n \in \mathbb{N}$, the canonical homomorphism $\pi_n : C \rightarrow C_n$ is surjective and induces an isomorphism of κ_C to κ_{C_n} .*

Proof. Since the maps $\pi_{n,m}$ are surjective, π_n is surjective (??). Let d_n be the length of C_n . By ??(a), the sequence d_n in $\mathbb{N} \cup \{+\infty\}$ is increasing; if it is stationary, there exists an integer n_0 such that $\pi_{n,m}$ is an isomorphism of C_m to C_n for $m \geq n \geq N_0$, hence C is isomorphic to C_{n_0} , a p -ring.

It then suffices to consider the case where d_n are all finite, and (d_n) tends to infinity. Let us endow the ring C with the trivial filtration. For each $n \in \mathbb{N}$, let I_n be the kernel of π_n , and put $I_n = C$ for $n < 0$. Endow C with the filtration $(I_n)_{n \in \mathbb{N}}$. It is then separated and complete, for the topology \mathcal{T} defined by the filtration is the inverse limit topology of the discrete topologies over C_n . Let $k \geq 1$ be an integer. We have $p^kC \subseteq \varprojlim_n (p^kC_n)$. Conversely, if $x = (x_n)_{n \in \mathbb{N}} \in \varprojlim_n (p^kC_n)$ and if we set $X_n = \{y \in C : \pi_n(p^ky) = x_n\}$, the sequence $(X_n)_{n \in \mathbb{N}}$ is a decreasing sequence of nonempty closed affine subsets of C . Since C/I_n is an Artinian C -module, the intersection of X_n is nonempty (??); for every $z \in \bigcap_{n \in \mathbb{N}} X_n$, we have $p^kz = x$. We have therefore proved that $p^kC = \varprojlim_n (p^kC_n)$ for each integer $k \geq 1$. In particular, the ideal p^kC of C is closed for the topology \mathcal{T} . For C , the p -adic topology is finer than the topology \mathcal{T} since we have $p^{d_n}C \subseteq I_n$. It then follows from ?? that C is separated and complete for the pC -adic topology. In addition we have $pC = \varprojlim pC_n = \pi_0^{-1}(pC_0)$ and therefore the surjective homomorphism of C/pC to C_0/pC_0 induced by π_0 is an isomorphism. This shows that pC is maximal and therefore C is a p -ring. The final assertion follows from ??(b). \square

Now let A be a separated and complete local ring, with residue field of characteristic p . A Cohen subring of A is defined to be a subring C of A that is a p -ring and such that $A = \mathfrak{m}_A + C$ (i.e., $A/\mathfrak{m}_A = C/(\mathfrak{m}_A \cap C)$). In this case, the ideal $\mathfrak{m}_A \cap C$ is maximal in C , hence equal to pC . The canonical map $\kappa_C = C/pC$ to $\kappa_A = A/\mathfrak{m}_A$ is then an isomorphism.

Example 16.2.7. Let C be a p -ring. The formal series ring $A = C[[X_1, \dots, X_n]]$ is then Noetherian, local, separated and complete, with maximal ideal generated by (p, T_1, \dots, T_n) . It is immediate that C is a Cohen subring of A . This applies in particular if C is equal to \mathbb{Z}_p , to $\mathbb{Z}/p^n\mathbb{Z}$, or a field with characteristic p .

Theorem 16.2.8. Let A be a separated and complete local ring, with residue field k of characteristic p . Let $\pi : A \rightarrow k$ be the canonical map, and S a subset of A such that π induces a bijection of S to a p -basis for k .

- (a) There exists a unique Cohen subring C of A containing S .
- (b) The subring C is closed in A , and the pC -adic topology of C is induced by the \mathfrak{m}_A -adic topology of A .
- (c) Every closed subring B of A , containing S , and such that $A = B + \mathfrak{m}_A$, contains C .

Proof. We divide the proof into several parts. First, we assume that \mathfrak{m}_A is nilpotent. Let n be a positive integer such that $\mathfrak{m}_A^{n+1} = \{0\}$. If Φ_n is the n -th Witt polynomial, the map $\rho : [a_0, \dots, a_n] \mapsto \Phi_n(a_0, \dots, a_n)$ is a homomorphism from $W_{n+1}(A)$ to A . Let B_n be the image of ρ_n and C_n the subring of A generated by $B_n \cup S$. We note that $pA \subseteq \mathfrak{m}_A$ and B_n consists of elements of the form $a_0^{p^n} + pa_1^{p^{n-1}} + \dots + p^n a_n$, where $a_0, \dots, a_n \in A$. Consequently, we have $\pi(B_n) = k^{p^n}$, and $\pi(C_n) = k^{p^n}[\pi(S)]$. Since $\pi(S)$ is a p -basis for k , we have $k = k^{p^n}[\pi(S)]$ (??), whence $\pi(C_n) = k$, and $C_n + \mathfrak{m}_A = A$.

Let B be a subring of A containing S . We claim that, for B to contain C_n , it is necessary and sufficient that $A = B + \mathfrak{m}_A$. First, if B contains C_n , then

$$B + \mathfrak{m}_A \supseteq C_n + \mathfrak{m}_A = A$$

so $B + \mathfrak{m}_A = A$. Conversely, suppose that $B + \mathfrak{m}_A = A$. Let a_0, \dots, a_n be elements of A ; there exists by hypothesis elements b_0, \dots, b_n of B such that $a_i \equiv b_i \pmod{\mathfrak{m}_A}$ for each i . By ?? and the hypothesis $\mathfrak{m}_A^{n+1} = 0$, we then have $\Phi_n(a_0, \dots, a_n) = \Phi_n(b_0, \dots, b_n) \in B$, whence $B_n \subseteq B$. Since C_n is the ring generated by $B_n \cup S$, we have $C_n \subseteq B$.

Let \mathcal{S} be the set of subrings B of A containing S and such that $B + \mathfrak{m}_A = A$. There exists by the above claim a smallest element C in \mathcal{S} (take the smallest integer n such that $\mathfrak{m}_A^{n+1} = \{0\}$, then C_n is the desired subring), and we have $C_n = C$ for any integer $n \geq 0$ such that $\mathfrak{m}_A^{n+1} = \{0\}$. We have $C + \mathfrak{m}_A = A$ by construction and $p1_C$ is nilpotent; clearly $pC \subseteq C \cap \mathfrak{m}_A$. We now prove that $C \cap \mathfrak{m}_A \subseteq pC$. Choose an integer $n \geq 1$ such that $\mathfrak{m}_A^n = \{0\}$, so $C = C_n = C_{n-1}$. Let Λ be the subset of $\mathbb{N}^{\oplus S}$ formed by elements $(\alpha_s)_{s \in S}$ with finite support such that $0 \leq \alpha_s < p^n$ for each $s \in S$. Since B_n contains $s^{p^n} = \Phi_n(s, 0, \dots, 0)$ for each $s \in S$, the monomials $z_\alpha = \prod_{s \in S} s^{\alpha_s}$, for $\alpha \in \Lambda$, generate the C_n as a B_n -module. Moreover, by the formula

$$\Phi_n(a_0, \dots, a_n) = a_0^{p^n} + p\Phi_{n-1}(a_1, \dots, a_n),$$

any element of B_n is of the form $a^{p^n} + pb$ where $a \in A$ and $b \in B_{n-1}$. Therefore any element of $C = C_n$ is of the form

$$x = \sum_{\alpha \in \Lambda} c_\alpha^{p^n} z_\alpha + py \quad (16.2.2)$$

where $c_\alpha \in A$ for $\alpha \in \Lambda$, and $y \in C_{n-1} = C$. If x is in $C \cap \mathfrak{m}_A$, we have $\pi(x) = 0$ whence $\sum_{\alpha \in \Lambda} \pi(c_\alpha)^{p^n} \pi(z_\alpha) = 0$. Since $\pi(S)$ is a p -basis for k , $\pi(c_\alpha) = 0$ for all $\alpha \in \Lambda$ (??). We then have $c_\alpha \in \mathfrak{m}_A$, whence $c_\alpha^n = 0$ and a fortiori $c_\alpha^{p^n} = 0$. By (??), we then get $x = py$, whence $C \cap \mathfrak{m}_A \subseteq pC$. With this, it is now clear that C is a Cohen subring of A . We have $p^n C = \mathfrak{m}_A^n = \{0\}$ for n large enough and assertion (b) is then trivial. Assertion (c) follows from the construction of C .

We now turn to the general case. For each integer $n \geq 0$, let A_n be the local ring A/\mathfrak{m}_A^{n+1} , $\mathfrak{m}_n = \mathfrak{m}_A/\mathfrak{m}_A n + 1$, and $\pi_n : A \rightarrow A_n$ the canonical homomorphism. By the previous case, there exists a unique Cohen subring C_n in A_n containing $\pi_n(S)$. For $0 \leq n \leq m$, we denote by $\pi_{n,m}$ the canonical map from A_m to A_n . By ??, $\pi_{n,m}(C_m)$ is a p -ring; we have $\pi_{n,m}(C_m) + \mathfrak{m}_n = A_n$, so $\pi_{n,m}(C_m)$ is equal to the Cohen subring C_n of A_n . By ??, the subring $\varprojlim C_n$ of $\varprojlim A_n$ is a p -ring. Put $C = \bigcap_{n \in \mathbb{N}} \pi_n^{-1}(C_n)$. Then C is the preimage of $\varprojlim C_n$ under the isomorphism $a \mapsto (\pi_n(a))_{n \in \mathbb{N}}$ of A to $\varprojlim A_n$, whence a closed subring of A , and a p -ring. We have $\pi_n(C) = C_n$ for each $n \in \mathbb{N}$ (??) and in particular $\pi_0(C) = A_0$, which means $\pi(C) = k$. Therefore C is a Cohen subring of A .

For each integer $n \geq 0$, let $J_n = C \cap \mathfrak{m}_A^n$. Since the local ring A is separated, we have $\bigcap_{n \in \mathbb{N}} J_n = \{0\}$, and by the structure of ideals of p -rings, every ideal of C is of the form $p^i C$ and contains some J_n . Conversely, J_n contains $p^n C$. Therefore the pC -adic topology is induced by the \mathfrak{m}_A -topology on A . This proves (b). Now let B be a closed subring of A containing S and such that $B + \mathfrak{m}_A = A$. Since B is closed, we have $B = \bigcap_{n \in \mathbb{N}} \pi_n^{-1}(\pi_n(B))$. We have $\pi_n(B) \supset \pi_n(S)$ and $\pi_n(B) + \mathfrak{m}_n = A_n$, whence $\pi_n(B) \supseteq C_n$ and therefore $B \supseteq C$. This proves (c) and completes the proof. \square

Remark 16.2.1. Suppose that $p1_A$ is not nilpotent (in particular if A is an integral domain with fraction field of characteristic 0). Then C is a DVR with fraction field of characteristic 0.

16.2.2 Existence and uniqueness of p -rings

Proposition 16.2.9. Let C_1 and C_2 be p -rings such that $l(C) \geq l(C_2)$, π_1 (resp. π_2) the canonical homomorphism of C_1 (resp. C_2) to κ_{C_1} (resp. κ_{C_2}). Let $(x_\lambda)_{\lambda \in \Lambda}$ (resp. $(y_\lambda)_{\lambda \in \Lambda}$) be a family of elements of C_1 (resp. C_2) whose image under π_1 (resp. π_2) is a p -basis of κ_{C_1} (resp. κ_{C_2}). Let η be an isomorphism of κ_{C_1} to κ_{C_2} such that $\eta(\pi_1(x_\lambda)) = \pi_2(y_\lambda)$ for every $\lambda \in \Lambda$. There then exists a unique homomorphism $\rho : C_1 \rightarrow C_2$ such that $\eta \circ \pi_1 = \pi_2 \circ \rho$ and $\rho(x_\lambda) = y_\lambda$ for every $\lambda \in \Lambda$. It is surjective, and an isomorphism if $l(C_1) = l(C_2)$.

Proof. Let A be the subring of $C_1 \times C_2$ formed by pairs (x, y) such that $\eta(\pi_1(x)) = \pi_2(y)$. The map $(x, y) \mapsto \pi_1(x)$ is a surjective homomorphism of A to κ_C . Its kernel \mathfrak{m} , equal to $pC_1 \times pC_2$, is then a maximal ideal of A . The subspace A of $C_1 \times C_2$ is closed, hence complete, and the topology induced on A from $C_1 \times C_2$ is the \mathfrak{m} -adic topology. Thus A is a separated and complete local ring with maximal ideal \mathfrak{m} . For each $\lambda \in \Lambda$, we have $(x_\lambda, y_\lambda) \in A$ by hypothesis; if ξ_λ is the class of (x_λ, y_λ) modulo \mathfrak{m} , the family $(\xi_\lambda)_{\lambda \in \Lambda}$ is a p -basis of A/\mathfrak{m} .

By ??, there exists a unique Cohen subring \tilde{C} of A containing (x_λ, y_λ) for each $\lambda \in \Lambda$. By definition, we have $l(\tilde{C}) = l(C_1) \geq l(C_2)$. The restriction to \tilde{C} of the projection of $C_1 \times C_2$ to C_1 is a homomorphism $h_1 : \tilde{C} \rightarrow C_1$ which induces an isomorphism of $\kappa_{\tilde{C}}$ to κ_{C_1} . By ??(c), h_1 is an isomorphism of \tilde{C} to C_1 . We also see that the restriction h_2 to \tilde{C} of the projection from $C_1 \times C_2$ to C_2 is a surjective homomorphism. Therefore, \tilde{C} is the graph of a surjective homomorphism $\rho = h_2 \circ h_1^{-1}$ of C_1 to C_2 , and we have $\eta \circ \pi_1 = \pi_2 \circ \rho$, $\rho(x_\lambda) = y_\lambda$ for $\lambda \in \Lambda$. Moreover, if $l(C_1) = l(C_2)$, ρ is an isomorphism.

Let ρ_1 be a homomorphism of C_1 to C_2 such that $\eta \circ \pi_1 = \pi_2 \circ \rho_1$, $\rho_1(x_\lambda) = y_\lambda$ for $\lambda \in \Lambda$, and let \tilde{C}' be the graph of ρ_1 . It is immediate that \tilde{C}' is a Cohen subring of A , containing (x_λ, y_λ) for $\lambda \in \Lambda$, whence $\tilde{C}' = \tilde{C}$ and $\rho_1 = \rho$. \square

Proposition 16.2.10. *Let k be a field of characteristic p , and let $n \geq 1$ an integer, or $+\infty$. Then there exists a p -ring of length n with residue field isomorphic to k .*

Proof. The ring $W(k)$ of Witt vectors with coefficients in k is an integral separated and complete local ring, with residue field isomorphic to k (??), and we have $p \cdot 1_{W(k)} \neq 0$. Let C be a Cohen subring of $W(k)$ (??). Then C is a p -ring of length $+\infty$ with residue field isomorphic to k , and, if $n \geq 1$ is an integer, the quotient C/p^nC is a p -ring of length n with residue field isomorphic to k . \square

Remark 16.2.2. Let $n \geq 1$ be an integer and S a p -basis of k . We can show that the subring $W_n(k)$ generated by $W_n(k^{p^n})$ and the elements $[\xi, 0, \dots, 0]$ (where $\xi \in S$), is a p -ring of length n with residue field isomorphic to k .

Corollary 16.2.11. *Let C be a p -ring of finite length n . Then there exists a p -ring \tilde{C} of infinite length such that C is isomorphic to $\tilde{C}/p^n\tilde{C}$.*

Proof. By ??, there exists a p -ring \tilde{C} of infinite length such that $\kappa_{\tilde{C}}$ is isomorphic to κ_C . Then $\tilde{C}/p^n\tilde{C} = \tilde{C}_n$ is a p -ring of length n , and $\kappa_{\tilde{C}_n}$ is isomorphic to $\kappa_{\tilde{C}}$, hence to κ_C . By ??, the ring C and \tilde{C}_n are isomorphic. \square

Proposition 16.2.12. *Let C be a p -ring, with residue field k perfect. Suppose that C has finite length n (resp. infinite length). There exists a unique isomorphism $\rho_C : W_n(k) \rightarrow C$ (resp. $\rho_C : W(k) \rightarrow C$) which induces by passing to quotient the identity map on k .*

Proof. Since $W_n(k)$ (resp. $W(k)$) is a p -ring with residue field k and length n (infinite), and \emptyset is a p -basis for k , ?? then implies the claim. \square

16.2.3 Multiplicative representatives

Theorem 16.2.13. *Let A be a separated and complete local ring, k the residue field, and $\pi : A \rightarrow k$ the canonical map. Suppose that k is perfect with characteristic p .*

- (a) *There exists a unique homomorphism $\rho : W(k) \rightarrow A$ such that $\pi(\rho(a)) = a_0$ for each $a = (a_n)_{n \in \mathbb{N}}$ in $W(k)$.*
- (b) *The homomorphism ρ is continuous if we endow $W(k)$ with the $pW(k)$ -adic topology, and its image is the unique Cohen subring of A .*

Proof. By ??, there exists a unique Cohen subring C of A . Let $\rho : W(k) \rightarrow A$ be a ring homomorphism such that $\pi(\rho(a)) = a_0$ for $a = (a_n)_{n \in \mathbb{N}}$ in $W(k)$; it is immediate that the image of ρ is a Cohen subring of A , hence equal to C . The existence and uniqueness of ρ then follows from ?? . The pC -adic topology on C is induced by the \mathfrak{m}_A -adic topology ??(b)), so ρ is continuous. \square

Proposition 16.2.14. *Retain the hypothesis and notations in Therorem ?? . There exist a multiplicative subset S of A such that π induces a bijection of S to k . For an element $a \in A$ to belong to S , it is necessary and sufficient that for each $n \in \mathbb{N}$, there exists an element a_n in A such that $a = a_n^{p^n}$. The set S consists of elements of the form $\rho(x, 0, 0, \dots)$.*

Proof. Let S be a multiplicative subset of A such that π induces a bijection of S to k . Let T be the set of elements of A which are p^n -th powers for every $n \in \mathbb{N}$. Let $a \in S$ and $n \in \mathbb{N}$; since k is perfect, there exists x_n in k such that $x_n^{p^n} = \pi(a)$. But $\pi(S) = k$, so there exists $a_n \in S$ such that $x_n = \pi(a_n)$. Then we have $\pi(a_n^{p^n}) = \pi(a)$ whence $a_n^{p^n} = a$ since π is injective on S . This shows $S \subseteq T$. Conversely, we show that the restriction of π on T is injective, which then proves that $S = T$: let a and b be elements of T such that $\pi(a) = \pi(b)$. Let $n \in \mathbb{N}$; there exists two elements a_n and b_n of A such that $a = a_n^{p^n}$, $b = b_n^{p^n}$. Then $\pi(a_n)^{p^n} = \pi(b_n)^{p^n}$, hence $\pi(a_n) = \pi(b_n)$. This means $a_n \equiv b_n \pmod{\mathfrak{m}_A}$, so $a_n^{p^n} \equiv b_n^{p^n} \pmod{\mathfrak{m}_A^{n+1}}$?? . But then $a \equiv b \pmod{\mathfrak{m}_A^{n+1}}$. Since n is arbitrary and A is separated, we conclude that $a = b$.

In the notations of Therorem ?? , let $\varphi = \rho \circ \tau_k$, which means

$$\varphi(x) = \rho(x, 0, 0, \dots)$$

for $x \in k$. Since τ_k is multiplicative ??, so is φ and it is clear that $\pi \circ \varphi$ is the identity map on k . The image of φ then satisfies the requirements. \square

The elements of S are often called the **multiplicative (or Teichmüller) representatives** of A . Recall that by ??, we have

$$a = \sum_{n=0}^{\infty} p^n \tau_k(a_n^{p^{-n}})$$

for $a = (a_n)_{n \in \mathbb{N}}$ in $W(k)$. Therefore

$$\rho(a) = \sum_{n=0}^{\infty} p^n \varphi(a_n^{p^{-n}})$$

because ρ is continuous. By this formula, the unique Cohen subring of A consists of elements of the form $\sum_{n=0}^{\infty} p^n s_n$ where $s_n \in S$ for each $n \in \mathbb{N}$.

Example 16.2.15 (Example of multiplicative representatives).

- (a) Let k be a perfect field of characteristic p . The multiplicative representatives of the ring $W(k)$ is the Witt vectors $\tau(x) = (x, 0, 0, \dots)$ for $x \in k$.
- (b) Let A be an integral separated and complete local ring. We suppose that the residue field k of A is finite, with $q = p^f$ elements, hence perfect with characteristic p . We have $x^q = x$ for each $x \in k$, hence $s^q = s$ for each multiplicative representative. It follows that the set of multiplicative representatives consists of 0 and the $q - 1$ -th roots of the unity in the field of fractions of A . If the field of fractions of A is locally compact, the existence of the multiplicative representatives follows also from ??.

In particular, consider the case $A = \mathbb{Z}_p$. Then the multiplicative representatives are 0 and the $(p - 1)$ -th roots of unity in the field \mathbb{Q}_p .

16.2.4 Structure of Noetherian complete local rings

We now use the results in this part to give a structural theorem for Noetherian complete local rings (such rings are separated by ??). Let A and C be Noetherian complete local rings and $\rho : C \rightarrow A$ a ring homomorphism which induces an isomorphism residue fields. Let (p_1, \dots, p_m) be a sequence generating the maximal ideal \mathfrak{m}_C of C , and let (t_1, \dots, t_n) be a sequence of elements of A . Let S denote the sequence $(\rho(p_1), \dots, \rho(p_m), t_1, \dots, t_n)$.

Lemma 16.2.16. *Let $B = C[[T_1, \dots, T_n]]$ and $\eta : B \rightarrow A$ be the unique homomorphism extending ρ that sends T_i to t_i for each i .*

- (a) *For η to be surjective, it is necessary and sufficient that S generates the ideal \mathfrak{m}_A of A , or its class modulo \mathfrak{m}_A^2 generates $\mathfrak{m}_A/\mathfrak{m}_A^2$ as a vector space over κ_A .*
- (b) *For η to make A a finite B -algebra, it is necessary and sufficient that S generates a defining ideal of the \mathfrak{m}_A -adic topology on A .*

Proof. Let \mathfrak{n} be the ideal of B generated by T_1, \dots, T_n . Any homomorphism η of B to A extending ρ and such that $\eta(T_i) = t_i$ maps \mathfrak{n} to \mathfrak{m}_A , hence is continuous when B is endowed with the \mathfrak{n} -adic topology. The existence and uniqueness of η then follows from ??.

The ring $B = C[[T_1, \dots, T_n]]$ is a Noetherian complete local ring, and \mathfrak{m}_B is generated by $p_1, \dots, p_m, T_1, \dots, T_n$. We then have $\eta(\mathfrak{m}_B) \subseteq \mathfrak{m}_A$ and η defines a homomorphism $\text{gr}(\eta) : \text{gr}(B) \rightarrow \text{gr}(A)$. But the ring $\text{gr}(A)$ is generated by $A/\mathfrak{m}_A = \kappa_A$ and $\mathfrak{m}_A/\mathfrak{m}_A^2$, $\text{gr}(\eta)$ induces an isomorphism of $\kappa_B = \kappa_C$ to κ_A , and the classes modulo \mathfrak{m}_B^2 of the elements $p_1, \dots, p_m, T_1, \dots, T_n$ generate $\mathfrak{m}_B/\mathfrak{m}_B^2$ as a κ_B -vector space; moreover η is surjective if and only if $\text{gr}(\eta)$ is surjective (??). These prove (a).

The ideal \mathfrak{a} of A generated by S is equal to $\eta(\mathfrak{m}_B)A$. Since \mathfrak{m}_A contains $\eta(\mathfrak{m}_B)$, A is a Zariski ring for the \mathfrak{a} -adic topology. The ring A/\mathfrak{a} is Artinian if and only if its length as an A -module is finite. But as any simple module of A is killed by \mathfrak{m}_A and, by hypothesis, A/\mathfrak{m}_A and B/\mathfrak{m}_B are isomorphic, this happens exactly when the dimension of the vector space A/\mathfrak{a} over B/\mathfrak{m}_B is finite. By ??, we see \mathfrak{a} is a defining ideal of A if and only if the dimension of A/\mathfrak{a} over B/\mathfrak{m}_B is finite. This is indeed the case if A is a finite B -algebra. Conversely, suppose then \mathfrak{a} is a defining ideal of A . The \mathfrak{m}_B -adic topology on the B -module A then coincides with the \mathfrak{m}_A -adic topology on A , hence is separated. Since A/\mathfrak{a} is a finitely generated B/\mathfrak{m}_B -module, A is a finitely generated B -module (??). This proves (c). \square

Lemma 16.2.17. *Suppose that C is regular and (p_1, \dots, p_m) is a system of parameters for C .*

- (a) *If the sequence S is secant for A , the homomorphism $\eta : B \rightarrow A$ is injective.*
- (b) *For η to be injective and make A a finite B -algebra, it is necessary and sufficient that S is a maximal secant sequence for A . In this case A has dimension $m + n$.*

Proof. For the sequence S to be maximal secant for A , it is necessary and sufficient that it generates a defining ideal of A , and in that case A has dimension $m + n$ (?? and ??). By ??(b), this is equivalent to that A is a finite B -algebra, and has dimension $m + n$. Now C is a Noetherian

integral domain with dimension m , so $B = C[[T_1, \dots, T_n]]$ is a Noetherian integral domain with dimension $m + n$. If A is a finite B -algebra, let \mathfrak{a} be the kernel of η , then $\dim(A) = \dim(B/\mathfrak{a})$. Since B is a Noetherian integral domain with finite dimension, we have $\dim(B/\mathfrak{a}) < \dim(B)$ if $\mathfrak{a} \neq \{0\}$. Thus, if A is a finite B algebra, η is injective if and only if A has dimension $m + n$. This proves (b).

Suppose that S is secant for A . We may add elements t_{n+1}, \dots, t_{n+r} of \mathfrak{m}_A to form a maximal secant sequence. By the argument above, there exists an injective homomorphism $\tilde{\eta}$ from $C[[T_1, \dots, T_n, T_{n+1}, \dots, T_{n+r}]] = B[[T_{n+1}, \dots, T_{n+r}]]$ which extends η and sends T_{n+j} to t_{n+j} for $1 \leq j \leq r$. Hence η is injective. This proves (a). \square

Theorem 16.2.18 (Cohen Structure Theorem). *Let A be a Noetherian complete local ring with residue field k of characteristic p . Let C be a p -ring of infinite length, whose residue field is isomorphic to k .*

- (a) *If n is the dimension of $\mathfrak{m}_A / (\mathfrak{m}_A^2 + pA)$ over k , there exists an ideal \mathfrak{a} of $C[[T_1, \dots, T_n]]$ such that A is isomorphic to $C[[T_1, \dots, T_n]]/\mathfrak{a}$.*
- (b) *Let d be the dimension of A . Suppose that $p1_A$ is not a divisor of zero in A . Then there exists a subring B of A isomorphic to $C[[T_1, \dots, T_{d-1}]]$ such that A is a finite B -algebra.*

Proof. Let \tilde{C} be a Cohen subring of A . Since C has infinite length, there exists a homomorphism of C to \tilde{C} . Consequently, there exists a local homomorphism $\rho : C \rightarrow A$. Choose elements t_1, \dots, t_n of \mathfrak{m}_A whose classes form a basis for $\mathfrak{m}_A / (\mathfrak{m}_A^2 + pA)$ over k . We have $\rho(p1_C) = p1_A$, and ??(b) proves the existence of a surjective homomorphism of $C[[T_1, \dots, T_m]]$ to A , which sends T_i to t_i for each i . This proves (a).

Suppose now that $p1_A$ is not a divisor of zero in A , hence secant for A . There exists by ?? elements t_1, \dots, t_{d-1} of \mathfrak{m}_A such that $(p1_A, t_1, \dots, t_{d-1})$ is a maximal secant sequence for A . The Noetherian local ring C is regular (it is a DVR), and $p1_C$ is a system of parameters of C . Assertion (b) now follows from ??(b). \square

16.2.5 Coefficient fields

Let A be a ring. Recall that the characteristic of A is defined if A contains a subfield. It is equal to 0 if and only if A contains a subfield isomorphic to \mathbb{Q} , and equal to a prime number p if and only if $p1_A = 0$. If the characteristic of A is defined and $\rho : A \rightarrow B$ is a nonzero ring homomorphism, then the characteristic of B is defined and equal to A .

Now let A be a local ring with maximal ideal \mathfrak{m} and residue field k . We distinguish two cases.

- (a) Suppose that k has characteristic 0. Then A contains a field and the characteristic of A is 0. In fact, the canonical homomorphism of \mathbb{Z} to A is injective since the canonical map of \mathbb{Z} to k factors through it, and for any nonzero integer n , $n1_A$ is invertible in A since it is not contained in the maximal ideal \mathfrak{m} .
- (b) Suppose that k has characteristic $p \neq 0$. Then A contains a field if and only if $p1_A = 0$ (since such a field must have characteristic p), and in this case A has characteristic p .

Now assume that A is an integral domain with fraction field K . We now the field K also has a characteristic, which is equal to that of A (if the characteristic of A is defined). Then A contains

a subfield if and only if the characteristic of K and k are equal. In this case, we say A is a local ring with **equal characteristic**. We also note that, if the characteristic of k and K are unequal, then k must have characteristic $p \neq 0$ with K characteristic 0. In this case we say A is a local ring with **unequal characteristic**.

Proposition 16.2.19. *Let k_0 be a field, A a k_0 -algebra and a separated and complete local ring, k a sub- k_0 -extension of κ_A which has a separable transcendental basis $(\xi)_{\lambda \in \Lambda}$ over k_0 . For any $\lambda \in \Lambda$, let x_λ be a representative of ξ_λ in A . Then there exists a unique subfield K of A , containing k_0 and the elements x_λ , such that the canonical homomorphism $\pi : A \rightarrow \kappa_A$ induces an isomorphism of K to k .*

Proof. Let φ be the k_0 -homomorphism of the polynomial ring $k_0[(X_\lambda)_{\lambda \in \Lambda}]$ to A which sends X_λ to x_λ for each λ . Let u be a nonzero element in $k_0[(X_\lambda)_{\lambda \in \Lambda}]$; we have $\pi(\varphi(u)) \neq 0$ since the family $(\xi)_\lambda$ is algebraically independent over k_0 in κ_A , so $\varphi(u)$ is invertible in A . It follows that φ extends to a homomorphism ψ of the field $k_1 = k_0((X_\lambda)_{\lambda \in \Lambda})$ to A . Then A is a k_1 -algebra, κ_A is an extension of k_1 and k a subextension of κ_A which is algebraic and separable over k_1 . It suffices to prove that there exists a unique subfield K of A containing $\psi(k_1)$ and such that $\pi(K) = k$.

Let \mathcal{S} be the set of subfield K of A , containing $\psi(k_1)$ and such that $\pi(K) \subseteq k$; this is inductive with respect to the inclusion relation. Let K be a maximal element in \mathcal{S} ; we consider k as an extension (algebraic and separable) of K . Let $\xi \in k$ and $P \in K[X]$ its minimal polynomial over k . Since ξ is a simple root of P , the Hensel lemma (??) then shows the existence of an element $x \in A$ such that $\pi(x) = \xi$ and $P(x) = 0$. The subfield $K(x)$ of A then belongs to \mathcal{S} , so $x \in K$ by the maximality of K . This shows $\xi \in \pi(K)$, so $\pi(K) = k$.

We now show the uniqueness of K . Let K and K' be subfields of A containing $\psi(k_1)$ and such that $\pi(K) = \pi(K') = k$. Let $\xi \in k$, and let $x \in K$, $x' \in K'$ be elements such that $\pi(x) = \pi(x') = \xi$. If $P \in k_1[X]$ is the minimal polynomial of ξ over k_1 , then ξ is a simple root of P , and we have $P(x) = P(x') = 0$. By Hensel lemma we then have $x = x'$, so $K = K'$. \square

Remark 16.2.3. The previous proof applies more generally to the case where we only assume that A is a Henselian local ring. The uniqueness proof uses the assumption that the local ring A is separate, but not that it is complete.

Let A be a local ring. A **coefficient field** of A is defined to be a subfield K of A such that the canonical homomorphism of A to κ_A induces an isomorphism of K to κ_A (in other words, $A = K + \mathfrak{m}_A$). It is clear that such a field exists only when the characteristic of A is defined. This condition is also sufficient when A is separated and complete. More precisely, we have the following theorem:

Theorem 16.2.20. *Let A be a separated and complete local ring of equal characteristic p .*

- (a) *Suppose $p = 0$ and let $(x_\lambda)_{\lambda \in \Lambda}$ be a family of elements of A whose class modulo \mathfrak{m}_A forms a transcendental basis for κ_A over \mathbb{Q} . Then there exists a unique coefficient field of A containing the elements x_λ .*
- (b) *Suppose $p \neq 0$ and let $(x_\lambda)_{\lambda \in \Lambda}$ be a family of elements of A whose class modulo \mathfrak{m}_A forms a p -basis for κ_A . Then there exists a unique coefficient field of A containing the elements x_λ , which is a Cohen subring of A .*

Proof. Suppose that $p = 0$, so A is a \mathbb{Q} -algebra. Since any transcendental basis for κ_A over \mathbb{Q} is separable, the assertion in (a) follows from ?? applied to $k_0 = \mathbb{Q}$, $k = \kappa_A$.

Now suppose that $p \neq 0$. Then we have $p1_A = 0$, and any Cohen subring C of A satisfies $pC = 0$. Therefore, the notations of coefficient field and Cohen subring coincide on A . Assertion (b) therefore follows from ??.

Corollary 16.2.21. *Let A be a separated and complete local ring, whose residue field is an algebraic extension of \mathbb{Q} . Then there exists a unique coefficient field of A .*

Proof. In fact, the ring A is of characteristic 0, and the empty set is a transcendental basis for κ_A over \mathbb{Q} . \square

Corollary 16.2.22. *Let A be a separated and complete local ring, with residue field of characteristic $p \neq 0$. Suppose that the residue field κ_A is perfect. Then there exists a unique coefficient field of A , which coincide with the multiplicative representatives.*

Proof. This follows from ??(b) and ??.

Theorem 16.2.23. *Let A be a Noetherian complete local ring with dimension d . Let K be a coefficient field of A , and m the dimension of $\mathfrak{m}_A/\mathfrak{m}_A^2$ over K .*

- (a) *There exists an ideal \mathfrak{a} of $K[[T_1, \dots, T_m]]$ such that A is isomorphic to $K[[T_1, \dots, T_m]]/\mathfrak{a}$.*
- (b) *There exists a sub- K -algebra B of A , isomorphic to $K[[T_1, \dots, T_d]]$, such that A is a finite B -algebra.*
- (c) *If A is regular then there exists an isomorphism of A to $K[[T_1, \dots, T_d]]$.*

Proof. Let t_1, \dots, t_m be elements of \mathfrak{m}_A whose classes in \mathfrak{m}_A form a basis for $\mathfrak{m}_A/\mathfrak{m}_A^2$. By ??, there exists a surjective K -homomorphism of $K[[T_1, \dots, T_m]]$ to A , sending T_i to t_i . This proves (a). Similarly, (b) follows from ??, and the existence of a maximal secant sequence for A . Finally, assertion (c) follows from ??.

16.3 Integral closure of complete local rings

16.3.1 Japanese rings

Let A be a Noetherian integral domain. It is called an **N-1 ring** if the integral closure of A in its fraction field K is a finite A -algebra. It is called an **N-2 ring** (or **Japanese**) if the integral closure of A in any finite extension of K is a finite A -algebra. In other words, A is Japanese if it satisfies the following condition: any integral A -algebra B integral over A , contained in a finite type extension of the field of fractions K of A , is a finite A -algebra. Indeed, the field of fractions L of B is an algebraic extension of K , so is of finite degree over K . The A -algebra B is contained in the integral closure of A in L , and is therefore finite if the latter is finite.

We note that Japanese rings are stable under localization. That is, if A is a Noetherian integral Japanese ring and S a multiplicative subset of A non containing 0, then the fraction ring $S^{-1}A$ is Japanese. This follows from the observation that the integral closure of $S^{-1}A$ is the localization of that of A , and hence is finite over A if A is Japanese.

Example 16.3.1. We recall that, by ??, any integral k -algebra of finite type, where k is a field, is a Japanese ring. Note that this is true for the coordinate ring of an algebraic variety.

Proposition 16.3.2. *Let A be a Noetherian integral domain, with fraction field K . Suppose that for any finite purely inseparable extension L of K , the integral closure of A in L is a finite A -algebra. Then A is Japanese.*

Proof. Let E be a finite extension of K . Let N be the normal closure of E over K and L the invariant field of $\text{Aut}_K(N)$. Then L is the purely inseparable closure of K in N and the extension L/N is separable (??). The integral closure B of A in L is then by hypothesis a finite A -algebra; the integral closure C of B in E is a finite B -algebra by ??, which contains the integral closure of A in E . Since A is Noetherian, this completes the proof. \square

Corollary 16.3.3. *Suppose that K is perfect (for example of characteristic 0). Then A is Japanese if and only if it is integrally closed.*

Proposition 16.3.4. *Let B be a Noetherian integral domain and A a Noetherian subring of B , such that B is a finite A -algebra. For A to be Japanese, it is necessary and sufficient that B is Japanese.*

Proof. Let K (resp. L) be the fraction field of A (resp. B). Suppose that A is Japanese, and let E be a finite extension of L . Let C be the integral closure of B in E . By ??, C is also the integral closure of A in E , hence is a finite A -algebra (since E is a finite extension of K and A is Japanese). A fortiori, C is a finite B -algebra, this proves B is Japanese.

Conversely, suppose that B is Japanese and let N be a finite extension of K . Let D be the integral closure of A in N . Let E be the composed extension of L and N (over K); since B is Japanese, the integral closure D' of B in E is a finite B -algebra, hence a finite A -algebra; the A -module D is a sub-module of D' , so is finite over A . This shows A is Japanese. \square

Theorem 16.3.5 (Tate). *Let A be an integrally closed Noetherian ring, a an element of A . Suppose that the ideal aA is prime, the ring A/aA is Japanese and A is complete for the aA -adic topology. Then the ring A is Japanese.*

Proof. Let K be the fraction field of A . The assertion is trivial if K has characteristic 0. We may therefore assume that K has characteristic $p > 0$. Also, suppose that $a \neq 0$. Let L be a finite purely inseparable extension of K and $q = p^e$ a power of p such that $L \subseteq K^{1/q}$. By extending our field, we may assume that $a = x^q$ for some $x \in L$. It then suffices to prove that the integral closure B of A is a finite A -algebra.

We claim that xB is the unique ideal of B lying over aA (such an ideal exists by going up theorem). In fact, let \mathfrak{P} be such an ideal. We have $x^q = a \in \mathfrak{P}$, hence $xB \subseteq \mathfrak{P}$ since \mathfrak{P} is prime. Conversely, let y be an element of \mathfrak{P} ; the element y^q of K is integral over A , hence belongs to A (recall that A is integrally closed). Since $\mathfrak{P} \cap A = aA$, there exist $b \in A$ such that $y^q = ab = x^q b$. Then the element y/x of L is integral over A , hence contained in B ; this means $y \in xB$, hence $\mathfrak{P} = xB$.

Recall that the ring B_{xB} is the integral closure of A_{aA} in L . By ??, A_{aA} is a DVR; we then deduce from the Krull-Akizuki theorem that B_{xB} is Noetherian (the ring $\kappa(xB)$ is a finite extension of $\kappa(aA)$). The ring B/xB is integral over the Japanese ring A/aA and its fraction field is a finite extension of that of A . Consequently, B/xB is a finitely generated (A/aA) -module. For each integer $i \geq 0$, this is isomorphic to the module $x^i B/x^{i+1} B$; so the (A/aA) -module B/aB possesses a composition sequence of length q with quotients finitely generated (A/aA) -modules, hence is also finitely generated over (A/aA) .

Endow the ring A with the aA -adic topology and B the aB -adic topology. Then A is complete by hypothesis; since B_{xB} is a Noetherian integral domain, the aB_{xB} -adic filtration on B_{xB} is separated (Krull intersection theorem); therefore we have $\bigcap_n a^n B \subseteq \bigcap_n a^n B_{xB} = \{0\}$, and the aB -adic filtration on B is separated. The $\text{gr}_{aA}(A)$ -module $\text{gr}_{aB}(B)$ is generated by $\text{gr}_0(B)$, hence is finitely generated. It then follows from ?? that B is a finitely generated A -module, which completes the proof. \square

Corollary 16.3.6. *Let R be a Noetherian integral domain and n an integer. If R is Japanese, so is the ring $R[[T_1, \dots, T_n]]$.*

Proof. Argue by induction, we may assume that $n = 1$. Let S be the integral closure of R . If R is Japanese, S is a finite R -algebra, hence a Japanese ring (??). The ring $S[[T]]$ is Noetherian and integrally closed, so by applying ?? on $A = S[[T]]$ and $a = T$, we deduce that $S[[T]]$ is Japanese. Thus $R[[T]]$ is Japanese (??). \square

Theorem 16.3.7 (Nagata). *Every Noetherian integral complete local ring is Japanese.*

Proof. By ??, there exist an integer $n \geq 0$, a ring R that is a field of a DVR whose fraction field has characteristic 0, and a subring B of A , isomorphic to $R[[T_1, \dots, T_n]]$, such that A is a finite B -algebra. Since R is Japanese, we see B is Japanese, hence so is A . \square

A Noetherian semi-local ring A is called **analytically unramified** if its completion \widehat{A} is reduced. A prime ideal \mathfrak{p} of A is said to be **analytically unramified** if $\widehat{A}/\widehat{\mathfrak{p}} = \widehat{A}/\mathfrak{p}$ is reduced. ?? then imply the following important result:

Corollary 16.3.8. *Let A be a semi-local Noetherian ring which is analytically unramified. Then the integral closure \bar{A} of A in the total fraction ring $Q(A)$ is a finite A -algebra.*

Proof. Suppose first that A is local and complete, with $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ the minimal prime ideals of A . For each $i = 1, \dots, n$, let K_i be the fraction field of A/\mathfrak{p}_i and \bar{A}_i the integral closure of A/\mathfrak{p}_i in K_i . Since A is reduced, $Q(A)$ is the product of the K_i and \bar{A} is the product of \bar{A}_i (??). Since the local ring A/\mathfrak{p}_i is integral and complete, it is Japanese (??), so \bar{A}_i is a finite A_i -algebra, and therefore \bar{A} is finite over A . If A is semi-local and complete, it is isomorphic to a finite product of complete local rings (??), and we conclude the claim by the previous case.

In the general case, note that the completion \widehat{A} of A is semi-local, complete, Noetherian and faithfully flat (A is a Zariski ring). Let S be the set of elements of A that are not zero divisors; we have $Q(A) = S^{-1}A$. Since \widehat{A} is flat over A , the elements of S are not zero divisors in \widehat{A} , and $S^{-1}\widehat{A}$ is identified with a subring of $Q(\widehat{A})$. Moreover, the ring $\bar{A} \otimes_A \widehat{A}$ is identified as a subring of $Q(A) \otimes_A \widehat{A}$, hence a subring T integral over \widehat{A} . The preceding arguments then apply and show that $\bar{A} \otimes_A \widehat{A}$ is a finitely generated \widehat{A} -module. Therefore, \bar{A} is a finitely generated A -module (??). \square

Recall that an algebra E over a field K is called separable if the ring $L \otimes_K E$ is reduced for any extension L of K . The following proposition generalizes ??.

Proposition 16.3.9. *Let A be a semi-local Noetherian integral ring, K its fraction field. If the K -algebra $K \otimes_A \widehat{A}$ is separable, the ring A is Japanese.*

Proof. Let L be a finite extension of K and B the integral closure of A in L . Let F be a finite subset of B such that $L = K[F]$ (??); let C be the A -algebra generated by F . Then L is the fraction field of C , the ring B is the integral closure of C , and it suffices to prove that B is a finite C -algebra. But C is a Noetherian semi-local ring, with completion identified with $C \otimes_A \widehat{A}$, which is a subring of the reduced ring $L \otimes_A \widehat{A} = L \otimes_K (K \otimes_A \widehat{A})$ and therefore reduced. The proposition then follows from ??.

Finally, before considering Nagata ring, we prove some lemmas which will be used latter.

Lemma 16.3.10. *Let A be a Noetherian semi-local ring and B a finite A -algebra. Then B is Noetherian and semi-local; let $\mathfrak{m}_1, \dots, \mathfrak{m}_n$ be its maximal ideals. The canonical homomorphism of B to $\prod_{i=1}^n \widehat{B}_{\mathfrak{m}_i}$ then extends to an isomorphism of $\widehat{A} \otimes_A B$ to $\prod_{i=1}^n \widehat{B}_{\mathfrak{m}_i}$.*

Proof. By ??, the ring B is semi-local and $\mathfrak{m}_A B$ is a defining ideal. By ??(b), the ring $\widehat{A} \otimes_A B$ is the completion of B for the topology defined by its Jacobson radical. Then we can apply ??.

Lemma 16.3.11. *Let A be a Noetherian ring and M an A -module. The canonical map of M into $\prod_{\mathfrak{p} \in \text{Ass}(M)} M_{\mathfrak{p}}$ is injective.*

Proof. Let m be a nonzero element in M ; then $\text{Ann}(m)$ is contained in a prime ideal \mathfrak{p} in $\text{Ass}(M)$ (??), and the image of m in $M_{\mathfrak{p}}$ is thus nonzero.

Lemma 16.3.12. *Let A be a Noetherian local ring and \mathfrak{p} a prime ideal of A . Assume that $A_{\mathfrak{p}}$ is a DVR and \mathfrak{p} is analytically unramified, then for any associated prime $\hat{\mathfrak{q}}$ of $\widehat{A}/\mathfrak{p}\widehat{A}$, the local ring $\widehat{A}_{\hat{\mathfrak{q}}}$ is a DVR.*

Proof. Let π be a uniformizer for $A_{\mathfrak{p}}$. Since \mathfrak{p} is analytically unramified, the associated primes of $\widehat{A}/\mathfrak{p}\widehat{A}$ are exactly the minimal primes of \widehat{A} over $\mathfrak{p}\widehat{A}$, and in particular there is no embedded associated prime. Also, since $\widehat{A}/\mathfrak{p}\widehat{A} = \widehat{A/\mathfrak{p}}$ is reduced, we have $\{0\} = \cap \hat{\mathfrak{q}}/\mathfrak{p}\widehat{A}$ in $\widehat{A}/\mathfrak{p}\widehat{A}$, where $\hat{\mathfrak{q}}$ runs through minimal primes over $\mathfrak{p}\widehat{A}$. Therefore in \widehat{A} we have

$$\mathfrak{p}\widehat{A} = \bigcap_{\hat{\mathfrak{q}} \in \text{Ass}(\widehat{A}/\mathfrak{p}\widehat{A})} \hat{\mathfrak{q}}$$

which is then the unique primary decomposition of $\mathfrak{p}\widehat{A}$ in \widehat{A} . By ??, the $\hat{\mathfrak{q}}$ -primary component of this decomposition is given by the saturation of $\mathfrak{p}\widehat{A}$ with respect to $\hat{\mathfrak{q}}$ (which is equal to $\hat{\mathfrak{q}}$), so in particular, we have

$$\hat{\mathfrak{q}}\widehat{A}_{\hat{\mathfrak{q}}} = \mathfrak{p}\widehat{A}_{\hat{\mathfrak{q}}} = \pi\widehat{A}_{\hat{\mathfrak{q}}}.$$

Since π is \widehat{A} -regular by the flatness of \widehat{A} over A , the local ring $\widehat{A}_{\hat{\mathfrak{q}}}$ is regular of dimension 1, hence a DVR.

Lemma 16.3.13. *Let A be a Noetherian semi-local integral domain with maximal ideal \mathfrak{m} . Let x be a nonzero element in \mathfrak{m} such that*

- (a) A/xA has no embedded primes;
- (b) for each associated prime \mathfrak{p} of A/xA , the local ring $A_{\mathfrak{p}}$ is regular and \mathfrak{p} is analytically unramified.

Then A is analytically unramified.

Proof. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ be the associated primes of A/xA . Since A/xA has no embedded primes, we see that each \mathfrak{p}_i has height 1, and is a minimal prime over xA . For each i , let $\hat{\mathfrak{q}}_{i,1}, \dots, \hat{\mathfrak{q}}_{i,n_i}$ be the associated primeds of the \widehat{A} -module $\widehat{A}/\mathfrak{p}_i\widehat{A}$. By ?? we see that $\widehat{A}_{\hat{\mathfrak{q}}_{i,j}}$ is a DVR for each i, j . Also, since \widehat{A} is flat over A , by ?? we have

$$\text{Ass}_{\widehat{A}}(\widehat{A}/x\widehat{A}) = \bigcup_{\mathfrak{p} \in \text{Ass}_A(A/xA)} \text{Ass}_{\widehat{A}}(\widehat{A}/\mathfrak{p}\widehat{A}) = \{\hat{\mathfrak{q}}_{i,j}\}.$$

Let y be a nonzero nilpotent in \widehat{A} . Then since $\widehat{A}_{\hat{\mathfrak{q}}_{i,j}}$ is integral for each i, j , the element y is mapped to zero in $\widehat{A}_{\hat{\mathfrak{q}}_{i,j}}$. By ??, the canonical image of y in $\widehat{A}/x\widehat{A}$ is then zero, which means $y = xy'$ for some $y' \in \widehat{A}$. But since the homothety with ratio x is injective on A , it is also injective on \widehat{A} , which implies $y' \in \widehat{A}$ is also a nonzero nilpotent element. Repeating this process, we conclude that $y \in \bigcap_n x^n \widehat{A} = \{0\}$ (??), which shows \widehat{A} is reduced and A is therefore analytically unramified. \square

16.3.2 Nagata rings

A Noetherian ring is called Nagata if for any prime ideal \mathfrak{p} of A , the Noetherian integral domain A/\mathfrak{p} is Japanese. In this part, we give some of the properties of Nagata rings, and a criterion of Noetherian semi-local rings to be Nagata.

Example 16.3.14 (Example of Nagata rings).

- (a) Any algebra of finite type over a field is a Nagata ring.
- (b) Any Noetherian complete local ring is a Nagata ring by ??.
- (c) The ring \mathbb{Z} is a Nagata ring since any quotient $\mathbb{Z}/p\mathbb{Z}$ is a perfect field (hence Japanese).
- (d) We will see that any algebra of finite type over a Nagata ring is Nagata.

Proposition 16.3.15. *Let A be a Nagata ring.*

- (a) *Any finite A -algebra is a Nagata ring.*
- (b) *For any multiplicative subset S of A , the ring $S^{-1}A$ is a Nagata ring.*

Proof. Let B be a finite A -algebra and $\rho : A \rightarrow B$ the homomorphism. For each prime ideal \mathfrak{P} of B , the ring B/\mathfrak{P} is a finite algebra of the Japanese ring A/\mathfrak{p}^c , hence is Japanese.

Let S be a multiplicative subset of A and \mathfrak{q} a prime ideal of $S^{-1}A$; then there exist a prime ideal \mathfrak{p} of A such that $\mathfrak{q} = S^{-1}\mathfrak{p}$. The ring $S^{-1}A/\mathfrak{q}$ is a fraction ring of the Japanese ring A/\mathfrak{p} , hence is Japanese. \square

Theorem 16.3.16 (Nagata). *Let A be a Noetherian semi-local integral domain. If A is a Nagata ring then it is analytically unramified.*

Proof. We use induction on $\dim(A)$. Let B be the integral closure of A in its fraction field K . Then B is finite over A , hence $\dim(A) = \dim(B)$ and for any $\mathfrak{P} \in \text{Spec}(B)$, the domain B/\mathfrak{P} is finite over A/\mathfrak{p} (where $\mathfrak{p} = \mathfrak{P} \cap A$), which is assumed to be Japanese. Thus B/\mathfrak{P} is Japanese and B is therefore a Nagata ring. Moreover, the defining ideal of B is given by the extension of that of A , which implies $\widehat{B} = B \otimes_A \widehat{A}$; since \widehat{A} is a flat A -module, we can then identify \widehat{A} as

a subring of \widehat{B} . Thus we can replace A by its integral closure and assume that A is integrally closed. In this case, for any nonzero element $x \in A$, the A -module A/xA has no embedded primes by ???. If $\mathfrak{p} \in \text{Ass}(A/xA)$, then A/\mathfrak{p} is a Nagata domain and $\dim(A/\mathfrak{p}) < \dim(A)$, hence \mathfrak{p} is analytically unramified by the induction hypothesis. Moreover, $A_{\mathfrak{p}}$ is a DVR because $\text{ht}(\mathfrak{p}) = 1$ (?? and ??). So the conditions of ?? are satisfied, and A is analytically unramified. \square

Theorem 16.3.17 (Zariski-Nagata). *Let A be a Noetherian semi-local ring. The following conditions are equivalent:*

- (i) A is a Nagata ring;
- (ii) for any prime ideal \mathfrak{p} of A , the $\kappa(\mathfrak{p})$ -algebra $\kappa(\mathfrak{p}) \otimes_A \widehat{A}$ is separable;
- (iii) for any reduced A -algebra R , the ring $R \otimes_A \widehat{A}$ is reduced.

Proof. We first demonstrate the equivalence of (ii) and (iii). The implication (iii) \Rightarrow (ii) is trivial; suppose conversely that A satisfies (ii). Then for any A -algebra K that is a field, the ring $K \otimes_A \widehat{A}$ is reduced since it is isomorphic to $K \otimes_{\kappa(\mathfrak{p})} \kappa(\mathfrak{p}) \otimes_A \widehat{A}$. Now let C be a reduced A -algebra of finite type, then the ring C , being Noetherian, is isomorphic to a subring of a finite product $K_1 \times \dots \times K_n$ of fields (??). Since \widehat{A} is flat over A , the ring $C \otimes_A \widehat{A}$ is isomorphic to a subring of the reduced ring $\prod_i (K_i \otimes_A \widehat{A})$, hence is reduced. Finally, since any reduced A -algebra R is the union of a family of finite type A -algebras (C_{α}) , we see $R \otimes_A \widehat{A}$ is reduced (it is the inductive limit of the family $(C_{\alpha} \otimes_A \widehat{A})$).

We now show that (ii) \Leftrightarrow (i). Let \mathfrak{p} be a prime ideal of A . The fraction field K of A/\mathfrak{p} is then identified with $\kappa(\mathfrak{p})$, and the K -algebra $K \otimes_{A/\mathfrak{p}} \widehat{A}/\mathfrak{p}$ is identified with $\kappa(\mathfrak{p}) \otimes_{A/\mathfrak{p}} \widehat{A}/\mathfrak{p}\widehat{A}$, hence to $\kappa(\mathfrak{p}) \otimes_A \widehat{A}$. If $\kappa(\mathfrak{p}) \otimes_A \widehat{A}$ is a separable $\kappa(\mathfrak{p})$ -algebra, then A/\mathfrak{p} is Japanese by ??.

Conversely, we prove (i) \Rightarrow (ii). Let A be a semi-local Noetherian Nagata ring, let \mathfrak{p} be an ideal of A , and L a finite extension of $\kappa(\mathfrak{p})$. It suffices to show that the ring $L \otimes_A \widehat{A}$ is reduced. Let B by the integral closure of A/\mathfrak{p} in L ; since A/\mathfrak{p} is Japanese, B is a finite A -algebra hence a semi-local Nagata ring (??). Let $\mathfrak{m}_1, \dots, \mathfrak{m}_r$ be the maximal ideal of B ; the ring $L \otimes_A \widehat{A}$ is identified with a fraction ring of $B \otimes_A \widehat{A}$, and with that of a product of complete local rings $\widehat{B}_{\mathfrak{m}_i}$ (??). It then suffices to prove that, for any maximal ideal \mathfrak{m} of B , the ring $\widehat{B}_{\mathfrak{m}}$ is reduced (??). But the ring $B_{\mathfrak{m}}$ is a Noetherian local ring that is Nagata (??), so by ?? it is analytically unramified, which is what we want. \square

Corollary 16.3.18 (Chevalley). *Let A be a reduced algebra of finite type over a field, and \mathfrak{p} a prime ideal of A . Then the local ring $A_{\mathfrak{p}}$ is analytically unramified.*

Proof. Since A is reduced, the local ring $A_{\mathfrak{p}}$ is reduced. The ring A is Nagata, hence $A_{\mathfrak{p}}$ is also Nagata, and ?? implies the claim. \square

Corollary 16.3.19. *Let k be a field of characteristic 0 and A a Noetherian local k -algebra. Then A is a Nagata ring if and only if every prime ideal \mathfrak{p} of A is analytically unramified.*

Proof. In fact, since the residue field $\kappa(\mathfrak{p})$ has characteristic 0, the $\kappa(\mathfrak{p})$ -algebra $\kappa(\mathfrak{p}) \otimes_A \widehat{A} = \kappa(\mathfrak{p}) \otimes_{A/\mathfrak{p}} \widehat{A}/\mathfrak{p}$ is separable if and only if it is reduced, which shows that the stated condition is sufficient (??, not that $\kappa(\mathfrak{p}) \otimes_{A/\mathfrak{p}} \widehat{A}/\mathfrak{p}$ is a localization of \widehat{A}/\mathfrak{p}). It is also necessary by take $R = A/\mathfrak{p}$ in ??(iii). \square

A Noetherian ring A is called **universally Japanese** if any integral A -algebra B of finite type over A is Japanese. It is clear that such a ring is Nagata, since A/\mathfrak{p} is a finite type A -algebra for any prime ideal \mathfrak{p} of A . We now prove the reverse direction, which is not at all obvious. For this, we first establish some lemmas.

Lemma 16.3.20. *Let A be a Noetherian integral domain and put $X = \text{Spec}(A)$. Suppose that there exist nonzero element f in A such that A_f is integrally closed, then the set*

$$\text{Nor}(X) = \{\mathfrak{p} \in \text{Spec}(A) : A_{\mathfrak{p}} \text{ is integrally closed}\}$$

is open in X .

Proof. If $f \notin \mathfrak{p}$ for some $\mathfrak{p} \in X$, then $A_{\mathfrak{p}}$ is a localization of A_f , hence integrally closed. This shows $D(f) \subseteq \text{Nor}(X)$. Now we put

$$E = \{\mathfrak{p} \in \text{Ass}_A(A/fA) : \text{either } \text{ht}(\mathfrak{p}) = 1 \text{ and } A_{\mathfrak{p}} \text{ is not regular, or } \text{ht}(\mathfrak{p}) > 1\}.$$

Since $\text{Ass}_A(A/fA)$ is finite, it is clear that E is a finite subset of X . On the other hand, by Serre's criterion for normality, if $\mathfrak{p} \notin \text{Nor}(X)$ then there exist a prime $\mathfrak{q} \subseteq \mathfrak{p}$ such that either $\text{ht}(\mathfrak{q}) \geq 2$ and $\text{depth}(A_{\mathfrak{q}}) < 2$, or $\text{ht}(\mathfrak{q}) = 1$ and $A_{\mathfrak{q}}$ is not regular. This in particular also means that $A_{\mathfrak{q}}$ is not integrally closed, and hence $f \in \mathfrak{q}$. In both cases we see \mathfrak{q} is an associated prime of A/fA . It then follows that $\text{Nor}(X) = X \setminus \bigcup_{\mathfrak{p} \in E} V(\mathfrak{p})$, so $\text{Nor}(X)$ is open in X . \square

Lemma 16.3.21. *Let A be a Noetherian domain with quotient field K . Then A is N-1 if and only if the following conditions hold:*

- (a) *there exists a nonzero element $f \in A$ such that A_f is integrally closed;*
- (b) *$A_{\mathfrak{m}}$ is N-1 for each maximal ideal \mathfrak{m} of A .*

Proof. First assume that A is N-1. Let \bar{A} be the integral closure of A in its fraction field K . By assumption we can find x_1, \dots, x_n in \bar{A} which generate it as an A -module. Since $\bar{A} \subseteq K$, we may further assume that $x_i = a_i/b$ for $a_i, b \in A$. Then $A_f \cong \bar{A}_f$, where the latter is the integral closure of the former. This shows A_f is integrally closed, hence (a) holds. It is clear that (b) follows, since the integral closure of $A_{\mathfrak{m}}$ is just $\bar{A}_{\mathfrak{m}}$.

Conversely, assume (a), (b) for A and let K be the fraction field of A . For a subring B of K that is finite over A , we set $X = \text{Spec}(A)$, $Y = \text{Spec}(B)$. Note that $A_f = B_f$ since A_f is integrally closed, so by ?? the set $\text{Nor}(Y)$ is open in Y . Since the canonical map $\bar{X} \rightarrow X$ is closed (??), the image of the complement of $\text{Nor}(Y)$ is then closed in X , we denote it by Z_B for such a ring B .

Pick a maximal ideal \mathfrak{m} of A and let $A_{\mathfrak{m}} \subseteq \bar{A}_{\mathfrak{m}}$ be the integral closure of the local ring in K . By assumption, this is a finite ring extension, so we can find finitely many elements x_1, \dots, x_n in \bar{A} such that $\bar{A}_{\mathfrak{m}}$ is generated by x_1, \dots, x_n over $A_{\mathfrak{m}}$. Let $B(\mathfrak{m}) = A[x_1, \dots, x_n] \subseteq K$, then $B(\mathfrak{m})$ is finite over A , hence is Noetherian. Let \mathfrak{M} be any prime ideal of $B(\mathfrak{m})$ lying over \mathfrak{m} , then

$$B(\mathfrak{m})_{\mathfrak{M}} \supset B(\mathfrak{m})_{\mathfrak{m}} \supset B(\mathfrak{m}),$$

and $B(\mathfrak{m})_{\mathfrak{m}} = \bar{A}_{\mathfrak{m}}$ is integrally closed. Thus $B(\mathfrak{m})_{\mathfrak{M}}$ is a localization of the integrally closed ring $\bar{A}_{\mathfrak{m}}$, so is integrally closed. This shows $\mathfrak{m} \notin Z_{B(\mathfrak{m})}$, so the intersection $\bigcap_{\mathfrak{m} \in \text{Max}(A)} Z_{B(\mathfrak{m})}$ contains

no closed point, and is therefore empty (since it is a closed subset of X). As X is quasi-compact, we can choose maximal ideals $\mathfrak{m}_1, \dots, \mathfrak{m}_r$ such that $\bigcap_i Z_{B(\mathfrak{m}_i)} = \emptyset$. Put $B_i = B(\mathfrak{m}_i)$ and let C be the ring generated by all of these; it is finite over A . We claim that $Z_C = \emptyset$, in other words, $C_{\mathfrak{Q}}$ is integrally closed for any $\mathfrak{Q} \in \text{Spec}(C)$. In fact, the contraction of any prime \mathfrak{Q} of C into A is not contained in some Z_{B_i} , so it lies over a prime \mathfrak{P}_i of B_i such that $(B_i)_{\mathfrak{P}_i}$ is integrally closed. This implies $C_{\mathfrak{Q}} = (B_i)_{\mathfrak{P}_i}$ is integrally closed too, so C is integrally closed (??). In other words, C is the integral closure of A in K , so A is N-1. \square

Theorem 16.3.22 (Nagata). *Let A be a Noetherian ring. The following are equivalent:*

- (i) A is a Nagata ring;
- (ii) any finite type A -algebra B is Nagata;
- (iii) A is universally Japanese.

Proof. It is clear that a Noetherian universally Japanese ring is universally Nagata (i.e., condition (ii) holds). Let A be a Noetherian Nagata ring. We will show that any finite type A -algebra B is Nagata. Note that the canonical image of A in B is also a Nagata ring, so we may assume that A is a subring of B . Then $B = A[x_1, \dots, x_n]$ with $x_i \in B$, and by induction it suffices to consider the case $B = A[x]$.

Let \mathfrak{P} be a prime ideal of B and $\mathfrak{p} = \mathfrak{P}^c$. Then $B/\mathfrak{P} = (A/\mathfrak{p})[\bar{x}]$, where A/\mathfrak{p} is a Nagata domain (??), and we must show that B/\mathfrak{P} is Japanese. We are then reduced to prove the following assertion:

(A1) If A is a Noetherian Nagata domain and $B = A[x]$ is an integral domain generated by a single x over A , then B is Japanese.

Let K be the fraction field of A and \bar{A} the integral closure of A in K . Let \bar{B} be the subring of the fraction field of B generated by \bar{A} and B . As \bar{A} is finite over A (by the Nagata property, since A is an integral domain), \bar{B} is finite over B . Since B is Noetherian, it then suffices to prove that \bar{B} is Japanese (??). Hence we can add the integrally closed assumption in (A1):

(A2) If A is an integrally closed Noetherian Nagata domain and $B = A[x]$ is an integral domain generated by a single x over A , then B is Japanese.

In this case, let L be the fraction field of B . We first note that, if x is transcendental over A , then $B = A[x]$ is isomorphic to the polynomial ring $A[X]$, hence is integrally closed. Thus if L has characteristic 0, we are done (??). Suppose otherwise L has characteristic $p > 0$, and take a finite purely inseparable extension E of $L = K(x)$. Let $q = p^e$ be a power of p such that $E \subseteq L^{1/q}$. Then there exists a finite purely inseparable extension K' of K such that $E \subseteq K'(x^{1/q})$. If \bar{A} (resp. \bar{B}) is the integral closure of A in K' (resp. of B in L), then $\bar{A}[x^{1/q}]$ is normal and we have $B = A[x] \subseteq \bar{B} \subseteq \bar{A}[x^{1/q}]$. Since $\bar{A}[x^{1/q}]$ is finite over B , we conclude that \bar{B} is finite over B .

Now assume that x is algebraic over A , so that L/K is a finite extension. Let E be a finite extension of L (hence finite over K) and \bar{A} be the integral closure of A in E . Then the integral closure \bar{B} of B in E is equal to the integral closure of $\bar{A}[x]$ in E . Also the fraction field of \bar{A} is E and \bar{A} is a finite A -algebra (A is Japanese). This implies that \bar{A} is a Nagata ring (??). To show that \bar{B} is finite over B is the same as showing that \bar{B} is finite over $\bar{A}[x]$, so by replacing A by \bar{A} and B by $\bar{B}[x]$, we are reduced to prove the following statement:

- (A3) If A is an integrally closed Noetherian Nagata domain with fraction field K and x is an element of K , the ring $B = A[x]$ is N-1.

We now proceed with proving (A3). Since $x \in K$, we can write $x = a/b$ where $a, b \in A$. Then $B_a = B[1/a] = A[1/a]$ is integrally closed since it is a localization of A . By ??, it then suffices to prove that $B_{\mathfrak{M}}$ is N-1 for any maximal ideal \mathfrak{M} of B . Now pick such a maximal ideal and set $\mathfrak{m} = \mathfrak{M} \cap A$. The residue field extension $\kappa(\mathfrak{m})/\kappa(\mathfrak{M})$ is finite and generated by the image of x . Hence there exists a monic polynomial $f(X) \in A[X]$ with $f(x) \in \mathfrak{M}$. Let E/K be a finite extension of fields such that $f(X)$ splits completely in $E[X]$. Let \bar{A} be the integral closure of A in E , and $\bar{B} \subseteq E$ be the subring generated by \bar{A} and x . As A is a Nagata ring, we see \bar{A} is finite over A and hence Nagata (??). Moreover, \bar{B} is finite over B . If for every maximal ideal $\bar{\mathfrak{M}}$ of \bar{B} the local ring $\bar{B}_{\bar{\mathfrak{M}}}$ is N-1, then $\bar{B}_{\bar{\mathfrak{M}}}$ is N-1 by ??, which in turn implies that $B_{\mathfrak{M}}$ is N-1 (by a proof similar to that of ??). Thus, after replacing A by \bar{A} and B by \bar{B} , and \mathfrak{M} by any of the maximal ideals $\bar{\mathfrak{M}}$ lying over \mathfrak{M} , we reach the situation where the polynomial $f(X)$ above split completely: $f(X) \prod_{i=1}^d (X - a_i)$ with $a_i \in A$. Since $f(x) \in \mathfrak{M}$ we see that $x - a_i \in \mathfrak{M}$ for some i . Finally, after replacing x by $x - a_i$ we may assume that $x \in \mathfrak{M}$.

We will show that ?? applies to the local ring $B_{\mathfrak{M}}$ and the element x . This will imply that $B_{\mathfrak{M}}$ is analytically unramified, whence it is N-1 by ?? . Let Q be the kernel of the canonical map $A[X] \rightarrow B$, where X is mapped to x . Then by ??, the ideal Q is generated by all linear forms $aX - b$, where $ax = b$ in K . Therefore, we see that

$$B/xB = A[X]/(x, \mathfrak{a}) = A/I$$

where I is the constant term of the elements in Q , or in other words $I = xA \cap A$. Now note that A is a Noetherian integrally closed domain and I is divisorial, so by ??, if $\text{div}(xA \cap A) = \sum_i n_i P_i$ with \mathfrak{p}_i the corresponding prime of P_i , then

$$I = xA \cap A = \bigcap_i \mathfrak{p}_i^{(n_i)}$$

is the unique reduced primary decomposition of I and the \mathfrak{p}_i are isolated primes of I . This shows $A/I = B/xB$ has no embedded primes, so does $B_{\mathfrak{M}}$, in view of ??.

Now let $\mathcal{P} \in \text{Spec}(B_{\mathfrak{M}})$ be an associated prime of $B_{\mathcal{P}}/xB_{\mathfrak{M}}$. Then $\mathfrak{p} = \mathcal{P} \cap A$ is an associated prime of $A/(xB_{\mathfrak{M}} \cap A) = A/I$ and $\text{ht}(\mathfrak{p}) = 1$ by ??, so $(B_{\mathfrak{M}})_{\mathcal{P}} = A_{\mathfrak{p}}$ is a DVR (??), hence regular. Finally, $B_{\mathfrak{M}}/\mathcal{P}$ is a localization of $B/(\mathcal{P} \cap B)$ (which is isomorphic to A/\mathfrak{p} since $x \in \mathfrak{p}$), so $B_{\mathfrak{M}}/\mathcal{P}$ is a Noetherian Nagata local domain, hence analytically unramified. Thus the conditions in ?? are verified and our proof is completed. \square

Chapter 17

Depth, regularity, and duality

17.1 Depth of modules

17.1.1 Homological definition of depth

Let A be a ring, \mathfrak{I} an ideal of A , and M an A -module. We define the **depth of M relative to \mathfrak{I}** , denoted by $\text{depth}_A(\mathfrak{I}, M)$ or $\text{depth}(\mathfrak{I}, M)$, to be the infimum in $\mathbb{N} \cup \{+\infty\}$ of the set of integers i such that $\text{Ext}_A^i(A/\mathfrak{I}, M)$ is nonzero. If A is a local ring, the depth of M relative to the maximum ideal \mathfrak{m}_A of A is simply called the **depth of M** and denoted by $\text{depth}_A(M)$ or $\text{depth}(M)$; the **depth of the local ring A** is defined to be the depth of the A -module A .

Example 17.1.1. Let M be a finitely generated A -module and such that $M = \mathfrak{I}M$, which means $\text{supp}(M) \cap V(\mathfrak{I}) = \emptyset$ (??). In this case, $\text{depth}_A(\mathfrak{I}, M)$ is equal to $+\infty$: in fact, the ideal $\text{Ann}(M) + \mathfrak{I}$ is equal to A (since $V(\text{Ann}(M) + \mathfrak{I}) = \emptyset$) and is contained in the annihilator of $\text{Ext}_A^i(A/\mathfrak{I}, M)$ for each i . Conversely, we shall see that (??) if \mathfrak{I} is a finitely generated ideal, then $\text{depth}_A(\mathfrak{I}, M) = +\infty$ implies $M = \mathfrak{I}M$.

Example 17.1.2. For $\text{depth}_A(\mathfrak{I}, M)$ to be zero, it is necessary and sufficient that $\text{Hom}_A(A/\mathfrak{I}, M)$ is zero, which signifies that M has a nonzero element annihilated by \mathfrak{I} . This in particular implies that $\text{Ass}(M) \cap V(\mathfrak{I}) \neq \emptyset$ (??). If A is Noetherian and the A -module M is finitely generated, the following conditions are equivalent (??):

- (i) $\text{depth}_A(\mathfrak{I}, M) = 0$;
- (ii) for any $x \in \mathfrak{I}$, the homothety x_M is not injective;
- (iii) $\text{Ass}(M) \cap V(\mathfrak{I}) \neq \emptyset$.

In particular, for a local ring to have zero depth, it is necessary and sufficient that there exist a nonzero element x of A such that $\mathfrak{m}_A x = 0$. If A is not a field, such an element is then not invertible, hence belongs to \mathfrak{m}_A and satisfies $x^2 = 0$. Therefore, any reduced local ring of dimension ≥ 1 has depth ≥ 1 .

Example 17.1.3. Let $\{M_i\}_{i \in I}$ be a family of A -module and \mathfrak{I} be an ideal of A . Then in view of (A, X, p.89, prop.7 ??), we have $\text{depth}_A(\mathfrak{I}) = \inf_{i \in I} \text{depth}_A(\mathfrak{I}, M_i)$.

Proposition 17.1.4. Let A be a ring, \mathfrak{I} an ideal of A , and let

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

be an exact sequence of A -modules. Then there are exactly three cases:

- (i) $\operatorname{depth}(\mathfrak{J}, M) = \operatorname{depth}(\mathfrak{J}, M') \leq \operatorname{depth}(\mathfrak{J}, M'')$;
- (ii) $\operatorname{depth}(\mathfrak{J}, M) = \operatorname{depth}(\mathfrak{J}, M'') \leq \operatorname{depth}(\mathfrak{J}, M')$;
- (iii) $\operatorname{depth}(\mathfrak{J}, M'') = \operatorname{depth}(\mathfrak{J}, M') - 1 < \operatorname{depth}(\mathfrak{J}, M'')$;

Proof. Consider the long exact sequence associated with this short exact sequence:

$$0 \rightarrow \operatorname{Ext}_A^0(A/\mathfrak{J}, M') \rightarrow \operatorname{Ext}_A^0(A/\mathfrak{J}, M) \rightarrow \operatorname{Ext}_A^0(A/\mathfrak{J}, M'') \rightarrow \operatorname{Ext}_A^1(A/\mathfrak{J}, M') \rightarrow \cdots$$

We may exclude the case $\operatorname{depth}(\mathfrak{J}, M') = \operatorname{depth}(\mathfrak{J}, M) = \operatorname{depth}(\mathfrak{J}, M'') = +\infty$. There then exists in this sequence a first non-zero module, and the next module is thus nonzero. The three cases then correspond to which module is the first to be nonzero. \square

Remark 17.1.1. Suppose that we have $\operatorname{depth}(\mathfrak{J}, M') = \operatorname{depth}(\mathfrak{J}, M)$ and the injection $u : M' \rightarrow M$ belongs to $\mathfrak{J}\operatorname{Hom}_A(M', M)$ (which is equivalent to that $1_{A/\mathfrak{J}} \otimes u$ is zero). We then have $\operatorname{depth}(\mathfrak{J}, M'') = \operatorname{depth}(\mathfrak{J}, M) - 1$: in fact, the hypotheses implies that $\operatorname{Ext}_A^i(1_{A/\mathfrak{J}}, u) = 0$ for any integer i , which means the homomorphism $\operatorname{Ext}_A^i(A/\mathfrak{J}, M'') \rightarrow \operatorname{Ext}_A^i(A/\mathfrak{J}, M')$ is surjective.

Proposition 17.1.5. Let A be a ring, \mathfrak{J} an ideal of A , M an A -module and N an A -module killed by a power of \mathfrak{J} . Then $\operatorname{Ext}_A^i(N, M) = 0$ for any integer $i < \operatorname{depth}_A(\mathfrak{J}, M)$.

Proof. We first assume that $\mathfrak{J}N = 0$ and prove by induction on the integer $i < \operatorname{depth}_A(\mathfrak{J}, M)$. The assertion is clear for $i < 0$; now consider N as a (A/\mathfrak{J}) -module and choose an exact sequence of (A/\mathfrak{J}) -modules

$$0 \longrightarrow K \longrightarrow (A/\mathfrak{J})^{\oplus I} \longrightarrow N \longrightarrow 0$$

We then deduce a long exact sequence of extension modules

$$\cdots \rightarrow \operatorname{Ext}_A^{i-1}(K, M) \rightarrow \operatorname{Ext}_A^i(N, M) \rightarrow \operatorname{Ext}_A^i((A/\mathfrak{J})^{\oplus I}, M) \rightarrow \operatorname{Ext}_A^i(K, M) \rightarrow \cdots$$

The A -module $\operatorname{Ext}_A^{i-1}(K, M)$ is zero by induction hypothesis, and $\operatorname{Ext}_A^i((A/\mathfrak{J})^{\oplus I}, M)$ is isomorphic to $\operatorname{Ext}_A^i((A/\mathfrak{J}), M)^{\oplus I}$ (A, X, p.89, prop.7 ??), which is zero by the definition of $\operatorname{depth}_A(\mathfrak{J}, M)$. We therefore conclude that $\operatorname{Ext}_A^i(N, M) = 0$.

Passing to the general case, we proceed by induction on the smallest number $m > 0$ such that $\mathfrak{J}^m N = 0$. We have proved the case $m = 1$, so assume that $m > 1$ and let $i < \operatorname{depth}_A(\mathfrak{J}, M)$ be an integer. Consider the sequence

$$\operatorname{Ext}_A^i(N/\mathfrak{J}N, M) \longrightarrow \operatorname{Ext}_A^i(N, M) \longrightarrow \operatorname{Ext}_A^i(\mathfrak{J}N, M)$$

The extremal modules are zero by induction hypothesis, since $N/\mathfrak{J}N$ and $\mathfrak{J}N$ are annihilated by \mathfrak{J}^{m-1} . We then deduce that $\operatorname{Ext}_A^i(N, M) = 0$, which completes the proof. \square

Corollary 17.1.6. Let $m > 0$ be an integer and \mathfrak{J}' be an ideal of A contained in \mathfrak{J}^m . Then for any A -module M we have $\operatorname{depth}_A(\mathfrak{J}, M) \leq \operatorname{depth}_A(\mathfrak{J}', M)$.

Proof. In fact, \mathfrak{I}^m annihilates the A -module A/\mathfrak{J}' , so $\text{Ext}_A^i(A/\mathfrak{J}', M) = 0$ for $i < \text{depth}_A(\mathfrak{J}, M)$ by ??.

Corollary 17.1.7. Suppose that the ideal \mathfrak{J} is finitely generated, and let \mathfrak{J}' be an ideal of A such that $V(\mathfrak{J}') \subseteq V(\mathfrak{J})$. Then we have $\text{depth}_A(\mathfrak{J}, M) \leq \text{depth}_A(\mathfrak{J}', M)$. Moreover, if the ideal \mathfrak{J}' is also finitely generated and $V(\mathfrak{J}) = V(\mathfrak{J}')$, then $\text{depth}_A(\mathfrak{J}, M) \leq \text{depth}_A(\mathfrak{J}', M)$.

Proof. By ??, there exists an integer $m > 0$ such that $\mathfrak{J}^m \subseteq \mathfrak{J}'$. The first assertion therefore follows from ??, and the second one follows also from this. \square

We conclude this paragraph by a cohomological application of depth. This result can also be served as our motivation of defining the depth via extension module.

Proposition 17.1.8. Let A be a ring, C_\bullet be a right bounded complex of A -modules, and n_0 be an integer. Suppose that for any integers $m \geq n \geq n_0$, the depth of the A -module C_m relative to the annihilator of $H_n(C_\bullet)$ is strictly bigger than $m - n$, then we have $H_n(C_\bullet) = 0$ for $n \geq n_0$.

Proof. Since C_\bullet is right bounded, we have $H_n(C_\bullet) = 0$ for $n \gg 0$. If the conclusion is false, then there exists an integer $m \geq n_0$ such that $H_n(C_\bullet) = 0$ for $n > m$ and $H_m(C_\bullet) \neq 0$. We denote by \mathfrak{J} the annihilator of $H_m(C_\bullet)$, so that $\text{depth}_A(\mathfrak{J}, H_m(C_\bullet)) = 0$ by ?? . Moreover, since $Z_m(C_\bullet)$ is a submodule of C_m , and by hypothesis we have $\text{depth}_A(\mathfrak{J}, C_m) > m - m = 0$, we conclude that $\text{depth}_A(\mathfrak{J}, Z_m(C_\bullet)) > 0$. From the exact sequence

$$0 \longrightarrow B_m(C_\bullet) \longrightarrow Z_m(C_\bullet) \longrightarrow H_m(C_\bullet) \longrightarrow 0$$

we then deduce that $\text{depth}_A(\mathfrak{J}, B_m(C_\bullet)) = 1$ (??). Now by the definition of m , we have $B_n(C_\bullet) = Z_n(C_\bullet) = 0$ for $n > m$, so from the canonical exact sequences

$$0 \longrightarrow B_n(C_\bullet) \longrightarrow C_n \longrightarrow B_{n-1}(C_\bullet) \longrightarrow 0$$

and the hypothesis $\text{depth}_A(\mathfrak{J}, C_n) > n - m$, we derive by induction that $\text{depth}_A(\mathfrak{J}, B_n(C_\bullet)) = n - m + 1$ for any $n \geq m$ (??). But this is absurd since $B_n(C_\bullet) = 0$ for $n \gg 0$. \square

Corollary 17.1.9. Let A be a ring, \mathfrak{J} be an ideal of A , C_\bullet be a right bounded complex of A -modules, and n_0 be an integer. Suppose that $\mathfrak{J}H_n(C_\bullet) = 0$ and $\text{depth}_A(\mathfrak{J}, C_n) > n - n_0$ for $n \geq n_0$. Then we have $H_n(C_\bullet) = 0$ for $n \geq n_0$.

Proof. In fact, for $n \geq n_0$ the annihilator \mathfrak{J}_n of $H_n(C_\bullet)$ contains \mathfrak{J} , so we have $\text{depth}_A(\mathfrak{J}_n, C_m) \geq \text{depth}_A(\mathfrak{J}, C_m)$ for $m \geq n \geq n_0$ (??), and ?? then proves the corollary. \square

Corollary 17.1.10. Let A be a local ring with maximal \mathfrak{m} , C_\bullet be a right bounded complex of A -modules, and n_0 be an integer. Suppose that for $n \geq n_0$, $H_n(C_\bullet)$ is of finite length and $\text{depth}_A(\mathfrak{m}, C_n) > n - n_0$. Then we have $H_n(C_\bullet) = 0$ for $n \geq n_0$.

Proof. The A -module $\bigoplus_{n \geq n_0} H_n(C_\bullet)$ is then of finite length. Let \mathfrak{J} be its annihilator, then A/\mathfrak{J} is Artinian, so \mathfrak{J} is contained in a power of the maximal ideal \mathfrak{m} of A . We then have $\text{depth}_A(\mathfrak{J}, C_n) \geq \text{depth}_A(\mathfrak{m}, C_n) > n - n_0$ (??), so we can apply ??.

17.1.2 Depth and Koszul complex

Let A be a ring, M be an A -module, and $\mathbf{x} = (x_i)_{i \in I}$ be a family of elements of A . Let $K^\bullet(\mathbf{x}, M)$ be the Koszul complex associated with \mathbf{x} and M . By definition, we have $K^p(\mathbf{x}, M) = 0$ for $p < 0$, and for $p \geq 0$ the A -module $K^p(\mathbf{x}, M) = \text{Hom}_A(\wedge^p A^{\oplus I}, M)$ is canonically identified with the A -module $C^p(M)$ formed by alternating maps from I^p to M , with differential $\partial^p : K^p(\mathbf{x}, M) \rightarrow K^{p+1}(\mathbf{x}, M)$ given by the formula

$$(\partial^p m)(\alpha_1, \dots, \alpha_{p+1}) = \sum_{j=1}^{p+1} (-1)^{j+1} x_{\alpha_j} \cdot m(\alpha_1, \dots, \hat{\alpha}_i, \dots, \alpha_{p+1})$$

where $m \in K^p(\mathbf{x}, M)$ and $(\alpha_1, \dots, \alpha_{p+1}) \in I^{p+1}$. In particular, the complex $K^\bullet(\mathbf{x}, M)$ only depends on the \mathbb{Z} -module structure on M and the endomorphisms $(x_i)_M$.

We denote by $H^\bullet(\mathbf{x}, M)$ the cohomology of the complex $K^\bullet(\mathbf{x}, M)$. By definition, the A -module $H^0(\mathbf{x}, M)$ is identified with $\text{Hom}_A(A/\mathfrak{I}, M)$, where \mathfrak{I} is the ideal generated by the x_i .

Theorem 17.1.11. *Let A be a ring, \mathfrak{I} be an ideal of A , $\mathbf{x} = (x_i)_{i \in I}$ be a family of generator of \mathfrak{I} , and M be an A -module. Then the depth of M relative to \mathfrak{I} is the infimum of integers i such that $H^i(\mathbf{x}, M) \neq 0$.*

Proof. We put $p = \text{depth}_A(\mathfrak{I}, M)$, and consider the complex $K^\bullet(\mathbf{x}, M)$. Its cohomology is annihilated by \mathfrak{I} (??), and the depth of $K^i(\mathbf{x}, M)$ relative to \mathfrak{I} is equal to p or $+\infty$ since $K^i(\mathbf{x}, M)$ is either isomorphic to a product of M or zero (??). By applying ?? to the relabeled complex $K_{-i}(\mathbf{x}, M)$ (defined by $K_{-i}(\mathbf{x}, M) = K^i(\mathbf{x}, M)$), we conclude that $H^i(\mathbf{x}, M) = 0$ for $i < p$. It then remains to prove that $H^p(\mathbf{x}, M)$ is nonzero if $p < +\infty$.

The case $p = 0$ is evident, so we suppose that $0 < p < +\infty$ and $H^p(\mathbf{x}, M) = 0$. Let L^\bullet be a free resolution of the A -module A/\mathfrak{I} , and denote by C^\bullet the Hom complex $\text{Hom}(L^\bullet, M)$. The A -module $H^i(C^\bullet)$ is then isomorphic to $\text{Ext}_A^i(A/\mathfrak{I}, M)$, and is hence zero for $i < p$. For each $i < p$, we then have a canonical exact sequence

$$0 \longrightarrow B^i(C^\bullet) \longrightarrow C^i \longrightarrow B^{i+1}(C^\bullet) \longrightarrow 0$$

Since each C^i is isomorphic to a product of M , by hypothesis we have $H^n(\mathbf{x}, C^i) = 0$ for $n \leq p$, and the connecting homomorphism $\partial^n : H^n(\mathbf{x}, B^{i+1}(C^\bullet)) \rightarrow H^{n+1}(\mathbf{x}, B^i(C^\bullet))$ in the induced long exact sequence is then injective for $n \leq p$ and $i < p$. As $B^0(C^\bullet) = 0$, it then follows that $H^{p-i}(\mathbf{x}, B^{i+1}(C^\bullet)) = 0$ for $i < p$, and in particular $H^1(\mathbf{x}, B^p(C^\bullet)) = 0$. Now from the exact sequence

$$0 \longrightarrow B^p(C^\bullet) \longrightarrow Z^p(C^\bullet) \longrightarrow H^p(C^\bullet) \longrightarrow 0$$

we then obtain a surjection $H^0(\mathbf{x}, Z^p(C^\bullet)) \rightarrow H^0(\mathbf{x}, H^p(C^\bullet))$. As the A -module $H^p(C^\bullet)$ is isomorphic to $\text{Ext}_A^p(A/\mathfrak{I}, M)$, which is nonzero and annihilated by \mathfrak{I} , we finally conclude that $H^0(\mathbf{x}, H^p(C^\bullet)) \neq 0$, whence $H^0(\mathbf{x}, Z^p(C^\bullet)) \neq 0$ and then $H^0(\mathbf{x}, C^p) \neq 0$. But then $H^0(\mathbf{x}, M) \neq 0$, which is a contradiction and therefore completes the proof. \square

Corollary 17.1.12. *Suppose that \mathfrak{I} is finitely generated and $\mathfrak{I}M \neq M$. Then $\text{depth}_A(\mathfrak{I}, M) \leq |I|$, and for the equality holds, it is necessary and sufficient that the family \mathbf{x} is complete secant for M .*

Proof. Suppose that I is finite and $n = |I|$. Then the A -module $H^n(x, M)$ is canonically isomorphic to $H_0(x, M)$, hence to $M/\mathfrak{I}M$. The inequality $\operatorname{depth}_A(\mathfrak{I}, M) \leq n$ then follows from ??.

For the equality holds, it is necessary and sufficient that the A -modules $H^i(x, M) = 0$ for $i < n$, which means that x is complete secant for M . \square

Corollary 17.1.13. *Let A be a local ring, \mathfrak{I} be a finitely generated proper ideal of A , and M be a finitely generated nonzero A -module. Put $r = [\mathfrak{I}/\mathfrak{m}_A\mathfrak{I} : \kappa_A]$, then we have $\operatorname{depth}_A(\mathfrak{I}, M) \leq r$, and the equality holds if and only if \mathfrak{I} is generated by a family that is complete secant for M . In this case, for a generating family of \mathfrak{I} to be complete secant for M , it is necessary and sufficient that it consists of r elements.*

Proof. By Nakayama's lemma, we have $\mathfrak{I}M \neq M$, and r is the minimal number of generators of \mathfrak{I} . The assertion then follows from ??.

Proposition 17.1.14. *Let A be a ring, \mathfrak{I} be a finitely generated ideal of A , and M be an A -module. Denote by Ω the set of maximal ideals of $V(\mathfrak{I}) \cap \operatorname{supp}(M)$. Then we have*

$$\operatorname{depth}_A(\mathfrak{I}, M) = \inf_{\mathfrak{p} \in \operatorname{Spec}(A)} \operatorname{depth}_{A_{\mathfrak{p}}}(I_{\mathfrak{p}}, M_{\mathfrak{p}}) = \inf_{\mathfrak{m} \in \Omega} \operatorname{depth}_{A_{\mathfrak{m}}}(\mathfrak{I}_{\mathfrak{m}}, M_{\mathfrak{m}}).$$

Proof. Let $x = (x_i)_{i \in I}$ be a finite generating family of \mathfrak{I} . Let \mathfrak{p} be a prime ideal of A , then the ideal $\mathfrak{I}_{\mathfrak{p}}$ is generated by the image $x_{\mathfrak{p}}$ of the family x in $A_{\mathfrak{p}}$. For any $p \geq 0$, the $A_{\mathfrak{p}}$ -module $(H^p(x, M))_{\mathfrak{p}}$ is isomorphic to $H^p(x_{\mathfrak{p}}, M_{\mathfrak{p}})$, so by ?? we have

$$\operatorname{depth}_A(\mathfrak{I}, M) \leq \inf_{\mathfrak{p} \in \operatorname{Spec}(A)} \operatorname{depth}_{A_{\mathfrak{p}}}(\mathfrak{I}_{\mathfrak{p}}, M_{\mathfrak{p}}) = \inf_{\mathfrak{m} \in \Omega} \operatorname{depth}_{A_{\mathfrak{m}}}(\mathfrak{I}_{\mathfrak{m}}, M_{\mathfrak{m}}).$$

Let p be an integer that is strictly smaller than $\operatorname{depth}_{A_{\mathfrak{m}}}(I_{\mathfrak{m}}, M_{\mathfrak{m}})$ for any $\mathfrak{m} \in \Omega$. We then have $H^p(x_{\mathfrak{m}}, M_{\mathfrak{m}}) = 0$ for any maximal ideal \mathfrak{m} of A : this follows from ?? if $\mathfrak{m} \in \Omega$, and from the fact that $M_{\mathfrak{m}} = 0$ if $\mathfrak{m} \notin \operatorname{supp}(M)$, and that $\mathfrak{I}A_{\mathfrak{m}} = A_{\mathfrak{m}}$ if $\mathfrak{m} \notin V(\mathfrak{I})$ (which annihilates $H^p(x_{\mathfrak{m}}, M_{\mathfrak{m}})$). We then conclude that $(H^p(x, M_{\mathfrak{m}}))_{\mathfrak{m}} = 0$ for any maximal ideal \mathfrak{m} of A , whence $H^p(x, M) = 0$. The proposition then follows from ??.

We now turn to another characterization of depth, which utilize the concept of regular sequences. Let A be a ring and M be an A -module. Recall that a sequence (x_1, \dots, x_r) of elements of A is said to be **regular for M** or **M -regular** if, for any $1 \leq i \leq r$, the homothety with ratio x_i is injective on the A -module $M_i = M/(x_1M + \dots + x_{i-1}M)$. If (x_1, \dots, x_r) is an M -regular sequence, then for any flat A -module N , the sequence (x_1, \dots, x_r) is $M \otimes N$ -regular, and for any flat ring homomorphism $\rho : A \rightarrow B$, the image $(\rho(x_1), \dots, \rho(x_r))$ is regular for the B -module $B \otimes_A M$. In particular, for any prime ideal \mathfrak{p} of A , the image of (x_1, \dots, x_r) in $A_{\mathfrak{p}}$ is $M_{\mathfrak{p}}$ -regular.

The importance of regular sequence is that the depth of M relative to an ideal \mathfrak{I} can also be characterized by the length of maximal regular M -sequences. Therefore, we can also define $\operatorname{depth}_A(\mathfrak{I}, M)$ by this maximal length. The connection of regular sequences with Ext modules then allows us to apply homological methods to commutative algebras.

Proposition 17.1.15. *Let A be a ring, \mathfrak{I} be an ideal of A , M be an A -module, and (x_1, \dots, x_r) be an M -regular sequence of elements of \mathfrak{I} . Then we have*

$$\operatorname{depth}_A(\mathfrak{I}, M) = r + \operatorname{depth}_A(\mathfrak{I}, M/(x_1M + \dots + x_rM)),$$

and in particular $\operatorname{depth}_A(\mathfrak{I}, M) \leq r$.

Proof. The case $n = 1$ follows from ??, since $M \xrightarrow{x_1} M$ is injective by hypothesis. The general case can then be proved by induction on n . \square

Theorem 17.1.16. *Let A be a Noetherian ring, \mathfrak{I} be an ideal of A , and M be a finitely generated A -module.*

- (a) *Suppose that $\operatorname{depth}_A(\mathfrak{I}, M)$ is finite, then any M -regular sequence of elements of \mathfrak{I} can be completed into an M -regular sequence of length $\operatorname{depth}_A(\mathfrak{I}, M)$ of elements of \mathfrak{I} .*
- (b) *The depth of M relative to \mathfrak{I} is the maximal length of M -regular sequences of elements of \mathfrak{I} .*
- (c) *For $\operatorname{depth}_A(\mathfrak{I}, M)$ to be finite, it is necessary and sufficient that the support of M meets $V(\mathfrak{I})$, or that we have $\mathfrak{I}M \neq M$.*

Proof. Let (x_1, \dots, x_r) be an M -regular sequence of elements of \mathfrak{I} . We have $n \leq \operatorname{depth}_A(\mathfrak{I}, M)$ by ??; suppose that this inequality is strict, and denote by N the A -module $M/(x_1M + \dots + x_rM)$. We then have $\operatorname{depth}_A(\mathfrak{I}, N) > 0$, so there exists an element x of \mathfrak{I} such that the homothety x_N is injective (??), which means (x_1, \dots, x_r, x) is M -regular. It then follows by recurrence that for any integer s such that $r \leq s \leq \operatorname{depth}_A(\mathfrak{I}, M)$ the sequence (x_1, \dots, x_r) can be extended into an M -regular sequence of length s , which proves the assertion (a) and (b). Assertion (c) then follows from ?? and ??.

Corollary 17.1.17. *For an M -regular (x_1, \dots, x_r) of elements of \mathfrak{I} , the following properties are equivalent:*

- (i) $r = \operatorname{depth}_A(\mathfrak{I}, M)$;
- (ii) the sequence (x_1, \dots, x_r) is a maximal M -regular sequence of elements of \mathfrak{I} ;
- (iii) the A -module $M/(x_1M + \dots + x_rM)$ possesses a nonzero element annihilated by \mathfrak{I} ;
- (iv) $\operatorname{Ass}(M/(x_1M + \dots + x_rM)) \cap V(\mathfrak{I}) \neq \emptyset$.

Proof. The equivalence of (i) and (ii) follows from ??, and that of (ii), (iii) and (iv) follows from ?? applied to $M/(x_1M + \dots + x_rM)$. \square

Corollary 17.1.18. *Let A be a Noetherian local ring and M be a finitely generated A -module. Then $\operatorname{depth}_A(M) \leq \dim_A(M) < +\infty$.*

Proof. In fact, any M -regular sequence of elements of \mathfrak{m}_A is complete secant for M (??), hence secant for M (??). \square

Proposition 17.1.19. *Let A be a Noetherian local ring, M be a finitely generated nonzero A -module, and \mathfrak{I} be a proper ideal of A . Then we have the inequalities*

$$\operatorname{depth}_A(\mathfrak{I}, M) \leq \operatorname{codim}(\operatorname{supp}(M) \cap V(\mathfrak{I}), \operatorname{supp}(M)) \leq \dim(M) - \dim(M/\mathfrak{I}M) \leq [\mathfrak{I}/\mathfrak{m}_A \mathfrak{I} : \kappa_A].$$

Proof. For any element \mathfrak{p} of $\text{supp}(M) \cap V(\mathfrak{J})$, $\text{depth}_A(\mathfrak{J}, M)$ is smaller than $\dim_{A_{\mathfrak{p}}}(M_{\mathfrak{p}})$ (?? and ??), which is equal to $\text{codim}(V(\mathfrak{p}), \text{supp}(M))$. If \mathfrak{p} runs through $\text{supp}(M) \cap V(\mathfrak{J})$, then $V(\mathfrak{p})$ runs through irreducible closed subsets of $\text{supp}(M) \cap V(\mathfrak{J})$, whence the first inequality. The second one follows from ?? . Moreover, we can always find a generating set of \mathfrak{J} with cardinality $[\mathfrak{J}/\mathfrak{m}_A \mathfrak{J} : \kappa_A]$, so the third inequality follows from (??). \square

Remark 17.1.2. Consider the inequalities in ??.

- (a) For that we have $\text{depth}_A(\mathfrak{J}, M) = [\mathfrak{J}/\mathfrak{m}_A \mathfrak{J} : \kappa_A]$, it is necessary and sufficient that \mathfrak{J} can be generated by a M -regular sequence (??).
- (b) The equality $\dim(M) - \dim(M/\mathfrak{J}M) = [\mathfrak{J}/\mathfrak{m}_A \mathfrak{J} : \kappa]$ signifies that \mathfrak{J} can be generated by a sequence that is secant for M (??).
- (c) If M is Cohen-Macaulay, then we have $\text{depth}_A(\mathfrak{J}, M) = \dim(M) - \dim(M/\mathfrak{J}M)$.

Lemma 17.1.20. *Let A be a Noetherian ring, $\mathfrak{p} \subseteq \mathfrak{p}_1 \subseteq \cdots \subseteq \mathfrak{p}_{r-1} \subseteq \mathfrak{q}$ be a saturated chain of length r of prime ideals of A , M be a finitely generated A -module, and n be an integer. If the A -module $\text{Ext}_{A_{\mathfrak{p}}}^n(\kappa(\mathfrak{p}), M_{\mathfrak{p}})$ is nonzero, then so is $\text{Ext}_{A_{\mathfrak{q}}}^{n+r}(\kappa(\mathfrak{q}), M_{\mathfrak{q}})$.*

Proof. It evidently suffices to prove the case $r = 1$; by replacing A , M , \mathfrak{p} and \mathfrak{q} with $A_{\mathfrak{q}}$, $M_{\mathfrak{q}}$, $\mathfrak{p}A_{\mathfrak{q}}$ and $\mathfrak{q}A_{\mathfrak{q}}$ respectively, we can then assume that A is local and $\mathfrak{q} = \mathfrak{m}_A$. Let x be an element of $\mathfrak{m}_A - \mathfrak{p}$. The $A_{\mathfrak{p}}$ -module $\text{Ext}_A^n(A/\mathfrak{p}, M) \otimes_A A_{\mathfrak{p}}$ is isomorphic to $\text{Ext}_{A_{\mathfrak{p}}}^n(\kappa(\mathfrak{p}), M_{\mathfrak{p}})$ (A, X, p.111, prop.10(b) ??), so is nonzero by hypothesis; a fortiori $\text{Ext}_A^n(A/\mathfrak{p}, M)$ is nonzero. The exact sequence

$$0 \longrightarrow A/\mathfrak{p} \xrightarrow{x_{A/\mathfrak{p}}} A/\mathfrak{p} \longrightarrow A/(\mathfrak{p} + xA) \longrightarrow 0$$

induces an exact sequence

$$\text{Ext}_A^n(A/\mathfrak{p}, M) \xrightarrow{u} \text{Ext}_A^n(A/\mathfrak{p}, M) \longrightarrow \text{Ext}_A^{n+1}(A/(\mathfrak{p} + xA), M)$$

where u is the homothety with ratio x . By Nakayama's lemma, this homomorphism is not surjective, so the A -module $\text{Ext}_A^{n+1}(A/(\mathfrak{p} + xA), M)$ is nonzero. Now if $\text{Ext}_A^{n+1}(\kappa_A, M)$ is zero, we then deduce, by induction on the length of N , that $\text{Ext}_A^{n+1}(N, M) = 0$ for any A -module N of finite length. But since $r = 1$, the unique prime ideal of A containing $\mathfrak{p} + xA$ is \mathfrak{m}_A , so the A -module $A/(\mathfrak{p} + xA)$ is of finite length. This contradicts the fact that $\text{Ext}_A^{n+1}(A/(\mathfrak{p} + xA), M) \neq 0$, which proves our claim. \square

Proposition 17.1.21. *Let A be a Noetherian ring, M be a finitely generated A -module, $\mathfrak{p} \subseteq \mathfrak{q}$ be prime ideals of $\text{supp}(M)$. Then we have*

$$\text{depth}_{A_{\mathfrak{q}}}(M_{\mathfrak{q}}) \leq \text{depth}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}) + \dim(A_{\mathfrak{q}}/\mathfrak{p}A_{\mathfrak{q}}).$$

More precisely, for any saturated chain of prime ideals $\mathfrak{p} \subseteq \mathfrak{p}_1 \subseteq \cdots \subseteq \mathfrak{p}_{r-1} \subseteq \mathfrak{q}$, we have

$$\text{depth}_{A_{\mathfrak{q}}}(M_{\mathfrak{q}}) \leq \text{depth}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}) + r.$$

Proof. Put $p = \text{depth}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}})$, it suffices to prove the second inequality. This is evident if $p = +\infty$; in the contrary case we have $\text{Ext}_{A_{\mathfrak{p}}}^p(\kappa(\mathfrak{p}), M_{\mathfrak{p}}) \neq 0$, so $\text{Ext}_{A_{\mathfrak{q}}}^{p+r}(\kappa(\mathfrak{q}), M_{\mathfrak{q}}) \neq 0$ by ??, which

implies $\operatorname{depth}_{A_{\mathfrak{q}}}(M_{\mathfrak{q}}) \leq p + r$. As $\dim(A_{\mathfrak{q}} + \mathfrak{p}A_{\mathfrak{q}})$ is the supremum of the length of saturated chains of prime ideals with endpoints \mathfrak{p} and \mathfrak{q} , the first assertion then follows. \square

Corollary 17.1.22. *We have the inequality*

$$\dim(M_{\mathfrak{q}}) - \operatorname{depth}_{A_{\mathfrak{q}}}(M_{\mathfrak{q}}) \geq \dim(M_{\mathfrak{p}}) - \operatorname{depth}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}) \geq 0.$$

Proof. This follows from ?? and $\dim(M_{\mathfrak{q}}) \geq \dim(M_{\mathfrak{p}}) + \dim(A_{\mathfrak{q}}/\mathfrak{p}A_{\mathfrak{q}})$??(a) and ??(b)). \square

Corollary 17.1.23. *Let A be a Noetherian local ring and M be a finitely generated A -module. Then we have the inequality*

$$\operatorname{depth}_A(M) \leq \inf_{\mathfrak{p} \in \operatorname{Ass}(M)} \dim(A/\mathfrak{p}).$$

Proof. Let \mathfrak{p} be an associated prime of M ; we have $\operatorname{depth}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}) = 0$ by ?.?. ?? applied to the ideals $\mathfrak{p} \subseteq \mathfrak{m}_A$ then implies the inequality $\operatorname{depth}_A(M) \leq \dim(A/\mathfrak{p})$, whence the corollary. \square

Remark 17.1.3. We have $\sup_{\mathfrak{p} \in \operatorname{Ass}(M)} \dim(A/\mathfrak{p}) = \dim(M)$ in view of ??, so ?? implies that $\operatorname{depth}_A(M) \leq \dim(M)$ for $M \neq 0$, which is ??.

17.1.3 Extension of scalars

Proposition 17.1.24. *Let $\rho : A \rightarrow B$ be a ring homomorphism, \mathfrak{I} be an ideal of A , and N be a B -module. Then we have $\operatorname{depth}_A(\mathfrak{I}, N) = \operatorname{depth}_B(\mathfrak{I}B, N)$.*

Proof. Let $\mathbf{x} = (x_i)_{i \in I}$ be a generating family of \mathfrak{I} ; the family $\rho(\mathbf{x}) = (\rho(x_i))_{i \in I}$ then generates $\mathfrak{I}B$. By construction the complex $K^{\bullet}(\rho(\mathbf{x}), N)$ is equal to $K^{\bullet}(\mathbf{x}, N)$, so the proposition follows from ??.

Proposition 17.1.25. *Let A be a ring, \mathfrak{I} be a finitely generated ideal of A and M be an A -module. Let $\rho : A \rightarrow B$ be a flat ring homomorphism.*

- (a) *We have $\operatorname{depth}_A(\mathfrak{I}, M) \leq \operatorname{depth}_B(\mathfrak{I}B, B \otimes_A M)$.*
- (b) *Suppose moreover that any maximal ideal of $\operatorname{supp}(M) \cap V(\mathfrak{I})$ belongs to the image of the canonical map $\operatorname{Spec}(B) \rightarrow \operatorname{Spec}(A)$. Then we have $\operatorname{depth}_A(\mathfrak{I}, M) = \operatorname{depth}_B(\mathfrak{I}B, B \otimes_A M)$. This is the case for example if the A -module B is faithfully flat.*

Proof. Let $\mathbf{x} = (x_i)_{i \in I}$ be a family of elements of A . For each integer $p \geq 0$, let $u^p : B \otimes_A C^p(M) \rightarrow C^p(B \otimes_A M)$ be the B -linear homomorphism give by

$$b \otimes m \mapsto ((\alpha_1, \dots, \alpha_p) \mapsto b \otimes m(\alpha_1, \dots, \alpha_p)).$$

The family (u^p) then defines an isomorphism of complexes $u : B \otimes_A K^{\bullet}(\mathbf{x}, M) \rightarrow K^{\bullet}(\mathbf{x}, B \otimes_A M)$. Now consider the canonical homomorphism

$$\gamma^p(B, K^{\bullet}(\mathbf{x}, M)) : B \otimes_A H^p(\mathbf{x}, M) \rightarrow H^p(B \otimes_A K^{\bullet}(\mathbf{x}, M))$$

By composing with $H^p(u)$, we obtain a homomorphism $v^p : B \otimes_A H^p(\mathbf{x}, M) \rightarrow H^p(\mathbf{x}, B \otimes_A M)$. It is clear from ?? that v^p is an isomorphism if B is flat over A , so assertion (a) follows from ??.

Suppose that p is an integer that is strictly smaller than $\operatorname{depth}_B(\mathfrak{I}B, B \otimes_A M)$, and let \mathfrak{m} be a maximal ideal of A belonging to $\operatorname{supp}(M) \cap V(\mathfrak{I})$. Let \mathbf{x} be a finite generating family of \mathfrak{I} . Under the hypothesis of (b), there exists a prime ideal \mathfrak{P} of B lying over \mathfrak{m} , and we have a canonical isomorphism

$$B_{\mathfrak{P}} \otimes_{A_{\mathfrak{m}}} (A_{\mathfrak{m}} \otimes_A H^p(\mathbf{x}, M)) \rightarrow B_{\mathfrak{P}} \otimes_B (B \otimes_A H^p(\mathbf{x}, M)).$$

Now $B \otimes_A H^p(\mathbf{x}, M)$ is isomorphic to $H^p(\rho(\mathbf{x}), B \otimes_A M)$, hence is zero; moreover $B_{\mathfrak{P}}$ is faithfully flat over $A_{\mathfrak{m}}$ (?? and ??), so we conclude that $A_{\mathfrak{m}} \otimes_A H^p(\mathbf{x}, M) = 0$, and therefore $p < \operatorname{depth}_{A_{\mathfrak{m}}}(\mathfrak{I}_{\mathfrak{m}}, M_{\mathfrak{m}})$. The first assertion of (b) then follows from ??, and the second one follows from ??.

Corollary 17.1.26. *Let A be a Noetherian ring, \mathfrak{I} be an ideal of A , M be a finitely generated A -module, \widehat{A} and \widehat{M} be the \mathfrak{I} -adic completion of A and M . Then we have $\operatorname{depth}_A(\mathfrak{I}, M) = \operatorname{depth}_{\widehat{A}}(\mathfrak{I}\widehat{A}, \widehat{M})$.*

Proof. In fact, the A -module \widehat{A} is flat and \widehat{M} is isomorphic to $\widehat{A} \otimes_A M$ (??); moreover, any maximal ideal of A containing \mathfrak{I} belongs to the image of the map $\operatorname{Spec}(\widehat{A}) \rightarrow \operatorname{Spec}(A)$ (??). \square

Lemma 17.1.27. *Let $\rho : A \rightarrow B$ be a homomorphism of Noetherian local rings, N be a finitely generated B -module, and y be an element of \mathfrak{m}_B . Then the following conditions are equivalent:*

- (i) *the A -module N/yN is flat and the homothety y_N is injective;*
- (ii) *the A -module N is flat and the homothety $y_{\kappa_A \otimes N}$ is injective.*

If these are satisfied, then the homothety $y_{M \otimes_A N}$ is injective for any A -module M .

Proof. Suppose that the hypotheses of (i) are satisfied, let us prove (ii) as well as the last assertion. Let M be an A -module; since the A -module N/yN is flat, we deduce from the exact sequence $0 \rightarrow N \xrightarrow{y_N} N \rightarrow N/yN \rightarrow 0$ the following exact sequences

$$\begin{aligned} 0 \longrightarrow M \otimes_A N &\xrightarrow{u} M \otimes_A N \longrightarrow M \otimes_A (N/yN) \longrightarrow 0 \\ 0 \longrightarrow \operatorname{Tor}_1^A(M, N) &\xrightarrow{v} \operatorname{Tor}_1^A(M, N) \longrightarrow 0 \end{aligned}$$

where $u = 1_M \otimes y_N$ and $v = \operatorname{Tor}_1^A(1_M, y_N)$. It then follows that the homothety with ratio y is injective on $M \otimes_A N$, and bijective on $\operatorname{Tor}_1^A(M, N)$. Suppose moreover that M is finitely generated, then the B -module $\operatorname{Tor}_1^A(M, N)$ is also finitely generated, and hence is zero by Nakayama's lemma. This implies the flatness of the A -module N .

Conversely, assume that the hypotheses in (ii) are satisfied. Consider the exact sequences of B -modules

$$0 \longrightarrow \ker y_N \longrightarrow N \xrightarrow{p} \operatorname{im} y_N \longrightarrow 0 \tag{17.1.1}$$

$$0 \longrightarrow \operatorname{im} y_N \xrightarrow{i} N \longrightarrow N/yN \longrightarrow 0 \tag{17.1.2}$$

where p and i are canonical homomorphisms. We then deduce that the homomorphism $1 \otimes p : \kappa_A \otimes_A N \rightarrow \kappa_A \otimes_A \operatorname{im} y_N$ is surjective, and (since N is flat) that the kernel of the homomorphism $1 \otimes i : \kappa_A \otimes_A \operatorname{im} y_N \rightarrow \kappa_A \otimes_A N$ is isomorphic to $\operatorname{Tor}_1^A(\kappa_A, N/yN)$. But the map $(1 \otimes i) \circ (1 \otimes p)$, equal to $y_{\kappa_A \otimes_A N}$, is injective by hypothesis; we then deduce that $1 \otimes p$ is bijective and $1 \otimes i$ is

injective, and therefore $\text{Tor}_1^A(\kappa_A, N/yN) = 0$. It then follows that the A -module N/yN is flat (?? and ??).

Since N and N/yN are flat over A , so is $\text{im } y_N$ (by exact sequence (??)). We then deduce from (??) that $\kappa_A \otimes_A \ker y_N$ is isomorphic to the kernel of $1 \otimes p$, which is then zero. The homothety y_N is then injective by Nakayama's lemma. \square

Proposition 17.1.28. *Let $\rho : A \rightarrow B$ be a homomorphism of Noetherian local rings, N be a finitely generated B -module, and $\mathbf{y} = (y_1, \dots, y_s)$ be a sequence of elements of \mathfrak{m}_B . Denote by \mathfrak{Y} the ideal of B generated by this sequence, then the following conditions are equivalent:*

- (i) *the A -module $N/\mathfrak{Y}N$ is flat and the sequence \mathbf{y} is N -regular.*
- (ii) *the A -module N is flat and the sequence \mathbf{y} is $(\kappa_A \otimes_A N)$ -regular.*

If these are satisfied, then for any A -module M , the sequence \mathbf{y} is $(M \otimes_A N)$ -regular.

Proof. We prove the equivalence by recurrence on s . The case $s = 0$ is evident, so suppose that $s \geq 1$. Denote by $\tilde{\mathbf{y}}$ the sequence (y_1, \dots, y_{s-1}) and $\tilde{\mathfrak{Y}}$ the ideal of B it generates. By ?? applied to the B -module $N/\tilde{\mathfrak{Y}}N$ and to the element y_s of B , we see that (i) is equivalent to the following condition:

- (i') *the A -module $N/\tilde{\mathfrak{Y}}N$ is flat, and sequence $\tilde{\mathbf{y}}$ is N -regular, and the homothety with ratio y_s is injective on $\kappa_A \otimes_A (N/\tilde{\mathfrak{Y}}N) = (\kappa_A \otimes N)/\tilde{\mathfrak{Y}}(\kappa_A \otimes_A N)$.*

This condition is equivalent to (ii) by the recurrence hypotheses. Finally, the last assertion follows similarly by recurrence on s and utilize ??.

Proposition 17.1.29. *Let $\rho : A \rightarrow B$ be a homomorphism of Noetherian local rings, M be a finitely generated A -module, and N be a finitely generated B -module. Suppose that the A -module N is flat.*

- (a) *Let (x_1, \dots, x_r) be an M -regular sequence of elements of \mathfrak{m}_A and (y_1, \dots, y_s) be a $(\kappa_A \otimes_A N)$ -regular sequence of elements of \mathfrak{m}_B . Then $(y_1, \dots, y_s, \rho(x_1), \dots, \rho(x_r))$ is an $(M \otimes_A N)$ -regular sequence of elements of \mathfrak{m}_B .*
- (b) *We have the equality*

$$\text{depth}_B(M \otimes_A N) = \text{depth}_A(M) + \text{depth}_B(\kappa_A \otimes_A N).$$

Proof. Denote by \mathfrak{x} the ideal of A generated by x and \mathfrak{Y} the ideal of B generated by \mathbf{y} . By ??, the sequence \mathbf{y} is $M \otimes_A N$ -regular and $N/\mathfrak{Y}N$ is flat for A , so that the sequence $\rho(\mathbf{x}) = (\rho(x_1), \dots, \rho(x_r))$ is regular for $M \otimes_A (N/\mathfrak{Y}N) = (M \otimes_A N)/\mathfrak{Y}(M \otimes_A N)$. This proves the assertion in (a).

To prove (b), we can suppose that M and N are nonzero. By Nakayama's lemma, $\kappa_A \otimes_A N$ is also nonzero, so that $\text{depth}_A(M)$ and $\text{depth}_B(\kappa_A \otimes N)$ are finite (??). Choose \mathbf{x} and \mathbf{y} to be maximal regular sequences, we then have $r = \text{depth}_A(M)$, $s = \text{depth}_B(\kappa_A \otimes N)$, and there exists an injective A -linear map $u : \kappa_A \rightarrow M/\mathfrak{x}M$ and an injective B -linear map $v : \kappa_B \rightarrow \kappa_A \otimes_A (N/\mathfrak{Y}N)$ (??). Since $N/\mathfrak{Y}N$ is flat over A , the B -linear map

$$(u \otimes 1_{N/\mathfrak{Y}N}) \circ v : \kappa_B \rightarrow (M/\mathfrak{x}M) \otimes_A (N/\mathfrak{Y}N) = (M \otimes_A N)/(\rho(\mathfrak{x}) + \mathfrak{Y})(M \otimes_A N)$$

is injective. This implies the equality $\text{depth}_B(M \otimes_A N) = r + s$, in view of ??.

\square

Remark 17.1.4. We note that under the hypotheses of ??, we have a similar equality for dimensions: (c.f. ??)

$$\dim_B(M \otimes_A N) = \dim_A(M) + \dim_B(\kappa_A \otimes_A N).$$

Corollary 17.1.30. Let $\rho : A \rightarrow B$ be a flat homomorphism of Noetherian local rings. Then we have

$$\begin{aligned} \operatorname{depth}(B) &= \operatorname{depth}(A) + \operatorname{depth}(\kappa_A \otimes_A B), \\ \dim(B) &= \dim(A) + \dim(\kappa_A \otimes_A B). \end{aligned}$$

Proof. In fact, the depth (resp. dimension) of the B -module $\kappa_A \otimes_A B$ is equal to the depth (resp. dimension) of the ring $\kappa_A \otimes_A B$ by ??.

□

17.1.4 Depth along a closed subset

We now introduce the concept of depth along a closed subset. Let A be a Noetherian ring, F be a closed subset of $\operatorname{Spec}(A)$, and M be an A -module. By ??, the integer $\operatorname{depth}_A(\mathfrak{I}, M)$ does not depend on the ideal \mathfrak{I} such that $F = V(\mathfrak{I})$, we then write $\operatorname{depth}_F(M)$ for this integer, and call it the **depth of M along F** . By ?? and ??, we see that the inequality $\operatorname{depth}_F(M) \geq r$ is equivalent to the property that for any finitely generated A -module N with support contained in F , we have $\operatorname{Ext}_A^i(N, M) = 0$ for $i < r$. Also, if M is finitely generated, then we have $\operatorname{depth}_F(M) = 0$ if and only if $\operatorname{Ass}(M) \cap F = \emptyset$, and $\operatorname{depth}_F(M) < +\infty$ if and only if $\operatorname{supp}(M) \cap F \neq \emptyset$.

Proposition 17.1.31. Let A be a Noetherian ring, F be a closed subset of $\operatorname{Spec}(A)$, and M be a finitely generated A -module. Then we have

$$\operatorname{depth}_F(M) = \inf_{\mathfrak{p} \in F} \operatorname{depth}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}) = \inf_{\mathfrak{p} \in \operatorname{supp}(M) \cap F} \operatorname{depth}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}).$$

Proof. This is clear if $\operatorname{depth}_F(M) = +\infty$. If $\operatorname{depth}_F(M) = 0$, there exists a prime ideal $\mathfrak{p} \in \operatorname{Ass}(M) \cap F$; we have $\mathfrak{p}A_{\mathfrak{p}} \in \operatorname{Ass}(M_{\mathfrak{p}})$ (??), so $\operatorname{depth}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}) = 0$, whence the assertion in this case.

Suppose that $0 < \operatorname{depth}_F(M) < +\infty$; let \mathfrak{I} be an ideal of A such that $V(\mathfrak{I}) = F$, and x be an element of \mathfrak{I} such that the homothety x_M is injective (??). For any prime ideal \mathfrak{p} , the homothety $x_{M_{\mathfrak{p}}}$ is injective, and by ?? we have

$$\operatorname{depth}_F(M/xM) = \operatorname{depth}_F(M) - 1, \quad \operatorname{depth}_{A_{\mathfrak{p}}}((M/xM)_{\mathfrak{p}}) = \operatorname{depth}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}) - 1.$$

The conclusion then follows by induction on $\operatorname{depth}_F(M)$.

□

Remark 17.1.5. If \mathfrak{q} is a point of $\operatorname{supp}(M)$, we then have $\operatorname{depth}_A(\mathfrak{q}, M) = \inf_{\mathfrak{p} \supseteq \mathfrak{q}} \operatorname{depth}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}})$. In particular, we have $\operatorname{depth}_A(\mathfrak{q}, M) \leq \operatorname{depth}_{A_{\mathfrak{q}}}(M_{\mathfrak{q}})$, and the equality holds if \mathfrak{q} is maximal. In the general case, we may have $\operatorname{depth}_A(\mathfrak{q}, M) < \operatorname{depth}_{A_{\mathfrak{q}}}(M_{\mathfrak{q}})$, or $\operatorname{depth}_A(\mathfrak{q}, M) < \inf \operatorname{depth}_{A_{\mathfrak{m}}}(M_{\mathfrak{m}})$ where \mathfrak{m} runs through maximal ideals of A containing \mathfrak{q} . For example, let \mathfrak{p} be a non maximal prime ideal of A , containing \mathfrak{q} and distinct from \mathfrak{q} ; put $M = A/\mathfrak{p}$, then we have $\operatorname{depth}_A(\mathfrak{q}, M) = 0$, $\operatorname{depth}_{A_{\mathfrak{q}}}(M_{\mathfrak{q}}) = +\infty$ and $\operatorname{depth}_{A_{\mathfrak{m}}}(M_{\mathfrak{m}}) > 0$ for any maximal ideal \mathfrak{m} of A .

Proposition 17.1.32. Let A be a Noetherian ring, M and N be finitely generated A -modules, and F be the support of N . Then $\operatorname{depth}_F(M)$ is the infimum of the integers i such that $\operatorname{Ext}_A^i(N, M) \neq 0$.

Proof. By ?? and ??, we have $\text{Ext}_A^i(N, M) = 0$ for $i < \text{depth}_F(M)$. It remains to prove that if $\text{depth}_F(M) = n <= \infty$, then $\text{Ext}_A^n(N, M) \neq 0$. Let \mathfrak{J} be the annihilator of N ; we then have $F = V(\mathfrak{J})$, so $\text{depth}_F(M) = \text{depth}_A(\mathfrak{J}, M)$. By ??, there exists an M -regular sequence (x_1, \dots, x_n) of length n formed by elements of \mathfrak{J} , and the depth of $\bar{M} = M/(x_1M + \dots + x_nM)$ relative to \mathfrak{J} is zero. By ??, it then suffices to prove that $\text{Hom}_A(N, \bar{M}) \neq 0$. Now by ??, there exists $\mathfrak{p} \in \text{supp}(M) \cap \text{supp}(N)$ such that $\text{depth}_{A_{\mathfrak{p}}}(\bar{M}_{\mathfrak{p}}) = 0$, which means $\text{Hom}_{A_{\mathfrak{p}}}(\kappa(\mathfrak{p}), \bar{M}_{\mathfrak{p}}) \neq 0$. Since $N_{\mathfrak{p}}$ is nonzero, the $\kappa(\mathfrak{p})$ -vector space $N_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} \kappa(\mathfrak{p})$ is nonzero (Nakayama's lemma), so there exists a surjective $A_{\mathfrak{p}}$ -linear map from $N_{\mathfrak{p}}$ to $\kappa(\mathfrak{p})$. It follows that $\text{Hom}_{A_{\mathfrak{p}}}(N_{\mathfrak{p}}, \bar{M}_{\mathfrak{p}}) \neq 0$, so $\text{Hom}_A(N, \bar{M}) \neq 0$ (??), which proves the assertion. \square

Remark 17.1.6. Let A be a Noetherian ring and N be a finitely generated A -module. The **grade** of N , denoted by $\text{grade}(N)$, is defined to be the infimum of the integers i such that $\text{Ext}_A^i(N, A) \neq 0$. By ??, this is also the depth of A along the support of N , and is the maximal length of A -regular sequences of elements of the annihilator of N . As for any prime ideal of A , the annihilator of $N_{\mathfrak{p}}$ is equal to $\text{Ann}(N)_{\mathfrak{p}}$, we deduce from ?? the equality

$$\text{grade}(N) = \inf_{\mathfrak{p} \in \text{Spec}(A)} \text{grade}(N_{\mathfrak{p}}) = \inf_{\mathfrak{m} \in \Omega} \text{grade}(N_{\mathfrak{m}})$$

where Ω denote the set of maximal ideals of A .

We recall that the localization operation is faithfully exact in the sense that a homomorphism $u : M \rightarrow N$ of A -modules is injective (resp. surjective) if and only if $u_{\mathfrak{p}} : M_{\mathfrak{p}} \rightarrow N_{\mathfrak{p}}$ is injective (resp. surjective) for any prime ideal \mathfrak{p} of A . Using the notion of depth, we can show that under certain conditions, this injectivity (resp. surjectivity) of $u_{\mathfrak{p}}$ only need to be checked on an open subset of $\text{Spec}(A)$.

Lemma 17.1.33. *Let A be a Noetherian ring, F be a closed subset of $\text{Spec}(A)$, U be its complement, and $u : M \rightarrow N$ be a homomorphism of finitely generated A -module.*

- (a) Suppose that $u_{\mathfrak{p}} : M_{\mathfrak{p}} \rightarrow N_{\mathfrak{p}}$ is injective for $\mathfrak{p} \in U$ and that $\text{depth}_F(M) \geq 1$, then u is injective.
- (b) Suppose that $u_{\mathfrak{p}} : M_{\mathfrak{p}} \rightarrow N_{\mathfrak{p}}$ is bijective for $\mathfrak{p} \in U$ and that $\text{depth}_F(M) \geq 2$ and $\text{depth}_F(N) \geq 1$, then u is bijective.

Proof. For the first assertion, we note that the hypotheses of (a) imply that $\text{supp}(\ker u) \subseteq F$, so that $\text{Hom}_A(\ker u, M) = 0$ by ???. We then conclude that $\ker u = 0$, since $\ker u$ is a submodule of M . As for the assertions in (b), we note that in this case u is injective by (a), and we have $\text{supp}(\text{coker } u) \subseteq F$. Again by ??, we have $\text{Hom}_A(\text{coker } u, N) = 0$ and $\text{Ext}_A^1(\text{coker } u, M) = 0$. From the exact sequence

$$\text{Hom}_A(\text{coker } u, N) \longrightarrow \text{Hom}_A(\text{coker } u, \text{coker } u) \longrightarrow \text{Ext}_A^1(\text{coker } u, M)$$

we deduce that $\text{Hom}_A(\text{coker } u, \text{coker } u) = 0$, whence $\text{coker } u = 0$. \square

Remark 17.1.7. Let A be a Noetherian ring, F be a closed subset of $\text{Spec}(A)$, and U be its complement. For that we have $\text{depth}_F(A) \geq 1$, it is necessary and sufficient that $\text{Ass}(A) \subseteq U$ (??). If this condition is satisfied, then any irreducible component of $\text{Spec}(A)$ meets U , so that U is dense in $\text{Spec}(A)$.

Theorem 17.1.34 (Hartshorne). *Let A be a Noetherian ring, F be a closed subset of A , and U be its complement. Suppose that $\text{depth}_F(A) \geq 2$, then for any connected component Y of $\text{Spec}(A)$, the set $Y \cap U$ is connected and dense in Y .*

Proof. Suppose first that $\text{Spec}(A)$ is connected. By ??, U is dense in $\text{Spec}(A)$ and it then suffices to prove that it is connected. To this end, suppose that there are two nonempty open disjoint subsets U_0 and U_1 of $\text{Spec}(A)$ whose union is U . As the set $\text{Ass}(A)$ is contained in U by ??, it is the union of the disjoint subsets $\text{Ass}(A) \cap U_0$ and $\text{Ass}(A) \cap U_1$. By ??, there exists ideals \mathfrak{I}_0 and \mathfrak{I}_1 of A such that

$$\text{Ass}(\mathfrak{I}_i) = \text{Ass}(A) \cap U_i, \quad \text{Ass}(A/\mathfrak{I}_i) = \text{Ass}(A) \cap U_{1-i} \quad (i = 0, 1).$$

The complement of U_i in $\text{Spec}(A)$ contains $\text{Ass}(A/\mathfrak{I}_i)$ and $\text{Ass}(\mathfrak{I}_j)$; as it is closed, it also contains $\text{supp}(A/\mathfrak{I}_i)$ and $\text{supp}(\mathfrak{I}_{1-i})$. For $\mathfrak{p} \in U_i$, we then conclude that $(A/\mathfrak{I}_i)_{\mathfrak{p}} = 0$ and $(\mathfrak{I}_{1-i})_{\mathfrak{p}} = 0$, which also implies that \mathfrak{I}_0 and \mathfrak{I}_1 are proper ideals of A . Now let B be the A -module $A/\mathfrak{I}_0 \times A/\mathfrak{I}_1$ and $u : A \rightarrow B$ be the canonical homomorphism. By the preceding remarks, the homomorphism $u_{\mathfrak{p}}$ is bijective for $\mathfrak{p} \in U$; on the other hand, we have $\text{Ass}(B) \subseteq U_0 \cup U_1 = U$ (??), so $\text{depth}_F(B) \geq 1$ in view of ???. ?? then implies that u is bijective, which contradicts the connectedness of $\text{Spec}(A)$.

In the general case, let \mathfrak{J} be an ideal of A such that $F = V(\mathfrak{J})$ and let Y be a connected component of $\text{Spec}(A)$. By ??, there exists an element f of A such that Y is identified with the subset $\text{Spec}(A_f)$ of $\text{Spec}(A)$ (for example, if $Y = V(e)$ for some idempotent e , then we can choose $f = 1 - e$). Then $Y \cap F$ is identified with $V(\mathfrak{J}_f)$, and we have $\text{depth}_{A_f}(\mathfrak{J}_f, A_f) \geq \text{depth}_A(\mathfrak{J}, A) \geq 2$ in view of ???. It then follows from our previous arguments that $Y \cap U = Y - (Y \cap F)$ is connected and dense in Y . \square

Corollary 17.1.35. *The map which associates each connected component of U with its closure in $\text{Spec}(A)$ is a bijection from the set of connected components of U onto the set of connected components of $\text{Spec}(A)$.*

Corollary 17.1.36. *For any Noetherian local ring B with $\text{depth}(B) \geq 2$, the space $\text{Spec}(B) - \{\mathfrak{m}_B\}$ is connected.*

Proof. This follows from the observation that any local ring has no idempotents, so $\text{Spec}(B)$ is connected and we can apply ?? to $F = V(\mathfrak{m}_B) = \{\mathfrak{m}_B\}$. \square

Corollary 17.1.37. *Under the hypotheses of ??, suppose that $\text{Spec}(A_{\mathfrak{p}})$ is irreducible (resp. $A_{\mathfrak{p}}$ is integral) for any $\mathfrak{p} \in U$. Then $\text{Spec}(A_{\mathfrak{p}})$ is irreducible (resp. $A_{\mathfrak{p}}$ is integral) for any $\mathfrak{p} \in \text{Spec}(A)$.*

Proof. Let $(Y_i)_{i \in I}$ be the (finite) family of irreducible components of $\text{Spec}(A)$. Let $\mathfrak{p} \in U$; as $\text{Spec}(A_{\mathfrak{p}})$ is irreducible, \mathfrak{p} contains a unique minimal prime ideal of A , so is contained in a unique Y_i . The intersection $Y_i \cap U$ is then nonempty disjoint and closed in U , dense in Y_i , and irreducible by ??, so they form the connected components of U . The closures Y_i are then the connected components of $\text{Spec}(A)$ by ???. This proves that the connected components of $\text{Spec}(A)$ are irreducible, so that $\text{Spec}(A_{\mathfrak{p}})$ is irreducible for any $\mathfrak{p} \in \text{Spec}(A_{\mathfrak{p}})$.

Now suppose that $A_{\mathfrak{q}}$ is integral for each $\mathfrak{q} \in U$. Let $\mathfrak{p} \in \text{Spec}(A)$, since $\text{Spec}(A_{\mathfrak{p}})$ is irreducible, the nilradical of $A_{\mathfrak{p}}$ is the unique prime ideal of $A_{\mathfrak{p}}$, and it therefore belongs to $\text{Ass}(A_{\mathfrak{p}})$ (??), and equal to $qA_{\mathfrak{p}}$, where q is an associated prime of A (??). We have $q \in U$ by ?? and $qA_{\mathfrak{q}} \in \text{Ass}(A_{\mathfrak{q}})$ (??); since $A_{\mathfrak{q}}$ is integral, q is then zero, so $A_{\mathfrak{p}}$ is also integral. \square

Corollary 17.1.38. Let A be a Noetherian ring with $\text{Spec}(A)$ connected. Suppose that there exists an integer $d \geq 1$ such that we have $\text{depth}(A_{\mathfrak{p}}) \geq 2$ for any prime ideal of A with $\text{ht}(\mathfrak{p}) > d$.

- (a) For any closed subset Z of $\text{Spec}(A)$ with $\text{codim}(Z) > d$, the space $\text{Spec}(A) - Z$ is connected.
- (b) Let Y and Y' be irreducible components of $\text{Spec}(A)$. Then there exists a sequence (X_1, \dots, X_n) of irreducible components of $\text{Spec}(A)$ such that $X_1 = Y$, $X_n = Y'$, and for each $i = 1, \dots, n-1$, we have $\text{codim}(X_i \cap X_{i+1}) \leq d$.

Proof. Let $Z \subseteq \text{Spec}(A)$ be a closed subset with $\text{codim}(Z) > d$. For any $\mathfrak{p} \in Z$, we have $\dim(A_{\mathfrak{p}}) > d$, so $\text{depth}(A_{\mathfrak{p}}) \geq 2$, which implies that $\text{depth}_Z(A) \geq 2$ (??). Then $\text{Spec}(A) - Z$ is connected by ??.

To prove (b), denote by Z the union of subsets $X' \cap X''$ where (X', X'') runs through subsets of couples of irreducible components of $\text{Spec}(A)$ such that $\text{codim}(X' \cap X'') > d$. In view of (a), the subset $\text{Spec}(A) - Z$ is then connected. All irreducible components of $\text{Spec}(A)$ meet $\text{Spec}(A) - Z$, and their trace over $\text{Spec}(A) - Z$ are the irreducible components of $\text{Spec}(A) - Z$ (??). Since $\text{Spec}(A) - Z$ is connected, there exists a sequence (X_1, \dots, X_n) of irreducible components of $\text{Spec}(A)$ such that $X_1 - Z = Y - Z$, $X_n - Z = Y' - Z$, and $(X_i - Z) \cap (X_{i+1} - Z) \neq \emptyset$ for $1 \leq i \leq n-1$. The construction of Z then implies that $X_1 = Y$, $X_n = Y'$, and $\text{codim}(X_i \cap X_{i+1}) \leq d$. \square

Example 17.1.39. Let k be a field and $S = k[T_1, T_2, T_3, T_4]$ be the polynomial ring. Recall that any maximal chain of prime ideals of S has length 4 (??). Let \mathfrak{m} be the maximal ideal of S generated by the T_i , and for $i \leq i < j \leq 4$, let $\mathfrak{p}_{ij} = (T_i, T_j)$. The ideals \mathfrak{p}_{ij} are then prime with height 2, and their sum is the maximal ideal \mathfrak{m} .

- (a) Let \mathfrak{a} be the ideal of S defined by $\mathfrak{a} = (T_1T_2, T_3T_4)$, and $A = S/\mathfrak{a}$. Then we have $\mathfrak{a} = \mathfrak{p}_{13} \cap \mathfrak{p}_{14} \cap \mathfrak{p}_{23} \cap \mathfrak{p}_{24}$, and the ring $A = S/\mathfrak{a}$ is reduced. The irreducible components of $\text{Spec}(A)$ are the subsets $X_{ij} = V(\mathfrak{p}_{ij}/\mathfrak{a})$ for $i = 1, 2$, $j = 3, 4$, which are all of dimension 2 and contain the closed point $\mathfrak{m}/\mathfrak{a}$. In particular, $\text{Spec}(A)$ is connected of dimension 2. The intersection of two distinct components X_{ij} and X_{kl} is reduced to $\{\mathfrak{m}/\mathfrak{a}\}$ if $\{i, j\} \cap \{k, l\} = \emptyset$, and is of dimension 1 otherwise. It then follows that the conclusion of ?? is valid for $d = 1$ (in fact A is Cohen-Macaulay).
- (b) Let \mathfrak{b} be the ideal of S defined by $\mathfrak{b} = (T_1T_2, T_1T_3, T_2T_4, T_3T_4)$, and $B = S/\mathfrak{b}$. Then we have $\mathfrak{b} = \mathfrak{p}_{14}\mathfrak{p}_{23} = \mathfrak{p}_{14} \cap \mathfrak{p}_{23}$, and the ring B is reduced. The space $\text{Spec}(B)$ is identified with the closed subset $X_{14} \cup X_{23}$ of $\text{Spec}(A)$, it have two irreducible components (of dimension 2), whose intersection reduces to $\{\mathfrak{m}/\mathfrak{b}\}$. The depth of B along this closed point is strictly positive because B is reduced, and smaller than 1 in view of ?? (since $\text{Spec}(B) - \{\mathfrak{m}/\mathfrak{b}\}$ is not connected), so it is equal to 1 (the ring B is not Cohen-Macaulay).

17.1.5 Serre's criterion for normality

Let A be a Noetherian ring. We denote by $(Y_i)_{i \in I}$ the finite family of connected components of $\text{Spec}(A)$. By ??, for each i there exists a unique idempotent e_i of A such that $Y_i = V(e_i)$, and the canonical homomorphism $A \rightarrow \prod_i A/Ae_i$ is bijective. The quotient rings A/Ae_i are called the canonical components of A . Put $f_i = 1 - e_i$, then we have $\sum_i f_i = 1$ and $(f_i)_{i \in I}$ is an orthogonal family of nonzero idempotents of A . It then follows that the image of f_i in A/Ae_i

is equal to 1 if $j = i$, and to 0 otherwise. The canonical homomorphism $A \rightarrow \prod_j A/Ae_j$ then induces a canonical isomorphism $A_{f_i} \rightarrow A/Ae_i$.

By ??, we see that the following conditions are equivalent:

- (i) the connected components of $\text{Spec}(A)$ are irreducible;
- (ii) each prime (resp. maximal) ideal of A belongs to a unique irreducible component of $\text{Spec}(A)$;
- (iii) each prime (resp. maximal) ideal of A contains a unique minimal prime ideal;
- (iv) for any prime (resp. maximal) ideal \mathfrak{p} of A , the topological space $\text{Spec}(A_{\mathfrak{p}})$ is irreducible;
- (v) for any canonical component C of A , the topological space $\text{Spec}(C)$ is irreducible.

Now note that if A is reduced, then each ring $A_{\mathfrak{p}}$ is reduced, and the converse is also true by ???. Applying ??, we then deduce that the following conditions are equivalent:

- (i) A is reduced and the connected components of $\text{Spec}(A)$ are irreducible;
- (ii) for any prime (resp. maximal) ideal of A , the ring $A_{\mathfrak{p}}$ is integral;
- (iii) the canonical components of A are integral.

The Noetherian ring A is called **locally integral** if it satisfies the above equivalent condition. Suppose that A is locally integral; let $u : A \rightarrow \prod_{j \in J} A_j$ be an isomorphism from A to a (finite) product of integral rings. Then there exists a bijection $\sigma : J \rightarrow I$ such that the map from $\text{Spec}(\prod_{j \in J} A_j)$ to $\text{Spec}(A)$ associated with u defines a homeomorphism from $\text{Spec}(A_j)$ to the connected component $Y_{\sigma(j)}$ of $\text{Spec}(A)$. We then deduce that u is an isomorphism from the canonical component $A/Ae_{\sigma(j)}$ to A_j .

Proposition 17.1.40. *Let A be a Noetherian ring. The following conditions are equivalent:*

- (i) A is reduced and integrally closed in its total fraction ring;
- (ii) A is isomorphic to the product of a finite family of integrally closed rings;
- (iii) the canonical components of A are integrally closed;
- (iv) for any prime (resp. maximal) ideal \mathfrak{p} of A , the ring $A_{\mathfrak{p}}$ is integrally closed.

Proof. The equivalence of (i) and (ii) follows from ??, and that of (ii) and (iii) follows from the preceding remarks. Let \mathfrak{p} be a prime ideal of A , then there exists a unique canonical component A' of A such that \mathfrak{p} belongs to the closed subset $\text{Spec}(A')$ of $\text{Spec}(A)$ and we have a canonical isomorphism $A_{\mathfrak{p}} \rightarrow A'_{\mathfrak{p}A'}$. The equivalence of (iii) and (iv) then follows from ?? and ???. \square

A ring A is called **normal** if it is Noetherian and it satisfies the equivalent conditions of ???. With this notion, a Noetherian ring is integrally closed if and only if it is integral and normal. It is clear that a local normal ring is integrally closed.

We now present Serre's criterion for the normality of a Noetherian ring. To begin with, we give the following characterization for a Noetherian ring to be reduced. Recall that for any prime ideal \mathfrak{p} of A , we have $\text{depth}(A_{\mathfrak{p}}) \leq \text{ht}(\mathfrak{p})$ by ??.

Proposition 17.1.41. *Let A be a Noetherian ring. Then A is reduced if and only if it satisfies the following conditions:*

- (R0) *for any minimal prime ideal, $A_{\mathfrak{p}}$ is a regular local ring;*
- (S1) *for any prime ideal \mathfrak{p} of A with $\text{ht}(\mathfrak{p}) \geq 1$, we have $\text{depth}(A_{\mathfrak{p}}) \geq 1$.*

Proof. Denote by \mathfrak{n} the nilradical of A . If A is reduced, then each local ring $A_{\mathfrak{p}}$ is reduced, so if $\text{ht}(\mathfrak{p}) = 0$, then $A_{\mathfrak{p}}$ is a field, and if $\text{ht}(\mathfrak{p}) \geq 1$, then $\text{depth}(A_{\mathfrak{p}}) \geq 1$ (??). Conversely, assume that A satisfies the above conditions. Then for any minimal prime ideal \mathfrak{p} of A , we have $\mathfrak{n}_{\mathfrak{p}} = 0$ by condition (R0), which means $\mathfrak{p} \notin \text{supp}(\mathfrak{n})$, and a fortiori $\mathfrak{p} \notin \text{Ass}(\mathfrak{n})$. For any $\mathfrak{p} \in \text{Spec}(A)$ with $\text{ht}(\mathfrak{p}) \geq 1$, by condition (S1) and ?? we have $\mathfrak{p}A_{\mathfrak{p}} \notin \text{Ass}_{A_{\mathfrak{p}}}(A_{\mathfrak{p}})$ and a fortiori $\mathfrak{p}A_{\mathfrak{p}} \notin \text{Ass}_{A_{\mathfrak{p}}}(\mathfrak{n}_{\mathfrak{p}})$ (??), so $\mathfrak{p} \notin \text{Ass}_A(\mathfrak{n})$ (??). We then conclude that $\text{Ass}(\mathfrak{n}) = \emptyset$, so $\mathfrak{n} = 0$ and A is reduced. \square

Proposition 17.1.42. *Let A be an integrally closed Noetherian ring, \mathfrak{I} be an ideal of A with $\text{ht}(\mathfrak{I}) \geq 2$, and M be a finitely generated reflexive A -module. Then we have $\text{depth}_A(\mathfrak{I}, M) \geq 2$.*

Proof. Since M is reflexive, we can choose a finite dimensional vector space V over the fraction field of A such that M is a lattice in V (??). The associated primes of V/M , being of height 1 (??), then do not belong to $V(\mathfrak{I})$. By ??, we then conclude that $\text{depth}_A(\mathfrak{I}, V/M) \geq 1$. On the other hand, the A -module V is divisible and torsion free, hence injective, and this implies $\text{depth}_A(\mathfrak{I}, V) = +\infty$ by ?. The inequality $\text{depth}_A(\mathfrak{I}, M) \geq 2$ then follows from ?. \square

Corollary 17.1.43. *An integrally closed Noetherian local ring of dimension ≥ 2 has depth ≥ 2 .*

Proof. This follows from the fact that an integrally closed Noetherian local ring A is Krull (??), hence reflexive (??), so we can apply ?? to $\mathfrak{I} = \mathfrak{m}_A$. \square

Theorem 17.1.44 (Serre's Normality Criterion). *Let A be a Noetherian ring. Then A is normal if and only if it satisfies the following properties:*

- (R1) *for any prime ideal \mathfrak{p} of A with $\text{ht}(\mathfrak{p}) \leq 1$, $A_{\mathfrak{p}}$ is a regular local ring.*
- (S2) *for any prime ideal \mathfrak{p} of A with $\text{ht}(\mathfrak{p}) \geq 2$, we have $\text{depth}(A_{\mathfrak{p}}) \geq 2$.*

Proof. By definition, if A is normal, then $A_{\mathfrak{p}}$ is integrally closed for any prime ideal \mathfrak{p} of A , so condition (S2) follows from ?. Also, it is clear that an integrally closed Noetherian local ring of dimension ≤ 1 must be regular. Conversely, assume that A satisfies condition (R1) and (S2). We show that $A_{\mathfrak{p}}$ is integrally closed by recurrence on $\text{ht}(\mathfrak{p})$. For $\text{ht}(\mathfrak{p}) \leq 1$, this follows directly from condition (R1). Suppose then that $\text{ht}(\mathfrak{p}) \geq 2$ and that $A_{\mathfrak{q}}$ is integrally closed for any prime ideal \mathfrak{q} with $\text{ht}(\mathfrak{q}) < \text{ht}(\mathfrak{p})$. By condition (S2), we have $\text{depth}(A_{\mathfrak{p}}) \geq 2$. By the recurrence hypotheses and ?? applied to the ring $A_{\mathfrak{p}}$ and the closed subset $\{\mathfrak{p}A_{\mathfrak{p}}\}$ of $\text{Spec}(A_{\mathfrak{p}})$, we then conclude that $A_{\mathfrak{p}}$ is integral. Let K be the fraction field of $A_{\mathfrak{p}}$ and B be a subring of K which is finite over $A_{\mathfrak{p}}$. It then suffices to prove that $B = A_{\mathfrak{p}}$, so let $i : A_{\mathfrak{p}} \rightarrow B$ be the canonical injection. As B is contained in K , it is a torsion-free $A_{\mathfrak{p}}$ -module, so $\text{depth}_{A_{\mathfrak{p}}}(B) \geq 1$. On the other hand, for any prime ideal \mathfrak{q} of $A_{\mathfrak{p}}$ distinct from $\mathfrak{p}A_{\mathfrak{p}}$, the homomorphism $i_{\mathfrak{q}} : (A_{\mathfrak{p}})_{\mathfrak{q}} \rightarrow B_{\mathfrak{q}}$ is bijective since $A_{\mathfrak{q}}$ is integrally closed by hypotheses. By ?? applied to the closed subset $F = \{\mathfrak{p}A_{\mathfrak{p}}\}$ of $\text{Spec}(A_{\mathfrak{p}})$, the homomorphism i is then bijective, which proves our assertion. \square

Remark 17.1.8. A convenient form of ?? is the following: let A be a Noetherian ring such that for any prime ideal \mathfrak{p} of A , the ring $A_{\mathfrak{p}}$ is either integrally closed, or $\text{depth}(A_{\mathfrak{p}}) \geq 2$, then A is normal. In fact, in this case if $\text{ht}(\mathfrak{p}) \leq 1$ then $\text{depth}(A_{\mathfrak{p}}) \leq \text{ht}(\mathfrak{p}) \leq 1$, so $A_{\mathfrak{p}}$ is integrally closed. If $\text{ht}(\mathfrak{p}) \geq 2$, then Noe integral closed depth geq 2 if dim geq 2 implies in both cases that $\text{depth}(A_{\mathfrak{p}}) \geq 2$, so we can apply ???. Similarly, we see if for any prime ideal \mathfrak{p} of A , the ring $A_{\mathfrak{p}}$ is either regular or $\text{depth}(A_{\mathfrak{p}}) \geq 1$, then A is reduced.

Corollary 17.1.45. Let A be a Noetherian ring, F be a closed subset of $\text{Spec}(A)$, and U be its complement. Suppose that $\text{depth}_F(A) \geq 2$ (resp. $\text{depth}_F(A) \geq 1$) and that for any $\mathfrak{p} \in U$, the ring $A_{\mathfrak{p}}$ is integrally closed (resp. reduced). Then A is normal (resp. reduced).

Proof. For any $\mathfrak{p} \in F$, we have $\text{depth}(A_{\mathfrak{p}}) \geq \text{depth}_F(A)$ by ??, so it suffices to apply the preceding remark. \square

Corollary 17.1.46. Let $\rho : A \rightarrow B$ be a flat homomorphism of rings.

- (a) If B is normal and faithfully flat over A , then A is normal.
- (b) Suppose that A is normal and the ring $\kappa(\mathfrak{p}) \otimes_A B$ is normal (resp. reduced) for any minimal prime ideal \mathfrak{p} (resp. prime ideal \mathfrak{p} of height one) of A . Then the ring B is normal.

Corollary 17.1.47. Let $\rho : A \rightarrow B$ be a homomorphism of Noetherian rings. Suppose that B is a flat A -module, that A is normal, and that $\kappa(\mathfrak{p}) \otimes_A B$ is normal for any $\mathfrak{p} \in \text{Spec}(A)$. Then B is normal.

17.2 Cohen-Macaulay rings and modules

17.2.1 Cohen-Macaulay modules

Let A be a Noetherian ring, M be a finitely generated A -module, and \mathfrak{p} be a prime ideal of A . If $\mathfrak{p} \notin \text{supp}(M)$, then we have $M_{\mathfrak{p}} = 0$, so $\text{depth}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}) = +\infty$. If $\mathfrak{p} \in \text{supp}(M)$, then by ??,

$$0 \leq \text{depth}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}) \leq \dim_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}) < +\infty.$$

The module M is called **Cohen-Macaulay** if for any maximal ideal $\mathfrak{m} \in \text{supp}(M)$, we have $\text{depth}_{A_{\mathfrak{m}}}(M_{\mathfrak{m}}) = \dim_{A_{\mathfrak{m}}}(M_{\mathfrak{m}})$. If A is local, then for a finitely generated nonzero A -module to be Cohen-Macaulay, it is necessary and sufficient that its depth is equal to its dimension.

Example 17.2.1. Any A -module of finite length is Cohen-Macaulay. In fact, if M is a module of finite length, then $M_{\mathfrak{m}}$ is of finite length for any maximal ideal \mathfrak{m} of A (??), and we have $\dim_{A_{\mathfrak{m}}}(M_{\mathfrak{m}}) = 0$ by ??, whence $\text{depth}_{A_{\mathfrak{m}}}(M_{\mathfrak{m}}) = 0$.

Example 17.2.2. Let N be a direct factor of a finitely generated Cohen-Macaulay A -module M . Then N is Cohen-Macaulay; in factm for any maximal ideal \mathfrak{m} of A , the $A_{\mathfrak{m}}$ -module $N_{\mathfrak{m}}$ is a direct factor of $M_{\mathfrak{m}}$, and we therefore have

$$\text{depth}_{A_{\mathfrak{m}}}(N_{\mathfrak{m}}) \geq \text{depth}_{A_{\mathfrak{m}}}(M_{\mathfrak{m}}) \geq \dim_{A_{\mathfrak{m}}}(M_{\mathfrak{m}}) \geq \dim_{A_{\mathfrak{m}}}(N_{\mathfrak{m}}),$$

where we use ?? and ??.

Example 17.2.3. Let M be a finitely generated Cohen-Macaulay A -module and (x_1, \dots, x_n) be an M -regular sequence for elements of A . Then the A -module $\bar{M} = M/(x_1M + \dots + x_nM)$ is Cohen-Macaulay. To see this, let \mathfrak{m} be a maximal ideal of A belonging to the support of \bar{M} ; we have $x_i \in \mathfrak{m}$ for each i since x_i annihilates \bar{M} , and the canonical image of x_i in $A_{\mathfrak{m}}$ form an $M_{\mathfrak{m}}$ -regular sequence of elements of $\mathfrak{m}A_{\mathfrak{m}}$. We then have the equalities (?? and ??)

$$\operatorname{depth}_{A_{\mathfrak{m}}}(\bar{M}_{\mathfrak{m}}) = \operatorname{depth}_{A_{\mathfrak{m}}}(M_{\mathfrak{m}}) - n, \quad \dim_{A_{\mathfrak{m}}}(\bar{M}_{\mathfrak{m}}) = \dim_{A_{\mathfrak{m}}}(M_{\mathfrak{m}}) - n.$$

whence our assertion.

Example 17.2.4. Let M be a finitely generated A -module and \mathfrak{a} be an ideal of A such that $\mathfrak{a}M = 0$. For that the A -module M to be Cohen-Macaulay, it is necessary and sufficient that it is Cohen-Macaulay as an (A/\mathfrak{a}) -module. In fact, put $B = A/\mathfrak{a}$; let \mathfrak{M} be the maximal ideal of B and \mathfrak{m} be its inverse image in A . We then have $\dim_{A_{\mathfrak{m}}}(M_{\mathfrak{m}}) = \dim_{B_{\mathfrak{M}}}(M_{\mathfrak{M}})$ and $\operatorname{depth}_{A_{\mathfrak{m}}}(M_{\mathfrak{m}}) = \operatorname{depth}_{B_{\mathfrak{M}}}(M_{\mathfrak{M}})$ (??).

Proof. This follows directly from ?? and depth of module prime ideal inclusion inequality. \square

Proposition 17.2.5. Let A be a Noetherian ring and M be a finitely generated A -module. Then the following conditions are equivalent:

- (i) the A -module M is Cohen-Macaulay;
- (ii) $\operatorname{depth}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}) = \dim_{A_{\mathfrak{p}}}(M_{\mathfrak{p}})$ for any $\mathfrak{p} \in \operatorname{supp}(M)$;
- (iii) $\operatorname{depth}_F(M) = \operatorname{codim}(\operatorname{supp}(M) \cap F, \operatorname{supp}(M))$ for any closed subset F of $\operatorname{Spec}(A)$.
- (iv) $\operatorname{depth}_A(\mathfrak{p}, M) = \dim_{A_{\mathfrak{p}}}(M_{\mathfrak{p}})$ for any $\mathfrak{p} \in \operatorname{supp}(M)$.

Proof. We first note that (i) \Rightarrow (ii) in view of ?? and depth of module prime ideal inclusion inequality. For (ii) \Rightarrow (iii), note that by ??, $\operatorname{depth}_F(M)$ is the infimum of the integers $\operatorname{depth}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}})$ for $\mathfrak{p} \in \operatorname{supp}(M) \cap F$. If M is Cohen-Macaulay, then for each such ideal \mathfrak{p} we have the equality

$$\operatorname{depth}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}) = \dim_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}) = \operatorname{codim}(V(\mathfrak{p}), \operatorname{supp}(M))$$

(??), so (iii) follows from the definition of $\operatorname{codim}(\operatorname{supp}(M) \cap F, \operatorname{supp}(M))$. On the other hand, it is clear that (iii) \Rightarrow (iv) by taking $F = V(\mathfrak{p})$ (??), and (iv) \Rightarrow (i) since we have the inequality $\operatorname{depth}_A(\mathfrak{p}, M) \leq \operatorname{depth}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}) \leq \dim(M_{\mathfrak{p}})$ for any $\mathfrak{p} \in \operatorname{supp}(M)$ (??). \square

Corollary 17.2.6. Let A be a Noetherian ring and M be a finitely generated A -module. Then for any prime ideals $\mathfrak{p} \subseteq \mathfrak{q}$ of $\operatorname{supp}(M)$, we have

$$\operatorname{depth}_{A_{\mathfrak{q}}}(M_{\mathfrak{q}}) - \operatorname{depth}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}) = \dim(A_{\mathfrak{q}}/\mathfrak{p}A_{\mathfrak{q}}) = \dim_{A_{\mathfrak{q}}}(M_{\mathfrak{q}}) - \dim_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}).$$

Proof. This follows from ?? and $\dim(M_{\mathfrak{q}}) \geq \dim(M_{\mathfrak{p}}) + \dim(A_{\mathfrak{q}}/\mathfrak{p}A_{\mathfrak{q}})$ (??(a) and ??(b)) \square

Corollary 17.2.7. Let S be a multiplicative subset of A and M be a finitely generated Cohen-Macaulay A -module. Then $S^{-1}M$ is a Cohen-Macaulay $S^{-1}A$ -module.

Proof. In fact, let $\mathfrak{q} \in \operatorname{Spec}(S^{-1}A)$, $i_A^S : A \rightarrow S^{-1}A$ be the canonical homomorphism, and $\mathfrak{p} = (i_A^S)^{-1}(\mathfrak{q})$. The ring $(S^{-1}A)_{\mathfrak{q}}$ is then identified with $A_{\mathfrak{p}}$, and the $A_{\mathfrak{p}}$ -module $(S^{-1}M)_{\mathfrak{q}}$ is identified with the $A_{\mathfrak{p}}$ -module $M_{\mathfrak{p}}$, whence the corollary. \square

Proposition 17.2.8. *Let A be a Noetherian ring and M be a finitely generated Cohen-Macaulay A -module.*

- (a) *The A -module M has no embedded associated prime ideals.*
- (b) *Let X be a closed irreducible subset of $\text{supp}(M)$ and Y be a closed subset of X , then we have*

$$\text{codim}(Y, X) + \text{codim}(X, \text{supp}(M)) = \text{codim}(Y, \text{supp}(M)).$$

- (c) *The topological space $\text{supp}(M)$ is catenary.*
- (d) *Let X_1 and X_2 be irreducible components of $\text{supp}(M)$ and Y be a closed subset of $X_1 \cap X_2$. Then we have $\text{codim}(Y, X_1) = \text{codim}(Y, X_2)$.*

Proof. Let $\mathfrak{p} \in \text{Ass}(M)$; we have $\text{depth}_A(\mathfrak{p}, M) = 0$ (??), so $\dim_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}) = 0$ by ??, this implies that \mathfrak{p} is a minimal element of $\text{supp}(M)$, so the assertion in (a) follows.

To prove the assertion in (b), it suffices to let X be a closed irreducible subset of $\text{supp}(M)$ and Y be a closed irreducible subset of X . Let \mathfrak{p} and \mathfrak{q} be prime ideals of $\text{supp}(M)$ such that $Y = V(\mathfrak{q})$ and $X = V(\mathfrak{p})$, then it follows from ?? that

$$\begin{aligned} \text{codim}(Y, X) &= \dim(A_{\mathfrak{q}}/\mathfrak{p}A_{\mathfrak{q}}) = \dim(M_{\mathfrak{q}}) - \dim(M_{\mathfrak{p}}) \\ &= \text{codim}(Y, \text{supp}(M)) - \text{codim}(X, \text{supp}(M)). \end{aligned}$$

Now if X, Y, Z are closed irreducible subsets of $\text{supp}(M)$ such that $Z \subseteq Y \subseteq X$, then the codimensions of them in $\text{supp}(M)$ are finite, and we deduce from (b) the equality

$$\text{codim}(Z, Y) + \text{codim}(Y, X) = \text{codim}(Z, X)$$

so we conclude from ?? that $\text{supp}(M)$ is catenary. Finally, if X_1 and X_2 are irreducible components of $\text{supp}(M)$ and Y be a closed subset of $X_1 \cap X_2$, then

$$\text{codim}(Y, X_1) = \text{codim}(Y, \text{supp}(M)) = \text{codim}(Y, X_2).$$

□

In particular, if there exists a finitely generated A -module M such that $\text{supp}(M) = \text{Spec}(A)$, then the ring A is catenary and any fraction ring of quotient ring of A is catenary.

Remark 17.2.1. Under the hypotheses of ??, it may happen that two components irreducible X_1 and X_2 of $\text{supp}(M)$ have an intersection Y reducing to a point and that $\dim(X_1) \neq \dim(X_2)$ and $\dim(X_2) \neq \text{codim}(Y, X_2)$. However, this cannot happen if A is local, as shown in the next corollary.

Corollary 17.2.9. *Let A be a Noetherian local ring and M be a finitely generated nonzero Cohen-Macaulay ring.*

- (a) *Any maximal chain of irreducible closed subsets of $\text{supp}(M)$ has length equal to $\dim(M)$.*
- (b) *For any closed subset X of $\text{supp}(M)$, we have*

$$\text{codim}(X, \text{supp}(M)) = \dim(\text{supp}(M)) - \dim(X).$$

- (c) All irreducible components of $\text{supp}(M)$ have the same dimension.
- (d) For any ideal \mathfrak{I} of A , we have

$$\text{depth}_A(\mathfrak{I}, M) = \dim(M) - \dim(M/\mathfrak{I}M).$$

Proof. A maximal chain of irreducible closed subsets of $\text{supp}(M)$ has smallest element $\{\mathfrak{m}_A\}$ and largest element a irreducible component X of $\text{supp}(M)$. Its length is therefore equal to the codimension of $\{\mathfrak{m}_A\}$ in X (??(c)); by ??(b), this is equal to $\text{codim}(\{\mathfrak{m}_A\}, \text{supp}(M))$, which is $\dim(M)$. Now (b) is a consequence of (a) if the subset X is irreducible, and the general case follows from (??), and (c) is a consequence of (b). Finally, for any ideal \mathfrak{I} , we have $\text{depth}_A(\mathfrak{I}, M) = \text{codim}(\text{supp}(M) \cap V(\mathfrak{I}), \text{supp}(M))$ by ??, so to prove (d) it suffices to apply (b) to $X = \text{supp}(M) \cap V(\mathfrak{I}) = \text{supp}(M/\mathfrak{I}M)$. \square

We now consider Cohen-Macaulay modules over Noetherian local rings. In particular, we will consider the behaviour of these modules with regular sequence of \mathfrak{m}_A .

Proposition 17.2.10. *Let A be a Noetherian local ring, M be a finitely generated nonzero A -module of dimension d . The following conditions are equivalent:*

- (i) the A -module M is Cohen-Macaulay;
- (ii) $\text{depth}(M) = d$;
- (iii) $\text{Ext}_A^i(\kappa_A, M) = 0$ for any integer $i < d$;
- (iv) $\text{Ext}_A^i(N, M) = 0$ for any A -module N of finite length and any integer $i < d$.
- (v) $\text{Ext}_A^i(N, M) = 0$ for any finitely generated A -module N and any integer $i < d - \dim(M \otimes_A N)$.
- (vi) there exists an M -regular sequence of elements of \mathfrak{m}_A of length d .

Proof. The equivalences of (i), (ii), (iii), (iv) and (vi) are clear from our definition. As for (v), note that it is clear that (v) \Rightarrow (iv), since if N is of finite length, then $\dim(M \otimes_A N) = 0$ for any finitely generated A -module M (?? and ??). Conversely, suppose that M is Cohen-Macaulay and let N be a finitely generated A -module. Put $F = \text{supp}(N)$, then by ?? we have $\text{supp}(M) \cap F = \text{supp}(M \otimes_A N)$, so that

$$\text{depth}_F(M) = \text{codim}(\text{supp}(M) \cap F, \text{supp}(M)) = \dim(M) - \dim(M \otimes_A N)$$

(?? and ??). The implication (i) \Rightarrow (v) then follows from ??.

A finitely generated module M over a Noetherian local ring is called **pure** if for any associated prime ideal \mathfrak{p} of M , we have $\dim(A/\mathfrak{p}) = \dim(M)$. This signifies that M has no embedded associated primes and that the irreducible components of M have the same dimension. As we have seen (??), every Cohen-Macaulay module over a Noetherian local ring is pure.

Lemma 17.2.11. *Let A be a Noetherian local ring, M be a finitely generated pure A -module, and x be an element of \mathfrak{m}_A . Then the following conditions are equivalent:*

- (i) $\dim(M/xM) = \dim(M) - 1$;
- (ii) the homothety x_M is injective.

Proof. We can suppose that M is nonzero. The assertion in (i) is equivalent to that fact that x does not belong to any minimal element \mathfrak{p} of $\text{supp}(M)$ such that $\dim(A/\mathfrak{p}) = \dim(M)$ (??), while the assertion in (ii) is equivalent to the fact that x does not belong to any associated prime ideal of M (??). Since M is pure, we see these two conditions are equivalent. \square

Let A be a Noetherian local ring and M be a finitely generated nonzero A -module. Recall that a sequence (x_1, \dots, x_r) of elements of \mathfrak{m}_A is said to be *secant* for M if we have

$$\dim(M/(x_1M + \dots + x_rM)) = \dim(M) - r.$$

Proposition 17.2.12. *Let A be a Noetherian local ring, M be a finitely generated nonzero A -module, and (x_1, \dots, x_r) be a sequence of elements of \mathfrak{m}_A that is secant for M . Then the following conditions are equivalent:*

- (i) the A -module M is Cohen-Macaulay;
- (ii) the sequence (x_1, \dots, x_r) is M -regular and the A -module $M/(x_1M + \dots + x_rM)$ is Cohen-Macaulay.

Proof. Suppose that the sequence (x_1, \dots, x_r) is M -regular. Then by ?? we have

$$\dim(M) = r + \dim(M/(x_1M + \dots + x_rM)), \quad \text{depth}(M) = r + \text{depth}(M/(x_1M + \dots + x_rM))$$

whence the implication (ii) \Rightarrow (i). Suppose now that the A -module M is Cohen-Macaulay, we prove (ii) by recurrence on r . The assertion is evident if $r = 0$, so assume that $r \geq 1$; the A -module $N = M/(x_1M + \dots + x_{r-1}M)$ is then Cohen-Macaulay by hypotheses and we have $\dim(N/x_rN) = \dim(N) - 1$ since the sequence (x_1, \dots, x_r) is secant. The hypothety $(x_r)_N$ is then injective by ??, and N/x_rN is Cohen-Macaulay by ??, whence (ii). \square

Theorem 17.2.13. *Let A be a Noetherian local ring, M be a finitely generated nonzero A -module of dimension d , $\mathbf{x} = (x_1, \dots, x_r)$ be a sequence of elements of \mathfrak{m}_A that is secant for M , and \mathfrak{I} be the ideal generated by \mathbf{x} . Then the following conditions are equivalent:*

- (i) the A -module M is Cohen-Macaulay;
- (ii) the sequence \mathbf{x} is regular for M ;
- (iii) the sequence \mathbf{x} is completely secant for M ;
- (iv) the multiplicity $e_{\mathfrak{I}}(M)$ of M relative to \mathfrak{I} is equal to the length of the A -module $M/\mathfrak{I}M$;
- (v) for each integer $1 \leq i \leq d$, the A -module $M/(x_1M + \dots + x_{i-1}M)$ is pure.

Proof. The equivalence of (ii) and (iii) follows from ?. The A -moduele $M/\mathfrak{I}M$ being of finite length (??), the equivalence of (iii) and (iv) follows from ??.

It remains to see that equivalence of (i), (ii) and (v). If M is Cohen-Macaulay, then each $M/(x_1M + \dots + x_{i-1}M)$ is Cohen-Macaulay by ??, hence pure. Conversely, by Noe local pure module secant element iff homothety injective, we see that (v) \Rightarrow (ii); the fact that (ii) \Rightarrow (i) follows from ??, since $M/\mathfrak{I}M$ is of finite length, hence Cohen-Macaulay. \square

17.2.2 Strongly secant subsets

Let A be a Noetherian ring, M be a finitely generated A -module, and S be a subset of A . We denote by SM the submodule $\sum_{s \in S} sM$ of M , and \mathfrak{S} the ideal generated by S .

Lemma 17.2.14. *Let $\bar{\mathfrak{S}}$ be the image of \mathfrak{S} in $A/\text{Ann}(M)$, then*

$$\text{ht}(\bar{\mathfrak{S}}) = \text{codim}(\text{supp}(M/SM), \text{supp}(M)).$$

If $M \neq SM$, then we have $\text{ht}(\bar{\mathfrak{S}}) \leq |S|$.

Proof. We denote by \mathfrak{a} the annihilator of M . By ??, the support of the A -module M/SM is equal to $V(\mathfrak{S} + \mathfrak{a})$, its codimension in $\text{supp}(M)$ is therefore equal to the codimension of $V(\mathfrak{S} + \mathfrak{a})$ in $V(\mathfrak{a})$, which is also the codimension of $V((\mathfrak{S} + \mathfrak{a})/\mathfrak{a})$ in $\text{Spec}(A/\mathfrak{a})$; this is exactly the height of $\bar{\mathfrak{S}}$. If $M \neq SM$, then the inequality $\text{ht}(\bar{\mathfrak{S}}) \leq |S|$ is evident if S is infinite, and follows from ?? if S is finite. \square

Let A be a Noetherian ring, M be a finitely generated A -module, and S be a subset of A . We say that the subset S is **strongly secant** for M if the reverse inequality of ?? is true, in other words, if we have

$$|S| \leq \text{codim}(\text{supp}(M/SM), \text{supp}(M)).$$

From this definition, we see that any finite subset S of A such that $SM = M$ is strongly secant for M , and if $SM \neq M$, then for a subset S to be strongly secant for M , it is necessary and sufficient that

$$|S| = \text{ht}(\bar{\mathfrak{S}}) = \text{codim}(\text{supp}(M/SM), \text{supp}(M)).$$

Example 17.2.15. If A is a Noetherian local ring and M is a nonzero finitely generated A -module, then any subset S of \mathfrak{m}_A that is strongly secant for M is secant for M . In fact, as the A -module M/SM is nonzero by Nakayama's lemma, we have

$$|S| \leq \text{codim}(\text{supp}(M/SM), \text{supp}(M)) \leq \dim(M) - \dim(M/SM)$$

(??), whence our assertion.

Proposition 17.2.16. *Let A be a Noetherian ring, M be a finitely generated A -module, and S be a finite subset of A . Then the following conditions are equivalent:*

- (i) S is strongly secant for M ;
- (ii) for any prime ideal \mathfrak{p} of $\text{supp}(M/SM)$, the canonical map $i_{\mathfrak{p}} : A \rightarrow A_{\mathfrak{p}}$ induces a bijection from S onto a subset of $\mathfrak{p}A_{\mathfrak{p}}$ that is secant for $M_{\mathfrak{p}}$.

Proof. Let $\mathfrak{p} \in \text{supp}(M/SM)$ and \tilde{S} be the image of S in $A_{\mathfrak{p}}$. The set \tilde{S} is contained in the maximal ideal $\mathfrak{p}A_{\mathfrak{p}}$, and we have ??

$$\dim(M_{\mathfrak{p}}/\tilde{S}M_{\mathfrak{p}}) = \text{codim}(V(\mathfrak{p}), \text{supp}(M/SM)).$$

If S is strongly secant for M , the inequality $|S| \leq \text{codim}(\text{supp}(M/SM), \text{supp}(M))$ and ??(b) implies the relations

$$|S| + \dim(M_{\mathfrak{p}}/\tilde{S}M_{\mathfrak{p}}) \leq \text{codim}(V(\mathfrak{p}), \text{supp}(M)) = \dim(M_{\mathfrak{p}}).$$

As $M_{\mathfrak{p}}$ is nonzero, we also have $\dim(M_{\mathfrak{p}}) \leq |\tilde{S}| + \dim(M_{\mathfrak{p}}/\tilde{S}M_{\mathfrak{p}})$ by formula (??). Condition (ii) then follows from the inequality $|\tilde{S}| \leq |S|$.

Conversely, assume condition (ii); we may suppose that $M \neq SM$. For any prime ideal $\mathfrak{p} \in \text{supp}(M/SM)$,

$$|S| = |\tilde{S}| \leq \dim(M_{\mathfrak{p}}) = \text{codim}(V(\mathfrak{p}), \text{supp}(M))$$

this implies (i) by passing to infimum, in view of the definition of codimensions. \square

Corollary 17.2.17. *Let A be a Noetherian ring and M be a finitely generated A -module. Then any M -regular sequence is strongly secant for M .*

Proof. Let x be a M -regular sequence, and \mathfrak{I} be the ideal it generates. For any prime ideal $\mathfrak{p} \in \text{supp}(M/\mathfrak{I}M)$, the image of x in $A_{\mathfrak{p}}$ is an M -regular sequence of elements of $\mathfrak{p}A_{\mathfrak{p}}$, so it is secant for $M_{\mathfrak{p}}$. \square

Proposition 17.2.18. *Let A be a Noetherian ring, M be a finitely generated Cohen-Macaulay A -module, and S be a finite subset of A that is strongly secant for M . Then the A -module M/SM is Cohen-Macaulay.*

Proof. For any maximal ideal $\mathfrak{m} \in \text{supp}(M/SM)$, the image of S in $A_{\mathfrak{m}}$ is secant for $M_{\mathfrak{m}}$ (??). Since $M_{\mathfrak{m}}$ is a Cohen-Macaulay $A_{\mathfrak{m}}$ -module (??), so is $(M/SM)_{\mathfrak{m}}$ by ??, whence the proposition. \square

Theorem 17.2.19 (Cohen-Macaulay). *Let A be a Noetherian ring and M be a finitely generated A -module. Then the following conditions are equivalent:*

- (i) *the A -module is Cohen-Macaulay;*
- (ii) *for any ideal \mathfrak{I} of A generated by an M -regular sequence of elements of A , the A -module $M/\mathfrak{I}M$ has no embedded associated prime ideals;*
- (iii) *for any finite subset S of A that is strongly secant for M , the A -module M/SM has no embedded associated prime ideals;*

Proof. If M is Cohen-Macaulay and S is a finite subset that is strongly secant for M , the A -module M/SM is Cohen-Macaulay by ??, and in particular it has no embedded associated primes by ?? This proves (i) \Rightarrow (iii), and (iii) \Rightarrow (ii) follows from ??.

Finally, to prove that (ii) \Rightarrow (i), let $\mathfrak{p} \in \text{supp}(M)$, we prove that the $A_{\mathfrak{p}}$ -module $M_{\mathfrak{p}}$ is Cohen-Macaulay by recurrence on the integer $\dim(M_{\mathfrak{p}})$. If $\dim(M_{\mathfrak{p}}) = 0$, then $M_{\mathfrak{p}}$ is of finite length, so is Cohen-Macaulay (??). Suppose that $\dim(M_{\mathfrak{p}}) > 0$, which means that \mathfrak{p} is not a minimal element of $\text{supp}(M)$. As M has no embedded associated primes, each element in $\text{Ass}(M)$ is minimal, so $V(\mathfrak{p}) \cap \text{Ass}(M) = \emptyset$ and there exists an element x of \mathfrak{p} such that the homothety x_M is injective (??). The homothety $x_{M_{\mathfrak{p}}}$ is then injective and we have $\dim(M_{\mathfrak{p}}/xM_{\mathfrak{p}}) < \dim(M_{\mathfrak{p}})$ by ?? By the recurrence hypotheses applied to the A -module M/xM and the prime ideal \mathfrak{p} of $\text{supp}(M/xM)$, the $A_{\mathfrak{p}}$ -module $M_{\mathfrak{p}}/xM_{\mathfrak{p}}$ is Cohen-Macaulay, which implies that the $A_{\mathfrak{p}}$ -module $M_{\mathfrak{p}}$ is Cohen-Macaulay (??). \square

17.2.3 Cohen-Macaulay rings

We say that a ring A is **Cohen-Macaulay** if it is Noetherian and the A -module A is Cohen-Macaulay.

Example 17.2.20 (Examples of Cohen-Macaulay rings).

- (a) Any Artinian ring is Cohen-Macaulay: this follows directly from ??.
- (b) A Cohen-Macaulay ring has no embedded associated primes (??). Conversely, let A is Noetherian ring of dimension ≤ 1 which has no embedded associated primes; for any nonempty finite strongly secant subset S of A , the A -module A/SA has dimension ≤ 0 , hence is Cohen-Macaulay; therefore A is Cohen-Macaulay by ?? . In particular, any reduced Noetherian ring of dimension ≤ 1 is Cohen-Macaulay (??).
- (c) A Noetherian normal ring of dimension ≤ 2 is Cohen-Macaulay by ?? . Conversely, let A be a Cohen-Macaulay ring whose local ring $A_{\mathfrak{p}}$ is integrally closed for any prime ideal \mathfrak{p} with $\text{ht}(\mathfrak{p}) \leq 1$; then A is normal by ??.
- (d) If A is Cohen-Macaulay, so is the localization $S^{-1}A$ for any multiplicative subset S of A (??). Conversely, it follows from our definition that, if $A_{\mathfrak{m}}$ is Cohen-Macaulay for any maximal ideal \mathfrak{m} of A , then A is Cohen-Macaulay.
- (e) Let A be a Noetherian ring and \mathfrak{I} be an ideal of A . For the ring A/\mathfrak{I} to be Cohen-Macaulay, it is necessary and sufficient that it is a Cohen-Macaulay A -module (??). On the other hand, if A is local and \mathfrak{I} is generated by an A -regular sequence, then A/\mathfrak{I} is Cohen-Macaulay if and only if A is (??).
- (f) For a Noetherian local ring to be Cohen-Macaulay, it is necessary and sufficient that it has a defining ideal that is generated by an A -regular sequence: this follows from ?? and the fact an A -regular sequence of \mathfrak{m}_A generates a defining ideal if and only if its length is equal to $\dim(A)$ (??). In particular, any Noetherian regular local ring is Cohen-Macaulay (??). More generally, the quotient of a Noetherian regular local ring A by an ideal generated by an A -regular sequence is Cohen-Macaulay (??)

Proposition 17.2.21. *For a Noetherian ring A , the following conditions are equivalent:*

- (i) *A is a Cohen-Macaulay ring;*
- (ii) *for any closed subset F of $\text{Spec}(A)$, we have $\text{depth}_F(A) = \text{codim}(F)$;*
- (iii) *any ideal \mathfrak{I} of A contains an A -regular sequence of length $\text{ht}(\mathfrak{I})$;*
- (iii') *any maximal ideal \mathfrak{m} of A contains an A -regular sequence of length $\text{ht}(\mathfrak{m})$;*
- (iv) *for any ideal \mathfrak{I} of A , we have $\text{Ext}_A^i(A/\mathfrak{I}, A) = 0$ for $i < \text{ht}(\mathfrak{I})$.*
- (iv') *for any maximal ideal \mathfrak{m} of A , we have $\text{Ext}_A^i(A/\mathfrak{m}, A) = 0$ for $i < \text{ht}(\mathfrak{m})$;*
- (v) *for any prime ideal \mathfrak{p} of A and any ideal \mathfrak{I} of $A_{\mathfrak{p}}$ generated by a maximal secant sequence for $A_{\mathfrak{p}}$, we have $e_{\mathfrak{I}}(A_{\mathfrak{p}}) = \ell(A_{\mathfrak{p}}/\mathfrak{I}A_{\mathfrak{p}})$.*

(v') for any maximal ideal \mathfrak{m} of A , there exists an ideal \mathfrak{I} of $A_{\mathfrak{m}}$, generated by a maximal secant sequence for $A_{\mathfrak{m}}$, satisfying $e_{\mathfrak{I}}(A_{\mathfrak{m}}) = \ell(A_{\mathfrak{m}}/\mathfrak{I}A_{\mathfrak{m}})$.

(vi) (**Cohen-Macaulay Criterion**) for any finite subset S of A such that the ideal \mathfrak{S} generated by S has height $|S|$, the A -module A/\mathfrak{S} has no embedded associated primes.

Proof. The equivalence of (i) and (ii) follows from ?? and the definition of depth, conditions (iii) and (iv) (resp. (iii') and (iv')) signifies that we have $\text{depth}_A(\mathfrak{I}, A) \geq \text{ht}(\mathfrak{I})$ for any ideal (resp. any maximal ideal) \mathfrak{I} of A . We then have

$$(i) \Leftrightarrow (ii) \Rightarrow (iii) \Leftrightarrow (iv) \Rightarrow (iii') \Leftrightarrow (iv').$$

But (iv') implies, for any maximal ideal \mathfrak{m} of A , $\text{Ext}_{A_{\mathfrak{m}}}^i(\kappa(\mathfrak{m}), A_{\mathfrak{m}}) = 0$ for $i < \dim(A_{\mathfrak{m}})$, whence $\text{depth}(A_{\mathfrak{m}}) = \dim(A_{\mathfrak{m}})$, and A is then Cohen-Macaulay. Finally, the equivalence of (i), (v) and (v') follows from ??, and that of (i) and (vi) from ?? \square

Remark 17.2.2. Let $\rho : A \rightarrow B$ be a ring homomorphism, and let $\mathfrak{p} \in \text{Spec}(A)$. Denote by \bar{B} the ring $\kappa(\mathfrak{p}) \otimes_A B$, which is identified with $S^{-1}B/\mathfrak{p}(S^{-1}B)$, where S is the multiplicative subset $\rho(A - \mathfrak{p})$ of B . The prime ideals of \bar{B} are then of the form $\mathfrak{P}\bar{B}$, where \mathfrak{P} is a prime ideal of B lying over \mathfrak{p} . For such a prime ideal \mathfrak{P} , we have $S \subseteq B - \mathfrak{P}$, so the local ring of \bar{B} at $\mathfrak{P}\bar{B}$ is identified with $B_{\mathfrak{P}}/\mathfrak{p}B_{\mathfrak{P}}$, which is $\kappa(\mathfrak{p}) \otimes_A B_{\mathfrak{P}}$.

Similarly, if N is an B -module, the $\bar{B}_{\mathfrak{P}\bar{B}}$ -module $(\kappa(\mathfrak{p}) \otimes_A N)_{\mathfrak{P}\bar{B}}$ is identified with $\kappa(\mathfrak{p}) \otimes_A N_{\mathfrak{P}}$. Suppose moreover that the B -module N is finitely generated, then by Nakayama's lemma, the condition $\kappa(\mathfrak{p}) \otimes_A N_{\mathfrak{P}} = 0$ is equivalent to $N_{\mathfrak{P}} = 0$. Therefore the support of \bar{B} -module $\kappa(\mathfrak{p}) \otimes_A N$ is formed by the prime ideals $\mathfrak{P}\bar{B}$, where \mathfrak{P} runs through the prime ideals of $\text{supp}_B(N)$ lying over \mathfrak{p} . In particular, for the module $\kappa(\mathfrak{p}) \otimes_A N$ to be nonzero, it is necessary and sufficient that there exists a prime ideal of $\text{supp}_B(N)$ lying over \mathfrak{p} .

Proposition 17.2.22. Let $\rho : A \rightarrow B$ be a homomorphism of Noetherian rings and N be a finitely generated B -module that is also a finitely generated A -module. For the A -module N to be Cohen-Macaulay, it is necessary and sufficient that the B -module N is Cohen-Macaulay and, for any couple $(\mathfrak{M}, \mathfrak{N})$ of maximal ideals of $\text{supp}_B(N)$ such that $\mathfrak{M}^c = \mathfrak{N}^c$, we have $\dim_{B_{\mathfrak{M}}}(N_{\mathfrak{M}}) = \dim_{B_{\mathfrak{N}}}(N_{\mathfrak{N}})$.

Proof. The A -module $B/\text{Ann}_B(N)$ is isomorphic to a sub- A -module of the finitely generated A -module $\text{End}_A(N)$, hence is finitely generated. By replacing A by $A/\text{Ann}_A(N)$ and B by $B/\text{Ann}_B(N)$, we may then reduce to the case where ρ is injective and B is a finite A -algebra, and where we have $\text{supp}_A(N) = \text{Spec}(A)$, $\text{supp}_B(N) = \text{Spec}(B)$. The map $f : \text{Spec}(B) \rightarrow \text{Spec}(A)$ induced by ρ is then surjective and a prime ideal \mathfrak{P} is maximal if and only if $f(\mathfrak{P})$ is maximal in A ??.

Let \mathfrak{m} be a maximal ideal of A . By ??, the prime ideals of the ring $B_{\mathfrak{m}}$ containing $\mathfrak{m}B_{\mathfrak{m}}$ are of the form $\mathfrak{P}B_{\mathfrak{m}}$ where \mathfrak{P} is an ideal of B (necessarily maximal) such that $f(\mathfrak{P}) = \mathfrak{m}$. By ?? and ??, have

$$\text{depth}_{A_{\mathfrak{m}}}(N_{\mathfrak{m}}) = \text{depth}_{B_{\mathfrak{m}}}(\mathfrak{m}B_{\mathfrak{m}}, N_{\mathfrak{m}}) = \inf_{\mathfrak{P} \in f^{-1}(\mathfrak{m})} \text{depth}_{B_{\mathfrak{P}}}(N_{\mathfrak{P}}),$$

while by ?? and ??,

$$\dim_{A_{\mathfrak{m}}}(N_{\mathfrak{m}}) = \dim_{B_{\mathfrak{m}}}(N_{\mathfrak{m}}) = \sup_{\mathfrak{P} \in f^{-1}(\mathfrak{m})} \dim_{B_{\mathfrak{P}}}(N_{\mathfrak{P}}).$$

As we have $\operatorname{depth}_{B_{\mathfrak{P}}}(N_{\mathfrak{P}}) \leq \dim_{B_{\mathfrak{P}}}(N_{\mathfrak{P}})$ for any $\mathfrak{P} \in f^{-1}(\mathfrak{m})$, the proposition then follows from these equalities. \square

Corollary 17.2.23. *Let $\rho : A \rightarrow B$ be a homomorphism of Noetherian ring. If B is a finite A -algebra and a Cohen-Macaulay A -module, then it is a Cohen-Macaulay ring. If ρ is also injective, we have $\operatorname{ht}(\mathfrak{a}^e) = \operatorname{ht}(\mathfrak{a})$ for any ideal \mathfrak{a} of A , and $\operatorname{ht}(\mathfrak{b}) = \operatorname{ht}(\mathfrak{b}^c)$ for any ideal \mathfrak{b} of B .*

Proof. The first assertion follows from ???. Suppose that ρ is injective, and let \mathfrak{a} be an ideal of A . We have $\operatorname{ht}(\mathfrak{a}) = \operatorname{depth}_A(\mathfrak{a}, B)$ and $\operatorname{ht}(\mathfrak{a}^e) = \operatorname{depth}_B(\mathfrak{a}^e, B)$ since the A -module B is Cohen-Macaulay with support equal to $\operatorname{Spec}(A)$???, and $\operatorname{depth}_A(\mathfrak{a}, B) = \operatorname{depth}_B(\mathfrak{a}^e, B)$ by ??, whence $\operatorname{ht}(\mathfrak{a}^e) = \operatorname{ht}(\mathfrak{a})$. Now if \mathfrak{b} is an ideal of B , then by the preceding arguments we have $\operatorname{ht}(\mathfrak{b}^c) = \operatorname{ht}(\mathfrak{b}^{ce})$. But \mathfrak{b}^{ce} is contained in \mathfrak{b} , hence has height smaller than $\operatorname{ht}(\mathfrak{b})$ and we have $\operatorname{ht}(\mathfrak{b}) \leq \operatorname{ht}(\mathfrak{b}^c)$ by ??(b). \square

Corollary 17.2.24. *Let A be a Noetherian integrally closed ring and B be a ring containing A . Suppose that B is a finitely generated torsion-free A -module. If B is a Cohen-Macaulay ring, the A -module B is Cohen-Macaulay.*

Proof. In fact, any prime ideals of B lying over the same ideal of A have the same height ???. We can then apply ?? with $N = B$. \square

Corollary 17.2.25. *Let A be an integrally closed ring, K be its fraction field, L be a finite K -algebra such that $[L : K]1_A$ is invertible in A , and B be a sub- A -algebra of L that is finite over A .*

- (a) *The sub- A -module $A1_B$ of B is a direct factor.*
- (b) *For any ideal \mathfrak{I} of A , we have $\operatorname{depth}_A(\mathfrak{I}, A) \geq \operatorname{depth}_B(\mathfrak{I}B, B)$.*
- (c) *If B is a Cohen-Macaulay ring, so is A .*

Proof. The K -linear map $\operatorname{tr}_{L/K} : L \rightarrow K$ sends B into A ???, so defines an A -linear map $t : B \rightarrow A$. For any $x \in A$, we have $t(x1_B) = [L : K]x$, whence the assertion of (a). By ??, we have $\operatorname{depth}_A(\mathfrak{I}, B) = \operatorname{depth}_B(\mathfrak{I}B, B)$; but by (a) and ??, we have $\operatorname{depth}_A(\mathfrak{I}, A) \geq \operatorname{depth}_A(\mathfrak{I}, B)$, whence (b).

If the ring B is Noetherian, so is A : in fact, by (a) we have $\mathfrak{a}B \cap A = \mathfrak{a}$ for any ideal \mathfrak{a} of A , so any increasing sequence (\mathfrak{a}_n) of ideals of A is stationary if the sequence (\mathfrak{a}_nB) is stationary. Under the hypothesis of (c), the A -module B is Cohen-Macaulay ???, and so is the A -module A ???. \square

Example 17.2.26. The result of ?? is applicable in the following situations:

- (a) Consider a Noetherian integrally closed ring A , a separable extension L of its fraction field, of finite degree n such that $n1_A$ is invertible in A , and let B be the integral closure of A in L ???.
- (b) Consider a Noetherian integrally closed ring B and a finite group G acting on B , such that $|G|$ is invertible in B . Let A be the ring of invariant elements of B under the action of G . Then we are in a particular case of (a): the group G acts on the fraction field L of B , and the invariant subfield of L for this action is the fraction field K of A ???. The extension L/K is Galois, and a fortiori separable; its Galois group is isomorphic to G , so $[L : K]$ is

equal to $|G|$. The inverse of $[L : K]1_B$ is invariant under G , so $[L : K]1_A$ is invertible in A . As B is integrally closed, the ring A , being equal to $K \cap B$, is then integrally closed and B is its integral closure in L (??).

In particular, if the ring B is Cohen-Macaulay, then so is the ring A .

17.2.4 Flat base change

Proposition 17.2.27. *Let $\rho : A \rightarrow B$ be a homomorphism of Noetherian rings, M be a finitely generated A -module and N be a finitely generated B -module that is flat over A . Let $f : \text{Spec}(B) \rightarrow \text{Spec}(A)$ be the associated map. The following conditions are equivalent:*

- (i) *the B -module $M \otimes_A N$ is Cohen-Macaulay;*
- (ii) *the $(\kappa(\mathfrak{p}) \otimes_A B)$ -module $\kappa(\mathfrak{p}) \otimes_A N$ is Cohen-Macaulay for any $\mathfrak{p} \in \text{supp}(M)$, and the $A_{\mathfrak{p}}$ -module $M_{\mathfrak{p}}$ is Cohen-Macaulay for any $\mathfrak{p} \in f(\text{supp}_B(N))$;*
- (iii) *for any maximal ideal of $\text{supp}_B(N)$ whose inverse image \mathfrak{p} in A belongs to $\text{supp}_A(M)$, the $A_{\mathfrak{p}}$ -module $M_{\mathfrak{p}}$ and the $(\kappa(\mathfrak{p}) \otimes_A B)$ -module $\kappa(\mathfrak{p}) \otimes_A N$ are Cohen-Macaulay.*

If the B -module is faithfully flat, these conditions imply that the A -module M is Cohen-Macaulay.

Proof. Let \mathfrak{P} be a prime ideal belonging to the support of $M \otimes_A N$, and put $\mathfrak{p} = \mathfrak{P}^c$. As the module $(M \otimes_A N)_{\mathfrak{P}}$ is identified with $M_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} N_{\mathfrak{P}}$, the modules $M_{\mathfrak{p}}$ and $N_{\mathfrak{P}}$ are necessarily nonzero, and so is the module $\kappa(\mathfrak{p}) \otimes_A N_{\mathfrak{P}}$ (??). The $A_{\mathfrak{p}}$ -module $N_{\mathfrak{P}}$, being isomorphic to a fraction module of $N_{\mathfrak{p}}$, is flat and by ??(b) and ?? we have the equalities

$$\begin{aligned}\text{depth}_{B_{\mathfrak{P}}}((M \otimes_A N)_{\mathfrak{P}}) &= \text{depth}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}) + \text{depth}_{B_{\mathfrak{P}}}(\kappa(\mathfrak{p}) \otimes_A N_{\mathfrak{P}}), \\ \dim_{B_{\mathfrak{P}}}((M \otimes_A N)_{\mathfrak{P}}) &= \dim_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}) + \dim_{B_{\mathfrak{P}}}(\kappa(\mathfrak{p}) \otimes_A N_{\mathfrak{P}}),\end{aligned}$$

where each term are nonnegative integers. In view of the fact that the $B_{\mathfrak{P}}$ -module $\kappa(\mathfrak{p}) \otimes_A N_{\mathfrak{P}}$ is Cohen-Macaulay if and only if so is it as a $(\kappa(\mathfrak{p}) \otimes_A B_{\mathfrak{P}})$ -module (??), we then duce the equivalence of the following conditions:

- (α) *the $B_{\mathfrak{P}}$ -module $(M \otimes_A N)_{\mathfrak{P}}$ is Cohen-Macaulay;*
- (β) *the $A_{\mathfrak{p}}$ -module $M_{\mathfrak{p}}$ and the $(\kappa(\mathfrak{p}) \otimes_A B_{\mathfrak{P}})$ -module $\kappa(\mathfrak{p}) \otimes_A N_{\mathfrak{P}}$ are Cohen-Macaulay.*

We now prove that (iii) implies (i). Let \mathfrak{N} be a maximal ideal of B belonging to the support of $M \otimes_A N$, and put $\mathfrak{p} = \mathfrak{N}^c$. By the preceding remarks, we have $\mathfrak{p} \in \text{supp}_A(M)$, so (iii) and ?? imply that the condition (β) above is satisfied for $\mathfrak{P} = \mathfrak{N}$, so the $B_{\mathfrak{N}}$ -module $(M \otimes_A N)_{\mathfrak{N}}$ is Cohen-Macaulay, whence (i).

The implication (ii) \Rightarrow (iii) is clear, so it remains to prove that (i) \Rightarrow (ii). Suppose that the B -module $M \otimes_A N$ is Cohen-Macaulay, and let \mathfrak{p} be an element of $\text{supp}_A(M)$. We can suppose that the $(\kappa(\mathfrak{p}) \otimes_A B)$ -module $\kappa(\mathfrak{p}) \otimes_A N$ is nonzero, which means there exists a prime ideal \mathfrak{P} of $\text{supp}_B(N)$ lying over \mathfrak{p} (??). The $B_{\mathfrak{P}}$ -module $(M \otimes_A N)_{\mathfrak{P}}$ is Cohen-Macaulay by hypotheses, so it follows from the implication (α) \Rightarrow (β) and ?? that the $A_{\mathfrak{p}}$ -module $M_{\mathfrak{p}}$ and the $(\kappa(\mathfrak{p}) \otimes_A B)$ -module $(\kappa(\mathfrak{p}) \otimes_A N)$ are Cohen-Macaulay, whence (ii).

If moreover N is faithfully flat over A , we have $\kappa(\mathfrak{p}) \otimes_A N \neq 0$ for any $p \in \text{Spec}(A)$, so $f(\text{supp}_B(N)) = \text{Spec}(A)$ by ??, and (ii) then implies that M is Cohen-Macaulay. \square

Corollary 17.2.28. Let $\rho : A \rightarrow B$ be a local homomorphism of Noetherian local rings, M be a finitely generated nonzero A -module and N be a finitely generated nonzero B -module which is flat over A . For the B -module $M \otimes_A N$ to be Cohen-Macaulay, it is necessary and sufficient that the A -module M and the $(\kappa_A \otimes_A B)$ -module $\kappa_A \otimes_A N$ are Cohen-Macaulay.

Proof. In fact, N is a faithfully flat A -module since $\kappa_A \otimes_A N$ is nonzero (??). \square

Corollary 17.2.29. Let A be a Noetherian ring, B be a finite and flat A -algebra, and M be a finitely generated Cohen-Macaulay A -module. Then the B -module $M \otimes_A B$ is Cohen-Macaulay.

Proof. For any $\mathfrak{p} \in \text{Spec}(A)$ the ring $\kappa(\mathfrak{p}) \otimes_A B$ is a finite $\kappa(\mathfrak{p})$ -algebra, hence Cohen-Macaulay, and we can apply ??.

Corollary 17.2.30. Let A be a Noetherian ring, \mathfrak{J} be an ideal of A and M be a finitely generated A -module. Let \widehat{A} and \widehat{M} be the \mathfrak{J} -adic completion of A and M , and S be the multiplicative subset $1 + \mathfrak{J}$ of A . Consider the following conditions:

- (i) the A -module M is Cohen-Macaulay;
- (ii) the \widehat{A} -module \widehat{M} is Cohen-Macaulay;
- (iii) the $S^{-1}A$ -module $S^{-1}M$ is Cohen-Macaulay;
- (iv) for any maximal ideal $\mathfrak{m} \in \text{supp}(M) \cap V(\mathfrak{J})$, the $A_{\mathfrak{m}}$ -module $M_{\mathfrak{m}}$ is Cohen-Macaulay;
- (v) for any prime ideal $\mathfrak{p} \in \text{supp}(M)$ not meeting S , the $A_{\mathfrak{p}}$ -module $M_{\mathfrak{p}}$ is Cohen-Macaulay and the ring $\kappa(\mathfrak{p}) \otimes_A \widehat{A}$ is Cohen-Macaulay.

Then conditions (ii) to (v) are equivalent, and are implied by (i). If the ideal \mathfrak{J} is contained in the Jacobson radical of A , then they are all equivalent.

Proof. We have seen that (i) implies (iii) in ??(d), and (iii) is identical to (i) if the ideal \mathfrak{J} is contained in the Jacobson radical of A (since the elements of S are then invertible). By ??, the ring \widehat{A} is Noetherian, and it is identified with the $S^{-1}\mathfrak{J}$ -adic completion of $S^{-1}A$ (??); similarly, \widehat{M} is identified with the $S^{-1}\mathfrak{J}$ -adic completion of $S^{-1}M$. To prove the equivalence of (ii) to (v), we can then replace A by $S^{-1}A$, \mathfrak{J} by $S^{-1}\mathfrak{J}$, and M by $S^{-1}M$. In other words, we can assume that \mathfrak{J} is contained in the Jacobson radical of A . The A -module \widehat{A} is then faithfully flat by ??.

Now it is clear that (v) implies (iv) and (iv) implies (i). Let \mathfrak{m} be a maximal ideal of A ; $\mathfrak{m}\widehat{A}$ is then a maximal ideal of \widehat{A} lying over \mathfrak{m} , and any maximal ideal of \widehat{A} is obtained in this way (??). The ring $\kappa(\mathfrak{m}) \otimes_A \widehat{A}$ is the residue field of \mathfrak{m} , hence Cohen-Macaulay. If the A -module M is Cohen-Macaulay, so is the \widehat{A} -module \widehat{M} (??); this proves (i) \Rightarrow (ii).

Finally, to see that (ii) \Rightarrow (v), note that \widehat{A} -module \widehat{M} is isomorphic to $M \otimes_A \widehat{A}$ (??); if it is Cohen-Macaulay, then by ?? we conclude that $\kappa(\mathfrak{p}) \otimes_A \widehat{A}$ is a Cohen-Macaulay ring for any $\mathfrak{p} \in \text{supp}(M)$, and that the A -module M is Cohen-Macaulay. \square

Proposition 17.2.31. Let $\rho : A \rightarrow B$ be a flat homomorphism of Noetherian rings. Then the following conditions are equivalent:

- (i) B is a Cohen-Macaulay ring;
- (ii) for any prime ideal \mathfrak{P} of B with $\mathfrak{p} = \mathfrak{P}^c$, the rings $A_{\mathfrak{p}}$ and $\kappa(\mathfrak{p}) \otimes_A B$ are Cohen-Macaulay;

(iii) for any maximal ideal \mathfrak{N} of B with $\mathfrak{p} = \mathfrak{N}^c$, the rings $A_{\mathfrak{p}}$ and $\kappa(\mathfrak{p}) \otimes_A B$ are Cohen-Macaulay;

If B is faithfully flat over A , these conditions imply that A is a Cohen-Macaulay ring.

Proof. This is a particular case of ??, where $M = A$ and $N = B$. \square

Corollary 17.2.32. Any finite and flat algebra over a Cohen-Macaulay ring is Cohen-Macaulay.

Corollary 17.2.33. Let A be a Cohen-Macaulay ring and $n > 0$ be a positive integer. Then the rings $A[X_1, \dots, X_n]$ and $A[\![X_1, \dots, X_n]\!]$ are Cohen-Macaulay.

Proof. It suffices to deal with the case where $n = 1$. The ring $A[T]$ is Noetherian by Hilbert basis theorem, and for any field k , the ring $k[T]$ is Cohen-Macaulay by ?? . The ring $A[T]$ is then Cohen-Macaulay by ?? and $A[\![T]\!]$ is Cohen-Macaulay by ?? . \square

Corollary 17.2.34. Any finitely generated algebra over a Cohen-Macaulay ring is catenary.

Proof. In fact, any such algebra is a quotient of a polynomial algebra over a Cohen-Macaulay ring, hence a quotient of Cohen-Macaulay ring, and therefore catenary (??). \square

17.3 Depth and homological dimension

17.3.1 Projective dimension and injective dimension

Let A be a ring, M be a A -module. Recall that the **projective dimension** of M , denoted by $\text{proj.dim}_A(M)$, is defined to be the infimum of the lengths of projective resolutions of M . We have $\text{proj.dim}_A(0) = -\infty$ and $\text{proj.dim}_A(M) \geq 0$ if M is nonzero. For the A -module M to be projective, it is necessary and sufficient that $\text{proj.dim}_A(M) \leq 0$.

Example 17.3.1. Let \mathfrak{J} be an ideal of A generated by an A -regular sequence $\mathbf{x} = (x_1, \dots, x_r)$. We have $\text{proj.dim}_A(A/\mathfrak{J}) \leq r$: in fact, this is clear if $A = 0$, and in the contrary case, the Koszul complex $K_{\bullet}(\mathbf{x}, A)$ is a free resolution of A/\mathfrak{J} of length r . Moreover, for any A -module N , the A -module $\text{Ext}_A^r(A/\mathfrak{J}, N)$ and $N/\mathfrak{J}N$ are isomorphic, so for that $\text{proj.dim}_A(A/\mathfrak{J}) = r$, it is necessary and sufficient that \mathfrak{J} is a proper ideal of A (A, X, p.134, prop.1 ??).

Similarly, we define the **injective dimension** of M , denoted by $\text{inj.dim}_A(M)$, to be the infimum of the lengths of injective resolutions of M . It is clear that we have $\text{inj.dim}_A(0) = -\infty$, and $\text{inj.dim}_A(M) \geq 0$ if $M \neq 0$. For the A -module M to be injective, it is necessary and sufficient that $\text{inj.dim}_A(M) \leq 0$.

Proposition 17.3.2. Let A be a ring, M be an A -module and $n \geq 0$ be an integer. The following conditions are equivalent:

- (i) $\text{inj.dim}_A(M) \leq n$;
- (ii) for any A -module N and any integer $i > n$, we have $\text{Ext}_A^i(N, M) = 0$;
- (iii) for any ideal \mathfrak{a} of A , we have $\text{Ext}_A^{n+1}(A/\mathfrak{a}, M) = 0$;

(iv) for any exact sequence of A -modules

$$0 \longrightarrow M \longrightarrow I^0 \longrightarrow \cdots \longrightarrow I^{n-1} \longrightarrow Q \longrightarrow 0$$

where the I^i are injective, the A -module Q is injective.

Proof. The implications (i) \Rightarrow (ii) \Rightarrow (iii) are clear from definition. In the situation of (iv), we have for any A -module N an isomorphism $\mathrm{Ext}_A^1(N, Q) \cong \mathrm{Ext}_A^{n+1}(N, M)$, so the hypothesis of (iii) implies that $\mathrm{Ext}_A^1(A/\mathfrak{a}, Q) = 0$ for any ideal \mathfrak{a} of A , and Q is therefore injective.

Finally, for (iv) \Rightarrow (i), consider the exact sequence

$$0 \longrightarrow M \longrightarrow I^0(M) \longrightarrow \cdots \longrightarrow I^{n-1}(M) \longrightarrow K^{n-1}(M) \longrightarrow 0$$

where $I^i(M)$ are free A -modules; if condition (iv) is satisfied, then the A -module $K^{n-1}(M)$ is injective, whence $\mathrm{inj.dim}_A(M) \leq n$. \square

Recall that the **homological dimension** (or **global dimension**) of the ring A , denoted by $\mathrm{gl.dim}(A)$, is the supremum of integers n such that there exists two A -modules M and N such that $\mathrm{Ext}_A^n(M, N)$ is nonzero. This is then also the supremum of the projective (or injective) dimensions of A -modules.

Proposition 17.3.3. *Let A a ring, M and N be A -modules, i an integer and S be a multiplicative subset of A . Then we have a canonical isomorphism of $S^{-1}A$ -modules*

$$S^{-1}\mathrm{Tor}_i^A(M, N) \rightarrow \mathrm{Tor}_{i-1}^{S^{-1}A}(S^{-1}M, S^{-1}N).$$

If the ring A is Noetherian and M is finitely generated, we have a canonical isomorphism

$$S^{-1}\mathrm{Ext}_A^i(M, N) \rightarrow \mathrm{Ext}_{S^{-1}A}^i(S^{-1}M, S^{-1}N).$$

Proof. As the A -module $S^{-1}A$ is flat, this follows from (A, X, p.110, prop.9 ??) and (A, X, p.111, prop.10 ??). \square

Corollary 17.3.4. *Let A be a ring, M and N be A -modules, and i be an integer.*

- (a) *The support of $\mathrm{Tor}_i^A(M, N)$ is contained in $\mathrm{supp}(M) \cap \mathrm{supp}(N)$, and so is the support of $\mathrm{Ext}_A^i(M, N)$ if A is Noetherian and M is finitely generated.*
- (b) *Suppose that A is Noetherian and the module M and N are finitely generated. If the A -module $M \otimes_A N$ is of finite length, so are the A -modules $\mathrm{Tor}_i^A(M, N)$ and $\mathrm{Ext}_A^i(M, N)$.*

Proof. If \mathfrak{p} is a prime ideal of A not belonging to $\mathrm{supp}(M) \cap \mathrm{supp}(N)$, then the modules $M_{\mathfrak{p}}$ and $N_{\mathfrak{p}}$ are nonzero, and this implies (a) in view of ???. Now for a finitely generated module over a Noetherian ring to be of finite length, it is necessary and sufficient that its support consists of maximal ideals (??). Under the hypotheses of (b), the A -modules $\mathrm{Tor}_i^A(M, N)$ and $\mathrm{Ext}_A^i(M, N)$ are finitely generated, so the assertions of (b) follow from (a) (??). \square

Proposition 17.3.5. *Let A be a Noetherian ring, M be a finitely generated A -module and N be an A -module. Then we have*

$$\text{proj.dim}_A(M) = \sup_{\mathfrak{p}} \text{proj.dim}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}), \quad \text{inj.dim}_A(M) = \sup_{\mathfrak{p}} \text{inj.dim}_{A_{\mathfrak{p}}}(N_{\mathfrak{p}}) \quad (17.3.1)$$

where \mathfrak{p} runs through prime (resp. maximal) ideals of A . Moreover, the map $\mathfrak{p} \mapsto \text{proj.dim}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}})$ on $\text{Spec}(A)$ is upper semi-continuous.

Proof. Let $n \geq 0$ be an integer, and suppose that we have $\text{proj.dim}_A(M) < n$. For any prime ideal \mathfrak{p} of A and any $A_{\mathfrak{p}}$ -module Q , the $A_{\mathfrak{p}}$ -module $\text{Ext}_{A_{\mathfrak{p}}}^n$ is isomorphic to $(\text{Ext}_A^n(M, Q))_{\mathfrak{p}}$ (??), hence is zero. In view of (A, X, p.134, prop.1 ??), we then deduce the inequality $\text{proj.dim}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}) \leq \text{proj.dim}_A(M)$. Suppose conversely that we have $\text{proj.dim}_{A_{\mathfrak{m}}}(M_{\mathfrak{m}}) < n$ for any maximal ideal \mathfrak{m} of A , and let R be an A -module. We have $(\text{Ext}_A^n(M, R))_{\mathfrak{m}} = 0$ for any \mathfrak{m} (??), so $\text{Ext}_A^n(M, R) = 0$, which implies $\text{proj.dim}_A(M) < n$. The first inequality of (??) then follows, and the second one can be proved similarly. As $\text{gl.dim}(A)$ is the supremum of injective dimensions of A -modules, the third one then follows.

Now let \mathfrak{p} be a prime ideal of A and $n = \text{proj.dim}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}})$; we want to show that there exists an open neighborhood U of \mathfrak{p} in $\text{Spec}(A)$ such that $\text{proj.dim}_{A_{\mathfrak{q}}}(M_{\mathfrak{q}}) \leq n$ for any $\mathfrak{q} \in U$. This is clear if $n = +\infty$, and if $n = -\infty$, this follows from the fact that $\text{supp}(M)$ is closed. Suppose now that n is finite and choose an exact sequence of A -modules

$$P_{n-1} \xrightarrow{d_{n-1}} P_{n-2} \longrightarrow \cdots \longrightarrow P_0 \xrightarrow{d_0} M \longrightarrow 0$$

where the P_i are free of finite rank. Put $P = \ker d_{n-1}$, which is an A -module of finite presentation. The $A_{\mathfrak{p}}$ -module $P_{\mathfrak{p}}$ is projective, so by ??, there exists an element f of $A - \mathfrak{p}$ such that A_f -module P_f is free; the $A_{\mathfrak{q}}$ -module $P_{\mathfrak{q}}$ is then free for $\mathfrak{q} \in D(f)$, which proves the second assertion. \square

Corollary 17.3.6. *For any multiplicative subset S of A , we have*

$$\text{proj.dim}_{S^{-1}A}(S^{-1}M) \leq \text{proj.dim}_A(M), \quad \text{inj.dim}_{S^{-1}A}(S^{-1}N) \leq \text{inj.dim}_A(N).$$

Corollary 17.3.7. *If $\text{proj.dim}_{A_{\mathfrak{m}}}(M_{\mathfrak{m}})$ is finite for any maximal ideal \mathfrak{m} of $\text{supp}(M)$, then we have $\text{proj.dim}_A(M) < +\infty$.*

Proof. In fact, the subspace X of $\text{supp}(M)$ formed by maximal ideals is quasi-compact (since $\text{Spec}(A)$ is Noetherian); the map $\mathfrak{m} \mapsto \text{proj.dim}_{A_{\mathfrak{m}}}(M_{\mathfrak{m}})$ from X to $\overline{\mathbb{R}}$ is upper semi-continuous, hence bounded. \square

17.3.2 Homological dimension of Noetherian rings

Let A be a Noetherian local ring and M be a finitely generated A -module. Recall that a resolution

$$\cdots \longrightarrow L_n \xrightarrow{d_n} L_{n-1} \longrightarrow \cdots \longrightarrow L_0 \xrightarrow{d_0} M \longrightarrow 0$$

of M is called a **minimal projective resolution** if each module L_i is free of finite rank, and if the complex $L \otimes_A \kappa_A$ has zero differential. For any integer $i \geq 0$, we then have (A, X, p103,

example 3)

$$[\mathrm{Ext}_A^i(M, \kappa_A) : \kappa_A] = [\mathrm{Tor}_i^A(M, \kappa_A) : \kappa_A] = \mathrm{rank}_A(L_i) \quad (17.3.2)$$

By (A, X, p.56, prop.10 ??), any finitely generated A -module admits a minimal projective resolution.

Proposition 17.3.8. *Let A be a Noetherian local ring, M be a finitely generated A -module and $n \geq 0$ be an integer. The following conditions are equivalent:*

- (i) $\mathrm{proj.dim}_A(M) < n$;
- (ii) $\mathrm{Tor}_n^A(M, \kappa_A) = 0$;
- (iii) $\mathrm{Ext}_A^n(M, \kappa_A) = 0$;
- (iv) any minimal projective resolution of M has length $< n$.

Proof. The assertions (i) \Rightarrow (ii) and (i) \Rightarrow (iii) are immediate. Let L be a minimal projective resolution of M ; if (ii) or (iii) are satisfied, we have $L_n = 0$ by ???. As any minimal projective resolution of M is isomorphic to L (A, X, p.54, prop.8 ??), we then deduce (iv). The implication (iv) \Rightarrow (i) is trivial. \square

Corollary 17.3.9. *Let A be a Noetherian local ring and $n \geq 0$ be an integer. The following conditions are equivalent:*

- (i) $\mathrm{gl.dim}(A) < n$;
- (ii) $\mathrm{Ext}_A^i(M, N) = 0$ and $\mathrm{Tor}_i^A(M, N) = 0$ for any couple (M, N) of A -modules and any integer $i \geq n$;
- (iii) $\mathrm{Tor}_n^A(\kappa_A, \kappa_A) = 0$;
- (iv) $\mathrm{Ext}_A^n(\kappa_A, \kappa_A) = 0$;
- (v) $\mathrm{proj.dim}_A(\kappa_A) < n$.

Proof. It is clear that (i) implies (ii) and (ii) implies (iii) and (iv). By ?? applied to the A -module κ_A , conditions (iii) and (iv) both implies (v). To see (v) \Rightarrow (i), we note that if $\mathrm{proj.dim}_A(\kappa_A) < n$, then $\mathrm{Tor}_n^A(M, \kappa_A) = 0$ for any A -module M ; then any finitely generated A -module has projective dimension $< n$ (??), so $\mathrm{gl.dim}(A) < n$. \square

Corollary 17.3.10. *For a Noetherian local ring, we have $\mathrm{gl.dim}(A) = \mathrm{proj.dim}_A(\kappa_A)$.*

Example 17.3.11. Let A be a local ring. Then the A -module $\mathrm{Tor}_1^A(\kappa_A, \kappa_A)$ is isomorphic to $\mathfrak{m}_A/\mathfrak{m}_A^2$ (A, X, p.72, example ??), so if A is Noetherian, for that $\mathrm{Tor}_1^A(\kappa_A, \kappa_A) = 0$, it is necessary and sufficient that $\mathfrak{m}_A = 0$, which means A is a field. ?? then implies that a Noetherian local ring with homological dimension being zero is a field.

Example 17.3.12. Let A be a Noetherian local ring, M be a finitely generated A -module with finite projective dimension n , and N be a nonzero finitely generated A -module. The A -module

$\mathrm{Ext}_A^n(M, N)$ is then nonzero: let L be a minimal projective resolution of M , and d be its differential. We then have an exact sequence

$$\mathrm{Hom}_A(L_{n-1}, N) \xrightarrow{\mathrm{Hom}(d_{n-1})} \mathrm{Hom}_A(L_n, N) \longrightarrow \mathrm{Ext}_A^n(M, N) \longrightarrow 0$$

As $d_n \otimes 1_{\kappa_A}$ is zero, we then deduce that tensoring with κ_A induces an isomorphism $\kappa_A \otimes_A \mathrm{Hom}_A(L_n, N) \rightarrow \kappa_A \otimes_A \mathrm{Ext}_A^n(M, N)$, whence in view of the formula ??,

$$[\kappa_A \otimes_A \mathrm{Ext}_A^n(M, N) : \kappa_A] = [\mathrm{Ext}_A^n(M, \kappa_A) : \kappa_A][\kappa_A \otimes_A N : \kappa_A];$$

which is nonzero by ?? and Nakayama's lemma. Therefore the projective dimension of M is the largest integer i such that $\mathrm{Ext}_A^i(M, N)$ is nonzero.

Example 17.3.13. Let A be a Noetherian ring, M be a finitely generated A -module with finite projective dimension, N be a finitely generated module with support equal to $\mathrm{Spec}(A)$. By ?? and ??, the projective dimension n of M is the largest integer i such that $\mathrm{Ext}_A^i(M, N) \neq 0$, and the support of the A -module $\mathrm{Ext}_A^n(M, N)$ is the set of elements $\mathfrak{p} \in \mathrm{Spec}(A)$ such that $\mathrm{proj.dim}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}) = n$.

Proposition 17.3.14. Let A be a Noetherian ring, M be a finitely generated A -module and $n \geq 0$ be an integer. Then the following conditions are equivalent:

- (i) $\mathrm{proj.dim}_A(M) < n$;
- (ii) for any maximal ideal \mathfrak{m} of A , we have $\mathrm{Ext}_A^n(M, A/\mathfrak{m}) = 0$ (resp. $\mathrm{Tor}_n^A(M, A/\mathfrak{m}) = 0$);
- (iii) for any maximal ideal \mathfrak{m} of A , we have $\mathrm{Ext}_{A_{\mathfrak{m}}}^n(M_{\mathfrak{m}}, A/\mathfrak{m}) = 0$ (resp. $\mathrm{Tor}_n^{A_{\mathfrak{m}}}(M_{\mathfrak{m}}, A/\mathfrak{m}) = 0$).

Proof. It is clear that (i) \Rightarrow (ii), and (ii) \Rightarrow (i) by ?. Finally, by ??, condition (iii) implies the inequality $\mathrm{proj.dim}_{A_{\mathfrak{m}}}(M_{\mathfrak{m}}) < n$ for any maximal ideal \mathfrak{m} of A , whence $\mathrm{proj.dim}_A(M) < n$ by ??.

Remark 17.3.1. Let A be a Noetherian ring and $n \geq 0$ be an integer. If \mathfrak{m} and \mathfrak{n} are two distinct maximal ideals of A , the A -modules $\mathrm{Ext}_A^n(A/\mathfrak{m}, A/\mathfrak{n})$ and $\mathrm{Tor}_n^A(A/\mathfrak{m}, A/\mathfrak{n})$ are then annihilated by $\mathfrak{m} + \mathfrak{n}$, whence is zero. By an argument similar to ??, we deduce from ?? the equivalence of the following conditions:

- (i) $\mathrm{gl.dim}_A(M) < n$;
- (ii) $\mathrm{Ext}_A^i(M, N) = 0$ and $\mathrm{Tor}_i^A(M, N) = 0$ for any couple (M, N) of A -modules and any integer $i \geq n$;
- (iii) $\mathrm{Tor}_n^A(A/\mathfrak{m}, A/\mathfrak{m}) = 0$ for any maximal ideal \mathfrak{m} of A ;
- (iv) $\mathrm{Ext}_A^n(A/\mathfrak{m}, A/\mathfrak{m}) = 0$ for any maximal ideal \mathfrak{m} of A ;
- (v) $\mathrm{proj.dim}_A(A/\mathfrak{m}) < n$ for any maximal ideal \mathfrak{m} of A .

In particular, we have $\mathrm{gl.dim}(A) = \sup_{\mathfrak{m}} \mathrm{proj.dim}_A(A/\mathfrak{m})$, where \mathfrak{m} runs through maximal ideals of A .

Proposition 17.3.15. *Let A be a Noetherian ring, N be an A -module, $n \geq 0$ be an integer. Then the following conditions are equivalent:*

- (i) $\text{inj.dim}_A(N) < n$;
- (ii) for any prime ideal \mathfrak{p} of A , we have $\text{Ext}_A^n(A/\mathfrak{p}, N) = 0$;
- (iii) for any prime ideal \mathfrak{p} of A , we have $\text{Ext}_{A_{\mathfrak{p}}}^n(\kappa(\mathfrak{p}), N_{\mathfrak{p}}) = 0$.

If the A -module N is finitely generated, these conditions are equivalent to:

- (iv) for any maximal ideal \mathfrak{m} of A , we have $\text{Ext}_A^i(A/\mathfrak{m}, N) = 0$ for $n \leq i \leq n + \text{ht}(\mathfrak{m})$.

Proof.

□

Remark 17.3.2. Let N be a finitely generated A -module; the condition $\text{Ext}_A^n(A/\mathfrak{m}, N) = 0$ for any maximal ideal \mathfrak{m} of A does not imply necessarily $\text{inj.dim}_A(N) < n$. For example, if A is local and not Gorenstein, we then have $\text{Ext}_A^n(A/\mathfrak{m}, A) = 0$ for $n < \text{depth}(A)$, but $\text{inj.dim}_A(A) = +\infty$.

Proposition 17.3.16. *Let A be a Noetherian ring, M be a finitely generated A -module. Then we have $\dim_A(M) \leq \text{inj.dim}_A(M)$.*

Proof. Let $r \leq \dim_A(M)$ be a positive integer. Then there exists a saturated chain of prime ideals $\mathfrak{p} \subseteq \mathfrak{p}_1 \subseteq \cdots \subseteq \mathfrak{p}_{r-1} \subseteq \mathfrak{q}$ such that \mathfrak{p} is a minimal element of $\text{supp}(M)$; the $A_{\mathfrak{p}}$ -module $M_{\mathfrak{p}}$ is then of finite length, so we have $\text{Hom}_{A_{\mathfrak{p}}}(\kappa(\mathfrak{p}), M_{\mathfrak{p}}) \neq 0$ (??), whence $\text{Ext}_{A_{\mathfrak{q}}}^r(\kappa(\mathfrak{q}), M_{\mathfrak{q}}) \neq 0$ (??); this implies $\text{inj.dim}_A(M) \geq r$ by ??.

□

Proposition 17.3.17. *Let A be a Noetherian local ring and M be a finitely generated nonzero A -module with finite injective dimension. Then we have $\text{inj.dim}_A(M) = \text{depth}(A)$.*

Proof. Put $r = \text{inj.dim}_A(M)$, we then have $\text{Ext}_A^i(\kappa_A, M) = 0$ for $i > r$, so $\text{Ext}_A^r(\kappa_A, M) \neq 0$ by ??(iv). Let $s = \text{depth}(A)$ and (x_1, \dots, x_s) be an A -regular sequence of elements of \mathfrak{m}_A ; put $N = A/(x_1A + \cdots + x_sA)$. By ??, we have $\text{proj.dim}_A(N) = s$ and $\text{Ext}_A^s(N, M) \neq 0$, so $s \leq \text{inj.dim}_A(M) = r$. But N is of zero depth (??), so there exists an exact sequence of A -modules

$$0 \longrightarrow \kappa_A \longrightarrow N \longrightarrow N' \longrightarrow 0$$

from which we deduce an exact sequence of extension modules

$$\text{Ext}_A^r(N, M) \longrightarrow \text{Ext}_A^r(\kappa_A, M) \longrightarrow \text{Ext}_A^{r+1}(N', M)$$

As we have $\text{Ext}_A^{r+1}(N', M) = 0$ and $\text{Ext}_A^r(\kappa_A, M) \neq 0$, we obtain that $\text{Ext}_A^r(N, M) \neq 0$, so $r \leq \text{proj.dim}_A(N) = s$; we then conclude that $r = s$, whence the proposition. □

17.3.3 Depth and projective dimension

Theorem 17.3.18 (Auslander-Buchsbaum). *Let A be a Noetherian local ring and M be a finitely generated A -module with finite projective dimension. Then we have*

$$\text{proj.dim}_A(M) + \text{depth}_A(M) = \text{depth}(A).$$

Proof. We prove by induction on $\text{proj.dim}_A(M)$. If $\text{proj.dim}_A(M) = 0$, then M is free of finite rank, so $\text{depth}_A(M) = \text{depth}(A)$ by [??](#). Now suppose that $\text{depth}_A(M) = 1$ and choose a minimal projective resolution of M :

$$0 \longrightarrow L_1 \xrightarrow{d_1} L_0 \xrightarrow{d_0} M \longrightarrow 0$$

The A -modules L_0 and L_1 are free of finite rank and nonzero, so their depths are equal to $\text{depth}(A)$ ([??](#)). The map $1_{\kappa_A} \otimes d_1 : \kappa_A \otimes_A L_1 \rightarrow \kappa_A \otimes_A L_0$ is zero, so d_1 belongs to $\mathfrak{m}_A \text{Hom}_A(L_1, L_0)$. By [??](#), we then have $\text{depth}_A(M) = \text{depth}(A) - 1$.

Finally, suppose that $\text{proj.dim}_A(M) > 1$. We choose an exact sequence

$$0 \longrightarrow N \longrightarrow L \longrightarrow M \longrightarrow 0$$

where L is a free A -module of finite rank. By [??](#), we then have $\text{depth}_A(L) = \text{depth}(A)$, and $\text{proj.dim}_A(N) = \text{proj.dim}_A(M) - 1$ in view of (A, X, p.135, cor.2(c) [??](#)); the induction hypotheses implies that $\text{depth}_A(N) = \text{depth}(A) - \text{proj.dim}_A(N)$, and in particular $\text{depth}_A(N) < \text{depth}_A(L)$. Now [??](#) implies that $\text{depth}(M) = \text{depth}_A(N) - 1$, which proves the assertion. \square

Remark 17.3.3. In view of [??](#), the formula [??](#) applied to the A -module κ_A implies that we are in exactly one of the following two cases:

- (i) $\text{proj.dim}_A(\kappa_A) = \text{gl.dim}(A) = +\infty$;
- (ii) $\text{proj.dim}_A(\kappa_A) = \text{gl.dim}(A) = \text{depth}(A) < +\infty$.

Later we shall see that case (ii) characterizes regular local rings.

Corollary 17.3.19. *Retain the hypotheses of [??](#).*

- (a) *We have $\text{proj.dim}_A(M) \leq \text{depth}(A)$, and for the equality holds, it is necessary and sufficient that $\mathfrak{m}_A \in \text{Ass}(M)$.*
- (b) *We have $\text{depth}_A(M) \leq \text{depth}(A)$, and for the equality holds, it is necessary and sufficient that M is free.*

Proof. This follows from the fact that $\text{depth}_A(M) = 0$ if and only if $\mathfrak{m}_A \in \text{Ass}(A)$ by [??](#), and $\text{proj.dim}_A(M) = 0$ if and only if M is projective, hence free. \square

Corollary 17.3.20. *Retain the hypotheses of [??](#) and suppose that A is a Cohen-Macaulay ring. Then $\text{proj.dim}_A(M)$ is the sum of the positive integers $\dim(A) - \dim_A(M)$ and $\dim_A(M) - \text{depth}(M)$.*

In particular, we have $\text{proj.dim}_A(M) \geq \dim(A) - \dim_A(M)$, and the equality holds if and only if M is Cohen-Macaulay.

Corollary 17.3.21. *Let A be a Noetherian ring, M be a finitely generated A -module with finite projective dimension, N be a finitely generated A -module, $i \geq 0$ be an integer, and F be the support of $\text{Ext}_A^i(M, N)$ (resp. $\text{Tor}_i^A(M, N)$). Then we have $\text{depth}_F(A) \geq i$.*

Proof. For $\mathfrak{p} \in F$ we have $\text{Ext}_{A_{\mathfrak{p}}}^i(M_{\mathfrak{p}}, N_{\mathfrak{p}}) \neq 0$ (resp. $\text{Tor}_i^{A_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}}) \neq 0$) by [??](#), so $i \leq \text{proj.dim}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}) \leq \text{proj.dim}_A(M) < +\infty$ in view of [??](#). The formula [??](#) then implies that $\text{depth}(A_{\mathfrak{p}}) \geq i$, and therefore $\text{depth}_F(A) = \inf_{\mathfrak{p} \in F} \text{depth}(A_{\mathfrak{p}}) \geq i$ ([??](#)). \square

With the terminologies of ??, the conclusion of ?? signifies that the modules $\text{Ext}_A^i(M, N)$ and $\text{Tor}_i^A(M, N)$ are of grade $\geq i$. This then implies that the codimension of their support in $\text{Spec}(A)$ is $\geq i$ (??).

Corollary 17.3.22. *Let A be a Noetherian Cohen-Macaulay ring and M be a finitely generated A -module with finite projective dimension.*

- (a) *For $\mathfrak{p} \in \text{Spec}(A)$, denote by $\mathcal{C}(\mathfrak{p})$ the set of irreducible components of $\text{supp}(M)$ containing \mathfrak{p} . Then we have*

$$\dim_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}) - \text{depth}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}) = \text{proj.dim}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}) - \inf_{X \in \mathcal{C}(\mathfrak{p})} \text{codim}(X, \text{Spec}(A)).$$

- (b) *The function $\mathfrak{p} \mapsto \dim_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}) - \text{depth}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}})$ on $\text{Spec}(A)$ is upper semi-continuous.*
- (c) *The set of prime ideals \mathfrak{p} of A such that the $A_{\mathfrak{p}}$ -module $M_{\mathfrak{p}}$ is Cohen-Macaulay is open and dense in $\text{Spec}(A)$. Its intersection with $\text{supp}(M)$ is dense in $\text{supp}(M)$.*

17.3.4 Gorenstein rings

We say that a ring A is **Gorenstein** if it is Noetherian and the $A_{\mathfrak{m}}$ -module $A_{\mathfrak{m}}$ is of finite injective dimension for any maximal ideal \mathfrak{m} of A . For a Noetherian local ring to be Gorenstein, it is then necessary and sufficient that $\text{inj.dim}_A(A) < +\infty$.

Proposition 17.3.23. *A Gorenstein ring A is Cohen-Macaulay and we have $\text{inj.dim}_A(A) = \dim(A)$.*

Proof. For any maximal ideal \mathfrak{m} of A , we have (??, ?? and ??)

$$\dim(A_{\mathfrak{m}}) \leq \text{inj.dim}_{A_{\mathfrak{m}}}(A_{\mathfrak{m}}), \quad \text{inj.dim}_{A_{\mathfrak{m}}}(A_{\mathfrak{m}}) = \text{depth}(A_{\mathfrak{m}}), \quad \text{depth}(A_{\mathfrak{m}}) \leq \dim(A_{\mathfrak{m}}).$$

It then follows that A is Cohen-Macaulay, and we have $\text{inj.dim}_A(A) = \dim(A)$ by passing to supremum (??). \square

Thus the Noetherian rings A such that $\text{inj.dim}_A(A)$ is finite are finite-dimensional Gorenstein rings (??), and the Noetherian rings such that the A -module A is injective are the Artinian Gorenstein rings.

Example 17.3.24. For any multiplicative subset S of a Gorenstein ring A , the fraction ring $S^{-1}A$ is Gorenstein: in fact, any maximal ideal of $S^{-1}A$ is of the form $S^{-1}\mathfrak{p}$, where \mathfrak{p} is a prime ideal of A not meeting S . Let \mathfrak{m} be a maximal ideal of A containing \mathfrak{p} , then the ring $B = (S^{-1}A)_{S^{-1}\mathfrak{p}}$ is isomorphic to $A_{\mathfrak{p}}$, hence is a fraction ring of $A_{\mathfrak{m}}$, and therefore satisfies $\text{inj.dim}_B(B) < +\infty$ (??).

Example 17.3.25. Let A be a Gorenstein ring and \mathfrak{I} be an ideal of A , generated by an A -regular sequence \mathbf{x} . The quotient ring A/\mathfrak{I} is a Gorenstein ring: for any maximal ideal of A containing \mathfrak{I} , the image of \mathbf{x} in $A_{\mathfrak{m}}$ is $A_{\mathfrak{m}}$ -regular and generates the ideal $\mathfrak{I}_{\mathfrak{m}}$, so $A_{\mathfrak{m}}/\mathfrak{I}_{\mathfrak{m}}$ is Gorenstein by (cor. de la prop. 7 n° ??). On the other hand, if A is a Noetherian local ring and \mathfrak{I} is an ideal generated by an A -regular sequence of elements of \mathfrak{m}_A , then A is Gorenstein if A/\mathfrak{I} is Gorenstein.

Example 17.3.26. Let A be a regular local ring, then A is Gorenstein. In fact, let \mathbf{x} be a system of parameters of A . Then \mathbf{x} is A -regular and generates the ideal \mathfrak{m}_A (??), so we can apply ??.

17.4 Regular rings

17.4.1 Homological properties of regular local rings

Proposition 17.4.1. *Let A be a Noetherian regular local ring of dimension n . Then $\text{gl.dim}(A) = n$, and for any integer $i \geq 0$,*

$$[\text{Ext}_A^i(\kappa_A, \kappa_A) : \kappa_A] = [\text{Tor}_i^A(\kappa_A, \kappa_A) : \kappa_A] = \binom{n}{i}$$

Proof. Let $\mathbf{x} = (x_1, \dots, x_n)$ be a system of parameters of A . The sequence \mathbf{x} generates \mathfrak{m}_A and is completely secant for A (??). The Koszul complex $K_{\bullet}(\mathbf{x}, A)$ is then a free resolution of κ_A , whose differentials are zero modulo \mathfrak{m}_A . For any integer $i \geq 0$, we have (formula ??))

$$[\text{Ext}_A^i(\kappa_A, \kappa_A) : \kappa_A] = [\text{Tor}_i^A(\kappa_A, \kappa_A) : \kappa_A] = \text{rank}_A(K_i(\mathbf{x}, A)) = \binom{n}{i}.$$

I then follows from ?? that we have $\text{proj.dim}_A(\kappa_A) \geq n$, whence $\text{proj.dim}_A(\kappa_A) = n$ and $\text{gl.dim}(A) = n$ by ??.

Proposition 17.4.2. *A Noetherian regular local ring is factorial.*

Proof. By ??, the minimal projective resolution of a finitely generated module over a Noetherian regular local ring has finite length. It then follows from ?? that this ring is factorial.

Proposition 17.4.3. *Let A be a Noetherian regular local ring and M be a finitely generated nonzero A -module. Then the projective dimension of M is finite and we have*

$$\text{proj.dim}_A(M) + \text{depth}_A(M) = \dim(A).$$

Proof. In fact, M has finite projective dimension by ??, and we have $\text{depth}(A) = \dim(A)$ since A is Cohen-Macaulay (??). We can therefore apply the formula ??.

Corollary 17.4.4. *We have $\text{proj.dim}_A(M) \geq \dim(A) - \dim(M)$, and the equality holds if and only if M is Cohen-Macaulay.*

Corollary 17.4.5. *For the A -module M to be free, it is necessary and sufficient it is Cohen-Macaulay and of dimension $\dim(A)$, or that it has depth $\geq \dim(A)$.*

Corollary 17.4.6. *Any finitely generated reflexive module over a Noetherian regular local ring of dimension 2 is free.*

Proof. In fact, a Noetherian regular local ring is integrally closed (??). The corollary then follows from ?? and ??.

Corollary 17.4.7. *Let $\rho : A \rightarrow B$ be a local homomorphism of Noetherian local rings. Suppose that A is regular and that B is a finitely generated A -module. Then we have $\text{proj.dim}_A(B) \geq \dim(A) - \dim(B)$, and the equality holds if and only if B is Cohen-Macaulay. In particular, for the ring B to be Cohen-Macaulay with dimension equal to $\dim(A)$, it is necessary and sufficient that the A -module B is free.*

Proof. In fact, we have $\dim(B) = \dim_A(B)$ by ??, so B is Cohen-Macaulay if and only if it is a Cohen-Macaulay A -module (??). It then suffices to apply ?? and ??.

17.4.2 Homological characterization for regularity

Theorem 17.4.8 (Serre). *For a Noetherian local ring to be regular, it is necessary and sufficient that it has finite homological dimension.*

Proof. We have seen that a Noetherian regular local ring has finite homological dimension. Conversely, let A be a Noetherian local ring with finite homological dimension n . By ?? and ??, we have

$$n = \text{gl.dim}(A) = \text{proj.dim}_A(\kappa_A) = \text{depth}(A).$$

If $n = 0$, the A -module κ_A is free, so $\mathfrak{m}_A = 0$ and A is a field. Suppose that $n > 0$ and we proceed by induction on n . Since $\text{depth}(A) > 0$, the ideal \mathfrak{m}_A is not associated with A (??), hence is not contained in the union of \mathfrak{m}_A^2 with the associated primes of A (??). By ??, we can therefore choose an element x of $\mathfrak{m}_A - \mathfrak{m}_A^2$ such that the homothety x_A is injective. Let B be the Noetherian local ring A/xA and consider the exact sequence

$$0 \longrightarrow \kappa_A \xrightarrow{i} \mathfrak{m}_A/x\mathfrak{m}_A \xrightarrow{p} \mathfrak{m}_B \longrightarrow 0$$

where i is the map induced by the homothety x_A on \mathfrak{m}_A and p is the canonical surjection. Since the class of x in the κ_A -vector space $\mathfrak{m}_A/\mathfrak{m}_A^2$ is nonzero, there exists an A -linear map $\phi : \mathfrak{m}_A \rightarrow \kappa_A$ with $\phi(x) = 1$; by passing to quotient, we then from ϕ a retraction of i , such that the preceding sequence splits. By (cor.2 de la prop.7 du §3, n4 ??) and (A, X, p.135, cor.1 ??), this implies the relations

$$\text{proj.dim}_B(\mathfrak{m}_B) \leq \text{proj.dim}_B(\mathfrak{m}_A/x\mathfrak{m}_A) = \text{proj.dim}_A(\mathfrak{m}_A) < +\infty.$$

(cor.2 de la prop.7 du §3, n4 ??) applied to the exact sequence $0 \rightarrow \mathfrak{m}_B \rightarrow B \rightarrow \kappa_B \rightarrow 0$ of B -modules then implies that $\text{proj.dim}_B(\kappa_B) < +\infty$. The ring B is then of finite homological dimension (??), and of depth $n - 1$ (??). It then follows from the induction hypotheses that B is regular, so A is regular by ??.

Therefore, if A is a Noetherian local ring, the following conditions are equivalent:

- (i) A is regular;
- (ii) the A -module κ_A has finite projective dimension;
- (iii) any finitely generated A -module has finite projective dimension.

Now we say that a ring A is **regular** if it is Noetherian and the local ring $A_{\mathfrak{m}}$ is regular for any maximal ideal \mathfrak{m} of A . From the above equivalence, it is not hard to derive the following characterization for regular rings.

Proposition 17.4.9. *Let A be a Noetherian ring. The following conditions are equivalent:*

- (i) A is regular;
- (ii) any finitely generated A -module has finite projective dimension;
- (iii) for any maximal ideal \mathfrak{m} of A , the projective dimension of A/\mathfrak{m} is finite;
- (iv) for any prime ideal \mathfrak{p} of A , the local ring $A_{\mathfrak{p}}$ is regular.

Proof. Let \mathfrak{p} be a prime ideal of A ; if the A -module A/\mathfrak{p} has finite projective dimension, so does the $A_{\mathfrak{p}}$ -module $\kappa(\mathfrak{p})$ (??), so the local ring $A_{\mathfrak{p}}$ is regular by Noe local regular iff finite gldim. We then reduce that (ii) \Rightarrow (iv) and (iii) \Rightarrow (i). The implications (iv) \Rightarrow (i) and (ii) \Rightarrow (iii) are clear.

To see that (i) \Rightarrow (ii), let M be a fintely generated A -module. Under the hypotheses of (i), we have $\text{proj.dim}_{A_{\mathfrak{m}}}(M_{\mathfrak{m}}) \leq \text{gldim}(A_{\mathfrak{m}}) < +\infty$ for any maximal ideal \mathfrak{m} of A (??), so M has finite projective dimension (??), whence (ii). \square

Example 17.4.10. If a ring A is regular, then so is $S^{-1}A$ for any multiplicative subset S of A : this follows for example by the characterization (iii) of ??.

Example 17.4.11. For a ring to be regular, it is necessary and sufficient that it is isomorphic to the product of a finite family of regular integral domains. This follows from the fact that any regular ring is locally integral, since regular local rings are integral.

Corollary 17.4.12. Let A be a Noetherian ring. Then the following conditions are equivalent:

- (i) $\text{gldim}(A) < +\infty$;
- (ii) A is regular and $\dim(A) < +\infty$.

If these conditions are satisfied, we have $\dim(A) = \text{gldim}(A)$.

Proof. If A is regular, for any maximal ideal \mathfrak{m} of A we have $\dim(A_{\mathfrak{m}}) = \text{gldim}(A_{\mathfrak{m}})$ (??), and then (?? and ??)

$$\text{gldim}(A) = \sup_{\mathfrak{m}} \text{gldim}(A_{\mathfrak{m}}) = \sup_{\mathfrak{m}} \dim(A_{\mathfrak{m}}) = \dim(A).$$

On the other hand, if $\text{gldim}(A) < +\infty$, the ring A is regular by ??, so the corollary follows. \square

Corollary 17.4.13. A regular ring is normal, Gorenstein and Cohen-Macaulay.

Proof. In fact, a regular local ring is integrally closed (??), Gorenstein (??), and Cohen-Macaulay (??(f)). \square

Corollary 17.4.14. Let A be a Noetherian ring, \mathfrak{I} be an ideal of A and \widehat{A} be the \mathfrak{I} -adic completion of A .

- (a) For the ring \widehat{A} to be regular, it is necessary and sufficient that, for any maximal ideal \mathfrak{m} of A containing \mathfrak{I} , the ring $A_{\mathfrak{m}}$ is regular.
- (b) If the ring A is regular, then \widehat{A} is regular. If the ring \widehat{A} is regular and \mathfrak{I} is contained in the Jacobson radical of A , then A is regular.

Proof. By ??, for the ring \widehat{A} to be regular, it is necessary and sufficient that $\widehat{A}_{\mathfrak{m}}$ is regular for any maximal ideal \mathfrak{m} of A containing \mathfrak{I} . As the local rings $\widehat{A}_{\mathfrak{m}}$ and $A_{\mathfrak{m}}$ are isomorphic, assertion of (a) then follows from ??, and (b) follows from (a). \square

Corollary 17.4.15. Let A be a regular ring and P be a fintely generated projective A -module. Then the symmetric algebra $S_A(P)$ is regular.

Proof. Let \mathfrak{P} be a prime ideal of $S_A(P)$ and \mathfrak{p} be its contraction in A . The local ring $S_A(P)_{\mathfrak{P}}$ is then a fraction field of $S_A(P)_{\mathfrak{p}}$, which is isomorphic to $S_{A_{\mathfrak{p}}}(P)_{\mathfrak{p}}$ by (A, III, p.72, prop.7 ??); it then suffices to prove that the later is regular. This allow us to reduce to the case where A is local, and then P is free of finite rank. By ?? and (A, X, p.143, cor.1 ??), we have

$$\mathrm{gl.dim}(S_A(P)) = \mathrm{gl.dim}(A) + \mathrm{rank}_A(P) < +\infty$$

and $S_A(P)$ is therefore regular by ??.

□

Corollary 17.4.16. *Let A be a regular ring and $(T_i)_{i \in I}$ be a finite family of indeterminates. Then rings $A[(T_i)_{i \in I}]$ and $A[\![(T_i)_{i \in I}]\!]$ are regular.*

Proof. This follows from ?? and ??(b).

□

Proposition 17.4.17. *Let $\rho : A \rightarrow B$ be a homomorphism of Noetherian rings and N be a B -module. Suppose that*

- (a) *the ring A is regular,*
- (b) *N is a finitely generated A -module,*
- (c) *the B -module N is Cohen-Macaulay,*
- (d) *any minimal prime ideal of $\mathrm{supp}_B(N)$ is lying over a minimal prime ideal of A .*

Then N is a (finitely generated) projective A -module.

Proof. It suffices to prove that, for any maximal ideal \mathfrak{m} of A , the $A_{\mathfrak{m}}$ -module $N_{\mathfrak{m}}$ is free (??). The A -module $B/\mathrm{Ann}_B(N)$ is isomorphic to a sub- A -module of the finitely generated A -module $\mathrm{End}_A(N)$, hence is finitely generated. By replacing B by $B/\mathrm{Ann}_B(N)$, we may then reduce to the case where B is a finite A -algebra and where $\mathrm{supp}_B(N) = \mathrm{Spec}(B)$.

Let \mathfrak{m} be a maximal ideal of $\mathrm{supp}_A(N)$, and put $n = \dim(A_{\mathfrak{m}})$. By ??, it suffices to prove that $N_{\mathfrak{m}}$ is a Cohen-Macaulay $A_{\mathfrak{m}}$ -module of dimension n . By ??, any maximal ideal of $B_{\mathfrak{m}}$ is of the form $\mathfrak{N}B_{\mathfrak{m}}$, where \mathfrak{N} is a maximal ideal of B lying over \mathfrak{m} . Let \mathfrak{P} be a minimal prime ideal of $\mathrm{supp}_B(N)$ which is contained in \mathfrak{N} . The closed subset $V(\mathfrak{P}B_{\mathfrak{N}})$ of $\mathrm{supp}_{B_{\mathfrak{N}}}(N_{\mathfrak{N}})$ then has zero codimension, and since the $B_{\mathfrak{N}}$ -module $N_{\mathfrak{N}}$ is Cohen-Macaulay, ?? implies that $\dim_{B_{\mathfrak{N}}}(N_{\mathfrak{N}}) = \dim(B_{\mathfrak{N}}/\mathfrak{P}B_{\mathfrak{N}})$. But \mathfrak{P} is lying over a minimal prime of A which is contained in \mathfrak{m} , so $\mathfrak{P}B_{\mathfrak{N}}$ is lying over a minimal prime ideal of $A_{\mathfrak{m}}$, which is zero since the local ring $A_{\mathfrak{m}}$ is regular (hence integral). The canonical map $A_{\mathfrak{m}} \rightarrow B_{\mathfrak{N}}/\mathfrak{P}B_{\mathfrak{N}}$ is then injective and it follows from ?? that we have

$$\dim_{B_{\mathfrak{N}}}(N_{\mathfrak{N}}) = \dim(B_{\mathfrak{N}}/\mathfrak{P}B_{\mathfrak{N}}) = \dim(A_{\mathfrak{m}}) = n.$$

In view of this, ?? then implies that the $A_{\mathfrak{m}}$ -module $N_{\mathfrak{m}}$ is Cohen-Macaulay, and the proposition then follows from the relations $\dim_{A_{\mathfrak{m}}}(N_{\mathfrak{m}}) = \dim_{B_{\mathfrak{m}}}(N_{\mathfrak{m}})$ (??) and $\dim_{B_{\mathfrak{m}}}(N_{\mathfrak{m}}) \geq \dim_{B_{\mathfrak{N}}}(N_{\mathfrak{N}}) = n$. □

Corollary 17.4.18. *Let B be an integral Noetherian ring and A be a subring of B which is regular. Suppose that B is a finie A -algebra, then for the ring B to be Cohen-Macaulay, it is necessary and sufficient that the A -module B is projective.*

Proof. If the ring B is Cohen-Macaulay, the A -module B is projective by [??](#). Conversely, assume that the A -module B is projective. We know that the A -module A is Cohen-Macaulay ([??](#)), and the A -module B is a direct factor of a free A -module of finite rank, hence Cohen-Macaulay ([??](#)), so we can apply [??](#). \square

Example 17.4.19. Let B be an integral algebra of finite type over a field k . By the normalization lemma ([??](#)), there exists a subring A of B which is isomorphic to a polynomial algebra and such that B is finite over A . By [??](#), we see that B is a Cohen-Macaulay ring if and only if the A -module B is projective (hence free).

17.4.3 Flat base change

Proposition 17.4.20. *Let $\rho : A \rightarrow B$ be a homomorphism of Noetherian rings such that B is a faithfully flat A -module.*

- (a) *For any finitely generated A -module M , we have $\text{proj.dim}_A(M) = \text{proj.dim}_B(B \otimes_A M)$.*
- (b) *If the ring B is regular, so is A .*

Proof. The B -module $B \otimes_A M$ is finitely generated, and for it to be nonzero, it is necessary and sufficient that M is nonzero ([??](#)). For it to be projective, it is necessary and sufficient that M is a projective A -module ([??](#)). This proves (a) if $\text{proj.dim}_A(M) \leq 0$, so suppose that $\text{proj.dim}_A(M) \geq 1$ (whence $\text{proj.dim}_B(B \otimes_A M) \geq 1$) and we proceed by induction on $\text{proj.dim}_A(M)$. Choose an exact sequence of A -modules

$$0 \longrightarrow N \longrightarrow L \longrightarrow M \longrightarrow 0$$

where L is a free A -module of finite rank. We have $\text{proj.dim}_A(N) = \text{proj.dim}_A(M) - 1$ (A, X, p.135, cor.2 de la prop.1). As B is flat over A , the sequence

$$0 \longrightarrow B \otimes_A N \longrightarrow B \otimes_A L \longrightarrow B \otimes_A M \longrightarrow 0$$

is exact and we have $\text{proj.dim}_B(B \otimes_A N) = \text{proj.dim}_B(B \otimes_A M) - 1$. Assertion (a) then follows from the induction hypotheses applied to N , and (b) follows from (a) and [??](#). \square

Corollary 17.4.21. *Let B be a regular domain and A be a Noetherian subring of B such that B is finite over A . Then the following conditions are equivalent:*

- (i) *A is regular;*
- (ii) *B is a projective A -module;*
- (iii) *B is a flat A -module;*
- (iv) *B is a faithfully flat A -module.*

Proof. The implication (i) \Rightarrow (ii) follows from [??](#), and (ii) \Rightarrow (iii) is clear. For any prime ideal \mathfrak{p} of A , we have $\mathfrak{p}^e \neq B$ ([??](#)), so it suffices to apply ([??](#)) to see that (iii) \Rightarrow (iv). Finally, (iv) implies (i) by [??\(b\)](#). \square

For any Noetherian local ring A , we denote by $\delta(A)$ the integer

$$\delta(A) = [\mathfrak{m}_A/\mathfrak{m}_A^2 : \kappa_A] - \dim(A).$$

Recall that $\delta(A)$ is always positive, and its vanishing characterize regular local rings. Now let $\rho : A \rightarrow B$ be a local homomorphism of Noetherian local rings; we then have a κ_A -linear homomorphism $\mathfrak{m}_A/\mathfrak{m}_A^2 \rightarrow \mathfrak{m}_B/\mathfrak{m}_B^2$, whence a κ_B -linear homomorphism

$$d\rho : \kappa_B \otimes_{\kappa_A} (\mathfrak{m}_A/\mathfrak{m}_A^2) \rightarrow \mathfrak{m}_B/\mathfrak{m}_B^2.$$

Lemma 17.4.22. *Under the above hypotheses, we have*

$$\delta(B) + [\ker d\rho : \kappa_B] = \delta_A + \delta(\kappa_A \otimes_A B) + (\dim(A) - \dim(B) + \dim(\kappa_A \otimes_A B)).$$

Proof. Let \bar{B} be the local ring $\kappa_A \otimes_A B$. Since $\mathfrak{m}_{\bar{B}} = \mathfrak{m}_B/\mathfrak{m}_A B$, we have the exact sequence of B -modules

$$B \otimes_A \mathfrak{m}_A \longrightarrow \mathfrak{m}_B \longrightarrow \mathfrak{m}_{\bar{B}} \longrightarrow 0$$

By tensoring with κ_B over B , we then obtain an exact sequence of κ_B -vector spaces

$$\kappa_B \otimes_{\kappa_A} (\mathfrak{m}_A/\mathfrak{m}_A^2) \xrightarrow{d\rho} \mathfrak{m}_B/\mathfrak{m}_B^2 \longrightarrow \mathfrak{m}_{\bar{B}}/\mathfrak{m}_{\bar{B}}^2 \longrightarrow 0 \quad (17.4.1)$$

where we use the equality

$$\kappa_B \otimes_B B \otimes_A \mathfrak{m}_A = \kappa_B \otimes_A \mathfrak{m}_A = \kappa_B \otimes_{\kappa_A} \kappa_A \otimes_A \mathfrak{m}_A = \kappa_B \otimes_{\kappa_A} (\mathfrak{m}_A/\mathfrak{m}_A^2).$$

Since $[\mathfrak{m}_{\bar{B}}/\mathfrak{m}_{\bar{B}}^2 : \kappa_B] = [\mathfrak{m}_B/\mathfrak{m}_B^2 : \kappa_B]$ and $[\kappa_B \otimes_{\kappa_A} (\mathfrak{m}_A/\mathfrak{m}_A^2) : \kappa_B] = [\mathfrak{m}_A/\mathfrak{m}_A^2 : \kappa_A]$, from ?? we then deduce the equality

$$[\ker d\rho : \kappa_B] + [\mathfrak{m}_B/\mathfrak{m}_B^2 : \kappa_B] = [\mathfrak{m}_A/\mathfrak{m}_A^2 : \kappa_A] + [\mathfrak{m}_{\bar{B}}/\mathfrak{m}_{\bar{B}}^2 : \kappa_{\bar{B}}],$$

which proves the lemma. □

Proposition 17.4.23. *Let $\rho : A \rightarrow B$ be a local homomorphism of Noetherian local rings. The following conditions are equivalent:*

- (i) B is regular and the κ_B -linear map $d\rho : \kappa_B \otimes_{\kappa_A} (\mathfrak{m}_A/\mathfrak{m}_A^2) \rightarrow \mathfrak{m}_B/\mathfrak{m}_B^2$ is injective;
- (ii) B and $\kappa_A \otimes_A B$ are regular and the A -module B is flat;
- (iii) A and $\kappa_A \otimes_A B$ are regular and the A -module B is flat.
- (iv) A and $\kappa_A \otimes_A B$ are regular and we have $\dim(B) = \dim(A) + \dim(\kappa_A \otimes_A B)$.

Proof. We have $\dim(B) \leq \dim(A) + \dim(\kappa_A \otimes_A B)$ by ??; the equivalence of (i) and (iv) then follows from ?. Under the hypotheses of (ii), the A -module B is faithfully flat since ρ is a local homomorphism and this implies (iii) by ?. The implication (iii) \Rightarrow (iv) follows from ??.

Now it suffices to prove that if the equivalent conditions of (i) and (iv) are satisfied, then the A -module B is flat. To this end, let x be a system of parameters of A . Since $d\rho$ is injective, the image of x under ρ can be extended into a system of parameters of B . The sequence x is then

completely secant for A and for B (??), and generates the ideal \mathfrak{m}_A of A . By ??, the A -module $\mathrm{Tor}_1^A(\kappa_A, B)$ is then isomorphic to $H_1(\mathbf{x}, B)$, hence is zero. It then follows that B is flat over A (?? and ??). \square

Example 17.4.24. Let X and Y be finite dimensional complex analytic varieties, $f : X \rightarrow Y$ be a morphism, and x be a point of X . Consider the local homomorphism $\rho : \mathcal{O}_{Y,f(x)} \rightarrow \mathcal{O}_{X,x}$ induced by f . The map $d\rho$ is then the tangent map $d_x f : T_x X \rightarrow T_{f(x)} Y$. The equivalent conditions of ?? are therefore equivalent in this case to the fact that f is a submersion at x .

Corollary 17.4.25. Let $\rho : A \rightarrow B$ be a flat homomorphism of Noetherian rings. If A is regular and $\kappa(\mathfrak{N}^c) \otimes_A B$ is regular for any maximal ideal \mathfrak{N} of B , then the ring B is regular.

Proof. In fact, for any maximal ideal \mathfrak{N} of B , the $A_{\mathfrak{N}^c}$ -module $B_{\mathfrak{N}}$ is flat (??), so the ring $B_{\mathfrak{N}}$ is regular by ??.

17.5 Complete intersections

17.5.1 Completely secant ideals

Let A be a ring and \mathfrak{I} be an ideal of A . We say that the ideal \mathfrak{I} is **completely secant at a point** \mathfrak{p} of $V(\mathfrak{I})$ if the ideal $\mathfrak{I}_{\mathfrak{p}}$ of $A_{\mathfrak{p}}$ is generated by a completely secant sequence for $A_{\mathfrak{p}}$. We say that \mathfrak{I} is **completely secant** if it is completely secant at every point of $V(\mathfrak{I})$.

If the ideal \mathfrak{I} is completely secant, so is the ideal $S^{-1}\mathfrak{I}$ of $S^{-1}A$ for any multiplicative subset S of A . Any ideal generated by a completely secant sequence is completely secant. More precisely, we have the following:

Proposition 17.5.1. Let A be a ring, \mathfrak{I} be an ideal of A generated by a finite sequence $\mathbf{x} = (x_1, \dots, x_r)$ of elements of A . Then the following conditions are equivalent:

- (i) the sequence \mathbf{x} is completely secant for A ;
- (ii) for any prime (resp. maximal) ideal $\mathfrak{p} \in V(\mathfrak{I})$, the image of \mathbf{x} in $A_{\mathfrak{p}}$ is completely secant for $A_{\mathfrak{p}}$;
- (iii) for any prime (resp. maximal) ideal $\mathfrak{p} \in V(\mathfrak{I})$, the ideal \mathfrak{I} is completely secant at \mathfrak{p} and the image of \mathbf{x} in the $\kappa(\mathfrak{p})$ -vector space $\kappa(\mathfrak{p}) \otimes_A \mathfrak{I}$ form a basis.

If A is Noetherian, these conditions are equivalent to:

- (iv) for any integer $n \geq 0$, the A/\mathfrak{I} -module $\mathfrak{I}^n/\mathfrak{I}^{n+1}$ is free and the image of the monomials $x_1^{\alpha_1} \cdots x_r^{\alpha_r}$ of total degree n form a basis.

Proof. Let \mathfrak{p} be a prime ideal of A , and denote by $\mathbf{x}_{\mathfrak{p}}$ the image of \mathbf{x} in $A_{\mathfrak{p}}$. For any integer $i \geq 0$, the $A_{\mathfrak{p}}$ -module $H_i(\mathbf{x}_{\mathfrak{p}}, A_{\mathfrak{p}})$ is isomorphic to $(H_i(\mathbf{x}, A))_{\mathfrak{p}}$ (??(b)); it is then zero if \mathfrak{p} does not contain \mathfrak{I} , since we then have $\mathfrak{I}_{\mathfrak{p}} = A_{\mathfrak{p}}$ (??). The equivalence of (ii) and (iii) then follows from ??(b). On the other hand, the equivalence of (ii) and (iii) follows from ??.

The implication (i) \Rightarrow (iv) is a consequence of ??(b). Finally, for any $\mathfrak{p} \in V(\mathfrak{I})$, condition (iv) implies by localization the analogous results for the $(A_{\mathfrak{p}}/\mathfrak{I}_{\mathfrak{p}})$ -module $\mathfrak{I}_{\mathfrak{p}}^n/\mathfrak{I}_{\mathfrak{p}}^{n+1}$, and this implies (i) by ??.

Remark 17.5.1. If A is Noetherian, we can replace "completely secant" by "regular" in condition (ii) of [??](#). However, this is not the case in condition (i): a completely secant sequence for A is not necessarily A -regular (exercise 1).

Remark 17.5.2. Let A be a ring, \mathfrak{I} be a finitely generated ideal of A and \mathfrak{p} be a prime ideal of A containing \mathfrak{I} . By [??](#), we have

$$\operatorname{depth}_{A_{\mathfrak{p}}}(\mathfrak{I}_{\mathfrak{p}}, A_{\mathfrak{p}}) \leq [\kappa(\mathfrak{p}) \otimes_A \mathfrak{I} : \kappa(\mathfrak{p})],$$

and the equality holds if and only if \mathfrak{I} is completely secant at \mathfrak{p} . Suppose that \mathfrak{I} is proper and generated by a completely secant sequence (x_1, \dots, x_r) . We then have ([??](#) and [??](#))

$$\operatorname{depth}_A(\mathfrak{I}, A) = \inf_{\mathfrak{p} \in V(\mathfrak{I})} \operatorname{depth}_{A_{\mathfrak{p}}}(\mathfrak{I}_{\mathfrak{p}}, A_{\mathfrak{p}}) = r.$$

If A is also Noetherian, then by [??](#) and [??](#) we have

$$\operatorname{depth}_A(\mathfrak{I}, A) \leq \operatorname{codim}(V(\mathfrak{I}), \operatorname{Spec}(A)) \leq r,$$

whence

$$\operatorname{depth}_A(\mathfrak{I}, A) = r = \operatorname{codim}(V(\mathfrak{I}), \operatorname{Spec}(A)) = \operatorname{ht}(\mathfrak{I}).$$

For an ideal \mathfrak{I} of a ring A , the graded A -module $\bigoplus_{n \in \mathbb{N}} \mathfrak{I}^n$ possesses a natural graded A -algebra structure, induced from the multiplication of the ring A . The identity map on \mathfrak{I} then extends to a canonical surjective homomorphism of graded A -algebras

$$\alpha_{\mathfrak{I}} : S_A(\mathfrak{I}) \rightarrow \bigoplus_{n \in \mathbb{N}} \mathfrak{I}^n.$$

By extension of scalars to the ring A/\mathfrak{I} , we also deduce a canonical surjective homomorphism of graded A/\mathfrak{I} -algebras

$$\beta_{\mathfrak{I}} : S_{A/\mathfrak{I}}(\mathfrak{I}/\mathfrak{I}^2) \rightarrow \operatorname{gr}_{\mathfrak{I}}(A).$$

As the case of [??](#), the injectivity of the canonical homomorphisms $\alpha_{\mathfrak{I}}$ and $\beta_{\mathfrak{I}}$ characterizes completely secant ideals. More precisely, we have the following theorem:

Theorem 17.5.2. *Let A be a Noetherian ring and \mathfrak{I} be an ideal of A . The following conditions are equivalent:*

- (i) *the ideal \mathfrak{I} is completely secant;*
- (ii) *the ideal \mathfrak{I} is completely secant at every maximal ideal $\mathfrak{m} \in V(\mathfrak{I})$;*
- (iii) *the A/\mathfrak{I} -module $\mathfrak{I}/\mathfrak{I}^2$ is projective and the canonical homomorphism $\alpha_{\mathfrak{I}} : S_A(\mathfrak{I}) \rightarrow \bigoplus_{n \in \mathbb{N}} \mathfrak{I}^n$ is bijective;*
- (iv) *the A/\mathfrak{I} -module $\mathfrak{I}/\mathfrak{I}^2$ is projective and the canonical homomorphism $\beta_{\mathfrak{I}} : S_{A/\mathfrak{I}}(\mathfrak{I}/\mathfrak{I}^2) \rightarrow \operatorname{gr}_{\mathfrak{I}}(A)$ is bijective.*

Proof. It is clear that (i) \Rightarrow (ii) and (iii) \Rightarrow (iv). Suppose that condition (ii) is satisfied; to see that (ii) \Rightarrow (iii), it suffices to prove that for any maximal ideal \mathfrak{m} of A , the $A_{\mathfrak{m}}/\mathfrak{I}_{\mathfrak{m}}$ -module $\mathfrak{I}_{\mathfrak{m}}/\mathfrak{I}_{\mathfrak{m}}^2$ is free and the homomorphism $\alpha_{\mathfrak{I}_{\mathfrak{m}}} : S_{A_{\mathfrak{m}}}(\mathfrak{I}_{\mathfrak{m}}) \rightarrow \bigoplus_{n \in \mathbb{N}} \mathfrak{I}_{\mathfrak{m}}^n$ is bijective ([??](#) and [??](#)). But this

assertion is trivial if \mathfrak{m} does not belong to $V(\mathfrak{I})$ (since we then have $\mathfrak{I}_{\mathfrak{m}} = A_{\mathfrak{m}}$), and follows from ?? if $\mathfrak{m} \in V(\mathfrak{I})$.

Finally, suppose that condition (iv) is satisfied, and let \mathfrak{p} be a prime ideal of A containing \mathfrak{I} . Then the $A_{\mathfrak{p}}/\mathfrak{I}_{\mathfrak{p}}$ -module $\mathfrak{I}_{\mathfrak{p}}/\mathfrak{I}_{\mathfrak{p}}^2$ is free. Let $\mathbf{x} = (x_1, \dots, x_r)$ be a sequence of elements of $\mathfrak{I}_{\mathfrak{p}}$ whose image forms a basis for $\mathfrak{I}_{\mathfrak{p}}/\mathfrak{I}_{\mathfrak{p}}^2$. The sequence \mathbf{x} generates $\mathfrak{I}_{\mathfrak{p}}$ by Nakayama's lemma, and satisfies condition (iv) of ?. Therefore the ideal $\mathfrak{I}_{\mathfrak{p}}$ of $A_{\mathfrak{p}}$ is completely secant, and \mathfrak{I} is then completely secant at \mathfrak{p} . \square

Remark 17.5.3. Suppose that the ideal \mathfrak{I} is completely secant and let $\mathbf{x} = (x_1, \dots, x_r)$ be a sequence of elements of \mathfrak{I} such that for any maximal ideal $\mathfrak{m} \in V(\mathfrak{I})$, the image of \mathbf{x} in $\mathfrak{I}/\mathfrak{m}\mathfrak{I}$ forms a basis for the A/\mathfrak{m} -vector space $\mathfrak{I}/\mathfrak{m}\mathfrak{I}$. Then the A/\mathfrak{I} -module $\mathfrak{I}/\mathfrak{I}^2$ is free and the canonical image of \mathbf{x} in $\mathfrak{I}/\mathfrak{I}^2$ forms a basis: in fact, it suffices to verify that for each $\mathfrak{m} \in V(\mathfrak{I})$, the image of \mathbf{x} forms a basis for the $A_{\mathfrak{m}}/\mathfrak{I}_{\mathfrak{m}}$ -module $\mathfrak{I}_{\mathfrak{m}}/\mathfrak{I}_{\mathfrak{m}}^2$, and this follows from ?? and ?? since the $A_{\mathfrak{m}}/\mathfrak{I}_{\mathfrak{m}}$ -module $\mathfrak{I}_{\mathfrak{m}}/\mathfrak{I}_{\mathfrak{m}}^2$ is projective by ??.

Corollary 17.5.3. Let $\rho : A \rightarrow B$ be a flat homomorphism of Noetherian rings, and \mathfrak{I} be an ideal of A .

- (a) If \mathfrak{I} is completely secant, the ideal $\mathfrak{I}B$ is completely secant.
- (b) Suppose that the ideal $\mathfrak{I}B$ is completely secant and that any maximal ideal $\mathfrak{m} \in V(\mathfrak{I})$ is the inverse image of a maximal ideal of B . Then the ideal \mathfrak{I} is completely secant. This is the case for example if B is faithfully flat over A .

Proof. Since the A -module B is flat, $\mathfrak{I}^n \otimes_A B$ is identified with $\mathfrak{I}^n B$ and $(\mathfrak{I}^n/\mathfrak{I}^{n+1}) \otimes_{A/\mathfrak{I}} (B/\mathfrak{I}B)$ is identified with $\mathfrak{I}^n B/\mathfrak{I}^{n+1} B$ for each integer $n \geq 0$. Assertion (a) then follows from criterion (iii) of ?. Under the hypotheses of (b), the A/\mathfrak{I} -module $B/\mathfrak{I}B$ is faithfully flat (?? and ??) and \mathfrak{I} is completely secant by criterion (iv) of ?. The last assertion follows from ?. \square

Corollary 17.5.4. Let A be a Noetherian ring, \mathfrak{I} be an ideal of A , \widehat{A} be the \mathfrak{I} -adic completion of A and $\widehat{\mathfrak{I}} = \mathfrak{I}\widehat{A}$ be the completion of \mathfrak{I} . For the ideal $\widehat{\mathfrak{I}}$ of \widehat{A} to be completely secant, it is necessary and sufficient that the ideal \mathfrak{I} of A is completely secant.

Proof. In fact, by ?? the condition of ??(b) is satisfied. \square

Corollary 17.5.5. Let A be a Noetherian ring and \mathfrak{I} be a completely secant ideal of A . If A is a Cohen-Macaulay (resp. Gorenstein) ring, so is A/\mathfrak{I} .

Proof. Let \mathfrak{m} be a maximal ideal of A containing \mathfrak{I} . The ideal $\mathfrak{I}_{\mathfrak{m}}$ of $A_{\mathfrak{m}}$ is generated by an $A_{\mathfrak{m}}$ -regular sequence, so $(A/\mathfrak{I})_{\mathfrak{m}}$ is Cohen-Macaulay (resp. Gorenstein) by ??(e) (resp. ??). \square

Remark 17.5.4. A Noetherian ring A is said to be **complete intersection** if for any maximal ideal \mathfrak{m} of A , the completion of the local ring $A_{\mathfrak{m}}$ is isomorphic to a quotient of a complete Noetherian regular local ring by a completely secant ideal. It then follows from ?? and ?? that such a ring is Gorenstein.

Proposition 17.5.6. Let A be a Noetherian ring. Then the following conditions are equivalent:

- (i) A is regular;
- (ii) any maximal ideal of A is completely secant;

(iii) any ideal \mathfrak{J} of A such that A/\mathfrak{J} is regular is completely secant.

Proof. Suppose that the ring A is regular; let \mathfrak{J} be an ideal of A such that A/\mathfrak{J} is regular and \mathfrak{p} be a prime ideal of A containing \mathfrak{J} . Then the local rings $A_{\mathfrak{p}}$ and $A_{\mathfrak{p}}/\mathfrak{J}_{\mathfrak{p}}$ are regular, so $\mathfrak{J}_{\mathfrak{p}}$ is generated by a completely secant sequence for $A_{\mathfrak{p}}$ (??), and this signifies that \mathfrak{J} is completely secant at \mathfrak{p} . This proves (i) \Rightarrow (iii), and (iii) \Rightarrow (ii) since a field is regular. Finally, under the hypotheses of (ii), let \mathfrak{m} be a maximal ideal of A . Then the maximal ideal $\mathfrak{m}A_{\mathfrak{m}}$ of $A_{\mathfrak{m}}$ is generated by a completely secant sequence for $A_{\mathfrak{m}}$, so $A_{\mathfrak{m}}$ is regular (??), whence (i). \square

Proposition 17.5.7. *Let A be a Noetherian ring and \mathfrak{J} be an ideal of A such that A/\mathfrak{J} is regular. Then the following conditions are equivalent:*

- (i) *the ideal \mathfrak{J} is completely secant;*
- (ii) *for any prime (resp. maximal) ideal \mathfrak{p} of A containing \mathfrak{J} , the ring $A_{\mathfrak{p}}$ is regular;*
- (iii) *the \mathfrak{J} -adic completion of A is regular.*

Proof. By ??, condition (i) signifies that for any prime (resp. maximal) ideal \mathfrak{p} of A containing \mathfrak{J} , the ideal $\mathfrak{J}_{\mathfrak{p}}$ of the local ring $A_{\mathfrak{p}}$ is generated by a completely secant sequence for $A_{\mathfrak{p}}$. Since $A_{\mathfrak{p}}/\mathfrak{J}_{\mathfrak{p}}$ is regular by hypothesis, this latter condition is equivalent to that $A_{\mathfrak{p}}$ is regular (??). This proves the equivalence of (i) and (ii), and the equivalence of (ii) and (iii) follows from ?? \square

Proposition 17.5.8. *Let A be a regular ring, \mathfrak{J} be an ideal of A and A_0 be a subring of A such that the canonical homomorphism $A_0 \rightarrow A/\mathfrak{J}$ is bijective.*

- (a) *The ideal \mathfrak{J} is completely secant, the A_0 -module $\mathfrak{J}/\mathfrak{J}^2$ is finitely generated and projective, and the ring A_0 is regular.*
- (b) *Let $\varphi : \mathfrak{J}/\mathfrak{J}^2 \rightarrow \mathfrak{J}$ be a A_0 -linear section of the canonical surjection $\mathfrak{J} \rightarrow \mathfrak{J}/\mathfrak{J}^2$. Then the A_0 -homomorphism $S_{A_0}(\mathfrak{J}/\mathfrak{J}^2) \rightarrow A$ extending φ then extends to an isomorphism from the completion of the graded ring $S_{A_0}(\mathfrak{J}/\mathfrak{J}^2)$ to the \mathfrak{J} -adic completion of A .*

Proof. Let \mathfrak{p} be a prime ideal of A containing \mathfrak{J} . We have $\mathfrak{p} = (\mathfrak{p} \cap A_0) \oplus \mathfrak{J}$ and therefore $\mathfrak{p}^2 = (\mathfrak{p} \cap A_0)^2 \oplus \mathfrak{p}\mathfrak{J}$, hence $\mathfrak{p}^2 \cap \mathfrak{J} = \mathfrak{p}\mathfrak{J}$. Denote by $i : \mathfrak{J} \rightarrow \mathfrak{p}$ be the canonical injection, then the map $i \otimes 1_{A/\mathfrak{p}} : \mathfrak{J} \otimes_A A/\mathfrak{p} \rightarrow \mathfrak{p} \otimes_A A/\mathfrak{p}$ is injective, and so is the map $i_{\mathfrak{p}} \otimes 1_{A/\mathfrak{p}} : \mathfrak{J}_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} \kappa(\mathfrak{p}) \rightarrow \mathfrak{p} \otimes_{A_{\mathfrak{p}}} \kappa(\mathfrak{p})$. Nakayama's lemma then implies that the ideal $\mathfrak{J}_{\mathfrak{p}}$ of $A_{\mathfrak{p}}$ is generated by a subset of a system of parameters of the regular local ring $A_{\mathfrak{p}}$. By ??, the ring $A_{\mathfrak{p}}/\mathfrak{J}_{\mathfrak{p}}$ is regular and the ideal $\mathfrak{J}_{\mathfrak{p}}$ is completely secant. By ??, \mathfrak{J} is then completely secant, the ring A_0 , isomorphic to A/\mathfrak{J} , is regular, and the A_0 -module $\mathfrak{J}/\mathfrak{J}^2$ is finitely generated and projective.

Now it follows from ?? that the canonical homomorphism $\beta_{\mathfrak{J}} : S_{A_0}(\mathfrak{J}/\mathfrak{J}^2) \rightarrow A$ is bijective. Let $f : S_{A_0}(\mathfrak{J}/\mathfrak{J}^2) \rightarrow A$ be the A_0 -homomorphism extending the A_0 -linear map $\varphi : \mathfrak{J}/\mathfrak{J}^2 \rightarrow \mathfrak{J}$. If we endow A with the \mathfrak{J} -adic filtration and $S_{A_0}(\mathfrak{J}/\mathfrak{J}^2)$ the graduation filtration, $\beta_{\mathfrak{J}}$ is identified with the homomorphism induced from f by passing to graded algebras, and assertion (b) follows from ?? \square

17.5.2 Graded regular rings

Let A_0 be a ring and P be a graded A_0 -module of type \mathbb{N} . Denote by A the ring $S_{A_0}(P)$, then there exists over A a unique graduation of type \mathbb{N} for which A_0 is of degree 0 and P is a graded submodule of A . Let $A_+ = \bigoplus_{n>0} A_n$ be the irrelevant ideal of A , then the canonical map $P \rightarrow A_+/A_+^2$ is an isomorphism of graded A_0 -modules (A, III, p76, prop.10 ??).

If the graded A_0 -module P is free and $(x_i)_{i \in I}$ is a basis for P formed by homogeneous elements, the graded A_0 -algebra $S_{A_0}(P)$ is then isomorphic to the polynomial algebra $A_0[(X_i)_{i \in I}]$, endowed with the graduation for which each X_i is homogeneous of degree $\deg(x_i)$. Any graded A_0 -algebra of type \mathbb{N} , isomorphic to the graded A_0 -algebra of this form, is said to be **polynomial**.

If A_0 is regular and the A_0 -module P is projective and finitely generated, then the ring $S_{A_0}(P)$ is regular by ???. Conversely, we have the following theorem which asserting that any regular graded A_0 -algebra is of this form.

Theorem 17.5.9. *Let A be a regular graded ring of type \mathbb{N} . Then the ring A_0 formed by elements of degree 0 in A is regular, and there exists a finitely generated projective graded A_0 -module P with positive degrees such that A is isomorphic to the graded A_0 -algebra $S_{A_0}(P)$.*

Proof. Denote by P the graded A_0 -module A_+/A_+^2 . By ??, the ring A_0 is regular and the A_0 -module P is projective and finitely generated. The homogeneous components of P are then projective and there exists an A_0 -linear section $\varphi : P \rightarrow A_+$ of the canonical surjection $A_+ \rightarrow P$, which is graded of degree 0. Let $f : S_{A_0}(P) \rightarrow A$ be the induced homomorphism of graded A_0 -algebras. Then by ??, $\hat{f} : \widehat{S_{A_0}(P)} \rightarrow \widehat{A}$ is an isomorphism, so f is injective and its image is dense in A for the A_+ -adic topology. But since the topologies induced over the homogeneous components of A are diecrete and the image of f is a graded submodule, this implies that f is bijective. \square

Corollary 17.5.10. *Let B be a regular graded ring of positive degrees. Suppose that any finitely generated projective B_0 -module is free.*

- (a) *The ring B_0 is integral and regular and the B_0 -algebra B is a polynomial graded B_0 -algebra of finite type.*
- (b) *Let A be a graded subring of B such that $A_0 = B_0$ and that B is a finitely generated A -module. Then the following conditions are equivalent:*
 - (i) *the graded A -module B is free;*
 - (ii) *the A -module B is flat;*
 - (iii) *A is a polynomial graded A_0 -algebra of finite type.*

Corollary 17.5.11. *Let k be a field, B be a polynomial graded k -algebra of finite type, and A be a graded subring of B . Then the following conditions are equivalent:*

- (i) *B is a graded free A -module;*
- (ii) *B is a flat A -module;*
- (iii) $\text{Tor}_1^A(k, B) = 0$;
- (iv) *the algebra A is a polynomial graded k -algebra of finite type, and any algebraically free generating sequence of A formed by homogeneous elements is B -regular.*

17.5.3 Extension of scalars

Proposition 17.5.12. Let $\rho : A \rightarrow B$ be a homomorphism of Noetherian rings and \mathfrak{I} be an ideal of B . Then the following conditions are equivalent:

- (i) the A -module B/\mathfrak{I} is flat and the ideal \mathfrak{I} is completely secant;
- (ii) for any $\mathfrak{P} \in V(\mathfrak{I})$, the A -module $B_{\mathfrak{P}}$ is flat and, for any A -algebra A' such that the ring $A' \otimes_A B$ is Noetherian, the ideal $\mathfrak{I}(A' \otimes_A B)$ is completely secant;
- (iii) for any maximal ideal \mathfrak{N} of B containing \mathfrak{I} , the A -module $B_{\mathfrak{N}}$ is flat and the ideal $\mathfrak{I}(\kappa(\mathfrak{N}^c) \otimes_A B_{\mathfrak{N}})$ of $\kappa(\mathfrak{N}^c) \otimes_A B_{\mathfrak{N}}$ is completely secant.

17.6 Geometric regularity and normality

17.6.1 Algebras essentially of finite type

Let k be a ring, A be a k -algebra, and $\mathbf{x} = (x_i)_{i \in I}$ be a family of elements of A ; denote by A' the subalgebra of A generated by the x_i . We say that \mathbf{x} is an **essentially generating family** of the k -algebra A if, for any element $a \in A$, there exists an element s of A' , invertible in A , such that $sa \in A'$. Equivalently, this amounts to saying that, for any $a \in A$, there exist polynomials P, Q of $k[(X_i)_{i \in I}]$ such that $Q(x)$ is invertible in A and that $a = P(x)Q(x)^{-1}$.

A k -algebra A is said to be **essentially of finite type** if it admits a finite essentially generating family. Equivalently, this means there exists a k -algebra A' of finite type and a multiplicative subset S of A' such that A is isomorphic to the k -algebra $S^{-1}A'$.

Example 17.6.1. To say that a field extension L/K is essentially of finite type signifies that it is a finitely generated field extension (note that this is not equivalent to that K is a k -algebra of finite type). In fact, if $\mathbf{x} = (x_i)_{i \in I}$ is an essentially generating family for L , then we obtain a surjective homomorphism from $K((x_i)_{i \in I}) \rightarrow L$. Also note that by Zariski's lemma (??), L is of finite type over K if and only if it is a finite extension of K .

Example 17.6.2. For a local k -algebra to be essentially of finite type, it is necessary and sufficient that it is isomorphic to a k -algebra of the form $A_{\mathfrak{p}}$, where A is a k -algebra of finite type and \mathfrak{p} is a prime ideal of A . In fact, if the local k -algebra is isomorphic to $S^{-1}A$, then by (??), it is isomorphic to $A_{\mathfrak{p}}$, where \mathfrak{p} is the inverse image of the maximal ideal of $S^{-1}A$ in A .

Proposition 17.6.3. If the ring k is Noetherian, then any k -algebra essentially of finite type is a Noetherian ring.

Proof. This follows from Hilbert basis theorem and ??.

□

Proposition 17.6.4. Let k be a ring.

- (a) Any quotient algebra of a k -algebra essentially of finite type is a k -algebra essentially of finite type.
- (b) Any fraction ring of a k -algebra essentially of finite type is a k -algebra essentially of finite type.
- (c) The product of a finite family of k -algebras essentially of finite type is a k -algebra essentially of finite type.

(d) Let $k \rightarrow k'$ be a ring homomorphism. For any k -algebra A essentially of finite type, the k' -algebra $A_{(k')} = k' \otimes_k A$ is essentially of finite type.

Proof. Since localization commutes with quotients, it is clear that assertion (a) is true, and assertion (b) follows from definition. Similarly, since localization commutes with base change, it is easy to conclude (d), and assertion (c) follows from the definition of essentially generating families. \square

Corollary 17.6.5. Let A be a k -algebra essentially of finite type and B be a Noetherian k -algebra. Then the ring $A \otimes_k B$ is Noetherian.

Proof. In fact, this is a B -algebra essentially of finite type, so the conclusion follows from ?? . \square

Proposition 17.6.6. Let k be a ring, A be a k -algebra and B be an A -algebra. If A is essentially of finite type over k and B is essentially of finite type over A , then B is essentially of finite type over k .

Proof. Denote by $\rho : A \rightarrow B$ the canonical map. Let $\mathbf{x} = (x_i)_{i \in I}$ be an essentially generating family of the k -algebra A and A' the subalgebra it generates; let $\mathbf{y} = (y_j)_{j \in J}$ be an essentially generating family of the A -algebra B , and B' be the subalgebra generated by the $\rho(x_i)$ and y_j . Let $b \in B$, then by hypothesis there exist polynomials P, Q of $A[(Y_j)_{j \in J}]$ such that $Q(\mathbf{y})$ is invertible in B and that we have $Q(\mathbf{y})b = P(\mathbf{y})$. The nonzero coefficients of P and Q are finite in number, so there exists a polynomial $R \in k[(X_i)_{i \in I}]$ such that $R(\mathbf{x})$ is invertible in A and that $R(\mathbf{x})P$ and $R(\mathbf{x})Q$ belongs to $A'[(Y_j)_{j \in J}]$. Then $\rho(R(\mathbf{x}))Q(\mathbf{y})$ is invertible in B , and we have

$$\rho(R(\mathbf{x}))Q(\mathbf{y})b = \rho(R(\mathbf{x}))P(\mathbf{y}) \in B'.$$

In other words, the elements $\rho(x_i)$ and y_j for an essentially generating family for the k -algebra B . \square

Corollary 17.6.7. The tensor product of two k -algebras essentially of finite type is a k -algebra essentially of finite type.

Proof. Let A and B be two k -algebras essentially of finite type. Then $A \otimes_k B$ is essentially of finite type over A ?? , and hence over k ?? . \square

17.6.2 Tensor product of Cohen-Macaulay algebras

17.6.3 Geometrically regular and geometrically normal algebras

Recall that a separable algebra A over a field k is defined by the property that for any finite field extension k' of k , the ring $A_{(k')}$ is reduced (in other words, the k -algebra A is *geometrically reduced*). In this paragraph, we develop similar notions for a k -algebra A , concerning the regularity (resp. normality) for any base change $A_{(k')}$ into a field extension k' of k . Such k -algebras are then called *geometrically regular* (resp. *geometrically normal*), and they will play an important role when we discuss smooth algebras.

Before giving the definition, we first recall if k' is a separable extension of k , then the base change $A_{(k')}$ of any reduced k -algebra is still reduced. We now prove a similar result for regular and normal algebras. However, for this to work, we need to impose some finite condition on the k -algebras. The technique points of our consideration are contained in the following two lemmas.

Lemma 17.6.8. *Let A be a Noetherian ring with an exhaustive filtration $(A_\alpha)_{\alpha \in I}$ of Noetherian subrings.*

- (a) *If the rings A_α are regular and A is a flat A_α -module for each $\alpha \in I$, then A is regular.*
- (b) *If the rings A_α are normal, then A is normal.*

Proof.

□

Lemma 17.6.9. *Let k be a field and K, L be field extensions of k such that one of them is separable over k . Suppose that K is finitely generated over k , then the ring $K \otimes_k L$ is regular.*

Proof.

□

Proposition 17.6.10. *Let k be a field, A be a k -algebra and K be an extension of k . Suppose that A is essentially of finite type or that the extension K is finitely generated.*

- (a) *If the ring $A_{(K)}$ is regular (resp. normal), then A is regular (resp. normal).*
- (b) *If the ring A is regular (resp. normal) and the extension K/k is separable, then $A_{(K)}$ is regular (resp. normal).*

Proof. Being the base change of the free k -module K by A , the A -module $A_{(K)}$ is free, hence faithfully flat. Assertion (a) then follows from ?? and ?? (resp. ??). Under the hypothesis of (b), for any prime ideal \mathfrak{p} of A , the ring $\kappa(\mathfrak{p}) \otimes_k K$ is regular by ??, and a fortiori normal (??). Assertion (b) then follows from ?? (resp. ??). □

Let k be a field and A be a k -algebra. We say that A is **geometrically regular** (resp. **geometrically normal**) if the ring $A_{(k')}$ is regular (resp. normal) for any finite extension k' of k . It is clear that any geometrically regular (resp. geometrically normal) k -algebra is itself regular (resp. normal), as we can take $k' = k$ in the definition. Also, in view of ??, to verify that a k -algebra is geometrically regular (resp. geometrically normal), it suffices to consider only finite purely separable extensions of k .

Example 17.6.11. If the field k is perfect, any regular (resp. normal) k -algebra is geometrically regular (resp. geometrically normal). This is an analogue for separable algebras, since in this case any reduced k -algebra is separable.

Example 17.6.12. Let A be an Artinian k -algebra. Then the following conditions are equivalent:

- A is separable;
- A is geometrically regular;
- A is geometrically normal.

In fact, if A is geometrically normal then it is separable, and if A is separable, for any finite extension k' of k , the ring $A_{(k')}$ is reduced and Artinian, hence isomorphic to a product of fields, and therefore regular (A, VIII, §8, n1, prop.2 ??). Note that if the k -algebra A is finite dimensional, then these conditions are also equivalent to that A is étale (A, V, p.34, th.4).

Example 17.6.13. Let A be a regular local k -algebra. If the extension κ_A of k is separable, then A is geometrically regular. In fact, let k' be a finite extension of k . The A -module $A_{(k')}$ is then free and finitely generated, hence flat, and each maximal ideal of $A_{(k')}$ is lying over \mathfrak{m}_A (??). The ring $\kappa_A \otimes_A A_{(k')}$, being isomorphic to $\kappa_A \otimes_k k'$, is regular by ???. ?? then implies that the ring $A_{(k')}$ is regular, so the k -algebra A is geometrically regular.

Proposition 17.6.14. Let k be a field and A be a Noetherian k -algebra.

- (a) If A is geometrically regular (resp. geometrically normal, resp. separable), so is the ring $S^{-1}A$ for any multiplicative subset S of A .
- (b) If $A_{\mathfrak{m}}$ is geometrically regular (resp. geometrically normal, resp. separable) for any maximal ideal \mathfrak{m} of A , then A is geometrically regular (resp. geometrically normal, resp. separable).

Proof. Assertion (a) follows from the fact that $(S^{-1}A)_{(k')}$ is isomorphic to a fraction ring of $A_{(k')}$ for any field extension k' of k . Now suppose that $A_{\mathfrak{m}}$ is geometrically regular (resp. geometrically normal, resp. separable) for any maximal ideal \mathfrak{m} of A . Let k' be a finite extension of k , and \mathfrak{m}' be a maximal ideal of $A_{(k')}$. It suffices to verify that the local ring $(A_{(k')})_{\mathfrak{m}'}$ is regular (resp. normal, resp. reduced).

The canonical homomorphism $A \rightarrow A_{(k')}$ is finite, so the maximal ideal \mathfrak{m}' is lying over a maximal ideal \mathfrak{m} of A (??) and $(A_{(k')})_{\mathfrak{m}'}$ is isomorphic to a fraction ring of the regular (resp. normal, resp. reduced) ring $(A_{\mathfrak{m}})_{(k')}$, hence is regular (resp. normal, resp. reduced). \square

Lemma 17.6.15. Let k be a field and K be a finitely generated extension of k . Then there exists a finite extension L of K and a sub- k -extension k' of L which is finite and purely inseparable over k , such that the extension L/k' is separable.

Proposition 17.6.16. Let k be a field, A and B be k -algebras such that one of them is essentially of finite type. Suppose that A is geometrically regular (resp. geometrically normal). If the ring B is regular (resp. normal), so is the ring $A \otimes_k B$.

Proof. \square

Corollary 17.6.17. Let k be a field. Then the tensor product of two geometrically regular (resp. geometrically normal) k -algebras, one of which is essentially of finite type, is a geometrically regular (resp. geometrically normal) k -algebra.

Proof. Let A and B be k -algebras satisfying the hypotheses of the corollary, and k' be a finite extension of k . The ring $B_{(k')}$ is regular (resp. normal), so $A \otimes_k B_{(k')}$ is regular (resp. normal) by ??, which is isomorphic to $(A \otimes_k B)_{(k')}$. \square

Corollary 17.6.18. Let k be a field, A be a geometrically regular (resp. geometrically normal) k -algebra and K be an extension of k . Suppose that A is essentially of finite type or the extension K/k is finitely generated.

- (a) The ring $A_{(K)}$ is regular (resp. normal).
- (b) If the extension K/k is separable, the k -algebra $A_{(K)}$ is geometrically regular (resp. geometrically normal).

Proof. Assertion (a) follows from ??, since a field is regular. Assertion (b) follows from ?? and ??.

Corollary 17.6.19. *Let k be a field, A be a k -algebra and K be an extension of k . Suppose that the k -algebra A is essentially of finite type or the extension K/k is finitely generated. For the k -algebra A to be geometrically regular (resp. geometrically normal), it is necessary and sufficient that so is the K -algebra $A_{(K)}$.*

Proof. Suppose that A is geometrically regular (resp. geometrically normal) and let K' be a finite extension of K . The ring $K' \otimes_K A_{(K)}$, isomorphic to $K' \otimes_k A$, is regular (resp. normal) by ??.

Suppose conversely that the K -algebra $A_{(K)}$ is geometrically regular (resp. geometrically normal) and let k' be a finite extension of k . Let L be the composition extension of k' and K , then the ring $A_{(L)}$ is identified with $L \otimes_K A_{(K)}$, hence is regular (resp. normal); therefore the ring $A_{(k')}$ is regular (resp. normal) by ??.

Corollary 17.6.20. *Let k be a field, A be a k -algebra essentially of finite type and K be a extension of k which is a perfect field. For the k -algebra A to be geometrically regular (resp. geometrically normal), it is necessary and sufficient that $A_{(K)}$ is regular (resp. normal).*

Proof. This follows from ?? and ??.

17.6.4 Characterization of geometrically regular algebras

Proposition 17.6.21. *Let k be a field and A be a k -algebra essentially of finite type. Let \mathfrak{I} be the kernel of the multiplication homomorphism $\mu : A \otimes_k A \rightarrow A$. Then the following conditions are equivalent:*

- (i) *the k -algebra A is geometrically regular;*
- (ii) *for any regular k -algebra R , the ring $A \otimes_k R$ is regular;*
- (iii) *the ring $A \otimes_k A$ is regular;*
- (iv) *the ideal \mathfrak{I} of $A \otimes_k A$ is completely secant.*

Proof. We denote by B the ring $A \otimes_k A$, which is endowed with the A -algebra structure defined by the homomorphism $\rho : A \rightarrow A \otimes_k A$ such that $\rho(x) = x \otimes 1$. Then μ is a homomorphism of A -algebras and induces an isomorphism from B/\mathfrak{I} to A . From ??, it is clear that (i) implies (ii), and (ii) implies (iii) by taking $C = k$ and $C = A$. Also, to see that (iii) \Rightarrow (iv), note that the A -module B is faithfully flat as the base change of the free k -module A by A . If the ring B is regular, then A is regular by ??, and the ideal \mathfrak{I} is then completely secant by ??.

Suppose that the ideal \mathfrak{I} is completely secant, we prove that A is regular. Let \mathfrak{m} be a maximal ideal of A and $\nu : (A/\mathfrak{m}) \otimes_k A \rightarrow A/\mathfrak{m}$ be the homomorphism induced by μ . The maximal ideal $\mathfrak{n} = \ker \nu$ is equal to $\mathfrak{I}((A/\mathfrak{m}) \otimes_k A)$, and by applying ?? to the ring homomorphism $B \rightarrow (A/\mathfrak{m}) \otimes_k A$ and use ??, we see that the ideal \mathfrak{n} of $(A/\mathfrak{m}) \otimes_k A$ is completely secant. Therefore ?? the local ring $((A/\mathfrak{m}) \otimes_k A)_{\mathfrak{n}}$ is regular. Denote by $j : A \rightarrow (A/\mathfrak{m}) \otimes_k A$ the homomorphism $x \mapsto 1 \otimes x$; since $\nu \circ j$ is the canonical homomorphism $A \rightarrow A/\mathfrak{m}$, we have $j^{-1}(\mathfrak{n}) = \mathfrak{m}$. Thus j extends to a local homomorphism $j_{\mathfrak{m}} : A_{\mathfrak{m}} \rightarrow ((A/\mathfrak{m}) \otimes_k A)_{\mathfrak{n}}$, which is faithfully flat since j is flat (the homomorphism $k \rightarrow A/\mathfrak{m}$ is free and j is its base change to A). By ??, the ring $A_{\mathfrak{m}}$ is then regular. This proves that the ring A is regular.

Now let k' be an extension of k . The kernel of the multiplication homomorphism $\mu' : A_{(k')} \otimes_{k'} A_{(k')} \rightarrow A_{(k')}$ is none other than $\mathfrak{I}B_{(k')}$, so it is completely secant in $A_{(k')}$ (??). The k' -algebra $A_{(k')}$ then satisfies condition (iv), and by our preceding arguments, it is therefore regular. This proves (i) and completes the proof. \square

Recall that (A, III, p.133-134) the quotient $\mathfrak{I}/\mathfrak{I}^2$ endowed with the A -module structure induced by ρ , is the differential module $\Omega_{A/k}$ of the k -algebra A . If the k -algebra A is essentially of finite type, the ring $A \otimes_k A$ is Noetherian (??), so $\Omega_{A/k}$ is a finitely generated A -module. We denote by $d : A \rightarrow \Omega_{A/k}$ the k -linear map which sends an element $x \in A$ to the class of $x \otimes 1 - 1 \otimes x$ in $\Omega_{A/k}$. Then d is a k -derivation, and for any A -module M and any k -derivation $D : A \rightarrow M$, there exists a unique A -linear map $\delta : \Omega_{A/k} \rightarrow M$ such that $D = \delta \circ d$ (A, III, p.133-134 prop.18).

If S is a multiplicative subset of A , the canonical $S^{-1}A$ -linear map (A, III, p.136) $S^{-1}\Omega_{A/k} \rightarrow \Omega_{S^{-1}A/k}$ is bijective: in fact, it suffices to verify that, for any $S^{-1}M$, any k -derivation $D : A \rightarrow M$ extends uniquely to a k -derivation $D : S^{-1}A \rightarrow M$, and this follows from (A, III, p.123, prop.4).

Now let k be a field and A be a local k -algebra essentially of finite type. Then by ??, A is isomorphic to a local ring $B_{\mathfrak{P}}$, where B is a finite type k -algebra and \mathfrak{P} is an ideal of B . Then we have $\dim(A) = \text{ht}(\mathfrak{P})$, and by ??(c), the dimension of the integral domain B/\mathfrak{P} is given by

$$\dim(B/\mathfrak{P}) = \text{tr.deg}_k(\kappa(\mathfrak{P})) = \text{tr.deg}_k(\kappa_A).$$

It then follows from ?? that $n := \dim(A) + \text{tr.deg}_k(\kappa_A)$ is equal to $\dim(B)$.

We now characterize geometrically regular local k -algebras by the differential module $\Omega_{A/k}$. We shall see that this property is equivalent to the freeness of $\Omega_{A/k}$ with rank equal to n . For simplicity, we single out the following two lemmas which will be used in our proof.

Lemma 17.6.22.

Lemma 17.6.23. *Let k*

Theorem 17.6.24. *Let k be a field and A be a local k -algebra essentially of finite type. Then we have*

$$[\kappa_A \otimes_A \Omega_{A/k} : \kappa_A] \geq n = \dim(A) + \text{tr.deg}_k(\kappa_A),$$

and the following conditions are equivalent:

- (i) *the k -algebra A is geometrically regular;*
- (ii) *the A -module $\Omega_{A/k}$ is free of rank n ;*
- (iii) $[\kappa_A \otimes_A \Omega_{A/k} : \kappa_A] = n$.

Proof. It is clear that (ii) implies (iii), so to prove the theorem, it suffices to prove the implications (i) \Rightarrow (ii) and (iii) \Rightarrow (i). Choose a finite type k -algebra B such that A is isomorphic to $B_{\mathfrak{P}}$, where \mathfrak{P} is a prime ideal of B . Then by the arguments above, we have $n = \dim(B)$, and by replacing B by a ring B_f , where $f \in B - \mathfrak{P}$, we can assume that B is connected of dimension n .

Suppose that the k -algebra A is

Example 17.6.25. Let K be a finitely generated extension of k , ?? then proves that the extension K/k is separable if and only if $\Omega_{K/k}$ has dimension $\text{tr.deg}_k(K)$, in view of ??.

Corollary 17.6.26. Let k be a field and A be k -algebra essentially of finite type. Then the set of elements \mathfrak{p} of $\text{Spec}(A)$ such that the k -algebra $A_{\mathfrak{p}}$ is geometrically regular is open in $\text{Spec}(A)$.

Proof. We can suppose that the k -algebra A is of finite type. The considered subset is then formed by the prime ideals \mathfrak{p} such that $[\kappa(\mathfrak{p}) \otimes_k \Omega_{A/k} : \kappa(\mathfrak{p})] \leq \dim_{\mathfrak{p}}(A)$. Now the function $\mathfrak{p} \mapsto \dim_{\mathfrak{p}}(A)$ is lower semi-continuous by definition, and the function $\mathfrak{p} \mapsto [\kappa(\mathfrak{p}) \otimes_k \Omega_{A/k} : \kappa(\mathfrak{p})]$ is upper semi-continuous (Nakayama's lemma and ??). \square

Corollary 17.6.27. Let k be a field and A be a k -algebra essentially of finite type. For A to be geometrically regular, it is necessary and sufficient that the A -module $\Omega_{A/k}$ is projective and for any minimal prime ideal \mathfrak{p} of A , the k -algebra $A_{\mathfrak{p}}$ is separable.

Proof. Suppose that A is geometrically regular. For any prime ideal of A , the k -algebra $A_{\mathfrak{p}}$ is geometrically regular, so the $A_{\mathfrak{p}}$ -module $\Omega_{A_{\mathfrak{p}}/k}$ is free (??). Moreover, if \mathfrak{p} is minimal, the local k -algebra $A_{\mathfrak{p}}$ is also Artinian, hence is a separable field extension of k (??).

Conversely, suppose that the A -module $\Omega_{A/k}$ is projective and $A_{\mathfrak{p}}$ is separable for any minimal prime \mathfrak{p} of A . Now let \mathfrak{p} be a prime ideal of A , and \mathfrak{q} be a minimal prime contained in \mathfrak{p} . Since the $A_{\mathfrak{p}}$ -module $(\Omega_{A/k})_{\mathfrak{p}}$ is free by ??, we have

$$[\kappa(\mathfrak{p}) \otimes_A \Omega_{A/k} : \kappa(\mathfrak{p})] = [\kappa(\mathfrak{q}) \otimes_A \Omega_{A/k} : \kappa(\mathfrak{q})].$$

The k -algebra $A_{\mathfrak{q}}$ is Artinian and separable, hence geometrically regular (??). ?? then implies that

$$[\kappa(\mathfrak{q}) \otimes_A \Omega_{A/k} : \kappa(\mathfrak{q})] = \text{tr.deg}_k(\kappa(\mathfrak{q})).$$

It then follows from ?? that the k -algebra $A_{\mathfrak{p}}$ is geometrically regular, whence the corollary. \square

Remark 17.6.1. Suppose that the k -algebra A essentially of finite type is geometrically regular. Then the total fraction ring $Q(A)$ of A is identified with the product of $A_{\mathfrak{p}}$, where \mathfrak{p} runs through minimal prime ideals of A (??); it is then a separable k -algebra. Conversely, suppose that $Q(A)$ is a separable k -algebra; for any minimal prime ideal \mathfrak{p} of A , the k -algebra $A_{\mathfrak{p}}$ is a fraction ring of $Q(A)$ (??), hence a separable k -algebra (??). If the A -module $\Omega_{A/k}$ is also projective, it then follows from ?? that the k -algebra A is geometrically regular.

17.7 Smooth algebras

17.7.1 Infinitesimal liftings

Let k be a ring, C be a k -algebra and N be a square zero ideal of C . We denote by $\pi : C \rightarrow C/N$ the canonical homomorphism; since $N^2 = 0$, the C -module N has a natural C/N -module structure. Let A be an k -algebra and $\varphi : A \rightarrow C/N$ be a homomorphism of k -algebra. Endow N with the A -module structure induced by φ , we define a **lifting** of φ (to C) to be a homomorphism $\tilde{\varphi} : A \rightarrow C$ of k -algebras such that $\pi \circ \tilde{\varphi} = \varphi$.

$$\begin{array}{ccc} k & \longrightarrow & C \\ \downarrow & \nearrow \tilde{\varphi} & \downarrow \pi \\ A & \xrightarrow{\varphi} & C/N \end{array}$$

Let $\tilde{\varphi}$ be such a lefting, and $\delta : A \rightarrow N$ be a map; we denote by $\delta + \tilde{\varphi}$ the map $x \mapsto \delta(x) + \tilde{\varphi}(x)$ from A into C .

Proposition 17.7.1. *If φ admits a lefting, the map $(\delta, \tilde{\varphi}) \mapsto \delta + \tilde{\varphi}$ defines a simply transitive action of the group $\text{Der}_k(A, N)$ of k -derivations from A to N on the set of leftings of φ .*

Proof. Let $\tilde{\varphi}_0 : A \rightarrow C$ be a lifting of φ . The map $\delta \mapsto \delta + \tilde{\varphi}_0$ induces a bijection from the set of maps $A \rightarrow N$ to the set of maps $\tilde{\varphi} : A \rightarrow C$ such that $\pi \circ \tilde{\varphi} = \varphi$. Fix δ , and put $\tilde{\delta} = \delta + \tilde{\varphi}_0$. For $\tilde{\delta}$ to be a homomorphism of k -algebras, it is necessary and sufficient that δ is a k -derivation: in fact, for $x, y \in A$ and $\lambda \in k$, we have the relations

$$\begin{aligned}\tilde{\varphi}(x+y) - \tilde{\varphi}(x) - \tilde{\varphi}(y) &= \delta(x+y) - \delta(x) - \delta(y) \\ \tilde{\varphi}(\lambda x) - \lambda \tilde{\varphi}(x) &= \delta(\lambda x) - \lambda \delta(x) \\ \tilde{\varphi}(xy) - \tilde{\varphi}(x)\tilde{\varphi}(y) &= \delta(xy) - \delta(x)\delta(y) - \delta(x)\tilde{\varphi}_0(y) - \tilde{\varphi}_0(x)\delta(y) \\ &= \delta(xy) - \varphi(x)\delta(y) - \varphi(y)\delta(x),\end{aligned}$$

the last equality resulting from the fact that N has zero square. \square

Example 17.7.2. Let B be a k -algebra, N be a B -module. In this section, we often endow the k -module $B \oplus N$ with the k -algebra structure defined by $(b, x)(d, y) = (bd, by + dx)$; N is then a square zero ideal of $B \oplus N$. Let $\varphi : A \rightarrow B$ be a homomorphism of k -algebras, then the liftings of φ to $B \oplus N$ are the maps $x \mapsto (\varphi(x), \delta(x))$, where δ is a k -derivation of A to N .

Let $\Omega_{A/k}$ be the module of k -differentials of the ring A , and $d : A \rightarrow \Omega_{A/k}$ be the universal k -derivation. Recall that for any A -module M , the map $v \mapsto v \circ d$ is an A -linear isomorphism from $\text{Hom}_A(\Omega_{A/k}, M)$ to $\text{Der}_k(A, M)$. Let \mathfrak{J} be an ideal of A . By (A, III, p.137 ??), we have an exact sequence of A/\mathfrak{J} -modules

$$\mathfrak{J}/\mathfrak{J}^2 \xrightarrow{\bar{d}} (A/\mathfrak{J}) \otimes_A \Omega_{A/k} \longrightarrow \Omega_{(A/\mathfrak{J})/k} \longrightarrow 0$$

where \bar{d} is the homomorphism induced by the restriction of d to \mathfrak{J} .

Let $\rho : A \rightarrow A/\mathfrak{J}^2$ and $\pi : A/\mathfrak{J}^2 \rightarrow A/\mathfrak{J}$ be the canonical homomorphisms. If

$$\alpha : (A/\mathfrak{J}) \otimes_A \Omega_{A/k} \rightarrow \mathfrak{J}/\mathfrak{J}^2$$

is a k -linear map, we then have an associated k -linear map $h_\alpha : A \rightarrow A/\mathfrak{J}^2$ defined by

$$h_\alpha(x) = \rho(x) - \alpha(1 \otimes dx).$$

If α is a retraction of \bar{d} (that is, $\alpha \circ \bar{d} = \text{id}_{\mathfrak{J}/\mathfrak{J}^2}$), then the map h_α factors into a k -linear map $h_\alpha : A/\mathfrak{J} \rightarrow A/\mathfrak{J}^2$. On the other hand, any k -linear map $h : A/\mathfrak{J} \rightarrow A/\mathfrak{J}^2$ induces a k -linear map $\psi_h : (A/\mathfrak{J}) \oplus (\mathfrak{J}/\mathfrak{J}^2) \rightarrow A/\mathfrak{J}^2$ given by $(x, y) \mapsto h(x) + y$.

Proposition 17.7.3. *Endow $(A/\mathfrak{J}) \oplus (\mathfrak{J}/\mathfrak{J}^2)$ with the k -algebra structure defined in ?? . Then the maps $\alpha \mapsto h_\alpha$ and $h \mapsto \psi_h$ induce bijections between the following sets:*

- (i) *the set of A/\mathfrak{J} -linear retractions α of \bar{d} ;*
- (ii) *the set of k -algebra homomorphisms $h : A/\mathfrak{J} \rightarrow A/\mathfrak{J}^2$ such that $\pi \circ h = \text{id}_{A/\mathfrak{J}}$.*

(iii) the set of k -algebra isomorphisms $\psi : (A/\mathfrak{I}) \oplus (\mathfrak{I}/\mathfrak{I}^2) \rightarrow A/\mathfrak{I}^2$ such that $\pi \circ \psi = \text{pr}_1$ and $\psi(0, z) = z$ for $z \in \mathfrak{I}/\mathfrak{I}^2$.

Proof. We apply ?? to $C = A/\mathfrak{I}^2$ and $N = \mathfrak{I}/\mathfrak{I}^2$. Let $\varphi : A \rightarrow A/\mathfrak{I}$ be the canonical surjection, then the homomorphism $\rho : A \rightarrow A/\mathfrak{I}^2$ is a lifting of φ to A/\mathfrak{I}^2 .

$$\begin{array}{ccc} k & \longrightarrow & A/\mathfrak{I}^2 \\ \downarrow & \nearrow \rho & \downarrow \pi \\ A & \xrightarrow{\varphi} & A/\mathfrak{I} \end{array}$$

The A -module $\text{Hom}_{A/\mathfrak{I}}((A/\mathfrak{I}) \otimes_A \Omega_{A/k}, \mathfrak{I}/\mathfrak{I}^2)$ is identified with $\text{Hom}_A(\Omega_{A/k}, \mathfrak{I}/\mathfrak{I}^2)$, so the map $\alpha \mapsto h_\alpha$ is a bijection from this set to the set of liftings of φ to A/\mathfrak{I}^2 (?). Now for $x \in \mathfrak{I}$ we have $1 \otimes dx = \bar{d}(\rho(x))$, so for the map h_α to factor through A/\mathfrak{I} , it is necessary and sufficient that $\alpha \circ \bar{d}$ is the identity on $\mathfrak{I}/\mathfrak{I}^2$, which means that α is a retraction of \bar{d} . This proves the correspondence of the sets in (i) and (ii) under the map $\alpha \mapsto h_\alpha$.

The map $h \mapsto \psi_h$ is a bijection from the set of k -linear homomorphisms $A/\mathfrak{I} \rightarrow A/\mathfrak{I}^2$ to the set of k -linear homomorphisms $\psi : (A/\mathfrak{I}) \oplus (\mathfrak{I}/\mathfrak{I}^2) \rightarrow A/\mathfrak{I}^2$ such that $\psi(0, z) = z$ for $z \in \mathfrak{I}/\mathfrak{I}^2$. For the equality $\pi \circ \psi_h = \text{pr}_1$ to be true, it is necessary and sufficient that $\pi \circ h = \text{id}_{A/\mathfrak{I}}$, which means $h(\pi(z)) \equiv z \pmod{\mathfrak{I}/\mathfrak{I}^2}$ for any $z \in A/\mathfrak{I}^2$. Suppose that this condition is satisfied, then for h to be a ring homomorphism, it is necessary and sufficient that ψ_h is a ring homomorphism. Moreover, in this case ψ_h is bijective: the inverse map is given by $z \mapsto (\pi(z), z - h(\pi(z)))$, where $z \in A/\mathfrak{I}^2$. This proves the correspondence of the sets in (ii) and (iii). \square

17.7.2 Formally smooth algebras

Let k be a ring and A be a linearly topologized k -algebra. We say that A is **formally smooth over k** , or that a **formally smooth k -algebra**, if it satisfies the following condition: for any k -algebra C and a square zero ideal N of C (endowed with the discrete topology), any continuous homomorphism from A into the k -algebra C/N can be lifted into a continuous homomorphism of A into the k -algebra C . We say that a k -algebra A is **formally smooth** if it is formally smooth endowed with the discrete topology, which is also the (0) -adic topology. In this case, it is then formally smooth for any \mathfrak{I} -adic topology.

Remark 17.7.1. Let k be a ring, A be a k -algebra and \mathfrak{I} be an ideal of A . Endow A with the \mathfrak{I} -adic topology. Let C be an k -algebra, N be a square zero ideal of C , and endow C and C/N with the discrete topology. Let $\varphi : A \rightarrow C/N$ be a continuous homomorphism of k -algebras. Then any lifting $\tilde{\varphi} : A \rightarrow C$ of φ is continuous: in fact, there exists an integer $n > 0$ such that $\varphi(\mathfrak{I}^n) = 0$, and we have $\tilde{\varphi}(\mathfrak{I}^n) \subseteq N$, whence $\tilde{\varphi}(\mathfrak{I}^{2n}) \subseteq N^2 = 0$. From this, we conclude that if A is formally smooth for the \mathfrak{I} -adic topology, it is also formally smooth for the \mathfrak{I}' -adic topology for any ideal \mathfrak{I}' containing \mathfrak{I} (note that a homomorphism from A to a discrete k -algebra is continuous if and only if its kernel is open in A).

Example 17.7.4. Let k be a ring, A be a k -algebra and \mathfrak{I} be an ideal of A . If the k -algebra A/\mathfrak{I} is formally smooth (for the discrete topology), then the identity map on A/\mathfrak{I} admits a lifting to

A/\mathfrak{I}^2 , so the sets of ?? are nonempty. In particular, the sequence

$$0 \rightarrow \mathfrak{I}/\mathfrak{I}^2 \xrightarrow{\bar{d}} (A/\mathfrak{I}) \otimes_A \Omega_{A/k} \longrightarrow \Omega_{(A/\mathfrak{I})/k} \longrightarrow 0$$

is then exact and split.

Remark 17.7.2. Let k be a ring, A be a linearly topologized formally smooth k -algebra, M be an A -module whose annihilator is open in A . Then *any derivation* $\delta : k \rightarrow M$ extends to a derivation $\tilde{\delta} : A \rightarrow M$. In fact, put $B = A/\text{Ann}(M)$, the map $\lambda \mapsto (\lambda 1_B, \delta(\lambda))$ then defines an algebra homomorphism from k to $B \oplus M$, which gives a k -algebra structure on $B \oplus M$. The canonical surjection $\varphi : A \rightarrow B$ is continuous, hence admits a lifting $\tilde{\varphi} : A \rightarrow B \oplus M$. By ??, $\text{pr}_2 \circ \tilde{\varphi}$ is then a derivation from A to M which extends δ .

Proposition 17.7.5. Let k be a ring.

- (a) Let A and B be linearly topologized k -algebras and $\rho : A \rightarrow B$ be a continuous homomorphism of k -algebras. If A is formally smooth over k and B is formally smooth over A , then B is formally smooth over k .
- (b) The product of a finite family of linearly topologized k -algebras is formally smooth over k if and only if each k -algebra is formally smooth over k .
- (c) Let A be a linearly topologized k -algebra and \hat{A} be the completion of A . For A to be formally smooth over k , it is necessary and sufficient that \hat{A} is formally smooth over k .

Proof. Let C be a k -algebra, N be a square zero ideal of C , and $\pi : C \rightarrow C/N$ be the canonical surjection. Endow C and C/N the discrete topology. In the situation of (a), let $\psi : B \rightarrow C/N$ be a continuous homomorphism of k -algebras. Since A is formally smooth over k , there exists a continuous homomorphism of k -algebras $\tilde{\varphi} : A \rightarrow C$ such that $\pi \circ \tilde{\varphi} = \psi \circ \rho$.

$$\begin{array}{ccccc} & & C & & \\ & \nearrow \tilde{\varphi} & \downarrow \pi & \searrow \tilde{\psi} & \\ A & \xrightarrow{\rho} & B & \xrightarrow{\psi} & C/N \end{array}$$

Consider C and C/N as A -algebras via the homomorphism $\tilde{\varphi}$, so that ψ is a homomorphism of A -algebras. Since B is formally smooth over A , there exists a continuous homomorphism $\tilde{\psi} : B \rightarrow C$ of A -algebras such that $\pi \circ \tilde{\psi} = \psi$, which proves (a). As for the assertion of (b), it suffices to note that giving a continuous k -homomorphism from a product $A = \prod_i A_i$ to C (resp. C/N) is equivalent to giving n continuous k -homomorphisms $A_i \rightarrow C$ (resp. $A_i \rightarrow C/N$) and that any continuous k -homomorphism $A_i \rightarrow C$ (resp. $A_i \rightarrow C/N$) gives by composition a continuous k -homomorphism $A \rightarrow A_i \rightarrow C$ (resp. $A \rightarrow A_i \rightarrow C/N$).

Finally, let $i : A \rightarrow \hat{A}$ be the canonical homomorphism. For any ring D , endowed with the discrete topology, the map which associates a continuous homomorphism $f : \hat{A} \rightarrow D$ the continuous homomorphism $f \circ i : A \rightarrow D$ is bijective, so the assertion of (c) follows. \square

Example 17.7.6. Assertion (c) of ?? is applicable in particular if the topology of A is the \mathfrak{I} -adic topology, where \mathfrak{I} is a finitely generated ideal; the closure $\widehat{\mathfrak{I}}$ of \mathfrak{I} in \hat{A} is then equal to $\mathfrak{I}\hat{A}$ and

the topology of \widehat{A} is the $\widehat{\mathfrak{I}}$ -topology (??). Therefore, it is equivalent to say that A is formally smooth for the \mathfrak{I} -adic topology or that the completion \widehat{A} is formally smooth for the $\widehat{\mathfrak{I}}$ -adic topology.

Proposition 17.7.7. *Let k be a ring, A and B be k -algebra, \mathfrak{I} be an ideal of A and \mathfrak{K} be an ideal of B .*

- (a) *Let S be a multiplicative subset of A and T be a multiplicative subset of k whose image in A is contained in S . If A is formally smooth over k for the \mathfrak{I} -adic topology, $S^{-1}A$ is formally smooth over $T^{-1}k$ for the $S^{-1}\mathfrak{I}$ -adic topology.*
- (b) *Let k' be a k -algebra. If A is formally smooth over k for the \mathfrak{I} -adic topology, the k' -algebra $A_{(k')}$ is formally smooth for the $\mathfrak{I}A_{(k')}$ -adic topology.*
- (c) *Let \mathfrak{R} be the ideal of $A \otimes_k B$ generated by the image of $\mathfrak{I} \otimes_k B$ and $A \otimes_k \mathfrak{K}$. If A and B are formally smooth over k for the \mathfrak{I} -adic topology and \mathfrak{K} -adic topology, respectively, then the k -algebra $A \otimes_k B$ is formally smooth for the \mathfrak{R} -adic topology.*

Proof. Under the hypotheses of (a), let C be a $T^{-1}k$ -algebra, N be a square zero ideal of C ; endow C and C/N the discrete topology and let $\pi : C \rightarrow C/N$ be the canonical homomorphism. Let $\varphi : S^{-1}A \rightarrow C/N$ be a continuous homomorphism of $T^{-1}k$ -algebras (for the $S^{-1}\mathfrak{I}$ -topology). Let $i : A \rightarrow S^{-1}A$ be the canonical homomorphism, then $\varphi \circ i$ is a continuous homomorphism from A to C/N (for the \mathfrak{I} -adic topology), so it admits a lifting $\tilde{\varphi}_0 : A \rightarrow C$. The elements of $\tilde{\varphi}_0(S)$ are invertible modulo N , hence are invertible (N has zero square), therefore there exists a ring homomorphism $\tilde{\varphi} : S^{-1}A \rightarrow C$ such that $\tilde{\varphi} \circ i = \tilde{\varphi}_0$, and the homomorphism $\tilde{\varphi}$ is $T^{-1}k$ -linear. We have $\pi \circ \tilde{\varphi} \circ i = \varphi \circ i$, whence $\pi \circ \tilde{\varphi} = \varphi$, and $\tilde{\varphi}$ is then a lifting of φ .

We now turn to the hypotheses of (b). Let C be a k' -algebra, N be a square zero ideal of C ; endow C and C/N the discrete topology. Let $\varphi : A_{(k')} \rightarrow C/N$ be a homomorphism of k' -algebras, continuous for the $\mathfrak{I}A_{(k')}$ -adic topology. Let $i : A \rightarrow A_{(k')}$ be the canonical homomorphism. The map $\varphi \circ i$ is then a homomorphism of k -algebras from A to C/N , continuous for the \mathfrak{I} -adic topology; if A is formally smooth over k for the \mathfrak{I} -adic topology, $\varphi \circ i$ admits a lifting $\tilde{\varphi} : A \rightarrow C$. The homomorphism $\tilde{\varphi} : A_{(k')} \rightarrow C$ induced by $\tilde{\varphi}$ is then a lifting of φ .

Finally, consider the hypotheses of (c). The B -algebra $A \otimes_k B$ is formally smooth for the $\mathfrak{I}(A \otimes_k B)$ -adic topology by (b), hence for the \mathfrak{R} -adic topology; moreover the canonical homomorphism $B \rightarrow A \otimes_k B$ is continuous if we endow B with the \mathfrak{K} -adic topology and $A \otimes_k B$ with the \mathfrak{R} -adic topology. Assertion (c) then follows from ??(a). \square

Example 17.7.8. Let A be a ring and \mathfrak{p} be a prime ideal of A . Then the local ring $A_{\mathfrak{p}}$ is formally smooth over A . To see this, let C be an A -algebra, N be a square zero ideal of C , and $\pi : C \rightarrow C/N$ be the canonical homomorphism. If $\varphi : A_{\mathfrak{p}} \rightarrow C/N$ is a homomorphism of A -algebras, then the image of elements of $\varphi(A - \mathfrak{p})$ are invertible in $C \mod N$, hence invertible. There then exists a unique lifting $\tilde{\varphi} : A_{\mathfrak{p}} \rightarrow C$ of φ , so $A_{\mathfrak{p}}$ is formally smooth over A (it is in fact formally étale over A).

Example 17.7.9. Let k be a ring, and P be a projective k -module. The symmetric k -algebra $S_k(P)$ is formally smooth for the discrete topology, and a fortiori for that defined by its graduation. In fact, for any k -algebra C and any ideal N of C , any algebra homomorphism from $S_k(P)$ to C (resp. C/N) corresponds to k -linear maps from P to C (resp. C/N), and the canonical map $\text{Hom}_k(P, C) \rightarrow \text{Hom}_k(P, C/N)$ is surjective since P is projective. In particular, by ?? the

k -algebra $\widehat{S}_k(P) = \prod_{n \in \mathbb{N}} S_k^n(P)$ is formally smooth (for the product topology of the discrete topology over $S_k^n(P)$).

Example 17.7.10. For any family $\mathbf{T} = (T_i)_{i \in I}$ of indeterminates, the k -algebra $k[\mathbf{T}]$ and $k[\![\mathbf{T}]\!]$, endowed with the canonical topology, are formally smooth over k : this follows from ???. If k is a field, the extension $k(\mathbf{T})$ is formally smooth over k by ??.

Example 17.7.11. Let $f \in k[T]$ be a polynomial in one indeterminate. For the k -algebra $k[T]/(f)$ to be formally smooth over k , it is necessary and sufficient that the following property is satisfied: for any k -algebra C and any square zero ideal N of C , any root of f in C/N extends to a root of f in C . This is the case if f and f' generate the unit ideal of $k[T]$; to see this, let α be a root of f in C/N and a be an element of C whose image in C/N is equal to α . Then $f(a)$ belongs to N and therefore $f'(a)$ is invertible in C ; the element $b = a - f'(a)^{-1}f(a)$ then has image α in C . Since $f'(a)^{-1}f(a)$ has zero square, we have

$$f(b) = f(a) - f'(a)f'(a)^{-1}f(a) = 0.$$

It is worth noting that in this case the lifting of any homomorphism $k[T]/(f) \rightarrow C$ is necessarily unique, which means $k[T]/(f)$ is always formally unramified.

Theorem 17.7.12 (Cohen). *Let k be a field and K be a separable extension of k . Then K is a formally smooth k -algebra.*

Proof. Let C be a k -algebra, N be a square zero ideal of C , $\pi : C \rightarrow C/N$ be the canonical homomorphism, and $\varphi : K \rightarrow C/N$ be a homomorphism of k -algebras. In order to construct a lifting of φ , we distinguish into two cases.

Suppose first that k has characteristic zero. Consider the couple $(K', \tilde{\varphi}')$, where K' is a sub-extension of K and $\tilde{\varphi} : K' \rightarrow C$ is a lifting of the restriction of φ to K' . The set of such couples, endowed with the order defined by extension relation, is inductive, so by Zorn lemma there exists a maximal couple $(K', \tilde{\varphi}')$. We now prove that $K' = K$, so let $x \in K - K'$. If x is transcendental over K' , then the K' -algebra $K'(x)$ is formally smooth (??). On the other hand, if x is algebraic over K' , its minimal polynomial $f \in K'[T]$ is separable, so it is coprime to its derivative (??) and the field $K'(x)$, isomorphic to $K'[T]/(f)$, is then formally smooth (??). In both cases, $K'(x)$ is formally smooth over K' , and there exists an extension of $\tilde{\varphi}'$ to $K'(x)$ which lifts the restriction of φ to $K'(x)$; this contradicts the maximality of $(K', \tilde{\varphi}')$.

Now suppose that k has characteristic $p > 0$. Consider the Frobenius homomorphism $F : C \rightarrow C$ defined by $F(x) = x^p$. Since $F(x) = 0$ for $x \in N$, so there exists a unique homomorphism $\lambda : C/N \rightarrow C$ such that $\lambda \circ \pi = F$. We have $\pi(\lambda(\pi(x))) = \pi(x^p) = \pi(x)^p$, and since π is surjective, this implies $\pi(\lambda(z)) = z^p$ for any element $z \in C/N$. Moreover, let $f : K \rightarrow K^p$ be the isomorphic $y \mapsto y^p$ and $f^{-1} : K^p \rightarrow K$ be the inverse isomorphism. Let $g : K^p \rightarrow C$ be the composition homomorphism

$$K^p \xrightarrow{f^{-1}} K \xrightarrow{\varphi} C/N \xrightarrow{\lambda} C$$

For any $x \in K$, we have $g(x^p) = \lambda(\varphi(x))$, and the map g is k -linear because $\lambda(\alpha z) = \alpha^p \lambda(z)$ for $\alpha \in k$ and $z \in C/N$. Since the extension K/k is separable, $k(K^p)$ is identified with $k \otimes_{k^p} K^p$ (A, V, p.119, remark ??), so there exists a unique k -homomorphism $\tilde{g} : k(K^p) \rightarrow C$ which extends

g. Let $(a_i)_{i \in I}$ be a p -basis for K over $k(K^p)$ (A, V, p.98, theorem 2 ??); for any $i \in I$, choose an element $b_i \in C$ such that $\pi(b_i) = \varphi(a_i)$. We have

$$\tilde{g}(a_i^p) = g(a_i^p) = \lambda(\varphi(a_i)) = \lambda(\pi(b_i)) = b_i^p.$$

By (A, V, p.94, remark ??), there is then a unique homomorphism $\tilde{\varphi} : K \rightarrow C$ which extends \tilde{g} and satisfies $\tilde{\varphi}(a_i) = b_i$ for each i . Note that $\pi(\tilde{\varphi}(a_i)) = \pi(b_i) = \varphi(a_i)$ and for each $x \in K$ we have

$$\pi(\tilde{\varphi}(x^p)) = \pi(h(x^p)) = \pi(g(x^p)) = \pi(\lambda(\varphi(x))) = \varphi(x^p);$$

so $\pi \circ \tilde{\varphi} = \varphi$ and this completes the proof of the theorem. \square

Corollary 17.7.13. *Let k be a field, K be a separable extension of k and A be a linearly topologized K -algebra. If A is formally smooth over K , it is formally smooth over k .*

Proof. This follows from ?? and ??.

\square

Remark 17.7.3. Let k be a field, then any étale k -algebra is formally smooth (A, V, p.34, th.4(d) ??). Later we shall see that any field extension that is formally smooth is absolutely regular, hence separable (§6, n4, example 2 ??).

We conclude this paragraph by establishing a stronger lifting property for formally smooth algebras. For this, we let k be a ring, C be a k -algebra, $(C_n)_{n \in \mathbb{Z}}$ be a filtration of C which is compatible with the k -algebra structure and such that $C_0 = C$. Suppose that C is separated and complete with respect to this filtration, so that the canonical homomorphism $C \rightarrow \varprojlim C/C_n$ is a homeomorphism. Let $m > 0$ be an integer and $\pi : C \rightarrow C/m$ be the canonical homomorphism.

Proposition 17.7.14. *Let A be a formally smooth linearly topologized k -algebra. Then any continuous k -homomorphism $\varphi : A \rightarrow C/C_m$ admits a continuous lifting to C .*

Proof. For any integer $n > m$, let $\pi_n : C/C_n \rightarrow C/C_{n-1}$ be the canonical homomorphism. Since C is identified with the limit $\varprojlim C/C_n$, giving a continuous lifting of φ to C is equivalent to giving a family $(\varphi_n)_{n > m}$ of continuous k -homomorphisms $\varphi_n : A \rightarrow C/C_n$ satisfying $\pi_n \circ \varphi_n = \varphi_{n-1}$. By induction on m , we are then reduced to proving the statement when $C_{m+1} = 0$. The ideal C_m then has square zero (since $2m \geq m+1$), and we can use the hypothesis that A is formally smooth over k . \square

Remark 17.7.4. The result of ?? if the filtration is defined by a nilpotent ideal N of C : that is, if $C_n = N^n$ and the ideal N is nilpotent. If A is a formally smooth linearly topologized k -algebra, we then conclude that any k -homomorphism $\varphi : A \rightarrow C/N$ admits a lifting to C . This can also be considered as a slightly stronger (but in fact equivalent) definition for formally smoothness, and justifies the name "infinitesimal lifting".

17.7.3 Criterion by associated graded algebras

Theorem 17.7.15. *Let k be a ring, A be a k -algebra and \mathfrak{I} be an ideal of A such that the k -algebra A/\mathfrak{I} is formally smooth. Endow A with the \mathfrak{I} -adic topology, the following conditions are equivalent:*

- (i) *the k -algebra A is formally smooth;*

(ii) the A/\mathfrak{J} -module $\mathfrak{J}/\mathfrak{J}^2$ is projective and the canonical homomorphism

$$\beta_{\mathfrak{J}} : S_{A/\mathfrak{J}}(\mathfrak{J}/\mathfrak{J}^2) \rightarrow \text{gr}_{\mathfrak{J}}(A)$$

is bijective;

(iii) the A/\mathfrak{J} -module $\mathfrak{J}/\mathfrak{J}^2$ is projective and the completion of A is k -linearly homeomorphic to the completion of the graded algebra $S_{A/\mathfrak{J}}(\mathfrak{J}/\mathfrak{J}^2)$.

If A is Noetherian, the these conditions are equivalent to

(iv) the ideal \mathfrak{J} is completely secant.

Proof. We first show that (iii) implies (i): in fact, under the hypotheses of (iii), the algebra $S_{A/\mathfrak{J}}(\mathfrak{J}/\mathfrak{J}^2)$, endowed with the topology given by its graduation, is formally smooth over A/\mathfrak{J} (??), hence over k (??(a)); assertion (i) then follows from ??(c). Also, if A is Noetherian, then (ii) is equivalent to (iv) by ??.

Let \widehat{A} be the completion of A and $\widehat{\mathfrak{J}}$ be the completion of \mathfrak{J} . The canonical homomorphism $i : A \rightarrow \widehat{A}$ induces an isomorphism $A/\mathfrak{J} \rightarrow \widehat{A}/\widehat{\mathfrak{J}}$, which admits a lifting $\varphi : A/\mathfrak{J} \rightarrow \widehat{A}$ (??). Let $\lambda : \widehat{\mathfrak{J}} \rightarrow \mathfrak{J}/\mathfrak{J}^2$ be the surjection induced by the canonical isomorphism $\mathfrak{J}/\mathfrak{J}^2 \rightarrow \widehat{\mathfrak{J}}/\widehat{\mathfrak{J}}^2$. For $a \in A$ and $z \in \widehat{\mathfrak{J}}$, we then have $\varphi(\bar{a}) \equiv i(a) \pmod{\widehat{\mathfrak{J}}}$, whence $\varphi(\bar{a}z) \equiv i(a)z \pmod{\widehat{\mathfrak{J}}^2}$ and

$$\lambda(\varphi(\bar{a})z) = \lambda(i(a)z) = \bar{a}\lambda(z).$$

In other words, λ is A/\mathfrak{J} -linear if we endow $\widehat{\mathfrak{J}}$ the A/\mathfrak{J} -module structure induced by φ . Suppose that the homomorphism λ admits a A/\mathfrak{J} -linear section $\sigma : \mathfrak{J}/\mathfrak{J}^2 \rightarrow \widehat{\mathfrak{J}}$.

$$\begin{array}{ccc} \widehat{\mathfrak{J}} & \xleftarrow[\lambda]{\sigma} & \mathfrak{J}/\mathfrak{J}^2 \\ \downarrow & \sim \nearrow & \downarrow \\ \widehat{\mathfrak{J}}/\widehat{\mathfrak{J}}^2 & & A/\mathfrak{J} \longrightarrow \widehat{A}/\widehat{\mathfrak{J}} \end{array}$$

Let S be the symmetric algebra $S_{A/\mathfrak{J}}(\mathfrak{J}/\mathfrak{J}^2)$ and \widehat{S} be its completion, and $\theta : S \rightarrow \widehat{A}$ be the k -homomorphism defined by $\theta(x) = \varphi(x)$ for $x \in S^0 = A/\mathfrak{J}$ and $\theta(x) = \sigma(x)$ for $x \in S^1 = \mathfrak{J}/\mathfrak{J}^2$. Since θ sends S^1 into $\widehat{\mathfrak{J}}$, it sends S^n into $\widehat{\mathfrak{J}}^n$ and therefore extends to a continuous homomorphism $\widehat{\theta} : \widehat{S} \rightarrow \widehat{A}$. The map $\text{gr}_1(\theta) : \mathfrak{J}/\mathfrak{J}^2 \rightarrow \widehat{\mathfrak{J}}/\widehat{\mathfrak{J}}^2$ is the composition of σ with the surjection $\widehat{\mathfrak{J}} \rightarrow \widehat{\mathfrak{J}}/\widehat{\mathfrak{J}}^2$, and since σ is a section of λ , $\text{gr}_1(\theta)$ then coincides with the canonical isomorphism $\mathfrak{J}/\mathfrak{J}^2 \rightarrow \widehat{\mathfrak{J}}/\widehat{\mathfrak{J}}^2$. The map $\text{gr}(\theta) : S \rightarrow \text{gr}_{\widehat{\mathfrak{J}}}(\widehat{A})$ is then the composition of the canonical surjection β with the canonical isomorphism $\text{gr}_{\mathfrak{J}}(A) \rightarrow \text{gr}_{\widehat{\mathfrak{J}}}(\widehat{A})$. Now under the hypothesis of (ii), the A/\mathfrak{J} -module $\mathfrak{J}/\mathfrak{J}^2$ is projective, so λ always admits a A/\mathfrak{J} -linear section. The induced homomorphism $\widehat{\theta} : \widehat{S} \rightarrow \widehat{A}$ is then an isomorphism by the preceding arguments, whence (iii).

Suppose that the k -algebra A is formally smooth, we prove that (i) \Rightarrow (ii). We first show that $\mathfrak{J}/\mathfrak{J}^2$ is projective. Let M be a A/\mathfrak{J} -module and $f : M \rightarrow \mathfrak{J}/\mathfrak{J}^2$ be a surjective A/\mathfrak{J} -linear map. It suffices to prove that f admits a A/\mathfrak{J} -linear section. Let $\pi : A/\mathfrak{J}^2 \rightarrow A/\mathfrak{J}$ be the canonical surjection. By ??, there exists an isomorphism $\psi : A/\mathfrak{J} \oplus \mathfrak{J}/\mathfrak{J}^2 \rightarrow A/\mathfrak{J}^2$ such that $\pi \circ \psi = \text{pr}_1$ and $\psi(0, z) = z$ for $z \in \mathfrak{J}/\mathfrak{J}^2$. Consider the k -algebra $(A/\mathfrak{J}) \oplus M$ and the map $u : (A/\mathfrak{J}) \oplus M \rightarrow A/\mathfrak{J}^2$ defined by $u(x, m) = \psi(x, f(m))$. This is a surjective homomorphism of k -algebras, whose kernel is the submodule $\ker f$ of M , hence has zero square. The canonical

surjection $\rho : A \rightarrow A/\mathfrak{J}^2$ is continuous, and as the k -algebra A is formally smooth, there exists a k -homomorphism $\tilde{\rho} : A \rightarrow (A/\mathfrak{J}) \oplus M$ such that $u \circ \tilde{\rho} = \rho$. As $\text{pr}_1 = \pi \circ \psi = \pi \circ u$, we have $\text{pr}_1 \circ \tilde{\rho} = \pi \circ u \circ \tilde{\rho} = \pi \circ \rho$, so $\text{pr}_1 \circ \tilde{\rho}$ is the canonical surjection $A \rightarrow A/\mathfrak{J}$.

$$\begin{array}{ccccc}
A/\mathfrak{J} & \longrightarrow & (A/\mathfrak{J}) \oplus M & \longrightarrow & M \\
\parallel & & \downarrow & \searrow u & \downarrow f \\
A/\mathfrak{J} & \xrightarrow{\tilde{\rho}} & (A/\mathfrak{J}) \oplus \mathfrak{J}/\mathfrak{J}^2 & \xrightarrow{u} & \mathfrak{J}/\mathfrak{J}^2 \\
& & \downarrow \psi & & \\
A & \xrightarrow{\rho} & A/\mathfrak{J}^2 & \xleftarrow{\pi} & A/\mathfrak{J}
\end{array}$$

We then have $\tilde{\mathfrak{J}}(\mathfrak{J}) \subseteq M$ and therefore $\tilde{\rho}(\mathfrak{J}^2) = 0$, so $\tilde{\rho}$ induces a A/\mathfrak{J} -linear map $s : \mathfrak{J}/\mathfrak{J}^2 \rightarrow M$. We have $u \circ \tilde{\rho} = \rho$ and $\text{pr}_2 \circ \psi^{-1} \circ u(y, m) = f(m)$ for $y \in A/\mathfrak{J}$ and $m \in M$. For $x \in \mathfrak{J}$, if \bar{x} is the class of x in $\mathfrak{J}/\mathfrak{J}^2$, we have

$$f(s(\bar{x})) = f(\text{pr}_2(\tilde{\rho}(x))) = \text{pr}_2(\psi^{-1}(\rho(x))) = \bar{x},$$

so s is a section of f .

Finally, it remains to prove that the homomorphism $\beta_{\mathfrak{J}}$ is injective. Since the A/\mathfrak{J} -module $\mathfrak{J}/\mathfrak{J}$ is injective, λ admits a A/\mathfrak{J} -linear section, and we have the induced homomorphism $\theta : S \rightarrow \widehat{A}$, the homomorphism $\text{gr}(\theta)$ is identified with β by our previous arguments. For each integer $m \geq 0$, let $S_m = \sum_{i>m} S^i$ and $\theta_m : S/S_m \rightarrow A/\mathfrak{J}^{m+1}$ be the induced homomorphism. The composition of θ_m with the canonical surjection $A/\mathfrak{J}^{m+1} \rightarrow A/\mathfrak{J}$ is then the canonical projection of S/S_m to $S^0 = A/\mathfrak{J}$, so its kernel is nilpotent. By ??, there exists a lifting $\psi_m : A \rightarrow S/S_m$ of the canonical surjection $A \rightarrow A/\mathfrak{J}^{m+1}$. As the composition of ψ_m with the projection $S/S_m \rightarrow A/\mathfrak{J}$ is the canonical surjection $A \rightarrow A/\mathfrak{J}$, $\psi_m(\mathfrak{J})$ is formed by elements of positive degrees. By passing to associated algebras, we then deduce a graded k -linear map $\text{gr}(\psi_m) : \text{gr}_{\mathfrak{J}}(A) \rightarrow S/S_m$ such that $\text{gr}_m(\theta) \circ \text{gr}_m(\psi_m) = \text{id}_{\mathfrak{J}^m/\mathfrak{J}^{m+1}}$. It then follows that $\text{gr}_m(\theta)$, and hence β_m , is injective. \square

Corollary 17.7.16. *Let k be a field and A be a Noetherian local k -algebra such that the extension κ_A of k is separable. Then the following conditions are equivalent:*

- (i) *the k -algebra A is formally smooth for the \mathfrak{m}_A -adic topology;*
- (ii) *the ring A is regular;*
- (iii) *the k -algebra A is geometrically regular;*
- (iv) *the k -algebra \widehat{A} is isomorphic to $\kappa_A[[T_1, \dots, T_n]]$, where $n = \dim(A)$.*

Proof. The conditions (ii) and (iii) are equivalent by (example 3 du §6, n4 ??), and amounts to saying that the ideal \mathfrak{m}_A is completely secant (??). Moreover, any isomorphism from \widehat{A} to $\kappa_A[[T_1, \dots, T_n]]$ is bicontinuous since they are local rings. As the k -algebra κ_A is formally smooth by ??, the corollary follows from ?? applied to $\mathfrak{J} = \mathfrak{m}_A$. \square

Corollary 17.7.17. *Let k be a field, A be a Noetherian k -algebra and \mathfrak{I} be an ideal contained in the Jacobson radical of A . Suppose that A is formally smooth over k for the \mathfrak{I} -adic topology, then it is geometrically regular.*

Proof. Let k' be a finite extension of k and $A' = A_{(k')}$. Then it suffices to prove that, for any maximal ideal \mathfrak{m}' of A' , the Noetherian local ring $A'_{\mathfrak{m}'}$ is regular. Now we have $\mathfrak{I}A' \subseteq \mathfrak{m}'$: in fact, the inverse image of \mathfrak{m}' in A is a maximal ideal of A (??), hence contains \mathfrak{I} . The k' -algebra A' is formally smooth for $\mathfrak{I}A'$ -adic topology by ??(b), and the k' -algebra $A'_{\mathfrak{m}'}$ is then formally smooth for the $\mathfrak{I}A'_{\mathfrak{m}'}$ -adic topology ??(a), hence also for the $\mathfrak{m}'A_{\mathfrak{m}'}$ -adic topology (??). Let k_0 be the prime subfield of k' , then $A'_{\mathfrak{m}'}$ is formally smooth over k_0 for the $\mathfrak{m}'A_{\mathfrak{m}'}$ -adic topology (??); as k_0 is perfect, $\kappa(\mathfrak{m}')$ is separable over k_0 , so the ring $A'_{\mathfrak{m}'}$ is regular (??). \square

Corollary 17.7.18. *Let k be a ring and A be a formally smooth k -algebra.*

- (a) *The A -module $\Omega_{A/k}$ is projective.*
- (b) *Suppose that the ring $A \otimes_k A$ is Noetherian and let \mathfrak{I} be the kernel of the multiplication map $A \otimes_k A \rightarrow A$. Then the ideal \mathfrak{I} is completely secant.*

Proof. The k -algebras A and $A \otimes_k A$ are formally smooth (??(c)), and A is isomorphic to a quotient algebra of $A \otimes_k A$ by the kernel \mathfrak{I} . By definition we have $\Omega_{A/k} = \mathfrak{I}$, so the corollary follows from ??.

17.7.4 Formally smoothness for local algebras

Proposition 17.7.19. *Let k_0 be a ring, k be a k_0 -algebra, A be a k -algebra, and \mathfrak{m} be a maximal ideal of A . Suppose that k and A/\mathfrak{m} are formally smooth over k_0 . For A to be formally smooth over k for the \mathfrak{m} -adic topology, it is necessary and sufficient that the following two conditions are satisfied*

- (i) *the canonical homomorphism $\alpha_{\mathfrak{m}} : S_{A/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2) \rightarrow \text{gr}_{\mathfrak{m}}(A)$ is bijective;*
- (ii) *the A/\mathfrak{m} -linear map*

$$\omega : A/\mathfrak{m} \otimes_k \Omega_{k/k_0} \rightarrow A/\mathfrak{m} \otimes_A \Omega_{A/k_0}$$

induced by the canonical map $k \rightarrow A$ is injective.

Proof. Let $d_k : k \rightarrow \Omega_{k/k_0}$ and $d_A : A \rightarrow \Omega_{A/k_0}$ be the universal k_0 -derivations, and endow A with the \mathfrak{m} -adic topology. Suppose first that A is formally smooth over k . Then A is formally smooth over k_0 (??), which is equivalent to (I) in view of ???. Moreover, the k_0 -derivation $\lambda \mapsto 1 \otimes d_k(\lambda)$ of k into $A/\mathfrak{m} \otimes_k \Omega_{k/k_0}$ can then be extended to a k_0 -derivation from A into $A/\mathfrak{m} \otimes_k \Omega_{k/k_0}$ (??), so there exists an A -linear map $u : \Omega_{A/k_0} \rightarrow A/\mathfrak{m} \otimes_k \Omega_{k/k_0}$ such that $u(d_A(\lambda 1_A)) = 1 \otimes d_k(\lambda)$ for $\lambda \in k$. The A/\mathfrak{m} -linear map $\bar{u} : A/\mathfrak{m} \otimes_A \Omega_{A/k_0} \rightarrow A/\mathfrak{m} \otimes_k \Omega_{k/k_0}$ induced by u is then a retraction of ω , which proves (ii).

Conversely, assume that conditions (i) and (ii) are satisfied. Then A is formally smooth over k_0 (??) and the A -module Ω_{A/k_0} is projective ???. Fix an integer $r \geq 0$ and consider the A/\mathfrak{m}^r -linear map

$$\omega_r : A/\mathfrak{m}^r \otimes \Omega_{k/k_0} \rightarrow A/\mathfrak{m}^r \otimes_A \Omega_{A/k_0}$$

induced by the canonical homomorphism $k \rightarrow A$. Let $(\lambda_i)_{i \in I}$ be a family of elements of k such that the $d_k(\lambda_i)$ form a basis for the k -vector space Ω_{k/k_0} . By condition (ii), the elements

$1 \otimes d_A(\lambda_i 1_A)$ are linearly independent in $A/\mathfrak{m} \otimes_A \Omega_{A/k_0}$. By ??, the $1 \otimes d_A(\lambda_i 1_A)$ then form a basis for a direct summand of the A/\mathfrak{m}^r -module $A/\mathfrak{m}^r \otimes_A \Omega_{A/k_0}$. There then exists a A/\mathfrak{m}^r -linear map

$$u_r : A/\mathfrak{m}^r \otimes_A \Omega_{A/k_0} \rightarrow A/\mathfrak{m}^r \otimes_k \Omega_{k/k_0}$$

such that $u_r(1 \otimes d_A(\lambda_i 1_A)) = 1 \otimes d_k(\lambda_i)$ for each i , and hence $u_r \circ \omega_r = \text{id}$.

We now verify that A is formally smooth over k . Let C be a k -algebra, N be a square zero ideal of C , and $\pi : C \rightarrow C/N$ be the canonical homomorphism. Endow C and C/N the discrete topology. Let $\varphi : A \rightarrow C/N$ be a continuous homomorphism of k -algebras. Since A is formally smooth over k_0 , there exists a k_0 -linear lifting $\tilde{\varphi}_0 : A \rightarrow C$ of φ . By ??, the k_0 -linear liftings $\tilde{\varphi} : A \rightarrow C$ of φ are given by the maps $x \mapsto v(d_A(x)) + \tilde{\varphi}_0(x)$, where $v \in \text{Hom}_A(\Omega_{A/k_0}, N)$, so we have to choose v so that $\tilde{\varphi}_0$ is a homomorphism of k -algebras. Now the map $\lambda \mapsto \lambda 1_C - \tilde{\varphi}_0(\lambda 1_A)$ is a k_0 -derivation from k to N ??, so can be written as $h \circ d_k$ where $h \in \text{Hom}_k(\Omega_{k/k_0}, N)$. We choose an integer $r \geq 0$ such that the kernel of φ contains \mathfrak{m}^r (note that $\ker \varphi$ is open in A). The A -module N is then annihilated by \mathfrak{m}^r and it suffices to choose v to be the composition of the homomorphisms

$$\Omega_{A/k_0} \longrightarrow A/\mathfrak{m}^r \otimes_A \Omega_{A/k_0} \xrightarrow{u_r} A/\mathfrak{m}^r \otimes_k \Omega_{k/k_0} \xrightarrow{\tilde{h}} N$$

where \tilde{h} is induced by h . In fact, in this case, for $\lambda \in k$ we have

$$v(d_A(\lambda 1_A)) = \tilde{h}(u_r(1 \otimes d_A(\lambda 1_A))) = \tilde{h}(1 \otimes d_k(\lambda 1_A)) = h(d_k(\lambda)) = \lambda 1_C - \tilde{\varphi}_0(\lambda 1_A). \quad \square$$

Remark 17.7.5. If A is a Noetherian ring, condition (i) of ?? then signifies that the local ring $A_{\mathfrak{m}}$ is regular ??.

Proposition 17.7.20. *Let k be a field and A be a Noetherian local k -algebra. The following conditions are equivalent:*

(i) *A is formally smooth for the \mathfrak{m}_A -adic topology;*

(ii) *A is regular and the κ_A -linear map*

$$\omega : \kappa_A \otimes_k \Omega_k \rightarrow \kappa_A \otimes_A \Omega_A$$

induced by the canonical map $k \rightarrow A$ is injective;

(iii) *A is geometrically regular.*

Proof. To see that (i) \Leftrightarrow (ii), it suffices to apply ?? and ??, where we choose k_0 to be the prime subfield of k . In fact, k and κ_A are then formally smooth over k_0 ???. Now (i) \Rightarrow (iii) follows from ??.

If k has characteristic zero, then it follows from ?? that (iii) implies (i), whence the proposition in this case. Suppose now that k has characteristic $p > 0$, and we prove that (iii) \Rightarrow (ii). Let k'/k be a finite purely inseparable extension of height ≤ 1 . If A and $A_{(k')}$ are regular, the canonical map $\kappa_A \otimes_{k'^p} \Omega_{k'^p/k^p} \rightarrow \kappa_A \otimes_A \Omega_A$ is injective by ???. By (A, V, p.97, th.1(b) ??) applied to the extension k/k^p , the k -vector space Ω_k , which coincides with Ω_{k/k^p} , is the union of the subspaces $k \otimes_{k'^p} \Omega_{k'^p/k^p}$, where k' runs through the set of finite purely inseparable extension of

k of height ≤ 1 in a fixed algebraic closure of k . From this, we see that condition (ii) is then satisfied. \square

17.7.5 Jacobian criterion

Let k be a ring, A be a k -algebra, \mathfrak{I} be an ideal of A and $\bar{d} : \mathfrak{I}/\mathfrak{I}^2 \rightarrow A/\mathfrak{I} \otimes_A \Omega_{A/k}$ be the canonical map. For each A/\mathfrak{I} -algebra R , we denote by

$$\bar{d}_R : R \otimes_{A/\mathfrak{I}} \mathfrak{I}/\mathfrak{I}^2 \rightarrow R \otimes_A \Omega_{A/k}$$

the R -linear map induced by \bar{d} . If the k -algebra A/\mathfrak{I} is formally smooth, \bar{d} then possesses an A -linear retraction (??) and \bar{d}_R possesses an R -linear retraction for any R . More generally, we have the following lemma.

Lemma 17.7.21. *Let \mathfrak{K} be an ideal of A containing \mathfrak{I} . Suppose that there exists an integer $m \geq 0$ such that $\mathfrak{I} \cap \mathfrak{K}^m$ is contained in $\mathfrak{I}\mathfrak{K}$ (this is satisfied if A is Noetherian). If A/\mathfrak{I} is formally smooth over k for the $\mathfrak{K}/\mathfrak{I}$ -adic topology, the map $\bar{d}_{A/\mathfrak{K}} : A/\mathfrak{K} \otimes_{A/\mathfrak{I}} \mathfrak{I}/\mathfrak{I}^2 \rightarrow A/\mathfrak{K} \otimes_A \Omega_{A/k}$ possesses an A -linear retraction.*

Lemma 17.7.22. *Suppose that A is formally smooth over k for the \mathfrak{I} -adic topology. For A/\mathfrak{I} to be formally smooth over k , it is necessary and sufficient that the canonical map $\bar{d} : \mathfrak{I}/\mathfrak{I}^2 \rightarrow A/\mathfrak{I} \otimes_A \Omega_{A/k}$ possesses an A -linear retraction.*

Theorem 17.7.23 (Jacobian Criterion). *Let k be a ring, A be a formally smooth k -algebra and \mathfrak{I} be a finitely generated ideal of A ; put $B = A/\mathfrak{I}$.*

(a) *Let \mathfrak{P} be a prime ideal of B and \mathfrak{p} be the prime ideal of A such that $\mathfrak{P} = \mathfrak{p}/\mathfrak{I}$. Then the following conditions are equivalent:*

- (i) *the k -algebra $B_{\mathfrak{P}}$ is formally smooth;*
- (ii) *there exists $f \in B - \mathfrak{P}$ such that the k -algebra B_f is formally smooth;*
- (iii) *the $\kappa(\mathfrak{P})$ -linear map*

$$\bar{d}_{\kappa(\mathfrak{P})} : \kappa(\mathfrak{P}) \otimes_B \mathfrak{I}/\mathfrak{I}^2 \rightarrow \kappa(\mathfrak{p}) \otimes_A \Omega_{A/k}$$

is injective;

- (iv) *there exist elements f_1, \dots, f_m of \mathfrak{I} whose images $(f_1)_{\mathfrak{p}}, \dots, (f_m)_{\mathfrak{p}}$ generate $\mathfrak{I}_{\mathfrak{p}}$ and k -derivations D_1, \dots, D_m of A such that $\det(D_j(f_i)) \notin \mathfrak{p}$.*

(b) *The set of prime ideals \mathfrak{P} of B satisfying the equivalent conditions of (a) is open in $\text{Spec}(B)$. For B to be formally smooth over k , it is necessary and sufficient that any prime (resp. maximal) ideal of B satisfies these conditions.*

(c) *If A is Noetherian, the conditions of (a) are equivalent to the following:*

- (v) *the k -algebra $B_{\mathfrak{P}}$ is formally smooth for the $\mathfrak{P}B_{\mathfrak{P}}$ -adic topology.*

Moreover, under condition (iv), the ideal $\mathfrak{I}_{\mathfrak{p}}$ is completely secant and $((f_1)_{\mathfrak{p}}, \dots, (f_m)_{\mathfrak{p}})$ is a completely secant sequence for $A_{\mathfrak{p}}$.

Proof. Put $M = \mathfrak{I}/\mathfrak{I}^2$ and $N = B \otimes_A \Omega_{A/k}$. The B -module M is finitely generated, and the B -module N is projective (??). \square

Corollary 17.7.24. *Let k_0 be a ring, k be a Noetherian formally smooth k_0 -algebra, and A be a local k -algebra essentially of finite type. If the k_0 -algebra A is formally smooth for the \mathfrak{m}_A -adic topology, it is formally smooth.*

Proof. By hypotheses, there exists an integer $n \geq 0$, a multiplicative subset S of $k[T_1, \dots, T_n]$, and a surjective k -homomorphism $S^{-1}k[T_1, \dots, T_n] \rightarrow A$. The k -algebra $S^{-1}k[T_1, \dots, T_n]$ is Noetherian and formally smooth by ?? and ??, and hence over k_0 (??). The corollary then follows from ??(c). \square

Corollary 17.7.25. *Let k_0 be a ring, k be a Noetherian formally smooth k_0 -algebra, and A be a local k -algebra essentially of finite type. The set U of prime ideals \mathfrak{p} of A such that the k_0 -algebra $A_{\mathfrak{p}}$ is formally smooth (for the discrete topology or the $\mathfrak{p}A_{\mathfrak{p}}$ -adic topology) is open in $\text{Spec}(A)$ and the following conditions are equivalent:*

- (i) $U = \text{Spec}(A)$;
- (ii) U contains every maximal ideal of A ;
- (iii) the k_0 -algebra A is formally smooth.

Proof. This is already contained in ??, in view of ??.

\square

Remark 17.7.6. The results of ?? and ?? are applicable if k_0 is a field and that we are in one of the following two cases:

- (a) A is an algebra essentially of finite type over a separable extension of k .
- (b) A is a complete Noetherian local algebra whose residue field κ_A is a separable extension of k_0 (in this case we choose k to be a formal series algebra over κ_A for which A is a quotient).

In each case, it follows from ??, in view of ?? and ??, that the k_0 -algebra A is formally smooth if and only if it is geometrically regular.

Corollary 17.7.26 (Zariski's Jacobian Criterion). *Let k be a field, A be a regular local k -algebra, and \mathfrak{I} be a proper ideal of A . Suppose that the k -algebra A is essentially of finite type or complete. For the local ring A/\mathfrak{I} to be regular, it is necessary and sufficient that there exists a generating family f_1, \dots, f_m of \mathfrak{I} and derivations D_1, \dots, D_m of A such that $\det(D_j(f_i)) \notin \mathfrak{m}_A$. In this case, the elements (f_1, \dots, f_m) is a subset of a system of parameters of A and the ideal \mathfrak{I} is prime.*

Proof. Let k_0 be the prime subfield of k . The k_0 -algebra A is geometrically regular by ??, hence formally smooth (??). By the same reasoning, the regularity of A/\mathfrak{I} is equivalent to its formally smoothness over k_0 . The first assertion then follows from ??, which also implies that the sequence (f_1, \dots, f_m) is completely secant for A in this case. We can then apply ?? to conclude the second assertion (note that A/\mathfrak{I} is then an integral domain). \square

Remark 17.7.7. Under the hypotheses of ??, the A -module $\Omega_{A/k}$ is projective by ??, hence free. Any derivation of A in κ_A is therefore lifted into a derivation of A . The condition of the statement can therefore be expressed as follows: there exists a generating system (f_1, \dots, f_m) of \mathfrak{I} and derivations D_1, \dots, D_m of A in κ_A such that $\det(D_j(f_i)) \neq 0$.

Corollary 17.7.27 (Zariski). *Let k be a field and A be a k -algebra essentially of finite type or a complete Noetherian local k -algebra. The set of prime ideals \mathfrak{p} of A such that $A_{\mathfrak{p}}$ is regular is open in $\text{Spec}(A)$.*

Proof. It suffices to apply ?? and choose k_0 to be the prime field of k . □

17.7.6 Smooth algebras

Lemma 17.7.28. *Let $\rho : A \rightarrow B$ be a local homomorphism of Noetherian local rings. Suppose that B is essentially of finite type over A . For the A -algebra to be formally smooth, it is necessary and sufficient that the A -module B is flat and the κ_A -algebra $\kappa_A \otimes_A B$ is geometrically regular.*

Proof. By hypotheses, there exists an integer $n \geq 0$, a prime ideal \mathfrak{q} of $A[T_1, \dots, T_n]$ and a surjective homomorphism $h : A[T_1, \dots, T_n]_{\mathfrak{q}} \rightarrow B$. Denote by C be the local A -algebra $A[T_1, \dots, T_n]_{\mathfrak{q}}$; it is a formally smooth (??) and flat over A , and we can identify B with the A -algebra C/\mathfrak{I} , where $\mathfrak{I} = \ker h$.

Put $\bar{C} = \kappa_A \otimes_A C$ and $\bar{B} = \kappa_A \otimes_A B$. Suppose that B is formally smooth over A . The κ_A -algebra \bar{C} is then formally smooth (??), hence geometrically regular (??). Moreover, since $\bar{C}/\mathfrak{I}\bar{C}$ is identified with \bar{B} and that the κ_A -algebra \bar{C} is formally smooth, the ideal $\mathfrak{I}\bar{C}$ of \bar{C} is completely secant (??). It then follows from ?? that the A -module B is flat.

Conversely, suppose that B is flat over A and the κ_A -algebra \bar{B} is geometrically regular. Then the local κ_A -algebra \bar{B} is formally smooth (??). Put $\bar{\mathfrak{I}} = \kappa_A \otimes_A \mathfrak{I}$; since B is a flat A -module, the canonical map $\bar{\mathfrak{I}} \rightarrow \mathfrak{I}\bar{C}$ is bijective and \bar{B} is identified with $\bar{C}/\bar{\mathfrak{I}}$. It then follows from ?? that the canonical map

$$\bar{\mathfrak{I}}/\bar{\mathfrak{I}}^2 \rightarrow \bar{B} \otimes_{\bar{C}} \Omega_{\bar{C}/\kappa_A}$$

is injective and admits a retraction. Now $\bar{\mathfrak{I}}/\bar{\mathfrak{I}}^2$ is identified with $\kappa_A \otimes_A \mathfrak{I}/\mathfrak{I}^2$, hence with $\bar{B} \otimes_B \mathfrak{I}/\mathfrak{I}^2$. On the other hand the \bar{C} -module $\Omega_{\bar{C}/\kappa_A}$ is canonically isomorphic to $\bar{C} \otimes_C \Omega_{C/A}$ (A, III, p.136, prop.20), hence $\bar{B} \otimes_{\bar{C}} \Omega_{\bar{C}/\kappa_A}$ is canonically isomorphic to $\bar{B} \otimes_C \Omega_{C/A}$. Passing to quotient by the maximal ideal of \bar{B} , we then obtain an injective homomorphism (by the existence of retraction)

$$\kappa_B \otimes_B \mathfrak{I}/\mathfrak{I}^2 \rightarrow \kappa_B \otimes_C \Omega_{C/A}$$

which is none other than \bar{d}_{κ_B} , so B is formally smooth over A (??). □

Theorem 17.7.29. *Let A be a Noetherian ring and B be an A -algebra essentially of finite type. Then the following conditions are equivalent:*

- (i) *the A -algebra B is formally smooth;*
- (ii) *for any prime ideal $\mathfrak{P} \in \text{Spec}(B)$, the A -algebra $B_{\mathfrak{P}}$ is formally smooth (resp. formally smooth for the $\mathfrak{P}B_{\mathfrak{P}}$ -adic topology);*
- (iii) *the A -module B is flat and for any $\mathfrak{p} \in \text{Spec}(A)$, the $\kappa(\mathfrak{p})$ -algebra $\kappa(\mathfrak{p}) \otimes_A B$ is geometrically regular;*

- (iv) the A -module B is flat and for any regular A -algebra R , the ring $R \otimes_A B$ is geometrically regular;
- (v) the A -module B is flat and the kernel of the multiplication homomorphism $\mu : B \otimes_A B \rightarrow B$ is completely secant.

Proof. The equivalence of (i) and (ii) follows from [??](#). To see that (i) \Rightarrow (v), suppose that B is formally smooth over A , let \mathfrak{P} be a prime ideal of B and \mathfrak{p} be its contraction into A . The $A_{\mathfrak{p}}$ -algebra $B_{\mathfrak{P}}$ is formally smooth [??](#), hence flat [??](#), so the A -module B is flat [??](#). On the other hand, the ring $B \otimes_A B$ is Noetherian [??](#), so the ideal \mathfrak{I} is completely secant in view of [??](#).

Suppose that condition (v) is satisfied and let $\mathfrak{p} \in \text{Spec}(A)$; we show that (v) \Rightarrow (iii). The map

$$1 \otimes \mu : \kappa(\mathfrak{p}) \otimes_A (B \otimes_A B) \rightarrow \kappa(\mathfrak{p}) \otimes_A B$$

is identified with the multiplication map

$$\mu_{\mathfrak{p}} : (\kappa(\mathfrak{p}) \otimes_A B) \otimes_{\kappa(\mathfrak{p})} (\kappa(\mathfrak{p}) \otimes_A B) \rightarrow \kappa(\mathfrak{p}) \otimes_A B$$

of the $\kappa(\mathfrak{p})$ -algebra $\kappa(\mathfrak{p}) \otimes_A B$, so the kernel $\ker \mu_{\mathfrak{p}}$ is identified with $\mathfrak{I}(\kappa(\mathfrak{p}) \otimes_A (B \otimes_A B))$. It is completely secant since the A -module B is flat [??](#). Condition (iii) then follows from [??](#).

Under the hypotheses of (iii), let \mathfrak{P} be a prime ideal of B and \mathfrak{p} be its contraction in A . The $A_{\mathfrak{p}}$ -module $B_{\mathfrak{P}}$ is flat, and the $\kappa(\mathfrak{p})$ -algebra $\kappa(\mathfrak{p}) \otimes_{A_{\mathfrak{p}}} B_{\mathfrak{P}}$, identified with a fraction ring of $\kappa(\mathfrak{p}) \otimes_A B$, is geometrically regular [??](#). It then follows from [??](#) that $B_{\mathfrak{P}}$ is formally smooth over $A_{\mathfrak{p}}$, hence over A [??](#); this proves (iii) \Rightarrow (ii).

It remains to prove the equivalence of (iii) and (iv). First consider the condition (iii). Let R be a regular A -algebra. The R -module $R \otimes_A B$ is flat by [??](#). Let \mathfrak{r} be a prime ideal of R and \mathfrak{p} be its contraction in A . The ring $\kappa(\mathfrak{r}) \otimes_R (R \otimes_A B)$, identified with $\kappa(\mathfrak{r}) \otimes_{\kappa(\mathfrak{p})} (\kappa(\mathfrak{p}) \otimes_A B)$, is regular by [??](#), so $R \otimes_A B$ is regular by [??](#). Conversely, if condition (iv) is satisfied, for any prime ideal \mathfrak{p} of A and any extension k of $\kappa(\mathfrak{p})$, the ring $k \otimes_{\kappa(\mathfrak{p})} (\kappa(\mathfrak{p}) \otimes_A B)$, which is identified with $k \otimes_A B$, is regular, whence (iii). \square

Let A be a Noetherian ring. We say that an A -algebra B is **smooth** if it is essentially of finite type and satisfies the equivalent conditions of [??](#).

Proposition 17.7.30. *Let A be a Noetherian ring.*

- (a) *Let A' be a Noetherian A -algebra and B be a smooth A -algebra. Then the A' -algebra $A' \otimes_A B$ is smooth.*
- (b) *Let B be a smooth A -algebra and C be a smooth B -algebra. Then the A -algebra C is smooth.*
- (c) *Let B and C be two smooth A -algebras. Then the A -algebra $B \otimes_A C$ is smooth.*

Proof. This follows from the analogues results for algebras essentially of finite type and formally smooth. \square

Example 17.7.31. The smooth algebras over a field k are the k -algebras essentially of finite type and geometrically regular. In particular, if k is a perfect field, then we see that these are exactly regular k -algebras essentially of finite type.

Example 17.7.32. Let A be a Noetherian ring and $(T_i)_{i \in I}$ be a finite family of indeterminates. Then the polynomial algebra $A[(T_i)_{i \in I}]$ is smooth over A . More generally, let F_1, \dots, F_m be elements of $A[(T_i)_{i \in I}]$, and $B = A[(T_i)_{i \in I}] / (F_1, \dots, F_m)$. If for any maximal ideal \mathfrak{N} of B , the class of the matrix $(\partial F_j / \partial T_i) \bmod \mathfrak{N}$ is of rank m , then the A -algebra B is smooth (??).

17.8 Duality of finite length modules

17.8.1 Indecomposable injective modules

Chapter 18

The language of schemes

18.1 Affine schemes

Let A be a ring. Recall that we have associate with A a topological space $\text{Spec}(A)$, called the spectrum of A . In this section we shall make $\text{Spec}(A)$ a locally ringed space and consider sheaf of modules over it; such spaces will be called **affine schemes**.

18.1.1 Sheaves associated with a module

Let A be a ring and M an A -module. For any element $f \in A$, let S_f be the multiplicative subset consisting of powers of f . Recall that the localization of M with respect to S_f is then denoted by M_f , and that of A by A_f . Let \bar{S}_f be the saturation of S_f , which is defined to be the complement of the union of prime ideals of A that are disjoint from S_f , or equivalently not contains f . By ??, the set \bar{S}_f is also characterized by

$$\bar{S}_f = \{x \in A : \text{there exist } n, m \geq 0 \text{ such that } f^n x = f^m\}.$$

Also, by ??, we have $\bar{S}_f A = A_f$ and $\bar{S}_f M = M_f$.

Lemma 18.1.1. *Let f, g be elements of A . Then the following conditions are equivalent:*

- (i) $g \in \bar{S}_f$, or equivalently $\bar{S}_g \subseteq \bar{S}_f$;
- (ii) $f \in \sqrt{(g)}$, or equivalently $\sqrt{(f)} \subseteq \sqrt{(g)}$;
- (iii) $D(f) \subseteq D(g)$, or equivalently $V(g) \subseteq V(f)$.

Proof. We first note that $g \in \bar{S}_f$ is equivalent to $S_g \subseteq \bar{S}_f$, so the equivalence in (i). Also, the equivalence of (ii) and (iii) follows from ?. Finally, if $g \in \bar{S}_f$, then there exist $n, m \geq 0$ such that $f^n g = f^m$, which is an element of (g) , and thus $f \in \sqrt{(g)}$. Conversely, if $D(f) \subseteq D(g)$, then by the descriptions $S_f = \bigcup_{f \notin \mathfrak{p}} \mathfrak{p}$ and $S_g = \bigcup_{g \notin \mathfrak{p}} \mathfrak{p}$, we conclude that $\bar{S}_g \subseteq \bar{S}_f$, whence the lemma. \square

If $D(g) \subseteq D(f)$ in $\text{Spec}(A)$, then by ??, we have $\bar{S}_f \subseteq \bar{S}_g$, so there is a canonical homomorphism $\rho_{g,f} : M_f \rightarrow M_g$; moreover, if $D(f) \supseteq D(g) \supseteq D(h)$, we then have

$$\rho_{h,g} \circ \rho_{g,f} = \rho_{h,f}.$$

As f runs through $A - \mathfrak{p}$ (where \mathfrak{p} is a point in $X = \text{Spec}(A)$), the set S_f then constitute a filtered set indexed by $A - \mathfrak{p}$, since any two element f, g of $A - \mathfrak{p}$ contains S_{fg} ; as the union of the S_f for $f \in A - \mathfrak{p}$ is $A - \mathfrak{p}$, we conclude from ?? that the $A_{\mathfrak{p}}$ -module $M_{\mathfrak{p}}$ is canonically identified with the direct limit $\varinjlim M_f$, relative to the family $(\rho_{g,f})$ of homomorphisms. For each $f \in A - \mathfrak{p}$, we denote the canonical homomorphism from M_f to $M_{\mathfrak{p}}$ by

$$\rho_{\mathfrak{p}}^f : M_f \rightarrow M_{\mathfrak{p}}.$$

We now define the **structural sheaf** of the prime spectrum $X = \text{Spec}(A)$, denoted by \tilde{A} , to be the sheaf of rings associated with the presheaf $D(f) \mapsto A_f$ over the basis \mathcal{B} of X , formed by $D(f)$ with $f \in A$. Similarly, for an A -module M , we define the **associated sheaf** \tilde{M} to be the sheaf associated presheaf $D(f) \mapsto M_f$ over the basis \mathcal{B} of X . By the property of sheafification, it is clear that the stalk $\tilde{A}_{\mathfrak{p}}$ (resp. $\tilde{M}_{\mathfrak{p}}$) is identified with the ring $A_{\mathfrak{p}}$ (resp. with $A_{\mathfrak{p}}$ -module $M_{\mathfrak{p}}$).

Theorem 18.1.2. *For each A -module M , the presheaf $D(f) \mapsto M_f$ is a sheaf on the basis \mathcal{B} of X , so for each $f \in A$ we have a canonical isomorphism*

$$M_f \rightarrow \Gamma(D(f), \tilde{M}).$$

In particular, M is canonically identified with $\Gamma(X, \tilde{M})$.

Proof. To show that the presheaf $D(f) \mapsto M_f$ is a sheaf on the basis \mathcal{B} of X , we need to check the sheaf condition for the standard open coverings. If $D(f) = \bigcup_{i=1}^n D(g_i)$, then the sheaf condition for this covering is equivalent with the exactness of the sequence

$$0 \longrightarrow M_f \longrightarrow \bigoplus_i M_{g_i} \longrightarrow \bigoplus_{i,j} M_{g_i g_j}$$

Note that $D(g_i) = D(fg_i)$, and hence we can rewrite this sequence as the sequence

$$0 \longrightarrow M_f \longrightarrow \bigoplus_i M_{fg_i} \longrightarrow \bigoplus_{i,j} M_{fg_i g_j}$$

In addition, since $D(g_i)$'s cover $D(f)$ (which is identified with $\text{Spec}(A_f)$), the elements g_1, \dots, g_n generate the unit ideal in A_f , so we may apply ?? to the module M_f over A_f and the elements g_1, \dots, g_n to conclude that the sequence is exact. \square

Corollary 18.1.3. *Let M, N be A -modules. The canonical homomorphism*

$$\text{Hom}_A(M, N) \rightarrow \text{Hom}_{\tilde{A}}(\tilde{M}, \tilde{N}), \quad \phi \mapsto \tilde{\phi}$$

is bijective. In particular, the relations $M = 0$ and $\tilde{M} = 0$ are equivalent.

Proof. Consider the canonical homomorphism $\text{Hom}_{\tilde{A}}(\tilde{M}, \tilde{N}) \rightarrow \text{Hom}_A(M, N)$, $\phi \mapsto \Gamma(\phi)$ (??). It suffices to show that $\phi \mapsto \tilde{\phi}$ and $\phi \mapsto \Gamma(\phi)$ are inverses of each other. Now, it is evident that $\Gamma(\tilde{\phi}) = \phi$, by the definition of $\tilde{\phi}$. On the other hand, if we put $\phi = \Gamma(\varphi)$ for $\varphi \in \text{Hom}_{\tilde{A}}(\tilde{M}, \tilde{N})$, the map $\varphi_{D(f)} : \Gamma(D(f), \tilde{M}) \rightarrow \Gamma(D(f), \tilde{N})$ induced by φ is making the following diagram

commutes:

$$\begin{array}{ccc} M & \xrightarrow{\phi} & N \\ \rho_{f,1} \downarrow & & \downarrow \rho_{f,1} \\ M_f & \xrightarrow{\varphi_{D(f)}} & N_f \end{array}$$

We then have necessarily $\varphi_{D(f)} = \phi_f$ for each $f \in A$, which shows $\widetilde{\Gamma(\varphi)} = \varphi$. \square

Proposition 18.1.4. *For each $f \in A$, the open set $D(f) \subseteq X$ is canonically identified with the spectrum $\text{Spec}(A_f)$, and the sheaf \tilde{M}_f associated with the A_f -module M_f is canonically identified with the restriction $\tilde{M}_{D(f)}$.*

Proof. The first assertion is proved in ???. Now for $D(g) \subseteq D(f)$, then M_g is identified with the localization of M_f with respect to the canonical image of g in A_f , so the canonical identification of \tilde{M}_f and $\tilde{M}_{D(f)}$ follows by definition. \square

Proposition 18.1.5. *The functor $M \mapsto \tilde{M}$ is an exact functor from the category of A -modules to the category of \tilde{A} -modules.*

Proof. Let M, N be two A -modules and $\phi : M \rightarrow N$ a homomorphism; for any $f \in A$, we have a corresponding homomorphism $\phi_f : M_f \rightarrow N_f$, and for $D(g) \subseteq D(f)$ the diagram

$$\begin{array}{ccc} M_f & \xrightarrow{\phi_f} & N_f \\ \rho_{g,f} \downarrow & & \downarrow \rho_{g,f} \\ M_g & \xrightarrow{\phi_g} & N_g \end{array}$$

is commutative. These then give a homomorphism of \tilde{A} -modules $\tilde{\phi} : \tilde{M} \rightarrow \tilde{N}$. Moreover, for each $\mathfrak{p} \in X$, $\tilde{\phi}_\mathfrak{p}$ is the direct limit of ϕ_f for $f \in A - \mathfrak{p}$, and consequently identified with the canonical homomorphism $\phi_\mathfrak{p} : M_\mathfrak{p} \rightarrow N_\mathfrak{p}$. If P is another A -module, $\psi : N \rightarrow P$ a homomorphism and $\eta = \psi \circ \underline{\phi}$, then it is immediate that $\eta_\mathfrak{p} = \psi_\mathfrak{p} \circ \phi_\mathfrak{p}$, hence $\tilde{\eta} = \tilde{\psi} \circ \tilde{\phi}$. We thus get a covariant functor $(-)$ from the category of A -modules to the category of \tilde{A} -modules. This functor is exact since for each $\mathfrak{p} \in X$, $M \mapsto M_\mathfrak{p}$ is an exact functor; furthermore, we have $\text{supp}(M) = \text{supp}(\tilde{M})$ by the definitions of these two members. \square

Corollary 18.1.6. *Let M and N be two A -modules.*

- (a) *If $\phi : M \rightarrow N$ is a homomorphism, then the sheaves associated with $\ker \phi$, $\text{im } \phi$, and $\text{coker } \phi$ are $\ker \tilde{\phi}$, $\text{im } \tilde{\phi}$, and $\text{coker } \tilde{\phi}$, respectively. In particular, $\tilde{\phi}$ is injective (resp. surjective, bijective) if and only if ϕ is injective (resp. surjective, bijective).*
- (b) *If M is a filtered limit (resp. direct sum) of a family $(M_i)_{i \in I}$ of A -modules, then \tilde{M} is a filtered limit (resp. direct sum) of the family (\tilde{M}_i) .*

Proof. For (a), it suffices to apply the exact functor $M \mapsto \tilde{M}$ to the following exact sequences:

$$0 \longrightarrow \ker \phi \longrightarrow M \longrightarrow \text{im } \phi \longrightarrow 0$$

$$0 \longrightarrow \text{im } \phi \longrightarrow N \longrightarrow \text{coker } \phi \longrightarrow 0$$

Now let (M_i, ρ_{ji}) be a filtered system of A -modules, with limit M , and let $\rho_i : M_i \rightarrow M$ be the canonical homomorphism. Since we have $\tilde{\rho}_{kj} \circ \tilde{\rho}_{ji} = \tilde{\rho}_{ji}$ and $\tilde{\rho}_i = \tilde{\rho}_j \circ \tilde{\rho}_{ji}$ for $i \leq j \leq k$, we see $(\tilde{M}, \tilde{\rho}_{ji})$ is a direct system of sheaves over X , and if we denote by $\eta_i : \tilde{M}_i \rightarrow \varinjlim \tilde{M}_i$ the canonical homomorphism, a unique homomorphism $\psi : \varinjlim \tilde{M}_i \rightarrow \tilde{M}$ such that $\psi \circ \eta_i = \tilde{\rho}_i$. For this ψ to be bijective, it suffices that for each $\mathfrak{p} \in X$, $\psi_{\mathfrak{p}}$ is a bijection from $(\varinjlim \tilde{M}_i)_{\mathfrak{p}}$ to $\tilde{M}_{\mathfrak{p}}$; but $\tilde{M}_{\mathfrak{p}} = M_{\mathfrak{p}}$ and

$$(\varinjlim \tilde{M}_i)_{\mathfrak{p}} = \varinjlim (\tilde{M}_i)_{\mathfrak{p}} = \varinjlim (M_i)_{\mathfrak{p}} = M_{\mathfrak{p}}$$

Also, it follows by definition that $(\tilde{\rho}_i)_{\mathfrak{p}}$ and $(\eta_i)_{\mathfrak{p}}$ are equal to the canonical homomorphism from $(M_i)_{\mathfrak{p}}$ to $M_{\mathfrak{p}}$; since $(\tilde{\rho}_i)_{\mathfrak{p}} = \psi_{\mathfrak{p}} \circ (\eta_i)_{\mathfrak{p}}$, $\psi_{\mathfrak{p}}$ is therefore the identity.

Finally, if M is a direct sum of two modules N and P , it is immediate that $\tilde{M} = \tilde{M} \oplus \tilde{P}$; by taking filtered limits, we then generalize this result for the direct sum of an arbitrary family. This completes the proof. \square

Remark 18.1.1. By ??, we conclude that the sheaves which are isomorphic to the sheaves associated with A -modules form an abelian category. Note also that it follows from ?? that if M is a finitely generated A -module, that is, if there exists a surjective homomorphism $A^n \rightarrow M$, then there exists a homomorphism surjective $\tilde{A}^n \rightarrow \tilde{M}$, in other words, the \tilde{A} -module \tilde{M} is generated by a finite family of sections over X , and vice versa.

Corollary 18.1.7. Let N and P be submodules of M . The sheaves \tilde{N} and \tilde{P} can be identified with sub- \tilde{A} -modules of \tilde{M} , and we have

$$\widetilde{N + P} = \tilde{N} + \tilde{P}, \quad \widetilde{N \cap P} = \tilde{N} \cap \tilde{P}.$$

In particular, if $\tilde{N} = \tilde{P}$, then $N = P$.

Proof. If N is a submodule of an A -module M , the canonical injection $N \rightarrow M$ induced an injective homomorphism $\tilde{N} \rightarrow \tilde{M}$, hence identifies \tilde{N} with a sub- \tilde{A} -module of \tilde{M} . Now note that $N + P$ is the image of the canonical homomorphism $\alpha : N \oplus P \rightarrow M$, so by ?? we have

$$\widetilde{N + P} = \widetilde{\text{im } \alpha} = \text{im } \tilde{\alpha} = \tilde{N} + \tilde{P}$$

since $\tilde{\alpha}$ is equal to the canonical homomorphism $\tilde{N} \oplus \tilde{P} \rightarrow \tilde{M}$. Similarly, since $N \cap P$ is the kernel of the canonical homomorphism $M \rightarrow (M/N) \oplus (M/P)$, we also have $\widetilde{N \cap P} = \tilde{N} \cap \tilde{P}$. \square

Corollary 18.1.8. Over the category of sheaves isomorphic to sheaves associated with A -modules, the global section functor Γ is exact.

Proof. In fact, let $\tilde{M} \xrightarrow{\tilde{\phi}} \tilde{N} \xrightarrow{\tilde{\psi}} \tilde{P}$ be an exact sequence corresponding to two homomorphisms $\phi : M \rightarrow N$ and $\psi : N \rightarrow P$. If $Q = \text{im } \phi$ and $R = \ker \psi$, we have

$$\tilde{Q} = \text{im } \tilde{\phi} = \ker \tilde{\psi} = \tilde{R}$$

by ??, so $Q = R$ and the sequence is exact. \square

Corollary 18.1.9. Let M and N be two A -modules.

- (a) The sheaf associated with $M \otimes_A N$ is canonically identified with $\tilde{M} \otimes_{\tilde{A}} \tilde{N}$.
- (b) If moreover M is finitely presented, the sheaf associated with $\text{Hom}_A(M, N)$ is canonically identified with $\mathcal{H}\text{om}_{\tilde{A}}(\tilde{M}, \tilde{N})$.

Proof. The sheaf $\mathcal{F} = \tilde{M} \otimes_{\tilde{A}} \tilde{N}$ is the sheafification of the presheaf

$$U \mapsto \mathcal{F}(U) = \Gamma(U, \tilde{M}) \otimes_{\Gamma(U, \tilde{A})} \Gamma(U, \tilde{N})$$

where U runs through the basis \mathcal{B} of X formed by $D(f)$, $f \in A$. Now, $\mathcal{F}(D(f))$ is canonically identified with $M_f \otimes_{A_f} N_f$ by ??, which is isomorphic to $\Gamma(D(f), \widetilde{M \otimes_A N})$. Moreover, it is immediately verified that the canonical isomorphisms

$$\mathcal{F}(D(f)) \cong \Gamma(D(f), \widetilde{M \otimes_A N})$$

is compatible with the restriction maps, so they define a canonical isomorphism $\widetilde{M \otimes_A N} \cong \widetilde{M} \otimes_{\tilde{A}} \tilde{N}$.

Now assume that M is finitely presented. The sheaf $\mathcal{G} = \mathcal{H}\text{om}_{\tilde{A}}(\tilde{M}, \tilde{N})$ is the sheafification of the presheaf

$$U \mapsto \mathcal{G}(U) = \text{Hom}_{\tilde{A}|_U}(\tilde{M}|_U, \tilde{N}|_U)$$

where U runs through the basis \mathcal{B} of X . By ?? and ??, the module $\mathcal{G}(D(f))$ is then identified with $\text{Hom}_{A_f}(M_f, N_f)$, which is isomorphic to $\Gamma(D(f), \widetilde{\text{Hom}_A(M, N)})$ by ???. It is clear that these isomorphisms are compatible with the restriction maps, so we conclude that $\mathcal{H}\text{om}_{\tilde{A}}(\tilde{M}, \tilde{N}) \cong \widetilde{\text{Hom}_A(M, N)}$. \square

Now consider an A -algebra B (commutative); this can be interpreted by saying that B is an A -module and that we are given an element $e \in B$ and an A -homomorphism $\varphi : B \otimes_A B \rightarrow B$ so that the diagrams

$$\begin{array}{ccc} B \otimes_A B \otimes_A B & \xrightarrow{\varphi \otimes 1} & B \otimes_A B \\ 1 \otimes \varphi \downarrow & & \downarrow \varphi \\ B \otimes_A B & \xrightarrow{\varphi} & B \end{array} \quad \begin{array}{ccc} B \otimes_A B & \xrightarrow{\sigma} & B \otimes_A B \\ & \searrow \varphi & \swarrow \varphi \\ & B & \end{array}$$

(where σ is the canonical symmetry) commute, and that $\varphi(e \otimes x) = \varphi(x \otimes e) = x$. In view of ??, the homomorphism $\tilde{\varphi} : \tilde{B} \otimes_{\tilde{A}} \tilde{B} \rightarrow \tilde{B}$ of \tilde{A} -modules satisfies similar conditions, hence defines a \tilde{A} -algebra structure on \tilde{B} . In the same way, the data of a B -module N amounts to giving an A -module N and an A -homomorphism $\psi : B \otimes_A N \rightarrow N$ such that the diagram

$$\begin{array}{ccc} B \otimes_A B \otimes_A N & \xrightarrow{\varphi \otimes 1} & B \otimes_A N \\ 1 \otimes \psi \downarrow & & \downarrow \psi \\ B \otimes_A N & \xrightarrow{\psi} & N \end{array}$$

commutes and $\psi(e \otimes n) = n$; the homomorphism $\tilde{\psi} : \tilde{B} \otimes_{\tilde{A}} \tilde{N} \rightarrow \tilde{N}$ then satisfies similar conditions, and defines on \tilde{N} a \tilde{B} -module structure.

If $\rho : B \rightarrow B'$ (resp. $\phi : N \rightarrow N'$) is a homomorphism of A -algebras (resp. a homomorphisms of B -modules), then $\tilde{\rho}$ (resp. $\tilde{\phi}$) is a homomorphism of \tilde{A} -algebras (resp. a homomorphisms of \tilde{B} -modules), $\ker \tilde{\rho}$ is an ideal of \tilde{B} (resp. $\ker \tilde{\phi}$, $\text{coker } \tilde{\phi}$, and $\text{im } \tilde{\phi}$ are \tilde{B} -modules). Moreover, by ??(b) if N is a B -module, then \tilde{N} is a finitely generated \tilde{B} -module if and only if N is finitely generated over B .

If M and N are two B -modules, the \tilde{B} -module $\tilde{M} \otimes_{\tilde{B}} \tilde{N}$ is canonically identified with $\widetilde{M \otimes_B N}$; similarly, $\mathcal{H}\text{om}_{\tilde{B}}(\tilde{M}, \tilde{N})$ is canonically identified with $\widetilde{\text{Hom}_B(M, N)}$ if M is finitely presented. If \mathfrak{b} is an ideal of B , then $\mathfrak{b}\tilde{N} = \tilde{\mathfrak{b}}\tilde{N}$.

Finally, if B is a graded A -algebra with (B_n) its graduation, the \tilde{A} -algebra \tilde{B} is then the direct sum of the sub- \tilde{A} -modules \tilde{B}_n (??), so (\tilde{B}_n) is a graduation of \tilde{B} . Similarly, if M is a graded B -module with graduation (M_n) , then \tilde{M} is a graded \tilde{B} -module with graduation (\tilde{M}_n) .

18.1.2 Functorial properties of the associated sheaf

We now consider the functorial properties of the operation $M \mapsto \tilde{M}$. Let A and B be rings and $\varphi : B \rightarrow A$ be a ring homomorphism. Then we have an associated map

$${}^a\varphi : X = \text{Spec}(A) \rightarrow Y = \text{Spec}(B)$$

We will define a canonical homomorphism

$$\varphi^\# : \mathcal{O}_Y \rightarrow {}^a\varphi_*(\mathcal{O}_X)$$

of sheaf of rings. For any $g \in B$, we set $f = \varphi(g)$; we have $\varphi^{-1}(D(g)) = D(f)$ by ???. Now the sections $\Gamma(D(g), \tilde{B})$ and $\Gamma(D(f), \tilde{A})$ are canonically identified with B_g and A_f , respectively, and we have an induced map $\varphi_g : B_g \rightarrow A_f$, which then gives a homomorphism of rings

$$\Gamma(D(g), \tilde{B}) \rightarrow \Gamma(\varphi^{-1}(D(g)), \tilde{A}) = \Gamma(D(g), {}^a\varphi_*(\tilde{A})).$$

Moreover, these homomorphisms satisfy the following compatible conditions: for $D(g) \supseteq D(g')$ in Y , the diagram

$$\begin{array}{ccc} \Gamma(D(g), \tilde{A}) & \longrightarrow & \Gamma(D(g), {}^a\varphi_*(\tilde{A})) \\ \downarrow & & \downarrow \\ \Gamma(D(g'), \tilde{A}) & \longrightarrow & \Gamma(D(g'), {}^a\varphi_*(\tilde{A})) \end{array}$$

is commutative; we then get a morphism of \mathcal{O}_Y -algebras, since $D(g)$ form a basis for the topological space Y . The couple $({}^a\varphi, \varphi^\#)$ is called the **canonical morphism** of the locally ringed spaces induced by φ .

We also note that, if $y = {}^a\varphi(x)$, the homomorphism $\varphi_x^\#$ is no other than the homomorphism

$$\varphi_x : B_y \rightarrow A_x$$

induced by the homomorphism $\varphi : B \rightarrow A$. In fact, for $b/g \in B_y$, where $b, g \in B$ and $g \notin \mathfrak{p}_y$,

$D(g)$ is then an open neighborhood of y in Y , and the homomorphism

$$\Gamma(D(g), \widetilde{B}) \rightarrow \Gamma(D(g), (^a\varphi)_*(\widetilde{A}))$$

induced by $\varphi^\#$ is just φ_g ; by considering the section $\xi \in \Gamma(D(g), \widetilde{B})$ corresponding to b/g , we then obtain that $\varphi_x^\#(\xi) = \varphi(b)/\varphi(g)$ in A_x .

Example 18.1.10. Let S be a multiplicative subset of A and $\varphi : A \rightarrow S^{-1}A$ the canonical homomorphism. We have seen in ?? that ${}^a\varphi$ is a homeomorphism from $Y = \text{Spec}(S^{-1}A)$ to the subspace $X = \text{Spec}(A)$ formed by x such that $\mathfrak{p}_x \cap S = \emptyset$. Moreover, for any x in this subspace, hence of the form ${}^a\varphi(y)$ where $y \in Y$, the homomorphism $\varphi_y^\# : \mathcal{O}_{Y,y} \rightarrow \mathcal{O}_{X,x}$ is bijective; therefore, \mathcal{O}_Y is identified with the sheaf induced over Y by \mathcal{O}_X .

Proposition 18.1.11. Let $\varphi : B \rightarrow A$ be a ring homomorphism and $\Phi : X \rightarrow Y$ the associated morphism. For any A -module M , there exists a canonical functorial isomorphism of the \mathcal{O}_Y -module $\widetilde{\varphi^*(M)}$ to its direct image $\Phi_*(\widetilde{M})$.

Proof. For $g \in B$, put $f = \varphi(g)$; the modules $\Gamma(D(g), \widetilde{\varphi^*(M)})$ and $\Gamma(D(f), \widetilde{M})$ are identified with $(\varphi^*(M))_g$ and M_f , respectively; moreover, the B_g -module $\varphi_g^*(M_f)$ is canonically isomorphic to $(\varphi^*(M))_g$. We then have a functorial isomorphism of $\Gamma(D(g), \widetilde{B})$ -modules:

$$\Gamma(D(g), \widetilde{\varphi^*(M)}) \cong \varphi^*(\Gamma(D(\varphi(g)), \widetilde{M}))$$

and this isomorphism satisfies the compatible conditions with restrictions, hence define an isomorphism of sheaves. \square

This proof also proves that for any A -algebra R , the canonical functorial isomorphism $\widetilde{\varphi^*(R)} \rightarrow \Phi_*(\widetilde{R})$ is an isomorphism of \mathcal{O}_Y -algebras. If M is an R -module, the canonical isomorphism $\varphi^*(M) \cong \Phi_*(\widetilde{M})$ is an isomorphism of $\Phi_*(\widetilde{R})$ -modules.

Corollary 18.1.12. The direct image functor Φ_* is exact on the category of quasi-coherent sheaves.

Proof. Recall that the functor φ^* is exact and $M \mapsto \widetilde{M}$ is an exact functor. \square

Proposition 18.1.13. Let $\varphi : B \rightarrow A$ be a ring homomorphism and $\Phi : X \rightarrow Y$ the associated morphism. Let N be a B -module and $\varphi_*(N)$ the A -module $N \otimes_B A$. Then there exist a canonical functorial isomorphism of the \mathcal{O}_X -module $\Phi^*(\widetilde{N})$ to $\widetilde{\varphi_*(N)}$.

Proof. We first remark that $j : z \mapsto z \otimes 1$ is a B -homomorphism from N to $\varphi^*\varphi_*(N)$: this holds because for $g \in B$, we have

$$(gz) \otimes 1 = z \otimes \varphi(g) = \varphi(g)(z \otimes 1).$$

By ??, this corresponds to a homomorphism $\tilde{j} : \widetilde{N} \rightarrow \widetilde{\varphi_*(N)}$ of \mathcal{O}_Y -modules, and via ?? we can think that \tilde{j} maps \widetilde{N} to $\Phi_*(\widetilde{\varphi_*(N)})$. From the adjointness of Φ^* and Φ_* , this canonically corresponds to a homomorphism

$$\theta : \Phi^*(\widetilde{N}) \rightarrow \widetilde{\varphi_*(N)}.$$

It then remains to show that θ is bijective, or equivalently that θ_x is bijective for every $x \in X$. For this, put $y = {}^a\varphi(x)$, choose $g \in B$ such that $y \in \widetilde{D(g)}$, and let $f = \varphi(g)$. Then the ring $\Gamma(D(f), \widetilde{A})$ is identified with A_f , the module $\Gamma(D(f), \varphi_*(N))$ is identified with $(\varphi_*(N))_f$, and $\Gamma(D(g), \widetilde{N})$ is identified with N_g . Let $s = n/g^p$ ($n \in N$) be a section of $\Gamma(D(g), \widetilde{N})$ and $t = a/f^q$ ($a \in A$) a section of $\Gamma(D(f), \widetilde{A})$. Then, since s is send to $(n \otimes 1)/f^p$ by \tilde{j} , by definition we have

$$\theta_x(s_x \otimes t_x) = t_x \cdot s_x.$$

Recall that we can identify $(\varphi_*(N))_f$ with $N_f \otimes_{B_g} \varphi^*(A_f)$, under which n/g^p is identified with $(n/g^p) \otimes 1$. So it is immediately seen that θ_x is none other than the canonical isomorphism

$$N_y \otimes_{B_y} \varphi_y^*(A_x) \cong (\varphi_*(N))_x = (N \otimes_B \varphi^*(A))_x.$$

Finally, let $v : N_1 \rightarrow N_2$ be a homomorphism of B -modules; since $\tilde{v}_y = v_y$ for any $y \in Y$, it follows immediately from the preceding argument that $\Phi^*(\tilde{v})$ is canonically identified to $\widetilde{v \otimes 1}$, which completes the proof. \square

If S is an B -algebra, the canonical isomorphism of $\Phi^*(\widetilde{S})$ to $\widetilde{\varphi_*(S)}$ is an isomorphism of \mathcal{O}_X -algebras; if moreover N is a S -module, the canonical isomorphism of $\Phi^*(\widetilde{N})$ to $\widetilde{\varphi_*(N)}$ is an isomorphism of $\Phi^*(\widetilde{S})$ -algebras.

Corollary 18.1.14. *The sections of $\Phi^*(\widetilde{N})$ which are canonical images of sections of the B -module $\Gamma(\widetilde{N})$, generate the A -module $\Gamma(\Phi^*(\widetilde{N}))$.*

Proof. In fact, these images are identified with the elements $z \otimes 1$ of $\varphi_*(N)$, if we identify N and $\varphi_*(N)$ with $\Gamma(\widetilde{N})$ and $\Gamma(\widetilde{\varphi_*(N)})$. \square

By the proof of ??, we see that the canonical map $\alpha : \widetilde{N} \rightarrow \Phi_*\Phi^*(\widetilde{N})$ is none other than the homomorphism \tilde{j} , where $j : N \rightarrow \varphi_*(N)$ is the canonical map $z \mapsto z \otimes 1$. Similarly, the canonical map $\beta : \Phi^*\Phi_*(\widetilde{M}) \rightarrow \widetilde{M}$ is none other than the homomorphism \tilde{p} , where $p : \varphi^*(M) \otimes_B \varphi^*(A) \rightarrow M$ is the canonical homomorphism that sends $m \otimes a$ to am .

Corollary 18.1.15. *Let N_1 and N_2 be B -modules and assume that N_1 is finitely presented. Then there is a canonical homomorphism*

$$\Phi^*(\mathcal{H}\text{om}_{\widetilde{B}}(\widetilde{N}_1, \widetilde{N}_2)) \rightarrow \mathcal{H}\text{om}_{\widetilde{A}}(\Phi^*(\widetilde{N}_1), \Phi^*(\widetilde{N}_2)).$$

This homomorphism is bijective if φ is a flat homomorphism.

Proof. By the above remarks and ??, this homomorphism is induced by the homomorphism

$$\text{Hom}_B(N_1, N_2) \otimes_B A \rightarrow \text{Hom}_A(N_1 \otimes_B A, N_2 \otimes_B A).$$

The last assertion follows from ??.

A locally ringed space (X, \mathcal{O}_X) is called an **affine scheme** if it is isomorphic to the spectrum of a ring A . In this case, the ring $\Gamma(X, \mathcal{O}_X)$ is canonically identified with A . By abusing language, we often call $\text{Spec}(A)$ an affine scheme, without mention the structural sheaf.

Let A and B be two rings and (X, \mathcal{O}_X) , (Y, \mathcal{O}_Y) the corresponding affine schemes. Then any ring homomorphism $\varphi : B \rightarrow A$ corresponds to a morphism $({}^a\varphi, \varphi^\#) : (X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$. Note that the homomorphism φ is completely determined by $({}^a\varphi, \varphi^\#)$, since by definition we have $\varphi = \Gamma(\varphi^\#) : \Gamma(\tilde{B}) \rightarrow \Gamma(\tilde{A})$.

Theorem 18.1.16. *Let (X, \mathcal{O}_X) and (Y, \mathcal{O}_Y) be two affine schemes. Then any morphism $(\psi, \psi^\#) : (X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$ of locally ringed spaces is of the form $({}^a\varphi, \varphi^\#)$, where $\varphi : \Gamma(Y, \mathcal{O}_Y) \rightarrow \Gamma(X, \mathcal{O}_X)$ is a ring homomorphism.*

Proof. Put $A = \Gamma(X, \mathcal{O}_X)$ and $B = \Gamma(Y, \mathcal{O}_Y)$. Let $(\psi, \psi^\#) : (X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$ be a morphism of locally ringed spaces. By definition, $\psi^\#$ is a homomorphism from \mathcal{O}_Y to $\psi_* \mathcal{O}_X$, and we then deduce a canonical homomorphism of rings

$$\varphi = \Gamma(\psi^\#) : B = \Gamma(Y, \mathcal{O}_Y) \rightarrow \Gamma(Y, \psi_* \mathcal{O}_X) = \Gamma(X, \mathcal{O}_X) = A.$$

Since $\psi_x^\#$ is a local homomorphism, by passing to quotients we deduce a monomorphism θ^x from the residue field $\kappa(\psi(x))$ into the residue field $\kappa(x)$ such that, for any section $f \in \Gamma(Y, \mathcal{O}_Y)$, we have $\theta^x(f(\psi(x))) = \varphi(f)(x)$ (we consider the elements of $\Gamma(Y, \mathcal{O}_Y)$ as functions on B). The relationship $f(\psi(x)) = 0$ is therefore equivalent to $\varphi(f)(x) = 0$, which means $\psi(x) = {}^a\varphi(x)$. Since this hold for any $x \in X$, we conclude that $\psi = {}^a\varphi$. We also know that the diagram

$$\begin{array}{ccc} B = \Gamma(Y, \mathcal{O}_Y) & \xrightarrow{{}^a\varphi} & \Gamma(X, \mathcal{O}_X) = A \\ \downarrow & & \downarrow \\ B_{\psi(x)} & \xrightarrow{\psi_x^\#} & A_x \end{array}$$

is commutative, so $\psi_x^\#$ is equal to the homomorphism $\varphi_x : B_{\psi(x)} \rightarrow A_x$ induced from φ . Since the morphism $\psi^\#$ is determined by $\psi_x^\#$, we obtain that $\psi^\# = \varphi^\#$. \square

Corollary 18.1.17. *Let (X, \mathcal{O}_X) and (Y, \mathcal{O}_Y) be two affine schemes. Then there is a canonical bijection*

$$\text{Mor}(X, Y) \rightarrow \text{Hom}_{\text{Ring}}(\Gamma(Y, \mathcal{O}_Y), \Gamma(X, \mathcal{O}_X))$$

which sends a morphism $(f, f^\#)$ to the global section of $f^\#$.

We can also say that the functors $(\text{Spec}(A), \tilde{A})$ in A and $\Gamma(X, \mathcal{O}_X)$ in (X, \mathcal{O}_X) define an equivalence of the opposite category of commutative rings and of the category of the category of affine schemes.

Corollary 18.1.18. *If $\varphi : B \rightarrow A$ is surjective, then the corresponding morphism Φ is a monomorphism of locally ringed spaces.*

Proof. The map ${}^a\varphi$ is injective by ??, and since φ is surjective, for any $x \in X$ the map $\varphi_x^\# : B_{\varphi(x)} \rightarrow A_x$, obtained by passing to localization, is surjective; these prove the assertion. \square

18.1.3 Quasi-coherent sheaves over affine schemes

Recall that we have defined the abstract notion of a quasi-coherent sheaf. In this paragraph we show that any quasi-coherent sheaf on an affine scheme $\text{Spec}(A)$ corresponds to the sheaf \tilde{M} associated with an A -module M .

Lemma 18.1.19. Let $X = \text{Spec}(A)$ and $V = \bigcup_{i=1}^n D(g_i)$ be a union of finitely many standard opens. Let \mathcal{F} be an \mathcal{O}_X -module satisfying the conditions:

- (a) For any $D(f) \subseteq D(g_i)$ (resp. $D(g_i g_j)$) and any section $s \in \Gamma(D(f), \mathcal{F})$, there exists $n \geq 0$ such that $f^n s$ can be extended to a section of \mathcal{F} on $D(g_i)$ (resp. on $D(g_i g_j)$).
- (b) For any $D(f) \subseteq D(g_i)$ (resp. $D(g_i g_j)$) and any section $t \in \Gamma(V, \mathcal{F})$ such that $t|_{D(f)} = 0$, there exists $n \geq 0$ such that $f^n t = 0$.

Then we have the stronger conditions:

- (α) For any $f \in A$ and any section $s \in \Gamma(D(f) \cap V, \mathcal{F})$, there exists $n \geq 0$ such that $f^n s$ can be extended to a section of \mathcal{F} on V .
- (β) For any $f \in A$ and any section $t \in \Gamma(V, \mathcal{F})$ such that $t|_{D(f) \cap V} = 0$, there exists $n \geq 0$ such that $f^n t = 0$.

Proof. First we prove condition (β). Since $D(f) \cap D(g_i) = D(fg_i)$, for each i we have an integer $n_i \geq 0$ such that $(fg_i)^{n_i} t$ restricts to zero on $D(g_i)$. Since g_i is invertible in $A_{g_i} = \Gamma(D(g_i), \mathcal{O}_X)$, this implies $f^{n_i} t = 0$ on $D(g_i)$. Take n such that $n \geq n_i$, then $f^n t = 0$ on each $D(g_i)$, whence $f^n t = 0$ and we get (β).

To show (α), we apply (a) on $\mathcal{F}|_{D(g_i)}$ to get an integer integers $n_i \geq 0$ and $s'_i \in \Gamma(D(g_i), \mathcal{F})$ such that

$$s'_i|_{D(fg_i)} = (fg_i)^{n_i} s|_{D(fg_i)}.$$

By inverting g_i , this produces sections $s_i \in \Gamma(D(g_i), \mathcal{F})$ such that

$$s_i|_{D(fg_i)} = f^{n_i} s|_{D(fg_i)}.$$

We may assume that all n_i take the same value n . Then each $s_i - s_j$ restricts to zero on $D(f) \cap D(g_i) \cap D(g_j) = D(fg_i g_j)$, so by applying (b) on $\mathcal{F}|_{D(g_i g_j)}$ we get an integer m_{ij} such that

$$(fg_i g_j)^{m_{ij}} (s_i - s_j)|_{D(g_i g_j)} = 0.$$

Then similarly, since $g_i g_j$ is invertible in $A_{g_i g_j}$, this implies

$$f^{m_{ij}} (s_i - s_j)|_{D(g_i g_j)} = 0.$$

We can also assume that all m_{ij} 's take the same value m , so that by the sheaf condition there exists a section $u \in \Gamma(V, \mathcal{F})$ such that $u|_{D(g_i)} = f^m s_i$. Then $f^n u$ extends $f^{m+n} s$, as desired. \square

Theorem 18.1.20. Let $X = \text{Spec}(A)$ be an affine scheme. Let V be a quasi-compact open subset and \mathcal{F} be an $\mathcal{O}_X|_V$ -module. Then the following are equivalent:

- (i) There is a A -module M such that \mathcal{F} is isomorphic to $\tilde{M}|_V$.
- (ii) There exists a finite open covering (V_i) of V by sets of the form $D(f_i)$ ($f_i \in A$) contained in V , such that, for each i , $\mathcal{F}|_{V_i}$ is isomorphic to a sheaf of the form \tilde{M}_i , where M_i is an A_{f_i} -module.
- (iii) The sheaf \mathcal{F} is quasi-coherent.
- (iv) (Serre's lifting criterion) The following conditions are satisfied:

- (a) For any $D(f) \subseteq V$ and any section $s \in \Gamma(D(f), \mathcal{F})$, there exists $n \geq 0$ such that $f^n s$ can be extended to a section of \mathcal{F} on V .
- (b) For any $D(f) \subseteq V$ and any section $t \in \Gamma(V, \mathcal{F})$ such that $t|_{D(f)} = 0$, there exists $n \geq 0$ such that $f^n t = 0$.

Proof. The implication (i) \Rightarrow (ii) is immediate from ?? since X can be covered by standard opens. Also, since any A -module is isomorphic to the kernel of a homomorphism $A^{\oplus I} \rightarrow A^{\oplus J}$, ?? shows that (ii) \Rightarrow (iii). Conversely, if \mathcal{F} is quasi-coherent, any point $x \in V$ possesses a neighborhood of the form $D(f) \subseteq V$ such that $\mathcal{F}|_{D(f)}$ is isomorphic to the cokernel of a homomorphism $(\tilde{A}_f)^{\oplus I} \rightarrow (\tilde{A}_f)^{\oplus J}$, hence to the sheaf associated with the cokernel of the corresponding homomorphism $A_f^{\oplus I} \rightarrow A_f^{\oplus J}$ (?? and ??); since V is quasi-compact, it then follows that (iii) implies (ii).

Now we prove that (ii) \Rightarrow (iv). First assume that $V = D(g)$ for some $g \in A$, and \mathcal{F} is isomorphic to \tilde{N} for some A_g -module N . Since $D(g)$ can be identified with $\text{Spec}(A_g)$, we can assume that $g = 1$ and $V = X$. In this case, the set $\Gamma(D(f), \mathcal{F})$ and N_f are canonically identified (??), and it is clear that conditions (a) and (b) in (iv) are satisfied. To prove the general case, since V is quasi-compact we can choose a finite covering by standard opens $D(g_i)$ with $\mathcal{F}|_{D(g_i)}$ isomorphic to \tilde{M}_i for some A_{g_i} -module M_i . Then \mathcal{F} satisfies the conditions (a) and (b) in ??, so by ??, \mathcal{F} also satisfies conditions (α) and (β), which is what we want.

Finally, we show that (iv) \Rightarrow (i). First we prove that, if (a) and (b) hold for \mathcal{F} , then they hold for $\mathcal{F}|_{D(g)}$ with $D(g) \subseteq V$. This is evident for condition (a); as for (b), if $t \in \Gamma(D(g), \mathcal{F})$ restricts to zero on $D(f) \subseteq D(g)$, then by condition (a) there is an integer $m \geq 0$ such that $g^m t$ can be extended to V . By applying condition (b) on the extension of $g^m t$, we get another integer $n \geq 0$ such that $f^n g^m t = 0$. Since g is invertible in A_g , this gives $f^n t = 0$ as desired.

This being done, since V is quasi-compact, by ?? we know that conditions (α) and (β) holds for \mathcal{F} . Now consider the module $M = \Gamma(V, \mathcal{F})$; we shall define a morphism $\varphi : \tilde{M} \rightarrow j_* \mathcal{F}$, where $j : V \hookrightarrow X$ is the inclusion. For this, it suffices to define

$$\varphi_f : M_f \rightarrow \Gamma(D(f), j_* \mathcal{F}) = \Gamma(D(f) \cap V, \mathcal{F})$$

for each $f \in A$. Since f is invertible in A_f and $\Gamma(D(f) \cap V, \mathcal{F})$ is a A_f -module, the restriction $M = \Gamma(V, \mathcal{F}) \rightarrow \Gamma(D(f) \cap V, \mathcal{F})$ factors into

$$M \longrightarrow M_f \xrightarrow{\varphi_f} \Gamma(D(f) \cap V, \mathcal{F})$$

which gives the desired maps φ_f . We now claim that the conditions (a) and (b) in (iv) imply that φ is an isomorphism. In fact, if $s \in \Gamma(D(f) \cap V, \mathcal{F})$, then by condition (a) there exist an integer $n \geq 0$ and $z \in \Gamma(V, \mathcal{F}) = M$ such that $z|_{D(f) \cap V} = f^n s$; then $\varphi_f(z/f^n) = s$, showing that φ is surjective. Similarly, if there is $z \in M$ such that $\varphi_f(z/1) = 0$ in $D(f) \cap V$, then by condition (b) there is $n \geq 0$ such that $f^n z = 0$, so that $z/1 = 0$ in M_f . This means φ_f is injective, so we get an isomorphism $\tilde{M} \cong j_* \mathcal{F}$. By restriction, we then conclude that $\mathcal{F} \cong \tilde{M}|_V$. \square

Corollary 18.1.21. *Let $X = \text{Spec}(A)$ be an affine scheme. Then the functors $M \mapsto \tilde{M}$ and $\mathcal{F} \mapsto \Gamma(X, \mathcal{F})$ define equivalences of categories between the category of quasi-coherent \mathcal{O}_X -modules and the category of A -modules.*

Proof. The space X it self is quasi-compact, so we can apply ??.

\square

Corollary 18.1.22. *Let $X = \text{Spec}(A)$ be an affine scheme. Then kernels and cokernels of maps of quasi-coherent \mathcal{O}_X -modules are quasi-coherent.*

Proof. This follows from the exactness of the functor \tilde{M} and ??.

□

Corollary 18.1.23. *For \tilde{M} to be a \mathcal{O}_X -module of finite type (resp. of finite presentation), it is necessary and sufficient that M is a finitely generated A -module (resp. of finite presentation).*

Proof. In view of the exactness of the functor $M \mapsto \tilde{M}$, it is immediate that if M is of finite type (resp. finite presentation), so is \tilde{M} . Conversely, if \tilde{M} is of finite type (resp. finite presentation), since X is quasi-compact, there exists finitely many $f_i \in A$ such that $D(f_i)$ cover X and M_{f_i} is of finite type (resp. finite presentation) over A_{f_i} . It then follows from ?? that M is of finite type (resp. finite presentation). □

Corollary 18.1.24. *For an A -module M , the \mathcal{O}_X -module is locally free of finite rank if and only if M is a finitely generated projective A -module.*

Proof. Since X is quasi-compact, this follows from ??.

□

Corollary 18.1.25. *Let $X = \text{Spec}(A)$ be an affine scheme. Then any quasi-coherent \mathcal{O}_X -algebra over X is isomorphic to an \mathcal{O}_X -algebra of the form \tilde{B} , where B is an algebra over A . Moreover, any quasi-coherent \tilde{B} -module is isomorphic to a \tilde{B} -module of the form \tilde{N} , where N is a B -module.*

Proof. In fact, a quasi-coherent \mathcal{O}_X -algebra is a quasi-coherent \mathcal{O}_X -module, hence of the form \tilde{B} , where B is an A -module. The fact that B is an A -algebra follows from the structural morphism $\tilde{B} \otimes_{\mathcal{O}_X} \tilde{B} \rightarrow \tilde{B}$ of \mathcal{O}_X -modules, which induces an A -algebra map $B \otimes_A B \rightarrow B$.

If \mathcal{G} is a quasi-coherent \tilde{B} -module, it suffices to show that \mathcal{G} is also a quasi-coherent \mathcal{O}_X -module to then conclude in the same way. As the question is local, we can, by restricting ourselves to an open set of X of the form $D(f)$, over which \mathcal{G} is the cokernel of a morphism $\tilde{B}^{\oplus I} \rightarrow \tilde{B}^{\oplus J}$ of \tilde{B} -modules (and a fortiori \mathcal{O}_X -modules). The claim then follows from ?? and ??.

□

Proposition 18.1.26. *Let $X = \text{Spec}(A)$ be an affine scheme. Suppose that*

$$0 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{F}_2 \longrightarrow \mathcal{F}_3 \longrightarrow 0$$

is a short exact sequence of \mathcal{O}_X -modules, and assume that \mathcal{F}_1 is quasi-coherent. Then the sequence

$$0 \longrightarrow \Gamma(X, \mathcal{F}_1) \longrightarrow \Gamma(X, \mathcal{F}_2) \longrightarrow \Gamma(X, \mathcal{F}_3) \longrightarrow 0$$

is exact.

Proof. We know already that Γ is a left-exact functor so we have only to show that the last map is surjective (which we denote by $\psi : \Gamma(X, \mathcal{F}_2) \rightarrow \Gamma(X, \mathcal{F}_3)$). Let $s \in \Gamma(X, \mathcal{F}_3)$ be a global section. Since the morphism $\mathcal{F}_2 \rightarrow \mathcal{F}_3$ is surjective, for any $x \in X$ there is an open neighborhood $D(f)$ of x such that

$$s|_{D(f)} = \psi(t)$$

where $t \in \mathcal{F}_2(D(f))$. We claim that for some $n > 0$, $f^n s = \psi(u)$ for some $u \in \Gamma(X, \mathcal{F}_2)$. Indeed, we can cover X with a finite number of open sets $D(g_i)$ such that for each i , $s|_{D(g_i)} = \psi(t_i)$ for

a section $t_i \in \mathcal{F}_2(D(g_i))$. Then by the exactness of the original sequence, on $D(f) \cap D(g_i) = D(fg_i)$ we have

$$(t - t_i)|_{D(fg_i)} \in \mathcal{F}_1(D(fg_i))$$

where we identify \mathcal{F}_1 as the kernel of ψ . Since \mathcal{F}_1 is quasi-coherent, by ??, there is an integer $n \geq 0$ such that the $f^n(t - t_i)|_{D(fg_i)}$ can be extended to a section $u_i \in \mathcal{F}_1(D(g_i))$. Let

$$\tilde{t}_i = f^n t_i + u_i \in \mathcal{F}_2(D(g_i)).$$

Then $\tilde{t}_i|_{D(fg_i)} = f^n t_i|_{D(fg_i)} + f^n(t - t_i)|_{D(fg_i)} = f^n t|_{D(fg_i)}$ and we have

$$f^n s|_{D(g_i)} = f^n \psi(t_i) = \psi(\tilde{t}_i - u_i) = \psi(\tilde{t}_i). \quad (18.1.1)$$

Now on $D(g_i g_j)$ the two sections \tilde{t}_i and \tilde{t}_j of \mathcal{F}_2 are mapped to $f^n s|_{D(g_i g_j)}$ by ψ , so $\tilde{t}_i - \tilde{t}_j \in \mathcal{F}_1(D(g_i g_j))$. Furthermore, since \tilde{t}_i and \tilde{t}_j are both equal to $f^n t|_{D(fg_i g_j)}$ on $D(fg_i g_j)$, by ?? there exists $m \geq 0$ such that $f^m(\tilde{t}_i - \tilde{t}_j) = 0$ on $D(g_i g_j)$, which we may take to be independent of i and j . Then the sections $f^m \tilde{t}_i$ glue to give a global section of \mathcal{F}_2 over X , which lifts $f^{m+n} s$ by ???. This proves the claim.

Now cover X by a finite number of open sets $D(f_i)$ such that $s|_{D(f_i)}$ lifts to a section of \mathcal{F}_2 over $D(f_i)$ for each i . Then by the previous proof, we can find an integer $n \geq 0$ (one for all i) and global sections $t_i \in \Gamma(X, \mathcal{F}_2)$ such that $\psi(t_i) = f^n s$. Since the open sets $D(f_i)$ cover X , the ideal (f_1^n, \dots, f_r^n) is the unit ideal of A , and we can write $1 = \sum a_i f_i^n$, with $a_i \in A$. Let $t = \sum a_i t_i$. Then t is a global section of \mathcal{F}_2 whose image under ψ is $\sum a_i f_i^n s = s$. \square

Proposition 18.1.27. *Let $X = \text{Spec}(A)$ be an affine scheme. Then the direct sum of an arbitrary collection of quasi-coherent sheaves on X is quasi-coherent. More generally, colimits of quasi-coherent sheaves are quasi-coherent.*

Proof. Let $\{\mathcal{F}_i\}_{i \in I}$ be a collection of quasi-coherent sheaves on X . By ?? we can write $\mathcal{F}_i = \tilde{M}_i$ for A -modules M_i , so the assertion follows from ???. \square

Proposition 18.1.28. *Let X be an affine scheme. Suppose that*

$$0 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{F}_2 \longrightarrow \mathcal{F}_3 \longrightarrow 0$$

is a short exact sequence of sheaves \mathcal{O}_X -modules. If two out of three are quasi-coherent then so is the third.

Proof. The statement about kernels and cokernels follows from the fact that the functor $M \mapsto \tilde{M}$ is exact and fully faithful from A -modules to quasi-coherent sheaves. Now let \mathcal{F}_1 and \mathcal{F}_3 be quasi-coherent. By ??, the corresponding sequence of global sections over X is exact, say $0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0$. Applying the functor \tilde{M} we get an exact commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \tilde{M}_1 & \longrightarrow & \tilde{M}_2 & \longrightarrow & \tilde{M}_3 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{F}_1 & \longrightarrow & \mathcal{F}_2 & \longrightarrow & \mathcal{F}_3 & \longrightarrow & 0 \end{array}$$

The two outside arrows are isomorphisms, since \mathcal{F}_1 and \mathcal{F}_3 are quasi-coherent. So by the five lemma, the middle one is also, showing that \mathcal{F}_2 is quasi-coherent. \square

Theorem 18.1.29. *Let $X = \text{Spec}(A)$ be an affine scheme where A is Noetherian. Let V be an open subset of X and \mathcal{F} an $\mathcal{O}_X|_V$ -module. Then the following conditions are equivalent:*

- (i) \mathcal{F} is coherent.
- (ii) \mathcal{F} is of finite type and quasi-coherent.
- (iii) There exists a finitely generated A -moduel M such that $\mathcal{F} \cong \tilde{M}|_V$.

Proof. It is clear that (i) implies (ii). To show (ii) implies (iii), we note that V is quasi-compact since X is Noetherian, so by ??, \mathcal{F} is isomorphic to $\tilde{M}|_V$, where M is an A -module. Now we have $M = \varinjlim M_\lambda$, where M_λ is the set of finitely generated sub- A -modules of M . Since the functor $(\widetilde{-})$ is exact, this implies $\mathcal{F} = \tilde{N}|_V = \varinjlim \tilde{M}_\lambda|_V$. But \mathcal{F} is of finite type and V is quasi-compact, so by ?? there exists an index λ such that $\mathcal{F} = \tilde{M}_\lambda|_V$ (note that the canonical homomorphism $\tilde{M}_\lambda \rightarrow \tilde{M}$ is injective). This proves (iii).

It remains to show that $\tilde{M}|_V$ is coherent if M is finitely generated. Since \mathcal{F} is clearly of finite type, it suffices to show that for every open $U \subseteq X$ and $s_1, \dots, s_n \in \mathcal{F}(U)$, the associated map $\bigoplus_{i=1}^n \mathcal{O}_U \rightarrow \mathcal{F}|_U$ is of finite type. Since the question is local, we may assume $V = D(f)$ for $f \in A$. Then it suffices to show the kernel of a morphism $\bigoplus_{i=1}^n \tilde{A}_f \rightarrow \tilde{M}$ is of finite type. But this morphism corresponds to a homomorphism $A_f^n \rightarrow M$, whose kernel is finitely generated since A_f is Noetherian, so the claim follows. \square

Corollary 18.1.30. *Let $X = \text{Spec}(A)$ be an affine scheme where A is Noetherian. Then any quasi-coherent \mathcal{O}_X -module \mathcal{F} is the inductive limit of coherent \mathcal{O}_X -modules.*

Proof. We have $\mathcal{F} = \tilde{M}$ for an A -module M , and M is the inductive limit of its finitely generated submodules. \square

Corollary 18.1.31. *Let $X = \text{Spec}(A)$ be an affine scheme where A is Noetherian. Then the functor $M \mapsto \tilde{M}$ gives an equivalence of categories between the category of finite generated A -modules and the category of coherent \mathcal{O}_X -modules.*

18.2 General schemes

18.2.1 Schemes and morphisms of schemes

Let (X, \mathcal{O}_X) be a ringed space. An oepn subset V of X is said to be **affine** if the ringed space $(V, \mathcal{O}_X|_V)$ is an affine scheme (i.e. isomorphic to the spectrum of a ring). We say (X, \mathcal{O}_X) is a **scheme** if every point of X admits an affine open neighborhood. If (X, \mathcal{O}_X) is a scheme, then affine open subsets of X form a basis for X (because the standard opens form a basis for a spectrum $\text{Spec}(A)$, and they are again affine), and in particular (X, \mathcal{O}_X) is a locally ringed space. With this, for any open subset U of X , the ringed space $(U, \mathcal{O}_X|_U)$ is also a scheme, called the scheme **induced** on U by X , or the **restriction** of (X, \mathcal{O}_X) on U .

Proposition 18.2.1. *The underlying space of a scheme is Kolmogoroff.*

Proof. In fact, if x and y are two points of a scheme X , then it is obvious that there exists an open neighborhood of one of these points not containing the other if x, y are not in a same open affine; and if they are in the same open affine, this follows from the fact that the underlying spaces of affine schemes are Kolmogoroff (??). \square

Proposition 18.2.2. *If (X, \mathcal{O}_X) is a scheme, any irreducible closed subset of X admits a unique generic point, and the map $x \mapsto \overline{\{x\}}$ is a bijection of X to the family of irreducible closed subsets of X .*

Proof. Let Y is an irreducible closed subset of X and $y \in Y$. If U is an affine open neighborhood of y in X , then $U \cap Y$ is dense in Y and is irreducible (??), so it is the closure in U of a point $x \in U$, and therefore $Y \subseteq \overline{U}$ is the closure of x in X . The uniqueness of the generic point of X follows from ?? and ??.

If Y is an irreducible closed subset of X and y its generic point, the local ring $\mathcal{O}_{X,y}$ is then denoted by $\mathcal{O}_{X,Y}$ and called the **local ring of X along Y** , or the **local ring of Y in X** . We say a scheme (X, \mathcal{O}_X) is **irreducible** (resp. **connected**) if the underlying space X is irreducible (resp. connected), and **integral** if it is irreducible and reduced. We say the scheme (X, \mathcal{O}_X) is **locally integral** if each point $x \in X$ admits an open neighborhoods U such that the scheme induced over U by (X, \mathcal{O}_X) is integral. If X is an irreducible scheme and x is its generic point, the local ring $\mathcal{O}_{X,x}$ is called **the ring of rational functions on X** .

Let (X, \mathcal{O}_X) and (Y, \mathcal{O}_Y) be schemes. A **morphism** (of schemes) from (X, \mathcal{O}_X) to (Y, \mathcal{O}_Y) is defined to be that of locally ringed space. That is, a pair $(f, f^\#)$ such that for each $x \in X$, the homomorphism $f_x^\#$ is local. In this case, by passing to quotients, $f_x^\#$ induces a monomorphism $f^x : \kappa(f(x)) \rightarrow \kappa(x)$, so $\kappa(x)$ can be considered as an extension of the field $\kappa(f(x))$.

The composition of two morphisms of schemes is defined in the same way with that of locally ringed spaces, and we then see that schemes form a category, denoted by **Sch**. Following the general notation, we denote by $\text{Hom}_{\mathbf{Sch}}(X, Y)$ the set of morphisms from a scheme X to a scheme Y .

Example 18.2.3. Let U be an open subset of X . Then the canonical injection of $(U, \mathcal{O}_X|_U)$ to (X, \mathcal{O}_X) is a morphism of schemes; it is moreover a monomorphism of ringed spaces (and a fortiori a monomorphism of schemes).

Proposition 18.2.4. *Let (X, \mathcal{O}_X) be a scheme and (Y, \mathcal{O}_Y) be an affine scheme. Then there exists a canonical bijection*

$$\text{Hom}_{\mathbf{Sch}}(X, Y) \rightarrow \text{Hom}_{\mathbf{Ring}}(\Gamma(X, \mathcal{O}_X), \Gamma(Y, \mathcal{O}_Y)).$$

Proof. Let $A = \Gamma(Y, \mathcal{O}_Y)$. Note first that, if (X, \mathcal{O}_X) and (Y, \mathcal{O}_Y) are any two ringed spaces, a morphism $(\psi, \psi^\#) : (X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$ canonically defines a homomorphism of rings

$$\Gamma(Y, \mathcal{O}_Y) \rightarrow \Gamma(Y, \psi_*(\mathcal{O}_X)) = \Gamma(X, \mathcal{O}_X).$$

It then remains to see that any homomorphism $\rho : A \rightarrow \Gamma(X, \mathcal{O}_X)$ is of the form $\Gamma(\psi^\#)$ for a unique morphism $(\psi, \psi^\#)$. However, there is by hypothesis a covering (V_α) of X by affine open sets. By considering the composition

$$A \xrightarrow{\rho} \Gamma(X, \mathcal{O}_X) \longrightarrow \Gamma(V_\alpha, \mathcal{O}_X|_{V_\alpha})$$

we obtain a homomorphism $\rho_\alpha : A \rightarrow \Gamma(V_\alpha, \mathcal{O}_X|_{V_\alpha})$, which corresponds to a morphism $(\psi_\alpha, \psi_\alpha^\#)$ from the scheme $(V_\alpha, \mathcal{O}_X|_{V_\alpha})$ to (Y, \mathcal{O}_Y) (??). Moreover, for each pair (α, β) of indices, any point of $V_\alpha \cap V_\beta$ admits an affine open neighborhood W contained in $V_\alpha \cap V_\beta$; it is clear that by composing ρ_α and ρ_β with the restriction homomorphism to W , we obtain the same homomorphism $A \rightarrow \Gamma(W, \mathcal{O}_X|_W)$, so, by virtue of the relation $(\psi_\alpha^\#)_x = (\rho_\alpha)_x$ for any $x \in V_\alpha$ and any α , the restrictions of $(\psi_\alpha, \psi_\alpha^\#)$ and $(\psi_\beta, \psi_\beta^\#)$ to W coincide. By gluing we then get a unique morphism $(\psi, \psi^\#) : (X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$ which extends $(\psi_\alpha, \psi_\alpha^\#)$ on each V_α . It is clear that $(\psi, \psi^\#)$ is a morphism of schemes, and we have $\Gamma(\psi^\#) = \rho$. \square

Remark 18.2.1. Let $\rho : A \rightarrow \Gamma(X, \mathcal{O}_X)$ be a ring homomorphism, and let $(\psi, \psi^\#)$ be the corresponding morphism $(X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$. For each $f \in A$, we have

$$\psi^{-1}(D(f)) = \{x \in X : f \notin \mathfrak{m}_{\psi(x)}\} = \{x \in X : (\rho(x))_x \notin \mathfrak{m}_x\} = X_{\rho(f)}.$$

Note that this can be viewed as a generalization of ??(a).

Proposition 18.2.5. Under the hypothesis of ??, let $\rho : A \rightarrow \Gamma(X, \mathcal{O}_X)$ be a ring homomorphism and $f : (X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$ the corresponding morphism of schemes. Let \mathcal{F} (resp. \mathcal{G}) be a quasi-coherent \mathcal{O}_X -module (resp. \mathcal{O}_Y -module). Then there exist a canonical bijection

$$\mathrm{Hom}_{\mathbf{Qcoh}(Y)}(\mathcal{G}, f_*(\mathcal{F})) \rightarrow \mathrm{Hom}_A(\Gamma(Y, \mathcal{G}), \rho^*(\Gamma(X, \mathcal{F}))).$$

Proof. Indeed, by reasoning as in ??, we are immediately reduced to the case where X is affine and the proposition then follows from ?? and ??.

We say a morphism $f : (X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$ is **open** (resp. **closed**) if for any open subset U of X (resp. any closed subset F of X), $f(U)$ is open in Y (resp. $f(F)$ is closed in Y). We say f is **dominant** if $f(X)$ is dense in Y , and **surjective** if f is surjective. It should be noted that these conditions only involve the continuous map f .

Proposition 18.2.6. Let $f : (X, \mathcal{O}_X)$ and $g : (Y, \mathcal{O}_Y) \rightarrow (Z, \mathcal{O}_Z)$ be morphisms of schemes.

- (a) If f and g are open (resp. closed, dominant, surjective), so is the composition $g \circ f$.
- (b) If f is surjective and $g \circ f$ is closed, g is closed.
- (c) If $g \circ f$ is surjective, g is surjective.

Proof. The assertions (a) and (c) are evident. Put $h = g \circ f$. If F is closed in Y , then $f^{-1}(F)$ is closed in X , so $h(f^{-1}(F))$ is closed in Z . But since f is surjective, we have $f(f^{-1}(F)) = F$, so $h(f^{-1}(F)) = g(F)$, which shows g is closed. \square

Proposition 18.2.7. Let $f : (X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$ be a morphism of schemes and (U_α) an open covering of Y . For f to be open (resp. closed, surjective, dominant), it is necessary and sufficient that for each U_α , the restrictions $f : f^{-1}(U_\alpha) \rightarrow U_\alpha$ is open (resp. closed, surjective, dominant).

Proof. The proposition follows immediately from the definitions, taking into account the fact that a subset F of Y is closed (resp. open, dense) in Y if and only if each of the sets $F \cap U_\alpha$ is closed (resp. open, dense) in U_α . \square

Let (X, \mathcal{O}_X) and (Y, \mathcal{O}_Y) be schemes; suppose that X and Y have a same finite number of irreducible components X_i (resp. Y_i) ($1 \leq i \leq n$); let ξ_i (resp. η_i) be the generic point of X_i (resp. Y_i). We say that a morphism

$$f : (X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$$

is **birational** if, for any i , we have $f^{-1}(\eta_i) = \{\xi_i\}$ and $f_{\xi_i}^* : \mathcal{O}_{Y, \eta_i} \rightarrow \mathcal{O}_{X, \xi_i}$ is an isomorphism. It is clear that any birational morphism is dominant, hence surjective if it is closed.

Remark 18.2.2. Throughout the remainder of this chapter and when there is no risk of creating confusion, we will omit in the notation of a scheme (resp. of a morphism) the structural sheaf (resp. the morphism of structural sheaf). If U is an open subset of the underlying space of a scheme X , when we speak of U as of a scheme, it will always be the scheme induced on U by X .

With the morphisms of schemes defined, we can talk about glueing schemes as in the case of ringed spaces. It follows immediately from the definition that any ringed space obtained by glueing schemes is again a scheme. In particular, since any scheme admits a basis of affine open subsets, we see any scheme is obtained by glueing affine schemes.

Example 18.2.8. Consider a field K , $A = K[s]$, $B = K[t]$ be two rings of polynomials over K with one indeterminate, and $X_1 = \text{Spec}(A)$, $X_2 = \text{Spec}(B)$. In X_1 (resp. X_2), let U_{12} (resp. U_{21}) be the affine open set $D(s)$ (resp. $D(t)$), whose ring A_s (resp. B_t) is formed by the rational fractions of the form $f(s)/s^m$ (resp. $g(t)/t^n$) with $f \in A$ (resp. $g \in B$). Let φ_{12} be the isomorphism of schemes $U_{21} \rightarrow U_{12}$ corresponding to the isomorphism of A and B such that, $f(s)/s^m$ is mapped to the rational fraction $f(1/t)/(1/t^m)$ (i.e. we map s to $1/t$). We can then glue X_1 and X_2 along U_{12} and U_{21} by the isomorphism u_{12} , which evidently satisfies the glueing condition. We will see later the scheme X thus obtained is a particular case of a general method of construction. We only show here that X is not an affine scheme, which will result from the fact that the ring $\Gamma(X, \mathcal{O}_X)$ is isomorphic to K , therefore has a spectrum reduced to a point. Indeed, a section of \mathcal{O}_X above X has a restriction over X_1 (resp. X_2), identified with an open affine of X , which is a polynomial $f(s)$ (resp. $g(t)$), and it follows from the definition of u_{12} that we must have $g(t) = f(1/t)$, which is not possible only if $f = g \in K$.

18.2.2 Local schemes

Let X be a scheme and $A = \Gamma(X, \mathcal{O}_X)$. We say X is a local scheme if X is affine and the ring X is local. In this case, there then exists a unique closed point ξ in X , and for any point $x \in X$ we have $\xi \in \overline{\{x\}}$.

Following this notation, for a general scheme Y and $y \in Y$, the scheme $\text{Spec}(\mathcal{O}_{Y,y})$ is called the **local scheme of Y at y** . Let V be an affine open subset of Y containing y , and B the ring of V . The local ring $\mathcal{O}_{Y,y}$ is then canonically identified with B_y , and the canonical homomorphism $B \rightarrow B_y$ then induces a morphism $\text{Spec}(\mathcal{O}_{Y,y}) \rightarrow V$ of schemes. If we compose this with the canonical injection of V into Y , we then get a morphism $\text{Spec}(\mathcal{O}_{Y,y}) \rightarrow Y$, which is independent of the choice of the affine open V containing y : in fact, if U is another affine neighborhood of y , there exists an affine open neighborhood W of y contained in $U \cap V$; we can then limit

ourselves to the case $U \subseteq V$, and if A is the ring of U , we have a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\quad} & B \\ & \searrow & \swarrow \\ & \mathcal{O}_{Y,y} & \end{array}$$

The morphism $\text{Spec}(\mathcal{O}_{Y,y}) \rightarrow Y$ thus defined is said to be **canonical**.

Proposition 18.2.9. *Let Y be a scheme, $y \in Y$, and $f : \text{Spec}(\mathcal{O}_{Y,y}) \rightarrow Y$ be the canonical morphism.*

- (a) *The map f is a homeomorphism of $\text{Spec}(\mathcal{O}_{Y,y})$ onto the subspace S_y of points $z \in Y$ such that $y \in \overline{\{z\}}$ (i.e. the set of generalizations of y).*
- (b) *For each $\mathfrak{p} \in \text{Spec}(\mathcal{O}_{Y,y})$, the homomorphism $f_{\mathfrak{p}}^{\#} : \mathcal{O}_{Y,f(\mathfrak{p})} \rightarrow (\mathcal{O}_{Y,y})_{\mathfrak{p}}$ is an isomorphism.*

In particular, f is a monomorphism of locally ringed spaces.

Proof. Since the unique closed point η of $\text{Spec}(\mathcal{O}_{Y,y})$ belongs to the closure of any other point in this space, and $f(\eta) = y$, the image of $\text{Spec}(\mathcal{O}_{Y,y})$ by the continuous map f is contained in S_y . As S_y is contained in any affine open neighborhood of y , we can reduce to the case where Y is an affine scheme; but in this case the proposition follows immediately. \square

Corollary 18.2.10. *There is a bijective correspondence between $\text{Spec}(\mathcal{O}_{Y,y})$ and irreducible closed subsets of Y containing y .*

Proof. This follows from ?? and the fact that every irreducible closed set in Y has a unique generic point. \square

Corollary 18.2.11. *For a point $y \in Y$ to be the generic point of an irreducible component of Y , it is necessary and sufficient that $\mathcal{O}_{Y,y}$ is zero-dimensional.*

Proof. This follows from the observation that y is the generic point of an irreducible component if and only if it is a maximal element under generalization, which is then equivalent by ?? to that $\text{Spec}(\mathcal{O}_{Y,y})$ is a singleton. \square

Proposition 18.2.12. *Let (X, \mathcal{O}_X) be a local scheme with $A = \Gamma(X, \mathcal{O}_X)$, ξ its unique closed point, and (Y, \mathcal{O}_Y) a scheme. Then any morphism $f : (X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$ factors through $\text{Spec}(\mathcal{O}_{Y,f(\xi)})$:*

$$f : X \rightarrow \text{Spec}(\mathcal{O}_{Y,f(\xi)}) \rightarrow Y$$

where the second one is the canonical morphism, and the first one corresponds to a local homomorphism $\mathcal{O}_{Y,f(\xi)} \rightarrow A$.

Proof. In fact, for any $x \in X$, we have $\xi \in \overline{\{x\}}$, hence $f(\xi) \in \overline{\{f(x)\}}$. It then follows that $f(X)$ is contained in any affine open neighborhood of $f(\xi)$ (in fact any open neighborhood of $f(\xi)$). We can then reduce to the case that (Y, \mathcal{O}_Y) is an affine scheme with ring $B = \Gamma(Y, \mathcal{O}_Y)$, and the morphism f corresponds to a ring homomorphism $\rho : B \rightarrow A$. We have $\rho^{-1}(\mathfrak{p}_{\xi}) = \mathfrak{p}_{f(\xi)}$, so the image under ρ of an element of $B - \mathfrak{p}_{f(\xi)}$ is invertible in the local ring A , and we get a canonical homomorphism $\rho_{\xi} : B_{f(\xi)} \rightarrow A$. \square

Corollary 18.2.13. *There is a canonical bijection between $\text{Hom}_{\mathbf{Sch}}(X, Y)$ to the set of local homomorphisms $\mathcal{O}_{Y,y} \rightarrow A$, where $y \in Y$.*

Proof. It suffices to note that any local homomorphism $\mathcal{O}_{Y,y} \rightarrow A$ corresponds to a unique morphism $f : X \rightarrow \text{Spec}(\mathcal{O}_{Y,y})$ such that $f(\xi) = y$, and by composing this with the canonical morphism $\text{Spec}(\mathcal{O}_{Y,y}) \rightarrow Y$, we get a morphism $X \rightarrow Y$. \square

Example 18.2.14. The affine scheme whose ring is a field K have an underlying space reduced to one point. If A is a local ring of maximal ideal \mathfrak{m} , any local homomorphism $A \rightarrow K$ has a kernel equal to \mathfrak{m} , so factors into $A \rightarrow A/\mathfrak{m} \rightarrow K$, where the second arrow is a monomorphism. The morphisms $\text{Spec}(K) \rightarrow \text{Spec}(A)$ correspond therefore bijectively to the field extensions $A/\mathfrak{m} \rightarrow K$.

Let (Y, \mathcal{O}_Y) be a scheme; for any $y \in Y$ and any ideal \mathfrak{a}_y of $\mathcal{O}_{Y,y}$, the canonical homomorphism $\mathcal{O}_{Y,y}/\mathfrak{a}_y$ defines a morphism $\text{Spec}(\mathcal{O}_{Y,y}/\mathfrak{a}_y) \rightarrow \text{Spec}(\mathcal{O}_{Y,y})$; by composing this with the canonical morphism $\text{Spec}(\mathcal{O}_{Y,y}) \rightarrow Y$, we obtain a morphism $\text{Spec}(\mathcal{O}_{Y,y}/\mathfrak{a}_y) \rightarrow Y$, also called canonical. If $\mathfrak{a}_y = \mathfrak{m}_y$ is the maximal ideal of $\mathcal{O}_{Y,y}$, then $\mathcal{O}_{Y,y} = \kappa(y)$ and ?? then imply the following result:

Corollary 18.2.15. *Let (X, \mathcal{O}_X) be a local scheme with $K = \Gamma(X, \mathcal{O}_X)$ a field, ξ its unique point, and (Y, \mathcal{O}_Y) be a scheme. Then any morphism $f : (X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$ factors into*

$$f : X \rightarrow \text{Spec}(\kappa(f(\xi))) \rightarrow Y$$

where the second arrow is the canonical morphism, and the first arrow corresponds to a field extension $\kappa(f(\xi)) \rightarrow K$. This establishes a canonical bijection between $\text{Hom}_{\mathbf{Sch}}(X, Y)$ to the set of field extensions $\kappa(y) \rightarrow K$, where $y \in Y$.

Corollary 18.2.16. *For any $y \in Y$, the canonical morphism $\text{Spec}(\mathcal{O}_{Y,y}/\mathfrak{a}_y) \rightarrow Y$ is a monomorphism of locally ringed spaces.*

Proof. This follows from ?? and ??.

\square

Remark 18.2.3. Let X be a local scheme, ξ its unique closed point. Since any affine open neighborhood of ξ is necessarily all of X , any invertible \mathcal{O}_X -module is necessarily isomorphic to \mathcal{O}_X (in other words, is trivial). This property does not hold in general for any affine scheme $\text{Spec}(A)$, but we will see that if A is a normal ring, this is true when A is factorial.

18.2.3 Schemes over a scheme

As in any category, for a scheme S we can define the category \mathbf{Sch}/S of S -objects in the category of schemes, which will be a morphism $\varphi : X \rightarrow S$ where X is a scheme. In this case we also say that X is a **scheme over S** , or an **S -scheme**. We say that S is the **base scheme** of the S -scheme X and φ is called the structural morphism of the S -scheme X . When S is an affine scheme of the ring A , we also say that X is a **scheme over A** or an **A -scheme**.

It follows from ?? that giving a scheme over a ring A is equivalent to giving a scheme (X, \mathcal{O}_X) , where \mathcal{O}_X is an A -algebra. In particular, any scheme can be considered as a scheme over \mathbb{Z} . In other words, the scheme $\text{Spec}(\mathbb{Z})$ is a final object in the category of schemes (also a final object in the category of locally ringed spaces).

If $\varphi : X \rightarrow S$ is the structural morphism of an S -scheme X , we say a point $x \in X$ is **lying over** a point $s \in S$ if $\varphi(x) = s$. We say X **dominates** S if the morphism φ is dominant. Let X and Y be two S -schemes; a morphism $u : X \rightarrow Y$ is called a **morphism of schemes over S** (or **S -morphism**) if the diagram

$$\begin{array}{ccc} X & \xrightarrow{u} & Y \\ & \searrow & \swarrow \\ & S & \end{array}$$

is commutative; in other words, if for any $s \in S$ and any $x \in X$ lying over s , the point $u(x)$ is lying over s . This definition immediately shows that the composite of two S -morphisms is an S -morphism, so the S -schemes thus form a category. We denote by $\text{Hom}_S(X, Y)$ the set of S -morphisms from an S -scheme X to an S -scheme Y ; the identity morphism of an S -scheme X is then denoted by 1_X or id_X . If S is an affine scheme S , we also say A -morphisms for S -morphisms.

If X is an S -scheme, $\varphi : X \rightarrow S$ the structural morphism, an **S -section** of X is defined to be an S -morphism of S to X , which is equivalently a morphism $\psi : S \rightarrow X$ of schemes such that $\varphi \circ \psi = \text{id}_S$. We denote by $\Gamma(X/S)$ the set of S -sections of X .

Example 18.2.17. If X is an S -scheme and $v : X' \rightarrow X$ a morphism of schemes, the composition scheme

$$X' \xrightarrow{v} X \longrightarrow S$$

then defines X' as an S -scheme; in particular, any scheme induced over an open subset U of X can be considered as an S -scheme by means of the canonical injection.

Example 18.2.18. Let $u : X \rightarrow Y$ be an S -morphism of S -schemes, the restriction of u on any open subset U of X is then an S -morphism $U \rightarrow Y$. Conversely, let (U_α) be a covering of X and for each α , let $u_\alpha : U_\alpha \rightarrow Y$ be an S -morphism; if for any pair (α, β) of indices, the restrictions of u_α and u_β on $U_\alpha \cap U_\beta$ coincide, then there exists a unique S -morphism $X \rightarrow Y$ whose restriction on U_α equals to u_α .

Let $S \rightarrow S'$ be a morphism of schemes; for any S' -scheme X , the composition morphism $X \rightarrow S' \rightarrow S$ then defines X as an S -scheme. Conversely, suppose that S' is the scheme induced over an open subset of S ; let X be an S -scheme and suppose that the structural morphism $X \rightarrow S$ has image contained in S' ; then we can consider X as an S' -scheme. In the latter case, if Y is an S -scheme whose structural morphism also maps the underlying space in S' , any S -morphism from X in Y is also an S' -morphism.

18.2.4 Quasi-coherent sheaves on schemes

Proposition 18.2.19. *Let X be a scheme. For an \mathcal{O}_X -module \mathcal{F} to be quasi-coherent, it is necessary and sufficient that, for any affine open subset V of X , the restriction $\mathcal{F}|_V$ is isomorphism to the sheaf associated with a $\Gamma(V, \mathcal{O}_X)$ -module.*

Proof. We recall that being quasi-coherent is a local property, and affine opens form a basis for X . Also, by ??, a quasi-coherent sheaf on an affine open V is isomorphic to \tilde{M} for some $\Gamma(V, \mathcal{O}_X)$ -module M . \square

Corollary 18.2.20. *Let X be an arbitrary scheme.*

- (i) *Let $0 \rightarrow \mathcal{F} \rightarrow \mathcal{G} \rightarrow \mathcal{H} \rightarrow 0$ be an exact sequence of \mathcal{O}_X -modules; if two of them are quasi-coherent, then so is the third one.*
- (ii) *The images, kernels and cokernels of homomorphisms of quasi-coherent \mathcal{O}_X -modules are quasi-coherent. The inductive limits and direct sums of quasi-coherent sheaves are quasi-coherent. If \mathcal{G} and \mathcal{H} are quasi-coherent \mathcal{O}_X -modules of a quasi-coherent \mathcal{O}_X -module \mathcal{F} , then $\mathcal{G} + \mathcal{H}$ and $\mathcal{G} \cap \mathcal{H}$ are quasi-coherent.*
- (iii) *Let $0 \rightarrow \mathcal{F}_1 \rightarrow \mathcal{F}_2 \rightarrow \mathcal{F}_3 \rightarrow \mathcal{F}_4 \rightarrow \mathcal{F}_5 \rightarrow 0$ be an exact sequence of \mathcal{O}_X -modules. If $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_4, \mathcal{F}_5$ are quasi-coherent, so is \mathcal{F}_3 .*
- (iv) *If \mathcal{F} and \mathcal{G} are quasi-coherent \mathcal{O}_X -modules, $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ is quasi-coherent. In particular, if \mathcal{I} is a quasi-coherent ideal of \mathcal{O}_X , $\mathcal{I}\mathcal{F}$ is a quasi-coherent \mathcal{O}_X -module.*
- (v) *Let \mathcal{F} be a quasi-coherent \mathcal{O}_X -module with finite presentation. Then for any quasi-coherent \mathcal{O}_X -module \mathcal{G} , $\mathcal{H}\text{om}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is quasi-coherent.*
- (vi) *If \mathcal{F} is a quasi-coherent \mathcal{O}_X -module of finite type, the annihilator \mathcal{I} of \mathcal{F} is a quasi-coherent ideal of \mathcal{O}_X .*

Proof. By ??, assertions (i) to (v) follow from ??, ??, and ?. To prove (vi), we can assume that $X = \text{Spec}(A)$ is affine, $\mathcal{F} = \tilde{M}$, where M is a finitely generated A -module, with generators t_1, \dots, t_r . The ideal \mathcal{I} is then the intersection of the annihilators of t_i . But the annihilator of t_i is by definition the kernel of the canonical morphism $\mathcal{O}_X \rightarrow \mathcal{F}$ corresponding to $s \mapsto st_i$ from A to M , hence quasi-coherent. It then follows that \mathcal{I} is quasi-coherent, as an intersection of quasi-coherent \mathcal{O}_X -modules. \square

Corollary 18.2.21. *Let X be a scheme, $0 \rightarrow \mathcal{F} \rightarrow \mathcal{G} \rightarrow \mathcal{H} \rightarrow 0$ an exact sequence of quasi-coherent \mathcal{O}_X -modules. If \mathcal{H} is finitely presented and \mathcal{G} is of finite type, then \mathcal{F} is of finite type.*

Proof. Since this question is local, we may assume that X is affine, and the corresponding result is then ?. \square

Proposition 18.2.22. *Let X be a scheme and \mathcal{B} be a quasi-coherent \mathcal{O}_X -algebra. For a \mathcal{B} -module \mathcal{F} to be quasi-coherent, it is necessary and sufficient that \mathcal{F} is a quasi-coherent \mathcal{O}_X -module. In particular, if \mathcal{F} and \mathcal{G} are quasi-coherent \mathcal{B} -modules, $\mathcal{F} \otimes_{\mathcal{B}} \mathcal{G}$ is a quasi-coherent \mathcal{B} -module; the same holds for $\mathcal{H}\text{om}_{\mathcal{B}}(\mathcal{F}, \mathcal{G})$ if \mathcal{F} is a finitely presented \mathcal{B} -module.*

Proof. Since the question is local, we can suppose that X is affine with ring A , and then $\mathcal{B} = \tilde{B}$, where B is an A -algebra. If \mathcal{F} is quasi-coherent over the space (X, \mathcal{B}) , we can write \mathcal{F} as the cokernel of \mathcal{B} -homomorphism $\mathcal{B}^{\oplus I} \rightarrow \mathcal{B}^{\oplus J}$; since this homomorphism is also an \mathcal{O}_X -homomorphism, and $\mathcal{B}^{\oplus I}, \mathcal{B}^{\oplus J}$ are quasi-coherent \mathcal{O}_X -modules, we conclude that \mathcal{F} is also quasi-coherent.

Conversely, if \mathcal{F} is a quasi-coherent \mathcal{O}_X -module, we have $\mathcal{F} = \tilde{M}$ where M is a B -module ??; M is isomorphic to the cokernel of a homomorphism $B^{\oplus I} \rightarrow B^{\oplus J}$, so \mathcal{F} is a \mathcal{B} -module isomorphic to the cokernel of the corresponding homomorphism $\mathcal{B}^{\oplus I} \rightarrow \mathcal{B}^{\oplus J}$. This completes the proof. \square

Let X be a scheme. A quasi-coherent \mathcal{O}_X -algebra \mathcal{B} is **of finite type** (resp. **of finite presentation**) if for all $x \in X$, there exists an open affine neighborhood U of x such that $\Gamma(U, \mathcal{B}) = B$ is an algebra of type finite (resp. of finite presentation¹) over $\Gamma(U, \mathcal{O}_X) = A$. If this is the case, we have $\mathcal{B}|_U = \tilde{B}$, and for all $f \in A$, the $(\mathcal{O}_X|_{D(f)})$ -algebra $\mathcal{B}|_{D(f)}$ induced on $D(f)$ is of finite type (resp. of finite presentation), because it is isomorphic to $B \otimes_A A_f$. As the $D(f)$ form a basis of the topology of X , we deduce that for any open set V of X , $\mathcal{B}|_V$ is a $(\mathcal{O}_X|_V)$ -algebra of finite type (resp. of finite presentation).

Proposition 18.2.23. *Let X be a scheme, \mathcal{E} a locally free \mathcal{O}_X -module of rank r , Z a finite subset of X contained in an affine open V . Then there exists an open neighborhood $U \subseteq V$ of Z such that $\mathcal{E}|_U$ is isomorphic to \mathcal{O}_U^r .*

Proof. By replacing X by V , we can assume that $X = \text{Spec}(A)$ is affine. For each $z_i \in Z$ there exists a closed point z'_i in the closure $\overline{\{z_i\}}$ (that is, a maximal ideal containing \mathfrak{p}_{z_i}); if Z' is the set of the z'_i , any neighborhood of Z' is a neighborhood of Z , and we can then suppose that Z is closed in X . Now, the subset Z of X is defined by an ideal \mathfrak{a} of A ; consider the scheme $\text{Spec}(A/\mathfrak{a})$, with Z its underlying space, and the injection $\iota : Z \rightarrow X$ corresponds to the canonical homomorphism $A \rightarrow A/\mathfrak{a}$. Then $\iota^*(\mathcal{E}) = \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}_Z$ is locally free with rank r over the discrete scheme Z , so is isomorphic to \mathcal{O}_Z^r . In other words, there exist sections s_1, \dots, s_r of $\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}_Z$ over Z such that the homomorphism $\mathcal{O}_Z^r \rightarrow \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}_Z$ defined by these sections is bijective. On the other hand, we have $\mathcal{E} = \tilde{M}$ where M is an A -module; then each s_i belongs to $M \otimes_A (A/\mathfrak{a})$, and is then the image of an element $t_i \in M = \Gamma(X, \mathcal{E})$. For each $z_j \in Z$, by ??, there then exists a neighborhood V_i of z_i in X such that the restrictions of t_i to V_j define an isomorphism $\mathcal{O}_X^r|_{V_j} \rightarrow \mathcal{E}|_{V_j}$; the union U of the V_j 's then satisfies the requirement. \square

Proposition 18.2.24. *Let X a scheme and \mathcal{F} be a quasi-coherent \mathcal{O}_X -module of finite type. Then there exists a closed subscheme Y of X with underlying space $\text{supp}(\mathcal{F})$ and a quasi-coherent \mathcal{O}_Y -module \mathcal{G} of finite type supported on Y such that, if $j : Y \rightarrow X$ is the canonical injection, \mathcal{F} is isomorphic to $j_*(\mathcal{G})$.*

Proof. It suffices to note that the annihilator \mathcal{J} of \mathcal{F} is a quasi-coherent ideal of \mathcal{O}_X (??), so if Y is the closed subscheme of X defined by \mathcal{J} , as $\mathcal{J}\mathcal{F} = 0$, \mathcal{F} is an $(\mathcal{O}_X/\mathcal{J})$ -module, and we can take $\mathcal{G} = j^*(\mathcal{F})$. \square

18.2.5 Noetherian schemes and locally Noetherian schemes

We say a scheme X is Noetherian (resp. locally Noetherian) if there is a finite covering (resp. a covering) of open affines V_α such that each ring $\Gamma(V_\alpha, \mathcal{O}_X)$ is Noetherian. The underlying space of a Noetherian (resp. locally Noetherian) scheme is then a Noetherian space (resp. locally Noetherian). Moreover, if X is locally Noetherian, the structural sheaf \mathcal{O}_X is coherent, any quasi-coherent \mathcal{O}_X -module of finite type is coherent (??), and any local ring $\mathcal{O}_{X,x}$ is Noetherian. Any quasi-coherent sub- \mathcal{O}_X -module (resp. any quasi-coherent \mathcal{O}_X -quotient) of a coherent \mathcal{O}_X -module \mathcal{F} is then coherent, because the question is local again, and we just apply ??, together with the fact that a sub-module (resp. quotient module) of a finitely generated module on a Noetherian ring is finitely generated. More particularly, any quasi-consistent ideal of \mathcal{O}_X is consistent.

¹Recall that an algebra B is finitely presented over A if it is isomorphic to the quotient of a polynomial ring over A in finitely many variables by a finitely generated ideal.

If a scheme X is a finite union (resp. a union) of open Noetherian (resp. locally Noetherian) subschemes W_λ , it is clear that X is then Noetherian (resp. locally Noetherian).

Proposition 18.2.25. *For a scheme X to be Noetherian, it is necessary and sufficient that it is locally Noetherian and its underlying space is quasi-compact.*

Proof. This follows from the definition, since a Noetherian space is quasi-compact. \square

Proposition 18.2.26. *Let X be an affine scheme with ring A . Then the following conditions are equivalent:*

- (i) X is Noetherian;
- (ii) X is locally Noetherian;
- (iii) A is Noetherian.

Proof. Since X is quasi-compact, it is clear that (i) and (ii) are equivalent. Also, (iii) implies (i) by definition. Now assume that X is Noetherian, then there is a finite covering (V_i) of X by affine opens where $A_i = \Gamma(V_i, \mathcal{O}_X)$ is Noetherian. Let (\mathfrak{a}_n) be an increasing sequence of ideals of A ; it corresponds to it canonically in a one-to-one way to an increasing sequence $(\tilde{\mathfrak{a}}_n)$ of ideals in $\tilde{A} = \mathcal{O}_X$; to see that the sequence (\mathfrak{a}_n) is stationary, it suffices to prove that the sequence $(\tilde{\mathfrak{a}}_n)$ is. However, the restriction $\tilde{\mathfrak{a}}_n|_{V_i}$ is a quasi-coherent ideal of $\mathcal{O}_X|_{V_i}$; $\tilde{\mathfrak{a}}_n|_{V_i}$ is then of the form $\tilde{\mathfrak{a}}_{n,i}$, where $\mathfrak{a}_{n,i}$ is an ideal of A_i . As A_i is Noetherian, the sequence $(\mathfrak{a}_{n,i})$ is stationary for all i , hence the proposition. \square

Note that the above reasoning also proves that if X is a Noetherian scheme, any increasing sequence of coherent ideals of \mathcal{O}_X is stationary.

Proposition 18.2.27. *Let X be a locally Noetherian scheme. Any quasi-coherent \mathcal{O}_X -algebra \mathcal{B} of finite type is a coherent shaf of rings and an \mathcal{O}_X -algebra of finite presentation.*

Proof. We can assume that $X = \text{Spec}(A)$ is affine, where A is a Noetherian ring, and $\mathcal{B} = \tilde{B}$, where B is an A -algebra of finite type. It then follows that B is finitely presented over A , so \mathcal{B} is of finite presentation. To show that \mathcal{B} is coherent, we must prove that the kernel \mathcal{N} of a \mathcal{B} -homomorphism $\mathcal{B}^m \rightarrow \mathcal{B}$ is a \mathcal{B} -module of finite type; but it is of the form \tilde{N} , where N is the kernel of the corresponding homomorphism $B^m \rightarrow B$ of B -modules. Since B is also Noetherian, N is a finitely generated B -module. There then exists a surjective B -homomorphism $B^n \rightarrow N$, so a surjective homomorphism $\mathcal{B}^n \rightarrow \mathcal{N}$, which proves our assertion. \square

Corollary 18.2.28. *Let X be a locally Noetherian scheme and \mathcal{B} be a quasi-coherent \mathcal{O}_X -algebra of finite type. For a \mathcal{B} -module \mathcal{F} to be coherent, it is necessary and sufficient that \mathcal{F} is a quasi-coherent \mathcal{O}_X -module and a \mathcal{B} -module of finite type, and if \mathcal{G} is a sub- \mathcal{B} -module or a quotient \mathcal{B} -module of \mathcal{F} , for \mathcal{G} to be a coherent \mathcal{B} -module, it is necessary and sufficient that \mathcal{G} is a quasi-coherent \mathcal{O}_X -module.*

Proof. Considering ??, the conditions on \mathcal{F} is necessary. To prove the sufficiency, we can assume that $X = \text{Spec}(A)$ is affine, where A is Noetherian, $\mathcal{B} = \tilde{B}$, where B is an A -algebra of finite type, and $\mathcal{F} = \tilde{M}$, where M is a B -module and there exists a surjective B -homomorphism $\mathcal{B}^m \rightarrow \mathcal{F}$. Then we get a corresponding homomorphism $B^m \rightarrow M$, so M is a finitely generated B -module; the kernel P of this homomorphism is finitely generated since B is Noetherian, and

\mathcal{F} is therefore the cokernel of a morphism $\mathcal{B}^n \rightarrow \mathcal{B}^m$, so it is coherent (since \mathcal{B} is a coherent sheaf of rings). The same reasoning shows that any quasi-coherent sub- \mathcal{B} -module (resp. quotient \mathcal{B} -module) of \mathcal{F} is of finite type, whence the second part of the corollary. \square

Proposition 18.2.29. *Let X be a locally Noetherian scheme and E be a subset of X . Any point $x \in E$ admits in E a maximal generalization y (i.e. y has no further generalization in E). In particular, if $E \neq \emptyset$, there exists a maximal element $y \in E$ under generalization.*

Proof. The generalizations of x in X lie in the points of $\text{Spec}(\mathcal{O}_{X,x})$ (??), where $\mathcal{O}_{X,x}$ is a Noetherian local ring. We then know that the lengths of chains of prime ideals in this ring are bounded by $\dim(\mathcal{O}_{X,x})$, and to prove the proposition it suffices to consider a chain of prime ideals belonging to E and having the greatest possible length. \square

Proposition 18.2.30. *Let X be a scheme. Then the following conditions are equivalent:*

- (i) $X = \text{Spec}(A)$ is affine and A is Artinian;
- (ii) X is Noetherian and has discrete underlying space;
- (iii) X is Noetherian and every point in X is closed (in other words, X is T1).

If these equivalent conditions hold, then X is finite and the ring A is a direct product of finitely many Artinian local rings.

Proof. We know that (i) implies the last assertion. It is clear that (i) \Rightarrow (ii) \Rightarrow (iii). To see that (iii) implies (i), let us first show that X is then finite; we can indeed reduce to case where X is affine, and we know that a Noetherian ring of which all the prime ideals are maximal is Artinian, hence our assertion. \square

Note that a Noetherian scheme can have an underlying space finite without being artinian, as shown by the example of a spectrum discrete valuation ring.

18.3 Product of schemes

Let (X_α) be a family of schemes, and X be the topological space which is the **coproduct** of the underlying spaces of X . Then X is the union of its open subspaces U_α , and for each α we have a embedding $\iota_\alpha : X_\alpha \rightarrow X$ with image equal to U_α . If we endow each U_α the sheaf $(\iota_\alpha)_*(\mathcal{O}_{X_\alpha})$, it is clear that X becomes a scheme, which we will call the **coproduct** of the family (X_α) , and denote by $\coprod_\alpha X_\alpha$. It is clear that the scheme X satisfies the universal property of coproducts of X_α 's: for any scheme Y and morphisms $f_\alpha : X_\alpha \rightarrow Y$, there exists a unique morphism $f : X \rightarrow Y$ such that $f \circ \iota_\alpha = f_\alpha$. In other words, we have a functorial bijection

$$\text{Hom}\left(\coprod_\alpha X_\alpha, Y\right) \rightarrow \prod_\alpha \text{Hom}(X_\alpha, Y).$$

This fact can be also stated that $\coprod_\alpha X_\alpha$ represents the covariant functor $\prod_\alpha \text{Hom}(X_\alpha, -)$ on the category of schemes. In particular, if X_α are S -schemes with structural morphisms ψ_α , then X is an S -scheme with structural morphism $\psi : X \rightarrow S$ such that $\psi \circ \iota_\alpha = \psi_\alpha$. We usually denote the coproduct of two schemes X and Y by $X \amalg Y$, and it is clear that if $X = \text{Spec}(A)$ and $Y = \text{Spec}(B)$, then $X \amalg Y$ is canonically identified with $\text{Spec}(A \times B)$.

In this section, we shall consider product of schemes, which is far more complicated than coproducts. We will see that fiber products plays a central role of many construction and operations on schemes.

18.3.1 Product of schemes

Let X and Y be S -schemes. Recall that the object $X \times_S Y$ represents by definition the contravariant functor

$$T \mapsto F(T) = \text{Hom}_S(T, X) \times \text{Hom}_S(T, Y)$$

on the category of S -schemes. To prove the existence of $X \times_S Y$, we shall apply the methods used in ???. We first verify condition (ii) in ???, which means F is a sheaf over the category **Sch**: this is evident since the functors $T \mapsto \text{Hom}_S(T, X)$ and $T \mapsto \text{Hom}_S(T, Y)$ are sheaves, and a projective limit of sheaves over **Sch** is again a sheaf over **Sch**.

This already allows us to bring ourselves back to the case that the scheme S is affine. In fact, let (S_α) is a covering of S by affine open sets. In view of the above fact and of ???, it suffices to show that each of the functors $F \times_{h_S} h_{S_\alpha}$ is representable. On the other hand, let $\varphi : X \rightarrow S$ and $\psi : Y \rightarrow S$ be the structural morphisms; it is immediate that when T is a locally ringed S_α -space (hence also an S -space), we have canonical identifies

$$\text{Hom}_S(T, X) \xrightarrow{\sim} \text{Hom}_{S_\alpha}(T, \varphi^{-1}(S_\alpha)), \quad \text{Hom}_S(T, Y) \xrightarrow{\sim} \text{Hom}_{S_\alpha}(T, \psi^{-1}(S_\alpha)).$$

Therefore, in view of ???, we only need to show that F is representable when it is restricted to the subcategory of locally ringed S_α -spaces.

With these being done, assume that S is affine and consider a covering (X_λ) (resp. (Y_μ)) of X (resp. Y) by affine opens. We shall verify the conditions (i) and (iii) of ?? for the subfunctors $F_{\lambda\mu} : T \mapsto \text{Hom}_S(T, X_\lambda) \times \text{Hom}_S(T, Y_\mu)$ of F . Let Z be a locally ringed S -space and (p, q) be an element of $F(Z)$, i.e. $p : Z \rightarrow X$ and $q : Z \rightarrow Y$ are S -morphisms. These determine by Yoneda Lemma a natural transform $h_Z \rightarrow F$ which associates a locally ringed S -space T the map

$$\text{Hom}_S(T, Z) \rightarrow F(T), \quad g \mapsto (p \circ g, q \circ g)$$

and every natural transform $h_Z \rightarrow F$ is of this form. We now show that the functor

$$T \mapsto F_{\lambda\mu}(T) \times_{F(T)} h_Z(T) \tag{18.3.1}$$

is representable by a locally ringed S -space induced by Z on an open subset of Z . In fact, an element of the right side of ?? (which is a fiber product of sets) is a triple (u_λ, v_μ, g) , where $g : T \rightarrow Z$, $u_\lambda : T \rightarrow X_\lambda$, and $v_\mu : T \rightarrow Y_\mu$ are S -morphisms such that the following diagram commutes:

$$\begin{array}{ccccc} T & & & & \\ & \searrow g & \swarrow v_\mu & & \\ & Z & \xrightarrow{q} & Y & \\ u_\lambda \downarrow & & p \downarrow & & \\ X & \longrightarrow & S & & \end{array}$$

Now this in particular implies that $g(T) \subseteq Z_{\lambda\mu} = p^{-1}(X_\lambda) \cap q^{-1}(Y_\mu)$, and conversely, any S -morphism $g : T \rightarrow Z$ verifying this condition corresponds to the unique triple $(p \circ g, q \circ g, g)$, since $p \circ g$ (resp $q \circ g$) can be viewed as a morphism from Z to X_λ (resp. Y_μ). In other word, we have a canonical bijection

$$F_{\lambda\mu}(T) \times_{F(T)} \text{Hom}_S(T, Z) \xrightarrow{\sim} \text{Hom}_S(T, Z_{\lambda\mu})$$

and the functor ?? is then represented by the couple $(Z_{\lambda\mu}, (p|_{Z_{\lambda\mu}}, q|_{Z_{\lambda\mu}}), j_{\lambda\mu})$, where $j_{\lambda\mu} : Z_{\lambda\mu} \rightarrow Z$ is the cannical injection. Since the $Z_{\lambda\mu}$ form an open covering of Z , this proves both of the conditions (i) and (iii) of ??.

It remains to show that the functors $F_{\lambda\mu}$ are representable, which means we need to construct $X \times_S Y$ when X , Y , and S are affine schemes. This is fairly easy, as we will now show.

Proposition 18.3.1. *Assume that $S = \text{Spec}(A)$, $X = \text{Spec}(B)$, and $Y = \text{Spec}(C)$, where B and C are A -algebras. Then the scheme $Z = \text{Spec}(B \otimes_A C)$, with p, q the S -morphisms corresponding to the canonical A -homomorphisms $B \rightarrow B \otimes_A C$ and $C \rightarrow B \otimes_A C$, represents the functor F in the category of locally rinfed S -spaces.*

Proof. In fact, in the category of rings, the tensor product $B \otimes_A C$ of two A -algebras B and C is a coproduct in the category of A -algebras, as can be easily verified. \square

We then conclude that fiber products exist in the category of schemes. As always, the notation $X \times_S Y$ will be used to denote this product for two S -schemes X and Y . If $S = \text{Spec}(A)$ is an affine scheme, we also write $X \times_A Y$. If $Y = \text{Spec}(B)$ is an affine scheme, in view of ??, we use $X \otimes_S B$ to denote this product, and $X \otimes_A B$ if $S = \text{Spec}(A)$ is also affine.

The general notations and results for fiber products in a category can be then used for the product of schemes. In particular, if $p_1 : X \times_X Y \rightarrow X$, $p_2 : X \times_S Y \rightarrow Y$ are the canonical projections, and $g : T \rightarrow X$, $h : T \rightarrow Y$ are two S -morphisms, we denote by $(g, h)_S$ the unique S -morphism fits into the following diagram:

$$\begin{array}{ccccc} T & \xrightarrow{(g,h)_S} & X \times_S Y & \xrightarrow{p_2} & Y \\ & \searrow g & \downarrow p_1 & & \downarrow \\ & & X & \longrightarrow & S \end{array}$$

If $S = \text{Spec}(A)$, $X = \text{Spec}(B)$, $Y = \text{Spec}(C)$, and $T = \text{Spec}(D)$ are all affine, and g, h correspond to homomorphisms $\rho : B \rightarrow D$, $\sigma : C \rightarrow D$ of A -algebras, then $(g, h)_S$ corresponds to the homomorphism $\tau : B \otimes_A C \rightarrow D$ such that

$$\tau(b \otimes c) = \rho(b)\sigma(c).$$

Again, if $S = \text{Spec}(A)$ is affine, we also write $(g, h)_A$ instead of $(g, h)_S$.

Corollary 18.3.2. *Let $Z = X \times_S Y$ be the product of two S -schemes, $p : Z \rightarrow X$, $q : Z \rightarrow Y$ the canonical projections, φ (resp ψ) the structural morphisms of X (resp. Y). Let U, V be open subsets*

of X, Y respectively, and W be an open subset of S such that $p(U) \subseteq W$ and $p(V) \subseteq W$. Then the product $U \times_W V$ is canonically identified with the scheme induced by Z on the subset $p^{-1}(V) \cap q^{-1}(W)$ (considered as a U -scheme). Moreover, if $g : T \rightarrow X, h : T \rightarrow Y$ are S -morphisms such that $g(T) \subseteq V, h(T) \subseteq W$, the U -morphism $(g, h)_S$ is identified with $(g, h)_S$, considered as morphisms from T to $p^{-1}(V) \cap q^{-1}(W)$.

Proof. We first note that, if U is an open set of S and $\varphi : X \rightarrow S, \psi : Y \rightarrow S$ are structural morphisms with images in U , then the fiber product $X \times_S Y$ is identified with $X \times_U Y$. Apply this to V and W , we conclude that $U \times_W V = U \times_V V$. So it suffices to prove that the subscheme $R = p^{-1}(U) \cap q^{-1}(V)$ with its restricted projections to U and V form a product of U and V . For this, we note that if T is an S -scheme, we can identify the S -morphisms $T \rightarrow R$ and the S -morphisms $T \rightarrow Z$ with image in R . If $g : T \rightarrow U, h : T \rightarrow V$ are two S -morphisms, we can consider them as S -morphisms of T in X and Y respectively, and by hypothesis there is therefore an S -morphism and there is a morphism $f : T \rightarrow Z$ such that $g = p \circ f, h = q \circ f$. Since $p(f(T)) \subseteq U$ and $q(f(T)) \subseteq V$, we have

$$f(T) \subseteq p^{-1}(U) \cap q^{-1}(V) = W$$

whence our claim. \square

Corollary 18.3.3. Let (X_λ) (resp. (Y_μ)) be a family of S -schemes and X (resp. Y) be their coproduct. Then $X \times_S Y$ is identified with the coproduct of the family $(X_\lambda \times_S Y_\mu)$.

Proof. In fact, in the notations of ??, the underlying space of $X \times_S Y$ is the disjoint union of open sets $p^{-1}(X_\lambda) \cap q^{-1}(Y_\mu)$, and it suffices to apply ??.

Remark 18.3.1. The product of two Noetherian S -schemes need not to be Noetherian, even if they are both spectrum of fields. For example, if k is a nonperfect field of characteristic $p > 0$, the tensor product $A = k^{p^\infty} \otimes_k k^{p^\infty}$ is not a Noetherian ring: in fact, for any integer $n > 0$, there exists $x_n \in k^{p^\infty}$ such that $x_n^{p^n} \in k$ and $x_n^{p^{n-1}} \notin k$. If we consider the element $z_n = 1 \otimes x_n - x_n \otimes 1$ of A , we then have $z_n^{p^n} = 0$, and $z_n^{p^{n-1}} \neq 0$ since 1 and $x_n^{p^{n-1}}$ are linearly independent over k . We then conclude that the nilradical of A is not nilpotent, so A is not Noetherian.

Remark 18.3.2. We should note that the underlying topological space of the fiber product $X \times_S Y$ is not the fiber product of the underlying topological spaces. This can be seen from the tensor product of two fields, which can not be a field.

18.3.2 Base change of schemes

The functor $X \times_S Y$ is covariant in both of its variables, and this follows from the following commutative diagram:

$$\begin{array}{ccccc} X \times Y & \xrightarrow{f \times 1} & X' \times Y & \xrightarrow{f' \times 1} & X'' \times Y \\ \downarrow & & \downarrow & & \downarrow \\ X & \xrightarrow{f} & X' & \xrightarrow{f'} & X'' \end{array}$$

Proposition 18.3.4. *For any S -scheme X , the first (resp. second) projection $X \times_S S$ (resp. $S \times_S X$) is a functorial isomorphism of $X \times_S S$ (resp. $S \times_S X$) to X , with inverse isomorphism $(1_X, \varphi)_S$ (resp. $(\varphi, 1_X)_S$), where $\varphi : X \rightarrow S$ is the structural morphism. We can therefore write*

$$X \times_S S = S \times_S X = X.$$

Proof. It suffices to prove that the triple $(X, 1_X, \varphi)$ form a product of X and S , which is immediate. \square

Corollary 18.3.5. *Let X and Y be S -schemes, $\varphi : X \rightarrow S$, $\psi : Y \rightarrow S$ the structural morphisms. If we identify canonically X with $X \times_S S$ and Y with $S \times_S Y$, the projections $X \times_S Y \rightarrow X$ and $X \times_S Y \rightarrow Y$ is identified respectively to $1_X \times \psi$ and $\varphi \times 1_Y$.*

We can define similarly the fiber product of S -schemes X_1, \dots, X_n , whose existence can be proved by induction on n , which is isomorphic to $(X_1 \times_S \dots \times_S X_{n-1}) \times_S X_n$. The uniqueness of the product entails, as in any category, its properties of commutativity and associativity. If, for example, p_1, p_2, p_3 denotes the projections of $X_1 \times_S X_2 \times_S X_3$, and if we identify this scheme with $(X_1 \times_S X_2) \times_S X_3$, the projection in $X_1 \times_S X_2$ is identified with $(p_1, p_2)_S$.

Let S, S' be two schemes, $\varphi : S \rightarrow S'$ an morphism, making S' an S -scheme. For any S -scheme X , consider the product $X \times_S S'$, and let p and π' the projections to X and S' respectively. Through the morphism π' , this product is an S' -scheme, which we may denoted by $X_{(S')}$ or $X_{(\varphi)}$, and the obtained scheme is called the **base change** of X from S to S' , or the inverse image of X via φ . We note that if π is the structural morphism of X and θ is the structural morphism of $X \times_S S'$, the following diagram is commutative:

$$\begin{array}{ccc} X_{(S')} & \xrightarrow{\pi'} & S' \\ p \downarrow & & \downarrow \varphi \\ X & \xrightarrow{\pi} & S \end{array}$$

For any S -morphism $f : X \rightarrow Y$, we denote by $f_{(S')}$ the S' -morphism $f \times_S 1 : X_{(S')} \rightarrow Y_{(S')}$ and call it the inverse image of f by φ . The operation $X_{(S')}$ is clearly a covariant functor on X , from the category **Sch/S** to **Sch/S'**.

Let S, S' be two affine schemes with rings A, A' ; a morphism $S' \rightarrow S$ corresponds to a homomorphism $A \rightarrow A'$. If X is an S -scheme, we then denote by $X_{(A')}$ of $X \otimes_A A'$ by the S' -scheme $X_{(S')}$; if X is also affine with ring B , then $X_{(A')}$ is affine with ring $B_{(A')} = B \otimes_A A'$.

We point out that the scheme $X_{(S')}$ satisfies the following universal property: any S' -scheme T is an S -scheme via the morphism φ , and for any S -morphism $g : T \rightarrow X$ there exists a unique S' -morphism $f : T \rightarrow X_{(S')}$ such that $g = p \circ f$.

Proposition 18.3.6 (Transitivity). *Let $\varphi' : S'' \rightarrow S'$ and $\varphi : S' \rightarrow S$ be morphism of schemes. For any S -scheme X , there is a canonical functorial isomorphism of the S'' -schemes $(X_{(\varphi)})_{(\varphi')}$ and $X_{(\varphi \circ \varphi')}$.*

Proof. In fact, let T be an S'' -scheme, ψ its structural morphism, $g : T \rightarrow X$ an S -morphism (T is an S -scheme via the morphism $\varphi \circ \varphi' \circ \psi$). Since T is an S' -scheme with structural morphism $\varphi' \circ \psi$, we can write $g = p \circ g'$, where $g' : T \rightarrow X_{(\varphi)}$ is an S' -morphism. Then $g' = p' \circ g''$,

where $g'' : T \rightarrow (X_{(\varphi)})_{\varphi'}$ is an S'' -morphism:

$$\begin{array}{ccccc} (X_{(\varphi)})_{\varphi'} & \xrightarrow{p'} & X_{(\varphi)} & \xrightarrow{p} & X \\ \downarrow \pi'' & & \downarrow \pi' & & \downarrow \pi \\ S'' & \xrightarrow{\varphi'} & S' & \xrightarrow{\varphi} & S \end{array}$$

The claim now follows from the definition of the universal property of $X_{(\varphi \circ \varphi')}$. \square

The previous result can also be written as $(X_{(S')})_{(S'')} = X_{(S'')}$, if there is no risk of confusion. Moreover precisely, we have

$$(X \times_S S') \times_{S'} S'' = X \times_S S'';$$

the functorial of the isomorphism in ?? also shows the transitive of inverse image of morphisms:

$$(f_{(S')})_{(S'')} = f_{(S'')}$$

for any S -morphism $f : X \rightarrow Y$.

Corollary 18.3.7. *If X and Y are S -schemes, there exists a canonical functorial isomorphism of S' -schemes $X_{(S')} \times_{S'} Y_{(S')}$ and $(X \times_S Y)_{S'}$.*

Proof. In fact, we have the following canonical isomorphisms:

$$(X \times_S S') \times_{S'} (Y \times_S S') = X \times_S (Y \times_S S') = (X \times_S Y) \times_S S'$$

where we use ?? and the associativity of fiber product. \square

Again, the functorial isomorphism in ?? also gives the isomorphism

$$(u_{(S')}, v_{(S')})_{(S')} = ((u, v)_S)_{S'}$$

for any S -morphisms $u : T \rightarrow X$, $v : T \rightarrow Y$. In other words, the inverse image functor $X_{(S')}$ commutes on the formation of the products; note that it also commutes to the formation of coproducts.

Corollary 18.3.8. *Let Y be an S -scheme, $f : X \rightarrow Y$ a morphism making X a Y -scheme (and also an S -scheme). Then the scheme $X_{(S')}$ is canonically identified with the product $X \times_Y Y_{(S')}$, and the projection $X \times_Y Y_{(S')} \rightarrow Y_{(S')}$ is identified with $f_{(S')}$.*

Proof. Let $\psi : Y \rightarrow S$ be the structural morphism of Y ; we have a commutative diagram

$$\begin{array}{ccccc} X_{(S')} & \xrightarrow{f_{(S')}} & Y_{(S')} & \xrightarrow{\psi_{(S')}} & S' \\ \downarrow & & \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y & \xrightarrow{\psi} & S \end{array}$$

Now $Y_{(S')}$ is identified with $S'_{(\psi)}$, and $X_{(S')}$ with $S'_{(\psi \circ f)}$, so by ?? and ??, we deduce the corollary. \square

Example 18.3.9. Let A be a ring, X an A -scheme, and \mathfrak{a} an ideal of A . Then $X_0 = X \otimes_A (A/\mathfrak{a})$ is an (A/\mathfrak{a}) -scheme, called the scheme obtained from X by **reduction mod \mathfrak{a}** .

Proposition 18.3.10. Let $f : X \rightarrow X'$, $g : Y \rightarrow Y'$ be two S -morphisms that are monomorphisms of schemes; then $f \times_S g$ is a monomorphism. In particular, for any extension $S' \rightarrow S$ of base scheme, the inverse image $f_{(S')} : X_{(S')} \rightarrow Y_{(S')}$ is a monomorphism.

Proof. In fact, if p, q are the projections of $X \times_S Y$, and p', q' that of $X' \times_S Y'$:

$$\begin{array}{ccccc} X \times_S Y & \xrightarrow{q} & Y & & \\ p \downarrow & \searrow f \times_S g & \downarrow g & & \\ & X' \times_S Y' & \xrightarrow{q'} & Y' & \\ & p' \downarrow & & \downarrow & \\ X & \xrightarrow{f} & X' & \longrightarrow & S \end{array}$$

then for any two morphisms $u, v : T \rightarrow X \times_S Y$, the relation $(f \times_S g) \circ u = (f \times_S g) \circ v$ implies

$$p' \circ (f \times_S g) \circ u = p' \circ (f \times_S g) \circ v$$

so $f \circ p \circ u = f \circ p \circ v$, and since f is a monomorphism, we conclude $p \circ u = p \circ v$. Similarly, since g is a monomorphism, we have $q \circ u = q \circ v$, whence $u = v$. \square

For any S -morphism $f : S' \rightarrow X$, the morphism $f' = (f, 1_{S'})_S$ is then an S' -morphism from S' to $X' = X_{(S')}$ such that $p \circ f' = f$, $\pi' \circ f' = 1_{S'}$, which is called an **S' -section** of X' :

$$\begin{array}{ccc} X' & \xrightarrow{\quad f' \quad} & S' \\ \pi' \downarrow & \swarrow f & \downarrow \varphi \\ X & \xrightarrow{\quad \pi \quad} & S \end{array}$$

Conversely if f' is an S' -section, then $f = p \circ f'$ is an S -morphism $S' \rightarrow X$. We then deduce the following canonical correspondence

$$\text{Hom}_S(S', X) \xrightarrow{\sim} \text{Hom}_{S'}(S', X') \tag{18.3.2}$$

The morphism f' is called the **graph** of f , and denoted by Γ_f . A particularly important case is $S' = X$ and $f = 1_X$, where corresponding morphism $X \rightarrow X \times_S X$ is called the **diagonal morphism** of X , and denoted by Δ_X . Also, if $f : X \rightarrow Y$ is a morphism of schemes, we denote by Δ_f the diagonal map from X to $X \times_Y X$.

Example 18.3.11. Since any scheme X can be considered as a \mathbb{Z} -scheme, we can consider the X -sections of $X \otimes_{\mathbb{Z}} \mathbb{Z}[T]$ (where T is an indeterminate) corresponding to the \mathbb{Z} -morphisms $X \rightarrow \text{Spec}(\mathbb{Z}[T])$. We claim that such X -sections correspond to sections of the structural sheaf \mathcal{O}_X of X . In fact, the morphisms $X \rightarrow \text{Spec}(\mathbb{Z}[T])$ correspond to ring homomorphisms $\mathbb{Z}[T] \rightarrow$

$\Gamma(X, \mathcal{O}_X)$, which in turn are entirely determined by the image of T , and can be an arbitrary element of $\Gamma(X, \mathcal{O}_X)$, whence our assertion.

18.3.3 Tensor product of quasi-coherent sheaves

Let S be a scheme, X, Y be two S -schemes, $Z = X \times_S Y$, and p, q be the projections of Z to X and Y , respectively. Let \mathcal{F} (resp. \mathcal{G}) be an \mathcal{O}_X -module (resp. an \mathcal{O}_Y -module). Then the tensor product $p^*(\mathcal{F}) \otimes_{\mathcal{O}_Z} q^*(\mathcal{G})$ is called the **tensor product of \mathcal{F} and \mathcal{G} over \mathcal{O}_S** (or **over S**) and denoted by $\mathcal{F} \otimes_{\mathcal{O}_S} \mathcal{G}$ (or $\mathcal{F} \otimes_S \mathcal{G}$). More generally, if $(X_i)_{1 \leq i \leq n}$ is a finite family of S -schemes and for each i , \mathcal{F}_i is an \mathcal{O}_{X_i} -module, we can define the tensor product $\mathcal{F}_1 \otimes_S \cdots \otimes_S \mathcal{F}_n$ over the scheme $Z = X_1 \times_S \cdots \times_S X_n$. This is a quasi-coherent \mathcal{O}_Z -module if each \mathcal{F}_i is quasi-coherent (??), and is coherent if each \mathcal{F}_i is coherent and Z is locally Noetherian in view of ??.

We note that if $X = Y = S$, the above definition coincide with the usual one of tensor product of \mathcal{O}_S -modules. Moreover, as $q^*(\mathcal{O}_Y) = \mathcal{O}_{X \times_S Y}$, the product $\mathcal{F} \otimes_S \mathcal{O}_Y$ is canonically identified with $p^*(\mathcal{F})$, and similarly $\mathcal{O}_X \otimes_S \mathcal{G}$ is identified with $q^*(\mathcal{G})$. In particular, if $Y = S$ and $f : X \rightarrow Y$ is the structural morphism, then $\mathcal{O}_X \otimes_Y \mathcal{G} = f^*(\mathcal{G})$: the ordinary tensor product and the inverse image therefore appears as a special case of the general tensor product. We also note that if X and Y are fixed, the operation $\mathcal{F} \otimes_S \mathcal{G}$ is a covariant bifunctor and is right exact on \mathcal{F} and \mathcal{G} .

Proposition 18.3.12. *Let S, X, Y be affine schemes with rings A, B, C , respectively, where B, C are A -algebras. Let M (resp. B) be a B -module (resp. C -module) and $\mathcal{F} = \tilde{M}$ (resp. $\mathcal{G} = \tilde{N}$) the associated quasi-coherent sheaf. Then $\mathcal{F} \otimes_S \mathcal{G}$ is canonically isomorphic to the sheaf associated with the $(B \otimes_A C)$ -module $M \otimes_A N$.*

Proof. In fact, in view of ?? and ??, $\mathcal{F} \otimes_S \mathcal{G}$ is canonically isomorphic to the sheaf associated with the $(B \otimes_A C)$ -module

$$(M \otimes_B (B \otimes_A C)) \otimes_{B \otimes_A C} ((B \otimes_A C) \otimes_C N)$$

and due to the canonical isomorphisms between tensor products, the latter is isomorphic to $M \otimes_B (B \otimes_A C) \otimes_C N = (M \otimes_B B) \otimes_A (C \otimes_C N) = M \otimes_A N$. \square

Proposition 18.3.13. *Let $f : T \rightarrow X, g : T \rightarrow Y$ be two S -morphisms, and \mathcal{F} (resp. \mathcal{G}) be an \mathcal{O}_X -module (resp. \mathcal{O}_Y -module). Then we have $(f, g)_S^*(\mathcal{F} \otimes_S \mathcal{G}) = f^*(\mathcal{F}) \otimes_{\mathcal{O}_T} g^*(\mathcal{G})$.*

Proof. If p, q are the projections of $X \times_S Y$, the assertion follows from the relations $(f, g)_S^* \circ p^* = f^*$ and $(f, g)_S^* \circ q^* = g^*$, and the fact that the inverse image operation commutes with tensor products. \square

Corollary 18.3.14. *Let $f : X \rightarrow X', g : Y \rightarrow Y'$ be two S -schemes and \mathcal{F}' (resp. \mathcal{G}') be an $\mathcal{O}_{X'}$ -module (resp. $\mathcal{O}_{Y'}$ -module). Then $(f \times_S g)^*(\mathcal{F}' \otimes_S \mathcal{G}') = f^*(\mathcal{F}') \otimes_S g^*(\mathcal{G}')$.*

Proof. This follows from ?? and the fact that $f \times_S g = (f \circ p, g \circ q)_S$, where p, q are the projections of $X \times_S Y$. \square

Corollary 18.3.15. *Let X, Y, Z be S -schemes and \mathcal{F} (resp. \mathcal{G}, \mathcal{H}) be an \mathcal{O}_X -module (resp. \mathcal{O}_Y -module, \mathcal{O}_Z -module). Then the sheaf $\mathcal{F} \otimes_S \mathcal{G} \otimes_S \mathcal{H}$ is the inverse image of $(\mathcal{F} \otimes_S \mathcal{G} \otimes_S \mathcal{H})$ under the canonical isomorphism $X \times_S Y \times_S Z \rightarrow (X \times_S Y) \times_S Z$, and $\mathcal{G} \otimes_S \mathcal{F}$ is the inverse image of $\mathcal{F} \otimes_S \mathcal{G}$ under the canonical isomorphism $X \times_S \rightarrow Y \times_S X$.*

Proof. The first isomorphism is $(p_1, p_2)_S \times_S p_3$, where p_1, p_2, p_3 are the projections of $X \times_S Y \times_S Z$, and second one is similarly. \square

Corollary 18.3.16. *If X is an S -scheme, any \mathcal{O}_X -module \mathcal{F} is the inverse image of $\mathcal{F} \otimes_S \mathcal{O}_S$ under the canonical isomorphism from X to $X \times_S S$.*

Let X be an S -scheme, \mathcal{F} be an \mathcal{O}_X -module, and $\varphi : S' \rightarrow S$ be a morphism. We denote by $\mathcal{F}_{(\varphi)}$ or $\mathcal{F}_{(S')}$ the sheaf $\mathcal{F} \otimes_S \mathcal{O}_{S'}$ over $X \times_S S' = X_{(\varphi)} = X_{(S')}$, so $\mathcal{F}_{(S')} = p^*(\mathcal{F})$, where p is the projection $X_{(S')} \rightarrow X$.

Proposition 18.3.17. *Let $\varphi' : S'' \rightarrow S'$ be a morphism. For any \mathcal{O}_X -module \mathcal{F} over the S -scheme X , $(\mathcal{F}_{(\varphi)})_{(\varphi')}$ is the inverse image of $\mathcal{F}_{(\varphi \circ \varphi')}$ under the canonical isomorphism $(X_{(\varphi)})_{(\varphi')} \rightarrow X_{(\varphi \circ \varphi')}$.*

Proof. This follows from the definition and the associativity of base change, since $(\mathcal{F} \otimes_S \mathcal{O}_{S'}) \otimes_{S'} \mathcal{O}_{S''} = \mathcal{F} \otimes_S \mathcal{O}_{S''}$. \square

Proposition 18.3.18. *Let Y be an S -scheme and $f : X \rightarrow Y$ be an S -morphism. For any \mathcal{O}_Y -module \mathcal{G} and any morphism $S' \rightarrow S$, we have $(f_{(S')})^*(\mathcal{G}_{(S')}) = (f^*(\mathcal{G}))_{(S')}$.*

Proof. This follows from the diagram

$$\begin{array}{ccc} X_{(S')} & \xrightarrow{f_{(S')}} & Y_{(S')} \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

and the functoriality of inverse images. \square

Corollary 18.3.19. *Let X, Y be S -schemes and \mathcal{F} (resp. \mathcal{G}) be an \mathcal{O}_X -module (resp. \mathcal{O}_Y -module). For any morphism $S' \rightarrow S$, the inverse image of the sheaf $(\mathcal{F}_{(S')}) \otimes_{S'} (\mathcal{G}_{(S')})$ under the canonical isomorphism $(X \times_S Y)_{(S')} \cong (X_{(S')}) \times_{S'} (Y_{(S')})$ is equal to $(\mathcal{F} \otimes_S \mathcal{G})_{(S')}$.*

Proof. If p, q are the projections of $X \times_S Y$, the isomorphism is given by $(p_{(S')}, q_{(S')})'_S$, so the corollary follows from ?? and ??.

Proposition 18.3.20. *Let X, Y be two S -schemes and \mathcal{F} (resp. \mathcal{G}) be an \mathcal{O}_X -module (resp. \mathcal{O}_Y -module). Let p, q be the projections of $Z = X \times_S Y$, z be a point of Z , and put $x = p(z)$, $y = q(z)$. Then the stalk $(\mathcal{F} \otimes_S \mathcal{G})_z$ is isomorphic to $(\mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{Z,z}) \otimes_{\mathcal{O}_{Z,z}} (\mathcal{G}_y \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{Z,z}) = \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{Z,z} \otimes_{\mathcal{O}_{Y,y}} \mathcal{G}_y$.*

Proof. Since the question is local, we can reduce to the affine case, and the assertion follows from ??.

Corollary 18.3.21. *With the notations in ??, if \mathcal{F} and \mathcal{G} are of finite type, then*

$$\text{supp}(\mathcal{F} \otimes_S \mathcal{G}) = p^{-1}(\text{supp}(\mathcal{F})) \cap q^{-1}(\text{supp}(\mathcal{G})).$$

Proof. As $p^*(\mathcal{F})$ and $q^*(\mathcal{G})$ are of finite type over \mathcal{O}_Z , in view of ?? and ??, we can reduce to the case $\mathcal{G} = \mathcal{O}_Y$, and the assertion then follows from the formula $\text{supp}(p^{-1}(\mathcal{F})) = p^{-1}(\text{supp}(\mathcal{F}))$. \square

18.3.4 Scheme valued points

Let X be a scheme; for any scheme T , we denote by $X(T)$ the set $\text{Hom}(T, X)$ of morphisms from T to X , and the elements of this set will be called **points of X with values in T** . The operation $T \mapsto X(T)$ is then a contravariant functor from the category of schemes to that of sets (in one word, we identify the scheme X with the induced functor h_X on \mathbf{Sch}). Moreover, any morphism $g : X \rightarrow Y$ of schemes defines a natural transform $X(T) \rightarrow Y(T)$, which send $v \in X(T)$ to $g \circ v \in Y(T)$. The product of two S -schemes X and Y is then defined by the canonical isomorphism

$$(X \times_S Y)(T) \xrightarrow{\sim} X(T) \times_{S(T)} Y(T) \quad (18.3.3)$$

where the maps $X(T) \rightarrow S(T)$ and $Y(T) \rightarrow S(T)$ corresponds to the structural morphisms $X \rightarrow S$ and $Y \rightarrow S$.

If we are given a scheme S and consider S -schemes and S -morphisms, we denote by $X(T)_S$ the set $\text{Hom}_S(T, X)$ of S -morphisms $T \rightarrow X$, and omit the index S if there is no risk of confusion. We also say the elements of $X(T)_S$ are the (S)-points of the S -scheme X with values in the S -scheme T . In particular, an S -section of X is none other than a point of X with values in S . The formula (??) is then written as

$$(X \times_S Y)(T)_S = X(T)_S \times Y(T)_S;$$

more generally, if Z is an S -scheme, X, Y, T are Z -schemes, we have

$$(X \times_Z Y)(T)_S = X(T)_S \times_{Z(T)_S} Y(T)_S.$$

We remark that for any morphism $S' \rightarrow S$, the set $X(S')_S = \text{Hom}_S(S', X)$ is identified with the set $\text{Hom}_{S'}(S', X')$, where $X' = X \times_S S'$, which is the set of S' -sections of X' .

In T (resp. S) is an affine scheme with ring B (resp. A), we replace T (resp. S) by B (resp. A) in the above notations, and we then refer points of X with values in the ring B , or points of the A -scheme X with values in the A -algebra B for the elements $X(B)$ of $X(B)_A$, respectively. We also call $X(T)_A$ the set of points of the A -scheme X with values in the A -scheme T .

Consider in particular the case where T is a local scheme $\text{Spec}(A)$, where A is a local ring; the elements $X(A)$ corresponds to local homomorphisms $\mathcal{O}_{X,x} \rightarrow A$ for $x \in X$ (??); we say the point x of the underlying topological space X is the **locality** of the point of X with values in A to which it corresponds (of course, several distinct points of X with values in A can have the same locality), or that the point of X with values in A which corresponds to x is **localized in x** .

Even more particularly, a point of X with values in a field K correspond to a point $x \in X$ and a field extension $\kappa(x) \rightarrow K$. If X is an S -scheme, saying that $S' = \text{Spec}(K)$ is an S -scheme means K is an extension of the residue field $\kappa(s)$ for an point $s \in S$; an element of $X(K)_S$, which is called **a point of X lying over s with values in K** , corresponds then to a $\kappa(s)$ -homomorphism $\kappa(x) \rightarrow K$, where x is a point of the topological space X lying over s (hence $\kappa(x)$ is an extension of $\kappa(s)$).

The points of X with values in an algebraically closed field K are called **geometric points**

of the scheme X ,² the field K is called the **value field** of the geometric point. If X is an S -scheme and s is a point of S , a **geometric point of X lying over s** is then a geometric point of X localized in a point of X lying over s . We then have a map $X(K) \rightarrow X$, which send a geometric point with values in K to the point it locates.

If $S = \text{Spec}(k)$ is the spectrum of a field k and X is an S -scheme, the S -points of X with values in k is identified with the S -sections of X , or with the points x of X such that the canonical homomorphism $k \rightarrow \kappa(x)$ is an isomorphism since only at such a point there exists a homomorphism $\kappa(x) \rightarrow k$ such that the composition $k \rightarrow \kappa(x) \rightarrow k$ is the identity. Such points are called the **rational points** over k of the k -scheme X . Note that if k' is an extension of k , the points of X with values in k' correspond to the points of $X' = X_{(k')}$ rational over k' (cf. (??)).

The example $X = \text{Spec}(K)$, where K is an nontrivial extension of k , shows that there do not necessarily exist in X rational points on k , even if X is nonempty. Still assuming that X is a k -scheme. For any point $x \in X$, there is always an extensions k' of k for which there is a point x' of $X' = X_{(k')}$ rational over k' and whose image by the canonical projection $X' \rightarrow X$ is x : it suffices to take for k' an extension of $\kappa(x)$, the k -monomorphism $\kappa(x) \rightarrow k'$ giving the sought point x' . When we thus passes from a point x to a rational point $x' \in X'$ over k' and above x , we say that we "make x rational."

Proposition 18.3.22. *Let $S = \text{Spec}(k)$ be the specturm of a field k , and X be an S -scheme. Then any k -rational point of X is closed in X .*

Proof. In fact, it suffices to show that the point x is closed in any open affine open set containing x , so we may assume that $X = \text{Spec}(A)$ is affine. In this case, since the composition homomorphism $k \rightarrow A \rightarrow \kappa(x)$ is an isomorphism (we know that $\kappa(x) = k$), we conclude in particular that $A/\mathfrak{p}_x \rightarrow k$ is an integral extension, which implies that A/\mathfrak{p}_x is a field (??). \square

Proposition 18.3.23. *Let $(X_i)_{1 \leq i \leq n}$ be S -schemes, s a point of X , and x_i a point of X_i lying over s for each i . For there to be a point y of the scheme $Y = X_1 \times_S \cdots \times_S X_n$ whose projections on X_i is x_i , it is necessary and sufficient that the x_i are over the same point s of S .*

Proof. This condition is clearly necessary. Now let s be an element of S and x_i a point of X_i lying over s . Then there exist $\kappa(s)$ -homomorphisms $\kappa(x_i) \rightarrow K$ where K is a common field. The composition $\kappa(s) \rightarrow \kappa(x_i) \rightarrow K$ are all identical, so the morphisms $\text{Spec}(K) \rightarrow X_i$ corresponding to $\kappa(x_i) \rightarrow K$ are S -morphisms, and we conclude that they define a unique morphism $\text{Spec}(K) \rightarrow Y$. If y is the corresponding point of Y , it is clear that its projection into each of the X_i is x_i . \square

In other words, if we denote by (X) the set underlying X , we see that we have a canonical surjective map $(X \times_S Y) \rightarrow (X) \times_{(S)} (Y)$; we have already pointed that this maps is not injective in general; that is, there can be multiple points distinct in $X \times_S Y$ having same projections to X and Y .

Corollary 18.3.24. *Let $f : X \rightarrow Y$ be an S -morphism, $f_{(S')} : X_{(S')} \rightarrow Y_{(S')}$ the S' -morphism induced by a base change $S' \rightarrow S$. Let p (resp. q) be the projection $X_{(S')} \rightarrow X$ (resp. $Y_{(S')} \rightarrow Y$); for any subset V of X , we have*

$$q^{-1}(f(M)) = f_{(S')}(p^{-1}(M)).$$

²This terminology is also sometimes used when K is only separably closed, but at that time we will explicitly clarify which convention we adapt.

Proof. By ??, $X_{(S')}$ is identified with the product $X \times_Y Y_{(S')}$ and we have the following commutative diagram

$$\begin{array}{ccc} X_{(S')} & \xrightarrow{f_{(S')}} & Y_{(S')} \\ p \downarrow & & \downarrow q \\ X & \xrightarrow{f} & Y \end{array}$$

By ??, the relation $q(y') = f(x)$ for $x \in V, y' \in Y_{(S')}$ is equivalent to the existence of a point $x' \in X_{(S')}$ such that $p(x') = x$ and $f_{(S')}(x') = y'$, whence the corollary. \square

Proposition 18.3.25. *Let X, Y be S -schemes and $x \in X, y \in Y$ two points lying over the same point $s \in S$. Then the set of points $X \times_S Y$ with projections x and y is in canonical correspondence with the set of types of the composition field extension of $\kappa(x)$ and $\kappa(y)$, considered as extensions of $\kappa(s)$.*

Proof. Let p (resp. q) be the projection of $X \times_S Y$ to X (resp. Y) and let E be the subspace $p^{-1}(x) \cap q^{-1}(y)$ of the underlying topological space of $X \times_S Y$. We first note that since x and y are lying over s , the morphisms $\text{Spec}(\kappa(x)) \rightarrow S$ and $\text{Spec}(\kappa(y)) \rightarrow S$ factor through $\text{Spec}(\kappa(s))$:

$$\begin{array}{ccc} \text{Spec}(\kappa(x)) & & \\ \downarrow & & \\ \text{Spec}(\kappa(y)) & \longrightarrow & \text{Spec}(\kappa(s)) \longrightarrow S \end{array}$$

since $\text{Spec}(\kappa(s)) \rightarrow S$ is a monomorphism by ??, it follows immediately that we have

$$P = \text{Spec}(\kappa(x)) \times_S \text{Spec}(\kappa(y)) = \text{Spec}(\kappa(x)) \times_{\text{Spec}(\kappa(s))} \text{Spec}(\kappa(y)) = \text{Spec}(\kappa(x) \otimes_{\kappa(s)} \kappa(y)).$$

Let $i : \text{Spec}(\kappa(x)) \rightarrow X$ and $j : \text{Spec}(\kappa(y)) \rightarrow Y$ be the canonical morphisms, we put $\alpha = i \times_S j : P \rightarrow E$ to be the map on the underlying topological space. On the other hand, any point $z \in E$ defines two $\kappa(s)$ -homomorphisms $\kappa(x) \rightarrow \kappa(z)$ and $\kappa(y) \rightarrow \kappa(z)$, hence a $\kappa(s)$ -homomorphism $\kappa(x) \otimes_{\kappa(s)} \kappa(y) \rightarrow \kappa(z)$, which corresponds to a morphism $\text{Spec}(\kappa(z)) \rightarrow P$; we take $\beta(z)$ to be the image of this morphism, which defines a map $\beta : E \rightarrow P$.

To verify that $\alpha \circ \beta$ and $\beta \circ \alpha$ are identity maps, we need the following commutative diagram

$$\begin{array}{ccccccc} \text{Spec}(\kappa(z)) & \swarrow & & \searrow & & & \\ & & P & \xrightarrow{\quad} & \text{Spec}(\kappa(y)) & \xrightarrow{j} & Y \\ & \searrow & \downarrow & & \downarrow & & \downarrow \\ & & \text{Spec}(\kappa(x)) & \xrightarrow{\alpha} & E & \xrightarrow{\quad} & Y \\ & & & \searrow & \downarrow & & \\ & & & & X & \xrightarrow{\quad} & S \end{array}$$

for $z \in E$. By the uniqueness part of the universal property of fiber products, the canonical morphism $\text{Spec}(\kappa(z)) \rightarrow E$ induced by $\text{Spec}(\kappa(z)) \rightarrow X$ and $\text{Spec}(\kappa(z)) \rightarrow Y$ is given by the composition $\text{Spec}(\kappa(z)) \rightarrow P \rightarrow E$, and also equal to the canonical morphism $\text{Spec}(\kappa(z)) \rightarrow E$ of the scheme $X \times_S Y$ at z (??). This means the image of $\beta(z)$ under α is exactly the image of the canonical morphism $\text{Spec}(\kappa(z)) \rightarrow E$, which is just z ; this shows $\alpha \circ \beta = 1_E$. As for $\beta \circ \alpha$,

we just note that if $z = \alpha(\mathfrak{p})$ for some $\mathfrak{p} \in P$ (a prime ideal), then the morphism α induces a field extensins $\kappa(z) \rightarrow \kappa(\mathfrak{p})$, which corresponds to morphism $\text{Spec}(\kappa(\mathfrak{p})) \rightarrow \text{Spec}(\kappa(z))$. Again by the uniqueness part of the fiber product P , we conclude that the canonical morphism $\text{Spec}(\kappa(\mathfrak{p})) \rightarrow P$ factors through $\text{Spec}(\kappa(z))$, which means $\beta(z) = \mathfrak{p}$, so $\beta \circ \alpha = 1_P$. Finally, we recall that the set P corresponds to composition fields of $\kappa(x)$ and $\kappa(y)$ over $\kappa(s)$. \square

18.3.5 Surjective morphisms

Let \mathcal{P} be a property for morphisms of schemes. We consider the following conditions:

- (i) If $f : X \rightarrow X'$ and $g : Y \rightarrow Y'$ are S -morphisms satisfying \mathcal{P} , then $f \times_S g$ also satisfies \mathcal{P} .
- (ii) If $f : X \rightarrow Y$ is an S -morphism satisfying \mathcal{P} and $S' \rightarrow S$ is a morphism, then $f_{(S')} : X_{(S')} \rightarrow Y_{(S')}$ also satisfies \mathcal{P} .

Since $f_{(S')} = f \times_S 1_{S'}$, we see if any identity morphism satisfies \mathcal{P} , then (i) implies (ii). On the other hand, since $f \times_S g$ is the following composition

$$X \times_S Y \xrightarrow{f \times 1_Y} X' \times_S Y \xrightarrow{1_{X'} \times g} X' \times_S Y'$$

it is clear that if the composition of two morphisms satisfying \mathcal{P} still satisfies \mathcal{P} , then (ii) implies (i) (in this case we say \mathcal{P} is stable under composition). In general, a property \mathcal{P} is called **stable under base change** if it satisfies the condition (ii). For example ?? just says that being a monomorphism is stable under base change. On the other hand, if \mathcal{P} is an arbitrary property of morphisms, we say a morphism $f : X \rightarrow S$ **satisfies \mathcal{P} universally** (or is **universally \mathcal{P}**), if for any morphism $S' \rightarrow S$ the inverse image $f_{(S')}$ satisfies \mathcal{P} .

Our first application of the above definition is that surjectivity is stable under base change:

Proposition 18.3.26. *Surjective morphisms of schemes are stable under base change.*

Proof. Note that it is clear that surjectivity is stable under composition, in fact we have the both conditions (i) and (ii) described above. But condition (ii) follows from ?? by setting $V = X$. \square

Proposition 18.3.27. *For a morphism $f : X \rightarrow Y$ of schemes to be surjective, it is necessary and sufficient that for any field K and any morphism $\text{Spec}(K) \rightarrow Y$, there exists an extension K' of K and a morphism $\text{Spec}(K') \rightarrow X$ fitting into the following diagram*

$$\begin{array}{ccc} \text{Spec}(K') & \longrightarrow & X \\ \downarrow & & \downarrow f \\ \text{Spec}(K) & \longrightarrow & Y \end{array}$$

Proof. The condition is sufficient, since for any $y \in Y$, we can apply the canonical morphism $\text{Spec}(\kappa(y)) \rightarrow Y$ to get a morphism $\text{Spec}(K) \rightarrow X$, which gives a inverse image of y in X . Conversely, suppose that f is surjective, and let $y \in Y$ be the image of $\text{Spec}(K)$ in Y ; there exists $x \in X$ such that $f(x) = y$. Consider the monomorphism $\kappa(y) \rightarrow \kappa(x)$ corresponding to f , and take an extension K' of $\kappa(y)$ containing $\kappa(x)$ and K ; the morphism $\text{Spec}(K') \rightarrow X$ corresponding to $\kappa(x) \rightarrow K'$ then satisfies the requirement. \square

Corollary 18.3.28. *For a morphism $f : X \rightarrow Y$ to be surjective, it is necessary and sufficient that, for any field K , there exist an algebraically closed extension K' of K such that the map $X(K') \rightarrow Y(K')$ corresponding to f is surjective.*

Proof. In view of ??, this condition is sufficient. Conversely, suppose that f is surjective and let K be a field. If p is the characteristic of K , let us take for K' an algebraically closed extension of K having over the prime field P a transcendence basis of strictly larger cardinality to the cardinals of all the transcendence bases on P of the residual fields of X and Y having characteristic p . It then remains to see, with the same notations as in ??, that any monomorphism $u : \kappa(y) \rightarrow K'$ factors into

$$\kappa(y) \xrightarrow{w} \kappa(x) \xrightarrow{v} K'$$

where $w = f^x$. Now, let L a purely transcendental extension of P contained in $\kappa(y)$ and over which $\kappa(y)$ is algebraic; if B is a transcendence basis of L over P , we can complete $w(B)$ into a transcendence basis B' of $\kappa(x)$ on P , and then (due to the assumption made on the trancendence bases of K') define a monomorphism $v_1 : P(B') \rightarrow K'$ such that $v_1 \circ (w|_L)$ coincides with $u|_L$. There is also an isomorphism $v_2 = u \circ w^{-1}$ from $w(\kappa(y))$ to $u(\kappa(y))$ such that v_2 and v_1 coincide in $w(L)$; as $w(\kappa(y))$ and $P(B' - w(B))$ are linearly disjoint on $w(L)$, we can extend v_1 and v_2 into a monomorphism v_0 of $M = P(B')(w(\kappa(y)))$ in K' ; as K' is algebraically closed and $\kappa(x)$ is algebraic over M , we can finally extend v_0 into the monomorphism $v : \kappa(x) \rightarrow K'$, which completes the proof. \square

18.3.6 Radical morphisms

Proposition 18.3.29. *Let $f : X \rightarrow Y$ be a morphism of schemes. The following conditions are equivalent:*

- (i) f is universally injective.
- (ii) The map f is injective and for any $x \in X$, the extension $f^x : \kappa(f(x)) \rightarrow \kappa(x)$ is purely inseparable.
- (iii) For any field K , the map $X(K) \rightarrow Y(K)$ corresponding to f is injective.
- (iv) For any field K , there exists an algebraically closed extension K' of K such that the map $X(K') \rightarrow X(K')$ corresponding to f is injective.
- (v) The diagonal morphism $\Delta_f : X \rightarrow X \times_Y X$ is surjective.

The morphism f is called **radical** if it satisfies the above equivalent conditions.

Proof. It is clear that (i) implies f is injective; on the other hand, if $\kappa(x)$ is not a purely inseparable extension of $\kappa(f(x))$, there exist two distinct $\kappa(f(x))$ -monomorphisms $\kappa(x) \rightarrow K$ into an algebraically closed extension K of $\kappa(x)$; hence we get two distinct morphisms g_1, g_2 of $\text{Spec}(K)$ to X , whose compositions $f \circ g_1, f \circ g_2$ equal to the same morphism $\text{Spec}(K) \rightarrow Y$. If we set $Y' = \text{Spec}(K)$, there then would be two distinct Y' -sections of $X_{(Y')}$; since K is a field, the Y' -sections of $X_{(Y')}$ correspond one-to-one to their images (the rational points of $X_{(Y')}$ over K), so $f_{(Y')} : X_{(Y')} \rightarrow Y'$ would not be injective, contrary to the assumption.

To show that (ii) implies (iii), we note that by ??, (iii) signifies that for any $y \in Y$ and a monomorphism $\kappa(y) \rightarrow K$ to a field K , there do not exist two distinct $\kappa(y)$ -monomorphisms

$\kappa(x_1) \rightarrow K$, $\kappa(x_2) \rightarrow K$, where x_1, x_2 are both lying over y . Now (ii) implies that if we have two such monomorphisms, they come from the same point x since f is injective; moreover, since $\kappa(x)$ is a purely inseparable extension of $\kappa(y)$, the two monomorphisms $\kappa(x) \rightarrow K$ are necessarily equal.

It is clear that (iii) implies (iv). Conversely, suppose that (iv) holds; let K be a field and K' be an algebraically closed extension of K ; then the diagram

$$\begin{array}{ccc} X(K) & \xrightarrow{\alpha} & Y(K) \\ \downarrow \varphi & & \downarrow \varphi' \\ X(K') & \xrightarrow{\alpha'} & Y(K') \end{array} \quad (18.3.4)$$

is commutative. Since the homomorphism $K \rightarrow K'$ is injective, φ is injective by ??, and by hypothesis we can choose K' such that α' is also injective. Then α is injective, which shows (iii).

To see that (iv) and (v) are equivalent, we note that, for the morphism Δ_f to be surjective, it is necessary and sufficient that, in view of ??, for any field K , there exists an algebraically closed extension K' of K such that the diagonal map

$$X(K') \rightarrow (X \times_Y X)(K') = X(K') \times_{Y(K')} X(K')$$

corresponding to Δ_f is surjective. But by the definition of this fiber product, this signifies that the map $X(K') \rightarrow Y(K')$ is injective, whence our claim.

Finally, we prove that (iii) implies (i). If (iii) is satisfied, then for any base change $Y' \rightarrow Y$, the map

$$(X \times_Y Y')(K) \rightarrow Y'(K)$$

is still injective, as we immediately verify by noting that $(X \times_Y Y')(K) = X(K) \times_{Y(K)} Y'(K)$ and that $X(K) \rightarrow Y(K)$ is injective. Therefore, it suffices to prove that if $X(K) \rightarrow Y(K)$ is injective for any field K , then f is injective. Now if x_1 and x_2 are two points of X such that $f(x_1) = f(x_2) = y$, there then exists a field extension K of $\kappa(y)$ and $\kappa(y)$ -monomorphisms $\kappa(x_1) \rightarrow K$, $\kappa(x_2) \rightarrow K$; the corresponding morphisms u_1, u_2 of $\text{Spec}(K)$ to X are then such that $f \circ u_1 = f \circ u_2$, and by hypothesis this implies $u_1 = u_2$, so $x_1 = x_2$. \square

Remark 18.3.3. We then obtain examples of injective morphisms (and even bijective) of schemes but not universally injective: it suffices to take a morphism $\text{Spec}(K) \rightarrow \text{Spec}(k)$, where K is a separable extension of k distinct from k .

Corollary 18.3.30. A monomorphism of schemes $f : X \rightarrow Y$ is radical. In particular, if A is a ring, S is a multiplicative subset of A , then the canonical morphism $\text{Spec}(S^{-1}A) \rightarrow \text{Spec}(A)$ is radical.

Proof. The first assertion follows from ??(iii), and the second from the fact that $\text{Spec}(S^{-1}A) \rightarrow \text{Spec}(A)$ is a monomorphism. \square

Corollary 18.3.31. Let $f : X \rightarrow Y$ be a radical morphism, $g : Y' \rightarrow Y$ a morphism, and $X' = X_{(Y')}$. Then the radical morphism $f_{(Y')}$ is a bijection from the underlying space X' to $g^{-1}(f(X))$. Moreover, for any field K , the set $X'(K)$ is identified with the inverse image in $Y'(K)$ under the map $Y'(K) \rightarrow Y(K)$ (corresponding to g) of the subset $X(K)$ of $Y(K)$.

Proof. The first assertion follows from ??(ii) and ??; the second one follows from the commutative diagram ??.

Proposition 18.3.32. *Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be two morphism of schemes.*

- (a) *If f and g are radical, so is $g \circ f$.*
- (b) *Conversely, if $g \circ f$ is radical, so is f .*

Proof. It suffices to apply the functors X, Y, Z on any field K , and use the characterization of ??(iii); the verification boils down to set-theoretic issues, which are straightforward. \square

Proposition 18.3.33. *Radical morphisms are stable under base changes. In particular, if $f : X \rightarrow X'$ and $g : Y \rightarrow Y'$ are radical S -morphisms, then so is $f \times_S g$.*

Proof. Since radical is equivalently to universally injective, the first assertion is clear. The second one follows from the first one since radical morphisms are stable under composition by ??.

18.3.7 Fibers of morphisms

Proposition 18.3.34. *Let $f : X \rightarrow Y$ be a morphism, y be a point of Y , and \mathfrak{a}_y be an ideal of $\mathcal{O}_{Y,y}$. Put $Y' = \text{Spec}(\mathcal{O}_{Y,y}/\mathfrak{a}_y)$, $X' = X \times_Y Y'$, and let $p : X' \rightarrow X$ be the canonical projection. Then p is a homeomorphism from X' onto the subspace $f^{-1}(Y')$ of X (where we identify Y' are a subspace of Y , cf. ??). Moreover, for any $x' \in X'$, the homomorphism $p_{x'}^\# : \mathcal{O}_{X,p(x')} \rightarrow \mathcal{O}_{X',x'}$ is surjective with kernel $\mathfrak{a}_y \mathcal{O}_{X,x}$.*

Proof. The morphism $\text{Spec}(\mathcal{O}_{Y,y}/\mathfrak{a}_y) \rightarrow Y$ is radical (??), so we conclude from ?? that p identifies the space $X' = X \times_Y \text{Spec}(\mathcal{O}_{Y,y}/\mathfrak{a}_y)$ with $f^{-1}(Y')$. It remains to show that p is a homeomorphism and identify its morphism on stalks. Since this question is local, we may assume that $X = \text{Spec}(B)$, $Y = \text{Spec}(A)$, where B is an A -algebra. Then the morphism p corresponds to the homomorphism $1 \otimes \rho : B \rightarrow B \otimes_A A'$, where $\rho : A \rightarrow A' = A_{\mathfrak{p}_y}/\mathfrak{a}_y$ is the canonical homomorphism. Now any element of $B \otimes_A A'$ is of the form

$$\sum_i b_i \otimes \rho(a_i)/\rho(s) = \rho \left(\sum_i a_i b_i \otimes 1 \right) (1 \otimes \rho(s))^{-1}$$

where $s \notin \mathfrak{p}_y$, so we can apply ?. The assertion on the homomorphism $p_{x'}^\#$ also follows from the equality. \square

We will mainly use ?? for the case $\mathfrak{a}_y = \mathfrak{m}_y$ is the maximal ideal of $\mathcal{O}_{Y,y}$. If there is no confusion, we denote by X_y the $\kappa(y)$ -scheme obtained by transporting of scheme structure from $X' = X \otimes_S \kappa(s)$ to $f^{-1}(y)$ via the projection p , and this is always the scheme that will be involved when we speak of the **fiber** $f^{-1}(y)$ of the morphism f as a scheme.

Let X, Y be two S -schemes and $f : X \rightarrow Y$ an S -morphism. By the transitivity of base change, we have the canonical isomorphism

$$X_s = X \times_Y Y_s$$

for any $s \in S$; the morphism $f_s : X_s \rightarrow Y_s$ induced by f by the base change $Y_s \rightarrow Y$ is such that, for any $y \in Y_s$, the fiber $f_s^{-1}(y)$ is identified with the $\kappa(y)$ -scheme $f^{-1}(y)$, since the residue field of Y_s at y is the same as that of Y at y , in view of ??

Proposition 18.3.35 (Transitivity of Fibers). *Let $f : X \rightarrow Y$, $g : Y' \rightarrow Y$ be two morphisms; put $X' = X_{(Y')}$ and $f' = f_{(Y')}$. For any $y' \in Y'$, if $y = g(y')$, then the scheme $X'_{y'}$ is canonically isomorphic to $X_y \otimes_{\kappa(y)} \kappa(y')$.*

Proof. In fact, by the transitivity of base change, we have canonical isomorphisms

$$(X \otimes_Y \kappa(y)) \otimes_{\kappa(y)} \kappa(y') \cong X \times_Y \text{Spec}(\kappa(y')) \cong (X \times_Y Y') \otimes_{Y'} \kappa(y')$$

The left one is $X_y \otimes_{\kappa(y)} \kappa(y')$, and the right one is $X'_{y'} \otimes_{Y'} \kappa(y')$, so our assertion follows. \square

Proposition 18.3.36. *Let $f : X \rightarrow Y$ be a monomorphism of schemes. Then for each $y \in Y$, the fiber X_y is a $\kappa(y)$ -scheme which is either empty or $\kappa(y)$ -isomorphic to $\text{Spec}(\kappa(y))$.*

Proof. By ??, X_y is reduced to a point, and hence affine. By ??, the morphism $f_y : X_y \rightarrow \text{Spec}(\kappa(y))$ induced by f under base change is still a monomorphism. If A is the ring of X_y , this signifies that the homomorphism $A \otimes_{\kappa(y)} A \rightarrow A$ which maps $a \times a'$ to aa' is bijective, and clearly implies that $A = \kappa(y)$, since otherwise there exist an element $a \in A$ not contained in $\kappa(y)$ and the image $a \otimes 1$ and $1 \otimes a$ are distinct, but both mapped to a . \square

Proposition 18.3.37. *Let $f : X \rightarrow Y$ be an S -morphism of S -schemes, $g : S' \rightarrow S$ a surjective morphism, and $f' = f_{(S')} : X' = X_{(S')} \rightarrow Y' = Y_{(S')}$. Consider the following properties:*

- (a) *surjective;*
- (b) *injective;*
- (c) *dominant;*
- (d) *finite fiber (as sets);*

Then if \mathcal{P} denotes one of the properties above and if f' satisfies \mathcal{P} , then so does f .

Proof. Since the projection $Y' \rightarrow Y$ is surjective by ??, we can, by virtue of ??, limiting ourselves to the case where $Y = S$, $Y' = S'$. For any $y' \in Y'$, let $y = g(y')$; we have the transitivity relation $X'_{y'} \cong X_y \otimes_{\kappa(y)} \kappa(y')$ (??). Since the morphism $\text{Spec}(\kappa(y')) \rightarrow \text{Spec}(\kappa(y))$ is surjective, so is the projection $X'_{y'} \rightarrow X_y$ (??). Thus, if $X'_{y'}$ is nonempty (resp. a singleton, resp. a finite set), the same holds for X_y . Since $S' \rightarrow S$ is surjective, this proves (a), (b), and (d). On the other hand, if f' is dominant, so is the composition $g \circ f' = f \circ g'$; but since $g' : X' \rightarrow X$ is surjective by ??, this implies f is dominant. \square

18.3.8 Universally open and universally closed morphisms

Following the usual terminology, we say a morphism $f : X \rightarrow Y$ is **universally open** (resp. **universally closed**, resp. a **universal embedding**, resp. a **universal homeomorphism**) if for any base change $Y' \rightarrow Y$, the morphism $f_{(Y')} : X_{(Y')} \rightarrow Y'$ is open (resp. closed, resp. an embedding, resp. a homeomorphism).

Proposition 18.3.38.

- (i) The composition of two universally open morphisms (resp. universally closed morphisms, resp. two universal embeddings, resp. two universal homeomorphisms) is universally open (resp. universally closed, resp. a universal embedding, resp. a universal homeomorphism).
- (ii) If $f : X \rightarrow X'$, $g : Y \rightarrow Y'$ are two universally open (resp. universally closed, resp. two universal embeddings, resp. two universal homeomorphisms) S -morphisms, so is the product $f \times_S g$.
- (iii) If $f : X \rightarrow Y$ is a universally open (resp. universally closed, resp. a universal embedding, resp. a universal homeomorphism) S -morphism, so is $f_{(S')} : X_{(S')} \rightarrow Y_{(S')}$ for any base change $S' \rightarrow S$.
- (iv) Let $f : X \rightarrow Y$, $g : Y \rightarrow Z$ be two morphism such that f is surjective; if $g \circ f$ is universally open (resp. universally closed, resp. a universal embedding, resp. a universal homeomorphism), so is g .
- (v) Let (U_α) be an open cover of Y . For a morphism $f : X \rightarrow Y$ to be universally open (resp. universally closed, resp. a universal embedding, resp. a universal homeomorphism), it is necessary and sufficient that, for each α , the restriction $f_\alpha : f^{-1}(U_\alpha) \rightarrow U_\alpha$ is universally open (resp. closed).

Proof. The assertion (i) follows from definiton, and so does (iii). We have already remareked that (i) and (iii) together imply (ii), since identity morphisms satisfies all the properties mentioned above. To prove (iv), we note that for any morphism $Z' \rightarrow Z$, the morphism $f_{(Z')} \rightarrow X_{(Z')}$ is surjective, so it suffices to prove that if $g \circ f$ is open (resp. closed, resp. an embedding, resp. a homeomorphism) and f is surjective, then so is g . For the case where $g \circ f$ is open or closed, the fact that g is open or closed result from ??; for the other two cases, we can limit ourselves to the case $g(f(X)) = g(Y) = Z$, so that $g \circ f$ is a homeomorphism from X to Z . As f is surjective, g is necessarily bijective, and as it is open by the first two cases already shown, g is then a homeomorphism from Y to Z .

Finally, the necessity in (v) follows from (iii) and ?. Conversely, suppose the condition in (v) and let $g : Y' \rightarrow Y$ be a morphism; then $g^{-1}(U_\alpha) = U'_\alpha$ form an open cover of Y' and if $f' = f_{(Y')}$, the restriction $f'^{-1}(U'_\alpha) \rightarrow U'_\alpha$ of f' is none other than $(f_\alpha)_{(U'_\alpha)}$ (?). We can then reduce to proving that f is open (resp. closed, resp. an embedding, resp. a homeomorphism) if each f_α is, which is immediate. \square

We recall that openness of a map is a local property, i.e., a map $f : X \rightarrow Y$ is open if and only if it is open at every point of X . Simialrly, the property of universally open morphisms is also local. To justify this, we define a morphism $f : X \rightarrow Y$ is **universally open at a point** $x \in X$ if for any base change $Y' \rightarrow Y$ the morphism $f_{(Y')} : X_{(Y')} \rightarrow Y_{(Y')}$ is open at any point x' of X' lying over x .

Proposition 18.3.39. Let $f : X \rightarrow Y$, $g : Y \rightarrow Z$ be two morphisms of schemes, x a point of X , and $y = f(x)$.

- (a) If f is universally open at x and g is universally open at y , then $g \circ f$ is universally open at x . Conversely, if $g \circ f$ is universally open at x , then g is universally open at y .
- (b) If $f : X \rightarrow Y$ is an S -morphism universally open at a point $x \in X$, then for any base change $S' \rightarrow S$, the morphism $f_{(S')} : X_{(S')} \rightarrow Y_{(S')}$ is universally open at any point of $X_{(S')}$ lying over x .

Proof. Assertion (b) is an immediate consequence of the definition of universally openness at a point and transitivity of base change. Also, it follows from ?? that to prove (a), we may drop the "universally" condition and prove the assertion for openness, which follows from ??.

Recall that openness is a local property for maps: a continuous map $f : X \rightarrow Y$ is open if and only if it is open at every point of X . Inspired by this, for a morphism $f : X \rightarrow Y$ of schemes, we may say f is universally open at a point $x \in X$ if for any base change $Y' \rightarrow Y$, the morphism $f_{(Y')} : X' \rightarrow Y'$ is open at any point $x' \in X'$ lying over x (where $X' = X_{(Y')}$).

Proposition 18.3.40. *Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be morphisms of schemes, x be a point of X , and $y = f(x)$.*

(a) *If f is universally open at a point x and g is universally open at y , then $g \circ f$ is universally open at x . Conversely, if $g \circ f$ is universally open at x , g is universally open at y .*

(b) *If f is an S -morphism universally open at a point x , then for any base change $S' \rightarrow S$, the morphism $f_{(S')} : X_{(S')} \rightarrow Y_{(S')}$ is universally open at any point of $X_{(S')}$ lying over x .*

Proof. In fact, (b) is a immediate consequence of (a) and the transitivity of base changes. To prove (a), it suffices to drop the "universal" condition and the result then follows from ??.

Proposition 18.3.41. *Let X, Y be two schemes, $f : X \rightarrow Y$ be a morphism, and x be a point of X . Let (Y_i) be a locally finite covering of Y by closed subschemes, and suppose that for each i such that $f(x) \in Y_i$, the restriction $f_i : f^{-1}(Y_i) \rightarrow Y_i$ of f is an open morphism (resp. universally open) at the point x . Then f is open (resp. universally open) at the point x .*

Proof. The assertion about openness is just ?. For the universal part, consider a morphism $g : Y' \rightarrow Y$ and in Y' the closed subschemes $Y'_i = g^{-1}(Y_i)$ (??), which underlying spaces form a locally finite covering of Y' . If $f' = f_{(Y')} : X_{(Y')} \rightarrow Y'$ is the base change of f , the restriction $f'_i : f'^{-1}(Y_i) \rightarrow Y'_i$ of f' equals to $(f_i)_{(Y')}$, so we can apply ?? to f' .

Proposition 18.3.42. *Let $f : X \rightarrow Y$ be a morphism of schemes, $(X_i)_{1 \leq i \leq n}$ (resp. $(Y_i)_{1 \leq i \leq n}$) be a finite family of closed subschemes of X (resp. Y), and $j_i : X_i \rightarrow X$ (resp. $h_i : Y_i \rightarrow Y$) be the canonical injections. Suppose that $(X_i)_{1 \leq i \leq n}$ covers X and for each i there exists a morphism $f_i : X_i \rightarrow Y_i$ fitting into the following diagram*

$$\begin{array}{ccc} X_i & \xrightarrow{f_i} & Y_i \\ j_i \downarrow & & \downarrow h_i \\ X & \xrightarrow{f} & Y \end{array}$$

Then for f to be closed (resp. universally closed), it is necessary and sufficient that each f_i is.

Proof. If f is closed, then each f_i is closed since Y_i are closed in Y . Conversely, if the f_i are closed, for any closed subset F of X , we have $f_i(F \cap X_i) = f(F \cap X_i)$ and it closed in Y_i , hence in Y , and as $f(F)$ is the union of $f(F \cap X_i)$, it is therefore closed in Y .

For the case of universally closed morphisms, the condition is necessary because j_i is a closed immersion (hence universally closed, cf. ??), and if f is universally closed, so is $f \circ j_i = h_i \circ f_i$. But h_i is a closed immersion, hence separated (??), it then follows that f_i is universally closed (??).

Conversely, suppose that each f_i is universally closed, and consider the scheme Z that is the coproduct of that X_i . Let $u : Z \rightarrow X$ be the induced morphism by the j_i 's. The restriction of $f \circ u$ to X_i is equal to $f \circ j_i = h_i \circ f$, hence universally closed (?? and ??(i)); we then deduce from ?? that $f \circ u$ is universally closed. But since u is surjective by hypotheses, we conclude that f is universally closed (??(iv)). \square

Remark 18.3.4. If X only have finitely many irreducible components, then we deduce from ?? that, to verify a morphism $f : X \rightarrow Y$ is closed (resp. universally closed), we can reduce ourselves to doing it for dominant morphisms of integral schemes. In fact, let $(X_i)_{1 \leq i \leq n}$ be the reduced subschemes of X with underlying spaces the irreducible components of X (??), which are then integral. Let Y_i be the unique reduced closed subscheme of Y with underlying space $\overline{f(X_i)}$ (??), which is irreducible (??). If $j_i : X_i \rightarrow X$ (resp. $h_i : Y_i \rightarrow Y$) is the canonical injection, there then exists a dominant morphism $f_i : X_i \rightarrow Y_i$ such that $f \circ g_i = h_i \circ f_i$ (??); we are then in the case of ??, so f is closed (resp. universally closed) if and only if each f_i is.

Proposition 18.3.43. Let X, Y be two S -schemes and $f : X \rightarrow Y$ be an S -morphism. For f to be universally closed, it suffices that, for any base change $S' \rightarrow S$ where $S' = S \otimes_{\mathbb{Z}} \mathbb{Z}[(T_{\lambda})_{\lambda \in I}]$ (denoted by $S[(T_{\lambda})_{\lambda \in I}]$), the morphism $f_{(S')} : X_{(S')} \rightarrow Y_{(S')}$ is closed.

Proof. We note that if (S_{α}) is an open cover of S and Y_{α} is the inverse image of S_{α} in Y , it suffices to prove that the restricted morphism $f_{\alpha} : f^{-1}(Y_{\alpha}) \rightarrow Y_{\alpha}$ is closed (??(v)). Now the inverse image of S_{α} in S' is $S_{\alpha}[(T_{\lambda})_{\lambda \in I}] = S'_{\alpha}$ and $(f_{\alpha})_{(S')}$ is the restriction $(f_{(S')})^{-1}(Y'_{\alpha}) \rightarrow Y'_{\alpha}$ of $f_{(S')}$, where Y'_{α} is the inverse image of S'_{α} in $Y_{(S')}$. If the proposition is proved for S_{α} and f_{α} , it is then true for S and f . We can then assume that S is affine.

Now let (U_{β}) be an open covering of Y ; for f to be universally closed, it suffices to prove that $f_{\beta} : f^{-1}(U_{\beta}) \rightarrow U_{\beta}$ is universally closed for each β (??(v)). Again, the morphism $(f_{\beta})_{(S')}$ is the restriction $(f_{(S')})^{-1}(U'_{\beta}) \rightarrow U'_{\beta}$ of $f_{(S')}$, where $U'_{\beta} = U_{\beta} \times_S S'$ is the inverse image of U_{β} in $Y_{(S')}$. If the proposition is proved for U_{β} and f_{β} , it then holds for Y and f . Therefore, we can further assume that Y is affine.

Let us first show that if $f_{(S')}$ is closed for any base change $S' \rightarrow S$, then f is universally closed. In fact, any Y -scheme Y' can be considered as an S -scheme, and as the morphism $Y \rightarrow S$ is separated (recall that Y and S are assumed to be affine), $X \times_Y Y'$ (resp. $Y \times_Y Y' = Y'$) is identified with a closed subscheme of $X \times_S Y'$ (resp. $Y \times_S Y' = Y'$) (??). In the following commutative diagram

$$\begin{array}{ccc} X \times_Y Y' & \xrightarrow{f \times 1_{Y'}} & Y \times_Y Y' = Y' \\ \downarrow & & \downarrow \\ X \times_S Y' & \xrightarrow{f_{(Y')}} & Y \times_S Y \end{array}$$

the vertical morphisms are closed immersions, so if $f_{(Y')}$ is a closed morphism, so is $f \times 1_{Y'}$.

It remains to prove that $f_{(S')}$ is closed for arbitrary base change $S' \rightarrow S$ if this is true for $S' = S[(T_{\lambda})]$. Now by hypotheses S is affine, and if (S'_{γ}) is an open covering of S' , we see in the same manner as before that for $f_{(S')}$ to be closed, it suffices to prove that $f_{(S'_{\gamma})}$ is closed. We can then assume S' to be affine. If $S = \text{Spec}(A)$, we have $S' = \text{Spec}(A')$, where A' is an A -algebra. Let $(t_{\lambda})_{\lambda \in I}$ be a generator for A' , which means there is a surjective A -homomorphism $A[(T_{\lambda})] \rightarrow A'$ identifying A' with $A[(T_{\lambda})]/\mathfrak{b}$, where \mathfrak{b} is an ideal. If $S'' = \text{Spec}(A[(T_{\lambda})])$, S'

is then a closed subscheme of S'' , and $X_{(S')}$ (resp. $Y_{(S')}$) is identified with a closed subscheme of $X_{(S'')}$ (resp. $Y_{(S'')}$). The morphism $f_{(S')}$ is the restriction of $f_{(S'')}$ on $X_{(S')}$, and since $f_{(S'')}$ is closed by hypotheses, we conclude that $f_{(S')}$ is closed. This completes the proof. \square

18.4 Subschemes and immersions

18.4.1 Subschemes

Proposition 18.4.1. *Let X be a scheme and \mathcal{I} be a quasi-coherent ideal of \mathcal{O}_X . The support Y of the sheaf $\mathcal{O}_X/\mathcal{I}$ is then closed, and if \mathcal{O}_Y is the sheaf induced on Y by $\mathcal{O}_Y/\mathcal{I}$, (Y, \mathcal{O}_Y) is a scheme.*

Proof. Since the problem is local, it suffices to consider the affine case and show that Y is closed and (X, \mathcal{O}_Y) is an affine scheme. In fact, if $X = \text{Spec}(A)$, we have $\mathcal{O}_X = \tilde{A}$ and $\mathcal{I} = \tilde{\mathfrak{a}}$, where \mathfrak{a} is an ideal of A (??); Y is then equal to the closed subset $V(\mathfrak{a})$ of X and is identified with the spectrum of $B = A/\mathfrak{a}$ (??). Moreover, if $\rho : A \rightarrow B = A/\mathfrak{a}$ is the canonical homomorphism, the direct image $\rho_*(\tilde{B})$ is canonically identified with the sheaf $\tilde{A}/\tilde{\mathfrak{a}} = \mathcal{O}_X/\mathcal{I}$ (?? and ??). These complete the proof. \square

We say (Y, \mathcal{O}_Y) is the **subscheme of (X, \mathcal{O}_X) defined by the quasi-coherent ideal \mathcal{I}** . More generally, we say a locally ringed space (Y, \mathcal{O}_Y) is a **subscheme** of a scheme (X, \mathcal{O}_X) if Y is a locally closed subspace of X and if U denote the largest open subset of X containing Y such that Y is open in U (in other words, the complement of $\bar{Y} - Y$, so $U = (X - \bar{Y}) \cup Y$), then (Y, \mathcal{O}_Y) is a subscheme of $(U, \mathcal{O}_X|_U)$ defined by a quasi-coherent ideal of $\mathcal{O}_X|_U$. We say the subscheme (Y, \mathcal{O}_Y) of (X, \mathcal{O}_X) is **closed** if Y is closed in X (in this case $U = X$). It follows from this definition and ?? that closed subschemes of X are in one-to-one correspondence with quasi-coherent ideals of \mathcal{O}_X , since if two such ideals \mathcal{I}, \mathcal{J} have the same support (closed) Y and the sheaf induced by $\mathcal{O}_X/\mathcal{I}$ and $\mathcal{O}_X/\mathcal{J}$ on Y are identical, then $\mathcal{I} = \mathcal{J}$.

Let (Y, \mathcal{O}_Y) be a subscheme of X , U the largest open subset of X containing Y such that Y is closed in U , V an open subset of X contained in U ; then $V \cap Y$ is closed in V . Moreover, if Y is defined by the quasi-coherent ideal \mathcal{I} of $\mathcal{O}_X|_U$, then $\mathcal{I}|_V$ is a quasi-coherent ideal of $\mathcal{O}_X|_V$, and it is immediate that the scheme induced by Y over $V \cap Y$ is the closed subscheme of V defined by the ideal $\mathcal{I}|_V$.

In particular, the scheme induced by X over an open subset of X is a subscheme of X ; such schemes are called **open subschemes** of X . One should note that a subscheme of X can have the underlying space being an open set U of X without being induced on this open subset by X : it is induced over U by X only if it is defined by the ideal 0 of $\mathcal{O}_X|_U$, and there are in general quasi-coherent ideals \mathcal{I} of $\mathcal{O}_X|_U$ such that $(\mathcal{O}_X|_U)/\mathcal{I}$ have support U but is nonzero.

Proposition 18.4.2. *Let (Y, \mathcal{O}_Y) be a locally ringed space such that Y is a subspace of X and there exists a covering (V_α) of Y by open sets of X such that for each α , $Y \cap V_\alpha$ is closed in V_α and the locally ringed space $(Y \cap V_\alpha, \mathcal{O}_Y|_{Y \cap V_\alpha})$ is a closed subscheme of the scheme induced over V_α by X . Then (Y, \mathcal{O}_Y) is a subscheme of X .*

Proof. The hypothesis implies that Y is locally closed in X and the largest open set U containing Y and in which Y is closed contains the V_α . We are then reduced to the case $U = X$ and Y is closed in X . We define a quasi-coherent ideal \mathcal{I} of \mathcal{O}_X by letting \mathcal{I}_{V_α} to be the sheaf of ideal of $\mathcal{O}_X|_{V_\alpha}$ that defines the closed subscheme $(Y \cap V_\alpha, \mathcal{O}_Y|_{Y \cap V_\alpha})$ and for any open set W of X not

meeting Y , $\mathcal{J}_W = \mathcal{O}_X|_W$. It is immediately verified that there exists a unique sheaf of ideals \mathcal{J} satisfying these conditions and that it defines the closed subscheme (Y, \mathcal{O}_Y) . \square

Proposition 18.4.3. *A (closed) subscheme of a (closed) subscheme of X is canonically identified with a (closed) subscheme of X .*

Proof. Since a locally closed subset of a locally closed subset of X is still locally closed in X , it is clear by ?? that the question is local and we may assume that X is affine. The proposition then follows from the identification A/\mathfrak{b} and $(A/\mathfrak{a})/(\mathfrak{b}/\mathfrak{a})$ where $\mathfrak{a}, \mathfrak{b}$ are ideals of the ring A such that $\mathfrak{a} \subseteq \mathfrak{b}$. \square

Let Y be a subscheme of X and denote by $\iota : Y \rightarrow X$ the canonical injection of the underlying space; we know the inverse image $\iota^*(\mathcal{O}_X)$ is the restriction $\mathcal{O}_X|_Y$. If for any $y \in Y$, we denote by ι_y the canonical homomorphism $\mathcal{O}_{X,y} \rightarrow \mathcal{O}_{Y,y}$, these homomorphisms are then the restrictions to the stalks of \mathcal{O}_X at the points of Y of a surjective homomorphism of sheaves of rings $\iota^\# : \mathcal{O}_X|_Y \rightarrow \mathcal{O}_Y$: it suffices indeed to check it locally on Y , so we can assume that X is affine and Y is a closed subscheme; in this case, if \mathcal{J} is the quasi-coherent ideal of \mathcal{O}_X which defines Y , the ι_y 's are nothing but the restrictions to the stalks of the canonical homomorphism $\mathcal{O}_X|_Y \rightarrow (\mathcal{O}_X/\mathcal{J})|_Y$. We have therefore defined a monomorphism $j_Y = (\iota, \iota^\#)$ of locally ringed spaces, which is called the **canonical injection morphism**. If $f : X \rightarrow Z$ is another morphism of schemes, we say the composition

$$Y \xrightarrow{j_Y} X \xrightarrow{f} Z$$

is the **restriction** of f to the subscheme Y of X .

A subscheme Y of a scheme X is considered as an X -scheme via the canonical injection $j_Y : Y \rightarrow X$. Two subschemes Y, Z of X that are X -isomorphic are then necessarily identical. In fact, if $u : Y \rightarrow Z$ is an X -isomorphism, the relation $j_Y = j_Z \circ u$ shows the underlying spaces of Y and Z are identical. Moreover if $U \supseteq Y$ is an open subset of X such that $Y = Z$ are closed in U , and $\mathcal{J}, \mathcal{J}'$ are the ideals of U defining respectively Y and Z , for each $x \in Y$ we then have a commutative diagram

$$\begin{array}{ccc} \mathcal{O}_{X,x}/\mathcal{J}_x & \xrightarrow{u_x^\#} & \mathcal{O}_{X,x}/\mathcal{J}'_x \\ & \searrow & \swarrow \\ & \mathcal{O}_{X,x} & \end{array}$$

Since u is an isomorphism, this implies $\mathcal{J}_x = \mathcal{J}'_x$, so $Y = Z$ and $u = 1_Y$.

According to the general definitions, we say a morphism $f : Z \rightarrow X$ is **dominated by the canonical injection** $j_Y : Y \rightarrow X$ of a subscheme Y of X , if f factors through j_Y :

$$Z \xrightarrow{g} Y \xrightarrow{j_Y} X$$

where g is a morphism of schemes. Since j_Y is a monomorphism, the morphism g is unique.

Proposition 18.4.4. *Let Y be a subscheme of a scheme X and $j : Y \rightarrow X$ be the canonical injection. For a morphism $f : Z \rightarrow X$ to be dominated by the injection j , it is necessary and sufficient that $f(Z) \subseteq Y$ and for each $z \in Z$, the homomorphism $f_z^\# : \mathcal{O}_{X,f(z)} \rightarrow \mathcal{O}_{Z,z}$ factors through $\mathcal{O}_{Y,f(z)}$ (or equivalently, the kernel of $\mathcal{O}_{X,f(z)} \rightarrow \mathcal{O}_{Z,z}$ is contained in that of $\mathcal{O}_{X,f(z)} \rightarrow \mathcal{O}_{Y,f(z)}$).*

Proof. The condition is clearly necessary. For the sufficiency, we may assume that Y is a closed subscheme of X , and replace X by an open subset U such that Y is closed in U . Y is then defined by a quasi-coherent ideal \mathcal{J} of \mathcal{O}_X . Let \mathcal{J} be the kernel of the homomorphism $f^*(\mathcal{O}_X) \rightarrow \mathcal{O}_Z$. In view of the properties of the functor f^* , the hypothesis implies that for each $z \in Z$ we have $(f^*(\mathcal{J}))_z \subseteq \mathcal{J}_z$, and consequently $f^*(\mathcal{J}) \subseteq \mathcal{J}$. Therefore the homomorphism $f^*(\mathcal{O}_X) \rightarrow \mathcal{O}_Z$ factors into

$$f^*(\mathcal{O}_X) \longrightarrow f^*(\mathcal{O}_X)/f^*(\mathcal{J}) = f^*(\mathcal{O}_X/\mathcal{J}) \xrightarrow{\theta_z} \mathcal{O}_Z$$

the first arrow being the canonical homomorphism. Let g be the continuous map of Z in Y coincide with f ; it is clear that $g^*(\mathcal{O}_Y) = f^*(\mathcal{O}_X/\mathcal{J})$; on the other hand, for any $z \in Z$, θ_z is obviously a local homomorphism, so $(g, \theta) : Z \rightarrow Y$ is a morphism of schemes which satisfies $f = j \circ g$, whence the proposition. \square

Corollary 18.4.5. *Let Y and Z be subschemes of X . For the canonical injection $Z \rightarrow X$ to be dominated by the injection $Y \rightarrow X$, it is necessary and sufficient that Z is a subscheme of Y .*

Due to this corollary, for two subschemes Y, Z of X we write $Y \preceq Z$ if Y is a subscheme of Z . It is clear that this defines an order relation on the set of subschemes of X , since two subschemes Y and Z are identical if $Y \preceq Z$ and $Z \preceq Y$.

18.4.2 Immersions of schemes

We say a morphism $f : Y \rightarrow X$ is an **immersion** (resp. a **closed immersion**, resp. an **open immersion**) if it is factorized into

$$Y \xrightarrow{g} Z \xrightarrow{j} X$$

where g is an isomorphism, Z is a subscheme (resp. a closed subscheme, resp. an open subscheme) of X , and j is the canonical injection. The subscheme Z and the isomorphism g are then uniquely determined since two X -isomorphic subschemes are identical. We say $f = i \circ g$ is the **canonical factorization** of the immersion f , and the subscheme Z and the isomorphism g is called **associated** with f . It is clear that an immersion is a monomorphism of schemes (since j is a monomorphism), and a fortiori a radical morphism (??). Also, it is clear from ?? that the composition of two immersions (resp. two open immersions, resp. two closed immersions) is an immersion (resp. an open immersion, resp. a closed immersion).

Again, one should note that an immersion $f : Y \rightarrow X$ such that $f(Y)$ is a open subset of X , in other words which is an open morphism, is not necessarily an open immersion.

Example 18.4.6. Let X be an affine scheme. Then from the definition of closed subschemes, we see that for a morphism $f : Y \rightarrow X$ to be a closed immersion, it is necessary and sufficient that Y is an affine scheme and $\Gamma(f) : \Gamma(X, \mathcal{O}_X) \rightarrow \Gamma(Y, \mathcal{O}_Y)$ is surjective.

Lemma 18.4.7. *Let $f : Y \rightarrow X$ be a morphism of schemes such that $f(Y)$ is closed and f is a homeomorphism onto $f(Y)$. Then for each point $x \in X$, there exists an affine open neighborhood U of x such that $f^{-1}(U)$ is an affine open of Y .*

Proof. Since $f(Y)$ is closed in X , the lemma is trivial if $x \notin f(Y)$, since it suffices to choose an affine open neighborhood of x disjoint from $f(Y)$. If $x \in f(Y)$, there exists a unique point

$y \in Y$ such that $f(y) = x$. Let W be an affine open neighborhood of x in X and V an affine open neighborhood of y in Y such that $f(V) \subseteq W$. By hypothesis $f(V)$ is an open neighborhood of x in $f(Y)$, so there exists an open neighborhood $U' \subseteq W$ of x such that $U' \cap f(Y) = f(V)$. Let U be an open neighborhood of x contained in U' and is of the form $D(s)$, where $s \in A = \Gamma(W, \mathcal{O}_X)$ (recall that W is chosen to be affine); in view of ??(b), $f^{-1}(U) \subseteq V$ is of the form $D(t)$, where t is the image of s in $B = \Gamma(V, \mathcal{O}_Y|_V)$, hence proves the lemma. \square

Lemma 18.4.8. *Let $f : Y \rightarrow X$ be a morphism of schemes and (U_λ) be an affine open covering of X such that for each λ , $f^{-1}(U_\lambda)$ is an affine open of Y . Then for any quasi-coherent \mathcal{O}_Y -module \mathcal{F} , the direct image $f_*(\mathcal{F})$ is a quasi-coherent \mathcal{O}_X -module.*

Proof. For each λ , put $V_\lambda = f^{-1}(U_\lambda)$, and let $f_\lambda : V_\lambda \rightarrow U_\lambda$ be the restriction of f to V_λ . Then the restriction $f_*(\mathcal{F})$ to U_λ is equal to $(f_\lambda)_*(\mathcal{F}_\lambda)$, where $\mathcal{F}_\lambda = \mathcal{F}|_{U_\lambda}$. But since U_λ and V_λ are affine by hypothesis, we see $(f_\lambda)_*(\mathcal{F}_\lambda)$ is quasi-coherent by ???. This proves the lemma. \square

Proposition 18.4.9. *Let $f : Y \rightarrow X$ be a morphism of schemes.*

- (a) *For f to be an open immersion, it is necessary and sufficient that f is a homeomorphism onto an open subset of X and for each $y \in Y$, the homomorphism $f_y^\# : \mathcal{O}_{X,f(y)} \rightarrow \mathcal{O}_{Y,y}$ is bijective.*
- (b) *For f to be an immersion (resp. a closed immersion), it is necessary and sufficient that f is a homeomorphism onto a locally closed (resp. closed) subset of X and for each $y \in Y$, the homomorphism $f_y^\# : \mathcal{O}_{X,f(y)} \rightarrow \mathcal{O}_{Y,y}$ is surjective.*

Proof. In the two cases, the conditions are clearly necessary, so we only need to prove the sufficiency. If the conditions in (a) holds, it is clear that $f^\#$ induces an isomorphism of \mathcal{O}_Y to $f^*(\mathcal{O}_X)$, and $f^*(\mathcal{O}_X)$ is the sheaf defined by the transport by structure by means of f^* from the induced sheaf $\mathcal{O}_X|_{f(Y)}$, hence the conclusion.

Suppose then the conditions in (b) holds. Let U_0 be the largest open set of X such that $Z = f(Y)$ is closed in U_0 ; by replacing X by the subscheme induced by X over U_0 , we may assume that $Z = f(Y)$ is closed in X . By ?? and ??, the sheaf $f_*(\mathcal{O}_Y)$ is a quasi-coherent \mathcal{O}_X -module. We have an exact sequence

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{O}_X \xrightarrow{f^\#} f_*(\mathcal{O}_Y) \longrightarrow 0$$

where two terms are quasi-coherent \mathcal{O}_X -modules; we then deduce that ?? \mathcal{I} is a quasi-coherent ideal of \mathcal{O}_X , and $f^\#$ factors into

$$\mathcal{O}_X \longrightarrow \mathcal{O}_X/\mathcal{I} \xrightarrow{\theta} f_*(\mathcal{O}_Y)$$

where the first arrow is the canonical homomorphism and θ is an isomorphism. If $\mathcal{O}_Z = (\mathcal{O}_X/\mathcal{I})|_Z$, (Z, \mathcal{O}_Z) is then a closed subscheme of (X, \mathcal{O}_X) and f factors through the canonical injection $j_Z : Z \rightarrow X$. Since the corresponding morphism $Y \rightarrow Z$ is just (f_0, θ_0) , where f_0 is the map f considered as a homeomorphism from Y to Z and θ_0 is the restriction of θ to \mathcal{O}_Z , it is clear that f is a closed immersion, which completes the proof. \square

Remark 18.4.1. It may happen that $f : Y \rightarrow X$ is a closed immersion and for all $y \in Y$, $f_y^\# : \mathcal{O}_{X,f(y)} \rightarrow \mathcal{O}_{Y,y}$ is bijective, without f being an open immersion (that is, $f(Y)$ is not necessarily

open in X). For example, let $X = \text{Spec}(A)$ be an affine scheme and $x \in X$ be a closed point of X that is not isolated. Then if $Y = \text{Spec}(A/\mathfrak{m}_x)$, the canonical morphism $Y \rightarrow X$ is a closed immersion satisfying the desired property, since the subspace $\{x\}$ is not open in X .

Corollary 18.4.10. *Let $f : Y \rightarrow X$ be a morphism of schemes.*

- (a) *Let (V_λ) be a covering of $f(Y)$ by open subsets of X . Then for f to be an immersion (an open immersion), it is necessary and sufficient that for each λ , the restriction $f^{-1}(V_\lambda) \rightarrow V_\lambda$ of f is an immersion (an open immersion).*
- (b) *Let (U_λ) be an open covering of X . Then for f to be a closed immersion, it is necessary and sufficient that for each λ , the restriction $f^{-1}(U_\lambda) \rightarrow U_\lambda$ of f is a closed immersion.*

Proof. In the case (a), $f_y^\#$ is surjective (resp. bijective) for every point $y \in Y$, and in case (b) it is surjective for every point $y \in Y$; it then suffices to verify that in case (a) f is a homeomorphism of Y onto a locally closed (resp. open) subset of X and in case (b), a homeomorphism onto a closed subset of X . Now, the hypothesis imply that f is clearly injective and maps each neighborhood of $y \in Y$ to a neighborhood of $f(y)$ in $f(Y)$. In case (a), $f(Y) \cap V_\lambda$ is locally closed (resp. open) in the union of the V_λ , and a fortiori in X ; in case (b), $f(Y) \cap U_\lambda$ is closed in U_λ , hence closed in X since $X = \bigcup_\lambda U_\lambda$. \square

Remark 18.4.2. We can generalize the notions of immersions to any ringed spaces. We define a **ringed subspace** of a ringed space (X, \mathcal{O}_X) to be a ringed space of the form $(Y, (\mathcal{O}_U/\mathcal{I})|_Y)$, where U is an open of X , \mathcal{I} an ideal of \mathcal{O}_U and Y the support of the sheaf of rings $\mathcal{O}_U/\mathcal{I}$ (support which is no longer necessarily closed in U). We can define the canonical injection $Y \rightarrow X$, and the definitions and results of ?? are valid without modification. We then define the notion of immersion (resp. of closed immersion) in the same manner. The characterizations of open (resp. closed) immersions given in ?? still hold (observing that if $f(Y)$ is closed in X , $f_*(\mathcal{O}_Y)$ has support $f(Y)$). The result of ?? can therefore be stated by saying that if a scheme Y is a ringed subspace of a scheme X , then Y is a subscheme of X .

18.4.3 Inverse image of subschemes

Proposition 18.4.11. *Let $f : X' \rightarrow X$ and $g : Y' \rightarrow Y$ be two S -morphisms. Let p, q be the projections of $X \times_S Y$.*

- (a) *If f and g are immersions (resp. open immersions, resp. closed immersions), then $f \times_S g$ is an immersion (resp. an open immersion, resp. a closed immersion).*
- (b) *If X' (resp. Y') is identified with a subscheme of X (resp. Y) via the immersion f (resp. g), then $f \times_S g$ identifies the underlying space of $X' \times_S Y'$ with the subspace $p^{-1}(X') \cap q^{-1}(Y')$ of the underlying space of $X \times_S Y$.*

Proof. We can restrict ourselves to the case where X' and Y' are subschemes and f and g are the canonical injection morphisms. The proposition has already been established for the subschemes induced on the open sets (??); as any subscheme is a closed subscheme of an open scheme, we are reduced in case X' and Y' are closed subschemes.

We can further assume that S is affine. In fact, let (S_λ) be an affine open cover of S , φ and ψ be the structural morphisms of X and Y , and let $X_\lambda = \varphi^{-1}(S_\lambda)$, $Y_\lambda = \psi^{-1}(S_\lambda)$. The

restriction X'_λ (resp. Y'_λ) of X' (resp Y') to $X_\lambda \cap X'$ (resp. $Y_\lambda \cap Y'$) is a closed subscheme of X_λ (resp. Y_λ), the schemes X_λ , Y_λ , X'_λ , Y'_λ can then be considered as S_λ -schemes and the product $X_\lambda \times_S Y_\lambda$ and $X_\lambda \times_{S_\lambda} Y_\lambda$ (resp. $X'_\lambda \times_S Y'_\lambda$ and $X'_\lambda \times_{S_\lambda} Y'_\lambda$) are identified (??). If the proposition is true when S is affine, the restriction of $f \times_S g$ to the $X'_\lambda \times_S Y'_\lambda$ will therefore be an immersion. As the product $X'_\lambda \times_S Y'_\mu$ (resp. $X_\lambda \times_S Y_\mu$) is identified with $(X'_\lambda \cap X'_\mu) \times_S (Y'_\lambda \cap Y'_\mu)$ (resp. $(X_\lambda \cap X_\mu) \times_S (Y_\lambda \cap Y_\mu)$), the restriction of $f \times_S g$ to $X'_\lambda \times_S Y'_\mu$ is also an immersion; it follows from ?? that $f \times_S g$ is an immersion.

Secondly, let's prove that we can also assume that X and Y are affine. In fact, let (U_i) (resp. (V_j)) be an affine open cover of X (resp. Y), and let X'_i (resp. Y'_j) be the restriction of X' (resp. Y') to $X' \cap U_i$ (resp. $Y' \cap V_j$), which is a closed subscheme of U_i (resp. V_j). Then $U_i \times_S V_j$ is identified with the restriction of $X \times_S Y$ to $p^{-1}(U_i) \cap q^{-1}(V_j)$ by ??, and similarly, if $p' : X' \times_S Y' \rightarrow X'$ and $q' : X' \times_S Y' \rightarrow Y'$ are the canonical projections, the product $X'_i \times_S Y'_j$ is identified with the restriction of $X' \times_S Y'$ to $p'^{-1}(X'_i) \cap q'^{-1}(Y'_j)$. Put $h = f \times_S g$, then since $X'_i = f^{-1}(U_i)$, $Y'_j = g^{-1}(V_j)$, we have

$$p'^{-1}(X'_i) \cap q'^{-1}(Y'_j) = h^{-1}(p^{-1}(U_i) \cap q^{-1}(V_j)) = h^{-1}(U_i \times_S V_j).$$

Again, by the same reasoning and using ??, we can show that h is an immersion.

Suppose then that X , Y , and S are affine, with rings B , C , and A , respectively. Then B and C are A -algebras, X' and Y' are affine subschemes with rings quotients B' , C' of B and C , respectively. Moreover, f and g are induced by ring homomorphisms $\rho : B \rightarrow B'$ and $\sigma : C \rightarrow C'$. With these, we see $X \times_S Y$ (resp. $X' \times_S Y'$) is the affine scheme with ring $B \otimes_A C$ (resp. $B' \otimes_A C'$), and $f \times_S g$ corresponds to the ring homomorphism $\rho \otimes \sigma : B \otimes_A C \rightarrow B' \otimes_A C'$. Since this homomorphism is surjective, $f \times_S g$ is an immersion. Moreover, if \mathfrak{b} (resp. \mathfrak{c}) is the kernel of ρ (resp. σ), the kernel of $\rho \otimes \sigma$ is $u(\mathfrak{b}) + v(\mathfrak{c})$, where u (resp. v) is the homomorphism $b \mapsto b \otimes 1$ (resp. $c \mapsto 1 \otimes c$). As p corresponds to the ring homomorphism u and q corresponds to v , this kernel corresponds, in the spectrum $\text{Spec}(B \otimes_A C)$, to the closed subset $p^{-1}(X') \cap q^{-1}(Y')$, which proves the demonstration. \square

Corollary 18.4.12. *If $f : X \rightarrow Y$ is an immersion (resp. an open immersion, resp. a closed immersion) and an S -morphism, then $f_{(S')}$ is an immersion (resp. an open immersion, resp. a closed immersion) for any extension $S' \rightarrow S$ of base schemes.*

Proof. This follows from the observation that the identity morphism is an immersion (resp. an open immersion, resp. a closed immersion). \square

Corollary 18.4.13. *An immersion (resp. a closed immersion, resp. an open immersion) is a universally embedding (resp. universally closed, resp. universally open).*

Proposition 18.4.14. *Let $f : X \rightarrow Y$ be a morphism, Y' a subscheme (resp. a closed subscheme, resp. an open subscheme) of Y , and $j : Y' \rightarrow Y$ the canonical injection.*

- (a) *The projection $p : X \times_Y Y' \rightarrow X$ is an immersion (resp. a closed immersion, resp. an open immersion), and the subscheme of X associated with p has underlying space $f^{-1}(Y')$. Moreover, if j' is the canonical injection of this subscheme, for a morphism $h : Z \rightarrow X$ to be such that $f \circ h : Z \rightarrow Y$ is dominated by j , it is necessary and sufficient that h is dominated by j' .*

- (b) If Z is a closed subscheme defined by a quasi-coherent ideal \mathcal{K} of \mathcal{O}_Y , the inverse image of Z by f is defined by the quasi-coherent ideal $f^*(\mathcal{K})\mathcal{O}_X$.

Proof. As $p = 1_X \times_Y j$, the first assertion in (a) follows from ???. The second one is a special case of ???. Finally, if we have $f \circ h = j \circ h'$, where $h' : Z \rightarrow Y'$ is a morphism, it follows from the universal property of product that we have $h = p \circ u$, where $u : Z \rightarrow X \times_Y Y'$ is a morphism, whence assertion (a).

To prove (b), since the question is local on X and Y , we may assume that X and Y are affine. It then suffices to note that if A is an B -algebra and \mathfrak{b} is an ideal of B , we have $A \otimes_B (B/\mathfrak{b}) = A/\mathfrak{b}A$, and apply ???. \square

We say the subscheme of X thus defined is the **inverse image** of the subscheme Y' of Y by the morphism f . We say the morphism $f \times 1_{Y'} : f^{-1}(Y') \rightarrow Y'$ is the restriction of f to $f^{-1}(Y')$. When we speak of $f^{-1}(Y')$ as a subscheme of X , it is always this subscheme that will be involved.

Example 18.4.15. If the scheme $f^{-1}(Y')$ and X are identical, $j' : f^{-1}(Y') \rightarrow X$ is then the identity and any morphism $h : Z \rightarrow X$ is dominated by j' ; hence the morphism $f : X \rightarrow Y$ factors into

$$X \xrightarrow{g} Y' \xrightarrow{j} Y$$

Example 18.4.16. If y is a closed point of Y and $Y' = \text{Spec}(\kappa(y))$ is the smallest closed subscheme of Y having $\{y\}$ as underlying space, the closed subscheme $f^{-1}(Y')$ is then canonically isomorphic to the **fiber** $f^{-1}(y)$.

Corollary 18.4.17. Retain the hypotheses of ??(b). Let X' be a closed subscheme of X defined by a quasi-coherent ideal \mathcal{I} of \mathcal{O}_X and $i : X' \rightarrow X$ be the canonical injection. For the restriction $f \circ i$ of f to X' is dominated by the injection $j : Y' \rightarrow Y$, it is necessary and sufficient that $f^*(\mathcal{K})\mathcal{O}_X \subseteq \mathcal{I}$.

Proof. This follows from ??(b) and (a). \square

Corollary 18.4.18. Let $f : X \rightarrow Y$, $g : Y \rightarrow Z$ be two morphisms, and $h = g \circ f$ be their composition. For any subscheme Z' of Z , the subscheme $f^{-1}(g^{-1}(Z'))$ and $h^{-1}(Z')$ of X are identical.

Proof. This follows from the transitivity of products and ???. \square

Corollary 18.4.19. Let X', X'' be two subschemes of X and $j' : X' \rightarrow X$, $j'' : X'' \rightarrow X$ be the canonical injections. Then $j'^{-1}(X'')$ and $j''^{-1}(X')$ are both equal to the infimum $\inf(X', X'')$ of X' and X'' for the ordered relation on subschemes, and is canonically isomorphic to $X' \times_S X''$.

Proof. This follows from ??, ??, and the universal property of products. \square

Corollary 18.4.20. Let $f : X \rightarrow Y$ be a morphism and Y', Y'' be two subschemes of Y . Then we have $f^{-1}(\inf(Y', Y'')) = \inf(f^{-1}(Y'), f^{-1}(Y''))$.

Proof. In fact, we have the canonical isomorphism of $(X \times_Y Y') \times_X (X \times_Y Y'')$ and $X \times_Y (Y' \times_Y Y'')$. \square

18.4.4 Local immersions and local isomorphisms

Let $f : X \rightarrow Y$ be a morphism of schemes. We say f is a **local immersion** (resp. a **local isomorphism**) at a point $x \in X$ if there exists an open neighborhood U of x in X and an open neighborhood V of $f(x)$ in Y such that the restriction of f to U is a closed immersion (resp. an open immersion) into V . We say f is a local immersion (resp. a local isomorphism) if f is a local immersion (resp. a local isomorphism) at every point of X .

An immersion (resp. closed immersion) $f : X \rightarrow Y$ can then be characterized as a local immersion such that f is a homeomorphism onto a subset of Y (resp. a closed subset of Y). An open immersion f can be characterized as an injective local isomorphism.

Proposition 18.4.21. *Let X be an irreducible scheme, $f : X \rightarrow Y$ be a injective dominant morphism. If f is a local immersion, then it is an immersion and $f(X)$ is open in Y .*

Proof. In fact, let $x \in X$ and U be an open neighborhood of x in X , V an open neighborhood of $f(x)$ in Y such that $f|_U$ is a closed immersion into V . As U is dense in X , $f(U)$ is also dense in Y by hypothesis, hence $f(U) = V$ and f is a homeomorphism from U to V . The hypothesis f is injective implies $f^{-1}(V) = U$, whence the proposition. \square

Proposition 18.4.22. *Let $f : X \rightarrow X'$, $g : Y \rightarrow Y'$ be two S -schemes.*

- (a) *The composition of two local immersions (resp. local isomorphisms) is a local immersion (resp. a local isomorphism).*
- (b) *If f and g are local immersions (resp. local isomorphisms), so is the product $f \times_S g$.*
- (c) *If f is a local immersion (resp. a local isomorphism), so is $f_{(S')}$ for any extension $S' \rightarrow S$ of base schemes.*

Proof. It suffices to prove (a) and (b). Now (a) follows from the transitivity of closed immersions (resp. open immersions) and the fact that if f is a homeomorphism of X to a closed subset Y , then for any open set $U \subseteq X$, $f(U)$ is open in $f(X)$, so there exists an open subset V of Y such that $f(U) = V \cap f(X)$, and $f(U)$ is therefore closed in V .

To prove (b), let p, q be the projections of $X \times_S Y$ and p', q' that of $X' \times_S Y'$. There exist by hypotheses open neighborhoods U, U', V, V' of $x = p(z), x' = p'(z'), y = q(z), y' = q'(z')$, respectively, such that $f|_U$ and $g|_V$ are closed immersions (resp. open immersions) onto U' and V' , respectively. As the underlying space of $U \times_S V$ is $p^{-1}(U) \cap q^{-1}(V)$ and that of $U' \times_S V'$ is $p'^{-1}(U') \cap q'^{-1}(V')$, which are neighborhoods of z and z' , respectively (??), the claim follows by ??.

Remark 18.4.3. A local isomorphism is clearly flat and universally open, and therefore is universally generalizing.

Proposition 18.4.23. *Let X be an irreducible scheme, Y an integral scheme, and $f : X \rightarrow Y$ be a morphism.*

- (a) *If f is dominant, then for any $x \in X$, the homomorphism $f_x^\# : \mathcal{O}_{Y,f(x)} \rightarrow \mathcal{O}_{X,x}$ is injective.*
- (b) *If f is dominant and a local immersion, then f is a local isomorphism (and therefore X is integral).*

Proof. Let ξ and η be the generic points of X and Y , respectively. If f is dominant, we then have $f(\xi) = \eta$; moreover $\mathcal{O}_{Y,\eta}$ is a field since Y is reduced, so $f_\xi^\# : \mathcal{O}_{Y,\eta} \rightarrow \mathcal{O}_{X,\xi}$ is a monomorphism. For any point $x \in X$, and any affine neighborhood U of $y = f(x)$, there exists an affine neighborhood V of x contained in $f^{-1}(U)$. The open set U (resp. V) contains η (resp. ξ), and the ring $\Gamma(U, \mathcal{O}_Y)$ is integral with fraction field $\mathcal{O}_{Y,\eta}$. If $\rho : \Gamma(U, \mathcal{O}_Y) \rightarrow \Gamma(V, \mathcal{O}_X)$ is the homomorphism corresponding to f , the composition

$$\Gamma(U, \mathcal{O}_Y) \xrightarrow{\rho} \Gamma(V, \mathcal{O}_X) \longrightarrow \mathcal{O}_{X,\xi}$$

is then the restriction of $f_\xi^\#$ to $\Gamma(U, \mathcal{O}_Y)$, so the homomorphism ρ . We now deduce that $f_x^\#$ is injective: in fact the hypotheses that X is irreducible implies that $\Gamma(V, \mathcal{O}_X)$ has a unique minimal \mathfrak{n} , which is its nilradical; the homomorphism $f_x^\#$ just send an element u/s (where $u, s \in \Gamma(U, \mathcal{O}_Y)$ and $s \neq 0$) to the element $\rho(u)/\rho(s) \in \mathcal{O}_{X,x}$, which is zero only if there exists $t \notin \mathfrak{p}_x$ such that $t\rho(u) = 0 \in \mathfrak{n}$. But as $t \notin \mathfrak{n}$, this then implies $\rho(u) \in \mathfrak{n}$, so $\rho(u)$ is nilpotent and since ρ is injective, this shows that u is nilpotent, which means $u = 0$ for $\Gamma(U, \mathcal{O}_Y)$ being integral.

To prove the second assertion, let f be dominant and a local immersion. We see $f(Y)$ is open in Y by ???. Since $f_x^\#$ is surjective for every point $x \in X$ (??), it follows that $f_x^\#$ is an isomorphism by (a), and this shows f is a local isomorphism, again by ?? . \square

Proposition 18.4.24. *Let Y be a reduced scheme such that the family of irreducible components of Y is locally finite. Let $j : X \rightarrow Y$ be an immersion. For j to be a local isomorphism at a point $x \in X$, it is necessary and sufficient that the homomorphism $j_x^\# : \mathcal{O}_{Y,j(x)} \rightarrow \mathcal{O}_{X,x}$ is flat.*

Proof. Since the question is local on Y , we may assume that Y is affine, that j is a closed immersion, and that all irreducible components of Y contain $j(x)$ (hence are finite in number), and we prove that j is an isomorphism in this case. If $Y = \text{Spec}(A)$, and if \mathfrak{p} is the prime ideal of A corresponding to $j(x)$, the morphism $\text{Spec}(A_{\mathfrak{p}}) \rightarrow \text{Spec}(A)$ is then dominant since \mathfrak{p} contains the minimal ideals of A (it is contained in every irreducible component of $\text{Spec}(A)$). As A is reduced, the homomorphism $A \rightarrow A_{\mathfrak{p}}$ is injective (??). If \mathcal{O} is the ideal of \mathcal{O}_Y defining the closed subscheme of Y associated with j , we have $\mathcal{I} = \tilde{\mathfrak{a}}$, where \mathfrak{a} is an ideal of A , and it is identified with a subset of $\mathfrak{a}_{\mathfrak{p}}$. If j is flat at x , then $(A/\mathfrak{a})_{\mathfrak{p}}$ is a flat $A_{\mathfrak{p}}$ -module, and since the homomorphism $A_{\mathfrak{p}} \rightarrow (A/\mathfrak{a})_{\mathfrak{p}}$ is local, it is faithfully flat (??). By ?? this implies $\mathfrak{a}_{\mathfrak{p}} = 0$, hence $\mathfrak{a} = 0$ and the claim follows. \square

18.4.5 Nilradical and associated reduced scheme

Proposition 18.4.25. *Let (X, \mathcal{O}_X) be a scheme and \mathcal{B} be a quasi-coherent \mathcal{O}_X -algebra. There exist a unique quasi-coherent ideal \mathcal{N} of \mathcal{B} such that for each point $x \in X$, the stalk \mathcal{N}_x is the nilradical of the ring \mathcal{B}_x . If $X = \text{Spec}(A)$ is affine and $\mathcal{B} = \tilde{B}$, where B is an A -algebra, then $\mathcal{N} = \tilde{\mathfrak{n}}$, where \mathfrak{n} is the nilradical of B .*

Proof. Since the question is local, we may assume that $X = \text{Spec}(A)$ is affine and $\mathcal{B} = \tilde{B}$. We know that $\tilde{\mathfrak{n}}$ is a quasi-coherent \mathcal{O}_X -module and for each point $x \in X$, the stalk \mathfrak{n}_x is an ideal of the fraction ring B_x . It suffices to show that the nilradical of B_x is contained in \mathfrak{n}_x , the opposite inclusion being evident. Now, let z/s be a nilpotent element in B_x , where $z \in B$ and $s \notin \mathfrak{p}_x$. By

hypotheses, there exist $k \geq 0$ such that $(z/s)^k = 0$, which means there exists $t \notin \mathfrak{p}_x$ such that $tz^k = 0$. We then conclude that $(tz)^k = 0$, so $z/s = (tz)/(ts)$ is indeed in \mathfrak{n}_x . \square

The quasi-coherent ideal \mathcal{N} is called the **nilradical** of the \mathcal{O}_X -algebra \mathcal{B} . In particular, we denote by \mathcal{N}_X the nilradical of \mathcal{O}_X .

Corollary 18.4.26. *Let X be a scheme. Then the closed subscheme of X defined by the quasi-coherent ideal \mathcal{N}_X is the unique reduced subscheme of X with underlying space X . It is also the smallest subscheme of X having X as underlying space.*

Proof. Since the structural sheaf of the closed subscheme Y defined by \mathcal{N}_X is $\mathcal{O}_X/\mathcal{N}_X$, it is immediate that Y is reduced and has X as underlying space, since $\mathcal{N}_x \neq \mathcal{O}_{X,x}$ for each $x \in X$. To prove the second claim, let Z be a subscheme of X with X as underlying space. Then Z is closed in X , so let \mathcal{I} be the ideal defining it. We can assume that X is affine, so $\mathcal{I} = \tilde{\mathfrak{a}}$ where \mathfrak{a} is an ideal of A . Then for each $x \in X$ we have $\mathfrak{a} \subseteq \mathfrak{p}_x$, so $\mathfrak{a} \subseteq \mathfrak{n}$, where \mathfrak{n} is the nilradical of A . This shows Y is the smallest subscheme of X with underlying space X , and if Z is distinct from Y , we necessarily have $\mathcal{I}_x \neq \mathcal{N}_x$ for some $x \in X$, and consequently Z is not reduced. \square

The reduced scheme defined by \mathcal{N}_X on X is called the **reduced scheme associated with X** , and denoted by X_{red} . To say that a schema X is reduced therefore means that $X_{\text{red}} = X$. Clearly, we have a canonical closed immersion $X_{\text{red}} \rightarrow X$, which is also a universal homeomorphism.

Proposition 18.4.27. *For the spectrum of a ring A to be reduced (resp. integral), it is necessary and sufficient that A is reduced (resp. integral).*

Proof. In fact, the condition $\mathcal{N} = 0$ is necessary and sufficient for $\text{Spec}(A)$ to be reduced, and the integral conditions follows from ??.

Proposition 18.4.28. *A scheme X is integral if and only if for each open subset U of X , the ring $\Gamma(U, \mathcal{O}_X)$ is integral.*

Proof. We first assume that $\Gamma(U, \mathcal{O}_X)$ is integral for any open set U . It is clear that X is reduced. If X is not irreducible, then one can find two nonempty disjoint open subsets U_1 and U_2 . Then $\Gamma(U_1 \cup U_2, \mathcal{O}_X) = \Gamma(U_1, \mathcal{O}_X) \times \Gamma(U_2, \mathcal{O}_X)$, which is not an integral domain. Thus X is an integral scheme.

Conversely, suppose that X is reduced and irreducible. Let $U \subseteq X$ be an open subset, and suppose that there are elements $f, g \in \Gamma(U, \mathcal{O}_X)$ with $fg = 0$. Let

$$Y = \{x \in U : f_x \in \mathfrak{m}_x\}, \quad Z = \{x \in U : g_x \in \mathfrak{m}_x\};$$

then Y and Z are closed subsets, and $Y \cup Z = U$. Since X is irreducible, so U is irreducible, and one of Y or Z is equal to U , say $Y = U$. But then the restriction of f to any open affine subset of U will be contained in every point of that subset, hence nilpotent and thus zero. This shows that $\Gamma(U, \mathcal{O}_X)$ is integral. \square

Proposition 18.4.29. *Let X be a scheme and x a point of X .*

- (a) *For x to belong to a unique irreducible component of X , it is necessary and sufficient that the nilradical of $\mathcal{O}_{X,x}$ is prime.*

- (b) If the nilradical of $\mathcal{O}_{X,x}$ is prime and the family of irreducible components of X is locally finite, there exists an open neighborhood U of x that is irreducible.
- (c) For X to be the coproduct of its irreducible components, it is necessary and sufficient that the family of irreducible components of X is locally finite and for each $x \in X$, the nilradical of $\mathcal{O}_{X,x}$ is prime.

Proof. To check that whether x belongs to distinct irreducible components of X , we may assume that $X = \text{Spec}(A)$ is affine (??). Then this signifies that \mathfrak{p}_x contains two distinct minimal prime ideals of A , and equivalently \mathfrak{m}_x contains two distinct minimal prime ideals of $\mathcal{O}_{X,x}$, which is a contradiction if and only if the nilradical of $\mathcal{O}_{X,x}$ is prime.

Now assume the conditions in (a). As the family of irreducible components of X is locally finite, the union of those of these components which do not contain x is closed, so its complement U is open and contained in the unique irreducible component of X containing x , and therefore irreducible (??). The assertion in (c) follows from (b) and ??.

Proposition 18.4.30. *For a scheme X to be locally integral, it is necessary and sufficient that the family of irreducible components is locally finite and for each point $x \in X$, the local ring $\mathcal{O}_{X,x}$ is integral. In this case, X is the coproduct of its irreducible components, which are integral subschemes.*

Proof. As the localizations of integral domains are integral, for a locally integral scheme X , its local rings $\mathcal{O}_{X,x}$ are integral. The set of the irreducible components of X is locally finite since each $x \in X$ admits an irreducible open neighborhoods; moreover, the irreducible components of X are all open and disjoint, so X is the coproduct of its irreducible components (??). Conversely, if X satisfies these conditions, then X is the coproduct of its irreducible components, which are open and integral. It follows immediately that X is locally integral.

Corollary 18.4.31. *Let X be a scheme whose set of irreducible components is locally finite (for example if X is locally Noetherian). Then for X to be integral, it is necessary and sufficient that it is connected and for each $x \in X$, the local ring $\mathcal{O}_{X,x}$ is integral. In particular, a locally integral and connected scheme is integral.*

Proposition 18.4.32. *Let X be a locally Noetherian scheme and let $x \in X$ be a point such that the nilradical \mathcal{N}_x of $\mathcal{O}_{X,x}$ is prime (resp. such that $\mathcal{O}_{X,x}$ is reduced, resp. such that $\mathcal{O}_{X,x}$ is integral). Then there exist an open neighborhood U of x that is irreducible (resp. reduced, resp. integral).*

Proof. It suffices to consider the case where \mathcal{N}_x is prime and where $\mathcal{N}_x = 0$, the third one is the conjunction of the first two cases. If \mathcal{N}_x is prime, the claim follows from ???. If $\mathcal{N}_x = 0$, we then have $\mathcal{N}_y = 0$ for y in a neighborhood of x , since \mathcal{N} is quasi-coherent, hence coherent since X is locally Noetherian and \mathcal{N} is of finite type, and the conclusion follows from ???.

Proposition 18.4.33. *For a Noetherian scheme X , the nilradical \mathcal{N}_X of \mathcal{O}_X is nilpotent.*

Proof. Since X is quasi-compact, We can cover X by a finite number of affine open sets U_i , and it suffices to prove that there exist integers n_i such that $(\mathcal{N}_X|_{U_i})^{n_i} = 0$. If n is the largest of the n_i , we will then have $\mathcal{N}_X^n = 0$. We are therefore reduced to the case where $X = \text{Spec}(A)$ is affine, A being a Noetherian ring. It then suffices to observe that the nilradical of A is nilpotent.

Corollary 18.4.34. *A Noetherian scheme X is affine if and only if X_{red} is affine.*

Proof. If X is affine it is clear that X_{red} is affine, regardless of X being Noetherian. Conversely, assume that X_{red} is affine and X is Noetherian. Then by ??, the nilradical \mathcal{N} of \mathcal{O}_X is nilpotent. For any quasi-coherent sheaf \mathcal{F} on X , consider the follows exact sequence (where $k \geq 0$)

$$0 \longrightarrow \mathcal{N}^k \mathcal{F} \longrightarrow \mathcal{N}^{k-1} \mathcal{F} \longrightarrow \mathcal{N}^{k-1} \mathcal{F} / \mathcal{N}^k \mathcal{F} \longrightarrow 0$$

Since $\mathcal{N}^{k-1} \mathcal{F} / \mathcal{N}^k \mathcal{F}$ is an $\mathcal{O}_X / \mathcal{N}$ -module and $(X, \mathcal{O}_X / \mathcal{N})$ is affine, by Serre's criterion we have $H^1(X, \mathcal{N}^{k-1} \mathcal{F} / \mathcal{N}^k \mathcal{F}) = 0$, so

$$H^1(X, \mathcal{N}^k \mathcal{F}) = 0 \Rightarrow H^1(X, \mathcal{N}^{k-1} \mathcal{F}) = 0.$$

Since \mathcal{N} is nilpotent, this shows $H^1(X, \mathcal{F}) = 0$, so (X, \mathcal{O}_X) is affine, by Serre's criterion again. \square

Let $f : X \rightarrow Y$ be a morphism of schemes; let $i : X_{\text{red}} \rightarrow X$ and $j : Y_{\text{red}} \rightarrow Y$ be the canonical injections. The homomorphism $f_x^\# : \mathcal{O}_{Y, f(x)} \rightarrow \mathcal{O}_{X, x}$ maps nilpotent elements of $\mathcal{O}_{Y, f(x)}$ into nilpotent elements of $\mathcal{O}_{X, x}$, so $f^*(\mathcal{N}_Y) \mathcal{O}_X \subseteq \mathcal{N}_X$. It then follows from ?? that $f \circ i$ factors through j , so we get an induced morphism $f_{\text{red}} : X_{\text{red}} \rightarrow Y_{\text{red}}$ and a commutative diagram

$$\begin{array}{ccc} X_{\text{red}} & \xrightarrow{f_{\text{red}}} & Y_{\text{red}} \\ i \downarrow & & \downarrow j \\ X & \xrightarrow{f} & Y \end{array} \quad (18.4.1)$$

In particular, if X is reduced, then the morphism f factors into

$$X \xrightarrow{f_{\text{red}}} Y_{\text{red}} \xrightarrow{j} Y$$

or in other words, f is dominated by the canonical injection j . We also conclude that Y_{red} satisfies the universal property that any morphism from a reduced scheme to Y factors through Y_{red} .

For two morphisms $f : X \rightarrow Y, g : Y \rightarrow Z$, it follows from the uniqueness of factorization that we have

$$(g \circ f)_{\text{red}} = g_{\text{red}} \circ f_{\text{red}};$$

we can equivalently say that the operation $X \mapsto X_{\text{red}}$ is a covariant functor on the category of schemes.

Proposition 18.4.35. *If X and Y are S -schemes, the schemes $X_{\text{red}} \times_{S_{\text{red}}} Y_{\text{red}}$ and $X_{\text{red}} \times_S Y_{\text{red}}$ are identical, and are canonically identified with a subscheme of $X \times_S Y$ having the same underlying space as this product.*

Proof. The fact that $X_{\text{red}} \times_S Y_{\text{red}}$ is identified with a subscheme of $X \times_S Y$ follows from the fact that if $f : X' \rightarrow X$ and $g : Y' \rightarrow Y$ are surjective S -immersions, then $f \times_S g : X' \times_S Y' \rightarrow X \times_S Y$ is a surjective immersion (?? and ??). On the other hand, if $\varphi : X_{\text{red}} \rightarrow S$ and $\psi : Y_{\text{red}} \rightarrow S$ are the structural morphisms, it is clear that they factors through S_{red} , and as $S_{\text{red}} \rightarrow S$ is a monomorphism, the second assertion follows. \square

Corollary 18.4.36. *The schemes $(X \times_S Y)_{\text{red}}$ and $(X_{\text{red}} \times_{S_{\text{red}}} Y_{\text{red}})_{\text{red}}$ are canonically identified.*

We note that if X and Y are reduced S -schemes, it is not necessarily the case that $X \times_S Y$ is reduced, since the tensor product of two reduced algebras (even two fields) may not be reduced.

Corollary 18.4.37. *For any morphism $f : X \rightarrow Y$ of schemes, the diagram ?? factors into*

$$\begin{array}{ccccc} X_{\text{red}} = (X \times_Y Y_{\text{red}})_{\text{red}} & \longrightarrow & X \times_Y Y_{\text{red}} & \longrightarrow & Y_{\text{red}} \\ \downarrow & & \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y & & \end{array} \quad (18.4.2)$$

Proof. For this, we only need to note that $(X \times_Y Y_{\text{red}})_{\text{red}} = (X_{\text{red}} \times_{Y_{\text{red}}} Y_{\text{red}})_{\text{red}} = X_{\text{red}}$. \square

Proposition 18.4.38. *Let X and Y be two schemes. Consider the following properties for a morphism $f : X \rightarrow Y$:*

- (i) *surjective;*
- (ii) *injective;*
- (iii) *open at the point x (resp. open);*
- (iv) *closed;*
- (v) *a homeomorphism onto its image;*
- (vi) *universally open at a point x (resp. universally open);*
- (vii) *universally closed;*
- (viii) *a universal embedding;*
- (ix) *a universal homeomorphism;*
- (x) *radical;*
- (xi) *generalizing at a point x (resp. generalizing);*
- (xii) *universally generalizing at a point x (resp. universally generalizing).*

Then, if \mathcal{P} denote one of the above properties, for f to possess the property \mathcal{P} , it is necessary and sufficient that f_{red} possess \mathcal{P} .

Proof. The proposition is evident for the properties (i), (ii), (iii), (iv), (v), (xi), which only depend on the map of the underlying spaces. For (x), the proposition follows from the fact that the fibers of f and f_{red} at a point $y \in Y$ have the same underlying space and the residue field at a point of X (resp. Y) is the same for X_{red} and Y_{red} . For the properties (vi), (vii), (viii), (ix) and (xii), if f possesses one of these properties, the same is true of f_{red} due to ?? and that the morphism $X_{\text{red}} \rightarrow X \times_Y Y_{\text{red}}$ is a universal homeomorphism. Conversely, if f_{red}

possesses one of these properties, it suffices to note that for any morphism $Y' \rightarrow Y$ we have $(X_{\text{red}} \times_{Y_{\text{red}}} Y'_{\text{red}})_{\text{red}} = (X \times_Y Y')_{\text{red}}$, so the morphism

$$(f_{(Y')})_{\text{red}} : (X \times_Y Y')_{\text{red}} \rightarrow Y'_{\text{red}}$$

possesses the "nonuniversal" version of the same property, and by what we have already seen, $f_{(Y')}$ then has the corresponding property. \square

Proposition 18.4.39. *Let X and Y be two schemes and x be a point of X . Consider the following properties for a morphism $f : X \rightarrow Y$:*

- (i) a monomorphism;
- (ii) an immersion;
- (iii) an open immersion;
- (iv) a closed immersion;
- (v) a local immersion at the point x ;
- (vi) a local isomorphism at the point x ;
- (vii) birational.

Then, if f possesses one of the above properties, f_{red} also possesses that property.

Proof. For the properties (ii) to (vii), the result follows from the observation that $(f_{\text{red}})_x^{\#} : \mathcal{O}_{Y,f(x)} \rightarrow \mathcal{O}_{X,x}$ is surjective (resp. bijective) if $f_x^{\#} : \mathcal{O}_{Y,f(x)} \rightarrow \mathcal{O}_{X,x}$ is surjective (resp. bijective) (??). For (i) it suffices to note that a monomorphism is universal (??), the diagram (??), and the fact that $X_{\text{red}} \rightarrow X \times_Y Y_{\text{red}}$ is a closed immersion, hence a monomorphism. \square

Note that if f_{red} is an immersion, it is not necessarily true that f is. For example, let $Y = \text{Spec}(k)$ where k is a field and $X = \text{Spec}(A)$, where $A = k[T]/(T^2)$. Then the canonical injection $\rho : k \rightarrow k[T]/(T^2)$ corresponds to a morphism $f : X \rightarrow Y$. It is clear that f is not an immersion (in fact any nonzero immersion into Y is automatically closed); but $A_{\text{red}} = k$ so f_{red} is an isomorphism.

Remark 18.4.4. To say that an immersion $f : Y \rightarrow X$ is surjective means it is closed and that the subscheme of X associated with f is defined by an ideal \mathcal{I} contained in the nilradical \mathcal{N}_X . In this case, we say f is a **nilimmersion**; f is then a homeomorphism from Y to X , and f_{red} is an isomorphism from Y_{red} to X_{red} . We say the nilimmersion f is **nilpotent** (resp. **locally nilpotent**) if the ideal \mathcal{I} is nilpotent (resp. locally nilpotent, i.e. that every $x \in X$ has an open neighborhood U such that $\mathcal{I}|_U$ is nilpotent). More precisely, we say f is **nilpotent of order n** if $\mathcal{I}^{n+1} = 0$. If Y is a subscheme of X and f is the canonical immersion, we say X is an **infinitesimal neighborhood** (resp. an **infinitesimal neighborhood of order n**) of Y if f is nilpotent (resp. nilpotent of order n).

18.4.6 Reduced scheme structure on closed subsets

Proposition 18.4.40. *For any locally closed subspace Y of the underlying space of a scheme X , there exists a unique reduced subscheme of X with underlying space Y .*

Proof. The uniqueness is immediate from ??, so we only need to construct a reduced scheme structure on Y . If X is affine with ring A and Y is closed in X , the proposition is immediate: $I(Y)$ is the largest ideal $\mathfrak{a} \subseteq A$ such that $V(\mathfrak{a}) = Y$, and is radical, hence the ring $A/I(Y)$ is reduced, and we can take the scheme structure $(Y, \widetilde{A/I(Y)})$ on Y .

In the general case, for any affine open $U \subseteq X$ such that $U \cap Y$ is closed in U , consider the closed subscheme Y_U of U defined by the quasi-coherent ideal associated with the ideal $I(U \cap Y)$ of $\Gamma(U, \mathcal{O}_X|_U)$, which is reduced. If V is an open affine of X contained in U , then Y_V is induced by Y_U on $V \cap Y$ since this induced scheme is a closed subscheme of V which is reduced and has $V \cap Y$ as underlying space; the uniqueness of Y_V therefore entails our assertion. \square

Corollary 18.4.41. *Let X be a scheme and Y be a locally closed subset of X . Then any point $x \in Y$ admits a maximal generalization y (i.e. y has no further generalization in Y). In particular, if $Y \neq \emptyset$, there exist a maximal element $y \in Y$ under generalization.*

Proof. It suffices to give Y a subscheme structure of X and take y to be the generic point of the irreducible components of Y containing x . \square

Example 18.4.42. Let X be a scheme and x be a closed point of X . Let U be an open neighborhood of x . Then $Z = (X - U) \cup \{x\}$ is a closed subset of X , so we can consider the reduced scheme structure on it. Let \mathcal{I} be the corresponding quasi-coherent ideal of \mathcal{O}_X , we want to determine the stalk \mathcal{I}_x . For this, we can assume that $X = \text{Spec}(A)$ is affine and $X - U = V(\mathfrak{a})$, where \mathfrak{a} is an ideal of A . Then the point x corresponds to a maximal ideal \mathfrak{m} of A such that $\mathfrak{a} \not\subseteq \mathfrak{m}$, and $Z = V(\mathfrak{a}) \cup \{x\} = V(\mathfrak{a} \cap \mathfrak{m})$. By definition, \mathcal{I} is the quasi-coherent ideal on X associated with $I(Z) = \sqrt{\mathfrak{a} \cap \mathfrak{m}}$, and therefore

$$\mathcal{I}_x = (\sqrt{\mathfrak{a} \cap \mathfrak{m}})_{\mathfrak{m}} = \sqrt{(\mathfrak{a} \cap \mathfrak{m})_{\mathfrak{m}}},$$

which is the intersection of prime ideals \mathfrak{p} containing $\mathfrak{a} \cap \mathfrak{m}$ and contained in \mathfrak{m} . But if a prime ideal $\mathfrak{p} \subseteq \mathfrak{m}$ of A contains $\mathfrak{a} \cap \mathfrak{m}$, then by ?? we have $\mathfrak{p} \supset \mathfrak{a}$ or $\mathfrak{p} \supset \mathfrak{m}$. If $\mathfrak{p} \neq \mathfrak{m}$, then this implies $\mathfrak{p} \supset \mathfrak{a}$ and therefore $\mathfrak{m} \supset \mathfrak{p} \supset \mathfrak{a}$, which is a contradiction (since x is not contained in $X - U = V(\mathfrak{a})$). From this, we conclude that $\mathcal{I}_x = \mathfrak{m}_x$.

Example 18.4.43. Let X be a reduced locally Noetherian scheme and X' be a reduced closed subscheme of X with underlying space an irreducible component of X . Then X' is defined by a quasi-coherent ideal \mathcal{I} of \mathcal{O}_X . Let x be the generic point of X' ; we claim that $\mathcal{I}_x = 0$. For this, we can assume that $X = \text{Spec}(A)$ is affine, where A is a reduced Noetherian ring, so $\mathcal{I} = \tilde{\mathfrak{p}}$ where \mathfrak{p} is a minimal prime ideal of A . By definition the stalk of \mathcal{I} at x is identified with $\mathfrak{p}A_{\mathfrak{p}}$, which is the maximal ideal of $A_{\mathfrak{p}} = \mathcal{O}_{X,x}$. But since A is reduced and \mathfrak{p} is minimal, $A_{\mathfrak{p}}$ is a reduced Artinian local ring, whence a field, and we then conclude that $\mathfrak{p}A_{\mathfrak{p}} = 0$, so $\mathcal{I}_x = 0$.

Proposition 18.4.44. *Let X be a reduced scheme, $f : X \rightarrow Y$ be a morphism, and Z be a closed subscheme of Y containing $f(X)$. Then f factors into*

$$X \xrightarrow{g} Z \xrightarrow{j} Y$$

where j is the canonical injection.

Proof. The hypotheses implies that the closed subscheme $f^{-1}(Z)$ of X has underlying space X (??). As X is reduced, this subscheme coincides with X by ??, and the proposition then follows from ??.

Corollary 18.4.45. *Let X be a reduced subscheme of a scheme Y . If Z is the reduced closed subscheme of Y with underlying space \bar{X} , then X is an open subscheme of Z .*

Proof. Since X is locally closed, there is an open set U of Y such that $X = U \cap \bar{X}$. By ??, X is then a reduced subscheme of Z , with underlying space open in Z . Since the scheme structure induced by Z is also reduced, we conclude that X is induced by Z , in view of the uniqueness part of ??.

Corollary 18.4.46. *Let $f : X \rightarrow Y$ be morphism and X' (resp. Y') be a closed subscheme of X (resp. Y) defined by a quasi-coherent ideal \mathcal{I} (resp. \mathcal{K}) of \mathcal{O}_X (resp. \mathcal{O}_Y). Suppose that X' is reduced and $f(X') \subseteq Y'$, then we have $f^*(\mathcal{K})\mathcal{O}_X \subseteq \mathcal{I}$.*

Proof. The restriction of f to X' factors into $X' \rightarrow Y' \rightarrow Y$ by ??, so it suffices to use ??.

18.5 Separated schemes and morphisms

18.5.1 Diagonal and graph of a morphism

Let X be an S -scheme; recall that the diagonal morphism $X \rightarrow X \times_S X$, denoted by $\Delta_{X/S}$ or Δ_X , is the S -morphism $(1_X, 1_X)_S$, which means Δ_X is the unique S -morphism such that

$$p_1 \circ \Delta_X = p_2 \circ \Delta_X = 1_X,$$

where p_1, p_2 are the canonical projections of $X \times_S X$. If $f : T \rightarrow X$ and $g : T \rightarrow Y$ are two S -morphisms, we verify that

$$(f, g)_S = (f \times_S g) \circ \Delta_{T/S}.$$

If $\varphi : X \rightarrow S$ is the structural morphism of X , we also write Δ_φ for $\Delta_{X/S}$.

Proposition 18.5.1. *Let X, Y be S -schemes. If we identify $(X \times_S Y) \times_S (X \times_S Y)$ and $(X \times_S X) \times_S (Y \times_S Y)$, the morphism $\Delta_{X \times Y}$ is identified with $\Delta_X \times \Delta_Y$.*

Proof. In fact, if p_1, q_1 are the canonical projections $X \times_S X \rightarrow X, Y \times_S Y \rightarrow Y$, the projection $(X \times_S Y) \times_S (X \times_S Y) \rightarrow X \times_S Y$ is identified with $p_1 \times q_1$, and we have

$$(p_1 \times q_1) \circ (\Delta_X \times \Delta_Y) = (p_1 \circ \Delta_X) \times (q_1 \circ \Delta_Y) = 1_{X \times Y}$$

similar for the projection to the second factor.

Corollary 18.5.2. *For any extension $S' \rightarrow S$ of base schemes, $\Delta_{X_{(S')}}$ is identified with $(\Delta_X)_{(S')}$.*

Proof. It suffices to remark that $(X \times_S X)_{(S')}$ is identified with $X_{(S')} \times_{S'} X_{(S')}$ canonically.

Proposition 18.5.3. Let X, Y be S -schemes and $S \rightarrow T$ be a morphism. Let $\varphi : X \rightarrow S$, $\psi : Y \rightarrow S$ be the structural morphisms, p, q the projection of $X \times_S Y$, and $\pi = \varphi \circ p = \psi \circ q$ the structural morphism $X \times_S Y \rightarrow S$. Then the diagram

$$\begin{array}{ccc} X \times_S Y & \xrightarrow{(p,q)_T} & X \times_T Y \\ \pi \downarrow & & \downarrow \varphi \times_T \psi \\ S & \xrightarrow{\Delta_{S/T}} & S \times_T S \end{array} \quad (18.5.1)$$

commutes and cartesian.

Proof. By the definition of products, we may prove the proposition in the category of sets, and replace X, Y, S by $X(Z)_T, Y(Z)_T, S(Z)_T$, where Z is an arbitrary T -scheme, and it is then immediate. \square

Corollary 18.5.4. If $f : X \rightarrow Y$ is an S -morphism, the diagram

$$\begin{array}{ccc} X & \xrightarrow{\Gamma_f} & X \times_S Y \\ f \downarrow & & \downarrow f \times_S 1_Y \\ Y & \xrightarrow{\Delta_Y} & Y \times_S Y \end{array} \quad (18.5.2)$$

is commutative and cartesian.

Proof. It suffices to apply ?? with S replaced by Y and T by S , and note that $X \times_Y Y = X$. \square

Proposition 18.5.5. For a morphism $f : X \rightarrow Y$ of schemes to be a monomorphism, it is necessary and sufficient that $\Delta_{X/Y}$ is an isomorphism from X to $X \times_Y X$.

Proof. In fact, f is monic means for any Y -scheme Z , the corresponding map $X(Z)_Y \rightarrow Y(Z)_Y$ is an injection, and as $Y(Z)_Y$ is reduced to a singleton, this means $X(Z)_Y$ is either empty or a singleton. But this is equivalent to saying that $X(Z)_Y \times X(Z)_Y$ is canonically isomorphic to $X(Z)_Y$ via the diagonal map, where the first set is $(X \times_Y X)(Z)_Y$, and this means $\Delta_{X/Y}$ is an isomorphism. \square

Proposition 18.5.6. The diagonal morphism is an immersion from X to $X \times_S X$, and the corresponding subscheme of $X \times_S X$ is called the **diagonal** of $X \times_S X$.

Proof. Let p_1, p_2 be the projections of $X \times_S X$. As the continuous maps p_1 and Δ_X are such that $p_1 \circ \Delta_X = 1_X$, Δ_X is a homeomorphism from X onto $\Delta_X(X)$. Similarly, the composition of the homomorphisms $\mathcal{O}_{X,x} \rightarrow \mathcal{O}_{\Delta_X(X),x} \rightarrow \mathcal{O}_{X,x}$ corresponding to p_1 and Δ_X is the identity, so the homomorphism corresponding to Δ_X on stalks are surjective. The proposition then follows from ?? . \square

Corollary 18.5.7. With the hypotheses of ??, the morphisms $(p, q)_T$ is an immersion.

Proof. This follows from ?? and ??.

Corollary 18.5.8. *Let X and Y be S -schemes and $f : X \rightarrow Y$ be an S -morphism. Then the graph morphism $\Gamma_f : X \rightarrow X \times_S Y$ of f is an immersion.*

Proof. This follows from the diagram (??) and ??.

The subscheme of $X \times_S Y$ associated with the immersion Γ_f is called the **graph** of the morphism f ; the subschemes of $X \times_S Y$ which are graphs of morphisms $X \rightarrow Y$ are characterized by the fact that the restriction of the projection $p_1 : X \times_S Y \rightarrow X$ to such a subscheme G is an isomorphism g from G to X : in fact, if this is the case, G is then the graph of the morphism $p_2 \circ g^{-1}$, where $p_2 : X \times_S Y \rightarrow Y$ is the second projection.

In particular, if $X = S$, the S -morphisms $S \rightarrow Y$, which are none other than the S -sections of Y , are equal to their graph morphisms; the subschemes of Y which are graphs of S -sections (in other words, those which are isomorphic to S by the restriction of the structural morphism $Y \rightarrow S$) are then called the **images of these sections**, or, by abuse of language, the S -sections of Y .

Proposition 18.5.9. *If $f : X \rightarrow Y$ is an S -morphism, the diagram*

$$\begin{array}{ccc} X & \xrightarrow{\Delta_X} & X \times_S X \\ f \downarrow & & \downarrow f \times_S f \\ Y & \xrightarrow{\Delta_Y} & Y \times_S Y \end{array} \quad (18.5.3)$$

is commutative (in other words, Δ_X is a functorial morphism on the category of schemes).

Proof. The morphisms $\Delta_Y \circ f$ satisfies the condition that

$$p_1 \circ (\Delta_Y \circ f) = p_2 \circ (\Delta_Y \circ f) = f$$

where p_1, p_2 are the projections of $Y \times_S Y$. Similarly, if q_1, q_2 are the projections of $X \times_S X$,

$$\begin{aligned} p_1 \circ (f \times_S f) \circ \Delta_X &= f \circ q_1 \circ \Delta_X = f, \\ p_2 \circ (f \times_S f) \circ \Delta_X &= f \circ q_2 \circ \Delta_X = f. \end{aligned}$$

It then follows from the universal property of products that $\Delta_Y \circ f = (f \times_S f) \circ \Delta_X$. □

Corollary 18.5.10. *If X is a subscheme of Y , the diagonal $\Delta_X(X)$ is identified with a subscheme of $\Delta_Y(Y)$ whose the underlying space is identified with*

$$\Delta_Y(Y) \cap p_1^{-1}(X) = \Delta_Y(Y) \cap p_2^{-1}(X)$$

where p_1, p_2 are the projections of $Y \times_S Y$.

Proof. Apply ?? to the immersion $f : X \rightarrow Y$, we see $(f \times_S f)$ is an immersion which identifies $X \times_S X$ with the subspace $p_1^{-1}(X) \cap p_2^{-1}(X)$ of $Y \times_S Y$ (??). Moreover, if $z \in \Delta_Y \cap p_1^{-1}(X)$, we have $z = \Delta_Y(y)$ and $y = p_1(z) \in X$, so $y = f(y)$, and $z = \Delta_Y(f(y))$ belongs to $\Delta_X(X)$ in view of the diagram (??). □

Proposition 18.5.11. *Let $u_1 : X \rightarrow Y$, $u_2 : X \rightarrow Y$ be two S -morphisms. Then the kernel $\ker(u_1, u_2)$ is canonically isomorphic to the inverse image in X of the diagonal $\Delta_Y(Y)$ under the morphism $(u_1, u_2)_S : X \rightarrow Y \times_S Y$.*

Proof. Let $Z \rightarrow X$ be the inverse image of $\Delta_Y(Y)$ under the morphism $(u_1, u_2)_S$. Then if $f : T \rightarrow X$ is an S -morphism such that $u_1 \circ f = u_2 \circ f$, then

$$\begin{aligned} p_1 \circ (u_1, u_2)_S \circ f &= u_1 \circ f = u_2 \circ f = p_1 \circ \Delta_Y \circ u_2 \circ f, \\ p_2 \circ (u_1, u_2)_S \circ f &= u_2 \circ f = p_2 \circ \Delta_Y \circ u_2 \circ f \end{aligned}$$

where p_1, p_2 are the projections of $Y \times_S Y$. We conclude that $(u_1, u_2)_S \circ f = \Delta_Y \circ u_2 \circ f$, and by the definition of Z , the morphism f factors uniquely through Z , which proves our claim. \square

Corollary 18.5.12. *For a point $x \in X$ to belong to $\ker(u_1, u_2)$, it is necessary and sufficient that $u_1(x) = u_2(x) = y$ and the homomorphisms $\kappa(y) \rightarrow \kappa(x)$ corresponding to u_1 and u_2 are equal.*

Proof. In fact, if $j : Z \rightarrow X$ is the kernel of u_1 and u_2 , to say $x \in Z$ signifies that the canonical morphism $h : \text{Spec}(\kappa(x)) \rightarrow X$ factors into $h = j \circ g$, where g is a morphism from $\text{Spec}(\kappa(x))$ to Z . This is equivalent to $u_1 \circ h = u_2 \circ h$, and by ?? to that $u_1(x) = u_2(x)$ and the field extensions $\kappa(y) \rightarrow \kappa(x)$ corresponding to u_1 and u_2 are equal. \square

Proposition 18.5.13. *Let X and Y be S -schemes and $f : X \rightarrow Y$, $g : X \rightarrow Y$ be S -morphisms. Then we have the following commutative diagram*

$$\begin{array}{ccccc} \ker(f, g) & \longrightarrow & X & \xrightarrow{g} & Y \\ \downarrow & & \downarrow \Gamma_g & & \downarrow \Delta_Y \\ X & \xrightarrow{\Gamma_f} & X \times_S Y & \xrightarrow{g \times_S \mathbf{1}_Y} & Y \times_S Y \\ f \downarrow & & \downarrow f \times_S \mathbf{1}_Y & & \\ Y & \xrightarrow{\Delta_Y} & Y \times_S Y & & \end{array}$$

where all squares are cartesian.

Proof. The fact that $\ker(f, g)$ is identified with the kernel of Γ_f and Γ_g can be deduced from by applying the projection $X \times_S Y \rightarrow Y$, or by Yoneda since this is clearly true for sets. The other two small squares are cartesian by ??, so the claim follows from the transitivity of products. \square

Proposition 18.5.14. *Let \mathcal{P} be a property for morphisms of schemes and consider the following conditions:*

- (i) *If $j : Z \rightarrow Y$ is an immersion and $g : Y \rightarrow Z$ is a morphism possessing the property \mathcal{P} , then $g \circ j$ possesses the property \mathcal{P} .*
- (ii) *If $f : X \rightarrow X'$ and $g : Y \rightarrow Y'$ are two S -morphisms possessing the property \mathcal{P} , then $f \times_S g$ possesses the property \mathcal{P} .*
- (iii) *If the composition $g \circ f$ of two morphisms $f : X \rightarrow Y$, $g : Y \rightarrow Z$ possesses the property \mathcal{P} , then f possesses the property \mathcal{P} .*

Then (iii) is a consequence of (i) and (ii).

Proof. The morphism f factors into

$$X \xrightarrow{\Gamma_f} X \times_Z Y \xrightarrow{p_2} Y$$

where p_2 is the projection. On the other hand, p_2 is identified with $(g \circ f) \times_Z 1_Y$, and by (ii) it possesses the property \mathcal{P} ; as Γ_f is an immersion, it then follows from (i) that f possesses \mathcal{P} . \square

Proposition 18.5.15. *Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be two morphisms. Consider the following properties for a morphism:*

- (i) *a monomorphism;*
- (ii) *an immersion;*
- (iii) *a local immersion;*
- (iv) *a universal embedding;*
- (v) *radical.*

Then, if $g \circ f$ possesses one of these properties, so does f .

Proof. The properties (i) and (v) have only been put for memory, because for (i) this is a property valid for any category, and for (v), the proposition has already been proven in ??.

An immersion has each of these properties, and the composition of two morphisms having one (determined) of these properties also possesses it; moreover, all the above properties are stable under products. Thus the claim follows from ??.

Corollary 18.5.16. *Let $j : X \rightarrow Y$ and $g : X \rightarrow Z$ be two S -morphisms. If j possesses one of the properties in ??, so does $(j, g)_S$.*

Proof. In fact, if $p : Y \times_S Z \rightarrow Y$ is the projection, we have $j = p \circ (j, g)_S$, and it suffices to apply ??.

18.5.2 Separated morphisms and schemes

A morphism $f : X \rightarrow Y$ of schemes is called **separated** if the diagonal morphism $\Delta_f : X \rightarrow X \times_Y X$ is a closed immersion. In this case, X is said to be a **separated Y -scheme**, or **separated over Y** . A scheme X is called **separated** if it is separated over \mathbb{Z} . In view of ??, for X to be separated over Y , it is necessary and sufficient that the diagonal is a closed subscheme of $X \times_Y X$.

Proposition 18.5.17. *Any morphism of affine schemes is separated. In particular, any affine scheme is separated.*

Proof. If $f : \text{Spec}(B) \rightarrow \text{Spec}(A)$ is a morphism of affine schemes, where B is an A -algebra, the diagonal morphism corresponds to the ring homomorphism $B \otimes_A B \rightarrow B$ given by $b \otimes b' \mapsto bb'$. Since this is surjective, we conclude that Δ_f is a closed immersion, so f is separated. \square

Proposition 18.5.18. *Let X, Y be S -schemes and $S \rightarrow T$ be a separated morphism. Then the canonical immersion $X \times_S Y \rightarrow X \times_T Y$ in (??) is closed.*

Proof. In fact, in the diagram (??), the diagonal $\Delta_{S/T}$ is a closed immersion, so its base change $\varphi \times_T \psi : X \times_S Y \rightarrow X \times_T Y$ is also closed. \square

Corollary 18.5.19. *Let X, Y be S -schemes and $f : X \rightarrow Y$ be an S -morphism. If Y is separated over S , the graph morphism $\Gamma_f : X \rightarrow X \times_S Y$ is a closed immersion. In particular, any S -section of Y is a closed immersion.*

Proof. This follows from (??) and ??.

Proposition 18.5.20. *Let Y be a separated S -scheme. Then for any S -morphisms $u_1 : X \rightarrow Y, u_2 : X \rightarrow Y$, the kernel of u_1 and u_2 is a closed subscheme of X .*

Proof. Recall that by ?? the kernel is the inverse image of $\Delta_Y(Y)$ under the morphism $(u_1, u_2)_S$. Since $\Delta_Y(Y)$ is closed, it follows from ?? that its inverse image is also closed. \square

Corollary 18.5.21. *Let S be an integral scheme, η its generic point, and X a separated S -scheme. If two S -sections u, v of X satisfy $u(\eta) = v(\eta)$, then $u = v$.*

Proof. In fact, if $x = u(\eta) = v(\eta)$, the corresponding homomorphisms $\kappa(x) \rightarrow \kappa(\eta)$ are necessarily identical, since their composition with the homomorphism $\kappa(\eta) \rightarrow \kappa(x)$ corresponding to the structural morphism $X \rightarrow S$ is the identity on $\kappa(\eta)$. We then deduce from ?? that $\eta \in \ker(u_1, u_2)$, and by hypothesis $\ker(u_1, u_2)$ is a closed subscheme of S (??). As S is reduced and η is its generic point, the unique closed subscheme of S containing η is S (??), so $u = v$. \square

Proposition 18.5.22. *Let \mathcal{P} be a property of morphisms of schemes, and consider the following properties:*

- (i) *If $j : X \rightarrow Y$ is a closed immersion and $g : Y \rightarrow Z$ is a morphism possessing the property \mathcal{P} , then $g \circ j$ possesses the property \mathcal{P} .*
- (ii) *If $f : X \rightarrow X'$ and $g : Y \rightarrow Y'$ are two S -morphisms possessing the property \mathcal{P} , then $f \times_S g$ possesses the property \mathcal{P} .*
- (iii) *If the composition $g \circ f$ of two morphisms $f : X \rightarrow Y, g : Y \rightarrow Z$ possesses the property \mathcal{P} and if g is separated, then f possesses the property \mathcal{P} .*
- (iv) *If $f : X \rightarrow Y$ possesses the property \mathcal{P} , so does f_{red} .*

Then, (iii) and (iv) are consequences of (i) and (ii).

Proof. For the property (iii), the demonstration is similar to ??, with the fact that Γ_f is a closed immersion by ??.

$$\begin{array}{ccc} X_{\text{red}} & \xrightarrow{f_{\text{red}}} & Y_{\text{red}} \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

the vertical morphisms are closed immersions, so we see that (iv) is a consequence of (i) and (iii), observing that a closed immersion is separated in view of the definition and ??.

Proposition 18.5.23. Let $f : X \rightarrow Y$ be a morphism and $g : Y \rightarrow Z$ be a separated morphism. Suppose that $g \circ f$ possesses one of the following properties:

- (i) universally closed;
- (ii) a closed immersion;

Then f possesses the same property.

Proof. In fact, these properties satisfy the conditions (i) and (ii) in ??.

□

Corollary 18.5.24. Let Z be a separated S -scheme and $f : X \rightarrow Y, g : X \rightarrow Z$ be two S -morphisms. If f is universally closed (resp. a closed immersion), so is $(f, g)_S : X \rightarrow Y \times_S Z$.

Proof. The morphism j factors into

$$X \xrightarrow{(f,g)_S} Y \times_S Z \xrightarrow{p} Y$$

and the projection $p : Y \times_S Z \rightarrow Y$ is a separate morphism by ?? (which do not use ??), so it suffices to apply ??.

□

Remark 18.5.1. From the diagram in ??, we conclude that a morphism $Y \rightarrow S$ is separated if and only if the following equivalent conditions holds:

- (i) The diagonal morphism $\Delta_{Y/S}$ is a closed immersion.
- (ii) For every S -scheme X and for any two S -morphisms $f, g : X \rightarrow Y$, the kernel $\ker(f, g)$ is a closed subscheme of X .
- (iii) For every S -scheme X and for any S -morphism $f : X \rightarrow Y$, the graph morphism Γ_f is a closed immersion.

Also, if the conclusion in ?? holds for the morphisms $\Delta_Y : Y \rightarrow Y \times_S Y$ and $p_2 : Y \times_S Y \rightarrow Y$, then Δ_Y is a closed immersion so Y is separated over S .

18.5.3 Criterion of separated morphisms

Proposition 18.5.25 (Properties of Separated Morphisms).

- (i) Any radical morphism (and in particular any monomorphism, hence any immersion) is a separated morphism.
- (ii) The composition of two separated morphisms is separated.
- (iii) If $f : X \rightarrow X'$ and $g : Y \rightarrow Y'$ are two separated S -morphisms, $f \times_S g$ is separated.
- (iv) If $f : X \rightarrow Y$ is a separated S -morphism, the S' -morphism $f_{(S')}$ is separated for any base change $S' \rightarrow S$.
- (v) If the composition $g \circ f$ of two morphisms is separated, then f is separated.
- (vi) For a morphism f to be separated, it is necessary and sufficient that f_{red} is separated.

Proof. A radical morphism, its diagonal morphism is surjective (??), so it is separated. If $f : X \rightarrow Y, g : Y \rightarrow Z$ are two morphisms, the diagram

$$\begin{array}{ccc} X & \xrightarrow{\Delta_{X/Z}} & X \times_Z X \\ & \searrow \Delta_{X/Y} & \swarrow j \\ & X \times_Y X & \end{array}$$

where j is the canonical immersion in (??), is commutative. If f and g are separated, $\Delta_{X/Y}$ is a closed immersion and j is a closed immersion by ??, so $\Delta_{X/Z}$ is closed, which proves (ii). With (i) and (ii), conditions (iii) and (iv) are equivalent, and it suffices to prove (iv). Now by transitivity, $X_{(S')} \times_{Y_{(S')}} X_{(S')}$ is identified with $(X \times_Y X) \times_Y Y_{(S')}$, and the diagonal morphism $\Delta_{X_{(S')}}$ is identified with $\Delta_X \times_Y 1_{Y_{(S')}}$. The assertion then follows from ??.

To prove (v), consider the factorization

$$X \xrightarrow{\Gamma_f} X \times_Z Y \xrightarrow{p_2} Y$$

of f , and note that $p_2 = (g \circ f) \times_Z 1_Y$; the hypothesis that $g \circ f$ is separated implies p_2 is separated by (iii), and as Γ_f is an immersion, Γ_f is separated by (i), hence f is separated by (ii). Finally, for (vi), we recall that the schemes $X_{\text{red}} \times_{Y_{\text{red}}} X_{\text{red}}$ and $X_{\text{red}} \times_Y X_{\text{red}}$ is canonically identified (??); if $j : X_{\text{red}} \rightarrow X$ is the canonical injection, the diagram

$$\begin{array}{ccc} X_{\text{red}} & \xrightarrow{\Delta_{X_{\text{red}}}} & X_{\text{red}} \times_Y X_{\text{red}} \\ j \downarrow & & \downarrow j \times_Y j \\ X & \xrightarrow{\Delta_X} & X \times_Y X \end{array}$$

is commutative, and the assertion follows from the fact that the vertical morphisms are homeomorphisms. \square

Corollary 18.5.26. *If $f : X \rightarrow Y$ is separated, the restriction of f to any subscheme of X is separated.*

Proof. This follows from ??(i) and (iii). \square

Corollary 18.5.27. *If X and Y are S -schemes and Y is separated over S , $X \times_S Y$ is separated over X .*

Proof. This is a particular case of ??(iv). \square

Proposition 18.5.28. *Let X be a scheme and suppose that $(X_i)_{1 \leq i \leq n}$ is a finite covering of X by closed subsets. Let $f : X \rightarrow Y$ be a morphism and for each i let Y_i be a closed subset of Y such that $f(X_i) \subseteq Y_i$. Consider the reduced subscheme structures on each X_i and Y_i and let $f_i : X_i \rightarrow Y_i$ be the restriction of f on X_i . Then for f to be separated, it is necessary and sufficient that each f_i is separated.*

Proof. The necessity follows from ??(i), (ii) and (v). Conversely, if each f_i is separated, then the restriction $X_i \rightarrow Y$ of f is separated (??). If p_1, p_2 are the projections of $X \times_Y X$, the subspace $\Delta_{X_i}(X_i)$ is identified with the subspace $\Delta_X(X) \cap p_1^{-1}(X_i)$ of $X \times_Y X$ (??). This subspace is closed in $X \times_Y X$ by hypothesis, and their union is $\Delta_X(X)$, so Δ_X is closed and f is separated. \square

Suppose in particular that X_i are the irreducible components of X ; then we can suppose that each Y_i is a irreducible closed subset of Y (??); ?? then enable us to reduce the separation problem to integral schemes.

Proposition 18.5.29. *Let (Y_λ) be an open covering of a scheme Y . For a morphism $f : X \rightarrow Y$ to be separated, it is necessary and sufficient that each of the restriction $f_\lambda : f^{-1}(Y_\lambda) \rightarrow Y_\lambda$ is separated.*

Proof. If we set $X_\lambda = f^{-1}(Y_\lambda)$ and identify $X_\lambda \times_Y X_\lambda$ and $X_\lambda \times_{Y_\lambda} X_\lambda$, the $X_\lambda \times_Y X_\lambda$ form an open covering of $X \times_Y X$. If $Y_{\lambda\mu} = Y_\lambda \cap Y_\mu$ and $X_{\lambda\mu} = X_\lambda \cap X_\mu = f^{-1}(Y_{\lambda\mu})$, then $X_\lambda \times_Y X_\mu$ is identified with $X_{\lambda\mu} \times_{Y_{\lambda\mu}} X_{\lambda\mu}$ by ??, hence with $X_{\lambda\mu} \times_Y X_{\lambda\mu}$, and finally to an open subset of $X_\lambda \times_Y X_\mu$, which establishes our assertion (??). \square

?? allows, by taking a covering of Y by open affines, to reduce the study of separated morphisms to separated morphisms with values in affine schemes.

Proposition 18.5.30. *Let Y be an affine scheme, X be a scheme, and (U_α) be an affine open covering of X . For a morphism $f : X \rightarrow Y$ to be separated, it is necessary and sufficient that, for any couple of indices (α, β) , the intersection $U_\alpha \cap U_\beta$ is affine, and the ring $\Gamma(U_\alpha \cap U_\beta, \mathcal{O}_X)$ is generated by the images of $\Gamma(U_\alpha, \mathcal{O}_X)$ and $\Gamma(U_\beta, \mathcal{O}_X)$.*

Proof. The open sets $U_\alpha \times_Y U_\beta$ form an open cover of $X \times_Y X$ (??). Let p, q be the projections of $X \times_Y X$, we have

$$\Delta_X^{-1}(U_\alpha \times_Y U_\beta) = \Delta_X^{-1}(p^{-1}(U_\alpha) \cap q^{-1}(U_\beta)) = U_\alpha \cap U_\beta.$$

It therefore amounts to show that the restriction of Δ_X to $U_\alpha \cap U_\beta$ is a closed immersion into $U_\alpha \times_Y U_\beta$. But this restriction is just $(j_\alpha, j_\beta)_Y$, where j_α (resp. j_β) is the canonical injection of $U_\alpha \cap U_\beta$ to U_α (resp. to U_β). As $U_\alpha \times_Y U_\beta$ is an affine scheme with ring isomorphic to $\Gamma(U_\alpha, \mathcal{O}_X) \otimes_{\Gamma(Y, \mathcal{O}_Y)} \Gamma(U_\beta, \mathcal{O}_X)$, we see that $U_\alpha \cap U_\beta$ is a closed subscheme of $U_\alpha \times_Y U_\beta$ if and only if it is affine and the ring homomorphism

$$\Gamma(U_\alpha, \mathcal{O}_X) \otimes_{\Gamma(Y, \mathcal{O}_Y)} \Gamma(U_\beta, \mathcal{O}_X) \rightarrow \Gamma(U_\alpha \cap U_\beta, \mathcal{O}_X), \quad h_\alpha \otimes h_\beta \mapsto h_\alpha h_\beta$$

is surjective (??), which proves our assertion. \square

Corollary 18.5.31. *Let Y be an affine scheme. For a morphism $f : X \rightarrow Y$ to be separated, it is necessary and sufficient that X is separated.*

Proof. In fact, the criterion in ?? does not depend on f . \square

Corollary 18.5.32. *For a morphism $f : X \rightarrow Y$ to be separated, it is necessary that for any open affine subscheme U that is separated, the open subscheme $f^{-1}(U)$ is separated, and it suffices that this ture for every affine open subset $U \subseteq Y$.*

Proof. The necessity follows from ?? and ??(ii). The sufficiency follows from ?? and ?. \square

Proposition 18.5.33. *Let Y be a separated scheme and $f : X \rightarrow Y$ be a morphism. For any affine open U of X and any affine open V of Y , $U \cap f^{-1}(V)$ is affine.*

Proof. Let p_1, p_2 be the projections of $X \times_{\mathbb{Z}} Y$. Using the universal property of Γ_f , the subspace $U \cap f^{-1}(V)$ can be characterized by

$$\Gamma_f(U \cap f^{-1}(V)) = \Gamma_f(X) \cap p_1^{-1}(U) \cap p_2^{-1}(V)$$

Now $p_1^{-1}(U) \cap p_2^{-1}(V)$ is identified with the product $U \times_{\mathbb{Z}} V$, and therefore is affine; as $\Gamma_f(X)$ is closed in $X \times_{\mathbb{Z}} Y$??, $\Gamma_f(X) \cap p_1^{-1}(U) \cap p_2^{-1}(V)$ is closed in $U \times_{\mathbb{Z}} V$, hence also affine. The assertion then follows from the fact that Γ_f is a closed immersion and ?? \square

Example 18.5.34. The scheme in ?? is separated. In fact, for the covering (X_1, X_2) of X by affine opens, $X_1 \cap X_2 = U_{12}$ is affine and $\Gamma(U_{12}, \mathcal{O}_X)$, the fraction ring of the form $f(s)/s^m$ where $f \in K[s]$, is generated by $K[s]$ and $1/s$, so the conditions in ?? are satisfied.

With the same choice of X_1, X_2, U_{12} and U_{21} as in ??, take this time for u_{12} the isomorphism which sends $f(s)$ to $f(t)$; this time we obtain by gluing together a non-separated integral scheme X , because the first condition of ?? holds, but the second fails. It is immediate here that $\Gamma(X, \mathcal{O}_X) \rightarrow \Gamma(X_1, \mathcal{O}_X) = K[s]$ is an isomorphism; the inverse isomorphism defines a morphism $f : X \rightarrow \text{Spec}(K[s])$ which is surjective, and for any $y \in \text{Spec}(K[s])$ such that $\mathfrak{p}_y \neq (0)$, $f^{-1}(y)$ is reduced to a singleton, but for $\mathfrak{p}_y = (0)$, $f^{-1}(y)$ consists of two distinct points (we say that X is the "affine line on K , where the point 0 is doubled").

We can also give examples where neither of the two conditions of ?? does not hold. Note first that in the prime spectrum Y of the ring of polynomials $A = K[s, t]$ in two indeterminates over a field K , the open set $U = D(s) \cup D(t)$ is not an affine open set. Indeed, if z is a section of \mathcal{O}_Y over U , there exist two integers $m, n \geq 0$ such that $s^m z$ and $t^n z$ are the restrictions to U of polynomials in s and t ??, which is obviously only possible if the section z extends into a section over the entire space Y , identified with a polynomial in s and t . If U were affine, the canonical injection $U \rightarrow Y$ would therefore be an isomorphism by ??, which is absurd since $U \neq Y$.

This being so, let us take two affine schemes Y_1, Y_2 , with rings $A_1 = K[x_1, t_1], A_2 = K[s_2, t_2]$. Let $U_{12} = D(s_1) \cup D(t_1)$, $U_{21} = D(s_2) \cup D(t_2)$, and let u_{12} be the restriction to U_{21} of the isomorphism $Y_2 \rightarrow Y_1$ corresponding to the isomorphism of rings, which sends $f(s_1, t_1)$ to $f(s_2, t_2)$. We thus obtain an example where none of the conditions of ?? is satisfied (the integral scheme thus obtained is called "affine plane over K , where point 0 is doubled").

18.6 Finiteness conditions for morphisms

We study, in this section, various "finiteness conditions" for a morphism $f : X \rightarrow Y$ of schemes. There are basically two notions of "global finiteness" on X : quasi-compactness and quasi-separateness. On the other hand, there are two notions of "local finiteness" on X : locally of finite type and locally of finite presentation. By combining these local notions with the previous global notions, we obtain the notions of morphism of finite type and of morphism of finite presentation.

18.6.1 Quasi-compact and quasi-separated morphisms

We say a morphism $f : X \rightarrow Y$ is **quasi-compact** if for any quasi-compact open subset U of Y , the inverse image $f^{-1}(U)$ is quasi-compact. It is clear that this condition is purely

topological, and if X is Noetherian, then any morphism $f : X \rightarrow Y$ is quasi-compact. We say a Y -scheme X is **quasi-compact over Y** if its structural morphism is quasi-compact.

If \mathcal{B} is a base of Y formed by quasi-compact open sets (for example, affine opens), for a morphism f to be quasi-compact, it is necessary and sufficient that for any open set $V \in \mathcal{B}$, $f^{-1}(V)$, since any quasi-compact open set of Y is a finite union of open sets in \mathcal{B} .

If $f : X \rightarrow Y$ is a quasi-compact morphism, it is clear that for any open set V of Y , the restriction $f^{-1}(V) \rightarrow V$ of f is quasi-compact. Conversely, if (U_α) is an open covering of Y and $f : X \rightarrow Y$ is a morphism such that the restriction $f^{-1}(U_\alpha) \rightarrow U_\alpha$ is quasi-compact, then f is quasi-compact, since there exist a basis of quasi-compact open sets for Y , each set of which is contained in at least one of the U_α . We conclude that if $f : X \rightarrow Y$ is an S -morphism of S -schemes, and if there is an open covering (S_λ) of S such that the restrictions $\varphi^{-1}(S_\lambda) \rightarrow \psi^{-1}(S_\lambda)$ of f (where $\varphi : X \rightarrow S$ and $\psi : Y \rightarrow S$ are the structural morphisms) are quasi-compact morphisms, then f is quasi-compact.

Proposition 18.6.1. *Let Y be a separated scheme. Then for a morphism $f : X \rightarrow Y$ to be quasi-compact, it is necessary and sufficient that X is quasi-compact.*

Proof. If X is quasi-compact, it is a union of finitely many affine opens U_i , and for any affine open V of Y , $U_i \cap f^{-1}(V)$ is an affine open by ??, hence quasi-compact; therefore $f^{-1}(V)$ is quasi-compact. Conversely, if f is a quasi-compact morphism, then since Y is quasi-compact open in Y , we see $X = f^{-1}(Y)$ is also quasi-compact. \square

Example 18.6.2. A closed immersion is quasi-compact since a closed subset of a quasi-compact set is again quasi-compact. However, open immersions are in general not quasi-compact: the standard example is the affine scheme $X = \text{Spec}(k[x_1, x_2, \dots])$ and consider $U = X - \{0\}$, where 0 is the point of X corresponding to the maximal ideal (x_1, x_2, \dots) . The canonical injection $j : U \rightarrow X$ is not quasi-compact because U is not quasi-compact. To see this, consider the covering $(D(x_i))_{i \in \mathbb{N}}$ of U ; for any finite subset J of \mathbb{N} , the family $(D(x_i))_{i \in J}$ can not cover U simply because the prime ideal \mathfrak{p}_J generated by x_i with $i \in J$ is contained in U but not in the union of the $D(x_i)$ for $i \in J$.

We say a morphism $f : X \rightarrow Y$ of schemes is **quasi-separated** (of X is an Y -scheme **quasi-separated over Y**) if the diagonal morphism $\Delta_f : X \rightarrow X \times_Y X$ is quasi-compact. A scheme X is called **quasi-separated** if it is quasi-separated over \mathbb{Z} . Since a closed immersion is quasi-compact, we see any separated morphism is quasi-separated. In particular, any separated scheme is quasi-separated.

Proposition 18.6.3 (Properties of Quasi-Compact Morphisms).

- (i) *An immersion $j : X \rightarrow Y$ is quasi-compact if it is closed, or Y is locally Noetherian, or X is Noetherian.*
- (ii) *The composition of two quasi-compact morphisms is quasi-compact.*
- (iii) *If $f : X \rightarrow Y$ is a quasi-compact S -morphism, $f_{(S')} : X_{(S')} \rightarrow Y_{(S')}$ is quasi-compact for any base change $g : S' \rightarrow S$.*
- (iv) *If $f : X \rightarrow X'$ and $g : Y \rightarrow Y'$ are two quasi-compact S -morphisms, then $f \times_S g$ is quasi-compact.*

(v) If the composition $g \circ f$ of two morphisms $f : X \rightarrow Y, g : Y \rightarrow Z$ is quasi-compact, and if f is quasi-separated or if X is locally Noetherian, then f is quasi-compact.

(vi) For a morphism f to be quasi-compact, it is necessary and sufficient that f_{red} is quasi-compact.

Proof. Assertion (vi) is evident, since the quasi-compactness for a morphism only depends on the map of the underlying spaces. Also, (ii) follows from the definition of quasi-compactness.

The assertion in (i) is clear if j is closed; if j is an immersion and Y is locally Noetherian, any quasi-compact open V of Y is Noetherian, so $j^{-1}(V) = X \cap V \subseteq V$ is quasi-compact (here we identify X as a subscheme of Y). If X is Noetherian, then any morphism from X is quasi-compact.

To prove (iii), we can assume that $S = Y$ by the transitivity of products; put $f' = f_{(S')}$, and let U' be a quasi-compact open subset of S' . For any $s' \in U'$, let T be an affine open neighborhood of $g(s')$ in S , and let W be an affine open neighborhood of s' contained in $U' \cap g^{-1}(T)$; it suffices to show that $f'^{-1}(W)$ is quasi-compact, or in other words, we only need to show that if S and S' are affine, then $X \times_S S'$ is quasi-compact. This is true because by hypothesis X is a finite union of affine opens V_j , and $X \times_S S'$ is then the union of the affine schemes $V_j \times_S S'$, hence quasi-compact. With (ii) and (iii), assertion (iv) then follows.

We now prove (v) in the case where X is locally Noetherian. Put $h = g \circ f$ and let U be a quasi-compact open of Y ; $g(U)$ is then quasi-compact in Z (not necessarily open), so it is contained in a finite union of quasi-compact opens V_j , and $f^{-1}(U)$ is contained in the union of the $h^{-1}(V_j)$, which are all quasi-compact by hypothesis. We then conclude that $f^{-1}(U)$ is a Noetherian space (??), and a fortiori quasi-compact.

To prove (v) in the case that g is quasi-separated, recall that f factors into

$$X \xrightarrow{\Gamma_f} X \times_Z Y \xrightarrow{p_2} Y$$

where p_2 is identified with $(g \circ f) \times_Z 1_Y$, and if $g \circ f$ is quasi-compact, so is p_2 by (iii). Finally, we have the following cartesian square (??)

$$\begin{array}{ccc} X & \xrightarrow{\Gamma_f} & X \times_Z Y \\ f \downarrow & & \downarrow f \times 1_Y \\ Y & \xrightarrow{\Delta_g} & Y \times_Z Y \end{array}$$

as by hypothesis Δ_g is quasi-compact, Γ_f is also quasi-compact, and by (ii) we conclude that f is quasi-compact. \square

Proposition 18.6.4. Let $f : X \rightarrow Y, g : Y \rightarrow Z$ be two morphisms. If $g \circ f$ is quasi-compact and f is surjective, then g is quasi-compact.

Proof. If fact, if V is a quasi-compact open of Z , $f^{-1}(g^{-1}(V))$ is quasi-compact by hypothesis, and we have $g^{-1}(V) = f(f^{-1}(g^{-1}(V)))$ since f is surjective, so $g^{-1}(V)$ is quasi-compact. \square

Proposition 18.6.5. Let f be a quasi-compact morphism of schemes.

(a) The following conditions are equivalent:

- (i) f is dominant;
 - (ii) for any maximal point $y \in Y$, $f^{-1}(y) \neq \emptyset$.
 - (iii) for any maximal point $y \in Y$, $f^{-1}(y)$ contains a maximal point of X .
- (b) If f is dominant, for any generalizing morphism $g : Y' \rightarrow Y$, the morphism $f_{(Y')} : X_{(Y')} \rightarrow Y'$ is quasi-compact and dominant.

Proof. It is clear that (iii) \Rightarrow (ii) \Rightarrow (i). Suppose that f is dominant and consider an affine open neighborhood U of y ; $f^{-1}(U)$ is quasi-compact, hence a union of finitely many affine opens V_i , and by hypothesis y belongs to the closure of $f(V_i)$ in U . We can evidently suppose that X and Y are reduced. As the closure in X of an irreducible component of V_i is an irreducible component of X (??), we can replace X by V_i , Y by the closed reduced subscheme $\overline{f(V_i)} \cap U$ of U , and we are thus reduced to proving the proposition when $X = \text{Spec}(B)$ and $Y = \text{Spec}(A)$ are affine and reduced. Since f is dominant, A is then a subring of B (??); the proposition then follows from the fact that any minimal prime ideal of A is the intersection with A of a minimal prime ideal of B (??).

If $f : X \rightarrow Y$ is quasi-compact and dominant, then $f' = f_{(Y')}$ is quasi-compact by ?? . On the other hand, a maximal point y' of Y' is hypothesis lying over a maximal point y of Y (??); as $f^{-1}(y)$ is nonempty by (i), the same holds for $f'^{-1}(y')$ (??), whence the conclusion. \square

Proposition 18.6.6. For a quasi-compact morphism $f : X \rightarrow Y$, the following conditions are equivalent:

- (i) The morphism f is closed.
- (ii) For any $x \in X$ and any specialization y' of $y = f(x)$ distinct from y , there exists a specialization x' of x such that $f(x') = y'$.

In particular, if $f : X \rightarrow Y$ is a quasi-compact immersion, for f to be a closed immersion, it is necessary and sufficient that X (considered as a subspace of Y) contains any specializations (in Y) of its points.

Proof. The condition (ii) expresses as $f(\overline{\{x\}}) = \overline{\{y\}}$, and is therefore a consequence of (i). To show that (ii) implies (i), consider a closed subset of X' of X ; let $Y' = \overline{f(X')}$ and we prove that $Y' = f(X')$. Endow X' and Y' the reduced subscheme structure, there then exists a morphism $f' : X' \rightarrow Y'$ such that the diagram

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

commutes. As f is quasi-compact, so is f' (??(i) and (v)). We are then reduced to proving that if f is a quasi-compact dominant morphism, then $f(X) = Y$. Now let y' be a point of y and let y be the generic point of a irreducible component of Y containing y' ; by (ii), it suffices to note that $f^{-1}(y)$ is nonempty, which follows from ?? . \square

Proposition 18.6.7 (Properties of Quasi-Separated Morphisms).

- (i) Any radical morphism $f : X \rightarrow Y$ (in particular, any monomorphism and any immersion) is quasi-separated.

- (ii) If $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are two quasi-separated morphisms, $g \circ f$ is quasi-separated.
- (iii) Let X, Y be two S -schemes and $f : X \rightarrow Y$ be a quasi-separated S -morphism. Then, for any base change $g : S' \rightarrow S$, the morphism $f_{(S')} : X_{(S')} \rightarrow Y_{(S')}$ is quasi-separated.
- (iv) If $f : X \rightarrow X'$, $g : Y \rightarrow Y'$ are two quasi-separated S -morphisms, $f \times_S g$ is quasi-separated.
- (v) If the composition $g \circ f$ of two morphisms $f : X \rightarrow Y$, $g : Y \rightarrow Z$ is quasi-separated, then f is quasi-separated; if moreover f is quasi-compact and surjective, g is also quasi-separated.
- (vi) For a morphism $f : X \rightarrow Y$ to be quasi-separated, it is necessary and sufficient that f_{red} is quasi-separated.

Proof. The assertion (i) follows from ??(i). To prove (iii), we may reduce to the case $Y = S$, and the assertion then follows from $\Delta_{f_{(S')}} = (\Delta_f)_{(S')}$?? and ??.

For assertion (ii), consider the projections p, q of $X \times_Y X$; if $\pi : X \times_Y X \rightarrow Y$ is the structural morphism and $j = (p, q)_Z$, we have the following cartesian square ??

$$\begin{array}{ccc} X \times_Y X & \xrightarrow{j} & X \times_Z X \\ \pi \downarrow & & \downarrow f \times_Z f \\ Y & \xrightarrow{\Delta_g} & Y \times_Z Y \end{array}$$

If g is quasi-separated then Δ_g is quasi-compact, so j is also quasi-compact by ??(iii). If f is quasi-separated, Δ_f is quasi-compact and so is $j \circ \Delta_f$, which equals to $\Delta_{g \circ f}$. With these, assertion (iv) then follows from (ii) and (iii).

Suppose now that $g \circ f$ is quasi-separated. Then with the preceding notations, $\Delta_{g \circ f} = j \circ \Delta_f$ is quasi-compact, so Δ_f is quasi-compact by ??(v) and f is then quasi-separated. If moreover f is quasi-compact and surjective, $f \times_Z f$ is also quasi-compact by ??(iv), and we conclude that $\Delta_g \circ \pi \circ \Delta_f$ is quasi-compact. Since $\pi \circ \Delta_f = f$ is surjective, it follows from ??(v) that Δ_g is quasi-compact, so g is quasi-separated.

Finally, for a morphism $f : X \rightarrow Y$, consider the following diagram

$$\begin{array}{ccc} X_{\text{red}} & \xrightarrow{f_{\text{red}}} & Y_{\text{red}} \\ j_X \downarrow & & \downarrow j_Y \\ X & \xrightarrow{f} & Y \end{array}$$

where j_X and j_Y are surjective closed immersions, and so quasi-separated and quasi-compact. From the equality $f \circ j_X = j_Y \circ f_{\text{red}}$ and (v), we see f is quasi-separated if and only if f_{red} is quasi-separated. \square

Corollary 18.6.8. Let X and Y be schemes.

- (i) If f is quasi-separated, any morphism $f : X \rightarrow Y$ is quasi-separated.
- (ii) If Y is quasi-separated, for a morphism $f : X \rightarrow Y$ to be quasi-separated, it is necessary and sufficient that the scheme X is quasi-separated.

(iii) Let X be a quasi-compact and Y be quasi-separated. Then any morphism $f : X \rightarrow Y$ is quasi-compact.

Proof. To show (i) we only need to note that any morphism $f : X \rightarrow Y$ is a \mathbb{Z} -morphism, and if X is quasi-separated, then for any morphism $f : X \rightarrow Y$ the composition $X \rightarrow Y \rightarrow \text{Spec}(\mathbb{Z})$ is quasi-separated, so f is quasi-separated by ??(v). Similarly, assertion (ii) follows from ??(ii) and (v). Assertion (iii) follows from ??(v). \square

Proposition 18.6.9. *Let $f : X \rightarrow Y$ be a morphism and (U_α) be a covering of Y by open subschemes that are quasi-separated. For f to be quasi-separated, it is necessary and sufficient that the open subschemes $f^{-1}(U_\alpha)$ is quasi-separated.*

Proof. The inverse image of U_α in $X \times_Y X$ is $X_\alpha \times_{U_\alpha} X_\alpha$, where $X_\alpha = f^{-1}(U_\alpha)$, and the restriction $X_\alpha \rightarrow X_\alpha \times_{U_\alpha} X_\alpha$ of Δ_f is just Δ_{f_α} , where f_α is the restriction $X_\alpha \rightarrow U_\alpha$ of f . Since quasi-compactness is local on target, we see f is quasi-separated if and only if each f_α is. But by hypothesis U_α is separated, so the conclusion follows from ??(ii). \square

By ??, to verify a morphism is quasi-separated, it suffices to verify the quasi-separateness of some subschemes. This can be done by the following simple criteria:

Proposition 18.6.10. *Let X be a scheme and (U_α) be a covering of X formed by quasi-compact open subsets. Then the following conditions are equivalent:*

- (i) X is a quasi-separated scheme.
- (ii) For any quasi-compact open subset U of X , the canonical injection $U \rightarrow X$ is quasi-compact (that is, U is retrocompact in X).
- (iii) The intersection of two quasi-compact open subsets of X is quasi-compact.
- (iv) For any couple of indices (α, β) , the intersection $U_\alpha \cap U_\beta$ is quasi-compact.

Proof. Properties (ii) and (iii) are equivalent by the definition of quasi-compactness. As a quasi-compact open is a finite union of affine open sets, for two quasi-compact open subsets U, V of X , $U \times_{\mathbb{Z}} V$ is a quasi-compact open subset of $X \times_{\mathbb{Z}} X$ (??), with inverse image $U \cap V$ under Δ_X , hence (i) implies (iii). It is clear that (iii) implies (iv); finally, if (iv) holds, the $U_\alpha \times_{\mathbb{Z}} U_\beta$ form a covering of $X \times_{\mathbb{Z}} X$ by quasi-compact open sets and the inverse image of $U_\alpha \times_{\mathbb{Z}} U_\beta$ under Δ_X is $U_\alpha \cap U_\beta$, hence quasi-compact. It then follows that Δ_X is quasi-compact, so (iv) implies (i). \square

Corollary 18.6.11. *Any locally Noetherian scheme X is quasi-separated, and any morphism $f : X \rightarrow Y$ is then quasi-separated.*

Proof. It suffices to note that any quasi-compact open subset of X is Noetherian, so X is quasi-separated by ?? and any morphism $f : X \rightarrow Y$ is quasi-separated by ??, since any open subscheme of X is again locally Noetherian. \square

Proposition 18.6.12. *Let $f : X \rightarrow Y$ be a morphism and $g : Y' \rightarrow Y$ be a base change that is surjective and quasi-compact. Put $f' = f_{(Y')}$ and consider the following properties:*

- (i) quasi-compact;

(ii) quasi-separated.

Then if \mathcal{P} denotes one of these properties and f' possesses the property \mathcal{P} , then f possesses the property \mathcal{P} .

Proof. Let $g' : X' \rightarrow X$ be the canonical projection, which is surjective and quasi-compact (?? and ??(iii)). If f' is quasi-compact, so is $g \circ f'$ and since $f \circ g' = g \circ f'$ we conclude that f is quasi-compact by ??(v).

Now assume that f' is quasi-separated. We have $X' \times_{Y'} X' = (X \times_Y X)_{(Y')}$ and $\Delta_{f'} = (\Delta_f)_{(Y')}$. The projection $X' \times_{Y'} X' \rightarrow X \times_Y X$ is quasi-compact and surjective by the same reasoning, and we can apply (i) to the morphism Δ_f . Since by hypothesis $\Delta_{f'}$ is quasi-compact, we conclude that Δ_f is quasi-compact, so f is quasi-separated. \square

Proposition 18.6.13. *Let $f_i : X_i \rightarrow Y$ be a finite family of morphisms and $f : X \rightarrow Y$ be their coproduct. Then for f to be quasi-compact (resp. quasi-separated), it is necessary and sufficient that each f_i is quasi-compact (resp. quasi-separated).*

Proof. The assertion about quasi-compactness follows from definition. We also note that $X \times_Y X$ is the coproduct of that $X_i \times_Y X_j$, and Δ_f is the morphism that coincides with Δ_{f_i} on each X_i , so the assertion for quasi-separatedness also follows. \square

Theorem 18.6.14. *Let X be a quasi-compact and quasi-separated scheme, \mathcal{L} be an invertible \mathcal{O}_X -module, s be a section over X , X_s the open subset of $x \in X$ such that $s(x) \neq 0$, and \mathcal{F} be a quasi-coherent \mathcal{O}_X -module.*

- (a) *If $t \in \Gamma(X, \mathcal{F})$ is such that $t|_{X_s} = 0$, there exists $n > 0$ such that $t \otimes s^{\otimes n} = 0$.*
- (b) *For any section $t \in \Gamma(X_s, \mathcal{F})$, there exists an integer $n > 0$ such that $t \otimes s^{\otimes n}$ can be extended to a section $t' \in \Gamma(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n})$.*

Proof. As the space X is a finite union of affine opens U_i such that $\mathcal{L}|_{U_i}$ is isomorphic to $\mathcal{O}_X|_{U_i}$, we can assume that X is affine and $\mathcal{L} = \mathcal{O}_X$. The assertion (a) then follows from ??(iv).

Now let t be a section of \mathcal{F} over X_s . Since $\mathcal{L}|_{U_i}$ is isomorphic to $\mathcal{O}_X|_{U_i}$, the restriction $(t \otimes s^{\otimes n})|_{U_i \cap X_s}$ is identified with $(t|_{U_i \cap X_s})(s|_{U_i \cap X_s})^n$ under the isomorphism $(\mathcal{F} \otimes \mathcal{L}^{\otimes n})|_{U_i} \cong \mathcal{F}|_{U_i}$. We then conclude from ??(iv) that there exists an integer $n \geq 0$ such that for each i , $(t \otimes s^{\otimes n})|_{U_i \cap X_s}$ extends to a section t_i of $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ over U_i . Let t_{ij} be the restriction of t_i to $U_i \cap U_j$; by definition we have $(t_{ij} - t_{ji})|_{X_s \cap U_i \cap U_j} = 0$. Since X is quasi-separated, $U_i \cap U_j$ is quasi-compact, so by ??(iv) there exists an integer $m \geq 0$ such that $(t_{ij} - t_{ji}) \otimes s^{\otimes m} = 0$. The sections $t_i \otimes s^{\otimes m}$ then glue together to give a section $t' \in \Gamma(X, \mathcal{F} \otimes \mathcal{L}^{\otimes(n+m)})$, which induces $t_i \otimes s^{\otimes m}$ over each U_i , and hence induces $t \otimes s^{\otimes(n+m)}$ over X_s . \square

Corollary 18.6.15. *With the hypotheses of ??, consider the ring $A = \Gamma_*(\mathcal{L})$ and the graded A -module $M = \Gamma_*(\mathcal{F}; \mathcal{L})$ of type \mathbb{Z} . Then for each $s \in A_n$, there exists a canonical isomorphism $\Gamma(X_s, \mathcal{F}) \cong M_{(s)}$, where $M_{(s)} = (M_s)_0$ is the degree zero part of the localization M_s .*

Proof. With the notations of ??(b), we see that any element $t \in \Gamma(X_s, \mathcal{F})$ corresponds to an element t'/s^n in $M_{(s)}$, which is independent of the integer n and the chosen extension t' , in view of ??(a). It is immediate that this defines a homomorphism, and is bijective. \square

Corollary 18.6.16. *Let X be a quasi-compact and quasi-separated scheme and \mathcal{F} a quasi-coherent \mathcal{O}_X -module. Then if $A = \Gamma(X, \mathcal{O}_X)$ and $M = \Gamma(X, \mathcal{F})$, the A_s -module $\Gamma(X_s, \mathcal{F})$ is canonically isomorphic to M_s .*

Proof. This is a special case of ??, by taking $\mathcal{L} = \mathcal{O}_X$. \square

Proposition 18.6.17. *Let X be a quasi-compact scheme, \mathcal{F} be a quasi-coherent \mathcal{O}_X -module of finite type, and \mathcal{I} be a quasi-coherent ideal of \mathcal{O}_X of finite type such that $\text{supp}(\mathcal{F})$ is contained in $\text{supp}(\mathcal{O}_X/\mathcal{I})$. Then there exists an integer $n \geq 0$ such that $\mathcal{I}^n \mathcal{F} = 0$.*

Proof. As X is a finite union of affine open subsets, we may assume that $X = \text{Spec}(A)$ is affine. Then $\mathcal{F} = \widetilde{M}$ and $\mathcal{I} = \widetilde{\mathfrak{a}}$, where M is a finitely generated A -module and \mathfrak{a} is a finitely generated ideal of A , and

$$\text{supp}(\mathcal{F}) = \text{supp}(M) = V(\text{Ann}(M)), \quad \text{supp}(\mathcal{O}_X/\mathcal{I}) = \text{supp}(A/\mathfrak{a}) = V(\mathfrak{a}).$$

By hypothesis we have $V(\text{Ann}(M)) \subseteq V(\mathfrak{a})$, so $\mathfrak{a} \subseteq \sqrt{\text{Ann}(M)}$. Since \mathfrak{a} is finitely generated, there exists an integer $n \geq 0$ such that $\mathfrak{a}^n \subseteq \text{Ann}(M)$, and therefore $\mathcal{I}^n \mathcal{F} = \widetilde{\mathfrak{a}^n M} = 0$. \square

Corollary 18.6.18. *Under the hypothesis of ??, there exists a closed subscheme Y of X with underlying space $\text{supp}(\mathcal{O}_X/\mathcal{I})$ such that, if $j : Y \rightarrow X$ is the canonical injection, we have $\mathcal{F} = j_*(j^*(\mathcal{F}))$.*

Proof. Note that the support of $\mathcal{O}_X/\mathcal{I}$ and $\mathcal{O}_X/\mathcal{I}^n$ are the same because if $\mathcal{I}_x = \mathcal{O}_{X,x}$, we also have $\mathcal{I}_x^n = \mathcal{O}_{X,x}$, and on the other hand we have $\mathcal{I}_x^n \subseteq \mathcal{I}_x$ for each $x \in X$. In view of ??, we may then suppose that $\mathcal{I}\mathcal{F} = 0$, so \mathcal{F} is also an $(\mathcal{O}_X/\mathcal{I})$ -module. If Y is the subscheme defined by \mathcal{I} , the conclusion is immediate. \square

18.6.2 Morphisms of finite type and of finite presentation

Let $f : X \rightarrow Y$ be a morphism of schemes, x be a point of X , and $y = f(x)$. We say f is **of finite type** (resp. **of finite presentation**) at the point x if there exist an affine neighborhood V of y and an affine neighborhood U of x such that $f(U) \subseteq V$ and $\Gamma(U, \mathcal{O}_X)$ is a $\Gamma(V, \mathcal{O}_Y)$ -algebra of finite type (resp. of finite presentation). We say f is **locally of finite type** (resp. **locally of finite presentation**) if it is of finite type (resp. of finite presentation) at every point of X . In this case, we say the Y -scheme X is locally of finite type (resp. locally of finite presentation) over Y .

Lemma 18.6.19. *Let $f : X \rightarrow Y$ be a morphism of schemes, x be a point of X , and $y = f(x)$. If there exist an affine neighborhood V of y and an affine neighborhood U of x such that $f(U) \subseteq V$ and $\Gamma(U, \mathcal{O}_X)$ is a $\Gamma(V, \mathcal{O}_Y)$ -algebra of finite type (resp. of finite presentation), then for any affine open neighborhoods U' of x and V' of y , there exist affine open neighborhoods $U_1 \subseteq U \cap U'$ of x and $V_1 \subseteq V \cap V'$ of y , respectively of the form $\text{Spec}(\Gamma(U', \mathcal{O}_X)_{s'})$ and $\text{Spec}(\Gamma(V', \mathcal{O}_Y)_{t'})$, such that $f(U_1) \subseteq V_1$ and $\Gamma(U_1, \mathcal{O}_X)$ is a $\Gamma(V_1, \mathcal{O}_Y)$ -algebra of finite type (resp. of finite presentation).*

Proof. Let $t' \in \Gamma(V', \mathcal{O}_Y)$ such that $V_1 = \text{Spec}(\Gamma(V', \mathcal{O}_Y)_{t'})$ is an affine neighborhood of y contained in $V \cap V'$ and choose $s'_0 \in \Gamma(U', \mathcal{O}_X)$ such that $U'' = \text{Spec}(\Gamma(U', \mathcal{O}_X)_{s'_0})$ is a neighborhood of x contained in $U \cap U' \cap f^{-1}(V_1)$. There then exists $s \in \Gamma(U, \mathcal{O}_X)$ such that $U_1 = \text{Spec}(\Gamma(U, \mathcal{O}_X)_s)$ is a neighborhood of x contained in U'' . If s'' is the image of s in $\Gamma(U'', \mathcal{O}_X)$, we then have $U_1 = \text{Spec}(\Gamma(U'', \mathcal{O}_X)_{s''})$, so there exists $s' \in \Gamma(U', \mathcal{O}_X)$ such that $U_1 = \text{Spec}(\Gamma(U', \mathcal{O}_X)_{s'})$.

Now $\Gamma(U_1, \mathcal{O}_X) = \Gamma(U, \mathcal{O}_X)[1/s]$, so it is a $\Gamma(U, \mathcal{O}_X)$ -algebra of finite presentation, and a fortiori a $\Gamma(V, \mathcal{O}_Y)$ -algebra of finite type (resp. of finite presentation). The homomorphism $\Gamma(V, \mathcal{O}_Y) \rightarrow \Gamma(U_1, \mathcal{O}_X)$ factors into

$$\Gamma(V, \mathcal{O}_Y) \longrightarrow \Gamma(V_1, \mathcal{O}_Y) \longrightarrow \Gamma(U_1, \mathcal{O}_X)$$

If $\Gamma(U_1, \mathcal{O}_X)$ is identified with a quotient algebra $\Gamma(V, \mathcal{O}_Y)[T_1, \dots, T_n]/\mathfrak{a}$, then it is also identified with the quotient algebra $\Gamma(V_1, \mathcal{O}_Y)[T_1, \dots, T_n]/\mathfrak{b}$, where \mathfrak{b} is the ideal generated by \mathfrak{a} . It then follows that $\Gamma(U_1, \mathcal{O}_X)$ is a $\Gamma(V_1, \mathcal{O}_Y)$ -algebra of finite type (resp. finite presentation). \square

Proposition 18.6.20. *If Y is locally Noetherian, then $f : X \rightarrow Y$ is locally of finite type if and only if it is locally of finite presentation. Moreover, if this holds, then X is also locally Noetherian.*

Proof. The first assertion is clear since we can take $\Gamma(V, \mathcal{O}_Y)$ to be Noetherian. The second one follows because $\Gamma(U, \mathcal{O}_X)$ is then also Noetherian. \square

Proposition 18.6.21 (Properties of Morphisms Locally of Finite Type).

- (i) *Any local immersion is locally of finite type.*
- (ii) *If $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are two morphisms locally of finite type, then $g \circ f$ is locally of finite type.*
- (iii) *If $f : X \rightarrow Y$ is an S -morphism locally of finite type, $f_{(S')} : X_{(S')} \rightarrow Y_{(S')}$ is locally of finite type for any base change $g : S' \rightarrow S$.*
- (iv) *If $f : X \rightarrow X'$ and $g : Y \rightarrow Y'$ are two S -morphisms locally of finite type, $f \times_S g$ is locally of finite type.*
- (v) *If the composition $g \circ f$ of two morphisms is locally of finite type, then f is locally of finite type.*
- (vi) *If a morphism f is locally of finite type, so is f_{red} .*

Proof. Assertion (vi) follows from the fact that if a ring homomorphism $A \rightarrow B$ is of finite type, then so is $A/\mathfrak{n}(A) \rightarrow B/\mathfrak{n}(B)$. Now in view of ??, it suffices to prove (i), (ii) and (iii). If $j : X \rightarrow Y$ is a local immersion, for any $x \in X$ there exists an affine open neighborhood V of $j(x)$ in Y and an affine open neighborhood U of x such that the restriction $U \rightarrow V$ of j is a closed immersion. Then $\Gamma(U, \mathcal{O}_X)$ is a quotient ring of $\Gamma(V, \mathcal{O}_Y)$, and is therefore of finite type.

To establish (iii), we may assume that $Y = S$; let $p : X_{(S')} \rightarrow X$ and $q : X_{(S')} \rightarrow S$ be the canonical projections, x' be a point of $X_{(S')}$, and $x = p(x')$, $s' = q(x')$, $s = f(p(x')) = g(q(x'))$. Let V be an affine neighborhood of s in S and U be an affine neighborhood of x in X such that $f(U) \subseteq V$ and $\Gamma(U, \mathcal{O}_X)$ is a $\Gamma(V, \mathcal{O}_S)$ -algebra of finite type. Let V' be an affine open neighborhood of s' in S' contained in $g^{-1}(V)$; then $p^{-1}(U) \cap q^{-1}(V)$ is an affine neighborhood of x' and is identified with $U \times_V V'$ (??). This is an affine scheme with ring $\Gamma(U, \mathcal{O}_X) \otimes_{\Gamma(V, \mathcal{O}_S)} \Gamma(V', \mathcal{O}_{S'})$; as this is a $\Gamma(V', \mathcal{O}_{S'})$ -algebra of finite type, we see (iii) follows.

Finally, to prove (ii), consider a point $x \in X$; there exists by hypothesis an affine open neighborhood W of $g(f(x))$ in Z and an affine open neighborhood V of $f(x)$ in Y such that $g(V) \subseteq W$ and $\Gamma(V, \mathcal{O}_Y)$ is a $\Gamma(W, \mathcal{O}_Z)$ -algebra of finite type. By ?? there exists an affine open neighborhood $V' \subseteq V$ of $f(x)$ and an affine open neighborhood $U \subseteq f^{-1}(V')$ of x such that

$\Gamma(V', \mathcal{O}_Y)$ is a $\Gamma(V, \mathcal{O}_Y)$ -algebra of finite type and $\Gamma(U, \mathcal{O}_X)$ is a $\Gamma(V', \mathcal{O}_Y)$ -algebra of finite type. We then conclude that $\Gamma(U, \mathcal{O}_X)$ is a $\Gamma(W, \mathcal{O}_Z)$ -algebra of finite type, so (ii) follows. \square

Corollary 18.6.22. *Let $f : X \rightarrow Y$ be a morphism locally of finite type. For any morphism $Y' \rightarrow Y$ such that Y' is locally Noetherian, $X \times_Y Y'$ is locally Noetherian.*

Proof. This follows from ??, since $f_{(Y')} : X \times_Y Y' \rightarrow Y'$ is locally of finite type by ??.

Proposition 18.6.23. *Let $\rho : A \rightarrow B$ be a homomorphism of rings. For the corresponding morphism $f : \text{Spec}(B) \rightarrow \text{Spec}(A)$ to be locally of finite type, it is necessary and sufficient that B is an A -algebra of finite type.*

Proof. This condition is clearly sufficient. Conversely, assume that f is locally of finite type. Then by ?? there exists a finite cover of $\text{Spec}(B)$ by open sets $D(g_i)$ (where $g_i \in B$) such that B_{g_i} is an A -algebra of finite type. Since the $D(g_i)$'s cover $\text{Spec}(B)$, we see g_i generate the ring B , and it follows from ?? that B is of finite type over A . \square

Proposition 18.6.24 (Properties of Morphisms Locally of Finite Presentation).

- (i) *Any local isomorphism is locally of finite presentation.*
- (ii) *If two morphisms $f : X \rightarrow Y, g : Y \rightarrow Z$ are locally of finite presentation, so is $g \circ f$.*
- (iii) *If $f : X \rightarrow Y$ is an S -morphism locally of finite presentation, $f_{(S')} : X_{(S')} \rightarrow Y_{(S')}$ is locally of finite presentation for any base change $g : S' \rightarrow S$.*
- (iv) *If $f : X \rightarrow X'$ and $g : Y \rightarrow Y'$ are two S -morphisms locally of finite presentation, $f \times_S g$ is locally of finite presentation.*
- (v) *If the composition $g \circ f$ of two morphisms $f : X \rightarrow Y, g : Y \rightarrow Z$ is locally of finite presentation and if g is locally of finite type, then f is locally of finite presentation.*

Proof. The first assertion is trivial, and to prove (ii), (iii), it suffices to replace the "algebra of finite type" in the proof of ?? by "algebra of finite presentation", and use ?? Again, assertion (iv) then follows from these. For (v), consider the cartesian square

$$\begin{array}{ccc} X & \xrightarrow{\Gamma_f} & X \times_Z Y \\ f \downarrow & & \downarrow f \times_Z 1_Y \\ Y & \xrightarrow{\Delta_g} & Y \times_Z Y \end{array}$$

If we can show that Δ_g is locally of finite presentation, then it follows from (iii) that Γ_f is also locally of finite presentation. But f factors into

$$X \xrightarrow{\Gamma_f} X \times_Z Y \xrightarrow{p_2} Y$$

and we have $p_2 = (g \circ f) \times_Z 1_Y$, which is locally of finite presentation by (iv) since $g \circ f$ is. We then deduce that f is locally of finite presentation.

It then suffices to prove that if $g : Y \rightarrow Z$ is a morphism locally of finite type, then the diagonal morphism $\Delta_g : Y \rightarrow Y \times_Z Y$ is locally of finite presentation. To do this we may

assume that $Z = \text{Spec}(A)$, $Y = \text{Spec}(B)$, and B is an A -algebra of finite type. The diagonal Δ_g corresponds to the homomorphism $\pi : B \otimes_A B \rightarrow B$ such that $\pi(x \otimes y) = xy$. Let \mathfrak{I} be the kernel of π . We claim that \mathfrak{I} is generated by the elements $1 \otimes s - s \otimes 1$, where s runs through a system of generators for the A -algebra B (this then proves the claim since B is of finite type over A). Now, it is clear that for any $x \in B$, we have $x \otimes 1 - 1 \otimes x \in \mathfrak{I}$; on the other hand, for $x, y \in B$, we have

$$x \otimes y = xy \otimes 1 + (x \otimes 1)(1 \otimes y - y \otimes 1)$$

If $\sum_i (x_i \otimes y_i) \in \mathfrak{I}$, we have by definition that $\sum_i x_i y_i = 0$, so

$$\sum_i (x_i \otimes y_i) = \sum_i (x_i \otimes 1)(1 \otimes y_i - y_i \otimes 1).$$

Moreover, if $x = st$, then

$$x \otimes 1 - 1 \otimes x = (s \otimes 1)(t \otimes 1 - 1 \otimes t) + (s \otimes 1 - 1 \otimes t)(1 \otimes t).$$

The claim then follows by induction on the number of factors of a product in B . \square

Corollary 18.6.25. *Let $g : Y \rightarrow Z$ be a morphism locally of finite type. Then the diagonal morphism $\Delta_g : Y \rightarrow Y \times_Z Y$ is locally of finite presentation.*

Proof. This is contained in the proof of ??.

Proposition 18.6.26. *Let A be a ring, B be an A -algebra, $B' = A[T_1, \dots, T_n]$, and $\rho : B' \rightarrow B$ be a surjective homomorphism of A -algebras. Then for B to be an A -algebra of finite presentation, it is necessary and sufficient that the kernel \mathfrak{a} of ρ is finitely generated in B' .*

Proof. The condition is sufficient by definition. Conversely, we note that the morphism $g : \text{Spec}(B') \rightarrow \text{Spec}(A)$ is locally of finite type; if B is an A -algebra of finite presentation, the morphism $f : \text{Spec}(B) \rightarrow \text{Spec}(B')$ corresponding to ρ and $g \circ f : \text{Spec}(B) \rightarrow \text{Spec}(A)$ are locally of finite presentation, so it follows from ??(v) that f is locally of finite presentation. Now it suffices to apply ??.

Corollary 18.6.27. *Let X, Y be two schemes, $j : X \rightarrow Y$ be an immersion, U an open subset of Y such that $j(X)$ is closed in U , and \mathcal{J} the quasi-coherent ideal of \mathcal{O}_U defining the closed subscheme of Y associated with j . For j to be locally of finite presentation, it is necessary and sufficient that \mathcal{J} is a \mathcal{O}_U -module of finite type.*

Proof. Since the question is local, we can assume that X and Y are affine. The assertion then reduces to ??.

Proposition 18.6.28. *Let $\rho : A \rightarrow B$ be a ring homomorphism. For the corresponding morphism $f : \text{Spec}(B) \rightarrow \text{Spec}(A)$ to be locally of finite presentation, it is necessary and sufficient that B is an A -algebra of finite presentation.*

Proof. The condition is clearly sufficient, so we only need to prove the necessity. If f is locally of finite presentation, then it follows from ?? that B is an A -algebra of finite type, so there exists a surjective homomorphism $\rho : C' = A[T_1, \dots, T_n] \rightarrow B$ of A -algebras. It then follows from ??(v) that the closed immersion $j : \text{Spec}(B) \rightarrow \text{Spec}(C')$ is locally of finite presentation, so, if \mathfrak{b}

is the kernel of ρ , the \tilde{B}' -module $\tilde{\mathfrak{b}}$ is of finite type, and \mathfrak{b} is therefore finitely generated in B' by ??.

Proposition 18.6.29. *Let $\rho : A \rightarrow B$ be a homomorphism of rings such that B is a finite A -algebra. For B to be an A -algebra of finite presentation, it is necessary and sufficient that B is an A -module of finite presentation.*

Proof. There exists a finite A -algebra B' of finite presentation such that B' is a free A -module, and a surjective A -homomorphism of A -algebras $u : B' \rightarrow B$??; we have a surjective homomorphism $v : B'' = A[T_1, \dots, T_m] \rightarrow B'$ of A -algebras whose kernel is finitely generated. If $w = v \circ u : B'' \rightarrow B$ and \mathfrak{b} (resp. \mathfrak{a}) is the kernel of w (resp. u), we have $\mathfrak{a} = v(\mathfrak{b})$ since v is surjective. If B is an A -algebra of finite presentation, \mathfrak{b} is a finitely generated ideal of B'' by ??, so \mathfrak{a} is a finitely generated ideal in B' , hence a finitely generated A -module since B' is a finite A -algebra. As B' is a free A -module, B is then an A -module of finite presentation. The converse is proved in ??.

Proposition 18.6.30. *Let $f : X \rightarrow Y$ be a local immersion of finite type at a point of $y \in Y$. The following conditions are equivalent:*

- (i) f is an open map at y .
- (ii) There exists an open neighborhood U of y in Y such that $f|_U$ is a nilimmersion over the open subscheme U .
- (iii) There exists an open neighborhood U of y in Y such that $f|_U$ is a nilpotent immersion over the open subscheme U .

Proof. It is clear that (iii) implies (ii) and (ii) implies (i). To show that (i) implies (iii), we can, by restricting f , suppose that f is a closed immersion from an affine open U of Y to an affine open V of X . Moreover, by choosing an irreducible component containing $f(y)$, we can further assume that V is irreducible. As f is a homeomorphism from U to $f(U)$, the hypothesis of (i) then implies that $f(U) = V$ since V is connected. If $V = \text{Spec}(A)$, $U = \text{Spec}(B)$, we have $B = A/\mathfrak{a}$, where \mathfrak{a} is a nilideal of A . On the other hand, in view of ??, we can, by replacing A with a fraction field A_s , suppose that B is an A -algebra of finite presentation. But then \mathfrak{a} is a finitely generated ideal of A by ??, so it is nilpotent.

Proposition 18.6.31. *Let $f : X \rightarrow Y$ be a morphism locally of finite type. Suppose that f admits a Y -section g , and for every $y \in Y$, the fiber $X_y = f^{-1}(y)$ is a $\kappa(y)$ -scheme $\kappa(y)$ -isomorphic to $\text{Spec}(\kappa(y))$ (and necessarily has underlying space $\{g(y)\}$). Then f is an isomorphism.*

Proof. In fact, g is a nilimmersion of Y to X ??, so the image $g(Y)$ has underlying space X and is defined by a nilideal \mathcal{I} of \mathcal{O}_X . As $f \circ g = 1_Y$ and f is locally of finite type, g is locally of finite presentation by ??(v), so \mathcal{I} is an ideal of finite type of \mathcal{O}_X ??). For any $y \in Y$, put $x = g(y)$, and consider the following exact sequence:

$$0 \longrightarrow \mathcal{I}_x \longrightarrow \mathcal{O}_{X,x} \xrightarrow[\underbrace{f_x^\#}_{\mathcal{I}}]{} \mathcal{O}_{Y,y} \longrightarrow 0$$

The relation $f \circ g = 1_Y$ implies $f_x^\# \circ g_y^\# = 1$, so the above exact sequence splits. By tensoring with $\kappa(y) = \mathcal{O}_{Y,y}/\mathfrak{m}_y$, we then get an isomorphism $\mathcal{O}_{X,x}/\mathfrak{m}_y\mathcal{O}_{X,x} \cong \kappa(y) \oplus (\mathcal{I}_x/\mathfrak{m}_y\mathcal{I}_x)$. But the hypothesis on X_y implies that $\kappa(y)$ -isomorphic to $\kappa(y)$, so we deduce that $\mathfrak{m}_y\mathcal{I}_x = \mathcal{I}_x$ and a fortiori $\mathfrak{m}_x\mathcal{I}_x = \mathcal{I}_x$. As \mathcal{I}_x is a finitely generated ideal of $\mathcal{O}_{X,x}$, by Nakayama lemma we conclude $\mathcal{I}_x = 0$, so $\mathcal{I} = 0$ and f is an isomorphism. \square

Corollary 18.6.32. *Let X, Y be two S -schemes and $f : X \rightarrow Y$ be an S -morphism. Suppose that X is locally of finite type over S . For each $s \in S$, let X_s, Y_s be the fiber of X and Y at the point s , and $f_s : X_s \rightarrow Y_s$ be the morphism induced by f under the base change $\text{Spec}(\kappa(s)) \rightarrow S$. Then if for each $s \in S$, f_s is a monomorphism, f is a monomorphism.*

Proof. If f_s is a monomorphism, so is $f_y : X_y \rightarrow \text{Spec}(\kappa(y))$. By hypothesis f is locally of finite type (??(v)), so we can limit ourselves to the case $Y = S$. To see that f is a monomorphism, it suffices to prove that the first projection $p : X \times_S X \rightarrow X$ is an isomorphism (??). Now the hypothesis on f_s implies that the projections $p_s : X_s \otimes_{\kappa(s)} X_s \rightarrow X_s$ are isomorphisms for all $s \in S$. Since p admits an S -section, namely the diagonal Δ_f , it follows from ?? that p is an isomorphism. \square

We now come to the definition of *morphisms of finite type*, which can be seen as a global version of morphisms locally of finite type. Briefly speaking, the notion of finite type concerns the "global finiteness" of a morphism: we have the following definition and proposition.

Proposition 18.6.33. *Let $f : X \rightarrow Y$ be a morphism and (U_α) be a covering of Y by affine opens. The following conditions are equivalent:*

- (i) *f is locally of finite type and quasi-compact.*
- (ii) *For each α , $f^{-1}(U_\alpha)$ is a finite union of affine opens $V_{\alpha,i}$ such that the ring $\Gamma(V_{\alpha,i}, \mathcal{O}_X)$ is a $\Gamma(U_\alpha, \mathcal{O}_Y)$ -algebra of finite type.*
- (iii) *For any affine open U of Y , $f^{-1}(U)$ is a finite union of affine opens V_j such that $\Gamma(V_j, \mathcal{O}_X)$ is a $\Gamma(U, \mathcal{O}_X)$ -algebra of finite type.*

We say the morphism f is **of finite type** if it satisfies the above equivalent conditions. In this case, we say X is of finite type over Y .

Proof. It is clear that (iii) \Rightarrow (ii) \Rightarrow (i). To prove that (i) implies (iii), we may assume that $Y = U$ is affine; then X is quasi-compact, hence is a finite union of affine opens V_j such that the restriction $V_j \rightarrow U$ of f is locally of finite type. By ??, we see $\Gamma(V_j, \mathcal{O}_X)$ is a $\Gamma(U, \mathcal{O}_X)$ -algebra of finite type. \square

Proposition 18.6.34. *Let $f : X \rightarrow Y$ be a morphism of finite type. If Y is Noetherian (resp. locally Noetherian), so is X .*

Proof. This follows from ?? and ??.

Proposition 18.6.35 (Properties of Morphisms of Finite Type).

- (i) *Any quasi-compact immersion is of finite type.*
- (ii) *The composition of two morphisms of finite type is of finite type.*

- (iii) If $f : X \rightarrow Y$ is an S -morphism of finite type, $f_{(S')} : X_{(S')} \rightarrow Y_{(S')}$ is of finite type for any base change $S' \rightarrow S$.
- (iv) If $f : X \rightarrow X'$ and $g : Y \rightarrow Y'$ are two S -morphisms of finite type, $f \times_S g$ is of finite type.
- (v) If the composition $g \circ f$ of two morphisms $f : X \rightarrow Y$, $g : Y \rightarrow Z$ is of finite type and if g is quasi-separated or X is Noetherian, then f is of finite type.
- (vi) If a morphism f is of finite type, so is f_{red} .

Proof. This follows directly from ?? and ??.

□

Corollary 18.6.36. Let $f : X \rightarrow Y$ be a morphism of finite type. For any morphism $Y' \rightarrow Y$ such that Y' is Noetherian, $X \times_Y Y'$ is Noetherian.

Proof. This follows from ??(iii) and ??.

□

Corollary 18.6.37. Let X be a scheme of finite type over a locally Noetherian scheme S . Then any S -morphism $f : X \rightarrow Y$ is of finite type.

Proof. The morphism f is locally of finite type by ??(v). To see it is quasi-compact, we can suppose that S is Noetherian. If $\varphi : X \rightarrow S$ and $\psi : Y \rightarrow S$ are the structural morphism, we have $\varphi = \psi \circ f$ and X is Noetherian by ??, so f is of finite type, by ??(v).

□

Let X and Y be two schemes. We say a morphism $f : X \rightarrow Y$ is **of finite presentation** if it satisfies the following conditions:

- (i) f is locally of presentation;
- (ii) f is quasi-compact;
- (iii) f is quasi-separated.

In this case, we say X is **of finite presentation over Y** , or is an **Y -scheme of finite presentation**. It is clear that condition (iii) is automatic if f is separated, or if X is locally Noetherian. If Y is locally Noetherian, then again, f is of finite type if and only if it is of finite presentation, and in this case X is also locally Noetherian.

Proposition 18.6.38 (Properties of Morphisms of Finite Presentation).

- (i) Any quasi-compact immersion that is locally of finite presentation (in particular any quasi-compact open immersion) is of finite presentation.
- (ii) The composition of two morphisms of finite presentation is of finite presentation.
- (iii) If $f : X \rightarrow Y$ be an S -morphism of finite presentation, $f_{(S')} : X_{(S')} \rightarrow Y_{(S')}$ is of finite presentation for any base change $S' \rightarrow S$.
- (iv) If $f : X \rightarrow X'$ and $g : Y \rightarrow Y'$ are two S -morphisms of finite presentation, $f \times_S g$ is of finite presentation.
- (v) If the composition $g \circ f$ of two morphisms $f : X \rightarrow Y$, $g : Y \rightarrow Z$ is of finite presentation and if g is quasi-separated and locally of finite type, then f is of finite presentation.

Proof. This follows from ??, ??, and ??.

It follows from ??(iii) that if f is a morphism of finite presentation and U is an open subset of Y , the restriction $f^{-1}(U) \rightarrow U$ of f is also of finite presentation. Conversely, let (U_α) be a covering of Y by affine opens and suppose that the restriction $f^{-1}(U_\alpha) \rightarrow U_\alpha$ of f is a morphism of finite presentation. Then it follows that f is of finite presentation, since f is clearly of finite presentation and quasi-compact and it is quasi-separated by ??.

If X is a quasi-separated scheme, any morphism $f : X \rightarrow Y$ is quasi-separated by ???. Therefore, if f is quasi-compact and locally of finite presentation, it is of finite presentation.

Corollary 18.6.39. *Let $\rho : A \rightarrow B$ be a ring homomorphism. For the corresponding morphism $f : \text{Spec}(B) \rightarrow \text{Spec}(A)$ to be of finite type (resp. of finite presentation), it is necessary and sufficient that B is an A -algebra of finite type (resp. of finite presentation).*

Proof. Since any morphism of affine schemes is quasi-compact and separated, this follows from ?? and ??.

Remark 18.6.1. In the definition of morphisms of finite presentation, the condition (iii) is not a consequence of the other two conditions. For example, let Y be a non-Noetherian affine scheme and let Y be a non-quasi-compact open subset of Y (an example for this is $Y = \text{Spec}(k[x_1, x_2, \dots])$) and $U = Y - \{0\}$, cf. ??). Let X be the scheme obtained by glueing two schemes Y_1, Y_2 isomorphic to Y along the open sets U_1, U_2 corresponding to U , so that X is the union of two affine opens isomorphic to Y_1, Y_2 , respectively, and $Y_1 \cap Y_2 = U$. Let $f : X \rightarrow Y$ be the morphism which coincides with the canonical isomorphism $Y_i \rightarrow Y$ on each Y_i . Then it is clearly locally of finite presentation, and is quasi-compact since the inverse image of a quasi-compact open of Y is the union of its images in Y_1 and Y_2 ; but as $Y_1 \cap Y_2 = U$ is not quasi-compact, it is not quasi-separated by ?? and ??(ii).

Proposition 18.6.40. *Let $f_i : X_i \rightarrow Y$ be a finite family of morphisms and $f : X \rightarrow Y$ be their coproduct, where $X = \coprod_i X_i$. Then for f to be of finite type (resp. finite presentation), it is necessary and sufficient that each f_i is.*

Proof. In view of ??, it suffices to note that the same assertion holds for morphisms of locally finite type and of finite presentation.

18.6.3 Algebraic schemes

We say a K -scheme is **algebraic** (resp. **locally algebraic**) if it is of finite type over K (resp. locally of finite type over K). The field K is called the **base field** of X .

Proposition 18.6.41. *Let K be a field. A locally algebraic (resp. algebraic) K -scheme is locally Noetherian (Noetherian). Moreover X is a Jacobson scheme and a point $x \in X$ is closed if and only if $\kappa(x)$ is a finite extension of K .*

Proof. The first assertion is clear, and X is Jacobson by ???. To characterize close points in X , we note that for a point $x \in X$ to be closed, it is necessary and sufficient that for an open covering (U_α) of X , x is closed in the U_α containing it. As there is a covering of X by affine opens U_α such that $\Gamma(U_\alpha, \mathcal{O}_X)$ is a K -algebra of finite type, we can then assume that $X = \text{Spec}(A)$ where A is a K -algebra of finite type. The closed points of X are then maximal ideals of A ; but then

$A/\mathfrak{p}_x = \kappa(x)$ is a finite extension by ???. Conversely, if $\kappa(x)$ is a finite K -algebra, so is the ring $A/\mathfrak{p}_x \subseteq \kappa(x)$, and as an integral K -algebra is also a field (??), we have $A/\mathfrak{p}_x = \kappa(x)$, so x is closed. \square

Corollary 18.6.42. *Let K be an algebraically closed field and X be a locally algebraic K -scheme. Then the closed points of X are exactly the rational points of X over K , which are identified with the K -points of X with values in K .*

Proposition 18.6.43. *Let K be a field and X be a locally algebraic scheme over K . Then the following conditions are equivalent:*

- (i) X is Artinian.
- (ii) The underlying space of X has only finitely many closed points.
- (iii) The underlying space of X is finite.
- (iv) X is isomorphic to $\text{Spec}(A)$ where A is K -algebra of finite dimension.

If X is algebraic over K , then these conditions are equivalent to the following:

- (v) The underlying space of X is discrete.
- (vi) The points of X are all closed.

Proof. We see (i) implies any other conditions, and (v) or (vi) implies (i) if X is Noetherian. Moreover, it is clear that (iv) implies (i), since a finite dimensional K -algebra is Artinian. In the condition of (ii), the set X_0 of closed points of X is then finite, closed and very dense in X , whence equal to X and X is therefore Artinian, since it is then Noetherian. \square

Corollary 18.6.44. *Let K be a field, X be a locally algebraic K -scheme, and x be a point of X . The following conditions are equivalent:*

- (i) x is isolated in X ;
- (ii) x is closed in X and $\mathcal{O}_{X,x}$ is Artinian;
- (iii) $\mathcal{O}_{X,x}$ is a finite dimensional K -algebra.

Proof. If $\mathcal{O}_{X,x}$ is a finite dimensional K -algebra, so is the residue field $\kappa(x) = \mathcal{O}_{X,x}/\mathfrak{m}_x$, so (iii) implies (ii) in view of ???. The local ring $\mathcal{O}_{X,x}$ is Artinian signifies that x is a maximal point of X , since a Noetherian local ring is Artinian if and only if it has a unique prime ideal. If x is moreover closed, the set $\{x\}$ is closed and stable under generalization, hence open (??), and this proves x is isolated in X . Finally, if x is isolated in X , there exists an affine open neighborhood U of x such that $U = \{x\}$ and $\Gamma(U, \mathcal{O}_X)$ is a finite type K -algebra. But then $\Gamma(U, \mathcal{O}_X) = \mathcal{O}_{X,x}$, so (i) implies (iii). \square

If the conditions in ???, we say X is a finite scheme over K , or a finite K -scheme. For such a scheme, we denote by $\dim_K(X)$ the dimension of the ring $\Gamma(X, \mathcal{O}_X)$ over K . If X and Y are two finite schemes over K , we have

$$\dim_K(X \amalg Y) = \dim_K(X) + \dim_K(Y), \quad \dim_K(X \times_K Y) = \dim_K(X) \dim_K(Y).$$

Corollary 18.6.45. *Let X be a finite scheme over a field K . For any extension K' of K , $X \otimes_K K'$ is a finite scheme over K' , with $\dim_{K'}(X') = \dim_K(X)$.*

Proof. In fact, if $X = \text{Spec}(A)$, we have $[A \otimes_K K' : K'] = [A : K]$, whence the claim. \square

Corollary 18.6.46. *Let X be a finite scheme over a field K . We put*

$$n = \sum_{x \in X} [\kappa(x) : K]_s$$

Then, for any algebraically closed extension Ω of K , the underlying space of $X \otimes_K \Omega$ has exactly n points, which are identified with the Ω -valued points of X .

Proof. By ??, we can assume that $A = \Gamma(X, \mathcal{O}_X)$ is local; let \mathfrak{m} be the maximal ideal of A , $L = A/\mathfrak{m}$ the residue field, which is a finite algebraic extension of K by ???. The Ω -points of X correspond bijectively to Ω -sections of $X \otimes_K \Omega$, and to the closed points of $X \otimes_K \Omega$ by ??, and finally to the points of this Artinian scheme (??). They also correspond to K -homomorphisms of L into Ω , and the assertion then follows from the definition of separable degree. \square

The number n defined in ?? is called the **separable rank** of A (or X) over K , or the **geometric number of points** of X . This is also the number of elements in $X(\Omega)_K$. It follows from this definition that for any extension K' of K , $X \otimes_K K'$ has the same geometric number of points as X . If we denote this number by $n(X)$, it is clear that, if X and Y are two finite schemes over K , we have

$$n(X \amalg Y) = n(X) + n(Y), \quad n(X \times_K Y) = n(X)n(Y).$$

Proposition 18.6.47. *Let $f : X \rightarrow Y$ be a morphism locally of finite type (resp. of finite type). Then, for any $y \in Y$, the fiber $X_y = f^{-1}(y)$ is a locally algebraic (resp. algebraic) scheme over $\kappa(y)$, and for each $x \in X_y$, $\kappa(x)$ is a extension of $\kappa(y)$ of finite type.*

Proof. As $X_y = X \otimes_Y \kappa(y)$, this follows from ??(iii) and ??(iii). \square

Proposition 18.6.48. *Let $f : X \rightarrow Y$, $g : Y' \rightarrow Y$ be two morphisms, and put $X' = X \times_Y Y'$, $f' = f_{(Y')} : X' \rightarrow Y'$. Let $y' \in Y'$ and $y = g(y')$. If the fiber $X_y = f^{-1}(y)$ is a finite scheme over $\kappa(y)$, then the fiber $X'_{y'} = f'^{-1}(y')$ is a finite scheme over $\kappa(y')$, and we have*

$$\dim_{\kappa(y')} (X'_{y'}) = \dim_{\kappa(y)} (X_y), \quad n(X'_{y'}) = n(X_y).$$

Proof. This follows from the observation $X'_{y'} = X_y \otimes_{\kappa(y)} \kappa(y')$. \square

?? shows that the morphisms of finite type (resp. locally of finite type) correspond intuitively to "algebraic families of algebraic varieties (resp. locally algebraic)", where Y plays the role of "parameters." Because of this, these morphisms are of significant geometric interests. The morphisms which are not locally of finite type will intervene them by the process of "base change", for example by localization and completion.

18.6.4 Local determination of morphisms

Proposition 18.6.49. *Let X and Y be S -schemes, $x \in X$, $y \in Y$ be points lying over the same point $s \in S$.*

- (a) Suppose that Y is locally of finite type over S at the point y . Then if two S -morphisms f, g from X to Y are such that $f(x) = g(x) = y$ and the $\mathcal{O}_{S,x}$ -homomorphisms $f_x^\#$ and $g_x^\#$ from $\mathcal{O}_{Y,y}$ to $\mathcal{O}_{X,x}$ coincide, then f and g coincide in an open neighborhood of x .
- (b) Suppose that Y is locally of finite presentation over S at the point y . Then, for any $\mathcal{O}_{X,x}$ -homomorphism $\varphi : \mathcal{O}_{Y,y} \rightarrow \mathcal{O}_{X,x}$ there exists an open neighborhood U of x in X and an S -morphism f from U to Y such that $f(x) = y$ and $f_x^\# = \varphi$.

Proof. We first consider case (a). The question is local over S, X and Y , so we can suppose that S, X, Y are affine with rings A, B, C , respectively. The morphisms f and g then correspond to A -homomorphisms ρ, σ from C to B such that $\rho^{-1}(\mathfrak{p}_x) = \sigma^{-1}(\mathfrak{p}_x) = \mathfrak{p}_y$, and the homomorphisms ρ_x and σ_x from C_y to B_x , deduced by ρ and σ , coincide. We can suppose that C is an A -algebra of finite type. Let $(c_i)_{1 \leq i \leq n}$ be generators of the A -algebra C , and put $b_i = \rho(c_i)$, $b'_i = \sigma(c_i)$. By hypothesis, we have $b_i/1 = b'_i/1$ in the ring B_x . This means there exist elements $s_i \in B - \mathfrak{p}_x$ such that $s_i(b_i - b'_i) = 0$ for each i , and we can evidently choose one $s \in B - \mathfrak{p}_x$ for all i . We then conclude that $b_i/1 = b'_i/1$ for each i in the ring B_s ; if $i_s : B \rightarrow B_s$ is the canonical homomorphism, we then have $i_s \circ \rho = i_s \circ \sigma$, so the restriction of f and g on $D(s)$ are identical.

We now come to case (b). Again we can suppose that S, X, Y are affine with rings A, B, C . Put $\mathfrak{p} = \mathfrak{p}_x, \mathfrak{q} = \mathfrak{p}_y$, and let $\varphi : C_q \rightarrow B_p$ be an A -homomorphism. We then get an A -homomorphism $\rho : C \rightarrow C_q \xrightarrow{\varphi} B_p$. Since we can consider B_p as an inductive limit of the filtered system of A -algebras B_s , where s runs through elements of $B - \mathfrak{p}$, and C is by hypothesis an A -algebra of finite presentation, we deduce from ?? that there exists $s \notin \mathfrak{p}$ and an A -homomorphism $\sigma : C \rightarrow B_s$ whose canonical image is ρ , that is, the following diagram is commutative

$$\begin{array}{ccc} C & \xrightarrow{\sigma} & B \\ \downarrow & & \downarrow \\ C_q & \xrightarrow{\varphi} & B_p \end{array} \tag{18.6.1}$$

It then suffices to take $U = D(s)$ and let f be the morphism induced by σ . \square

Corollary 18.6.50. Under the hypotheses of ??(ii), if moreover X is locally of finite type over S at the point of x , we can choose f to be of finite type.

Proof. To see this, we can assume that S, X, Y are affine, so that the structural morphisms $X \rightarrow S$ and $Y \rightarrow S$ are respectively of finite type and of finite presentation; then the results follows from ?? and ??(iv). \square

Corollary 18.6.51. Retain the hypotheses of ??(ii) and suppose that Y is integral and $\varphi : \mathcal{O}_{Y,y} \rightarrow \mathcal{O}_{X,x}$ is injective. Then we can suppose that U is affine and f factors into

$$U \xrightarrow{g} V \longrightarrow Y$$

where V is an affine open containing y and $g : U \rightarrow V$ is a morphism corresponding to a injective homomorphism $\Gamma(V, \mathcal{O}_Y) \rightarrow \Gamma(U, \mathcal{O}_X)$.

Proof. In fact, with the notations of ??(ii), C is integral and the canonical homomorphism $C \rightarrow C_g$ is then injective; the result then follows from the diagram ??, since σ is injective. \square

Proposition 18.6.52. *Let $f : X \rightarrow Y$ be a morphism, x be a point of X and $y = f(x)$.*

- (a) *Suppose that f is locally of finite type at the point x . For f to be a local immersion at the point x , it is necessary and sufficient that $f_x^\# : \mathcal{O}_{Y,y} \rightarrow \mathcal{O}_{X,x}$ is surjective.*
- (b) *Suppose that f is locally of finite presentation at the point x . For f to be a local isomorphism at the point x , it is necessary and sufficient that $f_x^\# : \mathcal{O}_{Y,y} \rightarrow \mathcal{O}_{X,x}$ is an isomorphism.*

Proof. We only need to prove the sufficiency, and we first consider case (b). Then X is locally of finite presentation over y at the point x , so by ??(i) and (ii), there exists an open neighborhood V of y and a morphism $g : V \rightarrow X$ such that $g \circ f$ (resp. $f \circ g$) is defined and coincide with the identity on an open neighborhood W of x (resp. an open neighborhood T of y). Put $T' = T \cap g^{-1}(W)$ and $W' = f^{-1}(T')$, we then verify that $g(T') \subseteq W'$, $f(W') \subseteq T'$ and $(g \circ f)|_{W'} = 1_{W'}$, whence f is a local isomorphism.

For (a), we can assume that X and Y are affine, with ring A and B . Then f corresponds to a homomorphism $\varphi : B \rightarrow A$ of finite type; we have $\varphi^{-1}(\mathfrak{p}_x) = \mathfrak{p}_y$, and the homomorphism $\varphi_x : B_y \rightarrow A_x$ induced by φ is surjective. Let $(t_i)_{1 \leq i \leq n}$ be a system of generators of the B -algebra A . The hypothesis on φ_x then implies $t_i/1 = \varphi(b_i)/\varphi(c)$ in the ring A_x , where $b_i \in B$ and $c \in B - \mathfrak{p}_y$, so we can find $a \in A - \mathfrak{p}_x$ such that

$$a(t_i\varphi(c) - \varphi(b_i)) = 0.$$

If we put $g = a\varphi(c)$, then $t_i/1 = a\varphi(b_i)/g$ in the ring A_g . Now there exists by hypothesis a polynomial $Q(X_1, \dots, X_n)$ with coefficients in $\varphi(B)$ such that $a = Q(t_1, \dots, t_n)$; write $Q(X_1/T, \dots, X_n/T) = P(X_1, \dots, X_n, T)/T^m$, where P is a polynomial of degree m . In the ring A_g , we then have

$$\begin{aligned} a/1 &= Q(t_1/1, \dots, t_n/1) = Q(a\varphi(b_1)/g, \dots, a\varphi(b_n)/g) \\ &= a^m P(\varphi(b_1), \dots, \varphi(b_n), \varphi(c))/g^m = a^m \varphi(d)/g^m \end{aligned}$$

where $d \in B$. Since $g/1 = (a/1)(\varphi(c)/1)$ is invertible in A_g by definition, so is $a/1$ and $\varphi(c)/1$, and we can then write $a/1 = (\varphi(d)/1)(\varphi(c)/1)^{-m}$. We conclude that $\varphi(d)/1$ is also invertible in A_g . Put $h = cd$, as $\varphi(h)/1$ is invertible in A_g , the composed homomorphism $B \rightarrow A \rightarrow A_g$ factors into

$$\begin{array}{ccc} B & \xrightarrow{\varphi} & A \longrightarrow A_g \\ & \searrow & \nearrow \gamma \\ & & B_h \end{array}$$

We claim that γ is surjective. For this, it suffices to verify that the image of B_h in A_g contains $t_i/1$ and $1/g$. Now we have

$$1/g = (\varphi(c)/1)^{m-1}(\varphi(d)/1)^{-1} = \gamma(c^m/h)$$

and $a/1 = \gamma(d^{m+1}/h^m)$, so $(a\varphi(b_i))/1 = \gamma(b_id^{m+1}/h^m)$, and as $t_i/1 = (a\varphi(b_i)/1)(g/1)^{-1}$, we conclude our assertion. The choice of h implies $f(D(g)) \subseteq D(h)$, and the restriction of f to $D(g)$ is induced by γ . Since γ is surjective, this restriction is a closed immersion from $D(g)$ to $D(h)$, so f is a local immersion at x . \square

Corollary 18.6.53. *With the notations of ??, suppose that f is a local immersion at the point x and is locally of finite presentation at x . For f to be open at x , it is necessary and sufficient that the kernel of $f_x^\#$ is nilpotent.*

Proof. In view of ??, it suffices to prove the sufficiency of the condition. We can suppose that $Y = \text{Spec}(A)$ and $X = \text{Spec}(A/\mathfrak{n})$, where \mathfrak{n} is a finitely generated ideal of A ??, and by hypothesis \mathfrak{n}_x is nilpotent. If $(s_i)_{1 \leq i \leq n}$ is a system of generators of \mathfrak{n} , we then have $s_i^m/1 = 0$ in A_x for an integer m and all i . Then there exists $t \in A - \mathfrak{p}_x$ such that $ts_i^m = 0$ for all i , so $(s_i/1)^m = 0$ in the ring A_t . This shows \mathfrak{n}_t is nilpotent, whence the conclusion. \square

18.6.5 Direct image of quasi-coherent sheaves

Proposition 18.6.54. *Let X, Y be two schemes and $f : X \rightarrow Y$ be a quasi-compact and quasi-separated morphism. Then for any quasi-coherent \mathcal{O}_X -module, $f_*(\mathcal{F})$ is quasi-coherent.*

Proof. Since the question is local over Y , we can assume that Y is affine. If f is quasi-compact, X is then a union of finitely many open affines X_i , and in view of ??(ii), X is a quasi-separated scheme, hence the intersections $X_i \cap X_j$ are quasi-compact ??.

We first assume that each intersection $X_i \cap X_j$ is affine. Put $\mathcal{F}_i = \mathcal{F}|_{X_i}$, $\mathcal{F}_{ij} = \mathcal{F}|_{X_i \cap X_j}$ and let \mathcal{F}'_i and \mathcal{F}'_{ij} be the inverse image of \mathcal{F}_i and \mathcal{F}_{ij} under the restriction of f to X_i and to $X_i \cap X_j$. We see that \mathcal{F}'_i and \mathcal{F}'_{ij} are quasi-coherent ??). We define a homomorphism

$$u : \bigoplus_i \mathcal{F}'_i \rightarrow \bigoplus_{i,j} \mathcal{F}'_{ij}$$

such that $f_*(\mathcal{F})$ is the kernel of u , and this then implies $f_*(\mathcal{F})$ is quasi-coherent by ???. For this, it suffices to define u as a homomorphism of presheaves, so for each open subset $W \subseteq Y$, we need a homomorphism

$$u_W : \bigoplus_i \Gamma(f^{-1}(W) \cap X_i, \mathcal{F}) \rightarrow \bigoplus_{ij} \Gamma(f^{-1}(W) \cap X_i \cap X_j, \mathcal{F})$$

so as to satisfy the compatibility for the restrictions to a smaller open subset. If for any section s_i of \mathcal{F} over $f^{-1}(W) \cap X_i$, we denote by s_{ij} its restriction to $f^{-1}(W) \cap X_i \cap X_j$, we set

$$u_W((s_i)) = (s_{ij} - s_{ji})$$

and the compatibility is evident. To identify the kernel \mathcal{R} of u , we define a homomorphism $v : f_*(\mathcal{F}) \rightarrow \mathcal{R}$ which sends a section s of \mathcal{F} over $f^{-1}(W)$ to the family (s_i) , where s_i is the restriction of s to $f^{-1}(W) \cap X_i$. By the sheaf axioms of \mathcal{F} , it is clear that v is bijective, which proves the assertion in this case.

In the general case, the same reasoning can be applied if we can show that each \mathcal{F}'_{ij} is quasi-coherent. But by hypotheses, $X_i \cap X_j$ is a union of finitely many affine opens X_{ijk} , and since each X_{ijk} are affine open subschemes of the affine scheme X_i , their intersections are again affine (affine schemes are separated), so we can apply the previous arguments to conclude that \mathcal{F}'_{ij} is quasi-coherent, and the proof is then complete. \square

Remark 18.6.2. We should note that even if X and Y are Noetherian schemes and $f : X \rightarrow Y$ is a morphism of finite type, the direct image $f_*(\mathcal{F})$ of a coherent \mathcal{O}_X -module \mathcal{F} is in general

not coherent. For example, let Y be the spectrum of a field K , $X = \text{Spec}(K[T])$, and choose $\mathcal{F} = \mathcal{O}_X$.

Proposition 18.6.55. *Let $f : X \rightarrow Y$ be a quasi-compact and quasi-separated morphism. Let (\mathcal{F}_λ) be a inductive system of quasi-coherent \mathcal{O}_X -modules and $\mathcal{F} = \varinjlim \mathcal{F}_\lambda$ be the inductive limit. Then $\varinjlim f_*(\mathcal{F}_\lambda) \cong f_*(\mathcal{F})$.*

Proof. For each affine open subset W of Y and any λ , we have a canonical homomorphism

$$u_{W,\lambda} : (f_*(\mathcal{F}_\lambda))|_W \rightarrow f_*(\mathcal{F})|_W$$

whence a canonical homomorphism

$$u_W : (\varinjlim f_*(\mathcal{F}_\lambda))|_W \rightarrow f_*(\mathcal{F})|_W$$

and this homomorphism is compatible with restrictions. Since f is quasi-compact and quasi-separated, by ?? $\varinjlim f_*(\mathcal{F}_\lambda)$ and $f_*(\mathcal{F})$ are quasi-coherent. Moreover, the homomorphism u_W corresponds by taking global section over W to the canonical homomorphism

$$\varphi_W : \Gamma(W, \varinjlim f_*(\mathcal{F}_\lambda)) \rightarrow \Gamma(f^{-1}(W), \mathcal{F}).$$

Since f is quasi-compact and quasi-separated, by ?? (Stack Project. Lemma 6.29.1) we have

$$\Gamma(W, \varinjlim f_*(\mathcal{F}_\lambda)) = \varinjlim \Gamma(W, f_*(\mathcal{F}_\lambda)) = \varinjlim \Gamma(f^{-1}(W), \mathcal{F}_\lambda) = \Gamma(f^{-1}(W), \mathcal{F})$$

and φ_W is therefore the identity homomorphism. By ??, it then follows that u_W is an isomorphism for each W , and the assertion then follows. \square

18.6.6 Extension of quasi-coherent sheaves

Let X be a topological space and \mathcal{F} be a sheaf of sets (resp. of groups, of rings) over X . Let U be an open subset of X with $j : U \rightarrow X$ the canonical injection, and let \mathcal{G} be a subsheaf of $\mathcal{F}|_U = j^{-1}(\mathcal{F})$. As the functor j_* is left exact, $j_*(\mathcal{G})$ is then a subsheaf of $j_*(j^{-1}(\mathcal{F}))$. Let $\rho_{\mathcal{F}} : \mathcal{F} \rightarrow j_*(j^{-1}(\mathcal{F}))$ be the canonical homomorphism associated with \mathcal{F} and consider the subsheaf $\bar{\mathcal{G}} = \rho_{\mathcal{F}}^{-1}(j_*(\mathcal{G}))$ of \mathcal{F} . It follows immediately from definition that, for any open subset V of X , $\Gamma(V, \bar{\mathcal{G}})$ is formed by sections $s \in \Gamma(V, \mathcal{F})$ whose restriction on $V \cap U$ is a section of \mathcal{G} over $V \cap U$. In particular, we have $\bar{\mathcal{G}}|_U = j^{-1}(\mathcal{G}) = \mathcal{G}$, and $\bar{\mathcal{G}}$ is the largest subsheaf of \mathcal{F} inducing \mathcal{G} on U . We say the subsheaf $\bar{\mathcal{G}}$ is the **canonical extension** of the subsheaf \mathcal{G} of $\mathcal{F}|_U$ to a subsheaf of \mathcal{F} .

Proposition 18.6.56. *Let X be a scheme and U be an open subset of X such that the canonical injection $j : U \rightarrow X$ is quasi-compact (in other words, U is retrocompact in X).*

- (a) *For any quasi-coherent $(\mathcal{O}_X|_U)$ -module \mathcal{G} , $j_*(\mathcal{G})$ is a quasi-coherent \mathcal{O}_X -module and we have $j_*(\mathcal{G})|_U = j^*(j_*(\mathcal{G})) = \mathcal{G}$.*
- (b) *For any quasi-coherent \mathcal{O}_X -module \mathcal{F} and any quasi-coherent sub- $(\mathcal{O}_X|_U)$ -module \mathcal{G} of $\mathcal{F}|_U$, the canonical extension $\bar{\mathcal{G}}$ of \mathcal{G} is a quasi-coherent sub- \mathcal{O}_X -module of \mathcal{F} .*

Proof. Assertion (a) is a special case of ?? since j is quasi-separated by ??(i), and the relation $j_*(\mathcal{G})|_U = j^*(j_*(\mathcal{G})) = \mathcal{G}$ can be checked directly. By the same reasoning, $j_*(j^*(\mathcal{F}))$ is quasi-coherent, and as $\bar{\mathcal{G}}$ is the inverse image of $j_*(\mathcal{G})$ under the homomorphism $\rho_{\mathcal{F}} : \mathcal{F} \rightarrow j_*(j^*(\mathcal{F}))$, assertion (b) follows from ??.

Corollary 18.6.57. *Let X be a scheme and U be a quasi-compact open subset of X such that the canonical injection $j : U \rightarrow X$ is quasi-compact. Suppose that any quasi-coherent \mathcal{O}_X -module is the inductive limit of its quasi-coherent sub- \mathcal{O}_X -modules of finite type (this is true if X is an affine scheme). Let \mathcal{F} be a quasi-coherent \mathcal{O}_X -module and \mathcal{G} be a quasi-coherent sub- $(\mathcal{O}_X|_U)$ -module of finite type of $\mathcal{F}|_U$. Then there exists a quasi-coherent sub- \mathcal{O}_X -module \mathcal{G}' of finite type such that $\mathcal{G}'|_U = \mathcal{G}$.*

Proof. We have $\mathcal{G} = \bar{\mathcal{G}}|_U$, and $\bar{\mathcal{G}}$ is quasi-coherent by ??, hence is the inductive limit of its quasi-coherent sub- \mathcal{O}_X -modules \mathcal{H}_{λ} of finite type. Then \mathcal{G} is the inductive limit of the $\mathcal{H}_{\lambda}|_U$, hence equals to one of $\mathcal{H}_{\lambda}|_U$ since they are of finite type (??).

Remark 18.6.3. Suppose that for any affine open $U \subseteq X$ the injection $U \rightarrow X$ is quasi-compact. Then if the conclusion of ?? holds for any affine open U and any quasi-coherent sub- $(\mathcal{O}_X|_U)$ -module of finite type \mathcal{G} of $\mathcal{F}|_U$, it follows that \mathcal{F} is the inductive limit of its quasi-coherent sub- \mathcal{O}_X -modules of finite type. In fact, for any affine open $U \subseteq X$, we have $\mathcal{F}|_U = \tilde{M}$, where M is a $\Gamma(U, \mathcal{O}_X)$ -module, and as the latter is the inductive limit of its finitely generated submodules, $\mathcal{F}|_U$ is the inductive limit of its quasi-coherent sub- $(\mathcal{O}_X|_U)$ -modules of finite type. Now, by hypotheses, such a submodule is induced over U by a quasi-coherent sub- \mathcal{O}_X -module of finite type $\mathcal{G}_{\lambda, U}$ of \mathcal{F} . The finite sums of $\mathcal{G}_{\lambda, U}$ are then quasi-coherent of finite type, since the question is local and we can assume that X is affine, where the conclusion is trivial. It then follows that \mathcal{F} is the inductive limit of these finite sums, whence our assertion.

Corollary 18.6.58. *Under the hypotheses of ??, if \mathcal{G} is a quasi-coherent $(\mathcal{O}_X|_U)$ -module of finite type, there exists a quasi-coherent \mathcal{O}_X -module \mathcal{G}' such that $\mathcal{G}'|_U = \mathcal{G}$.*

Proof. As $\mathcal{F} = j_*(\mathcal{G})$ is quasi-coherent (??) and $\mathcal{F}|_U = \mathcal{G}$, it suffices to apply ?? to \mathcal{F} . □

Lemma 18.6.59. *Let X be a scheme, $(V_{\lambda})_{\lambda \in L}$ be a covering of X by affine opens where L is well-ordered, and U be an open subset of X . For each $\lambda \in L$, let $W_{\lambda} = \bigcup_{\mu < \lambda} V_{\mu}$. Suppose that*

- (i) *for any $\lambda \in L$, $V_{\lambda} \cap W_{\lambda}$ is quasi-compact;*
- (ii) *the canonical injection $j : U \rightarrow X$ is quasi-compact.*

Then for any quasi-coherent \mathcal{O}_X -module \mathcal{F} and any quasi-coherent sub- $(\mathcal{O}_X|_U)$ -module of finite type \mathcal{G} of $\mathcal{F}|_U$, there exists a quasi-coherent sub- \mathcal{O}_X -module \mathcal{G}' of \mathcal{F} such that $\mathcal{G}'|_U = \mathcal{G}$.

Theorem 18.6.60. *Let X be a scheme and U be an open subset of X . Suppose that one of the following conditions is satisfied:*

- (a) *X is locally Noetherian;*
- (b) *X is quasi-compact and quasi-separated and U is quasi-compact.*

Then for any quasi-coherent \mathcal{O}_X -module \mathcal{F} and any quasi-coherent sub- $(\mathcal{O}_X|_U)$ -module of finite type \mathcal{G} of $\mathcal{F}|_U$, there exists a quasi-coherent sub- \mathcal{O}_X -module \mathcal{G}' of \mathcal{F} such that $\mathcal{G}'|_U = \mathcal{G}$.

Corollary 18.6.61. *With the conditions of ??, for any quasi-coherent $(\mathcal{O}_X|_U)$ -module \mathcal{G} of finite type, there exists a quasi-coherent \mathcal{O}_X -module \mathcal{G}' of finite type such that $\mathcal{G}'|_U = \mathcal{G}$.*

Corollary 18.6.62. *Let X be a locally Noetherian scheme or a quasi-compact and quasi-separated scheme. Then any quasi-coherent \mathcal{O}_X -module is the inductive limit of its quasi-coherent sub- \mathcal{O}_X -modules of finite type.*

Proof. This follows from ?? and ??.

□

Corollary 18.6.63. *Under the hypotheses of ??, if a quasi-coherent \mathcal{O}_X -module \mathcal{F} is such that any quasi-coherent sub- \mathcal{O}_X -module of finite type of \mathcal{F} is generated by its global sections, then \mathcal{F} is generated by its global sections.*

Proof. Let U be an affine neighborhood of a point $x \in X$, and let s be a section of \mathcal{F} over U . The sub- \mathcal{O}_X -module \mathcal{G} of $\mathcal{F}|_U$ generated by s is quasi-coherent and of finite type, hence there exists a quasi-coherent sub- \mathcal{O}_X -module of finite type \mathcal{G}' of \mathcal{F} such that $\mathcal{G}'|_U = \mathcal{G}$ (??). By hypotheses, there is then a finite number of sections t_i of \mathcal{G}' over X and sections a_i of \mathcal{O}_X over a neighborhood $V \subseteq U$ of x such that $s|_V = \sum_i a_i \cdot (t_i|_V)$, which proves the corollary. □

18.6.7 Scheme-theoretic image

Let $f : X \rightarrow Y$ be a morphism of schemes. If there exists a smallest closed subscheme Y' of Y such that the canonical injection $j : Y' \rightarrow Y$ dominates f (or equivalently, the inverse image $f^{-1}(Y')$ is equal to X), we then say that Y' is the **scheme-theoretic image** of X under f , or the **scheme-theoretic image of f** . If X is a subscheme of Y , the scheme-theoretic image of the canonical injection $j : X \rightarrow Y$ is called the **scheme-theoretic closure** of X .

Proposition 18.6.64 (Transitivity). *Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be morphisms. Suppose that Y' is the scheme-theoretic image of X under f , and if g' is the restriction of g to Y' , the scheme-theoretic image of Z' under g' exists. Then the scheme-theoretic image of X under $g \circ f$ is equal to Z' .*

Proof. To say that a closed subscheme Z_1 of Z is such that $(g \circ f)^{-1}(Z_1) = X$ signifies that $f^{-1}(g^{-1}(Z_1)) = X$, or that f is dominated by the canonical injection $g^{-1}(Z_1) \rightarrow Y$. Now, in view of the existence of the scheme-theoretic image Y' , for any closed subscheme Z_1 of Z having this property, $g^{-1}(Z_1)$ dominates Y' , which, if $j : Y' \rightarrow Y$ is the canonical injection, amounts to saying that $j^{-1}(g^{-1}(Z_1)) = g'^{-1}(Z_1) = Y'$. We then conclude that Z' is the smallest closed subscheme Z_1 having this property, whence our claim. □

Corollary 18.6.65. *Let $f : X \rightarrow Y$ be an S -morphism such that Y is the scheme-theoretic image of Y under f . Let Z be a separated S -scheme; if two S -morphisms $g_1, g_2 : Y \rightarrow Z$ are such that $g_1 \circ f = g_2 \circ f$, then $g_1 = g_2$.*

Proof. Let $h = (g_1, g_2)_S : Y \rightarrow Z \times_S Z$. As the diagonal $T = \Delta_Z(Z)$ is a closed subscheme of $Z \times_S Z$, $Y' = h^{-1}(T)$ is a closed subscheme of Y . Put $u = g_1 \circ f = g_2 \circ f$; we then have $h \circ f = (u, u)_S = \Delta_Z \circ u$. As $\Delta_Z^{-1}(T) = Z$, we have $(h \circ f)^{-1}(T) = u^{-1}(Z) = X$, so $f^{-1}(Y') = X$. We then conclude that the canonical injection $Y' \rightarrow Y$ dominates f , so $Y' = Y$ by hypothesis. Then by ??, h factors into $\Delta_Z \circ v$ where v is a morphism $Y \rightarrow Z$, which implies $g_1 = g_2 = v$. □

Let $f : X \rightarrow Y$ be a morphism and suppose that the scheme-theoretic image Y' of f exists. Then Y' is defined by a quasi-coherent ideal \mathcal{J}' of \mathcal{O}_Y , and by definition, \mathcal{J}' is the largest quasi-coherent ideal such that the homomorphism $f^\# : \mathcal{O}_Y \rightarrow f_*(\mathcal{O}_X)$ factors into $\mathcal{O}_Y \rightarrow \mathcal{O}_Y/\mathcal{J}' \rightarrow f_*(\mathcal{O}_X)$. This implies that $\mathcal{J}' \subseteq \ker f^\# = \mathcal{J}$; we therefore obtain a case where Y' exists, the one where \mathcal{J} is quasi-coherent, and where $\mathcal{J}' = \mathcal{J}$.

Proposition 18.6.66. *Let $f : X \rightarrow Y$ be a morphism. Then the scheme-theoretic image of X under f exists if one of the following conditions is satisfied:*

- (a) $f_*(\mathcal{O}_X)$ is quasi-coherent (which is the case if f is quasi-compact and quasi-separated).
- (b) X is reduced.

In this case, the underlying space of Y' is equal to $\overline{f(X)}$, and if f factors into

$$X \xrightarrow{f'} Y' \xrightarrow{\gamma} Y$$

where j is the canonical injection, f' is scheme-theoretic dominant. Moreover, if X is reduced (resp. integral), so is Y' .

Proof. The case (a) is immediate by our previous argument; moreover, as $\mathcal{O}_Y/\mathcal{J} \rightarrow f_*(\mathcal{O}_X)$ is then injective, this shows that f' is scheme-theoretic dominant. We still need to verify that the closed subscheme of Y defined by $\mathcal{J} = \ker f^\#$ has underlying space $\overline{f(X)}$. Since the support of $f_*(\mathcal{O}_X)$ is contained in $\overline{f(X)}$, we have $\mathcal{J}_y = \mathcal{O}_y$ for $y \notin \overline{f(X)}$, so the support pf $\mathcal{O}_Y/\mathcal{J}$ is contained in $\overline{f(X)}$. Moreover, this support is closed and contains $f(X)$: if $y \in f(X)$, the identity element of the ring $(f_*(\mathcal{O}_X))_y$ is nonzero, being the germ at y of the section $1 \in \Gamma(X, \mathcal{O}_X) = \Gamma(Y, f_*(\mathcal{O}_X))$; as this is the image of the identity element under $f^\#$, it is not contained in \mathcal{J}_y , so $\mathcal{O}_y/\mathcal{J}_y \neq 0$; this proves our first claim. The case (b) follows from ??, because there is a smallest closed subscheme Z with underlying space $\overline{f(X)}$ such that $f(X) \subseteq Z$. \square

Proposition 18.6.67. *Suppose the notations of ?? is satisfied, and let Y' be the scheme-theoretic image of X under f . For any open subset V of Y , let $f_V : f^{-1}(V) \rightarrow V$ be the restriction of f . Then the scheme-theoretic image of $f^{-1}(V)$ under f_V exists and is equal to the oepn subscheme $V \cap Y'$ of Y' .*

Proof. Put $X' = f^{-1}(V)$; as the direct image of $\mathcal{O}_{X'}$ is the restriction of $f_*(\mathcal{O}_X)$ to V , it is clear that the kernel of the homomorphism $\mathcal{O}_V \rightarrow (f_V)_*(\mathcal{O}_{X'})$ is the restriction of \mathcal{J} to V , whence the assertion. \square

Proposition 18.6.68. *Let Y be a subscheme of a scheme X , such that the canonical injection $j : Y \rightarrow X$ is quasi-compact. Then the scheme-theoretic closure of Y exists and has \overline{Y} as underlying space.*

Proof. It suffices to apply ?? to the injection j , which is separated (??) and quasi-compact by hypothesis. \square

With these notations, let \overline{Y} be the scheme-theoretic closure of Y in X . If the injection $\overline{Y} \rightarrow X$ is quasi-compact, and if \mathcal{J} is the quasi-coherent ideal of $\mathcal{O}_X|_{\overline{Y}}$ defining the closed subscheme Y of \overline{Y} , then the quasi-coherent ideal of \mathcal{O}_X defining \overline{Y} is the canonical extension (??) $\overline{\mathcal{J}}$ of \mathcal{J} , because it is evidently the largest quasi-coherent ideal of \mathcal{O}_X inducing \mathcal{J} over Y .

Corollary 18.6.69. *Under the hypothesis of ??, any section of $\mathcal{O}_{\bar{Y}}$ over an open subset V of \bar{Y} that is zero on $V \cap Y$ is zero.*

Proof. In view of ??, we can assume that $V = \bar{Y}$. If we consider sections of $\mathcal{O}_{\bar{Y}}$ over \bar{Y} as \bar{Y} -sections of $\bar{Y} \otimes_{\mathbb{Z}} \mathbb{Z}[T]$, which is separated over \bar{Y} , the assertion is then a particular case of ??.

□

18.7 Rational maps over schemes

18.7.1 Rational maps and rational functions

Let X and Y be two schemes, U and V be open dense sets of X , and f (resp. g) be a morphism from U (resp. V) to Y . We say the morphisms f and g are equivalent if they coincide over an open subset dense in $U \cap V$. As the intersection of finitely many open dense subsets of X is an open dense subset of X , it is clear that this relation is an equivalence relation.

Given two schemes X and Y , a **rational map** from X to Y is defined to be an equivalent class of morphisms from an open dense subset of X to Y . If X and Y are S -schemes, this class is called an **rational S -map** if there exists an S -morphism in it. An rational S -map from S to X is called an **rational S -section** of X . The rational X -sections of the X -scheme $X \otimes_{\mathbb{Z}} \mathbb{Z}[T]$ (where T is an indeterminate) are called the **rational functions over X** (cf. ??). A rational map f from X to Y is usually denoted by $f : X \dashrightarrow Y$.

Let $f : X \dashrightarrow Y$ be a rational map and U be an open subset of X . If f_1, f_2 are morphisms belonging to the class f , defined respectively over the open dense sets V_1, V_2 of X , the restrictions $f_1|_{U \cap V_1}$ and $f_2|_{U \cap V_2}$ coincide on $U \cap V_1 \cap V_2$, which is dense in U ; the class of morphisms f therefore defines a rational map $U \dashrightarrow Y$, called the **restriction** of f to U and denoted by $f|_U$.

It is clear that we have a canonical map from $\text{Hom}_S(X, Y)$ to the set of rational S -maps from X to Y , which associates any S -morphism $f : X \rightarrow Y$ to the rational S -map it belongs to. If we denote by $\Gamma_{\text{rat}}(X/Y)$ the set of rational Y -sections of X , we then have a canonical map $\Gamma(X/Y) \rightarrow \Gamma_{\text{rat}}(X/Y)$. It is also clear that if X and Y are two S -schemes, the set of rational S -maps from X to Y is canonically identified with $\Gamma_{\text{rat}}((X \times_S Y)/X)$.

In view of ??, we see the rational functions over X are canonically identified with the equivalent classes of sections of the structural sheaf \mathcal{O}_X over open dense sets of X , where two sections are equivalent if they coincide over an open dense subset of the intersection of their defining domain. In particular, we see the rational functions over X form a ring $K(X)$.

If X is an irreducible scheme, any nonempty open subset of X is dense; we can also say that the non-empty open sets of X are the open neighborhoods of the generic point x of X . To say that two morphisms from nonempty open subsets of X to Y are equivalent therefore means in this case that they have the same germ at the point x . In other words, rational maps (resp. rational S -maps) $X \dashrightarrow Y$ are identified with the germs of morphisms (resp. of S -morphisms) of non-empty open subsets of X to Y at the generic point x of X . In particular:

Proposition 18.7.1. *If X is an irreducible scheme, the ring $K(X)$ of rational functions over X is canonically identified with the local ring $\mathcal{O}_{X,x}$ at the generic point x of X . This is a local ring of zero dimension, and therefore an Artinian local ring if X is Noetherian. It is a field if X is integral, and is identified with the fraction field of $\Gamma(X, \mathcal{O}_X)$ if X is moreover affine.*

Proof. Since we can identify rational functions with sections over X , the first assertion follows from the definition of stalks. For the second one, we can assume that X is affine with ring A ; then \mathfrak{p}_x is the nilradical of A , and in particular $\mathcal{O}_{X,x}$ has zero dimension. If A is integral, $\mathfrak{p}_x = (0)$ and $\mathcal{O}_{X,x}$ is the fraction field of A . Finally, if A is Noetherian, then \mathfrak{p}_x is nilpotent and $\mathcal{O}_{X,x} = A_x$ is Artinian. \square

If X is integral, the ring $\mathcal{O}_{X,z}$ is integral for any $z \in X$. Any affine open U containing x must contain x as its generic point, and $\mathcal{O}_{X,z}$, equal to a fraction field of $\Gamma(U, \mathcal{O}_X)$, is identified with $K(X)$. We then conclude that $K(X)$ is identified with the fraction field of $\mathcal{O}_{X,z}$, and in this way, $\mathcal{O}_{X,z}$ is canonically identified with a subring of $K(X)$, so that a germ $s \in \mathcal{O}_{X,z}$ is canonically identified with a rational function over X .

Proposition 18.7.2. *Let X and Y be two S -schemes such that the family (X_λ) of irreducible components of X is locally finite. For each λ , let x_λ be the generic point of X_λ . If R_λ is the set of germs at x_λ of S -morphisms from open subsets of X to Y , the set of rational S -maps from X to Y is identified with the product of R_λ . In particular, the ring of rational functions over X is identified with the product of the local rings $\mathcal{O}_{X,x_\lambda}$.*

Proof. The set of the intersections $X_\lambda \cap X_\mu$ for $\lambda \neq \mu$ is then locally finite, so their union is closed and contains the maximal points of X . If we set $X'_\lambda = X_\lambda - \bigcup_{\mu \neq \lambda} X_\lambda \cap X_\mu$, then X'_λ is irreducible, with generic point equal to that of X_λ , and pairwise disjoint with union dense in X . For any open dense subset U of X , $U'_\lambda = U \cap X'_\lambda$ is a nonempty open dense subset of X'_λ , and U'_λ are pairwise disjoint with $U' = \bigcup_\lambda U'_\lambda$ closed in X . To give a morphism from U' to Y is then equivalent to giving (arbitrarily) a morphism from each of the U'_λ in Y , so the assertion follows. \square

Corollary 18.7.3. *Let A be a Noetherian ring and $X = \text{Spec}(A)$. The ring of rational function functions over X is identified with the total fraction ring $Q(A)$.*

Proof. Let S be the complement of the union of minimal prime ideals of A . Then by ??, the ring of sections $\Gamma(D(f), \mathcal{O}_X)$ is identified with A_f , so $D(f)$ with $f \in S$ form a cofinal subset of the open dense sets of X , and the ring of rational functions over X is then identified with the inductive limit of $A_f, f \in S$, which is exactly $Q(A)$. \square

Suppose that X is irreducible with generic point x . As any nonempty open set U of X contains x , and therefore contains any generalization $z \in X$, any morphism $U \rightarrow Y$ can be composed with the canonical morphism $\text{Spec}(\mathcal{O}_{X,x}) \rightarrow X$ (?). Two morphisms from nonempty open subsets of X to Y which coincide on a smaller open subset then give the same morphism $\text{Spec}(\mathcal{O}_{X,x}) \rightarrow Y$. In other words, to any rational map X to Y , there corresponds a well-defined morphism $\text{Spec}(\mathcal{O}_{X,x}) \rightarrow Y$.

Proposition 18.7.4. *Let X and Y be S -schemes. Suppose that X is irreducible with generic point x , and Y is of finite type over S . Then two rational S -maps X to Y corresponding to the same S -morphism $\text{Spec}(\mathcal{O}_{X,x}) \rightarrow Y$ are identical. If moreover S is locally of finite presentation over S , then any S -morphism $\text{Spec}(\mathcal{O}_{X,x}) \rightarrow Y$ corresponds to a rational S -map from X to Y .*

Proof. Given that every non-empty open subset of X is dense, this result follows immediately from ???. \square

Corollary 18.7.5. Suppose that X is irreducible with generic point x and Y is locally of finite presentation. Then the rational S -maps from X to Y are identified with the points of Y with values in the S -scheme $\text{Spec}(\mathcal{O}_{X,x})$.

Proof. This is just a reformulation of ??.

□

Corollary 18.7.6. Suppose that X is irreducible with generic point x and Y is locally of finite presentation. Let s be the image of x in S . Then giving a rational S -map from X to Y is equivalent to giving a point y of Y lying over s and a $\mathcal{O}_{S,s}$ -homomorphism $\mathcal{O}_{Y,y} \rightarrow \mathcal{O}_{X,x} = K(X)$.

Proof. This follows from ?? and ??.

□

Corollary 18.7.7. Suppose that X is irreducible with generic point x and Y is locally of finite presentation, then the rational S -maps from X to Y (with Y given) only depends on the S -scheme $\text{Spec}(\mathcal{O}_{X,x})$ and in particular remain the same if we replace X by $\text{Spec}(\mathcal{O}_{X,z})$, $z \in X$.

Proof. In fact, if $z \in \overline{\{x\}}$ then x is the generic point of $Z = \text{Spec}(\mathcal{O}_{X,z})$ and $\mathcal{O}_{X,x} = \mathcal{O}_{Z,x}$.

□

Corollary 18.7.8. Suppose that X is integral with generic point x and Y is locally of finite presentation. Let s be the image of x in S . Then following dates are equivalent:

- (i) a rational S -map from X to Y ;
- (ii) a point of $Y \otimes_S \kappa(s)$ with values in the extension $K(X)$ of $\kappa(s)$;
- (iii) a point $y \in Y$ over s and an $\kappa(s)$ -homomorphism $\kappa(y) \rightarrow \kappa(x) = K(X)$.

Proof. The points of Y over s belong to $Y \otimes_S \kappa(s)$ and the $\mathcal{O}_{S,s}$ -homomorphisms $\mathcal{O}_{Y,y} \rightarrow K(X)$ are $\kappa(s)$ -homomorphisms $\kappa(y) \rightarrow K(X)$, since $K(X)$ is a field.

□

Corollary 18.7.9. Let k be a field and X, Y be two schemes locally algebraic over k . Suppose that X is integral, then the rational k -maps from X to Y are identified with the points of Y with values in the extension $K(X)$ of k .

18.7.2 Defining domain of a rational map

Let X and Y be schemes, f a rational map from X to Y . We say f is **defined at a point** $x \in X$ if there exists an open dense subset U containing x and a morphism $U \rightarrow Y$ representing f . The set of points $x \in X$ where f is defined is called the **defining domain** of the rational map f . It is clearly an open dense subset of X .

Proposition 18.7.10. Let X and Y be two S -schemes such that X is reduced and Y is separated over S . Let f be a rational S -map from X to Y , and U_0 be its domain. There then exists a unique S -morphism $U_0 \rightarrow Y$ belonging to the class f .

Proof. For any morphism $U \rightarrow Y$ belonging to the class f , we necessarily have $U \subseteq U_0$, so we only need to prove that if U_1, U_2 are two dense subsets of X and $f_i : U_i \rightarrow Y$ ($i = 1, 2$) are two S -morphisms that coincide on an open subset $V \subseteq U_1 \cap U_2$, then f_1 and f_2 coincide on $U_1 \cap U_2$. For this, we can clearly assume that $X = U_1 = U_2$. As X (hence V) is reduced, X is smallest closed subscheme of X dominating V (?). Let $g = (f_1, f_2)_S : X \rightarrow Y \times_S Y$; as by hypothesis the diagonal $T = \Delta_Y(Y)$ is a closed subscheme of $Y \times_S Y$, $Z = g^{-1}(T)$ is a closed subscheme

of X . If $h : V \rightarrow Y$ is the restriction of f_1 and f_2 to V , the restriction of g to V is $\tilde{g} = (h, h)_S$, which factors into $\tilde{g} = \Delta_Y \circ h$. As $\Delta_Y^{-1}(T) = Y$, we have $\tilde{g}^{-1}(T) = V$, and Z is therefore a closed subscheme of X inducing the subscheme structure on V , hence dominates V , and this implies $Z = X$. From the relation $g^{-1}(T) = X$, we deduce that g factors into $\Delta_Y \circ f$, where f is a morphism $X \rightarrow Y$ (??), and we have $f_1 = f_2 = f$ from the definition of the diagonal morphism. \square

It is clear that the morphism $U_0 \rightarrow Y$ defined in ?? is the unique morphism in the class f that admits no further extension to open dense subsets of X containing U_0 . Under the conditions of ??, we can then identify the rational maps from X to Y with the morphisms unextendable to open dense subsets of X to Y .

Corollary 18.7.11. *Let X and Y be two S -schemes such that X is reduced and Y is separated over S . Let U be an open dense subset of X , then there is a canonical bijective correspondence between the S -morphisms from U to Y and the rational S -maps from X to Y defined at each point of U .*

Proof. In view of ??, for any S -morphism $f : U \rightarrow Y$, there exists a rational S -map \bar{f} from X to Y which extends f . \square

Corollary 18.7.12. *Let S be a separated scheme, X be a reduced S -scheme, Y be an S -scheme, and $f : U \rightarrow Y$ be an S -morphism from an open dense subset U of X to Y . If \bar{f} is the rational \mathbb{Z} -map from X to Y extending f , \bar{f} is an S -morphism (and therefore the rational S -map from X to Y extending f).*

Proof. In fact, if $\varphi : X \rightarrow S$ and $\psi : Y \rightarrow S$ are the structural morphisms, U_0 is the defining domain of \bar{f} , and $j : U_0 \rightarrow X$ is the injection, it suffices to prove that $\psi \circ \bar{f} = \varphi \circ j$, which follows from the proof of ??, since f is an S -morphism. \square

Corollary 18.7.13. *Let X and Y be S -schemes. Suppose that X is reduced and X, Y are separated over S . Let $p : Y \rightarrow X$ be an S -morphism, U be an open dense subset of X , and f be a U -section of Y . Then the rational map \bar{f} from X to Y extending f is a rational X -section of Y .*

Proof. We only need to prove that $p \circ \bar{f}$ is the identity on the defining domain of \bar{f} ; since X is separated over S , this follows from the proof of ??.

Corollary 18.7.14. *Let X be a reduced scheme and U be an open dense subset of X . There exists a canonical bijective correspondence between sections of \mathcal{O}_X over U and rational functions f over X defined on each point of U .*

Proof. It suffices to remark that the X -scheme $X \otimes_{\mathbb{Z}} \mathbb{Z}[T]$ is separated over X by ??.

Corollary 18.7.15. *Let Y be a reduced scheme, $f : X \rightarrow Y$ be a separated morphism, U be an open dense subset of Y , $g : U \rightarrow f^{-1}(U)$ be a U -section of $f^{-1}(U)$, and Z the reduced subscheme of X induced on $\overline{g(U)}$. For g to be the restriction of a Y -section of X , it is necessary and sufficient that the restriction of f to Z is an isomorphism from Z to Y .*

Proof. The restriction of f to $f^{-1}(U)$ is a separated morphism (??(i)), so g is a closed immersion by ??, and therefore $g(U) = Z \cap f^{-1}(U)$ coincides with the subscheme induced by Z over the open subset $g(U)$ of Z . It is then clear that the given condition is sufficient, since if $f_Z : Z \rightarrow Y$ is an isomorphism and $\bar{g} : Y \rightarrow Z$ is the inverse morphism, then \bar{g} extends g . Conversely, if g is the restriction to U of a Y -section h of X , h is then a closed immersion by ??, so $h(Y)$ is closed,

and as it contains $g(U)$ and we have (as h is continuous) $h(Y) = h(\bar{U}) \subseteq \overline{h(U)} = \overline{g(U)}$, we conclude that $h(Y) = Z$. It then follows from ?? that h is necessarily an isomorphism from Y to the closed subscheme Z of X , so $f|_Z$ is also an isomorphism. \square

Let X and Y be two S -schemes, where X is reduced and Y is separated over S . Let f be a rational S -map from X to Y , and let x be a point of X . We can compose f with the canonical morphism $\text{Spec}(\mathcal{O}_{X,x}) \rightarrow X$ provided that the trace on $\text{Spec}(\mathcal{O}_{X,x})$ of the defining domain of f is dense in $\text{Spec}(\mathcal{O}_{X,x})$ (identified with the set $z \in X$ such as $x \in \overline{\{z\}}$ (cf. ??). This happens if the family of irreducible components of X is *locally finite*:

Lemma 18.7.16. *Let X be a scheme such that the family of irreducible components of X is locally finite, and x be a point of X . The irreducible components of $\text{Spec}(\mathcal{O}_{X,x})$ are then the traces over $\text{Spec}(\mathcal{O}_{X,x})$ of the irreducible components of X containing x . For an open subset U of X to be such that $U \cap \text{Spec}(\mathcal{O}_{X,x})$ is dense in $\text{Spec}(\mathcal{O}_{X,x})$, it is necessary and sufficient that it meets the irreducible components of X containing x (and this is true in particular if U is dense in X).*

Proof. The second assertion clearly follows from the first one. As $\text{Spec}(\mathcal{O}_{X,x})$ is contained in any affine open U containing x and the irreducible components of U containing x are the traces of the irreducible components of X containing x on U (??), we can suppose that X is affine with ring A . Then the prime ideals of A_x correspond to prime ideals of A contained in \mathfrak{p}_x , so the minimal prime ideals of A_x correspond to minimal prime ideals of A contained in \mathfrak{p}_x , and the lemma follows from ??.

Suppose that we are under the assumption of ???. If U is the defining domain of definition of the rational S -map f , denote by f' the rational map from $\text{Spec}(\mathcal{O}_{X,x})$ to Y which coincides with f over $U \cap \text{Spec}(\mathcal{O}_{X,x})$; we will say that this rational map is **induced** by f .

Proposition 18.7.17. *Let S be a scheme, X be a reduced S -scheme, and Y be a separated S -scheme that is locally of finite presentation over S . Suppose that the family of irreducible components of X is locally finite. Let f be a rational S -map from X to Y and x be a point of X . For f to be defined at the point x , it is necessary and sufficient that the rational map f' from $\text{Spec}(\mathcal{O}_{X,x})$ to Y induced by f is a morphism.*

Proof. The conditions is clearly necessary since $\text{Spec}(\mathcal{O}_{X,x})$ is contained in any open subset containing x . We now prove the sufficiency, so suppose that f' is a morphism. In view of ??, there exists an open neighborhood U of x in X and an S -morphism $g : U \rightarrow Y$ inducing f' on $\text{Spec}(\mathcal{O}_{X,x})$. The point is that U is not necessarily dense in X , so we want to extend g to a morphism defined on an open dense subset of X . Now by ??, there are finitely many irreducible components X_i of X containing x , and we can assume that these are the only ones meeting U , by replacing U with a smaller open subset. As the generic points of X_i belong to the defining domain of f and to U , we see that f and g coincide over a non-empty open dense subset of each of the X_i (??). Consider the morphism f_1 defined on an open dense subset of $U \cup (X - \bar{U})$ which equals to g over U and to f over the intersection of $X - \bar{U}$ and the defining domain of f (we also note that each X_i is contained in \bar{U}). As $U \cup (X - \bar{U})$ is dense in X , f_1 and f coincide on an open dense subset of X , and f is an extension of f_1 . Since f_1 is defined at x , this shows f is defined at x . \square

18.7.3 Sheaf of rational functions

Let X be a scheme. For each open subset U of X , we denote by $K(U)$ the ring of rational functions over U , which is an $\Gamma(U, \mathcal{O}_X)$ -algebra. Moreover, if $V \subseteq U$ is a second open subset of X , any section of \mathcal{O}_X over a dense subset of U restricts to a section over a dense subset of V , and if two sections coincide over an open dense subset of U , their restriction also coincide over a smaller open dense subset of V . We then define a homomorphism of algebras $\text{Res}_V^U : K(U) \rightarrow K(V)$, and it is clear that for $U \supseteq V \supseteq W$ open in X we have $\text{Res}_W^U = \text{Res}_W^V \circ \text{Res}_V^U$. Therefore, we get a presheaf of algebras over X . The associated sheaf of \mathcal{O}_X -algebras over X is then called the **sheaf of rational functions** over the scheme X , and denoted by \mathcal{K}_X . For any open subset U of X , it is clear that the restriction $\mathcal{K}_X|_U$ is equal to \mathcal{K}_U .

Proposition 18.7.18. *Let X be a scheme such that the family (X_λ) of irreducible components of X is locally finite. Then the \mathcal{O}_X -module \mathcal{K}_X is quasi-coherent and for any open subset U of X , $K(U)$ is equal to $\Gamma(U, \mathcal{K}_X)$ and is identified with the product of the local rings of the generic points x_λ of the irreducible components X_λ such that $X_\lambda \cap U \neq \emptyset$.*

Proof. The fact that $K(U)$ is identified with the product follows from ???. We now show that the presheaf $U \mapsto K(U)$ is a sheaf. Consider an open subset U of X and an open covering (V_α) of U . If $s_\alpha \in K(V_\alpha)$ are such that s_α and s_β coincide over $V_\alpha \cap V_\beta$ for each pair of indices, we then conclude that for any index λ such that $U \cap X_\lambda \neq \emptyset$, the component in $K(X_\lambda)$ of all s_α such that $V_\alpha \cap X_\lambda \neq \emptyset$ are the same. Denoting by t_λ this component, it is clear that the element of $K(U)$ with component t_λ in $K(X_\lambda)$ has restriction s_α on each V_α . Finally, to see the sheaf \mathcal{K}_X is quasi-coherent, we can limit ourselves to the case $X = \text{Spec}(A)$ is affine with finitely many irreducible components; by taking for U the affine open sets of the form $D(f)$, where $f \in A$, it follows from the above argument that we have $\mathcal{K}_X = \tilde{M}$, where M is the direct sum of the A -modules A_{x_λ} . \square

Corollary 18.7.19. *Let X be a reduced scheme with irreducible components $(X_i)_{1 \leq i \leq n}$, endowed with the reduced subscheme structures. If $\iota_i : X_i \rightarrow X$ is the canonical injection, \mathcal{K}_X is the direct product of the \mathcal{O}_X -algebras $(\iota_i)_*(\mathcal{K}_{X_i})$.*

Proof. This is a particular case of ??, in view of the conditions in that proposition. \square

Corollary 18.7.20. *If X is irreducible, any quasi-coherent \mathcal{K}_X -module \mathcal{F} is a simple sheaf.*

Proof. It suffices to show that any $x \in X$ admits a neighborhood U such that $\mathcal{F}|_U$ is a simple sheaf, which means we can assume that X is affine. We can then suppose that \mathcal{F} is the cokernel of a homomorphism $\mathcal{K}_X^{\oplus I} \rightarrow \mathcal{K}_X^{\oplus J}$, and it all boils down to seeing that \mathcal{K}_X is a simple sheaf. But this is evident since $\Gamma(U, \mathcal{K}_X) = K(X)$ for any nonempty open subset U , since U contains the generic point of X . \square

Corollary 18.7.21. *If X is irreducible, for any quasi-coherent \mathcal{O}_X -module \mathcal{F} , $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{K}_X$ is a simple sheaf. If moreover X is reduced (hence integral), $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{K}_X$ is isomorphic to a sheaf of the form $\mathcal{K}_X^{\oplus I}$.*

Proof. The first claim follows from ??, and the second one follows from the fact that if X is integral then $K(X)$ is a field. \square

Proposition 18.7.22. *Let X be a scheme such that the family (X_λ) of irreducible components of X is locally finite. Then \mathcal{K}_X is a quasi-coherent \mathcal{O}_X -algebra. If X is reduced, the canonical homomorphism $\mathcal{O}_X \rightarrow \mathcal{K}_X$ is injective.*

Proof. Since the question is local, the first claim follows from ???. The second one follows from ???. \square

Let X and Y be integral schemes, so that \mathcal{K}_X (resp. \mathcal{K}_Y) is a quasi-coherent \mathcal{O}_X -module (resp. \mathcal{O}_Y -module). Let $f : X \rightarrow Y$ be a dominant morphism; then there exists a canonical homomorphism of \mathcal{O}_X -modules:

$$\tau : f^*(\mathcal{K}_Y) \rightarrow \mathcal{K}_X.$$

Suppose that $X = \text{Spec}(A)$ and $Y = \text{Spec}(B)$ are affine with integral domains A and B , then f corresponds to the injective homomorphism $B \rightarrow A$??, which extends to a monomorphism $L \rightarrow K$ of fraction fields. The homomorphism τ then corresponds to the canonical homomorphism $L \otimes_B A \rightarrow K$.

In the general case, for any couple of affine opens $U \subseteq X, V \subseteq Y$ such that $f(U) \subseteq V$, we define similarly a homomorphism $\tau_{U,V}$ and note that if $U' \subseteq U, V' \subseteq V$ and $f(U') \subseteq V'$, then $\tau_{U,V}$ extends $\tau_{U',V'}$. If x and y are the generic points of X and Y , respectively, then $f(x) = y$ and

$$(f^*(\mathcal{K}_Y))_x = \mathcal{O}_{Y,y} \otimes_{\mathcal{O}_{Y,y}} \mathcal{O}_{X,x} = \mathcal{O}_{X,x}$$

and τ_x is therefore an isomorphism. However, the homomorphism τ is usually not an isomorphism: for example, if $B = L$ is a field containing the integral domain A and A is not a field, the canonical homomorphism $L \otimes_B A \rightarrow K$ is then the canonical homomorphism $A \rightarrow K$, which is not bijective.

18.7.4 Torsion sheaves and torsion-free sheaves

Let X be a reduced scheme whose family of irreducible components is locally finite. For any \mathcal{O}_X -module \mathcal{F} , the canonical homomorphism $\mathcal{O}_X \rightarrow \mathcal{K}_X$ is injective by ??, and defines a homomorphism $\mathcal{F} \rightarrow \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{K}_X$ which on each stalk, is none other than the homomorphism $z \mapsto z \otimes 1$ from \mathcal{F}_x to $\mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{K}_{X,x}$. The kernel \mathcal{T} (also denoted by $\mathcal{T}(\mathcal{F})$) of this homomorphism is a sub- \mathcal{O}_X -module \mathcal{F} called the **torsion sheaf** of \mathcal{F} , which is quasi-coherent if \mathcal{F} is quasi-coherent ???. The sheaf \mathcal{F} is called **torsion-free** if $\mathcal{T} = 0$, and a **torsion sheaf** if $\mathcal{T} = \mathcal{F}$. For any \mathcal{O}_X -module \mathcal{F} , \mathcal{F}/\mathcal{T} is torsion-free.

Proposition 18.7.23. *If X is an integral scheme, for a quasi-coherent \mathcal{O}_X -module \mathcal{F} to be torsion-free, it is necessary and sufficient that it is isomorphic to a sub- \mathcal{O}_X -module \mathcal{G} of a simple sheaf of the form $\mathcal{K}_X^{\oplus I}$, generated (as a \mathcal{K}_X -module) by \mathcal{G} .*

Proof. This follows from ??, since \mathcal{F} is torsion-free if and only if the canonical homomorphism $\mathcal{F} \rightarrow \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{K}_X$ is injective. \square

The cardinality of the set I is called the **rank** of \mathcal{F} . For any affine open U of X , since the generic point of x is contained in U , the rank of \mathcal{F} is also equal to the rank of $\Gamma(U, \mathcal{F})$ as a $\Gamma(U, \mathcal{O}_X)$ -module.

Corollary 18.7.24. *Over an integral scheme X , any torsion-free quasi-coherent \mathcal{O}_X -module of rank 1 (and in particular any invertible \mathcal{O}_X -module) is isomorphic to a sub- \mathcal{O}_X -module of \mathcal{K}_X , and the converse is also true.*

Corollary 18.7.25. *Let X be an integral scheme, $\mathcal{L}, \mathcal{L}'$ be two torsion-free \mathcal{O}_X -module, s (resp. s') be two sections of \mathcal{L} (resp. \mathcal{L}') over X . For $s \otimes s' = 0$, it is necessary and sufficient that one of the sections s, s' is zero.*

Proof. Let x be the generic point of X . We have by hypothesis $(s \otimes s')_x = s_x \otimes s'_x = 0$. As \mathcal{L}_x and \mathcal{L}'_x are identified with sub- $\mathcal{O}_{X,x}$ -modules of the field $\mathcal{O}_{X,x}$, the preceding relation implies $s_x = 0$ or $s'_x = 0$, and therefore $s = 0$ or $s' = 0$ since \mathcal{L} and \mathcal{L}' are torsion-free (??). \square

Proposition 18.7.26. *Let X and Y be two integral schemes and $f : X \rightarrow Y$ be a dominant morphism. For any torsion-free quasi-coherent \mathcal{O}_X -module \mathcal{F} , $f_*(\mathcal{F})$ is a torsion-free \mathcal{O}_Y -module.*

Proof. As f_* is left exact, it suffices, in view of ??, to prove the proposition for $\mathcal{F} = \mathcal{K}_X^{\oplus I}$. Now any open subset U of Y contains the generic point of Y , hence $f^{-1}(U)$ contains the generic point of X , so we have $\Gamma(U, f_*(\mathcal{F})) = \Gamma(f^{-1}(U), \mathcal{F}) = K(X)^{\oplus I}$. Therefore $f_*(\mathcal{F})$ is the simple sheaf with stalk $K(X)^{\oplus I}$, considered as a \mathcal{K}_Y -module, and it is evidently torsion-free. \square

Proposition 18.7.27. *Let X be reduced scheme whose family of irreducible components is locally finite. For any quasi-coherent \mathcal{O}_X -module \mathcal{F} of finite type, the following conditions are equivalent:*

- (i) \mathcal{F} is a torsion sheaf.
- (ii) $\mathcal{F}_x = 0$ for every maximal point of x .
- (iii) $\text{supp}(\mathcal{F})$ contains no irreducible component of X .

Proof. Since the question is local, we may assume that X has finitely many irreducible components $(X_i)_{1 \leq i \leq n}$, with generic points x_i . Endow each X_i the reduced subscheme structure of X , and let $i_i : X_i \rightarrow X$ be the canonical injection. If we put $\mathcal{F} = i_i^*(\mathcal{F})$, we see immediately that (??) \mathcal{F} is torsion-free if and only if each \mathcal{F}_i is torsion-free. As $\mathcal{F}_{x_i} = (\mathcal{F}_i)_{x_i}$, to establish the equivalence of (i) and (ii), we can assume that X is integral. But then if x is the generic point of X , the relation $\mathcal{F}_x = 0$ and $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{K}_X = 0$ are equivalent by ?? and ???. The equivalence of (ii) and (iii) results from the fact that $\text{supp}(\mathcal{F})$ is closed in X (since \mathcal{F} is quasi-coherent) and that the conditions $\text{supp}(\mathcal{F}) \cap X_i = \emptyset$ and $x_i \notin \text{supp}(\mathcal{F})$ are then equivalent. \square

18.7.5 Separation criterion for integral schemes

Let X be an integral scheme, K its function field, identified with the local ring at the generic point ξ of X . For any $x \in X$, we can identify $\mathcal{O}_{X,x}$ as a subring of K , formed by the rational functions defined at the point x . For any rational function $f \in K$, the defining domain $\delta(f)$ of f is then the open subset of $x \in X$ such that $f \in \mathcal{O}_{X,x}$, and in view of ?? we have, for each open subset $U \subseteq X$, that

$$\Gamma(U, \mathcal{O}_X) = \bigcap_{x \in U} \mathcal{O}_{X,x}. \quad (18.7.1)$$

Given a field K , for any subring A of K , we denote by $L(A)$ the set of localizations $A_{\mathfrak{p}}$, where \mathfrak{p} runs through prime ideals of A ; they are identified with local subrings of K containing A . As $\mathfrak{p} = (\mathfrak{p}A_{\mathfrak{p}}) \cap A$, the map $\mathfrak{p} \mapsto A_{\mathfrak{p}}$ from $\text{Spec}(A)$ to $L(A)$ is bijective.

Lemma 18.7.28. *Let K be a field and A be a subring of K . For a local subring R to dominate a ring in $L(A)$, it is necessary and sufficient that $A \subseteq R$. In this case, the local ring $A_{\mathfrak{p}}$ dominated by R is then unique and corresponds to the prime ideal $\mathfrak{p} = \mathfrak{m}_R \cap A$, where \mathfrak{m}_R is the maximal ideal of R .*

Proof. In fact, if R dominates $A_{\mathfrak{p}}$, then $\mathfrak{m}_R \cap A_{\mathfrak{p}} = \mathfrak{p}A_{\mathfrak{p}}$ by ??, hence the uniqueness of \mathfrak{p} . On the other hand, if $A \subseteq R$, $\mathfrak{m}_R \cap A = \mathfrak{p}$ is a prime ideal of A , and as the elements of $A - \mathfrak{p}$ are then invertible in R , we have $A_{\mathfrak{p}} \subseteq R$, so $\mathfrak{p}A_{\mathfrak{p}} \subseteq \mathfrak{m}_R$ and R dominates $A_{\mathfrak{p}}$. \square

Lemma 18.7.29. *Let K be a field, A, B be two local subrings of K , and C be the subring of K generated by $A \cup B$. The following conditions are equivalent:*

- (i) *There exists a prime ideal \mathfrak{r} of C such that $\mathfrak{m}_A = \mathfrak{r} \cap A$ and $\mathfrak{m}_B = \mathfrak{r} \cap B$.*
- (ii) *The ideal \mathfrak{c} generated in C by $\mathfrak{m}_A \cup \mathfrak{m}_B$ is proper.*
- (iii) *There exists a local subring R of K dominating both A and B .*

Proof. It is clear that (i) implies (ii). Conversely, if \mathfrak{c} is proper, it is contained in a maximal ideal \mathfrak{n} of C , and $\mathfrak{n} \cap A$ contains \mathfrak{m}_A and is proper, so $\mathfrak{n} \cap A = \mathfrak{m}_A$ and similarly $\mathfrak{n} \cap B = \mathfrak{m}_B$. Finally, it is clear that if R dominates A and B then $C \subseteq R$ and $\mathfrak{m}_A = \mathfrak{m}_R \cap A = (\mathfrak{m}_R \cap C) \cap A$, $\mathfrak{m}_B = \mathfrak{m}_R \cap B = (\mathfrak{m}_R \cap C) \cap B$, so (iii) implies (i). the converse is clear since we can take $R = C_{\mathfrak{r}}$. \square

If the equivalent conditions in ?? hold, we say the two local subrings A and B are **related**.

Proposition 18.7.30. *Let A and B be subrings of a field K and C be the subring of K generated by $A \cup B$. The following conditions are equivalent:*

- (i) *For any local ring R containing A and B , we have $A_{\mathfrak{p}} = B_{\mathfrak{q}}$, where $\mathfrak{p} = \mathfrak{m}_R \cap A$ and $\mathfrak{q} = \mathfrak{m}_R \cap B$.*
- (ii) *For any prime ideal \mathfrak{r} of C , we have $A_{\mathfrak{p}} = B_{\mathfrak{q}}$, where $\mathfrak{p} = \mathfrak{r} \cap A$ and $\mathfrak{q} = \mathfrak{r} \cap B$.*
- (iii) *If $P \in L(A)$ and $Q \in L(B)$ are related, they are identical.*
- (iv) *We have $L(A) \cap L(B) = L(C)$.*

Proof. It follows from ?? and ?? that (i) and (iii) are equivalent, and (i) implies (ii) by applying (i) to the ring $R = C_{\mathfrak{r}}$. Conversely, (ii) implies (i) because if R contains $A \cup B$, it contains C , and if $\mathfrak{r} = \mathfrak{m}_R \cap C$, we have $\mathfrak{p} = \mathfrak{r} \cap A$ and $\mathfrak{q} = \mathfrak{r} \cap B$, so $A_{\mathfrak{p}} = B_{\mathfrak{q}}$. We also see that (iv) implies (i), because if R contains $A \cup B$, it then dominates a local ring $C_{\mathfrak{r}} \in L(C)$ by ??; we have by hypothesis that $L(C) = L(A) \cap L(B)$, and as R dominates a unique ring in $L(A)$ (resp. $L(B)$), we conclude that $C_{\mathfrak{r}} = A_{\mathfrak{p}} = B_{\mathfrak{q}}$.

Finally, we show that (iii) implies (iv). Let $R \in L(C)$; R then dominates a ring $P \in L(A)$ and a ring $Q \in L(B)$ by ??, so P and Q are related, hence identical by hypothesis. As we then have $C \subseteq P$, P dominates a ring $R' \in L(C)$ (??), so R dominates the ring R' , and by ?? we necessarily have $R = R' = P$, so $R \in L(A) \cap L(B)$. Conversely, if $R \in L(A) \cap L(B)$, we have $C \subseteq R$, so R dominates a ring $R'' \in L(C)$ by ?. The two subrings R and R'' are clearly related, and as $L(C) \subseteq L(A) \cap L(B)$, we conclude from condition (iii) that $R = R''$, so $R \in L(C)$ and the proof is complete. \square

Proposition 18.7.31. *Let X be an integral scheme and K be its field of rational functions. Then for X to be separated, it is necessary and sufficient that the relation " $\mathcal{O}_{X,x}$ and $\mathcal{O}_{X,y}$ are related" for two points $x, y \in X$ implies $x = y$.*

Proof. Suppose the given condition on X , we prove that X is separated. Let $U = \text{Spec}(A)$ and $V = \text{Spec}(B)$ be two distinct affine opens of X , with A, B identified as subrings of K . Then U (resp. V) is identified with the set $L(A)$ (resp. $L(B)$), and by ?? the hypothesis on X implies that, if C is the subring of K generated by $A \cup B$, $W = U \cap V$ is identified with $L(A) \cap L(B) = L(C)$. Moreover, we have seen from ?? that any subring R of K is equal to the intersection of the local rings belong to $L(R)$, so

$$C = \bigcap_{z \in W} \mathcal{O}_{X,z} = \Gamma(W, \mathcal{O}_X) \quad (18.7.2)$$

where we use formula (??). Consider then the subscheme induced by X over W . The identity homomorphism $\varphi : C \rightarrow \Gamma(W, \mathcal{O}_X)$ corresponds to a morphism $\psi : W \rightarrow \text{Spec}(C)$. In view of (??) and the relation $L(C) = L(A) \cap L(B)$, any prime ideal \mathfrak{r} of C is of the form $\mathfrak{r} = \mathfrak{m}_x \cap C$, where $x \in W$ is the point in $\text{Spec}(C)$ corresponding to \mathfrak{r} , and the map ψ just sends x to \mathfrak{r} , so it is bijective. On the other hand, for any $x \in W$, $\psi_x^\#$ is the canonical injection $C_{\mathfrak{r}} \rightarrow \mathcal{O}_{X,x}$, where $\mathfrak{r} = \mathfrak{m}_x \cap C$. Now the local ring $\mathcal{O}_{X,x}$ dominates $A_{\mathfrak{p}}, B_{\mathfrak{q}}$ and $C_{\mathfrak{r}}$, where $\mathfrak{p} = \mathfrak{m}_x \cap A$ and $\mathfrak{q} = \mathfrak{m}_x \cap B$, and as $C_{\mathfrak{r}} \in L(C) = L(A) \cap L(B)$, by ?? we then conclude that $C_{\mathfrak{r}} = A_{\mathfrak{p}} = B_{\mathfrak{q}}$ (we already have seen this in the proof of ??). But the local rings $A_{\mathfrak{p}}$ and $B_{\mathfrak{q}}$ are both identified with the stalk $\mathcal{O}_{X,x}$, so we see that $C_{\mathfrak{r}} = \mathcal{O}_{X,x}$ and $\psi_x^\#$ is bijective. It then remains to show that ψ is a homeomorphism, which amounts to show that, for any closed subset $F \subseteq W$, the image $\psi(F)$ is closed in $\text{Spec}(C)$. Now F is the intersection with W of a closed subset E of the form $V(\mathfrak{a})$, where \mathfrak{a} is an ideal of A . We claim that $\psi(F) = V(\mathfrak{a}C)$: in fact, the prime ideal of C containing $\mathfrak{a}C$ are the prime ideals of C containing \mathfrak{a} , hence the ideals of the form $\psi(x) = \mathfrak{m}_x \cap C$, where $\mathfrak{a} \subseteq \mathfrak{m}_x$ and $x \in W$. As $\mathfrak{a} \subseteq \mathfrak{m}_x$ is equivalent to $x \in V(\mathfrak{a}) = W \cap E$ for $x \in U$, we then get $\psi(F) = V(\mathfrak{a}C)$. In view of ??, we then conclude that X is separated, because $U \cap V$ is affine and the ring C is generated by $A \cup B$.

Conversely, suppose that X is separated, and let x, y be two points of X such that $\mathcal{O}_{X,x}$ and $\mathcal{O}_{X,y}$ are related. Let U (resp. V) be an open affine containing x (resp. y), with ring A (resp. B). We then see $U \cap V$ is affine and its ring C is generated by $A \cup B$ (??). If $\mathfrak{p} = \mathfrak{m}_x \cap A$ and $\mathfrak{q} = \mathfrak{m}_y \cap B$, we have $A_{\mathfrak{p}} = \mathcal{O}_{X,x}$ and $B_{\mathfrak{q}} = \mathcal{O}_{X,y}$, so $A_{\mathfrak{p}}$ and $B_{\mathfrak{q}}$ are related. Then by ?? there exists a prime ideal \mathfrak{r} of C such that $\mathfrak{p} = \mathfrak{r} \cap A$, $\mathfrak{q} = \mathfrak{r} \cap B$. But the prime ideal \mathfrak{r} then corresponds to a point $z \in U \cap V$ since $U \cap V$ is affine, and we have $x = z$ and $y = z$, so $x = y$. \square

Corollary 18.7.32. *Let X be a separated integral scheme and x, y be two points of X . For $x \in \overline{\{y\}}$, it is necessary and sufficient that $\mathcal{O}_{X,x} \subseteq \mathcal{O}_{X,y}$, which means the rational functions defined at x are also defined at y .*

Proof. This condition is clearly necessary since the defining domain $\delta(f)$ of a rational function is open, hence stable under generalization. To see it is also sufficient, assume that $\mathcal{O}_{X,x} \subseteq \mathcal{O}_{X,y}$, so there exists a prime ideal \mathfrak{p} of $\mathcal{O}_{X,x}$ such that $\mathcal{O}_{X,y}$ dominates $(\mathcal{O}_{X,x})_{\mathfrak{p}}$ (??). By ??, there exists $z \in X$ such that $x \in \overline{\{z\}}$ and $\mathcal{O}_{X,z} = (\mathcal{O}_{X,x})_{\mathfrak{p}}$; as $\mathcal{O}_{X,z}$ and $\mathcal{O}_{X,y}$ are then related, we have $z = y$ by ??, whence the corollary. \square

Corollary 18.7.33. *If X is a separated integral scheme, the map $x \mapsto \mathcal{O}_{X,x}$ is injective. In other words, if x, y are two distinct points of X , there exists a rational function defined at only one of these points.*

Proof. This follows from ?? and the T_0 -axiom. \square

Corollary 18.7.34. *Let X be a Noetherian separated integral scheme. The sets $\delta(f)$ for $f \in K(X)$ form a subbasis the topology of X .*

Proof. In fact, any closed subset of X is then a finite union of irreducible closed subsets, which are of the form $\overline{\{y\}}$. Now if $x \notin \overline{\{y\}}$, there exists a rational function f defined at x but not at y (??), which means $x \in \delta(f)$ and $\delta(f) \cap \overline{\{y\}} = \emptyset$. The complement of $\overline{\{y\}}$ is then a union of sets of the form $\delta(f)$, and in view of the previous remark, any open subset of X is a union of finite intersections of sets of the form $\delta(f)$. \square

Proposition 18.7.35. *Let X, Y be two integral schemes with rational function fields K and L , respectively. Suppose that Y is separated and let $f : X \rightarrow Y$ be a dominant morphism. Then L is identified with a subfield of K , and for every point $x \in X$, $\mathcal{O}_{Y,f(x)}$ is the unique local ring of Y dominated by $\mathcal{O}_{X,x}$.*

Proof. Ths first assertion is already proved in ?. Now for every $x \in X$, the local homomorphism $f_x^{\#} : \mathcal{O}_{Y,f(x)} \rightarrow \mathcal{O}_{X,x}$ is injective by ??, so, if we identify L as a subfield of K , $\mathcal{O}_{Y,f(x)}$ is dominated by $\mathcal{O}_{X,x}$. As Y is separated, two local rings of Y can not be dominated by the same local ring (??), so our assertion follows. \square

Proposition 18.7.36. *Let X be an irreducible scheme and $f : X \rightarrow Y$ be a local immersion (resp. a local isomorphism). Suppose that f is separated, then it is an immersion (resp. an open immersion).*

Proof. It suffices to prove that f is a homeomorphism from X to $f(X)$ (??). By replacing f with f_{red} , we may assume that X and Y are reduced. If Y' is the reduced subscheme of Y with underlying space $\overline{f(X)}$, f then factors into

$$X \xrightarrow{f'} Y' \xrightarrow{j} Y$$

where j is the canonical injection. Then f' is separated by ??(v) and is a local immersion by ??(iii), so we may reduce to the case that f is dominant. But then Y is irreducible by ??, and by ??, we see f is in fact a local isomorphism, so for each $x \in X$ the homomorphism $f_x^{\#}$ is an isomorphism. By ??, this implies that f is injective, so f is in fact a homeomorphism. \square

18.8 Formal schemes

18.8.1 Formal affine schemes and morphisms

Let A be a admissible topological ring, with a nilideal \mathfrak{I} (recall that this means \mathfrak{I} is open and (\mathfrak{I}^n) tends to 0 in A). The specturm $\text{Spec}(A/\mathfrak{I})$ is then a closed subscheme of $\text{Spec}(A)$, which is the set of open prime ideals of A . This topological space does not depend on the nilideal of \mathfrak{I} , and we denote it by \mathfrak{X} . Let (\mathfrak{J}_λ) be a system of fundamental neighborhood of 0 in A , formed by the nilideals of A , and for each λ , let \mathcal{O}_λ be the structural sheaf of $\text{Spec}(A/\mathfrak{J}_\lambda)$. This sheaf is induced over \mathfrak{X} by $\widetilde{A}/\widetilde{\mathfrak{J}}_\lambda$ (and is zero outside \mathfrak{X}). For $\mathfrak{J}_\mu \subseteq \mathfrak{J}_\lambda$, the canonical homomorphism $A/\mathfrak{J}_\mu \rightarrow A/\mathfrak{J}_\lambda$ defines a homomorphism $u_{\lambda\mu} : \mathcal{O}_\mu \rightarrow \mathcal{O}_\lambda$ of sheaves of rings, and (\mathcal{O}_λ) is a

projective system of sheaves of rings for these homomorphisms. As the topology of \mathfrak{X} admits a basis formed by quasi-compact open subsets, if we view each \mathcal{O}_λ as a sheaf of discrete rings, the \mathcal{O}_λ then form a projective system of sheaves of topological rings, and we denote by $\mathcal{O}_{\mathfrak{X}}$ the limit of this system (\mathcal{O}_λ). By (EGA 0I 3.2.6 ??), for any quasi-compact open subset U of \mathfrak{X} , $\Gamma(U, \mathcal{O}_{\mathfrak{X}})$ is then the limit topological ring of the discrete rings $\Gamma(U, \mathcal{O}_\lambda)$.

Given an admissible topological ring A , the closed subspace \mathfrak{X} of $\text{Spec}(A)$ formed by open prime ideals of A is called the **formal spectrum** of A and denoted by $\text{Spf}(A)$. A topologically ringed space is called a **formal affine scheme** if it is isomorphic to a formal spectrum $\text{Spf}(A) = \mathfrak{X}$ endowed with the sheaf of topological rings $\mathcal{O}_{\mathfrak{X}}$, which is the limit of the sheaf of discrete rings $(\widehat{A}/\mathfrak{J}_\lambda)|_{\mathfrak{X}}$, where \mathfrak{J}_λ runs through the filtered set of nilideals of A . When we speak of a formal spectrum $\mathfrak{X} = \text{Spf}(A)$ as a formal affine scheme, it will always be understood that the topologically ringed space $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ where $\mathcal{O}_{\mathfrak{X}}$ is defined as above. By an **adic** (resp. **Noetherian**) formal affine scheme, we mean a formal affine scheme which is isomorphic to a formal spectrum $\text{Spf}(A)$, where A is adic (resp. adic and Noetherian).

We note that any affine scheme $X = \text{Spec}(A)$ can be considered as a formal affine scheme in a unique way: consider A as a discrete topological ring, the rings $\Gamma(U, \mathcal{O}_X)$ are then discrete if U is quasi-compact (but not true in general if U is any open set of X).

Proposition 18.8.1. *If $\mathfrak{X} = \text{Spf}(A)$, where A is an admissible ring, then $\Gamma(X, \mathcal{O}_{\mathfrak{X}})$ is homeomorphic to A .*

Proof. In fact, as \mathfrak{X} is closed in $\text{Spec}(A)$, it is quasi-compact, and therefore $\Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ is homeomorphic to the limit of the discrete rings $\Gamma(\mathfrak{X}, \mathcal{O}_\lambda)$. But $\Gamma(\mathfrak{X}, \mathcal{O}_\lambda)$ is isomorphic to A/\mathfrak{J}_λ , and as A is separated and complete, this is homeomorphic to $\varinjlim A/\mathfrak{J}_\lambda$, whence the proposition. \square

Proposition 18.8.2. *Let A be an admissible ring, $\mathfrak{X} = \text{Spf}(A)$, and for $f \in A$, let $\mathfrak{D}(f) = D(f) \cap \mathfrak{X}$. Then the topologically ringed space $(\mathfrak{D}(f), \mathcal{O}_{\mathfrak{X}}|_{\mathfrak{D}(f)})$ is isomorphic to a formal affine spectrum $\text{Spf}(A_{\{f\}})$.*

Proof. For any nilideal \mathfrak{J} of A , the discrete ring A_f/\mathfrak{J}_f is canonically identified with $A_{\{f\}}/\mathfrak{J}_{\{f\}}$, so the topological space $\text{Spf}(A_{\{f\}})$ is canonically identified with $\mathfrak{D}(f)$. Moreover, for any quasi-compact open U of \mathfrak{X} contained in $\mathfrak{D}(f)$, $\Gamma(U, \mathcal{O}_\lambda)$ is identified with the module of sections of the structural sheaf of $\text{Spec}(A_f/\mathfrak{J}_\lambda)$ over U , so, if we put $\mathfrak{Y} = \text{Spf}(A_{\{f\}})$, $\Gamma(U, \mathcal{O}_{\mathfrak{X}})$ is identified with $\Gamma(U, \mathcal{O}_{\mathfrak{Y}})$, whence the proposition. \square

As a sheaf of rings, the stalk of the structural sheaf $\mathcal{O}_{\mathfrak{X}}$ of $\text{Spf}(A)$ for any $x \in X$ is, by ??, identified with the inductive limit $\varinjlim A_{\{f\}}$ for $f \notin \mathfrak{p}_x$. Therefore, by ?? and ??, we have the following:

Proposition 18.8.3. *For any $x \in \mathfrak{X} = \text{Spf}(A)$, the stalk $\mathcal{O}_{\mathfrak{X},x}$ is a local ring whose residue field is isomorphic to $\kappa(x)$. If A is adic and Noetherian, then $\mathcal{O}_{\mathfrak{X},x}$ is a Noetherian ring.*

As the field $\kappa(x)$ is not reduced to 0, we conclude in particular that the support of $\mathcal{O}_{\mathfrak{X}}$ is equal to \mathfrak{X} , and $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ is a locally topologically ringed space.

We now consider morphisms of formal affine schemes. Let A, B be admissible rings, and $\varphi : B \rightarrow A$ be a continuous homomorphism. The continuous map ${}^a\varphi : \text{Spec}(A) \rightarrow \text{Spec}(B)$ then maps $\mathfrak{X} = \text{Spf}(A)$ into $\mathfrak{Y} = \text{Spf}(B)$, because the inverse image of an open prime ideal of

A is an open prime ideal of B . On the other hand, for any $g \in B$, φ defines a continuous homomorphism $\Gamma(\mathfrak{D}(g), \mathcal{O}_{\mathfrak{Y}}) \rightarrow \Gamma(\mathfrak{D}(\varphi(g)), \mathcal{O}_{\mathfrak{X}})$ in view of ?? and ??; as these homomorphisms are compatible with restrictions and $\mathfrak{D}(\varphi(g)) = {}^a\varphi^{-1}(\mathfrak{D}(g))$, we obtain a continuous homomorphism of sheaves of topological rings $\mathcal{O}_{\mathfrak{Y}} \rightarrow {}^a\varphi_*(\mathcal{O}_{\mathfrak{X}})$, which we denoted by $\tilde{\varphi}$. We then get a morphism $({}^a\varphi, \tilde{\varphi}) : (X, \mathcal{O}_{\mathfrak{X}}) \rightarrow (\mathfrak{Y}, \mathcal{O}_{\mathfrak{Y}})$ of topologically ringed spaces.

Proposition 18.8.4. *Let A, B be admissible topological rings, and $\mathfrak{X} = \text{Spf}(A)$, $\mathfrak{Y} = \text{Spf}(B)$. For a morphism $\psi : \mathfrak{X} \rightarrow \mathfrak{Y}$ of topologically ringed spaces to be of the form $({}^a\varphi, \tilde{\varphi}) : \mathfrak{X} \rightarrow \mathfrak{Y}$, it is necessary and sufficient that for each $x \in X$, $f_x^\# : \mathcal{O}_{\mathfrak{Y}, \psi(x)} \rightarrow \mathcal{O}_{\mathfrak{X}, x}$ is a local homomorphism.*

Proof. This conditions is necessary: in fact, let $\mathfrak{p} = \mathfrak{p}_x \in \text{Spf}(A)$, and $\mathfrak{q} = \varphi^{-1}(\mathfrak{p}_x)$; if $g \notin \mathfrak{q}$, then $\varphi(g) \notin \mathfrak{p}$, and it is immediate that the homomorphism $B_{\{g\}} \rightarrow A_{\{\varphi(g)\}}$ induced from φ maps $\mathfrak{q}_{\{g\}}$ into $\mathfrak{p}_{\{\varphi(g)\}}$; by passing to inductive limit, we then see that $\tilde{\varphi}_x$ is a local homomorphism.

Conversely, let ψ be a morphism satisfying this condition. By ??, $\psi^\#$ defines a continuous homomorphism

$$\varphi = \Gamma(\psi^\#) : B = \Gamma(\mathfrak{Y}, \mathcal{O}_{\mathfrak{Y}}) \rightarrow \Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}) = A.$$

By the hypothesis on $\psi^\#$, for the section $\varphi(g)$ of $\mathcal{O}_{\mathfrak{X}}$ over \mathfrak{X} has invertible germ at a point x , it is necessary and sufficient that g has invertible germ at $\psi(x)$. But by ??, the sections of $\mathcal{O}_{\mathfrak{X}}$ (resp. $\mathcal{O}_{\mathfrak{Y}}$) over \mathfrak{X} (resp. \mathfrak{Y}) which have non-invertible germs at x (resp. $\psi(x)$) are exactly the elements of \mathfrak{p}_x (resp. $\mathfrak{p}_{\psi(x)}$), so we conclude that $\psi = {}^a\varphi$. Finally, for any $g \in B$, the diagram

$$\begin{array}{ccc} B = \Gamma(\mathfrak{Y}, \mathcal{O}_{\mathfrak{Y}}) & \xrightarrow{\varphi} & \Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}) = A \\ \downarrow & & \downarrow \\ B_{\{g\}} = \Gamma(\mathfrak{D}(g), \mathcal{O}_{\mathfrak{Y}}) & \xrightarrow{\Gamma(\psi_{\mathfrak{D}(g)}^\#)} & \Gamma(\mathfrak{D}(\varphi(g)), \mathcal{O}_{\mathfrak{X}})) = A_{\{\varphi(g)\}} \end{array}$$

is commutative. By the universal property of localization of complete rings ??, we conclude that $\psi_{\mathfrak{D}(g)}^\#$ is equal to $\tilde{\varphi}_{\mathfrak{D}(g)}$ for $g \in B$, so we have $\psi^\# = \tilde{\varphi}$. \square

We say a morphism $\psi : \mathfrak{X} \rightarrow \mathfrak{Y}$ satisfying the condition in ?? is a **morphism of formal affine schemes**. Then by ??, the functor $A \mapsto \text{Spf}(A)$ and $\mathfrak{X} \mapsto \Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ define an equivalence from the category of admissible topological rings to the opposite of the category of formal affine schemes.

As a particular case of ??, note that for $f \in A$, the canonical injection of the formal affine scheme over $\mathfrak{D}(f)$ induced by \mathfrak{X} corresponds to the canonical homomorphism $A \rightarrow A_{\{f\}}$. Under the hypothesis of ??, let h be an element of B and g be an element of A , which is a multiple of $\varphi(h)$. We then have $\psi(\mathfrak{D}(g)) \subseteq \mathfrak{D}(h)$; the restriction of ψ to $\mathfrak{D}(g)$, considered as a morphism $\mathfrak{D}(g) \rightarrow \mathfrak{D}(h)$, is the unique morphism η such that the diagram

$$\begin{array}{ccc} \mathfrak{D}(g) & \xrightarrow{\eta} & \mathfrak{D}(h) \\ \downarrow & & \downarrow \\ \mathfrak{X} & \xrightarrow{\psi} & \mathfrak{Y} \end{array}$$

This morphism corresponds to the unique continuous homomorphism $\tilde{\varphi} : B_{\{h\}} \rightarrow A_{\{g\}}$ such

that the diagram

$$\begin{array}{ccc} B & \xrightarrow{\varphi} & A \\ \downarrow & & \downarrow \\ B_{\{h\}} & \xrightarrow{\tilde{\varphi}} & A_{\{g\}} \end{array}$$

is commutative.

Let A be an admissible ring, \mathfrak{I} be an open ideal of A , and \mathfrak{X} be the formal affine scheme $\mathrm{Spf}(A)$. Let (\mathfrak{I}_λ) be the set of nilideals of A contained in \mathfrak{I} ; then $\widetilde{\mathfrak{I}}/\widetilde{\mathfrak{I}}_\lambda$ is a sheaf of ideals of $\widetilde{A}/\widetilde{\mathfrak{I}}_\lambda$. We denote by \mathfrak{I}^Δ the projective limit of the sheaves induced by $\widetilde{\mathfrak{I}}/\widetilde{\mathfrak{I}}_\lambda$ over \mathfrak{X} , which is considered as an ideal of $\mathcal{O}_{\mathfrak{X}}$. For any $f \in A$, $\Gamma(\mathfrak{D}(f), \mathfrak{I}^\Delta)$ is the projective limit of $\mathfrak{I}_f/(\mathfrak{I}_\lambda)_f$, which is identified with the open ideal $\mathfrak{I}_{\{f\}}$ of the ring $A_{\{f\}}$, and in particular $\Gamma(\mathfrak{X}, \mathfrak{I}^\Delta) = \mathfrak{I}$. We then conclude that (the $\mathfrak{D}(f)$ form a base of \mathfrak{X}) that we have

$$\mathfrak{I}^\Delta|_{\mathfrak{D}(f)} = (\mathfrak{I}_{\{f\}})^\Delta \quad (18.8.1)$$

With these notations, for $f \in A$ the canonical map of $A_{\{f\}} = \Gamma(\mathfrak{D}, \mathcal{O}_{\mathfrak{X}})$ in $\Gamma(\mathfrak{D}(f), (\widetilde{A}/\widetilde{\mathfrak{I}})|_{\mathfrak{X}}) = A_f/\mathfrak{I}_f$ is surjective with kernel $\Gamma(\mathfrak{D}(f), \mathfrak{I}^\Delta) = \mathfrak{I}_{\{f\}}$. These maps define a canonical continuous epimorphism from the sheaf $\mathcal{O}_{\mathfrak{X}}$ to the sheaf of discrete rings $(\widetilde{A}/\widetilde{\mathfrak{I}})|_{\mathfrak{X}}$, whose kernel is \mathfrak{I}^Δ ; this homomorphism is none other than the homomorphism $\tilde{\varphi}$, where φ is the canonical continuous homomorphism $A \rightarrow A/\mathfrak{I}$. The morphism $(^a\varphi, \tilde{\varphi}) : \mathrm{Spec}(A/\mathfrak{I}) \rightarrow \mathfrak{X}$ of the formal affine schemes is then called the canonical morphism. We then have a canonical isomorphism

$$\mathcal{O}_{\mathfrak{X}}/\mathfrak{I}^\Delta \xrightarrow{\sim} (\widetilde{A}/\widetilde{\mathfrak{I}})|_{\mathfrak{X}}. \quad (18.8.2)$$

It is clear (in view of $\Gamma(X, \mathfrak{I}^\Delta) = \mathfrak{I}$) that the map $\mathfrak{I} \mapsto \mathfrak{I}^\Delta$ is strictly increasing: in fact, for $\mathfrak{I} \subseteq \mathfrak{I}'$, the sheaf $\mathfrak{I}'^\Delta/\mathfrak{I}^\Delta$ is canonically isomorphic to $\widetilde{\mathfrak{I}'}/\widetilde{\mathfrak{I}} = \widetilde{\mathfrak{I}'}/\widetilde{\mathfrak{I}}$.

An ideal \mathscr{I} of $\mathcal{O}_{\mathfrak{X}}$ is called a **nilideal** of \mathfrak{X} if, for any $x \in \mathfrak{X}$, there exists an open neighborhood of x in \mathfrak{X} of the form $\mathfrak{D}(f)$, where $f \in A$, such that $\mathscr{I}|_{\mathfrak{D}(f)}$ is of the form \mathfrak{J}^Δ for a nilideal \mathfrak{J} of $A_{\{f\}}$. It is clear from our definition that for any $f \in A$, any nilideal of \mathfrak{X} induces a nilideal of $\mathfrak{D}(f)$.

Proposition 18.8.5. *If A is an admissible ring, any nilideal of $\mathfrak{X} = \mathrm{Spf}(A)$ is of the form \mathfrak{I}^Δ , where \mathfrak{I} is a nilideal of A .*

Proof. Let \mathscr{I} be a nilideal of \mathfrak{X} ; by hypothesis, and since \mathfrak{X} is quasi-compact, there exist finitely many elements $f_i \in A$ such that $\mathfrak{D}(f_i)$ cover \mathfrak{X} and such that $\mathscr{I}|_{\mathfrak{D}(f_i)} = \mathfrak{K}_i$, where \mathfrak{K}_i is a nilideal of $A_{\{f_i\}}$. For any i , there then exists an open ideal \mathfrak{R}_i of A such that $(\mathfrak{R}_i)_{\{f_i\}} = \mathfrak{K}_i$; let \mathfrak{R} be a nilideal of A contained in each the \mathfrak{R}_i . The canonical image of $\mathscr{I}/\mathfrak{R}^\Delta$ in the structural sheaf $(\widetilde{A}/\widetilde{\mathfrak{R}})$ of $\mathrm{Spec}(A/\mathfrak{R})$ is then such that its restriction to each $\mathfrak{D}(f_i)$ is equal to $\widetilde{\mathfrak{R}_i}/\widetilde{\mathfrak{R}}$; we then conclude that this canonical image is a quasi-coherent ideal over $\mathrm{Spec}(A/\mathfrak{R})$, hence is of the form $\widetilde{\mathfrak{I}}/\widetilde{\mathfrak{R}}$, where \mathfrak{I} is an ideal of A containing \mathfrak{R} , and whence $\mathscr{I} = \mathfrak{I}^\Delta$ by (??). Moreover, as for each i there exists an integer n_i such that $\mathfrak{K}_i^{n_i} \subseteq \mathfrak{R}_{\{f_i\}}$, we have $(\mathscr{I}/\mathfrak{R}^\Delta)^n = 0$ for n sufficiently large, and therefore $(\widetilde{\mathfrak{I}}/\widetilde{\mathfrak{R}})^n = 0$, and finally $(\mathfrak{I}/\mathfrak{R})^n = 0$, which proves that \mathfrak{I} is a nilideal of A . \square

Proposition 18.8.6. *Let A be an adic ring, \mathfrak{I} be a nilideal of A such that $\mathfrak{I}/\mathfrak{I}^2$ is an A/\mathfrak{I} of finite type. For any integer $n > 0$, we then have $(\mathfrak{I}^\Delta)^n = (\mathfrak{I}^n)^\Delta$.*

Proof. In fact, for any $f \in A$ we have (since \mathfrak{I}^n is an open ideal)

$$(\Gamma(\mathcal{D}(f), \mathfrak{I}^\Delta))^n = (\mathfrak{I}_{\{f\}})^n = (\mathfrak{I}^n)_{\{f\}} = \Gamma(\mathcal{D}(f^n), (\mathfrak{I}^n)^\Delta)$$

in view of (??) and ???. As $(\mathfrak{I}^\Delta)^n$ is associated with the presheaf $U \mapsto (\Gamma(U, \mathfrak{I}^\Delta))^n$, the corollary then follows since $\mathcal{D}(f)$ form a basis for \mathfrak{X} . \square

A family (\mathcal{J}_λ) of nilideals of \mathfrak{X} is called a **fundamental system of nilideals** if any nilideal of \mathcal{I} contains at least one of these \mathcal{J}_λ . As $\mathcal{J}_\lambda = \mathfrak{I}_\lambda^\Delta$, this is equivalent to saying that the \mathfrak{I}_λ form a fundamental neighborhood of 0 in A , where $\mathcal{J}_\lambda = \mathcal{I}_\lambda^\Delta$. Let (f_α) be a family of elements of A such that the $\mathcal{D}(f_\alpha)$ cover \mathfrak{X} . If (\mathcal{J}_λ) is a filtered decreasing family of ideals of $\mathcal{O}_{\mathfrak{X}}$ such that for any α , the family $(\mathcal{J}_\lambda|_{\mathcal{D}(f_\alpha)})$ is a fundamental system of nilideals of $\mathcal{D}(f_\alpha)$, then (\mathcal{J}_λ) is a fundamental system of nilideals of \mathfrak{X} . In fact, for any nilideal of \mathfrak{X} , there exists a finite covering of \mathfrak{X} by the $\mathcal{D}(f_i)$ such that, for any i , $\mathcal{J}_{\lambda_i}|_{\mathcal{D}(f_i)}$ is a nilideal of $\mathcal{D}(f_i)$ contained in $\mathfrak{I}|_{\mathcal{D}(f_i)}$. If μ is an index such that $\mathcal{J}_\mu \subseteq \mathcal{J}_{\lambda_i}$ for all i , then \mathcal{J}_μ is a nilideal of \mathfrak{X} which is evidently contained in \mathcal{I} , whence the assertion.

18.8.2 Formal schemes and morphisms

Given a topologically ringed space \mathfrak{X} , we say an open subset $U \subseteq \mathfrak{X}$ is a **formal affine open** (resp. **an adic formal affine open**, resp. **a Noetherian formal affine open**) if the topologically ringed space $(U, \mathcal{O}_{\mathfrak{X}}|_U)$ is a formal affine scheme (resp. an adic formal affine scheme, resp. a Noetherian formal affine scheme). We say $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ is a **formal scheme** (resp. **adic formal scheme**, resp. **locally Noetherian formal scheme**) if each of its point admits a formal affine open neighborhood (resp. an adic formal affine open, resp. a locally Noetherian formal affine open). We say that \mathfrak{X} is **Noetherian** if it is locally Noetherian and the underlying space is quasi-compact (hence Noetherian). As any affine scheme can be considered as a formal affine scheme, any scheme can be considered as a formal scheme.

Proposition 18.8.7. *If \mathfrak{X} is a formal scheme (resp. locally Noetherian formal scheme), the set of formal affine opens (resp. Noetherian formal affine opens) form a base for \mathfrak{X} .*

Proof. This follows from ??, and the fact that if A is a Noetherian adic ring, so is $A_{\{f\}}$ for any $f \in A$ (??). \square

Corollary 18.8.8. *If \mathfrak{X} is a formal scheme (resp. locally Noetherian formal scheme, resp. Noetherian formal scheme), the topological ringed space over any open subset of \mathfrak{X} is a formal scheme (resp. locally Noetherian formal scheme, resp. Noetherian formal scheme).*

Given two formal schemes $\mathfrak{X}, \mathfrak{Y}$, we say that $f : \mathfrak{X} \rightarrow \mathfrak{Y}$ is a morphism of formal schemes if it is a morphism of the underlying locally ringed spaces. That is, if $(f, f^\#)$ is a morphism of ringed spaces and $f_x^\# : \mathcal{O}_{\mathfrak{Y}, f(x)} \rightarrow \mathcal{O}_{\mathfrak{X}, x}$ is a local homomorphism. The composition of two morphisms are defined as the same and clearly a morphism of formal schemes. The formal schemes then form a category, which we denoted by \mathbf{Shf} , and we denote by $\mathrm{Hom}_{\mathbf{Shf}}(\mathfrak{X}, \mathfrak{Y})$ the set of morphisms of formal schemes $\mathfrak{X} \rightarrow \mathfrak{Y}$.

If U is an open subset of \mathfrak{X} , the canonical injection $U \rightarrow \mathfrak{X}$ is then a morphism of formal schemes, if we endow U the formal scheme structure induced by \mathfrak{X} . It is clear that this morphism is a monomorphism in the category \mathbf{Shf} .

Proposition 18.8.9. *Let \mathfrak{X} be a formal scheme, $\mathfrak{Y} = \mathrm{Spec}(A)$ be a formal affine scheme. Then there exists a canonical bijection*

$$\mathrm{Hom}_{\mathbf{Scf}}(\mathfrak{X}, \mathfrak{Y}) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{TopRing}}(A, \Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})).$$

Proof. We first note that, if $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ and $(\mathfrak{Y}, \mathcal{O}_{\mathfrak{Y}})$ are two topologically ringed spaces, a morphism $f : (\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}) \rightarrow (\mathfrak{Y}, \mathcal{O}_{\mathfrak{Y}})$ defines canonically a continuous homomorphism of rings $\varphi : \Gamma(\mathfrak{Y}, \mathcal{O}_{\mathfrak{Y}}) \rightarrow \Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$. In our case, we need to show that a continuous homomorphism $\varphi : A \rightarrow \Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ is of the form $\Gamma(\psi^{\#})$ for a unique morphism $(\psi, \psi^{\#})$. Now there exists by hypothesis a covering (V_{α}) of \mathfrak{X} by formal affine opens; by composing φ with the restriction homomorphisms $\Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}) \rightarrow \Gamma(V_{\alpha}, \mathcal{O}_{\mathfrak{X}}|_{V_{\alpha}})$, we obtain a continuous homomorphism $\varphi_{\alpha} : A \rightarrow \Gamma(V_{\alpha}, \mathcal{O}_{\mathfrak{X}}|_{V_{\alpha}})$, which corresponds to a unique morphism $\psi_{\alpha} : (V_{\alpha}, \mathcal{O}_{\mathfrak{X}}|_{V_{\alpha}}) \rightarrow (\mathfrak{Y}, \mathcal{O}_{\mathfrak{Y}})$, in view of ???. Moreover, for any couple (α, β) of indices, any point of $V_{\alpha} \cap V_{\beta}$ admits a formal affine open neighborhood W contained in $V_{\alpha} \cap V_{\beta}$ and it is clear that the compositions of φ_{α} and φ_{β} with the canonical restriction are the same continuous homomorphism $\Gamma(\mathfrak{Y}, \mathcal{O}_{\mathfrak{Y}}) \rightarrow \Gamma(W, \mathcal{O}_{\mathfrak{X}}|_W)$, so, in view of the relations $(\psi_{\alpha}^{\#})_x = (\tilde{\varphi}_{\alpha})_x$ for any $x \in V_{\alpha}$, the restrictions of ψ_{α} and ψ_{β} coincides on $V_{\alpha} \cap V_{\beta}$. We then conclude that there exists a unique morphism $\psi : (\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}) \rightarrow (\mathfrak{Y}, \mathcal{O}_{\mathfrak{Y}})$ whose restriction to V_{α} coincides with ψ_{α} , and it is clear that this is the unique morphism such that $\Gamma(\psi^{\#}) = \varphi$. \square

Given a formal scheme \mathfrak{S} , a **formal \mathfrak{S} -scheme** is defined to be a formal scheme \mathfrak{X} together with a morphism $\varphi : \mathfrak{X} \rightarrow \mathfrak{S}$, called the **structural morphism** of \mathfrak{X} . If $\mathfrak{S} = \mathrm{Spf}(A)$, where A is an admissible ring, we also say that the \mathfrak{S} -formal scheme \mathfrak{X} is a formal A -scheme or a formal scheme over A . Any formal scheme can be clearly considered as a formal scheme over \mathbb{Z} (endowed with the discrete topology).

If $\mathfrak{X}, \mathfrak{Y}$ are two formal \mathfrak{S} -schemes, we say a morphism $f : \mathfrak{X} \rightarrow \mathfrak{Y}$ is an **\mathfrak{S} -morphism** if the diagram

$$\begin{array}{ccc} \mathfrak{X} & \xrightarrow{f} & \mathfrak{Y} \\ & \searrow & \swarrow \\ & \mathfrak{S} & \end{array}$$

where the vertical arrows are structural morphisms, is commutative. With this definition, the \mathfrak{S} -schemes form (for \mathfrak{S} fixed) a category $\mathbf{Scf}_{\mathfrak{S}}$. We denote by $\mathrm{Hom}_{\mathfrak{S}}(\mathfrak{X}, \mathfrak{Y})$ the set of \mathfrak{S} -morphisms $\mathfrak{X} \rightarrow \mathfrak{Y}$. If $\mathfrak{S} = \mathrm{Spf}(A)$, we also say A -morphism for \mathfrak{S} -morphisms.

Let \mathfrak{X} be a formal scheme; we say an ideal \mathcal{I} of $\mathcal{O}_{\mathfrak{X}}$ is a nilideal of \mathfrak{X} if any $x \in \mathfrak{X}$ admits a formal affine open neighborhood U such that $\mathcal{I}|_U$ is a nilideal of the formal scheme U induced by \mathfrak{X} . In view of ??, for any open $V \subseteq \mathfrak{X}$, $\mathcal{I}|_V$ is then a nilideal of the formal scheme induced over V .

A family (\mathcal{I}_{λ}) of nilideals of \mathfrak{X} is called a **fundamental system of nilideals** if there exists a covering (U_{α}) of \mathfrak{X} by formal affine opens such that, for any α , the family $(\mathcal{I}_{\lambda}|_{U_{\alpha}})$ form a fundamental system of nilideals of U_{α} . For any open subset V of \mathfrak{X} , the family $(\mathcal{I}_{\lambda}|_V)$ then

forms a fundamental system of nilideals for V , in view of (??). If \mathfrak{X} is locally Noetherian, and \mathcal{I} is a nilideal of \mathfrak{X} , it then follows from ?? that the powers of \mathcal{I}^n form a fundamental system of nilideals of \mathfrak{X} .

Let \mathfrak{X} be a formal scheme, \mathcal{I} be a nilideal of \mathfrak{X} . Then the ringed space $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}/\mathcal{I})$ is a scheme, which is affine (resp. locally Noetherian, resp. Noetherian) if \mathfrak{X} is a formal affine scheme (resp. a locally Noetherian formal scheme, resp. a Noetherian formal scheme). Moreover, if $\varphi : \mathcal{O}_{\mathfrak{X}} \rightarrow \mathcal{O}_{\mathfrak{X}}/\mathcal{I}$ is the canonical homomorphism, then $(1_{\mathfrak{X}}, \varphi)$ is a morphism (called canonical) of formal schemes $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}/\mathcal{I}) \rightarrow (\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$.

Proposition 18.8.10. *Let \mathfrak{X} be a formal scheme, (\mathcal{I}_λ) be a fundamental system of nilideals of \mathfrak{X} . Then the sheaf $\mathcal{O}_{\mathfrak{X}}$ is the projective limit of the sheaf of discrete rings $\mathcal{O}_{\mathfrak{X}}/\mathcal{I}_\lambda$.*

Proof. As \mathfrak{X} admits a basis by quasi-compact open sets, we are reduced to the affine case, where the proposition follows from ?? and the definition of $\mathcal{O}_{\mathfrak{X}}$. \square

Proposition 18.8.11. *Let \mathfrak{X} be a locally Noetherian formal scheme. Then there exists a largest nilideal \mathcal{T} of \mathfrak{X} , which is the unique nilideal \mathcal{I} such that the scheme $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}/\mathcal{I})$ is reduced. If \mathcal{I} is a nilideal of \mathfrak{X} , then \mathcal{T} is the inverse image of the nilradical of $\mathcal{O}_{\mathfrak{X}}/\mathcal{I}$ under the homomorphism $\mathcal{O}_{\mathfrak{X}} \rightarrow \mathcal{O}_{\mathfrak{X}}/\mathcal{I}$. The reduced (usual) scheme $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}/\mathcal{T})$ is denoted by $\mathfrak{X}_{\text{red}}$.*

Proof. Suppose first that $\mathfrak{X} = \text{Spf}(A)$, where A is a Noetherian adic ring. The existence of \mathcal{T} and its properties then follows from ??, in view of ?? about the largest nilideal of A . To prove the existence of \mathcal{T} in the general case, it suffices to prove that if $V \subseteq U$ are two Noetherian formal affine opens of X , the largest nilideal \mathcal{T}_U of U induces the largest nilideal \mathcal{T}_V of V ; but as $\Gamma(V, (\mathcal{O}_{\mathfrak{X}}|_V)/(\mathcal{T}_U|_V))$ is reduced, this is immediate. \square

Corollary 18.8.12. *Let \mathfrak{X} be a locally Noetherian formal scheme, \mathcal{T} be the largest nilideal of \mathfrak{X} . Then for any open subset V of \mathfrak{X} , $\mathcal{T}|_V$ is the largest nilideal of V .*

Proof. This is already shown in the proof of ??.

Proposition 18.8.13. *Let $\mathfrak{X}, \mathfrak{Y}$ be formal schemes, \mathcal{I} (resp. \mathcal{K}) be the nilideal of \mathfrak{X} (resp. \mathfrak{Y}), $f : \mathfrak{X} \rightarrow \mathfrak{Y}$ be a morphism of formal schemes.*

(i) *If $f^*(\mathcal{K})\mathcal{O}_{\mathfrak{X}} \subseteq \mathcal{I}$, there exists a unique morphism*

$$f_{\text{red}} : (\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}/\mathcal{I}) \rightarrow (\mathfrak{Y}, \mathcal{O}_{\mathfrak{Y}}/\mathcal{K})$$

of schemes such that the following diagram

$$\begin{array}{ccc} (\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}/\mathcal{I}) & \xrightarrow{f_{\text{red}}} & (\mathfrak{Y}, \mathcal{O}_{\mathfrak{Y}}/\mathcal{K}) \\ \downarrow & & \downarrow \\ (\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}) & \xrightarrow{f} & (\mathfrak{Y}, \mathcal{O}_{\mathfrak{Y}}) \end{array} \tag{18.8.3}$$

(ii) *Suppose that $\mathfrak{X} = \text{Spf}(A)$, $\mathfrak{Y} = \text{Spf}(B)$ are formal affine schemes, $\mathcal{I} = \mathcal{J}^\Delta$, $\mathcal{K} = \mathcal{R}^\Delta$, where \mathcal{J} (resp. \mathcal{R}) is a nilideal of A (resp. B), and $f = (^a\varphi, \tilde{\varphi})$ where $\varphi : B \rightarrow A$ is a continuous homomorphism. For $f^*(\mathcal{K})\mathcal{O}_{\mathfrak{X}} \subseteq \mathcal{I}$, it is necessary and sufficient that $\varphi(\mathcal{R}) \subseteq \mathcal{J}$, and f_{red} is*

then the morphism $(^a\varphi', \tilde{\varphi}')$, where $\varphi' : B/\mathfrak{K} \rightarrow A/\mathfrak{I}$ is the induced homomorphism by passing to quotients.

Proof. In case (a), the hypothesis implies that the image of the ideal $f^{-1}(\mathcal{K})$ of $f^{-1}(\mathcal{O}_{\mathfrak{Y}})$ under $f^\# : f^{-1}(\mathcal{O}_{\mathfrak{Y}}) \rightarrow \mathcal{O}_{\mathfrak{X}}$ is contained in \mathcal{I} . By passing to quotients, we then deduce that $f^\#$ is a homomorphism

$$\omega : f^{-1}(\mathcal{O}_{\mathfrak{Y}})/\mathcal{K} = f^{-1}(\mathcal{O}_{\mathfrak{Y}})/f^{-1}(\mathcal{K}) \rightarrow \mathcal{O}_{\mathfrak{X}}/\mathcal{I};$$

moreover, as for any $x \in \mathfrak{X}$, $f_x^\#$ is a local homomorphism, so is ω_x . The morphism (f, ω^\flat) is then the unique morphism of ringed spaces satisfying the requirements.

With the assumptions of (b), the canonical correspondence between morphisms of formal affine schemes and continuous homomorphisms of ringed spaces shows that the relation $f^*(\mathcal{K})\mathcal{O}_{\mathfrak{X}} \subseteq \mathcal{I}$ implies that $f' = (^a\varphi', \tilde{\varphi}')$, where $\varphi' : B/\mathfrak{K} \rightarrow A/\mathfrak{I}$ is the unique homomorphism such that the diagram

$$\begin{array}{ccc} B & \xrightarrow{\varphi} & A \\ \downarrow & & \downarrow \\ B/\mathfrak{K} & \xrightarrow{\varphi'} & A/\mathfrak{I} \end{array} \tag{18.8.4}$$

The existence of φ' implies then that $\varphi(\mathfrak{K}) \subseteq \mathfrak{I}$. Conversely, if this condition is verified, we have a canonical homomorphism $\varphi' : B/\mathfrak{K} \rightarrow A/\mathfrak{I}$, whence the induced morphism $f' = (^a\varphi', \tilde{\varphi}')$ satisfies the commutativity of (??). By considering the homomorphisms ${}^a\varphi^*(\mathcal{O}_{\mathfrak{Y}}) \rightarrow \mathcal{O}_{\mathfrak{X}}$ and ${}^a\varphi'^*(\mathcal{O}_{\mathfrak{Y}}/\mathcal{K}) \rightarrow \mathcal{O}_{\mathfrak{X}}/\mathcal{I}$, we then see that $f^*(\mathcal{K})\mathcal{O}_{\mathfrak{X}} \subseteq \mathcal{I}$. \square

To conclude this paragraph, we discuss the fiber product of formal schemes. Let \mathfrak{S} be a formal scheme. It turns out that the fiber product of formal schemes has the same construction as that of usual schemes, provided that we replace tensor products by completed tensor products.

Proposition 18.8.14. *Let $\mathfrak{X} = \text{Spf}(B)$, $\mathfrak{Y} = \text{Spf}(C)$ be two formal affine schemes over a formal affine scheme $\text{Spf}(A)$. Let $\mathfrak{Z} = \text{Spf}(B \widehat{\otimes}_A C)$ and p_1, p_2 be the \mathfrak{S} -morphisms corresponding to the A -homomorphisms $\rho_1 : B \rightarrow B \widehat{\otimes}_A C$ and $\rho_2 : C \rightarrow B \widehat{\otimes}_A C$. Then (\mathfrak{Z}, p_1, p_2) forms a product in the category of the formal \mathfrak{S} -schemes \mathfrak{X} and \mathfrak{Y} .*

Proof. In view of ??, we note that for any continuous A -homomorphism $\varphi : B \widehat{\otimes}_A C \rightarrow D$, where D is an admissible ring that is a topological A -algebra, we can associate the couple $(\varphi \circ \rho_1, \varphi \circ \rho_2)$, so that we define a bijection

$$\text{Hom}_A(B \widehat{\otimes}_A C, D) \xrightarrow{\sim} \text{Hom}_A(B, D) \times \text{Hom}_A(C, D)$$

which follows from the universal property of the complete tensor product. \square

Proposition 18.8.15. *For any formal \mathfrak{S} -schemes $\mathfrak{X}, \mathfrak{Y}$, the product $\mathfrak{X} \times_{\mathfrak{S}} \mathfrak{Y}$ exists.*

Proof. The demonstration is similar as the case for usual schemes, where we replace affine schemes by formal affine schemes and use ??.

18.8.3 Inductive limits of schemes

Let \mathfrak{X} be a formal scheme, (\mathcal{I}_λ) be a fundamental system of nilideals of \mathfrak{X} ; for each λ , let $f_\lambda : (\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}/\mathcal{I}_\lambda) \rightarrow \mathfrak{X}$. For $\mathcal{I}_\mu \subseteq \mathcal{I}_\lambda$, the canonical homomorphism $\mathcal{O}_{\mathfrak{X}}/\mathcal{I}_\mu \rightarrow \mathcal{O}_{\mathfrak{X}}/\mathcal{I}_\lambda$ defines a canonical morphism

$$f_{\mu\lambda} : (\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}/\mathcal{I}_\lambda) \rightarrow (\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}/\mathcal{I}_\mu)$$

of (usual) schemes such that we have $f_\lambda = f_\mu \circ f_{\mu\lambda}$. The scheme $X_\lambda = (\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}/\mathcal{I}_\lambda)$ is the morphisms $f_{\mu\lambda}$ then constitutes an inductive system in the category of formal schemes.

Proposition 18.8.16. *The formal scheme and the morphisms f_λ constitute an inductive limit of the system $(X_\lambda, f_{\mu\lambda})$ in the category of formal schemes.*

Proof. Let \mathfrak{Y} be a formal scheme, and for each index λ , let

$$g_\lambda : X_\lambda \rightarrow \mathfrak{Y}$$

be a morphism such that $g_\lambda = g_\mu \circ f_{\mu\lambda}$ for $\mathcal{I}_\mu \subseteq \mathcal{I}_\lambda$. This last condition and the definition of X_λ then imply that the g_λ are equal to the same continuous map $g : \mathfrak{X} \rightarrow \mathfrak{Y}$ on the underlying space; moreover, the homomorphisms $g_\lambda^\# : g^{-1}(\mathcal{O}_{\mathfrak{Y}}) \rightarrow \mathcal{O}_{X_\lambda} = \mathcal{O}_{\mathfrak{X}}/\mathcal{I}_\lambda$ form a projective system of homomorphisms of sheaves of rings. By passing to projective limit, we then deduce a homomorphism $\omega : g^{-1}(\mathcal{O}_{\mathfrak{Y}}) \rightarrow \lim \mathcal{O}_{\mathfrak{X}}/\mathcal{I}_\lambda = \mathcal{O}_{\mathfrak{X}}$, and it is clear that (g, ω) is a morphism of ringed spaces such that the diagram

$$\begin{array}{ccc} X_\lambda & \xrightarrow{g_\lambda} & \mathfrak{Y} \\ & \searrow f_\lambda & \swarrow g \\ & \mathfrak{X} & \end{array} \tag{18.8.5}$$

It remains to prove that g is a morphism of formal schemes; the question is local on \mathfrak{X} and \mathfrak{Y} , we can assume that $\mathfrak{X} = \text{Spf}(A)$, $\mathfrak{Y} = \text{Spf}(B)$, where A, B are admissible rings, and $\mathcal{I}_\lambda = \mathcal{J}_\lambda^\Delta$ where (\mathcal{J}_λ) is a fundamental system of nilideals of A (??). As $A = \varprojlim A/\mathcal{I}_\lambda$, the existence of the morphism of formal affine schemes g fitting into the diagram (??) then follows from the one-to-one correspondence ?? between morphisms of formal affine schemes and continuous homomorphisms of rings, and of the definition of the projective limit. But the uniqueness of g as a morphism of ringed spaces shows that it coincides with the morphism at the beginning of the demonstration. \square

Proposition 18.8.17. *Let \mathfrak{X} be a topological space, (\mathcal{O}_i, u_{ji}) a projective system of sheaves of rings over \mathfrak{X} indexed by \mathbb{N} . Let \mathcal{I}_i be the kernel of $u_{0,i} : \mathcal{O}_i \rightarrow \mathcal{O}_0$ and suppose that*

- (a) *For each i , the ringed space $X_i = (\mathfrak{X}, \mathcal{O}_i)$ is a scheme.*
- (b) *For any $x \in \mathfrak{X}$ and any $i \in \mathbb{N}$, there exists an open neighborhood U_i of x in \mathfrak{X} such that the restriction $\mathcal{I}_i|_{U_i}$ is nilpotent.*
- (c) *The homomorphisms u_{ji} are surjective.*

Let $\mathcal{O}_{\mathfrak{X}}$ be the sheaf of topological rings which is the projective limit of the sheaf of discrete rings \mathcal{O}_i , and $u_i : \mathcal{O}_{\mathfrak{X}} \rightarrow \mathcal{O}_i$ be the canonical homomorphism. Then the topologically ringed space $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ is a formal

scheme, the homomorphisms u_i are surjective, whose kernel $\mathcal{J}^{(i)}$ form a fundamental system of nilideals of \mathfrak{X} , and $\mathcal{J}^{(0)}$ is the projective limit of the sheaf of ideals \mathcal{J}_i .

Proof. We first note that at each stalk, u_{ji} is a surjective homomorphism and a fortiori a local homomorphism, so $v_{ij} = (1_{\mathfrak{X}}, u_{ji}) : X_j \rightarrow X_i$ is a morphism of schemes for $i \geq j$. Suppose that each X_i is an affine scheme with ring A_i . Then there exists a homomorphism $\varphi_{ji} : A_i \rightarrow A_j$ such that $u_{ji} = \tilde{\varphi}_{ji}$, so the sheaf \mathcal{O}_j is a quasi-coherent \mathcal{O}_i -module over X_i , associated with A_j considered as an A_i -module via φ_{ji} . For each $f \in A_i$, let $f' = \varphi_{ji}(f)$; by hypothesis, the opens $D(f)$ and $D(f')$ are identical over \mathfrak{X} , and the homomorphism from $\Gamma(D(f), \mathcal{O}_i) = (A_i)_f$ to $\Gamma(D(f), \mathcal{O}_j) = (A_j)_{f'}$ corresponding to u_{ji} is none other than $(A_j)_{f'}$. But if we consider A_j as an A_i -module, $(A_j)_{f'}$ is the $(A_j)_f$ -module $(A_j)_f$, so we have $u_{ji} = \tilde{\varphi}_{ji}$, if φ_{ji} is considered as a homomorphism of A_i -modules. Then, as u_{ji} is surjective, so is the φ_{ji} and if \mathfrak{j}_i is the kernel of φ_{ji} , the kernel of u_{ji} is the quasi-coherent \mathcal{O}_i -module $\tilde{\mathfrak{j}}_{ji}$. In particular, we have $\mathcal{J}_i = \tilde{\mathfrak{j}}_i$, where \mathfrak{j}_i is the kernel of $\varphi_{0,i} : A_i \rightarrow A_0$. The hypothesis (b) implies that \mathcal{J}_i is nilpotent: in fact, as \mathfrak{X} is quasi-compact, we can cover X by finitely many opens U_k such that $(\mathcal{J}_i|_{U_k})^{n_k} = 0$ and by choosing n to be the largest n_k , we have $\mathcal{J}_i^n = 0$; we then conclude that each \mathfrak{j}_i is nilpotent. Then the ring $A = \varprojlim A_i$ is admissible by ??, the canonical homomorphisms $\varphi_i : A \rightarrow A_i$ is surjective and its kernel $\mathcal{J}^{(i)}$ is equal to the projective limit of the \mathfrak{j}_{ik} for $k \geq i$; the $\mathcal{J}^{(i)}$ form a fundamental system of neighborhoods of 0 in A . The assertion of the proposition then follows from the fact that $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}) = \text{Spf}(A)$. We also note that if $f = (f_i)$ is an element in the projective limit $A = \varprojlim A_i$, the open subsets $D(f_i)$ (affine open in X_i) is then identified with $\mathfrak{D}(f)$, and the scheme induced by X_i over $\mathfrak{D}(f)$ is then identified with the affine scheme $\text{Spec}((A_i)_{f_i})$.

In the general case, we remark that for any quasi-compact open U of \mathfrak{X} , the $\mathcal{J}_i|_U$ is nilpotent as we have seen. We claim that for any $x \in \mathfrak{X}$, there is an open neighborhood U of x in \mathfrak{X} which is an affine open for any X_i . In fact, let U be an affine open for X_0 , and observe that $\mathcal{O}_{X_0} = \mathcal{O}_X/\mathcal{J}_i$. As $\mathcal{J}_i|_U$ is nilpotent in view of the preceding arguments, U is also an affine open for X_i in view of ???. Now for any U satisfying this property, it follows from the same arguments that $(U, \mathcal{O}_{\mathfrak{X}}|_U)$ is a formal scheme such that the $\mathcal{J}^{(i)}|_U$ form a fundamental system of nilideals and $\mathcal{J}^{(0)}|_U$ is the projective limit of $\mathcal{J}_i|_U$, whence the conclusion. \square

Corollary 18.8.18. Suppose that for $i \geq j$, the kernel of u_{ji} is \mathcal{J}_i^{j+1} and that $\mathcal{J}_1/\mathcal{J}_1^2$ is of finite type over $\mathcal{O}_0 = \mathcal{O}_1/\mathcal{J}_1$. Then \mathfrak{X} is an adic formal scheme, and if $\mathcal{J}^{(n)}$ is the kernel of $\mathcal{O}_{\mathfrak{X}} \rightarrow \mathcal{O}_n$, we have $\mathcal{J}^{(n)} = \mathcal{J}^{n+1}$ and $\mathcal{J}/\mathcal{J}^2$ is isomorphic to \mathcal{J}_1 , where we put $\mathcal{J} = \mathcal{J}^{(0)}$. If moreover X_0 is locally Noetherian (resp. Noetherian), then \mathfrak{X} is locally Noetherian (resp. Noetherian).

Proof. As the underlying space of \mathfrak{X} and X_0 are the same, the question is local and we can assume that each X_i is affine. In view of the relations $\mathcal{J}_{ji} = \tilde{\mathfrak{j}}_{ji}$ (With the notations of ??), we are then reduced to the case of ??, and note that $\mathfrak{j}_1/\mathfrak{j}_1^2$ is then a finitely generated A_0 -module (??). \square

In particular, any locally Noetherian formal scheme \mathfrak{X} is the inductive limit of a sequence (X_n) of locally Noetherian (usual) schemes verifying the conditions of ?? and ???: it suffices to consider a nilideal \mathcal{J} of \mathfrak{X} (??) and put $X_n = (\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}/\mathcal{J}^{n+1})$ (??). More generally, the same is true if \mathfrak{X} is an adic formal scheme having a nilideal \mathcal{J} such that $\mathcal{J}/\mathcal{J}^2$ is a finitely generated $(\mathcal{O}_{\mathfrak{X}}/\mathcal{J})$ -module.

Corollary 18.8.19. Let A be an admissible ring. For the formal affine scheme $\mathfrak{X} = \text{Spf}(A)$ to be Noetherian, it is necessary and sufficient that A is adic and Noetherian.

Proof. This condition is clearly sufficient. Conversely, suppose that \mathfrak{X} is Noetherian, and let \mathcal{J} be a nilideal of A , $\mathcal{I} = \mathcal{J}^\Delta$ the nilideal of \mathfrak{X} . The (usual) schemes $X_n = (\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}/\mathcal{J}^{n+1})$ is then affine and Noetherian, so the ring $A_n = A/\mathcal{J}^{n+1}$ is Noetherian (??), and we conclude that $\mathcal{J}/\mathcal{J}^2$ is a finitely generated (A/\mathcal{J}) -module. As the \mathcal{J}^n form a fundamental system of nilideals of \mathfrak{X} , we have $\mathcal{O}_{\mathfrak{X}} = \varprojlim(\mathcal{O}_{\mathfrak{X}}/\mathcal{J}^n)$ (??). \square

Remark 18.8.1. With the notations of ??, let \mathcal{F}_i be an \mathcal{O}_i -module, and suppose that for $i \geq j$ we are given a v_{ij} -morphism $\theta_{ji} : \mathcal{F}_i \rightarrow \mathcal{F}_j$, such that $\theta_{kj} \circ \theta_{ji} = \theta_{ki}$ for $k \leq j \leq i$. As the underlying continuous map of v_{ij} is the identity, θ_{ji} is a homomorphism of sheaves of abelian groups over \mathfrak{X} . Moreover, if \mathcal{F} is the limit of the projective system (\mathcal{F}_i) of sheaves of abelian groups, the fact that each θ_{ji} is a v_{ij} -morphism permits us to define over \mathcal{F} an $\mathcal{O}_{\mathfrak{X}}$ -module structure by passing to projective limits. With this, we say that \mathcal{F} is the **projective limit** (for the θ_{ji}) of the system of \mathcal{O}_i -modules (\mathcal{F}_i) . In the particular case where $v_{ij}^*(\mathcal{F}_i) = \mathcal{F}_j$ and there θ_{ji} is the identity, we then say that \mathcal{F} is the limit of the system (\mathcal{F}_i) such that $v_{ij}^*(\mathcal{F}_i) = \mathcal{F}_j$ for $j \leq i$ (without mention of θ_{ji}).

Let $\mathfrak{X}, \mathfrak{Y}$ be two formal schemes, \mathcal{J} (resp. \mathcal{K}) be a nilideal of \mathfrak{X} (resp. \mathfrak{Y}), and $f : \mathfrak{X} \rightarrow \mathfrak{Y}$ be a morphism such that $f^*(\mathcal{K})\mathcal{O}_{\mathfrak{X}} \subseteq \mathcal{J}$. We then have for each integer $n > 0$ that $f^*(\mathcal{K}^n)\mathcal{O}_{\mathfrak{X}} = (f^*(\mathcal{K})\mathcal{O}_{\mathfrak{X}}) \subseteq \mathcal{J}^n$, so ?? deduce a morphism $f_n : X_n \rightarrow Y_n$ of (usual) schemes such that the diagram

$$\begin{array}{ccc} X_m & \xrightarrow{f_m} & Y_m \\ \downarrow & & \downarrow \\ X_n & \xrightarrow{f_n} & Y_n \end{array} \quad (18.8.6)$$

is commutative for $m \leq n$; in other words, (f_n) is a inductive system of morphisms.

Conversely, let (X_n) (resp. (Y_n)) be a inductive system of schemes satisfying the conditions (b), (c) of ??, and let \mathfrak{X} (resp. \mathfrak{Y}) be the inductive limits (whose existence is proved by ??). By the definition of inductive limits, any inductive system (f_n) of morphisms $f_n : X_n \rightarrow Y_n$ admits an inductive limit $f : \mathfrak{X} \rightarrow \mathfrak{Y}$, which is the unique morphism of formal schemes rendering the diagram

$$\begin{array}{ccc} X_n & \xrightarrow{f_n} & Y_n \\ \downarrow & & \downarrow \\ \mathfrak{X} & \xrightarrow{f} & \mathfrak{Y} \end{array}$$

Proposition 18.8.20. Let $\mathfrak{X}, \mathfrak{Y}$ be adic formal schemes, \mathcal{J} (resp. \mathcal{K}) be a nilideal of \mathfrak{X} (resp. \mathfrak{Y}). The map $f \mapsto (f_n)$ is a bijection from the set of morphisms $f : \mathfrak{X} \rightarrow \mathfrak{Y}$ such that $f^*(\mathcal{K})\mathcal{O}_{\mathfrak{X}} \subseteq \mathcal{J}$ to the set of sequences (f_n) of morphisms rendering the diagram ??.

Proof. If f is the inductive limit of this sequence, it is necessary to prove that $f^*(\mathcal{K})\mathcal{O}_{\mathfrak{X}} \subseteq \mathcal{J}$. This question is local over \mathfrak{X} and \mathfrak{Y} , so we can assume that $\mathfrak{X} = \text{Spf}(A)$, $\mathfrak{Y} = \text{Spf}(B)$ are affine, where A, B are adic, $\mathcal{J} = \mathcal{I}^\Delta$, $\mathcal{K} = \mathcal{R}^\Delta$, where \mathcal{I} (resp. \mathcal{R}) is a nilideal of A (resp. B). We then have $X_n = \text{Spec}(A_n)$, $Y_n = \text{Spec}(B_n)$, where $A_n = A/\mathcal{I}^{n+1}$ and $B_n = B/\mathcal{R}^{n+1}$, in view of ?. Then $f_n = (^a\varphi_n, \tilde{\varphi}_n)$, where $\varphi_n : B_n \rightarrow A_n$ is the homomorphism forming a projective system, so $f = (^a\varphi, \tilde{\varphi})$, where $\varphi = \varprojlim \varphi_n$. The commutative diagram ?? shows that

$\varphi_n(\mathfrak{K}/\mathfrak{K}^{n+1}) \subseteq \mathfrak{I}/\mathfrak{I}^{n+1}$ for each n (by take $m = 0$), so by passing to limit, $\varphi(\mathfrak{K}) \subseteq \mathfrak{I}$, and this implies $f^*(\mathcal{K})\mathcal{O}_{\mathfrak{X}} \subseteq \mathcal{I}$ (??). \square

Corollary 18.8.21. *Let $\mathfrak{X}, \mathfrak{Y}$ be locally Noetherian formal schemes, \mathcal{T} be the largest nilideal of \mathfrak{X} .*

- (i) *For any nilideal \mathcal{K} of \mathfrak{Y} and any morphism $f : \mathfrak{X} \rightarrow \mathfrak{Y}$, we have $f^*(\mathcal{K})\mathcal{O}_{\mathfrak{X}} \subseteq \mathcal{T}$.*
- (ii) *There exists a bijective correspondence between $\text{Hom}(\mathfrak{X}, \mathfrak{Y})$ and the set of sequences (f_n) of morphisms rendering the diagram (??), where $X_n = (\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}/\mathcal{I}^{n+1})$, $Y_n = (\mathfrak{Y}, \mathcal{O}_{\mathfrak{Y}}/\mathcal{K}^{n+1})$.*

Proof. It is clear that (ii) follows from (i) and ???. To prove (i), we can assume that $\mathfrak{X} = \text{Spf}(A)$, $\mathfrak{Y} = \text{Spf}(B)$, A, B being Noetherian and adic, $\mathcal{T} = \mathcal{T}^\Delta$, $\mathcal{K} = \mathfrak{K}^\Delta$, where \mathcal{T} is the largest nilideal of A and \mathfrak{K} is a nilideal of B . Let $f = (^a\varphi, \tilde{q})$ where $\varphi : B \rightarrow A$ is a continuous homomorphism; as the elements of \mathfrak{K} are topologically nilpotent, so are those of $\varphi(\mathfrak{K})$, so $\varphi(\mathfrak{K}) \subseteq \mathcal{T}$ since \mathcal{T} is the set of topologically nilpotent elements of A (EGA 0_I, 7.1.4 ??). The conclusion then follows from ??(ii). \square

Corollary 18.8.22. *Let $f : \mathfrak{X} \rightarrow \mathfrak{S}$ and $g : \mathfrak{Y} \rightarrow \mathfrak{S}$ be morphisms of adic formal schemes. Let \mathcal{I} (resp. \mathcal{K}, \mathcal{L}) be a nilideal of \mathfrak{S} (resp. $\mathfrak{X}, \mathfrak{Y}$), and suppose that $f^*(\mathcal{I})\mathcal{O}_{\mathfrak{X}} \subseteq \mathcal{K}$, $g^*(\mathcal{I})\mathcal{O}_{\mathfrak{Y}} = \mathcal{L}$. Let $S_n = (\mathfrak{S}, \mathcal{O}_{\mathfrak{S}}/\mathcal{I}^{n+1})$, $X_n = (\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}/\mathcal{K}^{n+1})$, $Y_n = (\mathfrak{Y}, \mathcal{O}_{\mathfrak{Y}}/\mathcal{L}^{n+1})$. Then there exists a bijective correspondence between $\text{Hom}_{\mathfrak{S}}(\mathfrak{X}, \mathfrak{Y})$ and the set of sequences (u_n) of S_n -morphisms $u_n : X_n \rightarrow Y_n$ rendering the diagram (??).*

Proof. For any \mathfrak{S} -morphism $u : \mathfrak{X} \rightarrow \mathfrak{Y}$, we have $f = g \circ u$ by definition, so

$$u^*(\mathcal{L})\mathcal{O}_{\mathfrak{X}} = u^*(g^*(\mathcal{I})\mathcal{O}_{\mathfrak{Y}})\mathcal{O}_{\mathfrak{X}} = f^*(\mathcal{I})\mathcal{O}_{\mathfrak{X}} \subseteq \mathcal{K},$$

so the corollary follows from ???. \square

We note that, for $m \leq n$, the datum of a morphism $f_n : X_n \rightarrow Y_n$ determines uniquely a morphism $f_m : X_m \rightarrow Y_m$ rendering the diagram (??), as we immediately see by reducing to the affine case; we have thus defined a map

$$\varphi_{mn} : \text{Hom}_{S_n}(X_n, Y_n) \rightarrow \text{hom}_{S_m}(X_m, Y_m)$$

and the $\text{Hom}_{S_n}(X_n, Y_n)$ form for the φ_{mn} a projective system of sets; ?? then shows that there exists a canonical bijection

$$\text{Hom}_{\mathfrak{S}}(\mathfrak{X}, \mathfrak{Y}) \xrightarrow{\sim} \varprojlim_n \text{Hom}_{S_n}(X_n, Y_n).$$

Remark 18.8.2. Let $\mathfrak{S}, \mathfrak{X}, \mathfrak{Y}$ be formal schemes and $f : \mathfrak{X} \rightarrow \mathfrak{S}, g : \mathfrak{Y} \rightarrow \mathfrak{S}$ be morphisms. Suppose that there are fundamental system of nilideals $(\mathcal{J}_\lambda), (\mathcal{K}_\lambda), (\mathcal{L}_\lambda)$ in $\mathfrak{S}, \mathfrak{X}, \mathfrak{Y}$, respectively, with the same index set I , such that $f^*(\mathcal{J}_\lambda)\mathcal{O}_{\mathfrak{X}} \subseteq \mathcal{K}_\lambda$ and $g^*(\mathcal{J}_\lambda)\mathcal{O}_{\mathfrak{Y}} \subseteq \mathcal{L}_\lambda$ for any λ . Put $S_\lambda = (\mathfrak{S}, \mathcal{O}_{\mathfrak{S}}/\mathcal{J}_\lambda)$, $X_\lambda = (\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}/\mathcal{K}_\lambda)$, $Y_\lambda = (\mathfrak{Y}, \mathcal{O}_{\mathfrak{Y}}/\mathcal{L}_\lambda)$; for $\mathcal{J}_\mu \subseteq \mathcal{J}_\lambda$, $\mathcal{K}_\mu \subseteq \mathcal{K}_\lambda$, $\mathcal{L}_\mu \subseteq \mathcal{L}_\lambda$, note that S_λ (resp. X_λ, Y_λ) is a closed subscheme of S_μ (X_μ, Y_μ) with the same underlying space. As $S_\lambda \rightarrow S_\mu$ is a monomorphism of schemes, we then see that the products $X_\lambda \times_{S_\lambda} Y_\lambda$ and $X_\lambda \times_{S_\mu} Y_\lambda$ are identical (??), because $X_\lambda \times_{S_\mu} Y_\lambda$ is identified with a closed subscheme of $X_\mu \times_{S_\mu} Y_\mu$ with the same underlying space. Now the product is the inductive limit of the schemes $X_\lambda \times_{S_\lambda} Y_\lambda$: in fact, we see as in ??, we can assume that $\mathfrak{S}, \mathfrak{X}, \mathfrak{Y}$ are formal affine

schemes. In view of ?? and our hypotheses, immediately see that our assertion follows from the definition of the completed tensor product of two algebras.

Moreover, let \mathfrak{Z} be a formal \mathfrak{S} -scheme, (\mathcal{M}_λ) be a fundamental system of nilideals of \mathfrak{Z} with index set I , $u : \mathfrak{Z} \rightarrow \mathfrak{X}$, $v : \mathfrak{Z} \rightarrow \mathfrak{Y}$ be morphisms such that $u^*(\mathcal{K}_\lambda)\mathcal{O}_{\mathfrak{Z}} \subseteq \mathcal{M}_\lambda$ and $v^*(\mathcal{L}_\lambda)\mathcal{O}_{\mathfrak{Z}} \subseteq \mathcal{M}_\lambda$. If we put $Z_\lambda = (\mathfrak{Z}, \mathcal{O}_{\mathfrak{Z}}/\mathcal{M}_\lambda)$, and if $u_\lambda : Z_\lambda \rightarrow X_\lambda$ and $v_\lambda : Z_\lambda \rightarrow Y_\lambda$ are the corresponding S_λ -morphisms, we then verify that $(u, v)_{\mathfrak{S}}$ is the inductive limits of the S_λ -morphisms $(u_\lambda, v_\lambda)_{S_\lambda}$.

18.8.4 Formal completion of schemes

Let X be a (usual) scheme, Z be a subscheme of X , U be an open subset of X containing Z and such that Z is a closed subscheme of U ; then Z is defined by a quasi-coherent ideal \mathcal{I}_U of \mathcal{O}_U . For any integer $n > 0$, and any quasi-coherent \mathcal{O}_X -module \mathcal{F} , $(\mathcal{F}|_U) \otimes_{\mathcal{O}_U} (\mathcal{O}_U/\mathcal{I}_U^n)$ is then a quasi-coherent \mathcal{O}_U -module whose support is contained in Z , which is therefore often identified with its restriction to Z . The family $\{(\mathcal{F}|_U) \otimes_{\mathcal{O}_U} (\mathcal{O}_U/\mathcal{I}_U^n)\}_{n \geq 1}$ then forms a projective system of sheaves of abelian groups. The limit $\varprojlim((\mathcal{F}|_U) \otimes_{\mathcal{O}_U} (\mathcal{O}_U/\mathcal{I}_U^n))$ is called the **completion of \mathcal{F} along the subscheme Z of X** , and denoted by $\widehat{\mathcal{F}}_Z$ or simply $\widehat{\mathcal{F}}$ (if there is no confusion). The sections of $\widehat{\mathcal{F}}$ over Z are called the **formal sections** of \mathcal{F} along Z .

This definition is justified by the fact that it obviously does not depend on the choice open subset U , because at every point x of $U - Z$, there is a neighborhood of x in which $\mathcal{O}_U/\mathcal{I}_U^n$ is zero for any integer n . We can therefore limit ourselves to the case where Z is a closed subscheme of X , and we will always assume this henceforth. Also, it is clear that for any open subset $U \subseteq X$, we have $(\mathcal{F}|_U)_{/(U \cap Z)} = (\mathcal{F}_Z)|_{U \cap Z}$.

By passing to projective limits, it is clear that $(\mathcal{O}_X)_Z$ is a sheaf of rings, and that \mathcal{F}_Z can be considered as an (\mathcal{O}_Z) -module. Furthermore, as there existss a basis for Z formed by quasi-compact opens, we can consider $(\mathcal{O}_X)_Z$ (resp. \mathcal{F}_Z) as a sheaf of topological rings (resp. topological groups) which is the projective limit of the sheaf of discrete rings $\mathcal{O}_X/\mathcal{I}^n$ (resp. the sheaf of groups $\mathcal{F} \otimes_{\mathcal{O}_X} (\mathcal{O}_X/\mathcal{I}^n) = \mathcal{F}/\mathcal{I}^n\mathcal{F}$), and by passing to projective limits, \mathcal{F}_Z is then a topological $(\mathcal{O}_X)_Z$ -module. Note that for any quasi-compact open subset U of X , $\Gamma(U \cap Z, (\mathcal{O}_X)_Z)$ (resp. $\Gamma(U \cap Z, \mathcal{F}_Z)$) is then the projective limit of the discrete rings (resp. groups) $\Gamma(U, \mathcal{O}_X/\mathcal{I})$ (resp. $\Gamma(U, \mathcal{F}/\mathcal{I}\mathcal{F})$).

Now let $u : \mathcal{F} \rightarrow \mathcal{G}$ be a homomorphism of \mathcal{O}_X -modules, we then deduce a canonical homomorphism

$$u_{\mathcal{F}^n} : \mathcal{F} \otimes_{\mathcal{O}_X} (\mathcal{O}_X/\mathcal{I}^n) \rightarrow \mathcal{G} \otimes_{\mathcal{O}_X} (\mathcal{O}_X/\mathcal{I}^n)$$

for any $n \geq 1$, and these homomorphisms form a projective system. By passing to projective limits and restricting to Z , we obtain a continuous $(\mathcal{O}_X)_Z$ -homomorphism $\mathcal{F}_Z \rightarrow \mathcal{G}_Z$, denoted by u_Z or \hat{u} , and is called the **completion of u along Z** . It is clear that if $v : \mathcal{G} \rightarrow \mathcal{H}$ is a second homomorphism of \mathcal{O}_X -modules, then $(v \circ u)_Z = (v_Z) \circ (u_Z)$, so \mathcal{F}_Z is a covariant additive functor on \mathcal{F} from the category of quasi-coherent \mathcal{O}_X -modules, with values in the category of $(\mathcal{O}_X)_Z$ -modules.

Proposition 18.8.23. *Let Z be a closed subscheme of X defined by a quasi-coherent ideal \mathcal{I} of finite type. Then the support of $(\mathcal{O}_X)_Z$ is Z , the topologically ringed space $(Z, (\mathcal{O}_X)_Z)$ is an adic formal scheme, and \mathcal{I}_Z is a nilideal of this formal scheme. If $X = \text{Spec}(A)$ is an affine scheme, $\mathcal{I} = \tilde{\mathcal{I}}$ where \mathcal{I} is an ideal of A , and $Z = V(\mathcal{I})$, then $(Z, (\mathcal{O}_X)_Z)$ is canonically identified with $\text{Spf}(\widehat{A})$, where \widehat{A} is*

the Hausdorff completion of A for the \mathfrak{I} -adic topology.

Proof. We can evidently assume that $X = \text{Spec}(A)$ is affine. By ?? the the Hausdorff completion $\widehat{\mathfrak{I}}$ of \mathfrak{I} for the \mathfrak{I} -adic topology is identified with the ideal $\mathfrak{I}\widehat{A}$ of \widehat{A} , and that \widehat{A} is a \widehat{A} -adic ring such that $\widehat{A}/\widehat{\mathfrak{I}}^n = A/\mathfrak{I}^n$. This last relation (for $n = 1$) proves that the open prime ideals of \widehat{A} are the ideals $\widehat{\mathfrak{p}} = \mathfrak{p}\widehat{A}$, where \mathfrak{p} is a prime ideal of A containing \mathfrak{I} , whence $\text{Spf}(\widehat{A}) = Z$. As $\mathcal{O}_X/\mathcal{I}^n = \widetilde{A/\mathfrak{I}^n}$, the proposition then follows from the definition of $\text{Spf}(\widehat{A})$. \square

The formal scheme therefore defined is called the **completion** of X along Z and denoted by X/Z or \widehat{X} . If $Z = X$, we can set $\mathcal{I} = 0$, and then $X/Z = X$. It is clear that if U is an open subscheme of X , then $U_{/(U \cap Z)}$ is canonically identified with the formal subscheme of X/Z induced over the open subset $U \cap Z$ of Z .

Corollary 18.8.24. *Under the hypothesis of ??, assume that X is locally Noetherian. Then the (usual) scheme \widehat{X}_{red} is the unique reduced subscheme of X with underlying space Z (??). For \widehat{X} to be Noetherian, it is necessary and sufficient that \widehat{X}_{red} is Noetherian, and it is sufficient that X is Noetherian.*

Proof. The determination of \widehat{X}_{red} is local (??), so we can assume that X is affine; with the notations of ??, the ideal \mathfrak{T} of topological nilpotent elements of \widehat{A} is the inverse image of the nilradical of A/\mathfrak{T} under the canonical map $\widehat{A} \rightarrow \widehat{A}/\widehat{\mathfrak{I}} = A/\mathfrak{I}$, so \widehat{A}/\mathfrak{T} is isomorphic to $(A/\mathfrak{I})_{\text{red}}$. The first assertion then follows from ?? and ?. If \widehat{X}_{red} is Noetherian, its underlying space Z is also Noetherian, so the $Z_n = \text{Spec}(\mathcal{O}_X/\mathcal{I}^n)$ are Noetherian and so is \widehat{X} (?); the converse of this is immeidate. \square

The canonical homomorphisms $\mathcal{O}_X \rightarrow \mathcal{O}_X/\mathcal{I}^n$ form a projective system and therefore gives, by passing to limit, a homomorphism of sheaves of rings $\theta : \mathcal{O}_X \rightarrow i_*((\mathcal{O}_X)/_Z) = \varprojlim(\mathcal{O}_X/\mathcal{I}^n)$, where $i : Z \rightarrow X$ is the canonical injection. We therefore obtain a morphism

$$(i, \theta) : X/Z \rightarrow X$$

of ringed spaces, called the **canonical morphism**. By tensoring, for any coherent \mathcal{O}_X -module \mathcal{F} , the canonical homomorphisms $\mathcal{O}_X \rightarrow \mathcal{O}_X/\mathcal{I}^n$ gives homomorphisms $\mathcal{F} \rightarrow \mathcal{F} \otimes_{\mathcal{O}_X} (\mathcal{O}_X/\mathcal{I}^n)$ of \mathcal{O}_X -modules which form a projective system, and hence defines a canonical functorial homomorphism $\gamma : \mathcal{F} \rightarrow i_*(\mathcal{F}/_Z)$ of \mathcal{O}_X -modules.

Example 18.8.25. Let Z, Z' be closed subschemes of X , defined by quasi-coherent ideals $\mathcal{I}, \mathcal{I}'$ of \mathcal{O}_X . Suppose that for any affine open U of X , the ideals $\mathcal{I}|_U, \mathcal{I}'|_U$ of \mathcal{O}_U are such that there exists integers $m, m' > 0$ such that $(\mathcal{I}|_U)^m \subseteq \mathcal{I}'|_U$ and $(\mathcal{I}'|_U)^{m'} \subseteq \mathcal{I}|_U$. It is clear that under this condition, for any quasi-coherent \mathcal{O}_X -module \mathcal{F} , the sheaf of abelian groups $\varprojlim(\mathcal{F} \otimes_{\mathcal{O}_X} (\mathcal{O}_X/\mathcal{I}^n))$ and $\varprojlim(\mathcal{F} \otimes_{\mathcal{O}_X} (\mathcal{O}_X/\mathcal{I}'^n))$ are canonically isomorphic, which means $\mathcal{F}/_Z = \mathcal{F}/_{Z'}$. Note that this condition on Z and Z' implies that these two closed subschemes have the same underlying space, but is not in general equivalent to the latter property.

However, if the quasi-coherent ideals \mathcal{I} and \mathcal{I}' are of finite type, then it follows from ?? that if the subschemes Z and Z' have the same underlying space, the above condition is satisfied. In particular, if X is locally Noetherian, so that any quasi-coherent ideal of \mathcal{O}_X is of finite type, then for any closed subset (or locally closed) Z of X , we can define $\mathcal{F}/_Z$ to be equal to $\mathcal{F}/_Y$ for any subscheme Y of X with underlying space Z .

Proposition 18.8.26. Suppose that X is locally Noetherian and let Z be a closed subset of X , \mathcal{F} be a coherent \mathcal{O}_X -module.

- (i) The functor $\mathcal{F}_{/Z}$ is exact on the category of coherent \mathcal{O}_X -modules.
- (ii) The functorial homomorphism $\gamma^\sharp : i^*(\mathcal{F}) \rightarrow \mathcal{F}_{/Z}$ of $(\mathcal{O}_X)_{/Z}$ -modules is an isomorphism.

Proof. To prove (i), it suffices to prove that if $0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow 0$ is an exact sequence of coherent \mathcal{O}_X -modules and U is an affine open of X with Noetherian ring $A = \Gamma(U, \mathcal{O}_X)$, the sequence

$$0 \longrightarrow \Gamma(U \cap Z, \mathcal{F}'_{/Z}) \longrightarrow \Gamma(U \cap Z, \mathcal{F}_{/Z}) \longrightarrow \Gamma(U \cap Z, \mathcal{F}''_{/Z}) \rightarrow 0$$

is exact. Then $\mathcal{F}|_U = \tilde{M}$, $\mathcal{F}'|_U = \tilde{M}'$, $\mathcal{F}''|_U = \tilde{M}''$, where M, M', M'' are A -modules of finite type such that the sequence $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ is exact. Let \mathcal{I} be a quasi-coherent ideal of \mathcal{O}_U defining a subscheme of U with underlying space $U \cap Z$, and \mathfrak{J} be the ideal of A such that $\mathcal{I} = \tilde{\mathfrak{J}}$. We have (??)

$$\Gamma(U \cap Z, \mathcal{F} \otimes_{\mathcal{O}_X} (\mathcal{O}_X / \mathcal{I}^n)) = M \otimes_A (A / \mathfrak{J}^n);$$

so by the definition of projective limit,

$$\Gamma(U \cap Z, \mathcal{F}_{/Z}) = \varprojlim(M \otimes_A (A / \mathfrak{J}^n)) = \widehat{M}$$

where \widehat{M} is the Hausdorff completion of M for the \mathfrak{J} -adic topology, and similarly

$$\Gamma(U \cap Z, \mathcal{F}'_{/Z}) = \widehat{M}', \quad \Gamma(U \cap Z, \mathcal{F}''_{/Z}) = \widehat{M}'';$$

our assertion then follows from the fact that A is Noetherian and the functor \widehat{M} on M is exact on the category of finitely generated A -modules (??).

For assertion (i), the assertion is local, so we can assume that there exists an exact sequence $\mathcal{O}_X^m \rightarrow \mathcal{O}_X^n \rightarrow \mathcal{F} \rightarrow 0$; as γ^\sharp is functorial, and the functors $i^*(\mathcal{F})$ and $\mathcal{F}_{/Z}$ are right exact, we have a commutative diagram

$$\begin{array}{ccccccc} i^*(\mathcal{O}_X^m) & \longrightarrow & i^*(\mathcal{O}_X^n) & \longrightarrow & i^*(\mathcal{F}) & \longrightarrow & 0 \\ \downarrow \gamma^\sharp & & \downarrow \gamma^\sharp & & \downarrow \gamma^\sharp & & \\ (\mathcal{O}_X^m)_{/Z} & \longrightarrow & (\mathcal{O}_X^n)_{/Z} & \longrightarrow & \mathcal{F}_{/Z} & \longrightarrow & 0 \end{array} \tag{18.8.7}$$

with exact rows. Moreover, the functors $i^*(\mathcal{F})$ and $\mathcal{F}_{/Z}$ commutes with finite sums, so we are reduced to the case where $\mathcal{F} = \mathcal{O}_X$. We then have $i^*(\mathcal{O}_X) = (\mathcal{O}_X)_{/Z} = \mathcal{O}_{\widehat{X}}$, and γ^\sharp is a homomorphism of $\mathcal{O}_{\widehat{X}}$ -modules; it then suffices to verify that γ^\sharp maps the unit section of $\mathcal{O}_{\widehat{X}}$ over an open subset of Z to itself, which is immediate and also shows that γ^\sharp is the identity. \square

Corollary 18.8.27. Under the hypotheses of ??, the morphism $i : \widehat{X} \rightarrow X$ is flat.

Corollary 18.8.28. Let X be a locally Noetherian scheme. If \mathcal{F} and \mathcal{G} are coherent \mathcal{O}_X -modules, there

exists a canonical functorial isomorphism

$$(\mathcal{F}/_Z) \otimes_{(\mathcal{O}_X)/_Z} (\mathcal{G}/_Z) \xrightarrow{\sim} (\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})/_Z, \quad (18.8.8)$$

$$(\mathcal{H}\text{om}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}))/_Z \xrightarrow{\sim} \mathcal{H}\text{om}_{(\mathcal{O}_X)/_Z}(\mathcal{F}/_Z, \mathcal{G}/_Z). \quad (18.8.9)$$

Proof. This follows from the canonical identification of $i^*(\mathcal{F})$ and $\mathcal{F}/_Z$; the existence of the first isomorphism is clear for any morphism of ringed spaces and the second is the homomorphism of (??), which is an isomorphism for any flat morphism. \square

Proposition 18.8.29. *Let X be a locally Noetherian scheme. For any coherent \mathcal{O}_X -module \mathcal{F} , the kernel of the canonical homomorphism $\Gamma(X, \mathcal{F}) \rightarrow \Gamma(Z, \mathcal{F}/_Z)$ induced from $\mathcal{F} \rightarrow \mathcal{F}/_Z$ is formed by the sections which are zero on an open neighborhood of Z .*

Proof. It follows from the definition of $\mathcal{F}/_Z$ that the canonical image of such a section is zero. Conversely, if the image of $s \in \Gamma(X, \mathcal{F})$ is zero in $\Gamma(Z, \mathcal{F}/_Z)$, it suffices to see that any $x \in Z$ admits an open neighborhood in X over which s is zero, and we can therefore assume that $X = \text{Spec}(A)$ is affine, A is Noetherian, $Z = V(\mathfrak{J})$ where \mathfrak{J} is an ideal of A , and $\mathcal{F} = \tilde{M}$, where M is a finitely generated A -module. Then $\Gamma(Z, \mathcal{F}/_Z)$ is the Hausdorff completion \hat{M} of M for the \mathfrak{J} -topology, and the homomorphism $\Gamma(X, \mathcal{F}) \rightarrow \Gamma(Z, \mathcal{F}/_Z)$ is the canonical homomorphism $M \rightarrow \hat{M}$. We have seen that the kernel of this homomorphism (??) consists of elements $z \in M$ which is annihilated by an element of $1 + \mathfrak{J}$. We then have $(1 + f)s = 0$ for $f \in \mathfrak{J}$, and for any $x \in Z$ we deduce that $(1_x + f_x)s_x = 0$; as $1_x + f_x$ is invertible in $\mathcal{O}_{X,x}$ ($\mathfrak{J}_x \mathcal{O}_{X,x}$ is contained in \mathfrak{m}_x), we then have $s_x = 0$, which proves the assertion. \square

Corollary 18.8.30. *Let X be a locally Noetherian scheme and $u : \mathcal{F} \rightarrow \mathcal{G}$ be a homomorphism of coherent \mathcal{O}_X -modules. For the homomorphism $u/_Z : \mathcal{F}/_Z \rightarrow \mathcal{G}/_Z$ to be zero, it is necessary and sufficient that u is zero on an open neighborhood of Z .*

Proof. In fact, by ??, $u/_Z$ is identified with $i^*(u)$, so if we consider u as a section of $\mathcal{H} = \mathcal{H}\text{om}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ over X , $u/_Z$ is the section of $i^*(\mathcal{H}) = \mathcal{H}/_Z$ over Z . It then suffices to apply ?? on \mathcal{H} . \square

Corollary 18.8.31. *Let X be a locally Noetherian scheme and $u : \mathcal{F} \rightarrow \mathcal{G}$ be a homomorphism of coherent \mathcal{O}_X -modules. For the homomorphism $u/_Z : \mathcal{F}/_Z \rightarrow \mathcal{G}/_Z$ to be a monomorphism (resp. epimorphism), it is necessary and sufficient that u is a monomorphism (resp. epimorphism) on an open neighborhood of Z .*

Proof. Let \mathcal{P} and \mathcal{N} be the kernel and cokernel of u , so that we have an exact sequence

$$0 \longrightarrow \mathcal{P} \longrightarrow \mathcal{F} \xrightarrow{u} \mathcal{G} \longrightarrow \mathcal{N} \longrightarrow 0$$

By applying $(-)/_Z$, we then get an exact sequence

$$0 \longrightarrow \mathcal{P}/_Z \longrightarrow \mathcal{F}/_Z \xrightarrow{u/_Z} \mathcal{G}/_Z \longrightarrow \mathcal{N}/_Z \longrightarrow 0$$

That $u/_Z$ is a monomorphism (resp. epimorphism) is equivalent to $\mathcal{P}/_Z = 0$ (resp. $\mathcal{N}/_Z = 0$), so we can apply ?? to get the conclusion. \square

Corollary 18.8.32. *Let X be a locally Noetherian scheme and \mathcal{F} be a coherent \mathcal{O}_X -module. For \mathcal{F}/Z to be locally free (resp. locally free of rank n), it is necessary and sufficient that there exists an open neighborhood U of Z such that $\mathcal{F}|_U$ is locally free (resp. locally free of rank n).*

Proof. To say that \mathcal{F}/Z is locally free signifies that any point $x \in Z$ admits an open neighborhood V in X such that there exists an isomorphism $v : (\mathcal{O}_X^n)/_Z \xrightarrow{\sim} \mathcal{F}/Z|_{V \cap Z}$. We can evidently assume that $V = X$, and then it follows from (??) that v is of the form u/Z , where u is a homomorphism $\mathcal{O}_X^n \rightarrow \mathcal{F}$; moreover, by replacing X with an open neighbourhood of Z , we can assume, in view of ??, that u is bijective, whence the corollary. \square

We now consider the induced morphisms between formal completions. Let X, Y be schemes, $f : X \rightarrow Y$ be a morphism, X' (resp. Y') be a closed subscheme of X (resp. Y), and $i : X' \rightarrow X$, $j : Y' \rightarrow Y$ be the canonical injections. Suppose that the composition morphism $f \circ i$ dominates j , so that we have commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ i \uparrow & & \uparrow j \\ X' & \xrightarrow{f'} & Y' \end{array}$$

where f' is the restriction of f to X' . If \mathcal{J} (resp. \mathcal{K}) is the quasi-coherent ideal defining X' (resp. Y'), then this means $f^*(\mathcal{K})\mathcal{O}_X \subseteq \mathcal{J}$ (??). We then have, for each integer $n > 0$, $f^*(\mathcal{K}^n)\mathcal{O}_X \subseteq \mathcal{J}^n$, so if we put $X'_n = (X', \mathcal{O}_X/\mathcal{J}^{n+1})$, $Y'_n = (Y', \mathcal{O}_Y/\mathcal{K}^{n+1})$, the morphism f induces a morphism $f_n : X'_n \rightarrow Y'_n$, and it is immediate that the f_n form a inductive system. The inductive limit of this system (??) is denote by $\hat{f} : X_{X'} \rightarrow Y_{Y'}$, and called the **completion of f along the subschemes X' and Y'** . It is clear from definition that the following diagram is commutative

$$\begin{array}{ccc} X_{X'} & \xrightarrow{\hat{f}} & Y_{Y'} \\ i_X \downarrow & & \downarrow i_Y \\ X & \xrightarrow{f} & Y \end{array} \tag{18.8.10}$$

where the vertical morphism are canonical morphism.

If $X = \text{Spec}(B)$ and $Y = \text{Spec}(A)$ are affine schemes, then $\mathcal{J} = \tilde{\mathfrak{I}}$, $\mathcal{K} = \tilde{\mathfrak{K}}$, where \mathfrak{I} (resp. \mathfrak{K}) is an ideal of B (resp. A), and f corresponds to a homomorphism $\varphi : A \rightarrow B$ such that $\varphi(\mathfrak{K}) \subseteq \mathfrak{I}$. The morphism \hat{f} then corresponds to the continuous homomorphism $\hat{\varphi} : \widehat{A} \rightarrow \widehat{B}$ (??), where \widehat{A} (resp. \widehat{B}) is the \mathfrak{K} -adic completion of A (resp. \mathfrak{I} -adic completion of B).

Let Z be a third scheme, $g : Y \rightarrow Z$ is a morphism, Z' is a closed subscheme of Z defined by a quasi-coherent ideal \mathcal{R} of \mathcal{O}_Z , and suppose that we have $g^*(\mathcal{R})\mathcal{O}_Y \subseteq \mathcal{K}$. Then, if \hat{g} is the completion of g along Y' and Z' , then it follows from our definition that $(\widehat{g \circ f}) = \hat{g} \circ \hat{f}$.

Now suppose that X and Y are locally Noetherian schemes, X', Y' are closed subsets of X, Y , respectively, and $f : X \rightarrow Y$ is a morphism such that $f(X') \subseteq Y'$. Then there exists a coherent ideal \mathcal{J} of \mathcal{O}_X (resp. \mathcal{K} of \mathcal{O}_Y) such that $\text{supp}(\mathcal{O}_X/\mathcal{J}) = X'$ (resp. $\text{supp}(\mathcal{O}_Y/\mathcal{K}) = Y'$) and that $f^*(\mathcal{K})\mathcal{O}_X \subseteq \mathcal{J}$: in fact, it suffices to choose \mathcal{J} to be the ideal defining the reduced subscheme structure of X' and \mathcal{K} to be the ideal defining that of Y' . The relation $f^*(\mathcal{K})\mathcal{O}_X \subseteq \mathcal{J}$ then follows from ???. We can then define the morphism $\hat{f} : X_{X'} \rightarrow Y_{Y'}$, and it follows from ?? that

\hat{f} does not depend on the choice of the idelas \mathcal{J} and \mathcal{K} .

Proposition 18.8.33. *Let X and Y be locally Noetherian S -schemes and suppose that Y is of finite type over S . Let X', Y' be closed subsets of X, Y , respectively, and f, g be two S -morphisms from X to Y such that $f(X') \subseteq Y', g(X') \subseteq Y'$. For that $\hat{f} = \hat{g}$, it is necessary and sufficient that f and g coincides in an open neighborhood of X' .*

Proof. This conditions is clearly sufficient (without the finiteness conditio on Y). To see that it is necessary, we first note that $\hat{f} = \hat{g}$ implies $f(x) = g(x)$ for any $x \in X'$. On the other hand, since the question is local, we can assume that $X = \text{Spec}(B)$ and $Y = \text{Spec}(A)$ are affine neighborhoods of x and $y = f(x) = g(x)$, respectively, with Noetherian rings, and that $S = \text{Spec}(R)$ is affine. Then A is an R -algebra of finite type (??), and f, g correspond to R -homomorphisms ρ, σ from A into B . By hypothesis, the induced homomorphisms $\hat{\rho}, \hat{\sigma}$ on completions are equal. We then conclude from ?? that for any section $s \in A$, the sections $\rho(s)$ and $\sigma(s)$ coincides in an open neighborhood of X' (dependent of s); as A is a finite type algebra over R , we then deduce that there exists an open neighborhood V of X' such that $\rho(s)$ and $\sigma(s)$ coincides over V for any section $s \in A$. If $h \in A$ is such that $D(h)$ is an open neighborhood of X' contained in V , we then conclude so f and g coincides on $D(h)$. \square

Proposition 18.8.34. *Let X and Y be locally Noetherian schemes, $f : X \rightarrow Y$ be a morphism, X', Y' be closed subsets of X, Y , respectively, such that $f(X') \subseteq Y'$. Then, for any coherent \mathcal{O}_Y -module \mathcal{G} , there exists a canonical isomorphism of $(\mathcal{O}_X)_{/X'}$ -modules*

$$(f^*(\mathcal{G}))_{/X'} \xrightarrow{\sim} \hat{f}^*(\mathcal{G}_{/Y'}). \quad (18.8.11)$$

Proof. If we identify canonically $(f^*(\mathcal{G}))_{/X'}$ with $i_X^*(f^*(\mathcal{G}))$ and $\hat{f}(\mathcal{G}_{/Y'})$ with $\hat{f}^*(i_Y^*(\mathcal{G}))$ (??), the proposition then follows from the commutative diagram (??). \square

Remark 18.8.3. Retain the hypotheses of ??, and let \mathcal{F} be a coherent \mathcal{O}_X -module, \mathcal{G} be a coherent \mathcal{O}_Y -module. If $u : \mathcal{G} \rightarrow \mathcal{F}$ is an f -morphism, then it correponds to a homomorphism $u^\sharp : f^*(\mathcal{G}) \rightarrow \mathcal{F}$, hence by completion a continuous $(\mathcal{O}_X)_{/X'}$ -homomorphism

$$(u^\sharp)_{/X'} : (f^*(\mathcal{G}))_{/X'} \rightarrow \mathcal{F}_{/X'}.$$

In view of (??), there exists a unique \hat{f} -morphism $v : \mathcal{G}_{/Y'} \rightarrow \mathcal{F}_{/X'}$ such that $v^\sharp = (u^\sharp)_{/X'}$. If we consider the triple (\mathcal{F}, X, X') (where \mathcal{F} is quasi-coherent \mathcal{O}_X -module and X' is a closed subset of X) as a category \mathcal{C} , with morphisms $(\mathcal{F}, X, X') \rightarrow (\mathcal{G}, Y, Y')$ consisting of a morphism $f : X \rightarrow Y$ of schemes such that $f(X') \subseteq Y'$ and an f -morphism $u : \mathcal{G} \rightarrow \mathcal{F}$, we can then say that $(X_{/X'}, \mathcal{F}_{/X'})$ is a functor from \mathcal{C} to the category of couples $(\mathfrak{Z}, \mathcal{H})$ formed by a locally Noetherian formal scheme \mathfrak{Z} and an $\mathcal{O}_{\mathfrak{Z}}$ -module \mathcal{H} , with morphisms given by morphisms g of formal schemes and g -morphisms of sheaves.

Proposition 18.8.35. *Let X and Y be S -schemes, S' be a closed subscheme of S and X', Y' be closed subschemes of X, Y , respectively, such that, if $\mathcal{J}, \mathcal{K}, \mathcal{L}$ are the nilideals of S', X', Y' , then $\varphi^*(\mathcal{J})\mathcal{O}_X \subseteq \mathcal{K}$ and $\psi^*(\mathcal{J})\mathcal{O}_Y \subseteq \mathcal{L}$ (where $\varphi : X \rightarrow S$ and $\psi : Y \rightarrow S$ are the structural morphisms). Let $Z = X \times_S Y$ and $Z' = p^{-1}(X') \cap q^{-1}(Y')$, where p, q are the canonical projections.*

- (a) *Then the completion $Z_{/Z'}$ is identified with the product of the formal $S_{/S'}$ -schemes $(X_{/X'}) \times_{S_{/S'}} (Y_{/Y'})$, the structural morphisms with $\hat{\varphi}, \hat{\psi}$, and the projections with \hat{p}, \hat{q} .*

- (b) If T is an S -scheme, $u : T \rightarrow X$, $v : T \rightarrow Y$ are S -morphisms, and T' is a closed subscheme of T such that, if \mathcal{M} is the ideal defining T' , then $u^*(\mathcal{K})\mathcal{O}_T \subseteq \mathcal{M}$ and $v^*(\mathcal{L})\mathcal{O}_T \subseteq \mathcal{M}$. Then the completion of $(u, v)_S$ along T' and Z' is canonically identified with $(\hat{u}, \hat{v})_{S, S'}$.

Proof. It is immediate that the question is local for S, X, Y , so we can assume that $S = \text{Spec}(A)$, $X = \text{Spec}(B)$, $Y = \text{Spec}(C)$, $S' = V(\mathfrak{J})$, $X' = V(\mathfrak{K})$, $Y' = V(\mathfrak{R})$, where $\mathfrak{J}, \mathfrak{K}, \mathfrak{R}$ are ideals such that $\rho(\mathfrak{J}) \subseteq \mathfrak{K}$ and $\sigma(\mathfrak{J}) \subseteq \mathfrak{R}$, where $\sigma : A \rightarrow B$ and $\sigma : A \rightarrow C$ are the corresponding homomorphisms. Then we see that $Z = \text{Spec}(B \otimes_A C)$ and that $Z' = V(\mathfrak{L})$, where \mathfrak{L} is the ideal generated by $\text{im}(\mathfrak{K} \otimes_A C) + \text{im}(B \otimes_A \mathfrak{R})$. The conclusion then follows from ?? and the fact that the complete tensor product $\widehat{B} \hat{\otimes}_{\widehat{A}} \widehat{C}$ is the completion of $B \otimes_A C$ for the \mathfrak{L} -adic topology. \square

Corollary 18.8.36. *With the hypotheses of ??, for any S -morphism $f : X \rightarrow Y$ satisfying $f^*(\mathcal{L})\mathcal{O}_X \subseteq \mathcal{K}$, the graph diagram Γ_f is identified with the completion $(\widehat{\Gamma_f})$ of the diagram morphism of f .*

Corollary 18.8.37. *Let X, Y be schemes, $f : X \rightarrow Y$ be a morphism, Y' be a closed subscheme of Y , and $X' = f^{-1}(Y')$. Then following commutative diagram is cartesian:*

$$\begin{array}{ccc} X_{/X'} & \xrightarrow{i_X} & X \\ f \downarrow & & \downarrow f \\ Y_{/Y'} & \xrightarrow{i_Y} & Y \end{array}$$

Proof. It suffices to apply ?? to the fiber product $X \times_Y Y'$, which is identified with X' . \square

18.8.5 Coherent sheaves over formal schemes

In this paragraph, we discuss coherent sheaves over locally Noetherian formal schemes, and characterize them as projective limits of coherent modules over (usual) schemes. For this, we shall first consider the affine case, so let A be an adic ring and \mathfrak{J} be a nilideal of A . Let $X = \text{Spec}(A)$, $\mathfrak{X} = \text{Spf}(A)$, whose underlying space is identified with the closed subset $V(\mathfrak{J})$ of $\text{Spec}(A)$. If $X' = \text{Spec}(A/\mathfrak{J})$ is the closed subscheme of X defined by $\tilde{\mathfrak{J}}$, it then follows from definition that \mathfrak{X} is identified with $X_{/X'}$. For any A -module M , the sheaf $M^\Delta = (\tilde{M})_{/X'}$ is then an $\mathcal{O}_{\mathfrak{X}}$ -module. Moreover, if $u : M \rightarrow N$ is a homomorphism of A -modules, it then corresponds to a homomorphism $\tilde{u} : \tilde{M} \rightarrow \tilde{N}$, and hence to a continuous homomorphism $\tilde{u}_{/X'} : (\tilde{M})_{/X'} \rightarrow (\tilde{N})_{/X'}$, which we denote by u^Δ . It is evident that $(v \circ u)^\Delta = v^\Delta \circ u^\Delta$, so we obtain an additive covariant functor $M \mapsto M^\Delta$ from the category of A -modules to the category of $\mathcal{O}_{\mathfrak{X}}$ -modules.

As A is an adic ring, the ideals \mathfrak{J}^n are open in A (EGA 0I, 7.1.9 ??), hence separated and complete. The ideal $(\mathfrak{J}^n)^\Delta$ of $\mathcal{O}_{\mathfrak{X}}$, with the preceding definition, is then equal to the ideal defined in ??, and if we put $\mathcal{J} = \mathfrak{J}^\Delta$, then $(\mathfrak{J}^n)^\Delta = \mathcal{J}^n$ if $\mathfrak{J}/\mathfrak{J}^2$ is a finitely generated A -module (??). Under this hypothesis, let $A_n = A/\mathfrak{J}^{n+1}$ and $X_n = \text{Spec}(A_n) = (\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}/\mathcal{J}^{n+1})$. If $u_{nm} : X_m \rightarrow X_n$ is the canonical morphism induced by $A_n \rightarrow A_m$ for $m \leq n$, the formal scheme \mathfrak{X} is then the inductive limit of X_n for the u_{nm} (??).

Proposition 18.8.38. *Let A be an adic Noetherian ring. Then the functor $M \mapsto M^\Delta$ is exact on the category of finitely generated A -modules, and we have a canonical isomorphism*

$$\Gamma(\mathfrak{X}, M^\Delta) = M.$$

Proof. The exactness of the functor $M \mapsto M^\Delta$ follows from that of $M \mapsto \tilde{M}$ (??) and $\mathcal{F} \rightarrow \mathcal{F}_{/X'}$ (??). By definition, $\Gamma(\mathfrak{X}, M^\Delta)$ is the \mathfrak{I} -adic completion of the A -module $\Gamma(X, \tilde{M}) = M$ (\mathfrak{I} is a nilideal of A). But as A is complete and M is finitely generated, we see that M is complete and separated (??), and this proves the proposition. \square

Proposition 18.8.39. *Let A be an adic Noetherian ring.*

(a) *If M and N are finitely generated A -modules, there exists a canonical isomorphism*

$$(M \otimes_A N)^\Delta \xrightarrow{\sim} M^\Delta \otimes_{\mathcal{O}_{\mathfrak{X}}} N^\Delta, \quad (18.8.12)$$

$$(\mathrm{Hom}_A(M, N))^\Delta \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}_{\mathfrak{X}}}(M^\Delta, N^\Delta). \quad (18.8.13)$$

(b) *The map $u \mapsto u^\Delta$ is a functorial isomorphism*

$$\mathrm{Hom}_A(M, N) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}_{\mathfrak{X}}}(M^\Delta, N^\Delta). \quad (18.8.14)$$

Proof. The first two isomorphisms follow from ??(a) and ?. Now as $\mathrm{Hom}_A(M, N)$ is a finitely generated A -module, we can apply ?? to identify $\Gamma(\mathfrak{X}, (\mathrm{Hom}_A(M, N))^\Delta)$ with $\mathrm{Hom}_A(M, N)$, and by (??), we see that (??) is an isomorphism. \square

Proposition 18.8.40. *If A is an adic Noetherian ring, $\mathcal{O}_{\mathfrak{X}}$ is a coherent sheaf of rings.*

Proof. If $f \in A$, we see that $A_{\{f\}}$ is an adic Noetherian ring (??) and as the question is local (??), we are reduced to prove that the kernel of a homomorphism $v : \mathcal{O}_{\mathfrak{X}}^n \rightarrow \mathcal{O}_{\mathfrak{X}}$ is an $\mathcal{O}_{\mathfrak{X}}$ -module of finite type. We then have $v = u^\Delta$, where $u : A^n \rightarrow A$ is a homomorphism (??). As A is Noetherian, the kernel of u is finitely generated, which means we have a homomorphism $w : A^m \rightarrow A^n$ such that the following sequence is exact:

$$A^m \xrightarrow{w} A^n \xrightarrow{u} A$$

We then conclude from ?? that the sequence

$$\mathcal{O}_{\mathfrak{X}}^m \xrightarrow{w^\Delta} \mathcal{O}_{\mathfrak{X}}^n \xrightarrow{v} \mathcal{O}_{\mathfrak{X}}$$

is exact, which means the kernel of v is of finite type. \square

With the preceding notations, let $A_n = A/\mathfrak{J}^{n+1}$ and X_n be the affine scheme $\mathrm{Spec}(A_n) = (\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}/\mathfrak{J}^{n+1})$, where $\mathcal{J} = \mathfrak{J}^\Delta$ is the nilideal of $\mathcal{O}_{\mathfrak{X}}$ corresponding to \mathfrak{J} . Let $u_{nm} : X_m \rightarrow X_n$ be the morphism induced by the ring homomorphism $A_n \rightarrow A_m$ for $m \leq n$. As we have remarked, \mathfrak{X} is then the inductive limit of the X_n (??).

Proposition 18.8.41. *Suppose that A is an adic Noetherian ring and let \mathcal{F} be an $\mathcal{O}_{\mathfrak{X}}$ -module. Then the following conditions are equivalent:*

(i) \mathcal{F} is a coherent $\mathcal{O}_{\mathfrak{X}}$ -module;

(ii) there exists a finitely generated A -module M (uniquely determined up to isomorphism) such that \mathcal{F} is isomorphic to M^Δ .

- (iii) \mathcal{F} is isomorphic to the projective limit of a sequence (\mathcal{F}_n) of coherent \mathcal{O}_{X_n} -modules such that $u_{nm}^*(\mathcal{F}_n) = \mathcal{F}_m$ for $m \leq n$, and the projective system (\mathcal{F}_n) is then isomorphic to the system $(\mathcal{F} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}_{X_n})$

Proof. By definition, we have $\mathcal{O}_{\mathfrak{X}} = A^\Delta$. If condition (ii) is satisfied, then M is the cokernel of a homomorphism $A^m \rightarrow A^n$, so it follows from ?? that M^Δ is the cokernel of a homomorphism $\mathcal{O}_{\mathfrak{X}}^m \rightarrow \mathcal{O}_{\mathfrak{X}}^n$. As the sheaf $\mathcal{O}_{\mathfrak{X}}$ is coherent ??, so is M^Δ ?? and this proves (ii) \Rightarrow (i).

Now assume the conditions in (iii). Then since each X_n is an affine (usual) scheme, we have $\mathcal{F}_n = \tilde{M}_n$, where M_n is a finitely generated A_n -module. Since $u_{mn}^*(\mathcal{F}_n) = \mathcal{F}_m$ for $m \leq n$, we have $M_m = M_n \otimes_{A_n} A_m$???. The modules M_n then form a projective system for the canonical bi-homomorphisms $M_n \rightarrow M_m$, and it follows from the definition of A_n that this projective system satisfies the conditions of ??, so its projective limit M is a finitely generated A -module such that $M_n = M \otimes_A A_n$ for each n . We then deduce that \mathcal{F}_n is induced over X_n by $\tilde{M} \otimes_{\mathcal{O}_{\mathfrak{X}}} (\mathcal{O}_{\mathfrak{X}}/\mathcal{J}^{n+1})$, so $\mathcal{F} = M^\Delta$ by definition. Conversely, if $\mathcal{F} = M^\Delta$ for a finitely generated A -module M , then by definition, \mathcal{F} is the projective limit of the system $\tilde{M}_n \otimes_{\mathcal{O}_{\mathfrak{X}}} (\mathcal{O}_{\mathfrak{X}}/\mathcal{J}^{n+1})$, and we have $u_{nm}^*(\mathcal{F}_n) = \mathcal{F}_m$ for $m \leq n$.

Finally, assume that \mathcal{F} is a coherent $\mathcal{O}_{\mathfrak{X}}$ -module. Considered as an $\mathcal{O}_{\mathfrak{X}}$ -module, we have $\mathcal{O}_{X_n} = \mathcal{O}_{\mathfrak{X}}/\mathcal{J}^{n+1} = \tilde{A}_n$, so $\mathcal{F}_n = \mathcal{F} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}_{X_n}$ is a coherent $\mathcal{O}_{\mathfrak{X}}$ -module ??, and since it is also a \mathcal{O}_{X_n} -module and \mathcal{J}^{n+1} is coherent (\mathcal{J} is finitely generated), we conclude that \mathcal{F}_n is a coherent \mathcal{O}_{X_n} -module ??, and it is immediate that $u_{mn}^*(\mathcal{F}_n) = \mathcal{F}_m$ for $m \leq n$. If $\mathcal{G} = \varprojlim \mathcal{F}_n$ is the projective limit, then it is a coherent $\mathcal{O}_{\mathfrak{X}}$ -module in view of the equivalence of (ii) and (iii), and it remains to prove that \mathcal{F} is isomorphic to \mathcal{G} . Now the canonical homomorphisms $\mathcal{F} \rightarrow \mathcal{F}_n$ form a projective system, hence induces a canonical homomorphism $w : \mathcal{F} \rightarrow \mathcal{G}$, so it suffices to prove that w is an isomorphism. Since this question is local, we can assume that \mathcal{F} is the cokernel of a homomorphism $\mathcal{O}_{\mathfrak{X}}^m \rightarrow \mathcal{O}_{\mathfrak{X}}^n$, which is of the form v^Δ , where v is a homomorphism $v : A^m \rightarrow A^n$ of A -modules ??, and \mathcal{F} is then isomorphic to M^Δ , where $M = \text{coker } v$. In view of ??, we then have

$$\mathcal{F}_n = M^\Delta \otimes_{\mathcal{O}_{\mathfrak{X}}} A_n^\Delta = (M \otimes_A A_n)^\Delta = (M \otimes_A A_n)^\Delta = \widetilde{M \otimes_A A_n},$$

since the \mathcal{J} -adic topology on $M \otimes_A A_n$ is discrete. This then implies $M^\Delta = \varprojlim \mathcal{F}_n = \mathcal{G}$, so w is an isomorphism. \square

Theorem 18.8.42. *Let A be an adic Noetherian ring. Then the functor $M \mapsto M^\Delta$ is an equivalence from the category of finitely generated A -modules to the category of coherent $\mathcal{O}_{\mathfrak{X}}$ -module.*

Proof. This follows from ??, ?? and ??.

Now let A, B be two adic Noetherian rings and $\varphi : B \rightarrow A$ be a continuous homomorphism. We denote by \mathcal{J} (resp. \mathfrak{K}) the nilideal of A (resp. B), so that $\varphi(\mathfrak{K}) \subseteq \mathcal{J}$, and we put $X = \text{Spec}(A)$, $Y = \text{Spec}(B)$, $\mathfrak{X} = \text{Spf}(A)$, $\mathfrak{Y} = \text{Spf}(B)$. Let $f : X \rightarrow Y$ be the corresponding morphism and $\hat{f} : \mathfrak{X} \rightarrow \mathfrak{Y}$ be the completion of f , which is also the morphism of formal schemes corresponding

to φ . We then have a commutative diagram

$$\begin{array}{ccc} \mathfrak{X} & \xrightarrow{\hat{f}} & \mathfrak{Y} \\ i_X \downarrow & & \downarrow i_Y \\ X & \xrightarrow{f} & Y \end{array} \quad (18.8.15)$$

Proposition 18.8.43. *For any finitely generated B -module N , there exists a canonical homomorphism*

$$\hat{f}^*(N^\Delta) \xrightarrow{\sim} (N \otimes_B A)^\Delta.$$

Proof. By ??, we have a canonical isomorphism $N^\Delta = i_Y^*(\tilde{N})$, so by ??

$$(N \otimes_B A)^\Delta = i_X^*(\widetilde{N \otimes_B A}) = i_X^*(f^*(\tilde{N})).$$

The proposition then follows from the commutative diagram ??.

□

Corollary 18.8.44. *For any ideal \mathfrak{b} of B , we have*

$$\hat{f}^*(\mathfrak{b}^\Delta)\mathcal{O}_{\mathfrak{X}} = (\mathfrak{b}A)^\Delta.$$

Proof. Let $j : \mathfrak{b} \rightarrow B$ be the canonical injection, which corresponds to the canonical injection $j^\Delta : \mathfrak{b}^\Delta \rightarrow \mathcal{O}_{\mathfrak{Y}}$. By definition, $\hat{f}^*(\mathfrak{b}^\Delta)\mathcal{O}_{\mathfrak{X}}$ is the image of the homomorphism $f^*(\mathfrak{b}^\Delta) : \hat{f}^*(\mathfrak{b}^\Delta) \rightarrow \mathcal{O}_{\mathfrak{X}} = \hat{f}^*(\mathcal{O}_{\mathfrak{Y}})$. But this homomorphism is identified with $(j \otimes 1)^\Delta : (\mathfrak{b} \otimes_B A)^\Delta \rightarrow \mathcal{O}_{\mathfrak{X}} = (B \otimes_B A)^\Delta$ in view of ???. Since the image of $j \otimes 1$ is the ideal $\mathfrak{b}A$ of A , the image of $(j \otimes 1)^\Delta$ is equal to $(\mathfrak{b}A)^\Delta$, whence the corollary. □

Proposition 18.8.45. *If \mathfrak{X} is a locally Noetherian formal scheme, the sheaf of rings $\mathcal{O}_{\mathfrak{X}}$ is coherent and any nilideal of \mathfrak{X} is coherent.*

Proof. In fact, this question is local, so we can assume that \mathfrak{X} is affine, and the proposition then follows from ?? and ??.

□

Let \mathfrak{X} be a locally Noetherian formal scheme, \mathcal{I} be a nilideal of \mathfrak{X} , and X_n be the (usual) scheme $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}/\mathcal{I}^{n+1})$, so that \mathfrak{X} is the inductive limit of (X_n) for the morphisms $u_{mn} : X_m \rightarrow X_n$ (??). With these notations, we have the following theorem:

Theorem 18.8.46. *For an $\mathcal{O}_{\mathfrak{X}}$ -module \mathcal{F} to be coherent, it is necessary and sufficient that it is isomorphic to the projective limit of a sequence (\mathcal{F}_n) of coherent \mathcal{O}_{X_n} -modules such that $u_{nm}^*(\mathcal{F}_n) = \mathcal{F}_m$ for $m \leq n$. In this case, the projective system (\mathcal{F}_n) is then isomorphic to the system $u_n^*(\mathcal{F}) = \mathcal{F} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}_{X_n}$, where $u_n : X_n \rightarrow \mathfrak{X}$ is the canonical morphism.*

Proof. This question is local, so we can assume that \mathfrak{X} is an affine formal scheme, and the theorem then follows from ??.

□

In view of ??, we can then say that giving a coherent $\mathcal{O}_{\mathfrak{X}}$ -module is equivalent to giving a projective system (\mathcal{F}_n) of coherent \mathcal{O}_{X_n} -modules such that $u_{mn}^*(\mathcal{F}_n) = \mathcal{F}_m$ for $m \leq n$.

Corollary 18.8.47. *Under the hypotheses of ??, if \mathcal{F} and \mathcal{G} are coherent $\mathcal{O}_{\mathfrak{X}}$ -modules, we have a canonical isomorphism*

$$\mathrm{Hom}_{\mathcal{O}_{\mathfrak{X}}}(\mathcal{F}, \mathcal{G}) \xrightarrow{\sim} \varprojlim \mathrm{Hom}_{\mathcal{O}_{X_n}}(\mathcal{F}_n, \mathcal{G}_n). \quad (18.8.16)$$

Proof. The transition homomorphism $\mathrm{Hom}_{\mathcal{O}_{X_n}}(\mathcal{F}_n, \mathcal{G}_n) \rightarrow \mathrm{Hom}_{\mathcal{O}_{X_m}}(\mathcal{F}_m, \mathcal{G}_m)$ is given by $\theta_n \mapsto u_{mn}^*(\theta_n)$ ($m \leq n$), and the homomorphism ?? sends an element $\theta \in \mathrm{Hom}_{\mathcal{O}_{\mathfrak{X}}}(\mathcal{F}, \mathcal{G})$ to the sequence $(u_n^*(\theta))$. In view of ??, we see that the inverse homomorphism is given by sending a projective system $(\theta_n) \in \varprojlim \mathrm{Hom}_{\mathcal{O}_{X_n}}(\mathcal{F}_n, \mathcal{G}_n)$ to its projective limit. \square

Corollary 18.8.48. *Under the hypotheses of ??, for a homomorphism $\theta : \mathcal{F} \rightarrow \mathcal{G}$ to be surjective, it is necessary and sufficient that the corresponding homomorphism $\theta_0 = u_0^*(\theta) : \mathcal{F}_0 \rightarrow \mathcal{G}_0$ is surjective.*

Proof. The question is local, so we can assume that $\mathfrak{X} = \mathrm{Spf}(A)$, where A is an adic Noetherian ring, $\mathcal{F} = M^\Delta$, $\mathcal{G} = N^\Delta$, and $\theta = u^\Delta$, where M, N are finitely generated A -modules and $u : M \rightarrow N$ is a homomorphism. We then have $\theta_0 = \tilde{u}_0$, where u_0 is the induced homomorphism

$$u \otimes 1 : M \otimes_A A/\mathfrak{J} \rightarrow N \otimes_A A/\mathfrak{J}.$$

The conclusion then follows if we can prove that u is surjective if and only if u_0 is surjective. To this end, recall that \mathfrak{J} is contained in the Jacobson radical of A (??(b)), so the assertion follows from Nakayama's lemma. \square

Remark 18.8.4. In view of ??, we see that any coherent $\mathcal{O}_{\mathfrak{X}}$ -module \mathcal{F} is canonically endowed with a structure of topological $\mathcal{O}_{\mathfrak{X}}$ -module, which is the projective limit of the sheaf of discrete groups \mathcal{F}_n . It then follows from ?? that any homomorphism $u : \mathcal{F} \rightarrow \mathcal{G}$ of coherent $\mathcal{O}_{\mathfrak{X}}$ -modules is automatically continuous. Moreover, if \mathcal{H} is a coherent sub- $\mathcal{O}_{\mathfrak{X}}$ -module of a coherent $\mathcal{O}_{\mathfrak{X}}$ -module \mathcal{F} , then for any open subset $U \subseteq \mathfrak{X}$, $\Gamma(U, \mathcal{H})$ is the kernel of the (continuous) homomorphism $\Gamma(U, \mathcal{F}) \rightarrow \Gamma(U, \mathcal{F}/\mathcal{H})$ (because the functor Γ is left exact). Since $\Gamma(U, \mathcal{F}/\mathcal{H})$ is a separated topological group, we conclude that $\Gamma(U, \mathcal{H})$ is a closed subgroup of $\Gamma(U, \mathcal{F})$.

Proposition 18.8.49. *Let X be a locally Noetherian formal scheme, \mathcal{F} and \mathcal{G} be coherent $\mathcal{O}_{\mathfrak{X}}$ -modules. Then there are canonical isomorphisms of topological $\mathcal{O}_{\mathfrak{X}}$ -modules*

$$\mathcal{F} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{G} \xrightarrow{\sim} \varprojlim (\mathcal{F}_n \otimes_{\mathcal{O}_{X_n}} \mathcal{G}_n), \quad (18.8.17)$$

$$\mathrm{Hom}_{\mathcal{O}_{\mathfrak{X}}}(\mathcal{F}, \mathcal{G}) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}_{X_0}}(\mathcal{F}_0, \mathcal{G}_0). \quad (18.8.18)$$

As $\mathrm{Hom}_{\mathcal{O}_{\mathfrak{X}}}(\mathcal{F}, \mathcal{G})$ is the global section of the topological $\mathcal{O}_{\mathfrak{X}}$ -module $\mathrm{Hom}_{\mathcal{O}_{\mathfrak{X}}}(\mathcal{F}, \mathcal{G})$, it is endowed with a canonical topology. If \mathfrak{X} is Noetherian, then it follows from ?? that a fundamental system of neighborhoods of 0 is given by the subgroups $\mathrm{Hom}_{\mathcal{O}_{\mathfrak{X}}}(\mathcal{F}, \mathcal{I}^n \mathcal{G})$ (c.f. EGA, 0, 7.8.2 ??).

Proposition 18.8.50. *Let \mathfrak{X} be a Noetherian formal scheme and \mathcal{F}, \mathcal{G} be coherent $\mathcal{O}_{\mathfrak{X}}$ -modules. Then in the topological group $\mathrm{Hom}_{\mathcal{O}_{\mathfrak{X}}}(\mathcal{F}, \mathcal{G})$, the set of surjective (resp. injective) homomorphisms is open.*

Proof. In view of ??, the set of surjective homomorphisms in $\mathrm{Hom}_{\mathcal{O}_{\mathfrak{X}}}(\mathcal{F}, \mathcal{G})$ is the inverse image of the set of surjective homomorphisms in $\mathrm{Hom}_{\mathcal{O}_{X_0}}(\mathcal{F}_0, \mathcal{G}_0)$ under the continuous map

$\text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \rightarrow \text{Hom}_{\mathcal{O}_{X_0}}(\mathcal{F}_0, \mathcal{G}_0)$. For the second assertion, we may cover X by finitely many open Noetherian affine formal schemes U_i . For an element $\theta \in \text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$, it is necessary and sufficient that its restriction in $\text{Hom}_{\mathcal{O}_{X|U_i}}(\mathcal{F}|_{U_i}, \mathcal{G}|_{U_i})$ is injective for each i . Since there are only finitely many U_i , we can then reduce to the affine case, and the assertion follows from (EGA, 0, 7.8.3 ??). \square

Chapter 19

Global properties of morphisms of schemes

19.1 Affine morphisms

19.1.1 Schemes affine over a scheme

Let S be a scheme and X be an S -scheme. If $f : X \rightarrow S$ is the structural morphism, then the direct image $f_*(\mathcal{O}_X)$ is an \mathcal{O}_S -algebra, which we denote by $\mathcal{A}(X)$ if there is no confusion. If U is an open subset of S , we have

$$\mathcal{A}(f^{-1}(U)) = \mathcal{A}(X)|_U.$$

Similarly, for any \mathcal{O}_X -module \mathcal{F} (resp. any \mathcal{O}_X -algebra \mathcal{B}), we denote by $\mathcal{A}(\mathcal{F})$ (resp. $\mathcal{A}(\mathcal{B})$) the direct image $f_*(\mathcal{F})$ (resp. $f_*(\mathcal{B})$) which is an $\mathcal{A}(X)$ -module (resp. $\mathcal{A}(X)$ -algebra), and also an \mathcal{O}_S -module (resp. \mathcal{O}_S -algebra).

Let Y be another S -scheme with $g : Y \rightarrow S$ the structural morphism, and $h : X \rightarrow Y$ be an S -morphism. We then have a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{h} & Y \\ & \searrow f & \swarrow g \\ & S & \end{array}$$

By definition we have a homomorphism $h^\# : \mathcal{O}_Y \rightarrow h_*(\mathcal{O}_X)$ of sheaves of rings, and we deduce from this a homomorphism of \mathcal{O}_S -algebras $g_*(h^\#) : g_*(\mathcal{O}_Y) \rightarrow g_*(h_*(\mathcal{O}_X)) = f_*(\mathcal{O}_X)$, which means, a homomorphism $\mathcal{A}(Y) \rightarrow \mathcal{A}(X)$ of \mathcal{O}_S -algebras, and we denote it by $\mathcal{A}(h)$. If $h' : Y \rightarrow Z$ is another S -morphism, it is immediate that $\mathcal{A}(h' \circ h) = \mathcal{A}(h) \circ \mathcal{A}(h')$. Therefore we have defined a contravariant functor $\mathcal{A}(X)$ from the category of S -schemes to the category of \mathcal{O}_S -algebras.

Now let \mathcal{F} be an \mathcal{O}_X -module, \mathcal{G} be an \mathcal{O}_Y -module, and $u : \mathcal{G} \rightarrow \mathcal{F}$ be an h -morphism, which is a homomorphism $\mathcal{G} \rightarrow h_*(\mathcal{F})$ of \mathcal{O}_Y -modules. Then $g_*(u) : g_*(\mathcal{G}) \rightarrow g_*(h_*(\mathcal{F})) = f_*(\mathcal{F})$ is a homomorphism $\mathcal{A}(\mathcal{G}) \rightarrow \mathcal{A}(\mathcal{F})$ of \mathcal{O}_S -modules, which we denote by $\mathcal{A}(u)$. The couple $(\mathcal{A}(h), \mathcal{A}(u))$ is then a bi-homomorphism of $\mathcal{A}(Y)$ -modules $\mathcal{A}(\mathcal{G})$ to the $\mathcal{A}(X)$ -module $\mathcal{A}(\mathcal{F})$.

If we fix S and consider the couples (X, \mathcal{F}) , where X is an S -scheme and \mathcal{F} is an \mathcal{O}_X -module, we then see that $(\mathcal{A}(X), \mathcal{A}(\mathcal{F}))$ defines a contravariant functor from the category of these couples to the category of couples of \mathcal{O}_S -algebras and modules of this algebra.

Consider now an S -scheme X and let $f : X \rightarrow S$ be a structural morphism. We say that X is **affine over S** if there exists a covering (S_α) of S by affine opens such that, for each α , the open subscheme $f^{-1}(S_\alpha)$ of X is affine. If this is true, we also say that X is an **affine S -scheme**, or the structural morphism f is affine.

Example 19.1.1. Any closed subscheme of S is an affine S -scheme. In fact, if Y is a closed subscheme of S , then for any affine open U of S , the intersection $U \cap Y$ is a closed subscheme of U , whence affine.

Remark 19.1.1. One should note that an affine S -scheme X is not necessarily an affine scheme (for example S is affine over S , but note that this is true if S itself is affine). On the other hand, if X is an S -scheme and is affine, it is not necessarily true that X is an affine S -scheme (we will see this later). However, if S is a separated scheme, then any affine scheme is affine over S by ??.

Proposition 19.1.2. *Any affine S -scheme is separated over S .*

Proof. Recall that separatedness is local on target ??, and if $f^{-1}(S_\alpha)$ is affine, then the restriction of f to $f^{-1}(S_\alpha)$ is a morphism between affine schemes, so is separated. \square

Proposition 19.1.3. *Let X be an affine S -scheme and $f : X \rightarrow S$ be the structural morphism. Then for any open subset $U \subseteq S$, $f^{-1}(U)$ is affine over U . In particular, if U is affine, so is $f^{-1}(U)$.*

Proof. In view of the definition, we can reduce to the case $S = \text{Spec}(A)$ and $X = \text{Spec}(B)$, so that f corresponds to a homomorphism $\rho : A \rightarrow B$. As the standard opens $D(g)$ with $g \in A$ form a basis for S , we only need to prove the assertion for $U = D(g)$. But recall that $f^{-1}(D(g)) = D(\rho(g))$, so our assertion follows. \square

Corollary 19.1.4. *Let S be an affine scheme. Then for an S -scheme X to be affine over S , it is necessary and sufficient that X is an affine scheme.*

Proposition 19.1.5. *Let X be an affine S -scheme and $f : X \rightarrow S$ be the structural morphism. For any quasi-coherent \mathcal{O}_X -module \mathcal{F} , $f_*(\mathcal{F})$ is a quasi-coherent \mathcal{O}_S -module. In particular, the \mathcal{O}_S -algebra $\mathcal{A}(X)$ is quasi-coherent.*

Proof. The morphism f is separated by ?? and quasi-compact by ?? (since any quasi-compact open subset is a finite union of affine opens), so we can apply ??.

Proposition 19.1.6. *Let X be an affine S -scheme. For any S -scheme Y , the map $h \mapsto \mathcal{A}(h)$ from $\text{Hom}_S(Y, X)$ to $\text{Hom}_{\mathcal{O}_S\text{-alg}}(\mathcal{A}(X), \mathcal{A}(Y))$ is bijective.*

Proof. Let $f : X \rightarrow S$, $g : Y \rightarrow S$ be the structural morphisms. Suppose first that $S = \text{Spec}(A)$ and $X = \text{Spec}(B)$ are affine; we must show that for any homomorphism $\omega : f_*(\mathcal{O}_X) \rightarrow g_*(\mathcal{O}_Y)$ of \mathcal{O}_S -algebras, there exists a unique S -morphism $h : Y \rightarrow X$ such that $\mathcal{A}(h) = \omega$. By definition, for any open subset $U \subseteq S$, ω defines a homomorphism $\omega_U : \Gamma(f^{-1}(U), \mathcal{O}_X) \rightarrow \Gamma(g^{-1}(U), \mathcal{O}_Y)$ of $\Gamma(U, \mathcal{O}_S)$ -algebras. In particular, for $U = S$, this gives a homomorphism $\varphi : \Gamma(X, \mathcal{O}_X) \rightarrow \Gamma(Y, \mathcal{O}_Y)$, which by ??, since X is affine, corresponds to a morphism $h : Y \rightarrow X$.

To see that $\mathcal{A}(h) = \omega$, we need to prove that for any open subset $U \subseteq S$, ω_U coincides with the algebra homomorphism φ_U , which corresponds to the S -morphism $h|_{g^{-1}(U)} : g^{-1}(U) \rightarrow f^{-1}(U)$. We may assume that $U = D(\lambda)$ where $\lambda \in A$; then, if $f : X \rightarrow S$ corresponds to the ring homomorphism $\rho : A \rightarrow B$, we have $f^{-1}(U) = D(\mu)$ where $\mu = \rho(\lambda)$, and $\Gamma(f^{-1}(U), \mathcal{O}_X)$ is the fraction ring B_μ . Now the following diagram commutes

$$\begin{array}{ccc}
 B = \Gamma(X, \mathcal{O}_X) & \xrightarrow{\varphi} & \Gamma(Y, \mathcal{O}_Y) \\
 \downarrow & \curvearrowright^{\varphi_U} & \downarrow \\
 B_\mu = \Gamma(f^{-1}(U), \mathcal{O}_X) & \xrightarrow{\omega_U} & \Gamma(g^{-1}(U), \mathcal{O}_Y)
 \end{array}$$

By the universal property of localization, we then conclude that $\varphi_U = \omega_U$, whence the assertion in this case.

In the general case, let (S_α) be a covering of S by affine opens such that $f^{-1}(S_\alpha)$ are affine. Then any homomorphism $\omega : \mathcal{A}(X) \rightarrow \mathcal{A}(Y)$ of \mathcal{O}_S -algebras restricts to a family of homomorphisms

$$\omega_\alpha : \mathcal{A}(f^{-1}(S_\alpha)) \rightarrow \mathcal{A}(g^{-1}(S_\alpha))$$

of \mathcal{O}_{S_α} -algebras, so there is a family of S_α -morphisms $h_\alpha : g^{-1}(S_\alpha) \rightarrow f^{-1}(S_\alpha)$ such that $\mathcal{A}(h_\alpha) = \omega_\alpha$. It all boils down to seeing that for any affine open U of the base $S_\alpha \cap S_\beta$, the restriction of h_α and h_β to $g^{-1}(U)$ coincide, which is immediate since these restrictions both correspond to the restriction homomorphism $\mathcal{A}(X)|_U \rightarrow \mathcal{A}(Y)|_U$ of ω . \square

Corollary 19.1.7. *Let X and Y be affine S -schemes. For an S -morphism $h : Y \rightarrow X$ to be an isomorphism, it is necessary and sufficient that $\mathcal{A}(h) : \mathcal{A}(X) \rightarrow \mathcal{A}(Y)$ is an isomorphism.*

Proof. This follows from ?? and the functoriality of $\mathcal{A}(X)$. \square

19.1.2 Affine S -scheme associated with an \mathcal{O}_S -algebra

Proposition 19.1.8. *Let S be a scheme. For any quasi-coherent \mathcal{O}_S -algebra \mathcal{B} , there exists an affine S -scheme X , defined up to S -isomorphisms, such that $\mathcal{A}(X) = \mathcal{B}$. The affine S -scheme X is said to be associated with the \mathcal{O}_S -algebra \mathcal{B} , and denoted by $\text{Spec}(\mathcal{B})$.*

Proof. The uniqueness follows from ??, so we only need to construct the affine S -scheme X . For any affine open $U \subseteq S$, let X_U be the scheme $\text{Spec}(\Gamma(U, \mathcal{B}))$; as $\Gamma(U, \mathcal{B})$ is an $\Gamma(U, \mathcal{O}_S)$ -algebra, X_U is an S -scheme, and is affine over U since U and X_U are both affine. Moreover, as \mathcal{B} is quasi-coherent, the \mathcal{O}_S -algebra $\mathcal{A}(X_U)$ is canonically identified with $\mathcal{B}|_U$ (?). Let V be another affine open of S , and $X_{U,V}$ be the open subscheme of X_U over $\varphi_U^{-1}(U \cap V)$, where $\varphi_U : X_U \rightarrow S$ is the structural morphism. Then $X_{U,V}$ and $X_{V,U}$ are affine over $U \cap V$ (?), and by definition $\mathcal{A}(X_{U,V})$ and $\mathcal{A}(X_{V,U})$ are canonically identified with $\mathcal{B}|_{U \cap V}$. There then exists (?) a canonical S -isomorphism $\theta_{U,V} : X_{V,U} \rightarrow X_{U,V}$; furthermore, if W is a third affine open of S , and if $\theta'_{U,V}, \theta'_{V,W}, \theta'_{U,W}$ are the restrictions of $\theta_{U,V}, \theta_{V,W}, \theta_{U,W}$ over the inverse images of $U \cap V \cap W$ in X_V, X_W, X_W , then $\theta'_{U,V} \circ \theta'_{V,W} = \theta'_{U,W}$. By glueing the $X_{U,V}$, there then exists a scheme X and an affine open cover (T_U) of X such that for each U there is an isomorphism $\varphi_U : X_U \rightarrow T_U$ such that φ_U maps $\varphi_U^{-1}(U \cap V)$ to $T_U \cap T_V$ and we have $\theta_{U,V} = \varphi_U^{-1} \circ \varphi_V$.

The morphism $g_U = \varphi_U \circ \varphi_U^{-1}$ then makes T_U an S -scheme, and the morphisms g_U and g_V coincide on $T_U \cap T_V$, so X is an S -scheme. It is clear by definition that X is affine over S and $\mathcal{A}(T_U) = \mathcal{B}|_U$, so $\mathcal{A}(X) = \mathcal{B}$. \square

Corollary 19.1.9. *Let S be a scheme. The functor $\mathcal{A}(X)$ defines an equivalence of categories between the category of affine S -schemes and the category of quasi-coherent \mathcal{O}_S -algebras.*

Proof. By ?? we know that $\mathcal{A}(X)$ is fully faithful, and ?? proves that it is essentially surjective, whence the claim. \square

Corollary 19.1.10. *Let S be a scheme. Then for any quasi-coherent \mathcal{O}_S -algebra \mathcal{B} , the contravariant functor*

$$Y \mapsto \text{Hom}_{\mathcal{O}_S\text{-alg}}(\mathcal{B}, \mathcal{A}(Y)) = \text{Hom}_{\mathcal{O}_Y\text{-alg}}(\mathcal{B} \otimes_{\mathcal{O}_S} \mathcal{O}_Y, \mathcal{O}_Y)$$

from the category of S -schemes to the category of sets, is represented by $\text{Spec}(\mathcal{B})$.

Proof. Let $X = \text{Spec}(\mathcal{B})$, then we know that $\mathcal{B} = \mathcal{A}(X)$, so the claim follows from ??.

Corollary 19.1.11. *Let X be an affine S -scheme and $f : X \rightarrow S$ be the structural morphism. For any affine open $U \subseteq S$, the open subscheme $f^{-1}(U)$ of X is an affine scheme with ring $\Gamma(U, \mathcal{A}(X))$.*

Proof. We can suppose that X is associated with the \mathcal{O}_S -algebra $\mathcal{A}(X)$, the corollary then follows from the construction of X in ??.

Example 19.1.12. Let S be the affine plane for a field K with the point 0 is doubled (??). With the notations there, S is the union of two affine opens Y_1, Y_2 . If f is the open immersion $Y_1 \rightarrow S$, then $f^{-1}(Y_2) = Y_1 \cap Y_2$ and we have already seen in ?? that this is not affine. So we obtain an example of an affine scheme not affine over a scheme S .

Remark 19.1.2. Let S be a scheme and $f : X \rightarrow S$ be a quasi-compact and quasi-separated morphism, so that $\mathcal{A}(X) = f_*(\mathcal{O}_X)$ is a quasi-coherent \mathcal{O}_S -algebra (??). The affine S -scheme

$$X^0 = \text{Aff}(X/S) = \text{Spec}(f_*(\mathcal{O}_X)) = \text{Spec}(\mathcal{A}(X))$$

is called the **affine envelope** of the S -scheme X . If $f^0 : X^0 \rightarrow S$ is the structural morphism, by ?? we then have

$$\mathcal{A}(X^0) = f_*^0(\mathcal{O}_{X^0}) = \mathcal{A}(X) = f_*(\mathcal{O}_X);$$

by ??, the identity homomorphism on $\mathcal{A}(X)$ therefore corresponds to a canonical S -morphism $i_X : X \rightarrow X^0$ such that f factors into

$$X \xrightarrow{i_X} X^0 \xrightarrow{f^0} S$$

This factorization for f is called the **Stein factorization** of f . For the morphism i_X to be an isomorphism, it is necessary and sufficient that the morphism f is affine. Moreover, for any S -scheme Y affine over S , the map $u \mapsto u \circ i_X$ is then a bijection

$$\text{Hom}_S(X^0, Y) \xrightarrow{\sim} \text{Hom}_S(X, Y). \quad (19.1.1)$$

which is functorial on Y : this follows from the canonical bijections

$$\mathrm{Hom}_S(X, Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}_S\text{-alg}}(\mathcal{A}(Y), \mathcal{A}(X)) = \mathrm{Hom}_{\mathcal{O}_S\text{-alg}}(\mathcal{A}(Y), \mathcal{A}(X^0)) \xrightarrow{\sim} \mathrm{Hom}_S(X^0, Y).$$

That is, the S -affine scheme X^0 satisfies the universal property that any S -morphism $f : X \rightarrow Y$ such that Y is affine over S must factors through X^0 , or equivalently that X^0 represents the covariant functor $Y \mapsto \mathrm{Hom}_S(X, Y)$ on the category of S -affine schemes. We also deduce that for S fixed, $X \mapsto \mathrm{Aff}(X/S)$ is a covariant functor from the category of S -schemes that are quasi-compact and quasi-separated over S to the category of S -schemes affine over S . Consider a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & X' \\ \iota_X \downarrow & & \downarrow \iota_{X'} \\ X^0 & \xrightarrow{f^0} & X'^0 \end{array}$$

The relation (??) can then be interpreted as the following: the functor $X \mapsto \mathrm{Aff}(X/S)$ is the left adjoint of the forgetful functor from the category of S -schemes affine over S to the category of S -schemes. We then conclude that this functor commutes with inductive limits, hence finite sums.

Corollary 19.1.13. *Let X be an affine S -scheme and Y be an X -scheme. For Y to be affine over S , it is necessary and sufficient that Y is affine over X .*

Proof. We can assume that S is affine, and then X is also affine by ???. Then Y is affine over S if and only if it is affine over X , if and only if it is affine, so our claim follows. \square

Let X be an affine S -scheme. Then by ??, to define of a scheme Y affine over X is equivalent to giving a scheme Y affine over S and an S -morphism $g : Y \rightarrow X$. In view of ?? and ??, this is equivalent to giving a quasi-coherent \mathcal{O}_S -algebra \mathcal{B} and a homomorphism $\mathcal{A}(X) \rightarrow \mathcal{B}$ of \mathcal{O}_S -algebras (which defines over \mathcal{B} an $\mathcal{A}(X)$ -algebra structure). If $f : X \rightarrow S$ is the structural morphism, we then have $\mathcal{B} = f_*(g_*(\mathcal{O}_Y))$.

Corollary 19.1.14. *Let X be an affine S -scheme. For X to be of finite type over S , it is necessary and sufficient that the quasi-coherent \mathcal{O}_X -algebra $\mathcal{A}(X)$ is of finite type.*

Proof. By definition, we can assume that S is affine. Then X is an affine scheme, hence quasi-compact; if $S = \mathrm{Spec}(A)$, $X = \mathrm{Spec}(B)$, $\mathcal{A}(X)$ is the \mathcal{O}_S -algebra \tilde{B} . As $\Gamma(X, \tilde{B}) = B$, the corollary follows from ???. \square

Corollary 19.1.15. *Let X be an affine S -scheme. For X to be reduced, it is necessary and sufficient that the quasi-coherent \mathcal{O}_X -algebra is reduced.*

Proof. The question is local on S so we can assume that S is affine, and the corollary then follows from ???. \square

19.1.3 Quasi-coherent sheaves over affine S -schemes

Proposition 19.1.16. *Let X be an affine S -scheme, Y be an S -scheme, and \mathcal{F} (resp. \mathcal{G}) be a quasi-coherent \mathcal{O}_X -module (resp. \mathcal{O}_Y -module). Then the map $(h, u) \mapsto (\mathcal{A}(h), \mathcal{A}(u))$ from the set of mor-*

phisms $(Y, \mathcal{G}) \rightarrow (X, \mathcal{F})$ to the set of bi-homomorphisms $(\mathcal{A}(X), \mathcal{A}(\mathcal{F})) \rightarrow (\mathcal{A}(Y), \mathcal{A}(\mathcal{G}))$ is bijective.

Proof. The proof is the same as in ??, by using ??.

Corollary 19.1.17. *Under the hypotheses of ??, suppose that Y is affine over S . Then for the couple (h, u) to be an isomorphism, it is necessary and sufficient that $(\mathcal{A}(h), \mathcal{A}(u))$ is a bi-isomorphism.*

Proposition 19.1.18. *For any couple $(\mathcal{B}, \mathcal{M})$ formed by a quasi-coherent \mathcal{O}_S -algebra \mathcal{B} and a quasi-coherent \mathcal{B} -module \mathcal{M} , there exists a couple (X, \mathcal{F}) formed by an affine S -scheme and a quasi-coherent \mathcal{O}_X -module \mathcal{F} such that $\mathcal{A}(X) = \mathcal{B}$ and $\mathcal{A}(\mathcal{F}) = \mathcal{M}$, and this couple is determined up to isomorphisms.*

Proof. The uniqueness part follows from ?. The existence for the scheme X follows from ?. To define \mathcal{M} , we can consider an affine open $U \subseteq S$ and set $\mathcal{F}|_{f^{-1}(U)} = \widetilde{\Gamma(U, \mathcal{M})}$, where $f : X \rightarrow S$ is the structural morphism. We will use $\widetilde{\mathcal{M}}$ to denote the quasi-coherent \mathcal{O}_X -module \mathcal{F} associated with \mathcal{M} . \square

Corollary 19.1.19. *In the category of quasi-coherent \mathcal{B} -modules, $\widetilde{\mathcal{M}}$ is an additive covariant functor which commutes with inductive limits and direct sums.*

Proof. We can in fact assume that S is affine, and the claim then reduces to the functor \widetilde{M} for B -modules, where $B = \Gamma(S, \mathcal{B})$. \square

Corollary 19.1.20. *Under the hypotheses of ??, assume that \mathcal{B} is an \mathcal{O}_X -algebra of finite type. Then for $\widetilde{\mathcal{M}}$ to be an \mathcal{O}_X -module of finite type, it is necessary and sufficient that \mathcal{M} is an \mathcal{B} -module of finite type.*

Proof. We can reduce to the case where $S = \text{Spec}(A)$ is affine. Then $\mathcal{B} = \widetilde{B}$ where B is an A -algebra of finite type, and $\mathcal{M} = \widetilde{M}$ where M is an B -module. Over the scheme X , \mathcal{O}_X is associated with the ring B and $\widetilde{\mathcal{M}}$ is associated with the B -module M . For $\widetilde{\mathcal{M}}$ to be of finite type, it is necessary and sufficient that M is of finite type, whence our claim. \square

Proposition 19.1.21. *Let Y be an affine S -scheme and X, X' be two schemes affine over Y . Let $\mathcal{B} = \mathcal{A}(Y)$, $\mathcal{A} = \mathcal{A}(X)$, and $\mathcal{A}' = \mathcal{A}(X')$. Then $X \times_Y X'$ is affine over Y and $\mathcal{A}(X \times_Y X')$ is identified with $\mathcal{A} \otimes_{\mathcal{B}} \mathcal{A}'$.*

Proof. In fact, $\mathcal{A} \otimes_{\mathcal{B}} \mathcal{A}'$ is a quasi-coherent \mathcal{B} -algebra ??, so is a quasi-coherent \mathcal{O}_S -algebra. Let Z be the spectral of $\mathcal{A} \otimes_{\mathcal{B}} \mathcal{A}'$. The canonical \mathcal{B} -homomorphisms $\mathcal{A} \rightarrow \mathcal{A} \otimes_{\mathcal{B}} \mathcal{A}'$ and $\mathcal{A}' \rightarrow \mathcal{A} \otimes_{\mathcal{B}} \mathcal{A}'$ corresponds to Y -morphisms $p : Z \rightarrow X$ and $p' : Z \rightarrow X'$?. To see that the triple (Z, p, p') is a product $X \times_Y X'$, we can reduce to the case $S = \text{Spec}(C)$ is affine. But then Y, X, X' are all affine schemes with rings B, A, A' , which are C -algebras such that $\mathcal{B} = \widetilde{B}$, $\mathcal{A} = \widetilde{A}$, $\mathcal{A}' = \widetilde{A}'$. We then see that $\mathcal{A} \otimes_{\mathcal{B}} \mathcal{A}'$ is identified with the \mathcal{O}_S -algebra $\widetilde{A} \otimes_B \widetilde{A}'$??, so the ring of Z is identified with $A \otimes_B A'$, and the morphisms p, p' correspond to the canonical homomorphisms $A \rightarrow A \otimes_B A'$ and $A' \rightarrow A \otimes_B A'$. The proposition then follows from ?. \square

Corollary 19.1.22. *Let \mathcal{F} (resp. \mathcal{F}') be a quasi-coherent \mathcal{O}_X -module (resp. $\mathcal{O}_{X'}$ -module). Then $\mathcal{A}(\mathcal{F} \otimes_Y \mathcal{F}')$ is canonically identified with $\mathcal{A}(\mathcal{F}) \otimes_{\mathcal{A}(Y)} \mathcal{A}(\mathcal{F}')$.*

Proof. The sheaf $\mathcal{F} \otimes_Y \mathcal{F}'$ is coherent over $X \times_Y X'$ by ?? Let $g : Y \rightarrow S, f : X \rightarrow Y, f' : X' \rightarrow Y$ be the structural morphisms, so the structural morphism $h : Z \rightarrow S$ is equal to $g \circ f \circ p$ and to $g \circ f' \circ p'$. We define a canonical homomorphism

$$\mathcal{A}(\mathcal{F}) \otimes_{\mathcal{A}(Y)} \mathcal{A}(\mathcal{F}') \rightarrow \mathcal{A}$$

by the following: for any open subset $U \subseteq S$, we have canonical homomorphisms

$$\Gamma(f^{-1}(g^{-1}(U)), \mathcal{F}) \rightarrow \Gamma(h^{-1}(U), p^*(\mathcal{F})), \quad \Gamma(f'^{-1}(g^{-1}(U)), \mathcal{F}') \rightarrow \Gamma(h^{-1}(U), p'^*(\mathcal{F}')),$$

whence a canonical homomorphism

$$\begin{array}{ccc} \Gamma(f^{-1}(g^{-1}(U)), \mathcal{F}) \otimes_{\Gamma(g^{-1}(U), \mathcal{O}_Y)} \Gamma(f'^{-1}(g^{-1}(U)), \mathcal{F}') & & \\ \downarrow & & \\ \Gamma(h^{-1}(U), p^*(\mathcal{F})) \otimes_{\Gamma(h^{-1}(U), \mathcal{O}_Z)} \Gamma(h^{-1}(U), p'^*(\mathcal{F}')) & & \end{array}$$

To see this is an isomorphism of $\mathcal{A}(Z)$ -modules, we can assume that S is affine, and $\mathcal{F} = M$, $\mathcal{F}' = \tilde{M}'$, where M (resp. M') is an A -module (resp. A' -module). Then $\mathcal{F} \otimes_Y \mathcal{F}'$ is identified with the sheaf over $X \times_Y X'$ associated with the $(A \otimes_B A')$ -module $M \otimes_B M'$ and the corollary follows from the canonical identification $\widetilde{M \otimes_B M'}$ with $\tilde{M} \otimes_{\tilde{B}} \tilde{M}'$. \square

Corollary 19.1.23. *Let X and Y be affine S -schemes and $f : Y \rightarrow X$ be an S -morphism. Let \mathcal{F} be a quasi-coherent \mathcal{O}_X -module, then $\mathcal{A}(f^*(\mathcal{F}))$ is identified with $\mathcal{A}(\mathcal{F}) \otimes_{\mathcal{A}(X)} \mathcal{A}(Y)$.*

Proof. This is a special case of ??, by replacing X' with Y and Y with X . \square

In particular, if $X = X' = Y$ (where X is an affine S -scheme), we see that if \mathcal{F}, \mathcal{G} are two quasi-coherent \mathcal{O}_X -modules, then

$$\mathcal{A}(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}) = \mathcal{A}(\mathcal{F}) \otimes_{\mathcal{A}(X)} \mathcal{A}(\mathcal{G})$$

If moreover \mathcal{F} is of finite presentation, then it follows from ?? and ?? that

$$\mathcal{A}(\mathcal{H}\text{om}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})) = \mathcal{H}\text{om}_{\mathcal{A}(X)}(\mathcal{A}(\mathcal{F}), \mathcal{A}(\mathcal{G})).$$

Proposition 19.1.24. *If X and X' are two affine S -schemes with $\mathcal{B} = \mathcal{A}(X)$ and $\mathcal{B}' = \mathcal{A}(X')$. Then the coproduct $X \amalg X'$ is affine over S with $\mathcal{A}(X \amalg X') = \mathcal{B} \times \mathcal{B}'$.*

Proof. The coproduct is affine over S since the product of two affine schemes is affine, and the second assertion also follows from this, and the fact that $\text{Spec}(A) \amalg \text{Spec}(A') = \text{Spec}(A \times A')$ for two rings A, A' . \square

Proposition 19.1.25. *Let S be a scheme, \mathcal{B} be a quasi-coherent \mathcal{O}_S -algebra, and $X = \text{Spec}(\mathcal{B})$. For any quasi-coherent ideal \mathcal{I} of \mathcal{B} , $\tilde{\mathcal{I}}$ is a quasi-coherent ideal of \mathcal{O}_X , and the closed subscheme of X defined by $\tilde{\mathcal{I}}$ is canonically isomorphic to $\text{Spec}(\mathcal{B}/\mathcal{I})$.*

Proof. In fact, it follows from ?? that Y is affine over S , and in view of ??, we can then assume that S is affine, and the proposition follows from the corresponding result in affine schemes. \square

We can also express the result of ?? by saying that if $h : \mathcal{B} \rightarrow \mathcal{B}'$ is a surjective homomorphism of quasi-coherent \mathcal{O}_S -algebras, then $\mathcal{A}(h) : \mathrm{Spec}(\mathcal{B}') \rightarrow \mathrm{Spec}(\mathcal{B})$ is a closed immersion.

Proposition 19.1.26. *Let S be a scheme, \mathcal{B} be a quasi-coherent \mathcal{O}_S -algebra, and $X = \mathrm{Spec}(\mathcal{B})$. For any quasi-coherent ideal \mathcal{K} of \mathcal{O}_S , we have (where $f : X \rightarrow S$ is the structural morphism) $f^*(\mathcal{K})\mathcal{O}_X = \mathcal{K}\mathcal{B}$.*

Proof. The question is local over S , so we can assume that $S = \mathrm{Spec}(A)$ is affine, and the proposition then follows ?.?. \square

19.1.4 Base change of affine S -schemes

Proposition 19.1.27. *Let X be an affine S -scheme. For any extension $g : S' \rightarrow S$ of base scheme, $X' = X_{(S')}$ is affine over S' .*

Proof. If $f' : X' \rightarrow S'$ is the projection, it suffices to prove that $f'^{-1}(U')$ is an affine open for any affine open subset U' of S' such that $g(U')$ is contained in an affine open U of S . We can then assume that S and S' are affine, so X is affine. But then X' is affine, so the claim follows. \square

Corollary 19.1.28. *Let $f : X \rightarrow S$ be the structural morphism, $f' : X' \rightarrow S'$, $g' : X' \rightarrow X$ the projections such that the diagram*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ f' \downarrow & & \downarrow f \\ S' & \xrightarrow{g} & S \end{array}$$

is commutative. For any quasi-coherent \mathcal{O}_X -module \mathcal{F} , there exists a canonical isomorphism $u : g^(f_*(\mathcal{F})) \rightarrow f'_*(g'^*(\mathcal{F}))$ of $\mathcal{O}_{S'}$ -modules. In particular, there exists a canonical isomorphism from $\mathcal{A}(X')$ to $g^*(\mathcal{A}(X))$.*

Proof. To define u , it suffices to define a homomorphism

$$v : f_*(\mathcal{F}) \rightarrow g_*(f'^*(\mathcal{F})) = f_*(g'_*(g'^*(\mathcal{F})))$$

and let u be the homomorphism corresponding to v (via the adjointness). We set $v = f_*(\rho)$, where $\rho : \mathcal{F} \rightarrow g'_*(g'^*(\mathcal{F}))$ is the canonical homomorphism. To prove that u is an isomorphism, we can assume that S and S' , hence X and X' , are affine. Let A, A', B, B' be the ring of X, X', S, S' , then $\mathcal{F} = \tilde{M}$ where M is an B -module. We then see that $g^*(f_*(\mathcal{F}))$ and $f'_*(g'^*(\mathcal{F}))$ are equal to the $\mathcal{O}_{S'}$ -module associated with the A' -module $A' \otimes_A M$, and u is the homomorphism associated with the identity. \square

Corollary 19.1.29. *For any affine S -scheme X and $s \in S$, the fiber X_s is an affine scheme.*

Proof. It suffices to apply ?? on $\mathrm{Spec}(k(s)) \rightarrow S$. \square

Corollary 19.1.30. *Let X be an S -scheme and S' be an affine S -scheme. Then $X' = X_{(S')}$ is affine over X . Moreover, if $f : X \rightarrow S$ is the structural homomorphism, there exists a canonical isomorphism of \mathcal{O}_X -algebras $\mathcal{A}(X') \cong f^*(\mathcal{A}(S'))$, and for any quasi-coherent $\mathcal{A}(S')$ -module \mathcal{M} , a canonical bi-isomorphism $f^*(\mathcal{M}) \cong \mathcal{A}(f'^*(\tilde{\mathcal{M}}))$, where $f' = f_{(S')}$ is the structural morphism $X' \rightarrow S'$.*

Proof. It suffices to apply ?? and ??, with the role of X and S' exchanged. \square

Let S, S' be two schemes, $q : S' \rightarrow S$ be a morphism, \mathcal{B} (resp. \mathcal{B}') be a quasi-coherent \mathcal{O}_S -algebra (resp. $\mathcal{O}_{S'}$ -algebra), and $u : \mathcal{B} \rightarrow \mathcal{B}'$ be a q -morphism (which means a homomorphism $\mathcal{B} \rightarrow q_*(\mathcal{B}')$ of \mathcal{O}_S -algebras). If $X = \text{Spec}(\mathcal{B})$ and $X' = \text{Spec}(\mathcal{B}')$, we deduce a canonical morphism $v = \text{Spec}(u) : X' \rightarrow X$ such that the diagram

$$\begin{array}{ccc} X' & \xrightarrow{v} & X \\ \downarrow & & \downarrow \\ S' & \xrightarrow{q} & S \end{array} \quad (19.1.2)$$

is commutative. In fact, the homomorphism u corresponds to a homomorphism $u^\sharp : q^*(\mathcal{B}) \rightarrow \mathcal{B}'$ by adjointness, and there then exists a canonical S' -morphism

$$w : \text{Spec}(\mathcal{B}') \rightarrow \text{Spec}(q^*(\mathcal{B}))$$

such that $\mathcal{A}(w) = u^\sharp$ (??). On the other hand, it follows from ?? that $\text{Spec}(q^*(\mathcal{B}))$ is canonically identified with $X \times_S S'$; the morphism v is defined to be the composition

$$X' \xrightarrow{w} X \times_S S' \xrightarrow{p_1} X$$

where p_1 is the projection, and the commutativity of (??) is easily verified. Let U (resp. U') be an affine open of S (resp. S') such that $q(U') \subseteq U$, $A = \Gamma(U, \mathcal{O}_S)$, $A' = \Gamma(U', \mathcal{O}_{S'})$, $B = \Gamma(U, \mathcal{B})$, $B' = \Gamma(U', \mathcal{B}')$. The restriction of u is a $(q|_U)$ -morphism $u|_U : \mathcal{B}|_U \rightarrow \mathcal{B}'|_{U'}$ corresponding to a bi-homomorphism $B \rightarrow B'$ of algebras. If V, V' are the inverse images of U, U' in X, X' , respectively, the morphism $V' \rightarrow V$, which is the restriction of v , corresponds to the preceding bi-homomorphism.

Now let \mathcal{M} be a quasi-coherent \mathcal{B} -module. There then exists a canonical isomorphism of $\mathcal{O}_{X'}$ -modules

$$v^*(\tilde{\mathcal{M}}) \cong (q^*(\mathcal{M}) \otimes_{q^*(\mathcal{B})} \mathcal{B}')^\sim. \quad (19.1.3)$$

In fact, the canonical isomorphism of ?? provides a canonical isomorphism of $p_1^*(\tilde{\mathcal{M}})$ with the sheaf over $\text{Spec}(q^*(\mathcal{B}))$ associated with $q^*(\mathcal{B})$ -module $q^*(\mathcal{M})$, and it suffices to apply ??.

Recall that we say a morphism $f : X \rightarrow Y$ is affine if X is an affine scheme over Y . The properties of affine S -schemes then translate into properties of affine morphisms.

Proposition 19.1.31 (Properties of Affine Morphisms).

- (i) A closed immersion is affine.
- (ii) The composition of two affine morphisms is affine.

- (iii) If $f : X \rightarrow Y$ is an affine S -morphism, then $f_{(S')}$ is affine for any base change $S' \rightarrow S$.
- (iv) If $f : X \rightarrow Y$ and $f' : X' \rightarrow Y'$ are two affine S -morphisms, then $f \times_S f'$ is affine.
- (v) If $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are two morphisms such that $g \circ f$ is affine and g is separated, then f is affine.
- (vi) If f is affine, so is f_{red} .

Proof. In view of ??, it suffices to prove (i), (ii), and (iii). Now (i) follows from ??, (ii) follows from ??, and (iii) follows from ??.

Corollary 19.1.32. *If X is an affine scheme and Y is a separated scheme, any morphism $f : X \rightarrow Y$ is affine.*

Proof. This is a direct consequence of ??(v), since the canonical morphism $X \rightarrow \text{Spec}(\mathbb{Z})$ is affine. \square

Proposition 19.1.33. *Let Y be a locally Noetherian scheme and $f : X \rightarrow Y$ be a morphism of finite type. Then for f to be affine, it is necessary and sufficient that f_{red} is affine.*

Proof. It suffices to prove that f is affine if f_{red} is affine. For this, we can assume that Y is affine and Noetherian, and show that X is affine. Now Y_{red} is affine, and X_{red} is therefore affine by hypothesis. Since X is Noetherian, the assertion follows from ??.

19.1.5 Vector bundles

Let A be a ring and E be an A -module. Recall that the symmetric algebra over E is the A -algebra $\mathbf{S}(E)$ (or $\mathbf{S}_A(E)$) which is the quotient of $\mathbf{T}(E)$ by the ideal generated by elements $x \otimes y - y \otimes x$, where x, y belongs to E . The algebra $\mathbf{S}(E)$ is characterized by the universal property that if $\sigma : E \rightarrow \mathbf{S}(E)$ is the canonical map, any A -linear map $E \rightarrow B$, where B is a commutative algebra, factors through $\mathbf{S}(E)$ and gives a homomorphism $\mathbf{S}(E) \rightarrow B$ of A -algebras. We deduce from this property that for two A -modules E, F , we have

$$\mathbf{S}(E \oplus F) = \mathbf{S}(E) \otimes \mathbf{S}(F).$$

Moreover, $\mathbf{S}(E)$ is a covariant functor on E from the category of A -modules to that of commutative A -algebras. Finally, the preceding characterization shows that if $E = \varinjlim E_\lambda$, then $\mathbf{S}(E) = \varinjlim \mathbf{S}(E_\lambda)$. By abuse of language, a product $\sigma(x_1) \cdots \sigma(x_n)$, where $x_i \in E$, is usually written as $x_1 \cdots x_n$. The algebra $\mathbf{S}(E)$ is graded, where $\mathbf{S}_n(E)$ is the set of linear combinations of products of n elements of E . In particular, the algebra $\mathbf{S}(A)$ is canonically isomorphic to the polynomial algebra $A[T]$ over A , and the algebra $\mathbf{S}(A^n)$ is the polynomial algebra $A[T_1, \dots, T_n]$ over A . More particularly, if E is free of rank 1, then $\mathbf{S}_n(E)$ is isomorphic to the tensor algebra $\mathbf{T}(E) = \bigoplus_{n \geq 0} E^{\otimes n}$.

Let $\varphi : A \rightarrow B$ be a ring homomorphism. If F is an B -module, the canonical map $F \rightarrow \mathbf{S}(F)$ then gives a canonical map $F_{(\varphi)} \rightarrow \mathbf{S}(F)_{(\varphi)}$, which factors into $F_{(\varphi)} \rightarrow \mathbf{S}(F_{(\varphi)}) \rightarrow \mathbf{S}(F)_{(\varphi)}$. The canonical homomorphism $\mathbf{S}(F_{(\varphi)}) \rightarrow \mathbf{S}(F)_{(\varphi)}$ is surjective, but not necessarily bijective. If E is an A -module, any bi-homomorphism $E \rightarrow F$ (which is an A -homomorphism $E \rightarrow$

$F_{(\varphi)}$) then gives an A -homomorphism $S(E) \rightarrow S(F_{(\varphi)}) \rightarrow S(F)_{(\varphi)}$ of algebras, which is a bi-homomorphism $S(E) \rightarrow S(F)$. Also, for any A -module E , $S(E \otimes_A B)$ is canonically identified with the algebra $S(E) \otimes_A B$, which follows from the universal property of $S(E)$.

Let R be a multiplicative subset of A . Then apply the previous arguments for the ring $B = R^{-1}A$, and recall that $R^{-1}E = E \otimes_A R^{-1}A$, we see that $S(R^{-1}E) = R^{-1}S(E)$. Moreover, if $R' \supseteq R$ is another multiplicative subset of A , the diagram

$$\begin{array}{ccc} R^{-1}E & \longrightarrow & R'^{-1}E \\ \downarrow & & \downarrow \\ S(R^{-1}E) & \longrightarrow & S(R'^{-1}E) \end{array}$$

is commutative.

Now let (S, \mathcal{A}) be a ringed space and \mathcal{E} be an \mathcal{A} -module over S . If for each open subset $U \subseteq S$, we associate the $\Gamma(U, \mathcal{A})$ -module $S(\Gamma(U, \mathcal{E}))$, we then define a presheaf of algebras. The associated sheaf is called the **symmetric \mathcal{A} -algebra** of the \mathcal{A} -module \mathcal{E} and denoted by $S(\mathcal{E})$. It follows immediately that $S(\mathcal{E})$ satisfies the following universal property: any homomorphism $\mathcal{E} \rightarrow \mathcal{B}$ of \mathcal{A} -modules, where \mathcal{B} is an \mathcal{A} -algebra, factors through $S(\mathcal{E})$ to give a homomorphism $S(\mathcal{E}) \rightarrow \mathcal{B}$ of \mathcal{A} -algebras. In particular, any homomorphism $u : \mathcal{E} \rightarrow \mathcal{F}$ of \mathcal{A} -modules defines a homomorphism $S(u) : S(\mathcal{E}) \rightarrow S(\mathcal{F})$ of \mathcal{A} -algebras and $S(\mathcal{E})$ is a covariant functor \mathcal{E} .

Now since the functor S commutes with inductive limits, we have $S(\mathcal{E})_s = S(\mathcal{E}_s)$ for any point $s \in S$. If \mathcal{E}, \mathcal{F} are two \mathcal{A} -module, $S(\mathcal{E} \oplus \mathcal{F})$ is canonically identified with $S(\mathcal{E}) \otimes_{\mathcal{A}} S(\mathcal{F})$, as we can check this for the corresponding presheaves.

We see that $S(\mathcal{E})$ is a graded \mathcal{A} -algebra, and $S_n(\mathcal{E})$ is the \mathcal{A} -module associated with the presheaf $U \mapsto S_n(\Gamma(U, \mathcal{E}))$. In particular, if $\mathcal{E} = \mathcal{A}$, then $S(\mathcal{A})$ is identified with $\mathcal{A}[T] = \mathcal{A} \otimes_{\mathbb{Z}} \mathbb{Z}[T]$, and if \mathcal{E} is an invertible sheaf, then $S(\mathcal{E})$ is isomorphic to the tensor algebra $T(\mathcal{E}) = \bigoplus_{n \geq 0} \mathcal{E}^{\otimes n}$.

Proposition 19.1.34. *Let $f : (S, \mathcal{A}) \rightarrow (T, \mathcal{B})$ be a morphism of ringed spaces. If \mathcal{F} is an \mathcal{B} -module, then $S(f^*(\mathcal{F}))$ is canonically identified with $f^*(S(\mathcal{F}))$*

Proof. To see this, we may make use the universal property of S . By definition, $S(f^*(\mathcal{F}))$ is defined to be the unique \mathcal{A} -algebra satisfying the following equality

$$\text{Hom}_{\mathcal{A}-\text{alg}}(S(f^*(\mathcal{F})), \mathcal{C}) = \text{Hom}_{\mathcal{A}}(f^*(\mathcal{F}), \mathcal{C})$$

for any \mathcal{A} -algebra \mathcal{C} . On the other hand, by the adjointness property of f_* and f^* , we have

$$\begin{aligned} \text{Hom}_{\mathcal{A}-\text{alg}}(f^*(S(\mathcal{F})), \mathcal{C}) &= \text{Hom}_{\mathcal{B}-\text{alg}}(S(\mathcal{F}), f_*(\mathcal{C})) \\ &= \text{Hom}_{\mathcal{B}}(\mathcal{F}, f_*(\mathcal{C})) = \text{Hom}_{\mathcal{A}}(f^*(\mathcal{F}), \mathcal{C}). \end{aligned}$$

This implies the desired isomorphism. \square

Proposition 19.1.35. *Let A be a ring, $S = \text{Spec}(A)$ be the spectrum, and $\mathcal{E} = \tilde{M}$ be the \mathcal{O}_S -module associated with an A -module M . Then the \mathcal{O}_S -algebra $S(\mathcal{E})$ is associated with the A -algebra $S(M)$.*

Proof. In fact, for any $f \in A$, $S(M_f) = S(M)_f$, so the proposition follows from the definition of $\widetilde{S(M)}$. \square

Corollary 19.1.36. *If S is a scheme and \mathcal{E} is a quasi-coherent \mathcal{O}_S -module. Then the \mathcal{O}_S -algebra $S(\mathcal{E})$ is quasi-coherent. If moreover \mathcal{E} is of finite type (resp. of finite presentation), then each \mathcal{O}_S -module $S_n(\mathcal{E})$ is of finite type (resp. finite presentation) and the \mathcal{O}_S -algebra $S(\mathcal{E})$ is of finite type (resp. of finite presentation).*

Proof. The first assertion is immediate by ?? The second one follows from the fact that, if E is a finitely generated A -module, $S_n(E)$ is also finitely generated. For the last assertion, we are reduced to the case $S = \text{Spec}(A)$ and $\mathcal{E} = \widetilde{E}$ where E is an A -module of finite type (resp. of finite presentation). Now if we have an exact sequence

$$0 \longrightarrow N \longrightarrow A^n \longrightarrow E \longrightarrow 0$$

then we deduce an exact sequence

$$0 \longrightarrow \mathfrak{I} \longrightarrow S(A^n) \longrightarrow S(E) \longrightarrow 0$$

where \mathfrak{I} is the ideal of $S(A^n)$ generated by $N \subseteq S_1(A^n)$, whence our conclusion. \square

Let S be a scheme and \mathcal{E} be a quasi-coherent \mathcal{O}_S -module. For any S -scheme T , with structural morphism $f : T \rightarrow S$, let $\mathcal{E}_{(T)} = f^*(\mathcal{E})$, which is a quasi-coherent \mathcal{O}_T -module. The map

$$T \mapsto F_{\mathcal{E}}(T) = \text{Hom}_{\mathcal{O}_T}(\mathcal{E}_{(T)}, \mathcal{O}_T) = \Gamma(T, \mathcal{E}_{(T)}^*)$$

then defines a contravariant functor from the category of S -schemes to that of sets if for any S -morphism $g : T' \rightarrow T$ we define $F_{\mathcal{E}}(g) : F_{\mathcal{E}}(T) \rightarrow F_{\mathcal{E}}(T')$ to be the map $g^* : u \mapsto g^*(u)$ (note that the structural morphism $T' \rightarrow S$ is $f \circ g$ and we have $\mathcal{E}_{(T')} = g^*(\mathcal{E}_{(T)})$ and $\mathcal{O}_{T'} = g^*(\mathcal{O}_T)$).

Proposition 19.1.37. *For any quasi-coherent \mathcal{O}_S -module \mathcal{E} , the contravariant functor $F_{\mathcal{E}}$ is represented by the couple formed by the affine S -scheme $\mathbb{V}(\mathcal{E}) = \text{Spec}(S(\mathcal{E}))$. The S -scheme $\mathbb{V}(\mathcal{E})$ is called the vector bundle over S defined by \mathcal{E} .*

Proof. This follows from the following canonical isomorphisms for any S -scheme T :

$$\begin{aligned} \text{Hom}_S(T, \mathbb{V}(\mathcal{E})) &= \text{Hom}_{\mathcal{O}_S-\text{alg}}(S(\mathcal{E}), \mathcal{A}(T)) = \text{Hom}_{\mathcal{O}_S}(\mathcal{E}, f_*(\mathcal{O}_T)) \\ &= \text{Hom}_{\mathcal{O}_T}(f^*(\mathcal{E}), \mathcal{O}_T) = \text{Hom}_{\mathcal{O}_T}(\mathcal{E}_{(T)}, \mathcal{O}_T). \end{aligned} \quad \square$$

The canonical $S(\mathcal{E})$ -homomorphism $\mathcal{E} \otimes_{\mathcal{O}_S} S(\mathcal{E}) \rightarrow S(\mathcal{E})$ induced by ?? is an $\mathcal{O}_{\mathbb{V}(\mathcal{E})}$ -homomorphism $\mathcal{E}_{(\mathbb{V}(\mathcal{E}))} \rightarrow \mathcal{O}_{\mathbb{V}(\mathcal{E})}$, which is a section over $\mathbb{V}(\mathcal{E})$ of dual sheaf $\mathcal{E}_{(\mathbb{V}(\mathcal{E}))}^*$ of $\mathcal{E}_{(\mathbb{V}(\mathcal{E}))}$, called the **universal section** of this dual. If $U = \text{Spec}(A)$ is an affine open of S , its inverse image in $\mathbb{V}(\mathcal{E})$ is identified with $\text{Spec}(S(M))$, if $\mathcal{E}|_U = \widetilde{M}$ where M is an A -module. Over the scheme $\text{Spec}(S(M))$, the universal section is identified with the homomorphism $m \otimes p \mapsto mp$ of $M \otimes_A S(M)$ to $S(M)$, where M is identified with the subset $S_1(M)$ of $S(M)$.

Consider in particular an open subset U of S . Then the S -morphisms $U \rightarrow \mathbb{V}(\mathcal{E})$ are the U -sections of the U -scheme induced by $\mathbb{V}(\mathcal{E})$ over $p^{-1}(U)$ (where $p : \mathbb{V}(\mathcal{E}) \rightarrow S$ is the structural morphism). By the definition of $\mathbb{V}(\mathcal{E})$, these U -sections correspond bijectively to sections of

the dual \mathcal{E}^* of \mathcal{E} over U . The functorial of \mathbb{V} shows that this interpretation is compatible with the restriction to an open subset $U' \subseteq U$, so we can say that the dual \mathcal{E}^* of \mathcal{E} is canonically identified with the sheaf of germs of S -sections of $\mathbb{V}(\mathcal{E})$. In particular, if $T = S$, the zero homomorphism $\mathcal{E} \rightarrow \mathcal{O}_S$ corresponds to an S -section of $\mathbb{V}(\mathcal{E})$, called the **zero section**.

If now we choose T to be the spectrum $\{\xi\}$ of a field K , the structural morphism $f : T \rightarrow S$ corresponds to a monomorphism $\kappa(s) \rightarrow K$, where $s = f(\xi)$ (??), and the S -morphisms $\{\xi\} \rightarrow \mathbb{V}(\mathcal{E})$ are none other than points of $\mathbb{V}(\mathcal{E})$ with values in the extension K of $\kappa(s)$, which all locate at some points of $p^{-1}(s)$. The set of these points, which is called the **rational fiber** of $\mathbb{V}(\mathcal{E})$ over K lying over the point s , is then identified (by the definition of $\mathbb{V}(\mathcal{E})$) with the dual of the K -vector space $\mathcal{E}_s \otimes_{\mathcal{O}_{S,s}} K = \mathcal{E}^s \otimes_{\kappa(s)} K$ where we set $\mathcal{E}^s = \mathcal{E}_s / \mathfrak{m}_s \mathcal{E}_s$. If \mathcal{E}^s and K are of finite rank over $\kappa(s)$, this dual is identified with $(\mathcal{E}^s)^* \otimes_{\kappa(s)} K$, where $(\mathcal{E}^s)^*$ is the dual space of the $\kappa(s)$ -vector space \mathcal{E}^s .

These properties justify the terminology of "vector bundle" introduced above, but note that the definition we obtained is dual to the classical definition, since one would expect to obtain the space $\mathcal{E}^s \otimes_{\kappa(s)} K$ for the fiber of $\mathbb{V}(\mathcal{E})$, rather than its dual. This distinction is imposed for the need of defining $\mathbb{V}(\mathcal{E})$ for any quasi-coherent \mathcal{O}_S -module \mathcal{E} , not only for locally free \mathcal{O}_S -modules of finite rank. We can indeed show that the functor $T \mapsto \Gamma(T, \mathcal{E}_{(T)})$ is only representable if \mathcal{E} is locally free of finite rank.

Proposition 19.1.38. *Let S be a scheme.*

- (i) *\mathbb{V} is a contravariant functor on \mathcal{E} from the category of quasi-coherent \mathcal{O}_S -modules to the category of affine S -schemes.*
- (ii) *If \mathcal{E} is of finite type (resp. of finite presentation), $\mathbb{V}(\mathcal{E})$ is of finite type (resp. of finite presentation) over S .*
- (iii) *If \mathcal{E} and \mathcal{F} are two quasi-coherent \mathcal{O}_S -modules, $\mathbb{V}(\mathcal{E} \oplus \mathcal{F})$ is canonically identified with $\mathbb{V}(\mathcal{E}) \times_S \mathbb{V}(\mathcal{F})$.*
- (iv) *Let $g : S' \rightarrow S$ be a morphism. For any quasi-coherent \mathcal{O}_S -module \mathcal{E} , $\mathbb{V}(g^*(\mathcal{E}))$ is canonically identified with $\mathbb{V}(\mathcal{E})_{(S')} = \mathbb{V}(\mathcal{E}) \times_S S'$.*
- (v) *A surjective homomorphism $\mathcal{E} \rightarrow \mathcal{F}$ of quasi-coherent \mathcal{O}_S -modules corresponds to a closed immersion $\mathbb{V}(\mathcal{F}) \rightarrow \mathbb{V}(\mathcal{E})$.*

Proof. Assertion (i) follows from ??, since for any homomorphism $\mathcal{E} \rightarrow \mathcal{F}$ of \mathcal{O}_S -modules we have a homomorphism $S(\mathcal{E}) \rightarrow S(\mathcal{F})$ of \mathcal{O}_S -algebras. Assertion (ii) follows immediately from ?? and ???. To prove (iii), it suffices to recall the canonical isomorphism $S(\mathcal{E} \oplus \mathcal{F}) \cong S(\mathcal{E}) \otimes_{\mathcal{O}_S} S(\mathcal{F})$ and apply ???. Similarly, to prove (iv), it suffices to remark that if the homomorphism $\mathcal{E} \rightarrow \mathcal{F}$ is surjective, so is the corresponding homomorphism $S(\mathcal{E}) \rightarrow S(\mathcal{F})$ of \mathcal{O}_S -algebras, and apply ???. \square

Example 19.1.39. Consider in particular $\mathcal{E} = \mathcal{O}_S$. The scheme $\mathbb{V}(\mathcal{O}_S)$ is the spectrum of the \mathcal{O}_S -algebra $S(\mathcal{O}_S)$, which is identified with $\mathcal{O}_S[T] = \mathcal{O}_S \otimes_{\mathbb{Z}} \mathbb{Z}[T]$. This is evident if $S = \text{Spec}(\mathbb{Z})$, in view of ??, and we pass from this to the general case by considering the structural morphism $S \rightarrow \text{Spec}(\mathbb{Z})$ and using ??(iv). Because of this result, we again set $\mathbb{V}(\mathcal{O}_S) = S[T]$, and we obtain the identification of the sheaf of germs of S -sections of $S[T]$ over \mathcal{O}_S as a particular case.

For any S -scheme X , by the definition of $\mathbb{V}(\mathcal{O}_S)$, the set $\text{Hom}_S(X, S[T])$ is canonically identified with $\text{Hom}_{\mathcal{O}_S}(\mathcal{O}_S, \mathcal{A}(X))$, which is canonically isomorphic to $\Gamma(S, \mathcal{A}(X))$ and therefore has a ring structure. Moreover, any S -morphism $h : X \rightarrow Y$ corresponds to a homomorphism $\Gamma(\mathcal{A}(h)) : \Gamma(S, \mathcal{A}(Y)) \rightarrow \Gamma(S, \mathcal{A}(X))$, so we obtain a contravariant functor $\text{Hom}_S(X, S[T])$ from the category of S -schemes to the category of rings. On the other hand, $\text{Hom}_S(X, \mathbb{V}(\mathcal{E}))$ is identified similarly with $\text{Hom}_{\mathcal{O}_S}(\mathcal{E}, \mathcal{A}(X))$ (where $\mathcal{A}(X)$ is considered as an \mathcal{O}_S -module); we can then endow this set a $\text{Hom}_S(X, S[T])$ -module structure, and the couple

$$(\text{Hom}_S(X, S[T]), \text{Hom}_S(X, \mathbb{V}(\mathcal{E})))$$

is a contravariant functor on X with values in the category of couples (A, M) formed by a ring A and an A -module M , with morphisms being the bi-homomorphisms. In view of this, we say that $S[T]$ is the **S -scheme of ring** and $\mathbb{V}(\mathcal{E})$ is the **S -scheme of module** over the S -scheme of ring $S[T]$.

19.2 Homogeneous specturm of graded algebras

Let S be a graded ring and S_+ be the irrelevant ideal. We say a subset \mathfrak{I} of S_+ is an **ideal of S_+** if it is an ideal of S , and it is called a **graded prime ideal of S_+** if it is the intersection with S_+ of a graded prime ideal of S not containing S_+ (in particular $\mathfrak{I} \neq S_+$, and this graded prime ideal of S is unique by ??). If \mathfrak{I} is an ideal of S_+ , the radical of \mathfrak{I} in S_+ , denoted by $\mathfrak{r}_+(\mathfrak{I})$, is the set of elements of S_+ which have some power contained in \mathfrak{I} , or equivalently, $\mathfrak{r}_+(\mathfrak{I}) = \sqrt{\mathfrak{I}} \cap S_+$. In particular the radical of 0 in S_+ is called the **nilradical of S_+** and denoted by \mathfrak{n}_+ : this is the subset of nilpotent elements of S_+ . If \mathfrak{I} is a graded ideal of S_+ , its radical $\mathfrak{r}_+(\mathfrak{I})$ is also graded: by passing to S/\mathfrak{I} , we may assume that $\mathfrak{I} = 0$, and note that if $x = x_h + x_{h+1} + \dots + x_k$ is nilpotent, then so is each $x_i \in S_i$; we can suppose that $x_k = 0$ and the top degree component of x^n is then x_k^n , so x_k is nilpotent, and we then proceed by induction on k . We say the graded ring S is essentially reduced if $\mathfrak{n}_+ = 0$, which means S_+ contains no nonzero nilpotent elements.

We note that in a graded ring S , if an element x is a zero divisor, so is its homogeneous component of top degree. We then say that the ring S is **essentially integral** if the ring S_+ (with the unit element) does not contain nonzero zero divisors; it suffices for this that a nonzero homogeneous element in S_+ is not divisor of 0 in this ring. It is clear that if \mathfrak{p} is a graded prime ideal of S_+ , S/\mathfrak{p} is essentially integral. Let S be an essentially integral graded ring, and let $x_0 \in S_0$. If there exists a homogeneous element $f \neq 0$ in S_+ such that $x_0f = 0$, we then have $x_0S_+ = 0$, because $(x_0g)f = (x_0f)g = 0$ for any $g \in S_+$, and the hypothesis on S implies that $x_0g = 0$. Therefore, for that S is integral, it is necessary and sufficient that S_0 is integral and the annihilator of S_+ in S_0 reduces to zero.

19.2.1 Localization of graded rings

Let S be a graded ring with positive degrees, f be a homogeneous element of S of degree $d > 0$. Then the fraction ring S_f is graded, where $(S_f)_n$ is the set of elements x/f^k , where $x \in S_{n+kd}$ with $k \geq 0$ (note that n can be an arbitrary integer). We denote by $S_{(f)}$ the subring $(S_f)_0$ of S_f formed by elements of degree 0.

If $f \in S_d$, the monomials $(f/1)^h$ in S_f (where h is an integer) form a linearly independent

system over the ring $S_{(f)}$, and the set of their linear combinations over $S_{(f)}$ is exactly the ring $(S^{(d)})_f$ (recall that $S^{(d)}$ is the direct sum of S_{nd}), and then we get an isomorphism

$$(S^{(d)})_f \cong S_{(f)}[T, T^{-1}] = S_{(f)} \otimes_{\mathbb{Z}} \mathbb{Z}[T, T^{-1}] \quad (19.2.1)$$

(where T is an indeterminate). In fact, if we have a relation

$$\sum_{h=-a}^a z_h (f/1)^h = 0$$

where $z_h = x_h/f^m \in S_{(f)}$, then there exists an integer $k > -a$ such that

$$\sum_{h=-a}^b f^{h+k} x_h = 0,$$

and as the degrees of these terms are distinct, we have $f^{h+k} x_h = 0$ for all h , so $z_h = 0$ for all h . Similarly, if M is a graded S -module, the localization M_f is a graded S_f -module with $(M_f)_n$ being the set of elements z/f^k where $z \in M_{n+kd}$. We denote by $M_{(f)}$ the set of elements of degree 0 in M_f . It is immediate that $M_{(f)}$ is an $S_{(f)}$ -module and we have $(M^{(d)})_f = M_{(f)} \otimes_{S_{(f)}} (S^{(d)})_f$.

Lemma 19.2.1. *Let $f \in S_d$ and $g \in S_e$ be two homogeneous elements of S with positive degrees. Then there exists a canonical isomorphism*

$$S_{(fg)} \cong (S_{(f)})_{g^d/f^e}.$$

If we identify these two rings, then for any S -module M , we have a canonical isomorphism

$$M_{(fg)} \cong (M_{(f)})_{g^d/f^e}.$$

Proof. Note that (fg) divides $f^e g^d$ and $f^e g^d$ divides $(fg)^{de}$, so the rings S_{fg} and $S_{f^e g^d}$ are canonically identified. On the other hand, $S_{f^e g^d}$ is also identified with $(S_{fe})_{g^d/1}$, and as $f^e/1$ is invertible in S_{fe} , $S_{f^e g^d}$ is also identified with $(S_{fe})_{g^d/f^e}$. Now the element g^e/f^e is of degree zero in S_{fe} , so we can conclude that the subring of $(S_{fe})_{g^d/f^e}$ formed by elements of degree zero is $(S_{fe})_{g^d/f^e}$, and as we have $S_{(fe)} = S_{(f)}$, we see the assertion follows. \square

With the hypotheses of ??, it is clear that the canonical homomorphism $S_f \rightarrow S_{fg}$, which maps x/f^k to $xg^k/(fg)^k$, is of degree 0 so restricts to a canonical homomorphism $S_{(f)} \rightarrow S_{(fg)}$, such that the diagram

$$\begin{array}{ccc} & S_{(f)} & \\ & \swarrow & \searrow \\ S_{(fg)} & \xrightarrow{\sim} & (S_{(f)})_{(g^d/f^e)} \end{array}$$

is commutative. We define similarly a canonical homomorphism $M_{(f)} \rightarrow M_{(fg)}$.

Lemma 19.2.2. *If f, g are two homogeneous elements of S_+ , the ring $S_{(fg)}$ is generated by the union of the canonical images of $S_{(f)}$ and $S_{(g)}$.*

Proof. In view of ??, it suffices to show that $1/(g^d/f^e) = f^{d+e}/(fg)^d$ belongs to the canonical image of $S_{(g)}$ in $S_{(fg)}$, which is evident from the definition. \square

Proposition 19.2.3. *Let $f \in S_d$ be a homogeneous element of positive degree. Then there exists a canonical isomorphism $S_{(f)} \cong S^{(d)}/(f-1)S^{(d)}$ of rings. If we identify these two rings, then for any S -module M , there exists a canonical isomorphism of modules $M_{(f)} \cong M^{(d)}/(f-1)M^{(d)}$*

Proof. The first isomorphism is defined by sending the element x/f^n , where $x \in S_{nd}$, to the element \bar{x} , the class of $x \bmod (f-1)S^{(d)}$. This map is well-defined, because we have the congruence $f^h x \equiv x \bmod (f-1)S^{(d)}$ for any $x \in S^{(d)}$, so if $f^h x = 0$ for some $h > 0$ then $\bar{x} = 0$. On the other hand, if $x \in S_{nd}$ is such that $x = (f-1)y$ with $y = y_{hd} + y_{(h+1)d} + \dots + y_{kd}$, where $y_{jd} \in S_{jd}$ and $y_{hd} \neq 0$, we have necessarily $h = n$ and $x = -y_{hd}$, as well as the relations $y_{(j+1)d} = fy_{jd}$ for $h \leq j \leq k-1$ and $y_{kd} = 0$; in particular, this implies $f^{k-n}x = 0$. We therefore have an inverse homomorphism from $S^{(d)}/(f-1)S^{(d)}$ to $S_{(f)}$ by corresponding a class $\bar{x} \bmod (f-1)S^{(d)}$ (where $x \in S_{nd}$) the element x/f^n of $S_{(f)}$, since the preceding remark shows that this map is well-defined. The first assertion is therefore proved, and the second one can be done similarly. \square

Corollary 19.2.4. *If S is Noetherian, so is $S_{(f)}$ for any homogeneous element f of positive degree.*

Proof. This follows from ?? and ??.

Let T be a multiplicative subset of S_+ formed by homogeneous elements; $T_0 = T \cup \{1\}$ is then a multiplicative subset of S . As the elements of T_0 are homogeneous, the ring $T_0^{-1}S$ is graded in a natural way, and we denote by $S_{(T)}$ the subring of $T_0^{-1}S$ formed by elements of degree 0. We know that $T_0^{-1}S$ is identified with the inductive limit of the rings S_f , where $f \in T$ (with the canonical homomorphisms $S_f \rightarrow S_{fg}$). As this identification preserves the degrees, it identifies $S_{(T)}$ as the inductive limit of $S_{(f)}$, where $f \in T$. For any graded S -module M , we define similarly the module $M_{(T)}$ (over the ring $S_{(T)}$) formed by degree zero elements of $T_0^{-1}M$, and we conclude that $M_{(T)}$ is the inductive limit of $M_{(f)}$ for $f \in T$.

If \mathfrak{p} is a graded prime ideal of S_+ , we denote by $S_{(\mathfrak{p})}$ and $M_{(\mathfrak{p})}$ the ring $S_{(T)}$ and the module $M_{(T)}$ respectively, where T is the homogeneous elements of S_+ not contained in \mathfrak{p} .

19.2.2 The homogeneous specturm of a graded ring

Given a graded ring S with positive degrees, we denote by $\text{Proj}(S)$ the **homogeneous specturm** of S , which is the set of graded prime ideals of S_+ , or, equivalently, the set of graded prime ideals of S not containing S_+ . We will define a scheme structure on $\text{Proj}(S)$, just as we have done for $\text{Spec}(A)$ for a ring A .

For a subset E of S , let $V_+(E)$ be the set of graded prime ideals of S containing E and not containing S_+ , which is also the subset $V(E) \cap \text{Proj}(S)$ of $\text{Spec}(S)$. We have immediately the following equalities:

$$\begin{aligned} V_+(0) &= \text{Proj}(S), \quad V_+(S) = V_+(S_+) = \emptyset, \\ V_+\left(\bigcup_\lambda E_\lambda\right) &= \bigcap_\lambda V_+(E_\lambda), \\ V_+(EF) &= V_+(E) \cup V_+(F). \end{aligned}$$

Again, the set $V_+(E)$ remain unchanged if we replace E by the graded ideal it generates; moreover, if \mathfrak{I} is a graded ideal of S , we have

$$V_+(\mathfrak{I}) = V_+\left(\bigcup_{i \geq n} (\mathfrak{I} \cap S_i)\right) \quad (19.2.2)$$

for any $n > 0$: in fact, if $\mathfrak{p} \in \text{Proj}(S)$ contains the homogeneous elements of \mathfrak{I} with degrees $\geq n$, as by hypothesis there exists a homogeneous element $f \in S_d$ not contained in \mathfrak{p} , for any $m \geq 0$ and any $x \in S_m \cap \mathfrak{I}$, we have $f^r x \in \mathfrak{I} \cap S_{m+rd}$ for r sufficiently large, hence $f^r x \in \mathfrak{p} \cap S_{m+rd}$, which implies $x \in \mathfrak{p} \cap S_m$. Finally, for any graded ideal \mathfrak{I} of S , we have

$$V_+(\mathfrak{I}) = V_+(\mathfrak{r}_+(\mathfrak{I}))$$

where $\mathfrak{r}_+(\mathfrak{I})$ is the radical of \mathfrak{I} in S_+ .

By definition, $V_+(E)$ is a closed subset of $X = \text{Proj}(S)$ for the topology induced by $\text{Spec}(S)$. For each element $f \in S$, we set

$$D_+(f) = D(f) \cap \text{Proj}(S) = \text{Proj}(S) \setminus V_+(f).$$

Then for two elements $f, g \in S$, $D_+(fg) = D_+(f) \cap D_+(g)$, and the subsets $D_+(f)$, with $f \in S_+$, form a basis for the topology of $X = \text{Proj}(S)$.

Let f be a homogeneous element of S_+ with degree $d > 0$. For any prime ideal \mathfrak{p} of S not containing f , we see the set of x/f^n , where $x \in \mathfrak{p}$ and $n \geq 0$, is a prime ideal of the fraction ring S_f . Its trace on $S_{(f)}$ is then a prime ideal of this ring, which we denote by $\psi_f(\mathfrak{p})$: this is the set of elements x/f^n , for $n \geq 0$, $x \in \mathfrak{p}_{nd}$. We have therefore defined a map

$$\psi_f : D_+(f) \rightarrow \text{Spec}(S_{(f)});$$

moreover, if g is another homogeneous element of S_+ with degree $e > 0$, we have a commutative diagram

$$\begin{array}{ccc} D_+(fg) & \xrightarrow{\psi_{fg}} & \text{Spec}(S_{(fg)}) \\ \downarrow & & \downarrow \\ D_+(f) & \xrightarrow{\psi_f} & \text{Spec}(S_{(f)}) \end{array} \quad (19.2.3)$$

where the left vertical maps are inclusions, and the right one is the map $\omega_{fg,f}$ induced from the canonical homomorphism $\omega_{fg,f} : S_{(f)} \rightarrow S_{(fg)}$. In fact, if $x/f^n \in \omega^{-1}(\psi_{fg}(\mathfrak{p}))$, where $fg \notin \mathfrak{p}$, we then have $g^n x / (fg)^n \in \psi_{fg}(\mathfrak{p})$, so $g^n x \in \mathfrak{p}$ and therefore $x \in \mathfrak{p}$, and the converse inclusion is evident.

Proposition 19.2.5. *The map ψ_f is a homeomorphism from $D_+(f)$ to $\text{Spec}(S_{(f)})$.*

Proof. For $h \in S_{nd}$ is such that $h/f^n \in \psi_f(\mathfrak{p})$, by definition it is necessary and sufficient that $h \in \mathfrak{p}$, so $\psi^{-1}(D(h/f^n)) = D_+(fh) = D_+(h) \cap D_+(f)$ and the map ψ_f is therefore continuous. Moreover, the sets $D_+(hf)$, where h runs through the set S_{nd} , form a basis of the topology of $D_+(f)$, so the preceding argument proves, in view of the T_0 -axiom for $D_+(f)$ and $\text{Spec}(S_{(f)})$, that ψ_f is injective and the inverse map $\psi_f(D_+(f)) \rightarrow D_+(f)$ is continuous. Finally, to show that ψ_f is surjective, we remark that, if \mathfrak{q}_0 is a prime ideal of $S_{(f)}$ and if, for any $n > 0$, we

denote by \mathfrak{p}_n the set of elements $x \in S_n$ such that $x^d/f^n \in \mathfrak{q}_n$, the \mathfrak{p}_n then verify the conditions ??: if $x \in S_n$, $y \in S_n$ are such that $x^d/f^n \in \mathfrak{q}_0$ and $y^d/f^n \in \mathfrak{q}_0$, we have $(x+y)^{2d}/f^{2n} \in \mathfrak{q}_0$, whence $(x+y)^d/f^n \in \mathfrak{q}_0$ since \mathfrak{q}_0 is prime; this proves that \mathfrak{p}_n is a subgroup of S_n , and the verification of other conditions of ?? is immediate. If \mathfrak{p} is the corresponding graded ideal of S , then $\psi_f(\mathfrak{p}) = \mathfrak{q}_0$, since if $x \in S_{nd}$, the relations $x/f^n \in \mathfrak{q}_0$ and $x^d/f^{nd} \in \mathfrak{q}_0$ are equivalent. \square

Corollary 19.2.6. *For $D_+(f) \neq \emptyset$, it is necessary and sufficient that f is nilpotent.*

Proof. For $\text{Spec}(S_{(f)}) = \emptyset$, it is necessary and sufficient that $S_{(f)} = 0$, which means $1 = 0$ in S_f , and this is equivalent to that f is nilpotent. \square

Corollary 19.2.7. *Let E be a subset of S_+ . The following conditions are equivalent:*

- (i) $V_+(E) = X = \text{Proj}(S)$.
- (ii) Every element of E is nilpotent.
- (iii) The homogeneous components of every element of E are nilpotent.

Proof. It is clear that (iii) \Rightarrow (ii) \Rightarrow (i). If \mathfrak{I} is the graded ideal of S generated by E , conditions (i) is equivalent to that $V_+(\mathfrak{I}) = X$, and a fortiori, (i) implies that any homogeneous element $f \in \mathfrak{I}$ is such that $V_+(f) = X$, so f is nilpotent by ??.

Corollary 19.2.8. *If \mathfrak{I} is a graded ideal of S_+ , $\mathfrak{r}_+(\mathfrak{I})$ is the intersection of graded prime ideals in $V_+(\mathfrak{I})$.*

Proof. By considering the ring S/\mathfrak{I} , we may assume that $\mathfrak{I} = 0$. It then suffices to prove that if $f \in S_+$ is not nilpotent, then there exists a graded prime ideal of S not containing f . Now, since there exists at least homogeneous component of f that is not nilpotent, we may assume that f is homogeneous, the result then follows from ??.

For any subset Y of $X = \text{Proj}(S)$, we denote by $I_+(Y)$ the subset of $f \in S_+$ such that $Y \subseteq V_+(f)$, which is in other words $I(Y) \cap S_+$; the set $I_+(Y)$ is then a radical ideal of S_+ .

Proposition 19.2.9. *Let E be a subset of S and Y be a subset of X .*

- (a) *The ideal $I_+(V_+(E))$ is the radical in S_+ of the graded ideal of S_+ generated by E .*
- (b) *The set $V_+(I_+(Y))$ is the closure of Y in X .*

Proof. If \mathfrak{I} is the graded ideal of S_+ generated by E , we have $V_+(E) = V_+(\mathfrak{I})$ and the first assertion follows from ??-. As for (b), since $V_+(\mathfrak{I}) = \bigcap_{f \in \mathfrak{I}} V_+(f)$, the relation $Y \subseteq V_+(\mathfrak{I})$ implies $Y \subseteq V_+(f)$ for any $f \in \mathfrak{I}$, and therefore $I_+(Y) \supseteq \mathfrak{I}$, so $V_+(I_+(Y)) \subseteq V_+(\mathfrak{I})$, which implies (b) by the definition of closure. \square

Corollary 19.2.10. *The closed subsets Y of $X = \text{Proj}(S)$ and the graded radical ideals of S_+ correspond bijectively via $Y \mapsto I_+(Y)$ and $\mathfrak{I} \mapsto V_+(\mathfrak{I})$. Also, the union $Y_1 \cup Y_2$ of two closed subsets of X corresponds to $I_+(Y_1) \cap I_+(Y_2)$, and the intersection of a family (Y_λ) of closed subsets corresponds to the radical of the sum of $I_+(Y_\lambda)$.*

Corollary 19.2.11. *Let (f_α) be a family of homogeneous elements of S_+ and f be an element of S_+ . The following conditions are equivalent:*

- (i) $D_+(f) \subseteq \bigcup_\alpha D_+(f_\alpha)$;
- (ii) $V_+(f) \supseteq \bigcap_\alpha V_+(f_\alpha)$;
- (iii) a power of f is contained in the ideal generated by the f_α .

In particular, if \mathfrak{I} is a graded ideal of S_+ , then $V_+(\mathfrak{I}) = \emptyset$ if and only if $\mathfrak{r}_+(\mathfrak{I}) = S_+$.

Corollary 19.2.12. For $X = \text{Proj}(S)$ to be empty, it is necessary and sufficient that every element of S_+ is nilpotent.

Corollary 19.2.13. The closed irreducible subset of $X = \text{Proj}(S)$ correspond to graded prime ideals of S_+ .

Proof. In fact, if $Y = Y_1 \cup Y_2$, where Y_1, Y_2 are closed and distinct in Y , then

$$I_+(Y) = I_+(Y_1) \cap I_+(Y_2)$$

the ideals $I_+(Y_1)$ and $I_+(Y_2)$ are distinct from $I_+(Y)$, so $I_+(Y)$ can not be prime. Conversely, if \mathfrak{I} is a graded non-prime ideal of S_+ , there exist elements f, g of S_+ such that $fg \in \mathfrak{I}$ but $f, g \notin \mathfrak{I}$. Then $V_+(f) \not\subseteq V_+(\mathfrak{I})$, $V_+(g) \not\subseteq V_+(\mathfrak{I})$, but $V_+(\mathfrak{I}) \subseteq V_+(f) \cup V_+(g)$. We then conclude that $V_+(\mathfrak{I})$ is the union of the closed subsets $V_+(f) \cap V_+(\mathfrak{I})$ and $V_+(g) \cap V_+(\mathfrak{I})$, both are distinct from $V_+(\mathfrak{I})$. \square

We now define the scheme structure on the homogeneous spectrum $\text{Proj}(S)$. Let f, g be two homogeneous elements of S_+ and consider the affine schemes $Y_f = \text{Spec}(S_{(f)})$, $Y_g = \text{Spec}(S_{(g)})$, and $Y_{fg} = \text{Spec}(S_{(fg)})$. In view of ??, the morphism $w_{fg,f} : Y_{fg} \rightarrow Y_f$ corresponding to the canonical homomorphism $\omega_{fg,f} : S_{(f)} \rightarrow S_{(fg)}$, is an open immersion. By the homeomorphism $\psi_f : D_+(f) \rightarrow Y_f$??, we can transport to $D_+(f)$ the affine scheme structure of Y_f ; in view of the commutative diagram ??, the affine scheme $D_+(fg)$ is identified with the subscheme induced over the open subset $D_+(fg)$ by the affine scheme $D_+(f)$. It is then clear that $X = \text{Proj}(S)$ is endowed with a unique scheme structure such that each $D_+(f)$ is an affine open subscheme of X . When we speak of the homogeneous spectrum $\text{Proj}(S)$ as a scheme, it will always be the structure defined in this way.

Proposition 19.2.14. The scheme $\text{Proj}(S)$ is separated.

Proof. By ??, it suffices to show that for any homogeneous elements f, g of S_+ , $D_+(f) \cap D_+(g) = D_+(fg)$ is affine and the canonical images of the rings of $D_+(f)$ and $D_+(g)$ in $D_+(fg)$ generate the ring of $D_+(fg)$. The first one is clear by definition, and the second one follows from ??, \square

Example 19.2.15. Let $S = K[T_1, T_2]$ where K is a field and T_1, T_2 are indeterminates. Then it follows from ?? that $\text{Proj}(S)$ is the union of $D_+(T_1)$ and $D_+(T_2)$. We see that each of these affine subscheme is isomorphic to $K[T]$, and that $\text{Proj}(S)$ is obtained by glueing these two schemes as described in ??.

Proposition 19.2.16. Let S be a graded ring with positive degrees and $X = \text{Proj}(S)$.

- (i) If \mathfrak{n}_+ is the nilradical of S_+ , the scheme X_{red} is canonically isomorphic to $\text{Proj}(S/\mathfrak{n}_+)$. In particular, if S is essentially reduced, then $\text{Proj}(S)$ is reduced.

(ii) Suppose that S is essentially reduced, then for X to be integral, it is necessary and sufficient that S is essentially integral.

Proof. Let $\bar{S} = S/\mathfrak{n}_+$, and denote by $x \mapsto \bar{x}$ the canonical homomorphism $S \rightarrow \bar{S}$, of degree 0. For any $f \in S_d$ ($d > 0$), the canonical homomorphism $S_f \rightarrow \bar{S}$ is surjective and of degree 0, hence restricts to a surjection $S_{(f)} \rightarrow \bar{S}_{(\bar{f})}$. If we suppose that $f \notin \mathfrak{n}_+$, then $\bar{S}_{(\bar{f})}$ is reduced and the kernel of the preceding homomorphism is the nilradical of $S_{(f)}$, whence $\bar{S}_{(\bar{f})} = (S_{(f)})_{\text{red}}$. This homomorphism then corresponds to a closed immersion $D_+(\bar{f}) \rightarrow D_+(f)$ which identifies $D_+(\bar{f})$ with $(D_+(f))_{\text{red}}$??, and in particular is a homeomorphism of affine scheme. Further, if $g \notin \mathfrak{n}_+$ is another homogeneous element of S_+ , the diagram

$$\begin{array}{ccc} S_{(f)} & \longrightarrow & \bar{S}_{(\bar{f})} \\ \downarrow & & \downarrow \\ S_{(fg)} & \longrightarrow & \bar{S}_{(\bar{f}\bar{g})} \end{array}$$

is commutative. As the sets $D_+(f)$ for f homogeneous in S_+ and $f \notin \mathfrak{n}_+$ form a covering for $X = \text{Proj}(S)$, we conclude that the morphisms $D_+(\bar{f}) \rightarrow D_+(f)$ glue together to a closed immersion $\text{Proj}(\bar{S}) \rightarrow \text{Proj}(S)$ which is a homeomorphism on the underlying spaces, whence the conclusion of (i) by ??.

Suppose now that S is essentially integral, which means (0) is a graded ideal of S_+ distinct from S_+ . Then X is reduced by (a) and irreducible by ?? Conversely, if S is essentially reduced and X is integral, then for any $f \neq 0$ homogeneous in S_+ , we have $D_+(f) \neq \emptyset$ by ??; the hypothesis that X is irreducible implies that $D_+(f) \cap D_+(g) \neq \emptyset$ for any f, g homogeneous and nonzero in S_+ , so in particular $fg \neq 0$, and we then conclude that S_+ is integral. \square

Proposition 19.2.17. Suppose that S is a graded A -algebra where A is a ring. Then over $X = \text{Proj}(S)$ the structural sheaf \mathcal{O}_X is an A -algebra, which means X is a scheme over $\text{Spec}(A)$.

Proof. It suffices to note that for any f homogeneous in S_+ , $S_{(f)}$ is an A -algebra and the homomorphism $S_{(f)} \rightarrow S_{(fg)}$ is an A -algebra homomorphism for any f, g homogeneous in S_+ . \square

Proposition 19.2.18. Let S be a graded ring with positive degrees.

- (a) For any integer $d > 0$, there exists a canonical isomorphism $\text{Proj}(S) \cong \text{Proj}(S^{(d)})$.
- (b) Let S' be the graded ring such that $S'_0 = \mathbb{Z}$ and $S'_n = S_n$ for $n > 0$, then there exists a canonical isomorphism $\text{Proj}(S) \cong \text{Proj}(S')$.

Proof. We have already seen in ?? that the map $\mathfrak{p} \mapsto \mathfrak{p} \cap S^{(d)}$ is a bijection from $\text{Proj}(S)$ to $\text{Proj}(S^{(d)})$. As for any f homogeneous in S_+ , we have $V_+(f) = V_+(f^d)$, this bijection is a homeomorphism of topological spaces. Finally, with the same notations, $S_{(f)}$ and $S_{(f^d)}$ are canonically identified by ??, so $\text{Proj}(S)$ and $\text{Proj}(S^{(d)})$ are canonically identified as schemes.

If to any $\mathfrak{p} \in \text{Proj}(S)$, we correspond the unique prime ideal $\mathfrak{p}' \in \text{Proj}(S')$ such that $\mathfrak{p}' \cap S_n = \mathfrak{p} \cap S_n$ for $n > 0$, then it is clear that this defines a homeomorphism $\text{Proj}(S) \cong \text{Proj}(S')$ of the underlying spaces, since $V_+(f)$ is the same set for S and S' if f is a homogeneous element of S_+ . We also note that $S_{(f)} = S'_{(f)}$: to see this it suffices to note that if $x/1$ is an element of $S_{(f)}$

with $x \in S_0$, then $x/1 = xf/f \in S'_{(f)}$; we then conclude that $\text{Proj}(S)$ and $\text{Proj}(S')$ are identified as schemes. \square

Corollary 19.2.19. *Let S be a graded A -algebra and S_A be the graded A -algebra such that $(S_A)_0 = A$ and $(S_A)_n = S_n$ for $n > 0$, then there exists a canonical isomorphism $\text{Proj}(S) \cong \text{Proj}(S_A)$.*

Proof. In fact, these two schemes are isomorphic to $\text{Proj}(S')$, where $S'_0 = \mathbb{Z}$ and $S'_n = S_n$ for $n > 0$, in view of ??.

19.2.3 Sheaf associated with a graded module

Let M be a graded S -module. For any homogeneous element f of S_+ , $M_{(f)}$ is an $S_{(f)}$ -module, and it therefore corresponds to a quasi-coherent sheaf $\widetilde{M}_{(f)}$ over the affine $\text{Spec}(S_{(f)})$, identified with $D_+(f)$.

Proposition 19.2.20. *There existss a unique quasi-coherent \mathcal{O}_X -module \tilde{M} such that for any homogeneous element $f \in S_+$, we have $\Gamma(D_+(f), \tilde{M}) = M_{(f)}$, and the restriction homomorphism $\Gamma(D_+(f), \tilde{M}) \rightarrow \Gamma(D_+(fg), \tilde{M})$ for f, g homogeneous in S_+ corresponds to the canonical homomorphism $M_{(f)} \rightarrow M_{(fg)}$.*

Proof. Suppose that $f \in S_d, g \in S_e$. As $D_+(fg)$ is identified with the prime specturm $(S_{(f)})_{g^d/f^e}$ by ??, the restriction of $\widetilde{M}_{(f)}$ to $D_+(fg)$ is canonically identified with the sheaf associated with the module $(M_{(f)})_{(g^d/f^e)}$, hence to $\widetilde{M}_{(fg)}$??). We then conclude that there is a canonical isomorphism

$$\theta_{g,f} : \widetilde{M}_{(f)}|_{D_+(fg)} \rightarrow \widetilde{M}_{(g)}|_{D_+(fg)}$$

such that, if g is another homogeneous element of S_+ , we have $\theta_{f,h} = \theta_{f,g} \circ \theta_{g,h}$ over $D_+(fgh)$. By glueing, there then exists a quasi-coherent sheaf \mathcal{F} over X such that for any homogeneous element $f \in S_+$, we have an isomorphism $\eta_f : \mathcal{F}|_{D_+(f)} \cong \widetilde{M}_{(f)}$ and $\theta_{g,f} = \eta_g \circ \eta_f^{-1}$. Since over $D_+(f)$ we have $\Gamma(D_+(f), \tilde{M})$, \mathcal{F} can be identified with the sheaf extended from the presheaf $D_+(f) \mapsto M_{(f)}$ over the basis of standard open sets of X , whence the assertions of the proposition. In particular, we have $\tilde{S} = \mathcal{O}_X$. \square

We say the quasi-coherent \mathcal{O}_X -module \tilde{M} is **associated** with the graded S -module M . Recall that the graded S -modules form a category whose morphisms are graded homomorphisms of degrees. With this convention:

Proposition 19.2.21. *The functor \tilde{M} is a covariant exact functor from the category of graded S -modules to the category of quasi-coherent \mathcal{O}_X -modules, which commutes with inductive limits and direct sums.*

Proof. Since the properties are local, it suffices to verify over the sheaf $\tilde{M}|_{D_+(f)} = \widetilde{M}_{(f)}$. Now the functor M_f on M , the functor N_0 on N , and the functor \tilde{P} on P all satisfy the stated properties, whence the claim. \square

We denote by $\tilde{u} : \tilde{M} \rightarrow \tilde{N}$ the homomorphism corresponding to a graded homomorphism $u : M \rightarrow N$ of degree 0. We also deduce from ?? that the results of ?? and ?? are also true for graded S -modules and homomorphism of degree 0, via the same demonstration.

Proposition 19.2.22. *For any $\mathfrak{p} \in X = \text{Proj}(S)$, we have $\tilde{M}_{\mathfrak{p}} = M_{(\mathfrak{p})}$.*

Proof. By definition we have $\tilde{M}_{\mathfrak{p}} = \varinjlim \Gamma(D_+(f), \tilde{M})$, where f runs through homogeneous elements $f \in S_+$ such that $f \notin \mathfrak{p}$. As $\Gamma(D_+(f), \tilde{M}) = M_{(f)}$, the proposition follows from the definition of $M_{(\mathfrak{p})}$. \square

In particular, the local ring $\mathcal{O}_{X,\mathfrak{p}}$ is just the ring $S_{(\mathfrak{p})}$, the set of elements x/f where f is homogeneous in S_+ and not contained in \mathfrak{p} , and x is homogeneous with the same degree as f . If moreover S is essentially integral, then $\text{Proj}(S) = X$ is integral (??), and if $\xi = (0)$ is the generic point of X , the rational function field $K(X) = \mathcal{O}_{X,\xi}$, is the field formed by f/g where f, g are homogeneous elements of S_+ and $g \neq 0$.

Proposition 19.2.23. *If, for any $z \in M$ and any homogeneous element $f \in S_+$, there exists a power of f annihilating z , then $\tilde{M} = 0$. This condition is also necessary if $S = S_0[S_1]$.*

Proof. The condition $\tilde{M} = 0$ is equivalent to $M_{(f)} = 0$ for any homogeneous element of S_+ . Now if $f \in S_d$, the condition $M_{(f)} = 0$ signifies that for any $z \in M$ homogeneous whose degree is a multiple of d , there exists power f^n such that $f^n z = 0$; this implies the first claim. Conversely, if moreover S is generated by S_1 , then condition then implies that $f^n z = 0$ for any $z \in M$ and any $f \in S_+$, since any element $f \in S_+$ is a finite linear combination of elements of S_1 . \square

Proposition 19.2.24. *Let $f \in S_d$ with $d > 0$. Then for any $n \in \mathbb{Z}$, the $(\mathcal{O}_X|_{D_+(f)})$ -module $\widetilde{S(nd)}|_{D_+(f)}$ is canonically isomorphic to $\mathcal{O}_X|_{D_+(f)}$.*

Proof. The multiplication by the invertible element $(f/1)^n$ of S_f defines a bijection from $S_{(f)} = (S_f)_0$ to the ring

$$(S_f)_{nd} = (S_f(nd))_0 = (S(nd)_f)_0 = S(nd)_{(f)},$$

whence the assertion. \square

Corollary 19.2.25. *Over the open subset $U = \bigcup_{f \in S_d} D_+(f)$, the restriction of the \mathcal{O}_X -module $\widetilde{S(nd)}$ is an invertible $(\mathcal{O}_X|_U)$ -module.*

Corollary 19.2.26. *If the ideal S_+ of S is generated by S_1 , then the \mathcal{O}_X -module $\widetilde{S(n)}$ is invertible for any $n \in \mathbb{Z}$.*

Proof. It suffices to note that under the hypothesis we have $X = \bigcup_{f \in S_1} D_+(f)$ by ??.

The quasi-coherent \mathcal{O}_X -modules $\widetilde{S(n)}$ is of particular interest in the theory of projective schemes, so for each $n \in \mathbb{Z}$, we put $\mathcal{O}_X(n) = \widetilde{S(n)}$ and for an open subset U of X and any $(\mathcal{O}_X|_U)$ -module \mathcal{F} , set

$$\mathcal{F}(n) = \mathcal{F} \otimes_{\mathcal{O}_X|_U} (\mathcal{O}_X(n)|_U).$$

If the ideal S_+ is generated by S_1 , the functor $\mathcal{F}(n)$ is exact on \mathcal{F} for any $n \in \mathbb{Z}$, since $\mathcal{O}_X(n)$ is then an invertible \mathcal{O}_X -module.

Example 19.2.27. Let k be a field and consider the graded algebra $S = k[x_0, \dots, x_n]$; let $X = \text{Proj}(S)$. Let d be an inter and consider the twist sheaf $\mathcal{O}_X(d)$. We compute the global sections for $\mathcal{O}_X(d)$: by definition, for each $x_i \in S$, the section of $\mathcal{O}_X(d)$ over $U_i = D_+(x_i)$ is given by

$$\Gamma(U_i, \mathcal{O}_X(d)) = S(d)_{(x_i)} = (S_{(x_i)})_d = \{f/x_i^n : f \in S_{n+d}\} = \{x_i^d f : f \in S^{(i)}\},$$

where $S^{(i)} = k[x_0/x_i, \dots, x_n/x_i]$. Therefore a section of $\mathcal{O}_X(d)$ is a family of rational polynomials (f_i) with $f_i \in S^{(i)}$ such that $x_i^d f_i = x_j^d f_j$ for each $i \neq j$; let f be this common rational polynomial. Then by construction, we have $f/x_i^d \in S^{(i)}$ for each i , which implies that f is a polynomial in S of degree d when $d \geq 0$. If on the other hand $d < 0$, then f can only have poles at x_i for each i , which is impossible, so there is no global sections for $\mathcal{O}_X(d)$ when $d < 0$.

Let M, N be graded S -modules. For any $f \in S_d$ we define a canonical homomorphism of $S_{(f)}$ -modules

$$\lambda_f : M_{(f)} \otimes_{S_{(f)}} N_{(f)} \rightarrow (M \otimes_S N)_{(f)}$$

by composing the homomorphism $M_{(f)} \otimes_{S_{(f)}} N_{(f)} \rightarrow M_f \otimes_{S_f} N_f$ (induced from the canonical injections) with the canonical isomorphism $M_f \otimes_{S_f} N_f \cong (M \otimes_S N)_{(f)}$, and note that by the definition of the grading of tensor products, these isomorphisms preserve degrees. Unwinding the definitions, for $x \in M_{md}, y \in N_{nd}$, we have

$$\lambda_f((x/f^m) \otimes (y/f^n)) = (x \otimes y)/f^{m+n}.$$

It then follows that, if $g \in S_e$ is another homogeneous element, the diagram

$$\begin{array}{ccc} M_{(f)} \otimes_{S_{(f)}} N_{(f)} & \xrightarrow{\lambda_f} & (M \otimes_S N)_{(f)} \\ \downarrow & & \downarrow \\ M_{(fg)} \otimes_{S_{(fg)}} N_{(fg)} & \xrightarrow{\lambda_{fg}} & (M \otimes_S N)_{(fg)} \end{array}$$

(where the vertical homomorphisms are canonical) is commutative. We then deduce that λ is a canonical homomorphism of \mathcal{O}_X -modules

$$\lambda : \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N} \rightarrow \widetilde{M \otimes_S N}.$$

Consider in particular two graded ideals $\mathfrak{I}, \mathfrak{K}$ of S . As $\widetilde{\mathfrak{I}}$ and $\widetilde{\mathfrak{K}}$ are two quasi-coherent ideals of \mathcal{O}_X , we have a canonical homomorphism $\widetilde{\mathfrak{I}} \otimes_{\mathcal{O}_X} \widetilde{\mathfrak{K}} \rightarrow \mathcal{O}_X$, and the diagram

$$\begin{array}{ccc} \widetilde{\mathfrak{I}} \otimes_{\mathcal{O}_X} \widetilde{\mathfrak{K}} & \xrightarrow{\lambda} & \widetilde{\mathfrak{I} \otimes_S \mathfrak{K}} \\ & \searrow & \swarrow \\ & \mathcal{O}_X & \end{array} \tag{19.2.4}$$

is commutative. Finally, note that if M, N, P are graded S -modules, the diagram

$$\begin{array}{ccc} \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N} \otimes_{\mathcal{O}_X} \widetilde{P} & \xrightarrow{\lambda \otimes 1} & \widetilde{M \otimes_S N} \otimes_{\mathcal{O}_X} \widetilde{P} \\ 1 \otimes \lambda \downarrow & & \downarrow \lambda \\ \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{N \otimes_S P} & \xrightarrow{\lambda} & (M \otimes_S N \otimes_S P)^{\sim} \end{array} \tag{19.2.5}$$

is commutative. Simialrly, we define a canonical homomorphism of $S_{(f)}$ -modules

$$\mu_f : \text{Hom}_S(M, N)_{(f)} \rightarrow \text{Hom}_{S_{(f)}}(M_{(f)}, N_{(f)})$$

which sends an element u/f^n , where u is a homomorphism of degree nd , the homomorphism $M_{(f)} \rightarrow N_{(f)}$ which sends x/f^m ($x \in M_{md}$) to $u(x)/f^{m+n}$. For $g \in S_e$, we have the following commutative diagram

$$\begin{array}{ccc} \text{Hom}_S(M, N)_{(f)} & \xrightarrow{\mu_f} & \text{Hom}_{S_{(f)}}(M_{(f)}, N_{(f)}) \\ \downarrow & & \downarrow \\ \text{Hom}_S(M, N)_{(fg)} & \xrightarrow{\mu_{fg}} & \text{Hom}_{S_{(fg)}}(M_{(fg)}, N_{(fg)}) \end{array} \quad (19.2.6)$$

We then conclude that the μ_f define a canonical homomorphism

$$\mu : (\text{Hom}_S(M, N))^{\sim} \rightarrow \mathcal{H}\text{om}_{\mathcal{O}_X}(\tilde{M}, \tilde{N}).$$

Proposition 19.2.28. *Suppose that the ideal S_+ is generated by S_1 . Then the homomorphism λ is an isomorphism; this holds for μ if the graded S -module M is of finite presentation.*

Proof. As X is the union of $D_+(f)$ for $f \in S_1$, we are reduced to prove tht λ_f and μ_f are isomorphisms for f homogeneous of degree 1. We then define a \mathbb{Z} -linear map $M_n \times N_n \rightarrow M_{(f)} \otimes_{S_{(f)}} N_{(f)}$ that send a pair (x, y) to the element $(x/f^m) \otimes (y/f^n)$. This then defines a \mathbb{Z} -linear map $M \otimes_{\mathbb{Z}} N \rightarrow M_{(f)} \otimes_{S_{(f)}} N_{(f)}$, and if $s \in S_q$, this map send $(sx) \otimes y$ to $(s/f^q)((x/f^m) \otimes (y/f^n))$ (where $x \in M_m$, $y \in N_n$), so we get a bi-homomorphism $\gamma_f : M \otimes_S N \rightarrow M_{(f)} \otimes_{S_{(f)}} N_{(f)}$ relative to the canonical homomorphism $S \rightarrow S_{(f)}$ (sending $s \in S_q$ to s/f^q). Suppose that for an element $\sum_i (x_i \otimes y_i)$ of $M \otimes_S N$ (where x_i, y_i are homogeneous elements of degrees m_i, n_i , respectively) we have $f^r \sum_i (x_i \otimes y_i) = 0$, which means $\sum_i (f^r x_i \otimes y_i) = 0$. Then we deduce from the isomorphism $M_f \otimes_{S_f} N_f \cong (M \otimes_S N)_f$ that $\sum_i (f^r x_i / f^{m_i+r}) \otimes (y_i / f^{n_i}) = 0$, which means $\gamma_i(\sum_i (x_i \otimes y_i)) = 0$. Therefore γ_f factors through $(M \otimes_S N)_f$ and give a homomorphism $\tilde{\gamma}_f : (M \otimes_S N)_f \rightarrow M_{(f)} \otimes_{S_{(f)}} N_{(f)}$; if $\tilde{\lambda}_f$ is the restriction of $\tilde{\gamma}_f$ to $(M \otimes_S N)_{(f)}$, we then verify that λ_f and $\tilde{\lambda}_f$ are inverses of each other, so the first assertion follows.

To demonstrate the second assertion, we now assume that M is of finite presentation, so is the cokernel of a homomorphism $P \rightarrow Q$ of graded S -module, P, Q being direct sums of finitely many modules of the form $S(n)$. By using the left exactness of $\text{Hom}_S(-, N)$ and the exactness of $M_{(f)}$ on M , we are reduced to prove that μ_f is an isomorphism in the case $M = S(n)$. Now for any homogeneous $z \in N$, let u_z be the homomorphism from $S(n)$ to N such that $u_z(1) = z$; we then see that $\eta : z \mapsto u_z$ is an isomorphism of degree 0 from $N(-n)$ to $\text{Hom}_S(S(n), N)$. It thus corresponds to an isomorphism

$$\eta_f : N(-n)_{(f)} \rightarrow \text{Hom}_S(S(n), N)_{(f)}.$$

On the other hand, let $\tilde{\eta}_f$ be the isomorphism $N_{(f)} \rightarrow \text{Hom}_{S_{(f)}}(S(n)_{(f)}, N_{(f)})$ which send $z' \in N_{(f)}$ to the homomorphism $v_{z'}$ such that $v_{z'}(s/f^k) = sz'/f^{n+k}$ (for $s \in S_{n+k} = S(n)_k$). We

consider the composition

$$N(-n)_{(f)} \xrightarrow{\eta_f} \text{Hom}_S(S(n), N)_{(f)} \xrightarrow{\mu_f} \text{Hom}_{S_{(f)}}(S(n)_{(f)}, N_{(f)}) \xrightarrow{\tilde{\eta}_f^{-1}} N_{(f)}$$

is the isomorphism $z/f^h \mapsto z/f^{h-n}$ from $N(-n)_{(f)} \rightarrow N_{(f)}$, so μ_f is an isomorphism. \square

If the ideal S_+ is generated by S_1 , we deduce from ?? that for any graded ideal \mathfrak{I} of S and any graded S -module M , we have $\widetilde{\mathfrak{I}M} = \widetilde{\mathfrak{I}}\widetilde{M}$.

Corollary 19.2.29. *Suppose that the ideal S_+ is generated by S_1 . Then for integers m, n , we have canonical isomorphisms*

$$\mathcal{O}_X(m) \otimes_{\mathcal{O}_X} \mathcal{O}_X(n) = \mathcal{O}_X(m+n), \quad \mathcal{O}_X(n) = (\mathcal{O}_X(1))^{\otimes n}.$$

Proof. The first formula follows from ?? and the existence of the canonical isomorphism $S(m) \otimes_S S(n) \cong S(m+n)$, which sends the element $1 \otimes 1 \in S(m)_{-m} \otimes S(n)_{-n}$ to the element $1 \in S(m+n)_{-(m+n)}$. It then suffices to demonstrate the second formula for $n = -1$, and in view of ??, this follows from the fact that $\text{Hom}_S(S(1), S)$ is canonically isomorphic to $S(-1)$. \square

Corollary 19.2.30. *Suppose that the ideal S_+ is generated by S_1 . For any graded S -module M and $n \in \mathbb{Z}$, we have a canonical isomorphism $\widetilde{M}(n) = \widetilde{M}(n)$.*

Proof. This follows from ?? and the canonical isomorphism $M(n) \cong M \otimes_S S(n)$ which send $z \in M(n)_h = M_{n+h}$ to $z \otimes 1 \in M_{n+h} \otimes S(n)_{-n} \subseteq (M \otimes_S S(n))_h$. \square

Example 19.2.31. Let S' be the graded ring such that $S'_0 = \mathbb{Z}$ and $S'_n = S_n$ for $n > 0$. Then if $f \in S_d$ ($d > 0$), we have $S(n)_{(f)} = S'(n)_{(f)}$ for any $n \in \mathbb{Z}$, because an element of $S'(n)_{(f)}$ is of the form x/f^k where $x \in S'_{n+kd}$ ($k > 0$), and we can always choose k such that $n + kd \neq 0$. As $X = \text{Proj}(S)$ and $X' = \text{Proj}(S')$ are canonically identified, we see that for any $n \in \mathbb{Z}$, $\mathcal{O}_X(n)$ and $\mathcal{O}_{X'}(n)$ are canonically isomorphic under this identification.

On the other hand, for any $d > 0$ and $n \in \mathbb{Z}$, we have

$$S^{(d)}(n)_h = S_{(n+h)d} = S(nd)_{hd};$$

so for any $f \in S_d$ we have $S^{(d)}(n)_{(f)} = S(nd)_{(f)}$. We have seen that the schemes $X = \text{Proj}(S)$ and $X^{(d)} = \text{Proj}(S^{(d)})$ are canonically identified, so under this identification, $\mathcal{O}_X(nd)$ and $\mathcal{O}_{X^{(d)}}(n)$ are canonically isomorphic, for any $n \in \mathbb{Z}$.

Proposition 19.2.32. *Let $d > 0$ be an integer and $U = \bigcup_{f \in S_d} D_+(f)$. Then the restriction to U of the canonical homomorphism $\mathcal{O}_X(nd) \otimes_{\mathcal{O}_X} \mathcal{O}_X(-nd) \rightarrow \mathcal{O}_X$ is an isomorphism for each $n \in \mathbb{Z}$.*

Proof. In view of ??, we can assume that $d = 1$, and the conclusion then follows the proof of ??.

19.2.4 Graded S -module associated with a sheaf

In this paragraph, for simplicity, we always assume that the ideal S_+ is generated by S_1 , which also means that $S = S_0[S_1]$ by ???. The \mathcal{O}_X -module $\mathcal{O}_X(1)$ is then invertible by ??; we then

put, for any \mathcal{O}_X -module \mathcal{F} , that

$$\Gamma_*(\mathcal{F}) = \Gamma_*(\mathcal{O}_X(1), \mathcal{F}) = \bigoplus_{n \in \mathbb{Z}} \Gamma(X, \mathcal{F}(n))$$

Recall that $\Gamma_*(\mathcal{O}_X)$ is endowed with a graded ring structure and $\Gamma_*(\mathcal{F})$ is a graded module over $\Gamma_*(\mathcal{O}_X)$. Since $\mathcal{O}_X(n)$ is locally free, $\Gamma_*(\mathcal{F})$ is a covariant left-exact functor on \mathcal{F} ; in particular, if \mathcal{I} is a sheaf of ideals of \mathcal{O}_X , $\Gamma_*(\mathcal{F})$ is canonically a graded ideal of $\Gamma_*(\mathcal{O}_X)$.

Suppose that M is a graded S -module. For any $f \in S_d$ with $d > 0$, $x \mapsto x/1$ is a homomorphism of abelian groups $M_0 \mapsto M_{(f)}$, and as $M_{(f)}$ is canonically identified with $\Gamma(D_+(f), \tilde{M})$, we obtain a homomorphism $\alpha_0^f : M_0 \rightarrow \Gamma(D_+(f), \tilde{M})$ of abelian groups. It is clear that, for any $g \in S_e$ with $e > 0$, the diagram

$$\begin{array}{ccc} & \Gamma(D_+(f), \tilde{M}) & \\ \alpha_0^f \nearrow & & \downarrow \\ M_0 & & \\ \alpha_0^{fg} \searrow & & \Gamma(D_+(fg), \tilde{M}) \end{array}$$

is commutative, and this signifies that for any $x \in M_0$, the sections $\alpha_0^f(x)$ and $\alpha_0^{fg}(x)$ of M coincide over $D_+(fg)$, and therefore there exists a unique section $\alpha_0(x) \in \Gamma(X, \tilde{M})$ whose restriction on $D_+(f)$ is $\alpha_0^f(x)$. We then define (under the hypothesis that S_+ is generated by S_1) a homomorphism

$$\alpha_0 : M_0 \rightarrow \Gamma(X, \tilde{M}).$$

By applying this result on each graded S -module $M(n)$ (where $n \in \mathbb{Z}$), we then obtain homomorphisms of abelian groups

$$\alpha_n : M_n = M(n)_0 \rightarrow \Gamma(X, \tilde{M}(n))$$

and therefore a homomorphism of graded abelian groups

$$\alpha : M \rightarrow \Gamma_*(\tilde{M})$$

(also denoted by α_M) such that α_M coincides with α_n on each M_n .

If we consider in particular $M = S$, then it is easy to see that (by the definition of the multiplication of $\Gamma_*(\mathcal{O}_X)$) $\alpha : S \rightarrow \Gamma_*(\mathcal{O}_X)$ is a homomorphism of graded rings, therefore, for any S -module M , α is a bi-homomorphism of graded modules.

Proposition 19.2.33. *For any $f \in S_d$ with $d > 0$, $D_+(f)$ is identified with the subset of $\mathfrak{p} \in X$ such that the section $\alpha_d(f)$ of $\mathcal{O}_X(d)$ is nonzero at \mathfrak{p} .*

Proof. As $X = \bigcup_{g \in S_1} D_+(g)$, it suffices to prove that for any $g \in S_d$, the set $\mathfrak{p} \in D_+(g)$ where $\alpha_d(f)$ is nonzero is identified with $D_+(fg)$. Now the restriction of $\alpha_d(f)$ to $D_+(g)$ is by definition the section corresponding to the element $f/1$ of $S(d)_{(g)}$; by the canonical isomorphism $S(d)_{(g)} \cong S_{(g)}$, this section of $\mathcal{O}_X(d)$ over $D_+(g)$ is identified with the section of \mathcal{O}_X over $D_+(g)$ corresponding to the element f/g^d of $S_{(g)}$. To see that this section is zero on $\mathfrak{p} \in D_+(g)$ then signifies that $f/g^d \in \mathfrak{q}$, where \mathfrak{q} is the prime ideal of $S_{(g)}$ corresponding to \mathfrak{p} ; by definition this means $f \in \mathfrak{p}$, whence the proposition. \square

Now let \mathcal{F} be an \mathcal{O}_X -module and put $M = \Gamma_*(\mathcal{F})$. In view of the homomorphism $\alpha : S \rightarrow \Gamma_*(\mathcal{O}_X)$ of graded rings, M can also be considered as a graded S -module. For any $f \in S_d$ ($d > 0$), it follows from ?? that the restriction of the section $\alpha_d(f)$ of $\mathcal{O}_X(d)$ to $D_+(f)$ is invertible, and so is the restriction to $D_+(f)$ of $\alpha_d(f^n)$ for any $n > 0$. Let $z \in M_{nd} = \Gamma(X, \mathcal{F}(nd))$, if there exists an integer $k \geq 0$ such that the restriction to $D_+(f)$ of $f^k z$, which is the section $(\alpha_d(f^k)z)|_{D_+(f)}$ of $\mathcal{F}((n+k)d)$, is zero, then we conclude that $z|_{D_+(f)} = 0$. This shows that we can define an $S_{(f)}$ -homomorphism $\beta_f : M_{(f)} \rightarrow \Gamma(D_+(f), \mathcal{F})$ which corresponds the element $z/f^n \in M_{(f)}$ the section $(z|_{D_+(f)})(\alpha_d(f^n)|_{D_+(f)})^{-1}$ of \mathcal{F} over $D_+(f)$. We also verify that for $g \in S_e$ ($e > 0$), the diagram

$$\begin{array}{ccc} M_{(f)} & \xrightarrow{\beta_f} & \Gamma(D_+(f), \mathcal{F}) \\ \downarrow & & \downarrow \\ M_{(fg)} & \xrightarrow{\beta_{fg}} & \Gamma(D_+(fg), \mathcal{F}) \end{array}$$

is commutative. Since $M_{(f)}$ is canonically identified with $\Gamma(D_+(f), \tilde{M})$ and the $D_+(f)$ form a basis for the topological space X , the homomorphisms β_f glue together to a unique canonical homomorphism of \mathcal{O}_X -modules

$$\beta : \widetilde{\Gamma_*(\mathcal{F})} \rightarrow \mathcal{F}$$

(also denoted by $\beta_{\mathcal{F}}$) which is evidently functorial.

Proposition 19.2.34. *Let M be a graded S -module and \mathcal{F} be an \mathcal{O}_X -module. Then the composition homomorphisms*

$$\tilde{M} \xrightarrow{\tilde{\alpha}} (\Gamma_*(\tilde{M}))^\sim \xrightarrow{\beta} \tilde{M} \tag{19.2.7}$$

$$\Gamma_*(\mathcal{F}) \xrightarrow{\alpha} \Gamma_*(\widetilde{\Gamma_*(\mathcal{F})}) \xrightarrow{\Gamma_*(\beta)} \Gamma_*(\mathcal{F}) \tag{19.2.8}$$

are isomorphisms.

Proof. The verification of ?? is local: on an open subset $D_+(f)$, this result follows from the definition and the fact that the action of β is determined by its action of sections over $D_+(f)$???. The verification of ?? can be done at each degree: if we put $M = \Gamma_*(\mathcal{F})$, we have $M_n = \Gamma(X, \mathcal{F}(n))$ and $\Gamma_*(\tilde{M})_n = \Gamma(X, \tilde{M}(n)) = \Gamma(X, \widetilde{M(n)})$. If $f \in S_1$ and $z \in M_n$, $\alpha_n^f(z)$ is the element $z/1$ of $M(n)_{(f)}$, and equals to $(f/1)^n(z/f^n)$; it then corresponds by β_f to the section

$$(\alpha_1(f)^n|_{D_+(f)})(z|_{D_+(f)}) (\alpha_1(f)^n|_{D_+(f)})^{-1}$$

over $D_+(f)$, which is the restriction of z to $D_+(f)$. □

In general, the homomorphisms α and β are not isomorphisms (for example, a graded S -module M can be nonzero with \tilde{M} being zero). To obtain some nice results about these two homomorphisms, we need to impose further conditions on the graded ring S and the graded S -module M .

Proposition 19.2.35. *Let S be a graded ring and A be a ring.*

(a) *If S is Noetherian, then $X = \text{Proj}(S)$ is a Noetherian scheme.*

(b) If S is a graded A -algebra of finite type, then $X = \text{Proj}(S)$ is a scheme of finite type over $Y = \text{Spec}(A)$.

Proof. If S is Noetherian, the ideal S_+ is generated by finitely many homogeneous elements $(f_i)_{1 \leq i \leq p}$, so the space X is the union of $D_+(f_i) = \text{Spec}(S_{(f_i)})$, and since each $S_{(f_i)}$ is Noetherian by ??, we see X is Noetherian.

Now assume that S is an A -algebra of finite type, then S_0 is an A -algebra of finite type and S is an S_0 -algebra of finite type, so S_+ is a finitely generated ideal by ???. We are then reduced to prove as in (a) that for any $f \in S_d$, $S_{(f)}$ is an A -algebra of finite type. In view of ??, it suffices to show that $S^{(d)}$ is an A -algebra of finite type, which follows from ???. \square

Let M be a graded S -module. We say M is **eventually null** if there exists an integer n such that $M_i = 0$ for $i \neq n$, and is **eventually finite** if there exists an integer n such that $\bigoplus_{i \geq n} M_i$ is a finitely generated S -module, or equivalently, that there exists a finitely generated graded sub- S -module M' of M such that M/M' is eventually null. We also note that if M is eventually null, then $M_{(f)} = 0$ for any homogeneous element f in S_+ , so $\tilde{M} = 0$.

Let M, N be two graded S -modules. We say a homomorphism $u : M \rightarrow N$ of degree 0 is **eventually injective** (resp. **eventually surjective**, **eventually bijective**) if there exists an integer n such that $u_i : M_i \rightarrow N_i$ is injective (resp. surjective, bijective) for $i \geq n$. Equivalently, the homomorphism u is eventually injective (resp. eventually surjective) if and only if $\ker u$ (resp. $\text{coker } u$) is eventually null. If u is eventually bijective, we say it is an **eventual isomorphism**.

Proposition 19.2.36. *Let S be a graded ring such that S_+ is finitely generated and M be a graded S -module.*

(a) *If M is eventually finite, the \mathcal{O}_X -module \tilde{M} is of finite type.*

(b) *Suppose that M is eventually finite, then for $\tilde{M} = 0$, it is necessary and sufficient that M is eventually null.*

Proof. If $M_n = 0$ for $n \geq n_0$ then $M_{(f)} = 0$ for any homogeneous element $f \in S_+$, so if M is eventually null, then $\tilde{M} = 0$. On the other hand, if M is eventually finite, then for $n \gg 0$ the graded submodule $M' = \bigoplus_{k \geq n} M_k$ is finitely generated by hypothesis, and M/M' is eventually null, so $(\widetilde{M/M'}) = 0$ and therefore $\tilde{M} = \widetilde{M'}$ by the exactness of the functor \widetilde{M} (??). Therefore, to prove that \tilde{M} is of finite type, we may assume that M is finitely generated. Now since this question is local, we only need to show that $M_{(f)}$ is finitely generated over $S_{(f)}$ for any homogeneous $f \in S_d$ with $d > 0$. But $M^{(d)}$ is a finitely generated $S^{(d)}$ module by ??, and the assertion follows from ??.

Suppose now that M is eventually finite and $\tilde{M} = 0$; then we have $\widetilde{M'} = 0$, so the condition that M' is eventually null is equivalent to that of M . We may therefore assume that M is finitely generated over S by homogeneous elements x_i ($1 \leq i \leq p$); let $(f_j)_{1 \leq j \leq q}$ be a system of generators of the ideal S_+ . We have by hypothesis $M_{(f_j)} = 0$ for any j , so there exists an integer n such that $f_j^n x_i = 0$ for any i, j . Let $n_j = \deg(f_j)$ and m be the supremum of $\sum_j r_j n_j$ for any finite system of integers (r_j) such that $\sum_j r_j \leq nq$. It is then clear that if $k > m$, we have $S_k x_i = 0$ for any i ; if d is the supremum of the degrees of x_i , we then conclude that $M_k = 0$ for $k > d + m$, which proves our assertion. \square

Corollary 19.2.37. *Let S be a graded ring such that S_+ is finitely generated. For $X = \text{Proj}(S) = \emptyset$, it is necessary and sufficient that S is eventually null.*

Proof. The condition $X = \emptyset$ is in fact equivalent to $\mathcal{O}_X = \tilde{S} = 0$, and S is clearly a finite generated S -module. \square

Example 19.2.38. To give a counterexample of ??, let k be a field and $S = k[x_1, \dots, x_n]$ be the polynomial ring with n variables. Consider the graded S -module M given by

$$M = k[x_1, x_2, \dots, x_n, \dots] / (x_1, x_2^2, \dots, x_n^n, \dots)$$

with the usual multiplication of polynomials. Then M is not finitely generated over S , and for every homogeneous polynomial f of degree 1, we see $f^N M = 0$ for sufficiently large N . Therefore $\tilde{M} = 0$. However, note that $M_n \neq 0$ for every n .

Theorem 19.2.39. Suppose that S is a graded ring such that the ideal S_+ is finitely generated by S_1 , and let $X = \text{Proj}(S)$. Then, for any quasi-coherent \mathcal{O}_X -module \mathcal{F} , the canonical homomorphisms $\beta : \widetilde{\Gamma_*}(\mathcal{F}) \rightarrow \mathcal{F}$ is an isomorphism.

Proof. If S_+ is generated by finitely many $f_i \in S_1$, then X is the union of $\text{Spec}(S_{(f_i)})$ which are quasi-compact, so X is quasi-compact. Also, $\mathcal{O}_X(n)$ is invertible for any $n \in \mathbb{Z}$ by ??, and since X is separated, by ?? and ??, we have for any $f \in S_d$ a canonical isomorphism $\Gamma_*(\mathcal{F})_{(\alpha_d(f))} \cong \Gamma(D_+(f), \mathcal{F})$ (the first module (considered as a $\Gamma_*(\mathcal{O}_X)$ -module) is none other than $\Gamma_*(\mathcal{F})_{(f)}$ (considered as an S -module)). If we trace the definition of this isomorphism, we see that it coincides with β_f , whence our assertion. \square

Corollary 19.2.40. Under the hypotheses of ??, any quasi-coherent \mathcal{O}_X -module (of finite type) is isomorphic to an \mathcal{O}_X -module of the form \tilde{M} , where M is a (finitely generated) graded S -module.

Proof. Let \mathcal{F} be a quasi-coherent \mathcal{O}_X -module, then $\mathcal{F} = \tilde{M}$ for a graded S -module M by ???. Let $(f_\lambda)_{\lambda \in I}$ be a system of homogeneous generators of M ; for each finite subset H of I , let M_H be the graded submodule of M generated by f_λ for $\lambda \in H$. It is clear that M is the inductive limit of the submodules M_H , so \mathcal{F} is the inductive limit of the sub- \mathcal{O}_X -modules \tilde{M}_H (??). If \mathcal{F} is of finite type, we conclude from ???. \square

Corollary 19.2.41. Under the hypotheses of ??, for any quasi-coherent \mathcal{O}_X -module \mathcal{F} of finite type, there exists an integer n_0 such that for any $n \geq n_0$, $\mathcal{F}(n)$ is isomorphic to a quotient of \mathcal{O}_X^r (where $r > 0$ depends on n), and therefore is generated by finitely many global sections.

Proof. By ??, we can assume that $\mathcal{F} = \tilde{M}$ where M is a quotient of a finite direct sum of $S(m_i)$. By ??, we are therefore reduced to the case where $M = S(m)$, so $\mathcal{F}(n) = (S(m+n))^\sim = \mathcal{O}_X(m+n)$. It then suffices to prove that for each $n \geq 0$ there exists r and a surjective homomorphism $\mathcal{O}_X^r \rightarrow \mathcal{O}_X(n)$. For this, it suffices to prove that, for a suitable r , there exists an eventually surjective homomorphism $u : S^r \rightarrow S(n)$ of degree zero. Now we have $S(n)_0 = S_n$, and by hypothesis $S_h = S_1^h$ for any $h > 0$, so $SS_n = \bigoplus_{h \geq n} S_h$. As S_n is a finitely generated S_0 -module (??), consider a system $(a_i)_{1 \leq i \leq r}$ of generators of this module, and let $u : S^r \rightarrow S(n)$ be the homomorphism that sends the i -th basis e_i of S^r to a_i . Then the image of u contains $\bigoplus_{h \geq 0} S(n)_h$, so u satisfies the requirement and the proof is complete. \square

Corollary 19.2.42. Under the hypotheses of ??, for any quasi-coherent \mathcal{O}_X -module \mathcal{F} of finite type, there exists an integer n_0 such that for any $n \geq n_0$, \mathcal{F} is isomorphic to a quotient of an \mathcal{O}_X -module of the form $\mathcal{O}_X(-n)^r$ (where $r > 0$ depends on n).

Proof. This follows from ?? by tensoring with the invertible sheaf $\mathcal{O}_X(-n)$, which preserves the exactness. \square

Proposition 19.2.43. *Assume the hypotheses of ?? and let M be a graded S -module.*

- (a) *The canonical homomorphism $\tilde{\alpha} : \tilde{M} \rightarrow (\Gamma_*(\tilde{M}))^\sim$ is an isomorphism.*
- (b) *Let \mathcal{G} be a quasi-coherent sub- \mathcal{O}_X -module of \tilde{M} and let N be the graded sub- S -module of M which is the inverse image of $\Gamma_*(\mathcal{G})$ under α . Then we have $\tilde{N} = \mathcal{G}$.*

Proof. As $\beta : (\Gamma_*(\tilde{M}))^\sim \rightarrow \tilde{M}$ is an isomorphism, $\tilde{\alpha}$ is its inverse isomorphism in view of ??, whence (a). Let P be the graded submodule $\alpha(M)$ of $\Gamma_*(\tilde{M})$; as \tilde{M} is an exact functor, the image of \tilde{M} under $\tilde{\alpha}$ is equal to \tilde{P} , so in view of (a), $\tilde{P} = (\Gamma_*(\tilde{M}))^\sim$. Put $Q = \Gamma_*(\mathcal{G}) \cap P$, so that $N = \alpha^{-1}(Q)$. Then by the preceding argument and ??, the image of \tilde{N} under $\tilde{\alpha}$ is \tilde{Q} , and we have $\tilde{Q} = \widetilde{\Gamma_*(\mathcal{G})}$. Since the image of $\widetilde{\Gamma_*(\mathcal{G})}$ under β is \mathcal{G} and $\tilde{\alpha}$ is the inverse of β , we conclude that $\tilde{N} = \mathcal{G}$. \square

19.2.5 Functorial properties of $\text{Proj}(S)$

Let S, S' be two graded rings with positive degree and $\varphi : S' \rightarrow S$ be a homomorphisms of graded rings. We denote by $G(\varphi)$ the open subset of $X = \text{Proj}(S)$ which is the complement of $V_+(\varphi(S'_+))$, or, the union of $D_+(\varphi(f'))$ where f' runs through homogeneous elements of S'_+ . The restriction to $G(\varphi)$ of the continuous map ${}^a\varphi : \text{Spec}(S') \rightarrow \text{Spec}(S)$ is then a continuous map from $G(\varphi)$ to $\text{Proj}(S')$, which is still denoted by ${}^a\varphi$. If $f' \in S'_+$ is homogeneous, we have

$${}^a\varphi^{-1}(D_+(f')) = D_+(\varphi(f')) \quad (19.2.9)$$

since ${}^a\varphi$ maps $G(\varphi)$ into $\text{Proj}(S')$. On the other hand, the homomorphism φ defines canonically a homomorphism of graded rings $S'_{f'} \rightarrow S_f$ of degree 0 (where $f = \varphi(f')$), whence a homomorphism $S'_{(f')} \rightarrow S_{(f)}$, which we denote by $\varphi_{(f)}$. It then corresponds to a morphism $({}^a\varphi_{(f)}, \tilde{\varphi}_{(f)}) : \text{Spec}(S_{(f)}) \rightarrow \text{Spec}(S'_{(f')})$ of affine schemes. If we identify $\text{Spec}(S_{(f)})$ with the open subscheme $D_+(f)$ of $\text{Proj}(S)$, we then obtain a morphism $\Phi_f : D_+(f) \rightarrow D_+(f')$ and ${}^a\varphi_{(f)}$ is identified with the restriction of ${}^a\varphi$ to $D_+(f)$. If g' is another homogeneous element of S'_+ and $g = \varphi(g')$, it is immediate that the diagram

$$\begin{array}{ccc} D_+(fg) & \xrightarrow{\Phi_{fg}} & D_+(f'g') \\ \downarrow & & \downarrow \\ D_+(f) & \xrightarrow{\Phi_f} & D_+(f') \end{array}$$

is commutative.

Proposition 19.2.44. *Let $\varphi : S' \rightarrow S$ be a homomorphism of graded rings. There exists a unique morphism $({}^a\varphi, \tilde{\varphi}) : G(\varphi) \rightarrow \text{Proj}(S')$ (called the morphism associated with φ and denoted by $\text{Proj}(\varphi)$), such that for any homogeneous element $f' \in S'_+$, the restriction of this morphism to $D_+(\varphi(f'))$ coincides with the morphism associated with the homomorphism $\varphi_{(f)} : S'_{(f')} \rightarrow S_{(\varphi(f'))}$.*

Proof. The morphism $({}^a\varphi, \tilde{\varphi})$ is obtained form glueing the morphisms Φ_f over $D_+(f)$, and the claim property is immediate. \square

Corollary 19.2.45. Let $\varphi : S' \rightarrow S$ be a homomorphism of graded rings.

- (a) The morphism $\text{Proj}(\varphi)$ is affine.
- (b) If $\ker \varphi$ is nilpotent (and in particular if φ is injective), the morphism $\text{Proj}(\varphi)$ is dominant.
- (c) If φ is eventually surjective, then $G(\varphi) = \text{Proj}(S)$.

Proof. The first assertion follows from ?? and the relation ${}^a\varphi^{-1}(D_+(f')) = D_+(\varphi(f'))$. On the other hand, if $\ker \varphi$ is nilpotent, for any f' homogeneous in S'_+ , we verify that $\ker \varphi_f$ is also nilpotent, and so is $\ker \varphi_{(f)}$. The conclusion then follows from ???. Finally, if φ is eventually surjective, then every homogeneous element $f \in S_+$ has some power contained in the image of φ , so by ?? we conclude that $G(\varphi) = \bigcup_{f' \in S'_+} D_+(\varphi(f')) = \text{Proj}(S)$, whence the claim. \square

Remark 19.2.1. Note that there are in general morphisms from $\text{Proj}(S)$ to $\text{Proj}(S')$ which are not affine, and therefore do not come from graded ring homomorphisms $S' \rightarrow S$; an example is the structural morphism $\text{Proj}(S) \rightarrow \text{Spec}(A)$, where A is field ($\text{Spec}(A)$ being identified with $\text{Proj}(A[T])$ (cf. ??)).

Let $\varphi' : S'' \rightarrow S'$ be another homomorphism of graded rings, and put $\varphi'' = \varphi \circ \varphi'$. Then by the formula ${}^a\varphi'' = {}^a\varphi' \circ {}^a\varphi$ and $G(\varphi'') \subseteq G(\varphi)$, if Φ , Φ' , and Φ'' are the associated morphisms of φ , φ' and φ'' , we have $\Phi'' = \Phi' \circ (\Phi|_{G(\varphi'')})$.

Suppose that S (resp. S') is a graded A -algebra (resp. a graded A' -algebra), and let $\psi : A' \rightarrow A$ be a homomorphism of rings such that the diagram

$$\begin{array}{ccc} A' & \xrightarrow{\psi} & A \\ \downarrow & & \downarrow \\ S' & \xrightarrow{\varphi} & S \end{array}$$

is commutative. We can then consider $G(\varphi)$ and $\text{Proj}(S')$ as schemes over $\text{Spec}(A)$ and $\text{Spec}(A')$, respectively. If Φ and Ψ are the associated morphisms of φ and ψ , respectively, the diagram

$$\begin{array}{ccc} G(\varphi) & \xrightarrow{\Phi} & \text{Proj}(S') \\ \downarrow & & \downarrow \\ \text{Spec}(A) & \xrightarrow{\Psi} & \text{Spec}(A') \end{array}$$

is commutative.

Now let M be a graded S -module and consider the S' -module $\varphi^*(M)$, which is clearly graded. Let f' be a homogeneous element in S'_+ , and set $f = \varphi(f')$. We then have a canonical isomorphism $(\varphi^*(M))_{f'} \cong \varphi_f^*(M_f)$, and it is clear that this isomorphism preserves degrees, so induces an isomorphism $(\varphi^*(M))_{(f')} \cong \varphi_{(f)}^*(M_{(f)})$. There is then canonically an isomorphism of sheaves $\widetilde{\varphi^*(M)}|_{D_+(f')} \cong (\Phi_f)_*(\widetilde{M}|_{D_+(f)})$ (??). Moreover, if g' is another homogeneous ele-

ment of S'_+ and $g = \varphi(g')$, the diagram

$$\begin{array}{ccc} (\varphi^*(M))_{(f')} & \xrightarrow{\sim} & (M_{(f)})_{(\varphi(f))} \\ \downarrow & & \downarrow \\ (\varphi^*(M))_{(f'g')} & \xrightarrow{\sim} & (M_{(fg)})_{(\varphi_{(fg)})} \end{array}$$

is commutative, whence we conclude that the isomorphism

$$\widetilde{\varphi^*(M)}|_{D_+(f'g')} \cong (\Phi_{fg})_*(\widetilde{M}|_{D_+(fg)})$$

is the restriction to $D_+(f'g')$ of the isomorphism $\widetilde{\varphi^*(M)}|_{D_+(f')} \cong (\Phi_f)_*(\widetilde{M}|_{D_+(f)})$. As Φ_f is the restriction of Φ on $D_+(f)$, we then obtain the following result:

Proposition 19.2.46. *There exists a canonical isomorphism $\widetilde{\varphi^*(M)} \cong \Phi_*(\widetilde{M}|_{G(\varphi)})$ of \mathcal{O}_X -modules.*

We also deduce a canonical functorial map from the set of φ -homomorphisms $M' \rightarrow M$ from a graded S' -module to a graded S -module M , to the set of Φ -morphisms $\widetilde{M}' \rightarrow \widetilde{M}|_{G(\varphi)}$. If $\varphi' : S'' \rightarrow S'$ is another ring homomorphism and M'' is a graded S'' -module, the composition of a φ -morphism $M' \rightarrow M$ and a φ' -morphism $M'' \rightarrow M'$ canonically corresponds to the composition of $\widetilde{M}'|_{G(\varphi')} \rightarrow \widetilde{M}|_{G(\varphi'')}$ and $\widetilde{M}'' \rightarrow \widetilde{M}'|_{G(\varphi')}$.

Proposition 19.2.47. *Let $\varphi : S' \rightarrow S$ be a homomorphism of graded rings and M' be a graded S' -module. Then there exists a canonical homomorphism $\nu : \Phi^*(\widetilde{M}') \rightarrow \widetilde{\varphi_*(M)}|_{G(\varphi)}$. If the ideal S'_+ is generated by S'_1 , then ν is an isomorphism.*

Proof. For $f' \in S'_d$ with $d > 0$, we define a canonical homomorphism of $S_{(f)}$ -modules (where $f = \varphi(f')$)

$$\nu_f : M'_{(f)} \otimes_{S'_{(f)}} S_{(f)} \rightarrow (M' \otimes_{S'} S)_{(f)}$$

by composing the homomorphism $M'_{(f)} \otimes_{S'_{(f)}} S_{(f)} \rightarrow M_{f'} \otimes_{S'_{f'}} S_f$ with the canonical homomorphism $M'_{f'} \otimes_{S'_{f'}} S_f \cong (M' \otimes_{S'} S)_f$. It is immediate to verify that compatibility of ν_f with the restriction homomorphisms $D_+(f)$ to $D_+(fg)$ (for $g' \in S'_+$ and $g = \varphi(g')$), so we obtain a homomorphism

$$\nu : \Phi^*(\widetilde{M}') \rightarrow \widetilde{\varphi_*(M)}|_{G(\varphi)}.$$

For the second assertion, it suffices to prove that ν_f is an isomorphism for each $f' \in S'_1$, since $G(\varphi)$ is the union of $D_+(\varphi(f'))$. We first define a \mathbb{Z} -bilinear map $M'_m \times S_n \rightarrow M'_{(f')} \otimes_{S'_{(f')}} S_{(f)}$ which sends (x', s) to the element $(x'/f'^m) \otimes (s/f^n)$. As in the proof of ??, this map then induces a bi-homomorphism

$$\eta_f : M' \otimes_{S'} S \rightarrow M'_{(f')} \otimes_{S'_{(f')}} S_{(f)}.$$

Moreover, if, for $r > 0$, we have $f^r \sum_i (x'_i \otimes s_i) = 0$, then $\sum_i (f'^r x'_i \otimes s_i) = 0$, so $\sum_i (f'^r x'_i / f'^{m_i+r}) \otimes (s_i / f^{n_i}) = 0$, which means $\eta_f(\sum_i x_i \otimes y_i) = 0$; the homomorphism then factors through $(M' \otimes_{S'} S)_f$ and gives a homomorphism $\tilde{\eta}_f : (M' \otimes_{S'} S)_f \rightarrow M'_{(f')} \otimes_{S'_{(f')}} S_{(f)}$. It is easy to verify that $\tilde{\eta}_f$ is the inverse of ν_f , whence our assertion. \square

In particular, since $\varphi_*(S'(n)) = S(n)$ for each $n \in \mathbb{Z}$, it follows from ?? that we have a canonical homomorphism $\Phi^*(\mathcal{O}_{X'}(n)) \rightarrow \mathcal{O}_X(n)|G(\varphi)$, and this an isomorphism if S'_+ is generated by S'_1 .

Remark 19.2.2. We note that it follows from ?? that the morphism Φ is unchanged if we replace S by $S^{(d)}$, S' by $S'^{(d)}$, and φ by $\varphi^{(d)}$. Also, it is also unchanged if we replace S_0 and S'_0 by \mathbb{Z} and φ_0 be the identity map.

Let A, A' be two rings and $\psi : A' \rightarrow A$ be a homomorphism of rings, which defines a morphism $\Psi : \text{Spec}(A) \rightarrow \text{Spec}(A')$. Let S' be an A' -algebra with positive degrees, and put $S = S' \otimes_{A'} A$, which is a graded A -algebra by setting $S_n = S'_n \otimes_{A'} A$. The map $s' \mapsto s' \otimes 1$ is then a homomorphism of graded rings and also a bi-homomorphism. Since S_+ is the A -module generated by $\varphi(S'_+)$, we have $G(\varphi) = \text{Proj}(S) = X$, so, if we put $X' = \text{Proj}(S')$, we have a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\Phi} & X' \\ p \downarrow & & \downarrow \\ Y & \xrightarrow{\Psi} & Y' \end{array} \quad (19.2.10)$$

Now let M' be a graded S' -module, and set $M = M' \otimes_{A'} A = M' \otimes_{S'} S$.

Proposition 19.2.48. *The commutative diagram ?? is cartesian and the canonical homomorphism $\nu : \Phi^*(\tilde{M}') \rightarrow \tilde{M}$ in ?? is an isomorphism.*

Proof. The first assertion follows if we can prove that for any f' homogeneous in S'_+ and $f = \varphi(f')$, the restriction of Φ and p to $D_+(f)$ identify this scheme as $D_+(f') \times_{Y'} Y$; in other words, it suffices to prove that $S_{(f)}$ is canonically identified with $S'_{(f')} \otimes_{A'} A$, which is immediate from the fact that the canonical isomorphism $S_f \cong S'_{f'} \otimes_{A'} A$ preserves degrees. The second assertion follows from the isomorphism $M'_{(f')} \otimes_{S'_{(f')}} S_{(f)} \cong M'_{(f')} \otimes_{A'} A$, and the later one is isomorphic to $M_{(f)}$ since M_f is canonically identified with $M'_{f'} \otimes_{A'} A$. \square

Corollary 19.2.49. *For any integer $n \in \mathbb{Z}$, $\tilde{M}(n)$ is identified with $\Phi^*(\tilde{M}'(n)) = \tilde{M}'(n) \otimes_{Y'} \mathcal{O}_Y$. In particular, $\mathcal{O}_X(n) = \Phi^*(\mathcal{O}_{X'}(n)) = \mathcal{O}_{X'}(n) \otimes_{Y'} \mathcal{O}_Y$.*

Proof. This follows from ?? and ??.

Now let \mathcal{F}' be an $\mathcal{O}_{X'}$ -module and $\mathcal{F} = \Phi^*(\mathcal{F}')$. Then we have for each $n \in \mathbb{Z}$ that $\mathcal{F}(n) = \Phi^*(\mathcal{F}'(n))$ in view of ?. Therefore, by the definition of Φ^* , we have a canonical homomorphism

$$\Gamma(\rho) : \Gamma(X', \mathcal{F}'(n)) \rightarrow \Gamma(X, \mathcal{F}(n))$$

which then gives a canonical bi-homomorphism $\Gamma_*(\mathcal{F}') \rightarrow \Gamma_*(\mathcal{F})$ of graded modules.

Suppose that the ideal S'_+ is generated by S'_1 and $\mathcal{F}' = \tilde{M}'$, so $\mathcal{F} = \tilde{M}$ where $M = M' \otimes_{A'} A$. If f' is homogeneous in S'_+ and $f = \varphi(f')$, we have $M_{(f)} = M'_{(f')} \otimes_{A'} A$ and the diagram

$$\begin{array}{ccc} M'_0 & \longrightarrow & M'_{(f')} = \Gamma(D_+(f), \tilde{M}') \\ \downarrow & & \downarrow \\ M_0 & \longrightarrow & M_{(f)} = \Gamma(D_+(f), \tilde{M}) \end{array}$$

is commutative. We then conclude from the definition of the homomorphism $\alpha : M \rightarrow \Gamma_*(\tilde{M})$ that the following diagram

$$\begin{array}{ccc} M' & \xrightarrow{\alpha_{M'}} & \Gamma_*(\tilde{M}') \\ \downarrow & & \downarrow \\ M & \xrightarrow{\alpha_M} & \Gamma_*(\tilde{M}) \end{array} \quad (19.2.11)$$

is commutative. Similarly, the diagram

$$\begin{array}{ccc} \widetilde{\Gamma_*(\mathcal{F}'')} & \xrightarrow{\beta_{\mathcal{F}''}} & \mathcal{F}'' \\ \downarrow & & \downarrow \\ \widetilde{\Gamma_*(\mathcal{F})} & \xrightarrow{\beta_{\mathcal{F}}} & \mathcal{F} \end{array} \quad (19.2.12)$$

is commutative (where the vertical is the canonical Φ -morphism $\mathcal{F}'' \rightarrow \Phi^*(\mathcal{F}'') = \mathcal{F}$).

Now let N' be another graded S' -module and $N = N' \otimes_{A'} A$. It is immediate that the canonical bi-homomorphisms $M' \rightarrow M$, $N' \rightarrow N$ give a bi-homomorphism $M' \otimes_{S'} N' \rightarrow M \otimes_S N$, and therefore an S -homomorphism $(M' \otimes_{S'} N') \otimes_{A'} A \rightarrow M \otimes_S N$ of degree 0, which then corresponds to an \mathcal{O}_X -homomorphism

$$\Phi^*((M' \otimes_{S'} N')^\sim) \rightarrow (M \otimes_S N)^\sim.$$

Moreover, it is immediaite to verify that the following diagram

$$\begin{array}{ccc} \Phi^*(\tilde{M}' \otimes_{\mathcal{O}_{X'}} \tilde{N}') & \xrightarrow{\sim} & \tilde{M} \otimes_{\mathcal{O}_X} \tilde{N} = \Phi^*(\tilde{M}') \otimes_{\mathcal{O}_X} \Phi^*(\tilde{N}') \\ \downarrow \Phi^*(\lambda) & & \downarrow \lambda \\ \Phi^*((M' \otimes_{S'} N')^\sim) & \longrightarrow & (M \otimes_S N)^\sim \end{array} \quad (19.2.13)$$

is commutative (where the first row is an isomorphism by (??)). If the ideal S'_+ is generated by S'_1 , it is clear that S_+ is generated by S_1 , so the two vertical homomorphisms are isomorphisms, so the second row is also an isomorphism.

We have similarly a cannical bi-homomorphism $\text{Hom}_{S'}(M', N') \rightarrow \text{Hom}_S(M, N)$, which sends a homomorphism u' of degree k the homomorphism $u' \otimes 1$, which is also of degree k . We then deduce an S -homomorphism of degree 0:

$$\text{Hom}_{S'}(M', N') \otimes_{A'} A \rightarrow \text{Hom}_S(M, N)$$

which corresponds to a homomorphism of \mathcal{O}_X -modules:

$$\Phi^*((\text{Hom}_{S'}(M', N'))^\sim) \rightarrow (\text{Hom}_S(M, N))^\sim.$$

Similarly, the diagram

$$\begin{array}{ccc}
 \Phi^*((\mathrm{Hom}_{S'}(M', N'))^\sim) & \longrightarrow & (\mathrm{Hom}_S(M, N))^\sim \\
 \downarrow \Phi^*(\mu) & & \downarrow \mu \\
 \Phi^*(\mathcal{H}\mathrm{om}_{\mathcal{O}_{X'}}(\tilde{M}', \tilde{N}')) & \longrightarrow & \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\tilde{M}, \tilde{N})
 \end{array} \tag{19.2.14}$$

is commutative (where the second row is the canonical homomorphism of ??).

19.2.6 Closed subschemes of $\mathrm{Proj}(S)$

Recall that if $\varphi : S \rightarrow S'$ is a homomorphism of graded rings, we say that φ is eventually surjective (resp. eventually injective, eventually bijective) if $\varphi_i : S_i \rightarrow S'_i$ is surjective (resp. injective, bijective) for sufficiently large i . It follows from ?? that the study of Φ can be reduced to the case where φ is surjective (resp. injective, bijective). Instead of saying that φ is eventually bijective, we also say that it is then an eventual isomorphism.

Proposition 19.2.50. *Let S, S' be graded rings with positive degrees and set $X = \mathrm{Proj}(S)$, $X' = \mathrm{Proj}(S')$.*

- (a) *If $\varphi : S \rightarrow S'$ is an eventually surjective homomorphism of graded rings, the corresponding morphism Φ is defined over $\mathrm{Proj}(S')$ and is a closed immersion. If \mathcal{I} is the kernel of φ , the closed subscheme of X associated with Φ is defined by the quasi-coherent ideal $\tilde{\mathcal{I}}$ of \mathcal{O}_X .*
- (b) *Suppose moreover that the ideal S_+ is finitely generated by S_1 . Let X' be a closed subscheme of X defined by a quasi-coherent ideal \mathcal{I} of \mathcal{O}_X . Let \mathcal{J} be the graded ideal of S which is the inverse image of $\Gamma_*(\mathcal{I})$ under the canonical homomorphism $\alpha : S \rightarrow \Gamma_*(\mathcal{O}_X)$, and put $S' = S/\mathcal{J}$. Then X' is the subscheme associated with the closed immersion $\mathrm{Proj}(S') \rightarrow X$ corresponding to the canonical homomorphism $S \rightarrow S'$ of graded rings.*

Proof. Let $\varphi : S \rightarrow S'$ be an eventually surjective homomorphism of graded rings. We can suppose that φ is surjective, so $\varphi(S_+)$ is generated by S'_+ , we have $G(\varphi) = \mathrm{Proj}(S')$. Now the second assertion in (a) can be verified locally over X ; let f be a homogeneous element of S_+ and put $f' = \varphi(f)$. As φ is a surjective homomorphism of rings, $\varphi_{(f')} : S_{(f)} \rightarrow S'_{(f')}$ is surjective with kernel $\mathcal{I}_{(f)}$, so the corresponding morphism is closed.

We now consider the case of (b); in view of (a), we only need to verify that the homomorphism $\tilde{j} : \tilde{\mathcal{I}} \rightarrow \mathcal{O}_X$ induced from the injection $j : \mathcal{I} \rightarrow S$ is an isomorphism from $\tilde{\mathcal{I}}$ to \mathcal{I} , which follows from ??(b). \square

Remark 19.2.3. Note that \mathcal{I} is the largest graded ideal \mathcal{J}' of S such that $\tilde{\mathcal{I}}' = \mathcal{I}$ (where we identify $\tilde{\mathcal{I}}'$ as a subsheaf of \mathcal{O}_X), since one immediately verify that this relation implies $\alpha(\mathcal{J}') \subseteq \Gamma_*(\mathcal{I})$.

Corollary 19.2.51. *Assume the hypotheses of ??(a) and that S_+ is generated by S_1 . Then $\Phi^*(\widetilde{S(n)})$ is canonically isomorphic to $\widetilde{S'(n)}$ for any $n \in \mathbb{Z}$, and therefore $\Phi^*(\mathcal{F}(n))$ is isomorphic to $\Phi^*(\mathcal{F})(n)$ for any \mathcal{O}_X -module \mathcal{F} .*

Proof. This is a particular case of ??, in view of the definition of $\mathcal{F}(n)$ and ??(a). \square

Corollary 19.2.52. *Assume the hypotheses of ??(a). Then for the closed subscheme X' of X to be integral, it is necessary and sufficient that the graded ideal \mathcal{I} is prime in S .*

Proof. As X' is isomorphic to $\text{Proj}(S/\mathcal{I})$, this condition is sufficient in view of ???. To see the necessity, assume that $\text{Proj}(S')$ is integral and consider the exact sequence $0 \rightarrow \mathcal{J} \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_X/\mathcal{J} \rightarrow 0$, which gives an exact sequence

$$0 \longrightarrow \Gamma_*(\mathcal{J}) \longrightarrow \Gamma_*(\mathcal{O}_X) \longrightarrow \Gamma_*(\mathcal{O}_X/\mathcal{J})$$

by the left-exactness of the global section functor. In view of the canonical homomorphism $\alpha : S/\mathcal{I} \rightarrow \Gamma_*(\mathcal{O}_X/\mathcal{J})$, it then suffices to prove that if $f \in S_m, g \in S_n$ are such that the image in $\Gamma_*(\mathcal{O}_X/\mathcal{J})$ of $\alpha_{n+m}(fg)$ is zero, then one of the images of $\alpha_m(f), \alpha_n(g)$ is zero. Now by definition, these images are sections of the invertible $(\mathcal{O}_X/\mathcal{J})$ -modules $\mathcal{L} = (\mathcal{O}_X/\mathcal{J})(m)$ and $\mathcal{L}' = (\mathcal{O}_X/\mathcal{J})(n)$ over the integral scheme X' . The hypotheses implies that their product is zero in $\mathcal{L} \otimes \mathcal{L}'$ (??), so one of them is zero by ???. \square

Corollary 19.2.53. *Let A be a ring, M be an A -module, and S be a graded A -algebra generated by S_1 . Let $u : M \rightarrow S_1$ be a surjective homomorphism of A -modules and $\bar{u} : S(M) \rightarrow S$ be the unique homomorphism of A -algebras extending u . Then the morphism corresponding to \bar{u} is a closed immersion from $\text{Proj}(S)$ to $\text{Proj}(S(M))$.*

Proof. The homomorphism \bar{u} is surjective by hypothesis, so it suffices to apply ???. \square

19.3 Homogeneous spectrum of sheaves of graded algebras

Let Y be a scheme and \mathcal{S} be an \mathcal{O}_Y -algebra. We say that \mathcal{S} is **graded** if \mathcal{S} is the direct sum of a family (\mathcal{S}_n) of \mathcal{O}_Y -algebras such that $\mathcal{S}_m \mathcal{S}_n \subseteq \mathcal{S}_{m+n}$. If \mathcal{S} is a graded \mathcal{O}_Y -algebra, by a **graded \mathcal{S} -module** \mathcal{M} we mean an \mathcal{S} -module \mathcal{M} which is the direct sum of a family $(\mathcal{M}_n)_{n \in \mathbb{Z}}$ such that $\mathcal{S}_m \mathcal{M}_n \subseteq \mathcal{M}_{m+n}$. We say \mathcal{S} is a graded \mathcal{O}_Y -algebra with positive degrees if $\mathcal{S}_n = 0$ for $n < 0$, and \mathcal{M} is a graded \mathcal{S} -module if $\mathcal{M}_n = 0$ for $n < 0$. In this section, without further specifications, we will only consider graded algebras with positive degree.

19.3.1 Homogeneous spectrum of a graded \mathcal{O}_Y -algebra

Let \mathcal{S} be a graded \mathcal{O}_Y -algebra (with positive degrees) and \mathcal{M} be a graded \mathcal{S} -module. If \mathcal{S} is quasi-coherent, each homogeneous component \mathcal{S}_n is also a quasi-coherent \mathcal{O}_Y -module, since it is the image of \mathcal{S} under the projection of \mathcal{S} onto \mathcal{S}_n . Similarly, if \mathcal{M} is quasi-coherent as an \mathcal{O}_Y -module, so is each of its homogeneous components, and the converse also holds. If $d > 0$ is an integer, we denote by $\mathcal{S}^{(d)}$ the direct sum of the \mathcal{O}_Y -modules \mathcal{S}_{nd} , which is quasi-coherent if \mathcal{S} is; for any integer k such that $0 \leq k \leq d-1$, we denote by $\mathcal{M}^{(d,k)}$ (or $\mathcal{M}^{(d)}$ if $k=0$) the direct sum of \mathcal{M}_{nd+k} (for $n \in \mathbb{Z}$). If \mathcal{S} and \mathcal{M} are quasi-coherent sheaves, $\mathcal{M}(n)$ is a quasi-coherent \mathcal{S} -module by ??.

We say that \mathcal{M} is a graded \mathcal{S} -module **of finite type** (resp. **of finite presentation**) if for any $y \in Y$, there exists an open neighborhood U of y and integers n_i (resp. integers m_i and n_i) such that there exists a surjective homomorphism $\bigoplus_{i=1}^r (\mathcal{S}(n_i)|_U) \rightarrow \mathcal{M}|_U$ of degree 0 (resp. such that $\mathcal{M}|_U$ is isomorphic to the cokernel of a homomorphism $\bigoplus_{i=1}^r \mathcal{S}(m_i)|_U \rightarrow \bigoplus_{j=1}^s \mathcal{S}(n_j)|_U$ of degree 0).

Let U be an affine open of Y and $A = \Gamma(U, \mathcal{O}_Y)$ by its ring. By hypothesis, the graded $(\mathcal{O}_Y|_U)$ -algebra $\mathcal{S}|_U$ is isomorphic to \tilde{S} where $S = \Gamma(U, \mathcal{S})$ is a graded A -algebra; we put $X_U = \text{Proj}(\Gamma(U, \mathcal{S}))$. Let $U' \subseteq U$ be another affine open subset of Y and $j : U' \rightarrow U$ the canonical injection, which corresponds to a homomorphism $A \rightarrow A'$, we have $\mathcal{S}|_{U'} = j^*(\mathcal{S}|_U)$, and therefore $S' = \Gamma(U', \mathcal{S})$ is identified with $S \otimes_A A'$ by ???. We then conclude from ?? that $X_{U'}$ is canonically identified with $X_U \times_U U'$, and therefore with $p_U^{-1}(U')$, where p_U is the structural morphism $X_U \rightarrow U$. Let $\sigma_{U',U}$ be the canonical isomorphism $p_U^{-1}(U') \cong X_{U'}$ thus defined, and $\rho_{U',U}$ be the open immersion $X_{U'} \rightarrow X_U$ obtained by composing $\sigma_{U',U}^{-1}$ with the canonical injectin $p_U^{-1}(U') \rightarrow X_U$. It is immediate that if $U'' \subseteq U'$ is a third affine open of Y , we have $\rho_{U'',U} = \rho_{U'',U'} \circ \rho_{U',U}$.

Proposition 19.3.1. *Let Y be a scheme and \mathcal{S} be a quasi-coherent graded \mathcal{O}_Y -algebra. Then there exists a unique scheme X over Y such that, if $p : X \rightarrow Y$ is the structural morphism, for any affine open U of Y , there exists an isomorphism $\eta_U : p^{-1}(U) \xrightarrow{\sim} X_U = \text{Proj}(\Gamma(U, \mathcal{S}))$ such that, if V is another affine open of Y contained in U , the following diagram*

$$\begin{array}{ccc} p^{-1}(V) & \xrightarrow{\eta_V} & X_V \\ \downarrow & & \downarrow \rho_{V,U} \\ p^{-1}(U) & \xrightarrow{\eta_U} & X_U \end{array}$$

is commutative. The scheme X is called the **homogeneous specturm** of the quasi-coherent \mathcal{O}_Y -algebra \mathcal{S} and denoted by $\text{Proj}(\mathcal{S})$.

Proof. For two affine opens U, V of Y , let $X_{U,V}$ be the scheme induced over $p_U^{-1}(U \cap V)$ by X_U ; we shall define a Y -isomorphism $\theta_{U,V} : X_{V,U} \rightarrow X_{U,V}$. For this, consider an affine open $W \subseteq U \cap V$; by composing the isomorphisms

$$p_U^{-1}(W) \xrightarrow{\sigma_{W,U}} X_W \xrightarrow{\sigma_{W,V}^{-1}} p_V^{-1}(W)$$

we obtain an isomorphism $\tau_W : p_U^{-1}(W) \rightarrow p_V^{-1}(W)$, and we can verify that if $W' \subseteq W$ is an affine open, $\tau_{W'}$ is the restriction of τ_W to $p_U^{-1}(W')$; the morphisms τ_W then glue together to a Y -isomorphism $\theta_{V,U}$, which is what we want. Moreover, if U, V, W are affine opens of Y and $\theta'_{U,V}, \theta'_{V,W}$, and $\theta'_{U,W}$ are restrictions of $\theta_{U,V}, \theta_{V,W}, \theta_{U,W}$ on the inverse images of $U \cap V \cap W$ in X_V, X_W, X_W , respectively, it follows from the preceding definition that $\theta'_{U,V} \circ \theta'_{V,W} = \theta'_{U,W}$. The existence of X then follows from glueing these schemes via the isomorphisms $\theta_{U,V}$, and the uniqueness is clear. \square

It is clear that the Y -scheme $\text{Proj}(\mathcal{S})$ is separated over Y since homogeneous specturms are separated. If \mathcal{S} is an \mathcal{O}_Y -algebra of finite type, it follows from ?? and ?? that $\text{Proj}(\mathcal{S})$ is of finite type over Y . If $p : X \rightarrow Y$ is the structural morphism, it is immediate that for any open subschem U of Y , $p^{-1}(U)$ is identified with the homogeneous specturm $\text{Proj}(\mathcal{S}|_U)$.

Proposition 19.3.2. *Let $f \in \Gamma(Y, \mathcal{S}_d)$ with $d > 0$ and $X = \text{Proj}(\mathcal{S})$. Then there exists an open subset X_f of X such that, for any affine open subset U of Y , we have $X_f \cap p^{-1}(U) = D_+(f|_U)$ in $X_U = \text{Proj}(\Gamma(U, \mathcal{S}))$ (where $p : X \rightarrow Y$ is the structural morphism). Moreover, the Y -scheme induced over X_f by X is canonically isomorphic to $\text{Spec}(\mathcal{S}^{(d)}) / (f - 1)\mathcal{S}^{(d)}$.*

Proof. For any affine open U , we have $f|_U \in \Gamma(U, \mathcal{S}_d) = \Gamma(U, \mathcal{S})_d$ since U is quasi-compact. If U, U' are two affine opens of Y such that $U' \subseteq U$, $f|_{U'}$ is the image of $f|_U$ by the restriction homomorphism $\Gamma(U, \mathcal{S}) \rightarrow \Gamma(U', \mathcal{S})$, so $D_+(f|_{U'})$ is equal to the open subscheme $\rho_{U', U}^{-1}(D_+(f|_U))$ of $X_{U'}$. The subset X_f can be then defined by glueing these subschemes, and the first assertion is then obvious. On the other hand, the open subscheme $D_+(f|_U)$ of X_U is canonically identified with $\text{Spec}(\Gamma(U, \mathcal{S})_{(f|_U)})$, and this identification is clearly compatible with restrictions; the second assertion then follows from ??.

Corollary 19.3.3. *If $f \in \Gamma(Y, \mathcal{S}_d)$ and $g \in \Gamma(Y, \mathcal{S}_e)$, we have $X_{fg} = X_f \cap X_g$.*

Proof. It suffices to consider the intersection of two members of $p^{-1}(U)$, where U is an affine open of Y , and the assertion follows from $D_+(fg) = D_+(f) \cap D_+(g)$ for a graded ring S . \square

Corollary 19.3.4. *Let (f_α) be a family of sections of \mathcal{S} over Y such that $f_\alpha \in \Gamma(Y, \mathcal{S}_{d_\alpha})$. If the sheaf of ideals of \mathcal{S} generated by this family contains all the \mathcal{S}_n for sufficiently large n , then the underlying space X is the union of X_{f_α} .*

Proof. In fact, for any affine open U of Y , $p^{-1}(U)$ is the union of $X_{f_\alpha} \cap p^{-1}(U)$ by ??, so the claim follows from the construction of X_{f_α} . \square

Corollary 19.3.5. *Let \mathcal{A} be a quasi-coherent \mathcal{O}_Y -algebra and put*

$$\mathcal{S} = \mathcal{A}[T] = \mathcal{A} \otimes_{\mathbb{Z}} \mathbb{Z}[T]$$

where T is an indeterminate. Then $X = \text{Proj}(\mathcal{S})$ is canonically identified with $\text{Spec}(\mathcal{A})$. In particular, $\text{Proj}(\mathcal{O}_Y[T])$ is identified with Y .

Proof. By applying ?? to the unique section $f \in \Gamma(Y, \mathcal{S})$ equal to T on each point of Y , we see that $X_f = X$. Moreover, we have $f \in \mathcal{S}_1$, and $\mathcal{S}^{(1)} / (f - 1)\mathcal{S}^{(1)} = \mathcal{S} / (f - 1)\mathcal{S}$ is canonically isomorphic to \mathcal{A} , whence the corollary \square

Let $g \in \Gamma(Y, \mathcal{O}_Y)$; if we put $\mathcal{S} = \mathcal{O}_Y[T]$, then $g \in \Gamma(Y, \mathcal{S}_0)$; let

$$h = gT \in \Gamma(Y, \mathcal{S}_1).$$

If $X = \text{Proj}(\mathcal{S})$, the canonical identification of ?? identifies X_h with the open subset Y_g of Y : in fact, we can assume that $Y = \text{Spec}(A)$ is affine, and this then follows from the fact that the ring A_g is canonically identified with $A[T]/(gT - 1)A[T]$.

Proposition 19.3.6. *Let \mathcal{S} be a quasi-coherent graded \mathcal{O}_Y -algebra.*

- (a) *For any $d > 0$, there exists a canonical Y -isomorphism from $\text{Proj}(\mathcal{S})$ to $\text{Proj}(\mathcal{S}^{(d)})$.*
- (b) *Let \mathcal{S}' be the graded \mathcal{O}_Y -algebra $\mathcal{O}_Y \oplus \bigoplus_{n>0} \mathcal{S}_n$, then the schemes $\text{Proj}(\mathcal{S}')$ and $\text{Proj}(\mathcal{S})$ are canonically Y -isomorphic.*
- (c) *Let \mathcal{L} be an invertible \mathcal{O}_Y -module and $\mathcal{S}_{(\mathcal{L})}$ be the graded \mathcal{O}_Y -algebra $\bigoplus_{d>0} \mathcal{S}_d \otimes \mathcal{L}^{\otimes d}$; then the schemes $\text{Proj}(\mathcal{S})$ and $\text{Proj}(\mathcal{S}_{(\mathcal{L})})$ are canonically Y -isomorphic.*

Proof. In all three cases, it suffices to define an isomorphism locally over Y and verifying the compatibility of restriction morphisms is immediate. We can then assume that Y is affine, and assertions (a) and (b) then follow from ???. As for (c), if the invertible sheaf \mathcal{L} is just isomorphic to \mathcal{O}_Y then the isomorphism between $\text{Proj}(\mathcal{S})$ and $\text{Proj}(\mathcal{S}_{(\mathcal{L})})$ is evident. To define a canonical isomorphism, let $Y = \text{Spec}(A)$, $\mathcal{S} = \tilde{S}$, where S is a graded A -algebra, and let c be a generator of the free A -module L such that $\mathcal{L} = \tilde{L}$. Then for any $n > 0$, $x_n \mapsto x_n \otimes c^{\otimes n}$ is an A -isomorphism from S_n to $S_n \otimes L^{\otimes n}$, and these A -isomorphisms define an A -isomorphism of graded algebras

$$p_c : S \rightarrow S_{(L)} = \bigoplus_{n \geq 0} S_n \otimes L^{\otimes n}.$$

Let $f \in S_+$ be homogeneous of degree d ; for any $x \in S_{nd}$, we have $(x \otimes c^{nd})/(f \otimes c^d)^n = (x \otimes (\varepsilon c)^{nd})/(f \otimes (\varepsilon c)^d)^n$ for any invertible element $\varepsilon \in A$, which implies that the isomorphism $S_{(f)} \rightarrow (S_{(L)})_{(f \otimes c^d)}$ induced by p_c is independent from the generator c of L , whence the assertion. \square

Recall that for the quasi-coherent graded \mathcal{O}_Y -algebra \mathcal{L} to be *generated* by the \mathcal{O}_Y -module \mathcal{S}_1 , it is necessary and sufficient that there exists a covering (U_α) of Y by affine opens such that the graded algebra $\Gamma(U_\alpha, \mathcal{S})$ over $\Gamma(U_\alpha, \mathcal{O}_Y)$ is generated by $\Gamma(U_\alpha, \mathcal{S}_1)$. If this is true, then for any open subset V of Y , $\mathcal{S}|_V$ is then generated by $(\mathcal{O}_Y|_V)$ -algebra $\mathcal{S}_1|_V$.

Proposition 19.3.7. *Suppose that there exists a finite affine open cover (U_i) of Y such that the graded algebra $\Gamma(U_i, \mathcal{S})$ is of finite type over $\Gamma(U_i, \mathcal{O}_Y)$. Then there exists $d > 0$ such that $\mathcal{S}^{(d)}$ is generated by \mathcal{S}_d , which is an \mathcal{O}_Y -module of finite type.*

Proof. In fact, it follows from ?? that for each i , there exist an integer m_i such that $\Gamma(U_i, \mathcal{S}_{nm_i}) = (\Gamma(U_i, \mathcal{S}_{m_i}))^n$ for all $n > 0$; it suffices to take d a common multiple of the m_i . \square

Corollary 19.3.8. *Under the hypotheses of ??, $\text{Proj}(\mathcal{S})$ is Y -isomorphic to a homogeneous spectrum $\text{Proj}(\mathcal{S}')$, where \mathcal{S}' is a graded \mathcal{O}_Y -algebra generated by \mathcal{S}'_1 , where \mathcal{S}' is an \mathcal{O}_Y -algebra of finite type.*

Proof. It suffices to take $\mathcal{S}' = \mathcal{S}^{(d)}$, where d is determined by the properties of ??, and apply ??(a). \square

If \mathcal{S} is a quasi-coherent graded \mathcal{O}_Y -algebra, we have seen in ?? that its nilradical is a quasi-coherent \mathcal{O}_Y -module. We say that $\mathcal{N}_+ = \mathcal{N} \cap \mathcal{S}_+$ is the nilradical of \mathcal{S}_+ , which is a quasi-coherent graded \mathcal{S}_0 -module, since this is the case if Y is affine. For any $y \in Y$, $(\mathcal{N}_+)_y$ is then the nilradical of $(\mathcal{S}_+)_y = (\mathcal{S}_y)_+$. Similar to the local case, we say the graded \mathcal{O}_Y -algebra \mathcal{S} is **essentially reduced** if $\mathcal{N}_+ = 0$, which means \mathcal{S}_y is an essentially reduced graded $\mathcal{O}_{Y,y}$ -algebra for any $y \in Y$. It is clear that for any quasi-coherent graded \mathcal{O}_Y -algebra \mathcal{S} , $\mathcal{S}/\mathcal{N}_+$ is essentially reduced. Finally, we say \mathcal{S} is **integral** if \mathcal{S}_y is an integral ring for each $y \in Y$ and if $(\mathcal{S}_+)_y = (\mathcal{S}_y)_+ \neq 0$ for all $y \in Y$.

Proposition 19.3.9. *Let \mathcal{S} be a graded \mathcal{O}_Y -algebra. If $X = \text{Proj}(\mathcal{S})$, the Y -scheme X_{red} is canonically isomorphic to $\text{Proj}(\mathcal{S}/\mathcal{N}_+)$. In particular, if \mathcal{S} is essentially reduced, then X is reduced.*

Proof. The fact that $X' = \text{Proj}(\mathcal{S}/\mathcal{N}_+)$ is reduced follows from ??, since the question is local. Moreover, for any affine open $U \subseteq Y$, $p'^{-1}(U)$ is equal to $(p^{-1}(U))_{\text{red}}$ (where p and p' are the structural morphisms $X \rightarrow Y$, $X' \rightarrow Y$, respectively); we also verify that the canonical

U -morphisms $p'^{-1}(U) \rightarrow p^{-1}(U)$ is compatible with restrictions and define therefore a closed immersion $X' \rightarrow X$, which is a homeomorphism on underlying spaces. Our assertion then follows from ??.

Proposition 19.3.10. *Let Y be an integral scheme and \mathcal{S} be a quasi-coherent graded \mathcal{O}_Y -algebra such that $\mathcal{S}_0 = \mathcal{O}_Y$.*

- (a) *If \mathcal{S} is integral then $X = \text{Proj}(\mathcal{S})$ is integral and the structural morphism $p : X \rightarrow Y$ is dominant.*
- (b) *Suppose moreover that \mathcal{S} is essentially reduced. Then, conversely, if X is integral and p is dominant, then \mathcal{S} is integral.*

Proof. We first assume that \mathcal{S} is integral. Then if (U_α) is a basis of Y formed by affine opens, it suffices to prove for Y being replaced by U_α and \mathcal{S} by $\mathcal{S}|_{U_\alpha}$: in fact, if this is true, the underlying space $p^{-1}(U_\alpha)$ is an open irreducible subset of X such that $p^{-1}(U_\alpha) \cap p^{-1}(U_\beta) \neq \emptyset$ for any couple of indices α, β (since $U_\alpha \cap U_\beta$ contains an U_γ and \mathcal{S} is integral), so X is irreducible by ??; it is clear that X is reduced since \mathcal{S} is reduced, so X is integral. It is clear that $p(X)$ is dense in Y since this holds for each U_α .

Suppose then that $Y = \text{Spec}(A)$ where A is integral (??) and $\mathcal{S} = \tilde{S}$, where S is a graded A -algebra; the hypotheses on \mathcal{S} is that for any $y \in Y$, $\tilde{S}_y = S_y$ is an integral graded ring such that $(S_y)_+ \neq 0$. It then suffices to prove that S is an integral ring, since then $S_+ \neq 0$ and we can apply ?. Now, let f, g be two nonzero elements of S and suppose that $fg = 0$; for any $y \in Y$ we have $(f/1)(g/1) = 0$ in S_y , so $f/1 = 0$ or $g/1 = 0$ by hypothesis. Suppose for example that $f/1 = 0$ in S_y , so there exists $a \in A$ such that $a \notin \mathfrak{p}_y$ and $af = 0$. We then see that for each $z \in Y$, $(a/1)(f/1) = 0$ in the integral ring S_z , and as $a/1 \neq 0$ (since A is integral), $f/1 = 0$, which implies $f = 0$.

Now consider the hypothesis in (b) and assume that X is integral and p is dominant. Since the question is local over Y , we can assume that $Y = \text{Spec}(A)$, where A is an integral ring, and $\mathcal{S} = \tilde{S}$. By hypothesis for any $y \in Y$, $(S_y)_+$ is reduced, and so is $(S_0)_y = A_y$ by hypothesis, so S_y is a reduced ring and we conclude that S is reduced. The hypothesis that X is integral implies that S is essentially integral (??). The proposition then boils down to see that the annihilator \mathfrak{J} of S_+ over $A = S_0$ is reduced to zero. In the contrary case, we would have $(S_h)_+ = 0$ for an $h \neq 0$ in \mathfrak{J} , which implies $p^{-1}(D(h)) = \emptyset$ by ??, and $p(X)$ is then not dense in Y , contradicting the hypothesis (since $D(h) \neq \emptyset$, h is not nilpotent). We then see that the ring S is integral, which conclude our assertion. \square

19.3.2 Sheaves associated with a graded \mathcal{S} -module

Let Y be a scheme, \mathcal{S} be a quasi-coherent \mathcal{O}_Y -algebra, and \mathcal{M} be a quasi-coherent graded \mathcal{S} -module over the ringed space (Y, \mathcal{S}) . With the notations of ??, we denote by $\tilde{\mathcal{M}}_U$ the quasi-coherent \mathcal{O}_{X_U} -module $\widetilde{\Gamma(U, \mathcal{M})}$. For $U' \subseteq U$, $\Gamma(U', \mathcal{M})$ is canonically identified with $\Gamma(U, \mathcal{M}) \otimes_A A'$ by ??, so $\tilde{\mathcal{M}}_U = \rho_{U', U}^*(\tilde{\mathcal{M}}_U)$ by ??.

Proposition 19.3.11. *There exists over $\text{Proj}(\mathcal{S}) = X$ a unique quasi-coherent \mathcal{O}_X -module $\tilde{\mathcal{M}}$ such that, for any affine open U of Y , we have $\eta_U^*(\tilde{\mathcal{M}}_U) = \tilde{\mathcal{M}}|_{p^{-1}(U)}$, where $p : X \rightarrow Y$ is the structural morphism and η_U is the isomorphism $p^{-1}(U) \cong \text{Proj}(\Gamma(U, \mathcal{S}))$. We say that $\tilde{\mathcal{M}}$ is the \mathcal{O}_X -module associated with \mathcal{M} .*

Proof. As $\rho_{U',U}$ is identified with the injection morphism $p^{-1}(U') \rightarrow p^{-1}(U)$, the proposition follows from the relation $\tilde{M}_{U'} = \rho_{U',U}^*(\tilde{\mathcal{M}}_U)$ and glueing the $\tilde{\mathcal{M}}_U$. \square

Proposition 19.3.12. *Let \mathcal{M} be a quasi-coherent \mathcal{S} -module and $f \in \Gamma(Y, \mathcal{S}_d)$ with $d > 0$. If ξ_f is the canonical isomorphism from X_f to the Y -scheme $Z_f = \text{Spec}(\mathcal{S}^{(d)} / (f - 1)\mathcal{S}^{(d)})$ in ??, then $(\xi_f)_*(\tilde{\mathcal{M}}|_{X_f})$ is the \mathcal{O}_{Z_f} -module $(\mathcal{M}^{(d)} / (f - 1)\mathcal{M}^{(d)})^\sim$.*

Proof. The question is local over Y we we are reduced to ??, and its compatibility with restrictions. \square

Proposition 19.3.13. *The \mathcal{O}_X -module $\tilde{\mathcal{M}}$ is an additive exact covariant functor from the category of quasi-coherent graded \mathcal{S} -modules to the category of quasi-coherent \mathcal{O}_X -modules, which commutes with direct sums and inductive limits.*

Proof. This follows from ?? and ??, since the question is local on Y . \square

In particular, if \mathcal{N} is a quasi-coherent graded sub- \mathcal{S} -module of \mathcal{M} , $\tilde{\mathcal{N}}$ is cannically identified with a quasi-coherent sub- \mathcal{O}_X -module of $\tilde{\mathcal{M}}$; if we take $\mathcal{M} = \mathcal{S}$, then for any quasi-coherent ideal \mathcal{I} of \mathcal{S} , $\tilde{\mathcal{I}}$ is a quasi-coherent ideal of \mathcal{O}_X .

If \mathcal{M} is a quasi-coherent graded \mathcal{S} -module and \mathcal{I} is a quasi-coherent ideal of \mathcal{O}_Y , then $\mathcal{I}\mathcal{M}$ is a quasi-coherent graded sub- \mathcal{S} -module of \mathcal{M} and we have $\widetilde{\mathcal{I}\mathcal{M}} = \mathcal{I} \cdot \tilde{\mathcal{M}}$: it suffices to verify this formula if $Y = \text{Spec}(A)$, $\mathcal{S} = \tilde{S}$, where S is a graded A -algebra, $\mathcal{M} = \tilde{M}$ where M is a graded S -module, and $\mathcal{I} = \tilde{\mathfrak{a}}$, where \mathfrak{a} is an ideal of A . For any homogeneous element f of S_+ , the restriction to $D_+(f) = \text{Spec}(S_{(f)})$ of $\widetilde{\mathcal{I}\mathcal{M}}$ is the assocaited sheaf of $(\mathfrak{a}M)_{(f)} = \mathfrak{a} \cdot M_{(f)}$, and the identification is compatible with restrictions.

Proposition 19.3.14. *Let $f \in \Gamma(Y, \mathcal{S}_d)$ with $d > 0$. Over the open subset X_f , the $(\mathcal{O}_X|_{X_f})$ -module $\widetilde{\mathcal{S}(nd)}|_{X_f}$ is canonically isomorphic to $\mathcal{O}_X|_{X_f}$ for any $n \in \mathbb{Z}$. In particular, if the \mathcal{O}_Y -algebra \mathcal{S} is generated by \mathcal{S}_1 , the \mathcal{O}_X -module $\widetilde{\mathcal{S}(n)}$ is invertible for all $n \in \mathbb{Z}$.*

Proof. For any affine open U of Y , by ?? we have a canonical isomorphism $\widetilde{\mathcal{S}(nd)}|_{X_f \cap p^{-1}(U)} \cong \mathcal{O}_X|_{X_f \cap p^{-1}(U)}$, in view of ?? (where $p : X \rightarrow Y$ is the structural morphism). It is immediate that this isomorphism is compatible with restrictions, whence the first assertion. For the second one, if \mathcal{S} is generated by \mathcal{S}_1 , there exists an affine open cover (U_α) of Y such that $\Gamma(U_\alpha, \mathcal{S})$ is generated by $\Gamma(U_\alpha, \mathcal{S}_1) = \Gamma(U_\alpha, \mathcal{S})_1$, and we can then use ?? \square

Again, for any integer $n \in \mathbb{Z}$ and any \mathcal{O}_X -module \mathcal{F} , we set

$$\mathcal{O}_X(n) = \widetilde{\mathcal{S}(n)}, \quad \mathcal{F}(n) = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n).$$

It follows from this definition that, for any open subset U of Y ,

$$\widetilde{\mathcal{S}|_U(n)} = \mathcal{O}_X|_{p^{-1}(U)},$$

where $p : X \rightarrow Y$ is the structural morphism.

Proposition 19.3.15. *Let \mathcal{M} and \mathcal{N} be quasi-coherent graded \mathcal{S} -modules. Then there exists cannical homomorphisms*

$$\begin{aligned} \lambda : \widetilde{M} \otimes_{\mathcal{O}_X} \widetilde{\mathcal{N}} &\rightarrow (\mathcal{M} \otimes_{\mathcal{S}} \mathcal{N})^\sim \\ \mu : (\mathcal{H}\text{om}_{\mathcal{S}}(\mathcal{M}, \mathcal{N}))^\sim &\rightarrow \mathcal{H}\text{om}_{\mathcal{O}_X}(\widetilde{\mathcal{M}}, \widetilde{\mathcal{N}}). \end{aligned}$$

If \mathcal{S} is generated by \mathcal{S}_1 , then λ is an isomorphism; if moreover \mathcal{M} is of finite presentation, μ is an isomorphism.

Proof. The isomorphisms λ and μ are defined in the arguments before ?? if Y is affine, and this definition is local and then glue together to define global morphisms, in view of the diagrams ?? and ??.

Corollary 19.3.16. *If \mathcal{S} is generated by \mathcal{S}_1 , for any integers $m, n \in \mathbb{Z}$, we have*

$$\mathcal{O}_X(m) \otimes_{\mathcal{O}_X} \mathcal{O}_X(n) = \mathcal{O}_X(m+n), \quad \mathcal{O}_X(n) = \mathcal{O}_X^{\otimes 1}.$$

Corollary 19.3.17. *If \mathcal{S} is generated by \mathcal{S}_1 , for any quasi-coherent graded \mathcal{S} -module \mathcal{M} and $n \in \mathbb{Z}$, we have*

$$\widetilde{\mathcal{M}(n)} = \widetilde{\mathcal{M}}(n).$$

Remark 19.3.1. If $\mathcal{S} = \mathcal{A}[T]$ where \mathcal{A} is a quasi-coherent \mathcal{O}_Y -algebra, we verify immediately that the invertible \mathcal{O}_X -module $\mathcal{O}_X(n)$ is canonically isomorphic to \mathcal{O}_X . Moreover, let \mathcal{N} be a quasi-coherent \mathcal{A} -module, and put $\mathcal{M} = \mathcal{N} \otimes_{\mathcal{A}} \mathcal{A}[T]$. It follows from ?? and ?? that under the canonical isomorphism of $X = \text{Proj}(\mathcal{A}[T])$ and $X' = \text{Spec}(\mathcal{A})$, the \mathcal{O}_X -module $\widetilde{\mathcal{M}}$ is identified with the \mathcal{O}_X -module $\widetilde{\mathcal{N}}$.

Remark 19.3.2. Let \mathcal{S} be a quasi-coherent graded \mathcal{O}_Y -algebra and define \mathcal{S}' to be the \mathcal{O}_Y -algebra such that $\mathcal{S}' = \mathcal{O}_Y$ and $\mathcal{S}'_n = \mathcal{S}_n$ for $n > 0$. Then the canonical isomorphism of $X = \text{Proj}(\mathcal{S})$ and $\text{Proj}(\mathcal{S}')$ identifies $\mathcal{O}_X(n)$ and $\mathcal{O}_{X'}(n)$ for any $n \in \mathbb{Z}$: this follows from the corresponding result in the affine case ?? and the fact that this identification is compatible with restrictions. Similarly, let $X^{(d)} = \text{Proj}(\mathcal{S}^{(d)})$; the canonical isomorphism of X and $X^{(d)}$ identifies $\mathcal{O}_X(nd)$ with $\mathcal{O}_{X^{(d)}}(n)$ for any $n \in \mathbb{Z}$.

Proposition 19.3.18. *Let \mathcal{L} be an invertible \mathcal{O}_Y -module and ψ be the structural morphism from $X_{(\mathcal{L})} = \text{Proj}(\mathcal{S}_{(\mathcal{L})})$ to $X = \text{Proj}(\mathcal{S})$. Then for any integer $n \in \mathbb{Z}$, $\psi_*(\mathcal{O}_{X_{(\mathcal{L})}}(n))$ is canonically isomorphic to $\mathcal{O}_X(n) \otimes_Y \mathcal{L}^{\otimes n}$.*

Proof. Suppose first that Y is affine with ring A and $\mathcal{L} = \widetilde{L}$, where L is a free A -module of rank 1. With the notations of ??(c), we define for each $f \in S_d$ an isomorphism from $S(n)_{(f)} \otimes_A L^{\otimes n}$ to $S_{(L)}(n)_{(f \otimes c^d)}$ which sends $(x/f^k) \otimes c^n$, where $x \in S_{kd+n}$, to the element $(x \otimes c^{n+kd})/(f \otimes c^d)^k$. It is immediate that this isomorphism is independent of the generator c of L , and is compatible with restrictions $D_+(f) \rightarrow D_+(fg)$. The general case then follows from glueing these isomorphisms. \square

19.3.3 Graded \mathcal{S} -module associated with a sheaf

For simplicity, in the following discussion, we always assume that the quasi-coherent \mathcal{O}_Y -algebra \mathcal{S} is generated by \mathcal{S}_1 , which by ?? is not at all essential if we impose the finiteness conditions of ?? on Y . Let $p : X \rightarrow Y$ be the structural morphism where $X = \text{Proj}(\mathcal{S})$, which is separated by ???. For any \mathcal{O}_X -module \mathcal{F} , we define

$$\Gamma_*(\mathcal{F}) = \bigoplus_{n \in \mathbb{Z}} p_*(\mathcal{F}(n))$$

and in particular

$$\Gamma_*(\mathcal{O}_X) = \bigoplus_{n \in \mathbb{Z}} p_*(\mathcal{O}_X(n)).$$

We have seen in (??) that there exists a canonical homomorphism

$$p_*(\mathcal{F}) \otimes_{\mathcal{O}_Y} p_*(\mathcal{G}) \rightarrow p_*(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})$$

for any \mathcal{O}_X -modules \mathcal{F} and \mathcal{G} , so we deduce from ?? that $\Gamma_*(\mathcal{O}_X)$ is endowed with a graded \mathcal{O}_Y -algebra structure and $\Gamma_*(\mathcal{F})$ is a graded module over $\Gamma_*(\mathcal{O}_X)$.

In view of ?? and the left-exactness of the functor f_* , $\Gamma_*(\mathcal{F})$ is an additive left-exact covariant functor from the category of \mathcal{O}_X -modules to the category of graded \mathcal{O}_Y -modules. In particular, if \mathcal{I} is a sheaf of ideals of \mathcal{O}_X , $\Gamma_*(\mathcal{I})$ is identified with a sheaf of graded ideals of $\Gamma_*(\mathcal{O}_X)$.

Now let \mathcal{M} be a quasi-coherent graded \mathcal{S} -module. For any affine open U of Y , we have defined a homomorphism of abelian groups

$$\alpha_{0,U} : \Gamma(U, \mathcal{M}_0) \rightarrow \Gamma(p^{-1}(U), \tilde{\mathcal{M}}).$$

It is immediate that these homomorphisms commutes with restrictions and define (which do not use the hypothesis that \mathcal{S} is generated by \mathcal{S}_1) a homomorphism of sheaf of abelian groups

$$\alpha_0 : \mathcal{M}_0 \rightarrow \tilde{\mathcal{M}}.$$

Apply this result to $\mathcal{M}_n = (\mathcal{M}(n))_0$ and use ??, we define a homomorphism of abelian groups

$$\alpha_n : \mathcal{M}_n \rightarrow p_*(\tilde{\mathcal{M}}(n)) \tag{19.3.1}$$

for each $n \in \mathbb{Z}$, whence a functorial homomorphism of graded sheaves of abelian groups

$$\alpha : \mathcal{M} \rightarrow \Gamma_*(\tilde{\mathcal{M}}) \tag{19.3.2}$$

(we also denote it by $\alpha_{\mathcal{M}}$). In the particular case $\mathcal{M} = \mathcal{S}$, we verify that $\alpha : \mathcal{S} \rightarrow \Gamma_*(\mathcal{O}_X)$ is a homomorphism of graded \mathcal{O}_Y -algebra and is a bi-homomorphism of graded modules, relative to this homomorphism of graded homomorphism of algebras.

We also remark that the homomorphism α_n corresponds to a canonical homomorphism of \mathcal{O}_X -modules

$$\alpha_n^\sharp : p^*(\mathcal{M}_n) \rightarrow \tilde{\mathcal{M}}(n).$$

Moreover, it is easy to verify that this homomorphism is none other than the associated homomorphism (by ??) of the canonical homomorphism $\mathcal{M}_n \otimes_{\mathcal{O}_Y} \mathcal{S} \rightarrow \mathcal{M}(n)$ of \mathcal{O}_Y -modules, where the \mathcal{O}_Y -module $\mathcal{M}_n \otimes_{\mathcal{O}_Y} \mathcal{S}$ is given the natural graduation. To see this, we can in fact assume that $Y = \text{Spec}(A)$ is affine, $\mathcal{M} = \tilde{M}$ and $\mathcal{S} = \tilde{S}$, where S is a graded A -algebra generated by S_1 . Returning to the definition of α , we see that the restriction to $D_+(f)$ of the homomorphism α_n^\sharp corresponds to the homomorphism $M_n \otimes_A S_{(f)} \rightarrow M(n)_{(f)}$, where $x \otimes 1$ is mapped to $x/1$.

Proposition 19.3.19. *For any section $f \in \Gamma(Y, \mathcal{S}_d)$ with $d > 0$, X_f is identified with the set of points of X on which $\alpha_d(f)$ is nonzero.*

Proof. The element $\alpha_d(f)$ is a section of $p_*(\mathcal{O}_X(d))$ over Y , and by definition is then a section of

$\mathcal{O}_X(d)$ over X . The definition of X_f (??) proves our claim in the affine case, in view of ??.

We shall henceforth suppose, in addition to the hypothesis at the beginning of this paragraph, that for every quasi-coherent \mathcal{O}_X -module \mathcal{F} , $p_*(\mathcal{F}(n))$ is quasi-coherent over Y , and therefore $\Gamma_*(\mathcal{F}) = \bigoplus_{n \in \mathbb{Z}} p_*(\mathcal{F}(n))$ is also a quasi-coherent \mathcal{O}_Y -module; this circumstance will always occur if X is of finite type on Y (??). We then conclude that $\widetilde{\Gamma_*}(\mathcal{F})$ is defined and is a quasi-coherent \mathcal{O}_X -module. For any affine open subset U of Y , we have (??, ??, and note that U is quasi-compact)

$$\begin{aligned} (\Gamma(U, \bigoplus_{n \in \mathbb{Z}} p_*(\mathcal{F}(n))))^\sim &= \bigoplus_{n \in \mathbb{Z}} (\Gamma(U, p_*(\mathcal{F}(n))))^\sim = \bigoplus_{n \in \mathbb{Z}} (\Gamma(p^{-1}(U), \mathcal{F}(n)))^\sim \\ &= \left(\bigoplus_{n \in \mathbb{Z}} \Gamma(p^{-1}(U), \mathcal{F}(n)) \right)^\sim = (\Gamma_*(\mathcal{F}|_{p^{-1}(U)}))^\sim \end{aligned}$$

and therefore a canonical homomorphism

$$\beta_U : (\Gamma(U, \bigoplus_{n \in \mathbb{Z}} p_*(\mathcal{F}(n))))^\sim \rightarrow \mathcal{F}|_{p^{-1}(U)}.$$

Moreover, the diagram (??) shows that these homomorphisms are compatible with restrictions on Y , so we deduce a canonical homomorphism

$$\beta : \widetilde{\Gamma_*}(\mathcal{F}) \rightarrow \mathcal{F}$$

(also denoted by $\beta_{\mathcal{F}}$) for the quasi-coherent \mathcal{O}_X -modules.

Proposition 19.3.20. *Let \mathcal{M} be a quasi-coherent \mathcal{S} -module and \mathcal{F} be a quasi-coherent \mathcal{O}_X -module. Then the composition homomorphisms*

$$\tilde{\mathcal{M}} \xrightarrow{\tilde{\alpha}} (\Gamma_*(\tilde{\mathcal{M}}))^\sim \xrightarrow{\beta} \tilde{\mathcal{M}} \tag{19.3.3}$$

$$\Gamma_*(\mathcal{F}) \xrightarrow{\alpha} \Gamma_*(\widetilde{\Gamma_*}(\mathcal{F})) \xrightarrow{\Gamma_*(\beta)} \Gamma_*(\mathcal{F}) \tag{19.3.4}$$

Proof. The question is local over Y , so we can apply ??.

Again, the homomorphisms α and β are in general not isomorphisms, and further finiteness conditions must be imposed. We note also that the homomorphism β is not always defined, unlike the affine case. However, we shall see that if \mathcal{S} is of finite type and generated by \mathcal{S}_1 , the homomorphisms α and β are well defined and the corresponding results of the affine cases carry over without difficulties.

Proposition 19.3.21. *Let Y be a scheme and \mathcal{S} be a quasi-coherent \mathcal{O}_Y -algebra generated by \mathcal{S}_1 . Suppose that \mathcal{S}_1 is of finite type, then $X = \text{Proj}(\mathcal{S})$ is of finite type over Y .*

Proof. Again we can assume that Y is affine with ring A , so $\mathcal{S} = \tilde{S}$ where S is a graded A -algebra generated by S_1 , and S_1 is a finitely generated A -module by hypothesis. Then S is an A -algebra of finite type, and the proposition follows from ??.

Let \mathcal{S} be a quasi-coherent graded \mathcal{O}_Y -algebra and \mathcal{M} be a quasi-coherent \mathcal{S} -module. We say that \mathcal{M} is **eventually null** if there exists an integer n such that $\mathcal{M}_k = 0$ for $k \geq n$, and is

eventually finite if there exists an integer n such that the \mathcal{S} -module $\bigoplus_{k \geq n} \mathcal{M}_k$ is of finite type. If \mathcal{M} is eventually null, it is clear that $\tilde{\mathcal{M}} = 0$, as in the affine case.

Let \mathcal{M} and \mathcal{N} be quasi-coherent graded \mathcal{S} -modules. We say a homomorphism $u : \mathcal{M} \rightarrow \mathcal{N}$ of degree 0 is eventually injective (resp. eventually surjective, eventually bijective) if there exists an integer n such that $u_k : \mathcal{M}_k \rightarrow \mathcal{N}_k$ is injective (resp. surjective, bijective) for $k \geq n$. It is clear that in this case, $\tilde{u} : \tilde{\mathcal{M}} \rightarrow \tilde{\mathcal{N}}$ is injective (resp. surjective, bijective), since this can be checked locally over Y and we can apply ???. If u is eventually bijective, we also say that it is an eventual isomorphism.

Proposition 19.3.22. *Let Y be a scheme and \mathcal{S} be a quasi-coherent \mathcal{O}_Y -algebra generated by \mathcal{S}_1 and such that \mathcal{S}_1 is of finite type. Let \mathcal{M} be a quasi-coherent graded \mathcal{S} -module.*

- (a) *If \mathcal{M} is eventually finite, $\tilde{\mathcal{M}}$ is of finite type.*
- (b) *If \mathcal{M} is eventually finite, for $\tilde{\mathcal{M}} = 0$, it is necessary and sufficient that \mathcal{M} is eventually null.*

Proof. The question is local over Y , so we can assume that $Y = \text{Spec}(A)$, and the proposition then follows from ???. \square

Theorem 19.3.23. *Let Y be a scheme and \mathcal{S} be a quasi-coherent \mathcal{O}_Y -algebra generated by \mathcal{S}_1 and such that \mathcal{S}_1 is of finite type. Let $X = \text{Proj}(\mathcal{S})$, then for any quasi-coherent \mathcal{O}_X -module \mathcal{F} , the canonical homomorphism $\beta : \Gamma(\tilde{\mathcal{F}}) \rightarrow \mathcal{F}$ is an isomorphism.*

Proof. We first note that the homomorphism β is defined because of ???. To see that β is an isomorphism, we can assume $Y = \text{Spec}(A)$ is affine, and then apply ???. \square

Corollary 19.3.24. *Under the hypotheses of ???, any quasi-coherent \mathcal{O}_X -module \mathcal{F} is isomorphic to an \mathcal{O}_X -module of the form $\tilde{\mathcal{M}}$, where \mathcal{M} is a quasi-coherent \mathcal{S} -module. If moreover \mathcal{F} is of finite type, and if we suppose that Y is a quasi-compact scheme, then we can choose \mathcal{M} to be of finite type.*

Proof. The first assertion follows from ?? by take $\mathcal{M} = \Gamma_*(\mathcal{F})$. For the second one, it suffices to prove that \mathcal{M} is the inductive limit of graded sub- \mathcal{S} -modules of finite type \mathcal{N}_λ : in fact, it then follows that $\tilde{\mathcal{M}}$ is the inductive limit of the $\tilde{\mathcal{N}}_\lambda$ (??), hence \mathcal{F} is the inductive limit of the $\beta(\mathcal{N}_\lambda)$. As X is quasi-compact (??) and \mathcal{F} is of finite type, \mathcal{F} then necessarily equal to one of the $\beta(\tilde{\mathcal{N}}_\lambda)$ (??).

To define the \mathcal{N}_λ , it suffices to consider for each $n \in \mathbb{Z}$ the quasi-coherent \mathcal{O}_Y -module \mathcal{M}_n , which is the inductive limit of its sub- \mathcal{O}_Y -modules $\mathcal{M}_n^{(\mu_n)}$ of finite type (by ??). It is immediate that $\mathcal{P}_{\mu_n} = \mathcal{S} \cdot \mathcal{M}_n^{(\mu_n)}$ is a graded \mathcal{S}_n -module of finite type, and \mathcal{M} is then the inductive limit of finite direct sums of these \mathcal{S} -modules. \square

Corollary 19.3.25. *Suppose the hypotheses of ?? and that Y is quasi-compact. Let \mathcal{F} be a quasi-coherent \mathcal{O}_X -module of finite type. There exists an integer n_0 such that for $n \geq n_0$, the canonical homomorphism $\sigma : p^*(p_*(\mathcal{F}(n))) \rightarrow \mathcal{F}(n)$ is surjective.*

Proof. For any $y \in Y$, let U be an affine open neighborhood of y in Y . There then exists an integer $n_0(U)$ such that, for $n \geq n_0(U)$, $\mathcal{F}(n)|_{p^{-1}(U)}$ is generated by finitely many sections over $p^{-1}(U)$ (??); but these are canonical images of sections of $p^*(p_*(\mathcal{F}(n)))$ over $p^{-1}(U)$, so $\mathcal{F}(n)|_{p^{-1}(U)}$ is equal to the canonical image of $p^*(p_*(\mathcal{F}(n)))|_{p^{-1}(U)}$. Finally, as Y is quasi-compact, there is a finite affine open cover (U_i) of Y , and we can choose n_0 to be the largest of the $n_0(U_i)$. \square

Remark 19.3.3. If $p : X \rightarrow Y$ is a morphism of ringed spaces and \mathcal{F} is an \mathcal{O}_X -module, the fact that the canonical homomorphism $\sigma : p^*(p_*(\mathcal{F})) \rightarrow \mathcal{F}$ is surjective is explained as follows: for any $x \in X$ and any sections of \mathcal{F} over an open neighborhood V of x , there exists an open neighborhood U of $p(x)$ in Y , finitely many sections $(t_i)_{1 \leq i \leq m}$ of \mathcal{F} over $p^{-1}(U)$, a neighborhood $W \subseteq V \cap p^{-1}(U)$ of x and sections $(a_i)_{1 \leq i \leq m}$ of \mathcal{O}_X over W such that

$$s|_W = \sum_i a_i \cdot (t_i|_W).$$

If Y is an affine scheme and $p_*(\mathcal{F})$ is *quasi-coherent*, this condition is equivalent to the fact that \mathcal{F} is generated by its sections over X : in fact, if $Y = \text{Spec}(A)$, we can suppose that $U = D(f)$ with $f \in A$. Since $p_*(\mathcal{F})$ is quasi-coherent, by ?? there exists an integer $n > 0$ and sections s_i of \mathcal{F} over X such that $g^n t_i$ is the restriction of s_i (where $g = \rho(f)$, and $\rho : A \rightarrow \Gamma(X, \mathcal{O}_X)$ is the homomorphism corresponding to p by ??) to $p^{-1}(U)$. As g is invertible over $p^{-1}(U)$, we then have

$$s|_W = \sum_i b_i \cdot (s_i|_W)$$

where $b_i = a_i \cdot (g|_W)^{-n}$, whence our assertion. Therefore, if Y is affine, ?? then recovers ??, in view of ??.

We finally conclude that if Y is a scheme, then the following three conditions are equivalent for a quasi-coherent \mathcal{O}_X -module \mathcal{F} such that $p_*(\mathcal{F})$ is quasi-coherent:

- (i) The canonical homomorphism $\sigma : p^*(p_*(\mathcal{F})) \rightarrow \mathcal{F}$ is surjective.
- (ii) There is a quasi-coherent \mathcal{O}_Y -module \mathcal{G} and a surjective homomorphism $p^*(\mathcal{G}) \rightarrow \mathcal{F}$.
- (iii) For any affine open U of Y , $\mathcal{F}|_{p^{-1}(U)}$ is generated by its sections over $p^{-1}(U)$.

We have already established the equivalence of (i) and (iii), and (i) clearly implies (ii). conversely, any homomorphism $u : p^*(\mathcal{G}) \rightarrow \mathcal{F}$ factors into $p^*(\mathcal{G}) \rightarrow p^*(p_*(\mathcal{F})) \xrightarrow{\sigma} \mathcal{F}$ by ??, so if u is surjective, so is the canonical homomorphism $\sigma : p^*(p_*(\mathcal{F})) \rightarrow \mathcal{F}$.

Corollary 19.3.26. Suppose the hypotheses of ?? and that Y is quasi-compact. Let \mathcal{F} be a quasi-coherent \mathcal{O}_X -module of finite type. There then exists an integer n_0 such that for $n \geq n_0$, \mathcal{F} is isomorphic to a quotient of an \mathcal{O}_X -module of the form $(p^*(\mathcal{G}))(-n)$, where \mathcal{G} is a quasi-coherent \mathcal{O}_Y -module of finite type (dependent on n).

Proof. As the structural morphism $X \rightarrow Y$ is separated and of finite type, $p_*(\mathcal{F}(n))$ is quasi-coherent by ??, and so is the inductive limits of its sub- \mathcal{O}_Y -modules of finite type, in view of ???. Since p^* commutes with inductive limits, we deduce from ?? and ?? that $\mathcal{F}(n)$ is the canonical image under $\sigma_{\mathcal{F}(n)}$ of an \mathcal{O}_X -module of the form $p^*(\mathcal{G})$, where \mathcal{G} is a quasi-coherent sub- \mathcal{O}_Y -module of $p_*(\mathcal{F}(n))$ of finite type. The corollary then follows from ?? and ???. \square

19.3.4 Functorial properties of $\text{Proj}(\mathcal{S})$

Let Y be a scheme and $\mathcal{S}, \mathcal{S}'$ be two quasi-coherent graded \mathcal{O}_Y -algebras with positive degrees. Let $X = \text{Proj}(\mathcal{S})$, $X' = \text{Proj}(\mathcal{S}')$, and p, p' be the structural morphisms of X and X' , respectively. Let $\varphi : \mathcal{S}' \rightarrow \mathcal{S}$ be an \mathcal{O}_Y -homomorphism of graded algebras. For any affine open U of Y , let $S_U = \Gamma(U, \mathcal{S})$, $S'_U = \Gamma(U, \mathcal{S}')$; the homomorphism φ defines a homomorphism

$\varphi_U : S'_U \rightarrow S_U$ of graded A_U -algebras, where $A_U = \Gamma(U, \mathcal{O}_Y)$. It then corresponds to an open subset $G(\varphi_U)$ of $p^{-1}(U)$ and a morphism $\Phi_U : G(\varphi_U) \rightarrow p'^{-1}(U)$. Moreover, if $V \subseteq U$ is another affine open subset, the diagram

$$\begin{array}{ccc} S'_U & \xrightarrow{\varphi_U} & S_U \\ \downarrow & & \downarrow \\ S'_V & \xrightarrow{\varphi_V} & S_V \end{array} \quad (19.3.5)$$

is commutative, and we also verify, by the definition of $G(\varphi_U)$, that

$$G(\varphi_V) = G(\varphi_U) \cap p^{-1}(V)$$

and that Φ_V is the restriction of Φ_U to $G(\varphi_V)$. We thus define an open subset $G(\varphi)$ of X such that $G(\varphi) \cap p^{-1}(U) = G(\varphi_U)$ for any affine open $U \subseteq Y$, and an affine Y -morphism $\Phi : G(\varphi) \rightarrow X'$, which is called the morphism associated with φ and denoted by $\text{Proj}(\varphi)$. If for any $y \in Y$, there exists an affine open neighborhood U of y such that $\Gamma(U, \mathcal{O}_Y)$ -module $\Gamma(U, \mathcal{S}_+)$ is generated by $\varphi(\Gamma(U, \mathcal{S}'_+))$, we then have $G(\varphi_U) = p^{-1}(U)$, and thus $G(\varphi) = X$.

Proposition 19.3.27. *Let \mathcal{M} (resp. \mathcal{M}') be a quasi-coherent graded \mathcal{S} -module (resp. \mathcal{S}' -module). Then there exist a canonical isomorphism $\widetilde{\varphi^*(\mathcal{M})} \xrightarrow{\sim} \Phi_*(\widetilde{\mathcal{M}}|_{G(\varphi)})$ of $\mathcal{O}_{X'}$ -modules and a canonical homomorphism $v : \Phi^*(\widetilde{\mathcal{M}'}) \rightarrow \widetilde{\varphi_*(\mathcal{M}')|_{G(\varphi)}}$. If \mathcal{S}' is generated by \mathcal{S}'_1 , v is an isomorphism.*

Proof. The homomorphisms considered are in fact already defined locally over Y (see ?? and ??), and the general case then follows from their compatibility with restrictions, and diagram (??). \square

In particular, for any $n \in \mathbb{Z}$, we have a canonical homomorphism $\Phi^*(\mathcal{O}_{X'}(n)) \rightarrow \mathcal{O}_X(n)|_{G(\varphi)}$, and this is a homomorphism if \mathcal{S}' is generated by \mathcal{S}'_1 .

Proposition 19.3.28. *Let Y, Y' be schemes, $\psi : Y' \rightarrow Y$ be a morphism, \mathcal{S} be a quasi-coherent \mathcal{O}_Y -algebra, and put $\mathcal{S}' = \psi^*(\mathcal{S})$. Then the Y' -scheme $X' = \text{Proj}(\mathcal{S}')$ is canonically identified with $\text{Proj}(\mathcal{S}) \times_Y Y'$. Moreover, if \mathcal{M} is a quasi-coherent graded \mathcal{S} -module, the $\mathcal{O}_{X'}$ -module $\widetilde{\psi^*(\mathcal{M})}$ is identified with $\widetilde{\mathcal{M}} \otimes_Y \mathcal{O}_{Y'}$.*

Proof. We first note that $\psi^*(\mathcal{S})$ and $\psi^*(\mathcal{M})$ are quasi-coherent $\mathcal{O}_{Y'}$ -modules. Let U be an affine open of Y , $U' \subseteq \psi^{-1}(U)$ an affine open of Y' , and A, A' the ring of U, U' , respectively. We then have $\mathcal{S}|_U = \widetilde{S}$ where S is a graded A -algebra, and $\mathcal{S}'|_{U'} = \widetilde{S \otimes_A A'}$ by ???. The first assertion then follows from ?? and ??, since we can easily verify that the projection $\text{Proj}(\mathcal{S}'|_{U'}) \rightarrow \text{Proj}(\mathcal{S}|_U)$ defined by this identification is compatible with restrictions over U and U' and therefore define a morphism $\text{Proj}(\mathcal{S}') \rightarrow \text{Proj}(\mathcal{S})$. Now let $p : \text{Proj}(\mathcal{S}) \rightarrow Y$ and $p' : \text{Proj}(\mathcal{S}') \rightarrow Y'$ be the structural morphisms; $p'^{-1}(U')$ is identified with $p^{-1}(U) \times_U U'$, and the two sheaves $\widetilde{\psi^*(\mathcal{M})}|_{p'^{-1}(U')}$ and $\widetilde{\mathcal{M}} \otimes_Y \mathcal{O}_{Y'}|_{p'^{-1}(U')}$ are then canonically identified to $\widetilde{M \otimes_A A'}$, where $M = \Gamma(U, \mathcal{M})$, in view of ?? and ??; whence the second assertion, since these identifications are compatible with restrictions. \square

Corollary 19.3.29. *With the notations of ??, $\mathcal{O}_{X'}(n)$ is canonically identified with $\mathcal{O}_X(n) \otimes_Y \mathcal{O}_{Y'}$ for any $n \in \mathbb{Z}$.*

Proof. With the notations of ??, it is clear that $\psi^*(\mathcal{S}(n)) = \mathcal{S}'(n)$ for any $n \in \mathbb{Z}$, whence the corollary. \square

Retain the notations in ??, denote by $\Psi : X' \rightarrow X$ the canonical projection, and put $\mathcal{M}' = \psi^*(\mathcal{M})$. We suppose that \mathcal{S} is generated by \mathcal{S}_1 and that X is of finite type over Y (for example if \mathcal{S}_1 is of finite type, cf. ??). Then \mathcal{S}' is generated by \mathcal{S}'_1 (as can be checked locally on affine opens of Y) and X' is of finite type over Y by ???. Let \mathcal{F} be an \mathcal{O}_X -module and set $\mathcal{F}' = \Psi^*(\mathcal{F})$; it then follows from ?? that we have $\mathcal{F}'(n) = \Psi^*(\mathcal{F})$ for each $n \in \mathbb{Z}$. We define a canonical Ψ -homomorphism $\theta_n : p_*(\mathcal{F}(n)) \rightarrow p'_*(\mathcal{F}'(n))$ as follows: from the commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{\Psi} & X \\ p' \downarrow & & \downarrow p \\ Y' & \xrightarrow{\psi} & Y \end{array}$$

we see that it suffices to define a homomorphism

$$p_*(\mathcal{F}(n)) \rightarrow \psi_*(q'_*(\Psi^*(\mathcal{F}(n)))) = q_*(\Psi_*(\Psi^*(\mathcal{F}(n)))),$$

and we can take $\theta_n = p_*(\rho_n)$, where ρ_n is the canonical homomorphism $\rho_n : \mathcal{F}(n) \rightarrow \Psi_*(\Psi^*(\mathcal{F}(n)))$. It is immediate that for any affine open U of Y and any affine open U' of Y such that $U' \subseteq \psi^{-1}(U)$, the homomorphism θ_n thus defined gives a canonical homomorphism $\Gamma(p^{-1}(U), \mathcal{F}(n)) \rightarrow \Gamma(p'^{-1}(U'), \mathcal{F}'(n))$, and the commutative diagram ?? shows that if \mathcal{F} is quasi-coherent, the diagram

$$\begin{array}{ccc} \widetilde{\Gamma_*(\mathcal{F})} & \xrightarrow{\tilde{\theta}} & \widetilde{\Gamma_*(\mathcal{F}')} \\ \beta_{\mathcal{F}} \downarrow & & \downarrow \beta_{\mathcal{F}'} \\ \mathcal{F} & \xrightarrow{\rho} & \mathcal{F}' \end{array}$$

is commutative (where the send row is the canonical Ψ -morphism $\mathcal{F} \rightarrow \Phi^*(\mathcal{F})$).

Similarly, the commutative diagram ?? shows that the diagram

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\rho} & \mathcal{M}' \\ \alpha_{\mathcal{M}} \downarrow & & \downarrow \alpha_{\mathcal{M}'} \\ \Gamma_*(\widetilde{\mathcal{M}}) & \xrightarrow{\theta} & \Gamma_*(\widetilde{\mathcal{M}}') \end{array}$$

is commutative (where the first row is the canonical ψ -morphism $\mathcal{M} \rightarrow \psi^*(\mathcal{M})$).

Consider now a morphism $\psi : Y' \rightarrow Y$ of schemes, a quasi-coherent graded \mathcal{O}_Y -algebra (resp. $\mathcal{O}_{Y'}$ -algebra) \mathcal{S} (resp. \mathcal{S}'), and a ψ -morphism $u : \mathcal{S} \rightarrow \mathcal{S}'$ of graded algebras. This is equivalent to giving an $\mathcal{O}_{Y'}$ -homomorphism of graded algebras $u^\sharp : \psi_*(\mathcal{S}) \rightarrow \mathcal{S}'$, and we deduce from u^\sharp an Y' -morphism

$$w = \text{Proj}(u^\sharp) : G(u^\sharp) \rightarrow \text{Proj}(\psi^*(\mathcal{S})),$$

where $G(u^\sharp)$ is an open subset of $X' = \text{Proj}(\mathcal{S}')$. On the other hand, $\text{Proj}(\psi^*(\mathcal{S}))$ is canonically

identified with $X \times_Y Y'$, where $X = \text{Proj}(\mathcal{S})$ (??). By composing the morphism $\text{Proj}(u^\sharp)$ with the first projection $\pi : X \times_Y Y' \rightarrow X$, we then obtain a morphism $v = \text{Proj}(u) : G(u^\sharp) \rightarrow X$ such that the diagram

$$\begin{array}{ccc} G(u^\sharp) & \xrightarrow{v} & X \\ \downarrow & & \downarrow \\ Y' & \xrightarrow{\psi} & Y \end{array} \quad (19.3.6)$$

is commutative. Moreover, for any quasi-coherent \mathcal{O}_Y -module \mathcal{M} , we have a canonical v -morphism

$$v : \widetilde{\mathcal{M}} \rightarrow (\psi^*(\mathcal{M}) \otimes_{\psi^*(\mathcal{S})} \mathcal{S}')^\sim|_{G(u^\sharp)} \quad (19.3.7)$$

such that v^\sharp is the composition

$$v^*(\widetilde{\mathcal{M}}) = w^*(\pi^*(\widetilde{\mathcal{M}})) \xrightarrow{\sim} w^*(\widetilde{\psi^*(\mathcal{M})}) \xrightarrow{v} (\psi^*(\mathcal{M}) \otimes_{\psi^*(\mathcal{M})} \mathcal{S}')^\sim|_{G(u^\sharp)} \quad (19.3.8)$$

where the first arrow is the isomorphism in ?? and the second one is the homomorphism v of ???. If \mathcal{S} is generated by \mathcal{S}_1 , then it follows from ?? that v^\sharp is an isomorphism. As a particular case, for any $n \in \mathbb{Z}$ we have a canonical v -homomorphism

$$v : \mathcal{O}_X(n) \rightarrow \mathcal{O}_{X'}(n)|_{G(u^\sharp)}. \quad (19.3.9)$$

19.3.5 Closed subschemes of $\text{Proj}(\mathcal{S})$

Let Y be a scheme and $\varphi : \mathcal{S} \rightarrow \mathcal{S}'$ be a homomorphism of quasi-coherent graded \mathcal{O}_Y -algebras of degree 0. We say that φ is **eventually injective** (resp. **eventually surjective**, **eventually bijective**) if there exists an integer n such that, for $k \geq n$, $\varphi_k : \mathcal{S}_k \rightarrow \mathcal{S}'_k$ is surjective (resp. injective, bijective). If this is the case, we can then reduce the study of the morphism $\Phi : \text{Proj}(\mathcal{S}') \rightarrow \text{Proj}(\mathcal{S})$ to the case where φ is surjective (resp. injective, bijective) (this follows from ??). If φ is eventually bijective, we also say that φ is an **eventual isomorphism**.

Proposition 19.3.30. *Let Y be a scheme, \mathcal{S} be a quasi-coherent \mathcal{O}_Y -algebra, and $X = \text{Proj}(\mathcal{S})$.*

- (a) *If $\varphi : \mathcal{S} \rightarrow \mathcal{S}'$ is a eventually surjective homomorphism of graded \mathcal{O}_Y -algebra, the corresponding morphism $\Phi = \text{Proj}(\varphi)$ is defined over $\text{Proj}(\mathcal{S}')$ and is a closed immersion from $\text{Proj}(\mathcal{S}')$ into X . If \mathcal{J} is the kernel of φ , the closed subscheme of X associated with Φ is defined by the quasi-coherent ideal $\widetilde{\mathcal{J}}$ of \mathcal{O}_X .*
- (b) *Suppose moreover that $\mathcal{S}_0 = \mathcal{O}_Y$ and \mathcal{S} is generated by \mathcal{S}_1 where \mathcal{S}_1 is of finite type. Let X' be a closed subscheme of X defined by a quasi-coherent ideal \mathcal{J} of \mathcal{O}_X . Let \mathcal{I} be the inverse image of $\Gamma_*(\mathcal{J})$ under the canonical homomorphism $\alpha : \mathcal{S} \rightarrow \Gamma_*(\mathcal{O}_X)$ and put $\mathcal{S}' = \mathcal{S}/\mathcal{I}$. Then X' is the subscheme associated with the closed immersion $\text{Proj}(\mathcal{S}') \rightarrow X$ corresponding to the canonical homomorphism $\mathcal{S} \rightarrow \mathcal{S}'$ of graded \mathcal{O}_Y -algebras.*

Proof. For the assertion of (a), we can assume that φ is surjective. Then for any affine open U of Y , $\Gamma(U, \mathcal{S}) \rightarrow \Gamma(U, \mathcal{S}')$ is surjective by ??, so we have $G(\varphi) = X$. We are immediately reduced to the case where Y is affine, and the assertion follows from ??(a).

As for case (b), we are reduced to prove that the homomorphism $\tilde{\mathcal{J}} \rightarrow \mathcal{O}_X$ induced from the canonical injection $\mathcal{J} \rightarrow \mathcal{S}$ is an isomorphism from $\tilde{\mathcal{J}}$ to \mathcal{J} . As the question is local, we can assume that Y is affine with ring A , which implies $\mathcal{S} = \tilde{S}$, where S is a graded A -algebra generated by S_1 and S_1 is of finite type over A . It then suffices to apply ??(b). \square

Corollary 19.3.31. *Under the hypotheses of ??(a), suppose that \mathcal{S} is generated by \mathcal{S}_1 . Then $\Phi^*(\mathcal{O}_X(n))$ is canonically identified with $\mathcal{O}_{X'}(n)$ for any $n \in \mathbb{Z}$.*

Proof. We have defined such an isomorphism if Y is affine; in the general case, it suffices to verify that these isomorphisms are compatible with restrictions. \square

Corollary 19.3.32. *Let Y be a scheme, \mathcal{S} be a quasi-coherent \mathcal{O}_Y -algebra generated by \mathcal{S}_1 , \mathcal{M} be a quasi-coherent \mathcal{O}_Y -module, and $u : \mathcal{M} \rightarrow \mathcal{S}_1$ be a surjective \mathcal{O}_Y -homomorphism. If $\bar{u} : S(\mathcal{M}) \rightarrow \mathcal{S}$ is the canonical homomorphism of \mathcal{O}_Y -algebras extending u , then the morphism corresponding to \bar{u} is a closed immersion from $\text{Proj}(\mathcal{S})$ into $\text{Proj}(S(\mathcal{M}))$.*

Proof. In fact, \bar{u} is surjective by hypothesis, so we can apply ??(a). \square

19.3.6 Morphisms into $\text{Proj}(\mathcal{S})$

Let $q : X \rightarrow Y$ be a morphism of schemes, \mathcal{L} be an invertible \mathcal{O}_X -module, and \mathcal{S} be a quasi-coherent \mathcal{O}_Y -algebra with positive degrees. Then $q^*(\mathcal{S})$ is a quasi-coherent graded \mathcal{O}_X -algebra with positive degrees. Suppose that we are given a graded homomorphism of \mathcal{O}_X -algebras

$$\psi : q^*(\mathcal{S}) \rightarrow S(\mathcal{L}) = \bigoplus_{n \geq 0} \mathcal{L}^{\otimes n}$$

(recall that for an invertible \mathcal{O}_X -module \mathcal{L} we have $T(\mathcal{L}) = S(\mathcal{L})$) or equivalently, a \mathcal{O}_Y -homomorphism of graded algebras

$$\psi^\flat : \mathcal{S} \rightarrow q_*(S(\mathcal{L})).$$

Since \mathcal{L} is invertible and $T(\mathcal{O}_X) = S(\mathcal{O}_X) = \mathcal{O}_X[T]$, by ?? and ?? we know that $\text{Proj}(S(\mathcal{L}))$ is canonically identified with X . We then conclude that the homomorphism ψ induces a Y -morphism

$$r_{\mathcal{L}, \psi} : G(\psi) \rightarrow \text{Proj}(\mathcal{S}) = P,$$

where $G(\psi)$ is an open subset of X . Recall that this morphism is by definition obtained by composing the first projection $\pi : \text{Proj}(q^*(\mathcal{S})) = P \times_Y X \rightarrow P$ with the Y -morphism $\tau = \text{Proj}(\psi) : G(\psi) \rightarrow \text{Proj}(q^*(\mathcal{S}))$, which is shown in the following diagram:

$$\begin{array}{ccccc}
& & r_{\mathcal{L}, \psi} & & \\
& \swarrow & \curvearrowright & \searrow & \\
P \times_Y X & \xrightarrow{\pi} & P & & \\
\downarrow & \nearrow \tau = \text{Proj}(\psi) & & & \downarrow p \\
G(\psi) & \xrightarrow{q} & X & \xrightarrow{q} & Y
\end{array}$$

Remark 19.3.4. Let us explain the morphism $r = r_{\mathcal{L}, \psi}$ when $Y = \text{Spec}(A)$ is affine, and therefore $\mathcal{S} = \widetilde{S}$, where S is a graded A -algebra with positive degrees. First suppose that $X = \text{Spec}(B)$ is also affine and $\mathcal{L} = \widetilde{L}$, where L is a free B -module of rank 1. We then have $q^*(\mathcal{S}) = \widetilde{S \otimes_A B}$ by [??](#). If c is a generator of L , the homomorphism $\psi_n : q^*(\mathcal{S}_n) \rightarrow \mathcal{L}^{\otimes n}$ then corresponds to a B -homomorphism

$$w_n : S_n \otimes_A B \rightarrow L^{\otimes n}, \quad s \otimes b \mapsto bv_n(s)c^{\otimes n}, \quad (19.3.10)$$

where $v_n : S_n \rightarrow B$ is the n -th component of a homomorphism $v : S \rightarrow B$ of algebras. Let $f \in S_d$ with positive degree and put $g = v_d(f)$. We have $\pi^{-1}(D_+(f)) = D_+(f \otimes 1)$ in view of [??](#) and the identification of $D_+(f)$ with $\text{Spec}(S_{(f)})$. On the other hand, the formula [\(??\)](#) and [\(??\)](#) shows that (using the canonical isomorphism of X and $\text{Proj}(S(\mathcal{L}))$)

$$\tau^{-1}(D_+(f \otimes 1)) = D(g)$$

whence $r^{-1}(D_+(f)) = D(g)$. Furthermore, the restriction of the morphism $\tau = \text{Proj}(\psi)$ to $D(g)$ corresponds to the homomorphism $(S \otimes_A B)_{(f \otimes 1)} \rightarrow B_g$, which sends $(s \otimes 1)/(f \otimes 1)^n$ (for $s \in S_{nd}$) to $v_{nd}(s)/g^n$, and the restriction of the projection π to $D_+(f \otimes 1)$ corresponds to the homomorphism $S_{(f)} \rightarrow (S \otimes_A B)_{(f \otimes 1)}$ given by $s/f^n \mapsto (s \otimes 1)/(f \otimes 1)^n$. We then conclude that the restriction of the morphism r to $D(g)$ corresponds to the homomorphisms $\omega : S_{(f)} \rightarrow B_g$ of A -algebras such that $\omega(s/f^n) = v_{nd}(s)/g^n$ for $s \in S_{nd}$.

Proposition 19.3.33. *Let $Y = \text{Spec}(A)$ be affine and $\mathcal{S} = \widetilde{S}$, where S is a graded A -algebra. For any $f \in S_d = \Gamma(Y, \mathcal{S}_d)$, we have (where $\psi^\flat(f) \in \Gamma(X, \mathcal{L}^{\otimes d})$)*

$$r_{\mathcal{L}, \psi}^{-1}(D_+(f)) = X_{\psi^\flat(f)}. \quad (19.3.11)$$

Moreover, under the canonical isomorphism of X and $\text{Proj}(S(\mathcal{L}))$, the restriction morphism $r_{\mathcal{L}, \psi} : X_{\psi^\flat(f)} \rightarrow D_+(f) = \text{Spec}(S_{(f)})$ corresponds to the homomorphism

$$\psi_{(f)}^\flat : S_{(f)} \rightarrow \Gamma(X_{\psi^\flat(f)}, \mathcal{O}_X)$$

such that, for any $s \in S_{nd} = \Gamma(Y, \mathcal{S}_{nd})$, we have

$$\psi_{(f)}^\flat(s/f^n) = (\psi^\flat(s)|_{X_{\psi^\flat(f)}})(\psi^\flat(f)|_{X_{\psi^\flat(f)}})^{-n}.$$

Proof. This follows from [??](#) by passing to the general case. □

We say the morphism $r_{\mathcal{L}, \psi}$ is **everywhere defined** if $G(\psi) = X$. For this to be the case, it is necessary and sufficient that $G(\psi) \cap q^{-1}(U) = q^{-1}(U)$ for any affine open $U \subseteq Y$, so this question is local over Y . If Y is affine, $G(\psi)$ is then the union of $r^{-1}(D_+(f))$ for $f \in S_+$, so by [\(??\)](#) the $X_{\psi^\flat(f)}$ then form a covering of X . In other words:

Corollary 19.3.34. *Under the hypotheses of [??](#), for the morphism $r_{\mathcal{L}, \psi}$ to be everywhere defined, it is necessary and sufficient that for any $x \in X$, there exists an integer $n > 0$ and a section $s \in S_n$ such that $t = \psi^\flat(s) \in \Gamma(X, \mathcal{L}^{\otimes n})$ is nonzero at x . In particular, this is true if ψ is eventually surjective.*

Corollary 19.3.35. *Under the hypotheses of ??, for the morphism $r_{\mathcal{L},\psi}$ to be dominant, it is necessary and sufficient that for any integer $n > 0$, any section $s \in S_n$ such that $\psi^\flat(s) \in \Gamma(X, \mathcal{L}^{\otimes n})$ is locally nilpotent, is itself nilpotent.*

Proof. We must check that $r_{\mathcal{L},\psi}^{-1}(D_+(s))$ is nonempty if $D_+(s)$ is nonempty, and the corollary follows from (??) and ??.

Proposition 19.3.36. *Let $q : X \rightarrow Y$ be a morphism, \mathcal{L} be an invertible \mathcal{O}_X -module, and $\mathcal{S}, \mathcal{S}'$ be quasi-coherent graded \mathcal{O}_Y -algebras. Let $u : \mathcal{S}' \rightarrow \mathcal{S}$ be a homomorphism of graded algebras, $\psi : q^*(\mathcal{S}) \rightarrow S(\mathcal{L})$ be a homomorphism of graded algebras, and $\psi' = \psi \circ q^*(u)$ be the composition.*

- (i) *If $r_{\mathcal{L},\psi'}$ is everywhere defined, then $r_{\mathcal{L},\psi}$ is everywhere defined;*
- (ii) *If u is eventually surjective and $r_{\mathcal{L},\psi'}$ is dominant, then $r_{\mathcal{L},\psi}$ is dominant;*
- (iii) *If u is eventually injective and $r_{\mathcal{L},\psi'}$ is dominant, then $r_{\mathcal{L},\psi}$ is dominant.*

Proof. We have $G(\psi') \subseteq G(\psi)$, whence the first assertion. If u is eventually surjective, $\text{Proj}(u) : \text{Proj}(S) \rightarrow \text{Proj}(S')$ is everywhere defined and is a closed immersion; as $r_{\mathcal{L},\psi'}$ is the composition of $\text{Proj}(u)$ and the restriction of $r_{\mathcal{L},\psi}$ to $G(\psi')$, we then conclude that if $r_{\mathcal{L},\psi'}$ is dominant, so is $r_{\mathcal{L},\psi}$. Finally, if u is eventually injective, then $\text{Proj}(u)$ is a dominant morphism from $G(u)$ into $\text{Proj}(\mathcal{S}')$ (??); as $G(\psi')$ is the inverse image of $G(u)$ under $r_{\mathcal{L},\psi}$, we see that if $r_{\mathcal{L},\psi'}$ is dominant, so is $r_{\mathcal{L},\psi}$. \square

Proposition 19.3.37. *Let Y be a quasi-compact scheme, $q : X \rightarrow Y$ be a quasi-compact morphism, \mathcal{L} be an invertible \mathcal{O}_X -module, and \mathcal{S} be a quasi-coherent \mathcal{O}_Y -algebra which is the filtered limit of a system (\mathcal{S}^λ) of quasi-coherent \mathcal{O}_Y -algebras. Let $\varphi_\lambda : \mathcal{S}^\lambda \rightarrow \mathcal{S}$ be the canonical homomorphism, $\psi : q^*(\mathcal{S}) \rightarrow S(\mathcal{L})$ be a homomorphism of graded algebras, and put $\psi_\lambda = \psi \circ q^*(\varphi_\lambda)$. Then for the morphism $r_{\mathcal{L},\psi}$ to be everywhere defined, it is necessary and sufficient that there exists an index λ such that $r_{\mathcal{L},\psi_\lambda}$ is everywhere defined; in this case, $r_{\mathcal{L},\psi_\mu}$ is everywhere defined for $\mu \geq \lambda$.*

Proof. The condition is sufficient in view of ???. Conversely, suppose that $r_{\mathcal{L},\psi}$ is everywhere defined; we can assume that Y is affine, because if for any affine open $U \subseteq Y$ there exists $\lambda(U)$ such that the restriction of $r_{\mathcal{L},\psi_{\lambda(U)}}$ to $q^{-1}(U)$ is defined everywhere, it then suffices to cover Y by finitely many affine opens U_i (recall that Y is quasi-compact) and choose $\lambda \geq \lambda(U_i)$ for all i , by ???. If Y is affine (so $\mathcal{S} = \tilde{S}$ where $S = \Gamma(Y, \mathcal{S})$) the hypotheses implies that for any $x \in X$, there exists a section $s^{(x)} \in S_n$ for some integer n such that, if $t^{(x)} = \psi^\flat(s^{(x)})$, then $t^{(x)}(x) \neq 0$ (where $t^{(x)}$ is a section of $\mathcal{L}^{\otimes n}$ over X), which implies $t^{(x)}(z) \neq 0$ for z in a neighborhood $V(x)$ of X . As the morphism $q : X \rightarrow Y$ is quasi-compact, we see X is quasi-compact, so we can cover X by finitely many $V(x_i)$ and let $s^{(i)}$ be the corresponding section of S . There is then an index λ such that $s^{(i)}$ is of the form $\varphi_\lambda(s_\lambda^{(i)})$, where $s_\lambda^{(i)} \in S^\lambda$ for all i , and it follows from ?? that $r_{\mathcal{L},\psi_\lambda}$ is everywhere defined. The second assertion is trivial by ???. \square

Corollary 19.3.38. *Under the hypotheses of ??, if the morphisms $r_{\mathcal{L},\psi_\lambda}$ are dominant, so is $r_{\mathcal{L},\psi}$. The converse is also true if the homomorphisms φ_λ are eventually injective.*

Proof. The second assertion is a particular case of ???. To show that $r_{\mathcal{L},\psi}$ is dominant if each $r_{\mathcal{L},\psi_\lambda}$ is, we can assume that Y is affine and thus $\mathcal{S} = \tilde{S}$ where $S = \Gamma(Y, \mathcal{S})$. If $s \in S$ is such that $\psi^\flat(s)$ is locally nilpotent, as we can write $s = \varphi_\lambda(s_\lambda)$ for some λ , from the definition of ψ_λ and

by ??, we conclude that s_λ is nilpotent, so s is nilpotent, and the assertion follows by applying ??.

Remark 19.3.5. With the hypotheses and notations of ??, for each $n \in \mathbb{Z}$ we have a homomorphism

$$\nu : r_{\mathcal{L}, \psi}^*(\mathcal{O}_P(n)) \rightarrow \mathcal{L}^{\otimes n}|_{G(\psi)} \quad (19.3.12)$$

which is in fact the homomorphism ν defined in ?? on $\mathcal{O}_P(n)$. We also see that under the hypotheses of ??, the restriction of ν to $X_{\psi^\flat(f)}$ corresponds to the homomorphism sending the element s/f^k (with $s \in S_{n+ka}$) to the section

$$(\psi^\flat(s)|_{X_{\psi^\flat(f)}})(\psi^\flat(f)|_{X_{\psi^\flat(f)}})^{-k} \in \Gamma(X_{\psi^\flat(f)}, \mathcal{L}^{\otimes n}),$$

where we also use the notations of ??.

Remark 19.3.6. Let \mathcal{F} be a quasi-coherent \mathcal{O}_X -module and suppose that q is quasi-compact and quasi-separated, so for each $n \geq 0$, $q_*(\mathcal{F} \otimes \mathcal{L}^{\otimes n})$ is a quasi-coherent \mathcal{O}_Y -module ??). Let $\mathcal{M}' = \bigoplus_{n \geq 0} \mathcal{F} \otimes \mathcal{L}^{\otimes n}$, which is a quasi-coherent \mathcal{O}_Y -module, and consider the image $\mathcal{M} = q_*(\mathcal{M}') = \bigoplus_{n \geq 0} q_*(\mathcal{F} \otimes \mathcal{L}^{\otimes n})$ (which is a quasi-coherent \mathcal{S} -module via the homomorphism ψ^\flat). We shall see that there is a canonical homomorphism of \mathcal{O}_X -modules

$$\xi : r_{\mathcal{L}, \psi}^*(\tilde{\mathcal{M}}) \rightarrow \mathcal{F}|_{G(\psi)}. \quad (19.3.13)$$

For this, recall that we have defined a canonical homomorphism ??:

$$\nu : r_{\mathcal{L}, \psi}^*(\tilde{\mathcal{M}}) \rightarrow (q^*(\mathcal{M}) \otimes_{q^*(\mathcal{S})} \mathbf{S}(\mathcal{L}))^\sim|_{G(\psi)},$$

where the right hand side is considered as a quasi-coherent sheaf over X . On the other hand, for any quasi-coherent graded $\mathbf{S}(\mathcal{L})$ -module \mathcal{M}' , we have a canonical homomorphism

$$q^*(q_*(\mathcal{M}')) \otimes_{q^*(\mathcal{S})} \mathbf{S}(\mathcal{L}) \rightarrow \mathcal{M}'$$

which, for any open subset U of X , any section t' of $q^*(q_*(\mathcal{M}'_h))$ over U and any section b' of $\mathcal{L}^{\otimes k}$ over U , sends the section $t' \otimes b'$ to the section $b'\sigma(t')$ of \mathcal{M}'_{h+k} , where σ is the canonical homomorphism $q^*(q_*(\mathcal{M}')) \rightarrow \mathcal{M}'$. We then conclude a canonical homomorphism

$$(q^*(q_*(\mathcal{M}')) \otimes_{q^*(\mathcal{S})} \mathbf{S}(\mathcal{L}))^\sim|_{G(\psi)} \rightarrow \tilde{\mathcal{M}}'|_{G(\psi)}$$

and as $\tilde{\mathcal{M}}'$ is canonically identified with \mathcal{F} by ??, we obtain the canonical homomorphism ξ .

Under the hypotheses and notations of ??, the restriction of this homomorphism to $X_{\psi^\flat(f)}$ is defined as follows: giving a section t_{nd} of $\mathcal{F} \otimes \mathcal{L}^{\otimes d}$ over X (which is also a section of $q_*(\mathcal{F} \otimes \mathcal{L}^{\otimes n})$ over Y), we send the element t_{nd}/f^n to the sectin $(t_{nd}|_{X_{\psi^\flat(f)}})(\psi^\flat(f)|_{X_{\psi^\flat(f)}})^{-n}$ of \mathcal{F} over $X_{\psi^\flat(f)}$.

We now consider the important question that whether the induced morphism $r_{\mathcal{L}, \psi}$ is an immersion (reps. an open immersion, a closed immersion). It is clear that this question is local over Y , and we shall give a criterion in this situation together with the condition that $r_{\mathcal{L}, \psi}$ is defined everywhere.

Proposition 19.3.39. *Under the hypothesis and notations of ??, for the morphism $r_{\mathcal{L},\psi}$ be everywhere defined and an immersion, it is necessary and sufficient that there exists a family of sections $s_\alpha \in S_{n_\alpha}$ (with $n_\alpha > 0$) such that, if $f_\alpha = \psi^\flat(s_\alpha)$, the following conditions are satisfied:*

- (i) *The X_{f_α} form a covering of X .*
- (ii) *The X_{f_α} are affine open subset of X .*
- (iii) *For any index α and any section $t \in \Gamma(X_{f_\alpha}, \mathcal{O}_X)$, there exists an integer $n > 0$ and $s \in S_{mn_\alpha}$ such that $t = (\psi^\flat(s)|_{X_{f_\alpha}})(f_\alpha|_{X_{f_\alpha}})^{-m}$.*
- (iv) *For any integer $m > 0$ and any $s \in S_{mn_\alpha}$ such that $\psi^\flat(s)|_{X_{f_\alpha}} = 0$, there exists an integer $n > 0$ such that $s_\alpha^n s = 0$.*

For the morphism $r_{\mathcal{L},\psi}$ to be everywhere defined and an open immersion, it is necessary and sufficient that the family (s_α) satisfies the following additional condition:

- (v) *The $D_+(s_\alpha)$ form a covering of $P = \text{Proj}(S)$.*

Proof. By ??, for the morphism $r = r_{\mathcal{L},\psi}$ to be an immersion (resp. a closed immersion), it is necessary and sufficient that there exists a covering of $r(G(\psi))$ (resp. of P) by the sets $D_+(s_\alpha)$ such that if $V_\alpha = r^{-1}(D_+(s_\alpha))$, the restriction of r on V_α is a closed immersion of V_α into $D_+(s_\alpha)$ (cf. ??). Now condition (i) just means that r is everywhere defined and that $D_+(s_\alpha)$ cover $r(X)$, by (??). As each $D_+(s_\alpha)$ is affine, condition (ii) and (iii) express that the restriction of r to X_{f_α} is a closed immersion into $D_+(s_\alpha)$ (??). Finally, as (iii) and (iv) means the ring homomorphism $\psi_{(s_\alpha)}^\flat : S_{(s_\alpha)} \rightarrow \Gamma(X_{f_\alpha}, \mathcal{O}_X)$ is an isomorphism, (ii), (iii), (iv) mean that the restriction of r to X_{f_α} is an isomorphism from X_{f_α} to $D_+(s_\alpha)$ for each α , so together with (i), they mean that r is an open immersion. \square

Corollary 19.3.40. *Under the hypothesis and notations of ??, if $r_{\mathcal{L},\psi'}$ is everywhere defined and is an immersion, so is $r_{\mathcal{L},\psi}$. If we suppose that u is eventually surjective and if $r_{\mathcal{L},\psi'}$ is everywhere defined and is a closed immersion (resp. open), then so is $r_{\mathcal{L},\psi}$.*

Proof. We first suppose that $r_{\mathcal{L},\psi'}$ is everywhere defined and is an immersion. Then by ??, there is a family $s'_\alpha \in S'_{n_\alpha}$ such that, if $f_\alpha = \psi'^\flat(s'_\alpha)$, the conditions (i), (ii), (iii) are satisfied. Set $s_\alpha = u(s'_\alpha)$, then $f_\alpha = \psi^\flat(s_\alpha)$, and we have a commutative diagram

$$\begin{array}{ccc} S'_{(s'_\alpha)} & \xrightarrow{\psi'^\flat_{(s'_\alpha)}} & \Gamma(X_{f_\alpha}, \mathcal{O}_X) \\ u_{(s'_\alpha)} \downarrow & \nearrow \psi^\flat_{(s_\alpha)} & \\ S_{(s_\alpha)} & & \end{array}$$

The hypothesis then implies that $\psi'^\flat_{(s'_\alpha)}$ is surjective, so the homomorphism $\psi^\flat_{(s_\alpha)}$ is also surjective. This shows that $r_{\mathcal{L},\psi}$ is everywhere defined and is an immersion, in view of ??.

If $r_{\mathcal{L},\psi'}$

is moreover an open immersion, then the homomorphism $\psi'^b_{(s'_\alpha)}$ is also injective, and this implies the homomorphism $\psi^b_{(s_\alpha)}$ is injective if u is eventually surjective, since in this case the homomorphism $u_{(s'_\alpha)}$ is just surjective.

Finally, if $r_{\mathcal{L},\psi'}$ is a closed immersion, then condition (v) is satisfied for (s'_α) , and hence satisfied for (s_α) if u is eventually surjective (since in this case $\text{Proj}(u)$ is a closed immersion from $\text{Proj}(S)$ to $\text{Proj}(S')$); this implies $r_{\mathcal{L},\psi}$ is a closed immersion. \square

Proposition 19.3.41. *Suppose the hypotheses of ?? and moreover that $q : X \rightarrow Y$ is a morphism of finite type. Then, for the morphism $r_{\mathcal{L},\psi}$ to be everywhere defined and an immersion, it is necessary and sufficient that there exists an index λ such that $r_{\mathcal{L},\psi_\lambda}$ is everywhere defined and is an immersion; in this case, $r_{\mathcal{L},\psi_\lambda}$ is everywhere defined and an immersion for $\mu \geq \lambda$.*

Proof. By ??, it suffices to prove that if $r_{\mathcal{L},\psi}$ is defined everywhere and is an immersion, then there exists an index λ such that $r_{\mathcal{L},\psi_\lambda}$ is defined everywhere and is an immersion. By the same reasoning of ?? and using the quasi-compactness of Y , we are reduced to the case where $Y = \text{Spec}(A)$ is affine. As X is then quasi-compact, ?? shows that there exists a finite family (s_i) of elements of S ($s_i \in S_{n_i}$) satisfying the conditions (i), (ii), and (iii). The morphism $X_{f_i} \rightarrow Y$ (where $f_i = \psi^b(s_i)$) is of finite type since X_{f_i} is affine and the morphism $q : X \rightarrow Y$ is locally of finite type. The ring B_i of X_{f_i} is therefore an A -algebra of finite type by ??, and we choose (t_{ij}) to be a family of generators of this algebra. There are then elements $s_{ij} \in S_{m_{ij}n_i}$ such that

$$t_{ij} = (\psi^b(s_{ij})|_{X_{f_i}})(\psi^b(s_i)|_{X_{f_i}})^{-m_{ij}}$$

We can choose an index λ and elements $s_{i\lambda} \in S_{n_i}^\lambda$, $s_{ij\lambda} \in S_{m_{ij}n_i}^\lambda$ such that their images under φ_λ is s_i and s_{ij} , respectively. It is then clear that the family $(s_{i\lambda})$ satisfies the conditions (i), (ii), and (iii), so $r_{\mathcal{L},\psi}$ is everywhere defined and an immersion. \square

Proposition 19.3.42. *Let Y be a quasi-compact scheme, $q : X \rightarrow Y$ be a morphism of finite type, \mathcal{L} be an invertible \mathcal{O}_X -module, \mathcal{S} be a quasi-coherent graded \mathcal{O}_Y -algebra, and $\psi : q^*(\mathcal{S}) \rightarrow \mathcal{S}(\mathcal{L})$ be a homomorphism of graded algebras. For the morphism $r_{\mathcal{L},\psi}$ to be defined everywhere and an immersion, it is necessary and sufficient that there exists an integer $n > 0$ and a quasi-coherent sub- \mathcal{O}_Y -module \mathcal{E} of \mathcal{S}_n of finite type such that:*

- (a) *the homomorphism $\psi_n \circ q^*(j_n) : q^*(\mathcal{E}) \rightarrow \mathcal{L}^{\otimes n}$ (where $j_n : \mathcal{E} \rightarrow \mathcal{S}_n$ is the canonical injection) is surjective.*
- (b) *if \mathcal{S}' is the graded sub- \mathcal{O}_Y -algebra of \mathcal{S} generated by \mathcal{E} and ψ' is the homomorphism $\psi \circ q^*(j')$, where $j' : \mathcal{S}' \rightarrow \mathcal{S}$ is the canonical injection, $r_{\mathcal{L},\psi'}$ is everywhere defined and an immersion.*

If these are true, any quasi-coherent sub- \mathcal{O}_Y -module \mathcal{E}' of \mathcal{S}_n containing \mathcal{E} possesses the same properties, and so does the sub- \mathcal{O}_Y -module \mathcal{S}'_k of \mathcal{S}_{kn} for any $k > 0$.

Proof. The sufficiency of these conditions is a particular case of ?? in view of the isomorphism between $\text{Proj}(\mathcal{S})$ and $\text{Proj}(\mathcal{S}^{(d)})$ (??). We now prove the necessity, so let (U_i) be a finite affine open covering of Y and set $A_i = \Gamma(U_i, \mathcal{O}_Y)$. As $q^{-1}(U_i)$ is compact, the hypotheses that $r_{\mathcal{L},\psi}$ is an immersion defined on X implies by ?? the existence of a finite family (s_{ij}) of elements of $S^{(i)} = \Gamma(U_i, \mathcal{S})$ (where $s_{ij} \in S_{n_{ij}}^{(i)}$) satisfying conditions (i), (ii), and (iii). Since $q : X \rightarrow Y$ is of finite type, the restricted homomorphism $X_{f_{ij}} \rightarrow U_i$ is of finite type (where $f_{ij} = \psi^b(s_{ij})$), so

the ring B_{ij} of $X_{f_{ij}}$ is an A_i -algebra of finite type, and we choose $(\psi^\flat(t_{ijk})|_{X_{f_{ij}}})(f_{ij}|_{X_{f_{ij}}})^{-m_{ijk}}$ to be a system of generators of B_{ij} , where $t_{ijk} \in S_{m_{ijk}n_{ij}}^{(i)}$. Let n be a common multiple of all the $m_{ijk}n_{ij}$ and put $s'_{ij} = s_{ij}^{h_{ij}} \in S_n^{(i)}$, where $h_{ij} = n/n_{ij}$. For any given pair (i, j, k) , the element $t'_{ij} = s_{ij}^{h-m_{ijk}}t_{ijk}$ belongs to $S_n^{(i)}$, and it is clear that the $(\psi^\flat(t'_{ijk})|_{X_{f'_{ij}}})(f'_{ij}|_{X_{f'_{ij}}})^{-1}$ also generate B_{ij} (where $f'_{ij} = \psi^\flat(s'_{ij})$, and we note that $X_{f'_{ij}} = X_{f_{ij}}$). Let E_i be the sub- A_i -module of $S^{(i)}$ generated by these s'_{ij} and t'_{ijk} ; then there exists a quasi-coherent sub- \mathcal{O}_Y -module \mathcal{E}_i of \mathcal{S}_n of finite type such that $\mathcal{E}_i|_{U_i} = \tilde{E}_i$ (??). It is then clear that the sub- \mathcal{O}_Y -module \mathcal{E} of \mathcal{E} , which is the sum of the \mathcal{E}_i , satisfies the required properties. \square

Remark 19.3.7. The point of ?? is that, for a scheme X of finite type over a quasi-compact scheme Y , if X can be embedded into $\text{Proj}(\mathcal{S})$ via a morphism $r_{\mathcal{L}, \psi}$, then we can choose \mathcal{S} so that it is generated by \mathcal{S}_1 and \mathcal{S}_1 of finite type (we already know that in this case the twisted sheaves over $\text{Proj}(\mathcal{S})$ have nice properties).

19.4 Projective bundles and ample sheaves

19.4.1 Projective bundles

Let Y be a scheme, \mathcal{E} be a quasi-coherent \mathcal{O}_Y -module, and $S(\mathcal{E})$ be the symmetric \mathcal{O}_Y -algebra of \mathcal{E} , which is quasi-coherent by ???. The **projective bundle** over Y associated with \mathcal{E} is defined to be the Y -scheme $P = \text{Proj}(S(\mathcal{E}))$. The \mathcal{O}_P -module $\mathcal{O}_P(1)$ is called the **fundamental sheaf** of P .

If $\mathcal{E} = \mathcal{O}_Y^n$, we then denote by \mathbb{P}_Y^{n-1} instead of $\mathbb{P}(\mathcal{E})$; if moreover Y is affine with ring A , we then denote this scheme by \mathbb{P}_A^{n-1} . As $S(\mathcal{O}_Y)$ is canonically isomorphic to $\mathcal{O}_Y[T]$, we see \mathbb{P}_Y^0 is canonically identified with Y .

If $Y = \text{Spec}(A)$ and $\mathcal{E} = \tilde{E}$ where E is an A -module, we also denote by $\mathbb{P}(E)$ the projective bundle $\mathbb{P}(E)$. The simplest example is $\mathbb{P}(E)$ where E is a vector space over a field k . In this case, we see $\mathbb{P}(E)$ is isomorphic to \mathbb{P}_k^{n-1} , where n is the dimension of E .

Let \mathcal{E}, \mathcal{F} be two quasi-coherent \mathcal{O}_Y -modules and $u : \mathcal{E} \rightarrow \mathcal{F}$ be an \mathcal{O}_Y -homomorphism. Then u corresponds canonically to a homomorphism $S(u) : S(\mathcal{E}) \rightarrow S(\mathcal{F})$ of graded \mathcal{O}_Y -algebras. If u is surjective, so is $S(u)$, and therefore $\text{Proj}(S(u))$ is a closed immersion $\mathbb{P}(\mathcal{F}) \rightarrow \mathbb{P}(\mathcal{E})$, which we then denote by $\mathbb{P}(u)$. We can then say that $\mathbb{P}(\mathcal{E})$ is a contravariant functor on the category of quasi-coherent \mathcal{O}_Y -modules with *surjective homomorphisms*. Suppose that u is surjective and put $P = \mathbb{P}(\mathcal{E})$, $Q = \mathbb{P}(\mathcal{F})$, and $j = \mathbb{P}(u)$. We then have an isomorphism

$$j^*(\mathcal{O}_P(n)) = \mathcal{O}_Q(n)$$

by ??.

If $\psi : Y' \rightarrow Y$ is a morphism and $\mathcal{E}' = \psi^*(\mathcal{E})$, we then have $S_{\mathcal{O}_{Y'}}(\mathcal{E}') = \psi^*(S_{\mathcal{O}_Y}(\mathcal{E}))$ by ??, so from ?? we deduce that

$$\mathbb{P}(\psi^*(\mathcal{E})) = \mathbb{P}(\mathcal{E}) \times_Y Y'. \quad (19.4.1)$$

Moreover, it is clear that $\psi^*((S_{\mathcal{O}_Y}(\mathcal{E}))(n)) = (S_{\mathcal{O}_{Y'}}(\mathcal{E}'))(n)$ for each $n \in \mathbb{Z}$, so if $P = \mathbb{P}(\mathcal{E})$ and

$P' = \mathbb{P}(\mathcal{E}')$, we have a canonical isomorphism

$$\mathcal{O}_{P'}(n) = \mathcal{O}_P(n) \otimes_Y \mathcal{O}_{Y'}. \quad (19.4.2)$$

Proposition 19.4.1. *Let \mathcal{L} be an invertible \mathcal{O}_Y -module. For any quasi-coherent \mathcal{O}_Y -module, there exists a canonical Y -isomorphism $i_{\mathcal{L}} : Q = \mathbb{P}(\mathcal{E} \otimes \mathcal{L}) \xrightarrow{\sim} P = \mathbb{P}(\mathcal{E})$. Moreover, $(i_{\mathcal{L}})_*(\mathcal{O}_Q(n))$ is canonically isomorphic to $\mathcal{O}_P(n) \otimes_Y \mathcal{L}^{\otimes n}$ for any $n \in \mathbb{Z}$.*

Proof. If A is a ring, E is an A -module, L is an A -module free of rank 1, we can define a canonical homomorphism

$$S_n(E \otimes L) \rightarrow S_n(E) \otimes L^{\otimes n}$$

which maps an element $(x_1 \otimes y_1) \cdots (x_n \otimes y_n)$ to

$$(x_1 \cdots x_n) \otimes (y_1 \otimes \cdots \otimes y_n)$$

where $x_i \in E$ and $y_i \in L$. This is easily seen to be an isomorphism, so we get an isomorphism $S(E \otimes L) \cong \bigoplus_{n \geq 0} S_n(E) \otimes L^{\otimes n}$. In the situation of the proposition, the preceding remark allows us to define a canonical isomorphism of graded \mathcal{O}_Y -algebras

$$S(\mathcal{E} \otimes_{\mathcal{O}_Y} \mathcal{L}) \xrightarrow{\sim} \bigoplus_{n \geq 0} S_n(\mathcal{E}) \otimes_{\mathcal{O}_Y} \mathcal{L}^{\otimes n}$$

The proposition then follows from ?? and ??.

□

Let $P = \mathbb{P}(\mathcal{E})$ and denote by $p : P \rightarrow Y$ the structural morphism. As by definition $\mathcal{E} = (S(\mathcal{E}))_1$, we have a canonical homomorphism $\alpha_1 : \mathcal{E} \rightarrow p_*(\mathcal{O}_P(1))$, and therefore a canonical homomorphism

$$\alpha_1^\sharp : p^*(\mathcal{E}) \rightarrow \mathcal{O}_P(1).$$

Proposition 19.4.2. *The canonical homomorphism α_1^\sharp is surjective.*

Proof. We have seen that α_1^\sharp corresponds to the functorial homomorphism $\mathcal{E} \otimes_{\mathcal{O}_Y} S(\mathcal{E}) \rightarrow (S(\mathcal{E}))(1)$ (see the remark before ??). Since \mathcal{E} generates $S(\mathcal{E})$, this homomorphism is surjective, whence our assertion in view of ??.

□

19.4.2 Morphisms into $\mathbb{P}(\mathcal{E})$

With the notations of the last subsection, we now let X be an Y -scheme, $q : X \rightarrow Y$ be the structural morphism, and $r : X \rightarrow P$ be an Y -morphism, which gives a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{r} & P \\ & \searrow q & \downarrow p \\ & & Y \end{array}$$

As the functor r^* is right-exact, we deduce from the surjective homomorphism α_1^\sharp in ?? a surjective homomorphism

$$r^*(\alpha_1^\sharp) : r^*(p^*(\mathcal{E})) \rightarrow r^*(\mathcal{O}_P(1)).$$

But $r^*(p^*(\mathcal{E})) = q^*(\mathcal{E})$ and $r^*(\mathcal{O}_P(1))$ is locally isomorphic to $r^*(\mathcal{O}_P) = \mathcal{O}_X$, which is then an invertible sheaf \mathcal{L}_r over X , so we obtain a canonical surjective \mathcal{O}_X -homomorphism

$$\varphi_r : q^*(\mathcal{E}) \rightarrow \mathcal{L}_r.$$

If $Y = \text{Spec}(A)$ is affine and $\mathcal{E} = \tilde{E}$, we can explicitly explain this homomorphism: given $f \in E$, it follows from ?? that

$$r^{-1}(D_+(f)) = X_{\varphi_r(f)}.$$

Let V be an affine open of X contained in $r^{-1}(D_+(f))$, and let B be its ring, which is an A -algebra; put $S = S_A(E)$. The restriction of r to V then corresponds to an A -homomorphism $\omega : S_{(f)} \rightarrow B$, and we have $q^*(\mathcal{E})|_V = \widetilde{E \otimes_A B}$ and $\mathcal{L}_r|_V = \tilde{L}_r$, where by ??, $L_r = S(1)_{(f)} \otimes_{S_{(f)}} B$. In view of the definition of α_1 , the restriction of φ_r to $q^*(\mathcal{E})|_V$ therefore corresponds to the B -homomorphism

$$u : E \otimes_A B \rightarrow L_r, \quad x \otimes 1 \mapsto (x/1) \otimes 1 = (f/1) \otimes \omega(x/f)$$

The canonical extension of φ_r to a homomorphism of \mathcal{O}_X -algebras (recall that $(\mathcal{O}_P(1))^{\otimes n} = \mathcal{O}_P(n)$ by ??)

$$\psi_r : q^*(S(\mathcal{E})) = S(q^*(\mathcal{E})) \rightarrow S(\mathcal{L}_r) = \bigoplus_{n \geq 0} \mathcal{L}_r^{\otimes n} = \bigoplus_{n \geq 0} r^*(\mathcal{O}_P(n))$$

is then such that the restriction of ψ_r to $q^*(S_n(\mathcal{E}))|_V$ corresponds to the B -homomorphism

$$S_n(\mathcal{E} \otimes_A B) = S_n(E) \otimes_A B \rightarrow L_r^{\otimes n} = (S(1)_{(f)})^{\otimes n} \otimes_{S_{(f)}} B$$

which send the element $s \otimes 1$ to $(f/1)^{\otimes n} \otimes \omega(s/f^n)$.

Conversely, given an invertible \mathcal{O}_X -module \mathcal{L} and a quasi-coherent \mathcal{O}_Y -module \mathcal{E} , then any homomorphism $\varphi : q^*(\mathcal{E}) \rightarrow \mathcal{L}$ corresponds to a canonical homomorphism of quasi-coherent \mathcal{O}_X -algebras

$$\psi : S(q^*(\mathcal{E})) = q^*(S(\mathcal{E})) \rightarrow S(\mathcal{L}) = \bigoplus_{n \geq 0} \mathcal{L}^{\otimes n}$$

which then gives an Y -morphism $r_{\mathcal{L}, \psi} : G(\psi) \rightarrow \text{Proj}(S(\mathcal{E})) \rightarrow \mathbb{P}(\mathcal{E})$, which we also denoted by $r_{\mathcal{L}, \varphi}$. If φ is surjective, then so is ψ and by ?? the morphism $r_{\mathcal{L}, \psi}$ is everywhere defined.

Proposition 19.4.3. *Let $q : X \rightarrow Y$ be a morphism and \mathcal{E} be a quasi-coherent \mathcal{O}_Y -module. Then the maps $r \mapsto (\mathcal{L}_r, \varphi_r)$ and $(\mathcal{L}, \varphi) \mapsto r_{\mathcal{L}, \varphi}$ form a bijective correspondence between the set of Y -morphisms $r : X \rightarrow \mathbb{P}(\mathcal{E})$ to the set of equivalence classes of couples (\mathcal{L}, φ) formed by an invertible \mathcal{O}_X -module and a surjective homomorphism $\varphi : q^*(\mathcal{E}) \rightarrow \mathcal{L}$, where two couples $(\mathcal{L}, \varphi), (\mathcal{L}', \varphi')$ are equivalent if there exists an \mathcal{O}_X -isomorphism $\tau : \mathcal{L} \rightarrow \mathcal{L}'$ such that $\varphi' = \tau \circ \varphi$.*

Proof. Let us start with an Y -morphism $r : X \rightarrow \mathbb{P}(\mathcal{E})$, form \mathcal{L}_r and φ_r , and put $r' = r_{\mathcal{L}_r, \varphi_r}$. To see the morphisms r and r' coincide, we may assume that $Y = \text{Spec}(A)$ is affine, so $\mathcal{E} = \tilde{E}$, and let $S = S_A(E)$. Let $V = \text{Spec}(B)$ be an affine open of X contained in $r^{-1}(D_+(f))$, where $f \in E$. Then as we have already seen, the restriction of r to V corresponds to an A -homomorphism $\omega : S_{(f)} \rightarrow B$, and the restriction of ψ_r to $q^*(S_n(\mathcal{E}))|_V$ corresponds to the B -homomorphism $S_n(E) \otimes_A B \rightarrow L_r^{\otimes n}$ which sends $s \otimes 1$ to $(f/1)^{\otimes n} \otimes \omega(s/f^n)$. The restriction of

ψ_r^\flat to $S(\mathcal{E})|_V$ then corresponds to the homomorphism $S_n(E) \rightarrow L_r^{\otimes n}$ which sends $s \in S_n(E)$ to $(f/1)^{\otimes n} \otimes \omega(s/f^n)$, and by ??, the restriction of $r_{\mathcal{L}, \psi_r}$ to V corresponds to the homomorphism $(\psi_r^\flat)_{(f)}$, which send $s \in S_n$ to

$$(\psi_r^\flat(s))(\psi_r^\flat(f))^{-n} = [(f/1)^{\otimes n} \otimes \omega(s/f^n)][(f/1) \otimes 1]^{-n} = 1 \otimes \omega(s/f^n).$$

Therefore, under the identification of X with $\text{Proj}(S(\mathcal{L}))$, $r_{\mathcal{L}, \psi_r}$ coincides with r over V , so they coincide on X .

Conversely, let (\mathcal{L}, φ) be a couple and form $r = r_{\mathcal{L}, \varphi}$, \mathcal{L}_r , and φ_r . We show that there is a canonical isomorphism $\tau : \mathcal{L}_r \rightarrow \mathcal{L}$ such that $\varphi = \tau \circ \varphi_r$. For this, we can assume that $Y = \text{Spec}(A)$ and $X = \text{Spec}(B)$ are affine, and (with the notations of ??) define τ be sending an element $(x/1) \otimes 1$ of L_r (where $x \in E$) to the element $v_1(x)c$ of L . It is easy to verify that τ is independent of the choice of the generator c of L . As v_1 is surjective, to show that τ is an isomorphism, it suffices to prove that if $x/1 = 0$ in $S(1)_{(f)}$, then $v_1(x)/1 = 0$ in B_g . But the first condition means that $f^n x = 0$ in S_{n+1} for some n , and we then deduce that $v_{n+1}(f^n x) = g^n v_1(x) = 0$ in B , whence the conclusion. Finally, it is immediate that for two equivalent couples $(\mathcal{L}, \varphi), (\mathcal{L}', \varphi')$, we have $r_{\mathcal{L}, \varphi} = r_{\mathcal{L}', \varphi'}$. \square

Theorem 19.4.4. *The set of Y -sections of $\mathbb{P}(\mathcal{E})$ is in bijective correspondence to the set of quasi-coherent sub- \mathcal{O}_Y -modules \mathcal{F} of \mathcal{E} such that \mathcal{E}/\mathcal{F} is invertible.*

Proof. This is a particular case of ?? by taking $X = Y$ and note that if two pairs (φ, \mathcal{L}) and (φ', \mathcal{L}') are equivalent, then $\ker \varphi$ and $\ker \varphi'$ are identical. \square

Note that this property corresponds to the classical definition of the "projective space" as the set of hyperplanes of a vector space (the classical case corresponding to $Y = \text{Spec}(k)$, where k is a field, and $\mathcal{E} = \tilde{E}$, E being a finite dimensional k -vector. The sheaves \mathcal{F} having the property stated in ?? corresponds then to the hyperplanes of E .

Remark 19.4.1. As there is a canonical correspondence between Y -morphisms from X to P and their graph morphisms, which are X -sections of $P \times_Y X$, we see conversely that ?? can be deduced from ?. Let $\text{Hyp}_Y(X, \mathcal{E})$ be the set of quasi-coherent sub- \mathcal{O}_X -modules \mathcal{F} of $\mathcal{E} \otimes_Y \mathcal{O}_X = q^*(\mathcal{E})$ such that $q^*(\mathcal{E})/\mathcal{F}$ is an invertible \mathcal{O}_X -module. If $g : X' \rightarrow X$ is an Y -morphism, then $g^*(q^*(\mathcal{E})/\mathcal{F}) = g^*(q^*(\mathcal{E}))/g^*(\mathcal{F})$ by the right exactness of g^* , so the second sheaf is invertible, and therefore $\text{Hyp}_Y(X, \mathcal{E})$ is a covariant functor over the category of Y -schemes. We can then interpret ?? by saying that the Y -scheme $\mathbb{P}(\mathcal{E})$ representes the functor $\text{Hyp}_Y(-, \mathcal{E})$. This also provides a characterization of the projective bundle $P = \mathbb{P}(\mathcal{E})$ by the following universal property, more close to the geometric intuition that the constructions of $r_{\mathcal{L}, \psi}$: for any morphism $q : X \rightarrow Y$ and any invertible \mathcal{O}_X -module \mathcal{L} which is a quotient of $q^*(\mathcal{E})$, there exists a unique Y -morphism $r : X \rightarrow P$ such that \mathcal{L} is isomorphic to $r^*(\mathcal{O}_P(1))$.

Corollary 19.4.5. *Suppose that any invertible \mathcal{O}_Y -module is trivial. Let $E = \text{Hom}_{\mathcal{O}_Y}(\mathcal{E}, \mathcal{O}_Y)$ considered as a module over the ring $A = \Gamma(Y, \mathcal{O}_Y)$, and let E^\times be the subset of E formed by surjective homomorphisms. Then the set of Y -sections of $\mathbb{P}(\mathcal{E})$ is canonically identified with E^\times / A^\times , where A^\times is the group of units of A .*

Example 19.4.6. Let Y be a scheme, y be a point of Y , and $Y' = \text{Spec}(\kappa(y))$. The fiber $p^{-1}(y)$ of $\mathbb{P}(\mathcal{E})$ is, in view of ??, identified with $\mathbb{P}(\mathcal{E}^y)$, where $\mathcal{E}^y = \mathcal{E}_y \otimes_{\mathcal{O}_{Y,y}} \kappa(y) = \mathcal{E}_y / \mathfrak{m}_y \mathcal{E}_y$ is

considered as a vector space over $\kappa(y)$. More generally, if K is an extension of $\kappa(y)$, $p^{-1}(y) \otimes_{\kappa(y)} K$ is identified with $\mathbb{P}(\mathcal{E}^y \otimes_{\kappa(y)} K)$. Since any invertible sheaf over a local scheme is trivial, ?? shows that the points of $\mathbb{P}(\mathcal{E})$ lying over y with values in K , which are called the **rational geometric fibers** of $\mathbb{P}(\mathcal{E})$ over K lying over y , is identified with the projective space of the dual of the vector K -space $\mathcal{E}^y \otimes_{\kappa(y)} K$.

Example 19.4.7. Suppose now that Y is affine with ring A , and any invertible sheaf on Y is trivial; we put $\mathcal{E} = \mathcal{O}_Y^n$. Then with the notations of ??, E is identified with A^n by ?? and E^\times is identified with the set of systems $(f_i)_{1 \leq i \leq n}$ of elements of A which generate the unit ideal of A . By ??, two such systems determine the same Y -section of $\mathbb{P}_Y^{n-1} = \mathbb{P}_A^{n-1}$, which means the same point of \mathbb{P}_A^{n-1} with values in A , if and only if one is deduced from the other by multiplication by an invertible element of A .

Remark 19.4.2. Let $u : X' \rightarrow X$ be a morphism. If the Y -morphism $r : X \rightarrow P$ corresponds to the homomorphism $\varphi : q^*(\mathcal{E}) \rightarrow \mathcal{L}$, then by definition, the morphism $r \circ u$ corresponds to $u^*(\varphi) : u^*(q^*(\mathcal{E})) \rightarrow u^*(\mathcal{L})$. On the other hand, let \mathcal{E}, \mathcal{F} be two quasi-coherent \mathcal{O}_Y -modules and $j = \mathbb{P}(v)$ be the closed immersion $\mathbb{P}(\mathcal{F}) \rightarrow \mathbb{P}(\mathcal{E})$ corresponding to a surjective homomorphism $v : \mathcal{E} \rightarrow \mathcal{F}$. If the Y -morphism $r : X \rightarrow \mathbb{P}(\mathcal{E})$ corresponds to the homomorphism $\varphi : q^*(\mathcal{E}) \rightarrow \mathcal{L}$, then $j \circ r$ corresponds to the composition

$$q^*(\mathcal{E}) \xrightarrow{q^*(v)} q^*(\mathcal{F}) \xrightarrow{\varphi} \mathcal{L}$$

Let $\psi : Y' \rightarrow Y$ be a morphism and put $\mathcal{E}' = \psi^*(\mathcal{E})$. If the Y -morphism $r : X \rightarrow P$ corresponds to the homomorphism $\varphi : q^*(\mathcal{E}) \rightarrow \mathcal{L}$, the Y' -morphism

$$r_{(Y')} : X_{(Y')} \rightarrow P' = \mathbb{P}(\mathcal{E}')$$

correspond to $\varphi_{(Y')} : q_{(Y')}^*(\mathcal{E}') = q^*(\mathcal{E}) \otimes_Y \mathcal{O}_{Y'} \rightarrow \mathcal{L} \otimes_Y \mathcal{O}_{Y'}$. In fact, by (??), we have the following commutative diagram

$$\begin{array}{ccccc} X_{(Y')} & \xrightarrow{r_{(Y')}} & P' = P_{(Y')} & \xrightarrow{p_{(Y')}} & Y' \\ \downarrow v & & \downarrow u & & \downarrow \psi \\ X & \xrightarrow{r} & P & \xrightarrow{p} & Y \end{array}$$

In view of (??), we have

$$(r_{(Y')})^*(\mathcal{O}_{P'}(1)) = (r_{(Y')})^*(u^*(\mathcal{O}_P(1))) = v^*(r^*(\mathcal{O}_P(1))) = v^*(\mathcal{L}) = \mathcal{L} \otimes_Y \mathcal{O}_{Y'}.$$

Also, $u^*(\alpha_1^\sharp)$ is equal to the canonical homomorphism $\alpha_1^\sharp : (p_{(Y')})^*(\mathcal{E}') \rightarrow \mathcal{O}_{P'}(1)$, in view of the definition of α_1 , whence our assertion.

19.4.3 The Segre morphism

Let Y be a scheme and \mathcal{E}, \mathcal{F} be two quasi-coherent \mathcal{O}_Y -modules. Put $P_1 = \mathbb{P}(\mathcal{E})$, $P_2 = \mathbb{P}(\mathcal{F})$, and denote by p_1, p_2 their morphisms; let $Q = P_1 \times_Y P_2$ and q_1, q_2 be the canonical projections.

The \mathcal{O}_Q -module

$$\mathcal{L} = \mathcal{O}_{P_1}(1) \times_Y \mathcal{O}_{P_2}(1) = q_1^*(\mathcal{O}_{P_1}(1)) \times_{\mathcal{O}_Q} q_2^*(\mathcal{O}_{P_2}(1))$$

is invertible as a tensor product of invertible modules. On the other hand, if $r = p_1 \circ q_1 = p_2 \circ q_2$ is the structural morphism of Q , we have $r^*(\mathcal{E} \otimes_{\mathcal{O}_Y} \mathcal{F}) \otimes_{\mathcal{O}_Q} q_2^*(p_2^*(\mathcal{F}))$; the canonical surjective homomorphism $p_1^*(\mathcal{E}) \rightarrow \mathcal{O}_{P_1}(1)$ and $p_2^*(\mathcal{F}) \rightarrow \mathcal{O}_{P_2}(1)$ then give a canonical surjective homomorphism

$$s : r^*(\mathcal{E} \otimes_{\mathcal{O}_Y} \mathcal{F}) \rightarrow \mathcal{L} \quad (19.4.3)$$

we then deduce a canonical homomorphism, called the Segre morphism:

$$\zeta : \mathbb{P}(\mathcal{E}) \times_Y \mathbb{P}(\mathcal{F}) \rightarrow \mathbb{P}(\mathcal{E} \otimes_{\mathcal{O}_Y} \mathcal{F}). \quad (19.4.4)$$

To explain this morphism ζ , let us consider the case where $Y = \text{Spec}(A)$ is affine, $\mathcal{E} = \tilde{E}$, $\mathcal{F} = \tilde{F}$, where E and F are two A -module; whence $\mathcal{E} \otimes_{\mathcal{O}_Y} \mathcal{F} = \widetilde{E \otimes_A F}$. Put $R = S(E)$, $S = S(F)$, and $T = S(E \otimes_A F)$. Let $f \in E$, $g \in F$, and consider the affine open

$$D_+(f) \times_Y D_+(g) = \text{Spec}(B)$$

of Q , where $B = R_{(f)} \otimes_A S_{(g)}$. The restriction of \mathcal{L} on this affine open is \widetilde{L} , where

$$L = (R(1)_{(f)}) \otimes_A (S(1)_{(g)})$$

and the element $c = (f/1) \otimes (g/1)$ is a generator of L as a free B -module (??). The homomorphism (??) then corresponds to the homomorphism

$$(x \otimes y) \otimes b \mapsto b((x/1) \otimes (y/1))$$

from $(E \otimes_A F) \otimes_A B$ to L . With the notations of ??, we then have $v_1(x \otimes y) = (x/f) \otimes (y/g)$, so the restriction of the morphism ζ to $D_+(f) \times_Y D_+(g)$ is a morphism from this affine scheme to $D_+(f \otimes g)$, which corresponds to the ring homomorphism

$$\omega((x \otimes y)/(f \otimes g)) = (x/f) \otimes (y/g) \quad (19.4.5)$$

for $x \in E$ and $y \in F$.

From ??, there is a canonical isomorphism

$$\tau : \zeta^*(\mathcal{O}_P(1)) \xrightarrow{\sim} \mathcal{O}_{P_1}(1) \otimes_Y \mathcal{O}_{P_2}(1)$$

where we put $P = \mathbb{P}(\mathcal{E} \otimes_{\mathcal{O}_Y} \mathcal{F})$. Moreover, for $x \in \Gamma(Y, \mathcal{E})$ and $y \in \Gamma(Y, \mathcal{F})$, we have

$$\tau(\alpha_1(x \otimes y)) = \alpha_1(x) \otimes \alpha_1(y) \quad (19.4.6)$$

To see this, we can assume that Y is affine, so with the notations above and the definition of α_1 , we have $\alpha_1^{f \otimes g}(x \otimes y) = (x \otimes y)/1$, $\alpha_1^f(x) = x/1$, and $\alpha_1^g(y) = y/1$. The definition of τ given in the proof of ?? says τ maps $(x/1) \otimes 1$ to $v_1(x)c$. Since we have seen that $v_1(x \otimes y) = (x/f) \otimes (y/g)$, this implies the assertion by a simple computation. From this, we then deduce

the formula

$$\zeta^{-1}(P_{x \otimes y}) = (P_1)_x \times_Y (P_2)_y \quad (19.4.7)$$

where we need to use the following lemma:

Lemma 19.4.8. *Let B, B' be two A -algebras, and let $Y = \text{Spec}(A)$, $Z = \text{Spec}(B)$, $Z' = \text{Spec}(B')$. Then for $t \in B$, $t' \in B'$, we have $D(t \otimes t') = D(t) \times_Y D(t')$.*

Proof. Let p, p' be the canonical projections of $Z \times_Y Z'$. Then it follows from ?? that $p^{-1}(D(t)) = D(t \otimes 1)$ and $p'^{-1}(D(t')) = D(1 \otimes t')$. ?? then implies the lemma, since $(t \otimes 1)(1 \otimes t') = t \otimes t'$. \square

Proposition 19.4.9. *The Segre morphism is a closed immersion.*

Proof. Since the question is local on Y , we can assume that Y is affine. With the previous notations, the $D_+(f \otimes g)$ form a basis for P , since the elements $f \otimes g$ generate T for $f \in E$, $g \in F$. On the other hand, we have $\zeta^{-1}(D_+(f \otimes g)) = D_+(f) \times_Y D_+(g)$ in view of (??). It then suffices to use ?? to prove that the restriction of ζ to $D_+(f) \times_Y D_+(g)$ is a closed immersion into $D_+(f \otimes g)$. But this is a morphism between affine schemes whose corresponding ring homomorphism ω is surjective in view of the formula (??), so our assertion follows. \square

The Segre morphism is functorial on \mathcal{E} and \mathcal{F} if we restrict ourselves to quasi-coherent \mathcal{O}_Y -modules with *surjective* homomorphisms. To see this, it suffices to consider a surjective \mathcal{O}_Y -homomorphism $\mathcal{E} \rightarrow \mathcal{E}'$ and prove that the following diagram is commutative

$$\begin{array}{ccc} \mathbb{P}(\mathcal{E}') \times \mathbb{P}(\mathcal{F}) & \xrightarrow{j \times 1} & \mathbb{P}(\mathcal{E}') \times \mathbb{P}(\mathcal{F}) \\ \zeta \downarrow & & \downarrow \zeta \\ \mathbb{P}(\mathcal{E}' \otimes \mathcal{F}) & \longrightarrow & \mathbb{P}(\mathcal{E} \otimes \mathcal{F}) \end{array}$$

where j is the canonical closed immersion $\mathbb{P}(\mathcal{E}') \rightarrow \mathbb{P}(\mathcal{E})$. Put $P'_1 = \mathbb{P}(\mathcal{E}')$ and retain the previous notations; $j \times 1$ is a closed immersion by ?? and we have

$$(j \times 1)^*(\mathcal{O}_{P_1}(1) \otimes \mathcal{O}_{P_2}(1)) = j^*(\mathcal{O}_{P_1}(1)) \otimes \mathcal{O}_{P_2}(1) = \mathcal{O}_{P'_1}(1) \otimes \mathcal{O}_{P_2}(1)$$

in view of (??) and ?. The assertion then follows from ??.

Proposition 19.4.10. *Let $\psi : Y \rightarrow Y'$ be a morphism and put $\mathcal{E}' = \psi^*(\mathcal{E})$, $\mathcal{F}' = \psi^*(\mathcal{F})$. Then the Segre morphism $\mathbb{P}(\mathcal{E}') \times \mathbb{P}(\mathcal{F}') \rightarrow \mathbb{P}(\mathcal{E}' \otimes \mathcal{F}')$ is identified with $\zeta_{(Y')}$.*

Proof. Put $P'_1 = \mathbb{P}(\mathcal{E}')$ and $P'_2 = \mathbb{P}(\mathcal{F}')$. Then by ??, P'_i is identified with $(P_i)_{(Y')}$ for $i = 1, 2$, so the structural morphism $P'_1 \times_{Y'} P'_2 \rightarrow Y'$ is identified with $r_{(Y')}$, where r is the structural morphism of $P_1 \times_Y P_2$. On the other hand, $\mathcal{E}' \otimes \mathcal{F}'$ is identified with $\psi^*(\mathcal{E} \otimes \mathcal{F})$, so $\mathbb{P}(\mathcal{E}' \otimes \mathcal{F}')$ is identified with $\mathbb{P}(\mathcal{E} \otimes \mathcal{F})_{(Y')}$ by ?. Finally, $\mathcal{O}_{P'_1}(1) \otimes_{Y'} \mathcal{O}_{P'_2}(1) = \mathcal{L}$ is identified with $\mathcal{L} \otimes_Y \mathcal{O}_{Y'}$ in view of (??) and ?. The canonical homomorphism $(r_{(Y')})^*(\mathcal{E}' \otimes \mathcal{F}') \rightarrow \mathcal{L}$ is then identified with $s_{(Y')}$, and our assertion follows from ?. \square

Remark 19.4.3. The coproduct of $\mathbb{P}(\mathcal{E})$ and $\mathbb{P}(\mathcal{F})$ is similarly canonical isomorphic to a closed subscheme of $\mathbb{P}(\mathcal{E} \oplus \mathcal{F})$. In fact, the surjective homomorphisms $\mathcal{E} \oplus \mathcal{F} \rightarrow \mathcal{E}$, $\mathcal{E} \oplus \mathcal{F} \rightarrow \mathcal{F}$ correspond to closed immersions $\mathbb{P}(\mathcal{E}) \rightarrow \mathbb{P}(\mathcal{E} \oplus \mathcal{F})$, $\mathbb{P}(\mathcal{F}) \rightarrow \mathbb{P}(\mathcal{E} \oplus \mathcal{F})$; it then boils down to showing that the underlying spaces of these closed subschemes of $\mathbb{P}(\mathcal{E} \oplus \mathcal{F})$ thus obtained have no common point. The question being local on Y , we can assume that Y is affine adapt our previous notations. Now $S_n(E)$ and $S_n(F)$ are identified with submodules of $S_n(E \oplus F)$ with intersection reduced to 0, and if \mathfrak{p} is a graded prime ideal of $S(E)$ such that $\mathfrak{p} \cap S_n(E) \neq S_n(E)$ for all $n \geq 0$, then it corresponds to a unique graded prime ideal in $S(E \oplus F)$ whose trace on $S_n(E)$ is $\mathfrak{p} \cap S_n(E)$, but which contains $S_n(F)$. Therefore, two distinct points of $\text{Proj}(S(E))$ and $\text{Proj}(S(F))$ can not have same image in $\text{Proj}(S(E \oplus F))$.

19.4.4 Very ample sheaves

Proposition 19.4.11. *Let Y be a quasi-compact scheme, $q : X \rightarrow Y$ a morphism of finite type, and \mathcal{L} be an invertible \mathcal{O}_X -module.*

- (a) *Let \mathcal{S} be a quasi-coherent graded \mathcal{O}_Y -algebra and $\psi : q^*(\mathcal{S}) \rightarrow S(\mathcal{L})$ be a graded homomorphism of algebras. For the morphism $r_{\mathcal{L},\psi}$ to be everywhere defined and an immersion, it is necessary and sufficient that there exists an integer $n \geq 0$ and a quasi-coherent sub- \mathcal{O}_Y -module of finite type \mathcal{E} of \mathcal{S}_n such that the homomorphism $\varphi' = \psi_n \circ q^*(j) : q^*(\mathcal{E}) \rightarrow S(\mathcal{L}) = \mathcal{L}'$ ($j : \mathcal{E} \rightarrow \mathcal{S}_n$ being the canonical injection) is surjective and the morphism $r_{\mathcal{L}',\varphi'} : X \rightarrow \mathbb{P}(\mathcal{E})$ is an immersion.*
- (b) *Let \mathcal{F} be a quasi-coherent \mathcal{O}_Y -module and $\varphi : q^*(\mathcal{F}) \rightarrow \mathcal{L}$ be a surjective homomorphism. For the morphism $r_{\mathcal{L},\varphi} : X \rightarrow \mathbb{P}(\mathcal{F})$ to be an immersion, it is necessary and sufficient that there exists a quasi-coherent sub- \mathcal{O}_Y -module of finite type \mathcal{E} of \mathcal{F} such that the homomorphism $\varphi' = \varphi \circ q(j) : q^*(\mathcal{E}) \rightarrow \mathcal{L}$ (where $j : \mathcal{E} \rightarrow \mathcal{F}$ is the canonical injection) is surjective and such that $r_{\mathcal{L},\varphi'} : X \rightarrow \mathbb{P}(\mathcal{E})$ is an immersion.*

Proof. We first consider case (a). The fact that $r_{\mathcal{L},\psi}$ is everywhere defined and an immersion is equivalent by ?? to the existence of an integer $n > 0$ and \mathcal{E} such that, if \mathcal{S}' is the subalgebra of \mathcal{S} generated by \mathcal{E} , the homomorphism $q^*(\mathcal{E}) \rightarrow \mathcal{S}'^{\otimes n}$ is surjective and $r_{\mathcal{L},\psi'} : X \rightarrow \text{Proj}(\mathcal{S}')$ is everywhere defined and an immersion. We also have a closed immersion corresponding to the surjective homomorphism $S(\mathcal{E}) \rightarrow \mathcal{S}'$, so these the morphism $X \rightarrow \mathbb{P}(\mathcal{E})$ is an immersion.

Now consider the situation of (b). As \mathcal{F} is the inductive limit of its quasi-coherent submodules of finite type \mathcal{E}_λ (??), $S(\mathcal{F})$ is the inductive limit of the $S(\mathcal{E}_\lambda)$, so by ?? there exists λ such that $r_{\mathcal{L},\varphi_\mu}$ is everywhere defined and an immersion for $\mu \geq \lambda$. Also, since the functor f^* is left-adjoint, it commutes with inductive limits and therefore $q^*(\mathcal{F})$ is the inductive limits of the $q^*(\mathcal{E}_\lambda)$. Since \mathcal{L} is an \mathcal{O}_X -module of finite type and $q^*(\mathcal{F}) \rightarrow \mathcal{L}$ is surjective, it follows from ?? that there exists λ' such that $q^*(\mathcal{E}_\mu) \rightarrow \mathcal{L}$ is surjective for $\mu \geq \lambda'$. It then suffices to choose $\mathcal{E} = \mathcal{E}_\mu$ for $\mu \geq \lambda$ and $\mu \geq \lambda'$. \square

Let Y be a scheme and $q : X \rightarrow Y$ be a morphism. We say an invertible \mathcal{O}_X -module \mathcal{L} is **very ample for q** (or **very ample relative to q** , or simple **very ample**) if there exists a quasi-coherent \mathcal{O}_Y -module \mathcal{E} and a Y -immersion $r : X \rightarrow \mathbb{P}(\mathcal{E})$ such that \mathcal{L} is isomorphic to $r^*(\mathcal{O}_{\mathbb{P}}(1))$. In view of ??, this is equivalent to the existence of a quasi-coherent \mathcal{O}_Y -module \mathcal{E} and a surjective homomorphism $\varphi : q^*(\mathcal{E}) \rightarrow \mathcal{L}$ such that the associated morphism $r_{\mathcal{L},\varphi} : X \rightarrow \mathbb{P}(\mathcal{E})$ is an immersion. We also note that the existence of a very ample \mathcal{O}_X -module relative to Y implies that q is separated (?? and ??).

Corollary 19.4.12. Suppose that there exists a quasi-coherent \mathcal{O}_Y -algebra \mathcal{S} generated by \mathcal{S}_1 and a Y -immersion $r : X \rightarrow P = \text{Proj}(\mathcal{S})$ such that $\mathcal{L} \cong r^*(\mathcal{O}_P(1))$. Then \mathcal{L} is very ample relative to q .

Proof. If $\mathcal{F} = \mathcal{S}_1$, the canonical homomorphism $S(\mathcal{F}) \rightarrow \mathcal{S}$ is surjective, so by composing r with the corresponding closed immersion $\text{Proj}(\mathcal{S}) \rightarrow \mathbb{P}(\mathcal{F})$, we obtain an immersion $r' : X \rightarrow \mathbb{P}(\mathcal{F}) = P'$ such that \mathcal{L} is isomorphic to $r'^*(\mathcal{O}_{P'}(1))$. \square

Proposition 19.4.13. Let $q : X \rightarrow Y$ be a quasi-compact morphism and \mathcal{L} be an invertible \mathcal{O}_X -module. Then \mathcal{L} is very ample relative to q if and only if $q_*(\mathcal{L})$ is quasi-coherent, the canonical homomorphism $\sigma : q^*(q_*(\mathcal{L})) \rightarrow \mathcal{L}$ is surjective, and the morphism $r_{\mathcal{L}, \sigma} : X \rightarrow \mathbb{P}(q_*(\mathcal{L}))$ is an immersion.

Proof. As q is quasi-compact, $q_*(\mathcal{L})$ is quasi-coherent if q is separated (??). By ??, the existence of a surjective homomorphism $\varphi : q^*(\mathcal{E}) \rightarrow \mathcal{L}$ (\mathcal{E} being a quasi-coherent \mathcal{O}_Y -module) implies that σ is surjective. Moreover, the factorization $\varphi : q^*(\mathcal{E}) \rightarrow q^*(q_*(\mathcal{L})) \xrightarrow{\sigma} \mathcal{L}$ of (??) corresponds to a canonical factorization (recall that q^* commutes with S)

$$q^*(S(\mathcal{E})) \longrightarrow q^*(S(q_*(\mathcal{L}))) \longrightarrow S(\mathcal{L})$$

so by ?? the hypothesis that $r_{\mathcal{L}, \varphi}$ is an immersion implies that $j = r_{\mathcal{L}, \sigma}$ is an immersion. Moreover, by ??, \mathcal{L} is isomorphic to $j^*(\mathcal{O}_{P'}(1))$ where $P' = \mathbb{P}(q_*(\mathcal{L}))$. The converse of this is clear by the definition of very ampleness. \square

Corollary 19.4.14. Let $q : X \rightarrow Y$ be a quasi-compact morphism and \mathcal{L} be an invertible \mathcal{O}_X -module. For \mathcal{L} to be very ample relative to Y , it is necessary and sufficient that there exists an open covering (U_α) of Y such that $\mathcal{L}|_{q^{-1}(U_\alpha)}$ is very ample relative to U_α for each α .

Proof. This follows from the fact that the criterion of ?? is local over Y . \square

Proposition 19.4.15. Let Y be a quasi-compact scheme, $q : X \rightarrow Y$ a morphism of finite type, and \mathcal{L} be an invertible \mathcal{O}_X -module. Then the following conditions are equivalent:

- (i) \mathcal{L} is very ample relative to Y .
- (ii) There exists a quasi-coherent \mathcal{O}_Y -module \mathcal{E} of finite type and a surjective homomorphism $\varphi : q^*(\mathcal{E}) \rightarrow \mathcal{L}$ such that $r_{\mathcal{L}, \varphi}$ is an immersion.
- (iii) There exists a coherent sub- \mathcal{O}_Y -module of finite type \mathcal{E} of $q_*(\mathcal{L})$ and a surjective homomorphism $\varphi : q^*(\mathcal{E}) \rightarrow \mathcal{L}$ such that $r_{\mathcal{L}, \varphi}$ is an immersion.

Proof. It is clear that (ii) or (iii) implies (i); but (i) implies (ii) by ??, and similarly (i) implies (iii) in view of ??.

Corollary 19.4.16. Suppose that Y is a quasi-compact scheme. If \mathcal{L} is very ample relative to Y , there exists a quasi-coherent graded \mathcal{O}_Y -algebra \mathcal{S} generated by \mathcal{S}_1 and such that \mathcal{S}_1 is of finite type, and a dominant open Y -immersion $i : X \rightarrow P = \text{Proj}(\mathcal{S})$ such that \mathcal{L} is isomorphic to $i^*(\mathcal{O}_P(1))$.

Proof. Since \mathcal{L} is very ample, by ?? there exists a coherent sub- \mathcal{O}_Y -module of finite type \mathcal{E} of $q_*(\mathcal{L})$ and a surjective homomorphism $\varphi : q^*(\mathcal{E}) \rightarrow \mathcal{L}$ such that $r_{\mathcal{L}, \varphi}$ is an immersion. The structural morphism $p : P' = \mathbb{P}(\mathcal{E}) \rightarrow Y$ is then separated and of finite type (??), so P' is

a quasi-compact scheme if Y is quasi-compact. Let Z be scheme theoretic image of X in P' , with underlying space $\overline{j(X)}$, where $j = r_{\mathcal{L}, \varphi}$; then j factors through Z into a dominant open immersion $i : X \rightarrow Z$. But Z is identified with the scheme $\text{Proj}(\mathcal{S})$, where \mathcal{S} is the quotient graded \mathcal{O}_Y -algebra of $S(\mathcal{E})$ by a quasi-coherent graded ideal ??, and it is clear that \mathcal{S}_1 is generated by \mathcal{S} (since $S(\mathcal{E})$ satisfies this condition). Moreover, by ??, $\mathcal{O}_Z(1)$ is the inverse image of $\mathcal{O}_{P'}(1)$ under the canonical injection, so $\mathcal{L} = i^*(\mathcal{O}_Z(1))$. \square

Proposition 19.4.17. *Let $q : X \rightarrow Y$ be a morphism, \mathcal{L} be a very ample \mathcal{O}_X -module relative to q , and \mathcal{L}' be an invertible \mathcal{O}_X -module such that there exists a quasi-coherent \mathcal{O}_Y -module \mathcal{E}' and a surjective homomorphism $q^*(\mathcal{E}') \rightarrow \mathcal{L}'$. Then $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{L}'$ is very ample relative to q .*

Proof. The hypothesis on \mathcal{L}' implies the existence of an Y -morphism $r' : X \rightarrow P' = \mathbb{P}(\mathcal{E}')$ such that \mathcal{L}' is isomorphic to $r'^*(\mathcal{O}_{P'}(1))$??). There is by hypothesis a quasi-coherent \mathcal{O}_Y -module \mathcal{E} and a Y -immersion $r : X \rightarrow P = \mathbb{P}(\mathcal{E})$ such that \mathcal{L} is isomorphic to $r^*(\mathcal{O}_P(1))$. Consider the Segre morphism $\zeta : P \times_Y P' \rightarrow Q$ where $Q = \mathbb{P}(\mathcal{E} \otimes \mathcal{E}')$. As r is an immersions, so is the morphism $(r, r')_Y : X \rightarrow P \times_Y P'$ by ??, and therefore we get an immersion

$$r'' : X \xrightarrow{(r, r')_Y} P \times_Y P' \xrightarrow{\zeta} Q.$$

Since $\zeta^*(\mathcal{O}_Q(1))$ is isomorphic to $\mathcal{O}_P(1) \otimes_Y \mathcal{O}_{P'}(1)$, we conclude from ?? that $r''^*(\mathcal{O}_Q(1))$ is isomorphic to $\mathcal{L} \otimes \mathcal{L}'$, this proves the assertion. \square

Remark 19.4.4. Note that $q^*(\mathcal{O}_Y^{\oplus I}) = \mathcal{O}_X^{\oplus I}$ and there exists a surjection $\mathcal{O}_X^{\oplus I} \rightarrow \mathcal{L}'$ if and only if \mathcal{L}' is generated by global sections, so ?? is applicable if \mathcal{L}' is generated by global sections.

Corollary 19.4.18. *Let $q : X \rightarrow Y$ be a morphism.*

- (a) *Let \mathcal{L} be an invertible \mathcal{O}_X -module and \mathcal{K} be an invertible \mathcal{O}_Y -module. For \mathcal{L} to be very ample relative to q , it is necessary and sufficient that $\mathcal{L} \otimes q^*(\mathcal{K})$ is very ample relative to q .*
- (b) *If \mathcal{L} and \mathcal{L}' are two invertible \mathcal{O}_X -modules that are very ample relative to q , then so is $\mathcal{L} \otimes \mathcal{L}'$. In particular, $\mathcal{L}^{\otimes n}$ is very ample relative to q for any $n > 0$.*

Proof. The assertions in (b) is an immediate consequence of ??, so is the half implication of (a). Now assume that $\mathcal{L} \otimes q^*(\mathcal{K})$ is very ample; then so is $(\mathcal{L} \otimes q^*(\mathcal{K})) \otimes q^*(\mathcal{K}^{-1})$ by ??, which is isomorphic to \mathcal{L} . \square

Proposition 19.4.19. *Let $f : X \rightarrow Y$ be a quasi-compact morphism where Y is quasi-compact, and \mathcal{L} be a very ample \mathcal{O}_X -module relative to f . Let \mathcal{F} be a quasi-coherent \mathcal{O}_X -module of finite type and for each $n \in \mathbb{Z}$, set $\mathcal{F}(n) = \mathcal{F} \otimes \mathcal{L}^{\otimes n}$. Then there exists an integer n_0 such that for $n \geq n_0$, the canonical homomorphism $\sigma : f^*(f_*(\mathcal{F}(n))) \rightarrow \mathcal{F}(n)$ is surjective.*

Proof. \square

Proposition 19.4.20. *Let $f : X \rightarrow Y$ be a quasi-compact morphism where Y is quasi-compact, and \mathcal{L} be a very ample \mathcal{O}_X -module relative to f . Let \mathcal{F} be a quasi-coherent \mathcal{O}_X -module of finite type. Then there exists an integer n_0 such that, for $n \geq n_0$, \mathcal{F} is isomorphic to a quotient of an \mathcal{O}_X -module $f^*(\mathcal{G}) \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes(-n)}$, where \mathcal{G} is a quasi-coherent \mathcal{O}_Y -module of finite type (dependent on n).*

Proof. Since \mathcal{L} is very ample, f is separated and by ?? the canonical homomorphism $\sigma : f^*(f_*(\mathcal{F}(n))) \rightarrow \mathcal{F}(n)$ is surjective for n sufficiently large. The proposition is then a generalization of ??, and can be proved similarly. \square

Proposition 19.4.21 (Properties of Very Ample Sheaves).

- (i) For any scheme Y , any invertible \mathcal{O}_Y -module \mathcal{L} is very ample relative to the identity morphism 1_Y .
- (ii) Let $f : X \rightarrow Y$ be a morphism and $j : X' \rightarrow X$ be an immersion. If \mathcal{L} is a very ample \mathcal{O}_X -module relative to f , then $j^*(\mathcal{L})$ is very ample relative to $f \circ j$.
- (iii) Let $f : X \rightarrow Y$ be a morphism of finite type and $g : Y \rightarrow Z$ be a quasi-compact morphism where Z is quasi-compact. Let \mathcal{L} a very ample \mathcal{O}_X -module relative to f and \mathcal{K} be a very ample \mathcal{O}_Y -module relative to g . Then there exists an integer $n_0 > 0$ such that for any $n \geq n_0$, $\mathcal{L} \otimes f^*(\mathcal{K}^{\otimes n})$ is very ample relative to $g \circ f$.
- (iv) Let $f : X \rightarrow Y$, $g : Y' \rightarrow Y$ be two morphisms, and put $X' = X_{(Y')}$. If \mathcal{L} is a very ample \mathcal{O}_X -module relative to f , $\mathcal{L}' = \mathcal{L} \otimes_Y \mathcal{O}_{Y'}$ is very ample relative to $f_{(Y')}$.
- (v) Let $f_i : X_i \rightarrow Y_i$ ($i = 1, 2$) be two S -morphisms. If \mathcal{L}_i is a very ample \mathcal{O}_{X_i} -module relative to f_i , then $\mathcal{L}_1 \otimes_S \mathcal{L}_2$ is very ample relative to $f_1 \times_S f_2$.
- (vi) Let $f : X \rightarrow Y$, $g : Y \rightarrow Z$ be two morphisms. If an \mathcal{O}_X -module \mathcal{L} is very ample relative to $g \circ f$, then \mathcal{L} is very ample relative to f .
- (vii) Let $f : X \rightarrow Y$ be a morphism and $j : X_{\text{red}} \rightarrow X$ be the canonical injection. If \mathcal{L} is a very ample \mathcal{O}_X -module relative to f , then $j^*(\mathcal{L})$ is very ample relative to f_{red} .

Proof. The property (ii) follows from the definition and it is immediate that (vii) is deduced from (ii) and (vi). To prove (vi), consider the factorization

$$X \xrightarrow{\Gamma_f} X \times_Z Y \xrightarrow{p_2} Y$$

and note that $p_2 = (g \circ f) \times 1_Y$. It follows from the hypothesis and from (i) and (v) that $\mathcal{L} \otimes_{\mathcal{O}_Z} \mathcal{O}_Y$ is very ample relative to p_2 . On the other hand, we have $\mathcal{L} = \Gamma_f^*(\mathcal{L} \otimes_{\mathcal{O}_Z} \mathcal{O}_Y)$ by ??, and Γ_f is an immersion (??), so we can apply (ii). As for (i), we can apply the definition with $\mathcal{E} = \mathcal{L}$, and note that $\mathbb{P}(\mathcal{L})$ is identified with Y (??).

We now prove (iv). Under the hypothesis of (iv), there exists a quasi-coherent \mathcal{O}_Y -module \mathcal{E} and a Y -immersion $i : X \rightarrow P = \mathbb{P}(\mathcal{E})$ such that $\mathcal{L} = i^*(\mathcal{O}_P(1))$. Then, if $\mathcal{E}' = g^*(\mathcal{E})$, \mathcal{E}' is a quasi-coherent \mathcal{O}_Y -module and we have $P' = \mathbb{P}(\mathcal{E}') = P_{(Y')}$, $i_{(Y')}$ is an immersion from $X_{(Y')}$ to P' , and \mathcal{L}' is isomorphic to $(i_{(Y')})^*(\mathcal{O}_{P'}(1))$ (??).

To prove (v), remark that there exists by hypothesis a Y_i -immersion $r_i : X_i \rightarrow P_i = \mathbb{P}(\mathcal{E}_i)$, where \mathcal{E}_i is a quasi-coherent \mathcal{O}_{Y_i} -module, and $\mathcal{L}_i = r_i^*(\mathcal{O}_{P_i}(1))$. Then $r_1 \times_S r_2$ is an S -immersion of $X_1 \times_S X_2$ to $P_1 \times_S P_2$ (??) and the inverse image of $\mathcal{O}_{P_1}(1) \otimes_S \mathcal{O}_{P_2}(1)$ by this immersion is $\mathcal{L}_1 \otimes_S \mathcal{L}_2$. On the other hand, put $T = Y_1 \times_S Y_2$, and let p_1, p_2 be the projection of T , respectively. If $P'_i = \mathbb{P}(p_i^*(\mathcal{E}_i))$, we have $P'_i = P_i \times_{Y_i} T$, whence

$$P'_1 \times_T P'_2 = (P_1 \times_{Y_1} T) \times_T (P_2 \times_{Y_2} T) = P_1 \times_{Y_1} (T \times_{Y_2} P_2) = P_1 \times_{Y_1} (Y_1 \times_S P_2) = P_1 \times_S P_2.$$

Similarly, we have $\mathcal{O}_{P'_i}(1) = \mathcal{O}_{P'_i}(1) \otimes_{Y_i} \mathcal{O}_T$, and $\mathcal{O}_{P'_1}(1) \otimes_T \mathcal{O}_{P'_2}(1)$ is identified with $\mathcal{O}_{P_1}(1) \otimes_S \mathcal{O}_{P_2}(1)$. We can then consider $r_1 \times_S r_2$ as an T -immersion from $X_1 \times_S X_2$ to $P'_1 \times_T P'_2$, the inverse image of $\mathcal{O}_{P'_1}(1) \otimes_T \mathcal{O}_{P'_2}(1)$ by this immersion being $\mathcal{L}_1 \otimes_S \mathcal{L}_2$. We can then conclude as in ?? that $\mathcal{L}_1 \otimes_S \mathcal{L}_2$ is very ample.

It remains to prove (iii). We can first restrict to the case where Z is an affine scheme, since there exists a finite covering (U_i) of Z by affine opens; if the property is proved for $\mathcal{K}|_{g^{-1}(U_i)}$, $\mathcal{L}|_{f^{-1}(g^{-1}(U_i))}$ and an integer n_i , it suffices to choose n_0 to be the largest n_i to prove the property for \mathcal{K} and \mathcal{L} (??). The hypotheses imply that f, g are separated morphisms, so X and Y are quasi-compact schemes. Since \mathcal{L} is very ample relative to f , there exists an immersion $r : X \rightarrow P = \mathbb{P}(\mathcal{E})$, where \mathcal{E} is a quasi-coherent \mathcal{O}_Y -module of finite type and $\mathcal{L} = r^*(\mathcal{O}_P(1))$, in view of ???. We claim that there exists an integer m_0 such that for any $m \geq m_0$, there is a very ample \mathcal{O}_P -module \mathcal{M} relative to the composition morphism $j : P \rightarrow Y \xrightarrow{g} Z$ such that $\mathcal{O}_P(1)$ is isomorphic to $\mathcal{M} \otimes_Y \mathcal{K}^{\otimes(-m)}$. For $n \geq m + 1$, $\mathcal{O}_P(1) \otimes_Y \mathcal{K}^{\otimes n}$ will then be very ample relative to Z in view of the hypothesis and applying (v) to the morphism $j : P \rightarrow Z$ and 1_Z ; as r is an immersion and $\mathcal{L} \otimes f^*(\mathcal{K}^{\otimes n}) = r^*(\mathcal{O}_P(1) \otimes_Y \mathcal{K}^{\otimes n})$, the conclusion then follows from (ii).

To establish the claim, we can use ?? to obtain a closed immersion j_1 of P to $P_1 = \mathbb{P}(g^*(\mathcal{F}) \otimes \mathcal{K}^{\otimes(-m)})$ such that $\mathcal{O}_P(1)$ is isomorphic to $j_1^*(\mathcal{O}_{P_1}(1))$ (??). On the other hand, there is an isomorphism from P_1 to $P_2 = \mathbb{P}(g^*(\mathcal{F}))$, identifying $\mathcal{O}_{P_1}(1)$ with $\mathcal{O}_{P_2}(1) \otimes_Y \mathcal{K}^{\otimes(-m)}$ (??); we then have a closed immersion $j_2 : P \rightarrow P_2$ such that $\mathcal{O}_P(1)$ is isomorphic to $j_2^*(\mathcal{O}_{P_2}(1)) \otimes_Y \mathcal{K}^{\otimes(-m)}$. Finally, P_2 is identified with $P_3 \times_Z Y$ where $P_3 = \mathbb{P}(\mathcal{F})$, and $\mathcal{O}_{P_2}(1)$ is identified with $\mathcal{O}_{P_3}(1) \otimes_Z \mathcal{O}_Y$ (??). By definition, $\mathcal{O}_{P_3}(1)$ is very ample for Z , and so is \mathcal{K} , so it follows from (v) that the \mathcal{O}_{P_2} -module $\mathcal{O}_{P_3}(1) \otimes_Y \mathcal{K}$ is very ample for Z . In view of (ii), $\mathcal{M} = j_2^*(\mathcal{O}_{P_3}(1) \otimes_Y \mathcal{K})$ is then very ample for Z , and $\mathcal{O}_P(1)$ is isomorphic to $\mathcal{M} \otimes_Y \mathcal{K}^{\otimes(-m-1)}$, whence the demonstration. \square

Proposition 19.4.22. *Let $f : X \rightarrow Y$, $f' : X' \rightarrow Y$ be two morphisms, $X'' = X \amalg X'$, and $f'' = f \amalg f'$. Let \mathcal{L} (resp. \mathcal{L}') be an invertible \mathcal{O}_X -module (resp. $\mathcal{O}_{X'}$ -module), and let \mathcal{L}'' be the invertible $\mathcal{O}_{X''}$ -module which coincides with \mathcal{L} over X and with \mathcal{L}' over X' . For \mathcal{L}'' to be very ample relative to f'' , it is necessary and sufficient that \mathcal{L} is very ample relative to f and \mathcal{L}' is very ample relative to f' .*

Proof. We can assume that Y is affine. If \mathcal{L}'' is very ample then so is \mathcal{L} and \mathcal{L}' in view of ??(ii). Conversely, if \mathcal{L} and \mathcal{L}' are very ample, it follows from ?? that \mathcal{L}'' is very ample. \square

19.4.5 Ample sheaves

Let X be a scheme, \mathcal{L} be an invertible \mathcal{O}_X -module, and \mathcal{F} be an \mathcal{O}_X -module. For any $n \in \mathbb{Z}$, we set $\mathcal{F}(n) = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}$ (if there is no confusion), and $S = \bigoplus_{n \geq 0} \Gamma(X, \mathcal{L}^{\otimes n})$. If we consider X as a \mathbb{Z} -scheme and let $p : X \rightarrow Y = \text{Spec}(\mathbb{Z})$ be the structural morphism, there are bijections

$$\text{Hom}_{\text{Qcoh}(X)}(p^*(\tilde{S}), \mathbf{S}(\mathcal{L})) \xrightarrow{\sim} \text{Hom}_{\text{Qcoh}(Y)}(\tilde{S}, p_*(\mathbf{S}(\mathcal{L}))) \xrightarrow{\sim} \text{Hom}_{\mathbb{Z}}(S, \Gamma(X, \mathbf{S}(\mathcal{L})))$$

where we use ???. The homomorphism $\varepsilon : p^*(\tilde{S}) \rightarrow \mathbf{S}(\mathcal{L})$ corresponding to the canonical injection of S into $\Gamma(X, \mathbf{S}(\mathcal{L}))$ is called the **canonical homomorphism associated with \mathcal{L}** . It then corresponds to a canonical morphism

$$r_{\mathcal{L}, \varepsilon} : G(\varepsilon) \rightarrow \text{Proj}(S). \quad (19.4.8)$$

Theorem 19.4.23. *Let X be a quasi-compact and quasi-separated scheme, \mathcal{L} be an invertible \mathcal{O}_X -module and $S = \bigoplus_{n \geq 0} \Gamma(X, \mathcal{L}^{\otimes n})$. The following conditions are equivalent:*

- (i) *The subsets X_f , as f runs through the set of homogeneous elements of S_+ , form a basis for X .*
- (i') *The subsets X_f which are affine, as f runs through the set of homogeneous elements of S_+ , form a basis for X .*
- (ii) *The canonical morphism $r_{\mathcal{L}, \varepsilon} : G(\varepsilon) \rightarrow \text{Proj}(S)$ is everywhere defined and a dominant open immersion.*
- (ii') *The canonical morphism $r_{\mathcal{L}, \varepsilon} : G(\varepsilon) \rightarrow \text{Proj}(S)$ is everywhere defined and a homeomorphism from X onto a subspace of $\text{Proj}(S)$.*
- (iii) *For any quasi-coherent \mathcal{O}_X -module \mathcal{F} , if \mathcal{F}_n is the sub- \mathcal{O}_X -module of $\mathcal{F}(n)$ generated by the sections of $\mathcal{F}(n)$ over X , then \mathcal{F} is the direct sum of the sub- \mathcal{O}_X -modules $\mathcal{F}_n(-n)$ for $n > 0$.*
- (iii') *Property (iii) holds for any quasi-coherent ideal of \mathcal{O}_X .*

Moreover, in this case, if (f_α) is a family of homogeneous elements of S_+ such that X_{f_α} is affine, then the restriction to $\bigcup_\alpha X_{f_\alpha}$ of the canonical morphism $r_{\mathcal{L}, \varepsilon} : X \rightarrow \text{Proj}(S)$ is an isomorphism from $\bigcup_\alpha X_{f_\alpha}$ to $\bigcup_\alpha (\text{Proj}(S))_{f_\alpha}$.

Proof. It is clear that (ii) implies (ii'), and (ii') implies (i) in view of the formula (??). Condition (i) implies (i'), because any $x \in X$ admits an affine neighborhood U such that $\mathcal{L}|_U$ is isomorphic to $\mathcal{O}_X|_U$; if $f \in \Gamma(X, \mathcal{L}^{\otimes n})$ is such that $x \in X_f \subseteq U$, X_f is also the set of $x' \in U$ such that $(f|_U)(x') \neq 0$, and this is an affine open subset of U (hence of X). To prove that (i') implies (ii), it suffices to show the last assertion of the statement, hence to check that if $X = \bigcup_\alpha X_{f_\alpha}$, the condition (iv) of ?? is satisfied; this follows immediately from ??(a). To see that $r_{\mathcal{L}, \varepsilon}$ is dominant, we note that for $f \in S_+$ homogeneous, X_f is the inverse image of $D_+(f)$ by $r_{\mathcal{L}, \varepsilon}$ and by ?? we have $\Gamma(X_f, \mathcal{O}_X) = S_{(f)}$ is nonzero if f is not nilpotent, so X_f is nonempty if $D_+(f)$ is not empty.

To prove that (i') implies (iii), note that if X_f is affine (where $f \in S_d$), $\mathcal{F}|_{X_f}$ is generated by its sections over X_f (??); on the other hand, by ?? such a section s is of the form $(t|_{X_f}) \otimes (f|_{X_f})^{-m}$ where $t \in \Gamma(X, \mathcal{F}(md))$. By definition, t is also a section of \mathcal{F}_{md} , so s is a section of $\mathcal{F}_{md}(-md)$ over X_f , which proves (iii). It is clear that (iii) implies (iii'), and it rests to show that (iii') implies (i). Now let U be an open neighborhood of $x \in X$, and let \mathcal{I} be a quasi-coherent ideal of \mathcal{O}_X defining a closed subscheme of X with underlying subspace $X - U$. The hypothesis of (iii') implies that there exists an integer $n > 0$ and a section $\mathcal{I}(n)$ over X such that $f(x) \neq 0$. Then we have evidently $f \in S_n$ and $x \in X_f \subseteq U$, this proves (i). \square

If X is a quasi-compact and quasi-separated scheme, the equivalent conditions of ?? implies that X is separated, since it is isomorphic to a subscheme of $\text{Proj}(S)$. We say an invertible \mathcal{O}_X -module \mathcal{L} is **ample** if X is a quasi-compact and quasi-separated scheme and the equivalent conditions of ?? are satisfied. It follows from ??(i) that if \mathcal{L} is an ample \mathcal{O}_X -module, then for any open subset U of X , $\mathcal{L}|_U$ is an ample $(\mathcal{O}_X|_U)$ -module.

Corollary 19.4.24. *Let \mathcal{L} be an ample \mathcal{O}_X -module. For any finite subspace Z of X and any open neighborhood U of Z , there exists an integer $n > 0$ and a section $f \in \Gamma(X, \mathcal{L}^{\otimes n})$ such that X_f is an affine neighborhood of Z contained in U .*

Proof. In view of ??(ii), we can view X as a subscheme of $\text{Proj}(S)$ and we only need to prove that for any finite subset Z' of $\text{Proj}(S)$ and any open neighborhood U of Z' , there exists a homogeneous element $f \in S_+$ such that $Z' \subseteq D_+(f) \subseteq U$. Now by definition the closed set Y , which is the complement of U in $\text{Proj}(S)$, is of the form $V_+(\mathfrak{I})$ where \mathfrak{I} is a graded ideal of S , not containing S_+ ; on the other hand, the points of Z' are by definition graded prime ideals \mathfrak{p}_i of S_+ not containing \mathfrak{I} . There then exists an element $f \in \mathfrak{I}$ not contained in each \mathfrak{p}_i ??, and as the \mathfrak{p}_i are graded, we can assume that f is homogeneous. This element f then satisfies the required. \square

Proposition 19.4.25. *Suppose that X is a quasi-compact and quasi-separated scheme. Then the conditions of ?? are equivalent to the following conditions:*

- (iv) *For any quasi-coherent \mathcal{O}_X -module \mathcal{F} of finite type, there exists an integer $n_0 > 0$ such that, for any $n \geq n_0$, $\mathcal{F}(n)$ is generated by its sections over X .*
- (iv') *For any quasi-coherent \mathcal{O}_X -module \mathcal{F} of finite type, there exists integers $n > 0, k > 0$ such that \mathcal{F} is isomorphic to a quotient of the \mathcal{O}_X -module $\mathcal{L}^{\otimes(-n)} \otimes \mathcal{O}_X^k$.*
- (iv'') *Property (iv') holds for any quasi-coherent ideal of \mathcal{O}_X of finite type.*

Proof. As X is quasi-compact, if a quasi-coherent \mathcal{O}_X -module \mathcal{F} of finite type is such that $\mathcal{F}(n)$ is generated by global sections, $\mathcal{F}(n)$ is then generated by finitely many global sections ??, so (iv) implies (iv') and it is clear that (iv') implies (iv''). As any quasi-coherent \mathcal{O}_X -module \mathcal{G} is the inductive limit of its sub- \mathcal{O}_X -modules of finite type ??, to verify condition (iii') of ??, it suffices to verify that for a quasi-coherent ideal of \mathcal{O}_X that is of finite type, and this is clear if condition (iv'') holds. It remains to prove that if \mathcal{L} is an ample \mathcal{O}_X -module, then condition (iv) holds. Consider a finite affine open covering (X_{f_i}) of X with $f_i \in S_{n_i}$; by changing f_i by its power, we can assume that the integers n_i equal to the same integer m . The sheaf $\mathcal{F}|_{X_{f_i}}$, being of finite type by hypotheses, is generated by a finitely number of sections h_{ij} over X_{f_i} ??). By ??, there then exists an integer k_0 such that the section $h_{ij} \otimes f_i^{k_0}$ extend to a section of $\mathcal{F}(k_0m)$ over X for any couple (i, j) . A fortiori the $h_{ij} \otimes f_i^k$ extend to sections of $\mathcal{F}(km)$ over X for $k \geq k_0$, and for such values of k , $\mathcal{F}(km)$ is then generated by global sections. For any integer p such that $0 < p < m$, $\mathcal{F}(p)$ is also of finite type, so there exist an integer k_p such that $\mathcal{F}(p)(km) = \mathcal{F}(p + km)$ is generated by global sections for $k \geq k_p$. Let n_0 be the largest of the $k_p m$ for $0 < p < m$; we then conclude that $\mathcal{F}(n)$ is generated by global sections for $n \geq n_0$. \square

Proposition 19.4.26. *Let X be a quasi-compact and quasi-separated scheme and \mathcal{L} be an invertible \mathcal{O}_X -module.*

- (a) *Let $n > 0$ be an integer. For \mathcal{L} to be ample, it is necessary and sufficient that $\mathcal{L}^{\otimes n}$ is ample.*
- (b) *Let \mathcal{L}' be an invertible \mathcal{O}_X -module such that, for any $x \in X$, there exists a section s' of $\mathcal{L}'^{\otimes n}$ over X such that $s'(x) \neq 0$. Then, if \mathcal{L} is ample, so is $\mathcal{L} \otimes \mathcal{L}'$.*

Proof. Property (a) is a consequence of (i) of ?? since $X_{f^{\otimes n}} = X_f$. On the other hand, if \mathcal{L} is ample, for any $x \in X$ and any neighborhood U of x , there exists $m > 0$ and $f \in \Gamma(X, f^{\otimes m})$ such that $x \in X_f \subseteq U$; if $f' \in \Gamma(X, \mathcal{L}'^{\otimes n})$ is such that $f'(x) \neq 0$, then $s(x) \neq 0$ for $s = f^{\otimes n} \otimes f'^{\otimes m} \in \Gamma(X, (\mathcal{L} \otimes \mathcal{L}')^{\otimes mn})$, so $x \in X_s \subseteq X_f \subseteq U$, and therefore $\mathcal{L} \otimes \mathcal{L}'$ is ample. \square

Corollary 19.4.27. *The tensor product of two ample \mathcal{O}_X -modules is ample.*

Proof. An ample \mathcal{O}_X -module satisfies the condition of ??(b). \square

Corollary 19.4.28. *Let \mathcal{L} be an ample \mathcal{O}_X -module and \mathcal{L}' be an invertible \mathcal{O}_X -module. There then exists an integer $n_0 > 0$ such that $\mathcal{L}^{\otimes n} \otimes \mathcal{L}'$ is ample and generated by global sections for $n \geq n_0$.*

Proof. It follows from ?? that there exists an integer m_0 such that for $m \geq m_0$, $\mathcal{L}^{\otimes m} \otimes \mathcal{L}'$ is generated by global sections, and therefore satisfies the condition of ??(b); we can then choose $n_0 = m_0 + 1$. \square

Remark 19.4.5. Let $P = \text{Pic}(X) = H^1(X, \mathcal{O}_X^\times)$ be the picard group of \mathcal{O}_X -modules, and let P^+ be the subset of P formed by ample sheaves. Suppose that P^+ is nonempty. Then it follows from ?? and ?? that we have

$$P^+ + P^+ \subseteq P^+, \quad P^+ - P^+ = P.$$

which means $P^+ \cup \{0\}$ is the set of positive elements of P for an order structure over P compatible with the group structure, which is archimedean in view of ??.

Proposition 19.4.29. *Let $q : X \rightarrow Y$ be a quasi-compact and quasi-separated morphism where Y is affine, and \mathcal{L} be an invertible \mathcal{O}_X -module.*

(a) *If \mathcal{L} is very ample relative to q then \mathcal{L} is ample.*

(b) *Suppose that q is of finite type. Then for \mathcal{L} to be ample, it is necessary and sufficient that it satisfies the following equivalent conditions:*

(v) *There exists $n_0 > 0$ such that for any integer $n \geq n_0$, $\mathcal{L}^{\otimes n}$ is very ample relative to q .*

(v') *There exists $n > 0$ such that $\mathcal{L}^{\otimes n}$ is very ample relative to q .*

Proof. The first assertion follows from the definition of very ample: if A is the ring of Y , there exists an A -module E and a surjective homomorphism

$$\psi : q^*(\widetilde{S(E)}) \rightarrow S(\mathcal{L})$$

such that $i = r_{\mathcal{L}, \psi}$ is an immersion from X to $P = \mathbb{P}(\widetilde{E})$ such that $\mathcal{L} \cong i^*(\mathcal{O}_P(1))$. As the $D_+(f)$ for $f \in S(E)_+$ homogeneous form a basis for P and $i^{-1}(D_+(f)) = X_{\psi(f)}$, we see that condition (i) of ?? holds, so \mathcal{L} is ample.

Now assume that q is of finite type and \mathcal{L} is ample. It follows from ??(ii) and ??(a) that there exists an integer $k_0 > 0$ such that $\mathcal{L}^{\otimes k_0}$ is very ample relative to q . On the other hand, in view of ??, there exists an integer m_0 such that, for $m \geq m_0$, $\mathcal{L}^{\otimes m}$ is generated by global sections. Put $n_0 = k_0 + m_0$; if $n \geq n_0$, we can write $n = k_0 + m$ where $m \geq m_0$, whence $\mathcal{L}^{\otimes n} = \mathcal{L}^{\otimes k_0} \otimes \mathcal{L}^{\otimes m}$. As $\mathcal{L}^{\otimes m}$ is generated by global sections, it follows from ?? and ?? that $\mathcal{L}^{\otimes n}$ is very ample relative to q . Finally, it is clear that (v) implies (v'), and (v') implies that \mathcal{L} is ample in view of (a) and ??.

Corollary 19.4.30. *Let $q : X \rightarrow Y$ be a quasi-compact and quasi-separated morphism of finite type where Y is affine, \mathcal{L} be an ample \mathcal{O}_X -module, and \mathcal{L}' be an invertible \mathcal{O}_X -module. Then there exists an integer $n_0 > 0$ such that for $n \geq n_0$, $\mathcal{L}^{\otimes n} \otimes \mathcal{L}'$ is very ample relative to q .*

Proof. In fact, there exists integer m_0 such that for $m \geq m_0$, $\mathcal{L}^{\otimes m} \otimes \mathcal{L}'$ is generated by global section ??; on the other hand, there exists k_0 such that $\mathcal{L}^{\otimes k}$ is very ample relative to q for $k \geq k_0$. Thus $\mathcal{L}^{\otimes(k+m_0)} \otimes \mathcal{L}'$ is very ample for $k \geq k_0$??). \square

Proposition 19.4.31. *Let X be a quasi-compact scheme, Z be a closed subscheme of X defined by a quasi-coherent nilpotent ideal \mathcal{I} of \mathcal{O}_X , and $j : Z \rightarrow X$ be the canonical injection. For an invertible \mathcal{O}_X -module \mathcal{L} to be ample, it is necessary and sufficient that $\mathcal{L}' = j^*(\mathcal{L})$ is an ample \mathcal{O}_Z -module.*

Proof. This condition is necessary. In fact, for any section f of $\mathcal{L}^{\otimes n}$ over X , let f' be the image $f \otimes 1$, which is a section of $\mathcal{L}'^{\otimes n} = \mathcal{L}^{\otimes n} \otimes_{\mathcal{O}_X} (\mathcal{O}_X / \mathcal{I})$ over the subspace Z (identified with X); it is clear that $Z_{f'} = X_f$, hence condition (i) of ?? shows that \mathcal{L}' is ample.

To prove the sufficiency, note first that we can reduce to the case $\mathcal{I}^2 = 0$ by considering the finite sequence of schemes $X_k = (X, \mathcal{O}_X / \mathcal{I}^{k+1})$, which is a closed subscheme of the previous one and is defined by a square zero ideal. Now X is quasi-separated if X_{red} is quasi-separated ??(vi)). Criterion (i) of ?? shows that it will suffice to prove that, if g is a section of $\mathcal{L}'^{\otimes n}$ over Z such that Z_g is affine, then there exists $m > 0$ such that $g^{\otimes m}$ is the canonical image of a section f of $\mathcal{L}^{\otimes nm}$ over X . For this, we consider the exact sequence

$$0 \longrightarrow \mathcal{I}(n) \longrightarrow \mathcal{O}_X(n) = \mathcal{L}^{\otimes n} \longrightarrow \mathcal{O}_Z(n) = \mathcal{L}'^{\otimes n} \longrightarrow 0$$

since $\mathcal{F}(n)$ is an exact functor on \mathcal{F} ; whence an exact sequence on cohomology:

$$0 \longrightarrow \Gamma(X, \mathcal{I}(n)) \longrightarrow \Gamma(X, \mathcal{L}^{\otimes n}) \longrightarrow \Gamma(X, \mathcal{L}'^{\otimes n}) \xrightarrow{\delta} H^1(X, \mathcal{I}(n))$$

which associates in particular g to an element $\delta g \in H^1(X, \mathcal{I}(n))$.

Note that since $\mathcal{I}^2 = 0$, \mathcal{I} can be considered as a quasi-coherent \mathcal{O}_Z -module and we have, for any k , $\mathcal{L}'^{\otimes k} \otimes_{\mathcal{O}_Z} \mathcal{I}(n) = \mathcal{I}(n+k)$. For any section $s \in \Gamma(X, \mathcal{L}^{\otimes k})$, tensoring by s is then a homomorphism $\mathcal{I}(n) \rightarrow \mathcal{I}(n+k)$ of \mathcal{O}_Z -modules, which gives a homomorphism $H^i(X, \mathcal{I}(n)) \rightarrow H^i(X, \mathcal{I}(n+k))$ of cohomology groups. We claim that

$$g^{\otimes m} \otimes \delta g = 0 \tag{19.4.9}$$

for $m > 0$ sufficiently large. In fact, Z_g is an affine open of Z and we have $H^1(Z_g, \mathcal{I}(n)) = 0$ where $\mathcal{I}(n)$ is considered as an \mathcal{O}_Z -module. In particular, if we put $g' = g|_{Z_g}$, and if we consider its image under $\delta : \Gamma(Z_g, \mathcal{L}'^{\otimes n}) \rightarrow H^1(Z_g, \mathcal{I}(n))$, we have $\delta g' = 0$. To explain this relation, observe that the first cohomology group of a sheaf coincides with the Čech cohomology; to form δg , it is necessary to consider an open covering (U_α) of X , which we can assume that is finite and formed by affine opens, and choose for each α a section $g_\alpha \in \Gamma(U_\alpha, \mathcal{L}^{\otimes n})$ whose image in $\Gamma(U_\alpha, \mathcal{L}'^{\otimes n})$ is $g|_{U_\alpha}$, and consider the class of cocycle $(g_{\alpha\beta} - g_{\beta\alpha})$, where $g_{\alpha\beta}$ is the restriction of g_α to $U_\alpha \cap U_\beta$ (a cocycle with values in $\mathcal{I}(n)$). We can moreover suppose that $\delta g'$ is in the same manner using the covering formed by $U_\alpha \cap Z_g$ and the restrictions $g_\alpha|_{U_\alpha \cap Z_g}$; the relation $\delta g' = 0$ signifies then that there exists for each α a section $h_\alpha \in \Gamma(U_\alpha \cap Z_g, \mathcal{I}(n))$ such that $(g_{\alpha\beta} - g_{\beta\alpha})|_{U_\alpha \cap U_\beta \cap Z_g} = h_{\alpha\beta} - h_{\beta\alpha}$, where $h_{\alpha\beta}$ denotes the restriction of h_α to $U_\alpha \cap U_\beta \cap Z_g$. Now by ?? there exists an integer $m > 0$ such that $g^{\otimes m} \otimes h_\alpha$ is the restriction to $U_\alpha \cap Z_g$ of a section $t_\alpha \in \Gamma(U_\alpha, \mathcal{I}(n+nm))$ for each α ; we then have $g^{\otimes m} \otimes (g_{\alpha\beta} - g_{\beta\alpha}) = t_{\alpha\beta} - t_{\beta\alpha}$ for any couple of indices, which proves $g^{\otimes m} \otimes \delta g = 0$.

We remark on the other hand that if $s \in \Gamma(X, \mathcal{O}_Z(p))$, $t \in \Gamma(X, \mathcal{O}_Z(q))$, we have, in the group $H^1(X, \mathcal{J}(p+q))$, that

$$\delta(s \otimes t) = (\delta s) \otimes t + s \otimes (\delta t). \quad (19.4.10)$$

For this, we can still consider an open cover (U_α) of X , and for each α a section $s_\alpha \in \Gamma(U_\alpha, \mathcal{O}_X(p))$ (resp. a section $t_\alpha \in \Gamma(U_\alpha, \mathcal{O}_X(q))$) whose canonical image in $\Gamma(U_\alpha, \mathcal{O}_Z(p))$ (resp. in $\Gamma(U_\alpha, \mathcal{O}_Z(q))$) is $s|_{U_\alpha}$; the relation (??) then follows from

$$(s_{\alpha\beta} \otimes t_{\alpha\beta}) - (s_{\beta\alpha} \otimes t_{\beta\alpha}) = (s_{\alpha\beta} - s_{\beta\alpha}) \otimes t_{\alpha\beta} + s_{\beta\alpha} \otimes (t_{\alpha\beta} - t_{\beta\alpha})$$

with the same notations before. By recurrence on k , we then have

$$\delta(g^{\otimes k}) = (kg^{\otimes(k-1)}) \otimes (\delta g) \quad (19.4.11)$$

and in view of (??) and (??), we have $\delta(g^{\otimes(m+1)}) = 0$, so $g^{\otimes(m+1)}$ is the canonical image of a section f of $\mathcal{L}^{\otimes n(m+1)}$ over X , whence our demonstration. \square

Corollary 19.4.32. *Let X be a Noetherian scheme and $j : X_{\text{red}} \rightarrow X$ be the canonical injection. For an invertible \mathcal{O}_X -module \mathcal{L} to be ample, it is necessary and sufficient that $j^*(\mathcal{L})$ is an ample $\mathcal{O}_{X_{\text{red}}}$ -module.*

Proof. The nilradical \mathcal{N}_X is nilpotent and we can apply ??.

19.4.6 Relatively ample sheaves

Let $f : X \rightarrow Y$ be a quasi-compact morphism and \mathcal{L} be an invertible \mathcal{O}_X -module. We say that \mathcal{L} is **ample relative to f** , or relative to Y , or **f -ample**, or **Y -ample**, if there exists an affine open covering (U_α) of Y such that if $X_\alpha = f^{-1}(U_\alpha)$, $\mathcal{L}|_{X_\alpha}$ is an ample \mathcal{O}_{X_α} -module for each α . Again, the existence of an f -ample \mathcal{O}_X -module implies that X is separated, so f is necessarily separated by ??.

Proposition 19.4.33. *Let $f : X \rightarrow Y$ be a quasi-compact morphism and \mathcal{L} be an invertible \mathcal{O}_X -module. If \mathcal{L} is very ample relative to f , then it is ample relative to f .*

Proof. If \mathcal{L} is very ample relative to f then the morphism f is separated, so by ??(a) the restriction $\mathcal{L}|_{f^{-1}(U)}$ for any affine open U of Y is very ample, hence ample. \square

Let $f : X \rightarrow Y$ be a quasi-compact morphism and \mathcal{L} be an invertible \mathcal{O}_X -module. We consider the graded \mathcal{O}_Y -algebra $\mathcal{S} = \bigoplus_{n \geq 0} f_*(\mathcal{L}^{\otimes n})$. Then the canonical homomorphisms $\sigma_n : f^*(f_*(\mathcal{L}^{\otimes n})) \rightarrow \mathcal{L}^{\otimes n}$ induce a canonical homomorphism

$$\sigma : f^*(\mathcal{S}) \rightarrow \bigoplus_{n \geq 0} \mathcal{L}^{\otimes n} = S(\mathcal{L}).$$

On the other hand, it is easy to see that σ^\flat is the canonical injection from \mathcal{S} into $f_*(S(\mathcal{L}))$. The homomorphism σ gives an Y -morphism

$$r_{\mathcal{L}, \sigma} : G(\sigma) \rightarrow \text{Proj}(\mathcal{S}) = P.$$

Proposition 19.4.34. *Let $f : X \rightarrow Y$ be a quasi-compact morphism and \mathcal{L} be an invertible \mathcal{O}_X -module. Then the following conditions are equivalent:*

- (i) \mathcal{L} is f -ample.
- (ii) \mathcal{S} is quasi-coherent and the Y -morphism $r_{\mathcal{L},\sigma}$ is everywhere defined and a dominant open immersion.
- (ii') The morphism f is separated, the Y -morphism $r_{\mathcal{L},\sigma}$ is everywhere defined and is a homeomorphism from X onto a subspace of $\text{Proj}(\mathcal{S})$.

Moreover, if these are satisfied, for any $n \in \mathbb{Z}$ the canonical homomorphism of ??

$$\nu : r_{\mathcal{L},\sigma}^*(\mathcal{O}_P(n)) \rightarrow \mathcal{L}^{\otimes n} \quad (19.4.12)$$

is an isomorphism. Finally, for any quasi-coherent \mathcal{O}_X -module \mathcal{F} , if we put $\mathcal{M} = \bigoplus_{n \geq 0} f_*(\mathcal{F} \otimes \mathcal{L}^{\otimes n})$, the canonical homomorphism of ??

$$\xi : r_{\mathcal{L},\sigma}^*(\tilde{\mathcal{M}}) \rightarrow \mathcal{F} \quad (19.4.13)$$

is an isomorphism.

Proof. We have remarked that (i) implies that f is separated, so \mathcal{S} is quasi-coherent by ?? As the fact that $r_{\mathcal{L},\sigma}$ is an open immersion everywhere defined is local over Y , to shows that (i) implies (ii), we can assume that Y is affine and \mathcal{L} is ample; the assertion then follows from ?? It is clear that (ii) implies (ii'); finally, to show that (ii') implies (i), it suffices to consider an affine open cover (U_α) of Y and use ??(ii') to $\mathcal{L}|_{X_\alpha}$.

To prove the last two assertions, we use the fact that σ^\flat is the canonical injection of \mathcal{S} to $f_*(\mathcal{S}(\mathcal{L}))$ and the expression of the morphisms ν and ξ in ?? and ?? It then follows that ν and ξ are injective. As for the surjectivity, we can assume that Y is affine, so \mathcal{L} is ample; the criterion of ??(iii) then shows that ν and ξ are surjective, whence the assertion. \square

Remark 19.4.6. From ?? and its proof, we conclude that if \mathcal{L} is f -ample then $f_*(\mathcal{S}(\mathcal{L}))$ is equal to \mathcal{S} , so the homomorphism σ^\flat is the identity on \mathcal{S} . This can also be seen from ?? since in this case f is separated.

Corollary 19.4.35. *Let (U_α) be an open covering of Y . For \mathcal{L} to be Y -ample, it is necessary and sufficient that $\mathcal{L}|_{f^{-1}(U_\alpha)}$ is U_α -ample for each α .*

Proof. This is ture since the condition (ii) of ?? is local over Y . \square

Corollary 19.4.36. *Let \mathcal{K} be an invertible \mathcal{O}_Y -module. For \mathcal{L} to be Y -ample, it is necessary and sufficient that $\mathcal{L} \otimes f^*(\mathcal{K})$ is Y -ample.*

Proof. This is a consequence of ?? by taking U_α to be such that $\mathcal{K}|_{U_\alpha}$ is isomorphic to $\mathcal{O}_Y|_{U_\alpha}$ for each α . \square

Corollary 19.4.37. *Suppose that Y is affine. For \mathcal{L} to be Y -ample, it is necessary and sufficient that \mathcal{L} is ample.*

Proof. This is immediate from the definition of Y -ample, and ??(ii) and ??(ii), since

$$\text{Proj}(\mathcal{S}) = \text{Proj}(\Gamma(Y, \mathcal{S})) = \text{Proj}\left(\bigoplus_{n \geq 0} \Gamma(Y, f_*(\mathcal{L}^{\otimes n}))\right) = \text{Proj}(S)$$

in this case (note that Y is quasi-compact). \square

Corollary 19.4.38. *Let $f : X \rightarrow Y$ be a quasi-compact morphism. Suppose that there exists a quasi-coherent \mathcal{O}_Y -module \mathcal{E} and a Y -morphism $i : X \rightarrow P = \mathbb{P}(\mathcal{E})$ which is a homeomorphism from X onto a subspace of P . Then $\mathcal{L} = i^*(\mathcal{O}_P(1))$ is Y -ample.*

Proof. We can assume that Y is affine, and the corollary then follows from the criterion (i) of ?? and the formula (??). \square

Proposition 19.4.39. *Let X be a quasi-compact and quasi-separated scheme and $f : X \rightarrow Y$ be a quasi-compact and quasi-separated morphism. For an invertible \mathcal{O}_X -module \mathcal{L} to be f -ample, it is necessary and sufficient that following equivalent conditions are satisfied:*

- (iii) *For any quasi-coherent \mathcal{O}_X -module \mathcal{F} of finite type, there exists an integer $n_0 > 0$ such that, for any $n \geq n_0$, the canonical homomorphism $\sigma : f^*(f_*(\mathcal{F} \otimes \mathcal{L}^{\otimes n})) \rightarrow \mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is surjective.*
- (iii') *For any ideal \mathcal{I} of \mathcal{O}_X of finite type, there exist an integer $n > 0$ such that the canonical homomorphism $\sigma : f^*(f_*(\mathcal{I} \otimes \mathcal{L}^{\otimes n})) \rightarrow \mathcal{I} \otimes \mathcal{L}^{\otimes n}$ is surjective.*

Proof. \square

Proposition 19.4.40. *Let $f : X \rightarrow Y$ be a quasi-compact morphism and \mathcal{L} be an invertible \mathcal{O}_X -module.*

- (a) *Let $n > 0$ be an integer. For \mathcal{L} to be f -ample, it is necessary and necessary that $\mathcal{L}^{\otimes n}$ is f -ample.*
- (b) *Let \mathcal{L}' be an invertible \mathcal{O}_X -module, and suppose that there exists an integer $n > 0$ such that the canonical homomorphism $\sigma : f^*(f_*(\mathcal{L}'^{\otimes n})) \rightarrow \mathcal{L}'^{\otimes n}$ is surjective. Then, if \mathcal{L} is f -ample, so is $\mathcal{L} \otimes \mathcal{L}'$.*

Corollary 19.4.41. *The tensor product of two f -ample \mathcal{O}_X -modules is f -ample.*

Proposition 19.4.42. *Let $f : X \rightarrow Y$ be a morphism of finite type where Y is quasi-compact, and \mathcal{L} be an invertible \mathcal{O}_X -module. For \mathcal{L} to be f -ample, it is necessary and sufficient that the following equivalent conditions are satisfied:*

- (iv) *There exists $n_0 > 0$ such that, for any integer $n \geq n_0$, $\mathcal{L}^{\otimes n}$ is very ample relative to f .*
- (iv') *There exist $n > 0$ such that $\mathcal{L}^{\otimes n}$ is very ample relative to f .*

Lemma 19.4.43. *Let $u : Z \rightarrow S$ be a morphism, \mathcal{L} be an invertible \mathcal{O}_S -module, s a section of \mathcal{L} over S , and t be the inverse image of s under u . Then $Z_t = u^{-1}(S_s)$.*

Proof. \square

Lemma 19.4.44. *Let Z, Z' be two S -schemes, p, p' be the projections of $T = Z \times_S Z'$, \mathcal{L} (resp. \mathcal{L}') be an invertible \mathcal{O}_Z -module (resp. $\mathcal{O}_{Z'}$ -module), t (resp. t') be a section of \mathcal{L} (resp. \mathcal{L}') over Z (resp. Z'), s (resp. s') be the inverse image of t (resp. t') under p (resp. p'). Then we have $T_{s \otimes s'} = Z_t \times_S Z'_{t'}$.*

Proof.

□

Proposition 19.4.45 (Properties of Relative Ample Sheaves).

- (i) For any scheme Y , any invertible \mathcal{O}_Y -module \mathcal{L} is relative ample relative to the identity morphism 1_Y .
- (ii) Let $f : X \rightarrow Y$ be a quasi-compact morphism and $j : X' \rightarrow X$ be a quasi-compact morphism that is a homeomorphism from X' onto a subspace of X . If \mathcal{L} is an \mathcal{O}_X -module relative to f , then $j^*(\mathcal{L})$ is relative ample relative to $f \circ j$.
- (iii) Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be quasi-compact morphisms where Z is quasi-compact. Let \mathcal{L} be an ample \mathcal{O}_X -module relative to f and \mathcal{K} be an ample \mathcal{O}_Y -module relative to g . Then there exists an integer $n_0 > 0$ such that for any $n \geq n_0$, $\mathcal{L} \otimes f^*(\mathcal{K}^{\otimes n})$ is ample relative to $g \circ f$.
- (iv) Let $f : X \rightarrow Y$ be a quasi-compact morphism, $g : Y' \rightarrow Y$ be a morphism, and put $X' = X_{(Y')}$. If \mathcal{L} is an ample \mathcal{O}_X -module relative to f , $\mathcal{L}' = \mathcal{L} \otimes_Y \mathcal{O}_{Y'}$ is ample relative to $f_{(Y')}$.
- (v) Let $f_i : X_i \rightarrow Y_i$ ($i = 1, 2$) be two quasi-compact S -morphisms. If \mathcal{L}_i is an ample \mathcal{O}_{X_i} -module relative to f_i , then $\mathcal{L}_1 \otimes_S \mathcal{L}_2$ is ample relative to $f_1 \times_S f_2$.
- (vi) Let $f : X \rightarrow Y$, $g : Y \rightarrow Z$ be two morphisms such that $g \circ f$ is quasi-compact. If an \mathcal{O}_X -module \mathcal{L} is ample relative to $g \circ f$ and if g is separated or X is locally Noetherian, then \mathcal{L} is ample relative to f .
- (vii) Let $f : X \rightarrow Y$ be a quasi-compact morphism and $j : X_{\text{red}} \rightarrow X$ be the canonical injection. If \mathcal{L} is an ample \mathcal{O}_X -module relative to f , then $j^*(\mathcal{L})$ is ample relative to f_{red} .

Proof.

□

Proposition 19.4.46. Let $f : X \rightarrow Y$ be a quasi-compact morphism, \mathcal{I} be a locally nilpotent ideal of \mathcal{O}_X , Z the closed subscheme of X defined by \mathcal{I} , and $j : Z \rightarrow X$ be the canonical injection. For an invertible \mathcal{O}_X -module to be ample relative to f , it is necessary and sufficient that $j^*(\mathcal{L})$ is ample relative to $f \circ j$.

Corollary 19.4.47. Let X be a locally Noetherian scheme and $f : X \rightarrow Y$ be a quasi-compact morphism. For an invertible \mathcal{O}_X -module \mathcal{L} to be ample relative to f , it is necessary and sufficient that its inverse image \mathcal{L}' under the canonical injection $X_{\text{red}} \rightarrow X$ is ample relative to f_{red} .

Proposition 19.4.48. Let $f : X \rightarrow Y$, $f' : X' \rightarrow Y$ be two morphisms, $X'' = X \amalg X'$, and $f'' = f \amalg f'$. Let \mathcal{L} (resp. \mathcal{L}') be an invertible \mathcal{O}_X -module (resp. $\mathcal{O}_{X'}$ -module), and let \mathcal{L}'' be the invertible $\mathcal{O}_{X''}$ -module which coincides with \mathcal{L} over X and with \mathcal{L}' over X' . For \mathcal{L}'' to be ample relative to f'' , it is necessary and sufficient that \mathcal{L} is ample relative to f and \mathcal{L}' is ample relative to f' .

Proposition 19.4.49. Let Y be a quasi-compact scheme, \mathcal{S} be a quasi-coherent graded \mathcal{O}_Y -algebra of finite type, $X = \text{Proj}(\mathcal{S})$, and $f : X \rightarrow Y$ be the structural morphism. Then f is of finite type and there exists an integer $n > 0$ such that $\mathcal{O}_X(n)$ is invertible and f -ample.

19.5 Quasi-affine, proper, and projective morphisms

19.5.1 Quasi-affine morphisms

We say a scheme is **quasi-affine** if it is isomorphic to the subscheme induced over a quasi-compact open subset of an affine scheme. We say a morphism $f : X \rightarrow Y$ is quasi-affine, or that X is a quasi-affine Y -scheme, if there exists an affine open cover (U_α) of Y such that $f^{-1}(U_\alpha)$ is a quasi-affine scheme. Since any quasi-compact open subscheme of an affine scheme is separated, it is clear that quasi-affine morphisms are separated and quasi-compact, and any affine morphism is quasi-affine.

Recall that for any scheme X , if $A = \Gamma(X, \mathcal{O}_X)$, the identify homomorphism $A \rightarrow A$ induces a canonical morphism $q : X \rightarrow \text{Spec}(A)$ by [??](#). This is also the morphism $r_{\mathcal{L}, \epsilon} : X \rightarrow \text{Proj}(S)$ in [\(??\)](#) defined for $\mathcal{L} = \mathcal{O}_X$, since $\Gamma(X, -)$ commutes with taking tensor product with \mathcal{O}_X (cf. [??](#)) and we have $S = \bigoplus_{n \geq 0} \Gamma(X, \mathcal{O}_X)^{\otimes n} = A[T]$, and $\text{Proj}(A[T])$ is canonically identified with $\text{Spec}(A)$.

Proposition 19.5.1. *Let X be a quasi-compact scheme and $A = \Gamma(X, \mathcal{O}_X)$. The following conditions are equivalent:*

- (i) *X is a quasi-affine scheme.*
- (ii) *The canonical morphism $q : X \rightarrow \text{Spec}(A)$ is an open immersion.*
- (ii') *The canonical morphism $q : X \rightarrow \text{Spec}(A)$ is a homeomorphism from X onto a subspace of $\text{Spec}(A)$.*
- (iii) *The \mathcal{O}_X -module \mathcal{O}_X is very ample relative to q .*
- (iii') *The \mathcal{O}_X -module \mathcal{O}_X is ample.*
- (iv) *The subsets X_f , as f runs through A , form a basis for X .*
- (iv') *The subsets X_f which are affine, as f runs through A , form a basis for X .*
- (v) *Any quasi-coherent \mathcal{O}_X -module is generated by its global sections.*
- (v') *Any quasi-coherent ideal of finite type of \mathcal{O}_X is generated by its global sections.*

Proof. It is clear that (ii) \Rightarrow (i) by definition, and (iii) \Rightarrow (iii') by [??](#). Since the canonical morphisms $q : X \rightarrow \text{Spec}(A)$ and $r_{\mathcal{O}_X, \epsilon} : X \rightarrow \text{Proj}(S)$ are identified, we see that (iii') \Leftrightarrow (ii) \Leftrightarrow (ii') \Leftrightarrow (iv) \Leftrightarrow (iv') by [??](#). Also, (iii') \Leftrightarrow (v) \Leftrightarrow (v') in view of [??](#).

We also note that if X is quasi-affine, then it can be identified as an open subscheme of an affine scheme $Y = \text{Spec}(B)$. Let $\varphi : B \rightarrow A$ be the correspond homomorphism [??](#). Since the affine opens $D(g)$, with $g \in B$, form a basis of Y , and we have $X_f = D(g) \cap X$ where $f = \varphi(g)$, it follows that the subsets X_f which are affine, with $f \in A$, form a basis for X , which proves (i) \Rightarrow (iv').

Finally, it remains to show that (i) \Rightarrow (iii). For this we first note that if X is quasi-affine then it is quasi-compact and separated, so by [??](#), for $f \in A$ we have $\Gamma(X_f, \mathcal{O}_X) = A_f$. Since we have $q^{-1}(D_+(f)) = X_f$, we conclude that the canonical morphism $q : X \rightarrow \text{Spec}(A)$ is of finite type, and by [??](#), since $\mathcal{O}_X^{\otimes n}$ is isomorphic to \mathcal{O}_X for any integer $n > 0$, \mathcal{O}_X is very ample relative to q . This completes the proof. \square

Remark 19.5.1. Let X be a quasi-affine scheme and $A = \Gamma(X, \mathcal{O}_X)$. By ?? we know that \mathcal{O}_X is very ample relative to $q : X \rightarrow \text{Spec}(A)$. Since X is separated and quasi-compact, by ?? we know that $q_*(\mathcal{O}_X)$ is quasi-coherent, and from $\Gamma(\text{Spec}(A), q_*(\mathcal{O}_X)) = A$ we conclude that $q_*(\mathcal{O}_X) = \tilde{A}$, so $q^*(q_*(\mathcal{O}_X)) = \mathcal{O}_X$ and the canonical homomorphism $\sigma : q^*(q_*(\mathcal{O}_X)) \rightarrow \mathcal{O}_X$ is identified with the identity on \mathcal{O}_X . This being so, the canonical morphism $r_{\mathcal{O}_X, \sigma} : X \rightarrow \mathbb{P}(q_*(\mathcal{O}_X))$ is then identified with $q : X \rightarrow \text{Spec}(A)$, because we have

$$\mathbb{P}(q_*(\mathcal{O}_X)) = \text{Proj}(\mathbf{S}(q_*(\mathcal{O}_X))) = \text{Proj}(\mathbf{S}(A)) = \text{Proj}(A[T]) = \text{Spec}(A);$$

and we conclude from ?? that this is an open immersion, which justifies ??.

Corollary 19.5.2. Let X be a quasi-compact scheme. If there exists a morphism $r : X \rightarrow Y$ from X into an affine scheme Y which is a homeomorphism onto an open subspace of Y , then X is quasi-affine.

Proof. In fact, there then exists a family (g_α) of sections of \mathcal{O}_Y over Y such that the $D(g_\alpha)$ form a basis for the topology of $r(X)$. If we put $f_\alpha = \theta(g_\alpha)$ where $\theta : \Gamma(Y, \mathcal{O}_Y) \rightarrow \Gamma(X, \mathcal{O}_X)$ is the corresponding ring homomorphism, then we have $X_{f_\alpha} = r^{-1}(D(g_\alpha))$, so the X_{f_α} form a basis for X , and by ?? X is then quasi-affine. \square

Corollary 19.5.3. If X is a quasi-affine scheme, any invertible \mathcal{O}_X -module is very ample (relative to the canonical morphism $q : X \rightarrow \text{Spec}(A)$) and a fortiori ample.

Proof. In fact any such module \mathcal{L} is generated by its global sections ??(v)), so $\mathcal{L} \otimes \mathcal{O}_X$ is very ample by ?. We also note that the morphism q is of finite type. \square

Corollary 19.5.4. Let X be a quasi-compact scheme. If there exists an invertible \mathcal{O}_X -module \mathcal{L} such that \mathcal{L} and \mathcal{L}^{-1} are ample, then X is quasi-affine.

Proof. In fact, $\mathcal{O}_X = \mathcal{L} \otimes \mathcal{L}^{-1}$ is then ample by ?. \square

Proposition 19.5.5. Let $f : X \rightarrow Y$ be a quasi-compact morphism. The following conditions are equivalent:

- (i) f is quasi-affine.
- (ii) The \mathcal{O}_Y -algebra $f_*(\mathcal{O}_X) = \mathcal{A}(X)$ is quasi-coherent and the canonical morphism $X \rightarrow \text{Spec}(\mathcal{A}(X))$ corresponding to the identity homomorphism of $\mathcal{A}(X)$ is an open immersion.
- (ii') The \mathcal{O}_Y -algebra $\mathcal{A}(X)$ is quasi-coherent and the canonical morphism $X \rightarrow \text{Spec}(\mathcal{A}(X))$ is a homeomorphism from X onto a subspace of $\text{Spec}(\mathcal{A}(X))$.
- (iii) The \mathcal{O}_X -module \mathcal{O}_X is very ample relative to f .
- (iii') The \mathcal{O}_X -module \mathcal{O}_X is ample relative to f .
- (iv) The morphism f is separated and for any quasi-coherent \mathcal{O}_X -module \mathcal{F} , the canonical homomorphism $\sigma : f^*(f_*(\mathcal{F})) \rightarrow \mathcal{F}$ is surjective.

Moreover, if f is quasi-affine, any invertible \mathcal{O}_X -module \mathcal{L} is very ample relative to f .

Proof. The equivalence of these properties follows from the fact that they are all local over Y and the criteria of ?. Also, we note that $f_*(\mathcal{F})$ is quasi-coherent if f is separated (?). The last assertion follows from ?. \square

Corollary 19.5.6. *Let Y be an affine scheme and $f : X \rightarrow Y$ be a quasi-compact morphism. For f to be quasi-affine, it is necessary and sufficient that X is quasi-affine scheme.*

Proof. This is an immediate consequence of ?? and ??.

□

Corollary 19.5.7. *Let Y be a quasi-compact and quasi-separated scheme, $f : X \rightarrow Y$ be a morphism of finite type. If f is quasi-affine, there exists a quasi-coherent sub- \mathcal{O}_Y -algebra \mathcal{B} of $\mathcal{A}(X)$ of finite type such that the morphism $X \rightarrow \text{Spec}(\mathcal{B})$ corresponding to the canonical injection $\mathcal{B} \rightarrow \mathcal{A}(X)$ is an immersion. Moreover, any quasi-coherent sub- \mathcal{O}_Y -algebra of finite type \mathcal{B}' of $\mathcal{A}(X)$, containing \mathcal{B} , has the same property.*

Proof. In fact, $\mathcal{A}(X)$ is the inductive limit of its quasi-coherent sub- \mathcal{O}_Y -algebras of finite type ??; the assertion is then a particular case of ??, in view of the identification of $\text{Spec}(\mathcal{A}(X))$ and $\text{Proj}(\mathcal{A}(X)[T])$?? and the canonical morphisms from X into them (cf. ??). □

Proposition 19.5.8 (Properties of Quasi-affine Morphisms).

- (i) *A quasi-compact morphism $f : X \rightarrow Y$ that is a homeomorphism from X onto a subspace of Y (and in particular a quasi-compact immersion) is quasi-affine.*
- (ii) *The composition of two quasi-affine morphisms is quasi-affine.*
- (iii) *If $f : X \rightarrow Y$ is a quasi-affine S -morphism, $f_{(S')} : X_{(S')} \rightarrow Y_{(S')}$ is a quasi-affine morphism for any base change $S' \rightarrow S$.*
- (iv) *If $f : X \rightarrow Y$ and $g : X' \rightarrow Y'$ are two quasi-affine S -morphisms, so is $f \times_S g$.*
- (v) *If $f : X \rightarrow Y$, $g : Y \rightarrow Z$ are two morphisms such that $g \circ f$ is quasi-affine and if g is separated or X is locally Noetherian, then f is quasi-affine.*
- (vi) *Let $f : X \rightarrow Y$ be a quasi-compact morphism and $j : X_{\text{red}} \rightarrow X$ the canonical injection. If an \mathcal{O}_X -module \mathcal{L} is ample relative to f , then $j^*(\mathcal{L})$ is ample relative to f_{red} .*

Proof. In view of the criterion (iii') of ??, (i), (iii), (iv), (v) and (vi) are consequences of ?? To prove (ii), we can assume that Z is affine, and the assertion then follows from ??(iii), applied to $\mathcal{L} = \mathcal{O}_X$ and $\mathcal{K} = \mathcal{O}_Y$. □

Remark 19.5.2. Let $f : X \rightarrow Y$, $g : Y \rightarrow Z$ be two morphisms such that $X \times_Z Y$ is locally Noetherian. Then the graph morphism $\Gamma_f : X \rightarrow X \times_Z Y$ is a quasi-compact immersion, hence quasi-affine, and the reasoning of ?? shows that the conclusion of (v) remains valid if we remove the hypothesis that g is separated.

Proposition 19.5.9. *Let $f : X \rightarrow Y$ be a quasi-compact morphism and $g : X' \rightarrow X$ be a quasi-affine morphism. If \mathcal{L} is an f -ample \mathcal{O}_X -module, then $g^*(\mathcal{L})$ is ample relative to $f \circ g$.*

Proof. By hypothesis $\mathcal{O}_{X'}$ is very ample relative to f , and since the question is local over Y , it follows from ??(iii) that there exists (for Y affine) an integer n such that $g^*(\mathcal{L}^{\otimes n}) = (g^*(\mathcal{L}))^{\otimes n}$ is ample relative to $f \circ g$, whence $g^*(\mathcal{L})$ is ample relative to $f \circ g$. □

19.5.2 Serre's criterion on affineness

Theorem 19.5.10 (Serre's criterion). *For a quasi-compact and quasi-separated scheme X , then the following conditions are equivalent:*

- (i) X is an affine scheme.
- (ii) There exists a family (f_α) of elements of $A = \Gamma(X, \mathcal{O}_X)$ such that X_{f_α} are affine and the ideal generated by the f_α equals to A .
- (iii) The functor $\Gamma(X, -)$ is exact on the category of quasi-coherent \mathcal{O}_X -modules.
- (iii') For any exact sequence of quasi-coherent \mathcal{O}_X -modules

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$$

where \mathcal{F} is isomorphic to a sub- \mathcal{O}_X -module of a finite product \mathcal{O}_X^n , the induced sequence

$$0 \longrightarrow \Gamma(X, \mathcal{F}') \longrightarrow \Gamma(X, \mathcal{F}) \longrightarrow \Gamma(X, \mathcal{F}'') \longrightarrow 0$$

is exact.

- (iv) $H^1(X, \mathcal{F}) = 0$ for any quasi-coherent \mathcal{O}_X -module \mathcal{F} .

- (iv') $H^1(X, \mathcal{I}) = 0$ for any quasi-coherent ideal \mathcal{I} of \mathcal{O}_X .

Proof. It is clear that (i) implies (ii); (ii) implies on the other hand that the X_{f_α} cover X , since by hypothesis the unit section 1 is a linear combination of f_α , and that the $D(f_\alpha)$ cover $\text{Spec}(A)$. The last assertion of ?? then implies that $X \rightarrow \text{Spec}(A)$ is an isomorphism.

We have seen that (i) implies (iii), and it is trivial that (iii) implies (iii'). On the other hand, (iii') implies that, for any closed point $x \in X$ and any open neighborhood U of x , there exists $f \in A$ such that $x \in X_f \subseteq X - U$. To see this, let \mathcal{I} (resp. \mathcal{I}') be the quasi-coherent ideal of \mathcal{O}_X defining the reduced closed subscheme of X with underlying space $X - U$ (resp. $(X - U) \cup \{x\}$). It is clear that $\mathcal{I}' \subseteq \mathcal{I}$, and the quotient $\mathcal{I}'' = \mathcal{I}/\mathcal{I}'$ is a quasi-coherent \mathcal{O}_X -module. By hypothesis, the stalk of $\mathcal{O}_X/\mathcal{I}$ and $\mathcal{O}_X/\mathcal{I}'$ are zero at any point $x \in U - \{x\}$. Moreover, since $\{x\}$ is closed in X , the subscheme $X - U$ is *open and closed* in $(X - U) \cup \{x\}$, so we conclude that $(\mathcal{O}_X/\mathcal{I})_z = (\mathcal{O}_X/\mathcal{I}')_z$ for $z \in X - U$, and therefore $\mathcal{I}''_z = 0$. At the point x , we have $\mathcal{I}_x = \mathcal{O}_X$, while $\mathcal{I}'_x = \mathfrak{m}_x$ (cf. ??), so \mathcal{I}'' is supported at $\{x\}$ and $\mathcal{I}''_x = \kappa(x)$. The hypothesis of (iii') applied to the exact sequence $0 \rightarrow \mathcal{I}' \rightarrow \mathcal{I} \rightarrow \mathcal{I}'' \rightarrow 0$ shows that $\Gamma(X, \mathcal{I}) \rightarrow \Gamma(X, \mathcal{I}'')$ is surjective, so the section of \mathcal{I}'' whose germ at x equals to 1_x is the image of a section $f \in \Gamma(X, \mathcal{I}) \subseteq \Gamma(X, \mathcal{O}_X)$, and we have by definition $f(x) = 1_x$ and $f(y) = 0$ over $X - U$, which proves the assertion. Moreover, if U is affine, so is X_f , and the union X' of these affine opens X_f (with $f \in A$) is then an open subset of X containing any closed point of X . As X is a quasi-compact Kolmogoroff space, we then have $X' = X$ (??). Since X is quasi-compact, there are finitely many elements $f_i \in A$ ($1 \leq i \leq n$) such that (X_{f_i}) is an affine open cover of X . Consider the homomorphism $\mathcal{O}_X^n \rightarrow \mathcal{O}_X$ defined by the sections f_i ; since for any $x \in X$ at least one of the $(f_i)_x$ is invertible, this homomorphism is surjective, and we then get an exact sequence

$$0 \longrightarrow \mathcal{R} \longrightarrow \mathcal{O}_X^n \longrightarrow \mathcal{O}_X \longrightarrow 0$$

where \mathcal{R} is a quasi-coherent sub- \mathcal{O}_X -module of \mathcal{O}_X . It then follows from (iii') that the corresponding homomorphism $\Gamma(X, \mathcal{O}_X^n) \rightarrow \Gamma(X, \mathcal{O}_X)$ is surjective, which proves (ii).

Finally, (i) implies (iv) and (iv) implies (iv'). We show that (iv') implies (iii'). Now, if \mathcal{F}' is a quasi-coherent sub- \mathcal{O}_X -module of \mathcal{O}_X^n , the filtration $0 \subseteq \mathcal{O}_X \subseteq \mathcal{O}_X^2 \cdots \subseteq \mathcal{O}_X^n$ defines over \mathcal{F}' a filtration of the form $\mathcal{F}'_k = \mathcal{F}' \cap \mathcal{O}_X^k$ ($0 \leq k \leq n$), which are quasi-coherent \mathcal{O}_X -modules (??(ii)), and $\mathcal{F}'_{k+1}/\mathcal{F}'_k$ is isomorphic to a quasi-coherent sub- \mathcal{O}_X -module of $\mathcal{O}_X^{k+1}/\mathcal{O}_X^k = \mathcal{O}_X$, which is thus a quasi-coherent ideal of \mathcal{O}_X . In the exact sequence

$$H^1(X, \mathcal{F}'_k) \longrightarrow H^1(X, \mathcal{F}'_{k+1}) \longrightarrow H^1(\mathcal{F}'_{k+1}/\mathcal{F}'_k) = 0$$

by hypothesis of (iv') we have $H^1(\mathcal{F}'_{k+1}/\mathcal{F}'_k) = 0$; since $H^0(X, \mathcal{F}'_0) = 0$, we conclude by recurrence on k that $H^1(X, \mathcal{F}'_k) = 0$ for each k , whence the claim. \square

Remark 19.5.3. Note that if X is a covering of (X_{f_i}) with X_{f_i} being affine, then X is automatically quasi-separated, since for any couple (i, j) of indices we have $X_{f_i} \cap X_{f_j} = D_{X_{f_i}}(f_j|_{X_{f_i}})$, which is an affine open of X_{f_i} and hence quasi-compact ??.

Remark 19.5.4. If X is a Noetherian scheme, then in conditions (iii') and (iv') we can replace "quasi-coherent" by "coherent." In fact, in the demonstration that (iii') implies (ii), \mathcal{I} and \mathcal{I}' are then coherent ideals, and moreover, any quasi-coherent submodule of a coherent module is coherent ??, whence the assertion.

Corollary 19.5.11. Let $f : X \rightarrow Y$ be a quasi-compact and quasi-separated morphism. Then the following conditions are equivalent:

- (i) f is an affine morphism.
- (ii) The functor f_* is exact on the category of quasi-coherent \mathcal{O}_X -modules.
- (iii) For any quasi-coherent \mathcal{O}_X -module \mathcal{F} , we have $R^1 f_*(\mathcal{F}) = 0$.
- (iii') For any quasi-coherent ideal \mathcal{I} of \mathcal{O}_X , we have $R^1 f_*(\mathcal{I}) = 0$.

Proof. Any of these conditions are local over Y , by the definition of $R^1 f_*(\mathcal{F})$ (that is, the sheaf associated with the presheaf $U \mapsto H^1(f^{-1}(U), \mathcal{F})$), so we may assume that Y is affine. If f is affine, X is then affine and (ii) follows from ???. Conversely, we prove that (ii) implies (i): for any quasi-coherent \mathcal{O}_X -module \mathcal{F} , $f_*(\mathcal{F})$ is a quasi-coherent \mathcal{O}_Y -module by ???. By hypothesis the functor f_* is exact; $\Gamma(Y, -)$ is exact since Y is affine, so we conclude that $\Gamma(Y, f_*(-)) = \Gamma(X, -)$ is exact, which proves that X is affine in view of ??.

If f is affine, $f^{-1}(U)$ is affine for any affine open U of Y , so $H^1(f^{-1}(U), \mathcal{F}) = 0$ by ??, which means $R^1 f_*(\mathcal{F}) = 0$. Finally, suppose that (iii') is satisfied; the exact sequence of low-degree terms in the Leray spectral sequence gives

$$0 \longrightarrow H^1(Y, f_*(\mathcal{I})) \longrightarrow H^1(X, \mathcal{I}) \longrightarrow H^0(Y, R^1 f_*(\mathcal{I}))$$

As Y is affine and $f_*(\mathcal{I})$ is quasi-coherent ??, we have $H^1(Y, f_*(\mathcal{I})) = 0$, so the hypothesis of (iii') implies that $H^1(X, \mathcal{I}) = 0$, and we conclude from ?? that X is an affine scheme. \square

Corollary 19.5.12. If $f : X \rightarrow Y$ is an affine morphism then for any quasi-coherent \mathcal{O}_X -module \mathcal{F} , the canonical homomorphism $H^1(Y, f_*(\mathcal{F})) \rightarrow H^1(X, \mathcal{F})$ is bijective.

Proof. In fact, we have an exact sequence

$$0 \longrightarrow H^1(Y, f_*(\mathcal{F})) \longrightarrow H^1(X, \mathcal{F}) \longrightarrow H^0(Y, R^1 f_*(\mathcal{F}))$$

which comes from the lower terms of the Leray spectral sequence, and the conclusion follows from ??.

□

19.5.3 Quasi-projective morphisms

We say a morphism $f : X \rightarrow Y$ is **quasi-projective**, or that X is **quasi-projective over Y** , or that X is a **quasi-projective Y -scheme**, if f is of finite type and there exists an invertible \mathcal{O}_X -module that is f -ample. It is clear that a quasi-projective morphism is necessarily separated. If Y is quasi-compact, it is also equivalent to say that f is of finite type and there exists a very ample \mathcal{O}_X -module relative to f (??).

Remark 19.5.5. It should be noted that this definition is not local over Y . There exist examples where X and Y are nonsingular algebraic schemes over an algebraically closed field such that any point of Y admits an affine neighborhood U such that $f^{-1}(U)$ is quasi-projective over U , but f is not quasi-projective.

Proposition 19.5.13. *Let Y be a quasi-compact and quasi-separated scheme and X be a Y -scheme. Then the following conditions are equivalent:*

- (i) X is a quasi-projective Y -scheme.
- (ii) X is of finite type over Y and there exists a quasi-coherent \mathcal{O}_Y -module \mathcal{E} of finite type such that X is Y -isomorphic to a subscheme of $\mathbb{P}(\mathcal{E})$.
- (iii) X is of finite type over Y and there exists a quasi-coherent graded \mathcal{O}_Y -algebra \mathcal{S} that is generated by \mathcal{S}_1 and \mathcal{S}_1 is of finite type such that X is isomorphic to a dense open subscheme of $\text{Proj}(\mathcal{S})$.

Proof. This follows from ??, ?? and ??.

□

Corollary 19.5.14. *Let Y be a quasi-compact and quasi-separated scheme such that there exists an ample \mathcal{O}_Y -module \mathcal{L} . For a Y -scheme X to be quasi-projective, it is necessary and sufficient that X is of finite type over Y and is isomorphic to a sub- Y -scheme of a projective bundle of the form \mathbb{P}_Y^r .*

Proof. By the hypothesis on Y , if \mathcal{E} is a quasi-coherent \mathcal{O}_Y -module of finite type, \mathcal{E} is isomorphic to a quotient of an \mathcal{O}_Y -module $\mathcal{L}^{\otimes(-n)} \otimes_{\mathcal{O}_Y} \mathcal{O}_Y^k$ (??), so $\mathbb{P}(\mathcal{E})$ is isomorphic to a closed subscheme of \mathbb{P}_Y^{k-1} (??).

□

Proposition 19.5.15 (Properties of Quasi-projective Morphisms).

- (i) A quasi-affine morphism of finite type (in particular a quasi-compact immersion or an affine morphism of finite type) is quasi-projective.
- (ii) If $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are quasi-projective and if Z is quasi-compact, $g \circ f$ is quasi-projective.
- (iii) If $f : X \rightarrow Y$ is a quasi-projective S -morphism, $f_{(S')} : X_{(S')} \rightarrow Y_{(S')}$ is quasi-projective for any base change $S' \rightarrow S$.

- (iv) If $f : X \rightarrow Y$ and $g : X' \rightarrow Y'$ are two quasi-projective S -morphisms, $f \times_S g$ is quasi-projective.
- (v) If $f : X \rightarrow Y$, $g : Y \rightarrow Z$ are two morphisms such that $g \circ f$ is quasi-projective and if g is separated or X is locally Noetherian, then f is quasi-projective.
- (vi) If f is quasi-projective, so is f_{red} .

Proof. Property (i) follows from ?? and ??(i). The other parts follows from the definition of quasi-projective morphism and ??, with the corresponding properties of morphisms of finite type ??). \square

Remark 19.5.6. Note that it may happen that f_{red} is quasi-projective without f being so, even we assume that Y is the spectrum of a finite dimensional algebra over C and f is proper.

Corollary 19.5.16. If X and X' are two quasi-projective Y -schemes, $X \amalg X'$ is quasi-projective over Y .

Proof. This follows from ?? \square

19.5.4 Universally closed and proper morphisms

As the terminology indicates, we say a morphism $f : X \rightarrow Y$ is **universally closed** if the projection $f \times_Y 1_{Y'} : X \times_Y Y' \rightarrow Y'$ is closed for any base change $Y' \rightarrow Y$. By ??, we know that a closed immersion is universally closed. We say a morphism $f : X \rightarrow Y$ is **proper** if it is separated, of finite type, and universally closed, and in this case X is said to be **proper over Y** , or a **proper Y -scheme**. It is clear that all these notations are local over Y . We also note that, to verify that the image of a closed subset Z of $X \times_Y Y'$ under the projection $q : X \times_Y Y' \rightarrow Y'$ is closed in Y' , it suffices to shows that $q(Z) \cap U'$ is closed in U' for any affine open subset U' of Y' . As $q(Z) \cap U' = q(Z \cap q^{-1}(U'))$ and $q^{-1}(U')$ is identified with $X \times_Y U'$??), we see that to verify the universally closedness of f , it suffices to limit the case where Y' is affine. We will see later that if Y is locally Noetherian, we can even assume that Y' is of finite type over Y .

Proposition 19.5.17 (Properties of Proper Morphisms).

- (i) A closed immersion is proper.
- (ii) The composition of two proper morphisms is proper.
- (iii) If $f : X \rightarrow Y$ is a proper S -morphism, then $f_{(S')} : X_{(S')} \rightarrow Y_{(S')}$ is proper for any base change $S' \rightarrow S$.
- (iv) If $f : X \rightarrow Y$ and $g : X' \rightarrow Y'$ are two proper S -morphisms, then $f \times_S g$ is proper.

Proof. It suffices to prove the first three properties. In view of ?? and ??, it suffices to verify the universally closedness in each cases. This is trivial in (i) since closed immersions are universal. For (ii), consider two proper morphisms $X \rightarrow Y$, $Y \rightarrow Z$, and a morphism $Z' \rightarrow Z$. We have $X \times_Z Z' = X \times_Y (Y \times_Z Z')$ and therefore the projection $X \times_Z Z' \rightarrow Z'$ factors into $X \times_Y (Y \times_Z Z') \rightarrow Y \times_Z Z' \rightarrow Z'$. By hypothesis, this is a composition of two closed morphisms, hence closed. Finally, in (iii), for any morphism $S' \rightarrow S$, $X_{(S')}$ is identified with $X \times_Y Y_{(S')}$; for any morphism $Z \rightarrow Y_{(S')}$, we have

$$X_{(S')} \times_{Y_{(S')}} Z = (X \times_Y Y_{(S')}) \times_{Y_{(S')}} Z = X \times_Y Z$$

and $X \times_Y Z \rightarrow Z$ is closed by hypothesis, so (iii) follows. \square

Corollary 19.5.18. *Let $f : X \rightarrow Y, g : Y \rightarrow Z$ be two morphisms such that $g \circ f$ is proper.*

- (a) *If g is separated, f is proper.*
- (b) *If g is separated and of finite type and f is surjective, then g is proper.*

Proof. The first claim follows from ???. To prove (b), we only need to verify that g is universally closed. For any morphism $Z' \rightarrow Z$, the diagram

$$\begin{array}{ccc} X \times_Z Z' & \xrightarrow{f \times 1_{Z'}} & Y \times_Z Z' \\ p \searrow & & \downarrow p' \\ & & Z' \end{array}$$

(where p and p' are projections) is commutative. Moreover, $f \times 1_{Z'}$ is surjective if f is ???, and p is a closed immersion by hypothesis. Any closed subset F of $Y \times_Z Z'$ is then the image under $f \times 1_{Z'}$ of a closed subset E of $X \times_Z Z'$, so $p'(F) = p(E)$ is closed in Z' by hypothesis, whence the corollary. \square

Corollary 19.5.19. *If X is a proper scheme over Y and \mathcal{S} is a quasi-coherent \mathcal{O}_Y -algebra, any Y -morphism $f : X \rightarrow \text{Proj}(\mathcal{S})$ is proper (and a fortiori closed).*

Proof. In fact, the structural morphism $p : \text{Proj}(\mathcal{S}) \rightarrow Y$ is separated, and $p \circ f$ is proper by hypothesis. \square

Corollary 19.5.20. *Let $f : X \rightarrow Y$ be a separated morphism of finite type. Let $(X_i)_{1 \leq i \leq n}$ (resp. $(Y_i)_{1 \leq i \leq n}$) be a family of closed subscheme of X (resp. Y), $j_i : X_i \rightarrow X$ (resp. $h_i : Y_i \rightarrow Y$) be the canonical injection. Suppose that (X_i) forms a covering of X and for each i , let $f_i : X_i \rightarrow Y_i$ be a morphism such that the diagram*

$$\begin{array}{ccc} X_i & \xrightarrow{f_i} & Y_i \\ j_i \downarrow & & \downarrow h_i \\ X & \xrightarrow{f} & Y \end{array}$$

is commutative. Then, for f to be proper, it is necessary and sufficient that each f_i is proper.

Proof. If f is proper, so is each $f \circ j_i$, since j_i is a closed immersion; as each h_i is a closed immersion, hence separated, f_i is proper by ???. Suppose conversely that each f_i is proper, and consider the sum Z of X_i ; let $u : Z \rightarrow X$ be the morphism that induces j_i on X_i . The restriction of $f \circ u$ to each X_i is equal to $f \circ j_i = h_i \circ f_i$, hence proper; it then follows that $f \circ u$ is proper. Since u is surjective by hypothesis, we conclude from ?? that f is proper. \square

Corollary 19.5.21. *Let $f : X \rightarrow Y$ be a separated morphism of finite type. For f to be proper, it is necessary and sufficient that $f_{\text{red}} : X_{\text{red}} \rightarrow Y_{\text{red}}$ is proper.*

Proof. This is a particular case where $n = 1$, $X_1 = X_{\text{red}}$ and $Y_1 = Y_{\text{red}}$. \square

If X and Y are Noetherian schemes and $f : X \rightarrow Y$ is a separated morphism of finite type, to verify that f is proper, we can reduce to dominant morphisms of integral schemes. In fact, let X_i ($1 \leq i \leq n$) be the irreducible components of X and consider for each i the unique reduced

closed subscheme structure on X_i . Let Y_i be the reduced closed subscheme with underlying space $\overline{f(X_i)}$. If $j_i : X_i \rightarrow X$ (resp. $h_i : Y_i \rightarrow Y$) are the canonical injections, we then have $f \circ j_i = h_i \circ f_i$, where f_i is a dominant morphism $f_i : X_i \rightarrow Y_i$. We then see that the conditions of ?? are satisfied, and for f to be proper, it is necessary and sufficient that each f_i is.

Corollary 19.5.22. *Let X and Y be separated S -schemes of finite type and $f : X \rightarrow Y$ be an S -morphism. For f to be proper, it is necessary and sufficient that for any S -scheme S' , the morphism $f_{(S')} : X_{(S')} \rightarrow Y_{(S')}$ is closed.*

Proof. We note that if $\varphi : X \rightarrow S$ and $\psi : Y \rightarrow S$ are the structural morphisms, we have $\varphi = \psi \circ f$, so f is separated and of finite type (?? and ??). If f is proper, so is $f_{(S')}$, and is a fortiori closed. Conversely, assume this condition and let Y' be a Y -scheme; Y' can be considered as an S -scheme, and the morphism $Y \rightarrow S$ is separated. In the commutative diagram

$$\begin{array}{ccc} X \times_Y Y' & \xrightarrow{f \times_Y 1_{Y'}} & Y \times_Y Y' = Y' \\ \downarrow & & \downarrow \\ X \times_S Y' & \xrightarrow{f_{(Y')}} & Y \times_S Y' \end{array}$$

the vertical morphisms are a closed immersion by ?. It follows from the assumption that $f_{(Y')}$ is closed, and so is $f \times_Y 1_{Y'}$. \square

Let $f : X \rightarrow Y$ be a morphism of finite type. We say that a closed subset Z of X is **proper over Y** (or **Y -proper**, or **f -proper**) if the restriction of f to a closed subscheme of X with underlying space Z is proper. As this restriction is then separated, it follows from ?? and ??(vi) the property that Z is proper over Y is independent of the closed subscheme structural chosen for Z .

Let Z be a proper subset of X for f and let $g : X' \rightarrow X$ be a proper morphism. Then $g^{-1}(Z)$ is then a proper subset of X' : if T is a subscheme of X with underlying space Z , it suffices to note that the restriction of g to the closed subscheme $g^{-1}(T)$ of X' is a proper morphism $g^{-1}(T) \rightarrow T$ by ??(iii), and we can apply ??(ii) to conclude that $g^{-1}(T)$ is proper.

On the other hand, if X'' is a Y -scheme of finite type and $h : X \rightarrow X''$ is a Y -morphism, $h(Z)$ is also a proper subset of X'' : in fact, for any reduced closed subscheme T of X with underlying space Z . The restriction of f to T is proper, and so is the restriction of h to T (??(a)), so $h(Z)$ is closed in X'' . Let T'' be a closed subscheme of X'' with underlying space $h(Z)$ so that the morphism $h|_T$ factors into (cf. ??)

$$T \xrightarrow{h|_T} T'' \xrightarrow{j} X''$$

where j is the canonical injection. Then $h|_T$ is proper by ?? and surjective. If $\psi : X'' \rightarrow Y$ is the structural morphism, $\psi|_{T''}$ is then separated of finite type (?? and ??), and we have $f|_T = (\psi|_{T''}) \circ (h|_T)$; it then follows from ??(ii) that $\psi|_{T''}$ is proper, whence the assertion.

In particular, for a Y -proper subset of X , we have the following:

- (a) For any closed subset X' of X , $Z \cap X'$ is a Y -proper subset of X' .
- (b) If X is a subscheme of a Y -scheme of finite type X'' , Z is also a Y -proper subset of X'' (and in particular is closed in X'').

19.5.5 Projective morphisms

Proposition 19.5.23. *Let X be a Y -scheme. The following conditions are equivalent:*

- (a) *X is Y -isomorphic to a closed subscheme of a projective bundle $\mathbb{P}(\mathcal{E})$, where \mathcal{E} is a quasi-coherent \mathcal{O}_Y -module of finite type.*
- (b) *There exists a quasi-coherent graded \mathcal{O}_Y -algebra such that \mathcal{S}_1 is of finite type and generates \mathcal{S} , and that X is Y -isomorphic to a $\text{Proj}(\mathcal{S})$.*

Proof. Condition (a) implies (b) by ??(b): if \mathcal{I} is the quasi-coherent graded ideal of $S(\mathcal{E})$, the quasi-coherent graded \mathcal{O}_Y -algebra $\mathcal{S} = S(\mathcal{E})/\mathcal{I}$ is generated by \mathcal{S}_1 and the later, which is the canonical image of \mathcal{E} , is an \mathcal{O}_X -module of finite type. Condition (b) implies (a) in view of ?? applied to the case where $\mathcal{M} \rightarrow \mathcal{S}_1$ is the identity homomorphism. \square

We say a Y -scheme X is **projective over Y** or a **projective Y -scheme** if it satisfies the equivalent conditions of ???. We say a morphism $f : X \rightarrow Y$ is **projective** if X is a projective Y -scheme via this morphism. It is clear that if $f : X \rightarrow Y$ is projective, then there exists a very ample \mathcal{O}_X -module relative to f ??.

Theorem 19.5.24. *Any projective morphism is quasi-projective and proper. Conversely, if Y is a quasi-compact and quasi-separated scheme, any quasi-projective and proper morphism $f : X \rightarrow Y$ is projective.*

Proof. It is clear that any projective morphism is of finite type and quasi-projective. On the other hand, it follows from ??(b) and ?? that if f is projective, so is $f \times_Y 1_{Y'} : X \times_Y Y' \rightarrow Y'$ for any morphism $Y' \rightarrow Y$. The proof that f is universally closed then boils down to show that a projective morphism f is closed. The question is local over Y , so we can suppose that $Y = \text{Spec}(A)$ is affine, so by ?? $X = \text{Proj}(S)$ where S is a graded A -algebra generated by finitely many elements of S_1 . For any $y \in Y$, the fiber $f^{-1}(y)$ is identified with $\text{Proj}(S \times_Y \text{Spec}(\kappa(y)))$, hence to $\text{Proj}(S \otimes_A \kappa(y))$???. Therefore, $f^{-1}(y)$ is empty if and only if $S \otimes_A \kappa(y)$ is eventually zero, which means $S_n \otimes_A \kappa(y) = 0$ for n sufficiently large. Now as $(S_n)_y$ is an $\mathcal{O}_{Y,y}$ -module of finite type, the preceding condition signifies that $(S_n)_y = 0$ for n sufficiently large, in view of Nakayama lemma. If \mathfrak{a}_n is the annihilator in A of the A -module S_n , the preceding condition is then equivalent to that $\mathfrak{a}_n \subseteq \mathfrak{p}_y$ for n sufficiently large. Now as $S_n S_1 = S_{n+1}$ by hypothesis, we have $\mathfrak{a}_n \subseteq \mathfrak{a}_{n+1}$, and if \mathfrak{a} is the sum of \mathfrak{a}_n , we then have $f(X) = V(\mathfrak{a})$, so $f(X)$ is closed in Y . If now X' is a closed subset of X , there exists a closed subscheme of X with underlying space X' and it is clear (by ??(a)) that the composition morphism $X' \rightarrow X \rightarrow Y$ is projective, so $f(X')$ is closed in Y .

Now conversely, assume that Y is quasi-compact and quasi-separated. The hypothesis that f is quasi-projective implies the existence of a quasi-coherent \mathcal{O}_Y -module of finite type \mathcal{E} and a Y -immersion $j : X \rightarrow \mathbb{P}(\mathcal{E})$???. Since f is proper and the structural morphism $\mathbb{P}(\mathcal{E}) \rightarrow Y$ is separated, j is proper (hence closed) by ??(a), so f is projective. \square

Remark 19.5.7. Let $f : X \rightarrow Y$ be a morphism such that

- (i) f is proper
- (ii) there exists a very ample \mathcal{O}_X -module \mathcal{L} relative to f

- (iii) the quasi-coherent \mathcal{O}_Y -module $\mathcal{E} = f_*(\mathcal{L})$ is of finite type

Then f is projective: there then exists a Y -immersion $j : X \rightarrow \mathbb{P}(\mathcal{E})$, and since f is proper, j is a closed immersion by ??(a). We will see that if Y is locally Noetherian, the last condition (iii) is a consequence of the others, and conditions (i) and (ii) characterize projective morphisms. If Y is Noetherian, we can further repalce in (ii) that there exists a *ample* \mathcal{O}_X -module relative to f (??). We also note that there are proper morphisms that is not projective.

Remark 19.5.8. Let Y be a quasi-compact scheme such that there exists an ample \mathcal{O}_Y -module. For a Y -scheme X to be projective, it is necessary and sufficient that it is isomorphic to a closed subscheme of a projective bundle of the form \mathbb{P}_Y^r . This conditions is clearly sufficient. Conversely, if X is projective over Y , it is quasi-projective, so there exists a Y -immersion $j : X \rightarrow \mathbb{P}_Y^r$ by ??, which is closed by ??(a) and ??.

Remark 19.5.9. The reasoning of ?? shows that for any scheme Y , and any integer $r \geq 0$, the structural morphism $\mathbb{P}_Y^r \rightarrow Y$ is surjective, because if we put $\mathcal{S}_{\mathcal{O}_Y} = S(\mathcal{O}_Y^{r+1})$, we have evidently $\mathcal{S}_y = S_{\kappa(y)}(\kappa(y)^{r+1})$, so $(\mathcal{S}_n)_y \neq 0$ for any $y \in Y$ and $n \geq 0$.

Proposition 19.5.25 (Properties of Projective Morphisms).

- (i) A closed immersion is projective.
- (ii) If $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are projective morphisms and if Z is quasi-compact and quasi-separated, then $g \circ f$ is projective.
- (iii) If $f : X \rightarrow Y$ is a projective morphism, $f_{(S')} : X_{(S')} \rightarrow Y_{(S')}$ is projective for any base change $S' \rightarrow S$.
- (iv) If $f : X \rightarrow Y$ and $g : X' \rightarrow Y'$ are projective S -morphisms, so is $f \times_S g$.
- (v) If $g \circ f$ is a projective morphism and if g is separated, f is projective.
- (vi) If f is projective, so is f_{red} .

Proof. Property (i) follows from ???. It is necessary here to prove (iii) and (iv) separately, because of the restriction introduced on Z in (ii). To prove (iii), we can reduce to the case where $S = Y$ (??) and the assertion then follows immediately from ??(b) and from ???. To prove (iv), we can assume that $X = \mathbb{P}(\mathcal{E})$, $X' = \mathbb{P}(\mathcal{E}')$, where \mathcal{E} (resp. \mathcal{E}') is a quasi-coherent \mathcal{O}_Y -module of finite type. Let p, p' be the projection of $T = Y \times_S Y'$ to Y and Y' respectively; by (??), we have $\mathbb{P}(p^*(\mathcal{E})) = \mathbb{P}(\mathcal{E}) \times_Y T$ and $\mathbb{P}(p'^*(\mathcal{E}')) = \mathbb{P}(\mathcal{E}') \times_{Y'} T$, whence

$$\begin{aligned} \mathbb{P}(p^*(\mathcal{E})) \times_T \mathbb{P}(p'^*(\mathcal{E}')) &= (\mathbb{P}(\mathcal{E}) \times_Y T) \times_T (T \times_{Y'} \mathbb{P}(\mathcal{E}')) = \mathbb{P}(\mathcal{E}) \times_Y (T \times_{Y'} \mathbb{P}(\mathcal{E}')) \\ &= \mathbb{P}(\mathcal{E}) \times_Y ((Y \times_S Y') \times_{Y'} \mathbb{P}(\mathcal{E}')) = \mathbb{P}(\mathcal{E}) \times_S \mathbb{P}(\mathcal{E}'). \end{aligned}$$

Now $p^*(\mathcal{E})$ and $p'^*(\mathcal{E}')$ are of finite type over T (??), and so is $p^*(\mathcal{E}) \otimes_{\mathcal{O}_T} p'^*(\mathcal{E}')$; as $\mathbb{P}(p^*(\mathcal{E})) \times_T \mathbb{P}(p'^*(\mathcal{E}'))$ is identified with a closed subscheme of $\mathbb{P}(p^*(\mathcal{E}) \otimes_{\mathcal{O}_T} p'^*(\mathcal{E}'))$ (??), this proves (iv). For (v) and (vi), we can apply ??, since any closed subscheme of a projective Y -scheme is projective by ??(a). \square

Proposition 19.5.26. If X and X' are two projective Y -schemes, so is $X \amalg X'$.

Proof. This is evident from ??.

□

Proposition 19.5.27. *Let X be a projective Y -scheme, \mathcal{L} be a Y -ample \mathcal{O}_Y -module. For any section f of \mathcal{L} over X , X_f is affine over Y .*

Proof. Since the question is local over Y , we can assume that $Y = \text{Spec}(A)$ is affine; since $X_{f^{\otimes n}} = X_f$, by replacing \mathcal{L} by $\mathcal{L}^{\otimes n}$ we can assume that \mathcal{L} is very ample for the structural morphism $q : X \rightarrow Y$ (??). The canonical homomorphism $\sigma : q^*(q_*(\mathcal{L})) \rightarrow \mathcal{L}$ is then surjective and the corresponding morphism

$$r = r_{\mathcal{L}, \sigma} : X \rightarrow P = \mathbb{P}(q_*(\mathcal{L}))$$

is an immersion such that $\mathcal{L} \cong r^*(\mathcal{O}_P(1))$ (??). Moreover, as X is proper over Y , the immersion r is closed by ?. By definition $f \in \Gamma(Y, q_*(\mathcal{L}))$ and σ^\flat is the identity of $q_*(\mathcal{L})$; it then follows from the formula (??) that we have $X_f = r^{-1}(D_+(f))$. Then X_f is a closed subscheme of the affine scheme $D_+(f)$, and therefore is affine. □

If we take $Y = X$ in ?, we obtain that, for any scheme X and any invertible \mathcal{O}_X -module \mathcal{L} , the open subset X_f is affine over X .

19.5.6 Chow's lemma

Theorem 19.5.28 (Chow's lemma). *Let X be a separated S -scheme of finite type and suppose that one of the following conditions is satisfied:*

- (a) S is Noetherian.
- (b) S is quasi-compact and X has finitely many irreducible components.

Then there exists a quasi-projective S -scheme X' and a surjective projective S -morphism $f : X' \rightarrow X$ that induces an isomorphism an isomorphism $f^{-1}(U) \cong U$ for some open dense subset of X . If X is reduced (resp. irreducible), we can also choose X' to be reduced (resp. irreducible).

Proof. The proof is divided into several steps. First of all, we can assume that X is irreducible. To see this, we note that in both cases the scheme X has finitely many irreducible components X_i . If the theorem is demonstrated for each reduced subscheme X_i , and if $f_i : X'_i : X_i$ is the corresponding homomorphism which induces an isomorphism $f_i^{-1}(U_i) \cong U_i$ with $U_i \subseteq X_i$, the sum $X' = \coprod_i X'_i$ is then quasi-projective over S (?? and ??) and the morphism $f : X' \rightarrow X$ whose restriction on X'_i equals to $j_i \circ f_i$ (where $j_i : X_i \rightarrow X$ is the canonical injection), is then surjective and projective (??); it is immediate to see that X' is reduced if each X'_i is. We now choose U to be the union of $U_i \cap (\bigcup_{j \neq i} X_j)^c$; since U_i is dense in X_i and X_i is maximal irreducible, we conclude that each $U_i \cap (\bigcup_{j \neq i} X_j)$ is nonempty. The open subset U is then dense in X and f clearly induces an isomorphism $f^{-1}(U) \cong U$.

So suppose now that X is irreducible. As the structural morphism $\eta : X \rightarrow S$ is of finite type, there exists a finite covering (S_i) of S by affine opens, and for each i there is a finite covering (T_{ij}) of $\eta^{-1}(S_i)$ by affine opens, with the morphism $T_{ij} \rightarrow S_i$ being affine and of finite type, hence quasi-projective (??(i)). As in both hypotheses the immersion $S_i \rightarrow S$ is quasi-compact, it is quasi-projective by ??(i), so the restriction of η to T_{ij} is quasi-projective (??(ii)). We relabel the T_{ij} by U_k with $1 \leq k \leq n$. There exists, for each index k , an open immersion

$\varphi_k : U_k \rightarrow P_k$, where P_k is projective over S (??). Let $U = \bigcap_k U_k$; as X is irreducible and the U_k is nonempty, U is nonempty, and therefore is dense in X ; the restrictions of φ_k to U together define a morphism

$$\varphi : U \rightarrow P = P_1 \times_S P_2 \times_S \cdots \times_S P_n$$

which fits into the diagram

$$\begin{array}{ccc} U & \xrightarrow{\varphi} & P \\ j_k \downarrow & & \downarrow p_k \\ U_k & \xrightarrow{\varphi_k} & P_k \end{array} \quad (19.5.1)$$

where j_k is the canonical injection and p_k is the canonical projection. If $j : U \rightarrow X$ is the canonical injection, the morphism $\psi = (j, \varphi)_S : U \rightarrow X \times_S P$ is then an immersion by ???. Under the hypotheses of (a), $X \times_S P$ is locally Noetherian (?? and ??), and under the hypotheses of (b), $X \times_S P$ is quasi-compact. In both cases the scheme-theoretic image X' of ψ in $X \times_S P$ exists (which is the closure of $\psi(U)$ in $X \times_S P$) and ψ factors into

$$\psi : U \xrightarrow{\psi'} X' \xrightarrow{h} X \times_S P$$

where ψ' is a dominant open immersion and h is a closed immersion. Let $q_1 : X \times_S P \rightarrow X$ and $q_2 : X \times_S P \rightarrow P$ be the canonical projections; we put

$$f : X' \xrightarrow{h} X \times_S P \xrightarrow{q_1} X, \quad g : X' \xrightarrow{h} X \times_S P \xrightarrow{q_2} P. \quad (19.5.2)$$

We shall verify that the scheme X' and the morphism f satisfy the requirements. First we show that f is projective and surjective, and that the restriction of $U' = f^{-1}(U)$ is an isomorphism from U' to U . As the P_k are projective over S , so is P (??(iv)), and $X \times_S P$ is projective over X by ??(iii); then X' is also projective over X , since it is a closed subscheme of $X \times_S P$. On the other hand, we have $f \circ \psi' = q_1 \circ (h \circ \psi') = q_1 \circ \psi = j$, so $f(X')$ contains the dense open subset U of X ; but f is proper by ??, so $f(X') = X$. Now $q_1^{-1}(U) = U \times_S P$ is an open subscheme of $X \times_S P$, and the the immersion ψ factors into

$$\psi : U \xrightarrow{\Gamma_\varphi} U \times_S P \xrightarrow{j \times 1} X \times_S P.$$

By ??, $U' = h^{-1}(U \times_S P)$ is the scheme-theoretic image of $\psi^{-1}(U \times_S P) = U$ under $\psi_U : U \rightarrow U \times_S P$, and therefore the closure of the image of Γ_φ in $U \times_S P$. As P is separated over S , Γ_φ is a closed immersion (??), so we conclude that $U' = \psi(U)$. As ψ is an immersion, the restriction of f to U' is then an isomorphism, with inverse ψ' . Finally, by definition, $U' = \psi(U) = \psi'(U)$ is open and dense in X' .

We now show that g is an immersion, which implies that X' is quasi-projective over S , since P is projective over S . Let $V_k = \varphi_k(U_k)$ be the image of U_k in P_k , $W_k = p_k^{-1}(V_k)$ be the inverse image in P , and put $U'_k = f^{-1}(U_k)$, $U''_k = g^{-1}(W_k)$. Since the U_k cover X , it clear that the U'_k form an open covering of X' ; we first shows that this is also ture for the U''_k , by proving that

$U'_k \subseteq U''_k$. For this, it suffices to establish the commutativity of the diagram

$$\begin{array}{ccc} U'_k & \xrightarrow{g|_{U'_k}} & P \\ f|_{U'_k} \downarrow & & \downarrow p_k \\ U_k & \xrightarrow{\varphi_k} & P_k \end{array} \quad (19.5.3)$$

Since $U'_k = h^{-1}(U_k \times_S P)$ and $\psi^{-1}(U_k \times_S P) = U$, by ?? U'_k is the scheme-theoretic image of U in $U_k \times_S P$ under the morphism $\psi_k : U \rightarrow U_k \times_S P$ induced by ψ . It then suffices to prove the commutativity of the diagram obtained by composing (??) with the morphism ψ_k (??), and this comes from the commutative diagram (??).

The W_k then form an open covering of $g(X')$, so to show that g is an immersion, it suffices to show the restriction $g|_{U''_k}$ is an immersion into W_k (??). For this, consider the morphism

$$u_k : W_k \xrightarrow{p_k} V_k \xrightarrow{\varphi_k^{-1}} U_k \rightarrow X$$

which fits into the following commutative diagram

$$\begin{array}{ccccccc} & & g|_{U''_k} & & & & \\ & & \curvearrowleft & & \curvearrowright & & \\ U' & \hookrightarrow & U'_k & \hookrightarrow & U''_k & \xleftarrow{h|_{U''_k}} & X \times_S W_k & \xrightarrow{\Gamma_{u_k}} & W_k \\ \searrow & & \searrow & & \searrow & & \downarrow q_1 & \searrow & \downarrow p_k \\ & & U & \hookrightarrow & U_k & \xrightarrow{f|_{U''_k}} & X & \xrightarrow{\cong} & V_k \\ & & & & & & \uparrow u_k & & \end{array}$$

By the definition of g (formula (??)), we have $U''_k = h^{-1}(X \times_S W_k) \subseteq X'$ and $\psi^{-1}(X \times_S W_k) = U$, so by ??, U''_k is the scheme-theoretic image of U under the morphism $U \rightarrow X \times_S W_k$ induced by ψ ; since $U' = \psi(U)$, it is therefore dense in U''_k . On the other hand, as X is separated over S , the graph morphism $\Gamma_{u_k} : W_k \rightarrow X \times_S W_k$ is a closed immersion, so the graph $T_k = \Gamma_{u_k}(W_k)$ is a closed subscheme of $X \times_S W_k$. If we can prove that T_k dominates the canonical image of the open subscheme U' in $X \times_S W_k$, it will then dominate the subscheme U''_k . As the restriction of q_2 to T_k is an isomorphism onto W_k and h is a closed immersion, the restriction of g to X''_k will then be an immersion in W_k , and our assertion will be proved. For this, we let $v_k : U' \rightarrow X \times_S W_k$ be the canonical injection, and $w_k = q_2 \circ v_k$; then from the definition of Γ_{u_k} we have $v_k = \Gamma_{u_k} \circ w_k$, and the image of U' in $X \times_S W_k$ is therefore contained in T_k , verifying our claim.

It is clear that U , and therefore U' , are irreducible, and so is X' by our construction, and that f is birational. If X is reduced, so is U' , and X' is then reduced (??). This completes the proof. \square

Corollary 19.5.29. Suppose the hypotheses of ???. For X to be proper over S , it is necessary and sufficient that there exists a projective S -scheme X' and a surjective S -morphism $f : X' \rightarrow X$ (which is projective

by ??(v)). If this is the case, we can choose an open dense subset U of X such that f induces an isomorphism $f^{-1}(U) \cong U$ and that $f^{-1}(U)$ is dense in X' . If X is irreducible (resp. reduced), we can choose X' to be irreducible (resp. reduced). If X and X' are irreducible, f is then a birational morphism.

Proof. The conditions is sufficient by ?? and ??(b). This is necessary because with the notations of ??, if X is proper over S , X' is then proper over S (since it is proper over X by ??), and our assertion follows from ??(ii). Moreover, as X' is quasi-projective over S , it is projective over S in view of ??.

Corollary 19.5.30. *Let S be a locally Noetherian scheme, X be an S -scheme of finite type, and $\varphi : X \rightarrow S$ be the structural morphism. For X to be proper over S , it is necessary and sufficient that for any morphism of finite type $S' \rightarrow S$, the morphism $\varphi_{(S')} : X_{(S')} \rightarrow S'$ is closed. Moreover, it suffices to verify this condition for any S -scheme of the form $S' = S \otimes_{\mathbb{Z}} \mathbb{Z}[T_1, \dots, T_n]$.*

Proof. The conditions is clearly necessary, and we now prove the sufficiency. The question is local over S and S' , so we may assume that S, S' are affine and Noetherian. By Chow's lemma, there exists a projective S -scheme P , an immersion $j : X' \rightarrow P$, and a projective surjective morphism $f : X' \rightarrow X$ such that the diagram

$$\begin{array}{ccccc} X' & \xrightarrow{\psi} & X \times_S P & \xrightarrow{q_1} & X \\ j \searrow & \swarrow f & q_2 \downarrow & & \downarrow \varphi \\ & & P & \xrightarrow{r} & S \end{array}$$

is commutative; let $\psi = (f, j)_S$. As P is of finite type over S , the projection $q_2 : X \times_S P \rightarrow P$ is a closed morphism by hypotheses. On the other hand, since f is projective and the projection $q_1 : X \times_S P \rightarrow X$ is separated (since P is separated over S), we conclude from ??(v) that ψ is projective, hence closed. Since the immersion j is the composition of q_2 with ψ , it is therefore a closed immersion, whence proper. Moreover, the structural morphism $r : P \rightarrow S$ is projective, hence proper (??), so $\varphi \circ f = r \circ j$ is proper. As f is surjective, we conclude by ??(b) that φ is proper.

To establish the second assertion of the proposition, it suffices to prove that it implies that $\varphi_{(S')}$ is closed for any morphism $S' \rightarrow S$ of finite type. Now if S' is affine and of finite type over $S = \text{Spec}(A)$, we have $S' = \text{Spec}(A[x_1, \dots, x_n])$, and S' is then isomorphic to a closed subscheme of $S'' = \text{Spec}(A[T_1, \dots, T_n])$ (where T_i are indeterminates). In the following commutative diagram

$$\begin{array}{ccc} X \times_S S' & \xrightarrow{1_X \times j} & X \times_S S'' \\ \varphi_{(S')} \downarrow & & \downarrow \varphi_{(S'')} \\ S' & \xrightarrow{j} & S'' \end{array}$$

where j and $1_X \times j$ are closed immersions (??) and $\varphi_{(S'')}$ is closed by hypothesis. We then conclude that $\varphi_{(S')}$ is closed, whence the claim. \square

19.6 Integral morphisms and finite morphisms

19.6.1 Integral and finite morphisms

Let X be an S -scheme and $f : X \rightarrow S$ be the structural morphism. We say that X is **integral over S** , or that f is an **integral morphism**, if there exists an affine open covering (S_α) of S such that for each α , the open subscheme $f^{-1}(S_\alpha)$ of X is affine and its ring B_α is an integral algebra over the ring A_α of S_α . We say that X is **finite over S** , or that f is a **finite morphism**, if X is integral and of finite type over S . If S is affine with ring A , we also say that X is integral or finite over A .

It is clear that any integral S -scheme is affine over S . Conversely, from the definition of integral morphisms, we see that for an affine S -scheme X to be integral over S (resp. finite), it is necessary and sufficient that the associated quasi-coherent \mathcal{O}_S -algebra $\mathcal{A}(X)$ is such that there exists an affine open covering (S_α) of S such that for each α , $\Gamma(S_\alpha, \mathcal{A}(X))$ is an integral algebra (resp. an integral algebra of finite type) over $\Gamma(S_\alpha, \mathcal{O}_S)$. A quasi-coherent \mathcal{O}_S -algebra satisfying this property is said to be **integral** (resp. **finite**) over \mathcal{O}_S . We note that a quasi-coherent \mathcal{O}_S -algebra \mathcal{B} is finite if and only if it is an \mathcal{O}_S -module of finite type; it amounts to the same thing to say that \mathcal{B} is an integral \mathcal{O}_S -algebra of finite type, because an integer algebra of finite type over a ring A is an A -module of finite type.

Proposition 19.6.1. *Let S be a locally Noetherian scheme. For an S -scheme X affine over S to be finite over S , it is necessary and sufficient that the \mathcal{O}_S -algebra $\mathcal{A}(X)$ is coherent.*

Proof. With the preceding remark, this follows from the fact that if S is locally Noetherian, then a quasi-coherent \mathcal{O}_S -module is of finite type if and only if it is coherent (??). \square

Proposition 19.6.2. *Let X be an integral (resp. finite) scheme over S with $f : X \rightarrow S$ the structural morphism. Then for any affine open subset $U \subseteq S$ with ring A , $f^{-1}(U)$ is affine and its ring B is an integral (resp. finite) algebra over A .*

Proof. To prove this proposition, we need ?? . We now that $f^{-1}(U)$ is affine by ?? . If $\varphi : A \rightarrow B$ is the corresponding homomorphism, there exists a finite covering of U by open subsets $D(g_i)$ ($g_i \in A$) such that, if $h_i = \varphi(g_i)$, then B_{h_i} is an integral (resp. finite) algebra over A_{g_i} . In fact, by assumption, there is a covering of U by affine open subsets $V_\alpha \subseteq U$ such that if $A_\alpha = \Gamma(V_\alpha, \mathcal{O}_S)$ and $B_\alpha = \Gamma(f^{-1}(V_\alpha), \mathcal{O}_X)$, then B_α is an integral (resp. finite) algebra over A_α . Any $x \in U$ belongs to one V_α , so there exists $g \in A$ such that $x \in D(g) \subseteq V_\alpha$. If g_α is the image of g in A_α , we have $\Gamma(D(g), \mathcal{O}_S) = A_g = (A_\alpha)_{g_\alpha}$; let $h = \varphi(g)$, and let h_α be the image of g_α in B_α . We have

$$\Gamma(D(h), \mathcal{O}_S) = B_h = (B_\alpha)_{h_\alpha}$$

and as B_α is integral over A_α , $(B_\alpha)_{h_\alpha}$ is integral (resp. finite) over $(A_\alpha)_{g_\alpha}$. Since U is quasi-compact, we obtain a finite cover.

If we suppose first that each B_{h_i} is a finite algebra over A_{g_i} , then as an A_{g_i} -module, B_{h_i} is finitely generated, so ?? shows that B is a finitely generated A -module. Now assume that each B_{h_i} is integral over A_{g_i} ; let $b \in B$, and let C be a sub- A -algebra of B generated by b . For each i , C_{h_i} is the A_{g_i} -algebra generated by $b/1$ over B_{h_i} . It then follows from the hypothesis that each C_{h_i} is finitely generated A_{g_i} -module, so by ?? C is a finitely generated A -module. This shows that B is integral over A . \square

Proposition 19.6.3 (Properties of Integral and Finite morphisms).

- (i) A closed immersion is finite (and a fortiori integral).
- (ii) The composition of two integral morphisms (resp. finite) is integral (resp. finite).
- (iii) If $f : X \rightarrow Y$ is an integral (resp. finite) S -morphism, $f_{(S')} : X_{(S')} \rightarrow Y_{(S')}$ is integral (resp. finite) for any base change $S' \rightarrow S$.
- (iv) If $f : X \rightarrow Y$ and $g : X' \rightarrow Y'$ are two integral (resp. finite) S -morphisms, so is $f \times_S g$.
- (v) If $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are two morphisms such that $g \circ f$ is integral (resp. finite) and g is separated, then f is integral (resp. finite).
- (vi) If $f : X \rightarrow Y$ is an integral (resp. finite) morphism, so is f_{red} .

Proof. In view of ??, it suffices to prove (i), (ii), and (iii). To prove that a closed immersion $X \rightarrow S$ is finite, we can assume that $S = \text{Spec}(A)$, and this then follows from the fact that a quotient ring A/\mathfrak{a} is a finitely generated A -module. To prove the composition of two integral (resp. finite) morphisms $X \rightarrow Y$, $Y \rightarrow Z$ is integral (finite), we can assume that Z (and therefore X and Y) is affine, and the assertion is then equivalent to that if B is an integral (resp. finite) A -algebra and C is an integral (resp. finite) B -algebra, then C is an integral (resp. finite) A -algebra, which is immediate. Finally, to prove (iii), we can Simialrly assume that $S = \text{Spec}(A)$, $S' = \text{Spec}(A')$; then X is affine with ring B (??), $X_{(S')}$ is affine with ring $A' \otimes_A B$, and it suffices to note that if B is an integral (resp. finite) A -algebra, then $A' \otimes_A B$ is an integral (resp. finite) A' -algebra. \square

Note also that if X and Y are two integral (resp. finite) S -schemes, the sum $X \amalg Y$ is an integral (resp. finite) over S , because a product of two integral (resp. finite) A -algebras is still integral (resp. finite).

Corollary 19.6.4. *If X is an integral (resp. finite) scheme over S , then for any open subset $U \subseteq S$, $f^{-1}(U)$ is integral (resp. finite) over U .*

Proof. This is a particular case of ??(iii). \square

Corollary 19.6.5. *Let $f : X \rightarrow Y$ be a finite morphism. Then for any $y \in Y$, the fiber $f^{-1}(y)$ is a finite algebraic scheme over $\kappa(y)$, and a fortiori with discrete and finite underlying space.*

Proof. The $\kappa(y)$ -scheme $f^{-1}(y)$ is identified with $X \times_Y \text{Spec}(\kappa(y))$, so is finite over $\kappa(y)$ by ??(iii). This is then an affine scheme whose ring is a finite dimensional $\kappa(y)$ -algebra, so is Artinian. The proposition then follows from ?. \square

Corollary 19.6.6. *Let X and S be integral schemes and $f : X \rightarrow S$ be a dominant morphism. If f is integral (resp. finite), then the rational function field $K(X)$ of X is an algebraic (resp. finite) extension of $K(S)$.*

Proof. Let s be the generic point of S ; the $\kappa(s)$ -scheme $f^{-1}(s)$ is integral (resp. finite) over $\text{Spec}(\kappa(s))$ by ??(iii), and contains by hypothesis the generic point of x of X ; the local ring of $f^{-1}(s)$ at x , equal to $\kappa(x)$ (?), is a localization of an integral (resp. finite) algebra over $\kappa(s)$, whence the corollary. \square

Proposition 19.6.7. *Any integral morphism is universally closed.*

Proof. Let $f : X \rightarrow Y$ be an integral morphism. In view of ??(iii), it suffices to prove that f is closed. Let Z be a closed subset of X . In view of ??(vi), we can suppose that X and Y are reduced; moreover, if T is the reduced closed subscheme of Y with underlying space $\overline{f(X)}$, we see that f factors into

$$f : X \xrightarrow{g} T \xrightarrow{j} Y,$$

where $j : T \rightarrow Y$ is the canonical injection, and as j is separated, it follows from ??(v) that g is an integral morphism. We can then assume that $f(X)$ is dense in Y , and prove that $f(X) = Y$. Since the question is local over Y , we can assume that $Y = \text{Spec}(A)$ is affine, so $X = \text{Spec}(B)$ where B is an integral algebra over A ??; moreover A is reduced and the hypothesis that $f(X)$ is dense in Y implies that the corresponding homomorphism $\varphi : A \rightarrow B$ is injective ??). The condition that $f(X) = Y$ then follows from ?? \square

Remark 19.6.1. The hypothesis that g is separated is essential for the validity of ??(v): in fact, if Y is not separated over Z , the identity 1_Y is the composition morphism

$$Y \xrightarrow{\Delta_Y} Y \times_Z Y \xrightarrow{p_1} Y$$

but Δ_Y is not integral, since it is not closed ??).

Corollary 19.6.8. *Any finite morphism $f : X \rightarrow Y$ is projective.*

Proof. As f is affine, \mathcal{O}_X is a very ample \mathcal{O}_X -module relative to f ; moreover $f_*(\mathcal{O}_X)$ is a quasi-coherent \mathcal{O}_Y -algebra of finite type by hypothesis. Finally, f is separated, of finite type, and universally closed ??, and we then have the conditions of ?? \square

Lemma 19.6.9. *Let Y be a scheme, \mathcal{E} be a locally free \mathcal{O}_Y -module of rank r , and Z be a finite subset of Y contained in an affine open subset V . Then there exists an open neighborhood $U \subseteq V$ of Z such that $\mathcal{E}|_U$ is isomorphic to $\mathcal{O}_Y^r|_U$.*

Proof. \square

Proposition 19.6.10. *Let $f : X' \rightarrow X$ be a finite morphism, and let $\mathcal{B} = f_*(\mathcal{O}_{X'})$ (which is a quasi-coherent finite \mathcal{O}_X -algebra). Let \mathcal{F}' be a quasi-coherent $\mathcal{O}_{X'}$ -module; for \mathcal{F}' to be locally free of rank r , it is necessary and sufficient that $f_*(\mathcal{F}')$ is a locally free \mathcal{B} -module of rank r .*

Proof. It is clear that if $f_*(\mathcal{F}')|_U$ is isomorphic to $\mathcal{B}^r|_U$ (where U is open in X), $\mathcal{F}'|_{f^{-1}(U)}$ is isomorphic to $\mathcal{O}_{X'}^r|_{f^{-1}(U)}$??). Conversely, suppose that \mathcal{F}' is locally free of rank r and we prove that $f_*(\mathcal{F}')$ is locally isomorphic to \mathcal{B}^r as \mathcal{B} -modules. Let x be a point of X ; if U runs through affine neighborhoods of x , $f^{-1}(U)$ form a fundamental system of affine neighborhoods ?? of the finite subset $f^{-1}(x)$, since f is closed ??). The proposition then follows from ?? \square

Proposition 19.6.11. *Let $g : X' \rightarrow X$ be an integral morphism of schemes, Y be a locally integral and normal scheme, f be a rational map from Y to X' such that $g \circ f$ is a everywhere defined rational map; then f is everywhere defined.*

Proof. Recall that we say a scheme X is normal if it is normal as a ringed space, which means the stalk $\mathcal{O}_{X,x}$ is an integrally closed domain for every $x \in X$. If f_1 and f_2 are two morphisms (densely defined from Y to X') in the class of f , it is clear that $g \circ f_1$ and $g \circ f_2$ are equivalent morphisms, which justifies the notation $g \circ f$ for their equivalent class. We recall also that if Y is locally Noetherian, then the hypothesis on Y implies that Y is locally integral (??).

To prove the proposition, we first note that the question is local over Y and we can assume that there exists a morphism $h : Y \rightarrow X$ in the class of $g \circ f$. Consider the inverse image $Y' = X'_{(h)} = X'_{(Y)}$, and note that the morphism $g' = g_{(Y)} : Y' \rightarrow Y$ is integral by ??(iii). Via the correspondence of rational maps from Y to X' with rational Y -sections of Y' , we see that we are reduced to the case $X = Y$. \square

Corollary 19.6.12. *Let X be a locally integral and normal scheme, $g : X' \rightarrow X$ be an integral morphism, and f be a rational X -section of X' . Then f is everywhere defined.*

Corollary 19.6.13. *Let X be a normal and integral scheme, X' be an integral scheme, and $g : X' \rightarrow X$ be an integral morphism. If there exists a rational X -section f of X' , g is an isomorphism.*

19.6.2 Quasi-finite morphisms

Proposition 19.6.14. *Let $f : X \rightarrow Y$ be a morphism locally of finite type, x be a point of X and $y = f(x)$. The following conditions are equivalent:*

- (i) *The point x is isolated in the fiber X_y .*
- (ii) *The point x is closed in X_y and there is no generalization of x in X_y .*
- (iii) *The $\mathcal{O}_{Y,y}$ -module $\mathcal{O}_{X,x}$ is quasi-finite.*

Proof. The question is local over X and Y , so we can suppose that $X = \text{Spec}(B)$ and $Y = \text{Spec}(A)$ are affine, where B is finite A -algebra. Moreover, replacing X by $X \times_Y \text{Spec}(\mathcal{O}_{Y,y})$ does not change the fiber X_y and the local ring $\mathcal{O}_{X,x}$ (??), so we can suppose that A is a local ring with maximal ideal \mathfrak{m} (which equals to the local ring $\mathcal{O}_{Y,y}$). The fiber X_y is then the affine scheme of the ring $B/\mathfrak{m}B$, of finite type over $\kappa(y) = A/\mathfrak{m}$ (??). Let \mathfrak{P} be the prime ideal of B corresponding to x .

We note that the fiber $X_y = X \times_Y \text{Spec}(\kappa(y)) = \text{Spec}(B \otimes_A \kappa(y))$ is Jacobson. If (i) is satisfied, then $\{x\}$ is an open subset of X_y , hence contains a closed point (by the Jacobson property) which must be x , and x is therefore closed in X_y . Also, since $\{x\}$ is open in X_y , it is clear that there is no further generalization x' of x (which means $x \in \overline{\{x'\}}$) in X_y ; this proves (i) \Rightarrow (ii).

We now consider the conditions in (ii) and (iii). Consider the ring $\bar{B} = B \otimes_A \kappa(y) = B/\mathfrak{m}B$ and let $\bar{\mathfrak{P}}$ be the prime ideal corresponding to \mathfrak{P} . If x is closed in X_y , we see that $\bar{\mathfrak{P}}$ is maximal in \bar{B} and by condition (ii) there is no prime ideal of \bar{B} contained in $\bar{\mathfrak{P}}$. This shows that $\bar{B}_{\bar{\mathfrak{P}}}$ is Artinian, so we conclude that the ring homomorphism $A \rightarrow B$ is quasi-finite at \mathfrak{P} ; that is, the $\mathcal{O}_{Y,y}$ -module $\mathcal{O}_{X,x}$ is quasi-finite. Conversely, if $\bar{B}_{\bar{\mathfrak{P}}}$ is Artinian, then $\bar{\mathfrak{P}}$ is maximal and there is no prime ideal of \bar{B} contained in $\bar{\mathfrak{P}}$; we then conclude that x is closed in X_y and there is no generalization of x in X_y . This shows that (ii) \Leftrightarrow (iii).

We finally prove that (ii) implies (i). If (ii) is satisfied, then the prime ideal \mathfrak{P} is both maximal and minimal in B since $\bar{B}_{\bar{\mathfrak{P}}}$ is Artinian. Let $\bar{\mathfrak{P}}_1 = \bar{\mathfrak{P}}, \bar{\mathfrak{P}}_2, \dots, \bar{\mathfrak{P}}_r$ be the minimal prime ideals

of \bar{B} . Then the intersection $\bigcap_{i=1}^r \bar{\mathfrak{P}}_i$ is equal to the nilradical of \bar{B} , and $\bigcap_{i=2}^r \bar{\mathfrak{P}}_i$ is contained in this intersection in view of ??, so there exists an element $\bar{b} \in \bigcap_{i=2}^r \bar{\mathfrak{P}}_i$ that is not nilpotent. The open subset $D_{\bar{B}}(\bar{b})$ of $X_y = \text{Spec}(\bar{B})$ then reduces to $\{x\}$, which shows that x is isolated. \square

Corollary 19.6.15. *Let $f : X \rightarrow Y$ be a morphism of finite type. Then the following conditions are equivalent:*

- (i) Any point $x \in X$ is isolated in the fiber $X_{f(x)}$ (that is, $X_{f(x)}$ is discrete).
- (ii) For any $x \in X$, $X_{f(x)}$ is a finite $\kappa(f(x))$ -scheme.
- (iii) For any $x \in X$, the $\mathcal{O}_{Y,y}$ -module $\mathcal{O}_{X,x}$ is quasi-finite.

Proof. The equivalence of (i) and (iii) follows from ???. On the other hand, as $X_{f(x)}$ is an algebraic $\kappa(f(x))$ -scheme, the equivalence of (i) and (ii) follows from ???. \square

We say a morphism $f : X \rightarrow Y$ of finite type is **quasi-finite**, or X is **quasi-finite** over Y , if it satisfies the equivalent conditions of ???. We say a morphism $f : X \rightarrow Y$ is quasi-finite at a point $x \in X$ if there exists an affine open neighborhood V of $y = f(x)$ and an affine open neighborhood U of x such that $f(U) \subseteq V$ and the restriction $f|_U : U \rightarrow V$ is quasi-finite. We say that $f : X \rightarrow Y$ is locally quasi-finite if it is quasi-finite at every point of X . From ??, it is clear that any finite morphism is quasi-finite.

Proposition 19.6.16 (Properties of Quasi-finite Morphisms).

- (i) Any quasi-compact immersion (in particular any closed immersion) is quasi-finite.
- (ii) If $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are quasi-finite morphisms, $g \circ f$ is quasi-finite.
- (iii) If $f : X \rightarrow Y$ is a quasi-finite S -morphism, $f_{(S')} : X_{(S')} \rightarrow Y_{(S')}$ is quasi-finite for any base change $S' \rightarrow S$.
- (iv) If $f : X \rightarrow Y$ and $g : X' \rightarrow Y'$ are quasi-finite S -morphisms, $f \times_S g$ is quasi-finite.
- (v) Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be two morphisms such that $g \circ f$ is quasi-finite; if g is separated, or X is Noetherian, or $X \times_Z Y$ is locally Noetherian, then f is quasi-finite.
- (vi) If f is quasi-finite, so is f_{red} .

Proof. If $f : X \rightarrow Y$ is an immersion, any fiber is reduced to a singleton, so (i) follows from ??(i). For (ii), we note that $h = g \circ f$ is of finite type by ??(ii); if $z = h(x)$ and $y = f(x)$, y is isolated in $g^{-1}(z)$, so there exists an open neighborhood V of y in Y not containing other points of $g^{-1}(z)$; $f^{-1}(V)$ is then an open neighborhood of x not containing other points of $f^{-1}(y')$, where $y' \neq y$ is in $g^{-1}(z)$, and therefore not containing points $x' \neq x$ in $h^{-1}(z)$ that is not in $f^{-1}(y)$. As x is isolated in $f^{-1}(y)$ by hypothesis, it is then isolated in $h^{-1}(z) = f^{-1}(g^{-1}(z))$. As for (iii), we can limit ourselves to the case where $Y = S$ (??); we first note that $f' = f_{(S')}$ is of finite type (??(iii)). On the other hand, if $x' \in X' = X_{(S')}$ and $y' = f'(x')$, $X'_{y'}$ is identified with $X_y \otimes_{\kappa(y)} \kappa(y')$ by ???. As X_y is of finite dimension over $\kappa(y)$ by hypothesis, $X'_{y'}$ is of finite dimension over $\kappa(y')$, hence discrete. The assertions (iv), (v), (vi) then follows from the first three assertions in view of the general principle ??, where in (v) we assume that g is separated. The other cases, we first remark that if x is isolated in $X_{g(f(x))}$, it is also isolated in $X_{f(x)}$; the fact that f is of finite type follows from ???. \square

Proposition 19.6.17. *Let A be a complete Noetherian local ring, $Y = \text{Spec}(A)$, X be a separated Y -scheme locally of finite type, x be a point over the closed point y of Y , and suppose that x is isolated in the fiber X_y . Then $\mathcal{O}_{X,x}$ is a finitely generated A -module and X is Y -isomorphic to the sum of $X' = \text{Spec}(\mathcal{O}_{X,x})$ (which is a finite Y -scheme) and an A -scheme X'' .*

Proof. It follows from ?? that $\mathcal{O}_{X,x}$ is a quasi-finite A -module. As $\mathcal{O}_{X,x}$ is Noetherian (??) and the homomorphism $A \rightarrow \mathcal{O}_{X,x}$ is local, the hypothesis that A is complete implies that $\mathcal{O}_{X,x}$ is a finitely generated A -module (??). Let $X' = \text{Spec}(\mathcal{O}_{X,x})$ be the local scheme of X at x and $g : X' \rightarrow X$ be the canonical morphism. The composition $f \circ g : X' \rightarrow Y$ is then finite, and since f is separated, g is finite by ??, so $g(X')$ is closed in X (??). On the other hand, as g is of finite type and A is Noetherian, it is of finite presentation, and hence a local immersion at the closed point x' of X' (?? and the definition of g). But X' is the only open neighborhood of x' in X' , so it follows that $g(X')$ is open in X , which proves our assertion. \square

Corollary 19.6.18. *Let A be a complete Noetherian local ring, $Y = \text{Spec}(A)$, $f : X \rightarrow Y$ be a quasi-finite and separated morphism. Then X is Y -isomorphic to a sum $X' \amalg X''$, where X' is a finite Y -scheme and X'' is a quasi-finite Y -scheme such that, if y is the closed point of y , $X'' \cap f^{-1}(y) \neq \emptyset$.*

Proof. The fiber $f^{-1}(y)$ is finite and discrete by hypothesis, and the corollary then follows by recurrence on the number of points of $f^{-1}(y)$, using ??.

19.6.3 Integral closure of a scheme

Proposition 19.6.19. *Let (X, \mathcal{A}) be a ringed space, \mathcal{B} be an \mathcal{A} -algebra, and f be a section of \mathcal{B} over X . The following properties are equivalent:*

- (i) *The sub- \mathcal{A} -algebra of \mathcal{B} generated by f is finite (that is, of finite type as an \mathcal{A} -module).*
- (ii) *There exists a sub- \mathcal{A} -algebra \mathcal{C} of \mathcal{B} , which is an \mathcal{A} -module of finite type, such that $f \in \Gamma(X, \mathcal{C})$.*
- (iii) *For any $x \in X$, f_x is integral over the fiber \mathcal{A}_x .*

*If these equivalent conditions are satisfied, the section f is said to be **integral** over \mathcal{A} .*

Proof. As the sub- \mathcal{A} -module of \mathcal{B} generated by f^n is an \mathcal{A} -algebra, it is clear that (i) implies (ii). On the other hand, (ii) implies that for any $x \in X$, the \mathcal{A}_x -module \mathcal{C}_x is of finite type, which implies that any element of the algebra \mathcal{C}_x , and in particular f_x , is integral over \mathcal{A}_x . Finally, if for any point $x \in X$, we have a relation

$$f_x^n + (a_1)_x f_x^{n-1} + \cdots + (a_n)_x = 0$$

where a_i are sections of \mathcal{A} over an open neighborhood U of x , the section $f^n|_U + a_1 \cdot f^{n-1}|_U + \cdots + a_n$ is zero over an open neighborhood $V \subseteq U$ of x , so $f^k|_V$ (for $k \geq 0$) is a linear combination over $\Gamma(V, \mathcal{A})$ of $f^j|_V$ with $0 \leq j \leq n-1$. We then conclude that (iii) implies (i). \square

Corollary 19.6.20. *Under the hypothesis of ??, there exists a (unique) sub- \mathcal{A} -algebra \mathcal{A}' of \mathcal{B} such that for any $x \in X$, \mathcal{A}'_x is the set of germs $f_x \in \mathcal{B}_x$ that is integral over \mathcal{A}_x . For any open subset $U \subseteq X$, the sections of \mathcal{A}' over U is the sections of $\Gamma(U, \mathcal{B})$ that is integral over $\mathcal{A}|_U$. We say that \mathcal{A}' is the **integral closure** of \mathcal{A} in \mathcal{B} .*

Proof. The existence of \mathcal{A}' is immediate, by setting $\Gamma(U, \mathcal{A}')$ to be the set of $f \in \Gamma(U, \mathcal{B})$ such that f_x is integral over \mathcal{A}_x for any $x \in U$. It is clear that \mathcal{A}' is an algebra, and the second assertion follows from ??.

Let $(X, \mathcal{A}), (Y, \mathcal{B})$ be two ringed space and $f : X \rightarrow Y$ be a morphism. Let \mathcal{C} (resp. \mathcal{D}) be an \mathcal{A} -algebra (resp. \mathcal{B} -algebra) and let $u : \mathcal{D} \rightarrow \mathcal{C}$ be a f -morphism. Then, if \mathcal{A}' (resp. \mathcal{B}') is the integral closure of \mathcal{A} (resp. \mathcal{B}) in \mathcal{C} (resp. \mathcal{D}), the restriction of u to \mathcal{B}' is then a f -morphism $u' : \mathcal{B}' \rightarrow \mathcal{A}'$. In fact, if j is the canonical injection $\mathcal{B}' \rightarrow \mathcal{D}$, it suffices to show that

$$v = u^\sharp \circ f^*(j) : f^*(\mathcal{B}') \rightarrow \mathcal{C}'$$

maps $f^*(\mathcal{B}')$ into \mathcal{A}' . Now an element of $(f^*(\mathcal{B}'))_x = \mathcal{B}'_{f(x)} \otimes_{\mathcal{B}(f(x))} \mathcal{A}_x$ is integral over \mathcal{A}_x by the definition of \mathcal{B}' , and hence so is its image under v_x , which proves our assertion.

Proposition 19.6.21. *Let X be a scheme and \mathcal{A} be a quasi-coherent \mathcal{O}_X -algebra. The integral closure \mathcal{O}'_X of \mathcal{O}_X in \mathcal{A} is then a quasi-coherent \mathcal{O}_X -algebra, and for any affine open U of X , $\Gamma(X, \mathcal{O}'_X)$ is the integral closure of $\Gamma(U, \mathcal{O}_X)$ in $\Gamma(U, \mathcal{A})$.*

Proof. We can assume that $X = \text{Spec}(B)$ is affine and $\mathcal{A} = \tilde{A}$, where A is an B -algebra. Let B' be the integral closure of B in A . It then boils down to seeing that for any $x \in X$, an element of A_x , integer over B_x , necessarily belongs to B'_x , which follows from the fact that taking integral closure commutes with localization ??). \square

Under the hypothesis of ??, the X -scheme $X' = \text{Spec}(\mathcal{O}'_X)$ is then called the **integral closure of X relative to \mathcal{A}** . We also deduce from ?? that if $f : X' \rightarrow X$ is the structural morphism, then for any open subset U of X , $f^{-1}(U)$ is the integral closure of the induced subscheme U by X , relative to $\mathcal{A}|_U$. In particular, we conclude that f is integral.

Let X and Y be schemes, $f : X \rightarrow Y$ be a morphism, \mathcal{A} (resp. \mathcal{B}) be a quasi-coherent \mathcal{O}_X -algebra (resp. a \mathcal{O}_Y -algebra), and $u : \mathcal{B} \rightarrow \mathcal{A}$ be an f -morphism. We have seen that we have an induced f -morphism $u' : \mathcal{O}'_Y \rightarrow \mathcal{O}'_X$, where \mathcal{O}'_X (resp. \mathcal{O}'_Y) is the integral closure of \mathcal{O}_X (resp. \mathcal{O}_Y) relative to \mathcal{A} (resp. \mathcal{B}), we deduce a canonical morphism $f' : \text{Spec}(u') : X' \rightarrow Y'$?? fitting into the following commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array} \tag{19.6.1}$$

Suppose that X has only finitely many irreducible components $(X_i)_{1 \leq i \leq r}$, with generic points $(\xi_i)_{1 \leq i \leq r}$, and consider in particular the integral closure of X relative to a quasi-coherent \mathcal{K}_X -algebra \mathcal{A} . By ?? and ??, \mathcal{A} is the direct product of r quasi-coherent \mathcal{O}_X -algebras \mathcal{A}_i , the support of \mathcal{A}_i being contained in X_i , and the induced sheaf of \mathcal{A}_i over X_i is the constant sheaf whose fiber A_i is an algebra over \mathcal{O}_{X, ξ_i} . It is clear that the integral closure \mathcal{O}'_X of \mathcal{O}_X is the direct product of the integral closures $\mathcal{O}_X^{(i)}$ of \mathcal{O}_X in each \mathcal{A}_i , and therefore the integral closure $X' = \text{Spec}(\mathcal{O}'_X)$ of X relative to \mathcal{A} is an X -scheme which is the sum of $\text{Spec}(\mathcal{O}_X^{(i)}) = X'_i$.

Now suppose that the \mathcal{O}_X -algebra \mathcal{A} is reduced, or equivalently, each algebra A_i is reduced, and therefore can be considered as an algebra over the field $\kappa(\xi_i)$ (equal to the rational function

field of the reduced subscheme X_i of X); then the X'_i is a reduced X -scheme and X' is also the integral closure of X_{red} . Suppose moreover that the algebras A_i is a direct product of finitely many field K_{ij} ($1 \leq j \leq s_i$); if \mathcal{K}_{ij} is the subalgebra of \mathcal{A}_i corresponding to K_{ij} , it is clear that $\mathcal{O}_X^{(ij)}$ is the direct product of integral closures $\mathcal{O}_X^{(ij)}$ of \mathcal{O}_X in \mathcal{K}_{ij} . Therefore, X'_i is the sum of $X'_{ij} = \text{Spec}(\mathcal{O}_X^{(ij)})$. Moreover, under this hypothesis, we have the following:

Proposition 19.6.22. *Each X'_{ij} is an integral and normal X -scheme, and its rational function field $K(X'_{ij})$ is canonically identified with the algebraic closure K'_{ij} of $\kappa(\xi_i)$ in K_{ij} .*

Proof. In view of the preceding remarks, we can assume that X is integral, so $r = 1, s_1 = 1$, so that the unique algebra A_1 is a field K ; let ξ be the generic point of X , and let $f : X' \rightarrow X$ be the structural morphism. For any nonempty affine open U of X , $f^{-1}(U)$ is identified with the integral closure B'_U in the field K of the integral ring $B_U = \Gamma(U, \mathcal{O}_X)$??; as the ring B'_U is integrally closed, so is its localization, and $f^{-1}(U)$ is by definition an integral and normal scheme. Moreover, as (0) is the unique prime ideal of B'_U lying over the prime ideal (0) of B_U , $f^{-1}(\xi)$ is reduced to a singleton ξ' , and $\kappa(\xi')$ is the fraction field K' of B'_U , which is none other than the algebraic closure of $\kappa(\xi)$ in K . Finally, X' is irreducible, because if U runs through the nonempty affine open subsets of X , the $f^{-1}(U)$ constitute an open covering of X' formed by irreducible open subsets; moreover the intersection $f^{-1}(U \cap V)$ two two opens contains ξ' , hence nonempty, and we conclude from ?? that X' is irreducible. \square

Corollary 19.6.23. *Let X be a reduced scheme with finitely many irreducible components (X_i) , and let ξ_i be the generic point of X_i . The integral closure X' of X relative to \mathcal{K}_X is the sum of r separated X -schemes X'_i which are integral and normal. If $f : X' \rightarrow X$ is the structural morphism, $f^{-1}(\xi_i)$ is reduced to the generic point ξ'_i of X'_i and we have $\kappa(\xi'_i) = \kappa(\xi_i)$, which means f is birational.*

Proof. This is a particular case of ?? by taking $K'_{ij} = \kappa(\xi_i)$. The rational function field of X'_i (which is $\kappa(\xi'_i)$) is then equal to $\kappa(\xi_i)$, whence our claim. \square

The integral closure X' of X relative to \mathcal{K}_X is called that **normalization** of the reduced scheme X . We note that the morphism $f : X' \rightarrow X$, being birational and integral, is closed by ??, hence surjective (recall that a birational morphism is dominant). For $X' = X$, it is necessary and sufficient that X is normal. If X is an integral scheme, it follows from ?? that its normalization X' is integral.

Let X, Y be integral schemes, $f : X \rightarrow Y$ be a dominant morphism, $L = K(X), K = K(Y)$ be the rational function field of X and Y . The morphism f corresponds to an injection $K \rightarrow L$, and if we identify K (resp. L) with the simple sheaf \mathcal{K}_Y (resp. \mathcal{K}_X), this injection is an f -morphism. Let K_1 (resp. L_1) be an extension of K (resp. L) and suppose that we are given a monomorphism $K_1 \rightarrow L_1$ such that the diagram

$$\begin{array}{ccc} K_1 & \longrightarrow & L_1 \\ \uparrow & & \uparrow \\ K & \longrightarrow & L \end{array}$$

is commutative; if K_1 (resp. L_1) is considered as a simple sheaf over Y (resp. X), hence a \mathcal{K}_Y -algebra (resp. a \mathcal{K}_X -algebra), this signifies that $K_1 \rightarrow L_1$ is an f -morphism. Now if X' (resp. Y') is the integral closure of X (resp. Y) relative to L_1 (resp. K_1), X' (resp. Y') is a normal and integral scheme ?? and its rational function field is canonically identified with the algebraic

closure L' (resp. K') of L (resp. K) in L_1 (resp. K_1), and there exists a canonical morphism (necessarily dominant) $f' : X' \rightarrow Y'$ rendering the diagram (??). The important case is that $L_1 = L$, K_1 is an extension of K contained in L , and where we suppose that X is integral and normal, hence $X' = X$.

The preceding arguments then show that if X is a normal scheme and Y' is integrally closure of Y relative to a field $K_1 \subseteq L = K(X)$, any dominant morphism $f : X \rightarrow Y$ factors into

$$f : X \xrightarrow{f'} Y' \rightarrow Y$$

where f' is dominant; if the monomorphism $K_1 \rightarrow L$ is fixed, f' is necessarily unique (this can be verified when X and Y are both affine). We then say that given Y , L , and a K -monomorphism $K_1 \rightarrow L$, the integral closure Y' of Y relative to K_1 is a universal object.

Remark 19.6.2. Retain the hypothesis of ?? and suppose moreover that each algebra A_i is of finite dimension over $\kappa(\xi_i)$ (which implies that A_i is a direct product of finitely many fields); we can prove that the structural morphism $X' \rightarrow X$ is finite. For this, we can reduce to the case where X is reduced and affine with ring C , and that C has finitely many minimal prime ideals \mathfrak{p}_i ($1 \leq i \leq r$) with $C_i = C/\mathfrak{p}_i$. Then by ?? X' is finite over X if the integral closure of each C_i in finite extension of its fraction field is a finitely generated C -module, or equivalently, if C_i is Japanese for each i . We know that this condition is true if C is an algebra of finite type over a field, or over \mathbb{Z} , or over a complete Noetherian local ring. We then conclude that $X' \rightarrow X$ is a finite morphism if X is a scheme of finite type over a field, or over \mathbb{Z} , or over a complete Noetherian local ring.

19.6.4 Determinant of an endomorphism of \mathcal{O}_X -modules

Let A be a ring, E be a free A -module of rank n , and $u : E \rightarrow E$ be an endomorphism of E ; recall that in order to define the characteristic polynomial of u , we consider the endomorphism $u \otimes 1$ of free the $A[T]$ -module $E \otimes_A A[T]$ (which is of rank n), and we put

$$P(u, T) = \det(T \cdot I - (u \otimes 1))$$

(I is the identity morphism on $E \otimes_A A[T]$). We have

$$P(u, T) = T^n - \sigma_1(u)T^{n-1} + \cdots + (-1)^n\sigma_n(u)$$

where $\sigma_i(u)$ is an element of A , equal to a homogeneous polynomial of degree i (with integer coefficients) with entries the elements of the matrix of u relative to any basis of E . We say that the $\sigma_i(u)$ are the **elementray symmetric functions** of u , and we have in particular $\sigma_1(u) = \text{tr}(u)$ and $\sigma_n(u) = \det(u)$. By Hamilton-Cayley theorem, we have

$$P(u, u) = u^n - \sigma_1(u)u^{n-1} + \cdots + (-1)^n\sigma_n(u) = 0 \tag{19.6.2}$$

which can also be written as

$$(\det(u)) \cdot 1_E = uQ(u) \tag{19.6.3}$$

(1_E is the identity morphism on E), where

$$Q(u) = (-1)^{n+1}(u^{n-1} - \sigma_1(u)u^{n-2} + \cdots + (-1)^{n-1}\sigma_{n-1}(u)) \quad (19.6.4)$$

Let $\varphi : A \rightarrow B$ be a homomorphism of rings; consider the B -module $E_{(B)} = E \otimes_A B$ which is free of rank n , and the extension $u \otimes 1$ of u to an endomorphism on $E_{(B)}$. It is immediate that we have $\sigma_i(u \otimes 1) = \varphi(\sigma_i(u))$ for all i .

Suppose now that A is an integral domain, with fraction field K , and E is a finitely generated A -module (not necessarily free now). Let n be the rank of E , which equals to the dimension of $E \otimes_A K$ over K . Any endomorphism u of E corresponds canonically to the endomorphism $u \otimes 1$ of $E \otimes_A K$. By abuse of language, we call $P(u \otimes 1, T)$ the characteristic polynomial of u and denoted by $P(u, T)$, and the coefficients $\sigma_i(u \otimes 1)$ is called the elementray symmetric functions of u and denoted by $\sigma_i(u)$. In particular the determinant $\det(u) = \det(u \otimes 1)$ is defiend. With these notations, the formulas (??) and (??) are meaningful and still valid, if we interpret the u^i as the homomorphism $E \rightarrow E \otimes_A K$ whici is the composition of the endomorphism $u^j \otimes 1 = (u \otimes 1)^j$ of $E \otimes_A K$ and the canonical homomorphism $x \mapsto x \otimes 1$.

If F is the torsion module of R and $E_0 = E/F$, we have $u(F) \subseteq F$, hence, by taking quotient, u induces an endomorphism u_0 of E_0 ; moreover $E \otimes_A K$ is identified with $E_0 \otimes_A K$ and $u \otimes 1$ is identified with $u_0 \otimes 1$, hence $\sigma_i(u) = \sigma_i(u_0)$ for $1 \leq i \leq n$.

If E is torsion-free, E is identified with a sub- A -module of $E \otimes_A K$, and the relation $u \otimes 1 = 0$ is equivalent to $u = 0$. If E is a free A -module, the two definitions of $\sigma_i(u)$ given above coincide according to the preceding remarks, which justifies the notations adopted. We also note that if E is a torsion module then $E_0 = \{0\}$, the exterior algebra of E_0 is reduced to K and the determinant of the endomorphism u_0 of E_0 is equal to 1.

Proposition 19.6.24. *Let A be an integral domain, E be a finitely generated A -module, u be an endomorphism of E . Then the elementray symmetric functions $\sigma_i(u)$ of u (and in particular $\det(u)$) are integral elements of K over A .*

Proof. This is a particular case of ??, where we set $B = K$ and note that condition (ii) is satisfied for $M = E$. \square

Corollary 19.6.25. *Under the hypothesis of ??, if A is normal, the $\sigma_i(u)$ belong to A .*

Proposition 19.6.26. *Let A be an integral domain, E be a finitely generated A -module, of rank n , and u be an endomorphism of E such that the $\sigma_i(u)$ belong to A for each i . For u to be an automorphism of E , it is necessary that $\det(u)$ is invertible in A ; this condition is sufficient if E is torsion free.*

Proof. This conditions is sufficient by (??) and (??), if E is torsion free, since E is then a sub- A -module of $E \otimes_A K$, and $(\det(u))^{-1}Q(u)$ is the inverse of u . Conversely, this is necessary, because if u is invertible, it follows from ?? that $\det(u^{-1})$ belongs to the integral closure A' of A in K , and is clearly the inverse of $\det(u)$ in A' . If $\det(u)$ is not invertible in A , then it belongs to a maximal ideal \mathfrak{m} of A , which is the contraction of a maximal ideal of A' (??), contradiction. \square

We note a generalization of the preceding results. Consider a reduced Noetherian ring A and let \mathfrak{p}_i ($1 \leq i \leq r$) be the minimal prime ideals of A , and K_i be the fractional field of $A_i = A/\mathfrak{p}_i$. Then the total fraction field K of A is the direct product of the fields K_i (??). Let

E be a finitely generated A -module, and suppose that $E \otimes_A K$ is a K -module of dimension n . Then each K_i -vector space $E_i = E \otimes_A K_i$ is of dimension n . If u is an endomorphism of E , we put $P(u, T) = P(u \otimes 1, T)$ and $\sigma_j(u) = \sigma_j(u \otimes 1)$, and in particular $\det(u) = \det(u \otimes 1)$; the $\sigma_j(u)$ are then elements of K . It is immediate that $E \otimes_A K$ is a direct sum of E_i and each of them is stable under $u \otimes 1$. The restriction of $u \otimes 1$ to E_i is just the extension of u to E_i , and we conclude that $\sigma_j(u)$ is the element of K with component in K_i being $\sigma_j(u_i)$. As the integral closure of A in K is the direct product of that of A in K_i (??), the $\sigma_j(u)$ are integral over A .

Lemma 19.6.27. *The sub- A -algebra of K generated by the elements $\sigma_j(u)$ ($1 \leq j \leq n$) for $u \in \text{Hom}_A(E, E)$, is a finitely generated A -module.*

Let (X, \mathcal{A}) be a ringed space, \mathcal{E} be a locally free \mathcal{A} -module (of finite rank). There is then by hypothesis a basis \mathfrak{B} of X such that for any $V \in \mathfrak{B}$, $\mathcal{E}|_V$ is isomorphic to $\mathcal{A}^n|_V$ (the integer n may vary with V). Let u be an endomorphism of \mathcal{E} ; for any $V \in \mathfrak{B}$, u_V is then an endomorphism of the $\Gamma(V, \mathcal{A})$ -module $\Gamma(V, \mathcal{E})$, which is free by hypothesis; the determinant of u_V is then defined and belongs to $\Gamma(V, \mathcal{A})$. Moreover, if e_1, \dots, e_n is a basis of $\Gamma(V, \mathcal{E})$, their restriction to any open subset $W \subseteq V$ form a basis of $\Gamma(W, \mathcal{E})$ over $\Gamma(W, \mathcal{A})$, so $\det(u_W)$ is the restriction of $\det(u_V)$ to W . There then exists a unique section of \mathcal{A} over X , which we denote by $\det(u)$ and call the **determinant** of u , such that the restriction of $\det(u)$ to any $V \in \mathfrak{B}$ is $\det(u_V)$. It is clear that for any $x \in X$, we have $\det(u)_x = \det(u_x)$; for two endomorphisms u, v of \mathcal{E} , we have

$$\det(u \circ v) = (\det(u))(\det(v)), \quad \det(1_{\mathcal{E}}) = 1_{\mathcal{A}}.$$

If \mathcal{E} is of rank n (for example if X is connected), we have

$$\det(s \cdot u) = s^n \det(u)$$

for any $s \in \Gamma(X, \mathcal{A})$ (we note that $\det(0) = 0_{\mathcal{A}}$ if $n \geq 1$, but $\det(0) = 1_{\mathcal{A}}$ if $n = 0$). Moreover, for u to be an automorphism of \mathcal{E} , it is necessary and sufficient that $\det(u)$ is invertible in $\Gamma(X, \mathcal{A})$ (??).

If \mathcal{E} is of rank n , we can similarly define the elementary symmetric functions $\sigma_i(u)$ for u , which are elements of $\Gamma(X, \mathcal{A})$, and we also have the relations (??) and (??).

We have then define a homomorphism $\det : \text{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{E}) \rightarrow \Gamma(X, \mathcal{A})$ of multiplicative monoids. Note that $\text{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{E}) = \Gamma(X, \mathcal{H}\text{om}_{\mathcal{A}}(\mathcal{E}, \mathcal{E}))$ by definition, so we can replace X by any open subset U in this definition of \det , and therefore obtain a homomorphism $\det : \mathcal{H}\text{om}_{\mathcal{A}}(\mathcal{E}, \mathcal{E}) \rightarrow \mathcal{A}$ of sheaves of multiplicative monoids. If \mathcal{E} is of constant rank, we can similarly define the homomorphisms $\sigma_i : \mathcal{H}\text{om}_{\mathcal{A}}(\mathcal{E}, \mathcal{E}) \rightarrow \mathcal{A}$ of sheaves of sets; for $i = 1$, the homomorphism $\sigma_1 = \text{tr}$ is a homomorphism of \mathcal{A} -modules.

Let (Y, \mathcal{B}) be a second ringed space and $f : (X, \mathcal{A}) \rightarrow (Y, \mathcal{B})$ be a morphism of ringed spaces; if \mathcal{F} is a locally free \mathcal{B} -module, $f^*(\mathcal{F})$ is a locally free \mathcal{A} -module (with the same rank of \mathcal{F}). For any endomorphism v of \mathcal{F} , $f^*(v)$ is then an endomorphism of $f^*(\mathcal{F})$, and it follows from these definitions that $\det(f^*(v))$ is the section of $\mathcal{A} = f^*(\mathcal{B})$ over X which corresponds canonically to $\det(v) \in \Gamma(Y, \mathcal{B})$. We can then say that the homomorphism $f^*(\det) : f^*(\mathcal{H}\text{om}_{\mathcal{B}}(\mathcal{F}, \mathcal{F})) \rightarrow f^*(\mathcal{B}) = \mathcal{A}$ is the composition

$$f^*(\mathcal{H}\text{om}_{\mathcal{B}}(\mathcal{F}, \mathcal{F})) \xrightarrow{\gamma^*} \mathcal{H}\text{om}_{\mathcal{A}}(f^*(\mathcal{F}), f^*(\mathcal{F})) \xrightarrow{\det} \mathcal{A} \tag{19.6.5}$$

(formula (??)). We have a similar result for σ_i .

Suppose now that X is a locally integral scheme, so its sheaf of rational function \mathcal{K}_X is locally simple over X (??) and quasi-coherent as \mathcal{O}_X -module. If \mathcal{E} is a quasi-coherent \mathcal{O}_X -module of finite type, $\mathcal{E}' = \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{K}_X$ is then a locally free \mathcal{K}_X -module (??). For any endomorphism u of \mathcal{E} , $u \otimes 1_{\mathcal{K}_X}$ is then an endomorphism of \mathcal{E}' , and $\det(u \otimes 1)$ is a section of \mathcal{K}_X over X , which is called the **determinant** of u and denoted by $\det(u)$. It follows from ?? that $\det(u)$ is a section of the integral closure of \mathcal{O}_X in \mathcal{K}_X ; if X is also normal, $\det(u)$ is then a section of \mathcal{O}_X over X , and if we suppose moreover that \mathcal{E} is torsion free, for u to be an automorphism of \mathcal{E} , it is necessary and sufficient that $\det(u)$ is invertible (??). The formulae (??) and (??) are still valid; the homomorphism $u \mapsto \det(u)$ then defines a homomorphism $\det : \mathcal{H}\text{om}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E}) \rightarrow \mathcal{K}_X$, which has values in \mathcal{O}_X if X is normal. We have analogous results for the elementary symmetric function functions $\sigma_j(u)$, if \mathcal{E}' has constant rank; if moreover X is normal, the $\sigma_j(u)$ are sections of \mathcal{O}_X over X .

Finally, let X and Y be integral schemes, and $f : X \rightarrow Y$ be a dominant morphism. We see that there exists a canonical homomorphism $f^*(\mathcal{K}_Y) \rightarrow \mathcal{K}_X$, whence induces, for any quasi-coherent \mathcal{O}_Y -module \mathcal{F} of finite type, a canonical homomorphism $\theta : f^*(\mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{K}_Y) \rightarrow f^*(\mathcal{F}) \otimes_{\mathcal{O}_X} \mathcal{K}_X$. If v is an endomorphism of \mathcal{F} , $f^*(v \otimes 1_{\mathcal{K}_Y})$ is an endomorphism of $f^*(\mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{K}_Y)$, and we have a commutative diagram

$$\begin{array}{ccc} f^*(\mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{K}_Y) & \xrightarrow{f^*(v \otimes 1)} & f^*(\mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{K}_Y) \\ \theta \downarrow & & \downarrow \theta \\ f^*(\mathcal{F}) \otimes_{\mathcal{O}_X} \mathcal{K}_X & \xrightarrow{f^*(v) \otimes 1} & f^*(\mathcal{F}) \otimes_{\mathcal{O}_X} \mathcal{K}_X \end{array}$$

We then conclude that $\det(f^*(v))$ is the canonical image of the section $\det(v)$ of \mathcal{K}_Y under the canonical homomorphism $f^*(\mathcal{K}_Y) \rightarrow \mathcal{K}_X$. In fact, it is immediate that we are reduced to the case where $X = \text{Spec}(B)$ and $Y = \text{Spec}(A)$ are affine, A, B being integral domains with fraction fields K, L respectively, the homomorphism $A \rightarrow B$ being injective and extends to a monomorphism $K \rightarrow L$. If $\mathcal{F} = \tilde{M}$ where M is a finitely generated A -module, the dimension of $M \otimes_A K$ is equal to that of $(M \otimes_A B) \otimes_B L$ over L , and $\det((u \otimes 1) \otimes 1)$ is the image of $\det(u \otimes 1)$ in L for any endomorphism u of M , whence our conclusion.

Finally, suppose that X is a reduced locally Noetherian scheme, whose sheaf of rational functions \mathcal{K}_X is quasi-coherent by ???. Let \mathcal{E} be a coherent \mathcal{O}_X -module such that $\mathcal{E}' = \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{K}_X$ is locally free of rank n . We can then define for each endomorphism u of \mathcal{E} the elementary symmetric functions $\sigma_j(u)$, which are sections of \mathcal{K}_X over X .

19.6.5 Norm of invertible sheaves

Let (X, \mathcal{A}) be a ringed space and \mathcal{B} be an \mathcal{A} -algebra. The \mathcal{A} -module \mathcal{B} is canonically identified with a sub- \mathcal{A} -module of $\mathcal{H}\text{om}_{\mathcal{A}}(\mathcal{B}, \mathcal{B})$, where a section f of \mathcal{B} over an open subset U of X is identified with the multiplication by this section. Assume that (X, \mathcal{A}) and \mathcal{B} satisfies the conditions given in the previous subsection, so that we can define $\det(f)$ (resp. $\sigma_j(f)$) to be a section of \mathcal{K}_X over U , which is called the **norm** of f (resp. the elementary symmetric functions) of f and denoted by $N_{\mathcal{B}/\mathcal{A}}(f)$. We suppose that one of the following conditions is satisfied:

- (α) \mathcal{B} is a locally free \mathcal{A} -module of finite rank n .
- (β) (X, \mathcal{A}) is a reduced locally Noetherian scheme, \mathcal{B} is a coherent \mathcal{A} -module such that $\mathcal{B} \otimes_{\mathcal{A}} \mathcal{K}_X$ is a locally free \mathcal{K}_X -module of rank n , and for any section $f \in \Gamma(U, \mathcal{B})$ over an open subset $U \subseteq X$, $\sigma_j(f)$ ($1 \leq j \leq n$) is a section of \mathcal{A} over U (this is true for example if X is normal).

The hypothesis that $\mathcal{B} \otimes_{\mathcal{A}} \mathcal{K}_X$ is locally free can be expressed by the following: denote by X_i the reduced closed subschemes of X with underlying space the irreducible components of X , which are then locally Noetherian integral schemes. Any $x \in X$ belongs to finitely many X_i , and $\mathcal{B} \otimes_{\mathcal{A}} \mathcal{K}_{X_i}$ is a locally free \mathcal{K}_{X_i} -module of constant rank k_i (??); to say that $\mathcal{B} \otimes_{\mathcal{A}} \mathcal{K}_X$ is locally free \mathcal{K}_X -module signifies that, for any $x \in X$, the ranks k_i such that $x \in X_i$ are all equal. This question is in fact local, and we can assume that $X = \text{Spec}(A)$, where A is a reduced Noetherian ring, and $\mathcal{B} = \tilde{B}$ where B is a finite A -algebra. If \mathfrak{p}_i ($1 \leq i \leq r$) are the minimal prime ideals of A , the total fraction ring K of A is then the direct product of K_i , where K_i is the fraction field of $A_i = A/\mathfrak{p}_i$, and $B \otimes_A K$ is then the direct sum of $B \otimes_A K_i$, whence our conclusion.

It is clear that under the hypotheses (α) or (β), we then define a homomorphism $N_{\mathcal{B}/\mathcal{A}} : \mathcal{B} \rightarrow \mathcal{A}$ of sheaves of multiplicative monoids, which is also denoted by N if there is no confusion, and called the norm homomorphism. For two sections f, g of \mathcal{B} over an open subset U , we then have

$$N_{\mathcal{B}/\mathcal{A}}(fg) = N_{\mathcal{B}/\mathcal{A}}(f)N_{\mathcal{B}/\mathcal{A}}(g), \quad N_{\mathcal{B}/\mathcal{A}}(1_{\mathcal{B}}) = 1_{\mathcal{A}} \quad (19.6.6)$$

for the corresponding sections of \mathcal{A} over U . Also, for any section s of \mathcal{A} over U , we have

$$N_{\mathcal{B}/\mathcal{A}}(s \cdot 1_{\mathcal{B}}) = s^n. \quad (19.6.7)$$

In case (α), for any $f \in \Gamma(U, \mathcal{B})$ to be invertible, it is necessary and sufficient that $N(f) \in \Gamma(U, \mathcal{A})$ is invertible; in case (β), this condition is necessary, and is sufficient if \mathcal{B} is a torsion free \mathcal{A} -module.

Suppose the one of the hypotheses (α), (β) is satisfied, and let \mathcal{L}' be an invertible \mathcal{B} -module. We can canonically associate an invertible \mathcal{A} -module by the following. Denote by \mathcal{A}^\times (resp. \mathcal{B}^\times) the subsheaf of \mathcal{A} (resp. \mathcal{B}) such that $\Gamma(U, \mathcal{A}^\times)$ (resp. $\Gamma(U, \mathcal{B}^\times)$) is the set of invertible elements of $\Gamma(U, \mathcal{A})$ (resp. $\Gamma(U, \mathcal{B})$) for any open subset $U \subseteq X$; this is a sheaf of multiplicative groups, and $N_{\mathcal{B}/\mathcal{A}}$, restricted to \mathcal{B}^\times , is a homomorphism $\mathcal{B}^\times \rightarrow \mathcal{A}^\times$ of sheaves of groups. Let \mathfrak{L} be the set of couples $(U_\lambda, \eta_\lambda)$, with the following property: U_λ is an open subset of X and $\eta_\lambda : \mathcal{L}'|_{U_\lambda} \xrightarrow{\sim} \mathcal{B}|_{U_\lambda}$ is an isomorphism of $(B|_{U_\lambda})$ -modules. By hypothesis, the U_λ for an open covering of X ; for two indices λ, μ , we put $\omega_{\lambda\mu} = (\eta_\lambda|_{U_\lambda \cap U_\mu}) \circ (\eta_\mu|_{U_\lambda \cap U_\mu})^{-1}$, which is an automorphism of $\mathcal{B}|_{U_\lambda \cap U_\mu}$, and canonically identified with a section of \mathcal{B}^\times over $U_\lambda \cap U_\mu$, and $(\omega_{\lambda\mu})$ is a 1-cocycle over the covering $\mathfrak{U} = (U_\lambda)$ with values in \mathcal{B}^\times . The fact that $N_{\mathcal{B}/\mathcal{A}} : \mathcal{B}^\times \rightarrow \mathcal{A}^\times$ is a homomorphism implies that $(N_{\mathcal{B}/\mathcal{A}}(\omega_{\lambda\mu}))$ is a 1-cocycle of \mathfrak{U} with values in \mathcal{A}^\times , which then corresponds (up to isomorphism) to an invertible \mathcal{A} -module. This invertible \mathcal{A} -module is denoted by $N_{\mathcal{B}/\mathcal{A}}(\mathcal{L}')$ and is called the norm of the invertible \mathcal{B} -module \mathcal{L}' .

Let \mathfrak{M} be a subset of \mathfrak{L} such that the U_λ form an open covering of X , and let \mathfrak{B} be a covering of X . The restriction of the cocycle $(\omega_{\lambda\mu})$ to \mathfrak{B} defines a 1-cocycle $(N_{\mathcal{B}/\mathcal{A}}(\omega_{\lambda\mu}))$, which is the

restriction of the 1-cocycle $(N_{\mathcal{B}/\mathcal{A}}(\omega_{\lambda\mu}))$ to \mathfrak{U} ; it is clear that there is a canonical isomorphism of the invertible \mathcal{A} -modules thus defined, and we can therefore define $N_{\mathcal{B}/\mathcal{A}}(\mathcal{L}')$ by a refinement of the covering \mathfrak{U} . This shows that, if \mathcal{L}' and \mathcal{K}' are two invertible \mathcal{B} -modules, by (??) we have

$$N(\mathcal{L}' \otimes_{\mathcal{B}} \mathcal{K}') = N(\mathcal{L}') \otimes_{\mathcal{A}} N(\mathcal{K}'), \quad N_{\mathcal{B}/\mathcal{A}}(\mathcal{B}) = \mathcal{A}, \quad (19.6.8)$$

and therefore $N(\mathcal{L}'^{-1}) = N(\mathcal{L}')^{-1}$. Also, it follows from (??) that if \mathcal{L} is an invertible \mathcal{A} -module, we have

$$N_{\mathcal{B}/\mathcal{A}}(\mathcal{L} \otimes_{\mathcal{A}} \mathcal{B}) = \mathcal{L}^{\otimes n}. \quad (19.6.9)$$

We show that $N_{\mathcal{B}/\mathcal{A}}$ is a covariant functor on the category of invertible \mathcal{B} -modules. Let $u' : \mathcal{L}' \rightarrow \mathcal{K}'$ be a homomorphism of invertible \mathcal{B} -modules, and let $\mathfrak{B} = (U_{\lambda})$ be an open covering of X such that for any λ , we have an isomorphism $\eta_{\lambda} : \mathcal{L}'|_{U_{\lambda}} \xrightarrow{\sim} \mathcal{B}|_{U_{\lambda}}$ and $\tau_{\lambda} : \mathcal{K}'|_{U_{\lambda}} \xrightarrow{\sim} \mathcal{B}|_{U_{\lambda}}$; there is then for each λ an endomorphism u'_{λ} of $\mathcal{B}|_{U_{\lambda}}$ such that $u'_{\lambda} \circ \eta_{\lambda} = \tau_{\lambda} \circ (u'|_{U_{\lambda}})$, and we can evidently identify u'_{λ} with a section of \mathcal{B} over U_{λ} . Hence, for any couple (λ, μ) of indices, the restriction of $(\tau_{\lambda})^{-1} \circ u'_{\lambda} \circ \eta_{\lambda}$ and $(\tau_{\mu})^{-1} \circ u'_{\mu} \circ \eta_{\mu}$ to $U_{\lambda} \cap U_{\mu}$ coincide. We then deduce for the 1-cocycle $(\omega_{\lambda\mu})$ corresponding to \mathcal{L}' and the 1-cocycle $(\gamma_{\lambda\mu})$ corresponding to \mathcal{K}' the relation

$$\gamma_{\lambda\mu} u'_{\mu} = u'_{\lambda} \omega_{\lambda\mu}.$$

If we put $u_{\lambda} = N(u'_{\lambda})$, we then have the analogous relation

$$N(\gamma_{\lambda\mu}) u_{\mu} = u_{\lambda} N(\omega_{\lambda\mu})$$

and therefore the u_{λ} define a homomorphism $N(\mathcal{L}') \rightarrow N(\mathcal{K}')$, which is denoted by $N_{\mathcal{B}/\mathcal{A}}(u)$ or $N(u)$. In view of ??, it is clear that under the hypothesis (α), u' is an isomorphism if and only if u is, and this is true under the hypothesis (β) if \mathcal{B} is moreover torsion free. In particular, if consider the homomorphisms $\mathcal{B} \rightarrow \mathcal{L}'$, which correspond to global sections of \mathcal{L}' , since $N_{\mathcal{B}/\mathcal{A}}(\mathcal{B}) = \mathcal{A}$, we get a canonical homomorphism

$$N_{\mathcal{B}/\mathcal{A}} : \Gamma(X, \mathcal{L}') \rightarrow \Gamma(X, N_{\mathcal{B}/\mathcal{A}}(\mathcal{L}')).$$

It also follows from (??) that if $f' \in \Gamma(X, \mathcal{L}')$, $g' \in \Gamma(X, \mathcal{K}')$, we have

$$N(f' \otimes g') = N(f') \otimes N(g'). \quad (19.6.10)$$

Also, for any invertible \mathcal{A} -module \mathcal{L} and any section $f \in \Gamma(X, \mathcal{L})$, we have

$$N_{\mathcal{B}/\mathcal{A}}(f \otimes 1_{\mathcal{B}}) = f^{\otimes n}. \quad (19.6.11)$$

Finally, for the homomorphism $\mathcal{B} \rightarrow \mathcal{L}'$ corresponding to a section f' of \mathcal{L}' over X to be an isomorphism, it is necessary and sufficient that f'_x generates \mathcal{L}'_x for any $x \in X$; under condition (α), this is equivalent to that $N(f')_x$ generates $(N(\mathcal{L}'))_x$ for any x , and this is true for condition (β) if \mathcal{B} is torsion free.

Let (X, \mathcal{A}) , (X', \mathcal{A}') be two ringed spaces and $\varphi : X' \rightarrow X$ be a morphism, \mathcal{B} be an \mathcal{A} -algebra, and $\mathcal{B}' = \varphi^*(\mathcal{B})$. Suppose that one of the following conditions is satisfied:

- (i) \mathcal{B} satisfies condition (α) .
- (ii) (X, \mathcal{A}) and \mathcal{B} satisfy condition (β) , (X', \mathcal{A}') is a reduced locally Noetherian scheme, and if we denote by X_α and X'_β the reduced closed subschemes of X and X' with underlying space the irreducible components of these spaces, the restriction of φ to X'_β is a dominant morphism from X'_β to X_α .

Under these conditions, we claim that \mathcal{B}' verifies the conditions (α) or (β) ; the first case is clear, and to prove the second one, it suffices to prove that for any $x' \in X'$, the ranks of $\mathcal{B}' \otimes_{\mathcal{O}_{X'}} \mathcal{K}_{X'_\beta}$ for the indices β such that $x' \in X'_\beta$ are the same. Now, if the restriction of φ to X'_β is a dominant morphism into X_α , the rank of $\mathcal{B}' \otimes_{\mathcal{O}_{X'}} \mathcal{K}_{X'_\beta}$ is equal to that of $\mathcal{B} \otimes_{\mathcal{O}_X} \mathcal{K}_{X_\alpha}$ (which can be seen from the affine case), whence our claim.

This being established, it follows that if f is a section of \mathcal{B} over an open subset $U \subseteq X$, and f' is the inverse image of f under φ , $N_{\mathcal{B}'/\mathcal{A}'}(f')$ is the section of \mathcal{A}' over $\varphi^{-1}(U)$ which is the inverse image of $N_{\mathcal{B}/\mathcal{A}}(f)$ under φ . If \mathcal{L} is an invertible \mathcal{B} -module and if $\mathcal{L}' = \varphi^*(\mathcal{L})$ (which is an invertible \mathcal{B}' -module), we have

$$N_{\mathcal{B}'/\mathcal{A}'}(\mathcal{L}') = \varphi^*(N_{\mathcal{B}/\mathcal{A}}(\mathcal{L})). \quad (19.6.12)$$

Suppose now that (X, \mathcal{A}) is a scheme. Then giving a quasi-coherent finite \mathcal{A} -algebra \mathcal{B} is equivalent to giving a finite morphism $\varphi : X' \rightarrow X$ such that $\varphi_*(\mathcal{O}_{X'}) = \mathcal{B}$, defined up to X -isomorphisms (??), and in this case X' is isomorphic to the affine spectrum $\text{Spec}(\mathcal{B})$. Moreover, if this morphism $\varphi : X' \rightarrow X$ is fixed, then giving a quasi-coherent $\mathcal{O}_{X'}$ -module \mathcal{F}' is equivalent to giving a quasi-coherent \mathcal{B} -module such that $\varphi_*(\mathcal{F}') = \mathcal{F}$ (??), and for \mathcal{F}' to be invertible, it is necessary and sufficient that \mathcal{F} is (??). To utilize the preceding results for the finite morphism φ , it is then necessary to assume that $\mathcal{B} = \varphi_*(\mathcal{O}_{X'})$ satisfies condition (α) or (β) . For any invertible $\mathcal{O}_{X'}$ -module \mathcal{L}' , we then set

$$N_{X'/X}(\mathcal{L}') := N_{\varphi_*(\mathcal{O}_{X'})/\mathcal{O}_X}(\varphi_*(\mathcal{L}')) \quad (19.6.13)$$

which is called the **norm** (relative to φ) of \mathcal{L}' . Similarly, if $u' : \mathcal{L}' \rightarrow \mathcal{K}'$ is a homomorphism of invertible $\mathcal{O}_{X'}$ -modules, we put

$$N_{X'/X}(u') = N_{\varphi_*(\mathcal{O}_{X'})/\mathcal{O}_X}(\varphi_*(u')) : N_{X'/X}(\mathcal{L}') \rightarrow N_{X'/X}(\mathcal{K}'). \quad (19.6.14)$$

In particular, if we consider homomorphisms $\mathcal{O}_{X'} \rightarrow \mathcal{L}'$, we obtain a canonical homomorphism

$$N_{X'/X} : \Gamma(X', \mathcal{L}') \rightarrow \Gamma(X, N_{X'/X}(\mathcal{L}')). \quad (19.6.15)$$

Proposition 19.6.28. *Let $\varphi : X' \rightarrow X$ be a finite morphism and suppose that condition (i) or (ii) is satisfied. For a homomorphism $u' : \mathcal{L}' \rightarrow \mathcal{K}'$ of invertible $\mathcal{O}_{X'}$ -modules to be an isomorphism, it is necessary and sufficient that, in the first case, that $N_{X'/X}(u')$ is an isomorphism; in the second case, this condition is necessary, and is sufficient if $\varphi_*(\mathcal{O}_{X'})$ is torsion free.*

Proof. This is a particular case of our previous discussions, where we put $\mathcal{B} = \varphi_*(\mathcal{O}_{X'})$, which is a quasi-coherent finite \mathcal{O}_X -algebra. We also note that by ??, for $\varphi_*(u')$ to be an isomorphism, it is necessary and sufficient that u' is an isomorphism. \square

Corollary 19.6.29. *Retain the hypothesis of ?? and suppose that $\varphi_*(\mathcal{O}_{X'})$ is torsion free. Let \mathcal{L}' be an invertible $\mathcal{O}_{X'}$ -module, f' be a section of \mathcal{L}' over X' , and $f = N_{X'/X}(f')$ the section of $\mathcal{L} = N_{X'/X}(\mathcal{L}')$ over X corresponding to f' . Then we have $\varphi(X' - X'_{f'}) = X - X_f$ and X_f is the largest open subset of X such that $\varphi^{-1}(U) \subseteq X'_{f'}$.*

Proof. In fact, $\varphi(X' - X'_{f'})$ is closed in X by ??, and it then suffices to prove the second assertion. Now the relation $U \subseteq X_f$ is equivalent to that the homomorphism $\mathcal{O}_X|_U \rightarrow \mathcal{L}|_U$ defined by $f|_U$ is an isomorphism. In view of ??, this is equivalent to that the homomorphism $\mathcal{O}_{X'}|_{\varphi^{-1}(U)} \rightarrow \mathcal{L}'|_{\varphi^{-1}(U)}$ defined by $f'|_{\varphi^{-1}(U)}$ is an isomorphism, which means $\varphi^{-1}(U) \subseteq X'_{f'}$. \square

Proposition 19.6.30. *Let $\varphi : X' \rightarrow X$ be a finite morphism, $\psi : Y \rightarrow X$ be a morphism; let $Y' = X'_{(Y)}$, $\varphi' = \varphi_{(Y)}$, $\psi' = \psi_{(X')}$ such that the following diagram is commutative*

$$\begin{array}{ccc} Y' & \xrightarrow{\psi'} & X' \\ \varphi' \downarrow & & \downarrow \varphi \\ Y & \xrightarrow{\psi} & X \end{array}$$

Assume the hypotheses of ??. Then for any invertible $\mathcal{O}_{X'}$ -module \mathcal{L}' , we have

$$N_{Y'/Y}(\psi'^*(\mathcal{L}')) = \psi^*(N_{X'/X}(\mathcal{L}')).$$

Proof. Note that we have $\psi^*(\varphi_*(\mathcal{L}')) = \varphi'_*(\psi'^*(\mathcal{L}'))$ in view of ??, and in particular $\varphi'_*(\mathcal{O}_{Y'}) = \psi^*(\varphi_*(\mathcal{O}_{X'}))$; if $\varphi_*(\mathcal{O}_{X'})$ is locally free, so is $\varphi'_*(\mathcal{O}_{Y'})$. The conclusion then follows from the definition of $N_{X'/X}$, $N_{Y'/Y}$, and ??). \square

19.6.6 A criterion for ample sheaves

Proposition 19.6.31. *Let $f : X \rightarrow Y$ be a quasi-compact morphism, $g : X' \rightarrow X$ be a finite and surjective morphism such that (X, \mathcal{O}_X) and $g_*(\mathcal{O}_{X'})$ satisfy condition (β) . Then, for an ample invertible $\mathcal{O}_{X'}$ -module \mathcal{L}' relative to $f \circ g$, $N_{X'/X}(\mathcal{L}') = \mathcal{L}$ is ample relative to f .*

Proof. We can suppose that Y is affine, and then, in view of ??, it suffices to prove that, if \mathcal{L}' is ample, then $\mathcal{L} = N_{X'/X}(\mathcal{L}')$ is ample. For this, we can assume that $g_*(\mathcal{O}_{X'})$ is torsion free. In fact, let \mathcal{T} be the kernel of the homomorphism $g_*(\mathcal{O}_{X'}) \rightarrow g_*(\mathcal{O}_{X'}) \otimes_{\mathcal{O}_X} \mathcal{K}_X$, which is a coherent ideal of $\mathcal{B} = g_*(\mathcal{O}_{X'})$ by hypothesis, and put $X'' = \text{Spec}(\mathcal{B}/\mathcal{T})$; we then have a commutative diagram

$$\begin{array}{ccc} X'' & \xrightarrow{j} & X' \\ & \searrow g' & \swarrow g \\ & X & \end{array}$$

where j is a closed immersion ??). Moreover, since \mathcal{T} is a torsion sheaf, by ?? and ?? we see that the support of \mathcal{T} is a closed subset that is rare in X , so for the generic point x of an irreducible component of X , there exists an affine open neighborhood U of x such that $\mathcal{B}|_U = (\mathcal{B}/\mathcal{T})|_U$. As g is by hypothesis surjective, we then conclude that $x \in g'(X'')$; g' is then dominant, and

hence surjective by ?? since it is a finite morphism. By definition we have

$$g'_*(\mathcal{O}_{X''}) \otimes \mathcal{K}_X = (\mathcal{B}/\mathcal{T}) \otimes_{\mathcal{O}_X} \mathcal{K}_X = g_*(\mathcal{O}_{X'}) \otimes_{\mathcal{O}_X} \mathcal{K}_X,$$

so (X, \mathcal{O}_X) and $g'_*(\mathcal{O}_{X''})$ satisfies condition (β) , and $g'_*(\mathcal{O}_{X''})$ is torsion free. Finally, $j^*(\mathcal{L}') = \mathcal{L}''$ is an ample $\mathcal{O}_{X''}$ -module ??(ii), and $N_{X''/X}(\mathcal{L}'') = N_{X'/X}(\mathcal{L}')$. To see this, we note that to define these two invertible \mathcal{O}_X -modules we can utilize an affine open covering (U_λ) of X such that $g_*(\mathcal{L}')$ and $g'_*(\mathcal{L}'')$ to U_λ are respectively isomorphic to $\mathcal{B}|_{U_\lambda}$ and $(\mathcal{B}/\mathcal{T})|_{U_\lambda}$. By ??, we immediately see that for any isomorphism $\eta_\lambda : g_*(\mathcal{L}')|_{U_\lambda} \rightarrow \mathcal{B}|_{U_\lambda}$ corresponds canonically to an isomorphism

$$\eta'_\lambda : g'_*(\mathcal{L}'')|_{U_\lambda} \rightarrow (\mathcal{B}/\mathcal{T})|_{U_\lambda}$$

so that, if $(\omega_{\lambda\mu})$ and $(\omega'_{\lambda\mu})$ are the 1-cocycles corresponding to the isomorphisms (η_λ) and (η'_λ) , $\omega'_{\lambda\mu}$ is the canonical image of $\omega_{\lambda\mu} \in \Gamma(U_\lambda \cap U_\mu, \mathcal{B})$ to $\Gamma(U_\lambda \cap U_\mu, \mathcal{B}/\mathcal{T})$. In view of the definition of \mathcal{T} , we conclude that

$$N_{\mathcal{B}/\mathcal{A}}(\omega_{\lambda\mu}) = N_{(\mathcal{B}/\mathcal{T})/\mathcal{A}}(\omega'_{\lambda\mu})$$

(where $\mathcal{A} = \mathcal{O}_X$), whence the equality.

Suppose then that $g_*(\mathcal{O}_{X'})$ is torsion free. It then suffices to prove that if f runs through the sections of $\mathcal{L}^{\otimes n}$ ($n > 0$) over X , the X_f form a basis of X ???. Now, let $x \in X$, and let U be an open neighborhood of x ; as $g^{-1}(x)$ is finite by ?? and \mathcal{L}' is ample, there exists an integer $n > 0$ and a section f' of $\mathcal{L}'^{\otimes n}$ over X' such that $X'_{f'}$ is an open neighborhood of $g^{-1}(x)$ contained in $g^{-1}(U)$. As we have $\mathcal{L}^{\otimes n} = N_{X'/X}(\mathcal{L}'^{\otimes n})$, it then suffices to choose $f = N_{X'/X}(f')$: in fact, we have $X - X_f = g(X' - X'_{f'})$ by ??, so $x \in X_f \subseteq U$. \square

Corollary 19.6.32. *Under the hypotheses of ??, for an invertible \mathcal{O}_X -module \mathcal{L} to be ample relative to f , it is necessary and sufficient that $\mathcal{L}' = g^*(\mathcal{L})$ is ample relative to $f \circ g$.*

Proof. This condition is necessary, since g is affine ???. To see the sufficiency, we can assume that Y is affine, so X and X' are quasi-compact and \mathcal{L}' is ample ???. Now the set of points $x \in X$ such that there is a neighborhood of x over which $g_*(\mathcal{O}_{X'})$ (resp. $g_*(\mathcal{O}_{X'}) \otimes_{\mathcal{O}_X} \mathcal{K}_X$) is of rank n is open and closed in X by hypotheses, so X is a finite sum of such open subschemes (recall that X is quasi-compact), and we can therefore suppose that it is equal to one of them ???. But we then have $N_{X'/X}(\mathcal{L}') = \mathcal{L}^{\otimes n}$, so $\mathcal{L}^{\otimes n}$ is ample in view of ??, and \mathcal{L} is therefore ample. \square

Corollary 19.6.33. *Under the hypotheses of ??, suppose moreover that $f : X \rightarrow Y$ is of finite type. Then, for f to be quasi-projective, it is necessary and sufficient that $f \circ g$ is quasi-projective. If we suppose that Y is quasi-compact and quasi-compact, then, for f to be projective, it is necessary and sufficient that $f \circ g$ is projective.*

Proof. The hypotheses implies that $f \circ g$ is of finite type. By the definition of quasi-projective morphisms, the first assertion then follows from ?? and ???. In view of this result and ??, it remains to prove that if f is quasi-projective, then for f to be proper, it is necessary and sufficient that $f \circ g$ is proper. But f is then separated and of fintie type, and as g is surjective, this follows from ??(ii). \square

Corollary 19.6.34. *Let X be a scheme of finite type over a field K and K' be finite extension of K . For X to be projective (resp. quasi-projective) over K , it is necessary and sufficient that $X' \otimes_K K'$ is projective (resp. quasi-projective) over K' .*

Proof. This condition is necessary by ??(iii) and ??(iii). Conversely, suppose that X' is projective (resp. quasi-projective), and let $g : X' \rightarrow X$ be the canonical projective. Since K' is finite over K , it is clear that g is a finite morphism by ?? and is surjective (??). Moreover, $g_*(\mathcal{O}_{X'})$ is a locally free \mathcal{O}_X -module, being isomorphic to $\mathcal{O}_X \otimes_K K'$ (??). It then follows from the hypotheses and ?? and ??(ii) that X' is projective (resp. quasi-projective) over K . We then deduce from ?? that X is projective (resp. quasi-projective) over K . \square

Remark 19.6.3. In fact, later we will see that the statement of ?? is valid for arbitrary extension K' of K .

The end of this subsection is devoted to the proof of the criterion in ??, which is a refinement of the techniques we have currently used.

Lemma 19.6.35. *Let X be a reduced Noetherian scheme and \mathcal{E} be a coherent \mathcal{O}_X -module such that $\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{K}_X$ is locally free of rank n . Then there exists a reduced Noetherian scheme Z and a birational and finite morphism $h : Z \rightarrow X$ such that the homomorphisms $\sigma_i : \mathcal{H}\text{om}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E}) \rightarrow \mathcal{K}_X$ send $\mathcal{H}\text{om}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E})$ into the coherent \mathcal{O}_X -algebra $h_*(\mathcal{O}_Z)$.*

Corollary 19.6.36. *Under the hypotheses of ??, let W be an open subset of X such that for any $x \in W$, either X is normal at x or \mathcal{E}_x is a free $\mathcal{O}_{X,x}$ -module. Then we can choose h so that the restriction of h to $h^{-1}(W)$ is an isomorphism $h^{-1}(W) \cong W$.*

Proof. In fact, the hypotheses imply that if $U \subseteq W$ is an affine open subset, we have, in the notations of ??, that $(\sigma_i(u))_x \in A_x$ for any $x \in U$ (??), hence $\sigma_i(u) \in A$, and the conclusion follows from the definition of h given in ??.

Proof. Let X be a reduced Noetherian scheme, $g : X' \rightarrow X$ be a finite and surjective morphism such that $\mathcal{B} = g_*(\mathcal{O}_{X'})$ is a coherent \mathcal{O}_X -module. Suppose that $\mathcal{B} \otimes_{\mathcal{O}_X} \mathcal{K}_X$ is locally free of rank n . Then we can apply ?? on $\mathcal{E} = \mathcal{B}$, with the same notations, to obtain a homomorphism $\sigma_n : \mathcal{H}\text{om}_{\mathcal{O}_X}(\mathcal{B}, \mathcal{B}) \rightarrow h_*(\mathcal{O}_Z)$, and by composing with the canonical injection $\mathcal{B} \rightarrow \mathcal{H}\text{om}_{\mathcal{O}_X}(\mathcal{B}, \mathcal{B})$, we obtain a homomorphism of sheaves of multiplicative monoids:

$$\tilde{N} : \mathcal{B} = g_*(\mathcal{O}_{X'}) \rightarrow h_*(\mathcal{O}_Z) = \mathcal{C}. \quad (19.6.16)$$

For any invertible $\mathcal{O}_{X'}$ -module \mathcal{L}' , $g_*(\mathcal{L}')$ is an invertible \mathcal{B} -module by ??, and using the same method, we can define an invertible \mathcal{C} -module $\tilde{N}(g_*(\mathcal{L}'))$, which is functorial on \mathcal{L}' . \square

Lemma 19.6.37. *Let X be a reduced Noetherian scheme, $g : X' \rightarrow X$ be a finite and surjective morphism such that $g^*(\mathcal{O}_{X'}) \otimes_{\mathcal{O}_X} \mathcal{K}_X$ is locally free of rank n . Then there exists a reduced Noetherian scheme Z and a finite and birational morphism $h : Z \rightarrow X$ such that for any ample \mathcal{O}_X -module \mathcal{L}' , the invertible \mathcal{O}_Z -module \mathcal{M} such that $h_*(\mathcal{M}) = \tilde{N}(g_*(\mathcal{L}'))$ is ample.*

Corollary 19.6.38. *Under the hypotheses of ??, for any invertible \mathcal{O}_X -module \mathcal{L} such that $g^*(\mathcal{L})$ is ample, $h^*(\mathcal{L})$ is ample.*

Proposition 19.6.39. *Let Y be an affine scheme, X be a reduced Noetherian scheme, $f : X \rightarrow Y$ be a quasi-compact morphism, and $g : X' \rightarrow X$ be a finite and surjective morphism. Let W be an open subset of X such that, for any $x \in W$, either X is normal at x , or there exists an open neighborhood $T \subseteq W$ of x such that $(g_*(\mathcal{O}_{X'}))|_T$ is a locally free $(\mathcal{O}_X|_T)$ -module. Then there exists a reduced Y -scheme Z and a finite and birational Y -morphism $h : Z \rightarrow X$ such that the restriction of h to $h^{-1}(W)$ is an isomorphism $h^{-1}(W) \cong W$ and satisfies the following property: for any invertible \mathcal{O}_X -module such that $g^*(\mathcal{L})$ is ample relative to $f \circ g$, $h^*(\mathcal{L})$ is ample relative to $f \circ h$.*

Corollary 19.6.40. *If in ?? we have $W = X$, then for an invertible \mathcal{O}_X -module \mathcal{L} to be ample relative to f , it is necessary and sufficient that $g^*(\mathcal{L})$ is ample relative to $f \circ g$.*

Remark 19.6.4. We shall see in ?? that if Y is Noetherian, f is of finite type, and if the restriction of f to the reduced closed subscheme of X having $X - W$ as underlying space is proper, then the conclusion of ?? is still valid. But we will also give examples of algebraic schemes X over a field K (the structural morphism $X \rightarrow \text{Spec}(K)$ not being proper) whose normalize X' is quasi-affine, but which is not quasi-affine (so that \mathcal{O}_X is not ample, although $\mathcal{O}_{X'}$ is, cf. ??, and that the morphism $g : X' \rightarrow X$ is finite and surjective (c.f. ??)). We will also see that this circumstance cannot occur when we replace "quasi-affine" by "affine" (by Chevalley's theorem).

19.6.7 Chevalley's theorem

Lemma 19.6.41. *Let X, Y be integral Noetherian schemes, x (resp. y) be the generic point of X (resp. Y), and $f : X \rightarrow Y$ be a finite and surjective morphism. Let \mathcal{L} be an invertible \mathcal{O}_X -module such that there exists an affine neighborhood U of y and a section $s \in \Gamma(X, \mathcal{L})$ such that $x \in X_s \subseteq f^{-1}(U)$. Then there exist integers $m, n > 0$, a homomorphism $u : \mathcal{O}_Y^m \rightarrow f_*(\mathcal{L}^{\otimes n})$ and an open neighborhood V of y such that the restriction $u|_V$ is an isomorphism $\mathcal{O}_Y^m|_V \xrightarrow{\sim} f_*(\mathcal{L}^{\otimes n})|_V$.*

Theorem 19.6.42 (Chevalley). *Let X be an affine scheme, Y be a Noetherian scheme, and $f : X \rightarrow Y$ be a finite and surjective morphism. Then Y is an affine scheme.*

Proof. It is clear that $f_{\text{red}} : X_{\text{red}} \rightarrow Y_{\text{red}}$ is finite (??(vi)); as X_{red} is an affine scheme, and Y is affine if and only if Y_{red} is (??), we can assume that X, Y are reduced. For any closed subset Y' of Y , there is then a unique reduced subscheme structure on Y' whose inverse image $f^{-1}(Y')$, canonically isomorphic to $X \times_Y Y'$, is affine as a closed subscheme of X , and the restriction of f to $f^{-1}(Y')$, which is identified with $f \times_Y 1_{Y'}$, is a finite and surjective morphism (?? and ??(iv)). In view of the Noetherian induction principle (??), we are then (in view of ??) reduced to prove the theorem under the hypothesis that for any closed subset $Y' \neq Y$, any closed subscheme of Y with underlying space Y' is affine. With this hypothesis, we first note that, for any coherent \mathcal{O}_Y -module \mathcal{F} whose support (closed) Z is distinct from Y , we have $H^1(Y, \mathcal{F}) = 0$. In fact, there exists a closed subscheme structure on Z such that, if $j : Z \rightarrow Y$ is the canonical injection, we have $\mathcal{F} = j_*(j^*(\mathcal{F}))$ (?? and ??), and therefore $H^1(Y, \mathcal{F}) = H^1(Z, j^*(\mathcal{F})) = 0$ (??, since j is affine).

Suppose first that Y is not irreducible, and let Y' be an irreducible component of Y ; we endow Y' with the reduced subscheme structure, and let $j : Y' \rightarrow Y$ be the canonical injection. Let \mathcal{F} be a coherent \mathcal{O}_Y -module, and consider the canonical homomorphism

$$\rho : \mathcal{F} \rightarrow \mathcal{F}' = j_*(j^*(\mathcal{F}));$$

Since j is proper and Y', Y are Noetherian schemes, \mathcal{F}' is a coherent \mathcal{O}_Y -module by ?? (since we have $j_*(\mathcal{O}_{Y'}) = \mathcal{O}_Y/\mathcal{I}$, where \mathcal{I} is the quasi-coherent ideal of \mathcal{O}_Y defining the subscheme Y'), so $\mathcal{G} = \ker \rho$ and $\mathcal{K} = \text{im } \rho$ are coherent \mathcal{O}_Y -modules (??). On the other hand, by definition the fiber \mathcal{F}'_y of \mathcal{F}' at the generic point y of Y' is equal to $\mathcal{F}_y/\mathcal{I}_y\mathcal{F}_y$, and hence to \mathcal{F}_y (??), so y is not contained in the support of \mathcal{G} and we conclude that $H^1(Y, \mathcal{G}) = 0$. Since the support of \mathcal{F}' (and a fortiori that of \mathcal{K}) is contained in Y' , it is distinct from Y , and we also conclude that $H^1(Y, \mathcal{K}) = 0$. From the exact sequence $0 \rightarrow \mathcal{G} \rightarrow \mathcal{F} \rightarrow \mathcal{K} \rightarrow 0$ we then get $H^1(Y, \mathcal{F}) = 0$, so by Serre's criterion Y is then affine.

Suppose now that Y is irreducible, and therefore integral. We can also assume that X is integral: in fact, if we denote by X_i the reduced closed subschemes of X with underlying spaces the irreducible components of X and by $f_i : X_i \rightarrow Y$ the restriction of f to X_i , then one of f_i is dominant (Y is irreducible, so if its generic point is contained in the image of f_i , then f_i is dominant, and we note that f is surjective), and as there are finite morphisms (??), it is surjective (??); as X_i is an affine scheme, we see that we can replace X by X_i . In this case, we can apply ?? to $\mathcal{L} = \mathcal{O}_X$, since X is affine, to obtain a homomorphism $u : \mathcal{O}_Y^m \rightarrow f_*(\mathcal{L}) = \mathcal{B}$ and an open neighborhood V of y such that $u|_V$ is an isomorphism. In view of Serre's criterion, it suffices to prove that for any coherent \mathcal{O}_Y -module \mathcal{F} such that $\mathcal{F} \subseteq \mathcal{O}_Y$, we have $H^1(Y, \mathcal{F}) = 0$. We note that then \mathcal{F} is torsion free since Y is integral, and we only need to show that $H^1(Y, \mathcal{F}) = 0$ for any torsion free coherent \mathcal{O}_Y -module \mathcal{F} . Now the homomorphism u defines a homomorphism

$$v : \mathcal{G} = \mathcal{H}\text{om}_{\mathcal{O}_Y}(\mathcal{B}, \mathcal{F}) \rightarrow \mathcal{H}\text{om}_{\mathcal{O}_Y}(\mathcal{O}_Y^m, \mathcal{F}) = \mathcal{F}^m.$$

By hypotheses the support of $\mathcal{T} = \text{coker } u$ does not meet V , so is a torsion \mathcal{O}_Y -module (??). From the exact sequence $\mathcal{O}_Y^m \rightarrow \mathcal{B} \rightarrow \mathcal{T} \rightarrow 0$ induces, by the left exactness of $\mathcal{H}\text{om}_{\mathcal{O}_Y}$, an exact sequence

$$0 \longrightarrow \mathcal{H}\text{om}_{\mathcal{O}_Y}(\mathcal{T}, \mathcal{F}) \longrightarrow \mathcal{G} \xrightarrow{v} \mathcal{F}^m$$

But as \mathcal{F} is torsion free and \mathcal{T} is torsion, we have $\mathcal{H}\text{om}_{\mathcal{O}_Y}(\mathcal{T}, \mathcal{F}) = 0$, so v is injective. We then obtain an exact sequence

$$0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{F}^m \longrightarrow \text{coker } v \longrightarrow 0$$

where \mathcal{G} and $\text{coker } v$ are coherent \mathcal{O}_Y -modules (??). In view of the exact sequence on cohomology, it suffices to prove that $H^1(Y, \mathcal{F}) = H^1(Y, \text{coker } v) = 0$ since this implies $H^1(Y, \mathcal{F}^m) = (H^1(Y, \mathcal{F}))^m = 0$, so $H^1(Y, \mathcal{F}) = 0$. Now the restriction $v|_V$ is an isomorphism, so the support of $\text{coker } v$ is distinct from Y , and we have $H^1(Y, \text{coker } v) = 0$ by our hypothesis. On the other hand, \mathcal{G} is a coherent \mathcal{B} -module by ??; as X is affine over Y , there exists a quasi-coherent \mathcal{O}_X -module \mathcal{K} such that \mathcal{G} is isomorphic to $f_*(\mathcal{K})$ (??), and since $H^1(X, \mathcal{K}) = 0$ (X is affine), we then have $H^1(Y, \mathcal{G}) = 0$ by ??, which completes the proof. \square

Corollary 19.6.43. *Let X be a Noetherian scheme and $(X_i)_{1 \leq i \leq n}$ be a finite covering of X by closed subsets. Then for X to be affine, it is necessary and sufficient that for each i , there exists a closed subscheme of X that is affine and has underlying space X_i .*

Proof. Let X' be the sum of the X_i , then it is clear that X' is affine if each X_i is affine, and we have a surjective morphism $f : X' \rightarrow X$ whose restriction to X_i is the canonical injection. To apply ??, it remains to verifying that f is finite, and this follows from ??(i). \square

19.7 Valuative criterion

In this section we give the valuative criterion of spartion and properness of a morphism, which are criteria which involve a auxiliary scheme $\text{Spec}(A)$, where A is a valuation ring. With a convenient "Noetherian" hypothesis, these criterion can be refined to the case where A is a discrete valuation ring, and this will probably be the only case that we will apply later.

19.7.1 Remainders for valuation rings

Among the vast properties that characterize valuation rings, we shall use the following one: a ring A is called a valuation ring if it is an integral domain which is not a field, and if in the set of proper local rinng contained in the fraction field K of A , A is maximal under the dominant relation. Recall that a valuation ring is integrally closed. If A is a valuation ring, then $A_{\mathfrak{p}}$ is also a valuation ring for any nonzero prime ideal $\mathfrak{p} \neq 0$.

Let K be a field, A be a proper local subring of K ; then there exists a valuation ring of K dominating A (??). On the other hand, let B be a valuation ring, k be its residue field, and K be the fraction field, L be an extension of k . Then there exists a complete valuation ring C dominating B with residue field equals to L . In fat, L is an algebraic extension of a purely transcendental extension $L' = k(T_{\mu})_{\mu \in M}$; we can extend the valuation of K corresponding to B to a valuation of $K' = K(T_{\mu})_{\mu \in M}$ with residue field L' ; replace B by this complete valuation ring C , we can assume then that B is complete that L is an algebraic closure of k . If \bar{K} is an algebraic closure of K , we can then extend the defining valuation of B to \bar{K} , and the corresponding residue field is an algebraic closure of k , as can be seen by lifting the coefficients of a monic polynomial of $k[T]$ to \bar{K} . We are therefore finally reduced to the case where $L = k$ and it suffices then to take for C the completion of B to answer the question.

Let K be a field and A be a subring of K ; the integral closure A' of A in K is the intersection of valuation rings of the fraction field of A containing A (??). The preceding argument then have the following geometric form:

Proposition 19.7.1. *Let Y be a scheme, $p : X \rightarrow Y$ be a morphism, x be a point of X , $y = p(x)$, and $y' \neq y$ be a specialization of y . Then there exists a local scheme Y' , which is the specturm of a valuation ring, and a separated morphism $f : Y' \rightarrow Y$ such that, if a is the unique closed point of Y' and b is the generic point of Y' , we have $f(a) = y'$ and $f(b) = y$. We can moreover suppose that one of the following additional conditions are satisfied:*

- (i) Y' is the specturm of a complete valuation ring whose residue field is algebraically closed, and there exists a $\kappa(y)$ -homomorphism $\kappa(x) \rightarrow \kappa(b)$.
- (ii) There exists a $\kappa(y)$ -isomorphism $\kappa(x) \cong \kappa(b)$.

Proof. Let Y_1 be the reduced closed subscheme of Y with $\overline{\{y\}}$ as underlying space, and X_1 be the closed subscheme $p^{-1}(Y_1)$; as $y' \in \overline{\{y\}}$ by hypothesis and $\kappa(x)$ is the same for X and X_1 , by replacing Y with Y_1 and X with X_1 , we can suppose that Y is integral with generic point y ; $\mathcal{O}_{Y,y'}$ is then an integral local ring which is not a field, whose fraction field is $\mathcal{O}_{Y,y} = \kappa(y)$, and $\kappa(x)$ is an extension of $\kappa(y)$. To realize the conditions $f(a) = y'$ and $f(b) = y$ with the additional condition (i) (resp. (ii)), we choose $Y' = \text{Spec}(A')$, where A' is a valuation ring dominating $\mathcal{O}_{Y,y'}$ and which is complete and with residue field an algebraically closed extension of $\kappa(x)$.

(resp. a valuation ring dominating $\mathcal{O}_{Y,Y'}$ with fraction field $\kappa(x)$); the existence of such rings are proved by the above remarks. \square

Recall that a local ring (A, \mathfrak{m}) is of dimension if and only if any prime ideal of A distinct from \mathfrak{m} is minimal; if A is integral, this means \mathfrak{m} and (0) are the only prime ideals, and $\mathfrak{m} \neq (0)$; equivalent, $Y = \text{Spec}(A)$ is reduced to two points a, b : a is the closed point, $\mathfrak{p}_a = \mathfrak{m}$, and $\kappa(a) = k$ is the residue field A/\mathfrak{m} ; b is the generic point of Y , $\mathfrak{p}_b = (0)$, the set $\{b\}$ is the unique nontrivial open subset of Y , and $\kappa(b) = K$ is the fraction field of A . For an integral Noetherian local ring A of dimension 1, the following conditions are then equivalent:

- (i) A is normal;
- (ii) A is regular;
- (iii) A is a valuation ring.

Moreover, if these are true, A is then a discrete valuation ring.

Proposition 19.7.2. *Let A be a Noetherian local integral domain which is not a field, K be its fraction field, L be an extension of K of finite type. There then exists a discrete valuation ring of L dominating A .*

Proof. \square

Corollary 19.7.3. *Let A be an integral Noetherian ring, K be its fraction field, and L be an extension of K of finite type. Then the integral closure of A in L is the intersection of discrete valuation rings of L containing A .*

Proposition 19.7.4. *Let Y be a locally Noetherian scheme, $p : X \rightarrow Y$ be a morphism locally of finite type, x be a point of X , $y = p(x)$, $y' \neq y$ be a specialization of y . Then there exists a local scheme Y' , which is a spectrum of a discrete valuation ring, and a separated morphism $f : Y' \rightarrow Y$ and rational Y -map $g : Y' \dashrightarrow X$ such that, if a is the closed point of Y' and b is the generic point, we have $f(a) = y'$, $f(b) = y$, $g(b) = x$, and in the following commutative diagram*

$$\begin{array}{ccc} & & \kappa(x) \\ & \swarrow \gamma & \uparrow \pi \\ \kappa(b) & \xleftarrow{\varphi} & \kappa(y) \end{array}$$

(where π, φ, γ are the homomorphisms corresponding to p, f and g), γ is a bijection.

Proof. \square

19.7.2 Valuative criterion of separation

Proposition 19.7.5 (Valuative Criterion of Separation). *Let Y be a scheme (resp. a locally Noetherian scheme), $f : X \rightarrow Y$ be a morphism (resp. a morphism locally of finite type). The following conditions are equivalent:*

- (i) f is separated.

- (ii) f is quasi-separated and for any Y -scheme of the form $Y' = \text{Spec}(A)$, where A is a valuation ring (resp. a discrete valuation ring), two Y -morphisms of Y' to X which coincide at the generic point of Y' are equal.
- (iii) f is quasi-separated and for any Y -scheme of the form $Y' = \text{Spec}(A)$, where A is a valuation ring (resp. a discrete valuation ring), two Y' -sections of $X' = X_{(Y')}$ which coincide at the generic point of Y' are equal.

Proof. The equivalence of (ii) and (iii) follows from the correspondence of Y -morphisms $Y' \rightarrow X$ and Y' -sections of X' . If X is separated over Y , condition (ii) follows from the proof of ??, since Y' is integral. It then remains to prove that condition (ii) implies that the diagonal morphism $\Delta : X \rightarrow X \times_Y X$ is closed, and for this we can use ?. Now, let z be a point of the diagonal $\Delta(X)$, $z' \neq z$ be a specialization of z in $X \times_Y X$. Then there exists by ?? a valuation ring A and a morphism $g : Y' \rightarrow X \times_Y X$ such that $g(a) = z'$, $g(b) = z$ (with the notations of ??, a is the closed point of Y' and b is the generic point of Y'); this morphism makes Y' an $(X \times_Y X)$ -scheme, and a fortiori a Y -scheme. If we compose g with the two projections of $X \times_Y X$, we obtain two Y -morphisms $g_1, g_2 : Y' \rightarrow X$, which by hypotheses send the point b to the same point in X ; in view of (ii), these two morphisms coincide with a morphism $h : Y' \rightarrow X$, which signifies that g factors into $g = \Delta \circ h$, and therefore $z' \in \Delta(X)$. If we suppose that Y is locally Noetherian and f is of finite type, $X \times_Y X$ is locally Noetherian by ??, and we can therefore replace ?? by ??.

The condition (ii) of ?? signifies that if $Y' = \text{Spec}(A)$ and $X' = \text{Spec}(K)$ where K is the fraction field of A , the canonical map

$$\text{Hom}_Y(Y', X) \rightarrow \text{Hom}_Y(X', X)$$

is injective. Equivalently, this means in the following diagram

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow f \\ Y' & \longrightarrow & Y \end{array}$$

the dashed morphism $Y' \rightarrow X$, if exists, is necessarily unique.

Remark 19.7.1. In the criterion (ii) of ??, we can restrict ourselves to valuation rings A which is complete whose residue field is algebraically closed; this follows from the additional condition (i) of ??.

19.7.3 Valuative criterion of properness

Proposition 19.7.6. *Let A be a valuation ring, $Y = \text{Spec}(A)$, b be the generic point of Y . Let X be an integral and separated scheme and $f : X \rightarrow Y$ be a closed morphism such that $f^{-1}(b)$ is reduced to a point x and the corresponding homomorphism $\kappa(b) \rightarrow \kappa(x)$ is bijective. Then f is an isomorphism.*

Proof. As f is closed and dominant, we have $f(X) = Y$; it then suffices to prove that for any $y' \neq b$ in Y , there exists a unique point x' such that $f(x') = y'$ and the corresponding

homomorphism $\mathcal{O}_{Y,y'} \rightarrow \mathcal{O}_{X,x'}$ is bijective, because f is then a homeomorphism. Now, if $f(x') = y'$, $\mathcal{O}_{X,x'}$ is a local ring contained in $K = \kappa(x) = \kappa(y)$ and dominates $\mathcal{O}_{Y,y'}$; the later is the local ring $A_{y'}$, which a valuation ring for the fraction field K of A . But $\mathcal{O}_{X,x'} \neq K$ since x' is not the generic point of X , and we then conclude that $\mathcal{O}_{X,x'} = \mathcal{O}_{Y,y'}$ by maximality. As X is an integral scheme, the relation $\mathcal{O}_{X,x'} = \mathcal{O}_{X,x''}$ implies $x' = x''$ by ??, which proves our claim. \square

Let A be a valuation ring, $Y = \text{Spec}(A)$, b the generic point of Y , so that $\mathcal{O}_{Y,b} = \kappa(b)$ is equal to the fraction field K of A . Let $f : X \rightarrow Y$ be a morphism. We have seen that the rational Y -sections of X correspond to the germs of Y -sections (defined over a neighborhood of b) at b , whence a canonical map

$$\Gamma_{\text{rat}}(X/Y) \rightarrow \Gamma(f^{-1}(b)/\text{Spec}(K)) \quad (19.7.1)$$

where the elements of $\Gamma(f^{-1}(b)/\text{Spec}(K))$ are identified with the rational points of $f^{-1}(b) = X \otimes_A K$ over K . If f is separated, it then follows from ?? that the map of ?? is injective, since Y is integral.

Composing ?? with the canonical map $\Gamma(X/Y) \rightarrow \Gamma_{\text{rat}}(X/Y)$, we then obtain a canonical map

$$\Gamma(X/Y) \rightarrow \Gamma(f^{-1}(b)/\text{Spec}(K)). \quad (19.7.2)$$

If f is separated, this map is injective by ??.

Proposition 19.7.7. *Let A be a valuation ring with fraction field K , $Y = \text{Spec}(A)$, b be the generic point of Y , and $f : X \rightarrow Y$ be a separated and closed morphism. Then the canonical map ?? is bijective.*

Proof. Let x be a rational point of $f^{-1}(b)$ over K . As f is separated, so is the morphism $f^{-1}(b) \rightarrow \text{Spec}(K)$ corresponding to f ??(iv)), since any section of $f^{-1}(b)$ a closed immersion by ??, $\{x\}$ is closed in $f^{-1}(b)$. Consider the reduced closed subscheme X' of X with underlying space $\overline{\{x\}}$ of $\{x\}$ in X . It is clear that the restriction of f to X' satisfies the conditions of ?? (note that since x is rational over K , we have $\kappa(x) = K$), hence an isomorphism from X' to Y , whose inverse isomorphism is the Y -section of X we want. \square

Recall that if F is a subset of the scheme Y , the codimension of F in Y is equal to the infimum of $\dim(\mathcal{O}_{Y,z})$ where $z \in F$ (this can be easily verified after reducing to affine case), and we denote this number by $\text{codim}_Y(F)$.

Corollary 19.7.8. *Let Y be a reduced locally Noetherian scheme such that the subset N of $y \in Y$ where Y is not regular has codimension ≥ 2 . Let $f : X \rightarrow Y$ be a separated and closed morphism of finite type and g be a rational Y -section of X . If Y' is the set of points of Y where g is not defined, then $\text{codim}_Y(Y') \geq 2$.*

Proof. It suffices to prove that g is defined at any point $y \in Y$ such that $\dim(\mathcal{O}_{Y,y}) \leq 1$. If $\dim(\mathcal{O}_{Y,y}) = 0$, then y is the generic point of an irreducible component of Y . For any open dense subset V of Y , by restricting to an affine neighborhood of y and apply ??, we conclude that V contains y . In particular, y belongs to the defining domain of g . Suppose now that $\dim(\mathcal{O}_{Y,y}) = 1$; then $\mathcal{O}_{Y,y}$ is a regular local ring, hence a discrete valuation ring. Let $Z = \text{Spec}(\mathcal{O}_{Y,y})$; as $U = Y - Y'$ is open and dense, by ?? and our preceding arguments, $U \cap Z$ is

nonempty (contains the generic of an irreducible component of Y containing y), so we can consider the rational map $g' : Z \dashrightarrow X$ induced by g . It then suffices to prove that g' is a morphism (??). Now, g' can be considered as a rational Z -section of the Z -scheme $f^{-1}(Z) = X \times_Y Z$; it is clear that the morphism $f^{-1}(Z) \rightarrow Z$ corresponding to f is closed, and is separated by ??(i). We then conclude from ?? that g' is everywhere defined, and as Z is reduced and X is separated over Y , g' is a morphism (??). \square

Corollary 19.7.9. *Let S be a locally Noetherian scheme, X, Y be S -schemes, and assume that X is proper over X . Suppose that Y is reduced and the subset N of $y \in Y$ where Y is not regular has codimension ≥ 2 . Let $f : Y \dashrightarrow X$ be a rational map and Y' be the set of points where f is not defined, then $\text{codim}_Y(Y') \geq 2$.*

Proof. The rational S -maps $Y \dashrightarrow X$ correspond to rational Y -sections of $X \times_S Y$; as the structural morphism $X \times_S Y \rightarrow Y$ is closed by ??, we can apply ??, whence the corollary. \square

Remark 19.7.2. The hypothesis on Y in ?? and ?? are satisfied in particular if Y is normal (by Serre's criterion for normality).

Theorem 19.7.10 (Valuative Criterion of Properness). *Let Y be a (resp. locally Noetherian) scheme and $f : X \rightarrow Y$ be a quasi-compact and separated morphism (resp. a quasi-compact morphism of finite type). The following conditions are equivalent:*

- (i) f is universally closed (resp. proper)
- (ii) For any Y -scheme $Y' = \text{Spec}(A)$ where A is a valuation ring (resp. a discrete valuation ring) with fraction field K , the canonical map (where $X' = \text{Spec}(K)$)

$$\text{Hom}_Y(Y', X) \rightarrow \text{Hom}_Y(\text{Spec}(K), X)$$

corresponding to the canonical injection $A \rightarrow K$, is surjective (resp. bijective).

- (iii) For any Y -scheme $Y' = \text{Spec}(A)$ where A is a valuation ring (resp. a discrete valuation ring) with fraction field K , the canonical map of ?? relative to the Y' -scheme $X' = X_{(Y')}$ is surjective (resp. bijective).

Proof. The equivalence of (ii) and (iii) follows from the correspondence of Y -morphisms $Y' \rightarrow X$ and Y' -morphisms $Y' \rightarrow X'$. If f is universally closed then $f_{(Y')}$ is closed and separated, and it then suffices to apply ?. It remains to prove that (ii) implies (i). Consider first the case where Y is arbitrary, f is separated and quasi-compact. If the condition of (ii) is satisfied for f , it is also true for $f_{(Y'')} : X_{(Y'')} \rightarrow Y''$, where Y'' is an arbitrary Y -scheme, in view of the equivalence of (ii) and (iii), and the fact that $X_{(Y'')} \times_{Y''} Y' = X \times_Y Y'$ for any morphism $Y' \rightarrow Y''$; as $f_{(Y'')}$ is also quasi-compact and separated, we then conclude that we only need to prove (ii) implies f is closed, and for this we shall use ?. Let $x \in X$, $y' \neq y$ be a specialization of $y = f(x)$; in view of ??, there is a scheme $Y' = \text{Spec}(A)$ where A is a valuation ring, and a separated morphism $g : Y' \rightarrow Y$ such that, if a is the closed point and b is the generic point of Y , we have $g(a) = y'$, $g(b) = y$, and there exists a $\kappa(y)$ -homomorphism $\kappa(x) \rightarrow \kappa(b)$. This homomorphism corresponds to a canonical Y -morphism $\text{Spec}(\kappa(b)) \rightarrow X$ (?), and it then

follows from condition (ii) that there exists a Y -morphism $h : Y' \rightarrow X$ which corresponds the previous morphism such that the following diagram commutes

$$\begin{array}{ccc} \mathrm{Spec}(\kappa(b)) & \longrightarrow & X \\ \downarrow & & \downarrow f \\ \mathrm{Spec}(A) & \xrightarrow{g} & Y \end{array}$$

We then have $h(b) = x$, and if we put $h(a) = x'$, x' is then a specialization of x , and we have $f(x') = f(h(a)) = g(a) = y'$.

If now Y is locally Noetherian and f is a quasi-compact morphism of finite type, then condition (ii) implies that f is separated (??), so the diagonal morphism $\Delta : X \rightarrow X \times_Y X$ is quasi-compact. Moreover, to verify that f is proper, it suffices to show that $f_{(Y'')} : X_{(Y'')} \rightarrow Y''$ is closed for any Y -scheme Y'' of finite type (??). As then Y'' is locally Noetherian, we can resume the reasoning given in the first case by taking for Y' the spectrum of discrete valuation ring, and applying ?? instead of ??.

Remark 19.7.3. We deduce from the criterion (iii) of ?? a new proof of the fact that a projective morphism $X \rightarrow Y$ is closed, which is closer to classical methods. We can in fact assume that Y is affine, and X is therefore a closed subscheme of a projective bundle \mathbb{P}_Y^n (??). To show that $X \rightarrow Y$ is closed, it suffices to verify that the structural morphism $\mathbb{P}_Y^n \rightarrow Y$ is closed. The criterion (iii) of ??, together with (??), show that we are reduced to proving the following fact: if Y is the spectrum of a valuation ring A with fraction field K , every point of \mathbb{P}_Y^n with values in K comes (by restriction to the generic point of Y) from a point of \mathbb{P}_Y^n with values in A . Now, any invertible \mathcal{O}_Y -module is trivial, so it follows from ?? that a point of \mathbb{P}_Y^n with values in K is identified with a class of elements $(\xi c_0, \xi c_1, \dots, \xi c_n)$ of K^{n+1} , where $\xi \neq 0$ and the c_i are elements of K which generate the unit ideal of K . By multiplying the c_i with an element of A , we can suppose that the c_i belong to A , and generate the unit ideal of A . But then (??) the system (c_0, \dots, c_n) defines a point of \mathbb{P}_Y^n with values in A , whence our assertion.

Remark 19.7.4. The criteria ?? and ?? are especially convenient when we consider a Y -scheme X as a functor

$$X(Y') = \mathrm{Hom}_Y(Y', X)$$

where Y' is a Y -scheme. These criteria will allow us, for example, to prove that under certain conditions the "Picard schemas" are proper.

Corollary 19.7.11. *Let Y be a separated integral scheme (resp. a separated integral locally Noetherian scheme) and $f : X \rightarrow Y$ be a dominant morphism.*

- (a) *If f is quasi-compact and universally closed, any valuation ring with fraction field the rational function field $K(X)$ and which dominates a local ring of Y , also dominates a local ring of X .*
- (b) *Conversely, suppose that f is of finite type, and the property of (a) is satisfied for any valuation ring (resp. any discrete valuation ring) with fraction field $K(X)$. Then f is proper.*

Proof. Assume the hypotheses of (a) and let $K = K(Y)$, $L = K(X)$, y be a point of Y , A be a valuation ring with L the fraction field and dominate $\mathcal{O}_{Y,y}$. The injection $\mathcal{O}_{Y,y} \rightarrow A$ is local, so it

defines a morphism

$$h : Y' = \text{Spec}(A) \rightarrow \text{Spec}(\mathcal{O}_{Y,y}) \rightarrow Y$$

(??) such that $h(a) = y$, where a is the closed point of Y' . Moreover, since $K \subseteq L$, the morphism $\text{Spec}(A) \rightarrow \text{Spec}(\mathcal{O}_{Y,y})$ is dominant, so if η is the generic point of Y (which is also that of $\text{Spec}(\mathcal{O}_{Y,y})$), we have $h(b) = \eta$, where b is the generic point of Y' . If ξ is the generic point of X , we have $\kappa(\xi) = \kappa(b) = L$ by hypothesis, so there is a Y -morphism $g : \text{Spec}(L) \rightarrow X$ such that $g(b) = \xi$. In view of ??, we obtain a Y -morphism $g' : Y' \rightarrow X$ such that $g(b) = \xi$. If we set $x = g'(a)$, then A dominates $\mathcal{O}_{X,x}$.

We now prove (b); since the question is local over Y , we can assume that Y is affine (resp. affine and Noetherian). As f is of finite type, we can apply Chow's lemma, so there exists a projective morphism $p : P \rightarrow Y$, an immersion $j : X' \rightarrow P$, and a projective and surjective birational morphism $g : X' \rightarrow X$ (where X' is integral) such that the diagram

$$\begin{array}{ccc} X' & \xrightarrow{j} & P \\ g \downarrow & & \downarrow \pi \\ X & \xrightarrow{f} & Y \end{array}$$

is commutative. It suffices to prove that j is a closed immersion, because then $f \circ g = p \circ j$ is projective, hence proper, and as g is surjective we conclude that f is proper by ???. Let Z be a reduced closed subscheme of P with underlying space $\overline{j(X')}$; as X' is integral, j factors into

$$j : X' \xrightarrow{h} Z \xrightarrow{i} P$$

where $i : Z \rightarrow P$ is the canonical injection and $h : X' \rightarrow Z$ is a dominant open immersion. Since Z is integral and projective over Y by ??, we are then reduced to the case where P is integral, j is dominant and birational, and prove that j is surjective. Now let $z \in P$; $\mathcal{O}_{Z,z}$ is an integral local ring (resp. Noetherian integral) whose fraction field is

$$L = K(P) = K(X') = K(X).$$

We can assume that z is not the generic point of P (since the later is contained in $j(Z)$ as j is dominant), so $\mathcal{O}_{Z,z} \neq L$ and by ?? and ??, there exists a valuation ring (resp. a discrete valuation ring) A with fraction field L that dominates $\mathcal{O}_{Z,z}$. A fortiori A dominates the local ring $\mathcal{O}_{Y,y}$ where $y = p(z)$, and by hypotheses there is then a point $x \in X$ such that A dominates $\mathcal{O}_{X,x}$. As the morphism g is proper, it satisfies the conditions of (a), so our previous arguments then prove that A also dominates $\mathcal{O}_{X,x'}$, for some $x' \in X'$. Then the local rings $\mathcal{O}_{Z,z}$ and $\mathcal{O}_{Z,j(x')} = \mathcal{O}_{X,x'}$ are related, and by ??, as P is separated, we have $z = j(x')$, which completes the proof. \square

Corollary 19.7.12. *Let A be an integral domain, $Y = \text{Spec}(A)$, and $f : X \rightarrow Y$ be a dominant morphism of integral schemes which is quasi-compact and universally closed. Then $\Gamma(X, \mathcal{O}_X)$ is canonically isomorphic to a subring of the integral closure of A in $K(X)$.*

Proof. Recall that by ??, $B = \Gamma(X, \mathcal{O}_X)$ is identified with the intersection of $\mathcal{O}_{X,x}$ for $x \in X$. If R is a valuation ring of $K(X)$ containing A , then it dominates the local ring $A_{\mathfrak{P}}$ where $\mathfrak{P} = \mathfrak{m}_R \cap B$, and therefore by ?? dominates a local ring of X . Then B is contained in R , and the conclusion

follows from ??.

□

Remark 19.7.5. Under the hypothesis of ??, if we suppose that $K(X)$ is a finite extension of $K(Y)$, then we can in many cases conclude that $\Gamma(X, \mathcal{O}_X)$ is a finitely generated module over the ring $B = \Gamma(Y, \mathcal{O}_X)$. This is the case for example if B is a Japanese ring. In particular, if $X = \text{Spec}(A)$ and $Y = \text{Spec}(k)$ where k is a algebraically closed field, then the the corresponding homomorphism $k \rightarrow A$ is injective by ?? and since the integral closure of k in $K(X)$ is equal to k , we conclude that $\Gamma(X, \mathcal{O}_X) = k$.

19.7.4 Algebraic curves

Let k be a field. In this subsection, all schemes are considered to be separated k -schemes of finite type, and any morphism are k -morphism.

Proposition 19.7.13. *Let X be a scheme of finite type over k ; let x_i ($1 \leq i \leq n$) be the generic points of the irreducible components X_i of X , and $K_i = \kappa(x_i)$. Then the following conditions are equivalent:*

- (i) *For each i , the transcendence degree of K_i over k is equal to 1.*
- (ii) *For any closed point x of X , the local ring $\mathcal{O}_{X,x}$ is of dimension 1.*
- (iii) *The closed irreducible subsets of X distinct from the X_i are closed points of X .*

Proof. As X is quasi-compact, any irreducible closed subset F of X contains a closed point (??). Let x be a closed point of X ; in view of ??, there is a correspondence between prime ideals of $\mathcal{O}_{X,x}$ and the irreducible closed subsets of X containing x . The equivalence of (ii) and (iii) then follows. On the other hand, if \mathfrak{p}_α ($1 \leq \alpha \leq r$) is the minimal prime ideals of the local Noetherian ring $\mathcal{O}_{X,x}$, the local ring $\mathcal{O}_{X,x}/\mathfrak{p}_\alpha$ are integral, whose fraction fields are the K_i such that $x \in X_i$. Moreover, the dimension of a k -algebra of finite type is equal to its transcendental degree over k . Finally, the dimension of $\mathcal{O}_{X,x}$ is the supremum of the $\mathcal{O}_{X,x}/\mathfrak{p}_\alpha$, and a k -algebra of finite type is catenary, so (i) and (ii) are equivalent. □

We note that under the conditions of ??, by ?? that set X is either empty or finite. We define an algebraic curve over k to be a nonempty scheme X over k satisfying the conditions of ???. Equivalently, we will see that this condition is equivalent to that the irreducible components of X has dimension 1. In particular, we note that if X is an algebraic curve over k , the reduced closed subschemes X_i of X with underlying spaces the irreducible components of X are also algebraic curves over k .

Corollary 19.7.14. *Let X be an irreducible algebraic curve. Then the only non-closed point of X is its generic point, the proper closed subsets of X are the finite subsets of X , which are also the non dense subsets of X .*

Proof. If a point $x \in X$ is not closed, then its closure is an irreducible closed subset of X , hence equal to X by ??, so x is the generic point of X . A proper closed subset F of X can not contain the generic point of X , so its points are all closed, hence T1, and by ?? we conclude that F is finite and discrete. The closure of an infinite subset of X is therefore necessarily equal to X , which proves the last assertion. □

If X is an arbitrary algebraic curve, then by applying ??, we conclude that the only non-closed points of X are the generic points of the irreducible components of X .

Corollary 19.7.15. *Let X and Y be irreducible algebraic curves over k and $f : X \rightarrow Y$ be a k -morphism. Then for f to be dominant, it is necessary and sufficient that $f^{-1}(y)$ is finite for any $y \in Y$.*

Proof. If f is not dominant, $f(X)$ is necessarily a finite subset of Y by ??, so it is not possible that $f^{-1}(y)$ is finite for any $y \in Y$ (since X is an infinite set). Conversely, if f is dominant, for any $y \in Y$ which is not the generic point η of Y , $f^{-1}(y)$ is closed in X since $\{y\}$ is closed in Y ??; on the other hand, by hypotheses, $f^{-1}(y)$ does not contain the generic point of X , so is finite by ?. Finally, to see that $f^{-1}(\eta)$ is finite, we note that the morphism f is of finite type by ??, so the fiber $f^{-1}(\eta)$ is an irreducible scheme of finite type over $\kappa(\eta)$ with generic point ξ ?. As $\kappa(\xi)$ and $\kappa(\eta)$ are extensions of k of finite type with transcendental degree 1, it follows that $\kappa(\xi)$ is a finite extension of $\kappa(\eta)$, so ξ is closed in $f^{-1}(\eta)$ by ??, and $f^{-1}(\eta)$ is therefore reduced to a point ξ . \square

Remark 19.7.6. We will see later that a proper morphism $f : X \rightarrow Y$ of Noetherian schemes, such that $f^{-1}(y)$ is finite for any $y \in Y$, is necessarily finite. It then follows from ?? that such a dominant proper morphism of irreducible algebraic curves is finite.

Corollary 19.7.16. *Let X be an algebraic curve over k . For X to be regular, it is necessary and sufficient that X is normal, or the local ring of its closed points are discrete valuation rings.*

Proof. This comes from conditions (ii) of ??.

Corollary 19.7.17. *Let X be a reduced algebraic curve, \mathcal{A} be a reduced coherent \mathcal{K}_X -algebra. Then the integral closure X' of X relative to \mathcal{A} is a normal algebraic curve, and the canonical morphism $X' \rightarrow X$ is finite.*

Proof. The fact that $X' \rightarrow X$ is finite follows from ??, and X' is then a normal algebraic scheme over k . Moreover, we note that if X is irreducible with generic point ξ and its integral closure X' has generic point ξ' , then $\kappa(\xi') = \kappa(\xi)$ by ??, so X' is also an algebraic curve over k . \square

Corollary 19.7.18. *For a reduced algebraic curve X to be proper over k (which is called **complete**), it is necessary and sufficient that the normalization X' of X is proper over k .*

Proof. The canonical morphism $f : X' \rightarrow X$ is finite by ??, hence proper ?? and surjective ???. If $g : X \rightarrow \text{Spec}(k)$ is the structural morphism, g and $g \circ f$ are then simultaneously proper, in view of ?? and ??.

Proposition 19.7.19. *Let X be a normal algebraic curve over k and Y be a proper algebraic scheme over k . Then any rational k -map $f : X \dashrightarrow Y$ is everywhere defined, hence a morphism.*

Proof. It follows from ?? that the set of points $x \in X$ where this rational map is not defined, the dimension of $\mathcal{O}_{X,x}$ is ≥ 2 , hence is empty. The assertion then follows from ??.

Corollary 19.7.20. *A normal algebraic curve over k is quasi-projective over k .*

Proof. As X is the sum of finitely many integral and normal algebraic curves (??), we can assume that X is integral (??). As X is quasi-compact, it can be covered by finitely many affine opens U_i ($1 \leq i \leq n$), and as each of this is of finite type over k , there exists an integer n_i and a k -immersion $f_i : U_i \rightarrow \mathbb{P}_k^{n_i}$ (?? and ??(i)). As U_i is dense in X (recall that X is integral by our assumption), it follows from ?? that f_i extends to a k -morphism $g_i : X \rightarrow \mathbb{P}_k^{n_i}$, and we obtain a k -morphism $g = (g_1, \dots, g_n)_k$ from X into the product P of the $\mathbb{P}_k^{n_i}$ over k . Moreover, for each index i , as the restriction of g_i to U_i is an immersion, so is the restriction of g to U_i (??). As the U_i cover X and g is separated by ??(v), g is an immersion from X into P by ???. The Segre morphism then provides from g an immersion of X into a projective bundle \mathbb{P}_k^n , so X is quasi-projective. \square

Corollary 19.7.21. *A normal algebraic curve X is isomorphic to a dense open subscheme of a normal and complete algebraic curve \widehat{X} , determined up to isomorphisms.*

Proof. If X_1, X_2 are two normal and complete algebraic curves containing X as open dense subscheme, it follows from ?? there is an isomorphism of X_1 and X_2 , whence the uniqueness of \widehat{X} . To prove the existence of \widehat{X} , it suffices to remark that we can consider X as a subscheme of a projective bundle \mathbb{P}_k^n (??). Let \bar{X} be the scheme-theoretic closure of X in \mathbb{P}_k^n (??); as X is an open and dense subscheme of \bar{X} , the generic points x_i of the irreducible components of X are those of \bar{X} , and the residue fields $\kappa(x_i)$ are the same for both schemes, so \bar{X} is an algebraic curve over k , which is reduced (??) and projective over k (??), hence complete by ???. We then take \widehat{X} to be the noramlization of \bar{X} , which is complete by ???. If $h : \widehat{X} \rightarrow \bar{X}$ is the canonical morphism, the restriction of h to $h^{-1}(X)$ is an isomorphism since X is normal (??), and as $h^{-1}(X)$ contains the generic point of the irreducible components of \widehat{X} (??), it is therefore dense in \widehat{X} , which proves the assertion. \square

Remark 19.7.7. We will later see that the conclusion of ?? is still valid without assuming the algebraic curve to be normal (or even reduced); we will also see that for an algebraic curve (reduced or not) to be affine, it is necessary and sufficient that its irreducible (reduced) components are not complete.

Corollary 19.7.22. *Let X be an irreducible normal algebraic curve with $L = K(X)$, Y be a complete and integral algebraic curve with $K = K(Y)$. Then there exists a canonical correspondence between dominant k -morphisms $X \rightarrow Y$ and k -monomorphisms $K \rightarrow L$.*

Proof. By ??, the rational k -maps $X \dashrightarrow Y$ are identified with k -morphisms $f : X \rightarrow Y$. The morphism f is dominante if and only if $f(x) = y$, where x and y are the generic points of X and Y , respectively. The corollary then follows from ???. \square

Example 19.7.23. We can precise the result of ?? if Y is the projective line $\mathbb{P}_k^1 = \text{Proj}(k[T_0, T_1])$, where T_0 and T_1 are indeterminates. This is an integral and separated k -scheme, and the induced subscheme $D_+(T_0)$ of Y is isomorphic to $\text{Spec}(k[T])$, so the generic point of Y is the ideal (0) of $k[T]$ and its rational function field is $k(T)$, which shows that Y is a complete algebraic cuver over k . Moreover, the only graded ideal of $S = k[T_0, T_1]$ containing T_0 and distinct from S_+ is the principal ideal (T_0) , so the complement of $D_+(T_0)$ in Y is reduced to a closed point, called the "infinite point" and denoted by ∞ .

Corollary 19.7.24. *Let X be an irreducible normal algebraic curve with $K = K(X)$. Then there exists a canonical correspondence between K and the set of morphisms $u : X \rightarrow \mathbb{P}_k^1$ which is distinct from the constant morphism with value ∞ . For u to be dominant, it is necessary and sufficient that the corresponding element in K is transcendental over k .*

Proof. By ?? and ??, the rational maps $X \dashrightarrow \mathbb{P}_k^1$ (hence morphisms $X \rightarrow \mathbb{P}_k^1$, in view of ??) correspond to points of \mathbb{P}_k^1 with values in K . For any element $\xi \in K$, we have a induced homomorphism $k[T_0] \rightarrow K$ which maps T_0 to ξ , and therefore a morphism $\text{Spec}(K) \rightarrow D_+(T_0)$. By composing with the canonical injection $D_+(T_0) \rightarrow \mathbb{P}_k^1$, we obtain a point of \mathbb{P}_k^1 with values in K which is not located at ∞ , which is completely determined by ξ in view of ???. On the other hand, by ??, any constant morphism $u : \text{Spec}(K) \rightarrow \mathbb{P}_k^1$ with value y factors into

$$u : \text{Spec}(K) \rightarrow \text{Spec}(\kappa(y)) \rightarrow \mathbb{P}_k^1.$$

Since $y \in D_+(T_0)$, the morphism $\text{Spec}(\kappa(y)) \rightarrow \mathbb{P}_k^1$ factors through the canonical injection $D_+(T_0) \rightarrow \mathbb{P}_k^1$, and u therefore is obtained by morphism $\text{Spec}(K) \rightarrow D_+(T_0)$, which corresponds to an element $\xi \in K$; this proves the first part of the corollary. For the morphism u to be dominant, it is necessary that the morphism $\text{Spec}(K) \rightarrow \text{Spec}(k[T_0]) \cong D_+(T_0)$ is dominant, which means the homomorphism $k[T] \rightarrow K$ is injective (??), and this is true if and only if ξ is transcendental. \square

Remark 19.7.8. With the notations of ??, we now determine the image of the morphism $\text{Spec}(K) \rightarrow \mathbb{P}_k^1$ induced by an algebraic element $\xi \in K$ over k . By definition, if $y \in \mathbb{P}_k^1$ is this image, the morphism factors into

$$\text{Spec}(K) \rightarrow \text{Spec}(\kappa(y)) \rightarrow \text{Spec}(k[T]) \rightarrow \mathbb{P}_k^1$$

where we write $\text{Spec}(k[T])$ for $D_+(T_0)$. We first note that any prime ideal in $k[T]$ is maximal, so $\kappa(y) = k[T]/\mathfrak{p}_y$, and the morphism $\text{Spec}(K) \rightarrow \text{Spec}(\kappa(y))$ is identified with the canonical injection $k[T]/\mathfrak{p}_y \rightarrow K$. By definition, this homomorphism is induced by the homomorphism $k[T] \rightarrow K, T \mapsto \xi$, so if $f(T)$ is the irreducible polynomial of ξ over k , we have $\mathfrak{p}_y = (f)$, and y is therefore the point of $\text{Spec}(k[T])$ corresponding to (f) . In particular, if $\xi \in k$, then $\mathfrak{p}_y = (T - \xi)$, and if k is an algebraically closed field, we conclude that any element $\xi \in k$ corresponds to a morphism $\text{Spec}(K) \rightarrow \mathbb{P}_k^1$ with image ξ (identified with its corresponding maximal ideal $(T - \xi)$), and any element $\xi \in K - k$ corresponds to a dominant morphism $\text{Spec}(K) \rightarrow \mathbb{P}_k^1$.

Corollary 19.7.25. *Let X and Y be normal, complete and irreducible algebraic curves, with $K = K(Y)$, $L = K(X)$. Then there exist a bijective correspondence between the set of k -isomorphisms $X \xrightarrow{\sim} Y$ and the set of k -isomorphisms $K \xrightarrow{\sim} L$.*

?? shows that a normal, complete and irreducible algebraic curves over k is determined by its rational function field K up to an isomorphism. By definition, K is a field of finite type over k with transcendental degree 1 (which is called a algebraic function field of one variable).

Proposition 19.7.26. *For any extension K of k of finite type and transcendental degree 1, there exists a normal, complete and irreducible algebraic curve X such that $K(X) = K$. The set of local rings of X is identified with the set formed by K and the valuation rings containing k with fraction field K .*

Proof. In fact, K is a finite extension of a purely transcendental extension $k(T)$ of k , which is identified with the rational function field of $Y = \mathbb{P}_k^1$. Let X be the integral closure of Y relative to K ; X is then a normal algebraic curve with field K (??), and it is complete since the morphism $X \rightarrow Y$ is finite (??). For $x \in X$, the local ring $\mathcal{O}_{X,x}$ is equal to K if x is the generic point of X , and otherwise it is a discrete valuation ring of K . Conversely, let A be a discrete valuation ring with fraction field K ; as the morphism $X \rightarrow \text{Spec}(k)$ is proper and A dominates k , it also dominates a local ring $\mathcal{O}_{X,x}$ of X by ??, and therefore equals to $\mathcal{O}_{X,x}$. \square

Remark 19.7.9. It follows from ?? and ?? that giving a normal, complete and irreducible algebraic curve over k is essentially equivalent to giving an extension K of k of finite type and transcendental degree 1. We note that if k' is an extension of k , $X \otimes_k k'$ is also a complete algebraic curve over k' (??(iii)), but in general it is neither reduced nor irreducible. However, this will be the case if K is a separable extension of k and if k is algebraically closed in K (which is expressed, in a classical terminology, that K is a "regular extension" of k). But even in this case, it may happen that $X \otimes_k k'$ is not normal.

19.8 Blow up of schemes, projective cones and closures

19.8.1 Blow up of schemes

Let Y be a scheme and $(\mathcal{I}_n)_{n \geq 0}$ be a decreasing sequence of quasi-coherent ideal of \mathcal{O}_Y . Suppose that the following conditions are satisfied:

$$\mathcal{I}_0 = \mathcal{O}_Y, \quad \mathcal{I}_n \mathcal{I}_n \subseteq \mathcal{I}_{m+n}$$

where m, n are integers. If this is true, we say that the sequence $(\mathcal{I}_n)_{n \geq 0}$ is **filtered**, or that $(\mathcal{I}_n)_{n \geq 0}$ is a **filtered sequence of quasi-coherent ideals of \mathcal{O}_Y** . We note that this hypothesis implies that $\mathcal{I}_1^n \subseteq \mathcal{I}_n$. Put

$$\mathcal{S} = \bigoplus_{n \geq 0} \mathcal{I}_n.$$

It follows the assumption that \mathcal{S} is a quasi-coherent \mathcal{O}_Y -algebra, hence defines a Y -scheme $X = \text{Proj}(\mathcal{S})$. If \mathcal{J} is an invertible ideal of \mathcal{O}_Y , $\mathcal{I}_n \otimes_{\mathcal{O}_Y} \mathcal{J}^{\otimes n}$ is canonically identified with $\mathcal{I}_n \mathcal{J}^n$, and if we replace \mathcal{I}_n by $\mathcal{I}_n \mathcal{J}^n$, then the obtained \mathcal{O}_Y -algebra $\mathcal{S}_{(\mathcal{J})}$ satisfies that $X_{(\mathcal{J})} = \text{Proj}(\mathcal{S}_{(\mathcal{J})})$ is canonically isomorphic to X (??).

Suppose that Y is locally integral, so that \mathcal{K}_Y is a quasi-coherent \mathcal{O}_Y -algebra (??). We say a sub- \mathcal{O}_Y -module \mathcal{J} of \mathcal{K}_Y is a **fractional ideal** of \mathcal{K}_Y if it is of finite type. Given a filtered sequence $(\mathcal{I}_n)_{n \geq 0}$ of quasi-coherent fractional ideal of \mathcal{K}_Y , we can then define the quasi-coherent graded \mathcal{O}_Y -algebra \mathcal{S} and the corresponding Y -scheme $X = \text{Proj}(\mathcal{S})$. We then see that for an invertible fractional ideal \mathcal{J} of \mathcal{K}_Y , there is a canonical isomorphism of X and $X_{(\mathcal{J})}$.

Let Y be a scheme (resp. a locally integral scheme), and \mathcal{J} be a quasi-coherent ideal of \mathcal{O}_Y (resp. a quasi-coherent fractional ideal of \mathcal{K}_Y); put $\mathcal{S} = \bigoplus_{n \geq 0} \mathcal{J}^n$. The Y -scheme $X = \text{Proj}(\mathcal{S})$ is said to be the scheme obtained by **blowing up along the ideal \mathcal{J}** , or the **blow up** of Y relative to \mathcal{J} . If \mathcal{J} is a quasi-coherent ideal of \mathcal{O}_Y and Y' is the closed subscheme defined by \mathcal{J} , we also say that X is the Y -scheme obtained by blowing up Y' . By definition, \mathcal{S} is generated by $\mathcal{S}_1 = \mathcal{J}$; if \mathcal{J} is a \mathcal{O}_Y -module of finite type, X is then projective over Y . By the hypotheses on \mathcal{J} , the \mathcal{O}_X -module $\mathcal{O}_X(1)$ is invertible (??) and very ample in view of ?? for the structural

morphism $j : X \rightarrow Y$. We also note that the restriction of f to $f^{-1}(Y - Y')$ is an isomorphism if \mathcal{I} is the quasi-coherent ideal of \mathcal{O}_Y defining Y' : in fact, this question is local over Y , so it suffice to suppose $\mathcal{I} = \mathcal{O}_Y$, and this then follows from ??.

If we replace \mathcal{I} by \mathcal{I}^d for some $d > 0$, the blow up Y -scheme X is then replaced by a Y -scheme canonically isomorphic to X' (??). Simialrly, for any invertible ideal (resp. fractional ideal) \mathcal{J} , the blow up scheme $X_{(\mathcal{J})}$ relative to $\mathcal{I}\mathcal{J}$ is canonically isomorphic to X . In particular, if \mathcal{I} is an invertible ideal (resp. fractional ideal), the blow up Y -scheme relative to \mathcal{I} is isomorphic to Y .

Proposition 19.8.1. *Let Y be an integral scheme.*

- (i) *For any filtered sequence (\mathcal{I}_n) of quasi-coherent fractional ideals of \mathcal{K}_Y , the Y -scheme $X = \text{Proj}(\bigoplus_{n \geq 0} \mathcal{I}_n)$ is integral and the structural morphism $f : X \rightarrow Y$ is dominant.*
- (ii) *Let \mathcal{I} be a quasi-coherent fractional ideal of \mathcal{K}_Y and X be the blow up Y -scheme relative to \mathcal{I} . If $\mathcal{I} \neq 0$, the structural morphism $f : X \rightarrow Y$ is surjective and birational.*

Proof. In case (i) the quasi-coherent graded \mathcal{O}_Y -algebra $\mathcal{S} = \bigoplus_{n \geq 0} \mathcal{I}_n$ is integral since for any $y \in Y$, $\mathcal{O}_{Y,y}$ is an integral domain, so the claim follows from ???. For (ii), we obtain from (i) that X is integral; if moreover x and y are generic points of X and Y , we have $f(x) = y$, and it is necessary to prove that $\kappa(x) = \kappa(y)$. Now x is also the generic point of the fiber $f^{-1}(y)$; if $\psi : Z \rightarrow Y$ is the canonical morphism, where $Z = \text{Spec}(\kappa(y))$, then the scheme $f^{-1}(y)$ is identified with $\text{Proj}(\mathcal{S}')$, where $\mathcal{S}' = \psi^*(\mathcal{S})$ (??). But it is clear that $\mathcal{S}' = \bigoplus_{n \geq 0} (\widetilde{\mathcal{I}}_y)^n$, and as \mathcal{I} is a nonzero quasi-coherent fractional ideal of \mathcal{K}_Y , $\mathcal{I}_y \neq 0$ (??), so $\mathcal{I}_y = \kappa(y)$ (since y is the generic point of Y , \mathcal{I}_y is a $\kappa(y)$ -vector space). The scheme $\text{Proj}(\mathcal{S}')$ is then identified with $\text{Spec}(\kappa(y))$ (??), whence the assertion. \square

Retain the notations of ???. By definition, the injection $\mathcal{I}_{n+1} \rightarrow \mathcal{I}_n$ defines for each $k \in \mathbb{Z}$ a injective homomorphism of degree 0 of graded \mathcal{S} -modules

$$u_k : \mathcal{S}_+(k+1) \rightarrow \mathcal{S}(k). \quad (19.8.1)$$

As $\mathcal{S}_+(k+1)$ and $\mathcal{S}(k+1)$ are eventually isomorphic \mathcal{S} -modules, the hommorphism u_k corresponds to a canonical injective homomorphism of \mathcal{O}_X -modules (??):

$$\tilde{u}_k : \mathcal{O}_X(k+1) \rightarrow \mathcal{O}_X(k). \quad (19.8.2)$$

Recall on the other hand that we have defined a canonical homomorphism

$$\lambda : \mathcal{O}_X(d) \otimes_{\mathcal{O}_X} \mathcal{O}_X(k) \rightarrow \mathcal{O}_X(d+k)$$

and as the diagram

$$\begin{array}{ccc} \mathcal{S}(d) \otimes_{\mathcal{S}} \mathcal{S}(k) \otimes_{\mathcal{S}} \mathcal{S}(l) & \longrightarrow & \mathcal{S}(d+k) \otimes_{\mathcal{S}} \mathcal{S}(l) \\ \downarrow & & \downarrow \\ \mathcal{S}(d) \otimes_{\mathcal{S}} \mathcal{S}(k+l) & \longrightarrow & \mathcal{S}(d+k+l) \end{array}$$

is commutative, it follows from the functoriality of λ that the homomorphism λ define a quasi-coherent graded \mathcal{O}_X -algebra structure on $\mathcal{S}_X = \bigoplus_{n \in \mathbb{Z}} \mathcal{O}_X(n)$. Moreover, the diagram

$$\begin{array}{ccc} \mathcal{S}(d) \otimes_{\mathcal{S}} \mathcal{S}_+(k+1) & \longrightarrow & \mathcal{S}_+(d+k+1) \\ 1 \otimes u_k \downarrow & & \downarrow u_{k+d} \\ \mathcal{S}(d) \otimes_{\mathcal{S}} \mathcal{S}(k) & \longrightarrow & \mathcal{S}(d+k) \end{array}$$

is commutative; the functoriality of λ shows that we have a commutative diagram

$$\begin{array}{ccc} \mathcal{O}_X(d) \otimes_{\mathcal{O}_X} \mathcal{O}_X(k+1) & \xrightarrow{\lambda} & \mathcal{O}_X(d+k+1) \\ 1 \otimes \tilde{u}_k \downarrow & & \downarrow \tilde{u}_{d+h} \\ \mathcal{O}_X(d) \otimes_{\mathcal{O}_X} \mathcal{O}_X(k) & \xrightarrow{\lambda} & \mathcal{O}_X(d+k) \end{array} \quad (19.8.3)$$

where the horizontal arrows are the canonical homomorphisms. We can then say that the \tilde{u}_k define an injective homomorphism (of degree 0) of graded \mathcal{S}_X -modules

$$\tilde{u} : \mathcal{S}_X(1) \rightarrow \mathcal{S}_X \quad (19.8.4)$$

Now we consider for each $n \geq 0$ the homomorphism $\tilde{v}_n = \tilde{u}_{n-1} \circ \cdots \circ \tilde{u}_0$, which is an injective homomorphism $\mathcal{O}_X(n) \rightarrow \mathcal{O}_X$; we denote its image by $\mathcal{I}_{n,X}$, which is a quasi-coherent ideal \mathcal{O}_X isomorphic to $\mathcal{O}_X(n)$. Moreover, the diagram

$$\begin{array}{ccc} \mathcal{O}_X(m) \otimes_{\mathcal{O}_X} \mathcal{O}_X(n) & \xrightarrow{\lambda} & \mathcal{O}_X(m+n) \\ \tilde{v}_m \otimes \tilde{v}_n \downarrow & & \downarrow \tilde{v}_{m+n} \\ \mathcal{O}_X & \xrightarrow{\text{id}} & \mathcal{O}_X \end{array}$$

is commutative for $m, n \geq 0$. We also conclude that $(\mathcal{I}_{n,X})_{n \geq 0}$ is a filtered sequence of quasi-coherent ideals of \mathcal{O}_X .

Proposition 19.8.2. *Let Y be a scheme, \mathcal{I} be a quasi-coherent ideal of \mathcal{O}_Y , and X be the blow up Y -scheme relative to \mathcal{I} . Then for each $n > 0$ we have a canonical isomorphism*

$$\mathcal{O}_X(n) \xrightarrow{\sim} \mathcal{I}^n \mathcal{O}_X = \mathcal{I}_{n,X}$$

and therefore $\mathcal{I}^n \mathcal{O}_X$ is a very ample invertible \mathcal{O}_X -module relative to Y for $n > 0$.

Proof. The last assertion is immediate since $\mathcal{O}_X(1)$ is invertible (??) and very ample relative to Y by definition. On the other hand, the image of the homomorphism $v_n : S_+(n) \rightarrow S$ is none other than $\mathcal{I}^n \mathcal{S}$, and the first assertion then follows from the exactness of the functor $\widetilde{\mathcal{M}}$ (??) and the formula $\widetilde{\mathcal{IM}} = \mathcal{I} \cdot \widetilde{\mathcal{M}}$. \square

Corollary 19.8.3. *Under the hypotheses of ??, if $f : X \rightarrow Y$ is the structural morphism and Y' is the closed subscheme of Y defined by \mathcal{I} , the closed subscheme $X' = f^{-1}(Y')$ of X is defined by $\mathcal{I}' \mathcal{O}_X$*

(isomorphic to $\mathcal{O}_X(1)$), so we have an exact sequence

$$0 \longrightarrow \mathcal{O}_X(1) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_{X'} \longrightarrow 0$$

Proof. This follows from ?? and ??(b). \square

Under the hypotheses of ??, we can specify that structure of $\mathcal{I}_{n,X}$. Note that that the homomorphism

$$\tilde{u}_{-1} : \mathcal{O}_X \rightarrow \mathcal{O}_X(-1)$$

corresponds canonically to a section s of $\mathcal{O}_X(-1)$ over X , which is called the canonical section (relative to \mathcal{I}). In the diagram ??, the horizontal homomorphisms are isomorphisms ??, so by replacing d by k and k by -1 in that diagram, we obtain $\tilde{u}_k = 1_k \otimes \tilde{u}_{-1}$ (where 1_k is the identity of $\mathcal{O}_X(k)$), which means the homomorphism \tilde{u}_k is none other than the tensor product by the canonical section k (for any $k \in \mathbb{Z}$). The homomorphism \tilde{u} of ?? can be interpreted in the same manner, and we then deduce that, for any $n \geq 0$, the homomorphism $\tilde{v}_n : \mathcal{O}_X(n) \rightarrow \mathcal{O}_X$ is the tensor product by $s^{\otimes n}$.

Corollary 19.8.4. *With the notations of ??, the underlying of X' is the set of $x \in X$ such that $s(x) = 0$, where s is the canonical section of $\mathcal{O}_X(-1)$.*

Proof. In fact, if c_x is a generator for the fiber $(\mathcal{O}_X(1))_x$ at a point x , $s_x \otimes c_x$ is canonically identified with a generator for the fiber $\mathcal{I}_{1,X}$ at the point x , and is therefore invertible if and only if $s_x \notin \mathfrak{m}_x(\mathcal{O}_X(-1))_x$, which means $s(x) \neq 0$. \square

Proposition 19.8.5. *Let Y be an integral scheme, \mathcal{I} be a quasi-coherent fractional ideal of \mathcal{K}_Y , and X be the blow up Y -scheme relative to \mathcal{I} . Then there is an isomorphism $\mathcal{O}_X \xrightarrow{\sim} \mathcal{I}\mathcal{O}_X$, and in particular $\mathcal{I}\mathcal{O}_X$ is a very ample invertible \mathcal{O}_X -module relative to Y .*

Proof. The question is local over Y ??, so we can assume that $Y = \text{Spec}(A)$ is affine, where A is an integral domain with fraction field K and $\mathcal{I} = \tilde{\mathfrak{I}}$, where \mathfrak{I} is a fractional ideal of K . Then there exists an element $a \neq 0$ such that $a\mathfrak{I} \subseteq A$. Put $S = \bigoplus_n \mathfrak{I}^n$; the map $x \mapsto ax$ is an A -isomorphism of $\mathfrak{I}^{n+1} = S(1)_n$ to $a\mathfrak{I}^{n+1} = a\mathfrak{I}S_n \subseteq \mathfrak{I}^n = S_n$, so defines a eventual isomorphism of degree 0 of graded S -modules $S_+(1) \rightarrow a\mathfrak{I}S$. But $x \mapsto a^{-1}x$ is an isomorphism of degree 0 of graded S -modules $a\mathfrak{I}S \xrightarrow{\sim} \mathfrak{I}S$, so we obtain an isomorphism $\mathcal{O}_X(1) \xrightarrow{\sim} \mathcal{I}\mathcal{O}_X$. As S is generated by $S_1 = \mathfrak{I}$, $\mathcal{O}_X(1)$ is invertible and very ample, whence the conclusion. \square

Proposition 19.8.6. *Let Y be a locally Noetherian scheme, \mathcal{I} ba a quasi-coherent ideal of \mathcal{O}_Y , and X be the blow up Y -scheme relative to \mathcal{I} . Let $f : X \rightarrow Y$ be the structural morphism. If $g : Z \rightarrow Y$ is any morphism such that $g^*(\mathcal{I})\mathcal{O}_Z$ is an invertible \mathcal{O}_Z -module, then there exists a unique morphism $\tilde{g} : Z \rightarrow X$ such that the following diagram is commutative:*

$$\begin{array}{ccc} Z & \xrightarrow{\tilde{g}} & X \\ & \searrow g & \downarrow f \\ & & Y \end{array}$$

Proof. The question is local over Y , so we can assume that $Y = \text{Spec}(A)$ is affine, where A is Noetherian, and $\mathcal{I} = \tilde{\mathfrak{I}}$ where \mathfrak{I} is an ideal of A . Then $X = \text{Proj}(S)$ where $S = \bigoplus_{n \geq 0} \mathfrak{I}^n$,

and we note that since A is Noetherian, \mathcal{I} is finitely generated, so S is of finite type over A . Let $a_0, \dots, a_n \in \mathcal{I}$ be a set of generators for \mathcal{I} , so that we have a surjective homomorphism $\varphi : A[T_0, \dots, T_n] \rightarrow S$ which maps T_i to a_i , and this gives a closed immersion $i : X \rightarrow \mathbb{P}_A^n$, and we can identify X with its image. If $g : Z \rightarrow Y$ is a morphism such that $\mathcal{L} = g^*(\mathcal{J})\mathcal{O}_Z$ is invertible, then the inverse images of the a_i , which are global sections of \mathcal{J} , give global sections s_0, \dots, s_n of \mathcal{L} which generate \mathcal{L} . Then by ?? $r : Z \rightarrow P = \mathbb{P}_A^n$ such that $\mathcal{L} \cong r^*(\mathcal{O}_P(1))$ and $s_i = r^{-1}(T_i)$. We now claim that this morphism factors through the closed subscheme X of \mathbb{P}_A^n : this follows from the fact that if $F(T_0, \dots, T_n)$ is a homogeneous polynomial of degree d of $\ker \varphi$, then $F(a_0, \dots, a_n) = 0$ in A and so $F(s_0, \dots, s_n) = 0$ in $\Gamma(Z, \mathcal{L}^{\otimes d})$. This gives the desired morphism $\tilde{g} : Z \rightarrow X$, and for any such morphism we necessarily have

$$g^*(\mathcal{J})\mathcal{O}_Z = \tilde{g}^*(f^*(\mathcal{J})\mathcal{O}_X)\mathcal{O}_Z = \tilde{g}^*(\mathcal{O}_X(1))\mathcal{O}_Z$$

by ??, so we obtain a surjective homomorphism $\tilde{g}^*(\mathcal{O}_X(1)) \rightarrow g^*(\mathcal{J})\mathcal{O}_Z = \mathcal{L}$, hence an isomorphism by ?. Clearly the sections s_i of \mathcal{L} are the inverse images of the sections T_i of $\mathcal{O}_P(1)$ on \mathbb{P}_A^n , so the uniqueness of \tilde{g} follows from ??.

Corollary 19.8.7. *Let $q : Y' \rightarrow Y$ be a morphism of locally Noetherian schemes and \mathcal{J} be a quasi-coherent ideal of \mathcal{O}_Y . Let X be the blow up Y -scheme relative to \mathcal{J} and X' be the blow up Y' -scheme relative to $\mathcal{J}' = q^*(\mathcal{J})\mathcal{O}_{Y'}$. Then there exists a unique morphism $p : X' \rightarrow X$ such that the diagram*

$$\begin{array}{ccc} X' & \xrightarrow{p} & X \\ \downarrow & & \downarrow \\ Y' & \xrightarrow{q} & Y \end{array} \tag{19.8.5}$$

is commutative. Moreover, if q is a closed immersion, so is p .

Proof. The existence and uniqueness of q follows from ?? and ?. To show that p is a closed immersion if q is, we trace the definition of the blow up: $X = \text{Proj}(\mathcal{S})$ where $\mathcal{S} = \bigoplus_{n \geq 0} \mathcal{I}^n$ and $X' = \text{Proj}(\mathcal{S}')$ where $\mathcal{S}' = \bigoplus_{n \geq 0} \mathcal{J}^n$. Since Y' is a closed subscheme of Y , we can consider \mathcal{S}' as a sheaf of graded algebras over Y . Then there exists a natural surjective homomorphism $\mathcal{S} \rightarrow \mathcal{S}'$, which gives rise to the closed immersion p . \square

In the situation of ??, if Y' is a closed subscheme of Y , we call the closed subscheme X' of X the **strict transform** of Y' under the blowing-up $f : X \rightarrow Y$.

Example 19.8.8. Let $Y = \mathbb{A}_k^n$ be the affine space over a field k and we consider the blow up of Y at the origin y of Y . Then $Y = \text{Spec}(A)$ where $A = k[X_1, \dots, X_n]$, y corresponds to the ideal $\mathcal{I} = (X_1, \dots, X_n)$, and $X = \text{Proj}(S)$ where $S = \bigoplus_{n \geq 0} \mathcal{I}^n$. We can define a surjective homomorphism

$$\varphi : A[Y_0, \dots, Y_n] \rightarrow S$$

of graded rings by sending Y_i to X_i as an element of degree 1 in S , which gives a closed immersion of X into \mathbb{P}_A^{n-1} . It is not hard to see that the kernel of φ is generated by the homogeneous polynomials $X_i Y_j - X_j Y_i$, where $i, j = 1, \dots, n$, so this definition is compatible with the definition of the blow up of the affine variety \mathbb{A}_k^n .

Now if Y' is a closed subscheme of Y passing through y , then the strict transform X' of Y' is a closed subscheme of X . Hence, provided that Y' is not reduced to the point y , we can recover X' as the closure of $f^{-1}(Y' - \{y\})$, where $f : f^{-1}(Y' - \{y\}) \rightarrow Y' - \{y\}$ is an isomorphism. Again this definition is compatible with the definition of blow up of closed subvarieties of \mathbb{A}_k^n .

Example 19.8.9. As an example of the general concept of blowing up a coherent sheaf of ideals, we show how to eliminate the points of indeterminacy of a rational map determined by an invertible sheaf. So let A be a ring, X be a Noetherian scheme over A , \mathcal{L} be an invertible sheaf on X , and s_0, \dots, s_n be global sections of \mathcal{L} . Let U be the open subset of X where the s_i generate the sheaf \mathcal{L} (that is, the subset where the corresponding homomorphism $\mathcal{O}_X^{n+1} \rightarrow \mathcal{L}$ is surjective, cf. ??(iii)). Then the invertible sheaf $\mathcal{L}|_U$ on U and the global sections s_0, \dots, s_n determine an A -morphism $\varphi : U \rightarrow \mathbb{P}_A^n$, which is also a rational map $X \dashrightarrow \mathbb{P}_A^n$. We will now show how to blow up a certain sheaf of ideals \mathcal{I} on X , whose corresponding closed subscheme Y has support equal to $X - U$ (i.e., the underlying topological space of Y is $X - U$), so that the morphism φ extends to a morphism $\tilde{\varphi} : \tilde{X} \rightarrow \mathbb{P}_A^n$.

$$\begin{array}{ccc} \tilde{X} & & \\ \pi \downarrow & \searrow \tilde{\varphi} & \\ X & & \mathbb{P}_A^n \\ \uparrow & \nearrow \varphi & \\ U & & \end{array}$$

Let \mathcal{F} be the quasi-coherent sub- \mathcal{O}_X -module of \mathcal{L} generated by s_0, \dots, s_n . We define a coherent ideal \mathcal{I} of \mathcal{O}_X as follows: for any open subset $V \subseteq X$ and an isomorphism $\psi : \mathcal{L}|_V \cong \mathcal{O}_X|_V$, we take $\mathcal{I}|_V = \psi(\mathcal{F}|_V)$. It is clear that \mathcal{I}_V is independent of the choice of ψ , so we get a well-defined coherent ideal \mathcal{I} of \mathcal{O}_X . We also note that $\mathcal{I}_x = \mathcal{O}_{X,x}$ if and only if $x \in U$, so the corresponding closed subscheme Y has support $X - U$. Let $\pi : \tilde{X} \rightarrow X$ be the corresponding blow up relative to \mathcal{I} . We claim that $\pi^*(\mathcal{I})$ is a coherent ideal of $\mathcal{O}_{\tilde{X}}$, so is invertible by ???. This can be verified on each affine open X_{s_i} . Then the global sections $\pi^*(s_i)$ of $\pi^*(\mathcal{L})$ generate an invertible sub- $\mathcal{O}_{\tilde{X}}$ -module \mathcal{L}' of $\pi^*(\mathcal{L})$. Now \mathcal{L}' and the sections $\pi^*(s_i)$ define a morphism $\tilde{\varphi} : \tilde{X} \rightarrow \mathbb{P}_A^n$ whose restriction on $\pi^{-1}(U)$ corresponds to the morphism φ under the isomorphism $\pi : \pi^{-1}(U) \xrightarrow{\sim} U$.

19.8.2 Homogenization of graded rings

Let S be a graded ring, which we do not suppose to be with positive degrees. We set

$$S^{\geqslant} = \bigoplus_{n \geq 0} S_n, \quad S^{\leqslant} = \bigoplus_{n \leq 0} S_n$$

which are subrings of S , with positive of negative degrees respectively. If f is a homogeneous element of degree d (positive or negative) of S , the localization $S_f = S'$ is endowed with a graded ring structure, where S'_n is the set of elements x/f^k , where $x \in S_{n+kd}$ ($k \geq 0$); we also note that $S_{(f)} = S'_0$, and we denote S_f^{\geqslant} and S_f^{\leqslant} for S'^{\geqslant} and S'^{\leqslant} , respectively. This notation is

justified by the fact that if $d > 0$, then we have

$$(S^{\geq})_f = S_f; \quad (19.8.6)$$

in fact, if $x \in S_{n+kd}$ where $n + kd < 0$, we can write $x/f^k = xf^h/f^{h+k}$ so that $n + (h+k)d > 0$ for $h > 0$ large enough. We then conclude by definition that

$$(S^{\geq})_{(f)} = (S_f^{\geq})_0 = S_{(f)}. \quad (19.8.7)$$

If M is a graded S -module, we put similarly

$$M^{\geq} = \bigoplus_{n \geq 0} M_n, \quad M^{\leq} = \bigoplus_{n \leq 0} M_n$$

which are respectively S^{\geq} -module and S^{\leq} -module, with intersection the S_0 -module M_0 . If $f \in S_d$, we also define M_f as the graded S_f -module such that $(M_f)_n$ is the set of elements z/f^k , where $z \in M_{n+kd}$. We denote by $M_{(f)}$ the set of degree 0 elements in M_f , which is an $S_{(f)}$ -module, and we write M_f^{\geq} and M_f^{\leq} for $(M_f)^{\geq}$ and $(M_f)^{\leq}$ respectively. If $d > 0$, we also have

$$(M^{\geq})_f = M_f, \quad (M^{\geq})_{(f)} = (M_f^{\geq})_0 = M_{(f)}. \quad (19.8.8)$$

Let z be an indeterminate, which is called the **homogenization variable**. If S is a graded ring, the polynomial algebra

$$\widehat{S} = S[z]$$

is a graded S -algebra, where for f homogeneous we put

$$\deg(fz^n) = n + \deg(f).$$

Lemma 19.8.10. *Let $f \in S_d$ with $d > 0$. We have canonical isomorphisms*

$$\widehat{S}_{(z)} \cong \widehat{S}/(z-1)\widehat{S} \cong S, \quad (19.8.9)$$

$$\widehat{S}_{(f)} \cong S_f^{\leq}. \quad (19.8.10)$$

Proof. The first isomorphism of (??) is already defined in ?? and the second one is trivial; the isomorphism $\widehat{S}_{(z)} \cong S$ thus defined then send an element xz^n/z^{n+k} (where $\deg(x) = k$ for $k \geq -n$) to the element x . The homomorphism (??) is defined by sending xz^n/f^k (where $\deg(x) = kd - n$) to the element x/f^k , of degree $-k$ in S_f^{\leq} , and it is easy to verify that this is an isomorphism. \square

Let M be a graded S -module. It is clear that the S -module

$$\widehat{M} = M \otimes_S \widehat{S} = M \otimes_S S[z]$$

is the direct sum of the S -modules $M \otimes Sz^n$, whence the abelian groups $M_k \otimes Sz^n$. We define over \widehat{M} an \widehat{S} -module structure by

$$\deg(x \otimes z^n) = n + \deg(x)$$

for x homogeneous in M .

Lemma 19.8.11. *Let $f \in S_d$ with $d > 0$. We have a canonical isomorphism*

$$\widehat{M}_{(z)} \cong \widehat{M}/(z-1)\widehat{M} \cong M, \quad (19.8.11)$$

$$\widehat{M}_{(f)} \cong M_f^{\leqslant}. \quad (19.8.12)$$

Proof. This can be proved as ?? by using the second part of ??.

Let S be a graded ring with positive degrees. Then for each $n \geq 0$, we can consider $S(n) = \bigoplus_{m \geq n} S_m$ as a graded ideal of S (in particular $S(0) = S$ and $S(1) = S_+$). As it is clear that $S(m)S(n) \subseteq S(m+n)$, we can then define a graded ring

$$S^\natural = \bigoplus_{n \geq 0} S(n)$$

whence $S_n^\natural = S(n)$. Then S_0^\natural is equal to S considered as a nongraded ring, and S^\natural is therefore an S -algebra. For any homogeneous element $f \in S_d$ with $d > 0$, we denote by f^\natural the element f considered as an element of $S(d) = S_d^\natural$.

Lemma 19.8.12. *Let S be a graded ring with positive degrees, f be a homogeneous element with $d > 0$. We have canonical isomorphisms:*

$$S_f \cong \bigoplus_{n \in \mathbb{Z}} S(n)_{(f)}, \quad (19.8.13)$$

$$(S_f^{\geqslant})_{f/1} \cong S_f, \quad (19.8.14)$$

$$S_{(f^\natural)}^\natural \cong S_f^{\geqslant}. \quad (19.8.15)$$

the first two of which are bi-isomorphisms of graded rings.

Proof. It is immediate that we have $(S_f)_n = (S(n)_f)_0 = S(n)_f$, whence the first isomorphism. On the other hand, as $f/1$ is invertible in S_f , there is a canonical isomorphism $S_f \cong S_f^{\geqslant} = (S_f)_{f/1}$ in view of ?? apply to S_f . Finally, if $x = \sum_{m \geq n} y_m$ is an element of $S(n)$, where $n = kd$, we can correspond the element $x/(f^\natural)^k$ to the element $\sum_m y_m/f^k$ of S_f^{\geqslant} , and we verify that this is an isomorphism. \square

If M is a graded S -module, we can similarly put for $n \in \mathbb{Z}$

$$M^\natural = \bigoplus_{n \in \mathbb{Z}} M(n)$$

as $S(m)M(n) \subseteq M(m+n)$. Then M^\natural is a graded S^\natural -module, and similarly we have the following:

Lemma 19.8.13. *Let $f \in S_d$ be a homogeneous element with $d > 0$. We have the following bi-homomorphisms*

$$M_f \cong \bigoplus_{n \in \mathbb{Z}} M(n)_{(f)}, \quad (19.8.16)$$

$$(M_f^{\geqslant})_{f/1} \cong M_f, \quad (19.8.17)$$

$$M_{(f^\natural)}^\natural \cong M_f^{\geqslant}. \quad (19.8.18)$$

the first two of which are bi-isomorphisms of graded modules.

Remark 19.8.1. We can think that S^\natural is obtained from S by adding a "phantom" element y of degree -1 . The component $S(n)$ can be then considered as the S -module $(Sy^n)_0$, which is the set of degree 0 elements in Sy^n . With this understanding, we can then relate the results of ?? and ??.

Lemma 19.8.14. Let S be a graded ring with positive degrees.

- (i) For S^\natural to be an S_0^\natural -algebra of finite type (resp. Noetherian), it is necessary and sufficient that S is an S_0 -algebra of finite type (resp. Noetherian).
- (ii) For $S_{n+1}^\natural = S_1^\natural S_n^\natural$ for $n \geq n_0$, it is necessary and sufficient that $S_{n+1} = S_1 S_n$ for $n \geq n_0$.
- (iii) For $S_n^\natural = (S_1^\natural)^n$ for $n \geq n_0$, it is necessary and sufficient that $S_n = S_1^n$ for $n \geq n_0$.
- (iv) If (f_α) is a set of homogeneous elements of S_+ such that the radical in S_+ of the ideal of S_+ generated by the f_α is equal to S_+ , then S_+^\natural is the radical in S_+^\natural of the ideal of S_+^\natural generated by the f_α^\natural .

Proof. If S^\natural is an S_0^\natural -algebra of finite type, $S_+ = S_1^\natural$ is a finitely generated module over $S = S_0^\natural$ by ??, so S is an S_0 -algebra of finite type by ?. If S^\natural is a Noetherian ring, so is the ring $S_0^\natural = S$ by ?. Conversely, if S is an S_0 -algebra of finite type, then by ? there exists $d > 0$ and $n_0 > 0$ such that $S_{n+d} = S_n S_n$ for any $n \geq n_0$; we can clearly suppose that $n_0 \geq d$. Moreover, the S_n are finitely generated S_0 -modules (??(c)), so if $n \geq n_0 + d$, we have $S_n^\natural = S_n S_{n-d}^\natural = S_d^\natural S_{n-d}^\natural$; and if $n < n_0 + d$, we have

$$S_n^\natural = S_n + \cdots + S_{n_0+d-1} + S_d E + S_d^2 E + \cdots$$

where $E = S_{n_0} + \cdots + S_{n_0+d-1}$. For $1 \leq n \leq n_0$, let G_n be the union of the generators of the S_0 -module S_i where $n \leq i \leq n_0 + d - 1$, considered as subsets of $S(n)$. For $n_0 + 1 \leq n \leq n_0 + d - 1$, similarly let G_n be the union of the generators of the S_0 -module S_i where $n \leq i \leq n_0 + d - 1$ and of $S_d E$, considered as subsets of $S(n)$. Then it is clear that we have $S_n^\natural = S_0^\natural G_n$ for $1 \leq n \leq n_0 + d - 1$, and therefore the union G of the G_n for $1 \leq n \leq n_0 + d - 1$ is a system of generators of the S_0^\natural -algebra S^\natural . We then conclude that if $S = S_0^\natural$ is Noetherian, then so is S^\natural .

It is clear that if $S_{n+1} = S_1 S_n$ for $n \geq n_0$, then we have $S_{n+1}^\natural = S_1^\natural S_n^\natural$, and a fortiori $S_{n+1}^\natural = S_1^\natural S_n^\natural$ for $n \geq n_0$. Conversely, the latter relation means that

$$S_{n+1} + S_{n+2} + \cdots = (S_1 + S_2 + \cdots)(S_n + S_{n+1} + \cdots);$$

by comparing the degree $n + 1$ component of both sides, we conclude that $S_{n+1} = S_1 S_n$. This proves the assertion (ii).

If $S_n = S_1^n$ for $n \geq n_0$, we have $S_n^\natural = \bigoplus_{m \geq n} S_1^m$, and as S_1^\natural contains $\bigoplus_{m \geq 1} S_1^m$, we then have $S_n^\natural \subseteq (S_1^\natural)^n$ for $n \geq n_0$. Conversely, the degree n component of $(S_1^\natural)^n = (S_1 + S_2 + \cdots)^n$ considered as elements of S is equal to S_1^n ; so the relation $S_n^\natural = (S_1^\natural)^n$ implies $S_n = S_1^n$.

Finally, to prove (iv), it suffices to show that if an element $g \in S_{k+d}$ is considered as an element of S_k^\natural ($k > 0, d \geq 0$), then there exists an integer $n > 0$ such that in S_{kn}^\natural , g^n is a linear

combination of the f_α^\natural with coefficients in S^\natural . By hypothesis, there exists an integer n_0 such that for $n \geq n_0$, we have $g^n = \sum_\alpha c_{\alpha n} f_\alpha$ in S , where the indices α appearing in this formula are independent of n . Moreover, we can evidently suppose that the $c_{\alpha n}$ are homogeneous, with

$$\deg(c_{\alpha n}) = n(k+d) - \deg(f_\alpha)$$

in S . Let n_0 be large enough such that we have $kn_0 > \deg(f_\alpha)$ for the f_α appearing in the formula of g^n ; for any α , let $c'_{\alpha n}$ be the element $c_{\alpha n}$ considered as an element of degree $kn - \deg(f_\alpha)$ in S^\natural . We then have $g^n = \sum_\alpha c'_{\alpha n} f_\alpha^\natural$ in S^\natural , which proves our assertion. \square

Consider the graded S_0 -algebra

$$S^\natural \otimes_S S_0 = S^\natural / S_+ S^\natural = \bigoplus_{n \geq 0} S(n) / S_+ S(n).$$

As S_n is a quotient S_0 -module of $S(n) / S_+ S(n)$, we have a canonical homomorphism of graded S_0 -algebras

$$S^\natural \otimes_S S_0 \rightarrow S \tag{19.8.19}$$

which is evidently surjective, and corresponds to a canonical closed immersion

$$\text{Proj}(S) \rightarrow \text{Proj}(S^\natural \otimes_S S_0) \tag{19.8.20}$$

Proposition 19.8.15. *The canonical morphism (??) is bijective. For the homomorphism (??) to be eventually bijective, it is necessary and sufficient that there exists an integer n_0 such that $S_{n+1} = S_1 S_n$ for $n \geq n_0$. If this is satisfied, then the morphism (??) is an isomorphism, and the converse of this is also true if S is Noetherian.*

Proof. To prove the first assertion, it suffices (??) to prove that the kernel \mathcal{I} of the homomorphism (??) is formed by nilpotent elements. Now if $f \in S(n)$ is an element whose class mod $S_+ S(n)$ belongs to this kernel, then $f \in S(n+1)$ (by the definition of (??)); the element f^{n+1} , considered as an element of $S(n(n+1))$, then belongs to $S_+ S(n(n+1))$, if we write it as $f \cdot f^n$. Then the class of f^{n+1} mod $S_+ S(n(n+1))$ is zero, which proves our assertion.

As the hypothesis $S_{n+1} = S_1 S_n$ for $n \geq n_0$ is equivalent to $S_{n+1}^\natural = S_1^\natural S_n^\natural$ for $n \geq n_0$ (??(ii)), these hypotheses are equivalent to that (??) is eventually injective, hence eventually bijective, and then (??) is an isomorphism by ??(a). Conversely, assume that S is Noetherian, hence so is S^\natural and $S^\natural \otimes_S S_0$ (??(i)). If (??) is an isomorphism, the sheaf $\tilde{\mathcal{F}}$ over $\text{Proj}(S^\natural \otimes_S S_0)$ is zero (??(a)); as $S^\natural \otimes_S S_0$ is Noetherian, we then conclude from ??(b) that \mathcal{I} is eventually null, so $S_{n+1}^\natural = S_1^\natural S_n^\natural$ for $n \geq n_0$. \square

Consider now the canonical injection $(S_+)^n \rightarrow S(n)$, which defines an injective homomorphism of degree 0 of graded rings

$$\bigoplus_{n \geq 0} (S_+)^n \rightarrow S^\natural. \tag{19.8.21}$$

Proposition 19.8.16. *For the homomorphism (??) to be an eventual isomorphism, it is necessary and sufficient that there exists an integer n_0 such that $S_n = S_1^n$ for $n \geq n_0$. If this is the case, the corre-*

sponding morphism

$$\mathrm{Proj}(S^\sharp) \rightarrow \mathrm{Proj}\left(\bigoplus_{n \geq 0} (S_+)^n\right) \quad (19.8.22)$$

is everywhere defined and an isomorphism, and the converse is also true if S is Noetherian.

Proof. The first two conditions are equivalent in view of ??(iii). The third assertion follows from ??(i), (iii) and ??.

Lemma 19.8.17. *Let S be a graded ring with positive degrees which is a S_0 -algebra of finite type. If the morphism corresponding to the canonical injection $S' = \bigoplus_{n \geq 0} S_1^n \rightarrow S$ is everywhere defined and an isomorphism, then there exists $n_0 > 0$ such that $S_n = S_1^n$ for $n \geq n_0$.*

Proof. In fact, let f_i ($1 \leq i \leq r$) be a system of generators for the S_0 -module S_1 . Then the hypotheses implies that the $D_+(f_i)$ cover $\mathrm{Proj}(S)$. Let $(g_j)_{1 \leq j \leq n}$ be a system of homogeneous elements of S_+ , with $n_j = \deg(g_j)$, which together with the f_i form a system of generators of the ideal S_+ , or a system of generators of S as an S_0 -algebra. The elements $g_j/f_i^{n_j}$ of the ring $S_{(f_i)}$ then by hypotheses belong to the subring $S'_{(f_i)}$, so there exists an integer k such that $S_1^k g_j \subseteq (S_1)^{k+n_j}$ for any j . We then conclude by recurrence on r that $S_1^k g_j^r \subseteq S'$ for any $r \geq 1$, and by the choice of the g_j , we then have $S_1^k S \subseteq S'$. On the other hand there exists for any j an integer m_j such that $g_j^{m_j}$ belongs to the ideal of S generated by the f_i (??), so $g_j^{m_j} \in S_1 S$, and $g_j^{m_j k} \in S_1^k S \subseteq S'$. Therefore there exists an integer $m_0 \geq k$ such that $g_j^m \in S_1^{mn_j}$ for $m \geq m_0$. Now if d is the largest of the n_j , the number $n_0 = dm_0 + k$ then satisfies the requirement. In fact, an element of S_n , for $n \geq n_0$, is a sum of elements of $S_1^\alpha u$, where u is a product of powers of g_j ; if $\alpha \geq k$, it follows from the choice of k that $S_1^\alpha u \subseteq S_1^n$; in the contrary case, at least one of the exponent of g_j is $\geq m_0$, so $u \in S_1^\beta v$ where $\beta \geq m_0 \geq k$ and v is a product of powers of g_j , so we are reduced to the previous case, and $S_1^\alpha u \subseteq S_1^n$. This completes the proof. \square

Remark 19.8.2. The condition $S_n = S_1^n$ for $n \geq n_0$ clearly implies that $S_{n+1} = S_1 S_n$ for $n \geq n_0$, but the converse is not true, even we assume that S is Noetherian. For example, let K be a field, $A = K[x]$, $B = K[y]/y^2K[y]$, where x, y are two indeterminates, with $\deg(x) = 1$ and $\deg(y) = 2$, and let $S = A \otimes_K B$, so that S is a graded algebra over K having a basis formed by the elements x^n and $x^n y$. It is immediate that $S_{n+1} = S_1 S_n$ for $n \geq 2$, but $S_1^n = Kx^n$ while $S_n = Kx^n + Kx^{n-2}y$ for $n \geq 2$.

19.8.3 Projective cones

Let Y be a scheme; in this subsection, we only consider Y -schemes and Y -morphisms. Let \mathcal{S} be a quasi-coherent graded \mathcal{O}_Y -algebra with positive degrees, which we suppose that $\mathcal{S}_0 = \mathcal{O}_Y$. According to the notations of the previous part, we put

$$\hat{\mathcal{S}} = \mathcal{S}[z] = \mathcal{S} \otimes_{\mathcal{O}_Y} \mathcal{O}_Y[z] \quad (19.8.23)$$

which we consider as a graded \mathcal{O}_Y -algebra with positive degrees, so that for any affine open subset U of Y , we have

$$\Gamma(U, \mathcal{S}) = \Gamma(U, \mathcal{S})[z].$$

In the following, we put

$$P = \text{Proj}(\mathcal{S}), \quad C = \text{Spec}(\mathcal{S}), \quad \widehat{C} = \text{Proj}(\widehat{\mathcal{S}})$$

(where in the definition of C , \mathcal{S} is considered as a nongraded \mathcal{O}_Y -algebra), and we say that C (resp. \widehat{C}) is the **affine cone** (resp. **projective cone**) defined by \mathcal{S} ; we will also say "cone" instead of "cone affine." By abuse of language, we say that C (resp. \widehat{C}) is the **affine projecting cone** of P (resp. the **projective projecting cone** of P), which P is understood to be given of the form $\text{Proj}(\mathcal{S})$. Finally, we say that \widehat{C} is the projective closure of C , where C is understood to be a scheme of the form $\text{Spec}(\mathcal{S})$.

Lemma 19.8.18. *Let S be a graded ring with positive degrees, $X = \text{Proj}(S)$, and f be a homogeneous element of S with degree $d > 0$. If f is not a divisor of zero in S , X is the smallest closed subscheme of $X_f = D_+(f)$.*

Proof. This question is clearly local on X ; for any homogeneous element $g \in S_h$ ($h > 0$), it suffices to prove that X_g is the smallest closed subscheme of X_g which dominates X_{fg} . It follows from the definition and ?? that this condition is equivalent to the fact that the canonical homomorphism $S_{(g)} \rightarrow S_{(fg)}$ is injective. Now this homomorphism is identified canonically with the homomorphism $S_{(g)} \rightarrow (S_{(g)})_{f^h/g^d}$ (??). But as f^h is not a zero divisor of S , f^h/g^d is not a zero divisor in S_g (and a fortiori in $S_{(g)}$), because the relation $(f^h/g^d)(t/g^m) = 0$ with $t \in S$ and $m > 0$ implies the existence of an integer $n > 0$ such that $g^n f^h t = 0$, whence $g^n t = 0$, and therefore $t/g^m = 0$ in S_g . This proves the claim. \square

Proposition 19.8.19. *There is a commutative diagram*

$$\begin{array}{ccccc} & & \widehat{C} & & \\ & \nearrow j & \downarrow & \swarrow i & \\ P & & C & & Y \\ & \searrow & \downarrow \varepsilon & \nearrow & \\ & & Y & & \end{array}$$

where ε and j are closed immersions and i is an affine morphism which is an open dominant immersion such that

$$i(C) = \widehat{C} - j(P). \tag{19.8.24}$$

Moreover \widehat{C} is the smallest closed subscheme of \widehat{C} dominating $i(C)$.

Proof. To define i , we consider the open subset $\widehat{C}_z = \text{Spec}(\widehat{\mathcal{S}}/(z-1)\widehat{\mathcal{S}})$ of \widehat{C} (by ??), where z is canonically identified with a section of $\widehat{\mathcal{S}}$ over Y . The isomorphism $i : C \xrightarrow{\sim} \widehat{C}_z$ corresponds then to the canonical isomorphism

$$\widehat{\mathcal{S}}/(z-1)\widehat{\mathcal{S}} \cong \mathcal{S}$$

of (??). The morphism ε corresponds to the augmentation homomorphism $\mathcal{S} \rightarrow \mathcal{S}_0 = \mathcal{O}_Y$ with kernel \mathcal{S}_+ , which is surjective so ε is a closed immersion (??). Finally, j corresponds similar to

the surjective homomorphism $\widehat{\mathcal{S}} \rightarrow \mathcal{S}$ of degree 0, which is the identity on \mathcal{S} and zero on $z\widehat{\mathcal{S}}$. By ?? it is clear that j is everywhere defined and a closed immersion.

To prove the other assertions of the proposition, we can evidently assume that $Y = \text{Spec}(A)$ is affine, $\mathcal{S} = \widehat{S}$ where S is a graded A -algebra, whence $\widehat{\mathcal{S}} = (\widehat{S})^\sim$; the homogeneous elements f of S_+ are then identified with the sections of $\widehat{\mathcal{S}}$ over Y , and the open subset $D_+(f)$ of \widehat{C} is identified with \widehat{C}_f ??; similarly the open subset $D(f)$ of C is identified with C_f . It then follows from ?? and the definition of $\widehat{\mathcal{S}}$ that the open subset $\widehat{C}_z = i(C)$ together with \widehat{C}_f (where f is homogeneous in S_+) constitute an open covering of \widehat{C} . Moreover, we have

$$i^{-1}(\widehat{C}_f) = C_f. \quad (19.8.25)$$

In fact, if we identify $i(C)$ with \widehat{C}_z , then

$$\widehat{C}_f \cap i(C) = \widehat{C}_f \cap \widehat{C}_z = \widehat{C}_{fz} = \text{Spec}(\widehat{S}_{(fz)}).$$

Now if $d = \deg(f)$, $\widehat{S}_{(fz)}$ is canonically isomorphic to $(\widehat{S}_{(z)})_{(f/z^d)}$??, and it follows from the definition of the isomorphism ?? that the image of $(\widehat{S}_{(z)})_{(f/z^d)}$ under the corresponding isomorphism is exactly S_f . As $C_f = \text{Spec}(S_f)$, this proves ?? and also shows that the morphism i is affine. Moreover, the restriction of i to C_f , considered as a morphism into \widehat{C}_f , corresponds ?? to the canonical homomorphism $\widehat{S}_{(f)} \rightarrow \widehat{S}_{(fz)} \cong S_f$. We also note that under the isomorphism ??, \widehat{C}_f is canonically identified with $\text{Spec}(S_f^\leq)$ and the morphism restriction $i|_{C_f} : C_f \rightarrow \widehat{C}_f$ corresponds to the canonical injection $S_f^\leq \rightarrow S_f$. The complement of \widehat{C}_z in $\widehat{C} = \text{Proj}(\widehat{S})$ is, by definition, the set of graded prime ideals of \widehat{S} containing z , which is $j(P)$ from the defintion of j , whence ??.

To prove the final assertion, we can still assume that Y is affine. With the preceding notations, we note that z is not a zero divisor in \widehat{S} , so we can apply ??.

We now identify the affine cone C with the open subscheme $i(C)$ of the projective cone \widehat{C} , which is dense in \widehat{C} . The closed subscheme of C associated with the closed immersion ε is called the **sommet scheme** of C . We also say that ε , which is a Y -section of C , is the **sommet section** or the **zero section** of C ; we can then identify Y with the sommet scheme of C via the morphism ε . The composition $i \circ \varepsilon$ is a Y -section of \widehat{C} , which is also a closed immersion ??, corresponding to the canonical surjective homomorphism $\widehat{\mathcal{S}} = \mathcal{S}[z] \rightarrow \mathcal{O}_Y[z]$ (cf. ??), with kernel $\mathcal{S}_+[z] = \widehat{\mathcal{S}}_+$. The closed subscheme of \widehat{C} associated with this closed immersion is called the **sommet scheme** of \widehat{C} , which can be identified with Y via $i \circ \varepsilon$, and $i \circ \varepsilon$ is called the **sommet section** of \widehat{C} . Finally, the closed subscheme of \widehat{C} associated with j is called the **place of infinity** of C , which is identified with P via j .

The subscheme of C (resp. \widehat{C}) induced respectively over the open subsets

$$E = C - \varepsilon(Y), \quad \widehat{E} = \widehat{C} - i(\varepsilon(Y)) \quad (19.8.26)$$

are called respectively (by abuse of language) the **blunt affine cone** (resp. **blunt projective cone**) defined by \mathcal{S} . We note that with this terminology, E is not necessarily affine over Y , nor is it projective over Y (cf. ??). If we identify C with $i(C)$, we then have

$$C \cup \widehat{E} = \widehat{C}, \quad C \cap \widehat{E} = E. \quad (19.8.27)$$

so that \widehat{C} can be considered as obtaining by gluing the open subschemes C and \widehat{E} along E ; moreover, in view of (??),

$$E = \widehat{E} - j(P). \quad (19.8.28)$$

If $Y = \text{Spec}(A)$ is affine, we then have (with the notations of ??),

$$E = \bigcup C_f, \quad \widehat{E} = \bigcup \widehat{C}_f, \quad C_f = C \cap \widehat{C}_f \quad (19.8.29)$$

where f runs through homogeneous elements of S_+ (or a family of homogeneous elements of S_+ generating the ideal S_+). The glueing of C and the \widehat{C}_f along the C_f is then determined by the injections $C_f \rightarrow C$, $C_f \rightarrow \widehat{C}_f$, which correspond to the canonical homomorphisms $S \rightarrow S_f$, $S_f^{\leqslant} \rightarrow S_f$. On the other hand, we note that $\bigcup \widehat{E}_f$ is the defining domain $G(\varphi)$ of the morphism associated with the canonical injection $\varphi : \mathcal{S} \rightarrow \widehat{\mathcal{S}} = \mathcal{S}[z]$, so we obtain a morphism $p : \widehat{E} \rightarrow P$.

Proposition 19.8.20. *The associated morphism $p : \widehat{E} \rightarrow P$ is an affine and surjective morphism (called the canonical retraction) such that*

$$p^{-1}(P_f) = \widehat{C}_f \quad (19.8.30)$$

and we have $p \circ j = 1_P$. Moreover, if Y is affine and $f \in S_1$, then \widehat{C}_f is isomorphic to $P_f \otimes_{\mathbb{Z}} \mathbb{Z}[T]$ (T is an indeterminate).

Proof. To prove the proposition we may assume that Y is affine, so $\mathcal{S} = \widetilde{S}$. For any $f \in S_+$ homogeneous, by (??) we have (??) and the restriction $p : \widehat{C}_f \rightarrow P_f$ corresponds to the canonical injection $S_{(f)} \rightarrow S_f^{\leqslant}$. The formula $p \circ j = 1_P$ and the fact that p is surjective follows from the fact that the composition $\mathcal{S} \rightarrow \widehat{\mathcal{S}} \rightarrow \mathcal{S}$ is the identity on \mathcal{S} . Finally, the last assertion follows from the fact that S_f^{\leqslant} is isomorphic to $S_{(f)}[T]$ (cf. (??)). \square

Corollary 19.8.21. *The restriction $\pi : E \rightarrow P$ of p to E is a surjective and affine morphism. If Y is affine and $f \in S_+$ is homogeneous, we have*

$$\pi^{-1}(P_f) = C_f \quad (19.8.31)$$

and the restriction of $\pi|_{C_f} : C_f \rightarrow P_f$ corresponds to the canonical injection $S_{(f)} \rightarrow S_f$. If $f \in S_1$, then C_f is isomorphic to $P_f \otimes_{\mathbb{Z}} \mathbb{Z}[T, T^{-1}]$ (T is an indeterminant).

Proof. The formula (??) follows from (??) and (??), which also proves the surjectivity of π . We also have seen that the canonical injection $C_f \rightarrow \widehat{C}_f$ corresponds to $S_{(f)} \rightarrow S_f$, whence the second assertion. Finally, the last assertion is a consequence of the fact that for $f \in S_1$, S_f is isomorphic to $S_{(f)}[T, T^{-1}]$ (cf. (??)). \square

Remark 19.8.3. If Y is affine, the elements of the underlying space of E are the prime ideals \mathfrak{p} (not necessarily graded) of S not containing S_+ , in view of the definition of the immersion ε . For such a prime ideal \mathfrak{p} , the intersections $\mathfrak{p} \cap S_n$ satisfy the conditions of ??, so there exists a graded prime ideal \mathfrak{q} of S such that $\mathfrak{q} \cap S_n = \mathfrak{p} \cap S_n$ for any n . The map $\pi : E \rightarrow P$ on the underlying topological space is then interpreted by the relation

$$\pi(\mathfrak{p}) = \mathfrak{q}.$$

In fact, to verify this relation, it suffices to consider a homogeneous element f of S_+ such that $\mathfrak{p} \in D(f)$, and we then observe that $\mathfrak{q}_{(f)}$ is the inverse image of \mathfrak{p}_f under the canonical injection $S_{(f)} \rightarrow S_f$.

Corollary 19.8.22. *If \mathcal{S} is generated by S_1 , the morphism p and π are of finite type. Moreover, for any $x \in P$, the fiber $p^{-1}(x)$ is isomorphic to $\text{Spec}(\kappa(x)[T])$ and $\pi^{-1}(x)$ is isomorphic to $\text{Spec}(\kappa(x)[T, T^{-1}])$.*

Proof. This follows from ?? and ??, since if Y is affine and S is generated by S_1 , then the P_f for $f \in S_1$ form an open covering of P . \square

Remark 19.8.4. The blunt affine cone E corresponding to the graded \mathcal{O}_Y -algebra $\mathcal{O}_Y[T]$ (where T is an indeterminate) is identified with $G_m = \text{Spec}(\mathcal{O}_Y[T, T^{-1}])$, since it is none other than C_T as we have seen in ???. This scheme is canonically endowed with an abelian group Y -scheme structure.

Example 19.8.23. Let k be a field, $k[T_0, \dots, T_n]$ be the polynomial ring, and \mathfrak{p} be a graded prime ideal of $k[T_0, \dots, T_n]$ not containing the irrelevant ideal. Consider the quotient graded ring $S = k[T_0, \dots, T_n]/\mathfrak{p}$, and set

$$P = \text{Proj}(S), \quad C = \text{Spec}(S), \quad \widehat{C} = \text{Spec}(\widehat{S}).$$

In the language of varieties, if $V \subseteq \mathbb{P}_k^n$ is the variety defined by S , C can be viewed as the affine cone obtained by considering the lines connecting the origin with points of V . Moreover, \widehat{C} is the closure of C in \mathbb{P}_k^{n+1} if we embed \mathbb{A}_k^{n+1} into \mathbb{P}_k^{n+1} via the map $(x_0, \dots, x_n) \rightarrow [x_0 : \dots : x_n : 1]$. Also, the morphism $j : P \rightarrow \widehat{C}$ corresponds to the injection $[x_0 : \dots : x_n] \mapsto [x_0 : \dots : x_n : 0]$. With these, the projective closure \widehat{C} of C in \mathbb{P}_k^{n+1} is given by the equivalent classes $[x_0 : \dots : x_n : x_{n+1}]$ in \mathbb{P}_k^{n+1} such that $[x_0 : \dots : x_n] \in P$, and we can divide into two cases depending on whether $x_{n+1} \neq 0$:

- (a) $(x_0, \dots, x_n) \in C, x_{n+1} \neq 0$;
- (b) $[x_0 : \dots : x_n] \in P, x_{n+1} = 0$.

Thus we see that the variety \widehat{C} can be viewed as a union of P with C , which justifies the formula (??). Also, by definition the blunt affine cone E is the subvariety of C obtained by removing the origin of \mathbb{A}_k^{n+1} , and \widehat{E} can be considered as a union of E and P , which is also the projective cone \widehat{C} removing the point $[0 : \dots : 0 : 1]$ in \mathbb{P}_k^{n+1} . We also remark that if we base change C through the structural morphism $P \rightarrow Y$, then the projection $C_{(P)} \rightarrow C$ can be viewed as the projection from $\mathbb{A}_k^{n+1} \times \mathbb{P}_k^n$ to \mathbb{P}_k^n which maps (x, ξ) to ξ (where the class of $x \in \mathbb{A}_k^{n+1}$ is equal to ξ), and this is the blow up map of \mathbb{A}_k^{n+1} at the origin.

Let Y be a scheme, \mathcal{E} be a quasi-coherent \mathcal{O}_Y -module. If \mathcal{S} is the graded \mathcal{O}_Y -algebra $S_{\mathcal{O}_Y}(\mathcal{E})$, then $\widehat{\mathcal{S}}$ is identified with $S_{\mathcal{O}_Y}(\mathcal{E} \oplus \mathcal{O}_Y)$. The affine cone $\text{Spec}(\mathcal{S})$ defined by \mathcal{S} is by definition the vector bundle $V(\mathcal{E})$, and $\text{Proj}(\mathcal{S})$ is by definition $\mathbb{P}(\mathcal{E})$, so we see that:

Proposition 19.8.24. *The projective closure of a vector bundle $V(\mathcal{E})$ over Y is canonically isomorphic to $\mathbb{P}(\mathcal{E} \oplus \mathcal{O}_Y)$, and the place of infinity of this is canonically isomorphic to $\mathbb{P}(\mathcal{E})$.*

Example 19.8.25. Put for example $\mathcal{E} = \mathcal{O}_Y^r$ where $r \geq 2$. Then the blunt cones E, \widehat{E} defined by \mathcal{S} are neither affine nor projective over Y if $Y \neq \emptyset$. The second assertion is immediate, since

$\widehat{C} = \mathbb{P}(\mathcal{O}_Y^{r+1})$ is projective over Y and the underlying spaces of E and \widehat{E} are not closed in \widehat{C} , so the canonical immersions $E \rightarrow \widehat{C}$ and $\widehat{E} \rightarrow \widehat{C}$ are not projective (?? and ??(v)). On the other hand, suppose that $Y = \text{Spec}(A)$ is affine and for example $r = 2$; we have $C = \text{Spec}(A[T_1, T_2])$ and E is the open subscheme $D(T_1) \cup D(T_2)$ of C , and we have seen that this is not affine (??); a fortiori \widehat{E} is not affine, since E is the open subset of \widehat{E} where the section z is nonzero (??).

Proposition 19.8.26. *Let \mathcal{L} be an invertible \mathcal{O}_Y -module, we have canonical isomorphisms for the blunt cones corresponding to $C = V(\mathcal{L})$:*

$$\text{Spec}(\bigoplus_{n \in \mathbb{Z}} \mathcal{L}^n) \xrightarrow{\sim} E, \quad V(\mathcal{L}^{-1}) \xrightarrow{\sim} \widehat{E}. \quad (19.8.32)$$

Moreover, there exists a canonical isomorphism from the projective closure of $V(\mathcal{L})$ to that of $V(\mathcal{L}^{-1})$, which transform the sommet scheme (resp. the place of infinity) of the first one to the place of infinity (resp. the sommet scheme) of the second one.

Proof. Here we have $\mathcal{S} = \bigoplus_{n \geq 0} \mathcal{L}^{\otimes n}$; the canonical injection $\mathcal{S} \rightarrow \bigoplus_{n \in \mathbb{Z}} \mathcal{L}^{\otimes n}$ defines a canonical dominant morphism

$$\text{Spec}(\bigoplus_{n \in \mathbb{Z}} \mathcal{L}^{\otimes n}) \rightarrow V(\mathcal{L}) = \text{Spec}(\bigoplus_{n \geq 0} \mathcal{L}^{\otimes n}) \quad (19.8.33)$$

and it suffices to prove that this morphism is an isomorphism from $\text{Spec}(\bigoplus_{n \in \mathbb{Z}} \mathcal{L}^{\otimes n})$ to E . The question is local over Y , so we can suppose that $Y = \text{Spec}(A)$ is affine and $\mathcal{L} = \mathcal{O}_Y$, so $\mathcal{S} = \widetilde{A[T]}$ and $\bigoplus_{n \in \mathbb{Z}} \mathcal{L}^{\otimes n} = \widetilde{A[T, T^{-1}]}$. Now $A[T, T^{-1}]$ is the fraction ring $A[T]_T$ of $A[T]$, so (??) identify $\text{Spec}(\bigoplus_{n \in \mathbb{Z}} \mathcal{L}^{\otimes n})$ as the open subscheme $D(T)$ of $C = V(\mathcal{L})$, which by definition is E .

The isomorphism $V(\mathcal{L}^{-1}) \cong \widehat{E}$ will on the other hand be a consequence of the last assertion, since $V(\mathcal{L}^{-1})$ is the complement of the place of infinity of its integral closure and \widehat{E} is the complement of the sommet scheme of projective closure of $C = V(\mathcal{L})$. Now these projective closures are respectively $\mathbb{P}(\mathcal{L}^{-1} \oplus \mathcal{O}_Y)$ and $\mathbb{P}(\mathcal{L} \oplus \mathcal{O}_Y)$; but we have

$$\mathcal{L} \oplus \mathcal{O}_Y = \mathcal{L} \oplus (\mathcal{L} \otimes \mathcal{L}^{-1}) = \mathcal{L} \otimes (\mathcal{L}^{-1} \oplus \mathcal{O}_Y),$$

so the existence of the isomorphism follows from ??, and it remains to see that this isomorphism exchanges the sommet scheme and the place of infinity. For this we can assume that $Y = \text{Spec}(A)$ is affine, $\mathcal{L} = \widetilde{L}$, with $L = Ac$, $L^{-1} = Ac'$, and the canonical isomorphism $L \otimes L^{-1} \rightarrow A$ sends $c \otimes c'$ to 1. Then

$$S(L \oplus A) = A[z] \otimes \bigoplus_{n \geq 0} Ac^{\otimes n}, \quad S(L^{-1} \oplus A) = A[z] \otimes \bigoplus_{n \geq 0} Ac'^{\otimes n},$$

and the isomorphism defined in ?? sends $z^h \otimes c'^{\otimes(n-h)}$ to the element $z^{n-h} \otimes c^{\otimes h}$. Now, in $\mathbb{P}(\mathcal{L}^{-1} \oplus \mathcal{O}_Y)$ the place of infinity is the set of points where the section z vanishes, and the sommet section is the set of points where the section c' vanishes. As we have a similar description for $\mathbb{P}(\mathcal{L} \oplus \mathcal{O}_Y)$, our conclusion follows immediately from the preceding explanations. \square

19.8.4 Functorial properties

Let Y, Y' be two schemes, $q : Y' \rightarrow Y$ be a morphism, \mathcal{S} (resp. \mathcal{S}') be a quasi-coherent \mathcal{O}_Y -algebra (resp. $\mathcal{O}_{Y'}$ -algebra) with positive degrees. Consider a q -morphism of graded algebras

$$\varphi : \mathcal{S} \rightarrow \mathcal{S}'.$$

We have seen that this corresponds canonically to a morphism

$$\Phi = \text{Spec}(\varphi) = \text{Spec}(\mathcal{S}') \rightarrow \text{Spec}(\mathcal{S})$$

such that the diagram

$$\begin{array}{ccc} C' & \xrightarrow{\Phi} & C \\ \downarrow & & \downarrow \\ Y' & \xrightarrow{q} & Y \end{array} \quad (19.8.34)$$

where $C = \text{Spec}(\mathcal{S}), C' = \text{Spec}(\mathcal{S}')$, is commutative. Suppose moreover that $\mathcal{S}_0 = \mathcal{O}_Y$ and $\mathcal{S}'_0 = \mathcal{O}_{Y'}$; let $\varepsilon : Y \rightarrow C$ and $\varepsilon' : Y' \rightarrow C'$ be the canonical immersions, we then have a commutative diagram

$$\begin{array}{ccc} C' & \xrightarrow{\Phi} & C \\ \varepsilon' \uparrow & & \uparrow \varepsilon \\ Y' & \xrightarrow{q} & Y \end{array} \quad (19.8.35)$$

which corresponds to the diagram

$$\begin{array}{ccc} \mathcal{S} & \xrightarrow{\varphi} & \mathcal{S}' \\ \downarrow & & \downarrow \\ \mathcal{O}_Y & \longrightarrow & \mathcal{O}_{Y'} \end{array}$$

where the vertical are augmentation homomorphisms, and the commutativity follows from the hypotheses that φ is a homomorphism of graded algebras.

Proposition 19.8.27. *If E (resp. E') is the blunt affine cone defined by \mathcal{S} (resp. \mathcal{S}'), we have $\Phi^{-1}(E) \subseteq E'$. Moreover, the morphism $\text{Proj}(\varphi) : G(\varphi) \rightarrow \text{Proj}(\mathcal{S})$ is everywhere defined (in other words $G(\varphi) = \text{Proj}(\mathcal{S}')$ if and only if $\Phi^{-1}(E) = E'$.*

Proof. The first assertion follows from (??), since $E = C - \varepsilon(Y)$ and $E' = C' - \varepsilon'(Y')$. To prove the second assertion, we can assume that $Y = \text{Spec}(A)$ and $Y' = \text{Spec}(A')$ are affine, $\mathcal{S} = \tilde{S}$, $\mathcal{S}' = \tilde{S}'$. For f homogeneous in S_+ , if we put $f' = \varphi(f)$, we have $\Phi^{-1}(C_f) = C'_{f'}$ (??); to say that $G(\varphi) = \text{Proj}(S')$ signifies that in S'_+ the radical of the ideal generated by the $f' = \varphi(f)$ is equal to S'_+ (??), and this is equivalent to that the $C'_{f'}$ cover E' (??). \square

The q -morphism φ extends canonically to a q -morphism of graded algebras

$$\hat{\varphi} : \widehat{\mathcal{S}} \rightarrow \widehat{\mathcal{S}'}$$

which satisfies $\hat{\varphi}(z) = z$. We then deduce a morphism

$$\hat{\Phi} = \text{Proj}(\hat{\varphi}) : G(\hat{\varphi}) \rightarrow \widehat{C} = \text{Proj}(\widehat{\mathcal{S}})$$

such that the diagram

$$\begin{array}{ccc} G(\hat{\varphi}) & \xrightarrow{\hat{\Phi}} & \widehat{C} \\ \downarrow & & \downarrow \\ Y' & \xrightarrow{q} & Y \end{array}$$

is commutative. It then follows from the definition that if we denote by $i : C \rightarrow \widehat{C}$ and $i' : C' \rightarrow \widehat{C}'$ are the canonical immersions, we have $i'(C') \subseteq G(\hat{\varphi})$ and the diagram

$$\begin{array}{ccc} G(\hat{\varphi}) & \xrightarrow{\hat{\Phi}} & \widehat{C} \\ i' \uparrow & & \uparrow i \\ C' & \xrightarrow{\Phi} & C \end{array} \tag{19.8.36}$$

is commutative. Finally, if we put $P = \text{Proj}(\mathcal{S})$, $P' = \text{Proj}(\mathcal{S}')$, and if $j : P \rightarrow \widehat{C}$, $j' : P' \rightarrow \widehat{C}'$ are the canonical closed immersions, we have $j'(G(\varphi)) \subseteq G(\hat{\varphi})$ and the diagram

$$\begin{array}{ccc} G(\hat{\varphi}) & \xrightarrow{\hat{\Phi}} & \widehat{C} \\ j \uparrow & & \uparrow j \\ G(\varphi) & \xrightarrow{\text{Proj}(\varphi)} & P \end{array} \tag{19.8.37}$$

is commutative.

Proposition 19.8.28. *If \widehat{E} (resp. \widehat{E}') is the blunt projective cone defined by \mathcal{S} (resp. \mathcal{S}'), we have $\hat{\Phi}^{-1}(\widehat{E}) \subseteq \widehat{E}'$. Moreover, if $p : \widehat{E} \rightarrow P$ and $p' : \widehat{E}' \rightarrow P'$ are the canonical retractions, we have $p'(\hat{\Phi}^{-1}(\widehat{E})) \subseteq G(\varphi)$, and the diagram*

$$\begin{array}{ccc} \hat{\Phi}^{-1}(\widehat{E}) & \xrightarrow{\hat{\Phi}} & E \\ p' \downarrow & & \downarrow p \\ G(\varphi) & \xrightarrow{\text{Proj}(\varphi)} & P \end{array}$$

is commutative. If $\text{Proj}(\varphi)$ is everywhere defined, so is $\hat{\Phi}$ and we have $\hat{\Phi}^{-1}(\widehat{E}) = \widehat{E}'$.

Proof. The first assertion follows from the commutative diagrams (??) and (??), and the next two follow from the definition of the canonical retraction, the definition of $\hat{\varphi}$, and the fact that \widehat{E} is the defining domain of the morphism induced by the canonical injection $\mathcal{S} \rightarrow \widehat{\mathcal{S}}$. On the other hand, to see that $\hat{\Phi}$ is everywhere defined if $\text{Proj}(\varphi)$ is, we can assume that $Y = \text{Spec}(A)$, $Y' = \text{Spec}(A')$ are affine, $\mathcal{S} = \widetilde{S}$, $\mathcal{S}' = \widetilde{S}'$; the hypothesis is that if f runs through homogeneous elements of S_+ , the ideal in S'_+ generated by the $\varphi(f)$ has radical in S'_+ equal to S'_+ . We then conclude that the radical of the ideal generated by z and the $\varphi(f)$ in $(S'[z])_+$ is equal to $(S'[z])_+$,

whence our assertion. This proves similarly that \widehat{E}' is the union of the $\widehat{C}'_{(\varphi(f))}$, which is equal to $\Phi^{-1}(\widehat{E})$. \square

Corollary 19.8.29. *If Φ is everywhere defined, the inverse image under $\widehat{\Phi}$ of underlying space of the place of infinity (resp. the sommet scheme) of \widehat{C}' is the underlying space of the place of infinity (resp. the sommet scheme) of \widehat{C} .*

Proof. This follows from ?? and ??, in view of the relations (??) and (??). \square

19.8.5 Blunt cones over a homogeneous spectrum

Let Y be a scheme, \mathcal{S} be a quasi-coherent \mathcal{O}_Y -algebra with positive degrees such that $\mathcal{S}_0 = \mathcal{O}_Y$, and $X = \text{Proj}(\mathcal{S})$. We now apply the previous results to the structure morphism $q : X \rightarrow Y$. Let

$$\mathcal{S}_X = \bigoplus_{n \in \mathbb{Z}} \mathcal{O}_X(n) \quad (19.8.38)$$

which is a quasi-coherent \mathcal{O}_X -algebra, the multiplication γ being defined by the canonical homomorphisms

$$\mathcal{O}_X(m) \otimes \mathcal{O}_X(n) \rightarrow \mathcal{O}_X(m+n)$$

which satisfies the associativity in view of ???. Let \mathcal{S}' be the quasi-coherent sub-algebra

$$\mathcal{S}_X^{\geq} = \bigoplus_{n \geq 0} \mathcal{O}_X(n)$$

of \mathcal{S}_X , with positive degrees. For each $n \in \mathbb{Z}$, we have a canonical q -morphisms $\alpha_n : \mathcal{S}_n \rightarrow \mathcal{O}_X(n)$ defined in (??), which together give a homomorphism

$$\alpha : \mathcal{S} \rightarrow \bigoplus_{n \geq 0} q_*(\mathcal{O}_X(n)).$$

By composing with the canonical homomorphism $\bigoplus_{n \geq 0} q_*(\mathcal{O}_X(n)) \rightarrow q_*(\mathcal{S}_X^{\geq})$, this gives a q -homomorphism $\mathcal{S} \rightarrow \mathcal{S}_X^{\geq}$, still denoted by α . We set

$$C_X = \text{Spec}(\mathcal{S}_X^{\geq}), \quad \widehat{C}_X = \text{Proj}(\mathcal{S}_X^{\geq}[z]), \quad P_X = \text{Proj}(\mathcal{S}_X^{\geq})$$

and denote by E_X and \widehat{E}_X the corresponding blunt cones. We then have the canonical morphisms

$$\begin{array}{ccccc} & & \widehat{C}_X & & \\ & j_X & \swarrow & \searrow i_X & \\ P_X & & \downarrow & & C_X \\ & & \varepsilon_X & & \end{array}$$

and $p_X : \widehat{E}_X \rightarrow P_X$, $\pi_X : E_X \rightarrow P_X$.

Proposition 19.8.30. *The structural morphism $\psi : P_X \rightarrow X$ is an isomorphism, and the morphism $\text{Proj}(\alpha)$ is everywhere defined and equals to ψ . The morphism $\text{Proj}(\hat{\alpha}) : \widehat{C}_X \rightarrow \widehat{C}$ is everywhere defined and its restriction to \widehat{E}_X and E_X are isomorphisms into \widehat{E} and E , respectively. Finally, if we identify P_X and X via ψ , the morphisms p_X and π_X are identified with the structural morphisms of the X -schemes \widehat{E}_X and E_X .*

Proof. We can clearly assume that $Y = \text{Spec}(A)$ is affine and $\mathcal{S} = \widetilde{S}$; then X is the union of the affine opens X_f , where $f \in S_+$ is homogeneous, the ring of X_f being $S_{(f)}$. It follows from the isomorphism (??) that

$$\Gamma(X_f, \mathcal{S}_X^{\geqslant}) = S_f^{\geqslant}. \quad (19.8.39)$$

We then have $\psi^{-1}(X_f) = \text{Proj}(S_f^{\geqslant})$. But if $f \in S_d$ with $d > 0$, $\text{Proj}(S_f^{\geqslant})$ is canonically isomorphic to $\text{Proj}((S_f^{\geqslant})^{(d)})$ by ??, and $(S_f^{\geqslant})^{(d)} = (S^{(d)})_f^{\geqslant}$ is identified with $S_{(f)}[T]$ by the map $T \mapsto f/1$ (cf. ??), so we conclude from ?? that the structural morphism $\psi^{-1}(X_f) \rightarrow X_f$ is an isomorphism, whence the first assertion. To prove the second one, we first note that $\text{Proj}(\alpha)$ is everywhere defined by ???. Since $\psi^{-1}(X_f) = (\psi^{-1}(X_f))_{f/1}$, it follows from (??) that the image of $\psi^{-1}(X_f)$ under $\text{Proj}(\alpha)$ is contained in X_f , and the restriction of $\text{Proj}(\alpha)$ to $\psi^{-1}(X_f)$, considered as a morphism into $X_f = \text{Spec}(S_{(f)})$, is identified with ψ . Finally, the formula (??) and (??) show that $p_X^{-1}(\psi^{-1}(X_f)) = \text{Spec}((S_f^{\geqslant})_{f/1}^{\leqslant})$, and this open subset is, by ?? and formula (??), the inverse image of $p^{-1}(X_f) = \text{Spec}(S_f^{\leqslant})$ under $\text{Proj}(\hat{\alpha})$. By the isomorphism (??), the restriction of $\text{Proj}(\hat{\alpha})$ to $p_X^{-1}(\psi^{-1}(X_f))$ corresponds to the isomorphism $S_f^{\leqslant} \cong (S_f^{\geqslant})_{f/1}^{\leqslant}$, whence the third assertion. The last assertion is clear by definition. \square

We note that by (??) the restriction of $\text{Proj}(\hat{\alpha})$ to C_X is equal to $\text{Spec}(\alpha)$.

Corollary 19.8.31. *Considered as X -schemes, \widehat{E}_X is canonically isomorphic to $\text{Spec}(\mathcal{S}_X^{\leqslant})$, E_X is canonically isomorphic to $\text{Spec}(\mathcal{S}_X)$, and C_X is canonically isomorphic to $\text{Spec}(\mathcal{S}_X^{\geqslant})$.*

Proof. As we have seen that p_X and π_X are affine, it suffices to verify the corollary if $Y = \text{Spec}(A)$ is affine and $\mathcal{S} = \widetilde{S}$. The first assertion follows from the canonical isomorphism $(S_f^{\geqslant})_{f/1}^{\leqslant} \cong S_f^{\leqslant}$, which are compatible with passage from f to fg (f, g homogeneous in S_+). Similarly, the formula (??), applied to π_X , shows that $\pi_X^{-1}(\psi^{-1}(X_f)) = \text{Spec}((S_f^{\geqslant})_{f/1})$ for f homogeneous in S_+ , and the second assertion then follows from the canonical isomorphism $(S_f^{\geqslant})_{f/1} \cong S_f$. \square

We can then say that \widehat{C}_X , considered as an X -scheme, is obtained by glueing the affine X -schemes $C_X = \text{Spec}(\mathcal{S}_X^{\geqslant})$ and $\widehat{E}_X = \text{Spec}(\mathcal{S}_X^{\leqslant})$ along their intersection $E_X = \text{Spec}(\mathcal{S}_X)$.

Corollary 19.8.32. *Suppose that $\mathcal{O}_X(1)$ is an invertible \mathcal{O}_X -module and that $\mathcal{S}_X \cong \bigoplus_{n \in \mathbb{Z}} (\mathcal{O}_X(1))^{\otimes n}$ (for example if \mathcal{S} is generated by \mathcal{S}_1). Then the blunt projective cone \widehat{E} is identified with the rank one vector bundle $\mathbf{V}(\mathcal{O}_X(-1))$ over X , and the bulk affine cone E is isomorphic to the open subscheme induced over the complement of the zero section in this vector bundle. With these identifications, the canonical retraction $\widehat{E} \rightarrow X$ is identified with the structural morphism of the X -scheme $\mathbf{V}(\mathcal{O}_X(-1))$. Finally, there exists a canonical Y -morphism $\mathbf{V}(\mathcal{O}_X(1)) \rightarrow C$, whose restriction to the complement of the zero section of $\mathbf{V}(\mathcal{O}_X(1))$ is an isomorphism from this complement to the blunt affine cone E .*

Proof. In fact, if $\mathcal{L} = \mathcal{O}_X(1)$, then $\mathcal{S}_X^{\geqslant}$ is identified with $S_{\mathcal{O}_X}(\mathcal{L})$ and $\mathcal{S}_X^{\leqslant}$ is identified with $S_{\mathcal{O}_X}(\mathcal{L}^{-1})$, so \widehat{E}_X is identified with $V(\mathcal{L}^{-1})$ in view of ?? and C_X is identified with $V(\mathcal{L})$. The morphism $V(\mathcal{L}) \rightarrow C$ is the restriction of $\text{Proj}(\hat{\alpha})$, and the assertion of the corollary is a particular case of ??.

We note that the inverse image of the sommet scheme of C under the morphism $V(\mathcal{O}_X(1)) \rightarrow C$ is the zero section of $V(\mathcal{O}_X(1))$ (??). But in general the corresponding subschemes of C and of $V(\mathcal{O}_X(1))$ are not isomorphic.

19.8.6 Blow up of projective cones

With the notations of the previous subsection, we have a commutative diagram

$$\begin{array}{ccc} \widehat{C}_X & \xrightarrow{r} & \widehat{C} \\ i_X \circ \varepsilon_X \uparrow & & \uparrow i \circ \varepsilon \\ X & \xrightarrow{q} & Y \end{array}$$

where $r = \text{Proj}(\hat{\alpha})$. Moreover, the restriction of r to the complement $\widehat{C}_X - i_X(\varepsilon_X(X))$ of the sommet section is an isomorphism to $\widehat{C} - i(\varepsilon(Y))$ of the sommet section in view of ???. If we suppose for simplicity that Y is affine, \mathcal{S} is of finite type and generated by \mathcal{S}_1 , X is projective over Y and \widehat{C}_X is projective over X , so \widehat{C}_X is projective over Y (??(ii)), and a fortiori over \widehat{C} (??(v)). We thus have a projective Y -morphism $r : \widehat{C}_X \rightarrow \widehat{C}$ (hence restricts to a projective Y -morphism $C_X \rightarrow C$) which contract X to Y and induces an isomorphism when restricted to the complement of X and of Y . We therefore have a relation between C_X and C , analogous to that which takes place between a blow up scheme and its initial scheme. We will effectively show that we can identify C_X with the homogeneous spectrum of a graded \mathcal{O}_C -algebra.

For each $n \geq 0$, we consider the quasi-coherent ideal

$$\mathcal{S}(n) = \bigoplus_{m \geq n} \mathcal{S}_m$$

of the graded \mathcal{O}_Y -algebra of \mathcal{S} . It is clear that $(\mathcal{S}(n))_{n \geq 0}$ is a filtered sequence of ideals of \mathcal{S} . Consider the \mathcal{O}_C -module associated with $\mathcal{S}(n)$, which is a quasi-coherent ideal of $\mathcal{O}_C = \widetilde{\mathcal{S}}$:

$$\mathcal{I}_n = \widetilde{\mathcal{S}(n)}.$$

Then (\mathcal{I}_n) is also a filtered sequence of quasi-coherent \mathcal{O}_C -ideals, so we can consider the quasi-coherent graded \mathcal{O}_C -algebra

$$\mathcal{S}^\natural = \bigoplus_{n \geq 0} \mathcal{I}_n = \left(\bigoplus_{n \geq 0} \mathcal{S}(n) \right)^\sim.$$

Proposition 19.8.33. *There exists a canonical C -isomorphism*

$$h : C_X \rightarrow \text{Proj}(\mathcal{S}^\natural). \quad (19.8.40)$$

Proof. Suppose first that $Y = \text{Spec}(A)$ is affine, so $\mathcal{S} = \widetilde{S}$ where S is a graded A -algebra with positive degrees and $C = \text{Spec}(S)$. We then have $\mathcal{S}^\natural = (\widetilde{S^\natural})$. To define the morphism h ,

consider an element $f \in S_d$ ($d > 0$) and the corresponding element $f^\natural \in S^\natural$; the S -isomorphism (??) defines a C -isomorphism

$$\mathrm{Spec}(S_f^\geqslant) \xrightarrow{\sim} \mathrm{Spec}(S_{(f^\natural)}^\natural). \quad (19.8.41)$$

But with the notations of ??, if $\varphi : C_X \rightarrow X$ is the structural morphism, it follows from (??) that $\varphi^{-1}(X_f) = \mathrm{Spec}(S_f^\geqslant)$. We have on the other hand $D_+(f^\natural) = \mathrm{Spec}(S_{(f^\natural)}^\natural)$, so that (??) define an isomorphism $v^{-1}(X_f) \rightarrow D_+(f^\natural)$. Moreover, if $g \in S_e$ with $e > 0$, the diagram

$$\begin{array}{ccc} \varphi^{-1}(X_{fg}) & \xrightarrow{\sim} & D_+(f^\natural g^\natural) \\ \downarrow & & \downarrow \\ \varphi^{-1}(X_f) & \xrightarrow{\sim} & D_+(f^\natural) \end{array}$$

is commutative, which is clear from the definition of (??). By definition S_+ is generated by these homogeneous elements F , so it follows from ??(iv) that the $D_+(f^\natural)$ form a covering of $\mathrm{Proj}(S^\natural)$ and the $\varphi^{-1}(X_f)$ form a covering of C_X , if X_f form a covering of X . These together gives a isomorphism $h : C_X \rightarrow \mathrm{Proj}(S^\natural)$.

To prove the proposition in the general case, it suffices to see that if U, U' are two affine opens of Y such that $U' \subseteq U$, with rings A and A' , and if $\mathcal{S}|_U = \tilde{S}$, $\mathcal{S}|_{U'} = \tilde{S}'$, the diagram

$$\begin{array}{ccc} C_{U'} & \longrightarrow & \mathrm{Proj}(S^\natural) \\ \downarrow & & \downarrow \\ C_U & \longrightarrow & \mathrm{Proj}(S^\natural) \end{array} \quad (19.8.42)$$

is commutative. But S' is canonically identified with $S \otimes_A A'$, so S'^\natural is identified with $S^\natural \otimes_S S' = S^\natural \otimes_A A'$ and we then have $\mathrm{Proj}(S'^\natural) = \mathrm{Proj}(S^\natural) \times_U U'$ (??). Similarly, if $X = \mathrm{Proj}(S)$ and $X' = \mathrm{Proj}(S')$, we have $X' = X \times_U U'$ and $\mathcal{S}_{X'} = \mathcal{S}_X \otimes_{\mathcal{O}_U} \mathcal{O}_{U'}$ (??), which means $\mathcal{S}_{X'} = j^*(\mathcal{S}_X)$, where $j : X' \rightarrow X$ is the projection. By ?? we then have $C_{U'} = C_U \times_X X' = C_U \times_U U'$, and the commutativity of (??) is immediate. \square

Remark 19.8.5. The end of the reasoning of ?? is immediately generalized in the following way.: let $g : Y' \rightarrow Y$ be a morphism, $\mathcal{S}' = g^*(\mathcal{S})$, $X' = \mathrm{Proj}(\mathcal{S}')$; we then have a commutative diagram

$$\begin{array}{ccc} C_{X'} & \longrightarrow & \mathrm{Proj}(\mathcal{S}'^\natural) \\ \downarrow & & \downarrow \\ C_X & \longrightarrow & \mathrm{Proj}(\mathcal{S}^\natural) \end{array} \quad (19.8.43)$$

On the other hand, let $\varphi : \mathcal{S}'' \rightarrow \mathcal{S}$ be a homomorphism of graded \mathcal{O}_Y -algebras such that the induced morphism $\Phi = \mathrm{Proj}(\varphi) : X \rightarrow X''$ is everywhere defined, where $X'' = \mathrm{Proj}(\mathcal{S}'')$. We have an Y -morphism $v : C \rightarrow C''$ (where $C'' = \mathrm{Spec}(\mathcal{S}'')$) such that $\mathcal{A}(v) = \varphi$, and as φ is a graded homomorphism, we deduce from φ a v -morphism $\psi : \mathcal{S}''^\natural \rightarrow \mathcal{S}^\natural$ (??). Moreover, it follows from ??(iv) and the hypotheses on φ that $\Psi = \mathrm{Proj}(\psi)$ is everywhere defined. Finally, in

view of (??), we have a canonical Φ -morphism $\mathcal{S}_{X''} \rightarrow \mathcal{S}_X$, whence a morphism $w : C_{X''} \rightarrow C_X$. The diagram

$$\begin{array}{ccc} C_{X''} & \xrightarrow{\sim} & \text{Proj}(\mathcal{S}^\natural) \\ \downarrow w & & \downarrow \Psi \\ C_X & \xrightarrow{\sim} & \text{Proj}(\mathcal{S}^\natural) \end{array}$$

is commutative, as can be verified in the case where Y is affine.

Remark 19.8.6. Recall that $(\mathcal{J}_n)_{n \geq 0}$ is a filtered sequence where $\mathcal{J}_n = \mathcal{S}(n)$, so we have $\mathcal{J}_1^n \subseteq \mathcal{J}_n \subseteq \mathcal{J}_1$ for any $n > 0$. Now by definition, $\mathcal{J}_1 = \widetilde{\mathcal{S}_+}$, so \mathcal{J}_1 defines in C the closed subscheme $\varepsilon(Y)$ (?? and ??). We then conclude that for any $n > 0$, the support of $\mathcal{O}_C/\mathcal{J}_n$ is contained in the underlying space of the sommet scheme $\varepsilon(Y)$. In the inverse image of the blunt affine cone E , the structural morphism $\text{Proj}(\mathcal{S}^\natural) \rightarrow C$ reduces to an isomorphism (as it follows from ?? and ??). Moreover, if we canonically identify C as a dense open subset of \widehat{C} , we can evidently extend the ideal \mathcal{J}_n of \mathcal{O}_C to an ideal \mathcal{J}_n of $\mathcal{O}_{\widehat{C}}$, such that it coincides with $\mathcal{O}_{\widehat{C}}$ on the open subset \widehat{E} of \widehat{C} . If we put $\mathcal{T} = \bigoplus_{n \geq 0} \mathcal{J}_n$, which is a graded $\mathcal{O}_{\widehat{C}}$ -algebra, we can then extend the isomorphism (??) into a \widehat{C} -isomorphism

$$\widehat{C}_X \xrightarrow{\sim} \text{Proj}(\mathcal{T}). \quad (19.8.44)$$

In fact, over \widehat{E} , it follows from the definition of \mathcal{J} that $\text{Proj}(\mathcal{T})$ is identified with \widehat{E} , and we therefore define the isomorphism (??) so that it coincides with the canonical isomorphism $\widehat{E}_X \rightarrow \widehat{E}$ on \widehat{E} (??); it is then clear that this isomorphism and (??) coincides over \widehat{E} .

Corollary 19.8.34. Suppose that there exists $n_0 > 0$ such that $\mathcal{S}_{n+1} = \mathcal{S}_1 \mathcal{S}_n$ for $n \geq n_0$. Then the sommet scheme of C_X (isomorphic to X) is the inverse image of the sommet scheme of C (isomorphic to Y) under the canonical morphism $r = \text{Proj}(\alpha) : C_X \rightarrow C$. The converse of this is true if moreover Y is Noetherian and \mathcal{S} is of finite type.

Proof. The first assertion is local over Y , so we can assume that $Y = \text{Spec}(A)$ is affine, $\mathcal{S} = \widetilde{S}$ where S is a graded A -algebra with positive degrees. This then follows from ??, because we have

$$\text{Proj}(S^\natural \otimes_S S_0) = \text{Proj}(S^\natural \otimes_S (S/S_+)) = C_X \times_C \varepsilon(Y)$$

(in view of the identification (??) and ??), which is also the inverse image of $\varepsilon(Y)$ in C_X under the morphism $r : C_X \rightarrow C$. The converse of this also follows from ?? if Y is Noetherian and affine and S is of finite type. If Y is Noetherian (not necessarily affine) and \mathcal{S} is of finite type, there exists a finite covering of Y by Noetherian affine covers U_i , and we then deduce that for each i , there is an integer n_i such that $\mathcal{S}_{n+1}|_{U_i} = (\mathcal{S}_1|_U)(\mathcal{S}_n|_U)$ for $n \geq n_i$; the largest integer n_0 of the n_i then satisfies the requirement. \square

We now consider the C -scheme \widetilde{C} which is obtained by blowing up the affine cone C along the sommet scheme $\varepsilon(Y)$. By definition this is $\text{Proj}(\bigoplus_{n \geq 0} (\mathcal{S}_+)^n)$; the canonical injection

$$\iota : \bigoplus_{n \geq 0} (\mathcal{S}_+)^n \rightarrow \mathcal{S}^\natural$$

defines (by the identification of ??) a canonical dominant C -morphism $\text{Proj}(\iota) : G(\iota) \rightarrow \tilde{C}$, where $G(\iota)$ is an open subset of C_X . We note that it is possible that $G(\iota) \neq C_X$; for example, $Y = \text{Spec}(k)$ where k is a field, $\mathcal{S} = \tilde{S}$ where $S = k[y]$ and y is an indeterminate of degree 2. If R_n is the set $(S_+)^n$, considered as a subset of $S(n) = S_n^\sharp$, then S_+^\sharp is not equal to the radical in S_+^\sharp of the ideal generated by the R_n .

Corollary 19.8.35. *Suppose that there exists $n_0 > 0$ such that $\mathcal{S}_n = \mathcal{S}_1^n$ for $n \geq n_0$. Then the canonical morphism $\text{Proj}(\iota) : G(\iota) \rightarrow \tilde{C}$ is everywhere defined and an isomorphism from C_X to \tilde{C} . The converse of this is also true if moreover Y is Noetherian and \mathcal{S} is of finite type.*

Proof. This assertion is local over Y , and therefore follows from ?? . The converse of this is also true if Y is Noetherian and \mathcal{S} is of finite type, as can be shown similarly to ?? . \square

Remark 19.8.7. As the condition of ?? implies that of ??, we see that if this condition is verified, not only C_X is identified with the scheme obtained by blowing up C along the sommet scheme (isomorphic to Y), but also the sommet scheme of C_X (isomorphic to X) is identified with the inverse image of the sommet scheme of C in C_X . Moreover, the hypothesis of ?? implies that over $X = \text{Proj}(\mathcal{S})$, the \mathcal{O}_X -modules $\mathcal{O}_X(n)$ are invertible ?? and we have $\mathcal{O}_X(n) = \mathcal{L}^{\otimes n}$, where $\mathcal{L} = \mathcal{O}_X(1)$?? . By definition C_X is then the vector bundle $V(\mathcal{L})$ over X , and the sommet scheme is the zero section of this vector bundle.

19.8.7 Ample sheaves and contractions

Let Y be a scheme, $f : X \rightarrow Y$ be a quasi-compact and quasi-separated morphism \mathcal{L} be an ample invertible \mathcal{O}_X -module relative to f . Consider the graded \mathcal{O}_Y -algebra with positive degrees

$$\mathcal{S} = \mathcal{O}_Y \oplus \bigoplus_{n \geq 1} f_*(\mathcal{L}^{\otimes n})$$

which is quasi-coherent by ?? . We have a canonical homomorphism of graded \mathcal{O}_X -algebras

$$\sigma : f^*(\mathcal{S}) \rightarrow \bigoplus_{n \geq 0} \mathcal{L}^{\otimes n}$$

which, for each $n \geq 1$, coincides with the canonical homomorphism $\sigma_n : f^*(f_*(\mathcal{L}^{\otimes n})) \rightarrow \mathcal{L}^{\otimes n}$, and for $n = 0$ is the identity on \mathcal{O}_X . The hypothesis that \mathcal{L} is f -ample implies that ?? and ?? the corresponding Y -morphism

$$r = r_{\mathcal{L}, \sigma} : X \rightarrow P = \text{Proj}(\mathcal{S})$$

is everywhere defined and a dominant open immersion, and we have $\mathcal{L}^{\otimes n} = r^*(\mathcal{O}_P(n))$ for $n \in \mathbb{Z}$.

Proposition 19.8.36. *Let $C = \text{Spec}(\mathcal{S})$ the affine cone defined by \mathcal{S} . If \mathcal{L} is f -ample, there exists a canonical Y -morphism*

$$g : V = V(\mathcal{L}) \rightarrow C \tag{19.8.45}$$

such that the diagram

$$\begin{array}{ccccc}
 X & \xrightarrow{j} & V(\mathcal{L}) & \xrightarrow{\pi} & X \\
 f \downarrow & & g \downarrow & & \downarrow f \\
 Y & \xrightarrow{\varepsilon} & C & \xrightarrow{\psi} & Y
 \end{array} \tag{19.8.46}$$

is commutative, where ψ and π are structural morphisms, j and ε are the canonical immersions which maps X and Y respectively to the zero section of $V(\mathcal{L})$ and the sommet scheme of C . Moreover, the restriction of g to $V(\mathcal{L}) - j(X)$ is an open immersion

$$V(\mathcal{L}) - j(X) \rightarrow E = C - \varepsilon(Y)$$

into the blunt affine cone E corresponding to \mathcal{S} .

19.8.8 Quasi-coherent sheaves over the projective cone

Let Y be a scheme, \mathcal{S} be a quasi-coherent graded \mathcal{O}_Y -algebra, $X = \text{Proj}(\mathcal{S})$, $C = \text{Spec}(\mathcal{S})$ and $\widehat{C} = \text{Proj}(\widehat{\mathcal{S}})$. Let \mathcal{M} be a quasi-coherent graded \mathcal{S} -module; to avoid any possible confusion, we denote by $\tilde{\mathcal{M}}$ the quasi-coherent \mathcal{O}_C -module associated with \mathcal{M} if \mathcal{M} is considered as a *nongraded* \mathcal{S} -module, and by $\text{Proj}_0(\mathcal{M})$ the quasi-coherent \mathcal{O}_X -module associated with \mathcal{M} , where \mathcal{M} is considered as a graded \mathcal{S} -module. We also set

$$\mathcal{M}_X = \text{Proj}(\mathcal{M}) = \bigoplus_{n \in \mathbb{Z}} \text{Proj}_0(\mathcal{M}(n));$$

with the quasi-coherent \mathcal{O}_X -algebra being defined by (??), $\text{Proj}(\mathcal{M})$ is endowed a quasi-coherent graded \mathcal{S}_X -module structure, via the canonical homomorphisms

$$\mathcal{O}_X(m) \otimes_{\mathcal{O}_X} \text{Proj}_0(\mathcal{M}(n)) \rightarrow \text{Proj}_0(\mathcal{S}(m) \otimes_{\mathcal{S}} \mathcal{M}(n)) \rightarrow \text{Proj}_0(\mathcal{M}(m+n))$$

which satisfies the axioms of modules in view of the commutative diagram (??). If $Y = \text{Spec}(A)$ is affine, $\mathcal{S} = \widetilde{S}$ and $\mathcal{M} = \widetilde{M}$, where S is a graded A -algebra and M is a graded S -module, then for any homogeneous element $f \in S_+$, we have

$$\Gamma(X_f, \text{Proj}(\mathcal{M})) = M_f$$

in view of the definition and (??).

Now consider the quasi-coherent graded $\widehat{\mathcal{S}}$ -module

$$\widehat{\mathcal{M}} = \mathcal{M} \otimes_{\mathcal{S}} \widehat{\mathcal{S}}$$

(where $\widehat{\mathcal{S}} = \mathcal{S}[T] = \mathcal{S} \otimes_{\mathbb{Z}} \mathbb{Z}[T]$); we then deduce a quasi-coherent graded $\mathcal{O}_{\widehat{C}}$ -module $\text{Proj}_0(\widehat{\mathcal{M}})$ (recall that $\widehat{C} = \text{Proj}(\widehat{\mathcal{S}})$), which we also denote by

$$\mathcal{M}^{\square} = \text{Proj}_0(\widehat{\mathcal{M}}).$$

It is clear that \mathcal{M}^{\square} is an exact functor on \mathcal{M} and commutes with inductive limits and direct sums.

Proposition 19.8.37. *With the notations of ?? and ??, we have canonical homomorphisms*

$$i^*(\mathcal{M}^\square) \xrightarrow{\sim} \tilde{\mathcal{M}}, \quad (19.8.47)$$

$$j^*(\mathcal{M}^\square) \rightarrow \mathcal{P}roj_0(\mathcal{M}), \quad (19.8.48)$$

$$p^*(\mathcal{P}roj_0(\mathcal{M})) \rightarrow \mathcal{M}^\square|_{\hat{E}} \quad (19.8.49)$$

Moreover, the homomorphism ?? is an isomorphism if \mathcal{S} is generated by \mathcal{S}_1 .

Proof. In fact, $i^*(\mathcal{M}^\square)$ is canonically identified with $(\hat{\mathcal{M}}/(z-1)\hat{\mathcal{M}})^\sim$ over $\text{Spec}(\hat{\mathcal{S}}/(z-1)\hat{\mathcal{S}})$ in view of ?? and the definition of i . The first isomorphism of ?? is then deduced from ?? and the canonical isomorphism $\hat{\mathcal{M}}/(z-1)\hat{\mathcal{M}} \cong \mathcal{M}$. On the other hand, the canonical immersion $j : X \rightarrow \hat{C}$ corresponds to the canonical homomorphism $\hat{\mathcal{S}} \rightarrow \mathcal{S}$ with kernel $z\hat{\mathcal{S}}$, so the second isomorphism is a particular case of the canonical homomorphism of ??, using the fact that $\hat{\mathcal{M}} \otimes_{\hat{\mathcal{S}}} \mathcal{S} = \mathcal{M}$. Finally, the homomorphism of ?? is a particular case of the homomorphisms v^\sharp defined in ???. If $Y = \text{Spec}(A)$ is affine, $\mathcal{S} = \tilde{S}$, $\mathcal{M} = \tilde{M}$, then we see from ?? that the restriction of ?? to $p^{-1}(X_f) = \hat{C}_f$ (for $f \in S_+$ homogeneous) corresponds to the canonical homomorphism

$$M_{(f)} \otimes_{S_{(f)}} S_f^\leq \rightarrow M_f^\leq$$

in view of ?? and ???. The last assertion also follows from ??.

□

By abuse of language, we say that \mathcal{M}^\square is the projective closure of the \mathcal{O}_C -module $\tilde{\mathcal{M}}$, where \mathcal{M} is understood to be a graded \mathcal{S} -module.

Let us consider a morphism $q : Y' \rightarrow Y$ and a q -homomorphism $\varphi : \mathcal{S} \rightarrow \mathcal{S}'$. It then follows from ?? that for any quasi-coherent graded \mathcal{S} -module \mathcal{M} , we have a canonical isomorphism

$$\Phi^*(\tilde{\mathcal{M}}) \xrightarrow{\sim} (q^*(\mathcal{M}) \otimes_{q^*(\mathcal{S})} \mathcal{S}')^\sim$$

of \mathcal{O}_C -modules, where $\Phi = \text{Spec}(\varphi)$. On the other hand, if $w = \text{Proj}(\varphi)$ and $\hat{\Phi} = \text{Proj}(\hat{\varphi})$, ?? gives a canonical w -homomorphism

$$\mathcal{P}roj_0(\mathcal{M}) \rightarrow (\mathcal{P}roj_0(q^*(\mathcal{M}) \otimes_{q^*(\mathcal{S})} \mathcal{S}'))|_{G(\varphi)} \quad (19.8.50)$$

and also a canonical $\hat{\Phi}$ -morphism

$$\mathcal{P}roj_0(\tilde{\mathcal{M}}) \rightarrow (\mathcal{P}roj_0(q^*(\tilde{\mathcal{M}}) \otimes_{q^*(\hat{\mathcal{S}})} \hat{\mathcal{S}}'))|_{G(\hat{\varphi})}. \quad (19.8.51)$$

Now we consider the situation of the structural morphism $q : X \rightarrow Y$, where $X = \text{Proj}(\mathcal{S})$, with the canonical q -homomorphism $\alpha : \mathcal{S} \rightarrow \mathcal{S}_X^\geq$. We then have a canonical isomorphism

$$q^*(\mathcal{M}) \otimes_{q^*(\mathcal{S})} \mathcal{S}_X^\geq \xrightarrow{\sim} \mathcal{M}_X^\geq \quad (19.8.52)$$

where $\mathcal{M}_X^\geq = \bigoplus_{n \geq 0} \mathcal{P}roj_0(\mathcal{M}(n))$. To see this, we can assume that $Y = \text{Spec}(A)$ is affine, $\mathcal{S} = \tilde{S}$ and $\mathcal{M} = \tilde{M}$, and define the isomorphism ?? in each affine open X_f (f is homogeneous in S_+), and verify the compatibility when passing to a homogeneous multiple of f . Now, the restriction of the left side of ?? to X_f is $\tilde{M}' = ((M \otimes_A S_{(f)}) \otimes_{S \otimes_A S_{(f)}} S_f^\geq)^\sim$ by ???. As we have a canonical isomorphism $M \otimes_A S_{(f)} \cong M \otimes_S (S \otimes_A S_{(f)})$, we conclude that $\tilde{M}' \cong (M \otimes_S S_f^\geq)^\sim$,

and this is canonically isomorphic to the restriction of \mathcal{M}_X^{\geq} by (??). The compatibility of this isomorphism with restrictions is clear.

By replace \mathcal{M} by $\widehat{\mathcal{M}}$, \mathcal{S} by $\widehat{\mathcal{S}}$ and \mathcal{S}_X by $(\mathcal{S}_X^{\geq})^{\wedge}$ in the preceding arguments, we obtain similarly a canonical isomorphism

$$q^*(\widehat{\mathcal{M}}) \otimes_{q^*(\widehat{\mathcal{S}})} (\mathcal{S}_X^{\geq})^{\wedge} \xrightarrow{\sim} (\mathcal{M}_X^{\geq})^{\wedge} \quad (19.8.53)$$

If we recall ?? that the structural morphism $\psi : \text{Proj}(\mathcal{S}_X^{\geq}) \rightarrow X$ is an isomorphism, we then deduce a canonical ψ -isomorphism

$$\text{Proj}_0(\mathcal{M}) \xrightarrow{\sim} \text{Proj}_0(\mathcal{M}_X^{\geq}) \quad (19.8.54)$$

as a particular case of (??). In fact, we observe that, in the notations of ??, that this reduces to the fact that canonical homomorphism $M_{(f)} \otimes_{S_{(f)}} (S_f^{\geq})^{(d)} \rightarrow (M_f^{\geq})^{(d)}$ is an isomorphism if $f \in S_d$ is homogeneous, which is immediate.

The isomorphism (??) permits us, by apply (??) to the canonical morphism $r = \text{Proj}(\hat{\alpha}) : \widehat{C}_X \rightarrow \widehat{C}$, to obtain a canonical r -homomorphism

$$\mathcal{M}^{\square} \rightarrow (\mathcal{M}_X^{\geq})^{\square}. \quad (19.8.55)$$

Now recall that the restrictions of r to the blunt cones \widehat{E}_X and E_X are isomorphisms onto \widehat{E} and E , respectively.

Proposition 19.8.38. *The restriction of the canonical r -homomorphism (??) to \widehat{E}_X and to E_X are isomorphisms*

$$\mathcal{M}^{\square}|_{\widehat{E}} \xrightarrow{\sim} (\mathcal{M}_X^{\geq})^{\square}|_{\widehat{E}_X}, \quad (19.8.56)$$

$$\mathcal{M}^{\square}|_E \xrightarrow{\sim} (\mathcal{M}_X^{\geq})^{\square}|_{E_X}. \quad (19.8.57)$$

Proof. We can assume that $Y = \text{Spec}(A)$ is affine as in the proof of ??; with the notations there, we must show that the canonical homomorphism

$$\widehat{M}_{(f)} \otimes_{\widehat{S}_{(f)}} (S_f^{\geq})_{(f/1)}^{\wedge} \rightarrow (M \otimes_S S_f^{\geq})_{(f/1)}^{\wedge}$$

is an isomorphism. But in view of (??) and (??), the left side is canonically identified with $M_f^{\geq} \otimes_{S_f^{\geq}} (S_f^{\geq})_{f/1}^{\leq}$, hence with M_f^{\geq} in view of (??); the right side is identified with $(M_f^{\geq})_{f/1}^{\leq}$, hence also to M_f^{\geq} by (??), whence our assertion about (??). The isomorphism (??) then follows from (??) and (??). \square

Corollary 19.8.39. *With the notations of ??, the restriction of $(\mathcal{M}_X^{\geq})^{\square}$ to \widehat{E}_X is identified with $\widetilde{\mathcal{M}_X^{\leq}}$ and its restriction to E_X is identified with $\widetilde{\mathcal{M}_X}$.*

Proof. We can clearly reduce to the affine case, and this follows from the identification of $(M_f^{\geq})_{f/1}^{\leq}$ with M_f^{\leq} and $(M_f^{\geq})_{f/1}$ with M_f (cf. (??)). \square

Proposition 19.8.40. *Under the hypotheses of ??, the canonical homomorphism (??) is an isomorphism.*

Proof. In view of the fact that the structural morphism $\text{Proj}(\mathcal{S}_X^{\geq}) \rightarrow X$ is an isomorphism and the isomorphisms (??) and (??), we only need to prove that the canonical homomorphism

$p_X^*(\mathcal{P}roj_0(\mathcal{M}_X^\geq)) \rightarrow (\mathcal{M}_X^\geq)^\square|_{E_X}$ is an isomorphism, which means that we can assume that \mathcal{S}_1 is an invertible \mathcal{O}_Y -module and \mathcal{S} is generated by \mathcal{S}_1 . With the notations of ??, we then have, for $f \in S_1$, $S_f^\leq = S_{(f)}[1/f]$ and the canonical isomorphism $M_{(f)} \otimes_{S_{(f)}} S_f^\leq \rightarrow M_f^\leq$ is an isomorphism by the definition of M_f^\geq . \square

We now consider the quasi-coherent \mathcal{S} -module $\mathcal{M}(n) = \bigoplus_{m \geq n} \mathcal{M}_m$ and the quasi-coherent graded \mathcal{S}^\natural -module

$$\mathcal{M}^\natural = \left(\bigoplus_{n \geq 0} \mathcal{M}(n) \right)^\sim.$$

By ?? we have a canonical C -isomorphism $h : C_X \xrightarrow{\sim} \text{Proj}(\mathcal{S}^\natural)$.

Proposition 19.8.41. *There exists a canonical h -isomorphism*

$$\mathcal{P}roj_0(\mathcal{M}^\natural) \xrightarrow{\sim} \widetilde{\mathcal{M}_X}. \quad (19.8.58)$$

Proof. This can be proved as ??, by using the bi-isomorphism ?? here instead of ??.

\square

Chapter 20

Cohomology of coherent sheaves over schemes

20.1 Cohomology of affine schemes

20.1.1 Čech cohomology and Koszul complex

Let X be a quasi-compact and quasi-separated scheme and \mathcal{F} be a quasi-coherent \mathcal{O}_X -module. Let $A = \Gamma(X, \mathcal{O}_X)$, $M = \Gamma(X, \mathcal{F})$, $f = (f_i)_{1 \leq i \leq r}$ be a family of elements of A , and $U_i = X_{f_i}$ be the open subset of X . Let $U = \bigcup_{i=1}^r U_i$ and $\mathfrak{U} = (U_i)$ be the covering of U . For any sequence $(i_0, i_1, \dots, i_p) \in I^{p+1}$ with $I = \{1, \dots, r\}$, we set

$$U_{i_0 \dots i_p} = \bigcap_{j=0}^p U_{i_j} = X_{f_{i_0} \dots f_{i_p}}.$$

By ??, we have $\Gamma(U_{i_0 \dots i_p}, \mathcal{F}) = M_{f_{i_0} \dots f_{i_p}}$, where $M = \Gamma(X, \mathcal{F})$. Note that the localization module $M_{f_{i_0} \dots f_{i_p}}$ is identified with the limit of the inductive system $\{M_{i_0 \dots i_p}^n\}_{n \geq 0}$, where $M_{i_0 \dots i_p}^n = M$ and the homomorphism $\varphi_{nm} : M_{i_0 \dots i_p}^m \rightarrow M_{i_0 \dots i_p}^n$ is given by multiplication by $(f_{i_0} \dots f_{i_p})^{n-m}$ for $m \leq n$.

For any $n \geq 0$, let $C_n^{p+1}(M)$ be the set of alternating maps from I^{p+1} to M , and consider the inductive system formed by these A -modules and the homomorphisms induced by φ_{nm} . If $C^p(\mathfrak{U}, \mathcal{F})$ is the group of alternating Čech p -cochains relative to the covering \mathfrak{U} with coefficients in \mathcal{F} , then we see that

$$C^p(\mathfrak{U}, \mathcal{F}) = \varinjlim_n C_n^{p+1}(M).$$

On the other hand, from the definition of $C_n^{p+1}(M)$ it is easy to see that it is canonically identified with the Koszul complex $K^{p+1}(f^n, M)$, and the homomorphism φ_{nm} is identified with the map

$$\varphi_{f^{n-m}} : K^\bullet(f^n, M) \rightarrow K^\bullet(f^m, M)$$

induced by the map $(x_1, \dots, x_r) \mapsto (f_1^{n-m}x_1, \dots, f_r^{n-m}x_r)$ on A^r . We then have, for any $p \geq 0$, a functorial isomorphism

$$C^p(\mathfrak{U}, \mathcal{F}) \xrightarrow{\sim} C^{p+1}(f, M), \tag{20.1.1}$$

where \mathfrak{f} is the ideal generated by f . Moreover, the definition of the differentials of $C^p(\mathfrak{U}, \mathcal{F})$ and $C_n^p(M)$ show that the isomorphism (??) is in fact a morphism of complexes.

Proposition 20.1.1. *If X is a quasi-compact and quasi-separated scheme, there exists a canonical functorial isomorphism*

$$H^p(\mathfrak{U}, \mathcal{F}) \xrightarrow{\sim} H^{p+1}(\mathfrak{f}, M) \text{ for } p \geq 1, \quad (20.1.2)$$

where \mathfrak{f} is the ideal generated by f . Moreover, we have a functorial exact sequence

$$0 \longrightarrow H^0(\mathfrak{f}, M) \longrightarrow M \longrightarrow H^0(\mathfrak{U}, \mathcal{F}) \longrightarrow H^1(\mathfrak{f}, M) \longrightarrow 0 \quad (20.1.3)$$

Proof. The relation (??) is in fact a consequence of (??). On the other hand, we have $C^0(\mathfrak{U}, \mathcal{F}) = C^1(\mathfrak{f}, M)$, so $H^0(\mathfrak{U}, \mathcal{F})$ is identified with a subgroup of 1-cocycles of $C^1(\mathfrak{f}, M)$. As $C^0(\mathfrak{f}, M) = M$, the exact sequence (??) follows from the definition of $H^0(\mathfrak{f}, M)$ and $H^1(\mathfrak{f}, M)$. \square

Corollary 20.1.2. *Suppose that the X_{f_i} are quasi-compact and there exists $g_i \in \Gamma(U, \mathcal{F})$ such that $\sum_i g_i(f_i|_U) = 1|_U$. Then for any quasi-coherent $(\mathcal{O}_X|_U)$ -module \mathcal{G} , we have $H^p(\mathfrak{U}, \mathcal{G}) = 0$ for $p > 0$. If moreover $U = X$, then the canonical homomorphism $M \rightarrow H^0(\mathfrak{U}, \mathcal{F})$ in (??) is bijective.*

Proof. By hypothesis $U_i = X_{f_i}$ is quasi-compact, so U is quasi-compact, and we can assume that $U = X$. Then the hypothesis implies that $\mathfrak{f} = A$, so by ?? we have $H^p(\mathfrak{f}, M) = 0$ for $p \geq 1$, and the corollary follows from (??) and (??). \square

Remark 20.1.1. Let X be an affine scheme, so that the $U_i = X_{f_i} = D(f_i)$ are affine opens, and so is each $U_{i_0 \dots i_p}$ (but U is not necessarily affine). In this case, the functors $\Gamma(X, \mathcal{F})$ and $\Gamma(U_{i_0 \dots i_p}, \mathcal{F})$ are exact by ?. If we have an exact sequence $0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow 0$ of quasi-coherent \mathcal{O}_X -modules, the sequence of complexes

$$0 \longrightarrow C^\bullet(\mathfrak{U}, \mathcal{F}') \longrightarrow C^\bullet(\mathfrak{U}, \mathcal{F}) \longrightarrow C^\bullet(\mathfrak{U}, \mathcal{F}'') \longrightarrow 0$$

is exact, so we obtain a long exact sequence of cohomology groups

$$\dots \longrightarrow H^p(\mathfrak{U}, \mathcal{F}') \longrightarrow H^p(\mathfrak{U}, \mathcal{F}) \longrightarrow H^p(\mathfrak{U}, \mathcal{F}'') \xrightarrow{\delta} H^{p+1}(\mathfrak{U}, \mathcal{F}') \longrightarrow 0$$

On the other hand, if we put $M' = \Gamma(X, \mathcal{F}')$, $M'' = \Gamma(X, \mathcal{F}'')$, $M = \Gamma(X, \mathcal{F})$, then the sequence $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ is exact; as $C^\bullet(\mathfrak{f}, M)$ is an exact functor on M , we then get a long exact sequence on cohomology

$$\dots \longrightarrow H^p(\mathfrak{f}, M') \longrightarrow H^p(\mathfrak{f}, M) \longrightarrow H^p(\mathfrak{f}, M'') \xrightarrow{\delta} H^{p+1}(\mathfrak{f}, M') \longrightarrow 0$$

Now as the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & C^\bullet(\mathfrak{U}, \mathcal{F}') & \longrightarrow & C^\bullet(\mathfrak{U}, \mathcal{F}) & \longrightarrow & C^\bullet(\mathfrak{U}, \mathcal{F}'') \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & C^\bullet(\mathfrak{f}, M') & \longrightarrow & C^\bullet(\mathfrak{f}, M) & \longrightarrow & C^\bullet(\mathfrak{f}, M'') \longrightarrow 0 \end{array}$$

is commutative, we conclude that the diagram

$$\begin{array}{ccc} H^p(\mathfrak{U}, \mathcal{F}'') & \xrightarrow{\delta} & H^{p+1}(\mathfrak{U}, \mathcal{F}') \\ \downarrow & & \downarrow \\ H^{p+1}(\mathfrak{f}, M'') & \xrightarrow{\delta} & H^{p+2}(\mathfrak{f}, M') \end{array} \quad (20.1.4)$$

is commutative for any $p > 0$.

20.1.2 Cohomology of affine schemes

Theorem 20.1.3. *Let X be an affine scheme. For any quasi-coherent \mathcal{O}_X -module \mathcal{F} , we have $H^p(X, \mathcal{F}) = 0$ for $p > 0$.*

Proof. Let \mathfrak{U} be a finite covering of X by affine opens $X_{f_i} = D(f_i)$ ($1 \leq i \leq r$); then the ideal generated by f_i is equal to $A = \Gamma(X, \mathcal{O}_X)$. We then conclude from ?? that we have $H^p(\mathfrak{U}, \mathcal{F}) = 0$ for any $p > 0$. As affine opens form a basis for X , we then conclude from the definition of the Čech cohomology that $\check{H}^p(X, \mathcal{F}) = 0$ for any $p > 0$. But this is also applicable on X_f for $f \in A$, so $\check{H}^p(X_f, \mathcal{F}) = 0$ for $p > 0$; as $X_f \cap X_g = X_{fg}$, we then conclude from Leray's vanishing theorem that $H^p(X, \mathcal{F}) = 0$ for $p > 0$. \square

Corollary 20.1.4. *Let $f : X \rightarrow Y$ be an affine morphism. Then for any quasi-coherent \mathcal{O}_X -module \mathcal{F} , we have $R^p f_*(\mathcal{F}) = 0$ for $p > 0$.*

Proof. By definition $R^p f_*(\mathcal{F})$ is defined to be the sheaf associated with the presheaf $U \mapsto H^p(f^{-1}(U), \mathcal{F})$, where U runs through open subsets of Y . Now the affine opens U form a basis for Y , and for such U , $f^{-1}(U)$ is affine, so $H^p(f^{-1}(U), \mathcal{F}) = 0$ by ??, so we conclude that $R^p f_*(\mathcal{F}) = 0$. \square

Corollary 20.1.5. *Let $f : X \rightarrow Y$ be an affine morphism. Then for any quasi-coherent \mathcal{O}_X -module \mathcal{F} , the canonical homomorphism $H^p(Y, f_*(\mathcal{F})) \rightarrow H^p(X, \mathcal{F})$ is bijective for any p .*

Proof. Consider the Leray spectral sequence

$$E_2^{p,q} = (R^p \Gamma \circ R^q f_*)(\mathcal{F}) = H^p(Y, R^q f_*(\mathcal{F})),$$

it follows from ?? that $E_2^{p,q} = 0$ for $q > 0$, so this sequence collapses at E_2 page, whence our assertion. \square

Corollary 20.1.6. *Let $f : X \rightarrow Y$ be an affine morphism and $g : Y \rightarrow Z$ be a morphism. Then for any quasi-coherent \mathcal{O}_X -module \mathcal{F} , the canonical homomorphism $R^p g_*(f_*(\mathcal{F})) \rightarrow R^p(g \circ f)_*(\mathcal{F})$ is bijective for any p .*

Proof. It suffices to remark that, by ??, for any affine open W of Z , the canonical homomorphism $H^p(g^{-1}(W), f_*(\mathcal{F})) \rightarrow H^p(f^{-1}(g^{-1}(W)), \mathcal{F})$ is bijective; this homomorphism of presheaves then defines a canonical homomorphism $R^p g_*(f_*(\mathcal{F})) \rightarrow R^p(g \circ f)_*(\mathcal{F})$ which is bijective. \square

20.1.3 Applications to cohomology of schemes

Proposition 20.1.7. *Let X be a separated scheme, $\mathfrak{U} = (U_\alpha)$ be a covering of X by affine opens. For any quasi-coherent \mathcal{O}_X -module \mathcal{F} , the cohomology module $H^\bullet(X, \mathcal{F})$ and $H^\bullet(\mathfrak{U}, \mathcal{F})$ (over $\Gamma(X, \mathcal{O}_X)$) are canonically isomorphic.*

Proof. In fact, as X is separated, any finite intersection V of open sets in the covering \mathfrak{U} is affine (??), so $H^q(V, \mathcal{F}) = 0$ for $q > 0$ in view of ?? . The proposition then follows from Leray's vanishing theorem. \square

Remark 20.1.2. We note that the conclusion of ?? is still valid if the finite intersections of U_α are affine, even if X is not necessarily separated.

Corollary 20.1.8. *Let X be a quasi-compact and separated scheme, $A = \Gamma(X, \mathcal{O}_X)$, $\mathbf{f} = (f_i)_{1 \leq i \leq r}$ be a sequence of elements of A such that the X_{f_i} are affine. Then for any quasi-coherent \mathcal{O}_X -module \mathcal{F} , we have a canonical functorial isomorphism*

$$H^p(U, \mathcal{F}) \xrightarrow{\sim} H^{p+1}(\mathfrak{f}, M) \text{ for } p > 0 \quad (20.1.5)$$

where \mathfrak{f} is the ideal generated by \mathbf{f} . Moreover, we have a functorial exact sequence

$$0 \longrightarrow H^0(\mathfrak{f}, M) \longrightarrow M \longrightarrow H^0(U, \mathcal{F}) \longrightarrow H^1(\mathfrak{f}, M) \longrightarrow 0 \quad (20.1.6)$$

If X is an affine scheme, it then follows from ?? and ?? that for any $q \geq 0$, the diagram

$$\begin{array}{ccc} H^p(U, \mathcal{F}'') & \xrightarrow{\delta} & H^{p+1}(U, \mathcal{F}') \\ \downarrow & & \downarrow \\ H^{p+1}(\mathfrak{f}, M'') & \xrightarrow{\delta} & H^{p+2}(\mathfrak{f}, M') \end{array} \quad (20.1.7)$$

corresponding to an exact sequence $0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow 0$ of quasi-coherent \mathcal{O}_X -modules, is commutative.

Proposition 20.1.9. *Let X be a quasi-compact and separated scheme, \mathcal{L} be an invertible \mathcal{O}_X -module, and consider the graded ring $A_* = \Gamma_*(\mathcal{L})$. Then $H^\bullet(\mathcal{F}, \mathcal{L}) = \bigoplus_{n \in \mathbb{Z}} H^\bullet(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n})$ is a graded A_* -module, and for any $f \in (A_*)_n$, we have a canonical isomorphism*

$$H^\bullet(X_f, \mathcal{F}) \xrightarrow{\sim} (H^\bullet(\mathcal{F}, \mathcal{L}))_{(f)} \quad (20.1.8)$$

of $(A_*)_{(f)}$ -modules.

Proof. As X is quasi-compact and separated, we can compute the cohomology of \mathcal{O}_X -module $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ using the same finite covering $\mathfrak{U} = (U_i)$ by open affine subsets such that $\mathcal{L}|_{U_i}$ is isomorphic to $\mathcal{O}_X|_{U_i}$ for each i (??). Also, since each $U_i \cap X_f$ is open and affine, we can also compute $H^\bullet(X_f, \mathcal{F} \otimes \mathcal{L}^{\otimes n})$ using the covering $\mathfrak{U}|_{X_f} = (U_i \cap X_f)$. Now for any $f \in A_n$, it is immediate that the multiplication by f defines a homomorphism $C^\bullet(\mathfrak{U}, \mathcal{F} \otimes \mathcal{L}^{\otimes m}) \rightarrow C^\bullet(\mathfrak{U}, \mathcal{F} \otimes \mathcal{L}^{\otimes(m+n)})$, whence a homomorphism $H^\bullet(\mathfrak{U}, \mathcal{F} \otimes \mathcal{L}^{\otimes m}) \rightarrow H^\bullet(\mathfrak{U}, \mathcal{F} \otimes \mathcal{L}^{\otimes(m+n)})$.

On the other hand, by ?? and ?? (ii), any $f \in A_n$ gives an isomorphism of $(A_*)_f$ -modules

$$C^\bullet(\mathfrak{U}|_{X_f}, \mathcal{F}) \xrightarrow{\sim} (C^\bullet(\mathfrak{U}, \bigoplus_{n \in \mathbb{Z}} \mathcal{F} \otimes \mathcal{L}^{\otimes n}))_{(f)}.$$

By passing to cohomologies, we then deduce the desired isomorphism ??, using the fact that $M \mapsto M_{(f)}$ is an exact functor on the category of graded modules. \square

Corollary 20.1.10. *If $A = \Gamma(X, \mathcal{O}_X)$, then for any $f \in A$ we have a canonical isomorphism $H^\bullet(X_f, \mathcal{F}) \xrightarrow{\sim} (H^\bullet(X, \mathcal{F}))_f$ of A_f -module.*

Corollary 20.1.11. *Let X be a quasi-compact and separated scheme and $f \in \Gamma(X, \mathcal{O}_X)$.*

- (a) *If X_f is affine, then for any quasi-coherent \mathcal{O}_X -module \mathcal{F} and any $\xi \in H^i(X, \mathcal{F})$ with $i > 0$, there exists an integer $n > 0$ such that $f^n \xi = 0$.*
- (b) *Conversely, suppose that X_f is quasi-compact and for any quasi-coherent ideal \mathcal{I} and any $\zeta \in H^1(X, \mathcal{I})$, there exists $n > 0$ such that $f^n \zeta = 0$. Then X_f is affine.*

Proof. First, if X_f is affine then $H^i(X_f, \mathcal{F}) = 0$ for $i > 0$, so (a) follows from the isomorphism of ???. Conversely, in case (b), in view of Serre's criterion, it suffices to prove that for any quasi-coherent ideal \mathcal{K} of $\mathcal{O}_X|_{X_f}$, we have $H^1(X_f, \mathcal{K}) = 0$. Now as X_f is a quasi-compact open subset of the quasi-compact scheme X , by ?? there exists a quasi-coherent ideal \mathcal{J} of \mathcal{O}_X such that $\mathcal{K} = \mathcal{J}|_{X_f}$. By ?? we have $H^1(X_f, \mathcal{K}) = (H^1(X, \mathcal{J}))_f$, and the hypothesis then implies our claim. \square

Lemma 20.1.12 (Induction principle). *Let X be a quasi-compact and quasi-separated scheme and \mathcal{P} be a property for quasi-compact open subsets of X . Assume that the following conditions are satisfied:*

- (a) \mathcal{P} holds for affine opens of X ,
- (b) *if U is a quasi-compact open subset of X , V is an affine open of X , and \mathcal{P} holds for $U, V, U \cap V$, then \mathcal{P} holds for $U \cup V$.*

Then \mathcal{P} holds for every quasi-compact open subset of X , and in particular holds for X .

Proof. We first prove that \mathcal{P} holds for separated quasi-compact open subset $W \subseteq X$. For this, note that W can be written as a union $W = U_1 \cup \dots \cup U_n$ of affine opens and we can applying induction on n with $U = U_1 \cup \dots \cup U_n$ and $V = U_n$. This is allowed because $U \cap V = (U_1 \cap U_n) \cup \dots \cup (U_{n-1} \cap U_n)$ is again a union of $n-1$ affine open subschemes. Now for any quasi-compact open subset $W \subseteq X$, we can induct on the number of affine opens needed to cover W using the same trick as before and using that the quasi-compact open $U_i \cap U_n$ is separated as an open subscheme of the affine scheme U_n . \square

Proposition 20.1.13. *Let $f : X \rightarrow Y$ be a quasi-compact and quasi-separated morphism. Then for any quasi-coherent \mathcal{O}_X -module \mathcal{F} , $R^p f_*(\mathcal{F})$ is quasi-coherent for $p \geq 0$.*

Proof. Since taking restriction commutes with higher direct images, we may assume that Y is affine. Then X is quasi-compact and quasi-separated. For a quasi-compact open subset $U \subseteq X$ and $f_U = f|_U$, we let $\mathcal{P}(U)$ be the property that $R^p(f_U)_*(\mathcal{F})$ is quasi-coherent for all quasi-coherent modules \mathcal{F} on U and $o \geq 0$. It then suffices to prove that the conditions of ?? hold.

If U is affine, then the morphism f_U is affine, so by ?? we have $R^p(f_U)_*(\mathcal{F}) = 0$ for $p > 0$, and $f_*(\mathcal{F})$ is quasi-coherent by ???. Now let $U \subseteq X$ be a quasi-compact open subset, $V \subseteq X$ be an affine open subset, and assume that property \mathcal{P} holds for U, V and $U \cap V$. Then for any quasi-coherent $\mathcal{O}_X|_{U \cup V}$ -module \mathcal{F} , we have the relative Mayer-Vietoris sequence

$$0 \rightarrow (f_{U \cup V})_*(\mathcal{F}) \rightarrow (f_U)_*(\mathcal{F}|_U) \oplus (f_V)_*(\mathcal{F}|_V) \rightarrow (f_{U \cap V})_*(\mathcal{F}|_{U \cap V}) \rightarrow R^1(f_{U \cup V})_*(\mathcal{F}) \rightarrow \cdots$$

It then follows from our assumption and ?? that $R^p(f_{U \cup V})_*(\mathcal{F})$ is quasi-coherent for $p \geq 0$, so the assertion follows by applying ???. \square

Corollary 20.1.14. *Let $f : X \rightarrow Y$ be a quasi-compact and quasi-separated morphism. Then for any quasi-coherent \mathcal{O}_X -module \mathcal{F} and any affine open $V \subseteq Y$, the canonical homomorphism*

$$H^p(f^{-1}(V), \mathcal{F}) \rightarrow H^0(V, R^p f_*(\mathcal{F}))$$

is an isomorphism for $p \geq 0$.

Proof. Since this question is local, we may assume that Y is affine. Consider the Leray spectral sequence

$$E_2^{p,q} = H^p(Y, R^q f_*(\mathcal{F})) \Rightarrow H^{p+q}(X, \mathcal{F}).$$

By ??, we see that the $R^q f_*(\mathcal{F})$ are quasi-coherent \mathcal{O}_Y -module, so we have $H^p(Y, R^q f_*(\mathcal{F})) = 0$ for $p > 0$ (??). The spectral sequence therefore collapses at E_2 page and we obtain an isomorphism $H^q(X, \mathcal{F}) \cong H^0(Y, R^q f_*(\mathcal{F}))$. \square

Corollary 20.1.15. *Let $f : X \rightarrow Y$ be a quasi-compact and separated morphism, and suppose that Y is quasi-compact. Then there exists an integer $r > 0$ such that for any quasi-coherent \mathcal{O}_X -module, we have $R^p f_*(\mathcal{F}) = 0$ for $p > r$. If Y is affine, then we can choose r so that there exists a covering of X by r affine opens.*

Proof. Since Y is a union of affine opens, it suffices to prove the second assertion, in view of ???. Now if \mathfrak{U} is a covering of X by r affine opens, then $H^p(\mathfrak{U}, \mathcal{F}) = 0$ for $p > r$, since the cochains can be taken to be alternating. The conclusion then follows from ???. \square

Corollary 20.1.16. *Under the hypothesis of ??, suppose that $Y = \text{Spec}(A)$ is affine. Then for any quasi-coherent \mathcal{O}_X -module \mathcal{F} and any $f \in A$, we have*

$$\Gamma(Y_f, R^p f_*(\mathcal{F})) = (\Gamma(Y, R^p f_*(\mathcal{F})))_f.$$

Proof. This follows from ?? and ???. \square

Proposition 20.1.17. *Let $f : X \rightarrow Y$ be a quasi-compact and quasi-separated morphism, $g : Y \rightarrow Z$ be an affine morphism. For any quasi-coherent \mathcal{O}_X -module \mathcal{F} , the canonical homomorphism $R^p(g \circ f)_*(\mathcal{F}) \rightarrow g_*(R^p f_*(\mathcal{F}))$ is bijective for $p \geq 0$.*

Proof. Consider the Grothendieck spectral sequence

$$E_2^{p,q} = R^p g_*(R^q f_*(\mathcal{F})) \Rightarrow R^{p+q}(g \circ f)_*(\mathcal{F}).$$

Since each $R_q f_*(\mathcal{F})$ is coherent by ??, it then suffices to prove that $R^p g_*(\mathcal{G}) = 0$ for any coherent \mathcal{O}_Y -module \mathcal{G} and $p \geq 0$. Since this question is local, we may assume that Z is affine; but then Y is also affine so the assertion follows from ??.

□

Proposition 20.1.18. *Let $f : X \rightarrow Y$ be a separated morphism of finite type and $g : Y' \rightarrow Y$ be a flat morphism. Let $f' = f_{(Y')} : X' = X_{(Y')} \rightarrow Y'$, and consider the commutative diagram*

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ g' \downarrow & & \downarrow g \\ X & \xrightarrow{f} & Y \end{array}$$

Let \mathcal{F} be a quasi-coherent \mathcal{O}_X -module and set $\mathcal{F}' = g'^(\mathcal{F})$. Then $R^p f'_*(\mathcal{F}')$ is canonically isomorphic to $g^*(R^p f_*(\mathcal{F}))$ for $p \geq 0$.*

Proof. The canonical homomorphism $\rho : \mathcal{F} \rightarrow g'_*(g'^*(\mathcal{F}))$ defines a homomorphism

$$R^p f_*(\mathcal{F}) \rightarrow R^p f'_*(g'_*(g'^*(\mathcal{F}))). \quad (20.1.9)$$

On the other hand, if $h = f \circ g' = g \circ f'$, we also have the canonical homomorphisms (EGA 0_{III}, 12.2.5.1 et 12.2.5.2 ??)

$$R^p f_*(g'_*(\mathcal{F}')) \rightarrow R^p h_*(\mathcal{F}') \rightarrow g_*(R^p f'_*(\mathcal{F}')), \quad (20.1.10)$$

Composing ?? with ??, we then obtain a canonical homomorphism

$$\psi : R^p f_*(\mathcal{F}) \rightarrow g_*(R^p f'_*(\mathcal{F}))$$

and finally a induced canonical homomorphism (without the flatness hypothesis on g)

$$\psi^\sharp : g^*(R^p f_*(\mathcal{F})) \rightarrow R^p f'_*(\mathcal{F}'). \quad (20.1.11)$$

It suffices to prove that ψ^\sharp is an isomorphism if g is flat. Since this question is local over Y and Y' , we may assume that $Y = \text{Spec}(A)$, $Y' = \text{Spec}(B)$, and g then corresponds to a flat ring homomorphism $\varphi : A \rightarrow B$. It follows from ?? and the definition of the homomorphism ?? that ψ corresponds to the composition homomorphism

$$H^p(X, \mathcal{F}) \xrightarrow{\rho_p} H^p(X, g'_*(g'^*(\mathcal{F}))) \xrightarrow{\theta_p} H^p(X', g'^*(g'_*(g'^*(\mathcal{F})))) \xrightarrow{\sigma_p} H^p(X', g'^*(\mathcal{F})) \quad (20.1.12)$$

where ρ_p and σ_p are induced by the canonical homomorphism $\rho_{\mathcal{F}}$ and $\sigma_{g'^*(\mathcal{F})}$, and θ_p is the φ -morphism of (EGA 0_{III}, 12.1.3.1 ??) relative to the \mathcal{O}_X -module $g'_*(g'^*(\mathcal{F}))$. Now by the functoriality of θ_p , we have a commutative diagram

$$\begin{array}{ccc} H^p(X, \mathcal{F}) & \xrightarrow{\rho_p} & H^p(X, g'_*(g'^*(\mathcal{F}))) \\ \theta_p \downarrow & & \downarrow \theta_p \\ H^p(X', g'^*(\mathcal{F})) & \xrightarrow{g'^*(\rho_p)} & H^p(X', g'^*(g'_*(g'^*(\mathcal{F})))) \end{array}$$

and as by definition $g'^*(\rho)$ is the inverse of σ , we conclude that the composition homomorphism (??) is none other than θ_p . The homomorphism ψ^\sharp is therefore the B -homomorphism $H^p(X, \mathcal{F}) \otimes_A B \rightarrow H^p(X', \mathcal{F}')$. As f is of finite type, X has a finite affine open covering $\mathfrak{U} = (U_i)$. Since g is now an affine morphism, so is g' (??), and the $U'_i = g'^{-1}(U_i)$ therefore form an open affine covering \mathfrak{U}' of X' . We then conclude from (EGA 0_{III}, 12.1.4.2 ??) that the diagram

$$\begin{array}{ccc} H^p(\mathfrak{U}, \mathcal{F}) & \xrightarrow{\theta_p} & H^p(\mathfrak{U}', \mathcal{F}') \\ \downarrow & & \downarrow \\ H^p(X, \mathcal{F}) & \xrightarrow{\theta_p} & H^p(X', \mathcal{F}') \end{array}$$

is commutative, where the columns are isomorphisms since X and X' are separated (??). It then suffices to prove that the associated B -homomorphism

$$H^p(\mathfrak{U}, \mathcal{F}) \otimes_A B \rightarrow H^p(\mathfrak{U}', \mathcal{F}')$$

is an isomorphism. Now in view of ??, the canonical map $C^p(\mathfrak{U}, \mathcal{F}) \otimes_A B \rightarrow C^p(\mathfrak{U}', \mathcal{F}')$ is bijective. As B is a flat A -module, it follows that the canonical map $H^p(\mathfrak{U}, \mathcal{F}) \otimes_A B \rightarrow H^p(\mathfrak{U}', \mathcal{F}')$ is an isomorphism. \square

Corollary 20.1.19. *Let A be a ring, X be an A -scheme of finite type, and B be a faithfully flat A -algebra. Then for X to be affine, it is necessary and sufficient that $X \otimes_A B$ is affine.*

Proof. The condition is clearly necessary. Conversely, as X is separated over A and the morphism $\text{Spec}(B) \rightarrow \text{Spec}(A)$ is flat, it follows from ?? that we have

$$H^i(X \otimes_A B, \mathcal{F} \otimes_A B) = H^i(X, \mathcal{F}) \otimes_A B$$

for $i \geq 0$ and any quasi-coherent \mathcal{O}_X -module \mathcal{F} . As $X \otimes_A B$ is affine, we see that $H^i(X \otimes_A B, \mathcal{F} \otimes_A B) = 0$ for $i > 0$, so $H^i(X, \mathcal{F}) = 0$ for $i > 0$ since B is faithfully flat over A . As X is quasi-compact, the conclusion then follows from Serre's criterion. \square

20.2 Cohomological properties of projective morphisms

20.2.1 Cohomology associated with an invertible sheaf

Let X be a scheme, \mathcal{L} be an invertible \mathcal{O}_X -module, and consider the graded ring

$$S = \Gamma_*(X, \mathcal{L}) = \bigoplus_{n \in \mathbb{Z}} \Gamma(X, \mathcal{L}^{\otimes n}).$$

Let $\mathbf{f} = (f_i)_{1 \leq i \leq r}$ be a family of homogeneous elements of S , where $f_i \in S_{d_i}$. Put $U_i = X_{f_i}$, $U = \bigcup_i U_i$, and let $\mathfrak{U} = (U_i)$ be the covering of U . For any quasi-coherent \mathcal{O}_X -module \mathcal{F} of X , we set

$$H^\bullet(\mathfrak{U}, \mathcal{F}; \mathcal{L}) := \bigoplus_{n \in \mathbb{Z}} H^\bullet(\mathfrak{U}, \mathcal{F} \otimes \mathcal{L}^{\otimes n}), \quad H^\bullet(U, \mathcal{F}; \mathcal{L}) := \bigoplus_{n \in \mathbb{Z}} H^\bullet(U, \mathcal{F} \otimes \mathcal{L}^{\otimes n}). \quad (20.2.1)$$

We note that the abelian groups in (??) are bigraded: for $m, n \in \mathbb{Z}$ we set

$$(H^\bullet(\mathfrak{U}, \mathcal{F}; \mathcal{L}))_{mn} = H^m(\mathfrak{U}, \mathcal{F} \otimes \mathcal{L}^n), \quad (H^\bullet(U, \mathcal{F}; \mathcal{L}))_{mn} = H^m(U, \mathcal{F} \otimes \mathcal{L}^n).$$

For any fixed $m \in \mathbb{Z}$, it is clear that $H^m(\mathfrak{U}, \mathcal{F}; \mathcal{L})$ and $H^m(U, \mathcal{F}; \mathcal{L})$ are graded S -modules. We now consider the graded S -module

$$M = \Gamma_*(\mathcal{F}; \mathcal{L}) = H^0(X, \mathcal{F}; \mathcal{L}) = \bigoplus_{n \in \mathbb{Z}} \Gamma(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n}).$$

If X is quasi-compact and quasi-separated, then it follows from ?? that for any sequence (i_0, i_1, \dots, i_p) , we have a canonical isomorphism

$$\Gamma(U_{i_0 \dots i_p}, \mathcal{F}; \mathcal{L}) = H^0(U_{i_0 \dots i_p}, \mathcal{F}; \mathcal{L}) = M_{f_{i_0} \dots f_{i_p}}.$$

Recall that $M_{f_{i_0} \dots f_{i_p}}$ is identified with $\varinjlim_n M_{i_0 \dots i_p}^n$, and this identification is an isomorphism of graded S -modules, if we define the degree of an element $z \in \varinjlim_n M_{i_0 \dots i_p}^n$ to be the number $m - n(d_{i_0} + \dots + d_{i_p})$ if z is the image of a homogeneous element $x \in M_{i_0 \dots i_p}^n = M$ of degree m (from the definition of the transition homomorphism, it follows that this definition does not depend on the choice of x). Let $C_n^p(M)$ be the set of alternating maps $I^{p+1} \rightarrow M$ (for any n), then we can similarly define a graded S -module structure on $\varinjlim_n C_n^p(M)$. Now the canonical isomorphism

$$C^p(\mathfrak{U}, \mathcal{F}; \mathcal{L}) = \varinjlim_n C_n^p(M)$$

is then an isomorphism of graded S -modules. By ??, we have isomorphism of graded S -modules

$$C^p(\mathfrak{U}, \mathcal{F}; \mathcal{L}) = C^{p+1}(\mathfrak{f}, M) = \varinjlim_n K^{p+1}(\mathbf{f}^n, M)$$

where the degree of an element in $\varinjlim_n K^{p+1}(\mathbf{f}^n, M)$ is defined similarly. It is easy to see that the isomorphisms above are compatible with differential maps, so from ?? we conclude the following:

Proposition 20.2.1. *Let X be a quasi-compact and quasi-separated scheme. Then there exists a canonical isomorphism of graded S -modules*

$$H^p(\mathfrak{U}, \mathcal{F}; \mathcal{L}) \xrightarrow{\sim} H^{p+1}(\mathfrak{f}, M) \text{ for } p \geq 1, \tag{20.2.2}$$

where \mathfrak{f} is the ideal generated by \mathbf{f} . Moreover, we have a functorial exact sequence

$$0 \longrightarrow H^0(\mathfrak{f}, M) \longrightarrow M \longrightarrow H^0(\mathfrak{U}, \mathcal{F}; \mathcal{L}) \longrightarrow H^1(\mathfrak{f}, M) \longrightarrow 0 \tag{20.2.3}$$

of homomorphisms of degree 0.

Corollary 20.2.2. *If X is quasi-compact and separated and $U_i = X_{f_i}$ are affine, then there exists a canonical isomorphism of degree 0:*

$$H^p(U, \mathcal{F}; \mathcal{L}) \xrightarrow{\sim} H^{p+1}(\mathfrak{f}, M) \text{ for } p > 0.$$

and we have a functorial exact sequence

$$0 \longrightarrow H^0(\mathfrak{f}, M) \longrightarrow M \longrightarrow H^0(\mathfrak{U}, \mathcal{F}; \mathcal{L}) \longrightarrow H^1(\mathfrak{f}, M) \longrightarrow 0$$

of homomorphisms of degree 0.

Corollary 20.2.3. Let S be a graded ring with positive degrees, $(f_i)_{1 \leq i \leq r}$ be homogeneous elements of S_+ with $f_i \in S_{d_i}$, and M be a graded S -module. Let $X = \text{Proj}(S)$, $U_i = D_+(f_i)$, and $\mathcal{L} = \mathcal{O}_X(1)$, then there exists a canonical isomorphism of degree 0:

$$H^p(U, \tilde{M}; \mathcal{L}) \xrightarrow{\sim} H^{p+1}(\mathfrak{f}, M) \quad \text{for } p > 0.$$

and we have a functorial exact sequence

$$0 \longrightarrow H^0(\mathfrak{f}, M) \longrightarrow M \longrightarrow H^0(\mathfrak{U}, \tilde{M}; \mathcal{L}) \longrightarrow H^1(\mathfrak{f}, M) \longrightarrow 0$$

of homomorphisms of degree 0.

Proof. In fact, we have $\Gamma(U_{i_0 \dots i_p}, \widetilde{M(n)}) = (M_{f_{i_0} \dots f_{i_p}})_n$, so $\Gamma(U_{i_0 \dots i_p}, \tilde{M}; \mathcal{L}) = M_{f_{i_0} \dots f_{i_p}}$. The conclusion then follows from ?? since X is quasi-compact and separated. \square

Remark 20.2.1. ?? is interesting if S is an A -algebra generated by S_1 where A is Noetherian. In fact, in this case any quasi-coherent \mathcal{O}_X -module \mathcal{F} is of the form \tilde{M} by ??.

Remark 20.2.2. Under the hypothesis of ??, the functor $\Gamma(U_{i_0 \dots i_p}, \tilde{M}; \mathcal{L}) = M_{f_{i_0} \dots f_{i_p}}$ is exact on M , and as in ??, if $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ is an exact sequence of graded S -modules, we have a commutative diagram for $p \geq 0$:

$$\begin{array}{ccc} H^p(U, \tilde{M}''; \mathcal{L}) & \xrightarrow{\partial} & H^{p+1}(U, \tilde{M}'; \mathcal{L}) \\ \downarrow & & \downarrow \\ H^{p+1}(\mathfrak{f}, M'') & \xrightarrow{\partial} & H^{p+2}(\mathfrak{f}, M') \end{array}$$

We now apply ?? to the polynomial ring $S = A[T_0, \dots, T_r]$, where A is a ring and T_i are indeterminates. Let $M = S$ and $f_i = T_i$, we are then reduced to compute $H^\bullet(\mathfrak{m}, S)$, where $\mathbf{T} = (T_i)_{0 \leq i \leq r}$ and \mathfrak{m} is the maximal ideal of S generated by T_0, \dots, T_r .

Lemma 20.2.4. If $S = A[T_0, \dots, T_r]$ and $\mathbf{T} = (T_i)_{0 \leq i \leq r}$, then

$$H^i(\mathbf{T}^n, S) = \begin{cases} 0 & i \neq r+1, \\ S/\mathfrak{m}^n & i = r+1. \end{cases} \quad (20.2.4)$$

Proof. This is an immediate from the fact that the sequence \mathbf{T} is regular. \square

By passing to inductive limits over n , we see that $H^i(\mathfrak{m}, S) = 0$ for $i \neq r+1$. If $i = r+1$, then the inductive system is formed by S/\mathfrak{m}^n and $\varphi_{nm} : S/\mathfrak{m}^n \rightarrow S/\mathfrak{m}^m$ for $0 \leq n \leq m$ is the multiplication by $(T_0 \cdots T_r)^{n-m}$. For any sequence $\alpha = (\alpha_0, \dots, \alpha_r)$ and integer $n \geq \sup_i \{\alpha_i\}$, we define

$$\xi_\alpha^n = T_0^{n-\alpha_0} \cdots T_r^{n-\alpha_r} \mod \mathfrak{m}^n.$$

Then $\varphi_{nm}(\xi_\alpha^n) = \xi_\alpha^m$, so the ξ_α^n form an element ξ_α in the inductive limit $H^{r+1}(\mathfrak{m}, S)$.

Corollary 20.2.5. *With the notations of ??, we have $H^i(\mathfrak{m}, S) = 0$ for $i \neq r + 1$, and $H^{r+1}(\mathfrak{m}, S)$ is a free A -module with basis formed by the elements ξ_α with $\alpha_i > 0$ for each i .*

Proof. In fact, for each $n \geq 0$, the elements ξ_α^n for $0 < \alpha_i \leq n$ form a basis of the A -module S/\mathfrak{m}^n , so the corollary follows from ??.

Proposition 20.2.6. *Let A be a ring, $r > 0$ be an integer, and $X = \mathbb{P}_A^r$.*

- (a) *We have $H^i(X, \mathcal{O}_X; \mathcal{O}_X(1)) = 0$ for $i \neq 0, r$.*
- (b) *The canonical homomorphism $\alpha : S \rightarrow H^0(X, \mathcal{O}_X; \mathcal{O}_X(1))$ is an isomorphism.*
- (c) *The A -module $H^r(X, \mathcal{O}_X; \mathcal{O}_X(1))$ is free with a basis formed by the elements ξ_α with $\alpha_i > 0$ for each i . Moreover, ξ_α is of degree $-\lvert\alpha\rvert = -(\alpha_0 + \dots + \alpha_r)$ and $T_i \cdot \xi_\alpha = \xi_{\alpha_0, \dots, \alpha_{i-1}, \dots, \alpha_r}$.*

Proof. By ?? we have $H^i(\mathfrak{m}, S) = 0$ for $i \neq r + 1$, so the assertion in (a) follows from ??). From the exact sequence ??), it is easy to see that $S \cong H^0(X, \mathcal{O}_X; \mathcal{O}_X(1))$, and the isomorphism is given by the canonical homomorphism α . The last assertion also follows from ??) and ??.

Corollary 20.2.7. *The values (i, n) such that $H^i(X, \mathcal{O}_X(n)) \neq 0$ are the following: $i = 0$ and $n \geq 0$, or $i = r$ and $n \leq -(r + 1)$.*

Proof. We note that if $A \neq 0$ then $H^i(X, \mathcal{O}_X(n)) \neq 0$ by the listed values of (i, n) .

Corollary 20.2.8. *The A -modules $H^i(X, \mathcal{O}_X(n))$ are free of finite rank. If $i > 0$, then they are zero for $n > 0$.*

Proposition 20.2.9. *Let Y be a scheme, \mathcal{E} be a locally free \mathcal{O}_Y -module of rank $r + 1$, and $X = \mathbb{P}(\mathcal{E})$ be the projective bundle defined by \mathcal{E} . Let $f : X \rightarrow Y$ be the structural morphism, then the values (n, i) such that $R^i f_*(\mathcal{O}_X(n)) \neq 0$ are the following: $i = 0$ and $n \geq 0$, or $i = r$ and $n \leq -(r + 1)$. Moreover, the canonical homomorphism*

$$\alpha : S_{\mathcal{O}_Y}(\mathcal{E}) \rightarrow \Gamma_*(\mathcal{O}_X) = \bigoplus_{n \in \mathbb{Z}} f_*(\mathcal{O}_X(n))$$

is an isomorphism.

Proof. This question is local over Y , so we can assume that $Y = \text{Spec}(A)$ is affine and $\mathcal{E} = \tilde{E}$, where $E = A^{r+1}$. We can then apply ??, in view of ??.

20.2.2 The coherence theorem for projective morphisms

Proposition 20.2.10. *Let A be a Noetherian ring and S be a graded A -algebra with positive degrees that is generated by $r + 1$ elements of S_1 . Let $X = \text{Proj}(S)$, and consider a coherent \mathcal{O}_X -module \mathcal{F} .*

- (a) *The A -module $H^p(X, \mathcal{F})$ is finitely generated.*
- (b) *We have $H^p(X, \mathcal{F}) = 0$ for $p > r$.*
- (c) *There exists an integer n_0 such that for $n \geq n_0$, we have $H^p(X, \mathcal{F}(n))$ for $p > 0$.*

(d) There exists an integer n_0 such that for $n \geq n_0$, $\mathcal{F}(n)$ is generated by global sections.

Proof. Note that X can be identified with a closed subscheme of $P = \mathbb{P}_A^r$ (??). Moreover, if $j : X \rightarrow P$ is the canonical injection, $j_*(\mathcal{F})$ is a coherent \mathcal{O}_P -module and we have $j_*(\mathcal{F}(n)) = (j_*(\mathcal{F}))(n)$ (?? and ??). In view of (G, II, cor. du th.4.9.1 ??), we only need to consider the case where $X = \mathbb{P}_A^r$ and $S = A[T_0, \dots, T_r]$. Now X can be covered by $r + 1$ affine opens $D_+(T_i)$, so (b) follows from ?? and ?. We also note that (d) is proved in ??.

We now prove (a) and (c). By ??, these assertions hold for $\mathcal{F} = \mathcal{O}_X(m)$, hence for direct sums of finitely many \mathcal{O}_X -module of the form $\mathcal{O}_X(m_j)$. On the other hand, (a) and (c) are trivial for $p > r$ in view of (b). We now proceed by descendent induction on p . Since \mathcal{F} is coherent, it is a quotient of direct sum of finitely many sheaves $\mathcal{O}_X(m_j)$ (??). That is, we have an exact sequence $0 \rightarrow \mathcal{R} \rightarrow \mathcal{E} \rightarrow \mathcal{F} \rightarrow 0$ where \mathcal{R} is coherent and \mathcal{E} satisfies (a) and (c). Since tensoring with $\mathcal{O}_X(n)$ is exact, we obtain an exact sequence

$$H^{p-1}(X, \mathcal{E}(n)) \longrightarrow H^{p-1}(X, \mathcal{F}(n)) \longrightarrow H^p(X, \mathcal{R}(n)).$$

As $\mathcal{E}(n)$ is a direct sum of $\mathcal{O}_X(m_j + n)$, we see $H^{p-1}(X, \mathcal{E}(n))$ is finitely generated by ??, and so is $H^p(X, \mathcal{R}(n))$ by induction hypothesis. As A is Noetherian, we then conclude that $H^{p-1}(X, \mathcal{F}(n))$ is finitely generated for any $n \in \mathbb{Z}$, and in particular for $n = 0$. On the other hand, by induction hypothesis there exists an integer n_0 such that for $n \geq n_0$ we have $H^p(X, \mathcal{R}(n)) = 0$, and we can choose n_0 such that $H^{p-1}(X, \mathcal{E}(n)) = 0$ for $n \geq n_0$, since \mathcal{E} satisfies (c). Therefore we see that $H^{p-1}(X, \mathcal{F}(n)) = 0$ for $n \geq n_0$, which completes the proof. \square

Theorem 20.2.11 (Serre). Let Y be a Noetherian scheme, $f : X \rightarrow Y$ be a projective morphism, and \mathcal{L} be an invertible \mathcal{O}_X -module that is ample for f . For any coherent \mathcal{O}_X -module \mathcal{F} , set $\mathcal{F}(n) = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}$.

- (a) The \mathcal{O}_Y -module $R^p f_*(\mathcal{F})$ is coherent for $p \geq 0$.
- (b) There exists an integer n_0 such that for $n \geq n_0$, we have $R^p f_*(\mathcal{F}(n)) \neq 0$ for $p > 0$.
- (c) There exists an integer n_0 such that for $n \geq n_0$, the canonical homomorphism $f^*(f_*(\mathcal{F}(n))) \rightarrow \mathcal{F}(n)$ is surjective.

Proof. We note that the theorem is unchanged if we replace \mathcal{L} by $\mathcal{L}^{\otimes d}$ for $d > 0$. In fact, we then have $\mathcal{F}(n) = (\mathcal{F} \otimes \mathcal{L}^{\otimes r}) \otimes \mathcal{L}^{\otimes kd}$ for $k > 0$ and $0 \leq r < d$, and by hypothesis for any r there is an integer n_r such that for $k \geq n_r$, the properties (b) and (c) holds for $\mathcal{F} \otimes \mathcal{L}^{\otimes r}$. Let n_0 be the supremum of dn_r , then assertion (b) and (c) hold for $n \geq n_0$. In view of ??, we may assume that \mathcal{L} is very ample relative to f , so there exists a dominant open immersion $i : X \rightarrow P$, where $P = \text{Proj}(\mathcal{S})$ for a quasi-coherent graded \mathcal{O}_Y -algebra \mathcal{S} with positive degrees that is finitely generated by \mathcal{S}_1 ; moreover, $\mathcal{L} \cong i^*(\mathcal{O}_P(1))$. Since f is proper, the morphism i is also proper by ??, so it is an isomorphism $X \cong P$. We can therefore assume that $X = \text{Proj}(\mathcal{S})$ and $\mathcal{L} = \mathcal{O}_X(1)$, and the theorem then follows from ??.

Corollary 20.2.12. Under the hypothesis of ??, let $0 \rightarrow \mathcal{F} \rightarrow \mathcal{G} \rightarrow \mathcal{H} \rightarrow 0$ be an exact sequence of coherent \mathcal{O}_X -modules. Then there exists an integer n_0 such that for $n \geq n_0$, the sequence

$$0 \longrightarrow f_*(\mathcal{F}(n)) \longrightarrow f_*(\mathcal{G}(n)) \longrightarrow f_*(\mathcal{H}(n))$$

is exact.

Proof. This follows from the long exact sequence of f_* and property (b) of ??.

Corollary 20.2.13. *Let Y be a Noetherian scheme, $f : X \rightarrow Y$ be a morphism of finite type, and \mathcal{L} be an invertible \mathcal{O}_X -module that is ample for f . Let $0 \rightarrow \mathcal{F} \rightarrow \mathcal{G} \rightarrow \mathcal{H} \rightarrow 0$ be an exact sequence of coherent \mathcal{O}_X -modules such that the supports of \mathcal{F} and \mathcal{H} are proper over Y . Then there exists an integer n_0 such that for $n \geq n_0$, the sequence*

$$0 \longrightarrow f_*(\mathcal{F}(n)) \longrightarrow f_*(\mathcal{G}(n)) \longrightarrow f_*(\mathcal{H}(n))$$

is exact.

Proof. The same reasoning of ?? show that we can assume that \mathcal{L} is very ample relative to f , so we can identify X as an open subscheme of $Z = \text{Proj}(\mathcal{S})$, where \mathcal{S} is a quasi-coherent graded \mathcal{O}_Y -algebra with positive degrees, such that \mathcal{S} is finitely generated by \mathcal{S}_1 , and $\mathcal{L} = i^*(\mathcal{O}_Z(1))$, where $i : X \rightarrow Z$ is the canonical immersion. Now as $\text{supp}(\mathcal{G})$ is closed in X and contained in $\text{supp}(\mathcal{F}) \cap \text{supp}(\mathcal{H})$, it is proper over Y ; the supports of $\mathcal{F}, \mathcal{G}, \mathcal{H}$ are then closed in Z . The sheaves $\mathcal{F}' = i_*(\mathcal{F})$, $\mathcal{G}' = i_*(\mathcal{G})$, and $\mathcal{H}' = i_*(\mathcal{H})$ are then coherent \mathcal{O}_Z -modules, and the sequence $0 \rightarrow \mathcal{F}' \rightarrow \mathcal{G}' \rightarrow \mathcal{H}' \rightarrow 0$ is exact. Moreover, if $g : Z \rightarrow Y$ is the structural morphism, then $f = g \circ i$, and it is clear that $\mathcal{F}'(n) = i_*(\mathcal{F}(n))$ and similarly for $\mathcal{G}', \mathcal{H}'$. The conclusion then follows from ??.

Remark 20.2.3. The assertion (a) of ?? is still valid if we only assume that Y is locally Noetherian. In fact, this property is local over Y , and the hypotheses in ?? imply that for any affine open $U \subseteq Y$, the restriction f to $f^{-1}(U)$ is projective and $\mathcal{L}|_{f^{-1}(U)}$ is ample for this restriction.

Remark 20.2.4. The assertion (a) of ?? is still valid, as we have seen, when we only assumes that X is a quasi-compact and quasi-separated scheme and $f : X \rightarrow Y$ is a quasi-compact and quasi-separated morphism (??). However, note that assertion (b) is not true if we suppose that Y is the spectrum of a field k and that f is quasi-projective. For example, let $X' = \text{Spec}(k[T_0, \dots, T_r])$ and X be the union of the affine opens $D(T_i)$ (so that X can be considered as the space \mathbb{A}_k^r with the origin removed). As the immersion $X \rightarrow X'$ is quasi-compact, the structural morphism $f : X \rightarrow Y$ is quasi-affine, so \mathcal{O}_X is very ample relative to f (??). But the ring $\Gamma(X, \mathcal{O}_X)$ is identified with the intersection of $K[T_0, \dots, T_r]_{T_i}$ for $0 \leq i \leq r$ (??), which is $K[T_0, \dots, T_r]$, so it follows from (??) that we have $H^r(X, \mathcal{O}_X^{\otimes n}) = H^r(X, \mathcal{O}_X) = A \neq 0$ for any $n \in \mathbb{Z}$.

Theorem 20.2.14. *Let Y be a Noetherian scheme and \mathcal{S} be a quasi-coherent \mathcal{O}_Y -algebra of finite type. Let $f : X \rightarrow Y$ be a projective morphism, $\mathcal{S}' = f^*(\mathcal{S})$, \mathcal{M} be a quasi-coherent \mathcal{S}' -module of finite type.*

- (a) *For any $p \geq 0$, $R^p f_*(\mathcal{M})$ is an \mathcal{S} -module of finite type.*
- (b) *Let \mathcal{L} be an f -ample \mathcal{O}_X -module, and put $\mathcal{M}(n) = \mathcal{M} \otimes \mathcal{L}^{\otimes n}$. Then there exists an integer n_0 such that for $n \geq n_0$, we have $R^p f_*(\mathcal{M}(n)) = 0$ for $p > 0$, and the canonical homomorphism $f^*(f_*(\mathcal{M}(n))) \rightarrow \mathcal{M}(n)$ is surjective.*

20.2.3 Applications to associated sheaves of graded modules

Theorem 20.2.15. *Let Y be a Noetherian scheme, \mathcal{S} be a quasi-coherent \mathcal{O}_Y -algebra of finite type with positive degrees, $X = \text{Proj}(\mathcal{S})$, and $q : X \rightarrow Y$ be the structural morphism. Let \mathcal{M} be a quasi-coherent*

graded \mathcal{S} -module that is eventually finite. Then there exists an integer n_0 such that for $n \geq n_0$, the canonical homomorphism of (EGA II, 8.14.5.1 ??)

$$\alpha_n : \mathcal{M}_n \rightarrow q_*(\text{Proj}_0(\mathcal{M}(n))) = q_*((\text{Proj}(\mathcal{M}))_n)$$

is bijective. In other words, the canonical homomorphism $\alpha : \mathcal{M} \rightarrow \Gamma_*(\text{Proj}(\mathcal{M}))$ is an eventual isomorphism.

Proof. By ??, we can assume that \mathcal{M} is an \mathcal{S} -module of finite type. As Y is quasi-compact, by ?? there exists an integer $d > 0$ such that $\mathcal{S}^{(d)}$ is generated by \mathcal{S}_d , which is of finite type. Now \mathcal{M} is the direct sum of $\mathcal{M}^{(d,k)}$ for $0 \leq k < d$, and each $\mathcal{M}^{(d,k)}$ is quasi-coherent $\mathcal{S}^{(d)}$ -module of finite type (??), so it suffices to prove that the canonical homomorphism $\alpha : \mathcal{M}^{(d,k)} \rightarrow \Gamma_*((\text{Proj}(\mathcal{M}))^{(d,k)})$ is an eventual isomorphism. In view of (EGA II, 8.14.13 ??) (and the diagram (EGA II, 8.14.13.4 ??)), we can therefore assume that \mathcal{S} is finitely generated by \mathcal{S}_1 . As Y is Noetherian, the same reasoning as in ?? shows that we can further assume that $Y = \text{Spec}(A)$, $\mathcal{S} = \tilde{\mathcal{S}}$, $\mathcal{M} = \widetilde{\mathcal{M}}$, where A is a Noetherian ring, S_1 is a finitely generated A -module, and M is a finitely generated S -module.

We also note that it suffices to prove the assertion for $M = S$. In fact, in the general case, we have an exact sequence $R \rightarrow L \rightarrow M \rightarrow 0$, where L and R are direct sums of graded modules of the form $S(m)$. If the assertion is proved for $M = S$, then it also holds for $M = S(m)$, hence for L and R . Consider the commutative diagram

$$\begin{array}{ccccccc} \widetilde{R}_n & \longrightarrow & \widetilde{L}_n & \longrightarrow & \widetilde{M}_n & \longrightarrow & 0 \\ \downarrow \alpha_n & & \downarrow \alpha_n & & \downarrow \alpha_n & & \\ q_*(\widetilde{R}(n)) & \longrightarrow & q_*(\widetilde{L}(n)) & \longrightarrow & q_*(\widetilde{M}(n)) & \longrightarrow & 0 \end{array}$$

The first and second vertical arrows are isomorphisms for $n \gg 0$, so by five lemma we conclude that the middle one is also an isomorphism for sufficiently large n , whence our assertion.

This being done, we are left to prove the theorem for $M = S$; for this, we first suppose that $S = A_0[T_0, \dots, T_r]$ (where T_i are indeterminates). In this case, the assertion follows from ??(b). In the general case, S is identified with a quotient of a ring $S' = A[T_0, \dots, T_n]$ by a graded ideal, so X is a closed subscheme of $X' = \mathbb{P}_A^r$. If $j : X \rightarrow X'$ is the canonical injection, then $j_*(\widetilde{S}(n))$ is equal to the $\mathcal{O}_{X'}$ -module $(\text{Proj}(\widetilde{S}))(n)$ where S is considered as a graded S' -module (??). As $j_*(\widetilde{S}(n))$ is a eventually finite $\mathcal{O}_{X'}$ -module, the canonical homomorphism $\alpha_n : S_n \rightarrow \Gamma(X', j_*(\widetilde{S}(n)))$ is bijective for $n \gg 0$, and this proves our assertion since $\Gamma(X', j_*(\widetilde{S}(n))) = \Gamma(X, \widetilde{S}(n))$. \square

Corollary 20.2.16. Under the hypotheses of ??, let $\mathcal{S}_X = \bigoplus_{n \in \mathbb{Z}} \mathcal{O}_X(n)$, and \mathcal{F} be a quasi-coherent graded \mathcal{S}_X -module of finite type. Then $\Gamma_*(\mathcal{F})$ is eventually finite.

Proof. We see that in the proof of ??, X is isomorphic to $\text{Proj}(\mathcal{S}^{(d)})$ which is of finite type over Y . It then follows from (EGA II, 8.14.9 ??) that \mathcal{F} is isomorphic to a graded \mathcal{S}_X -module of the form $\text{Proj}_0(\mathcal{M})$, where \mathcal{M} is a quasi-coherent \mathcal{S} -module of finite type. In view of ??, we see $\Gamma_*(\mathcal{F})$ is eventually isomorphic to \mathcal{M} , so is eventually finite. \square

Remark 20.2.5. Let Y be a Noetherian scheme, \mathcal{S} be a graded \mathcal{O}_Y -algebra satisfying the conditions of ??, and $X = \text{Proj}(\mathcal{S})$. Let $\mathcal{K}_{\mathcal{S}}$ be the abelian category of quasi-coherent graded \mathcal{S} -modules that are eventually finite, and $\mathcal{K}'_{\mathcal{S}}$ be the subcategory of $\mathcal{K}_{\mathcal{S}}$ consists of quasi-coherent graded \mathcal{S} -modules that are eventually null. Finally, let \mathcal{K}_X be the category of quasi-coherent graded \mathcal{S}_X -module of finite type (which amounts to saying, since \mathcal{S}_X is periodic by (EGA II, 8.14.4 ??) and (EGA II, 8.14.12 ??), that each \mathcal{F}_i is a coherent \mathcal{P}_X -module). Then in view of (EGA II, 8.14.8 ??), (EGA II, 8.14.10 ??) and ??, the functors $\mathcal{M} \mapsto \text{Proj}(\mathcal{M})$ and $\mathcal{F} \mapsto \Gamma_*(\mathcal{F})$ define an equivalence from the quotient category $\mathcal{K}_{\mathcal{S}}/\mathcal{K}'_{\mathcal{S}}$ to the category \mathcal{K}_X . If \mathcal{S} is generated by \mathcal{S}_1 , we can also replace \mathcal{K}_X by the category of coherent \mathcal{O}_X -modules.

Proposition 20.2.17. Let Y be a Noetherian scheme.

- (a) Let \mathcal{S} be a quasi-coherent graded \mathcal{O}_Y -algebra of finite type with positive degrees, $X = \text{Proj}(\mathcal{S})$, and $\mathcal{S}_X = \text{Proj}(\mathcal{S}) = \bigoplus_{n \in \mathbb{Z}} \mathcal{O}_X(n)$. Then \mathcal{S}_X is a periodic graded \mathcal{O}_X -algebra whose homogeneous components $(\mathcal{S}_X)_n = \mathcal{O}_X(n)$ are coherent \mathcal{O}_X -modules. If $d > 0$ is a period of \mathcal{S}_X , then $(\mathcal{S}_X)_d = \mathcal{O}_X(d)$ is an invertible \mathcal{O}_X -module that is Y -ample. Moreover, the canonical homomorphism $\alpha : \mathcal{S} \rightarrow \Gamma_*(\mathcal{S}_X)$ is a eventual isomorphism.
- (b) Conversely, let $q : X \rightarrow Y$ be a projective morphism, and \mathcal{S}' be a graded \mathcal{O}_X -algebra whose homogeneous components \mathcal{S}'_n are coherent \mathcal{O}_X -modules, and that admits a period $d > 0$ such that \mathcal{S}'_d is an invertible \mathcal{O}_X -module that is Y -ample. Then $\mathcal{S} = \bigoplus_{n \geq 0} q_*(\mathcal{S}'_n)$ is a quasi-coherent graded \mathcal{O}_Y -algebra of finite type with positive degrees, and there exists a Y -isomorphism $i : X \cong \text{Proj}(\mathcal{S})$ such that $i^*(\text{Proj}(\mathcal{S}))$ is isomorphic to \mathcal{S}' .

Proposition 20.2.18. Let Y be a Noetherian scheme, $q : X \rightarrow Y$ be a projective morphism, and \mathcal{L} be an invertible \mathcal{O}_X -module that is very ample for q . Then $\mathcal{S} = \bigoplus_{n \geq 0} q_*(\mathcal{L}^{\otimes n})$ is a quasi-coherent \mathcal{O}_Y -algebra of finite type such that $\mathcal{S}_n = \mathcal{S}_1^n$ for $n \gg 0$, and there exists an Y -isomorphism $r : X \cong P = \text{Proj}(\mathcal{S})$ such that $\mathcal{S} \cong r^*(\mathcal{O}_P(1))$.

Proposition 20.2.19. Let Y be a Noetherian integral scheme, X be an integral scheme, and $f : X \rightarrow Y$ be a projective birational morphism. Then there exists a coherent fractional ideal $\mathcal{I} \subseteq \mathcal{K}_Y$ such that X is Y -isomorphic to blow up Y -scheme relative to \mathcal{I} . Moreover, there exists an open subset $U \subseteq Y$ such that the restriction $f : f^{-1}(U) \rightarrow U$ is an isomorphic, and $\mathcal{I}|_U$ is invertible.

Proof. By ??, there exists an invertible \mathcal{O}_X -module \mathcal{L} that is very ample for f , so we can apply ??, hence identify X with $\text{Proj}(\mathcal{S})$, where $\mathcal{S} = \bigoplus_{n \geq 0} f_*(\mathcal{L}^{\otimes n})$. We also see that each $f_*(\mathcal{L}^{\otimes n})$ is a torsion-free \mathcal{O}_Y -module ??, so \mathcal{S} is canonically identified with a sub- \mathcal{O}_Y -module of $\mathcal{S} \otimes_{\mathcal{O}_Y} \mathcal{K}_Y$. By ?? this sheaf is simple, so is completely determined when we know its restriction to any non-empty open set, for example to a open nonempty $U' \subseteq U$ (here $U \subseteq Y$ is an open subset such that f is an isomorphic on $f^{-1}(U)$ such that $\mathcal{L}|_{f^{-1}(U')}$ is isomorphic to $\mathcal{O}_X|_{f^{-1}(U')}$. As by hypotheses $f_*(\mathcal{L}^{\otimes n})|_{U'}$ is then isomorphic to $\mathcal{O}_Y|_{U'}$, we conclude that $\mathcal{S} \otimes \mathcal{K}_Y$ is an \mathcal{K}_Y -module isomorphic to $\mathcal{K}_Y[T]$, where T is an indeterminate, and \mathcal{S} is eventually isomorphic to the sub- \mathcal{O}_Y -module generated by the canonical image of $f_*(\mathcal{L})$ in $\mathcal{S} \otimes_{\mathcal{O}_Y} \mathcal{K}_Y$??). But if we identify $\mathcal{S} \otimes_{\mathcal{O}_Y} \mathcal{K}_Y$ with $\mathcal{K}_Y[T]$, then the image of $f_*(\mathcal{L})$ is identified with $\mathcal{I} \cdot T$, where \mathcal{I} is a coherent sub- \mathcal{O}_Y -module of \mathcal{K}_Y ??, and its restriction to U' is isomorphic to $\mathcal{O}_Y|_{U'}$. We therefore conclude that $\mathcal{I}|_U$ is invertible and \mathcal{S} is eventually isomorphic to $\bigoplus_{n \geq 0} \mathcal{I}^n$, whence our assertion. \square

Corollary 20.2.20. *Under the hypothesis of ??, suppose that for any nontrivial coherent sub- \mathcal{O}_Y -module \mathcal{I} of \mathcal{K}_Y , there exists an invertible \mathcal{O}_Y -module \mathcal{L} such that*

$$\Gamma(Y, \mathcal{L} \otimes_{\mathcal{O}_Y} \mathcal{H}\text{om}_{\mathcal{O}_Y}(\mathcal{I}, \mathcal{O}_Y)) \neq 0.$$

Then, in the situation of ??, we can suppose that \mathcal{I} is an ideal of \mathcal{O}_Y . This additional condition is always verified if there exists an ample \mathcal{O}_Y -module.

Proof. We first note that

$$\mathcal{L} \otimes_{\mathcal{O}_Y} \mathcal{H}\text{om}_{\mathcal{O}_Y}(\mathcal{I}, \mathcal{O}_Y) = \mathcal{H}\text{om}_{\mathcal{O}_Y}(\mathcal{L}^{-1}, \mathcal{H}\text{om}_{\mathcal{O}_Y}(\mathcal{I}, \mathcal{O}_Y)) = \mathcal{H}\text{om}_{\mathcal{O}_Y}(\mathcal{I} \otimes_{\mathcal{O}_Y} \mathcal{L}^{-1}, \mathcal{O}_Y)$$

so the hypothesis signifies that there is a nonzero homomorphism $u : \mathcal{I} \otimes \mathcal{L}^{-1} \rightarrow \mathcal{O}_Y$. Now for any $y \in Y$, $(\mathcal{I} \otimes \mathcal{L}^{-1})_y$ is identified with a sub- $\mathcal{O}_{Y,y}$ -module of the fraction field $\mathcal{K}_{Y,y}$ of $\mathcal{O}_{Y,y}$, so u_y is necessarily injective, and u is therefore an isomorphism from $\mathcal{I} \otimes \mathcal{L}^{-1}$ onto an ideal \mathcal{I}' of \mathcal{O}_Y . As the blow up Y -scheme relative to \mathcal{I} and $\mathcal{I} \otimes \mathcal{L}^{-1}$ are isomorphic ??, this proves the corollary. The last remark is a direct consequence of ??.

Corollary 20.2.21. *Let X and Y be integral schemes that are projective over a field k , and let $f : X \rightarrow Y$ be a birational k -morphism. Then X is k -isomorphic to a blow up Y -scheme relative to a closed subscheme Y' (not necessarily reduced) of Y .*

Proof. In fact, f is projective by ??, and as Y is projective over k , the condition of ?? is satisfied. It then suffices to consider the closed subscheme of Y defined by the coherent ideal \mathcal{I} .

20.2.4 Euler characteristic and Hilbert polynomial

Let A be an Artinian ring, X be a projective scheme over $Y = \text{Spec}(A)$. For any coherent \mathcal{O}_X -module \mathcal{F} , the cohomology module $H^i(X, \mathcal{F})$ is finitely generated ??, hence is of finite length. We also see that $H^i(X, \mathcal{F}) = 0$ for $i \gg 0$, so the integer

$$\chi_A(\mathcal{F}) = \sum_{i=0}^{\infty} (-1)^i \ell_A(H^i(X, \mathcal{F}))$$

is defined for any coherent \mathcal{O}_X -module. If A is local Artinian, then we say that $\chi_A(\mathcal{F})$ is the Euler characteristic of \mathcal{F} (over the ring A). For $\mathcal{F} = \mathcal{O}_X$, the integer $\chi_A(\mathcal{O}_X)$ is called the **arithmetic genus** of X (over A).

It is clear that the map χ_A is additive on the category of coherent \mathcal{O}_X -modules. That is, for an exact sequence $0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow 0$, we have

$$\chi_A(\mathcal{F}) = \chi_A(\mathcal{F}') + \chi_A(\mathcal{F}'').$$

Theorem 20.2.22. *Let A be a local Artinian ring, X be a projective scheme over $Y = \text{Spec}(A)$, \mathcal{L} be a very ample \mathcal{O}_X -module relative to Y , and \mathcal{F} be a coherent \mathcal{O}_X -module. For $n \in \mathbb{Z}$, we set $\mathcal{F}(n) = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}$.*

- (a) *There exists a polynomial $P \in \mathbb{Q}[T]$ such that $\chi_A(\mathcal{F}(n)) = P(n)$ for $n \in \mathbb{Z}$ (the polynomial P is called the Hilbert polynomial of \mathcal{F} over A).*

(b) For $n \gg 0$, we have $\chi_A(\mathcal{F}(n)) = \ell_A(\Gamma(X, \mathcal{F}(n)))$.

(c) The leading coefficient of $\chi_A(\mathcal{F}(n))$ is positive.

Example 20.2.23. Let k be a field, $r > 0$ be an integer, and $X = \mathbb{P}_k^r$. Then we have $\chi_A(\mathcal{O}_X(n)) = \binom{n+r}{r}$ for $n \in \mathbb{Z}$. To see this, we divide into three cases.

- For $n > 0$, we have $\chi_A(\mathcal{O}_X(n)) = \dim_k(H^0(X, \mathcal{O}_X(n)))$, which is the number of homogeneous polynomials of degree n and is equal to $\binom{n+r}{r}$.
- For $n < -r$, we have $\chi_A(\mathcal{O}_X(n)) = (-1)^r \dim_k(H^r(X, \mathcal{O}_X(n)))$. If $n = -r - d$, the dimension of $H^r(X, \mathcal{O}_X(n))$ over k is the number of sequences of integers $(\alpha_0, \dots, \alpha_r)$ with $\alpha_i > 0$ and $|\alpha| = r + d$ (??), which is equal to the number $\binom{d+r-1}{r} = (-1)^r \binom{n+r}{r}$.
- For $-r \leq n \leq 0$, we have $\binom{n+r}{r} = 0$ and also $H^i(X, \mathcal{O}_X(n)) = 0$ for $i \geq 0$.

Corollary 20.2.24. Let A be a local Artinian ring, S be a graded A -algebra of finite type generated by S_1 , M be a graded S -module, and $X = \text{Proj}(S)$. Then we have $\chi_A(\tilde{M}(n)) = \ell_A(M_n)$ for $n \gg 0$.

Proof. This follows from ??, since $H^i(X, \tilde{M}(n)) = 0$ for $i > 0$ if $n \gg 0$. \square

20.2.5 Cohomological criterion for ampleness

Proposition 20.2.25. Let Y be a Noetherian scheme, $f : X \rightarrow Y$ be a proper morphism, and \mathcal{L} be an invertible \mathcal{O}_X -module. For any coherent \mathcal{O}_X -module \mathcal{F} , we set $\mathcal{F}(n) = \mathcal{F} \otimes \mathcal{L}^{\otimes n}$ for $n \in \mathbb{Z}$.

- (i) \mathcal{L} is ample for f .
- (ii) For any coherent \mathcal{O}_X -module \mathcal{F} , there exists an integer n_0 such that for $n \geq n_0$, we have $R^p f_*(\mathcal{F}(n)) = 0$ for $p > 0$.
- (iii) For any coherent ideal \mathcal{I} of \mathcal{O}_X , there exists an integer n_0 such that for $n \geq n_0$, we have $R^1 f_*(\mathcal{I}(n)) = 0$.

20.3 The finiteness theorem for proper morphisms

20.3.1 The dévissage lemma

Let \mathcal{A} be an abelian category. We say that a subset \mathfrak{E} of objects of \mathcal{A} is exact if $0 \in \mathfrak{E}$, and for any exact sequence $0 \rightarrow A' \rightarrow A \rightarrow A'' \rightarrow 0$ in \mathcal{A} such that two of A, A', A'' are in \mathfrak{E} , then the third one is in \mathfrak{E} . The following technique, called the *dévissage lemma*, is introduced by Alexander Grothendieck to prove statements about coherent sheaves on Noetherian schemes. One can think this method as an adaption of Noetherian induction.

Theorem 20.3.1 (Dévissage Lemma). Let X be a Noetherian scheme, \mathfrak{E} be an exact subset of the category \mathcal{A} of coherent \mathcal{O}_X -modules, and X' be a closed subset of X . Suppose that for any irreducible subset Y of X' , with generic point y , there exists a coherent \mathcal{O}_X -module $\mathcal{G} \in \mathfrak{E}$ such that \mathcal{G}_y is a $\kappa(y)$ -vector space of dimension 1. Then any coherent \mathcal{O}_X -module with support in X' belongs to \mathfrak{E} .

Proof. Consider the following property $\mathcal{P}(Y)$ on a closed subset Y of X : any coherent \mathcal{O}_X -module with support contained in Y belongs to \mathfrak{E} . In view of the Noetherian induction principle (??), we are reduced to prove the following: if Y is a closed subset of X' such that $\mathcal{P}(Y')$ is valid for any proper closed subset $Y' \subseteq Y$, then $\mathcal{P}(Y)$ is valid.

Now let \mathcal{F} be a coherent \mathcal{O}_X -module with support contained in Y , we show that $\mathcal{F} \in \mathfrak{E}$. In this case, we endow Y with the reduced subscheme structure of X , and let \mathcal{I} be the ideal of \mathcal{O}_X defining it. By ??, we see there exists an integer $n > 0$ such that $\mathcal{I}^n \mathcal{F} = 0$; for $1 \leq k \leq n$, we then have an exact sequence

$$0 \longrightarrow \mathcal{I}^{k-1} \mathcal{F} / \mathcal{I}^k \mathcal{F} \longrightarrow \mathcal{F} / \mathcal{I}^k \mathcal{F} \longrightarrow \mathcal{F} / \mathcal{I}^{k-1} \mathcal{F} \longrightarrow 0$$

of coherent \mathcal{O}_X -modules. As \mathfrak{E} is exact, by recurrence on k , it suffices to prove that $\mathcal{F}_k = \mathcal{I}^{k-1} \mathcal{F} / \mathcal{I}^k \mathcal{F}$ belongs to \mathfrak{E} ; in other words, we may also assume that $\mathcal{I} \mathcal{F} = 0$, which means $\mathcal{F} = j_*(j^*(\mathcal{F}))$, where $j : Y \rightarrow X$ is the canonical injection.

First suppose that Y is reducible, and let $Y = Y' \cup Y''$, where Y', Y'' are proper closed subsets of Y . We endow Y', Y'' with the reduced subscheme structure, and let $\mathcal{J}', \mathcal{J}''$ be the defining ideals of \mathcal{O}_X . Put $\mathcal{F}' = \mathcal{F} \otimes_{\mathcal{O}_X} (\mathcal{O}_X / \mathcal{J}')$ and $\mathcal{F}'' = \mathcal{F} \otimes_{\mathcal{O}_X} (\mathcal{O}_X / \mathcal{J}'')$. The canonical homomorphisms $\mathcal{F} \rightarrow \mathcal{F}'$, $\mathcal{F} \rightarrow \mathcal{F}''$ then define a homomorphism $u : \mathcal{F} \rightarrow \mathcal{F} \oplus \mathcal{F}''$ such that any point $z \notin Y' \cap Y''$, the induced homomorphism $u_z : \mathcal{F} \rightarrow \mathcal{F}'_z \oplus \mathcal{F}''_z$ is bijective. In fact, we have $\mathcal{J}' \cap \mathcal{J}'' = \mathcal{J}$; if $z \notin Y''$ then $\mathcal{J}'_z = \mathcal{J}_z$, so $\mathcal{F}'_z = \mathcal{F}_z$ and $\mathcal{F}''_z = 0$; and similarly if $z \notin Y'$. The kernel and cokernel of u , which belong to \mathcal{A} , are then supported in $Y' \cap Y''$, and hence belong to \mathfrak{E} by hypothesis. By the same reasoning, \mathcal{F}' and \mathcal{F}'' are in \mathfrak{E} , hence so is $\mathcal{F}' \oplus \mathcal{F}''$. We now conclude from the following exact sequences

$$\begin{aligned} 0 \longrightarrow \ker u &\longrightarrow \mathcal{F} \longrightarrow \operatorname{im} u \longrightarrow 0 \\ 0 \longrightarrow \operatorname{im} u &\longrightarrow \mathcal{F}' \oplus \mathcal{F}'' \longrightarrow \operatorname{coker} u \longrightarrow 0 \end{aligned}$$

that \mathcal{F} belongs to \mathfrak{E} .

If on the other hand Y is irreducible, then the subscheme Y of X is integral. If y is the generic point of Y , we have $\mathcal{O}_{Y,y} = \kappa(y)$, and as $j^*(\mathcal{F})$ is a coherent \mathcal{O}_Y -module, $\mathcal{F}_y = (j^*(\mathcal{F}))_y$ is a $\kappa(y)$ -vector space of finite dimension m . By hypothesis, there exists a coherent \mathcal{O}_X -module $\mathcal{G} \in \mathfrak{E}$ (necessarily supported in Y) such that \mathcal{G}_y is a $\kappa(y)$ -vector space of dimension 1, so there is an $\kappa(y)$ -isomorphism $(\mathcal{G}_y)^m \cong \mathcal{F}_y$, which is also an \mathcal{O}_Y -isomorphism, and as \mathcal{G}^m and \mathcal{F} are coherent, there exists an open neighborhood W of y in X and an isomorphism $\mathcal{G}^m|_W \cong \mathcal{F}|_W$ (??). Let \mathcal{H} be the graph of this isomorphism, which is a coherent sub- $(\mathcal{O}_X|_W)$ -module of $(\mathcal{G}^m \oplus \mathcal{F})|_W$. Then there is a sub- \mathcal{O}_X -module \mathcal{H}_0 of $\mathcal{G}^m \oplus \mathcal{F}$, inducing \mathcal{H} over W and zero over $X - Y$ (??). The restrictions $v : \mathcal{H}_0 \rightarrow \mathcal{G}^m$ and $w : \mathcal{H}_0 \rightarrow \mathcal{F}$ of the canonical projections of $\mathcal{G}^m \oplus \mathcal{F}$ are then homomorphisms of coherent \mathcal{O}_X -modules which are isomorphic over W and over $X - Y$. The kernel and cokernels of these homomorphisms are then supported in the proper closed subset $Y - (Y \cap W)$ of Y , so by hypotheses they belong to \mathfrak{E} . On the other hand, we have $\mathcal{G}^m \in \mathfrak{E}$ since $\mathcal{G} \in \mathfrak{E}$, so by the exactness of \mathfrak{E} we conclude that $\mathcal{H}_0 \in \mathfrak{E}$, hence $\mathcal{F} \in \mathfrak{E}$. \square

Corollary 20.3.2. Suppose that the exact subset \mathfrak{E} satisfies the additional property that any direct factor in \mathcal{A} of a coherent \mathcal{O}_X -module $\mathcal{F} \in \mathfrak{E}$ belongs to \mathfrak{E} . Then the conclusion of ?? is still valid if

we replace " \mathcal{G}_y is a $\kappa(y)$ -vector space of dimension 1" by the condition that $\mathcal{G}_y \neq 0$ (or equivalently $\text{supp}(\mathcal{G}) = Y$).

Proof. In fact, in this case the proof of ?? when Y is irreducible can be modified as follows: \mathcal{G}_y is a $\kappa(y)$ -vector space of dimension $q > 0$, and we then have an isomorphism $(\mathcal{G}_y)^m \cong (\mathcal{F}_y)^q$. By the same reasoning, we obtain that $\mathcal{F}^q \in \mathfrak{E}$, so $\mathcal{F} \in \mathfrak{E}$ by our additional assumption on \mathfrak{E} . \square

20.3.2 The finiteness theorem for proper morphisms

Theorem 20.3.3. *Let Y be a locally Noetherian scheme and $f : X \rightarrow Y$ be a proper morphism. Then for any coherent \mathcal{O}_X -module \mathcal{F} , the \mathcal{O}_Y -modules $R^p f_*(\mathcal{F})$ are coherent for $p \geq 0$.*

Proof. Since this question is local over Y , we can suppose that Y is Noetherian, and hence X is Noetherian (??). The coherent \mathcal{O}_X -module \mathcal{F} satisfying the conclusion of ?? is easily seen to form an exact subset \mathfrak{E} of the category \mathcal{A} of coherent \mathcal{O}_X -modules. In fact, let $0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow 0$ be an exact sequence of coherent \mathcal{O}_X -modules, and suppose for example that $\mathcal{F}', \mathcal{F}$ belong to \mathfrak{E} . Then we have an exact sequence

$$R^{p-1}f_*(\mathcal{F}'') \xrightarrow{\partial} R^p f_*(\mathcal{F}) \longrightarrow R^p f_*(\mathcal{F}) \longrightarrow R^p f_*(\mathcal{F}'') \xrightarrow{\partial} R^{p+1} f_*(\mathcal{F}')$$

in which the four outer terms are coherent. It then follows from ?? that $R^p f_*(\mathcal{F})$ is coherent. We also note that any direct factor $\mathcal{F}' \in \mathcal{A}$ of a coherent \mathcal{O}_X -module $\mathcal{F} \in \mathfrak{E}$ belongs to \mathfrak{E} : in fact, $R^p f_*(\mathcal{F}')$ is then a direct factor of $R^p f_*(\mathcal{F})$ (G, II 4.4.4 ??), so is of finite type, and since it is quasi-coherent (??), it is coherent (note that Y is Noetherian). In view of ??, we only need to prove that if X is irreducible with generic point x , then there exists a coherent \mathcal{O}_X -module $\mathcal{F} \in \mathfrak{E}$ such that $\mathcal{F}_x \neq 0$. In fact, for any irreducible closed subscheme Y of X with canonical injection $j : Y \rightarrow X$, the composition $f \circ j$ is proper (??), and if \mathcal{G} is a coherent \mathcal{O}_Y -module supported in Y , then $j_*(\mathcal{G})$ is a coherent \mathcal{O}_X -module such that $R^p(f \circ j)_*(\mathcal{G}) = R^p f_*(j_*(\mathcal{G}))$ (G, II, 4.9.1 ??), so we can apply our result on Y .

By Chow's lemma (??), there exists an irreducible scheme X' and a surjective projective morphism $g : X' \rightarrow X$ such that $f \circ g : X' \rightarrow Y$ is projective. By ??, there exists a g -ample $\mathcal{O}_{X'}$ -module \mathcal{L} , so by ?? we see that there exists an integer n_0 such that $\mathcal{F} = g_*(\mathcal{O}_{X'}(n))$ is a coherent \mathcal{O}_X -module and $R^p g_*(\mathcal{O}_{X'}(n)) = 0$ for $p > 0$ and $n \geq n_0$. Moreover, if x (resp. x') is the generic point of X (resp. X'), there exists an open subset U of x such that g is an isomorphism from $g^{-1}(U)$ onto U , so $\mathcal{F}_x \cong (\mathcal{O}_{X'}(n))_{x'} \neq 0$. On the other hand, since $f \circ g$ is projective and Y is Noetherian, the $R^p(f \circ g)_*(\mathcal{O}_{X'}(n))$ are coherent by ???. Consider the Grothendieck's spectral sequence:

$$E_2^{p,q} = R^p f_*(R^q g_*(\mathcal{O}_{X'}(n))) \Rightarrow R^*(f \circ g)_*(\mathcal{O}_{X'}(n)).$$

We have already remarked that for $n \gg 0$ we have $R^q g_*(\mathcal{O}_{X'}(n)) = 0$ for $q > 0$, so this sequence collapse at E_2 and we obtain an isomorphism $E_2^{p,0} = R^p f_*(\mathcal{F}) \cong R^p(f \circ g)_*(\mathcal{O}_{X'}(n))$, which implies $\mathcal{F} \in \mathfrak{E}$ and completes the proof. \square

Corollary 20.3.4. *Let A be a Noetherian ring, X be a proper scheme over A . For any coherent \mathcal{O}_X -module \mathcal{F} , the $H^p(X, \mathcal{F})$ are finitely generated A -modules, and there exists an integer r such that $H^p(X, \mathcal{F}) = 0$ for any coherent \mathcal{O}_X -module \mathcal{F} and $p > r$.*

Proof. The second assertion is proved in ??, and the first one follows from ??, in view of ???. \square

Corollary 20.3.5. *Let Y be a locally Noetherian scheme, $f : X \rightarrow Y$ be a morphism of finite type. For any coherent \mathcal{O}_X -module with support proper over Y , the \mathcal{O}_Y -modules $R^p f_*(\mathcal{F})$ are coherent for $p \geq 0$.*

Proof. Since this question is local over Y , we may assume that Y is Noetherian, and then so is X . By hypothesis, any closed subscheme Z of X with underlying space $\text{supp}(\mathcal{F})$ is proper over Y , so if $j : Y \rightarrow X$ is the canonical injection, $f \circ j$ is proper. Now we may choose Z such that $\mathcal{F} = j_*(\mathcal{G})$, where $\mathcal{G} = j^*(\mathcal{F})$ is a coherent \mathcal{O}_Z -module $(??)$. Since we have $R^p f_*(\mathcal{F}) = R^p(f \circ j)_*(\mathcal{G})$ by $??$, the conclusion follows from $??$. \square

Proposition 20.3.6. *Let Y be a Noetherian scheme, \mathcal{S} be a quasi-coherent graded \mathcal{O}_Y -algebra of finite type with positive degrees, $Y' = \text{Proj}(\mathcal{S})$, and $g : Y' \rightarrow Y$ be the structural morphism. Let $f : X \rightarrow Y$ be a proper morphism, $\mathcal{S}' = f^*(\mathcal{S})$, and \mathcal{M} be a quasi-coherent graded \mathcal{S}' -module of finite type. Then the $R^p f_*(\mathcal{M})$ are graded \mathcal{S} -modules of finite type for $p \geq 0$. Suppose moreover that \mathcal{S} is generated by \mathcal{S}_1 , then for any $p \in \mathbb{N}$, there exists an integer k_p such that for $k \geq k_p$ and $r \geq 0$, we have*

$$R^p f_*(\mathcal{M}_{k+r}) = \mathcal{S}_r R^p f_*(\mathcal{M}_k). \quad (20.3.1)$$

Proof. The first assertion follows from $??$ since in its proof the condition on f is only used to derive the coherence of the $\mathcal{O}_{Y'}$ -modules $R^p f'_*(\tilde{\mathcal{M}})$. With the hypotheses of $??$, f' is proper $(??\text{(iii)})$, so we can now utilize $??$ to complete the proof. \square

Corollary 20.3.7. *Let A be a Noetherian ring, \mathfrak{I} be an ideal of A , X be a proper A -scheme, and \mathcal{F} be a coherent \mathcal{O}_X -module. Then for any $p \geq 0$, the direct sum $\bigoplus_{k \geq 0} H^p(X, \mathfrak{I}^k \mathcal{F})$ is a finitely generated module over $S = \bigoplus_{k \geq 0} \mathfrak{I}^k$. In particular, there exists an integer $k_p \geq 0$ such that for $k \geq k_p$, $r \geq 0$, we have*

$$H^p(X, \mathfrak{I}^{k+r} \mathcal{F}) = \mathfrak{I}^r H^p(X, \mathfrak{I}^k \mathcal{F}). \quad (20.3.2)$$

Proof. It suffices to apply $??$ to $Y = \text{Spec}(A)$, $\mathcal{S} = \tilde{S}$, and $\mathcal{M}_k = \mathfrak{I}^k \mathcal{F}$. \square

20.3.3 Generalization to formal schemes

20.4 Zariski's main theorem and applications

20.4.1 Grothendieck's comparison theorem

Let X, Y be Noetherian schemes, $f : X \rightarrow Y$ be a proper morohism, Y' be a closed subscheme of Y , and X' be the inverse image $f^{-1}(Y')$. We denote by \widehat{X} and \widehat{Y} the formal completion $X_{/X'}$ and $Y_{/Y'}$ of X and Y along X' and Y' , respectively, and let $\widehat{f} : \widehat{X} \rightarrow \widehat{Y}$ be the extension of f to completions. For any coherent \mathcal{O}_X -module \mathcal{F} , let $\widehat{\mathcal{F}}$ be the completion $\mathcal{F}_{/X'}$ of \mathcal{F} along X' , which is coherent by $??$.

Let \mathcal{I} be a coherent ideal of \mathcal{O}_Y defining Y' , then by $??\text{(b)}$ the coherent ideal $\mathcal{K} = f^*(\mathcal{I})\mathcal{O}_X$ defines the closed subscheme X' of X . For each $k \geq 0$, we consider the coherent \mathcal{O}_X -module

$$\mathcal{F}_k = \mathcal{F} \otimes_{\mathcal{O}_Y} (\mathcal{O}_Y / \mathcal{I}^{k+1}) = \mathcal{F} / \mathcal{K}^{k+1} \mathcal{F}.$$

The \mathcal{O}_Y -modules $R^p f_*(\mathcal{F})$ and $R^p f_*(\mathcal{F}_k)$ are coherent for $p \geq 0$ $(??)$. For any $k \geq 0$ and $p \geq 0$,

the canonical homomorphism $\mathcal{F} \rightarrow \mathcal{F}_k$ defines by functoriality a homomorphism

$$R^p f_*(\mathcal{F}) \rightarrow R^p f_*(\mathcal{F}_k) \quad (20.4.1)$$

Moreover, as \mathcal{F}_k is an $\mathcal{O}_X/\mathcal{K}^{k+1}$ -module, $R^p f_*(\mathcal{F}_k)$ is an $\mathcal{O}_Y/\mathcal{I}^{k+1}$ -module (EGA 0_{III}, 12.2.1 ??) and we then deduce from (??) a homomorphism

$$R^p f_*(\mathcal{F}) \otimes_{\mathcal{O}_Y} (\mathcal{O}_Y/\mathcal{I}^{k+1}) \rightarrow R^p f_*(\mathcal{F}_k). \quad (20.4.2)$$

The two sides of (??) form two projective systems, and the projective limit of the left side is just the completion $(R^p f_*(\mathcal{F}))_{/\widehat{Y'}}$, which we also denote by $\widehat{R^p f_*(\mathcal{F})}$. Moreover, it is immediate that the homomorphisms in (??) form a projective system, so by passing to projective limit we obtain a canonocal homomorphism

$$\varphi_p : \widehat{R^p f_*(\mathcal{F})} \rightarrow \varprojlim_k R^p f_*(\mathcal{F}_k). \quad (20.4.3)$$

Since (??) is a homomorphism of $(\mathcal{O}_Y/\mathcal{I}^{k+1})$ -modules, and can be considered as a continuous homomorphism of discrete $\mathcal{O}_{\widehat{Y}}$ -modules, we see that the homomorphism φ_p is a continuous homomorphism of topological \mathcal{O}_Y -modules.

On the other hand, let $i_X : \widehat{X} \rightarrow X$ be the canonical morphism, which fits into the commutative diagram

$$\begin{array}{ccc} X_k & \xrightarrow{h_k} & \widehat{X} \\ & \searrow i_k & \downarrow i_X \\ & & X \end{array} \quad (20.4.4)$$

where X_k is the closed subscheme of X defined by the ideal \mathcal{K}^{k+1} , $i_k : X_k \rightarrow X$ is the canonical injection, and h_k is given the canonical homomorphism $\mathcal{O}_{\widehat{X}} \rightarrow \mathcal{O}_X/\mathcal{K}^{k+1}$. By ??, we have $\widehat{\mathcal{F}} = i_X^*(\mathcal{F})$. Since $\mathcal{F}_k = (i_k)_*(i_k^*(\mathcal{F}_k))$ (G, II, 4.9.1 ??), we see that

$$H^p(X_k, i_k^*(\mathcal{F}_k)) = H^p(X, \mathcal{F}_k). \quad (20.4.5)$$

The canonical homomorphism $H^p(\widehat{X}, \widehat{\mathcal{F}}) \rightarrow H^p(X_k, h_k^*(\widehat{\mathcal{F}}))$ can then be identified as the following homomorphism:

$$H^p(\widehat{X}, \widehat{\mathcal{F}}) \rightarrow H^p(X, \mathcal{F}_k). \quad (20.4.6)$$

These homomorphisms evidently form a projective system, so by passing to projective limit we obtain a canonical homomorphism

$$\psi_{p,X} : H^p(\widehat{X}, \widehat{\mathcal{F}}) \rightarrow \varprojlim_k H^p(X, \mathcal{F}_k). \quad (20.4.7)$$

Now in view of ??, by replacing X with open subsets of the form $f^{-1}(V)$, where V is an affine open of Y , we also obtain homomorphisms

$$\psi_{p,V} : H^p(\widehat{X} \cap f^{-1}(V), \widehat{\mathcal{F}}) \rightarrow \varprojlim_k \Gamma(V, R^p f_*(\mathcal{F}_k)).$$

It is clear that the homomorphisms $\psi_{p,V}$ are compatible with restrictions, so by shefification, we obtain an induced canonical homomorphism

$$\psi_p : R^p \hat{f}_*(\widehat{\mathcal{F}}) \rightarrow \varprojlim_k R^p f_*(\mathcal{F}_k). \quad (20.4.8)$$

Finally, let $i_Y : \widehat{Y} \rightarrow Y$ be the canonical morphism; as $R^p f_*(\mathcal{F})$ is a coherent \mathcal{O}_Y -module by ??, we have $i_Y^*(R^p f_*(\mathcal{F})) = \widehat{R^p f_*(\mathcal{F})}$??, and therefore a canonical homomorphism

$$\rho_p : \widehat{R^p f_*(\mathcal{F})} = i_Y^*(R^p f_*(\mathcal{F})) \rightarrow R^p \hat{f}_*(i_Y^*(\mathcal{F})) = R^p \hat{f}_*(\widehat{\mathcal{F}}), \quad (20.4.9)$$

which is defined in the same way as the canonical homomorphism ???. From the commutative diagram ???, we then obtain a commutative diagram

$$\begin{array}{ccc} \widehat{R^p f_*(\mathcal{F})} & \xrightarrow{\rho_p} & R^p \hat{f}_*(\widehat{\mathcal{F}}) \\ \varphi_p \searrow & & \swarrow \psi_p \\ \varprojlim_k R^p f_*(\mathcal{F}_k) & & \end{array} \quad (20.4.10)$$

Theorem 20.4.1 (Grothendieck's Comparison Theorem). *Let $f : X \rightarrow Y$ be a proper morphism between Noetherian schemes, Y' be a closed subset of Y , and $X' = f^{-1}(Y')$. Then for any coherent $\mathcal{O}_{X'}$ -module \mathcal{F} , $R^p \hat{f}_*(\widehat{\mathcal{F}})$ is a coherent $\mathcal{O}_{X'}$ -module and the homomorphisms in ?? are homeomorphisms for $p \geq 0$.*

The fact that ρ_p is an isomorphism signifies that the formation of $R^p f_*$ commutes with that of completion, so ?? gives a comparision result between formal geometry and algebraic geometry. We being its proof by establishing the following affine case:

Corollary 20.4.2. *Under the hypotheses of ??, assume that $Y = \text{Spec}(A)$, where A is Noetherian, and $\mathcal{I} = \widetilde{\mathcal{I}}$ is an ideal of A , so that $\mathcal{F}_k = \mathcal{F}/\mathcal{I}^{k+1}\mathcal{F}$. Then for each $p \geq 0$, the projective system $(H^p(X, \mathcal{F}_k))_{k \geq 0}$ satisfies the Mittag-Leffler condition, and the canonical homomorphism*

$$\psi_p : H^p(\widehat{X}, \widehat{\mathcal{F}}) \rightarrow \varprojlim_k H^p(X, \mathcal{F}_k) \quad (20.4.11)$$

is an isomorphism. Moreover, the filtration on $H^p(X, \mathcal{F})$ defined by the kernel of the canonical homomorphisms $H^p(X, \mathcal{F}) \rightarrow H^p(X, \mathcal{F}_k)$ is \mathcal{I} -good and the canonical homomorphisms

$$\varphi_p : H^p(\widehat{X}, \widehat{\mathcal{F}}) \rightarrow \varprojlim_k H^p(X, \mathcal{F}_k) \quad (20.4.12)$$

is a homeomorphism (where the left side is the \mathcal{I} -adic completion of $H^p(X, \mathcal{F})$).

Proof. Fix an integer $p \geq 0$, and we simplfy the notation by setting

$$H = H^p(X, \mathcal{F}), \quad H_k = H^p(X, \mathcal{F}_k), \quad R_k = \ker(H \rightarrow H_k) \subseteq H.$$

The exact sequence on cohomology

$$H^p(X, \mathcal{I}^{k+1}\mathcal{F}) \longrightarrow H^p(X, \mathcal{F}) \longrightarrow H^p(X, \mathcal{F}_k) \xrightarrow{\partial} H^{p+1}(X, \mathcal{I}^{k+1}\mathcal{F}) \longrightarrow H^{p+1}(X, \mathcal{F})$$

shows that R_k is also the image of the homomorphism $H^p(X, \mathcal{J}^{k+1}\mathcal{F}) \rightarrow H^p(X, \mathcal{F})$; we set

$$Q_k = \ker(H^{p+1}(X, \mathcal{J}^{k+1}\mathcal{F}) \rightarrow H^{p+1}(X, \mathcal{F})) = \text{im}(H^p(X, \mathcal{F}_k) \rightarrow H^{p+1}(X, \mathcal{J}^{k+1}\mathcal{F})) \quad (20.4.13)$$

so that there is an exact sequence

$$0 \longrightarrow R_k \longrightarrow H \longrightarrow H_k \longrightarrow Q_k \longrightarrow 0$$

Let x be an element of \mathcal{J}^m for $m \geq 0$; the multiplication by x on $\mathcal{J}^k\mathcal{F}$ is a homomorphism $\mathcal{J}^k\mathcal{F} \rightarrow \mathcal{J}^{k+m}\mathcal{F}$, and therefore gives a homomorphism

$$\mu_{x,m} : H^p(X, \mathcal{J}^k\mathcal{F}) \rightarrow H^p(X, \mathcal{J}^{k+m}\mathcal{F}).$$

If we denote by S the graded A -algebra $\bigoplus_{k \geq 0} \mathcal{J}^k$, then the multiplications $\mu_{x,m}$ define over $E = \bigoplus_{k \geq 0} H^p(X, \mathcal{J}^k\mathcal{F})$ a finitely generated graded S -module structure (??), which is Noetherian since S is Noetherian (??).

We begin by showing that the submodules (R_k) define a \mathcal{J} -good filtration on H . First, for any $x \in \mathcal{J}^m$, the diagram

$$\begin{array}{ccc} \mathcal{J}^{k+1}\mathcal{F} & \xrightarrow{x} & \mathcal{J}^{k+m+1}\mathcal{F} \\ \downarrow & & \downarrow \\ \mathcal{F} & \xrightarrow{x} & \mathcal{F} \end{array}$$

is commutative, so the corresponding diagram

$$\begin{array}{ccc} H^p(X, \mathcal{J}^{k+1}\mathcal{F}) & \xrightarrow{\mu_{x,m}} & H^p(X, \mathcal{J}^{k+m+1}\mathcal{F}) \\ \downarrow & & \downarrow \\ H^p(X, \mathcal{F}) & \xrightarrow{\mu_{x,0}} & H^p(X, \mathcal{F}) \end{array} \quad (20.4.14)$$

is commutative, which proves, in view of the interpretation of R_k as the image of $H^p(X, \mathcal{J}^{k+1}\mathcal{F})$ in $H^p(X, \mathcal{F})$, that $\mathcal{J}^m R_k \subseteq R_{k+m}$ and that the graded S -module $R = \bigoplus_{k \geq 0} R_k$ is a quotient of the sub- S -module $M = \bigoplus_{k \geq 0} H^p(X, \mathcal{J}^{k+1}\mathcal{F})$ of E . Since M is also a finitely generated S -module by ??, the S -module R is then finitely generated, which is equivalent to the condition that (R_k) is \mathcal{J} -good (??).

Consider now the graded S -module $N = \bigoplus_{k \geq 0} H^{p+1}(X, \mathcal{J}^{k+1}\mathcal{F})$, which is a finitely generated S -module in view of ???. For each $k \geq 0$, by (??) we have $Q_k \subseteq N_k$, and by replacing p by $p+1$ in the diagram (??) we see that $S_m Q_k = \mathcal{J}^m Q_k \subseteq Q_{k+m}$. In other words, $Q = \bigoplus_{k \geq 0} Q_k$ is a graded sub- S -module of N , and is therefore finitely generated. We denote by $\alpha_m : \mathcal{J}^m \rightarrow A$ the canonical injection, which can also be written as $S_m \rightarrow S_0$. Since $\mathcal{J}^{k+1}\mathcal{F}_k = 0$, the A -module $H^p(X, \mathcal{F}_k)$ is annihilated by \mathcal{J}^{k+1} , so Q_k , as the image of the A -homomorphism $H^p(X, \mathcal{F}_k) \rightarrow H^{p+1}(X, \mathcal{J}^{k+1}\mathcal{F})$, is also annihilated by \mathcal{J}^{k+1} . This signifies that, in the S -module Q , we have

$$\alpha_{k+1}(S_{k+1})Q_k = 0. \quad (20.4.15)$$

As Q is a finitely generated S -module, there exists an integers k_0 and d such that $Q_{k+d} = S_d Q_k$

for $k \geq k_0$ (??); we then deduce from this relation and (??) that there exists an integer $r > 0$ such that

$$\alpha_r(S_r)Q = 0. \quad (20.4.16)$$

Now note that the canonical injection $\mathfrak{I}^{k+m}\mathcal{F} \rightarrow \mathfrak{I}^k\mathcal{F}$ gives by passing to cohomology an A -homomorphism

$$\nu_m : H^{p+1}(X, \mathfrak{I}^{k+m}\mathcal{F}) \rightarrow H^{p+1}(X, \mathfrak{I}^k\mathcal{F})$$

and, for any $x \in \mathfrak{I}^m$, we have evidently a factorization

$$\mu_{x,0} : H^{p+1}(X, \mathfrak{I}^k\mathcal{F}) \xrightarrow{\mu_{x,m}} H^{p+1}(X, \mathfrak{I}^{k+m}\mathcal{F}) \xrightarrow{\nu_m} H^{p+1}(X, \mathfrak{I}^k\mathcal{F})$$

from which we conclude that, for any sub- A -module P of $H^{p+1}(X, \mathfrak{I}^k\mathcal{F})$, we have, in the S -module N ,

$$\nu_m(S_m P) = \alpha_m(S_m)P. \quad (20.4.17)$$

If we choose $m \geq r$ to be a multiple of d , then as $Q_{k+m} = S_m Q_k$ for $k \geq k_0$ (by our choice of d), we derive from (??) and (??) that $\nu_m(Q_{k+m}) = \alpha_m(S_m)Q_k \subseteq \alpha_r(S_r)Q_k = 0$ for $k \geq k_0$.

Consider now the commutative diagram

$$\begin{array}{ccccccc} H^p(X, \mathcal{F}) & \longrightarrow & H^p(X, \mathcal{F}_{k+m}) & \xrightarrow{\partial} & H^{p+1}(X, \mathfrak{I}^{k+1}\mathcal{F}) & \longrightarrow & H^{p+1}(X, \mathcal{F}) \\ \parallel & & \downarrow & & \downarrow & & \downarrow \\ H^p(X, \mathcal{F}) & \longrightarrow & H^p(X, \mathcal{F}_k) & \xrightarrow{\partial} & H^{p+1}(X, \mathfrak{I}^{k+1}\mathcal{F}) & \longrightarrow & H^{p+1}(X, \mathcal{F}) \end{array}$$

induced from the homomorphisms $\mathfrak{I}^{k+m+1}\mathcal{F} \rightarrow \mathfrak{I}^k\mathcal{F}$ and $\mathcal{F}_{k+m} \rightarrow \mathcal{F}_k$. From this, we conclude the following commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & R_{k+m} & \longrightarrow & H & \longrightarrow & H_{k+m} \longrightarrow Q_{k+m} \longrightarrow 0 \\ & & \downarrow & & \parallel & & \downarrow \\ 0 & \longrightarrow & R_k & \longrightarrow & H & \longrightarrow & H_k \longrightarrow Q_k \longrightarrow 0 \end{array}$$

whose rows are exact. As the last vertical arrow is zero for $k \geq k_0$, the image of H_{k+m} in H_k is contained in $\ker(H_k \rightarrow Q_k) = \text{im}(H \rightarrow H_k)$, whence equals to $\text{im}(H \rightarrow H_k)$ by the commutativity of the diagram. This image then equals to the image of $H_{k'}$ in H_k for $k' \geq k+m$, so the projective system $(H_k)_{k \geq 0}$ satisfies the Mittag-Leffler condition. Moreover, for any affine open U of X , we have $H^i(U, \mathcal{F}_k) = 0$ for $i > 0$, and the map $H^0(U, \mathcal{F}_{k+m}) \rightarrow H^0(U, \mathcal{F}_k)$ is surjective for $m \geq 0$ (??). We can then apply (EGA 0_{III}, 13.3.1 ??) to conclude that the canonical homomorphism $H^p(\widehat{X}, \widehat{\mathcal{F}}) \rightarrow \varprojlim_k H^p(X, \mathcal{F}_k)$ is bijective for each $p \geq 0$.

Now since the projective system (H/R_k) also satisfies the Mittag-Leffler condition, taking

projective limit preserves the exactness of the sequence

$$0 \longrightarrow H/R_k \longrightarrow H_k \longrightarrow Q_k \longrightarrow 0$$

As $v_m(Q_{k+m}) = 0$, we have $\varprojlim_k Q_k = 0$, so we obtain an isomorphism $\varprojlim_k (H/R_k) \cong \varprojlim_k H_k$. But the filtration (R_k) of H is \mathfrak{I} -good, so it defines the \mathfrak{I} -adic topology on H , and $\varprojlim_k (H/R_k)$ is therefore the \mathfrak{I} -adic completion of H . \square

Proof of ??. We now return to the proof of ???. For any affine open V of Y , since $R^p f_*(\mathcal{F})$ is coherent, we see that $\Gamma(V, \widehat{R^p f_*(\mathcal{F})})$ is equal to the \mathfrak{I} -adic completion of $\Gamma(V, R^p f_*(\mathcal{F}))$, and $\Gamma(V, \varprojlim_k R^p f_*(\mathcal{F}_k))$ is equal to $\varprojlim_k \Gamma(V, R^p f_*(\mathcal{F}_k))$ (EGA 0 I, 3.2.6 ??). The fact that φ_p is a homeomorphism then follows from ?? and ???. Similarly, we see that each $\psi_{p,V}$ is an isomorphism, so ψ_p is an isomorphism by the definition of $R^p \widehat{f}_*(\widehat{\mathcal{F}})$, and hence a homeomorphism by . \square

Corollary 20.4.3. *Under the hypotheses of ??, for any affine open V of Y , the canonical homomorphism*

$$H^p(\widehat{X} \cap f^{-1}(V), \widehat{\mathcal{F}}) \rightarrow \Gamma(\widehat{Y} \cap V, R^p \widehat{f}_*(\widehat{\mathcal{F}}))$$

is bijective.

Proof. This follows from the isomorphism ψ_p in ?? and ???. \square

Remark 20.4.1. Let $f : X \rightarrow Y$ be a morphism of finite type between Noetherian schemes, and \mathcal{F} be a coherent \mathcal{O}_Y -module with support proper over Y . Then we see from ?? that $R^p f_*(\mathcal{F})$ is coherent for $p \geq 0$. Moreover, we can also suppose that $\mathcal{F} = j_*(\mathcal{G})$, where $\mathcal{G} = j^*(\mathcal{F})$ is a coherent \mathcal{O}_Z -module, Z being a closed subscheme of X with underlying space $\text{supp}(\mathcal{F})$, and $j : Z \rightarrow X$ is the canonical injection (??). If we set $\mathcal{G}_k = \mathcal{G} \otimes_{\mathcal{O}_Y} (\mathcal{O}_Y/\mathcal{I}^{k+1})$, then

$$\mathcal{G}_k = j^*(\mathcal{F}_k), \quad R^p f_*(\mathcal{F}_k) = R^p(f \circ j)_*(\mathcal{G}_k), \quad R^p f_*(\mathcal{F}) = R^p(f \circ j)_*(\mathcal{G})$$

by ???. On the other hand, in view of ??, we also have

$$R^p \widehat{f}_*(\widehat{\mathcal{F}}) = R^p \widehat{(f \circ j)}_*(\mathcal{G}).$$

We can then apply ?? to conclude that the result of ?? is also valid for \mathcal{F} and f .

Proposition 20.4.4 (Formal Function Theorem). *Let Y be a locally Noetherian scheme, $f : X \rightarrow Y$ be a proper morphism, and \mathcal{F} be a coherent \mathcal{O}_X -module. Then for any $y \in Y$ and $p \geq 0$, the $\mathcal{O}_{Y,y}$ -module $(R^p f_*(\mathcal{F}))_y$ is finitely generated, separated for the \mathfrak{m}_y -adic topology, and we have a homeomorphism*

$$(R^p \widehat{f}_*(\widehat{\mathcal{F}}))_y \xrightarrow{\sim} \varprojlim_k H^p(f^{-1}(y), \mathcal{F} \otimes_{\mathcal{O}_Y} (\mathcal{O}_{Y,y}/\mathfrak{m}_y^k))$$

where the left side is the \mathfrak{m}_y -adic completion of $(R^p f_(\mathcal{F}))_y$ and $f^{-1}(y)$ is considered as the underlying space of $X \times_Y \text{Spec}(\mathcal{O}_{Y,y}/\mathfrak{m}_y^k)$ (??).*

Proof. As $\mathcal{O}_{Y,y}$ is a Noetherian local ring and $(R^p f_*(\mathcal{F}))_y$ is finitely generated by ??, the \mathfrak{m}_y -adic topology on $(R^p f_*(\mathcal{F}))_y$ is separated (??). The assertions therefore result from ?? if Y is

Noetherian and the point y is closed, since we can then replace Y be an affine neighborhood of y and put $Y' = \{y\}$, in view of (G, II, 4.9.1 ??). In the general case, we set

$$Y_1 = \text{Spec}(\mathcal{O}_{Y,y}), \quad X_1 = X \times_Y Y_1, \quad \mathcal{F}_1 = \mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{O}_{Y_1}, \quad f_1 = f \times 1_{Y_1} : X_1 \rightarrow Y_1.$$

Then Y_1 is Noetherian, f_1 is proper, and \mathcal{F}_1 is coherent. Let y_1 be the unique closed point in Y_1 ; the proposition is then valid for f_1 , \mathcal{F}_1 and y_1 . We have $\mathcal{O}_{Y_1,y_1} = \mathcal{O}_{Y,y}$, $f_1^{-1}(y_1) = f^{-1}(y)$??, and the schemes $X \times_Y \text{Spec}(\mathcal{O}_{Y,y}/\mathfrak{m}_y^k)$ and $X_1 \times_{Y_1} \text{Spec}(\mathcal{O}_{Y_1,y_1}/\mathfrak{m}_{y_1}^k)$ are canonically identified ???. Moreover, $\mathcal{F}_1 \otimes_{\mathcal{O}_{Y_1}} (\mathcal{O}_{Y_1,y_1}/\mathfrak{m}_{y_1}^k)$ is identified with $\mathcal{F} \otimes_{\mathcal{O}_Y} (\mathcal{O}_{Y,y}/\mathfrak{m}_y^k)$???. It then remains to see that $R^p(f_1)_*(\mathcal{F}_1)$ is canonically isomorphic to $R^p f_*(\mathcal{F}) \otimes_{\mathcal{O}_Y} \mathcal{O}_{Y_1}$, which follows from ??, since the local morphism $\text{Spec}(\mathcal{O}_{Y,y}) \rightarrow Y$ is flat. \square

Corollary 20.4.5. *Let Y be a locally Noetherian scheme. $f : X \rightarrow Y$ be a proper morphism, y be a point of Y , and r be the dimension of $f^{-1}(y)$. Then for any coherent \mathcal{O}_X -module \mathcal{F} , the sheaf $R^p f_*(\mathcal{F})$ vanishes in a neighborhood of y for $p > r$.*

Proof. In fact, if $p > r$, we then have $H^p(f^{-1}(y), \mathcal{F} \otimes (\mathcal{O}_{Y,y}/\mathfrak{m}_y^k)) = 0$ (by Leray's vanishing theorem) for any k , so the \mathfrak{m}_y -adic completion of $(R^p f_*(\mathcal{F}))_y$ is zero. As this topology is separated by ??, we then have $(R^p f_*(\mathcal{F}))_y = 0$, so the conclusion follows from ???. \square

Corollary 20.4.6. *Under the hypothesis of ??, we have a canonical homeomorphism*

$$\widehat{(f_*(\mathcal{F}))}_y \xrightarrow{\sim} \varprojlim_k \Gamma(f^{-1}(y), \mathcal{F}_y / \mathfrak{m}_y^k \mathcal{F}_y).$$

Remark 20.4.2. Many applications of ?? use only the case $p = 0$, in which case the right-hand side is equal to $\Gamma(\widehat{X}, \widehat{\mathcal{F}})$, where \widehat{X} is the formal completion of X along X_y , and $\widehat{\mathcal{F}} = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_{\widehat{X}}$. In particular, if $\mathcal{F} = \mathcal{O}_X$, we have $\Gamma(\widehat{X}, \mathcal{O}_{\widehat{X}})$, which is the ring of **formal-regular functions** (also called **holomorphic functions**) on X along X_y .

20.4.2 Zariski's connectedness theorem

Theorem 20.4.7 (Zariski's Connectednes Theorem). *Let Y be a locally Noetherian scheme, $f : X \rightarrow Y$ be a proper morphism, and*

$$X \xrightarrow{f'} Y' \xrightarrow{g} Y$$

be the Stein factorization of f . Then g is finite, f' is proper, $f'_(\mathcal{O}_X)$ is isomorphic to $\mathcal{O}_{Y'}$, and the fibers $f'^{-1}(y')$ are nonempty and connected for any $y' \in Y'$.*

Proof. Let $\theta : \mathcal{O}_{Y'} \rightarrow f'_*(\mathcal{O}_X)$ be the morphism induced by f' . Then since $g_*(\mathcal{O}_{Y'}) = \mathcal{A}(Y')$ and $f_*(\mathcal{O}_X) = \mathcal{A}(X)$, the homomorphism $g_*(\theta) : g_*(\mathcal{O}_{Y'}) \rightarrow g_*(f'_*(\mathcal{O}_X)) = f_*(\mathcal{O}_X)$ is an isomorphism, so θ is an isomorphism by ???. Since $\mathcal{A}(X)$ is coherent by ??, we see f' is finite, and it is proper by ???. It then remains to prove the following assertion: if $f : X \rightarrow Y$ is a proper morphism with $f_*(\mathcal{O}_X) \cong \mathcal{O}_Y$ and Y being locally Noetherian, then the fibers $f^{-1}(y)$ are nonempty and connected for any $y \in Y$. To this end, we note that the hypothesis $f_*(\mathcal{O}_X) \cong \mathcal{O}_Y$ implies $(f_*(\mathcal{O}_X))_y \cong \mathcal{O}_{Y,y} \neq 0$ for any $y \in Y$, so f is dominant and hence is surjective since f is closed. We may then, as in ??, reduce to the case where y is closed in Y . Then $f^{-1}(y)$ is a Noetherian space with finitely many connected components, and is equal to the underlying space of

the completion \widehat{X} of X along $f^{-1}(y)$. If $(Z_i)_{1 \leq i \leq n}$ is its connected components, it is clear that $\Gamma(\widehat{X}, \mathcal{O}_{\widehat{X}})$ is the direct product of the rings $\Gamma(Z_i, \mathcal{O}_{\widehat{X}})$, and each of them is nonzero since the unit section is nonzero at any point of \widehat{X} . Now if we apply ?? to $\mathcal{F} = \mathcal{O}_X$, whose completion along $f^{-1}(y)$ is $\mathcal{O}_{\widehat{X}}$, then we see that $\Gamma(\widehat{X}, \mathcal{O}_{\widehat{X}})$ is isomorphic to the \mathfrak{m}_y -adic completion $\widehat{\mathcal{O}_{Y,y}}$ of the local ring $\mathcal{O}_{\widehat{X}}$; this is a local ring which can not be a direct product of nonzero proper rings, since otherwise there are elements e_1, e_2 such that $e_1 + e_2 = 1$ and $e_1 e_2 = 0$. But then e_1, e_2 are nonunits so are contained in the maximal ideal, and their sum cannot be 1. We therefore have $n = 1$, which proves our assertion. \square

Corollary 20.4.8. *Under the hypothesis of ??, if $f_*(\mathcal{O}_X) \cong \mathcal{O}_Y$, then the fibers $f^{-1}(y)$ are connected and nonempty for any $y \in Y$.*

Corollary 20.4.9. *Under the hypothesis of ??, for any $y \in Y$, the set of connected components of the fiber $f^{-1}(y)$ is in one-to-one correspondence to the fiber $g^{-1}(y)$ (in other words, the set of maximal ideals of $(f_*(\mathcal{O}_X))_y$).*

Proof. Since Y' is finite over Y , it has finite fiber at y (??). As $f^{-1}(y) = f'^{-1}(g^{-1}(y))$, the corollary then follows from ??.

Remark 20.4.3. Let k be an extension field of $\kappa(y)$; if the scheme $f^{-1}(y) \otimes_{\kappa(y)} k = X \times_Y \text{Spec}(k)$ is connected, so is $f^{-1}(y)$, since it is the image under a projection. For a morphism $f : X \rightarrow Y$ of scheme and a point $y \in Y$, we say that the fiber $f^{-1}(y)$ is **geometrically connected** if for any field extension k of $\kappa(y)$, the scheme $f^{-1}(y) \otimes_{\kappa(y)} k = X \times_Y \text{Spec}(k)$ is connected. Under the hypothesis of ??, we can then conclude that the fibers $f^{-1}(y)$ are in fact geometrically connected. To see this, observe that for any extension k of $\kappa(y)$, there exists a Noetherian local ring A and a local homomorphism $\varphi : \mathcal{O}_{Y,y} \rightarrow A$ which is flat and such that the residue field of A is $\kappa(y)$ -isomorphic to k (EGA 0_{III}, 10.3.1 ??). Then let $Y_1 = \text{Spec}(A)$ and $h : Y_1 \rightarrow Y$ be the local morphism corresponding to φ , sending the unique closed point y_1 of Y_1 to y ; put $X_1 = X \times_Y Y_1$ and $f_1 = f \times 1_{Y_1}$. Then f_1 is proper and $f_1^{-1}(y_1)$ is $\kappa(y_1)$ -isomorphic to $X \times_Y \text{Spec}(k)$. It then remains to show that $(f_1)_*(\mathcal{O}_{X_1}) = \mathcal{O}_{Y_1}$; now since g is flat, we have $(f_1)_*(\mathcal{O}_{X_1}) = h^*(f_*(\mathcal{O}_X)) = h^*(\mathcal{O}_Y) = \mathcal{O}_Y$, in view of ?? applied to $p = 0$.

In the general case of ??, the same reasoning shows that we have $(f_1)_*(\mathcal{O}_{X_1}) = h^*(g_*(\mathcal{O}_{Y'}))$ (??), and the Stein factorization $f_1 = g_1 \circ f'_1$ of f_1 is such that we have the Cartesian square

$$\begin{array}{ccccc} X_1 & \xrightarrow{f'_1} & Y'_1 & \xrightarrow{g_1} & Y_1 \\ \downarrow & & \downarrow & & \downarrow h \\ X & \xrightarrow{f'} & Y' & \xrightarrow{g} & Y \end{array}$$

By the transitivity of fibers, we then see that the number of connected components of $f_1^{-1}(y_1)$ is, in view of ??, equal to the number of elements of $g_1^{-1}(y_1) = g^{-1}(y) \otimes_{\kappa(y)} k$. If we choose k to be an algebraic closure of $\kappa(y)$, this is also the number of geometric points of $g^{-1}(y)$, or the sum of the separable degrees $[\kappa(y'_i) : \kappa(y)]_s$, where y'_i runs through the finite set $g^{-1}(y)$ (??). Note that the $\kappa(y'_i)$ are none other than the residual fields of the semi-local ring $(f_*(\mathcal{O}_X))_y$.

Proposition 20.4.10. *Let X and Y be two locally Noetherian integral schemes and $f : X \rightarrow Y$ be a proper and dominante morphism. For any $y \in Y$, the number of connected components of $f^{-1}(y)$ is*

also equal to the number of maximal ideals of the integral closure $\mathcal{O}'_{Y,y}$ of $\mathcal{O}_{Y,y}$ in the rational function field $K(X)$.

Proof. Recall that for any open subset U of Y , $\Gamma(U, f_*(\mathcal{O}_X)) = \Gamma(f^{-1}(U), \mathcal{O}_X)$ is the intersection of the local rings $\mathcal{O}_{X,x}$ for $x \in f^{-1}(U)$ (formula (??)). We then conclude that the fiber $(f_*(\mathcal{O}_X))_y$ is a subring of $K(X)$ containing $\mathcal{O}_{Y,y}$. Moreover, as $f_*(\mathcal{O}_X)$ is a coherent \mathcal{O}_X -module, $(f_*(\mathcal{O}_X))_y$ is a finitely generated $\mathcal{O}_{Y,y}$ -module, hence contained in $\mathcal{O}'_{Y,y}$. By ??, any maximal ideal of the ring A is the intersection of A of a maximal ideal of $\mathcal{O}'_{Y,y}$, whence the proposition. \square

A local ring A is called **unibranch** if A_{red} is an integral ring and the integral closure of A_{red} is a local ring. We say a point y of an integral scheme Y is unibranch if the local ring $\mathcal{O}_{Y,y}$ is **unibranch** (this is the case if Y is normal at y). Let A be an integral local ring, and K be its fractional field. For A to be unibranch, it is then necessary and sufficient that any subring R of K , containing A and is a finite A -algebra, is a local ring. In fact, if A' is the integral closure A , then any such ring is contained in A' and any maximal ideal of R is trace of a maximal ideal of A' along R , so if A' is local, so is R . Conversely, A' is the inductive limit of the filtered family of such finite A -algebras A_α of A' , and if each A_α is a local ring, then for $A_\alpha \subseteq A_\beta$, the maximal ideal of A_α is the trace over A_α of the maximal ideal of A_β , so A' is a local ring.

We also note that if the completion of a Noetherian local ring A is integral (in this case A is called **analytically integral**), then A is unibranch. In fact, let \mathfrak{m} be the maximal ideal of A , K be its fraction field, and L be that of \widehat{A} ; we then have $L = K \otimes_A \widehat{A}$. Let B be a finite sub- A -algebra of K , then the subring R of L generated by \widehat{A} and B is isomorphic to $B \otimes_A \widehat{A}$; this is a finitely generated \widehat{A} -module, and is the \mathfrak{m} -adic completion of B . As B is a semi-local ring (??) and this completion is integral, we conclude from ?? that B has a unique maximal ideal, whence our assertion.

Corollary 20.4.11. *Under the hypotheses of ??, suppose that $[K(Y) : K(X)]_s = n$ and that $y \in Y$ is unibranch. Then the fiber $f^{-1}(y)$ has at most n connected components. In particular, if $K(X)$ is purely inseparable over $K(Y)$, then $f^{-1}(y)$ is connected.*

Proof. Let $\mathcal{O}'_{Y,y}$ be the integral closure of $\mathcal{O}_{Y,y}$, then the integral closure R of $\mathcal{O}_{Y,y}$ in $K(X)$ is also that of $\mathcal{O}'_{Y,y}$. If $\mathcal{O}'_{Y,y}$ is a local ring, then R is a semi-local ring with at most n maximal ideals (??). \square

Remark 20.4.4. ?? is essentially the "connectedness theorem" proved by Zariski for algebraic schemes. Note that in ?? if we suppose that Y is normal at y , then the fiber $f^{-1}(y)$ is geometrically connected, since (with the notation of ??) $g^{-1}(y)$ then reduces to a point y' and $\kappa(y')$ is purely inseparable over $\kappa(y)$.

Corollary 20.4.12. *Under the hypothesis of ??, suppose moreover that $K(Y)$ is algebraically closed in $K(X)$, and let y be a normal point of Y . Then $f^{-1}(y)$ is geometrically connected, and there exists an open neighborhood U of y such that $f_*(\mathcal{O}_X|_{f^{-1}(U)})$ is isomorphic to $\mathcal{O}_Y|_U$. In particular, if we suppose that Y is normal (and $K(Y)$ is algebraically closed in $K(X)$), then $f_*(\mathcal{O}_X)$ is isomorphic to \mathcal{O}_Y .*

Proof. The first assertion concerning $f^{-1}(y)$ is a particular case of ?. If $f : X \xrightarrow{f'} Y' \xrightarrow{g} Y$ is the Stein factorization, then $g^{-1}(y)$ is reduced to a single point y' . Moreover, we have $\mathcal{O}_{Y,y} \subseteq \mathcal{O}_{Y',y'} = (f_*(\mathcal{O}_X))_y \subseteq K(X)$, and as $\mathcal{O}_{Y',y'}$ is finite over $\mathcal{O}_{Y,y}$ (and a fortiori over $K(Y)$), it is contained in $K(Y)$ by our hypothesis; as y is normal, we necessarily have $\mathcal{O}_{Y',y'} = \mathcal{O}_{Y,y}$. We

then conclude that g is a local isomorphism at the point y' (??), which completes the proof of the first part of the corollary. There seconds one follows from the first, because the additional assumption entails that g is an isomorphism in the neighborhood of any point of Y' , whence an isomorphism. \square

Proposition 20.4.13. *Let A be a unibranch Noethreian local ring, \mathfrak{a} be a defining ideal of A , $A_0 = A/\mathfrak{a}$, and $S = \text{gr}_{\mathfrak{a}}(A)$. Then $\text{Proj}(S)$ is a connected A_0 -scheme.*

Proof. Let \mathfrak{m} be the maximal ideal of A ; $Y = \text{Spec}(A)$ is an integral scheme with unique closed point y . By hypothesis, we have $\mathfrak{m}^k \subseteq \mathfrak{a} \subseteq \mathfrak{m}$ for an integer k , so $V(\mathfrak{a}) = \{\mathfrak{m}\}$. Let $S' = \bigoplus_{n \geq 0} \mathfrak{a}^n$, and $X = \text{Proj}(S')$, which is the blow up Y -scheme relative to \mathfrak{a} . Then X is integral and the structural morphism $f : X \rightarrow Y$ is birational and projective (??). ?? is then applicable and shows that $f^{-1}(y)$ is connected. But $f^{-1}(y)$ is the underlying space of $\text{Proj}(S' \otimes_A A_0)$ (?? and ??), and as $S' \otimes_A A_0 = S$ by definition, the proposition follows. \square

20.4.3 Zariski's "Main Theorem"

Proposition 20.4.14. *Let Y be a locally Noetherian scheme and $f : X \rightarrow Y$ be a proper morphism. Let X' be the set of points $x \in X$ which is isolated in the fiber $f^{-1}(f(x))$. Then X' is open in X , and if $f = g \circ f'$ is the Satein factorization of f , the restriction of f' to X' is an isomorphism from X' onto an open subscheme of Y' , and we have $X' = f'^{-1}(U)$.*

Proof. As $g^{-1}(f(x))$ is finite and discrete (??), for x to be isolated in $f^{-1}(f(x))$, it is necessary and sufficient that it is isolated in $f'^{-1}(f'(x))$. We may then assume that $f' = f$, so $f_*(\mathcal{O}_X) = \mathcal{O}_Y$. Then, if $x \in X'$, the fiber $f^{-1}(f(x))$ is connected by ??, and is necessarily reduced to the point x . As f is closed, for any open neighborhood V of x , $f(X - V)$ is closed in Y and does not contain $y = f(x)$, since $f^{-1}(y) = \{x\}$; if U is the complement of $f(X - V)$ in Y , then we have $f^{-1}(U) \subseteq V$, so we conclude that the inverse images under f of a fundamental system of open neighborhoods of y form a fundamental system of open neighborhoods of x . The hypothesis $f_*(\mathcal{O}_X) = \mathcal{O}_Y$ then implies that the homomorphism $f_y^\# : \mathcal{O}_{Y,y} \rightarrow \mathcal{O}_{X,x}$ is an isomorphism, so by ?? there exists an open neighborhood V of x and an open neighborhood U of y such that $f^{-1}(U) \subseteq V$ and the restriction $f|_V : V \rightarrow U$ is an isomorphism. Moreover, by the remarks above, we may also assume that $V = f^{-1}(U)$, so that $V \subseteq X'$, which completes the proof. \square

Proposition 20.4.15. *Let Y be a locally Noetherian scheme, $f : X \rightarrow Y$ be a morphism. Then the following conditions are equivalent:*

- (i) f is finite.
- (ii) f is affine and proper.
- (iii) f is proper, and for any $y \in Y$, the fiber $f^{-1}(y)$ is a finite set.

Proof. We see that (i) \Rightarrow (ii) by ???. If f is proper and affine, then so is the induced morphism $f^{-1}(y) \rightarrow \text{Spec}(\kappa(y))$ (?? and ??), and the finiteness theorem ??, applied to the structural sheaf of $f^{-1}(y)$, shows that $f^{-1}(y) = \text{Spec}(A)$, where A is a finite dimensional $\kappa(y)$ -algebra. Then $f^{-1}(y)$ is a finite $\kappa(y)$ -scheme, so is a finite set (??), and we see that (ii) \Rightarrow (iii). Finally, as $f^{-1}(y)$ is an algebraic scheme, the hypothesis that $f^{-1}(y)$ is a finite set implies that it is discrete (??). With the notations of ??, we then have $X' = X$, and $f' : X' \rightarrow Y'$ is an isomorphism; as g is a finite morphism, we then conclude that (iii) \Rightarrow (i). \square

Theorem 20.4.16 (Zariski's "Main Theorem"). Let Y be a Noetherian scheme, $f : X \rightarrow Y$ be a quasi-projective morphism, and X' be the set of points $x \in X$ that is isolated in the fiber $f^{-1}(f(x))$. Then X' is an open subset of X , and the open subscheme X' of X is isomorphic via f to an open subscheme Y' of Y that is finite over Y .

Proof. The hypotheses implies that there exists a projective Y -scheme P such that X is Y -isomorphic to an open subscheme of P . We then reduce to prove the theorem if f is a projective morphism, hence proper ??, and this follows from ??.

Remark 20.4.5. If X is reduced (resp. irreducible and X' is nonempty), we can suppose, in the situation of ??, that Y' is reduced (resp. irreducible). In fact, we can replace Y' by the scheme-theoretic closure $\overline{X'}$ of X' in Y' , and it is reduced if X' is ??). If X' is nonempty, then it is irreducible if X is, and then $\overline{X'}$ is also irreducible.

Corollary 20.4.17. Let Y be a locally Noetherian and separated scheme, $f : X \rightarrow Y$ be morphism of finite type, and $x \in X$ be a point that is isolated in $f^{-1}(f(x))$. then there exists an open neighborhood U of x that is isomorphic to an open subscheme of Y that is finite over Y .

Proof. Let $y = f(x)$, U be an affine open neighborhood of Y , V an affine open neighborhood of x contained in $f^{-1}(U)$. As Y is separated, the injection $U \rightarrow Y$ is affine ??, and as V is affine over U (again by ??), the restriction of f to V is an affine morphism $V \rightarrow Y$??(ii)); a fortiori, this restriction is quasi-projective since it is of finite type ??(i) and ??). It then suffices to apply ??.

Corollary 20.4.18. Let A be a Noetherian ring, B be an A -algebra of finite type, \mathfrak{P} be a prime ideal of B , \mathfrak{p} be its contraction in A . Suppose that \mathfrak{P} is both maximal and minimal among prime ideals of B with contraction \mathfrak{p} , then there exists $g \in B - \mathfrak{P}$, a finite A -algebra A' and an element $f' \in A'$ such that the A -algebra B_g and $A'_{f'}$ are isomorphic.

Proof. It suffices to apply ?? to $Y = \text{Spec}(A)$ and $X = \text{Spec}(B)$, since the hypothesis on \mathfrak{P} signified that it is isolated in the fiber.

Corollary 20.4.19. Let A be a Noetherian local ring, B be an A -algebra of finite type, \mathfrak{M} be a maximal ideal of B whose contraction in A is the maximal ideal \mathfrak{m} of A . Suppose that \mathfrak{M} and minimal among prime ideals of B with contraction \mathfrak{m} . Then there exists a finite A -algebra A' and a maximal ideal \mathfrak{m}' of A' (whose contraction is \mathfrak{m}) such that $B_{\mathfrak{M}}$ is isomorphic to the A -algebra $A'_{\mathfrak{m}'}$.

Proof. This follows from ?? by taking stalks at \mathfrak{m} and \mathfrak{M} .

Corollary 20.4.20. Under the hypothesis of ??, suppose that A and B are integral with the same fraction field K . Then, if A is integrally closed, we have $B = A$.

Proof. By ?? we can suppose that A' is integral with fraction field K . The hypothesis over A implies that $A' = A$, so $B_{\mathfrak{M}} = A$. As $A \subseteq B \subseteq B_{\mathfrak{M}}$, we conclude that $A = B$.

Corollary 20.4.21. Let Y be a locally Noetherian integral scheme, $f : X \rightarrow Y$ be a separated morphism that is of finite type and birational. Suppose that Y is normal and the fibers $f^{-1}(y)$ are finite for $y \in Y$. Then f is an open immersion; if moreover f is closed, then it is an isomorphism.

Proof. Let $x \in X$ and put $y = f(x)$. As $f^{-1}(y)$ is an algebraic scheme over $\kappa(y)$, the hypothesis that it is finite implies that it is discrete (??); moreover $\mathcal{O}_{Y,y}$ is integrally closed and $\mathcal{O}_{X,x}$ and $\mathcal{O}_{Y,y}$ have the same fraction field. We can then apply ??, and the homomorphism $f_y^\sharp : \mathcal{O}_{Y,y} \rightarrow \mathcal{O}_{X,x}$ is bijective. We then conclude that f is a local isomorphism. \square

Proposition 20.4.22. *Let Y be a locally Noetherian scheme and $f : X \rightarrow Y$ be a morphism locally of finite type. Then the set X' of $x \in X$ isolated in the fiber $f^{-1}(f(x))$ is open in X .*

Proof. This question is local over X and Y , so we can assume that X and Y are affine Noetherian; f is then affine and of finite type, hence quasi-projective (??(i)), and it suffices to apply ??.

Corollary 20.4.23. *Let Y be a locally Noetherian scheme, $f : X \rightarrow Y$ be a proper morphism. Then the set U of points $y \in Y$ such that $f^{-1}(y)$ is discrete is open in Y , and the restriction morphism $f|_{f^{-1}(U)} : f^{-1}(U) \rightarrow U$ is finite. In particular, a proper and quasi-finite morphism is finite.*

Proof. In fact, the complement of U in Y is the image of $X - X'$ under f , which is closed in X by ???. As f is a closed map, U is then open in Y . Moreover, it follows from ?? that $f^{-1}(y)$ is finite for any $y \in U$; as the restriction morphism $f|_{f^{-1}(U)} : f^{-1}(U) \rightarrow U$ is proper (??), it is finite in view of ??.

Chapter 21

Local study of schemes and morphisms of schemes

21.1 Unramified morphisms, smooth morphisms and étale morphisms

In this section, we introduce the notions of unramified morphisms, smooth morphisms and étale morphisms between schemes. These three classes of morphisms are analogues of the following types of maps of manifolds in differential geometry.

- **Submersions** are maps inducing surjections of tangent spaces everywhere. They are useful in the notion of a fibration. (Perhaps a more relevant notion from differential geometry, allowing singularities, is "locally on the source a smooth fibration".)
- **Immersions** are maps inducing injections of tangent spaces. They can be thought as a generalized notion for submanifolds.
- **Local isomorphisms** are maps inducing isomorphisms of tangent spaces. They are be viewed as covering spaces of manifolds.

21.1.1 Formally unramification and formally smooth

Let $f : X \rightarrow Y$ be a morphism of schemes. We say that f is formally smooth (resp. formally unramified) if for any affine scheme Y' over Y and any closed subscheme Y'_0 of Y' defined by a square zero ideal \mathcal{I} of $\mathcal{O}_{Y'}$, the canonical map

$$\mathrm{Hom}_Y(Y', X) \rightarrow \mathrm{Hom}_Y(Y'_0, X) \tag{21.1.1}$$

is surjective (resp. injective). In this case, we say that X is formally smooth (resp. formally unramified) over S . If X is both formally smooth and formally unramified over S , then it is said to be **étale** over S .

Example 21.1.1. Suppose that $Y = \mathrm{Spec}(A)$ and $X = \mathrm{Spec}(B)$ are affine schemes, so that the morphism f corresponds to a ring homomorphism $\varphi : A \rightarrow B$. Then X is formally smooth (resp. formally unramified) over Y if and only if B is a formally smooth (resp. formally unramified) A -algebra.

Example 21.1.2. In view of the proof of ??, if a morphism $f : X \rightarrow Y$ is formally smooth (resp. formally unramified), then for any affine scheme Z over Y and any closed subscheme Z_0 of Z defined by a nilpotent ideal \mathcal{J} of \mathcal{O}_Z , the canonical map

$$\mathrm{Hom}_Y(Z, X) \rightarrow \mathrm{Hom}_Y(Z_0, X)$$

is surjective (resp. injective), so we can also take this as the definition of formally unramification (resp. formally smoothness). We also note that if f is formally smooth (resp. formally étale), then for an *arbitrary* scheme Z over Y and any closed subscheme Z_0 of Z defined by a locally nilpotent ideal \mathcal{J} of \mathcal{O}_Z , the canonical map

$$\mathrm{Hom}_Y(Z, X) \rightarrow \mathrm{Hom}_Y(Z_0, X)$$

is surjective (resp. bijective). To see this, let (U_α) be an open affine covering of Z such that the ideal $\mathcal{J}|_{U_\alpha}$ is nilpotent, and for each α , let U_α^0 be the inverse image of U_α in Z_0 , which is the closed subscheme of U_α defined by $\mathcal{J}|_{U_\alpha}$. Let $f_0 : Z_0 \rightarrow X$ be a Y -morphism, then by the hypotheses and ??, for any α , there is a (resp. unique) Y -morphism $f_\alpha : U_\alpha \rightarrow X$ whose restriction to U_α^0 equals to $f_0|_{U_\alpha^0}$. Since f_α and f_β then coincide on any affine open of $U_\alpha \cap U_\beta$, we conclude that there exists a (resp. unique) morphism $f : Z \rightarrow X$ whose restriction to Z_0 is equal to f_0 .

Proposition 21.1.3 (Properties of Formally Unramified and Formally Smooth Morphisms).

- (i) A monomorphism is formally unramified, and an open immersion is formally étale.
- (ii) The composition of two formally smooth (resp. formally unramified) morphisms is formally smooth (resp. formally unramified).
- (iii) If $f : X \rightarrow Y$ is a formally smooth (resp. formally unramified) S -morphism, so is $f_{(S')} : X_{(S')} \rightarrow Y_{(S')}$ for any base change $S' \rightarrow S$.
- (iv) If $f : X \rightarrow X'$ and $g : Y \rightarrow Y'$ are two formally smooth (resp. formally unramified) S -morphisms, $f \times_S g$ is formally smooth (resp. formally unramified).
- (v) Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be two morphisms. If $g \circ f$ is formally unramified, so is f .
- (vi) If $f : X \rightarrow Y$ is a formally unramified morphism, so is $f_{\mathrm{red}} : X_{\mathrm{red}} \rightarrow Y_{\mathrm{red}}$.

Proof. In view of ?? and ??, it suffices to prove (i), (ii) and (iii). The two assertions of (i) are trivial by definition. To prove (ii), consider two morphisms $f : X \rightarrow Y$, $g : Y \rightarrow Z$, an affine scheme Z' over Z , a closed subscheme Z'_0 of Z' defined by a square zero ideal. Suppose that f and g are formally smooth, and consider a Z -morphism $u_0 : Z'_0 \rightarrow X$.

$$\begin{array}{ccc} Z'_0 & \xrightarrow{u_0} & X \\ j \downarrow & \nearrow u & \downarrow f \\ Z' & \xrightarrow{v} & Y \\ & \nearrow v & \downarrow g \\ & & Z \end{array}$$

The hypothesis over g implies that there exists a Z -morphism $v : Z' \rightarrow Y$ such that $f \circ u_0 = v \circ j$ (where $j : Z'_0 \rightarrow Z'$ is the canonical injection), and the hypothesis over f implies that there exists a morphism $u : Z \rightarrow X$ such that $f \circ u = v$ and $u \circ j = u_0$, so $(g \circ f) \circ u$ is equal to the structural morphism $Z' \rightarrow Z$ and $u \circ j = u_0$, which proves that $g \circ f$ is formally smooth. By a similar reasoning, we see that $f \circ g$ is formally unramified if both morphisms are formally unramified.

Finally, to prove (iii), put $X' = X_{(S')}$, $Y' = Y_{(S')}$, $f' = f_{(S')}$. Consider an affine scheme Y'' over Y' and a closed subscheme Y''_0 of Y'' defined by a square zero ideal. Then $\text{Hom}_{Y'}(Y'', X')$ is canonically identified with $\text{Hom}_Y(Y'', X)$, and $\text{Hom}_{Y'}(Y''_0, X')$ with $\text{Hom}_Y(Y'', X)$, so our conclusion follows from the definition. \square

Proposition 21.1.4. *Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be morphisms and suppose that g is formally unramified. Then, if $g \circ f$ is formally smooth (resp. formally étale), so is f .*

Proof. Let Y' be an affine scheme, $h : Y' \rightarrow Y$ be a morphism, Y'_0 be a closed subscheme of Y' defined by a square zero ideal, and $u_0 : Y'_0 \rightarrow X$ be a Y -morphism.

$$\begin{array}{ccccc} Y'_0 & \xrightarrow{u_0} & X & & \\ j \downarrow & \nearrow u & \downarrow f & & \\ Y' & \xrightarrow{h} & Y & \xrightarrow{g} & Z \end{array}$$

If $g \circ f$ is formally smooth, there exists a Z -morphism $u : Y' \rightarrow X$ such that $u \circ j = u_0$ (where $j : Y'_0 \rightarrow Y'$ is the canonical injection) and $(g \circ f) \circ u = g \circ h$. But this then implies that $f \circ u$ and h are Z -morphisms from Y' to Y such that $(f \circ u) \circ j = h \circ j$, so since g is formally unramified, we conclude that $f \circ u = h$, so u is a Y -morphism. In view of ??(v), the proposition is therefore proved. \square

Corollary 21.1.5. *Suppose that g is formally étale. Then for the composition $g \circ f$ to be formally smooth (resp. formally unramified), it is necessary and sufficient that f is formally smooth (resp. formally unramified).*

Proof. This follows from ?? and ??(ii) and (v). \square

Proposition 21.1.6. *Let $f : X \rightarrow Y$ be a morphism of schemes.*

- (i) *Let (U_α) be an open covering of X and, for each α , let $i_\alpha : U_\alpha \rightarrow X$ be the canonical injection. For f to be formally smooth (resp. formally unramified), it is necessary and sufficient that each morphism $f \circ i_\alpha$ is formally smooth (resp. formally unramified).*
- (ii) *Let (V_λ) be an open covering of Y . For f to be formally smooth (resp. formally unramified), it is necessary and sufficient that each restriction $f|_{f^{-1}(V_\lambda)} : f^{-1}(V_\lambda) \rightarrow V_\lambda$ is formally smooth (resp. formally unramified).*

Proof. We first note that (ii) is a consequence of (i): in fact, if $j_\lambda : V_\lambda \rightarrow Y$ and $i_\lambda : f^{-1}(V_\lambda) \rightarrow X$ are the canonical injections, the restriction $f_\lambda : f^{-1}(V_\lambda) \rightarrow V_\lambda$ satisfies $j_\lambda \circ f_\lambda = f \circ i_\lambda$; if f is formally smooth (resp. formally unramified), then so is $f \circ i_\lambda$ (?); but as j_λ is formally étale, this implies that f_λ is formally smooth (resp. formally unramified) by ?. Conversely, if each restriction f_λ is formally smooth (resp. formally unramified), then so is $j_\lambda \circ f_\lambda$ (?), and hence

is f in view of (i). Now since each i_α is formally étale, it then boils down to prove that if the $f \circ i_\alpha$ is formally smooth (resp. formally unramified), then so is f .

Let Y' be an affine scheme, Y'_0 be a closed subscheme of Y' defined by a square zero ideal, and $g : Y' \rightarrow Y$ be a morphism. Let $u_0 : Y'_0 \rightarrow X$ be a morphism, and denote by W_α (resp. W_α^0) the open subscheme of Y' (resp. Y'_0) induced on $u_0^{-1}(U_\alpha)$ (note that Y' and Y'_0 have the same underlying space). First suppose that $f \circ i_\alpha$ is formally unramified. We show that if u_1 and u_2 are two Y -morphisms from Y' to X whose restriction on Y'_0 coincide, then $u_1 = u_2$. To see this, note that by ??(iv), the hypothesis that $f \circ i_\alpha$ is formally unramified implies that $u_1|_{W_\alpha} = u_2|_{W_\alpha}$ for each α , so the assertion is true in this case.

Now suppose that $f \circ i_\alpha$ is formally smooth for each α , we prove that there exists a Y -morphism $u : Y' \rightarrow X$ whose restriction to Y'_0 is u_0 . Now since Y' is affine, we can apply (EGA IV, 16.5.17 ??), whose conclusion precisely proves the existence of u . \square

21.1.2 Differential properties and characterizations

Proposition 21.1.7. *For a morphism $f : X \rightarrow Y$ to be formally unramified, it is necessary and sufficient that $\Omega_{X/Y} = 0$.*

Proof. By ??, the question is local on source and target, so we can assume that $X = \text{Spec}(B)$ and $Y = \text{Spec}(A)$ are affine, hence reduce to proving that the A -algebra B is formally unramified if and only if $\Omega_{B/A} = 0$. For this, recall that $\text{Hom}_B(\Omega_{B/A}, M)$ is isomorphic to $\text{Der}_A(B, M)$ for any B -module M , so if $\Omega_{B/A} = 0$, ?? implies that B is formally unramified over A . Conversely, assume that B is formally unramified over A and consider the multiplication map $\mu : B \otimes_A B \rightarrow B$ of the A -algebra B . Let \mathfrak{J} be the kernel of μ and set $C = (B \otimes_A B)/\mathfrak{J}^2$, $N = \mathfrak{J}/\mathfrak{J}^2$. Then N is a square zero ideal of C , and since B is formally unramified over A , we get $\text{Der}_A(B, N) = 0$ by ???. But N is by definition the differential module $\Omega_{B/A}$, so we conclude that $\text{Hom}_B(\Omega_{B/A}, \Omega_{B/A}) = 0$, whence $\Omega_{B/A} = 0$. \square

Corollary 21.1.8. *Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be two morphism. For f to be formally unramified, it is necessary and sufficient that the canonical homomorphism $f^*(\Omega_{Y/Z}) \rightarrow \Omega_{X/Z}$ is surjective.*

Proof. This is an immediate consequence of ?? and the exact sequence

$$f^*(\Omega_{Y/Z}) \longrightarrow \Omega_{X/Z} \longrightarrow \Omega_{X/Y} \longrightarrow 0 \quad \square$$

Proposition 21.1.9. *Let $f : X \rightarrow Y$ be a formally smooth morphism.*

- (i) *The \mathcal{O}_X -module $\Omega_{X/Y}$ is locally projective. If f is locally of finite type, then $\Omega_{X/Y}$ is locally free of finite rank.*
- (ii) *For any morphism $g : Y \rightarrow Z$, the sequence of \mathcal{O}_X -modules*

$$0 \longrightarrow f^*(\Omega_{Y/Z}) \longrightarrow \Omega_{X/Z} \longrightarrow \Omega_{X/Y} \longrightarrow 0 \quad (21.1.2)$$

is exact. Moreover, for any $x \in X$, there exists an open neighborhood U of x such that the restriction of these homomorphisms to U form a split exact sequence.

Proof. If f is locally of finite type, then the diagonal map $\Delta_{X/Y} : X \rightarrow X \times_Y X$ is of finite presentation (??), so $\Omega_{X/Y}$ is an \mathcal{O}_X -module of finite type in view of ???. To see that $\Omega_{X/Y}$ is

locally projective, it suffices to assume that $Y = \text{Spec}(A)$ and $X = \text{Spec}(B)$ are affine, and the conclusion then follows from [??](#). Now to prove (ii), we can also assume that X, Y, Z are affine schemes, and in this case the conclusion then follows from the interpretation of the modules appearing in the sequence [\(??\)](#) and (EGA 0_{IV}, 20.5.7, [??](#)). \square

Corollary 21.1.10. *If $f : X \rightarrow Y$ is a formally étale morphism, then for any morphism $g : Y \rightarrow Z$, the canonical homomorphism $f^*(\Omega_{Y/Z}) \rightarrow \Omega_{X/Z}$ is bijective.*

Proof. This follows from the exact sequence [\(??\)](#) and the fact that we have $\Omega_{X/Y} = 0$ [\(??\)](#). \square

Proposition 21.1.11. *Let $f : X \rightarrow Y$ be a morphism, X' be a closed subscheme of X such that the composition morphism $X' \rightarrow X \rightarrow Y$ is formally smooth. Then the sequence of $\mathcal{O}_{X'}$ -modules*

$$0 \longrightarrow \mathcal{N}_{X'/X} \longrightarrow \Omega_{X/Y} \otimes_{\mathcal{O}_X} \mathcal{O}_{X'} \longrightarrow \Omega_{X'/Y} \longrightarrow 0 \quad (21.1.3)$$

is exact. Moreover, for any $x \in X$, there exists an open neighborhood U of x such that the restriction of these homomorphisms to U form a split exact sequence.

Proof. In view of [??](#), we can assume that $X = \text{Spec}(B)$, $X' = \text{Spec}(B/\mathfrak{I})$ and $Y = \text{Spec}(A)$ are affine, where \mathfrak{I} is an ideal of B . Then the conormal sheaf $\mathcal{N}_{X'/X}$ corresponds to the B -module $\mathfrak{I}/\mathfrak{I}^2$, and the conclusion follows from [??](#). \square

Proposition 21.1.12. *Let $f : X \rightarrow Y$ be a morphism locally of finite type. Then the following conditions are equivalent:*

- (i) f is a monomorphism;
- (ii) f is radical and formally unramified;
- (iii) for any $y \in Y$, the fiber $f^{-1}(y)$ is either empty or $\kappa(y)$ -isomorphic to $\text{Spec}(\kappa(y))$.

Proof. The fact that (i) implies (iii) follows from [??](#). It is clear that (iii) implies that f is radical, and we prove that $\Omega_{X/Y} = 0$ in this case. Note that the \mathcal{O}_X -module $\Omega_{X/Y}$ is quasi-coherent of finite type (EGA IV, 16.3.9), so it follows from [??](#) that, for $(\Omega_{X/Y})_x = 0$, it is necessary and sufficient that if we put $Y_1 = \text{Spec}(\kappa(y))$, $X_1 = f^{-1}(y) = X \times_Y Y_1$, then $(\Omega_{X_1/Y_1})_x = 0$. But as the morphism $f_1 : X_1 \rightarrow Y_1$ induced from f is formally unramified in view of the hypotheses of (iii) [\(??\)](#), the conclusion follows from [??](#).

Finally, we show that (ii) implies (i). For this, consider the diagonal morphism $g = \Delta_f : X \rightarrow X \times_Y X$; since f is radical, g is surjective [??](#). On the other hand, $\Omega_{X/Y}$ is defined by the conormal sheaf of the immersion g , and the hypothesis that f is unramified implies $\Omega_{X/Y} = 0$. Moreover, g is locally of finite presentation [??](#), so it is an open immersion by (EGA IV, 16.1.10). Being surjective, this immersion is then an isomorphism, so f is a monomorphism by [??](#). \square

21.1.3 Unramified morphisms and smooth morphisms

We say that a morphism $f : X \rightarrow Y$ is **smooth** (resp. **unramified**) if it is locally of finite presentation and formally smooth (resp. formally unramified), and **étale** if it is both smooth and unramified.

Example 21.1.13. Let A be a ring and B be an A -algebra. Then B is a smooth (resp. unramified) algebra if the corresponding morphism $\text{Spec}(B) \rightarrow \text{Spec}(A)$ is smooth (resp. unramified).

Proposition 21.1.14 (Properties of Unramified and Smooth Morphisms).

- (i) An open immersion is étale. For an immersion to be unramified, it is necessary and sufficient that it is locally of finite presentation.
- (ii) The composition of two smooth (resp. unramified) morphisms is formally smooth (resp. formally unramified).
- (iii) If $f : X \rightarrow Y$ is a smooth (resp. unramified) S -morphism, so is $f_{(S')} : X_{(S')} \rightarrow Y_{(S')}$ for any base change $S' \rightarrow S$.
- (iv) If $f : X \rightarrow X'$ and $g : Y \rightarrow Y'$ are two smooth (resp. unramified) S -morphisms, $f \times_S g$ is smooth (resp. unramified).
- (v) Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be two morphisms. If g is locally of finite type and $g \circ f$ is unramified, so is f .

Proof. This follows from ?? and ??.

□

Proposition 21.1.15. Let $f : X \rightarrow Y$, $g : Y \rightarrow Z$ be morphisms, and suppose that g is unramified. Then, if $g \circ f$ is smooth (resp. étale), so is f .

Proof. In fact, as g and $g \circ f$ are locally of finite presentation, so is f (??(v)), and the conclusion then follows from ??.

□

Corollary 21.1.16. Suppose that g is étale, then for f to be smooth (resp. étale), it is necessary and sufficient that $g \circ f$ is smooth (resp. étale).

Proof. This follows from ?? and ??(ii).

□

Proposition 21.1.17. Let $g : Y \rightarrow S$ and $h : X \rightarrow S$ be morphisms locally of finite presentation. For an S -morphism $f : X \rightarrow Y$ to be unramified, it is necessary and sufficient that the canonical homomorphism $f^*(\Omega_{Y/S}) \rightarrow \Omega_{X/S}$ is surjective.

Proof. As f is then locally of finite presentation (??(ii)), the proposition follows from ??.

□

In view of ?? and the local nature of morphisms locally of finite presentation, we say a morphism $f : X \rightarrow Y$ is smooth (resp. unramified) at a point $x \in X$ if there exists an open neighborhood U of x in X such that the restriction $f|_U$ is a smooth (resp. unramified) morphism from U into Y . Then a morphism $f : X \rightarrow Y$ is smooth (resp. unramified) if and only if it is smooth (resp. unramified) at every point of X . Moreover, it is clear from this definition that the points of X where f is smooth (resp. unramified) is open in X .

Proposition 21.1.18. For any scheme Y and any locally free \mathcal{O}_X -module \mathcal{E} of finite rank, the vector bundle $V(\mathcal{E})$ is a smooth Y -scheme.

Proof. In fact, by ?? we can assume that $Y = \text{Spec}(A)$ is affine and $V(\mathcal{E}) = \text{Spec}(A[T_1, \dots, T_n])$. As $A[T_1, \dots, T_n]$ is formally smooth over A and finitely presented, this proves the proposition.

□

Corollary 21.1.19. Under the hypothesis of ??, the projective bundle $\mathbb{P}(\mathcal{E})$ is a smooth Y -scheme.

Proof. We can assume that $Y = \text{Spec}(A)$ and $\mathbb{P}(\mathcal{E}) = \mathbb{P}_Y^n$. Then there is a finite open covering of \mathbb{P}_A^n by the $D_+(T_i)$. But the ring of $D_+(T_i)$, being $A[T_1, \dots, T_n]_{(T_i)}$, is isomorphic to the polynomial ring $A[T_0, \dots, \hat{T}_i, \dots, T_n]$, and hence smooth, so the corollary follows from ??.

□

21.1.4 Characterization of unramification and smoothness

Theorem 21.1.20. *Let $f : X \rightarrow Y$ be a morphism locally of finite presentation and x be a point of X . Then the following conditions are equivalent:*

- (i) f is unramified at x ;
- (ii) the diagonal morphism $\Delta_f : X \rightarrow X \times_Y X$ is a local isomorphism at x .
- (ii') if $Z = X \times_Y X$ and $z = \Delta_f(x)$, the morphism $(\Delta_f)_x^\# : \mathcal{O}_{Z,z} \rightarrow \mathcal{O}_{X,x}$ is bijective;
- (ii'') for any morphism $g : Y' \rightarrow Y$ and any point $y' \in Y'$ lying over $y = f(x)$, any Y' -section s' of $X' = X \times_Y Y'$ with $x' = s'(y')$ lying over x is a local isomorphism at y' ;
- (iii) $(\Omega_{X/Y})_x = 0$;
- (iv) the $\kappa(y)$ -scheme X_y is unramified over $\kappa(y)$ at x ;
- (iv') the point x is isolated in X_y and the local ring $\mathcal{O}_{X_y,x}$ is a field and separable over $\kappa(y)$;
- (iv'') $\mathcal{O}_{X_y,x} = \mathcal{O}_{X,x}/\mathfrak{m}_y \mathcal{O}_{X,x}$ is a field and a finite separable extension of $\kappa(y)$;
- (v) $\mathcal{O}_{X,x}$ is a formally unramified $\mathcal{O}_{Y,y}$ -algebra.

Proof. As f is locally of finite type, the \mathcal{O}_X -module $\Omega_{X/Y}$ is of finite type (EGA IV, 16.3.9 ??), so $(\Omega_{X/Y})_x = 0$ if and only if there exists an open neighborhood U of x such that $\Omega_{X/Y}|_U = 0$. In view of ??, this proves the equivalence of (i) and (iii). On the other hand, if we put $A = \mathcal{O}_{Y,y}$, $B = \mathcal{O}_{X,x}$, then $(\Omega_{X/Y})_x = \Omega_{B/A}$ (EGA IV, 16.4.15 ??), so the equivalence of (iii) and (v) also follows from the affine case of ??.

By the very definition of unramified morphism at x , we see that (i) is equivalent to (iv). Also, as properties (iv) and (iv') only involve the morphism $X_y \rightarrow \text{Spec}(\kappa(y))$, this also implies the equivalence of (iv) and (iv'). On the other hand, (iv') and (iv'') are equivalent, because it amounts to the same thing to say that $\mathcal{O}_{X_y,x}$ is a finite $\kappa(y)$ -algebra or that x is an isolated point of X_y , since X_y is a $\kappa(y)$ -scheme locally of type ??.

We now prove the equivalence of (ii) and (ii'). We can limit ourselves to the case where $Y = \text{Spec}(R)$, $X = \text{Spec}(S)$ are affine and f is finitely presented. Then we have $Z = \text{Spec}(S \otimes_R S)$ and Δ_f corresponds to the multiplication map $S \otimes_R S \rightarrow S$, whose kernel \mathfrak{I} is a finitely generated ideal (EGA 0_{IV}, 20.4.4 ??). If we put $\mathcal{J} = \tilde{\mathfrak{I}}$, the \mathcal{O}_Z -module $\Delta_f^*(\mathcal{O}_X) = \mathcal{O}_Z/\mathcal{J}$ is then of finite presentation, and the hypothesis that the homomorphism $(\Delta_f)_x^\# : \mathcal{O}_{Z,z} \rightarrow \mathcal{O}_{X,x}$ is bijective implies that, by replacing X by an open neighborhood of x , the homomorphism $(\Delta_f)^\# : \mathcal{O}_Z \rightarrow \Delta_f^*(\mathcal{O}_X)$ is itself bijective (??). This then proves that (ii') implies (ii), and the converse is evident.

On the other hand, the equivalence of (ii) and (ii'') follows from ?? even without the finiteness hypothesis on f : in fact, giving a Y' -section $s' : Y' \rightarrow X'$ is equivalent to giving a Y -morphism $h = g' \circ s' : Y' \rightarrow X$ (where $g' : Y' \rightarrow X$ is the canonical projection) such that $s' = \Gamma_h$, and the diagram

$$\begin{array}{ccc} Y' & \xrightarrow{s'} & X' = Y' \times_Y X \\ h \downarrow & & \downarrow h \times_X 1_X \\ X & \xrightarrow{\Delta_f} & X \times_Y X \end{array}$$

is then cartesian. Therefore if Δ_f is a local isomorphism at x , then s' is a local isomorphism at y' (since $x = h(y')$), and this proves (ii) \Rightarrow (ii''). The converse is obtained by applying (ii'') to the case where $Y' = X$, $y' = x$, $g = f$ and $s' = \Delta_f$.

To finish the proof of the theorem, it then suffices to prove the following implications

$$(iv'') \Rightarrow (iii) \Rightarrow (ii) \Rightarrow (iv'').$$

First, as $\Omega_{X/Y}$ is an \mathcal{O}_X -module of finite type, it follows from Nakayama lemma that condition (iii) is equivalent to $(\Omega_{X/Y})_x / \mathfrak{m}_y(\Omega_{X/Y})_x = 0$, which means $(\Omega_{X_y/\text{Spec}(\kappa(y))})_x = 0$ (EGA IV, 16.4.5 ??). We are therefore reduced to the case where Y is the spectrum of a field k and X is an algebraic k -scheme. The hypothesis that $\mathcal{O}_{X,x}$ is a finite field extension k' of k implies that x is closed in X (??), and hence isolated in X . But then the hypothesis that k' is separable over k implies that $\Omega_{k'/k} = 0$ (EGA 0_{IV}, 20.6.20 ??), which proves (iii). Moreover, in this case $\Omega_{X/Y}|_U = 0$ for an open neighborhood of x in X , so assertion (ii) follows from the definition of $\Omega_{X/Y}$.

Finally, to show that (ii) \Rightarrow (iv''), we can, by replacing X by an open neighborhood of x , suppose that Δ_f is an open immersion. If we denote by $f_y : X_y \rightarrow \text{Spec}(\kappa(y))$ the morphism induced by f on fiber, then Δ_{f_y} is also an open immersion (??), and as condition (iv'') only concerns the $\kappa(y)$ -scheme X_y , we can therefore assume that $Y = \text{Spec}(k)$ and $X = \text{Spec}(A)$, where k is a field and A is a finite type k -algebra. Condition (iv'') is then established if we can show that A is a finite and separable k -algebra. If K is an algebraic closure of k , this amounts to saying that $A \otimes_k K$ is a finite and separable K -algebra (EGA IV, 4.6.1 ??), so we can further assume that k is algebraically closed. We first prove that A is a finite k -algebra. For this, it suffices to show that every closed point x of X is isolated, since then the set of such points is open in X and discrete, hence finite (X is Noetherian), and the assertion then follows from ??.

Now for such a point x , we have $\kappa(x) = k$ since k is algebraically closed (??), so by ?? there is a Y -section of X such that $s(Y) = \{x\}$, and in view of (EGA IV, 17.4.1.1 ??), $\{x\}$ is the inverse image of the diagonal $\Delta_X(X)$ under a morphism $X \rightarrow X \times_Y X$, hence is open in X in view of the hypothesis of (ii). The k -algebra A is therefore finite, hence is isomorphic to a direct product of finite local k -algebras. Since our question is local, we can then assume that A is a finite local k -algebra, so that $X = \text{Spec}(A)$ is reduced to a singleton. The residue field of A , being a finite extension of k , is necessarily equal to k , so in view of ??, the product $X \times_k X$ is reduced to a singleton, and Δ_f is then an isomorphism. Since Δ_f corresponds to the multiplication map $\mu : A \otimes_k A \rightarrow A$, we conclude that μ is an isomorphism, which means $\Omega_{A/k} = 0$ and A is separable over k . \square

Remark 21.1.1. If we only assume that f is locally of finite type, then $\Omega_{X/Y}$ is an \mathcal{O}_X -module of finite type (EGA IV, 16.3.9 ??), and Δ_f is a morphism locally of finite presentation (??). The proof of ?? is still valid, provided that condition (i) is replaced by the following: the restriction of f to an open neighborhood of x is formally unramified. We also see that in this case the restriction of f to an open neighborhood of x is a locally quasi-finite morphism.

In fact, many authors require unramified morphisms to be (locally of finite type), rather than (locally) of finite presentation. The benefits of this definition is that any closed immersion is then unramified, which is reasonable under our intuition about unramification. The requirement of being locally of finite presentation, however, is necessary when we consider

étale morphisms, so we add it to the definition of unramified morphism for consistency (following Grothendieck's definition). However, one should note that most of the proofs involving unramified morphisms work through finite type cases, and the restriction of being locally of finite presentation in fact unnecessary in most of the cases when we talk about unramified morphisms.

Corollary 21.1.21. *Let $f : X \rightarrow Y$ be a morphism locally of finite presentation. Then the following conditions are equivalent:*

- (i) f is unramified;
- (ii) the diagonal morphism $\Delta_f : X \rightarrow X \times_Y X$ is an open immersion;
- (ii') for any morphism $Y' \rightarrow Y$, any Y' -section of $X' \times_Y Y'$ is an open immersion;
- (iii) $\Omega_{X/Y} = 0$;
- (iv) for any $y \in Y$, the $\kappa(y)$ -scheme X_y is unramified over $\kappa(y)$;
- (iv') for any $y \in Y$, the $\kappa(y)$ -scheme X_y is isomorphic to $\coprod_{\lambda \in I} \text{Spec}(K_\lambda)$, where for each λ , K_λ is a finite separable extension of $\kappa(y)$;
- (v) for any $x \in X$, the local ring $\mathcal{O}_{X,x}$ is a formally unramified $\mathcal{O}_{Y,f(x)}$ -algebra.

Proof. This follows from the observation that a morphism $f : X \rightarrow Y$ is unramified if and only if it is unramified at every point of X . \square

Corollary 21.1.22. *If $f : X \rightarrow Y$ is unramified, then it is locally quasi-finite.*

Proof. By ??, for any point $x \in X$, the $\kappa(y)$ -algebra $\mathcal{O}_{X_y,x}$ is finite, so x is isolated in X_y and the conclusion follows from (EGA IV, 13.1.4 ??). \square

Proposition 21.1.23. *Let Y be a locally Noetherian scheme, $f : X \rightarrow Y$ be a morphism locally of finite type, x be a point of X . If $A = \mathcal{O}_{Y,y}$ and $\mathcal{O}_{X,x}$, then the conditions of ?? are equivalent to the following:*

- (vi) $\widehat{B} \otimes_{\widehat{A}} \kappa_A$ is a finite and separable field extension of κ_A (which implies that \widehat{B} is a finite \widehat{A} -algebra);
- (vi') \widehat{B} is a formally unramified \widehat{A} -algebra.

Moreover, if $\kappa(x) = \kappa(y)$, or if k is separably closed, these conditions are also equivalent to:

- (vi'') the homomorphism $\widehat{A} \rightarrow \widehat{B}$ is surjective.

Proof. Note that by the same reasoning of (EGA 0_{IV}, 19.3.6 ??), the A -algebra B is formally unramified if and only if \widehat{B} is formally unramified over \widehat{A} for the adic topology. On the other hand, since f is locally of finite type, $\Omega_{B/A}$ is a finitely generated B -module (EGA IV, 16.3.9 ??), $\Omega_{B/A} = 0$ if and only if $\widehat{\Omega}_{B/A} = 0$ (??), so B is a formally unramified A -algebra if and only if it is formally unramified over A for the adic topology (EGA 0_{IV}, 20.7.4 ??), and this proves the equivalence of (v) and (vi').

Since $\kappa_A = A/\mathfrak{m}_A = \widehat{A}/\mathfrak{m}_A \widehat{A}$, we have $\widehat{B} \otimes_{\widehat{A}} \kappa_A = \widehat{B}/\mathfrak{m}_A \widehat{B} = \widehat{B} \otimes_B (B/\mathfrak{m}_A B)$, and $\widehat{B}/\mathfrak{m}_A \widehat{B}$ is therefore the completion of $B/\mathfrak{m}_A B = B \otimes_A \kappa_A$ for the \mathfrak{m}_B -adic topology (??). Since \widehat{B} is a faithfully flat B -module, we see that $B/\mathfrak{m}_A B$ is a field if and only if $\widehat{B}/\mathfrak{m}_A \widehat{B}$ is a field. Moreover,

since completion does not change the residue field, in this case $B/\mathfrak{m}_A B$ is separable over κ_A if and only if $\widehat{B}/\mathfrak{m}_A \widehat{B}$ is separable over κ_A ; this proves the equivalence of (iv'') and (vi).

Finally, if $\kappa(x) = \kappa(y)$ or κ_A is separably closed, condition (vi) implies that the homomorphism $\widehat{A}/\mathfrak{m}_A \widehat{A} \rightarrow \widehat{B}/\mathfrak{m}_A \widehat{B}$ is bijective and that \widehat{B} is a quasi-finite \widehat{A} -algebra (EGA 0_I, 7.4.4 ??), hence finite (since \widehat{A} is complete, \widehat{B} is separated for the \mathfrak{m}_A -adic topology, and $\mathfrak{m}_A \widehat{B}$ is a defining ideal of \widehat{B} (EGA 0_I, 7.4.1 ??)). The homomorphism $\widehat{A} \rightarrow \widehat{B}$ is then surjective by Nakayama lemma, so (vi) implies (vi''); the converse of this is evident. \square

Given an S -scheme Y and two S -morphisms $f : X \rightarrow Y, g : X \rightarrow Y$, we define the **coincidence scheme** of f and g to be the inverse image of the diagonal $\Delta_{Y/S}$ under the S -morphism $(f, g)_S$. Moreover, by ??, this subscheme is canonically identified with the kernel $\ker(f, g)$.

Proposition 21.1.24. *Let $h : Y \rightarrow S$ be an unramified morphism and $f : X \rightarrow Y, g : X \rightarrow Y$ be two S -morphisms. Then the coincidence scheme of f and g is an open subscheme of X .*

Proof. In fact, since $\Delta_h : Y \rightarrow Y \times_S Y$ is an open immersion by ??, the inverse image of $\Delta_{Y/S}$ under $(f, g)_S$ is an open subscheme of X . \square

Corollary 21.1.25. *Under the hypotheses of ??, let x be a point of X such that the following diagram commutes*

$$\text{Spec}(\kappa(x)) \longrightarrow X \xrightarrow{\quad f \quad} Y \\ \downarrow g$$

Then there exists an open neighborhood U of x such that $f|_U = g|_U$. If Y is also separated over S , then there also exists an open neighborhood Z of x such that $f|_Z = g|_Z$. In particular, if X is also connected, then $f = g$.

Proof. By ??, x belongs to the subscheme $\ker(f, g)$, and the assertion then follows from ?? . \square

Corollary 21.1.26. *Under the hypotheses of ??, suppose that the structural morphism $\varphi : X \rightarrow S$ is closed. Let s be a point of X and suppose that the compositions the following diagram commutes*

$$X_s \longrightarrow X \xrightarrow{\quad f \quad} Y \\ \downarrow g$$

Then there exists an open neighborhood V of s in S such that $f|_{\varphi^{-1}(V)} = g|_{\varphi^{-1}(V)}$. If Y is also separated over S and φ is open, then we can choose V to be clopen. In particular, if S is also connected, then $f = g$.

Proof. It follows from ?? that the subscheme $C = \ker(f, g)$ is open in X and contains X_s . As φ is closed, there exists an open neighborhood V of s such that $\varphi^{-1}(V) \subseteq C$. If Y is also separated over S , then C is also closed, so $\varphi(X - C)$ is clopen in S , and its complement V in S is then a clopen neighborhood of s such that $\varphi^{-1}(V) \subseteq C$. \square

Proposition 21.1.27. *Let Y be a connected scheme, $f : X \rightarrow Y$ be a unramified and separated morphism. Then any Y -section g of X is an isomorphism from Y onto a connected component of X , and the map $g \mapsto g(Y)$ is a bijection from $\Gamma(X/Y)$ to the set of connected components Z of X (necessarily open in X) such that the restriction of f to Z is an isomorphism from Z onto Y . In particular, if g_1 and g_2 are two Y -sections of X such that $g_1(y) = g_2(y)$ for a point $y \in Y$, then $g_1 = g_2$.*

Proof. It follows from ?? that any Y -section s of X is an open immersion, and as X is a separated Y -scheme, s is also a closed immersion (??). Then s is an isomorphism from Y onto a clopen subscheme of X , and as $s(Y)$ is connected, this is necessarily a connected component of X . The rest of the proposition is then immediate. \square

Example 21.1.28. Let k be a field with $\text{char}(k) \neq 2$ and consider the normalization $f : \mathbb{A}_k^1 \rightarrow C$ morphism, where $C = \text{Spec}(k[T^2, T^3])$ and $\mathbb{A}_k^1 = \text{Spec}(k[T])$. The morphism f corresponds to the ring homomorphism

$$\varphi : k[T^2, T^3] \rightarrow k[T], \quad (X, Y) \mapsto (T^2, T^3).$$

If we set $A = k[T^2, T^3]$ and $B = k[T]$, then it is easy to see that $B = A[X]/(X^2 - T^2)$, so the B -module $\Omega_{B/A}$ is generated by the symbol dX subject to the relation

$$0 = d(X^2 - T^2) = 2XdX - dT^2 = 2XdX.$$

and it is therefore isomorphic, as a B -module, to

$$A[X]/(2X, X^2 - T^2) = k[T^2, T^3, X]/(2X, X^2 - T^2) = k[T]/(2T) = k[T]/(T).$$

From this, we see that the support of $\Omega_{B/A}$ is equal to $\{(T)\}$, so by ?? the unramified locus of f is $\mathbb{A}_k^1 - \{0\}$. This ramification can also be detected from the local homomorphism $\varphi_{\mathfrak{p}} : A_{\mathfrak{p}} \rightarrow B_{\mathfrak{P}}$, where $\mathfrak{p} = (T^2)$ and $\mathfrak{P} = (T)$. The induced homomorphism on residue fields is an isomorphism on $k(T)$, but the image of the maximal ideal of $\mathfrak{p}A_{\mathfrak{p}}$ to $\mathfrak{P}B_{\mathfrak{P}}$ is equal to $\mathfrak{P}^2B_{\mathfrak{P}}$, so it is not unramified.

Geometrically, the ramification of f at origin is resulted from the fact that the tangent vectors (two different directions) of \mathbb{A}_k^1 are both bended to the same tangent vector (the same direction) of the origin of C . This justifies our intuition of unramification by unicity of tangent vector liftings.

Example 21.1.29. Let k be a field and consider the morphism $f : \mathbb{A}_k^1 \rightarrow \mathbb{A}_k^1$ corresponding to the ring homomorphism

$$\varphi : k[X] \rightarrow k[Y], \quad X \mapsto Y^2.$$

This can be considered as the projection of the parabola onto \mathbb{A}_k^1 . Let $A = k[X]$ and $B = k[Y]$, then we have $B = A[T]/(T^2 - X)$, so the B -module $\Omega_{B/A}$ is isomorphic to $k[X]/(2T) = k[Y]/(Y)$. Therefore the ramification locus of f is again $\mathbb{A}_k^1 - \{0\}$. This is not surprising, since geometrically the tangent map of f at the origin is zero, so it does not satisfy the unicity of tangent vector liftings.

We also remark that the morphism f is flat. In fact, for any $\lambda \in k$, the fiber of f over the closed point $(X - \lambda)$ is $\text{Spec}(k[Y]/(Y^2 - \lambda))$, which is the disjoint union of 2 points if $\lambda \neq 0$, and is the tangent vector $\text{Spec}(k[\varepsilon]/(\varepsilon^2))$ if $\lambda = 0$. In all cases, we see that $\dim_k(k[Y]/(Y^2 - \lambda)) = 2$, so f is flat. Another way to see this is to use the miracle flatness: since f is a morphism between regular schemes of equal dimension, it must be flat.

Theorem 21.1.30. Let $f : X \rightarrow Y$ be a morphism locally of finite presentation, x be a point of X and $y = f(x)$. Then the following conditions are equivalent:

- (i) f is smooth at x ;

- (ii) f is flat at x and the $\kappa(y)$ -scheme X_y is smooth over $\kappa(y)$ at x ;
- (ii') f is flat and geometrically regular at x ;
- (iii) $\mathcal{O}_{X,x}$ is a formally flat $\mathcal{O}_{Y,y}$ -algebra.

Proof. By the hypothesis on f , we can assume that $Y = \text{Spec}(A)$, $X = \text{Spec}(C)$, where $C = B/\mathfrak{I}$, $B = A[T_1, \dots, T_n]$ being a polynomial algebra and \mathfrak{I} a finitely generated ideal of B . The equivalence of (i) and (iii) then follows from ???. On the other hand, apply this result to the morphism $f_y : X_y \rightarrow \text{Spec}(\kappa(y))$ (which is locally of finite type), we see that the equivalence of (i), (ii) and (ii') follows from ???. \square

Corollary 21.1.31. *Let $f : X \rightarrow Y$ be a morphism locally of finite presentation. For f to be smooth, it is necessary and sufficient that f is flat and for any $y \in Y$, X_y is a geometrically regular $\kappa(y)$ -scheme.*

Proof. This follows from the definition of flat morphisms and ??.

Proposition 21.1.32. *Let Y be a locally Noetherian scheme, $f : X \rightarrow Y$ be a morphism locally of finite type, x be a point of X and $y = f(x)$. If $A = \mathcal{O}_{Y,y}$ and $\mathcal{O}_{X,x}$, then the conditions of ?? are equivalent to the following:*

- (iv) B is a formally smooth A -algebra;
- (iv') \widehat{B} is a formally unramified \widehat{A} -algebra.

Moreover, if $\kappa(x) = \kappa(y)$, these conditions are also equivalent to:

- (iv'') \widehat{B} is isomorphic to a power series \widehat{A} -algebra $\widehat{A}[[T_1, \dots, T_n]]$.

Proof. The equivalence of condition (iii) of ?? and (iv) follows from Jacobian criterion (??), and that of (iv) and (iv') follows from (EGA 0_{IV} 19.3.6). On the other hand, (iv'') implies (iv') without the hypothesis on residue fields (??). Finally, if \mathfrak{m} is the maximal ideal of \widehat{A} , condition (iv') implies that $\widehat{B}/\mathfrak{m}\widehat{B}$ is a formally smooth complete Noetherian local $\kappa(y)$ -algebra. The hypothesis $\kappa(x) = \kappa(y)$ then shows that $\widehat{B}/\mathfrak{m}\widehat{B}$ is $\kappa(y)$ -isomorphic to a formal series algebra $\kappa(y)[[T_1, \dots, T_n]]$ (??). On the other hand, as $\widehat{A}[[T_1, \dots, T_n]]$ is a flat \widehat{A} -module and a complete Noetherian local \widehat{A} -algebra, we conclude from (EGA 0_{IV}, 19.6.4 ??) that it is isomorphic to \widehat{B} , so (iv') implies (iv''). \square

Proposition 21.1.33. *Let Y be a locally Noetherian scheme, $f : X \rightarrow Y$ be a morphism locally of finite type, x be a point of X and $y = f(x)$. Suppose that Y is reduced at y . Then for f to be smooth at x , it is necessary and sufficient that f is universally open in an open neighborhood of x in X_y and that the $\kappa(y)$ -scheme X_y is a geometrically regular at x .*

Proof. In view of, it amounts to show that if X_y is geometrically regular at x , then f is flat at x if and only if it is universally open in an open neighborhood of x in X_y . Now if f is flat at x , so is it in an open neighborhood of x in X (EGA IV, 11.1.1 ??) and hence universally open in this neighborhood (EGA IV, 2.4.6 ??). Conversely, the hypotheses that X_y is geometrically regular at x and f is universally open in an open neighborhood of x in X_y together imply that f is flat at x , since $\mathcal{O}_{Y,y}$ is reduced (EGA IV, 15.2.2 ??). \square

Corollary 21.1.34. *Let Y be a locally Noetherian scheme, $f : X \rightarrow Y$ be a morphism locally of finite type, x be a point of X and $y = f(x)$. Suppose that Y is reduced and geometrically unibranch at y . Then for f to be smooth at x , it is necessary and sufficient that f is equidimensional at x and that the $\kappa(y)$ -scheme X_y is a geometrically regular at x .*

Proof. Since the set of points where f is equidimensional is open (EGA IV, 13.3.2 ??), this follows from ?? and Chevalley's criterion (EGA IV, 14.4.4 ??). \square

Proposition 21.1.35. *Let $f : X \rightarrow Y$ be a morphism locally of finite presentation which is smooth at a point $x \in X$, and $y = f(x)$. Then, for the local ring $\mathcal{O}_{X,x}$ to be reduced (resp. integrally closed, resp. geometrically unibranch), it is necessary and sufficient that $\mathcal{O}_{Y,y}$ is.*

Proposition 21.1.36. *Let Y be a locally Noetherian scheme, $f : X \rightarrow Y$ be a morphism locally of finite type which is smooth at a point $x \in X$. Put $y = f(x)$, then*

- $\dim(\mathcal{O}_{X,x}) = \dim(\mathcal{O}_{Y,y}) + \dim(\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{Y,y}} \kappa(y));$
- $\text{coprof}(\mathcal{O}_{X,x}) = \text{coprof}(\mathcal{O}_{Y,y});$
- for the local ring $\mathcal{O}_{X,x}$ to possesses property (S_n) or (R_n) , it is necessary and sufficient that $\mathcal{O}_{Y,y}$ to possesses property (S_n) or (R_n) . In particular, if $\mathcal{O}_{X,x}$ is regular (resp. normal), so is $\mathcal{O}_{Y,y}$.

Theorem 21.1.37. *Let $f : X \rightarrow Y$ be a morphism locally of finite presentation, x be a point of X and $y = f(x)$. then the following conditions are equivalent:*

- (i) f is étale at x .
- (i') f is smooth and unramified at x ;
- (ii) f is smooth and quasi-finite at x ;
- (iii) f is flat and unramified at x ;
- (iii') f is flat at x and the ring $\mathcal{O}_{X,x}/\mathfrak{m}_y \mathcal{O}_{X,x}$ is a finite and separable field extension of $\kappa(y)$;
- (iv) $\mathcal{O}_{X,x}$ is a formally étale $\mathcal{O}_{Y,y}$ -algebra.

Proof. The equivalence of (i) and (i') follows from definition, and that of (i) and (iv) follows from ?? and ?. The equivalence of (iii) and (iii') also follows from ?. The fact that (i') \Rightarrow (iii) follows from ?. Conversely, if (iii') is satisfied, then f is geometrically regular at x by ??, and hence smooth at x by ?. Also, the implication (i) \Rightarrow (ii) follows from ?.

It then remains to prove that (ii) \Rightarrow (iii), and as we have seen that f is flat at x (?), it suffices to show that X_y is unramified over $\kappa(y)$. In other words, we are reduced to the case where $Y = \text{Spec}(k)$. As the question is local over X , we can also assume that $X = \text{Spec}(A)$, where A is a finite local k -algebra (EGA 0, 7.4.1). In view of the hypothesis (ii), A is a formally smooth k -algebra, and hence geometrically regular (?). Since it is Artinian, we then conclude $\mathfrak{m}_A = 0$, so A a finite and separable extension of k . \square

Corollary 21.1.38. *Let f be a morphism locally of finite presentation. Then the following conditions are equivalent:*

- (i) f is étale.

- (i') f is smooth and unramified;
- (ii) f is smooth and locally quasi-finite;
- (iii) f is flat and unramified;
- (iii') f is flat and for any $y \in Y$, the fiber X_y is a sum of spectra of finite and separable field extensions of $\kappa(y)$;
- (iii'') f is flat and for any $y \in Y$ and any separably closed extension k' of $\kappa(y)$, the geometric fiber $X_y \otimes_{\kappa(y)} k'$ is a sum of spectra of fields isomorphic to k' .

Proposition 21.1.39. Let Y be a locally Noetherian scheme, $f : X \rightarrow Y$ be a morphism locally of finite presentation, x be a point of X and $y = f(x)$. If $A = \mathcal{O}_{Y,y}$ and $\mathcal{O}_{X,x}$, then the conditions of ?? are equivalent to the following:

- (iv) \widehat{B} is a formally étale \widehat{A} -algebra.
- (iv') \widehat{B} is a free \widehat{A} -module and $\widehat{B} \otimes_{\widehat{A}} \kappa_A$ is a finite and separable field extension of κ_A (which implies that \widehat{B} is a finite \widehat{A} -algebra).

Moreover, if $\kappa(x) = \kappa(y)$ or $\kappa(y)$ is separably closed, these conditions are also equivalent to:

- (iv'') the canonical homomorphism $\widehat{A} \rightarrow \widehat{B}$ is bijective.

Proposition 21.1.40. Let Y be a locally Noetherian scheme, $f : X \rightarrow Y$ be a morphism locally of finite type, x be a point of X and $y = f(x)$. If f is étale at x , then $\dim(\mathcal{O}_{X,x}) = \dim(\mathcal{O}_{Y,y})$.

Proof. This follows from ?? since x is isolated in the fiber X_y . □

21.2 The étale fundamental group

21.2.1 Finite group quotients of schemes

Let G be a (fixed) finite group and S be a scheme. By a **G -scheme** over S , we mean an S -scheme X with a right action of G on X . By definition, this means we have a homomorphism $\rho : G \rightarrow \text{Aut}_S(X)$ from G into the set of S -automorphisms of X . For any S -scheme Z , G has an induced left action on the set $\text{Hom}_S(X, Z)$, so we can consider the set $\text{Hom}_S(X, Z)^G$ of G -invariant S -morphisms. Since this set depends functorially by Z , we then obtain a functor $\text{Hom}(X, -)^G$, for which we can ask the representability. By Yoneda lemma, this is equivalent to the existence of an S -scheme Y and a G -invariant S -morphism $p : X \rightarrow Y$ such that, for any S -scheme Z , the corresponding map

$$\text{Hom}_S(Y, Z) \rightarrow \text{Hom}_S(X, Z)^G, \quad g \mapsto g \circ p$$

is bijective. In this case, we say that (Y, p) (or the S -morphism $p : X \rightarrow Y$) is a **quotient scheme** of X by G , and denote it by X/G . It is clear that the pair (Y, p) is determined up to isomorphism.

If the scheme X is affine, then the quotient scheme of X always exists and has a simple interpretation. In fact, in this case G has a left action on the ring A of X , and the invariant subring A^G then provides such a quotient.

Proposition 21.2.1. *Let R be a ring, A be an R -ring with an R -linear action by G . Let $X = \text{Spec}(A)$, $Y = \text{Spec}(A^G)$, and $p : X \rightarrow Y$ be the canonical morphism.*

- (a) *The morphism p is integral and surjective.*
- (b) *The fibers of p are orbits of G , and p is a quotient map.*
- (c) *Let $x \in X$, $y = p(x)$, and G_x be the stabilizer of x . Then $\kappa(x)$ is a normal extension of $\kappa(y)$ and the canonical map $G_x \rightarrow \text{Gal}(\kappa(x)/\kappa(y))$ is surjective.*
- (d) *(Y, p) is a quotient scheme of X by G .*

Proof. The assertions (a), (b) and (c) follows from ??, ?? and the fact that an integral morphism is closed (??). Finally, assertion (d) follows from ?? and the fact that for any ring B , we have a canonical bijection

$$\text{Hom}_{\mathbf{Ring}}(B, A)^G \xrightarrow{\sim} \text{Hom}_{\mathbf{Ring}}(B, A^G).$$

□

Corollary 21.2.2. *Under the hypotheses of ??, the canonical homomorphism $\mathcal{O}_Y \rightarrow p_*(\mathcal{O}_X)^G$ is an isomorphism.*

Proof. By ??, the sheaf $p_*(\mathcal{O}_X)$ corresponds to the A^G -module A , so the corollary follows from the isomorphism $(S^{-1}A)^G = S^{-1}A^G$ of ??.

□

Proposition 21.2.3. *Let S be a scheme, X be a G -scheme over S and $p : X \rightarrow Y$ be a G -invariant affine S -morphism such that $\mathcal{O}_Y \cong p_*(\mathcal{O}_X)^G$. Then the conditions of ?? are satisfied.*

Proof. Since conditions (a), (b) and (c) are local on X and Y , we can assume that $Y = \text{Spec}(B)$ and $X = \text{Spec}(A)$ are both affine (since the morphism p is affine). Then the hypothesis implies that $B = A^G$, so it suffices to apply ?. As for assertion (d), it suffices to note that $p : X \rightarrow Y$ is a quotient map.

□

Corollary 21.2.4. *Under the hypotheses of ??, for any open subset U of Y , the restriction $p|_{p^{-1}(U)} : p^{-1}(U) \rightarrow U$ is a quotient U by G .*

Proof. This follows from the fact that the restriction $p|_{p^{-1}(U)} : p^{-1}(U) \rightarrow U$ also satisfies the hypotheses of ??.

□

Corollary 21.2.5. *Retain the hypotheses of ??.*

- (a) *For X to be affine (resp. separated) over S , it is necessary and sufficient that Y is affine (resp. separated) over S .*
- (b) *If X is of finite type over S , then it is finite over Y . If S is also locally Noetherian, then Y is of finite type over S .*

Proof. As X is affine (and a fortiori separated) over Y , we see that Y is affine (resp. separated) over S if and only if X is (?? and ??). Now if X is of finite type over Z , so is it over Y , and hence finite over Y (since $p : X \rightarrow Y$ is integral). If S is also locally Noetherian, then since Y is already quasi-compact over S (??), it suffices to show that Y is locally of finite type over S , so we may assume that $X = \text{Spec}(A)$, $Y = \text{Spec}(B)$ and $S = \text{Spec}(R)$ are affine. Then the ring A is of finite type over R and the conclusion follows from ??.

□

Let X be a G -scheme over a scheme S . If there exists a quotient scheme (Y, p) of X by G , we then say that X is **admissible** (or an **admissible G -scheme** over S). By ??, we see that any affine G -scheme over S is admissible. In the general case, we have the following characterization for admissible G -schemes.

Proposition 21.2.6. *Let X be a G -scheme over a scheme S . Then the following conditions are equivalent:*

- (i) X is admissible;
- (ii) X is a union of affine open G -invariant subsets;
- (iii) any orbit of G in X is contained in an affine open subset.

Proof. It is clear that (ii) \Rightarrow (iii), and conversely, if an orbit T of G is contained in an affine open subset U , then the intersection $U' = \bigcap_{g \in G} g \cdot U$ is a G -invariant open subset containing T and contained in the affine open U . As in U , any finite subset has a fundamental system of open affine neighborhoods, there exists an open affine neighborhood V of T contained in U' . The transforms of V by G are therefore affine and contained in U' , which is separate, so their intersection U'' is an affine open subset which is invariant under G and contains T . Since X is the union of orbits of G , we conclude that (iii) \Rightarrow (ii).

We note that condition (ii) is necessary for (i), since if $p : X \rightarrow Y$ is a quotient scheme of X and (V_α) is an affine open covering of Y , then $U_\alpha = p^{-1}(V_\alpha)$ is G -invariant and affine in X , and they cover X . Conversely, if (X_α) is a covering of X by G -invariant affine opens, then by ??, we can form the quotient $Y_\alpha = X_\alpha/G$; in each Y_i , the image of $X_i \cap X_j$ is an open subset Y_{ij} , which is identified with X_{ij}/G in view of ?. In particular, we deduce an isomorphism $Y_{ij} \cong Y_{ji}$, so we can glue Y_i to construct Y . \square

Corollary 21.2.7. *If X is an admissible G -scheme, then it is admissible for any subgroup H of G .*

Corollary 21.2.8. *Let S be a scheme and suppose that X is a G -scheme over S which is affine over S . Then X is admissible; in fact, if X is defined by the quasi-coherent \mathcal{O}_S -algebra \mathcal{A} , then its quotient Y is defined by the quasi-coherent algebra \mathcal{A}^G .*

Proof. We may assume that $X = \text{Spec}(A)$ and $S = \text{Spec}(R)$ are affine, where A is an R -algebra. Then by ?? the quotient is defined by the R -algebra A^G , so the conclusion follows. \square

Proposition 21.2.9. *Let X be an admissible G -scheme over S and $p : X \rightarrow Y$ be its quotient scheme. Consider a base change $S' \rightarrow S$ and put $X' = X \times_S S'$, $Y' = Y \times_S S'$, so that X' is a G -scheme over S' and the morphism $p' : X' \rightarrow Y'$ is G -invariant. If S' is flat over S , then (Y', p') is the quotient of X' by G . In other words, we have an isomorphism*

$$(X/G) \times_S S' \cong (X \times_S S')/G$$

Proof. We can evidently assume that S , X and Y are affine. It then suffices to prove that, if A is an R -algebra acted by G and $B = A^G$ is the invariant subring, then for any flat R -algebra R' , the invariant subring of $A' = A \times_R R'$ is identified with $B' = B \otimes_R R'$. To see this, note that the subring B is characterized by the exact sequence

$$0 \longrightarrow B \longrightarrow A \xrightarrow{\varphi_A} \prod_{g \in G} A \longrightarrow 0$$

where the homomorphism φ_A is defined by $x \mapsto (gx - x)_{g \in G}$. Since the induced homomorphism $\varphi_A \otimes 1_{R'} : A' \rightarrow A'$ is identified with $\varphi_{A'}$ and R' is flat over R , the conclusion follows. \square

Remark 21.2.1. We note that the flatness assumption is essential for ???. For example, if Y' is a closed subscheme of Y and X' is its inverse image under p , then Y' is not necessarily isomorphic to the quotient X'/G . However, as we shall see, this is true if X is étale over Y .

21.2.2 Decomposition groups and inertial groups

Let G be a finite group and X be a G -scheme. For $x \in X$, the stabilizer subgroup of x is called the **decomposition group** of x , and denoted by $G^Z(x)$. This group has a canonical action on the residue field $\kappa(x)$, and the kernel of the canonical homomorphism $G^Z(x) \rightarrow \text{Aut}(\kappa(x))$ is called the **inertia group** of x , and denoted by $G^T(x)$.

Suppose that X is admissible and let Y be a scheme over a scheme S .

Proposition 21.2.10. *Let X be an admissible G -scheme, $p : X \rightarrow Y$ be its quotient, and suppose that Y is locally Noetherian and p is finite. Let H be a subgroup of G and consider $X' = X/H$. Let $x \in X$, x' be its image in X' , and $y = p(x)$.*

- (a) *If $H \supseteq G^T(x)$, then the homomorphism $p_{x'}^\# : \mathcal{O}_{Y,y} \rightarrow \mathcal{O}_{X',x'}$ is étale.*
- (b) *If $H \supseteq G^Z(x)$, then the homomorphism $p_{x'}^\# : \mathcal{O}_{Y,y} \rightarrow \mathcal{O}_{X',x'}$ induces an isomorphism on completions.*

Proof. Since Y is locally Noetherian, the local scheme $Y' = \text{Spec}(\widehat{\mathcal{O}_{Y,y}})$ is flat over Y , so by considering the base change $Y' \rightarrow Y$ and use ??, we may assume that Y is the spectrum of a complete Noetherian local ring and X is the spectrum of a finite A -algebra B . If $H \supseteq G^Z(x)$, then by ??(b), $\kappa(x')$ is identified with $\kappa(y)$ and \mathfrak{m}_y generates the maximal ideal of $\mathfrak{m}_{x'}$, so the homomorphism $p_{x'}^\# : \mathcal{O}_{Y,y} \rightarrow \mathcal{O}_{X',x'}$ induces an isomorphism on completions. Assertion (a) then follows from (b) if we base change Y so that $G^Z(x) = G^T(x)$. \square

Corollary 21.2.11. *Under the conditions of ??, suppose that $G^T(x)$ is trivial. Then X is étale over Y at x .*

Corollary 21.2.12. *Suppose that X is connected and the action of G is faithful on X . For X to be étale over Y , it is necessary and sufficient that the inertia groups of X are trivial. In this case, G is identified with the group of Y -automorphisms of X .*

Proof. In view of ??, it suffices to suppose that X is étale over Y . Let $x \in X$ and $g \in G^T(x)$, then it follows from ?? that g acts trivially on X , and hence equals to the identity since G acts faithfully on X . \square

Proposition 21.2.13. *Let S be a locally Noetherian scheme, X be a separated and étale scheme of finite type over S , and G be a finite group of S -automorphisms of X . Then the G -scheme X is admissible and the quotient scheme X/G is étale over S .*

Proof. Since X is separated and étale over S , it is quasi-projective over S (EGA IV, 8.11.2 ??), so the existence of X/G follows from ??(iii). To see that X/G is étale over S , we may assume that G acts transitively on the set of connected components of X , and by considering the stabilizer

of a connected component, that X is connected. Finally, we can assume that G acts faithfully on X . But then the inertia groups of X are trivial, so it follows from ?? that $p : X \rightarrow X/G$ is étale. To see that X/G is then étale over S , we consider a point $x \in X$ and let $y = p(x)$, $s = \varphi(x)$ (where $\varphi : X \rightarrow S$ is the structural morphism). By taking a flat base change, we may assume that $\kappa(s)$ is separably closed, so by ??, the induced homomorphisms $\widehat{\mathcal{O}_{Y,y}} \rightarrow \widehat{\mathcal{O}_{X,x}}$ and $\widehat{\mathcal{O}_{S,s}} \rightarrow \widehat{\mathcal{O}_{X,x}}$ are bijective, so $\widehat{\mathcal{O}_{S,s}} \rightarrow \widehat{\mathcal{O}_{Y,y}}$ is bijective and X/S is étale over S by ??.

Corollary 21.2.14. *If X is finite étale over S , then X/G is finite étale over S .*

Proof. By ??, if X is defined by the quasi-coherent \mathcal{O}_S -algebra \mathcal{A} , then X/G is defined by \mathcal{A}^G . By assumption \mathcal{A} is a finite \mathcal{O}_S -algebra, so its subalgebra \mathcal{A}^G is also finite over S , since S is locally Noetherian. \square

21.2.3 Galois categories