Algebra

Xiaolong Pan

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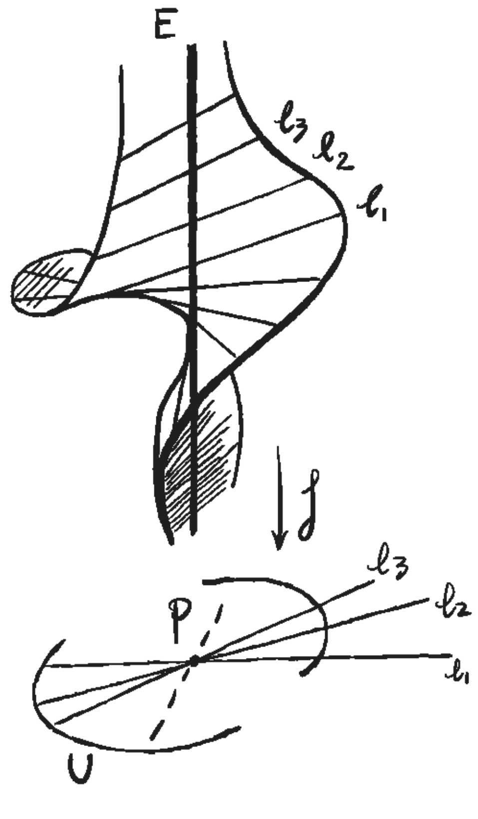


Contents

Chapter 1

Sheaves and cohomology of sheaves

1.1 Sheaves of sets

In this section, X always denotes a topological space.

1.1.1 Presheaves and sheaves

Definition 1.1.1. A **presheaf** $\mathscr{F}$ on a topological space X is the following data.

- To each open set $U \subseteq X$, we have a set $\mathscr{F}(U)$.
- For each inclusion $U \subseteq V$ of open sets, we have a **restriction map** $\mathrm{res}^V_U : \mathscr{F}(V) \to \mathscr{F}(U)$.

The data is required to satisfy the following two conditions.

- The map res^U_U is the identity: $\mathrm{res}^U_U = \mathbf{1}_{\mathscr{F}(U)}$.
- If $U \subseteq V \subseteq W$ are inclusions of open sets, then the restriction maps commute,

$$\begin{tikzcd} \mathscr{F}(W) \arrow[rr, "\mathrm{res}^W_U"] \arrow[dr, "\mathrm{res}^W_V"'] & & \mathscr{F}(U) \\ & \mathscr{F}(V) \arrow[ur, "\mathrm{res}^V_U"'] & \end{tikzcd}$$

Definition 1.1.2. A **morphism of presheaves** $\varphi : \mathscr{F} \to \mathscr{G}$ is a family of maps $\varphi(U) : \mathscr{F}(U) \to \mathscr{G}(U)$ for all open subsets $U \subseteq X$ behaving well with respect to restriction: if $U \hookrightarrow V$ then

$$\begin{tikzcd} \mathscr{F}(U) \arrow[r, "\varphi(U)"] \arrow[d, "\mathrm{res}^U_V"'] & \mathscr{G}(U) \arrow[d, "\mathrm{res}^U_V"] \\ \mathscr{F}(V) \arrow[r, "\varphi(V)"] & \mathscr{G}(V) \end{tikzcd} \tag{1.1.1}$$

commutes. Composition of morphisms φ and ψ of presheaves is defined in the obvious way: $(\varphi \circ \psi)_U := \varphi_U \circ \psi_U$. We obtain the category $\mathbf{PSh}(X)$ of presheaves on X.

If $U \subseteq V$ are open subset of X and $s \in \mathscr{F}(V)$, we will usually write $s|_U$ instead of $\mathrm{res}^V_U(s)$. The elements of $\mathscr{F}(U)$ are called **sections of $\mathscr{F}$ over U**. By convention, if the U is omitted, it is implicitly taken to be X: sections of $\mathscr{F}$ means sections of $\mathscr{F}$ over X. These are also called **global sections**. Very often we will also write $\Gamma(U, \mathscr{F})$ instead of $\mathscr{F}(U)$.

Remark 1.1.1. Given any topological space X, we have a category of open sets $\mathcal{T}_X$, where the objects are the open sets and the morphisms are set inclusions. A presheaf is a contravariant funtor from this category to **Set**:

$$\mathscr{F} : (\mathcal{T}_X)^{op} \to \mathbf{Set}$$

and a morphism of presheaves is a natural transforml of presheaves. With this observation, for any category $\mathcal{C}$ we can define a **presheaf $\mathscr{F}$ with values in $\mathcal{C}$** to be a contravariant funtor $(\mathcal{T}_X)^{op} \to \mathcal{C}$. A morphism $\mathscr{F} \to \mathscr{G}$ of presheaves with values in $\mathcal{C}$ is then simply a morphism of functors.

Example 1.1.1 (presheaf of functions). Let E be a set. For each open subset U of X, let $\mathrm{Map}_E(U)$ be the set of all maps $U \to E$. For an open subset $V \subseteq U$ we define res^U_V : $\mathrm{Map}_E(U) \to \mathrm{Map}_E(V)$ as the usual restriction of maps. Then Map_E is a presheaf on X.

More generally, a family $\mathscr{F}$ of subsets $\mathscr{F}(U) \subseteq \mathrm{Map}_E(U)$, where U runs through the open subsets of X, is called a **presheaf of E-valued functions on X**, if it is stable under restriction, i.e., for all open sets $V \subseteq U$ and all $f \in \mathscr{F}(U)$ one has $f|_V \in \mathscr{F}(V)$. Then $\mathscr{F}$ together with the restriction maps is a presheaf of sets.

If E is a group (respectively an R-module for some ring R, respectively an A-algebra for some commutative ring A), then $\mathscr{F}$ is a presheaf of groups (respective of R-modules, respective of A-algebras).

Example 1.1.2 (Examples of presheaf of functions).

(a) Let Y be a topological space. For open subset $U \subseteq X$, define

$$C(U, Y) := \{f : U \to Y \mid f \text{ is continuous}\}.$$

Then $C(-, Y)$ is a presheaf of Y-valued functions on X. If $Y = \mathbb{R}$, then $C(-, Y)$ a presheaf of $\mathbb{R}$-algebras.

(b) Let V be a finite-dimensional R-vector spaces and let X be an open subspace of V. Let $0 \leq n \leq \infty$. For an open subset U of X, define

$$C^n(U) := \{f : U \to W, f \text{ is a } C^n\text{-map}\}.$$

Then C^n is a presheaf of functions on X. It is a presheaf of $\mathbb{R}$-algebras.

Definition 1.1.3. A presheaf is a **sheaf** if it satisfies two more axioms.

- **Identity axiom**. If $\{U_i\}_{i\in I}$ is an open cover of U, and $s_1, s_2 \in \mathscr{F}(U)$, and $s_1|_{U_i} = s_2|_{U_i}$ for all i, then $s_1 = s_2$.
- **Gluability axiom**. If $\{U_i\}_{i\in I}$ is an open cover of U, then given $s_i \in \mathscr{F}(U_i)$ for all i, such that
$$s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j}$$
for all i and j, then there is some $s \in \mathscr{F}(U)$ such that $s|_{U_i} = s_i$ for all i.

A **morphism of sheaves** is a morphism of presheaves, and we obtain the category of sheaves on the topological space X, which we denote by $\mathrm{Sh}(X)$.

Remark 1.1.2. If $\mathscr{F}$ is a sheaf on X, then by using the covering of the empty set $U = \varnothing$ with empty index set $I = \varnothing$, we see $\mathscr{F}(\varnothing)$ is a set consisting of one element. In general, if $U, V \subseteq X$ are disjoint open subsets, then

$$\mathscr{F}(U \cup V) = \mathscr{F}(U) \times \mathscr{F}(V).$$

Example 1.1.3 (**Sheaf of *E*-valued functions**)**.** Let E be a set and let $\mathscr{F}$ be a presheaf of E-valued functions on X. Then $\mathscr{F}$ is a sheaf if and only if the following condition holds:

> For every open subset U of X, for every open covering $\{U_i\}_{i\in I}$ of U and for every map $f : U \to E$ such that $f|_{U_i} \in \mathscr{F}(U_i)$ for all i, one has $f \in \mathscr{F}(U)$.

In particular, the presheaves $C(-, Y)$ and C^n are in fact sheaves.

Example 1.1.4 (**Constant sheaf**)**.** The presheaf of constant functions with values in some set is in general not a sheaf: if U_1 and U_2 are disjoint non-empty open subsets and if $f_1 : U_1 \to E$ and $f_2 : U_2 \to E$ are constant maps that take different values, then there does not exist a constant map f on $U = U_1 \cup U_2$ whose restriction to U_i is f_i for $i = 1, 2$.

If one takes instead the sheaf of locally constant functions with values in some set E, then this is a sheaf. This comes from the simple observation: endow E with the discrete topology, then locally constant maps are exactly continuous maps from U to E. This is called the **constant sheaf associated to *E***. We denote this sheaf E_X.

1.1.2 Stalks of presheaves and sheaves

Let X be a topological space, $\mathscr{F}$ be a presheaf on X, and let $x \in X$ be a point. The **stalk** of $\mathscr{F}$ in x is defined by the direct limit

$$\mathscr{F}_x := \varinjlim_{U \ni x} \mathscr{F}(U).$$

More concretely, one has

$$\mathscr{F}_x = \{(U, s) : U \text{ is an open neighborhood of } x \text{ and } s \in \mathscr{F}(U)\}/\sim .$$

where two pairs $(U_1, s_1) \sim (U_2, s_2)$ are equivalent if there exists an open neighborhood V of x with $V \subseteq U_1 \cap U_2$ such that $s_1|_V = s_2|_V$. Then for each open neighborhood U of x we have a canonical map

$$\mathscr{F}(U) \mapsto \mathscr{F}_x, \quad s \mapsto s_x,$$

which sends $s \in \mathscr{F}(U)$ to the class of (U, s) in $\mathscr{F}_x$. We call s_x the **germ** of s in x.

Remark 1.1.3. If $\mathscr{F}$ is a presheaf on X with values in $\mathcal{C}$, where $\mathcal{C}$ is any category in which filtered colimits exist, then the stalk $\mathscr{F}_x$ is an object in $\mathcal{C}$ and we obtain a functor $\mathscr{F} \to \mathscr{F}_x$ from the category of presheaves on X with values in $\mathcal{C}$ to the category $\mathcal{C}$.

Example 1.1.5 (**Stalk of the sheaf of continuous functions**)**.** Let X be a topological space, let C_X be the sheaf of continuous $\mathbb{R}$-valued functions on X, and let $x \in X$. Then

$$C_{X,x} = \{(U, f) : x \in U \subseteq X \text{ open}, f : U \to \mathbb{R} \text{ is continuous}\}/\sim,$$

where $(U, f) \sim (V, g)$ if there exists $x \in W \subseteq U \cap V$ open such that $f|_W = g|_W$. In particular we see that for $x \in V \subseteq U$ open and $f : U \to \mathbb{R}$ continuous one has $(U, f) \sim (V, f|_V)$. As C_X is a sheaf of $\mathbb{R}$-algebras, $C_{X,x}$ is an $\mathbb{R}$-algebra.

If the germ $s \in C_{X,x}$ of a continuous function at x is represented by (U, f), then $s(x) := f(x) \in \mathbb{R}$ is independent of the choice of the representative (U, f). We obtain an $\mathbb{R}$-algebra homomorphism

$$\mathrm{ev}_x : C_{X,x} \to \mathbb{R}, \quad s \mapsto s(x),$$

which is surjective because $C_{X,x}$ contains in particular the germs of all constant functions. Let $\mathfrak{m}_x = \ker \mathrm{ev}_x = \{s \in C_{X,x} : s(x) = 0\}$. Then $\mathfrak{m}_x$ is a maximal ideal because $C_{X,x}/\mathfrak{m}_x \cong \mathbb{R}$ is a field. Let $s \in C_{X,x} \setminus \mathfrak{m}_x$ be represented by (U, f). Then $f(x) \neq 0$. By shrinking U we may assume that $f(y) \neq 0$ for all $y \in Y$ because f is continuous. Hence $1/f$ exists and s is a unit in $C_{X,x}$. Therefore the complement of $\mathfrak{m}_x$ consists of units of $C_{X,x}$. This shows that $C_{X,x}$ is a local ring with maximal ideal $\mathfrak{m}_x$. The same argument shows that for every open subspace X of a finite-dimensional $\mathbb{R}$-vector space the stalk $C^n_{X,x}$ is a local ring with residue field $\mathbb{R}$.

Now let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism of presheaves on X, we then have an induced map $\varphi_x : \mathscr{F}_x \to \mathscr{G}_x$ defined by

$$\varphi_x := \varinjlim_{U \ni x} \varphi_U,$$

which sends the equivalence class of (U, f) in $\mathscr{F}_x$ to the class of $(U, \varphi_U(f))$ in $\mathscr{G}_x$. This defines a functor $\mathscr{F} \mapsto \mathscr{F}_x$ from the category of presheaves on X to the category of sets such that for every open neighborhood U of X one has a commutative diagram

$$\begin{tikzcd} \mathscr{F}(U) \arrow[r, "f \mapsto f_x"] \arrow[d, "\varphi_U"'] & \mathscr{F}_x \arrow[d, "\varphi_x"] \\ \mathscr{G}(U) \arrow[r, "g \mapsto g_x"] & \mathscr{G}_x \end{tikzcd} \tag{1.1.2}$$

Now we introduct a construction frequently apprears in the theory of sheaves. The **Godement construction**

$$\mathrm{God} : \mathbf{Psh}(X) \to \mathrm{Sh}(X)$$

is the functor given by sending a presheaf $\mathscr{F} \in \mathbf{PSh}(X)$ to the sheaf $\mathrm{God}(\mathscr{F})$ defined by sending an open set U to

$$\mathrm{God}(\mathscr{F})(U) := \prod_{x \in U} \mathscr{F}_x,$$

with the restriction morphisms given by product projections (it is easy to see this defines a sheaf on X). The assignment of God on morphisms is given by sending a morphism of presheaves $\phi : \mathscr{F} \to \mathscr{G}$ to the morphism

$$\mathrm{God}(\phi)_U : \prod_{x \in U} \mathscr{F}_x \to \prod_{x \in U} \mathscr{G}_x.$$

Proposition 1.1.1. *Let $\mathscr{F}$ be a sheaf of sets. For every open $U \subseteq X$ the map*

$$\mathscr{F}(U) \to \prod_{x \in U} \mathscr{F}_x, \quad s \mapsto (s_x)_{x \in U}.$$

identifies $\mathscr{F}(U)$ with the elements $(s_x)_{x \in U}$ with the property

> *For any $x \in U$ there exists a open subset V containing x and a section $s \in \mathscr{F}(V)$ such that for all $y \in V$ we have $s_y = (V, s)$ in $\mathscr{F}_y$.*

*If an element (s_x) satisfies the condition above, we say it **consists of compatible germs**. Thus the Godement construction identifies $\mathscr{F}$ as a subsheaf of* $\mathrm{God}(\mathscr{F})$.

Proof. Let $s, t \in \mathscr{F}(U)$ such that $s_x = t_x$ for all $x \in U$. Then for all $x \in U$ there exists an open neighborhood $V_x \subseteq U$ of x such that $s|_{V_x} = t|_{V_x}$. Clearly, $U = \bigcup_{x \in U} V_x$ and therefore $s = t$ by identity axiom.

Clearly any section s of $\mathscr{F}$ over U gives a choice of compatible germs for U. Conversely, if $(s_x)_{x \in U}$ consists of compatible germs, that is, there is an open cover $\{U_i\}$ of U, and sections $s_i \in \mathscr{F}(U_i)$, such that $(s_x)_{x \in U_i}$ is given by (U_i, s_i). Then by gluability there is a section $\sigma \in \mathscr{F}(U)$ such that $\sigma|_{U_i} = s_i$, and it is clear from the condition that $\sigma_x = s_x$. □

The importance of stalks is contained in the following result, which says a morphism between sheaves is determined by its value on stalks.

Proposition 1.1.2. *Let X be a topological space, $\mathscr{F}$ and $\mathscr{G}$ presheaves on X, and let $\varphi, \psi : \mathscr{F} \to \mathscr{G}$ be two morphisms of presheaves.*

(a) If $\mathscr{F}$ is a sheaf, the induced maps on stalks $\varphi_x : \mathscr{F}_x \to \mathscr{G}_x$ are injective for all $x \in X$ if and only of $\varphi_U : \mathscr{F}(U) \to \mathscr{G}(U)$ is injective for all open subsets $U \subseteq X$.

(b) If $\mathscr{F}$ and $\mathscr{G}$ are both sheaves, the maps $\varphi_x : \mathscr{F}_x \to \mathscr{G}_x$ are bijective for all $x \in X$ if and only of $\varphi_U : \mathscr{F}(U) \to \mathscr{G}(U)$ is injective for all open subsets $U \subseteq X$.

(c) If $\mathscr{F}$ and $\mathscr{G}$ are both sheaves, the morphism φ and ψ are equal if and only of $\varphi_x = \psi_x$ for all $x \in X$.

Proof. First we show that taking stalks preserves injectivity and surjectivity. Assume that φ_U is injective for all U. Let $s_0, t_0 \in \mathscr{F}_x$ such that $\varphi_x(s_0) = \varphi_x(t_0)$. Let s_0 be represented by (s, U) and t_0 by (t, V). By shrinking U and V we may assume $U = V$. From diagram (1.1.2), we see

$$\varphi_U(s)_x = \varphi_x(s_0) = \varphi_x(t_0) = \varphi_U(t)_x,$$

so there exists an open neighborhood $W \subseteq U$ containing such that

$$\varphi_W(s|_W) = (\varphi_U(s))|_W = (\varphi_U(t))|_W = \varphi_W(t|_W).$$

As φ_W is injective, we find $s|_W = t|_W$ and hence $s_0 = s_x = t_x = t_0$. Thus φ_x is injective. If on the other hand φ_U is surjective for all $U \subseteq X$, let t_0 be any elememt in $\mathscr{G}_x$, which is represented by (t, U). Then there is a $s \in \mathscr{F}(U)$ such that $\varphi_U(s) = t$, and (1.1.2) implies

$$\varphi_x(s_x) = \big(\varphi_U(s)\big)_x = t_x.$$

so φ_x is surjective.

For (a), assume that $\mathscr{F}$ is a sheaf, and consider the commutative diagram

$$\begin{tikzcd}
\mathscr{F}(U) \arrow[r] \arrow[d, "\varphi_U"] & \prod_{x \in U} \mathscr{F}_x \arrow[d, "\prod_{x\in U} \varphi_x"] \\
\mathscr{G}(U) \arrow[r] & \prod_{x \in U} \mathscr{G}_x
\end{tikzcd}$$

By Proposition 1.1.1 the map $\mathscr{F}(U) \to \prod_{x \in U} \mathscr{F}_x$ is injective, thus if φ_x are all injective, so is φ_U. Assertion (c) can also be derived from this diagram.

Then we prove (b). Assume φ_x are bijective, we show that φ_U is surjective. Let $t \in \mathscr{G}(U)$. For any $x \in U$, since φ_x is surjective, there exist $s_x \in \mathscr{F}_x$ such that $\varphi_x(s_x) = t_x$. Let s_x be represented by a section $s(x)$ on a neighborhood V_x of x. Then by (1.1.2),

$$\big(\varphi_{V_x}(s(x))\big)_x = \varphi_x(s_x) = (t|_{V_x})_x.$$

By shrinking V_x we may assume that $\varphi_{V_x}(s(x)) = t|_{V_x}$.

Now U is covered by such open sets V_x, and on each V_x we have a section $s(x) \in \mathscr{F}(V_x)$. For two distinct points $x, y \in U$, $s(x)|_{V_x \cap V_y}$ and $s(y)|_{V_x \cap V_y}$ are both sent to $t|_{V_x \cap V_y}$ by φ, so by the injectivity of φ we just proved, $s(x)|_{V_x \cap V_y} = s(y)|_{V_x \cap V_y}$. Therefore, the gulability axiom produces a section $s \in \mathscr{F}(U)$ such that $s|_{V_x} = s(x)$, and from diagram (1.1.1) and the construction, we have

$$(\varphi_U(s))|_{V_x} = \varphi_{V_x}(s|_{V_x}) = \varphi_{V_x}(s(x)) = t|_{V_x},$$

so the identity axiom implies $\varphi_U(s) = t$. $\square$

Definition 1.1.4. Let $\mathscr{F}$ be a sheaf of abelian groups on a topological space X, $U \subseteq X$ open and $s \in \mathscr{F}(U)$ a section. The **support** of s is defined by

$$\operatorname{supp}(s) = \{x \in U : s_x \neq 0\}.$$

The **support** of $\mathscr{F}$ is defined to be

$$\operatorname{supp}(\mathscr{F}) = \{x \in X : \mathscr{F}_x \neq 0\}.$$

Proposition 1.1.3. *Let $\mathscr{F}$ be a sheaf of abelian groups on a topological space, $U \subseteq X$ open, and $s \in \mathscr{F}(U)$ a section. Then $\operatorname{supp}(s)$ is closed in U.*

Proof. For $x \in U \setminus \operatorname{supp}(s)$ we have $s_x = 0$, so there exists an open subset $V \subseteq U$ with $s|_V = 0$. This implies $s_y = 0$ for every $y \in V$ and therefore $V \subseteq U \setminus \operatorname{supp}(s)$. Hence $U \setminus \operatorname{supp}(s)$ is open. □

Example 1.1.6. Let X be a topological space. Let C_X be the sheaf of continuous $\mathbb{R}$-valued functions on X. Let $U \subseteq X$ be open and $s \in C_X(U)$ a continuous function $U \to \mathbb{R}$. In the proof of Proposition 1.1.3 we have just seen that $U \setminus \operatorname{supp}(s)$ is the interior of $\{x \in U : s(x) = 0\}$. Therefore we have

$$\operatorname{supp}(s) = \overline{\{x \in U : s(x) \neq 0\}}$$

which coincides with usual definition of the support of a continuous function.

1.1.3 Sheafification

In this part, we give a functorial way to attach to a presheaf a sheaf. This can be seen as the left adjoint of the forgetful functor from $\mathrm{Sh}(X)$ to $\mathbf{Psh}(X)$.

Definition 1.1.5 (Universal property of sheafification). If $\mathscr{F}$ is a presheaf on X, then a morphism of presheaves $\iota_{\mathscr{F}} : \mathscr{F} \to \mathscr{F}^{\#}$ on X is called a **sheafification** of $\mathscr{F}$ if $\mathscr{F}^{\#}$ is a sheaf, and for any sheaf $\mathscr{G}$, and any presheaf morphism $\varphi : \mathscr{F} \to \mathscr{G}$, there exists a unique morphism of sheaves $\tilde{\varphi} : \mathscr{F}^{\#} \to \mathscr{G}$ making the diagram

$$\begin{tikzcd}
\mathscr{F} \arrow[r, "\iota_{\mathscr{F}}"] \arrow[dr, "\varphi"'] & \mathscr{F}^{\#} \arrow[d, "\tilde{\varphi}"] \\
& \mathscr{G}
\end{tikzcd}$$

commute.

As a universal construction, the sheafification functor $\mathscr{F} \to \mathscr{F}^{\#}$ is left-adjoint to the forgetful functor from sheaves on X to presheaves on X: there is a canonical isomorphism

$$\operatorname{Mor}_{\mathrm{Sh}(X)}(\mathscr{F}^{\#}, \mathscr{G}) \cong \operatorname{Mor}_{\mathbf{Psh}(X)}(\mathscr{F}, \mathscr{G})$$

for any presheaf $\mathscr{F}$ and sheaf $\mathscr{G}$.

In Proposition 1.1.1 we see that if $\mathscr{F}$ is a sheaf, then $\mathscr{F}(U)$ can be identified as elements in $\prod_{x \in U} \mathscr{F}_x$ consists of compatible germs. This turns out to be a appropriate way to define the sheafification.

Proposition 1.1.4. *Let $\mathscr{F}$ be a presheaf on a topological space X. Then there exists a sheafification $\iota_{\mathscr{F}} : \mathscr{F} \to \mathscr{F}^{\#}$ such that the following properties hold:*

(a) For all $x \in X$ the map on stalks $\iota_{\mathscr{F},x} : \mathscr{F}_x \to \mathscr{F}^{\#}_x$ is bijective.

(b) For every presheaf $\mathscr{G}$ on X and every morphism of presheaves $\varphi : \mathscr{F} \to \mathscr{G}$ there exists a unique morphism $\tilde{\varphi} : \mathscr{F}^\# \to \mathscr{G}^\#$ making the diagram commutes:

$$\begin{tikzcd} \mathscr{F} \arrow[r, "\iota_{\mathscr{F}}"] \arrow[d, "\varphi"'] & \mathscr{F}^\# \arrow[d, "\tilde{\varphi}"] \\ \mathscr{G} \arrow[r, "\iota_{\mathscr{G}}"] & \mathscr{G}^\# \end{tikzcd} \tag{1.1.3}$$

In particular, $\mathscr{F} \to \mathscr{F}^\#$ is a functor from the category of presheaves on X to the category of sheaves on X. The sheafification $\iota_{\mathscr{F}} : \mathscr{F} \to \mathscr{F}^\#$ is unique up to isomorphism.

Proof. Suppose $\mathscr{F}$ is a presheaf. Define $\mathscr{F}^\#$ by declearing $\mathscr{F}^\#(U)$ as the set of compatible germs of the presheaf $\mathscr{F}$ over U. Explicitly:

$$\begin{aligned} \mathscr{F}^\#(U) &= \{(s_x)_{x\in U} \in \prod_{x\in U} \mathscr{F}_x : (s_x)_{x\in U} \text{ consists of compatible germs}\} \\ &= \{(s_x)_{x\in U} \in \prod_{x\in U} \mathscr{F}_x : \forall x \in U, \exists V \subseteq U \text{ and } \tilde{s} \in \mathscr{F}(V) \text{ s.t. } \tilde{s}_y = s_x \text{ for all } y \in V\}. \end{aligned}$$

For $U \subseteq V$ the restriction map $\mathscr{F}^\#(V) \to \mathscr{F}^\#(U)$ is induced by the natural projection. With this, for any covering $U = \bigcup_i U_i$ and sections $(s_x)_{x\in U_i}$ in $\mathscr{F}^\#$, the condition

$$((s_x)_{x\in U_i})|_{U_i\cap U_j} = ((s_x)_{x\in U_j})|_{U_i\cap U_j}$$

implies that $(s_x)_{x\in U_i}$ and $(s_x)_{x\in U_j}$ have common germs on their common domains. Thus we can construct a unique section $(s_x)_{x\in U}$ by just gathering their germs. It is clear that such a section $(s_x)_{x\in U}$ has compatible germs, hence belongs to $\mathscr{F}^\#$. This shows $\mathscr{F}^\#$ is a sheaf. For $U \subseteq X$ open, we define $\iota_{\mathscr{F},U} : \mathscr{F}(U) \to \mathscr{F}^\#(U)$ by $s \mapsto (s_x)_{x\in U}$. The definition of $\mathscr{F}^\#$ shows that, for $x \in X$, $\mathscr{F}^\#_x = \mathscr{F}_x$ and that $\iota_{\mathscr{F},x}$ is the identity.

Now let $\mathscr{G}$ be a presheaf on X and let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism. Sending $(s_x)_{x\in U}$ to $(\varphi_x(s_x))_{x\in U}$ defines a morphism $\mathscr{F}^\# \to \mathscr{G}^\#$. By Proposition 1.1.2(c) this is the unique morphism making the diagram (1.1.3) commutative.

If we assume in addition that $\mathscr{G}$ is a sheaf, then the morphism of sheaves $\iota_{\mathscr{G}}$, which is bijective on stalks, is an isomorphism by Proposition 1.1.2(b). Composing the morphism $\mathscr{F}^\# \to \mathscr{G}^\#$ with $\iota_{\mathscr{G}}^{-1}$, we obtain the morphism $\tilde{\varphi} : \mathscr{F}^\# \to \mathscr{G}$. Finally, the uniqueness of $\iota_{\mathscr{F}} : \mathscr{F} \to \mathscr{F}^\#$ is a formal consequence of universal property. $\square$

Proposition 1.1.5. *The sheafification of an injection (resp. surjection) of presheaves of sets is an injection (resp. surjection).*

Proof. This follows from the fact that sheafification does not change the stalk. $\square$

From Proposition 1.1.2, we get the following characterization of the sheafification.

Proposition 1.1.6. *Let $\mathscr{F}$ be a presheaf and $\mathscr{G}$ be a sheaf. Then $\mathscr{G}$ is isomorphic to the sheafification of $\mathscr{F}$ if and only and if there exists a morphism $\iota : \mathscr{F} \to \mathscr{G}$ such that ι_x is bijective for all $x \in X$.*

Proof. One direction is trivial, assume the converse. Then there is a $\tilde{\iota} : \mathscr{F}^\# \to \mathscr{G}$ such that the following diagram commutes

$$\begin{tikzcd} \mathscr{F}^\# \arrow[r, "\tilde{\iota}"] & \mathscr{G} \\ \mathscr{F} \arrow[u] \arrow[ur, "\iota"'] & \end{tikzcd}$$

By Proposition 1.1.2 $\tilde{\iota}$ induced isomorphisms on stalks, hence is an isomorphism $\mathscr{F}^\# \cong \mathscr{G}$. $\square$

Example 1.1.7. Let E be a set and let $\mathscr{F}$ be a presheaf of functions with values in E. Then its sheafification is given by

$$\mathscr{F}^\#(U) = \{f : U \to E \mid \exists \text{ open covering } (U_i)_i \text{ of } U \text{ such that } f|_{U_i} \in \mathscr{F}(U_i) \text{ for all } i\}.$$

Indeed, this is a sheaf by Example 1.1.3 and the inclusions $\mathscr{F}(U) \hookrightarrow \mathscr{F}^\#(U)$ for U open define a morphism of presheaves $\mathscr{F} \to \mathscr{F}^\#$ that is bijective on stalks. Hence we can apply Proposition 1.1.6.

Example 1.1.8. From the previous example, let's consider the constant presheaves. Let E_X be the sheaf of locally constant functions with values in E:

$$E_X = \{f : U \to E \mid \forall x \in U, \exists V \ni x \text{ open s.t. } f \text{ is constant on } V\}$$

then by Proposition 1.1.6, E_X is the sheafification of the presheaf of constant functions with values in E.

Example 1.1.9 (Open subset as a sheaf). Let $U \subseteq X$ be an open subset, then we can define a presheaf $\mathscr{U}$

$$\mathscr{U}(V) = \begin{cases} \{i_V\} & \text{if } V \subseteq U \\ \varnothing & \text{otherwise} \end{cases}$$

where $i_V : V \hookrightarrow U$ is the inclusion map. Let $\mathscr{U}^\#$ be its sheafification.

Let $\mathscr{F}$ be a sheaf and $s \in \mathscr{F}(U)$, then we can define a morphism φ_s by setting $\varphi_{s,U}(\mathbf{1}_U) = s$ and others by restriction. Then we get a map

$$\mathscr{F}(U) \to \mathrm{Mor}_{\mathbf{Psh}(X)}(\mathscr{U}, \mathscr{F})$$

which can be shown is an isomorphism. Together with the adjointness of sheafification, we get isomorphisms

$$\mathscr{F}(U) \cong \mathrm{Mor}_{\mathbf{Psh}(X)}(\mathscr{U}, \mathscr{F}) \cong \mathrm{Mor}_{\mathrm{Sh}(X)}(\mathscr{U}^\#, \mathscr{F}).$$

1.1.4 Direct and inverse images of sheaves

In this part $f : X \to Y$ denotes a continuous map of topological spaces. We will now see how to use f in order to attach to a sheaf on X a sheaf on Y (direct image) and to a sheaf on Y a sheaf an X (inverse image).

Definition 1.1.6 (Direct image of a presheaf). Let $f : X \to Y$ be a continuous map. Let $\mathscr{F}$ be a presheaf on X. We define a presheaf $f_*\mathscr{F}$ on Y by (for $V \subseteq Y$ open)

$$(f_*\mathscr{F})(V) = \mathscr{F}\big(f^{-1}(V)\big)$$

the restriction maps given by the restriction maps for $\mathscr{F}$. Then $f_*\mathscr{F}$ is called the **direct image** of $\mathscr{F}$ under f or the **pushforward presheaf** of $\mathscr{F}$ by f. Whenever $\varphi : \mathscr{F}_1 \to \mathscr{F}_2$ is a morphism of presheaves, the family of maps $f_*(\varphi)_V := \varphi_{f^{-1}(V)}$ for $V \subseteq Y$ open is a morphism $f_*(\varphi) : f_*\mathscr{F}_1 \to f_*\mathscr{F}_2$. We thus obtain a functor f_* from the category of presheaves on X to the category of presheaves on Y.

Proposition 1.1.7. *Let $f : X \to Y$ be a continuous map*

(a) If $\mathscr{F}$ is a sheaf on X, then $f_*\mathscr{F}$ is a sheaf on Y. Therefore f_* also defines a functor $f_* : \mathrm{Sh}(Y) \to \mathrm{Sh}(X)$.

(b) If $g : Y \to Z$ is a second continuous map, then $(g \circ f)_* = g_* \circ f_*$.

Proof. This first statement immediately follows from the fact that if $V = \bigcup V_i$ is an open covering in Y, then $f^{-1}(V) = \bigcup_i f^{-1}(V_i)$ is an open covering in X. The second claim is a computation:

$$(g \circ f)_*\mathscr{F}(W) = \mathscr{F}\big((g \circ f)^{-1}(W)\big) = \mathscr{F}\big(f^{-1} \circ g^{-1}(W)\big) = g_*\mathscr{F}\big(f^{-1}(W)\big) = g_* \circ f_*\mathscr{F}(W).$$

where $W \subseteq Y$ is open. $\square$

We now define the inverse image of a sheaf. Let $\mathscr{G}$ be a presheaf of sets on Y. The **pullback presheaf** $f^p\mathscr{G}$ of a given presheaf $\mathscr{G}$ is defined as the left adjoint of the pushforward f_* on presheaves. In other words, it should be a presheaf $f^p\mathscr{G}$ on X such that

$$\mathrm{Mor}_{\mathbf{Psh}(X)}(f^p\mathscr{G}, \mathscr{F}) \cong \mathrm{Mor}_{\mathbf{Psh}(X)}(\mathscr{G}, f_*\mathscr{F})$$

It turns out that this actually exists.

Proposition 1.1.8 (Inverse image of a presheaf). *Let $f : X \to Y$ be a continuous map and let $\mathscr{G}$ be a presheaf on Y. There exists a functor $f^p : \mathbf{Psh}(Y) \to \mathbf{Psh}(X)$ which is left adjoint to f_*. For a presheaf $\mathscr{G}$ it is determined by*

$$f^p\mathscr{G}(U) = \varinjlim_{\substack{V \supseteq f(U) \\ V \subseteq Y \text{ open}}} \mathscr{G}(V).$$

the restriction maps being induced by the restriction maps of $\mathscr{G}$.

Proof. The colimit is over the partially ordered set consisting of open subset $V \subseteq Y$ which contain $f(U)$ with ordering by reverse inclusion. This is a directed partially ordered set, and if $U_1 \subseteq U_2$, then every open neighbourhood of $f(U_2)$ is an open neighbourhood of $f(U_1)$. Hence the system defining $f^p\mathscr{G}(U_2)$ is a subsystem of the one defining $f^p\mathscr{G}(U_1)$ and we obtain a restriction map. Note that the construction of the colimit is clearly functorial in $\mathscr{G}$, and similarly for the restriction mappings. Hence we have defined f^p as a functor. Now we turn to the proof of the adjointness. For this, we need to define the unit map and the counit map, as follows.

- There exists a canonical map $\mathscr{G}(V) \to f^p\mathscr{G}\big(f^{-1}(V)\big)$ for any open subset $V \subseteq Y$, because the system of open neighbourhoods of $f\big(f^{-1}(V)\big)$ contains the element V:

  $$\rho_{\mathscr{G},V} : \mathscr{G}(V) \longrightarrow f^p\mathscr{G}(f^{-1}(V)) = \varinjlim_{U \supseteq f(f^{-1}(V))} \mathscr{G}(U)$$

  This is compatible with restriction mappings, so there is a canonical map $\rho_{\mathscr{G}} : \mathscr{G} \to f_*f^p\mathscr{G}$.

- There exists a canonical map $f^pf_*\mathscr{F}(U) \to \mathscr{F}(U)$ for any open subset $U \subseteq X$:

  $$\sigma_{\mathscr{F},U} : f^pf_*\mathscr{F}(U) = \varinjlim_{V \supseteq f(U)} \mathscr{F}(f^{-1}(V)) \longrightarrow \mathscr{F}(U)$$

  where the map is given by the restriction of $\mathscr{F}(f^{-1}(V))$ to $\mathscr{F}(U)$. One easily verifies that the maps are compatible with restriction maps and thus there is a canonical map $\sigma_{\mathscr{F}} : f^pf_*\mathscr{F} \to \mathscr{F}$.

The maps we get are illustrated in the following diagram:

$$\begin{tikzcd}
X \arrow[d, "f"] & & f^p\mathscr{G} \arrow[r, "\psi"'] \arrow[dl, dashed] \arrow[rr, bend left, "f^p\varphi"] & \mathscr{F} \arrow[d, dashed] & f^pf_*\mathscr{F} \arrow[l, "\sigma_{\mathscr{F}}"'] \\
Y & f_*f^p\mathscr{G} \arrow[rr, bend right, "f_*\psi"'] & \mathscr{G} \arrow[l, "\rho_{\mathscr{G}}"'] \arrow[u, dashed] \arrow[r, "\varphi"] & f_*\mathscr{F} \arrow[ur, dashed] &
\end{tikzcd}$$

Now let $\mathscr{F}$ be a presheaf of sets on X. Suppose that $\psi : f^p\mathscr{G} \to \mathscr{F}$ is a map of presheaves of sets. The corresponding map $\psi^\flat : \mathscr{G} \to f_*\mathscr{F}$ is the composition

$$\psi^\flat : \mathscr{G} \xrightarrow{\rho_{\mathscr{G}}} f_*f^p\mathscr{G} \xrightarrow{f_*\psi} f_*\mathscr{F}$$

Suppose that $\varphi : \mathscr{G} \to f_*\mathscr{F}$ is a map of presheaves of sets. The map $\varphi^\sharp : f^p\mathscr{G} \to \mathscr{F}$ is then the composition

$$\varphi^\sharp : f^p\mathscr{G} \xrightarrow{f^p\varphi} f^pf_*\mathscr{F} \xrightarrow{\sigma_{\mathscr{F}}} \mathscr{F}$$

It can be verified that these tow maps are inverse of each other. Let $U \subseteq X$, then the map $(\psi^\flat)^\sharp : f^p\mathscr{G} \to \mathscr{F}$ is given by

$$f^p\mathscr{G}(U) = \varinjlim_{V\supseteq f(U)} \mathscr{G}(V) \to \varinjlim_{V\supseteq f(U)} \varinjlim_{V'\supseteq f(f^{-1}(V))} \mathscr{G}(V') \xrightarrow{\psi_{f^{-1}(V)}} \varinjlim_{V\supseteq f(U)} \mathscr{F}(f^{-1}(V)) \xrightarrow{\mathrm{res}^*_U} \mathscr{F}(U)$$

Let $s \in f^p\mathscr{G}(U)$ be represented by $v \in \mathscr{G}(V)$ for some $V \supseteq f(U)$. Since $f^{-1}(V) \supseteq U$, from the diagram

$$\begin{array}{ccc} f^p\mathscr{G}(f^{-1}(V)) & \xrightarrow{\psi_{f^{-1}(V)}} & \mathscr{F}(f^{-1}(V)) \\ {\scriptstyle \mathrm{res}^{f^{-1}(V)}_U}\downarrow & & \downarrow{\scriptstyle \mathrm{res}^{f^{-1}(V)}_U} \\ f^p\mathscr{G}(U) & \xrightarrow{\psi_U} & \mathscr{F}(U) \end{array}$$

we obtain

$$\begin{aligned} (\psi^\flat)^\sharp(s) &= \mathrm{res}^*_U\big[\psi_{f^{-1}(V)}\big([v]_{f^{-1}(V)}\big)\big]_U = \mathrm{res}^{f^{-1}(V)}_U \circ \psi_{f^{-1}(V)}\big([v]_{f^{-1}(V)}\big) \\ &= \psi_U\big(\mathrm{res}^{f^{-1}(V)}_U([v]_{f^{-1}(V)})\big) = \psi_U([v]_U) = \psi_U(s). \end{aligned}$$

Thus $(\psi^\flat)^\sharp = \psi$. A similar argument gives $(\varphi^\sharp)^\flat = \varphi$, hence we are done. $\square$

Remark 1.1.4. We will almost never use the concrete description of $f^p\mathscr{G}$ in the sequel. Very often we are given f, $\mathscr{F}$, and $\mathscr{G}$ as in the Proposition 1.1.8, and a morphism of sheaves $\varphi : \mathscr{G} \to f_*\mathscr{F}$. Then usually it will be sufficient to understand for each $x \in X$ the map

$$\varphi^\sharp_x : (f^p\mathscr{G})_x = \mathscr{G}_{f(x)} \to \mathscr{F}_x$$

induced by $\varphi^\sharp : f^p\mathscr{G} \to \mathscr{F}$ on stalks. The proof of Proposition 1.1.8 shows that we can describe this map in terms of as follows. For every open neighborhood $V \subseteq Y$ of $f(x)$, we have maps

$$\mathscr{G}(V) \xrightarrow{\varphi_V} \mathscr{F}(f^{-1}(V)) \longrightarrow \mathscr{F}_x$$

and taking the colimit over all V we obtain the map $\varphi^\sharp_x : \mathscr{G}_{f(x)} \to \mathscr{F}_x$.

Proposition 1.1.9. *Let $f : X \to Y$ be a continuous map. Let $x \in X$. Let $\mathscr{G}$ be a presheaf of sets on Y. There is a canonical bijection of stalks $(f^p\mathscr{G})_x = \mathscr{G}_{f(x)}$.*

Proof. This is obtained as follows

$$(f^p\mathscr{G})_x = \varinjlim_{U\ni x} f^p\mathscr{G}(U) = \varinjlim_{U\ni x} \varinjlim_{V\supseteq f(U)} \mathscr{G}(V) = \varinjlim_{V\ni f(x)} \mathscr{G}(V) = \mathscr{G}_{f(x)}.$$

Here we have used the fact that any V open in Y containing $f(x)$ occurs in the description above. The case for sheaves is obtained from Proposition 1.1.4. $\square$

Let $\mathscr{G}$ be a sheaf of sets on Y. The **pullback sheaf** $f^{-1}\mathscr{G}$ is defined by the formula

$$f^{-1}\mathscr{G} = (f^p\mathscr{G})^{\#}.$$

Sheafification is a left adjoint to the inclusion of sheaves in presheaves, and f^p is a left adjoint to f_* on presheaves. As a formal consequence we obtain that f^{-1} is a left adjoint of pushforward on sheaves: for sheaves $\mathscr{F}$ and $\mathscr{G}$, we have

$$\begin{aligned}\mathrm{Mor}_{\mathrm{Sh}(X)}(f^{-1}\mathscr{G},\mathscr{F}) &= \mathrm{Mor}_{\mathrm{Sh}(X)}\big((f^p\mathscr{G})^{\#},\mathscr{F}\big) \cong \mathrm{Mor}_{\mathbf{Psh}(X)}\big(f^p\mathscr{G},\mathscr{F}\big)\\ &\cong \mathrm{Mor}_{\mathbf{Psh}(X)}(\mathscr{G},f_*\mathscr{F}) = \mathrm{Mor}_{\mathrm{Sh}(X)}(\mathscr{G},f_*\mathscr{F})\end{aligned}$$

Proposition 1.1.10. *There are canonical maps*

$$f^{-1}f_*\mathscr{F} \to \mathscr{F}, \quad \mathscr{G} \to f_*f^{-1}\mathscr{G}$$

for sheaves $\mathscr{F}$ on X and $\mathscr{G}$ on Y.

Proof. We already have maps

$$\sigma_{\mathscr{F}} : f^p f_*\mathscr{F} \to \mathscr{F}, \quad \rho_{\mathscr{G}} : \mathscr{G} \to f^p f_*\mathscr{G}.$$

The map $f^{-1}f_*\mathscr{F} \to \mathscr{F}$ is given by the universal property of sheafification:

$$\begin{tikzcd} f^p f_*\mathscr{F} \arrow[r, "sh"] \arrow[dr, "\sigma_{\mathscr{F}}"'] & f^{-1}f_*\mathscr{F} \arrow[d, dashed] \\ & \mathscr{F} \end{tikzcd}$$

and the map $\mathscr{G} \to f_*f^{-1}\mathscr{G}$ is given by the composition

$$\mathscr{G}(V) \xrightarrow{\rho_{\mathscr{G},V}} f^p\mathscr{G}(f^{-1}(V)) \xrightarrow{sh} f^{-1}\mathscr{G}(f^{-1}(V))$$

for $V \subseteq Y$ open. $\square$

Proposition 1.1.11 (Inverse image and composition). *Let $f : X \to Y$ and $g : Y \to Z$ be continuous maps of topological spaces. The functors $(g \circ f)^{-1}$ and $f^{-1} \circ g^{-1}$ are canonically isomorphic. Similarly $(g \circ f)^p \cong f^p \circ g^p$ on presheaves.*

Proof. This comes from the formal consequence

$$\begin{aligned}\mathrm{Mor}_{\mathbf{Psh}(X)}\big((g\circ f)^p\mathscr{G},\mathscr{F}\big) &\cong \mathrm{Mor}_{\mathbf{Psh}(X)}\big(\mathscr{G},(g\circ f)_*\mathscr{F}\big) = \mathrm{Mor}_{\mathbf{Psh}(X)}\big(\mathscr{G},g_*\circ f_*\mathscr{F}\big)\\ &\cong \mathrm{Mor}_{\mathbf{Psh}(X)}\big(g^p\mathscr{G},f_*\mathscr{F}\big) \cong \mathrm{Mor}_{\mathbf{Psh}(X)}\big(f^p\circ g^p\mathscr{G},\mathscr{F}\big)\end{aligned}$$

By the uniqueness of adjoint functors, we obtain $(g \circ f)^p \cong f^p \circ g^p$. A similar computation holds for $(g \circ f)^{-1}$. $\square$

To conclude this part, we use the direct image functor to produce an adjoint of the stalk functor. First, we need the following concept of a skyscraper sheaf.

Definition 1.1.7. Let $x \in X$ be a point. Denote $i_x : \{x\} \mapsto X$ the inclusion map. Let A be a set and think of A as a sheaf on the one point space $\{x\}$. We define the **skyscraper sheaf** at x with value A as the pushforward sheaf $i_{x,*}A$. Explicitly,

$$i_{x,*}A(U) = \begin{cases} A & \text{if } x \in U, \\ \varnothing & \text{if } x \notin U. \end{cases}$$

We say a sheaf $\mathscr{F}$ is a skyscraper sheaf if $\mathscr{F} \cong i_{x,*}A$ for some $x \in X$ and a set A.

Lemma 1.1.12. *Let X be a topological space, $x \in X$ a point, and A a set. For any point $y \in X$ the stalk of the skyscraper sheaf at x with value A at y is*

$$(i_{x,*}A)_y = \begin{cases} A & \text{if } y \in \overline{\{x\}} \\ \{*\} & \text{otherwise} \end{cases}$$

Proof. If $x \notin \overline{\{x\}}$, then there exist arbitrarily small open neighbourhoods U of y which do not contain x. Because $\mathscr{F}$ is a sheaf we have $\mathscr{F}(i_x^{-1}(U)) = \{*\}$ for any such U. □

Proposition 1.1.13. *Let X be a topological space, and let $x \in X$ be a point. The functors $\mathscr{F} \mapsto \mathscr{F}_x$ and $A \mapsto i_{x,*}A$ are adjoint. In a formula,*

$$\mathrm{Mor}_{\mathbf{Set}}(\mathscr{F}_x, A) \cong \mathrm{Mor}_{\mathrm{Sh}(X)}(\mathscr{F}, i_{x,*}A)$$

Proof. Consider the pull back functor of the map $i_x : \{x\} \to X$: for a sheaf $\mathscr{F}$ on X,

$$i_x^{-1}\mathscr{F}(\{x\}) = \varinjlim_{U \ni x} \mathscr{F}(U),$$

which is exactly the stalk of $\mathscr{F}$ at x. Thus the adjointness comes from that of i_x^{-1} and $i_{x,*}$. □

1.1.5 Locally closed subspaces

In this paragraph, we shall be concerned with a locally closed subspace W of X. This means that any point $x \in W$ has an open neighbourhood V in X such that $W \cap V$ is closed relative to V. The inclusion of W in X will be denoted $f : W \to X$.

Definition 1.1.8. For a sheaf $\mathscr{E}$ over W we define $f_!(\mathscr{E})$ to be the sheaf on X whose section over an open subset U of X is given by

$$\Gamma(U, f_!(\mathscr{E})) = \{s \in \Gamma(W \cap U, \mathscr{E}) : \mathrm{supp}(s) \text{ is closed relative to } U\}.$$

The restriction maps of $f_!(\mathscr{E})$ are induced from $f_!(\mathscr{E})$, of which it is a subsheaf. We therefore obtain a canonical monomorphism $f_!(\mathscr{E}) \to f_*(\mathscr{E})$.

On the other hand, from the description of the sections of $f_!(\mathscr{E})$, it is not hard to see, in view of the fact that W is locally closed, that we have

$$(f_!(\mathscr{E}))_x = \begin{cases} \mathscr{E}_x & \text{if } x \in W, \\ 0 & \text{if } x \notin W. \end{cases}$$

which in particular shows that $f_!$ is exact.

Proposition 1.1.14. *The functor $f_! : \mathrm{Sh}(W) \to \mathrm{Sh}(X)$ is an equivalence between the category of sheaves on W and the full subcategory of $\mathrm{Sh}(X)$ consisting of sections of $\mathscr{F}$ that are supported in W. The inverse functor is given by f^{-1}.*

Proof. For a sheaf $\mathscr{E}$ on W, we note the following simple identity among sheaves on W:

$$f^{-1}(f_!(\mathscr{E})) = \mathscr{E}.$$

Next, consider a sheaf $\mathscr{F}$ on X whose stalks vanish outside W. A close examination reveals that the adjunction morphism $\mathscr{F} \to f_*(f^{-1}(\mathscr{F}))$ may be factored through $f_!(f^{-1}\mathscr{F})$, so that we obtain an isomorphism $\mathscr{F} \cong f_!(f^{-1}(\mathscr{F}))$. □

Proposition 1.1.15. *The functor $f_! : \mathrm{Sh}(W) \to \mathrm{Sh}(X)$ admits a right adjoint $f^! : \mathrm{Sh}(X) \to \mathrm{Sh}(W)$.*

Proof. For a sheaf $\mathscr{F}$ over X, we let $\mathscr{F}^W$ denote the sheaf on X whose section over an open subset U is given by

$$\Gamma(U, \mathscr{F}^W) = \{s \in \Gamma(U, \mathscr{F}) : \operatorname{supp}(s) \subseteq W\}.$$

The stalks of $\mathscr{F}^W$ are then zero at any poitns outside W, so if $f^!(\mathscr{F}) = f^{-1}(\mathscr{F}^W)$, then by the last assertion of Proposition 1.1.14 applied to $\mathscr{F}^W$, we have

$$\Gamma(U, f_!(f^!(\mathscr{F}))) = \Gamma(U, \mathscr{F}^W) = \{s \in \Gamma(U, \mathscr{F}) : \operatorname{supp}(s) \subseteq W\}. \tag{1.1.4}$$

This formula provides a monomorphism $f_!(f^!(\mathscr{F})) \to \mathscr{F}$, and it follows from (1.1.4) that any morphism $\mathscr{G} \to \mathscr{F}$, where $\mathscr{G}$ is a sheaf on X whose stalks are zero outside W, may by factored through the monomorphism $f_!(f^!(\mathscr{F})) \to \mathscr{F}$. In particular, for a sheaf $\mathscr{E}$ on W, we get accordingly

$$\operatorname{Hom}(f_!(\mathscr{E}), \mathscr{F}) = \operatorname{Hom}(f_!(\mathscr{E}), f_!(f^!(\mathscr{F}))).$$

On the other hand, the first assertion of Proposition 1.1.14 implies that we have an isomorphism

$$f_! : \operatorname{Hom}(\mathscr{E}, f^!(\mathscr{F})) \cong \operatorname{Hom}(f_!(\mathscr{E}), f_!(f^!(\mathscr{F}))),$$

whence the claim. □

Proposition 1.1.16. *The functor $f^! : \mathrm{Sh}(X) \to \mathrm{Sh}(W)$ is left exact and transforms injective sheaves into injective sheaves.*

Proof. The left exactness is a formal consequence of the adjunction. The second statement is a formal consequence of the exactness of $f_!$. □

1.1.5.1 Open and closed subspaces

We shall now make a close investigation of the special case of open and closed subsets.

Proposition 1.1.17. *Let X be a topological space.*

(a) *For the inclusion $i : Z \to X$ of a closed subspace, we have $i_! = i_*$.*

(b) *For the inclusion $j : U \to X$ of an open subspace, we have $j^{-1} = j^!$.*

Proof. For a sheaf $\mathscr{E}$ on Z, we have a natural map $i_!(\mathscr{E}) \to i_*(\mathscr{E})$ which is an isomorphism as one checks by localization, since Z is closed in X. On the other hand, for a sheaf $\mathscr{F}$ on X, we have according to the proof of Proposition 1.1.15 that $j^!(\mathscr{F}) = j^{-1}(\mathscr{F}^U)$. Combined with the inclusion $\mathscr{F}^U \to \mathscr{F}$, this yields a monomorphism $j^!(\mathscr{F}) \to j^{-1}(\mathscr{F})$, which is an isomorphism because for any open subset V of U, we have

$$\Gamma(V, j^!(\mathscr{F})) = \Gamma(V, \mathscr{F}^U) = \Gamma(V, \mathscr{F}).$$ □

Corollary 1.1.18. *The functor $i_* : \mathrm{Sh}(Z) \to \mathrm{Sh}(X)$ is exact, and $j^{-1} : \mathrm{Sh}(X) \to \mathrm{Sh}(U)$ transforms an injective sheaf $\mathscr{J}$ on X into an injective sheaf $j^{-1}(\mathscr{J})$ on U.*

Proof. This follows from Proposition 1.1.17 and Proposition 1.1.16. □

Proposition 1.1.19. *In the case where U and Z are complementary subspaces of X, a sheaf $\mathscr{F}$ on X give rise to exact sequences*

$$0 \longrightarrow j_!j^{-1}(\mathscr{F}) \longrightarrow \mathscr{F} \longrightarrow i_*i^{-1}(\mathscr{F}) \longrightarrow 0 \tag{1.1.5}$$

$$0 \longrightarrow i_*i^!(\mathscr{F}) \longrightarrow \mathscr{F} \longrightarrow j_*(j^{-1}(\mathscr{F})) \tag{1.1.6}$$

of sheaves on X, where the two morphism are adjunction morphisms. Moreover, the last morphism in (1.1.6) is surjective if $\mathscr{F}$ is injective.

Proof. The exactness is seen by taking stalks. For example, if $x \in U$, then we have $j_!(j^{-1}(\mathscr{F}))_x = (j^{-1}(\mathscr{F}))_x = \mathscr{F}_x$ and $i_*(i^{-1}(\mathscr{F})) = 0$, and a similar result holds for $x \in Z$. $\square$

In the situation of Proposition 1.1.19, we have adjoint pairs $(j_!, j^! = j^{-1}, j_*)$ and $(i^{-1}, i_* = i_!, i^!)$, which are presented in the following diagram

$$\begin{tikzcd} \mathrm{Sh}(U) \arrow[r, bend left, "j_!"] \arrow[r, bend right, "j_*"'] & \mathrm{Sh}(X) \arrow[l, "j^{-1}=j^!"'] \arrow[r, bend left, "i^{-1}"] \arrow[r, bend right, "i^!"'] & \mathrm{Sh}(Z) \arrow[l, "i_*=i_!"'] \end{tikzcd}$$

The functors in the lower row are left exact and preserves injectives, that in the middle row are exact and preserves injectives, and that in the upper row are exact. Moreover, the adjoint relationship between the various functions can be read off from this diagram, and we have identities

$$j^{-1}j_* = 1, \quad j^{-1}j_! = 1, \quad i^!i_* = 1, \quad i^{-1}i_* = 1,$$
$$j^{-1}i_* = 0, \quad i^{-1}j_! = 0, \quad i^!j_* = 0.$$

1.1.6 Glueing sheaves

It is quite often that we want to glue a bunch of objects defined on the members of a covering of X to create a new one. In this paragraph we will see how to do this for morphisms and sheaves.

Proposition 1.1.20. *Let X be a topological space. Let $X = \bigcup_i U_i$ be an open covering. Let $\mathscr{F}, \mathscr{G}$ be sheaves of sets on X. Given a collection*

$$\varphi_i : \mathscr{F}|_{U_i} \to \mathscr{G}|_{U_i}$$

of maps of sheaves such that for all $i, j \in I$ the maps φ_i, φ_j restrict to the same map $\mathscr{F}|_{U_i \cap U_j} \to \mathscr{G}|_{U_i \cap U_j}$, there exists a unique map of sheaves

$$\varphi : \mathscr{F} \to \mathscr{G}$$

whose restriction to each U_i agrees with φ_i.

Proof. For $V \subseteq X$ open, we have $V = \bigcup_i (V \cap U_i)$, so for $s \in \mathscr{F}(V)$ we define $\varphi_V(s)$ by the equations

$$(\varphi_V(s))|_{V \cap U_i} = (\varphi_i)_{V \cap U_i}(s|_{V \cap U_i}) \quad \text{for } \ i \in I. \tag{1.1.7}$$

By the condition of sheaf, this is well-defined, and indeed defines a morphism of sheaves. It is clear that φ restrict to φ_i on each U_i. To see the uniqueness, if $\psi : \mathscr{F} \to \mathscr{G}$ is another morphism restricting to φ_i on each i, then for each $s \in \mathscr{F}(U)$, $\psi(s)$ also satisfies (1.1.7), so it must coincides with $\varphi(s)$ by the condition of sheaf. This shows $\varphi = \psi$, as desired. $\square$

The previous proposition implies that given two sheaves $\mathscr{F}, \mathscr{G}$ on the topological space X the rule

$$U \mapsto \mathrm{Mor}_{\mathrm{Sh}(X)}(\mathscr{F}|_U, \mathscr{G}|_U)$$

defines a sheaf. This is a kind of **internal hom sheaf**. It is seldom used in the setting of sheaves of sets, and more usually in the setting of sheaves of modules.

Let X be a topological space. Let $X = \bigcup_i U_i$ be an open covering. For each $i \in I$ let $\mathscr{F}_i$ be a sheaf of sets on U_i. For each pair $i, j \in I$, let

$$\varphi_{ij} : \mathscr{F}_i|_{U_i \cap U_j} \to \mathscr{F}_j|_{U_i \cap U_j}$$

be an isomorphism of sheaves of sets. Assume in addition that for every triple of indices $i, j, k \in I$ the following diagram is commutative

$$\begin{tikzcd}
\mathscr{F}_i|_{U_i \cap U_j \cap U_k} \arrow[rr, "\varphi_{ik}"] \arrow[dr, "\varphi_{ij}"'] & & \mathscr{F}_k|_{U_i \cap U_j \cap U_k} \\
& \mathscr{F}_j|_{U_i \cap U_j \cap U_k} \arrow[ur, "\varphi_{jk}"'] &
\end{tikzcd}$$

We will call such a collection of data $(\mathscr{F}_i, \varphi_{ij})$ a **glueing data** for sheaves of sets with respect to the covering $X = \bigcup_i X_i$.

Proposition 1.1.21. *Let X be a topological space. Let $X = \bigcup_i U_i$ be an open covering. Given any glueing data $(\mathscr{F}_i, \varphi_{ij})$ for sheaves of sets with respect to the covering, there exists a sheaf of sets $\mathscr{F}$ on X together with isomorphisms*

$$\varphi_i : \mathscr{F}|_{U_i} \to \mathscr{F}_i$$

such that the diagrams

$$\begin{tikzcd}
\mathscr{F}|_{U_i \cap U_j} \arrow[r, "\varphi_i"] \arrow[d, equal] & \mathscr{F}_i|_{U_i \cap U_j} \arrow[d, "\varphi_{ij}"] \\
\mathscr{F}|_{U_i \cap U_j} \arrow[r, "\varphi_j"] & \mathscr{F}_j|_{U_i \cap U_j}
\end{tikzcd}$$

are commutative.

Proof. Actually we can write a formula for the set of sections of $\mathscr{F}$ over an open $W \subseteq X$. Namely, we define

$$\mathscr{F}(W) = \{(s_i)_{i \in I} \mid s_i \in \mathscr{F}_i(W \cap U_i) \text{ and } \varphi_{ij}(s_i|_{W \cap U_i \cap U_j}) = s_j|_{W \cap U_i \cap U_j}\}.$$

Restriction mappings for $W' \subseteq W$ are defined by the restricting each of the (s_i) to $W' \cap U_i$. The sheaf condition for $\mathscr{F}$ follows immediately from the sheaf condition for each of the $\mathscr{F}_i$.

We still have to prove that $\mathscr{F}|_{U_i}$ maps isomorphically to $\mathscr{F}_i$. Let $W \subseteq U_i$; then the commutativity of the diagrams in the definition of a glueing data assures that we may start with any section $s \in \mathscr{F}_i(W)$ and obtain a compatible collection by setting $s_i = s$ and $s_j = \varphi_{ij}(s_i|_{W \cap U_j})$. Thus the claim follows. $\square$

Corollary 1.1.22. *Let X be a topological space. Let $X = \bigcup_i U_i$ be an open covering. The functor which associates to a sheaf of sets $\mathscr{F}$ the following collection of glueing data*

$$(\mathscr{F}|_{U_i}, (\mathscr{F}_i)|_{U_i \cap U_j} \to (\mathscr{F}_j)|_{U_i \cap U_j})$$

with respect to the covering $X = \bigcup_i U_i$ defines an equivalence of categories between $\mathrm{Sh}(X)$ and the category of glueing data.

This result means that if the sheaf $\mathscr{F}$ was constructed from the glueing data $(\mathscr{F})_i, \varphi_{ij})$ and if $\mathscr{G}$ is a sheaf on X, then a morphism $f : \mathscr{F} \to \mathscr{G}$ is given by a collection of morphisms of sheaves

$$f_i : \mathscr{F}_i \to \mathscr{G}$$

compatible with the glueing maps φ_{ij}. Similarly, to give a morphism of sheaves $g : \mathscr{G} \to \mathscr{F}$ is the same as giving a collection of morphisms of sheaves

$$g_i : \mathscr{G}|_{U_i} \to \mathscr{F}_i$$

compatible with the glueing maps φ_{ij}.

1.1.7 Preheaves and sheaves over a basis

Sometimes there exists a basis for the topology consisting of opens that are easier to work with than general opens. For convenience we give here some definitions and simple lemmas in order to facilitate working with (pre)sheaves in such a situation.

Let X be a topological space. Let $\mathcal{B}$ be a basis for the topology on X. A **presheaf $\mathscr{F}$ of sets on $\mathcal{B}$** is a rule which assigns to each $U \in \mathcal{B}$ a set $\mathscr{F}(U)$ and to each inclusion $V \subseteq U$ of elements of $\mathcal{B}$ a map $\mathrm{res}_V^U : \mathscr{F}(U) \to \mathscr{F}(V)$ such that whenever $W \subseteq V \subseteq U$ in $\mathcal{B}$ we have $\mathrm{res}_V^U \circ \mathrm{res}_W^V = \mathrm{res}_W^U$. If $\mathscr{F}$ be a presheaf over the basis $\mathcal{B}$. We can associate $\mathscr{F}$ with a sheaf $\mathscr{F}_{\mathcal{B}}$ by defining

$$\mathscr{F}_{\mathcal{B}}(U) = \{(s_V) \in \prod_{\substack{V\in\mathcal{B}\\ V\subseteq U}} \mathscr{F}(V) : \text{for all } W, V \in \mathcal{B}, W \subseteq V, s_V|_W = s_W\} = \varprojlim_{\substack{V\in\mathcal{B}\\ V\subseteq U}} \mathscr{F}(V). \quad (1.1.8)$$

for any open subset U of X. If U and U' are two open sets of X such that $U \subseteq U'$, we define $\mathrm{res}_U^{U'}$ as the inverse limit (for $V \subseteq U$) of the canonical morphisms $\mathscr{F}_{\mathcal{B}}(U') \to \mathscr{F}(V)$, in other words, the unique morphism $\mathscr{F}_{\mathcal{B}}(U') \to \mathscr{F}_{\mathcal{B}}(U)$ which, composed with the canonical morphisms $\mathscr{F}_{\mathcal{B}}(U) \to \mathscr{F}(V)$, gives the canonical morphisms $\mathscr{F}_{\mathcal{B}}(U') \to \mathscr{F}(V)$; it is then immediate that the transitivity holds. Moreover, if $U \in \mathcal{B}$, the canonical morphism $\mathscr{F}_{\mathcal{B}}(U) \to \mathscr{F}(U)$ is an isomorphism, allowing us to identify these two sets.

Proposition 1.1.23. *The following conditions are equivalent:*

(i) The presheaf $\mathscr{F}_{\mathcal{B}}$ is a sheaf on X.

(ii) For any covering (U_α) of $U \in \mathcal{B}$ given by elements of $\mathcal{B}$, the set $\mathscr{F}(U)$ corresponds bijectively to $(s_\alpha) \in \prod_\alpha \mathscr{F}(U_\alpha)$ such that $s_\alpha|_V = s_\beta|_V$ for any $V \in \mathcal{B}$ and $V \subseteq U_\alpha \cap U_\beta$.

(iii) For any covering (U_α) of $U \in \mathcal{B}$ given by elements of $\mathcal{B}$ and $(U_{\alpha\beta\gamma})$ of $U_{\alpha\beta} = U_\alpha \cap U_\beta$ by elements of $\mathcal{B}$, the set $\mathscr{F}(U)$ corresponds bijectively to $(s_\alpha) \in \prod_\alpha \mathscr{F}(U_\alpha)$ such that $s_\alpha|_{U_{\alpha\beta\gamma}} = s_\beta|_{U_{\alpha\beta\gamma}}$ for all γ.

Proof. It is clear that (i)⇒(iii)⇒(ii). Now assume (ii) and let (U_α) be a covering of $U \in \mathcal{B}$ by elements of $\mathcal{B}$ and $(U_{\alpha\beta\gamma})$ a covering of $U_{\alpha\beta}$ by elements of $\mathcal{B}$. Let $(s_\alpha) \in \prod_\alpha \mathscr{F}(U_\alpha)$ be a family such that $s_\alpha|_{U_{\alpha\beta\gamma}} = s_\beta|_{U_{\alpha\beta\gamma}}$ for all γ. Let $V \subseteq U_\alpha \cap U_\beta$ be an basis element in $\mathcal{B}$. For each index γ, let $(V_{\mu\gamma})$ be a covering of $V \cap U_{\alpha\beta\gamma}$ by elements of $\mathcal{B}$, so that the family $(V_{\mu\gamma})_{\mu,\gamma}$ is a covering of V by elements of $\mathcal{B}$. For each pair (μ, γ) of indices, set

$$t_{\mu\gamma}^\alpha = s_\alpha|_{V_{\mu\gamma}}, \quad t_{\mu\gamma}^\beta = s_\beta|_{V_{\mu\gamma}}.$$

By hypothesis $s_\alpha|_{U_{\alpha\beta\gamma}} = s_\beta|_{U_{\alpha\beta\gamma}}$ and $V_{\mu\gamma} \subseteq U_{\alpha\beta\gamma}$, so we have $t_{\mu\gamma}^\alpha = t_{\mu\gamma}^\beta$. Moreover, for each $W \in \mathcal{B}$ and $W \subseteq V_{\mu\gamma} \cap V_{\tilde{\mu}\tilde{\gamma}}$, since $V_{\mu\gamma}$ and $V_{\tilde{\mu}\tilde{\gamma}}$ are both contained in $U_{\alpha\beta}$,

$$t_{\mu\gamma}^\alpha|_W = s_\alpha|_W = t_{\tilde{\mu}\tilde{\gamma}}^\alpha|_W, \quad t_{\mu\gamma}^\beta|_W = s_\beta|_W = t_{\tilde{\mu}\tilde{\gamma}}^\beta|_W.$$

Applying (ii) on the open set V and the covering $(V_{\mu\gamma})$, we then conclude that $s_\alpha|_V = s_\beta|_V$, which again by condition (ii) implies that (s_α) corresponds to a section on $\mathscr{F}(U)$. This proves (ii)⇒(iii).

Now we prove the implication (iii)⇒(i). Before this, we first note that, if (iii) holds and $\mathcal{B}'$ is a basis of X contained in $\mathcal{B}$, then the presheaf $\mathscr{F}_{\mathcal{B}'}$ associated to the presheaf $(\mathscr{F}(U))_{U\in\mathcal{B}'}$ is canonically isomorphic to the presheaf $\mathscr{F}_{\mathcal{B}}$ associated to $(\mathscr{F}(U))_{U\in\mathcal{B}}$. Indeed, first of all the inverse limit (for $V \in \mathcal{B}' \subseteq \mathcal{B}$, $V \subseteq U$) of the canonical morphisms $\mathscr{F}_{\mathcal{B}}(U) \to \mathscr{F}(V)$ gives a morphism

$$\mathscr{F}_{\mathcal{B}}(U) \to \mathscr{F}_{\mathcal{B}'}(U)$$

for any open set U. We claim that this is an isomorphism if $U \in \mathcal{B}$. To see this, let (U_α) be a covering of U by elements of $\mathcal{B}'$ and for each (α, β), choose a covering $(U_{\alpha\beta\gamma})$ of $U_{\alpha\beta} = U_\alpha \cap U_\beta$ by elements of $\mathcal{B}'$. For each γ, we have a commutative diagram

$$\begin{tikzcd}
 & & \mathscr{F}(U_\alpha) \arrow[dr] & \\
\mathscr{F}_{\mathcal{B}'}(U) \arrow[r, dashed] \arrow[urr, bend left] \arrow[drr, bend right] & \mathscr{F}(U) \arrow[ur] \arrow[dr] & & \mathscr{F}(U_{\alpha\beta\gamma}) \\
 & & \mathscr{F}(U_\beta) \arrow[ur] &
\end{tikzcd}$$

so condition (iii) shows that the canonical morphism $\mathscr{F}_{\mathcal{B}'}(U) \to \mathscr{F}(U_\alpha)$ factors through $\mathscr{F}(U)$. It is immediate that the morphisms $\mathscr{F}_{\mathcal{B}}(U) \to \mathscr{F}_{\mathcal{B}'}(U)$ and $\mathscr{F}_{\mathcal{B}'}(U) \to \mathscr{F}_{\mathcal{B}}(U)$ thus defined are inverses of each other. This being so, for all open set U of X, the morphisms

$$\mathscr{F}_{\mathcal{B}'}(U) \to \mathscr{F}_{\mathcal{B}'}(W) = \mathscr{F}_{\mathcal{B}}(W) = \mathscr{F}(W)$$

for $W \in \mathcal{B}$, $W \subseteq U$ satisfy the conditions characterizing the inverse limit of the $\mathscr{F}(W)$'s, so our claim follows from the uniqueness of the inverse limit.

Now let U be any open set of X, (U_α) a covering of U by open sets contained in U, and let $\mathcal{B}'$ be the subfamily of $\mathcal{B}$ consisting of the sets of $\mathcal{B}$ contained in at least one U_α. It is clear that $\mathcal{B}'$ is still a basis of the topology of X, so $\mathscr{F}_{\mathcal{B}}(U)$ (resp. $\mathscr{F}_{\mathcal{B}}(U_\alpha)$) is the inverse limit of the $\mathscr{F}(V)$ for $V \in \mathcal{B}'$ and $V \subseteq U$ (resp. $V \subseteq U_\alpha$); the sheaf axiom is then verified immediately by virtue of the definition of the inverse limit. $\square$

We say $\mathscr{F}$ is a **sheaf on** $\mathcal{B}$ if it satisfies the equivalent conditions in Proposition 1.1.23.

Corollary 1.1.24. *Let X be a topological space. Let $\mathcal{B}$ be a basis for the topology on X. Assume that for every pair $U, U' \in \mathcal{B}$ we have $U \cap U' \in \mathcal{B}$. Let $\mathscr{F}$ be a presheaf of sets on $\mathcal{B}$. The following are equivalent:*

(i) The presheaf $\mathscr{F}$ is a sheaf on $\mathcal{B}$.

(ii) For every $U \in \mathcal{B}$ and for every family of sections $s_i \in \mathscr{F}(U_i)$ such that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ there exists a unique section $s \in \mathscr{F}(U)$ which restricts to s_i on U_i.

Proof. This is a reformulation of Proposition 1.1.23, as we can take V to be $U_\alpha \cap U_\beta$. $\square$

Note that for any $x \in X$ we have $\mathscr{F}_x = (\mathscr{F}_{\mathcal{B}})_x$ in the situation of the proposition. This is so because the collection of elements of $\mathcal{B}$ containing x forms a fundamental system of open neighbourhoods of x.

Let $\mathscr{F}$, $\mathscr{G}$ be two presheaves over $\mathcal{B}$. A morphism $\varphi : \mathscr{F} \to \mathscr{G}$ is defined to be a family $(\varphi_V)_{V \in \mathcal{B}}$ of morphisms $\varphi_V : \mathscr{F}(V) \to \mathscr{F}(V)$ satisfying the compatible conditions with restriction maps. By passing to inverse limits, we get a morphism $\varphi_{\mathcal{B}} : \mathscr{F}_{\mathcal{B}} \to \mathscr{G}_{\mathcal{B}}$ of presheaves (it is easy to verify the compatible conditions with restriction maps).

Theorem 1.1.25. *Let X be a topological space. Let $\mathcal{B}$ be a basis for the topology on X. Denote* $\mathrm{Sh}(\mathcal{B})$ *the category of sheaves on $\mathcal{B}$. There is an equivalence of categories*

$$\mathrm{Sh}(X) \to \mathrm{Sh}(\mathcal{B})$$

which assigns to a sheaf on X its restriction to the members of $\mathcal{B}$.

Proof. If $\mathscr{F}$ is a sheaf on $\mathcal{B}$, then the sheaf $\mathscr{F}_{\mathcal{B}}$ satisfies $\mathscr{F}(U) = \mathscr{F}_{\mathcal{B}}(U)$ for $U \in \mathcal{B}$, thus the restriction of $\mathscr{F}_{\mathcal{B}}$ equals $\mathscr{F}$. Conversely, if $\mathscr{F}$ is a sheaf on X, then the restriction $\mathscr{F}|_{\mathcal{B}}$ induces a sheaf $\mathscr{F}' := (\mathscr{F}|_{\mathcal{B}})_{\mathcal{B}}$ on X. Then $\mathscr{F}$ and $\mathscr{F}'$ has the same stalk, so we conclude $\mathscr{F} \cong \mathscr{F}'$. Moreover, by looking at stalks, we see the Hom sets are canonically identified, whence the claim. $\square$

1.1.8 The category of presheaves and sheaves

In this subsection, we derive some result for the categories $\mathbf{Psh}(X)$ and $\mathrm{Sh}(X)$.

1.1.8.1 The category of presheaves We first consider the category of presheaves, with morphisms are defined to be morphisms of presheaves. As we will see, this category behaves much like the base category **Set**.

Example 1.1.10 (Final object of presheaves). Let X be a topological space. Consider a rule $\mathscr{F}$ that associates to every open subset a singleton set. Since every set has a unique map into a singleton set, there exist unique restriction maps res^U_V. The resulting structure is a presheaf of sets. It is a final object in the category of presheaves of sets, by the property of singleton sets mentioned above. Hence it is also unique up to unique isomorphism. We will sometimes write $*$ for this presheaf.

Proposition 1.1.26. *Let $\mathcal{I}$ be a small category and let $\mathscr{F} : \mathcal{I} \to \mathbf{Psh}(X)$ be an $\mathcal{I}$-diagram of presheaves on X. Then the limit and colimit of $\mathscr{F}$ both exist, whichi are given by*

$$(\varprojlim_i \mathscr{F}_i)(U) := \varprojlim_i \mathscr{F}_i(U), \quad (\varinjlim_i \mathscr{F}_i)(U) := \varinjlim_i \mathscr{F}_i(U).$$

and the restriction maps are induced by that of the $\mathscr{F}_i$'s.

Proof. The given constructions are clearly meaningful and define presheaves, where the restriction map is given by the limit (or colimit) of that of $\mathscr{F}_i$. For the colimit, if we are given morphisms $\varphi_i : \mathscr{F}_i \to G$ such that the following diagram commutes

$$\begin{tikzcd}
\mathscr{F}_i \arrow[rr] \arrow[dr] & & \mathscr{F}_j \arrow[dl] \\
& \mathscr{G} &
\end{tikzcd}$$

Then for each $U \subseteq X$ open we can take limit of the system $(\varphi_{i,U})$ to get a morphism $\varphi : \varinjlim_i \mathscr{F}_i(U) \to \mathscr{G}(U)$. Moreover, these morphisms are compatible with the restriction maps of $\mathscr{F}_i$, hence compatible with that of $\varinjlim_i \mathscr{F}_i$. This gives a morphism $\varinjlim_i \mathscr{F}_i \to \mathscr{G}$, so $\varinjlim_i \mathscr{F}_i$ satisfies the universal property of colimits. Similarly, we can show that $\varprojlim_i \mathscr{F}_i$ satisfies the universal property of limits. □

Definition 1.1.9. Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism of presheaves of sets.

(a) We say that φ is **injective** if for every open subset $U \subseteq X$ the map $\varphi_U : \mathscr{F}(U) \to \mathscr{G}(U)$ is injective.

(b) We say that φ is **surjective** if for every open subset $U \subseteq X$ the map $\varphi_U : \mathscr{F}(U) \to \mathscr{G}(U)$ is injective.

We show that the injectivity and surjectivity gives monomorphisms and epimorphisms in the category of presheaves.

Proposition 1.1.27. *The injective (resp. surjective) maps defined above are exactly the monomorphisms (resp. epimorphisms) of $\mathbf{Psh}(X)$. A map is an isomorphism if and only if it is both injective and surjective.*

Proof. We shall show that $\varphi : \mathscr{F} \to \mathscr{G}$ is injective if and only if it is an monomorphism of $\mathbf{Psh}(X)$. Indeed, the only if direction is straightforward, so let us show the if direction. If φ is a monomorphism, let $U \subseteq X$ be an open subset; we are going to show that φ_U is a monomorphism in the category **Set**. For this, consider any two maps $f, g : A \to \mathscr{F}(U)$ such that $\varphi_U \circ f = \varphi_U \circ g$. We define a presheaf $\mathscr{A}$ by

$$\mathscr{A}(V) = \begin{cases} A & V \subseteq U, \\ \varnothing & \text{otherwise.} \end{cases}$$

Then we have induced morphism of presheaves ψ_f and ψ_g given by the diagram

$$\begin{tikzcd} \mathscr{A}(U) \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[d, equal] & \mathscr{F}(U) \arrow[d, "\mathrm{res}^U_V"] \\ \mathscr{A}(V) \arrow[r, dashed, "\psi_V"] & \mathscr{F}(V) \end{tikzcd}$$

Then we can see $\varphi \circ \psi_f = \varphi \circ \psi_g$, which implies $\psi_f = \psi_g$ since φ is monic. This gives $f = g$ by our construction, so φ_U is monic in the category of sets, hence injective.

Now we show that $\varphi : \mathscr{F} \to \mathscr{G}$ is surjective if and only if it is an epimorphism of $\mathbf{Psh}(X)$. Similarly, the only if direction is straightforward, so let us show the if direction. Assume that φ is an epimorphism, and we show φ_U is epic in **Set**. For any maps $f, g : \mathscr{G}(U) \to B$ such that $f \circ \varphi_U = g \circ \varphi_U$, we define a presheaf $\mathscr{B}$ by

$$\mathscr{B}(V) = \begin{cases} B & V \supseteq U, \\ \varnothing & \text{othewise.} \end{cases}$$

Similarly, we can define morphism of presheaves by the diagram

$$\begin{tikzcd} \mathscr{G}(V) \arrow[r, dashed, "\psi_V"] \arrow[d, "\mathrm{res}^U_V"] & \mathscr{B}(V) \arrow[d, equal] \\ \mathscr{G}(U) \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & \mathscr{B}(U) \end{tikzcd}$$

and we have $\psi_f \circ \varphi_U = \psi_g \circ \varphi_U$, which implies $f = g$, so φ_U is surjective. □

1.1.8.2 The category of sheaves We have already seen that limits and colimits exist in the category of presheaves. Now we consider them in the category of sheaves.

Proposition 1.1.28. *Let $\mathcal{I}$ be a small category and let $\mathscr{F} : \mathcal{I} \to \mathbf{Psh}(X)$ be an $\mathcal{I}$-diagram of sheaves on X. Then the limit and colimit of $\mathscr{F}$ both exist, whichi are given by*

$$\varprojlim_{i, \mathrm{Sh}(X)} \mathscr{F}_i = \varprojlim_{i, \mathbf{Psh}(X)} \mathscr{F}_i, \quad \varinjlim_{i, \mathrm{Sh}(X)} \mathscr{F}_i := (\varinjlim_{i, \mathbf{Psh}(X)} \mathscr{F}_i)^{\#}.$$

Proof. Since the limit and colimit exist in $\mathbf{Psh}(X)$, we can define

$$\varprojlim_{i, \mathrm{Sh}(X)} \mathscr{F}_i := (\varprojlim_{i, \mathbf{Psh}(X)} \mathscr{F}_i)^{\#}, \quad \varinjlim_{i, \mathrm{Sh}(X)} \mathscr{F}_i := (\varinjlim_{i, \mathbf{Psh}(X)} \mathscr{F}_i)^{\#}.$$

Using the universal property of sheafification, we can show that the construction above indeed gives the colimits in the category of sheaves. Now we note that, since the forgetful functor $\iota : \mathrm{Sh}(X) \to \mathbf{Psh}(X)$ has a left adjoint given by the sheafification, it commutes with limits. In other words, if $\mathcal{I}$ is a small category and $\mathscr{F} : \mathcal{I} \to \mathbf{Psh}(X)$ be an $\mathcal{I}$-diagram of sheaves on X. Then the limit $\varprojlim_i \mathscr{F}_i$ satisfies

$$\iota(\varprojlim_{i,\mathrm{Sh}(X)} \mathscr{F}_i) = \varprojlim_{i,\mathbf{Psh}(X)} \iota(\mathscr{F}_i).$$

But the forgetful functor does nothing actually, so the limit $\varprojlim_i \mathscr{F}_i$ is in fact given by the limit in $\mathbf{Psh}(X)$, i.e.,

$$\varprojlim_{i,\mathrm{Sh}(X)} \mathscr{F}_i = \varprojlim_{i,\mathbf{Psh}(X)} \mathscr{F}_i.$$

However, since ι may not commutes with colimits, the colimit of $\mathscr{F}$ should be defined as the sheafification of that in $\mathbf{Psh}(X)$:

$$\varinjlim_{i,\mathrm{Sh}(X)} \mathscr{F}_i := (\varinjlim_{i,\mathbf{Psh}(X)} \mathscr{F}_i)^{\#}.$$

This completes the proof. $\square$

Proposition 1.1.29. *Let $x \in X$ be a point. Then there are maps*

$$(\varprojlim_i \mathscr{F}_i)_x \to \varprojlim_i (\mathscr{F}_i)_x, \quad (\varinjlim_i \mathscr{F}_i)_x \cong \varinjlim_i (\mathscr{F}_i)_x$$

for limits and colimits of (pre)sheaves. The second map is always bijective, and the first map becomes an bijection when $\mathcal{I}$ is finite,

Proof. First we consider (co)limit of presheaves. The maps $\mathscr{F}_i(U) \to (\mathscr{F}_i)_x$ yield maps

$$\varprojlim_i \mathscr{F}_i(U) \to \varprojlim_i (\mathscr{F}_i)_x \quad \text{and} \quad \varinjlim_i \mathscr{F}_i(U) \to \varinjlim_i (\mathscr{F}_i)_x.$$

Taking the filtered colimit over the open neighborhoods of x we obtain maps

$$(\varprojlim_i \mathscr{F}_i)_x \to \varprojlim_i (\mathscr{F}_i)_x, \quad (\varinjlim_i \mathscr{F}_i)_x \to \varinjlim_i (\mathscr{F}_i)_x$$

As filtered colimits commute with finite limits, the first map is an isomorphism if $\mathcal{I}$ is finite. And since colimits commute with each other, the second is always bijective.

In the case of sheaves, we need to take sheafification. Since the sheafification does not change stalk, the result is the same. $\square$

Proposition 1.1.30. *Suppose $\varphi : \mathscr{F} \to \mathscr{G}$ is a morphism of sheaves of sets on a topological space X. Then the following are equivalent.*

(a) φ is a monomorphism in the category of sheaves.

(b) φ is injective on the level of stalks: $\varphi_x : \mathscr{F}_x \to \mathscr{G}_x$ is injective for all $x \in X$.

(c) φ is injective on the level of open sets: $\varphi(U) : \mathscr{F}(U) \to \mathscr{G}(U)$ is injective for all open $U \subseteq X$.

*If these conditions hold, we say that $\mathscr{F}$ is a **subsheaf** of $\mathscr{G}$.*

Proof. The last two statements are equivalent by Proposition 1.1.2, and the implication (c)⇒(a) is immediate, since a monomorphism in $\mathbf{Psh}(X)$ is clearly monic in $\mathrm{Sh}(X)$.

Now assume that φ is a monomorphism, and let $s, t \in \mathscr{F}(U)$ be such that $\varphi_U(s) = \varphi_U(t)$. Then by Example 1.1.9 there are morphisms $\psi_s, \psi_t : \mathscr{U}^{\#} \to \mathscr{F}$. From $\varphi_U(s) = \varphi_U(t)$ we see $\varphi \circ \psi_s = \varphi \circ \psi_t$, which implies $\psi_s = \psi_t$. By the isomorphism in Example 1.1.9, we conclude $s = t$, so φ_U is injective. $\square$

Proposition 1.1.31. *Suppose $\varphi : \mathscr{F} \to \mathscr{G}$ is a morphism of sheaves of sets on a topological space X. Then the following are equivalent.*

(a) φ is a epimorphism in the category of sheaves.

(b) φ is surjective on the level of stalks: $\varphi_x : \mathscr{F}_x \to \mathscr{G}_x$ is injective for all $x \in X$.

(c) For any open subsets $U \subseteq X$ and every $t \in \mathscr{G}(U)$ there exist an open covering $U = \bigcup_i U_i$ and sections $s_i \in \mathscr{F}(U_i)$ such that $\varphi_{U_i}(s_i) = t|_{U_i}$, i.e., we can locally find preimage of t.

*If these conditions hold, we say that $\mathscr{G}$ is a **quotient sheaf** of $\mathscr{F}$.*

Proof. If $\varphi_x : \mathscr{F}_x \to \mathscr{G}_x$ is surjective for all $x \in X$, then by Proposition 1.1.2 and the diagram

$$\begin{tikzcd}
\mathscr{F}(U) \arrow[r, "\varphi_U"] \arrow[d, hook] & \mathscr{G} \arrow[r, "\psi_U"] \arrow[d, hook] & \mathscr{H}(U) \arrow[d, hook] \\
\prod_{x\in U} \mathscr{F}_x \arrow[r, "\varphi_x"] & \prod_{x\in U} \mathscr{G}_x \arrow[r, "\psi_x"] & \prod_{x\in U} \mathscr{H}_x
\end{tikzcd}$$

we see φ is an epimorphism.

Now assume that φ is an epimorphism, and let $f, g : \mathscr{G}_x \to B$ be two maps such that $f \circ \varphi_x = g \circ \varphi_x$. Then by Proposition 1.1.13 we have induced morphism of sheaves $\psi_f, \psi_g : \mathscr{G} \to i_{x,*}(B)$ such that $\psi_f \circ \varphi = \psi_g \circ \varphi$. Since φ is an epic, this implies $f = g$, which means φ_x is surjective.

Finally, the condition (b) clearly implies (c), and if (c) holds, let $t_x \in \mathscr{G}_x$ with $t_x = (t, V)$ for some $V \subseteq X$ open and $t \in \mathscr{G}_x$. Then there is a covering $V = \bigcup_i V_i$ and $s_i \in \mathscr{F}(U_i)$ such that $\varphi_{V_i}(s_i) = t|_{V_i}$. Choose a $x \in V_i$, then $\varphi_x\big((s_i)_x\big) = t_x$. Thus φ_x is surjective. □

Remark 1.1.5. The condition for φ in Proposition 1.1.31 does not imply that φ_U is surjective for all open sets U of X as Example 1.1.11 shows. In other words, being epic in the category of sheaves is a weaker that begin surjective on objects.

Example 1.1.11. Let $\mathscr{O}_X$ be the sheaf of holomorphic functions on an open subset X of $\mathbb{C}$. For every open subspace $U \subseteq X$ and $f \in \mathscr{O}_X(U)$ we let $D_U(f) = f'$ be the derivative. We obtain a morphism $D : \mathscr{O}_X \to \mathscr{O}_X$ of sheaves of $\mathbb{C}$-vector spaces. Then D is an epimorphism, because locally every holomorphic function has a primitive. But there exist open subsets U of X and functions f on U that have no primitive., for instance $U = \mathbb{B}(z_0) \setminus \{z_0\} \subseteq \mathbb{C}$ contained in X and $f = 1/(z - z_0)$. More precisely, by complex analysis we know that D_U is surjective if and only if every connected component of U is simply connected. Thus D is not surjective.

1.2 Sheaf of modules

1.2.1 Presheaf of moudles

Definition 1.2.1. Let X be a topological space, and let $\mathscr{O}$ be a presheaf of rings on X.

- A presheaf of $\mathscr{O}$-modules is given by an abelian presheaf $\mathscr{F}$ together with a map of presheaves of sets
$$\mathscr{O} \times \mathscr{F} \to \mathscr{F}$$
such that for every open $U \subseteq X$ the map $\mathscr{O}(U) \times \mathscr{F}(U) \to \mathscr{F}(U)$ defines the structure of an $\mathscr{O}(U)$-module structure on the abelian group $\mathscr{F}(U)$.

- A morphism $\varphi : \mathscr{F} \to \mathscr{G}$ of presheaves of $\mathscr{O}$-modules is a morphism of abelian presheaves $\varphi : \mathscr{F} \to \mathscr{G}$ such that the diagram

$$\begin{tikzcd} \mathscr{O} \times \mathscr{F} \arrow[r] \arrow[d, "\mathbf{1}\times\varphi"'] & \mathscr{F} \arrow[d, "\varphi"] \\ \mathscr{O} \times \mathscr{G} \arrow[r] & \mathscr{G} \end{tikzcd}$$

  commutes.

- The set of $\mathscr{O}$-module morphisms as above is denoted $\mathrm{Hom}_{\mathscr{O}}(\mathscr{F}, \mathscr{G})$. The category of presheaves of $\mathscr{O}$-modules is denoted $\mathbf{PMod}(\mathscr{O})$.

Suppose that $\mathscr{O}_1 \to \mathscr{O}_2$ is a morphism of presheaves of rings on X. In this case, if $\mathscr{F}$ is a presheaf of $\mathscr{O}_2$-modules then we can think of $\mathscr{F}$ as a presheaf of $\mathscr{O}_2$-modules by using the composition

$$\mathscr{O}_1 \times \mathscr{F} \to \mathscr{O}_2 \times \mathscr{F} \to \mathscr{F}.$$

We sometimes denote this by $\mathscr{F}_{\mathscr{O}_1}$ to indicate the restriction of rings. We call this the **restriction of $\mathscr{F}$**. We obtain the restriction functor

$$\mathbf{PMod}(\mathscr{O}_2) \to \mathbf{PMod}(\mathscr{O}_1).$$

On the other hand, given a presheaf of $\mathscr{O}_1$-modules $\mathscr{G}$ we can construct a presheaf of $\mathscr{O}_2$-modules $\mathscr{O}_2 \otimes_{p,\mathscr{O}_1} \mathscr{F}$ by the rule

$$(\mathscr{O}_2 \otimes_{p,\mathscr{O}_1} \mathscr{F})(U) = \mathscr{O}_2(U) \otimes_{p,\mathscr{O}_1} \mathscr{F}(U).$$

The index p stands for presheaf. This presheaf is called the **tensor product presheaf**. We obtain the change of rings functor

$$\mathbf{PMod}(\mathscr{O}_1) \to \mathbf{PMod}(\mathscr{O}_2).$$

Proposition 1.2.1. *With $\mathscr{O}_1, \mathscr{O}_2$ and $\mathscr{F}, \mathscr{G}$ as above there exists a canonical bijection*

$$\mathrm{Hom}_{\mathscr{O}_1}(\mathscr{G}, \mathscr{F}_{\mathscr{O}_1}) = \mathrm{Hom}_{\mathscr{O}_2}(\mathscr{O}_2 \otimes_{p,\mathscr{O}_1} \mathscr{G}, \mathscr{F}).$$

In other words, the restriction and change of rings functors are adjoint to each other.

Proof. This follows from the fact that for a ring map $A \to B$ the restriction functor and the change of ring functor are adjoint to each other. □

The stalk of a sheaf of $\mathscr{O}$-module is defined in the same as that of sheaf of sets.

Proposition 1.2.2. *Let X be a topological space. Let $\mathscr{O}$ be a presheaf of rings on X. Let $\mathscr{F}$ be a presheaf of $\mathscr{O}$-modules. Let $x \in X$. The canonical map $\mathscr{O}_x \times \mathscr{F}_x \to \mathscr{F}_x$ coming from the multiplication map $\mathscr{O} \times \mathscr{F} \to \mathscr{F}$ defines a $\mathscr{O}_x$-module structure on the abelian group $\mathscr{F}_x$.*

Proposition 1.2.3. *Let X be a topological space. Let $\mathscr{O} \to \mathscr{O}'$ be a morphism of presheaves of rings on X. Let $\mathscr{F}$ be a presheaf of $\mathscr{O}$-modules. Let $x \in X$. We have*

$$\mathscr{F}_x \otimes_{\mathscr{O}_x} \mathscr{O}'_x = (\mathscr{F} \otimes_{p,\mathscr{O}} \mathscr{O}')_x.$$

as $\mathscr{O}'_x$-modules.

Proof. Tensor product is left-adjoint, so it commutes with colimit. □

1.2.2 Sheaf of moudles

Definition 1.2.2. Let X be a topological space. Let $\mathcal{O}$ be a sheaf of rings on X.

- A sheaf of $\mathcal{O}$-modules is a presheaf of $\mathcal{O}$-modules $\mathcal{F}$, such that the underlying presheaf of abelian groups $\mathcal{F}$ is a sheaf.
- A morphism of sheaves of $\mathcal{O}$-modules is a morphism of presheaves of $\mathcal{O}$-modules.
- Given sheaves of $\mathcal{O}$-modules $\mathcal{F}$ and $\mathcal{G}$ we denote $\mathrm{Hom}_{\mathcal{O}}(\mathcal{F},\mathcal{G})$ the set of morphism of sheaves of $\mathcal{O}$-modules. The category of sheaves of $\mathcal{O}$-modules is denoted $\mathbf{Mod}(\mathcal{O})$.

1.2.3 Sheafification of presheaves of modules

Proposition 1.2.4. *Let X be a topological space. Let $\mathcal{O}$ be a presheaf of rings on X. Let $\mathcal{F}$ be a presheaf $\mathcal{O}$-modules. Let $\mathcal{O}^{\#}$ be the sheafification of $\mathcal{O}$. Let $\mathcal{F}^{\#}$ be the sheafification of $\mathcal{F}$ as a presheaf of abelian groups. There exists a map of sheaves of sets*

$$\mathcal{O}^{\#} \times \mathcal{F}^{\#} \to \mathcal{F}^{\#}.$$

which makes the diagram

$$\begin{tikzcd} \mathcal{O} \times \mathcal{F} \arrow[r] \arrow[d] & \mathcal{F} \arrow[d] \\ \mathcal{O}^{\#} \times \mathcal{F}^{\#} \arrow[r] & \mathcal{F}^{\#} \end{tikzcd}$$

commute and which makes $\mathcal{F}^{\#}$ into a sheaf of $\mathcal{O}^{\#}$-modules. In addition, if $\mathcal{G}$ is a presheaf of $\mathcal{O}$-modules, then any morphism of presheaves of $\mathcal{O}$-modules $\mathcal{F} \to \mathcal{G}$ induced a unique morphism $\mathcal{F}^{\#} \to \mathcal{G}^{\#}$ of sheaf of $\mathcal{O}^{\#}$-modules.

Proof. Since finite product and coproduct coincide in the category of modules, the sheafification commutes with both of them. Thus we can apply the universal property of sheafification on the map $\mathcal{O} \times \mathcal{F} \to \mathcal{F}$ to get a map

$$\mathcal{O}^{\#} \times \mathcal{F}^{\#} \to \mathcal{F}^{\#}$$

Moreover, for a morphism of $\mathcal{O}$-modules $\mathcal{F} \to \mathcal{G}$, apply sheafification on the diagram

$$\begin{tikzcd} \mathcal{O} \times \mathcal{F} \arrow[r] \arrow[d] & \mathcal{F} \arrow[d] \\ \mathcal{O} \times \mathcal{G} \arrow[r] & \mathcal{G} \end{tikzcd}$$

we get the desired morphism. ☐

This actually means that the functor $\iota : \mathbf{Mod}(\mathcal{O}^{\#}) \to \mathbf{PMod}(\mathcal{O})$ and the sheafification functor of the lemma $\# : \mathbf{PMod}(\mathcal{O}) \to \mathbf{Mod}(\mathcal{O}^{\#})$ are adjoint. In a formula

$$\mathrm{Hom}_{\mathbf{Mod}(\mathcal{O}^{\#})}(\mathcal{F}^{\#},\mathcal{G}) = \mathrm{Hom}_{\mathbf{PMod}(\mathcal{O})}(\mathcal{F},\iota\mathcal{G}).$$

Let X be a topological space. Let $\mathcal{O}_1 \to \mathcal{O}_2$ be a morphism of sheaves of rings on X. We defined a restriction functor and a change of rings functor on presheaves of modules associated to this situation. If $\mathcal{F}$ is a sheaf of $\mathcal{O}_2$-modules then the restriction $\mathcal{F}_{\mathcal{O}_2}$ of $\mathcal{F}$ is clearly a sheaf of $\mathcal{O}_1$-modules. We obtain the restriction functor

$$\mathbf{Mod}(\mathcal{O}_2) \to \mathbf{Mod}(\mathcal{O}_1).$$

On the other hand, given a sheaf of $\mathcal{O}_1$-modules $\mathcal{G}$ the presheaf of $\mathcal{O}_2$-modules $\mathcal{O}_2 \otimes_{p,\mathcal{O}_1} \mathcal{G}$ is in general not a sheaf. Hence we define the tensor product sheaf $\mathcal{O}_2 \otimes_{\mathcal{O}_1} \mathcal{G}$ by the formula

$$\mathcal{O}_2 \otimes_{\mathcal{O}_1} \mathcal{G} = (\mathcal{O}_2 \otimes_{p,\mathcal{O}_1} \mathcal{G})^{\#}.$$

Proposition 1.2.5. *With X, $\mathcal{O}_1, \mathcal{O}_2$, $\mathcal{F}$ and $\mathcal{G}$ as above there exists a canonical bijection*

$$\mathrm{Hom}_{\mathcal{O}_1}(\mathcal{G}, \mathcal{F}_{\mathcal{O}_1}) = \mathrm{Hom}_{\mathcal{O}_2}(\mathcal{O}_2 \otimes_{\mathcal{O}_1} \mathcal{G}, \mathcal{F}).$$

In other words, the restriction and change of rings functors are adjoint to each other.

Proof. This follows from Proposition 1.2.1 and the fact that

$$\mathrm{Hom}_{\mathcal{O}_2}(\mathcal{O}_2 \otimes_{\mathcal{O}_1} \mathcal{G}, \mathcal{F}) = \mathrm{Hom}_{\mathcal{O}_2}(\mathcal{O}_2 \otimes_{p,\mathcal{O}_1} \mathcal{G}, \mathcal{F}, \mathcal{F})$$

by the property of sheafification. $\square$

Proposition 1.2.6. *Let X be a topological space. Let $\mathcal{O} \to \mathcal{O}'$ be a morphism of sheaves of rings on X. Let $\mathcal{F}$ be a sheaf $\mathcal{O}$-modules. Let $x \in X$. We have*

$$\mathcal{F}_x \otimes_{\mathcal{O}_x} \mathcal{O}'_x = (\mathcal{F} \otimes_{\mathcal{O}} \mathcal{O}')_x.$$

1.2.4 Continuous maps

The case of sheaves of modules is more complicated. First we state a few obvious lemmas.

Lemma 1.2.7. *Let $f : X \to Y$ be a continuous map of topological spaces. Let $\mathcal{O}$ be a presheaf of rings on X. Let $\mathcal{F}$ be a presheaf of $\mathcal{O}$-modules. There is a natural map of underlying presheaves of sets*

$$f_*\mathcal{O} \times f_*\mathcal{F} \to f_*\mathcal{F}$$

which turns $f_*\mathcal{F}$ into a presheaf of $f_*\mathcal{O}$-modules. This construction is functorial in $\mathcal{F}$.

Proof. Let $V \subseteq Y$ is open. We define the map of the lemma to be the map

$$f_*\mathcal{O}(V) \times f_*\mathcal{F}(V) = \mathcal{O}(f^{-1}(V)) \times \mathcal{F}(f^{-1}(V)) \to \mathcal{F}(f^{-1}(V)) = f_*\mathcal{F}(V).$$

Here the arrow in the middle is the multiplication map on X. $\square$

Lemma 1.2.8. *Let $f : X \to Y$ be a continuous map of topological spaces. Let $\mathcal{O}$ be a presheaf of rings on Y. Let G be a presheaf of $\mathcal{O}$-modules. There is a natural map of underlying presheaves of sets*

$$f^p\mathcal{O} \times f^p\mathcal{G} \to \mathcal{G}$$

which turns $f^p\mathcal{G}$ into a presheaf of $f^p\mathcal{O}$-modules. This construction is functorial in $\mathcal{G}$.

Proof. Let $U \subseteq X$ is open. We define the map of the lemma to be the map

$$\begin{aligned} f^p\mathcal{O}(U) \times f^p\mathcal{G}(U) &= \varinjlim_{f(U)\subseteq V} \mathcal{O}(V) \times \varinjlim_{f(U)\subseteq V} \mathcal{G}(V) = \varinjlim_{f(U)\subseteq V} \big(\mathcal{O}(V) \times \mathcal{G}(V)\big) \\ &\to \varinjlim_{f(U)\subseteq V} \mathcal{G}(V) = f^p\mathcal{G}(V) \end{aligned}$$

Here the arrow in the middle is the multiplication map on Y. The second equality holds because directed colimits commute with finite limits. $\square$

Let $f : X \to Y$ be a continuous map. Let $\mathcal{O}_X$ be a presheaf of rings on X and let $\mathcal{O}_Y$ be a presheaf of rings on Y. So at the moment we have defined functors

$$f_* : \mathbf{PMod}(\mathcal{O}_X) \to \mathbf{PMod}(f_*\mathcal{O}_X), \quad f^p : \mathbf{PMod}(\mathcal{O}_Y) \to \mathbf{PMod}(f^p\mathcal{O}_Y).$$

These satisfy some compatibilities as follows.

Proposition 1.2.9. *Let $f : X \to Y$ be a continuous map of topological spaces. Let $\mathcal{O}$ be a presheaf of rings on Y. Let $\mathcal{G}$ be a presheaf of $\mathcal{O}$-modules. Let $\mathcal{F}$ be a presheaf of $f^p\mathcal{O}$-modules. Then*

$$\mathrm{Hom}_{\mathbf{PMod}(f^p\mathcal{O})}(f^p\mathcal{G}, \mathcal{F}) = \mathrm{Hom}_{\mathbf{PMod}(\mathcal{O})}(\mathcal{G}, f_*\mathcal{F})$$

Here we think of $f_*\mathcal{F}$ as an $\mathcal{O}$-module via the map $\rho_{\mathcal{O}} : \mathcal{O} \to f_*f^p\mathcal{O}$.

Proof. Note that we have

$$\mathrm{Mor}_{\mathbf{PAb}(X)}(f^p\mathcal{G}, \mathcal{F}) = \mathrm{Mor}_{\mathbf{PAb}(Y)}(\mathcal{G}, f_*\mathcal{F}).$$

So what we have to prove is that under this correspondence, the subsets of module maps correspond. In addition, the correspondence is determined by the rule

$$(\psi : f^p\mathcal{G} \to \mathcal{F}) \to (f_*\psi \circ \rho_{\mathcal{G}} : \mathcal{G} \to f_*\mathcal{F})$$

and in the other direction by the rule

$$(\varphi : \mathcal{G} \to f_*\mathcal{F}) \mapsto (\sigma_{\mathcal{F}} \circ f^p\varphi : f^p\mathcal{F} \to \mathcal{G})$$

Hence, using the functoriality of f_* and f^p we see that it suffices to check that the maps $\rho_{\mathcal{G}} : \mathcal{G} \to f_*f^p\mathcal{G}$ and $\sigma_{\mathcal{F}} : f^pf_*\mathcal{F} \to \mathcal{F}$ are compatible with module structures, which can be done by tracing definitions. $\square$

Proposition 1.2.10. *Let $f : X \to Y$ be a continuous map of topological spaces. Let $\mathcal{O}$ be a presheaf of rings on X. Let $\mathcal{F}$ be a presheaf of $\mathcal{O}$-modules. Let $\mathcal{G}$ be a presheaf of $f_*\mathcal{O}$-modules. Then*

$$\mathrm{Hom}_{\mathbf{PMod}(\mathcal{O})}(\mathcal{O} \otimes_{p,f^pf_*\mathcal{O}} f^p\mathcal{G}, \mathcal{F}) = \mathrm{Hom}_{\mathbf{PMod}(f_*\mathcal{O})}(\mathcal{G}, f_*\mathcal{F}).$$

Here we use the map $\sigma_{\mathcal{O}} : f^pf_*\mathcal{O} \to \mathcal{O}$ in the definition of the tensor product.

Proof. This follows from the equalities

$$\begin{aligned}\mathrm{Hom}_{\mathbf{PMod}(\mathcal{O})}(\mathcal{O} \otimes_{p,f^pf_*\mathcal{O}} f^p\mathcal{G}, \mathcal{F}) &= \mathrm{Hom}_{\mathbf{PMod}(f^pf_*\mathcal{O})}(f^p\mathcal{G}, \mathcal{F}_{f^pf_*\mathcal{O}}) \\ &= \mathrm{Hom}_{\mathbf{PMod}(f_*\mathcal{O})}(\mathcal{G}, f_*(\mathcal{F}_{f^pf_*\mathcal{O}})) \\ &= \mathrm{Hom}_{\mathbf{PMod}(f_*\mathcal{O})}(\mathcal{G}, f_*\mathcal{F}).\end{aligned}$$

For the third equality, note that $\mathbf{1}_{f_*\mathcal{O}}$ corresponds to $\sigma_{\mathcal{O}}$ under the adjunction described in the proof of Proposition 1.2.9 and thus we have the equality $\mathbf{1}_{f_*\mathcal{O}} = f_*\sigma_{\mathcal{O}} \circ \rho_{f_*\mathcal{O}}$. Now consider the module structures:

$$\begin{array}{ll} \mathcal{F}_{f^pf_*\mathcal{O}} : & f^pf_*\mathcal{O} \times \mathcal{F} \xrightarrow{\sigma_{\mathcal{O}} \times \mathbf{1}} \mathcal{O} \times \mathcal{F} \longrightarrow \mathcal{F} \\ f_*(\mathcal{F}_{f^pf_*\mathcal{O}}) : & f_*\mathcal{O} \times f_*\mathcal{F} \xrightarrow{\rho_{f_*\mathcal{O}} \times \mathbf{1}} f_*f^pf_*\mathcal{O} \times f_*\mathcal{F} \xrightarrow{f_*\sigma_{\mathcal{O}} \times \mathbf{1}} f_*\mathcal{O} \times f_*\mathcal{F} \longrightarrow f_*\mathcal{F} \end{array}$$

we conclude that $f_*(\mathcal{F}_{f^pf_*\mathcal{O}}) = f_*\mathcal{F}$. $\square$

Now we consider the case of sheaves.

Lemma 1.2.11. *Let $f : X \to Y$ be a continuous map of topological spaces. Let $\mathcal{O}$ be a sheaf of rings on X. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}$-modules. The pushforward $f_*\mathcal{F}$ is a sheaf of $f_*\mathcal{O}$-modules.*

Lemma 1.2.12. *Let $f : X \to Y$ be a continuous map of topological spaces. Let $\mathcal{O}$ be a sheaf of rings on Y. Let $\mathcal{G}$ be a sheaf of $\mathcal{O}$-modules. There is a natural map of underlying presheaves of sets*

$$f^{-1}\mathcal{O} \times f^{-1}\mathcal{G} \to f^{-1}\mathcal{G}$$

which turns $f^{-1}\mathcal{G}$ into a sheaf of $f^{-1}\mathcal{O}$-modules.

Proof. Recall that f^{-1} is defined as the composition of the functor f^p and sheafification. Thus the lemma is a combination of Lemma 1.2.8 and Proposition 1.2.4. □

Let $f : X \to Y$ be a continuous map. Let $\mathcal{O}_X$ be a sheaf of rings on X and let $\mathcal{O}_Y$ be a sheaf of rings on Y. So now we have defined functors

$$f_* : \mathbf{Mod}(\mathcal{O}_X) \to \mathbf{Mod}(f_*\mathcal{O}_X), \quad f^{-1} : \mathbf{Mod}(\mathcal{O}_Y) \to \mathbf{Mod}(f^{-1}\mathcal{O}_Y).$$

These satisfy some compatibilities as follows.

Proposition 1.2.13. *Let $f : X \to Y$ be a continuous map of topological spaces. Let $\mathcal{O}$ be a sheaf of rings on Y. Let $\mathcal{G}$ be a sheaf of $\mathcal{O}$-modules. Let $\mathcal{F}$ be a sheaf of $f^{-1}\mathcal{O}$-modules. Then*

$$\mathrm{Hom}_{\mathbf{Mod}(f^{-1}\mathcal{O})}(f^{-1}\mathcal{G}, \mathcal{F}) = \mathrm{Hom}_{\mathbf{Mod}(\mathcal{O})}(\mathcal{G}, f_*\mathcal{F}).$$

Here we think of $f_*\mathcal{F}$ as an $\mathcal{O}$-module by restriction via $\mathcal{O} \to f_*f^{-1}\mathcal{O}$.

Proof. Argue by the equalities

$$\mathrm{Hom}_{\mathbf{Mod}(f^{-1}\mathcal{O})}(f^{-1}\mathcal{G}, \mathcal{F}) = \mathrm{Hom}_{\mathbf{Mod}(f^p\mathcal{O})}(f^p\mathcal{G}, \mathcal{F}) = \mathrm{Hom}_{\mathbf{Mod}(\mathcal{O})}(\mathcal{G}, f_*\mathcal{F}).$$

which holds by the property of sheafification. □

Proposition 1.2.14. *Let $f : X \to Y$ be a continuous map of topological spaces. Let $\mathcal{O}$ be a sheaf of rings on X. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}$-modules. Let $\mathcal{G}$ be a sheaf of $f_*\mathcal{O}$-modules. Then*

$$\mathrm{Hom}_{\mathbf{Mod}(\mathcal{O})}(\mathcal{O} \otimes_{f^{-1}f_*\mathcal{O}} f^{-1}\mathcal{G}, \mathcal{F}) = \mathrm{Hom}_{\mathbf{Mod}(f_*\mathcal{O})}(\mathcal{G}, f_*\mathcal{F}).$$

Here we use the canonical map $f^{-1}f_*\mathcal{O} \to \mathcal{O}$ in the definition of the tensor product.

Proof. This follows from the equalities

$$\begin{aligned}
\mathrm{Hom}_{\mathbf{Mod}(\mathcal{O})}(\mathcal{O} \otimes_{f^{-1}f_*\mathcal{O}} f^{-1}\mathcal{G}, \mathcal{F}) &= \mathrm{Hom}_{\mathbf{Mod}(f^{-1}f_*\mathcal{O})}(f^{-1}\mathcal{G}, \mathcal{F}_{f^{-1}f_*\mathcal{O}}) \\
&= \mathrm{Hom}_{\mathbf{Mod}(f_*\mathcal{O})}(\mathcal{G}, f_*(\mathcal{F}_{f^{-1}f_*\mathcal{O}})) \\
&= \mathrm{Hom}_{\mathbf{Mod}(f_*\mathcal{O})}(\mathcal{G}, f_*\mathcal{F}).
\end{aligned}$$

where the last equality is obtained in Proposition 1.2.10. □

1.2.5 Supports of modules and sections

Definition 1.2.3. Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules.

- The support of $\mathcal{F}$ is the set of points $x \in X$ such that $\mathcal{F}_x = 0$. We denote it by $\mathrm{supp}(\mathcal{F})$.
- Let $s \in \Gamma(X, \mathcal{F})$ be a global section. The support of s is the set of points $x \in X$ such that the image $s_x \in \mathcal{F}_x$ of s is not zero.

Of course the support of a local section is then defined also since a local section is a global section of the restriction of $\mathcal{F}$.

Lemma 1.2.15. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. Let $U \subseteq X$ open.*

- *The support of $s \in \mathcal{F}(U)$ is closed in U.*
- *The support of fs is contained in the intersections of the supports of $f \in \mathcal{O}_X(U)$ and $s \in \mathcal{F}(U)$.*
- *The support of $s + s'$ is contained in the union of the supports of $s, s' \in \mathcal{F}(U)$.*

- *The support of $\mathcal{F}$ is the union of the supports of all local sections of $\mathcal{F}$.*
- *If $\varphi : \mathcal{F} \to \mathcal{G}$ is a morphism of $\mathcal{O}_X$-modules, then the support of $\varphi(s)$ is contained in the support of $s \in \mathcal{F}(U)$.*

In general the support of a sheaf of modules is not closed. Namely, the sheaf could be an abelian sheaf on $\mathbb{R}$ (with the usual archimedean topology) which is the direct sum of infinitely many nonzero skyscraper sheaves each supported at a single point p_i of $\mathbb{R}$. Then the support would be the set of points pi which may not be closed.

Another example is to consider the open immersion $j : U = (0, \infty) \to \mathbb{R} = X$, and the abelian sheaf $j_!\mathbb{Z}_U$. By **??** the support of this sheaf is exactly U.

Lemma 1.2.16. *Let X be a topological space. The support of a sheaf of rings is closed.*

Proof. This is true because a ring is 0 if and only if $1 = 0$, and hence the support of a sheaf of rings is the support of the unit section. □

1.2.6 Modules generated by sections

Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. Then there is an canonical identification $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{F}) = \Gamma(X, \mathcal{F})$ which associate a global section $s \in \Gamma(X, \mathcal{F})$ with the unique homomorphism of $\mathcal{O}_X$-modules $\mathcal{O}_X \to \mathcal{F}, f \mapsto fs$. That is, a local section f of $\mathcal{O}_X$, i.e., a section f over some open U, is mapped to the multiplication of f with the restriction of s to U. We say that $\mathcal{F}$ is **generated by global sections** if there exist a set I and global sections $s_i \in \Gamma(X, \mathcal{F})$, $i \in I$ such that the homomorphism

$$\mathcal{O}_X^{\oplus I} \to \mathcal{F}$$

which is the homomorphism associated to s_i on the summand corresponding to i, is surjective. In this case we say that the sections s_i **generate** $\mathcal{F}$.

Proposition 1.2.17. *Let $(X, \mathcal{O}_X)$ be a ringed space and $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. Let $(s_i)_{i \in I}$ be a family of global sections of $\mathcal{F}$. Then the sections s_i generate $\mathcal{F}$ if and only if for any point $x \in X$ the elements $s_{i,x} \in \mathcal{F}_x$ generate the $\mathcal{O}_{X,x}$-module $\mathcal{F}_x$.*

Proof. The homomorphism $\mathcal{O}_X^{\oplus I} \to \mathcal{F}$ is surjective if and only if for each point $x \in X$ the homomorphism $(\mathcal{O}_X^{\oplus I})_x \to \mathcal{F}_x$ is surjective. Since taking stalk commutes with colimit, we have $(\mathcal{O}_X)_x^{\oplus I}\mathcal{O}_{X,x}^{\oplus I}$, which implies the claim. □

Proposition 1.2.18. *Let $(X, \mathcal{O}_X)$ be a ringed space and $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. Let $(s_i)_{i \in I}$ be a collection of local sections of $\mathcal{F}$, i.e., $s_i \in \Gamma(U_i, \mathcal{F})$ for some open subset U_i of X. Then there exists a unique smallest sub-$\mathcal{O}_X$-module $\mathcal{G}$ of $\mathcal{F}$ such that each s_i corresponds to a local section of $\mathcal{G}$, which is called the **sub-$\mathcal{O}_X$-module generated by the s_i**.*

Proof. Consider the subpresheaf of $\mathcal{F}$ defined by the rule

$$U \mapsto \{\sum_{i \in J} f_i(s_i|_U) : J \subseteq I \text{ is finite, } U \subseteq U_i \text{ for every } i \in J \text{ and } f_i \in \Gamma(U, \mathcal{O}_X)\}.$$

Let $\mathcal{G}$ be the sheafification of this subpresheaf. This is a subsheaf of $\mathcal{F}$ by Proposition 1.1.5. Since all the finite sums clearly have to be in $\mathcal{F}$ this is the smallest subsheaf as desired. □

Proposition 1.2.19. *Let $(X, \mathcal{O}_X)$ be a ringed space and $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. Let $(s_i)_{i \in I}$ be a family of local sections of $\mathcal{F}$ and $\mathcal{G}$ be the subsheaf generated by the s_i and let $x \in X$. Then $\mathcal{G}_x$ is the $\mathcal{O}_{X,x}$-submodule of $\mathcal{F}_x$ generated by the elements $s_{i,x}$ for those i such that s_i is defined at x.*

Proof. This is clear from the construction of $\mathcal{G}$ in the proof of Proposition 1.2.18. □

Example 1.2.1. Consider the open immersion $j : U = (0, \infty) \to \mathbb{R} = X$, and the abelian sheaf $j_!(\mathbb{Z}_U)$. By **??** the stalk of $j_!(\mathbb{Z}_U)$ at $x = 0$ is 0. In fact the sections of this sheaf over any open interval containing 0 are 0. Thus there is no open neighbourhood of the point 0 over which the sheaf can be generated by global sections.

Let $(X, \mathscr{O}_X)$ be a ringed space. Let $\mathscr{F}$ be a sheaf of $\mathscr{O}_X$-modules. We say that $\mathscr{F}$ is **locally generated by sections** if for every $x \in X$ there exists an open neighbourhood U such that $\mathscr{F}|_U$ is globally generated as a sheaf of $\mathscr{O}_U$-modules. In other words there exists a set I and for each i a section $s_i \in \mathscr{F}(U)$ such that the associated map

$$\bigoplus_{i \in I} \mathscr{O}_U \to \mathscr{F}|_U$$

is surjective.

Proposition 1.2.20. *Let $(f, f^\#) : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ be a morphism of ringed spaces. Let $\mathscr{G}$ be a $\mathscr{O}_Y$-module. The pullback $f^*\mathscr{G}$ is locally generated by sections if $\mathscr{G}$ is locally generated by sections.*

Proof. Given an open subspace V of Y we may consider the commutative diagram of ringed spaces

$$\begin{tikzcd}
(f^{-1}(V), \mathscr{O}_{f^{-1}(V)}) \arrow[r, "i"] \arrow[d, "\tilde{f}"] & (X, \mathscr{O}_X) \arrow[d, "f"] \\
(V, \mathscr{O}_V) \arrow[r, "j"] & (Y, \mathscr{O}_Y)
\end{tikzcd}$$

We know that $(f^*\mathscr{G})|_{f^{-1}(V)} \cong (\tilde{f})^*(\mathscr{G}|_V)$, so we may assume that $\mathscr{G}$ is globally generated. We have seen that f^* commutes with all colimits, and is right exact. Thus if we have a surjection $\mathscr{O}_Y^{\oplus I} \to \mathscr{G} \to 0$, then upon applying f^* we obtain the surjection $\mathscr{O}_X^{\oplus I} \to f^*\mathscr{G} \to 0$, where we use the observation that

$$f^*\mathscr{O}_Y = \mathscr{O}_X \otimes_{f^{-1}\mathscr{O}_Y} f^{-1}\mathscr{O}_Y = \mathscr{O}_X.$$

This implies the assertion. $\square$

1.2.7 Tensor product

Let $(X, \mathscr{O}_X)$ be a ringed space. Let $\mathscr{F}, \mathscr{G}$ be $\mathscr{O}_X$-modules. We define first the tensor product presheaf

$$\mathscr{F} \otimes_{p, \mathscr{O}_X} \mathscr{G}$$

as the rule which assigns to $U \subseteq X$ open the $\mathscr{O}_X(U)$-module $\mathscr{F}(U)_{\mathscr{O}_X(U)}\mathscr{G}(U)$. Having defined this we define the tensor product sheaf as the sheafification of the above:

$$\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G} = (\mathscr{F} \otimes_{p, \mathscr{O}_X} \mathscr{G})^\#$$

This can be characterized as the sheaf of $\mathscr{O}_X$-modules such that for any third sheaf of $\mathscr{O}_X$-modules $\mathscr{H}$ we have

$$\mathrm{Hom}_{\mathscr{O}_X}(\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G}, \mathscr{H}) = \mathrm{Bilin}_{\mathscr{O}_X}(\mathscr{F} \times \mathscr{G}, \mathscr{H}).$$

Here the right hand side indicates the set of bilinear maps of sheaves of $\mathscr{O}_X$-modules.

The tensor product of modules M, N over a ring R satisfies symmetry, hence the same holds for tensor products of sheaves of modules, i.e., we have

$$\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G} = \mathscr{G} \otimes_{\mathscr{O}_X} \mathscr{F}$$

functorial in $\mathscr{F}$, $\mathscr{G}$. And since tensor product of modules satisfies associativity we also get canonical functorial isomorphisms

$$(\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G}) \otimes_{\mathscr{O}_X} \mathscr{H} = \mathscr{F} \otimes_{\mathscr{O}_X} (\mathscr{G} \otimes_{\mathscr{O}_X} \mathscr{H}).$$

functorial in $\mathscr{F}, \mathscr{G}$, and $\mathscr{H}$.

Proposition 1.2.21. *Let $(X, \mathscr{O}_X)$ be a ringed space. Let $\mathscr{F}$, $\mathscr{G}$ be $\mathscr{O}_X$-modules. Let $x \in X$. There is a canonical isomorphism of $\mathscr{O}_{X,x}$-modules*

$$(\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G})_x = \mathscr{F}_x \otimes_{\mathscr{O}_{X,x}} \mathscr{G}_x.$$

functorial in $\mathscr{F}$ and $\mathscr{G}$.

Proposition 1.2.22. *Let $(X, \mathscr{O}_X)$ be a ringed space. Let $\mathscr{F}', \mathscr{G}'$ be presheaves of $\mathscr{O}_X$-modules with sheafifications $\mathscr{F}, \mathscr{G}$. Then $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G} = (\mathscr{F}' \otimes_{p, \mathscr{O}_X} \mathscr{G}')^\#$.*

Proof. On stalks we have

$$(\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G})_x = \mathscr{F}_x \otimes_{\mathscr{O}_{X,x}} \mathscr{G}_x = \mathscr{F}'_x \otimes_{\mathscr{O}_{X,x}} \mathscr{G}'_x = \mathscr{F}'_x \otimes_{p, \mathscr{O}_{X,x}} \mathscr{G}'_x$$

Thus by Proposition 1.1.6 we conclude the result. □

Proposition 1.2.23. *Let $(X, \mathscr{O}_X)$ be a ringed space. Let $\mathscr{G}$ be an $\mathscr{O}_X$-module. If $\mathscr{F}_1 \to \mathscr{F}_2 \to \mathscr{F}_3 \to 0$ is an exact sequence of $\mathscr{O}_X$-modules then the induced sequence*

$$\mathscr{F}_1 \otimes_{\mathscr{O}_X} \mathscr{G} \to \mathscr{F}_2 \otimes_{\mathscr{O}_X} \mathscr{G} \to \mathscr{F}_3 \to 0$$

is exact.

Proof. This follows from the fact that exactness may be checked at stalks, the description of stalks and the corresponding result for tensor products of modules □

Proposition 1.2.24. *Let $f : X \to Y$ be a morphism of ringed spaces. Let $\mathscr{F}, \mathscr{G}$ be $\mathscr{O}_Y$-modules. Then $f^*(\mathscr{F} \otimes_{\mathscr{O}_Y} \mathscr{G}) = f^*\mathscr{F} \otimes_{\mathscr{O}_X} f^*\mathscr{G}$ functorially in $\mathscr{F}, \mathscr{G}$.*

Proof. Let $x \in X$, we check that

$$\begin{aligned}
\big(f^*(\mathscr{F} \otimes_{\mathscr{O}_Y} \mathscr{G})\big)_x &= \mathscr{O}_{X,x} \otimes_{\mathscr{O}_{Y,f(x)}} (\mathscr{F} \otimes_{\mathscr{O}_Y} \mathscr{G})_{f(x)} = \mathscr{O}_{X,x} \otimes_{\mathscr{O}_{Y,f(x)}} \big(\mathscr{F}_{f(x)} \otimes_{\mathscr{O}_{Y,f(x)}} \mathscr{G}_{f(x)}\big) \\
&= \mathscr{O}_{X,x} \otimes_{\mathscr{O}_{X,x}} \mathscr{O}_{X,x} \otimes_{\mathscr{O}_{Y,f(x)}} \mathscr{F}_{f(x)} \otimes_{\mathscr{O}_{Y,f(x)}} \mathscr{G}_{f(x)} \\
&= \mathscr{O}_{X,x} \otimes_{\mathscr{O}_{Y,f(x)}} \mathscr{F}_{f(x)} \otimes_{\mathscr{O}_{X,x}} \mathscr{O}_{X,x} \otimes_{\mathscr{O}_{Y,f(x)}} \mathscr{G}_{f(x)} \\
&= (f^*\mathscr{F})_x \otimes_{X,x} (f^*\mathscr{G})_x.
\end{aligned}$$

as desired. □

Proposition 1.2.25. *Let $(X, \mathscr{O}_X)$ be a ringed space. Let $\mathscr{F}, \mathscr{G}$ be $\mathscr{O}_X$-modules.*

(i) If $\mathscr{F}, \mathscr{G}$ are locally generated by sections, so is $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G}$.

(ii) If $\mathscr{F}, \mathscr{G}$ are of finite type, so is $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G}$.

(iii) If $\mathscr{F}, \mathscr{G}$ are quasi-coherent, so is $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G}$.

(iv) If $\mathscr{F}, \mathscr{G}$ are of finite presentation, so is $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G}$.

(v) If $\mathscr{F}$ is of finite presentation and $\mathscr{G}$ is coherent, then $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G}$ is coherent.

(vi) If $\mathscr{F}, \mathscr{G}$ are coherent, so is $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G}$.

(vii) If $\mathscr{F}, \mathscr{G}$ are locally free, so is $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G}$.

Proof. We first prove that the tensor product of locally free $\mathscr{O}_X$-modules is locally free. This follows if we show that

$$(\bigoplus_{i\in I} \mathscr{O}_X) \otimes_{\mathscr{O}_X} (\bigoplus_{j\in J} \mathscr{O}_X) \cong \bigoplus_{(i,j)\in I\times J} \mathscr{O}_X.$$

The sheaf $\bigoplus_{i\in I} \mathscr{O}_X$ is the sheaf associated to the presheaf $U \mapsto \bigoplus_{i\in I} \mathscr{O}_X(U)$. Hence the tensor product is the sheaf associated to the presheaf

$$U \mapsto \Big(\bigoplus_{i\in I} \mathscr{O}_X(U)\Big) \otimes_{\mathscr{O}_X(U)} \Big(\bigoplus_{j\in J} \mathscr{O}_X(U)\Big).$$

We deduce what we want since for any ring R we have $(\bigoplus_{i\in I} R) \otimes_R (\bigoplus_{j\in J} R) \cong \bigoplus_{(i,j)\in I\times J} R$.

If $\mathscr{F}_2 \to \mathscr{F}_1 \to \mathscr{F} \to 0$ is exact, then by Proposition 1.2.23 the complex $\mathscr{F}_2 \otimes \mathscr{G} \to \mathscr{F}_1 \otimes \mathscr{G} \to \mathscr{F} \otimes \mathscr{G} \to 0$ is exact. Using this we can prove (v). Namely, in this case there exists locally such an exact sequence with $\mathscr{F}_i, i = 1, 2$ finite free. Hence the two terms $\mathscr{F}_i \otimes \mathscr{G}$ are isomorphic to finite direct sums of $\mathscr{G}$. Since finite direct sums are coherent sheaves, these are coherent and so is the cokernel of the map.

If we also have another exact sequence $\mathscr{G}_2 \to \mathscr{G}_1 \to \mathscr{G} \to 0$, then tensoring together we get an exact sequence

$$(\mathscr{F}_2 \otimes \mathscr{G}_1) \oplus (\mathscr{F}_1 \otimes \mathscr{G}_2) \longrightarrow \mathscr{F}_1 \otimes \mathscr{G}_1 \longrightarrow \mathscr{F} \otimes \mathscr{G} \longrightarrow 0$$

This can be used to prove $(i), (ii), (iii), (iv), (vi)$. □

Proposition 1.2.26. *Let $(X, \mathscr{O}_X)$ be a ringed space. For any $\mathscr{O}_X$-module $\mathscr{F}$ the functor $\mathscr{F} \otimes_{\mathscr{O}_X}$ commutes with arbitrary colimits.*

Proof. Let I be a partially ordered set and let $\{\mathscr{G}_i\}$ be a system over I. Set $\mathscr{G} = \varinjlim_i \mathscr{G}_i$. Recall that $\mathscr{G}$ is the sheaf associated to the presheaf $\mathscr{G}' : U \mapsto \varinjlim_i \mathscr{G}_i(U)$. By the tensor product $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G}$ is the sheafification of the presheaf

$$U \mapsto \mathscr{F}(U) \otimes_{\mathscr{O}_X(U)} \varinjlim_i \mathscr{G}_i(U) = \varinjlim_i \mathscr{F}(U) \otimes_{\mathscr{O}_X(U)} \mathscr{G}_i(U)$$

where the equality sign follows from the property of tensor product of modules. Hence the lemma follows from the description of colimits in $\mathbf{Mod}(\mathscr{O}_X)$. □

1.2.8 Internal Hom

Let $(X, \mathscr{O}_X)$ be a ringed space. Let $\mathscr{F}, \mathscr{G}$ be $\mathscr{O}_X$-modules. Consider the rule

$$U \mapsto \mathrm{Hom}_{\mathscr{O}_X|_U}(\mathscr{F}|_U, \mathscr{G}|_U)$$

It follows from Proposition 1.1.20 that this is a sheaf of abelian groups. In addition, given an element $\varphi \in \mathrm{Hom}_{\mathscr{O}_X|_U}(\mathscr{F}|_U, \mathscr{G}|_U)$ and a section $f \in \mathscr{O}_X(U)$ then we can define $f\varphi \in \mathrm{Hom}_{\mathscr{O}_X|_U}(\mathscr{F}|_U, \mathscr{G}|_U)$ by either precomposing with multiplication by f on $\mathscr{F}|_U$ or postcomposing with multiplication by f on $\mathscr{G}|_U$ (it gives the same result). Hence we in fact get a sheaf of $\mathscr{O}_X$-modules. We will denote this sheaf $\mathcal{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})$. For every $x \in X$ there is also a canonical morphism

$$\mathcal{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})_x \to \mathrm{Hom}_{\mathscr{O}_{X,x}}(\mathscr{F}_x, \mathscr{G}_x)$$

which is rarely an isomorphism.

Proposition 1.2.27. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathscr{F}, \mathscr{G}, \mathscr{H}$ be $\mathcal{O}_X$-modules. There is a canonical isomorphism*

$$\mathcal{H}om_{\mathcal{O}_X}(\mathscr{F} \otimes_{\mathcal{O}_X} \mathscr{G}, \mathscr{H}) \to \mathcal{H}om_{\mathcal{O}_X}(\mathscr{F}, \mathcal{H}om_{\mathcal{O}_X}(\mathscr{G}, \mathscr{H}))$$

which is functorial in all three entries. In particular, to give a morphism $\mathscr{F} \otimes_{\mathcal{O}_X} \mathscr{G} \to \mathscr{H}$ is the same as giving a morphism $\mathscr{F} \to \mathcal{H}om_{\mathcal{O}_X}(\mathscr{G}, \mathscr{H})$.

Proof. This is the analogue of that for modules. □

Due to this proposition, the functors $\mathcal{H}om(-, \mathscr{G})$ and $\mathcal{H}om(\mathscr{F}, -)$ are left-exact.

Proposition 1.2.28. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathscr{F}, \mathscr{G}$ be $\mathcal{O}_X$-modules.*

- *If $\mathscr{F}_2 \to \mathscr{F}_1 \to \mathscr{F} \to 0$ is an exact sequence of $\mathcal{O}_X$-modules, then*

$$0 \longrightarrow \mathcal{H}om(\mathscr{F}, \mathscr{G}) \longrightarrow \mathcal{H}om(\mathscr{F}_1, \mathscr{G}) \longrightarrow \mathcal{H}om(\mathscr{F}_2, \mathscr{G})$$

  is exact.

- *If $0 \to \mathscr{G} \to \mathscr{G}_1 \to \mathscr{G}_2 \to 0$ is an exact sequence of $\mathcal{O}_X$-modules, then*

$$0 \longrightarrow \mathcal{H}om(\mathscr{F}, \mathscr{G}) \longrightarrow \mathcal{H}om(\mathscr{F}, \mathscr{G}_1) \longrightarrow \mathcal{H}om(\mathscr{F}, \mathscr{G}_2)$$

  is exact.

Proposition 1.2.29. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathscr{F}, \mathscr{G}$ be $\mathcal{O}_X$-modules. If $\mathscr{F}$ is finitely presented then the canonical map*

$$\mathcal{H}om_{\mathcal{O}_X}(\mathscr{F}, \mathscr{G})_x \to \operatorname{Hom}_{\mathcal{O}_{X,x}}(\mathscr{F}_x, \mathscr{G}_x)$$

is an isomorphism.

Proof. By localizing on X we may assume that $\mathscr{F}$ has a presentation

$$\bigoplus_{j=1}^m \mathcal{O}_X \longrightarrow \bigoplus_{i=1}^n \mathcal{O}_X \longrightarrow \mathscr{F} \longrightarrow 0$$

Then this gives an exact sequence

$$0 \longrightarrow \mathcal{H}om_{\mathcal{O}_X}(\mathscr{F}, \mathscr{G}) \longrightarrow \bigoplus_{i=1}^n \mathscr{G} \longrightarrow \bigoplus_{j=1}^m \mathscr{G}$$

Taking stalks we get an exact sequence

$$0 \longrightarrow \mathcal{H}om_{\mathcal{O}_X}(\mathscr{F}, \mathscr{G})_x \longrightarrow \bigoplus_{i=1}^n \mathscr{G}_x \longrightarrow \bigoplus_{j=1}^m \mathscr{G}_x$$

The result now follows since $\mathscr{F}_x$ sits in an exact sequence

$$\bigoplus_{j=1}^m \mathcal{O}_{X,x} \longrightarrow \bigoplus_{i=1}^n \mathcal{O}_{X,x} \longrightarrow \mathscr{F}_x \longrightarrow 0$$

which induces the exact sequence

$$0 \longrightarrow \operatorname{Hom}_{\mathcal{O}_{X,x}}(\mathscr{F}_x, \mathscr{G}_x) \longrightarrow \bigoplus_{i=1}^n \mathscr{G}_x \longrightarrow \bigoplus_{j=1}^m \mathscr{G}_x$$

which is the same as the one above. □

Proposition 1.2.30. *Let $f : X \to Y$ be a morphism of ringed spaces. Let $\mathscr{F}, \mathscr{G}$ be $\mathcal{O}_Y$-modules. If $\mathscr{F}$ is finitely presented then the canonical map*

$$f^*\mathcal{H}om_{\mathcal{O}_Y}(\mathscr{F}, \mathscr{G}) \to \mathcal{H}om_{\mathcal{O}_X}(f^*\mathscr{F}, f^*\mathscr{G})$$

is an isomorphism.

Proof. Note that $f^*\mathscr{F}$ is also finitely presented. Let $x \in X$ map to $y \in Y$. Looking at the stalks at x we get an isomorphism by Proposition 1.2.29 and that in this case Hom commutes with base change by $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$. □

Proposition 1.2.31. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathscr{F}, \mathscr{G}$ be $\mathcal{O}_X$-modules. If $\mathscr{F}$ is finitely presented then the sheaf $\mathcal{H}om_{\mathcal{O}_X}(\mathscr{F}, \mathscr{G})$ is locally a kernel of a map between finite direct sums of copies of $\mathscr{G}$. In particular, if $\mathscr{G}$ is coherent then $\mathcal{H}om_{\mathcal{O}_X}(\mathscr{F}, \mathscr{G})$ is coherent too.*

Proof. The first assertion we saw in the proof of Proposition 1.2.29. And the result for coherent sheaves then follows from Proposition 1.4.17. □

Proposition 1.2.32. *Let X be a topological space. Let $\mathcal{O}_1 \to \mathcal{O}_2$ be a homomorphism of sheaves of rings. Then we have*

$$\mathrm{Hom}_{\mathcal{O}_1}(\mathscr{F}_{\mathcal{O}_1}, \mathscr{G}) = \mathrm{Hom}_{\mathcal{O}_2}(\mathscr{F}, \mathcal{H}om_{\mathcal{O}_1}(\mathcal{O}_2, \mathscr{G}))$$

*bifunctorially in $\mathscr{F} \in$ **Mod**$(\mathcal{O}_2)$ and $\mathscr{G} \in$ **Mod**$(\mathcal{O}_1)$.*

Proof. This is the analogue of the result for modules. □

1.2.9 The abelian category of sheaves of modules

Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathscr{F}, \mathscr{G}$ be sheaves of $\mathcal{O}_X$-modules. Let $\varphi, \psi : \mathscr{F} \to \mathscr{G}$ be morphisms of sheaves of $\mathcal{O}_X$-modules. We define $\varphi + \psi : \mathscr{F} \to \mathscr{G}$ to be the map which on each open $U \subseteq X$ is the sum of the maps induced by φ, ψ. This is clearly again a map of sheaves of $\mathcal{O}_X$-modules. It is also clear that composition of maps of $\mathcal{O}_X$-modules is bilinear with respect to this addition. Thus **Mod**$(\mathcal{O}_X)$ is a pre-additive category. We will denote 0 the sheaf of $\mathcal{O}_X$-modules which has constant value $\{0\}$ for all open $U \subseteq X$. Clearly this is both a final and an initial object of **Mod**$(\mathcal{O}_X)$. Moreover, given a pair $\mathscr{F}, \mathscr{G}$ of sheaves of $\mathcal{O}_X$-modules we may define the direct sum as

$$\mathscr{F} \oplus \mathscr{G} = \mathscr{F} \times \mathscr{G}$$

Thus **Mod**$(\mathcal{O}_X)$ is an additive category.

Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism of $\mathcal{O}_X$-modules. We may define the **presheaf kernel** $\ker \varphi$ and the **presheaf cokernel** to be

$$(\ker_p \varphi)(U) = \ker \varphi_U, \quad (\operatorname{coker}_p \varphi)(U) = \operatorname{coker} \varphi_U.$$

for open subsets $U \subseteq X$. We define $\operatorname{coker} \varphi$ be the sheafification of $\operatorname{coker}_p \varphi$.

Proposition 1.2.33. *The presheaf kernel is a presheaf of $\mathcal{O}_X$-modules, so is the presheaf cokernel.*

Proof. For $U \subseteq V$ open in X, consider the diagram

$$\begin{tikzcd} 0 \arrow[r] & \ker_p \varphi_V \arrow[r, hook] \arrow[d, dashed] & \mathscr{F}(V) \arrow[r, "\varphi_V"] \arrow[d, "\mathrm{res}_U^V"] & \mathscr{G}(V) \arrow[d, "\mathrm{res}_U^V"] \\ 0 \arrow[r] & \ker_p \varphi_U \arrow[r, hook] & \mathscr{F}(U) \arrow[r, "\varphi_U"] & \mathscr{G}(U) \end{tikzcd} \tag{1.2.1}$$

By the comutativity, there is a unique map $\mathrm{res}_U^V : \ker \varphi_V \to \ker \varphi_U$ fitting in the diagram. This makes $\ker_p \varphi$ into a presheaf of $\mathcal{O}_X$-modules, and essentially the same argument works for $\operatorname{coker}_p \varphi$. □

Proposition 1.2.34. *The presheaf (co)kernel satisfies the universal property of (co)kernel in the category of presheaves of $\mathscr{O}_X$-modules.*

Proof. Let $\psi : \mathscr{G} \to \mathscr{H}$ be another morphism of presheaves of $\mathscr{O}_X$-modules such that $(\psi \circ \varphi)_U = \psi_U \circ \varphi_U = 0$ for any $U \subseteq X$. Then we have an induced map $\tilde{\psi}_U : \ker \varphi_U \to \mathscr{H}(U)$ for each U. For $U \subseteq V$ open in X, it is easy to verify that the following commutative diagram:

$$\begin{tikzcd} \ker \varphi_V \arrow[r, "\tilde{\psi}_V"] \arrow[d, "\mathrm{res}^V_U"'] & \mathscr{H}(V) \arrow[d, "\mathrm{res}^V_U"] \\ \ker \varphi_U \arrow[r, "\tilde{\psi}_U"] & \mathscr{H}(U) \end{tikzcd}$$

so $\ker \varphi$ is the kernel in the category of presheaves. A similar verification works for cokernels. $\square$

Proposition 1.2.35. *Suppose $\varphi : \mathscr{F} \to \mathscr{G}$ is a morphism of sheaves. Then the presheaf kernel $\ker_p \varphi$ satisfies the universal property of kernels in $\mathbf{Mod}(\mathscr{O}_X)$, and the sheafification of $\operatorname{coker}_p \varphi$ satisfies the universal property of cokernels in $\mathbf{Mod}(\mathscr{O}_X)$.*

Proof. Let $U = \bigcup_i U_i$ be an open covering in X, and $s_i \in \ker \varphi(U_i)$ such that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for all i, j. Then by the sheaf condition of $\mathscr{F}$, there is a unique $s \in \mathscr{F}(U)$ such that $s|_{U_i} = s_i$. Moreover, from the diagram (1.2.1) we get

$$\big(\varphi_U(s)\big)_{U_i} = \varphi_{U_i}(s|_{U_i}) = 0.$$

Thus by the sheaf condition of $\mathscr{G}$, we conclude $\varphi_U(s) = 0$. This implies $s \in \varphi_U$, so $\ker \varphi$ is a sheaf.

For the cokernel, given any sheaf $\mathscr{E}$ and a diagram

$$\begin{tikzcd} \mathscr{F} \arrow[r, "\varphi"] & \mathscr{G} \arrow[r, "\psi"] \arrow[d] & \mathscr{E} \\ & \operatorname{coker}_p \varphi \arrow[r] \arrow[ur] & (\operatorname{coker}_p \varphi)^{\#} \arrow[u, dashed, "\tilde{\psi}"'] \end{tikzcd}$$

we construct the map $\tilde{\psi}$ by using the universal property of $\operatorname{coker}_p \varphi$ and that of sheafification. $\square$

In view of Proposition 1.2.35, for a morphism $\varphi : \mathscr{F} \to \mathscr{G}$, we define the kernel and cokernel of φ by

$$\ker \varphi = \ker_p \varphi, \quad \operatorname{coker} \varphi = (\operatorname{coker}_p \varphi)^{\#}.$$

Since taking stalks commutes with taking sheafification, the following result is immediate.

Proposition 1.2.36. *Suppose $\mathscr{F} \to \mathscr{G}$ is a morphism of sheaves of $\mathscr{O}_X$-modules, then for all $x \in X$, we have canonical isomorphisms*

$$(\ker \varphi)_x \cong \ker \varphi_x, \quad (\operatorname{coker} \varphi)_x \cong \operatorname{coker} \varphi_x.$$

Proof. Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves. We have the commutative diagram:

$$\begin{tikzcd} \mathscr{F}(U) \arrow[r, "\varphi_U"] \arrow[d] & \mathscr{G}(U) \arrow[d] \\ \mathscr{F}_x \arrow[r, "\varphi_x"] & \mathscr{G}_x \end{tikzcd}$$

Let $s \in \ker \varphi_U$, then its image $\bar{s}$ is mapped to zero by φ_x, so $\bar{s} \in \ker \varphi_x$. Conversely, if $\bar{s} \in \ker \varphi_x$ and $\bar{s} = (U, s)$ where $s \in \mathscr{F}(U)$, then the image of $t = \varphi(s) \in \mathscr{G}(U)$ is zero in $\mathscr{G}_x$. Therefore in some neighborhood, say $V \subseteq U$, $t|_V = 0$. Then we have the commutative diagram

$$\begin{tikzcd} \mathscr{F}(V) \arrow[r, "\varphi_V"] \arrow[d] & \mathscr{G}(V) \arrow[d] \\ \mathscr{F}_x \arrow[r, "\varphi_x"] & \mathscr{G}_x \end{tikzcd}$$

where now $\varphi_V(s|_V) = 0$. This shows $\bar{s} \in (\ker \varphi)_p$. The proof is similar for cokernels. □

Now that we kernel and cokernels, we can prove that $\mathbf{Mod}(\mathscr{O}_X)$ is an abelian category.

Theorem 1.2.37. *Let $(X, \mathscr{O}_X)$ be a ringed space. The category $\mathbf{Mod}(\mathscr{O}_X)$ is an abelian category. Moreover, a complex $\mathscr{F} \to \mathscr{G} \to \mathscr{H}$ is exact at $\mathscr{G}$ if and only if for all $x \in X$ the complex $\mathscr{F}_x \to \mathscr{G}_x \to \mathscr{H}_x$ is exact at $\mathscr{G}_x$.*

Proof. We have to show that image and coimage agree. By Proposition 1.1.2 it is enough to show that image and coimage have the same stalk at every $x \in X$. By the constructions of kernels and cokernels above these stalks are the coimage and image in the categories of $\mathscr{O}_{X,x}$-modules. Thus we get the result from the fact that the category of modules over a ring is abelian. □

Actually the category $\mathbf{Mod}(\mathscr{O}_X)$ has many more properties. Here are two constructions we can do.

- Given any set I and for each $i \in I$ a $\mathscr{O}_X$-module we can form the product

$$\prod_{i \in I} \mathscr{F}_i$$

  which is the sheaf that associates to each open U the product of the modules $\mathscr{F}_i(U)$.

- Given any set I and for each $i \in I$ a $\mathscr{O}_X$-module we can form the direct sum

$$\bigoplus_{i \in I} \mathscr{F}_i$$

  which is the sheafification of the presheaf that associates to each open U the direct sum of the modules $\mathscr{F}_i(U)$.

Using these we conclude that all limits and colimits exist in $\mathbf{Mod}(\mathscr{O}_X)$.

Proposition 1.2.38. *Let $(X, \mathscr{O}_X)$ be a ringed space.*

(a) All limits exist in $\mathbf{Mod}(\mathscr{O}_X)$. Limits are the same as the corresponding limits of presheaves of $\mathscr{O}_X$-modules.

(b) All colimits exist in $\mathbf{Mod}(\mathscr{O}_X)$. Colimits are the sheafification of the corresponding colimit in the category of presheaves. Taking colimits commutes with taking stalks.

(c) Filtered colimits are exact.

(d) Finite direct sums are the same as the corresponding finite direct sums of presheaves of $\mathscr{O}_X$-modules.

Proof. As $\mathbf{Mod}(\mathscr{O}_X)$ is abelian it has all finite limits and colimits. Thus the existence of limits and colimits and their description follows from the existence of products and coproducts and their description. Since sheafification commutes with taking stalks we see that colimits commute with taking stalks. Part (c) signifies that given a system $0 \to \mathscr{F}_i \to \mathscr{G}_i \to \mathscr{H}_i \to 0$ of exact sequences of $\mathscr{O}_X$-modules over a directed partially ordered set I the sequence $0 \to \varinjlim_i \mathscr{F}_i \to \varinjlim_i \mathscr{G}_i \to \varinjlim_i \mathscr{H}_i \to 0$ is exact as well. Since we can check exactness on stalks, this follows from the case of modules. Part (d) comes from the fact that finite direct sum coincides with finite product in an Abelian category. □

Remark 1.2.1. For an arbitrary direct sum $\bigoplus_{i\in I} \mathscr{O}_X$, by the construction of sheafification we see that an element $s \in \bigoplus_{i\in I} \mathscr{O}_X$ satisfies the following property: For any $x \in X$ there is a neighborhood of x such that $s|_U$ is a finite sum $\sum_{i\in I'} f_i$ with $f_i \in \mathscr{O}_X(U)$.

The existence of limits and colimits allows us to consider exactness properties of functors defined on the category of $\mathscr{O}_X$-modules in terms of limits and colimits.

Proposition 1.2.39. *Let $(f, f^\#) : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ be a morphism of ringed spaces.*

(a) The functor $f_* : \mathbf{Mod}(\mathscr{O}_X) \to \mathbf{Mod}(\mathscr{O}_Y)$ is left exact. In fact it commutes with all limits.

(b) The functor $f^* : \mathbf{Mod}(\mathscr{O}_X) \to \mathbf{Mod}(\mathscr{O}_Y)$ is right exact. In fact it commutes with all colimits.

(c) The functor $f^{-1} : \mathbf{Ab}(Y) \to \mathbf{Ab}(X)$ on abelian sheaves is exact.

Proof. Recall that (f^*, f_*) is an adjoint pair of functors. The last part holds because exactness can be checked on stalks and the description of stalks of the pullback. □

Proposition 1.2.40. *Let $j : U \to X$ be an open immersion of topological spaces. The functor $j_!$: $\mathbf{Ab}(U) \to \mathbf{Ab}(X)$ is exact.*

Proof. This follows from the description of stalks given in **??**. □

Proposition 1.2.41. *Let $(X, \mathscr{O}_X)$ be a ringed space. Let I be a set. For $i \in I$, let $\mathscr{F}_i$ be a sheaf of $\mathscr{O}_X$-modules. For $U \subseteq X$ quasi-compact open the map*

$$\bigoplus_{i\in I} \mathscr{F}_i(U) \to \Big(\bigoplus_{i\in I} \mathscr{F}_i\Big)(U)$$

is bijective.

Proof. If s is an element of the right hand side, then there exists an open covering $U = \bigcup_{j\in J} U_j$ such that $s|_{U_j}$ is a finite sum $\sum_{i\in I_j} s_{ji}$ with $s_{ji} \in \mathscr{F}_i(U_j)$. Because U is quasi-compact we may assume that the covering is finite, i.e., that J is finite. Then $I' = \bigcup_{j\in J} I_j$ is a finite subset of I. Clearly, s is a section of the subsheaf $\bigoplus_{i\in I'} \mathscr{F}_i$. The result follows from the fact that for a finite direct sum sheafification is not needed. □

1.3 Ringed spaces

1.3.1 Ringed spaces, $\mathscr{A}$-modules, and $\mathscr{A}$-algebras

A **ringed space** (resp. **topologically ringed space**) is defiend to be a couple $(X, \mathscr{A})$ formed by a topological space X and a sheaf of rings (resp. a sheaf of topological rings) $\mathscr{A}$ on X. We call X the topological space underling the ringed space $(X, \mathscr{A})$, and $\mathscr{A}$ is the structural sheaf. We only denote by $\mathscr{O}_X$ the structural sheaf, and for $x \in X$, $\mathscr{O}_{X,x}$ denotes the stalk of $\mathscr{O}_X$ at x. If $\mathscr{A}$ is a sheaf of commutative rings, we say $(X, \mathscr{A})$ is a **commutative ringed space**. Without further specifications, we only consider commutative ringed spaces.

The ringed spaces (resp. topologically ringed spaces) form a category, if we define a morphism $(X, \mathscr{A}) \to (Y, \mathscr{B})$ as a couple $(f, f^\#)$ formed by a continuous map $f : X \to Y$ and a f-morphism $f^\# : \mathscr{B} \to \mathscr{A}$ (that is, a morphism from $\mathscr{B}$ to $f_*(\mathscr{A})$) of sheaf of rings (resp. sheaf of topological rings). As the category of rings admits inductive limits, for any $x \in X$ we have a homomorphism $f^\#_x : \mathscr{B}_{f(x)} \to \mathscr{A}_x$. The composition of two morphisms $(f, f^\#) : (X, \mathscr{A}) \to (Y, \mathscr{B})$ and $(g, g^\#) : (Y, \mathscr{B}) \to (Z, \mathscr{C})$ is then defined to be the couple $(h, h^\#)$, where $h = g \circ f$ and $g^\#$ is the composition of $f^\#$ and $g^\#$ (which equals to $g_*(f^\#) \circ g^\#$). For any $x \in X$, we then have $h^\#_x = g^\#_{f(x)} \circ f^\#_x$, so if the homomorphisms $f^\#$ and $f^\#$ are injective (resp. surjective), then so is $h^\#$. We then verify that, if f is a injective continuous map and $f^\#$ is a surjective homomorphism of sheaf of rings, then the morphism $(f, f^\#)$ is a monomorphism of category of ringed spaces.

By abuse of language, we only replace $(f, f^\#)$ by f, and say that $f : (X, \mathscr{A}) \to (Y, \mathscr{B})$ is a morphism of ringed spaces. In this case, it is understood that the homomorphism $f^\#$ is also given.

For any subset U of X, the couple $(U, \mathscr{A}|_U)$ is clearly a ringe space, called induced over U by $(X, \mathscr{A})$, or the restriction of $(X, \mathscr{A})$ to U. If $j : U \to X$ is the injection and $j^\# : \mathscr{A} \to j_*(\mathscr{A}|_U)$ is the canonical homomorphism, we then have a *monomorphism* $(j, j^\#) : (U, \mathscr{A}|_U) \to (X, \mathscr{A})$ of ringed spaces, called the **canonical injection**. The composition of a morphism $f : (X, \mathscr{A}) \to (Y, \mathscr{B})$ with this canonical injection is said to be the restriction of f to U, and denoted by $f|_U$.

Example 1.3.1. Let $f : X \to Y$ be a continuous map of topological spaces. Consider the sheaves of continuous real valued functions C^0_X on X and C^0_Y on Y. We claim that there is a natural f-map $f^\# : C^0_Y \to C^0_X$ associated to f. Namely, we simply define it by the rule

$$C^0_Y(V) \to C^0_X(f^{-1}(V)), \quad h \mapsto h \circ f$$

Strictly speaking we should write $f^\#(h) = h \circ f|_{f^{-1}(V)}$. It is clear that this is an f-map of sheaves of $\mathbb{R}$-algebras.

Of course there are lots of other situations where there is a canonical morphism of ringed spaces associated to a geometrical type of morphism. For example, if M, N are C^∞-manifolds and $f : M \to N$ is a smooth map, then f induces a canonical morphism of ringed spaces $(M, C^\infty_M) \to (N, C^\infty_N)$.

We will not review the definition of the $\mathscr{A}$-modules for a ringed space $(X, \mathscr{A})$. If $\mathscr{A}$ is a sheaf of commutative rings, and we replace the module structure by the algebra structure in the definition of $\mathscr{A}$-modules, we obtain the definition of an $\mathscr{A}$-algebra over X. In other words, an $\mathscr{A}$-algebra (not necessarily commutative) is an $\mathscr{A}$-module $\mathscr{C}$ endowed with a homomorphism of $\mathscr{A}$-modules $\varphi : \mathscr{C} \otimes_{\mathscr{A}} \mathscr{C} \to \mathscr{C}$ and of a section e above X, such that:

(i) the diagram

$$\begin{tikzcd} \mathscr{C} \otimes_{\mathscr{A}} \mathscr{C} \otimes_{\mathscr{A}} \mathscr{C} \arrow[r, "\varphi\otimes 1"] \arrow[d, "1\otimes\varphi"'] & \mathscr{C} \otimes_{\mathscr{A}} \mathscr{C} \arrow[d, "\varphi"] \\ \mathscr{C} \otimes_{\mathscr{A}} \mathscr{C} \arrow[r, "\varphi"] & \mathscr{C} \end{tikzcd}$$

is commutative;

(ii) for any open subset $U \subseteq X$ and any section $s \in \Gamma(U, \mathscr{C})$, we have $\varphi((e|_U) \otimes s) = \varphi(s \otimes (e|_U)) = s$.

Saying that $\mathscr{C}$ is a commutative $\mathscr{A}$-algebra amounts to the fact that the diagram

$$\begin{tikzcd} \mathscr{C} \otimes_{\mathscr{A}} \mathscr{C} \arrow[rr, "\sigma"] \arrow[dr, "\varphi"'] & & \mathscr{C} \otimes_{\mathscr{A}} \mathscr{C} \arrow[dl, "\varphi"] \\ & \mathscr{C} & \end{tikzcd}$$

is commutative, where σ is the canonical symmetry of the tensor producr $\mathscr{C} \otimes_{\mathscr{A}} \mathscr{C}$.

The homomorphism of $\mathscr{A}$ are defined just as that of $\mathscr{A}$-modules, where we replace the "modules" by "algebras". If $\mathscr{M}$ is a sub-$\mathscr{A}$-module of an $\mathscr{A}$-algebra $\mathscr{C}$, the sub-$\mathscr{A}$-algebra of $\mathscr{C}$ **generated by** $\mathscr{M}$ is the sum of images of the homomorphisms $\otimes^n \mathscr{M} \to \mathscr{C}$ (for $n \geq 0$). This is the sheaf associated with the presheaf $U \mapsto \mathscr{B}(U)$ of algebras, where $\mathscr{B}(U)$ is the sub-algebra of $\mathscr{C}(U)$ generated by the sub-module $\mathscr{M}(U)$.

We say a sheaf of rings $\mathscr{A}$ over a topological space X is **reduced** (resp. **integral**) at a point x of X if the stalk $\mathscr{A}_x$ is a reduced ring (resp. integral ring). We say that $\mathscr{A}$ is reduced if it is reduced at every point of X. Recall that a ring A is called regular if for any prime ideal $\mathfrak{p}$ of A, the local ring $A_{\mathfrak{p}}$ is a regular local ring. We say a sheaf of ring $\mathscr{A}$ over X is **regular at a point x** (resp. **regular**) if the stalk $\mathscr{A}_x$ is a regular local ring (resp. if $\mathscr{A}$ is regular at every point). Finally, we say that a sheaf of rings $\mathscr{A}$ over X is **normal at a point x** (resp. **normal**) if the stalk $\mathscr{A}_x$ is an integrally closed ring (resp. if $\mathscr{A}$ is normal at every point). We say the ringed space $(X, \mathscr{A})$ is reduced (resp. normal, regular) if the structural sheaf $\mathscr{A}$ satisfies this property.

A **sheaf of graded rings** is by definition a sheaf or rings $\mathscr{A}$ which is the direct sum of a family $(\mathscr{A}_n)_{n\in\mathbb{Z}}$ of sheaves of abelian groups which satisfies the condition $\mathscr{A}_n\mathscr{A}_m \subseteq \mathscr{A}_{m+n}$. A graded $\mathscr{A}$-module is an $\mathscr{A}$-module $\mathscr{F}$ which is a direct sum of a family $(\mathscr{F}_n)_{n\in\mathbb{Z}}$ of sheaves of abelian groups, satisfying $\mathscr{A}_m\mathscr{F}_n \subseteq \mathscr{F}_{m+n}$.

Given a ringed space $(X, \mathscr{A})$ (commutative, and we shall not specify this condition further), we recall that definition of the bifunctors $\mathscr{F} \otimes_{\mathscr{A}} \mathscr{G}$, $\mathcal{H}om_{\mathscr{A}}(\mathscr{F}, \mathscr{G})$, and $\mathrm{Hom}_{\mathscr{A}}(\mathscr{F}, \mathscr{G})$ on the category of $\mathscr{A}$-modules, with values in the category of sheaves of abelian groups. the stalk $(\mathscr{F} \otimes_{\mathscr{A}} \mathscr{G})_x$ at any point $x \in X$ is identified canonically with $\mathscr{F}_x \otimes_{\mathscr{A}_x} \mathscr{G}_x$ and we define a functorial homomorphism $(\mathcal{H}om_{\mathscr{A}}(\mathscr{F}, \mathscr{G}))_x \mapsto \mathrm{Hom}_{\mathscr{A}_x}(\mathscr{F}_x, \mathscr{G}_x)$ which is, in general, neither injective nor surjective. These bifunctors are additive and, in particular, commutes with finite direct sums. The functor $\mathscr{F} \otimes_{\mathscr{A}} \mathscr{G}$ is right exact on $\mathscr{F}$ and $\mathscr{G}$, commutes with inductive limits, and $\mathscr{A} \otimes_{\mathscr{A}} \mathscr{G}$ (resp. $\mathscr{F} \otimes_{\mathscr{A}} \mathscr{A}$) is canonically identified with $\mathscr{G}$ (resp. $\mathscr{F}$). The functor $\mathcal{H}om_{\mathscr{A}}(\mathscr{F}, \mathscr{G})$ and $\mathrm{Hom}_{\mathscr{A}}(\mathscr{F}, \mathscr{G})$ are left exact on $\mathscr{F}$ and $\mathscr{G}$ (but note that $\mathcal{H}om_{\mathscr{A}}(-, \mathscr{G})$ and $\mathrm{Hom}_{\mathscr{A}}(\mathscr{F}, \mathscr{G})$ are contravariant). Moreover, $\mathcal{H}om_{\mathscr{A}}(\mathscr{A}, \mathscr{G})$ is canonically identified with $\mathscr{G}$, and for open subset $U \subseteq X$, we have

$$\Gamma(U, \mathcal{H}om_{\mathscr{A}}(\mathscr{F}, \mathscr{G})) = \mathrm{Hom}_{\mathscr{A}|_U}(\mathscr{F}|_U, \mathscr{G}|_U).$$

In particular, $\mathrm{Hom}_{\mathscr{A}}(\mathscr{A}, \mathscr{G})$ is identified with $\Gamma(X, \mathscr{G})$. For any $\mathscr{A}$-module $\mathscr{F}$, we denote by $\mathscr{F}^*$ the **dual** of $\mathscr{F}$, which is $\mathcal{H}om_{\mathscr{A}}(\mathscr{F}, \mathscr{A})$.

Finally, if $\mathscr{A}$ is a sheaf of rings and $\mathscr{F}$ is an $\mathscr{A}$-module, then $U \mapsto \bigwedge^p \Gamma(U, \mathscr{F})$ is a presheaf whose associated sheaf is an $\mathscr{A}$-module and is denoted by $\bigwedge^p \mathscr{F}$, called the **p-th exterior power** of $\mathscr{F}$. We can easily verify that the cannical map from the presheaf $U \mapsto \bigwedge^p \Gamma(U, \mathscr{F})$ to $\bigwedge^p \mathscr{F}$ is injective, and for $x \in X$, we have $(\bigwedge^p \mathscr{F})_x = \bigwedge^p (\mathscr{F}_x)$. It is clear that $\bigwedge^p \mathscr{F}$ is a covariant functor on $\mathscr{F}$. We can similarly define the functors $\boldsymbol{T}_p(\mathscr{F})$ and $\boldsymbol{S}_p(\mathscr{F})$, which are the **$p$-th tensor power** and **p-th symmetric power** of $\mathscr{F}$.

Let $\mathscr{I}$ be an ideal of $\mathscr{A}$ and $\mathscr{F}$ be an $\mathscr{A}$-module. Then we note that $\mathscr{I}\mathscr{F}$, the image of $\mathscr{I} \otimes_{\mathscr{A}} \mathscr{F}$ by the canonical map $\mathscr{I} \otimes_{\mathbb{Z}} \mathscr{F} \to \mathscr{F}$, is a sub-$\mathscr{A}$-module of $\mathscr{F}$. It is clear that for any $x \in X$, we have $(\mathscr{I}\mathscr{F})_x = \mathscr{I}_x\mathscr{F}_x$. It is immediate that $\mathscr{I}\mathscr{F}$ is also the $\mathscr{A}$-module associated sheaf of the presheaf $U \mapsto \Gamma(U, \mathscr{I})\Gamma(U, \mathscr{F})$. If $\mathscr{I}_1$, $\mathscr{I}_2$ are ideals of $\mathscr{A}$, we have $\mathscr{I}_1(\mathscr{I}_2\mathscr{F}) = (\mathscr{I}_1\mathscr{I}_2)\mathscr{F}$.

Let $(X_\lambda, \mathscr{A}_\lambda)_{\lambda\in L}$ be a family of ringed spaces; for each couple (λ, μ), suppose that we are given an open subset $V_{\lambda\mu} \subseteq X_\lambda$, and an isomorphism $\varphi_{\lambda\mu} : (V_{\mu\lambda}, \mathscr{A}_\mu|_{V_{\mu\lambda}}) \to (V_{\lambda\mu}, \mathscr{A}_\lambda|_{V_{\lambda\mu}})$ of ringed spaces, with $V_{\lambda\lambda} = X_\lambda$ and $\varphi_{\lambda\lambda}$ being the identity. Suppose moreover that, for any triple (λ, μ, ν), we have $\varphi_{\lambda\nu} = \varphi_{\lambda\mu} \circ \varphi_{\mu\nu}$ on the open subset $V_{\lambda\mu} \cap V_{\lambda\nu}$ (glueing condition for $\varphi_{\lambda\mu}$). We can then consider the topological space obtained by glueing (via the morphism $\varphi_{\lambda\mu}$) the X_λ along $V_{\lambda\mu}$. If we identify X_λ with the corresponding open subset X'_λ of X, the hypotheses implies that $V_{\lambda\mu} \cap V_{\lambda\nu}$, $V_{\mu\nu} \cap V_{\mu\lambda}$, $V_{\nu\lambda} \cap V_{\nu\mu}$ are identified with $X'_\lambda \cap X'_\mu \cap X'_\nu$. We an then transport the ringed space structure of X_λ to X'_λ, and if $\mathscr{A}'_\lambda$ is the sheaf of rings transported by

$\mathscr{A}_\lambda$, the $\mathscr{A}'_\lambda$ satisfies the glueing condition for sheaves and define a sheaf of rings $\mathscr{A}$ over X. We say that $(X, \mathscr{A})$ is the ringed space obtained by glueing $(X_\lambda, \mathscr{A}_\lambda)$ along $V_{\lambda\mu}$ via the morphisms $\varphi_{\lambda\mu}$.

Let $(X, \mathscr{O}_X)$ be a ringed space. For a section $s \in \Gamma(U, \mathscr{I}_X)$ over an open subset U of X to be **invertible**, it is necessary and sufficient that for any open cover (U_α) of U, the restriction of s to U_α is invertible in $\Gamma(U_\alpha, \mathscr{O}_X)$, in view of the uniqueness of inverse element. There then exists a sub-sheaf of multiplication groups $\mathscr{O}_X^\times$ of $\mathscr{O}_X$ such that, for any open subset U of X, $\Gamma(U, \mathscr{O}_X^\times)$ is the group of invertible elements of the ring $\Gamma(U, \mathscr{O}_X)$. For any $x \in X$, the stalk $(\mathscr{O}_X^\times)_x$ is the set of invertible elements of the ring $\mathscr{O}_{X,x}$, because if $s_x \in \mathscr{O}_{X,x}$ admits an inverse t_x in this ring, s_x and t_x are the germs of two sections s, t of $\mathscr{O}_X$ over a neighborhood V of x, and the relation $(st)_x = 1_x$ implies $st|_W = 1$ over a smaller neighborhood $W \subseteq V$ of x.

On the other hand, with the same notations, if s is a regular element (that is, not a zero divisor) of $\Gamma(U, \mathscr{O}_X)$ and V is an open subset of U, $s|_V$ is not necessarily regular in $\Gamma(V, \mathscr{O}_X)$, because if $(s|_V)t = 0$ for a section $t \in \Gamma(V, \mathscr{O}_X)$, t does not necessarily admits an extension to U. We denote by $\mathscr{S}(\mathscr{O}_X)$ the presheaf of sets such that $\mathscr{S}(\mathscr{O}_X)(U)$, for each open subset U of X, is the set of sections $s \in \Gamma(U, \mathscr{O}_X)$ whose restriction to *any* open subset $V \subseteq U$ is a regular element of the ring $\Gamma(V, \mathscr{O}_X)$. From this definition, it is clear that $\mathscr{S}(\mathscr{O}_X)$ is a sheaf, since if s is a section of $\mathscr{O}_X$ over U and $s|_V$ is regular in $\Gamma(V, \mathscr{O}_X)$ is regular for some open subset $V \subseteq U$, then s is regular in $\Gamma(U, \mathscr{O}_X)$. For a section $s \in \Gamma(U, \mathscr{O}_X)$, to be in $\mathscr{S}(\mathscr{O}_X)(U)$ amount to saying that for any open subset $V \subseteq U$, the map $t \mapsto (s|_V)$ on $\Gamma(V, \mathscr{O}_X)$ is injective (recall that we always assume that $\mathscr{O}_X$ is commutative). In this case, by the exactness of the functor $\varinjlim$ on the category of modules, it follows that for any $x \in U$, the germ s_x is regular in $\mathscr{O}_{X,x}$. Conversely, if this does not hold, then for some $x \in U$, then there exists a section $t \in \Gamma(V, \mathscr{O}_X)$ for some open subset $V \subseteq U$ such that $(s|_V)t = 0$, so $s \notin \mathscr{S}(\mathscr{O}_X)(U)$. We then say that $\mathscr{S}(\mathscr{O}_X)$ is the sheaf of sets defined by the condition that $\Gamma(U, \mathscr{O}_X)$ is the set of sections of $\Gamma(U, \mathscr{O}_X)$ whose germ at every point $x \in U$ is regular. We also note that the stalk $(\mathscr{S}(\mathscr{O}_X))_x$, which is contained in the set of regular elements of $\mathscr{O}_{X,x}$, does not necessarily equal to this set.

We say a ringed space $(X, \mathscr{O}_X)$ is **local** if for each $x \in X$, the stalk $\mathscr{O}_{X,x}$ is a local ring. In this case, we say $(X, \mathscr{O}_X)$ is a **locally ringed space**. We denote by $\mathfrak{m}_x$ the maximal idea lof $\mathscr{O}_{X,x}$ and $\kappa(x) = \mathscr{O}_{X,x}/\mathfrak{m}_x$ the residue field. For any $\mathscr{O}_X$-module $\mathscr{F}$, any open subset $U \subseteq X$, any point $x \in U$ and any section $f \in \Gamma(U, \mathscr{F})$, we denote by $f(x) \in \kappa(x)$ the class of the germ $f_x \in \mathscr{F}_x$ modulo $\mathfrak{m}_x\mathscr{F}_x$, and we say $f(x)$ is the **value** of f at x. The relation $f(x) = 0$ then signifies $f_x \in \mathfrak{m}_x\mathscr{F}_x$ (do not confuse with the condition $f_x = 0_x$). We denote by U_f the set of $x \in U$ such that $f(x) \neq 0$ (or equivalently $f_x \notin \mathfrak{m}_x\mathscr{F}_x$). We note that if $f \in \Gamma(X, \mathscr{F})$, then X_f is contained in $\operatorname{supp}(\mathscr{F})$.

Proposition 1.3.1. *Let $(X, \mathscr{O}_X)$ be a locally ringed space, $\mathscr{F}$ and $\mathscr{F}$ be two $\mathscr{O}_X$-module. For any $x \in X$, $(\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G})_x/\mathfrak{m}_x(\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G})_x$ is canonically identified with $(\mathscr{F}_x/\mathfrak{m}_x\mathscr{F}_x) \otimes_{\kappa(x)} (\mathscr{G}_x/\mathfrak{m}_x\mathscr{G}_x)$. If s (resp. t) is a section of $\mathscr{F}$ (resp. $\mathscr{G}$) over X, $(s \otimes t)(x)$ is identified with $s(x) \otimes t(x)$ and we have $X_{s\otimes t} = X_s \cap X_t$.*

Proof. In fact $(\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G})_x = \mathscr{F}_x \otimes_{\mathscr{O}_{X,x}} \mathscr{G}_x$, and $(\mathscr{F}_x \otimes_{\mathscr{O}_{X,x}} \mathscr{G}_x) \otimes_{\mathscr{O}_{X,x}} (\mathscr{O}_{X,x}/\mathfrak{m}_x)$ is canonically isomorphic to $(\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G})_x/\mathfrak{m}_x(\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G})_x$, hence to $(\mathscr{F}_x/\mathfrak{m}_x\mathscr{F}_x) \otimes_{\kappa(x)} (\mathscr{G}_x/\mathfrak{m}_x\mathscr{G}_x)$. The relation $X_{s\otimes t} = X_s \cap X_t$ then follows from the fact that a product $a \otimes b$ in a tensor product of vector spaces is nonzero if and only if a and b are both nonzero. □

Proposition 1.3.2. *Let $(X, \mathscr{O}_X)$ be a locally ringed space. The set of idempotents of the ring $\Gamma(X, \mathscr{O}_X)$ corresponds to the set of clopen subsets of X. In particular, for X to be connected, it is necessary and sufficent that $\Gamma(X, \mathscr{O}_X)$ has no nontrivial idempotents.*

Proof. Let U be a clopen subset of X. Then it corresponds to the section $s \in \Gamma(X, \mathscr{O}_X)$ such that $s|_V = 1$ for any open subset $V \subseteq U$ and $s|_V = 0$ for any open subset $V \subseteq X - U$; these open subsets by hypotheses form a base for the topology of X, so we define a section $s = e_U$ of $\mathscr{O}_X$

over X, which is an idempotent of $\Gamma(X, \mathscr{O}_X)$. Conversely, if s is an idempotent, for any $x \in X$, s_x is an idempotent of $\mathscr{O}_{X,x}$, hence equal to 0_x or 1_x because $\mathscr{O}_{X,x}$ is a local ring ($s_x(1-s_x) = 0$, and if $s_x \in \mathfrak{m}_x$, then $1 - s_x$ is invertible so $s_x = 0$). It is clear that the set U of $x \in X$ such that $s_x = 1_x$ is open, and so is the set $X - U$ of $x \in X$ such that $s_x = 0_x$, so U is a clopen subset of X and we have $s = e_U$. □

A morphism of locally ringed space $f : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ is defined to be a morphism of ringed spaces such that for each $x \in X$ the homomorphism $f_x^\# : \mathscr{O}_{Y,f(x)} \to \mathscr{O}_{X,x}$ is local. Note that by this definition, the category of locally ringed spaces is not a full subcategory of that of ringed spaces.

Proposition 1.3.3. *Let $f : X \to Y$ be a morphism of locally ringed spaces, x be a point of X, and $y = f(x)$. For any $\mathscr{O}_Y$-module $\mathscr{G}$, $(f^*(\mathscr{G}))_x/\mathfrak{m}_x(f^*(\mathscr{G}))_x$ is canonically identified with $(\mathscr{G}_y/\mathfrak{m}_y\mathscr{G}_y) \otimes_{\kappa(y)} \kappa(x)$. If t is a section of $\mathscr{G}$ over Y and $s = \rho_{\mathscr{G}}(t)$ is the corresponding section of $f^*(\mathscr{G})$ over X, then $s(x)$ is identified with $t(y) \otimes 1$ and we have $X_s = f^{-1}(Y_t)$.*

Proof. We have $(f^*(\mathscr{G}))_x = \mathscr{G}_y \otimes_{\mathscr{O}_{Y,y}} \mathscr{O}_X$, so $(f^*(\mathscr{G}))_x/\mathfrak{m}_x(f^*(\mathscr{G}))_x$ is identified with $\mathscr{G}_y \otimes_{\mathscr{O}_{Y,y}} \kappa(x)$. The homomorphism $\mathscr{O}_{Y,y} \to \mathscr{O}_{X,x}$ is local by hypotheses, so $\mathfrak{m}_y$ annihilates $\kappa(x)$ and $\mathscr{G}_y \otimes_{\mathscr{O}_{Y,y}} \kappa(x)$ is isomorphic to $(\mathscr{G}_y/\mathfrak{m}_y\mathscr{G}_y) \otimes_{\kappa(y)} \kappa(x)$. The last assertion follows from the fact that $t(y) \otimes 1 = 0$ is equivalent to $t(y) = 0$. □

1.3.2 Direct image of $\mathscr{A}$-modules

Let $(X, \mathscr{A})$ and $(Y, \mathscr{B})$ be ringed spaces, f be a morphism $(X, \mathscr{A}) \to (Y, \mathscr{B})$. Then $f_*(\mathscr{A})$ is a sheaf of rings over Y, and $f^\#$ is a homomorphism $\mathscr{B} \to f_*(\mathscr{A})$ of sheaf of rings. Let $\mathscr{F}$ be an $\mathscr{A}$-module; the direct image $f_*(\mathscr{F})$ is then a sheaf of abelian groups over Y. Moreover, for any open subset $U \subseteq Y$,

$$\Gamma(U, f_*(\mathscr{F})) = \Gamma(f^{-1}(U), \mathscr{F})$$

is endowed with a module structure over the ring $\Gamma(U, f_*(\mathscr{A})) = \Gamma(f^{-1}(U), \mathscr{A})$. These bilinear maps are compatible with restrictions, so $f_*(\mathscr{F})$ becomes an $f_*(\mathscr{A})$-module. The homomorphism $f^\# : \mathscr{B} \to f_*(\mathscr{A})$ then makes $f_*(\mathscr{F})$ a $\mathscr{B}$-module. We say that this $\mathscr{B}$-module is the direct image of $\mathscr{F}$ under the morphism f, still denoted by $f_*(\mathscr{F})$. If $\mathscr{F}_1, \mathscr{F}_2$ are two $\mathscr{A}$-modules over X and $u : \mathscr{F}_1 \to \mathscr{F}_2$ is an $\mathscr{A}$-homomorphism, it is immediate that $f_*(u)$ is an $f_*(\mathscr{A})$-homomorphism $f_*(\mathscr{F}_1) \to f_*(\mathscr{F}_2)$, and a fortiori a $\mathscr{B}$-homomorphism, also denoted by $f_*(u)$. We then see that f_* is a covariant funtor from the category of $\mathscr{A}$-modules to that of $\mathscr{B}$-modules. Moreover, it is immediate that this functor is left exact.

Over $f_*(\mathscr{A})$, the $\mathscr{B}$-module structure and the sheaf of rings structure define a structure of $\mathscr{B}$-algebras; we denote by $f_*(\mathscr{A})$ this $\mathscr{B}$-algebra. Let $(Z, \mathscr{C})$ be a third ringed space, $g : (Y, \mathscr{B}) \to (Z, \mathscr{C})$ be a morphism; if $h = g \circ f$ is the composition morphism, then we have $h_* = g_* \circ f_*$.

Let $\mathscr{M}, \mathscr{N}$ be two $\mathscr{A}$-modules. For any open subset U of Y, we have a canonical map

$$\Gamma(f^{-1}(U), \mathscr{M}) \times \Gamma(f^{-1}(U), \mathscr{N}) \to \Gamma(f^{-1}(U), \mathscr{M} \otimes_{\mathscr{A}} \mathscr{N})$$

which is bilinear over the ring $\Gamma(f^{-1}(U), \mathscr{A}) = \Gamma(U, f_*(\mathscr{A}))$, and a fortiori over $\Gamma(U, \mathscr{B})$; this defines a homomorphism

$$\Gamma(U, f_*(\mathscr{M})) \otimes_{\Gamma(U,\mathscr{B})} \Gamma(U, f_*(\mathscr{N})) \to \Gamma(U, f_*(\mathscr{M} \otimes_{\mathscr{A}} \mathscr{N}))$$

and as we can verify that these homomorphisms are compatible with restrictions, we obtain a canonical homomorphism of $\mathscr{B}$-modules

$$f_*(\mathscr{M}) \otimes_{\mathscr{B}} f_*(\mathscr{N}) \to f_*(\mathscr{M} \otimes_{\mathscr{A}} \mathscr{N}) \tag{1.3.1}$$

which in general is neither injective nor surjective. If $\mathscr{P}$ is a third $\mathscr{A}$-module, we verify that the diagram

$$\begin{array}{ccc} f_*(\mathscr{M}) \otimes_{\mathscr{B}} f_*(\mathscr{N}) \otimes_{\mathscr{B}} f_*(\mathscr{P}) & \longrightarrow & f_*(\mathscr{M} \otimes_{\mathscr{A}} \mathscr{N}) \otimes_{\mathscr{B}} f_*(\mathscr{P}) \\ \downarrow & & \downarrow \\ f_*(\mathscr{M}) \otimes_{\mathscr{B}} f_*(\mathscr{N} \otimes_{\mathscr{A}} \mathscr{P}) & \longrightarrow & f_*(\mathscr{M} \otimes_{\mathscr{A}} \mathscr{N} \otimes_{\mathscr{A}} \mathscr{P}) \end{array}$$

is commutative.

Let $\mathscr{M}, \mathscr{N}$ be two $\mathscr{A}$-modules. For any open $U \subseteq Y$, we have by definition

$$\Gamma(f^{-1}(U), \mathcal{H}om_{\mathscr{A}}(\mathscr{M}, \mathscr{N})) = \mathrm{Hom}_{\mathscr{A}|_V}(\mathscr{M}|_V, \mathscr{N}|_V),$$

where we put $V = f^{-1}(U)$. The map $u \mapsto f_*(u)$ is a homomorphism

$$\mathrm{Hom}_{\mathscr{A}|_V}(\mathscr{M}|_V, \mathscr{N}|_V) \to \mathrm{Hom}_{\mathscr{B}|_U}(f_*(\mathscr{M})|_U, f_*(\mathscr{N})|_U)$$

for the structure of $\Gamma(U, \mathscr{B})$-modules. These homomorphisms are compatible with restrictions, so define a canonical homomorphism of $\mathscr{B}$-modules

$$f_*(\mathcal{H}om_{\mathscr{A}}(\mathscr{M}, \mathscr{N})) \to \mathcal{H}om_{\mathscr{B}}(f_*(\mathscr{M}), f_*(\mathscr{N})).$$

If $\mathscr{C}$ is an $\mathscr{A}$-algebra, the composition homomorphism

$$f_*(\mathscr{C}) \otimes_{\mathscr{B}} f_*(\mathscr{C}) \to f_*(\mathscr{C} \otimes_{\mathscr{A}} \mathscr{C}) \to f_*(\mathscr{C})$$

defines on $f_*(\mathscr{C})$ a $\mathscr{B}$-algebra structure. We see that if $\mathscr{M}$ is a $\mathscr{C}$-module, $f_*(\mathscr{M})$ is an $f_*(\mathscr{C})$-module.

Consider in particular the case where X is a closed subspace of Y and f is the canonical injection $j : X \to Y$. If $\mathscr{B}' = \mathscr{B}|_X = j^{-1}(\mathscr{B})$ is the restriction of $\mathscr{B}$ to X, an $\mathscr{A}$-module $\mathscr{M}$ can be considered as a $\mathscr{B}'$-module via the homomorphism $f^\sharp : \mathscr{B}' \to \mathscr{A}$; $f_*(\mathscr{M})$ is then the $\mathscr{B}$-module inducing $\mathscr{M}$ on X and 0 elsewhere. If $\mathscr{N}$ is a second $\mathscr{A}$-module, $f_*(\mathscr{M}) \otimes_{\mathscr{B}} f_*(\mathscr{N})$ is identified with $f_*(\mathscr{M} \otimes_{\mathscr{B}'} \mathscr{N})$ and $\mathcal{H}om_{\mathscr{B}}(f_*(\mathscr{M}), f_*(\mathscr{N}))$ with $f_*(\mathcal{H}om_{\mathscr{B}'}(\mathscr{M}, \mathscr{N}))$.

Let $(S, \mathscr{O}_S)$ be a ringed space and let $f : (X, \mathscr{O}_X) \to (S, \mathscr{O}_S)$ be a morphism of ringed spaces. If we fix S, we denote by $\mathscr{A}(X)$ (if there is no confusion) the direct image $f_*(\mathscr{O}_X)$, which is an $\mathscr{O}_S$-algebra. For any open subset U of S, we have $\mathscr{A}(f^{-1}(U)) = \mathscr{A}(X)|_U$. Simialrly, for any $\mathscr{O}_X$-module $\mathscr{F}$ (resp. any $\mathscr{O}_X$-algebra $\mathscr{B}$), we denote by $\mathscr{A}(\mathscr{F})$ (resp. $\mathscr{A}(\mathscr{B})$) the direct image $f_*(\mathscr{F})$ (resp. $f_*(\mathscr{B})$), which is an $\mathscr{A}(X)$-module (resp. an $\mathscr{A}(X)$-algebra) and also an $\mathscr{O}_S$-module (resp. $\mathscr{O}_S$-algebra).

We can define $\mathscr{A}(X)$ as a contravariant functor on X, from the category of S-ringed spaces to the category of $\mathscr{O}_S$-algebras. In fact, consider two morphisms $f : X \to S$, $g : Y \to S$ and let $h : X \to Y$ be a morphism, such that the diagram

$$\begin{tikzcd} X \arrow[rr, "h"] \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & S & \end{tikzcd}$$

is commutative. Then by definition $h^\# : \mathscr{O}_Y \to h_*(\mathscr{O}_X)$ is a homomorphism of sheaves of rings; we also deduce a homomorphism $g_*(h^\#) : g_*(\mathscr{O}_Y) \to g_*(h_*(\mathscr{O}_X)) = f_*(\mathscr{O}_X)$ of $\mathscr{O}_S$-algebras, which is a homomorphism $\mathscr{A}(Y) \to \mathscr{A}(X)$, denoted by $\mathscr{A}(h)$. If $h' : Y \to Z$ is a second S-morphism, then we have $\mathscr{A}(h' \circ h) = \mathscr{A}(h) \circ \mathscr{A}(h')$.

Now let $\mathscr{F}$ be an $\mathscr{O}_X$-module, $\mathscr{G}$ be an $\mathscr{O}_Y$-module, and $u : \mathscr{G} \to h_*(\mathscr{F})$ be a homomorphism of $\mathscr{O}_Y$-modules. Then $g_*(u) : g_*(\mathscr{G}) \to g_*(h_*(\mathscr{F})) = f_*(\mathscr{F})$ is a homomorphism $\mathscr{A}(\mathscr{G}) \to$

$\mathscr{A}(\mathscr{F})$ of $\mathscr{O}_S$-modules, which we denote by $\mathscr{A}(u)$. Moreover, the couple $(\mathscr{A}(h), \mathscr{A}(u))$ is a bi-homomorphism from $\mathscr{A}(Y)$-module $\mathscr{A}(\mathscr{G})$ to the $\mathscr{A}(X)$-module $\mathscr{A}(\mathscr{F})$. The ringed space S being fixed, we can consider the couples $(X, \mathscr{F})$, where X is a S-ringed space and $\mathscr{F}$ is an $\mathscr{O}_X$-module, which form a category, and define a morphism $(X, \mathscr{F}) \to (Y, \mathscr{G})$ to be a couple (h, u), where $h : X \to Y$ is an S-morphism and $u : \mathscr{G} \to h_*(\mathscr{F})$ is a homomorphism of $\mathscr{O}_Y$-modules. We can then say that $(X, \mathscr{F}) \mapsto (\mathscr{A}(X), \mathscr{A}(\mathscr{F}))$ is a contravariant functor from this category to the cateogry of couples formed by an $\mathscr{O}_S$-algebra and a module of this algebra.

1.3.3 Inverse image of a $\mathscr{B}$-module

Let $f : (X, \mathscr{A}) \to (Y, \mathscr{B})$ be a morphism of ringed spaces. Let $\mathscr{G}$ be a $\mathscr{B}$-module and $f^{-1}(\mathscr{G})$ be the inverse image of $\mathscr{G}$, which is a sheaf of abelian groups over X. The definition of sections of $f^{-1}(\mathscr{G})$ and of $f^{-1}(\mathscr{B})$ shows that $f^{-1}(\mathscr{G})$ is canonically endowed with an $f^{-1}(\mathscr{B})$-module structure. On the other hand, the homomorphism $f^\sharp : f^{-1}(\mathscr{B}) \to \mathscr{A}$ endows $\mathscr{A}$ with an $f^{-1}(\mathscr{B})$-module structure. The tensor product $f^{-1}(\mathscr{G}) \otimes_{f^{-1}(\mathscr{B})} \mathscr{A}$ is then an $\mathscr{A}$-module, called the inverse image of $\mathscr{G}$ under the morphism $(f, f^\#)$ and denoted by $f^*(\mathscr{G})$. If $\mathscr{G}_1, \mathscr{G}_2$ are two $\mathscr{B}$-modules over Y and $v : \mathscr{G}_1 \to \mathscr{G}_2$ is a $\mathscr{B}$-homomorphism, $f^{-1}(v)$ is then an $f^{-1}(\mathscr{B})$-homomorphism from $f^{-1}(\mathscr{G}_1)$ to $f^{-1}(\mathscr{G}_2)$; therefore $f^{-1}(v) \otimes 1_{\mathscr{A}}$ is an $\mathscr{A}$-homomorphism $f^*(\mathscr{G}_1) \to f^*(\mathscr{G}_2)$, which we denote by $f^*(v)$. We then define f^* as a covariant functor from the category of $\mathscr{B}$-modules to that of $\mathscr{A}$-modules. Note that this functor (contrary to f^{-1}) is not in general exact, and is only right exact, since the tensor product with $\mathscr{A}$ is only right exact. We will see that f^* is the left adjoint of the functor f_*. For any $x \in X$, we have $(f^*(\mathscr{G}))_x = \mathscr{G}_{f(x)} \otimes_{\mathscr{B}_{f(x)}} \mathscr{A}_x$ in view of the formula for the stalk of tensor products. The support of $f^*(\mathscr{G})$ is tens contained in $f^{-1}(\mathrm{supp}(\mathscr{G}))$.

Let $(Z, \mathscr{C})$ be a third ringed space and $g : (Y, \mathscr{B}) \to (Z, \mathscr{C})$ be a morphism. Then if $h = g \circ f$ is the composition morphism, then it follows from the definitions that $h^* = f^* \circ g^*$.

Let $(\mathscr{G}_\lambda)$ be a inductive system of $\mathscr{B}$-modules, and $\mathscr{G} = \varinjlim \mathscr{G}_\lambda$ be the inductive limit. The canonical homomorphisms $\mathscr{G}_\lambda \to \mathscr{G}$ define an $f^{-1}(\mathscr{B})$-homomorphism $f^{-1}(\mathscr{G}_\lambda) \to f^{-1}(\mathscr{G})$, which gives a canonical homomorphism $\varinjlim f^{-1}(\mathscr{G}_\lambda) \to f^{-1}(\mathscr{G})$. As taking stalk commutes with inductive limits, this canonical homomorphism is bijective. Moreover, tensor product also commutes with inductive limits, and we then have a canonical functorial isomorphism $\varinjlim f^*(\mathscr{G}_\lambda) \cong f^*(\varinjlim \mathscr{G}_\lambda)$ of $\mathscr{A}$-modules.

On the other hand, for a finite direct sum $\bigoplus_i \mathscr{G}_i$ of $\mathscr{B}$-modules, it is clear that

$$f^*(\bigoplus_i \mathscr{G}_i) = \bigoplus_i f^*(\mathscr{G}_i).$$

By passing to inductive limits, we then deduce that, in view of the preceding, that this equality holds for arbitrary direct sums.

Let $\mathscr{G}_1, \mathscr{G}_2$ be two $\mathscr{B}$-modules; from the definition of inverse images of sheaves of abelian groups we deduce a canonical homomorphism

$$f^{-1}(\mathscr{G}_1) \otimes_{f^{-1}(\mathscr{B})} f^{-1}(\mathscr{G}_2) \to f^{-1}(\mathscr{G}_1 \otimes_{\mathscr{B}} \mathscr{G}_2)$$

of $f^{-1}(\mathscr{B})$, and since the stalk of a tensor product is the rensor product of stalks, this homomorphism is an isomorphism. By tensoring with $\mathscr{A}$, we then deduce a canonical isomorphism

$$f^*(\mathscr{G}_1) \otimes_{\mathscr{A}} f^*(\mathscr{G}_2) \xrightarrow{\sim} f^*(\mathscr{G}_1 \otimes \mathscr{G}_2). \tag{1.3.2}$$

Let $\mathscr{C}$ be a $\mathscr{B}$-algebra. The algebra structure over $\mathscr{C}$ is given by a $\mathscr{B}$-homomorphism $\mathscr{C} \otimes_{\mathscr{B}} \mathscr{C} \to \mathscr{C}$ satisfying the associativity and commutativity conditions (which can be verified on stalks); the isomorphism (1.3.2) then provides a homomorphism $f^*(\mathscr{C}) \otimes_{\mathscr{A}} f^*(\mathscr{C}) \to f^*(\mathscr{C})$ satisfying the same conditions, whence $f^*(\mathscr{C})$ is endowed with an $\mathscr{A}$-algebra structure. In particular, it follows from the definitions that the $\mathscr{A}$-algebra $f^*(\mathscr{B})$ is equal to $\mathscr{A}$. If $\mathscr{C}_1, \mathscr{C}_2$ are two $\mathscr{B}$-algebras and $v : \mathscr{C}_1 \to \mathscr{C}_2$ is a homomorphism of $\mathscr{B}$-algebras, then $f^*(v) : f^*(\mathscr{C}_1) \to f^*(\mathscr{C}_2)$ is a homomorphism of $\mathscr{A}$-algebras.

Similarly, if $\mathscr{M}$ is a $\mathscr{C}$-module, then a $\mathscr{B}$-module structure is given by a $\mathscr{B}$-homomorphism $\mathscr{C} \otimes_{\mathscr{B}} \mathscr{M} \to \mathscr{M}$ satisfying the associativity condition; by transporting structure, we see that $f^*(\mathscr{C})$ is endowed with a $f^*(\mathscr{C})$-module structure.

Let $\mathscr{I}$ be an ideal of $\mathscr{B}$; as the functor f^{-1} is exact, the $f^{-1}(\mathscr{B})$-module $f^{-1}(\mathscr{I})$ is canonically identified an ideal of $f^{-1}(\mathscr{B})$; the canonical injection $f^{-1}(\mathscr{I}) \to f^{-1}(\mathscr{B})$ then gives a homomorphism of $\mathscr{A}$-modules

$$f^*(\mathscr{I}) = f^{-1}(\mathscr{I}) \otimes_{f^{-1}(\mathscr{B})} \mathscr{A} \to \mathscr{A};$$

we denote by $f^*(\mathscr{I})\mathscr{A}$, or simply $\mathscr{I}\mathscr{A}$ if there is no confusion, the image of $f^*(\mathscr{I})$ under this hommorphism. We then have $\mathscr{I}\mathscr{A} = f^\sharp(f^{-1}(\mathscr{I}))\mathscr{A}$ and in particular, for any $x \in X$, $(\mathscr{I}\mathscr{A})_x = f_x^\sharp(\mathscr{I}_{f(x)})\mathscr{A}_x$, in view of the canonical identification of the stalk $f^{-1}(\mathscr{I})$. If $\mathscr{I}_1, \mathscr{I}_2$ are two ideals of $\mathscr{B}$, we have $(\mathscr{I}_1\mathscr{I}_2)\mathscr{A} = \mathscr{I}_1(\mathscr{I}_2\mathscr{A}) = (\mathscr{I}_1\mathscr{A})(\mathscr{I}_2\mathscr{A})$. If $\mathscr{F}$ is an $\mathscr{A}$-module, we put $\mathscr{I}\mathscr{F} = (\mathscr{I}\mathscr{A})\mathscr{F}$.

Remark 1.3.1. Over any topological space, we can define a canonical sheaf of rings $\mathbb{Z}_X$, which is the constant sheaf associated the presheaf $U \mapsto \mathbb{Z}$. It is clear thet the sheaves of abelian groups over X are identified with the $\mathbb{Z}_X$-modules over the ringed space $(X, \mathbb{Z}_X)$, and we can in particular consider the tensor product $\mathscr{F} \otimes_{\mathbb{Z}_X} \mathscr{G}$ of two sheaves of rings $\mathscr{F}, \mathscr{G}$ over X. If $f : X \to Y$ is a continuous map, for any open subset V of Y, we have a canonical homomorphism $\mathbb{Z} \to \Gamma(f^{-1}(V), \mathbb{Z}_X)$ of rings and this defines a homomorphism $f^\# : \mathbb{Z}_Y \to f_*(\mathbb{Z}_X)$ of sheaves of rings over Y. We then obtain a morphism $(f, f^\#) : (X, \mathbb{Z}_X) \to (Y, \mathbb{Z}_Y)$ of ringed spaces. If $\mathscr{F}$ is a sheaf of abelian groups over Y, $f^*(\mathscr{F})$ is canonically identified with $f^{-1}(\mathscr{F})$; if $\mathscr{F}, \mathscr{G}$ are two sheves of abelian groups over Y, we then have a canonical isomorphism $f^{-1}(\mathscr{F} \otimes_{\mathbb{Z}_Y} \mathscr{G}) = f^{-1}(\mathscr{F}) \otimes_{\mathbb{Z}_X} f^{-1}(\mathscr{G})$. We then deduce that if f is a quasi-homeomorphism, f^{-1} is not only an equivalence from the category of sheaf of abelian groups over Y to that of sheaf of abelian groups over X, but also an equivalence from the category of sheaf of rings over Y to that of sheaf of rings over X.

Given two ringed spaces $(X, \mathscr{O}_X)$, $(Y, \mathscr{O}_Y)$, we say a morphism $f : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ is a **quasi-isomorphism** if f is a quasi-homeomorphism and $f^\sharp : f^{-1}(\mathscr{O}_Y) \to \mathscr{O}_X$ is an isomorphism of sheaf of rings. If this is the case, the ringed space $(X, \mathscr{O}_X)$ is entirely determined up to isomorphism, by $(Y, \mathscr{O}_Y)$, the space X, and the quasi-homeomorphism.

If f is a quasi-isomorphism of ringed spaces, the functor $\mathscr{F} \to f^*(\mathscr{F})$ is an equivalence from category of $\mathscr{O}_Y$-modules to the category of $\mathscr{O}_X$-modules, since $f^*(\mathscr{F})$ is identified with $f^{-1}(\mathscr{F})$. We then conclude for example the isomorphisms of bi-∂-functors

$$\mathrm{Ext}^i_{\mathscr{O}_Y}(\mathscr{F}, \mathscr{G}) \xrightarrow{\sim} \mathrm{Ext}^i_{\mathscr{O}_X}(f^*(\mathscr{F}), f^*(\mathscr{G})).$$

In general, we can say that the usual constructions of the sheaf theory and homological algebra on the ringed space Y or X, are equivalent.

1.3.4 Relations of direct images and inverse images

Again we let $f : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ be a morphism of ringed spaces. By definition, a homomorphism $u : \mathscr{G} \to f_*(\mathscr{F})$ of $\mathscr{B}$-modules is called an f**-morphism** from $\mathscr{G}$ to $\mathscr{F}$ and we denote it by $u : \mathscr{G} \to \mathscr{F}$ if there is no confusion. Given such a homomorphism, for any couple (U, V) where U is an open subset of X and V is an open subset of Y such that $f(U) \subseteq V$, a **homomorphism** $u_{U,V} : \Gamma(V, \mathscr{G}) \to \Gamma(U, \mathscr{F})$ of $\Gamma(V, \mathscr{B})$ modules, where $\Gamma(U, \mathscr{F})$ is considered as a $\Gamma(V,\mathscr{B})$-module via the ring homomorphism $f^\sharp_{U,V}:\Gamma(V,\mathscr{B})\to\Gamma(U,\mathscr{A})$. The homomorphisms $u_{U,V}$ fits into the commutative diagram

$$\begin{tikzcd}
\mathscr{G}(V) \arrow[r, "u_{U,V}"] \arrow[d] & \mathscr{F}(U) \arrow[d] \\
\mathscr{G}(V') \arrow[r, "u_{U',V'}"] & \mathscr{F}(U')
\end{tikzcd}$$

for $U'\subseteq U$, $V'\subseteq V$, $f(U')\subseteq V'$. Moreover, for the homomorphism u, it suffices to define $u_{U,V}$ for U (resp. V) in a base $\mathcal{B}$ (resp. $\mathcal{B}'$) of the topology of X (resp. Y). Let $g:(Y,\mathscr{B})\to(Z,\mathscr{C})$ be another morphism and $h=g\circ f$. Let $\mathscr{H}$ be a $\mathscr{C}$-module, and $v:\mathscr{H}\to g_*(\mathscr{G})$ be a g-morphism; then

$$w:\mathscr{H}\xrightarrow{v}g_*(\mathscr{G})\xrightarrow{g_*(u)}g_*(f_*(\mathscr{F}))$$

is an h-morphism which is called the **composition** of u and v.

We will now see that there is a canonical isomorphism of bifunctors on $\mathscr{F}$ and $\mathscr{G}$

$$\mathrm{Hom}_{\mathscr{A}}(f^*(\mathscr{G}),\mathscr{F})\xrightarrow{\sim}\mathrm{Hom}_{\mathscr{B}}(\mathscr{G},f_*(\mathscr{F}))\tag{1.3.3}$$

which we denote by $v\mapsto v^\flat$, and the inverse of this isomorphism is denoted by $u\mapsto u^\sharp$. The definition is the folowing: for an $\mathscr{A}$-homomorphism $v:f^*(\mathscr{G})\to\mathscr{F}$, by composing with the canonical homomorphism $f^{-1}(\mathscr{G})\to f^*(\mathscr{G})$, we obtain a hommorphism $\tilde{v}:f^{-1}(\mathscr{G})\to\mathscr{F}$ of sheaves of abelian groups, which is also a homomorphism of $f^{-1}(\mathscr{B})$-modules. We then deduce a homomorphism $\tilde{v}^\flat:\mathscr{G}\to f_*(\mathscr{F})$ by the adjointness of f_* and f^{-1}, which is a homomorphism of $\mathscr{B}$-modules, and we denote by $v^\flat$. Similarly, for a $\mathscr{B}$-homomorphism $u:\mathscr{G}\to f_*(\mathscr{F})$, we deduce a homomorphism $u^\sharp:f^{-1}(\mathscr{G})\to\mathscr{F}$ of $f^{-1}(\mathscr{B})$-modules, whence by tensoring with $\mathscr{A}$ a homomorphism of $\mathscr{A}$-modules $f^*(\mathscr{G})\to\mathscr{F}$, still denoted by $u^\sharp$. It is immediate that $(u^\sharp)^\flat=u$ and $(v^\flat)^\sharp=v$, as well as the functoriality in $\mathscr{F}$ of the isomorphism $v\mapsto v^\flat$. The functoriality in $\mathscr{G}$ of $u\mapsto u^\sharp$ can be then deduced formally and we then see that f^* is left adjoint to f_*.

If we choose v to be the identify homomorphism of $f^*(\mathscr{G})$, then $v^\flat$ is a homomorphism

$$\rho_{\mathscr{G}}:\mathscr{G}\to f_*(f^*(\mathscr{G}));$$

if we choose u to be the identify homomorphism on $f_*(\mathscr{F})$, then $u^\sharp$ is a homomorphism

$$\sigma_{\mathscr{F}}:f^*(f_*(\mathscr{F}))\to\mathscr{F};$$

these homomorphism are called canonical, and are in general neither injective nor surjective. As always, for a homomorphism $v:\mathscr{G}\to f_*(\mathscr{F})$, we have a canonical factorization

$$v^\flat:\mathscr{G}\xrightarrow{\rho_{\mathscr{G}}}f_*(f^*(\mathscr{G}))\xrightarrow{f_*(v)}f_*(\mathscr{F})\tag{1.3.4}$$

and for a homomorphism $u:\mathscr{G}\to f_*(\mathscr{F})$, we have a canonical factorization

$$u^\sharp:f^*(\mathscr{G})\xrightarrow{f^*(u)}f^*(f_*(\mathscr{F}))\xrightarrow{\sigma_{\mathscr{F}}}\mathscr{F}\tag{1.3.5}$$

Now let $g:(Y,\mathscr{B})\to(Z,\mathscr{C})$ be another morphism and $h=g\circ f$ be the composition. Let $\mathscr{H}$ be a $\mathscr{C}$-module, $u:\mathscr{G}\to\mathscr{F}$ and $v:\mathscr{H}\to\mathscr{G}$ be homomorphisms, and $w=g_*(u)\circ v$ be the composition of u and v. Then $w^\sharp$ is the composition homomorphism

$$w^\sharp:f^*(g^*(\mathscr{H}))\xrightarrow{f^*(v^\sharp)}f^*(\mathscr{G})\xrightarrow{u^\sharp}\mathscr{F}.\tag{1.3.6}$$

To verify this, we use the description of $w^\sharp$ in (1.3.5): $w^\sharp$ is given by the following diagram

$$\begin{tikzcd}
f^*(g^*(\mathscr{H})) \arrow[r, "f^*(g^*(v))"] \arrow[rr, bend left, "f^*(g^*(w))"] \arrow[dr, "f^*(v^\sharp)"'] & f^*(g^*(g_*(\mathscr{G}))) \arrow[r, "f^*(g^*(g_*(u)))"] \arrow[d, "f^*(\sigma_{\mathscr{G}})"'] & f^*(g^*(g_*(f_*(\mathscr{F})))) \arrow[d, "f^*(\sigma_{f_*(\mathscr{F})})"] \arrow[dr, "\sigma_{\mathscr{F}}"] & \\
 & f^*(\mathscr{G}) \arrow[r, "f^*(u)"] \arrow[rr, bend right, "u^\sharp"'] & f^*(f_*(\mathscr{F})) \arrow[r, "\sigma_{\mathscr{F}}"] & \mathscr{F}
\end{tikzcd}$$

The central square is immediately verified to be commutative by the naturality of σ, whence the assertion.

We note that if s is a section of $\mathscr{G}$ over an open subset V of Y, $\rho_{\mathscr{G}}(s)$ is the section $s' \otimes 1$ of $f^*(\mathscr{G})$ over $f^{-1}(V)$, where s' is the section such that $s'_x = s_{f(x)}$ for any $x \in f^{-1}(V)$. We say that $\rho_{\mathscr{G}}(s)$ is the **inverse image** of s under f. Note also that if $u : \mathscr{G} \to f_*(\mathscr{F})$ is a homomorphism, it defines for each $x \in X$ a homomorphism $u_x : \mathscr{G}_{f(x)} \to \mathscr{F}_x$ over stalks, obtained by composing $(u^\sharp)_x : (f^*(\mathscr{G}))_x \to \mathscr{F}_x$ and the canonical homomorphism $s_x \mapsto s_x \otimes 1$ from $\mathscr{G}_{f(x)}$ to $(f^*(\mathscr{G}))_x = \mathscr{G}_{f(x)} \otimes_{\mathscr{B}_{f(x)}} \mathscr{A}_x$. The homomorphism u_x is also obtained by taking inductive limit of the homomorphisms $\Gamma(V, \mathscr{G}) \xrightarrow{u} \Gamma(f^{-1}(V), \mathscr{F}) \to \mathscr{F}_x$, where V runs through neighborhood of $f(x)$.

Let $\mathscr{F}_1, \mathscr{F}_2$ be $\mathscr{A}$-modules, $\mathscr{G}_1, \mathscr{G}_2$ be $\mathscr{B}$-modules, u_i ($i = 1, 2$) be a homomorphism from $\mathscr{G}_i$ to $\mathscr{F}_i$. We denote by $u_1 \otimes u_2$ the homomorphism $u : \mathscr{G}_1 \otimes_{\mathscr{B}} \mathscr{G}_2 \to \mathscr{F}_1 \otimes_{\mathscr{A}} \mathscr{F}_2$ such that $u^\sharp = (u_1)^\sharp \otimes (u_2)^\sharp$; we verify that u is also the composition

$$\mathscr{G}_1 \otimes_{\mathscr{B}} \mathscr{G}_2 \to f_*(\mathscr{F}_1) \otimes_{\mathscr{B}} f_*(\mathscr{F}_2) \to f_*(\mathscr{F}_1 \otimes_{\mathscr{A}} \mathscr{F}_2),$$

where the first morphism is the ordinary tensor product $u_1 \otimes_{\mathscr{B}} u_2$ and the second homomorphism is the canonical homomorphism (1.3.1).

Let $(\mathscr{G}_\lambda)_{\lambda \in L}$ be an inductive system of $\mathscr{B}$-modules, and, for $\lambda \in L$, let u_λ be a homomorphism $\mathscr{G}_\lambda \to f_*(\mathscr{F})$, which form an inductive system; put $\mathscr{G} = \varinjlim \mathscr{G}_\lambda$ and $u = \varinjlim u_\lambda$. Then $(u^\lambda)^\sharp$ form an inductive system of homomorphisms $f^*(\mathscr{G}_\lambda) \to \mathscr{F}$, and the inductive limit of this system is just $u^\sharp$, since taking tensor products commutes with inductive limits.

Let $\mathscr{M}, \mathscr{N}$ be two $\mathscr{B}$-modules, V be an open subset of Y, and $U = f^{-1}(V)$. The map $v \mapsto f^*(v)$ is a homomorphism

$$\mathrm{Hom}_{\mathscr{B}|_V}(\mathscr{M}|_V, \mathscr{N}|_V) \to \mathrm{Hom}_{\mathscr{A}|_U}(f^*(\mathscr{M})|_U, f^*(\mathscr{N})|_U)$$

for the structure of $\Gamma(V, \mathscr{B})$-modules: note that $\mathrm{Hom}_{\mathscr{A}|_U}(f^*(\mathscr{M})|_U, f^*(\mathscr{N})|_U)$ is endowed with a $\Gamma(U, f^{-1}(\mathscr{B}))$-module structure, and thanks to the canonical homomorphism $\Gamma(V, \mathscr{B}) \to \Gamma(U, f^{-1}(\mathscr{B}))$ obtained from the definition of f^{-1}, this is then a $\Gamma(V, \mathscr{B})$-module. We also verify that these homomorphisms are compatible with restrictions, and therefore define a canonical homomorphism

$$\gamma : \mathcal{H}om_{\mathscr{B}}(\mathscr{M}, \mathscr{N}) \to f_*\big(\mathcal{H}om_{\mathscr{A}}(f^*(\mathscr{M}), f^*(\mathscr{N}))\big)$$

which corresponds to a homomorphism

$$\gamma^\sharp : f^*(\mathcal{H}om_{\mathscr{B}}(\mathscr{M}, \mathscr{N})) \to \mathcal{H}om_{\mathscr{A}}(f^*(\mathscr{M}), f^*(\mathscr{N})) \tag{1.3.7}$$

and this canonical homomorphisms are functorial on $\mathscr{M}$ and $\mathscr{N}$.

Suppose that $\mathscr{F}$ (resp. $\mathscr{G}$) is an $\mathscr{A}$-algebra (resp. a $\mathscr{B}$-algebra). If $u : \mathscr{G} \to f_*(\mathscr{F})$ is a homomorphisms of $\mathscr{B}$-algebras, $u^\sharp$ is a homomorphism $f^*(\mathscr{G}) \to \mathscr{F}$ of $\mathscr{A}$-algebras; this follows from the diagram

$$\begin{tikzcd} \mathscr{G} \otimes_{\mathscr{B}} \mathscr{G} \arrow[r] \arrow[d] & \mathscr{G} \arrow[d, "u"] \\ f_*(\mathscr{F} \otimes_{\mathscr{A}} \mathscr{F}) \arrow[r] & f_*(\mathscr{F}) \end{tikzcd}$$

Similarly, if $v : f^*(\mathscr{G}) \to \mathscr{F}$ is a homormorphism of $\mathscr{A}$-algebras, then $v^\flat : \mathscr{G} \to f_*(\mathscr{F})$ is a homomorphisms of $\mathscr{B}$-algebras. We then get an isomorphism of bifunctors

$$\mathrm{Hom}_{\mathscr{A}-\mathbf{alg}}(f^*(\mathscr{G}), \mathscr{F}) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{B}-\mathbf{alg}}(\mathscr{G}, f_*(\mathscr{F})).$$

We can then say that f^* is the left adjoint of the functor f^* from the category of $\mathscr{A}$-algebras to that of $\mathscr{B}$-algebras.

1.3.5 Open immersions and representability

The set of open immersions in the category **Rsp** is closed under composition and fiber product, so we can speak of natrual transformations $F \to G$ (where F and G are contravariant functors from **Rsp** to **Set**) which are **representable by open immersions**. The same is true when if we consider the category $\mathbf{Rsp}_{/S}$, where S is a base ringed space.

Proposition 1.3.4. *Let S be a ringed space, $F : (\mathbf{Rsp}_{/S})^\circ \to \mathbf{Set}$ a contravariant functor, and $(F_i)_{i\in I}$ be a family of subfunctors of F. Suppose that the following conditions are satisfied:*

(i) For each i, the canonical natrual transformation $u_i : F_i \to F$ are representable by an open immersion.

(ii) For any S-ringed space X, the map $U \mapsto F(U)$, where U is an open subset of X, is a sheaf of sets over X (i.e., F is a sheaf over $\mathbf{Rsp}_{/S}$).

(iii) For any S-ringed space Z and any natrual transformation $h_Z \to F$, if Z_i is the S-ringed space representating the functor $F_i \times_F h_Z$ and U_i is the image of morphism $Z_i \to Z$, then (U_i) form an open covering of Z.

(iv) For each i, the functor F_i is representable by an S-ringed space X_i.

Then the functor F is representable by an S-ringed space X, and the images of X_i under the morphism $X_i \to X$ (which is open by condition (i)) form an open covering of X.

1.4 Quasi-coherent sheaves and coherent sheaves

1.4.1 Quasi-coherent sheaves

In this subsection we introduce an abstract notion of quasi-coherent $\mathscr{O}_X$-module. This notion is very useful in algebraic geometry, since quasi-coherent modules on a scheme have a good description on any affine open. However, in the general setting of locally ringed spaces this notion is not well behaved at all. The category of quasi-coherent sheaves is not abelian in general, infinite direct sums of quasi-coherent sheaves aren't quasi-coherent, etc, etc.

Let $(X, \mathscr{O}_X)$ be a ringed space. Let $\mathscr{F}$ be a sheaf of $\mathscr{O}_X$-modules. We say that $\mathscr{F}$ is a **quasi-coherent sheaf** of $\mathscr{O}_X$-modules if for each $x \in X$, there exists an open neighbourhood U of x such that $\mathscr{F}|_U$ is isomorphic to the cokernel of a homomorphism of the form $\mathscr{O}_X^{\oplus J}|_U \to \mathscr{O}_X^{\oplus I}|_U$, where I and J are arbitrary index sets. It is clear that $\mathscr{O}_X$ itself is a quasi-coherent $\mathscr{O}_X$-module, and a finite direct sum of quasi-coherent modules is quasi-coherent. The category of quasi-coherent $\mathscr{O}_X$-modules is denoted $\mathbf{QCoh}(\mathscr{O}_X)$. We say an $\mathscr{O}_X$-lagebra $\mathscr{A}$ is quasi-coherent if it is a quasi-coherent $\mathscr{O}_X$-module.

The definition of quasi-coherence amounts to saying that locally the sheaf $\mathscr{F}$ admits a *presentation* by the structural sheaf $\mathscr{O}_X$. This definition is inspired by following ideal: for any module M over a ring A, M admits a presentation of the form

$$A^{\oplus J} \longrightarrow A^{\oplus I} \longrightarrow M \longrightarrow 0$$

where I and J are arbitrary index sets (and in particular may not be finite). Therefore, quasi-coherent sheaves can be seen as a "real module" over the ringed space $(X, \mathscr{O}_X)$, and this idea really makes sense in the realm of algebraic geometry.

Proposition 1.4.1. *Let $(f, f^\#) : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ be a morphism of ringed spaces. Then the pullback $f^*(\mathscr{G})$ of a quasi-coherent $\mathscr{O}_Y$-module $\mathscr{G}$ is quasi-coherent.*

Proof. Since the question is local, we may assume that $\mathscr{G}$ has a global presentation by $\mathscr{O}_Y$. We have seen that f^* commutes with all colimits, and is right exact, so if we have an exact sequence

$$\mathscr{O}_Y^{\oplus J} \longrightarrow \mathscr{O}_Y^{\oplus I} \longrightarrow \mathscr{G} \longrightarrow 0$$

then upon applying f^* we obtain the exact sequence

$$\mathscr{O}_X^{\oplus J} \longrightarrow \mathscr{O}_X^{\oplus I} \longrightarrow f^*(\mathscr{G}) \longrightarrow 0$$

This implies the assertion. $\square$

Proposition 1.4.2. *Let $(X, \mathscr{O}_X)$ be ringed space. Let $\rho : A \to \Gamma(X, \mathscr{O}_X)$ be a ring homomorphism from a ring A into the ring of global sections on X and M be an A-module. Then the following three constructions give canonically isomorphic $\mathscr{O}_X$-modules:*

(i) Let $(\pi, \pi^\#) : (X, \mathscr{O}_X) \to (\{*\}, A)$ be the morphism of ringed spaces where $\pi : X \to \{*\}$ is the unique map and $\pi^\#$ is the given homomorphism $\rho : A \to \Gamma(X, \mathscr{O}_X)$. Set $\mathscr{F}_1 = \pi^*(M)$.

(ii) Choose a presentation $A^{\oplus J} \to A^{\oplus I} \to M \to 0$ and set

$$\mathscr{F}_2 = \mathrm{coker}(\mathscr{O}_X^{\oplus J} \to \mathscr{O}_X^{\oplus I})$$

where the homomorphism is induced by ρ and the matrix coefficients of the homomorphism in the presentation of M.

(iii) Let $\mathscr{F}_3$ be the sheaf associated to the presheaf $U \mapsto \Gamma(U, \mathscr{O}_X) \otimes_A M$, where the homomorphism $A \to \Gamma(X, \mathscr{O}_X)$ is the composition of ρ with the restriction map $\Gamma(X, \mathscr{O}_X) \to \Gamma(U, \mathscr{O}_X)$.

This construction has the following properties:

(a) The resulting $\mathscr{O}_X$-modules $\mathscr{F}_M = \mathscr{F}_1 = \mathscr{F}_2 = \mathscr{F}_3$ is quasi-coherent.

(b) The construction gives a functor from the category of A-modules to the category of quasi-coherent sheaves on X which commutes with arbitrary colimits.

(c) For any point $x \in X$ we have $\mathscr{F}_{M,x} = \mathscr{O}_{X,x} \otimes_A M$ which is functorial in M.

(d) For any $\mathscr{O}_X$-module $\mathscr{G}$ we have

$$\mathrm{Hom}_{\mathscr{O}_X}(\mathscr{F}_M, \mathscr{G}) = \mathrm{Hom}_A(M, \Gamma(X, \mathscr{G}))$$

where the A-module structure on $\Gamma(X, \mathscr{G})$ is induced from the $\Gamma(X, \mathscr{O}_X)$-module structure via α.

Proof. The isomorphism between $\mathscr{F}_1$ and $\mathscr{F}_3$ comes from the fact that π^* is defined as the sheafification of the presheaf in (iii). The isomorphism between the constructions in (ii) and (i) comes from the fact that the functor π^* is right exact, so the sequence

$$\pi^*(A^{\oplus I}) \longrightarrow \pi^*(A^{\oplus J}) \longrightarrow \pi^*(M) \to 0$$

is exact, that π^* commutes with arbitrary direct sums, and the fact that $\pi^*(A) = \mathscr{O}_X$.

Now assertion (a) is clear from construction (ii), so is (b) since π^* has these properties. Assertion (c) follows from the description of stalks of pullback sheaves, and (d) follows from adjointness of π^* and π_*. $\square$

In the situation of Proposition 1.4.2 we say $\mathscr{F}_M$ is the **sheaf associated to the module M and the ring map ρ**. If $A = \Gamma(X, \mathscr{O}_X)$ and $\rho = 1_A$, we simply say that $\mathscr{F}_M$ is the sheaf associated to the A-module M.

Proposition 1.4.3. *Let $(X, \mathscr{O}_X)$ be a ringed space and $A = \Gamma(X, \mathscr{O}_X)$. Let M be an A-module and $\mathscr{F}_M$ be the quasi-coherent sheaf of $\mathscr{O}_X$-modules associated to M. If $(f, f^\#) : (Y, \mathscr{O}_Y) \to (X, \mathscr{O}_X)$ is a morphism of ringed spaces, then $f^*(\mathscr{F}_M)$ is the sheaf associated to the $\Gamma(Y, \mathscr{O}_Y)$-module $\Gamma(Y, \mathscr{O}_Y) \otimes_A M$.*

Proof. In view of the following diagram of ringed spaces

$$\begin{tikzcd}
(Y, \mathscr{O}_Y) \arrow[r, "\pi"] \arrow[d, "f"'] & (\{*\}, \Gamma(Y, \mathscr{O}_Y)) \arrow[d, "\text{induced by } f^\#"] \\
(X, \mathscr{O}_X) \arrow[r, "\pi"] & (\{*\}, \Gamma(X, \mathscr{O}_X))
\end{tikzcd}$$

the assertion follows from the first description of $\mathscr{F}_M$ in Proposition 1.4.2 as $\pi^*(M)$. $\square$

To conclude this part, we prove an important result which will be used when we consider quasi-coherent sheaf on affine schemes. We state it in a general manner.

Proposition 1.4.4. *Let $(X, \mathscr{O}_X)$ be a ringed space and x be a point of X. suppose that x has a fundamental system of quasi-compact neighbourhoods, then for any quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, there exists an open neighbourhood U of x such that $\mathscr{F}|_U$ is isomorphic to the sheaf of modules $\mathscr{F}_M$ on $(U, \mathscr{O}_U)$ associated to a $\Gamma(U, \mathscr{O}_U)$-module M.*

Proof. Since $\mathscr{F}$ is quasi-coherent, we may replace X by an open neighbourhood of x and assume that $\mathscr{F}$ is isomorphic to the cokernel of a map $\mathscr{O}_X^{\oplus J} \to \mathscr{O}_X^{\oplus I}$. The problem is that this map may not be given by a matrix, because the global sections of a direct sum is in general different from the direct sum of the global sections.

Let U be a quasi-compact neighbourhood of x. We proceed as in the proof of Proposition 1.2.41. For each $j \in J$ denote $s_j \in \Gamma(X, \bigoplus_{i \in I} \mathscr{O}_X)$ the image of the section 1 in the summand $\mathscr{O}_X$ corresponding to j. There exists a finite collection of opens $U_{jk}, k \in K_j$ such that $U = \bigcup_{k \in K_j} U_{jk}$ and such that each restriction $s_j|_{U_{jk}}$ is a finite sum $\sum_{i \in I_{jk}} f_{jki}$ with $I_{jk} \subseteq I$. Let $I_j = \bigcup_{k \in K_j} I_{jk}$. This is a finite set since there are finitely many U_{jk} and each I_{jk} is finite. Since $U = \bigcup_{k \in K_j} U_{jk}$ the section $s_j|_U$ is a section of the finite direct sum $\bigoplus_{i \in I_j} \mathscr{O}_X$. Then by Proposition 1.2.38 we see that actually $s_j|_U$ is a sum $\sum_{i \in I_j} f_{ij}$ with $f_{ij} \in \mathscr{O}_X(U) = \Gamma(U, \mathscr{O}_U)$. At this point we can define a module M as the cokernel of the map

$$\bigoplus_{j \in J} \Gamma(U, \mathscr{O}_U) \to \bigoplus_{i \in I} \Gamma(U, \mathscr{O}_U).$$

with matrix given by the (f_{ij}). By construction (ii) of Proposition 1.4.2 we see that $\mathscr{F}_M$ has the same presentation as $\mathscr{F}|_U$ and therefore $\mathscr{F}_M \cong \mathscr{F}|_U$. $\square$

1.4.2 Sheaves of finite type

Let $(X, \mathscr{O}_X)$ be a ringed space and $\mathscr{F}$ be an $\mathscr{O}_X$-modules. We say that $\mathscr{F}$ is **of finite type** if for every point $x \in X$ there exists an open neighbourhood U of x such that $\mathscr{F}|_U$ is generated by a finite family of sections over U, whence isomorphic to a quotient sheaf of a sheaf of the form $\mathscr{O}_X^n|_U$. It is clear that any quotient of a sheaf of finite type is of finite type, and a finite direct sum of sheaves of finite type is of finite type.

Proposition 1.4.5. *Let $(f, f^\#) : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ be a morphism of ringed spaces. The pullback $f^*\mathscr{G}$ of a finite type $\mathscr{O}_Y$-module is a finite type $\mathscr{O}_X$-module.*

Proof. Since the question is local, we may assume $\mathscr{G}$ is globally generated by finitely many sections. We have seen that f^* commutes with all colimits, and is right exact, so if we have a surjection $\bigoplus_{i=1}^n \mathscr{O}_Y \to \mathscr{G} \to 0$, then by applying f^* we obtain the surjection $\bigoplus_{i=1}^n \mathscr{O}_X \to f^*\mathscr{G} \to 0$. □

Proposition 1.4.6. *Let $\mathscr{F}$ be an $\mathscr{O}_X$-module of finite type.*

(i) If $(s_i)_{1 \le i \le n}$ are sections of $\mathscr{F}$ over an open neighborhood U of a point x and if the $s_{i,x}$ generate $\mathscr{F}$, then there exists an open neighborhood $V \subseteq U$ of x such that the $s_{i,y}$ generate $\mathscr{F}_y$ for any $y \in V$.

(ii) If $\varphi : \mathscr{F} \to \mathscr{G}$ is a homomorphism such that $\varphi_x = 0$, there exists an open neighborhood U of x such that $\varphi|_U = 0$.

(iii) If $\psi : \mathscr{G} \to \mathscr{F}$ is a homomorphism such that ψ_x is surjective, then there exists an open neighborhood V of x such that $\psi|_V$ is surjective.

(iv) The support of $\mathscr{F}$ is closed.

Proof. We first prove (i), so let $(t_j)_{1 \le j \le m}$ be a family of sections of $\mathscr{F}$ over an open neighborhood $U' \subseteq U$ of x generating $\mathscr{F}|_{U'}$. Since $(s_{i,x})$ generates $\mathscr{F}_x$, there exists sections a_{ij} of $\mathscr{O}_X$ over an open neighborhood $U'' \subseteq U$ of x such that $t_{j,x} = \sum_{i=1}^n a_{ij,x} s_{i,x}$ for each j. We then conclude that there is an open neighborhood $V \subseteq U''$ of x such that for each $y \in V$, we have $t_{j,y} = \sum_{i=1}^n a_{ij,y} s_{i,y}$, so $(s_{i,y})$ generates $\mathscr{F}_y$ for $y \in V$.

Assertion (iv) follows from (i) by taking $n = 1$ and $s_1 = 0$; also, (iii) follows from (iv) by considering coker ψ, which is of finite type. Finally, (ii) follows from (iv) by considering im φ, which is also of finite type. □

Corollary 1.4.7. *Let $(X, \mathscr{O}_X)$ be a locally ringed space and $\mathscr{F}$ be an $\mathscr{O}_X$-module of finite type. Then $\operatorname{supp}(\mathscr{F})$ is the set of $x \in X$ such that $\mathscr{F}_x / \mathfrak{m}_x \mathscr{F}_x \neq 0$.*

Proof. In fact, $\mathscr{F}_x$ is a finitely generated $\mathscr{O}_{X,x}$, so $\mathscr{F}_x = 0$ if and only if $\mathfrak{m}_x \mathscr{F}_x = \mathscr{F}_x$ by Nakayama's lemma. □

Corollary 1.4.8. *Let $(X, \mathscr{O}_X)$ be a locally ringed space and $\mathscr{F}, \mathscr{G}$ be $\mathscr{O}_X$-modules of finite type. Then*

$$\operatorname{supp}(\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G}) = \operatorname{supp}(\mathscr{F}) \cap \operatorname{supp}(\mathscr{G}).$$

Proof. As the tensor product of two $\kappa(x)$-vector spaces is nonzero if both of them are nonzero, this follows from Corollary 1.4.7. □

Corollary 1.4.9. *Let $f : X \to Y$ be a morphism of locally ringed spaces. For any $\mathscr{O}_Y$-module $\mathscr{G}$ of finite type, we have*

$$\operatorname{supp}(f^*(\mathscr{G})) = f^{-1}(\operatorname{supp}(\mathscr{G})).$$

Proof. This follows from Corollary 1.4.7 and **??**. □

Proposition 1.4.10. *Suppose that X is quasi-compact and let $\mathscr{F}, \mathscr{G}$ be two $\mathscr{O}_X$-modules such that $\mathscr{G}$ is of finite type and $\mathscr{F}$ is the filtered limit $(\mathscr{F}_\lambda)$ of $\mathscr{O}_X$-modules. Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a surjective homomorphism, then there exists an index λ such that the homomorphism $\mathscr{F}_\lambda \to \mathscr{G}$ is surjective.*

Proof. For any $x \in X$, there exists a finite system of sections s_i of $\mathscr{G}$ over an open neighborhood $U(x)$ of x such that $s_{i,y}$ generates $\mathscr{G}_y$ for any $y \in U(x)$. Then there is an open neighborhood $V(x) \subseteq U(x)$ of x and sections t_i of $\mathscr{F}$ over $V(x)$ such that $s_i|_{V(x)} = \varphi(t_i)$ for each i. We can then suppose that the t_i are the images of sections of a single sheaf $\mathscr{F}_{\lambda(x)}$ over $V(x)$. Since X is quasi-compact, it can be covered by finitely many $V(x_k)$, and let λ be the supremum of the $\lambda(x_k)$ the assertion then follows. $\square$

Corollary 1.4.11. *Suppose that X is quasi-compact and let $\mathscr{F}$ be an $\mathscr{O}_X$-module of finite type that is generated by global sections. Then $\mathscr{F}$ is generated by finitely many global sections.*

Proof. It suffices to cover X be finitely many open neighborhoods U_k such that for each k, there exists finitely many sections s_{ik} of $\mathscr{F}$ over X whose restrictions to U_k generate $\mathscr{F}|_{U_k}$. It is clear that the s_{ik} then generate $\mathscr{F}$. $\square$

Proposition 1.4.12. *Let X be a ringed space. Let*

$$0 \longrightarrow \mathscr{F} \xrightarrow{\varphi} \mathscr{G} \xrightarrow{\psi} \mathscr{H} \longrightarrow 0$$

be a short exact sequence of $\mathscr{O}_X$-modules. If $\mathscr{F}$ and $\mathscr{H}$ are of finite type, so is $\mathscr{G}$.

Proof. Since the question is local, we may assume that $\mathscr{F}$ (resp. $\mathscr{H}$) is generated by finitely many global sections $(s_i)_{1 \le i \le n}$ (resp. $(t_j)_{1 \le j \le m}$), and there are sections $(t'_j)_{1 \le j \le m}$ of $\mathscr{G}$ over X such that $t_j = \psi(t'_j)$ for each j. It is then clear that $\mathscr{G}$ is generated by the sections $\varphi(s_i)$ and t'_j. $\square$

Let $(X, \mathscr{O}_X)$ be a ringed space. Let $\mathscr{F}$ be a sheaf of $\mathscr{O}_X$-modules. We say that $\mathscr{F}$ is **of finite presentation** if for every point $x \in X$ there exists an open neighbourhood $x \in U \subseteq X$, and $n, m \in \mathbb{N}$ such that $\mathscr{F}|_U$ is isomorphic to the cokernel of a homomorphism $\bigoplus_{j=1}^m \mathscr{O}_U \to \bigoplus_{j=1}^n \mathscr{O}_U$. This means that X is covered by open sets U such that $\mathscr{F}|_U$ has a presentation of the form

$$\textstyle\bigoplus_{j=1}^m \mathscr{O}_U \longrightarrow \bigoplus_{j=1}^n \mathscr{O}_U \longrightarrow \mathscr{F}|_U \longrightarrow 0$$

As in the case of $\mathscr{O}_X$-modules, the pullback of a $\mathscr{O}_X$-module of finite presentation is of finite presentation. We also note that any $\mathscr{O}_X$-module of finite presentation is in particular quasi-coherent.

Proposition 1.4.13. *Let $(X, \mathscr{O}_X)$ be a ringed space. Let $\mathscr{F}$ be a $\mathscr{O}_X$-module of finite presentation.*

(a) If $\psi : \mathscr{O}_X^{\oplus r} \to \mathscr{F}$ is a surjective homomorphism, then $\ker \psi$ *is of finite type.*

(b) If $\varphi : \mathscr{G} \to \mathscr{F}$ is a surjective homomorphism with $\mathscr{G}$ of finite type, then $\ker \theta$ *is of finite type.*

Proof. $\square$

Proposition 1.4.14. *Let $(X, \mathscr{O}_X)$ be a ringed space and $\mathscr{F}$ be an $\mathscr{O}_X$-module of finite presentation. Then for any $\mathscr{O}_X$-module $\mathscr{H}$, the canonical homomorphism*

$$(\mathcal{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{H}))_x \to \mathrm{Hom}_{\mathscr{O}_{X,x}}(\mathscr{F}_x, \mathscr{G}_x)$$

is bijective.

Proof. $\square$

Proposition 1.4.15. *Let $(X, \mathscr{O}_X)$ be a ringed space and $\mathscr{F}, \mathscr{G}$ be finitely presented $\mathscr{O}_X$-module. If for a point $x \in X$, $\mathscr{F}_x$ and $\mathscr{G}_x$ are isomorphic $\mathscr{O}_{X,x}$-modules, then there exists an open neighbourhood U of x such that $\mathscr{F}|_U \cong \mathscr{G}|_U$.*

Proof. If $\varphi : \mathscr{F}_x \to \mathscr{G}_x$ and $\psi : \mathscr{G}_x \to \mathscr{F}_x$ are the isomorphisms, there exists, by Proposition 1.4.14, an open neighborhood V of x and a section u (resp. v) of $\mathcal{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})$ (resp. $\mathcal{H}om_{\mathscr{O}_X}(\mathscr{G} \circ \mathscr{F})$) over V such that $u_x = \varphi$ (resp. $v_x = \psi$). As $(u \circ v)_x$ and $(v \circ u)_x$ are the identities, by Proposition 1.4.14 again there exists an open neighborhood $U \subseteq V$ of x such that $(u \circ v)|_U$ and $(v \circ u)|_U$ are idnetities, whence the proposition. $\square$

1.4.3 Coherent sheaves

Let $(X, \mathscr{O}_X)$ be a ringed space. Let $\mathscr{F}$ be a sheaf of $\mathscr{O}_X$-modules. We say that $\mathscr{F}$ is a **coherent** $\mathscr{O}_X$-module if $\mathscr{F}$ is of finite type and for every open $U \subseteq X$ and every finite collection $s_1, \dots, s_n$ of sections of $\mathscr{F}$ over U, the kernel of the associated map $\bigoplus_{i=1}^n \mathscr{O}_U \to \mathscr{F}|_U$ is of finite type. The category of coherent $\mathscr{O}_X$-modules is denoted $\mathbf{Coh}(\mathscr{O}_X)$. This is a more reasonable object than the category of quasi-coherent sheaves, in the sense that it is at least an abelian subcategory of $\mathbf{Mod}(\mathscr{O}_X)$ no matter what X is. However, the pullback of a coherent module is almost never coherent in the general setting of ringed spaces.

Proposition 1.4.16. *Let $(X, \mathscr{O}_X)$ be a ringed space. Any coherent $\mathscr{O}_X$-module is of finite presentation and hence quasi-coherent.*

Proof. Let $\mathscr{F}$ be a coherent sheaf on X and let x be a point of X. We may find an open neighbourhood U and sections $(s_i)_{1 \le i \le n}$ of $\mathscr{F}$ over U such that the associated homomorphism $\varphi : \bigoplus_{i=1}^n \mathscr{O}_U \to \mathscr{F}|_U$ is surjective. Since $\ker \varphi$ is also of finite type, we may find an open neighbourhood $x \in V \subseteq U$ and sections $(t_j)_{1 \le j \le m}$ of $\bigoplus_{i=1}^m \mathscr{O}_V$ which generate the kernel of $\varphi|_V$. Then over V we get the presentation

$$\bigoplus_{j=1}^m \mathscr{O}_V \longrightarrow \bigoplus_{i=1}^n \mathscr{O}_V \longrightarrow \mathscr{F}|_V \longrightarrow 0$$

which shows that $\mathscr{F}$ is of finite presentation. $\square$

Example 1.4.1. Suppose that X is a point. In this case the definition above gives a notion for modules over rings. What does the definition of coherent mean? It is closely related to the notion of Noetherian, but it is not the same: namely, the ring $A = \mathbb{C}[x_1, x_2, x_3, \dots]$ is coherent as a module over itself but not Noetherian as a module over itself.

Proposition 1.4.17. *Let $(X, \mathscr{O}_X)$ be a ringed space.*

(a) Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a homomorphism from an $\mathscr{O}_X$-module $\mathscr{F}$ of finite type to a coherent $\mathscr{O}_X$-module $\mathscr{G}$. Then $\ker \varphi$ is of finite type.

(b) Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a homomorphism of coherent $\mathscr{O}_X$-modules. Then $\ker \varphi$ and $\operatorname{coker} \varphi$ are coherent.

(c) Given a short exact sequence of $\mathscr{O}_X$-modules $0 \to \mathscr{F}_1 \to \mathscr{F}_2 \to \mathscr{F}_3 \to 0$ if two out of three are coherent so is the third.

(d) The category of coherent $\mathscr{O}_X$-modules is abelian and the inclusion functor $\mathbf{Coh}(\mathscr{O}_X) \to \mathbf{Mod}(\mathscr{O}_X)$ is exact.

Proof. Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a homomorphism where $\mathscr{F}$ is of finite type and $\mathscr{G}$ is coherent. Let us show that $\ker \varphi$ is of finite type. Pick $x \in X$ and choose an open neighbourhood U of x in X such that $\mathscr{F}|_U$ is generated by $s_1, \dots, s_n$. By definition the kernel $\mathscr{K}$ of the induced map

$\bigoplus_{i=1}^n \mathscr{O}_U \to \mathscr{G}, e_i \mapsto \varphi(s_i)$ is of finite type. Hence $\ker\varphi$ which is the image of the composition $\mathscr{K} \to \bigoplus_{i=1}^n \mathscr{O}_U \to \mathscr{F}$ is of finite type.

Now consider the case of assertion (b). By assertion (a) the kernel of φ is of finite type and hence is coherent as a subsheaf of $\mathscr{F}$. With the same hypotheses let us show that $\operatorname{coker}\varphi$ is coherent. Since $\mathscr{G}$ is of finite type so is $\operatorname{coker}\varphi$. Let $U \subseteq X$ be open and let $\bar{s}_i \in \operatorname{coker}\varphi(U), 1 \leq i \leq n$ be sections. We have to show that the kernel of the associated morphism $\bar{\psi} : \bigoplus_{i=1}^n \mathscr{O}_U \to \operatorname{coker}\varphi$ has finite type. There exists an open covering of U such that on each open all the sections $\bar{s}_i$ lift to sections s_i of $\mathscr{G}$. Hence we may assume this is the case over U. Thus $\bar{\psi}$ lifts to $\psi : \bigoplus_{i=1}^n \mathscr{O}_U \to \mathscr{G}$. Consider the following diagram

$$\begin{tikzcd}
0 \arrow[r] & \ker\psi \arrow[r] \arrow[d] & \bigoplus_{i=1}^n \mathscr{O}_U \arrow[r] \arrow[d, equal] & \mathscr{G} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \ker\bar{\psi} \arrow[r] & \bigoplus_{i=1}^n \mathscr{O}_U \arrow[r] & \operatorname{coker}\psi \arrow[r] & 0
\end{tikzcd}$$

By the snake lemma we have $\operatorname{im}\varphi \cong \operatorname{coker}(\ker\psi \to \ker\bar{\psi})$, thus there is a short exact sequence $0 \to \ker\psi \to \ker\bar{\psi} \to \operatorname{im}\varphi \to 0$. Hence by Proposition 1.4.12 we see that $\ker\bar{\psi}$ is of finite type.

Let $0 \to \mathscr{F}_1 \to \mathscr{F}_2 \to \mathscr{F}_3 \to 0$ be a short exact sequence of $\mathscr{O}_X$-modules. By part (b) it suffices to prove that if $\mathscr{F}_1$ and $\mathscr{F}_3$ are coherent so is $\mathscr{F}_2$. By Proposition 1.4.12 we see that $\mathscr{F}_2$ has finite type. Let $s_1, \ldots, s_n$ be finitely many local sections of $\mathscr{F}_2$ defined over a common open U of X. We have to show that the module of relations $\mathscr{K}$ between them is of finite type. Consider the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] \arrow[d] & \bigoplus_{i=1}^n \mathscr{O}_U \arrow[r] \arrow[d] & \bigoplus_{i=1}^n \mathscr{O}_U \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathscr{F}_1 \arrow[r] & \mathscr{F}_2 \arrow[r] & \mathscr{F}_3 \arrow[r] & 0
\end{tikzcd}$$

with obvious notation. By the snake lemma we get a short exact sequence $0 \to \mathscr{K} \to \mathscr{K}_3 \to \mathscr{F}_1$ where $\mathscr{K}_3$ is the module of relations among the images of the sections s_i in $\mathscr{F}_3$. Since $\mathscr{F}_3$ is coherent we see that $\mathscr{K}_3$ is finite type. Since $\mathscr{F}_1$ is coherent we see that the image $\mathscr{J}$ of $\mathscr{K}_3 \to \mathscr{F}_1$ is coherent. Hence $\mathscr{K}$ is the kernel of the map $\mathscr{K} \to \mathscr{J}$ between a finite type sheaf and a coherent sheaves and hence finite type by (b). $\square$

Corollary 1.4.18. *Let $\mathscr{F}_1 \xrightarrow{u} \mathscr{F}_2 \xrightarrow{v} \mathscr{F}_3 \xrightarrow{w} \mathscr{F}_4 \xrightarrow{t} \mathscr{F}_5$ be an exact sequence of $\mathscr{O}_X$-modules. If $\mathscr{F}_1, \mathscr{F}_2, \mathscr{F}_4$ and $\mathscr{F}_5$ are coherent, then $\mathscr{F}_3$ is coherent.*

Proof. In fact, $\operatorname{coker} u = \mathscr{F}_2 / \ker v$ and $\operatorname{im} w = \ker t$ are coherent, and it suffices to consider the exact sequence

$$0 \longrightarrow \operatorname{coker} u \longrightarrow \mathscr{F}_3 \longrightarrow \operatorname{im} w \longrightarrow 0$$

and apply Proposition 1.4.17. $\square$

Corollary 1.4.19. *Let $\mathscr{F}$ and $\mathscr{G}$ be two coherent subsheaves of a coherent sheaf $\mathscr{K}$. The sheaves $\mathscr{F} + \mathscr{G}$ and $\mathscr{F}/\mathscr{G}$ are coherent.*

Proof. The sheaf $\mathscr{F} + \mathscr{G}$ is a subsheaf of $\mathscr{K}$ of fintie type, so it is coherent by definition. As for $\mathscr{F} \cap \mathscr{G}$, it is the kernel of $\mathscr{F} \to \mathscr{K}/\mathscr{G}$, so is coherent. $\square$

Corollary 1.4.20. *If $\mathscr{F}$ and $\mathscr{G}$ are coherent $\mathscr{O}_X$-modules, so are $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G}$ and $\mathcal{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})$.*

Proof. Since the question is local, we can assume that $\mathscr{F}$ is the cokernel of a homomorphism $\mathscr{O}_X^n \to \mathscr{O}_X^m$. Then $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G}$ is isomorphic to the cokernel of the homomorphism $\mathscr{O}_X^n \otimes_{\mathscr{O}_X} \mathscr{G} \to \mathscr{O}_X^m \otimes_{\mathscr{O}_X} \mathscr{G}$, which is identified with the cokernel of $\mathscr{G}^n \to \mathscr{G}^m$. Since $\mathscr{G}$ is coherent, we then see $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G}$ is also coherent.

Similarly, in view of Proposition 1.4.14, we have an exact sequence □

Corollary 1.4.21. *Let $\mathscr{F}$ be a coherent $\mathscr{O}_X$-module and $\mathscr{J}$ be an coherent ideal of $\mathscr{O}_X$. Then the $\mathscr{O}_X$-module $\mathscr{J}\mathscr{F}$ is coherent.*

Proof. The image of the canonical homomorphism $\mathscr{J} \otimes_{\mathscr{O}_X} \mathscr{F} \to \mathscr{F}$ is $\mathscr{J}\mathscr{F}$, so this follows from Corollary 1.4.20. □

Corollary 1.4.22. *Let X be a ringed space and $\varphi : \mathscr{F} \to \mathscr{G}$ be a homomorphism of $\mathscr{O}_X$-modules. Assume $\mathscr{F}$ of finite type, $\mathscr{G}$ is coherent and the homomorphism $\varphi : \mathscr{F}_x \to \mathscr{G}_x$ is injective for a point $x \in X$. Then there exists an open neighbourhood $x \in U \subseteq X$ such that $\varphi|_U$ is injective.*

Proof. Denote by $\mathscr{K} \subseteq \mathscr{F}$ the kernel of φ. By Proposition 1.4.17 we see that $\mathscr{K}$ is a finite type $\mathscr{O}_X$-module. Our assumption is that $\mathscr{K}_x = 0$, so by Proposition 1.4.6(iv) there exists an open neighbourhood U of x such that $\mathscr{K}|_U = 0$. □

We say an $\mathscr{O}_X$-algebra $\mathscr{A}$ is **coherent** if $\mathscr{A}$ is a coherent $\mathscr{O}_X$-module. In particular, $\mathscr{O}_X$ is a coherent $\mathscr{O}_X$-algebra if, for any open subset $U \subseteq X$ and any homomorphism $\mathscr{O}_X^n|_U \to \mathscr{O}_X|U$ of $\mathscr{O}_U$-modules, the kernel of this homomorphism is of finite type. We then say that $\mathscr{O}_X$ is a coherent sheaf of rings. If $\mathscr{O}_X$ is a coherent sheaf of rings, any $\mathscr{O}_X$-module of finite presentation is coherent in view of Proposition 1.4.17.

Example 1.4.2. The annihilator of a $\mathscr{O}_X$-module $\mathscr{F}$ is the kernel $\mathscr{J}$ of the homomorphism $\mathscr{O}_X \to \mathcal{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{F})$ which sends a section $s \in \Gamma(U, \mathscr{O}_X)$ to the multiplication by s in $\mathrm{Hom}(\mathscr{F}|_U, \mathscr{F}|_U)$. If $\mathscr{O}_X$ is a coherent sheaf of rings and if $\mathscr{F}$ is a coherent $\mathscr{O}_X$-module, $\mathscr{J}$ is coherent by Proposition 1.4.17, and it follows from Proposition 1.4.14 that for each $x \in X$, $\mathscr{J}_x$ is the annihilator of the $\mathscr{O}_{X,x}$-module $\mathscr{F}_x$.

Proposition 1.4.23. *Suppose that $\mathscr{O}_X$ is a coherent sheaf of rings and let $\mathscr{J}$ be a coherent ideal of $\mathscr{J}$. For an $(\mathscr{O}_X/\mathscr{J})$-module $\mathscr{F}$ to be coherent, it is necessary and sufficient that as an $\mathscr{O}_X$-module, $\mathscr{F}$ is coherent. In particular, $\mathscr{O}_X/\mathscr{J}$ is a coherent sheaf of rings.*

Proof. We note that $\mathscr{O}_X/\mathscr{J}$ is a coherent $\mathscr{O}_X$-module. If $\mathscr{F}$ is a coherent $(\mathscr{O}_X/\mathscr{J})$, any point of X admits an open neighborhood U such that $\mathscr{F}|_U$ is the cokernel of a homomorphism $(\mathscr{O}_X/\mathscr{J})^m|_U \to (\mathscr{O}_X/\mathscr{J})^n|_U$, so $\mathscr{F}$ is a coherent $\mathscr{O}_X$-module.

Conversely, suppose that $\mathscr{F}$, as an $\mathscr{O}_X$-module, is coherent. First, since $\mathscr{F}$ is an $\mathscr{O}_X$-module of finite type, it is an $(\mathscr{O}_X/\mathscr{J})$-module of finite type. On ther other hand, let U be an open subset of X and $u : (\mathscr{O}_X/\mathscr{J})^n|_U \to \mathscr{F}|_U$ is an $(\mathscr{O}_X/\mathscr{J})$-homomorphism; by composing with the canonical homomorphism $v : \mathscr{O}_X^n|_U \to (\mathscr{O}_X/\mathscr{J})^n|_U$, we obtain a homomorphism $u \circ v : \mathscr{O}_X^n|_U \to \mathscr{F}|_U$, and $\ker(u \circ v)$ is by hypothesis of finite type. But since v is surjective, $\ker u$ is the image of $\ker(u \circ v)$ by v, so is of finite type. □

Proposition 1.4.24. *Let $f : X \to Y$ be a morphism of ringed spaces and suppose that $\mathscr{O}_X$ is a coherent rings. Then for any coherent $\mathscr{O}_Y$-module $\mathscr{G}$, $f^*(\mathscr{G})$ is a coherent $\mathscr{O}_X$-module.*

Proof. Let V be an open subset of Y such that $\mathscr{G}|_V$ is the cokernel of a homomorphism $v : \mathscr{O}_Y^n|_V \to \mathscr{O}_Y^n|_V$. As f_U^* is right exact, we have $f^*(\mathscr{G})|_U = f_U^*(\mathscr{G}|_V)$ (where $U = f^{-1}(V)$) and the cokernel of the homomorphism $f_U^*(v) : \mathscr{O}_X^m|_U \to \mathscr{O}_X^n|_U$ is coherent by Proposition 1.4.17. □

1.4.4 Locally free sheaves

Let $(X, \mathscr{O}_X)$ be a ringed space. We say an $\mathscr{O}_X$-module $\mathscr{F}$ is **locally free** if, for any $x \in X$, there exists an open neighborhood U of x such that $\mathscr{F}|_U$ is isomorphic to an $(\mathscr{O}_X|_U)$-module of the form $\mathscr{O}_X^{\oplus I}|_U$. If for any open subset U the set I is finite, we say that $\mathscr{F}$ is **of finite rank**. If for any open subset U, the set I has n elements, we say that $\mathscr{F}$ is **of rank n**. If $\mathscr{F}$ is a locally free $\mathscr{O}_X$-module of finite rank, for any point $x \in X$, $\mathscr{F}_x$ is a free $\mathscr{O}_X$-module of finite rank $n(x)$, and there exists an open neighborhood U of x such that $\mathscr{F}|_U$ is of rank $n(x)$. If X is connected, we then see $n(x)$ is constant.

It is clear that any locally finite sheaf is quasi-coherent, and if $\mathscr{O}_X$ is a coherent sheaf of rings, any locally free $\mathscr{O}_X$-module of finite rank is coherent. If $\mathscr{E}$ is locally free, $\mathscr{E} \otimes_{\mathscr{O}_X} \mathscr{F}$ is an exact functor on the category of $\mathscr{O}_X$-modules.

We will mostly consider locally free $\mathscr{O}_X$-modules of finite rank, so when we mention the notation of a locally free $\mathscr{O}_X$-modules, it should be understood that we mean locally free $\mathscr{O}_X$-modules of finite rank.

Example 1.4.3. Let $\mathscr{F}$ be an $\mathscr{O}_X$-module of finite presentation. Then if for a point $x \in X$, $\mathscr{F}_x$ is a free module of rank n, there exists an open neighborhood U of x such that $\mathscr{F}|_U$ is locally free of rank n (Proposition 1.4.15).

Proposition 1.4.25. *Let $\mathscr{E}$, $\mathscr{F}$ be $\mathscr{O}_X$-modules and consider the canonical functorial homomorphism*

$$\mathscr{E}^* \otimes_{\mathscr{O}_X} \mathscr{F} = \mathcal{H}om_{\mathscr{O}_X}(\mathscr{E}, \mathscr{O}_X) \otimes_{\mathscr{O}_X} \mathscr{F} \to \mathcal{H}om_{\mathscr{O}_X}(\mathscr{E}, \mathscr{F})$$

If $\mathscr{F}$ or $\mathscr{E}$ is locally free of finite rank, then this homomorphism is bijective.

Proof. The homomorphism is defined by the following: for any open subset U and a couple (u, t), where $u \in \Gamma(U, \mathscr{E}) = \mathrm{Hom}(\mathscr{E}|_U, \mathscr{O}_X|_U)$ and $t \in \Gamma(U, \mathscr{F})$, we associate the element of $\mathrm{Hom}(\mathscr{E}|_U, \mathscr{F}|_U)$ whose stalk at each point $x \in U$ send $s_x \in \mathscr{E}_x$ to the element $u_x(s_x)t_s \in \mathscr{F}_x$. Since the question is local, we may assume that $\mathscr{E} = \mathscr{O}_X^n$ or $\mathscr{F} = \mathscr{O}_X^n$. As for any $\mathscr{O}_X$-module $\mathscr{G}$, $\mathcal{H}om_{\mathscr{O}_X}(\mathscr{O}_X^n, \mathscr{G})$ is canonically isomorphic to $\mathscr{G}^n$ and $\mathcal{H}om_{\mathscr{O}_X}(\mathscr{G}, \mathscr{O}_X^n)$ is isomorphic to $(\mathscr{G}^n)^*$, we are reduced to the case $\mathscr{E} = \mathscr{F} = \mathscr{O}_X$, and the claim is then immediate. □

If $\mathscr{L}$ is locally free of rank 1, so is its dual $\mathscr{L}^* = \mathcal{H}om_{\mathscr{O}_X}(\mathscr{L}, \mathscr{O}_X)$, because this is ture for $\mathscr{L} = \mathscr{O}_X$ and the question is local. Moreover, we have a canonical isomorphism

$$\mathcal{H}om_{\mathscr{O}_X}(\mathscr{L}, \mathscr{O}_X) \otimes_{\mathscr{O}_X} \mathscr{L} \cong \mathscr{O}_X.$$

In fact, it suffices to prove that the canonical homomorphism $\mathscr{O}_X \to \mathcal{H}om_{\mathscr{O}_X}(\mathscr{L}, \mathscr{L})$ is bijective, and for this we may assume that $\mathscr{L} = \mathscr{O}_X$, and then the claim is immediate. Due to this, we put $\mathscr{L}^{-1} = \mathcal{H}om_{\mathscr{O}_X}(\mathscr{L}, \mathscr{O}_X)$, and $\mathscr{L}^{-1}$ is called the **inverse** of $\mathscr{L}$.

If $\mathscr{L}$ and $\mathscr{L}'$ are two $\mathscr{O}_X$-modules locally free of rank 1, so is their tensor product $\mathscr{L} \otimes_{\mathscr{O}_X} \mathscr{L}'$, since locally we have $\mathscr{O}_X \otimes_{\mathscr{O}_X} \mathscr{O}_X \cong \mathscr{O}_X$. For any integer $n \geq 1$, we denote by $\mathscr{L}^{\otimes n}$ the n-fold tensor product $\mathscr{L}$, which is also a locally free $\mathscr{O}_X$-module of rank 1; by convention, we set $\mathscr{L}^{\otimes 0} = \mathscr{O}_X$ and $\mathscr{L}^{\otimes(-n)} = (\mathscr{L}^{-1})^{\otimes n}$. Then there exists a canonical isomorphism

$$\mathscr{L}^{\otimes m} \otimes_{\mathscr{O}_X} \mathscr{L}^{\otimes n} \cong \mathscr{L}^{\otimes(n+m)}$$

Proposition 1.4.26. *Let $f : Y \to X$ be a morphism of ringed spaces. If $\mathscr{E}$ is a locally free $\mathscr{O}_X$-module (resp. locally free $\mathscr{O}_X$-module of rank n), then $f^*(\mathscr{E})$ is a locally free $\mathscr{O}_Y$-module (resp. locally free $\mathscr{O}_Y$-module of rank n). Moreover, we have a canonical isomorphism $f^*(\mathscr{E}^*) = (f^*(\mathscr{E}))^*$.*

Proof. The first assertion from the fact that f^* commutes with direct sums and $f^*(\mathscr{O}_X) = \mathscr{O}_Y$, the second assertin can be checked for the case $\mathscr{E} = \mathscr{O}_X$, since the question is local. □

Let $\mathscr{L}$ be a locally free $\mathscr{O}_X$-module of rank 1. We denote by $\Gamma_*(X,\mathscr{L})$ or simple $\Gamma_*(\mathscr{L})$ the abelian group of the direct sum $\bigoplus_{n\in\mathbb{Z}}\Gamma(X,\mathscr{L}^{\otimes n})$. We endow it a graded ring structure by defining the product of a couple (s_n,s_m), where $s_n\in\Gamma(X,\mathscr{L}^{\otimes n})$ and $s_m\in\Gamma(X,\mathscr{L}^{\otimes m})$, the section of $\mathscr{L}^{\otimes(n+m)}$ over X corresponding to the section $s_n\otimes s_m$ of $\mathscr{L}^{\otimes n}\otimes_{\mathscr{O}_X}\mathscr{L}^{\otimes m}$. The associative of this multiplication is immediately verified, and it is clear that $\Gamma_*(X,\mathscr{L})$ is a covariant functor on $\mathscr{L}$ with values in the category of graded rings of type $\mathbb{Z}$.

If now $\mathscr{F}$ is an $\mathscr{O}_X$-module, we set

$$\Gamma_*(\mathscr{F};\mathscr{L})=\bigoplus_{n\in\mathbb{Z}}\Gamma(X,\mathscr{F}\otimes_{\mathscr{O}_X}\mathscr{L}^{\otimes n}).$$

We endow this abelian group a graded $\Gamma_*(\mathscr{L})$-module structure by the following: for any couple (s_n,u_m), where $s_n\in\Gamma(X,\mathscr{L}^{\otimes n})$ and $u_m\in\Gamma(X,\mathscr{F}\otimes\mathscr{L}^{\otimes n})$, we associate the section of $\mathscr{F}\otimes\mathscr{L}^{\otimes(n+m)}$ that corresponds to $s_n\otimes u_m$; the verification that this defines a module structure is immediate. For fixed X and $\mathscr{L}$, we see $\Gamma_*(\mathscr{F};\mathscr{L})$ is a covariant functor on $\mathscr{F}$ with values in the category of graded $\Gamma_*(\mathscr{L})$-modules. For X and $\mathscr{F}$ fixed, this is a covariant functor on $\mathscr{L}$ iwth values in the category of abelian groups.

Let $f:Y\to X$ be a morphism of ringed spaces. The canonical homomorphism $\rho:\mathscr{L}^{\otimes n}\to f_*(f^*(\mathscr{L}^{\otimes n}))$ defines a homomorphism $\Gamma(X,\mathscr{L}^{\otimes n})\to\Gamma(Y,f^*(\mathscr{L}^{\otimes n}))$ of abelian groups, and as $f^*(\mathscr{L}^{\otimes n})=(f^*(\mathscr{L}))^{\otimes n}$, it gives a homomorphism $\Gamma_*(\mathscr{L})\to\Gamma_*(f^*(\mathscr{L}))$ of graded rings. Similarly, the canonical homomorphism $\mathscr{F}\otimes\mathscr{L}^{\otimes n}\to f_*(f^*(\mathscr{F}\otimes\mathscr{L}^{\otimes n}))$ defines a homomorphism $\Gamma(X,\mathscr{F}\otimes\mathscr{L}^{\otimes n})\to\Gamma(Y,f^*(\mathscr{F}\otimes\mathscr{L}^{\otimes n}))$ and as

$$f^*(\mathscr{F}\times\mathscr{L}^{\otimes n})=f^*(\mathscr{F})\otimes(f^*(\mathscr{L}))^{\otimes n}$$

these homomorphisms give rise to a bi-homomorphism $\Gamma_*(\mathscr{F};\mathscr{L})\to\Gamma_*(f^*(\mathscr{L}),f^*(\mathscr{F}))$ of graded modules.

Proposition 1.4.27. *Let $f:X\to Y$ be a morphism of ringed spaces, $\mathscr{F}$ be an $\mathscr{O}_X$-module, and $\mathscr{E}$ be a locally free $\mathscr{O}_Y$-module of finite rank. Then there exists a canonical isomorphism*

$$f_*(\mathscr{F})\otimes_{\mathscr{O}_Y}\mathscr{E}\cong f_*(\mathscr{F}\otimes_{\mathscr{O}_X}f^*(\mathscr{E})).$$

Proof. For any $\mathscr{O}_Y$-module $\mathscr{E}$, we have canonical homomorphisms

$$f_*(\mathscr{F})\otimes_{\mathscr{O}_Y}\mathscr{E}\xrightarrow{1\otimes\rho_{\mathscr{E}}}f_*(\mathscr{E})\otimes_{\mathscr{O}_Y}f_*(f^*(\mathscr{E}))\xrightarrow{\alpha}f_*(\mathscr{F}\otimes_{\mathscr{O}_X}f^*(\mathscr{E}))$$

Since $\mathscr{E}$ is locally free and the question is local on Y, we may assume that $\mathscr{E}=\mathscr{O}_Y^n$; as f_* and f^* commutes with finite direct sums, we can also suppose that $n=1$, and in this case, the proposition follows directly from the definition and the relation $f^*(\mathscr{O}_Y)=\mathscr{O}_X$. □

Proposition 1.4.28. *Let $f:X\to Y$ be a morphism of ringed spaces, $\mathscr{F},\mathscr{G}$ be two $\mathscr{O}_Y$-modules, and suppose that $\mathscr{F}$ is locally free of finite rank. Then the canonical homomorphism*

$$f^*(\mathcal{H}om_{\mathscr{O}_Y}(\mathscr{F},\mathscr{G}))\to\mathcal{H}om_{\mathscr{O}_X}(f^*(\mathscr{F}),f^*(\mathscr{G}))$$

is an isomorphism.

Proof. Since the question is local on Y, we can assume that $\mathscr{F}=\mathscr{O}_Y^n$; then $\mathcal{H}om_{\mathscr{O}_Y}(\mathscr{O}_Y,\mathscr{G})$ is identified with $\mathscr{G}^n$, $f^*(\mathscr{F})$ is identified with $\mathscr{O}_X^n$, and $\mathcal{H}om_{\mathscr{O}_X}(f^*(\mathscr{F}),f^*(\mathscr{G}))$ is identified with $(f^*(\mathscr{G}))^n$, whence the assertion. □

Remark 1.4.1. Let X be a ringed space. For any integer $n>0$, we prove that there is a *set* $\mathfrak{M}_n$ (denoted also by $\mathfrak{M}_n(X)$) of locally free $\mathscr{O}_X$-module of rank n such that any locally free $\mathscr{O}_X$-module of rank n is isomorphic to an element of $\mathfrak{M}_n$. For this, we consider the set $\mathfrak{R}_n$ of couples $(\mathfrak{U},\Theta)$, where $\mathfrak{U}$ is an open covering of X and Θ is a family $(\theta_{UV})_{(U,V)\in\mathfrak{U}\times\mathfrak{U}}$, where θ_{UV} is an automorphism of $\mathscr{O}_X^n|_{(U\cap V)}$, θ_{UU} is the identity automorphism of $\mathscr{O}_U^n$, and the family (θ_{UV}) satisfies the cocycle conditions. Then any element $(\mathfrak{U},\Theta)$ of $\mathfrak{R}_n$ corresponds to a well-defined locally free $\mathscr{O}_X$-module of rank n. If we denote by $\mathfrak{L}_n$ the set of these $\mathscr{O}_X$-modules, then any locally free $\mathscr{O}_X$-module of rank n is isomorphic to one of the elements in $\mathfrak{L}_n$; it then suffices to choose for $\mathfrak{M}_n$ a system of representatives of $\mathfrak{L}_n$ for the equivalence condition: $\mathscr{E}$ and $\mathscr{E}'$ are equivalent if they are isomorphic. For any locally free $\mathscr{O}_X$-module $\mathscr{E}$ of rank n, we denote by $\mathrm{cl}(\mathscr{E})$ the uique element of $\mathfrak{M}_n$ which is isomorphic to $\mathscr{E}$.

We can define a composition law on the set $\mathfrak{M}_1$ by associating two elements $\mathscr{L}$, $\mathscr{L}'$ of $\mathfrak{M}_1(X)$ the element $\mathrm{cl}(\mathscr{L}\otimes\mathscr{L}')$. It is clear that this law is associative and commutative and with identity element $\mathrm{cl}(\mathscr{O}_X)$. Moreover for any $\mathscr{L}\in\mathfrak{M}_1(X)$, $\mathrm{cl}(\mathscr{L}^{-1})$ is the inverse of $\mathscr{L}$ for this composition law. We then define a commutative group structure on $\mathfrak{M}_1(X)$, and this group is called the **Picard group** of the ringed space X and denoted by $\mathrm{Pic}(X)$. We also note that there exists a canonical isomorphism

$$\varphi_X : H^1(X,\mathscr{O}_X^\times) \xrightarrow{\sim} \mathrm{Pic}(X).$$

For this, we note that for any open subset U of X the multiplicative group $\Gamma(U,\mathscr{O}_X^\times)$ is canonically identified with the grop of automorphisms of $\mathscr{O}_U$-module $\mathscr{O}_U$, which send each section ε of $\mathscr{O}_X^\times$ over U to the automorphism $u:\mathscr{O}_U\to\mathscr{O}_U$ such that $u_x(s_x)=\varepsilon_x s_x$ for any $x\in U$ and any $s_x\in\mathscr{O}_{X,x}$. Let $\mathfrak{U}=(U_\lambda)$ be an open covering of X; the datum that, given any couple (λ,μ) of indices, an automorphism θ_λ of $\mathscr{O}_X|_{U_\lambda\cap U_\mu}$ which satisfy the cocycle conditions, is then equivalent to giving a 1-cochain of the covering $\mathfrak{U}$, with values in $\mathscr{O}_X^\times$. Similarly, the datum that, given an index λ, an automorphism ω_λ of $\mathscr{O}_{U_\lambda}$, is equivalent to giving a 0-cochain of $\mathfrak{U}$ with values in $\mathscr{O}_X^\times$, and the coboundary of this cochain corresponds to the automorphisms $(\omega_\lambda|_{U_\lambda\cap U_\mu})\circ(\omega_\mu|_{U_\lambda\cap U_\mu})^{-1}$. We then corresponds any 1-cocycle $(\theta_{\lambda\mu})$ of $\mathfrak{U}$ with values in $\mathscr{O}_X^\times$, the element $\mathrm{cl}(X)$ of $\mathrm{Pic}(X)$, where $\mathscr{L}$ is the locally free $\mathscr{O}_X$-module of rank 1 defined by the family $\theta=(\theta_{\lambda\mu})$; two cohomologous cocycles then correspond to the same element of $\mathrm{Pic}(X)$, so we obtain a map $\varphi_{\mathfrak{U}}: H^1(\mathfrak{U},\mathscr{O}_X^\times)\to\mathrm{Pic}(X)$. Moreover, if $\mathfrak{V}$ is a second open covering of X, which is a refinement of $\mathfrak{U}$, the diagram

$$\begin{tikzcd}
H^1(\mathfrak{U},\mathscr{O}_X^\times) \arrow[dd] \arrow[dr, "\varphi_{\mathfrak{U}}"] & \\
& \mathrm{Pic}(X) \\
H^1(\mathfrak{V},\mathscr{O}_X^\times) \arrow[ur, "\varphi_{\mathfrak{V}}"] &
\end{tikzcd}$$

where the vertical arrow is the canonical homomorphism, is commutative. By passing to limit, we then obtain a map $\varphi_X : H^1(X,\mathscr{O}_X^\times)\to\mathrm{Pic}(X)$, where the Čech cohomology group $\check{H}^1(X,\mathscr{O}_X^\times)$ is canonically identified with $H^1(X,\mathscr{O}_X^\times)$. The map φ_X is clearly surjective, since any locally free sheaf of rank 1 is defined by a 1-cocycle. It is injective, because it suffices to show that the maps $\varphi_{\mathfrak{U}}$ are injective, and this follows from the definition of $H^1(\mathfrak{U},\mathscr{O}_X^\times)$. It remains to prove that $\varphi_{\mathfrak{U}}$ is a homomorphism of groups. Let $\mathscr{L},\mathscr{L}'$ be two locally free $\mathscr{O}_X$-modules of rank 1 such that, for any λ, $\mathscr{L}|_{U_\lambda}$ and $\mathscr{L}'|_{U_\lambda}$ are isomorphic to $\mathscr{O}_{U_\lambda}$. There then exists for each λ an element a_λ (resp. a'_λ) of $\Gamma(U_\lambda,\mathscr{L})$ (resp. $\Gamma(U_\lambda,\mathscr{L}')$) such that the elements of $\Gamma(U_\lambda,\mathscr{L})$ (resp. $\Gamma(U_\lambda,\mathscr{L}')$) are of the form $s_\lambda\cdot a_\lambda$ (resp. $s_\lambda\cdot a'_\lambda$) where s_λ runs through $\Gamma(U_\lambda,\mathscr{O}_X)$. The corresponding cocycles $(\varepsilon_{\lambda\mu})$, $(\varepsilon'_{\lambda\mu})$ are such that the relation $s_\lambda\cdot a_\lambda = s_\mu\cdot a_\mu$ (resp. $s_\lambda\cdot a'_\lambda = s_\mu\cdot a'_\mu$) over $U_\lambda\cap U_\mu$ is equivalent to $s_\lambda=\varepsilon_{\lambda\mu}s_\mu$ (resp. $s_\lambda=\varepsilon'_{\lambda\mu}s_\mu$) over $U_\lambda\cap U_\mu$. As the sections of $\mathscr{L}\otimes\mathscr{L}'$ over U_λ are the finite sums of $s_\lambda s'_\lambda\cdot(a_\lambda\otimes a'_\lambda)$ where s_λ, s'_λ runs through $\Gamma(U_\lambda,\mathscr{O}_X)$, it is clear that the cocycle $(\varepsilon_{\lambda\mu}\varepsilon'_{\lambda\mu})$ corresponds to $\mathscr{L}\otimes\mathscr{L}'$, which completes the proof.

Let $f : X' \to X$ be a morphism of ringed spaces. If $\mathscr{L}_1$, $\mathscr{L}_2$ are two locally free $\mathscr{O}_X$-modules of rank 1 and are isomorphic, the $\mathscr{O}_{X'}$-modules $f^*(\mathscr{L}_1)$ and $f^*(\mathscr{L}_2)$ are isomorphic. On the other hand, for any $\mathscr{O}_X$-modules $\mathscr{F}, \mathscr{G}$, we have $f^*(\mathscr{F} \otimes \mathscr{G}) = f^*(\mathscr{F}) \otimes f^*(\mathscr{G})$. We then conclude that the morphism f defines a canonical homomorphism of abelian groups

$$\mathrm{Pic}(f) : \mathrm{Pic}(X) \to \mathrm{Pic}(X').$$

On the other hand, we have a canonical homomorphism

$$H^1(f) : H^1(X, \mathscr{O}_X^\times) \to H^1(X', \mathscr{O}_{X'}^\times)$$

corresponds the restriction of the homomorphism $f^\# : \mathscr{O}_X \to f_*(\mathscr{O}_{X'})$ to $\mathscr{O}_X^\times$. We claim that the diagram

$$\begin{tikzcd} H^1(X, \mathscr{O}_X^\times) \arrow[r, "H^1(f)"] \arrow[d, "\varphi_X"', "\sim"] & H^1(X', \mathscr{O}_{X'}^\times) \arrow[d, "\sim"', "\varphi_{X'}"] \\ \mathrm{Pic}(X) \arrow[r, "\mathrm{Pic}(f)"] & \mathrm{Pic}(X') \end{tikzcd}$$

is commutative. In fact, if $\mathscr{L}$ is given by the cocycle $(\varepsilon_{\lambda\mu})$ of an open covering (U_λ) of X, it suffices to prove that $f^*(\mathscr{L})$ is defined by a cocycle whose class is cohomologous to the image of $(\varepsilon_{\lambda\mu})$ under $H^1(f)$. But if $\theta_{\lambda\mu}$ is the automorphism of $\mathscr{O}_X|_{U_\lambda \cap U_\mu}$ corresponding to $\varepsilon_{\lambda\mu}$, it is clear that $f^*(\mathscr{L})$ is obtained by glueing $\mathscr{O}_{X'}|_{\psi^{-1}(U_\lambda)}$ with the automorphisms $g^*(\theta_{\lambda\mu})$, and it sufficies to verify that this corresponds to the cocycle $(f^\#(\varepsilon_{\lambda\mu}))$; this follows also from the definition by identify $\varepsilon_{\lambda\mu}$ with its image under $\rho_{\mathscr{O}_X}$, which is a section of $\psi^{-1}(\mathscr{O}_X)$ over $\psi^{-1}(U_\lambda \cap U_\mu)$.

We now consider locally free sheaves over *locally* ringed spaces. The point of locality is that we can consider sections of a locally free sheaf as functions over X. More precisely, if $s \in \Gamma(U, \mathscr{E})$ for some locally free $\mathscr{O}_X$-module $\mathscr{E}$ of rank r, then for $x \in U$ we can define $s(x)$ to be the image of s_x in $\mathscr{E}_x / \mathfrak{m}_x \mathscr{E}_x$ (this is called the *value* of s at x, written as $s(x)$, do not confuse it with the *stalk* s_x at x). Since $\mathscr{E}_x \cong \mathscr{O}_{X,x}^r$, we can then think s as a function on x with r components. As we shall see, this is the common situation when we talk about schemes.

Proposition 1.4.29. *Let X be a locally ringed space and $\mathscr{E}$ be a locally free $\mathscr{O}_X$-module of finite rank. Then for any section s of $\mathscr{E}$ over X, the set X_s of $x \in X$ such that $s(x) \neq 0$ is open in X and s is invertible over X_s.*

Proof. Since the question is local, we can assume that $\mathscr{E} = \mathscr{O}_X^n$. If $(s_j)_{1 \leq j \leq n}$ is the projection of s to the n-th component, we see $s(x) \neq 0$ if and only if $s_j(x) \neq 0$ for some j, so X_s is the union of the X_{s_j}, and we are reduced to the case $n = 1$. But for $s \in \Gamma(X, \mathscr{O}_X)$, to say that $s(x) \neq 0$ amounts to that $s_x \notin \mathfrak{m}_x$, hence s_x is inveritble in $\mathscr{O}_{X,x}$. By then there then existes an open neighborhood U of x in X such that $s|_U$ is invertible in $\Gamma(U, \mathscr{O}_X)$, and therefore $s(y) \neq 0$ for any $y \in U$. □

Corollary 1.4.30. *Let X be a locally ringed space and $\mathscr{E}$ be a locally free $\mathscr{O}_X$-module of finite rank. Let $s_1, \dots, s_p$ be sections of $\mathscr{E}$ over X. Then the set of $x \in X$ such that $s_1(x), \dots, s_p(x)$ are lineraly independent over $\mathscr{E}_x / \mathfrak{m}_x \mathscr{E}_x$ is open in X.*

Proof. We know that $\bigwedge^p \mathscr{E}$ is a locally free $\mathscr{O}_X$-module of rank $\binom{n}{p}$. Moreover, for any $x \in X$, the stalk $(\bigwedge^p \mathscr{E})_x / \mathfrak{m}_x (\bigwedge^p \mathscr{E})_x$ is identified with $\bigwedge^p (\mathscr{E}_x / \mathfrak{m}_x \mathscr{E}_x)$. If $s = s_1 \wedge \cdots \wedge s_p$, then $s(x)$ is identified with $s_1(x) \wedge \cdots \wedge s_p(x)$, and $s(x) \neq 0$ if and only if $s_1(x), \dots, s_p(x)$ are linearly independent, whence the assertion by Proposition 1.4.29. □

Corollary 1.4.31. *Let X be a locally ringed space, $\mathscr{E}$ be a locally free $\mathscr{O}_X$-module of rank n, and $s_1, \dots, s_n$ be sections of $\mathscr{E}$ over X such that, for any $x \in X$, $s_1(x), \dots, s_n(x)$ are lineraly independent. Then the homomorphism $u : \mathscr{O}_X^n \to \mathscr{E}$ defined by the sections s_i is bijective.*

Proof. Again we can assume that $\mathscr{E} = \mathscr{O}_X^n$ and canonically identity $\bigwedge^n \mathscr{E}$ with $\mathscr{O}_X$. The section $s = s_1 \wedge \cdots \wedge s_n$ is then a section of $\mathscr{O}_X$ over X such that $s(x) \neq 0$ for all $x \in X$, and thus invertible in $\Gamma(X, \mathscr{O}_X)$. We can then define a homomorphism inverse to u by Cramer's rule. □

Proposition 1.4.32. *Let X be a locally ringed space, $\mathscr{F}$ be an $\mathscr{O}_X$-module of finite type, $\mathscr{E}$ a locally free $\mathscr{O}_X$-module of finite rank, and $u : \mathscr{F} \to \mathscr{E}$ be a homomorphism, and x be a point of X. Then the following conditions are equivalent:*

(i) The homomorphism u_x is left invertible (which means u_x is injective and its image in $\mathscr{E}_x$ is a direct factor).

(ii) The homomorphism $u_x \otimes 1 : \mathscr{F}_x/\mathfrak{m}_x\mathscr{F}_x \to \mathscr{E}_x/\mathfrak{m}_x\mathscr{E}_x$ of vector spaces induced by u_x is injective.

(iii) There exists an open neighborhood U of x such that $u|_U : \mathscr{F}|_U \to \mathscr{E}|_U$ is left invertible, the image $u(\mathscr{F})|_U$ is a locally free $\mathscr{O}_U$-module isomorphic to $\mathscr{F}|_U$, which admits a locally free complement in $\mathscr{E}|_U$.

Moreover, the set of $x \in X$ satisfying these equivalent conditions are open.

Proof. The equivalence of (i) and (ii) is a general result in algebras, and it is clear that (iii) implies (i). We now prove that (i) implies (iii); there exists by hypothesis a homomorphism $w : \mathscr{E}_x \to \mathscr{F}_x$ such that $w \circ u_x$ is the identify automorphism on $\mathscr{F}_x$. As $\mathscr{E}$ is locally free of finite rank, hence of finite presentation, it follows from Proposition 1.4.14 that there exists an open neigbourhood U of x and a homomorphism $v : \mathscr{E}|_U \to \mathscr{F}|_U$ such that $w = v_x$, hence $(v \circ (u|_U))_x = v_x \circ u_x$ is the identity automorphism. We then conclude that, by restricting U, we can suppose that $v \circ (u|_U)$ is the identity on $\mathscr{F}|_U$, which means $u|_U$ is left invertible. We then know that $p = (u|_U) \circ v$ is a projection from $\mathscr{E}|_U$ to $u(\mathscr{F})|_U$, and that $u|_U$ is an isomorphism from $\mathscr{F}|_U$ to $u(\mathscr{F})|_U$. We claim that (aftering shrinking U if necessary), $u(\mathscr{E})_U$ and $\ker p$ are locally free $\mathscr{O}_U$-modules (supplementary in $\mathscr{F}|_U$). In fact, $p_x : \mathscr{F}_x \to (u(\mathscr{F}))_x$ is a projection, hence so is $p_x \otimes 1 : \mathscr{E}_x/\mathfrak{m}_x\mathscr{E}_x \to (u(\mathscr{F}))_x/\mathfrak{m}_x(u(\mathscr{F}))_x$. There exists sections s_j $(1 \le j \le n)$ of $\mathscr{F}$ over U such that the first m sections s_j $(1 \le j \le m)$ are sections of $u(\mathscr{F})$, and the later $n - m$ are sections of $\ker p$, and that the $s_j(x)$ $(1 \le j \le n)$ form a base for $\mathscr{F}_x/\mathfrak{m}_x\mathscr{F}_x$. In view of Corollary 1.4.31, by shrinking U, we can suppose that the $\mathscr{O}_X$-module $\mathscr{M}$ generated by s_j for $1 \le j \le n$ and the $\mathscr{O}_X$-module $\mathscr{N}$ generated by the s_j for $m+1 \le j \le n$ are free and supplementary in $\mathscr{E}|_U$. We have evidently $\mathscr{M} \subseteq u(\mathscr{F})|_U$ and $\mathscr{N} \subseteq \ker p$; on the other hand, if $i : \mathscr{M} \to u(\mathscr{F})|_U$ and $j : \mathscr{N} \to \ker p$ are the canonical injections, then the choice of s_j implies that i_x and j_x are bijections. As $u(\mathscr{F})|_U$ and $\ker p$ are $\mathscr{O}_U$-modules of finite type (the second being the image of $\mathscr{E}|_U$ under $1 - p$), we conclude from Proposition 1.4.6 (by shrinking U if necessary) that $\mathscr{M} = u(\mathscr{E})|_U$ and $\mathscr{N} = \ker p$. □

Corollary 1.4.33. *With the hypotheses in Proposition 1.4.32, the following conditions are equivalent:*

(i) For any morphism $g : X' \to X$ of locally ringed spaces, the homomorphism $g^*(u) : g^*(\mathscr{F}) \to g^*(\mathscr{E})$ is injective.

(ii) For any $x \in X$, the homomorphism $u_x \otimes 1 : \mathscr{F}_x/\mathfrak{m}_x\mathscr{F}_x \to \mathscr{E}_x/\mathfrak{m}_x\mathscr{E}_x$ is injective.

(iii) For any $x \in X$, there exists an open neighborhood U of x such that $u|_U : \mathscr{F}|_U \to \mathscr{E}|_U$ is left invertible.

Moreover, if the conditions are satisfied, $\mathscr{F}$ is a locally free $\mathscr{O}_X$-module of finite rank.

Proof. The equivalence of (ii) and (iii) follows from Proposition 1.4.32, so does the last one. The fact that (iii) implies (i) follows from follows from the fact that we can reduce ourselves to the case where $\mathscr{E} = \mathscr{O}_X^n$ and $\mathscr{F} = \mathscr{O}_X^m$ and that g^* is left exact, since the question is local on X. Finally, we show that (ii) is a particular case of (i): it suffices to consider the locally ringed space X' reduced to a point x, with sheaf of rings $\kappa(x)$ (that is, $\mathrm{Spec}(\kappa(x))$). We let $g : X' \to X$ be the canonical morphism which maps x to x and $g^\# : \mathscr{O}_X \to \kappa(x)$ is the cannical homomorphism $\Gamma(U, \mathscr{O}_X) \to \mathscr{O}_{X,x} \to \kappa(x)$ for any open neigbourhood U of x. It is then easily verified that $g^*(u)$ is the homomorphism $u_x \otimes 1$. □

Remark 1.4.2. If u satisfies the conditions of Corollary 1.4.33, then we say that it is **universally injective**.

Corollary 1.4.34. *Let X be a locally ringed space, $\mathscr{F}$, $\mathscr{E}$ be two locally free $\mathscr{O}_X$-modules of finite rank, $u : \mathscr{F} \to \mathscr{E}$ be a homomorphism, and x be a point of X. The following conditions are equivalent:*

(i) *The homomorphism $u_x \otimes 1 : \mathscr{F}_x/\mathfrak{m}_x\mathscr{F}_x \to \mathscr{E}_x/\mathfrak{m}_x\mathscr{E}_x$ is surjective.*

(ii) *The homomorphism u_x is surjective.*

(iii) *The homomorphism $\bigwedge^m u_x : \bigwedge^m \mathscr{F}_x \to \bigwedge^m \mathscr{E}_x$ (where m is the rank of $\mathscr{F}_x$) is surjective.*

(iv) *The homomorphism $u_x^t : \mathscr{E}_x^* \to \mathscr{F}_x^*$ is left invertible.*

Moreover, the set S of $x \in X$ satisfying these conditions is open in X, $\ker(u)|_S$ is a locally free $\mathscr{O}_S$-module and any $x \in S$ admits an open neighborhood $U \subseteq S$ such that $\ker(u)|_U$ admits in $\mathscr{F}|_U$ a locally free complement (isomorphic to $\mathscr{E}|_U$).

Proof. The equivalence of (i) and (ii) follows from Nakayama's Lemma. Similarly, (iii) is equivalent to that

$$(\textstyle\bigwedge^m u_x) \otimes 1 : (\bigwedge^m \mathscr{F}_x)/\mathfrak{m}_x(\bigwedge^m \mathscr{F}_x) \to (\bigwedge^m \mathscr{E}_x)/\mathfrak{m}_x(\bigwedge^m \mathscr{E}_x)$$

is surjective; but $(\bigwedge^m u_x) \otimes 1$ is identified with

$$\textstyle\bigwedge^m (u_x \otimes 1) : \bigwedge^m (\mathscr{F}_x/\mathfrak{m}_x\mathscr{F}_x) \to \bigwedge^m (\mathscr{E}_x/\mathfrak{m}_x\mathscr{E}_x),$$

and as $\mathscr{E}_x/\mathfrak{m}_x\mathscr{E}_x$ is a vector space of dimension m over $\kappa(x)$, $\bigwedge^m (u_x \otimes 1)$ is surjective if $u_x \otimes 1$ is surjective, and is zero in the contrary case, whence the equivalence of (i) and (iii). On the other hand, as $(\mathscr{F}^*)^* = \mathscr{F}$, $(\mathscr{E}^*)^* = \mathscr{E}$ and $(u^t)^t = u$, it is the same to say that $u_x \otimes 1$ is surjective and that $(u_x \otimes 1)^t = (u_x)^t \otimes 1 : \mathscr{E}_x^*/\mathfrak{m}_x\mathscr{E}_x^* \to \mathscr{F}_x^*/\mathfrak{m}_x\mathscr{F}_x^*$ is injective, whence the equivalence of (i) and (iv) in view of Proposition 1.4.32. The fact that S is open follows from Proposition 1.4.32. We can then reduce to the case where $S = X$, and the other assertions of the statement are deduced by transposing the conclusions of Proposition 1.4.32 applied to u^t. □

Corollary 1.4.35. *With the notations of Corollary 1.4.34, suppose moreover that $\mathscr{F}$ and $\mathscr{E}$ have the same rank at each point. Then, for any $x \in X$, the following conditions are equivalent:*

(i) *u_x is left invertible;*

(ii) *u_x is surjective;*

(iii) *u_x is bijective.*

Moreover, the set of $x \in x$ satisfying these conditions is open in X.

Corollary 1.4.36. *With the notations of Corollary 1.4.34, let $f : X' \to X$ be a morphism of locally ringed spaces and put $\mathscr{F}' = f^*(\mathscr{F})$, $\mathscr{E}' = f^*(\mathscr{E})$, which are locally free $\mathscr{O}_{X'}$-modules of finite rank. Let $u' = f^*(u) : \mathscr{F}' \to \mathscr{E}'$. Then for a point $x' \in X'$, $u'_{x'}$ is surjective (resp. left invertible) if and only if at the point $x = f(x')$, u_x is surjective (resp. left invertible).*

Proof. In fact, we have $\mathscr{F}'_{x'} = \mathscr{F}_x \otimes_{\mathscr{O}_{X,x}} \mathscr{O}_{X',x'}$, $\mathscr{E}'_{x'} = \mathscr{F}_x \otimes_{\mathscr{O}_{X,x}} \mathscr{O}_{X',x'}$, and $u'_{x'}$ is deduced from u_x be base changing $\mathscr{O}_{X,x}$ to $\mathscr{O}_{X',x'}$. If k and k' are the residue fields of x and x', we then have $\mathscr{F}'_{x'} \otimes k' = (\mathscr{F}_x \otimes k) \otimes_k k'$, $\mathscr{F}'_{x'} \otimes k' = (\mathscr{F}_x \otimes k) \otimes_k k'$, and the homomorphism $u'_{x'} \otimes 1 : \mathscr{F}'_{x'} \otimes k' \to \mathscr{E}'_{x'} \otimes k'$ is then deduced from $u_x \otimes 1_k : \mathscr{F}_x \otimes k \to \mathscr{E}_x \otimes k$ by base changing from k to k'. The conclusion then follows from the fact that this base change is faithfully flat, the Nakayama lemma, and Proposition 1.4.32. □

Proposition 1.4.37. *Let X be a locally ringed space, $\mathscr{L}$ be an $\mathscr{O}_X$-module of finite type. For that $\mathscr{L}$ to be locally free of rank 1, it is necessary and sufficient that there exists an $\mathscr{O}_X$-module $\mathscr{F}$ such that $\mathscr{L} \otimes_{\mathscr{O}_X} \mathscr{F}$ is isomorphic to $\mathscr{O}_X$. Moreover, any $\mathscr{O}_X$-module with this property is isomorphic to $\mathscr{L}^{-1}$.*

Proof. We have seen that if $\mathscr{L}$ is locally of rank 1 then $\mathscr{L}^{-1} \otimes_{\mathscr{O}_X} \mathscr{L} \cong \mathscr{O}_X$. Moreover, in this case $\mathscr{L}^{-1}$ is isomorphic to $\mathscr{L}^{-1} \otimes_{\mathscr{O}X} \mathscr{O}_X$, hence to $\mathscr{L}^{-1} \otimes_{\mathscr{O}_X} (\mathscr{L} \otimes_{\mathscr{O}_X} \mathscr{F})$ and therefore to $\mathscr{F}$. It then remains to prove that if $\mathscr{L} \otimes_{\mathscr{O}_X} \mathscr{F}$ is isomorphic to $\mathscr{O}_X$ then $\mathscr{L}$ is locally free of rank 1. Let $x \in X$ and put $A = \mathscr{O}_{X,x}$ (which is a local ring with maximal ideal $\mathfrak{m}$), $M = \mathscr{L}_x$, $N = \mathscr{F}_x$. The hypothesis implies that $M \otimes_A N$ is isomorphic to A, and as $(A/\mathfrak{m}) \otimes_A (M \otimes_A N)$ is identified with $(M/\mathfrak{m}M) \otimes_{A/\mathfrak{m}} (N/\mathfrak{m}N)$, this tensor product is isomorphic to $A/\mathfrak{m}$ over the field $A/\mathfrak{m}$, which shows that $M/\mathfrak{m}M$ and $N/\mathfrak{m}N$ are of dimension 1 over $A/\mathfrak{m}$. For any element $z \in M$ not belonging to $\mathfrak{m}M$, we then have $M = Az + \mathfrak{m}M$, which implies $M = Az$ by Nakayama's Lemma. Also, the annihilator of z also annihilates $M \otimes_A N$, which is isomorphic to A, so this annihilator is zero and M is isomorphic to A. There is then a section s of $\mathscr{L}$ over an open neigbourhood U of x such that $t_x \mapsto t_x s_x$ is an isomorphism from $\mathscr{O}_{X,x}$ to $\mathscr{L}_x$. Since $\mathscr{L}$ is of finite type, we can, by shrinking U, suppose that s generates $\mathscr{L}|_U$ (Proposition 1.4.6), which means we have a surjective homomorphism $u : \mathscr{O}_U \to \mathscr{L}|_U$. Moreover, for any $y \in U$, the homomorphism $\mathscr{O}_{X,y}/\mathfrak{m}_y \to \mathscr{L}_y/\mathfrak{m}_y\mathscr{L}_y$ deduced from u is bijective, hence so is u (**??**). □

The $\mathscr{O}_X$-modules locally free of rank 1 over a locally ringed space X are then called the **invertible** $\mathscr{O}_X$-modules.

Proposition 1.4.38. *Let X be a locally ringed space, $\mathscr{L}$ be an invertible $\mathscr{O}_X$-module, and f be a section of $\mathscr{L}$ over X. For any $x \in X$, the following conditions are equivalent:*

(i) *f_x generates the $\mathscr{O}_{X,x}$-module $\mathscr{L}_x$.*

(ii) *$f_x \notin \mathfrak{m}_x\mathscr{L}_x$ (that is, $f(x) \neq 0$).*

(iii) *There exists a section g of $\mathscr{L}^{-1}$ over an open neigbourhood V of x such that the canonical image of $(f|_V) \otimes g$ in $\Gamma(V, \mathscr{O}_X)$ is the unit element.*

Moreover, the set X_f of $x \in X$ satisfying these conditions is open in X.

Proof. The question is local on X so we can assume that $\mathscr{L} = \mathscr{O}_X$, and the proposition then follows. □

Chapter 2

Cohomology group of sheaves

In this chapter we consider the cohomology of shaves of modules. First we have a proposition.

Proposition 2.0.1. *A sequence of sheaves of $\mathscr{O}_X$-modules on a space X*

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{G} \longrightarrow \mathscr{H} \longrightarrow 0$$

is exact if and only if for all points $x \in X$ the sequence of stalks is exact. This is equivalent to

(a) For all open sets $U \subseteq X$ the sequence

$$0 \longrightarrow \mathscr{F}(U) \longrightarrow \mathscr{G}(U) \longrightarrow \mathscr{H}(U)$$

is exact.

(b) For any $s \in \mathscr{H}(U)$, we can find a covering $U = \bigcup_i U_i$ by open sets and $s_i \in \mathscr{F}(U_i)$ such that $s|_{U_i} = s_i$.

Applied to $U = X$ this tells us that the functor of global sections $\mathscr{F} \mapsto \mathscr{F}(X)$ is left-exact. It turns out that the category of sheaves of $\mathscr{O}_X$-modules has enough injectives, thus the right derived functors $\mathcal{R}^p\Gamma(X,-)$ exist, and for every sheaf $\mathscr{F}$ on X, the cohomology groups $\mathcal{R}^p\Gamma(X,-)(\mathscr{F})$ are defined. These groups denoted by $H^p(X,\mathscr{F})$ are called the **cohomology groups of the sheaf $\mathscr{F}$** or **the cohomology groups of X with values in $\mathscr{F}$**.

2.1 Definition of sheaf cohomology

We first show that the category $\mathbf{Mod}(\mathscr{O}_X)$ has enough injectives.

Proposition 2.1.1. *Let $(X,\mathscr{O}_X)$ be a ringed space. Then the category $\mathbf{Mod}(\mathscr{O}_X)$ of sheaves of $\mathscr{O}_X$-modules has enough injectives.*

Proof. Let $\mathscr{F}$ be a sheaf of $\mathscr{O}_X$-modules. For each point $x \in X$, the stalk $\mathscr{F}_x$ is an $\mathscr{O}_{X,x}$-module. Therefore there is an injection $\mathscr{F}_x \to I_x$, where $\mathscr{I}_x$ is an injective $\mathscr{O}_{X,x}$-module. For each point x, let i_x denote the inclusion of the one-point space $\{x\}$ into X, and consider the sheaf $\mathscr{I} = \prod_{X\in X} i_{x,*}(I_x)$. Here we consider I_x as a sheaf on the one-point space $\{x\}$.

Now for any sheaf $\mathscr{G}$ of $\mathscr{O}_X$-modules, we have

$$\mathrm{Hom}_{\mathscr{O}_X}(\mathscr{G},\mathscr{I}) = \prod_{x\in X} \mathrm{Hom}_{\mathscr{O}_X}(\mathscr{G}, i_{x,*}(I_x))$$

by definition of the direct product. On the other hand, for each point $x \in X$, since $i_{x,*}$ is the adjoint of taking stalk, we have

$$\mathrm{Hom}_{\mathscr{O}_X}(\mathscr{G}, i_{x,*}(I_x)) = \mathrm{Hom}_{\mathscr{O}_{X,x}}(\mathscr{G}_x, I_x)$$

Thus we conclude first that there is a natural morphism of sheaves of $\mathscr{O}_X$-modules $\mathscr{F} \to \mathscr{I}$ obtained from the local maps $\mathscr{F}_x \to \mathscr{I}_x$. It is clearly injective. Second, the functor $\mathrm{Hom}_{\mathscr{O}_X}(-, \mathscr{I})$ is the direct product over all $x \in X$ of the stalk functor $\mathscr{G} \to \mathscr{G}$ which is exact, followed by $\mathrm{Hom}_{\mathscr{O}_{X,x}}(-, I_x)$, which is exact, since I_x is a injective. Hence $\mathrm{Hom}(-, \mathscr{I})$ is an exact functor, and therefore $\mathscr{I}$ is an injective $\mathscr{O}_X$-module. □

Corollary 2.1.2. *If X is any topological space, then the category $\mathbf{Ab}(X)$ of sheaves of abelian groups on X has enough injectives.*

Definition 2.1.1. Let X be a topological space, and let $\Gamma(X, -) : \mathbf{Ab}(X) \to \mathbf{Ab}(X)$ be the global section functor. The cohomology groups of the sheaf $\mathscr{F}$ or the cohomology groups of X with values in $\mathscr{F}$, denoted by $H^i(X, \mathscr{F})$, are the groups $\mathcal{R}^i\Gamma(X, -)(\mathscr{F})$ induced by the right derived functor $\mathcal{R}^i\Gamma(X, -)$.

Similarly, we can define cohomology groups of a $\mathscr{O}_X$-module to be the right-derived functor $\mathcal{R}^i\Gamma(X, -)$, where $\Gamma(X, -)$ is viewed as a functor from $\mathbf{Mod}(\mathscr{O}_X)$ to $\mathbf{Ab}(X)$. However, it turns out that this definition is unnecessary: the dericed functor of $\Gamma(X, -)$ in $\mathbf{Ab}(X)$ and $\mathbf{Mod}(\mathscr{O}_X)$ coincide.

2.2 Flasque sheaves

Definition 2.2.1. A sheaf $\mathscr{F}$ on a topological space X is **flasque** if for every open subset U of X the restriction map $\mathscr{F}(X) \to \mathscr{F}(U)$ is surjective.

We will see shortly that injective sheaves are flasque. Although this is not obvious from the definition, the notion of being flasque is local.

Proposition 2.2.1. *Let $\mathscr{F}$ be an $\mathscr{O}_X$-module. If $\mathscr{F}$ is flasque, so is $\mathscr{F}|_U$ for every open subset U of X. Conversely, if for every $x \in X$, there is a neighborhood U such that $\mathscr{F}|_U$ is flasque, then $\mathscr{F}$ is flasque.*

Proof. The first statement is trivial, let us prove the converse. Given any open set V of X, let s be a section of $\mathscr{F}$ over V. Let T be the set of all pairs (U, σ), where U is an open in X containing V, and σ is an extension of s to U. Partially order T by saying that $(U_1, \sigma_1) \leq (U_2, \sigma_2)$ if $U_1 \subseteq U_2$ and σ_2 extends σ_1, and observe that T is inductive, which means that every chain has an upper bound. Zorn's lemma provides us with a maximal extension of s to a section σ over an open set U_0. Were U_0 not X, there would exist an open set W in X not contained in U_0 such that $F|_W$ is flasque. Thus we could extend the section $\sigma|_{U_0 \cap W}$ to a section σ_0 of $\mathscr{F}$. Since σ and σ_0 agree on $U_0 \cap W$ by construction, their common extension to $U_0 \cup W$ extends s, a contradiction. □

Proposition 2.2.2. *Every $\mathscr{O}_X$-module may be embedded in a canonical functorial way into a flasque $\mathscr{O}_X$-module. Consequently, every $\mathscr{O}_X$-module has a canonical flasque resolution.*

Proof. Let $\mathscr{F}$ be an $\mathscr{O}_X$-module, and consider the Godement construction

$$U \mapsto \prod_{x \in U} \mathscr{F}_x$$

which we denote by $C^0(X, \mathscr{F})$. It is immediate that we have an injection of $\mathscr{O}_X$-modules $\mathscr{F} \to C^0(X, \mathscr{F})$ by Proposition 1.1.1. An element of $C^0(X, \mathscr{F})$ over any open set U is a collection (s_x) of elements indexed by U, each s_x lying over the $\mathscr{O}_{X,x}$-module $\mathscr{F}_x$. Such a sheaf is flasque because every U-indexed sequence s_x can be extended to an X-indexed sequence by assigning any arbitrary element of $\mathscr{F}_x$ to any $x \in X - U$. Hence $\mathbf{Mod}(\mathscr{O}_X)$ possesses enough flasque sheaves.

If $\mathscr{Z}_1$ is the cokernel of the canonical injection $\mathscr{F} \to C^0(X, \mathscr{F})$, we define $C^1(X, \mathscr{F})$ to be the flasque sheaf $C^0(X, \mathscr{Z}_1)$. In general,

$$\mathscr{Z}_n = \operatorname{coker}\left(\mathscr{Z}_{n-1} \hookrightarrow C^0(X, \mathscr{Z}_{n-1})\right) \quad \text{and} \quad C^n(X, \mathscr{F}) = C^0(X, \mathscr{Z}_n).$$

Putting all this information together, we obtain the desired flasque resolution of $\mathscr{F}$

$$0 \longrightarrow \mathscr{F} \longrightarrow C^0(X, \mathscr{F}) \longrightarrow C^1(X, \mathscr{F}) \longrightarrow C^2(X, \mathscr{F}) \longrightarrow \cdots$$

as claimed. □

Remark 2.2.1. The resolution of $\mathscr{F}$ constructed above will be called the **canonical flasque resolution** of $\mathscr{F}$ or the **Godement resolution** of $\mathscr{F}$.

Here is the principal property of flasque sheaves.

Theorem 2.2.3. *Let $0 \to \mathscr{F}_1 \to \mathscr{F}_2 \to \mathscr{F}_3 \to 0$ be an exact sequence of $\mathscr{O}_X$-modules, and assume $\mathscr{F}_1$ is flasque. Then this sequence is exact as a sequence of presheaves. If both $\mathscr{F}_1$ and $\mathscr{F}_2$ are flasque, so is $\mathscr{F}_3$. Finally, any direct summand of a flasque sheaf is flasque.*

Proof. Given any open set U, we must prove that

$$0 \longrightarrow \mathscr{F}_1(U) \xrightarrow{\varphi} \mathscr{F}_2(U) \xrightarrow{\psi} \mathscr{F}_3(U) \longrightarrow 0$$

is exact. By Proposition 1.1.30 and Proposition 1.1.31, the sole problem is to prove that $\mathscr{F}_2(U) \to \mathscr{F}_3(U)$ is surjective. By restricting we only need to prove the case for X. Let t be a global section of $\mathscr{F}_3$, then by Proposition 1.1.31, locally t may be lifted to sections of $\mathscr{F}$. Let T be the family of all pairs (U, σ) where U is an open in X, and σ is a section of $\mathscr{F}$ over U whose image in $\mathscr{F}_3(U)$ equal $t|_U$. Partially order T as in the proof of Proposition 2.2.1 and observe that T is inductive. Zorn's lemma provides us with a maximal lifting of t to a section $\sigma \in \mathscr{F}(U_0)$.

Were U_0 not X, there would exist $x \in X - U_0$, a neighborhood V of x, and a section τ of $\mathscr{F}$ over V which is a local lifting of $t|_V$. The sections $\sigma|_{V \cap U_0}$ and $\tau|_{V \cap U_0}$ have the same image in $\mathscr{F}_3(U_0 \cap V)$ under the map ψ, so their difference maps to 0. Since $\operatorname{im} \varphi = \ker \psi$, there is a section s of $\mathscr{F}_1(U_0 \cap V)$ such that

$$\sigma|_{U_0 \cap V} = \tau|_{V \cap U_0} + \varphi(s).$$

Since $\mathscr{F}_1$ is flasque, the section s is the restriction of a section $s_0 \in \mathscr{F}_1(V)$. Upon replacing τ by $\tau + \varphi(t_0)$ (which does not affect the image in $\mathscr{F}_3(V)$), we may assume that $\sigma|_{V \cap U_0} = \tau|_{V \cap U_0}$; that is, τ and σ agree on the overlap. Then we may extend σ to $U_0 \cup V$, contradicting the maximality of (U_0, σ); hence, $U_0 = X$.

Now suppose that $\mathscr{F}_1$ and $\mathscr{F}_2$ are flasque. If $t \in \mathscr{F}_3(U)$, then by the above, there is a section $s \in \mathscr{F}_2(U)$ mapping onto t. Since $\mathscr{F}_2$ is also flasque, we may lift s to a global section s_0 of $\mathscr{F}$. The image t_0 of s_0 in $\mathscr{F}_3(X)$ is the required extension of t to a global section of $\mathscr{F}_3$.

Finally, assume that $\mathscr{F}$ is flasque, and that $\mathscr{F} = \mathscr{F}_1 \oplus \mathscr{F}_2$ for some sheaf $\mathscr{F}_1, \mathscr{F}_2$. For any open subset U of X and any section $s \in \mathscr{F}_1(U)$, we can make s into a section $\tilde{s} \in \mathscr{F}(U)$ by setting the component of $\tilde{s}(U)$ in $\mathscr{F}_2(U)$ equal to the zero section. Since $\mathscr{F}$ is flasque, there is some section $t \in \mathscr{F}(X)$ such that $t|_U = \tilde{s}$. Write $t = t_1 + t_2$ for some unique $t_1 \in \mathscr{F}_1(X)$ and $t_2 \in \mathscr{F}_2(X)$, then

$$s + 0 = \tilde{s} = t|_U = (t_1)|_U + (t_2)|_U$$

with $(t_1)|_U \in \mathscr{F}_1(U)$ and $(t_2)|_U \in \mathscr{F}_2(U)$, so $s = (t_1)|_U$ with $t_1 \in \mathscr{F}_1(X)$, which shows that $\mathscr{F}_1$ is flasque. □

The following general proposition from Tohoku implies that flasque sheaves are $\Gamma(X,-)$-acyclic. It will also imply that soft sheaves are $\Gamma(X,-)$-acyclic. Since the only functor involved is the global section functor, it is customary to abbreviate $\Gamma(X,-)$-acyclic to acyclic.

Theorem 2.2.4. *Let $\mathscr{F}$ be an additive functor from the abelian category $\mathcal{C}$ to the abelian category $\mathcal{D}$, and suppose that $\mathcal{C}$ has enough injectives. Let $\mathfrak{X}$ be a class of objects in $\mathcal{C}$ which satisfies the following conditions:*

- *$\mathcal{C}$ possesses enough $\mathfrak{X}$-objects.*
- *If $0 \to A_1 \to A_2 \to A_3 \to 0$ is exact and if A_1 belongs to $\mathfrak{X}$, then $0 \to \mathscr{F}(A_1) \to \mathscr{F}(A_2) \to \mathscr{F}(A_3) \to 0$ is exact.*
- *If $0 \to A_1 \to A_2 \to A_3 \to 0$ is exact and A_1, A_2 belongs to $\mathfrak{X}$, then A_3 belongs to $\mathfrak{X}$.*
- *If A is an object of $\mathcal{C}$ and A is a direct summand of some object in $\mathfrak{X}$, then A belongs to $\mathfrak{X}$.*

Then every injective object belongs to $\mathfrak{X}$, for each M in $\mathfrak{X}$ we have $\mathcal{R}^i\mathscr{F}(M) = 0$ for $i > 0$, and finally the functors $\mathcal{R}^i\mathscr{F}$ may be computed by taking $\mathfrak{X}$-resolutions.

Proof. Let I be an injective of $\mathcal{C}$. Then I admits a monomorphism into some object M of the class $\mathfrak{X}$. We have an exact sequence

$$0 \longrightarrow I \xrightarrow{\varphi} M \longrightarrow \operatorname{coker}\varphi \longrightarrow 0$$

and as I is injective this sequence split. Thus I is a direct summand of M and thus belongs to $\mathfrak{X}$ be the condition.

Let us now show that $\mathcal{R}^n\mathscr{F}(M) = 0$ for $i > 0$ if M lies in $\mathfrak{X}$. Now, $\mathcal{C}$ possesses enough injectives, so if we have an exact sequence

$$0 \longrightarrow M \longrightarrow I \longrightarrow K \longrightarrow 0$$

where I is injective. Then from the long exact sequence of $\mathcal{R}^\bullet\mathscr{F}$ and the fact $\mathcal{R}^i\mathscr{F}(I) = 0$ for $i \geq 1$ we conclude

$$\mathcal{R}^1\mathscr{F}(M) = 0 \quad \text{and} \quad \mathcal{R}^i\mathscr{F}(M) = \mathcal{R}^{i-1}\mathscr{F}(K) \textit{ for } i \geq 2.$$

Since M and I are both in $\mathfrak{X}$, K is also in $\mathfrak{X}$. Therefore by applying the same argument on K we also get $\mathcal{R}^1\mathscr{F}(K) = 0$. Now the claim follows by an induction. □

This result tells us that flasque sheaves are acyclic for the functor $\Gamma(X,-)$. Hence we can calculate cohomology using flasque resolutions.

Proposition 2.2.5. *Flasque sheaves are acyclic, that is $H^i(X,\mathscr{F}) = 0$ for every flasque sheaf $\mathscr{F}$ and all $i \geq 1$, and the cohomology groups $H^i(X,\mathscr{F})$ of any arbitrary sheaf $\mathscr{F}$ can be computed using flasque resolutions.*

Proof. Apply Theorem 2.2.4 on the functor $\Gamma(X,-) : \mathbf{Ab}(X) \to \mathbf{Ab}(X)$. □

In view of the proposition above, we also have the following result.

Proposition 2.2.6. *Let $(X,\mathscr{O}_X)$ be a ringed space. Then the derived functors of the functor $\Gamma(X,-)$ from $\mathbf{Mod}(\mathscr{O}_X)$ to $\mathbf{Ab}(X)$ coincide with the cohomology functors $H^i(X,-)$.*

Proof. Let I be an injective $\mathscr{O}_X$-module, then $\mathscr{I}$ is a flasque $\mathscr{O}_X$-module by Theorem 2.2.4. Therefore an injective resolution in $\mathbf{Mod}(\mathscr{O}_X)$ is a flasque resolution in $\mathbf{Ab}(X)$, hence computes the sheaf cohomology. □

2.3 Locality of cohomology

we first sate a useful result in abelian categories.

Theorem 2.3.1. *Let (F, G) be an adjoint pair between abelian categories $\mathcal{C}$ and $\mathcal{D}$, in the sense that*

$$\mathrm{Hom}_{\mathcal{D}}(F(X), Y) = \mathrm{Hom}_{\mathcal{C}}(X, G(Y)).$$

Assume that F is exact, then G maps injectives to injectives.

Proof. Let I be injective in $\mathcal{D}$. Assume that we have an exact diagram in $\mathcal{C}$:

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r] \arrow[d] & Y \\ & G(I) & \end{tikzcd}$$

Then by applying the exact functor F we get an exact diagram in $\mathcal{D}$:

$$\begin{tikzcd} 0 \arrow[r] & F(X) \arrow[r] \arrow[d] & F(Y) \arrow[d, dashed] \\ & FG(I) \arrow[r] & I \end{tikzcd}$$

where $FG(I) \to I$ is the unit map of the adjunction (F, G). Since I is injective, this extends to a map $F(Y) \to I$. Applying G again and compose the counit map $I \to GF(Y)$ we get the desired map $Y \to G(I)$, so $G(I)$ is injective. □

Proposition 2.3.2. *Let $f : X \to Y$ be a morphism of ringed spaces. Assume f is flat. Then $f_*\mathcal{I}$ is an injective $\mathcal{O}_Y$-module for any injective $\mathcal{O}_X$-module $\mathcal{I}$. In particular, the pushforward $f_* : \mathbf{Ab}(X) \to \mathbf{Ab}(Y)$ transforms injective abelian sheaves into injective abelian sheaves.*

Proof. In this case f^* is exact, and we have an adjoint pair (f^*, f_*). Now apply Theorem 2.3.1 we get the claim. □

The following lemma says there is no ambiguity in defining the cohomology of a sheaf $\mathcal{F}$ over an open.

Lemma 2.3.3. *Let X be a ringed space. Let $U \subseteq X$ be an open subspace.*

(a) If $\mathcal{I}$ is an injective $\mathcal{O}_X$-module then $\mathcal{I}|_U$ is an injective $\mathcal{U}$-module.

(b) For any sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ we have $H^p(U, \mathcal{F}) = H^p(U, \mathcal{F}|_U)$.

Proof. Denote $j : U \to X$ the open immersion. Then $(j_!, j^{-1})$ satisfies the condition of Theorem 2.3.1. By definition $H^p(U, \mathcal{F}) = H^p(\Gamma(U, \mathcal{I}^\bullet))$ where $\mathcal{F} \to \mathcal{I}^\bullet$ is an injective resolution in $\mathbf{Mod}(\mathcal{O}_X)$. By the above we see that $\mathcal{F}|_U \to \mathcal{I}|_U$ is an injective resolution in $\mathbf{Mod}(\mathcal{O}_U)$. Hence $H^p(U, \mathcal{F}|_U)$ is equal to $H^p(\Gamma(U, \mathcal{I}^\bullet|_U))$. Of course $\Gamma(U, \mathcal{F}) = \Gamma(U, \mathcal{F}|_U)$ for any sheaf $\mathcal{F}$ on X. Hence the equality in (b). □

Let $f : X \to Y$ be a continuous. Since the functor f_* is left-exact, for an injective resolution $0 \to \mathcal{F} \to \mathcal{I}^\bullet$ we define

$$\mathcal{R}^i f_*\mathcal{F} = H^i(f_*\mathcal{I}^\bullet)$$

to be the ***i*-th higher direct image of $\mathcal{F}$**.

Proposition 2.3.4. *Let $f : X \to Y$ be a morphism of ringed spaces. Let $\mathcal{F}$ be a $\mathcal{O}_X$-module. The sheaves $\mathcal{R}^i f_*\mathcal{F}$ are the sheaves associated to the presheaves*

$$U \mapsto H^i(f^{-1}(U), \mathcal{F}).$$

Proof. Let $0 \to \mathcal{F} \to \mathcal{I}^\bullet$ be an injective resolution. Then $\mathcal{R}^i f_*\mathcal{F}$ is by definition the i-th cohomology sheaf of the complex

$$f_*\mathcal{I}^0 \longrightarrow f_*\mathcal{I}^1 \longrightarrow f_*\mathcal{I}^2 \longrightarrow \cdots$$

By definition of the abelian category structure on $\mathcal{O}_Y$-modules this cohomology sheaf is the sheaf associated to the presheaf

$$V \mapsto \frac{\ker(f_*\mathcal{I}^i(V) \to f_*\mathcal{I}^{i+1}(V))}{\operatorname{im}(f_*\mathcal{I}^{i-1}(V) \to f_*\mathcal{I}^i(V))}$$

and this is obviously equal to

$$\frac{\ker\left(\mathcal{I}^i(f^{-1}(V)) \to \mathcal{I}^{i+1}(f^{-1}(V))\right)}{\operatorname{im}\left(\mathcal{I}^{i-1}(f^{-1}(V)) \to \mathcal{I}^i(f^{-1}(V))\right)}$$

which is equal to $H^i(f^{-1}(V), \mathcal{F})$ and we win. □

Corollary 2.3.5. *Let $f : X \to Y$ be a morphism of ringed spaces. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. If $\mathcal{F}$ is flasque, then $\mathcal{R}^p f_*\mathcal{F} = 0$ for $p > 0$.*

Proof. This follows from Proposition 2.2.5 and Proposition 2.3.4. □

Proposition 2.3.6. *Let $f : X \to Y$ be a morphism of ringed spaces. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Let $V \subseteq Y$ be an open subspace. Denote $g : f^{-1}(V) \to V$ the restriction of f. Then we have*

$$\mathcal{R}^i g_*(\mathcal{F}|_{f^{-1}(V)}) = (\mathcal{R}^i f_*\mathcal{F})|_V.$$

There is a similar statement for the derived image $\mathcal{R}^i f_*\mathcal{F}$ where $\mathcal{F}^\bullet$ is a bounded below complex of $\mathcal{O}_X$-modules.

Proof. Choose an injective resolution $0 \to \mathcal{F} \to \mathcal{I}^\bullet$ and use that $\mathcal{F}|_{f^{-1}(V)} \to \mathcal{I}|_{f^{-1}(V)}$ is an injective resolution also. □

Chapter 3

Čech cohomology

3.1 The Čech cohomology group with respect to a covering

Let X be a topological space and $\mathcal{U} = (U_i)_{i\in I}$ be an open covering of X. Given any finite sequence $(i_0, \ldots, i_p)$ of elements of I, we let

$$U_{i_0\ldots i_p} := U_{i_0} \cap \cdots \cap U_{i_p}.$$

Also, we denote by $U_{i_0\ldots\widehat{i_j}\ldots i_p}$ the intersection

$$U_{i_0\ldots\widehat{i_j}\ldots i_p} = U_{i_0} \cap \cdots \cap \widehat{U}_{i_j} \cap \cdots \cap U_{i_p}.$$

Then we have $p+1$ inclusion maps

$$\delta_j^p : U_{i_0\ldots i_p} \to U_{i_0\ldots\widehat{i_j}\ldots i_p} \quad \text{for} \;\; 0 \leq j \leq p.$$

Definition 3.1.1. Given a topological space X, an open cover $\mathcal{U} = (U_i)_{i\in I}$ of X, and a abelian presheaf $\mathscr{F}$ on X, the **Čech p-cochains** $C^p(\mathcal{U}, \mathscr{F})$ is the set of all functions f with domain I^{p+1} such that $f(i_0, \ldots, i_p) \in \mathscr{F}(U_{i_0} \ldots i_p)$; in other words,

$$C^p(\mathcal{U}, \mathscr{F}) = \prod_{(i_0,\ldots,i_p)\in I^{p+1}} \mathscr{F}(U_{i_0\ldots i_p})$$

the set of all I^{p+1}-indexed families $(f_{i_0\ldots i_p})$ with $f_{i_0\ldots i_p} \in \mathscr{F}(U_{i_0\ldots i_p})$.

Remark 3.1.1. Since $\mathscr{F}(\varnothing) = 0$, for any cochain $f \in C^p(\mathcal{U}, \mathscr{F})$, if $U_{i_0\ldots i_p} = \varnothing$, then $f_{i_0\ldots i_p} = 0$. Therefore, we could define $C^p(\mathcal{U}, \mathscr{F})$ as the set of families $f_{i_0\ldots i_p}$ corresponding to tuples $(i_0, \ldots, i_p) \in I^{p+1}$ such that $U_{i_0\ldots i_p} \neq \varnothing$.

Each inclusion map $\delta_j^p : U_{i_0\ldots i_p} \to U_{i_0\ldots\widehat{i_j}\ldots i_p}$ induces a map

$$\mathscr{F}(\delta_j^p) : \mathscr{F}(U_{i_0\ldots\widehat{i_j}\ldots i_p}) \to \mathscr{F}(U_{i_0\ldots i_p})$$

which is none other that the restriction map.

Definition 3.1.2. Given a topological space X, an open cover $\mathcal{U} = (U_i)_{i\in I}$ of X, and a abelian presheaf $\mathscr{F}$ on X, the coboundary maps $d : C^p(\mathcal{U}, \mathscr{F}) \to C^{p+1}(\mathcal{U}, \mathscr{F})$ are given by

$$d = \sum_{j=0}^{p+1} (-1)^j \mathscr{F}(\delta_j^{p+1}).$$

More explicitly, for any p-cochain $f \in C^p(\mathcal{U}, \mathscr{F})$, for any sequence $(i_0, \ldots, i_{p+1}) \in I^{p+2}$, we have

$$(df)_{i_0\ldots i_{p+1}} = \sum_{j=0}^{p+1} (-1)^j (f_{i_0\ldots\widehat{i_j}\ldots i_{p+1}})|_{U_{i_0\cdots i_{p+1}}}.$$

Proposition 3.1.1. *With the notations above, we have $d^2 = 0$. Thus we obtain a complex $C^\bullet(\mathcal{U}, \mathscr{F})$.*

Proof. This is a typical computation. Let $f \in C^p(\mathcal{U}, \mathscr{F})$,

$$\begin{aligned}
(d^2 f)_{i_0 \ldots i_{p+2}} &= \sum_{k=0}^{p+2} (-1)^k \big((df)_{i_0 \ldots \widehat{i_k} \ldots i_{p+2}}\big)|_{U_{i_0 \cdots i_{p+2}}} \\
&= \sum_{j<k} (-1)^k \big((-1)^j (f_{i_0 \ldots \widehat{i_j} \ldots \widehat{i_k} \ldots i_{p+1}})|_{U_{i_0 \cdots \widehat{i_k} \cdots i_{p+2}}}\big)|_{U_{i_0 \cdots i_{p+2}}} \\
&+ \sum_{j>k} (-1)^k \big((-1)^{j-1} (f_{i_0 \ldots \widehat{i_k} \ldots \widehat{i_j} \ldots i_{p+1}})|_{U_{i_0 \cdots \widehat{i_k} \cdots i_{p+2}}}\big)|_{U_{i_0 \cdots i_{p+2}}} \\
&= \sum_{j<k} (-1)^k (-1)^j (f_{i_0 \ldots \widehat{i_j} \ldots \widehat{i_k} \ldots i_{p+1}})|_{U_{i_0 \cdots i_{p+2}}} \\
&- \sum_{j>k} (-1)^k (-1)^j (f_{i_0 \ldots \widehat{i_k} \ldots \widehat{i_j} \ldots i_{p+1}})|_{U_{i_0 \cdots i_{p+2}}} \\
&= 0
\end{aligned}$$

as desired. □

Therefore, we can form the Čech cohomology groups with respect to $\mathcal{U}$ as follows.

Definition 3.1.3. Given a topological space X, an open cover $\mathcal{U} = (U_i)_{i \in I}$ of X, and a abelian presheaf $\mathscr{F}$ on X, we define

$$\check{H}^p(\mathcal{U}, \mathscr{F}) = H^p(C^\bullet(\mathcal{U}, \mathscr{F}))$$

to be the ***p*-th Čech-cohomology group with respect to the covering** $\mathcal{U}$.

First of all, we note that $\check{H}^0(\mathcal{U}, \mathscr{F})$ can be easily computed.

Proposition 3.1.2. *Given a topological space X, an open cover $\mathcal{U}$ of X, and a sheaf $\mathscr{F}$ on X, then*

$$\check{H}^0(\mathcal{U}, \mathscr{F}) = \mathscr{F}(X) = \Gamma(X, \mathscr{F}).$$

More generally, if $\mathscr{F}$ is an abelian presheaf, then the following are equivalent

- *$\mathscr{F}$ is a sheaf.*
- *For every open covering $\mathcal{U} = (U_i)_{i \in I}$ the natural map*

  $$\mathscr{F}(U) \to \check{H}^0(\mathcal{U}, \mathscr{F})$$

  is bijective.

Proof. By definition, $\check{H}^0(\mathcal{U}, \mathscr{F}) = \ker(C^0(\mathcal{U}, \mathscr{F}) \to C^1(\mathcal{U}, \mathscr{F}))$. If $f \in C^0(\mathcal{U}, \mathscr{F})$ is given by (f_i), then for each $(i, j) \in I^2$, $(df)_{i,j} = f_j - f_i$. So $df = 0$ says the sections f_i and f_j agree on $U_i \cap U_j$. Thus it follows that $\ker d = \Gamma(X, \mathscr{F})$ if and only if $\mathscr{F}$ is a sheaf. □

An element of $C^p(\mathcal{U}, \mathscr{F})$ is called a ***p*-cochain**. We say that a p-cochain $f \in C^p(\mathcal{U}, \mathscr{F})$ is **alternating** if

- $f_{i_0 \ldots i_p} = 0$ whenever any two of the indices $i_0, \ldots, i_p$ are equal.
- For every permutation σ of the indices, we have $f_{i_{\sigma(0)} \ldots i_{\sigma(p)}} = (-1)^\sigma f_{i_0 \ldots i_p}$.

It is clear that if $f \in C^p(\mathcal{U}, \mathscr{F})$ is alternating, then df is also alternating, hence we get a complex $C^\bullet_{alt}(\mathcal{U}, \mathscr{F})$.

Definition 3.1.4. Let X be a topological space. Let $\mathcal{U} = (U_i)_{i\in I}$ be an open covering. Let $\mathscr{F}$ be an abelian presheaf on X. The complex $C^\bullet_{alt}(\mathcal{U}, \mathscr{F})$ is the **alternating Čech complex** associated to $\mathscr{F}$ and the open covering $\mathcal{U}$.

Let us endow the set of indices I with a total ordering and set

$$C^p_{ord}(\mathcal{U}, \mathscr{F}) := \prod_{i_0 < \cdots < i_p} \mathscr{F}(U_{i_0 \ldots i_p})$$

It can immediately be verified that the differential d of $C^\bullet(\mathcal{U}, \mathscr{F})$ induces a differential on $C^\bullet_{ord}(\mathcal{U}, \mathscr{F})$, by restriction.

Definition 3.1.5. Let X be a topological space. Let $\mathcal{U} = (U_i)_{i\in I}$ be an open covering. Let $\mathscr{F}$ be an abelian presheaf on X. Assume given a total ordering on I. The complex $C^\bullet_{ord}(\mathcal{U}, \mathscr{F})$ is the **ordered Čech complex** associated to $\mathscr{F}$, the open covering $\mathcal{U}$ and the given total ordering on I.

There is an obvious comparison map between the ordered Čech complex and the alternating Čech complex. Namely, consider the map

$$c : C^\bullet_{ord}(\mathcal{U}, \mathscr{F}) \to C^\bullet_{alt}(\mathcal{U}, \mathscr{F})$$

given by the rule

$$c(s)_{i_0 \ldots i_p} = \begin{cases} (-1)^\sigma s_{i_{\sigma(0)},\ldots,i_{\sigma(p)}} & \text{if } i_{\sigma(0)} < \cdots < i_{\sigma(p)}, \\ 0 & \text{if } i_n = i_m \text{ for some } n \neq m. \end{cases}$$

The alternating and ordered Čech complexes are often identified in the literature via the map c. Namely we have the following easy lemma.

Lemma 3.1.3. *Let X be a topological space. Let $\mathcal{U}$ be an open covering. Assume I comes equipped with a total ordering. The map c is a morphism of complexes. In fact it induces an isomorphism*

$$c : C^\bullet_{ord}(\mathcal{U}, \mathscr{F}) \to C^\bullet_{alt}(\mathcal{U}, \mathscr{F})$$

of complexes.

There is also a map $\pi : C^\bullet(\mathcal{U}, \mathscr{F}) \to C^\bullet_{ord}(\mathcal{U}, \mathscr{F})$ which is described by the rule

$$\pi(s)_{i_0 \ldots i_p} = s_{i_0 \ldots i_p}$$

whenever $i_0 < \cdots < i_p$. The following result is immediate.

Lemma 3.1.4. *Let X be a topological space. Let $\mathcal{U}$ be an open covering. Assume I comes equipped with a total ordering. The map $\pi : C^\bullet(\mathcal{U}, \mathscr{F}) \to C^\bullet_{ord}(\mathcal{U}, \mathscr{F})$ is a morphism of complexes which is a left inverse to the morphism c. Moreover, it induces an isomorphism*

$$\tilde{\pi} : C^\bullet_{alt}(\mathcal{U}, \mathscr{F}) \to C^\bullet_{ord}(\mathcal{U}, \mathscr{F})$$

of complexes.

It turns out that the maps π and c give a homotopy equivalence between $C^\bullet(\mathcal{U}, \mathscr{F})$ and $C^\bullet_{alt}(\mathcal{U}, \mathscr{F})$. Therefore, we can use the alternating complex to compute the Čech cohomology.

Theorem 3.1.5. *Let X be a topological space. Let $\mathcal{U}$ be an open covering. Assume I comes equipped with a total ordering. The map $c \circ \pi$ is homotopic to the identity on $C^\bullet(\mathcal{U}, \mathscr{F})$. In particular the inclusion map $C^\bullet_{alt}(\mathcal{U}, \mathscr{F}) \to C^\bullet(\mathcal{U}, \mathscr{F})$ is a homotopy equivalence.*

Corollary 3.1.6. *Given a total ordering on I, there exists a canonical isomorphism*

$$H^\bullet(C^\bullet_{ord}(\mathcal{U}, \mathscr{F})) \cong H^\bullet(C^\bullet(\mathcal{U}, \mathscr{F})).$$

Corollary 3.1.7. *If $\mathcal{U}$ is made up of n open subsets, then $\check{H}^p(\mathcal{U}, \mathscr{F}) = 0$ for every $p \geq n$.*

Proof. Indeed, if $p \geq n$, there does not exist any strictly increasing $(p+1)$-uple of indices $i_0, \ldots, i_p$. Hence $C^p_{ord}(\mathcal{U}, \mathscr{F}) = 0$, whence $\check{H}^p(\mathcal{U}, \mathscr{F}) = 0$. $\square$

3.2 Čech cohomology as a functor on presheaves

Warning: In this subsection we work almost exclusively with presheaves and categories of presheaves and the results are completely wrong in the setting of sheaves and categories of sheaves!

Let X be a ringed space. Let $\mathcal{U} = (U_i)_{i\in I}$ be an open covering. Let $\mathscr{F}$ be a presheaf of $\mathscr{O}_X$-modules. We have the Čech complex $C^\bullet(\mathcal{U},\mathscr{F})$ of $\mathscr{F}$ just by thinking of $\mathscr{F}$ as a presheaf of abelian groups. However, each term $C^p(\mathcal{U},\mathscr{F})$ has a natural structure of a $\mathscr{O}_X(X)$-module and the differential is given by $\mathscr{O}_X(X)$-module maps. Moreover, it is clear that the construction

$$\mathscr{F} \to C^\bullet(\mathcal{U},\mathscr{F})$$

is functorial in $\mathscr{F}$. In fact, it is a functor

$$C^\bullet(\mathcal{U},-) : \mathbf{PMod}(\mathscr{O}_X) \to \mathcal{C}^+(\mathbf{Mod}(\mathscr{O}_X(X)))$$

Proposition 3.2.1. *The functor $C^\bullet(\mathcal{U},-)$ is an exact functor.*

Proof. For any open $U \subseteq X$ the functor $\mathscr{F} \to \mathscr{F}(U)$ is an additive exact functor from $\mathbf{PMod}(\mathscr{O}_X)$ to $\mathbf{Mod}(\mathscr{O}_X(X))$. The terms $C^p(\mathcal{U},\mathscr{F})$ of the complex are products of these exact functors and hence exact. Moreover a sequence of complexes is exact if and only if the sequence of terms in a given degree is exact. Hence the lemma follows. □

Theorem 3.2.2. *Let X be a ringed space. Let $\mathcal{U} = (U_i)_{i\in I}$ be an open covering. The functors $\check{H}^p(\mathcal{U},-)$ form a δ-functor from the abelian category of presheaves of $\mathscr{O}_X$-modules to the category of $\mathscr{O}_X(X)$-modules*

Proof. By Proposition 3.2.1 a short exact sequence of presheaves of $\mathscr{O}_X$-modules is turned into a short exact sequence of complexes of $\mathscr{O}_X(X)$-modules. Hence we can get a long exact sequence. □

Proposition 3.2.3. *Let X be a ringed space. Let $\mathcal{U}$ be an open covering of X. The Čech cohomology functors $\check{H}^p(\mathcal{U},-)$ are canonically isomorphic as a δ-functor to the right derived functors of the functor*

$$\check{H}^0(\mathcal{U},-) : \mathbf{PMod}(\mathscr{O}_X) \to \mathbf{Mod}(\mathscr{O}_X(X)).$$

Moreover, there is a functorial quasi-isomorphism

$$C^\bullet(\mathcal{U},\mathscr{F}) \to \mathcal{R}\check{H}^0(\mathcal{U},\mathscr{F}).$$

Proof. This comes from the universal property of the δ-functor. □

3.3 The Čech cohomology groups

Our next goal is to define Čech cohomology groups $\check{H}^p(X,\mathscr{F})$ that are independent of the open cover $\mathcal{U}$ chosen for X.

We say that a covering $\mathcal{V} = (V_j)_{j\in J}$ of X is a **refinement** of another covering $\mathcal{U} = \{U_i\}_{i\in I}$ if there exists a map $\tau : J \to I$ such that $V_j \subseteq U_{\tau(j)}$ for every $j \in J$. We then have a homomorphism, which we also denote by τ:

$$C^p(\mathcal{U},\mathscr{F}) \to C^p(\mathcal{V},\mathscr{F})$$

defined by

$$\tau(f)_{j_0\ldots j_p} = f_{\tau(j_0)\ldots\tau(j_p)}|_{V_{j_0\cdots j_p}}$$

This homomorphism commutes with the differentials and therefore induces a homomorphism

$$\tau^* : \check{H}^p(\mathcal{U},\mathscr{F}) \to \check{H}^p(\mathcal{V},\mathscr{F}).$$

Proposition 3.3.1. *The homomorphisms $\tau^* : \check{H}^p(\mathcal{U},\mathscr{F}) \to \check{H}^p(\mathcal{V},\mathscr{F})$ depend only on $\mathcal{U}$ and $\mathcal{V}$ and not on the chosen mapping τ.*

Proof. Let τ_1 and τ_2 be two mappings from I to J such that $V_j \subseteq U_{\tau_1(j)}$ and $V_j \subseteq U_{\tau_2(j)}$, we have to show that $\tau_1^* = \tau_2^*$.

Let $f \in C^p(\mathcal{U},\mathscr{F})$, define a map $\kappa : C^p(\mathcal{U},\mathscr{F}) \to C^{p-1}(\mathcal{V},\mathscr{F})$ by setting

$$(\kappa f)_{j_0\cdots j_{p-1}} = \sum_{k=0}^{p-1}(-1)^k (f_{\tau_1(j_1)\ldots\tau_1(j_k)\tau_2(j_k)\ldots\tau_2(j_{p-1})})|_{V_{j_0\cdots j_{p-1}}}.$$

Then, it can be verified that

$$\kappa(df) + d(\kappa f) = \tau_2(f) - \tau_1(f).$$

Thus k defines a homotopy from τ_2^* to τ_1^*, which implies the claim. $\square$

Corollary 3.3.2. *If $\mathcal{V}$ is a refinement of $\mathcal{U}$ and if $\mathcal{U}$ is a refinement of $\mathcal{V}$, then $\check{H}^p(\mathcal{U},\mathscr{F}) \to \check{H}^p(\mathcal{V},\mathscr{F})$ is an isomorphism.*

Proof. Let us keep the notation above. If $\mathcal{U}$ is a refinement of a covering $\mathcal{W}$ with a map σ such that $U_i \subseteq W_{\sigma(i)}$, then $\mathcal{V}$ is a refinement of $\mathcal{W}$ because $V_j \subseteq W_{\sigma\circ\tau(j)}$. Moreover, $(\sigma\circ\tau)^* = \sigma^*\circ\tau^*$ in an obvious way. Let us now take $\mathcal{W} = \mathcal{V}$. Then $\sigma^*\circ\tau^*$ and $\tau^*\circ\sigma^*$ coincides with $\mathbf{1}^*$ by the result above. Hence τ^* is bijective. $\square$

The relation $\mathcal{U}$ is a refinement of $\mathcal{V}$ (which we denote henceforth by $\mathcal{U} \prec \mathcal{V}$) is a relation of a preorder between coverings of X; moreover, this relation is filtered, since if $\mathcal{U} = \{U_i\}_{i\in I}$ and $\mathcal{V} = \{V_j\}_{j\in J}$ are two coverings, then the covering $\mathcal{W} = \{U_i \cap V_j\}_{(i,j)\in I\times J}$ is a refinement of $\mathcal{U}$ and $\mathcal{V}$. Consequently, it appears that the family $(\check{H}^p(\mathcal{U},\mathscr{F}))_{\mathcal{U}}$ is a direct mapping family of groups indexed by the directed set of open covers of X. However, there is a set-theoretic diffculty, which is that the family of open covers of X is not a set because it allows arbitrary index sets.

A way to circumvent this difficulty is provided by Serre. The key observation is that any covering $(U_i)_{i\in I}$ is equivalent to a covering $(U'_j)_{j\in L}$ whose index set L is a subset of 2^X. Indeed, we can take for $(U'_j)_{j\in L}$ the set of all open subsets of X that belong to the family $(U_i)_{i\in I}$.

As we noted earlier, if $\mathcal{U}$ and $\mathcal{V}$ are equivalent, then there is an isomorphism between $\check{H}^p(\mathcal{U},\mathscr{F})$ and $\check{H}^p(\mathcal{V},\mathscr{F})$, so we can define

$$\check{H}^p(X,\mathscr{F}) = \varinjlim \check{H}^p(\mathcal{U},\mathscr{F})$$

with respect to coverings $\mathcal{U}$ whose index set is a subset of 2^X. In summary, we have the following definition.

Definition 3.3.1. Given a topological space X and a abelian presheaf $\mathscr{F}$ on X, the **Čech cohomology groups** $\check{H}^p(X,\mathscr{F})$ with values in $\mathscr{F}$ are defined by

$$\check{H}^p(X,\mathscr{F}) = \varinjlim \check{H}^p(\mathcal{U},\mathscr{F})$$

with respect to coverings $\mathcal{U}$ whose index set is a subset of 2^X.

3.4 Long exact sequence of Čech-cohomology

3.4.1 General case

Let $0 \to \mathscr{A} \to \mathscr{B} \to \mathscr{C} \to 0$ be an exact sequence of sheaves. If $\mathcal{U}$ is a covering of X, the sequence

$$0 \longrightarrow C^\bullet(\mathcal{U},\mathscr{A}) \xrightarrow{\alpha} C^\bullet(\mathcal{U},\mathscr{B}) \xrightarrow{\beta} C^\bullet(\mathcal{U},\mathscr{C})$$

is obviously exact, but the homomorphism β need not be surjective in general. Denote by $C_0^\bullet(\mathcal{U},\mathscr{C})$ the image of this homomorphism; it is a subcomplex of $C^\bullet(\mathcal{U},\mathscr{C})$ whose cohomology groups will be denoted by $\check{H}_0^p(\mathcal{U},\mathscr{C})$. The exact sequence of complexes:

$$0 \longrightarrow C^\bullet(\mathcal{U},\mathscr{A}) \xrightarrow{\alpha} C^\bullet(\mathcal{U},\mathscr{B}) \xrightarrow{\beta} C_0^\bullet(\mathcal{U},\mathscr{C}) \longrightarrow 0$$

giving rise to a long exact sequence of cohomology

$$\cdots \longrightarrow \check{H}^p(\mathcal{U},\mathscr{A}) \longrightarrow \check{H}^p(\mathcal{U},\mathscr{B}) \longrightarrow \check{H}_0^p(\mathcal{U},\mathscr{C}) \xrightarrow{\delta} \check{H}^{p+1}(\mathcal{U},\mathscr{A}) \longrightarrow \cdots$$

where the coboundary operator δ is defined as usual.

Now let $\mathcal{U} = (U_i)_{i\in I}$ and $\mathcal{V} = (V_j)_{j\in J}$ be two coverings and let $\tau : J \to I$ be such that $V_j \subseteq U_{\tau(j)}$; we thus have $\mathcal{V} \prec \mathcal{U}$. The commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & C^\bullet(\mathcal{U},\mathscr{A}) \arrow[r, "\alpha"] \arrow[d, "\tau"] & C^\bullet(\mathcal{U},\mathscr{B}) \arrow[r, "\beta"] \arrow[d, "\tau"] & C^\bullet(\mathcal{U},\mathscr{C}) \arrow[d, "\tau"] \\
0 \arrow[r] & C^\bullet(\mathcal{V},\mathscr{A}) \arrow[r, "\alpha"] & C^\bullet(\mathcal{V},\mathscr{B}) \arrow[r, "\beta"] & C^\bullet(\mathcal{V},\mathscr{C})
\end{tikzcd}$$

shows that τ maps $C_0^\bullet(\mathcal{U},\mathscr{C})$ into $C_0^\bullet(\mathcal{V},\mathscr{C})$, thus defining the homomorphisms

$$\tau^* : \check{H}_0^p(\mathcal{U},\mathscr{C}) \to \check{H}_0^p(\mathcal{V},\mathscr{C}).$$

Moreover, the homomorphisms τ are independent of the choice of the mapping τ: this follows from the fact that, if $f \in C_0^p(\mathcal{U},\mathscr{C})$, we have $\kappa f \in C_0^{p-1}(\mathcal{V},\mathscr{C})$, with the notations of Proposition 3.3.1. We have thus obtained canonical homomorphisms $H_0^p(\mathcal{U},\mathscr{C}) \to \check{H}_0^p(\mathcal{V},\mathscr{C})$; we might then define $\check{H}_0^p(X,\mathscr{C})$ as the inductive limit of the groups $\check{H}^p(X,\mathscr{C})$.

Because an inductive limit of exact sequences is an exact sequence, we obtain:

Proposition 3.4.1. *The sequence*

$$\cdots \longrightarrow \check{H}^p(X,\mathscr{A}) \xrightarrow{\alpha^*} \check{H}^p(X,\mathscr{B}) \xrightarrow{\beta^*} \check{H}_0^p(X,\mathscr{C}) \xrightarrow{\delta} \check{H}^{p+1}(X,\mathscr{A}) \longrightarrow \cdots$$

is exact.

To apply the preceding proposition, it is convenient to compare the groups $\check{H}_0^p(X,\mathscr{C})$ and $\check{H}^p(X,\mathscr{C})$. The inclusion of $C_0^\bullet(\mathcal{U},\mathscr{C})$ in $C^\bullet(\mathcal{U},\mathscr{C})$ defines the homomorphisms

$$\check{H}_0^p(\mathcal{U},\mathscr{C}) \to \check{H}^p(\mathcal{U},\mathscr{C}),$$

hence, by passing to the limit with $\mathcal{U}$, the homomorphisms:

$$\check{H}_0^p(X,\mathscr{C}) \to \check{H}^p(X,\mathscr{C})$$

We first prove a lemma.

Lemma 3.4.2. *Let $\mathcal{U} = (U_i)_{i\in I}$ be a covering and let $f = (f_j)$ be an element of $C^0(\mathcal{U},\mathscr{C})$. There exists a covering $\mathcal{V} = (V_j)_{j\in J}$ and a mapping $\tau : J \to I$ such that $V_j \subseteq U_{\tau(j)}$ and $\tau(f) \in C_0^0(\mathcal{V},\mathscr{C})$.*

Proof. For any $x \in X$, take a $\tau(x) \in I$ such that $x \in U_{\tau(x)}$. Since $f_{\tau(x)}$ is a section of $\mathscr{C}$ over $U_{\tau(x)}$, by the surjectivity of β there exists an open neighborhood V_x of x, contained in $U_{\tau(x)}$ and a section b_x of $\mathscr{B}$ over V_x such that $\beta(b_x) = f_{\tau(x)}|_{V_x}$ on V_x. The $(V_x)_{x\in X}$ form a covering $\mathcal{V}$ of X, and the b_x form a 0-chain b of $\mathcal{V}$ with values in $\mathcal{U}$. From the construction we have $\tau(f) = \beta(b)$, so that $\tau(f) \in C_0^0(\mathcal{V},\mathscr{C})$. □

Now we can prove the following result.

Theorem 3.4.3. *The canonical homomorphism $\check{H}_0^p(X,\mathscr{C}) \to \check{H}^p(X,\mathscr{C})$ is bijective for $p=0$ and injective for $p=1$.*

Proof. By Lemma 3.4.2 the bijectivity for $p=0$ is immediate. We now show that

$$\check{H}_0^1(X,\mathscr{C}) \to \check{H}^1(X,\mathscr{C})$$

is injective. Let $[z]$ be in the kernel of this map, which is represented by a 1-cocycle $z=(z_{i_0i_1}) \in C_0^1(\mathcal{U},\mathscr{C})$. Then since $[z]=0$ in $\check{H}^1(X,\mathscr{C})$, there exists an $f=(f_i)\in C^0(\mathcal{U},\mathscr{C})$ with $df=z$; applying Lemma 3.4.2 (and its notations) to f yields a covering $\mathcal{V}$ such that $\tau(f)\in C_0^0(\mathcal{V},\mathscr{C})$, which shows that $\tau(z)$ is cohomologous to 0 in $C_0^1(\mathcal{V},\mathscr{C})$, thus its image $[z]$ in $\check{H}_0^1(X,\mathscr{C})$ is 0. This shows the claim. □

Corollary 3.4.4. *With notations above, we have an exact sequence:*

$$0 \to \Gamma(X,\mathscr{A}) \to \Gamma(X,\mathscr{B}) \to \Gamma(X,\mathscr{C}) \xrightarrow{\delta} \check{H}^1(X,\mathscr{A}) \to \check{H}^1(X,\mathscr{B}) \to \check{H}^1(X,\mathscr{C})$$

Corollary 3.4.5. *If $\check{H}^1(X,\mathscr{A})=0$, then $\Gamma(X,\mathscr{B})\to\Gamma(X,\mathscr{C})$ is surjective.*

3.4.2 Paracompact space

Recall that a space X is said to be paracompact if any covering of X admits a locally finite refinement. On paracompact spaces, we can extend Theorem 3.4.3 for all values of p:

Theorem 3.4.6. *If X is paracompact, the canonical homomorphism*

$$\check{H}_0^p(X,\mathscr{C}) \to \check{H}^p(X,\mathscr{C})$$

is bijective for all $p\geq 0$.

This Proposition is an immediate consequence of the following lemma, analogous to Lemma 3.4.2:

Lemma 3.4.7. *Let X be a paracompact space. Let $\mathcal{U}=(U_i)_{i\in I}$ be a covering, and let $f=(f_{i_0\ldots i_p})$ be an element of $C^p(\mathcal{U},\mathscr{C})$. Then there exists a covering $\mathcal{V}=(V_j)_{j\in J}$ and a mapping $\tau: J\to I$ such that $V_j\subseteq U_{\tau(j)}$ and $\tau(f)\in C_0^p(\mathcal{V},\mathscr{C})$.*

Proof. Since X is paracompact, we might assume that $\mathcal{U}$ is locally finite. For every $x\in X$, we can choose an open neighborhood V_x of x such that

(a) If $x\in U_i$, then $V_x\subseteq U_i$.

(b) If $V_x\cap U_i\neq\varnothing$, then $V_x\subseteq U_i$.

(c) If $x\in U_{i_0\ldots i_p}$, there exists a section b of $\mathscr{B}$ over V_x such that $\beta(b)=f_{i_0\ldots i_p}|_{V_x}$.

The condition (c) can be satisfid due to the surjectivity of β and to the fact that x belongs to a finite number of sets $U_{i_0\ldots i_p}$. Having (c) satisfied, it suffices to restrict V_x to satisfy (a) and (b).

The family $(V_x)_{x\in X}$ forms a covering $\mathcal{V}$; for any $x\in X$, choose $\tau(x)\in I$ such that $x\in U_{\tau(x)}$. We now check that $\tau(f)$ belongs to $C_0^p(\mathcal{V},\mathscr{C})$. If $V_{x_0\ldots x_p}$ is empty, this is obvious; if not, we have $V_{x_0}\cap U_{x_k}\neq\varnothing$ for $0\leq k\leq p$, and then

$$V_{x_0}\cap U_{\tau(x_k)}\neq\varnothing \quad \text{for } \ 0\leq k\leq p,$$

which implies by (b) that $V_{x_0}\subseteq U_{\tau(x_k)}$ for all k, and hence $x_0\in U_{\tau(x_0)\ldots\tau(x_p)}$. We then apply (c) to get a section b of $\mathscr{B}$ over V_{x_0} such that $\beta(b)=f_{\tau(x_0)\ldots\tau(x_p)}|_{V_{x_0}}$. Thus $\tau(f)\in C_0^p(\mathcal{V},\mathscr{C})$, which completes the proof. □

Corollary 3.4.8. *If X is paracompact, we have an exact sequence:*

$$\cdots \longrightarrow \check{H}^p(X,\mathscr{A}) \xrightarrow{\alpha^*} \check{H}^p(X,\mathscr{B}) \xrightarrow{\beta^*} \check{H}^p(X,\mathscr{C}) \xrightarrow{\delta} \check{H}^{p+1}(X,\mathscr{A}) \longrightarrow \cdots$$

The exact sequence mentioned above is called the long exact sequence of cohomology defined by a given exact sequence of sheaves $0 \to \mathscr{A} \to \mathscr{B} \to \mathscr{C} \to 0$. More generally, it exists whenever we can show that $\check{H}^p_0(X,\mathscr{C}) \to \check{H}^p(X,\mathscr{C})$ is bijective.

3.5 Čech resolution and Leray Acyclic Theorem

We will now compare the Čech cohomology and derived functor cohomology of a sheaf $\mathscr{F}$ on a topological space X. We will see that in some cases we are able to conclude that these two cohomologies coincide. In order to compare Čech cohomology with derived functor cohomology, we will need to consider first a sheafified version of the Čech complex.

Fix a space X, an open cover $\mathcal{U} = (U_i)_{i\in I}$ and a sheaf $\mathscr{F}$ on X. For every open set U of X, let $j_U : U \hookrightarrow X$ denote the inclusion. Define a sheaf $\mathscr{C}^p(\mathcal{U},\mathscr{F})$ by

$$\mathscr{C}^p(\mathcal{U},\mathscr{F}) = \prod_{(i_0,\ldots,i_p)} (j_{U_{i_0\ldots i_p}})_*(\mathscr{F}|_{U_{i_0\ldots i_p}}).$$

Explicitly, for an open subset $V \subseteq X$ we have

$$\Gamma(V,\mathscr{C}^p(\mathcal{U},\mathscr{F})) = \prod_{(i_0,\ldots,i_p)} \mathscr{F}(V \cap U_{i_0\ldots i_p}).$$

For each inclusion $\delta^p_j : U_{i_0\ldots i_p} \to U_{i_0\ldots\widehat{i_j}\ldots i_p}$, we also have a restriction map

$$\mathscr{C}(\delta^p_j) : (j_{U_{i_0\ldots\widehat{i_j}\ldots i_p}})_*(\mathscr{F}|_{U_{i_0\ldots\widehat{i_j}\ldots i_p}}) \to (j_{U_{i_0\ldots i_p}})_*(\mathscr{F}|_{U_{i_0\ldots i_p}})$$

so we can define the differential $d : \mathscr{C}^p(\mathcal{U},\mathscr{F}) \to \mathscr{C}^{p+1}(\mathcal{U},\mathscr{F})$ by

$$d = \sum_{j=0}^{p+1}(-1)^j\mathscr{C}(\delta^{p+1}_j).$$

Thus we get complex $\mathscr{C}^\bullet(\mathcal{U},\mathscr{F})$. By definition, we have

$$\Gamma(X,\mathscr{C}^\bullet(\mathcal{U},\mathscr{F})) = C^\bullet(\mathcal{U},\mathscr{F})$$

Also, there is a product map $\epsilon : \mathscr{F} \to \mathscr{C}^0(\mathcal{U},\mathscr{F})$ defined by

$$\mathscr{F} \to \prod_i (j_{U_i})_*(j_{U_i})^{-1}\mathscr{F} = \prod_i (j_{U_i})_*(\mathscr{F}|_{U_i}).$$

Proposition 3.5.1. *For an open cover $\mathcal{U}$ and a sheaf $\mathscr{F}$, there is an exact sequence of sheaves*

$$0 \longrightarrow \mathscr{F} \xrightarrow{\epsilon} \mathscr{C}^0(\mathcal{U},\mathscr{F}) \longrightarrow \mathscr{C}^1(\mathcal{U},\mathscr{F}) \longrightarrow \mathscr{C}^2(\mathcal{U},\mathscr{F}) \longrightarrow \cdots$$

This is called the ***Čech resolution*** *of $\mathscr{F}$ with respect to the cover $\mathcal{U}$.*

Proof. The facts that ϵ is injective and that $\operatorname{im}\epsilon = \ker d^0$ follow directly from $\mathscr{F}$ being a sheaf.

It remains to be shown that the proposed sequence is exact in degrees $p > 0$. For this, it suffices to work at the level of stalks.

To prove the exactness at $\mathscr{C}^p(\mathcal{U},\mathscr{F})$, we need to check the sequence of stalks $\mathscr{C}^\bullet(\mathcal{U},\mathscr{F})$ is exact for all $x \in X$. Since $\mathcal{U}$ covers X, we can choose an open subset U_j containing x. Take $f_x \in \mathscr{C}^p(\mathcal{U},\mathscr{F})_x$ which is represented by (V,f), where V is a neighborhood of x and $f \in \Gamma(V,\mathscr{C}^p(\mathcal{U},\mathscr{F}))$. We may assume $V \subseteq U_j$. Then we observe that for any index $(i_0,\dots,i_{p-1}) \in I^p$ we have

$$V \cap U_{i_0\dots i_{p-1}} = V \cap U_{j,i_0\dots i_{p-1}}$$

Thus, we may define an element $\theta f \in \Gamma(V,\mathscr{C}^{p-1}(\mathcal{U},\mathscr{F}))$ by the formula

$$(\theta f)_{i_0\dots i_{p-1}} := f_{j,i_0\dots i_{p-1}}$$

This gives a map

$$\theta^p : \mathscr{C}^p(\mathcal{U},\mathscr{F})_x \to \mathscr{C}^{p-1}(\mathcal{U},\mathscr{F})_x, \quad f_x \mapsto (\theta f)_x$$

Now we check that

$$(\theta(df))_{i_0\dots i_p} = (df)_{j,i_0\dots i_p} = f_{i_0\dots i_p} - \sum_{k=0}^{p}(-1)^k (f_{j,i_0\dots\widehat{i_k}\dots i_p})|_{U_{i_0\cdots i_p}}.$$

and

$$(d(\theta f))_{i_0\dots i_p} = \sum_{k=0}^{p}(-1)^k((\theta f)_{i_0\dots\widehat{i_k}\dots i_p})|_{U_{i_0\cdots i_p}} = \sum_{k=0}^{p}(-1)^k(f_{j,i_0\dots\widehat{i_k}\dots i_p})|_{U_{i_0\cdots i_p}}.$$

Therefore

$$\theta^{p+1}\big((df)_x\big) = (\theta(df))_x = \big(f - d(\theta f)\big)_x = f_x - d((\theta f)_x) = f_x - d(\theta^p f_x).$$

Hence θ^p is a homotopy from the identity of $\mathscr{C}^p(\mathcal{U},\mathscr{F})_x$ to the zero map, which shows the sequence of stalks is exact. □

One of the motivitions of defining the Čech resolution is the following proposition.

Proposition 3.5.2. *The Čech resolution computes the Čech cohomology. In the sense that the Čech cohomology groups can be derived from the Čech cohomology by appying the global section and taking cohomology.*

Proof. Applying the global section on the complex $0 \to \mathscr{F} \to \mathscr{C}^\bullet(\mathcal{U},\mathscr{F})$ gives the complex

$$0 \longrightarrow \mathscr{F}(X) \longrightarrow C^0(\mathcal{U},\mathscr{F}) \longrightarrow C^1(\mathcal{U},\mathscr{F}) \longrightarrow C^2(\mathcal{U},\mathscr{F}) \longrightarrow \cdots$$

which is exactly the Čech complex. Thus its cohomology gives the Čech cohomology. □

Corollary 3.5.3. *With $X, \mathcal{U}, \mathscr{F}$ as above, there is a canonical map*

$$\check{H}^\bullet(\mathcal{U},\mathscr{F}) \to H^\bullet(X,\mathscr{F}).$$

Proof. Let $0 \to \mathscr{F} \to \mathscr{I}^\bullet$ be an injective resolution of $\mathscr{F}$. Then since $0 \to \mathscr{F} \to \mathscr{C}^\bullet(\mathcal{U},\mathscr{F})$ is also a resolution of $\mathscr{F}$, we have a canonical induced map $\mathscr{C}^\bullet(\mathcal{U},\mathscr{F}) \to \mathscr{I}^\bullet$ induced by the identity of $\mathscr{F}$. The induced map on cohomology is what we want. □

We will now state some results stating sufficient conditions for these canonical morphisms to actually be isomorphisms, thus enabling us to calculate sheaf cohomology via Čech cohomology.

Proposition 3.5.4. *If $\mathscr{F}$ is a flasque sheaf on X, then $\check{H}^p(\mathcal{U},\mathscr{F}) = 0$ for all open covers $\mathcal{U}$ of X and all $p > 0$.*

Proof. Let $\mathscr{C}^\bullet(\mathcal{U},\mathscr{F})$ be the Čech resolution of $\mathscr{F}$ with respect to $\mathcal{U}$. Recall that $\mathscr{C}^p(\mathcal{U},\mathscr{F})$ is defined for any open $V \subseteq X$ by

$$\Gamma(V,\mathscr{C}^p(\mathcal{U},\mathscr{F})) = \prod \mathscr{F}(V \cap U_{i_0\dots i_p}).$$

For each of these $U_{i_0\dots i_p}$, the sheaf $V \mapsto \mathscr{F}(V \cap U_{i_0\dots i_p})$ is flasque and since products of flasque sheaves are flasque, the entire sheaf $\mathscr{C}^p(\mathcal{U},\mathscr{F})$ is flasque. Therefore $\mathscr{C}^p(\mathcal{U},\mathscr{F})$ is acyclic by Proposition 2.2.5, so the Čech resolution can used to compute sheaf cohomology:

$$\check{H}^p(\mathcal{U},\mathscr{F}) = H^p(X,\mathscr{F}) = 0$$

as desired. □

Corollary 3.5.5. *For any sheaf $\mathscr{F}$ on X, the map $\check{H}^1(X,\mathscr{F}) \to H^1(X,\mathscr{F})$ is an isomorphism.*

Proof. Let $\mathscr{G}$ be a flasque sheaf and $0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{Q} \to 0$ be a short exact sequence of sheaves on X. Then $\check{H}^1(X,\mathscr{G}) = 0$, and by Corollary 3.4.4 there is an exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \Gamma(X,\mathscr{F}) \arrow[r] \arrow[d, "1"] & \Gamma(X,\mathscr{G}) \arrow[r] \arrow[d, "1"] & \Gamma(X,\mathscr{Q}) \arrow[r] \arrow[d, "1"] & \check{H}^1(X,\mathscr{F}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Gamma(X,\mathscr{F}) \arrow[r] & \Gamma(X,\mathscr{G}) \arrow[r] & \Gamma(X,\mathscr{Q}) \arrow[r] & H^1(X,\mathscr{F}) \arrow[r] & 0
\end{tikzcd}$$

Thus the map $\check{H}^1(X,\mathscr{F}) \to H^1(X,\mathscr{F})$ is an isomorphism by the five lemma. □

Proposition 3.5.6. *Let X be a paracompact space, and $\mathscr{F}$ be a sheaf on X. Then the canonical maps*

$$\check{H}^p(X,\mathscr{F}) \to H^p(X,\mathscr{F})$$

are isomorphisms for $p \geq 0$.

Proof. This follows from Corollary 3.4.8 and an induction. □

Definition 3.5.1. A sheaf $\mathscr{F}$ on X is **acyclic for an open cover** $\mathcal{U} = (U_i)_{i\in I}$ if for all $p > 0$ we have

$$H^p(U_{i_0\dots i_p}, \mathscr{F}|_{U_{i_0\dots i_p}}) = 0.$$

Theorem 3.5.7 (Leray). *If $\mathscr{F}$ is a sheaf on X which is acyclic for an open cover $\mathcal{U}$, then the canonical maps*

$$\check{H}^p(\mathcal{U},\mathscr{F}) \to H^p(X,\mathscr{F})$$

are isomorphisms for all $p \geq 0$.

Proof. We proceed by induction on the degree p. For $p = 0$ we already know that the result is true thanks to Proposition 3.1.2.

Now, assume $\check{H}^j(U,\mathscr{G}) \to H^j(X,\mathscr{G})$ is an isomorphism for all $j \leq p$ and all sheaves $\mathscr{G}$ acyclic for $\mathcal{U}$. Embed $\mathscr{F}$ in a flasque sheaf $\mathscr{G}$ and let $\mathscr{H}$ be the quotient, so that we have an exact sequence

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{G} \longrightarrow \mathscr{H} \longrightarrow 0$$

For each finite sequence $\sigma = (i_0,\dots,i_p)$, by hypothesis we have $H^i(U_\sigma,\mathscr{F}|_{U_\sigma}) = 0$ and $H^i(U_\sigma,\mathscr{G}|_{U_\sigma}) = 0$. Therefore $H^i(U_\sigma,\mathscr{H}|_{U_\sigma})$ is zero by the long exact sequence of $H^i(U_\sigma.-)$, and by taking the product over all such U_σ, we conclude that $\mathscr{H}$ is also acyclic for $\mathcal{U}$.

Now by the argument above, the sequence

$$0 \longrightarrow \mathscr{F}(U_\sigma) \longrightarrow \mathscr{G}(U_\sigma) \longrightarrow \mathscr{H}(U_\sigma) \longrightarrow 0$$

is exact. Hence by taking product the corresponding short sequence of Čech complexes

$$0 \longrightarrow \mathscr{C}^\bullet(\mathcal{U}, \mathscr{F}) \longrightarrow \mathscr{C}^\bullet(\mathcal{U}, \mathscr{G}) \longrightarrow \mathscr{C}^\bullet(\mathcal{U}, \mathscr{H}) \longrightarrow 0$$

is exact, so that we get a long exact sequence in Čech cohomology

$$\cdots \longrightarrow \check{H}^p(\mathcal{U}, \mathscr{F}) \longrightarrow \check{H}^p(\mathcal{U}, \mathscr{G}) \longrightarrow \check{H}^p(\mathcal{U}, \mathscr{H}) \longrightarrow \check{H}^{p+1}(\mathcal{U}, \mathscr{F}) \longrightarrow \cdots$$

Now since $\mathscr{G}$ is flasque, Proposition 3.5.4 shows that $\check{H}^p(\mathcal{U}, \mathscr{G}) = 0$. Therefore we have a diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \check{H}^p(\mathcal{U}, \mathscr{H}) \arrow[r] \arrow[d] & \check{H}^{p+1}(\mathcal{U}, \mathscr{F}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H^p(\mathcal{U}, \mathscr{H}) \arrow[r] & H^{p+1}(\mathcal{U}, \mathscr{F}) \arrow[r] & 0
\end{tikzcd}$$

By induction, the left column is an isomorphism, so $\check{H}^{p+1}(X, \mathscr{F}) \to H^{p+1}(X, \mathscr{F})$ is also an isomorphism. □

3.6 Čech vs. Sheaf cohomology

Proposition 3.6.1. *Let X be a ringed space. Let*

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{G} \longrightarrow \mathscr{H} \longrightarrow 0$$

be a short exact sequence of $\mathscr{O}_X$-modules. Let $U \subseteq X$ be an open subset. If there exists a cofinal system of open coverings $\mathcal{U}$ of U such that $\check{H}^1(\mathcal{U}, \mathscr{F}) = 0$, then the map $\mathscr{G}(U) \to \mathscr{H}(U)$ is surjective.

Proof. Take an element $s \in \mathscr{H}(U)$. Choose an open covering $\mathcal{U} : U = (U_i)_{i \in I}$ such that

(a) $\check{H}^1(\mathcal{U}, \mathscr{F}) = 0$.

(b) $s|_{U_i}$ is the image of a section $s_i \in \mathscr{G}(U_i)$.

Since we can certainly find a covering such that (b) holds by the exactness of the sequence, it follows from the assumptions that we can find a covering such that (a) and (b) both hold. Consider the sections

$$s_{i_0 i_1} = s_{i_1}|_{U_{i_0 i_1}} - s_{i_0}|_{U_{i_0 i_1}}$$

Since s_i lifts s we see that $s_{i_0 i_1} \in \mathscr{F}(U_{i_0 i_1})$. By the vanishing of $\check{H}^1(\mathcal{U}, \mathscr{F})$ we can find sections $t_i \in \mathscr{F}(U_i)$ such that

$$s_{i_0 i_1} = t_{i_1}|_{U_{i_0 i_1}} - t_{i_0}|_{U_{i_0 i_1}}$$

Then clearly the sections $s_i - t_i$ satisfy the sheaf condition and glue to a section of $\mathscr{G}$ over U which maps to s. Hence we win. □

Proposition 3.6.2. *Let X be a ringed space. Let $\mathscr{F}$ be an $\mathscr{O}_X$-module such that*

$$\check{H}^p(\mathcal{U}, \mathscr{F}) = 0$$

for all $p > 0$ and any open covering $\mathcal{U} : U = \bigcup_{i \in I} U_i$ of an open subset U of X. Then $H^p(U, \mathscr{F}) = 0$ for all $p > 0$ and any open $U \subseteq X$.

Proof. Choose an embedding $\mathcal{F} \to \mathcal{I}$ into an injective $\mathcal{O}_X$-module. By Proposition 3.5.4 $\mathcal{I}$ has vanishing higher Čech cohomology for any open covering. Let $\mathcal{Q} = \mathcal{I}/\mathcal{F}$ so that we have a short exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{I} \longrightarrow \mathcal{Q} \longrightarrow 0$$

By Proposition3.6.1 and our assumptions this sequence is actually exact as a sequence of presheaves! In particular we have a long exact sequence of Čech cohomology groups for any open covering $\mathcal{U}$, see Theorem 3.2.2 for example. This implies that $\mathcal{Q}$ is also an $\mathcal{O}_X$-module with vanishing higher Čech cohomology for all open coverings.

Next, we look at the long exact cohomology sequence

$$\cdots \longrightarrow H^p(U,\mathcal{F}) \longrightarrow H^p(U,\mathcal{I}) \longrightarrow H^p(U,\mathcal{Q}) \longrightarrow H^{p+1}(U,\mathcal{F}) \longrightarrow \cdots$$

for any open $U \subseteq X$. Since $\mathcal{I}$ is injective we have $H^p(U,\mathcal{I}) = 0$ for $p > 0$. By the above we see that $H^0(U,\mathcal{I}) \to H^0(U,\mathcal{Q})$ is surjective, thus $H^1(U,\mathcal{F}) = 0$. Also, $H^p(U,\mathcal{Q}) = H^{p+1}(U,\mathcal{F})$ for $p \geq 1$. Thus the claim follows by an induction. □

Proposition 3.6.3. *Let X be a ringed space. Let $\mathcal{B}$ be a basis for the topology on X. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Assume there exists a set of open coverings* Cov *with the following properties:*

(i) For every $\mathcal{U} \in$ Cov with $\mathcal{U} : U = \bigcup_{i\in I} U_i$ we have $U, U_i \in \mathcal{B}$ and every $U_{i_0\ldots i_p} \in \mathcal{B}$.

(ii) For every $U \in \mathcal{B}$ the open coverings of U occurring in Cov *is a cofinal system of open coverings of U.*

(iii) For every $\mathcal{U} \in$ Cov we have $\check{H}^p(\mathcal{U},\mathcal{F}) = 0$ for all $p > 0$.

Then $H^p(U,\mathcal{F}) = 0$ for all $p > 0$ and any $U \in \mathcal{B}$.

Proof. Let $\mathcal{F}$ and Cov be as in the lemma. We will indicate this by saying $\mathcal{F}$ has vanishing higher Čech cohomology for any $\mathcal{U} \in$ Cov. Choose an embedding $\mathcal{F} \hookrightarrow \mathcal{I}$ into an injective $\mathcal{O}_X$-module. By Proposition 3.5.4 has vanishing higher Čech cohomology for any $\mathcal{U} \in$ Cov. Let $\mathcal{Q} = \mathcal{I}/\mathcal{F}$ so that we have a short exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{I} \longrightarrow \mathcal{Q} \longrightarrow 0$$

By Proposition 3.6.1 and our assumption (ii) this sequence gives rise to an exact sequence

$$0 \longrightarrow \mathcal{F}(U) \longrightarrow \mathcal{I}(U) \longrightarrow \mathcal{Q}(U) \longrightarrow 0$$

for every $U \in \mathcal{B}$. Hence for any $\mathcal{U} \in$ Cov we get a short exact sequence of Čech complexes

$$0 \longrightarrow \mathcal{C}^\bullet(\mathcal{U},\mathcal{F}) \longrightarrow \mathcal{C}^\bullet(\mathcal{U},\mathcal{I}) \longrightarrow \mathcal{C}^\bullet(\mathcal{U},\mathcal{Q}) \longrightarrow 0$$

since each term in the Čech complex is made up out of a product of values over elements of $\mathcal{B}$ by assumption (i). In particular we have a long exact sequence of Čech cohomology groups for any open covering $\mathcal{U} \in$ Cov. This implies that $\mathcal{Q}$ is also an $\mathcal{O}_X$-module with vanishing higher Čech cohomology for all $\mathcal{U} \in$ Cov.

Next, we look at the long exact cohomology sequence

$$\cdots \longrightarrow H^p(U,\mathcal{F}) \longrightarrow H^p(U,\mathcal{I}) \longrightarrow H^p(U,\mathcal{Q}) \longrightarrow H^{p+1}(U,\mathcal{F}) \longrightarrow \cdots$$

for any open $U \in \mathcal{B}$. Since $\mathcal{I}$ is injective we have $H^n(U,\mathcal{I}) = 0$ for $n > 0$. By the above we see that $H^0(U,\mathcal{I}) \to H^0(U,\mathcal{Q})$ is surjective and hence $H^1(U,\mathcal{F}) = 0$. Since $\mathcal{F}$ was an arbitrary $\mathcal{O}_X$-module with vanishing higher Čech cohomology for all $\mathcal{U} \in$ Cov we conclude that also $H^1(U,\mathcal{Q}) = 0$ since $\mathcal{Q}$ is another of these sheaves (see above). Also, $H^p(U,\mathcal{Q}) = H^{p+1}(U,\mathcal{F})$ for $p \geq 1$. Thus the claim follows by an induction. □

Proposition 3.6.4. *Let $f : X \to Y$ be a morphism of ringed spaces. Let $\mathscr{I}$ be an injective $\mathscr{O}_X$-module. Then*

- $\check{H}^p(\mathcal{V}, f_*\mathscr{I}) = 0$ *for all $p > 0$ and any open covering $\mathcal{V} = (V_j)_{j \in J}$ of Y.*
- $H^p(V, f_*\mathscr{I}) = 0$ *for all $p > 0$ and every open $V \subseteq Y$.*

In other words, $f_*\mathscr{I}$ is right acyclic for $\Gamma(U, -)$ for any $U \subseteq X$ open.

Proof. Set $\mathcal{U} = f^{-1}(\mathcal{V})$. It is an open covering of X and

$$C^\bullet(\mathcal{V}, f_*\mathscr{I}) = C^\bullet(\mathcal{U}, \mathscr{I}).$$

This is true because

$$f_*\mathscr{I}(V_{j_0 \dots j_p}) = \mathscr{I}(f^{-1}(V_{j_0 \dots j_p})) = \mathscr{I}\big(f^{-1}(V_{j_0}) \cap \dots \cap f^{-1}(V_{j_p})\big) = \mathscr{I}(U_{j_0 \dots j_p}).$$

Thus the first statement of the lemma follows from Proposition 3.5.4. The second statement follows from the first and Proposition 3.6.2. □

3.7 Cohomology with compact support

3.7.1 The sheaf of sections with compact support

Let X be a locally compact Hausdorff space. For a sheaf $\mathscr{F}$ on X, we define the sections of $\mathscr{F}$ over X with compact support to be

$$\Gamma_c(X, \mathscr{F}) = \{s \in \Gamma(X, \mathscr{F}) : \mathrm{supp}(s)\}$$

This define a left exact functor $\Gamma_c(X, -) : \mathrm{Sh}(X) \to \mathbf{Ab}$, whose i-th derived functor is denoted by $H^i_c(X, \mathscr{F})$ and called the i-the cohomology group with compact support with coefficient in $\mathscr{F}$.

Consider a proper map $f : X \to Y$ between locally compact Hausdorff spaces, i.e. such that the inverse image of any compact subset of Y is compact in X. For a sheaf $\mathscr{G}$ over X, the adjunction map $\Gamma(Y, \mathscr{G}) \to \Gamma(X, f^*(\mathscr{G}))$ will transform a section of G with compact support into a section of $f^*(\mathscr{G})$ with compact support, and hence induces a natural map

$$\Gamma_c(Y, \mathscr{G}) \to \Gamma_c(X, f^*(\mathscr{G})). \tag{3.7.1}$$

Since f^* is exact, we can extend this into a morphism

$$f^* : H^*_c(Y, \mathscr{G}) \to H^*_c(X, f^*(\mathscr{G}))$$

by choosing an injective resolution $\mathscr{G} \to \mathscr{J}^\bullet$ and an injective resolution $f^*(\mathscr{J}^\bullet) \to \mathscr{I}^\bullet$. The compostion

$$\Gamma_c(Y, \mathscr{J}^\bullet) \to \Gamma_c(X, f^*(\mathscr{J}^\bullet)) \to \Gamma_c(X, \mathscr{I}^\bullet)$$

then represents (3.7.1) on the chain level, noticing that $\mathscr{I}$ is an injective resolution of $f^*(\mathscr{G})$.

By the functoriality, we can derive from an exact sequence of sheaves on Y

$$0 \longrightarrow \mathscr{E} \longrightarrow \mathscr{F} \longrightarrow \mathscr{G} \longrightarrow 0$$

a commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & H^p_c(Y, \mathscr{E}) \arrow[r] \arrow[d] & H^p_c(Y, \mathscr{F}) \arrow[r] \arrow[d] & H^p_c(Y, \mathscr{G}) \arrow[r] \arrow[d] & H^{p+1}_c(Y, \mathscr{E}) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & H^p_c(X, f^*(\mathscr{E})) \arrow[r] & H^p_c(X, f^*(\mathscr{F})) \arrow[r] & H^p_c(X, f^*(\mathscr{G})) \arrow[r] & H^{p+1}_c(X, f^*(\mathscr{E})) \arrow[r] & \cdots
\end{tikzcd}$$

Consider in particular the inclusion $i : Z \to X$ of a closed subspace. A sheaf $\mathscr{E}$ on Z givs rise to a canonical isomorphism

$$H_c^*(Z, \mathscr{E}) = H_c^*(X, i_*(\mathscr{E})).$$

For closed subspaces A and B of X, we also have a Mayer-Vietoris sequence

$$\cdots \longrightarrow H_c^p(A \cup B, \mathscr{F}) \longrightarrow H_c^p(A, \mathscr{F}) \oplus H_c^p(\mathscr{B}, \mathscr{F}) \longrightarrow H_c^p(A \cap B, \mathscr{F}) \longrightarrow \cdots$$

whose proof is similar to the case of usual cohomology groups.

3.7.2 Soft sheaves

Let X denote a locally compact Hausdorff space. We shall introduce an important class of $\Gamma_c(X, -)$-acyclic sheaves on X.

Proposition 3.7.1. *Let $i : Z \to X$ be the inclusion of a compact subset into a locally compact Hausdorff space X. For any sheaf $\mathscr{F}$ on X, there is a canonical isomorphism*

$$\varinjlim_U \Gamma(U, \mathscr{F}) \xrightarrow{\sim} \Gamma(Z, i^{-1}(\mathscr{F})),$$

where the limit is taken over all open subsets $U \supseteq Z$.

Proof. We note that the canonical map $\varinjlim \Gamma(U, \mathscr{F}) \to \Gamma(Z, i^{-1}(\mathscr{F}))$ is injective for trivial reasons. To see that it is surjective, consider a section $s \in \Gamma(Z, i^{-1}(\mathscr{F}))$. Since Z is compact, there exists a finite family of open subsets $(U_i)_{i \in I}$ and sections $s_i \in \Gamma(U_i, \mathscr{F})$ such that $s|_{U_i \cap Z} = s_i|_{U_i \cap Z}$ and $Z \subseteq \bigcup_i U_i$. We can then find a family of open subsets $(V_i)_{i \in I}$ such that $\bar{V_i} \subseteq U_i$ and $Z \subseteq \bigcup_i V_i$. For $x \in X$, let $I(x) = \{i \in I : x \in \bar{V_i}\}$ and define

$$W = \{x \in \bigcup_i V_i : (s_i)_x = (s_j)_x \text{ for any } i, j \in I(x)\}.$$

Then $I(x)$ is finite and each x has a neighborhood W_x such that $I(y) \subseteq I(x)$ for any $y \in W_x$. We then conclude that W is open and contains Z by its construction. Since $s_i|_{W \cap V_i \cap V_j} = s_j|_{W \cap V_i \cap V_j}$, there exists $\tilde{s} \in \Gamma(W, \mathscr{F})$ such that $\tilde{s}|_{W \cap V_i} = s_i|_{W \cap V_i}$, which is our desired section. □

Definition 3.7.1. A sheaf $\mathscr{F}$ on X is called **soft** if for any compact subset K of X, the restriction map $\Gamma(X, \mathscr{F}) \to \Gamma(K, \mathscr{F})$ is surjective.

In view of Proposition 3.7.1, softness can also be described by the following conditions: for any compact subset K, any open subset U of X containing K, and any section s of $\mathscr{F}$ over U, there exists a section t of $\mathscr{F}$ over X such that s and t have the same restriction to some open neighborhood of K contained in U. Therefore, flasque sheaves, and in particular injective sheaves, are soft. We also note that softness can be detected by sections with compact support.

Proposition 3.7.2. *Let $\mathscr{F}$ be a sheaf over X. Then X is soft if and only if for any closed subset Z of X, the restriction map $\Gamma_c(X, \mathscr{F}) \to \Gamma_c(Z, \mathscr{F}|_Z)$ is surjective.*

Proof. If K is compact, then $\Gamma(K, \mathscr{F}) = \Gamma_c(K, \mathscr{F}|_K)$, so it suffices to prove the converse. Let $\mathscr{F}$ be soft and $s \in \Gamma(Z, \mathscr{F}|_Z)$ with compact support K. Let U be a relatively compact open neighborhood of K in X and define a section $\tilde{s} \in \Gamma(\partial U \cup (Z \cap \bar{U}), \mathscr{F})$ by setting $\tilde{s}|_{Z \cap \bar{U}} = s$ and $\tilde{s}|_{\partial U} = 0$.If we extend $\tilde{s}$ to a section $t \in \Gamma(X, \mathscr{F})$, then since $t = 0$ on a neighborhood of ∂U, we may assume that t is supported by $\bar{U}$, which is compact by hypothesis. □

Corollary 3.7.3. *Let W be a locally closed subspace of X and $\mathscr{F}$ be a soft sheaf on X. Then $\mathscr{F}|_W$ is soft on W.*

Proof. If W is open, this is clear, and the closed case follows from Proposition 3.7.2. □

Theorem 3.7.4. *A soft sheaf $\mathscr{F}$ on a locally compact Hausdorff space X is $\Gamma_c(X, -)$ acyclic, i.e.*

$$H_c^n(X, \mathscr{F}) = 0 \quad \text{for} \quad n > 0.$$

Proof. Consider an exact sequence

$$0 \longrightarrow \mathscr{E} \xrightarrow{\varphi} \mathscr{F} \xrightarrow{\psi} \mathscr{G} \longrightarrow 0$$

with $\mathscr{E}$ soft. We shall prove that the projection $\Gamma_c(X, \mathscr{F}) \to \Gamma_c(X, \mathscr{G})$ is surjective. First assume that X is compact, so let there be given a section $s \in \Gamma(X, \mathscr{G})$. For each point $x \in X$, we can choose a compact neighborhood K_x of x and a section $t_x \in \Gamma(K_x, \mathscr{F})$ which projects onto the restriction of s to K_x. We may refine the covering $(\operatorname{Int} K_x)_{X \in X}$ to a finite covering, which leads us to the following construction: Given compact subsets K and L of X, $u \in \Gamma(K, \mathscr{F})$ and $v \in \Gamma(L, \mathscr{F})$ and $v \in \Gamma(L, \mathscr{F})$ which projects onto s. The section $u|_{K \cap L} - v|_{K \cap L}$ represents a section of $\mathscr{E}$ over $L \cap K$ and can as such be extended to a section $w \in \Gamma(L, \mathscr{E})$. We can now consider the sections $u \in \Gamma(K, \mathscr{F})$ and $v + \varphi(w) \in \Gamma(L, \mathscr{F})$ and glue them toghther to obtain a section of $\mathscr{F}$ over $K \cap L$ which projects onto s. Now apply this construction to our finite covering, we get a section of F over X which projects onto $s \in \Gamma(X, \mathscr{G})$.

In the case where X is not compact, we can choose a compact subset K so that $\operatorname{Int} K$ contains $\operatorname{supp}(s)$. Choose a section t of F over K which projects onto the restriction of s to K. The restriction of t to ∂K represents a section of E over ∂K which we extend to a section of $\mathscr{E}$ over K, and subtract from t. Thus we may assume that the restriction of t to ∂K is zero, and hence extend t by zero outside K, to obtain a global section of F which projects onto s. We also note that these considerations proves that if we add the assumption that $\mathscr{F}$ is soft, then G is soft.

We now return to a general soft sheaf $\mathscr{F}$ over X and choose an exact sequence

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{I} \longrightarrow \mathscr{T} \longrightarrow 0$$

with $\mathscr{I}$ injetive. Using that $\mathscr{I}$ is flasque, we conclude first that $\mathscr{I}$ is soft and second that $\mathscr{T}$ is soft. We can now conclude from our previous arguments that $H_c^1(X, \mathscr{F}) = 0$, and the general case follows by an induction. □

Corollary 3.7.5. *Let $0 \to \mathscr{E} \to \mathscr{F} \to \mathscr{G} \to 0$ be an exact sequence of sheaves over X. If $\mathscr{E}$ and $\mathscr{F}$ are soft, then $\mathscr{G}$ is soft.*

Proof. This is remarked in the proof of Theorem 3.7.4. □

Proposition 3.7.6. *Let $\mathscr{F}$ be a sheaf over a locally compact Hausdorff space X. If every point of X has an open neighborhood U such that the restriction of $\mathscr{F}$ to U is a soft sheaf on U, then $\mathscr{F}$ is soft.*

Proof. We first treat the case where X is compact. By assumption we can find a finite number of compact subsets $X_1, \ldots, X_n$ of X such that restrictions of $\mathscr{F}$ to each of these is soft. To prove that $\mathscr{F}$ is soft, consider a compact subset K of X and $s \in \Gamma(K, \mathscr{F})$. Put $X^i = K \cup X_1 \cup \cdots \cup X_i$, where $i = 1, \ldots, n$. Suppose that we have already extended $s = s^0$ to a section s^i of s over X^i. By assumption we can extend the restriction of s^i to $X^i \cap X_{i+1}$ into a section σ^{i+1} of $\mathscr{F}$ over X_{i+1}. By glueing, we obtain a section s^{i+1} of $\mathscr{F}$ over X^{i+1} which extends s^i and σ^{i+1}. We can now proceed by induction to obtain an extension of s to X.

We now consider the general case. We know that the restriction of $\mathscr{F}$ to any compact subset of X is soft. Let us attempt to extend a section s of $\mathscr{F}$ over the compact subset K to a global section. Choose a compact subset L of X with $K \subseteq \operatorname{Int} L$. We can then extend the section over $K \cup \partial L$ which is s on K and zero on ∂L to a section of $\mathscr{F}$ over L, and extend it by zero outside L. □

Proposition 3.7.7. *Let $\mathscr{A}$ be a soft sheaf of commutative rings. Then any sheaf $\mathscr{M}$ of A-modules is soft.*

Proof. Let K be a compact subset of X and L be a compact neighborhood of K. Then there exists $\sigma \in \Gamma(X, \mathscr{A})$ such that $\sigma = 1$ in an open neighborhood of K and such that the restriction of σ to the complement of $\operatorname{Int} L$ is zero. To see this, let us consider the section on $K \cup \partial L$ which is 1 on K and 0 on ∂L. Extend this to a section τ of $\mathscr{A}$ over L and let σ be the extension of τ by zero outside L, we then get the desired section.

Now let us show that $\mathscr{M}$ is soft, so let s be a section of $\mathscr{F}$ over an open neighborhood U of K. Let L be a compact neighborhood of K contained and σ be a global section of $\mathscr{A}$ as above. If τ is the restriction of σ to L, then the section τs of $\mathscr{F}$ over L is zero on ∂L, and therefore can be extended by zero to X. □

Proposition 3.7.8. *Let $\mathscr{F}$ and $\mathscr{G}$ be sheaves on a locally compact Hausdorff space X and $f : \Gamma(X, \mathscr{F}) \to \Gamma(X, \mathscr{G})$ be a linear map which satisfies the following conditions:*

$$\operatorname{supp}(f(s)) \subseteq \operatorname{supp}(s) \quad \text{for} \quad s \in \Gamma_c(X, \mathscr{F}).$$

If $\mathscr{F}$ is soft, then there exists a unique morphism of sheaves $\varphi : \mathscr{F} \to \mathscr{G}$ which induces f on the level of lobal sections.

Proof. For any point x of X the restriction map $\Gamma_c(X, \mathscr{F}) \to \mathscr{F}_x$ is surjective, so the uniqueness of φ follows. Given an open subset U of X and $s \in \Gamma(U, \mathscr{F})$, we cover U with open subsets U_i such that there exists $s_i \in \Gamma_c(X, \mathscr{F})$ whose restriction to U_i equals to that of s. In view of the condition on supports, we see that the sections $f(s_i)|_{U_i}$ coincide on overlappings, so we can glue them together to get a section of $\mathscr{G}$ over U. It is clear that this construction is compatible with restriction to smaller opens, so we obtain a morphism $\varphi : \mathscr{F} \to \mathscr{G}$. It is then easy to check that φ induces f on the level of global sections. □

3.7.3 Soft sheaves on $\mathbb{R}^n$

A basic sheaf on $\mathbb{R}^n$ is the sheaf $\mathscr{C}^\infty$ of smooth $\mathbb{R}$-valued functions, which we now prove that is soft.

Theorem 3.7.9. *The sheaf $\mathscr{C}^\infty$ of smooth functions on $\mathbb{R}^n$ is soft.*

Proof. Let s be a section of $\mathscr{C}^\infty$ over an open neighborhood U of the compact set K. We can choose a compact neighborhood V of K contained in U and a smooth function on $\mathbb{R}^n$ which is zero outside V and is 1 in a neighborhood of K. The function φs can then be extended into a section of $\mathbb{R}^n$. □

Corollary 3.7.10. *The sheaf $\mathscr{C}$ of continuous real functions on $\mathbb{R}^n$ is soft. More generally, the sheaf $\mathscr{C}^{(p)}$ of p-times continuously differentiable real functions on $\mathbb{R}^n$ is soft.*

Proof. Note that $\mathscr{C}^{(p)}$ is a sheaf of module over $\mathscr{C}^\infty$, so we can apply Proposition 3.7.7. □

Example 3.7.1. The fundamental example of a locally compact Hausdorff space is the real line $\mathbb{R}$. The constant sheaf $\underline{\mathbb{R}}$ has the "Calculus resolution" which is soft:

$$0 \longrightarrow \underline{\mathbb{R}} \longrightarrow \mathscr{C}^{(1)} \xrightarrow{d} \mathscr{C} \longrightarrow 0$$

where d is the ordinary differentiation. To calculate the cohomology, we consider the exact sequence

$$0 \longrightarrow \Gamma_c(\mathbb{R}, \mathscr{C}^{(1)}) \xrightarrow{d} \Gamma_c(\mathbb{R}, \mathscr{C}) \xrightarrow{\int} \mathbb{R} \longrightarrow 0$$

where $\int$ is given by the formula $\int f = \int_{\mathbb{R}} f(x) dx$. We therefore conclude that

$$H^i_c(\mathbb{R}, \underline{\mathbb{R}}) = \begin{cases} 0 & i \neq 1, \\ \mathbb{R} & i = 1. \end{cases}$$

Proposition 3.7.11.

Chapter 4

Category

4.1 Functors

4.1.1 Definition of Functors

Let $\mathcal{C}, \mathcal{D}$ be two categories. A **covariant functor**

$$F : \mathcal{C} \to \mathcal{D}$$

is an assignment of an object $F(A) \in \mathrm{Ob}(\mathcal{D})$ for every $A \in \mathrm{Ob}(\mathcal{C})$ and of a function

$$\mathrm{Mor}_{\mathcal{C}}(A, B) \to \mathrm{Mor}_{\mathcal{D}}(F(A), F(B))$$

for every pair of objects A, B in $\mathcal{C}$ such that $\forall f \in \mathrm{Mor}_{\mathcal{C}}(A, B), g \in \mathrm{Mor}_{\mathcal{C}}(B, C)$ we have

$$F(\mathrm{id}_A) = \mathrm{id}_{F_A}, \quad F(g \circ f) = F(g) \circ F(f).$$

Example 4.1.1. If R is a ring, we have denoted by $R^\times$ the group of units in R; every ring homomorphism $R \to S$ induces a group homomorphism $R^\times \to S^\times$, and this assignment is compatible with compositions; therefore this operation defines a covariant functor **Ring** $\to$ **Ab**.

4.1.2 Equivalence of Category

The structure of an object in a category is adequately carried by its isomorphism class, and a natural notion of equivalence of categories should aim at matching isomorphism classes, rather than individual objects. The *morphisms* are a more essential piece of information; the quality of a functor is first of all measured on how it acts on morphisms.

Definition 4.1.1. Let $\mathcal{C}, \mathcal{D}$ be two categories. Let $F : \mathcal{C} \to \mathcal{D}$ be a covariant functor.

(a) F is **faithful** if for all objects A, B of $\mathcal{C}$, the induced function

$$\mathrm{Mor}_{\mathcal{C}}(A, B) \to \mathrm{Mor}_{\mathcal{D}}(F(A), F(D))$$

is injective.

(b) F is **full** if this function is surjective, for all objects A, B.

(c) F is called essentially surjective if for any object $B \in \mathrm{Ob}(\mathcal{B})$ there exists an object $A \in \mathrm{Ob}(\mathcal{A})$ such that $F(A)$ is isomorphic to B in $\mathcal{B}$.

Lemma 4.1.1. *Let $F : \mathcal{C} \to \mathcal{D}$ be a fully faithful functor. If A, B are objects in $\mathcal{C}$, then $A \cong B$ in $\mathcal{C}$ if and only if $F(A) \cong F(B)$ in $\mathcal{D}$.*

Proof. Assume F is covariant. Since we have $F(f \circ g) = F(f) \circ G(g)$, if $A \cong B$, we have $F(A) \cong \mathscr{B}$. Conversely, if $F(A) \cong F(B)$, assume f, g are isomorphisms between them with $g = f^{-1}$. Since F is full, there are two morphisms f', g' such that $F(f') = f, F(g') = g$. Then we have

$$F(f' \circ g') = f \circ g = \mathrm{id}, \quad F(g' \circ f') = g \circ f = \mathrm{id}$$

Since $F(\mathrm{id}) = \mathrm{id}$, and F is faithful, we get $f' \circ g' = g' \circ f' = \mathrm{id}$, so $A \cong B$ follows. □

Definition 4.1.2. Let $\mathcal{C}, \mathcal{D}$ be categories, and let F, G be functors $\mathcal{C} \to \mathcal{D}$. A **natural transformation** $\nu : F \to G$ is the datum of a morphism $\nu_X : F(X) \to G(X)$ in $\mathcal{D}$ for every object X in $\mathcal{C}$, such that $\forall \alpha : X \to Y$ in $\mathcal{C}$ the diagram

$$\begin{tikzcd} F(X) \arrow[r, "\nu_X"] \arrow[d, "F(\alpha)"'] & G(X) \arrow[d, "G(\alpha)"] \\ F(Y) \arrow[r, "\nu_Y"] & G(Y) \end{tikzcd}$$

commutes. A **natural isomorphism** is a natural transformation μ such that μ_X is an isomorphism for every X.

A natural transformation is often written as

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\nu"] & \mathcal{D} \end{tikzcd}$$

In addition, given a morphism of functors $\mu : F \to G$ and a morphism of functors $\nu : \mathscr{E} \to F$ then the composition $\nu \circ \mu$ is defined by the rule

$$(\nu \circ \mu)_X := \nu_X \circ \mu_X.$$

for $X \in \mathrm{Ob}(\mathcal{A})$. It is easy to verify that this is indeed a morphism of functors from $\mathscr{E}$ to G. In this way, given categories $\mathcal{A}$ and $\mathcal{B}$ we obtain a new category, namely the category of functors between $\mathcal{A}$ and $\mathcal{B}$.

Definition 4.1.3. An **equivalence of categories** $F : \mathcal{A} \to \mathcal{B}$ is a functor such that there exists a functor $G : \mathcal{B} \to \mathcal{A}$ such that the compositions $F \circ G$ and $G \circ F$ are isomorphic to the identity functors $\mathrm{id}_{\mathcal{B}}$, respectively $\mathrm{id}_{\mathcal{A}}$. In this case we say that G is a **quasi-inverse** to F.

Proposition 4.1.2. *Let $F : \mathcal{A} \to \mathcal{B}$ be a fully faithful functor. Suppose for every $X \in \mathrm{Ob}(\mathcal{B})$ we are given an object $G(X)$ of A and an isomorphism $i_X : X \to F(G(X))$. Then there is a unique functor $G : \mathcal{B} \to \mathcal{A}$ such that G extends the rule on objects, and the isomorphisms i_X define an isomorphism of functors $\mathrm{id}_{\mathcal{B}} \to F \circ G$. Moreover, G and F are quasi-inverse equivalences of categories.*

Proof. The action of G on objects is defined. For $X, Y \in \mathrm{Ob}(\mathcal{B})$ and $f \in \mathrm{Mor}_{\mathcal{B}}(X, Y)$, we have the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "i_X"] & Y \arrow[d, "i_Y"] \\ F(G(X)) \arrow[r, "i_Y \circ f \circ i_X^{-1}"] & F(G(Y)) \\ G(X) \arrow[u, "F"] \arrow[r, dashed] & G(Y) \arrow[u, "F"'] \end{tikzcd}$$

where the dashed map is induced by the bijection

$$\mathrm{Mor}_{\mathcal{A}}(X, Y) \cong \mathrm{Mor}_{\mathcal{B}}(F(X), F(Y))$$

by the functoriality, this defines a functor $G : \mathcal{B} \to \mathcal{A}$. Moreover, the upper half of the diagram means i_X define an isomorphism of functors $\mathrm{id}_{\mathcal{B}} \to F \circ G$.

For $X \in \mathrm{Ob}(\mathcal{A})$, we have an isomorphism $i_{F(X)} : F(X) \to F \circ G \circ F(X)$. Since F is full and faithful, there is an isomorphism $\mu_X : X \to G \circ F(X)$ by Lemma 4.1.1. The naturality of μ_X follows from that of $i_{F(X)}$ and the faithfulness of F. Thus μ is an isomorphism $\mathrm{id}_{\mathcal{A}} \to G \circ F$. □

Corollary 4.1.3. *A functor is an equivalence of categories if and only if it is both fully faithful and essentially surjective.*

Proof. Let $F : \mathcal{A} \to \mathcal{B}$ be essentially surjective and fully faithful. As by convention all categories are small and as F is essentially surjective we can, using the axiom of choice, choose for every $X \in \mathrm{Ob}(\mathcal{B})$ an object $G(X)$ of $\mathcal{A}$ and an isomorphism $i_X : X \to F(G(X))$. Then we apply Proposition 4.1.2. □

4.1.3 Yoneda's Lemma

Definition 4.1.4. Given a category $\mathcal{C}$ the opposite category $\mathcal{C}^{op}$ is the category with the same objects as $\mathcal{C}$ but all morphisms reversed.

Definition 4.1.5. Let $\mathcal{A}, \mathcal{B}$ be categories. A contravariant functor F from $\mathcal{A}$ to $\mathcal{B}$ is a functor $\mathcal{A}^{op} \to \mathcal{B}$.

Concretely, a contravariant functor F satisfies the property that, given another morphism $f : X \to Y$ and $g : Y \to Z$, we have

$$F(g \circ f) = F(g) \circ F(f)$$

as morphism from $F(Z)$ to $F(X)$.

Definition 4.1.6. Let $\mathcal{C}$ be a category. A **presheaf of sets on** $\mathcal{C}$ or simply a presheaf is a contravariant functor F from $\mathcal{C}$ to **Set**. The category of presheaves is denoted $\mathbf{Psh}(\mathcal{C})$.

Example 4.1.2 (Functor of points). For any $U \in \mathrm{Ob}(\mathcal{C})$ there is a contravariant functor

$$h_X : \mathcal{C} \to \mathbf{Set}, \quad Y \mapsto \mathrm{Mor}_{\mathcal{C}}(Y, X).$$

In other words h_X is a presheaf. We will always denote this presheaf $h_X : \mathcal{C}^{op} \to \mathbf{Set}$. It is called the **representable presheaf** associated to X.

Note that given a morphism $s : X \to Y$ in $\mathcal{C}$ we get a corresponding natural transformation of functors $h(s) : h_X \to h_Y$ defined simply by composing with the morphism $U \to V$. It is trivial to see that this turns composition of morphisms in $\mathcal{C}$ into composition of transformations of functors. In other words we get a functor

$$h : \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{op}, \mathbf{Set}) =: \widehat{\mathcal{C}}.$$

Lemma 4.1.4 (Yoneda lemma). *The functor h is fully faithful. More generally, given any contravariant functor F and any object X of $\mathcal{C}$ we have a natural bijection*

$$\mathrm{Mor}_{\widehat{\mathcal{C}}}(h_X, F) \to F(X), \quad \alpha \mapsto \alpha_X(\mathrm{id}_X).$$

Proof. An element $f \in h_X(Y) = \mathrm{Mor}_{\mathcal{C}}(Y, X)$ can be viewed as a morphism $f^* : \mathrm{Mor}_{\mathcal{C}}(X, X) \to \mathrm{Mor}_{\mathcal{C}}(Y, X)$. Note that $f^*(\mathrm{id}_X) = f$, so if there is a natural transformation $\alpha : h_X \to F$, then from the diagram

$$\begin{tikzcd} \mathrm{Mor}_{\mathcal{C}}(X,X) \arrow[r, "\alpha_X"] \arrow[d, "f^*"] & F(X) \arrow[d, "F(f)"] \\ \mathrm{Mor}_{\mathcal{C}}(Y,X) \arrow[r, "\alpha_X"] & F(Y) \end{tikzcd}$$

we obtain

$$\alpha_Y(f) = \alpha_Y\big(f^*(\mathrm{id}_X)\big) = F(f)\big(\alpha_X(\mathrm{id}_X)\big).$$

That is, α is simply determined by $\alpha_X(\mathrm{id}_X)$. Conversely, given $\xi \in F(X)$, we can define a natural transformation by the formula above:

$$\beta_Y : h_X(Y) \to F(Y), \quad \beta_Y(f) = F(f)(\xi).$$

It follows that these two map are inverse of each other. $\square$

Definition 4.1.7. A contravariant functor $F : \mathcal{C} \to \mathbf{Set}$ is said to be **representable** if it is isomorphic to the functor of points h_X for some object X of $\mathcal{C}$.

Let $\mathcal{C}$ be a category and let $F : \mathcal{C}^{op} \to \mathbf{Set}$ be a representable functor. Choose an object X of $\mathcal{C}$ and an isomorphism $\alpha : h_X \to F$. The Yoneda lemma guarantees that the pair (X, α) is unique up to unique isomorphism. The object X is called an object representing F.

4.1.4 Limits and colimits

The various universal properties encountered along the way are all particular cases of the notion of categorical **limit**, which is worth mentioning explicitly. Let $F : I \to C$ be a *covariant functor*, where one thinks of $\mathcal{I}$ as a category of indices. The **limit** of F is an object L of $\mathcal{C}$, endowed with morphisms $\lambda_I : L \to F(I)$ for all objects I of $\mathcal{I}$, satisfying the following properties:

1. If $\alpha : I \to J$ is a morphism in $\mathcal{I}$, then $\lambda_J = F(\alpha) \circ \lambda_I$:

$$\begin{tikzcd} & L \arrow[dl, "\lambda_I"'] \arrow[dr, "\lambda_J"] & \\ F(I) \arrow[rr, "F(\alpha)"'] & & F(J) \end{tikzcd}$$

2. L is final with respect to this property: that is, if M is another object, endowed with morphisms μ_I, also satisfying the previous requirement, then there exists a unique morphism $M \to L$ making all relevant diagrams commute

Example 4.1.3 (Products). Let $\mathcal{I}$ be the discrete category consisting of two objects $\mathbf{1}$, $\mathbf{2}$, with only identity morphisms, and let $\mathscr{A}$ be a functor from $\mathcal{I}$ to any category $\mathcal{C}$; let $A_1 = \mathscr{A}(\mathbf{1})$, $A_2 = \mathscr{A}(\mathbf{2})$ be the two objects of $\mathcal{C}$ indexed by $\mathcal{I}$. Then $\varprojlim \mathscr{A}$ is nothing but the product of A_1 and A_2 in $\mathcal{C}$: a limit exists if and only if a product of A_1 and A_2 exists in $\mathcal{C}$.

We can similarly define the product of any (possibly infinite) family of objects in a category as the limit over the corresponding discrete indexing category, provided of course that this limit exists.

The limit notion is a little more interesting if the indexing category $\mathcal{I}$ carries more structure.

Example 4.1.4 (Equalizers and kernels). Let $\mathcal{I}$ again be a category with two objects $\mathbf{1}$, $\mathbf{2}$, but assume that morphisms look like this:

$$\begin{tikzcd} \mathbf{1} \arrow[loop left] \arrow[r, bend left, "\alpha"] \arrow[r, bend right, "\beta"'] & \mathbf{2} \arrow[loop right] \end{tikzcd}$$

That is, add to the discrete category two parallel morphisms α, β from one of the objects to the other. A functor $\mathscr{K} : \mathcal{I} \to \mathcal{C}$ amounts to the choice of two objects A_1, A_2 in $\mathcal{C}$ and two parallel morphisms between them. Limits of such functors are called **equalizers**. For a concrete example, assume $\mathcal{C} = R$-**Mod** is the category of R-modules for some ring R; let $\varphi : A_2 \to A_1$ be a homomorphism, and choose $\mathscr{K}$ as above, with $\mathscr{K}(\alpha) = \varphi$ and $\mathscr{K}(\beta) =$ the zero-morphism. Then $\varprojlim \mathscr{K}$ is nothing but the kernel of φ.

Example 4.1.5 (Limits over chains). In another typical situation, $\mathcal{I}$ may consist of a totally ordered set, for example:

$$\cdots \longrightarrow \mathbf{4} \longrightarrow \mathbf{3} \longrightarrow \mathbf{2} \longrightarrow \mathbf{1}$$

(that is, the objects are $\boldsymbol{i}$, for all positive integers i, and there is a unique morphism $\boldsymbol{i} \to \boldsymbol{j}$ whenever $i \geq j$; we are only drawing the morphisms $\boldsymbol{j+1} \to \boldsymbol{j}$). Choosing $F : \mathcal{I} \to \mathcal{C}$ is equivalent to choosing objects A_i of $\mathcal{C}$ for all positive integers i and morphisms $\varphi_{ji} : A_i \to A_j$ for all $i \geq j$, with the requirement that $\varphi_{ii} = 1_{A_i}$, and $\varphi_{kj} \circ \varphi_{ji} = \varphi_{ki}$ for all $i \geq j \geq k$. That is, the choice of F amounts to the choice of a diagram

$$\cdots \xrightarrow{\varphi_{45}} A_4 \xrightarrow{\varphi_{34}} A_3 \xrightarrow{\varphi_{23}} A_2 \xrightarrow{\varphi_{12}} A_1$$

in $\mathcal{C}$. An inverse limit $\varprojlim F$ (which may also be denoted $\varprojlim_i A_i$, when the morphisms φ_{ji} are evident from the context) is then an object A endowed with morphisms $\varphi_i : A \to A_i$ such that the whole diagram

$$\begin{tikzcd} & & & A \arrow[dlll, dashed, "\cdots"'] \arrow[dll, "\varphi_4"'] \arrow[dl, "\varphi_3"] \arrow[d, "\varphi_2"] \arrow[dr, "\varphi_1"] & \\ \cdots \arrow[r, "\varphi_{45}"] & A_4 \arrow[r, "\varphi_{34}"] & A_3 \arrow[r, "\varphi_{23}"] & A_2 \arrow[r, "\varphi_{12}"] & A_1 \end{tikzcd}$$

commutes and such that any other object satisfying this requirement factors uniquely through A.

Such limits exist in many standard situations. For example, let $C = R$-**Mod** be the category of left-modules over a fixed ring R, and let A_i, φ_{ji} be as above.

Proposition 4.1.5. *The limit $\varprojlim_i A_i$ exists in R-**Mod**.*

Proof. The product $\prod_i A_i$ consists of arbitrary sequences $(a_i)_{i>0}$ of elements $a_i \in A_i$. Say that a sequence $(a_i)_{i>0}$ is *coherent* if for all $i > 0$ we have $a_i = \varphi_{i,i+1}(a_{i+1})$. Coherent sequences form an R-submodule A of $\prod_i A_i$; the canonical projections restrict to R-module homomorphisms $\varphi_i : A \to A_i$. The reader will check that A is a limit $\varprojlim_i A_i$. $\square$

This example easily generalizes to families indexed by more general posets.

The dual notion to limit is the **colimit** of a functor $F : \mathcal{I} \to \mathcal{C}$. The colimit is an object C of $\mathcal{C}$, endowed with morphisms $\gamma_I : F(I) \to \mathcal{C}$ for all objects I of $\mathcal{I}$, such that $\gamma_I = \gamma_J \circ F(\alpha)$ for all $\alpha : I \to J$ and that C is *initial* with respect to this requirement.

Example 4.1.6. For a typical situation consider again the case of a totally ordered set $\mathcal{I}$, for example:

$$\mathbf{1} \longrightarrow \mathbf{2} \longrightarrow \mathbf{3} \longrightarrow \mathbf{4} \longrightarrow \cdots$$

A functor $F : \mathcal{I} \to \mathcal{C}$ consists of the choice of objects and morphisms

$$A_1 \xrightarrow{\psi_{12}} A_2 \xrightarrow{\psi_{23}} A_3 \xrightarrow{\psi_{34}} A_4 \xrightarrow{\psi_{45}} \cdots$$

and the direct limit $\varinjlim_i A_i$ will be an object A with morphisms $\psi_i : A_i \to A$ such that the diagram

$$\begin{tikzcd}
A_1 \arrow[r, "\psi_{12}"] \arrow[drr, "\psi_1"'] & A_2 \arrow[r, "\psi_{23}"] \arrow[dr, "\psi_2"] & A_3 \arrow[r, "\psi_{34}"] \arrow[d, "\psi_3"] & A_4 \arrow[r, "\psi_{45}"] \arrow[dl, "\psi_4"'] & \cdots \arrow[dll, dashed, "\cdots"'] \\
 & & A & &
\end{tikzcd}$$

commutes and such that A is initial with respect to this requirement.

Example 4.1.7. If $\mathcal{C} = \mathbf{Set}$ and all the ψ_{ij} are injective, we are talking about a nested sequence of sets:

$$A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$$

the direct limit of this sequence would be the infinite union $\bigcup_i A_i$.

4.1.5 Exact functors

Definition 4.1.8. Let $F : \mathcal{A} \to \mathcal{B}$ be a functor

(a) Suppose all finite limits exist in $\mathcal{A}$. We say F is **left exact** if it commutes with all finite limits.

(b) Suppose all finite colimits exist in $\mathcal{A}$. We say F is **right exact** if it commutes with all finite colimits.

(c) We say F is **exact** if it is both left and right exact.

Proposition 4.1.6. *Let $F : \mathcal{A} \to \mathcal{B}$ be a functor. Suppose all finite limits exist in $\mathcal{A}$. The following are equivalent:*

(a) *F is left exact,*

(b) *F commutes with finite products and equalizers.*

(c) *F transforms a final object of $\mathcal{A}$ into a final object of $\mathcal{B}$, and commutes with fibre products.*

4.1.6 Adjunction

Definition 4.1.9. Let $\mathcal{C}, \mathcal{D}$ be categories, and let $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{C}$ be functors. We say that F and G are **adjoint** (and we say that G is right-adjoint to F and F is left-adjoint to G) if there are natural isomorphisms

$$\tau_{XY} : \mathrm{Mor}_{\mathcal{C}}(X, G(Y)) \xrightarrow{\sim} \mathrm{Mor}_{\mathcal{D}}(F(X), Y)$$

for all objects X of $\mathcal{C}$ and Y of $\mathcal{D}$. More precisely, there should be a natural isomorphism of bifunctors

$$\mathcal{C}^{op} \times \mathcal{D} \to \mathbf{Set} : \mathrm{Mor}_{\mathcal{C}}(-, G(-)) \xrightarrow{\sim} \mathrm{Mor}_{\mathcal{D}}(F(-), -)$$

Proposition 4.1.7. *For each Y there is a map $\eta_Y : FG(Y) \to Y$ so that for any for any $f : X \to G(Y)$, the corresponding map $\tau_{XY}(f) : F(X) \to Y$ is given by the composition*

$$F(X) \xrightarrow{F(f)} FG(Y) \xrightarrow{\eta_Y} Y$$

Similarly, there is a map $\theta_X : X \to GF(X)$ for each X so that $g : F(X) \to Y$, the corresponding $\tau_{XY}^{-1}(g) : X \to G(Y)$ is given by the composition

$$X \xrightarrow{\theta_X} GF(Y) \xrightarrow{G(g)} G(Y)$$

So the information of τ_{XY} is the same as these two maps.

Proof. We deal with the first case. Let $f : X \to G(Y)$ be a map, consider the follwing diagram

$$\begin{tikzcd} \mathrm{Mor}_{\mathcal{C}}(X, G(Y)) \arrow[r, "\tau_{XY}"] & \mathrm{Mor}_{\mathcal{D}}(F(X), Y) \\ \mathrm{Mor}_{\mathcal{C}}(G(Y), G(Y)) \arrow[u, "f^*"] \arrow[r, "\tau_{G(Y)Y}"] & \mathrm{Mor}_{\mathcal{D}}(FG(Y), Y) \arrow[u, "F(f)^*"'] \end{tikzcd}$$

Set η_Y to be the image of $\mathrm{id}_{G(Y)}$ under $\tau_{G(Y)Y}$ we get the claim. The second can be done similarly. $\square$

Proposition 4.1.8. *Let F be a left adjoint to G. Then*

(a) *F is fully faithful if and only if $\mathrm{id}_{\mathcal{C}} \cong G \circ F$.*

(b) *G is fully faithful if and only if $F \circ G \cong \mathrm{id}_{\mathcal{D}}$.*

Proof. Assume F is fully faithful. We have to show the adjunction map $X \to G \circ F(X)$ is an isomorphism. Let $X' \to G \circ F(X)$ be any morphism. By adjointness this corresponds to a morphism $F(X') \to F(X)$. By fully faithfulness of F this corresponds to a morphism $X' \to X$. Thus we see that $X \mapsto F \circ G(X)$ defines a bijection

$$\mathrm{Mor}_{\mathcal{C}}(X', X) \to \mathrm{Mor}(X', GF(X))$$

Hence it is an isomorphism. Conversely, if $\mathrm{id}_{\mathcal{C}} \cong G \circ F$ then F has to be fully faithful, as G defines an left-inverse on morphism sets. The other case is the dual part. $\square$

Proposition 4.1.9. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between categories. If for each $Y \in \mathrm{Ob}(\mathcal{D})$ the functor $\mathrm{Mor}_{\mathcal{D}}(F(-), Y)$ is representable, then F has a right adjoint.*

Proof. For each Y we choose an object $G(Y)$ and an isomorphism

$$\mathrm{Mor}_{\mathcal{C}}(-, G(Y)) \xrightarrow{\sim} \mathrm{Mor}_{\mathcal{D}}(F(-), Y)$$

of functors. By Yoneda's lemma for any morphism $g : Y \to Y'$ the transformation of functors

$$\mathrm{Mor}_{\mathcal{C}}(-, G(Y)) \xrightarrow{\sim} \mathrm{Mor}_{\mathcal{D}}(F(-), Y) \longrightarrow \mathrm{Mor}_{\mathcal{D}}(F(-), Y') \xrightarrow{\sim} \mathrm{Mor}_{\mathcal{C}}(-, G(Y'))$$

corresponds to a unique morphism $G(g) : G(Y) \to G(Y')$. The functoriality of G comes from that of F. $\square$

Example 4.1.8. The construction of the free group on a given set is concocted so that giving a set-function from a set A to a group G is the same as giving a group homomorphism from $F(A)$ to G. What this really means is that for all sets A and all groups G there are natural identifications

$$\mathrm{Mor}_{\mathbf{Set}}(A, S(G)) \xrightarrow{\sim} \mathrm{Mor}_{\mathbf{Grp}}(F(A), G)$$

where $S(G)$ forgets the group structure of G. That is, the functor $F : \mathbf{Set} \to \mathbf{Grp}$ constructing free groups is left-adjoint to the forgetful functor $S : \mathbf{Grp} \to \mathbf{Set}$. This of course applies to every other construction of free objects we have encountered: the free functor is, as a rule, left-adjoint to the forgetful functor.

In fact, the very fact that a functor has an adjoint will endow that functor with convenient features. We say that F is a **left-adjoint functor** if it has a right adjoint, and that G is a **right-adjoint functor** if it has a left-adjoint.

Theorem 4.1.10. *Let F be a left adjoint to G.*

(a) *Suppose that $\mathscr{A} : \mathcal{I} \to \mathcal{C}$ is a diagram, and suppose that $\varprojlim \mathscr{A}$ exists in $\mathcal{C}$. Then*

$$G(\varprojlim \mathscr{A}) = \varprojlim (G \circ \mathscr{A})$$

In other words, G commutes with limits.

(b) *Suppose that $\mathscr{A} : \mathcal{I} \to \mathcal{C}$ is a diagram, and suppose that $\varinjlim \mathscr{A}$ exists in $\mathcal{C}$. Then*

$$F(\varinjlim \mathscr{A}) = \varinjlim (F \circ \mathscr{A})$$

In other words, F commutes with colimits.

Proof. A morphism from a colimit into an object is the same as a compatible system of morphisms from the constituents of the limit into the object, so

$$\mathrm{Mor}_{\mathcal{C}}(X, G(\varprojlim \mathscr{A})) \cong \mathrm{Mor}_{\mathcal{D}}(F(X), \varprojlim \mathscr{A}) = \varprojlim \mathrm{Mor}_{\mathcal{D}}(F(X), \mathscr{A}_i) = \varprojlim \mathrm{Mor}_{\mathcal{D}}(X, G(\mathscr{A}_i))$$

proves that $G(\varprojlim \mathscr{A})$ is the limit we are looking for. A similar argument works for the other statement. $\square$

Corollary 4.1.11. *Let F be a left adjoint to G.*

(a) *If $\mathcal{C}$ has finite colimits, then F is right exact.*

(b) *If $\mathcal{D}$ has finite limits, then G is right exact.*

4.1.7 Exercise

Exercise 4.1.1. Let $F : \mathcal{C} \to \mathcal{D}$ be a covariant functor, and assume that both $\mathcal{C}$ and $\mathcal{D}$ have products. Prove that for all objects A, B of $\mathcal{C}$, there is a unique morphism $F(A \times B) \to F(A) \times F(B)$ such that the relevant diagram involving natural projections commutes.

If $\mathcal{D}$ has coproducts (denoted $\amalg$) and $G : \mathcal{C} \to \mathcal{D}$ is contravariant, prove that there is a unique morphism $G(A) \amalg G(B) \to G(A \times B)$ (again, such that an appropriate diagram commutes).

Proof. Apply the functor F yields:

$$\begin{tikzcd}
 & A \times B \arrow[dl] \arrow[dr] & & & F(A \times B) \arrow[dl] \arrow[dr] & \\
A & & B \arrow[r, phantom, "\overset{F}{\Longrightarrow}"] & F(A) & & F(B)
\end{tikzcd}$$

by the universal property of $F(A) \times F(B)$, there is a unique morphism:

$$F(A \times B) \xrightarrow{\exists!} F(A) \times F(B)$$

Similar for coproducts:

$$\begin{tikzcd}
& A \times B \arrow[dl] \arrow[dr] & {} \arrow[r, Rightarrow, "G"] & {} & G(A \times B) & \\
A & & B & G(A) \arrow[ur] & & G(B) \arrow[ul]
\end{tikzcd}$$

By the universal property of $G(A) \amalg G(B)$, we get

$$G(A) \amalg G(B) \to G(A \times B)$$

$\square$

Exercise 4.1.2. Let $\mathcal{C}$ be a small category. Prove that $\mathcal{C}$ is equivalent to the subcategory of representable functors in $\mathbf{Set}^{\mathcal{C}^\circ}$. Thus, every (*small*) category is equivalent to a subcategory of a functor category.

Proof. For $\varphi : A \to B$, there is an induced natural transformation:

$$\varphi : \mathrm{Hom}_{\mathcal{C}}(-, A) \to \mathrm{Hom}_{\mathcal{C}}(-, B)$$

from Yoneda lemma, there is a bijection from $\mathrm{Hom}(h_A, h_B)$ to $h_B(A) = \mathrm{Hom}(A, B)$. So h is a fully faithful covariant functor. For any representable F, there is a functor h_X and a natural isomorphism $F \cong h_X$. This shows h is a equivalence of categories. $\square$

Exercise 4.1.3. Let R be a commutative ring, and let $I \subseteq R$ be an ideal. Note that $I^n \subseteq I^m$ if $n \geq m$, and hence we have natural homomorphisms $\varphi_{mn} : R/I^n \to R/I^m$ for $n \geq m$.

1. Prove that the inverse limit $\widehat{R}_I := \varprojlim_n R/I^n$ exists as a commutative ring. This is called the I-adic completion of R.

2. By the universal property of inverse limits, there is a unique homomorphism $R \to \widehat{R}_I$. Prove that the kernel of this homomorphism is $\bigcap_n I^n$.

3. Let $I = (x)$ in $R[x]$. Prove that the completion $\widehat{R[x]}_I$ is isomorphic to the power series ring $R[[x]]$.

Proof. We first prove that limit exists in **Ring**. Let $\mathcal{I}$ be a poset $(\mathcal{I}, \leq)$. Choose $\{R_i\}_{i \in \mathcal{I}}$ and $\{\varphi_{ij} : R_i \to R_j\}$ such that

$$i \geq j \geq k \Rightarrow \varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}$$

A sequence $(r_i)_{i \in \mathcal{I}}$ is coherent if $\varphi_{ij}(r_i) = r_j$. Define the limit $\varprojlim_i R_i$ to be

$$\varprojlim_i R_i := \{(r_i)_{i \in \mathcal{I}} \mid (r_i) \text{ is coherent}\}.$$

Since $I^n \subseteq I^m$ for $n \geq m$, there is a well defined quotient homomorphism:

$$\varphi_{nm} : R/I^n \to R/I^m, \quad a + I^n \mapsto a + I^m$$

So the limit $\varprojlim_i R/I^n$ is well defined.

For the homomorphism $\psi_n : R \to R/I^n$, it is clear that

$$\varphi_{nm} \circ \psi_n = \psi_m$$

so we get the unique homomorphsim $\psi : R \to \widehat{R}_I$, defined by $\psi(r) = (\psi_i(r))_{i\in\mathcal{I}}$. It follows that

$$\psi(r) = 0 \iff \psi_i(r) = 0 \iff r \in \bigcap_n I^n.$$

Finally, if $I = (x)$, then $i^n = (x^n)$. So

$$\widehat{R[x]}_I = \{(r_i)_{i\in\mathbb{N}} : \deg r_i < i, r_i = r_{i-1} + a_{i-1}x^{i-1}\}$$

this set equals $R[[x]]$. □

Exercise 4.1.4. An important example of the construction presented in Exercise 4.1.3 is the ring $\mathbb{Z}_p$ of p-adic integers: this is the limit $\varprojlim_r \mathbb{Z}/p^r\mathbb{Z}$, for a positive prime integer p.

The field of fractions of $\mathbb{Z}_p$ is denoted $\mathbb{Q}_p$; elements of $\mathbb{Q}_p$ are called **p-adic numbers**.

1. Show that giving a p-adic integer A is equivalent to giving a sequence of integers $A_r, r \geq 1$, such that $0 \leq A_r < p^r$, and that $A_s \equiv A_r \bmod p^s$ if $s \leq r$.
2. Equivalently, show that every p-adic integer has a unique infinite expansion $A = a_0 + a_1 \cdot p + a_2 \cdot p^2 + a_3 \cdot p^3 + \cdots$, where $0 \leq a_i \leq p - 1$. The arithmetic of p-adic integers may be carried out with these expansions in precisely the same way as ordinary arithmetic is carried out with ordinary decimal expansions.
3. With notation as in the previous point, prove that $A \in \mathbb{Z}_p$ is invertible if and only if $a_0 \neq 0$.
4. Prove that $\mathbb{Z}_p$ is a local domain, with maximal ideal generated by (the image in $\mathbb{Z}_p$ of) p.
5. Prove that $\mathbb{Z}_p$ is a DVR. (There is an evident valuation on $\mathbb{Q}_p$.)

Proof. Every A_r is in $\mathbb{Z}/p^r\mathbb{Z}$, so $0 \leq A_r < p^r$. From the construction, $A_s + p^s = \varphi_{rs}(A_r + p^r) = A_r + p^s$ for $s \leq r$, we see that $A_r \equiv A_s \bmod p^s$ if $s \leq r$.

Similar to the example $R[[x]]$. Giving a sequence is the same as giving a truncation of a series. And the third point is the same as series $R[[x]]$. From the previous point, we find that $\mathbb{Z}_p/p\mathbb{Z}_p$ is a field, so $p\mathbb{Z}_p$ is a maximal ideal.

Define a function $v_p(x) := \sup\{n : x \in p^n\mathbb{Z}_p\} = \inf\{n : x_n \neq 0\}$. Then for any ideal $I \subseteq \mathbb{Z}_p$, let $n := \min\{v_p(x) : x \in I\}$, then $I \subseteq p^n\mathbb{Z}_p$. Now let $y = p^n x \in I$, then x is invertible, so $(p^n x) = p^n\mathbb{Z}_p \subseteq I$. This shows every ideal in $\mathbb{Z}_p$ has the form $p^n\mathbb{Z}_p$, and $p\mathbb{Z}_p$ is the unique maximal ideal. □

Exercise 4.1.5. If m, n are positive integers and $m \mid n$, then $(n) \subseteq (m)$, and there is an onto ring homomorphism $\mathbb{Z}/n\mathbb{Z} \twoheadrightarrow \mathbb{Z}/m\mathbb{Z}$. The limit ring $\varprojlim \mathbb{Z}/n\mathbb{Z}$ exists and is denoted by $\widehat{\mathbb{Z}}$. Prove that $\widehat{\mathbb{Z}} = \mathrm{End}_{\mathbf{Ab}}(\mathbb{Q}/\mathbb{Z})$.

Proof. Every $f \in \mathrm{End}_{\mathbf{Ab}}(\mathbb{Q}/\mathbb{Z})$ is uniquely determined by $f(\frac{1}{n})$. Since $n \cdot f(\frac{1}{n}) = f(1) = f(0) = 0$, $f(\frac{1}{n}) = \frac{g(n)}{n}$ for some integer $g(n)$. Since we are deal with $\mathbb{Q}/\mathbb{Z}$, we may choose $0 \leq g(n) < n$.

For $m \mid n$, we have $n = am$, so

$$f(\frac{1}{n}) = f(\frac{1}{am}) = \frac{g(n)}{am}, \quad f(\frac{1}{m}) = \frac{g(m)}{m}$$

and

$$a \cdot f(\frac{1}{n}) = \frac{g(n)}{m} = f(\frac{1}{m}) = \frac{g(m)}{m}$$

so we have $g(n) \equiv g(n) \bmod m$. This means the sequence

$$(f(\frac{1}{n}))_{i\in\mathbb{N}}$$

is an element of $\widehat{\mathbb{Z}}$. Conversely, any element in $\widehat{\mathbb{Z}}$ uniquely defines an endomorphism of $\mathbb{Q}/\mathbb{Z}$. So we have $\widehat{\mathbb{Z}} = \mathrm{End}_{\mathbf{Ab}}(\mathbb{Q}/\mathbb{Z})$. □

Exercise 4.1.6. Let $\widehat{\mathbb{Z}}$ be as in Exercise 4.1.5.

1. If R is a commutative ring endowed with homomorphisms $R \to \mathbb{Z}/p^r\mathbb{Z}$ for all primes p and all r, compatible with all projections $\mathbb{Z}/p^r\mathbb{Z} \to \mathbb{Z}/p^s\mathbb{Z}$ for $s \le r$, prove that there are ring homomorphisms $R \to \mathbb{Z}/n\mathbb{Z}$ for all n, compatible with all projections $\mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ for $m \mid n$.

2. Deduce that $\widehat{\mathbb{Z}}$ satisfies the universal property for the product of $\mathbb{Z}_p$, as p ranges over all positive prime integers.

It follows that $\prod_p \mathbb{Z}_p \cong \widehat{\mathbb{Z}} \cong \mathrm{End}_{\mathbf{Ab}}(\mathbb{Q}/\mathbb{Z})$.

Proof. For any $n = p_1^{r_1} \cdots p_i^{r_i}$, $m = p_1^{r'_1} \cdots p_i^{r'_i}$ with $m \mid n$, by Chinese remainder theorem we have a commutative diagram

$$\begin{array}{ccc} \prod_i \mathbb{Z}/p_i^{r_i}\mathbb{Z} & \xrightarrow{\sim} & \mathbb{Z}/n\mathbb{Z} \\ \downarrow & & \downarrow \\ \prod_i \mathbb{Z}/p_i^{r'_i}\mathbb{Z} & \xrightarrow{\sim} & \mathbb{Z}/m\mathbb{Z} \end{array}$$

so we get the first result.

Note that giving a morphism from R to $\mathbb{Z}_p$ is the same as giving morphisms from R to $\mathbb{Z}/p^r\mathbb{Z}$ fro all r. So if there is a ring R with morphisms to $\mathbb{Z}_p$ for all prime p, then we get morphisms to $\mathbb{Z}/p^r\mathbb{Z}$ for all prime p, all r. From the previous point, there are morphisms $R \to \mathbb{Z}/n\mathbb{Z}$ for all n, compatible with all projection $\mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}m\mathbb{Z}$. From the definition of $\widehat{\mathbb{Z}}$, there is a unique morphism from R to $\widehat{\mathbb{Z}}$. So $\widehat{\mathbb{Z}}$ satisfies the universal property of $\prod_p \mathbb{Z}_p$. □

Exercise 4.1.7. Let R, S be rings. An additive covariant functor $F : R$-**Mod** $\to S$-**Mod** is **faithfully exact** if

$$F(A) \xrightarrow{F(\varphi)} F(B) \xrightarrow{F(\psi)} F(C)$$

is exact in S-**Mod** if and only if

$$A \xrightarrow{\varphi} B \xrightarrow{\psi} C$$

is exact in R-**Mod**. Prove that an exact functor $F : R$-**Mod** $\to S$-**Mod** is faithfully exact if and only if $F(M) \neq 0$ for every nonzero R-module M, if and only if $F(\varphi) \neq 0$ for every nonzero morphism φ in R-Mod.

Proof.

1. One direction is easy: If F is faithfully exact. Assume $F(M) = 0$, then the sequence $0 \to F(M) \to 0$ is exact, but $0 \to M \to 0$ is not exact unless $M = 0$, so we find $M = 0$. If $F(\varphi) = 0$, then

   $$F(M) \xrightarrow{F(\varphi)} F(N) \xrightarrow{id_{F(N)}} F(N)$$

   is exact, but

   $$M \xrightarrow{\varphi} N \xrightarrow{id_N} N$$

   is exact only if $\varphi = 0$.

2. Then we show that F reflects zero objects if and only if F reflects zero morphisms: If F reflects zero morphisms, assume $F(X) = 0$, then $\mathrm{id}_{F(X)} = 0$, so $\mathrm{id}_X = 0$. But $id_X = 0$ if and only if $X = 0$, so $X = 0$.

   Now assume F reflects zero objects. For $\varphi : A \to B$ such that $F(\varphi) = 0$. Consider the exact sequence:
$$A \xrightarrow{\varphi} \operatorname{im}\varphi \xrightarrow{\psi} \operatorname{coker}\varphi$$
   since F is exact, we also have a exact sequence
$$F(A) \xrightarrow{F(\varphi)} F(\operatorname{im}\varphi) \xrightarrow{F(\psi)} F(\operatorname{coker}\varphi)$$
   Note that $F(\varphi) = 0$, so $F(\psi)$ is monic. But $\psi = 0$ so $F(\psi) = 0$, we conclude that $F(\operatorname{im}\varphi) = 0$. This means $\operatorname{im}\varphi = 0$, so we have $\varphi = 0$.

3. Now we show the last direction. Let F reflects zero morphisms, first we show that F reflects monomorphisms and epimorphisms. In deed, suppose
$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z$$
   is exact; then
$$0 \longrightarrow F(X) \longrightarrow F(Y) \longrightarrow F(Z)$$
   is exact. If $F(Y) \to F(Z)$ is monic, then $F(X) = 0$, so $X = 0$. This shows F reflects monomorphisms, the dual argument shows that F reflects epimorphisms. Now, in an abelian category, f is an isomorphism if and only if f is both monic and epic, so this implies F reflects isomorphisms. Now suppose $X \to Y \to Z$ is given and
$$0 \longrightarrow F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z)$$
   is exact. Since $F(X) \to \ker F(g)$ is an isomorphism, and $\ker F(g) = F(\ker g)$, $X \to \ker g$ is also an isomorphism, so $X \to Y \to Z$ is exact.

□

Exercise 4.1.8. Prove that localization is an exact functor.
In fact, prove that localization preserves homology: if
$$M_\bullet : \qquad \cdots \longrightarrow M_{i+1} \xrightarrow{d_{i+1}} M_i \xrightarrow{d_i} M_{i-1} \longrightarrow \cdots$$
is a complex of R-modules and S is a multiplicative subset of R, then the localization of the i-th homology of $M_\bullet$ is the i-th homology $H_i(S^{-1}M_\bullet)$ of the localized complex
$$S^{-1}M_\bullet : \qquad \cdots \longrightarrow S^{-1}M_{i+1} \xrightarrow{S^{-1}d_{i+1}} S^{-1}M_i \xrightarrow{S^{-1}d_i} S^{-1}M_{i-1} \longrightarrow \cdots$$

Proof. Since
$$d_i(\frac{m}{s}) = d(\frac{ms'}{ss'})$$
we have
$$\ker S^{-1}d_i = \{\frac{m}{s} \mid \exists r \in S, rm \in \ker d_i\}, \quad \operatorname{im} S^{-1}d_{i+1} = \{\frac{m}{s} \mid \exists r \in s, rm \in \operatorname{im} d_{i+1}\}$$

Concerning the quotient, first we observet that, in the construction of $S^{-1}H_i(M_\bullet)$:

$$\begin{aligned}\frac{a+\operatorname{im} d_{i+1}}{s} = \frac{a'+\operatorname{im} d_{i+1}}{s'} &\iff (\exists r\in S)\quad r[(a+\operatorname{im} d_{i+1})s' - (a'+\operatorname{im} d_{i+1})s] = 0 \text{ in } H_i(M_\bullet)\\ &\iff (\exists r\in S)\quad r(as'-a's)\in \operatorname{im} d_{i+1}\end{aligned}$$

While in the quotient $H_i(S^{-1}M_\bullet)$:

$$\frac{a}{s}+\operatorname{im} S^{-1}d_{i+1} = \frac{a'}{s'}+\operatorname{im} S^{-1}d_{i+1} \iff \frac{a}{s}-\frac{a'}{s'}\in \operatorname{im} S^{-1}d_{i+1} \iff (\exists r\in S)\, r(as'-a's)\in\in d_{i+1}$$

So there is a natural homomorphism:

$$\psi : S^{-1}H_i(M_\bullet)\to H_i(S^{-1}M_\bullet),\qquad \frac{a+\operatorname{im} d_{i+1}}{s}\mapsto \frac{a}{s}+\operatorname{im} S^{-1}d_{i+1}$$

this is an isomorphism from the observation above. □

Exercise 4.1.9. Suppose M is a finitely presented R-module and N is an arbitrary R-module. Show the followsing holds

$$S^{-1}\mathrm{Hom}_R(M,N)\xrightarrow{\sim}\mathrm{Hom}_{S^{-1}R}(S^{-1}M,S^{-1}N)$$

But note that this does not holds for any module.

Proof. First we have

$$S^{-1}\mathrm{Hom}_R(R,N)\xrightarrow{\sim}\mathrm{Hom}_{S^{-1}R}(S^{-1}R,S^{-1}N)$$

for any N. And we have a natural homomorphism

$$S^{-1}\mathrm{Hom}_A(M,N)\to\mathrm{Hom}_{S^{-1}A}(S^{-1}M,S^{-1}N).$$

Consider the diagram:

$$\begin{tikzcd}
0 \arrow[r] & S^{-1}\mathrm{Hom}_R(M,N) \arrow[r]\arrow[d] & S^{-1}\mathrm{Hom}_R(R^m,N) \arrow[r]\arrow[d] & S^{-1}\mathrm{Hom}_R(R^n,N)\arrow[d]\\
0 \arrow[r] & \mathrm{Hom}_{S^{-1}R}(S^{-1}M,S^{-1}N) \arrow[r] & \mathrm{Hom}_{S^{-1}R}(S^{-1}R^m,S^{-1}N) \arrow[r] & \mathrm{Hom}_{S^{-1}R}(S^{-1}R^n,S^{-1}N)
\end{tikzcd}$$

The right two vertical maps are isomorphisms, so we get the isomorphism.

For $R=N=\mathbb{Z}$, $M=\mathbb{Q}$, $S=\mathbb{Z}-\{0\}$, we have

$$S^{-1}\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q},\mathbb{Z}) = S^{-1}\{0\}=0,\quad \mathrm{Hom}_{S^{-1}\mathbb{Z}}(S^{-1}\mathbb{Q},S^{-1}\mathbb{Z}) = \mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q},\mathbb{Q})=\mathbb{Q}$$

□

Exercise 4.1.10. Suppose $F:\mathcal{A}\to\mathcal{B}$ is a covariant functor of abelian categories, and $C^\bullet$ is a complex in $\mathcal{A}$.

(a) If F is right-exact, describe a natural morphism $FH^\bullet\to H^\bullet F$.

(b) If F is right-exact, describe a natural morphism $FH^\bullet\leftarrow H^\bullet F$.

(c) If F is exact, show that the morphisms of (a) and (b) are inverses and thus isomorphisms.

Proof. First we recall that if F is right-exact, then F commutes with cokernels: For we have the following exact sequence

$$0 \longrightarrow F(C^i) \xrightarrow{F(d^i)} F(C^{i+1}) \longrightarrow F(\operatorname{coker} d^i) \longrightarrow 0$$

which is obtained from the corresponding short exact sequence. Hence

$$F(\operatorname{coker} d^i) \cong \operatorname{coker} F(d^i)$$

(a) Consider the exact sequence

$$0 \longrightarrow \operatorname{im} d^i \longrightarrow C^{i+1} \longrightarrow \operatorname{coker} d^i \longrightarrow 0$$

Applying F on this gives us

$$F \operatorname{im} d^i \longrightarrow F(C^{i+1}) \longrightarrow F \operatorname{coker} d^i \longrightarrow 0$$

Together with the similar sequence in $F(C^\bullet)$ we get a diagram

$$\begin{tikzcd}
& F \operatorname{im} d^i \arrow[r] \arrow[d, dashed, "\alpha"] & F(C^{i+1}) \arrow[r] \arrow[d, equal] & F \operatorname{coker} d^i \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & \operatorname{im} F(d^i) \arrow[r] & F(C^{i+1}) \arrow[r] & F \operatorname{coker} d^i \arrow[r] & 0
\end{tikzcd}$$

Then we can show there is an induced map $\alpha : F \operatorname{im} d^i \to \operatorname{im} f(d^i)$. Further, by the snake lemma, this induced map α is an epimorphism. Now consider another sequence

$$0 \longrightarrow H^i(C^\bullet) \longrightarrow \operatorname{coker} d^{i-1} \longrightarrow \operatorname{im} d^i \longrightarrow 0$$

Applying F gives

$$FH^i(C^\bullet) \longrightarrow F \operatorname{coker} d^{i-1} \longrightarrow F \operatorname{im} d^i \longrightarrow 0$$

Similarly, with the counterpart in $F(C^\bullet)$, there is a diagram

$$\begin{tikzcd}
& FH^i(C^\bullet) \arrow[r] \arrow[d, dashed, "\beta"] & F \operatorname{coker} d^{i-1} \arrow[r] \arrow[d, "\cong"] & F \operatorname{im} d^i \arrow[r] \arrow[d, "\alpha"] & 0 \\
0 \arrow[r] & H^i F(C^\bullet) \arrow[r] & \operatorname{coker} F(d^{i-1}) \arrow[r] & \operatorname{im} F(d^i) \arrow[r] & 0
\end{tikzcd}$$

Together with α, we get our desired map

$$\beta : FH^i(C^\bullet) \to H^i F(C^\bullet)$$

(b) Instead of (a), we may use the sequence for kernels:

$$0 \longrightarrow \ker d^i \longrightarrow C^i \longrightarrow \operatorname{im} d^i \longrightarrow 0$$

and

$$0 \longrightarrow \operatorname{im} d^{i-1} \longrightarrow \ker d^i \longrightarrow H^i(C^\bullet) \longrightarrow 0$$

with the identification

$$\ker F(d^i) \cong F(\ker d^i)$$

(c) With the exactness hypothesis, the map we obtained all becomes isomorphisms.

□

4.2 Presheaves of sets

In this section, we consider the category of presheaves of sets over a category $\mathcal{C}$, and prove some of its properties. In order to avoid set-theoretic issues, we fix once for all a universe $\mathscr{U}$ which has an element with infinite cardinality. A set is said to be **$\mathscr{U}$-small** (or simply **small** if there is no confusion) if it is isomorphic to an element of $\mathscr{U}$. We also use the following terminology: small group, small ring, small category. We often assume that the schemes, topological spaces, sets of indices, with which we work are $\mathscr{U}$-small, or at least have cardinality belonging to $\mathscr{U}$. A category $\mathcal{C}$ is called a **$\mathscr{U}$-category** if for any objects x, y in $\mathcal{C}$, the set $\mathrm{Hom}_{\mathcal{C}}(x,y)$ is $\mathscr{U}$-small, and is called $\mathscr{U}$-small if the set $\mathrm{Ob}(\mathcal{D})$ is also contained in the universe $\mathscr{U}$. For two categories $\mathcal{C}$, $\mathcal{D}$, we denote by $\mathcal{H}om(\mathcal{C},\mathcal{D})$ the category of (covariant) functors from $\mathcal{C}$ to $\mathcal{D}$. It is then easy to verify the following two conditions:

- If $\mathcal{C}$ and $\mathcal{D}$ are elements of $\mathscr{U}$ (resp. $\mathscr{U}$-small), then $\mathcal{H}om(\mathcal{C},\mathcal{D})$ is an element of $\mathscr{U}$ (resp. $\mathscr{U}$-small).
- If $\mathcal{C}$ is a $\mathscr{U}$-small category and $\mathcal{D}$ is a $\mathscr{U}$-category, $\mathcal{H}om(\mathcal{C},\mathcal{D})$ is a $\mathscr{U}$-category.

However, note that if $\mathcal{D}$ is a $\mathscr{U}$-small category and $\mathcal{C}$ is a $\mathscr{U}$-category, then $\mathcal{H}om(\mathcal{C},\mathcal{D})$ is not $\mathscr{U}$-small in general. For example, the category $\mathcal{H}om(\mathcal{C},\mathscr{U}\text{-}\mathbf{Set})$. It should be noted that $\mathscr{U}$-smallness is really a restrictive condition for categories, and there are many interesting examples where this condition is not satisfied in general.

4.2.1 The category of presheaves of sets

Let $\mathcal{C}$ be a category. We define the **category of presheaves of sets over $\mathcal{C}$ relative to the universe** $\mathscr{U}$ (or, if there is no confusion, the category of presheaves of sets over $\mathcal{C}$) to be the category of contravariant functors from $\mathcal{C}$ to the category of $\mathscr{U}$-sets, and denote it by $\mathrm{PSh}(\mathcal{C})_{\mathscr{U}}$ (or simply $\mathrm{PSh}(\mathcal{C})$ if there is no risk of confusion). The objects of $\mathrm{PSh}(\mathcal{C})_{\mathscr{U}}$ are called **$\mathscr{U}$-presheaves** (of simply presheaves) over $\mathcal{C}$. If $\mathcal{C}$ is $\mathscr{U}$-small, then $\mathrm{PSh}(\mathcal{C})_{\mathscr{U}}$ is a $\mathscr{U}$-category. However, this is not true in general if $\mathcal{C}$ is only assumed to be a $\mathscr{U}$-category.

Let x be an object of a $\mathscr{U}$-category $\mathcal{C}$. We can associate with x a presheaf $h_x : \mathcal{C}^{\mathrm{op}} \to \mathscr{U}\text{-}\mathbf{Set}$, defined in the following way:

- If $\mathrm{Hom}_{\mathcal{C}}(y,x)$ is an element of $\mathscr{U}$, then we set $h_x(y) = \mathrm{Hom}_{\mathcal{C}}(y,x)$.
- Suppose that $\mathrm{Hom}_{\mathcal{C}}(y,x)$ is not an element of $\mathscr{U}$ and let $R(Z)$ be the relation "the set Z is the target of an isomorphism $\mathrm{Hom}_{\mathcal{C}}(y,x) \xrightarrow{\sim} Z$". We then put $h_x(y) = \tau_Z(R(Z))$.

Let $R'(u)$ be the relation "u is a bijection from $\mathrm{Hom}_{\mathcal{C}}(y,x)$ to $h_x(y)$" and set $\varphi(y,x) = \tau_u(R'(u))$. Then in both cases, we have a canonical isomorphism

$$\varphi(y,x) : \mathrm{Hom}_{\mathcal{C}}(y,x) \xrightarrow{\sim} h_y(x).$$

Now let $u : y \to y'$ be a morphism of $\mathcal{C}$. Then by composition, u defines a map

$$\mathrm{Hom}_{\mathcal{C}}(u,x) : \mathrm{Hom}_{\mathcal{C}}(y',x) \to \mathrm{Hom}_{\mathcal{C}}(y,x)$$

and we define $h_x(u)$ to be the composition

$$h_x(u) = \varphi(y,x)\mathrm{Hom}_{\mathcal{C}}(x,u)\varphi(y,x)^{-1}.$$

It is immediate to verify that h_x then defines a functor $\mathcal{C}^{\mathrm{op}} \to \mathscr{U}\text{-}\mathbf{Set}$.

4.2.2 Projective limits and inductive limits

4.2.3 Exactness properties of the category of presheaves

4.2.4 The functors $\mathcal{H}om$ and $\mathcal{I}so$

Let $\mathcal{C}$ be a category and F, G be objects of $\mathrm{PSh}(\mathcal{C})$. We define an object $\mathcal{H}om(F, G)$ of $\mathrm{PSh}(\mathcal{C})$ in the following way:

$$\mathcal{H}om(F,G)(S) = \mathrm{Hom}_{\mathrm{PSh}(\mathcal{C}_{/S})}(F_S, G_S) \cong \mathrm{Hom}_{\mathrm{PSh}(\mathcal{C}_{/S})}(F \times h_S, G \times h_S) \cong \mathrm{Hom}_{\mathrm{PSh}(\mathcal{C})}(F \times h_S, G).$$

It is easy to verify that $\mathcal{H}om(F, G)$ possesses the following properties:

- $\mathcal{H}om(e, G) \cong G$,
- If E is an object of $\mathrm{PSh}(\mathcal{C})$, then
$$\mathcal{H}om(E, F \times G) \cong \mathcal{H}om(E, F) \times \mathcal{H}om(E, G). \tag{4.2.1}$$
- The functor Hom commutes with base change:
$$\mathcal{H}om(F_S, G_S) \cong \mathcal{H}om(F, G)_S. \tag{4.2.2}$$
- $(F, G) \mapsto \mathcal{H}om(F, G)$ is a bifunctor which is contravariant on F and covariant on G.

Now we consider an object E of $\mathrm{PSh}(\mathcal{C})$. Let $\phi : E \times F \to G$ be a morphism, we want to associates with ϕ a morphism from E into $\mathrm{Hom}(F, G)$. For this, consider a morphism $S' \to S$ of $\mathcal{C}$. We then have the following induced maps:

$$E(S) \times F(S') \to E(S') \times F(S') \xrightarrow{\phi(S')} G(S').$$

Any element e of $E(S)$ therefore defines a map $F(S') \to G(S')$, which is functorial on S'; that is, an element $\theta_\phi(e)$ of $\mathcal{H}om(F, G)(S)$. We therefore obtain a map

$$\mathrm{Hom}(E \times F, G) \to \mathrm{Hom}(E, \mathcal{H}om(F, G)), \quad \phi \mapsto \theta_\phi$$

which is functorial on E.

Proposition 4.2.1. *Let E, F, G be objects of $\mathrm{PSh}(\mathcal{C})$. Then the map $\phi \mapsto \theta_\phi$ is a bijection:*

$$\mathrm{Hom}_{\mathrm{PSh}(\mathcal{C})}(E \times F, G) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{PSh}(\mathcal{C})}(E, \mathcal{H}om(F, G)), \tag{4.2.3}$$

and we obtain an isomorphism of functors

$$\mathcal{H}om(E \times F, G) \xrightarrow{\sim} \mathcal{H}om(E, \mathcal{H}om(F, G)). \tag{4.2.4}$$

Proof. We consider the two members of (4.2.3) as functors of E. The first assertion is then valid for $E = h_X$, which follows directly from the definition of $\mathcal{H}om(F, G)$. On the other hand, since the two functors both transforms inductive limits to projective limits and any object of $\mathrm{PSh}(\mathcal{C})$ can be written as an inductive limits of h_X, where X runs through $\mathcal{C}_{/E}$, we conclude that (4.2.3) is a bijection.

We can also give a direct proof of (4.2.3). To any element $\theta \in \mathrm{Hom}(E, \mathcal{H}om(F, G))$, we associate an element ϕ_θ of $\mathrm{Hom}(E \times F, G)$ as follows. For any $S \in \mathcal{C}$, we have a map

$$\theta(S) : E(S) \to \mathcal{H}om(F, G)(S) = \mathrm{Hom}(F \times S, G)$$

which is functorial on S. If $(e, f) \in E(S) \times F(S)$, then f can be considered as a morphism $S \to F$, so $f \times \mathrm{id}_S$ is a morphism $S \to F \times S$. On the other hand, $\theta(S)(e)$ is a morphism $F \times S \to G$, so by composing we obtain a morphism

$$\theta(S)(e) \circ (f \times \mathrm{id}_S) : S \to G,$$

which is an element $\phi_\theta(e, f)$ of $G(S)$. We verify immediately that the correspondence $S \mapsto \phi_\theta(S)$ is functorial on S, so we get a morphism $\phi_\theta : E \times F \to G$. It then remains to check that $\theta \mapsto \phi_\theta$ and $\phi \mapsto \theta_\phi$ are inverses of each other, which is straightforward from definition.

We now prove the isomorphism (4.2.4). If $S \in \mathcal{C}$, then by (4.2.2) and (4.2.3) applied to $\mathcal{C}_{/S}$, we have

$$\begin{aligned}\mathcal{H}om(E, \mathcal{H}om(F, G))(S) &\cong \mathrm{Hom}_S(E_S, \mathcal{H}om_S(F_S, G_S)) \cong \mathrm{Hom}_S(E_S \times_S F_S, G_S) \\ &\cong \mathrm{Hom}(E \times F \times S, G) \cong \mathcal{H}om(E \times F, G)(S),\end{aligned}$$

and these isomorphisms are functorial on S. □

Corollary 4.2.2. *We have the following isomorphisms:*

$$\mathrm{Hom}(E, \mathcal{H}om(F, G)) \cong \mathrm{Hom}(F, \mathcal{H}om(E, G)), \tag{4.2.5}$$

$$\mathcal{H}om(E, \mathcal{H}om(F, G)) \cong \mathcal{H}om(F, \mathcal{H}om(E, G)). \tag{4.2.6}$$

Proof. The first isomorphism follows from (4.2.3) and the fact that $E \times F \cong F \times E$, and the second one follows from (4.2.2). □

In particular, if $E = e$ is the final object, then since $\mathcal{H}om(e, G) \cong G$, we have

$$\Gamma(\mathcal{H}om(F, G)) = \mathrm{Hom}(e, \mathcal{H}om(F, G)) \cong \mathrm{Hom}(F, \mathcal{H}om(e, G)) \cong \mathrm{Hom}(F, G).$$

We also note that the composition of Hom induces a functorial morphsim

$$\circ : \mathcal{H}om(F, G) \times \mathcal{H}om(G, H) \to \mathcal{H}om(F, H).$$

In other words, with the operation $\mathcal{H}om$ and $\times$, the category $\mathrm{PSh}(\mathcal{C})$ is self-enriched.

If F and G are objects of $\mathrm{PSh}(\mathcal{C})$, we denote by $\mathrm{Iso}(F, G)$ the subset of $\mathrm{Hom}(F, G)$ formed by isomorphisms from F to G. We define a subobject $\mathcal{I}so(F, G)$ of $\mathcal{H}om(F, G)$ by

$$\mathcal{I}so(F, G)(S) = \mathrm{Iso}(F_S, G_S).$$

We then have the following isomorphisms

$$\Gamma(\mathcal{I}so(F, G)) \cong \mathrm{Iso}(F, G), \quad \mathrm{Iso}(F, G) \cong \mathrm{Iso}(G, F).$$

In the particular case where $F = G$, we put

$$\begin{aligned}\mathcal{E}nd(F) &= \mathcal{H}om(F, F), &\quad \mathrm{End}(F) &= \mathrm{Hom}(F, F) \cong \Gamma(\mathcal{E}nd(F)), \\ \mathcal{A}ut(F) &= \mathcal{I}so(F, F), &\quad \mathrm{Aut}(F) &= \mathrm{Iso}(F, F) \cong \Gamma(\mathcal{A}ut(F)).\end{aligned}$$

It is clear that the formations of $\mathcal{I}so$, $\mathcal{A}ut$, $\mathcal{E}nd$ also commutes with base changes.

Remark 4.2.1. Note that we can construct an object isomorphic to $\mathrm{Iso}(F, G)$ in the following way: we have a morphism

$$\mathcal{H}om(F, G) \times \mathcal{H}om(G, F) \to \mathcal{E}nd(F);$$

By permuting F and G, we then deduce a morphism

$$\mathrm{Hom}(F,G) \times \mathrm{Hom}(G,F) \to \mathcal{E}nd(F) \times \mathcal{E}nd(G).$$

On the other hand, the identity morphism of F is an element of $\mathrm{End}(F)$ and defines a morphism $e \to \mathcal{E}nd(F)$. By composition, we then obtain a morphism

$$e \mapsto \mathcal{E}nd(F) \times \mathcal{E}nd(G).$$

It it then immediate to see that the fiber product of e and $\mathcal{H}om(F,G) \times \mathcal{H}om(G,F)$ is isomorphic to $\mathrm{Iso}(F,G)$.

The definitions above are applicable in particular if $F = h_X$ and $G = h_Y$. In the case where $\mathcal{H}om(h_X, h_Y)$ is representable by an object of $\mathcal{C}$, we denote this object by $\mathcal{H}om(X,Y)$. It possesses the following property: if $Z \times X$ is representable, then

$$\mathrm{Hom}(Z, \mathcal{H}om(X,Y)) \cong \mathrm{Hom}(Z \times X, Y).$$

We can also define the objects $\mathcal{I}so(X)$, $\mathcal{E}nd(X)$ and $\mathcal{A}ut(X)$. The preceding argumants also applies to the categories of the form $\mathcal{C}_{/S}$, and in this case, the corresponding objects are denoted by $\mathcal{H}om_S$, $\mathcal{I}so_S$, etc.

4.3 Abelian Category

4.3.1 Additive categories

4.3.1.1 Preaditive category

Definition 4.3.1. A category $\mathcal{A}$ is called **preadditive** if each morphism set $\mathrm{Mor}_{\mathcal{A}}(X,Y)$ is endowed with the structure of an abelian group such that the compositions

$$\mathrm{Mor}_{\mathcal{A}}(X,Y) \times \mathrm{Mor}_{\mathcal{A}}(Y,Z) \to \mathrm{Mor}_{\mathcal{A}}(X,Z)$$

are bilinear. A functor $F : \mathcal{A} \to \mathcal{B}$ of preadditive categories is called an **additive functor** if and only if

$$F : \mathrm{Mor}_{\mathcal{A}}(X,Y) \to \mathrm{Mor}_{\mathcal{B}}(F(X), F(Y))$$

is a homomorphism of abelian groups for all $X, Y \in \mathrm{Ob}(\mathcal{A})$.

In particular for every X, Y there exists at least one morphism $X \to Y$, namely the **zero map**.

Lemma 4.3.1. *Let $\mathcal{A}$ be a preadditive category. Let X be an object of $\mathcal{A}$. The following are equivalent:*

(a) *X is a initial object.*

(b) *X is a final object.*

(c) $\mathrm{id}_X = 0$ *in* $\mathrm{Mor}_{\mathcal{A}}(X,X)$.

Furthermore, if such an object 0 exists, then a morphism $f : X \to Y$ factors through 0 if and only if $f = 0$.

Proof. Clearly if X is a final or initial object, then $\mathrm{id}_X = 0$ is the unique morphism $X \to X$. Now assume $\mathrm{id}_X = 0$ holds, then

$$f \in \mathrm{Mor}_{\mathcal{A}}(X,Y) \Rightarrow f = f \circ \mathrm{id}_X = 0, \quad \text{and} \quad g \in \mathrm{Mor}_{\mathcal{A}}(Y,X) \Rightarrow g = \mathrm{id}_X \circ g = 0.$$

Thus X is final and initial. □

Definition 4.3.2. In a preadditive category $\mathcal{A}$ we call **zero object**, and we denote it 0 any final and initial object as in Lemma 4.3.1 above.

Proposition 4.3.2. *Let $\mathcal{A}$ be a preadditive category. Let $X, Y \in \mathrm{Ob}(\mathcal{A})$. Then the product $X \times Y$ exists if and only if the coproduct $X \amalg Y$ exists. In this case $X \times Y \cong X \amalg Y$.*

Proof. Suppose that $X \times Y$ exists with projections $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$. Denote $i_1 : X \to X \times Y$ the morphism corresponding to $(0, 1)$:

$$\begin{tikzcd}
 & X \arrow[d, dashed, "i_1"] \arrow[ddl, bend right, "1"'] \arrow[ddr, bend left, "0"] & \\
 & X \times Y \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
X & & Y
\end{tikzcd}$$

Similarly, denote $i_2 : Y \to X \times Y$ the morphism corresponding to $(0, 1)$. Thus we have the commutative diagram

$$\begin{tikzcd}
X \arrow[rr, "1"] \arrow[dr, "i_1"] & & X \\
 & X \times Y \arrow[ur, "\pi_1"] \arrow[dr, "\pi_2"] & \\
Y \arrow[rr, "1"] \arrow[ur, "i_2"] & & Y
\end{tikzcd}$$

where the diagonal compositions are zero. It follows that $i_1 \circ \pi_1 + i_2 \circ \pi_2$ is the identity since it is a morphism which upon composing with π_1 gives π_1 and upon composing with π_2 gives π_2. Suppose given morphisms $f : X \to Z$ and $g : Y \to Z$. Then we can form the map $f \circ \pi_1 + g \circ \pi_2 : X \times Y \to Z$. In this way we get a bijection $\mathrm{Mor}_{\mathcal{A}}(X \times Y, Z) = \mathrm{Mor}_{\mathcal{A}}(X, Z) \times \mathrm{Mor}_{\mathcal{A}}(Y, Z)$ which show that $X \times Y \cong X \amalg Y$. The coproduet case can be done similarly. $\square$

Definition 4.3.3. Given a pair of objects X, Y in a preadditive category $\mathcal{A}$ we call **direct sum**, and we denote it $X \oplus Y$ the product $X \times Y$ endowed with the morphisms π_1, π_2, i_1, i_2 as in Proposition 4.3.2 above.

Proposition 4.3.3. *Let $\mathcal{A}, \mathcal{B}$ be preadditive categories. Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor. Then F transforms direct sums to direct sums and zero to zero.*

Proof. Suppose F is additive. A direct sum Z of X and Y is characterized by having morphisms

$$i_1 : X \to Z,\ i_2 : Y \to Z,\ \pi_1 : Z \to X,\ \pi_2 : Z \to Y$$

such that

$$\pi_1 \circ i_1 = \mathrm{id}_X, \pi_2 \circ i_2 = \mathrm{id}_Y, \pi_2 \circ i_1 = 0, \pi_1 \circ i_2 = 0 \quad \text{and} \quad i_1 \circ \pi_1 + i_2 \circ \pi_2 = \mathrm{id}_Z\,.$$

Clearly $F(X), F(Y), F(Z)$ and the morphisms $F(i_1), F(i_2), F(\pi_1), F(\pi_1)$ satisfy exactly the same relations (by additivity) and we see that $F(Z)$ is a direct sum of $F(X)$ and $F(Y)$. $\square$

4.3.1.2 Additive category

Definition 4.3.4. A category $\mathcal{A}$ is called **additive** if it is preadditive and finite products exist, in other words it has a zero object and direct sums.

Namely the empty product is a finite product and if it exists, then it is a final object.

Definition 4.3.5. Let $\varphi : A \to B$ be a morphism in an additive category $\mathcal{A}$. A morphism $\iota : K \to A$ is a **kernel** of φ if $\varphi \circ \iota = 0$ and for all morphisms $\zeta : Z \to A$ such that $\varphi \circ \zeta = 0$ there exists a unique $\widetilde{\zeta} : Z \to K$ making the diagram

$$\begin{tikzcd}
Z \arrow[r, "\zeta"'] \arrow[rr, bend left, "0"] \arrow[dr, dashed, "\exists!\widetilde{\zeta}"'] & A \arrow[r, "\varphi"'] & B \\
& K \arrow[u, "\iota"] &
\end{tikzcd}$$

commute.

A morphism $\psi : B \to C$ is a **cokernel** of φ if $\psi \circ \varphi = 0$ and for all morphisms $\beta : B \to Z$ such that $\beta \circ \varphi = 0$ there exists a unique $\widetilde{\beta} : C \to Z$ making the diagram

$$\begin{tikzcd}
& C \arrow[dr, dashed, "\exists!\widetilde{\beta}"] & \\
A \arrow[r, "\varphi"] \arrow[rr, bend right, "0"'] & B \arrow[u, "\psi"] \arrow[r, "\beta"] & Z
\end{tikzcd}$$

commute.

Definition 4.3.6. If a kernel of $\varphi : A \to B$ exists, then a **coimage** of φ is a cokernel for the morphism $\ker \varphi \to A$. If a cokernel of $\varphi : A \to B$ exists, then the **image** of φ is a kernel of the morphism $B \to \operatorname{coker} \varphi$.

Lemma 4.3.4. *In any additive category, kernels are monomorphisms and cokernels are epimorphisms.*

Proof. Let $\varphi : A \to B$ be a morphism in an additive category $\mathcal{A}$, and let $\ker \varphi : K \to A$ be its kernel. Let $\zeta : Z \to K$ be a morphism such that $\ker \varphi \circ \zeta = 0$. Then the composition $\varphi \circ (\ker \varphi \circ \zeta)$ is 0 and by definition of kernel, $\ker \varphi \circ \zeta$ factors uniquely through K:

$$\begin{tikzcd}
Z \arrow[r, bend left, "\zeta"] \arrow[r, bend right, dashed, "\exists!"'] & K \arrow[r, "\ker \varphi"] & A \arrow[r, "\varphi"] & B
\end{tikzcd}$$

since $\ker \varphi \circ \zeta = 0 = \ker \varphi \circ 0$, the uniqueness of the decomposition gives $\zeta = 0$.

The proof that cokernels are epimorphisms is analogous. □

Now we relate the direct sum to kernels as follows.

Proposition 4.3.5. *Let $\mathcal{A}$ be a preadditive category. Let $X \oplus Y$ with morphisms as in Propostion 4.3.2 be a direct sum in $\mathcal{A}$. Then $i_1 : X \to X \oplus Y$ is a kernel of $\pi_2 : X \oplus Y \to Y$. Dually, π_1 is a cokernel for i_2.*

Proof. Let $f : Z \to X \oplus Y$ be a morphism such that $\pi_2 \circ f = 0$. We have to show that there exists a unique morphism $g : Z \to X$ such that $f = i_1 \circ g$:

$$\begin{tikzcd}
X \arrow[rr, "1"] \arrow[dr, "i_1"] & & X \\
Z \arrow[r, "f"] & X \times Y \arrow[ur, "\pi_1"] \arrow[dr, "\pi_2"] & \\
Y \arrow[rr, "1"] \arrow[ur, "i_2"] & & Y
\end{tikzcd}$$

Since $i_1 \circ \pi_1 + i_2 \circ \pi_2$ is the identity on $X \oplus Y$ we see that

$$f = (i_1 \circ \pi_1 + i_2 \circ \pi_2) \circ f = i_1 \circ \pi_1 \circ f$$

and hence $g = \pi_1 \circ f$ works. Uniquess holds because $\pi_1 \circ i_1$ is the identity on X. The proof of the second statement is dual. $\square$

Theorem 4.3.6. *Let $\varphi : A \to B$ be a morphism in a preadditive category such that the kernel, cokernel, image and coimage all exist. Then φ can be factored uniquely*

$$\begin{tikzcd} A \arrow[r] \arrow[rrr, bend left, "\varphi"] & \operatorname{coim}\varphi \arrow[r] & \operatorname{im}\varphi \arrow[r] & B \end{tikzcd}$$

Proof. There is a canonical morphism $\operatorname{coim}\varphi \to B$ because $\ker\varphi \to A \to B$ is zero,

$$\begin{tikzcd} & & \operatorname{im}\varphi & \\ \ker\varphi \arrow[r] & A \arrow[r, "\varphi"] \arrow[d] & B \arrow[r] & \operatorname{coker}\varphi \\ & \operatorname{coim}\varphi \arrow[ur, dashed] & & \end{tikzcd}$$

The composition $\operatorname{coim}\varphi \to B \to \operatorname{coker}\varphi$ is zero, because it is the unique morphism which gives rise to the morphism $A \to B \to \operatorname{coker}\varphi$ which is zero. Hence $\operatorname{coim}\varphi \to B$ factors uniquely through $\operatorname{im}\varphi \to B$, which gives us the desired map. $\square$

4.3.2 Abelian categories

An abelian category is a category satisfying just enough axioms so the snake lemma holds. An axiom is that the canonical map $\operatorname{coim}\varphi \to \operatorname{im}\varphi$ of Theorem 4.3.6 is always an isomorphism.

Definition 4.3.7. A category $\mathcal{A}$ is **abelian** if it is additive, if all kernels and cokernels exist, and if the natural map $\operatorname{coim}\varphi \to \operatorname{im}\varphi$ is an isomorphism for all morphisms φ of $\mathcal{A}$.

Definition 4.3.8. Let $\varphi : A \to B$ be a morphism in an abelian category.

(a) We say φ is **injective** if $\ker\varphi = 0$.

(b) We say φ is **surjective** if $\operatorname{coker}\varphi = 0$.

Proposition 4.3.7. *Let $\varphi : A \to B$ be a morphism in an abelian category. Then*

(a) *φ is **injective** if and only if f is a monomorphism.*

(b) *φ is **surjective** if and only if f is a epimorphism.*

Proof. The condition for monomorphism can be interpreted as: If $\psi : Z \to A$ is any morphism such that $\varphi \circ \psi = 0$, then ψ factors through $0 \to A$. So φ is a monomorphism if and only if $0 \to A$ is its kernel. The same holds for epimorphism. $\square$

Proposition 4.3.8. *Let $\mathcal{A}$ be an abelian category. All finite limits and finite colimits exist in $\mathcal{A}$.*

Proof. To show that finite limits exist it suffices to show that finite products and equalizers exist. Finite products exist by definition and the equalizer of $f, g : X \to Y$ is the kernel of $a - b$. The argument for finite colimits is similar but dual to this. $\square$

Example 4.3.1. Let $\mathcal{A}$ be an abelian category. Pushouts and fibre products in $\mathcal{A}$ have the following simple descriptions:

(a) If $f : X \to Y, g : Z \to Y$ are morphisms in $\mathcal{A}$, then we have the fibre product: $X \times_Y Z = \ker((f, -g) : X \oplus Z \to Y)$.

(b) If $f : Y \to X, g : Y \to Z$ are morphisms in $\mathcal{A}$, then we have the pushout: $X \amalg_Y Z = \operatorname{coker}((f, -g) : Y \oplus X \to Z)$.

Lemma 4.3.9. *In an abelian category $\mathcal{A}$, every kernel is the kernel of its cokernel; every cokernel is the cokernel of its kernel.*

Proof. Let $\varphi : K \to A$ be the kernel of some morphism $A \to B$; since $\mathcal{A}$ is abelian, φ has a cokernel $\psi : A \to C$. The composition $K \to A \to B$ is 0, so $A \to B$ factors through ψ by definition of cokernel:

$$\begin{tikzcd}
C \arrow[dr, dashed] & \\
A \arrow[u, "\psi"] \arrow[r] & B \\
K \arrow[u, "\varphi"] &
\end{tikzcd}$$

Now let $Z \to A$ be a morphism such that the composition $Z \to A \to C$ is the zero-morphism; then so is the composition $Z \to A \to B$. Therefore $Z \to A$ factors through a unique morphism $Z \to K$,

$$\begin{tikzcd}
& C \arrow[dr, dashed] & \\
Z \arrow[r] \arrow[dr, dashed] & A \arrow[u, "\psi"] \arrow[r] & B \\
& K \arrow[u, "\varphi"] &
\end{tikzcd}$$

since φ is the kernel of $A \to B$. But this shows that $\varphi : A \to B$ satisfies the property defining the kernel of its cokernel $A \to C$, as stated. ☐

Proposition 4.3.10. *Let $\varphi : A \to B$ be a morphism in an abelian category $\mathcal{A}$.*

(a) *φ is a monomorphism if and only if φ has a left-invere.*

(b) *φ is a epimorphism if and only if φ has a right-invere.*

Thus φ is an isomorphism if and only if it is a monomorphism and a epimorphism.

Proof. If φ has a left-inverse, then clearly it is monic. Conversely, if φ is a monomorphis, then the kernel of φ is $0 \to A$. Further, φ is the cokernel of $0 \to A$. Now consider the identity $A \to A$:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "\varphi"] \arrow[d, "\mathrm{id}_A"'] & B \\
& A &
\end{tikzcd}$$

Since $0 \to A \to A$ is the zero morphism and φ is the cokernel of $0 \to A$, we obtain a unique morphism $\psi : B \to A$ making the diagram commute:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "\varphi"] \arrow[d, "\mathrm{id}_A"'] & B \arrow[dl, "\psi"] \\
& A &
\end{tikzcd}$$

As $\psi \circ \varphi = \mathrm{id}_A$, this shows that φ has a right-inverse. The part (b) can be done similarly. $\square$

4.3.3 Exact sequence in Abelian category

Definition 4.3.9. Let $\mathcal{A}$ be an additive category. We say a sequence of morphisms

$$\cdots \longrightarrow A \xrightarrow{\varphi} B \xrightarrow{\psi} C \longrightarrow \cdots$$

in $\mathcal{A}$ is a **complex** if the composition of any two arrows is zero. If $\mathcal{A}$ is abelian then we say a sequence as above is **exact at B** if $\operatorname{im} \psi = \ker \varphi$. We say it is exact if it is exact at every object. A **short exact sequence** is an exact complex of the form

$$0 \longrightarrow A \xrightarrow{\varphi} B \xrightarrow{\psi} C \longrightarrow 0$$

Proposition 4.3.11. *Let $\mathcal{A}$ be an abelian category. Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be a complex of $\mathcal{A}$.*

(a) $M_1 \to M_2 \to M_3 \to 0$ *is exact if and only if*

$$0 \longrightarrow \operatorname{Hom}_{\mathcal{A}}(M_3, N) \longrightarrow \operatorname{Hom}_{\mathcal{A}}(M_2, N) \longrightarrow \operatorname{Hom}_{\mathcal{A}}(M_1, N)$$

is an exact sequence of abelian groups for all objects N of $\mathcal{A}$.

(b) $0 \to M_1 \to M_2 \to M_3$ *is exact if and only if*

$$\operatorname{Hom}_{\mathcal{A}}(N, M_1) \longrightarrow \operatorname{Hom}_{\mathcal{A}}(N, M_2) \longrightarrow \operatorname{Hom}_{\mathcal{A}}(N, M_3) \longrightarrow 0$$

is an exact sequence of abelian groups for all objects N of $\mathcal{A}$.

Example 4.3.2. For a slightly more interesting example, consider a diagram

$$\begin{tikzcd} D \arrow[r, "\varphi'"] \arrow[d, "\psi'"'] & B \arrow[d, "\psi"] \\ A \arrow[r, "\varphi"'] & C \end{tikzcd}$$

and the associated sequence

$$D \xrightarrow{(\psi', \varphi')} A \oplus B \xrightarrow{(\varphi, -\psi)} C$$

obtained by letting $A \oplus B$ play both roles of product and coproduct. Then

1. the diagram is commutative if and only if this sequence is a complex;
2. the sequence obtained by adding a 0 to the left,
$$0 \longrightarrow D \longrightarrow A \oplus B \longrightarrow C$$
is exact if and only if D may be identified with the fibered product $A \times_C B$. If this holds, we say the diagram is **cartesian**.
3. likewise, the sequence
$$D \longrightarrow A \oplus B \longrightarrow C \longrightarrow 0$$
is exact if and only if C may be identified with the fibered coproduct $A \amalg_D B$. If this holds, we say the diagram is **cocartesian**.

Lemma 4.3.12. *Let $\mathcal{A}$ be an abelian category. Let*

$$\begin{tikzcd} D \arrow[r, "\varphi'"] \arrow[d, "\psi'"'] & B \arrow[d, "\psi"] \\ A \arrow[r, "\varphi"'] & C \end{tikzcd}$$

be a commutative diagram.

(a) *If the diagram is cartesian, then the morphism* $\ker\varphi' \to \ker\varphi$ *induced by* ψ' *is an isomorphism.*

(b) *If the diagram is cocartesian, then the morphism* $\operatorname{coker}\varphi' \to \operatorname{coker}\varphi$ *induced by* ψ *is an isomorphism.*

Proof. Suppose the diagram is cartesian. Let $\epsilon : \ker\varphi' \to \ker\varphi$ be induced by ψ'. Let $i : \ker\varphi \to A$ and $j : \ker\varphi' \to D$ be the canonical injections. Consider the map $\alpha : \ker\varphi \to D$ determined by the morphisms $(i, 0)$:

$$\psi' \circ \alpha = i, \quad \varphi' \circ \alpha = 0.$$

Then there is an induced morphism $\gamma : \ker\varphi \to \ker\varphi'$:

$$\begin{tikzcd} \ker\varphi' \arrow[r, "j"] \arrow[d, "\epsilon"] & D \arrow[r, "\varphi'"] \arrow[d, "\psi'"'] & B \arrow[d, "\psi"] \\ \ker\varphi \arrow[r, "i"] \arrow[u, bend left, "\gamma"] \arrow[ur, "\alpha"] & A \arrow[r, "\varphi"'] & C \end{tikzcd}$$

It follows that

$$\psi' \circ j \circ \gamma \circ \epsilon = \psi' \circ \alpha \circ \epsilon = i \circ \epsilon = \psi' \circ j \quad \text{and} \quad \varphi' \circ j \circ \gamma \circ \epsilon = \varphi' \circ \alpha \circ \epsilon = 0 = \psi' \circ j.$$

By the universal property of pull back, we claim $j \circ \gamma \circ \epsilon = j$. Since j is a monomorphism, this means $\gamma \circ \varepsilon = \mathrm{id}_{\ker\varphi'}$.

Furthermore, we have

$$i \circ \epsilon \circ \gamma = \psi' \circ j \circ \gamma = \psi' \circ \alpha = i.$$

Since i is a monomorphism this implies $\epsilon \circ \gamma = \mathrm{id}_{\ker\varphi}$. This proves (a). Now, (b) follows by duality. □

Lemma 4.3.13. *Let $\mathcal{A}$ be an abelian category. Let*

$$\begin{tikzcd} D \arrow[r, "\varphi'"] \arrow[d, "\psi'"'] & B \arrow[d, "\psi"] \\ A \arrow[r, "\varphi"'] & C \end{tikzcd}$$

be a commutative diagram.

(a) *If the diagram is cartesian and φ is an epimorphism, then the diagram is cocartesian and φ' is an epimorphism.*

(b) *If the diagram is cocartesian and φ' is an monomorphism, then the diagram is cartesian and φ is an epimorphism.*

Proof. Suppose the diagram is cartesian and φ is an epimorphism. Let $\alpha = (\psi', \varphi') : D \to A \oplus B$ and let $\beta = (\varphi, -\psi) : A \oplus B \to C$. As φ is an epimorphism, α is an epimorphism, too. Therefore by Example 4.3.2 the diagram is cocartesian. Finally, φ' is an epimorphism by Lemma 4.3.12. This proves (1), and (2) follows by duality. □

Corollary 4.3.14. *Let $\mathcal{A}$ be an abelian category.*

(a) *If $X \to Y$ is surjective, then for every $Z \to Y$ the projection $X \times_Y Z \to Z$ is surjective.*

(b) *If $X \to Y$ is injective, then for every $X \to Z$ the morphism $Z \to Z \amalg_X Y$ is injective.*

Lemma 4.3.15. *Let* $X' \xrightarrow{f} X \xrightarrow{g} X''$ *be a complex. Then the conditions below are equivalent:*

(i) *the complex* $X' \xrightarrow{f} X \xrightarrow{g} X''$ *is exact.*

(ii) *the induced morphism $X' \to \ker g$ is an epimorphism.*

(iii) *for any morphism $h : S \to X$ such that $g \circ h = 0$, there exist an epimorphism $f' : S' \twoheadrightarrow S$ and a commutative diagram*

$$\begin{tikzcd} S' \arrow[r, two heads, "f'"] \arrow[d] & S \arrow[d, "h"] \arrow[dr, "0"] & \\ X' \arrow[r, "f"] & X \arrow[r, "g"] & X'' \end{tikzcd}$$

Proof. $(i) \iff (ii)$: the exactness is saying $\ker g = \operatorname{im} f$. If $X' \to \ker g$ is epic, by Exercise 4.3.1, $\ker g = \operatorname{im} f$ as needed. Conversely, if $\ker g =\in f$, by Lemma **??**, $X' \to \ker g$ is epic.

$(i) \Rightarrow (iii)$: It is enough to choose $X' \times_{\ker g} S$ as S'. Since $X' \to \ker g$ is an epimorphism, $S' \to S$ is an epimorphism by Lemma 4.3.12.

$(iii) \Rightarrow (ii)$: Choose $S = \ker g$. Then the diagram becomes

$$\begin{tikzcd} S' \arrow[r, two heads, "f'"] \arrow[d] & \ker g \arrow[d] \arrow[dr, "0"] & \\ X' \arrow[r, "f"] \arrow[ur, dashed] & X \arrow[r, "g"] & X'' \end{tikzcd}$$

since $g \circ f = 0$, by the universal property of $\ker g$, there is a unique morphism $X' \to \ker g$. It follows that the composition $S' \to X' \to \ker g$ is an epimorphism. Hence $X' \to \ker g$ is an epimorphism. □

4.3.4 Exercise

Exercise 4.3.1. Let $\varphi : A \to B$ be a morphism in an abelian category, and assume φ decomposes as an epimorphism π followed by a monomorphism i:

$$\begin{tikzcd} A \arrow[r, two heads, "\pi"] \arrow[rr, bend right, "\varphi"'] & C \arrow[r, tail, "i"] & B \end{tikzcd}$$

Prove that necessarily $\pi = \operatorname{coim} \varphi$ and $i = \operatorname{im} \varphi$.

Proof. From the universal property of image and coimage, we have the following commutative diagram:

$$\begin{tikzcd}
 & \operatorname{coim}\varphi \arrow[dr, bend left, tail] & \\
A \arrow[ur, bend left, two heads] \arrow[r, two heads, "\pi"] \arrow[dr, bend right, two heads] & C \arrow[u, dashed, "\exists!\nu"] \arrow[r, tail, "i"] & B \\
 & \operatorname{im}\varphi \arrow[u, dashed, "\exists!\mu"] \arrow[ur, bend right, tail] &
\end{tikzcd}$$

By simple observation, we find μ and ν are both monomorphic and epimorphic, hence are isomorphisms. $\square$

4.4 Triangulated categories

4.4.1 Localization of categories

Consider a category $\mathcal{C}$ and a family $\mathcal{S}$ of morphisms in $\mathcal{C}$. The aim of localization is to find a new category $\mathcal{C}_{\mathcal{S}}$ and a functor $Q : \mathcal{C} \to \mathcal{C}_{\mathcal{S}}$ which sends the morphisms belonging to $\mathcal{C}$ to isomorphisms in $\mathcal{C}_{\mathcal{S}}$, $(\mathcal{C}_{\mathcal{S}}, Q)$ being "universal" for such a property. We also discuss with some details the localization of functors. When considering a functor F from $\mathcal{C}$ to a category $\mathcal{A}$ which does not necessarily send the morphisms in $\mathcal{S}$ to isomorphisms in $\mathcal{A}$, it is possible to define the right (resp. left) localization of F, a functor $R_{\mathcal{S}}F$ (resp. $L_{\mathcal{S}}F$) from $\mathcal{C}_{\mathcal{S}}$ to $\mathcal{A}$. Such a right localization always exists if $\mathcal{A}$ admits filtrant inductive limits.

Let $\mathcal{C}$ be a category and $\mathcal{S}$ be a family of morphisms in $\mathcal{C}$. A **localization** of $\mathcal{C}$ by $\mathcal{S}$ is the data of a category $\mathcal{C}_{\mathcal{S}}$ and a functor $Q : \mathcal{C} \to \mathcal{C}_{\mathcal{S}}$ satisfying:

(L1) for all $s \in \mathcal{S}$, $Q(s)$ is an isomorphism;

(L2) for any category $\mathcal{A}$ and any functor $F : \mathcal{C} \to \mathcal{A}$ such that $F(s)$ is an isomorphism for all $s \in \mathcal{S}$, there exist a functor $F_{\mathcal{S}} : \mathcal{C}_{\mathcal{S}} \to \mathcal{A}$ and an isomorphism $F \cong F_{\mathcal{S}} \circ Q$ visualized by the diagram

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[d, "Q"'] & \mathcal{A} \\
\mathcal{C}_{\mathcal{S}} \arrow[ur, dashed, "F_{\mathcal{S}}"'] &
\end{tikzcd}$$

(L3) if G and G' are two functors from $\mathcal{C}_{\mathcal{S}}$ to $\mathcal{A}$, then the natural map

$$\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}_{\mathcal{S}},\mathcal{A})}(G, G') \to \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathcal{A})}(G \circ Q, G' \circ Q) \tag{4.4.1}$$

is bijective.

Note that condition (L3) means that the functor $Q^\star : \mathrm{Fun}(\mathcal{C}_{\mathcal{S}}, A) \to \mathrm{Fun}(\mathcal{C}, A)$ induced by composition is fully faithful. In particular, this implies that $F_{\mathcal{S}}$ in (L2) is unique up to isomorphism.

Proposition 4.4.1. *Let $\mathcal{C}$ be a category and $\mathcal{S}$ be a family of morphisms in $\mathcal{C}$.*

(a) *If $\mathcal{C}_{\mathcal{S}}$ exists, it is unique up to equivalence of categories.*

(b) *If $\mathcal{C}_{\mathcal{S}}$ exists, then, denoting by $\mathcal{S}^{\mathrm{op}}$ the image of $\mathcal{S}$ in $\mathcal{C}^{\mathrm{op}}$, $(\mathcal{C}^{\mathrm{op}})_{\mathcal{S}^{\mathrm{op}}}$ exists and there is an equivalence of categories $(\mathcal{C}_{\mathcal{S}})^{\mathrm{op}} \cong (\mathcal{C}^{\mathrm{op}})_{\mathcal{S}^{\mathrm{op}}}$.*

Proof. If $(\mathcal{C}_{\mathcal{S}}, Q)$ and $(\mathcal{C}'_{\mathcal{S}}, Q')$ are two localizations of $\mathcal{C}$ by $\mathcal{S}$, then since $Q'(s)$ is an isomorphism for any $s \in \mathcal{S}$, we obtain a funcor $G : \mathcal{C}_{\mathcal{S}} \to \mathcal{C}'_{\mathcal{S}}$ such that $GQ \cong Q'$; similarly, there is a functor $G' : \mathcal{C}'_{\mathcal{S}} \to \mathcal{C}_{\mathcal{S}}$ such that $G'Q' \cong Q$. Since $G'GQ \cong Q$, we conclude from (L3) that $G'G \cong \mathrm{id}_{\mathcal{C}_{\mathcal{S}}}$, and similarly $GG' \cong \mathrm{id}_{\mathcal{C}'_{\mathcal{S}}}$. The second assertion follows immediately from (a) and a easy verification by reversing the arrows. $\square$

The existence of the localization $\mathcal{C}_{\mathcal{S}}$ is generally true, since we can construct $\mathcal{C}_{\mathcal{S}}$ by adding virtue inverses to $\mathcal{C}$ (like the construction of free groups). More precisely, we have $\mathrm{Ob}(\mathcal{C}_{\mathcal{S}}) = \mathrm{Ob}(\mathcal{C})$, and the morphisms in $\mathcal{C}_{\mathcal{S}}$ are of the form

$$\cdots bt^{-1}as^{-1}\cdots = \left(\begin{tikzcd}[column sep=small] \cdots \arrow[dr] & & Y \arrow[dl, "s"'] \arrow[dr, "a"] & & W \arrow[dl, "t"'] \arrow[dr, "b"] & & \cdots \arrow[dl] \\ & X & & Z & & U & \end{tikzcd}\right)$$

where $a, b \in \mathrm{Mor}(\mathcal{C})$ and $s, t \in \mathcal{S}$, with the composition map defined in the obvious way subject to the relations

$$s^{-1}t^{-1} = (ts)^{-1}, \quad ss^{-1} = \mathrm{id}, \quad s^{-1}s = \mathrm{id}\,.$$

The problem is that the equivalence relation in $\mathcal{C}_{\mathcal{S}}$ is now hard to manipulate: we can not tell which morphisms f, g in $\mathcal{C}$ satisfy $Q(f) = Q(g)$. Due to this failure, we now impose some additional conditions on the system $\mathcal{S}$, so that the resulting localization $\mathcal{C}_{\mathcal{S}}$ is way more easiler to describe.

Definition 4.4.1. The family $\mathcal{S}$ is called a **right multiplicative system** if it satisfies the following axioms:

(S1) For any object X of $\mathcal{C}$, id_X belongs to $\mathcal{S}$.

(S2) If two morphisms $f : X \to Y$ and $g : Y \to Z$ belong to $\mathcal{S}$, then $g \circ f$ belongs to $\mathcal{S}$.

(S3) Given two morphisms $f : X \to Y$ and $s : X \to X'$ with $s \in \mathcal{S}$, there exist $t : Y \to Y'$ and $g : X' \to Y'$ with $t \in \mathcal{S}$ such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, tail, "s"'] & Y \arrow[d, tail, dashed, "t"] \\ X' \arrow[r, dashed, "g"] & Y' \end{tikzcd}$$

(S4) Let $f, g : X \rightrightarrows Y$ be two morphisms in $\mathcal{C}$. If there exists a morphism $s : Z \to X$ in $\mathcal{S}$ such that $fs = gs$, then there exists $t : Y \to W$ in $\mathcal{S}$ such that $tf = tg$. This is visualized by the diagram:

$$\begin{tikzcd} Z \arrow[r, tail, "s"] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, tail, dashed, "t"] & W \end{tikzcd}$$

Remark 4.4.1. Axioms (S1)-(S2) asserts that there exists a half-full subcategory $\widetilde{\mathcal{S}}$ of $\mathcal{C}$ with $\mathrm{Ob}(\widetilde{\mathcal{S}}) = \mathrm{Ob}(\mathcal{C})$ and $\mathrm{Mor}(\widetilde{\mathcal{S}}) = \mathcal{S}$. With these axioms, the notion of a right multiplicative system is stable by equivalence of categories.

Remark 4.4.2. The notion of a **left multiplicative system** is defined similarly by reversing the arrows. This means that the condition (S3) and (S4) are replaced by the conditions (S3′) and (S4′) below:

(S3′) Given two morphisms $f : X \to Y$ and $t : Y' \to Y$ with $t \in \mathcal{S}$, there exist $s : X' \to X$ and $g : X' \to Y'$ with $s \in \mathcal{S}$ such that the following diagram commutes:

$$\begin{tikzcd} X' \arrow[r, dashed, "g"] \arrow[d, dashed, tail, "s"'] & Y' \arrow[d, tail, "t"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

(S4′) Let $f, g : X \rightrightarrows Y$ be two morphisms in $\mathcal{C}$. If there exists a morphism $t : Y \to W$ in $\mathcal{S}$ such that $tf = tg$, then there exists $s : Z \to X$ in $\mathcal{S}$ such that $fs = gs$. This is visualized by the diagram:

$$\begin{tikzcd} Z \arrow[r, dashed, tail, "s"] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, tail, "t"] & W \end{tikzcd}$$

A collection $\mathcal{S}$ is simply called a **multiplicative system** if it is both a left multiplicative system and a right multiplicative system.

Let $\mathcal{S}$ be a system of morphisms of $\mathcal{C}$ satisfying axioms (S1)-(S2) and $X \in \mathrm{Ob}(\mathcal{C})$. We define $\mathcal{S}_{/X}$ (resp. $\mathcal{S}_{X/}$) to be the full subcategory of $\mathcal{C}_{/X}$ (resp. $\mathcal{C}_{X/}$) with objects (morphisms in $\mathcal{C}$) belonging to $\mathcal{S}$.

Proposition 4.4.2. *If $\mathcal{S}$ is a left (resp. right) multiplicative system. Then the category $\mathcal{S}_{/X}$ (resp. $\mathcal{S}_{X/}$) is cofiltrant (resp. filtrant).*

Proof. Note that $(\mathcal{S}^{\mathrm{op}})_{X/} = (\mathcal{S}_{/X})^{\mathrm{op}}$, so we only need to consider right multiplicative systems. For any objects $s : X \to Z$ and $s' : X \to Z'$ of $\mathcal{S}_{X/}$, by (S3) we have a commutative diagram

$$\begin{tikzcd} X \arrow[r, "s'"] \arrow[d, "s"'] & Z' \arrow[d, "t'"] \\ Z \arrow[r, "t"] & Y \end{tikzcd}$$

with $t \in S$. Then $ts \in \mathcal{S}$ by (S2) and the composition $ts : X \to Y$ belongs to $\mathcal{S}_{X/}$. Now consider two morphisms $f, g : Z \rightrightarrows Z'$ such that $fs = gs = s'$. Then by (S4) there exists $t : Z' \to W$ such that $tf = tg$, so $t \circ s' : X \to W$ belongs to $\mathcal{S}_{X/}$ and the compositions

$$\begin{tikzcd} (Z, s) \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & (Z', s') \arrow[r, "t"] & (W, t \circ s') \end{tikzcd}$$

coincides; this completes the proof. □

Let X, Y be objects of $\mathcal{C}$. For left (resp. right) multiplicative system $\mathcal{S}$, we define a collection $M^l_{X,Y}$ (resp. $M^r_{X,Y}$), which will be considered to be the "morphisms" from X to Y in our localization category.

- If $\mathcal{S}$ is a left multiplicative system, we denote by $M^l_{X,Y}$ the collection of diagrams of the form

$$\begin{tikzcd} & Z \arrow[dl, "s"'] \arrow[dr, "a"] & \\ X & & Y \end{tikzcd}$$

  where $s \in \mathcal{S}$ (such a diagram will be denoted by $(Z; s, a)$).

- If $\mathcal{S}$ is a right multiplicative system, we denote by $M^r_{X,Y}$ the collection of diagrams of the form

$$\begin{tikzcd} X \arrow[dr, "a"'] & & Y \arrow[dl, "s"] \\ & Z & \end{tikzcd}$$

where $s \in \mathcal{S}$ (such a diagram will be denoted by $(Z; a, s)$).

We define an equivalence relation on $M^l_{X,Y}$ (resp. $M^r_{X,Y}$) as follows: $(Z; s, a) \sim (Z'; s', a')$ (resp. $(Z; a, s) \sim (Z'; a', s')$) if there eixsts a commutative diagram of the form

$$\begin{tikzcd} & Z \arrow[dl, tail, "s"'] \arrow[dr, "a"] & \\ X & W \arrow[l, tail] \arrow[r] \arrow[u] \arrow[d] & Y \\ & Z' \arrow[ul, tail, "s'"] \arrow[ur, "a'"'] & \end{tikzcd} \quad \text{resp.} \quad \begin{tikzcd} & Z \arrow[d] & \\ X \arrow[ur, "a"] \arrow[r] \arrow[dr, "a'"'] & W & Y \arrow[l, tail] \arrow[ul, tail, "s"'] \arrow[dl, tail, "s'"] \\ & Z' \arrow[u] & \end{tikzcd}$$

(we use $\rightarrowtail$ to indicates a morphism in $\mathcal{S}$.) We also note that for any morphism $f : X \to Y$, axiom (S1) implies that $(X; \mathrm{id}_X, f) \in M^l_{X,Y}$ and $(X; f, \mathrm{id}_Y) \in M^r_{X,Y}$.

Lemma 4.4.3. *Let $\mathcal{S}$ be a left (resp. right) multiplicative system. For any object X, Y of $\mathcal{C}$, we have a bijection*

$$M^l_{X,Y}/\sim \xrightarrow{\sim} \varinjlim_{(Z\to X)\in \mathrm{Ob}(\mathcal{S}^{\mathrm{op}}_{/X})} \mathrm{Hom}(Z, Y), \quad [Z; s, a] \mapsto [a : Z \to Y],$$
$$M^r_{X,Y}/\sim \xrightarrow{\sim} \varinjlim_{(Y\to Z)\in \mathrm{Ob}(\mathcal{S}_{Y/})} \mathrm{Hom}(X, Z), \quad [Z; a, s] \mapsto [a : X \to Z].$$

Proof. We consider the functor $\alpha : \mathcal{S}^{\mathrm{op}}_{/X} \to \mathbf{Set}$ given by $(Z \to X) \mapsto \mathrm{Hom}(Z, Y)$. Then by definition we have

$$M^l_{X,Y} = \coprod_{Z\to X} \alpha(Z \to X).$$

On the other hand, it is not hard to see that the equivalence relation $\sim$ on $M^l_{X,Y}$ is induced from the limit $\varinjlim \mathrm{Hom}(Z, Y)$, so the claim follows. □

For a left multiplicative system $\mathcal{S}$, any objects X, Y of $\mathcal{C}$ and $(U; s, a) \in M^l_{X,Y}$ and $(V; t, b) \in M^l_{Y,Z}$, by axiom (S3) we have a commutative diagram

$$\begin{tikzcd} & & W \arrow[dl, tail, "r"'] \arrow[dr, "c"] & & \\ & U \arrow[dl, tail, "s"'] \arrow[dr, "a"] & & V \arrow[dl, tail, "t"'] \arrow[dr, "b"] & \\ X & & Y & & Z \end{tikzcd} \tag{4.4.2}$$

We now define the composition of $(U; s, a)$ and $(V; t, b)$ to be the equivalent class of $(W; sr, bc)$.

Proposition 4.4.4. *The composition law defined above is associative and only depends on the equivalent class of $(U; s, a)$ and $(V; t, b)$. Also, a similar result holds if $\mathcal{S}$ is a right multiplicative system.*

Proof. We first fix the diagram $(V;t,b)$. In view of the definition, it suffices to prove that in the following diagram

$$\begin{tikzcd}
& U \arrow[dl, "s"'] \arrow[dr, "a"] & W \arrow[l, "r"'] \arrow[dr, "c"] & & \\
X & & Y & V \arrow[l, "t"'] \arrow[r, "b"] & Z \\
& U' \arrow[uu, "x"] \arrow[ul, "s'"] \arrow[ur, "a'"'] & W' \arrow[l, "r'"] \arrow[ur, "c'"'] & &
\end{tikzcd} \tag{4.4.3}$$

we have $(W;sr,bc) \sim (W';s'r',bc')$. For this, we apply axiom (S3) twice to obtain the following solid diagram:

$$\begin{tikzcd}
U \arrow[dr, "a"] & W \arrow[l, "r"'] \arrow[dr, "c"] & & & & \\
& Y & V \arrow[l, "t"'] & & & \\
U' \arrow[uu, "x"] \arrow[ur, "a'"'] & W' \arrow[l, "r'"] \arrow[ur, "c'"'] & & & & \\
& & & R \arrow[uuull, dashed, bend right, "q"'] \arrow[ulll, dashed, bend left, "p"] & Q \arrow[l, dashed, "h"'] \arrow[ulll, dashed, bend right, "k"'] & P \arrow[l, dashed, "w"']
\end{tikzcd} \tag{4.4.4}$$

From the diagram (4.4.3), we then conclude that $tc'k = a'r'k$, so by (S4) there is a morphism $w : P \to Q$ in $\mathcal{S}$ such that $c'kw = cqhw$. Now, it is not hard to verify that the following diagram commutes:

$$\begin{tikzcd}
& W \arrow[dl, "sr"'] \arrow[dr, "bc"] & \\
X & P \arrow[u, "qhw"] \arrow[l, "s'phw"'] \arrow[r, "bc'kw"] \arrow[d, "kw"] & Z \\
& W' \arrow[ul, "s'r'"] \arrow[ur, "bc'"'] &
\end{tikzcd}$$

so $(W;sr,bc) \sim (W';s'r',bc')$. A similar argument proves the case where $(U;s,a)$ is fixed, and the same result holds for right multiplicative systems.

We now verify that associativity, so let $(A;s,a) \in M^l_{X,Y}$, $(B;t,b) \in M^l_{Y,Z}$, and $(C;u,c) \in M^l_{Z,W}$. Apply axiom (S3) three times, we obtain a diagram

$$\begin{tikzcd}
& & & \bullet \arrow[dl] \arrow[dr] & & & \\
& & \bullet \arrow[dl] \arrow[dr] & & \bullet \arrow[dl] \arrow[dr] & & \\
& A \arrow[dl, "s"'] \arrow[dr, "a"] & & B \arrow[dl, "t"'] \arrow[dr, "b"] & & C \arrow[dl, "u"'] \arrow[dr, "c"] & \\
X & & Y & & Z & & W
\end{tikzcd}$$

which can be considered as an element of $M^l_{X,W}$ and witnesses the associativity:

$$[C;u,c] \circ ([B;t,b] \circ [A;s,a]) = ([C;u,c] \circ [B;t,c]) \circ [A;s,a]. \qquad \square$$

Definition 4.4.2. Let $\mathcal{S}$ be a left multiplicative system of a category $\mathcal{C}$. We define $\mathcal{C}^l_{\mathcal{S}}$ to be the (big) category with objects $\mathrm{Ob}(\mathcal{C})$, and morphisms from X to Y given by $M^l_{X,Y}/\sim$. The identity morphism of X is given by $[X; \mathrm{id}_X, \mathrm{id}_X]$, and the composition law is determined by Proposition 4.4.4. Moreover, we define a functor $Q^l : \mathcal{C} \to \mathcal{C}^l_{\mathcal{S}}$ such that it is the identity on objects and sends a morphism $f : X \to Y$ to $[X; \mathrm{id}_X, f]$. Similarly, if $\mathcal{S}$ be a right multiplicative system, we can define a functor $Q^r : \mathcal{C} \to \mathcal{C}^r_{\mathcal{S}}$.

It then remains to check that $(\mathcal{C}^l_{\mathcal{S}}, Q^l)$ (resp. $(\mathcal{C}^r_{\mathcal{S}}, Q^r)$) is our desired localization of $\mathcal{C}$ with respect to $\mathcal{S}$. For this, we shall use the following lemma:

Lemma 4.4.5. *Let $Q : \mathcal{C} \to \mathcal{C}'$ and $G : \mathcal{C}' \to \mathcal{A}$ be two functors. Assume that for any $X \in \mathrm{Ob}(\mathcal{C}')$, there exist $Y \in \mathrm{Ob}(\mathcal{C})$ and a morphism $s : X \to Q(Y)$ which satisfy the following two properties:*

(a) *$G(s)$ is an isomorphism;*

(b) *for any $Y' \in \mathcal{C}$ and any morphism $t : X \to Q(Y')$, there exists $Y'' \in \mathcal{C}$ and morphisms $s' : Y' \to Y''$, $t' : Y \to Y''$ in $\mathcal{C}$ such that $G(Q(s'))$ is an isomorphism and the following diagram commutes:*

$$\begin{tikzcd} X \arrow[r, "s"] \arrow[d, "t"'] & Q(Y) \arrow[d, "Q(t')"] \\ Q(Y') \arrow[r, "Q(s')"] & Q(Y'') \end{tikzcd}$$

Then the canonical map

$$\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}',\mathcal{A})}(F, G) \to \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathcal{A})}(F \circ Q, G \circ Q) \tag{4.4.5}$$

is bijective for any functor $F : \mathcal{C}' \to \mathcal{A}$.

Proof. Let θ_1 and θ_2 be two morphisms from F to G and assume that $\theta_1(Q(Y)) = \theta_2(Q(Y))$ for all $Y \in \mathcal{C}$. For $X \in \mathcal{C}'$, choose a morphism $s : X \to Q(Y)$ such that $G(s)$ is an isomorphism, and consider the commutative diagram for $i = 1, 2$:

$$\begin{tikzcd} F(X) \arrow[r, "\theta_i(X)"] \arrow[d, "F(s)"'] & G(X) \arrow[d, "G(s)"] \\ F(Q(Y)) \arrow[r, "\theta_i(Q(Y))"] & G(Q(Y)) \end{tikzcd}$$

Since $G(s)$ is an isomorphism, we conclude from the hypothesis that $\theta_1(X) = \theta_2(X)$, so (4.4.5) is injective (it is given by horizontal composition).

Now let $\theta : F \circ Q \to G \circ Q$ be a morphism of functors. For each $X \in \mathcal{C}'$, we choose a morphism $s : X \to Q(Y)$ satisfying conditions (a) and (b), and define a morphism $\gamma(X) : F(X) \to G(X)$ by

$$\gamma(X) = (G(s))^{-1} \circ \theta(Y) \circ F(s).$$

Let us prove that this construction is functorial, and in particular, does not depend on the choice of the morphism $s : X \to Q(Y)$ (take $f = \mathrm{id}_X$ in the proof). Once this is done, we obtain a morphism $\gamma : F \to G$ of functors which satisfies $\gamma(Q(Y)) = \theta(Y)$ (since we can choose $s = \mathrm{id}_{Q(Y)}$ in this case), so (4.4.5) is surjective.

To this end, let $f : X_1 \to X_2$ be a morphism in $\mathcal{C}'$. For any choice of morphisms $s_1 : X_1 \to Q(Y_1)$ and $s_2 : X_2 \to Q(Y_2)$ satisfying the given conditions, we can apply (b) to the morphisms $s_1 : X_1 \to Q(Y_1)$ and $s_2 \circ f : X_1 \to Q(Y_2)$; we then obtain morphisms $t_1 : Y_1 \to Y_3$ and

$t_2 : Y_2 \to Y_3$ such that $G(Q(t_2))$ is an isomorphism and $Q(t_1) \circ s_1 = Q(t_2) \circ s_2 \circ f$. We then obtain a commutative diagram

$$\begin{tikzcd}
F(X_1) \arrow[rrrrr, "\gamma(X_1)"] \arrow[dr, "F(s_1)"'] \arrow[dddd, "F(f)"'] & & & & & G(X_1) \arrow[dl, "G(s_1)"', "\sim"] \arrow[dddd, "G(f)"] \\
& F(Q(Y_1)) \arrow[rrr, "\theta(Y_1)"] \arrow[dr, "F(Q(t_1))"] & & & G(Q(Y_1)) \arrow[dl, "G(Q(t_1))"'] & \\
& & F(Q(Y_3)) \arrow[r, "\theta(Y_3)"] & G(Q(Y_3)) & & \\
& F(Q(Y_2)) \arrow[rrr, "\theta(Y_2)"] \arrow[ur, "F(Q(t_2))"'] & & & G(Q(Y_2)) \arrow[ul, "G(Q(t_2))"] & \\
F(X_2) \arrow[rrrrr, "\gamma(X_2)"] \arrow[ur, "F(s_2)"'] & & & & & G(X_2) \arrow[ul, "G(s_2)", "\sim"']
\end{tikzcd}$$

Since all the internal diagrams commute, the outer square also commutes, which proves our assertion. $\square$

Theorem 4.4.6 (P. Gabriel, M. Zisman). *Assume that $\mathcal{S}$ is a left multiplicative system. Then $(\mathcal{C}^l_{\mathcal{S}}, Q^l)$ (resp. $(\mathcal{C}^r_{\mathcal{S}}, Q^r)$) define a localization of $\mathcal{C}$ with respect to $\mathcal{S}$.*

Proof. It suffices to verify the universal properties for $(\mathcal{C}^l_{\mathcal{S}}, Q^l)$. Write $Q = Q^l$, then if $f : X \to Y$ belongs to $\mathcal{S}$, the diagram

$$\begin{tikzcd}
& X \arrow[dl, "f"'] \arrow[d, dashed, "f"] \arrow[dr, "f"] & \\
Y \arrow[r, equal] & Y \arrow[r, equal] & Y
\end{tikzcd}$$

suggests that $[X; f, f] = [Y, \mathrm{id}_Y, \mathrm{id}_Y]$; on the other hand, the following diagram

$$\begin{tikzcd}[column sep=small]
& & X \arrow[dl, equal] \arrow[dr, equal] & & & & & X \arrow[dl, equal] \arrow[dr, equal] & & \\
& X \arrow[dl, equal] \arrow[dr, "f"] & & X \arrow[dl, "f"'] \arrow[dr, equal] & & & X \arrow[dl, "f"'] \arrow[dr, equal] & & X \arrow[dl, equal] \arrow[dr, "f"] & \\
X & & Y & & X & X & & Y & & X
\end{tikzcd}$$

proves that $[X; f, \mathrm{id}_X] = [X; \mathrm{id}_X, f]^{-1}$, so the functor Q sends the elements in $\mathcal{S}$ to isomorphisms.

Now consider a functor $F : \mathcal{C} \to \mathcal{A}$ such that $F(s)$ is an isomorphism for $s \in \mathcal{S}$. For any $X \in \mathrm{Ob}(\mathcal{C}_{\mathcal{S}}) = \mathrm{Ob}(\mathcal{C})$, we define $F_{\mathcal{S}}(X) = F(X)$, and consider the morphisms

$$F_{\mathcal{S}} : \mathrm{Hom}_{\mathcal{C}^l_{\mathcal{S}}}(X, Y) \to \mathrm{Hom}_{\mathcal{A}}(F(X), F(Y)), \quad [U; s, a] \mapsto F(a)(F(s))^{-1}.$$

It is clear that $F_{\mathcal{S}}$ sends identities to identities, and by applying F to the diagram (4.4.2), we see that $F_{\mathcal{S}}$ preserves compositions, so we obtain a functor $F_{\mathcal{S}} : \mathcal{C}_{\mathcal{S}} \to \mathcal{A}$. It is clear that $F_{\mathcal{S}} \circ Q \cong F$, and $F_{\mathcal{S}}$ is unique up to isomorphisms.

Finally, with the notations of Lemma 4.4.5, we can choose $Y \in \mathrm{Ob}(\mathcal{C})$ such that $X = Q(Y)$ and $s = \mathrm{id}_{Q(Y)}$. Then any morphism $t : Q(Y) \to Q(Y')$ is given by morphisms

$$Y \xrightarrow{t'} Y'' \xleftarrow{s'} Y'$$

and the diagram in Lemma 4.4.5 commutes. $\square$

Remark 4.4.3. If $\mathcal{S}$ is both a left multiplicative system and right multiplicative system, then by Proposition 4.4.1, the two localizations of $\mathcal{C}$ are equivalent, and we simply denote it by $\mathcal{C}_{\mathcal{S}}$.

Corollary 4.4.7. *Let $\mathcal{S}$ be a left (resp. right) multiplicative system, then two morphisms $f, g : X \rightrightarrows Y$ satisfy $Q^l(f) = Q^l(g)$ (resp. $Q^r(f) = Q^r(g)$) if and only if there exists $s \in \mathcal{S}$ such that $fs = gs$ (resp. $sf = sg$).*

Proof. Let $\mathcal{S}$ be a left multiplicative system. Then if $Q^l(f) = Q^l(g)$, we have a commutative diagram

$$\begin{tikzcd} & X \arrow[dr, "f"] & \\ X \arrow[ur, equal] \arrow[dr, equal] & U \arrow[l, "s"'] \arrow[u] \arrow[d] \arrow[r, "a"] & Y \\ & X \arrow[ur, "g"'] & \end{tikzcd}$$

It then follows that $fs = a = gs$, whence the corollary. □

Corollary 4.4.8. *Let $\mathcal{S}$ be a left (resp. right) multiplicative system. Then the functor Q^l (resp. Q^r) sends monomorphisms to monomorphisms, and epimorphisms to epimorphisms.*

Proof. We only consider a left multiplicative system $\mathcal{S}$. Let $f : X \to Y$ be a monomorphism in $\mathcal{C}$ and $\alpha, \beta : Q^l(W) \to Q^l(X)$ be two morphisms in $\mathcal{C}^l_{\mathcal{S}}$ such that $(Q^l(f))\alpha = (Q^l(f))\beta$. Then by (S2) and (S3), we can write $\alpha = (U; s, a)$ and $\beta = (U; s, b)$, and it then follows that $Q^l(fa) = Q^l(fb)$, so by Corollary 4.4.7, there exists $t \in \mathcal{S}$ such that $fat = fbt$, whence $at = bt$ and $\alpha = \beta$. □

Remark 4.4.4. We note that the category $\mathcal{C}^l_{\mathcal{S}}$ (resp. $\mathcal{C}^r_{\mathcal{S}}$) may not be small, since the collection $M^l_{X,Y}$ (resp. $M^r_{X,Y}$) is too big. However, if the collection $\mathcal{S}$ has a cofinal subset, then by Lemma 4.4.3, we can restrict the inductive limit to this set, and then $\mathcal{C}^l_{\mathcal{S}}$ (resp. $\mathcal{C}^r_{\mathcal{S}}$) will be small. This is the case if $\mathcal{C}$ itself is already small.

We now give some properties of the localization functor Q. For this, assume that $\mathcal{S}$ is a left (resp. right) multiplicative system and let $X \in \mathcal{C}$. We define a functor

$$\theta_{/X} : \mathcal{S}_{/X} \to \mathcal{C}_{Q(X)/} \quad (\text{resp. } \theta_{X/} : \mathcal{S}_{X/} \to \mathcal{C}_{/Q(X)})$$

by associating a morphism $s : Y \to X$ in $\mathcal{S}_{/X}$ (resp. a morphism $s : X \to Y$ in $\mathcal{S}_{X/}$) with the morphism $Q(s)^{-1} : Q(X) \to Q(Y)$ in $\mathcal{C}_{Q(X)/}$ (resp. the morphism $Q(s)^{-1} : Q(Y) \to Q(X)$ in $\mathcal{C}_{Q(X)/}$).

Lemma 4.4.9. *Assume that $\mathcal{S}$ is a left (resp. right) multiplicative system and let $X \in \mathcal{C}$. Then the functor $\theta^{\mathrm{op}}_{/X}$ (resp. $\theta_{X/}$) is cofinal.*

Proof. We only consider right multiplicative systems. □

Proposition 4.4.10. *Let $\mathcal{S}$ be a left (resp. right) multiplicative system and $Q : \mathcal{C} \to \mathcal{C}_{\mathcal{S}}$ be the corresponding localization functor.*

(a) *The functor Q is left (resp. right) exact.*

(b) *Let $\alpha : I \to \mathcal{C}$ be a projective (resp. inductive) system in $\mathcal{C}$ indexed by a finite category I. Assume that $\varprojlim \alpha$ (resp. $\varinjlim \alpha$) exists in $\mathcal{C}$, then $\varprojlim(Q \circ \alpha)$ (resp. $\varinjlim(Q \circ \alpha)$) exists in $\mathcal{C}_{\mathcal{S}}$ and is isomorphic to $Q(\varprojlim \alpha)$ (resp. $Q(\varinjlim \alpha)$).*

(c) *Assume that $\mathcal{C}$ admits kernels (resp. cokernels). Then $\mathcal{C}_{\mathcal{S}}$ admits kernels (resp. cokernels) and Q commutes with kernels (resp. cokernels).*

(d) *Assume that $\mathcal{C}$ admits finite products (resp. coproducts). Then $\mathcal{C}_{\mathcal{S}}$ admits finite products (resp. coproducts) and Q commutes with finite products (resp. coproducts).*

(e) *If $\mathcal{C}$ admits finite projective (resp. inductive) limits, then so does $\mathcal{C}_{\mathcal{S}}$.*

Proposition 4.4.11. *Let $\mathcal{C}$ be a category, $\mathcal{I}$ be a full subcategory, $\mathcal{S}$ be a left (resp. right) multiplicative system in $\mathcal{C}$, and $\mathcal{T}$ be the family of morphisms in $\mathcal{I}$ which belong to $\mathcal{S}$.*

(a) *Assume that $\mathcal{T}$ is a left (resp. right) multiplicative system in $\mathcal{I}$. Then there is a well-defined functor $\mathcal{I}^l_{\mathcal{T}} \to \mathcal{C}^l_{\mathcal{S}}$ (resp. $\mathcal{I}^r_{\mathcal{T}} \to \mathcal{C}^r_{\mathcal{S}}$).*

(b) *Assume that for every $f : X \to Y$ in $\mathcal{S}$ with $Y \in \mathcal{I}$ (resp. $X \in \mathcal{I}$), there exist a morphism $g : W \to X$ with $W \in \mathcal{I}$ and $fg \in \mathcal{S}$ (resp. a morphism $g : Y \to W$ with $W \in \mathcal{I}$ and $gf \in \mathcal{S}$). Then $\mathcal{T}$ is a left (resp. right) multiplicative system and the functor $\mathcal{I}^l_{\mathcal{T}} \to \mathcal{C}^l_{\mathcal{S}}$ (resp. $\mathcal{I}^r_{\mathcal{T}} \to \mathcal{C}^r_{\mathcal{S}}$) is fully faithful.*

Proof. Assertion (a) is clear from the definition, and as for (b), it is easy to verify that $\mathcal{T}$ is a left multiplicative system under the corresonding assumption. For $X \in \mathcal{I}$, we define the category $\mathcal{T}_{/X}$ as the full subcategory of $\mathcal{S}_{/X}$ whose objects are morphisms $s : Y \to X$ with $Y \in \mathcal{I}$. The hypothesis in (b) then amounts to saying that the functor $\mathcal{T}_{/X} \to \mathcal{S}_{/X}$ is cofinal, so the result follows from **??**. □

Corollary 4.4.12. *Let $\mathcal{C}$ be a category, $\mathcal{I}$ a full subcategory, $\mathcal{S}$ be a left (resp. right) multiplicative system in $\mathcal{C}$, $\mathcal{T}$ the family of morphisms in $\mathcal{I}$ which belong to $\mathcal{S}$. Assume that for any $X \in \mathcal{C}$ there exists a morphism $s : I \to X$ with $I \in \mathcal{I}$ and $s \in \mathcal{S}$ (resp. a morphism $s : X \to I$ with $I \in \mathcal{I}$ and $s \in \mathcal{S}$). Then $\mathcal{T}$ is a left (resp. right) multiplicative system and $\mathcal{I}^l_{\mathcal{T}}$ (resp. $\mathcal{I}^r_{\mathcal{T}}$) is equivalent to $\mathcal{C}^l_{\mathcal{S}}$ (resp. $\mathcal{C}^r_{\mathcal{S}}$).*

Proof. The natural functor $\mathcal{I}^l_{\mathcal{T}} \to \mathcal{C}^l_{\mathcal{S}}$ (resp. $\mathcal{I}^r_{\mathcal{T}} \to \mathcal{C}^r_{\mathcal{S}}$) is fully faithful by Proposition 4.4.11, and essentially surjective by hypothesis. □

Theorem 4.4.13. *Let $\mathcal{C}$ be a pre-additive category and $\mathcal{S}$ be a left (resp. right) multiplicative system.*

(a) *The localization $\mathcal{C}^l_{\mathcal{S}}$ (resp. $\mathcal{C}^r_{\mathcal{S}}$) has a canonical structure of a pre-additive category, so that Q^l (resp. Q^r) is an additive functor.*

(b) *If $\mathcal{C}$ is additive and $\mathcal{S}$ be a multiplicative system, then $\mathcal{C}_{\mathcal{S}}$ is an additive category.*

The same result is true if we replace additive by k-linear, where k is a commutative ring.

Proof. As for (a), it suffices to consider right multiplicative systems. We now define an addition for the Hom set of $\mathcal{C}_{\mathcal{S}}$. If $f, g \in \mathrm{Hom}_{\mathcal{C}_{\mathcal{S}}}(X, Y)$, then, since $\mathcal{S}^{\mathrm{op}}_{/X}$ is filtrant, there exist $s : U \rightarrowtail X$ and $a_1, a_2 : U \to Y$ such that $f = [U; s, a_1]$ and $g = [U; s, a_2]$. We can therefore define $f + g$ by

$$f + g := [U; s, a_1 + a_2] \in \mathrm{Hom}_{\mathcal{C}_{\mathcal{S}}}(X, Y).$$

In particular, the zero morphism can be written as $[U; s, 0]$, and $-[U; s, a] = [U; s, -a]$. It is then a simple matter to show that this definition is independent of the choices of a_1 and a_2, which follows easily from the filtrant property of $\mathcal{S}^{\mathrm{op}}_{/X}$. Finally, with this definition, it is then easy to check that Q is an additive functor, and the second assertion follows from Proposition 4.4.10. □

4.4.2 Kan extensions along a localization

Let $\mathcal{C}$ be a category, $\mathcal{S}$ a (right, say) multiplicative system in $\mathcal{C}$ and $F : \mathcal{C} \to \mathcal{A}$ a functor. We consider the existence of the following factorization diagram:

$$\begin{tikzcd} \mathcal{C} \arrow[d, "Q"'] \arrow[dr, bend left, "F"] & \\ \mathcal{C}_{\mathcal{S}} \arrow[r, dashed, "\exists ?"] & \mathcal{A} \end{tikzcd}$$

In general, F does not send morphisms in $\mathcal{S}$ to isomorphisms in $\mathcal{A}$, so it does not factorize through $\mathcal{C}_{\mathcal{S}}$. It is however possible in some cases to define a localization of F as a "best approximation", in the following sense:

Definition 4.4.3. Let $\mathcal{S}$ be a family of morphisms in $\mathcal{C}$ and assume that the localization $Q : \mathcal{C} \to \mathcal{C}_{\mathcal{S}}$ exists.

(a) We say that F is **right localizable** if the left Kan extension $\mathrm{Lan}_Q F$ of F with respect to Q exists. In such a case, we say that $\mathrm{Lan}_Q F$ is a **right localization** of F and we denote it by $R_{\mathcal{S}}F$. In other words, the **right localization** of F is a functor $R_{\mathcal{S}}F : \mathcal{C}_{\mathcal{S}} \to \mathcal{A}$ together with a morphism of functors $\eta : F \to R_{\mathcal{S}}F \circ Q$ such that for any functor $G : \mathcal{C}_{\mathcal{S}} \to \mathcal{A}$, the map

$$\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}_{\mathcal{S}},\mathcal{A})}(R_{\mathcal{S}}F, G) \to \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathcal{A})}(F, G \circ Q)$$

is bijective (This map is given by composing with η).

(b) We say that F is universally right localizable if for any functor $K : \mathcal{A} \to \mathcal{B}$, the functor $K \circ F$ is localizable and $R_{\mathcal{S}}(K \circ F) \xrightarrow{\sim} K \circ R_{\mathcal{S}}F$.

We can similarly define left localizations of F by right Kan extensions, and consider universally left localizable functors. That is, the left localization of F is a functor $L_{\mathcal{S}}F : \mathcal{C}_{\mathcal{S}} \to \mathcal{A}$ together with a morphism $\varepsilon : L_{\mathcal{S}}F \circ Q \to F$ such that for any functor $G : \mathcal{C}_{\mathcal{S}} \to \mathcal{A}$, ε induces a bijection

$$\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}_{\mathcal{S}},\mathcal{A})}(G, L_{\mathcal{S}}F) \to \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathcal{A})}(G \circ Q, F).$$

One should be aware that even if F admits both a right and a left localization, the two localizations are not isomorphic in general. However, when the localization $Q : \mathcal{C} \to \mathcal{C}_{\mathcal{S}}$ exists and F is right and left localizable, the canonical morphisms of functors $L_{\mathcal{S}}F \circ Q \to F \to R_{\mathcal{S}}F \circ Q$ together with the isomorphism $\mathrm{Hom}(L_{\mathcal{S}}F \circ Q, R_{\mathcal{S}}F \circ Q) \cong \mathrm{Hom}(L_{\mathcal{S}}F, R_{\mathcal{S}}F)$ in (L3) gives a canonical morphism of functors $L_{\mathcal{S}}F \to R_{\mathcal{S}}F$. From now on, we shall concentrate on right localizations.

Proposition 4.4.14. *Let $\mathcal{C}$ be a category, $\mathcal{I}$ be a full subcategory, $\mathcal{S}$ be a left (resp. right) multiplicative system in $\mathcal{S}$, $\mathcal{T}$ be the family of morphisms in $\mathcal{I}$ which belong to $\mathcal{S}$. Let $F : \mathcal{C} \to \mathcal{A}$ be a functor. Assume that the following "resolution condition" is satisfied:*

(a) *for any $X \in \mathcal{C}$, there exists $s : I \to X$ (resp. $s : X \to I$) with $I \in \mathcal{I}$ and $s \in \mathcal{S}$;*

(b) *for any $t \in \mathcal{T}$, $F(t)$ is an isomorphism.*

Then F is universally left (resp. right) localizable and the composition

$$\mathcal{I} \longrightarrow \mathcal{C} \xrightarrow{Q} \mathcal{C}_{\mathcal{S}} \xrightarrow{L_{\mathcal{S}}F \text{ or } R_{\mathcal{S}}F} \mathcal{A}$$

is isomorphic to the restriction of F to $\mathcal{I}$. Moreover, we have canonical isomorphisms

$$(L_{\mathcal{S}}F)(Q(X)) \xrightarrow{\sim} \varprojlim_{[Y \rightarrowtail X] \in \mathrm{Ob}(\mathcal{S}^{\mathrm{op}}_{/X})} F(Y), \tag{4.4.6}$$

$$(R_{\mathcal{S}}F)(Q(X)) \xrightarrow{\sim} \varinjlim_{[X\to Y]\in \mathrm{Ob}(\mathcal{S}_{X/})} F(Y). \tag{4.4.7}$$

and the morphism $\varepsilon : L_{\mathcal{S}}F \circ Q \to F$ *(resp.* $\eta : F \to R_{\mathcal{S}}F \circ Q$*) is given by projecting to the term* $F(X)$ *corresponding to the identity morphism* $\mathrm{id}_X \in \mathrm{Ob}(\mathcal{S}^{\mathrm{op}}_{/X})$ *(resp.* $\mathrm{id}_X \in \mathrm{Ob}(\mathcal{S}_{X/})$*).*

Proof. It suffices to consider right multiplicative systems. Denote by $\iota : \mathcal{I} \to \mathcal{C}$ the natural functor. By condition (a) and Corollary 4.4.12, $\iota_Q : \mathcal{I}_{\mathcal{T}} \to \mathcal{C}_{\mathcal{S}}$ is an equivalence, and condition (b) implies that the localization $F_{\mathcal{T}}$ of $F \circ \iota$ exists. We consider the solid diagram

$$\begin{tikzcd}
 & \mathcal{C} \arrow[dr, "Q_{\mathcal{S}}"] \arrow[drr, bend left, "F"] & & \\
\mathcal{I} \arrow[ur, "\iota"] \arrow[dr, "Q_{\mathcal{T}}"'] & & \mathcal{C}_{\mathcal{S}} \arrow[r, dashed, "RF"] & \mathcal{A} \\
 & \mathcal{I}_{\mathcal{T}} \arrow[ur, "\iota_Q"] \arrow[urr, bend right, "F_{\mathcal{T}}"'] & &
\end{tikzcd}$$

Denote by ι_Q^{-1} a quasi-inverse of ι_Q and set $RF = F_{\mathcal{T}} \circ \iota_Q^{-1}$. Then the above diagram commutes, except the triangle $(\mathcal{C}, \mathcal{C}_{\mathcal{S}}, \mathcal{A})$. We now prove that RF is the right localization of F.

Let $G : \mathcal{C}_{\mathcal{S}} \to \mathcal{A}$ be a functor; we have the chain of a morphism and isomorphisms:

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathcal{A})}(F, G\circ Q_{\mathcal{S}}) &\xrightarrow{\lambda} \mathrm{Hom}_{\mathrm{Fun}(\mathcal{I},\mathcal{A})}(F\circ\iota, G\circ Q_{\mathcal{S}}\circ\iota) \\
&\cong \mathrm{Hom}_{\mathrm{Fun}(\mathcal{I},\mathcal{A})}(F_{\mathcal{T}}\circ Q_{\mathcal{T}}, G\circ \iota_Q\circ Q_{\mathcal{T}}) \\
&\cong \mathrm{Hom}_{\mathrm{Fun}(\mathcal{I}_{\mathcal{T}},\mathcal{A})}(F_{\mathcal{T}}, G\circ\iota_Q) \\
&\cong \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}_{\mathcal{S}},\mathcal{A})}(F_{\mathcal{T}}\circ\iota_Q^{-1}, G) \\
&\cong \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}_{\mathcal{S}},\mathcal{A})}(RF, G).
\end{aligned} \tag{4.4.8}$$

The second isomorphism follows from the fact that $Q_{\mathcal{T}}$ satisfies axiom (L3). To conclude, it remains to prove that the morphism λ is bijective. Let us check that Lemma 4.4.5 applies to $\iota : \mathcal{I} \to \mathcal{C}$ and $Q_{\mathcal{S}} : \mathcal{C} \to \mathcal{C}_{\mathcal{S}}$, and hence to $\iota : \mathcal{I} \to \mathcal{C}$ and $G \circ Q_{\mathcal{S}} : \mathcal{C} \to \mathcal{A}$. Let $X \in \mathrm{Ob}(\mathcal{C})$; by hypothesis, there exists $Y \in \mathcal{I}$ and $s : X \to \iota(Y)$ with $s \in \mathcal{S}$. Then $F(s)$ is an isomorphism and condition (a) of Lemma 4.4.5 is satisfied. On the other hand, condition (b) follows from axiom (S3′) and the fact that ι is fully faithful. Finally, to see that the limit of (4.4.7) exists, we can assume that $X \in \mathrm{Ob}(\mathcal{I})$, but the limit is then isomorphic to $F(X)$, since id_X is initial in $\mathrm{Ob}(\mathcal{S}_{X/})$. In view of the general construction of $\mathrm{Lan}_Q F$ and Lemma 4.4.9, it follows that $R_{\mathcal{S}}F$ is isomorphic to the limit in (4.4.7).

If $K : \mathcal{A} \to \mathcal{A}'$ is another functor, $K \circ F(t)$ will be an isomorphism for any $t \in \mathcal{T}$. Hence, $K \circ F$ is localizable and we have

$$R_{\mathcal{S}}(K\circ F) \cong (K\circ F)_{\mathcal{T}} \circ \iota_Q^{-1} \cong K \circ F_{\mathcal{T}} \circ \iota_Q^{-1} \cong K \circ R_{\mathcal{S}}F. \qquad \square$$

Corollary 4.4.15. *Let* $\mathcal{A}$ *be a category which admits small filtrant inductive limits. Let* $\mathcal{S}$ *be a left (resp. right) multiplicative system and assume that for each* $X \in \mathcal{C}$*, the category* $\mathcal{S}^{\mathrm{op}}_{/X}$ *(resp.* $\mathcal{S}_{X/}$*) is cofinally small.*

(a) $\mathcal{C}_{\mathcal{S}}$ *is a* $\mathcal{U}$*-category.*

(b) *The functor* $Q^\star$ *admits a right adjoint* ${}_\star Q$ *(resp. left adjoint functor* $Q_\star$*).*

(c) *Any functor* $F : \mathcal{C} \to \mathcal{A}$ *is left (resp. right) localizable and*

$$(L_{\mathcal{S}}F)(Q(X)) \xrightarrow{\sim} \varprojlim_{[Y\to X]\in\mathrm{Ob}(\mathcal{S}^{\mathrm{op}}_{/X})} F(Y), \quad (R_{\mathcal{S}}F)(Q(X)) \xrightarrow{\sim} \varinjlim_{[X\to Y]\in\mathrm{Ob}(\mathcal{S}_{X/})} F(Y).$$

Proof. Assertion (a) is obvious and (b), (c) follow from Lemma 4.4.9, since we may apply Proposition 4.4.14 to construct ${}_\star Q$ (resp. $Q_\star$). $\square$

4.4.3 Triangulated categories

Triangulated categories are additive categories with a collection of distingushied triangles. They arise naturally from the derived category of an abelian category and is important for the study of properties of derived categories. To begin with, we first consider categories with a translation functor.

Definition 4.4.4. A **category with translation** $(\mathcal{D}, T)$ is a category $\mathcal{D}$ endowed with an equivalence of categories $T : \mathcal{D} \to \mathcal{D}$. The functor T is called the **translation functor**.

- A functor of categories with translation $F : (\mathcal{D}, T) \to (\mathcal{D}', T')$ is a functor $F : \mathcal{D} \to \mathcal{D}'$ together with an isomorphism $F \circ T \cong T' \circ F$. If $\mathcal{D}$ and $\mathcal{D}'$ are additive categories and F is additive, we say that F is a functor of additive categories with translation.

- Let $F, F' : (\mathcal{D}, T) \to (\mathcal{D}', T')$ be two functors of categories with translation. A morphism $\theta : F \to F'$ of functors of categories with translation is a morphism of functors such that the diagram below commutes

$$\begin{tikzcd} F \circ T \arrow[r, "\theta \circ T"] \arrow[d, "\sim"'] & F' \circ T \arrow[d, "\sim"] \\ T' \circ F \arrow[r, "T' \circ \theta"] & T' \circ F' \end{tikzcd}$$

- A subcategory with translation $(\mathcal{D}', T')$ of $(\mathcal{D}, T)$ is a category with translation such that $\mathcal{D}'$ is a subcategory of $\mathcal{D}$ and the translation functor T' is the restriction of T.

- Let $(\mathcal{D}, T)$, $(\mathcal{D}', T')$ and $(\mathcal{D}'', T'')$ be additive categories with translation. A bifunctor of additive categories with translation $F : \mathcal{D} \times \mathcal{D}' \to \mathcal{D}''$ is an additive bifunctor endowed with functorial isomorphisms

$$\theta_{X,Y} : F(T(X), Y) \xrightarrow{\sim} T''(F(X, Y)), \quad \lambda_{X,Y} : F(X, T'(Y)) \xrightarrow{\sim} T''(F(X, Y))$$

for $(X, Y) \in \mathcal{D} \times \mathcal{D}'$ such that the diagram below anti-commutes:

$$\begin{tikzcd} F(T(X), T'(Y)) \arrow[r, "\theta_{X,T'(Y)}"] \arrow[d, "\lambda_{T(X),Y}"'] & T''(F(X, T'(Y))) \arrow[d, "T''(\lambda_{X,Y})"] \\ T''(F(T(X), Y)) \arrow[r, "T''(\theta_{X,Y})"] & T''^2(F(X, Y)) \end{tikzcd}$$

If $(\mathcal{D}, T)$ is a category with translation, we shall denote by T^{-1} a quasi-inverse of T. Then T^n is well defined for $n \in \mathbb{Z}$. These functors are unique up to unique isomorphism. If there is no risk of confusion, we shall write $\mathcal{D}$ instead of $(\mathcal{D}, T)$ and $X[1]$ (resp. $X[-1]$) instead of $T(X)$ (resp. $T^{-1}(X)$).

Example 4.4.1. Let $\mathcal{A}$ be an additive category and $\mathrm{Ch}(A)$ be the category of chain complexes of $\mathcal{A}$. Then we have the shift functor $T : X \mapsto X[1]$ defined by $X[1]^n = X^{n+1}$ and $d[1]^n = -d^{n+1}$, so $(\mathrm{Ch}(\mathcal{A}), T)$ is an additive category with translation.

Definition 4.4.5. Let $(\mathcal{D}, T)$ be an additive category with translations. A **triangle** in $\mathcal{D}$ is a sequence of morphisms

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

A morphism of triangles is a commutative diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\alpha"] & Y \arrow[r, "g"] \arrow[d, "\beta"] & Z \arrow[r, "h"] \arrow[d, "\gamma"] & X[1] \arrow[d, "\alpha[1]"] \\
X' \arrow[r, "f'"] & Y' \arrow[r, "g'"] & Z' \arrow[r, "h'"] & X'[1]
\end{tikzcd}$$

Remark 4.4.5. Let $(\mathcal{D}, T)$ be a k-linear category with translations and

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

be a triangle. Let $\varepsilon, \zeta, \eta \in k^{\times}$. If $\varepsilon\zeta\eta = 1$, then the original triangle is isomorphic to the following:

$$X \xrightarrow{\varepsilon f} Y \xrightarrow{\zeta g} Z \xrightarrow{\eta h} X[1]$$

In fact, we have a commutative diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r, "g"] \arrow[d, "\varepsilon"] & Z \arrow[r, "h"] \arrow[d, "\varepsilon\zeta"] & X[1] \arrow[d, "\varepsilon\zeta\eta"] \\
X \arrow[r, "\varepsilon f"] & Y \arrow[r, "\zeta g"] & Z \arrow[r, "\eta h"] & X[1]
\end{tikzcd}$$

Definition 4.4.6. A **triangulated category** is an additive category $(\mathcal{D}, T)$ with translation endowed with a family of triangles, called **distinguished triangles**, satisfying the axioms below:

(TR0) A triangle isomorphic to a distinguished triangle is a distinguished triangle.

(TR1) The triangle $X \xrightarrow{\mathrm{id}_X} X \to 0 \to X[1]$ is a distinguished triangle.

(TR2) For any morphism $f : X \to Y$, there exists a distinguished triangle $X \xrightarrow{f} Y \to Z \to X[1]$.

(TR3) A triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ is a distinguished triangle if and only if its "rotation"

$$Y \xrightarrow{-g} Z \xrightarrow{-h} X[1] \xrightarrow{-f[1]} Y[1]$$

is a distinguished triangle.

(TR4) Given a solid diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "\alpha"] & Y \arrow[r] \arrow[d, "\beta"] & Z \arrow[r] \arrow[d, dashed, "\gamma"] & X[1] \arrow[d, dashed, "\alpha[1]"] \\
X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & X'[1]
\end{tikzcd}$$

with both rows being distinguished triangles, there exists a morphism $\gamma : Z \to Z'$ giving rise to a morphisms of distinguished triangles.

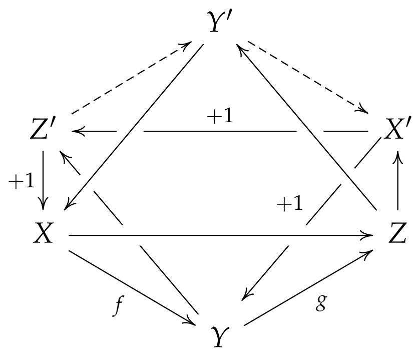
(TR5) Given three distinguished triangles

$$X \xrightarrow{f} Y \to Z' \to X[1],$$
$$Y \xrightarrow{g} Z \to X' \to Y[1],$$
$$X \xrightarrow{gf} Z \to Y' \to X[1],$$

there exists a distinguished triangle $Z' \to Y' \to X' \to Z'[1]$ making the following diagram commutative:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r] \arrow[d, "g"] & Z' \arrow[r] \arrow[d, dashed] & X[1] \arrow[d, equal] \\
X \arrow[r, "gf"] \arrow[d, "f"] & Z \arrow[r] \arrow[d, equal] & Y' \arrow[r] \arrow[d, dashed] & X[1] \arrow[d] \\
Y \arrow[r, "g"] \arrow[d] & Z \arrow[r] \arrow[d] & X' \arrow[r] \arrow[d, equal] & Y[1] \arrow[d] \\
Z' \arrow[r, dashed] & Y' \arrow[r, dashed] & X' \arrow[r, dashed] & Z'[1]
\end{tikzcd} \tag{4.4.9}$$

Diagram (4.4.9) is often called the **octahedron diagram**. Indeed, it can be written using the vertices of an octahedron:



Here we use $X' \overset{+1}{\to} Y$ to denote a morphism $X' \to Y[1]$.

An additive category $(\mathcal{D}, T)$ satisfying (TR0)–(TR4) is called a **pretriangulated category**. One should note that the morphism γ in (TR4) is not unique, and is unique up to *non-unique* isomorphisms.

Definition 4.4.7. A **triangulated functor** of triangulated categories $F : (\mathcal{D}, T) \to (\mathcal{D}', T')$ is a functor of additive categories with translation sending distinguished triangles to distinguished triangles. If moreover F is an equivalence of categories, F is called an **equivalence of triangulated categories**. If $F, F' : (\mathcal{D}, T) \to (\mathcal{D}', T')$ are triangulated functors, a morphism $\theta : F \to F'$ of triangulated functors is a morphism of functors of additive categories with translation.

A triangulated subcategory $(\mathcal{D}', T')$ of $(\mathcal{D}, T)$ is an additive subcategory with translation of $\mathcal{D}$ (i.e., the functor T' is the restriction of T) such that it is triangulated and that the inclusion functor is triangulated.

Remark 4.4.6. A triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ is called **anti-distinguished** if the triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{-h} X[1]$ is distinguished. Then $(\mathcal{D}, T)$ endowed with the family of anti-distinguished triangles is triangulated. If we denote by $(\mathcal{D}^{ant}, T)$ this triangulated category, then $(\mathcal{D}^{ant}, T)$ and $(\mathcal{D}, T)$ are equivalent as triangulated categories.

Remark 4.4.7. Consider the contravariant functor op : $\mathcal{D} \to \mathcal{D}^{\mathrm{op}}$, and define

$$T^{\mathrm{op}} = \mathrm{op} \circ T^{-1} \circ \mathrm{op} : \mathcal{D}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$$

(we use the fact that $\mathrm{op}^2 = \mathrm{id}_{\mathcal{D}}$.) A triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} T^{\mathrm{op}}(X)$ in $\mathcal{D}^{\mathrm{op}}$ is called distinguished if its image

$$Z^{\mathrm{op}} \xrightarrow{g^{\mathrm{op}}} Y^{\mathrm{op}} \xrightarrow{f^{\mathrm{op}}} X^{\mathrm{op}} \xrightarrow{T(h^{\mathrm{op}})} T(Z^{\mathrm{op}})$$

by op is distinguished. With this definition, it is easy to check that $(\mathcal{D}^{\mathrm{op}}, T^{\mathrm{op}})$ is a triangulated category.

Proposition 4.4.16. *If $X \xrightarrow{f} Y \xrightarrow{g} Z \to X[1]$ is a distinguished triangle, then $gf = 0$.*

Proof. Applying (TR1) and (TR4) we get a commutative diagram

$$\begin{tikzcd}
X \arrow[r, equal] \arrow[d, equal] & X \arrow[r] \arrow[d, "f"] & 0 \arrow[r] \arrow[d] & X[1] \arrow[d, equal] \\
X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r] & X[1]
\end{tikzcd}$$

Then gf factorizes through 0. □

Definition 4.4.8. Let $(\mathcal{D}, T)$ be a pretriangulated category and $\mathcal{C}$ an abelian category. An additive functor $F : \mathcal{D} \to \mathcal{C}$ is called **cohomological** if for any distinguished triangle $X \to Y \to Z \to X[1]$ in $\mathcal{D}$, the sequence $F(X) \to F(Y) \to F(Z)$ is exact in $\mathcal{C}$.

If F is a cohomological functor $F : \mathcal{D} \to \mathcal{C}$, then for any distinguished triangle $X \to Y \to Z \to X[1]$ in $\mathcal{D}$, by rotating the triangle by (TR3), we obtain a long exact sequence

$$\cdots \longrightarrow F(Z[-1]) \longrightarrow F(X) \longrightarrow F(Y) \longrightarrow F(Z) \longrightarrow F(X[1]) \longrightarrow \cdots$$

A basic example of cohomological functors is the Hom functor:

Proposition 4.4.17. *Let $(\mathcal{D}, T)$ be a pretriangulated category and S be an object of $\mathcal{D}$. Then the functors $\mathrm{Hom}_{\mathcal{D}}(S, -)$ and $\mathrm{Hom}_{\mathcal{D}}(-, S)$ are cohomological.*

Proof. Let $X \to Y \to Z \to X[1]$ be a distinguished triangle. We want to show that

$$\mathrm{Hom}(S, X) \xrightarrow{f_*} \mathrm{Hom}(S, Y) \longrightarrow \mathrm{Hom}(S, Z)$$

is exact, i.e. for any morphism $\varphi : S \to Y$ such that $g \circ \varphi = 0$, there exists a morphism $\psi : S \to X$ such that $\varphi = f \circ \psi$. This is equivalent to say that the solid diagram below may be completed:

$$\begin{tikzcd}
S \arrow[r, equal] \arrow[d, dashed] & S \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & S[1] \arrow[d, dashed] \\
X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r] & X[1]
\end{tikzcd}$$

and this follows from (TR4) and (TR3). By replacing $\mathcal{D}$ with $\mathcal{D}^{\mathrm{op}}$, we obtain the assertion for $\mathrm{Hom}_{\mathcal{D}}(-, S)$. □

Corollary 4.4.18. *For a distinguished triangle $X \xrightarrow{f} Y \to 0 \to X[1]$ in a pretriangulated category, f must be an isomorphism.*

Proof. For every object S of $\mathcal{D}$, by Proposition 4.4.17 we have an exact sequence

$$\mathrm{Hom}(S,0[-1]) = 0 \longrightarrow \mathrm{Hom}(S,X) \xrightarrow{f_*} \mathrm{Hom}(S,Y) \longrightarrow \mathrm{Hom}(S,0) = 0$$

so f_* is an isomorphism, which means f is an isomorphism. $\square$

Proposition 4.4.19. *Let $(\mathcal{D},T)$ be a pretriangulated category and consider a morphism of distinguished triangle:*

$$\begin{tikzcd} X \arrow[r] \arrow[d, "\alpha"] & Y \arrow[r] \arrow[d, "\beta"] & Z \arrow[r] \arrow[d, dashed, "\gamma"] & X[1] \arrow[d, dashed, "\alpha[1]"] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & X'[1] \end{tikzcd}$$

If two of α,β,γ are isomorphisms, then so is the third one.

Proof. By rotating the triangle, we may assume that α,γ are isomorphisms. To show that β is an isomorphism, it suffices to show that for any object S of $\mathcal{D}$, the map $\beta_* : \mathrm{Hom}(S,Y) \to \mathrm{Hom}(S,Y')$ is an isomorphism. Now by Proposition 4.4.17 we have a commutative diagram with exact rows:

$$\begin{tikzcd} \mathrm{Hom}(S,Z[-1]) \arrow[r] \arrow[d, "\sim"] & \mathrm{Hom}(S,X) \arrow[r] \arrow[d, "\sim"] & \mathrm{Hom}(S,Y) \arrow[r] \arrow[d, "\beta_*"] & \mathrm{Hom}(S,Z) \arrow[r] \arrow[d, "\sim"] & \mathrm{Hom}(S,X[1]) \arrow[d, "\sim"] \\ \mathrm{Hom}(S,Z'[-1]) \arrow[r] & \mathrm{Hom}(S,X') \arrow[r] & \mathrm{Hom}(S,Y') \arrow[r] & \mathrm{Hom}(S,Z') \arrow[r] & \mathrm{Hom}(S,X'[1]) \end{tikzcd}$$

so the claim follows from five lemma. $\square$

Corollary 4.4.20. *Let $\mathcal{D}'$ be a full pretriangulated subcategory of $\mathcal{D}$.*

(a) *Consider a triangle $X \xrightarrow{f} Y \to Z \to X[1]$ in $\mathcal{D}'$ and assume that this triangle is distinguished in $\mathcal{D}$. Then it is distinguished in $\mathcal{D}'$.*

(b) *Consider a distinguished triangle $X \to Y \to Z \to X[1]$ in $\mathcal{D}$ with X,Y in $\mathcal{D}'$. Then Z is isomorphic to an object of $\mathcal{D}'$.*

Proof. In the situation of (a), there exists a distinguished triangle $X \xrightarrow{f} Y \to Z' \to X[1]$ in $\mathcal{D}'$, and $X \xrightarrow{f} Y \to Z \to X[1]$ is isomorphic to it in $\mathcal{D}$ in view of axiom (TR4) and Proposition 4.4.19. The second assertion follows from (a). $\square$

Corollary 4.4.21. *The distinguished triangle $X \xrightarrow{f} Y \to Z \to X[1]$ in (TR2) is unique up to (non-canonical) isomorphisms.*

Proof. For distinguished triangles $X \xrightarrow{f} Y \to Z \to X[1]$ and $X \xrightarrow{f} Y \to Z \to X[1]$, axiom (TR4) gives a morphism γ such that the diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, equal] & Y \arrow[r] \arrow[d, equal] & Z \arrow[r] \arrow[d, dashed, "\gamma"] & X[1] \arrow[d, equal] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & X'[1] \end{tikzcd}$$

is commutative. It then suffices to apply Proposition 4.4.19. $\square$

By Corollary 4.4.21, we see that the object Z given in (TR2) is unique up to isomorphism. As already mentioned, the fact that this isomorphism is not unique is the source of many difficulties (e.g., gluing problems in sheaf theory). Let us give a criterion which ensures, in some very special cases, the uniqueness of the third term of a distinguished triangle.

Proposition 4.4.22. *In the situation of (TR4), assume that* $\mathrm{Hom}_{\mathcal{D}}(Y, X') = 0$ *and* $\mathrm{Hom}_{\mathcal{D}}(X[1], Y') = 0$. *Then* γ *is unique.*

Proof. We may replace α and β by the zero morphisms and prove that in this case, γ is zero:

$$\begin{tikzcd} X \arrow[r] \arrow[d, "0"] & Y \arrow[r, "f"] \arrow[d, "0"] & Z \arrow[r, "g"] \arrow[d, "\gamma"] & X[1] \arrow[d, "0"] \\ X' \arrow[r, "f'"] & Y' \arrow[r, "g'"] & Z' \arrow[r, "h'"] & X'[1] \end{tikzcd}$$

Since $h'\gamma = 0$, by Proposition 4.4.17 the morphism γ factorizes through g', i.e. there exists $u : Z \to Y'$ with $\gamma = g' \circ u$. Similarly, since $\gamma g = 0$, γ factorizes through h so there exists $v : X[1] \to Z'$ with $\gamma = vh$. By (TR4), there then exists a morphism $w : Y[1] \to X'[1]$ defining a morphism

$$\begin{tikzcd} Y \arrow[r, "g"] & Z \arrow[r, "h"] \arrow[dl, "u"'] & X[1] \arrow[r, "-f[1]"] \arrow[dl, "v"'] & Y[1] \arrow[dl, "w"'] \\ Y' \arrow[r, "g'"] & Z' \arrow[r, "h'"] & X'[1] \arrow[r] & Y'[1] \end{tikzcd}$$

By hypothesis we have $w = 0$, so v factorizes through Y', and this implies $v = 0$ by our hypothesis, whence $\gamma = 0$. $\square$

Proposition 4.4.23. *Let* $F : \mathcal{D} \to \mathcal{D}'$ *be a triangulated functor between pretriangulated categories. Then* F *is exact.*

Proof. Since $F^{\mathrm{op}} : \mathcal{D}'^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$ is also a triangulated functor between pretriangulated categories, it suffices to prove that F is left exact, that is, for any $X \in \mathcal{D}'$, the category $\mathcal{D}_{/X}$ is filtrant.

The category $\mathcal{D}_{/X}$ is nonempty since it contains the object $0 \to X$, and if (Y_1, s_1) and (Y_2, s_2) are two objects of $\mathcal{D}_{/X}$ with $Y_i \in \mathcal{D}$ and $s_i : F(Y_i) \to X$, $i = 1, 2$, we obtain a morphism $s : F(Y_1 \oplus Y_2) \to X$, whence morphisms $(Y_i, s_i) \to (Y_1 \oplus Y_2, s)$ for $i = 1, 2$. Finally, consider morphisms $f, g : (Y, s) \rightrightarrows (Y', s')$ in $\mathcal{D}_{/X}$. We can embed $f - g : Y \to Y'$ into a distinguished triangle

$$Y \xrightarrow{f-g} Y' \xrightarrow{h} Z \longrightarrow Y[1]$$

Since $s'F(f) = s'F(g)$, Proposition 4.4.17 implies that the morphism $s' : F(Y') \to X$ factorizes as $F(Y') \to F(Z) \xrightarrow{t} X$, so the compositions $(Y, s) \rightrightarrows (Y', s') \to (Z, t)$ coincide, and this proves that $\mathcal{D}_{/X}$ is filtrant. $\square$

Proposition 4.4.24. *Let* $\mathcal{D}$ *be a pretriangulated category which admits direct sums (resp. products) indexed by a set* I. *Then direct sums indexed by* I *commute with the translation functor* T, *and a direct sum (resp. products) of distinguished triangles indexed by* I *is a distinguished triangle.*

Proof. The first assertion is obvious since T is an equivalence of categories. Now let $D_i : X_i \to Y_i \to Z_i \to X_i[1]$ be a family of distinguished triangles indexed by I, and D be the triangle

$$\bigoplus_i D_i : \bigoplus_i X_i \to \bigoplus_i Y_i \to \bigoplus_i Z_i \to \bigoplus_i X_i[1].$$

By (TR2) there exists a distinguished triangle $D' : \bigoplus_i X_i \to \bigoplus_i Y_i \to Z \to (\bigoplus_i X_i)[1]$, and by (TR3) there exists morphisms of triangles $D_i \to D'$ such that they induces a morphism $D \to D'$. Let $S \in \mathcal{D}$, we show that the morphism $\mathrm{Hom}_{\mathcal{D}}(D', S) \to \mathrm{Hom}_{\mathcal{D}}(D, S)$ is an isomorphism, which then implies the isomorphism $D \cong D'$. Consider the commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}((\bigoplus_i Y_i)[1], S) \arrow[r] \arrow[d] & \mathrm{Hom}((\bigoplus_i X_i)[1], S) \arrow[r] \arrow[d] & \mathrm{Hom}(Z, S) \arrow[r] \arrow[d] & \mathrm{Hom}(\bigoplus_i Y_i, S) \arrow[r] \arrow[d] & \mathrm{Hom}(\bigoplus_i X_i, S) \arrow[d] \\
\mathrm{Hom}(\bigoplus_i Y_i[1], S) \arrow[r] & \mathrm{Hom}(\bigoplus_i X_i[1], S) \arrow[r] & \mathrm{Hom}(\bigoplus_i Z_i, S) \arrow[r] & \mathrm{Hom}(\bigoplus_i Y_i, S) \arrow[r] & \mathrm{Hom}(\bigoplus_i X_i, S)
\end{tikzcd}$$

The first row is exact since the functor Hom is cohomological, and the second row is isomorphic to

$$\prod_i \mathrm{Hom}(Y_i[1], S) \to \prod_i \mathrm{Hom}(X_i[1], S) \to \prod_i \mathrm{Hom}(Z_i, S) \to \prod_i \mathrm{Hom}(Y_i, S) \to \prod_i \mathrm{Hom}(X_i, S)$$

Since the functor $\prod_i$ is exact on **Ab**, this complex is exact. Now the vertical arrows except the middle one are isomorphisms, so the middle one is an isomorphism by five lemma. □

Corollary 4.4.25. *Let $\mathcal{D}$ be a pretriangulated category. Then a triangle of the form $X \xrightarrow{\iota_1} X \oplus Y \xrightarrow{p_2} Y \xrightarrow{0} X[1]$ is distinguished. Conversely, if a morphisn in a distinguished triangle is zero, then this triangle comes from a direct sum.*

Proof. To prove the first assertion, it suffices to apply Proposition 4.4.24 to the distinguished triangles $X \xrightarrow{\mathrm{id}_X} X \to 0 \to X[1]$ and $0 \to Y \xrightarrow{\mathrm{id}_Y} Y \to 0$. Now consider the second assertion; by rotating the triangle, it suffices to consider a distinguished triangle of the form

$$X \to M \to Y \xrightarrow{0} X[1].$$

By (TR4), we then obtain a morphism of distinguished triangles:

$$\begin{tikzcd}
X \arrow[r, "\iota_1"] \arrow[d, equal] & X \oplus Y \arrow[r, "p_2"] \arrow[d, "\alpha"] & Y \arrow[r, "0"] \arrow[d, equal] & X[1] \arrow[d, equal] \\
X \arrow[r] & M \arrow[r] & Y \arrow[r, "0"] & X[1]
\end{tikzcd}$$

and it follows from Proposition 4.4.19 that α is an isomorphism. □

Corollary 4.4.26. *Let $F : (\mathcal{D}, T) \to (\mathcal{D}', T')$ be a functor between pretriangulated categories such that F sends distinguished triangles to distinguished triangles. Then F is additive, so it is a triangulated functor.*

Proof. From the distinguished triangle $0 \to 0 \to 0 \to 0$, we obtain a distinguished triangle $F(0) \xrightarrow{\mathrm{id}} F(0) \xrightarrow{\mathrm{id}} F(0) \xrightarrow{\mathrm{id}} F(0)$ in $\mathcal{D}'$, so Proposition 4.4.16 implies that $\mathrm{id}_{F(0)} = \mathrm{id}_{F(0)} \circ \mathrm{id}_{F(0)} = 0$, and therefore $F(0) \cong 0$. This also shows that F sends zero morphisms to zero morphisms.

Now consider a distinguished triangle $X \xrightarrow{\iota_1} X \oplus Y \xrightarrow{p_2} Y \xrightarrow{0} X[1]$ in $\mathcal{D}$. By applying F, we obtain a distinguished triangle

$$F(X) \xrightarrow{F(\iota_1)} F(X \oplus Y) \longrightarrow F(Y) \xrightarrow{0} T(F(X))$$

in $\mathcal{D}'$. From Corollary 4.4.25 and its proof, it is easy to see that the canonical morphism $F(X) \oplus F(Y) \to F(X \oplus Y)$ is an isomorphism, so F is additive. □

Proposition 4.4.27 (Verdier's Exercise). *Let $\mathcal{D}$ be a triangulated category. Then any commutative diagram*

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ X' \arrow[r] & Y' \end{tikzcd}$$

can be extended into a diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & X[1] \arrow[d] \\ X' \arrow[r] \arrow[d] & Y' \arrow[r] \arrow[d] & Z' \arrow[r] \arrow[d] & X'[1] \arrow[d] \\ X'' \arrow[r] \arrow[d] & Y'' \arrow[r] \arrow[d] & Z'' \arrow[r] \arrow[d] \arrow[dr, phantom, "*"] & X''[1] \arrow[d] \\ X[1] \arrow[r] & Y[1] \arrow[r] & Z[1] \arrow[r] & X[2] \end{tikzcd}$$

so that every rows and columns are distinguished triangles and every square is commutative except the one labeled by $*$, which is anti-commutative.

4.4.4 Verdier quotient

Let $\mathcal{D}$ be a triangulated category and $\mathcal{N}$ a full saturated subcategory. Recall that $\mathcal{N}$ is saturated if $X \in \mathcal{D}$ belongs to $\mathcal{N}$ whenever X is isomorphic to an object of $\mathcal{N}$.

Lemma 4.4.28. *Let $\mathcal{N}$ be a full saturated triangulated subcategory of $\mathcal{D}$. Then $\mathrm{Ob}(\mathcal{N})$ satisfies the following conditions:*

(N1) $0 \in \mathcal{N}$.

(N2) $X \in \mathcal{N}$ *if and only if* $X[1] \in \mathcal{N}$.

(N3) *If $X \to Y \to Z \to X[1]$ is a distinguished triangle in $\mathcal{D}$ and $X, Z \in \mathcal{N}$, then $Y \in \mathcal{N}$.*

Conversely, let $\mathcal{N}$ be a full saturated subcategory of $\mathcal{D}$ and assume that $\mathrm{Ob}(\mathcal{N})$ satisfies conditions (N1)–(N3) above. Then the restriction of T and the collection of distinguished triangles $X \to Y \to Z \to X[1]$ with X, Y, Z in $\mathcal{N}$ make $\mathcal{N}$ a full saturated triangulated subcategory of $\mathcal{D}$. Moreover it satisfies

(N3') *If $X \to Y \to Z \to X[1]$ is a distinguished triangle in $\mathcal{D}$ and two objects among X, Y, Z belong to $\mathcal{N}$, then so does the third one.*

Proof. Assume that $\mathcal{N}$ is a full saturated triangulated subcategory of $\mathcal{D}$. Then (N1) and (N2) are clearly satisfied. Moreover, (N3) follows from Corollary 4.4.20 and the hypothesis that $\mathcal{N}$ is saturated.

Conversely, let $\mathcal{N}$ be a full subcategory of $\mathcal{D}$ satisfying (N1)–(N3); then (N3') follows from (N2) and (N3) by rotating the triangle. We now show that $\mathcal{N}$ is saturated, so let $f : X \to Y$ be an isomorphism with $X \in \mathcal{N}$. The triangle $X \xrightarrow{f} Y \to 0 \to X[1]$ is then isomorphic to the distinguished triangle $X \xrightarrow{\mathrm{id}_X} X \to 0 \to X[1]$:

$$\begin{tikzcd} X \arrow[r, "\mathrm{id}_X"] \arrow[d, equal] & X \arrow[r] \arrow[d, "f", "\sim"'] & 0 \arrow[r] \arrow[d, equal] & X[1] \arrow[d, equal] \\ X \arrow[r, "f"] & Y \arrow[r] & 0 \arrow[r] & X[1] \end{tikzcd}$$

and hence is distinguished. It then follows from (N3) that $Y \in \mathcal{N}$. On the other hand, since $X \to X \oplus Y \to Y \xrightarrow{0} X[1]$ is a distinguished triangle for $X, Y \in \mathcal{N}$, we find that $X \oplus Y \in \mathcal{N}$, so $\mathcal{N}$ is a full additive subcategory of $\mathcal{D}$. The axioms of triangulated categories are then easily checked. □

A **null system** in $\mathcal{D}$ is a full saturated subcategory N such that $\mathrm{Ob}(\mathcal{N})$ satisfies the conditions (N1)–(N3) in Lemma 4.4.28. By Lemma 4.4.28, $\mathcal{N}$ can be then considered as a triangulated subcategory of $\mathcal{D}$. We associate a family of morphisms to a null system as follows:

$$\mathcal{N}Q = \{f : X \to Y : \text{there exists a distinguished triangle } X \to Y \to Z \to X[1] \text{ with } Z \in \mathcal{N}\}. \tag{4.4.10}$$

The morphisms in $\mathcal{N}Q$ turn out to form a multiplicative system of $\mathcal{C}$ that is compatible with the distinguished triangles in $\mathcal{D}$. To make this precise, we introduce the following definition:

Definition 4.4.9. Let $\mathcal{S}$ be a multiplicative system of a triangulated category $\mathcal{D}$. Then $\mathcal{D}$ is said to be **compatible with the distinguished triangles** in $\mathcal{D}$ if it satisfies the following conditions:

(ST1) For any morphism $s : X \to Y$ in $\mathcal{D}$, $s \in \mathcal{S}$ if and only if $s[1] \in \mathcal{S}$.

(ST2) Consider a solid diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "\alpha"] & Y \arrow[r] \arrow[d, "\beta"] & Z \arrow[r] \arrow[d, dashed, "\gamma"] & X[1] \arrow[d, dashed] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & X'[1] \end{tikzcd}$$

if $\alpha, \beta \in \mathcal{S}$, then there exists a morphism $\gamma \in \mathcal{S}$ giving rise to a morphisms of distinguished triangles.

The importance of the compatibility of $\mathcal{S}$ with distinguished triangles is contained in the following proposition:

Proposition 4.4.29. *Let $\mathcal{S}$ be a multiplicative system of $\mathcal{D}$ that is compatible with the distinguished triangles. Then the localization $\mathcal{D}_{\mathcal{S}}$ has a uniquely determined triangulated category structure so that the localization functor $Q : \mathcal{D} \to \mathcal{D}_{\mathcal{S}}$ is triangulated.*

Proof. Since $\mathcal{D}$ is additive, it follows from Theorem 4.4.13 that the localization $\mathcal{D}/\mathcal{N}$ is additive, and $Q : \mathcal{D} \to \mathcal{D}/\mathcal{N}$ is an additive functor. As for the uniqueness of the triangulated category structure, it suffices to note that for any distinguished triangle $X \xrightarrow{f} Y \to Z \to X[1]$, by adjusting using isomorphisms in $\mathcal{C}_{\mathcal{S}}$, we can assume that $f : X \to Y$ is a morphism in $\mathcal{D}$. But it then follows from Proposition 4.4.19 that this triangle is isomorphic to a distinguished triangle in $\mathcal{D}$.

We now define the distinguished triangles of $\mathcal{C}_{\mathcal{S}}$ as the images of that of $\mathcal{D}$ under Q. Axioms (TR0)–(TR3) follow directly from that of $\mathcal{D}$, so let's prove (TR4). With the notations of (TR3) and (Exercise), we may assume that there exists a commutative diagram in $\mathcal{D}$ of solid arrows, with horizontal arrows belong to $\mathcal{D}$:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\tilde{\alpha}"] & Y \arrow[r, "g"] \arrow[d, "\tilde{\beta}"] & Z \arrow[r] \arrow[d, dashed, "\tilde{\gamma}"] & X[1] \arrow[d, dashed, "\tilde{\alpha}[1]"] \\
A \arrow[r] & B \arrow[r, dashed] & C \arrow[r, dashed] & A[1] \\
X' \arrow[r, "f'"] \arrow[u, "s"] & Y' \arrow[r, "g'"] \arrow[u, "t"] & Z' \arrow[r] \arrow[u, dashed, "u"] & X'[1] \arrow[u, dashed, "s[1]"]
\end{tikzcd}$$

Now by applying (TR2) to the morphism $A \to B$, we obtain a distinguished triangle $A \to B \to C \to A[1]$, and by (ST2) there is a morphism $u : Z' \to C$ in $\mathcal{S}$ completing the lower square. Also, by (TR4) there is a morphism $\tilde{\gamma} : Z \to C$ completing the upper square, and we have construct the desired morphism of distinguished triangles in $\mathcal{D}_{\mathcal{S}}$. Finally, consider two morphisms $f : X \to Y$ and $g : Y \to Z$ in $\mathcal{D}_{\mathcal{S}}$, which we may assume to belong to $\mathcal{D}$. Then by applying (TR5) and take the iamge in $\mathcal{D}_{\mathcal{S}}$ of the octahedron diagram, we conclude that (TR5) holds for $\mathcal{D}_{\mathcal{S}}$. □

Theorem 4.4.30 (**Verdier**). *Let $\mathcal{N}$ be a null system in a triangulated category $\mathcal{D}$.*

(i) *$\mathcal{N}Q$ is a multiplicative system compatible with distinguished triangles in $\mathcal{D}$.*

(ii) *Denote by $\mathcal{D}/\mathcal{N}$ the localization of $\mathcal{D}$ by $\mathcal{N}Q$ and by $Q : \mathcal{D} \to \mathcal{D}/\mathcal{N}$ the localization functor. Then $\mathcal{D}/\mathcal{N}$ is an additive category endowed with an automorphism (the image of T, still denoted by T), and there is a canonical triangulated structure on $\mathcal{D}_{\mathcal{S}}$ so that $\mathcal{D}/\mathcal{Q}$ is a triangulated category and Q is a triangulated functor.*

(iii) *For a morphism $f : X \to Y$ in $\mathcal{D}$, we have $Q(f) = 0$ if and only if f factorizes through an object of $\mathcal{N}$. In particular, $Q(N) = 0$ for $N \in \mathrm{Ob}(\mathcal{N})$.*

(iv) *For any pretriangulated category $\mathcal{D}'$ and any triangulated functor $F : \mathcal{D} \to \mathcal{D}'$ such that $F(X) \cong 0$ for any $X \in \mathcal{N}$, F factors uniquely through Q.*

(v) *For any abelian category $\mathcal{A}$ and any cohomological functor $H : \mathcal{D} \to \mathcal{A}$, if $H(N) = 0$ for $N \in \mathrm{Ob}(\mathcal{N})$, then H factors uniquely through Q.*

Proof. Since the opposite category of $\mathcal{D}$ is again triangulated and $\mathcal{N}^{\mathrm{op}}$ is a null system in $\mathcal{D}^{\mathrm{op}}$, it is enough to check that $\mathcal{N}Q$ is a right multiplicative system.

(S1) If $X \in \mathrm{Ob}(\mathcal{D})$, then $X \xrightarrow{\mathrm{id}_X} X \to 0 \to X[1]$ is distinguished in $\mathcal{D}$ by (TR1), so $\mathrm{id}_X \in \mathcal{N}Q$.

(S2) Let $f : X \to Y$ and $g : Y \to Z$ be in $\mathcal{N}Q$. By (TR3), there are distinguished triangles

$$X \xrightarrow{f} Y \to Z' \to X[1],$$
$$Y \xrightarrow{g} Z \to X' \to Y[1],$$
$$X \xrightarrow{gf} Z \to Y' \to X[1],$$

where we can assume that $Z', X' \in \mathcal{N}$. By (TR5), there exists a distinguished triangle $Z' \to Y' \to X' \to Z'[1]$, so $Y' \in \mathcal{N}$ in view of (N3).

(S3') Let $f : X \to Y$ and $s : X \to X'$ be two morphisms with $s \in \mathcal{N}Q$. Then there exists a distinguished triangle $W \xrightarrow{h} X \xrightarrow{s} X' \to W[1]$ with $W \in \mathcal{N}$. By (TR2), there also exists a distinguished triangle $W \xrightarrow{fh} Y \xrightarrow{t} Z \to W[1]$, and we obtain a commutative diagram in view of (TR4):

$$\begin{tikzcd} W \arrow[r, "h"] \arrow[d, equal] & X \arrow[r, "s"] \arrow[d, "f"] & X' \arrow[r] \arrow[d] & W[1] \arrow[d, equal] \\ W \arrow[r, "fh"] & Y \arrow[r, "t"] & Z \arrow[r] & W[1] \end{tikzcd}$$

Since $W \in \mathcal{N}$, we conclude that $t \in \mathcal{N}Q$.

(S4') Replacing f by $f - g$, it is enough to check that if there exists $s \in \mathcal{N}Q$ with $fs = 0$, then there exists $t \in \mathcal{N}Q$ with $tf = 0$. Consider the solid diagram

$$\begin{tikzcd} X' \arrow[r, "s"] & X \arrow[r] \arrow[dr, "f"] & Z \arrow[r] \arrow[d, dashed, "h"] & X'[1] \\ & & Y \arrow[d, dashed, "t"] & \\ & & Y' & \end{tikzcd}$$

where the row is a distinguished triangle with $Z \in \mathcal{N}$. Since $fs = 0$, the morphism f factors thorugh Z in view of Proposition 4.4.17. There then exists a distinguished triangle $Z \to Y \xrightarrow{h} Y' \to Z[1]$ by (TR2), and we obtain that $t \in \mathcal{N}Q$ since $Z \in \mathcal{N}$. Finally, $th = 0$ implies that $tf = 0$ (cf. Proposition 4.4.16).

It remains to see that $\mathcal{N}Q$ is compatible with distinguished triangles. For this, since $\mathcal{N}$ is closed under T, it is easy to see that $\mathcal{N}Q$ satisfies (ST1). Moreover, consider the diagram of (ST2) and assume that $\alpha, \beta \in \mathcal{N}Q$. Then by Verdier's Exercise, we have a commutative diagram

$$\begin{tikzcd} {} & {} & {} & \\ X'' \arrow[r] \arrow[u, "+1"] & Y'' \arrow[r] \arrow[u, "+1"] & Z'' \arrow[r, "+1"] \arrow[u, "+1"] & {} \\ X' \arrow[r] \arrow[u] & Y' \arrow[r] \arrow[u] & Z' \arrow[r, "+1"] \arrow[u] & {} \\ X \arrow[r] \arrow[u, "\alpha"] & Y \arrow[r] \arrow[u, "\beta"] & Z \arrow[r, "+1"] \arrow[u, "\gamma"] & {} \end{tikzcd}$$

so that every rows and columns are distinguished triangles and every square is commutative. By the saturality of $\mathcal{N}$ and Corollary 4.4.21, we see that $X'', Y'' \in \mathrm{Ob}(\mathcal{N})$, so it follows from (N3') that $Z'' \in \mathcal{N}$, whence $\gamma \in \mathcal{N}Q$. Now from Proposition 4.4.29, we see that $\mathcal{D}/\mathcal{N}$ has a canonical structure of a triangulated category, and $Q : \mathcal{D} \to \mathcal{D}/\mathcal{N}$ is a triangulated functor.

As for (iii), consider a distinguished triangle $0 \to N \to N \to 0$, where $N \in \mathcal{N}$. Then the morphism $0 \to X$ belongs to $\mathcal{N}Q$, and hence is an isomorphism under Q. In particular, if $f : X \to Y$ can be decomposed into $X \to N \to Y$ with $N \in \mathcal{N}$, then $Q(f) = 0$. Conversely, if $Q(f) = 0$, then there exists a morphism $s : M \to X$ such that $s \in \mathcal{N}Q$ and $fs = 0$

(Corollary 4.4.7). From the definition of $\mathcal{N}Q$, we have a solid commutative diagram

$$\begin{tikzcd} M \arrow[r, "s"] \arrow[d] & X \arrow[r] \arrow[d, "f"] & N \arrow[r] \arrow[d, dashed] & M[1] \arrow[d, dashed] \\ 0 \arrow[r] & Y \arrow[r, "\mathrm{id}_Y"] & Y \arrow[r] & 0 \end{tikzcd}$$

By (TR4), there exists a morphism $N \to Y$ giving rise to the commutative diagram, and we then obtain a decomposition $X \to N \to Y$ of f.

Now let $F : \mathcal{D} \to \mathcal{D}'$ be a triangulated functor, where $\mathcal{D}'$ is a pretriangulated category. Then for $s \in \mathcal{N}Q$, we have a distinguished triangle $X \xrightarrow{s} Y \to N \to X[1]$ such that $N \in \mathrm{Ob}(\mathcal{N})$, whence a distinguished triangle $F(X) \xrightarrow{F(s)} F(Y) \to 0 \to F(X)[1]$ in $\mathcal{D}'$. By Corollary 4.4.18, we conclude that $F(s)$ is an isomorphism, so there is a uniquely determined factorization $F = \bar{F} \circ Q$, where $\bar{F}$ is an additive functor. From the description of distinguished triangles in $\mathcal{N}Q$, it is easy to see that $\bar{F}$ is a triangulated functor.

Finally, let $H : \mathcal{D} \to \mathcal{A}$ be a cohomological functor. By considering a distinguished triangle $X \xrightarrow{s} Y \to N \to X[1]$ such that $N \in \mathrm{Ob}(\mathcal{N})$, we obtain an exact sequence

$$0 = H(N[-1]) \longrightarrow H(X) \xrightarrow{H(s)} H(Y) \longrightarrow H(N) = 0$$

so $H(s)$ is an isomorphism and we obtain a uniquely determined factorization $H = \overline{H} \circ Q$. Similarly, it is immediate to check that $\overline{H}$ is a cohomological functor. □

Now consider a full triangulated subcategory $\mathcal{I}$ of $\mathcal{D}$. We shall write $\mathcal{N} \cap \mathcal{I}$ for the full subcategory whose objects are $\mathrm{Ob}(\mathcal{N}) \cap \mathrm{Ob}(\mathcal{I})$. This is clearly a null system in $\mathcal{I}$.

Proposition 4.4.31. *Let $\mathcal{D}$ be a triangulated category, $\mathcal{N}$ a null system, $\mathcal{I}$ a full triangulated subcategory of $\mathcal{D}$. Assume that one of the following conditions is true:*

(a) *any morphism $Y \to Z$ with $Y \in \mathcal{I}$ and $Z \in \mathcal{N}$ factorizes as $Y \to Z' \to Z$ with $Z' \in \mathcal{N} \cap \mathcal{I}$;*

(b) *any morphism $Z \to Y$ with $Y \in \mathcal{I}$ and $Z \in \mathcal{N}$ factorizes as $Z \to Z' \to Y$ with $Z' \in \mathcal{N} \cap \mathcal{I}$.*

Then $\mathcal{I}/(\mathcal{N} \cap \mathcal{I}) \to \mathcal{D}/\mathcal{N}$ is fully faithful.

Proof. We may assume (b), the case (a) being deduced by considering $\mathcal{D}^{\mathrm{op}}$. We shall apply Proposition 4.4.11. Let $f : X \to Y$ is a morphism in $\mathcal{N}Q$ with $X \in \mathcal{I}$, we show that there exists $g : Y \to W$ with $W \in \mathcal{I}$ and $g \circ f \in \mathcal{N}Q$. By definition, the morphism f is embedded in a distinguished triangle $X \to Y \to Z \to X[1]$ with $Z \in \mathcal{N}$, and the hypothesis implies that the morphism $Z \to X[1]$ factorizes through an object $Z' \in \mathcal{N} \cap \mathcal{I}$. We may embed $Z' \to TX$ in a distinguished triangle in $\mathcal{I}$ and obtain a commutative diagram of distinguished triangles by (TR4):

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r] \arrow[d, dashed, "g"] & Z \arrow[r] \arrow[d] & X[1] \arrow[d, equal] \\ X \arrow[r, "g \circ f"] & W \arrow[r] & Z' \arrow[r] & X[1] \end{tikzcd}$$

Since Z' belongs to $\mathcal{N}$, we conclude that $gf \in \mathcal{N}Q \cap \mathrm{Mor}(\mathcal{I})$. □

Proposition 4.4.32. *Let $\mathcal{D}$ be a triangulated category, $\mathcal{N}$ be a null system, $\mathcal{I}$ be a full triangulated subcategory of $\mathcal{D}$, and assume that one of the following conditions is true:*

(a) *for any $X \in \mathrm{Ob}(\mathcal{D})$, there exists a morphism $X \to Y$ in $\mathcal{N}Q$ with $Y \in \mathcal{I}$;*

(b) *for any $X \in \mathrm{Ob}(\mathcal{D})$, there exists a morphism $Y \to X$ in $\mathcal{N}Q$ with $Y \in \mathcal{I}$;*

Then $\mathcal{I}/(\mathcal{N} \cap \mathcal{I}) \to \mathcal{D}/\mathcal{N}$ is an equivalence of categories.

Proof. Apply Corollary 4.4.12. □

Example 4.4.2. Let H be a cohomological functor on $\mathcal{D}$. We define $\mathcal{N}_H$ to be the collection of objects $X \in \mathcal{D}$ such that $H(X[n]) = 0$ for $n \in \mathbb{Z}$. Then it is easy to verify that $\mathcal{N}_H$ satisfies conditions (N1)–(N3), so we can form the localization $\mathcal{D}/\mathcal{N}_H$.

Proposition 4.4.33. *Let $\mathcal{D}$ be a triangulated category admitting direct sums indexed by a set I and let $\mathcal{N}$ be a null system closed by such direct sums. Let $Q : \mathcal{D} \to \mathcal{D}/\mathcal{N}$ denote the localization functor. Then $\mathcal{D}/\mathcal{N}$ admits direct sums indexed by I and the localization functor $Q : \mathcal{D} \to \mathcal{D}/\mathcal{N}$ commutes with such direct sums.*

Proof. Let $\{X_i\}_{i \in I}$ be a family of objects in $\mathcal{D}$. It is enough to show that $Q(\bigoplus_i X_i)$ is the direct sum of the family $Q(X_i)$, i.e., the map

$$\mathrm{Hom}_{\mathcal{D}/\mathcal{N}}(Q(\bigoplus_i X_i), Y) \to \prod_i \mathrm{Hom}_{\mathcal{D}/\mathcal{N}}(Q(X_i), Y)$$

is bijective for any $Y \in \mathcal{D}$. To this end, we first consider morphisms $u_i \in \mathrm{Hom}_{\mathcal{D}/\mathcal{N}}(Q(X_i), Y)$. Then u_i is represented by a pair $(X_i'; s, u_i')$, where $u_i' : X_i' \to Y$ is a morphism in $\mathcal{D}$ and we have a distinguished triangle

$$X_i' \xrightarrow{s} X_i \longrightarrow Z_i \longrightarrow X_i'[1]$$

in $\mathcal{D}$ with $Z_i \in \mathcal{N}$. We then get a morphism $\bigoplus_i X_i' \to Y$ and a distinguished triangle $\bigoplus_i X_i' \to \bigoplus_i X_i \to \bigoplus_i Z_i \to (\bigoplus_i X_i')[1]$ in $\mathcal{D}$ with $\bigoplus_i Z_i \in \mathcal{N}$.

Now assume that the composition $Q(X_i) \to Q(\bigoplus_i X_i) \xrightarrow{u} Q(Y)$ is zero for each $i \in I$. By definition, the morphism u is represented by a pair $(Y'; s, u')$, where $u' : \bigoplus_i X_i \to Y'$ is a morphism in $\mathcal{D}$ and $s : Y \to Y'$ is a morphism in $\mathcal{N}Q$. Using the result of Theorem 4.4.30(iii), we can find $Z_i \in \mathcal{N}$ such that $u_i' : X_i \to Y'$ factroizes as $X_i \to Z_i \to Y'$. Then the induced morphism $\bigoplus_i X_i \to Y'$ factorizes as $\bigoplus_i X_i \to \bigoplus_i Z_i \to Y'$. Since $\bigoplus_i Z_i \in \mathcal{N}$, we conclude that $Q(u) = 0$, whence the proposition. □

4.4.5 Localization of triangulated functors

Let $F : \mathcal{D} \to \mathcal{D}'$ be a functor of triangulated categories, $\mathcal{N}$ and $\mathcal{N}'$ be null systems in $\mathcal{D}$ and $\mathcal{D}'$, respectively. The left (resp. right) localization of F (when it exists) is then defined, by replacing "functor" by "triangulated functor". In the sequel, $\mathcal{D}$ (resp. $\mathcal{D}'$, $\mathcal{D}''$) is a triangulated category and $\mathcal{N}$ (resp. $\mathcal{N}'$, $\mathcal{N}''$) is a null system in this category. We denote by $Q : \mathcal{D} \to \mathcal{D}/\mathcal{N}$ (resp. $Q' : \mathcal{D}' \to \mathcal{D}'/\mathcal{N}'$, $Q'' : \mathcal{D}'' \to \mathcal{D}''/\mathcal{N}''$) the localization functor and by $\mathcal{N}Q$ (resp. $\mathcal{N}'Q$, $\mathcal{N}''Q$) the family of morphisms in $\mathcal{D}$ (resp. $\mathcal{D}'$, $\mathcal{D}''$) defined in (4.4.10).

Definition 4.4.10. A triangulated functor $F : \mathcal{D} \to \mathcal{D}'$ is called **left (resp. right) localizable with respect to** $(\mathcal{N}, \mathcal{N}')$ if $Q' \circ F : \mathcal{D} \to \mathcal{D}'/\mathcal{N}'$ is universally left (resp. right) localizable with respect to the multiplicative system $\mathcal{N}Q$. If there is no risk of confusion, we simply say that F is left (resp. right) localizable or that LF (resp. RF) exists.

Definition 4.4.11. Let $F : \mathcal{D} \to \mathcal{D}'$ be a triangulated functor of triangulated categories, $\mathcal{N}$ and $\mathcal{N}'$ null systems in $\mathcal{D}$ and $\mathcal{D}'$, and $\mathcal{I}$ a full triangulated subcategory of $\mathcal{D}$. Consider the following conditions:

(a) For any $X \in \mathcal{D}$, there exists a morphism $X \to Y$ in $\mathcal{N}Q$ with $Y \in \mathcal{I}$.

(b) For any $X \in \mathcal{D}$, there exists a morphism $Y \to X$ in $\mathcal{N}Q$ with $Y \in \mathcal{I}$.

(c) For any $Y \in \mathcal{N} \cap \mathcal{I}$, $F(Y) \in \mathcal{N}'$.

Then if (a) and (b) (resp. (b) and (c)) are satisfied, we say that the subcategory $\mathcal{I}$ is ***F*-injective** (resp. ***F*-projective**) with respect to $\mathcal{N}$ and $\mathcal{N}'$. If there is no risk of confusion, we often omit "with respect to $\mathcal{N}$ and $\mathcal{N}'$".

Note that if $F(\mathcal{N}) \subseteq \mathcal{N}'$, then D is both F-injective and F-projective.

Proposition 4.4.34. *Let $F : \mathcal{D} \to \mathcal{D}'$ be a triangulated functor of triangulated categories, $\mathcal{N}$ and $\mathcal{N}'$ null systems in $\mathcal{D}$ and $\mathcal{D}'$, and $\mathcal{I}$ a full triangulated category of $\mathcal{D}$.*

(a) *If $\mathcal{I}$ is F-injective with respect to $\mathcal{N}$ and $\mathcal{N}$, then F is right localizable and its right localization is a triangulated functor.*

(b) *If $\mathcal{I}$ is F-projective with respect to $\mathcal{N}$ and $\mathcal{N}'$, then F left localizable and its left localization is a triangulated functor.*

Proof. By Proposition 4.4.14, the existence of the localizations is clear. To verify that they are triangulated, it suffices to apply Theorem 4.4.30 to check this in $\mathcal{D}$ and $\mathcal{D}'$, and this follows from the hypothesis on F. □

We denote by $R^{\mathcal{N}'}_{\mathcal{N}}F : \mathcal{D}/\mathcal{N} \to \mathcal{D}'/\mathcal{N}'$ (resp. $L^{\mathcal{N}'}_{\mathcal{N}}F$) the right (resp. left) localization of F with respect to $(\mathcal{N}, \mathcal{N}')$. If there is no risk of confusion, we simply write RF (resp. LF) instead of $R^{\mathcal{N}'}_{\mathcal{N}}F$ (resp. $L^{\mathcal{N}'}_{\mathcal{N}}F$). If $\mathcal{I}$ is F-injective, then RF can be defined by the diagram

$$\begin{tikzcd}
& \mathcal{D} \arrow[r] & \mathcal{D}/\mathcal{N} \arrow[dd, dashed, "R^{\mathcal{N}'}_{\mathcal{N}}F"] \\
\mathcal{I} \arrow[ur] \arrow[r] \arrow[drr] & \mathcal{I}/(\mathcal{I}\cap\mathcal{N}) \arrow[ur, "\sim"] \arrow[dr] & \\
& & \mathcal{D}'/\mathcal{N}'
\end{tikzcd}$$

and we have

$$R^{\mathcal{N}'}_{\mathcal{N}}F(X) \cong F(Y) \quad \text{for } (X \to Y) \in \mathcal{N}Q \text{ with } Y \in \mathcal{I}. \tag{4.4.11}$$

Similarly, if $\mathcal{I}$ is F-projective, then the diagram above defines LF and we have

$$L^{\mathcal{N}'}_{\mathcal{N}}F(X) \cong F(Y) \quad \text{for } (Y \to X) \in \mathcal{N}Q \text{ with } Y \in \mathcal{I}. \tag{4.4.12}$$

Proposition 4.4.35. *Let $F : \mathcal{D} \to \mathcal{D}'$ and $F' : \mathcal{D}' \to \mathcal{D}''$ be triangulated functors of triangulated categories and let $\mathcal{N}$, $\mathcal{N}'$ and $\mathcal{N}''$ be null systems in $\mathcal{D}$, $\mathcal{D}'$ and $\mathcal{D}''$, respectively.*

(a) *Assume that $R^{\mathcal{N}'}_{\mathcal{N}}F$, $R^{\mathcal{N}'}_{\mathcal{N}}F$ and $R^{\mathcal{N}'}_{\mathcal{N}}F$ exist. Then there is a canonical morphism of functors:*

$$R^{\mathcal{N}''}_{\mathcal{N}}(F' \circ F) \to R^{\mathcal{N}''}_{\mathcal{N}'}F' \circ R^{\mathcal{N}'}_{\mathcal{N}}F. \tag{4.4.13}$$

(b) *Let $\mathcal{I}$ and $\mathcal{I}'$ be full triangulated subcategories of $\mathcal{D}$ and $\mathcal{D}'$, respectively. Assume that I is F-injective with respect to $\mathcal{N}$ and $\mathcal{N}'$, $\mathcal{I}'$ is F'-injective with respect to $\mathcal{N}'$ and $\mathcal{N}''$, and $F(\mathcal{I}) \subseteq \mathcal{I}'$. Then $\mathcal{I}$ is $(F' \circ F)$-injective with respect to $\mathcal{N}$ and $\mathcal{N}''$ and (4.4.13) is an isomorphism*

Proof. By definition, there exists a bijection

$$\mathrm{Hom}(R^{\mathcal{N}''}_{\mathcal{N}}F' \circ F, R^{\mathcal{N}''}_{\mathcal{N}'}F' \circ R^{\mathcal{N}'}_{\mathcal{N}}F) \cong \mathrm{Hom}(Q'' \circ F' \circ F, R^{\mathcal{N}''}_{\mathcal{N}'}F' \circ R^{\mathcal{N}'}_{\mathcal{N}}F \circ Q),$$

and the natural morphism of functors

$$Q'' \circ F' \to R^{\mathcal{N}''}_{\mathcal{N}'} F' \circ Q', \quad Q' \circ F \to R^{\mathcal{N}'}_{\mathcal{N}} F \circ Q.$$

We then deduce the canonical morphisms

$$Q'' \circ F' \circ F \to R^{\mathcal{N}''}_{\mathcal{N}'} F' \circ Q' \circ F \to R^{\mathcal{N}''}_{\mathcal{N}'} F' \circ R^{\mathcal{N}'}_{\mathcal{N}} F \circ Q$$

whence the morphism in (a). Now assume the conditions in (b); the fact that $\mathcal{I}$ is $(F' \circ F)$-injective follows immediately from the definition. Let $X \in \mathcal{D}$ and consider a morphism $X \to Y$ in $\mathcal{N}Q$ with $Y \in \mathcal{I}$. Then $R^{\mathcal{N}'}_{\mathcal{N}} F(X) \cong F(Y)$ by (4.4.11) and $F(Y) \in \mathcal{I}'$ by our hypothesis. It then follows from (4.4.11) that $(R^{\mathcal{N}''}_{\mathcal{N}'} F')(F(Y)) \cong F'(F(Y))$, and we conclude that

$$(R^{\mathcal{N}''}_{\mathcal{N}'} F)(R^{\mathcal{N}'}_{\mathcal{N}} F(X)) \cong F'(F(Y)).$$

On the other hand, $R^{\mathcal{N}'}_{\mathcal{N}}(F' \circ F)(X) \cong F'(F(Y))$ by (4.4.11), since $\mathcal{I}$ is $(F' \circ F)$-injective. $\square$

We now restrict our notations of localizations to triangulated bifunctors. Let $(\mathcal{D}, T)$, $(\mathcal{D}', T')$ and $(\mathcal{D}'', T'')$ be triangulated categories. A **triangulated bifunctor** $F : \mathcal{D} \times \mathcal{D}' \to \mathcal{D}''$ is a bifunctor of additive categories with translations which sends distinguished triangles in each arguments to distinguished triangles.

Definition 4.4.12. Let $\mathcal{D}$, $\mathcal{D}'$ and $\mathcal{D}''$ be triangulated categories and $\mathcal{N}$, $\mathcal{N}'$ and $\mathcal{N}''$ be null systems in $\mathcal{D}$, $\mathcal{D}'$ and $\mathcal{D}''$, respectively. We say that a triangulated bifunctor $F : \mathcal{D} \times \mathcal{D}' \to \mathcal{D}''$ is **right (resp. left) localizable with respect to** $(\mathcal{N}, \mathcal{N}, \mathcal{N}'')$ if $Q'' \circ F : \mathcal{D} \times \mathcal{D}' \to \mathcal{D}''/\mathcal{N}''$ is universally right (resp. left) localizable with respect to the multiplicative system $\mathcal{N}Q \times \mathcal{N}'Q$. If there is no risk of confusion, we simply say that F is right (resp. left) localizable.

If $\mathcal{I}, \mathcal{I}'$ are full triangulated subcategories of $\mathcal{D}$ and $\mathcal{D}'$, respectively, then the pair $(\mathcal{I}, \mathcal{I}')$ is F**-injective** with respect to $(\mathcal{N}, \mathcal{N}', \mathcal{N}'')$ if

(i) $\mathcal{I}'$ is $F(Y, -)$-injective with respect to $\mathcal{N}'$ and $\mathcal{N}''$ for any $Y \in \mathcal{I}$.

(ii) $\mathcal{I}$ is $F(-, Y')$-injective with respect to $\mathcal{N}$ and $\mathcal{N}''$ for any $Y' \in \mathcal{I}'$.

Equivalently, this amounts to saying that

(a) for any $X \in \mathcal{D}$, there exists a morphism $X \to Y$ in $\mathcal{N}Q$ with $Y \in \mathcal{I}$;

(b) for any $X' \in \mathcal{D}'$, there exists a morphism $X' \to Y'$ in $\mathcal{N}'Q$ with $Y' \in \mathcal{I}'$;

(c) $F(X, X')$ belongs to $\mathcal{N}''$ for $X \in \mathcal{I}$, $X' \in \mathcal{I}'$ as soon as X belongs to $\mathcal{N}$ or X' belongs to $\mathcal{N}'$.

The property for $(\mathcal{I}, \mathcal{I}')$ of being F**-projective** is defined similarly.

We denote by $R^{\mathcal{N}''}_{\mathcal{N} \times \mathcal{N}'}$ the right localization of F with respect to $(\mathcal{N} \times \mathcal{N}', \mathcal{N}'')$ if it exists. If there is no risk of confusion, we simply write RF. We use similar notations for the left localization.

Proposition 4.4.36. *Let $\mathcal{D}$, $\mathcal{D}'$ and $\mathcal{D}''$ be triangulated categories and $\mathcal{N}$, $\mathcal{N}'$ and $\mathcal{N}''$ be null systems in $\mathcal{D}$, $\mathcal{D}'$ and $\mathcal{D}''$, respectively. Let $F : \mathcal{D} \times \mathcal{D}' \to \mathcal{D}$ be a triangulated bifunctor and $\mathcal{I}, \mathcal{I}'$ be full triangulated subcategories of $\mathcal{D}$ and $\mathcal{D}'$ such that $(\mathcal{I}, \mathcal{I}')$ is F-injective with respect to $(\mathcal{N}, \mathcal{N}')$. Then F is right localizable, its right localization $R^{\mathcal{N}''}_{\mathcal{N} \times \mathcal{N}'} F$ is a triangulated bifunctor*

$$R^{\mathcal{N}''}_{\mathcal{N} \times \mathcal{N}'} : \mathcal{D}/\mathcal{N} \times \mathcal{D}'/\mathcal{N}' \to \mathcal{D}''/\mathcal{N}'',$$

and moreover,

$$R^{\mathcal{N}''}_{\mathcal{N} \times \mathcal{N}'} F(X, X') \cong F(Y, Y') \tag{4.4.14}$$

for $(X \to Y) \in \mathcal{N}Q$ and $(X' \to Y') \in \mathcal{N}'Q$ with $Y \in \mathcal{I}$, $Y' \in \mathcal{I}'$. There exists a similar result by replacing "injective" with "projective" and reversing the arrows.

Proof. By definition, $Q'' \circ F$ sends $\mathcal{N}Q \cap \mathrm{Mor}(\mathcal{I}) \times (\mathcal{N}'Q \cap \mathrm{Mor}(\mathcal{I}'))$ to isomorphisms in $\mathcal{D}''$, so it follows from Proposition 4.4.14 that the right localization $R^{\mathcal{N}''}_{\mathcal{N}\times\mathcal{N}'}$ exists. The fact that $R^{\mathcal{N}''}_{\mathcal{N}\times\mathcal{N}'}$ is triangulated follows from the hypothsis on F, in view of Theorem 4.4.30. The last equation is a concequence of Proposition 4.4.35. □

Corollary 4.4.37. *Retain the notations of Proposition 4.4.36 and assume that*

(a) $F(\mathcal{I}, \mathcal{N}') \subseteq \mathcal{N}''$;

(b) *for any* $X' \in \mathcal{D}'$, $\mathcal{I}$ *is* $F(-, X')$*-injective with respect to* $\mathcal{N}$.

Then F is right localizable and we have

$$R^{\mathcal{N}''}_{\mathcal{N}\times\mathcal{N}'} F(X, X') \cong R^{\mathcal{N}''}_{\mathcal{N}} F(-, X')(X).$$

Again, there is a similar statement by replacing "injective" with "projective".

Proof. Under our hypothesis, for any fixed object $X' \in \mathcal{D}'$, the functor $F(-, X')$ is right localizable, and the last claim follows from (4.4.13) and (4.4.14). □

4.5 Derived categories

In this section, we apply the previous results of triangulated categories on the derived category of an abelian category $\mathcal{A}$, which is defined to be the localization of $K(\mathcal{A})$ with respect to quasi-isomorphisms. Our main refrence will be [**?**].

4.5.1 Derived categories

Let $(\mathcal{A}, T)$ be an abelian category with translation. Recall that the cohomology functor $H : \mathcal{A}_c \to \mathcal{A}$ induces a cohomological functor

$$H : K_c(\mathcal{A}) \to \mathcal{A}.$$

Let $\mathcal{N}$ be the full subcategory of $K_c(\mathcal{A})$ consisting of objects X such that $H(X) \cong 0$, that is, X is quasi-isomorphic to 0. Since H is cohomological, the category $\mathcal{N}$ is a triangulated subcategory of $K_c(\mathcal{A})$. We denote by $D_c(\mathcal{A})$ the category $K_c(\mathcal{A})/\mathcal{N}$, and call it the **derived category** of $(\mathcal{A}, T)$. Note that $D_c(\mathcal{A})$ is triangulated by **??**. By the properties of the localization, a quasi-isomorphism in $K_c(\mathcal{A})$ (or in $\mathcal{A}_c$) becomes an isomorphism in $D_c(\mathcal{A})$. One shall be aware that the category $D_c(\mathcal{A})$ may be a big category.

From now on, we shall restrict our study to the case where $\mathcal{A}_c$ is the category of complexes of an abelian category $\mathcal{A}$. Recall that the categories $C^*(\mathcal{A})$ are defined for $* \in \{+, -, b, \varnothing\}$, and we have full subcategories $K^*(\mathcal{A})$ of $K(\mathcal{A})$. For $* \in \{+, -, b, \varnothing\}$, we define

$$N^*(\mathcal{A}) = \{X \in K^*(\mathcal{A}) : H^i(X) \cong 0 \text{ for all } i\}.$$

Clearly, $N^*(\mathcal{A})$ is a null system in $K^*(\mathcal{C})$.

Definition 4.5.1. The triangulated categories $D^*(\mathcal{A})$ are defined as $K^*(\mathcal{A})/N^*(\mathcal{A})$ and are called the **derived categories** of $\mathcal{A}$.

Recall that to a null system $\mathcal{N}$ we have associated in (4.4.10) a multiplicative system denoted by $\mathcal{N}Q$. It will be more intuitive to use here another notation for $\mathcal{N}Q$ when $\mathcal{N} = N(\mathcal{A})$:

$$\mathrm{Qis} = \{f \in \mathrm{Mor}(K(\mathcal{A})) : f \text{ is a quasi-isomorphism}\}.$$

With this notation, we then have

$$\begin{aligned}\mathrm{Hom}_{D(\mathcal{A})}(X,Y) &\cong \varinjlim_{(X'\to X)\in \mathrm{Qis}} \mathrm{Hom}_{K(\mathcal{A})}(X',Y) \cong \varinjlim_{(Y'\to Y)\in \mathrm{Qis}} \mathrm{Hom}_{K(\mathcal{A})}(X,Y') \\ &\cong \varinjlim_{\substack{(X'\to X)\in \mathrm{Qis} \\ (Y\to Y')\in \mathrm{Qis}}} \mathrm{Hom}_{K(\mathcal{A})}(X',Y').\end{aligned}$$

Remark 4.5.1. Let $X \in K(\mathcal{A})$, and let $Q(X)$ denote its image in $D(\mathcal{A})$. Then it follows from our definition of $\mathcal{N}(\mathcal{A})$ that $Q(X) = 0$ if and only if $H^n(X) = 0$ for all $n \in \mathbb{Z}$. Also, if $f : X \to Y$ is a morphism in $\mathrm{Ch}(\mathcal{A})$, then by Theorem 4.4.30, $f = 0$ in $D(\mathcal{A})$ if and only if there exist X' and a quasi-isomorphism $g : X' \to X$ such that fg is homotopic to 0, or else, if and only if there exist Y' and a quasi-isomorphism $h : Y \to Y'$ such that hf is homotopic to 0.

Proposition 4.5.1. *Let $\mathcal{A}$ be an abelian category and $D(\mathcal{A})$ be its derived category.*

(a) *For $n \in \mathbb{Z}$, the functor $H^n : D(\mathcal{A}) \to \mathcal{A}$ is well defined and is a cohomological functor.*

(b) *A morphism $f : X \to Y$ in $D(\mathcal{A})$ is an isomorphism if and only if $H^n(f) : H^n(X) \to H^n(Y)$ is an isomorphism for all $n \in \mathbb{Z}$.*

(c) *For $n \in \mathbb{Z}$, the functors $\tau_{\leq n}, \tau^{\leq n} : D(\mathcal{A}) \to D^-(\mathcal{A})$, as well as the functors $\tau_{\geq n}, \tau^{\geq n} : D(\mathcal{A}) \to D^+(\mathcal{A})$, are well defined and isomorphic.*

(d) *For $n \in \mathbb{Z}$, the functor $\tau^{\leq n}$ induces a functor $D^+(\mathcal{A}) \to D^b(\mathcal{A})$ and $\tau^{\geq n}$ induces a functor $D^-(\mathcal{A}) \to D^b(\mathcal{A})$.*

Proof. Since $H^n(X) = 0$ for $X \in N(\mathcal{A})$, the first assertion is clear, and the second one follows from Theorem 4.4.30 and the definition of $\mathcal{N}Q$ for $\mathcal{N} = N(\mathcal{A})$: in fact, if $Q(f)$ is an isomorphism, then from the following commutative diagram

$$\begin{tikzcd} Q(X) \arrow[r, "Q(f)"] \arrow[d, "Q(f)"] & Q(Y) \arrow[r] \arrow[d, equal] & Q(M(f)) \arrow[r] \arrow[d] & X[1] \arrow[d] \\ Q(Y) \arrow[r, equal] & Q(Y) \arrow[r] & 0 \arrow[r] & Q(Y)[1] \end{tikzcd}$$

we conclude that $Q(M(f)) \to 0$ is an isomorphism (Proposition 4.4.19), so $H^n(f)$ is an isomorphism for each $n \in \mathbb{Z}$.

Now if $f : X \to Y$ is a quasi-isomorphism in $K(\mathcal{A})$, then $\tau^{\leq n}(f)$ and $\tau^{\geq n}(f)$ are quasi-isomorphism. Moreover, for $X \in K(\mathcal{A})$, the morphisms $\tau^{\leq n}(X) \to \tau_{\leq n}(X)$ and $\tau^{\geq n}(X) \to \tau_{\geq n}(X)$ are also quasi-isomorphism (**??**), so assertion (c) follows from (d), and (d) is then obvious. □

To a distinguished triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \to X[1]$ in $\mathcal{D}(\mathcal{A})$, the cohomological functor H^0 associates a long exact sequence in $\mathcal{A}$:

$$\cdots \longrightarrow H^i(X) \longrightarrow H^i(Y) \longrightarrow H^i(Z) \longrightarrow H^{i+1}(X) \longrightarrow \cdots$$

For $X \in K(\mathcal{A})$, recall that the categories $\mathrm{Qis}_{/X}$ and $\mathrm{Qis}_{X/}$ are filtrant (or cofiltrant) categories of $K(\mathcal{C})_{/X}$ and $K(\mathcal{A})_{X/}$, respectively. If $\mathcal{J}$ is a subcategory of $K(\mathcal{C})_{/X}$, we denote by $\mathrm{Qis}_{/X} \cap \mathcal{J}$ the full subcategory of $\mathrm{Qis}_{/X}$ consisting of objects which belong to $\mathcal{J}$. We use similar notations for $\mathrm{Qis}_{X/}$ and $K(\mathcal{C})_{X/}$.

Lemma 4.5.2. *Let $\mathcal{A}$ be an abelian category and n be an integer.*

(a) *For* $X \in K^{\leq n}(\mathcal{A})$, *the categories* $\mathrm{Qis}_{/X} \cap K^{\leq n}(\mathcal{A})_{/X}$ *and* $\mathrm{Qis}_{/X} \cap K^{-}(\mathcal{A})_{/X}$ *are co-cofinal to* $\mathrm{Qis}_{/X}$.

(b) *For* $X \in K^{\geq n}(\mathcal{A})$, *the categories* $\mathrm{Qis}_{X/} \cap K^{\geq n}(\mathcal{A})_{X/}$ *and* $\mathrm{Qis}_{X/} \cap K^{+}(\mathcal{A})_{X/}$ *are co-cofinal to* $\mathrm{Qis}_{X/}$.

Proof. The two statements are equivalent by reversing the arrows, so we only prove (b). The category $\mathrm{Qis}_{X/} \cap K^{\geq n}(\mathcal{A})_{X/}$ is a full subcategory of a filtrant category $\mathrm{Qis}_{X/}$, and for any object $(X \to Y)$ in $\mathrm{Qis}_{X/}$, there exists a canonical morphism $(X \to Y) \to (X \to \tau^{\geq n} Y)$. □

Proposition 4.5.3. *Let* $n \in \mathbb{Z}$ *and* $X \in K^{\leq n}(\mathcal{A})$, $Y \in K^{\geq n}(\mathcal{A})$. *Then we have*

$$\mathrm{Hom}_{D(\mathcal{A})}(X, Y) \cong \mathrm{Hom}_{\mathcal{C}}(H^n(X), H^n(Y)) \tag{4.5.1}$$

Proof. The map $\mathrm{Hom}_{\mathrm{Ch}(\mathcal{A})}(X, Y) \to \mathrm{Hom}_{K(\mathcal{A})}(X, Y)$ is an isomorphism by our hypothesis and

$$\begin{aligned}\mathrm{Hom}_{\mathrm{Ch}(\mathcal{A})}(X, Y) &\cong \{f \in \mathrm{Hom}_{\mathcal{A}}(X^n, Y^n) : u \circ d_X^{n-1} = 0, d_Y^n \circ f = 0\} \\ &\cong \mathrm{Hom}_{\mathcal{A}}(\operatorname{coker} d_X^{n-1}, \ker d_Y^n) \cong \mathrm{Hom}_{\mathcal{A}}(H^n(X), H^n(Y))\end{aligned}$$

so we conclude that $\mathrm{Hom}_{K(\mathcal{A})}(X, Y) \cong \mathrm{Hom}_{\mathcal{A}}(H^n(X), H^n(Y))$. On the other hand, in view of Lemma 4.5.2, we have

$$\mathrm{Hom}_{D(\mathcal{A})}(X, Y) \cong \varinjlim_{(Y \to Y') \in \mathrm{Qis} \cap K^{\geq n}(\mathcal{A})} \mathrm{Hom}_{K(\mathcal{A})}(X, Y') \cong \mathrm{Hom}_{\mathcal{A}}(H^n(X), H^n(Y))$$

so the proposition follows. □

For $-\infty \leq a \leq b \leq +\infty$, we denote by $D^{[a,b]}(\mathcal{A})$ the full subcategory of $D(\mathcal{A})$ consisting of objects X satisfying $H^i(X) = 0$ for $i \notin [a, b]$. With this notation, we set $D^{\leq a}(\mathcal{A}) := D^{[-\infty,a]}(\mathcal{A})$ and $D^{\geq a}(\mathcal{A}) := D^{[a,+\infty]}(\mathcal{A})$.

Proposition 4.5.4. *Let* $\mathcal{A}$ *be an abelian category.*

(a) *For* $* \in \{+, -, b\}$, *the triangulated category* $D^*(\mathcal{A})$ *is equivalent to the full triangulated subcategory of* $\mathcal{D}(\mathcal{A})$ *consisting of objects* X *satisfying* $H^i(X) = 0$ *for* $i \ll 0$ *(resp.* $i \gg 0$*, resp.* $|i| \gg 0$*).*

(b) *For* $-\infty \leq a \leq b \leq +\infty$, *the canonical functor* $Q : K^{[a,b]}(\mathcal{A}) \to D^{[a,b]}(\mathcal{A})$ *is essentially surjective.*

(c) *The category* $\mathcal{A}$ *is equivalent to the full subcategory* $D^{\leq 0}(\mathcal{A}) \cap D^{\geq 0}(\mathcal{A})$.

(d) *For any* $n \in \mathbb{Z}$ *and* $X, Y \in D(\mathcal{C})$, *we have*

$$\mathrm{Hom}_{D(\mathcal{A})}(\tau^{\leq n} X, \tau^{\geq n} Y) \cong \mathrm{Hom}_{\mathcal{A}}(H^n(X), H^n(Y)).$$

In particular, $\mathrm{Hom}_{D(\mathcal{A})}(\tau^{\leq n} X, \tau^{\geq n+1} Y) = 0$.

Proof. As for assertion (a), let us treat the case $* = +$, the other cases being similar. For $Y \in K^{\geq a}(\mathcal{A})$ and $Z \in N(\mathcal{A})$, any morphism $Z \to Y$ in $K(\mathcal{A})$ factors through $\tau^{\geq n} Z \in N(\mathcal{A}) \cap K^{\geq n}(\mathcal{A})$. Applying Proposition 4.4.11, we find that the natural functor $D^+(\mathcal{A}) \to D(\mathcal{A})$ is fully faithful, and it is clear that if $Y \in D(\mathcal{A})$ belongs to the image of the functor $D^+(\mathcal{A}) \to D(\mathcal{A})$, then $H^i(X) = 0$ for $i \ll 0$. Conversely, let $X \in K(\mathcal{A})$ with $H^i(X) = 0$ for $i < a$. Then $\tau^{\geq a} X \in K^+(\mathcal{A})$ and the morphism $X \to \tau^{\geq a} X$ in $K(\mathcal{A})$ is a quasi-isomorphism, whence an isomorphism in $D(\mathcal{A})$. We therefore conclude (a), and the proof of (b) can be done similarly. Finally, by Proposition 4.5.3, the functor $\mathcal{A} \to D(\mathcal{A})$ is fully faithful and essentially surjective by (b); this proves (c), and (d) follows from (b) and Proposition 4.5.3. □

Proposition 4.5.5. *Let $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ be an exact sequence in $\mathrm{Ch}(\mathcal{A})$. Then there exists a distinguished triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \to X[1]$ and Z is isomorphic to $M(f)$ in $D(\mathcal{A})$.*

Proof. We define a morphism $\varphi : M(f) \to Z$ in $\mathrm{Ch}(\mathcal{A})$ by $\varphi^n = (0, g^n)$. By **??**, φ is then a quasi-isomorphism, whence an isomorphism in $D(\mathcal{A})$. □

Remark 4.5.2. Let $0 \to X \to Y \to Z \to 0$ be an exact sequence in $\mathcal{A}$. By Proposition 4.5.5, we then get a morphism $\gamma : Z \to X[1]$ in $D(\mathcal{A})$. The morphism $H^i(\gamma) : H^i(Z) \to H^{i+1}(X)$ is zero for all $i \in \mathbb{Z}$, although γ is not the zero morphism in $D(\mathcal{A})$ in general (this happens only if the short exact sequence splits). The morphism γ may be described in $K(\mathcal{A})$ by the morphisms with φ a quasi-isomorphism:

$$X[1] \xleftarrow{\beta(f)} M(f) \xrightarrow{\varphi} Z.$$

Proposition 4.5.6. *If $X \in D(\mathcal{A})$, there are distinguished triangles in $D(\mathcal{A})$:*

$$\tau^{\leq n} X \longrightarrow X \longrightarrow \tau^{\geq n+1} X \xrightarrow{+1}, \tag{4.5.2}$$

$$\tau^{\leq n-1} X \longrightarrow \tau^{\leq n} X \longrightarrow H^n(X)[-n] \xrightarrow{+1}, \tag{4.5.3}$$

$$H^n(X)[-n] \longrightarrow \tau^{\geq n} X \longrightarrow \tau^{\geq n+1} X \xrightarrow{+1}, \tag{4.5.4}$$

Moreover, we have canonical isomorphisms

$$H^n(X)[-n] \cong \tau^{\leq n}\tau^{\geq n}(X) \cong \tau^{\geq n}\tau^{\leq n} X. \tag{4.5.5}$$

Proof. This is a direct concequence of **??** and **??**. □

Proposition 4.5.7. *The functor $\tau^{\leq n} : D(\mathcal{A}) \to D^{\leq n}(\mathcal{A})$ is a right adjoint to the natural inclusion $D^{\leq n}(\mathcal{A}) \to D(\mathcal{A})$ and $\tau^{\geq n} : D(\mathcal{A}) \to D^{\geq n}(\mathcal{A})$ is a left adjoint to the natural functor $D^{\geq n}(\mathcal{A}) \to D(\mathcal{A})$. In other words, there are functorial isomorphisms*

$$\mathrm{Hom}_{D(\mathcal{A})}(X, Y) \cong \mathrm{Hom}_{D^{\leq n}(\mathcal{A})}(X, \tau^{\leq n} Y) \quad \textit{for} \quad X \in D^{\leq n}(\mathcal{A}), Y \in D(\mathcal{A}),$$
$$\mathrm{Hom}_{D(\mathcal{A})}(X, Y) \cong \mathrm{Hom}_{D^{\geq n}(\mathcal{A})}(\tau^{\geq n} X, Y) \quad \textit{for} \quad X \in D(\mathcal{A}), Y^{\geq n} \in D(\mathcal{A}).$$

Proof. Let $X \in D^{\leq n}(\mathcal{A})$, then by the distinguished triangle (4.5.2) for Y, we have an exact sequence

$$\mathrm{Hom}_{D(\mathcal{A})}(X, \tau^{\geq n+1} Y[-1]) \longrightarrow \mathrm{Hom}_{D(\mathcal{A})}(X, \tau^{\leq n} Y) \longrightarrow \mathrm{Hom}_{D(\mathcal{A})}(X, Y) \longrightarrow \mathrm{Hom}_{D(\mathcal{A})}(X, \tau^{\geq n+1} Y) \tag{4.5.6}$$

Since $\tau^{\geq n+1} Y[-1]$ and $\tau^{\geq n+1} Y$ belong to $D^{\geq n+1}(\mathcal{A})$, the first and fourth terms of (4.5.6) are zero by Proposition 4.5.4 (d). The second isomorphism follows by reversing the arrows. □

4.5.2 Resolutions

The derived category $D^*(\mathcal{A})$ is often a big category and this causes many problems. In this paragraph, by considering resolutions in the category $\mathrm{Ch}(\mathcal{A})$, we show that in some case $D^*(\mathcal{A})$ is equivalent to the homotopy category of a subcategory of $\mathcal{A}$, and hence a $\mathscr{U}$-category, where $\mathscr{U}$ is the chosen universe.

Lemma 4.5.8. *Let $\mathcal{J}$ be a full additive subcategory of $\mathcal{A}$ and $X^\bullet \in \mathrm{Ch}^{\geq n}(\mathcal{A})$ for some $n \in \mathbb{Z}$. Assume that one of the following conditions holds:*

(a) *$\mathcal{J}$ is cogenerating in $\mathcal{A}$ (i.e. for any $Y \in \mathcal{A}$ there exists a monomorphism $Y \to I$ with $I \in \mathcal{J}$);*

(b) *$\mathcal{J}$ is closed under extensions and cokernels of monomorphisms, and for any monomorphism $I' \to Y$ in $\mathcal{A}$ with $I' \in \mathcal{J}$, there exists a morphism $Y \to I$ with $I \in \mathcal{J}$ such that the composition $I' \to I$ is a monomorphism. Moreover, $H^i(X^\bullet) \in \mathcal{J}$ for all $i \in \mathbb{Z}$.*

Then there exists $Y^\bullet \in \mathrm{Ch}^{\geq n}(\mathcal{J})$ and a quasi-isomorphism $X^\bullet \to Y^\bullet$.

Proof. □

Let $\mathcal{J}$ be a full additive subcategory of $\mathcal{A}$. It is clear that for $* \in \{+, -, b, \varnothing\}$, the $N^*(\mathcal{J}) := N(\mathcal{A}) \cap K^*(\mathcal{J})$ is a null system in $K^*(\mathcal{J})$. We say that $\mathcal{A}$ has **finite $\mathcal{J}$-dimension** if there exists a non-negative integer d such that, for any exact sequence

$$Y_d \longrightarrow \cdots \longrightarrow Y_1 \longrightarrow Y \longrightarrow 0$$

with $Y_i \in \mathcal{J}$ for $1 \leq i \leq d$, we have $Y \in \mathcal{J}$.

Proposition 4.5.9. *Assume that $\mathcal{J}$ is cogenerating in $\mathcal{C}$, then the natural functor*

$$\theta^+ : K^+(\mathcal{J})/N^+(\mathcal{J}) \to D^+(\mathcal{C})$$

is an equivalence of categories. If $\mathcal{A}$ has finite $\mathcal{J}$-dimension, then $\theta^b : K^b(\mathcal{J})/N^b(\mathcal{J}) \to D^b(\mathcal{C})$ is also an equivalence of categories.

Proof. Let $X \in K^+(\mathcal{A})$. By Lemma 4.5.8, there exists $Y \in K^+(\mathcal{J})$ and a quasi-isomorphism $X \to Y$, so the first assertion follows from Proposition 4.4.31. Now assume that $\mathcal{A}$ has finite $\mathcal{J}$-dimension and that $X^i = 0$ for $i \geq n$, where $n \in \mathbb{Z}$. Then $\tau^{\leq i} Y \to Y$ is a quasi-isomorphism for $i \geq n$ and the hypothesis implies that $\tau^{\leq i} Y$ belongs to $K^b(\mathcal{J})$ for $i > n + d$. This proves the second assertion in view of Proposition 4.4.31. □

Let us apply the preceding proposition to the full subcategory of injective objects: $\mathcal{I}_\mathcal{A} = \{X \in \mathcal{A} : X \text{ is injective}\}$.

Proposition 4.5.10. *Assume that $\mathcal{A}$ admits enough injectives. Then the functor $K^+(\mathcal{I}_\mathcal{A}) \to D^+(\mathcal{A})$ is an equivalence of categories. If moreover $\mathcal{A}$ has finite injective dimension, then $K^b(\mathcal{I}_\mathcal{A}) \to D^b(\mathcal{A})$ is an equivalence of categories.*

Proof. By Proposition 4.5.9, it is enough to prove that if $X^\bullet \in \mathrm{Ch}^+(\mathcal{I}_\mathcal{A})$ is quasi-isomorphic to 0, then $X^\bullet$ is homotopic to 0. This is a particular case of the lemma below (choose $f = \mathrm{id}_{X^\bullet}$ in the lemma). □

Lemma 4.5.11. *Let $f : X^\bullet \to I^\bullet$ be a morphism in $\mathrm{Ch}(\mathcal{A})$. Assume that $I^\bullet$ belongs to $\mathrm{Ch}^+(\mathcal{I}_\mathcal{A})$ and $X^\bullet$ is exact. Then f is homotopic to 0.*

Corollary 4.5.12. *Let $\mathcal{A}$ be an abelian $\mathscr{U}$-category with enough injectives. Then $D^+(\mathcal{A})$ is a $\mathscr{U}$-category.*

Proposition 4.5.13. *Let $\mathcal{J}$ be a full additive subcategory of $\mathcal{A}$ and assume that $\mathcal{J}$ is cogenerating and $\mathcal{A}$ has finite $\mathcal{J}$-dimension. Then for any $X \in \mathrm{Ch}(\mathcal{A})$, there exists $Y \in \mathrm{Ch}(\mathcal{J})$ and a quasi-isomorphism $X \to Y$. In particular, there is an equivalence of triangulated categories $K(\mathcal{J})/N(\mathcal{J}) \xrightarrow{\sim} D(\mathcal{A})$.*

An important class of examples are given by Serre subcategories of $\mathcal{A}$: let $\mathcal{T}$ be a weak Serre subcategory of $\mathcal{A}$. For $* \in \{+, -, b, \varnothing\}$, we denote by $D^*_\mathcal{T}(\mathcal{A})$ the full additive subcategory of $D^*(\mathcal{A})$ consisting of objects X such that $H^i(X) \in \mathcal{T}$ for all $i \in \mathbb{Z}$. This is clearly a triangulated subcategory of $D(\mathcal{A})$, and there is a natural functor

$$\delta^* : D^*(\mathcal{T}) \to D^*_\mathcal{T}(\mathcal{A}). \tag{4.5.7}$$

Theorem 4.5.14. *Let $\mathcal{T}$ be a Serre subcategory of $\mathcal{A}$ and assume that for any monomorphism $Y \to X$, with $Y \in \mathcal{T}$, there exists a morphism $X \to Y'$ with $Y' \in \mathcal{T}$ such that the composition $Y \to Y'$ is a monomorphism. Then the functors δ^+ and δ^b in (4.5.7) are equivalences of categories.*

Proof. The result for δ^+ is an immediate consequence of Corollary 4.4.12 and Lemma 4.5.8 (b). The case of δ^b follows since $D^b(\mathcal{T})$ is equivalent to the full subcategory of $D^+(\mathcal{T})$ of objects with bounded cohomology, and similarly for $D^b_{\mathcal{T}}(\mathcal{A})$. □

Note that, by reversing the arrows in Theorem 4.5.14, the functors δ^- and δ^b in (4.5.7) are equivalences of categories if for any epimorphism $X \to Y$ with $Y \in \mathcal{T}$, there exists a morphism $Y' \to X$ with $Y' \in \mathcal{T}$ such that the composition $Y' \to Y$ is an epimorphism.

4.5.3 Bounded functors and the way-out lemma

We now introduce an important result on how a triangulated functor on derived categories is determined by its values on the underlying abelian category. This is useful when one want to show that some natural map is a functorial isomorphism.

In this paragraph, we consider abelian categories $\mathcal{A}$ and $\mathcal{A}'$, and additive functors between subcategories of $D(\mathcal{A})$ and $D(\mathcal{A}')$. We choose a weak Serre subcategory $\mathcal{T}$ of $\mathcal{A}$ and denote by $D^*_{\mathcal{T}}(\mathcal{A})$ the full additive subcategory of $D^*(\mathcal{A})$ consisting of objects X such that $H^i(X) \in \mathcal{T}$ for all $i \in \mathbb{Z}$. If $\mathcal{E}$ is a subcategory of $D(\mathcal{A})$, we write $\mathcal{E}^{\geq n}$ (resp. $\mathcal{E}^{\geq n}$) for the subcategories of $\mathcal{E}$ whose objects are complexes X such that $H^i(X) = 0$ for $i < n$ (resp. $i > n$).

Definition 4.5.2. Let $\mathcal{E}$ be a subcategory of $D(\mathcal{A})$ and let $F : \mathcal{E} \to D(\mathcal{A}')$ be an additive functor. The **upper dimension** $\dim^+$ and **lower dimension** $\dim^-$ of the functor F are defined by

$$\dim^+(F) := \inf\{d \in \mathbb{Z} : F(\mathcal{E}^{\leq n}) \subseteq D^{\leq n+d}(\mathcal{A}') \text{ for all } n \in \mathbb{Z}\},$$
$$\dim^-(F) := \inf\{d \in \mathbb{Z} : F(\mathcal{E}^{\geq n}) \subseteq D^{\geq n-d}(\mathcal{A}') \text{ for all } n \in \mathbb{Z}\}.$$

The functor F is called **bounded above** (resp. **bounded below**) if $\dim^+(F) < +\infty$ (resp. $\dim^-(F) < +\infty$), and **bounded** if it is both bounded above and bounded below.

Remark 4.5.3. If $F : \mathcal{E} \to D(\mathcal{A}')$ is compatible with the translation functors of $D(\mathcal{A})$ and $D(\mathcal{A}')$, then we see that $F(\mathcal{E}^{\geq n}) \subseteq D^{\geq n+d}(\mathcal{A}')$ holds for all $n \in \mathbb{Z}$ as soon as it holds for one single n, for example $n = 0$. Therefore, in this case we can also define $\dim^+(F)$ to be the smallest integer d such that $F(\mathcal{E}^{\leq 0}) \subseteq D^{\leq d}(\mathcal{A}')$, and similarly for $\dim^-(F)$.

Example 4.5.1. If $\mathcal{E}$ is a triangulated subcategory of $D(\mathcal{A})$ such that $\tau^{\geq n}(\mathcal{E}) \subseteq \mathcal{E}$ and $\tau^{\leq n}(\mathcal{E}) \subseteq \mathcal{E}$ (for example, if $\mathcal{E} = D^*_{\mathcal{T}}(\mathcal{A})$), and if F is a triangulated functor, then $\dim^+(F) \leq d$ if and only if for any $X \in \mathcal{E}$, $n \in \mathbb{Z}$, and $i \geq n + d$, the canonical morphism

$$H^i(F(X)) \to H^i F(\tau^{\geq n} X)$$

is an isomorphism. In fact, the implication $\Rightarrow$ follows from the exact sequence induced from the distinguished triangle (4.5.2), since we have $H^i(F(\tau^{n-1} X)) = 0$ in this case. The converse implication is obtained by taking X to be an arbitray complex in $\mathcal{E}^{\leq n-1}$. An equivalent condition is that if $f : X \to Y$ is a morphism in $\mathcal{E}$ such that $H^i(f)$ is an isomorphism for all $i \geq n$, (that is, if f induces an isomorphism $\tau^{\geq n} X \to \tau^{\geq n} Y$), then $H^i(F(f))$ is an isomorphism for all $i \geq n + d$. Similarly, we have $\dim^-(F) \leq d$ if and only if the canonical morphism

$$H^i(F(\tau^{\leq n} X)) \to H^i F(X)$$

is an isomorphism.

In particular, if $\mathcal{E} = \mathcal{T}$ is a weak Serre subcategory of $\mathcal{A}$ (also considered as a subcategory of $D(\mathcal{A})$), then $\mathcal{E}^{\geq 0} = \mathcal{E} = \mathcal{E}^{\leq 0}$, and we have

$$\dim^+(F) \leq d \Leftrightarrow H^i(F(X)) = 0 \text{ for all } i > d \text{ and } X \in \mathcal{T},$$
$$\dim^-(F) \leq d \Leftrightarrow H^i(F(X)) = 0 \text{ for all } i < -d \text{ and } X \in \mathcal{T}.$$

Proposition 4.5.15. *If $\mathcal{E} = D^*_{\mathcal{T}}(\mathcal{A})$ and F is a triangulated functor, then*

$$\dim^+(F) = \dim^+(F_0), \quad \dim^-(F) = \dim^-(F_0),$$

where F_0 is the restriction of F to $\mathcal{T}$.

Proof. We deal with the case for $\dim^+(F)$, the case for $\dim^-(F)$ can be done similarly. First, we note that $\dim^+(F_0) \leq \dim^+(F)$ since $F'(\mathcal{E}^{\leq n}) \subseteq F'(\mathcal{E}^{\leq n})$ for each $n \in \mathbb{Z}$. For the reverse inequality, we assume that $\dim^+(F_0) \leq d < +\infty$ and fix an integer $n \in \mathbb{Z}$. We prove that $H^i(F(X)) = 0$ for any $X \in \mathcal{E}^{\leq n}$ and $i > n + d$ by induction on the number $\nu = \nu(X)$ of non-vanishing cohomology objects of X. Since the case $\nu = 0$ is trivial, we may assume that $\nu \geq 1$. If $\nu = 1$, say $H = H^m(X) \neq 0$ for some $m \leq n$, and we have

$$X \cong \tau^{\leq m}\tau^{\geq m}X \cong H[-m]$$

by (4.5.5). Since F is a triangulated functor, we conclude that $F(X) \cong F(H)[-m]$, so by definition of $\dim^+(F_0)$,

$$H^i(F(X)) \cong H^{i-m}(F(H)) = H^{i-m}(F_0(H)) = 0 \quad \text{for} \quad i - m > d$$

whence the conclusion. If $\nu > 1$, we choose an integer s such that there exists integers $p < s \leq q$ with $H^p(X) \neq 0$ and $H^q(X) \neq 0$. Then $\nu(\tau^{\leq s-1}X) < \nu(X)$ and $\nu(\tau^{\geq s}X) < \nu(X)$, so by applying the induction hypothesis, we have

$$H^i(F(\tau^{\leq s-1}X)) = H^i(\tau^{\geq s}X) = 0 \quad \text{for} \quad i > n + d.$$

The inductive step then follows from the long exact sequence induced by the distinguished triangle (4.5.2). □

Proposition 4.5.16 (Way-out Lemma). *Let $\mathcal{T}$ be a weak Serre subcategory of $\mathcal{A}$ and $* \in \{+, b, \varnothing\}$. Consider triangulated functors $F, G : D^*_{\mathcal{T}}(\mathcal{A}) \to D(\mathcal{A}')$ and a morphism of functors $\eta : F \to G$, so that $\eta(X) : F(X) \to G(X)$ is an isomorphism for any $X \in \mathcal{T}$. If one of the following conditions holds, then η is an isomorphism:*

(i) $* = b$;

(ii) $* = +$ *and F, G are bounded below;*

(iii) $* = -$ *and F, G are bounded above;*

(iv) $* = \varnothing$ *and F, G are bounded.*

Moreover, if $\mathfrak{I}$ (resp. $\mathfrak{P}$) is a subset of $\mathrm{Ob}(\mathcal{T})$ *such that for any $X \in \mathcal{T}$ there exists a monomorphism $X \hookrightarrow I$ with $I \in \mathfrak{I}$ (resp. an epimorphism $P \to X$ with $P \in \mathfrak{P}$), then η is an isomorphism if $\eta(X) : F(X) \to G(X)$ is an isomorphism for each $X \in \mathfrak{I}$, and one of conditions (i), (ii) (resp. (i), (iii)) is satisfied.*

Proof. We first deal with the case $* = b$. Since η is a morphism of triangulated functors, we see by induction on $|n|$ that $\eta(X[n])$ is an isomorphism for each $X \in \mathcal{T}$ and $n \in \mathbb{Z}$. To see that $\eta(X)$ is an isomorphism for any $X \in D^*_{\mathcal{T}}(\mathcal{A})$, we may replace X with the isomorphic complex $\tau^{\leq n}(X)$ with some integer n large enough. From (4.5.3), we obtain a morphism of triangles, induced by η:

$$\begin{tikzcd}
F(H^n(X)[-n-1]) \arrow[r] \arrow[d] & F(\tau^{\leq n-1}X) \arrow[r] \arrow[d] & F(\tau^{\leq n}X) \arrow[r] \arrow[d] & F(H^n(X)[-n]) \arrow[d] \\
G(H^n(X)[-n-1]) \arrow[r] & G(\tau^{\leq n-1}X) \arrow[r] & G(\tau^{\leq n}X) \arrow[r] & G(H^n(X)[-n])
\end{tikzcd}$$

and then we can conclude the proposition by Proposition 4.4.19 and induction on the number of non-vanishing cohomology objects of X (a number which is less for $\tau^{\leq n-1}X$ than for X whenever n is finite).

As for the case of (ii), by Proposition 4.5.1, it suffices to show that $\eta(X)$ induces an isomorphism from $H^i(F(X))$ to $H^i(G(X))$ for any $X \in D^+_{\mathcal{T}}(\mathcal{A})$ and all $i \in \mathbb{Z}$. For this, we may apply Example 4.5.1 to replace X by $\tau^{\leq i+d}X \in D^b_{\mathcal{T}}(\mathcal{A})$ for $d \geq \max\{\dim^-(F), \dim^-(G)\}$, and then we can apply the conclusion of (i). The case for (iii) can be proved similarly.

We now consider the case where $* = \varnothing$. In view of (4.5.2), we have a morphism of triangles, induced by η:

$$\begin{tikzcd}
F(\tau^{\geq 0}(X)[-1]) \arrow[r] \arrow[d] & F(\tau^{\leq -1}X) \arrow[r] \arrow[d] & F(X) \arrow[r] \arrow[d] & F(\tau^{\geq 0}(X)) \arrow[d] \\
G(\tau^{\geq 0}(X)[-1]) \arrow[r] & G(\tau^{\leq -1}X) \arrow[r] & G(X) \arrow[r] & G(\tau^{\geq 0}(X))
\end{tikzcd}$$

and by induction on the number of non-vanishing cohomology objects, we may assume that the vertical morphisms, except $F(X) \to G(X)$, are all isomorphisms. It then follows from Proposition 4.4.19 that $F(X) \to G(X)$ is also an isomorphism.

Finally, as for the last assertion (we consider the case for $\mathfrak{I}$, the other case can be proved similarly), it suffices to show that $\eta(X)$ is an isomorphism for any $X \in \mathcal{T}$. We take an exact sequence $0 \to X \to I^0 \to I^1 \to \cdots$ so that each $I^i \in \mathfrak{I}$. Then gives rise to a quasi-isomorphism $X \to I$, so it remains to show that $\eta(I)$ is an isomorphism. For this, we consider the stupid truncations $\sigma^{\leq n}I$ and $\sigma^{\geq n+1}I$, which fit into an exact sequence

$$0 \longrightarrow \sigma^{\geq n+1} \longrightarrow I \longrightarrow \tau^{\leq n}I \longrightarrow 0$$

and we have a corresponding distinguished triangle in $D(\mathcal{A})$; it then suffices to mimic the proof of (ii). $\square$

4.5.4 Derived functors

Let $\mathcal{A}$, $\mathcal{A}'$ and $\mathcal{A}''$ be abelian categories and $F : \mathcal{A} \to \mathcal{A}'$ be an additive functor. Then F defines naturally a triangulated functor

$$K^*(F) : K^*(\mathcal{A}) \to K^*(\mathcal{A}').$$

For short, we often write F instead of $K^*(F)$. We shall denote by $Q : K^*(\mathcal{A}) \to D^*(\mathcal{A})$ the localization functor, and similarly with Q', Q'', when replacing $\mathcal{A}$ with $\mathcal{A}'$, $\mathcal{A}''$.

Definition 4.5.3. Let $* \in \{+, b, \varnothing\}$. We say that the functor F is **right derivable** (or F admits a **right derived functor**) on $K^*(\mathcal{A})$ if the triangulated functor $K^*(F) : K^*(\mathcal{A}) \to K^*(\mathcal{A}')$ is universally right localizable with respect to $N^*(\mathcal{A})$ and $N^*(\mathcal{A}')$. In such a case the localization of F is denoted by R^*F and $H^n \circ R^*F$ is denoted by R^nF. The functor $R^*F : D^*(\mathcal{A}) \to D^*(\mathcal{A}')$ is called the **right derived functor** of F and R^nF the **n-th derived functor** of F.

By definition, the functor F admits a right derived functor on $K^*(\mathcal{A})$ if the ind-object

$$\varinjlim_{\substack{(X\to X')\in \mathrm{Qis}\\ X'\in K^*(\mathcal{A})}} Q'\circ K(F)(X')$$

is representable in $D^*(\mathcal{A}')$ for all $X\in K^*(\mathcal{A})$. In such a case, this object is isomorphic to $R^*F(X)$. Note that R^*F is a triangulated functor from $D^*(\mathcal{A})$ to $D^*(\mathcal{A})$ if it exists, and R^nF is a cohomological functor from $D^*(\mathcal{A})$ to $\mathcal{A}'$. Morever, if RF exists, then R^+F exists and R^+F is the restriction of RF to $D^+(\mathcal{A})$.

Definition 4.5.4. Let $\mathcal{J}$ be a full additive subcategory of $\mathcal{A}$. We say for short that $\mathcal{J}$ is ***F*-injective** if the subcategory $K^+(\mathcal{J})$ of $K^+(\mathcal{A})$ is $K^+(F)$-injective with respect to $N^+(\mathcal{A})$ and $N^+(\mathcal{A}')$. We shall also say that $\mathcal{J}$ is injective with respect to F. We define similarly the notion of an F-projective subcategory.

By the definition, $\mathcal{J}$ is F-injective if and only if for any $X\in K^+(\mathcal{A})$, there exists a quasi-isomorphism $X\to Y$ with $Y\in K^+(\mathcal{J})$ and $F(Y)$ is exact for any exact complex $Y\in K^+(\mathcal{J})$. If F is right (resp. left) derivable, an object X of $\mathcal{A}$ such that $R^nF(X)=0$ (resp. $L^nF(X)=0$) for all $n\neq 0$ is called **right *F*-acyclic** (resp. **left *F*-acyclic**). If $\mathcal{J}$ is an F-injective subcategory, then any object of $\mathcal{J}$ is right F-acyclic.

From Proposition 4.4.35, it is immediate that we have the following result concerning composition of derived functors:

Proposition 4.5.17. *Let $F:\mathcal{A}\to\mathcal{A}'$ and $F':\mathcal{A}'\to\mathcal{A}''$ be additive functors of abelian categories. Let $*\in\{+,b,\varnothing\}$ and assume that the right derived functors R^*F, R^*F' and $R^*(F'\circ F)$ exist. Then there is a canonical morphism of functors*

$$R^*(F'\circ F)\to R^*(F')\circ R^*(F). \tag{4.5.8}$$

Assume that there exist full additive subcategories $\mathcal{J}\subseteq\mathcal{A}$ and $\mathcal{J}'\subseteq\mathcal{A}'$ such that $\mathcal{A}$ is F-injective, $\mathcal{J}'$ is F' injective and $F(\mathcal{J})\subseteq\mathcal{J}'$. Then $\mathcal{J}$ is $F'\circ F$-injective and (4.5.8) induces an isomorphism

$$R^+(F'\circ F)\xrightarrow{\sim} R^+F'\circ R^+F.$$

Note that in many cases (even if F is exact), F may not send injective objects of $\mathcal{A}$ to injective objects of $\mathcal{A}'$. This is a reason why the notion of an "F-injective" category is useful.

Proposition 4.5.18. *Let $F:\mathcal{A}\to\mathcal{A}'$ be an additive functor of abelian categories and let $\mathcal{J}$ be a full additive subcategory of $\mathcal{A}$.*

(a) *If $\mathcal{J}$ is F-injective, then $R^+F: D^+(\mathcal{A})\to D^+(\mathcal{A}')$ exists and $R^+F(X)\cong F(Y)$ for any quasi-isomorphism $X\to Y$ with $Y\in K^+(\mathcal{J})$.*

(b) *If F is left exact, then $\mathcal{J}$ is F-injective if and only if it satisfies the following conditions:*

  (i) *the category $\mathcal{J}$ is cogenerating in $\mathcal{A}$;*

  (ii) *for any exact sequence $0\to X'\to X\to X''\to 0$, the sequence $0\to F(X')\to F(X)\to F(X'')\to 0$ is exact as soon as $X\in\mathcal{J}$ and there exists an exact sequence*

$$0\longrightarrow Y^0\longrightarrow\cdots\longrightarrow Y^n\longrightarrow X'\longrightarrow 0$$

  with $Y^i\in\mathcal{J}$.

Proof. The first assertion follows from Proposition 4.4.34 and (4.4.11), so assume that F is left exact. If $\mathcal{J}$ is F-injective, then for $X \in \mathcal{A}$, there exists a quasi-isomorphism $X \to Y$ with $Y \in K^+(\mathcal{J})$. The composition $X \to \ker d_Y^0 \to H^0(Y)$ is then an isomorphism, so $X \to Y^0$ is a monomorphism and this proves that $\mathcal{J}$ is cogenerating in $\mathcal{A}$. By Lemma 4.5.8, there then exists an exact sequence $0 \to X'' \to Z^0 \to Z^1 \to \cdots$ with $Z^i \in \mathcal{J}$ for all i. The sequence

$$0 \longrightarrow Y^0 \longrightarrow \cdots \longrightarrow Y^n \longrightarrow X \longrightarrow Z^0 \longrightarrow Z^1 \longrightarrow \cdots$$

is then exact and belongs to $K^+(\mathcal{J})$, so $F(X) \to F(Z^0) \to F(Z^1)$ is exact. Since F is left exact, $F(X'') \cong \ker(F(Z^0) \to F(Z^1))$ and this implies that $F(X) \to F(X'')$ is an epimorphism.

Conversely, assume the two conditions in (b). By Lemma 4.5.8, for any $X \in K^+(\mathcal{A})$ there exists a quasi-isomorphism $X \to Y$ with $Y \in K^+(\mathcal{J})$, so it suffices to show that $F(X)$ is exact if $X \in K^+(\mathcal{J})$ is exact. To this end, we note that for each $n \in \mathbb{Z}$, the sequences

$$\cdots \longrightarrow X^{n-2} \longrightarrow X^{n-1} \longrightarrow \ker d_X^n \longrightarrow 0$$

$$0 \longrightarrow \ker d_X^n \longrightarrow X^n \longrightarrow \ker d_X^{n+1} \longrightarrow 0$$

are exact, so by condition (ii), the sequence $0 \to F(\ker d_X^n) \to F(X^n) \to F(\ker d_X^{n+1}) \to 0$ is exact, and this proves that $F(X)$ is exact. $\square$

Remark 4.5.4. Note that for $X \in \mathcal{A}$, $R^nF(X) = 0$ for $n < 0$ and assuming that F is left exact, $R^0F(X) \cong F(X)$. Indeed, for $X \in \mathcal{A}$ and any quasi-isomorphism $X \to Y$, the composition $X \to Y \to \tau^{\geq 0}Y$ is a quasi-isomorphism.

Example 4.5.2. If $\mathcal{A}$ has enough injectives, then the full subcategory $\mathcal{I}_{\mathcal{A}}$ of injective objects of $\mathcal{A}$ is F-injective for any additive functor $F : \mathcal{A} \to \mathcal{A}'$. Indeed, any exact complex in $\mathrm{Ch}^+(\mathcal{I})$ is homotopic to zero by Lemma 4.5.11, whence its image under F. In particular, $R^+F : D^+(\mathcal{A}) \to D^+(\mathcal{A}')$ exists in this case.

We shall now give a sufficient condition in order that $\mathcal{J}$ is F-injective, which is especially useful if the category $\mathcal{A}$ does not have enough injectives.

Theorem 4.5.19. *Let $\mathcal{J}$ be a full additive subcategory of $\mathcal{A}$ and let $F : \mathcal{A} \to \mathcal{A}'$ be a left exact functor. Assume that*

(a) *$\mathcal{J}$ is cogenerating in $\mathcal{A}$;*

(b) *for any monomorphism $Y' \hookrightarrow X$ with $Y' \in \mathcal{J}$, there exists an exact sequence $0 \to Y' \to Y \to Y'' \to 0$ with $Y, Y'' \in \mathcal{J}$ such that $Y' \to Y$ factors through X and the sequence $0 \to F(Y') \to F(Y) \to F(Y'') \to 0$ is exact.*

Then $\mathcal{J}$ is F-injective.

Condition (b) of Theorem 4.5.19 may be visualized as

$$\begin{tikzcd}
0 \arrow[r] & Y' \arrow[r] \arrow[d, equal] & X \arrow[d, dashed] & & \\
0 \arrow[r] & Y' \arrow[r, dashed] & Y \arrow[r, dashed] & Y'' \arrow[r] & 0
\end{tikzcd}$$

Since this condition is rather intricate, the often consider the following particular case of Theorem 4.5.19, which is sufficient for most applications.

Corollary 4.5.20. *Let $\mathcal{J}$ be a full additive subcategory of $\mathcal{A}$ and let $F : \mathcal{A} \to \mathcal{A}'$ be a left exact functor. Assume that*

(a) $\mathcal{J}$ *is cogenerating in* $\mathcal{A}$*;*

(b) $\mathcal{J}$ *is closed under cokernels of monomorphisms;*

(c) *for any exact sequence* $0 \to X' \to X \to X'' \to 0$ *in* $\mathcal{A}$ *with* $X', X \in \mathcal{J}$*, the sequence* $0 \to F(X') \to F(X) \to F(X'') \to 0$ *is exact.*

Then $\mathcal{J}$ *is F-injective.*

Proof. For any monomorphism $Y' \to X$ with $Y' \in \mathcal{J}$, we can embedd X into an object $Y \in \mathcal{J}$ and take Y'' to be the cokernel of Y by Y':

$$\begin{tikzcd} 0 \arrow[r] & Y' \arrow[r] \arrow[d, equal] & X \arrow[d, hook] & & \\ 0 \arrow[r] & Y' \arrow[r] & Y \arrow[r] & Y'' \arrow[r] & 0 \end{tikzcd}$$

By hypothesis, we have $Y'' \in \mathcal{J}$, and the sequence $0 \to F(X') \to F(X) \to F(X'') \to 0$ is exact, so we can apply Theorem 4.5.19. □

Corollary 4.5.21. *Let* $F : \mathcal{A} \to \mathcal{A}'$ *be a left exact functor of abelian categories and let* $\mathcal{J}$ *be an F-injective full subcategory of* $\mathcal{A}$*. Let* $\mathcal{J}_F$ *be the full subcategory of* $\mathcal{A}$ *consisting of right F-acyclic objects, then* $\mathcal{J}_F$ *contains* $\mathcal{J}$ *and* $\mathcal{J}_F$ *satisfies the conditions of Corollary 4.5.20. In particular,* $\mathcal{J}_F$ *is F-injective.*

Proof. Since $\mathcal{J}_F$ contains $\mathcal{J}$, $\mathcal{J}_F$ is cogenerating. Consider an exact sequence $0 \to X' \to X \to X'' \to 0$ in $\mathcal{A}$ with $X', X \in \mathcal{J}_F$. The exact sequences $R^iF(X) \to R^iF(X'') \to R^{i+1}F(X')$ for $i \geq 0$ imply that $R^iF(X'') = 0$ for $i > 0$. Moreover, there is an exact sequence $0 \to F(X') \to F(X) \to F(X'') \to 0$ since $R^1F(X') = 0$. □

Therefore, a full additive subcategory $\mathcal{J}$ of $\mathcal{A}$ is F-injective if and only if it is cogenerating and any object of $\mathcal{J}$ is F-acyclic (assuming the right derivability of F). Note that even if F is right derivable, there may not exist an F-injective subcategory, since we do not know that whether the subcategory of F-acyclic objects is cogenerating.

Example 4.5.3. Let A be a ring and let N be a right A-module. The full additive subcategory of $\mathbf{Mod}(A)$ consisting of flat A-modules is $(N \otimes_A -)$-projective. In fact, this subcategory satisfies the dual conditions of Corollary 4.5.20.

We now turn to the proof of Theorem 4.5.19, which we decompose into several lemmas.

Lemma 4.5.22. *With the assumptions of Theorem 4.5.19, let* $0 \to Y' \to X \to X''$ *be an exact sequence in* $\mathcal{A}$ *with* $Y' \in \mathcal{J}$*. Then the sequence* $0 \to F(Y') \to F(X) \to F(X'')$ *is exact.*

Proof. Choose an exact sequence $0 \to Y' \to Y \to Y''$ as in Theorem 4.5.19. We get the commutative exact diagram:

$$\begin{tikzcd} 0 \arrow[r] & Y' \arrow[r] \arrow[d, equal] & X \arrow[r] \arrow[d] & X'' \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & Y' \arrow[r] & Y \arrow[r] & Y'' \arrow[r] & 0 \end{tikzcd}$$

where the right square is Cartesian. Since F is left exact, it transforms this square to a Cartesian square and the bottom row to an exact row. Hence, the result follows from ([?] lemma 8.3.11). □

Lemma 4.5.23. *With the assumptions of Theorem 4.5.19, let* $X^\bullet \in \mathrm{Ch}^+(\mathcal{A})$ *be an exact complex, and assume* $X^i = 0$ *for* $i < n$ *and* $X^n \in \mathcal{J}$*. There exist an exact complex* $Y^\bullet \in \mathrm{Ch}^+(\mathcal{J})$ *and a morphism* $f : X^\bullet \to Y^\bullet$ *such that* $Y^i = 0$ *for* $i < n$*,* $f^n : X^n \to Y^n$ *is an isomorphism, and* $\operatorname{im} d_Y^i \in \mathcal{J}$ *for all* i*.*

Proof. We argue by induction. By the hypothesis of Theorem 4.5.19, there exists a commutative exact diagram:

$$\begin{tikzcd} 0 \arrow[r] & X^n \arrow[r] \arrow[d, "\sim"] & X^{n+1} \arrow[d] & & \\ 0 \arrow[r] & Y^n \arrow[r] & Y^{n+1} \arrow[r] & Z^{n+2} \arrow[r] & 0 \end{tikzcd}$$

with Y^{n+1}, Z^{n+2} in $\mathcal{J}$. Now suppose that we have already constructed a diagram

$$\begin{tikzcd} 0 \arrow[r] & X^n \arrow[r] \arrow[d, "\sim"] & \cdots \arrow[r] & X^m \arrow[d] & & \\ 0 \arrow[r] & Y^n \arrow[r] & \cdots \arrow[r] & Y^m \arrow[r] & Z^{m+1} \arrow[r] & 0 \end{tikzcd}$$

where the bottom row is exact and belongs to $\mathrm{Ch}^+(\mathcal{J})$, and $\operatorname{im} d_Y^i$ belongs to $\mathcal{J}$ for $n \le i \le m-1$. Define $W^{m+1} = X^{m+1} \oplus_{\operatorname{coker} d_X^{m-1}} Z^{m+1}$, so that we have a Cartesian exact diagram

$$\begin{tikzcd} 0 \arrow[r] & \operatorname{coker} d_X^{m-1} \arrow[r] \arrow[d] & X^{m+1} \arrow[d] \\ 0 \arrow[r] & Z^{m+1} \arrow[r] & W^{m+1} \end{tikzcd}$$

By the hypotheses, there exists an exact commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & Z^{m+1} \arrow[r] \arrow[d, equal] & W^{m+1} \arrow[d] & & \\ 0 \arrow[r] & Z^{m+1} \arrow[r] & Y^{m+1} \arrow[r] & Z^{m+2} \arrow[r] & 0 \end{tikzcd}$$

with Y^{m+1} and Z^{m+2} in $\mathcal{J}$. If we define d_Y^m to be the composition $Y^m \to Z^{m+1} \to Y^{m+1}$, then $\operatorname{im} d_Y^m \cong Z^{m+1} \in \mathcal{J}$, and this completes the induction process. $\square$

Proof of Theorem 4.5.19. Let $X^\bullet \in \mathrm{Ch}^+(\mathcal{J})$ be an exact complex, we have to prove that $F(X^\bullet)$ is exact. Let us show by induction on $m-n$ that $H^m(F(X^\bullet)) = 0$ if $X \in \mathrm{Ch}^{\ge n}(\mathcal{J})$. If $m < n$, this is clear, so we may assume that $m \ge n$. By Lemma 4.5.23, there exists a morphism of complexes $f : X^\bullet \to Y^\bullet$ in $\mathrm{Ch}^+(\mathcal{J})$ such that $Y^\bullet \in \mathrm{Ch}^{\ge n}(\mathcal{J})$, $f^n : X^n \to Y^n$ is an isomorphism and $F(Y^\bullet)$ is exact. Let $\sigma^{\ge n+1}$ denote the stupid truncated complexes and W denote the mapping cone of the morphism

$$\sigma^{\ge n+1}(f) : \sigma^{\ge n+1}X^\bullet \to \sigma^{\ge n+1}Y^\bullet.$$

Then $W^i = (\sigma^{\ge n+1}X^\bullet)^{i+1} \oplus (\sigma^{\ge n+1}Y^\bullet)^i = 0$ for $i < n$, and we have a distinguished triangle in $K(\mathcal{J})$:

$$W \longrightarrow M(f) \longrightarrow M(X^n[-n] \to Y^n[-n]) \longrightarrow W[1].$$

Since $X^n \to Y^n$ is an isomorphism, the mapping cone $M(X^n[-n] \to Y^n[-n])$ is exact and therefore $W \to M(f)$ is an isomorphism in $K(\mathcal{A})$. Applying the functor F, we then obtain an isomorphism $F(W) \cong F(M(f))$ in $K(\mathcal{A}')$, so $H^i(F(W)) \cong H^i(F(M(f)))$ for each i. On the other hand, there is a distinguished triangle in $K^+(\mathcal{A}')$:

$$F(X) \longrightarrow F(Y) \longrightarrow F(M(f)) \longrightarrow F(X)[1]$$

and $F(Y)$ is exact by our hypothesis, whence $H^m(F(X)) \cong H^{m-1}(F(M(f))) \cong H^{m-1}(F(W))$. Since W is an exact complex and belongs to $\mathrm{Ch}^{\ge n}(\mathcal{J})$, the induction hypothesis implies that $H^{m-1}(F(W)) = 0$, so we conclude that $H^m(F(X)) = 0$. $\square$

4.5.4.1 Derived projective limit As an application of Theorem 4.5.19 we shall discuss the existence of the derived functor of projective limits. Let $\mathcal{A}$ be an abelian $\mathcal{U}$-category. Recall that $\mathrm{Pro}(\mathcal{A})$ is an abelian category admitting small projective limits, and small filtrant projective limits as well as small products are exact. Assume that $\mathcal{A}$ admits small projective limits. Then the natural exact functor $\mathcal{A} \to \mathrm{Pro}(\mathcal{A})$ admits a right adjoint

$$\pi_{\mathcal{A}} : \mathrm{Pro}(\mathcal{A}) \to \mathcal{A}$$

defined as follows: if $\beta : I^{\mathrm{op}} \to \mathcal{A}$ is a functor with I small and filtrant, then $\pi_{\mathcal{A}}$ transforms $"\varprojlim"\beta$ (as a pro-object) to the limit $\varprojlim \beta$ in $\mathcal{A}$. The functor $\pi_{\mathcal{A}}$ is left exact and we shall give a condition in order that it is right derivable.

For a full additive subcategory $\mathcal{J}$ of $\mathcal{A}$, we define a full additive subcategory $\mathcal{J}_{\mathrm{pro}}$ of $\mathrm{Pro}(\mathcal{A})$ by

$$\mathcal{J}_{\mathrm{pro}} = \{X \in \mathrm{Pro}(\mathcal{A}) : X \cong "\prod_{i\in I}"X_i \text{ for a small set } I \text{ and } X_i \in \mathcal{J}\}.$$

Here the product $"\prod"$ is taken in the category $\mathrm{Pro}(\mathcal{A})$, so for $X_i, Y \in \mathcal{A}$, we have a canonical bijection

$$\mathrm{Hom}_{\mathrm{Pro}(\mathcal{A})}("\prod_{i\in I}"X_i, Y) \xrightarrow{\sim} \bigoplus_{i\in I} \mathrm{Hom}_{\mathcal{A}}(X_i, Y).$$

Proposition 4.5.24. *Let $\mathcal{A}$ be an abelian category admitting small projective limits and let $\mathcal{J}$ be a full additive subcategory of $\mathcal{A}$ satisfying:*

(a) *$\mathcal{J}$ is cogenerating in $\mathcal{A}$;*

(b) *$\mathcal{J}$ is closed under cokernels of monomorphisms;*

(c) *if $0 \to Y_i' \to Y_i \to Y_i'' \to 0$ is a family of exact sequences in $\mathcal{J}$ indexed by a small set I, then the sequence $0 \to Y_i' \to Y_i \to Y_i'' \to 0$ is exact.*

Then the category $\mathcal{J}_{\mathrm{pro}}$ is $\pi_{\mathcal{A}}$-injective and the left exact functor $\pi_{\mathcal{A}}$ admits a right derived functor

$$R^+\pi_{\mathcal{A}} : D^+(\mathrm{Pro}(\mathcal{A})) \to D^+(\mathcal{A})$$

which satisfies $R^n\pi_{\mathcal{A}}("\prod_i"X_i) = 0$ for $n > 0$ and $X_i \in \mathcal{J}$. Moreover, the composition

$$D^+(\mathcal{A}) \longrightarrow D^+(\mathrm{Pro}(\mathcal{A})) \xrightarrow{R^+\pi_{\mathcal{A}}} D^+(\mathcal{A})$$

is isomorphic to the identity.

Proof. We shall verify the conditions of Theorem 4.5.19. The category $\mathcal{J}_{\mathrm{pro}}$ is cogenerating in $\mathrm{Pro}(\mathcal{A})$ since for $A = "\varprojlim"_{i\in I}\alpha(i) \in \mathrm{Pro}(\mathcal{A})$, we obtain a monomorphism $A \hookrightarrow "\prod"_i X_i$ by choosing a monomorphism $\alpha(i) \hookrightarrow X_i$ with $X_i \in \mathcal{J}$ for each $i \in I$. Now consider a monomorphism $Y \hookrightarrow A$ in $\mathrm{Pro}(\mathcal{A})$ with $A \in \mathrm{Pro}(\mathcal{A})$ and $Y = "\prod"_i Y_i$, $Y_i \in \mathcal{J}$. Applying the dual version of ([**?**] proposition 8.6.9), for each i we can find $X_i \in \mathcal{A}$ and a commutative exact diagram

$$\begin{tikzcd} 0 \arrow[r] & Y \arrow[r] \arrow[d] & A \arrow[d] \\ 0 \arrow[r] & Y_i \arrow[r] & X_i \end{tikzcd}$$

Since $\mathcal{J}$ is cogenerating, we may assume that $X_i \in \mathcal{J}$. Let $Z_i = \mathrm{coker}(Y_i \to X_i)$, which is in $\mathcal{J}$ by hypothesis. By hypothesis, functor $\prod$ is exact on $\mathcal{J}$, so we obtain a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & Y \arrow[r] \arrow[d, equal] & A \arrow[d] & & \\ 0 \arrow[r] & "\prod"_i Y_i \arrow[r] & "\prod"_i X_i \arrow[r] & "\prod"_i Z_i \arrow[r] & 0 \end{tikzcd}$$

with exact rows. Applying $\pi_{\mathcal{A}}$ to the second row, we then obtain the sequence $0 \to \prod_i Y_i \to \prod_i X_i \to \prod_i Z_i \to 0$ in $\mathcal{A}$, which is exact by hypothesis (c). By Theorem 4.5.19, we then conclude that $\mathcal{J}_{\text{pro}}$ is $\pi_{\mathcal{A}}$-injective, so $\pi_{\mathcal{A}}$ admits a right derived functor (Proposition 4.4.34), and we have $R^n\pi_{\mathcal{A}}(\text{"}\prod\text{"}_i X_i) = 0$ for $n > 0$ and $X_i \in \mathcal{J}$.

Finally, by assumption $\mathcal{J}$ is injective with respect to the exact functor $\mathcal{A} \to \text{Pro}(\mathcal{A})$. Since the functor $\mathcal{A} \to \text{Pro}(\mathcal{A})$ sends $\mathcal{J}$ to $\mathcal{J}_{\text{pro}}$, the last assertion follows from Proposition 4.5.17. □

Corollary 4.5.25. *Let $\mathcal{J}$ be a full additive subcategory of $\mathcal{A}$ satisfying the conditions of Proposition 4.5.24. If (X_n) is a projective system in $\mathcal{J}$ indexed by $\mathbb{N}$ and $X = \text{"}\varprojlim\text{"} X_n$, then $R^p\pi_{\mathcal{A}}(X) = 0$ for $p > 1$, and we have a canonical isomorphism*

$$R^1\pi_{\mathcal{A}}(X) \xrightarrow{\sim} \operatorname{coker}\left(\prod_n X_n \xrightarrow{\Delta_X} \prod_n X_n\right)$$

where $\Delta_X := T_X - \mathrm{id} : \prod_n X_n \to \prod_n X_n$.

Proof. We have an exact sequence in $\text{Pro}(\mathcal{A})$:

$$0 \longrightarrow \text{"}\varprojlim_n\text{"} X_n \longrightarrow \text{"}\prod\text{"}_n X_n \xrightarrow{\Delta_X} \text{"}\prod\text{"}_n X_n \longrightarrow 0$$

Applying the functor $R^+\pi_{\mathcal{A}}$, we then get a long exact sequence and the results follows since $R^p\pi_{\mathcal{A}}(\prod_n X_n) = 0$ for $p > 0$. □

Example 4.5.4. If A is a ring and $\mathcal{A} = \mathbf{Mod}(A)$, we may choose $\mathcal{J} = \mathcal{A}$ in Proposition 4.5.24. In fact, for a family of objects $(X_i)_{i\in I}$ in $\mathcal{A}$, the product $\text{"}\prod\text{"}_i X_i$ can be considered as the limit of the functor $\alpha : I \to \mathcal{A}$ with I being considered as a discrete category.

4.5.4.2 Derived bifunctors

We shall now apply the previous results to bifunctors between abelian categories. The most important examples in mind will be Hom and tensor functors.

Theorem 4.5.26. *Let $\mathcal{A}$ be an abelian category and $X, Y \in D(\mathcal{A})$. Assume that the functor*

$$\mathrm{Hom}^\bullet_{\mathcal{A}} : K(\mathcal{A})^{\text{op}} \times K(\mathcal{A}) \to K(\mathbf{Mod}(\mathbb{Z})), \quad (X', Y') \mapsto \mathrm{Hom}^\bullet_{\mathcal{A}}(X', Y')$$

is right localizable at (X, Y), then for any $n \in \mathbb{Z}$, we have

$$R^n\mathrm{Hom}_{\mathcal{A}}(X, Y) \cong \mathrm{Hom}_{D(\mathcal{A})}(X, Y[n]).$$

Proof. By hypothesis, we have

$$R\mathrm{Hom}_{\mathcal{A}}(X, Y) \xrightarrow{\sim} \varinjlim_{\substack{(X'\to X)\in\mathrm{Qis},\\ (Y\to Y')\in\mathrm{Qis}}} \mathrm{Hom}^\bullet_{\mathcal{A}}(X', Y').$$

Applying the functor H^n and recalling that $\varinjlim$ commutes with H^n, we conclude from ([?] Proposition 11.7.3) that

$$R^n\mathrm{Hom}_{\mathcal{A}}(X, Y) \cong \varinjlim H^n(\mathrm{Hom}^\bullet_{\mathcal{A}}(X', Y')) \cong \varinjlim \mathrm{Hom}_{K(\mathcal{A})}(X', Y'[n]) \cong \mathrm{Hom}_{D(\mathcal{A})}(X, Y[n]).$$

□

Consider now three abelian categories $\mathcal{A}$, $\mathcal{A}'$, $\mathcal{A}''$ and an additive bifunctor

$$F : \mathcal{A} \times \mathcal{A}' \to \mathcal{A}''.$$

By ([?] Proposition 11.6.3), the triangulated functor

$$K^+F : K^+(\mathcal{A}) \times K^+(\mathcal{A}') \to K^+(\mathcal{A}'')$$

is naturally defined by setting

$$K^+F(X,X') = \mathrm{Tot}(F(X,X')).$$

Similarly to the case of functors, if the triangulated bifunctor K^+F is universally right localizable with respect to $(N^+(\mathcal{A}) \times N^+(\mathcal{A}'), N^+(\mathcal{A}''))$, then F is said to be **right derivable** and its localization is denoted by R^+F. We set $R^nF = H^n \circ R^+F$ and call it the n-th **derived bifunctor** of F.

Definition 4.5.5. Let $\mathcal{J}$ and $\mathcal{J}'$ be full additive subcategories of $\mathcal{A}$ and $\mathcal{A}'$ respectively. We say for short that $(\mathcal{J},\mathcal{J}')$ is ***F*-injective** if $(K^+(\mathcal{J}), K^+(\mathcal{J}'))$ is K^+F-injective.

Proposition 4.5.27. *Let $\mathcal{J}$ and $\mathcal{J}'$ be full additive subcategories of $\mathcal{A}$ and $\mathcal{A}'$ respectively. Assume that $(\mathcal{J},\mathcal{J}')$ is F-injective, then F is right derivable and for $(X,X') \in D^+(\mathcal{A}) \times D^+(\mathcal{A}')$ we have*

$$R^+F(X,X') \cong Q'' \circ K^+F(Y,Y')$$

for $(X \to Y) \in \mathrm{Qis}$ and $(X' \to Y') \in \mathrm{Qis}$ with $Y \in K^+(\mathcal{J})$ and $Y' \in K^+(\mathcal{J}')$.

Proof. It suffices to apply Proposition 4.4.36 to the functor $Q'' \circ K^+F$. □

Proposition 4.5.28. *Let $\mathcal{J}$ and $\mathcal{J}'$ be full additive subcategories of $\mathcal{A}$ and $\mathcal{A}'$ respectively. Assume that*

(a) *for any $Y \in \mathcal{J}$, $\mathcal{J}'$ is $F(Y,-)$-injective;*

(b) *for any $Y' \in \mathcal{J}'$, $\mathcal{J}$ is $F(-,Y')$-injective.*

Then $(\mathcal{J},\mathcal{J}')$ is F-injective.

Proof. Let $(Y,Y') \in K^+(\mathcal{J}) \times K^+(\mathcal{J}')$. If either Y or Y' is quasi-isomorphic to zero, then $\mathrm{Tot}(F(Y,Y'))$ is quasi-isomorphic to zero by ([?] Proposition 12.5.5), so $(\mathcal{J},\mathcal{J}')$ is F-injective. □

Corollary 4.5.29. *Let $\mathcal{J}$ be a full additive cogenerating subcategory of $\mathcal{A}$ and assume:*

(a) *for any $X \in \mathcal{J}$, $F(X,-)$ is exact;*

(b) *for any $X' \in \mathcal{A}'$, $\mathcal{J}$ is $F(-,X')$-injective.*

Then F is right derivable and for $X \in K^+(\mathcal{A})$, $X' \in K^+(\mathcal{A}')$,

$$R^+F(X,X') \cong Q'' \circ K^+F(Y,X')$$

for any $(X \to Y) \in \mathrm{Qis}$ with $Y \in K^+(\mathcal{J})$. In particular, for $X \in \mathcal{A}$ and $X' \in \mathcal{A}'$, $R^+F(X,X')$ is the derived functor of $F(-,X')$ calculated at X, that is, we have

$$R^+F(X,X') = (R^+F(-,X'))(X).$$

Proof. The first assertion follows from Proposition 4.5.28 by setting $\mathcal{J}' = \mathcal{A}'$, and the second one follows from Corollary 4.4.37. □

Corollary 4.5.30. *Let $\mathcal{A}$ be an abelian category and assume that there are subcategories $\mathcal{P},\mathcal{J}$ in $\mathcal{A}$ such that $(\mathcal{P}^{\mathrm{op}},\mathcal{J})$ is injective with respect to the functor $\mathrm{Hom}_{\mathcal{A}}$. Then the functor $\mathrm{Hom}_{\mathcal{A}}$ admits a right derived functor $R^+\mathrm{Hom}_{\mathcal{A}} : D^-(\mathcal{A})^{\mathrm{op}} \times D^+(\mathcal{A}) \to D^+(\mathbb{Z})$. In particular, $D^b(\mathcal{A})$ is a $\mathcal{U}$-category.*

Proof. The first assertion follows from Proposition 4.5.28, and the second one is a concequence of Theorem 4.5.26, since $R^0\mathrm{Hom}$ takes its values in $\mathcal{U}$-sets. □

Example 4.5.5. Assume that $\mathcal{A}$ has enough injectives. Then by Corollary 4.5.29, the derived Hom functor

$$R^+\mathrm{Hom}_{\mathcal{A}} : D^-(\mathcal{A})^{\mathrm{op}} \times D^+(\mathcal{A}) \to D^+(\mathbb{Z})$$

exists and may be calculated as follows. Let $X \in D^-(\mathcal{A})$ and $Y \in D^+(\mathcal{A})$. Then there exists a quasi-isomorphism $Y \to I$ in $K^+(\mathcal{A})$, with the I^i's being injective, and

$$R^+\mathrm{Hom}_{\mathcal{A}}(X, Y) \cong \mathrm{Hom}^{\bullet}_{\mathcal{A}}(X, I).$$

If $\mathcal{A}$ has enough projectives, then $R^+\mathrm{Hom}_{\mathcal{A}}$ also exists, and for a quasi-isomorphism $P \to X$ with P^i's being projective, we have

$$R^+\mathrm{Hom}_{\mathcal{A}}(X, Y) \cong \mathrm{Hom}^{\bullet}_{\mathcal{A}}(P, Y).$$

These isomorphisms hold in $D^+(\mathbb{Z})$, which means $R^+\mathrm{Hom}_{\mathcal{A}}(X, Y) \in D^+(\mathbb{Z})$ is represented by the simple complex associated with the double complex $\mathrm{Hom}^{\bullet,\bullet}_{\mathcal{A}}(X, I)$ or $\mathrm{Hom}^{\bullet,\bullet}_{\mathcal{A}}(P, Y)$.

Example 4.5.6. Let A be a k-algebra, with k being a ring. Since the category $\mathbf{Mod}(A)$ has enough projectives, the left derived functor of the functor $- \otimes_A -$ is well defined:

$$- \otimes^L_A - : D^-(A^{\mathrm{op}}) \times D^-(A) \to D^-(k).$$

This functor may by calculated as follows:

$$N \otimes^L_A M \cong \mathrm{Tot}(N \otimes_A P) \cong \mathrm{Tot}(Q \otimes_A M) \cong \mathrm{Tot}(Q \otimes_A P)$$

where P is a complex of projective A-modules quasi-isomorphic to M and Q is a complex of projective A^{op}-modules quasi-isomorphic to N. A classical notation is $\mathrm{Tor}^A_n(N, M) := H_n(N \otimes^L_A M)$.

Proposition 4.5.31. *Let $F : \mathcal{A} \to \mathcal{A}' : G$ be an adjoint pair of additive functors. Assume that $\mathcal{A}$ has enough projectives and $\mathcal{A}'$ has enough injective objects, then there exists a canonical isomorphism in $D^+(\mathbb{Z})$:*

$$R\mathrm{Hom}_{\mathcal{A}'}(L^-F(X), Y) \xrightarrow{\sim} R\mathrm{Hom}_{\mathcal{A}'}(X, R^+G(Y)),$$

where $X \in D^-(\mathcal{A})$ and $Y \in D^+(\mathcal{A}')$. In particular, we have canonical isomorphisms

$$\mathrm{Hom}_{D(\mathcal{A}')}(L^-F(X), Y) \xrightarrow{\sim} \mathrm{Hom}_{D(\mathcal{A})}(X, R^+G(Y))$$

Proof. We take a projective resolution $P \to X$ and an injective resolution $Y \to I$. By Example 4.5.5, we have

$$R\mathrm{Hom}_{\mathcal{A}'}(L^-F(X), Y) \cong \mathrm{Hom}^{\bullet}(K^-F(P), I).$$

By the adjointness, in view of the definition of K^-F and the Hom complex, the RHS is isomorphic to $\mathrm{Hom}^{\bullet}(P, K^+G(I))$, which is $R\mathrm{Hom}_{\mathcal{A}}(X, R^+G(Y))$. The last assertion follows from Theorem 4.5.26 by taking H^0. □

Note that the functors L^-F and R^+G are not adjoint functors, since they are functors between different pair of categories. This problem shall be resolved after we introduce the unbounded version of derived functors.

4.6 Unbounded derived categories

In this section we study the unbounded derived categories of Grothendieck categories. We prove the existence of enough homotopically injective objects in order to define unbounded right derived functors, and we prove that these triangulated categories satisfy the hypotheses of the Brown representability theorem. We also study unbounded derived functors in particular for pairs of adjoint functors. We start this study in the framework of abelian categories with translation, then we apply it to the case of the categories of unbounded complexes in abelian categories.

4.6.1 Derived categories of abelian categories with translation

Let $(\mathcal{A}, T)$ be an abelian category with translation. Recall that, denoting by $\mathcal{N}$ the triangulated subcategory of the homotopy category $K_c(\mathcal{A})$ consisting of objects X quasi-isomorphic to 0, the derived category $D_c(\mathcal{A})$ of $(\mathcal{A}, T)$ is the localization $K_c(\mathcal{A})/\mathcal{N}$. Also recall that an object X is quasi-isomorphic to 0 if and only the sequence

$$T^{-1}(X) \xrightarrow{T^{-1}(d_X)} X \xrightarrow{d_X} T(X)$$

is exact.

For $X \in \mathcal{A}_c$, the differential $d_X : X \to T(X)$ is a morphism in $\mathcal{A}_c$, so its cohomology $H(X)$ is regarded as an object of $\mathcal{A}_c$ and similarly for $\ker d_X$ and $\operatorname{im} d_X$. Note that their differentials vanish.

Proposition 4.6.1. *Assume that $\mathcal{A}$ admits direct sums indexed by a set I and that such direct sums are exact. Then $\mathcal{A}_c$, $K_c(\mathcal{A})$ and $D_c(\mathcal{A})$ admit such direct sums and the two functors $\mathcal{A}_c \to K_c(\mathcal{A})$ and $K_c(\mathcal{A}) \to D_c(\mathcal{A})$ commute with such direct sums.*

Proof. The result concerning $\mathcal{A}_c$ and $K_c(\mathcal{A})$ is immediate, and that concerning $D_c(\mathcal{A})$ follows from Proposition 4.4.33. □

For an object X of $\mathcal{A}$, we denote by $M(X)$ the mapping cone of $\mathrm{id}_{T^{-1}(X)}$, regarding $T^{-1}(X)$ as an object of $\mathcal{A}_c$ with zero differential. Hence $M(X)$ is the object $X \oplus T^{-1}(X)$ of $\mathcal{A}_c$ with the differential

$$d_{M(X)} = \begin{pmatrix} 0 & 0 \\ \mathrm{id}_X & 0 \end{pmatrix} : X \oplus T^{-1}(X) \to T(X) \oplus X.$$

The functor $M : \mathcal{A} \to \mathcal{A}_c$ is easily seen to be exact, and M is a left adjoint of the forgetful functor $\mathcal{A}_c \to \mathcal{A}$, as seen by the following lemma.

Lemma 4.6.2. *For $Y \in \mathcal{A}$ and $X \in \mathcal{A}_c$, we have the isomorphism*

$$\mathrm{Hom}_{\mathcal{A}_c}(M(Y), X) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{A}}(Y, X)$$

Proof. Any morphism $(u, v) : M(Y) \to X$ in $\mathcal{A}_c$ satisfies $d_X \circ (u, v) = T(u, v) \circ d_{M(X)}$, which reads as $d_X \circ u = Tv$ and $d_X \circ v = 0$. Therefore, it is completely determined by $u : Y \to X$. □

Proposition 4.6.3. *Let $\mathcal{A}$ be a Grothendieck category. Then $\mathcal{A}_c$ is again a Grothendieck category.*

Proof. The category $\mathcal{A}_c$ is abelian and admits small inductive limits, and small filtrant inductive limits in $\mathcal{A}_c$ are clearly exact. Moreover, if G is a generator in $\mathcal{A}$, then $M(G)$ is a generator in $\mathcal{A}_c$ by Lemma 4.6.2. □

Definition 4.6.1. Let $(\mathcal{A}, T)$ be an abelian category with translation.

- An object $I \in K_c(\mathcal{A})$ is called **homotopically injective** if $\mathrm{Hom}_{K_c(\mathcal{A})}(X, I) = 0$ for any $X \in K_c(\mathcal{A})$ that is quasi-isomorphic to 0.
- An object $P \in K_c(\mathcal{A})$ is called **homotopically projective** if $\mathrm{Hom}_{K_c(\mathcal{A})}(P, X) = 0$ for any $X \in K_c(\mathcal{A})$ that is quasi-isomorphic to 0.

We shall denote by $K_{c,hi}(\mathcal{A})$ the full subcategory of $K_c(\mathcal{A})$ consisting of homotopically injective objects and by $\iota : K_{c,hi}(\mathcal{A}) \to K_c(\mathcal{A})$ the inclusion functor. Similarly, we denote by $K_{c,hp}(\mathcal{A})$ the full subcategory of $K_c(\mathcal{A})$ consisting of homotopically projective objects. Note that $K_{c,hi}(\mathcal{A})$ is obviously a full triangulated subcategory of $K_c(\mathcal{A})$.

Lemma 4.6.4. *Let $(\mathcal{A}, T)$ be an abelian category with translation. If $I \in K_c(\mathcal{A})$ is homotopically injective, then*

$$\mathrm{Hom}_{K_c(\mathcal{A})}(X, I) \xrightarrow{\sim} \mathrm{Hom}_{D_c(\mathcal{A})}(X, I)$$

for all $X \in K_c(\mathcal{A})$.

Proof. Let $X \in K_c(\mathcal{A})$ and $X' \to X$ be a quasi-isomorphism. Then for $I \in K_{c,hi}(\mathcal{A})$, the morphism

$$\mathrm{Hom}_{K_c(\mathcal{A})}(X, I) \to \mathrm{Hom}_{K_c(\mathcal{A})}(X', I)$$

is an isomorphism since there is a distinguished triangle $X' \to X \to N \to T(X)$ with N quasi-isomorphic to zero and

$$\mathrm{Hom}_{K_c(\mathcal{A})}(N, I) \cong \mathrm{Hom}_{K_c(\mathcal{A})}(T^{-1}(N), I) = 0.$$

Therefore, for any $X \in K_c(\mathcal{A})$ and $I \in K_{c,hi}(\mathcal{A})$, we have

$$\mathrm{Hom}_{D_c(\mathcal{A})}(X, I) \cong \varinjlim_{(X' \to X) \in \mathrm{Qis}} \mathrm{Hom}_{K_c(\mathcal{A})}(X', I) \cong \mathrm{Hom}_{K_c(\mathcal{A})}(X, I). \qquad \square$$

We now introduce the following notation:

$$QM = \{f \in \mathrm{Mor}(\mathcal{A}_c) : f \text{ is both a quasi-isomorphism and a monomorphism}\}.$$

Recall that an object I of $\mathcal{A}_c$ is called QM-injective if for any morphism $f : X \to Y$ in QM, the induced map

$$f^* : \mathrm{Hom}_{\mathcal{A}_c}(Y, I) \to \mathrm{Hom}_{\mathcal{A}_c}(X, I)$$

is surjective.

Proposition 4.6.5. *Let $I \in \mathcal{A}_c$, then I is QM-injective if and only if it satisfies the following conditions:*

(a) *I is homotopically injective,*

(b) *I is injective as an object of $\mathcal{A}$.*

Proof. $\square$

We shall now prove the following theorem, which asserts that any Grothendieck category has enough QM-injective objects.

Theorem 4.6.6. *Let $(\mathcal{A}, T)$ be an abelian category with translation where $\mathcal{A}$ is a Grothendieck category. Then, for any $X \in \mathcal{A}_c$, there exists $u : X \to I$ such that $u \in QM$ and I is QM-injective.*

Corollary 4.6.7. *Let $(\mathcal{A}, T)$ be an abelian category with translation where $\mathcal{A}$ is a Grothendieck category. Then for any $X \in \mathcal{A}_c$, there exists a quasi-isomorphism $X \to I$ such that I is homotopically injective.*

Corollary 4.6.8. *Let $(\mathcal{A}, T)$ be an abelian category with translation where $\mathcal{A}$ is a Grothendieck category. Then*

(a) *the localization functor $Q : K_c(\mathcal{A}) \to D_c(\mathcal{A})$ induces an equivalence $Q \circ \iota : K_{c,hi}(\mathcal{A}) \xrightarrow{\sim} D_c(\mathcal{A})$;*

(b) *the category $D_c(\mathcal{A})$ is a $\mathcal{U}$-category;*

(c) *the functor $Q : K_c(\mathcal{A}) \to D_c(\mathcal{A})$ admits a right adjoint $R : D_c(\mathcal{A}) \to K_c(\mathcal{A})$ so that $Q \circ R \cong$ id, and R is the composition of $\iota : K_{c,hi}(\mathcal{A}) \to K_c(\mathcal{A})$ and a quasi-inverse of $Q \circ \iota$;*

(d) *for any triangulated category D, any triangulated functor $F : K_c(\mathcal{A}) \to \mathcal{D}$ admits a right localization $RF : D_c(\mathcal{A}) \to \mathcal{D}$, and $RF \cong F \circ R$.*

Proof. The functor $Q : K_{c,hi}(\mathcal{A})$ is fully faithful by Lemma 4.6.4 and essentially surjective by Corollary 4.6.7, whence assertion (a), and (b), (c) follow immediately: in fact, for $X \in K_c(\mathcal{A})$ and $Y \in D_c(\mathcal{A})$, we have $(Q \circ \iota)^{-1}(Y) \in K_{c,hi}(\mathcal{A})$, so by Lemma 4.6.4,

$$\begin{aligned}
\mathrm{Hom}_{K_c(\mathcal{A})}(X, \iota \circ (Q \circ \iota)^{-1}(Y)) &\cong \mathrm{Hom}_{K_c(\mathcal{A})}(X, (Q \circ \iota)^{-1}(Y)) \\
&\cong \mathrm{Hom}_{D_c(\mathcal{A})}(Q(X), (Q \circ \iota) \circ (Q \circ \iota)^{-1}(Y)) \\
&\cong \mathrm{Hom}_{D_c(\mathcal{A})}(Q(X), Y).
\end{aligned}$$

Finally, (d) follows from Proposition 4.4.14 and (c). □

4.6.2 The Brown representability theorem

We shall show that the hypotheses of the Brown representability theorem are satisfied for $D_c(\mathcal{A})$ when $\mathcal{A}$ is a Grothendieck abelian category with translation. Note that $D_c(\mathcal{A})$ admits small direct sums and the localization functor $Q : K_c(\mathcal{A}) \to D_c(\mathcal{A})$ commutes with such direct sums by Proposition 4.6.1.

Theorem 4.6.9. *Let $(\mathcal{A}, T)$ be an abelian category with translation where $\mathcal{A}$ is a Grothendieck category. Then the triangulated category $D_c(\mathcal{A})$ admits small direct sums and a system of t-generators.*

Applying ([**?**] corollary 10.5.2 and corollary 10.5.3), we then obtain the following corollaries.

Corollary 4.6.10. *Let $(\mathcal{A}, T)$ be an abelian category with translation where $\mathcal{A}$ is a Grothendieck category. Let $G : D^c(\mathcal{A})^{\mathrm{op}} \to \mathbf{Mod}(\mathbb{Z})$ be a cohomological functor which commutes with small products, then G is representable.*

Corollary 4.6.11. *Let $(\mathcal{A}, T)$ be an abelian category with translation where $\mathcal{A}$ is a Grothendieck category. Let $\mathcal{D}$ be a triangulated category and $F : D_c(\mathcal{A}) \to \mathcal{D}$ be a triangulated functor. Assume that F commutes with small direct sums, then F admits a right adjoint.*

We shall prove a slightly more general statement than Theorem 4.6.9. Let $\mathcal{I}$ be a full subcategory of $\mathcal{A}$ closed by subobjects, quotients and extensions in $\mathcal{A}$, and also by small direct sums. Let us denote by $D_{c,\mathcal{I}}(\mathcal{A})$ the full subcategory of $D_c(\mathcal{A})$ consisting of objects $X \in D_c(\mathcal{A})$ such that $H(X) \in \mathcal{I}$. Then $D_{c,\mathcal{I}}(\mathcal{A})$ is a full triangulated subcategory of $D_c(\mathcal{A})$ closed by small direct sums.

Proposition 4.6.12. *The triangulated category $D_{c,\mathcal{I}}(\mathcal{A})$ admits a system of t-generators.*

4.6.3 Unbounded derived category

From now on and until the end of this section, we consider abelian categories $\mathcal{A}$, $\mathcal{A}'$, etc. We shall apply the results in the preceding paragraphs to the abelian category with translation given by shifting. Then we have $\mathcal{A}_c \cong \mathrm{Ch}(\mathcal{A})$, $K_c(\mathcal{A}) \cong K(\mathcal{A})$ and $D_c(\mathcal{A}) \cong D(\mathcal{A})$. Assume that $\mathcal{A}$ admits direct sums indexed by a set I and that such direct sums are exact. Then, clearly, $\mathcal{A}_c$ has the same properties, so it follows from Proposition 4.6.1 that $\mathrm{Ch}(\mathcal{A})$, $K(\mathcal{A})$ and $D(\mathcal{A})$ also admit such direct sums and the two functors $\mathrm{Ch}(\mathcal{A}) \to K(\mathcal{A})$ and $K(\mathcal{A}) \to D(\mathcal{A})$ commute with such direct sums.

We shall write $K_{hi}(\mathcal{A})$ for $K_{c,hi}(\mathcal{A})$, so $K_{hi}(\mathcal{A})$ is the full subcategory of $K(\mathcal{A})$ consisting of homotopically injective objects. Let us denote by $\iota : K_{hi}(\mathcal{A}) \to K(\mathcal{A})$ the inclusion functor. Similarly we denote by $K_{hp}(\mathcal{A})$ the full subcategory of $K(\mathcal{A})$ consisting of homotopically projective objects. Recall that $I \in K(\mathcal{A})$ is homotopically injective if and only if $\mathrm{Hom}_{K(\mathcal{A})}(X, I) = 0$ for all $X \in K(\mathcal{A})$ that is quasi-isomorphism to 0. Note that an object $I \in K^+(\mathcal{A})$ whose components are all injective is homotopically injective in view of Lemma 4.5.11.

Let $\mathcal{A}$ be a Grothendieck abelian category. Then $\mathrm{Ch}(\mathcal{A})$ is also a Grothendieck category, so applying Corollary 4.6.7 and Theorem 4.6.9, we get the following theorem.

Theorem 4.6.13. *Let $\mathcal{A}$ be a Grothendieck category.*

(i) *if $I \in K(\mathcal{A})$ is homotopically injective, then we have an isomorphism*

$$\mathrm{Hom}_{K(\mathcal{A})}(X, I) \xrightarrow{\sim} \mathrm{Hom}_{D(\mathcal{A})}(X, I)$$

for any $X \in K(\mathcal{A})$;

(ii) *for any $X \in \mathrm{Ch}(\mathcal{A})$, there exists a quasi-isomorphism $X \to I$ such that I is homotopically injective;*

(iii) *the localization functor $Q : K(\mathcal{A}) \to D(\mathcal{A})$ induces an equivalence $K_{hi}(\mathcal{A}) \xrightarrow{\sim} D(\mathcal{A})$;*

(iv) *the category $D(\mathcal{A})$ is a $\mathcal{U}$-category;*

(v) *the functor $Q : K(\mathcal{A}) \to D(\mathcal{A})$ admits a right adjoint $R : D(\mathcal{A}) \to K(\mathcal{A})$ such that $Q \circ R \cong \mathrm{id}$ and R is the composition of $\iota : K_{hi}(\mathcal{A}) \to K(\mathcal{A})$ and a quasi-inverse of $Q \circ \iota$;*

(vi) *for any triangulated category $\mathcal{D}$, any triangulated functor $F : K(\mathcal{A}) \to \mathcal{D}$ admits a right localization $RF : D(\mathcal{A}) \to \mathcal{D}$ and $RF \cong F \circ R$;*

(vii) *the triangulated category $D(\mathcal{A})$ admits small direct sums and a system of t-generators;*

(viii) *any cohomological functor $G : D(\mathcal{A})^{\mathrm{op}} \to \mathbf{Mod}(\mathbb{Z})$ is representable as soon as G commutes with small products;*

(ix) *for any triangulated category $\mathcal{D}$, any triangulated functor $F : D(\mathcal{A}) \to \mathcal{D}$ admits a right adjoint as soon as F commutes with small direct sum.*

Corollary 4.6.14. *Let k be a commutative ring and let $\mathcal{A}$ be a Grothendieck k-abelian category. Then $(K(\mathcal{A})^{\mathrm{op}}, K_{hi}(\mathcal{A}))$ is $\mathrm{Hom}_{\mathcal{A}}$-injective and the functor $\mathrm{Hom}_{\mathcal{A}}$admits a right derived functor $R\mathrm{Hom}_{\mathcal{A}} : D(\mathcal{A})^{\mathrm{op}} \times D(\mathcal{A}) \to D(k)$. Moreover, we have*

$$H^n(R\mathrm{Hom}_{\mathcal{A}}(X, Y)) \cong \mathrm{Hom}_{D(\mathcal{A})}(X, Y)$$

for $X, Y \in D(\mathcal{A})$.

Proof. The functor $\mathrm{Hom}_{\mathcal{A}}$ induces a functor $\mathrm{Hom}^{\bullet}_{\mathcal{A}} : K(\mathcal{A})^{\mathrm{op}} \times K(\mathcal{A}) \to K(k)$ and by ([**?**], proposition 11.7.3) we have

$$H^n(\mathrm{Hom}^{\bullet}_{\mathcal{A}}(X, Y)) = \mathrm{Hom}_{K(\mathcal{A})}(X, Y[n]).$$

Let $I \in K_{hi}(\mathcal{A})$; if $X \in K(\mathcal{A})$ is quasi-isomorphic to zero, then $\mathrm{Hom}_{K(\mathcal{A})}(X, I) = 0$. Moreover, if $I \in K_{hi}(\mathcal{A})$ is quasi-isomorphic to zero, then I is isomorphic to zero. Therefore $(K(\mathcal{A})^{\mathrm{op}}, K_{hi}(\mathcal{A}))$ is $\mathrm{Hom}_{\mathcal{A}}$-injective and we can apply Proposition 4.5.27. The last assertion follows from Theorem 4.5.26. □

Remark 4.6.1. Let $\mathcal{I}$ be a full subcategory of a Grothendieck category $\mathcal{A}$ and assume that $\mathcal{I}$ is closed by subobjects, quotients and extensions in $\mathcal{A}$, and also by small direct sums. Then by Proposition 4.6.12, the triangulated category $D_{\mathcal{I}}(\mathcal{A})$ admits small direct sums and a system of t-generators. Hence $D_{\mathcal{I}}(\mathcal{A}) \to D(\mathcal{A})$ has a right adjoint.

4.6.4 Left derived functors

We now give a criterion for the existence of the left derived functor $LG : D(\mathcal{A}) \to D(\mathcal{A}')$ of an additive functor $G : \mathcal{A} \to \mathcal{A}'$ of abelian categories, assuming that G admits a right adjoint. For this, we shall assume throughout this paragraph that $\mathcal{A}$ admits small direct sums and small direct sums are exact in $\mathcal{A}$. Hence, by Proposition 4.6.1, $\mathrm{Ch}(\mathcal{A})$, $K(\mathcal{A})$ and $D(\mathcal{A})$ admit small direct sums. Note that Grothendieck categories satisfy these conditions.

4.7 Truncations and recollements

4.7.1 Abelian subcategories

Let $\mathcal{D}$ be a triangulated category. For $X, Y \in \mathcal{D}$ and $i \in \mathbb{Z}$, we denote by $\mathrm{Hom}^i(X,Y) := \mathrm{Hom}(X, Y[i])$ the morphisms from X to Y of **degree** $\boldsymbol{i}$.

Proposition 4.7.1. *Let (X,Y,Z) and (X',Y',Z') be distinguished triangles, and $\beta : Y \to Y'$ be a morphism such that we have a solid commutative diagram*

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, dashed, "\alpha"] \arrow[dr, phantom, "(A)"] & Y \arrow[r, "g"] \arrow[d, "\beta"] \arrow[dr, phantom, "(B)"] & Z \arrow[r, "h"] \arrow[d, dashed, "\gamma"] & X[1] \\
X \arrow[r, "f'"] & Y \arrow[r, "g'"] & Z \arrow[r, "h'"] & X'[1]
\end{tikzcd}$$

Then the following conditions are equivalent:

(i) $g'\beta f = 0$;

(ii) *there eixsts a morphism $\alpha : X \to X'$ rendering the commutative square (A);*

(iii) *there eixsts a morphism $\gamma : Z \to Z'$ rendering the commutative square (B);*

(iv) *there exists a morphism of triangles (α, β, γ).*

If these conditions are verified and $\mathrm{Hom}^{-1}(X, Z') = 0$*, then the morphisms α, β are unique.*

Proof. The exactness of the sequence

$$\mathrm{Hom}^{-1}(X,Z') \longrightarrow \mathrm{Hom}(X,X') \longrightarrow \mathrm{Hom}(X,Y') \longrightarrow \mathrm{Hom}(X,Z')$$

applied to $\beta f \in \mathrm{Hom}(X,Y')$, proves the equivalence of (i) and (ii), with the uniqueness of α if $\mathrm{Hom}^{-1}(X,Z) = 0$. The implication (ii)⇒(iv) follows from (TR2): if α satisfies (ii), then there exists $\gamma : Z \to Z'$ such that (α, β, γ) is a morphism of triangles; the converse of this is trivial. Finally, a dual argument shows that (i)⇔(iii), and the uniqueness of γ if $\mathrm{Hom}^{-1}(X,Z') = 0$. □

Corollary 4.7.2. *Let $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ be a distinguished triangle. If* $\mathrm{Hom}^{-1}(X,Z) = 0$*, then*

(a) *the cone of f is unique up to unique isomorphisms;*

(b) *the morphism $h : Z \to X[1]$ is the unique morphism such that the triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \to X[1]$ is distinguished.*

Proof. If in Proposition 4.7.1, we set $X = X'$, $Y = Y'$ and let f, g be the identity, then Z is isomorphic to Z' and $\mathrm{Hom}^{-1}(X,Z') = 0$, so (a) follows from the uniqueness of γ. For (b), we can apply triangle cat morphism extension to triangle iff to the diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r, "g"] \arrow[d, equal] & Z \arrow[r, "h"] & X[1] \\
X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r, "h'"] & X[1]
\end{tikzcd}$$

We necessarily have $\gamma = \mathrm{id}_Z$, whence $h = h'$. □

Let $\mathcal{C}$ be a (fixed) full subcategory of $\mathcal{D}$ such that $0 \in \mathcal{C}$. We say that $\mathcal{C}$ is **right leaning** if $\operatorname{Hom}^i_{\mathcal{D}}(X,Y) := \operatorname{Hom}_{\mathcal{D}}(X, Y[i])$ is null for any $i < 0$ and $X, Y \in \mathcal{C}$. For such a category $\mathcal{C}$, by Corollary 4.7.2, we see that a morphism (α, β, γ) of distinguished triangles in $\mathcal{C}$ is uniquely determined by the morphism β.

Example 4.7.1. Let $\mathcal{D}$ be the derived category of an abelian category $\mathcal{A}$ and $\mathcal{C} = \mathcal{A}$, identified with a full subcategory of $\mathcal{D}$. It is clear that for $X, Y \in \mathcal{A}$ we have $\operatorname{Hom}(X, Y[i]) = 0$ unless $i = 0$, so $\mathcal{A}$ is right leaning in $\mathcal{D}$.

Proposition 4.7.3. *Let $f : X \to Y$ be a morphism in $\mathcal{C}$. Complete f into a distinguished triangle $X \xrightarrow{f} Y \to S$ and suppose that there is a distinguished triangle $N[1] \to S \to C$ with $N, C \in \mathcal{C}$, so that we have a commutative diagram*

$$\begin{tikzcd}
N[1] \arrow[d, "\alpha"'] \arrow[dr] & & C \arrow[ll, "+1"'] \\
& S \arrow[ur] \arrow[dl, "+1"] & \\
X \arrow[rr, "f"'] & & Y \arrow[ul] \arrow[uu, "\beta"']
\end{tikzcd} \tag{4.7.1}$$

Then $\alpha[-1] : N \to X$ is a kernel of f in $\mathcal{C}$ and $\beta : Y \to C$ is a cokernel of f in $\mathcal{C}$.

Proof. For $Z \in \mathcal{C}$, the long exact sequence of Hom gives the following exact sequences

$$0 \longrightarrow \operatorname{Hom}^{-1}(Z, S) \longrightarrow \operatorname{Hom}(Z, X) \longrightarrow \operatorname{Hom}(Z, Y)$$

$$0 \longrightarrow \operatorname{Hom}(Z, N) \longrightarrow \operatorname{Hom}^{-1}(Z, S) \longrightarrow \operatorname{Hom}^{-1}(Z, C) = 0$$

which proves that $(N, \alpha[-1])$ is a kernel of f (recall the hypothesis on $\mathcal{C}$). A dual argument proves that (C, β) is a cokernel of f. ☐

Example 4.7.2. Let $\mathcal{A}$ be an abelian category and $\mathcal{D} = D(\mathcal{A})$. Then the cone S of $f : X \to Y$ is the complex $X \xrightarrow{f} Y$, with X at degree -1 and Y at degree 0. It has a subcomplex $(\ker f)[1] = H^{-1}(S)[1]$ and the quotient $X/\ker f \to Y$ is quasi-isomorphic to the complex $\operatorname{coker} f = H^0(S)$, placed at degree 0; we thus obtain the diagram (4.7.1).

Example 4.7.3. In the situation of Proposition 4.7.3, if f is a monomorphism, then we have $N = 0$, so $S \cong C$ and (4.7.1) is reduced to the distinguished triangle (X, Y, C). On the other hand, if f is an epimorphic, then $C = 0$, so $N[1] \cong S$ and (4.7.1) is reduced to the distinguished triangle (N, X, Y).

A morphism $f : X \to Y$ in $\mathcal{C}$ is called **$\mathcal{C}$-admissible**, or simply **admissible** (if there is no ambiguity on $\mathcal{C}$), if it is the base of a diagram (4.7.1). For any distinguished triangle

$$X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow X[1]$$

with X, Y, Z in $\mathcal{C}$ and f, g admissible, we see that f is a kernel of g and g is a cokernel of f. By Corollary 4.7.2, the morphism $Z \to X[1]$ is uniquely determined by f and g.

A sequence $X \to Y \to Z$ in $\mathcal{C}$ is called an **admissible short exact sequence** if it is deduced from a distinguished triangle by removing the arrow of degree 1. In other words, $X \to Y \to Z$ is admissible if it can be extended into a distinguished triangle in $\mathcal{D}$.

Proposition 4.7.4. *Suppose that $\mathcal{C}$ is stable under finite direct sums. Then the following conditions are equivalent:*

(i) *$\mathcal{C}$ is an abelian category and every short exact sequence is admissible.*

(ii) *Any morphism of $\mathcal{C}$ is $\mathcal{C}$-admissible.*

Proof. We first assume that any morphism in $\mathcal{C}$ is admissible. By Proposition 4.7.3, any morphism of $\mathcal{C}$ has a kernel and cokernel, so to prove that $\mathcal{C}$ is abelian, it suffices to verify that $\operatorname{coim} f \cong \operatorname{im} f$ for any morphism $f : X \to Y$ in $\mathcal{C}$. Regarding (4.7.1) as the cap of an octahedron and apply (TR4), we obtain an octahedron

$$\begin{tikzcd}
C \arrow[dd, "+1"'] & & Y \arrow[ll, "\beta"'] \arrow[dl] \\
& S \arrow[ul] \arrow[dr, "+1"] & \\
N[1] \arrow[ur] \arrow[rr, "\alpha[1]"'] & & X \arrow[uu, "f"']
\end{tikzcd}
\qquad
\begin{tikzcd}
C \arrow[dd, "+1"'] \arrow[dr] & & Y \arrow[ll, "\beta"'] \\
& I \arrow[ur] \arrow[dl, "+1"] & \\
N[1] \arrow[rr, "\alpha[1]"'] & & X \arrow[uu, "f"'] \arrow[ul]
\end{tikzcd}$$

By Proposition 4.7.3, we see that β is an (admissible) epimorphism (as the cokernel of f). Since the triangle (I, Y, C) is distinguished, we conclude from Example 4.7.3 that $I \in \mathcal{C}$ and it is the the kernel of β, i.e. the image of f. Dually, the distinguished triangle (N, X, I), obtained by rotating, shows that I is the coimage of f. Finally, by Example 4.7.3, we see that any short exact sequence is admissible.

Conversely, assume the codition (i). The kernel N, cokernel C and image I of $f : X \to Y$ then give short exact sequences

$$0 \longrightarrow N \longrightarrow X \longrightarrow I \longrightarrow 0 \qquad 0 \longrightarrow I \longrightarrow Y \longrightarrow C \longrightarrow 0$$

By hypothesis, these two sequences are admissible, and the two thus obtained triangles form the upper cap of an octahedron. We can then apply (TR4) to obtain an octahedron of the above form, so f is admissible. $\square$

A full subcategory $\mathcal{C}$ of $\mathcal{D}$ is called **admissibly abelian** if it is right leaning and satisfies the equivalent conditions of Proposition 4.7.4. By Example 4.7.2, we see that if $\mathcal{D}$ is the derived category of an abelian category $\mathcal{A}$, then $\mathcal{A}$ is admissibly abelian in $\mathcal{D}$.

4.7.2 t-structures

Let $\mathcal{D}$ be a triangulated category. A ***t*-structure** on $\mathcal{D}$ is a pair of strictly full subcategories $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ satisfying the following conditions: If we put $\mathcal{D}^{\leq n} = T^{-n}(\mathcal{D}^{\leq 0})$ and $\mathcal{D}^{\geq n} = T^{-n}(\mathcal{D}^{\geq 0})$, then

(t1) $\mathcal{D}^{\leq 0} \subseteq \mathcal{D}^{\leq 1}$ and $\mathcal{D}^{\geq 0} \supseteq \mathcal{D}^{\geq 1}$;

(t2) $\operatorname{Hom}_{\mathcal{D}}(X, Y) = 0$ for $X \in \mathcal{D}^{\leq 0}$ and $Y \in \mathcal{D}^{\geq 1}$;

(t3) for any $X \in \mathcal{D}$, there exists a distinguished triangle (A, X, B) with $A \in \mathcal{D}^{\leq 0}$ and $B \in \mathcal{D}^{\geq 1}$.

The **core** (or **heart**) of the t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ is defined to be the full subcategory $\mathcal{C} = \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$.

Example 4.7.4 (Examples of t-structures).

(a) Let $\mathcal{A}$ be an abelian category and $\mathcal{D} = \mathcal{D}(A)$. The natural t-structure on $\mathcal{D}(A)$ is then defined so that $X \in \mathcal{D}^{\leq n}$ (resp. $X \in \mathcal{D}^{\geq n}$) if and only if $H^i(X) = 0$ for $i > n$ (resp. for $i < n$). To verify (t3), we note that for any complex X, the truncations $\tau^{\leq 0}X$ and $\tau^{\geq 1}X$ are in $\mathcal{D}^{\leq 0}$ and $\mathcal{D}^{\geq 1}$, respectively, and we have a distinguished triangle $(\tau^{\leq 0}X, X, \tau^{\geq 1}X)$.

(b) If $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ is a t-structure on $\mathcal{D}$, then for any integer n, $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ is also a t-structure. We say that this t-structure is induced from the previous one by *translation*.

(c) If $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ is a t-structure on $\mathcal{D}$, then $((\mathcal{D}^{\leq 0})^{\mathrm{op}}, (\mathcal{D}^{\geq 0})^{\mathrm{op}})$ is a t-structure on $\mathcal{D}^{\mathrm{op}}$, called the dual t-structure.

A triangulated category $\mathcal{D}$, endowed with a t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$, is called a ***t*-category**.

Proposition 4.7.5. *Let $\mathcal{D}$ be a t-category. Then for integers $m < n$, we have $\mathrm{Hom}_{\mathcal{D}}(X, Y) = 0$ if $X \in \mathcal{D}^{\leq m}$ and $Y \in \mathcal{D}^{\geq n}$.*

Proof. □

Proposition 4.7.6. *Let $\mathcal{D}$ be a t-category.*

(a) *The inclusion functor $\mathcal{D}^{\leq n} \to \mathcal{D}$ admits a right adjoint $\tau^{\leq n}$, and $\mathcal{D}^{\geq n} \to \mathcal{D}$ admits a left adjoint $\tau^{\geq n}$.*

(b) *For any $X \in \mathcal{D}$, there eixsts a unique morphism in $\mathrm{Hom}^1(\tau^{\geq 1}X, \tau^{\leq 0}X)$ such that the triangle*

$$\tau^{\leq 0}X \longrightarrow X \longrightarrow \tau^{\geq 1}X \xrightarrow{+1}$$

is distinguished. Moreover, up to isomorphisms, this is the unique distinguished (A, X, B) such that $A \in \mathcal{D}^{\leq 0}$ and $B \in \mathcal{D}^{\geq 1}$.

Proof. By duality and translation, it suffices to prove (a) for $\mathcal{D}^{\leq 0}$. For each $X \in \mathcal{D}$, we need to find $A \in \mathcal{D}^{\leq 0}$ with a morphism $A \to X$ (the value of $\tau^{\leq 0}$ at X), such that for any $T \in \mathcal{D}^{\leq 0}$ we have an isomorphism $\mathrm{Hom}(T, A) \cong \mathrm{Hom}(T, X)$. Let (A, X, B) be a triangle as in (t3). The long exact sequence of Hom, together with (t1), (t2), shows that $\mathrm{Hom}(T, A) \cong \mathrm{Hom}(T, X)$, so we set $A = \tau^{\leq 0}X$. A similar argument shows that $B = \tau^{\geq 1}X$, so there is a distinguished triangle $(\tau^{\leq 0}X, X, \tau^{\geq 1}X)$ and any distinguished triangle in (t3) is isomorphic to this triangle. The uniqueness of these isomorphisms follows from (t2) and Proposition 4.7.1. □

The distinguished triangle $(\tau^{\leq 0}X, X, \tau^{\geq 1}X)$ shows that the following conditions are equivalent:

(i) $\tau^{\leq 0}X = 0$;

(ii) $\mathrm{Hom}(T, X) = 0$ for $T \in \mathcal{D}^{\leq 0}$;

(iii) $X \cong \tau^{\geq 1}X$.

The equivalence (ii)⇔(iii) means that $\mathcal{D}^{\geq 1}$ is the right orthogonal of $\mathcal{D}^{\leq 0}$, which shows that $\mathcal{D}^{\geq 1}$ is stable under exensions[1]. Dually, we see that $\tau^{\geq 1}X = 0$ if and only if $X \in \mathcal{D}^{\leq 0}$, and $\mathcal{D}^{\leq 0}$ is the left orthogonal of $\mathcal{D}^{\geq 1}$, which is stable under extensions. In particular, $\mathcal{D}^{\leq 0}$ and $\mathcal{D}^{\geq 1}$ are stable under finite direct sums.

For $a \leq b$, we have $\mathcal{D}^{\leq a} \subseteq \mathcal{D}^{\leq b}$, so there exists a unique morphism $\tau^{\leq a}X \to \tau^{\leq b}X$ so that we have a commutative diagram

$$\begin{tikzcd} \tau^{\leq a} \arrow[rr] \arrow[dr] & & \tau^{\leq b}X \arrow[dl] \\ & X & \end{tikzcd}$$

[1] We recall that an object Y is called an extensin of Z by X if there is a distinguished triangle (X, Y, Z) in $\mathcal{D}$, and a subcategory $\mathcal{D}'$ of $\mathcal{D}$ is stable under extensions if for any distinguished triangle (X, Y, Z) with $X, Z \in \mathcal{D}'$, we have $Y \in \mathcal{D}'$.

which identifies $\tau^{\le a}X$ with $\tau^{\le a}\tau^{\le b}X$. Dually, we have $\tau^{\ge a}X \to \tau^{\ge b}X$, which identifies $\tau^{\ge b}X$ with $\tau^{\ge b}\tau^{\ge a}X$.

For any integer a, we write $\tau^{>a}$ for $\tau^{\ge a+1}$ and $\tau^{<a}$ for $\tau^{<a-1}$. We deduce by translation that $X \in \mathcal{D}^{\le a}$ if and only if $\tau^{>a}X = 0$. If $b > a$, then we have $\tau^{\ge b}X = 0$ in this case, and for $b \le a$ we have $\tau^{>a}\tau^{\ge b}X = \tau^{>a}X = 0$, and hence $\tau^{\ge b}$ sends $\mathcal{D}^{\le a}$ into itself.

Proposition 4.7.7. *Let $a \le b$ be integers. For $X \in \mathcal{D}$, there exists a unique isomorphism $\tau^{\ge a}\tau^{\le b}X \to \tau^{\le b}\tau^{\ge a}X$ rendering the following commutative diagram*

$$\begin{tikzcd}
\tau^{\le b}X \arrow[r] \arrow[d] & X \arrow[r] & \tau^{\ge a}X \\
\tau^{\ge a}\tau^{\le b}X \arrow[rr] & & \tau^{\le b}\tau^{\ge a}X \arrow[u]
\end{tikzcd} \tag{4.7.2}$$

Proof. Since $\tau^{\ge a}X \in \mathcal{D}^{\ge a}$, we see that the canonical morphism $\tau^{\le b}X \to \tau^{\ge a}X$ is obtained by the composition $\tau^{\le b}X \to \tau^{\ge a}\tau^{\le b}X \to \tau^{\ge a}X$. Also, since $\tau^{\ge a}\tau^{\le b}X \in \mathcal{D}^{\le b}$, the morphism $\tau^{\le b}X \to \tau^{\ge a}X$ factors through $\tau^{\le b}\tau^{\ge a}X$, so we obtain the diagram (4.7.2). Applying (TR4) to $\tau^{<a}X \to \tau^{\le b}X \to X$, we obtain an octahedron

$$\begin{tikzcd}
 & & & & {} & & \\
 & & & Y \arrow[ur] \arrow[dr] & & & \\
 & & \tau^{\le b}X \arrow[ur] \arrow[dr] & & \tau^{\ge a}X \arrow[r] \arrow[dr] & {} & \\
 & & & X \arrow[ur] \arrow[drr] & & & \\
 & \tau^{<a}X \arrow[uur] \arrow[urr] & & & & \tau^{>b}X \arrow[r] \arrow[d] & {} \\
 & & & & & {} &
\end{tikzcd} \tag{4.7.3}$$

In this octahedron, Y is both isomorphic to $\tau^{\ge a}\tau^{\le b}X$ (because of the distinguished triangle $(\tau^{<a}X, \tau^{\le b}X, Y)$, in which $\tau^{<a}X = \tau^{<a}\tau^{\le b}X$) and to $\tau^{\ge b}\tau^{\le a}X$ (because of the distinguished triangle $(Y, \tau^{\ge a}X, \tau^{>b}X)$). □

Corollary 4.7.8. *Let $m, n \in \mathbb{Z}$ with $m < n$. Then we have $\tau^{\le m}\tau^{\ge n} = \tau^{\ge n}\tau^{\le m} = 0$.*

Proof. This follows from Proposition 4.7.5. □

Because of Proposition 4.7.7, we write $\tau^{[a,b]}X := \tau^{\ge a}\tau^{\le b}X \cong \tau^{\le b}\tau^{\ge a}X$. It is clear that we thus obtain a functor $\tau^{[a,b]} : \mathcal{D} \to \mathcal{D}^{[a,b]} := \mathcal{D}^{\le b} \cap \mathcal{D}^{\ge a}$.

Let $\mathcal{A}$ be an abelian category and $\mathcal{D} = \mathcal{D}(A)$ be the derived category of $\mathcal{A}$. Then the natural t-structure $(\mathcal{D}^{\le 0}, \mathcal{D}^{\ge 0})$ has the property that $\mathcal{A} \cong \mathcal{D}^{\le 0} \cap \mathcal{D}^{\ge 0}$. We now prove that this is true for any t-structure of a triangulated category $\mathcal{D}$. That is, the heart $\mathcal{C} = \mathcal{D}^{\le 0} \cap \mathcal{D}^{\ge 0}$ is an abelian category.

Theorem 4.7.9. *The heart $\mathcal{C} = \mathcal{D}^{\le 0} \cap \mathcal{D}^{\ge 0}$ of a t-category $\mathcal{D}$ is an admissible abelian category of $\mathcal{D}$, which is stable under extensions. Moreover, the functor $H^0 := \tau^{\ge 0}\tau^{\le 0} : \mathcal{D} \to \mathcal{C}$ is a cohomological functor.*

Proof. Let $X, Y \in \mathcal{C}$ and $f : X \to Y$ be a morphism with cone S. The distinguished triangle $(Y, S, X[1])$ then shows that S is in $\mathcal{D}^{\le 0} \cap \mathcal{D}^{\ge -1}$. The truncations $\tau^{\ge 0}S$ and $\tau^{\le -1}S$ are hence in $\mathcal{C}$ and $\mathcal{C}[1]$, respectively, and the distinguished triangle $(\tau^{\le -1}S, S, \tau^{\ge 0}S)$ fits into a diagram (4.7.3). This proves that $\mathcal{C}$ is admissibly abelian, and it is stable under extensions as we have remarked.

It remains to prove that for any distinguished triangle (X, Y, Z), the sequence $H^0(X) \to H^0(Y) \to H^0(Z)$ is exact. To this end, we first assume that X, Y and Z are in $\mathcal{D}^{\leq 0}$. For $U \in \mathcal{D}^{\leq 0}$ and $V \in \mathcal{D}^{\geq 0}$, we note that $H^0(U) = \tau^{\geq 0}U$ and $H^0(V) = \tau^{\leq 0}V$, so there are isomorphisms

$$\mathrm{Hom}_{\mathcal{D}}(H^0(U), H^0(V)) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}}(U, H^0(V)) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}}(U, V).$$

For $T \in \mathcal{D}^{\geq 0}$, the long exact sequence of Hom and (t2) then give an exact sequence

$$0 \longrightarrow \mathrm{Hom}(Z, T) \longrightarrow \mathrm{Hom}(Y, T) \longrightarrow \mathrm{Hom}(X, T)$$

so the sequence $0 \to \mathrm{Hom}(H^0(Z), T) \to \mathrm{Hom}(H^0(Y), T) \to \mathrm{Hom}(H^0(X), T)$ is exact. Since this is true for any T, we conclude that $H^0(X) \to H^0(Y) \to H^0(Z) \to 0$ is exact.

We now show that the above conclusion is still valid if we only assume that $X \in \mathcal{D}^{\leq 0}$. For this, we note that for $T \in \mathcal{D}^{\geq 1}$, the long exact sequence of Hom gives an isomorphism $\mathrm{Hom}(Z, T) \cong \mathrm{Hom}(Y, T)$, so we have $\tau^{\geq 1}Y \cong \tau^{\geq 1}Z$. Apply (TR4) to $Y \to Z \to \tau^{\geq 1}Z$,

$$\begin{tikzcd}
 & & & {} & & \\
 & & \tau^{\leq 0}Z \arrow[ur] \arrow[dr] & & {} & \\
 & \tau^{\leq 0}Y \arrow[ur] \arrow[dr] & & Z \arrow[ur] \arrow[dr] & & \\
 & & Y \arrow[ur] \arrow[drr] & & & \\
X \arrow[uur] \arrow[urr] & & & & \tau^{\geq 1}Z \arrow[r] \arrow[dr] & {} \\
 & & & & & {}
\end{tikzcd}$$

we then obtain a distinguished triangle $(X, \tau^{\leq 0}Y, \tau^{\leq 0}Z)$, on which we can apply the preceding arguments to conclude that $H^0(X) \to H^0(\tau^{\leq 0}Y) \to H^0(\tau^{\leq 0}Z) \to 0$ is exact. This proves our claim since $H^0\tau^{\leq 0} \cong H^0$. Dually, we conclude that if $Z \in \mathcal{D}^{\leq 0}$, then the sequence $0 \to H^0(X) \to H^0(Y) \to H^0(Z)$ is exact.

Finally, we deal with the general case. By (TR4) we have an octahedron

$$\begin{tikzcd}
 & & & {} & & \\
 & & \tau^{\geq 1}X \arrow[ur] \arrow[dr] & & {} & \\
 & X \arrow[ur] \arrow[dr] & & U \arrow[ur] \arrow[dr] & & \\
 & & Y \arrow[ur] \arrow[drr] & & & \\
\tau^{\leq 0}X \arrow[uur] \arrow[urr] & & & & Z \arrow[r] \arrow[dr] & {} \\
 & & & & & {}
\end{tikzcd}$$

From the distinguished triangle $(\tau^{\leq 0}X, Y, U)$, we see that the sequence $H^0(X) \to H^0(Y) \to H^0(U) \to 0$ is exact, and $(U, Z, \tau^{\geq 1}X[1])$ shows that $0 \to H^0(U) \to H^0(Z)$ is exact. We then conclude the exactness of $H^0(X) \to H^0(Y) \to H^0(Z)$, which completes the proof. □

Proposition 4.7.10. *Let X be an object of $\mathcal{D}$ and $n \in \mathbb{Z}$.*

(a) *$H^p(i) : H^p(\tau^{\leq n}X) \to H^p(X)$ is an isomorphism for $p \leq n$ and $H^p(\tau^{\leq n}X) = 0$ for $p > 0$.*

(b) *$H^p(j) : H^p(X) \to H^p(\tau^{\geq n}X)$ is an isomorphism for $p \geq n$ and $H^p(\tau^{\geq n}X) = 0$ for $p < n$.*

Proof. By duality, it suffices to prove (a). If $p > n$, then by Corollary 4.7.8 we have

$$H^p(\tau^{\leq n}X) = \tau^{\leq p}\tau^{\geq p}\tau^{\leq n}X[p] = 0.$$

On the other hand, if $p \leq n$, then $\tau^{\leq p}\tau^{\leq n}X \to \tau^{\leq p}X$ is an isomorphism, and therefore

$$H^p(\tau^{\leq n}X) = \tau^{\geq p}\tau^{\leq p}\tau^{\leq n}X[p] \to \tau^{\geq p}\tau^{\leq p}X[p] = H^p(X)[p]$$

is an isomorphism. □

Corollary 4.7.11. *Let $n \in \mathbb{Z}$ and X be an object in $\mathcal{D}^{\leq n}$ (resp. $\mathcal{D}^{\geq n}$). Then $H^p(X) = 0$ for $p > n$ (resp. $p < n$).*

Proof. If X is an object in $\mathcal{D}^{\leq n}$, then $\tau^{\leq n}X \to X$ is an isomorphism, so $H^p(\tau^{\leq n}X) \to H^p(X)$ is an isomorphism for all $p \in \mathbb{Z}$. On the other hand, by Proposition 4.7.10, $H^p(\tau^{\leq n}X) = 0$ for $p > n$. The other part of the corollary follows by duality. □

We say the t-structure of $\mathcal{D}$ is **non-degenerate** if the intersection of the $\mathcal{D}^{\leq n}$, and that of the $\mathcal{D}^{\geq n}$, both reduce to the zero object. For each integer $i \in \mathbb{Z}$, we put $H^i(X) := H^0(X[i])$.

Proposition 4.7.12. *If the t-structure of $\mathcal{D}$ is non-degenerate, then the system of functors H^i is conservative, and for an object $X \in \mathcal{D}$ to belong to $\mathcal{D}^{\leq 0}$ (resp. $\mathcal{D}^{\geq 0}$), it is necessary and sufficient that $H^i(X) = 0$ for $i > 0$ (resp. $i < 0$).*

Proof. Let $X \in \mathcal{D}$. We first prove that $H^i(X) = 0$ for all $i \in \mathbb{Z}$ if and only if $X = 0$. If $X \in \mathcal{D}^{\leq 0}$, the hypothesis $H^0(X) = 0$ shows that $\tau^{\geq 0}X = 0$, whence $X \in \mathcal{D}^{\leq 1}$. Inductively, we conclude that $X \in \bigcap_n \mathcal{D}^{\leq n}$, whence is zero by the hypothesis. Dually, we also conclude that if $X \in \mathcal{D}^{\geq 0}$, then $X = 0$ if and only if $H^i(X) = 0$ for any $i \in \mathbb{Z}$. For the general case, the values of $\tau^{\leq 0}X$ and $\tau^{\geq 1}X$ under H^i are all zero, hence they are zero. We then conclude the claim by the distinguished triangle $(\tau^{\leq 0}X, X, \tau^{\geq 1}X)$.

If a morphism $f : X \to Y$, with cone Z, induces isomorphisms $H^i(X) \cong H^i(Y)$ for each i, then the long exact sequence of H^i shows that $H^i(Z) = 0$ for all i, so $Z = 0$ and f is an isomorphism. Finally, if $H^i(X) = 0$ for $i > 0$, then $H^i(\tau^{>0}X) = 0$ for any $i \in \mathbb{Z}$, so $\tau^{>0}X = 0$ and we conclude that $X \in \mathcal{D}^{\leq 0}$. The dual argument shows that $X \in \mathcal{D}^{\geq 0}$ if $H^i(X) = 0$ for $i < 0$. □

Proposition 4.7.13. *Let $\mathcal{D}$ be a t-category. Then the following conditins are equivalent:*

(i) *the union of the $\mathcal{D}^{\leq n}$ and that of the $\mathcal{D}^{\geq n}$ both equal to $\mathcal{D}$;*

(ii) *the t-structure is non-degenerate and for any $X \in \mathcal{D}$, $H^p(X)$ are nonzero for finitely many $p \in \mathbb{Z}$.*

The t-structure $\mathcal{D}$ is called ***bounded*** *if it satisfies the above conditions.*

Proof. Assume the conditions of (i) and let X be an object of $\mathcal{D}$ such that $H^p(X) = 0$ for all $p \in \mathbb{Z}$. By assumption, there exists $n, m \in \mathbb{Z}$ such that $X \in \mathcal{D}^{\leq n} \cap \mathcal{D}^{\geq m}$. By considering the distinguished triangle $(\tau^{\leq p-1}X, X, \tau^{\geq p}X)$ for any $p \in \mathbb{Z}$, we conclude that $X \in \mathcal{D}^{\leq n} \cap \mathcal{D}^{\geq n}$ for all $n \in \mathbb{Z}$. In particular, $X \in \mathcal{D}^{\leq -1} \cap \mathcal{D}^{\geq 0}$, which means $\mathrm{Hom}(X, X) = 0$ and therefore $X = 0$. This shows that the t-structure on $\mathcal{D}$ is non-degenerate. Let X be an arbitrary object in $\mathcal{D}$. Then $X \in \mathcal{D}^{\leq n} \cap \mathcal{D}^{\geq m}$ for some $m, n \in \mathbb{Z}$. By Proposition 4.7.10, $H^p(X) = 0$ for $p > n$ and $p < m$, so $H^p(X) \neq 0$ for finitely many $p \in \mathbb{Z}$.

Conversely, let X be an object in $\mathcal{D}$. Then there exists $n \in \mathbb{N}$ such thar $H^p(X) = 0$ for $|p| > n$. By , this implies that $H^p(\tau^{\leq -n}X) = 0$ and $H^p(\tau^{\geq n}X) = 0$ for all $p \in \mathbb{Z}$. Since the t-structure is non-degenerate, $\tau^{\leq -n}X = \tau^{\geq n}X = 0$, so $X \in \mathcal{D}^{\geq -n+1}$ and $\mathcal{D}^{\leq n-1}$. □

Example 4.7.5. Let $\mathcal{A}$ be an abelian category. Then the standard t-structure on the bounded derived category Db(A) is bounded. The standard t-structures on $D^+(\mathcal{A})$, $D^-(\mathcal{A})$ and $D(\mathcal{A})$ are not bounded.

Let $\mathcal{D}$ be a triangulated category with a nondegenerate t-structure. Let Db be the full subcategory consisting of all X in $\mathcal{D}$ such that $H^p(X) \neq 0$ for finitely many $p \in \mathbb{Z}$. Clearly, $\mathcal{D}^b$ is strictly full subcategory. Assume that (X, Y, Z) is a distinguished triangle in $\mathcal{D}$ and that two of its vertices are in $\mathcal{D}^b$. Then, from the long exact sequence of cohomology we see that the third vertex is also in $\mathcal{D}^b$. Therefore, $\mathcal{D}^b$ is a triangulated subcategory. Let X be an object in $\mathcal{D}^b$. Then, by Proposition 4.7.10, $\tau^{\leq n}X$ and $\tau^{\geq n}X$ are also in Db for all $n \in \mathbb{Z}$. This implies that $(\mathcal{D}^b \cap \mathcal{D}^{\leq 0}, \mathcal{D}^b \cap \mathcal{D}^{\geq 0})$ is a t-structure on $\mathcal{D}^b$. Clearly, the truncation functors and the cohomology functor H^0 for this t-structure are the restrictions of the corresponding functors on $\mathcal{D}$. Also, from the above result we see that this t-structure on $\mathcal{D}^b$ is bounded. We call $\mathcal{D}^b$ the subcategory of **cohomologically bounded objects** in $\mathcal{D}$.

Let $\mathcal{D}$ be a triangulated category and denote by $\mathrm{Iso}(\mathcal{D})$ the collection of sets of isomorphism classes of objects of $\mathcal{D}$ (for $X \in \mathcal{D}$, we denote by $[X]$ its isomorphism class). We define an operation $*$ on $\mathrm{Iso}(\mathcal{D})$ as follows: for $A, B \in \mathrm{Iso}(\mathcal{D})$, $A * B$ is defined to be

$$A * B = \{[X] : \text{there exists a distinguished triangle } (U, X, V) \text{ with } [U] \in A \text{ and } [V] \in B\}.$$

Lemma 4.7.14. *The operation $*$ is associative.*

Proof. It suffices to prove that for $X, Y, Z \in \mathcal{D}$, we have

$$(\{[X]\} * \{[Y]\}) * \{[Z]\} = \{[X]\} * (\{[Y]\} * \{[Z]\}).$$

For that $[T]$ belongs to the left side (resp. right side), it is necessary and sufficient that T fits into a diagram of the upper cap (resp. lower cap)

$$\begin{tikzcd}
Z \arrow[dd] \arrow[dr, "+1"] & & T \arrow[ll] \\
& U \arrow[ur] \arrow[dl] & \\
Y \arrow[rr, "+1"'] & & X \arrow[uu] \arrow[ul]
\end{tikzcd}
\quad \text{resp.} \quad
\begin{tikzcd}
Z \arrow[dd, "+1"'] & & T \arrow[ll] \arrow[dl] \\
& V \arrow[ul] \arrow[dr, "+1"] & \\
Y \arrow[rr] \arrow[ur] & & X \arrow[uu]
\end{tikzcd}$$

The lemma then follows from axiom (TR4) and the inverse of (TR4'). □

Lemma 4.7.14 permits us to define the $*$-product $A_1 * \cdots * A_p$ for a sequence of elements in $\mathrm{Iso}(\mathcal{D})$, without using the parentheses. It will be convenient for us to define the $*$-product of the empty sequence as being $\{[0]\}$.

Example 4.7.6. Let $\mathcal{A}$ be a strictly full subcategory of $\mathcal{A}$ and $E\mathcal{A}$ be the smallest strictly full subcategory of $\mathcal{D}$ containing $\mathcal{A}$, the zero object, and is stable under extensions. We have

$$[E\mathcal{A}] = \bigcup_{n \geq 0} \underbrace{[\mathcal{A}] * \cdots * [\mathcal{A}]}_{n\text{-factors}}. \tag{4.7.4}$$

We also note that the condition that any morphisms in $\mathcal{A}$ are $\mathcal{A}$-admissible can be reformulated as

$$[\mathcal{A}] * [\mathcal{A}[1]] \subseteq [\mathcal{A}[1]] * [\mathcal{A}]. \tag{4.7.5}$$

We now consider an admissible abelian subcategory $\mathcal{C}$ of $\mathcal{D}$ which is stable under extensions. Let $\mathcal{D}^b$ (resp. $\mathcal{D}^{b,\leq 0}$, resp. $\mathcal{D}^{b,\geq 0}$, resp. $\mathcal{D}^{b,I}$, where I is an integer of $\mathbb{Z}$) be the smallest strictly full subcategory of $\mathcal{D}$ containing the $\mathcal{C}[n]$ for $n \in \mathbb{Z}$ (resp. $-n \leq 0$, resp. $-n \geq 0$, resp. $-n \in I$) which is stable under extensions[2].

[2]If I is the empty interval, it is more natural to define $\mathcal{D}^{b,I}$ to be the category of zero objects.

Proposition 4.7.15. *The pari $(\mathcal{D}^{b,\leq 0}, \mathcal{D}^{b,\geq 0})$ is a bounded t-structure on $\mathcal{D}^b$. For $m \leq n$, we have $\mathcal{D}^{b,[m,n]} = \mathcal{D}^{b,\geq m} \cap \mathcal{D}^{b,\leq n}$. In particular, $\mathcal{C} = \mathcal{D}^{b,\leq 0} \cap \mathcal{D}^{b,\geq 0}$.*

Proof. The axiom (t1) is trivially satisfied, and (t2) follows from the long exact sequence of Hom and $\mathrm{Hom}^i(X,Y) = 0$ for $i < 0$ and $X, Y \in \mathcal{C}$. Since $\mathcal{C}$ is stable under extension, for any interver $I = [m,n]$ of $\mathbb{Z}$ ($m \leq n$), we have

$$[\mathcal{D}^{b,I}] = [\mathcal{C}[-m]] * \cdots * [\mathcal{C}[-n]]. \tag{4.7.6}$$

In view of this, for intervals J, K such that $I = J \cup K$, we then have

$$[\mathcal{D}^{b,I}] = [\mathcal{D}^{b,J}] * [\mathcal{D}^{b,K}]. \tag{4.7.7}$$

In particular, for $J = [m,0]$ and $K = [1,n]$, we then conclude that any object of $\mathcal{D}^{b,I}$ is the extension of an object of $\mathcal{D}^{b,K} \subseteq \mathcal{D}^{b,\geq 1}$ by an object of $\mathcal{D}^{b,J} \subseteq \mathcal{D}^{b,\leq 0}$. Since $\mathcal{D}^b$ is the union of $\mathcal{D}^{b,I}$, we conclude axiom (t3).

By (4.7.6), if X is in $\mathcal{D}^{b,I}$, there exists a sequence of distinguished triangles (X_i, X_{i+1}, A_{i+1}) ($m \leq i \leq n$) with $X_m \in \mathcal{C}[-m]$, $A_i \in \mathcal{C}[-j]$, and $X_n = X$. We may also set $X_{m-1} = 0$ and $A_m = X_m$ and obtain a distinguished triangle $(X_{m-1}, X_m, A_m) = (0, X_m, X_m)$. Apply the long exact sequence of cohomology to these triangles, we see by recurrence over j that, for any $m \leq i \leq n$, we have

$$H^i(X_j) = \begin{cases} 0 & i \notin [m,j], \\ A_i[i] & i \in [m,j]. \end{cases}$$

In particular, $[X] \in \{[H^m(X)[-m]]\} * \cdots * \{[H^n(X)[-n]]\}$; this proves the boundedness of the t-structure (if $H^i(X) = 0$ for all i, then $X = 0$) and the second statement of the proposition (if $X \in \mathcal{C}^{b,\geq m} \cap \mathcal{D}^{b,\leq n}$, the $H^i(X) = 0$ for $i \notin [m,n]$ and therefore X is in $\mathcal{D}^{b,[m,n]}$). □

Remark 4.7.1. Let $\mathcal{D}$ be a triangulated category and $\mathcal{C}$ be a right leaning full subcategory such that any morphism in $\mathcal{C}$ is admissible. The proof of Proposition 4.7.4 shows that any morphism of $\mathcal{C}$ has a kernel and a cokernel, that $\mathrm{im}\, f = \mathrm{coim}\, f$ and that any short exact sequence of $\mathcal{C}$ is admissible. For $\mathcal{C}$ to be abelian, it is necessary and sufficient that $\mathcal{C}$ admits finite direct sums.

Let $\mathcal{C}'$ be the category $E\mathcal{C}$ of successive extensions of objects of $\mathcal{C}$. The long exact sequence of Hom shows that $\mathcal{C}'$ is right leaning. Moreover, we deduce from (4.7.4), (4.7.5) and the associativity of $*$ that $[\mathcal{C}'] * [C'[1]] \subseteq [\mathcal{C}'[1]] * [\mathcal{C}']$, i.e. that every morphism of $\mathcal{C}$ is admissible. Therefore $\mathcal{C}$ is an admissible abelian subcategory of $\mathcal{D}$. Applying Proposition 4.7.15 to $\mathcal{C}'$, we conclude that, for $\mathcal{D}^b$, $\mathcal{D}^{b,\leq 0}$ and $\mathcal{D}^{b,\leq 0}$ defined as above, $(\mathcal{D}^{b,\leq 0}, \mathcal{D}^{b,\geq 0})$ is a t-structure of $\mathcal{D}^b$ with heart $\mathcal{C}'$.

4.7.2.1 t-exact functors

Proposition 4.7.16. *Let $\mathcal{D}$ be a t-category, $K \in \mathcal{D}$, and consider an short exact sequence in $\mathcal{C}$:*

$$0 \longrightarrow A \longrightarrow H^0(K) \longrightarrow B \longrightarrow 0$$

(a) *There exists $K' \in \mathcal{D}^{\leq 0}$, with a morphism $i : K' \to K$, such that, over $\mathcal{D}^{\leq 0}$, K' represents the functor*

$$L \mapsto \ker(\mathrm{Hom}(L,K) \to \mathrm{Hom}(H^0(L), B)).$$

Moreover, for a couple (K', i) to represents this functor, it is necessary and sufficient that for $K' \in \mathcal{D}^{\leq 0}$, $\tau^{<0}K' \cong \tau^{<0}K$ and $H^0(K') \cong A$.

(b) *Dually, we obtain a morphhism $j : K \to K''$ with $K'' \in \mathcal{D}^{\geq 0}$ and the couple (K'', j) represents the functor*

$$L \mapsto \mathrm{coker}(\mathrm{Hom}(L,K) \to \mathrm{Hom}(H^0(L), B)).$$

Moreover, the couple (K'', j) is characterized by $K'' \in \mathcal{D}^{\geq 0}$, $\tau^{>0}K \cong \tau^{>0}K''$ and $H^0(K'') = B$.

(c) *There exists a unique morphism $d : K'' \to K'$ such that the triangle (K', K, K'') is distinguished.*

Proof. $\square$

Let $\mathcal{D}_i$ $(i = 1, 2)$ be t-categories, $\mathcal{C}_i$ be the heart of $\mathcal{D}_i$, and denote by $\varepsilon : \mathcal{C}_i \to \mathcal{D}_i$ the inclusion functor. Let $T : \mathcal{D}_1 \to \mathcal{D}_2$ be an exact functor of triangulated categories (in the usual sense for triangulated categories, that is, up to a natural equivalence it commutes with translation and preserves distinguished triangles); we say that T is **right t-exact** if $T(\mathcal{D}^{\leq 0}) \subseteq D^{\leq 0}$, **left t-exact** if $T(\mathcal{D}^{\geq 0}) \subseteq D^{\geq 0}$, and t-exact if it is both left t-exact and right t-exact.

Proposition 4.7.17. *Let $T : \mathcal{D}_1 \to \mathcal{D}_2$ be an exact functor.*

(a) *If T is left (resp. right) t-exact, the functor ${}^pT := H^0 \circ T \circ \varepsilon : \mathcal{C}_1 \to \mathcal{C}_2$ is left (resp. right) exact.*

(b) *If T is left (resp. right) t-exact and $K \in \mathcal{D}_1^{\geq 0}$ (resp. $K \in \mathcal{D}_1^{\leq 0}$), we have ${}^pT(H^0(K)) \cong H^0(T(K))$ (resp. $H^0(T(K)) \cong {}^pT(H^0(K))$).*

(c) *Let $T^* : \mathcal{D}_2 \to \mathcal{D}_1 : T_*$ be a pair of adjoint functors. For T^* to be right t-exact, it is necessary and sufficient that T_* is left t-exact, and in this case $({}^pT^*, {}^pT_*)$ is an adjoint pair.*

(d) *If $T_1 : \mathcal{D}_1 \to \mathcal{D}_2$ and $T_2 : \mathcal{D}_2 \to \mathcal{D}_3$ are left (resp. right) t-exact functors, then $T_2 \circ T_1$ is also left (resp. right) t-exact and ${}^p(T_2 \circ T_1) = {}^pT_2 \circ {}^pT_1$.*

Proof. If T is left t-exact, for any short exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathcal{C}_1$, the long exact sequence on cohomology of the distinguished triangle $(T(X), T(Y), T(Z))$ gives an exact sequence

$$0 \to H^0(T(X)) \to H^0(T(Y)) \to H^0(T(Z)),$$

since $T(Z)$ is in $\mathcal{D}_2^{\geq 0}$, so ${}^pT$ is left exact.

For $K \in \mathcal{D}_1^{\geq 0}$, the distinguished triangle $(H^0(K), K, \tau^{>0}K)$ gives a distinguished triangle $(T(H^0(K)), T(K), T(\tau^{>0}K))$ with $T(\tau^{>0}K) \in \mathcal{D}_2^{>0}$, so the long exact sequence on cohomology shows that $H^0(T(H^0(K))) \cong H^0(T(K))$. This (and the dual argument) proves the assertions of (a) and (b).

Let $T^* : \mathcal{D}_2 \to \mathcal{D}_1 : T_*$ be a pair of adjoint functors. If T_* is left t-exact, for $U \in \mathcal{D}_1^{>0}$ and $V \in \mathcal{D}_2^{\leq 0}$, we have $\mathrm{Hom}(T^*(V), U) = \mathrm{Hom}(V, T_*(U)) = 0$. Since this is valid for any U, we have $\tau^{>0}T^*(V) = 0$, i.e. $T^*(V)$ is in $\mathcal{D}_1^{\leq 0}$, so T^* is right t-exact. For $A \in \mathcal{C}_1$ and $B \in \mathcal{C}_2$, we then have $H^0(T^*(B)) = \tau^{\geq 0}T^*(B)$ and $H^0(T_*(A)) = \tau^{\leq 0}T_*(A)$, whence a functorial isomorphism

$$\mathrm{Hom}(H^0(T^*(B)), A) \xrightarrow{\sim} \mathrm{Hom}(T^*(B), A) = \mathrm{Hom}(A, T_*(B)) \xleftarrow{\sim} \mathrm{Hom}(A, H^0(T_*(B))).$$

This (resp. its dual form) prove the assertions of (c).

Finally, if T_1 and T_2 are left t-exact and that $A \in \mathcal{C}_1$, we have $T_1(A) \in \mathcal{D}_2^{\geq 0}$ and

$${}^p(T_2 \circ T_1)(A) = H^0(T_2(T_1(A))) = H^0(T_2(H^0(T_1(A))))$$

in view of (b). This, together with its dual form, proves (d). $\square$

Remark 4.7.2. Let $\mathcal{D}_1^+ = \bigcup_n \mathcal{D}_1^{\geq n}$ and $\mathcal{D}_2^- = \bigcup_n \mathcal{D}_2^{\leq n}$. Then the result of Proposition 4.7.17 (c) is still valid for functors $T^* : \mathcal{D}_2^- \to \mathcal{D}_1$ and $T_* : \mathcal{D}_1^+ \to \mathcal{D}_2$ which are adjoint in the sense that $\mathrm{Hom}(T^*(V), U) = \mathrm{Hom}(V, T_*(U))$ for $V \in \mathcal{D}_2^-$ and $U \in \mathcal{D}_2^+$. The proof is the same.

Remark 4.7.3. In the situation of Proposition 4.7.17 (c), for $A \in \mathcal{C}_1$ and $B \in \mathcal{C}_2$, the adjoint morphisms of (T^*, T_*) and $({}^pT^*, {}^pT_*)$ fit into the commmutative diagram

$$\begin{tikzcd}
T^{*p}T_*(A) \arrow[r] \arrow[d] & {}^pT^{*p}T_*(A) \arrow[d] \\
T^*T_*(A) \arrow[r] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
B \arrow[r] \arrow[d] & T_*T^*(B) \arrow[d] \\
{}^pT_*{}^pT^*(B) \arrow[r] & T_*{}^pT^*(B)
\end{tikzcd}$$

Example 4.7.7. Let $T : \mathcal{D}' \to \mathcal{D}$ be a fully faithful exact functor between triangulated categories. For a triangle (X, Y, Z) of $\mathcal{D}'$ to be distinguished, it is necessary and sufficient that its image under T is diatinguished: by (TR2) we can find a distinguished triangle (X, Y, Z') of $\mathcal{D}'$, whose image under T is then isomorphic to the distinguished triangle $(T(X), T(Y), T(Z))$ by Proposition 4.4.19. Since T is fully faithful, this implies that (X, Y, Z) is isomorphic to (X, Y, Z'), so it is distinguished.

Suppose that $\mathcal{D}$ and $\mathcal{D}'$ are endowed with t-structures and that T is t-exact. For $X \in \mathcal{D}'$ to belong to $\mathcal{D}'^{\leq 0}$ (resp. $\mathcal{D}'^{\geq 0}$), it is it is necessary and sufficient that $T(X)$ is in $\mathcal{D}^{\leq 0}$ (resp. $\mathcal{D}^{\geq 0}$). This follows from the fact that $X \in \mathcal{D}'^{\leq 0}$ if and only if $\tau^{>0}X = 0$, and that T commutes with $\tau^{<0}$ (resp. the dual argument).

Conversely, if $\mathcal{D}'$ is a full triangulated subcategory of a triangulated category $\mathcal{D}$ and that $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ is a t-structure over $\mathcal{D}$, for that $(\mathcal{D}'^{\leq 0}, \mathcal{D}'^{\geq 0}) := (\mathcal{D}' \cap \mathcal{D}^{\leq 0}, \mathcal{D}' \cap \mathcal{D}^{\geq 0})$ is a t-structure over $\mathcal{D}'$, it is necessary and sufficient that $\mathcal{D}'$ is stable under the functor $\tau^{\leq 0}$. If this conditions is satisfied, this t-structure over $\mathcal{D}'$ is called the **induced t-structure**. For $\mathcal{D}'$ endowed with the induced t-structure, the inclusion functor $\mathcal{D}' \to \mathcal{D}$ is then t-exact: we have $\mathcal{C}' = \mathcal{D}' \cap \mathcal{C}$, and the restriction of the functors $\tau^{\leq n}$, $\tau^{\geq n}$ or H^p of $\mathcal{D}$ is identified with the same functors of $\mathcal{D}'$.

Let $(\mathcal{D}_i)_{i \in I}$ be a finite family of triangulated categories and $T : \prod_i \mathcal{D}_i \to \mathcal{D}$ be an exact multifunctor to a triangulated category $\mathcal{D}$. Suppose that the $\mathcal{D}_i$ and $\mathcal{D}$ are endowed with t-structures. We say that T is **left t-exact** (resp. **right t-exact**) if it sends the product of the $\mathcal{D}_i^{\geq 0}$ (resp. $\mathcal{D}_i^{\leq 0}$) into $\mathcal{D}^{\geq 0}$ (resp. $\mathcal{D}^{\leq 0}$), and **t-exact** if it is both left t-exact and right t-exact. If T is left t-exact (resp. right t-exact, resp. t-exact) and fix certain variables to be an object of $\mathcal{D}_i^{\geq 0}$ (resp. $\mathcal{D}_i^{\leq 0}$, resp. $\mathcal{C}_i$, the herat of $\mathcal{D}_i$), the functor obtained in the remaining variables is still left t-exact (resp. right t-exact, resp. t-exact). This allows us to apply some proven results for functors of one variable. For example: let $\varepsilon_i : \mathcal{C}_i \to \mathcal{D}_i$ be the inclusion functor. Put ${}^pT = H^0 \circ T \circ (\varepsilon_i)_{i \in I}$. If T is left (resp. right) t-exact, the additive multifunctor ${}^pT$ is then left (resp. right) exact.

Suppose that T is left t-exact. For $(K_i) \in \prod_i \mathcal{D}_i^{\geq 0}$, we conclude from Proposition 4.7.17 (b) that ${}^pT(H^0(K_i)) \cong H^0(T(K_i))$. Therefore, for $(K_i) \in \prod_i \mathcal{D}_i$, the morphisms $K_i \to \tau^{\geq 0}K_i$ then give a morphism

$$H^0(T(K_i)) \to H^0(T(\tau^{\geq 0}K_i)) \xleftarrow{\sim} {}^pT(H^0(K_i)) \tag{4.7.8}$$

By translating, we then obtain, for $\sum n_i = n$, a morphism

$$H^n(T(K_i)) \to {}^pT(H^{n_i}(K_i)) \tag{4.7.9}$$

There is a problem of signs here, which does not appear if we consider instead

$$H^n(T(K_i)) \to H^nT((H^{n_i}(K_i)[-n_i])) \tag{4.7.10}$$

For T right t-exact, we have $H^0T(K_i) \cong {}^pTH^0(K_i)$ for $(K_i) \in \prod_i \mathcal{D}_i^{\leq 0}$, so the morphisms $\tau^{\leq 0}K_i \to K_i$ provides a morphism ${}^pT(H^0(K_i)) \to H^0(T(K_i))$, and by translating, we obtain morphisms

$$H^nT((H^{n_i}(K_i)[-n_i])) \to H^nT(K_i). \tag{4.7.11}$$

where $n = \sum_i n_i$.

If T is t-exact, we have both (4.7.10) and (4.7.11). Applying H^nT to the commutative diagram

$$\begin{tikzcd}
\tau^{\leq n_i}K_i \arrow[r] \arrow[d] & H^{n_i}K_i[-n_i] \arrow[d] \\
K_i \arrow[r] & \tau^{\geq n_i}K_i
\end{tikzcd}$$

we conclude that the composition of (4.7.10) and (4.7.11) is the identity. Now if $\sum_i n_i = \sum_i m_i = n$ and that $(n_i)_{i \in I} \neq (m_i)_{i \in I}$, there exists $i \in I$ such that $n_i < m_i$. The composition $\tau^{\leq n_i}K_i \to$

$K_i \to \tau^{\geq m_i} K_i$ is zero, and we deduce that the composition of (4.7.10) for (m_i) and (4.7.11) for n_i is zero.

Proposition 4.7.18. *If T is t-exact and $(K_i) \in \prod_i \mathcal{D}_i^b$, the morphisms (4.7.10) and (4.7.11) are inverses of each other, so that we have an isomorphism*

$$H^n(T(K_i)) = \bigoplus H^n T((H^{n_i} K_i)[-n_i]) \tag{4.7.12}$$

where the sum is taken over $\sum_i n_i = n$.

Proof. We have already seen that these morphisms make the second member a direct factor of the first. Both members of (4.7.12) are cohomological functors in each K_i (for the right-hand side, thanks to the exactness of ${}^pT$) and the morphisms (4.7.10) and (4.7.11) are morphisms of cohomoloaic functors. By dévissage, we are reduced to assume each $K_i \in \mathcal{C}_i$, where the assertion is trivial. □

4.7.3 Recollement

4.7.3.1 Six functors for topological spaces For X a topological space, endowed with a sheaf of rings $\mathcal{O}_X$, we denote by $D(X, \mathcal{O}_X)$ the derived category of the abelian category $\mathbf{Mod}(\mathcal{O}_X)$ of sheaf of left $\mathcal{O}_X$-modules over X. As usual, $D^+(X, \mathcal{O}_X)$ is the full subcategory consisting of lower bounded complexes.

Let U be an open subset of X and Z be its complement. We denote by $j : U \to X$ and $i : Z \to X$ the canonical inclusions, and by $\mathcal{O}_U$, $\mathcal{O}_Z$ the inverse image of the structural sheaf $\mathcal{O}_X$ over U and Z, respectively. We now describe the construction of glueing a t-structure over $D^+(U, \mathcal{O}_U)$ and that of $D^+(Z, \mathcal{O}_Z)$.

The categories $\mathbf{Mod}(\mathcal{O}_X)$, $\mathbf{Mod}(\mathcal{O}_U)$, and $\mathbf{Mod}(\mathcal{O}_Z)$ are related by the functors

$$\begin{tikzcd} \mathbf{Mod}(\mathcal{O}_U) \arrow[r, bend left, "j_!"] \arrow[r, bend right, "j_*"'] & \mathbf{Mod}(\mathcal{O}_X) \arrow[l, "j^*=j^!"'] \arrow[r, bend left, "i^*"] \arrow[r, bend right, "i^!"'] & \mathbf{Mod}(\mathcal{O}_Z) \arrow[l, "i_*=i_!"'] \end{tikzcd}$$

so that we have adjoint pairs $(j_!, j^*, j_*)$ and $(i^*, i_*, i^!)$. We also have identities

$$j^* j_* = 1, \quad j^* j_! = 1, \quad i^! i_* = 1, \quad i^* i_* = 1, \tag{4.7.13}$$

$$j^* i_* = 0, \quad i^* j_! = 0, \quad i^! j_* = 0. \tag{4.7.14}$$

and for a sheaf $\mathcal{F}$ over X, the adjoint morphisms fit into exact sequences

$$0 \longrightarrow j_! j^*(\mathcal{F}) \longrightarrow \mathcal{F} \longrightarrow i_* i^*(\mathcal{F}) \longrightarrow 0 \tag{4.7.15}$$

$$0 \longrightarrow i_* i^!(\mathcal{F}) \longrightarrow \mathcal{F} \longrightarrow j_* j^*(\mathcal{F}) \tag{4.7.16}$$

where the last arrow is surjective if $\mathcal{F}$ is injective.

For an adjoint pair (T^*, T_*), the adjunction morphism $T^*T_* \to \mathrm{id}$ (resp. $\mathrm{id} \to T_*T^*$) is an isomorphism if and onyl if T_* (resp. T^*) is fully faithful. The assertion (4.7.13) is then equivalent to that i_*, j_* and $j_!$ are fully faithful.

For an exact functor T between abelian categories, it trivially extends to the derived categories. We shall use the same symbol for this extension. This extension coincides with the left derived functor LT and the right derived functor RT, whenever they are defined.

The functors described above induce functors over $D^+(X, \mathcal{O}_X)$, $D^+(U, \mathcal{O}_U)$ and $D^+(Z, \mathcal{O}_Z)$, which form adjoint pairs $(j_!, j^*, Rj_*)$ and $(i^*, i_*, Ri^!)$. We have $j^* i_* = 0$, whence by adjunction

$i^*j_! = 0$ and $Ri^!Rj_* = 0$. For $K \in K^+(X, \mathcal{O}_X)$, the exact sequences (4.7.15) and (4.7.15) then gives distinguished triangles $(j_!j^*K, K, i_*i^*K)$ and $(i_*Ri^!K, K, Rj_*j^*K)$. Finally, for $K \in D^+(Z, \mathcal{O}_Z)$ (resp. $K \in D^+(U, \mathcal{O}_U)$), by (4.7.13) we have isomorphisms

$$i_*i^*K \cong K \cong Ri^!i_*K, \qquad (\text{resp.} \quad j^*Rj_*K \cong K \cong j^*j_!K.)$$

In fact, j_* and i_* transform injectives to injectives, so we can apply Grothendieck's spectral sequence.

The properties listed above are all that we need to glue t-structures, and we meet them in various contexts: for example, for ℓ-adic derived categories, which do not come strictly within the scope of 4.7.3.1. To cover these cases, we will place ourselves in a more general framework. In this context, the sheaf categories no longer appear (only the triangulated categories appear) and we take advantage of this to lighten the notation by simply writing j_* and i^* for the derived functors Rj_* and $Ri^!$.

4.7.3.2 Recollement of t-structures Let $\mathcal{T}$ be a triangulated category endowed with strictly full subcategories $\mathcal{U}, \mathcal{V}$, which are stable under translations. Suppose that for $U \in \mathcal{U}$ and $V \in \mathcal{V}$ we have $\mathrm{Hom}(U, V) = 0$, and that any $X \in \mathcal{T}$ fits into a distinguished triangle (U, X, V), with $U \in \mathcal{U}$ and $V \in \mathcal{V}$. The pair $(\mathcal{U}, \mathcal{V})$ is then a t-structure over $\mathcal{T}$. By Proposition 4.7.5, $\mathcal{V}$ is the right orthogonal of $\mathcal{U}$ and $\mathcal{U}$ is the left orthogonal of $\mathcal{V}$. In particular, $\mathcal{U}$ and $\mathcal{V}$ are Serre subcategories of $\mathcal{T}$. The hypothesis (i), (ii) of ([?] 6.4, p.25) are then satisfied and by ([?] 6.4, p.23-p.26), the projection $\mathcal{T} \to \mathcal{T}/\mathcal{U}$ admits a fully faithful right adjoint, with image $\mathcal{V}$, and the projection $\mathcal{T} \to \mathcal{T}/\mathcal{V}$ admits a fully faithful left adjoint, with image $\mathcal{U}$. In other words, the inclusion $u : \mathcal{U} \to \mathcal{T}$ (resp. $v : \mathcal{V} \to \mathcal{T}$) has a right adjoint $u_\bullet$ (resp. a left adjoint $v^\bullet$) and the sequences

$$0 \longrightarrow \mathcal{U} \xrightarrow{u} \mathcal{T} \xrightarrow{v^\bullet} \mathcal{V} \longrightarrow 0$$

$$0 \longrightarrow \mathcal{V} \xrightarrow{v} \mathcal{T} \xrightarrow{u_\bullet} \mathcal{U} \longrightarrow 0$$

are "exact" in the sense that $v^\bullet$ (resp. $u_\bullet$) identified $\mathcal{V}$ (resp. $\mathcal{U}$) with the quotient of $\mathcal{T}$ by the Serre subcategory $\mathcal{U}$ (resp. $\mathcal{V}$).

Now consider triangulated categories $\mathcal{D}$, $\mathcal{D}_U$ and $\mathcal{D}_Z$ such that we have exact functors

$$\mathcal{D}_Z \xrightarrow{i_*} \mathcal{D} \xrightarrow{j^*} \mathcal{D}_U$$

It is convenient to put $i_! := i_*$ and $j^! = j^*$. Suppose that the following *recollement conditions* are satisfied[3]:

(R1) i_* admits a left adjoint functor i^* and an exact right adjoint functor $i^!$.

(R2) j^* admits a right adjoint functor j_* and an exact left adjoint functor $j_!$.

(R3) $j^*i_* = i^*j_! = i^!j_* = 0$, and therefore for $A \in \mathcal{D}_Z$ and $B \in \mathcal{D}_U$,

$$\mathrm{Hom}(j_!(B), i_*(A)) = \mathrm{Hom}(i_*(A), j_*(B)) = 0.$$

(R4) There are distinguished triangles $(j_!j^*(K), K, i_*i^*(K))$ and $(i_*i^!(K), K, j_*j^*(K))$ for $K \in \mathcal{D}$.

(R5) $i^*i_* \to \mathrm{id} \to i^!i_*$ and $j^*j_* \to \mathrm{id} \to j^*j_!$ are isomorphisms, or equivalently, i_*, $j_!$ and j_* are fully faithful.

[3]We note that this formalism is selfdual by exchanging $j_!$ with j_* and i^* with $i^!$.

Applying the preceding arguments to $\mathcal{T} = \mathcal{D}$, and choose for $(\mathcal{U}, \mathcal{V})$ the pairs of subcategories $(i_*\mathcal{D}_Z, j_*\mathcal{D}_U)$ and $(j_!\mathcal{D}_U, i_*\mathcal{D}_Z)$, we obtain the following exact sequences

$$
\begin{array}{ccccccccc}
0 & \longleftarrow & \mathcal{D}_Z & \xleftarrow{i^*} & \mathcal{D} & \xleftarrow{j_!} & \mathcal{D}_U & \longleftarrow & 0 \\
0 & \longrightarrow & \mathcal{D}_Z & \xrightarrow{i_*} & \mathcal{D} & \xrightarrow{j^*} & \mathcal{D}_U & \longrightarrow & 0 \\
0 & \longleftarrow & \mathcal{D}_Z & \xleftarrow{i^!} & \mathcal{D} & \xleftarrow{j_*} & \mathcal{D}_U & \longleftarrow & 0
\end{array}
$$

Since the functor i_* is fully faithful, the composition of the adjunction morphisms $i_*i^! \to \mathrm{id} \to i_*i^*$ gives a unique morphism of functors

$$i^! \to i^*. \tag{4.7.17}$$

If we apply this to $i_*(X)$ and identify $i^!i_*(X)$ and $i^*i_*(X)$ with X, we obtain the identity morphism of X.

The functor j^* being a quotient functor (it identifies $\mathcal{D}_U$ with the quotient category), the composition of the adjunction morphisms $j_!j^* \to \mathrm{id} \to j_*j^*$ defines a unique morphism of functors

$$j_! \to j_*. \tag{4.7.18}$$

If we identify $j^*j_!$ and j^*j_* with the identity functor, then by applying j^* to (4.7.18), we obtain the identity morphism of the idnetity functor.

For $X \in \mathcal{D}_U$, the cone of $j_!(X) \to j_*(X)$ is then annihilated by j^*, so it belongs to $i_*\mathcal{D}_F$. By condition (R3) and Proposition 4.7.1, the distinguished triangle with base $j_!(X) \to j_*(X)$ is uniquely determined up to unique isomorphisms, so we obtain a functor $j_*/j_! : \mathcal{D}_U \to \mathcal{D}_F$ which fits into a functorial distinguished triangle

$$(j_!, j_*, i_*(j_*/j_!)). \tag{4.7.19}$$

The dual construction provides a functor $T : \mathcal{D}_U \to \mathcal{D}_F$, which is characterized by a distinguished triangle $(i_*T, j_!, j_*)$. A triangle of this type is deduced from (4.7.19) by rotation, so we have an isomorphism $T = (j_*/j_!)[-1]$. Applying i^* and $i^!$ to the triangle (4.7.19) (and its rotation), and noting that $i^*j_! = i^!j_* = 0$, we obtain isomorphisms

$$i^*j_* \xrightarrow{\sim} j_*/j_! \xrightarrow{\sim} i^!j_![1]. \tag{4.7.20}$$

Let $X \in \mathcal{D}$ and apply (TR4) to the adjuction morphims $j_!j^*(X) \to X \to j_*j^*(X)$, we obtain an octahedron

$$
\begin{tikzcd}
 & & & A \arrow[r] \arrow[dr] & {} & & \\
 & & X \arrow[ur] \arrow[dr] & & B \arrow[r] \arrow[ddr] & {} & \\
 & & & j_*j^*(X) \arrow[ur] \arrow[drr] & & & \\
 & j_!j^*(X) \arrow[uur] \arrow[urr] & & & & C \arrow[r] \arrow[d] & {} \\
 & & & & & {} &
\end{tikzcd}
$$

By (R3), (R4) and Proposition 4.7.1, there exists a unique isomorphism $A \cong i_*i^*(X)$, which identified $X \to A$ with the adjunction morphism. It also identifies $(j_!j^*X, X, A)$ to the distinguished triangle $(j_!j^*(X), X, i_*i^*(X))$ of (R4).

The same argument, applied to $j_*j^*(X)$, whose image under j^* is X, we see that B is identified with $i_*i^*j_*j^*(X) = i_*(j_*/j_!)j^*(X)$ (cf. (4.7.20)) and $j_*j^*(X) \to B$ is identified with the adjunction morphism of (i^*, i_*). By Proposition 4.7.1, there is a unique morphism $A \to B$ rendering the upper cap of an octahedron, i.e. a morphism $i_*i^*(X) \to i_*i^*j_*j^*(X)$; this is then deduced from the morphism $X \to j_*j^*(X)$ by adjunction.

Dually, there exists a unique isomorphism from C to $i_*i^!(X)[1]$, which identified the morphism $j_*j^*(X) \to C$ with the $+1$ shift of the adjunction morphism $i_*i^!(X) \to X$ (the triangle $(X, j_*j^*(X), C)$ is obtained by rotating $(i_*i^!(X), X, j_*j^*(X))$). The morphism $B \to C$, which is the unique morphism rendering the commutative square $(B, C, j_!j^*(X), X)$, is then identified, via the isomorphism (4.7.20), with the morphism

$$i_*(j_*/j_!)j^*(X) = i_*i^!j_!j^*(X)[1] \to i_*i^!(X)[1]$$

induced from $j_!j^*(X) \to X$ by functoriality.

We have thus determined all the vertices, and all the arrows of the octahedron ($C \to A$ is the composition $C \to X \to A$), and proves its functoriality. If we replace A, B, C by their values, the octahedron is then written as

$$\begin{tikzcd}
& & i_*i^*(X) \arrow[r] \arrow[dr] & {} & \\
& X \arrow[ur] \arrow[dr] & & i_*(j_*/j_!)j^*(X) \arrow[r] \arrow[ddr] & {} & \\
& & j_*j^*(X) \arrow[ur] \arrow[drr] & & & \\
j_!j^*(X) \arrow[uur] \arrow[urr] & & & & i_*i^!(X)[1] \arrow[r] \arrow[dr] & {} \\
& & & & & {}
\end{tikzcd} \tag{4.7.21}$$

Since i_* is fully faithful, the distinguished triangle $(i_*i^*(X), i_*(j_*/j_!)j^*(X), i_*i^!(X)[1])$ is then the image under i_* of a distinguished triangle $(i^*(X), (j_*/j_!)j^*(X), i^!(X)[1])$. The image under i_* of the $+1$ morphism of this triangle is the composition $i_*i^!X[1] \to X \to i_*i^*(X)$, so $d : i^!(X)[1] \to i^*(X)[1]$ is the morphism (4.7.17) for $X[1]$ (the translation of (4.7.17) for X). Rotating the triangle, with a sign change for the morphism of degree $+1$ and erasing i_* (cf. Example 4.7.7), we obtain a functorial distinguished triangle

$$(i^!, i^*, (j_*/j_!)j^*). \tag{4.7.22}$$

Now let $(\mathcal{D}_U^{\leq 0}, \mathcal{D}_U^{\geq 0})$ be a t-structure over $\mathcal{D}_U$ and $(\mathcal{D}_Z^{\leq 0}, \mathcal{D}_Z^{\geq 0})$ be a t-structure over $\mathcal{D}_Z$. We define subcategories of $\mathcal{D}$ by

$$\mathcal{D}^{\leq 0} := \{K \in \mathcal{D} : j^*(K) \in \mathcal{D}_U^{\leq 0} \text{ and } i^*(K) \in \mathcal{D}_Z^{\leq 0}\},$$
$$\mathcal{D}^{\geq 0} := \{K \in \mathcal{D} : j^*(K) \in \mathcal{D}_U^{\geq 0} \text{ and } i^!(K) \in \mathcal{D}_Z^{\geq 0}\}.$$

Theorem 4.7.19. *With the preceding hypotheses, $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ is a t-structure over $\mathcal{D}$.*

Proof. Let $X \in \mathcal{D}^{\leq 0}$ and $Y \in \mathcal{D}^{\geq 1}$. The triangle $(j_!j^*(X), X, i_*i^*(X))$ gives the exact sequence

$$\mathrm{Hom}(i^*(X), i^!(Y)) = \mathrm{Hom}(i_*i^*(X), Y) \to \mathrm{Hom}(X, Y) \to \mathrm{Hom}(j_!j^*(X), Y) = \mathrm{Hom}(j^*(X), j^*(Y)).$$

We then conclude from axiom (t2) of $\mathcal{D}_U$ and $\mathcal{D}_Z$ that $\mathrm{Hom}(X, Y) = 0$. Also, in view of the definition above, the axiom (t1) for $\mathcal{D}$ follows from that of $\mathcal{D}_U$ and $\mathcal{D}_Z$.

For $X \in \mathcal{D}$, to verify axiom (t3), we choose objects Y and A so that we have distinguished triangles $(Y, X, j_*\tau^{>0}j^*(X))$ and $(A, Y, i_*\tau^{>0}i^*(Y))$, and apply (TR4):

$$\begin{tikzcd}[column sep=small]
 & & & {} & & & \\
 & & i_*\tau^{>0}i^*(Y) \arrow[ur] \arrow[dr] & & & & \\
 & Y \arrow[ur] \arrow[dr] & & B \arrow[r] \arrow[dr] & {} & & \\
 & & X \arrow[ur] \arrow[rr] & & j_*\tau^{>0}j^*(X) \arrow[r] \arrow[dr] & {} & \\
A \arrow[uur] \arrow[urr] & & & & & {} &
\end{tikzcd}$$

Applying j^*, i^*, $i^!$ to the distinguished triangles in this octahedron, we conclude from (R3) and (R5) that

$$\begin{aligned}
&j^*(i_*\tau^{>0}i^*(Y), B, j_*\tau^{>0}j^*(X)) = (0, j^*(B), \tau^{>0}j^*(X)) && \text{whence } j^*(B) \xrightarrow{\sim} \tau^{>0}j^*(X),\\
&j^*(A, X, B) = (j^*(A), j^*(X), \tau^{>0}j^*(X)) && \text{whence } j^*(A) \xrightarrow{\sim} \tau^{\leq 0}j^*(X),\\
&i^*(A, Y, i_*\tau^{>0}i^*(Y)) = (i^*(A), i^*(Y), \tau^{>0}i^*(Y)) && \text{whence } i^*(A) \xrightarrow{\sim} \tau^{\leq 0}i^*(Y),\\
&i^!(i_*\tau^{>0}i^*(Y), B, j_*\tau^{>0}j^*(X)) = (\tau^{>0}i^*(Y), i^!(B), 0) && \text{whence } \tau^{>0}i^*(Y) \xrightarrow{\sim} i^!(B).
\end{aligned}$$

We then conclude that $A \in \mathcal{D}^{\leq 0}$ and $B \in \mathcal{D}^{\geq 1}$, and this proves axiom (t3). □

Proposition 4.7.20. *Under the hypothesis and notations of Theorem 4.7.19, let $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ be a t-structure over $\mathcal{D}$. Then the following conditions are equivalent:*

(i) *$j_!j^*$ is right t-exact;*

(i') *j_*j^* is left t-exact;*

(ii) *the t-structure of $\mathcal{D}$ is obtained by glueing.*

Proof. The equivalence of (i) and (i') follows from Proposition 4.7.17 (c), and (i), (i') are equivalent to axiom (t2) for $(j^*\mathcal{D}^{\leq 0}, j^*\mathcal{D}^{\geq 0})$. The distinguished triangle $(j_!j^*, \mathrm{id}, i_*i^*)$ and $(i_*i^!, \mathrm{id}, j_*j^*)$ shows respectively that (i) and (i') implies that i_*i^* is right t-exact and $i_*i^!$ is left t-exact, which are equivalent by Proposition 4.7.17 (c), and signifies that $(i^*\mathcal{D}^{\leq 0}, i^!\mathcal{D}^{\geq 0})$ verifies axiom (t2).

It is clear that (ii)⇒(i), (i'), and that the t-structure over $\mathcal{D}_U$ and $\mathcal{D}_Z$ are such that $\mathcal{D}$ is the glueing of $(j^*\mathcal{D}^{\leq 0}, j^*\mathcal{D}^{\geq 0})$ over $\mathcal{D}_U$ and $(i^*\mathcal{D}^{\leq 0}, i^!\mathcal{D}^{\geq 0})$ over $\mathcal{D}_Z$. Conversely, if (i) and (i') are satisfied, we verify successively that

(a) $(j^*\mathcal{D}^{\leq 0}, j^*\mathcal{D}^{\geq 0})$ is a t-structure over $\mathcal{D}_U$. This follows from the fact that j^* is essentially surjective.

(b) $(i^*\mathcal{D}^{\leq 0}, i^!\mathcal{D}^{\geq 0})$ is a t-structure over $\mathcal{D}_Z$. Only axiom (t3) is nontrivial: for $X \in \mathcal{D}_Z$, the t-exactness of j^* shows that $j^*\tau^{\leq 0}i_*(X) = \tau^{\leq 0}j^*i_*(X) = 0$, and that similarly $j^*\tau^{>0}i_*(X) = 0$[4]. The truncations $\tau^{\leq 0}i_*(X)$ and $\tau^{>0}i_*(X)$ are therefore in $i_*\mathcal{D}_Z$, and we obtain a distinguished triangle $(i^*\tau^{\leq 0}i_*(X), X, i^*\tau^{>0}i_*(X))$, which proves axiom (t3).

(c) The identity functor of $\mathcal{D}$, endowed with the t-structure over $\mathcal{D}$ and the t-structure glueing $(j^*\mathcal{D}^{\leq 0}, j^*\mathcal{D}^{\geq 0})$ and $(i^*\mathcal{D}^{\leq 0}, i^!\mathcal{D}^{\geq 0})$, is t-exact.

By Example 4.7.7, the original t-structure of $\mathcal{D}$ is therefore obtained by glueing $(j^*\mathcal{D}^{\leq 0}, j^*\mathcal{D}^{\geq 0})$ and $(i^*\mathcal{D}^{\leq 0}, i^!\mathcal{D}^{\geq 0})$. □

[4] The truncations functor are that for the t-structure $(j^*\mathcal{D}^{\leq 0}, j^*\mathcal{D}^{\geq 0})$ over $\mathcal{D}_U$.

Suppose that we are given a t-structure over $\mathcal{D}_Z$, and apply Theorem 4.7.19 to the degenerate t-structure $(\mathcal{D}_U, 0)$ over $\mathcal{D}_U$ and the given t-structure over $\mathcal{D}_Z$. The functor $\tau^{\leq n}$ relative to the t-structure obtained over $\mathcal{D}$ is then denoted by $\tau_Z^{\leq n}$, which is right adjoint to the inclusion of the full subcategory of $\mathcal{D}$ formed by $X \in \mathcal{D}$ such that $i^*(X) \in \mathcal{D}_Z^{\leq n}$. The proof of axiom (t3) in Theorem 4.7.19 shows that we have a distinguished triangle

$$(\tau_Z^{\leq n} X, X, i_* \tau^{>n} i^*(X)) \tag{4.7.23}$$

(with the notations of Theorem 4.7.19, we have $X = Y$ since $j_* \tau^{>0} j^*(X) = 0$). The cohomology functor H^n for this t-structure is then given by $i_* H^n i^*(X)$.

Dually, we define $\tau_Z^{\geq n}$ via the degenerate t-structure $(0, \mathcal{D}_U)$ over $\mathcal{D}_U$, which is left adjoint to the inclusion of the full subcategory of $\mathcal{D}$ formed by $X \in \mathcal{D}$ such that $i^!(X) \in \mathcal{D}_Z^{\geq n}$. We have a distinguished triangle

$$(i_* \tau^{<n} i^!(X), X, \tau_Z^{\geq n} X), \tag{4.7.24}$$

and the functor H^n is given by $i_* H^n i^!(X)$.

Similarly, if we are given a t-structure over $\mathcal{D}_U$, and we endow $\mathcal{D}_Z$ with the t-structure $(\mathcal{D}_Z, 0)$ (resp. $(0, \mathcal{D}_Z)$), we can define t-structure over $\mathcal{D}$, and truncation functor $\tau_U^{\leq n}$ (resp. $\tau_U^{\geq n}$), which fit into distinguished triangles

$$(\tau_U^{\leq n} X, X, j_* \tau^{>n} j^*(X)), \qquad (\text{resp.} \quad (j_! \tau^{<n} j^*(X), X, \tau_U^{\geq n} X)). \tag{4.7.25}$$

The cohomology functor H^n is given by $j_* H^n j^*(X)$ (resp. $j_! H^n j^*(X)$).

The proof of axiom (t3) in Theorem 4.7.19 shows that $\tau^{\leq 0} = \tau_Z^{\leq 0} \tau_U^{\leq 0}$. By translating and duality, we obtain

$$\tau^{\leq n} = \tau_Z^{\leq n} \tau_U^{\leq n}, \quad \tau^{\geq n} = \tau_Z^{\geq n} \tau_U^{\geq n}. \tag{4.7.26}$$

Example 4.7.8. In the situation of 4.7.3.1 and for the natural t-structure of $D^+(Z, \mathscr{O}_Z)$, the functor $\tau_Z^{\leq n}$ is deduced from the functor of the category of complexes of sheaves into itself which to a complex K associates the subcomplex which coincides with K over U, and with the subcomplex $\tau^{\leq n} K$ over Z.

A **prolongation** of an object Y of $\mathcal{D}_U$ is defined to be an object X of $\mathcal{D}$ endowed with an isomorphism $j^*(X) \cong Y$. Such an isomorphism gives by adjunction morphisms $j_!(Y) \to X \to j_*(Y)$. If $n \in \mathbb{Z}$ is an integer, then from the distinguished triangle (4.7.23) (resp. (4.7.24)) and (R3), (R5), we see that $\tau_Z^{\geq n} j_!(Y)$ (resp. $\tau_Z^{\leq n} j_*(Y)$) is a prolongation of Y. If a prolongation X is isomorphic as a prolongation to $\tau_Z^{\geq n} j_!(Y)$ (resp. $\tau_Z^{\leq n} j_*(Y)$), the isomorphism $j^*(X) \cong Y$ is then uniquely determined, and we simply write $X = \tau_Z^{\geq n} j_!(Y)$ (resp. $\tau_Z^{\leq n} j_*(Y)$).

Proposition 4.7.21. *Let $Y \in \mathcal{D}_U$ and n be an integer. There exists, up to unique isomorphisms, a unique prolongation X of Y such that $i^*(X) \in \mathcal{D}_Z^{\leq n-1}$ and $i^!(X) \in \mathcal{D}_Z^{\geq n+1}$. This prolongation is given by $\tau_Z^{\leq n-1} j_*(Y)$, and we have $\tau_Z^{\leq n-1} j_*(Y) = \tau_Z^{\geq n+1} j_!(Y)$.*

Proof. Let X be a prolongation of Y. The distinguished triangle $(i^*(X), (j_*/j_!)(Y), i^!(X)[1])$ obtained by rotating (4.7.22) shows that the following conditions are equivalent:

(i) $i^*(X) \in \mathcal{D}_Z^{\leq n-1}$ and $i^!(X) \in \mathcal{D}_Z^{\geq n+1}$;

(ii) $i^!(X)[1] = \tau^{\geq n}(j_*/j_!)(Y) = \tau^{\geq n} i^* j_*(Y)$;

(ii’) $i^*(X) = \tau^{\leq n-1}(j_*/j_!)(Y) = \tau^{\leq n} i^! j_!(Y)[1]$.

The distinguished triangles $(X, j_*(Y), i_* i^!(X)[1])$ of (4.7.21) and $(\tau_Z^{\leq n-1}, \mathrm{id}, i_* \tau^{>n-1} i^*)$ imply that condition (ii) is equivalent to $X = \tau_F^{\leq n-1} j_*(Y)$, and the triangles $(j_!(Y), X, i_* i^*(X))$ of (4.7.21) and $(i_* \tau^{<n+1} i^!, \mathrm{id}, \tau_Z^{\geq n+1})$ imply that condition (ii’) is equivalent to $X = \tau_Z^{\geq n+1} j_!(Y)$; we therefore conclude the proposition. ☐

Remark 4.7.4. Let n be an integer and $\mathcal{D}_m$ be the full subcategory of $\mathcal{D}$ formed by objects X such that $i^*(X) \in \mathcal{D}_Z^{\leq n-1}$ and $i^!(X) \in \mathcal{D}_Z^{\geq n+1}$. The functor j^* then induces an equivalence of categories $\mathcal{D}_m \xrightarrow{\sim} \mathcal{D}_U$. It admits $\tau_Z^{\leq n-1} j_*$ as a quasi-inverse, which we often denote by ${}^p j_{!*}$.

Let $\mathcal{C}$, $\mathcal{C}_U$ and $\mathcal{C}_Z$ be the hearts of the t-categories $\mathcal{D}$, $\mathcal{D}_U$ and $\mathcal{D}_Z$, respectively, where $\mathcal{D}_U$ and $\mathcal{D}_Z$ are endowed with given t-structures, and $\mathcal{D}$ with the glueing t-structure. Denote by ε the inclusions of $\mathcal{C}$, $\mathcal{C}_U$ or $\mathcal{C}_Z$ into $\mathcal{D}$, $\mathcal{D}_U$ or $\mathcal{D}_Z$, and for T the functors $j_!, j^*, j_*, i^*, i_*, i^!$, we denote by ${}^pT$ the functor $H^0 \circ T \circ \varepsilon$. By the definition of the t-structure of $\mathcal{D}$, j^* is t-exact, i^* is right t-exact, and $i^!$ is left t-exact. Applying Proposition 4.7.17 (c) and (d), we obtain the following result:

Proposition 4.7.22. *Let $\mathcal{D}$, $\mathcal{D}_U$ and $\mathcal{D}_Z$ be triangulated categories as above.*

(a) *The functors $j_!$ and i^* (resp. j_* and $i^!$, resp. j^* and i_*) are right t-exact (resp. left t-exact, resp. t-exact), and we have adjoint triples $({}^pj_!, {}^pj^*, {}^pj_*)$ and $({}^pi^*, {}^pi_*, {}^pi^!)$.*

(b) *The composition ${}^pj^* \circ {}^pi_*$, ${}^pi^* \circ {}^pj_!$ and ${}^pi^! \circ {}^pj_*$ are zero. For $A \in \mathcal{C}_Z$ and $B \in \mathcal{C}_U$, we have*

$$\mathrm{Hom}({}^pj_!(B), {}^pi_*(A)) = \mathrm{Hom}({}^pi_*(A), {}^pj_*(B)) = 0.$$

(c) *For $A \in \mathcal{C}$, the sequences*

$$ {}^pj_!{}^pj^*(A) \longrightarrow A \longrightarrow {}^pi_*{}^pi^*(A) \longrightarrow 0$$

$$0 \longrightarrow {}^pi_*{}^pi^!(A) \longrightarrow A \longrightarrow {}^pj_*{}^pj^*(A)$$

are exact.

(d) *The functors ${}^pi_*$, ${}^pj_!$ and ${}^pj_*$ are fully faithful, or equivalently, ${}^pi^*{}^pi_* \to \mathrm{id} \to {}^pi^!{}^pi_*$ and ${}^pj^*{}^pj_* \to \mathrm{id} \to {}^pj^*{}^pj_!$ are isomorphisms.*

Proof. Only the exact sequences in (c) are nontrivial. For this, we note that by Proposition 4.7.17 we have ${}^pT_2 \circ {}^pT_1 = H^0(T_2 \circ T_1)$, so for $A \in \mathcal{C}$ we have a long exact sequence

$$\cdots \xrightarrow{+1} H^0(j_!j^*(A)) = {}^pj_!{}^pj^*(A) \longrightarrow A \longrightarrow {}^pi_*{}^pi^*(A) = H^0(i_*i^*(A)) \xrightarrow{+1} \cdots$$

To see that the morphism $A \to H^0(i_*i^*(A))$ is surjective, we note that since $j_!$ is right t-exact and j^* is t-exact, we have $j_!j^*(A) \in \mathcal{D}^{\leq 0}$, so $H^1(j_!j^*(A)) = 0$ and we obtain the first exact sequence in (c). The second one can be deduced similarly, using the fact that j_* is left t-exact and j^* is t-exact (so $j_*j^*(A) \in \mathcal{D}^{\geq 0}$). □

Remark 4.7.5. By ${}^pj^*{}^pi_* = 0$ and the exact sequence of (c), for $X \in \mathcal{C}$ to be in the essential image $\bar{\mathcal{C}}_Z$ of ${}^pi_*$, it is necessary and sufficient that ${}^pj^*(X) = 0$. Since the functor ${}^pj^*$ is exact, this essential image is a Serre subcategory of $\mathcal{C}$. If we identify $\mathcal{C}_Z$ with $\bar{\mathcal{C}}_Z$ by the fully faithful functor ${}^pi_*$, the adjunctions $({}^pi^*, {}^pi_*)$ and $({}^pi_*, {}^pi_!)$ show that for $X \in \mathcal{C}$, ${}^pi^*(X)$ is the largest quotient of X which is in $\mathcal{C}_Z$, and ${}^pi^!(X)$ is the largest sub-object of X that is in $\mathcal{C}_Z$.

Proposition 4.7.23. *The functor ${}^pj^*$ identifies $\mathcal{C}_U$ with the quotient of $\mathcal{C}$ by the Serre subcategory $\mathcal{C}_Z$ (or, more precisely, its iamge $\bar{\mathcal{C}}_Z$).*

Proof. Let $Q : \mathcal{C} \to \mathcal{C}/\mathcal{C}_Z$ be the quotient functor. The exact functor ${}^pj^*$ admits a factorization $T \circ Q$:

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "{}^pj^*"] \arrow[dr, "Q"'] & & \mathcal{C}_U \\ & \mathcal{C}/\mathcal{C}_Z \arrow[ur, "T"'] & \end{tikzcd}$$

We note that T is faithful: if a morphism f in $\mathcal{C}/\mathcal{C}_Z$ comes from a morphism f_1 of $\mathcal{C}$, and f is killed by T, then f_1 is killed by ${}^p j^*$. Since ${}^p j^*$ is exact, this implies ${}^p j^*(\mathrm{im}(f_1)) = \mathrm{im}({}^p j^*(f_1)) = 0$, so $\mathrm{im}(f_1) \in \bar{\mathcal{C}}_Z$ and f_1 is killed by Q. Since $\mathrm{id} \xrightarrow{\sim} {}^p j^* {}^p j_! = T \circ Q \circ {}^p j_!$, we see that T is essentially surjective, so it remains to verify that T is fully faithful, whence an equivalence of categories.

We note that for $A \in \mathcal{C}$ there is an exact sequence

$$0 \longrightarrow {}^p i_* H^{-1} i^*(A) \longrightarrow {}^p j_! {}^p j^*(A) \longrightarrow A \longrightarrow {}^p i_* {}^p i^*(A) \longrightarrow 0$$

(Note that i_* is t-exact and i^* is right t-exact, so we have $H^{-1}(i_* i^*(A)) = {}^p i_* H^{-1} i^*(A)$ by Proposition 4.7.17 (b).) The kernel and cokernel of the morphism ${}^p j_! {}^p j^*(A) \to A$ are therefore in the image of ${}^p i_*$, and any object of $\mathcal{C}/\bar{\mathcal{C}}_Z$ is then contained in the essential image of ${}^p j_!$. For ${}^p j_!(X)$ and ${}^p j_!(Y)$ in this image, the map

$$T : \mathrm{Hom}(Q {}^p j_!(X), Q {}^p j_!(Y)) \to \mathrm{Hom}(TQ {}^p j_!(X), TQ {}^p j_!(Y)) = \mathrm{Hom}(X, Y)$$

admits a section $Q {}^p j_!$, and hence is surjective. This completes the proof. $\square$

Remark 4.7.6. In the situation of 4.7.3.1, if we endow $D^+(U, \mathscr{O}_U)$ and $D^+(Z, \mathscr{O}_Z)$ the natural t-structures, then the glueing t-structure on $D^+(X, \mathscr{O}_X)$ is the natural t-structure, and the abelian categories $\mathcal{C}$, $\mathcal{C}_U$ and $\mathcal{C}_Z$ are $\mathbf{Mod}(\mathscr{O}_X)$, $\mathbf{Mod}(\mathscr{O}_U)$ and $\mathbf{Mod}(\mathscr{O}_Z)$. In the general case, however, the functor ${}^p j_!$ is only right exact, ${}^p i_*$ is only left exact, and the first sequence of Proposition 4.7.22 (c) is not left exact.

Since the functor ${}^p i_*$ is fully faithful, the composition of the adjunction morphisms ${}^p i_* {}^p i^! \to \mathrm{id} \to {}^p i_* {}^p i^*$ is the image under ${}^p i_*$ of a unique morphism of functors

$$ {}^p i^! \to {}^p i^*. \tag{4.7.27}$$

The diagrams of Remark 4.7.3 for (i^*, i_*) and $(i_*, i^!)$, and the t-exactness of i_*, imply that for $A \in \mathcal{C}$ we have a commutative diagram

$$\begin{tikzcd}
{}^p i_* {}^p i_! \arrow[r] \arrow[d, equal] & \mathrm{id} \arrow[r] \arrow[dd, equal] & {}^p i_* {}^p i^* \arrow[d, equal] \\
i_* {}^p i_! \arrow[d] & & i_* {}^p i^* \\
i_* i^! \arrow[r] & \mathrm{id} \arrow[r] & i_* i^* \arrow[u]
\end{tikzcd}$$

Therefore, for $A \in \mathcal{C}$, the morphism ${}^p i^!(A) \to {}^p i^*(A)$ in (4.7.27) is given by the composition

$$ {}^p i^!(A) \to i^!(A) \xrightarrow{(4.7.17)} i^*(A) \to {}^p i^*(A). \tag{4.7.28}$$

By Proposition 4.7.22(d), if we apply (4.7.27) to ${}^p i_*(A)$ (for $A \in \mathcal{C}_Z$), then we obtain the identity morphism of A.

On the other hand, since the functor ${}^p j^*$ is identified with the quotient functor (Proposition 4.7.23), the composition of the adjunction morphisms ${}^p j_! {}^p j^* \to \mathrm{id} \to {}^p j_* {}^p j^*$ provides a unique morphism of functors

$$ {}^p j_! \to {}^p j_*. \tag{4.7.29}$$

Similarly, the diagrams of Remark 4.7.3 for (j^*, j_*) and $(j_!, j^*)$, together with the t-exactness of j^*, imply that for $A \in \mathcal{C}$ we have a commutative diagram

$$\begin{tikzcd}
{}^p j_! {}^p j^* \arrow[r] \arrow[d, equal] & \mathrm{id} \arrow[r] \arrow[dd, equal] & {}^p j_* {}^p j^* \arrow[d, equal] \\
{}^p j_! j^* & & {}^p j_* j^* \arrow[d] \\
j_! j^* \arrow[u] \arrow[r] & \mathrm{id} \arrow[r] & j_* j^*
\end{tikzcd}$$

Therefore, for $B \in \mathcal{C}_U$, the morphism $j_!(B) \to j_*(B)$ of (4.7.29) is the composition

$$j_!(B) \to \tau^{\geq 0} j_!(B) = {}^p j_!(B) \xrightarrow{(4.7.18)} {}^p j_*(B) = \tau^{\leq 0} j_*(B) \to j_*(B). \tag{4.7.30}$$

By Proposition 4.7.22(d), if we apply ${}^p j_*$ to (4.7.29), then we obtain the identity morphism. In particular, for $B \in \mathcal{C}_U$, the kernel and cokernel of ${}^p j_!(B) \to {}^p j_*(B)$ are in ${}^p i_* \mathcal{C}_Z$.

Definition 4.7.1. The functor $j_{!*} : \mathcal{C}_U \to \mathcal{C}$ is defined to be the functor which to $B \in \mathcal{C}_U$ associates the image of ${}^p j_!(B)$ into ${}^p j_*(B)$.

For $B \in \mathcal{C}_U$, (4.7.30) shows that we have the following factorization of the morphism $j_!(B) \to j_*(B)$ of (4.7.18):

$$j_!(B) \to {}^p j_!(B) \to j_{!*}(B) \to {}^p j_*(B) \to j_*(B) \tag{4.7.31}$$

Proposition 4.7.24. *For $B \in \mathcal{C}_U$, we have*

$$\begin{aligned}
{}^p j_!(B) &= \tau_Z^{\geq 0} j_!(B) = \tau_Z^{\leq -2} j_*(B), \\
j_{!*}(B) &= \tau_Z^{\geq 1} j_!(B) = \tau_Z^{\leq -1} j_*(B), \\
{}^p j_*(B) &= \tau_Z^{\geq 2} j_!(B) = \tau_Z^{\leq 0} j_*(B).
\end{aligned}$$

More precisely, ${}^p j_!(B)$, endowed with the morphism $j_!(B) \to {}^p j_!(B)$, is isomorphic to $\tau_Z^{\geq 0} j_!(B)$, and so on.

Proof. Since $j^* j_!(B) \cong B$ is in $\mathcal{C}_U$, we see that $j_!(B)$ is stable under $\tau_U^{\geq 0}$. By (4.7.26) and Proposition 4.7.22 (a), we then have ${}^p j_!(B) = \tau^{\geq 0} j_!(B) = \tau_Z^{\geq 0} j_!(B)$, and Proposition 4.7.21 shows that $\tau_Z^{\geq 0} j_!(B) = \tau_Z^{\leq -2} j_*(B)$. Similarly, since $j^* j_*(B) \cong B$, we have ${}^p j_*(B) = \tau_Z^{\leq 0} j_*(B) = \tau_Z^{\geq 2} j_!(B)$.

The determination $H^n = i_* H^n i^!$ for the t-structure defining $\tau_Z^{\geq n}$ shows that we have a distinguished triangle

$$(i_* H^0 i^! j_!(B), \tau_Z^{\geq 0} j_!(B), \tau_Z^{\geq 1} j_!(B)) = (i_* H^0 i_! j_!(B), {}^p j_!(B), \tau_Z^{\geq 1} j_!(B)),$$

which implies that $\tau_Z^{\geq 1} j_!(B) \in \mathcal{D}^{[-1,0]}$. A dual argument provides a distinguished triangle

$$(\tau_Z^{\leq -1} j_*(B), \tau_F^{\leq 0} j_*(B), i_* H^0 i^* j_*(B)) = (\tau_Z^{\leq -1} j_*(B), {}^p j_*(B), i_* H^0 i^*(B)),$$

which shows that $\tau_Z^{\leq -1} j_*(B) \in \mathcal{D}^{[0,1]}$. By Proposition 4.7.21, we then conclude that $\tau_Z^{\geq 1} j_!(B) = \tau_Z^{\leq -1} j_*(B)$ belongs to $\mathcal{C}$, and the above triangles produce short exact sequences

$$0 \longrightarrow i_* H^0 i^! j_!(B) \longrightarrow {}^p j_!(B) \longrightarrow \tau_Z^{\geq 1} j_!(B) \longrightarrow 0$$

$$0 \longrightarrow \tau_Z^{\leq -1} j_*(B) \longrightarrow {}^p j_*(B) \longrightarrow i_* H^0 i^* j_*(B) \longrightarrow 0$$

These together show that $\tau_Z^{\geq 1} j_!(B) = \tau_Z^{\leq -1} j_*(B)$ is the image $j_{!*}(B)$ of ${}^p j_!(B)$ in ${}^p j_*(B)$. $\square$

Corollary 4.7.25. *For $B \in \mathcal{C}_U$, $j_{!*}(B)$ is the unique prolongation X of B in $\mathcal{D}$ such that $i^*(X) \in \mathcal{D}_{\bar{Z}}^{\leq -1}$ and $i^!(X) \in \mathcal{D}_{\bar{Z}}^{\geq 1}$.*

Proof. This follows from Proposition 4.7.21. Similarly, ${}^p j_!(B)$ (resp. ${}^p j_*(B)$) is the unique prolongation X such that $i^*(X) \in \mathcal{D}_{\bar{Z}}^{\leq -2}$ (resp. $\mathcal{D}_{\bar{Z}}^{\leq 0}$) and $i^!(X) \in \mathcal{D}_{\bar{Z}}^{\geq 0}$ (Resp. $\mathcal{D}_{\bar{Z}}^{\geq 2}$). □

Corollary 4.7.26. *For $B \in \mathcal{C}_U$, $j_{!*}(B)$ is the unique prolongation X of B in $\mathcal{C}$ with no nontrivial sub-object or quotient in the essential image $\bar{\mathcal{C}}_Z$ of $\mathcal{C}_Z$ under ${}^p i_*$.*

Proof. By definition, $j_{!*}(B)$ is in $\mathcal{C} \subseteq \mathcal{D}$. For any prolongation $X \in \mathcal{C}$ of B, we have $i^*(X) \in \mathcal{D}_{\bar{Z}}^{\leq 0}$, and $i^*(X) \in \mathcal{D}_{\bar{Z}}^{\leq -1}$ if and only if ${}^p i^*(X) = 0$. Dually, $i^!(X) \in \mathcal{D}_{\bar{Z}}^{\geq 0}$, and $i^!(X) \in \mathcal{D}_{\bar{Z}}^{\geq 1}$ if and only if ${}^p i^!(X) = 0$. Identify $\mathcal{C}_Z$ with $\bar{\mathcal{C}}_Z$ via ${}^p i_*$. Since ${}^p i^*(X)$ (resp. ${}^p i^!(X)$) is the largets quotient (resp. sub-object) of X which is in $\mathcal{C}_Z$ (cf. Remark 4.7.5), the characterization Corollary 4.7.26 of $j_{!*}(B)$ follows from Corollary 4.7.25. □

Proposition 4.7.27. *The simple objects of $\mathcal{C}$ are the ${}^p i_*(S)$, for S simple in $\mathcal{C}_Z$, and the $j_{!*}(S)$, for S simple in $\mathcal{C}_U$.*

Proof. Since the essential image $\bar{\mathcal{C}}_Z$ of $\mathcal{C}_Z$ under ${}^p i_*$ is a Serre subcategory of $\mathcal{C}$, for an object $X \in \mathcal{C}$ to be simple, it is necessary and sufficient that one of the following conditions is satisfied:

(a) $X \in \bar{\mathcal{C}}_Z$ and is simple in $\bar{\mathcal{C}}_Z$;

(b) the image of X in $\mathcal{C}/\bar{\mathcal{C}}_Z$ is simple, and X has no nontrial sub-object or quotient in $\bar{\mathcal{C}}_Z$.

The case (a) corresponds to $X = {}^p i_*(S)$, for S simple in $\mathcal{C}_Z$, and by Corollary 4.7.26 and Proposition 4.7.23, case (b) corresponds to $X = j_{!*}(S)$, for S simple in $\mathcal{C}_U$. □

4.8 Perverse sheaves

4.8.1 Stratified spaces

Let X be a topological space endowed with a structural sheaf of rings $\mathcal{O}_X$, $\mathcal{S}$ be a finite partition of X by locally closed subsets (a *stratification*), and $p : \mathcal{S} \to \mathbb{Z}$ be a function (called the **perversity**). By definition, the stratum S is nonempty. We further suppose that the closure of any strarum is a union of strata.

For a continuous map $f : X \to Y$, we shall simply write $f_!$, f_*, $f^!$, f^* for the derived functors $Rf_!$, Rf_*, $Rf^!$, Lf^*. This notation is motivated by the fact that we often work with the derived categories, rather than the module categories. The corresponding functors on module categories will be denoted by ${}^0 f_!$, ${}^0 f_*$, ${}^0 f^!$, ${}^0 f^*$.

Definition 4.8.1. We denote by ${}^p D^{\leq 0}(X, \mathcal{O}_X)$ (resp. ${}^p D^{\geq 0}(X, \mathcal{O}_X)$) the subcategory of $D(X, \mathcal{O}_X)$ formed by complexes $K \in D(X, \mathcal{O}_X)$ (resp. $K \in D^+(X, \mathcal{O}_X)$) such that for any strata $S \in \mathcal{S}$, we have $H^n(i_S^*(K)) = 0$ for $n > p(S)$ (resp. $H^n(i_S^!(K)) = 0$ for $n < p(S)$), where $i_S : S \to X$ denote the inclusion map.

The exactness of the functor ${}^0 i^*$ allows us to give another definition for ${}^p D^{\leq 0}(X, \mathcal{O}_X)$: for K to be in ${}^p D^{\leq 0}(X, \mathcal{O}_X)$, it is necessary and sufficient that the restriction of $H^i(K)$ to S is zero for $i > p(S)$. The truncation functors $\tau^{\leq n}$ and $\tau^{\geq n}$, relative to the natrual t-structure of $D(X, \mathcal{O}_X)$, therefore sends ${}^p D^{\leq 0}(X, \mathcal{O}_X)$ into itself.

Remark 4.8.1. If the functors ${}^0 i_S^!$ have finite cohomological dimensions, then $i_S^! : D^+(X, \mathcal{O}_X) \to D^+(S, \mathcal{O}_S)$ has a natural extension $D^+(X, \mathcal{O}_X) \to D^+(S, \mathcal{O}_S)$, still denoted by $i_S^!$. In this case, the condition "$H^n(i_S^!(K)) = 0$ for $n < p(S)$" makes sense for any complex K in $D(X, \mathcal{O}_X)$. In fact, this condition implies that $K \in D^+(X, \mathcal{O}_X)$, or more precisely, that $K \in D^{\geq n}(X, \mathcal{O}_X)$ for

$p \geq n$. To see this, we shall prove by descendent induction on the strata $\mathcal{S}$ (for the order $S_1 \subseteq \bar{S}_2$) that the restriction of $H^i(K)$ to S is zero for $i < n$. We first note that the distinguished triangle $(\tau^{<n}K, K, \tau^{\geq n}K)$ and the left exactness of ${}^0 i_S^!$ imply that $H^i(i_S^! \tau^{<n}K) \xrightarrow{\sim} H^i(i_S^!(K))$. The induction hypothesis shows that $H^j(\tau^{<n}K)$ is zero over the strata $T \neq S$ such that $S \subseteq \overline{T}$, so S admits an neighborhood in which $H^j(\tau^{<n}K)$ is supported in S. We then have

$$H^i(K)|_S = H^i(i_S^* \tau^{<n}K) = H^i(i_S^! \tau^{<n}(K)),$$

and the last member is zero by our hypothesis.

Without the hypothesis on cohomological dimension, the same argument is applicable for $K \in K^+(X, \mathcal{O}_X)$, and for integers $a \leq p \leq b$, we have

$$D^{\leq a}(X, \mathcal{O}_X) \subseteq {}^p D^{\leq 0}(X, \mathcal{O}_X) \subseteq D^{\leq b}(X, \mathcal{O}_X), \tag{4.8.1}$$

$$D^{\geq a}(X, \mathcal{O}_X) \supseteq {}^p D^{\geq 0}(X, \mathcal{O}_X) \supseteq D^{\geq b}(X, \mathcal{O}_X). \tag{4.8.2}$$

We denote by ${}^p D^{+,\leq 0}(X, \mathcal{O}_X)$ the intersection of $D^+(X, \mathcal{O}_X)$ with ${}^p D^{\leq 0}(X, \mathcal{O}_X)$, and similarly for $+$ replaced by $-, b$ and 0 replaced by $n \in \mathbb{Z}$.

Proposition 4.8.1. *For any perversity $p : \mathcal{S} \to \mathbb{Z}$, the pair $({}^p D^{+,\leq 0}(X, \mathcal{O}_X), {}^p D^{+,\geq 0}(X, \mathcal{O}_X))$ is a t-structure over $D^+(X, \mathcal{O}_X)$.*

Proof. We preceed by induction on the number N of the stratum. If $N = 0$, we have $X = \varnothing$, and the assertion is trivial. If $N = 1$, then we obtain the natrual t-structure on $D^+(X, \mathcal{O}_X)$, translated by $p(X)$. For $N \geq 2$, let Z be a proper closed subset of X which is a union of strata, and U be its complement. The induction hypothesis, applied to Z and U, endowed with the induced stratification, gives t-structures over $D^+(U, \mathcal{O}_U)$ and $D^+(Z, \mathcal{O}_Z)$. The t-structure considered over $D^+(X, \mathcal{O}_X)$ is then obtained by glueing: this is clear for ${}^p D^{+,\leq 0}(X, \mathcal{O}_X)$, since i_S^* is exact. As for ${}^p D^{+,\geq 0}(X, \mathcal{O}_X)$, we note that for $K \in D^+(X, \mathcal{O}_X)$ and $S \in \mathcal{S}$, we have

$$\begin{aligned} H^n(i_{S\cap U}^! j^*(K)) &= H^n(i_{S\cap U}^! j^!(K)) = H^n(i_{S\cap U}^!(K)), \\ H^n(i_{S\cap Z}^! i^!(K)) &= H^n(i_{S\cap Z}^!(K)). \end{aligned}$$

On the other hand, since Z is a union of strata, S is either disjoint from Z or contained in Z, so we see that $H^n(i_S^!(K)) = 0$ if and only if $H^n(i_{S\cap U}^! j^*(K)) = H^n(i_{S\cap Z}^! i^!(K)) = 0$. We can then apply Theorem 4.7.19 to conclude the proposition. □

Corollary 4.8.2. *The pair $({}^p D^{\leq 0}(X, \mathcal{O}_X), {}^p D^{\geq 0}(X, \mathcal{O}_X))$ is a t-structure over $D(X, \mathcal{O}_X)$. It induces a t-structure over $D^*(X, \mathcal{O}_X)$ for $* \in \{+, -, b\}$.*

Proof. Let a, b be integers such that $a \leq p \leq b$. For $K \in {}^p \mathcal{D}^{\leq 0}$ □