Algebra

Long

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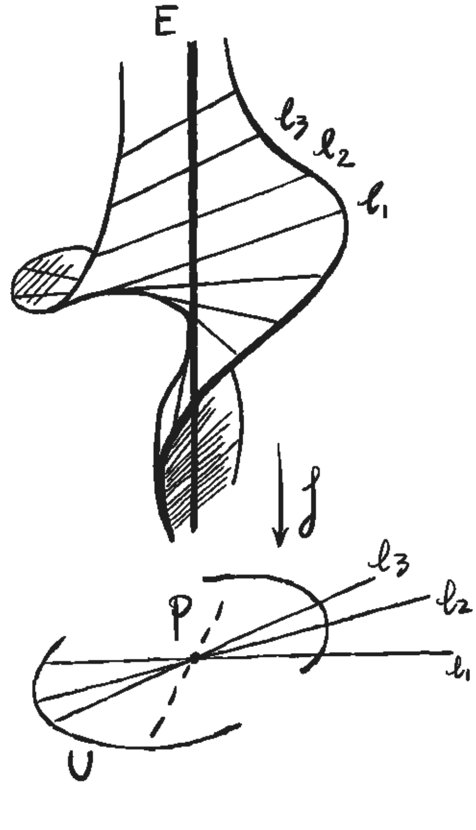

Contents

Chapter 1

Sheaves and cohomology of sheaves

1.1 Sheaves of sets

In this section, X always denotes a topological space.

1.1.1 Presheaves and sheaves

Definition 1.1.1. *A* ***presheaf*** $\mathscr{F}$ *on a topological space* X *is the following data.*

- *To each open set* $U \subseteq X$*, we have a set* $\mathscr{F}(U)$*.*
- *For each inclusion* $U \subseteq V$ *of open sets, we have a* ***restriction map*** $\mathrm{res}_U^V : \mathscr{F}(V) \to \mathscr{F}(U)$*.*

The data is required to satisfy the following two conditions.

- *The map* res_U^U *is the identity:* $\mathrm{res}_U^U = \mathbf{1}_{\mathscr{F}(U)}$*.*
- *If* $U \subseteq V \subseteq W$ *are inclusions of open sets, then the restriction maps commute,*

$$\begin{tikzcd} \mathscr{F}(W) \arrow[rr, "\mathrm{res}_U^W"] \arrow[dr, "\mathrm{res}_V^W"'] & & \mathscr{F}(U) \\ & \mathscr{F}(V) \arrow[ur, "\mathrm{res}_U^V"'] & \end{tikzcd}$$

Definition 1.1.2. *A* ***morphism of presheaves*** $\varphi : \mathscr{F} \to \mathscr{G}$ *is a family of maps* $\varphi(U) : \mathscr{F}(U) \to \mathscr{G}(U)$ *for all open subsets* $U \subseteq X$ *behaving well with respect to restriction: if* $U \hookrightarrow V$ *then*

$$\begin{tikzcd} \mathscr{F}(U) \arrow[r, "\varphi(U)"] \arrow[d, "\mathrm{res}_V^U"'] & \mathscr{G}(U) \arrow[d, "\mathrm{res}_V^U"] \\ \mathscr{F}(V) \arrow[r, "\varphi(V)"] & \mathscr{G}(V) \end{tikzcd} \tag{1.1.1}$$

commutes. Composition of morphisms φ *and* ψ *of presheaves is defined in the obvious way:* $(\varphi \circ \psi)_U := \varphi_U \circ \psi_U$*. We obtain the category* $\mathbf{PSh}(X)$ *of presheaves on* X*.*

If $U \subseteq V$ are open subset of X and $s \in \mathscr{F}(V)$, we will usually write $s|_U$ instead of $\mathrm{res}_U^V(s)$. The elements of $\mathscr{F}(U)$ are called **sections of $\mathscr{F}$ over U**. By convention, if the U is omitted, it is implicitly taken to be X: sections of $\mathscr{F}$ means sections of $\mathscr{F}$ over X. These are also called **global sections**. Very often we will also write $\Gamma(U, \mathscr{F})$ instead of $\mathscr{F}(U)$.

Remark 1.1.1. Given any topological space X, we have a category of open sets $\mathcal{T}_X$, where the objects are the open sets and the morphisms are set inclusions. A presheaf is a contravariant funtor from this category to **Set**:

$$\mathscr{F} : (\mathcal{T}_X)^{op} \to \mathbf{Set}$$

and a morphism of presheaves is a natural transforml of presheaves. With this observation, for any category C we can define a **presheaf $\mathscr{F}$ with values in C** to be a contravariant funtor $(\mathcal{T}_X)^{op} \to C$. A morphism $\mathscr{F} \to \mathscr{G}$ of presheaves with values in C is then simply a morphism of functors.

Example 1.1.3 (presheaf of functions). Let E be a set. For each open subset U of X, let $\mathrm{Map}_E(U)$ be the set of all maps $U \to E$. For an open subset $V \subseteq U$ we define $\mathrm{res}_V^U : \mathrm{Map}_E(U) \to \mathrm{Map}_E(V)$ as the usual restriction of maps. Then Map_E is a presheaf on X.

More generally, a family $\mathscr{F}$ of subsets $\mathscr{F}(U) \subseteq \mathrm{Map}_E(U)$, where U runs through the open subsets of X, is called a **presheaf of E-valued functions on X**, if it is stable under restriction, i.e., for all open sets $V \subseteq U$ and all $f \in \mathscr{F}(U)$ one has $f|_V \in \mathscr{F}(V)$. Then $\mathscr{F}$ together with the restriction maps is a presheaf of sets.

If E is a group (respectively an R-module for some ring R, respectively an A-algebra for some commutative ring A), then $\mathscr{F}$ is a presheaf of groups (respective of R-modules, respective of A-algebras).

Example 1.1.4 (Examples of presheaf of functions).

(a) Let Y be a topological space. For open subset $U \subseteq X$, define

$$C(U, Y) := \{f : U \to Y \mid f \text{ is continuous}\}.$$

Then $C(-, Y)$ is a presheaf of Y-valued functions on X. If $Y = \mathbb{R}$, then $C(-, Y)$ a presheaf of $\mathbb{R}$-algebras.

(b) Let V be a finite-dimensional R-vector spaces and let X be an open subspace of V. Let $0 \leq n \leq \infty$. For an open subset U of X, define

$$C^n(U) := \{f : U \to W, f \text{ is a } C^n\text{-map}\}.$$

Then C^n is a presheaf of functions on X. It is a presheaf of $\mathbb{R}$-algebras.

Definition 1.1.5. *A presheaf is a **sheaf** if it satisfies two more axioms.*

- ***Identity axiom.*** *If $\{U_i\}_{i \in I}$ is an open cover of U, and $s_1, s_2 \in \mathscr{F}(U)$, and $s_1|_{U_i} = s_2|_{U_i}$ for all i, then $s_1 = s_2$.*

- ***Gluability axiom.*** *If $\{U_i\}_{i \in I}$ is an open cover of U, then given $s_i \in \mathscr{F}(U_i)$ for all i, such that*

$$s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$$

for all i and j, then there is some $s \in \mathscr{F}(U)$ such that $s|_{U_i} = s_i$ for all i.

*A **morphism of sheaves** is a morphism of presheaves, and we obtain the category of sheaves on the topological space X, which we denote by* $\mathbf{Shv}(X)$.

Remark 1.1.2. If $\mathscr{F}$ is a sheaf on X, then by using the covering of the empty set $U = \varnothing$ with empty index set $I = \varnothing$, we see $\mathscr{F}(\varnothing)$ is a set consisting of one element. In general, if $U, V \subseteq X$ are disjoint open subsets, then

$$\mathscr{F}(U \cup V) = \mathscr{F}(U) \times \mathscr{F}(V).$$

Example 1.1.6 (Sheaf of E-valued functions). Let E be a set and let $\mathscr{F}$ be a presheaf of E-valued functions on X. Then $\mathscr{F}$ is a sheaf if and only if the following condition holds:

> For every open subset U of X, for every open covering $\{U_i\}_{i\in I}$ of U and for every map $f : U \to E$ such that $f|_{U_i} \in \mathscr{F}(U_i)$ for all i, one has $f \in \mathscr{F}(U)$.

In particular, the presheaves $C(-, Y)$ and C^n are in fact sheaves.

Example 1.1.7 (Constant sheaf). The presheaf of constant functions with values in some set is in general not a sheaf: if U_1 and U_2 are disjoint non-empty open subsets and if $f_1 : U_1 \to E$ and $f_2 : U_2 \to E$ are constant maps that take different values, then there does not exist a constant map f on $U = U_1 \cup U_2$ whose restriction to U_i is f_i for $i = 1, 2$.

If one takes instead the sheaf of locally constant functions with values in some set E, then this is a sheaf. This comes from the simple observation: endow E with the discrete topology, then locally constant maps are exactly continuous maps from U to E. This is called the **constant sheaf associated to E**. We denote this sheaf E_X.

1.1.2 Stalks of presheaves and sheaves

Let X be a topological space, $\mathscr{F}$ be a presheaf on X, and let $x \in X$ be a point. The **stalk** of $\mathscr{F}$ in x is defined by the direct limit

$$\mathscr{F}_x := \varinjlim_{U \ni x} \mathscr{F}(U).$$

More concretely, one has

$$\mathscr{F}_x = \{(U, s) : U \text{ is an open neighborhood of } x \text{ and } s \in \mathscr{F}(U)\}/\sim .$$

where two pairs $(U_1, s_1) \sim (U_2, s_2)$ are equivalent if there exists an open neighborhood V of x with $V \subseteq U_1 \cap U_2$ such that $s_1|_V = s_2|_V$. Then for each open neighborhood U of x we have a canonical map

$$\mathscr{F}(U) \mapsto \mathscr{F}_x, \quad s \mapsto s_x,$$

which sends $s \in \mathscr{F}(U)$ to the class of (U, s) in $\mathscr{F}_x$. We call s_x the **germ** of s in x.

Remark 1.1.3. If $\mathscr{F}$ is a presheaf on X with values in C, where C is any category in which filtered colimits exist, then the stalk $\mathscr{F}_x$ is an object in C and we obtain a functor $\mathscr{F} \to \mathscr{F}_x$ from the category of presheaves on X with values in C to the category C.

Example 1.1.8 (Stalk of the sheaf of continuous functions). Let X be a topological space, let C_X be the sheaf of continuous $\mathbb{R}$-valued functions on X, and let $x \in X$. Then

$$C_{X,x} = \{(U, f) : x \in U \subseteq X \text{ open}, f : U \to \mathbb{R} \text{ is continuous}\}/\sim,$$

where $(U, f) \sim (V, g)$ if there exists $x \in W \subseteq U \cap V$ open such that $f|_W = g|_W$. In particular we see that for $x \in V \subseteq U$ open and $f : U \to \mathbb{R}$ continuous one has $(U, f) \sim (V, f|_V)$. As C_X is a sheaf of $\mathbb{R}$-algebras, $C_{X,x}$ is an $\mathbb{R}$-algebra.

If the germ $s \in C_{X,x}$ of a continuous function at x is represented by (U, f), then $s(x) := f(x) \in \mathbb{R}$ is independent of the choice of the representative (U, f). We obtain an $\mathbb{R}$-algebra homomorphism

$$\mathrm{ev}_x : C_{X,x} \to \mathbb{R}, \quad s \mapsto s(x),$$

which is surjective because $C_{X,x}$ contains in particular the germs of all constant functions. Let $\mathfrak{m}_x = \ker \mathrm{ev}_x = \{s \in C_{X,x} : s(x) = 0\}$. Then $\mathfrak{m}_x$ is a maximal ideal because $C_{X,x}/\mathfrak{m}_x \cong \mathbb{R}$ is a field. Let $s \in C_{X,x} \setminus \mathfrak{m}_x$ be represented by (U, f). Then $f(x) \neq 0$. By shrinking U we may assume that $f(y) \neq 0$ for all $y \in Y$ because f is continuous. Hence $1/f$ exists and s is a unit in $C_{X,x}$. Therefore the complement of $\mathfrak{m}_x$ consists of units of $C_{X,x}$. This shows that $C_{X,x}$ is a local ring with maximal ideal $\mathfrak{m}_x$. The same argument shows that for every open subspace X of a finite-dimensional $\mathbb{R}$-vector space the stalk $C^n_{X,x}$ is a local ring with residue field $\mathbb{R}$.

Now let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism of presheaves on X, we then have an induced map $\varphi_x : \mathscr{F}_x \to \mathscr{G}_x$ defined by

$$\varphi_x := \varinjlim_{U \ni x} \varphi_U,$$

which sends the equivalence class of (U, f) in $\mathscr{F}_x$ to the class of $(U, \varphi_U(f))$ in $\mathscr{G}_x$. This defines a functor $\mathscr{F} \mapsto \mathscr{F}_x$ from the category of presheaves on X to the category of sets such that for every open neighborhood U of X one has a commutative diagram

$$\begin{tikzcd} \mathscr{F}(U) \arrow[r, "f \mapsto f_x"] \arrow[d, "\varphi_U"'] & \mathscr{F}_x \arrow[d, "\varphi_x"] \\ \mathscr{G}(U) \arrow[r, "g \mapsto g_x"] & \mathscr{G}_x \end{tikzcd} \tag{1.2.1}$$

Now we introduct a construction frequently apprears in the theory of sheaves. The **Godement construction**

$$\mathrm{God} : \mathbf{Psh}(X) \to \mathbf{Shv}(X)$$

is the functor given by sending a presheaf $\mathscr{F} \in \mathbf{PSh}(X)$ to the sheaf $\mathrm{God}(\mathscr{F})$ defined by sending an open set U to

$$\mathrm{God}(\mathscr{F})(U) := \prod_{x \in U} \mathscr{F}_x,$$

with the restriction morphisms given by product projections (it is easy to see this defines a sheaf on X). The assignment of God on morphisms is given by sending a morphism of presheaves $\phi : \mathscr{F} \to \mathscr{G}$ to the morphism

$$\mathrm{God}(\phi)_U : \prod_{x \in U} \mathscr{F}_x \to \prod_{x \in U} \mathscr{G}_x.$$

Proposition 1.1.9. *Let $\mathscr{F}$ be a sheaf of sets. For every open $U \subseteq X$ the map*

$$\mathscr{F}(U) \to \prod_{x \in U} \mathscr{F}_x, \quad s \mapsto (s_x)_{x \in U}.$$

identifies $\mathscr{F}(U)$ with the elements $(s_x)_{x \in U}$ with the property

> *For any $x \in U$ there exists a open subset V containing x and a section $s \in \mathscr{F}(V)$ such that for all $y \in V$ we have $s_y = (V, s)$ in $\mathscr{F}_y$.*

If an element (s_x) satisfies the condition above, we say it ***consists of compatible germs****. Thus the Godement construction identifies $\mathscr{F}$ as a subsheaf of* God$(\mathscr{F})$.

Proof. Let $s, t \in \mathscr{F}(U)$ such that $s_x = t_x$ for all $x \in U$. Then for all $x \in U$ there exists an open neighborhood $V_x \subseteq U$ of x such that $s|_{V_x} = t|_{V_x}$. Clearly, $U = \bigcup_{x \in U} V_x$ and therefore $s = t$ by identity axiom.

Clearly any section s of $\mathscr{F}$ over U gives a choice of compatible germs for U. Conversely, if $(s_x)_{x \in U}$ consists of compatible germs, that is, there is an open cover $\{U_i\}$ of U, and sections $s_i \in \mathscr{F}(U_i)$, such that $(s_x)_{x \in U_i}$ is given by (U_i, s_i). Then by gluability there is a section $\sigma \in \mathscr{F}(U)$ such that $\sigma|_{U_i} = s_i$, and it is clear from the condition that $\sigma_x = s_x$. □

The importance of stalks is contained in the following result, which says a morphism between sheaves is determined by its value on stalks.

Proposition 1.1.10. *Let X be a topological space, $\mathscr{F}$ and $\mathscr{G}$ presheaves on X, and let $\varphi, \psi : \mathscr{F} \to \mathscr{G}$ be two morphisms of presheaves.*

(a) If $\mathscr{F}$ is a sheaf, the induced maps on stalks $\varphi_x : \mathscr{F}_x \to \mathscr{G}_x$ are injective for all $x \in X$ if and only of $\varphi_U : \mathscr{F}(U) \to \mathscr{G}(U)$ is injective for all open subsets $U \subseteq X$.

(b) If $\mathscr{F}$ and $\mathscr{G}$ are both sheaves, the maps $\varphi_x : \mathscr{F}_x \to \mathscr{G}_x$ are bijective for all $x \in X$ if and only of $\varphi_U : \mathscr{F}(U) \to \mathscr{G}(U)$ is injective for all open subsets $U \subseteq X$.

(c) If $\mathscr{F}$ and $\mathscr{G}$ are both sheaves, the morphism φ and ψ are equal if and only of $\varphi_x = \psi_x$ for all $x \in X$.

Proof. First we show that taking stalks preserves injectivity and surjectivity. Assume that φ_U is injective for all U. Let $s_0, t_0 \in \mathscr{F}_x$ such that $\varphi_x(s_0) = \varphi_x(t_0)$. Let s_0 be represented by (s, U) and t_0 by (t, V). By shrinking U and V we may assume $U = V$. From diagram (1.2.1), we see

$$\varphi_U(s)_x = \varphi_x(s_0) = \varphi_x(t_0) = \varphi_U(t)_x,$$

so there exists an open neighborhood $W \subseteq U$ containing such that

$$\varphi_W(s|_W) = (\varphi_U(s))|_W = (\varphi_U(t))|_W = \varphi_W(t|_W).$$

As φ_W is injective, we find $s|_W = t|_W$ and hence $s_0 = s_x = t_x = t_0$. Thus φ_x is injective. If on the other hand φ_U is surjective for all $U \subseteq X$, let t_0 be any elememt in $\mathscr{G}_x$, which is represented by

(t, U). Then there is a $s \in \mathscr{F}(U)$ such that $\varphi_U(s) = t$, and (1.2.1) implies

$$\varphi_x(s_x) = \big(\varphi_U(s)\big)_x = t_x.$$

so φ_x is surjective.

For (a), assume that $\mathscr{F}$ is a sheaf, and consider the commutative diagram

$$\begin{tikzcd}
\mathscr{F}(U) \arrow[r] \arrow[d, "\varphi_U"] & \prod_{x\in U} \mathscr{F}_x \arrow[d, "\prod_{x\in U}\varphi_x"] \\
\mathscr{G}(U) \arrow[r] & \prod_{x\in U} \mathscr{G}_x
\end{tikzcd}$$

By Proposition 1.1.9 the map $\mathscr{F}(U) \to \prod_{x\in U} \mathscr{F}_x$ is injective, thus if φ_x are all injective, so is φ_U. Assertion (c) can also be derived from this diagram.

Then we prove (b). Assume φ_x are bijective, we show that φ_U is surjective. Let $t \in \mathscr{G}(U)$. For any $x \in U$, since φ_x is surjective, there exist $s_x \in \mathscr{F}_x$ such that $\varphi_x(s_x) = t_x$. Let s_x be represented by a section $s(x)$ on a neighborhood V_x of x. Then by (1.2.1),

$$\big(\varphi_{V_x}(s(x))\big)_x = \varphi_x(s_x) = (t|_{V_x})_x.$$

By shrinking V_x we may assume that $\varphi_{V_x}(s(x)) = t|_{V_x}$.

Now U is covered by such open sets V_x, and on each V_x we have a section $s(x) \in \mathscr{F}(V_x)$. For two distinct points $x, y \in U$, $s(x)|_{V_x\cap V_y}$ and $s(y)|_{V_x\cap V_y}$ are both sent to $t|_{V_x\cap V_y}$ by φ, so by the injectivity of φ we just proved, $s(x)|_{V_x\cap V_y} = s(y)|_{V_x\cap V_y}$. Therefore, the gulability axiom produces a section $s \in \mathscr{F}(U)$ such that $s|_{V_x} = s(x)$, and from diagram (1.1.1) and the construction, we have

$$(\varphi_U(s))|_{V_x} = \varphi_{V_x}(s|_{V_x}) = \varphi_{V_x}(s(x)) = t|_{V_x},$$

so the identity axiom implies $\varphi_U(s) = t$. □

Definition 1.1.11. *Let $\mathscr{F}$ be a sheaf of abelian groups on a topological space X, $U \subseteq X$ open and $s \in \mathscr{F}(U)$ a section. The **support** of s is defined by*

$$\operatorname{supp}(s) = \{x \in U : s_x \neq 0\}.$$

*The **support** of $\mathscr{F}$ is defined to be*

$$\operatorname{supp}(\mathscr{F}) = \{x \in X : \mathscr{F}_x \neq 0\}.$$

Proposition 1.1.12. *Let $\mathscr{F}$ be a sheaf of abelian groups on a topological space, $U \subseteq X$ open, and $s \in \mathscr{F}(U)$ a section. Then $\operatorname{supp}(s)$ is closed in U.*

Proof. For $x \in U \setminus \operatorname{supp}(s)$ we have $s_x = 0$, so there exists an open subset $V \subseteq U$ with $s|_V = 0$. This implies $s_y = 0$ for every $y \in V$ and therefore $V \subseteq U \setminus \operatorname{supp}(s)$. Hence $U \setminus \operatorname{supp}(s)$ is open. □

Example 1.1.13. Let X be a topological space. Let C_X be the sheaf of continuous $\mathbb{R}$-valued functions on X. Let $U \subseteq X$ be open and $s \in C_X(U)$ a continuous function $U \to \mathbb{R}$. In the proof of Proposition 1.1.12 we have just seen that $U \setminus \mathrm{supp}(s)$ is the interior of $\{x \in U : s(x) = 0\}$. Therefore we have

$$\mathrm{supp}(s) = \overline{\{x \in U : s(x) \neq 0\}}$$

which coincides with usual definition of the support of a continuous function.

1.1.3 Sheafification

In this part, we give a functorial way to attach to a presheaf a sheaf. This can be seen as the left adjoint of the forgetful functor from **Shv**(X) to **Psh**(X).

Definition 1.1.14 (Universal property of sheafification). *If $\mathscr{F}$ is a presheaf on X, then a morphism of presheaves $\iota_{\mathscr{F}} : \mathscr{F} \longrightarrow \mathscr{F}^{\#}$ on X is called a* ***sheafification*** *of $\mathscr{F}$ if $\mathscr{F}^{\#}$ is a sheaf, and for any sheaf $\mathscr{G}$, and any presheaf morphism $\varphi : \mathscr{F} \longrightarrow \mathscr{G}$, there exists a unique morphism of sheaves $\tilde{\varphi} : \mathscr{F}^{\#} \longrightarrow \mathscr{G}$ making the diagram*

$$\begin{tikzcd} \mathscr{F} \arrow[r, "\iota_{\mathscr{F}}"] \arrow[dr, "\varphi"'] & \mathscr{F}^{\#} \arrow[d, "\tilde{\varphi}"] \\ & \mathscr{G} \end{tikzcd}$$

commute.

As a universal construction, the sheafification functor $\mathscr{F} \longrightarrow \mathscr{F}^{\#}$ is left-adjoint to the forgetful functor from sheaves on X to presheaves on X: there is a canonical isomorphism

$$\mathrm{Mor}_{\mathbf{Shv}(X)}(\mathscr{F}^{\#}, \mathscr{G}) \cong \mathrm{Mor}_{\mathbf{Psh}(X)}(\mathscr{F}, \mathscr{G})$$

for any presheaf $\mathscr{F}$ and sheaf $\mathscr{G}$.

In Proposition 1.1.9 we see that if $\mathscr{F}$ is a sheaf, then $\mathscr{F}(U)$ can be identified as elements in $\prod_{x\in U} \mathscr{F}_x$ consists of compatible germs. This turns out to be a appropriate way to define the sheafification.

Proposition 1.1.15. *Let $\mathscr{F}$ be a presheaf on a topological space X. Then there exists a sheafification $\iota_{\mathscr{F}} : \mathscr{F} \longrightarrow \mathscr{F}^{\#}$ such that the following properties hold:*

(a) For all $x \in X$ the map on stalks $\iota_{\mathscr{F},x} : \mathscr{F}_x \longrightarrow \mathscr{F}^{\#}_x$ is bijective.

(b) For every presheaf $\mathscr{G}$ on X and every morphism of presheaves $\varphi : \mathscr{F} \longrightarrow \mathscr{G}$ there exists a unique morphism $\tilde{\varphi} : \mathscr{F}^{\#} \longrightarrow \mathscr{G}^{\#}$ making the diagram commutes:

$$\begin{tikzcd} \mathscr{F} \arrow[r, "\iota_{\mathscr{F}}"] \arrow[d, "\varphi"'] & \mathscr{F}^{\#} \arrow[d, "\tilde{\varphi}"] \\ \mathscr{G} \arrow[r, "\iota_{\mathscr{G}}"] & \mathscr{G}^{\#} \end{tikzcd} \tag{1.3.1}$$

In particular, $\mathscr{F} \longrightarrow \mathscr{F}^{\#}$ is a functor from the category of presheaves on X to the category of sheaves on X. The sheafification $\iota_{\mathscr{F}} : \mathscr{F} \longrightarrow \mathscr{F}^{\#}$ is unique up to isomorphism.

Proof. Suppose $\mathscr{F}$ is a presheaf. Define $\mathscr{F}^{\#}$ by declearing $\mathscr{F}^{\#}(U)$ as the set of compatible germs of the presheaf $\mathscr{F}$ over U. Explicitly:

$$\begin{aligned}\mathscr{F}^{\#}(U) &= \{(s_x)_{x\in U} \in \prod_{x\in U} \mathscr{F}_x : (s_x)_{x\in U} \text{ consists of compatible germs}\}\\ &= \{(s_x)_{x\in U} \in \prod_{x\in U} \mathscr{F}_x : \forall x \in U, \exists V \subseteq U \text{ and } \tilde{s} \in \mathscr{F}(V) \text{ s.t. } \tilde{s}_y = s_x \text{ for all } y \in V\}.\end{aligned}$$

For $U \subseteq V$ the restriction map $\mathscr{F}^{\#}(V) \to \mathscr{F}^{\#}(U)$ is induced by the natural projection. With this, for any covering $U = \bigcup_i U_i$ and sections $(s_x)_{x\in U_i}$ in $\mathscr{F}^{\#}$, the condition

$$\big((s_x)_{x\in U_i}\big)|_{U_i\cap U_j} = \big((s_x)_{x\in U_j}\big)|_{U_i\cap U_j}$$

implies that $(s_x)_{x\in U_i}$ and $(s_x)_{x\in U_j}$ have common germs on their common domains. Thus we can construct a unique section $(s_x)_{x\in U}$ by just gathering their germs. It is clear that such a section $(s_x)_{x\in U}$ has compatible germs, hence belongs to $\mathscr{F}^{\#}$. This shows $\mathscr{F}^{\#}$ is a sheaf. For $U \subseteq X$ open, we define $\iota_{\mathscr{F},U} : \mathscr{F}(U) \to \mathscr{F}^{\#}(U)$ by $s \mapsto (s_x)_{x\in U}$. The definition of $\mathscr{F}^{\#}$ shows that, for $x \in X$, $\mathscr{F}^{\#}_x = \mathscr{F}_x$ and that $\iota_{\mathscr{F},x}$ is the identity.

Now let $\mathscr{G}$ be a presheaf on X and let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism. Sending $(s_x)_{x\in U}$ to $\big(\varphi_x(s_x)\big)_{x\in U}$ defines a morphism $\mathscr{F}^{\#} \to \mathscr{G}^{\#}$. By Proposition 1.1.10(c) this is the unique morphism making the diagram (1.3.1) commutative.

If we assume in addition that $\mathscr{G}$ is a sheaf, then the morphism of sheaves $\iota_{\mathscr{G}}$, which is bijective on stalks, is an isomorphism by Proposition 1.1.10(b). Composing the morphism $\mathscr{F}^{\#} \to \mathscr{G}^{\#}$ with $\iota_{\mathscr{G}}^{-1}$, we obtain the morphism $\tilde{\varphi} : \mathscr{F}^{\#} \to \mathscr{G}$. Finally, the uniqueness of $\iota_{\mathscr{F}} : \mathscr{F} \to \mathscr{F}^{\#}$ is a formal consequence of universal property. □

Proposition 1.1.16. *The sheafification of an injection (resp. surjection) of presheaves of sets is an injection (resp. surjection).*

Proof. This follows from the fact that sheafification does not change the stalk. □

From Proposition 1.1.10, we get the following characterization of the sheafification.

Proposition 1.1.17. *Let $\mathscr{F}$ be a presheaf and $\mathscr{G}$ be a sheaf. Then $\mathscr{G}$ is isomorphic to the sheafification of $\mathscr{F}$ if and only and if there exists a morphism $\iota : \mathscr{F} \to \mathscr{G}$ such that ι_x is bijective for all $x \in X$.*

Proof. One direction is trivial, assume the converse. Then there is a $\tilde{\iota} : \mathscr{F}^{\#} \to \mathscr{G}$ such that the following diagram commutes

$$\begin{tikzcd} \mathscr{F}^{\#} \arrow[r, "\tilde{\iota}"] & \mathscr{G} \\ \mathscr{F} \arrow[u] \arrow[ur, "\iota"'] & \end{tikzcd}$$

By Proposition 1.1.10 $\tilde{\iota}$ induced isomorphisms on stalks, hence is an isomorphism $\mathscr{F}^{\#} \cong \mathscr{G}$. □

Example 1.1.18. Let E be a set and let $\mathscr{F}$ be a presheaf of functions with values in E. Then its sheafification is given by

$$\mathscr{F}^{\#}(U) = \{f : U \to E \mid \exists \text{ open covering } (U_i)_i \text{ of } U \text{ such that } f|_{U_i} \in \mathscr{F}(U_i) \text{ for all } i\}.$$

Indeed, this is a sheaf by Example 1.1.6 and the inclusions $\mathscr{F}(U) \hookrightarrow \mathscr{F}^{\#}(U)$ for U open define a morphism of presheaves $\mathscr{F} \to \mathscr{F}^{\#}$ that is bijective on stalks. Hence we can apply Proposition 1.1.17.

Example 1.1.19. From the previous example, let's consider the constant presheaves. Let E_X be the sheaf of locally constant functions with values in E:

$$E_X = \{f : U \to E \mid \forall x \in U, \exists V \ni x \text{ open s.t. } f \text{ is constant on } V\}$$

then by Proposition 1.1.17, E_X is the sheafification of the presheaf of constant functions with values in E.

Example 1.1.20 (Open subset as a sheaf). Let $U \subseteq X$ be an open subset, then we can define a presheaf $\mathscr{U}$

$$\mathscr{U}(V) = \begin{cases} \{i_V\} & \text{if } V \subseteq U \\ \varnothing & \text{otherwise} \end{cases}$$

where $i_V : V \hookrightarrow U$ is the inclusion map. Let $\mathscr{U}^{\#}$ be its sheafification.

Let $\mathscr{F}$ be a sheaf and $s \in \mathscr{F}(U)$, then we can define a morphism φ_s by setting $\varphi_{s,U}(\mathbf{1}_U) = s$ and others by restriction. Then we get a map

$$\mathscr{F}(U) \to \mathrm{Mor}_{\mathbf{Psh}(X)}(\mathscr{U}, \mathscr{F})$$

which can be shown is an isomorphism. Together with the adjointness of sheafification, we get isomorphisms

$$\mathscr{F}(U) \cong \mathrm{Mor}_{\mathbf{Psh}(X)}(\mathscr{U}, \mathscr{F}) \cong \mathrm{Mor}_{\mathbf{Shv}(X)}(\mathscr{U}^{\#}, \mathscr{F}).$$

1.1.4 Direct and inverse images of sheaves

In this part $f : X \to Y$ denotes a continuous map of topological spaces. We will now see how to use f in order to attach to a sheaf on X a sheaf on Y (direct image) and to a sheaf on Y a sheaf an X (inverse image).

Definition 1.1.21 (Direct image of a presheaf). *Let $f : X \to Y$ be a continuous map. Let $\mathscr{F}$ be a presheaf on X. We define a presheaf $f_*\mathscr{F}$ on Y by (for $V \subseteq Y$ open)*

$$(f_*\mathscr{F})(V) = \mathscr{F}\left(f^{-1}(V)\right)$$

*the restriction maps given by the restriction maps for $\mathscr{F}$. Then $f_*\mathscr{F}$ is called the **direct image** of $\mathscr{F}$ under f or the **pushforward presheaf** of $\mathscr{F}$ by f. Whenever $\varphi : \mathscr{F}_1 \to \mathscr{F}_2$ is a morphism of presheaves, the family of maps $f_*(\varphi)_V := \varphi_{f^{-1}(V)}$ for $V \subseteq Y$ open is a morphism $f_*(\varphi) : f_*\mathscr{F}_1 \to f_*\mathscr{F}_2$. We thus obtain a functor f_* from the category of presheaves on X to the category of presheaves on Y.*

Proposition 1.1.22. *Let $f : X \to Y$ be a continuous map*

(a) If $\mathscr{F}$ is a sheaf on X, then $f_*\mathscr{F}$ is a sheaf on Y. Therefore f_* also defines a functor $f_* : \mathbf{Shv}(Y) \to \mathbf{Shv}(X)$.

(b) If $g : Y \to Z$ is a second continuous map, then $(g \circ f)_* = g_* \circ f_*$.

Proof. This first statement immediately follows from the fact that if $V = \bigcup V_i$ is an open covering in Y, then $f^{-1}(V) = \bigcup_i f^{-1}(V_i)$ is an open covering in X. The second claim is a computation:

$$(g \circ f)_*\mathscr{F}(W) = \mathscr{F}\left((g \circ f)^{-1}(W)\right) = \mathscr{F}\left(f^{-1} \circ g^{-1}(W)\right) = g_*\mathscr{F}\left(f^{-1}(W)\right) = g_* \circ f_*\mathscr{F}(W).$$

where $W \subseteq Y$ is open. $\square$

We now define the inverse image of a sheaf. Let $\mathscr{G}$ be a presheaf of sets on Y. The **pullback presheaf** $f^p\mathscr{G}$ of a given presheaf $\mathscr{G}$ is defined as the left adjoint of the pushforward f_* on presheaves. In other words, it should be a presheaf $f^p\mathscr{G}$ on X such that

$$\mathrm{Mor}_{\mathbf{Psh}(X)}(f^p\mathscr{G}, \mathscr{F}) \cong \mathrm{Mor}_{\mathbf{Psh}(X)}(\mathscr{G}, f_*\mathscr{F})$$

It turns out that this actually exists.

Proposition 1.1.23 (Inverse image of a presheaf). *Let $f : X \to Y$ be a continuous map and let $\mathscr{G}$ be a presheaf on Y. There exists a functor $f^p : \mathbf{Psh}(Y) \to \mathbf{Psh}(X)$ which is left adjoint to f_*. For a presheaf $\mathscr{G}$ it is determined by*

$$f^p\mathscr{G}(U) = \varinjlim_{\substack{V \supseteq f(U) \\ V \subseteq Y \text{ open}}} \mathscr{G}(V).$$

the restriction maps being induced by the restriction maps of $\mathscr{G}$.

Proof. The colimit is over the partially ordered set consisting of open subset $V \subseteq Y$ which contain $f(U)$ with ordering by reverse inclusion. This is a directed partially ordered set, and if $U_1 \subseteq U_2$, then every open neighbourhood of $f(U_2)$ is an open neighbourhood of $f(U_1)$. Hence the system defining $f^p\mathscr{G}(U_2)$ is a subsystem of the one defining $f^p\mathscr{G}(U_1)$ and we obtain a restriction map. Note that the construction of the colimit is clearly functorial in $\mathscr{G}$, and similarly for the restriction mappings. Hence we have defined f^p as a functor. Now we turn to the proof of the adjointness. For this, we need to define the unit map and the counit map, as follows.

- There exists a canonical map $\mathscr{G}(V) \to f^p\mathscr{G}\left(f^{-1}(V)\right)$ for any open subset $V \subseteq Y$, because the system of open neighbourhoods of $f\left(f^{-1}(V)\right)$ contains the element V:

  $$\rho_{\mathscr{G},V} : \mathscr{G}(V) \longrightarrow f^p\mathscr{G}(f^{-1}(V)) = \varinjlim_{U \supseteq f(f^{-1}(V))} \mathscr{G}(U)$$

  This is compatible with restriction mappings, so there is a canonical map $\rho_{\mathscr{G}} : \mathscr{G} \to f_*f^p\mathscr{G}$.

- There exists a canonical map $f^p f_*\mathscr{F}(U) \to \mathscr{F}(U)$ for any open subset $U \subseteq X$:

  $$\sigma_{\mathscr{F},U} : f^p f_*\mathscr{F}(U) = \varinjlim_{V \supseteq f(U)} \mathscr{F}(f^{-1}(V)) \longrightarrow \mathscr{F}(U)$$

where the map is given by the restriction of $\mathscr{F}(f^{-1}(V))$ to $\mathscr{F}(U)$. One easily verifies that the maps are compatible with restriction maps and thus there is a canonical map $\sigma_{\mathscr{F}}$: $f^p f_*\mathscr{F} \to \mathscr{F}$.

The maps we get are illustrated in the following diagram:

$$
\begin{tikzcd}
X \arrow[d, "f"] & & f^p\mathscr{G} \arrow[r, "\psi"'] \arrow[dl, dashed] \arrow[rr, bend left, "f^p\varphi"] & \mathscr{F} \arrow[d, dashed] & f^p f_*\mathscr{F} \arrow[l, "\sigma_{\mathscr{F}}"'] \\
Y & f_*f^p\mathscr{G} \arrow[rr, bend right, "f_*\psi"'] & \mathscr{G} \arrow[l, "\rho_{\mathscr{G}}"'] \arrow[u, dashed] \arrow[r, "\varphi"] & f_*\mathscr{F} \arrow[ur, dashed] &
\end{tikzcd}
$$

Now let $\mathscr{F}$ be a presheaf of sets on X. Suppose that $\psi : f^p\mathscr{G} \to \mathscr{F}$ is a map of presheaves of sets. The corresponding map $\psi^\flat : \mathscr{G} \to f_*\mathscr{F}$ is the composition

$$\psi^\flat : \mathscr{G} \xrightarrow{\rho_{\mathscr{G}}} f_*f^p\mathscr{G} \xrightarrow{f_*\psi} f_*\mathscr{F}$$

Suppose that $\varphi : \mathscr{G} \to f_*\mathscr{F}$ is a map of presheaves of sets. The map $\varphi^\sharp : f^p\mathscr{G} \to \mathscr{F}$ is then the composition

$$\varphi^\sharp : f^p\mathscr{G} \xrightarrow{f^p\varphi} f^p f_*\mathscr{F} \xrightarrow{\sigma_{\mathscr{F}}} \mathscr{F}$$

It can be verified that these tow maps are inverse of each other. Let $U \subseteq X$, then the map $(\psi^\flat)^\sharp : f^p\mathscr{G} \to \mathscr{F}$ is given by

$$f^p\mathscr{G}(U) = \varinjlim_{V \supseteq f(U)} \mathscr{G}(V) \to \varinjlim_{V \supseteq f(U)} \varinjlim_{V' \supseteq f(f^{-1}(V))} \mathscr{G}(V') \xrightarrow{\psi_{f^{-1}(V)}} \varinjlim_{V \supseteq f(U)} \mathscr{F}(f^{-1}(V)) \xrightarrow{\mathrm{res}_U^*} \mathscr{F}(U)$$

Let $s \in f^p\mathscr{G}(U)$ be represented by $v \in \mathscr{G}(V)$ for some $V \supseteq f(U)$. Since $f^{-1}(V) \supseteq U$, from the diagram

$$
\begin{tikzcd}
f^p\mathscr{G}(f^{-1}(V)) \arrow[r, "\psi_{f^{-1}(V)}"] \arrow[d, "\mathrm{res}_U^{f^{-1}(V)}"'] & \mathscr{F}(f^{-1}(V)) \arrow[d, "\mathrm{res}_U^{f^{-1}(V)}"] \\
f^p\mathscr{G}(U) \arrow[r, "\psi_U"] & \mathscr{F}(U)
\end{tikzcd}
$$

we obtain

$$
\begin{aligned}
(\psi^\flat)^\sharp(s) = \mathrm{res}_U^*\left[\psi_{f^{-1}(V)}\big([v]_{f^{-1}(V)}\big)\right]_U &= \mathrm{res}_U^{f^{-1}(V)} \circ \psi_{f^{-1}(V)}\big([v]_{f^{-1}(V)}\big) \\
&= \psi_U\big(\mathrm{res}_U^{f^{-1}(V)}([v]_{f^{-1}(V)})\big) = \psi_U([v]_U) = \psi_U(s).
\end{aligned}
$$

Thus $(\psi^\flat)^\sharp = \psi$. A similar argument gives $(\varphi^\sharp)^\flat = \varphi$, hence we are done. □

Remark 1.1.4. We will almost never use the concrete description of $f^p\mathscr{G}$ in the sequel. Very often we are given f, $\mathscr{F}$, and $\mathscr{G}$ as in the Proposition 1.1.23, and a morphism of sheaves φ :

$\mathscr{G} \to f_*\mathscr{F}$. Then usually it will be sufficient to understand for each $x \in X$ the map

$$\varphi_x^\sharp : (f^p\mathscr{G})_x = \mathscr{G}_{f(x)} \to \mathscr{F}_x$$

induced by $\varphi^\sharp : f^p\mathscr{G} \to \mathscr{F}$ on stalks. The proof of Proposition 1.1.23 shows that we can describe this map in terms of as follows. For every open neighborhood $V \subseteq Y$ of $f(x)$, we have maps

$$\mathscr{G}(V) \xrightarrow{\varphi_V} \mathscr{F}(f^{-1}(V)) \longrightarrow \mathscr{F}_x$$

and taking the colimit over all V we obtain the map $\varphi_x^\sharp : \mathscr{G}_{f(x)} \to \mathscr{F}_x$.

Proposition 1.1.24. *Let $f : X \to Y$ be a continuous map. Let $x \in X$. Let $\mathscr{G}$ be a presheaf of sets on Y. There is a canonical bijection of stalks $(f^p\mathscr{G})_x = \mathscr{G}_{f(x)}$.*

Proof. This is obtained as follows

$$(f^p\mathscr{G})_x = \varinjlim_{U \ni x} f^p\mathscr{G}(U) = \varinjlim_{U \ni x} \varinjlim_{V \supseteq f(U)} \mathscr{G}(V) = \varinjlim_{V \ni f(x)} \mathscr{G}(V) = \mathscr{G}_{f(x)}.$$

Here we have used the fact that any V open in Y containing $f(x)$ occurs in the description above. The case for sheaves is obtained from Proposition 1.1.15. □

Let $\mathscr{G}$ be a sheaf of sets on Y. The **pullback sheaf** $f^{-1}\mathscr{G}$ is defined by the formula

$$f^{-1}\mathscr{G} = (f^p\mathscr{G})^\#.$$

Sheafification is a left adjoint to the inclusion of sheaves in presheaves, and f^p is a left adjoint to f_* on presheaves. As a formal consequence we obtain that f^{-1} is a left adjoint of pushforward on sheaves: for sheaves $\mathscr{F}$ and $\mathscr{G}$, we have

$$\begin{aligned} \mathrm{Mor}_{\mathbf{Shv}(X)}(f^{-1}\mathscr{G}, \mathscr{F}) &= \mathrm{Mor}_{\mathbf{Shv}(X)}\big((f^p\mathscr{G})^\#, \mathscr{F}\big) \cong \mathrm{Mor}_{\mathbf{Psh}(X)}\big(f^p\mathscr{G}, \mathscr{F}\big) \\ &\cong \mathrm{Mor}_{\mathbf{Psh}(X)}(\mathscr{G}, f_*\mathscr{F}) = \mathrm{Mor}_{\mathbf{Shv}(X)}(\mathscr{G}, f_*\mathscr{F}) \end{aligned}$$

Proposition 1.1.25. *There are canonical maps*

$$f^{-1}f_*\mathscr{F} \to \mathscr{F}, \quad \mathscr{G} \to f_*f^{-1}\mathscr{G}$$

for sheaves $\mathscr{F}$ on X and $\mathscr{G}$ on Y.

Proof. We already have maps

$$\sigma_{\mathscr{F}} : f^p f_*\mathscr{F} \to \mathscr{F}, \quad \rho_{\mathscr{G}} : \mathscr{G} \to f^p f_*\mathscr{G}.$$

The map $f^{-1}f_*\mathscr{F} \to \mathscr{F}$ is given by the universal property of sheafification:

$$\begin{tikzcd} f^p f_*\mathscr{F} \arrow[r, "sh"] \arrow[dr, "\sigma_{\mathscr{F}}"'] & f^{-1}f_*\mathscr{F} \arrow[d, dashed] \\ & \mathscr{F} \end{tikzcd}$$

and the map $\mathscr{G} \to f_* f^{-1}\mathscr{G}$ is given by the composition

$$\mathscr{G}(V) \xrightarrow{\rho_{\mathscr{G},V}} f^p\mathscr{G}(f^{-1}(V)) \xrightarrow{sh} f^{-1}\mathscr{G}(f^{-1}(V))$$

for $V \subseteq Y$ open. □

Proposition 1.1.26 (Inverse image and composition). *Let $f : X \to Y$ and $g : Y \to Z$ be continuous maps of topological spaces. The functors $(g \circ f)^{-1}$ and $f^{-1} \circ g^{-1}$ are canonically isomorphic. Similarly $(g \circ f)^p \cong f^p \circ g^p$ on presheaves.*

Proof. This comes from the formal consequence

$$\begin{aligned}\mathrm{Mor}_{\mathbf{Psh}(X)}\big((g \circ f)^p\mathscr{G}, \mathscr{F}\big) &\cong \mathrm{Mor}_{\mathbf{Psh}(X)}\big(\mathscr{G}, (g \circ f)_*\mathscr{F}\big) = \mathrm{Mor}_{\mathbf{Psh}(X)}\big(\mathscr{G}, g_* \circ f_*\mathscr{F}\big)\\ &\cong \mathrm{Mor}_{\mathbf{Psh}(X)}\big(g^p\mathscr{G}, f_*\mathscr{F}\big) \cong \mathrm{Mor}_{\mathbf{Psh}(X)}\big(f^p \circ g^p\mathscr{G}, \mathscr{F}\big)\end{aligned}$$

By the uniqueness of adjoint functors, we obtain $(g \circ f)^p \cong f^p \circ g^p$. A similar computation holds for $(g \circ f)^{-1}$. □

To conclude this part, we use the direct image functor to produce an adjoint of the stalk functor. First, we need the following concept of a skyscraper sheaf.

Definition 1.1.27. *Let $x \in X$ be a point. Denote $i_x : \{x\} \mapsto X$ the inclusion map. Let A be a set and think of A as a sheaf on the one point space $\{x\}$. We define the* ***skyscraper sheaf*** *at x with value A as the pushforward sheaf $i_{x,*}A$. Explicitly,*

$$i_{x,*}A(U) = \begin{cases} A & \text{if } x \in U, \\ \varnothing & \text{if } x \notin U. \end{cases}$$

We say a sheaf $\mathscr{F}$ is a skyscraper sheaf if $\mathscr{F} \cong i_{x,*}A$ for some $x \in X$ and a set A.

Lemma 1.1.28. *Let X be a topological space, $x \in X$ a point, and A a set. For any point $y \in X$ the stalk of the skyscraper sheaf at x with value A at y is*

$$(i_{x,*}A)_y = \begin{cases} A & \text{if } y \in \overline{\{x\}} \\ \{*\} & \text{otherwise} \end{cases}$$

Proof. If $x \notin \overline{\{x\}}$, then there exist arbitrarily small open neighbourhoods U of y which do not contain x. Because $\mathscr{F}$ is a sheaf we have $\mathscr{F}(i_x^{-1}(U)) = \{*\}$ for any such U. □

Proposition 1.1.29. *Let X be a topological space, and let $x \in X$ be a point. The functors $\mathscr{F} \mapsto \mathscr{F}_x$ and $A \mapsto i_{x,*}A$ are adjoint. In a formula,*

$$\mathrm{Mor}_{\mathbf{Set}}(\mathscr{F}_x, A) \cong \mathrm{Mor}_{\mathbf{Shv}(X)}(\mathscr{F}, i_{x,*}A)$$

Proof. Consider the pull back functor of the map $i_x : \{x\} \to X$: for a sheaf $\mathscr{F}$ on X,

$$i_x^{-1}\mathscr{F}(\{x\}) = \varinjlim_{U \ni x} \mathscr{F}(U),$$

which is exactly the stalk of $\mathcal{F}$ at x. Thus the adjointness comes from that of i_x^{-1} and $i_{x,*}$. □

1.1.5 Open immersions and closed immersions

1.1.5.0.1 Open immersions and sheaves Let $j : U \hookrightarrow X$ be an open immersion (that is, an embedded of U into an open subset of X). It turns out that there is a functor $j_!$ which is left adjoint to j^{-1}, so that we get a triple $(j_!, j^{-1}, j_*)$ in which each consecutive pair is an adjunction and j^{-1} is exact. But first let us point out that j^{-1} has a particularly simple description in the case of an open immersion.

Proposition 1.1.30. *Let X be a topological space. Let $j : U \to X$ be the inclusion of an open subset U into X.*

(a) Let $\mathcal{G}$ be a presheaf of sets on X. The presheaf $j^p\mathcal{G}$ is given by the rule $V \mapsto \mathcal{G}(V)$ for $V \subseteq U$ open.

(b) Let $\mathcal{G}$ be a sheaf of sets on X. The sheaf $j^{-1}\mathcal{G}$ is given by the rule $V \mapsto \mathcal{G}(V)$ for $V \subseteq U$ open.

(c) On the category of presheaves of U we have $j^p j_* = \mathbf{1}$, and on the category of sheaves of U we have $j^{-1}j_* = \mathbf{1}$.

Proof. Note that $j^{-1}j_*\mathcal{F}(V) = j^p j_*\mathcal{F}(V) = \mathcal{F}(V)$ for open subsets $V \subseteq U$, so the claims follow immediately. □

Definition 1.1.31. *Let X be a topological space. Let $j : U \to X$ be the inclusion of an open subset.*

- *Let $\mathcal{G}$ be a presheaf of sets on X. The presheaf $j^p\mathcal{G}$ is called the **restriction** of $\mathcal{G}$ to U and denoted $\mathcal{G}|_U$.*
- *Let $\mathcal{G}$ be a sheaf of sets on X. The presheaf $j^{-1}\mathcal{G}$ is called the **restriction** of $\mathcal{G}$ to U and denoted $\mathcal{G}|_U$.*

Definition 1.1.32. *Let X be a topological space. Let $j : U \hookrightarrow X$ be the inclusion of an open subset.*

*(a) Let $\mathcal{F}$ be a presheaf of sets on U. We define the **extension of $\mathcal{F}$ by the empty set** $j_{p!}\mathcal{F}$ to be the presheaf of sets on X defined by the rule*

$$j_{p!} = \begin{cases} \mathcal{F}(V) & \text{if } V \subseteq U, \\ \varnothing & \text{otherwise.} \end{cases}$$

with obvious restriction mappings.

*(b) Let $\mathcal{F}$ be a sheaf of sets on U. We define the **extension of** $\mathcal{F}$ by the empty set $j_!\mathcal{F}$ to be the sheafification of the presheaf $j_{p!}\mathcal{F}$.*

Proposition 1.1.33. *Let X be a topological space. Let $j : U \to X$ be the inclusion of an open subset.*

(a) The functor $j_{p!}$ is a left adjoint to the restriction functor j^p:

$$\mathrm{Mor}_{\mathbf{Psh}(X)}(j_{p!}\mathcal{F}, \mathcal{G}) \cong \mathrm{Mor}_{\mathbf{Psh}(U)}(\mathcal{F}, j^p\mathcal{G}) = \mathrm{Mor}_{\mathbf{Psh}(X)}(\mathcal{F}, \mathcal{G}|_U).$$

(b) The functor $j_!$ is a left adjoint to restriction,

$$\mathrm{Mor}_{\mathbf{Shv}(X)}(j_!\mathscr{F}, \mathscr{G}) \cong \mathrm{Mor}_{\mathbf{Shv}(U)}(\mathscr{F}, j^{-1}\mathscr{G}) = \mathrm{Mor}_{\mathbf{Shv}(U)}(\mathscr{F}, \mathscr{G}|_U).$$

(c) Let $\mathscr{F}$ be a sheaf of sets on U. The stalks of the sheaf $j_!\mathscr{F}$ are described as follows

$$(j_!\mathscr{F})_x = \begin{cases} \mathscr{F}_x & \text{if } x \in U, \\ \emptyset & \text{otherwise.} \end{cases}$$

Therefore the functor $j_!$ is exact.

(e) On the category of presheaves of U we have $j^p j_{p!} = \mathbf{1}$, and on the category of sheaves of U we have $j^{-1} j_! = \mathbf{1}$.

Proof. To map $j_{p!}\mathscr{F}$ into $\mathscr{G}$ it is enough to map $\mathscr{F}(V) \to \mathscr{G}(V)$ whenever $V \subseteq U$ compatibly with restriction mappings. And the same description holds for maps $\mathscr{F} \mapsto \mathscr{G}|_U$. The adjointness of $j_!$ and restriction follows from this and the properties of sheafification. The identification of stalks is obvious from the definition of the extension by the empty set and the definition of a stalk.

Finally, if $\mathscr{F}$ is a sheaf on U, consider the canonical maps

$$\mathscr{F} \mapsto j^p j_{p!}\mathscr{F}, \quad \mathscr{F} \mapsto j^{-1} j_! \mathscr{F}.$$

Since the induced maps on stalks are isomorphisms, the claim follows. □

Theorem 1.1.34. *Let X be a topological space. Let $j : U \to X$ be the inclusion of an open subset. The functor*

$$j_! : \mathbf{Shv}(U) \to \mathbf{Shv}(X)$$

is fully faithful. Its essential image consists exactly of those sheaves $\mathscr{G}$ whose support is contained in U.

Proof. Fully faithfulness follows formally from $j^{-1} j_! = \mathbf{1}$. We have seen that any sheaf in the image of the functor has the property on the stalks mentioned in the lemma. Conversely, suppose that $\mathscr{G}$ has the indicated property, then the canonical map

$$j_! j^{-1}\mathscr{G} \to \mathscr{G}$$

is an isomorphism on all stalks and hence an isomorphism. □

1.1.5.0.2 Closed immersions and sheaves Let X be a topological space. Let $i : Z \to X$ be the inclusion of a closed subset Z into X. In a similar fashion, we can extend the adjunction (i^{-1}, i_*). First we state an important result for the direct image i_*.

Lemma 1.1.35. *Let X be a topological space. Let $i : Z \hookrightarrow X$ be the inclusion of a closed subset Z into X. Let $\mathscr{F}$ be a sheaf of sets on Z. The stalks of $i_*\mathscr{F}$ are described By*

$$(i_*\mathscr{F})_x = \begin{cases} \mathscr{F}_x & x \in Z, \\ \{*\} & \text{otherwise.} \end{cases}$$

Proof. This follows from the definition of $i_*\mathscr{F}$, and the fact that Z is closed. □

Theorem 1.1.36. *Let X be a topological space. Let $i : Z \to X$ be the inclusion of a closed subset. The functor*

$$i_* : \mathbf{Shv}(Z) \to \mathbf{Shv}(X)$$

is fully faithful. Its essential image consists exactly of those sheaves $\mathscr{G}$ whose support is contained in Z. The functor i^{-1} is a left inverse to i_*.

Proof. Fully faithfulness follows formally from $i^{-1}i_* = \mathbf{1}$. We have seen that any sheaf in the image of the functor has the property on the stalks mentioned in the lemma. Conversely, suppose that $\mathscr{G}$ has the indicated property. Then the map $\mathscr{G} \to i_*i^{-1}\mathscr{G}$ is an isomorphism on stalks, and hence an isomrophism. □

Now we define the functor $i^!$ for a closed immersion.

Proposition 1.1.37. *Let X be a topological space. Let $Z \subseteq X$ be a closed subset. Let $\mathscr{F}$ be a sheaf on X. Define a sheaf $\mathscr{H}_Z(\mathscr{F})$ by*

$$\Gamma(U, \mathscr{H}_Z(\mathscr{F})) = \{s \in \mathscr{F}(U) : \operatorname{supp}(s) \subseteq Z \cap U\}.$$

Then $\mathscr{H}_Z(\mathscr{F})$ is a subsheaf of $\mathscr{F}$. It is the largest subsheaf of $\mathscr{F}$ whose support is contained in Z. The construction $\mathscr{F} \mapsto \mathscr{H}_Z(\mathscr{F})$ is functorial in the sheaf $\mathscr{F}$.

Proof. □

Definition 1.1.38. *Let $i : Z \to X$ be the inclusion of a closed subset. For a sheaf $\mathscr{F}$ on X, define the* ***inverse image of $\mathscr{F}$ supported on Z*** *to be the sheaf $i^!\mathscr{F} = i^{-1}\mathscr{H}_Z(\mathscr{F})$.*

Proposition 1.1.39. *For a closed immersion $i : Z \hookrightarrow X$, the functor $i^!$ is right adjoint to i_*. In a formula*

$$\mathrm{Mor}_{\mathbf{Shv}(X)}(i_*\mathscr{G}, \mathscr{F}) = \mathrm{Mor}_{\mathbf{Shv}(Z)}(\mathscr{G}, i^!\mathscr{F})$$

Proof. Note that $i_*i^!\mathscr{F} = \mathscr{H}_Z(\mathscr{F})$ by Theorem 1.1.36. Since i_* is fully faithful we are reduced to showing that

$$\mathrm{Mor}_{\mathbf{Shv}(X)}(i_*\mathscr{G}, \mathscr{F}) = \mathrm{Mor}_{\mathbf{Shv}(Z)}(i_*\mathscr{G}, \mathscr{H}_Z(\mathscr{F})).$$

This follows since the support of the image via any homomorphism of a section of $i_*\mathscr{G}$ is contained in Z, by Theorem 1.1.36. □

Proposition 1.1.40. *Let $i : Z \hookrightarrow X$ be a closed embedding, set $U = Z^c$ and let $j : U \hookrightarrow X$ be the corresponding open embedding. Then for any sheaf $\mathscr{F}$ on X, the sequence*

$$0 \longrightarrow j_!j^{-1}\mathscr{F} \longrightarrow \mathscr{F} \longrightarrow j_*j^{-1}\mathscr{F} \longrightarrow 0$$

is exact. Moreover, for any morphism $\varphi : \mathscr{F} \to \mathscr{G}$ of sheaves on X we obtain a morphism of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & j_!j^{-1}\mathscr{F} \arrow[r] \arrow[d] & \mathscr{F} \arrow[r] \arrow[d] & j_*j^{-1}\mathscr{F} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & j_!j^{-1}\mathscr{G} \arrow[r] & \mathscr{G} \arrow[r] & j_*j^{-1}\mathscr{G} \arrow[r] & 0
\end{tikzcd}$$

Proof. The functoriality of the short exact sequence is immediate from the naturality of the adjunction mappings. We may check exactness on stalks: for $x \in X$, the sequence is

$$\begin{cases} 0 \to \mathscr{F}_x \to \mathscr{F}_x \to 0 \to 0 & x \in U, \\ 0 \to 0 \to \mathscr{F}_x \to \mathscr{F}_x \to 0 & x \in Z. \end{cases}$$

so the claim follows. □

1.1.6 Glueing sheaves

It is quite often that we want to glue a bunch of objects defined on the members of a covering of X to create a new one. In this paragraph we will see how to do this for morphisms and sheaves.

Proposition 1.1.41. *Let X be a topological space. Let $X = \bigcup_i U_i$ be an open covering. Let $\mathscr{F}, \mathscr{G}$ be sheaves of sets on X. Given a collection*

$$\varphi_i : \mathscr{F}|_{U_i} \to \mathscr{G}|_{U_i}$$

of maps of sheaves such that for all $i, j \in I$ the maps φ_i, φ_j restrict to the same map $\mathscr{F}|_{U_i \cap U_j} \to \mathscr{G}|_{U_i \cap U_j}$, there exists a unique map of sheaves

$$\varphi : \mathscr{F} \to \mathscr{G}$$

whose restriction to each U_i agrees with φ_i.

Proof. For $V \subseteq X$ open, we have $V = \bigcup_i (V \cap U_i)$, so for $s \in \mathscr{F}(V)$ we define $\varphi_V(s)$ by the equations

$$(\varphi_V(s))|_{V \cap U_i} = (\varphi_i)_{V \cap U_i}(s|_{V \cap U_i}) \quad \text{for} \quad i \in I. \tag{1.6.1}$$

By the condition of sheaf, this is well-defined, and indeed defines a morphism of sheaves. It is clear that φ restrict to φ_i on each U_i. To see the uniqueness, if $\psi : \mathscr{F} \to \mathscr{G}$ is another morphism restricting to φ_i on each i, then for each $s \in \mathscr{F}(U)$, $\psi(s)$ also satisfies (1.6.1), so it must coincides with $\varphi(s)$ by the condition of sheaf. This shows $\varphi = \psi$, as desired. □

The previous proposition implies that given two sheaves $\mathscr{F}, \mathscr{G}$ on the topological space X the rule

$$U \mapsto \mathrm{Mor}_{\mathbf{Shv}(X)}(\mathscr{F}|_U, \mathscr{G}|_U)$$

defines a sheaf. This is a kind of **internal hom sheaf**. It is seldom used in the setting of sheaves of sets, and more usually in the setting of sheaves of modules.

Let X be a topological space. Let $X = \bigcup_i U_i$ be an open covering. For each $i \in I$ let $\mathscr{F}_i$ be a sheaf of sets on U_i. For each pair $i, j \in I$, let

$$\varphi_{ij} : \mathscr{F}_i|_{U_i \cap U_j} \to \mathscr{F}_j|_{U_i \cap U_j}$$

be an isomorphism of sheaves of sets. Assume in addition that for every triple of indices $i, j, k \in$

I the following diagram is commutative

$$\begin{tikzcd} \mathscr{F}_i|_{U_i\cap U_j\cap U_k} \arrow[rr, "\varphi_{ik}"] \arrow[dr, "\varphi_{ij}"'] & & \mathscr{F}_k|_{U_i\cap U_j\cap U_k} \\ & \mathscr{F}_j|_{U_i\cap U_j\cap U_k} \arrow[ur, "\varphi_{jk}"'] & \end{tikzcd}$$

We will call such a collection of data $(\mathscr{F}_i, \varphi_{ij})$ a **glueing data** for sheaves of sets with respect to the covering $X = \bigcup_i X_i$.

Proposition 1.1.42. *Let X be a topological space. Let $X = \bigcup_i U_i$ be an open covering. Given any glueing data $(\mathscr{F}_i, \varphi_{ij})$ for sheaves of sets with respect to the covering, there exists a sheaf of sets $\mathscr{F}$ on X together with isomorphisms*

$$\varphi_i : \mathscr{F}|_{U_i} \to \mathscr{F}_i$$

such that the diagrams

$$\begin{tikzcd} \mathscr{F}|_{U_i\cap U_j} \arrow[r, "\varphi_i"] \arrow[d, equal] & \mathscr{F}_i|_{U_i\cap U_j} \arrow[d, "\varphi_{ij}"] \\ \mathscr{F}|_{U_i\cap U_j} \arrow[r, "\varphi_j"] & \mathscr{F}_j|_{U_i\cap U_j} \end{tikzcd}$$

are commutative.

Proof. Actually we can write a formula for the set of sections of $\mathscr{F}$ over an open $W \subseteq X$. Namely, we define

$$\mathscr{F}(W) = \{(s_i)_{i\in I} \mid s_i \in \mathscr{F}_i(W \cap U_i) \text{ and } \varphi_{ij}(s_i|_{W\cap U_i\cap U_j}) = s_j|_{W\cap U_i\cap U_j}\}.$$

Restriction mappings for $W' \subseteq W$ are defined by the restricting each of the (s_i) to $W' \cap U_i$. The sheaf condition for $\mathscr{F}$ follows immediately from the sheaf condition for each of the $\mathscr{F}_i$.

We still have to prove that $\mathscr{F}|_{U_i}$ maps isomorphically to $\mathscr{F}_i$. Let $W \subseteq U_i$; then the commutativity of the diagrams in the definition of a glueing data assures that we may start with any section $s \in \mathscr{F}_i(W)$ and obtain a compatible collection by setting $s_i = s$ and $s_j = \varphi_{ij}(s_i|_{W\cap U_j})$. Thus the claim follows. □

Corollary 1.1.43. *Let X be a topological space. Let $X = \bigcup_i U_i$ be an open covering. The functor which associates to a sheaf of sets $\mathscr{F}$ the following collection of glueing data*

$$\left(\mathscr{F}|_{U_i}, (\mathscr{F}_i)|_{U_i\cap U_j} \to (\mathscr{F}_j)|_{U_i\cap U_j}\right)$$

with respect to the covering $X = \bigcup_i U_i$ defines an equivalence of categories between $\mathbf{Shv}(X)$ *and the category of glueing data.*

This result means that if the sheaf $\mathscr{F}$ was constructed from the glueing data $(\mathscr{F})_i, \varphi_{ij})$ and if $\mathscr{G}$ is a sheaf on X, then a morphism $f : \mathscr{F} \to \mathscr{G}$ is given by a collection of morphisms of sheaves

$$f_i : \mathscr{F}_i \to \mathscr{G}$$

compatible with the glueing maps φ_{ij}. Similarly, to give a morphism of sheaves $g : \mathscr{G} \to \mathscr{F}$ is the same as giving a collection of morphisms of sheaves

$$g_i : \mathscr{G}|_{U_i} \to \mathscr{F}_i$$

compatible with the glueing maps φ_{ij}.

1.1.7 Preheaves and sheaves over a basis

Sometimes there exists a basis for the topology consisting of opens that are easier to work with than general opens. For convenience we give here some definitions and simple lemmas in order to facilitate working with (pre)sheaves in such a situation.

Let X be a topological space. Let $\mathcal{B}$ be a basis for the topology on X. A **presheaf $\mathscr{F}$ of sets on $\mathcal{B}$** is a rule which assigns to each $U \in \mathcal{B}$ a set $\mathscr{F}(U)$ and to each inclusion $V \subseteq U$ of elements of $\mathcal{B}$ a map $\mathrm{res}^U_V : \mathscr{F}(U) \to \mathscr{F}(V)$ such that whenever $W \subseteq V \subseteq U$ in $\mathcal{B}$ we have $\mathrm{res}^U_V \circ \mathrm{res}^V_W = \mathrm{res}^U_W$. If $\mathscr{F}$ be a presheaf over the basis $\mathcal{B}$. We can associate $\mathscr{F}$ with a sheaf $\mathscr{F}_{\mathcal{B}}$ by defining

$$\mathscr{F}_{\mathcal{B}}(U) = \{(s_V) \in \prod_{\substack{V\in\mathcal{B}\\ V\subseteq U}} \mathscr{F}(V) : \text{for all } W, V \in \mathcal{B}, W \subseteq V, s_V|_W = s_W\} = \varprojlim_{\substack{V\in\mathcal{B}\\ V\subseteq U}} \mathscr{F}(V). \tag{1.7.1}$$

for any open subset U of X. If U and U' are two open sets of X such that $U \subseteq U'$, we define $\mathrm{res}^{U'}_U$ as the inverse limit (for $V \subseteq U$) of the canonical morphisms $\mathscr{F}_{\mathcal{B}}(U') \to \mathscr{F}(V)$, in other words, the unique morphism $\mathscr{F}_{\mathcal{B}}(U') \to \mathscr{F}_{\mathcal{B}}(U)$ which, composed with the canonical morphisms $\mathscr{F}_{\mathcal{B}}(U) \to \mathscr{F}(V)$, gives the canonical morphisms $\mathscr{F}_{\mathcal{B}}(U') \to \mathscr{F}(V)$; it is then immediate that the transitivity holds. Moreover, if $U \in \mathcal{B}$, the canonical morphism $\mathscr{F}_{\mathcal{B}}(U) \to \mathscr{F}(U)$ is an isomorphism, allowing us to identify these two sets.

Proposition 1.1.44. *The following conditions are equivalent:*

(i) The presheaf $\mathscr{F}_{\mathcal{B}}$ is a sheaf on X.

(ii) For any covering (U_α) of $U \in \mathcal{B}$ given by elements of $\mathcal{B}$, the set $\mathscr{F}(U)$ corresponds bijectively to $(s_\alpha) \in \prod_\alpha \mathscr{F}(U_\alpha)$ such that $s_\alpha|_V = s_\beta|_V$ for any $V \in \mathcal{B}$ and $V \subseteq U_\alpha \cap U_\beta$.

(iii) For any covering (U_α) of $U \in \mathcal{B}$ given by elements of $\mathcal{B}$ and $(U_{\alpha\beta\gamma})$ of $U_{\alpha\beta} = U_\alpha \cap U_\beta$ by elements of $\mathcal{B}$, the set $\mathscr{F}(U)$ corresponds bijectively to $(s_\alpha) \in \prod_\alpha \mathscr{F}(U_\alpha)$ such that $s_\alpha|_{U_{\alpha\beta\gamma}} = s_\beta|_{U_{\alpha\beta\gamma}}$ for all γ.

Proof. It is clear that (i)⇒(iii)⇒(ii). Now assume (ii) and let (U_α) be a covering of $U \in \mathcal{B}$ by elements of $\mathcal{B}$ and $(U_{\alpha\beta\gamma})$ a covering of $U_{\alpha\beta}$ by elements of $\mathcal{B}$. Let $(s_\alpha) \in \prod_\alpha \mathscr{F}(U_\alpha)$ be a family such that $s_\alpha|_{U_{\alpha\beta\gamma}} = s_\beta|_{U_{\alpha\beta\gamma}}$ for all γ. Let $V \subseteq U_\alpha \cap U_\beta$ be an basis element in $\mathcal{B}$. For each index γ, let $(V_{\mu\gamma})$ be a covering of $V \cap U_{\alpha\beta\gamma}$ by elements of $\mathcal{B}$, so that the family $(V_{\mu\gamma})_{\mu,\gamma}$ is a covering of V by elements of $\mathcal{B}$. For each pair (μ, γ) of indices, set

$$t^\alpha_{\mu\gamma} = s_\alpha|_{V_{\mu\gamma}}, \quad t^\beta_{\mu\gamma} = s_\beta|_{V_{\mu\gamma}}.$$

By hypothesis $s_\alpha|_{U_{\alpha\beta\gamma}} = s_\beta|_{U_{\alpha\beta\gamma}}$ and $V_{\mu\gamma} \subseteq U_{\alpha\beta\gamma}$, so we have $t^\alpha_{\mu\gamma} = t^\beta_{\mu\gamma}$. Moreover, for each

$W \in \mathcal{B}$ and $W \subseteq V_{\mu\gamma} \cap V_{\tilde{\mu}\tilde{\gamma}}$, since $V_{\mu\gamma}$ and $V_{\tilde{\mu}\tilde{\gamma}}$ are both contained in $U_{\alpha\beta}$,

$$t^{\alpha}_{\mu\gamma}|_W = s_\alpha|_W = t^{\alpha}_{\tilde{\mu}\tilde{\gamma}}|_W, \quad t^{\beta}_{\mu\gamma}|_W = s_\beta|_W = t^{\beta}_{\tilde{\mu}\tilde{\gamma}}|_W.$$

Applying (ii) on the open set V and the covering $(V_{\mu\gamma})$, we then conclude that $s_\alpha|_V = s_\beta|_V$, which again by condition (ii) implies that (s_α) corresponds to a section on $\mathscr{F}(U)$. This proves (ii)⇒(iii).

Now we prove the implication (iii)⇒(i). Before this, we first note that, if (iii) holds and $\mathcal{B}'$ is a basis of X contained in $\mathcal{B}$, then the presheaf $\mathscr{F}_{\mathcal{B}'}$ associated to the presheaf $(\mathscr{F}(U))_{U\in\mathcal{B}'}$ is canonically isomorphic to the presheaf $\mathscr{F}_{\mathcal{B}}$ associated to $(\mathscr{F}(U))_{U\in\mathcal{B}}$. Indeed, first of all the inverse limit (for $V \in \mathcal{B}' \subseteq \mathcal{B}$, $V \subseteq U$) of the canonical morphisms $\mathscr{F}_{\mathcal{B}}(U) \to \mathscr{F}(V)$ gives a morphism

$$\mathscr{F}_{\mathcal{B}}(U) \to \mathscr{F}_{\mathcal{B}'}(U)$$

for any open set U. We claim that this is an isomorphism if $U \in \mathcal{B}$. To see this, let (U_α) be a covering of U by elements of $\mathcal{B}'$ and for each (α,β), choose a covering $(U_{\alpha\beta\gamma})$ of $U_{\alpha\beta} = U_\alpha \cap U_\beta$ by elements of $\mathcal{B}'$. For each γ, we have a commutative diagram

$$\begin{tikzcd}
 & & \mathscr{F}(U_\alpha) \arrow[dr] & \\
\mathscr{F}_{\mathcal{B}'}(U) \arrow[r, dashed] \arrow[urr, bend left] \arrow[drr, bend right] & \mathscr{F}(U) \arrow[ur] \arrow[dr] & & \mathscr{F}(U_{\alpha\beta\gamma}) \\
 & & \mathscr{F}(U_\beta) \arrow[ur] &
\end{tikzcd}$$

so condition (iii) shows that the canonical morphism $\mathscr{F}_{\mathcal{B}'}(U) \to \mathscr{F}(U_\alpha)$ factors through $\mathscr{F}(U)$. It is immediate that the morphisms $\mathscr{F}_{\mathcal{B}}(U) \to \mathscr{F}_{\mathcal{B}'}(U)$ and $\mathscr{F}_{\mathcal{B}'}(U) \to \mathscr{F}_{\mathcal{B}}(U)$ thus defined are inverses of each other. This being so, for all open set U of X, the morphisms

$$\mathscr{F}_{\mathcal{B}'}(U) \to \mathscr{F}_{\mathcal{B}'}(W) = \mathscr{F}_{\mathcal{B}}(W) = \mathscr{F}(W)$$

for $W \in \mathcal{B}$, $W \subseteq U$ satisfy the conditions characterizing the inverse limit of the $\mathscr{F}(W)$'s, so our claim follows from the uniqueness of the inverse limit.

Now let U be any open set of X, (U_α) a covering of U by open sets contained in U, and let $\mathcal{B}'$ be the subfamily of $\mathcal{B}$ consisting of the sets of $\mathcal{B}$ contained in at least one U_α. It is clear that $\mathcal{B}'$ is still a basis of the topology of X, so $\mathscr{F}_{\mathcal{B}}(U)$ (resp. $\mathscr{F}_{\mathcal{B}}(U_\alpha)$) is the inverse limit of the $\mathscr{F}(V)$ for $V \in \mathcal{B}'$ and $V \subseteq U$ (resp. $V \subseteq U_\alpha$); the sheaf axiom is then verified immediately by virtue of the definition of the inverse limit. □

We say $\mathscr{F}$ is a **sheaf on** $\mathcal{B}$ if it satisfies the equivalent conditions in Proposition 1.1.44.

Corollary 1.1.45. *Let X be a topological space. Let $\mathcal{B}$ be a basis for the topology on X. Assume that for every pair $U, U' \in \mathcal{B}$ we have $U \cap U' \in \mathcal{B}$. Let $\mathscr{F}$ be a presheaf of sets on $\mathcal{B}$. The following are equivalent:*

(i) The presheaf $\mathscr{F}$ is a sheaf on $\mathcal{B}$.

(ii) For every $U \in \mathcal{B}$ and for every family of sections $s_i \in \mathscr{F}(U_i)$ such that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ there exists a unique section $s \in \mathscr{F}(U)$ which restricts to s_i on U_i.

Proof. This is a reformulation of Proposition 1.1.44, as we can take V to be $U_\alpha \cap U_\beta$. □

Note that for any $x \in X$ we have $\mathscr{F}_x = (\mathscr{F}_{\mathcal{B}})_x$ in the situation of the proposition. This is so because the collection of elements of $\mathcal{B}$ containing x forms a fundamental system of open neighbourhoods of x.

Let $\mathscr{F}, \mathscr{G}$ be two presheaves over $\mathcal{B}$. A morphism $\varphi : \mathscr{F} \to \mathscr{G}$ is defined to be a family $(\varphi_V)_{V \in \mathcal{B}}$ of morphisms $\varphi_V : \mathscr{F}(V) \to \mathscr{F}(V)$ satisfying the compatible conditions with restriction maps. By passing to inverse limits, we get a morphism $\varphi_{\mathcal{B}} : \mathscr{F}_{\mathcal{B}} \to \mathscr{G}_{\mathcal{B}}$ of presheaves (it is easy to verify the compatible conditions with restriction maps).

Theorem 1.1.46. *Let X be a topological space. Let $\mathcal{B}$ be a basis for the topology on X. Denote* **Shv**$(\mathcal{B})$ *the category of sheaves on $\mathcal{B}$. There is an equivalence of categories*

$$\mathbf{Shv}(X) \to \mathbf{Shv}(\mathcal{B})$$

which assigns to a sheaf on X its restriction to the members of $\mathcal{B}$.

Proof. If $\mathscr{F}$ is a sheaf on $\mathcal{B}$, then the sheaf $\mathscr{F}_{\mathcal{B}}$ satisfies $\mathscr{F}(U) = \mathscr{F}_{\mathcal{B}}(U)$ for $U \in \mathcal{B}$, thus the restriction of $\mathscr{F}_{\mathcal{B}}$ equals $\mathscr{F}$. Conversely, if $\mathscr{F}$ is a sheaf on X, then the restriction $\mathscr{F}|_{\mathcal{B}}$ induces a sheaf $\mathscr{F}' := (\mathscr{F}|_{\mathcal{B}})_{\mathcal{B}}$ on X. Then $\mathscr{F}$ and $\mathscr{F}'$ has the same stalk, so we conclude $\mathscr{F} \cong \mathscr{F}'$. Moreover, by looking at stalks, we see the Hom sets are canonically identified, whence the claim. □

1.1.8 The category of presheaves and sheaves

In this subsection, we derive some result for the categories **Psh**(X) and **Shv**(X).

1.1.8.0.1 The category of presheaves We first consider the category of presheaves, with morphisms are defined to be morphisms of presheaves. As we will see, this category behaves much like the base category **Set**.

Example 1.1.47 (Final object of presheaves). Let X be a topological space. Consider a rule $\mathscr{F}$ that associates to every open subset a singleton set. Since every set has a unique map into a singleton set, there exist unique restriction maps res^U_V. The resulting structure is a presheaf of sets. It is a final object in the category of presheaves of sets, by the property of singleton sets mentioned above. Hence it is also unique up to unique isomorphism. We will sometimes write $*$ for this presheaf.

Proposition 1.1.48. *Let $\mathcal{I}$ be a small category and let $\mathscr{F} : \mathcal{I} \to$* **Psh**$(X)$ *be an $\mathcal{I}$-diagram of presheaves on X. Then the limit and colimit of $\mathscr{F}$ both exist, whichi are given by*

$$(\varprojlim_i \mathscr{F}_i)(U) := \varprojlim_i \mathscr{F}_i(U), \quad (\varinjlim_i \mathscr{F}_i)(U) := \varinjlim_i \mathscr{F}_i(U).$$

and the restriction maps are induced by that of the $\mathscr{F}_i$'s.

Proof. The given constructions are clearly meaningful and define presheaves, where the restriction map is given by the limit (or colimit) of that of $\mathscr{F}_i$. For the colimit, if we are given morphisms $\varphi_i : \mathscr{F}_i \to G$ such that the following diagram commutes

$$\begin{tikzcd} \mathscr{F}_i \arrow[rr] \arrow[dr] & & \mathscr{F}_j \arrow[dl] \\ & \mathscr{G} & \end{tikzcd}$$

Then for each $U \subseteq X$ open we can take limit of the system $(\varphi_{i,U})$ to get a morphism $\varphi : \varinjlim_i \mathscr{F}_i(U) \to \mathscr{G}(U)$. Moreover, these morphisms are compatible with the restriction maps of $\mathscr{F}_i$, hence compatible with that of $\varinjlim_i \mathscr{F}_i$. This gives a morphism $\varinjlim_i \mathscr{F}_i \to \mathscr{G}$, so $\varinjlim_i \mathscr{F}_i$ satisfies the universal property of colimits. Similarly, we can show that $\varprojlim_i \mathscr{F}_i$ satisfies the universal property of limits. □

Definition 1.1.49. *Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism of presheaves of sets.*

*(a) We say that φ is **injective** if for every open subset $U \subseteq X$ the map $\varphi_U : \mathscr{F}(U) \to \mathscr{G}(U)$ is injective.*

*(b) We say that φ is **surjective** if for every open subset $U \subseteq X$ the map $\varphi_U : \mathscr{F}(U) \to \mathscr{G}(U)$ is injective.*

We show that the injectivity and surjectivity gives monomorphisms and epimorphisms in the category of presheaves.

Proposition 1.1.50. *The injective (resp. surjective) maps defined above are exactly the monomorphisms (resp. epimorphisms) of* **Psh**(X)*. A map is an isomorphism if and only if it is both injective and surjective.*

Proof. We shall show that $\varphi : \mathscr{F} \to \mathscr{G}$ is injective if and only if it is an monomorphism of **Psh**(X). Indeed, the only if direction is straightforward, so let us show the if direction. If φ is a monomorphism, let $U \subseteq X$ be an open subset; we are going to show that φ_U is a monomorphism in the category **Set**. For this, consider any two maps $f, g : A \to \mathscr{F}(U)$ such that $\varphi_U \circ f = \varphi_U \circ g$. We define a presheaf $\mathscr{A}$ by

$$\mathscr{A}(V) = \begin{cases} A & V \subseteq U, \\ \varnothing & \text{otherwise.} \end{cases}$$

Then we have induced morphism of presheaves ψ_f and ψ_g given by the diagram

$$\begin{tikzcd} \mathscr{A}(U) \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[d, equal] & \mathscr{F}(U) \arrow[d, "\mathrm{res}^U_V"] \\ \mathscr{A}(V) \arrow[r, dashed, "\psi_V"] & \mathscr{F}(V) \end{tikzcd}$$

Then we can see $\varphi \circ \psi_f = \varphi \circ \psi_g$, which implies $\psi_f = \psi_g$ since φ is monic. This gives $f = g$ by our construction, so φ_U is monic in the category of sets, hence injective.

Now we show that $\varphi : \mathscr{F} \to \mathscr{G}$ is surjective if and only if it is an epimorphism of $\mathbf{Psh}(X)$. Similarly, the only if direction is straightforward, so let us show the if direction. Assume that φ is an epimorphism, and we show φ_U is epic in **Set**. For any maps $f, g : \mathscr{G}(U) \to B$ such that $f \circ \varphi_U = g \circ \varphi_U$, we define a presheaf $\mathscr{B}$ by

$$\mathscr{B}(V) = \begin{cases} B & V \supseteq U, \\ \varnothing & \text{othewise.} \end{cases}$$

Similarly, we can define morphism of presheaves by the diagram

$$\begin{tikzcd} \mathscr{G}(V) \arrow[r, dashed, "\psi_V"] \arrow[d, "\mathrm{res}^U_V"'] & \mathscr{B}(V) \arrow[d, equal] \\ \mathscr{G}(U) \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & \mathscr{B}(U) \end{tikzcd}$$

and we have $\psi_f \circ \varphi_U = \psi_g \circ \varphi_U$, which implies $f = g$, so φ_U is surjective. □

1.1.8.0.2 The category of sheaves We have already seen that limits and colimits exist in the category of presheaves. Now we consider them in the category of sheaves.

Proposition 1.1.51. *Let $\mathcal{I}$ be a small category and let $\mathscr{F} : \mathcal{I} \to \mathbf{Psh}(X)$ be an $\mathcal{I}$-diagram of sheaves on X. Then the limit and colimit of $\mathscr{F}$ both exist, whichi are given by*

$$\varprojlim_{i,\mathbf{Shv}(X)} \mathscr{F}_i = \varprojlim_{i,\mathbf{Psh}(X)} \mathscr{F}_i, \qquad \varinjlim_{i,\mathbf{Shv}(X)} \mathscr{F}_i := (\varinjlim_{i,\mathbf{Psh}(X)} \mathscr{F}_i)^{\#}.$$

Proof. Since the limit and colimit exist in $\mathbf{Psh}(X)$, we can define

$$\varprojlim_{i,\mathbf{Shv}(X)} \mathscr{F}_i := (\varprojlim_{i,\mathbf{Psh}(X)} \mathscr{F}_i)^{\#}, \qquad \varinjlim_{i,\mathbf{Shv}(X)} \mathscr{F}_i := (\varinjlim_{i,\mathbf{Psh}(X)} \mathscr{F}_i)^{\#}.$$

Using the universal property of sheafification, we can show that the construction above indeed gives the colimits in the category of sheaves. Now we note that, since the forgetful functor $\iota : \mathbf{Shv}(X) \to \mathbf{Psh}(X)$ has a left adjoint given by the sheafification, it commutes with limits. In other words, if $\mathcal{I}$ is a small category and $\mathscr{F} : \mathcal{I} \to \mathbf{Psh}(X)$ be an $\mathcal{I}$-diagram of sheaves on X. Then the limit $\varprojlim_i \mathscr{F}_i$ satisfies

$$\iota(\varprojlim_{i,\mathbf{Shv}(X)} \mathscr{F}_i) = \varprojlim_{i,\mathbf{Psh}(X)} \iota(\mathscr{F}_i).$$

But the forgetful functor does nothing actually, so the limit $\varprojlim_i \mathscr{F}_i$ is in fact given by the limit in $\mathbf{Psh}(X)$, i.e.,

$$\varprojlim_{i,\mathbf{Shv}(X)} \mathscr{F}_i = \varprojlim_{i,\mathbf{Psh}(X)} \mathscr{F}_i.$$

However, since ι may not commutes with colimits, the colimit of $\mathscr{F}$ should be defined as the sheafification of that in **Psh**(X):

$$\varinjlim_{i,\mathbf{Shv}(X)} \mathscr{F}_i := (\varinjlim_{i,\mathbf{Psh}(X)} \mathscr{F}_i)^{\#}.$$

This completes the proof. □

Proposition 1.1.52. *Let $x \in X$ be a point. Then there are maps*

$$(\varprojlim_i \mathscr{F}_i)_x \to \varprojlim_i (\mathscr{F}_i)_x, \quad (\varinjlim_i \mathscr{F}_i)_x \cong \varinjlim_i (\mathscr{F}_i)_x$$

for limits and colimits of (pre)sheaves. The second map is always bijective, and the first map becomes an bijection when $\mathcal{I}$ is finite,

Proof. First we consider (co)limit of presheaves. The maps $\mathscr{F}_i(U) \to (\mathscr{F}_i)_x$ yield maps

$$\varprojlim_i \mathscr{F}_i(U) \to \varprojlim_i (\mathscr{F}_i)_x \quad \text{and} \quad \varinjlim_i \mathscr{F}_i(U) \to \varinjlim_i (\mathscr{F}_i)_x.$$

Taking the filtered colimit over the open neighborhoods of x we obtain maps

$$(\varprojlim_i \mathscr{F}_i)_x \to \varprojlim_i (\mathscr{F}_i)_x, \quad (\varinjlim_i \mathscr{F}_i)_x \to \varinjlim_i (\mathscr{F}_i)_x$$

As filtered colimits commute with finite limits, the first map is an isomorphism if $\mathcal{I}$ is finite. And since colimits commute with each other, the second is always bijective.

In the case of sheaves, we need to take sheafification. Since the sheafification does not change stalk, the result is the same. □

Proposition 1.1.53. *Suppose $\varphi : \mathscr{F} \to \mathscr{G}$ is a morphism of sheaves of sets on a topological space X. Then the following are equivalent.*

(a) φ is a monomorphism in the category of sheaves.

(b) φ is injective on the level of stalks: $\varphi_x : \mathscr{F}_x \to \mathscr{G}_x$ is injective for all $x \in X$.

(c) φ is injective on the level of open sets: $\varphi(U) : \mathscr{F}(U) \to \mathscr{G}(U)$ is injective for all open $U \subseteq X$.

*If these conditions hold, we say that $\mathscr{F}$ is a **subsheaf** of $\mathscr{G}$.*

Proof. The last two statements are equivalent by Proposition 1.1.10, and the implication (c)⇒(a) is immediate, since a monomorphism in **Psh**(X) is clearly monic in **Shv**(X).

Now assume that φ is a monomorphism, and let $s, t \in \mathscr{F}(U)$ be such that $\varphi_U(s) = \varphi_U(t)$. Then by Example 1.1.20 there are morphisms $\psi_s, \psi_t : \mathscr{U}^{\#} \to \mathscr{F}$. From $\varphi_U(s) = \varphi_U(t)$ we see $\varphi \circ \psi_s = \varphi \circ \psi_t$, which implies $\psi_s = \psi_t$. By the isomorphism in Example 1.1.20, we conclude $s = t$, so φ_U is injective. □

Proposition 1.1.54. *Suppose $\varphi : \mathscr{F} \to \mathscr{G}$ is a morphism of sheaves of sets on a topological space X. Then the following are equivalent.*

(a) *φ is a epimorphism in the category of sheaves.*

(b) *φ is surjective on the level of stalks: $\varphi_x : \mathscr{F}_x \longrightarrow \mathscr{G}_x$ is injective for all $x \in X$.*

(c) *For any open subsets $U \subseteq X$ and every $t \in \mathscr{G}(U)$ there exist an open covering $U = \bigcup_i U_i$ and sections $s_i \in \mathscr{F}(U_i)$ such that $\varphi_{U_i}(s_i) = t|_{U_i}$, i.e., we can locally find preimage of t.*

*If these conditions hold, we say that $\mathscr{G}$ is a **quotient sheaf** of $\mathscr{F}$.*

Proof. If $\varphi_x : \mathscr{F}_x \longrightarrow \mathscr{G}_x$ is surjective for all $x \in X$, then by Proposition 1.1.10 and the diagram

$$\begin{tikzcd}
\mathscr{F}(U) \arrow[r, "\varphi_U"] \arrow[d, hook] & \mathscr{G} \arrow[r, "\psi_U"] \arrow[d, hook] & \mathscr{H}(U) \arrow[d, hook] \\
\prod_{x\in U} \mathscr{F}_x \arrow[r, "\varphi_x"] & \prod_{x\in U} \mathscr{G}_x \arrow[r, "\psi_x"] & \prod_{x\in U} \mathscr{H}_x
\end{tikzcd}$$

we see φ is an epimorphism.

Now assume that φ is an epimorphism, and let $f, g : \mathscr{G}_x \longrightarrow B$ be two maps such that $f \circ \varphi_x = g \circ \varphi_x$. Then by Proposition 1.1.29 we have induced morphism of sheaves $\psi_f, \psi_g : \mathscr{G} \longrightarrow i_{x,*}(B)$ such that $\psi_f \circ \varphi = \psi_g \circ \varphi$. Since φ is an epic, this implies $f = g$, which means φ_x is surjective.

Finally, the condition (b) clearly implies (c), and if (c) holds, let $t_x \in \mathscr{G}_x$ with $t_x = (t, V)$ for some $V \subseteq X$ open and $t \in \mathscr{G}_x$. Then there is a covering $V = \bigcup_i V_i$ and $s_i \in \mathscr{F}(U_i)$ such that $\varphi_{V_i}(s_i) = t|_{V_i}$. Choose a $x \in V_i$, then $\varphi_x\big((s_i)_x\big) = t_x$. Thus φ_x is surjective. □

Remark 1.1.5. The condition for φ in Proposition 1.1.54 does not imply that φ_U is surjective for all open sets U of X as Example 1.1.55 shows. In other words, being epic in the category of sheaves is a weaker that begin surjective on objects.

Example 1.1.55. Let $\mathcal{O}_X$ be the sheaf of holomorphic functions on an open subset X of $\mathbb{C}$. For every open subspace $U \subseteq X$ and $f \in \mathcal{O}_X(U)$ we let $D_U(f) = f'$ be the derivative. We obtain a morphism $D : \mathcal{O}_X \to \mathcal{O}_X$ of sheaves of $\mathbb{C}$-vector spaces. Then D is an epimorphism, because locally every holomorphic function has a primitive. But there exist open subsets U of X and functions f on U that have no primitive., for instance $U = \mathbb{B}(z_0) \setminus \{z_0\} \subseteq \mathbb{C}$ contained in X and $f = 1/(z - z_0)$. More precisely, by complex analysis we know that D_U is surjective if and only if every connected component of U is simply connected. Thus D is not surjective.

1.2 Sheaf of modules

1.2.1 Presheaf of moudles

Definition 1.2.1. *Let X be a topological space, and let $\mathscr{O}$ be a presheaf of rings on X.*

- *A presheaf of $\mathscr{O}$-modules is given by an abelian presheaf $\mathscr{F}$ together with a map of presheaves of sets*

$$\mathscr{O} \times \mathscr{F} \longrightarrow \mathscr{F}$$

such that for every open $U \subseteq X$ the map $\mathscr{O}(U) \times \mathscr{F}(U) \longrightarrow \mathscr{F}(U)$ defines the structure of an $\mathscr{O}(U)$-module structure on the abelian group $\mathscr{F}(U)$.

- *A morphism $\varphi : \mathscr{F} \to \mathscr{G}$ of presheaves of $\mathscr{O}$-modules is a morphism of abelian presheaves $\varphi : \mathscr{F} \to \mathscr{G}$ such that the diagram*

$$\begin{tikzcd} \mathscr{O} \times \mathscr{F} \arrow[r] \arrow[d, "1\times\varphi"'] & \mathscr{F} \arrow[d, "\varphi"] \\ \mathscr{O} \times \mathscr{G} \arrow[r] & \mathscr{G} \end{tikzcd}$$

  commutes.

- *The set of $\mathscr{O}$-module morphisms as above is denoted* $\mathrm{Hom}_{\mathscr{O}}(\mathscr{F}, \mathscr{G})$. *The category of presheaves of $\mathscr{O}$-modules is denoted* $\mathbf{PMod}(\mathscr{O})$.

Suppose that $\mathscr{O}_1 \to \mathscr{O}_2$ is a morphism of presheaves of rings on X. In this case, if $\mathscr{F}$ is a presheaf of $\mathscr{O}_2$-modules then we can think of $\mathscr{F}$ as a presheaf of $\mathscr{O}_2$-modules by using the composition

$$\mathscr{O}_1 \times \mathscr{F} \to \mathscr{O}_2 \times \mathscr{F} \to \mathscr{F}.$$

We sometimes denote this by $\mathscr{F}_{\mathscr{O}_1}$ to indicate the restriction of rings. We call this the **restriction of** $\mathscr{F}$. We obtain the restriction functor

$$\mathbf{PMod}(\mathscr{O}_2) \to \mathbf{PMod}(\mathscr{O}_1).$$

On the other hand, given a presheaf of $\mathscr{O}_1$-modules $\mathscr{G}$ we can construct a presheaf of $\mathscr{O}_2$-modules $\mathscr{O}_2 \otimes_{p,\mathscr{O}_1} \mathscr{F}$ by the rule

$$(\mathscr{O}_2 \otimes_{p,\mathscr{O}_1} \mathscr{F})(U) = \mathscr{O}_2(U) \otimes_{p,\mathscr{O}_1} \mathscr{F}(U).$$

The index p stands for presheaf. This presheaf is called the **tensor product presheaf**. We obtain the change of rings functor

$$\mathbf{PMod}(\mathscr{O}_1) \to \mathbf{PMod}(\mathscr{O}_2).$$

Proposition 1.2.2. *With $\mathscr{O}_1, \mathscr{O}_2$ and $\mathscr{F}, \mathscr{G}$ as above there exists a canonical bijection*

$$\mathrm{Hom}_{\mathscr{O}_1}(\mathscr{G}, \mathscr{F}_{\mathscr{O}_1}) = \mathrm{Hom}_{\mathscr{O}_2}(\mathscr{O}_2 \otimes_{p,\mathscr{O}_1} \mathscr{G}, \mathscr{F}).$$

In other words, the restriction and change of rings functors are adjoint to each other.

Proof. This follows from the fact that for a ring map $A \to B$ the restriction functor and the change of ring functor are adjoint to each other. □

The stalk of a sheaf of $\mathscr{O}$-module is defined in the same as that of sheaf of sets.

Proposition 1.2.3. *Let X be a topological space. Let $\mathscr{O}$ be a presheaf of rings on X. Let $\mathscr{F}$ be a presheaf of $\mathscr{O}$-modules. Let $x \in X$. The canonical map $\mathscr{O}_x \times \mathscr{F}_x \to \mathscr{F}_x$ coming from the multiplication map $\mathscr{O} \times \mathscr{F} \to \mathscr{F}$ defines a $\mathscr{O}_x$-module structure on the abelian group $\mathscr{F}_x$.*

Proposition 1.2.4. *Let X be a topological space. Let $\mathscr{O} \to \mathscr{O}'$ be a morphism of presheaves of rings on X. Let $\mathscr{F}$ be a presheaf of $\mathscr{O}$-modules. Let $x \in X$. We have*

$$\mathscr{F}_x \otimes_{\mathscr{O}_x} \mathscr{O}'_x = (\mathscr{F} \otimes_{p,\mathscr{O}} \mathscr{O}')_x.$$

as $\mathscr{O}'_x$-modules.

Proof. Tensor product is left-adjoint, so it commutes with colimit. □

1.2.2 Sheaf of moudles

Definition 1.2.5. *Let X be a topological space. Let $\mathcal{O}$ be a sheaf of rings on X.*

- *A sheaf of $\mathcal{O}$-modules is a presheaf of $\mathcal{O}$-modules $\mathcal{F}$, such that the underlying presheaf of abelian groups $\mathcal{F}$ is a sheaf.*
- *A morphism of sheaves of $\mathcal{O}$-modules is a morphism of presheaves of $\mathcal{O}$-modules.*
- *Given sheaves of $\mathcal{O}$-modules $\mathcal{F}$ and $\mathcal{G}$ we denote $\mathrm{Hom}_{\mathcal{O}}(\mathcal{F},\mathcal{G})$ the set of morphism of sheaves of $\mathcal{O}$-modules. The category of sheaves of $\mathcal{O}$-modules is denoted $\mathbf{Mod}(\mathcal{O})$.*

1.2.3 Sheafification of presheaves of modules

Proposition 1.2.6. *Let X be a topological space. Let $\mathcal{O}$ be a presheaf of rings on X. Let $\mathcal{F}$ be a presheaf $\mathcal{O}$-modules. Let $\mathcal{O}^{\#}$ be the sheafification of $\mathcal{O}$. Let $\mathcal{F}^{\#}$ be the sheafification of $\mathcal{F}$ as a presheaf of abelian groups. There exists a map of sheaves of sets*

$$\mathcal{O}^{\#} \times \mathcal{F}^{\#} \to \mathcal{F}^{\#}.$$

which makes the diagram

$$\begin{tikzcd} \mathcal{O} \times \mathcal{F} \arrow[r] \arrow[d] & \mathcal{F} \arrow[d] \\ \mathcal{O}^{\#} \times \mathcal{F}^{\#} \arrow[r] & \mathcal{F}^{\#} \end{tikzcd}$$

commute and which makes $\mathcal{F}^{\#}$ into a sheaf of $\mathcal{O}^{\#}$-modules. In addition, if $\mathcal{G}$ is a presheaf of $\mathcal{O}$-modules, then any morphism of presheaves of $\mathcal{O}$-modules $\mathcal{F} \to \mathcal{G}$ induced a unique morphism $\mathcal{F}^{\#} \to \mathcal{G}^{\#}$ of sheaf of $\mathcal{O}^{\#}$-modules.

Proof. Since finite product and coproduct coincide in the category of modules, the sheafification commutes with both of them. Thus we can apply the universal property of sheafification on the map $\mathcal{O} \times \mathcal{F} \to \mathcal{F}$ to get a map

$$\mathcal{O}^{\#} \times \mathcal{F}^{\#} \to \mathcal{F}^{\#}$$

Moreover, for a morphism of $\mathcal{O}$-modules $\mathcal{F} \to \mathcal{G}$, apply sheafification on the diagram

$$\begin{tikzcd} \mathcal{O} \times \mathcal{F} \arrow[r] \arrow[d] & \mathcal{F} \arrow[d] \\ \mathcal{O} \times \mathcal{G} \arrow[r] & \mathcal{G} \end{tikzcd}$$

we get the desired morphism. □

This actually means that the functor $\iota : \mathbf{Mod}(\mathcal{O}^{\#}) \to \mathbf{PMod}(\mathcal{O})$ and the sheafification functor of the lemma $\# : \mathbf{PMod}(\mathcal{O}) \to \mathbf{Mod}(\mathcal{O}^{\#})$ are adjoint. In a formula

$$\mathrm{Hom}_{\mathbf{Mod}(\mathcal{O}^{\#})}(\mathcal{F}^{\#},\mathcal{G}) = \mathrm{Hom}_{\mathbf{PMod}(\mathcal{O})}(\mathcal{F}, \iota\mathcal{G}).$$

Let X be a topological space. Let $\mathcal{O}_1 \to \mathcal{O}_2$ be a morphism of sheaves of rings on X. We defined a restriction functor and a change of rings functor on presheaves of modules associated to this situation. If $\mathcal{F}$ is a sheaf of $\mathcal{O}_2$-modules then the restriction $\mathcal{F}_{\mathcal{O}_2}$ of $\mathcal{F}$ is clearly a sheaf of $\mathcal{O}_1$-modules. We obtain the restriction functor

$$\mathbf{Mod}(\mathcal{O}_2) \to \mathbf{Mod}(\mathcal{O}_1).$$

On the other hand, given a sheaf of $\mathcal{O}_1$-modules $\mathcal{G}$ the presheaf of $\mathcal{O}_2$-modules $\mathcal{O}_2 \otimes_{p,\mathcal{O}_1} \mathcal{G}$ is in general not a sheaf. Hence we define the tensor product sheaf $\mathcal{O}_2 \otimes_{\mathcal{O}_1} \mathcal{G}$ by the formula

$$\mathcal{O}_2 \otimes_{\mathcal{O}_1} \mathcal{G} = (\mathcal{O}_2 \otimes_{p,\mathcal{O}_1} \mathcal{G})^{\#}.$$

Proposition 1.2.7. *With X, $\mathcal{O}_1, \mathcal{O}_2$, $\mathcal{F}$ and $\mathcal{G}$ as above there exists a canonical bijection*

$$\mathrm{Hom}_{\mathcal{O}_1}(\mathcal{G}, \mathcal{F}_{\mathcal{O}_1}) = \mathrm{Hom}_{\mathcal{O}_2}(\mathcal{O}_2 \otimes_{\mathcal{O}_1} \mathcal{G}, \mathcal{F}).$$

In other words, the restriction and change of rings functors are adjoint to each other.

Proof. This follows from Proposition 1.2.2 and the fact that

$$\mathrm{Hom}_{\mathcal{O}_2}(\mathcal{O}_2 \otimes_{\mathcal{O}_1} \mathcal{G}, \mathcal{F}) = \mathrm{Hom}_{\mathcal{O}_2}(\mathcal{O}_2 \otimes_{p,\mathcal{O}_1} \mathcal{G}, \mathcal{F}, \mathcal{F})$$

by the property of sheafification. □

Proposition 1.2.8. *Let X be a topological space. Let $\mathcal{O} \to \mathcal{O}'$ be a morphism of sheaves of rings on X. Let $\mathcal{F}$ be a sheaf $\mathcal{O}$-modules. Let $x \in X$. We have*

$$\mathcal{F}_x \otimes_{\mathcal{O}_x} \mathcal{O}'_x = (\mathcal{F} \otimes_{\mathcal{O}} \mathcal{O}')_x.$$

1.2.4 Continuous maps

The case of sheaves of modules is more complicated. First we state a few obvious lemmas.

Lemma 1.2.9. *Let $f : X \to Y$ be a continuous map of topological spaces. Let $\mathcal{O}$ be a presheaf of rings on X. Let $\mathcal{F}$ be a presheaf of $\mathcal{O}$-modules. There is a natural map of underlying presheaves of sets*

$$f_*\mathcal{O} \times f_*\mathcal{F} \to f_*\mathcal{F}$$

which turns $f_*\mathcal{F}$ into a presheaf of $f_*\mathcal{O}$-modules. This construction is functorial in $\mathcal{F}$.

Proof. Let $V \subseteq Y$ is open. We define the map of the lemma to be the map

$$f_*\mathcal{O}(V) \times f_*\mathcal{F}(V) = \mathcal{O}(f^{-1}(V)) \times \mathcal{F}(f^{-1}(V)) \to \mathcal{F}(f^{-1}(V)) = f_*\mathcal{F}(V).$$

Here the arrow in the middle is the multiplication map on X. □

Lemma 1.2.10. *Let $f : X \to Y$ be a continuous map of topological spaces. Let $\mathcal{O}$ be a presheaf of rings on Y. Let G be a presheaf of $\mathcal{O}$-modules. There is a natural map of underlying presheaves of sets*

$$f^p\mathcal{O} \times f^p\mathcal{G} \to \mathcal{G}$$

which turns $f^p\mathcal{G}$ into a presheaf of $f^p\mathcal{O}$-modules. This construction is functorial in $\mathcal{G}$.

Proof. Let $U \subseteq X$ is open. We define the map of the lemma to be the map

$$\begin{aligned} f^p\mathcal{O}(U) \times f^p\mathcal{G}(U) &= \varinjlim_{f(U)\subseteq V} \mathcal{O}(V) \times \varinjlim_{f(U)\subseteq V} \mathcal{G}(V) = \varinjlim_{f(U)\subseteq V} \big(\mathcal{O}(V) \times \mathcal{G}(V)\big) \\ &\to \varinjlim_{f(U)\subseteq V} \mathcal{G}(V) = f^p\mathcal{G}(V) \end{aligned}$$

Here the arrow in the middle is the multiplication map on Y. The second equality holds because directed colimits commute with finite limits. $\square$

Let $f : X \to Y$ be a continuous map. Let $\mathcal{O}_X$ be a presheaf of rings on X and let $\mathcal{O}_Y$ be a presheaf of rings on Y. So at the moment we have defined functors

$$f_* : \mathbf{PMod}(\mathcal{O}_X) \to \mathbf{PMod}(f_*\mathcal{O}_X), \quad f^p : \mathbf{PMod}(\mathcal{O}_Y) \to \mathbf{PMod}(f^p\mathcal{O}_Y).$$

These satisfy some compatibilities as follows.

Proposition 1.2.11. *Let $f : X \to Y$ be a continuous map of topological spaces. Let $\mathcal{O}$ be a presheaf of rings on Y. Let $\mathcal{G}$ be a presheaf of $\mathcal{O}$-modules. Let $\mathcal{F}$ be a presheaf of $f^p\mathcal{O}$-modules. Then*

$$\mathrm{Hom}_{\mathbf{PMod}(f^p\mathcal{O})}(f^p\mathcal{G}, \mathcal{F}) = \mathrm{Hom}_{\mathbf{PMod}(\mathcal{O})}(\mathcal{G}, f_*\mathcal{F})$$

Here we think of $f_*\mathcal{F}$ as an $\mathcal{O}$-module via the map $\rho_\mathcal{O} : \mathcal{O} \to f_*f^p\mathcal{O}$.

Proof. Note that we have

$$\mathrm{Mor}_{\mathbf{PAb}(X)}(f^p\mathcal{G}, \mathcal{F}) = \mathrm{Mor}_{\mathbf{PAb}(Y)}(\mathcal{G}, f_*\mathcal{F}).$$

So what we have to prove is that under this correspondence, the subsets of module maps correspond. In addition, the correspondence is determined by the rule

$$(\psi : f^p\mathcal{G} \to \mathcal{F}) \to (f_*\psi \circ \rho_\mathcal{G} : \mathcal{G} \to f_*\mathcal{F})$$

and in the other direction by the rule

$$(\varphi : \mathcal{G} \to f_*\mathcal{F}) \mapsto (\sigma_\mathcal{F} \circ f^p\varphi : f^p\mathcal{F} \to \mathcal{G})$$

Hence, using the functoriality of f_* and f^p we see that it suffices to check that the maps $\rho_\mathcal{G} : \mathcal{G} \to f_*f^p\mathcal{G}$ and $\sigma_\mathcal{F} : f^pf_*\mathcal{F} \to \mathcal{F}$ are compatible with module structures, which can be done by tracing definitions. $\square$

Proposition 1.2.12. *Let $f : X \to Y$ be a continuous map of topological spaces. Let $\mathcal{O}$ be a presheaf of rings on X. Let $\mathcal{F}$ be a presheaf of $\mathcal{O}$-modules. Let $\mathcal{G}$ be a presheaf of $f_*\mathcal{O}$-modules. Then*

$$\mathrm{Hom}_{\mathbf{PMod}(\mathcal{O})}(\mathcal{O} \otimes_{p,f^pf_*\mathcal{O}} f^p\mathcal{G}, \mathcal{F}) = \mathrm{Hom}_{\mathbf{PMod}(f_*\mathcal{O})}(\mathcal{G}, f_*\mathcal{F}).$$

Here we use the map $\sigma_\mathcal{O} : f^pf_*\mathcal{O} \to \mathcal{O}$ in the definition of the tensor product.

Proof. This follows from the equalities

$$\begin{aligned}\mathrm{Hom}_{\mathbf{PMod}(\mathcal{O})}(\mathcal{O}\otimes_{p,f^pf_*\mathcal{O}} f^p\mathcal{G},\mathcal{F}) &= \mathrm{Hom}_{\mathbf{PMod}(f^pf_*\mathcal{O})}(f^p\mathcal{G},\mathcal{F}_{f^pf_*\mathcal{O}})\\ &= \mathrm{Hom}_{\mathbf{PMod}(f_*\mathcal{O})}(\mathcal{G},f_*(\mathcal{F}_{f^pf_*\mathcal{O}}))\\ &= \mathrm{Hom}_{\mathbf{PMod}(f_*\mathcal{O})}(\mathcal{G},f_*\mathcal{F}).\end{aligned}$$

For the third equality, note that $\mathbf{1}_{f_*\mathcal{O}}$ corresponds to $\sigma_{\mathcal{O}}$ under the adjunction described in the proof of Proposition 1.2.11 and thus we have the equality $\mathbf{1}_{f_*\mathcal{O}} = f_*\sigma_{\mathcal{O}}\circ\rho_{f_*\mathcal{O}}$. Now consider the module structures:

$$\begin{array}{ll}\mathcal{F}_{f^pf_*\mathcal{O}}: & f^pf_*\mathcal{O}\times\mathcal{F}\xrightarrow{\sigma_{\mathcal{O}}\times\mathbf{1}}\mathcal{O}\times\mathcal{F}\longrightarrow\mathcal{F}\\ f_*(\mathcal{F}_{f^pf_*\mathcal{O}}): & f_*\mathcal{O}\times f_*\mathcal{F}\xrightarrow{\rho_{f_*\mathcal{O}}\times\mathbf{1}} f_*f^pf_*\mathcal{O}\times f_*\mathcal{F}\xrightarrow{f_*\sigma_{\mathcal{O}}\times\mathbf{1}} f_*\mathcal{O}\times f_*\mathcal{F}\longrightarrow f_*\mathcal{F}\end{array}$$

we conclude that $f_*(\mathcal{F}_{f^pf_*\mathcal{O}}) = f_*\mathcal{F}$. □

Now we consider the case of sheaves.

Lemma 1.2.13. *Let $f : X\to Y$ be a continuous map of topological spaces. Let $\mathcal{O}$ be a sheaf of rings on X. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}$-modules. The pushforward $f_*\mathcal{F}$ is a sheaf of $f_*\mathcal{O}$-modules.*

Lemma 1.2.14. *Let $f : X\to Y$ be a continuous map of topological spaces. Let $\mathcal{O}$ be a sheaf of rings on Y. Let $\mathcal{G}$ be a sheaf of $\mathcal{O}$-modules. There is a natural map of underlying presheaves of sets*

$$f^{-1}\mathcal{O}\times f^{-1}\mathcal{G}\to f^{-1}\mathcal{G}$$

which turns $f^{-1}\mathcal{G}$ into a sheaf of $f^{-1}\mathcal{O}$-modules.

Proof. Recall that f^{-1} is defined as the composition of the functor f^p and sheafification. Thus the lemma is a combination of Lemma 1.2.10 and Proposition 1.2.6. □

Let $f : X\to Y$ be a continuous map. Let $\mathcal{O}_X$ be a sheaf of rings on X and let $\mathcal{O}_Y$ be a sheaf of rings on Y. So now we have defined functors

$$f_* : \mathbf{Mod}(\mathcal{O}_X)\to\mathbf{Mod}(f_*\mathcal{O}_X),\quad f^{-1} : \mathbf{Mod}(\mathcal{O}_Y)\to\mathbf{Mod}(f^{-1}\mathcal{O}_Y).$$

These satisfy some compatibilities as follows.

Proposition 1.2.15. *Let $f : X\to Y$ be a continuous map of topological spaces. Let $\mathcal{O}$ be a sheaf of rings on Y. Let $\mathcal{G}$ be a sheaf of $\mathcal{O}$-modules. Let $\mathcal{F}$ be a sheaf of $f^{-1}\mathcal{O}$-modules. Then*

$$\mathrm{Hom}_{\mathbf{Mod}(f^{-1}\mathcal{O})}(f^{-1}\mathcal{G},\mathcal{F}) = \mathrm{Hom}_{\mathbf{Mod}(\mathcal{O})}(\mathcal{G},f_*\mathcal{F}).$$

Here we think of $f_*\mathcal{F}$ as an $\mathcal{O}$-module by restriction via $\mathcal{O}\to f_*f^{-1}\mathcal{O}$.

Proof. Argue by the equalities

$$\mathrm{Hom}_{\mathbf{Mod}(f^{-1}\mathcal{O})}(f^{-1}\mathcal{G},\mathcal{F}) = \mathrm{Hom}_{\mathbf{Mod}(f^p\mathcal{O})}(f^p\mathcal{G},\mathcal{F}) = \mathrm{Hom}_{\mathbf{Mod}(\mathcal{O})}(\mathcal{G},f_*\mathcal{F}).$$

which holds by the property of sheafification. □

Proposition 1.2.16. *Let $f : X \to Y$ be a continuous map of topological spaces. Let $\mathcal{O}$ be a sheaf of rings on X. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}$-modules. Let $\mathcal{G}$ be a sheaf of $f_*\mathcal{O}$-modules. Then*

$$\mathrm{Hom}_{\mathbf{Mod}(\mathcal{O})}(\mathcal{O} \otimes_{f^{-1}f_*\mathcal{O}} f^{-1}\mathcal{G}, \mathcal{F}) = \mathrm{Hom}_{\mathbf{Mod}(f_*\mathcal{O})}(\mathcal{G}, f_*\mathcal{F}).$$

Here we use the canonical map $f^{-1}f_*\mathcal{O} \to \mathcal{O}$ in the definition of the tensor product.

Proof. This follows from the equalities

$$\begin{aligned}\mathrm{Hom}_{\mathbf{Mod}(\mathcal{O})}(\mathcal{O} \otimes_{f^{-1}f_*\mathcal{O}} f^{-1}\mathcal{G}, \mathcal{F}) &= \mathrm{Hom}_{\mathbf{Mod}(f^{-1}f_*\mathcal{O})}(f^{-1}\mathcal{G}, \mathcal{F}_{f^{-1}f_*\mathcal{O}}) \\ &= \mathrm{Hom}_{\mathbf{Mod}(f_*\mathcal{O})}\big(\mathcal{G}, f_*(\mathcal{F}_{f^{-1}f_*\mathcal{O}})\big) \\ &= \mathrm{Hom}_{\mathbf{Mod}(f_*\mathcal{O})}(\mathcal{G}, f_*\mathcal{F}).\end{aligned}$$

where the last equality is obtained in Proposition 1.2.12. □

1.2.5 Supports of modules and sections

Definition 1.2.17. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules.*

- *The support of $\mathcal{F}$ is the set of points $x \in X$ such that $\mathcal{F}_x = 0$. We denote it by* $\mathrm{supp}(\mathcal{F})$*.*
- *Let $s \in \Gamma(X, \mathcal{F})$ be a global section. The support of s is the set of points $x \in X$ such that the image $s_x \in \mathcal{F}_x$ of s is not zero.*

Of course the support of a local section is then defined also since a local section is a global section of the restriction of $\mathcal{F}$.

Lemma 1.2.18. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. Let $U \subseteq X$ open.*

- *The support of $s \in \mathcal{F}(U)$ is closed in U.*
- *The support of fs is contained in the intersections of the supports of $f \in \mathcal{O}_X(U)$ and $s \in \mathcal{F}(U)$.*
- *The support of $s + s'$ is contained in the union of the supports of $s, s' \in \mathcal{F}(U)$.*
- *The support of $\mathcal{F}$ is the union of the supports of all local sections of $\mathcal{F}$.*
- *If $\varphi : \mathcal{F} \to \mathcal{G}$ is a morphism of $\mathcal{O}_X$-modules, then the support of $\varphi(s)$ is contained in the support of $s \in \mathcal{F}(U)$.*

In general the support of a sheaf of modules is not closed. Namely, the sheaf could be an abelian sheaf on $\mathbb{R}$ (with the usual archimedean topology) which is the direct sum of infinitely many nonzero skyscraper sheaves each supported at a single point p_i of $\mathbb{R}$. Then the support would be the set of points pi which may not be closed.

Another example is to consider the open immersion $j : U = (0, \infty) \to \mathbb{R} = X$, and the abelian sheaf $j_!\mathbb{Z}_U$. By Proposition 1.1.33 the support of this sheaf is exactly U.

Lemma 1.2.19. *Let X be a topological space. The support of a sheaf of rings is closed.*

Proof. This is true because a ring is 0 if and only if $1 = 0$, and hence the support of a sheaf of rings is the support of the unit section. □

1.2.6 Modules generated by sections

Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. Then there is an canonical identification $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{F}) = \Gamma(X, \mathcal{F})$ which associate a global section $s \in \Gamma(X, \mathcal{F})$ with the unique homomorphism of $\mathcal{O}_X$-modules $\mathcal{O}_X \to \mathcal{F}, f \mapsto fs$. That is, a local section f of $\mathcal{O}_X$, i.e., a section f over some open U, is mapped to the multiplication of f with the restriction of s to U. We say that $\mathcal{F}$ is **generated by global sections** if there exist a set I and global sections $s_i \in \Gamma(X, \mathcal{F})$, $i \in I$ such that the homomorphism

$$\mathcal{O}_X^{\oplus I} \to \mathcal{F}$$

which is the homomorphism associated to s_i on the summand corresponding to i, is surjective. In this case we say that the sections s_i **generate** $\mathcal{F}$.

Proposition 1.2.20. *Let $(X, \mathcal{O}_X)$ be a ringed space and $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. Let $(s_i)_{i\in I}$ be a family of global sections of $\mathcal{F}$. Then the sections s_i generate $\mathcal{F}$ if and only if for any point $x \in X$ the elements $s_{i,x} \in \mathcal{F}_x$ generate the $\mathcal{O}_{X,x}$-module $\mathcal{F}_x$.*

Proof. The homomorphism $\mathcal{O}_X^{\oplus I} \to \mathcal{F}$ is surjective if and only if for each point $x \in X$ the homomorphism $(\mathcal{O}_X^{\oplus I})_x \to \mathcal{F}_x$ is surjective. Since taking stalk commutes with colimit, we have $(\mathcal{O}_X)_x^{\oplus I} \mathcal{O}_{X,x}^{\oplus I}$, which implies the claim. □

Proposition 1.2.21. *Let $(X, \mathcal{O}_X)$ be a ringed space and $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. Let $(s_i)_{i\in I}$ be a collection of local sections of $\mathcal{F}$, i.e., $s_i \in \Gamma(U_i, \mathcal{F})$ for some open subset U_i of X. Then there exists a unique smallest sub-$\mathcal{O}_X$-module $\mathcal{G}$ of $\mathcal{F}$ such that each s_i corresponds to a local section of $\mathcal{G}$, which is called the **sub-$\mathcal{O}_X$-module generated by the s_i**.*

Proof. Consider the subpresheaf of $\mathcal{F}$ defined by the rule

$$U \mapsto \{\sum_{i\in J} f_i(s_i|_U) : J \subseteq I \text{ is finite}, U \subseteq U_i \text{ for every } i \in J \text{ and } f_i \in \Gamma(U, \mathcal{O}_X)\}.$$

Let $\mathcal{G}$ be the sheafification of this subpresheaf. This is a subsheaf of $\mathcal{F}$ by Proposition 1.1.16. Since all the finite sums clearly have to be in $\mathcal{F}$ this is the smallest subsheaf as desired. □

Proposition 1.2.22. *Let $(X, \mathcal{O}_X)$ be a ringed space and $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. Let $(s_i)_{i\in I}$ be a family of local sections of $\mathcal{F}$ and $\mathcal{G}$ be the subsheaf generated by the s_i and let $x \in X$. Then $\mathcal{G}_x$ is the $\mathcal{O}_{X,x}$-submodule of $\mathcal{F}_x$ generated by the elements $s_{i,x}$ for those i such that s_i is defined at x.*

Proof. This is clear from the construction of $\mathcal{G}$ in the proof of Proposition 1.2.21. □

Example 1.2.23. Consider the open immersion $j : U = (0, \infty) \to \mathbb{R} = X$, and the abelian sheaf $j_!(\mathbb{Z}_U)$. By Proposition 1.1.33 the stalk of $j_!(\mathbb{Z}_U)$ at $x = 0$ is 0. In fact the sections of this sheaf over any open interval containing 0 are 0. Thus there is no open neighbourhood of the point 0 over which the sheaf can be generated by global sections.

Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. We say that $\mathcal{F}$ is **locally generated by sections** if for every $x \in X$ there exists an open neighbourhood U such that $\mathcal{F}|_U$

is globally generated as a sheaf of $\mathcal{O}_U$-modules. In other words there exists a set I and for each i a section $s_i \in \mathcal{F}(U)$ such that the associated map

$$\bigoplus_{i\in I} \mathcal{O}_U \to \mathcal{F}|_U$$

is surjective.

Proposition 1.2.24. *Let $(f, f^{\#}) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. Let $\mathcal{G}$ be a $\mathcal{O}_Y$-module. The pullback $f^*\mathcal{G}$ is locally generated by sections if $\mathcal{G}$ is locally generated by sections.*

Proof. Given an open subspace V of Y we may consider the commutative diagram of ringed spaces

$$\begin{tikzcd}
(f^{-1}(V), \mathcal{O}_{f^{-1}(V)}) \arrow[r, "i"] \arrow[d, "\tilde{f}"] & (X, \mathcal{O}_X) \arrow[d, "f"] \\
(V, \mathcal{O}_V) \arrow[r, "j"] & (Y, \mathcal{O}_Y)
\end{tikzcd}$$

We know that $(f^*\mathcal{G})|_{f^{-1}(V)} \cong (\tilde{f})^*(\mathcal{G}|_V)$ by Proposition **??**. Thus we may assume that $\mathcal{G}$ is globally generated. We have seen that f^* commutes with all colimits, and is right exact. Thus if we have a surjection $\mathcal{O}_Y^{\oplus I} \to \mathcal{G} \to 0$, then upon applying f^* we obtain the surjection $\mathcal{O}_X^{\oplus I} \to f^*\mathcal{G} \to 0$, where we use the observation that

$$f^*\mathcal{O}_Y = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{O}_Y = \mathcal{O}_X.$$

This implies the assertion. □

1.2.7 Tensor product

Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{F}, \mathcal{G}$ be $\mathcal{O}_X$-modules. We define first the tensor product presheaf

$$\mathcal{F} \otimes_{p,\mathcal{O}_X} \mathcal{G}$$

as the rule which assigns to $U \subseteq X$ open the $\mathcal{O}_X(U)$-module $\mathcal{F}(U)_{\mathcal{O}_X(U)}\mathcal{G}(U)$. Having defined this we de

ne the tensor product sheaf as the sheafification of the above:

$$\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} = (\mathcal{F} \otimes_{p,\mathcal{O}_X} \mathcal{G})^{\#}$$

This can be characterized as the sheaf of $\mathcal{O}_X$-modules such that for any third sheaf of $\mathcal{O}_X$-modules $\mathcal{H}$ we have

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}, \mathcal{H}) = \mathrm{Bilin}_{\mathcal{O}_X}(\mathcal{F} \times \mathcal{G}, \mathcal{H}).$$

Here the right hand side indicates the set of bilinear maps of sheaves of $\mathcal{O}_X$-modules.

The tensor product of modules M, N over a ring R satisfies symmetry, hence the same holds for tensor products of sheaves of modules, i.e., we have

$$\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} = \mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{F}$$

functorial in $\mathscr{F}$, $\mathscr{G}$. And since tensor product of modules satisfies associativity we also get canonical functorial isomorphisms

$$(\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G}) \otimes_{\mathscr{O}_X} \mathscr{H} = \mathscr{F} \otimes_{\mathscr{O}_X} (\mathscr{G} \otimes_{\mathscr{O}_X} \mathscr{H}).$$

functorial in $\mathscr{F}, \mathscr{G}$, and $\mathscr{H}$.

Proposition 1.2.25. *Let $(X, \mathscr{O}_X)$ be a ringed space. Let $\mathscr{F}$, $\mathscr{G}$ be $\mathscr{O}_X$-modules. Let $x \in X$. There is a canonical isomorphism of $\mathscr{O}_{X,x}$-modules*

$$(\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G})_x = \mathscr{F}_x \otimes_{\mathscr{O}_{X,x}} \mathscr{G}_x.$$

functorial in $\mathscr{F}$ and $\mathscr{G}$.

Proposition 1.2.26. *Let $(X, \mathscr{O}_X)$ be a ringed space. Let $\mathscr{F}', \mathscr{G}'$ be presheaves of $\mathscr{O}_X$-modules with sheafifications $\mathscr{F}, \mathscr{G}$. Then $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G} = (\mathscr{F}' \otimes_{p,\mathscr{O}_X} \mathscr{G}')^{\#}$.*

Proof. On stalks we have

$$(\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G})_x = \mathscr{F}_x \otimes_{\mathscr{O}_{X,x}} \mathscr{G}_x = \mathscr{F}'_x \otimes_{\mathscr{O}_{X,x}} \mathscr{G}'_x = \mathscr{F}'_x \otimes_{p,\mathscr{O}_{X,x}} \mathscr{G}'_x$$

Thus by Proposition 1.1.17 we conclude the result. ◻

Proposition 1.2.27. *Let $(X, \mathscr{O}_X)$ be a ringed space. Let $\mathscr{G}$ be an $\mathscr{O}_X$-module. If $\mathscr{F}_1 \to \mathscr{F}_2 \to \mathscr{F}_3 \to 0$ is an exact sequence of $\mathscr{O}_X$-modules then the induced sequence*

$$\mathscr{F}_1 \otimes_{\mathscr{O}_X} \mathscr{G} \to \mathscr{F}_2 \otimes_{\mathscr{O}_X} \mathscr{G} \to \mathscr{F}_3 \to 0$$

is exact.

Proof. This follows from the fact that exactness may be checked at stalks, the description of stalks and the corresponding result for tensor products of modules ◻

Proposition 1.2.28. *Let $f : X \to Y$ be a morphism of ringed spaces. Let $\mathscr{F}, \mathscr{G}$ be $\mathscr{O}_Y$-modules. Then $f^*(\mathscr{F} \otimes_{\mathscr{O}_Y} \mathscr{G}) = f^*\mathscr{F} \otimes_{\mathscr{O}_X} f^*\mathscr{G}$ functorially in $\mathscr{F}, \mathscr{G}$.*

Proof. Let $x \in X$, we check that

$$\begin{aligned}
\big(f^*(\mathscr{F} \otimes_{\mathscr{O}_Y} \mathscr{G})\big)_x &= \mathscr{O}_{X,x} \otimes_{\mathscr{O}_{Y,f(x)}} (\mathscr{F} \otimes_{\mathscr{O}_Y} \mathscr{G})_{f(x)} = \mathscr{O}_{X,x} \otimes_{\mathscr{O}_{Y,f(x)}} (\mathscr{F}_{f(x)} \otimes_{\mathscr{O}_{Y,f(x)}} \mathscr{G}_{f(x)}) \\
&= \mathscr{O}_{X,x} \otimes_{\mathscr{O}_{X,x}} \mathscr{O}_{X,x} \otimes_{\mathscr{O}_{Y,f(x)}} \mathscr{F}_{f(x)} \otimes_{\mathscr{O}_{Y,f(x)}} \mathscr{G}_{f(x)} \\
&= \mathscr{O}_{X,x} \otimes_{\mathscr{O}_{Y,f(x)}} \mathscr{F}_{f(x)} \otimes_{\mathscr{O}_{X,x}} \mathscr{O}_{X,x} \otimes_{\mathscr{O}_{Y,f(x)}} \mathscr{G}_{f(x)} \\
&= (f^*\mathscr{F})_x \otimes_{X,x} (f^*\mathscr{G})_x.
\end{aligned}$$

as desired. ◻

Proposition 1.2.29. *Let $(X, \mathscr{O}_X)$ be a ringed space. Let $\mathscr{F}, \mathscr{G}$ be $\mathscr{O}_X$-modules.*

(i) If $\mathscr{F}, \mathscr{G}$ are locally generated by sections, so is $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G}$.

(ii) *If $\mathscr{F}, \mathscr{G}$ are of finite type, so is $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G}$.*

(iii) *If $\mathscr{F}, \mathscr{G}$ are quasi-coherent, so is $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G}$.*

(iv) *If $\mathscr{F}, \mathscr{G}$ are of finite presentation, so is $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G}$.*

(v) *If $\mathscr{F}$ is of finite presentation and $\mathscr{G}$ is coherent, then $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G}$ is coherent.*

(vi) *If $\mathscr{F}, \mathscr{G}$ are coherent, so is $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G}$.*

(vii) *If $\mathscr{F}, \mathscr{G}$ are locally free, so is $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G}$.*

Proof. We first prove that the tensor product of locally free $\mathscr{O}_X$-modules is locally free. This follows if we show that

$$(\bigoplus_{i\in I} \mathscr{O}_X) \otimes_{\mathscr{O}_X} (\bigoplus_{j\in J} \mathscr{O}_X) \cong \bigoplus_{(i,j)\in I\times J} \mathscr{O}_X.$$

The sheaf $\bigoplus_{i\in I} \mathscr{O}_X$ is the sheaf associated to the presheaf $U \mapsto \bigoplus_{i\in I} \mathscr{O}_X(U)$. Hence the tensor product is the sheaf associated to the presheaf

$$U \mapsto \Big(\bigoplus_{i\in I} \mathscr{O}_X(U)\Big) \otimes_{\mathscr{O}_X(U)} \Big(\bigoplus_{j\in J} \mathscr{O}_X(U)\Big).$$

We deduce what we want since for any ring R we have $(\bigoplus_{i\in I} R) \otimes_R (\bigoplus_{j\in J} R) \cong \bigoplus_{(i,j)\in I\times J} R$.

If $\mathscr{F}_2 \to \mathscr{F}_1 \to \mathscr{F} \to 0$ is exact, then by Proposition 1.2.27 the complex $\mathscr{F}_2 \otimes \mathscr{G} \to \mathscr{F}_1 \otimes \mathscr{G} \to \mathscr{F} \otimes \mathscr{G} \to 0$ is exact. Using this we can prove (v). Namely, in this case there exists locally such an exact sequence with $\mathscr{F}_i, i = 1, 2$ finite free. Hence the two terms $\mathscr{F}_i \otimes \mathscr{G}$ are isomorphic to finite direct sums of $\mathscr{G}$. Since finite direct sums are coherent sheaves, these are coherent and so is the cokernel of the map.

If we also have another exact sequence $\mathscr{G}_2 \to \mathscr{G}_1 \to \mathscr{G} \to 0$, then tensoring together we get an exact sequence

$$(\mathscr{F}_2 \otimes \mathscr{G}_1) \oplus (\mathscr{F}_1 \otimes \mathscr{G}_2) \longrightarrow \mathscr{F}_1 \otimes \mathscr{G}_1 \longrightarrow \mathscr{F} \otimes \mathscr{G} \longrightarrow 0$$

This can be used to prove $(i), (ii), (iii), (iv), (vi)$. □

Proposition 1.2.30. *Let $(X, \mathscr{O}_X)$ be a ringed space. For any $\mathscr{O}_X$-module $\mathscr{F}$ the functor $\mathscr{F} \otimes_{\mathscr{O}_X}$ commutes with arbitrary colimits.*

Proof. Let I be a partially ordered set and let $\{\mathscr{G}_i\}$ be a system over I. Set $\mathscr{G} = \varinjlim_i \mathscr{G}_i$. Recall that $\mathscr{G}$ is the sheaf associated to the presheaf $\mathscr{G}' : U \mapsto \varinjlim_i \mathscr{G}_i(U)$. By the tensor product $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G}$ is the sheafification of the presheaf

$$U \mapsto \mathscr{F}(U) \otimes_{\mathscr{O}_X(U)} \varinjlim_i \mathscr{G}_i(U) = \varinjlim_i \mathscr{F}(U) \otimes_{\mathscr{O}_X(U)} \mathscr{G}_i(U)$$

where the equality sign follows from the property of tensor product of modules. Hence the lemma follows from the description of colimits in $\mathbf{Mod}(\mathscr{O}_X)$. □

1.2.8 Internal Hom

Let $(X, \mathscr{O}_X)$ be a ringed space. Let $\mathscr{F}, \mathscr{G}$ be $\mathscr{O}_X$-modules. Consider the rule

$$U \mapsto \operatorname{Hom}_{\mathscr{O}_X|_U}(\mathscr{F}|_U, \mathscr{G}|_U)$$

It follows from Proposition 1.1.41 that this is a sheaf of abelian groups. In addition, given an element $\varphi \in \operatorname{Hom}_{\mathscr{O}_X|_U}(\mathscr{F}|_U, \mathscr{G}|_U)$ and a section $f \in \mathscr{O}_X(U)$ then we can define $f\varphi \in \operatorname{Hom}_{\mathscr{O}_X|_U}(\mathscr{F}|_U, \mathscr{G}|_U)$ by either precomposing with multiplication by f on $\mathscr{F}|_U$ or postcomposing with multiplication by f on $\mathscr{G}|_U$ (it gives the same result). Hence we in fact get a sheaf of $\mathscr{O}_X$-modules. We will denote this sheaf $\mathcal{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})$. For every $x \in X$ there is also a canonical morphism

$$\mathcal{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})_x \to \operatorname{Hom}_{\mathscr{O}_{X,x}}(\mathscr{F}_x, \mathscr{G}_x)$$

which is rarely an isomorphism.

Proposition 1.2.31. *Let $(X, \mathscr{O}_X)$ be a ringed space. Let $\mathscr{F}, \mathscr{G}, \mathscr{H}$ be $\mathscr{O}_X$-modules. There is a canonical isomorphism*

$$\mathcal{H}om_{\mathscr{O}_X}(\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G}, \mathscr{H}) \to \mathcal{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathcal{H}om_{\mathscr{O}_X}(\mathscr{G}, \mathscr{H}))$$

which is functorial in all three entries. In particular, to give a morphism $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G} \to \mathscr{H}$ is the same as giving a morphism $\mathscr{F} \to \mathcal{H}om_{\mathscr{O}_X}(\mathscr{G}, \mathscr{H})$.

Proof. This is the analogue of that for modules. □

Due to this proposition, the functors $\mathcal{H}om(-, \mathscr{G})$ and $\mathcal{H}om(\mathscr{F}, -)$ are left-exact.

Proposition 1.2.32. *Let $(X, \mathscr{O}_X)$ be a ringed space. Let $\mathscr{F}, \mathscr{G}$ be $\mathscr{O}_X$-modules.*

- *If $\mathscr{F}_2 \to \mathscr{F}_1 \to \mathscr{F} \to 0$ is an exact sequence of $\mathscr{O}_X$-modules, then*

$$0 \longrightarrow \mathcal{H}om(\mathscr{F}, \mathscr{G}) \longrightarrow \mathcal{H}om(\mathscr{F}_1, \mathscr{G}) \longrightarrow \mathcal{H}om(\mathscr{F}_2, \mathscr{G})$$

  is exact.

- *If $0 \to \mathscr{G} \to \mathscr{G}_1 \to \mathscr{G}_2 \to 0$ is an exact sequence of $\mathscr{O}_X$-modules, then*

$$0 \longrightarrow \mathcal{H}om(\mathscr{F}, \mathscr{G}) \longrightarrow \mathcal{H}om(\mathscr{F}, \mathscr{G}_1) \longrightarrow \mathcal{H}om(\mathscr{F}, \mathscr{G}_2)$$

  is exact.

Proposition 1.2.33. *Let $(X, \mathscr{O}_X)$ be a ringed space. Let $\mathscr{F}, \mathscr{G}$ be $\mathscr{O}_X$-modules. If $\mathscr{F}$ is finitely presented then the canonical map*

$$\mathcal{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})_x \to \operatorname{Hom}_{\mathscr{O}_{X,x}}(\mathscr{F}_x, \mathscr{G}_x)$$

is an isomorphism.

Proof. By localizing on X we may assume that $\mathscr{F}$ has a presentation

$$\bigoplus_{j=1}^m \mathscr{O}_X \longrightarrow \bigoplus_{i=1}^n \mathscr{O}_X \longrightarrow \mathscr{F} \longrightarrow 0$$

Then this gives an exact sequence

$$0 \longrightarrow \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{G}) \longrightarrow \bigoplus_{i=1}^n \mathcal{G} \longrightarrow \bigoplus_{j=1}^m \mathcal{G}$$

Taking stalks we get an exact sequence

$$0 \longrightarrow \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})_x \longrightarrow \bigoplus_{i=1}^n \mathcal{G}_x \longrightarrow \bigoplus_{j=1}^m \mathcal{G}_x$$

The result now follows since $\mathcal{F}_x$ sits in an exact sequence

$$\bigoplus_{j=1}^m \mathcal{O}_{X,x} \longrightarrow \bigoplus_{i=1}^n \mathcal{O}_{X,x} \longrightarrow \mathcal{F}_x \longrightarrow 0$$

which induces the exact sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{F}_x,\mathcal{G}_x) \longrightarrow \bigoplus_{i=1}^n \mathcal{G}_x \longrightarrow \bigoplus_{j=1}^m \mathcal{G}_x$$

which is the same as the one above. □

Proposition 1.2.34. *Let $f : X \to Y$ be a morphism of ringed spaces. Let $\mathcal{F},\mathcal{G}$ be $\mathcal{O}_Y$-modules. If $\mathcal{F}$ is finitely presented then the canonical map*

$$f^*\mathcal{H}om_{\mathcal{O}_Y}(\mathcal{F},\mathcal{G}) \to \mathcal{H}om_{\mathcal{O}_X}(f^*\mathcal{F}, f^*\mathcal{G})$$

is an isomorphism.

Proof. Note that $f^*\mathcal{F}$ is also finitely presented. Let $x \in X$ map to $y \in Y$. Looking at the stalks at x we get an isomorphism by Proposition 1.2.33 and that in this case Hom commutes with base change by $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$. □

Proposition 1.2.35. *Let $(X,\mathcal{O}_X)$ be a ringed space. Let $\mathcal{F},\mathcal{G}$ be $\mathcal{O}_X$-modules. If $\mathcal{F}$ is finitely presented then the sheaf $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})$ is locally a kernel of a map between finite direct sums of copies of $\mathcal{G}$. In particular, if $\mathcal{G}$ is coherent then $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})$ is coherent too.*

Proof. The first assertion we saw in the proof of Proposition 1.2.33. And the result for coherent sheaves then follows from Proposition 1.4.18. □

Proposition 1.2.36. *Let X be a topological space. Let $\mathcal{O}_1 \to \mathcal{O}_2$ be a homomorphism of sheaves of rings. Then we have*

$$\mathrm{Hom}_{\mathcal{O}_1}(\mathcal{F}_{\mathcal{O}_1},\mathcal{G}) = \mathrm{Hom}_{\mathcal{O}_2}(\mathcal{F},\mathcal{H}om_{\mathcal{O}_1}(\mathcal{O}_2,\mathcal{G}))$$

*bifunctorially in $\mathcal{F} \in$ **Mod**$(\mathcal{O}_2)$ and $\mathcal{G} \in$ **Mod**$(\mathcal{O}_1)$.*

Proof. This is the analogue of the result for modules. □

1.2.9 The abelian category of sheaves of modules

Let $(X,\mathcal{O}_X)$ be a ringed space. Let $\mathcal{F},\mathcal{G}$ be sheaves of $\mathcal{O}_X$-modules. Let $\varphi,\psi : \mathcal{F} \to \mathcal{G}$ be morphisms of sheaves of $\mathcal{O}_X$-modules. We define $\varphi + \psi : \mathcal{F} \to \mathcal{G}$ to be the map which on each open $U \subseteq X$ is the sum of the maps induced by φ,ψ. This is clearly again a map of sheaves of $\mathscr{O}_X$-modules. It is also clear that composition of maps of $\mathscr{O}_X$-modules is bilinear with respect to this addition. Thus $\mathbf{Mod}(\mathscr{O}_X)$ is a pre-additive category. We will denote 0 the sheaf of $\mathscr{O}_X$-modules which has constant value $\{0\}$ for all open $U \subseteq X$. Clearly this is both a final and an initial object of $\mathbf{Mod}(\mathscr{O}_X)$. Moreover, given a pair $\mathscr{F}, \mathscr{G}$ of sheaves of $\mathscr{O}_X$-modules we may define the direct sum as

$$\mathscr{F} \oplus \mathscr{G} = \mathscr{F} \times \mathscr{G}$$

Thus $\mathbf{Mod}(\mathscr{O}_X)$ is an additive category.

Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism of $\mathscr{O}_X$-modules. We may define the **presheaf kernel** $\ker \varphi$ and the **presheaf cokernel** to be

$$(\ker_p \varphi)(U) = \ker \varphi_U, \quad (\operatorname{coker}_p \varphi)(U) = \operatorname{coker} \varphi_U.$$

for open subsets $U \subseteq X$. We define $\operatorname{coker} \varphi$ be the sheafification of $\operatorname{coker}_p \varphi$.

Proposition 1.2.37. *The presheaf kernel is a presheaf of $\mathscr{O}_X$-modules, so is the presheaf cokernel.*

Proof. For $U \subseteq V$ open in X, consider the diagram

$$\begin{tikzcd} 0 \arrow[r] & \ker_p \varphi_V \arrow[r, hook] \arrow[d, dashed] & \mathscr{F}(V) \arrow[r, "\varphi_V"] \arrow[d, "\mathrm{res}^V_U"] & \mathscr{G}(V) \arrow[d, "\mathrm{res}^V_U"] \\ 0 \arrow[r] & \ker_p \varphi_U \arrow[r, hook] & \mathscr{F}(U) \arrow[r, "\varphi_U"] & \mathscr{G}(U) \end{tikzcd} \tag{2.9.1}$$

By the comutativity, there is a unique map $\mathrm{res}^V_U : \ker \varphi_V \to \ker \varphi_U$ fitting in the diagram. This makes $\ker_p \varphi$ into a presheaf of $\mathscr{O}_X$-modules, and essentially the same argument works for $\operatorname{coker}_p \varphi$. □

Proposition 1.2.38. *The presheaf (co)kernel satisfies the universal property of (co)kernel in the category of presheaves of $\mathscr{O}_X$-modules.*

Proof. Let $\psi : \mathscr{G} \to \mathscr{H}$ be another morphism of presheaves of $\mathscr{O}_X$-modules such that $(\psi \circ \varphi)_U = \psi_U \circ \varphi_U = 0$ for any $U \subseteq X$. Then we have an induced map $\tilde{\psi}_U : \ker \varphi_U \to \mathscr{H}(U)$ for each U. For $U \subseteq V$ open in X, it is easy to verify that the following commutative diagram:

$$\begin{tikzcd} \ker \varphi_V \arrow[r, "\tilde{\psi}_V"] \arrow[d, "\mathrm{res}^V_U"] & \mathscr{H}(V) \arrow[d, "\mathrm{res}^V_U"] \\ \ker \varphi_U \arrow[r, "\tilde{\psi}_U"] & \mathscr{H}(U) \end{tikzcd}$$

so $\ker \varphi$ is the kernel in the category of presheaves. A similar verification works for cokernels. □

Proposition 1.2.39. *Suppose $\varphi : \mathscr{F} \to \mathscr{G}$ is a morphism of sheaves. Then the presheaf kernel* $\ker_p \varphi$ *satisfies the universal property of kernels in* $\mathbf{Mod}(\mathscr{O}_X)$*, and the sheafification of* $\operatorname{coker}_p \varphi$ *satisfies the universal property of cokernels in* $\mathbf{Mod}(\mathscr{O}_X)$.

Proof. Let $U = \bigcup_i U_i$ be an open covering in X, and $s_i \in \ker \varphi(U_i)$ such that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for all i, j. Then by the sheaf condition of $\mathscr{F}$, there is a unique $s \in \mathscr{F}(U)$ such that $s|_{U_i} = s_i$.

Moreover, from the diagram (2.9.1) we get

$$\big(\varphi_U(s)\big)_{U_i} = \varphi_{U_i}(s|_{U_i}) = 0.$$

Thus by the sheaf condition of $\mathscr{G}$, we conclude $\varphi_U(s) = 0$. This implies $s \in \varphi_U$, so $\ker\varphi$ is a sheaf.

For the cokernel, given any sheaf $\mathscr{E}$ and a diagram

$$\begin{tikzcd}
\mathscr{F} \arrow[r, "\varphi"] & \mathscr{G} \arrow[r, "\psi"] \arrow[d] & \mathscr{E} \\
& \operatorname{coker}_p\varphi \arrow[r] \arrow[ur] & (\operatorname{coker}_p\varphi)^{\#} \arrow[u, dashed, "\tilde{\psi}"']
\end{tikzcd}$$

we construct the map $\tilde{\psi}$ by using the universal property of $\operatorname{coker}_p\varphi$ and that of sheafification. □

In view of Proposition 1.2.39, for a morphism $\varphi : \mathscr{F} \longrightarrow \mathscr{G}$, we define the kernel and cokernel of φ by

$$\ker\varphi = \ker_p\varphi, \quad \operatorname{coker}\varphi = (\operatorname{coker}_p\varphi)^{\#}.$$

Since taking stalks commutes with taking sheafification, the following result is immediate.

Proposition 1.2.40. *Suppose $\mathscr{F} \longrightarrow \mathscr{G}$ is a morphism of sheaves of $\mathscr{O}_X$-modules, then for all $x \in X$, we have canonical isomorphisms*

$$(\ker\varphi)_x \cong \ker\varphi_x, \quad (\operatorname{coker}\varphi)_x \cong \operatorname{coker}\varphi_x.$$

Proof. Let $\varphi : \mathscr{F} \longrightarrow \mathscr{G}$ be a morphism of sheaves. We have the commutative diagram:

$$\begin{tikzcd}
\mathscr{F}(U) \arrow[r, "\varphi_U"] \arrow[d] & \mathscr{G}(U) \arrow[d] \\
\mathscr{F}_x \arrow[r, "\varphi_x"] & \mathscr{G}_x
\end{tikzcd}$$

Let $s \in \ker\varphi_U$, then its image $\bar{s}$ is mapped to zero by φ_x, so $\bar{s} \in \ker\varphi_x$. Conversely, if $\bar{s} \in \ker\varphi_x$ and $\bar{s} = (U, s)$ where $s \in \mathscr{F}(U)$, then the image of $t = \varphi(s) \in \mathscr{G}(U)$ is zero in $\mathscr{G}_x$. Therefore in some neighborhood, say $V \subseteq U$, $t|_V = 0$. Then we have the commutative diagram

$$\begin{tikzcd}
\mathscr{F}(V) \arrow[r, "\varphi_V"] \arrow[d] & \mathscr{G}(V) \arrow[d] \\
\mathscr{F}_x \arrow[r, "\varphi_x"] & \mathscr{G}_x
\end{tikzcd}$$

where now $\varphi_V(s|_V) = 0$. This shows $\bar{s} \in (\ker\varphi)_p$. The proof is similar for cokernels. □

Now that we kernel and cokernels, we can prove that $\mathbf{Mod}(\mathscr{O}_X)$ is an abelian category.

Theorem 1.2.41. *Let $(X, \mathscr{O}_X)$ be a ringed space. The category $\mathbf{Mod}(\mathscr{O}_X)$ is an abelian category. Moreover, a complex $\mathscr{F} \longrightarrow \mathscr{G} \longrightarrow \mathscr{H}$ is exact at $\mathscr{G}$ if and only if for all $x \in X$ the complex $\mathscr{F}_x \longrightarrow \mathscr{G}_x \longrightarrow \mathscr{H}_x$ is exact at $\mathscr{G}_x$.*

Proof. We have to show that image and coimage agree. By Proposition 1.1.10 it is enough to show that image and coimage have the same stalk at every $x \in X$. By the constructions of kernels and cokernels above these stalks are the coimage and image in the categories of $\mathscr{O}_{X,x}$-modules. Thus we get the result from the fact that the category of modules over a ring is abelian. □

Actually the category $\mathbf{Mod}(\mathscr{O}_X)$ has many more properties. Here are two constructions we can do.

- Given any set I and for each $i \in I$ a $\mathscr{O}_X$-module we can form the product

  $$\prod_{i \in I} \mathscr{F}_i$$

  which is the sheaf that associates to each open U the product of the modules $\mathscr{F}_i(U)$.

- Given any set I and for each $i \in I$ a $\mathscr{O}_X$-module we can form the direct sum

  $$\bigoplus_{i \in I} \mathscr{F}_i$$

  which is the sheafification of the presheaf that associates to each open U the direct sum of the modules $\mathscr{F}_i(U)$.

Using these we conclude that all limits and colimits exist in $\mathbf{Mod}(\mathscr{O}_X)$.

Proposition 1.2.42. *Let $(X, \mathscr{O}_X)$ be a ringed space.*

(a) All limits exist in $\mathbf{Mod}(\mathscr{O}_X)$. Limits are the same as the corresponding limits of presheaves of $\mathscr{O}_X$-modules.

(b) All colimits exist in $\mathbf{Mod}(\mathscr{O}_X)$. Colimits are the sheafification of the corresponding colimit in the category of presheaves. Taking colimits commutes with taking stalks.

(c) Filtered colimits are exact.

(d) Finite direct sums are the same as the corresponding finite direct sums of presheaves of $\mathscr{O}_X$-modules.

Proof. As $\mathbf{Mod}(\mathscr{O}_X)$ is abelian it has all finite limits and colimits. Thus the existence of limits and colimits and their description follows from the existence of products and coproducts and their description. Since sheafification commutes with taking stalks we see that colimits commute with taking stalks. Part (c) signifies that given a system $0 \to \mathscr{F}_i \to \mathscr{G}_i \to \mathscr{H}_i \to 0$ of exact sequences of $\mathscr{O}_X$-modules over a directed partially ordered set I the sequence $0 \to \varinjlim_i \mathscr{F}_i \to \varinjlim_i \mathscr{G}_i \to \varinjlim_i \mathscr{H}_i \to 0$ is exact as well. Since we can check exactness on stalks, this follows from the case of modules. Part (d) comes from the fact that finite direct sum coincides with finite product in an Abelian category. □

Remark 1.2.1. For an arbitrary direct sum $\bigoplus_{i\in I}\mathscr{O}_X$, by the construction of sheafification we see that an element $s \in \bigoplus_{i\in I}\mathscr{O}_X$ satisfies the following property: For any $x \in X$ there is a neighborhood of x such that $s|_U$ is a finite sum $\sum_{i\in I'} f_i$ with $f_i \in \mathscr{O}_X(U)$.

The existence of limits and colimits allows us to consider exactness properties of functors defined on the category of $\mathscr{O}_X$-modules in terms of limits and colimits.

Proposition 1.2.43. *Let $(f, f^{\#}) : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ be a morphism of ringed spaces.*

(a) The functor $f_* : \mathbf{Mod}(\mathscr{O}_X) \to \mathbf{Mod}(\mathscr{O}_Y)$ is left exact. In fact it commutes with all limits.

(b) The functor $f^* : \mathbf{Mod}(\mathscr{O}_X) \to \mathbf{Mod}(\mathscr{O}_Y)$ is right exact. In fact it commutes with all colimits.

(c) The functor $f^{-1} : \mathbf{Ab}(Y) \to \mathbf{Ab}(X)$ on abelian sheaves is exact.

Proof. Recall that (f^*, f_*) is an adjoint pair of functors. The last part holds because exactness can be checked on stalks and the description of stalks of the pullback. $\square$

Proposition 1.2.44. *Let $j : U \to X$ be an open immersion of topological spaces. The functor $j_!$: $\mathbf{Ab}(U) \to \mathbf{Ab}(X)$ is exact.*

Proof. This follows from the description of stalks given in Proposition 1.1.33. $\square$

Proposition 1.2.45. *Let $(X, \mathscr{O}_X)$ be a ringed space. Let I be a set. For $i \in I$, let $\mathscr{F}_i$ be a sheaf of $\mathscr{O}_X$-modules. For $U \subseteq X$ quasi-compact open the map*

$$\bigoplus_{i\in I}\mathscr{F}_i(U) \to \Big(\bigoplus_{i\in I}\mathscr{F}_i\Big)(U)$$

is bijective.

Proof. If s is an element of the right hand side, then there exists an open covering $U = \bigcup_{j\in J} U_j$ such that $s|_{U_j}$ is a finite sum $\sum_{i\in I_j} s_{ji}$ with $s_{ji} \in \mathscr{F}_i(U_j)$. Because U is quasi-compact we may assume that the covering is finite, i.e., that J is finite. Then $I' = \bigcup_{j\in J} I_j$ is a finite subset of I. Clearly, s is a section of the subsheaf $\bigoplus_{i\in I'}\mathscr{F}_i$. The result follows from the fact that for a finite direct sum sheafification is not needed. $\square$

1.3 Ringed spaces

1.3.1 Ringed spaces, $\mathscr{A}$-modules, and $\mathscr{A}$-algebras

A **ringed space** (resp. **topologically ringed space**) is defiend to be a couple $(X, \mathscr{A})$ formed by a topological space X and a sheaf of rings (resp. a sheaf of topological rings) $\mathscr{A}$ on X. We call X the topological space underling the ringed space $(X, \mathscr{A})$, and $\mathscr{A}$ is the structural sheaf. We only denote by $\mathscr{O}_X$ the structural sheaf, and for $x \in X$, $\mathscr{O}_{X,x}$ denotes the stalk of $\mathscr{O}_X$ at x. If $\mathscr{A}$ is a sheaf of commutative rings, we say $(X, \mathscr{A})$ is a **commutative ringed space**. Without further specifications, we only consider commutative ringed spaces.

The ringed spaces (resp. topologically ringed spaces) form a category, if we define a morphism $(X, \mathscr{A}) \to (Y, \mathscr{B})$ as a couple $(f, f^{\#})$ formed by a continuous map $f : X \to Y$ and a

f-morphism $f^{\#} : \mathscr{B} \to \mathscr{A}$ (that is, a morphism from $\mathscr{B}$ to $f_*(\mathscr{A})$) of sheaf of rings (resp. sheaf of topological rings). As the category of rings admits inductive limits, for any $x \in X$ we have a homomorphism $f_x^{\#} : \mathscr{B}_{f(x)} \to \mathscr{A}_x$. The composition of two morphisms $(f, f^{\#}) : (X, \mathscr{A}) \to (Y, \mathscr{B})$ and $(g, g^{\#}) : (Y, \mathscr{B}) \to (Z, \mathscr{C})$ is then defined to be the couple $(h, h^{\#})$, where $h = g \circ f$ and $g^{\#}$ is the composition of $f^{\#}$ and $g^{\#}$ (which equals to $g_*(f^{\#}) \circ g^{\#}$). For any $x \in X$, we then have $h_x^{\#} = g_{f(x)}^{\#} \circ f_x^{\#}$, so if the homomorphisms $f^{\#}$ and $f^{\#}$ are injective (resp. surjective), then so is $h^{\#}$. We then verify that, if f is a injective continuous map and $f^{\#}$ is a surjective homomorphism of sheaf of rings, then the morphism $(f, f^{\#})$ is a monomorphism of category of ringed spaces.

By abuse of language, we only replace $(f, f^{\#})$ by f, and say that $f : (X, \mathscr{A}) \to (Y, \mathscr{B})$ is a morphism of ringed spaces. In this case, it is understood that the homomorphism $f^{\#}$ is also given.

For any subset U of X, the couple $(U, \mathscr{A}|_U)$ is clearly a ringe space, called induced over U by $(X, \mathscr{A})$, or the restriction of $(X, \mathscr{A})$ to U. If $j : U \to X$ is the injection and $j^{\#} : \mathscr{A} \to j_*(\mathscr{A}|_U)$ is the canonical homomorphism, we then have a *monomorphism* $(j, j^{\#}) : (U, \mathscr{A}|_U) \to (X, \mathscr{A})$ of ringed spaces, called the **canonical injection**. The composition of a morphism $f : (X, \mathscr{A}) \to (Y, \mathscr{B})$ with this canonical injection is said to be the restriction of f to U, and denoted by $f|_U$.

Example 1.3.1. Let $f : X \to Y$ be a continuous map of topological spaces. Consider the sheaves of continuous real valued functions C_X^0 on X and C_Y^0 on Y. We claim that there is a natural f-map $f^{\#} : C_Y^0 \to C_X^0$ associated to f. Namely, we simply define it by the rule

$$C_Y^0(V) \to C_X^0(f^{-1}(V)), \quad h \mapsto h \circ f$$

Strictly speaking we should write $f^{\#}(h) = h \circ f|_{f^{-1}(V)}$. It is clear that this is an f-map of sheaves of $\mathbb{R}$-algebras.

Of course there are lots of other situations where there is a canonical morphism of ringed spaces associated to a geometrical type of morphism. For example, if M, N are C^∞-manifolds and $f : M \to N$ is a smooth map, then f induces a canonical morphism of ringed spaces $(M, C_M^\infty) \to (N, C_N^\infty)$.

We will not review the definition of the $\mathscr{A}$-modules for a ringed space $(X, \mathscr{A})$. If $\mathscr{A}$ is a sheaf of commutative rings, and we replace the module structure by the algebra structure in the definition of $\mathscr{A}$-modules, we obtain the definition of an $\mathscr{A}$-algebra over X. In other words, an $\mathscr{A}$-algebra (not necessarily commutative) is an $\mathscr{A}$-module $\mathscr{C}$ endowed with a homomorphism of $\mathscr{A}$-modules $\varphi : \mathscr{C} \otimes_{\mathscr{A}} \mathscr{C} \to \mathscr{C}$ and of a section e above X, such that:

(i) the diagram

$$\begin{tikzcd} \mathscr{C} \otimes_{\mathscr{A}} \mathscr{C} \otimes_{\mathscr{A}} \mathscr{C} \arrow[r, "\varphi\otimes 1"] \arrow[d, "1\otimes\varphi"'] & \mathscr{C} \otimes_{\mathscr{A}} \mathscr{C} \arrow[d, "\varphi"] \\ \mathscr{C} \otimes_{\mathscr{A}} \mathscr{C} \arrow[r, "\varphi"] & \mathscr{C} \end{tikzcd}$$

is commutative;

(ii) for any open subset $U \subseteq X$ and any section $s \in \Gamma(U, \mathscr{C})$, we have $\varphi((e|_U) \otimes s) = \varphi(s \otimes (e|_U)) = s$.

Saying that $\mathscr{C}$ is a commutative $\mathscr{A}$-algebra amounts to the fact that the diagram

$$\begin{tikzcd}
\mathscr{C} \otimes_{\mathscr{A}} \mathscr{C} \arrow[rr, "\sigma"] \arrow[dr, "\varphi"'] & & \mathscr{C} \otimes_{\mathscr{A}} \mathscr{C} \arrow[dl, "\varphi"] \\
& \mathscr{C} &
\end{tikzcd}$$

is commutative, where σ is the canonical symmetry of the tensor producr $\mathscr{C} \otimes_{\mathscr{A}} \mathscr{C}$.

The homomorphism of $\mathscr{A}$ are defined just as that of $\mathscr{A}$-modules, where we replace the "modules" by "algebras". If $\mathscr{M}$ is a sub-$\mathscr{A}$-module of an $\mathscr{A}$-algebra $\mathscr{C}$, the sub-$\mathscr{A}$-algebra of $\mathscr{C}$ **generated by** $\mathscr{M}$ is the sum of images of the homomorphisms $\otimes^n \mathscr{M} \to \mathscr{C}$ (for $n \geq 0$). This is the sheaf associated with the presheaf $U \mapsto \mathscr{B}(U)$ of algebras, where $\mathscr{B}(U)$ is the sub-algebra of $\mathscr{C}(U)$ generated by the sub-module $\mathscr{M}(U)$.

We say a sheaf of rings $\mathscr{A}$ over a topological space X is **reduced** (resp. **integral**) at a point x of X if the stalk $\mathscr{A}_x$ is a reduced ring (resp. integral ring). We say that $\mathscr{A}$ is reduced if it is reduced at every point of X. Recall that a ring A is called regular if for any prime ideal $\mathfrak{p}$ of A, the local ring $A_{\mathfrak{p}}$ is a regular local ring. We say a sheaf of ring $\mathscr{A}$ over X is **regular at a point x** (resp. **regular**) if the stalk $\mathscr{A}_x$ is a regular local ring (resp. if $\mathscr{A}$ is regular at every point). Finally, we say that a sheaf of rings $\mathscr{A}$ over X is **normal at a point x** (resp. **normal**) if the stalk $\mathscr{A}_x$ is an integrally closed ring (resp. if $\mathscr{A}$ is normal at every point). We say the ringed space $(X, \mathscr{A})$ is reduced (resp. normal, regular) if the structural sheaf $\mathscr{A}$ satisfies this property.

A **sheaf of graded rings** is by definition a sheaf or rings $\mathscr{A}$ which is the direct sum of a family $(\mathscr{A}_n)_{n\in\mathbb{Z}}$ of sheaves of abelian groups which satisfies the condition $\mathscr{A}_n\mathscr{A}_m \subseteq \mathscr{A}_{m+n}$. A graded $\mathscr{A}$-module is an $\mathscr{A}$-module $\mathscr{F}$ which is a direct sum of a family $(\mathscr{F}_n)_{n\in\mathbb{Z}}$ of sheaves of abelian groups, satisfying $\mathscr{A}_m\mathscr{F}_n \subseteq \mathscr{F}_{m+n}$.

Given a ringed space $(X, \mathscr{A})$ (commutative, and we shall not specify this condition further), we recall that definition of the bifunctors $\mathscr{F} \otimes_{\mathscr{A}} \mathscr{G}$, $\mathscr{H}om_{\mathscr{A}}(\mathscr{F}, \mathscr{G})$, and $\mathrm{Hom}_{\mathscr{A}}(\mathscr{F}, \mathscr{G})$ on the category of $\mathscr{A}$-modules, with values in the category of sheaves of abelian groups. the stalk $(\mathscr{F} \otimes_{\mathscr{A}} \mathscr{G})_x$ at any point $x \in X$ is identified canonically with $\mathscr{F}_x \otimes_{\mathscr{A}_x} \mathscr{G}_x$ and we define a functorial homomorphism $(\mathscr{H}om_{\mathscr{A}}(\mathscr{F}, \mathscr{G}))_x \mapsto \mathrm{Hom}_{\mathscr{A}_x}(\mathscr{F}_x, \mathscr{G}_x)$ which is, in general, neither injective nor surjective. These bifunctors are additive and, in particular, commutes with finite direct sums. The functor $\mathscr{F} \otimes_{\mathscr{A}} \mathscr{G}$ is right exact on $\mathscr{F}$ and $\mathscr{G}$, commutes with inductive limits, and $\mathscr{A} \otimes_{\mathscr{A}} \mathscr{G}$ (resp. $\mathscr{F} \otimes_{\mathscr{A}} \mathscr{A}$) is canonically identified with $\mathscr{G}$ (resp. $\mathscr{F}$). The functor $\mathscr{H}om_{\mathscr{A}}(\mathscr{F}, \mathscr{G})$ and $\mathrm{Hom}_{\mathscr{A}}(\mathscr{F}, \mathscr{G})$ are left exact on $\mathscr{F}$ and $\mathscr{G}$ (but note that $\mathscr{H}om_{\mathscr{A}}(-, \mathscr{G})$ and $\mathrm{Hom}_{\mathscr{A}}(\mathscr{F}, \mathscr{G})$ are contravariant). Moreover, $\mathscr{H}om_{\mathscr{A}}(\mathscr{A}, \mathscr{G})$ is canonically identified with $\mathscr{G}$, and for open subset $U \subseteq X$, we have

$$\Gamma(U, \mathscr{H}om_{\mathscr{A}}(\mathscr{F}, \mathscr{G})) = \mathrm{Hom}_{\mathscr{A}|_U}(\mathscr{F}|_U, \mathscr{G}|_U).$$

In particular, $\mathrm{Hom}_{\mathscr{A}}(\mathscr{A}, \mathscr{G})$ is identified with $\Gamma(X, \mathscr{G})$. For any $\mathscr{A}$-module $\mathscr{F}$, we denote by $\mathscr{F}^*$ the **dual** of $\mathscr{F}$, which is $\mathscr{H}om_{\mathscr{A}}(\mathscr{F}, \mathscr{A})$.

Finally, if $\mathscr{A}$ is a sheaf of rings and $\mathscr{F}$ is an $\mathscr{A}$-module, then $U \mapsto \bigwedge^p \Gamma(U, \mathscr{F})$ is a presheaf whose associated sheaf is an $\mathscr{A}$-module and is denoted by $\bigwedge^p \mathscr{F}$, called the **p-th exterior power** of $\mathscr{F}$. We can easily verify that the cannical map from the presheaf $U \mapsto \bigwedge^p \Gamma(U, \mathscr{F})$ to $\bigwedge^p \mathscr{F}$ is injective, and for $x \in X$, we have $(\bigwedge^p \mathscr{F})_x = \bigwedge^p (\mathscr{F}_x)$. It is clear that $\bigwedge^p \mathscr{F}$ is a covariant functor on $\mathscr{F}$. We can similarly define the functors $\boldsymbol{T}_p(\mathscr{F})$ and $\boldsymbol{S}_p(\mathscr{F})$, which are the **$p$-th tensor power** and p**-th symmetric power** of $\mathscr{F}$.

Let $\mathscr{I}$ be an ideal of $\mathscr{A}$ and $\mathscr{F}$ be an $\mathscr{A}$-module. Then we note that $\mathscr{I}\mathscr{F}$, the image of $\mathscr{I}\otimes_{\mathscr{A}}\mathscr{F}$ by the canonical map $\mathscr{I}\otimes_{\mathbb{Z}}\mathscr{F}\to\mathscr{F}$, is a sub-$\mathscr{A}$-module of $\mathscr{F}$. It is clear that for any $x\in X$, we have $(\mathscr{I}\mathscr{F})_x=\mathscr{I}_x\mathscr{F}_x$. It is immediate that $\mathscr{I}\mathscr{F}$ is also the $\mathscr{A}$-module associated sheaf of the presheaf $U\mapsto\Gamma(U,\mathscr{I})\Gamma(U,\mathscr{F})$. If $\mathscr{I}_1$, $\mathscr{I}_2$ are ideals of $\mathscr{A}$, we have $\mathscr{I}_1(\mathscr{I}_2\mathscr{F})=(\mathscr{I}_1\mathscr{I}_2)\mathscr{F}$.

Let $(X_\lambda,\mathscr{A}_\lambda)_{\lambda\in L}$ be a family of ringed spaces; for each couple (λ,μ), suppose that we are given an open subset $V_{\lambda\mu}\subseteq X_\lambda$, and an isomorphism $\varphi_{\lambda\mu}:(V_{\mu\lambda},\mathscr{A}_\mu|_{V_{\mu\lambda}})\to(V_{\lambda\mu},\mathscr{A}_\lambda|_{V_{\lambda\mu}})$ of ringed spaces, with $V_{\lambda\lambda}=X_\lambda$ and $\varphi_{\lambda\lambda}$ being the identity. Suppose moreover that, for any triple (λ,μ,ν), we have $\varphi_{\lambda\nu}=\varphi_{\lambda\mu}\circ\varphi_{\mu\nu}$ on the open subset $V_{\lambda\mu}\cap V_{\lambda\nu}$ (glueing condition for $\varphi_{\lambda\mu}$). We can then consider the topological space obtained by glueing (via the morphism $\varphi_{\lambda\mu}$) the X_λ along $V_{\lambda\mu}$. If we identify X_λ with the corresponding open subset X'_λ of X, the hypotheses implies that $V_{\lambda\mu}\cap V_{\lambda\nu}$, $V_{\mu\nu}\cap V_{\mu\lambda}$, $V_{\nu\lambda}\cap V_{\nu\mu}$ are identified with $X'_\lambda\cap X'_\mu\cap X'_\nu$. We an then transport the ringed space structure of X_λ to X'_λ, and if $\mathscr{A}'_\lambda$ is the sheaf of rings transported by $\mathscr{A}_\lambda$, the $\mathscr{A}'_\lambda$ satisfies the glueing condition for sheaves and define a sheaf of rings $\mathscr{A}$ over X. We say that $(X,\mathscr{A})$ is the ringed space obtained by glueing $(X_\lambda,\mathscr{A}_\lambda)$ along $V_{\lambda\mu}$ via the morphisms $\varphi_{\lambda\mu}$.

Let $(X,\mathscr{O}_X)$ be a ringed space. For a section $s\in\Gamma(U,\mathscr{I}_X)$ over an open subset U of X to be **invertible**, it is necessary and sufficient that for any open cover (U_α) of U, the restriction of s to U_α is invertible in $\Gamma(U_\alpha,\mathscr{O}_X)$, in view of the uniqueness of inverse element. There then exists a sub-sheaf of multiplication groups $\mathscr{O}_X^\times$ of $\mathscr{O}_X$ such that, for any open subset U of X, $\Gamma(U,\mathscr{O}_X^\times)$ is the group of invertible elements of the ring $\Gamma(U,\mathscr{O}_X)$. For any $x\in X$, the stalk $(\mathscr{O}_X^\times)_x$ is the set of invertible elements of the ring $\mathscr{O}_{X,x}$, because if $s_x\in\mathscr{O}_{X,x}$ admits an inverse t_x in this ring, s_x and t_x are the germs of two sections s,t of $\mathscr{O}_X$ over a neighborhood V of x, and the relation $(st)_x=1_x$ implies $st|_W=1$ over a smaller neighborhood $W\subseteq V$ of x.

On the other hand, with the same notations, if s is a regular element of $\Gamma(U,\mathscr{O}_X)$ and V is an open subset of U, $s|_V$ is not necessarily regular in $\Gamma(V,\mathscr{O}_X)$, because if $(s|_V)t=0$ for a section $t\in\Gamma(V,\mathscr{O}_X)$, t does not necessarily admits an extension to U. We denote by $\mathscr{S}(\mathscr{O}_X)$ the presheaf of sets such that $\mathscr{S}(\mathscr{O}_X)(U)$, for each open subset U of X, is the set of sections $s\in\Gamma(U,\mathscr{O}_X)$ whose restriction to *any* open subset $V\subseteq U$ is a regular element of the ring $\Gamma(V,\mathscr{O}_X)$. From this definition, it is clear that $\mathscr{S}(\mathscr{O}_X)$ is a sheaf, since if s is a section of $\mathscr{O}_X$ over U and $s|_V$ is regular in $\Gamma(V,\mathscr{O}_X)$ is regular for some open subset $V\subseteq U$, then s is regular in $\Gamma(U,\mathscr{O}_X)$. For a section $s\in\Gamma(U,\mathscr{O}_X)$, to be in $\mathscr{S}(\mathscr{O}_X)(U)$ amount to saying that for any open subset $V\subseteq U$, the map $t\mapsto(s|_V)$ on $\Gamma(V,\mathscr{O}_X)$ is injective (recall that we always assume that $\mathscr{O}_X$ is commutative). In this case, by the exactness of the functor $\varinjlim$ on the category of modules, it follows that for any $x\in U$, the germ s_x is regular in $\mathscr{O}_{X,x}$. Conversely, if this does not hold, then for some $x\in U$, then there exists a section $t\in\Gamma(V,\mathscr{O}_X)$ for some open subset $V\subseteq U$ such that $(s|_V)t=0$, so $s\notin\mathscr{S}(\mathscr{O}_X)(U)$. We then say that $\mathscr{S}(\mathscr{O}_X)$ is the sheaf of sets defined by the condition that $\Gamma(U,\mathscr{O}_X)$ is the set of sections of $\Gamma(U,\mathscr{O}_X)$ whose germ at every point $x\in U$ is regular. We also note that the stalk $(\mathscr{S}(\mathscr{O}_X))_x$, which is contained in the set of regular elements of $\mathscr{O}_{X,x}$, does not necessarily equal to this set.

We say a ringed space $(X,\mathscr{O}_X)$ is **local** if for each $x\in X$, the stalk $\mathscr{O}_{X,x}$ is a local ring. In this case, we say $(X,\mathscr{O}_X)$ is a **locally ringed space**. We denote by $\mathfrak{m}_x$ the maximal idea lof $\mathscr{O}_{X,x}$ and $\kappa(x)=\mathscr{O}_{X,x}/\mathfrak{m}_x$ the residue field. For any $\mathscr{O}_X$-module $\mathscr{F}$, any open subset $U\subseteq X$, any point $x\in U$ and any section $f\in\Gamma(U,\mathscr{F})$, we denote by $f(x)\in\kappa(x)$ the class of the germ $f_x\in\mathscr{F}_x$

modulo $\mathfrak{m}_x\mathscr{F}_x$, and we say $f(x)$ is the **value** of f at x. The relation $f(x) = 0$ then signifies $f_x \in \mathfrak{m}_x\mathscr{F}_x$ (do not confuse with the condition $f_x = 0_x$). We denote by U_f the set of $x \in U$ such that $f(x) \neq 0$ (or equivalently $f_x \notin \mathfrak{m}_x\mathscr{F}_x$). We note that if $f \in \Gamma(X, \mathscr{F})$, then X_f is contained in $\mathrm{supp}(\mathscr{F})$.

Proposition 1.3.2. *Let $(X, \mathscr{O}_X)$ be a locally ringed space, $\mathscr{F}$ and $\mathscr{F}$ be two $\mathscr{O}_X$-module. For any $x \in X$, $(\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G})_x/\mathfrak{m}_x(\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G})_x$ is canonically identified with $(\mathscr{F}_x/\mathfrak{m}_x\mathscr{F}_x) \otimes_{\kappa(x)} (\mathscr{G}_x/\mathfrak{m}_x\mathscr{G}_x)$. If s (resp. t) is a section of $\mathscr{F}$ (resp. $\mathscr{G}$) over X, $(s \otimes t)(x)$ is identified with $s(x) \otimes t(x)$ and we have $X_{s\otimes t} = X_s \cap X_t$.*

Proof. In fact $(\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G})_x = \mathscr{F}_x \otimes_{\mathscr{O}_{X,x}} \mathscr{G}_x$, and $(\mathscr{F}_x \otimes_{\mathscr{O}_{X,x}} \mathscr{G}_x) \otimes_{\mathscr{O}_{X,x}} (\mathscr{O}_{X,x}/\mathfrak{m}_x)$ is canonically isomorphic to $(\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G})_x/\mathfrak{m}_x(\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G})_x$, hence to $(\mathscr{F}_x/\mathfrak{m}_x\mathscr{F}_x) \otimes_{\kappa(x)} (\mathscr{G}_x/\mathfrak{m}_x\mathscr{G}_x)$. The relation $X_{s\otimes t} = X_s \cap X_t$ then follows from the fact that a product $a \otimes b$ in a tensor product of vector spaces is nonzero if and only if a and b are both nonzero. ◻

Proposition 1.3.3. *Let $(X, \mathscr{O}_X)$ be a locally ringed space. The set of idempotents of the ring $\Gamma(X, \mathscr{O}_X)$ corresponds to the set of clopen subsets of X. In particular, for X to be connected, it is necessary and sufficent that $\Gamma(X, \mathscr{O}_X)$ has no nontrivial idempotents.*

Proof. Let U be a clopen subset of X. Then it corresponds to the section $s \in \Gamma(X, \mathscr{O}_X)$ such that $s|_V = 1$ for any open subset $V \subseteq U$ and $s|_V = 0$ for any open subset $V \subseteq X - U$; these open subsets by hypotheses form a base for the topology of X, so we define a section $s = e_U$ of $\mathscr{O}_X$ over X, which is an idempotent of $\Gamma(X, \mathscr{O}_X)$. Conversely, if s is an idempotent, for any $x \in X$, s_x is an idempotent of $\mathscr{O}_{X,x}$, hence equal to 0_x or 1_x because $\mathscr{O}_{X,x}$ is a local ring ($s_x(1 - s_x) = 0$, and if $s_x \in \mathfrak{m}_x$, then $1 - s_x$ is invertible so $s_x = 0$). It is clear that the set U of $x \in X$ such that $s_x = 1_x$ is open, and so is the set $X - U$ of $x \in X$ such that $s_x = 0_x$, so U is a clopen subset of X and we have $s = e_U$. ◻

A morphism of locally ringed space $f : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ is defined to be a morphism of ringed spaces such that for each $x \in X$ the homomorphism $f_x^\# : \mathscr{O}_{Y,f(x)} \to \mathscr{O}_{X,x}$ is local. Note that by this definition, the category of locally ringed spaces is not a full subcategory of that of ringed spaces.

Proposition 1.3.4. *Let $f : X \to Y$ be a morphism of locally ringed spaces, x be a point of X, and $y = f(x)$. For any $\mathscr{O}_Y$-module $\mathscr{G}$, $(f^*(\mathscr{G}))_x/\mathfrak{m}_x(f^*(\mathscr{G}))_x$ is canonically identified with $(\mathscr{G}_y/\mathfrak{m}_y\mathscr{G}_y) \otimes_{\kappa(y)} \kappa(x)$. If t is a section of $\mathscr{G}$ over Y and $s = \rho_{\mathscr{G}}(t)$ is the corresponding section of $f^*(\mathscr{G})$ over X, then $s(x)$ is identified with $t(y) \otimes 1$ and we have $X_s = f^{-1}(Y_t)$.*

Proof. We have $(f^*(\mathscr{G}))_x = \mathscr{G}_y \otimes_{\mathscr{O}_{Y,y}} \mathscr{O}_X$, so $(f^*(\mathscr{G}))_x/\mathfrak{m}_x(f^*(\mathscr{G}))_x$ is identified with $\mathscr{G}_y \otimes_{\mathscr{O}_{Y,y}} \kappa(x)$. The homomorphism $\mathscr{O}_{Y,y} \to \mathscr{O}_{X,x}$ is local by hypotheses, so $\mathfrak{m}_y$ annihilates $\kappa(x)$ and $\mathscr{G}_y \otimes_{\mathscr{O}_{Y,y}} \kappa(x)$ is isomorphic to $(\mathscr{G}_y/\mathfrak{m}_y\mathscr{G}_y) \otimes_{\kappa(y)} \kappa(x)$. The last assertion follows from the fact that $t(y) \otimes 1 = 0$ is equivalent to $t(y) = 0$. ◻

1.3.2 Direct image of $\mathscr{A}$-modules

Let $(X, \mathscr{A})$ and $(Y, \mathscr{B})$ be ringed spaces, f be a morphism $(X, \mathscr{A}) \to (Y, \mathscr{B})$. Then $f_*(\mathscr{A})$ is a sheaf of rings over Y, and $f^\#$ is a homomorphism $\mathscr{B} \to f_*(\mathscr{A})$ of sheaf of rings. Let $\mathscr{F}$ be an

$\mathscr{A}$-module; the direct image $f_*(\mathscr{F})$ is then a sheaf of abelian groups over Y. Moreover, for any open subset $U \subseteq Y$,

$$\Gamma(U, f_*(\mathscr{F})) = \Gamma(f^{-1}(U), \mathscr{F})$$

is endowed with a module structure over the ring $\Gamma(U, f_*(\mathscr{A})) = \Gamma(f^{-1}(U), \mathscr{A})$. These bilinear maps are compatible with restrictions, so $f_*(\mathscr{F})$ becomes an $f_*(\mathscr{A})$-module. The homomorphism $f^{\#} : \mathscr{B} \to f_*(\mathscr{A})$ then makes $f_*(\mathscr{F})$ a $\mathscr{B}$-module. We say that this $\mathscr{B}$-module is the direct image of $\mathscr{F}$ under the morphism f, still denoted by $f_*(\mathscr{F})$. If $\mathscr{F}_1$, $\mathscr{F}_2$ are two $\mathscr{A}$-modules over X and $u : \mathscr{F}_1 \to \mathscr{F}_2$ is an $\mathscr{A}$-homomorphism, it is immediate that $f_*(u)$ is an $f_*(\mathscr{A})$-homomorphism $f_*(\mathscr{F}_1) \to f_*(\mathscr{F}_2)$, and a fortiori a $\mathscr{B}$-homomorphism, also denoted by $f_*(u)$. We then see that f_* is a covariant funtor from the category of $\mathscr{A}$-modules to that of $\mathscr{B}$-modules. Moreover, it is immediate that this functor is left exact.

Over $f_*(\mathscr{A})$, the $\mathscr{B}$-module structure and the sheaf of rings structure define a structure of $\mathscr{B}$-algebras; we denote by $f_*(\mathscr{A})$ this $\mathscr{B}$-algebra. Let $(Z, \mathscr{C})$ be a third ringed space, $g : (Y, \mathscr{B}) \to (Z, \mathscr{C})$ be a morphism; if $h = g \circ f$ is the composition morphism, then we have $h_* = g_* \circ f_*$.

Let $\mathscr{M}, \mathscr{N}$ be two $\mathscr{A}$-modules. For any open subset U of Y, we have a canonical map

$$\Gamma(f^{-1}(U), \mathscr{M}) \times \Gamma(f^{-1}(U), \mathscr{N}) \to \Gamma(f^{-1}(U), \mathscr{M} \otimes_{\mathscr{A}} \mathscr{N})$$

which is bilinear over the ring $\Gamma(f^{-1}(U), \mathscr{A}) = \Gamma(U, f_*(\mathscr{A}))$, and a fortiori over $\Gamma(U, \mathscr{B})$; this defines a homomorphism

$$\Gamma(U, f_*(\mathscr{M})) \otimes_{\Gamma(U,\mathscr{B})} \Gamma(U, f_*(\mathscr{N})) \to \Gamma(U, f_*(\mathscr{M} \otimes_{\mathscr{A}} \mathscr{N}))$$

and as we can verify that these homomorphisms are compatible with restrictions, we obtain a canonical homomorphism of $\mathscr{B}$-modules

$$f_*(\mathscr{M}) \otimes_{\mathscr{B}} f_*(\mathscr{N}) \to f_*(\mathscr{M} \otimes_{\mathscr{A}} \mathscr{N}) \tag{3.2.1}$$

which in general is neither injective nor surjective. If $\mathscr{P}$ is a third $\mathscr{A}$-module, we verify that the diagram

$$\begin{tikzcd}
f_*(\mathscr{M}) \otimes_{\mathscr{B}} f_*(\mathscr{N}) \otimes_{\mathscr{B}} f_*(\mathscr{P}) \arrow[r] \arrow[d] & f_*(\mathscr{M} \otimes_{\mathscr{A}} \mathscr{N}) \otimes_{\mathscr{B}} f_*(\mathscr{P}) \arrow[d] \\
f_*(\mathscr{M}) \otimes_{\mathscr{B}} f_*(\mathscr{N} \otimes_{\mathscr{A}} \mathscr{P}) \arrow[r] & f_*(\mathscr{M} \otimes_{\mathscr{A}} \mathscr{N} \otimes_{\mathscr{A}} \mathscr{P})
\end{tikzcd}$$

is commutative.

Let $\mathscr{M}, \mathscr{N}$ be two $\mathscr{A}$-modules. For any open $U \subseteq Y$, we have by definition

$$\Gamma(f^{-1}(U), \mathscr{H}om_{\mathscr{A}}(\mathscr{M}, \mathscr{N})) = \operatorname{Hom}_{\mathscr{A}|_V}(\mathscr{M}|_V, \mathscr{N}|_V),$$

where we put $V = f^{-1}(U)$. The map $u \mapsto f_*(u)$ is a homomorphism

$$\operatorname{Hom}_{\mathscr{A}|_V}(\mathscr{M}|_V, \mathscr{N}|_V) \to \operatorname{Hom}_{\mathscr{B}|_U}(f_*(\mathscr{M})|_U, f_*(\mathscr{N})|_U)$$

for the structure of $\Gamma(U, \mathscr{B})$-modules. These homomorphisms are compatible with restrictions, so define a canonical homomorphism of $\mathscr{B}$-modules

$$f_*(\mathscr{H}om_{\mathscr{A}}(\mathscr{M},\mathscr{N})) \to \mathscr{H}om_{\mathscr{B}}(f_*(\mathscr{M}), f_*(\mathscr{N})).$$

If $\mathscr{C}$ is an $\mathscr{A}$-algebra, the composition homomorphism

$$f_*(\mathscr{C}) \otimes_{\mathscr{B}} f_*(\mathscr{C}) \to f_*(\mathscr{C} \otimes_{\mathscr{A}} \mathscr{C}) \to f_*(\mathscr{C})$$

defines on $f_*(\mathscr{C})$ a $\mathscr{B}$-algebra structure. We see that if $\mathscr{M}$ is a $\mathscr{C}$-module, $f_*(\mathscr{M})$ is an $f_*(\mathscr{C})$-module.

Consider in particular the case where X is a closed subspace of Y and f is the canonical injection $j : X \to Y$. If $\mathscr{B}' = \mathscr{B}|_X = j^{-1}(\mathscr{B})$ is the restriction of $\mathscr{B}$ to X, an $\mathscr{A}$-module $\mathscr{M}$ can be considered as a $\mathscr{B}'$-module via the homomorphism $f^\sharp : \mathscr{B}' \to \mathscr{A}$; $f_*(\mathscr{M})$ is then the $\mathscr{B}$-module inducing $\mathscr{M}$ on X and 0 elsewhere. If $\mathscr{N}$ is a second $\mathscr{A}$-module, $f_*(\mathscr{M}) \otimes_{\mathscr{B}} f_*(\mathscr{N})$ is identified with $f_*(\mathscr{M} \otimes_{\mathscr{B}'} \mathscr{N})$ and $\mathscr{H}om_{\mathscr{B}}(f_*(\mathscr{M}), f_*(\mathscr{N}))$ with $f_*(\mathscr{H}om_{\mathscr{B}'}(\mathscr{M},\mathscr{N}))$.

Let $(S, \mathscr{O}_S)$ be a ringed space and let $f : (X, \mathscr{O}_X) \to (S, \mathscr{O}_S)$ be a morphism of ringed spaces. If we fix S, we denote by $\mathscr{A}(X)$ (if there is no confusion) the direct image $f_*(\mathscr{O}_X)$, which is an $\mathscr{O}_S$-algebra. For any open subset U of S, we have $\mathscr{A}(f^{-1}(U)) = \mathscr{A}(X)|_U$. Simialrly, for any $\mathscr{O}_X$-module $\mathscr{F}$ (resp. any $\mathscr{O}_X$-algebra $\mathscr{B}$), we denote by $\mathscr{A}(\mathscr{F})$ (resp. $\mathscr{A}(\mathscr{B})$) the direct image $f_*(\mathscr{F})$ (resp. $f_*(\mathscr{B})$), which is an $\mathscr{A}(X)$-module (resp. an $\mathscr{A}(X)$-algebra) and also an $\mathscr{O}_S$-module (resp. $\mathscr{O}_S$-algebra).

We can define $\mathscr{A}(X)$ as a contravariant functor on X, from the category of S-ringed spaces to the category of $\mathscr{O}_S$-algebras. In fact, consider two morphisms $f : X \to S$, $g : Y \to S$ and let $h : X \to Y$ be a morphism, such that the diagram

$$\begin{tikzcd} X \arrow[rr, "h"] \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & S & \end{tikzcd}$$

is commutative. Then by definition $h^\# : \mathscr{O}_Y \to h_*(\mathscr{O}_X)$ is a homomorphism of sheaves of rings; we also deduce a homomorphism $g_*(h^\#) : g_*(\mathscr{O}_Y) \to g_*(h_*(\mathscr{O}_X)) = f_*(\mathscr{O}_X)$ of $\mathscr{O}_S$-algebras, which is a homomorphism $\mathscr{A}(Y) \to \mathscr{A}(X)$, denoted by $\mathscr{A}(h)$. If $h' : Y \to Z$ is a second S-morphism, then we have $\mathscr{A}(h' \circ h) = \mathscr{A}(h) \circ \mathscr{A}(h')$.

Now let $\mathscr{F}$ be an $\mathscr{O}_X$-module, $\mathscr{G}$ be an $\mathscr{O}_Y$-module, and $u : \mathscr{G} \to h_*(\mathscr{F})$ be a homomorphism of $\mathscr{O}_Y$-modules. Then $g_*(u) : g_*(\mathscr{G}) \to g_*(h_*(\mathscr{F})) = f_*(\mathscr{F})$ is a homomorphism $\mathscr{A}(\mathscr{G}) \to \mathscr{A}(\mathscr{F})$ of $\mathscr{O}_S$-modules, which we denote by $\mathscr{A}(u)$. Moreover, the couple $(\mathscr{A}(h), \mathscr{A}(u))$ is a bi-homomorphism from $\mathscr{A}(Y)$-module $\mathscr{A}(\mathscr{G})$ to the $\mathscr{A}(X)$-module $\mathscr{A}(\mathscr{F})$. The ringed space S being fixed, we can consider the couples $(X, \mathscr{F})$, where X is a S-ringed space and $\mathscr{F}$ is an $\mathscr{O}_X$-module, which form a category, and define a morphism $(X, \mathscr{F}) \to (Y, \mathscr{G})$ to be a couple (h, u), where $h : X \to Y$ is an S-morphism and $u : \mathscr{G} \to h_*(\mathscr{F})$ is a homomorphism of $\mathscr{O}_Y$-modules. We can then say that $(X, \mathscr{F}) \mapsto (\mathscr{A}(X), \mathscr{A}(\mathscr{F}))$ is a contravariant functor from this category to the cateogry of couples formed by an $\mathscr{O}_S$-algebra and a module of this algebra.

1.3.3 Inverse image of a $\mathscr{B}$-module

Let $f : (X, \mathscr{A}) \to (Y, \mathscr{B})$ be a morphism of ringed spaces. Let $\mathscr{G}$ be a $\mathscr{B}$-module and $f^{-1}(\mathscr{G})$ be the inverse image of $\mathscr{G}$, which is a sheaf of abelian groups over X. The definition of sections of $f^{-1}(\mathscr{G})$ and of $f^{-1}(\mathscr{B})$ shows that $f^{-1}(\mathscr{G})$ is canonically endowed with an $f^{-1}(\mathscr{B})$-module structure. On the other hand, the homomorphism $f^\sharp : f^{-1}(\mathscr{B}) \to \mathscr{A}$ endows $\mathscr{A}$ with an $f^{-1}(\mathscr{B})$-module structure. The tensor product $f^{-1}(\mathscr{G}) \otimes_{f^{-1}(\mathscr{B})} \mathscr{A}$ is then an $\mathscr{A}$-module, called the inverse image of $\mathscr{G}$ under the morphism $(f, f^\#)$ and denoted by $f^*(\mathscr{G})$. If $\mathscr{G}_1$, $\mathscr{G}_2$ are two $\mathscr{B}$-modules over Y and $v : \mathscr{G}_1 \to \mathscr{G}_2$ is a $\mathscr{B}$-homomorphism, $f^{-1}(v)$ is then an $f^{-1}(\mathscr{B})$-homomorphism from $f^{-1}(\mathscr{G}_1)$ to $f^{-1}(\mathscr{G}_2)$; therefore $f^{-1}(v) \otimes 1_{\mathscr{A}}$ is an $\mathscr{A}$-homomorphism $f^*(\mathscr{G}_1) \to f^*(\mathscr{G}_2)$, which we denote by $f^*(v)$. We then define f^* as a covariant functor from the category of $\mathscr{B}$-modules to that of $\mathscr{A}$-modules. Note that this functor (contrary to f^{-1}) is not in general exact, and is only right exact, since the tensor product with $\mathscr{A}$ is only right exact. We will see that f^* is the left adjoint of the functor f_*. For any $x \in X$, we have $(f^*(\mathscr{G}))_x = \mathscr{G}_{f(x)} \otimes_{\mathscr{B}_{f(x)}} \mathscr{A}_x$ in view of the formula for the stalk of tensor products. The support of $f^*(\mathscr{G})$ is tens contained in $f^{-1}(\operatorname{supp}(\mathscr{G}))$.

Let $(Z, \mathscr{C})$ be a third ringed space and $g : (Y, \mathscr{B}) \to (Z, \mathscr{C})$ be a morphism. Then if $h = g \circ f$ is the composition morphism, then it follows from the definitions that $h^* = f^* \circ g^*$.

Let $(\mathscr{G}_\lambda)$ be a inductive system of $\mathscr{B}$-modules, and $\mathscr{G} = \varinjlim \mathscr{G}_\lambda$ be the inductive limit. The canonical homomorphisms $\mathscr{G}_\lambda \to \mathscr{G}$ define an $f^{-1}(\mathscr{B})$-homomorphism $f^{-1}(\mathscr{G}_\lambda) \to f^{-1}(\mathscr{G})$, which gives a canonical homomorphism $\varinjlim f^{-1}(\mathscr{G}_\lambda) \to f^{-1}(\mathscr{G})$. As taking stalk commutes with inductive limits, this canonical homomorphism is bijective. Moreover, tensor product also commutes with inductive limits, and we then have a canonical functorial isomorphism $\varinjlim f^*(\mathscr{G}_\lambda) \cong f^*(\varinjlim \mathscr{G}_\lambda)$ of $\mathscr{A}$-modules.

On the other hand, for a finite direct sum $\bigoplus_i \mathscr{G}_i$ of $\mathscr{B}$-modules, it is clear that

$$f^*(\bigoplus_i \mathscr{G}_i) = \bigoplus_i f^*(\mathscr{G}_i).$$

By passing to inductive limits, we then deduce that, in view of the preceding, that this equality holds for arbitrary direct sums.

Let $\mathscr{G}_1, \mathscr{G}_2$ be two $\mathscr{B}$-modules; from the definition of inverse images of sheaves of abelian groups we deduce a canonical homomorphism

$$f^{-1}(\mathscr{G}_1) \otimes_{f^{-1}(\mathscr{B})} f^{-1}(\mathscr{G}_2) \to f^{-1}(\mathscr{G}_1 \otimes_{\mathscr{B}} \mathscr{G}_2)$$

of $f^{-1}(\mathscr{B})$, and since the stalk of a tensor product is the rensor product of stalks, this homomorphism is an isomorphism. By tensoring with $\mathscr{A}$, we then deduce a canonical isomorphism

$$f^*(\mathscr{G}_1) \otimes_{\mathscr{A}} f^*(\mathscr{G}_2) \xrightarrow{\sim} f^*(\mathscr{G}_1 \otimes \mathscr{G}_2). \tag{3.3.1}$$

Let $\mathscr{C}$ be a $\mathscr{B}$-algebra. The algebra structure over $\mathscr{C}$ is given by a $\mathscr{B}$-homomorphism $\mathscr{C} \otimes_{\mathscr{B}} \mathscr{C} \to \mathscr{C}$ satisfying the associativity and commutativity conditions (which can be verified on stalks); the isomorphism (3.3.1) then provides a homomorphism $f^*(\mathscr{C}) \otimes_{\mathscr{A}} f^*(\mathscr{C}) \to f^*(\mathscr{C})$ satisfying the same conditions, whence $f^*(\mathscr{C})$ is endowed with an $\mathscr{A}$-algebra structure. In particular, it follows from the definitions that the $\mathscr{A}$-algebra $f^*(\mathscr{B})$ is equal to $\mathscr{A}$. If $\mathscr{C}_1, \mathscr{C}_2$ are two $\mathscr{B}$-

algebras and $v : \mathscr{C}_1 \to \mathscr{C}_2$ is a homomorphism of $\mathscr{B}$-algebras, then $f^*(v) : f^*(\mathscr{C}_1) \to f^*(\mathscr{C}_2)$ is a homomorphism of $\mathscr{A}$-algebras.

Similarly, if $\mathscr{M}$ is a $\mathscr{C}$-module, then the structure of a $\mathscr{B}$-module is given by a $\mathscr{B}$-homomorphism $\mathscr{C} \otimes_{\mathscr{B}} \mathscr{M} \to \mathscr{M}$ satisfying the associativity condition; by transporting structure, we see that $f^*(\mathscr{C})$ is endowed with a $f^*(\mathscr{C})$-module structure.

Let $\mathscr{I}$ be an ideal of $\mathscr{B}$; as the functor f^{-1} is exact, the $f^{-1}(\mathscr{B})$-module $f^{-1}(\mathscr{I})$ is canonically identified an ideal of $f^{-1}(\mathscr{B})$; the canonical injection $f^{-1}(\mathscr{I}) \to f^{-1}(\mathscr{B})$ then gives a homomorphism of $\mathscr{A}$-modules

$$f^*(\mathscr{I}) = f^{-1}(\mathscr{I}) \otimes_{f^{-1}(\mathscr{B})} \mathscr{A} \to \mathscr{A};$$

we denote by $f^*(\mathscr{I})\mathscr{A}$, or simply $\mathscr{I}\mathscr{A}$ if there is no confusion, the image of $f^*(\mathscr{I})$ under this hommorphism. We then have $\mathscr{I}\mathscr{A} = f^\sharp(f^{-1}(\mathscr{I}))\mathscr{A}$ and in particular, for any $x \in X$, $(\mathscr{I}\mathscr{A})_x = f_x^\sharp(\mathscr{I}_{f(x)})\mathscr{A}_x$, in view of the canonical identification of the stalk $f^{-1}(\mathscr{I})$. If $\mathscr{I}_1, \mathscr{I}_2$ are two ideals of $\mathscr{B}$, we have $(\mathscr{I}_1\mathscr{I}_2)\mathscr{A} = \mathscr{I}_1(\mathscr{I}_2\mathscr{A}) = (\mathscr{I}_1\mathscr{A})(\mathscr{I}_2\mathscr{A})$. If $\mathscr{F}$ is an $\mathscr{A}$-module, we put $\mathscr{I}\mathscr{F} = (\mathscr{I}\mathscr{A})\mathscr{F}$.

Remark 1.3.1. Over any topological space, we can define a canonical sheaf of rings $\mathbb{Z}_X$, which is the constant sheaf associated the presheaf $U \mapsto \mathbb{Z}$. It is clear thet the sheaves of abelian groups over X are identified with the $\mathbb{Z}_X$-modules over the ringed space $(X, \mathbb{Z}_X)$, and we can in particular consider the tensor product $\mathscr{F} \otimes_{\mathbb{Z}_X} \mathscr{G}$ of two sheaves of rings $\mathscr{F}, \mathscr{G}$ over X. If $f : X \to Y$ is a continuous map, for any open subset V of Y, we have a canonical homomorphism $\mathbb{Z} \to \Gamma(f^{-1}(V), \mathbb{Z}_X)$ of rings and this defines a homomorphism $f^\# : \mathbb{Z}_Y \to f_*(\mathbb{Z}_X)$ of sheaves of rings over Y. We then obtain a morphism $(f, f^\#) : (X, \mathbb{Z}_X) \to (Y, \mathbb{Z}_Y)$ of ringed spaces. If $\mathscr{F}$ is a sheaf of abelian groups over Y, $f^*(\mathscr{F})$ is canonically identified with $f^{-1}(\mathscr{F})$; if $\mathscr{F}, \mathscr{G}$ are two sheves of abelian groups over Y, we then have a canonical isomorphism $f^{-1}(\mathscr{F} \otimes_{\mathbb{Z}_Y} \mathscr{G}) = f^{-1}(\mathscr{F}) \otimes_{\mathbb{Z}_X} f^{-1}(\mathscr{G})$. We then deduce that if f is a quasi-homeomorphism, f^{-1} is not only an equivalence from the category of sheaf of abelian groups over Y to that of sheaf of abelian groups over X, but also an equivalence from the category of sheaf of rings over Y to that of sheaf of rings over X.

Given two ringed spaces $(X, \mathscr{O}_X)$, $(Y, \mathscr{O}_Y)$, we say a morphism $f : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ is a **quasi-isomorphism** if f is a quasi-homeomorphism and $f^\sharp : f^{-1}(\mathscr{O}_Y) \to \mathscr{O}_X$ is an isomorphism of sheaf of rings. If this is the case, the ringed space $(X, \mathscr{O}_X)$ is entirely determined up to isomorphism, by $(Y, \mathscr{O}_Y)$, the space X, and the quasi-homeomorphism.

If f is a quasi-isomorphism of ringed spaces, the functor $\mathscr{F} \to f^*(\mathscr{F})$ is an equivalence from category of $\mathscr{O}_Y$-modules to the category of $\mathscr{O}_X$-modules, since $f^*(\mathscr{F})$ is identified with $f^{-1}(\mathscr{F})$. We then conclude for example the isomorphisms of bi-∂-functors

$$\operatorname{Ext}^i_{\mathscr{O}_Y}(\mathscr{F}, \mathscr{G}) \xrightarrow{\sim} \operatorname{Ext}^i_{\mathscr{O}_X}(f^*(\mathscr{F}), f^*(\mathscr{G})).$$

In general, we can say that the usual constructions of the sheaf theory and homological algebra on the ringed space Y or X, are equivalent.

1.3.4 Relations of direct images and inverse images

Again we let $f : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ be a morphism of ringed spaces. By definition, a homomorphism $u : \mathscr{G} \to f_*(\mathscr{F})$ of $\mathscr{B}$-modules is called an f**-morphism** from $\mathscr{G}$ to $\mathscr{F}$ and we denote it by $u : \mathscr{G} \to \mathscr{F}$ if there is no confusion. Given such a homomorphism, for any couple (U, V) where U is an open subset of X and V is an open subset of Y such that $f(U) \subseteq V$, a **homomorphism** $u_{U,V} : \Gamma(V, \mathscr{G}) \to \Gamma(U, \mathscr{F})$ of $\Gamma(V, \mathscr{B})$ modules, where $\Gamma(U, \mathscr{F})$ is considered as a $\Gamma(V, \mathscr{B})$-module via the ring homomorphism $f^\sharp_{U,V} : \Gamma(V, \mathscr{B}) \to \Gamma(U, \mathscr{A})$. The homomorphisms $u_{U,V}$ fits into the commutative diagram

$$\begin{tikzcd}
\mathscr{G}(V) \arrow[r, "u_{U,V}"] \arrow[d] & \mathscr{F}(U) \arrow[d] \\
\mathscr{G}(V') \arrow[r, "u_{U',V'}"] & \mathscr{F}(U')
\end{tikzcd}$$

for $U' \subseteq U$, $V' \subseteq V$, $f(U') \subseteq V'$. Moreover, for the homomorphism u, it suffices to define $u_{U,V}$ for U (resp. V) in a base $\mathcal{B}$ (resp. $\mathcal{B}'$) of the topology of X (resp. Y). Let $g : (Y, \mathscr{B}) \to (Z, \mathscr{C})$ be another morphism and $h = g \circ f$. Let $\mathscr{H}$ be a $\mathscr{C}$-module, and $v : \mathscr{H} \to g_*(\mathscr{G})$ be a g-morphism; then

$$w : \mathscr{H} \xrightarrow{v} g_*(\mathscr{G}) \xrightarrow{g_*(u)} g_*(f_*(\mathscr{F}))$$

is an h-morphism which is called the **composition** of u and v.

We will now see that there is a canonical isomorphism of bifunctors on $\mathscr{F}$ and $\mathscr{G}$

$$\mathrm{Hom}_{\mathscr{A}}(f^*(\mathscr{G}), \mathscr{F}) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{B}}(\mathscr{G}, f_*(\mathscr{F})) \tag{3.4.1}$$

which we denote by $v \mapsto v^\flat$, and the inverse of this isomorphism is denoted by $u \mapsto u^\sharp$. The definition is the folowing: for an $\mathscr{A}$-homomorphism $v : f^*(\mathscr{G}) \to \mathscr{F}$, by composing with the canonical homomorphism $f^{-1}(\mathscr{G}) \to f^*(\mathscr{G})$, we obtain a hommorphism $\tilde{v} : f^{-1}(\mathscr{G}) \to \mathscr{F}$ of sheaves of abelian groups, which is also a homomorphism of $f^{-1}(\mathscr{B})$-modules. We then deduce a homomorphism $\tilde{v}^\flat : \mathscr{G} \to f_*(\mathscr{F})$ by the adjointness of f_* and f^{-1}, which is a homomorphism of $\mathscr{B}$-modules, and we denote by $v^\flat$. Similarly, for a $\mathscr{B}$-homomorphism $u : \mathscr{G} \to f_*(\mathscr{F})$, we deduce a homomorphism $u^\sharp : f^{-1}(\mathscr{G}) \to \mathscr{F}$ of $f^{-1}(\mathscr{B})$-modules, whence by tensoring with $\mathscr{A}$ a homomorphism of $\mathscr{A}$-modules $f^*(\mathscr{G}) \to \mathscr{F}$, still denoted by $u^\sharp$. It is immediate that $(u^\sharp)^\flat = u$ and $(v^\flat)^\sharp = v$, as well as the functoriality in $\mathscr{F}$ of the isomorphism $v \mapsto v^\flat$. The functoriality in $\mathscr{G}$ of $u \mapsto u^\sharp$ can be then deduced formally and we then see that f^* is left adjoint to f_*.

If we choose v to be the identify homomorphism of $f^*(\mathscr{G})$, then $v^\flat$ is a homomorphism

$$\rho_{\mathscr{G}} : \mathscr{G} \to f_*(f^*(\mathscr{G}));$$

if we choose u to be the identify homomorphism on $f_*(\mathscr{F})$, then $u^\sharp$ is a homomorphism

$$\sigma_{\mathscr{F}} : f^*(f_*(\mathscr{F})) \to \mathscr{F};$$

these homomorphism are called canonical, and are in general neither injective nor surjective.

As always, for a homomorphism $v : \mathscr{G} \to f_*(\mathscr{F})$, we have a canonical factorization

$$v^\flat : \mathscr{G} \xrightarrow{\rho_{\mathscr{G}}} f_*(f^*(\mathscr{G})) \xrightarrow{f_*(v)} f_*(\mathscr{F}) \tag{3.4.2}$$

and for a homomorphism $u : \mathscr{G} \to f_*(\mathscr{F})$, we have a canonical factorization

$$u^\sharp : f^*(\mathscr{G}) \xrightarrow{f^*(u)} f^*(f_*(\mathscr{F})) \xrightarrow{\sigma_{\mathscr{F}}} \mathscr{F} \tag{3.4.3}$$

Now let $g : (Y, \mathscr{B}) \to (Z, \mathscr{C})$ be another morphism and $h = g \circ f$ be the composition. Let $\mathscr{H}$ be a $\mathscr{C}$-module, $u : \mathscr{G} \to \mathscr{F}$ and $v : \mathscr{H} \to \mathscr{G}$ be homomorphisms, and $w = g_*(u) \circ v$ be the composition of u and v. Then $w^\sharp$ is the composition homomorphism

$$w^\sharp : f^*(g^*(\mathscr{H})) \xrightarrow{f^*(v^\sharp)} f^*(\mathscr{G}) \xrightarrow{u^\sharp} \mathscr{F}. \tag{3.4.4}$$

To verify this, we use the description of $w^\sharp$ in (3.4.3) to obtain that $w^\sharp$ is given by the following diagram

$$\begin{tikzcd}
f^*(g^*(\mathscr{H})) \arrow[r, "f^*(g^*(v))"] \arrow[rr, bend left, "f^*(g^*(w))"] \arrow[dr, "f^*(v^\sharp)"'] & f^*(g^*(g_*(\mathscr{G}))) \arrow[r, "f^*(g^*(g_*(u)))"] \arrow[d, "f^*(\sigma_{\mathscr{G}})"'] & f^*(g^*(g_*(f_*(\mathscr{F})))) \arrow[d, "f^*(\sigma_{f_*(\mathscr{F})})"] \arrow[dr, "\sigma_{\mathscr{F}}"] & \\
& f^*(\mathscr{G}) \arrow[r, "f^*(u)"] \arrow[rr, bend right, "u^\sharp"'] & f^*(f_*(\mathscr{F})) \arrow[r, "\sigma_{\mathscr{F}}"] & \mathscr{F}
\end{tikzcd}$$

The central square is immediately verified to be commutative by the naturality of σ, whence the assertion.

We note that if s is a section of $\mathscr{G}$ over an open subset V of Y, $\rho_{\mathscr{G}}(s)$ is the section $s' \otimes 1$ of $f^*(\mathscr{G})$ over $f^{-1}(V)$, where s' is the section such that $s'_x = s_{f(x)}$ for any $x \in f^{-1}(V)$. We say that $\rho_{\mathscr{G}}(s)$ is the **inverse image** of s under f. Note also that if $u : \mathscr{G} \to f_*(\mathscr{F})$ is a homomorphism, it defines for each $x \in X$ a homomorphism $u_x : \mathscr{G}_{f(x)} \to \mathscr{F}_x$ over stalks, obtained by composing $(u^\sharp)_x : (f^*(\mathscr{G}))_x \to \mathscr{F}_x$ and the canonical homomorphism $s_x \mapsto s_x \otimes 1$ from $\mathscr{G}_{f(x)}$ to $(f^*(\mathscr{G}))_x = \mathscr{G}_{f(x)} \otimes_{\mathscr{B}_{f(x)}} \mathscr{A}_x$. The homomorphism u_x is also obtained by taking inductive limit of the homomorphisms $\Gamma(V, \mathscr{G}) \xrightarrow{u} \Gamma(f^{-1}(V), \mathscr{F}) \to \mathscr{F}_x$, where V runs through neighborhood of $f(x)$.

Let $\mathscr{F}_1, \mathscr{F}_2$ be $\mathscr{A}$-modules, $\mathscr{G}_1, \mathscr{G}_2$ be $\mathscr{B}$-modules, u_i $(i = 1, 2)$ be a homomorphism from $\mathscr{G}_i$ to $\mathscr{F}_i$. We denote by $u_1 \otimes u_2$ the homomorphism $u : \mathscr{G}_1 \otimes_{\mathscr{B}} \mathscr{G}_2 \to \mathscr{F}_1 \otimes_{\mathscr{A}} \mathscr{F}_2$ such that $u^\sharp = (u_1)^\sharp \otimes (u_2)^\sharp$; we verify that u is also the composition

$$\mathscr{G}_1 \otimes_{\mathscr{B}} \mathscr{G}_2 \to f_*(\mathscr{F}_1) \otimes_{\mathscr{B}} f_*(\mathscr{F}_2) \to f_*(\mathscr{F}_1 \otimes_{\mathscr{A}} \mathscr{F}_2),$$

where the first morphism is the ordinary tensor product $u_1 \otimes_{\mathscr{B}} u_2$ and the second homomorphism is the canonical homomorphism (3.2.1).

Let $(\mathscr{G}_\lambda)_{\lambda\in L}$ be an inductive system of $\mathscr{B}$-modules, and, for $\lambda \in L$, let u_λ be a homomorphism $\mathscr{G}_\lambda \to f_*(\mathscr{F})$, which form an inductive system; put $\mathscr{G} = \varinjlim \mathscr{G}_\lambda$ and $u = \varinjlim u_\lambda$. Then $(u^\lambda)^\sharp$ form an inductive system of homomorphisms $f^*(\mathscr{G}_\lambda) \to \mathscr{F}$, and the inductive limit of this system is just $u^\sharp$, since taking tensor products commutes with inductive limits.

Let $\mathscr{M}, \mathscr{N}$ be two $\mathscr{B}$-modules, V be an open subset of Y, and $U = f^{-1}(V)$. The map $v \mapsto f^*(v)$ is a homomorphism

$$\mathrm{Hom}_{\mathscr{B}|_V}(\mathscr{M}|_V, \mathscr{N}|_V) \to \mathrm{Hom}_{\mathscr{A}|_U}(f^*(\mathscr{M})|_U, f^*(\mathscr{N})|_U)$$

for the structure of $\Gamma(V,\mathscr{B})$-modules: $\mathrm{Hom}_{\mathscr{A}|_U}(f^*(\mathscr{M})|_U, f^*(\mathscr{N})|_U)$ is endowed with a $\Gamma(U, f^{-1}(\mathscr{B}))$-module structure, and thanks to the canonical homomorphism $\Gamma(V,\mathscr{B}) \to \Gamma(U, f^{-1}(\mathscr{B}))$ obtained from the definition of f^{-1}, this is then a $\Gamma(V,\mathscr{B})$-module. We also verify that these homomorphisms are compatible with restrictions, and therefore define a canonical homomorphism

$$\gamma : \mathscr{H}om_{\mathscr{B}}(\mathscr{M}, \mathscr{N}) \to f_*\big(\mathscr{H}om_{\mathscr{A}}(f^*(\mathscr{M}), f^*(\mathscr{N}))\big)$$

which corresponds to a homomorphism

$$\gamma^\sharp : f^*(\mathscr{H}om_{\mathscr{B}}(\mathscr{M}, \mathscr{N})) \to \mathscr{H}om_{\mathscr{A}}(f^*(\mathscr{M}), f^*(\mathscr{N})) \tag{3.4.5}$$

and this canonical homomorphisms are functorial on $\mathscr{M}$ and $\mathscr{N}$.

Suppose that $\mathscr{F}$ (resp. $\mathscr{G}$) is an $\mathscr{A}$-algebra (resp. a $\mathscr{B}$-algebra). If $u : \mathscr{G} \to f_*(\mathscr{F})$ is a homomorphisms of $\mathscr{B}$-algebras, $u^\sharp$ is a homomorphism $f^*(\mathscr{G}) \to \mathscr{F}$ of $\mathscr{A}$-algebras; this follows from the diagram

$$\begin{tikzcd}
\mathscr{G} \otimes_{\mathscr{B}} \mathscr{G} \arrow[r] \arrow[d] & \mathscr{G} \arrow[d, "u"] \\
f_*(\mathscr{F} \otimes_{\mathscr{A}} \mathscr{F}) \arrow[r] & f_*(\mathscr{F})
\end{tikzcd}$$

Similarly, if $v : f^*(\mathscr{G}) \to \mathscr{F}$ is a homormorphism of $\mathscr{A}$-algebras, then $v^\flat : \mathscr{G} \to f_*(\mathscr{F})$ is a homomorphisms of $\mathscr{B}$-algebras. We then get an isomorphism of bifunctors

$$\mathrm{Hom}_{\mathscr{A}-\mathbf{alg}}(f^*(\mathscr{G}), \mathscr{F}) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{B}-\mathbf{alg}}(\mathscr{G}, f_*(\mathscr{F})).$$

We can then say that f^* is the left adjoint of the functor f^* from the category of $\mathscr{A}$-algebras to that of $\mathscr{B}$-algebras.

1.3.5 Open immersions and representable natrual transformations

The set of open immersions in the category **Rsp** is closed under composition and fiber product, so we can speak of natrual transformations $F \to G$ (where F and G are contravariant functors from **Rsp** to **Set**) which are **representable by open immersions**. The same is true when if we consider the category $\mathbf{Rsp}_{/S}$, where S is a base ringed space.

Proposition 1.3.5. *Let S be a ringed space, $F : (\mathbf{Rsp}_{/S})^\circ \to \mathbf{Set}$ a contravariant functor, and $(F_i)_{i\in I}$ be a family of subfunctors of F. Suppose that the following conditions are satisfied:*

(i) For each i, the canonical natrual transformation $u_i : F_i \to F$ are representable by an open immersion.

(ii) For any S-ringed space X, the map $U \mapsto F(U)$, where U is an open subset of X, is a sheaf of sets over X (i.e., F is a sheaf over $\mathbf{Rsp}_{/S}$).

(iii) For any S-ringed space Z and any natrual transformation $h_Z \to F$, if Z_i is the S-ringed space representating the functor $F_i \times_F h_Z$ and U_i is the image of morphism $Z_i \to Z$, then (U_i) form an open covering of Z.

(iv) For each i, the functor F_i is representable by an S-ringed space X_i.

Then the functor F is representable by an S-ringed space X, and the images of X_i under the morphism $X_i \to X$ (which is open by condition (i)) form an open covering of X.

1.4 Quasi-coherent sheaves and coherent sheaves

1.4.1 Quasi-coherent sheaves

In this subsection we introduce an abstract notion of quasi-coherent $\mathscr{O}_X$-module. This notion is very useful in algebraic geometry, since quasi-coherent modules on a scheme have a good description on any affine open. However, in the general setting of locally ringed spaces this notion is not well behaved at all. The category of quasi-coherent sheaves is not abelian in general, infinite direct sums of quasi-coherent sheaves aren't quasi-coherent, etc, etc.

Let $(X, \mathscr{O}_X)$ be a ringed space. Let $\mathscr{F}$ be a sheaf of $\mathscr{O}_X$-modules. We say that $\mathscr{F}$ is a **quasi-coherent sheaf** of $\mathscr{O}_X$-modules if for each $x \in X$, there exists an open neighbourhood U of x such that $\mathscr{F}|_U$ is isomorphic to the cokernel of a homomorphism of the form $\mathscr{O}_X^{\oplus J}|_U \to \mathscr{O}_X^{\oplus I}|_U$, where I and J are arbitrary index sets. It is clear that $\mathscr{O}_X$ itself is a quasi-coherent $\mathscr{O}_X$-module, and a finite direct sum of quasi-coherent modules is quasi-coherent. The category of quasi-coherent $\mathscr{O}_X$-modules is denoted $\mathbf{QCoh}(\mathscr{O}_X)$. We say an $\mathscr{O}_X$-lagebra $\mathscr{A}$ is quasi-coherent if it is a quasi-coherent $\mathscr{O}_X$-module.

The definition of quasi-coherence amounts to saying that locally the sheaf $\mathscr{F}$ admits a *presentation* by the structural sheaf $\mathscr{O}_X$. This definition is inspired by following ideal: for any module M over a ring A, M admits a presentation of the form

$$A^{\oplus J} \longrightarrow A^{\oplus I} \longrightarrow M \longrightarrow 0$$

where I and J are arbitrary index sets (and in particular may not be finite). Therefore, quasi-coherent sheaves can be seen as a "real module" over the ringed space $(X, \mathscr{O}_X)$, and this idea really makes sense in the realm of algebraic geometry.

Proposition 1.4.1. *Let $(f, f^\#) : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ be a morphism of ringed spaces. Then the pullback $f^*(\mathscr{G})$ of a quasi-coherent $\mathscr{O}_Y$-module $\mathscr{G}$ is quasi-coherent.*

Proof. Since the question is local, we may assume that $\mathscr{G}$ has a global presentation by $\mathscr{O}_Y$. We have seen that f^* commutes with all colimits, and is right exact, so if we have an exact sequence

$$\mathscr{O}_Y^{\oplus J} \longrightarrow \mathscr{O}_Y^{\oplus I} \longrightarrow \mathscr{G} \longrightarrow 0$$

then upon applying f^* we obtain the exact sequence

$$\mathscr{O}_X^{\oplus J} \longrightarrow \mathscr{O}_X^{\oplus I} \longrightarrow f^*(\mathscr{G}) \longrightarrow 0$$

This implies the assertion. □

Proposition 1.4.2. *Let $(X, \mathscr{O}_X)$ be ringed space. Let $\rho : A \to \Gamma(X, \mathscr{O}_X)$ be a ring homomorphism from a ring A into the ring of global sections on X and M be an A-module. Then the following three constructions give canonically isomorphic $\mathscr{O}_X$-modules:*

(i) Let $(\pi, \pi^\#) : (X, \mathscr{O}_X) \to (\{*\}, A)$ be the morphism of ringed spaces where $\pi : X \to \{*\}$ is the unique map and $\pi^\#$ is the given homomorphism $\rho : A \to \Gamma(X, \mathscr{O}_X)$. Set $\mathscr{F}_1 = \pi^*(M)$.

(ii) Choose a presentation $A^{\oplus J} \to A^{\oplus I} \to M \to 0$ and set

$$\mathscr{F}_2 = \operatorname{coker}(\mathscr{O}_X^{\oplus J} \to \mathscr{O}_X^{\oplus I})$$

where the homomorphism is induced by ρ and the matrix coefficients of the homomorphism in the presentation of M.

(iii) Let $\mathscr{F}_3$ be the sheaf associated to the presheaf $U \mapsto \Gamma(U, \mathscr{O}_X) \otimes_A M$, where the homomorphism $A \to \Gamma(X, \mathscr{O}_X)$ is the composition of ρ with the restriction map $\Gamma(X, \mathscr{O}_X) \to \Gamma(U, \mathscr{O}_X)$.

This construction has the following properties:

(a) The resulting $\mathscr{O}_X$-modules $\mathscr{F}_M = \mathscr{F}_1 = \mathscr{F}_2 = \mathscr{F}_3$ is quasi-coherent.

(b) The construction gives a functor from the category of A-modules to the category of quasi-coherent sheaves on X which commutes with arbitrary colimits.

(c) For any point $x \in X$ we have $\mathscr{F}_{M,x} = \mathscr{O}_{X,x} \otimes_A M$ which is functorial in M.

(d) For any $\mathscr{O}_X$-module $\mathscr{G}$ we have

$$\operatorname{Hom}_{\mathscr{O}_X}(\mathscr{F}_M, \mathscr{G}) = \operatorname{Hom}_A(M, \Gamma(X, \mathscr{G}))$$

where the A-module structure on $\Gamma(X, \mathscr{G})$ is induced from the $\Gamma(X, \mathscr{O}_X)$-module structure via α.

Proof. The isomorphism between $\mathscr{F}_1$ and $\mathscr{F}_3$ comes from the fact that π^* is defined as the sheafification of the presheaf in (iii). The isomorphism between the constructions in (ii) and (i) comes from the fact that the functor π^* is right exact, so the sequence

$$\pi^*(A^{\oplus I}) \longrightarrow \pi^*(A^{\oplus J}) \longrightarrow \pi^*(M) \to 0$$

is exact, that π^* commutes with arbitrary direct sums, and the fact that $\pi^*(A) = \mathscr{O}_X$.

Now assertion (a) is clear from construction (ii), so is (b) since π^* has these properties. Assertion (c) follows from the description of stalks of pullback sheaves, and (d) follows from adjointness of π^* and π_*. □

In the situation of Proposition 1.4.2 we say $\mathscr{F}_M$ is the **sheaf associated to the module M and the ring map ρ**. If $A = \Gamma(X, \mathscr{O}_X)$ and $\rho = 1_A$, we simply say that $\mathscr{F}_M$ is the sheaf associated to the A-module M.

Proposition 1.4.3. *Let $(X, \mathscr{O}_X)$ be a ringed space and $A = \Gamma(X, \mathscr{O}_X)$. Let M be an A-module and $\mathscr{F}_M$ be the quasi-coherent sheaf of $\mathscr{O}_X$-modules associated to M. If $(f, f^\#) : (Y, \mathscr{O}_Y) \to (X, \mathscr{O}_X)$ is a morphism of ringed spaces, then $f^*(\mathscr{F}_M)$ is the sheaf associated to the $\Gamma(Y, \mathscr{O}_Y)$-module $\Gamma(Y, \mathscr{O}_Y) \otimes_A M$.*

Proof. In view of the following diagram of ringed spaces

$$\begin{tikzcd}
(Y, \mathscr{O}_Y) \arrow[r, "\pi"] \arrow[d, "f"'] & (\{*\}, \Gamma(Y, \mathscr{O}_Y)) \arrow[d, "\text{induced by } f^\#"] \\
(X, \mathscr{O}_X) \arrow[r, "\pi"] & (\{*\}, \Gamma(X, \mathscr{O}_X))
\end{tikzcd}$$

the assertion follows from the first description of $\mathscr{F}_M$ in Proposition 1.4.2 as $\pi^*(M)$. □

To conclude this part, we prove an important result which will be used when we consider quasi-coherent sheaf on affine schemes. We state it in a general manner.

Proposition 1.4.4. *Let $(X, \mathscr{O}_X)$ be a ringed space and x be a point of X. suppose that x has a fundamental system of quasi-compact neighbourhoods, then for any quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, there exists an open neighbourhood U of x such that $\mathscr{F}|_U$ is isomorphic to the sheaf of modules $\mathscr{F}_M$ on $(U, \mathscr{O}_U)$ associated to a $\Gamma(U, \mathscr{O}_U)$-module M.*

Proof. Since $\mathscr{F}$ is quasi-coherent, we may replace X by an open neighbourhood of x and assume that $\mathscr{F}$ is isomorphic to the cokernel of a map $\mathscr{O}_X^{\oplus J} \to \mathscr{O}_X^{\oplus I}$. The problem is that this map may not be given by a matrix, because the global sections of a direct sum is in general different from the direct sum of the global sections.

Let U be a quasi-compact neighbourhood of x. We proceed as in the proof of Proposition 1.2.45. For each $j \in J$ denote $s_j \in \Gamma(X, \bigoplus_{i \in I} \mathscr{O}_X)$ the image of the section 1 in the summand $\mathscr{O}_X$ corresponding to j. There exists a finite collection of opens $U_{jk}, k \in K_j$ such that $U = \bigcup_{k \in K_j} U_{jk}$ and such that each restriction $s_j|_{U_{jk}}$ is a finite sum $\sum_{i \in I_{jk}} f_{jki}$ with $I_{jk} \subseteq I$. Let $I_j = \bigcup_{k \in K_j} I_{jk}$. This is a finite set since there are finitely many U_{jk} and each I_{jk} is finite. Since $U = \bigcup_{k \in K_j} U_{jk}$ the section $s_j|_U$ is a section of the finite direct sum $\bigoplus_{i \in I_j} \mathscr{O}_X$. Then by Proposition 1.2.42 we see that actually $s_j|_U$ is a sum $\sum_{i \in I_j} f_{ij}$ with $f_{ij} \in \mathscr{O}_X(U) = \Gamma(U, \mathscr{O}_U)$. At this point we can define a module M as the cokernel of the map

$$\bigoplus_{j \in J} \Gamma(U, \mathscr{O}_U) \to \bigoplus_{i \in I} \Gamma(U, \mathscr{O}_U).$$

with matrix given by the (f_{ij}). By construction (ii) of Proposition 1.4.2 we see that $\mathscr{F}_M$ has the same presentation as $\mathscr{F}|_U$ and therefore $\mathscr{F}_M \cong \mathscr{F}|_U$. □

1.4.2 Sheaves of finite type

Let $(X, \mathscr{O}_X)$ be a ringed space and $\mathscr{F}$ be an $\mathscr{O}_X$-modules. We say that $\mathscr{F}$ is **of finite type** if for every point $x \in X$ there exists an open neighbourhood U of x such that $\mathscr{F}|_U$ is generated by a finite family of sections over U, whence isomorphic to a quotient sheaf of a sheaf of the form $\mathscr{O}_X^n|_U$. It is clear that any quotient of a sheaf of finite type is of finite type, and a finite direct sum of sheaves of finite type is of finite type.

Proposition 1.4.5. *Let $(f, f^\#) : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ be a morphism of ringed spaces. The pullback $f^*\mathscr{G}$ of a finite type $\mathscr{O}_Y$-module is a finite type $\mathscr{O}_X$-module.*

Proof. Since the question is local, we may assume $\mathscr{G}$ is globally generated by finitely many sections. We have seen that f^* commutes with all colimits, and is right exact, so if we have a surjection $\bigoplus_{i=1}^n \mathscr{O}_Y \to \mathscr{G} \to 0$, then by applying f^* we obtain the surjection $\bigoplus_{i=1}^n \mathscr{O}_X \to f^*\mathscr{G} \to 0$. □

Proposition 1.4.6. *Let $\mathscr{F}$ be an $\mathscr{O}_X$-module of finite type.*

- *(i) If $(s_i)_{1\le i\le n}$ are sections of $\mathscr{F}$ over an open neighborhood U of a point x and if the $s_{i,x}$ generate $\mathscr{F}$, then there exists an open neighborhood $V \subseteq U$ of x such that the $s_{i,y}$ generate $\mathscr{F}_y$ for any $y \in V$.*
- *(ii) If $\varphi : \mathscr{F} \to \mathscr{G}$ is a homomorphism such that $\varphi_x = 0$, there exists an open neighborhood U of x such that $\varphi|_U = 0$.*
- *(iii) If $\psi : \mathscr{G} \to \mathscr{F}$ is a homomorphism such that ψ_x is surjective, then there exists an open neighborhood V of x such that $\psi|_V$ is surjective.*
- *(iv) The support of $\mathscr{F}$ is closed.*

Proof. We first prove (i), so let $(t_j)_{1\le j\le m}$ be a family of sections of $\mathscr{F}$ over an open neighborhood $U' \subseteq U$ of x generating $\mathscr{F}|_{U'}$. Since $(s_{i,x})$ generates $\mathscr{F}_x$, there exists sections a_{ij} of $\mathscr{O}_X$ over an open neighborhood $U'' \subseteq U$ of x such that $t_{j,x} = \sum_{i=1}^n a_{ij,x}s_{i,x}$ for each j. We then conclude that there is an open neighborhood $V \subseteq U''$ of x such that for each $y \in V$, we have $t_{j,y} = \sum_{i=1}^n a_{ij,y}s_{i,y}$, so $(s_{i,y})$ generates $\mathscr{F}_y$ for $y \in V$.

Assertion (iv) follows from (i) by taking $n = 1$ and $s_1 = 0$; also, (iii) follows from (iv) by considering $\operatorname{coker}\psi$, which is of finite type. Finally, (ii) follows from (iv) by considering $\operatorname{im}\varphi$, which is also of finite type. □

Corollary 1.4.7. *Let $(X, \mathscr{O}_X)$ be a locally ringed space and $\mathscr{F}$ be an $\mathscr{O}_X$-module of finite type. Then* $\operatorname{supp}(\mathscr{F})$ *is the set of $x \in X$ such that $\mathscr{F}_x/\mathfrak{m}_x\mathscr{F}_x \neq 0$.*

Proof. In fact, $\mathscr{F}_x$ is a finitely generated $\mathscr{O}_{X,x}$, so $\mathscr{F}_x = 0$ if and only if $\mathfrak{m}_x\mathscr{F}_x = \mathscr{F}_x$ by Nakayama's lemma. □

Corollary 1.4.8. *Let $(X, \mathscr{O}_X)$ be a locally ringed space and $\mathscr{F}, \mathscr{G}$ be $\mathscr{O}_X$-modules of finite type. Then*

$$\operatorname{supp}(\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G}) = \operatorname{supp}(\mathscr{F}) \cap \operatorname{supp}(\mathscr{G}).$$

Proof. As the tensor product of two $\kappa(x)$-vector spaces is nonzero if both of them are nonzero, this follows from Corollary 1.4.7. □

Corollary 1.4.9. *Let $f : X \to Y$ be a morphism of locally ringed spaces. For any $\mathscr{O}_Y$-module $\mathscr{G}$ of finite type, we have*

$$\mathrm{supp}(f^*(\mathscr{G})) = f^{-1}(\mathrm{supp}(\mathscr{G})).$$

Proof. This follows from Corollary 1.4.7 and Proposition **??**. □

Proposition 1.4.10. *Suppose that X is quasi-compact and let $\mathscr{F}, \mathscr{G}$ be two $\mathscr{O}_X$-modules such that $\mathscr{G}$ is of finite type and $\mathscr{F}$ is the filtered limit $(\mathscr{F}_\lambda)$ of $\mathscr{O}_X$-modules. Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a surjective homomorphism, then there exists an index λ such that the homomorphism $\mathscr{F}_\lambda \to \mathscr{G}$ is surjective.*

Proof. For any $x \in X$, there exists a finite system of sections s_i of $\mathscr{G}$ over an open neighborhood $U(x)$ of x such that $s_{i,y}$ generates $\mathscr{G}_y$ for any $y \in U(x)$. Then there is an open neighborhood $V(x) \subseteq U(x)$ of x and sections t_i of $\mathscr{F}$ over $V(x)$ such that $s_i|_{V(x)} = \varphi(t_i)$ for each i. We can then suppose that the t_i are the images of sections of a single sheaf $\mathscr{F}_{\lambda(x)}$ over $V(x)$. Since X is quasi-compact, it can be covered by finitely many $V(x_k)$, and let λ be the supremum of the $\lambda(x_k)$ the assertion then follows. □

Corollary 1.4.11. *Suppose that X is quasi-compact and let $\mathscr{F}$ be an $\mathscr{O}_X$-module of finite type that is generated by global sections. Then $\mathscr{F}$ is generated by finitely many global sections.*

Proof. It suffices to cover X be finitely many open neighborhoods U_k such that for each k, there exists finitely many sections s_{ik} of $\mathscr{F}$ over X whose restrictions to U_k generate $\mathscr{F}|_{U_k}$. It is clear that the s_{ik} then generate $\mathscr{F}$. □

Proposition 1.4.12. *Let X be a ringed space. Let*

$$0 \longrightarrow \mathscr{F} \xrightarrow{\varphi} \mathscr{G} \xrightarrow{\psi} \mathscr{H} \longrightarrow 0$$

be a short exact sequence of $\mathscr{O}_X$-modules. If $\mathscr{F}$ and $\mathscr{H}$ are of finite type, so is $\mathscr{G}$.

Proof. Since the question is local, we may assume that $\mathscr{F}$ (resp. $\mathscr{H}$) is generated by finitely many global sections $(s_i)_{1\le i\le n}$ (resp. $(t_j)_{1\le j\le m}$), and there are sections $(t'_j)_{1\le j\le m}$ of $\mathscr{G}$ over X such that $t_j = \psi(t'_j)$ for each j. It is then clear that $\mathscr{G}$ is generated by the sections $\varphi(s_i)$ and t'_j. □

Let $(X, \mathscr{O}_X)$ be a ringed space. Let $\mathscr{F}$ be a sheaf of $\mathscr{O}_X$-modules. We say that $\mathscr{F}$ is **of finite presentation** if for every point $x \in X$ there exists an open neighbourhood $x \in U \subseteq X$, and $n, m \in \mathbb{N}$ such that $\mathscr{F}|_U$ is isomorphic to the cokernel of a homomorphism $\bigoplus_{j=1}^m \mathscr{O}_U \to \bigoplus_{j=1}^n \mathscr{O}_U$. This means that X is covered by open sets U such that $\mathscr{F}|_U$ has a presentation of the form

$$\bigoplus_{j=1}^m \mathscr{O}_U \longrightarrow \bigoplus_{j=1}^n \mathscr{O}_U \longrightarrow \mathscr{F}|_U \longrightarrow 0$$

As in the case of $\mathscr{O}_X$-modules, the pullback of a $\mathscr{O}_X$-module of finite presentation is of finite presentation. We also note that any $\mathscr{O}_X$-module of finite presentation is in particular quasi-coherent.

Proposition 1.4.13. *Let $(X, \mathscr{O}_X)$ be a ringed space. Let $\mathscr{F}$ be a $\mathscr{O}_X$-module of finite presentation.*

(a) If $\psi : \mathscr{O}_X^{\oplus r} \longrightarrow \mathscr{F}$ is a surjective homomorphism, then $\ker \psi$ *is of finite type.*

(b) If $\varphi : \mathscr{G} \longrightarrow \mathscr{F}$ is a surjective homomorphism with $\mathscr{G}$ of finite type, then $\ker \theta$ *is of finite type.*

Proof. □

Proposition 1.4.14. *Let $(X, \mathscr{O}_X)$ be a ringed space and $\mathscr{F}$ be an $\mathscr{O}_X$-module of finite presentation. Then for any $\mathscr{O}_X$-module $\mathscr{H}$, the canonical homomorphism*

$$(\mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{H}))_x \longrightarrow \mathrm{Hom}_{\mathscr{O}_{X,x}}(\mathscr{F}_x, \mathscr{G}_x)$$

is bijective.

Proof. □

Proposition 1.4.15. *Let $(X, \mathscr{O}_X)$ be a ringed space and $\mathscr{F}, \mathscr{G}$ be finitely presented $\mathscr{O}_X$-module. If for a point $x \in X$, $\mathscr{F}_x$ and $\mathscr{G}_x$ are isomorphic $\mathscr{O}_{X,x}$-modules, then there exists an open neighbourhood U of x such that $\mathscr{F}|_U \cong \mathscr{G}|_U$.*

Proof. If $\varphi : \mathscr{F}_x \longrightarrow \mathscr{G}_x$ and $\psi : \mathscr{G}_x \longrightarrow \mathscr{F}_x$ are the isomorphisms, there exists, by Proposition 1.4.14, an open neighborhood V of x and a section u (resp. v) of $\mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})$ (resp. $\mathscr{H}om_{\mathscr{O}_X}(\mathscr{G} \circ \mathscr{F})$) over V such that $u_x = \varphi$ (resp. $v_x = \psi$). As $(u \circ v)_x$ and $(v \circ u)_x$ are the identities, by Proposition 1.4.14 again there exists an open neighborhood $U \subseteq V$ of x such that $(u \circ v)|_U$ and $(v \circ u)|_U$ are idnetities, whence the proposition. □

1.4.3 Coherent sheaves

Let $(X, \mathscr{O}_X)$ be a ringed space. Let $\mathscr{F}$ be a sheaf of $\mathscr{O}_X$-modules. We say that $\mathscr{F}$ is a **coherent** $\mathscr{O}_X$-module if $\mathscr{F}$ is of finite type and for every open $U \subseteq X$ and every finite collection $s_1, \ldots, s_n$ of sections of $\mathscr{F}$ over U, the kernel of the associated map $\bigoplus_{i=1}^n \mathscr{O}_U \to \mathscr{F}|_U$ is of finite type. The category of coherent $\mathscr{O}_X$-modules is denoted $\mathbf{Coh}(\mathscr{O}_X)$. This is a more reasonable object than the category of quasi-coherent sheaves, in the sense that it is at least an abelian subcategory of $\mathbf{Mod}(\mathscr{O}_X)$ no matter what X is. However, the pullback of a coherent module is almost never coherent in the general setting of ringed spaces.

Proposition 1.4.16. *Let $(X, \mathscr{O}_X)$ be a ringed space. Any coherent $\mathscr{O}_X$-module is of finite presentation and hence quasi-coherent.*

Proof. Let $\mathscr{F}$ be a coherent sheaf on X and let x be a point of X. We may find an open neighbourhood U and sections $(s_i)_{1 \le i \le n}$ of $\mathscr{F}$ over U such that the associated homomorphism $\varphi : \bigoplus_{i=1}^n \mathscr{O}_U \to \mathscr{F}|_U$ is surjective. Since $\ker \varphi$ is also of finite type, we may find an open neighbourhood $x \in V \subseteq U$ and sections $(t_j)_{1 \le j \le m}$ of $\bigoplus_{i=1}^m \mathscr{O}_V$ which generate the kernel of $\varphi|_V$. Then over V we get the presentation

$$\bigoplus_{j=1}^m \mathscr{O}_V \longrightarrow \bigoplus_{i=1}^n \mathscr{O}_V \longrightarrow \mathscr{F}|_V \longrightarrow 0$$

which shows that $\mathscr{F}$ is of finite presentation. □

Example 1.4.17. Suppose that X is a point. In this case the definition above gives a notion for modules over rings. What does the definition of coherent mean? It is closely related to the notion of Noetherian, but it is not the same: namely, the ring $A = \mathbb{C}[x_1, x_2, x_3, \dots]$ is coherent as a module over itself but not Noetherian as a module over itself.

Proposition 1.4.18. *Let $(X, \mathscr{O}_X)$ be a ringed space.*

(a) Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a homomorphism from an $\mathscr{O}_X$-module $\mathscr{F}$ of finite type to a coherent $\mathscr{O}_X$-module $\mathscr{G}$. Then $\ker \varphi$ *is of finite type.*

(b) Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a homomorphism of coherent $\mathscr{O}_X$-modules. Then $\ker \varphi$ *and* $\operatorname{coker} \varphi$ *are coherent.*

(c) Given a short exact sequence of $\mathscr{O}_X$-modules $0 \to \mathscr{F}_1 \to \mathscr{F}_2 \to \mathscr{F}_3 \to 0$ if two out of three are coherent so is the third.

(d) The category of coherent $\mathscr{O}_X$-modules is abelian and the inclusion functor $\mathbf{Coh}(\mathscr{O}_X) \to \mathbf{Mod}(\mathscr{O}_X)$ *is exact.*

Proof. Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a homomorphism where $\mathscr{F}$ is of finite type and $\mathscr{G}$ is coherent. Let us show that $\ker \varphi$ is of finite type. Pick $x \in X$ and choose an open neighbourhood U of x in X such that $\mathscr{F}|_U$ is generated by $s_1, \dots, s_n$. By definition the kernel $\mathscr{K}$ of the induced map $\bigoplus_{i=1}^n \mathscr{O}_U \to \mathscr{G}, e_i \mapsto \varphi(s_i)$ is of finite type. Hence $\ker \varphi$ which is the image of the composition $\mathscr{K} \to \bigoplus_{i=1}^n \mathscr{O}_U \to \mathscr{F}$ is of finite type.

Now consider the case of assertion (b). By assertion (a) the kernel of φ is of finite type and hence is coherent as a subsheaf of $\mathscr{F}$. With the same hypotheses let us show that $\operatorname{coker} \varphi$ is coherent. Since $\mathscr{G}$ is of finite type so is $\operatorname{coker} \varphi$. Let $U \subseteq X$ be open and let $\bar{s}_i \in \operatorname{coker} \varphi(U), 1 \leq i \leq n$ be sections. We have to show that the kernel of the associated morphism $\overline{\psi} : \bigoplus_{i=1}^n \mathscr{O}_U \to \operatorname{coker} \varphi$ has finite type. There exists an open covering of U such that on each open all the sections $\bar{s}_i$ lift to sections s_i of $\mathscr{G}$. Hence we may assume this is the case over U. Thus $\overline{\psi}$ lifts to $\psi : \bigoplus_{i=1}^n \mathscr{O}_U \to \mathscr{G}$. Consider the following diagram

$$\begin{tikzcd}
0 \arrow[r] & \ker \psi \arrow[r] \arrow[d] & \bigoplus_{i=1}^n \mathscr{O}_U \arrow[r] \arrow[d, equal] & \mathscr{G} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \ker \overline{\psi} \arrow[r] & \bigoplus_{i=1}^n \mathscr{O}_U \arrow[r] & \operatorname{coker} \psi \arrow[r] & 0
\end{tikzcd}$$

By the snake lemma we have $\operatorname{im} \varphi \cong \operatorname{coker}(\ker \psi \to \ker \overline{\psi})$, thus there is a short exact sequence $0 \to \ker \psi \to \ker \overline{\psi} \to \operatorname{im} \varphi \to 0$. Hence by Proposition 1.4.12 we see that $\ker \overline{\psi}$ is of finite type.

Let $0 \to \mathscr{F}_1 \to \mathscr{F}_2 \to \mathscr{F}_3 \to 0$ be a short exact sequence of $\mathscr{O}_X$-modules. By part (b) it suffices to prove that if $\mathscr{F}_1$ and $\mathscr{F}_3$ are coherent so is $\mathscr{F}_2$. By Proposition 1.4.12 we see that $\mathscr{F}_2$ has finite type. Let $s_1, \dots, s_n$ be finitely many local sections of $\mathscr{F}_2$ defined over a common open U of X. We have to show that the module of relations $\mathscr{K}$ between them is of finite type. Consider the following commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & 0 \arrow[r] \arrow[d] & \bigoplus_{i=1}^n \mathcal{O}_U \arrow[r] \arrow[d] & \bigoplus_{i=1}^n \mathcal{O}_U \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathscr{F}_1 \arrow[r] & \mathscr{F}_2 \arrow[r] & \mathscr{F}_3 \arrow[r] & 0 \end{tikzcd}$$

with obvious notation. By the snake lemma we get a short exact sequence $0 \to \mathscr{K} \to \mathscr{K}_3 \to \mathscr{F}_1$ where $\mathscr{K}_3$ is the module of relations among the images of the sections s_i in $\mathscr{F}_3$. Since $\mathscr{F}_3$ is coherent we see that $\mathscr{K}_3$ is finite type. Since $\mathscr{F}_1$ is coherent we see that the image $\mathscr{I}$ of $\mathscr{K}_3 \to \mathscr{F}_1$ is coherent. Hence $\mathscr{K}$ is the kernel of the map $\mathscr{K} \to \mathscr{I}$ between a finite type sheaf and a coherent sheaves and hence finite type by (b). $\square$

Corollary 1.4.19. *Let $\mathscr{F}_1 \xrightarrow{u} \mathscr{F}_2 \xrightarrow{v} \mathscr{F}_3 \xrightarrow{w} \mathscr{F}_4 \xrightarrow{t} \mathscr{F}_5$ be an exact sequence of $\mathcal{O}_X$-modules. If $\mathscr{F}_1, \mathscr{F}_2, \mathscr{F}_4$ and $\mathscr{F}_5$ are coherent, then $\mathscr{F}_3$ is coherent.*

Proof. In fact, $\operatorname{coker} u = \mathscr{F}_2/\ker v$ and $\operatorname{im} w = \ker t$ are coherent, and it suffices to consider the exact sequence

$$0 \longrightarrow \operatorname{coker} u \longrightarrow \mathscr{F}_3 \longrightarrow \operatorname{im} w \longrightarrow 0$$

and apply Proposition 1.4.18. $\square$

Corollary 1.4.20. *Let $\mathscr{F}$ and $\mathscr{G}$ be two coherent subsheaves of a coherent sheaf $\mathscr{K}$. The sheaves $\mathscr{F} + \mathscr{G}$ and $\mathscr{F}/\mathscr{G}$ are coherent.*

Proof. The sheaf $\mathscr{F} + \mathscr{G}$ is a subsheaf of $\mathscr{K}$ of fintie type, so it is coherent by definition. As for $\mathscr{F} \cap \mathscr{G}$, it is the kernel of $\mathscr{F} \to \mathscr{K}/\mathscr{G}$, so is coherent. $\square$

Corollary 1.4.21. *If $\mathscr{F}$ and $\mathscr{G}$ are coherent $\mathcal{O}_X$-modules, so are $\mathscr{F} \otimes_{\mathcal{O}_X} \mathscr{G}$ and $\mathscr{H}om_{\mathcal{O}_X}(\mathscr{F}, \mathscr{G})$.*

Proof. Since the question is local, we can assume that $\mathscr{F}$ is the cokernel of a homomorphism $\mathcal{O}_X^n \to \mathcal{O}_X^m$. Then $\mathscr{F} \otimes_{\mathcal{O}_X} \mathscr{G}$ is isomorphic to the cokernel of the homomorphism $\mathcal{O}_X^n \otimes_{\mathcal{O}_X} \mathscr{G} \to \mathcal{O}_X^m \otimes_{\mathcal{O}_X} \mathscr{G}$, which is identified with the cokernel of $\mathscr{G}^n \to \mathscr{G}^m$. Since $\mathscr{G}$ is coherent, we then see $\mathscr{F} \otimes_{\mathcal{O}_X} \mathscr{G}$ is also coherent.

Similarly, in view of Proposition 1.4.14, we have an exact sequence $\square$

Corollary 1.4.22. *Let $\mathscr{F}$ be a coherent $\mathcal{O}_X$-module and $\mathscr{I}$ be an coherent ideal of $\mathcal{O}_X$. Then the $\mathcal{O}_X$-module $\mathscr{I}\mathscr{F}$ is coherent.*

Proof. The image of the canonical homomorphism $\mathscr{I} \otimes_{\mathcal{O}_X} \mathscr{F} \to \mathscr{F}$ is $\mathscr{I}\mathscr{F}$, so this follows from Corollary 1.4.21. $\square$

Corollary 1.4.23. *Let X be a ringed space and $\varphi : \mathscr{F} \to \mathscr{G}$ be a homomorphism of $\mathcal{O}_X$-modules. Assume $\mathscr{F}$ of finite type, $\mathscr{G}$ is coherent and the homomorphism $\varphi : \mathscr{F}_x \to \mathscr{G}_x$ is injective for a point $x \in X$. Then there exists an open neighbourhood $x \in U \subseteq X$ such that $\varphi|_U$ is injective.*

Proof. Denote by $\mathscr{K} \subseteq \mathscr{F}$ the kernel of φ. By Proposition 1.4.18 we see that $\mathscr{K}$ is a finite type $\mathcal{O}_X$-module. Our assumption is that $\mathscr{K}_x = 0$, so by Proposition 1.4.6(iv) there exists an open neighbourhood U of x such that $\mathscr{K}|_U = 0$. $\square$

We say an $\mathscr{O}_X$-algebra $\mathscr{A}$ is **coherent** if $\mathscr{A}$ is a coherent $\mathscr{O}_X$-module. In particular, $\mathscr{O}_X$ is a coherent $\mathscr{O}_X$-algebra if, for any open subset $U \subseteq X$ and any homomorphism $\mathscr{O}_X^n|_U \to \mathscr{O}_X|U$ of $\mathscr{O}_U$-modules, the kernel of this homomorphism is of finite type. We then say that $\mathscr{O}_X$ is a coherent sheaf of rings. If $\mathscr{O}_X$ is a coherent sheaf of rings, any $\mathscr{O}_X$-module of finite presentation is coherent in view of Proposition 1.4.18.

Example 1.4.24. The annihilator of a $\mathscr{O}_X$-module $\mathscr{F}$ is the kernel $\mathscr{I}$ of the homomorphism $\mathscr{O}_X \to \mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{F})$ which sends a section $s \in \Gamma(U, \mathscr{O}_X)$ to the multiplication by s in $\mathrm{Hom}(\mathscr{F}|_U, \mathscr{F}|_U)$. If $\mathscr{O}_X$ is a coherent sheaf of rings and if $\mathscr{F}$ is a coherent $\mathscr{O}_X$-module, $\mathscr{I}$ is coherent by Proposition 1.4.18, and it follows from Proposition 1.4.14 that for each $x \in X$, $\mathscr{I}_x$ is the annihilator of the $\mathscr{O}_{X,x}$-module $\mathscr{F}_x$.

Proposition 1.4.25. *Suppose that $\mathscr{O}_X$ is a coherent sheaf of rings and let $\mathscr{I}$ be a coherent ideal of $\mathscr{I}$. For an $(\mathscr{O}_X/\mathscr{I})$-module $\mathscr{F}$ to be coherent, it is necessary and sufficient that as an $\mathscr{O}_X$-module, $\mathscr{F}$ is coherent. In particular, $\mathscr{O}_X/\mathscr{I}$ is a coherent sheaf of rings.*

Proof. We note that $\mathscr{O}_X/\mathscr{I}$ is a coherent $\mathscr{O}_X$-module. If $\mathscr{F}$ is a coherent $(\mathscr{O}_X/\mathscr{I})$, any point of X admits an open neighborhood U such that $\mathscr{F}|_U$ is the cokernel of a homomorphism $(\mathscr{O}_X/\mathscr{I})^m|_U \to (\mathscr{O}_X/\mathscr{I})^n|_U$, so $\mathscr{F}$ is a coherent $\mathscr{O}_X$-module.

Conversely, suppose that $\mathscr{F}$, as an $\mathscr{O}_X$-module, is coherent. First, since $\mathscr{F}$ is an $\mathscr{O}_X$-module of finite type, it is an $(\mathscr{O}_X/\mathscr{I})$-module of finite type. On ther other hand, let U be an open subset of X and $u : (\mathscr{O}_X/\mathscr{I})^n|_U \to \mathscr{F}|_U$ is an $(\mathscr{O}_X/\mathscr{I})$-homomorphism; by composing with the canonical homomorphism $v : \mathscr{O}_X^n|_U \to (\mathscr{O}_X/\mathscr{I})^n|_U$, we obtain a homomorphism $u \circ v : \mathscr{O}_X^n|_U \to \mathscr{F}|_U$, and $\ker(u \circ v)$ is by hypothesis of finite type. But since v is surjective, $\ker u$ is the image of $\ker(u \circ v)$ by v, so is of finite type. □

Proposition 1.4.26. *Let $f : X \to Y$ be a morphism of ringed spaces and suppose that $\mathscr{O}_X$ is a coherent rings. Then for any coherent $\mathscr{O}_Y$-module $\mathscr{G}$, $f^*(\mathscr{G})$ is a coherent $\mathscr{O}_X$-module.*

Proof. Let V be an open subset of Y such that $\mathscr{G}|_V$ is the cokernel of a homomorphism $v : \mathscr{O}_Y^n|_V \to \mathscr{O}_Y^n|_V$. As f_U^* is right exact, we have $f^*(\mathscr{G})|_U = f_U^*(\mathscr{G}|_V)$ (where $U = f^{-1}(V)$) and the cokernel of the homomorphism $f_U^*(v) : \mathscr{O}_X^m|_U \to \mathscr{O}_X^n|_U$ is coherent by Proposition 1.4.18. □

1.4.4 Locally free sheaves

Let $(X, \mathscr{O}_X)$ be a ringed space. We say an $\mathscr{O}_X$-module $\mathscr{F}$ is **locally free** if, for any $x \in X$, there exists an open neighborhood U of x such that $\mathscr{F}|_U$ is isomorphic to an $(\mathscr{O}_X|_U)$-module of the form $\mathscr{O}_X^{\oplus I}|_U$. If for any open subset U the set I is finite, we say that $\mathscr{F}$ is **of finite rank**. If for any open subset U, the set I has n elements, we say that $\mathscr{F}$ is **of rank n**. If $\mathscr{F}$ is a locally free $\mathscr{O}_X$-module of finite rank, for any point $x \in X$, $\mathscr{F}_x$ is a free $\mathscr{O}_X$-module of finite rank $n(x)$, and there exists an open neighborhood U of x such that $\mathscr{F}|_U$ is of rank $n(x)$. If X is connected, we then see $n(x)$ is constant.

It is clear that any locally finite sheaf is quasi-coherent, and if $\mathscr{O}_X$ is a coherent sheaf of rings, any locally free $\mathscr{O}_X$-module of finite rank is coherent. If $\mathscr{E}$ is locally free, $\mathscr{E} \otimes_{\mathscr{O}_X} \mathscr{F}$ is an exact functor on the category of $\mathscr{O}_X$-modules.

We will mostly consider locally free $\mathscr{O}_X$-modules of finite rank, so when we mention the notation of a locally free $\mathscr{O}_X$-modules, it should be understood that we mean locally free $\mathscr{O}_X$-modules of finite rank.

Example 1.4.27. Let $\mathscr{F}$ be an $\mathscr{O}_X$-module of finite presentation. Then if for a point $x \in X$, $\mathscr{F}_x$ is a free module of rank n, there exists an open neighborhood U of x such that $\mathscr{F}|_U$ is locally free of rank n (Proposition 1.4.15).

Proposition 1.4.28. *Let $\mathscr{E}$, $\mathscr{F}$ be $\mathscr{O}_X$-modules and consider the canonical functorial homomorphism*

$$\mathscr{E}^* \otimes_{\mathscr{O}_X} \mathscr{F} = \mathscr{H}om_{\mathscr{O}_X}(\mathscr{E}, \mathscr{O}_X) \otimes_{\mathscr{O}_X} \mathscr{F} \to \mathscr{H}om_{\mathscr{O}_X}(\mathscr{E}, \mathscr{F})$$

If $\mathscr{F}$ or $\mathscr{E}$ is locally free of finite rank, then this homomorphism is bijective.

Proof. The homomorphism is defined by the following: for any open subset U and a couple (u, t), where $u \in \Gamma(U, \mathscr{E}^*) = \mathrm{Hom}(\mathscr{E}|_U, \mathscr{O}_X|_U)$ and $t \in \Gamma(U, \mathscr{F})$, we associate the element of $\mathrm{Hom}(\mathscr{E}|_U, \mathscr{F}|_U)$ whose stalk at each point $x \in U$ send $s_x \in \mathscr{E}_x$ to the element $u_x(s_x)t_s \in \mathscr{F}_x$. Since the question is local, we may assume that $\mathscr{E} = \mathscr{O}_X^n$ or $\mathscr{F} = \mathscr{O}_X^n$. As for any $\mathscr{O}_X$-module $\mathscr{G}$, $\mathscr{H}om_{\mathscr{O}_X}(\mathscr{O}_X^n, \mathscr{G})$ is canonically isomorphic to $\mathscr{G}^n$ and $\mathscr{H}om_{\mathscr{O}_X}(\mathscr{G}, \mathscr{O}_X^n)$ is isomorphic to $(\mathscr{G}^n)^*$, we are reduced to the case $\mathscr{E} = \mathscr{F} = \mathscr{O}_X$, and the claim is then immediate. □

If $\mathscr{L}$ is locally free of rank 1, so is its dual $\mathscr{L}^* = \mathscr{H}om_{\mathscr{O}_X}(\mathscr{L}, \mathscr{O}_X)$, because this is ture for $\mathscr{L} = \mathscr{O}_X$ and the question is local. Moreover, we have a canonical isomorphism

$$\mathscr{H}om_{\mathscr{O}_X}(\mathscr{L}, \mathscr{O}_X) \otimes_{\mathscr{O}_X} \mathscr{L} \cong \mathscr{O}_X.$$

In fact, it suffices to prove that the canonical homomorphism $\mathscr{O}_X \to \mathscr{H}om_{\mathscr{O}_X}(\mathscr{L}, \mathscr{L})$ is bijective, and for this we may assume that $\mathscr{L} = \mathscr{O}_X$, and then the claim is immediate. Due to this, we put $\mathscr{L}^{-1} = \mathscr{H}om_{\mathscr{O}_X}(\mathscr{L}, \mathscr{O}_X)$, and $\mathscr{L}^{-1}$ is called the inverse of $\mathscr{L}$.

If $\mathscr{L}$ and $\mathscr{L}'$ are two $\mathscr{O}_X$-modules locally free of rank 1, so is their tensor product $\mathscr{L} \otimes_{\mathscr{O}_X} \mathscr{L}'$, since locally we have $\mathscr{O}_X \otimes_{\mathscr{O}_X} \mathscr{O}_X \cong \mathscr{O}_X$. For any integer $n \geq 1$, we denote by $\mathscr{L}^{\otimes n}$ the n-fold tensor product $\mathscr{L}$, which is also a locally free $\mathscr{O}_X$-module of rank 1; by convention, we set $\mathscr{L}^{\otimes 0} = \mathscr{O}_X$ and $\mathscr{L}^{\otimes(-n)} = (\mathscr{L}^{-1})^{\otimes n}$. Then there exists a canonical isomorphism

$$\mathscr{L}^{\otimes m} \otimes_{\mathscr{O}_X} \mathscr{L}^{\otimes n} \cong \mathscr{L}^{\otimes(n+m)}$$

Proposition 1.4.29. *Let $f : Y \to X$ be a morphism of ringed spaces. If $\mathscr{E}$ is a locally free $\mathscr{O}_X$-module (resp. locally free $\mathscr{O}_X$-module of rank n), then $f^*(\mathscr{E})$ is a locally free $\mathscr{O}_Y$-module (resp. locally free $\mathscr{O}_Y$-module of rank n). Moreover, we have a canonical isomorphism $f^*(\mathscr{E}^*) = (f^*(\mathscr{E}))^*$.*

Proof. The first assertion from the fact that f^* commutes with direct sums and $f^*(\mathscr{O}_X) = \mathscr{O}_Y$, the second assertin can be checked for the case $\mathscr{E} = \mathscr{O}_X$, since the question is local. □

Let $\mathscr{L}$ be a locally free $\mathscr{O}_X$-module of rank 1. We denote by $\Gamma_*(X, \mathscr{L})$ or simple $\Gamma_*(\mathscr{L})$ the abelian group of the direct sum $\bigoplus_{n \in \mathbb{Z}} \Gamma(X, \mathscr{L}^{\otimes n})$. We endow it a graded ring structure by defining the product of a couple (s_n, s_m), where $s_n \in \Gamma(X, \mathscr{L}^{\otimes n})$ and $s_m \in \Gamma(X, \mathscr{L}^{\otimes m})$, the section of $\mathscr{L}^{\otimes(n+m)}$ over X corresponding to the section $s_n \otimes s_m$ of $\mathscr{L}^{\otimes n} \otimes_{\mathscr{O}_X} \mathscr{L}^{\otimes m}$. The associative of this multiplication is immediately verified, and it is clear that $\Gamma_*(X, \mathscr{L})$ is a covariant functor on $\mathscr{L}$ with values in the category of graded rings of type $\mathbb{Z}$.

If now $\mathscr{F}$ is an $\mathscr{O}_X$-module, we set

$$\Gamma_*(\mathscr{L}, \mathscr{F}) = \bigoplus_{n\in\mathbb{Z}} \Gamma(X, \mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{L}^{\otimes n}).$$

We endow this abelian group a graded $\Gamma_*(\mathscr{L})$-module structure by the following: for any couple (s_n, u_m), where $s_n \in \Gamma(X, \mathscr{L}^{\otimes n})$ and $u_m \in \Gamma(X, \mathscr{F}\otimes\mathscr{L}^{\otimes n})$, we associate the section of $\mathscr{F}\otimes\mathscr{L}^{\otimes(n+m)}$ that corresponds to $s_n \otimes u_m$; the verification that this defines a module structure is immediate. For fixed X and $\mathscr{L}$, we see $\Gamma_*(\mathscr{L}, \mathscr{F})$ is a covariant functor on $\mathscr{F}$ with values in the category of graded $\Gamma_*(\mathscr{L})$-modules. For X and $\mathscr{F}$ fixed, this is a covariant functor on $\mathscr{L}$ iwth values in the category of abelian groups.

Let $f : Y \to X$ be a morphism of ringed spaces. The canonical homomorphism $\rho : \mathscr{L}^{\otimes n} \to f_*(f^*(\mathscr{L}^{\otimes n}))$ defines a homomorphism $\Gamma(X, \mathscr{L}^{\otimes n}) \to \Gamma(Y, f^*(\mathscr{L}^{\otimes n}))$ of abelian groups, and as $f^*(\mathscr{L}^{\otimes n}) = (f^*(\mathscr{L}))^{\otimes n}$, it gives a homomorphism $\Gamma_*(\mathscr{L}) \to \Gamma_*(f^*(\mathscr{L}))$ of graded rings. Similarly, the canonical homomorphism $\mathscr{F}\otimes\mathscr{L}^{\otimes n} \to f_*(f^*(\mathscr{F}\otimes\mathscr{L}^{\otimes n}))$ defines a homomorphism $\Gamma(X, \mathscr{F} \otimes \mathscr{L}^{\otimes n}) \to \Gamma(Y, f^*(\mathscr{F} \otimes \mathscr{L}^{\otimes n}))$ and as

$$f^*(\mathscr{F} \times \mathscr{L}^{\otimes n}) = f^*(\mathscr{F}) \otimes (f^*(\mathscr{L}))^{\otimes n}$$

these homomorphisms give rise to a bi-homomorphism $\Gamma_*(\mathscr{L}, \mathscr{F}) \to \Gamma_*(f^*(\mathscr{L}), f^*(\mathscr{F}))$ of graded modules.

Proposition 1.4.30. *Let $f : X \to Y$ be a morphism of ringed spaces, $\mathscr{F}$ be an $\mathscr{O}_X$-module, and $\mathscr{E}$ be a locally free $\mathscr{O}_Y$-module of finite rank. Then there exists a canonical isomorphism*

$$f_*(\mathscr{F}) \otimes_{\mathscr{O}_Y} \mathscr{E} \cong f_*(\mathscr{F} \otimes_{\mathscr{O}_X} f^*(\mathscr{E})).$$

Proof. For any $\mathscr{O}_Y$-module $\mathscr{E}$, we have canonical homomorphisms

$$f_*(\mathscr{F}) \otimes_{\mathscr{O}_Y} \mathscr{E} \xrightarrow{1\otimes\rho_{\mathscr{E}}} f_*(\mathscr{E}) \otimes_{\mathscr{O}_Y} f_*(f^*(\mathscr{E})) \xrightarrow{\alpha} f_*(\mathscr{F} \otimes_{\mathscr{O}_X} f^*(\mathscr{E}))$$

Since $\mathscr{E}$ is locally free and the question is local on Y, we may assume that $\mathscr{E} = \mathscr{O}_Y^n$; as f_* and f^* commutes with finite direct sums, we can also suppose that $n = 1$, and in this case, the proposition follows directly from the definition and the relation $f^*(\mathscr{O}_Y) = \mathscr{O}_X$. □

Proposition 1.4.31. *Let $f : X \to Y$ be a morphism of ringed spaces, $\mathscr{F}, \mathscr{G}$ be two $\mathscr{O}_Y$-modules, and suppose that $\mathscr{F}$ is locally free of finite rank. Then the canonical homomorphism*

$$f^*(\mathscr{H}om_{\mathscr{O}_Y}(\mathscr{F}, \mathscr{G})) \to \mathscr{H}om_{\mathscr{O}_X}(f^*(\mathscr{F}), f^*(\mathscr{G}))$$

is an isomorphism.

Proof. Since the question is local on Y, we can assume that $\mathscr{F} = \mathscr{O}_Y^n$; then $\mathscr{H}om_{\mathscr{O}_Y}(\mathscr{O}_Y, \mathscr{G})$ is identified with $\mathscr{G}^n$, $f^*(\mathscr{F})$ is identified with $\mathscr{O}_X^n$, and $\mathscr{H}om_{\mathscr{O}_X}(f^*(\mathscr{F}), f^*(\mathscr{G}))$ is identified with $(f^*(\mathscr{G}))^n$, whence the assertion. □

Let X be a ringed space. For any integer $n > 0$, we prove that there is a *set* $\mathfrak{M}_n$ (denoted also by $\mathfrak{M}_n(X)$) of locally free $\mathscr{O}_X$-module of rank n such that any locally free $\mathscr{O}_X$-module of rank n is isomorphic to an element of $\mathfrak{M}_n$. For this, we consider the set $\mathfrak{R}_n$ of couples $(\mathfrak{U}, \Theta)$, where $\mathfrak{U}$ is an open covering of X and Θ is a family $(\theta_{UV})_{(U,V)\in\mathfrak{U}\times\mathfrak{U}}$, where θ_{UV} is an automorphism of $\mathscr{O}_X^n|_{(U\cap V)}$, θ_{UU} is the identity automorphism of $\mathscr{O}_U^n$, and the family (θ_{UV}) satisfies the cocycle conditions. Then any element $(\mathfrak{U}, \Theta)$ of $\mathfrak{R}_n$ corresponds to a well-defined locally free $\mathscr{O}_X$-module of rank n. If we denote by $\mathfrak{L}_n$ the set of these $\mathscr{O}_X$-modules, then any locally free $\mathscr{O}_X$-module of rank n is isomorphic to one of the elements in $\mathfrak{L}_n$; it then suffices to choose for $\mathfrak{M}_n$ a system of representatives of $\mathfrak{L}_n$ for the equivalence condition: $\mathscr{E}$ and $\mathscr{E}'$ are equivalent if they are isomorphic. For any locally free $\mathscr{O}_X$-module $\mathscr{E}$ of rank n, we denote by $\mathrm{cl}(\mathscr{E})$ the uique element of $\mathfrak{M}_n$ which is isomorphic to $\mathscr{E}$.

We can define a composition law on the set $\mathfrak{M}_1$ by associating two elements $\mathscr{L}, \mathscr{L}'$ of $\mathfrak{M}_1(X)$ the element $\mathrm{cl}(\mathscr{L}\otimes\mathscr{L}')$. It is clear that this law is associative and commutative and with identity element $\mathrm{cl}(\mathscr{O}_X)$. Moreover for any $\mathscr{L} \in \mathfrak{M}_1(X)$, $\mathrm{cl}(\mathscr{L}^{-1})$ is the inverse of $\mathscr{L}$ for this composition law. We then define a commutative group structure on $\mathfrak{M}_1(X)$, and this group is called the **Picard group** of the ringed space X and denoted by $\mathrm{Pic}(X)$. We also note that there exists a canonical isomorphism

$$\varphi_X : H^1(X, \mathscr{O}_X^\times) \xrightarrow{\sim} \mathrm{Pic}(X).$$

For this, we note that for any open subset U of X the multiplicative group $\Gamma(U, \mathscr{O}_X^\times)$ is canonically identified with the grop of automorphisms of $\mathscr{O}_U$-module $\mathscr{O}_U$, which send each section ε of $\mathscr{O}_X^\times$ over U to the automorphism $u : \mathscr{O}_U \to \mathscr{O}_U$ such that $u_x(s_x) = \varepsilon_x s_x$ for any $x \in U$ and any $s_x \in \mathscr{O}_{X,x}$. Let $\mathfrak{U} = (U_\lambda)$ be an open covering of X; the datum that, given any couple (λ, μ) of indices, an automorphism θ_λ of $\mathscr{O}_X|_{U_\lambda\cap U_\mu}$ which satisfy the cocycle conditions, is then equivalent to giving a 1-cochain of the covering $\mathfrak{U}$, with values in $\mathscr{O}_X^\times$. Similarly, the datum that, given an index λ, an automorphism ω_λ of $\mathscr{O}_{U_\lambda}$, is equivalent to giving a 0-cochain of $\mathfrak{U}$ with values in $\mathscr{O}_X^\times$, and the coboundary of this cochain corresponds to the automorphisms $(\omega_\lambda|_{U_\lambda\cap U_\mu}) \circ (\omega_\mu|_{U_\lambda\cap U_\mu})^{-1}$. We then corresponds any 1-cocycle $(\theta_{\lambda\mu})$ of $\mathfrak{U}$ with values in $\mathscr{O}_X^\times$, the element $\mathrm{cl}(X)$ of $\mathrm{Pic}(X)$, where $\mathscr{L}$ is the locally free $\mathscr{O}_X$-module of rank 1 defined by the family $\theta = (\theta_{\lambda\mu})$; two cohomologous cocycles then correspond to the same element of $\mathrm{Pic}(X)$, so we obtain a map $\varphi_{\mathfrak{U}} : H^1(\mathfrak{U}, \mathscr{O}_X^\times) \to \mathrm{Pic}(X)$. Moreover, if $\mathfrak{V}$ is a second open covering of X, which is a refinement of $\mathfrak{U}$, the diagram

$$\begin{tikzcd}
H^1(\mathfrak{U}, \mathscr{O}_X^\times) \arrow[dd] \arrow[dr, "\varphi_{\mathfrak{U}}"] & \\
& \mathrm{Pic}(X) \\
H^1(\mathfrak{V}, \mathscr{O}_X^\times) \arrow[ur, "\varphi_{\mathfrak{V}}"] &
\end{tikzcd}$$

where the vertical arrow is the canonical homomorphism, is commutative. By passing to limit, we then obtain a map $\varphi_X : H^1(X, \mathscr{O}_X^\times) \to \mathrm{Pic}(X)$, where the Čech cohomology group $\check{H}^1(X, \mathscr{O}_X^\times)$ is canonically identified with $H^1(X, \mathscr{O}_X^\times)$. The map φ_X is clearly surjective, since any locally free sheaf of rank 1 is defined by a 1-cocycle. It is injective, because it suffices to show that the maps

$\varphi_{\mathfrak{U}}$ are injective, and this follows from the definition of $H^1(\mathfrak{U}, \mathscr{O}_X^\times)$. It remains to prove that $\varphi_{\mathfrak{U}}$ is a homomorphism of groups. Let $\mathscr{L}, \mathscr{L}'$ be two locally free $\mathscr{O}_X$-modules of rank 1 such that, for any λ, $\mathscr{L}|_{U_\lambda}$ and $\mathscr{L}'|_{U_\lambda}$ are isomorphic to $\mathscr{O}_{U_\lambda}$. There then exists for each λ an element a_λ (resp. a'_λ) of $\Gamma(U_\lambda, \mathscr{L})$ (resp. $\Gamma(U_\lambda, \mathscr{L}')$) such that the elements of $\Gamma(U_\lambda, \mathscr{L})$ (resp. $\Gamma(U_\lambda, \mathscr{L}')$) are of the form $s_\lambda \cdot a_\lambda$ (resp. $s_\lambda \cdot a'_\lambda$) where s_λ runs through $\Gamma(U_\lambda, \mathscr{O}_X)$. The corresponding cocycles $(\varepsilon_{\lambda\mu})$, $(\varepsilon'_{\lambda\mu})$ are such that the relation $s_\lambda \cdot a_\lambda = s_\mu \cdot a_\mu$ (resp. $s_\lambda \cdot a'_\lambda = s_\mu \cdot a'_\mu$) over $U_\lambda \cap U_\mu$ is equivalent to $s_\lambda = \varepsilon_{\lambda\mu} s_\mu$ (resp. $s_\lambda = \varepsilon'_{\lambda\mu} s_\mu$) over $U_\lambda \cap U_\mu$. As the sections of $\mathscr{L} \otimes \mathscr{L}'$ over U_λ are the finite sums of $s_\lambda s'_\lambda \cdot (a_\lambda \otimes a'_\lambda)$ where s_λ, s'_λ runs through $\Gamma(U_\lambda, \mathscr{O}_X)$, it is clear that the cocycle $(\varepsilon_{\lambda\mu}\varepsilon'_{\lambda\mu})$ corresponds to $\mathscr{L} \otimes \mathscr{L}'$, which completes the proof.

Let $f : X' \to X$ be a morphism of ringed spaces. If $\mathscr{L}_1$, $\mathscr{L}_2$ are two locally free $\mathscr{O}_X$-modules of rank 1 and are isomorphic, the $\mathscr{O}_{X'}$-modules $f^*(\mathscr{L}_1)$ and $f^*(\mathscr{L}_2)$ are isomorphic. On the other hand, for any $\mathscr{O}_X$-modules $\mathscr{F}$, $\mathscr{G}$, we have $f^*(\mathscr{F} \otimes \mathscr{G}) = f^*(\mathscr{F}) \otimes f^*(\mathscr{G})$. We then conclude that the morphism f defines a canonical homomorphism of abelian groups

$$\mathrm{Pic}(f) : \mathrm{Pic}(X) \to \mathrm{Pic}(X').$$

On the other hand, we have a canonical homomorphism

$$H^1(f) : H^1(X, \mathscr{O}_X^\times) \to H^1(X', \mathscr{O}_{X'}^\times)$$

corresponds the restriction of the homomorphism $f^\# : \mathscr{O}_X \to f_*(\mathscr{O}_{X'})$ to $\mathscr{O}_X^\times$. We claim that the diagram

$$\begin{tikzcd} H^1(X, \mathscr{O}_X^\times) \arrow[r, "H^1(f)"] \arrow[d, "\varphi_X"', "\sim"] & H^1(X', \mathscr{O}_{X'}^\times) \arrow[d, "\sim"', "\varphi_{X'}"] \\ \mathrm{Pic}(X) \arrow[r, "\mathrm{Pic}(f)"] & \mathrm{Pic}(X') \end{tikzcd}$$

is commutative. In fact, if $\mathscr{L}$ is given by the cocycle $(\varepsilon_{\lambda\mu})$ of an open covering (U_λ) of X, it suffices to prove that $f^*(\mathscr{L})$ is defined by a cocycle whose class is cohomologous to the image of $(\varepsilon_{\lambda\mu})$ under $H^1(f)$. But if $\theta_{\lambda\mu}$ is the automorphism of $\mathscr{O}_X|_{U_\lambda \cap U_\mu}$ corresponding to $\varepsilon_{\lambda\mu}$, it is clear that $f^*(\mathscr{L})$ is obtained by glueing $\mathscr{O}_{X'}|_{\psi^{-1}(U_\lambda)}$ with the automorphisms $g^*(\theta_{\lambda\mu})$, and it sufficies to verify that this corresponds to the cocycle $(f^\#(\varepsilon_{\lambda\mu}))$; this follows also from the definition by identify $\varepsilon_{\lambda\mu}$ with its image under $\rho_{\mathscr{O}_X}$, which is a section of $\psi^{-1}(\mathscr{O}_X)$ over $\psi^{-1}(U_\lambda \cap U_\mu)$.

1.4.5 Locally free sheaves over locally ringed spaces

Proposition 1.4.32. *Let X be a locally ringed space and $\mathscr{E}$ be a locally free $\mathscr{O}_X$-module of finite rank. Then for any section s of $\mathscr{E}$ over X, the set X_s of $x \in X$ such that $s(x) \neq 0$ is open in X and s is invertible over X_s.*

Proof. Since the question is local, we can assume that $\mathscr{E} = \mathscr{O}_X^n$. If $(s_j)_{1 \le j \le n}$ is the projection of s to the n-th component, we see $s(x) \neq 0$ if and only if $s_j(x) \neq 0$ for some j, so X_s is the union of the X_{s_j}, and we are reduced to the case $n = 1$. But for $s \in \Gamma(X, \mathscr{O}_X)$, to say that $s(x) \neq 0$ amounts to that $s_x \notin \mathfrak{m}_x$, hence s_x is inveritble in $\mathscr{O}_{X,x}$. By then there then existes an open neighborhood U of x in X such that $s|_U$ is invertible in $\Gamma(U, \mathscr{O}_X)$, and therefore $s(y) \neq 0$ for any $y \in U$. □

Corollary 1.4.33. *Let X be a locally ringed space and $\mathscr{E}$ be a locally free $\mathscr{O}_X$-module of finite rank. Let $s_1, \ldots, s_p$ be sections of $\mathscr{E}$ over X. Then the set of $x \in X$ such that $s_1(x), \ldots, s_p(x)$ are lineraly independent over $\mathscr{E}_x/\mathfrak{m}_x\mathscr{E}_x$ is open in X.*

Proof. In fact, $\bigwedge^p \mathscr{E}$ is a locally free $\mathscr{O}_X$-module of rank $\binom{n}{p}$. Moreover, for any $x \in X$, $(\bigwedge^p \mathscr{E})_x/\mathfrak{m}_x(\bigwedge^p \mathscr{E})_x$ is identified with $\bigwedge^p(\mathscr{E}_x/\mathfrak{m}_x\mathscr{E}_x)$. If $s = s_1 \wedge \cdots \wedge s_p$, then $s(x)$ is identified with $s_1(x) \wedge \cdots \wedge s_p(x)$, and $s(x) \neq 0$ if and only if $s_1(x), \ldots, s_p(x)$ are linearly independent, whence the assertion by Proposition 1.4.32. □

Corollary 1.4.34. *Let X be a locally ringed space, $\mathscr{E}$ be a locally free $\mathscr{O}_X$-module of rank n, and $s_1, \ldots, s_n$ be sections of $\mathscr{E}$ over X such that, for any $x \in X$, $s_1(x), \ldots, s_n(x)$ are lineraly independent. Then the homomorphism $u : \mathscr{O}_X^n \to \mathscr{E}$ defined by the sections s_i is bijective.*

Proof. Again we can assume that $\mathscr{E} = \mathscr{O}_X^n$ and canonically identity $\bigwedge^n \mathscr{E}$ with $\mathscr{O}_X$. The section $s = s_1 \wedge \cdots \wedge s_n$ is then a section of $\mathscr{O}_X$ over X such that $s(x) \neq 0$ for all $x \in X$, and thus invertible in $\Gamma(X, \mathscr{O}_X)$. We can then define a homomorphism inverse to u by Cramer's rule. □

Proposition 1.4.35. *Let X be a locally ringed space, $\mathscr{F}$ be an $\mathscr{O}_X$-module of finite type, $\mathscr{E}$ a locally free $\mathscr{O}_X$-module of finite rank, and $u : \mathscr{F} \to \mathscr{E}$ be a homomorphism, and x be a point of X. Then the following conditions are equivalent:*

(i) The homomorphism u_x is left invertible (which means u_x is injective and its image in $\mathscr{E}_x$ is a direct factor).

(ii) The homomorphism $u_x \otimes 1 : \mathscr{F}_x/\mathfrak{m}_x\mathscr{F}_x \to \mathscr{E}_x/\mathfrak{m}_x\mathscr{E}_x$ of vector spaces induced by u_x is injective.

(iii) There exists an open neighborhood U of x such that $u|_U : \mathscr{F}|_U \to \mathscr{E}|_U$ is left invertible, the image $u(\mathscr{F})|_U$ is a locally free $\mathscr{O}_U$-module isomorphic to $\mathscr{F}|_U$, which admits a locally free complement in $\mathscr{E}|_U$.

Moreover, the set of $x \in X$ satisfying these equivalent conditions are open.

Proof. The equivalence of (i) and (ii) is a general result in algebras, and it is clear that (iii) implies (i). We now prove that (i) implies (iii); there exists by hypothesis a homomorphism $w : \mathscr{E}_x \to \mathscr{F}_x$ such that $w \circ u_x$ is the identify automorphism on $\mathscr{F}_x$. As $\mathscr{E}$ is locally free of finite rank, hence of finite presentation, it follows from Proposition 1.4.14 that there exists an open neigbourhood U of x and a homomorphism $v : \mathscr{E}|_U \to \mathscr{F}|_U$ such that $w = v_x$, hence $(v \circ (u|_U))_x = v_x \circ u_x$ is the identity automorphism. We then conclude that, by restricting U, we can suppose that $v \circ (u|_U)$ is the identity on $\mathscr{F}|_U$, which means $u|_U$ is left invertible. We then know that $p = (u|_U) \circ v$ is a projection from $\mathscr{E}|_U$ to $u(\mathscr{F})|_U$, and that $u|_U$ is an isomorphism from $\mathscr{F}|_U$ to $u(\mathscr{F})|_U$. We claim that (aftering shrinking U if necessary), $u(\mathscr{E})_U$ and $\ker p$ are locally free $\mathscr{O}_U$-modules (supplementary in $\mathscr{F}|_U$). In fact, $p_x : \mathscr{F}_x \to (u(\mathscr{F}))_x$ is a projection, hence so is $p_x \otimes 1 : \mathscr{E}_x/\mathfrak{m}_x\mathscr{E}_x \to (u(\mathscr{F}))_x/\mathfrak{m}_x(u(\mathscr{F}))_x$. There exists sections s_j $(1 \leq j \leq n)$ of $\mathscr{F}$ over U such that the first m sections s_j $(1 \leq j \leq m)$ are sections of $u(\mathscr{F})$, and the later $n - m$ are sections of $\ker p$, and that the $s_j(x)$ $(1 \leq j \leq n)$ form a base for $\mathscr{F}_x/\mathfrak{m}_x\mathscr{F}_x$. In view of Corollary 1.4.34, by shrinking U, we can suppose that the $\mathscr{O}_X$-module $\mathscr{M}$ generated by s_j for $1 \leq j \leq n$ and the $\mathscr{O}_X$-module $\mathscr{N}$ generated by the s_j for $m+1 \leq j \leq n$ are free and supplementary in $\mathscr{E}|_U$. We have evidently $\mathscr{M} \subseteq u(\mathscr{F})|_U$ and $\mathscr{N} \subseteq \ker p$; on the other hand, if $i : \mathscr{M} \to u(\mathscr{F})|_U$ and $j : \mathscr{N} \to \ker p$ are the canonical injections, then the choice of s_j implies that i_x and j_x are bijections. As $u(\mathscr{F})|_U$ and $\ker p$ are $\mathscr{O}_U$-modules of finite type (the second being the image of $\mathscr{E}|_U$ under $1-p$), we conclude from Proposition 1.4.6 (by shrinking U if necessary) that $\mathscr{M} = u(\mathscr{E})|_U$ and $\mathscr{N} = \ker p$. □

Corollary 1.4.36. *With the hypotheses in Proposition 1.4.35, the following conditions are equivalent:*

(i) For any morphism $g : X' \to X$ of locally ringed spaces, the homomorphism $g^*(u) : g^*(\mathscr{F}) \to g^*(\mathscr{E})$ is injective.

(ii) For any $x \in X$, the homomorphism $u_x \otimes 1 : \mathscr{F}_x/\mathfrak{m}_x\mathscr{F}_x \to \mathscr{E}_x/\mathfrak{m}_x\mathscr{E}_x$ is injective.

(iii) For any $x \in X$, there exists an open neighborhood U of x such that $u|_U : \mathscr{F}|_U \to \mathscr{E}|_U$ is left invertible.

Moreover, if the conditions are satisfied, $\mathscr{F}$ is a locally free $\mathscr{O}_X$-module of finite rank.

Proof. The equivalence of (ii) and (iii) follows from Proposition 1.4.35, so does the last one. The fact that (iii) implies (i) follows from follows from the fact that we can reduce ourselves to the case where $\mathscr{E} = \mathscr{O}_X^n$ and $\mathscr{F} = \mathscr{O}_X^m$ and that g^* is left exact, since the question is local on X. Finally, we show that (ii) is a particular case of (i): it suffices to consider the locally ringed space X' reduced to a point x, with sheaf of rings $\kappa(x)$ (that is, $\mathrm{Spec}(\kappa(x))$). We let $g : X' \to X$ be the canonical morphism which maps x to x and $g^\# : \mathscr{O}_X \to \kappa(x)$ is the cannical homomorphism $\Gamma(U, \mathscr{O}_X) \to \mathscr{O}_{X,x} \to \kappa(x)$ for any open neigbourhood U of x. It is then easily verified that $g^*(u)$ is the homomorphism $u_x \otimes 1$. □

Remark 1.4.1. If u satisfies the conditions of Corollary 1.4.36, then we say that it is **universally injective**.

Corollary 1.4.37. *Let X be a locally ringed space, $\mathscr{F}$, $\mathscr{E}$ be two locally free $\mathscr{O}_X$-modules of finite rank, $u : \mathscr{F} \to \mathscr{E}$ be a homomorphism, and x be a point of X. The following conditions are equivalent:*

(i) The homomorphism $u_x \otimes 1 : \mathscr{F}_x/\mathfrak{m}_x\mathscr{F}_x \to \mathscr{E}_x/\mathfrak{m}_x\mathscr{E}_x$ is surjective.

(ii) The homomorphism u_x is surjective.

(iii) The homomorphism $\bigwedge^m u_x : \bigwedge^m \mathscr{F}_x \to \bigwedge^m \mathscr{E}_x$ (where m is the rank of $\mathscr{F}_x$) is surjective.

(iv) The homomorphism $u_x^t : \mathscr{E}_x^* \to \mathscr{F}_x^*$ is left invertible.

Moreover, the set S of $x \in X$ satisfying these conditions is open in X, $\ker(u)|_S$ is a locally free $\mathscr{O}_S$-module and any $x \in S$ admits an open neighborhood $U \subseteq S$ such that $\ker(u)|_U$ admits in $\mathscr{F}|_U$ a locally free complement (isomorphic to $\mathscr{E}|_U$).

Proof. The equivalence of (i) and (ii) follows from Nakayama's Lemma. Similarly, (iii) is equivalent to that

$$\left(\bigwedge^m u_x\right) \otimes 1 : \left(\bigwedge^m \mathscr{F}_x\right)/\mathfrak{m}_x\left(\bigwedge^m \mathscr{F}_x\right) \to \left(\bigwedge^m \mathscr{E}_x\right)/\mathfrak{m}_x\left(\bigwedge^m \mathscr{E}_x\right)$$

is surjective; but $(\bigwedge^m u_x) \otimes 1$ is identified with

$$\bigwedge^m(u_x \otimes 1) : \bigwedge^m(\mathscr{F}_x/\mathfrak{m}_x\mathscr{F}_x) \to \bigwedge^m(\mathscr{E}_x/\mathfrak{m}_x\mathscr{E}_x),$$

and as $\mathscr{E}_x/\mathfrak{m}_x\mathscr{E}_x$ is a vector space of dimension m over $\kappa(x)$, $\bigwedge^m(u_x \otimes 1)$ is surjective if $u_x \otimes 1$ is surjective, and is zero in the contrary case, whence the equivalence of (i) and (iii). On the other hand, as $(\mathscr{F}^*)^* = \mathscr{F}$, $(\mathscr{E}^*)^* = \mathscr{E}$ and $(u^t)^t = u$, it is the same to say that $u_x \otimes 1$ is surjective and that $(u_x \otimes 1)^t = (u_x)^t \otimes 1 : \mathscr{E}_x^*/\mathfrak{m}_x\mathscr{E}_x^* \to \mathscr{F}_x^*/\mathfrak{m}_x\mathscr{F}_x^*$ is injective, whence the equivalence of (i) and (iv) in view of Proposition 1.4.35. The fact that S is open follows from Proposition 1.4.35. We can then reduce to the case where $S = X$, and the other assertions of the statement are deduced by transposing the conclusions of Proposition 1.4.35 applied to u^t. ◻

Corollary 1.4.38. *With the notations of Corollary 1.4.37, suppose moreover that $\mathscr{F}$ and $\mathscr{E}$ have the same rank at each point. Then, for any $x \in X$, the following conditions are equivalent:*

(i) u_x is left invertible;

(ii) u_x is surjective;

(iii) u_x is bijective.

Moreover, the set of $x \in x$ satisfying these conditions is open in X.

Corollary 1.4.39. *With the notations of Corollary 1.4.37, let $f : X' \to X$ be a morphism of locally ringed spaces and put $\mathscr{F}' = f^*(\mathscr{F})$, $\mathscr{E}' = f^*(\mathscr{E})$, which are locally free $\mathscr{O}_{X'}$-modules of finite rank. Let $u' = f^*(u) : \mathscr{F}' \to \mathscr{E}'$. Then for a point $x' \in X'$, $u'_{x'}$ is surjective (resp. left invertible) if and only if at the point $x = f(x')$, u_x is surjective (resp. left invertible).*

Proof. In fact, we have $\mathscr{F}'_{x'} = \mathscr{F}_x \otimes_{\mathscr{O}_{X,x}} \mathscr{O}_{X',x'}$, $\mathscr{E}'_{x'} = \mathscr{F}_x \otimes_{\mathscr{O}_{X,x}} \mathscr{O}_{X',x'}$, and $u'_{x'}$ is deduced from u_x be base changing $\mathscr{O}_{X,x}$ to $\mathscr{O}_{X',x'}$. If k and k' are the residue fields of x and x', we then have $\mathscr{F}'_{x'} \otimes k' = (\mathscr{F}_x \otimes k) \otimes_k k'$, $\mathscr{F}'_{x'} \otimes k' = (\mathscr{F}_x \otimes k) \otimes_k k'$, and the homomorphism $u'_{x'} \otimes 1 : \mathscr{F}'_{x'} \otimes k' \to \mathscr{E}'_{x'} \otimes k'$ is then deduced from $u_x \otimes 1_k : \mathscr{F}_x \otimes k \to \mathscr{E}_x \otimes k$ by base changing from k to k'. The conclusion then follows from the fact that this base change is faithfully flat, the Nakayama lemma, and Proposition 1.4.35. ◻

Proposition 1.4.40. *Let X be a locally ringed space, $\mathscr{L}$ be an $\mathscr{O}_X$-module of finite type. For that $\mathscr{L}$ to be locally free of rank 1, it is necessary and sufficient that there exists an $\mathscr{O}_X$-module $\mathscr{F}$ such that $\mathscr{L} \otimes_{\mathscr{O}_X} \mathscr{F}$ is isomorphic to $\mathscr{O}_X$. Moreover, any $\mathscr{O}_X$-module with this property is isomorphic to $\mathscr{L}^{-1}$.*

Proof. We have seen that if $\mathscr{L}$ is locally of rank 1 then $\mathscr{L}^{-1} \otimes_{\mathscr{O}_X} \mathscr{L} \cong \mathscr{O}_X$. Moreover, in this case $\mathscr{L}^{-1}$ is isomorphic to $\mathscr{L}^{-1} \otimes_{\mathscr{O}X} \mathscr{O}_X$, hence to $\mathscr{L}^{-1} \otimes_{\mathscr{O}_X} (\mathscr{L} \otimes_{\mathscr{O}_X} \mathscr{F})$ and therefore to $\mathscr{F}$. It then remains to prove that if $\mathscr{L} \otimes_{\mathscr{O}_X} \mathscr{F}$ is isomorphic to $\mathscr{O}_X$ then $\mathscr{L}$ is locally free of rank 1. Let $x \in X$ and put $A = \mathscr{O}_{X,x}$ (which is a local ring with maximal ideal $\mathfrak{m}$), $M = \mathscr{L}_x$, $N = \mathscr{F}_x$. The hypothesis implies that $M \otimes_A N$ is isomorphic to A, and as $(A/\mathfrak{m}) \otimes_A (M \otimes_A N)$ is identified with $(M/\mathfrak{m}M) \otimes_{A/\mathfrak{m}} (N/\mathfrak{m}N)$, this tensor product is isomorphic to $A/\mathfrak{m}$ over the field $A/\mathfrak{m}$, which shows that $M/\mathfrak{m}M$ and $N/\mathfrak{m}N$ are of dimension 1 over $A/\mathfrak{m}$. For any element $z \in M$ not belonging to $\mathfrak{m}M$, we then have $M = Az + \mathfrak{m}M$, which implies $M = Az$ by Nakayama's Lemma. Also, the annihilator of z also annihilates $M \otimes_A N$, which is isomorphic to A, so this annihilator is zero and M is isomorphic to A. There is then a section s of $\mathscr{L}$ over an open neigbourhood U of x such that $t_x \mapsto t_x s_x$ is an isomorphism from $\mathscr{O}_{X,x}$ to $\mathscr{L}_x$. Since $\mathscr{L}$ is of finite type, we can, by shrinking U, suppose that s generates $\mathscr{L}|_U$ (Proposition 1.4.6), which means we have a surjective homomorphism $u : \mathscr{O}_U \to \mathscr{L}|_U$. Moreover, for any $y \in U$, the homomorphism $\mathscr{O}_{X,y}/\mathfrak{m}_y \to \mathscr{L}_y/\mathfrak{m}_y\mathscr{L}_y$ deduced from u is bijective, hence so is u (Proposition **??**). ◻

The $\mathscr{O}_X$-modules locally free of rank 1 over a locally ringed space X are then called the **invertible** $\mathscr{O}_X$-modules.

Proposition 1.4.41. *Let X be a locally ringed space, $\mathscr{L}$ be an invertible $\mathscr{O}_X$-module, and f be a section of $\mathscr{L}$ over X. For any $x \in X$, the following conditions are equivalent:*

(i) f_x generates the $\mathscr{O}_{X,x}$-module $\mathscr{L}_x$.

(ii) $f_x \notin \mathfrak{m}_x\mathscr{L}_x$ (that is, $f(x) \neq 0$).

(iii) There exists a section g of $\mathscr{L}^{-1}$ over an open neigbourhood V of x such that the canonical image of $(f|_V) \otimes g$ in $\Gamma(V, \mathscr{O}_X)$ is the unit element.

Moreover, the set X_f of $x \in X$ satisfying these conditions is open in X.

Proof. The question is local on X so we can assume that $\mathscr{L} = \mathscr{O}_X$, and the proposition then follows. ◻

Chapter 2

Cohomology group of sheaves

In this chapter we consider the cohomology of shaves of modules. First we have a proposition.

Proposition 2.0.1. *A sequence of sheaves of $\mathcal{O}_X$-modules on a space X*

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$$

is exact if and only if for all points $x \in X$ the sequence of stalks is exact. This is equivalent to

(a) For all open sets $U \subseteq X$ the sequence

$$0 \longrightarrow \mathcal{F}(U) \longrightarrow \mathcal{G}(U) \longrightarrow \mathcal{H}(U)$$

is exact.

(b) For any $s \in \mathcal{H}(U)$, we can find a covering $U = \bigcup_i U_i$ by open sets and $s_i \in \mathcal{F}(U_i)$ such that $s|_{U_i} = s_i$.

Applied to $U = X$ this tells us that the functor of global sections $\mathcal{F} \mapsto \mathcal{F}(X)$ is left-exact. It turns out that the category of sheaves of $\mathcal{O}_X$-modules has enough injectives, thus the right derived functors $\mathcal{R}^p\Gamma(X, -)$ exist, and for every sheaf $\mathcal{F}$ on X, the cohomology groups $\mathcal{R}^p\Gamma(X, -)(\mathcal{F})$ are defined. These groups denoted by $H^p(X, \mathcal{F})$ are called the **cohomology groups of the sheaf $\mathcal{F}$** or **the cohomology groups of X with values in $\mathcal{F}$**.

2.1 Definition of sheaf cohomology

We first show that the category $\mathbf{Mod}(\mathcal{O}_X)$ has enough injectives.

Proposition 2.1.1. *Let $(X, \mathcal{O}_X)$ be a ringed space. Then the category $\mathbf{Mod}(\mathcal{O}_X)$ of sheaves of $\mathcal{O}_X$-modules has enough injectives.*

Proof. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. For each point $x \in X$, the stalk $\mathcal{F}_x$ is an $\mathcal{O}_{X,x}$-module. Therefore there is an injection $\mathcal{F}_x \to I_x$, where $\mathcal{I}_x$ is an injective $\mathcal{O}_{X,x}$-module. For each point x, let i_x denote the inclusion of the one-point space $\{x\}$ into X, and consider the sheaf $\mathcal{I} = \prod_{X \in X} i_{x,*}(I_x)$. Here we consider I_x as a sheaf on the one-point space $\{x\}$.

Now for any sheaf $\mathscr{G}$ of $\mathscr{O}_X$-modules, we have

$$\mathrm{Hom}_{\mathscr{O}_X}(\mathscr{G}, \mathscr{I}) = \prod_{x \in X} \mathrm{Hom}_{\mathscr{O}_X}(\mathscr{G}, i_{x,*}(I_x))$$

by definition of the direct product. On the other hand, for each point $x \in X$, by Proposition **??** we have

$$\mathrm{Hom}_{\mathscr{O}_X}(\mathscr{G}, i_{x,*}(I_x)) = \mathrm{Hom}_{\mathscr{O}_{X,x}}(\mathscr{G}_x, I_x)$$

Thus we conclude first that there is a natural morphism of sheaves of $\mathscr{O}_X$-modules $\mathscr{F} \to \mathscr{I}$ obtained from the local maps $\mathscr{F}_x \to \mathscr{I}_x$. It is clearly injective. Second, the functor $\mathrm{Hom}_{\mathscr{O}_X}(-, \mathscr{I})$ is the direct product over all $x \in X$ of the stalk functor $\mathscr{G} \to \mathscr{G}$ which is exact, followed by $\mathrm{Hom}_{\mathscr{O}_{X,x}}(-, I_x)$, which is exact, since I_x is a injective. Hence $\mathrm{Hom}(-, \mathscr{I})$ is an exact functor, and therefore $\mathscr{I}$ is an injective $\mathscr{O}_X$-module. □

Corollary 2.1.2. *If X is any topological space, then the category $\mathbf{Ab}(X)$ of sheaves of abelian groups on X has enough injectives.*

Definition 2.1.3. *Let X be a topological space, and let $\Gamma(X, -) : \mathbf{Ab}(X) \to \mathbf{Ab}(X)$ be the global section functor. The cohomology groups of the sheaf $\mathscr{F}$ or the cohomology groups of X with values in $\mathscr{F}$, denoted by $H^i(X, \mathscr{F})$, are the groups $\mathcal{R}^i\Gamma(X, -)(\mathscr{F})$ induced by the right derived functor $\mathcal{R}^i\Gamma(X, -)$.*

Similarly, we can define cohomology groups of a $\mathscr{O}_X$-module to be the right-derived functor $\mathcal{R}^i\Gamma(X, -)$, where $\Gamma(X, -)$ is viewed as a functor from $\mathbf{Mod}(O_X)$ to $\mathbf{Ab}(X)$. However, it turns out that this definition is unnecessary: the dericed functor of $\Gamma(X, -)$ in $\mathbf{Ab}(X)$ and $\mathbf{Mod}(\mathscr{O}_X)$ coincide.

2.2 Flasque sheaves

Definition 2.2.1. *A sheaf $\mathscr{F}$ on a topological space X is **flasque** if for every open subset U of X the restriction map $\mathscr{F}(X) \to \mathscr{F}(U)$ is surjective.*

We will see shortly that injective sheaves are flasque. Although this is not obvious from the definition, the notion of being flasque is local.

Proposition 2.2.2. *Let $\mathscr{F}$ be an $\mathscr{O}_X$-module. If $\mathscr{F}$ is flasque, so is $\mathscr{F}|_U$ for every open subset U of X. Conversely, if for every $x \in X$, there is a neighborhood U such that $\mathscr{F}|_U$ is flasque, then $\mathscr{F}$ is flasque.*

Proof. The first statement is trivial, let us prove the converse. Given any open set V of X, let s be a section of $\mathscr{F}$ over V. Let T be the set of all pairs (U, σ), where U is an open in X containing V, and σ is an extension of s to U. Partially order T by saying that $(U_1, \sigma_1) \leq (U_2, \sigma_2)$ if $U_1 \subseteq U_2$ and σ_2 extends σ_1, and observe that T is inductive, which means that every chain has an upper bound. Zorn's lemma provides us with a maximal extension of s to a section σ over an open set U_0. Were U_0 not X, there would exist an open set W in X not contained in U_0 such that $F|_W$ is flasque. Thus we could extend the section $\sigma|_{U_0 \cap W}$ to a section σ_0 of $\mathscr{F}$. Since σ and σ_0 agree on $U_0 \cap W$ by construction, their common extension to $U_0 \cup W$ extends s, a contradiction. □

Proposition 2.2.3. *Every $\mathscr{O}_X$-module may be embedded in a canonical functorial way into a flasque $\mathscr{O}_X$-module. Consequently, every $\mathscr{O}_X$-module has a canonical flasque resolution.*

Proof. Let $\mathscr{F}$ be an $\mathscr{O}_X$-module, and consider the Godement construction

$$U \mapsto \prod_{x \in U} \mathscr{F}_x$$

which we denote by $C^0(X, \mathscr{F})$. It is immediate that we have an injection of $\mathscr{O}_X$-modules $\mathscr{F} \to C^0(X, \mathscr{F})$ by Proposition 1.1.9. An element of $C^0(X, \mathscr{F})$ over any open set U is a collection (s_x) of elements indexed by U, each s_x lying over the $\mathscr{O}_{X,x}$-module $\mathscr{F}_x$. Such a sheaf is flasque because every U-indexed sequence s_x can be extended to an X-indexed sequence by assigning any arbitrary element of $\mathscr{F}_x$ to any $x \in X - U$. Hence $\mathbf{Mod}(\mathscr{O}_X)$ possesses enough flasque sheaves.

If $\mathscr{Z}_1$ is the cokernel of the canonical injection $\mathscr{F} \to C^0(X, \mathscr{F})$, we define $C^1(X, \mathscr{F})$ to be the flasque sheaf $C^0(X, \mathscr{Z}_1)$. In general,

$$\mathscr{Z}_n = \operatorname{coker}\left(\mathscr{Z}_{n-1} \hookrightarrow C^0(X, \mathscr{Z}_{n-1})\right) \quad \text{and} \quad C^n(X, \mathscr{F}) = C^0(X, \mathscr{Z}_n).$$

Putting all this information together, we obtain the desired flasque resolution of $\mathscr{F}$

$$0 \longrightarrow \mathscr{F} \longrightarrow C^0(X, \mathscr{F}) \longrightarrow C^1(X, \mathscr{F}) \longrightarrow C^2(X, \mathscr{F}) \longrightarrow \cdots$$

as claimed. □

Remark 2.2.1. The resolution of $\mathscr{F}$ constructed above will be called the **canonical flasque resolution** of $\mathscr{F}$ or the **Godement resolution** of $\mathscr{F}$.

Here is the principal property of flasque sheaves.

Theorem 2.2.4. *Let $0 \to \mathscr{F}_1 \to \mathscr{F}_2 \to \mathscr{F}_3 \to 0$ be an exact sequence of $\mathscr{O}_X$-modules, and assume $\mathscr{F}_1$ is flasque. Then this sequence is exact as a sequence of presheaves. If both $\mathscr{F}_1$ and $\mathscr{F}_2$ are flasque, so is $\mathscr{F}_3$. Finally, any direct summand of a flasque sheaf is flasque.*

Proof. Given any open set U, we must prove that

$$0 \longrightarrow \mathscr{F}_1(U) \xrightarrow{\varphi} \mathscr{F}_2(U) \xrightarrow{\psi} \mathscr{F}_3(U) \longrightarrow 0$$

is exact. By Proposition 1.1.53 and 1.1.54, the sole problem is to prove that $\mathscr{F}_2(U) \to \mathscr{F}_3(U)$ is surjective. By restricting we only need to prove the case for X. Let t be a global section of $\mathscr{F}_3$, then by Proposition 1.1.54, locally t may be lifted to sections of $\mathscr{F}$. Let T be the family of all pairs (U, σ) where U is an open in X, and σ is a section of $\mathscr{F}$ over U whose image in $\mathscr{F}_3(U)$ equal $t|_U$. Partially order T as in the proof of Proposition 2.2.2 and observe that T is inductive. Zorn's lemma provides us with a maximal lifting of t to a section $\sigma \in \mathscr{F}(U_0)$.

Were U_0 not X, there would exist $x \in X - U_0$, a neighborhood V of x, and a section τ of $\mathscr{F}$ over V which is a local lifting of $t|_V$. The sections $\sigma|_{V \cap U_0}$ and $\tau|_{V \cap U_0}$ have the same image in

$\mathscr{F}_3(U_0 \cap V)$ under the map ψ, so their difference maps to 0. Since $\operatorname{im}\varphi = \ker\psi$, there is a section s of $\mathscr{F}_1(U_0 \cap V)$ such that

$$\sigma|_{U_0 \cap V} = \tau|_{V \cap U_0} + \varphi(s).$$

Since $\mathscr{F}_1$ is flasque, the section s is the restriction of a section $s_0 \in \mathscr{F}_1(V)$. Upon replacing τ by $\tau + \varphi(t_0)$ (which does not affect the image in $\mathscr{F}_3(V)$), we may assume that $\sigma|_{V\cap U_0} = \tau|_{V\cap U_0}$; that is, τ and σ agree on the overlap. Then we may extend σ to $U_0 \cup V$, contradicting the maximality of (U_0, σ); hence, $U_0 = X$.

Now suppose that $\mathscr{F}_1$ and $\mathscr{F}_2$ are flasque. If $t \in \mathscr{F}_3(U)$, then by the above, there is a section $s \in \mathscr{F}_2(U)$ mapping onto t. Since $\mathscr{F}_2$ is also flasque, we may lift s to a global section s_0 of $\mathscr{F}$. The image t_0 of s_0 in $\mathscr{F}_3(X)$ is the required extension of t to a global section of $\mathscr{F}_3$.

Finally, assume that $\mathscr{F}$ is flasque, and that $\mathscr{F} = \mathscr{F}_1 \oplus \mathscr{F}_2$ for some sheaf $\mathscr{F}_1, \mathscr{F}_2$. For any open subset U of X and any section $s \in \mathscr{F}_1(U)$, we can make s into a section $\tilde{s} \in \mathscr{F}(U)$ by setting the component of $\tilde{s}(U)$ in $\mathscr{F}_2(U)$ equal to the zero section. Since $\mathscr{F}$ is flasque, there is some section $t \in \mathscr{F}(X)$ such that $t|_U = \tilde{s}$. Write $t = t_1 + t_2$ for some unique $t_1 \in \mathscr{F}_1(X)$ and $t_2 \in \mathscr{F}_2(X)$, then

$$s + 0 = \tilde{s} = t|_U = (t_1)|_U + (t_2)|_U$$

with $(t_1)|_U \in \mathscr{F}_1(U)$ and $(t_2)|_U \in \mathscr{F}_2(U)$, so $s = (t_1)|_U$ with $t_1 \in \mathscr{F}_1(X)$, which shows that $\mathscr{F}_1$ is flasque. $\square$

The following general proposition from Tohoku implies that flasque sheaves are $\Gamma(X, -)$-acyclic. It will also imply that soft sheaves are $\Gamma(X, -)$-acyclic. Since the only functor involved is the global section functor, it is customary to abbreviate $\Gamma(X, -)$-acyclic to acyclic.

Theorem 2.2.5. *Let $\mathscr{F}$ be an additive functor from the abelian category C to the abelian category $\mathcal{D}$, and suppose that C has enough injectives. Let $\mathfrak{X}$ be a class of objects in C which satisfies the following conditions:*

- *C possesses enough $\mathfrak{X}$-objects.*
- *If $0 \to A_1 \to A_2 \to A_3 \to 0$ is exact and if A_1 belongs to $\mathfrak{X}$, then $0 \to \mathscr{F}(A_1) \to \mathscr{F}(A_2) \to \mathscr{F}(A_3) \to 0$ is exact.*
- *If $0 \to A_1 \to A_2 \to A_3 \to 0$ is exact and A_1, A_2 belongs to $\mathfrak{X}$, then A_3 belongs to $\mathfrak{X}$.*
- *If A is an object of C and A is a direct summand of some object in $\mathfrak{X}$, then A belongs to $\mathfrak{X}$.*

Then every injective object belongs to $\mathfrak{X}$, for each M in $\mathfrak{X}$ we have $\mathcal{R}^i\mathscr{F}(M) = 0$ for $i > 0$, and finally the functors $\mathcal{R}^i\mathscr{F}$ may be computed by taking $\mathfrak{X}$-resolutions.

Proof. Let I be an injective of C. Then I admits a monomorphism into some object M of the class $\mathfrak{X}$. We have an exact sequence

$$0 \longrightarrow I \xrightarrow{\varphi} M \longrightarrow \operatorname{coker}\varphi \longrightarrow 0$$

and as I is injective this sequence split. Thus I is a direct summand of M and thus belongs to $\mathfrak{X}$ be the condition.

Let us now show that $\mathcal{R}^n\mathcal{F}(M) = 0$ for $i > 0$ if M lies in $\mathfrak{X}$. Now, C possesses enough injectives, so if we have an exact sequence

$$0 \longrightarrow M \longrightarrow I \longrightarrow K \longrightarrow 0$$

where I is injective. Then from the long exact sequence of $\mathcal{R}^\bullet\mathcal{F}$ and the fact $\mathcal{R}^i\mathcal{F}(I) = 0$ for $i \geq 1$ we conclude

$$\mathcal{R}^1\mathcal{F}(M) = 0 \quad \text{and} \quad \mathcal{R}^i\mathcal{F}(M) = \mathcal{R}^{i-1}\mathcal{F}(K) \textit{ for } i \geq 2.$$

Since M and I are both in $\mathfrak{X}$, K is also in $\mathfrak{X}$. Therefore by applying the same argument on K we also get $\mathcal{R}^1\mathcal{F}(K) = 0$. Now the claim follows by an induction. □

This result tells us that flasque sheaves are acyclic for the functor $\Gamma(X, -)$. Hence we can calculate cohomology using flasque resolutions.

Proposition 2.2.6. *Flasque sheaves are acyclic, that is $H^i(X, \mathcal{F}) = 0$ for every flasque sheaf $\mathcal{F}$ and all $i \geq 1$, and the cohomology groups $H^i(X, \mathcal{F})$ of any arbitrary sheaf $\mathcal{F}$ can be computed using flasque resolutions.*

Proof. Apply Theorem 2.2.5 on the functor $\Gamma(X, -) : \mathbf{Ab}(X) \to \mathbf{Ab}(X)$. □

In view of the proposition above, we also have the following result.

Proposition 2.2.7. *Let $(X, \mathcal{O}_X)$ be a ringed space. Then the derived functors of the functor $\Gamma(X, -)$ from $\mathbf{Mod}(\mathcal{O}_X)$ to $\mathbf{Ab}(X)$ coincide with the cohomology functors $H^i(X, -)$.*

Proof. Let I be an injective $\mathcal{O}_X$-module, then $\mathcal{I}$ is a flasque $\mathcal{O}_X$-module by Theorem 2.2.5. Therefore an injective resolution in $\mathbf{Mod}(\mathcal{O}_X)$ is a flasque resolution in $\mathbf{Ab}(X)$, hence computes the sheaf cohomology. □

2.3 Locality of cohomology

we first sate a useful result in abelian categories.

Theorem 2.3.1. *Let (F, G) be an adjoint pair between abelian categories C and $\mathcal{D}$, in the sense that*

$$\mathrm{Hom}_{\mathcal{D}}(F(X), Y) = \mathrm{Hom}_C(X, G(Y)).$$

Assume that F is exact, then G maps injectives to injectives.

Proof. Let I be injective in $\mathcal{D}$. Assume that we have an exact diagram in C:

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r] \arrow[d] & Y \\ & G(I) & \end{tikzcd}$$

Then by applying the exact functor F we get an exact diagram in $\mathcal{D}$:

$$\begin{tikzcd} 0 \arrow[r] & F(X) \arrow[r] \arrow[d] & F(Y) \arrow[d, dashed] \\ & FG(I) \arrow[r] & I \end{tikzcd}$$

where $FG(I) \longrightarrow I$ is the unit map of the adjunction (F, G). Since I is injective, this extends to a map $F(Y) \longrightarrow I$. Applying G again and compose the counit map $I \longrightarrow GF(Y)$ we get the desired map $Y \longrightarrow G(I)$, so $G(I)$ is injective. □

Proposition 2.3.2. *Let $f : X \longrightarrow Y$ be a morphism of ringed spaces. Assume f is flat. Then $f_*\mathcal{I}$ is an injective $\mathcal{O}_Y$-module for any injective $\mathcal{O}_X$-module $\mathcal{I}$. In particular, the pushforward $f_* :$ **Ab**$(X) \longrightarrow$ **Ab**(Y) transforms injective abelian sheaves into injective abelian sheaves.*

Proof. In this case f^* is exact, and we have an adjoint pair (f^*, f_*). Now apply Theorem 2.3.1 we get the claim. □

The following lemma says there is no ambiguity in defining the cohomology of a sheaf $\mathcal{F}$ over an open.

Lemma 2.3.3. *Let X be a ringed space. Let $U \subseteq X$ be an open subspace.*

(a) If $\mathcal{I}$ is an injective $\mathcal{O}_X$-module then $\mathcal{I}|_U$ is an injective $\mathcal{U}$-module.

(b) For any sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ we have $H^p(U, \mathcal{F}) = H^p(U, \mathcal{F}|_U)$.

Proof. Denote $j : U \longrightarrow X$ the open immersion. Then $(j_!, j^{-1})$ satisfies the condition of Theorem 2.3.1. By definition $H^p(U, \mathcal{F}) = H^p(\Gamma(U, \mathcal{I}^\bullet))$ where $\mathcal{F} \longrightarrow \mathcal{I}^\bullet$ is an injective resolution in **Mod**$(\mathcal{O}_X)$. By the above we see that $\mathcal{F}|_U \longrightarrow \mathcal{I}|_U$ is an injective resolution in **Mod**$(\mathcal{O}_U)$. Hence $H^p(U, \mathcal{F}|_U)$ is equal to $H^p(\Gamma(U, \mathcal{I}^\bullet|_U))$. Of course $\Gamma(U, \mathcal{F}) = \Gamma(U, \mathcal{F}|_U)$ for any sheaf $\mathcal{F}$ on X. Hence the equality in (b). □

Let $f : X \longrightarrow Y$ be a continuous. Since the functor f_* is left-exact, for an injective resolution $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{I}^\bullet$ we define

$$\mathcal{R}^i f_*\mathcal{F} = H^i(f_*\mathcal{I}^\bullet)$$

to be the ***i*-th higher direct image of** $\mathcal{F}$.

Proposition 2.3.4. *Let $f : X \longrightarrow Y$ be a morphism of ringed spaces. Let $\mathcal{F}$ be a $\mathcal{O}_X$-module. The sheaves $\mathcal{R}^i f_*\mathcal{F}$ are the sheaves associated to the presheaves*

$$U \mapsto H^i(f^{-1}(U), \mathcal{F}).$$

Proof. Let $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{I}^\bullet$ be an injective resolution. Then $\mathcal{R}^i f_*\mathcal{F}$ is by definition the i-th cohomology sheaf of the complex

$$f_*\mathcal{I}^0 \longrightarrow f_*\mathcal{I}^1 \longrightarrow f_*\mathcal{I}^2 \longrightarrow \cdots$$

By definition of the abelian category structure on $\mathcal{O}_Y$-modules this cohomology sheaf is the sheaf associated to the presheaf

$$V \mapsto \frac{\ker(f_*\mathcal{I}^i(V) \to f_*\mathcal{I}^{i+1}(V))}{\operatorname{im}(f_*\mathcal{I}^{i-1}(V) \to f_*\mathcal{I}^i(V))}$$

and this is obviously equal to

$$\frac{\ker\left(\mathcal{I}^i(f^{-1}(V)) \to \mathcal{I}^{i+1}(f^{-1}(V))\right)}{\operatorname{im}\left(\mathcal{I}^{i-1}(f^{-1}(V)) \to \mathcal{I}^i(f^{-1}(V))\right)}$$

which is equal to $H^i(f^{-1}(V), \mathcal{F})$ and we win. □

Corollary 2.3.5. *Let $f : X \to Y$ be a morphism of ringed spaces. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. If $\mathcal{F}$ is flasque, then $\mathcal{R}^p f_*\mathcal{F} = 0$ for $p > 0$.*

Proof. This follows from Proposition 2.2.6 and Proposition 2.3.4. □

Proposition 2.3.6. *Let $f : X \to Y$ be a morphism of ringed spaces. Let $\mathcal{F}$ be an O_X-module. Let $V \subseteq Y$ be an open subspace. Denote $g : f^{-1}(V) \to V$ the restriction of f. Then we have*

$$\mathcal{R}^i g_*(\mathcal{F}|_{f^{-1}(V)}) = (\mathcal{R}^i f_*\mathcal{F})|_V.$$

There is a similar statement for the derived image $\mathcal{R}^i f_*\mathcal{F}$ where $\mathcal{F}^\bullet$ is a bounded below complex of O_X-modules.

Proof. Choose an injective resolution $0 \to \mathcal{F} \to \mathcal{I}^\bullet$ and use that $\mathcal{F}|_{f^{-1}(V)} \to \mathcal{I}|_{f^{-1}(V)}$ is an injective resolution also. □

Chapter 3

Čech cohomology

3.1 The Čech cohomology group with respect to a covering

Let X be a topological space and $\mathcal{U} = (U_i)_{i\in I}$ be an open covering of X. Given any finite sequence $(i_0, \ldots, i_p)$ of elements of I, we let

$$U_{i_0\ldots i_p} := U_{i_0} \cap \cdots \cap U_{i_p}.$$

Also, we denote by $U_{i_0\ldots\widehat{i_j}\ldots i_p}$ the intersection

$$U_{i_0\ldots\widehat{i_j}\ldots i_p} = U_{i_0} \cap \cdots \cap \widehat{U}_{i_j} \cap \cdots \cap U_{i_p}.$$

Then we have $p+1$ inclusion maps

$$\delta_j^p : U_{i_0\ldots i_p} \to U_{i_0\ldots\widehat{i_j}\ldots i_p} \quad \text{for } 0 \leq j \leq p.$$

Definition 3.1.1. *Given a topological space X, an open cover $\mathcal{U} = (U_i)_{i\in I}$ of X, and a abelian presheaf $\mathscr{F}$ on X, the* ***Čech p-cochains*** *$C^p(\mathcal{U}, \mathscr{F})$ is the set of all functions f with domain I^{p+1} such that $f(i_0, \ldots, i_p) \in \mathscr{F}(U_{i_0} \ldots i_p)$; in other words,*

$$C^p(\mathcal{U}, \mathscr{F}) = \prod_{(i_0,\ldots,i_p)\in I^{p+1}} \mathscr{F}(U_{i_0\ldots i_p})$$

the set of all I^{p+1}-indexed families $(f_{i_0\ldots i_p})$ with $f_{i_0\ldots i_p} \in \mathscr{F}(U_{i_0\ldots i_p})$.

Remark 3.1.1. Since $\mathscr{F}(\emptyset) = 0$, for any cochain $f \in C^p(\mathcal{U}, \mathscr{F})$, if $U_{i_0\ldots i_p} = \emptyset$, then $f_{i_0\ldots i_p} = 0$. Therefore, we could define $C^p(\mathcal{U}, \mathscr{F})$ as the set of families $f_{i_0\ldots i_p}$ corresponding to tuples $(i_0, \ldots, i_p) \in I^{p+1}$ such that $U_{i_0\ldots i_p} \neq \emptyset$.

Each inclusion map $\delta_j^p : U_{i_0\ldots i_p} \to U_{i_0\ldots\widehat{i_j}\ldots i_p}$ induces a map

$$\mathscr{F}(\delta_j^p) : \mathscr{F}(U_{i_0\ldots\widehat{i_j}\ldots i_p}) \to \mathscr{F}(U_{i_0\ldots i_p})$$

which is none other that the restriction map.

Definition 3.1.2. *Given a topological space X, an open cover $\mathcal{U} = (U_i)_{i\in I}$ of X, and a abelian presheaf*

$\mathscr{F}$ on X, the coboundary maps $d : C^p(\mathcal{U},\mathscr{F}) \to C^{p+1}(\mathcal{U},\mathscr{F})$ are given by

$$d = \sum_{j=0}^{p+1} (-1)^j \mathscr{F}(\delta_j^{p+1}).$$

More explicitly, for any p-cochain $f \in C^p(\mathcal{U},\mathscr{F})$, for any sequence $(i_0,\dots,i_{p+1}) \in I^{p+2}$, we have

$$(df)_{i_0\dots i_{p+1}} = \sum_{j=0}^{p+1} (-1)^j (f_{i_0\dots \widehat{i_j}\dots i_{p+1}})|_{U_{i_0\cdots i_{p+1}}}.$$

Proposition 3.1.3. *With the notations above, we have $d^2 = 0$. Thus we obtain a complex $C^\bullet(\mathcal{U},\mathscr{F})$.*

Proof. This is a typical computation. Let $f \in C^p(\mathcal{U},\mathscr{F})$,

$$\begin{aligned}
(d^2 f)_{i_0\dots i_{p+2}} &= \sum_{k=0}^{p+2} (-1)^k \big((df)_{i_0\dots\widehat{i_k}\dots i_{p+2}}\big)|_{U_{i_0\cdots i_{p+2}}} \\
&= \sum_{j<k} (-1)^k \big((-1)^j (f_{i_0\dots\widehat{i_j}\dots\widehat{i_k}\dots i_{p+1}})|_{U_{i_0\cdots\widehat{i_k}\cdots i_{p+2}}}\big)|_{U_{i_0\cdots i_{p+2}}} \\
&+ \sum_{j>k} (-1)^k \big((-1)^{j-1} (f_{i_0\dots\widehat{i_k}\dots\widehat{i_j}\dots i_{p+1}})|_{U_{i_0\cdots\widehat{i_k}\cdots i_{p+2}}}\big)|_{U_{i_0\cdots i_{p+2}}} \\
&= \sum_{j<k} (-1)^k (-1)^j (f_{i_0\dots\widehat{i_j}\dots\widehat{i_k}\dots i_{p+1}})|_{U_{i_0\cdots i_{p+2}}} \\
&- \sum_{j>k} (-1)^k (-1)^j (f_{i_0\dots\widehat{i_k}\dots\widehat{i_j}\dots i_{p+1}})|_{U_{i_0\cdots i_{p+2}}} \\
&= 0
\end{aligned}$$

as desired. □

Therefore, we can form the Čech cohomology groups with respect to $\mathcal{U}$ as follows.

Definition 3.1.4. *Given a topological space X, an open cover $\mathcal{U} = (U_i)_{i\in I}$ of X, and a abelian presheaf $\mathscr{F}$ on X, we define*

$$\check{H}^p(\mathcal{U},\mathscr{F}) = H^p(C^\bullet(\mathcal{U},\mathscr{F}))$$

to be the ***p-th Čech-cohomology group with respect to the covering $\mathcal{U}$****.*

First of all, we note that $\check{H}^0(\mathcal{U},\mathscr{F})$ can be easily computed.

Proposition 3.1.5. *Given a topological space X, an open cover $\mathcal{U}$ of X, and a sheaf $\mathscr{F}$ on X, then*

$$\check{H}^0(\mathcal{U},\mathscr{F}) = \mathscr{F}(X) = \Gamma(X,\mathscr{F}).$$

More generally, if $\mathscr{F}$ is an abelian presheaf, then the following are equivalent

- *$\mathscr{F}$ is a sheaf.*
- *For every open covering $\mathcal{U} = (U_i)_{i\in I}$ the natural map*

$$\mathscr{F}(U) \to \check{H}^0(\mathcal{U},\mathscr{F})$$

is bijective.

Proof. By definition, $\check{H}^0(\mathcal{U}, \mathscr{F}) = \ker(C^0(\mathcal{U}, \mathscr{F}) \to C^1(\mathcal{U}, \mathscr{F}))$. If $f \in C^0(\mathcal{U}, \mathscr{F})$ is given by (f_i), then for each $(i, j) \in I^2$, $(df)_{i,j} = f_j - f_i$. So $df = 0$ says the sections f_i and f_j agree on $U_i \cap U_j$. Thus it follows that $\ker d = \Gamma(X, \mathscr{F})$ if and only if $\mathscr{F}$ is a sheaf. □

An element of $C^p(\mathcal{U}, \mathscr{F})$ is called a ***p*-cochain**. We say that a p-cochain $f \in C^p(\mathcal{U}, \mathscr{F})$ is **alternating** if

- $f_{i_0 \dots i_p} = 0$ whenever any two of the indices $i_0, \dots, i_p$ are equal.
- For every permutation σ of the indices, we have $f_{i_{\sigma(0)} \dots i_{\sigma(p)}} = (-1)^\sigma f_{i_0 \dots i_p}$.

It is clear that if $f \in C^p(\mathcal{U}, \mathscr{F})$ is alternating, then df is also alternating, hence we get a complex $C^\bullet_{alt}(\mathcal{U}, \mathscr{F})$.

Definition 3.1.6. *Let X be a topological space. Let $\mathcal{U} = (U_i)_{i \in I}$ be an open covering. Let $\mathscr{F}$ be an abelian presheaf on X. The complex $C^\bullet_{alt}(\mathcal{U}, \mathscr{F})$ is the **alternating Čech complex** associated to $\mathscr{F}$ and the open covering $\mathcal{U}$.*

Let us endow the set of indices I with a total ordering and set

$$C^p_{ord}(\mathcal{U}, \mathscr{F}) := \prod_{i_0 < \dots < i_p} \mathscr{F}(U_{i_0 \dots i_p})$$

It can immediately be verified that the differential d of $C^\bullet(\mathcal{U}, \mathscr{F})$ induces a differential on $C^\bullet_{ord}(\mathcal{U}, \mathscr{F})$, by restriction.

Definition 3.1.7. *Let X be a topological space. Let $\mathcal{U} = (U_i)_{i \in I}$ be an open covering. Let $\mathscr{F}$ be an abelian presheaf on X. Assume given a total ordering on I. The complex $C^\bullet_{ord}(\mathcal{U}, \mathscr{F})$ is the **ordered Čech complex** associated to $\mathscr{F}$, the open covering $\mathcal{U}$ and the given total ordering on I.*

There is an obvious comparison map between the ordered Čech complex and the alternating Čech complex. Namely, consider the map

$$c : C^\bullet_{ord}(\mathcal{U}, \mathscr{F}) \to C^\bullet_{alt}(\mathcal{U}, \mathscr{F})$$

given by the rule

$$c(s)_{i_0 \dots i_p} = \begin{cases} (-1)^\sigma s_{i_{\sigma(0)}, \dots, i_{\sigma(p)}} & \text{if } i_{\sigma(0)} < \dots < i_{\sigma(p)}, \\ 0 & \text{if } i_n = i_m \text{ for some } n \neq m. \end{cases}$$

The alternating and ordered Čech complexes are often identified in the literature via the map c. Namely we have the following easy lemma.

Lemma 3.1.8. *Let X be a topological space. Let $\mathcal{U}$ be an open covering. Assume I comes equipped with a total ordering. The map c is a morphism of complexes. In fact it induces an isomorphism*

$$c : C^\bullet_{ord}(\mathcal{U}, \mathscr{F}) \to C^\bullet_{alt}(\mathcal{U}, \mathscr{F})$$

of complexes.

There is also a map $\pi : C^\bullet(\mathcal{U}, \mathscr{F}) \to C^\bullet_{ord}(\mathcal{U}, \mathscr{F})$ which is described by the rule

$$\pi(s)_{i_0 \dots i_p} = s_{i_0 \dots i_p}$$

whenever $i_0 < \cdots < i_p$. The following result is immediate.

Lemma 3.1.9. *Let X be a topological space. Let $\mathcal{U}$ be an open covering. Assume I comes equipped with a total ordering. The map $\pi : C^\bullet(\mathcal{U}, \mathscr{F}) \to C^\bullet_{ord}(\mathcal{U}, \mathscr{F})$ is a morphism of complexes which is a left inverse to the morphism c. Moreover, it induces an isomorphism*

$$\tilde{\pi} : C^\bullet_{alt}(\mathcal{U}, \mathscr{F}) \to C^\bullet_{ord}(\mathcal{U}, \mathscr{F})$$

of complexes.

It turns out that the maps π and c give a homotopy equivalence between $C^\bullet(\mathcal{U}, \mathscr{F})$ and $C^\bullet_{alt}(\mathcal{U}, \mathscr{F})$. Therefore, we can use the alternating complex to compute the Čech cohomology.

Theorem 3.1.10. *Let X be a topological space. Let $\mathcal{U}$ be an open covering. Assume I comes equipped with a total ordering. The map $c \circ \pi$ is homotopic to the identity on $C^\bullet(\mathcal{U}, \mathscr{F})$. In particular the inclusion map $C^\bullet_{alt}(\mathcal{U}, \mathscr{F}) \to C^\bullet(\mathcal{U}, \mathscr{F})$ is a homotopy equivalence.*

Corollary 3.1.11. *Given a total ordering on I, there exists a canonical isomorphism*

$$H^\bullet(C^\bullet_{ord}(\mathcal{U}, \mathscr{F})) \cong H^\bullet(C^\bullet(\mathcal{U}, \mathscr{F})).$$

Corollary 3.1.12. *If $\mathcal{U}$ is made up of n open subsets, then $\check{H}^p(\mathcal{U}, \mathscr{F}) = 0$ for every $p \geq n$.*

Proof. Indeed, if $p \geq n$, there does not exist any strictly increasing $(p+1)$-uple of indices $i_0, \dots, i_p$. Hence $C^p_{ord}(\mathcal{U}, \mathscr{F}) = 0$, whence $\check{H}^p(\mathcal{U}, \mathscr{F}) = 0$. □

3.2 Čech cohomology as a functor on presheaves

Warning: In this subsection we work almost exclusively with presheaves and categories of presheaves and the results are completely wrong in the setting of sheaves and categories of sheaves!

Let X be a ringed space. Let $\mathcal{U} = (U_i)_{i \in I}$ be an open covering. Let $\mathscr{F}$ be a presheaf of $\mathcal{O}_X$-modules. We have the Čech complex $C^\bullet(\mathcal{U}, \mathscr{F})$ of $\mathscr{F}$ just by thinking of $\mathscr{F}$ as a presheaf of abelian groups. However, each term $C^p(\mathcal{U}, \mathscr{F})$ has a natural structure of a $\mathcal{O}_X(X)$-module and the differential is given by $\mathcal{O}_X(X)$-module maps. Moreover, it is clear that the construction

$$\mathscr{F} \to C^\bullet(\mathcal{U}, \mathscr{F})$$

is functorial in $\mathscr{F}$. In fact, it is a functor

$$C^\bullet(\mathcal{U}, -) : \mathbf{PMod}(\mathscr{O}_X) \to C^+(\mathbf{Mod}_{\mathscr{O}_X(X)})$$

Proposition 3.2.1. *The functor $C^\bullet(\mathcal{U}, -)$ is an exact functor.*

Proof. For any open $U \subseteq X$ the functor $\mathscr{F} \to \mathscr{F}(U)$ is an additive exact functor from $\mathbf{PMod}(\mathcal{O}_X)$ to $\mathbf{Mod}_{\mathscr{O}_X(X)}$. The terms $C^p(\mathcal{U}, \mathscr{F})$ of the complex are products of these exact functors and hence exact. Moreover a sequence of complexes is exact if and only if the sequence of terms in a given degree is exact. Hence the lemma follows. □

Theorem 3.2.2. *Let X be a ringed space. Let $\mathcal{U} = (U_i)_{i \in I}$ be an open covering. The functors $\check{H}^p(\mathcal{U}, -)$ form a δ-functor from the abelian category of presheaves of $\mathscr{O}_X$-modules to the category of $\mathscr{O}_X(X)$-modules*

Proof. By Proposition 3.2.1 a short exact sequence of presheaves of $\mathscr{O}_X$-modules is turned into a short exact sequence of complexes of $\mathscr{O}_X(X)$-modules. Hence we can get a long exact sequence. □

Proposition 3.2.3. *Let X be a ringed space. Let $\mathcal{U}$ be an open covering of X. The Čech cohomology functors $\check{H}^p(\mathcal{U}, -)$ are canonically isomorphic as a δ-functor to the right derived functors of the functor*

$$\check{H}^0(\mathcal{U}, -) : \mathbf{PMod}(\mathscr{O}_X) \to \mathbf{Mod}_{\mathscr{O}_X(X)}.$$

Moreover, there is a functorial quasi-isomorphism

$$C^\bullet(\mathcal{U}, \mathscr{F}) \to \mathcal{R}\check{H}^0(\mathcal{U}, \mathscr{F}).$$

Proof. This comes from the universal property of the δ-functor. □

3.3 The Čech cohomology groups

Our next goal is to define Čech cohomology groups $\check{H}^p(X, \mathscr{F})$ that are independent of the open cover $\mathcal{U}$ chosen for X.

We say that a covering $\mathcal{V} = (V_j)_{j \in J}$ of X is a **refinement** of another covering $\mathcal{U} = \{U_i\}_{i \in I}$ if there exists a map $\tau : J \to I$ such that $V_j \subseteq U_{\tau(j)}$ for every $j \in J$. We then have a homomorphism, which we also denote by τ:

$$C^p(\mathcal{U}, \mathscr{F}) \to C^p(\mathcal{V}, \mathscr{F})$$

defined by

$$\tau(f)_{j_0 \dots j_p} = f_{\tau(j_0) \dots \tau(j_p)}|_{V_{j_0 \cdots j_p}}$$

This homomorphism commutes with the differentials and therefore induces a homomorphism

$$\tau^* : \check{H}^p(\mathcal{U}, \mathscr{F}) \to \check{H}^p(\mathcal{V}, \mathscr{F}).$$

Proposition 3.3.1. *The homomorphisms $\tau^* : \check{H}^p(\mathcal{U}, \mathscr{F}) \to \check{H}^p(\mathcal{V}, \mathscr{F})$ depend only on $\mathcal{U}$ and $\mathcal{V}$ and not on the chosen mapping τ.*

Proof. Let τ_1 and τ_2 be two mappings from I to J such that $V_j \subseteq U_{\tau_1(j)}$ and $V_j \subseteq U_{\tau_2(j)}$, we have to show that $\tau_1^* = \tau_2^*$.

Let $f \in C^p(\mathcal{U}, \mathcal{F})$, define a map $\kappa : C^p(\mathcal{U}, \mathcal{F}) \to C^{p-1}(\mathcal{V}, \mathcal{F})$ by setting

$$(\kappa f)_{j_0 \dots j_{p-1}} = \sum_{k=0}^{p-1} (-1)^k (f_{\tau_1(j_1) \dots \tau_1(j_k) \tau_2(j_k) \dots \tau_2(j_{p-1})})|_{V_{j_0 \dots j_{p-1}}}.$$

Then, it can be verified that

$$\kappa(df) + d(\kappa f) = \tau_2(f) - \tau_1(f).$$

Thus k defines a homotopy from τ_2^* to τ_1^*, which implies the claim. □

Corollary 3.3.2. *If $\mathcal{V}$ is a refinement of $\mathcal{U}$ and if $\mathcal{U}$ is a refinement of $\mathcal{V}$, then $\check{H}^p(\mathcal{U}, \mathcal{F}) \to \check{H}^p(\mathcal{V}, \mathcal{F})$ is an isomorphism.*

Proof. Let us keep the notation above. If $\mathcal{U}$ is a refinement of a covering $\mathcal{W}$ with a map σ such that $U_i \subseteq W_{\sigma(i)}$, then $\mathcal{V}$ is a refinement of $\mathcal{W}$ because $V_j \subseteq W_{\sigma \circ \tau(j)}$. Moreover, $(\sigma \circ \tau)^* = \sigma^* \circ \tau^*$ in an obvious way. Let us now take $\mathcal{W} = \mathcal{V}$. Then $\sigma^* \circ \tau^*$ and $\tau^* \circ \sigma^*$ coincides with $\mathbf{1}^*$ by the result above. Hence τ^* is bijective. □

The relation $\mathcal{U}$ is a refinement of $\mathcal{V}$ (which we denote henceforth by $\mathcal{U} \prec \mathcal{V}$) is a relation of a preorder between coverings of X; moreover, this relation is filtered, since if $\mathcal{U} = \{U_i\}_{i \in I}$ and $\mathcal{V} = \{V_j\}_{j \in J}$ are two coverings, then the covering $\mathcal{W} = \{U_i \cap V_j\}_{(i,j) \in I \times J}$ is a refinement of $\mathcal{U}$ and $\mathcal{V}$. Consequently, it appears that the family $(\check{H}^p(\mathcal{U}, \mathcal{F}))_{\mathcal{U}}$ is a direct mapping family of groups indexed by the directed set of open covers of X. However, there is a set-theoretic diffculty, which is that the family of open covers of X is not a set because it allows arbitrary index sets.

A way to circumvent this difficulty is provided by Serre. The key observation is that any covering $(U_i)_{i \in I}$ is equivalent to a covering $(U'_j)_{j \in L}$ whose index set L is a subset of 2^X. Indeed, we can take for $(U'_j)_{j \in L}$ the set of all open subsets of X that belong to the family $(U_i)_{i \in I}$.

As we noted earlier, if $\mathcal{U}$ and $\mathcal{V}$ are equivalent, then there is an isomorphism between $\check{H}^p(\mathcal{U}, \mathcal{F})$ and $\check{H}^p(\mathcal{V}, \mathcal{F})$, so we can define

$$\check{H}^p(X, \mathcal{F}) = \varinjlim \check{H}^p(\mathcal{U}, \mathcal{F})$$

with respect to coverings $\mathcal{U}$ whose index set is a subset of 2^X. In summary, we have the following definition.

Definition 3.3.3. *Given a topological space X and a abelian presheaf $\mathcal{F}$ on X, the **Čech cohomology groups** $\check{H}^p(X, \mathcal{F})$ with values in $\mathcal{F}$ are defined by*

$$\check{H}^p(X, \mathcal{F}) = \varinjlim \check{H}^p(\mathcal{U}, \mathcal{F})$$

with respect to coverings $\mathcal{U}$ whose index set is a subset of 2^X.

3.4 Long exact sequence of Čech-cohomology

3.4.1 General case

Let $0 \to \mathscr{A} \to \mathscr{B} \to \mathscr{C} \to 0$ be an exact sequence of sheaves. If $\mathcal{U}$ is a covering of X, the sequence

$$0 \longrightarrow C^\bullet(\mathcal{U},\mathscr{A}) \xrightarrow{\alpha} C^\bullet(\mathcal{U},\mathscr{B}) \xrightarrow{\beta} C^\bullet(\mathcal{U},\mathscr{C})$$

is obviously exact, but the homomorphism β need not be surjective in general. Denote by $C_0^\bullet(\mathcal{U},\mathscr{C})$ the image of this homomorphism; it is a subcomplex of $C^\bullet(\mathcal{U},\mathscr{C})$ whose cohomology groups will be denoted by $\check{H}_0^p(\mathcal{U},\mathscr{C})$. The exact sequence of complexes:

$$0 \longrightarrow C^\bullet(\mathcal{U},\mathscr{A}) \xrightarrow{\alpha} C^\bullet(\mathcal{U},\mathscr{B}) \xrightarrow{\beta} C_0^\bullet(\mathcal{U},\mathscr{C}) \longrightarrow 0$$

giving rise to a long exact sequence of cohomology

$$\cdots \longrightarrow \check{H}^p(\mathcal{U},\mathscr{A}) \longrightarrow \check{H}^p(\mathcal{U},\mathscr{B}) \longrightarrow \check{H}_0^p(\mathcal{U},\mathscr{C}) \xrightarrow{\delta} \check{H}^{p+1}(\mathcal{U},\mathscr{A}) \longrightarrow \cdots$$

where the coboundary operator δ is defined as usual.

Now let $\mathcal{U} = (U_i)_{i\in I}$ and $\mathcal{V} = (V_j)_{j\in J}$ be two coverings and let $\tau : J \to I$ be such that $V_j \subseteq U_{\tau(j)}$; we thus have $\mathcal{V} \prec \mathcal{U}$. The commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & C^\bullet(\mathcal{U},\mathscr{A}) \arrow[r, "\alpha"] \arrow[d, "\tau"] & C^\bullet(\mathcal{U},\mathscr{B}) \arrow[r, "\beta"] \arrow[d, "\tau"] & C^\bullet(\mathcal{U},\mathscr{C}) \arrow[d, "\tau"] \\
0 \arrow[r] & C^\bullet(\mathcal{V},\mathscr{A}) \arrow[r, "\alpha"] & C^\bullet(\mathcal{V},\mathscr{B}) \arrow[r, "\beta"] & C^\bullet(\mathcal{V},\mathscr{C})
\end{tikzcd}$$

shows that τ maps $C_0^\bullet(\mathcal{U},\mathscr{C})$ into $C_0^\bullet(\mathcal{V},\mathscr{C})$, thus defining the homomorphisms

$$\tau^* : \check{H}_0^p(\mathcal{U},\mathscr{C}) \to \check{H}_0^p(\mathcal{V},\mathscr{C}).$$

Moreover, the homomorphisms τ are independent of the choice of the mapping τ: this follows from the fact that, if $f \in C_0^p(\mathcal{U},\mathscr{C})$, we have $\kappa f \in C_0^{p-1}(\mathcal{V},\mathscr{C})$, with the notations of Proposition 3.3.1. We have thus obtained canonical homomorphisms $H_0^p(\mathcal{U},\mathscr{C}) \to \check{H}_0^p(\mathcal{V},\mathscr{C})$; we might then define $\check{H}_0^p(X,\mathscr{C})$ as the inductive limit of the groups $\check{H}^p(X,\mathscr{C})$.

Because an inductive limit of exact sequences is an exact sequence, we obtain:

Proposition 3.4.1. *The sequence*

$$\cdots \longrightarrow \check{H}^p(X,\mathscr{A}) \xrightarrow{\alpha^*} \check{H}^p(X,\mathscr{B}) \xrightarrow{\beta^*} \check{H}_0^p(X,\mathscr{C}) \xrightarrow{\delta} \check{H}^{p+1}(X,\mathscr{A}) \longrightarrow \cdots$$

is exact.

To apply the preceding proposition, it is convenient to compare the groups $\check{H}_0^p(X,\mathscr{C})$ and $\check{H}^p(X,\mathscr{C})$. The inclusion of $C_0^\bullet(\mathcal{U},\mathscr{C})$ in $C^\bullet(\mathcal{U},\mathscr{C})$ defines the homomorphisms

$$\check{H}_0^p(\mathcal{U},\mathscr{C}) \to \check{H}^p(\mathcal{U},\mathscr{C}),$$

hence, by passing to the limit with $\mathcal{U}$, the homomorphisms:

$$\check{H}^p_0(X,\mathscr{C}) \to \check{H}^p(X,\mathscr{C})$$

We first prove a lemma.

Lemma 3.4.2. *Let $\mathcal{U} = (U_i)_{i\in I}$ be a covering and let $f = (f_j)$ be an element of $C^0(\mathcal{U},\mathscr{C})$. There exists a covering $\mathcal{V} = (V_j)_{j\in J}$ and a mapping $\tau : J \to I$ such that $V_j \subseteq U_{\tau(j)}$ and $\tau(f) \in C^0_0(\mathcal{V},\mathscr{C})$.*

Proof. For any $x \in X$, take a $\tau(x) \in I$ such that $x \in U_{\tau(x)}$. Since $f_{\tau(x)}$ is a section of $\mathscr{C}$ over $U_{\tau(x)}$, by the surjectivity of β there exists an open neighborhood V_x of x, contained in $U_{\tau(x)}$ and a section b_x of $\mathscr{B}$ over V_x such that $\beta(b_x) = f_{\tau(x)}|_{V_x}$ on V_x. The $(V_x)_{x\in X}$ form a covering $\mathcal{V}$ of X, and the b_x form a 0-chain b of $\mathcal{V}$ with values in $\mathcal{U}$. From the construction we have $\tau(f) = \beta(b)$, so that $\tau(f) \in C^0_0(\mathcal{V},\mathscr{C})$. □

Now we can prove the following result.

Theorem 3.4.3. *The canonical homomorphism $\check{H}^p_0(X,\mathscr{C}) \to \check{H}^p(X,\mathscr{C})$ is bijective for $p = 0$ and injective for $p = 1$.*

Proof. By Lemma 3.4.2 the bijectivity for $p = 0$ is immediate. We now show that

$$\check{H}^1_0(X,\mathscr{C}) \to \check{H}^1(X,\mathscr{C})$$

is injective. Let $[z]$ be in the kernel of this map, which is represented by a 1-cocycle $z = (z_{i_0 i_1}) \in C^1_0(\mathcal{U},\mathscr{C})$. Then since $[z] = 0$ in $\check{H}^1(X,\mathscr{C})$, there exists an $f = (f_i) \in C^0(\mathcal{U},\mathscr{C})$ with $df = z$; applying Lemma 3.4.2 (and its notations) to f yields a covering $\mathcal{V}$ such that $\tau(f) \in C^0_0(\mathcal{V},\mathscr{C})$, which shows that $\tau(z)$ is cohomologous to 0 in $C^1_0(\mathcal{V},\mathscr{C})$, thus its image $[z]$ in $\check{H}^1_0(X,\mathscr{C})$ is 0. This shows the claim. □

Corollary 3.4.4. *With notations above, we have an exact sequence:*

$$0 \longrightarrow \Gamma(X,\mathscr{A}) \longrightarrow \Gamma(X,\mathscr{B}) \longrightarrow \Gamma(X,\mathscr{C}) \xrightarrow{\delta} \check{H}^1(X,\mathscr{A}) \longrightarrow \check{H}^1(X,\mathscr{B}) \longrightarrow \check{H}^1(X,\mathscr{C})$$

Corollary 3.4.5. *If $\check{H}^1(X,\mathscr{A}) = 0$, then $\Gamma(X,\mathscr{B}) \to \Gamma(X,\mathscr{C})$ is surjective.*

3.4.2 Paracompact space

Recall that a space X is said to be paracompact if any covering of X admits a locally finite refinement. On paracompact spaces, we can extend Proposition 3.4.3 for all values of p:

Theorem 3.4.6. *If X is paracompact, the canonical homomorphism*

$$\check{H}^p_0(X,\mathscr{C}) \to \check{H}^p(X,\mathscr{C})$$

is bijective for all $p \geq 0$.

This Proposition is an immediate consequence of the following lemma, analogous to Lemma 3.4.2:

Lemma 3.4.7. *Let X be a paracompact space. Let $\mathcal{U} = (U_i)_{i\in I}$ be a covering, and let $f = (f_{i_0\dots i_p})$ be an element of $C^p(\mathcal{U}, \mathscr{C})$. Then there exists a covering $\mathcal{V} = (V_j)_{j\in J}$ and a mapping $\tau : J \to I$ such that $V_j \subseteq U_{\tau(j)}$ and $\tau(f) \in C_0^p(\mathcal{V}, \mathscr{C})$.*

Proof. Since X is paracompact, we might assume that $\mathcal{U}$ is locally finite. For every $x \in X$, we can choose an open neighborhood V_x of x such that

(a) If $x \in U_i$, then $V_x \subseteq U_i$.

(b) If $V_x \cap U_i \neq \varnothing$, then $V_x \subseteq U_i$.

(c) If $x \in U_{i_0\dots i_p}$, there exists a section b of $\mathscr{B}$ over V_x such that $\beta(b) = f_{i_0\dots i_p}|_{V_x}$.

The condition (c) can be satisfid due to the surjectivity of β and to the fact that x belongs to a finite number of sets $U_{i_0\dots i_p}$. Having (c) satisfied, it suffices to restrict V_x to satisfy (a) and (b).

The family $(V_x)_{x\in X}$ forms a covering $\mathcal{V}$; for any $x \in X$, choose $\tau(x) \in I$ such that $x \in U_{\tau(x)}$. We now check that $\tau(f)$ belongs to $C_0^p(\mathcal{V}, \mathscr{C})$. If $V_{x_0\dots x_p}$ is empty, this is obvious; if not, we have $V_{x_0} \cap U_{x_k} \neq \varnothing$ for $0 \leq k \leq p$, and then

$$V_{x_0} \cap U_{\tau(x_k)} \neq \varnothing \quad \text{for} \quad 0 \leq k \leq p,$$

which implies by (b) that $V_{x_0} \subseteq U_{\tau(x_k)}$ for all k, and hence $x_0 \in U_{\tau(x_0)\dots\tau(x_p)}$. We then apply (c) to get a section b of $\mathscr{B}$ over V_{x_0} such that $\beta(b) = f_{\tau(x_0)\dots\tau(x_p)}|_{V_{x_0}}$. Thus $\tau(f) \in C_0^p(\mathcal{V}, \mathscr{C})$, which completes the proof. □

Corollary 3.4.8. *If X is paracompact, we have an exact sequence:*

$$\cdots \longrightarrow \check{H}^p(X, \mathscr{A}) \xrightarrow{\alpha^*} \check{H}^p(X, \mathscr{B}) \xrightarrow{\beta^*} \check{H}^p(X, \mathscr{C}) \xrightarrow{\delta} \check{H}^{p+1}(X, \mathscr{A}) \longrightarrow \cdots$$

The exact sequence mentioned above is called the long exact sequence of cohomology defined by a given exact sequence of sheaves $0 \to \mathscr{A} \to \mathscr{B} \to \mathscr{C} \to 0$. More generally, it exists whenever we can show that $\check{H}_0^p(X, \mathscr{C}) \to \check{H}^p(X, \mathscr{C})$ is bijective.

3.5 Čech resolution and Leray Acyclic Theorem

We will now compare the Čech cohomology and derived functor cohomology of a sheaf $\mathscr{F}$ on a topological space X. We will see that in some cases we are able to conclude that these two cohomologies coincide. In order to compare Čech cohomology with derived functor cohomology, we will need to consider first a sheafified version of the Čech complex.

Fix a space X, an open cover $\mathcal{U} = (U_i)_{i\in I}$ and a sheaf $\mathscr{F}$ on X. For every open set U of X, let $j_U : U \hookrightarrow X$ denote the inclusion. Define a sheaf $\mathscr{C}^p(\mathcal{U}, \mathscr{F})$ by

$$\mathscr{C}^p(\mathcal{U}, \mathscr{F}) = \prod_{(i_0,\dots,i_p)} (j_{U_{i_0\dots i_p}})_*(\mathscr{F}|_{U_{i_0\dots i_p}}).$$

Explicitly, for an open subset $V \subseteq X$ we have

$$\Gamma(V, \mathscr{C}^p(\mathcal{U}, \mathscr{F})) = \prod_{(i_0,\dots,i_p)} \mathscr{F}(V \cap U_{i_0\dots i_p}).$$

For each inclusion $\delta_j^p : U_{i_0 \dots i_p} \to U_{i_0 \dots \widehat{i_j} \dots i_p}$, we also have a restriction map

$$\mathscr{C}(\delta_j^p) : (j_{U_{i_0 \dots \widehat{i_j} \dots i_p}})_*(\mathscr{F}|_{U_{i_0 \dots \widehat{i_j} \dots i_p}}) \to (j_{U_{i_0 \dots i_p}})_*(\mathscr{F}|_{U_{i_0 \dots i_p}})$$

so we can define the differential $d : \mathscr{C}^p(\mathcal{U}, \mathscr{F}) \to \mathscr{C}^{p+1}(\mathcal{U}, \mathscr{F})$ by

$$d = \sum_{j=0}^{p+1} (-1)^j \mathscr{C}(\delta_j^{p+1}).$$

Thus we get complex $\mathscr{C}^\bullet(\mathcal{U}, \mathscr{F})$. By definition, we have

$$\Gamma(X, \mathscr{C}^\bullet(\mathcal{U}, \mathscr{F})) = C^\bullet(\mathcal{U}, \mathscr{F})$$

Also, there is a product map $\epsilon : \mathscr{F} \to \mathscr{C}^0(\mathcal{U}, \mathscr{F})$ defined by

$$\mathscr{F} \to \prod_i (j_{U_i})_*(j_{U_i})^{-1}\mathscr{F} = \prod_i (j_{U_i})_*(\mathscr{F}|_{U_i}).$$

Proposition 3.5.1. *For an open cover $\mathcal{U}$ and a sheaf $\mathscr{F}$, there is an exact sequence of sheaves*

$$0 \longrightarrow \mathscr{F} \xrightarrow{\epsilon} \mathscr{C}^0(\mathcal{U}, \mathscr{F}) \longrightarrow \mathscr{C}^1(\mathcal{U}, \mathscr{F}) \longrightarrow \mathscr{C}^2(\mathcal{U}, \mathscr{F}) \longrightarrow \cdots$$

This is called the ***Čech resolution*** *of $\mathscr{F}$ with respect to the cover $\mathcal{U}$.*

Proof. The facts that ϵ is injective and that $\operatorname{im} \epsilon = \ker d^0$ follow directly from $\mathscr{F}$ being a sheaf.

It remains to be shown that the proposed sequence is exact in degrees $p > 0$. For this, it suffices to work at the level of stalks.

To prove the exactness at $\mathscr{C}^p(\mathcal{U}, \mathscr{F})$, we need to check the sequence of stalks $\mathscr{C}^\bullet(\mathcal{U}, \mathscr{F})$ is exact for all $x \in X$. Since $\mathcal{U}$ covers X, we can choose an open subset U_j containing x. Take $f_x \in \mathscr{C}^p(\mathcal{U}, \mathscr{F})_x$ which is represented by (V, f), where V is a neighborhood of x and $f \in \Gamma(V, \mathscr{C}^p(\mathcal{U}, \mathscr{F}))$. We may assume $V \subseteq U_j$. Then we observe that for any index $(i_0, \dots, i_{p-1}) \in I^p$ we have

$$V \cap U_{i_0 \dots i_{p-1}} = V \cap U_{j, i_0 \dots i_{p-1}}$$

Thus, we may define an element $\theta f \in \Gamma(V, \mathscr{C}^{p-1}(\mathcal{U}, \mathscr{F}))$ by the formula

$$(\theta f)_{i_0 \dots i_{p-1}} := f_{j, i_0 \dots i_{p-1}}$$

This gives a map

$$\theta^p : \mathscr{C}^p(\mathcal{U}, \mathscr{F})_x \to \mathscr{C}^{p-1}(\mathcal{U}, \mathscr{F})_x, \quad f_x \mapsto (\theta f)_x$$

Now we check that

$$(\theta(df))_{i_0 \dots i_p} = (df)_{j, i_0 \dots i_p} = f_{i_0 \dots i_p} - \sum_{k=0}^{p} (-1)^k (f_{j, i_0 \dots \widehat{i_k} \dots i_p})|_{U_{i_0 \dots i_p}}.$$

and

$$(d(\theta f))_{i_0 \dots i_p} = \sum_{k=0}^{p} (-1)^k ((\theta f)_{i_0 \dots \widehat{i_k} \dots i_p})|_{U_{i_0 \dots i_p}} = \sum_{k=0}^{p} (-1)^k (f_{j, i_0 \dots \widehat{i_k} \dots i_p})|_{U_{i_0 \dots i_p}}.$$

Therefore

$$\theta^{p+1}\big((df)_x\big) = (\theta(df))_x = \big(f - d(\theta f)\big)_x = f_x - d((\theta f)_x) = f_x - d(\theta^p f_x).$$

Hence θ^p is a homotopy from the identity of $\mathscr{C}^p(\mathcal{U}, \mathscr{F})_x$ to the zero map, which shows the sequence of stalks is exact. □

One of the motivitions of defining the Čech resolution is the following proposition.

Proposition 3.5.2. *The Čech resolution computes the Čech cohomology. In the sense that the Čech cohomology groups can be derived from the Čech cohomology by appying the global section and taking cohomology.*

Proof. Applying the global section on the complex $0 \to \mathscr{F} \to \mathscr{C}^\bullet(\mathcal{U}, \mathscr{F})$ gives the complex

$$0 \longrightarrow \mathscr{F}(X) \longrightarrow C^0(\mathcal{U}, \mathscr{F}) \longrightarrow C^1(\mathcal{U}, \mathscr{F}) \longrightarrow C^2(\mathcal{U}, \mathscr{F}) \longrightarrow \cdots$$

which is exactly the Čech complex. Thus its cohomology gives the Čech cohomology. □

Corollary 3.5.3. *With X, $\mathcal{U}$, $\mathscr{F}$ as above, there is a canonical map*

$$\check{H}^\bullet(\mathcal{U}, \mathscr{F}) \to H^\bullet(X, \mathscr{F}).$$

Proof. Let $0 \to \mathscr{F} \to \mathscr{I}^\bullet$ be an injective resolution of $\mathscr{F}$. Then since $0 \to \mathscr{F} \to \mathscr{C}^\bullet(\mathcal{U}, \mathscr{F})$ is also a resolution of $\mathscr{F}$, we have a canonical induced map $\mathscr{C}^\bullet(\mathcal{U}, \mathscr{F}) \to \mathscr{I}^\bullet$ induced by the identity of $\mathscr{F}$. The induced map on cohomology is what we want. □

We will now state some results stating sufficient conditions for these canonical morphisms to actually be isomorphisms, thus enabling us to calculate sheaf cohomology via Čech cohomology.

Proposition 3.5.4. *If $\mathscr{F}$ is a flasque sheaf on X, then $\check{H}^p(\mathcal{U}, \mathscr{F}) = 0$ for all open covers $\mathcal{U}$ of X and all $p > 0$.*

Proof. Let $\mathscr{C}^\bullet(\mathcal{U}, \mathscr{F})$ be the Čech resolution of $\mathscr{F}$ with respect to $\mathcal{U}$. Recall that $\mathscr{C}^p(\mathcal{U}, \mathscr{F})$ is defined for any open $V \subseteq X$ by

$$\Gamma(V, \mathscr{C}^p(\mathcal{U}, \mathscr{F})) = \prod \mathscr{F}(V \cap U_{i_0 \dots i_p}).$$

For each of these $U_{i_0 \dots i_p}$, the sheaf $V \mapsto \mathscr{F}(V \cap U_{i_0 \dots i_p})$ is flasque and since products of flasque sheaves are flasque, the entire sheaf $\mathscr{C}^p(\mathcal{U}, \mathscr{F})$ is flasque. Therefore $\mathscr{C}^p(\mathcal{U}, \mathscr{F})$ is acyclic by Proposition 2.2.6, so the Čech resolution can used to compute sheaf cohomology:

$$\check{H}^p(\mathcal{U}, \mathscr{F}) = H^p(X, \mathscr{F}) = 0$$

as desired. □

Corollary 3.5.5. *For any sheaf $\mathscr{F}$ on X, the map $\check{H}^1(X,\mathscr{F}) \to H^1(X,\mathscr{F})$ is an isomorphism.*

Proof. Let $\mathscr{G}$ be a flasque sheaf and $0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{Q} \to 0$ be a short exact sequence of sheaves on X. Then $\check{H}^1(X,\mathscr{G}) = 0$, and by Corollary 3.4.4 there is an exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \Gamma(X,\mathscr{F}) \arrow[r] \arrow[d, "1"] & \Gamma(X,\mathscr{G}) \arrow[r] \arrow[d, "1"] & \Gamma(X,\mathscr{Q}) \arrow[r] \arrow[d, "1"] & \check{H}^1(X,\mathscr{F}) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \Gamma(X,\mathscr{F}) \arrow[r] & \Gamma(X,\mathscr{G}) \arrow[r] & \Gamma(X,\mathscr{Q}) \arrow[r] & H^1(X,\mathscr{F}) \arrow[r] & 0 \end{tikzcd}$$

Thus the map $\check{H}^1(X,\mathscr{F}) \to H^1(X,\mathscr{F})$ is an isomorphism by the five lemma. □

Proposition 3.5.6. *Let X be a paracompact space, and $\mathscr{F}$ be a sheaf on X. Then the canonical maps*

$$\check{H}^p(X,\mathscr{F}) \to H^p(X,\mathscr{F})$$

are isomorphisms for $p \geq 0$.

Proof. This follows from Corollary 3.4.8 and an induction. □

Definition 3.5.7. *A sheaf $\mathscr{F}$ on X is **acyclic for an open cover $\mathcal{U} = (U_i)_{i\in I}$** if for all $p > 0$ we have*

$$H^p(U_{i_0\cdots i_p}, \mathscr{F}|_{U_{i_0\cdots i_p}}) = 0.$$

Theorem 3.5.8 (Leray). *If $\mathscr{F}$ is a sheaf on X which is acyclic for an open cover $\mathcal{U}$, then the canonical maps*

$$\check{H}^p(\mathcal{U},\mathscr{F}) \to H^p(X,\mathscr{F})$$

are isomorphisms for all $p \geq 0$.

Proof. We proceed by induction on the degree p. For $p = 0$ we already know that the result is true thanks to Proposition 3.1.5.

Now, assume $\check{H}^j(U,\mathscr{G}) \to H^j(X,\mathscr{G})$ is an isomorphism for all $j \leq p$ and all sheaves $\mathscr{G}$ acyclic for $\mathcal{U}$. Embed $\mathscr{F}$ in a flasque sheaf $\mathscr{G}$ and let $\mathscr{H}$ be the quotient, so that we have an exact sequence

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{G} \longrightarrow \mathscr{H} \longrightarrow 0$$

For each finite sequence $\sigma = (i_0,\dots,i_p)$, by hypothesis we have $H^i(U_\sigma,\mathscr{F}|_{U_\sigma}) = 0$ and $H^i(U_\sigma,\mathscr{G}|_{U_\sigma}) = 0$. Therefore $H^i(U_\sigma,\mathscr{H}|_{U_\sigma})$ is zero by the long exact sequence of $H^i(U_\sigma.-)$, and by taking the product over all such U_σ, we conclude that $\mathscr{H}$ is also acyclic for $\mathcal{U}$.

Now by the argument above, the sequence

$$0 \longrightarrow \mathscr{F}(U_\sigma) \longrightarrow \mathscr{G}(U_\sigma) \longrightarrow \mathscr{H}(U_\sigma) \longrightarrow 0$$

is exact. Hence by taking product the corresponding short sequence of Čech complexes

$$0 \longrightarrow \mathscr{C}^\bullet(\mathcal{U},\mathscr{F}) \longrightarrow \mathscr{C}^\bullet(\mathcal{U},\mathscr{G}) \longrightarrow \mathscr{C}^\bullet(\mathcal{U},\mathscr{H}) \longrightarrow 0$$

is exact, so that we get a long exact sequence in Čech cohomology

$$\cdots \longrightarrow \check{H}^p(\mathcal{U},\mathscr{F}) \longrightarrow \check{H}^p(\mathcal{U},\mathscr{G}) \longrightarrow \check{H}^p(\mathcal{U},\mathscr{H}) \longrightarrow \check{H}^{p+1}(\mathcal{U},\mathscr{F}) \longrightarrow \cdots$$

Now since $\mathscr{G}$ is flasque, Proposition 3.5.4 shows that $\check{H}^p(\mathcal{U},\mathscr{G}) = 0$. Therefore we have a diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & \check{H}^p(\mathcal{U},\mathscr{H}) \arrow[r] \arrow[d] & \check{H}^{p+1}(\mathcal{U},\mathscr{F}) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & H^p(\mathcal{U},\mathscr{H}) \arrow[r] & H^{p+1}(\mathcal{U},\mathscr{F}) \arrow[r] & 0 \end{tikzcd}$$

By induction, the left column is an isomorphism, so $\check{H}^{p+1}(X,\mathscr{F}) \to H^{p+1}(X,\mathscr{F})$ is also an isomorphism. □

3.6 Čech vs. Sheaf cohomology

Proposition 3.6.1. *Let X be a ringed space. Let*

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{G} \longrightarrow \mathscr{H} \longrightarrow 0$$

be a short exact sequence of $\mathscr{O}_X$-modules. Let $U \subseteq X$ be an open subset. If there exists a cofinal system of open coverings $\mathcal{U}$ of U such that $\check{H}^1(\mathcal{U},\mathscr{F}) = 0$, then the map $\mathscr{G}(U) \to \mathscr{H}(U)$ is surjective.

Proof. Take an element $s \in \mathscr{H}(U)$. Choose an open covering $\mathcal{U} : U = (U_i)_{i\in I}$ such that

(a) $\check{H}^1(\mathcal{U},\mathscr{F}) = 0$.

(b) $s|_{U_i}$ is the image of a section $s_i \in \mathscr{G}(U_i)$.

Since we can certainly find a covering such that (b) holds by the exactness of the sequence, it follows from the assumptions that we can find a covering such that (a) and (b) both hold. Consider the sections

$$s_{i_0 i_1} = s_{i_1}|_{U_{i_0 i_1}} - s_{i_0}|_{U_{i_0 i_1}}$$

Since s_i lifts s we see that $s_{i_0 i_1} \in \mathscr{F}(U_{i_0 i_1})$. By the vanishing of $\check{H}^1(\mathcal{U},\mathscr{F})$ we can find sections $t_i \in \mathscr{F}(U_i)$ such that

$$s_{i_0 i_1} = t_{i_1}|_{U_{i_0 i_1}} - t_{i_0}|_{U_{i_0 i_1}}$$

Then clearly the sections $s_i - t_i$ satisfy the sheaf condition and glue to a section of $\mathscr{G}$ over U which maps to s. Hence we win. □

Proposition 3.6.2. *Let X be a ringed space. Let $\mathscr{F}$ be an $\mathcal{O}_X$-module such that*

$$\check{H}^p(\mathcal{U},\mathscr{F}) = 0$$

for all $p > 0$ and any open covering $\mathcal{U} : U = \bigcup_{i\in I} U_i$ of an open subset U of X. Then $H^p(U,\mathscr{F}) = 0$ for all $p > 0$ and any open $U \subseteq X$.

Proof. Choose an embedding $\mathcal{F} \to \mathcal{I}$ into an injective $\mathcal{O}_X$-module. By Proposition 3.5.4 $\mathcal{I}$ has vanishing higher Čech cohomology for any open covering. Let $\mathcal{Q} = \mathcal{I}/\mathcal{F}$ so that we have a short exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{I} \longrightarrow \mathcal{Q} \longrightarrow 0$$

By Proposition3.6.1 and our assumptions this sequence is actually exact as a sequence of presheaves! In particular we have a long exact sequence of Čech cohomology groups for any open covering $\mathcal{U}$, see Proposition 3.2.2 for example. This implies that $\mathcal{Q}$ is also an $\mathcal{O}_X$-module with vanishing higher Čech cohomology for all open coverings.

Next, we look at the long exact cohomology sequence

$$\cdots \longrightarrow H^p(U, \mathcal{F}) \longrightarrow H^p(U, \mathcal{I}) \longrightarrow H^p(U, \mathcal{Q}) \longrightarrow H^{p+1}(U, \mathcal{F}) \longrightarrow \cdots$$

for any open $U \subseteq X$. Since $\mathcal{I}$ is injective we have $H^p(U, \mathcal{I}) = 0$ for $p > 0$. By the above we see that $H^0(U, \mathcal{I}) \to H^0(U, \mathcal{Q})$ is surjective, thus $H^1(U, \mathcal{F}) = 0$. Also, $H^p(U, \mathcal{Q}) = H^{p+1}(U, \mathcal{F})$ for $p \geq 1$. Thus the claim follows by an induction. □

Proposition 3.6.3. *Let X be a ringed space. Let $\mathcal{B}$ be a basis for the topology on X. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Assume there exists a set of open coverings* Cov *with the following properties:*

(i) For every $\mathcal{U} \in \mathrm{Cov}$ with $\mathcal{U} : U = \bigcup_{i \in I} U_i$ we have $U, U_i \in \mathcal{B}$ and every $U_{i_0 \ldots i_p} \in \mathcal{B}$.

(ii) For every $U \in \mathcal{B}$ the open coverings of U occurring in Cov *is a cofinal system of open coverings of U.*

(iii) For every $\mathcal{U} \in \mathrm{Cov}$ we have $\check{H}^p(\mathcal{U}, \mathcal{F}) = 0$ for all $p > 0$.

Then $H^p(U, \mathcal{F}) = 0$ for all $p > 0$ and any $U \in \mathcal{B}$.

Proof. Let $\mathcal{F}$ and Cov be as in the lemma. We will indicate this by saying $\mathcal{F}$ has vanishing higher Čech cohomology for any $\mathcal{U} \in \mathrm{Cov}$. Choose an embedding $\mathcal{F} \hookrightarrow \mathcal{I}$ into an injective $\mathcal{O}_X$-module. By Proposition 3.5.4 has vanishing higher Čech cohomology for any $\mathcal{U} \in \mathrm{Cov}$. Let $\mathcal{Q} = \mathcal{I}/\mathcal{F}$ so that we have a short exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{I} \longrightarrow \mathcal{Q} \longrightarrow 0$$

By Proposition 3.6.1 and our assumption (ii) this sequence gives rise to an exact sequence

$$0 \longrightarrow \mathcal{F}(U) \longrightarrow \mathcal{I}(U) \longrightarrow \mathcal{Q}(U) \longrightarrow 0$$

for every $U \in \mathcal{B}$. Hence for any $\mathcal{U} \in \mathrm{Cov}$ we get a short exact sequence of Čech complexes

$$0 \longrightarrow \mathcal{C}^\bullet(\mathcal{U}, \mathcal{F}) \longrightarrow \mathcal{C}^\bullet(\mathcal{U}, \mathcal{I}) \longrightarrow \mathcal{C}^\bullet(\mathcal{U}, \mathcal{Q}) \longrightarrow 0$$

since each term in the Čech complex is made up out of a product of values over elements of $\mathcal{B}$ by assumption (i). In particular we have a long exact sequence of Čech cohomology groups for any open covering $\mathcal{U} \in \mathrm{Cov}$. This implies that $\mathcal{Q}$ is also an $\mathcal{O}_X$-module with vanishing higher Čech cohomology for all $\mathcal{U} \in \mathrm{Cov}$.

Next, we look at the long exact cohomology sequence

$$\cdots \longrightarrow H^p(U, \mathcal{F}) \longrightarrow H^p(U, \mathcal{I}) \longrightarrow H^p(U, \mathcal{Q}) \longrightarrow H^{p+1}(U, \mathcal{F}) \longrightarrow \cdots$$

for any open $U \in \mathcal{B}$. Since $\mathcal{I}$ is injective we have $H^n(U, \mathcal{I}) = 0$ for $n > 0$. By the above we see that $H^0(U, \mathcal{I}) \to H^0(U, \mathcal{Q})$ is surjective and hence $H^1(U, \mathcal{F}) = 0$. Since $\mathcal{F}$ was an arbitrary $\mathcal{O}_X$-module with vanishing higher Čech cohomology for all $\mathcal{U} \in \mathrm{Cov}$ we conclude that also $H^1(U, \mathcal{Q}) = 0$ since $\mathcal{Q}$ is another of these sheaves (see above). Also, $H^p(U, \mathcal{Q}) = H^{p+1}(U, \mathcal{F})$ for $p \geq 1$. Thus the claim follows by an induction. □

Proposition 3.6.4. *Let $f : X \to Y$ be a morphism of ringed spaces. Let $\mathcal{I}$ be an injective $\mathcal{O}_X$-module. Then*

- *$\check{H}^p(\mathcal{V}, f_*\mathcal{I}) = 0$ for all $p > 0$ and any open covering $\mathcal{V} = (V_j)_{j \in J}$ of Y.*
- *$H^p(V, f_*\mathcal{I}) = 0$ for all $p > 0$ and every open $V \subseteq Y$.*

In other words, $f_*\mathcal{I}$ is right acyclic for $\Gamma(U, -)$ for any $U \subseteq X$ open.

Proof. Set $\mathcal{U} = f^{-1}(\mathcal{V})$. It is an open covering of X and

$$C^\bullet(\mathcal{V}, f_*\mathcal{I}) = C^\bullet(\mathcal{U}, \mathcal{I}).$$

This is true because

$$f_*\mathcal{I}(V_{j_0 \ldots j_p}) = \mathcal{I}(f^{-1}(V_{j_0 \ldots j_p})) = \mathcal{I}\left(f^{-1}(V_{j_0}) \cap \cdots \cap f^{-1}(V_{j_p})\right) = \mathcal{I}(U_{j_0 \ldots j_p}).$$

Thus the first statement of the lemma follows from Proposition 3.5.4. The second statement follows from the first and Proposition 3.6.2. □

Chapter 4

The language of schemes

4.1 Affine schemes

Let A be a ring. Recall that we have associate with A a topological space $\mathrm{Spec}(A)$, called the spectrum of A. In this section we shall make $\mathrm{Spec}(A)$ a locally ringed space and consider sheaf of modules over it; such spaces will be called **affine schemes**.

4.1.1 Sheaves associated with a module

Let A be a ring and M an A-module. For any element $f \in A$, let S_f be the multiplicative subset consisting of powers of f. Recall that the localization of M with respect to S_f is then denoted by M_f, and that of A by A_f. Let $\bar{S}_f$ be the saturation of S_f, which is defined to be the complement of the union of prime ideals of A that are disjoint from S_f, or equivalently not contains f. By Proposition **??**, the set $\bar{S}_f$ is also characterized by

$$\bar{S}_f = \{x \in A : \text{there exist } n, m \geq 0 \text{ such that } f^n x = f^m\}.$$

Also, by Proposition **??**, we have $\bar{S}_f A = A_f$ and $\bar{S}_f M = M_f$.

Lemma 4.1.1. *Let f, g be elements of A. Then the following conditions are equivalent:*

(i) $g \in \bar{S}_f$*, or equivalently* $\bar{S}_g \subseteq \bar{S}_f$*;*

(ii) $f \in \sqrt{(g)}$*, or equivalently* $\sqrt{(f)} \subseteq \sqrt{(g)}$*;*

(iii) $D(f) \subseteq D(g)$*, or equivalently* $V(g) \subseteq V(f)$*.*

Proof. We first note that $g \in \bar{S}_f$ is equivalent to $S_g \subseteq \bar{S}_f$, so the equivalence in (i). Also, the equivalence of (ii) and (iii) follows from Proposition **??**. Finally, if $g \in \bar{S}_f$, then there exist $n, m \geq 0$ such that $f^n g = f^m$, which is an element of (g), and thus $f \in \sqrt{(g)}$. Conversely, if $D(f) \subseteq D(g)$, then by the discriptions $S_f = \bigcup_{f \notin \mathfrak{p}} \mathfrak{p}$ and $S_g = \bigcup_{g \notin \mathfrak{p}} \mathfrak{p}$, we conclude that $\bar{S}_g \subseteq \bar{S}_f$, whence the lemma. □

If $D(g) \subseteq D(f)$ in $\mathrm{Spec}(A)$, then by Lemma 4.1.1, we have $\bar{S}_f \subseteq \bar{S}_g$, so there is a canonical homomorphism $\rho_{g,f} : M_f \to M_g$; moreover, if $D(f) \supseteq D(g) \supseteq D(h)$, we then have

$$\rho_{h,g} \circ \rho_{g,f} = \rho_{h,f}.$$

As f runs through $A-\mathfrak{p}$ (where $\mathfrak{p}$ is a point in $X=\mathrm{Spec}(A)$), the set S_f then constitute a filtered set indexed by $A-\mathfrak{p}$, since any two element f,g of $A-\mathfrak{p}$ contains S_{fg}; as the union of the S_f for $f\in A-\mathfrak{p}$ is $A-\mathfrak{p}$, we conclude from Proposition **??** that the $A_\mathfrak{p}$-module $M_\mathfrak{p}$ is canonically identified with the direct limit $\varinjlim M_f$, relative to the family $(\rho_{g,f})$ of homomorphisms. For each $f\in A-\mathfrak{p}$, we denote the canonical homomorphism from M_f to $M_\mathfrak{p}$ by

$$\rho_\mathfrak{p}^f : M_f\to M_\mathfrak{p}.$$

We now define the **structural sheaf** of the prime spectrum $X=\mathrm{Spec}(A)$, denoted by $\widetilde{A}$, to be the sheaf of rings associated with the presheaf $D(f)\mapsto A_f$ over the basis $\mathcal{B}$ of X, formed by $D(f)$ with $f\in A$. Simialrly, for an A-module M,we define the **associated sheaf** $\widetilde{M}$ to be the sheaf associated presheaf $D(f)\mapsto M_f$ over the basis $\mathcal{B}$ of X. By the property of sheafification, it is clear that the stalk $\widetilde{A}_\mathfrak{p}$ (resp. $\tilde{M}_\mathfrak{p}$) is identified with the ring $A_\mathfrak{p}$ (resp. with $A_\mathfrak{p}$-module $M_\mathfrak{p}$).

Theorem 4.1.2. *For each A-module M, the presheaf $D(f)\to M_f$ is a sheaf on the basis $\mathcal{B}$ of X, so for each $f\in A$ we have a canonical isomorphism*

$$M_f\to\Gamma(D(f),\widetilde{M}).$$

In particular, M is canonically identified with $\Gamma(X,\widetilde{M})$.

Proof. To show that the presheaf $D(f)\mapsto M_f$ is a sheaf on the basis $\mathcal{B}$ of X, we need to check the sheaf condition for the standard open coverings. If $D(f)=\bigcup_{i=1}^n D(g_i)$, then the sheaf condition for this covering is equivalent with the exactness of the sequence

$$0\longrightarrow M_f\longrightarrow \textstyle\bigoplus_i M_{g_i}\longrightarrow \bigoplus_{i,j} M_{g_ig_j}$$

Note that $D(g_i)=D(fg_i)$, and hence we can rewrite this sequence as the sequence

$$0\longrightarrow M_f\longrightarrow \textstyle\bigoplus_i M_{fg_i}\longrightarrow \bigoplus_{i,j} M_{fg_ig_j}$$

In addition, since $D(g_i)$'s cover $D(f)$ (which is identified with $\mathrm{Spec}(A_f)$), the elements $g_1,\dots,g_n$ generate the unit ideal in A_f, so we may apply Proposition **??** to the module M_f over A_f and the elements $g_1,\dots,g_n$ to conclude that the sequence is exact. □

Corollary 4.1.3. *Let M,N be A-modules. The canonical homomorphism*

$$\mathrm{Hom}_A(M,N)\to\mathrm{Hom}_{\widetilde{A}}(\widetilde{M},\widetilde{N}),\quad \phi\mapsto\tilde{\phi}$$

is bijective. In particular, the relations $M=0$ and $\widetilde{M}=0$ are equivalent.

Proof. Consider the canonical homomorphism $\mathrm{Hom}_{\widetilde{A}}(\widetilde{M},\widetilde{N})\to\mathrm{Hom}_A(M,N)$, $\varphi\mapsto\Gamma(\varphi)$ (Theorem 4.1.2). It suffices to show that $\phi\mapsto\tilde{\phi}$ and $\varphi\mapsto\Gamma(\varphi)$ are inverses of each other. Now, it is evident that $\Gamma(\tilde{\phi})=\phi$, by the definition of $\tilde{\phi}$. On the other hand, if we put $\phi=\Gamma(\varphi)$ for $\varphi\in\mathrm{Hom}_{\widetilde{A}}(\widetilde{M},\widetilde{N})$, the map $\varphi_{D(f)}:\Gamma(D(f),\widetilde{M})\to\Gamma(D(f),\widetilde{N})$ induced by φ is making the following diagram commutes:

$$\begin{tikzcd} M \arrow[r, "\phi"] \arrow[d, "\rho_{f,1}"'] & N \arrow[d, "\rho_{f,1}"] \\ M_f \arrow[r, "\varphi_{D(f)}"] & N_f \end{tikzcd}$$

We then have necessarily $\varphi_{D(f)} = \phi_f$ for each $f \in A$, which shows $\widetilde{\Gamma(\varphi)} = \varphi$. □

Proposition 4.1.4. *For each $f \in A$, the open set $D(f) \subseteq X$ is canonically identified with the spectrum* Spec(A_f)*, and the sheaf $\widetilde{M}_f$ associated with the A_f-module M_f is canonically identified with the restriction $\widetilde{M}_{D(f)}$.*

Proof. The first assertion is proved in Proposition **??**. Now for $D(g) \subseteq D(f)$, then M_g is identified with the localization of M_f with respect to the canonical image of g in A_f, so the canonical identification of $\tilde{M}_f$ and $\widetilde{M}_{D(f)}$ follows by definition. □

Proposition 4.1.5. *The functor $M \mapsto \widetilde{M}$ is an exact functor from the category of A-modules to the category of $\widetilde{A}$-modules.*

Proof. Let M, N be two A-modules and $\phi : M \to N$ a homomorphism; for any $f \in A$, we have a corresponding homomorphism $\phi_f : M_f \to N_f$, and for $D(g) \subseteq D(f)$ the diagram

$$\begin{tikzcd} M_f \arrow[r, "\phi_f"] \arrow[d, "\rho_{g,f}"'] & N_f \arrow[d, "\rho_{g,f}"] \\ M_g \arrow[r, "\phi_g"] & N_g \end{tikzcd}$$

is commutative. These then give a homomorphism of $\widetilde{A}$-modules $\tilde{\phi} : \widetilde{M} \to \widetilde{N}$. Moreover, for each $\mathfrak{p} \in X$, $\tilde{\phi}_x$ is the direct limit of ϕ_f for $f \in A - \mathfrak{p}$, and concequently identified with the canonical homomorphism $\phi_{\mathfrak{p}} : M_{\mathfrak{p}} \to N_{\mathfrak{p}}$. If P is another A-module, $\psi : N \to P$ a homomorphism and $\eta = \psi \circ \phi$, then it is immediate that $\eta_{\mathfrak{p}} = \psi_{\mathfrak{p}} \circ \phi_{\mathfrak{p}}$, hence $\tilde{\eta} = \tilde{\psi} \circ \tilde{\phi}$. We thus get a covariant functor $\widetilde{(-)}$ from the category of A-modules to the category of $\widetilde{A}$-modules. This functor is exact since for each $\mathfrak{p} \in X$, $M \mapsto M_{\mathfrak{p}}$ is an exact functor; furthermore, we have $\mathrm{supp}(M) = \mathrm{supp}(\widetilde{M})$ by the definitions of these two members. □

Corollary 4.1.6. *Let M and N be two A-modules.*

(a) If $\phi : M \to N$ is a homomorphism, then the sheaves associated with $\ker\phi$, $\operatorname{im}\phi$, *and* $\operatorname{coker}\phi$ *are* $\ker\tilde{\phi}$, $\operatorname{im}\tilde{\phi}$, *and* $\operatorname{coker}\tilde{\phi}$*, respectively. In particular, $\tilde{\phi}$ is injective (resp. surjective, bijective) if and only if ϕ is injective (resp. surjective, bijective).*

(b) If M is a filtered limit (resp. direct sum) of a family $(M_i)_{i \in I}$ of A-modules, then $\widetilde{M}$ is a filtered limit (resp. direct sum) of the family $(\widetilde{M}_i)$.

Proof. For (a), it suffices to apply the exact functor $M \mapsto \widetilde{M}$ to the following exact sequences:

$$0 \longrightarrow \ker\phi \longrightarrow M \longrightarrow \operatorname{im}\phi \longrightarrow 0$$
$$0 \longrightarrow \operatorname{im}\phi \longrightarrow N \longrightarrow \operatorname{coker}\phi \longrightarrow 0$$

Now let (M_i, ρ_{ji}) be a filtered system of A-modules, with limit M, and let $\rho_i : M_i \to M$ be the canonical homomorphism. Since we have $\tilde{\rho}_{kj} \circ \tilde{\rho}_{ji} = \tilde{\rho}_{ji}$ and $\tilde{\rho}_i = \tilde{\rho}_j \circ \tilde{\rho}_{ji}$ for $i \leq j \leq k$, we see $(\widetilde{M}, \tilde{\rho}_{ji})$ is a direct system of sheaves over X, and if we denote by $\eta_i : \widetilde{M}_i \to \varinjlim \widetilde{M}_i$ the canonical homomorphism, a unique homomorphism $\psi : \varinjlim \widetilde{M}_i \to \widetilde{M}$ such that $\psi \circ \eta_i = \tilde{\rho}_i$. For this ψ to be bijective, it suffices that for each $\mathfrak{p} \in X$, $\psi_{\mathfrak{p}}$ is a bijection from $(\varinjlim \widetilde{M}_i)_{\mathfrak{p}}$ to $\widetilde{M}_{\mathfrak{p}}$; but $\widetilde{M}_{\mathfrak{p}} = M_{\mathfrak{p}}$ and

$$(\varinjlim \widetilde{M}_i)_{\mathfrak{p}} = \varinjlim (\widetilde{M}_i)_{\mathfrak{p}} = \varinjlim (M_i)_{\mathfrak{p}} = M_{\mathfrak{p}}$$

Also, it follows by definition that $(\tilde{\rho}_i)_{\mathfrak{p}}$ and $(\eta_i)_{\mathfrak{p}}$ are equal to the canonical homomorphism from $(M_i)_{\mathfrak{p}}$ to $M_{\mathfrak{p}}$; since $(\tilde{\rho}_i)_{\mathfrak{p}} = \psi_{\mathfrak{p}} \circ (\eta_i)_{\mathfrak{p}}$, $\psi_{\mathfrak{p}}$ is therefore the identity.

Finally, if M is a direct sum of two modules N and P, it is immediate that $\widetilde{M} = \widetilde{M} \oplus \widetilde{P}$; by taking filtered limits, we then generalize this result for the direct sum of an arbitrary family. This completes the proof. □

Remark 4.1.1. By Proposition 4.1.5, we conclude that the sheaves which are isomorphic to the sheaves associated with A-modules form an abelian category. Note also that it follows from Corollary 4.1.6 that if M is a finitely generated A-module, that is, if there exists a surjective homomorphism $A^n \to M$, then there exists a homomorphism surjective $\widetilde{A}^n \to \widetilde{M}$, in other words, the $\widetilde{A}$-module $\widetilde{M}$ is generated by a finite family of sections over X, and vice versa.

Corollary 4.1.7. *Let N and P be submodules of M. The sheaves $\widetilde{N}$ and $\widetilde{P}$ can be identified with sub-$\widetilde{A}$-modules of $\widetilde{M}$, and we have*

$$\widetilde{N+P} = \widetilde{N} + \widetilde{P}, \quad \widetilde{N \cap P} = \widetilde{N} \cap \widetilde{P}.$$

In particular, if $\widetilde{N} = \widetilde{P}$, then $N = P$.

Proof. If N is a submodule of an A-module M, the canonical injection $N \to M$ induced an injective homomorphism $\widetilde{N} \to \widetilde{M}$, hence identifies $\widetilde{N}$ with a sub-$\widetilde{A}$-module of $\widetilde{M}$. Now note that $N + P$ is the image of the canonical homomorphism $\alpha : N \oplus P \to M$, so by Corollary 4.1.6 we have

$$\widetilde{N+P} = \widetilde{\operatorname{im} \alpha} = \operatorname{im} \tilde{\alpha} = \widetilde{N} + \widetilde{P}$$

since $\tilde{\alpha}$ is equal to the canonical homomorphism $\widetilde{N} \oplus \widetilde{P} \to \widetilde{M}$. Similarly, since $N \cap P$ is the kenrel of the canonical homomorphism $M \to (M/N) \oplus (M/P)$, we also have $\widetilde{N \cap P} = \widetilde{N} \cap \widetilde{P}$. □

Corollary 4.1.8. *Over the category of sheaves isomorphic to sheaves associated with A-modules, the global section functor Γ is exact.*

Proof. In fact, let $\widetilde{M} \xrightarrow{\tilde{\phi}} \widetilde{N} \xrightarrow{\tilde{\psi}} P$ be an exact sequence corresponding to two homomorphisms $\phi : M \to N$ and $\psi : N \to P$. If $Q = \operatorname{im} \phi$ and $R = \ker \psi$, we have

$$\widetilde{Q} = \operatorname{im} \tilde{\phi} = \ker \tilde{\psi} = \widetilde{R}$$

by Corollary 4.1.6, so $Q = R$ and the sequence is exact. □

Corollary 4.1.9. *Let M and N be two A-modules.*

(a) *The sheaf associated with $M \otimes_A N$ is canonically identified with $\widetilde{M} \otimes_{\widetilde{A}} \widetilde{N}$.*

(b) *If moreover M is finitely presented, the sheaf associated with $\mathrm{Hom}_A(M,N)$ is canonically identified with $\mathscr{H}om_{\widetilde{A}}(\widetilde{M},\widetilde{N})$.*

Proof. The sheaf $\mathscr{F} = \widetilde{M} \otimes_{\widetilde{A}} \widetilde{N}$ is the sheafification of the presheaf

$$U \mapsto \mathscr{F}(U) = \Gamma(U,\widetilde{M}) \otimes_{\Gamma(U,\widetilde{A})} \Gamma(U,\widetilde{N})$$

where U runs through the basis $\mathscr{B}$ of X formed by $D(f)$, $f \in A$. Now, $\mathscr{F}(D(f))$ is canonically identified with $M_f \otimes_{A_f} N_f$ by Therorem 4.1.2, which is canonically isomorphic to $\Gamma(D(f), \widetilde{M \otimes_A N})$. Moreover, it is immediately verified that the canonical isomorphisms

$$\mathscr{F}(D(f)) \cong \Gamma(D(f), \widetilde{M \otimes_A N})$$

is compatible with the restriction maps, so they define a canonical isomorphism $\widetilde{M} \otimes_{\widetilde{A}} \widetilde{N} \cong \widetilde{M \otimes_A N}$.

Now assume that M is finitely presented. The sheaf $\mathscr{G} = \mathscr{H}om_{\widetilde{A}}(\widetilde{M},\widetilde{N})$ is the sheafification of the presheaf

$$U \mapsto \mathscr{G}(U) = \mathrm{Hom}_{\widetilde{A}|_U}(\widetilde{M}|_U, \widetilde{N}|_U)$$

where U runs through the basis $\mathscr{B}$ of X. By Proposition 4.1.4 and Corollary 4.1.3, the module $\mathscr{G}(D(f))$ is then canonically identified with $\mathrm{Hom}_{A_f}(M_f, N_f)$, which is isomorphic to $\Gamma(D(f), \widetilde{\mathrm{Hom}_A(M,N)})$ by Proposition **??**. It is clear that these isomorphisms are compatible with the restriction maps, so we conclude that $\mathscr{H}om_{\widetilde{A}}(\widetilde{M},\widetilde{N}) \cong \widetilde{\mathrm{Hom}_A(M,N)}$. □

Now consider an A-algebra B (commutative); this can be interpreted by saying that B is an A-module and that we are given an element $e \in B$ and an A-homomorphism $\varphi : B \otimes_A B \to B$ so that the diagrams

$$\begin{tikzcd} B \otimes_A B \otimes_A B \arrow[r, "\varphi\otimes 1"] \arrow[d, "1\otimes\varphi"'] & B \otimes_A B \arrow[d, "\varphi"] \\ B \otimes_A B \arrow[r, "\varphi"] & B \end{tikzcd} \qquad \begin{tikzcd} B \otimes_A B \arrow[rr, "\sigma"] \arrow[dr, "\varphi"'] & & B \otimes_A B \arrow[dl, "\varphi"] \\ & B & \end{tikzcd}$$

(where σ is the canonical symmetry) commute, and that $\varphi(e \otimes x) = \varphi(x \otimes e) = x$. In view of Corollary 4.1.9, the homomorphism $\tilde{\varphi} : \widetilde{B} \otimes_{\widetilde{A}} \widetilde{B} \to \widetilde{B}$ of $\widetilde{A}$-modules satisfies similar conditions, hence defines a $\widetilde{A}$**-algebra** structure on $\widetilde{B}$. In the same way, the data of a B-module N amounts to giving an A-module N and an A-homomorphism $\psi : B \otimes_A N \to N$ such that the diagram

$$\begin{tikzcd} B \otimes_A B \otimes_A N \arrow[r, "\varphi\otimes 1"] \arrow[d, "1\otimes\psi"'] & B \otimes_A N \arrow[d, "\psi"] \\ B \otimes_A N \arrow[r, "\psi"] & N \end{tikzcd}$$

commutes and $\psi(e \otimes n) = n$; the homomorphism $\tilde{\psi} : \widetilde{B} \otimes_{\widetilde{A}} \widetilde{N} \to \widetilde{N}$ then satisfies similar conditions, and defines on $\widetilde{N}$ a $\widetilde{B}$-module structure.

If $\rho : B \to B'$ (resp. $\phi : N \to N'$) is a homomorphism of A-algebras (resp. a homomorphisms of B-modules), then $\tilde{\rho}$ (resp. $\tilde{\phi}$) is a homomorphism of $\widetilde{A}$-algebras (resp. a homomorphisms of $\widetilde{B}$-modules), $\ker\tilde{\rho}$ is an ideal of $\widetilde{B}$ (resp. $\ker\tilde{\phi}$, $\operatorname{coker}\tilde{\phi}$, and $\operatorname{im}\tilde{\phi}$ are $\widetilde{B}$-modules). Moreover, by Proposition **??**(b) if N is a B-module, then $\widetilde{N}$ is a finitely generated $\widetilde{B}$-module if and only if N is finitely generated over B.

If M and N are two B-modules, the $\widetilde{B}$-module $\widetilde{M}\otimes_{\widetilde{B}}\widetilde{N}$ is canonically identified with $\widetilde{M\otimes_B N}$; similarly, $\mathscr{H}om_{\widetilde{B}}(\widetilde{M},\widetilde{N})$ is canonically identified with $\widetilde{\operatorname{Hom}_B(M,N)}$ if M is finitely presented. If $\mathfrak{b}$ is an ideal of B, then $\widetilde{\mathfrak{b}N} = \tilde{\mathfrak{b}}\widetilde{N}$.

Finally, if B is a graded A-algebra with (B_n) its graduation, the $\widetilde{A}$-algebra $\widetilde{B}$ is then the direct sum of the sub-$\widetilde{A}$-modules $\widetilde{B}_n$ (Corollary 4.1.6), so $(\widetilde{B}_n)$ is a graduation of $\widetilde{B}$. Similarly, if M is a graded B-module with graduation (M_n), then $\widetilde{M}$ is a graded $\widetilde{B}$-module with graduation $(\widetilde{M}_n)$.

4.1.2 Functorial properties of the associated sheaf

We now consider the functorial properties of the operation $M \mapsto \widetilde{M}$. Let A and B be rings and $\varphi : B \to A$ be a ring homomorphism. Then we have an associated map

$$^a\varphi : X = \operatorname{Spec}(A) \to Y = \operatorname{Spec}(B)$$

We will define a canonical homomorphism

$$\varphi^\# : \mathscr{O}_Y \to {}^a\varphi_*(\mathscr{O}_X)$$

of sheaf of rings. For any $g \in B$, we set $f = \varphi(g)$; we have $\varphi^{-1}(D(g)) = D(f)$ by Proposition **??**. Now the sections $\Gamma(D(g),\widetilde{B})$ and $\Gamma(D(f),\widetilde{A})$ are canonically identified with B_g and A_f, respectively, and we have an induced map $\varphi_g : B_g \to A_f$, which then gives a homomorphism of rings

$$\Gamma(D(g),\widetilde{B}) \to \Gamma(\varphi^{-1}(D(g)),\widetilde{A}) = \Gamma(D(g),{}^a\varphi_*(\widetilde{A})).$$

Moreover, these homomorphisms satisfy the following compatible conditions: for $D(g) \supseteq D(g')$ in Y, the diagram

$$\begin{tikzcd}
\Gamma(D(g),\widetilde{A}) \arrow[r] \arrow[d] & \Gamma(D(g),{}^a\varphi_*(\widetilde{A})) \arrow[d] \\
\Gamma(D(g'),\widetilde{A}) \arrow[r] & \Gamma(D(g'),{}^a\varphi_*(\widetilde{A}))
\end{tikzcd}$$

is commutative; we then get a morphism of $\mathscr{O}_Y$-algebras, since $D(g)$ form a basis for the topological space Y. The couple $({}^a\varphi,\varphi^\#)$ is called the **canonical morphism** of the locally ringed spaces induced by φ.

We also note that, if $y = {}^a\varphi(x)$, the homomorphism $\varphi_x^\#$ is no other that the homomorphism

$$\varphi_x : B_y \to A_x$$

induced by the homomorphism $\varphi : B \to A$. In fact, for $b/g \in B_y$, where $b, g \in B$ and $g \notin \mathfrak{p}_y$, $D(g)$ is then an open neighborhood of y in Y, and the homomorphism

$$\Gamma(D(g),\widetilde{B}) \to \Gamma(D(g),({}^a\varphi)_*(\widetilde{A}))$$

induced by $\varphi^{\#}$ is just φ_g; by considering the section $\xi \in \Gamma(D(g), \widetilde{B})$ corresponding to b/g, we then obtain that $\varphi_x^{\#}(\xi) = \varphi(b)/\varphi(g)$ in A_x.

Example 4.1.10. Let S be a miltiplicative subset of A and $\varphi : A \to S^{-1}A$ the canonical homomorphism. We have seen in Proposition **??** that ${}^a\varphi$ is a homeomorphism from $Y = \operatorname{Spec}(S^{-1}A)$ to the subspace $X = \operatorname{Spec}(A)$ formed by x such that $\mathfrak{p}_x \cap S = \varnothing$. Moreover, for any x in this subspace, hence of the form ${}^a\varphi(y)$ where $y \in Y$, the homomorphism $\varphi_y^{\#} : \mathscr{O}_{Y,y} \to \mathscr{O}_{X,x}$ is bijective; therefore, $\mathscr{O}_Y$ is identified with the sheaf induced over Y by $\mathscr{O}_X$.

Proposition 4.1.11. *Let $\varphi : B \to A$ be a ring homomorphism and $\Phi : X \to Y$ the associated morphism. For any A-module M, there exists a canonical functorial isomorphism of the $\mathscr{O}_Y$-module $\widetilde{\varphi^*(M)}$ to its direct image $\Phi_*(\widetilde{M})$.*

Proof. For $g \in B$, put $f = \varphi(g)$; the modules $\Gamma(D(g), \widetilde{\varphi^*(M)})$ and $\Gamma(D(f), \widetilde{M})$ are identified with $(\varphi^*(M))_g$ and M_f, respectively; moreover, the B_g-module $\varphi_g^*(M_f)$ is canonically isomorphic to $(\varphi^*(M))_g$. We then have a functorial isomorphism of $\Gamma(D(g), \widetilde{B})$-modules:

$$\Gamma(D(g), \widetilde{\varphi^*(M)}) \cong \varphi^*(\Gamma(D(\varphi(g)), \widetilde{M}))$$

and this isomorphism satisfies the compatible conditions with restrictions, hence define an isomorphism of sheaves. ◻

This proof also proves that for any A-algebra R, the canonical functorial isomorphism $\widetilde{\varphi^*(R)} \to \Phi_*(\widetilde{R})$ is an isomorphism of $\mathscr{O}_Y$-algebras. If M is an R-module, the canonical isomorphism $\widetilde{\varphi^*(M)} \cong \Phi_*(\widetilde{M})$ is an isomorphism of $\Phi_*(\widetilde{R})$-modules.

Corollary 4.1.12. *The direct image functor Φ_* is exact on the category of quasi-coherent sheaves.*

Proof. Recall that the functor φ^* is exact and $M \mapsto \widetilde{M}$ is an exact functor. ◻

Proposition 4.1.13. *Let $\varphi : B \to A$ be a ring homomorphism and $\Phi : X \to Y$ the associated morphism. Let N be a B-module and $\varphi_*(N)$ the A-module $N \otimes_B A$. Then there exist a canonical functorial isomorphism of the $\mathscr{O}_X$-module $\Phi^*(\widetilde{N})$ to $\widetilde{\varphi_*(N)}$.*

Proof. We first remark that $j : z \mapsto z \otimes 1$ is a B-homomorphism from N to $\varphi^*\varphi_*(N)$: this holds because for $g \in B$, we have

$$(gz) \otimes 1 = z \otimes \varphi(g) = \varphi(g)(z \otimes 1).$$

By Proposition 4.1.3, this corresponds to a homomorphism $\tilde{j} : \widetilde{N} \to \widetilde{\varphi^*(\varphi_*(N))}$ of $\mathscr{O}_Y$-modules, and via Proposition 4.1.11 we can think that $\tilde{j}$ maps $\widetilde{N}$ to $\Phi_*(\widetilde{\varphi_*(N)})$. From the adjointness of Φ^* and Φ_*, this canonically corresponds to a homomorphism

$$\theta : \Phi^*(\widetilde{N}) \to \widetilde{\varphi_*(N)}.$$

It then remains to show that θ is bijective, or equivalently that θ_x is bijective for every $x \in X$. For this, put $y = {}^a\varphi(x)$, choose $g \in B$ such that $y \in D(g)$, and let $f = \varphi(g)$. Then the ring $\Gamma(D(f), \widetilde{A})$ is identified with A_f, the module $\Gamma(D(f), \widetilde{\varphi_*(N)})$ is identified with $(\varphi_*(N))_f$, and

$\Gamma(D(g), \widetilde{N})$ is identified with N_g. Let $s = n/g^p$ $(n \in N)$ be a section of $\Gamma(D(g), \widetilde{N})$ and $t = a/f^q$ $(a \in A)$ a section of $\Gamma(D(f), \widetilde{A})$. Then, since s is send to $(n \otimes 1)/f^p$ be $\tilde{j}$, by definition we have

$$\theta_x(s_x \otimes t_x) = t_x \cdot s_x.$$

Recall that we can identify $(\varphi_*(N))_f$ with $N_f \otimes_{B_g} \varphi^*(A_f)$, under which n/g^p is identified with $(n/g^p) \otimes 1$. So it is immediately seen that θ_x is none other than the canonical isomorphism

$$N_y \otimes_{B_y} \varphi_y^*(A_x) \cong (\varphi_*(N))_x = (N \otimes_B \varphi^*(A))_x.$$

Finally, let $v : N_1 \to N_2$ be a homomorphism of B-modules; since $\tilde{v}_y = v_y$ for any $y \in Y$, it follows immediately from the preceding argument that $\Phi^*(\tilde{v})$ is canonically identified to $\widetilde{v \otimes 1}$, which completes the proof. □

If S is an B-algebra, the canonical isomorphism of $\Phi^*(\widetilde{S})$ to $\widetilde{\varphi_*(S)}$ is an isomorphism of $\mathscr{O}_X$-algebras; if moreover N is a S-module, the canonical isomorphism of $\Phi^*(\widetilde{N})$ to $\widetilde{\varphi_*(N)}$ is an isomorphism of $\Phi^*(\widetilde{S})$-algebras.

Corollary 4.1.14. *The sections of $\Phi^*(\widetilde{N})$ which are cannical images of section of the B-module $\Gamma(\widetilde{N})$, generate the A-module $\Gamma(\Phi^*(\widetilde{N}))$.*

Proof. In fact, these images are identified with the elements $z \otimes 1$ of $\varphi_*(N)$, if we identify N and $\varphi_*(N)$ with $\Gamma(\widetilde{N})$ and $\Gamma(\widetilde{\varphi_*(N)})$. □

By the proof of Proposition 4.1.13, we see that the canonical map $\alpha : \widetilde{N} \to \Phi_*\Phi^*(\widetilde{N})$ is none other than the homomorphism $\tilde{j}$, where $j : N \to \varphi_*(N)$ is the canonical map $z \mapsto z \otimes 1$. Similarly, the canonical map $\beta : \Phi^*\Phi_*(\widetilde{M}) \to \widetilde{M}$ is none other than the homomorphism $\tilde{p}$, where $p : \varphi^*(M) \otimes_B \varphi^*(A) \to M$ is the canonical homomorphism that sends $m \otimes a$ to am.

Corollary 4.1.15. *Let N_1 and N_2 be B-modules and assume that N_1 is finitely presented. Then there is a canonical homomorphism*

$$\Phi^*(\mathscr{H}om_{\widetilde{B}}(\widetilde{N}_1, \widetilde{N}_2)) \to \mathscr{H}om_{\widetilde{A}}(\Phi^*(\widetilde{N}_1), \Phi^*(\widetilde{N}_2)).$$

This homomorphism is bijective if φ is a flat homomorphism.

Proof. By the above remarks and Corollary 4.1.9, this homomorphism is induced by the homomorphism

$$\mathrm{Hom}_B(N_1, N_2) \otimes_B A \to \mathrm{Hom}_A(N_1 \otimes_B A, N_2 \otimes_B A).$$

The last assertion follows from Proposition **??**. □

A locally ringed space $(X, \mathscr{O}_X)$ is called an **affine scheme** if it is isomorphic to the spectrum of a ring A. In this case, the ring $\Gamma(X, \mathscr{O}_X)$ is canonically identified with A. By absuing language, we often call $\mathrm{Spec}(A)$ an affine scheme, without mention the structural sheaf.

Let A and B be two rings and $(X, \mathscr{O}_X)$, $(Y, \mathscr{O}_Y)$ the corresponding affine schemes. Then any ring homomorphism $\varphi : B \to A$ corresponds to a morphism $({}^a\varphi, \varphi^\#) : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$. Note that the homomorphism φ is completely determined by $({}^a\varphi, \varphi^\#)$, since by definition we have $\varphi = \Gamma(\varphi^\#) : \Gamma(\widetilde{B}) \to \Gamma(\widetilde{A})$.

Theorem 4.1.16. *Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be two affine schemes. Then any morphism $(\psi, \psi^\#)$: $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ of locally ringed spaces is of the form $({}^a\varphi, \varphi^\#)$, where $\varphi : \Gamma(Y, \mathcal{O}_Y) \to \Gamma(X, \mathcal{O}_X)$ is a ring homomorphism.*

Proof. Put $A = \Gamma(X, \mathcal{O}_X)$ and $B = \Gamma(Y, \mathcal{O}_Y)$. Let $(\psi, \psi^\#) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of locally ringed spaces. By definition, $\psi^\#$ is a homomorphism from $\mathcal{O}_Y$ to $\psi_*\mathcal{O}_X$, and we then deduce a canonical homomorphism of rings

$$\varphi = \Gamma(\psi^\#) : B = \Gamma(Y, \mathcal{O}_Y) \to \Gamma(Y, \psi_*\mathcal{O}_X) = \Gamma(X, \mathcal{O}_X) = A.$$

Since $\psi_x^\#$ is a local homomorphism, by passing to quotients we deduce a monomorphism θ^x from the residue field $\kappa(\psi(x))$ into the residue field $\kappa(x)$ such that, for any section $f \in \Gamma(Y, \mathcal{O}_Y)$, we have $\theta^x(f(\psi(x))) = \varphi(f)(x)$ (we consider the elements of $\Gamma(Y, \mathcal{O}_Y)$ as functions on B). The relationship $f(\psi(x)) = 0$ is therefore equivalent to $\varphi(f)(x) = 0$, which means $\psi(x) = {}^a\varphi(x)$. Since this hold for any $x \in X$, we conclude that $\psi = {}^a\varphi$. We also know that the diagram

$$\begin{tikzcd} B = \Gamma(Y, \mathcal{O}_Y) \arrow[r, "{}^a\varphi"] \arrow[d] & \Gamma(X, \mathcal{O}_X) = A \arrow[d] \\ B_{\psi(x)} \arrow[r, "\psi_x^\#"] & A_x \end{tikzcd}$$

is commutative, so $\psi_x^\#$ is equal to the homomorphism $\varphi_x : B_{\psi(x)} \to A_x$ induced from φ. Since the morphism $\psi^\#$ is determined by $\psi_x^\#$, we obtain that $\psi^\# = \varphi^\#$. □

Corollary 4.1.17. *Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be two affine schemes. Then there is a canonical bijection*

$$\mathrm{Mor}(X, Y) \to \mathrm{Hom}_{\mathbf{Ring}}(\Gamma(Y, \mathcal{O}_Y), \Gamma(X, \mathcal{O}_X))$$

which sends a morphism $(f, f^\#)$ to the global section of $f^\#$.

We can also say that the functors $(\mathrm{Spec}(A), \widetilde{A})$ in A and $\Gamma(X, \mathcal{O}_X)$ in $(X, \mathcal{O}_X)$ define an equivalence of the opposite category of commutative rings and of the category of the category of affine schemes.

Corollary 4.1.18. *If $\varphi : B \to A$ is surjective, then the corresponding morphism Φ is a monomorphism of locally ringed spaces.*

Proof. The map ${}^a\varphi$ is injective by Proposition **??**, and since φ is surjective, for any $x \in x$ the map $\varphi_x^\# : B_{{}^a\varphi(x)} \to A_x$, obtained by passing to localization, is surjective; these prove the assertion. □

4.1.3 Quasi-coherent sheaves over affine schemes

Recall that we have defined the abstract notion of a quasi-coherent sheaf. In this part we show that any quasi-coherent sheaf on an affine scheme $\mathrm{Spec}(A)$ corresponds to the sheaf $\widetilde{M}$ associated with an A-module M.

Lemma 4.1.19. *Let $X = \mathrm{Spec}(A)$ and $V = \bigcup_{i=1}^n D(g_i)$ be a union of finitely may standard opens. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module satisfying the conditions:*

(a) *For any $D(f) \subseteq D(g_i)$ (resp. $D(g_ig_j)$) and any section $s \in \Gamma(D(f), \mathscr{F})$, there exists $n \geq 0$ such that $f^n s$ can be extended to a section of $\mathscr{F}$ on $D(g_i)$ (resp. on $D(g_ig_j)$).*

(b) *For any $D(f) \subseteq D(g_i)$ (resp. $D(g_ig_j)$) and any section $t \in \Gamma(V, \mathscr{F})$ such that $t|_{D(f)} = 0$, there exists $n \geq 0$ such that $f^n t = 0$.*

Then we have the stronger conditions:

(α) *For any $f \in A$ and any section $s \in \Gamma(D(f) \cap V, \mathscr{F})$, there exists $n \geq 0$ such that $f^n s$ can be extended to a section of $\mathscr{F}$ on V.*

(β) *For any $f \in A$ and any section $t \in \Gamma(V, \mathscr{F})$ such that $t|_{D(f)\cap V} = 0$, there exists $n \geq 0$ such that $f^n t = 0$.*

Proof. First we prove condition (β). Since $D(f) \cap D(g_i) = D(fg_i)$, for each i we have an integer $n_i \geq 0$ such that $(fg_i)^{n_i} t$ restricts to zero on $D(g_i)$. Since g_i is invertible in $A_{g_i} = \Gamma(D(g_i), \mathscr{O}_X)$, this implies $f^{n_i} t = 0$ on $D(g_i)$. Take n such that $n \geq n_i$, then $f^n t = 0$ on each $D(g_i)$, whence $f^n t = 0$ and we get (β).

To show (α), we apply (a) on $\mathscr{F}|_{D(g_i)}$ to get an integer integers $n_i \geq 0$ and $s_i' \in \Gamma(D(g_i), \mathscr{F})$ such that

$$s_i'|_{D(fg_i)} = (fg_i)^{n_i} s|_{D(fg_i)}.$$

By inverting g_i, this produces sections $s_i \in \Gamma(D(g_i), \mathscr{F})$ such that

$$s_i|_{D(fg_i)} = f^{n_i} s|_{D(fg_i)}.$$

We may assume that all n_i take the same value n. Then each $s_i - s_j$ restricts to zero on $D(f) \cap D(g_i) \cap D(g_j) = D(fg_ig_j)$, so by applying (b) on $\mathscr{F}|_{D(g_ig_j)}$ we get an integer m_{ij} such that

$$(fg_ig_j)^{m_{ij}}(s_i - s_j)|_{D(g_ig_j)} = 0.$$

Then similarly, since g_ig_j is invertible in $A_{g_ig_j}$, this implies

$$f^{m_{ij}}(s_i - s_j)|_{D(g_ig_j)} = 0.$$

We can also assume that all m_{ij}'s take the same value m, so that by the sheaf condition there exists a section $u \in \Gamma(V, \mathscr{F})$ such that $u|_{D(g_i)} = f^m s_i$. Then $f^n u$ extends $f^{m+n} s$, as desired. □

Theorem 4.1.20. *Let $X = \mathrm{Spec}(A)$ be an affine scheme. Let V be a quasi-compact open subset and $\mathscr{F}$ be an $\mathscr{O}_X|_V$-module. Then the following are equivalent:*

(i) *There is a A-module M such that $\mathscr{F}$ is isomorphic to $\widetilde{M}|_V$.*

(ii) *There exists a finite open covering (V_i) of V by sets of the form $D(f_i)$ ($f_i \in A$) contained in V, such that, for each i, $\mathscr{F}|_{V_i}$ is isomorphic to a sheaf of the form $\widetilde{M}_i$, where M_i is an A_{f_i}-module.*

(iii) *The sheaf $\mathscr{F}$ is quasi-coherent.*

(iv) *(**Serre's lifting criterion**) The following conditions are satisfied:*

(a) *For any $D(f) \subseteq V$ and any section $s \in \Gamma(D(f), \mathscr{F})$, there exists $n \geq 0$ such that $f^n s$ can be extended to a section of $\mathscr{F}$ on V.*

(b) *For any $D(f) \subseteq V$ and any section $t \in \Gamma(V, \mathscr{F})$ such that $t|_{D(f)} = 0$, there exists $n \geq 0$ such that $f^n t = 0$.*

Proof. The implication (i)⇒(ii) is immediate from Proposition 4.1.4 since X can be covered by standard opens. Also, since any A-module is isomorphic to the kernel of a homomorphism $A^{\oplus I} \to A^{\oplus J}$, Corollary 4.1.6 shows that (ii)⇒(iii). Conversely, if $\mathscr{F}$ is quasi-coherent, any point $x \in V$ possesses a neighborhood of the form $D(f) \subseteq V$ such that $\mathscr{F}|_{D(f)}$ is isomorphic to the cokernel of a homomorphism $(\widetilde{A}_f)^{\oplus I} \to (\widetilde{A}_f)^{\oplus J}$, hence to the sheaf associated with the cokernel of the corresponding homomorphism $A_f^{\oplus I} \to A_f^{\oplus J}$ (Corollary 4.1.3 and Corollary 4.1.6); since V is quasi-compact, it then follows that (iii) implies (ii).

Now we prove that (ii)⇒(iv). First assume that $V = D(g)$ for some $g \in A$, and $\mathscr{F}$ is isomorphic to $\widetilde{N}$ for some A_g-module N. Since $D(g)$ can be identified with $\mathrm{Spec}(A_g)$, we can assume that $g = 1$ and $V = X$. In this case, the set $\Gamma(D(f), \mathscr{F})$ and N_f are canonically identified (Theorem 4.1.2), and it is clear that conditions (a) and (b) in (iv) are satisfied. To prove the general case, since V is quasi-compact we can choose a finite covering by standard opens $D(g_i)$ with $\mathscr{F}|_{D(g_i)}$ isomorphic to $\widetilde{M}_i$ for some A_{g_i}-module M_i. Then $\mathscr{F}$ satisfies the conditions (a) and (b) in Lemma 4.1.19, so by Lemma 4.1.19, $\mathscr{F}$ also satisfies conditions (α) and (β), which is what we want.

Finally, we show that (iv)⇒(i). First we prove that, if (a) and (b) hold for $\mathscr{F}$, then they hold for $\mathscr{F}|_{D(g)}$ with $D(g) \subseteq V$. This is evident for condition (a); as for (b), if $t \in \Gamma(D(g), \mathscr{F})$ restricts to zero on $D(f) \subseteq D(g)$, then by condition (a) there is an integer $m \geq 0$ such that $g^m t$ can be extended to V. By applying condition (b) on the extension of $g^m t$, we get another integer $n \geq 0$ such that $f^n g^m t = 0$. Since g is invertible in A_g, this gives $f^n t = 0$ as desired.

This being done, since V is quasi-compact, by Lemma 4.1.19 we know that conditions (α) and (β) holds for $\mathscr{F}$. Now consider the module $M = \Gamma(V, \mathscr{F})$; we shall define a morphism $\varphi : \widetilde{M} \to j_*\mathscr{F}$, where $j : V \hookrightarrow X$ is the inclusion. For this, it suffices to define

$$\varphi_f : M_f \to \Gamma(D(f), j_*\mathscr{F}) = \Gamma(D(f) \cap V, \mathscr{F})$$

for each $f \in A$. Since f is invertible in A_f and $\Gamma(D(f) \cap V, \mathscr{F})$ is a A_f-module, the restriction $M = \Gamma(V, \mathscr{F}) \to \Gamma(D(f) \cap V, \mathscr{F})$ factors into

$$M \longrightarrow M_f \xrightarrow{\varphi_f} \Gamma(D(f) \cap V, \mathscr{F})$$

which gives the desired maps φ_f. We now claim that the conditions (a) and (b) in (iv) imply that φ is an isomorphism. In fact, if $s \in \Gamma(D(f) \cap V, \mathscr{F})$, then by condition (a) there exist an integer $n \geq 0$ and $z \in \Gamma(V, \mathscr{F}) = M$ such that $z|_{D(f)\cap V} = f^n s$; then $\varphi_f(z/f^n) = s$, showing that φ is surjective. Similarly, if there is $z \in M$ such that $\varphi_f(z/1) = 0$ in $D(f) \cap V$, then by condition (b) there is $n \geq 0$ such that $f^n z = 0$, so that $z/1 = 0$ in M_f. This means φ_f is injective, so we get an isomorphism $\widetilde{M} \cong j_*\mathscr{F}$. By restriction, we then conclude that $\mathscr{F} \cong \widetilde{M}|_V$. □

Corollary 4.1.21. *Let $X = \mathrm{Spec}(A)$ be an affine scheme. Then the functors $M \mapsto \widetilde{M}$ and $\mathscr{F} \mapsto$*

$\Gamma(X,\mathscr{F})$ define equivalences of categories between the category of quasi-coherent $\mathscr{O}_X$-modules and the category of A-modules.

Proof. The space X it self is quasi-compact, so we can apply Theorem 4.1.20. □

Corollary 4.1.22. *Let $X = \operatorname{Spec}(A)$ be an affine scheme. Then kernels and cokernels of maps of quasi-coherent $\mathscr{O}_X$-modules are quasi-coherent.*

Proof. This follows from the exactness of the functor $\widetilde{M}$ and Corollary 4.1.6. □

Corollary 4.1.23. *For $\widetilde{M}$ to be a $\mathscr{O}_X$-module of finite type (resp. of finite presentation), it is necessary and sufficient that M is a finitely generated A-module (resp. of finite presentation).*

Proof. In view of the exactness of the functor $M \mapsto \widetilde{M}$, it is immediate that if M is of finite type (resp. finite presentation), so is $\widetilde{M}$. Conversely, if $\widetilde{M}$ is of finite type (resp. finite presentation), since X is quasi-compact, there exists finitely many $f_i \in A$ such that $D(f_i)$ cover X and M_{f_i} is of finite type (resp. finite presentation) over A_{f_i}. It then follows from Proposition **??** that M is of finite type (resp. finite presentation). □

Corollary 4.1.24. *For an A-module M, the $\mathscr{O}_X$-module is locally free of finite rank if and only if M is a finitely generated projective A-module.*

Proof. Since X is quasi-compact, this follows from Theorem **??**. □

Corollary 4.1.25. *Let $X = \operatorname{Spec}(A)$ be an affine scheme. Then any quasi-coherent $\mathscr{O}_X$-algebra over X is isomorphic to an $\mathscr{O}_X$-algebra of the form $\widetilde{B}$, where B is an algebra over A. Moreover, any quasi-coherent $\widetilde{B}$-module is isomorphic to a $\widetilde{B}$-module of the form $\widetilde{N}$, where N is a B-module.*

Proof. In fact, a quasi-coherent $\mathscr{O}_X$-algebra is a quasi-coherent $\mathscr{O}_X$-module, hence of the form $\widetilde{B}$, where B is an A-module. The fact that B is an A-algebra follows from the structural morphism $\widetilde{B}\otimes_{\mathscr{O}_X}\widetilde{B}\to\widetilde{B}$ of $\mathscr{O}_X$-modules, which induces an A-algebra map $B\otimes_A B\to B$.

If $\mathscr{G}$ is a quasi-coherent $\widetilde{B}$-module, it suffices to show that $\mathscr{G}$ is also a quasi-coherent $\mathscr{O}_X$-module to then conclude in the same way. As the question is local, we can, by restricting ourselves to an open set of X of the form $D(f)$, over which $\mathscr{G}$ is the cokernel of a morphism $\widetilde{B}^{\oplus I}\to\widetilde{B}^{\oplus J}$ of $\widetilde{B}$-modules (and a fortiori $\mathscr{O}_X$-modules). The claim then follows from Corallary 4.1.3 and Corollary 4.1.6. □

Proposition 4.1.26. *Let $X = \operatorname{Spec}(A)$ be an affine scheme, Suppose that*

$$0 \longrightarrow \mathscr{F}_1 \longrightarrow \mathscr{F}_2 \longrightarrow \mathscr{F}_3 \longrightarrow 0$$

is a short exact sequence of $\mathscr{O}_X$-modules, and assume that $\mathscr{F}_1$ is quasi-coherent. Then the sequence

$$0 \longrightarrow \Gamma(X,\mathscr{F}_1) \longrightarrow \Gamma(X,\mathscr{F}_2) \longrightarrow \Gamma(X,\mathscr{F}_3) \longrightarrow 0$$

is exact.

Proof. We know already that Γ is a left-exact functor so we have only to show that the last map is surjective (which we denot by $\psi : \Gamma(X, \mathscr{F}_2) \to \Gamma(X, \mathscr{F}_3)$). Let $s \in \Gamma(X, \mathscr{F}_3)$ be a global section. Since the morphism $\mathscr{F}_2 \to \mathscr{F}_3$ is surjective, for any $x \in X$ there is an open neighborhood $D(f)$ of x such that

$$s|_{D(f)} = \psi(t)$$

where $t \in \mathscr{F}_2(D(f))$. We claim that for some $n > 0$, $f^n s = \psi(u)$ for some $u \in \Gamma(X, \mathscr{F}_2)$. Indeed, we can cover X with a finite number of open sets $D(g_i)$ such that for each i, $s|_{D(g_i)} = \psi(t_i)$ for a section $t_i \in \mathscr{F}_2(D(g_i))$. Then by the exactness of the original sequence, on $D(f) \cap D(g_i) = D(fg_i)$ we have

$$(t - t_i)|_{D(fg_i)} \in \mathscr{F}_1(D(fg_i))$$

where we identify $\mathscr{F}_1$ as the kernel of ψ. Since $\mathscr{F}_1$ is quasi-coherent, by Proposition 4.1.21, there is an integer $n \geq 0$ such that the $f^n(t - t_i)|_{D(fg_i)}$ can be extended to a section $u_i \in \mathscr{F}_1(D(g_i))$. Let

$$\tilde{t}_i = f^n t_i + u_i \in \mathscr{F}_2(D(g_i)).$$

Then $\tilde{t}_i|_{D(fg_i)} = f^n t_i|_{D(fg_i)} + f^n(t - t_i)|_{D(fg_i)} = f^n t|_{D(fg_i)}$ and we have

$$f^n s|_{D(g_i)} = f^n \psi(t_i) = \psi(\tilde{t}_i - u_i) = \psi(\tilde{t}_i). \tag{1.3.1}$$

Now on $D(g_i g_j)$ the two sections $\tilde{t}_i$ and $\tilde{t}_j$ of $\mathscr{F}_2$ are mapped to $f^n s|_{D(g_i g_j)}$ by ψ, so $\tilde{t}_i - \tilde{t}_j \in \mathscr{F}_1(D(g_i g_j))$. Furthermore, since $\tilde{t}_i$ and $\tilde{t}_j$ are both equal to $f^n t|_{D(fg_i g_j)}$ on $D(fg_i g_j)$, by Proposition 4.1.21 there exists $m \geq 0$ such that $f^m(\tilde{t}_i - \tilde{t}_j) = 0$ on $D(g_i g_j)$, which we may take to be independent of i and j. Then the sections $f^m \tilde{t}_i$ glue to give a global section of $\mathscr{F}_2$ over X, which lifts $f^{m+n} s$ by (1.3.1). This proves the claim.

Now cover X by a finite number of open sets $D(f_i)$ such that $s|_{D(f_i)}$ lifts to a section of $\mathscr{F}_2$ over $D(f_i)$ for each i. Then by the previous proof, we can find an integer $n \geq 0$ (one for all i) and global sections $t_i \in \Gamma(X, \mathscr{F}_2)$ such that $\psi(t_i) = f^n s$. Since the open sets $D(f_i)$ cover X, the ideal $(f_1^n, \ldots, f_r^n)$ is the unit ideal of A, and we can write $1 = \sum a_i f_i^n$, with $a_i \in A$. Let $t = \sum a_i t_i$. Then t is a global section of $\mathscr{F}_2$ whose image under ψ is $\sum a_i f_i^n s = s$. □

Proposition 4.1.27. *Let $X = \mathrm{Spec}(A)$ be an affine scheme. Then the direct sum of an arbitrary collection of quasi-coherent sheaves on X is quasi-coherent. More generally, colimits of quasi-coherent sheaves are quasi-coherent.*

Proof. Let $\{\mathscr{F}_i\}_{i \in I}$ be a collection of quasi-coherent sheaves on X. By Theorem 4.1.20 we can write $\mathscr{F}_i = \widetilde{M}_i$ for A-modules M_i, so the assertion follows from Corollary 4.1.6. □

Proposition 4.1.28. *Let X be an affine scheme. Suppose that*

$$0 \longrightarrow \mathscr{F}_1 \longrightarrow \mathscr{F}_2 \longrightarrow \mathscr{F}_3 \longrightarrow 0$$

is a short exact sequence of sheaves $\mathscr{O}_X$-modules. If two out of three are quasi-coherent then so is the third.

Proof. The statement about kernels and cokernels follows from the fact that the functor $M \mapsto \widetilde{M}$ is exact and fully faithful from A-modules to quasi-coherent sheaves. Now let $\mathscr{F}_1$ and $\mathscr{F}_3$ be quasi-coherent. By Proposition 4.1.26, the corresponding sequence of global sections over X is exact, say $0 \to M_1 \to M_2 \to M_3 \to 0$. Applying the functor $\widetilde{M}$ we get an exact commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \widetilde{M}_1 \arrow[r] \arrow[d] & \widetilde{M}_2 \arrow[r] \arrow[d] & \widetilde{M}_3 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathscr{F}_1 \arrow[r] & \mathscr{F}_2 \arrow[r] & \mathscr{F}_3 \arrow[r] & 0
\end{tikzcd}$$

The two outside arrows are isomorphisms, since $\mathscr{F}_1$ and $\mathscr{F}_3$ are quasi-coherent. So by the five lemma, the middle one is also, showing that $\mathscr{F}_2$ is quasi-coherent. □

Theorem 4.1.29. *Let $X = \operatorname{Spec}(A)$ be an affine scheme where A is Noetherian. Let V be an open subset of X and $\mathscr{F}$ an $\mathscr{O}_X|_V$-module. Then the following conditions are equivalent:*

(i) $\mathscr{F}$ is coherent.

(ii) $\mathscr{F}$ is of finite type and quasi-coherent.

(iii) There exists a finitely generated A-moduel M such that $\mathscr{F} \cong \widetilde{M}|_V$.

Proof. It is clear that (i) implies (ii). To show (ii) implies (iii), we note that V is quasi-compact since X is Noetherian, so by Theorem 4.1.20, $\mathscr{F}$ is isomorphic to $\widetilde{M}|_V$, where M is an A-module. Now we have $M = \varinjlim M_\lambda$, where M_λ is the set of finitely generated sub-A-modules of M. Since the functor $\widetilde{(-)}$ is exact, this implies $\mathscr{F} = \widetilde{N}|_V = \varinjlim \widetilde{M_\lambda}|_V$. But $\mathscr{F}$ is of finite type and V is quasi-compact, so by Proposition 1.4.10 there exists an index λ such that $\mathscr{F} = \widetilde{M_\lambda}|_V$ (note that the canonical homomorphism $\widetilde{M}_\lambda \to \widetilde{M}$ is injective). This proves (iii).

It remains to show that $\widetilde{M}|_V$ is coherent if M is finitely generated. Since $\mathscr{F}$ is clearly of finite type, it suffices to show that for every open $U \subseteq X$ and $s_1, \dots, s_n \in \mathscr{F}(U)$, the associated map $\bigoplus_{i=1}^n \mathscr{O}_U \to \mathscr{F}|_U$ is of finite type. Since the question is local, we may assume $V = D(f)$ for $f \in A$. Then it suffices to show the kernel of a morphism $\bigoplus_{i=1}^n \widetilde{A}_f \to \widetilde{M}$ is of finite type. But this morphism corresponds to a homomorphism $A_f^n \to M$, whose kernel is finitely generated since A_f is Noetherian, so the claim follows. □

Corollary 4.1.30. *Let $X = \operatorname{Spec}(A)$ be an affine scheme where A is Noetherian. Then any quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$ is the inductive limit of coherent $\mathscr{O}_X$-modules.*

Proof. We have $\mathscr{F} = \widetilde{M}$ for an A-module M, and M is the inductive limit of its finitely generated submodules. □

Corollary 4.1.31. *Let $X = \operatorname{Spec}(A)$ be an affine scheme where A is Noetherian. Then the functor $M \mapsto \widetilde{M}$ gives an equivalence of categories between the category of finite generated A-modules and the category of coherent $\mathscr{O}_X$-modules.*

4.2 General schemes

4.2.1 Schemes and morphisms of schemes

Let $(X, \mathscr{O}_X)$ be a ringed space. An oepn subset V of X is said to be **affine** if the ringed space $(V, \mathscr{O}_X|_V)$ is an affine scheme (i.e. isomorphic to the spectrum of a ring). We say $(X, \mathscr{O}_X)$ is a

scheme if every point of X admits an affine open neighborhood. If $(X, \mathcal{O}_X)$ is a scheme, then affine open subsets of X form a basis for X (because the standard opens form a basis for a spectrum $\operatorname{Spec}(A)$, and they are again affine), and in particular $(X, \mathcal{O}_X)$ is a locally ringed space. With this, for any open subset U of X, the ringed space $(U, \mathcal{O}_X|_U)$ is also a scheme, called the scheme **induced** on U by X, or the **restriction** of $(X, \mathcal{O}_X)$ on U.

Proposition 4.2.1. *The underlying space of a scheme is Kolmogoroff.*

Proof. In fact, if x and y are two points of a scheme X, then it is obvious that there exists an open neighborhood of one of these points not containing the other if x, y are not not in a same open affine; and if they are in the same open affine, this follows from the fact that the underlying spaces of affine schemes are Kolmogoroff (Proposition **??**). □

Proposition 4.2.2. *If $(X, \mathcal{O}_X)$ is a scheme, any irreducible closed subset of X admits a unique generic point, and the map $x \mapsto \overline{\{x\}}$ is a bijection of X to the family of irreducible closed subsets of X.*

Proof. Let Y is an irreducible closed subset of X and $y \in Y$. If U is an affine open neighborhood of y in X, then $U \cap Y$ is dense in Y and is irreducible (Proposition **??**), so it is the closure in U of a point $x \in U$, and therefore $Y \subseteq \overline{U}$ is the closure of x in X. The uniqueness of the generic point of X follows from Proposition 4.2.1 and Proposition **??**. □

If Y is an irreducible closed subset of X and y its generic point, the local ring $\mathcal{O}_{X,y}$ is then denoted by $\mathcal{O}_{X,Y}$ and called the **local ring of X along Y**, or the **local ring of Y in X**. We say a scheme $(X, \mathcal{O}_X)$ is **irreducible** (resp. **connected**) if the underlying space X is irreducible (resp. connected), and **integral** if it is irreducible and reduced. We say the scheme $(X, \mathcal{O}_X)$ is **locally integral** if each point $x \in X$ admits an open neighborhoods U such that the scheme induced over U by $(X, \mathcal{O}_X)$ is integral. If X is an irreducible scheme and x is its generic point, the local ring $\mathcal{O}_{X,x}$ is called **the ring of rational functions on X**.

Let $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$ be schemes. A **morphism** (of schemes) from $(X, \mathcal{O}_X)$ to $(Y, \mathcal{O}_Y)$ is defined to be that of locally ringed space. That is, a pair $(f, f^\#)$ such that for each $x \in X$, the homomorphism $f_x^\#$ is local. In this case, by passing to quotients, $f_x^\#$ induces a monomorphism $f^x : \kappa(f(x)) \to \kappa(x)$, so $\kappa(x)$ can be considered as an extension of the field $\kappa(f(x))$.

The composition of two morphisms of schemes is defined in the same way with that of locally ringed spaces, and we then see that schemes form a category, denoted by **Sch**. Following the general notation, we denote by $\operatorname{Hom}_{\mathbf{Sch}}(X, Y)$ the set of morphisms from a scheme X to a scheme Y.

Example 4.2.3. Let U be an open subset of X. Then the canonical injection of $(U, \mathcal{O}_X|_U)$ to $(X, \mathcal{O}_X)$ is a morphism of schemes; it is moreover a monomorphism of ringed spaces (and a fortiori a monomorphism of schemes).

Proposition 4.2.4. *Let $(X, \mathcal{O}_X)$ be a scheme and $(Y, \mathcal{O}_Y)$ be an affine scheme. Then there exists a canonical bijection*

$$\operatorname{Hom}_{\mathbf{Sch}}(X, Y) \to \operatorname{Hom}_{\mathbf{Ring}}(\Gamma(X, \mathcal{O}_X), \Gamma(X, \mathcal{O}_X)).$$

Proof. Let $A = \Gamma(Y, \mathscr{O}_Y)$. Note first that, if $(X, \mathscr{O}_X)$ and $(Y, \mathscr{O}_Y)$ are any two ringed spaces, a morphism $(\psi, \psi^\#) : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ canonically defines a homomorphism of rings

$$\Gamma(Y, \mathscr{O}_Y) \to \Gamma(Y, \psi_*(\mathscr{O}_X)) = \Gamma(X, \mathscr{O}_X).$$

It then remains to see that any homomorphism $\rho : A \to \Gamma(X, \mathscr{O}_X)$ is of the form $\Gamma(\psi^\#)$ for a unique morphism $(\psi, \psi^\#)$. However, there is by hypothesis a covering (V_α) of X by affine open sets. By considering the composition

$$A \xrightarrow{\rho} \Gamma(X, \mathscr{O}_X) \longrightarrow \Gamma(V_\alpha, \mathscr{O}_X|_{V_\alpha})$$

we obtain a homomorphism $\rho_\alpha : A \to \Gamma(V_\alpha, \mathscr{O}_X|_{V_\alpha})$, which corresponds to a morphism $(\psi_\alpha, \psi_\alpha^\#)$ from the scheme $(V_\alpha, \mathscr{O}_X|_{V_\alpha})$ to $(Y, \mathscr{O}_Y)$ (Proposition 4.1.16). Moreover, for each pair (α, β) of indices, any point of $V_\alpha \cap V_\beta$ admits an affine open neighborhood W contained in $V_\alpha \cap V_\beta$; it is clear that by composing ρ_α and ρ_β with the restriction homomorphism to W, we obtain the same homomorphism $A \to \Gamma(W, \mathscr{O}_X|_W)$, so, by virtue of the relation $(\psi_\alpha^\#)_x = (\rho_\alpha)_x$ for any $x \in V_\alpha$ and any α, the restrictions of $(\psi_\alpha, \psi_\alpha^\#)$ and $(\psi_\beta, \psi_\beta^\#)$ to W coincide. By gluing we then get a unique morphism $(\psi, \psi^\#) : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ which extending $(\psi_\alpha, \psi_\alpha^\#)$ on each V_α. It is clear that $(\psi, \psi^\#)$ is a morphism of schemes, and we have $\Gamma(\psi^\#) = \rho$. □

Remark 4.2.1. Let $\rho : A \to \Gamma(X, \mathscr{O}_X)$ be a ring homomorphism, and let $(\psi, \psi^\#)$ be the corresponding morphism $(X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$. For each $f \in A$, we have

$$\psi^{-1}(D(f)) = \{x \in X : f \notin \mathfrak{m}_{\psi(x)}\} = \{x \in X : (\rho(x))_x \notin \mathfrak{m}_x\} = X_{\rho(f)}.$$

Note that this can be viewed as a generalization of Proposition **??**(a).

Proposition 4.2.5. *Under the hypothesis of Proposition 4.2.4, let $\rho : A \to \Gamma(X, \mathscr{O}_X)$ be a ring homomorphism and $f : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ the corresponding morphism of schemes. Let $\mathscr{F}$ (resp. $\mathscr{G}$) be a quasi-coherent $\mathscr{O}_X$-module (resp. $\mathscr{O}_Y$-module). Then there exist a canonical bijection*

$$\mathrm{Hom}_{\mathbf{Qcoh(Y)}}(\mathscr{G}, f_*(\mathscr{F})) \to \mathrm{Hom}_A(\Gamma(Y, \mathscr{G}), \rho^*(\Gamma(X, \mathscr{F}))).$$

Proof. Indeed, by reasoning as in Proposition 4.2.4, we are immediately reduced to the case where X is affine and the proposition then follows from Proposition 4.1.3 and Proposition 4.1.11. □

We say a morphism $f : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ is **open** (resp. **closed**) if for any open subset U of X (resp. any closed subset F of X), $f(U)$ is open in Y (resp. $f(F)$ is closed in Y). We say f is **dominant** if $f(X)$ is dense in Y, and **surjective** if f is surjective. It should be noted that these conditions only involve the continuous map f.

Proposition 4.2.6. *Let $f : (X, \mathscr{O}_X)$ and $g : (Y, \mathscr{O}_Y) \to (Z, \mathscr{O}_Z)$ be morphisms of schemes.*

(a) If f and g are open (resp. closed, dominant, surjective), so is the composition $g \circ f$.

(b) If f is surjective and $g \circ f$ is closed, g is closed.

(c) If $g \circ f$ is surjective, g is surjective.

Proof. The assertions (a) and (c) are evident. Put $h = g \circ f$. If F is closed in Y, then $f^{-1}(F)$ is closed in X, so $h(f^{-1}(F))$ is closed in Z. But since f is surjective, we have $f(f^{-1}(F)) = F$, so $h(f^{-1}(F)) = g(F)$, which shows g is closed. □

Proposition 4.2.7. *Let $f : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ be a morphism of schemes and (U_α) an open covering of Y. For f to be open (resp. closed, surjective, dominant), it is necessary and sufficient that for each U_α, the restrictions $f : f^{-1}(U_\alpha) \to U_\alpha$ is open (resp. closed, surjective, dominant).*

Proof. The proposition follows immediately from the definitions, taking into account the fact that a subset F of Y is closed (resp. open, dense) in Y if and only if each of the sets $F \cap U_\alpha$ is closed (resp. open, dense) in U_α. □

Let $(X, \mathscr{O}_X)$ and $(Y, \mathscr{O}_Y)$ be schemes; suppose that X and Y have a same finite number of irreducible components X_i (resp. Y_i) ($1 \le i \le n$); let ξ_i (resp. η_i) be the generic point of X_i (resp. Y_i). We say that a morphism

$$f : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$$

is **birational** if, for any i, we have $f^{-1}(\eta_i) = \{\xi_i\}$ and $f^{\#}_{\xi_i} : \mathscr{O}_{Y,\eta_i} \to \mathscr{O}_{X,\xi_i}$ is an isomorphism. It is clear that any birational morphism is dominant, hence surjective if it is closed.

Remark 4.2.2. Throughout the remainder of this chapter and when there is no risk of creating confusion, we will omit in the notation of a scheme (resp. of a morphism) the structural sheaf (resp. the morphism of structural sheaf). If U is an open subset of the underlying space of a scheme X, when we speak of U as of a scheme, it will always be the scheme induced on U by X.

With the morphisms of schemes defined, we can talk about glueing schemes as in the case of ringed spaces. It follows immediately from the definition that any ringed space obtained by glueing schemes is again a scheme. In particular, since any scheme admits a basis of affine open subsets, we see any scheme is obtained by glueing affine schemes.

Example 4.2.8. Cosnider a field K, $A = K[s]$, $B = K[t]$ be two rings of polynomials over K with one indeterminate, and $X_1 = \mathrm{Spec}(A)$, $X_2 = \mathrm{Spec}(B)$. In X_1 (resp. X_2), let U_{12} (resp. U_{21}) be the affine open set $D(s)$ (resp. $D(t)$), whose ring A_s (resp. B_t) is formed by the rational fractions of the form $f(s)/s^m$ (resp. $g(t)/t^n$) with $f \in A$ (resp. $g \in B$). Let φ_{12} be the isomorphism of of schemes $U_{21} \to U_{12}$ corresponding to the isomorphism of A and B such that, $f(s)/s^m$ is mapped to the rational fraction $f(1/t)/(1/t^m)$ (i.e. we map s to $1/t$). We can then glue X_1 and X_2 along U_{12} and U_{21} by the isomorphism u_{12}, which evidently satisfies the glueing condition. We will see later the schem X thus obtained is a particular case of a general method of construction. We only show here that X is not an affine scheme, which will result from the fact that the ring $\Gamma(X, \mathscr{O}_X)$ is isomorphic to K, therefore has a spectrum reduced to a point. Indeed, a section of $\mathscr{O}_X$ above X has a restriction over X_1 (resp. X_2), identified with an open affine of X, which is a polynomial $f(s)$ (resp. $g(t)$), and it follows from the definition of u_{12} that we must have $g(t) = f(1/t)$, which is not possible only if $f = g \in K$.

4.2.2 Local schemes

Let X be a scheme and $A = \Gamma(X, \mathcal{O}_X)$. We say X is a local scheme if X is affine and the ring X is local. In this case, there then exists a unique closed point ξ in X, and for any point $x \in X$ we have $\xi \in \overline{\{x\}}$.

Following this notation, for a general scheme Y and $y \in Y$, the scheme $\operatorname{Spec}(\mathcal{O}_{Y,y})$ is called the **local scheme of Y at y**. Let V be an affine open subset of Y containing y, and B the ring of V. The local ring $\mathcal{O}_{Y,y}$ is then canonically identified with B_y, and the canonical homomorphism $B \to B_y$ then induces a morphism $\operatorname{Spec}(\mathcal{O}_{Y,y}) \to V$ of schemes. If we compose this with the canonical injection of V into Y, we then get a morphism $\operatorname{Spec}(\mathcal{O}_{Y,y}) \to Y$, which is independent of the choice of the affine open V containing y: in fact, if U is another affine neighborhood of y, there exists an affine open neighborhood W of y contained in $U \cap V$; we can then limit ourselves to the case $U \subseteq V$, and if A is the ring of U, we have a commutative diagram

$$\begin{tikzcd} A \arrow[rr] \arrow[dr] & & B \arrow[dl] \\ & \mathcal{O}_{Y,y} & \end{tikzcd}$$

The morphism $\operatorname{Spec}(\mathcal{O}_{Y,y}) \to Y$ thus defined is said to be **canonical**.

Proposition 4.2.9. *Let Y be a scheme, $y \in Y$, and $f : \operatorname{Spec}(\mathcal{O}_{Y,y}) \to Y$ be the canonical morphism.*

(a) The map f is a homeomorphism of $\operatorname{Spec}(\mathcal{O}_{Y,y})$ onto the subspace S_y of points $z \in Y$ such that $y \in \overline{\{z\}}$ (i.e. the set of generalizations of y).

(b) For each $\mathfrak{p} \in \operatorname{Spec}(\mathcal{O}_{Y,y})$, the homomorphism $f_{\mathfrak{p}}^{\#} : \mathcal{O}_{Y,f(\mathfrak{p})} \to (\mathcal{O}_{Y,y})_{\mathfrak{p}}$ is an isomorphism.

In particular, f is a monomorphism of locally ringed spaces.

Proof. Since the unique closed point η of $\operatorname{Spec}(\mathcal{O}_{Y,y})$ belongs to the closure of any other point in this space, and $f(\eta) = y$, the image of $\operatorname{Spec}(\mathcal{O}_{Y,y})$ by the continuous map f is contained in S_y. As S_y is contained in any affine open neighborhood of y, we can reduce to the case where Y is an affine scheme; but in this case the proposition follows immediately. □

Corollary 4.2.10. *There is a bijective correspondence between $\operatorname{Spec}(\mathcal{O}_{Y,y})$ and irreducible closed subsets of Y containing y.*

Proof. This follows from Proposition 4.2.9 and the fact that every irreducible closed set in Y has a unique generic point. □

Corollary 4.2.11. *For a point $y \in Y$ to be the generic point of an irreducible component of Y, it is necessary and suffcient that $\mathcal{O}_{Y,y}$ is zero-dimensional.*

Proof. This follows from the observation that y is the generic point of an irreducible component if and only if it is a maximal element under generalization, which is then equivalent by Corollary 4.2.10 to that $\operatorname{Spec}(\mathcal{O}_{Y,y})$ is a singleton. □

Proposition 4.2.12. *Let $(X, \mathscr{O}_X)$ be a local scheme with $A = \Gamma(X, \mathscr{O}_X)$, ξ its unique closed point, and $(Y, \mathscr{O}_Y)$ a scheme. Then any morphism $f : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ factors through* $\operatorname{Spec}(\mathscr{O}_{Y,f(\xi)})$*:*

$$f : X \to \operatorname{Spec}(\mathscr{O}_{Y,f(\xi)}) \to Y$$

where the second one is the canonical morphism, and the first one corresponds to a local homomorphism $\mathscr{O}_{Y,f(\xi)} \to A$.

Proof. In fact, for any $x \in X$, we have $\xi \in \overline{\{x\}}$, hence $f(\xi) \in \overline{\{f(x)\}}$. It then follows that $f(X)$ is contained in any affine open neighborhood of $f(\xi)$ (in fact any open neighborhood of $f(\xi)$). We can then reduce to the case that $(Y, \mathscr{O}_Y)$ is an affine scheme with ring $B = \Gamma(Y, \mathscr{O}_Y)$, and the morphism f corresponds to a ring homomorphism $\rho : B \to A$. We have $\rho^{-1}(\mathfrak{p}_\xi) = \mathfrak{p}_{f(\xi)}$, so the image under ρ of an element of $B - \mathfrak{p}_{f(\xi)}$ is invertible in the local ring A, and we get a canonical homomorphism $\rho_\xi : B_{f(\xi)} \to A$. □

Corollary 4.2.13. *There is a canonical bijection between* $\operatorname{Hom}_{\mathbf{Sch}}(X, Y)$ *to the set of local homomorphisms $\mathscr{O}_{Y,y} \to A$, where $y \in Y$.*

Proof. It suffices to note that any local homomorphism $\mathscr{O}_{Y,y} \to A$ corresponds to a unique morphism $f : X \to \operatorname{Spec}(\mathscr{O}_{Y,y})$ such that $f(\xi) = y$, and by composing this with the canonical morphism $\operatorname{Spec}(\mathscr{O}_{Y,y}) \to Y$, we get a morphism $X \to Y$. □

Example 4.2.14. The affine scheme whose ring is a field K have an underlying space reduced to one point. If A is a local ring of maximal ideal $\mathfrak{m}$, any local homomorphism $A \to K$ has a kernel equal to $\mathfrak{m}$, so factors into $A \to A/\mathfrak{m} \to K$, where the second arrow is a monomorphism. The morphisms $\operatorname{Spec}(K) \to \operatorname{Spec}(A)$ correspond therefore bijectively to the field extensions $A/\mathfrak{m} \to K$.

Let $(Y, \mathscr{O}_Y)$ be a scheme; for any $y \in Y$ and any ideal $\mathfrak{a}_y$ of $\mathscr{O}_{Y,y}$, the canonical homomorphism $\mathscr{O}_{Y,y}/\mathfrak{a}_y$ defines a morphism $\operatorname{Spec}(\mathscr{O}_{Y,y}/\mathfrak{a}_y) \to \operatorname{Spec}(\mathscr{O}_{Y,y})$; by composing this with the canonical morphism $\operatorname{Spec}(\mathscr{O}_{Y,y}) \to Y$, we obtain a morphism $\operatorname{Spec}(\mathscr{O}_{Y,y}/\mathfrak{a}_y) \to Y$, also called canonical. If $\mathfrak{a}_y = \mathfrak{m}_y$ is the maximal ideal of $\mathscr{O}_{Y,y}$, then $\mathscr{O}_{Y,y} = \kappa(y)$ and Corollary 4.2.13 then imply the following result:

Corollary 4.2.15. *Let $(X, \mathscr{O}_X)$ be a local scheme with $K = \Gamma(X, \mathscr{O}_X)$ a field, ξ its unique point, and $(Y, \mathscr{O}_Y)$ be a scheme. Then any morphism $f : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ factors into*

$$f : X \to \operatorname{Spec}(\kappa(f(\xi))) \to Y$$

where the second arrow is the canonical morphism, and the first arrow corresponds to a field extension $\kappa(f(\xi)) \to K$. This establishes a canonical bijection between $\operatorname{Hom}_{\mathbf{Sch}}(X, Y)$ *to the set of field extensions $\kappa(y) \to K$, where $y \in Y$.*

Corollary 4.2.16. *For any $y \in Y$, the canonical morphism* $\operatorname{Spec}(\mathscr{O}_{Y,y}/\mathfrak{a}_y) \to Y$ *is a monomorphism of locally ringed spaces.*

Proof. This follows from Proposition 4.2.9 and Corollary 4.1.18. □

Remark 4.2.3. Let X be a local scheme, ξ its unique closed point. Since any affine open neighborhood of ξ is necessarily all of X, any invertible $\mathcal{O}_X$-module is necessarily isomorphic to $\mathcal{O}_X$ (in other words, is trivial). This property does not hold in general for any affine scheme $\mathrm{Spec}(A)$, but we will see that if A is a normal ring, this is true when A is factorial.

4.2.3 Shemes over a scheme

As in any category, for a scheme S we can define the category **Sch**$/S$ of S-objects in the category of schemes, which will be a morphism $\varphi : X \to S$ where X is a scheme. In this case we also say that X is a **scheme over *S***, or an ***S*-scheme**. We say that S is the **base scheme** of the S-scheme X and φ is called the structural morphism of the S-scheme X. When S is an affine scheme of the ring A, we also say that X is a **scheme over *A*** or an ***A*-scheme**.

It follows from Proposition 4.2.4 that giving a scheme over a ring A is equivalent to giving a scheme $(X, \mathcal{O}_X)$, where $\mathcal{O}_X$ is an A-algebra. In particular, any scheme can be considered as a scheme over $\mathbb{Z}$. In other words, the scheme $\mathrm{Spec}(\mathbb{Z})$ is a final object in the category of schemes (also a final object in the category of locally ringed spaces).

If $\varphi : X \to S$ is the structural morphism of an S-scheme X, we say a point $x \in X$ is **lying over** a point $s \in S$ if $\varphi(x) = s$. We say X **dominates** S if the morphism φ is dominant. Let X and Y be two S-schemes; a morphism $u : X \to Y$ is called a **morphism of schemes over *S*** (or ***S*-morphism**) if the diagram

$$\begin{tikzcd} X \arrow[rr, "u"] \arrow[dr] & & Y \arrow[dl] \\ & S & \end{tikzcd}$$

is commutative; in other words, if for any $s \in S$ and any $x \in X$ lying over s, the point $u(x)$ is lying over s. This definition immediately shows that the composite of two S-morphisms is an S-morphism, so the S-schemes thus form a category. We denote by $\mathrm{Hom}_S(X, Y)$ the set of S-morphisms from an S-scheme X to an S-scheme Y; the identity morphism of an S-scheme X is then denoted by 1_X or id_X. If S is an affine scheme S, we also say A-morphisms for S-morphisms.

If X is an S-scheme, $\varphi : X \to S$ the structural morphism, an ***S*-section** of X is defined to be an S-morphism of S to X, which is equivalently a morphism $\psi : S \to X$ of schemes such that $\varphi \circ \psi = \mathrm{id}_S$. We denote by $\Gamma(X/S)$ the set of S-sections of X.

Example 4.2.17. If X is an S-scheme and $v : X' \to X$ a morphism of schemes, the composition scheme

$$X' \xrightarrow{v} X \longrightarrow S$$

then defines X' as an S-scheme; in particular, any scheme induceed over an open subset U of X can be considered as an S-scheme by means of the canonical injection.

Example 4.2.18. Let $u : X \to Y$ be an S-morphism of S-schemes, the restriction of u on any open subset U of X is then an S-morphism $U \to Y$. Conversely, let (U_α) be a covering of X and for each α, let $u_\alpha : U_\alpha \to Y$ be an S-morphism; if for any pair (α, β) of indices, the restrictions of u_α and u_β on $U_\alpha \cap U_\beta$ coincide, then there exists a unique S-morphism $X \to Y$ whose restriction on U_α equals to u_α.

Let $S \to S'$ be a morphism of schemes; for any S'-scheme X, the composition morphism $X \to S' \to S$ then defines X as an S-scheme. Conversely, suppose that S' is the scheme induced over an open subset of S; let X be an S-scheme and suppose that the structural morphism $X \to S$ has image contained in S'; then we can consider X as an S'-scheme. In the latter case, if Y is an S-scheme whose structural morphism also maps the underlying space in S', any S-morphism from X in Y is also an S'-morphism.

4.2.4 Quasi-coherent sheaves on schemes

Proposition 4.2.19. *Let X be a scheme. For an $\mathscr{O}_X$-module $\mathscr{F}$ to be quasi-coherent, it is necessary and sufficient that, for any affine open subset V of X, the restriction $\mathscr{F}|_V$ is isomorphism to the sheaf associated with a $\Gamma(V,\mathscr{O}_X)$-module.*

Proof. We recall that being quasi-coherent is a local property, and affine opens form a basis for X. Also, by Proposition 4.1.20, a quasi-coherent sheaf on an affine open V is isomorphic to $\widetilde{M}$ for some $\Gamma(V,\mathscr{O}_X)$-module M. □

Corollary 4.2.20. *Let X be an arbitrary scheme.*

(i) Let $0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{H} \to 0$ be an exact sequence of $\mathscr{O}_X$-modules; if two of them are quasi-coherent, then so is the third one.

(ii) The images, kernels and cokernels of homomorphisms of quasi-coherent $\mathscr{O}_X$-modules are quasi-coherent. The inductive limits and direct sums of quasi-coherent sheaves are quasi-coherent. If $\mathscr{G}$ and $\mathscr{H}$ are quasi-coherent $\mathscr{O}_X$-modules of a quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, then $\mathscr{G}+\mathscr{H}$ and $\mathscr{G}\cap\mathscr{H}$ are quasi-coherent.

(iii) Let $0 \to \mathscr{F}_1 \to \mathscr{F}_2 \to \mathscr{F}_3 \to \mathscr{F}_4 \to \mathscr{F}_5 \to 0$ be an exact sequence of $\mathscr{O}_X$-modules. If $\mathscr{F}_1,\mathscr{F}_2,\mathscr{F}_4,\mathscr{F}_5$ are quasi-coherent, so is $\mathscr{F}_3$.

(iv) If $\mathscr{F}$ and $\mathscr{G}$ are quasi-coherent $\mathscr{O}_X$-modules, $\mathscr{F}\otimes_{\mathscr{O}_X}\mathscr{G}$ is quasi-coherent. In particular, if $\mathscr{J}$ is a quasi-coherent ideal of $\mathscr{O}_X$, $\mathscr{J}\mathscr{F}$ is a quasi-coherent $\mathscr{O}_X$-module.

(v) Let $\mathscr{F}$ be a quasi-coherent $\mathscr{O}_X$-module with finite presentation. Then for any quasi-coherent $\mathscr{O}_X$-module $\mathscr{G}$, $\mathscr{H}om_{\mathscr{O}_X}(\mathscr{F},\mathscr{G})$ is quasi-coherent.

(vi) If $\mathscr{F}$ is a quasi-coherent $\mathscr{O}_X$-module of finite type, the annihilator $\mathscr{J}$ of $\mathscr{F}$ is a quasi-coherent ideal of $\mathscr{O}_X$.

Proof. By Proposition 4.2.19, assertions (i) to (v) follow from Proposition 4.1.28, Corollary 4.1.6, and Corollary 4.1.9. To prove (vi), we can assume that $X=\operatorname{Spec}(A)$ is affine, $\mathscr{F}=\widetilde{M}$, where M is a finitely generated A-module, with generators $t_1,\dots,t_r$. The ideal $\mathscr{J}$ is then the intersection of the annihilators of t_i. But the annihilator of t_i is by definition the kernel of the canonical morphism $\mathscr{O}_X \to \mathscr{F}$ corresponding to $s \mapsto st_i$ from A to M, hence quasi-coherent. It then follows that $\mathscr{J}$ is quasi-coherent, as an intersection of quasi-coherent $\mathscr{O}_X$-modules. □

Corollary 4.2.21. *Let X be a scheme, $0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{H} \to 0$ an exact sequence of quasi-coherent $\mathscr{O}_X$-modules. If $\mathscr{H}$ is finitely presented and $\mathscr{G}$ is of finite type, then $\mathscr{F}$ is of finite type.*

Proof. Since this question is local, we may assume that X is affine, and the corresponding result is then Proposition **??**. □

Proposition 4.2.22. *Let X be a scheme and $\mathscr{B}$ be a quasi-coherent $\mathscr{O}_X$-algebra. For a $\mathscr{B}$-module $\mathscr{F}$ to be quasi-coherent, it is necessary and sufficient that $\mathscr{F}$ is a quasi-coherent $\mathscr{O}_X$-module. In particular, if $\mathscr{F}$ and $\mathscr{G}$ are quasi-coherent $\mathscr{B}$-modules, $\mathscr{F} \otimes_{\mathscr{B}} \mathscr{G}$ is a quasi-coherent $\mathscr{B}$-module; the same holds for $\mathscr{H}om_{\mathscr{B}}(\mathscr{F}, \mathscr{G})$ if $\mathscr{F}$ is a finitely presented $\mathscr{B}$-module.*

Proof. Since the question is local, we can suppose that X is affine with ring A, and then $\mathscr{B} = \widetilde{B}$, where B is an A-algebra. If $\mathscr{F}$ is quasi-coherent over the space $(X, \mathscr{B})$, we can write $\mathscr{F}$ as the cokernel of $\mathscr{B}$-homomorphism $\mathscr{B}^{\oplus I} \to \mathscr{B}^{\oplus J}$; since this homomorphism is also an $\mathscr{O}_X$-homomorphism, and $\mathscr{B}^{\oplus I}$, $\mathscr{B}^{\oplus J}$ are quasi-coherent $\mathscr{O}_X$-modules, we conclude that $\mathscr{F}$ is also quasi-coherent.

Conversely, if $\mathscr{F}$ is a quasi-coherent $\mathscr{O}_X$-module, we have $\mathscr{F} = \widetilde{M}$ where M is a B-module (Proposition 4.1.25); M is isomorphic to the cokernel of a homomorphism $B^{\oplus I} \to B^{\oplus J}$, so $\mathscr{F}$ is a $\mathscr{B}$-module isomorphic to the cokernel of the corresponding homomorphism $\mathscr{B}^{\oplus I} \to \mathscr{B}^{\oplus J}$. This completes the proof. □

Let X be a scheme. A quasi-coherent $\mathscr{O}_X$-algebra $\mathscr{B}$ is **of finite type** (resp. **of finite presentation**) if for all $x \in X$, there exists an open affine neighborhood U of x such that $\Gamma(U, \mathscr{B}) = B$ is an algebra of type finite (resp. of finite presentation[1]) over $\Gamma(U, \mathscr{O}_X) = A$. If this is the case, we have $\mathscr{B}|_U = \widetilde{B}$, and for all $f \in A$, the $(\mathscr{O}_X|_{D(f)})$-algebra $\mathscr{B}|_{D(f)}$ induced on $D(f)$ is of finite type (resp. of finite presentation), because it is isomorphic to $\widetilde{B \otimes_A A_f}$. As the $D(f)$ form a basis of the topology of X, we deduce that for any open set V of X, $\mathscr{B}|_V$ is a $(\mathscr{O}_X|_V)$-algebra of finite type (resp. of finite presentation).

Proposition 4.2.23. *Let X be a scheme, $\mathscr{E}$ a locally free $\mathscr{O}_X$-module of rank r, Z a finite subset of X contained in an affine open V. Then there exists an open neighborhood $U \subseteq V$ of Z such that $\mathscr{E}|_U$ is isomorphic to $\mathscr{O}_U^r$.*

Proof. By replacing X by V, we can assume that $X = \operatorname{Spec}(A)$ is affine. For each $z_i \in Z$ there exists a closed point z_i' in the closure $\overline{\{z_i\}}$ (that is, a maximal ideal containing $\mathfrak{p}_{z_i}$); if Z' is the set of the z_i', any neighborhood of Z' is a neighborhood of Z, and we can then suppose that Z is closed in X. Now, the subset Z of X is defined by an ideal $\mathfrak{a}$ of A; consider the scheme $\operatorname{Spec}(A/\mathfrak{a})$, with Z its underlying space, and the injection $\iota : Z \to X$ corresponds to the canonical homomorphism $A \to A/\mathfrak{a}$. Then $\iota^*(\mathscr{E}) = \mathscr{E} \otimes_{\mathscr{O}_X} \mathscr{O}_Z$ is locally free with rank r over the discrete scheme Z, so is isomorphic to $\mathscr{O}_Z^r$. In other words, there exist sections $s_1, \dots, s_r$ of $\mathscr{E} \otimes_{\mathscr{O}_X} \mathscr{O}_Z$ over Z such that the homomorphism $\mathscr{O}_Z^r \to \mathscr{E} \otimes_{\mathscr{O}_X} \mathscr{O}_Z$ defined by these sections is bijective. On the other hand, we have $\mathscr{E} = \widetilde{M}$ where M is an A-module; then each s_i belongs to $M \otimes_A (A/\mathfrak{a})$, and is then the image of an element $t_i \in M = \Gamma(X, \mathscr{E})$. For each $z_j \in Z$, by Corollary 1.4.38, there then exists a neighborhood V_i of z_i in X such that the restrictions of t_i to V_j define an isomorphism $\mathscr{O}_X^r|_{V_j} \to \mathscr{E}|_{V_j}$; the union U of the V_j's then satisfies the requirement. □

[1] Recall that an algebra B is finitely presented over A if it is isomorphic to the quotient of a polynomial ring over A in finitely many variables by a finitely generated ideal.

Proposition 4.2.24. *Let X a scheme and $\mathscr{F}$ be a quasi-coherent $\mathscr{O}_X$-module of finite type. Then there exists a closed subscheme Y of X with underlying space* $\operatorname{supp}(\mathscr{F})$ *and a quasi-coherent $\mathscr{O}_Y$-module $\mathscr{G}$ of finite type supported on Y such that, if $j : Y \to X$ is the canonical injection, $\mathscr{F}$ is isomorphic to $j_*(\mathscr{G})$.*

Proof. It suffices to note that the annihilator $\mathscr{I}$ of $\mathscr{F}$ is a quasi-coherent ideal of $\mathscr{O}_X$ (Corollary 4.2.20), so if Y is the closed subscheme of X defined by $\mathscr{I}$, as $\mathscr{I}\mathscr{F} = 0$, $\mathscr{F}$ is an $(\mathscr{O}_X/\mathscr{I})$-module, and we can take $\mathscr{G} = j^*(\mathscr{F})$. □

4.2.5 Noetherian schemes and locally Noetherian schemes

We say a scheme X is Noetherian (resp. locally Noetherian) if there is a finite covering (resp. a covering) of open affines V_α such that each ring $\Gamma(V_\alpha, \mathscr{O}_X)$ is Noetherian. The underlying space of a Noetherian (resp. locally Noetherian) scheme is then a Noetherian space (resp. locally Noetherian). Moreover, if X is locally Noetherian. the structural sheaf $\mathscr{O}_X$ is coherent, any quasi-coherent $\mathscr{O}_X$-module of finite type is coherent (Proposition 4.1.29), and any local ring $\mathscr{O}_{X,x}$ is Noetherian. Any quasi-coherent sub-$\mathscr{O}_X$-module (resp. any quasi-coherent $\mathscr{O}_X$-quotient) of a coherent $\mathscr{O}_X$-module $\mathscr{F}$ is then coherent, because the question is local again, and we just apply Proposition 4.1.29, together with the fact that a sub-module (resp. quotient module) of a finitely generated module on a Noetherian ring is finitely generated. More particularly, any quasi-consistent ideal of $\mathscr{O}_X$ is consistent.

If a scheme X is a finite union (resp. a union) of open Noetherian (resp. locally Noetherian) subschemes W_λ, it is clear that X is then Noetherian (resp. locally Noetherian).

Proposition 4.2.25. *For a scheme X to be Noetherian, it is necessary and sufficient that it is locally Noetherian and its underlying space is quasi-compact.*

Proof. This follows from the definition, since a Noetherian space is quasi-compact. □

Proposition 4.2.26. *Let X be an affine scheme with ring A. Then the following conditions are equivalent:*

(i) X is Noetherian;

(ii) X is locally Noetherian;

(iii) A is Noetherian.

Proof. Since X is quasi-compact, it is clear that (i) and (ii) are equivalent. Also, (iii) implies (i) by definition. Now assume that X is Noetherian, then there is a finite covering (V_i) of X by affine opens where $A_i = \Gamma(V_i, \mathscr{O}_X)$ is Noetherian. Let $(\mathfrak{a}_n)$ be an increasing sequence of ideals of A; it corresponds to it canonically in a one-to-one way to an increasing sequence $(\tilde{\mathfrak{a}}_n)$ of ideals in $\widetilde{A} = \mathscr{O}_X$; to see that the sequence $(\mathfrak{a}_n)$ is stationary, it suffices to prove that the sequence $(\tilde{\mathfrak{a}}_n)$ is. However, the restriction $\tilde{\mathfrak{a}}_n|_{V_i}$ is a quasi-coherent ideal of $\mathscr{O}_X|_{V_i}$; $\tilde{\mathfrak{a}}_n|_{V_i}$ is then of the form $\tilde{\mathfrak{a}}_{n,i}$, where $\mathfrak{a}_{n,i}$ is an ideal of A_i. As A_i is Noetherian, the sequence $(\mathfrak{a}_{n,i})$ is stationary for all i, hence the proposition. □

Note that the above reasoning also proves that if X is a Noetherian scheme, any increasing sequence of coherent ideals of $\mathscr{O}_X$ is stationary.

Proposition 4.2.27. *Let X be a locally Noetherian scheme. Any quasi-coherent $\mathscr{O}_X$-algebra $\mathscr{B}$ of finite type is a coherent shaf of rings and an $\mathscr{O}_X$-algebra of finite presentation.*

Proof. We can assume that $X = \operatorname{Spec}(A)$ is affine, where A is a Noetherian ring, and $\mathscr{B} = \widetilde{B}$, where B is an A-algebra of finite type. It then follows that B is finitely presented over A, so $\mathscr{B}$ is of finite presentation. To show that $\mathscr{B}$ is coherent, we must prove that the kernel $\mathscr{N}$ of a $\mathscr{B}$-homomorphism $\mathscr{B}^m \to \mathscr{B}$ is a $\mathscr{B}$-module of finite type; but it is of the form $\widetilde{N}$, where N is the kernel of the corresponding homomorphism $B^m \to B$ of B-modules. Since B is also Noetherian, N is a finitely generated B-module. There then exists a surjective B-homomorphism $B^n \to N$, so a surjective homomorphism $\mathscr{B}^n \to \mathscr{N}$, which proves our assertion. □

Corollary 4.2.28. *Let X be a locally Noetherian scheme and $\mathscr{B}$ be a quasi-coherent $\mathscr{O}_X$-algebra of finite type. For a $\mathscr{B}$-module $\mathscr{F}$ to be coherent, it is necessary and sufficient that $\mathscr{F}$ is a quasi-coherent $\mathscr{O}_X$-module and a $\mathscr{B}$-module of finite type, and if $\mathscr{G}$ is a sub-$\mathscr{B}$-module or a quotient $\mathscr{B}$-module of $\mathscr{F}$, for $\mathscr{G}$ to be a coherent $\mathscr{B}$-module, it is necessary and sufficient that $\mathscr{G}$ is a quasi-coherent $\mathscr{O}_X$-module.*

Proof. Considering Proposition 4.2.22, the conditions on $\mathscr{F}$ is necessary. To prove the sufficiency, we can assume that $X = \operatorname{Spec}(A)$ is affine, where A is Noetherian, $\mathscr{B} = \widetilde{B}$, where B is an A-algebra of finite type, and $\mathscr{F} = \widetilde{M}$, where M is a B-module and there exists a surjective B-homomorphism $\mathscr{B}^m \to \mathscr{F}$. Then we get a corresponding homomorphism $B^m \to M$, so M is a finitely generated B-module; the kernel P of thie homomorphism is finitely generated since B is Noetherian, and $\mathscr{F}$ is therefore the cokernel of a morphism $\mathscr{B}^n \to \mathscr{B}^m$, so it is coherent (since $\mathscr{B}$ is a coherent sheaf of rings). The same reasoning shows that any quasi-coherent sub-$\mathscr{B}$-module (resp. quotient $\mathscr{B}$-module) of $\mathscr{F}$ is of finite type, whence the second part of the corollary. □

Proposition 4.2.29. *Let X be a locally Noetherian scheme and E be a subset of X. Any point $x \in E$ admits in E a maximal generalization y (i.e. y has no further generalization in Y). In particular, if $E \neq \varnothing$, there exists a maximal element $y \in E$ under generalization.*

Proof. The generalizations of x in X lie in the points of $\operatorname{Spec}(\mathscr{O}_{X,x})$ (Proposition 4.2.10), where $\mathscr{O}_{X,x}$ is a Noetherian local ring. We then know that the lengths of chains of prime ideals in this ring are bounded by $\dim(\mathscr{O}_{X,x})$, and to prove the proposition it suffices to consider a chain of prime ideals belonging to E and having the greatest possible length. □

Proposition 4.2.30. *Let X be a scheme. Then the following conditions are equivalent:*

(i) $X = \operatorname{Spec}(A)$ *is affine and A is Artinian;*

(ii) X is Noetherian and has discrete underlying space;

(iii) X is Noetherian and every point in X is closed (in other words, X is T1).

If these equivalent conditions hold, then X is finite and the ring A is a direct product of finitely many Artinian local rings.

Proof. We know that (i) implies the last asserion. It is clear that (i)⇒(ii)⇒(iii). To see that (iii) implies (i), let us first show that X is then finite; we can indeed reduce to case where X is affine, and we know that a Noetherian ring of which all the prime ideals are maximal is Artinian, hence our assertion. □

Note that a Noetherian scheme can have an underlying space finite without being artinian, as shown by the example of a spectrum discrete valuation ring.

4.3 Product of schemes

Let (X_α) be a family of schemes, and X be the topological space which is the **coproduct** of the underlying spaces of X. Then X is the union of its open subspaces U_α, and for each α we have a embedding $\iota_\alpha : X_\alpha \to X$ with image equal to U_α. If we endow each U_α the sheaf $(\iota_\alpha)_*(\mathscr{O}_{X_\alpha})$, it is clear that X becomes a scheme, which we will call the **coproduct** of the family (X_α), and denote by $\coprod_\alpha X_\alpha$. It is clear that the scheme X satisfies the universal property of coproducts of X_α's: for any scheme Y and morphisms $f_\alpha : X_\alpha \to Y$, there exists a unique morphism $f : X \to Y$ such that $f \circ \iota_\alpha = f_\alpha$. In other words, we have a functorial bijection

$$\mathrm{Hom}(\coprod_\alpha X_\alpha, Y) \to \prod_\alpha \mathrm{Hom}(X_\alpha, Y).$$

This fact can be also stated that $\coprod_\alpha X_\alpha$ represents the covariant functor $\prod_\alpha \mathrm{Hom}(X_\alpha, -)$ on the category of schemes. In particular, if X_α are S-schemes with structural morphisms ψ_α, then X is an S-scheme with structural morphism $\psi : X \to S$ such that $\psi \circ \iota_\alpha = \psi_\alpha$. We usually denote the coproduct of two schemes X and Y by $X \amalg Y$, and it is clear that if $X = \mathrm{Spec}(A)$ and $Y = \mathrm{Spec}(B)$, then $X \amalg Y$ is canonically identified with $\mathrm{Spec}(A \times B)$.

In this section, we shall consider product of schemes, which is far more complicated than coproducts. We will see that fiber products plays a central role of many construction and operations on schemes.

4.3.1 Product of schemes

Let X and Y be S-schemes. Recall that the object $X \times_S Y$ represents by definition the contravariant functor

$$T \mapsto F(T) = \mathrm{Hom}_S(T, X) \times \mathrm{Hom}_S(T, Y)$$

on the category of S-schemes. To prove the existence of $X \times_S Y$, we shall apply the methods used in Proposition 1.3.5. We first verify condition (ii) in Proposition 1.3.5, which means F is a sheaf over the category **Sch**: this is evident since the functors $T \mapsto \mathrm{Hom}_S(T, X)$ and $T \mapsto \mathrm{Hom}_S(T, Y)$ are sheaves, and a projective limit of sheaves over **Sch** is again a sheaf over **Sch** (Remark **??**).

This already allows us to bring ourselves back to the case that the scheme S is affine. In fact, let (S_α) is a covering os S by affine open sets. In view of the above fact and of Corollary **??**, it suffices to show that each of the functors $F \times_{h_S} h_{S_\alpha}$ is representable. On the other hand, let $\varphi : X \to S$ and $\psi : Y \to S$ be the structural morphisms; it is immediate that when T is a locally ringed S_α-space (hence also an S-space), we have canonical identifies

$$\mathrm{Hom}_S(T, X) \xrightarrow{\sim} \mathrm{Hom}_{S_\alpha}(T, \varphi^{-1}(S_\alpha)), \quad \mathrm{Hom}_S(T, Y) \xrightarrow{\sim} \mathrm{Hom}_{S_\alpha}(T, \psi^{-1}(S_\alpha)).$$

Therefore, in view of Remark **??**, we only need to show that F is representable when it is restricted to the subcategory of locally ringed S_α-spaces.

With these being done, assume that S is affine and consider a covering (X_λ) (resp. (Y_μ)) of X (resp. Y) by affine opens. We shall verify the conditions (i) and (iii) of Proposition 1.3.5 for the subfunctors $F_{\lambda\mu} : T \mapsto \mathrm{Hom}_S(T, X_\lambda) \times \mathrm{Hom}_S(T, Y_\mu)$ of F. Let Z be a locally ringed S-space and (p, q) be an element of $F(Z)$, i.e. $p : Z \to X$ and $q : Z \to Y$ are S-morphisms. These determine by Yoneda Lemma a natural transform $h_Z \to F$ which associates a locally ringed S-space T the map

$$\mathrm{Hom}_S(T, Z) \to F(T), \quad g \mapsto (p \circ g, q \circ g)$$

and every natural transform $h_Z \to F$ is of this form. We now show that the functor

$$T \mapsto F_{\lambda\mu}(T) \times_{F(T)} h_Z(T) \tag{3.1.1}$$

is representable by a locally ringed S-space induced by Z on an open subset of Z. In fact, an element of the right side of (3.1.1) (which is a fiber product of sets) is a triple (u_λ, v_μ, g), where $g : T \to Z$, $u_\lambda : T \to X_\lambda$, and $v_\mu : T \to Y_\mu$ are S-morphisms such that the following diagram commutes:

$$\begin{tikzcd}
T \arrow[dr, "g"] \arrow[drr, bend left, "v_\mu"] \arrow[ddr, bend right, "u_\lambda"'] & & \\
& Z \arrow[r, "q"] \arrow[d, "p"'] & Y \arrow[d] \\
& X \arrow[r] & S
\end{tikzcd}$$

Now this in particular implies that $g(T) \subseteq Z_{\lambda\mu} = p^{-1}(X_\lambda) \cap q^{-1}(Y_\mu)$, and conversely, any S-morphism $g : T \to Z$ verifying this condition corresponds to the unique triple $(p \circ g, q \circ g, g)$, since $p \circ g$ (resp $q \circ g$) can be viewed as a morphism from Z to X_λ (resp. Y_μ). In other word, we have a canonical bijection

$$F_{\lambda\mu}(T) \times_{F(T)} \mathrm{Hom}_S(T, Z) \xrightarrow{\sim} \mathrm{Hom}_S(T, Z_{\lambda\mu})$$

and the functor (3.1.1) is then represented by the couple $(Z_{\lambda\mu}, (p|_{Z_{\lambda\mu}}, q|_{Z_{\lambda,\mu}}), j_{\lambda\mu})$, where $j_{\lambda\mu} : Z_{\lambda\mu} \to Z$ is the cannical injection. Since the $Z_{\lambda\mu}$ form an open covering of Z, this proves both of the conditions (i) and (iii) of Proposition 1.3.5.

It remains to show that the functors $F_{\lambda\mu}$ are representable, which means we need to construct $X \times_S Y$ when X, Y, and S are affine schemes. This is fairly easy, as we will now show.

Proposition 4.3.1. *Assume that* $S = \mathrm{Spec}(A)$, $X = \mathrm{Spec}(B)$, *and* $Y = \mathrm{Spec}(C)$, *where* B *and* C *are* A*-algebras. Then the scheme* $Z = \mathrm{Spec}(B \otimes_A C)$, *with* p, q *the* S*-morphisms corresponding to the canonical* A*-homomorphisms* $B \to B \otimes_A C$ *and* $C \to B \otimes_A C$, *representes the functor* F *in the category of locally rinfed* S*-spaces.*

Proof. In fact, in the category of rings, the tensor product $B \otimes_A C$ of two A-algebras B and C is a coproduct in the category of A-algebras, as can be easily verified. □

We then conclude that fiber products exist in the category of schemes. As always, the notation $X \times_S Y$ will be used to denote this product for two S-schemes X and Y. If $S = \mathrm{Spec}(A)$

is an affine scheme, we also write $X \times_A Y$. If $Y = \mathrm{Spec}(B)$ is an affine scheme, in view of Proposition 4.3.1, we use $X \otimes_S B$ to denote this product, and $X \otimes_A B$ if $S = \mathrm{Spec}(A)$ is also affine.

The general notations and results for fiber products in a category can be then used for the product of schemes. In particular, if $p_1 : X \times_X Y \to X$, $p_2 : X \times_S Y \to Y$ are the canonical projections, and $g : T \to X$, $h : T \to Y$ are two S-morphisms, we denote by $(g, h)_S$ the unique S-morphism fits into the following diagram:

$$\begin{tikzcd}
T \arrow[dr, dashed, "{(g,h)_S}"] \arrow[drr, bend left, "h"] \arrow[ddr, bend right, "g"'] & & \\
& X \times_S Y \arrow[r, "p_2"] \arrow[d, "p_1"'] & Y \arrow[d] \\
& X \arrow[r] & S
\end{tikzcd}$$

If $S = \mathrm{Spec}(A)$, $X = \mathrm{Spec}(B)$, $Y = \mathrm{Spec}(C)$, and $T = \mathrm{Spec}(D)$ are all affine, and g, h correspond to homomorphisms $\rho : B \to D$, $\sigma : C \to D$ of A-algebras, then $(g, h)_S$ corresponds to the homomorphism $\tau : B \otimes_A C \to D$ such that

$$\tau(b \otimes c) = \rho(b)\sigma(c).$$

Again, if $S = \mathrm{Spec}(A)$ is affine, we also write $(g, h)_A$ instead of $(g, h)_S$.

Corollary 4.3.2. *Let $Z = X \times_S Y$ be the product of two S-schemes, $p : Z \to X$, $q : Z \to Y$ the canonical projections, φ (resp ψ) the structural morphisms of X (resp. Y). Let U, V be open subsets of X, Y respectively, and W be an open subset of S such that $p(U) \subseteq W$ and $p(V) \subseteq W$. Then the product $U \times_W V$ is canonically identified with the scheme induced by Z on the subset $p^{-1}(V) \cap q^{-1}(W)$ (considered as a U-scheme). Moreover, if $g : T \to X$, $h : T \to Y$ are S-morphisms such that $g(T) \subseteq V$, $h(T) \subseteq W$, the U-morphism $(g, h)_S$ is identified with $(g, h)_S$, considered as morphisms from T to $p^{-1}(V) \cap q^{-1}(W)$.*

Proof. We first note that, if U is an open set of S and $\varphi : X \to S$, $\psi : Y \to S$ are structural morphisms with images in U, then the fiber product $X \times_S Y$ is identified with $X \times_U Y$. Apply this to V and W, we conclude that $U \times_W V = U \times_W V$. So it suffices to prove that the subscheme $R = p^{-1}(U) \cap q^{-1}(V)$ with its restricted projections to U and V form a product of U and V. For this, we note that if T is an S-scheme, we can identify the S-morphisms $T \to R$ and the S-morphisms $T \to Z$ with image in R. If $g : T \to U$, $h : T \to V$ are two S-morphisms, we can consider them as S-morphisms of T in X and Y respectively, and by hypothesis there is therefore an S-morphism and there is a morphism $f : T \to Z$ such that $g = p \circ f$, $h = q \circ f$. Since $p(f(T)) \subseteq U$ and $q(f(T)) \subseteq V$, we have

$$f(T) \subseteq p^{-1}(U) \cap q^{-1}(V) = W$$

whence our claim. □

Corollary 4.3.3. *Let (X_λ) (resp. (Y_μ)) be a family of S-schemes and X (resp. Y) be their coproduct. Then $X \times_S Y$ is identified with the coproduct of the family $(X_\lambda \times_S Y_\mu)$.*

Proof. In fact, in the notations of Corollary 4.3.2, the underlying space of $X \times_S Y$ is the disjoint union of open sets $p^{-1}(X_\lambda) \cap q^{-1}(Y_\mu)$, and it suffices to apply Corollary 4.3.2. □

Remark 4.3.1. The product of two Noetherian S-schemes need not to be Noetherian, even if they are both spectrum of fields. For example, if k is a nonperfect field of characteristic $p > 0$, the tensor product $A = k^{p^{-\infty}} \otimes_k k^{p^{-\infty}}$ is not a Noetherian ring: in fact, for any integer $n > 0$, there exists $x_n \in k^{p^{-\infty}}$ such that $x_n^{p^n} \in k$ and $x_n^{p^{n-1}} \notin k$. If we consider the element $z_n = 1 \otimes x_n - x_n \otimes 1$ of A, we then have $z_n^{p^n} = 0$, and $z_n^{p^{n-1}} \neq 0$ since 1 and $x_n^{p^{n-1}}$ are linearly independent over k. We then conclude that the nilradical of A is not nilpotent, so A is not Noetherian.

Remark 4.3.2. We should note that the underlying topological space of the fiber product $X \times_S Y$ is not the fiber product of the underlying topological spaces. This can be seen from the tensor product of two fields, which can not be a field.

4.3.2 Base change of schemes

The functor $X \times_S Y$ is covariant in both of its variables, and this follows from the following commutative diagram:

$$\begin{tikzcd}
X \times Y \arrow[r, "f \times 1"] \arrow[d] & X' \times Y \arrow[r, "f' \times 1"] \arrow[d] & X'' \times Y \arrow[d] \\
X \arrow[r, "f"] & X' \arrow[r, "f'"] & X''
\end{tikzcd}$$

Proposition 4.3.4. *For any S-scheme X, the first (resp. second) projection $X \times_S S$ (resp. $S \times_S X$) is a functorial isomorphism of $X \times_S S$ (resp. $S \times_S X$) to X, with inverse isomorphism $(1_X, \varphi)_S$ (resp. $(\varphi, 1_X)_S$), where $\varphi : X \to S$ is the structural morphism. We can therefore write*

$$X \times_S S = S \times_S X = X.$$

Proof. It suffices to prove that the triple $(X, 1_X, \varphi)$ form a product of X and S, which is immediate. □

Corollary 4.3.5. *Let X and Y be S-schemes, $\varphi : X \to S$, $\psi : Y \to S$ the structural morphisms. If we identify canonically X with $X \times_S S$ and Y with $S \times_S Y$, the projections $X \times_S Y \to X$ and $X \times_S Y \to Y$ is identified respectively to $1_X \times \psi$ and $\varphi \times 1_Y$.*

We can define similarly the fiber product of S-schemes $X_1, \dots, X_n$, whose existence can be proved by induction on n, which is isomorphic to $(X_1 \times_S \cdots \times_S X_{n-1}) \times_S X_n$. The uniqueness of the product entails, as in any category, its properties of commutativity and associativity. If, for example, p_1, p_2, p_3 denotes the projections of $X_1 \times_S X_2 \times_S X_3$, and if we identify this scheme with $(X_1 \times_S X_2) \times_S X_3$, the projection in $X_1 \times_S X_2$ is identified with $(p_1, p_2)_S$.

Let S, S' be two schemes, $\varphi : S \to S'$ an morphism, making S' an S-scheme. For any S-scheme X, consider the product $X \times_S S'$, and let p and π' the projections to X and S' respectively. Through the morphism π', this product is an S'-scheme, which we may denoted by $X_{(S')}$ or $X_{(\varphi)}$, and the obtained scheme is called the **base change** of X from S to S', or the inverse image of

X via φ. We note that if π is the structural morphism of X and θ is the structural morphism of $X \times_S S'$, the following diagram is commutative:

$$\begin{tikzcd} X_{(S')} \arrow[r, "\pi'"] \arrow[d, "p"'] & S' \arrow[d, "\varphi"] \\ X \arrow[r, "\pi"] & S \end{tikzcd}$$

For any S-morphism $f : X \to Y$, we denote by $f_{(S')}$ the S'-morphism $f \times_S 1 : X_{(S')} \to Y_{(S')}$ and call it the inverse image of f by φ. The operation $X_{(S')}$ is clearly a covariant functor on X, from the category **Sch**$/S$ to **Sch**$/S'$.

Let S, S' be two affine schemes with rings A, A'; a morphism $S' \to S$ corresponds to a homomorphism $A \to A'$. If X is an S-scheme, we then denote by $X_{(A')}$ of $X \otimes_A A'$ by the S'-scheme $X_{(S')}$; if X is also affine with ring B, then $X_{(A')}$ is affine with ring $B_{(A')} = B \otimes_A A'$.

We point out that the scheme $X_{(S')}$ satisfies the following universal property: any S'-scheme T is an S-scheme via the morphism φ, and for any S-morphism $g : T \to X$ there exists a unique S'-morphism $f : T \to X_{(S')}$ such that $g = p \circ f$.

Proposition 4.3.6 (Transitivity). *Let $\varphi' : S'' \to S'$ and $\varphi : S' \to S$ be morphism of schemes. For any S-scheme X, there is a canonical functorial isomorphism of the S''-schemes $(X_{(\varphi)})_{(\varphi')}$ and $X_{(\varphi\circ\varphi')}$.*

Proof. In fact, let T be an S''-scheme, ψ its structural morphism, $g : T \to X$ an S-morphism (T is an S-scheme via the morphism $\varphi \circ \varphi' \circ \psi$). Since T is an S'-scheme with structural morphism $\varphi' \circ \psi$, we can write $g = p \circ g'$, where $g' : T \to X_{(\varphi)}$ is an S'-morphism. Then $g' = p' \circ g''$, where $g'' : T \to (X_{(\varphi)})_{\varphi'}$ is an S''-morphism:

$$\begin{tikzcd} (X_{(\varphi)})_{\varphi'} \arrow[r, "p'"] \arrow[d, "\pi''"] & X_{(\varphi)} \arrow[r, "p"] \arrow[d, "\pi'"] & X \arrow[d, "\pi"] \\ S'' \arrow[r, "\varphi'"] & S' \arrow[r, "\varphi"] & S \end{tikzcd}$$

The claim now follows from the definition of the universal property of $X_{(\varphi\circ\varphi')}$. ◻

The previous result can also be written as $(X_{(S')})_{(S'')} = X_{(S'')}$, if there is no risk of confusion. Moreover precisely, we have

$$(X \times_S S') \times_{S'} S'' = X \times_S S'';$$

the functorial of the isomorphism in Proposition 4.3.6 also shows the transitive of inverse image of morphisms:

$$(f_{(S')})_{(S'')} = f_{(S'')}$$

for any S-morphism $f : X \to Y$.

Corollary 4.3.7. *If X and Y are S-schemes, there exists a canonical functorial isomorphism of S'-schemes $X_{(S')} \times_{S'} Y_{(S')}$ and $(X \times_S Y)_{S'}$.*

Proof. In fact, we have, the following canonical isomorphisms:

$$(X \times_S S') \times_{S'} (Y \times_S S') = X \times_S (Y \times_S S') = (X \times_S Y) \times_S S'$$

where we use Proposition 4.3.6 and the associativity of fiber product. □

Again, the functorial isomorphism in Corollary 4.3.7 also gives the isomorphism

$$(u_{(S')}, v_{(S')})_{(S')} = ((u, v)_S)_{S'}$$

for any S-morphisms $u : T \to X$, $v : T \to Y$. In other words, the inverse image functor $X_{(S')}$ commutes on the formation of the products; note that it also commutes to the formation of coproducts.

Corollary 4.3.8. *Let Y be an S-scheme, $f : X \to Y$ a morphism making X a Y-scheme (and also an S-scheme). Then the scheme $X_{(S')}$ is canonically identified with the product $X \times_Y Y_{(S')}$, and the projection $X \times_Y Y_{(S')} \to Y_{(S')}$ is identified with $f_{(S')}$.*

Proof. Let $\psi : Y \to S$ be the structural morphism of Y; we have a commutative diagram

$$\begin{tikzcd}
X_{(S')} \arrow[r, "f_{(S')}"] \arrow[d] & Y_{(S')} \arrow[r, "\psi_{(S')}"] \arrow[d] & S' \arrow[d] \\
X \arrow[r, "f"] & Y \arrow[r, "\psi"] & S
\end{tikzcd}$$

Now $Y_{(S')}$ is identified with $S'_{(\psi)}$, and $X_{(S')}$ with $S'_{(\psi\circ f)}$, so by Proposition 4.3.6 and Corollary 4.3.4, we deduce the corollary. □

Example 4.3.9. Let A be a ring, X an A-scheme, and $\mathfrak{a}$ an ideal of A. Then $X_0 = X \otimes_A (A/\mathfrak{a})$ is an $(A/\mathfrak{a})$-scheme, called the scheme obtained from X by **reduction mod** $\mathfrak{a}$.

Proposition 4.3.10. *Let $f : X \to X'$, $g : Y \to Y'$ be two S-morphisms that are monomorphisms of schemes; then $f \times_S g$ is a monomorphism. In particular, for any extension $S' \to S$ of base scheme, the inverse image $f_{(S')} : X_{(S')} \to Y_{(S')}$ is a monomorphism.*

Proof. In fact, if p, q are the projections of $X \times_S Y$, and p', q' that of $X' \times_S Y'$:

$$\begin{tikzcd}
X \times_S Y \arrow[rr, "q"] \arrow[dr, "f\times_S g"] \arrow[dd, "p"'] & & Y \arrow[d, "g"] \\
& X' \times_S Y' \arrow[r, "q'"] \arrow[d, "p'"'] & Y' \arrow[d] \\
X \arrow[r, "f"] & X' \arrow[r] & S
\end{tikzcd}$$

then for any two morphisms $u, v : T \to X \times_S Y$, the relation $(f \times_S g) \circ u = (f \times_S g) \circ v$ implies

$$p' \circ (f \times_S g) \circ u = p' \circ (f \times_S g) \circ v$$

so $f \circ p \circ u = f \circ p \circ v$, and since f is a monomorphism, we conclude $p \circ u = p \circ v$. Similarly, since g is a monomorphism, we have $q \circ u = q \circ v$, whence $u = v$. □

For any S-morphism $f : S' \to X$, the morphism $f' = (f, 1_{S'})_S$ is then an S'-morphism from S' to $X' = X_{(S')}$ such that $p \circ f' = f$, $\pi' \circ f' = 1_{S'}$, which is called an ***S'*-section** of X':

$$\begin{tikzcd}
X' \arrow[r, "\pi'"] \arrow[d, "p"'] & S' \arrow[l, bend right, "f'"'] \arrow[d, "\varphi"] \arrow[dl, "f"] \\
X \arrow[r, "\pi"] & S
\end{tikzcd}$$

Conversely if f' is an S'-section, then $f = p \circ f'$ is an S-morphism $S' \to X$. We then deduce the following canonical correspondence

$$\mathrm{Hom}_S(S', X) \xrightarrow{\sim} \mathrm{Hom}_{S'}(S', X') \tag{3.2.1}$$

The morphism f' is called the **graph** of f, and denoted by Γ_f. A particularly important case is $S' = X$ and $f = 1_X$, where corresponding morphism $X \to X \times_S X$ is called the **diagonal morphism** of X, and denoted by Δ_X. Also, if $f : X \to Y$ is a morphism of schemes, we denote by Δ_f the diagonal map from X to $X \times_Y X$.

Example 4.3.11. Since any scheme X can be considered as a $\mathbb{Z}$-scheme, we can consider the X-sections of $X \otimes_{\mathbb{Z}} \mathbb{Z}[T]$ (where T is an indeterminate) corresponding to the $\mathbb{Z}$-morphisms $X \to \mathrm{Spec}(\mathbb{Z}[T])$. We claim that such X-sections correspond to sections of the structural sheaf $\mathscr{O}_X$ of X. In fact, the morphisms $X \to \mathrm{Spec}(\mathbb{Z}[T])$ correspond to ring homomorphisms $\mathbb{Z}[T] \to \Gamma(X, \mathscr{O}_X)$, which in turn are entirely determined by the image of T, and can be an arbitrary element of $\Gamma(X, \mathscr{O}_X)$, whence our assertion.

4.3.3 Tensor product of quasi-coherent sheaves

Let S be a scheme, X, Y be two S-schemes, $Z = X \times_S Y$, and p, q be the projections of Z to X and Y, respectively. Let $\mathscr{F}$ (resp. $\mathscr{G}$) be an $\mathscr{O}_X$-module (resp. an $\mathscr{O}_Y$-module). Then the tensor product $p^*(\mathscr{F}) \otimes_{\mathscr{O}_Z} q^*(\mathscr{G})$ is called the **tensor product of $\mathscr{F}$ and $\mathscr{G}$ over $\mathscr{O}_S$** (or **over *S***) and denoted by $\mathscr{F} \otimes_{\mathscr{O}_S} \mathscr{G}$ (or $\mathscr{F} \otimes_S \mathscr{G}$). More generaly, if $(X_i)_{1 \le i \le n}$ is a finite family of S-schemes and for each i, $\mathscr{F}_i$ is an $\mathscr{O}_X$-module, we can define the tensor product $\mathscr{F}_1 \otimes_S \cdots \otimes_S \mathscr{F}_n$ over the scheme $Z = X_1 \times_S \cdots \times_S X_n$. This is a quasi-coherent $\mathscr{O}_Z$-module if each $\mathscr{F}_i$ is quasi-coherent (Corollary 4.2.20), and is coherent if each $\mathscr{F}_i$ is coherent and Z is locally Noetherian in view of Proposition 1.4.26.

We note that if $X = Y = S$, the above definition coincide with the usual one of tensor product of $\mathscr{O}_S$-modules. Moreover, as $q^*(\mathscr{O}_Y) = \mathscr{O}_{X \times_S Y}$, the product $\mathscr{F} \otimes_S \mathscr{O}_Y$ is canonically identified with $p^*(\mathscr{F})$, and similarly $\mathscr{O}_X \otimes_S \mathscr{G}$ is identified with $q^*(\mathscr{G})$. In particular, if $Y = S$ and $f : X \to Y$ is the structural morphism, then $\mathscr{O}_X \otimes_Y \mathscr{G} = f^*(\mathscr{G})$: the ordinary tensor product and the inverse image therefore appears as a special case of the general tensor product. We also note that if X and Y are fixed, the operation $\mathscr{F} \otimes_S \mathscr{G}$ is a covariant bifunctor and is right exact on $\mathscr{F}$ and $\mathscr{G}$.

Proposition 4.3.12. *Let S, X, Y be affine schemes with rings A, B, C, respectively, where B, C are A-algebras. Let M (resp. B) be a B-module (resp. C-module) and $\mathscr{F} = \widetilde{M}$ (resp. $\mathscr{G} = \widetilde{N}$) the associated* *quasi-coherent sheaf. Then $\mathscr{F} \otimes_S \mathscr{G}$ is canonically isomorphic to the sheaf associated with the $(B \otimes_A C)$-module $M \otimes_A N$.*

Proof. In fact, in view of Proposition 4.1.13 and Proposition 4.1.9, $\mathscr{F} \otimes_S \mathscr{G}$ is canonically isomorphic to the sheaf associated with the $(B \otimes_A C)$-module

$$(M \otimes_B (B \otimes_A C)) \otimes_{B \otimes_A C} ((B \otimes_A C) \otimes_C N)$$

and due to the canonical isomorphisms between tensor products, the latter is isomorphic to $M \otimes_B (B \otimes_A C) \otimes_C N = (M \otimes_B B) \otimes_A (C \otimes_C N) = M \otimes_A N$. □

Proposition 4.3.13. *Let $f : T \to X$, $g : T \to Y$ be two S-morphisms, and $\mathscr{F}$ (resp. $\mathscr{G}$) be an $\mathscr{O}_X$-module (resp. $\mathscr{O}_Y$-module). Then we have $(f, g)_S^*(\mathscr{F} \otimes_S \mathscr{G}) = f^*(\mathscr{F}) \otimes_{\mathscr{O}_T} g^*(\mathscr{G})$.*

Proof. If p, q are the projections of $X \times_S Y$, the assertion follows from the relations $(f, g)_S^* \circ p^* = f^*$ and $(f, g)_S^* \circ q^* = g^*$, and the fact that the inverse image operation commutes with tensor products. □

Corollary 4.3.14. *Let $f : X \to X'$, $g : Y \to Y'$ be two S-schemes and $\mathscr{F}'$ (resp. $\mathscr{G}'$) be an $\mathscr{O}_{X'}$-module (resp. $\mathscr{O}_{Y'}$-module). Then $(f \times_S g)^*(\mathscr{F}' \otimes_S \mathscr{G}') = f^*(\mathscr{F}') \otimes_S g^*(\mathscr{G}')$.*

Proof. This follows from Proposition 4.3.13 and the fact that $f \times_S g = (f \circ p, g \circ q)_S$, where p, q are the projections of $X \times_S Y$. □

Corollary 4.3.15. *Let X, Y, Z be S-schemes and $\mathscr{F}$ (resp. $\mathscr{G}$, $\mathscr{H}$) be an $\mathscr{O}_X$-module (resp. $\mathscr{O}_Y$-module, $\mathscr{O}_Z$-module). Then the sheaf $\mathscr{F} \otimes_S \mathscr{G} \otimes_S \mathscr{H}$ is the inverse image of $(\mathscr{F} \otimes_S \mathscr{G} \otimes_S \mathscr{H})$ under the canonical isomorphism $X \times_S Y \times_S Z \to (X \times_S Y) \times_S Z$, and $\mathscr{G} \otimes_S \mathscr{F}$ is the inverse image of $\mathscr{F} \otimes_S \mathscr{G}$ under the canonical isomorphism $X \times_S \to Y \times_S X$.*

Proof. The first isomorphism is $(p_1, p_2)_S \times_S p_3$, where p_1, p_2, p_3 are the projections of $X \times_S Y \times_S Z$, and second one is similarly. □

Corollary 4.3.16. *If X is an S-scheme, any $\mathscr{O}_X$-module $\mathscr{F}$ is the inverse image of $\mathscr{F} \otimes_S \mathscr{O}_S$ under the canonical isomorphism from X to $X \times_S S$.*

Let X be an S-scheme, $\mathscr{F}$ be an $\mathscr{O}_X$-module, and $\varphi : S' \to S$ be an morphism. We denote by $\mathscr{F}_{(\varphi)}$ or $\mathscr{F}_{(S')}$ the sheaf $\mathscr{F} \otimes_S \mathscr{O}_{S'}$ over $X \times_S S' = X_{(\varphi)} = X_{(S')}$, so $\mathscr{F}_{(S')} = p^*(\mathscr{F})$, where p is the projection $X_{(S')} \to X$.

Proposition 4.3.17. *Let $\varphi' : S'' \to S'$ be a morphism. For any $\mathscr{O}_X$-module $\mathscr{F}$ over the S-scheme X, $(\mathscr{F}_{(\varphi)})_{(\varphi')}$ is the inverse image of $\mathscr{F}_{(\varphi\circ\varphi)}$ under the canonical isomorphism $(X_{(\varphi)})_{(\varphi')} \to X_{(\varphi\circ\varphi')}$.*

Proof. This follows from the definition and the associativity of base change, since $(\mathscr{F} \otimes_S \mathscr{O}_{S'}) \otimes_{S'} \mathscr{O}_{S''} = \mathscr{F} \otimes_S \mathscr{O}_{S''}$. □

Proposition 4.3.18. *Let Y be an S-scheme and $f : X \to Y$ be an S-morphism. For any $\mathscr{O}_Y$-module $\mathscr{G}$ and any morphism $S' \to S$, we have $(f_{(S')})^*(\mathscr{G}_{(S')}) = (f^*(\mathscr{G}))_{(S')}$.*

Proof. This follows from the diagram

$$\begin{tikzcd} X_{(S')} \arrow[r, "f_{(S')}"] \arrow[d] & Y_{(S')} \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

and the functoriality of inverse images. □

Corollary 4.3.19. *Let X, Y be S-schemes and $\mathscr{F}$ (resp. $\mathscr{G}$) be an $\mathscr{O}_X$-module (resp. $\mathscr{O}_Y$-module). For any morphism $S' \to S$, the inverse image of the sheaf $(\mathscr{F}_{(S')}) \otimes_{S'} (\mathscr{G}_{(S')})$ under the canonical isomorphism $(X \times_S Y)_{(S')} \cong (X_{(S')}) \times_{S'} (Y_{(S')})$ is equal to $(\mathscr{F} \otimes_S \mathscr{G})_{(S')}$.*

Proof. If p, q are the projections of $X \times_S Y$, the isomorphism is given by $(p_{(S')}, q_{(S')})'_S$, so the corollary follows from Proposition 4.3.13 and Corollary 4.3.18. □

Proposition 4.3.20. *Let X, Y be two S-schemes and $\mathscr{F}$ (resp. $\mathscr{G}$) be an $\mathscr{O}_X$-module (resp. $\mathscr{O}_Y$-module). Let p, q be the projections of $Z = X \times_S Y$, z be a point of Z, and put $x = p(z)$, $y = q(z)$. Then the stalk $(\mathscr{F} \otimes_S \mathscr{G})_z$ is isomorphic to $(\mathscr{F}_x \otimes_{\mathscr{O}_{X,x}} \mathscr{O}_{Z,z}) \otimes_{\mathscr{O}_{Z,z}} (\mathscr{G}_y \otimes_{\mathscr{O}_{Y,y}} \mathscr{O}_{Z,z}) = \mathscr{F}_x \otimes_{\mathscr{O}_{X,x}} \mathscr{O}_{Z,z} \otimes_{\mathscr{O}_{Y,y}} \mathscr{G}_y$.*

Proof. Since the question is local, we can reduce to the affine case, and the assertion follows from Proposition 4.1.9. □

Corollary 4.3.21. *With the notations in Proposition 4.3.20, if $\mathscr{F}$ and $\mathscr{G}$ are of finite type, then*

$$\operatorname{supp}(\mathscr{F} \otimes_S \mathscr{G}) = p^{-1}(\operatorname{supp}(\mathscr{F})) \cap q^{-1}(\operatorname{supp}(\mathscr{G})).$$

Proof. As $p^*(\mathscr{F})$ and $q^*(\mathscr{G})$ are of finite type over $\mathscr{O}_Z$, in view of Proposition 4.3.20 and Proposition **??**, we can reduce to the case $\mathscr{G} = \mathscr{O}_Y$, and the assertion then follows from the formula $\operatorname{supp}(p^{-1}(\mathscr{F})) = p^{-1}(\operatorname{supp}(\mathscr{F}))$. □

4.3.4 Scheme valued points

Let X be a scheme; for any scheme T, we denote by $X(T)$ the set $\mathrm{Hom}(T, X)$ of morphisms from T to X, and the elements of this set will be called **points of X with values in T**. The operation $T \mapsto X(T)$ is then a contravariant functor from the category of schemes to that of sets (in one word, we identify the scheme X with the induced functor h_X on **Sch**). Moreover, any morphism $g : X \to Y$ of schemes defines a natural transform $X(T) \to Y(T)$, which send $v \in X(T)$ to $g \circ v \in Y(T)$. The product of two S-schemes X and Y is then defined by the canonical isomorphism

$$(X \times_S Y)(T) \xrightarrow{\sim} X(T) \times_{S(T)} Y(T) \tag{3.4.1}$$

where the maps $X(T) \to S(T)$ and $Y(T) \to S(T)$ corresponds to the structural morphisms $X \to S$ and $Y \to S$.

If we are given a scheme S and consider S-schemes and S-morphisms, we denote by $X(T)_S$ the set $\mathrm{Hom}_S(T, X)$ of S-morphisms $T \to X$, and omit the index S if there is no risk of confusion.

We also say the elements of $X(T)_S$ are the (S-)points of the S-scheme X with values in the S-scheme T. In particular, an S-section of X is none other than a point of X with values in S. The formula (3.4.1) is then written as

$$(X \times_S Y)(T)_S = X(T)_S \times Y(T)_S;$$

more generally, if Z is an S-scheme, X, Y, T are Z-schemes, we have

$$(X \times_Z Y)(T)_S = X(T)_S \times_{Z(T)_S} Y(T)_S.$$

We remark that for any morphism $S' \to S$, the set $X(S')_S = \mathrm{Hom}_S(S', X)$ is identified with the set $\mathrm{Hom}_{S'}(S', X')$, where $X' = X \times_S S'$, which is the set of S'-sections of X'.

In T (resp. S) is an affine scheme with ring B (resp. A), we replace T (resp. S) by B (resp. A) in the above notations, and we then refer points of X with values in the ring B, or points os the A-scheme X with values in the A-algebra B for the elements $X(B)$ of $X(B)_A$, respectively. We also call $X(T)_A$ the set of points of the A-scheme X with values in the A-scheme T.

Consider in particular the case where T is a local scheme $\mathrm{Spec}(A)$, where A is a local ring; the elements $X(A)$ corresponds to local homomorphisms $\mathscr{O}_{X,x} \to A$ for $x \in X$ (Proposition 4.2.13); we say the point x of the underlying topological space X is the **locality** of the point of X with values in A to which it corresponds (of course, several distinct points of X with values in A can have the same locality), or that the point of X with values in A which corresponds to x is **localizaed in x**.

Even more particularly, a point of X with values in a field K correspond to a point $x \in X$ and a field extension $\kappa(x) \to K$. If X is an S-scheme, saying that $S' = \mathrm{Spec}(K)$ is an S-scheme means K is an extension of the residue field $\kappa(s)$ for an point $s \in S$; an element of $X(K)_S$, which is called **a point of X lying over s with values in K**, corresponds then to a $\kappa(s)$-homomorphism $\kappa(x) \to K$, where x is a point of the topological space X lying over s (hence $\kappa(x)$ is an extension of $\kappa(s)$).

The points of X with values in an algebraically closed field K are called **geometric points** of the scheme X;[2] the field K is called the **value field** of the geometric point. If X is an S-scheme and s is an point of S, a **geometric point of X lying over s** is then a geometric point of X localizaed in a point of X lying over s. We then have a map $X(K) \to X$, which send a geometric point with values in K to the point it locates.

If $S = \mathrm{Spec}(k)$ is the spectrum of a field k and X is an S-scheme, the S-points of X with values in k is identified with the S-sections of X, or with the points x of X such that the canonical homomorphism $k \to \kappa(x)$ is an isomorphism since only at such a point there exists a homomorphism $\kappa(x) \to k$ such that the composition $k \to \kappa(x) \to k$ is the identity. Such points are called the **rational points** over k of the k-scheme X. Note that if k' is an extension of k, the points of X with values in k' correspond to the points of $X' = X_{(k')}$ rational over k' (cf. (3.2.1)).

The example $X = \mathrm{Spec}(K)$, where K is an nontrivial extension of k, shows that there do not necessarily exist in X rational points on k, even if X is nonempty. Still assuming that X is a k-scheme. For any point $x \in X$, there is always an extensions k' of k for which there is a point

[2] This terminology is also sometimes used when K is only separably closed, but at that time we will explicitly clarify which convention we adapt.

x' of $X' = X_{(k')}$ rational over k' and whose image by the canonical projection $X' \to X$ is x: it suffices to take for k' an extension of $\kappa(x)$, the k-monomorphism $\kappa(x) \to k'$ giving the sought point x'. When we thus passes from a point x to a rational point $x' \in X'$ over k' and above x, we say that we "make x rational".

Proposition 4.3.22. *Let $S = \mathrm{Spec}(k)$ be the specturm of a field k, and X be an S-scheme. Then the points of X rational over k are closed in X.*

Proof. In fact, it suffices to show that the point x is closed in any open affine open set containing x, so we may assume that $X = \mathrm{Spec}(A)$ is affine. In this case, since the composition homomorphism $k \to A \to \kappa(x)$ is an isomorphism (we know that $\kappa(x) = k$), we conclude in particular that $A/\mathfrak{p}_x \to k$ is an integral extension, which implies that $A/\mathfrak{p}_x$ is a field (Proposition **??**). □

Proposition 4.3.23. *Let $(X_i)_{1\le i\le n}$ be S-schemes, s a point of X, and x_i a point of X_i lying over s for each i. For there to be a point y of the scheme $Y = X_1 \times_S \cdots \times_S X_n$ whose projections on X_i is x_i, it is necessary and sufficient that the x_i are over the same point s of S.*

Proof. This condition is clearly necessary. Now let s be an element of S and x_i a point of X_i lying over s. Then there exist $\kappa(s)$-homomorphisms $\kappa(x_i) \to K$ where K is a common field. The composition $\kappa(s) \to \kappa(x_i) \to K$ are all identical, so the morphisms $\mathrm{Spec}(K) \to X_i$ corresponding to $\kappa(x_i) \to K$ are S-morphisms, and we conclude that they define a unique morphism $\mathrm{Spec}(K) \to Y$. If y is the corresponding point of Y, it is clear that its projection into each of the X_i is x_i. □

In other words, if we denote by (X) the set underlying X, we see that we have a canonical surjective map $(X \times_S Y) \to (X) \times_{(S)} (Y)$; we have already pointed that this maps is not injective in general; that is, there can be multiple points distinct in $X \times_S Y$ having same projections to X and Y.

Corollary 4.3.24. *Let $f : X \to Y$ be an S-morphism, $f_{(S')} : X_{(S')} \to Y_{(S')}$ the S'-morphism induced by a base change $S' \to S$. Let p (resp. q) be the projection $X_{(S')} \to X$ (resp. $Y_{(S')} \to Y$); for any subset V of X, we have*

$$q^{-1}(f(M)) = f_{(S')}(p^{-1}(M)).$$

Proof. By Corollary 4.3.8, $X_{(S')}$ is identified with the product $X \times_Y Y_{(S')}$ and we have the following commutative diagram

$$\begin{tikzcd} X_{(S')} \arrow[r, "f_{(S')}"] \arrow[d, "p"'] & Y_{(S')} \arrow[d, "q"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

By Proposition 4.3.23, the relation $q(y') = f(x)$ for $x \in V$, $y' \in Y_{(S')}$ is equivalent to the existence of a point $x' \in X_{(S')}$ such that $p(x') = x$ and $f_{(S')}(x') = y'$, whence the corollary. □

Proposition 4.3.25. *Let X, Y be S-schemes and $x \in X$, $y \in Y$ two points that are over the same point $s \in S$. The set of points $X \times_S Y$ having projections x and y is in canonical one-to-one correspondence with the set of types of the composition extension of $\kappa(x)$ and $\kappa(y)$, considered as extensions of $\kappa(s)$.*

Proof. Let p (resp. q) be the projection of $X \times_S Y$ to X (resp. Y) and let E be the subspace $p^{-1}(x) \cap q^{-1}(y)$ of the underlying topological space of $X \times_S Y$. We first note that since x and y are lying over s, the morphisms $\mathrm{Spec}(\kappa(x)) \to S$ and $\mathrm{Spec}(\kappa(y)) \to S$ factor through $\mathrm{Spec}(\kappa(s))$:

$$\begin{tikzcd} & \mathrm{Spec}(\kappa(x)) \arrow[d] & \\ \mathrm{Spec}(\kappa(y)) \arrow[r] & \mathrm{Spec}(\kappa(s)) \arrow[r] & S \end{tikzcd}$$

since $\mathrm{Spec}(\kappa(s)) \to S$ is a monomorphism by Corollary 4.2.16, it follows immediately that we have

$$P = \mathrm{Spec}(\kappa(x)) \times_S \mathrm{Spec}(\kappa(y)) = \mathrm{Spec}(\kappa(x)) \times_{\mathrm{Spec}(\kappa(s))} \mathrm{Spec}(\kappa(y)) = \mathrm{Spec}(\kappa(x) \otimes_{\kappa(s)} \kappa(y)).$$

Let $i : \mathrm{Spec}(\kappa(x)) \to X$ and $j : \mathrm{Spec}(\kappa(y)) \to Y$ be the canonical morphisms, we put $\alpha = i \times_S j : P \to E$ to be the map on the underlying topological space. On the other hand, any point $z \in E$ defines two $\kappa(s)$-homomorphisms $\kappa(x) \to \kappa(z)$ and $\kappa(y) \to \kappa(z)$, hence a $\kappa(s)$-homomorphism $\kappa(x) \otimes_{\kappa(s)} \kappa(y) \to \kappa(z)$, which corresponds to a morphism $\mathrm{Spec}(\kappa(z)) \to P$; we take $\beta(z)$ to be the image of this morphism, which defines a map $\beta : E \to P$.

To verify that $\alpha \circ \beta$ and $\beta \circ \alpha$ are identity maps, we need the following commutative diagram

$$\begin{tikzcd} \mathrm{Spec}(\kappa(z)) \arrow[dr] \arrow[drr, bend left] \arrow[ddr, bend right] & & & \\ & P \arrow[r] \arrow[d] \arrow[dr, "\alpha"] & \mathrm{Spec}(\kappa(y)) \arrow[dr, "j"] & \\ & \mathrm{Spec}(\kappa(x)) \arrow[dr, "i"'] & E \arrow[r] \arrow[d] & Y \arrow[d] \\ & & X \arrow[r] & S \end{tikzcd}$$

for $z \in E$. By the uniqueness part of the universal property of fiber products, the canonical morphism $\mathrm{Spec}(\kappa(z)) \to E$ induced by $\mathrm{Spec}(\kappa(z)) \to X$ and $\mathrm{Spec}(\kappa(z)) \to Y$ is given by the composition $\mathrm{Spec}(\kappa(z)) \to P \to E$, and also equal to the canonical morphism $\mathrm{Spec}(\kappa(z)) \to E$ of the scheme $X \times_S Y$ at z (Corollary 4.2.15). This means the image of $\beta(z)$ under α is exactly the image of the canonical morphism $\mathrm{Spec}(\kappa(z)) \to E$, which is just z; this shows $\alpha \circ \beta = 1_E$. As for $\beta \circ \alpha$, we just note that if $z = \alpha(\mathfrak{p})$ for some $\mathfrak{p} \in P$ (a prime ideal), then the morphism α induces a field extensins $\kappa(z) \to \kappa(\mathfrak{p})$, which corresponds to morphism $\mathrm{Spec}(\kappa(\mathfrak{p})) \to \mathrm{Spec}(\kappa(z))$. Again by the uniqueness part of the fiber product P, we conclude that the canonical morphism $\mathrm{Spec}(\kappa(\mathfrak{p})) \to P$ factors through $\mathrm{Spec}(\kappa(z))$, which means $\beta(z) = \mathfrak{p}$, so $\beta \circ \alpha = 1_P$. Finally, we recall that the set P corresponds to composition fields of $\kappa(x)$ and $\kappa(y)$ over $\kappa(s)$. □

4.3.5 Surjective morphisms

Let $\mathcal{P}$ be a property for morphisms of schemes. We consider the following conditions:

(i) If $f : X \to X'$ and $g : Y \to Y'$ are S-morphisms satisfying $\mathcal{P}$, then $f \times_S g$ also satisfies $\mathcal{P}$.

(ii) If $f : X \to Y$ is an S-morphism satisfying $\mathcal{P}$ and $S' \to S$ is a morphism, then $f_{(S')} : X_{(S')} \to Y_{(S')}$ also satisfies $\mathcal{P}$.

Since $f_{(S')} = f \times_S 1_{S'}$, we see if any identity morphism satisfies $\mathcal{P}$, then (i) implies (ii). On the other hand, since $f \times_S g$ is the following composition

$$X \times_S Y \xrightarrow{f \times 1_Y} X' \times_S Y \xrightarrow{1_{X'} \times g} X' \times_S Y'$$

it is clear that if the composition of two morphisms satisfying $\mathcal{P}$ still satisfies $\mathcal{P}$, then (ii) implies (i) (in this case we say $\mathcal{P}$ is stable under composition). In general, a property $\mathcal{P}$ is called **stable under base change** if it satisfies the condition (ii). For example Proposition 4.3.10 just says that being a monomorphism is stable under base change. On the other hand, if $\mathcal{P}$ is an arbitrary property of morphisms, we say a morphism $f : X \to S$ **satisfies** $\mathcal{P}$ **universally** (or is **universally** $\mathcal{P}$), if for any morphism $S' \to S$ the inverse image $f_{(S')}$ satisfies $\mathcal{P}$.

Our first application of the above definition is that surjectivity is stable under base change:

Proposition 4.3.26. *Surjective morphisms of schemes are stable under base change.*

Proof. Note that it is clear that surjectivity is stable under composition, in fact we have the both conditions (i) and (ii) described above. But condition (ii) follows from Corollary 4.3.24 by setting $V = X$. □

Proposition 4.3.27. *For a morphism $f : X \to Y$ of schemes to be surjective, it is necessary and sufficient that for any field K and any morphism $\mathrm{Spec}(K) \to Y$, there exists an extension K' of K and a morphism $\mathrm{Spec}(K') \to X$ fitting into the following diagram*

$$\begin{tikzcd} \mathrm{Spec}(K') \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \mathrm{Spec}(K) \arrow[r] & Y \end{tikzcd}$$

Proof. The condition is sufficient, since for any $y \in Y$, we can apply the canonical morphism $\mathrm{Spec}(\kappa(y)) \to Y$ to get a morphism $\mathrm{Spec}(K) \to X$, which gives a inverse image of y in X. Conversely, suppose that f is surjective, and let $y \in T$ be the image of $\mathrm{Spec}(K)$ in Y; there exists $x \in X$ such that $f(x) = y$. Consider the monomorphism $\kappa(y) \to \kappa(x)$ corresponding to f, and take an extension K' of $\kappa(y)$ containing $\kappa(x)$ and K; the morphism $\mathrm{Spec}(K') \to X$ corresponding to $\kappa(x) \to K'$ then satisfies the requirement. □

Corollary 4.3.28. *For a morphism $f : X \to Y$ to be surjective, it is necessary and sufficient that, for any field K, there exist an algebraically closed extension K' of K such that the map $X(K') \to Y(K')$ corresponding to f is surjective.*

Proof. In view of Proposition 4.3.27, this condition is sufficient. Conversely, suppose that f is surjective and let K be a field. If p is the characteristic of K, let us take for K' an algebraically closed extension of K having over the prime field P a transcendence basis of strictly larger cardinality to the cardinals of all the transcendence bases on P of the residual fields of X and Y having characteristic p. It then remains to see, with the same notations as in Proposition 4.3.27, that any monomorphism $u : \kappa(y) \to K'$ factors into

$$\kappa(y) \xrightarrow{w} \kappa(x) \xrightarrow{v} K'$$

where $w = f^x$. Now, let L a purely transcendental extension of P contained in $\kappa(y)$ and over which $\kappa(y)$ is algebraic; if B is a transcendence basis of L over P, we can complete $w(B)$ into a transcendence basis B' of $\kappa(x)$ on P, and then (due to the assumption made on the trancendence bases of K') define a monomorphism $v_1 : P(B') \to K'$ such that $v_1 \circ (w|_L)$ coincides with $u|_L$. There is also an isomorphism $v_2 = u \circ w^{-1}$ from $w(\kappa(y))$ to $u(\kappa(y))$ such that v_2 and v_1 coincide in $w(L)$; as $w(\kappa(y))$ and $P(B' - w(B))$ are linearly disjoint on $w(L)$, we can extend v_1 and v_2 into a monomorphism v_0 of $M = P(B')(w(\kappa(y)))$ in K'; as K' is algebraically closed and $\kappa(x)$ is algebraic over M, we can finally extend v_0 into the monomorphism $v : \kappa(x) \to K'$, which completes the proof. □

4.3.6 Radical morphisms

Proposition 4.3.29. *Let $f : X \to Y$ be a morphism of schemes. The following conditions are equivalent:*

(i) f is universally injective.

(ii) The map f is injective and for any $x \in X$, the extension $f^x : \kappa(f(x)) \to \kappa(x)$ is purely inseparable.

(iii) For any field K, the map $X(K) \to Y(K)$ corresponding to f is injective.

(iv) For any field K, there exists an algebraically closed extension K' of K such that the map $X(K') \to X(K')$ corresponding to f is injective.

(v) The diagonal morphism $\Delta_f : X \to X \times_Y X$ is surjective.

*The morphism f is called **radical** if it satisfies the above equivalent conditions.*

Proof. It is clear that (i) implies f is injective; on the other hand, if $\kappa(x)$ is not a purely inseparable extension of $\kappa(f(x))$, there exist two distinct $\kappa(f(x))$-monomorphisms $\kappa(x) \to K$ into an algebraically closed extension K of $\kappa(x)$; hence we get two distinct morphisms g_1, g_2 of $\mathrm{Spec}(K)$ to X, whose compositions $f \circ g_1$, $f \circ g_2$ equal to the same morphism $\mathrm{Spec}(K) \to Y$. If we set $Y' = \mathrm{Spec}(K)$, there then would be two distinct Y'-sections of $X_{(Y')}$; since K is a field, the Y'-sections of $X_{(Y')}$ correspond one-to-one to their images (the rational points of $X_{(Y')}$ over K), so $f_{(Y')} : X_{(Y')} \to Y'$ would not be injective, contrary to the assumption.

To show that (ii) implies (iii), we note that by Corollary 4.2.15, (iii) signifies that for any $y \in Y$ and a monomorphism $\kappa(y) \to K$ to a field K, there do not exist two distinct $\kappa(y)$-monomorphisms $\kappa(x_1) \to K$, $\kappa(x_2) \to K$, where x_1, x_2 are both lying over y. Now (ii) implis that if we have two such monomorphisms, they come from the same point x since f is injective; moreover, since $\kappa(x)$ is a purely inseparable extension of $\kappa(y)$, the two monomorphisms $\kappa(x) \to K$ are necessarily equal.

It is clear that (iii) implies (iv). Conversely, suppose that (iv) holds; let K be a field and K' be an algebraically closed extension of K; then the diagram

$$\begin{tikzcd} X(K) \arrow[r, "\alpha"] \arrow[d, "\varphi"] & Y(K) \arrow[d, "\varphi'"] \\ X(K') \arrow[r, "\alpha'"] & Y(K') \end{tikzcd} \tag{3.6.1}$$

is commutative. Since the homomorphism $K \to K'$ is injective, φ is injective by Corollary 4.2.15, and by hypothesis we can choose K' such that α' is also injective. Then α is injective, which shows (iii).

To see that (iv) and (v) are equivalent, we note that, for the morphism Δ_f to be surjective, it is necessary and sufficient that, in view of Corollary 4.3.28, for any field K, there exists an algebraically closed extension K' of K such that the diagonal map

$$X(K') \to (X \times_Y X)(K') = X(K') \times_{Y(K')} X(K')$$

corresponding to Δ_f is surjective. But by the definition of this fiber product, this signifies that the map $X(K') \to Y(K')$ is injective, whence our claim.

Finally, we prove that (iii) implies (i). If (iii) is satisfied, then for any base change $Y' \to Y$, the map

$$(X \times_Y Y')(K) \to Y'(K)$$

is still injective, as we immediately verify by noting that $(X \times_Y Y')(K) = X(K) \times_{Y(K)} Y'(K)$ and that $X(K) \to Y(K)$ is injective. Therefore, it suffices to prove that if $X(K) \to Y(K)$ is injective for any field K, then f is injective. Now if x_1 and x_2 are two points of X such that $f(x_1) = f(x_2) = y$, there then exists a field extension K of $\kappa(y)$ and $\kappa(y)$-monomorphisms $\kappa(x_1) \to K$, $\kappa(x_2) \to K$; the corresponding morphisms u_1, u_2 of $\mathrm{Spec}(K)$ to X are then such that $f \circ u_1 = f \circ u_2$, and by hypothesis this implies $u_1 = u_2$, so $x_1 = x_2$. □

Remark 4.3.3. We then obtain examples of injective morphisms (and even bijective) of schemes but not universally injective: it suffices to take a morphism $\mathrm{Spec}(K) \to \mathrm{Spec}(k)$, where K is a separable extension of k distinct from k.

Corollary 4.3.30. *A monomorphism of schemes $f : X \to Y$ is radical. In particular, if A is a ring, S is a multiplicative subset of A, then the canonical morphism $\mathrm{Spec}(S^{-1}A) \to \mathrm{Spec}(A)$ is radical.*

Proof. The first assertion follows from Proposition 4.3.29(iii), and the second from the fact that $\mathrm{Spec}(S^{-1}A) \to \mathrm{Spec}(A)$ is a monomorphism. □

Corollary 4.3.31. *Let $f : X \to Y$ be a radical morphism, $g : Y' \to Y$ a morphism, and $X' = X_{(Y')}$. Then the radical morphism $f_{(Y')}$ is a bijection from the underlying space X' to $g^{-1}(f(X))$. Moreover, for any field K, the set $X'(K)$ is identified with the inverse image in $Y'(K)$ under the map $Y'(K) \to Y(K)$ (corresponding to g) of the subset $X(K)$ of $Y(K)$.*

Proof. The first assertion follows from Proposition 4.3.29(ii) and Corollary 4.3.24; the second one follows from the commutative diagram (3.6.1). □

Proposition 4.3.32. *Let $f : X \to Y$ and $g : Y \to Z$ be two morphism of schemes.*

(a) If f and g are radical, so is $g \circ f$.

(b) Conversely, if $g \circ f$ is radical, so is f.

Proof. It suffices to apply the functors X, Y, Z on any field K, and use the characterization of Proposition 4.3.29(iii); the verification boils down to set-theoretic issues, which are straightforward. □

Proposition 4.3.33. *Radical morphisms are stable under base changes. In particular, if $f : X \to X'$ and $g : y \to Y'$ are radical S-morphisms, then so is $f \times_S g$.*

Proof. Since radical is equivalently to universally injective, the first assertion is clear. The second one follows from the first one since radical morphisms are stable under composition by Proposition 4.3.32. ◻

4.3.7 Fibers of morphisms

Proposition 4.3.34. *Let $f : X \to Y$ be a morphism, y be a point of Y, and $\mathfrak{a}_y$ be an ideal of $\mathscr{O}_{Y,y}$. Put $Y' = \mathrm{Spec}(\mathscr{O}_{Y,y}/\mathfrak{a}_y)$, $X' = X \times_Y Y'$, and let $p : X' \to X$ be the canonical projection. Then p is a homeomorphism from X' onto the subspace $f^{-1}(Y')$ of X (where we identify Y' are a subspace of Y, cf. Proposition 4.2.10). Moreover, for any $x' \in X'$, the homomorphism $p^{\#}_{x'} : \mathscr{O}_{X,p(x')} \to \mathscr{O}_{X',x'}$ is surjective with kernel $\mathfrak{a}_y\mathscr{O}_{X,x}$.*

Proof. The morphism $\mathrm{Spec}(\mathscr{O}_{Y,y}/\mathfrak{a}_y) \to Y$ is radical (Proposition 4.3.29), so we conclude from Proposition 4.3.31 that p identifies the space $X' = X \times_Y \mathrm{Spec}(\mathscr{O}_{Y,y}/\mathfrak{a}_y)$ with $f^{-1}(Y')$. It remains to show that p is a homeomorphism and identify its morphism on stalks. Since this question is local, we may assume that $X = \mathrm{Spec}(B)$, $Y = \mathrm{Spec}(A)$, where B is an A-algebra. Then the morphism p corresponds to the homomorphism $1 \otimes \rho : B \to B \otimes_A A'$, where $\rho : A \to A' = A_{\mathfrak{p}_y}/\mathfrak{a}_y$ is the canonical homomorphism. Now any element of $B \otimes_A A'$ is of the form

$$\sum_i b_i \otimes \rho(a_i)/\rho(s) = \rho\Big(\sum_i a_i b_i \otimes 1\Big)(1 \otimes \rho(s))^{-1}$$

where $s \notin \mathfrak{p}_y$, so we can apply Proposition **??**. The assertion on the homomorphism $p^{\#}_{x'}$ also follows from the equality. ◻

We will mainly use Proposition 4.3.34 for the case $\mathfrak{a}_y = \mathfrak{m}_y$ is the maximal ideal of $\mathscr{O}_{Y,y}$. If there is no confusion, we denote by X_y the $\kappa(y)$-scheme obtained by transporting of scheme structure from $X' = X \otimes_S \kappa(s)$ to $f^{-1}(y)$ via the projection p, and this is always the scheme that will be involved when we speak of the **fiber** $f^{-1}(y)$ of the morphism f as a scheme.

Let X, Y be two S-schemes and $f : X \to Y$ an S-morphism. By the transitivity of base change, we have the canonical isomorphism

$$X_s = X \times_Y Y_s$$

for any $s \in S$; the morphism $f_s : X_s \to Y_s$ induced by f by the base change $Y_s \to Y$ is such that, for any $y \in Y_s$, the fiber $f_s^{-1}(y)$ is identified with the $\kappa(y)$-scheme $f^{-1}(y)$, since the residue field of Y_s at y is the same as that of Y at y, in view of Proposition 4.3.34

Proposition 4.3.35 (Transitivity of Fibers). *Let $f : X \to Y$, $g : Y' \to Y$ be two morphisms; put $X' = X_{(Y')}$ and $f' = f_{(Y')}$. For any $y' \in Y'$, if $y = g(y')$, then the scheme $X'_{y'}$ is canonically isomorphic to $X_y \otimes_{\kappa(y)} \kappa(y')$.*

Proof. In fact, by the transitivity of base change, we have canonical isomorphisms

$$(X \otimes_Y \kappa(y)) \otimes_{\kappa(y)} \kappa(y') \cong X \times_Y \mathrm{Spec}(\kappa(y')) \cong (X \times_Y Y') \otimes_{Y'} \kappa(y')$$

The left one is $X_y \otimes_{\kappa(y)} \kappa(y')$, and the right one is $X'_{y'} \otimes_{Y'} \kappa(y')$, so our assertion follows. □

Proposition 4.3.36. *Let $f : X \to Y$ be a monomorphism of schemes. Then for each $y \in Y$, the fiber X_y is a $\kappa(y)$-scheme which is either empty or $\kappa(y)$-isomorphic to* $\mathrm{Spec}(\kappa(y))$.

Proof. By Proposition 4.3.34, X_y is reduced to a point, and hence affine. The morphism $f_y : X_y \to \mathrm{Spec}(\kappa(y))$ induced by f under base change is still a monomorphism by Proposition 4.3.10. If A is the ring of X_y, this signifies that the homomorphism $A \otimes_{\kappa(y)} A \to A$ which maps $a \times a'$ to aa' is bijective. This clearly implies that $A = \kappa(y)$, since otherwise there exist an element $a \in A$ not contained in $\kappa(y)$ and the image $a \otimes 1$ and $1 \otimes a$ are distinct, but both mapped to a. □

Proposition 4.3.37. *Let $f : X \to Y$ be an S-morphism of S-schemes, $g : S' \to S$ a surjective morphism, and $f' = f_{(S')} : X' = X_{(S')} \to Y' = Y_{(S')}$. Consider the following properties:*

(a) surjective;

(b) injective;

(c) dominant;

(d) finite fiber (as sets);

Then if $\mathcal{P}$ denotes one of the properties above and if f' satisfies $\mathcal{P}$, then so does f.

Proof. Since the projection $Y' \to Y$ is surjective by Proposition 4.3.26, we can, by virtue of Corollary 4.3.8, limiting ourselves to the case where $Y = S$, $Y' = S'$. For any $y' \in Y'$, let $y = g(y')$; we have the transitivity relation $X'_{y'} \cong X_y \otimes_{\kappa(y)} \kappa(y')$ (Proposition 4.3.35). Since the morphism $\mathrm{Spec}(\kappa(y')) \to \mathrm{Spec}(\kappa(y))$ is surjective, so is the projection $X'_{y'} \to X_y$ (Proposition 4.3.26). Thus, if $X'_{y'}$ is nonempty (resp. a singleton, resp. a finite set), the same holds for X_y. Since $S' \to S$ is surjective, this proves (a), (b), and (d). On the other hand, if f' is dominant, so is the composition $g \circ f' = f \circ g'$; but since $g' : X' \to X$ is surjective by Proposition 4.3.26, this implies f is dominant. □

4.3.8 Universally open and universally closed morphisms

Following the usual terminology, we say a morphism $f : X \to Y$ is **universally open** (resp. **universally closed**, resp. a **universal embedding,** resp. a **universal homeomorphism**) if for any base change $Y' \to Y$, the morphism $f_{(Y')} : X_{(Y')} \to Y'$ is open (resp. closed, resp. an embedding, resp. a homeomorphism).

Proposition 4.3.38.

(i) The composition of two universally open morphisms (resp. universally closed morphisms, resp. two universal embeddings, resp. two universal homeomorphisms) is universally open (resp. universally closed, resp. a universal embedding, resp. a universal homeomorphism).

(ii) If $f : X \to X'$, $g : Y \to Y'$ are two universally open (resp. universally closed, resp. two universal embeddings, resp. two universal homeomorphisms) S-morphisms, so is the product $f \times_S g$.

(iii) *If $f : X \to Y$ is a universally open (resp. universally closed, resp. a universal embedding, resp. a universal homeomorphism) S-morphism, so is $f_{(S')} : X_{(S')} \to Y_{(S')}$ for any base change $S' \to S$.*

(iv) *Let $f : X \to Y$, $g : Y \to Z$ be two morphism such that f is surjective; if $g \circ f$ is universally open (resp. universally closed, resp. a universal embedding, resp. a universal homeomorphism), so is g.*

(v) *Let (U_α) be an open cover of Y. For a morphism $f : X \to Y$ to be universally open (resp. universally closed, resp. a universal embedding, resp. a universal homeomorphism), it is necessary and sufficient that, for each α, the restriction $f_\alpha : f^{-1}(U_\alpha) \to U_\alpha$ is universally open (resp. closed).*

Proof. The assertion (i) follows from definiton, and so does (iii). We have already remareked that (i) and (iii) together imply (ii), since identity morphisms satisfies all the properties mentioned above. To prove (iv), we note that for any morphism $Z' \to Z$, the morphism $f_{(Z')} \to X_{(Z')}$ is surjective, so it suffices to prove that if $g \circ f$ is open (resp. closed, resp. an embedding, resp. a homeomorphism) and f is surjective, then so is g. For the case where $g \circ f$ is open or closed, the fact that g is open or closed result from Proposition **??**; for the other two cases, we can limit ourselves to the case $g(f(X)) = g(Y) = Z$, so that $g \circ f$ is a homeomorphism from X to Z. As f is surjective, g is necessarily bijective, and as it is open by the first two cases already shown, g is then a homeomorphism from Y to Z.

Finally, the necessity in (v) follows from (iii) and Proposition 4.3.2. Conversely, suppose the condition in (v) and let $g : Y' \to Y$ be a morphism; then $g^{-1}(U_\alpha) = U'_\alpha$ form an open cover of Y' and if $f' = f_{(Y')}$, the restriction $f'^{-1}(U'_\alpha) \to U'_\alpha$ of f' is none other than $(f_\alpha)_{(U'_\alpha)}$ (Proposition 4.3.2). We can then reduce to proving that f is open (resp. closed, resp. an embedding, resp. a homeomorphism) if each f_α is, which is immediate. □

We recall that openness of a map is a local property, i.e., a map $f : X \to Y$ is open if and only if it is open at every point of X. Simialrly, the property of universally open morphisms is also local. To justify this, we define a morphism $f : X \to Y$ is **universally open at a point** $x \in X$ if for any base change $Y' \to Y$ the morphism $f_{(Y')} : X_{(Y')} \to Y_{(Y')}$ is open at any point x' of X' lying over x.

Proposition 4.3.39. *Let $f : X \to Y$, $g : Y \to Z$ be two morphisms of schemes, x a point of X, and $y = f(x)$.*

(a) *If f is universally open at x and g is universally open at y, then $g \circ f$ is universally open at x. Conversely, if $g \circ f$ is universally open at x, then g is universally open at y.*

(b) *If $f : X \to Y$ is an S-morphism universally open at a point $x \in X$, then for any base change $S' \to S$, the morphism $f_{(S')} : X_{(S')} \to Y_{(S')}$ is universally open at any point of $X_{(S')}$ lying over x.*

Proof. Assertion (b) is an immediate concequence of the definition of universally openness at a point and transitivity of base change. Also, it follows from Corollary 4.3.24 that to prove (a), we may drop the "universally" condition and prove the assertion for openness, which follows from Proposition **??**. □

Recall that openness is a local property for maps: a continuous map $f : X \to Y$ is open if and only if it is open at every point of X. Inspired by this, for a morphism $f : X \to Y$ of schemes, we may say f is universally open at a point $x \in X$ if for any base change $Y' \to Y$, the morphism $f_{(Y')} : X' \to Y'$ is open at any point $x' \in X'$ lying over x (where $X' = X_{(Y')}$).

Proposition 4.3.40. *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of schemes, x be a point of X, and $y = f(x)$.*

(a) If f is universally open at a point x and g is universally open at y, then $g \circ f$ is universally open at x. Conversely, if $g \circ f$ is universally open at x, g is universally open at y.

(b) If f is an S-morphism universally open at a point x, then for any base change $S' \to S$, the morphism $f_{(S')} : X_{(S')} \to Y_{(S')}$ is universally open at any point of $X_{(S')}$ lying over x.

Proof. In fact, (b) is a immediate concequence of (a) and the transitivity of base changes. To prove (a), it suffices to drop the "universal" condition and the result then follows from Proposition **??**. ◻

Proposition 4.3.41. *Let X, Y be two schemes, $f : X \to Y$ be a morphism, and x be a point of X. Let (Y_i) be a locally finite covering of Y by closed subschemes, and suppose that for each i such that $f(x) \in Y_i$, the restriction $f_i : f^{-1}(Y_i) \to Y_i$ of f is an open morphism (resp. universally open) at the point x. Then f is open (resp. universally open) at the point x.*

Proof. The assertion about openness is just Proposition **??**. For the universal part, consider a morphism $g : Y' \to Y$ and in Y' the closed subschemes $Y_i' = g^{-1}(Y_i)$ (Proposition 4.4.14), which underlying spaces form a locally finite covering of Y'. If $f' = f_{(Y')} : X_{(Y')} \to Y'$ is the base change of f, the restriction $f_i' : f'^{-1}(Y_i) \to Y_i'$ of f' equals to $(f_i)_{(Y')}$, so we can apply Proposition **??** to f'. ◻

Proposition 4.3.42. *Let $f : X \to Y$ be a morphism of schemes, $(X_i)_{1 \le i \le n}$ (resp. $(Y_i)_{1 \le i \le n}$) be a finite family of closed subschemes of X (resp. Y), and $j_i : X_i \to X$ (resp. $h_i : Y_i \to Y$) be the canonical injections. Suppose that $(X_i)_{1 \le i \le n}$ covers X and for each i there exists a morphism $f_i : X_i \to Y_i$ fitting into the following diagram*

$$\begin{tikzcd} X_i \arrow[r, "f_i"] \arrow[d, "j_i"'] & Y_i \arrow[d, "h_i"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

Then for f to be closed (resp. universally closed), it is necessary and sufficient that each f_i is.

Proof. If f is closed, then each f_i is closed since Y_i are closed in Y. Conversely, if the f_i are closed, for any closed subset F of X, we have $f_i(F \cap X_i) = f(F \cap X_i)$ and it closed in Y_i, hence in Y, and as $f(F)$ is the union of $f(F \cap X_i)$, it is therefore closed in Y.

For the case of universally closed morphisms, the condition is necessary because j_i is a closed immersion (hence universally closed, cf. Proposition 4.4.12), and if f is universally closed, so is $f \circ j_i = h_i \circ f_i$. But as h_i is a closed immersion, hence separated (Proposition 4.5.25), it follows that f_i is universally closed (Proposition 4.5.23).

Conversely, suppose that each f_i is universally closed, and consider the scheme Z that is the coproduct of that X_i. Let $u : Z \to X$ be the induced morphism by the j_i's. The restriction of $f \circ u$ to X_i is equal to $f \circ j_i = h_i \circ f$, hence universally closed (Proposition 4.4.12 and Proposition 4.3.38(i)); we then deduce from Corollary 4.3.3 that $f \circ u$ is universally closed. But since u is surjective by hypotheses, we conclude that f is universally closed (Proposition 4.3.38(iv)). □

Remark 4.3.4. If X only have finitely many irreducible components, then we deduce from Proposition 4.3.42 that, to verify a morphism $f : X \to Y$ is closed (resp. universally closed), we can reduce ourselves to doing it for dominant morphisms of integral schemes. In fact, let $(X_i)_{1 \le i \le n}$ be the reduced subschemes of X with underlying spaces the irreducible components of X (Proposition 4.4.40), which are then integral. Let Y_i be the unique reduced closed subscheme of Y with underlying space $\overline{f(X_i)}$ (Proposition 4.4.40), which iss irreducible (Proposition **??**). If $j_i : X_i \to X$ (resp. $h_i : Y_i \to Y$) is the canonical injection, there then exists a dominant morphism $f_i : X_i \to Y_i$ such that $f \circ g_i = h_i \circ f_i$ (Proposition 4.4.44); we are then in the case of Proposition 4.3.42, so f is closed (resp. universally closed) if and only if each f_i is.

Proposition 4.3.43. *Let X, Y be two S-schemes and $f : X \to Y$ be an S-morphism. For f to be universally closed, it suffices that, for any base change $S' \to S$ where $S' = S \otimes_{\mathbb{Z}} \mathbb{Z}[(T_\lambda)_{\lambda \in I}]$ (denoted by $S[(T_\lambda)_{\lambda \in I}]$), the morphism $f_{(S')} : X_{(S')} \to Y_{(S')}$ is closed.*

Proof. We note that if (S_α) is an open cover of S and Y_α is the inverse image of S_α in Y, it suffices to prove that the restricted morphism $f_\alpha : f^{-1}(Y_\alpha) \to Y_\alpha$ is closed (Proposition 4.3.38(v)). Now the inverse image of S_α in S' is $S_\alpha[(T_\lambda)_{\lambda \in I}] = S'_\alpha$ and $(f_\alpha)_{(S')}$ is the restriction $(f_{(S')})^{-1}(Y'_\alpha) \to Y'_\alpha$ of $f_{(S')}$, where Y'_α is the inverse image of S'_α in $Y_{(S')}$. If the proposition is proved for S_α and f_α, it is then true for S and f. We can then assume that S is affine.

Now let (U_β) be an open covering of Y; for f to be universally closed, it suffices to prove that $f_\beta : f^{-1}(U_\beta) \to U_\beta$ is universally closed for each β (Proposition 4.3.38(v)). Again, the morphism $(f_\beta)_{(S')}$ is the restriction $(f_{(S')})^{-1}(U'_\beta) \to U'_\beta$ of $f_{(S')}$, where $U'_\beta = U_\beta \times_S S'$ is the inverse image of U_β in $Y_{(S')}$. If the proposition is proved for U_β and f_β, it then holds for Y and f. Therefore, we can further assume that Y is affine.

Let us first show that if $f_{(S')}$ is closed for any base change $S' \to S$, then f is universally closed. In fact, any Y-scheme Y' can be considered as an S-scheme, and as the morphism $Y \to S$ is separated (recall that Y and S are assumed to be affine), $X \times_Y Y'$ (resp. $Y \times_Y Y' = Y'$) is identified with a closed subscheme of $X \times_S Y'$ (resp. $Y \times_S Y'$) (Proposition 4.4.11). In the following commutative diagram

$$\begin{tikzcd} X \times_Y Y' \arrow[r, "f \times 1_{Y'}"] \arrow[d] & Y \times_Y Y' = Y' \arrow[d] \\ X \times_S Y' \arrow[r, "f_{(Y')}"] & Y \times_S Y \end{tikzcd}$$

the vertical morphisms are closed immersions, so if $f_{(Y')}$ is a closed morphism, so is $f \times 1_{Y'}$.

It remains to prove that $f_{(S')}$ is closed for arbitrary base change $S' \to S$ if this is ture for $S' = S[(T_\lambda)]$. Now by hypotheses S is affine, and if (S'_γ) is an open covering of S', we see in the same manner as before that for $f_{(S')}$ to be closed, it suffices to prove that $f_{(S'_\gamma)}$ is closed. We can then assume S' to be affine. If $S = \mathrm{Spec}(A)$, we have $S' = \mathrm{Spec}(A')$, where A' is an A-algebra. Let $(t_\lambda)_{\lambda\in I}$ be a generator for A', which means there is a surjective A-homomorphism $A[(T_\lambda)] \to A'$ identifying A' with $A[(T_\lambda)]/\mathfrak{b}$, where $\mathfrak{b}$ is an ideal. If $S'' = \mathrm{Spec}(A[(T_\lambda)])$, S' is then a closed subscheme of S'', and $X_{(S')}$ (resp. $Y_{(S')}$) is identified with a closed subscheme of $X_{(S'')}$ (resp. $Y_{(S'')}$). The morphism $f_{(S')}$ is the restriction of $f_{(S'')}$ on $X_{(S')}$, and since $f_{(S'')}$ is closed by hypotheses, we conclude that $f_{(S')}$ is closed. This completes the proof. □

4.4 Subschemes and immersions

4.4.1 Subschemes

Proposition 4.4.1. *Let X be a scheme and $\mathscr{I}$ be a quasi-coherent ideal of $\mathscr{O}_X$. The support Y of the sheaf $\mathscr{O}_X/\mathscr{I}$ is then closed, and if $\mathscr{O}_Y$ is the sheaf induced on Y by $\mathscr{O}_Y/\mathscr{I}$, $(Y, \mathscr{O}_Y)$ is a scheme.*

Proof. Since the problem is local, it suffices to consider the affine case and show that Y is closed and $(X, \mathscr{O}_Y)$ is an affine scheme. In fact, if $X = \mathrm{Spec}(A)$, we have $\mathscr{O}_X = \widetilde{A}$ and $\mathscr{I} = \widetilde{\mathfrak{a}}$, where $\mathfrak{a}$ is an ideal of A (Proposition 4.1.20); Y is then equal to the closed subset $V(\mathfrak{a})$ of X and is identified with the spectrum of $B = A/\mathfrak{a}$ (Proposition **??**). Moreover, if $\rho : A \to B = A/\mathfrak{a}$ is the canonical homomorphism, the direct image ${}^a\rho_*(\widetilde{B})$ is canonically identified with the sheaf $\widetilde{A}/\widetilde{\mathfrak{a}} = \mathscr{O}_X/\mathscr{I}$ (Proposition 4.1.6 and Proposition 4.1.11). These complete the proof. □

We say $(Y, \mathscr{O}_Y)$ is the **subscheme of** $(X, \mathscr{O}_X)$ **defined by the quasi-coherent ideal** $\mathscr{I}$. More generally, we say a locally ringed space $(Y, \mathscr{O}_Y)$ is a **subscheme** of a scheme $(X, \mathscr{O}_X)$ if Y is a locally closed subspace of X and if U denote the largest open subset of X containing Y such that Y is open in U (in other words, the complement of $\overline{Y} - Y$, so $U = (X - \overline{Y}) \cup Y$), then $(Y, \mathscr{O}_Y)$ is a subscheme of $(U, \mathscr{O}_X|_U)$ defined by a quasi-coherent ideal of $\mathscr{O}_X|_U$. We say the subscheme $(Y, \mathscr{O}_Y)$ of $(X, \mathscr{O}_X)$ is **closed** if Y is closed in X (in this case $U = X$). It follows from this definition and Proposition 4.4.1 that closed subschemes of X are in one-to-one correspondence with quasi-coherent ideals of $\mathscr{O}_X$, since if two such ideals $\mathscr{I}$, $\mathscr{J}$ have the same support (closed) Y and the sheaf induced by $\mathscr{O}_X/\mathscr{I}$ and $\mathscr{O}_X/\mathscr{J}$ on Y are identical, then $\mathscr{I} = \mathscr{J}$.

Let $(Y, \mathscr{O}_Y)$ be a subscheme of X, U the largest open subset of X containing Y such that Y is closed in U, V an open subset of X contained in U; then $V \cap Y$ is closed in V. Moreover, if Y is defined by the quasi-coherent ideal $\mathscr{I}$ of $\mathscr{O}_X|_U$, then $\mathscr{I}|_V$ is a quasi-coherent ideal of $\mathscr{O}_X|_V$, and it is immediate that the scheme induced by Y over $Y \cap V$ is the closed subscheme of V defined by the ideal $\mathscr{I}|_V$.

In particular, the scheme induced by X over an open subset of X is a subscheme of X; such schemes are called **open subschemes** of X. One should note that a subscheme of X can have the underlying space being an open set U of X without being induced on this open subset by X: it is induced over U by X only if it is defined by the ideal 0 of $\mathscr{O}_X|_U$, and there are in general quasi-coherent ideals $\mathscr{I}$ of $\mathscr{O}_X|_U$ such that $(\mathscr{O}_X|_U)/\mathscr{I}$ have support U but is nonzero.

Proposition 4.4.2. *Let $(Y, \mathscr{O}_Y)$ be a locally ringed space such that Y is a subspace of X and there exists a covering (V_α) of Y by open sets of X such that for each α, $Y \cap V_\alpha$ is closed in V_α and the locally ringed space $(Y_\cap V_\alpha, \mathscr{O}_Y|_{Y\cap V_\alpha})$ is a closed subscheme of the scheme induced over V_α by X. Then $(Y, \mathscr{O}_Y)$ is a subscheme of X.*

Proof. The hypothesis implies that Y is locally closed in X and the largest open set U containing Y and in which Y is closed contains the V_α. We are then reduced to the case $U = X$ and Y is closed in X. We define a quasi-coherent ideal $\mathscr{I}$ of $\mathscr{O}_X$ by letting $\mathscr{I}_{V_\alpha}$ to be the sheaf of ideal of $\mathscr{O}_X|_{V_\alpha}$ that defines the closed subscheme $(Y \cap V_\alpha, \mathscr{O}_Y|_{Y\cap V_\alpha})$ and for any open set W of X not meeting Y, $\mathscr{I}_W = \mathscr{O}_X|_W$. It is immediately verified that there exists a unique sheaf of ideals $\mathscr{I}$ satisfying these conditions and that it defines the closed subscheme $(Y, \mathscr{O}_Y)$. □

Proposition 4.4.3. *A (closed) subscheme of a (closed) subscheme of X is canonically identified with a (closed) subscheme of X.*

Proof. Since a locally closed subset of a locally closed subset of X is still locally closed in X, it is clear by Proposition 4.4.2 that the question is local and we may assume that X is affine. The proposition then follows from the identification $A/\mathfrak{b}$ and $(A/\mathfrak{a})/(\mathfrak{b}/\mathfrak{a})$ where $\mathfrak{a}, \mathfrak{b}$ are ideals of the ring A such that $\mathfrak{a} \subseteq \mathfrak{b}$. □

Let Y be a subscheme of X and denote by $\iota : Y \to X$ the canonical injection of the underlying space; we know the inverse image $\iota^*(\mathscr{O}_X)$ is the restriction $\mathscr{O}_X|_Y$. If for any $y \in Y$, we denote by ι_y the canonical homomorphism $\mathscr{O}_{X,y} \to \mathscr{O}_{Y,y}$, these homomorphisms are then the restrictions to the stalks of $\mathscr{O}_X$ at the points of Y of a surjective homomorphism of sheaves of rings $\iota^\# : \mathscr{O}_X|_Y \to \mathscr{O}_Y$: it suffices indeed to check it locally on Y, so we can assume that X is affine and Y is a closed subscheme; in this case, if $\mathscr{I}$ is the quasi-coherent ideal of $\mathscr{O}_X$ which defines Y, the ι_y's are nothing but the restrictions to the stalks of the canonical homomorphism $\mathscr{O}_X|_Y \to (\mathscr{O}_X/\mathscr{I})|_Y$. We have therefore defined a monomorphism $j_Y = (\iota, \iota^\#)$ of locally ringed spaces, which is called the **canonical injection morphism**. If $f : X \to Z$ is another morphism of schemes, we say the composition

$$Y \xrightarrow{j_Y} X \xrightarrow{f} Z$$

is the **restriction** of f to the subscheme Y of X.

A subscheme Y of a scheme X is considered as an X-scheme via the canonical injection $j_Y : Y \to X$. Two subschemes Y, Z of X that are X-isomorphic are then necessary identical. In fact, if $u : Y \to Z$ is an X-isomorphism, the relation $j_Y = j_Z \circ u$ shows the underlying spaces of Y and Z are identical. Moreover if $U \supseteq Y$ is an open subset of X such that $Y = Z$ are closed in U, and $\mathscr{I}, \mathscr{J}$ are the ideals of U defining respectively Y and Z, for each $x \in Y$ we then have a commutative diagram

$$\begin{tikzcd}
\mathscr{O}_{X,x}/\mathscr{J}_x \arrow[rr, "u_x^\#"] & & \mathscr{O}_{X,x}/\mathscr{I}_x \\
& \mathscr{O}_{X,x} \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Since u is an isomorphism, this implies $\mathscr{I}_x = \mathscr{J}_x$, so $Y = Z$ and $u = 1_Y$.

According to the general definitions, we say a morphism $f : Z \to X$ is **dominated by the canonical injection** $j_Y : Y \to X$ of a subscheme Y of X, if f factors through j_Y:

$$Z \xrightarrow{g} Y \xrightarrow{j_Y} X$$

where g is a morphism of schemes. Since j_Y is a monomorphism, the morphism g is unique.

Proposition 4.4.4. *Let Y be a subscheme of a scheme X and $j : Y \to X$ be the cannical injection. For a morphism $f : Z \to X$ to be dominated by the injection j, it is necessary and sufficient that $f(Z) \subseteq Y$ and for each $z \in Z$, the homomorphism $f_z^\# : \mathscr{O}_{X,f(z)} \to \mathscr{O}_{Z,z}$ factors through $\mathscr{O}_{Y,f(z)}$ (or equivalently, the kernel of $\mathscr{O}_{X,f(z)} \to \mathscr{O}_{Z,z}$ is contained in that of $\mathscr{O}_{X,f(z)} \to \mathscr{O}_{Y,f(z)}$).*

Proof. The condition is cleraly necessary. For the sufficiency, we may assume that Y is a closed subscheme of X, and replace X by an open subset U such that Y is closed in U. Y is then defined by a quasi-coherent ideal $\mathscr{I}$ of $\mathscr{O}_X$. Let $\mathscr{J}$ be the kernel of the homomorphism $f^*(\mathscr{O}_X) \to \mathscr{O}_Z$. In view of the properties of the functor f^*, the hypothesis implies tht for each $z \in Z$ we have $(f^*(\mathscr{I}))_z \subseteq \mathscr{J}_z$, and concequently $f^*(\mathscr{I}) \subseteq \mathscr{J}$. Therefore the homomorphism $f^*(\mathscr{O}_X) \to \mathscr{O}_Z$ factors into

$$f^*(\mathscr{O}_X) \longrightarrow f^*(\mathscr{O}_X)/f^*(\mathscr{I}) = f^*(\mathscr{O}_X/\mathscr{I}) \xrightarrow{\theta_z} \mathscr{O}_Z$$

the first arrow being the canonical homomorphism. Let g be the continuous map of Z in Y coincide with f; it is clear that $g^*(\mathscr{O}_Y) = f^*(\mathscr{O}_X/\mathscr{I})$; on the other hand, for any $z \in Z$, θ_z is obviously a local homomorphism, so $(g,\theta) : Z \to Y$ is a morphism of schemes which satisfies $f = j \circ g$, whence the proposition. □

Corollary 4.4.5. *Let Y and Z be subschemes of X. For the canonical injection $Z \to X$ to be dominated by the injection $Y \to X$, it is necessary and sufficient that Z is a subscheme of Y.*

Due to this corollary, for two subschemes Y, Z of X we write $Y \le Z$ if Y is a subscheme of Z. It is clear that this defines an order relation on the set of subschemes of X, since two subschemes Y and Z are identical if $Y \le Z$ and $Z \le Y$.

4.4.2 Immersions of schemes

We say a morphism $f : Y \to X$ is an **immersion** (resp. a **closed immersion**, resp. an **open immersion**) if it is factorized into

$$Y \xrightarrow{g} Z \xrightarrow{j} X$$

where g is an isomorphism, Z is a subscheme (resp. a closed subscheme, resp. an open subscheme) of X, and j is the canonical injection. The subscheme Z and the isomorphism g are then uniquely determined since two X-isomorphic subschemes are identical. We say $f = i \circ g$ is the **canonical factorization** of the immersion f, and the subscheme Z and the isomorphism g is called **associated** with f. It is clera that an immersion is a monomorphism of schemes (since j is a monomorphism), and a fortiori a radical morphism (Proposition 4.3.30). Also, it is clear from Proposition 4.4.3 that the composition of two immersions (resp. two open immersions, resp. two closed immersions) is an immersion (resp. an open immersion, resp. a closed immersion).

Again, one should note that an immersion $f : Y \to X$ such that $f(Y)$ is a open subset of X, in other words which is an open morphism, is not necessarily an open immersion.

Example 4.4.6. Let X be an affine scheme. Then from the definition of closed subschemes, we see that for a morphism $f : Y \to X$ to be a closed immersion, it is necessary and sufficient that Y is an affine scheme and $\Gamma(f) : \Gamma(X,\mathscr{O}_X) \to \Gamma(Y,\mathscr{O}_Y)$ is surjective.

Lemma 4.4.7. *Let $f : Y \to X$ be a morphism of schemes such that $f(Y)$ is closed and f is a homeomorphism onto $f(Y)$. Then for each point $x \in X$, there exists an affine open neighborhood U of x such that $f^{-1}(U)$ is an affine open of Y.*

Proof. Since $f(Y)$ is closed in X, the lemma is trivial if $x \notin f(Y)$, since it suffices to choose an affine open neighborhood of x disjoint from $f(Y)$. If $x \in f(Y)$, there exists a unique point $y \in Y$ such that $f(y) = x$. Let W be an affine open neighborhood of x in X and V an affine open neighborhood of y in Y such that $f(V) \subseteq W$. By hypothesis $f(V)$ is an open neighborhood of x in $f(Y)$, so there exists an open neighborhood $U' \subseteq W$ of x such that $U' \cap f(Y) = f(V)$. Let U be an open neighborhood of x contained in U' and is of the form $D(s)$, where $s \in A = \Gamma(W, \mathscr{O}_X)$ (recall that W is chosen to be affine); in view of Proposition **??**(b), $f^{-1}(U) \subseteq V$ is of the form $D(t)$, where t is the image of s in $B = \Gamma(V, \mathscr{O}_Y|_V)$, hence proves the lemma. □

Lemma 4.4.8. *Let $f : Y \to X$ be a morphism of schemes and (U_λ) be an affine open covering of X such that for each λ, $f^{-1}(U_\lambda)$ is an affine open of Y. Then for any quasi-coherent $\mathscr{O}_Y$-module $\mathscr{F}$, the direct image $f_*(\mathscr{F})$ is a quasi-coherent $\mathscr{O}_X$-module.*

Proof. For each λ, put $V_\lambda = f^{-1}(U_\lambda)$, and let $f_\lambda : V_\lambda \to U_\lambda$ be the restriction of f to V_λ. Then the restriction $f_*(\mathscr{F})$ to U_λ is equal to $(f_\lambda)_*(\mathscr{F}_\lambda)$, where $\mathscr{F}_\lambda = \mathscr{F}|_{U_\lambda}$. But since U_λ and V_λ are affine by hypothesis, we see $(f_\lambda)_*(\mathscr{F}_\lambda)$ is quasi-coherent by Proposition 4.1.11. This proves the lemma. □

Proposition 4.4.9. *Let $f : Y \to X$ be a morphism of schemes.*

(a) For f to be an open immersion, it is necessary and sufficient that f is a homeomorphism onto an open subset of X and for each $y \in Y$, the homomorphism $f_y^\# : \mathscr{O}_{X,f(y)} \to \mathscr{O}_{Y,y}$ is bijective.

(b) For f to be an immersion (resp. a closed immersion), it is necessary and sufficient that f is a homeomorphism onto a locally closed (resp. closed) subset of X and for each $y \in Y$, the homomorphism $f_y^\# : \mathscr{O}_{X,f(y)} \to \mathscr{O}_{Y,y}$ is surjective.

Proof. In the two cases, the conditions are clearly necessary, so we only need to prove the sufficiency. If the conditions in (a) holds, it is clear that $f^\#$ induces an isomorphism of $\mathscr{O}_Y$ to $f^*(\mathscr{O}_X)$, and $f^*(\mathscr{O}_X)$ is the sheaf defined by the transport by structure by means of f^* from the induced sheaf $\mathscr{O}_X|_{f(Y)}$, hence the conclusion.

Suppose then the conditions in (b) holds. Let U_0 be the largest open set of X such that $Z = f(Y)$ is closed in U_0; by replacing X by the subscheme induced by X over U_0, we may assume that $Z = f(Y)$ is closed in X. By Lemma 4.4.7 and Lemma 4.4.8, the sheaf $f_*(\mathscr{O}_Y)$ is a quasi-coherent $\mathscr{O}_X$-module. We have an exact sequence

$$0 \longrightarrow \mathscr{I} \longrightarrow \mathscr{O}_X \xrightarrow{f^\#} f_*(\mathscr{O}_Y) \longrightarrow 0$$

where two terms are quasi-coherent $\mathscr{O}_X$-modules; we then deduce that (Corollary 4.2.20) $\mathscr{I}$ is a quasi-coherent ideal of $\mathscr{O}_X$, and $f^\#$ factors into

$$\mathscr{O}_X \longrightarrow \mathscr{O}_X/\mathscr{I} \xrightarrow{\theta} f_*(\mathscr{O}_Y)$$

where the first arrow is the canonical homomorphism and θ is an isomorphism. If $\mathscr{O}_Z = (\mathscr{O}_X/\mathscr{I})|_Z$, $(Z, \mathscr{O}_Z)$ is then a closed subscheme of $(X, \mathscr{O}_X)$ and f factors through the canonical injection $j_Z : Z \to X$. Since the corresponding morphism $Y \to Z$ is just (f_0, θ_0), where f_0 is the map f considered as a homeomorphism from Y to Z and θ_0 is the restriction of θ to $\mathscr{O}_Z$, it is clear that f is a closed immersion, which completes the proof. □

Remark 4.4.1. It may happen that $f : Y \to X$ is a closed immersion and for all $y \in Y$, $f_y^\# : \mathscr{O}_{X,f(y)} \to \mathscr{O}_{Y,y}$ is bijective, without f being an open immersion (that is, $f(Y)$ is not necessarily opened in X). For example, let $X = \operatorname{Spec}(A)$ be an affine scheme and $x \in X$ be a closed point of X that is not isolated. Then if $Y = \operatorname{Spec}(A/\mathfrak{m}_x)$, the canonical morphism $Y \to X$ is a closed immersion satisfying the desired property, since the subspace $\{x\}$ is not open in X.

Corollary 4.4.10. *Let $f : Y \to X$ be a morphism of schemes.*

(a) Let (V_λ) be a covering of $f(Y)$ by open subsets of X. Then for f to be an immersion (an open immersion), it is necessary and sufficient that for each λ, the restriction $f^{-1}(V_\lambda) \to V_\lambda$ of f is an immersion (an open immersion).

(b) Let (U_λ) be an open covering of X. Then for f to be a closed immersion, it is necessary and sufficient that for each λ, the restriction $f^{-1}(U_\lambda) \to U_\lambda$ of f is a closed immersion.

Proof. In the case (a), $f_y^\#$ is surjective (resp. bijective) for every point $y \in Y$, and in case (b) it is surjective for every point $y \in Y$; it then suffices to verify that in case (a) f is a homeomorphism of Y onto a locally closed (resp. open) subset of X and in case (b), a homeomorphism onto a closed subset of X. Now, the hypothesis imply that f is clearly injective and maps each neighborhood of $y \in Y$ to a neighborhood of $f(y)$ in $f(Y)$. In case (a), $f(Y) \cap V_\lambda$ is locally closed (resp. open) in the union of the V_λ, and a fortiori in X; in case (b), $f(Y) \cap U_\lambda$ is closed in U_λ, hence closed in X since $X = \bigcup_\lambda U_\lambda$. □

Remark 4.4.2. We can generalize the notions of this part to any ringed spaces. We define a **ringed subspace** of a ringed space $(X, \mathscr{O}_X)$ to be a ringed space of the form $(Y, (\mathscr{O}_U/\mathscr{I})|_Y)$, where U is an open of X, $\mathscr{I}$ an ideal of $\mathscr{O}_U$ and Y the support of the sheaf of rings $\mathscr{O}_U/\mathscr{I}$ (support which is no longer necessarily closed in U). We can define the canonical injection $Y \to X$, and the definitions and results of Proposition 4.4.4 are valid without modification. We then define the notion of immersion (resp. of closed immersion) in the same manner. The characterizations of open (resp. closed) immersions given in Proposition 4.4.9 still hold (observing that if $f(Y)$ is closed in X, $f_*(\mathscr{O}_Y)$ has support $f(Y)$). The result of Proposition 4.4.9 can therefore be stated by saying that if a scheme Y is a ringed subspace of a scheme X, then Y is a subscheme of X.

4.4.3 Inverse image of subschemes

Proposition 4.4.11. *Let $f : X' \to X$ and $g : Y' \to Y$ be two S-morphisms. Let p, q be the projections of $X \times_S Y$.*

(a) If f and g are immersions (resp. open immersions, resp. closed immersions), then $f \times_S g$ is an immersion (resp. an open immersion, resp. a closed immersion).

(b) *If X' (resp. Y') is identified with a subscheme of X (resp. Y) via the immersion f (resp. g), then $f \times_S g$ identifies the underlying space of $X' \times_S Y'$ with the subspace $p^{-1}(X') \cap q^{-1}(Y')$ of the underlying space of $X \times_S Y$.*

Proof. We can restrict ourselves to the case where X' and Y' are subschemas and f and g are the canonical injection morphisms. The proposition has already been established for the subschemes induced on the open sets (Corollary 4.3.2); as any subscheme is a closed subscheme of an open scheme, we are reduced in case X' and Y' are closed subschemes.

We can further assume that S is affine. In fact, let (S_λ) be an affine open cover of S, φ and ψ be the structural morphisms of X and Y, and let $X_\lambda = \varphi^{-1}(S_\lambda)$, $Y_\lambda = \psi^{-1}(S_\lambda)$. The restriction X'_λ (resp. Y'_λ) of X' (resp Y') to $X_\lambda \cap X'$ (resp. $Y_\lambda \cap Y'$) is a closed subscheme of X_λ (resp. Y_λ), the schemes X_λ, Y_λ, X'_λ, Y'_λ can then be considered as S_λ-schemes and the product $X_\lambda \times_S Y_\lambda$ and $X_\lambda \times_{S\lambda} Y_\lambda$ (resp. $X'_\lambda \times_S Y'_\lambda$ and $X'_\lambda \times_{S_\lambda} Y'_\lambda$) are identified (Corollary 4.3.2). If the proposition is true when S is affine, the restriction of $f \times_S g$ to the $X'_\lambda \times_S Y'_\lambda$ will therefore be an immersion. As the product $X'_\lambda \times_S Y'_\mu$ (resp. $X_\lambda \times_S Y_\mu$) is identified with $(X'_\lambda \cap X'_\mu) \times_S (Y'_\lambda \cap Y'_\mu)$ (resp. $(X_\lambda \cap X_\mu) \times_S (Y_\lambda \cap Y_\mu)$), the restriction of $f \times_S g$ to $X'_\lambda \times_S Y'_\mu$ is also an immersion; it follows from Corollary 4.4.10 that $f \times_S g$ is an immersion.

Secondly, let's prove that we can also assume that X and Y are affine. In fact, let (U_i) (resp. (V_j)) be an affine open cover of X (resp. Y), and let X'_i (resp. Y'_j) be the restriction of X' (resp. Y') to $X' \cap U_i$ (resp. $Y' \cap V_j$), which is a closed subscheme of U_i (resp. V_j). Then $U_i \times_S V_j$ is identified with the restriction of $X \times_S Y$ to $p^{-1}(U_i) \cap q^{-1}(V_j)$ by Corollary 4.3.2, and similarly, if $p' : X' \times_S Y' \to X'$ and $q' : X' \times_S Y' \to Y'$ are the canonical projections, the product $X'_i \times_S Y'_j$ is identified with the restriction of $X' \times_S Y'$ to $p'^{-1}(X'_i) \cap q'^{-1}(Y'_j)$. Put $h = f \times_S g$, then since $X'_i = f^{-1}(U_i)$, $Y'_j = g^{-1}(V_j)$, we have

$$p'^{-1}(X'_i) \cap q'^{-1}(Y'_j) = h^{-1}(p^{-1}(U_i) \cap q^{-1}(V_j)) = h^{-1}(U_i \times_S V_j).$$

Again, by the same reasoning and using Corollary 4.4.10, we can show that h is an immersion.

Suppose then that X, Y, and S are affine, with rings B, C, and A, respectively. Then B and C are A-algebras, X' and Y' are affine subschemes with rings quotients B', C' of B and C, respectively. Moreover, f and g are induced by ring homomorphisms $\rho : B \to B'$ and $\sigma : C \to C'$. With these, we see $X \times_S Y$ (resp. $X' \times_S Y'$) is the affine scheme with ring $B \otimes_A C$ (resp. $B' \otimes_A C'$), and $f \times_S g$ corresponds to the ring homomorphism $\rho \otimes \sigma : B \otimes_A C \to B' \otimes_A C'$. Since this homomorphism is surjective, $f \times_S g$ is an immersion. Moreover, if $\mathfrak{b}$ (resp. $\mathfrak{c}$) is the kernel of ρ (resp. σ), the kernel of $\rho \otimes \sigma$ is $u(\mathfrak{b}) + v(\mathfrak{c})$, where u (resp. v) is the homormorphism $b \mapsto b \otimes 1$ (resp. $c \mapsto 1 \otimes c$). As p corresponds to the ring homomorphism u and q corresponds to v, this kernel corresponds, in the spectrum $\operatorname{Spec}(B \otimes_A C)$, to the closed subset $p^{-1}(X') \cap q^{-1}(Y')$, which proves the demonstration. □

Corollary 4.4.12. *If $f : X \to Y$ is an immersion (resp. an open immersion, resp. a closed immersion) and an S-morphism, then $f_{(S')}$ is an immersion (resp. an open immersion, resp. a closed immersion) for any extension $S' \to S$ of base schemes.*

Proof. This follows from the observation that the identity morphism is an immersion (resp. an open immersion, resp. a closed immersion). □

Corollary 4.4.13. *An immersion (resp. a closed immersion, resp. an open immersion) is a universally embedding (resp. universally closed, resp. universally open).*

Proposition 4.4.14. *Let $f : X \to Y$ be a morphism, Y' a subscheme (resp. a closed subscheme, resp. an open subscheme) of Y, and $j : Y' \to Y$ the canonical injection.*

(a) The projection $p : X \times_Y Y' \to X$ is an immersion (resp. a closed immersion, resp. an open immersion), and the subscheme of X associated with p has underlying space $f^{-1}(Y')$. Moreover, if j' is the canonical injection of this subscheme, for a morphism $h : Z \to X$ to be such that $f \circ h : Z \to Y$ is dominated by j, it is necessary and sufficient that h is dominated by j'.

(b) If Z is a closed subscheme defined by a quasi-coherent ideal $\mathscr{K}$ of $\mathscr{O}_Y$, the inverse image of Z by f is defined by the quasi-coherent ideal $f^*(\mathscr{K})\mathscr{O}_X$.

Proof. As $p = 1_X \times_Y j$, the first assertion in (a) follows from Proposition 4.4.11. The second one is a special case of Corollary 4.3.31. Finally, if we have $f \circ h = j \circ h'$, where $h' : Z \to Y'$ is a morphism, it follows from the universal property of product that we have $h = p \circ u$, where $u : Z \to X \times_Y Y'$ is a morphism, whence assertion (a).

To prove (b), since the question is local on X and Y, we may assume that X and Y are affine. It then suffices to note that if A is an B-algebra and $\mathfrak{b}$ is an ideal of B, we have $A \otimes_B (B/\mathfrak{b}) = A/\mathfrak{b}A$, and apply Proposition 4.1.13. □

We say the subscheme of X thus defined is the **inverse image** of the subscheme Y' of Y by the morphism f. We say the morphism $f \times 1_{Y'} : f^{-1}(Y') \to Y'$ is the restriction of f to $f^{-1}(Y')$. When we speak of $f^{-1}(Y')$ as a subscheme of X, it is always this subscheme that will be involved.

Example 4.4.15. If the scheme $f^{-1}(Y')$ and X are identical, $j' : f^{-1}(Y') \to X$ is then the identity and any morphism $h : Z \to X$ is dominated by j'; hence the morphism $f : X \to Y$ factors into

$$X \xrightarrow{g} Y' \xrightarrow{j} Y$$

Example 4.4.16. If y is a closed point of Y and $Y' = \operatorname{Spec}(\kappa(y))$ is the smallest closed subscheme of Y having $\{y\}$ as underlying space, the closed subscheme $f^{-1}(Y')$ is then canonically isomorphic to the **fiber** $f^{-1}(y)$.

Corollary 4.4.17. *Retain the hypotheses of Proposition 4.4.14(b). Let X' be a closed subscheme of X defined by a quasi-coherent ideal $\mathscr{J}$ of $\mathscr{O}_X$ and $i : X' \to X$ be the canonical injection. For the restriction $f \circ i$ of f to X' is dominated by the injection $j : Y' \to Y$, it is necessary and sufficient that $f^*(\mathscr{K})\mathscr{O}_X \subseteq \mathscr{J}$.*

Proof. This follows from Proposition 4.4.14(b) and (a). □

Corollary 4.4.18. *Let $f : X \to Y$, $g : Y \to Z$ be two morphisms, and $h = g \circ f$ be their composition. For any subscheme Z' of Z, the subscheme $f^{-1}(g^{-1}(Z'))$ and $h^{-1}(Z')$ of X are identical.*

Proof. This follows from the transitivity of products and Proposition 4.4.14. □

Corollary 4.4.19. *Let X', X'' be two subschemes of X and $j' : X' \to X$, $j'' : X'' \to X$ be the canonical injections. Then $j'^{-1}(X'')$ and $j''^{-1}(X')$ are both equal to the infimum $\inf(X', X'')$ of X' and X'' for the ordered relation on subschemes, and is canonically isomorphic to $X' \times_S X''$.*

Proof. This follows from Proposition 4.4.14, Corollary 4.4.3, and the universal property of products. ◻

Corollary 4.4.20. *Let $f : X \to Y$ be a morphism and Y', Y'' be two subschemes of Y. Then we have $f^{-1}(\inf(Y', Y'')) = \inf(f^{-1}(Y'), f^{-1}(Y''))$.*

Proof. In fact, we have the canonical isomorphism of $(X \times_Y Y') \times_X (X \times_Y Y'')$ and $X \times_Y (Y' \times_Y Y'')$. ◻

4.4.4 Local immersions and local isomorphisms

Let $f : X \to Y$ be a morphism of schemes. We say f is a **local immersion** (resp. a **local isomorphism**) **at a point** $x \in X$ if there exists an open neighborhood U of x in X and an open neighborhood V of $f(x)$ in Y such that the restriction of f to U is a closed immersion (resp. an open immersion) into V. We say f is a local immersion (resp. a local isomorphism) if f is a local immersion (resp. a local isomorphism) at every point of X.

An immersion (resp. closed immersion) $f : X \to Y$ can then be characterized as a local immersion such that f is a homeomorphism onto a subset of Y (resp. a closed subset of Y). An open immersion f can be characterized as an injective local isomorphism.

Proposition 4.4.21. *Let X be an irreducible scheme, $f : X \to Y$ be a injective dominant morphism. If f is a local immersion, then it is an immersion and $f(X)$ is open in Y.*

Proof. In fact, let $x \in X$ and U be an open neighborhood of x in X, V an open neighborhood of $f(x)$ in Y such that $f|_Y$ is a closed immersion into V. As U is dense in X, $f(U)$ is also dense in Y by hypothesis, hence $f(U) = V$ and f is a homeomorphism from U to V. The hypothesis f is injective implies $f^{-1}(V) = U$, whence the proposition. ◻

Proposition 4.4.22. *Let $f : X \to X'$, $g : Y \to Y'$ be two S-schemes.*

(a) The composition of two local immersions (resp. local isomorphisms) is a local immersion (resp. a local isomorphism).

(b) If f and g are local immersions (resp. local isomorphisms), so is the product $f \times_S g$.

(c) If f is a local immersion (resp. a local isomorphism), so is $f_{(S')}$ for any extension $S' \to S$ of base schemes.

Proof. It suffices to prove (a) and (b). Now (a) follows from the transitivity of closed immersions (resp. open immersions) and the fact that if f is a homeomorphism of X to a closed subset Y, then for any open set $U \subseteq X$, $f(U)$ is open in $f(X)$, so there exists an open subset V of Y such that $f(U) = V \cap f(X)$, and $f(U)$ is therefore closed in V.

To prove (b), let p, q be the projections of $X \times_S Y$ and p', q' that of $X' \times_S Y'$. There exist by hypotheses open neighborhoods U, U', V, V' of $x = p(z)$, $x' = p'(z')$, $y = q(z)$, $y' = q'(z')$, respectively, such that $f|_U$ and $g|_V$ are closed immersions (resp. open immersions) onto U' and V', respectively. As the underlying space of $U \times_S V$ is $p^{-1}(U) \cap q^{-1}(V)$ and that of $U' \times_S V'$ is $p'^{-1}(U') \cap q'^{-1}(V')$, which are neighborhoods of z and z', respectively (Corollary 4.3.2), the claim follows by Proposition 4.4.12. ◻

Remark 4.4.3. A local isomorphism is clearly flat and universally open, and therefore is universally generalizing.

Proposition 4.4.23. *Let X be an irreducible scheme, Y an integral scheme, and $f : X \to Y$ be a morphism.*

(a) If f is dominant, then for any $x \in X$, the homomorphism $f_x^\# : \mathscr{O}_{Y,f(x)} \to \mathscr{O}_{X,x}$ is injective.

(b) If f is dominant and a local immersion, then f is a local isomorphism (and therefore X is integral).

Proof. Let ξ and η be the generic points of X and Y, respectively. If f is dominant, we then have $f(\xi) = \eta$; moreover $\mathscr{O}_{Y,\eta}$ is a field since Y is reduced, so $f_\xi^\# : \mathscr{O}_{Y,\eta} \to \mathscr{O}_{X,\xi}$ is a monomorphism. For any point $x \in X$, and any affine neighborhood U of $y = f(x)$, there exists an affine neighborhood V of x contained in $f^{-1}(U)$. The open set U (resp. V) contains η (resp. ξ), and the ring $\Gamma(U, \mathscr{O}_Y)$ is integral with fraction field $\mathscr{O}_{Y,\eta}$. If $\rho : \Gamma(U, \mathscr{O}_Y) \to \Gamma(V, \mathscr{O}_X)$ is the homomorphism corresponding to f, the composition

$$\Gamma(U, \mathscr{O}_Y) \xrightarrow{\rho} \Gamma(V, \mathscr{O}_X) \longrightarrow \mathscr{O}_{X,\xi}$$

is then the restriction of $f_\xi^\#$ to $\Gamma(U, \mathscr{O}_Y)$, so the homomorphism ρ. We now deduce that $f_x^\#$ is injective: in fact the hypotheses that X is irreducible implies that $\Gamma(V, \mathscr{O}_X)$ has a unique minimal $\mathfrak{n}$, which is its nilradical; the homomorphism $f_x^\#$ just send an element u/s (where $u, s \in \Gamma(U, \mathscr{O}_Y)$ and $s \neq 0$) to the element $\rho(u)/\rho(s) \in \mathscr{O}_{X,x}$, which is zero only if there exists $t \notin \mathfrak{p}_x$ such that $t\rho(u) = 0 \in \mathfrak{n}$. But as $t \notin \mathfrak{n}$, this then implies $\rho(u) \in \mathfrak{n}$, so $\rho(u)$ is nilpotent and since ρ is injective, this shows that u is nilpotent, which means $u = 0$ for $\Gamma(U, \mathscr{O}_Y)$ being integral.

To prove the second assertion, let f be dominant and a local immersion. We see $f(Y)$ is open in Y by Proposition 4.4.21. Since $f_x^\#$ is surjective for every point $x \in X$ (Proposition 4.4.9), it follows that $\mathfrak{f}_x^\#$ is an isomorphism by (a), and this shows f is a local isomorphism, again by Proposition 4.4.9 . ◻

Proposition 4.4.24. *Let Y be a reduced scheme such that the family of irreducible components of Y is locally finite. Let $j : X \to Y$ be an immersion. For j to be a local isomorphism at a point $x \in X$, it is necessary and sufficient that the homomorphism $j_x^\# : \mathscr{O}_{Y,j(x)} \to \mathscr{O}_{X,x}$ is flat.*

Proof. Since the question is local on Y, we may assume that Y is affine, that j is a closed immersion, and that all irreducible components of Y contain $j(x)$ (hence are finite in number), and we prove that j is an isomorphism in this case. If $Y = \operatorname{Spec}(A)$, and if $\mathfrak{p}$ is the prime ideal of A corresponding to $j(x)$, the morphism $\operatorname{Spec}(A_\mathfrak{p}) \to \operatorname{Spec}(A)$ is then dominant since $\mathfrak{p}$ contains the minimal ideals of A (it is contaiend in every irreducible component of $\operatorname{Spec}(A)$). As A is reduced, the homomorphism $A \to A_\mathfrak{p}$ is injective (Corollary **??**). If $\mathscr{O}$ is the ideal of $\mathscr{O}_Y$ defining the closed subscheme of Y associated with j, we have $\mathscr{I} = \widetilde{\mathfrak{a}}$, where $\mathfrak{a}$ is an ideal of A, and it is identified with a subset of $\mathfrak{a}_\mathfrak{p}$. If j is flat at x, then $(A/\mathfrak{a})_\mathfrak{p}$ is a flat $A_\mathfrak{p}$-module, and since the homomorphism $A_\mathfrak{p} \to (A/\mathfrak{a})_\mathfrak{p}$ is local, it is faithfully flat (Proposition **??**). By Proposition **??** this implies $\mathfrak{a}_\mathfrak{p} = 0$, hence $\mathfrak{a} = 0$ and the claim follows. □

4.4.5 Nilradical and associated reduced scheme

Proposition 4.4.25. *Let $(X, \mathscr{O}_X)$ be a scheme and $\mathscr{B}$ be a quasi-coherent $\mathscr{O}_X$-algebra. There exist a unique quasi-coherent ideal $\mathscr{N}$ of $\mathscr{B}$ such that for each point $x \in X$, the stalk $\mathscr{N}_x$ is the nilradical of the ring $\mathscr{B}_x$. If $X = \mathrm{Spec}(A)$ is affine and $\mathscr{B} = \widetilde{B}$, where B is an A-algebra, then $\mathscr{N} = \widetilde{\mathfrak{n}}$, where $\mathfrak{n}$ is the nilradical of B.*

Proof. Since the question is loca, we may assume that $X = \mathrm{Spec}(A)$ is affine and $\mathscr{B} = \widetilde{B}$. We know that $\widetilde{\mathfrak{n}}$ is a quasi-coherent $\mathscr{O}_X$-module and for each point $x \in X$, the stalk $\mathfrak{n}_x$ is an ideal of the fraction ring B_x. It suffices to show that the nilradical of B_x is contained in $\mathfrak{n}_x$, the opposite inclusion being evident. Now, let z/s be a nilpotent element in B_x, where $z \in B$ and $s \notin \mathfrak{p}_x$. By hypotheses, there exist $k \geq 0$ such that $(z/s)^k = 0$, which means there exists $t \notin \mathfrak{p}_x$ such that $tz^k = 0$. We then conclude that $(tz)^k = 0$, so $z/s = (tz)/(ts)$ is indeed in $\mathfrak{n}_x$. □

The quasi-coherent ideal $\mathscr{N}$ is called the **nilradical** of the $\mathscr{O}_X$-algebra $\mathscr{B}$. In particular, we denote by $\mathscr{N}_X$ the nilradical of $\mathscr{O}_X$.

Corollary 4.4.26. *Let X be a scheme. Then the closed subscheme of X defined by the quasi-coherent ideal $\mathscr{N}_X$ is the unique reduced subscheme of X with underlying space X. It is also the smallest subscheme of X having X as underlying space.*

Proof. Since the structural sheaf of the closed subscheme Y defined by $\mathscr{N}_X$ is $\mathscr{O}_X/\mathscr{N}_X$, it is immediate that Y is reduced and has X as underlying space, since $\mathscr{N}_x \neq \mathscr{O}_{X,x}$ for each $x \in X$. To prove the second claim, let Z be a subscheme of X with X as underlying space. Then Z is closed in X, so let $\mathscr{I}$ be the ideal defining it. We can assume that X is affine, so $\mathscr{I} = \widetilde{\mathfrak{a}}$ where $\mathfrak{a}$ is an ideal of A. Then for each $x \in X$ we have $\mathfrak{a} \subseteq \mathfrak{p}_x$, so $\mathfrak{a} \subseteq \mathfrak{n}$, where $\mathfrak{n}$ is the nilradical of A. This shows Y is the smallest subscheme of X with underlying space X, and if Z is distinct from Y, we necessarily have $\mathscr{I}_x \neq \mathscr{N}_x$ for some $x \in X$, and concequently Z is not reduced. □

The reduced scheme defined by $\mathscr{N}_X$ on X is called the **reduced scheme associated with X**, and denoted by X_{red}. To say that a schema X is reduced therefore means that $X_{\mathrm{red}} = X$. Clearly, we have a canonical closed immersion $X_{\mathrm{red}} \to X$, which is also a universal homeomorphism.

Proposition 4.4.27. *For the spectrum of a ring A to be reduced (resp. integral), it is necessary and sufficient that A is reduced (resp. integral).*

Proof. In fact, the condition $\mathscr{N} = 0$ is necessary and sufficient for $\mathrm{Spec}(A)$ to be reduced, and the integral conditions follows from Corollary **??**. □

Proposition 4.4.28. *A scheme X is integral if and only if for each open subset U of X, the ring $\Gamma(U, \mathscr{O}_X)$ is integral.*

Proof. We first assume that $\Gamma(U, \mathscr{O}_X)$ is integral for any open set U. It is clear that X is reduced. If X is not irreducible, then one can find two nonempty disjoint open subsets U_1 and U_2. Then $\Gamma(U_1 \cup U_2, \mathscr{O}_X) = \Gamma(U_1, \mathscr{O}_X) \times \Gamma(U_2, \mathscr{O}_X)$, which is not an integral domain. Thus X is an integral scheme.

Conversely, suppose that X is reduced and irreducible. Let $U \subseteq X$ be an open subset, and suppose that there are elements $f, g \in \Gamma(U, \mathscr{O}_X)$ with $fg = 0$. Let

$$Y = \{x \in U : f_x \in \mathfrak{m}_x\}, \quad Z = \{x \in U : g_x \in \mathfrak{m}_x\};$$

then Y and Z are closed subsets, and $Y \cup Z = U$. Since X is irreducible, so U is irreducible, and one of Y or Z is equal to U, say $Y = U$. But then the restriction of f to any open affine subset of U will be contained in every point of that subset, hence nilpotent and thus zero. This shows that $\Gamma(U, \mathscr{O}_X)$ is integral. □

Proposition 4.4.29. *Let X be a scheme and x a point of X.*

(a) For x to belong to a unique irreducible component of X, it is necessary and sufficient that the nilradical of $\mathscr{O}_{X,x}$ is prime.

(b) If the nilradical of $\mathscr{O}_{X,x}$ is prime and the family of irreducible components of X is locally finite, there exists an open neighborhood U of x that is irreducible.

(c) For X to be the coproduct of its irreducible components, it is necessary and sufficient that the family of irreducible components of X is locally finite and for each $x \in X$, the nilradical of $\mathscr{O}_{X,x}$ is prime.

Proof. To check that whether x belongs to distinct irreducible components of X, we may assume that $X = \mathrm{Spec}(A)$ is affine (Proposition **??**). Then this signifies that $\mathfrak{p}_x$ contains two distinct minimal prime ideals of A, and equivalently $\mathfrak{m}_x$ contains two distinct minimal prime ideals of $\mathscr{O}_{X,x}$, which is a contradiction if and only if the nilradical of $\mathscr{O}_{X,x}$ is prime.

Now assume the conditions in (a). As the family of irreducible components of X is locally finite, the union of those of these components which do not contain x is closed, so its complement U is open and contained in the unique irreducible component of X containing x, and therefore irreducible (Proposition **??**). The assertion in (c) follows from (b) and Proposition **??**. □

Proposition 4.4.30. *For a scheme X to be locally integral, it is necessary and sufficient that the family of irreducible components is locally finite and for each point $x \in X$, the local ring $\mathscr{O}_{X,x}$ is integral. In this case, X is the coproduct of its irreducible components, which are integral subschemes.*

Proof. As the localizations of integral domains are integral, for a locally integral scheme X, its local rings $\mathscr{O}_{X,x}$ are inegral. The set of the irreducible components of X is locally finite since each $x \in X$ admits an irreducible open neighborhoods; moreover, the irreducible components of X are all open and disjoint, so X is the coproduct of its irreducible components (Proposition **??**). Conversely, if X satisfies these conditions, then X is the coproduct of its irreducible components, which are open and integral. It follows immediately that X is locally integral. □

Corollary 4.4.31. *Let X be a scheme whose set of irreducible components is locally finite (for example if X is locally Noetherian). Then for X to be integral, it is necessary and sufficient that it is connected and for each $x \in X$, the local ring $\mathscr{O}_{X,x}$ is integral. In particular, a locally integral and connected scheme is integral.*

Proposition 4.4.32. *Let X be a locally Noetherian scheme and let $x \in X$ be a point such that the nilradical $\mathscr{N}_x$ of $\mathscr{O}_{X,x}$ is prime (resp. such that $\mathscr{O}_{X,x}$ is reduced, resp. such that $\mathscr{O}_{X,x}$ is integral). Then there exist an open neighborhood U of x that is irreducible (resp. reduced, resp. integral).*

Proof. It suffices to consider the case where $\mathscr{N}_x$ is prime and where $\mathscr{N}_x = 0$, the third one is the conjunction of the first two cases. If $\mathscr{N}_X$ is prime, the claim follows from Proposition 4.4.29. If $\mathscr{N}_x = 0$, we then have $\mathscr{N}_y = 0$ for y in a neighborhood of x, since $\mathscr{N}$ is quasi-coherent, hence coherent since X is locally Noetherian and $\mathscr{N}$ is of finite type, and the conclusion follows from Proposition **??**. □

Proposition 4.4.33. *For a Noetherian scheme X, the nilradical $\mathscr{N}_X$ of $\mathscr{O}_X$ is nilpotent.*

Proof. Since X is quasi-compact, We can cover X by a finite number of affine open sets U_i, and it suffices to prove that there exist integers n_i such that $(\mathscr{N}_X|_{U_i})^{n_i} = 0$. If n is the largest of the n_i, we will then have $\mathscr{N}_X^n = 0$. We are therefore reduced to the case where $X = \operatorname{Spec}(A)$ is affine, A being a Noetherian ring. It then suffices to observe that the nilradical of A is nilpotent. □

Corollary 4.4.34. *A Noetherian sheme X is affine if and only if X_{red} is affine.*

Proof. If X is affine it is clear that X_{red} is affine, regardless of X being Noetherian. Conversely, assume that X_{red} is affine and X is Noetherian. Then by Proposition 4.4.33, the nilradical $\mathscr{N}$ of $\mathscr{O}_X$ is nilpotent. For any quasi-coherent sheaf $\mathscr{F}$ on X, consider the follows exact sequence (where $k \geq 0$)

$$0 \longrightarrow \mathscr{N}^k\mathscr{F} \longrightarrow \mathscr{N}^{k-1}\mathscr{F} \longrightarrow \mathscr{N}^{k-1}\mathscr{F}/\mathscr{N}^k\mathscr{F} \longrightarrow 0$$

Since $\mathscr{N}^{k-1}\mathscr{F}/\mathscr{N}^k\mathscr{F}$ is an $\mathscr{O}_X/\mathscr{N}$-module and $(X, \mathscr{O}_X/\mathscr{N})$ is affine, by Serre's criterion we have $H^1(X, \mathscr{N}^{k-1}\mathscr{F}/\mathscr{N}^k\mathscr{F}) = 0$, so

$$H^1(X, \mathscr{N}^k\mathscr{F}) = 0 \Rightarrow H^1(X, \mathscr{N}^{k-1}\mathscr{F}) = 0.$$

Since $\mathscr{N}$ is nilpotent, this shows $H^1(X, \mathscr{F}) = 0$, so $(X, \mathscr{O}_X)$ is affine, by Serre's criterion again. □

Let $f : X \to Y$ be a morphism of schemes; let $i : X_{\mathrm{red}} \to X$ and $j : Y_{\mathrm{red}} \to Y$ be the canonical injections. The homomorphism $f_x^\# : \mathscr{O}_{Y,f(x)} \to \mathscr{O}_{X,x}$ maps nilpotent elements of $\mathscr{O}_{Y,f(x)}$ into nilpotent elements of $\mathscr{O}_{X,x}$, so $f^*(\mathscr{N}_Y)\mathscr{O}_X \subseteq \mathscr{N}_X$. It then follows from Corollary 4.4.4 that $f \circ i$ factors through j, so we get an induced morphism $f_{\mathrm{red}} : X_{\mathrm{red}} \to Y_{\mathrm{red}}$ and a commutative diagram

$$\begin{tikzcd} X_{\mathrm{red}} \arrow[r, "f_{\mathrm{red}}"] \arrow[d, "i"'] & Y_{\mathrm{red}} \arrow[d, "j"] \\ X \arrow[r, "f"] & Y \end{tikzcd} \tag{4.5.1}$$

In particular, if X is reduced, then the morphism f factors into

$$X \xrightarrow{f_{\mathrm{red}}} Y_{\mathrm{red}} \xrightarrow{j} Y$$

or in other words, f is dominated by the canonical injection j. We also conclude that Y_{red} satisfies the universal property that any morphism from a reduced scheme to Y factors through Y_{red}.

For two morphisms $f : X \to Y$, $g : Y \to Z$, it follows from the uniqueness of factorization that we have

$$(g \circ f)_{\mathrm{red}} = g_{\mathrm{red}} \circ f_{\mathrm{red}};$$

we can equivalently say that the operation $X \mapsto X_{\mathrm{red}}$ is a covariant functor on the category of schemes.

Proposition 4.4.35. *If X and Y are S-schemes, the schemes $X_{\mathrm{red}} \times_{S_{\mathrm{red}}} Y_{\mathrm{red}}$ and $X_{\mathrm{red}} \times_S Y_{\mathrm{red}}$ are identical, and are canonically identified with a subscheme of $X \times_S Y$ having the same underlying space as this product.*

Proof. The fact that $X_{\mathrm{red}} \times_S Y_{\mathrm{red}}$ is identified with a subscheme of $X \times_S Y$ follows from the fact that if $f : X' \to X$ and $g : Y' \to Y$ are surjective S-immersions, then $f \times_S g : X' \times_S Y' \to X \times_S Y$ is a surjective immersion (Proposition 4.4.11 and Proposition 4.3.26). On the other hand, if $\varphi : X_{\mathrm{red}} \to S$ and $\psi : Y_{\mathrm{red}} \to S$ are the structural morphisms, it is clear that they factors through S_{red}, and as $S_{\mathrm{red}} \to S$ is a monomorphism, the second assertion follows. □

Corollary 4.4.36. *The schemes $(X \times_S Y)_{\mathrm{red}}$ and $(X_{\mathrm{red}} \times_{S_{\mathrm{red}}} Y_{\mathrm{red}})_{\mathrm{red}}$ are canonically identified.*

We note that if X and Y are reduced S-schemes, it is not necessarily that case that $X \times_S Y$ is reduced, since the tensor product of two reduced algebras (even two fields) may not be reduced.

Corollary 4.4.37. *For any morphism $f : X \to Y$ of schemes, the diagram (4.5.1) factors into*

$$\begin{tikzcd} X_{\mathrm{red}} = (X \times_Y Y_{\mathrm{red}})_{\mathrm{red}} \arrow[r] & X \times_Y Y_{\mathrm{red}} \arrow[r] \arrow[d] & Y_{\mathrm{red}} \arrow[d] \\ & X \arrow[r, "f"] & Y \end{tikzcd} \tag{4.5.2}$$

Proof. For this, we only need to note that $(X \times_Y Y_{\mathrm{red}})_{\mathrm{red}} = (X_{\mathrm{red}} \times_{Y_{\mathrm{red}}} Y_{\mathrm{red}})_{\mathrm{red}} = X_{\mathrm{red}}$. □

Proposition 4.4.38. *Let X and Y be two schemes. Consider the following properties for a morphism $f : X \to Y$:*

(i) surjective;

(ii) injective;

(iii) open at the point x (resp. open);

(iv) closed;

(v) a homeomorphism onto its image;

(vi) universally open at a point x (resp. universally open);

(vii) universally closed;

(viii) a universal embedding;

(ix) a universal homeomorphism;

(x) radical;

(xi) generalizing at a point x (resp. generalizing);

(xii) universally generalizing at a point x (resp. universally generalizing).

Then, if $\mathcal{P}$ denote one of the above properties, for f to possess the property $\mathcal{P}$, it is necessary and sufficient that f_{red} possess $\mathcal{P}$.

Proof. The proposition is evident for the properties (i), (ii), (iii), (iv), (v), (xi), which only depend on the map of the underlying spaces. For (x), the proposition follows from the fact that the fibers of f and f_{red} at a point $y \in Y$ have the same underlying space and the residue field at a point of X (resp. Y) is the same for X_{red} and Y_{red}. For the properties (vi), (vii), (viii), (ix) and (xii), if f possesses one of these properties, the same is true of f_{red} due to Corollary 4.4.37 and that the morphism $X_{\mathrm{red}} \to X \times_Y Y_{\mathrm{red}}$ is a universal homeomorphism. Conversely, if f_{red} possesses one of these properties, it suffices to note that for any morphism $Y' \to Y$ we have $(X_{\mathrm{red}} \times_{Y_{\mathrm{red}}} Y'_{\mathrm{red}})_{\mathrm{red}} = (X \times_Y Y')_{\mathrm{red}}$, so the morphism

$$(f_{(Y')})_{\mathrm{red}} : (X \times_Y Y')_{\mathrm{red}} \to Y'_{\mathrm{red}}$$

possesses the "nonuniversal" version of the same property, and by what we have already seen, $f_{(Y')}$ then has the corresponding property. □

Proposition 4.4.39. *Let X and Y be two schemes and x be a point of X. Consider the following properties for a morphism $f : X \to Y$:*

(i) a monomorphism;

(ii) an immersion;

(iii) an open immersion;

(iv) a closed immersion;

(v) a local immersion at the point x;

(vi) a local isomorphism at the point x;

(vii) birational.

Then, if f possesses one of the above properties, f_{red} also possesses that property.

Proof. For the properties (ii) to (vii), the result follows from the observation that $(f_{\text{red}})^{\#}_x : \mathscr{O}_{Y,f(x)} \to \mathscr{O}_{X,x}$ is surjective (resp. bijective) if $f^{\#}_x : \mathscr{O}_{Y,f(x)} \to \mathscr{O}_{X,x}$ is surjective (resp. bijective) (Proposition 4.4.9). For (i) it suffices to note that a monomorphism is universal (Proposition 4.3.10), the diagram (4.5.2), and the fact that $X_{\text{red}} \to X \times_Y Y_{\text{red}}$ is a closed immersion, hence a monomorphism. □

Note that if f_{red} is an immersion, it is not necessarily true that f is. For example, let $Y = \operatorname{Spec}(k)$ where k is a field and $X = \operatorname{Spec}(A)$, where $A = k[T]/(T^2)$. Then the canonical injection $\rho : k \to k[T]/(T^2)$ corresponds to a morphism $f : X \to Y$. It is clear that f is not an immersion (in fact any nonzero immersion into Y is automatically closed); but $A_{\text{red}} = k$ so f_{red} is an isomorphism.

Remark 4.4.4. To say that an immersion $f : Y \to X$ is surjective means it is closed and that the subscheme of X associated with f is defined by an ideal $\mathscr{I}$ contained in the nilradical $\mathscr{N}_X$. In this case, we say f is a nilimmersion; f is then a homeomorphism from Y to X, and f_{red} is an isomorphism from Y_{red} to X_{red}. We say the nilimmersion f is **nilpotent** (resp. **locally nilpotent**) if the ideal $\mathscr{I}$ is nilpotent (resp. locally nilpotent, i.e. that every $x \in X$ has an open neighborhood U such that $\mathscr{I}|_U$ is nilpotent). More precisely, we say f is **nilpotent of order n** if $\mathscr{I}^{n+1} = 0$. If Y is a subscheme of X and f is the canonical immersion, we say X is an **infinitesimal neighborhood** (resp. an **infinitesimal neighborhood of order n**) of Y if f is nilpotent (resp. nilpotent of order n).

4.4.6 Reduced scheme structure on closed subsets

Proposition 4.4.40. *For any locally closed subspace Y of the underlying space of a scheme X, there exists a unique reduced subscheme of X with underlying space Y.*

Proof. The uniqueness is immediate from Corollary 4.4.26, so we only need to construct a reduced scheme structure on Y. If X is affine with ring A and Y is closed in X, the proposition is immediate: $I(Y)$ is the largest ideal $\mathfrak{a} \subseteq A$ such that $V(\mathfrak{a}) = Y$, and is radical, hence the ring $A/I(Y)$ is reduced, and we can take the scheme structure $(Y, \widetilde{A/I(Y)})$ on Y.

In the general case, for any affine open $U \subseteq X$ such that $U \cap Y$ is closed in U, consider the closed subscheme Y_U of U defined by the quasi-coherent ideal associated with the ideal $I(U \cap Y)$ of $\Gamma(U, \mathscr{O}_X|_U)$, which is reduced. If V is an open affine of X contained in U, then Y_V is induced by Y_U on $V \cap Y$ since this induced scheme is a closed subscheme of V which is reduced and has $V \cap Y$ as underlying space; the uniqueness of Y_V therefore entails our assertion. □

Corollary 4.4.41. *Let X be a scheme and Y be a locally closed subset of X. Then any point $x \in Y$ admits a maximal generalization y (i.e. y has no further generalization in Y). In particular, if $Y \neq \varnothing$, there exist a maximal element $y \in Y$ under generalization.*

Proof. It suffices to give Y a subscheme structure of X and take y to be the generic point of the irreducible components of Y containing x. □

Example 4.4.42. Let X be a scheme and x be a closed point of X. Let U be an open neighborhood of x. Then $Z = (X - U) \cup \{x\}$ is a closed subset of X, so we can consider the reduced scheme structure on it. Let $\mathscr{I}$ be the corresponding quasi-coherent ideal of $\mathscr{O}_X$, we want to determine the stalk $\mathscr{I}_x$. For this, we can assume that $X = \operatorname{Spec}(A)$ is affine and $X - U = V(\mathfrak{a})$, where $\mathfrak{a}$ is an ideal of A. Then the point x corresponds to a maximal ideal $\mathfrak{m}$ of A such that $\mathfrak{a} \not\subseteq \mathfrak{m}$, and $Z = V(\mathfrak{a}) \cup \{x\} = V(\mathfrak{a} \cap \mathfrak{m})$. By definition, $\mathscr{I}$ is the quasi-coherent ideal on X associated with $I(Z) = \sqrt{\mathfrak{a} \cap \mathfrak{m}}$, and therefore

$$\mathscr{I}_x = (\sqrt{\mathfrak{a} \cap \mathfrak{m}})_{\mathfrak{m}} = \sqrt{(\mathfrak{a} \cap \mathfrak{m})_{\mathfrak{m}}},$$

which is the intersection of prime ideals $\mathfrak{p}$ containing $\mathfrak{a} \cap \mathfrak{m}$ and contained in $\mathfrak{m}$. But if a prime ideal $\mathfrak{p} \subseteq \mathfrak{m}$ of A contains $\mathfrak{a} \cap \mathfrak{m}$, then by Proposition **??** we have $\mathfrak{p} \supset \mathfrak{a}$ or $\mathfrak{p} \supset \mathfrak{m}$. If $\mathfrak{p} \neq \mathfrak{m}$, then this implies $\mathfrak{p} \supset \mathfrak{a}$ and therefore $\mathfrak{m} \supset \mathfrak{p} \supset \mathfrak{a}$, which is a contradiction (since x is not contained in $X - U = V(\mathfrak{a})$). From this, we conclude that $\mathscr{I}_x = \mathfrak{m}_x$.

Example 4.4.43. Let X be a reduced locally Noetherian scheme and X' be a reduced closed subscheme of X with underlying space an irreducible component of X. Then X' is defined by a quasi-coherent ideal $\mathscr{I}$ of $\mathscr{O}_X$. Let x be the generic point of X'; we claim that $\mathscr{I}_x = 0$. For this, we can assume that $X = \operatorname{Spec}(A)$ is affine, where A is a reduced Noetherian ring, so $\mathscr{I} = \tilde{\mathfrak{p}}$ where $\mathfrak{p}$ is a minimal prime ideal of A. By definition the stalk of $\mathscr{I}$ at x is indeitified with $\mathfrak{p}A_{\mathfrak{p}}$, which is the maximal ideal of $A_{\mathfrak{p}} = \mathscr{O}_{X,x}$. But since A is reduced and $\mathfrak{p}$ is minimal, $A_{\mathfrak{p}}$ is a reduced Artinian local ring, whence a field, and we then conclude that $\mathfrak{p}A_p = 0$, so $\mathscr{I}_x = 0$.

Proposition 4.4.44. *Let X be a reduced scheme, $f : X \to Y$ be a morphism, and Z be a closed subscheme of Y containing $f(X)$. Then f factors into*

$$X \xrightarrow{g} Z \xrightarrow{j} Y$$

where j is the canonical injection.

Proof. The hypotheses implies that the closed subscheme $f^{-1}(Z)$ of X has underlying space X (Proposition 4.4.14). As X is reduced, this subscheme coincides with X by Corollary 4.4.26, and the proposition then follows from Proposition 4.4.14. □

Corollary 4.4.45. *Let X be a reduced subscheme of a scheme Y. If Z is the reduced closed subscheme of Y with underlying space $\overline{X}$, then X is an open subscheme of Z.*

Proof. Since X is locally closed, there is an open set U of Y such that $X = U \cap \overline{X}$. By Proposition 4.4.44, X is then a reduced subscheme of Z, with underlying space open in Z. Since the scheme structure induced by Z is also reduced, we conclude that X is induced by Z, in view of the uniqueness part of Proposition 4.4.40. □

Corollary 4.4.46. *Let $f : X \to Y$ be morphism and X' (resp. Y') be a closed subscheme of X (resp. Y) defined by a quasi-coherent ideal $\mathscr{I}$ (resp. $\mathscr{K}$) of $\mathscr{O}_X$ (resp. $\mathscr{O}_Y$). Suppose that X' is reduced and $f(X') \subseteq Y'$, then we have $f^*(\mathscr{K})\mathscr{O}_X \subseteq \mathscr{I}$.*

Proof. The restriction of f to X' factors into $X' \to Y' \to Y$ by Proposition 4.4.44, so it suffices to use Corollary 4.4.17. □

4.5 Separated schemes and morphisms

4.5.1 Diagonal and graph of a morphism

Let X be an S-scheme; recall that the diagonal morphism $X \to X \times_S X$, denoted by $\Delta_{X/S}$ or Δ_X, is the S-morphism $(1_X, 1_X)_S$, which means Δ_X is the unique S-morphism such that

$$p_1 \circ \Delta_X = p_2 \circ \Delta_X = 1_X,$$

where p_1, p_2 are the canonical projections of $X \times_S X$. If $f : T \to X$ and $g : T \to Y$ are two S-morphisms, we verify that

$$(f, g)_S = (f \times_S g) \circ \Delta_{T/S}.$$

If $\varphi : X \to S$ is the structural morphism of X, we also write Δ_φ for $\Delta_{X/S}$.

Proposition 4.5.1. *Let X, Y be S-schemes. If we identify $(X \times_S Y) \times_S (X \times_S Y)$ and $(X \times_S X) \times_S (Y \times_S Y)$, the morphism $\Delta_{X \times Y}$ is identified with $\Delta_X \times \Delta_Y$.*

Proof. In fact, if p_1, q_1 are the canonical projections $X \times_S X \to X$, $Y \times_S Y \to Y$, the projection $(X \times_S Y) \times_S (X \times_S Y) \to X \times_S Y$ is identified with $p_1 \times q_1$, and we have

$$(p_1 \times q_1) \circ (\Delta_X \times \Delta_Y) = (p_1 \circ \Delta_X) \times (q_1 \circ \Delta_Y) = 1_{X \times Y}$$

similar for the projection to the second factor. □

Corollary 4.5.2. *For any extension $S' \to S$ of base schemes, $\Delta_{X_{(S')}}$ is identified with $(\Delta_X)_{(S')}$.*

Proof. It suffices to remark that $(X \times_S X)_{(S')}$ is identified with $X_{(S')} \times_{S'} X_{(S')}$ canonically. □

Proposition 4.5.3. *Let X, Y be S-schemes and $S \to T$ be a morphism. Let $\varphi : X \to S$, $\psi : Y \to S$ be the structural morphisms, p, q the projection of $X \times_S Y$, and $\pi = \varphi \circ p = \psi \circ q$ the structural morphism $X \times_S Y \to S$. Then the diagram*

$$\begin{tikzcd}
X \times_S Y \arrow[r, "(p,q)_T"] \arrow[d, "\pi"'] & X \times_T Y \arrow[d, "\varphi \times_T \psi"] \\
S \arrow[r, "\Delta_{S/T}"] & S \times_T S
\end{tikzcd} \tag{5.1.1}$$

commutes and cartesian.

Proof. By the definition of products, we may prove the proposition in the category of sets, and replace X, Y, S by $X(Z)_T$, $Y(Z)_T$, $S(Z)_T$, where Z is an arbitrary T-scheme, and it is then immediate. □

Corollary 4.5.4. *If $f : X \to Y$ is an S-morphism, the diagram*

$$\begin{tikzcd}
X \arrow[r, "\Gamma_f"] \arrow[d, "f"'] & X \times_S Y \arrow[d, "f \times_S 1_Y"] \\
Y \arrow[r, "\Delta_Y"] & Y \times_S Y
\end{tikzcd} \tag{5.1.2}$$

is commutative and cartesian.

Proof. It suffcies to apply Proposition 4.5.3 with S replaced by Y and T by S, and note that $X \times_Y Y = X$. □

Proposition 4.5.5. *For a morphism $f : X \to Y$ of schemes to be a monomorphism, it is necessary and sufficient that $\Delta_{X/Y}$ is an isomorphism from X to $X \times_Y X$.*

Proof. In fact, f is a monomorphism means for any Y-scheme Z, the corresponding map $X(Z)_Y \to Y(Z)_Y$ is an injection, and as $Y(Z)_Y$ is reduced to a singleton, this means $X(Z)_Y$ is either empty or a singleton. But this is equivalent to saying that $X(Z)_Y \times X(Z)_Y$ is canonically isomorphic to $X(Z)_Y$ via the diagonal map, where the first set is $(X \times_Y X)(Z)_Y$, and this means $\Delta_{X/Y}$ is an isomorphism. □

Proposition 4.5.6. *The diagonal morphism is an immersion from X to $X \times_S X$, and the corresponding subscheme of $X \times_S X$ is called the **diagonal** of $X \times_S X$.*

Proof. Let p_1, p_2 be the projections of $X \times_S X$. As the continuous maps p_1 and Δ_X are such that $p_1 \circ \Delta_X = 1_X$, Δ_X is a homeomorphism from X onto $\Delta_X(X)$. Simialrly, the composition of the homomorphisms $\mathscr{O}_{X,x} \to \mathscr{O}_{\Delta_X(X),x} \to \mathscr{O}_{X,x}$ corresponding to p_1 and Δ_X is the identity, so the homomorphism corresponding to Δ_X on stalks are surjective. The proposition then follows from Proposition 4.4.9 . □

Corollary 4.5.7. *With the hypotheses of Proposition 4.5.3, the morphisms $(p, q)_T$ is an immersion.*

Proof. This follows from Proposition 4.5.3 and Proposition 4.4.12. □

Corollary 4.5.8. *Let X and Y be S-schemes and $f : X \to Y$ be an S-morphism. Then the graph morphism $\Gamma_f : X \to X \times_S Y$ of f is an immersion.*

Proof. This follows from the diagram (5.1.2) and Proposition 4.4.12. □

The subscheme of $X \times_S Y$ associated with the immersion Γ_f is called the **graph** of the morphism f; the subschemes of $X \times_S Y$ which are graphs of morphisms $X \to Y$ are characterized by the fact that the restriction of the projection $p_1 : X \times_S Y \to X$ to such a subscheme G is an isomorphism g from G to X: in fact, if this is the case, G is then the graph of the morphism $p_2 \circ g^{-1}$, where $p_2 : X \times_S Y \to Y$ is the second projection.

In particular, if $X = S$, the S-morphisms $S \to Y$, which are none other than the S-sections of Y, are equal to their graph morphisms; the subschemes of Y which are graphs of S-sections (in other words, those which are isomorphic to S by the restriction of the structural morphism $Y \to S$) are then called the **images of these sections**, or, by abuse of language, the S-sections of Y.

Proposition 4.5.9. *If $f : X \to Y$ is an S-morphism, the diagram*

$$\begin{tikzcd} X \arrow[r, "\Delta_X"] \arrow[d, "f"'] & X \times_S X \arrow[d, "f \times_S f"] \\ Y \arrow[r, "\Delta_Y"] & Y \times_S Y \end{tikzcd} \tag{5.1.3}$$

is commutative (in other words, Δ_X is a functorial morphism on the category of schemes).

Proof. The morphisms $\Delta_Y \circ f$ satisfies the condition that

$$p_1 \circ (\Delta_Y \circ f) = p_2 \circ (\Delta_Y \circ f) = f$$

where p_1, p_2 are the projections of $Y \times_S Y$. Similarly, if q_1, q_2 are the projections of $X \times_S X$,

$$p_1 \circ (f \times_S f) \circ \Delta_X = f \circ q_1 \circ \Delta_X = f,$$
$$p_2 \circ (f \times_S f) \circ \Delta_X = f \circ q_2 \circ \Delta_X = f.$$

It then follows from the universal property of products that $\Delta_Y \circ f = (f \times_S f) \circ \Delta_X$. □

Corollary 4.5.10. *If X is a subscheme of Y, the diagonal $\Delta_X(X)$ is identified with a subscheme of $\Delta_Y(Y)$ whose the underlying space is identified with*

$$\Delta_Y(Y) \cap p_1^{-1}(X) = \Delta_Y(Y) \cap p_2^{-1}(X)$$

where p_1, p_2 are the projections of $Y \times_S Y$.

Proof. Apply Proposition 4.5.9 to the immersion $f : X \to Y$, we see $(f \times_S f)$ is an immersion which identifies $X \times_S X$ with the subspace $p_1^{-1}(X) \cap p_2^{-1}(X)$ of $Y \times_S Y$ (Proposition 4.4.11). Moreover, if $z \in \Delta_Y \cap p_1^{-1}(X)$, we have $z = \Delta_Y(y)$ and $y = p_1(z) \in X$, so $y = f(y)$, and $z = \Delta_Y(f(y))$ belongs to $\Delta_X(X)$ in view of the diagram (5.1.3). □

Proposition 4.5.11. *Let $u_1 : X \to Y$, $u_2 : X \to Y$ be two S-morphisms. Then the kernel* $\ker(u_1, u_2)$ *is canonically isomorphic to the inverse image in X of the diagonal $\Delta_Y(Y)$ under the morphism $(u_1, u_2)_S : X \to Y \times_S Y$.*

Proof. Let $Z \to X$ be the inverse image of $\Delta_Y(Y)$ under the morphism $(u_1, u_2)_S$. Then if $f : T \to X$ is an S-morphism such that $u_1 \circ f = u_2 \circ f$, then

$$p_1 \circ (u_1, u_2)_S \circ f = u_1 \circ f = u_2 \circ f = p_1 \circ \Delta_Y \circ u_2 \circ f,$$
$$p_2 \circ (u_1, u_2)_S \circ f = u_2 \circ f = p_2 \circ \Delta_Y \circ u_2 \circ f$$

where p_1, p_2 are the projections of $Y \times_S Y$. We conclude that $(u_1, u_2)_S \circ f = \Delta_Y \circ u_2 \circ f$, and by the definition of Z, the morphism f factors uniquely through Z, which proves our claim. □

Corollary 4.5.12. *For a point $x \in X$ to belong to* $\ker(u_1, u_2)$*, it is necessary and sufficient that $u_1(x) = u_2(x) = y$ and the homomorphisms $\kappa(y) \to \kappa(x)$ corresponding to u_1 and u_2 are equal.*

Proof. In fact, if $j : Z \to X$ is the kernel of u_1 and u_2, to say $x \in Z$ signifies that the canonical morphism $h : \mathrm{Spec}(\kappa(x)) \to X$ factors into $h = j \circ g$, where g is a morphism from $\mathrm{Spec}(\kappa(x))$ to Z. This is equivalent to $u_1 \circ h = u_2 \circ h$, and by Proposition 4.2.15 to that $u_1(x) = u_2(x)$ and the field extensions $\kappa(y) \to \kappa(x)$ corresponding to u_1 and u_2 are equal. □

Proposition 4.5.13. *Let X and Y be S-schemes and $f : X \to Y$, $g : X \to Y$ be S-morphisms. Then we have the following commutative diagram*

$$\begin{tikzcd}
\ker(f,g) \arrow[r] \arrow[d] & X \arrow[r, "g"] \arrow[d, "\Gamma_g"] & Y \arrow[d, "\Delta_Y"] \\
X \arrow[r, "\Gamma_f"] \arrow[d, "f"] & X \times_S Y \arrow[r, "g \times_S \mathbf{1}_Y"] \arrow[d, "f \times_S \mathbf{1}_Y"] & Y \times_S Y \\
Y \arrow[r, "\Delta_Y"] & Y \times_S Y &
\end{tikzcd}$$

where all squares are cartesian.

Proof. The fact that $\ker(f,g)$ is identified with the kernel of Γ_f and Γ_g can be deduced from by applying the projection $X \times_S Y \to Y$, or by Yoneda since this is clearly true for sets. The other two small squares are cartesian by Corollary 4.5.4, so the claim follows from the transitivity of products. □

Proposition 4.5.14. *Let $\mathcal{P}$ be a property for morphisms of schemes and consider the following conditions:*

(i) If $j : X \to Y$ is an immersion and $g : Y \to Z$ is a morphism possessing the property $\mathcal{P}$, then $g \circ j$ possesses the property $\mathcal{P}$.

(ii) If $f : X \to X'$ and $g : y \to Y'$ are two S-morphisms possessing the property $\mathcal{P}$, then $f \times_S g$ possesses the property $\mathcal{P}$.

(iii) If the composition $g \circ f$ of two morphisms $f : X \to Y$, $g : Y \to Z$ possesses the property $\mathcal{P}$, then f possesses the property $\mathcal{P}$.

Then (iii) is a concequence of (i) and (ii).

Proof. The morphism f factors into

$$X \xrightarrow{\Gamma_f} X \times_Z Y \xrightarrow{p_2} Y$$

where p_2 is the projection. On the other hand, p_2 is idnetified with $(g \circ f) \times_Z 1_Y$, and by (ii) it possesses the property $\mathcal{P}$; as Γ_f is an immersion, it then follows from (i) that f possesses $\mathcal{P}$. □

Proposition 4.5.15. *Let $f : X \to Y$ and $g : Y \to Z$ be two morphisms. Consider the following properties for a morphism:*

(i) a monomorphism;

(ii) an immersion;

(iii) a local immersion;

(iv) a universal embedding;

(v) radical.

Then, if $g \circ f$ possesses one of these properties, so does f.

Proof. The properties (i) and (v) have only been put for memory, because for (i) this is a property valid for any category, and for (v), the proposition has already been proven in Proposition 4.3.29.

An immersion has each of these properties, and the composition of two morphisms having one (determined) of these properties also possesses it; moreover, all the above properties are stable under products. Thus the claim follows from Proposition 4.5.14. □

Corollary 4.5.16. *Let $j : X \to Y$ and $g : X \to Z$ be two S-morphisms. If j possesses one of the properties in Proposition 4.5.15, so does $(j, g)_S$.*

Proof. In fact, if $p : Y \times_S Z \to Y$ is the projection, we have $j = p \circ (j, g)_S$, and it suffices to apply Proposition 4.5.15. □

4.5.2 Separated morphisms and schemes

A morphism $f : X \to Y$ of schemes is called **separated** if the diagonal morphism $\Delta_f : X \to X \times_Y X$ is a closed immersion. In this case, X is said to be a **separated Y-scheme**, or **separated over Y**. A scheme X is called **separated** if it is separated over $\mathbb{Z}$. In view of Proposition 4.5.6, for X to be separated over Y, it is necessary and sufficient that the diagonal is a closed subscheme of $X \times_Y X$.

Proposition 4.5.17. *Any morphism of affine schemes is separated. In particular, any affine scheme is separated.*

Proof. If $f : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is a morphism of affine schemes, where B is an A-algebra, the diagonal morphism corresponds to the ring homomorphism $B \otimes_A B \to B$ given by $b \otimes b' \mapsto bb'$. Since this is surjective, we conclude that Δ_f is a closed immersion, so f is separated. □

Proposition 4.5.18. *Let X, Y be S-schemes and $S \to T$ be a separated morphism. Then the canonical immersion $X \times_S Y \to X \times_T Y$ in (5.1.1) is closed.*

Proof. In fact, in the diagram (5.1.1), the diagonal $\Delta_{S/T}$ is a closed immersion, so its base change $\varphi \times_T \psi : X \times_S Y \to X \times_T Y$ is also closed. □

Corollary 4.5.19. *Let X, Y be S-schemes and $f : X \to Y$ be an S-morphism. If Y is spearated over S, the graph morphism $\Gamma_f : X \to X \times_S Y$ is a closed immersion. In particular, any S-section of Y is a closed immersion.*

Proof. This follows from (5.1.2) and Proposition 4.4.12. □

Proposition 4.5.20. *Let Y be a separated S-scheme. Then for any S-morphisms $u_1 : X \to Y$, $u_2 : X \to Y$, the kernel of u_1 and u_2 is a closed subscheme of X.*

Proof. Recall that by Proposition 4.5.11 the kernel is the inverse image of $\Delta_Y(Y)$ under the morphism $(u_1, u_2)_S$. Since $\Delta_Y(Y)$ is closed, it follows from Proposition 4.4.12 that its inverse image is also closed. □

Corollary 4.5.21. *Let S be an integral scheme, η its generic point, and X a separated S-scheme. If two S-sections u, v of X satisfy $u(\eta) = v(\eta)$, then $u = v$.*

Proof. In fact, if $x = u(\eta) = v(\eta)$, the corresponding homomorphisms $\kappa(x) \to \kappa(\eta)$ are necessarily identical, since their composition with the homomorphism $\kappa(\eta) \to \kappa(x)$ corresponding to the structural morphism $X \to S$ is the idnetity on $\kappa(\eta)$. We then deduce from Proposition 4.5.12 that $\eta \in \ker(u_1, u_2)$, and by hypothesis $\ker(u_1, u_2)$ is a closed subscheme of S (Proposition 4.5.20). As S is reduced and η is its generic point, the unique closed subscheme of S containing η is S (Corollary 4.4.26), so $u = v$. □

Proposition 4.5.22. *Let $\mathcal{P}$ be a property of morphisms of schemes, and consider the following properties:*

(i) If $j : X \to Y$ is a closed immersion and $g : Y \to Z$ is a morphism possessing the property $\mathcal{P}$, then $g \circ j$ possesses the property $\mathcal{P}$.

(ii) If $f : X \to X'$ and $g : Y \to Y'$ are two S-morphisms possessing the property $\mathcal{P}$, then $f \times_S g$ possesses the property $\mathcal{P}$.

(iii) If the composition $g \circ f$ of two morphisms $f : X \to Y$ and $g : Y \to Z$ possesses the property $\mathcal{P}$ and if g is separated, then f possesses the property $\mathcal{P}$.

(iv) If $f : X \to Y$ possesses the property $\mathcal{P}$, so does f_{red}.

Then, (iii) and (iv) are concequences of (i) and (ii).

Proof. For the property (iii), the demonstration is similar to Proposition 4.5.14, with the fact that Γ_f is a closed immersion by Corollary 4.5.19. On the other hand, in the commutative diagram

$$\begin{tikzcd} X_{\mathrm{red}} \arrow[r, "f_{\mathrm{red}}"] \arrow[d] & Y_{\mathrm{red}} \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

the vertical morphisms are closed immersions, so we see that (iv) is a consequence of (i) and (iii), observing that a closed immersion is separated in view of the definition and Proposition 4.5.5. □

Proposition 4.5.23. *Let $f : X \to Y$ be a morphism and $g : Y \to Z$ be a separated morphism. Suppose that $g \circ f$ possesses one of the following properties:*

(i) universally closed;

(ii) a closed immersion;

Then f possesses the same property.

Proof. In fact, these properties satisfy the conditions (i) and (ii) in Proposition 4.5.22. □

Corollary 4.5.24. *Let Z be a separated S-scheme and $f : X \to Y$, $g : X \to Z$ be two S-morphisms. If f is universally closed (resp. a closed immersion), so is $(f, g)_S : X \to Y \times_S Z$.*

Proof. The morphism j factors into

$$X \xrightarrow{(f,g)_S} Y \times_S Z \xrightarrow{p} Y$$

and the projection $p : Y \times_S Z \to Y$ is a separate morphism by Proposition 4.5.25 (which do not use Corollary 4.5.24), so it suffices to apply Proposition 4.5.23. □

Remark 4.5.1. From the diagram in Proposition 4.5.13, we conclude that a morphism $Y \to S$ is separated if and only if the following equivalent conditions holds:

(i) The diagonal morphism $\Delta_{Y/S}$ is a closed immersion.

(ii) For every S-scheme X and for any two S-morphisms $f, g : X \to Y$, the kernel $\ker(f, g)$ is a closed subscheme of X.

(iii) For every S-scheme X and for any S-morphism $f : X \to Y$, the graph morphism Γ_f is a closed immersion.

Also, if the conclusion in Proposition 4.5.23 holds for the morphisms $\Delta_Y : Y \to Y \times_S Y$ and $p_2 : Y \times_S Y \to Y$, then Δ_Y is a closed immersion so Y is separated over S.

4.5.3 Criterion of separated morphisms

Proposition 4.5.25 (Propoerties of Seperated Morphisms).

(i) Any radical morphism (and in particular any monomorphism, hence any immersion) is a separated morphism.

(ii) The composition of two separated morphisms is separated.

(iii) If $f : X \to X'$ and $g : Y \to Y'$ are two separated S-morphisms, $f \times_S g$ is separated.

(iv) If $f : X \to Y$ is a separated S-morphism, the S'-morphism $f_{(S')}$ is separated for any base change $S' \to S$.

(v) If the composition $g \circ f$ of two morphisms is separated, then f is separated.

(vi) For a morphism f to be separated, it is necessary and sufficient that f_{red} is separated.

Proof. Recall that for a radical morphism the diagonal morphism is surjective (Proposition 4.3.29), so it is separated. If $f : X \to Y$, $g : Y \to Z$ are two morphisms, the diagram

$$\begin{tikzcd} X \arrow[rr, "\Delta_{X/Z}"] \arrow[dr, "\Delta_{X/Y}"'] & & X \times_Z X \\ & X \times_Y X \arrow[ur, "j"'] & \end{tikzcd}$$

where j is the canonical immersion in (5.1.1), is commutative. If f and g are separated, $\Delta_{X/Y}$ is a closed immersion and j is a closed immersion by Proposition 4.5.18, so $\Delta_{X/Z}$ is closed, which proves (ii). With (i) and (ii), conditions (iii) and (iv) are equivalent, and it suffices to prove (iv).

Now by transitivity, $X_{(S')} \times_{Y_{(S')}} X_{(S')}$ is identified with $(X \times_Y X) \times_Y Y_{(S')}$, and the diagonal morphism $\Delta_{X_{(S')}}$ is identified with $\Delta_X \times_Y 1_{Y_{(S')}}$. The assertion then follows from Proposition 4.4.12.

To prove (v), consider the factorization

$$X \xrightarrow{\Gamma_f} X \times_Z Y \xrightarrow{p_2} Y$$

of f, and note that $p_2 = (g \circ f) \times_Z 1_Y$; the hypothesis that $g \circ f$ is separated implies p_2 is separated by (iii), and as Γ_f is an immersion, Γ_f is separated by (i), hence f is separated by (ii). Finally, for (vi), we recall that the schemes $X_{\mathrm{red}} \times_{Y_{\mathrm{red}}} X_{\mathrm{red}}$ and $X_{\mathrm{red}} \times_Y X_{\mathrm{red}}$ is canonically identified (Proposition 4.4.35); if $j : X_{\mathrm{red}} \to X$ is the canonical injection, the diagram

$$\begin{tikzcd}
X_{\mathrm{red}} \arrow[r, "\Delta_{X_{\mathrm{red}}}"] \arrow[d, "j"'] & X_{\mathrm{red}} \times_Y X_{\mathrm{red}} \arrow[d, "j \times_Y j"] \\
X \arrow[r, "\Delta_X"] & X \times_Y X
\end{tikzcd}$$

is commutative, and the assertion follows from the fact that the vertical morphisms are homeomorphisms. □

Corollary 4.5.26. *If $f : X \to Y$ is separated, the restriction of f to any subscheme of X is separated.*

Proof. This follows from Proposition 4.5.25(i) and (iii). □

Corollary 4.5.27. *If X and Y are S-schemes and Y is separated over S, $X \times_S Y$ is separated over X.*

Proof. This is a particular case of Proposition 4.5.25(iv). □

Proposition 4.5.28. *Let X be a scheme and suppose that $(X_i)_{1 \le i \le n}$ is a finite covering of X by closed subsets. Let $f : X \to Y$ be a morphism and for each i let Y_i be a closed subset of Y such that $f(X_i) \subseteq Y_i$. Consider the reduced subscheme structures on each X_i and Y_i and let $f_i : X_i \to Y_i$ be the restriction of f on X_i. Then for f to be separated, it is necessary and sufficient that each f_i is separated.*

Proof. The necessity follows from Proposition 4.5.25(i), (ii) and (v). Conversely, if each f_i is separated, then the restriction $X_i \to Y$ of f is separated (Proposition 4.5.25). If p_1, p_2 are the projections of $X \times_Y X$, the subspace $\Delta_{X_i}(X_i)$ is identified with the subspace $\Delta_X(X) \cap p_1^{-1}(X_i)$ of $X \times_Y X$ (Proposition 4.5.10). This subspace is closed in $X \times_Y X$ by hypothesis, and their union is $\Delta_X(X)$, so Δ_X is closed and f is separated. □

Suppose in particular that X_i are the irreducible components of X; then we can suppose that each Y_i is a irreducible closed subset of Y (Proposition **??**); Proposition 4.5.28 then enable us to reduce the separation problem to integral schemes.

Proposition 4.5.29. *Let (Y_λ) be an open covering of a scheme Y. For a morphism $f : X \to Y$ to be separated, it is necessary and sufficient that each of the restriction $f_\lambda : f^{-1}(Y_\lambda) \to Y_\lambda$ is separated.*

Proof. If we set $X_\lambda = f^{-1}(Y_\lambda)$ and identify $X_\lambda \times_Y X_\lambda$ and $X_\lambda \times_{Y_\lambda} X_\lambda$, the $X_\lambda \times_Y X_\lambda$ form an open covering of $X \times_Y X$. If $Y_{\lambda\mu} = Y_\lambda \cap Y_\mu$ and $X_{\lambda\mu} = X_\lambda \cap X_\mu = f^{-1}(Y_{\lambda\mu})$, then $X_\lambda \times_Y X_\mu$ is identified with $X_{\lambda\mu} \times_{Y_{\lambda\mu}} X_{\lambda\mu}$ by Corollary 4.3.2, hence with $X_{\lambda\mu} \times_Y X_{\lambda\mu}$, and finally to an open subset of $X_\lambda \times_Y X_\lambda$, which establishes our assertion (Proposition 4.4.10). □

Proposition 4.5.29 allows, by taking a covering of Y byopen affines, to reduce the study of separated morphisms to separated morphisms with values in affine schemes.

Proposition 4.5.30. *Let Y be an affine scheme, X be a scheme, and (U_α) be an affine open covering of X. For a morphism $f : X \to Y$ to be separated, it ie necessary and sufficient that, for any couple of indices (α, β), the intersection $U_\alpha \cap U_\beta$ is affine, and the ring $\Gamma(U_\alpha \cap U_\beta, \mathscr{O}_X)$ is generated by the images of $\Gamma(U_\alpha, \mathscr{O}_X)$ and $\Gamma(U_\beta, \mathscr{O}_X)$.*

Proof. The open sets $U_\alpha \times_Y U_\beta$ form an open cover of $X \times_Y X$ (Proposition 4.3.2). Let p, q be the projections of $X \times_Y X$, we have

$$\Delta_X^{-1}(U_\alpha \times_Y U_\beta) = \Delta_X^{-1}(p^{-1}(U_\alpha) \cap q^{-1}(U_\beta)) = U_\alpha \cap U_\beta.$$

It therefore amounts to show that the restriction of Δ_X to $U_\alpha \cap U_\beta$ is a closed immersion into $U_\alpha \times_Y U_\beta$. But this restriction is just $(j_\alpha, j_\beta)_Y$, where j_α (resp. j_β) is the canonical injection of $U_\alpha \cap U_\beta$ to U_α (resp. to U_β). As $U_\alpha \times_Y U_\beta$ is an affine scheme with ring isomorphic to $\Gamma(U_\alpha, \mathscr{O}_X) \otimes_{\Gamma(Y, \mathscr{O}_Y)} \Gamma(U_\beta, \mathscr{O}_X)$, we see that $U_\alpha \cap U_\beta$ is a closed subscheme of $U_\alpha \times_Y U_\beta$ if and only if it is affine and the ring homomorphism

$$\Gamma(U_\alpha, \mathscr{O}_X) \otimes_{\Gamma(Y, \mathscr{O}_Y)} \Gamma(U_\beta, \mathscr{O}_X) \to \Gamma(U_\alpha \cap U_\beta, \mathscr{O}_X), \quad h_\alpha \otimes h_\beta \to h_\alpha h_\beta$$

is surjective (Example 4.4.6), which proves our assertion. □

Corollary 4.5.31. *Let Y be an affine scheme. For a morphism $f : X \to Y$ to be separated, it is necessary and sufficient that X is separated.*

Proof. In fact, the criterion in Proposition 4.5.30 does not depend on f. □

Corollary 4.5.32. *For a morphism $f : X \to Y$ to be separated, it is necessary that for any open affine subscheme U that is separated, the open subscheme $f^{-1}(U)$ is separated, and it suffices that this ture for every affine open subset $U \subseteq Y$.*

Proof. The necessity follows from Proposition 4.5.28 and Proposition 4.5.25(ii). The sufficiency follows from Proposition 4.5.29 and Corollary 4.5.31. □

Proposition 4.5.33. *Let Y be a separated scheme and $f : X \to Y$ be a morphism. For any affine open U of X and any affine open V of Y, $U \cap f^{-1}(V)$ is affine.*

Proof. Let p_1, p_2 be the projections of $X \times_{\mathbb{Z}} Y$. Using the universal property of Γ_f, the subspace $U \cap f^{-1}(V)$ can be characterized by

$$\Gamma_f(U \cap f^{-1}(V)) = \Gamma_f(X) \cap p_1^{-1}(U) \cap p_2^{-1}(V)$$

Now $p_1^{-1}(U) \cap p_2^{-1}(V)$ is idnetified with the product $U \times_{\mathbb{Z}} V$, and therefore is affine; as $\Gamma_f(X)$ is closed in $X \times_{\mathbb{Z}} Y$ (Corollary 4.5.19), $\Gamma_f(X) \cap p_1^{-1}(U) \cap p_2^{-1}(V)$ is closed in $U \times_{\mathbb{Z}} V$, hence also affine. The assertion then follows from the fact that Γ_f is a closed immersion and Example 4.4.6. □

Example 4.5.34. The scheme in Example 4.2.8 is separated. In fact, for the covering (X_1, X_2) of X by affine opens, $X_1 \cap X_2 = U_{12}$ is affine and $\Gamma(U_{12}, \mathcal{O}_X)$, the fraction ring of the form $f(s)/s^m$ where $f \in K[s]$, is generated by $K[s]$ and $1/s$, so the conditions in Proposition 4.5.30 are satisfied.

With the same choice of X_1, X_2, U_{12} and U_{21} as in Example 4.2.8, take this time for u_{12} the isomorphism which sends $f(s)$ to $f(t)$; this time we obtain by gluing together a non-separated integral scheme X, because the first condition of Proposition 4.5.30 holds, but the second fails. It is immediate here that $\Gamma(X, \mathcal{O}_X) \to \Gamma(X_1, \mathcal{O}_X) = K[s]$ is an isomorphism; the inverse isomorphism defines a morphism $f : X \to \operatorname{Spec}(K[s])$ which is surjective, and for any $y \in \operatorname{Spec}(K[s])$ such that $\mathfrak{p}_y \neq (0)$, $f^{-1}(y)$ is reduced to a singleton, but for $\mathfrak{p}_y = (0)$, $f^{-1}(y)$ consists of two distinct points (we say that X is the "affine line on K, where the point 0 is doubled").

We can also give examples where neither of the two conditions of Proposition 4.5.30 does not hold. Note first that in the prime spectrum Y of the ring of polynomials $A = K[s, t]$ in two indeterminates over a field K, the open set $U = D(s) \cup D(t)$ is not an affine open set. Indeed, if z is a section of $\mathcal{O}_Y$ over U, there exist two integers $m, n \geq 0$ such that $s^m z$ and $t^n z$ are the restrictions to U of polynomials in s and t (Proposition 4.1.20), which is obviously only possible if the section z extends into a section over the entire space Y, identified with a polynomial in s and t. If U were affine, the canonical injection $U \to Y$ would therefore be an isomorphism by Proposition 4.1.16, which is absurd since $U \neq Y$.

This being so, let us take two affine schemes Y_1, Y_2, with rings $A_1 = K[x_1, t_1]$, $A_2 = K[s_2, t_2]$. Let $U_{12} = D(s_1) \cup D(t_1)$, $U_{21} = D(s_2) \cup D(t_2)$, and let u_{12} be the restriction to U_{21} of the isomorphism $Y_2 \to Y_1$ corresponding to the isomorphism of rints, which sends $f(s_1, t_1)$ to $f(s_2, t_2)$. We thus obtain an example where none of the conditions of Proposition 4.1.16 is satisfied (the integral scheme thus obtained is called "affine plane over K, where point 0 is doubled").

4.6 Finiteness conditions for morphisms

We study, in this section, various "finiteness conditions" for a morphism $f : X \to Y$ of schemes. There are basically two notions of "global finiteness" on X: quasi-compactness and quasi-separateness. On the other hand, there is two notions of "local finiteness" on X: locally of finite type and locally of finite presentation. By combining these local notions with the previous global notions, we obtain the notions of morphism of finite type and of morphism of finite presentation.

4.6.1 Quasi-compact and quasi-separated morphisms

We say a morphism $f : X \to Y$ is **quasi-compact** is for any quasi-compact open subset U of Y, the inverse image $f^{-1}(U)$ is quasi-compact. It is clear that this condition is purely topological, and if X is Noetherian, then any morphism $f : X \to Y$ is quasi-compact. We say a Y-scheme X is **quasi-compact over Y** if its structural morphism is quasi-compact.

If $\mathcal{B}$ is a base of Y formed by quasi-compact open sets (for example, affine opens), for a morphism f to be quasi-compact, it is necessary and sufficient that for any open set $V \in \mathcal{B}$, $f^{-1}(V)$, since any quasi-compact open set of Y is a finite union of open sets in $\mathcal{B}$.

If $f : X \to Y$ is a quasi-compact morphism, it is clear that for any open set V of Y, the restriction $f^{-1}(V) \to V$ of f is quasi-compact. Conversely, if (U_α) is an open covering of Y and $f : X \to Y$ is a morphism such that the restriction $f^{-1}(U_\alpha) \to U_\alpha$ is quasi-compact, then f is quasi-compact, since there exist a basis of quasi-compact open sets for Y, each set of which is contained in at least one of the U_α. We conclude that if $f : X \to Y$ is an S-morphism of S-schemes, and if there is an open covering (S_λ) of S such that the restrictions $\varphi^{-1}(S_\lambda) \to \psi^{-1}(S_\lambda)$ of f (where $\varphi : X \to S$ and $\psi : Y \to S$ are the structural morphisms) are quasi-compact morphisms, then f is quasi-compact.

Proposition 4.6.1. *Let Y be a separated scheme. Then for a morphism $f : X \to Y$ to be quasi-compact, it is necessary and sufficient that X is quasi-compact.*

Proof. If X is quasi-compact, it is a union of finitely many affine opens U_i, and for any affine open V of Y, $U_i \cap f^{-1}(V)$ is an affine open by Proposition 4.5.33, hence quasi-compact; therefore $f^{-1}(V)$ is quasi-compact. Conversely, if f is a quasi-compact morphism, then since Y is quasi-compact open in Y, we see $X = f^{-1}(Y)$ is also quasi-compact. □

Example 4.6.2. A closed immersion is quasi-compact since a closed subset of a quasi-compact set is again quasi-compact. However, open immersions are in general not quasi-compact: the standard example is the affine scheme $X = \mathrm{Spec}(k[x_1, x_2, \dots])$ and consider $U = X - \{0\}$, where 0 is the point of X corresponding to the maximal ideal $(x_1, x_2, \dots)$. The canonical injection $j : U \to X$ is not quasi-compact because U is not quasi-compact. To see this, consider the covering $(D(x_i))_{i\in\mathbb{N}}$ of U; for any finite subset J of $\mathbb{N}$, the family $(D(x_i))_{i\in J}$ can not cover U simply because the prime ideal $\mathfrak{p}_J$ generated by x_i with $i \in J$ is contained in U but not in the union of the $D(x_i)$ for $i \in J$.

We say a morphism $f : X \to Y$ of schemes is **quasi-separated** (of X is an **Y-scheme quasi-separated over Y**) if the diagonal morphism $\Delta_f : X \to X \times_Y X$ is quasi-compact. A scheme X is called **quasi-separated** if it is quasi-separated over $\mathbb{Z}$. Since a closed immersion is quasi-compact, we see any separated morphism is quasi-separated. In particular, any separated scheme is quasi-separated.

Proposition 4.6.3 (Properties of Quasi-Compact Morphisms).

(i) An immersion $j : X \to Y$ is quasi-compact if it is closed, or Y is locally Noetherian, or X is Noetherian.

(ii) The composition of two quasi-compact morphisms is quasi-compact.

(iii) If $f : X \to Y$ is a quasi-compact S-morphism, $f_{(S')} : X_{(S')} \to Y_{(S')}$ is quasi-compact for any base change $g : S' \to S$.

(iv) If $f : X \to X'$ and $g : Y \to Y'$ is two quasi-compact S-morphisms, then $f \times_S g$ is quasi-compact.

(v) If the composition $g \circ f$ of two morphisms $f : X \to Y$, $g : Y \to Z$ is quasi-compact, and if f is quasi-separated or if X is locally Noetherian, then f is quasi-compact.

(vi) For a morphism f to be quasi-compact, it is necessary and sufficient that f_{red} is quasi-compact.

Proof. Assertion (vi) is evident, since the quasi-compactness for a morphism only depends on the map of the underlying spaces. Also, (ii) follows from the definition of quasi-compactness.

The assertion in (i) is clear if j is closed; if j is an immersion and Y is locally Noetherian, any quasi-compact open V of Y is Noetherian, so $j^{-1}(V) = X \cap V \subseteq V$ is quasi-compact (here we identify X as a subscheme of Y). If X is Noetherian, then any morphism from X is quasi-compact.

To prove (iii), we can assume that $S = Y$ by the transitivity of products; put $f' = f_{(S')}$, and let U' be a quasi-compact open subset of S'. For any $s' \in U'$, let T be an affine open neighborhood of $g(s')$ in S, and let W be an affine open neighborhood of s' contained in $U' \cap g^{-1}(T)$; it suffices to show that $f'^{-1}(W)$ is quasi-compact, or in other words, we only need to show that if S and S' are affine, then $X \times_S S'$ is quasi-compact. This is true because by hypothesis X is a finite union of affine opens V_j, and $X \times_S S'$ is then the union of the affine schemes $V_j \times_S S'$, hence quasi-compact. With (ii) and (iii), asserion (iv) then follows.

We now prove (v) in the case where X is locally Noetherian. Put $h = g \circ f$ and let U be a quasi-compact open of Y; $g(U)$ is then quasi-compact in Z (not necessarily open), so it is contained in a finite union of quasi-compact opens V_j, and $f^{-1}(U)$ is contained in the union of the $h^{-1}(V_j)$, which are all quasi-compact by hypothesis. We then conclude that $f^{-1}(U)$ is a Noetherian space (Proposition **??**), and a fortiori quasi-compact.

To prove (v) in the case that g is quasi-separated, recall that f factors into

$$X \xrightarrow{\Gamma_f} X \times_Z Y \xrightarrow{p_2} Y$$

where p_2 is identified with $(g \circ f) \times_Z 1_Y$, and if $g \circ f$ is quasi-compact, so is p_2 by (iii). Finally, we have the following cartesian square (Corollary 4.5.4)

$$\begin{tikzcd} X \arrow[r, "\Gamma_f"] \arrow[d, "f"'] & X \times_Z Y \arrow[d, "f \times 1_Y"] \\ Y \arrow[r, "\Delta_g"] & Y \times_Z Y \end{tikzcd}$$

as by hypothesis Δ_g is quasi-compact, Γ_f is also quasi-compact, and by (ii) we conclude that f is quasi-compact. □

Proposition 4.6.4. *Let $f : X \to Y$, $g : Y \to Z$ be two morphisms. If $g \circ f$ is quasi-compact and f is surjective, then g is quasi-compact.*

Proof. If fact, if V is a quasi-compact open of Z, $f^{-1}(g^{-1}(V))$ is quasi-compact by hypothesis, and we have $g^{-1}(V) = f(f^{-1}(g^{-1}(V)))$ since f is surjective, so $g^{-1}(V)$ is quasi-compact. □

Proposition 4.6.5. *Let f be a quasi-compact morphism of schemes.*

(a) The following conditions are equivalent:

(i) f is dominant;

(ii) for any maximal point $y \in Y$, $f^{-1}(y) \neq \varnothing$.

(iii) for any maximal point $y \in Y$, $f^{-1}(y)$ contains a maximal point of X.

(b) *If f is dominant, for any generalizing morphism $g : Y' \to Y$, the morphism $f_{(Y')} : X_{(Y')} \to Y'$ is quasi-compact and dominant.*

Proof. It is clear that (iii)⇒(ii)⇒(i). Suppose that f is dominant and consider an affine open neighborhood U of y; $f^{-1}(U)$ is quasi-compact, hence a union of finitely many affine opens V_i, and by hypothesis y belongs to the closure of $f(V_i)$ in U. We can evidently suppose that X and Y are reduced. As the closure in X of an irreducible component of V_i is an irreducible component of X (Proposition **??**), we can replace X by V_i, Y by the closed reduced subscheme $\overline{f(V_i)} \cap U$ of U, and we are thus reduced to proving the proposition when $X = \operatorname{Spec}(B)$ and $Y = \operatorname{Spec}(A)$ are affine and reduced. Since f is dominant, A is then a subring of B (Proposition **??**); the proposition then follows from the fact that any minimal prime ideal of A is the intersection with A of a minimal prime ideal of B (Proposition **??**).

If $f : X \to Y$ is quasi-compact and dominant, then $f' = f_{(Y')}$ is quasi-compact by Proposition 4.6.3. On the other hand, a maximal point y' of Y' is hypothesis lying over a maximal point y of Y (Proposition **??**); as $f^{-1}(y)$ is nonempty by (i), the same holds for $f'^{-1}(y')$ (Proposition 4.3.35), whence the conclusion. □

Proposition 4.6.6. *For a quasi-compact morphism $f : X \to Y$, the following conditions are equivalent:*

(i) *The morphism f is closed.*

(ii) *For any $x \in X$ and any specialization y' of $y = f(x)$ distinct from y, there exists a specialization x' of x such that $f(x') = y'$.*

In particular, if $f : X \to Y$ is a quasi-compact immersion, for f to be a closed immersion, it is necessary and sufficient that X (considered as a subspace of Y) contains any specializations (in Y) of its points.

Proof. The condition (ii) expresses as $f(\overline{\{x\}}) = \overline{\{y\}}$, and is therefore a concequence of (i). To show that (ii) implies (i), consider a closed subset of X' of X; let $Y' = \overline{f(X')}$ and we prove that $Y' = f(X')$. Endow X' and Y' the reduced subscheme structure, there then exists a morphism $f' : X' \to Y'$ such that the diagram

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d] & Y' \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

commutes. As f is quasi-compact, so is f' (Proposition 4.6.3(i) and (v)). We are then reduced to proving that if f is a quasi-compact dominant morphism, then $f(X) = Y$. Now let y' be a point of y and let y be the generic point of a irreducible component of Y containing y'; by (ii), it suffices to note that $f^{-1}(y)$ is nonempty, which follows from Proposition 4.6.5. □

Proposition 4.6.7 (Properties of Quasi-Separated Morphisms).

(i) *Any radical morphism $f : X \to Y$ (in particular, any monomorphism and any immersion) is quasi-separated.*

(ii) *If $f : X \to Y$ and $g : Y \to Z$ are two quasi-separated morphisms, $g \circ f$ is quasi-separated.*

(iii) *Let X, Y be two S-schemes and $f : X \to Y$ be a quasi-separated S-morphism. Then, for any base change $g : S' \to S$, the morphism $f_{(S')} : X_{(S')} \to Y_{(S')}$ is quasi-separated.*

(iv) *If $f : X \to X'$, $g : Y \to Y'$ are two quasi-separated S-morphisms, $f \times_S g$ is quasi-separated.*

(v) *If the composition $g \circ f$ of two morphisms $f : X \to Y$, $g : Y \to Z$ is quasi-separated, then f is quasi-separated; if moreover f is quasi-compact and surjective, g is also quasi-separated.*

(vi) *For a morphism $f : X \to Y$ to be quasi-separated, it is necessary and sufficient that f_{red} is quasi-separated.*

Proof. The assertion (i) follows from Proposition 4.5.25(i). To prove (iii), we may reduce to the case $Y = S$, and the assertion then follows from $\Delta_{f_{(S')}} = (\Delta_f)_{(S')}$ (Proposition 4.5.2) and Proposition 4.6.3.

For assertion (ii), consider the projections p, q of $X \times_Y X$; if $\pi : X \times_Y X \to Y$ is the structural morphism and $j = (p, q)_Z$, we have the following cartesian square (Proposition 5.1.1)

$$\begin{tikzcd} X \times_Y X \arrow[r, "j"] \arrow[d, "\pi"'] & X \times_Z X \arrow[d, "f \times_Z f"] \\ Y \arrow[r, "\Delta_g"] & Y \times_Z Y \end{tikzcd}$$

If g is quasi-separated then Δ_g is quasi-compact, so j is also quasi-compact by Proposition 4.6.3(iii). If f is quasi-separated, Δ_f is quasi-compact and so is $j \circ \Delta_f$, which equals to $\Delta_{g \circ f}$. With these, assertion (iv) then follows from (ii) and (iii).

Suppose now that $g \circ f$ is quasi-separated. Then with the preceding notations, $\Delta_{g \circ f} = j \circ \Delta_f$ is quasi-compact, so Δ_f is quasi-compact by Proposition 4.6.3(v) and f is then quasi-separated. If moreover f is quasi-compact and surjective, $f \times_{\mathbb{Z}} f$ is also quasi-compact by Proposition 4.6.3(iv), and we conclude that $\Delta_g \circ \pi \circ \Delta_f$ is quasi-compact. Since $\pi \circ \Delta_f = f$ is surjective, it follows from Proposition 4.6.3(v) that Δ_g is quasi-compact, so g is quasi-separated.

Finally, for a morphism $f : X \to Y$, consider the following diagram

$$\begin{tikzcd} X_{\mathrm{red}} \arrow[r, "f_{\mathrm{red}}"] \arrow[d, "j_X"'] & Y_{\mathrm{red}} \arrow[d, "j_Y"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

where j_X and j_Y are surjective closed immersions, and so quasi-separated and quasi-compact. From the equality $f \circ j_X = j_Y \circ f_{\mathrm{red}}$ and (v), we see f is quasi-separated if and only if f_{red} is quasi-separated. □

Corollary 4.6.8. *Let X and Y be schemes.*

(i) *If f is quasi-separated, any morphism $f : X \to Y$ is quasi-separated.*

(ii) *If Y is quasi-separated, for a morphism $f : X \to Y$ to be quasi-separated, it is necessary and sufficient that the scheme X is quasi-separated.*

(iii) *Let X be a quasi-compact and Y be quasi-separated. Then any morphism $f : X \to Y$ is quasi-compact.*

Proof. To show (i) we only need to note that any morphism $f : X \to Y$ is a $\mathbb{Z}$-morphism, and if X is quasi-separated, then for any morphism $f : X \to Y$ the composition $X \to Y \to \mathrm{Spec}(\mathbb{Z})$ is quasi-separated, so f is quasi-separated by Proposition 4.6.7(v). Similarly, assertion (ii) follows from Proposition 4.6.7(ii) and (v). Assertion (iii) follows from Proposition 4.6.3(v). □

Proposition 4.6.9. *Let $f : X \to Y$ be a morphism and (U_α) be a covering of Y by open subschemes that are quasi-separated. For f to be quasi-separated, it is necessary and sufficient that the open subschemes $f^{-1}(U_\alpha)$ is quasi-separated.*

Proof. The inverse image of U_α in $X \times_Y X$ is $X_\alpha \times_{U_\alpha} X_\alpha$, where $X_\alpha = f^{-1}(U_\alpha)$, and the restriction $X_\alpha \to X_\alpha \times_{U_\alpha} X_\alpha$ of Δ_f is just Δ_{f_α}, where f_α is the restriction $X_\alpha \to U_\alpha$ of f. Since quasi-compactness is local on target, we see f is quasi-separated if and only if each f_α is. But by hypothesis U_α is separated, so the conclusion follows from Proposition 4.6.8(ii). □

By Proposition 4.6.9, to verify a morphism is quasi-separated, it suffices to verify the quasi-separateness of some subschemes. This can be done by the following simple criteria:

Proposition 4.6.10. *Let X be a scheme and (U_α) be a covering of X formed by quasi-compact open subsets. Then the following conditions are equivalent:*

(i) X is a quasi-separated scheme.

(ii) For any quasi-compact open subset U of X, the canonical injection $U \to X$ is quasi-compact (that is, U is retrocompact in X).

(iii) The intersection of two quasi-compact open subsets of X is quasi-compact.

(iv) For any couple of indices (α, β), the intersection $U_\alpha \cap U_\beta$ is quasi-compact.

Proof. Properties (ii) and (iii) are equivalent by the definition of quasi-compactness. As a quasi-compact open is a finie union of affine open sets, for two quasi-compact open subsets U, V of X, $U \times_{\mathbb{Z}} V$ is a quasi-compact open subset of $X \times_{\mathbb{Z}} X$ (Corollary 4.3.2), with inverse image $U \cap V$ under Δ_X, hence (i) implies (iii). It is clear that (iii) implies (iv); finally, if (iv) holds, the $U_\alpha \times_{\mathbb{Z}} U_\beta$ form a covering of $X \times_{\mathbb{Z}} X$ by quasi-compact open sets and the inverse image of $U_\alpha \times_{\mathbb{Z}} U_\beta$ under Δ_X is $U_\alpha \cap U_\beta$, hence quasi-compact. It then follows that Δ_X is quasi-compact, so (iv) implies (i). □

Corollary 4.6.11. *Any locally Noetherian scheme X is quasi-separated, and any morphism $f : X \to Y$ is then quasi-separated.*

Proof. It suffices to note that any quasi-compact open subset of X is Noetherian, so X is quasi-separated by Proposition 4.6.10 and any morphism $f : X \to Y$ is quasi-separated by Proposition 4.6.9, since any open subscheme of X is again locally Noetherian. □

Proposition 4.6.12. *Let $f : X \to Y$ be a morphism and $g : Y' \to Y$ be a base change that is surjective and quasi-compact. Put $f' = f_{(Y')}$ and consider the following properties:*

(i) quasi-compact;

(ii) quasi-separated.

Then if $\mathcal{P}$ denotes one of these properties and f' possesses the property $\mathcal{P}$, then f possesses the property $\mathcal{P}$.

Proof. Let $g' : X' \to X$ be the cannical projection, which is surjective and quasi-compact (Proposition 4.3.26 and Proposition 4.6.3(iii)). If f' is quasi-compact, so is $g \circ f'$ and since $f \circ g' = g \circ f'$ we conclude that f is quasi-compact by Proposition 4.6.3(v).

Now assume that f' is quasi-separated. We have $X' \times_{Y'} X' = (X \times_Y X)_{(Y')}$ and $\Delta_{f'} = (\Delta_f)_{(Y')}$. The projection $X' \times_{Y'} X' \to X \times_Y X$ is quasi-compact and surjective by the same reasoning, and we can apply (i) to the morphism Δ_f. Since by hypothesis $\Delta_{f'}$ is quasi-compact, we conclude that Δ_f is quasi-compact, so f is quasi-separated. □

Proposition 4.6.13. *Let $f_i : X_i \to Y$ be a finite family of morphisms and $f : X \to Y$ be their coproduct. Then for f to be quasi-compact (resp. quasi-separated), it is necessary and sufficient that each f_i is quasi-compact (resp. quasi-separated).*

Proof. The assertion about quasi-compactness follows from definition. We also note that $X \times_Y X$ is the coproduct of that $X_i \times_Y X_j$, and Δ_f is the morphism that coincides with Δ_{f_i} on each X_i, so the assertion for quasi-separatedness also follows. □

Theorem 4.6.14. *Let X be a quasi-compact and quasi-separated scheme, $\mathscr{L}$ be an invertible $\mathscr{O}_X$-module, s be a section over X, X_s the open subset of $x \in X$ such that $s(x) \neq 0$, and $\mathscr{F}$ be a quasi-coherent $\mathscr{O}_X$-module.*

- *(a) If $t \in \Gamma(X, \mathscr{F})$ is such that $t|_{X_s} = 0$, there exists $n > 0$ such that $t \otimes s^{\otimes n} = 0$.*
- *(b) For any section $t \in \Gamma(X_s, \mathscr{F})$, there exists an integer $n > 0$ such that $t \otimes s^{\otimes n}$ can be extended to a section $t' \in \Gamma(X, \mathscr{F} \otimes \mathscr{L}^{\otimes n})$.*

Proof. As the space X is a finite union of affine opens U_i such that $\mathscr{L}|_{U_i}$ is isomorphic to $\mathscr{O}_X|_{U_i}$, we can assume that X is affine and $\mathscr{L} = \mathscr{O}_X$. The assertion (a) then follows from Proposition 4.1.20(iv).

Now let t be a sectin of $\mathscr{F}$ over X_s. Since $\mathscr{L}|_{U_i}$ is isomorphic to $\mathscr{O}_X|_{U_i}$, the restriction $(t \otimes s^{\otimes n})|_{U_i \cap X_s}$ is identified with $(t|_{U_i \cap X_s})(s|_{U_i \cap X_s})^n$ under the isomorphism $(\mathscr{F} \otimes \mathscr{L}^{\otimes n})|_{U_i} \cong \mathscr{F}|_{U_i}$. We then conclude from Proposition 4.1.20(iv) that there exists an integer $n \geq 0$ such that for each i, $(t \otimes s^{\otimes n})|_{U_i \cap X_s}$ extends to a section t_i of $\mathscr{F} \otimes \mathscr{L}^{\otimes n})$ over U_i. Let t_{ij} be the restriction of t_i to $U_i \cap U_j$; by definition we have $(t_{ij} - t_{ji})|_{X_s \cap U_i \cap X_j} = 0$. Since X is quasi-separated, $U_i \cap U_j$ is quasi-compact, so by Proposition 4.1.20(iv) there exists an integer $m \geq 0$ such that $(t_{ij} - t_{ji}) \otimes s^{\otimes m} = 0$. The sections $t_i \otimes s^{\otimes m}$ then glue together to give a section $t' \in \Gamma(X, \mathscr{F} \otimes \mathscr{L}^{\otimes(n+m)})$, which induces $t_i \otimes s^{\otimes m}$ over each U_i, and hence induces $t \otimes s^{\otimes(n+m)}$ over X_s. □

Corollary 4.6.15. *With the hypotheses of Theorem 4.6.14, consider the ring $A = \Gamma_*(\mathscr{L})$ and the graded A-module $M = \Gamma_*(\mathscr{L}, \mathscr{F})$ of type $\mathbb{Z}$. Then for each $s \in A_n$, there exists a canonical isomorphism $\Gamma(X_s, \mathscr{F}) \cong M_{(s)}$, where $M_{(s)} = (M_s)_0$ is the degree zero part of the localization M_s.*

Proof. With the notations of Theorem 4.6.14(b), we see that any element $t \in \Gamma(X_s, \mathscr{F})$ corresponds to an element t'/s^n in $M_{(s)}$, which is independent of the integer n and the chosen extension t', in view of Theorem 4.6.14(a). It is immediate that this defines a homomorphism, and is bijective. □

Corollary 4.6.16. *Let X be a quasi-compact and quasi-separated scheme and $\mathscr{F}$ a quasi-coherent $\mathscr{O}_X$-module. Then if $A = \Gamma(X, \mathscr{O}_X)$ and $M = \Gamma(X, \mathscr{F})$, the A_s-module $\Gamma(X_s, \mathscr{F})$ is canonically isomorphic to M_s.*

Proof. This is a special case of Corollary 4.6.15, by taking $\mathscr{L} = \mathscr{O}_X$. □

Proposition 4.6.17. *Let X be a quasi-compact scheme, $\mathscr{F}$ be a quasi-coherent $\mathscr{O}_X$-module of finite type, and $\mathscr{I}$ be a quasi-coherent ideal of $\mathscr{O}_X$ of finite type such that* $\operatorname{supp}(\mathscr{F})$ *is contained in* $\operatorname{supp}(\mathscr{O}_X/\mathscr{I})$. *Then there exists an integer $n \geq 0$ such that $\mathscr{I}^n\mathscr{F} = 0$.*

Proof. As X is a finite union of affine open subsets, we may assume that $X = \operatorname{Spec}(A)$ is affine. Then $\mathscr{F} = \widetilde{M}$ and $\mathscr{I} = \tilde{\mathfrak{a}}$, where M is a finitely generated A-module and $\mathfrak{a}$ is a finitely generated ideal of A, and

$$\operatorname{supp}(\mathscr{F}) = \operatorname{supp}(M) = V(\operatorname{Ann}(M)), \quad \operatorname{supp}(\mathscr{O}_X/\mathscr{I}) = \operatorname{supp}(A/\mathfrak{a}) = V(\mathfrak{a}).$$

By hypothesis we have $V(\operatorname{Ann}(M)) \subseteq V(\mathfrak{a})$, so $\mathfrak{a} \subseteq \sqrt{\operatorname{Ann}(M)}$. Since $\mathfrak{a}$ is finitely generated, there exists an integer $n \geq 0$ such that $\mathfrak{a}^n \subseteq \operatorname{Ann}(M)$, and therefore $\mathscr{I}^n\mathscr{F} = \widetilde{\mathfrak{a}^n M} = 0$. □

Corollary 4.6.18. *Under the hypothesis of Proposition 4.6.17, there exists a closed subscheme Y of X with underlying space* $\operatorname{supp}(\mathscr{O}_X/\mathscr{I})$ *such that, if $j : Y \to X$ is the canonical injection, we have $\mathscr{F} = j_*(j^*(\mathscr{F}))$.*

Proof. Note that the support of $\mathscr{O}_X/\mathscr{I}$ and $\mathscr{O}_X/\mathscr{I}^n$ are the same because if $\mathscr{I}_x = \mathscr{O}_{X,x}$, we also have $\mathscr{I}_x^n = \mathscr{O}_{X,x}$, and on the other hand we have $\mathscr{I}_n^x \subseteq \mathscr{I}_x$ for each $x \in X$. We then conclude from Proposition 4.6.17 to suppose that $\mathscr{I}\mathscr{F} = 0$, so $\mathscr{F}$ is also an $(\mathscr{O}_X/\mathscr{I})$-module. If Y is the subscheme defined by $\mathscr{I}$, the conclusion is immediate. □

4.6.2 Morphisms of finite type and of finite presentation

Let $f : X \to Y$ be a morphism of schemes, x be a point of X, and $y = f(x)$. We say f is **of finite type** (resp. **of finite presentation**) **at the point x** if there exist an affine neighborhood V of y and an affine neighborhood U of x such that $f(U) \subseteq V$ and $\Gamma(U, \mathscr{O}_X)$ is a $\Gamma(V, \mathscr{O}_Y)$-algebra of finite type (resp. of finite presentation). We say f is **locally of finite type** (resp. **locally of finite presentation**) if it is of finite type (resp. of finite presentation) at every point of X. In this case, we say the Y-scheme X is locally of finite type (resp. locally of finite presentation) over Y.

Lemma 4.6.19. *Let $f : X \to Y$ be a morphism of schemes, x be a point of X, and $y = f(x)$. If there exist an affine neighborhood V of y and an affine neighborhood U of x such that $f(U) \subseteq V$ and $\Gamma(U, \mathscr{O}_X)$ is a $\Gamma(V, \mathscr{O}_Y)$-algebra of finite type (resp. of finite presentation), then for any affine open neighborhoods U' of x and V' of y, there exist affine open neighborhoods $U_1 \subseteq U \cap U'$ of x and $V_1 \subseteq V \cap V'$ of y, respectively of the form* $\operatorname{Spec}(\Gamma(U', \mathscr{O}_X)_{s'})$ *and* $\operatorname{Spec}(\Gamma(V', \mathscr{O}_Y)_{t'})$, *such that $f(U_1) \subseteq V_1$ and $\Gamma(U_1, \mathscr{O}_X)$ is a $\Gamma(V_1, \mathscr{O}_Y)$-algebra of finite type (resp. of finite presentation).*

Proof. Let $t' \in \Gamma(V', \mathscr{O}_Y)$ such that $V_1 = \operatorname{Spec}(\Gamma(V', \mathscr{O}_Y)_t)$ is an affine neighborhood of y contained in $V \cap V'$ and choose $s'_0 \in \Gamma(U', \mathscr{O}_X)$ such that $U'' = \operatorname{Spec}(\Gamma(U', \mathscr{O}_X)_{s'_0})$ is a neighborhood of x contained in $U \cap U' \cap f^{-1}(V_1)$. There then exists $s \in \Gamma(U, \mathscr{O}_X)$ such that $U_1 = \operatorname{Spec}(\Gamma(U, \mathscr{O}_X)_s)$

is a neighborhood of x contained in U''. If s'' is the image of s in $\Gamma(U'', \mathscr{O}_X)$, we then have $U_1 = \operatorname{Spec}(\Gamma(U'', \mathscr{O}_X)_{s''})$, so there exists $s' \in \Gamma(U', \mathscr{O}_X)$ such that $U_1 = \operatorname{Spec}(\Gamma(U', \mathscr{O}_X)_{s'})$. Now $\Gamma(U_1, \mathscr{O}_X) = \Gamma(U, \mathscr{O}_X)[1/s]$, so it is a $\Gamma(U, \mathscr{O}_X)$-algebra of finite presentation, and a fortiori a $\Gamma(V, \mathscr{O}_Y)$-algebra of finite type (resp. of finite presentation). The homomorphism $\Gamma(V, \mathscr{O}_Y) \to \Gamma(U_1, \mathscr{O}_X)$ factors into

$$\Gamma(V, \mathscr{O}_X) \longrightarrow \Gamma(V_1, \mathscr{O}_Y) \longrightarrow \Gamma(U_1, \mathscr{O}_X)$$

If $\Gamma(U_1, \mathscr{O}_X)$ is identified with a quotient algebra $\Gamma(V, \mathscr{O}_Y)[T_1, \dots, T_n]/\mathfrak{a}$, then it is also identified with the quotient algebra $\Gamma(V_1, \mathscr{O}_Y)[T_1, \dots, T_n]/\mathfrak{b}$, where $\mathfrak{b}$ is the ideal generated by $\mathfrak{a}$. It then follows that $\Gamma(U_1, \mathscr{O}_X)$ is a $\Gamma(V_1, \mathscr{O}_Y)$-algebra of finite type (resp. finite presentation). □

Proposition 4.6.20. *If Y is locally Noetherian, then $f : X \to Y$ is locally of finite type if and only if it is locally of finite presentation. Moreover, if this holds, then X is also locally Noetherian.*

Proof. The first assertion is clear since we can take $\Gamma(V, \mathscr{O}_Y)$ to be Noetherian. The second one follows because $\Gamma(U, \mathscr{O}_X)$ is then also Noetherian. □

Proposition 4.6.21 (Properties of Morphisms Locally of Finite Type).

(i) Any local immersion is locally of finite type.

(ii) If $f : X \to Y$ and $g : Y \to Z$ are two morphisms locally of finite type, then $g \circ f$ is locally of finite type.

(iii) If $f : X \to Y$ is an S-morphism locally of finite type, $f_{(S')} : X_{(S')} \to Y_{(S')}$ is locally of finite type for any base change $g : S' \to S$.

(iv) If $f : X \to X'$ and $g : Y \to Y'$ are two S-morphisms locally of finite type, $f \times_S g$ is locally of finite type.

(v) If the composition $g \circ f$ of two morphisms is locally of finite type, then f is locally of finite type.

(vi) If a morphism f is locally of finite type, so is f_{red}.

Proof. Assertion (vi) follows from the fact that if a ring homomorphism $A \to B$ is of finite type, then so is $A/\mathfrak{n}(A) \to B/\mathfrak{n}(B)$. Now in view of Proposition 4.5.14, it suffices to prove (i), (ii) and (iii). If $j : X \to Y$ is a local immersion, for any $x \in X$ there exists an affine open neighborhood V of $j(x)$ in Y and an affine open neighborhood U of x such that the restriction $U \to V$ of j is a closed immersion. Then $\Gamma(U, \mathscr{O}_X)$ is a quotient ring of $\Gamma(V, \mathscr{O}_Y)$, and is therefore of finite type.

To establish (iii), we may assume that $Y = S$; let $p : X_{(S')} \to X$ and $q : X_{(S')} \to S$ be the canonical projections, x' be a point of $X_{(S')}$, and $x = p(x')$, $s' = q(x')$, $s = f(p(x')) = g(q(x'))$. Let V be an affine neighborhood of s in S and U be an affine neighborhood of x in X such that $f(U) \subseteq V$ and $\Gamma(U, \mathscr{O}_X)$ is a $\Gamma(V, \mathscr{O}_S)$-algebra of finite type. Let V' be an affine open neighborhood of s' in S' contained in $g^{-1}(V)$; then $p^{-1}(U) \cap q^{-1}(V)$ is an affine neighborhood of x' and is identified with $U \times_V V'$ (Corollary 4.3.2). This is an affine scheme with ring $\Gamma(U, \mathscr{O}_X) \otimes_{\Gamma(V, \mathscr{O}_S)} \Gamma(V', \mathscr{O}_{S'})$; as this is a $\Gamma(V', \mathscr{O}_{S'})$-algebra of finite type, we see (iii) follows.

Finally, to prove (ii), consider a point $x \in X$; there exists by hypothesis an affine open neighborhood W of $g(f(x))$ in Z and an affine open neighborhood V of $f(x)$ in Y such that

$g(V) \subseteq W$ and $\Gamma(V, \mathscr{O}_Y)$ is a $\Gamma(W, \mathscr{O}_Z)$-algebra of finite type. By Lemma 4.6.19 there exists an affine open neighborhood $V' \subseteq V$ of $f(x)$ and an affine open neighborhood $U \subseteq f^{-1}(V')$ of x such that $\Gamma(V', \mathscr{O}_Y)$ is a $\Gamma(V, \mathscr{O}_Y)$-algebra of finite type and $\Gamma(U, \mathscr{O}_X)$ is a $\Gamma(V', \mathscr{O}_Y)$-algebra of finite type. We then conclude that $\Gamma(U, \mathscr{O}_X)$ is a $\Gamma(W, \mathscr{O}_Z)$-algebra of finite type, so (ii) follows. □

Corollary 4.6.22. *Let $f : X \to Y$ be a morphism locally of finite type. For any morphism $Y' \to Y$ such that Y' is locally Noetherian, $X \times_Y Y'$ is locally Noetherian.*

Proof. This follows from Proposition 4.6.20, since $f_{(Y')} : X \times_Y Y' \to Y'$ is locally of finite type by Proposition 4.6.21. □

Proposition 4.6.23. *Let $\rho : A \to B$ be a homomorphism of rings. For the corresponding morphism $f : \operatorname{Spec}(B) \to \operatorname{Spec}(A)$ to be locally of finite type, it is necessary and sufficient that B is an A-algebra of finite type.*

Proof. This condition is clearly sufficient. Conversely, assume that f is locally of finite type. Then by Lemma 4.6.19 there exists a finite cover of $\operatorname{Spec}(B)$ by open sets $D(g_i)$ (where $g_i \in B$) such that B_{g_i} is an A-algebra of finite type. Since the $D(g_i)$'s cover $\operatorname{Spec}(B)$, we see g_i generate the ring B, and it follows from Corollary **??** that B is of finite type over A. □

Proposition 4.6.24 (Properties of Morphisms Locally of Finite Presentation).

(i) Any local isomorphism is locally of finite presentation.

(ii) If two morphisms $f : X \to Y$, $g : Y \to Z$ are locally of finite presentation, so is $g \circ f$.

(iii) If $f : X \to Y$ is an S-morphism locally of finite presentation, $f_{(S')} : X_{(S')} \to Y_{(S')}$ is locally of finite presentation for any base change $g : S' \to S$.

(iv) If $f : X \to X'$ and $g :\to Y'$ are two S-morphisms locally of finite presentation, $f \times_S g$ is locally of finite presentation.

(v) If the compositin $g \circ f$ of two morphisms $f : X \to Y$, $g : Y \to Z$ is locally of finite presentation and if g is locally of finite type, then f is locally of finite presentation.

Proof. The first assertion is trivial, and to prove (ii), (iii), it suffices to replace the "algebra of finite type" in the proof of Proposition 4.6.21 by "algebra of finite presentation", and use Lemma 4.6.19. Again, assertion (iv) then follows from these. For (v), consider the cartesian square

$$\begin{tikzcd} X \arrow[r, "\Gamma_f"] \arrow[d, "f"'] & X \times_Z Y \arrow[d, "f \times_Z 1_Y"] \\ Y \arrow[r, "\Delta_g"] & Y \times_Z Y \end{tikzcd}$$

If we can show that Δ_g is locally of finite presentation, then it follows from (iii) that Γ_f is also locally of finite presentation. But f factors into

$$X \xrightarrow{\Gamma_f} X \times_Z Y \xrightarrow{p_2} Y$$

and we have $p_2 = (g \circ f) \times_Z 1_Y$, which is locally of finite presentation by (iv) since $g \circ f$ is. We then deduce that f is locally of finite presentation.

It then suffices to prove that is $g : Y \to Z$ is a morphism locally of finite type, then the diagonal morphism $\Delta_g : Y \to Y \times_Z Y$ is locally of finite presentation. To do this we may assume that $Z = \mathrm{Spec}(A)$, $Y = \mathrm{Spec}(B)$, and B is an A-algebra of finite type. The diagonal Δ_g corresponds to the homomorphism $\pi : B \otimes_A B \to B$ such that $\pi(x \otimes y) = xy$. Let $\mathfrak{I}$ be the kernel of π. We claim that $\mathfrak{I}$ is generated by the elements $1 \otimes s - s \otimes 1$, where s runs through a system of generators for the A-algebra B (this then proves the claim since B is of finite type over A). Now, it is clear that for any $x \in B$, we have $x \otimes 1 - 1 \otimes x \in \mathfrak{I}$; on the other hand, for $x, y \in B$, we have

$$x \otimes y = xy \otimes 1 + (x \otimes 1)(1 \otimes y - y \otimes 1)$$

If $\sum_i (x_i \otimes y_i) \in \mathfrak{I}$, we have by definition that $\sum_i x_i y_i = 0$, so

$$\sum_i (x_i \otimes y_i) = \sum_i (x_i \otimes 1)(1 \otimes y_i - y_i \otimes 1).$$

Moreover, if $x = st$, then

$$x \otimes 1 - 1 \otimes x = (s \otimes 1)(t \otimes 1 - 1 \otimes t) + (s \otimes 1 - 1 \otimes t)(1 \otimes t).$$

The claim then follows by indcution on the number of factors of a product in B. □

Corollary 4.6.25. *Let $g : Y \to Z$ be a morphism locally of finite type. Then the diagonal morphism $\Delta_g : Y \to Y \times_Z Y$ is locally of finite presentation.*

Proof. This is contained in the proof of Proposition 4.6.24. □

Proposition 4.6.26. *Let A be a ring, B be an A-algebra, $B' = A[T_1, \dots, T_n]$, and $\rho : B' \to B$ be a surjective homomorphism of A-algebras. Then for B to be an A-algebra of finite presentation, it is necessary and sufficient that the kernel $\mathfrak{a}$ of ρ is finitely generated in B'.*

Proof. The condition is sufficient by definition. Conversely, we note that the morphism $g : \mathrm{Spec}(B') \to \mathrm{Spec}(A)$ is locally of finite type; if B is an A-algebra of finite presentation, the morphism $f : \mathrm{Spec}(B) \to \mathrm{Spec}(B')$ corresponding to ρ and $g \circ f : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ are locally of finite presentation, so it follows from Proposition 4.6.24(v) that f is locally of finite presentation. Now to show that $\mathfrak{a}$ is finitely generated, it suffices to apply Proposition **??**. □

Corollary 4.6.27. *Let X, Y be two schemes, $j : X \to Y$ be an immersion, U an open subset of Y such that $j(X)$ is closed in U, and $\mathscr{I}$ the quasi-coherent ideal of $\mathscr{O}_U$ defining the closed subscheme of Y associated with j. For j to be locally of finite presentation, it is necessary and sufficient that $\mathscr{I}$ is a $\mathscr{O}_U$-module of finite type.*

Proof. Since the question is local, we can assume that X and Y are affine. The assertion then reduces to Proposition 4.6.26. □

Proposition 4.6.28. *Let $\rho : A \to B$ be a ring homomorphism. For the corresponding morphism $f : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ to be locally of finite presentation, it is necessary and sufficient that B is an A-algebra of finite presentation.*

Proof. The condition is clearly sufficient, so we only need to prove the necessity. If f is locally of finite presentation, then it follows from Proposition 4.6.23 that B is an A-algebra of finite type, so there exists a surjective homomorphism $\rho : C' = A[T_1, \dots, T_n] \to B$ of A-algebras. It then follows from Proposition 4.6.24(v) that the closed immersion $j : \mathrm{Spec}(B) \to \mathrm{Spec}(B')$ is locally of finite presentation, so, if $\mathfrak{b}$ is the kernel of ρ, the $\widetilde{B'}$-module $\tilde{\mathfrak{b}}$ is of finite type, and $\mathfrak{b}$ is therefore finitely generated in B' by Proposition 4.1.23. □

Proposition 4.6.29. *Let $\rho : A \to B$ be a homomorphism of rings such that B is a finite A-algebra. For B to be an A-algebra of finite presentation, it is necessary and sufficient that B is an A-module of finite presentation.*

Proof. There exists a finite A-algebra B' of finite presentation B' that is a free A-module, and a surjective A-homomorphism of A-algebras $u : B' \to B$ (Proposition **??**); we have a surjective A-homomorphism $v : B'' = A[T_1, \dots, T_m] \to B'$ with kernel finitely generated. If $w = v \circ u : B'' \to B$ and $\mathfrak{b}$ (resp. $\mathfrak{a}$) is the kernel of w (resp. u), we have $\mathfrak{a} = v(\mathfrak{b})$ since v is surjective. If B is an A-algebra of finite presentation, $\mathfrak{b}$ is a finitely generated ideal of B'' by Proposition 4.6.26, so $\mathfrak{a}$ is a finitely generated ideal in B', hence a finitely generated A-module since B' is a finite A-algebra. As B' is a free A-module, B is then an A-module of finite presentation. The converse is proved in Corollary **??**. □

Proposition 4.6.30. *Let $f : X \to Y$ be a local immersion of finite type at a point of $y \in Y$. The following conditions are equivalent:*

- *(i) f is an open map at y.*
- *(ii) There exists an open neighborhood U of y in Y such that $f|_U$ is a nilimmersion over the open subscheme U.*
- *(iii) There exists an open neighborhood U of y in Y such that $f|_U$ is a nilpotent immersion over the open subscheme U.*

Proof. It is clear that (iii) implies (ii) and (ii) implies (i). To show that (i) implies (iii), we can, by restricting f, suppose that f is a closed immersion from an affine open U of Y to an affine open V of X. Moreover, by choosing an irreducible component containing $f(y)$, we can further assume that V is irreducible. As f is a homeomorphism from U to $f(U)$, the hypothesis of (i) then implies that $f(U) = V$ since V is connected. If $V = \mathrm{Spec}(A)$, $U = \mathrm{Spec}(B)$, we have $B = A/\mathfrak{a}$, where $\mathfrak{a}$ is a nilideal of A. On the other hand, in view of Lemma 4.6.19, we can, by replacing A with a fraction field A_s, suppose that B is an A-algebra of finite presentation. But then $\mathfrak{a}$ is a finitely generated ideal of A by Proposition 4.6.26, so it is nilpotent. □

Proposition 4.6.31. *Let $f : X \to Y$ be a morphism locally of finite type. Suppose that f admits a Y-section g, and for every $y \in Y$, the fiber $X_y = f^{-1}(y)$ is a $\kappa(y)$-scheme $\kappa(y)$-isomorphic to* $\mathrm{Spec}(\kappa(y))$ *(and necessarily has underlying space $\{g(y)\}$). Then f is an isomorphism.*

Proof. In fact, g is a nilimmersion of Y to X (Corollary 4.5.8), so the image $g(Y)$ has underlying space X and is defined by a nilideal $\mathscr{I}$ of $\mathscr{O}_X$. As $f \circ g = 1_Y$ and f is locally of finite type, g

is locally of finite presentation by Proposition 4.6.24(v), so $\mathscr{I}$ is an ideal of finite type of $\mathscr{O}_X$ (Proposition 4.6.27). For any $y \in Y$, put $x = g(y)$, and consider the following exact sequence:

$$\begin{tikzcd} 0 \arrow[r] & \mathscr{I}_x \arrow[r] & \mathscr{O}_{X,x} \arrow[r, shift left, "g_y^{\#}"] & \mathscr{O}_{Y,y} \arrow[l, shift left, "f_x^{\#}"] \arrow[r] & 0 \end{tikzcd}$$

The relation $f \circ g = 1_Y$ implies $f_x^{\#} \circ g_y^{\#} = 1$, so the above exact sequence splits. By tensoring with $\kappa(y) = \mathscr{O}_{Y,y}/\mathfrak{m}_y$, we then get an isomorphism $\mathscr{O}_{X,x}/\mathfrak{m}_y\mathscr{O}_{X,x} \cong \kappa(y) \oplus (\mathscr{I}_x/\mathfrak{m}_y\mathscr{I}_x)$. But the hypothesis on X_y implies that $\kappa(y)$-isomorphic to $\kappa(y)$, so we deduce that $\mathfrak{m}_y\mathscr{I}_x = \mathscr{I}_x$ and a fortiori $\mathfrak{m}_x\mathscr{I}_x = \mathscr{I}_x$. As $\mathscr{I}_x$ is a finitely generated ideal of $\mathscr{O}_{X,x}$, by Nakayama's lemma we conclude $\mathscr{I}_x = 0$, so $\mathscr{I} = 0$ and f is an isomorphism. □

Corollary 4.6.32. *Let X, Y be two S-schemes and $f : X \to Y$ be an S-morphism. Suppose that X is locally of finite type over S. For each $s \in S$, let X_s, Y_s be the fiber of X and Y at the point s, and $f_s : X_s \to Y_s$ be the morphism induced by f under the base change* $\mathrm{Spec}(\kappa(s)) \to S$*. Then if for each $s \in S$, f_s is a monomorphism, f is a monomorphism.*

Proof. If f_s is a monomorphism, so is $f_y : X_y \to \mathrm{Spec}(\kappa(y))$. By hypothesis f is locally of finite type (Proposition 4.6.21(v)), so we can limit ourselves to the case $Y = S$. To see that f is a monomorphism, it suffices to prove that the first projection $p : X \times_S X \to X$ is an isomorphism (Proposition 4.5.5). Now the hypothesis on f_s implies that the projections $p_s : X_s \otimes_{\kappa(s)} X_s \to X_s$ are isomorphisms for all $s \in S$. Since p admits an S-section, namely the diagonal Δ_f, it follows from Proposition 4.6.31 that p is an isomorphism. □

We now come to the definition of *morphisms of finite type*, which can be seen as a global version of morphisms locally of finite type. Briefly speaking, the notion of finite type concerns the "global finiteness" of a morphism: we have the following definition and proposition.

Proposition 4.6.33. *Let $f : X \to Y$ be a morphism and (U_α) be a covering of Y by affine opens. The following conditions are equivalent:*

(i) f is locally of finite type and quasi-compact.

(ii) For each α, $f^{-1}(U_\alpha)$ is a finite union of affine opens $V_{\alpha,i}$ such that the ring $\Gamma(V_{\alpha,i}, \mathscr{O}_X)$ is a $\Gamma(U_\alpha, \mathscr{O}_Y)$-algebra of finite type.

(iii) For any affine open U of Y, $f^{-1}(U)$ is a finite union of affine opens V_j such that $\Gamma(V_j, \mathscr{O}_X)$ is a $\Gamma(U, \mathscr{O}_X)$-algebra of finite type.

*We say the morphism f is **of finite type** if it satisfies the above equivalent conditions. In this case, we say X is of finite type over Y.*

Proof. It is clear that (iii)⇒(ii)⇒(i). To prove that (i) implies (iii), we may assume that $Y = U$ is affine; then X is quasi-compact, hence is a finte union of affine opens V_j such that the restriction $V_j \to U$ of f is locally of finite type. By Proposition 4.6.23, we see $\Gamma(V_j, \mathscr{O}_X)$ is a $\Gamma(U, \mathscr{O}_X)$-algebra of finite type. □

Proposition 4.6.34. *Let $f : X \to Y$ be a morphism of finite type. If Y is Noetherian (resp. locally Noetherian), so is X.*

Proof. This follows from Proposition 4.6.20 and Proposition **??**. □

Proposition 4.6.35 (Properties of Morphisms of Finite Type).

(i) Any quasi-compact immersion is of finite type.

(ii) The composition of two morphisms of finite type is of finite type.

(iii) If $f : X \to Y$ is an S-morphism of finite type, $f_{(S')} : X_{(S')} \to Y_{(S')}$ is of finite type for any base change $S' \to S$.

(iv) If $f : X \to X'$ and $g : Y \to Y'$ are two S-morphisms of finite type, $f \times_S g$ is of finite type.

(v) If the composition $g \circ f$ of two morphisms $f : X \to Y$, $g : Y \to Z$ is of finite type and if g is quasi-separated or X is Noetherian, then f is of finite type.

(vi) If a morphism f is of finite type, so is f_{red}.

Proof. This follows directly from Proposition 4.6.21 and Proposition 4.6.3. □

Corollary 4.6.36. *Let $f : X \to Y$ be a morphism of finite type. For any morphism $Y' \to Y$ such that Y' is Noetherian, $X \times_Y Y'$ is Noetherian.*

Proof. This follows from Proposition 4.6.35(iii) and Proposition 4.6.34. □

Corollary 4.6.37. *Let X be a scheme of finite type over a locally Noetherian scheme S. Then any S-morphism $f : X \to Y$ is of finite type.*

Proof. The morphism f is locally of finite type by Proposition 4.6.21(v). To see it is quasi-compact, we can suppose that S is Noetherian. If $\varphi : X \to S$ and $\psi : Y \to S$ are the structural morphism, we have $\varphi = \psi \circ f$ and X is Noetherian by Proposition 4.6.34, so f is of finite type, by Proposition 4.6.35(v). □

Let X and Y be two schemes. We say a morphism $f : X \to Y$ is **of finite presentation** if it satisfies the following conditions:

(i) f is locally of presentation;

(ii) f is quasi-compact;

(iii) f is quasi-separated.

In this case, we say X is **of finite presentation over Y**, or is an **Y-scheme of finite presentation**. It is clear that condition (iii) is automatic if f is separated, or if X is locally Noetherian. If Y is locally Noetherian, then again, f is of finite type if and only if it is of finite presentation, and in this case X is also locally Noetherian.

Proposition 4.6.38 (Properties of Morphisms of Finite Presentation).

(i) *Any quasi-compact immersion that is locally of finite presentation (in particular any quasi-compact open immersion) is of finite presentation.*

(ii) *The composition of two morphisms of finite presentation is of finite presentation.*

(iii) *If $f : X \to Y$ be an S-morphism of finite presentation, $f_{(S')} : X_{(S')} \to Y_{(S')}$ is of finite presentation for any base change $S' \to S$.*

(iv) *If $f : X \to X'$ and $g : Y \to Y'$ are two S-morphisms of finite presentation, $f \times_S g$ is of finite presentation.*

(v) *If the composition $g \circ f$ of two morphisms $f : X \to Y$, $g : Y \to Z$ is of finite presentation and if g is quasi-separated and locally of finite type, then f is of finite presentation.*

Proof. This follows immediately from Proposition 4.6.3, Proposition 4.6.7, and Proposition 4.6.24. □

It follows from Proposition 4.6.38(iii) that if f is a morphism of finite presentation and U is an open subset of Y, the restriction $f^{-1}(U) \to U$ of f is also of finite presentation. conversely, let (U_α) be a covering of Y by affine opens and suppose that the restriction $f^{-1}(U_\alpha) \to U_\alpha$ of f is a morphism of finite presentation. Then it follows that f is of finite presentation, since f is clearly of finite presentation and quasi-compact and it is quasi-separated by Proposition 4.6.9.

If X is a quasi-separated scheme, any morphism $f : X \to Y$ is quasi-separated by Corollary 4.6.8. Therefore, if f is quasi-compact and locally of finite presentation, it is of finite presentation.

Corollary 4.6.39. *Let $\rho : A \to B$ be a ring homomorphism. For the corresponding morphism f :* $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ *to be of finite type (resp. of finite presentation), it is necessary and sufficient that B is an A-algebra of finite type (resp. of finite presentation).*

Proof. Since any morphism of affine schemes is quasi-compact and separated, this follows from Proposition 4.6.23 and Proposition 4.6.28. □

Remark 4.6.1. In the definition of morphisms of finite presentation, the condition (iii) is not a concequence of the other two conditions. For example, let Y be a non-Noetherian affine scheme and let Y be a non-quasi-compact open subset of Y (an example for this is $Y = \mathrm{Spec}(k[x_1, x_2, \dots])$ and $U = Y - \{0\}$, cf. Example 4.6.2). Let X be the scheme obtained by glueing two schemes Y_1, Y_2 isomorphic to Y along the open sets U_1, U_2 corresponding to U, so that X is the union of two affine opens isomorphic to Y_1, Y_2, respectively, and $Y_1 \cap Y_2 = U$. Let $f : X \to Y$ be the morphism which coincides with the canonical isomorphism $Y_i \to Y$ on each Y_i. Then it is clearly locally of finite presentation, and is quasi-compact since the inverse image of a quasi-compact open of Y is the union of its images in Y_1 and Y_2; but as $Y_1 \cap Y_2 = U$ is not quasi-compact, it is not quasi-separated by Proposition 4.6.10 and Corollary 4.6.8(ii).

Proposition 4.6.40. *Let $f_i : X_i \to Y$ be a finite family of morphisms and $f : X \to Y$ be their coproduct, where $X = \coprod_i X_i$. Then for f to be of finite type (resp. finite presentation), it s necessary and sufficient that each f_i is.*

Proof. In view of Proposition 4.6.13, it suffices to note that the same assertion holds for morphisms of locally finite type and of finite presentation. □

4.6.3 Algebraic schemes

We say a K-scheme is **algebraic** (resp. **locally algebraic**) if it is of finite type over K (resp. locally of finite type over K). The field K is called the **base field** of X.

Proposition 4.6.41. *Let K be a field. A locally algebraic (resp. algebraic) K-scheme is locally Noetherian (Noetherian). Moreover X is a Jacobson scheme and a point $x \in X$ is closed if and only if $\kappa(x)$ is a finite extension of K.*

Proof. The first assertion is clear, and X is Jacobson by Corollary **??**. To characterize close points in X, we note that for a point $x \in X$ to be closed, it is necessary and sufficient that for an open covering (U_α) of X, s is closed in the U_α containing it. As there is a covering of X by affine opens U_α such that $\Gamma(U_\alpha, \mathscr{O}_X)$ is a K-algebra of finite type, we can then assume that $X = \mathrm{Spec}(A)$ where A is a K-algebra of finite type. The closed points of X are then maximal ideals of A; but then $A/\mathfrak{p}_x = \kappa(x)$ is a finite extension by Theorem **??**. Conversely, if $\kappa(x)$ is a finite K-algebra, so is the ring $A/\mathfrak{p}_x \subseteq \kappa(x)$, and as an integral K-algebra is also a field (Corollary **??**), we have $A/\mathfrak{p}_x = \kappa(x)$, so x is closed. □

Corollary 4.6.42. *Let K be an algebraically closed field and X be a locally algebraic K-scheme. Then the closed points of X are exactly the rational points of X over K, which are identified with the K-points of X with values in K.*

Proposition 4.6.43. *Let K be a field and X be a locally algebraic scheme over K. Then the following conditions are equivalent:*

(i) X is Artinian.

(ii) The underlying space of X has only finitely many closed points.

(iii) The underlying space of X is finite.

(iv) X is isomorphic to $\mathrm{Spec}(A)$ *where A is K-algebra of finite dimension.*

If X is algebraic over K, then these conditions are equivalent to the following:

(v) The underlying space of X is discrete.

(vi) The points of X are all closed.

Proof. We see (i) implies any other conditions, and (v) or (vi) implies (i) if X is Noetherian. Moreover, it is clear that (iv) implies (i), since a finite dimensional K-algebra is Artianian. In the condition of (ii), the set X_0 of closed points of X is then finite, closed and very dense in X, whence equal to X and X is therefore Artinian, since it is then Noetherian. □

Corollary 4.6.44. *Let K be a field, X be a locally algebraic K-scheme, and x be a point of X. The following conditions are equivalent:*

(i) x is isolated in X;

(ii) x is closed in X and $\mathscr{O}_{X,x}$ is Artinian;

(iii) $\mathscr{O}_{X,x}$ *is a finite dimensional* K*-algebra.*

Proof. If $\mathscr{O}_{X,x}$ is a finite dimensional K-algebra, so is the residue field $\kappa(x) = \mathscr{O}_{X,x}/\mathfrak{m}_x$, so (iii) implies (ii) in view of Proposition 4.6.41. The local ring $\mathscr{O}_{X,x}$ is Artinian signifies that x is a maximal point of X, since a Noetherian local ring is Artinian if and only if it has a unique primei deal. If x is moreover closed, the set $\{x\}$ is closed and stable under generalization, hence open (Proposition **??**), and this proves x is isolated in X. Finally, if x is isolated in X, there exists an affine open neighborhood U of x such that $U = \{x\}$ and $\Gamma(U, \mathscr{O}_X)$ is a finite type K-algebra. But then $\Gamma(U, \mathscr{O}_X) = \mathscr{O}_{X,x}$, so (i) implies (iii). □

If the conditions in Proposition 4.6.43, we say X is a finite scheme over K, or a finite K-scheme. For such a scheme, we denote by $\dim_K(X)$ the dimension of the ring $\Gamma(X, \mathscr{O}_X)$ over K. If X and Y are two finite schemes over K, we have

$$\dim_K(X \amalg Y) = \dim_K(X) + \dim_K(Y), \quad \dim_K(X \times_K Y) = \dim_K(X)\dim_K(Y).$$

Corollary 4.6.45. *Let X be a finite scheme over a field K. For any extension K' of K, $X \otimes_K K'$ is a finite scheme over K', with* $\dim_{K'}(X') = \dim_K(X)$.

Proof. In fact, if $X = \mathrm{Spec}(A)$, we have $[A \otimes_K K' : K'] = [A : K]$, whence the claim. □

Corollary 4.6.46. *Let X be a finite scheme over a field K. We put*

$$n = \sum_{x \in X} [\kappa(x) : K]_s$$

Then, for any algebraically closed extension Ω of K, the underlying space of $X \otimes_K \Omega$ has exactly n points, which are identified with the Ω-valued points of X.

Proof. By Proposition 4.2.30, we can assume that $A = \Gamma(X, \mathscr{O}_X)$ is local; let $\mathfrak{m}$ be the maximal ideal of A, $L = A/\mathfrak{m}$ the residue field, which is a finite algebraic extension of K by Proposition 4.6.43. The Ω-points of X correspond bijectively to Ω-sections of $X \otimes_K \Omega$, and to the closed points of $X \otimes_K \Omega$ by Corollary 4.6.42, and finally to the points of this Artinian scheme (Proposition 4.2.30). They also correspond to K-homomorphisms of L into Ω, and the assertion then follows from the definition of separable degree. □

The number n defined in Corollary 4.6.46 is called the **separable rank** of A (or X) over K, or the **geometric number of points** of X. This is also the number of elements in $X(\Omega)_K$. It follows from this definition that for any extension K' of K, $X \otimes_K K'$ has the same geometric number of points as X. If we denote this number by $n(X)$, it is clear that, if X and Y are two finite schemes over K, we have

$$n(X \amalg Y) = n(X) + n()Y, \quad n(X \times_K Y) = n(X)n(Y).$$

Proposition 4.6.47. *Let $f : X \to Y$ be a morphism locally of finite type (resp. of finite type). Then, for any $y \in Y$, the fiber $X_y = f^{-1}(y)$ is a locally algebraic (resp. algebraic) scheme over $\kappa(y)$, and for each $x \in X_y$, $\kappa(x)$ is a extension of $\kappa(y)$ of finite type.*

Proof. As $X_y = X \otimes_Y \kappa(y)$, the claim follows from Proposition 4.6.21(iii) and Proposition 4.6.35(iii). □

Proposition 4.6.48. *Let $f : X \to Y$, $g : Y' \to Y$ be two morphisms, and put $X' = X \times_Y Y'$, $f' = f_{(Y')} : X' \to Y'$. Let $y' \in Y'$ and $y = g(y')$. If the fiber $X_y = f^{-1}(y)$ is a finite scheme over $\kappa(y)$, then the fiber $X'_{y'} = f'^{-1}(y')$ is a finite scheme over $\kappa(y')$, and we have*

$$\dim_{\kappa(y')}(X'_{y'}) = \dim_{\kappa(y)}(X_y), \quad n(X'_{y'}) = n(X_y).$$

Proof. This follows from the observation $X'_{y'} = X_y \otimes_{\kappa(y)} \kappa(y')$. □

Proposition 4.6.47 shows that the morphisms of finite type (resp. locally of finite type) correspond intuitively to "algebraic families of algebraic varieties (resp. locally algebraic)", where Y plays the role of "parameters". Because of this, these morphisms are of significant geometric interests. The morphisms which are not locally of finite type will intervene them by the process of "base change", for example by localization and completion.

4.6.4 Local determination of morphisms

Proposition 4.6.49. *Let X and Y be S-schemes, $x \in X$, $y \in Y$ be points lying over the same point $s \in S$.*

(a) Suppose that Y is locally of finite type over S at the point y. Then if two S-morphisms f, g from X to Y are such that $f(x) = g(x) = y$ and the $\mathscr{O}_{S,s}$-homomorphisms $f_x^\#$ and $g_x^\#$ from $\mathscr{O}_{Y,y}$ to $\mathscr{O}_{X,x}$ coincide, then f and g coincide in an open neighborhood of x.

(b) Suppose that Y is locally of finite presentation over S at the point y. Then, for any $\mathscr{O}_{X,x}$-homomorphism $\varphi : \mathscr{O}_{Y,y} \to \mathscr{O}_{X,x}$ there exists an open neighborhood U of x in X and an S-morphism f from U to Y such that $f(x) = y$ and $f_x^\# = \varphi$.

Proof. We first consider case (a). The question is local over S, X and Y, so we can suppose that S, X, Y are affine with rings A, B, C, respectively. The morphisms f and g then correspond to A-homomorphisms ρ, σ from C to B such that $\rho^{-1}(\mathfrak{p}_x) = \sigma^{-1}(\mathfrak{p}_x) = \mathfrak{p}_y$, and the homomorphisms ρ_x and σ_x from C_y to B_x, deduced by ρ and σ, coincide. We can morphism suppose that C is an A-algebra of finite type. Let $(c_i)_{1\leq i\leq n}$ be generators of the A-algebra C, and put $b_i = \rho(c_i)$, $b'_i = \sigma(c_i)$. By hypothesis, we have $b_i/1 = b'_i/1$ in the ring B_x. This means there exist elements $s_i \in B - \mathfrak{p}_x$ such that $s_i(b_i - b'_i) = 0$ for each i, and we can evidently choose one $s \in B - \mathfrak{p}_x$ for all i. We then conclude that $b_i/1 = b'_i/1$ for each i in the ring B_s; if $i_s : B \to B_s$ is the canonical homomorphism, we then have $i_s \circ \rho = i_s \circ \sigma$, so the restriction of f and g on $D(s)$ are identical.

We now come to case (b). Again we can suppose that S, X, Y are affine with rings A, B, C. Put $\mathfrak{p} = \mathfrak{p}_x$, $\mathfrak{q} = \mathfrak{p}_y$, and let $\varphi : C_\mathfrak{q} \to B_\mathfrak{p}$ be an A-homomorphism. We then get an A-homomorphism $\rho : C \to C_\mathfrak{q} \xrightarrow{\varphi} B_\mathfrak{p}$. Since we can consider $B_\mathfrak{p}$ as an inductive limit of the filtered system of A-algebras B_s, where s runs through elements of $B - \mathfrak{p}$, and C is by hypothesis an A-algebra of finite presentation, we deduce from Proposition **??** that there exists $s \notin \mathfrak{p}$ and an A-homomorphism $\sigma : C \to B_s$ whose canonical image is ρ, that is, the following diagram is commutative

$$\begin{tikzcd} C \arrow[r, "\sigma"] \arrow[d] & B \arrow[d] \\ C_\mathfrak{q} \arrow[r, "\varphi"] & B_\mathfrak{p} \end{tikzcd} \tag{6.4.1}$$

It then suffices to take $U = D(s)$ and let f be the morphism induced by σ. □

Corollary 4.6.50. *Under the hypotheses of Proposition 4.6.49(ii), if moreover X is locally of finite type over S at the point of x, we can choose f to be of finite type.*

Proof. To see this, we can assume that S, X, Y are affine, so that the structural morphisms $X \to S$ and $Y \to S$ are respectively of finite type and of finite presentation; then the results follows from Lemma 4.6.19 and Proposition 4.6.21(iv). □

Corollary 4.6.51. *Retain the hypotheses of Proposition 4.6.49(ii) and suppose that Y is integral and $\varphi : \mathscr{O}_{Y,y} \to \mathscr{O}_{X,x}$ is injective. Then we can suppose that U is affine and f factors into*

$$U \xrightarrow{g} V \longrightarrow Y$$

where V is an affine open containing y and $g : U \to V$ is a morphism corresponding to a injective homomorphism $\Gamma(V, \mathscr{O}_Y) \to \Gamma(U, \mathscr{O}_X)$.

Proof. In fact, with the notations of Proposition 4.6.49(ii), C is integral and the canonical homomorphism $C \to C_g$ is then injective; the result then follows from the diagram (6.4.1), since σ is injective. □

Proposition 4.6.52. *Let $f : X \to Y$ be a morphism, x be a point of X and $y = f(x)$.*

(a) Suppose that f is locally of finite type at the point x. For f to be a local immersion at the point x, it is necessary and sufficient that $f_x^\# : \mathscr{O}_{Y,y} \to \mathscr{O}_{X,x}$ is surjective.

(b) Suppose that f is locally of finite presentation at the point x. For f to be a local isomorphism at the point x, it is necessary and sufficient that $f_x^\# : \mathscr{O}_{Y,y} \to \mathscr{O}_{X,x}$ is an isomorphism.

Proof. We only need to prove the sufficiency, and we first consider case (b). Then X is locally of finite presentation over y at the point x, so by Proposition 4.6.49(i) and (ii), there exists an open neighborhood V of y and a morphism $g : V \to X$ such that $g \circ f$ (resp. $f \circ g$) is defined and coincide with the identity on an open neighborhood W of x (resp. an open neighborhood T of y). Put $T' = T \cap g^{-1}(w)$ and $W' = f^{-1}(T')$, we then verify that $g(T') \subseteq W'$, $f(W') \subseteq T'$ and $(g \circ f)|_{W'} = 1_{W'}$, whence f is a local isomorphism.

For (a), we can assume that X and Y are affine, with ring A and B. Then f corresponds to a homomorphism $\varphi : B \to A$ of finite type; we have $\varphi^{-1}(\mathfrak{p}_x) = \mathfrak{p}_y$, and the homomorphism $\varphi_x : B_y \to A_x$ induced by φ is surjective. Let $(t_i)_{1 \le i \le n}$ be a system of generators of the B-algebra A. The hypothesis on φ_x then implies $t_i/1 = \varphi(b_i)/\varphi(c)$ in the ring A_x, where $b_i \in B$ and $c \in B - \mathfrak{p}_y$, so we can find $a \in A - \mathfrak{p}_x$ such that

$$a(t_i\varphi(c) - \varphi(b_i)) = 0.$$

If we put $g = a\varphi(c)$, then $t_i/1 = a\varphi(b_i)/g$ in the ring A_g. Now there exists by hypothesis a polynomial $Q(X_1, \dots, X_n)$ with coefficients in $\varphi(B)$ such that $a = Q(t_1, \dots, t_n)$; write

$Q(X_1/T,\dots,X_n/T)=P(X_1,\dots,X_n,T)/T^m$, where P is a polynomial of degree m. In the ring A_g, we then have

$$\begin{aligned}a/1&=Q(t_1/1,\dots,t_n/1)=Q(a\varphi(b_1)/g,\dots,a\varphi(b_n)/g)\\&=a^mP(\varphi(b_1),\dots,\varphi(b_n),\varphi(c))/g^m=a^m\varphi(d)/g^m\end{aligned}$$

where $d\in B$. Since $g/1=(a/1)(\varphi(c)/1)$ is invertible in A_g by definition, so is $a/1$ and $\varphi(c)/1$, and we can then write $a/1=(\varphi(d)/1)(\varphi(c)/1)^{-m}$. We conclude that $\varphi(d)/1$ is also invertible in A_g. Put $h=cd$, as $\varphi(h)/1$ is invertible in A_g, the composed homomorphism $B\to A\to A_g$ factors into

$$\begin{tikzcd} B \arrow[r, "\varphi"] \arrow[dr] & A \arrow[r] & A_g \\ & B_h \arrow[ur, "\gamma"'] & \end{tikzcd}$$

We claim that γ is surjective. For this, it suffices to verify that the image of B_h in A_g contains $t_i/1$ and $1/g$. Now we have

$$1/g=(\varphi(c)/1)^{m-1}(\varphi(d)/1)^{-1}=\gamma(c^m/h)$$

and $a/1=\gamma(d^{m+1}/h^m)$, so $(a\varphi(b_i))/1=\gamma(b_id^{m+1}/h^m)$, and as $t_i/1=(a\varphi(b_i)/1)(g/1)^{-1}$, we conclude our assertion. The choice of h implies $f(D(g))\subseteq D(h)$, and the restriction of f to $D(g)$ is induced by γ. Since γ is surjective, this restriction is a closed immersion from $D(g)$ to $D(h)$, so f is a local immersion at x. □

Corollary 4.6.53. *With the notations of Proposition 4.6.52, suppose that f is a local immersion at the point x and is locally of finite presentation at x. For f to be open at x, it is necessary and sufficient that the kernel of $f_x^\#$ is nilpotent.*

Proof. In view of Proposition 4.6.30, it suffices to prove the sufficiency of the condition. We can suppose that $Y=\operatorname{Spec}(A)$ and $X=\operatorname{Spec}(A/\mathfrak{n})$, where $\mathfrak{n}$ is a finitely generated ideal of A (Proposition 4.6.27), and by hypothesis $\mathfrak{n}_x$ is nilpotent. If $(s_i)_{1\le i\le n}$ is a system of generators of $\mathfrak{n}$, we then have $s_i^m/1=0$ in A_x for an integer m and all i. Then there exists $t\in A-\mathfrak{p}_x$ such that $ts_i^m=0$ for all i, so $(s_i/1)^m=0$ in the ring A_t. This shows $\mathfrak{n}_t$ is nilpotent, whence the conclusion. □

4.6.5 Direct image of quasi-coherent sheaves

Proposition 4.6.54. *Let X,Y be two schemes and $f:X\to Y$ be a quasi-compact and quasi-separated morphism. Then for any quasi-coherent $\mathscr{O}_X$-module, $f_*(\mathscr{F})$ is quasi-coherent.*

Proof. Since the question is local over Y, we can assume that Y is affine. If f is quasi-compact, X is then a union of finitely many open affines X_i, and in view of Proposition 4.6.7(ii), X is a quasi-separated scheme, hence the intersections $X_i\cap X_j$ are qausi-compact (Proposition 4.6.10).

We first assume that each intersection $X_i\cap X_j$ is affine. Put $\mathscr{F}_i=\mathscr{F}|_{X_i}$, $\mathscr{F}_{ij}=\mathscr{F}|_{X_i\cap X_j}$ and let $\mathscr{F}_i'$ and $\mathscr{F}_{ij}'$ be the inverse image of $\mathscr{F}_i$ and $\mathscr{F}_j$ under the restriction of f to X_i and to $X_{i\cap X_j}$. We see that $\mathscr{F}'_i$ and $\mathscr{F}'_{ij}$ are quasi-coherent (Proposition 1.4.1). We define a homomorphism

$$u : \bigoplus_i \mathscr{F}'_i \to \bigoplus_{i,j} \mathscr{F}'_{ij}$$

such that $f_*(\mathscr{F})$ is the kernel of u, and this then implies $f_*(\mathscr{F})$ is quasi-coherent by Proposition **??**. For this, it suffices to define u as a homomorphism of presheaves, so for each open subset $W \subseteq Y$, we need a homomorphism

$$u_W : \bigoplus_i \Gamma(f^{-1}(W) \cap X_i, \mathscr{F}) \to \bigoplus_{ij} \Gamma(f^{-1}(W) \cap X_i \cap X_j, \mathscr{F})$$

so as to satisfy the compatibility for the restrictions to a smaller open subset. If for any section s_i of $\mathscr{F}$ over $f^{-1}(W) \cap X_i$, we denote by s_{ij} its restriction to $f^{-1}(W) \cap X_i \cap X_j$, we set

$$u_W((s_i)) = (s_{ij} - s_{ji})$$

and the compatibility is evident. To identity the kernel $\mathscr{R}$ of u, we define a homomorphism $v : f_*(\mathscr{F}) \to \mathscr{R}$ which sends a section s of $\mathscr{F}$ over $f^{-1}(W)$ to the family (s_i), where s_i is the restriction of s to $f^{-1}(W) \cap X_i$. By the sheaf axoims of $\mathscr{F}$, it is clear that v is bijective, which proves the assertion in this case.

In the general case, the same reasoning can be applied if we can show that each $\mathscr{F}'_{ij}$ is quasi-coherent. But by hypotheses, $X_i \cap X_j$ is a union of finitely many affine opens X_{ijk}, and since each X_{ijk} are affine open subschemes of the affine scheme X_i, their intersections are again affine (affine schemes are separated), so we can apply the previous arguments to conclude that $\mathscr{F}'_{ij}$ is quasi-coherent, and the proof is then complete. □

Remark 4.6.2. We should note that even if X and Y are Noetherian schemes and $f : X \to Y$ is a morphism of finite type, the direct image $f_*(\mathscr{F})$ of a coherent $\mathscr{O}_X$-module $\mathscr{F}$ is in general not coherent. For example, let Y be the spectrum of a field K, $X = \mathrm{Spec}(K[T])$, and choose $\mathscr{F} = \mathscr{O}_X$.

Proposition 4.6.55. *Let $f : X \to Y$ be a quasi-compact and quasi-separated morphism. Let $(\mathscr{F}_\lambda)$ be a inductive system of quasi-coherent $\mathscr{O}_X$-modules and $\mathscr{F} = \varinjlim \mathscr{F}_\lambda$ be the inductive limit. Then $\varinjlim f_*(\mathscr{F}_\lambda) \cong f_*(\mathscr{F})$.*

Proof. For each affine open subset W of Y and any λ, we have a canonical homomorphism

$$u_{W,\lambda} : (f_*(\mathscr{F}_\lambda))|_W \to f_*(\mathscr{F})|_W$$

whence a canonical homomorphism

$$u_W : (\varinjlim f_*(\mathscr{F}_\lambda))|_W \to f_*(\mathscr{F})|_W$$

and this hommorphism is compatible with restrictions. Since f is quasi-compact and quasi-separated, by Proposition 4.6.54 $\varinjlim f_*(\mathscr{f}_\lambda)$ and $f_*(\mathscr{F})$ are quasi-coherent. Moreover, the homomorphism u_W corresponds by taking global section over W to the canonical homomorphism

$$\varphi_W : \Gamma(W, \varinjlim f_*(\mathscr{F}_\lambda)) \to \Gamma(f^{-1}(W), \mathscr{F}).$$

Since f is quasi-compact and quasi-separated, by Proposition **??** (Stack Project. Lemma 6.29.1) we have

$$\Gamma(W, \varinjlim f_*(\mathscr{F}_\lambda)) = \varinjlim \Gamma(W, f_*(\mathscr{F}_\lambda)) = \varinjlim \Gamma(f^{-1}(W), \mathscr{F}_\lambda) = \Gamma(f^{-1}(W), \mathscr{F})$$

and φ_W is therefore the identity homomorphism. By Proposition 4.1.21, it then follows that u_W is an isomorphism for each W, and the assertion then follows. □

4.6.6 Extension of quasi-coherent sheaves

Let X be a topological space and $\mathscr{F}$ be a sheaf of sets (resp. of groups, of rings) over X. Let U be an open subset of X with $j : U \to X$ the canonical injection, and let $\mathscr{G}$ be a subsheaf of $\mathscr{F}|_U = j^{-1}(\mathscr{F})$. As the functor j_* is left exact, $j_*(\mathscr{G})$ is then a subsheaf of $j_*(j^{-1}(\mathscr{F}))$. Let $\rho_{\mathscr{F}} : \mathscr{F} \to j_*(j^{-1}(\mathscr{F}))$ be the canonical homomorphism associated with $\mathscr{F}$ and consider the subsheaf $\overline{\mathscr{G}} = \rho_{\mathscr{F}}^{-1}(j_*(\mathscr{G}))$ of $\mathscr{F}$. It follows immediately from definition that, for any open subset V of X, $\Gamma(V, \overline{\mathscr{G}})$ is formed by sections $s \in \Gamma(V, \mathscr{F})$ whose restriction on $V \cap U$ is a section of $\mathscr{G}$ over $V \cap U$. In particular, we have $\overline{\mathscr{G}}|_U = j^{-1}(\overline{\mathscr{G}}) = \mathscr{G}$, and $\overline{\mathscr{G}}$ is the largest subsheaf of $\mathscr{F}$ inducing $\mathscr{G}$ on U. We say the subsheaf $\overline{\mathscr{G}}$ is the **canonical extension** of the subsheaf $\mathscr{G}$ of $\mathscr{F}|_U$ to a subsheaf of $\mathscr{F}$.

Proposition 4.6.56. *Let X be a scheme and U be an open subset of X such that the canonical injection $j : U \to X$ is quasi-compact (in other words, U is retrocompact in X).*

(a) For any quasi-coherent $(\mathscr{O}_X|_U)$-module $\mathscr{G}$, $j_*(\mathscr{G})$ is a quasi-coherent $\mathscr{O}_X$-module and we have $j_*(\mathscr{G})|_U = j^*(j_*(\mathscr{G})) = \mathscr{G}$.

(b) For any quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$ and any quasi-coherent sub-$(\mathscr{O}_X|_U)$-module $\mathscr{G}$ of $\mathscr{F}|_U$, the canonical extension $\overline{\mathscr{G}}$ of $\mathscr{G}$ is a quasi-coherent sub-$\mathscr{O}_X$-module of $\mathscr{F}$.

Proof. Assertion (a) is a special case of Proposition 4.6.54 since j is quasi-separated by Proposition 4.6.7(i), and the relation $j_*(\mathscr{G})|_U = j^*(j_*(\mathscr{G})) = \mathscr{G}$ can be checked directly. By the same reasoning, $j_*(j^*(\mathscr{F}))$ is quasi-coherent, and as $\overline{G}$ is the inverse image of $j_*(\mathscr{G})$ under the homomorphism $\rho_{\mathscr{F}} : \mathscr{F} \to j_*(j^*(\mathscr{F}))$, assertion (b) follows from Corollary 4.2.20. □

Corollary 4.6.57. *Let X be a scheme and U be a quasi-compact open subset of X such that the canonical injection $j : U \to X$ is quasi-compact. Suppose that any quasi-coherent $\mathscr{O}_X$-module is the inductive limit of its quasi-coherent sub-$\mathscr{O}_X$-modules of finite type (this is ture if X is an affine scheme). Let $\mathscr{F}$ be a quasi-coherent $\mathscr{O}_X$-module and $\mathscr{G}$ be a quasi-coherent sub-$(\mathscr{O}_X|_U)$-module of finite type of $\mathscr{F}|_U$. Then there exists a quasi-coherent sub-$\mathscr{O}_X$-module $\mathscr{G}'$ of finite type such that $\mathscr{G}'|_U = \mathscr{G}$.*

Proof. We have $\mathscr{G} = \overline{\mathscr{G}}|_U$, and $\overline{\mathscr{G}}$ is quasi-coherent by Proposition 4.6.56, hence is the inductive limit of its quasi-coherent sub-$\mathscr{O}_X$-modules $\mathscr{H}_\lambda$ of finite type. Then $\mathscr{G}$ is the inductive limit of the $\mathscr{H}_\lambda|_U$, hence equals to one of $\mathscr{H}_\lambda|_U$ since they are of finite type (Proposition 1.4.10). □

Remark 4.6.3. Suppose that for any affine open $U \subseteq X$ the injection $U \to X$ is quasi-compact. Then if the conclusion of Corollary 4.6.57 holds for any affine open U and any quasi-coherent sub-$(\mathcal{O}_X|_U)$-module of finite type $\mathcal{G}$ of $\mathcal{F}|_U$, it follows that $\mathcal{F}$ is the inductive limit of its quasi-coherent sub-$\mathcal{O}_X$-modules of finite type. In fact, for any affine open $U \subseteq X$, we have $\mathcal{F}|_U = \widetilde{M}$, where M is a $\Gamma(U, \mathcal{O}_X)$-module, and as the latter is the inductive limit of its finitely generated sub-modules, $\mathcal{F}|_U$ is the inductive limit of its quasi-coherent sub-$(\mathcal{O}_X|_U)$-modules of finite type. Now, by hypotheses, such a submodule is induced over U by a quasi-coherent sub-$\mathcal{O}_X$-module of finite type $\mathcal{G}_{\lambda,U}$ of $\mathcal{F}$. The finite sums of $\mathcal{G}_{\lambda,U}$ are then quasi-coherent of finite type, since the question is local and we can assume that X is affine, where the conclusion is trivial. It then follows that $\mathcal{F}$ is the inductive limit of these finite sums, whence our assertion.

Corollary 4.6.58. *Under the hypotheses of Corollary 4.6.57, if $\mathcal{G}$ is a quasi-coherent $(\mathcal{O}_X|_U)$-module of finite type, there exists a quasi-coherent $\mathcal{O}_X$-module $\mathcal{G}'$ such that $\mathcal{G}'|_U = \mathcal{G}$.*

Proof. As $\mathcal{F} = j_*(\mathcal{G})$ is quasi-coherent (Proposition 4.6.56) and $\mathcal{F}|_U = \mathcal{G}$, it suffices to apply Corollary 4.6.57 to $\mathcal{F}$. □

Lemma 4.6.59. *Let X be a scheme, $(V_\lambda)_{\lambda\in L}$ be a covering of X by affine opens where L is well-ordered, and U be an open subset of X. For each $\lambda \in L$, let $W_\lambda = \bigcup_{\mu<\lambda} V_\mu$. Suppose that*

(i) for any $\lambda \in L$, $V_\lambda \cap W_\lambda$ is quasi-compact;

(ii) the canonical injection $j : U \to X$ is quasi-compact.

Then for any quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ and any quasi-coherent sub-$(\mathcal{O}_X|_U)$-module of finite type $\mathcal{G}$ of $\mathcal{F}|_U$, there exists a quasi-coherent sub-$\mathcal{O}_X$-module $\mathcal{G}'$ of $\mathcal{F}$ such that $\mathcal{G}'|_U = \mathcal{G}$.

Theorem 4.6.60. *Let X be a scheme and U be an open subset of X. Suppose that one of the following conditions is satisfied:*

(a) X is locally Noetherian;

(b) X is quasi-compact and quasi-separated and U is quasi-compact.

Then for any quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ and any quasi-coherent sub-$(\mathcal{O}_X|_U)$-module of finite type $\mathcal{G}$ of $\mathcal{F}|_U$, there exists a quasi-coherent sub-$\mathcal{O}_X$-module $\mathcal{G}'$ of $\mathcal{F}$ such that $\mathcal{G}'|_U = \mathcal{G}$.

Corollary 4.6.61. *With the conditions of Theorem 4.6.60, for any quasi-coherent $(\mathcal{O}_X|_U)$-module $\mathcal{G}$ of finite type, there exists a quasi-coherent $\mathcal{O}_X$-module $\mathcal{G}'$ of finite type such that $\mathcal{G}'|_U = \mathcal{G}$.*

Corollary 4.6.62. *Let X be a locally Noetherian scheme or a quasi-compact and quasi-separated scheme. Then any quasi-coherent $\mathcal{O}_X$-module is the inductive limit of its quasi-coherent sub-$\mathcal{O}_X$-modules of finite type.*

Proof. This follows from Theorem 4.6.60 and Remark 4.6.3. □

Corollary 4.6.63. *Under the hypotheses of Corollary 4.6.62, if a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ is such that any quasi-coherent sub-$\mathcal{O}_X$-module of finite type of $\mathcal{F}$ is generated by its global sections, then $\mathcal{F}$ is generated by its global sections.*

Proof. Let U be an affine neighborhood of a point $x \in X$, and let s be a section of $\mathcal{F}$ over U. The sub-$\mathcal{O}_X$-module $\mathcal{G}$ of $\mathcal{F}|_U$ generated by s is quasi-coherent and of finite type, hence there exists a quasi-coherent sub-$\mathcal{O}_X$-module of finite type $\mathcal{G}'$ of $\mathcal{F}$ such that $\mathcal{G}'|_U = \mathcal{G}$ (Theorem 4.6.60). By hypotheses, there is then a finite number of sections t_i of $\mathcal{G}'$ over X and sections a_i of $\mathcal{O}_X$ over a neighborhood $V \subseteq U$ of x such that $s|_V = \sum_i a_i \cdot (t_i|_V)$, which proves the corollary. □

4.6.7 Scheme-theoretic image

Let $f : X \to Y$ be a morphism of schemes. If there exists a smallest closed subscheme Y' of Y such that the canonical injection $j : Y' \to Y$ dominates f (or equivalently, the inverse image $f^{-1}(Y')$ is equal to X), we then say that Y' is the **scheme-theoretic image** of X under f, or the **scheme-theoretic image of f**. If X is a subscheme of Y, the scheme-theoretic image of the canonical injection $j : X \to Y$ is called the **scheme-theoretic closure** of X.

Proposition 4.6.64 (Transitivity). *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms. Suppose that Y' is the scheme-theoretic image of X under f, and if g' is the restriction of g to Y', the scheme-theoretic image of Z' under g' exists. Then the scheme-theoretic image of X under $g \circ f$ is equal to Z'.*

Proof. To say that a closed subscheme Z_1 of Z is such that $(g \circ f)^{-1}(Z_1) = X$ signifies that $f^{-1}(g^{-1}(Z_1)) = X$, or that f is dominated by the canonical injection $g^{-1}(Z_1) \to Y$. Now, in view of the existence of the scheme-theoretic image Y', for any closed subscheme Z_1 of Z having this property, $g^{-1}(Z_1)$ dominates Y', which, if $j : Y' \to Y$ is the canonical injection, amounts to saying that $j^{-1}(g^{-1}(Z_1)) = g'^{-1}(Z_1) = Y'$. We then conclude that Z' is the smallest closed subscheme Z_1 having this property, whence our claim. □

Corollary 4.6.65. *Let $f : X \to Y$ be an S-morphism such that Y is the scheme-theoretic image of Y under f. Let Z be a separated S-scheme; if two S-morphisms $g_1, g_2 : Y \to Z$ are such that $g_1 \circ f = g_2 \circ f$, then $g_1 = g_2$.*

Proof. Let $h = (g_1, g_2)_S : Y \to Z \times_S Z$. As the diagonal $T = \Delta_Z(Z)$ is a closed subscheme of $Z \times_S Z$, $Y' = h^{-1}(T)$ is a closed subscheme of Y. Put $u = g_1 \circ f = g_2 \circ f$; we then have $h \circ f = (u, u)_S = \Delta_Z \circ u$. As $\Delta_Z^{-1}(T) = Z$, we have $(h \circ f)^{-1}(T) = u^{-1}(Z) = X$, so $f^{-1}(Y') = X$. We then conclude that the canonical injection $Y' \to Y$ dominates f, so $Y' = Y$ by hypothesis. Then by Proposition 4.4.14, h factors into $\Delta_Z \circ v$ where v is a morphism $Y \to Z$, which implies $g_1 = g_2 = v$. □

Let $f : X \to Y$ be a morphism and suppose that the scheme-theoretic image Y' of f exists. Then Y' is defined by a quasi-coherent ideal $\mathscr{I}'$ of $\mathscr{O}_Y$, and by definition, $\mathscr{J}'$ is the largest quasi-coherent ideal such that the homomorphism $f^\# : \mathscr{O}_Y \to f_*(\mathscr{O}_X)$ factors into $\mathscr{O}_Y \to \mathscr{O}_Y/\mathscr{J}' \to f_*(\mathscr{O}_X)$. This implies that $\mathscr{J}' \subseteq \ker f^\# = \mathscr{J}$; we therefore obtain a case where Y' exists, the one where $\mathscr{J}$ is quasi-coherent, and where $\mathscr{J}' = \mathscr{J}$.

Proposition 4.6.66. *Let $f : X \to Y$ be a morphism. Then the scheme-theoretic image of X under f exists if one of the following conditions is satisfied:*

(a) *$f_*(\mathscr{O}_X)$ is quasi-coherent (which is the case if f is quasi-compact and quasi-separated).*

(b) *X is reduced.*

In this case, the underlying space of Y' is equal to $\overline{f(X)}$, and if f factors into

$$X \xrightarrow{f'} Y' \xrightarrow{Y} Y$$

where j is the canonical injection, f' is scheme-theoretic dominant. Moreover, if X is reduced (resp. integral), so is Y'.

Proof. The case (a) is immediate by our previous argument; moreover, as $\mathscr{O}_Y/\mathscr{J} \to f_*(\mathscr{O}_X)$ is then injective, this shows that f' is scheme-theoretic dominant. We still need to verify that the closed subscheme of Y defined by $\mathscr{J} = \ker f^\#$ has underlying space $\overline{f(X)}$. Since the support of $f_*(\mathscr{O}_X)$ is contained in $\overline{f(X)}$, we have $\mathscr{J}_y = \mathscr{O}_y$ for $y \notin \overline{f(X)}$, so the support pf $\mathscr{O}_Y/\mathscr{J}$ is contained in $\overline{f(X)}$. Moreover, this support is closed and contains $f(X)$: if $y \in f(X)$, the identity element of the ring $(f_*(\mathscr{O}_X))_y$ is nonzero, being the germ at y of the section $1 \in \Gamma(X, \mathscr{O}_X) = \Gamma(Y, f_*(\mathscr{O}_X))$; as this is the image of the identity element under $f^\#$, it is not contained in $\mathscr{J}_y$, so $\mathscr{O}_y/\mathscr{J}_y \neq 0$; this proves our first claim. The case (b) follows from Proposition 4.4.44, because there is a smallest closed subscheme Z with underlying space $\overline{f(X)}$ such that $f(X) \subseteq Z$. □

Proposition 4.6.67. *Suppose the notations of Proposition 4.6.66 is satisfied, and let Y' be the scheme-theoretic image of X under f. For any open subset V of Y, let $f_V : f^{-1}(V) \to V$ be the restriction of f. Then the scheme-theoretic image of $f^{-1}(V)$ under f_V exists and is equal to the oepn subscheme $V \cap Y'$ of Y'.*

Proof. Put $X' = f^{-1}(V)$; as the direct image of $\mathscr{O}_{X'}$ is the restriction of $f_*(\mathscr{O}_X)$ to V, it is clear that the kernel of the homomorphism $\mathscr{O}_V \to (f_V)_*(\mathscr{O}_{X'})$ is the restriction of $\mathscr{J}$ to V, whence the assertion. □

Proposition 4.6.68. *Let Y be a subscheme of a scheme X, such that the canonical injection $j : Y \to X$ is quasi-compact. Then the scheme-theoretic closure of Y exists and has $\overline{Y}$ as underlying space.*

Proof. It suffices to apply Proposition 4.6.66 to the injection j, which is separated (Proposition 4.5.25) and quasi-compact by hypothesis. □

With these notations, let $\overline{Y}$ be the scheme-theoretic closure of Y in X. If the injection $\overline{Y} \to X$ is quasi-compact, and if $\mathscr{I}$ is the quasi-coherent ideal of $\mathscr{O}_X|_{\overline{Y}}$ defining the closed subscheme Y of $\overline{Y}$, then the quasi-coherent ideal of $\mathscr{O}_X$ defining $\overline{Y}$ is the canonical extension (Proposition 4.6.56) $\overline{\mathscr{I}}$ of $\mathscr{I}$, because it is evidently the largest quasi-coherent ideal of $\mathscr{O}_X$ inducing $\mathscr{I}$ over Y.

Corollary 4.6.69. *Under the hypothesis of Proposition 4.6.68, any section of $\mathscr{O}_{\overline{Y}}$ over an open subset V of $\overline{Y}$ that is zero on $V \cap Y$ is zero.*

Proof. In view of Proposition 4.6.67, we can assume that $V = \overline{Y}$. If we consider sections of $\mathscr{O}_{\overline{Y}}$ over $\overline{Y}$ as $\overline{Y}$-sections of $\overline{Y} \otimes_{\mathbb{Z}} \mathbb{Z}[T]$, which is separated over $\overline{Y}$, the assertion is then a particular case of Corollary 4.6.65. □

4.7 Rational maps over schemes

4.7.1 Rational maps and rational functions

Let X and Y be two schemes, U and V be open dense sets of X, and f (resp. g) be a morphism from U (resp. V) to Y. We say the morphisms f and g are equivalent if they coincide over an open subset dense in $U \cap V$. As the intersection of finitely many open dense subsets of X is an open dense subset of X, it is clear that this relation is an equivalence relation.

Given two schemes X and Y, a **rational map** from X to Y is defined to be an equivalent class of morphisms from an open dense subset of X to Y. If X and Y are S-schemes, this class is called an **rational S-map** is there exists an S-morphism in it. An rational S-map from S to X is called an **rational S-section** of X. The rational X-sections of the X-scheme $X \otimes_{\mathbb{Z}} \mathbb{Z}[T]$ (where T is an indeterminate) are called the **rational functions over** X (cf. Example 4.3.11). A rational map f from X to Y is usually denoted by $f : X \dashrightarrow Y$.

Let $f : X \dashrightarrow Y$ be a rational map and U be an open subset of X. If f_1, f_2 are morphisms belonging to the class f, defined respectively over the open dense sets V_1, V_2 of X, the restrictions $f_1|_{U\cap V_1}$ and $f_2|_{U\cap V_2}$ coincide on $U \cap V_1 \cap V_2$, which is dense in U; the class of morphisms f therefore defines a rational map $U \dashrightarrow Y$, called the **restriction** of f to U and denoted by $f|_U$.

It is clear that we have a canonical map from $\mathrm{Hom}_S(X, Y)$ to the set of rational S-maps from X to Y, which associates any S-morphism $f : X \to Y$ to the rational S-map it belongs to. If we denote by $\Gamma_{\mathrm{rat}}(X/Y)$ the set of rational Y-sections of X, we then have a canonical map $\Gamma(X/Y) \to \Gamma_{\mathrm{rat}}(X/Y)$. It is also clear that if X and Y are two S-schems, the set of rational S-maps from X to Y is canonically identified with $\Gamma_{\mathrm{rat}}((X \times_S Y)/X)$.

In view of Example 4.3.11, we see the rational functions over X are canonically identified with the equivalent classes of sections of the structural sheaf $\mathscr{O}_X$ over open dense sets of X, where two sections are equivalent if they coincide over an open dense subset of the intersection of their defining domain. In partiuclar, we see the rational functions over X form a ring $K(X)$.

If X is an irreducible scheme, any nonempty open subset of X is dense; we can also say that the non-empty open sets of X are the open neighborhoods of the generic point x of X. To say that two morphisms from nonempty open subsets of X to Y are equivalent therefore means in this case that they have the same germ at the point x. In other words, rational maps (resp. rational S-maps) $X \dashrightarrow Y$ are identified with the germs of morphisms (resp. of S-morphisms) of non-empty open subsets of X to Y at the generic point x of X. In particular:

Proposition 4.7.1. *If X is an irreducible scheme, the ring $K(X)$ of rational functions over X is canonically identified with the local ring $\mathscr{O}_{X,x}$ at the generic point x of X. This is a local ring of zero dimension, and therefore an Artinian local ring if X is Noetherian. It is a field if X is integral, and is identified with the fraction field of $\Gamma(X, \mathscr{O}_X)$ if X is moreover affine.*

Proof. Since we can identify rational functions with sections over X, the first assertion follows from the definiton of stalks. For the second one, we can assume that X is affine with ring A; then $\mathfrak{p}_x$ is the nilradical of A, and in particular $\mathscr{O}_{X,x}$ has zero dimension. If A is integral, $\mathfrak{p}_x = (0)$ and $\mathscr{O}_{X,x}$ is the fraction field of A. Finally, if A is Noetherian, then $\mathfrak{p}_x$ is nilpotent and $\mathscr{O}_{X,x} = A_x$ is Artinian. □

If X is integral, the ring $\mathscr{O}_{X,z}$ is integral for any $z \in X$. Any affine open U containing x must contain x as its generic point, and $\mathscr{O}_{X,z}$, equal to a fraction field of $\Gamma(U, \mathscr{O}_X)$, is identified with $K(X)$. We then conclude that $K(X)$ is identified with the fraction field of $\mathscr{O}_{X,z}$, and in this way, $\mathscr{O}_{X,z}$ is canonically identified with a subring of $K(X)$, so that a germ $s \in \mathscr{O}_{X,z}$ is canonically identified with a rational function over X.

Proposition 4.7.2. *Let X and Y be two S-schemes such that the family (X_λ) of irreducible components of X is locally finite. For each λ, let x_λ be the generic point of X_λ. If R_λ is the set of germs at x_λ of*

S-morphisms from open subsets of X to Y, the set of rational S-maps from X to Y is identified with the product of R_λ. In particular, the ring of rational functions over X is identified with the product of the local rings $\mathscr{O}_{X,x_\lambda}$.

Proof. The set of the intersections $X_\lambda \cap X_\mu$ for $\lambda \neq \mu$ is then locally finite, so their union is closed and contains the maximal points of X. If we set $X'_\lambda = X_\lambda - \bigcup_{\mu\neq\lambda} X_\lambda \cap X_\mu$, then X'_λ is irreducible, with generic point equal to that of X_λ, and pairwise disjoint with union dense in X. For any open dense subset U of X, $U'_\lambda = U \cap X'_\lambda$ is a nonempty open dense subset of X'_λ, and U'_λ are pairwise disjoint with $U' = \bigcup_\lambda U'_\lambda$ closed in X. To give a morphism from U' to Y is then equivalent to giving (arbitrarily) a morphism from each of the U'_λ in Y, so the assertion follows. □

Corollary 4.7.3. *Let A be a Noetherian ring and $X = \mathrm{Spec}(A)$. The ring of rational function functions over X is identified with the total fraction ring $Q(A)$.*

Proof. Let S be the complement of the union of minimal prime ideals of A. Then by Proposition **??**, the ring of sections $\Gamma(D(f), \mathscr{O}_X)$ is identified with A_f, so $D(f)$ with $f \in S$ form a cofinal subset of the open dense sets of X, and the ring of rational functions over X is then identified with the inductive limit of A_f, $f \in S$, which is exactly $Q(A)$. □

Suppose that X is irreducible with generic point x. As any nonempty open set U of X contains x, and therefore contains any generalization $z \in X$, any morphism $U \to Y$ can be composed with the canonical morphism $\mathrm{Spec}(\mathscr{O}_{X,x}) \to X$ (Proposition 4.2.10). Two morphisms from nonempty open subsets of X to Y which coincide on a smaller open subset then give the same morphism $\mathrm{Spec}(\mathscr{O}_{X,x}) \to Y$. In other words, to any rational map X to Y, there corresponds a well-defined morphism $\mathrm{Spec}(\mathscr{O}_{X,x}) \to Y$.

Proposition 4.7.4. *Let X and Y be S-schemes. Suppose that X is irreducible with generic point x, and Y is of finite type over S. Then two rational S-maps X to Y corresponding to the same S-morphism $\mathrm{Spec}(\mathscr{O}_{X,x}) \to Y$ are identical. If moreover S is locally of finite presentation over S, then any S-morphism $\mathrm{Spec}(\mathscr{O}_{X,x}) \to Y$ corresponds to a rational S-map from X to Y.*

Proof. Given that every non-empty open subset of X is dense, this result follows immediately from Proposition 4.6.49. □

Corollary 4.7.5. *Suppose that X is irreducible with generic point x and Y is localy of finite presentation. Then the rational S-maps from X to Y are identified with the points of Y with values in the S-scheme $\mathrm{Spec}(\mathscr{O}_{X,x})$.*

Proof. This is just a reformulation of Proposition 4.7.4. □

Corollary 4.7.6. *Suppose that X is irreducible with generic point x and Y is localy of finite presentation. Let s be the image of x in S. Then giving a rational S-map from X to Y is equivalent to giving a point y of Y lying over s and a $\mathscr{O}_{S,s}$-homomorphism $\mathscr{O}_{Y,y} \to \mathscr{O}_{X,x} = K(X)$.*

Proof. This follows from Proposition 4.7.4 and Proposition 4.2.12. □

Corollary 4.7.7. *Suppose that X is irreducible with generic point x and Y is localy of finite presentation, then the rational S-maps from X to Y (with Y given) only depends on the S-scheme* $\operatorname{Spec}(\mathscr{O}_{X,x})$ *and in particular remain the same if we replace X by* $\operatorname{Spec}(\mathscr{O}_{X,z})$, *$z \in X$.*

Proof. In fact, if $z \in \overline{\{x\}}$ then x is the generic point of $Z = \operatorname{Spec}(\mathscr{O}_{X,z})$ and $\mathscr{O}_{X,x} = \mathscr{O}_{Z,x}$. □

Corollary 4.7.8. *Suppose that X is integral with generic point x and Y is localy of finite presentation. Let s be the image of x in S. Then following dates are equivalent:*

(i) a rational S-map from X to Y;

(ii) a point of $Y \otimes_S \kappa(s)$ with values in the extension $K(X)$ of $\kappa(s)$;

(iii) a point $y \in Y$ over s and an $\kappa(s)$-homomorphism $\kappa(y) \to \kappa(x) = K(X)$.

Proof. The points of Y over s belong to $Y \otimes_S \kappa(s)$ and the $\mathscr{O}_{S,s}$-homomorphisms $\mathscr{O}_{Y,y} \to K(X)$ are $\kappa(s)$-homomorphisms $\kappa(y) \to K(X)$, since $K(X)$ is a field. □

Corollary 4.7.9. *Let k be a field and X, Y be two schemes locally algebraic over k. Suppose that X is integral, then the rational k-maps from X to Y are identified with the points of Y with values in the extension $K(X)$ of k.*

4.7.2 Defining domain of a rational map

Let X and Y be schemes, f a rational map from X to Y. We say f is **defined at a point** $x \in X$ if there exists an open dense subset U containing x and a morphism $U \to Y$ representing f. The set of points $x \in X$ where f is defined is called the **defining domain** of the rational map f. It is clearly an open dense subset of X.

Proposition 4.7.10. *Let X and Y be two S-schemes such that X is reduced and Y is separated over S. Let f be a rational S-map from X to Y, and U_0 be its domain. There then exists a unique S-morphism $U_0 \to Y$ belonging to the class f.*

Proof. For any morphism $U \to Y$ belonging to the class f, we necessarily have $U \subseteq U_0$, so we only need to prove that if U_1, U_2 are two dense subsets of X and $f_i : U_i \to Y$ ($i = 1, 2$) are two S-morphisms that coincide on an open subset $V \subseteq U_1 \cap U_2$, then f_1 and f_2 coincide on $U_1 \cap U_2$. For this, we can clearly assume that $X = U_1 = U_2$. As X (hence V) is reduced, X is smallest closed subscheme of X dominating V (Proposition 4.4.44). Let $g = (f_1, f_2)_S : X \to Y \times_S Y$; as by hypothesis the diagonal $T = \Delta_Y(Y)$ is a closed subscheme of $Y \times_S Y$, $Z = g^{-1}(T)$ is a closed subscheme of X. If $h : V \to Y$ is the restriction of f_1 and f_2 to V, the restriction of g to V is $\tilde{g} = (h, h)_S$, which factors into $\tilde{g} = \Delta_Y \circ h$. As $\Delta_Y^{-1}(T) = Y$, we have $\tilde{g}^{-1}(T) = V$, and Z is therefore a closed subscheme of X inducing the subscheme structure on V, hence dominates V, and this implies $Z = X$. From the relation $g^{-1}(T) = X$, we deduce that g factors into $\Delta_Y \circ f$, where f is a morphism $X \to Y$ (Proposition 4.4.14), and we have $f_1 = f_2 = f$ from the definition of the diagonal morphism. □

It is clear that the morphism $U_0 \to Y$ defined in Proposition 4.7.10 is the unique morphism in the class f that admits no further extension to open dense subsets of X containing U_0. Under the conditions of Proposition 4.7.10, we can then identify the rational maps from X to Y with the morphisms unextendable to open dense subsets of X to Y.

Corollary 4.7.11. *Let X and Y be two S-schemes such that X is reduced and Y is separated over S. Let U be an open dense subset of X, then there is a canonical bijective correspondence between the S-morphisms from U to Y and the rational S-maps from X to Y defined at each point of U.*

Proof. In view of Proposition 4.7.10, for any S-morphism $f : U \to Y$, there exists a rational S-map $\bar{f}$ from X to Y which extends f. □

Corollary 4.7.12. *Let S be a separated scheme, X be a reduced S-scheme, Y be an S-scheme, and $f : U \to Y$ be an S-morphism from an open dense subset U of X to Y. If $\bar{f}$ is the rational $\mathbb{Z}$-map from X to Y extending f, $\bar{f}$ is an S-morphism (and therefore the rational S-map from X to Y extending f).*

Proof. In fact, if $\varphi : X \to S$ and $\psi : Y \to S$ are the structural morphisms, U_0 is the defining domain of $\bar{f}$, and $j : U_0 \to X$ is the injection, it suffices to prove that $\psi \circ \bar{f} = \varphi \circ j$, which follows from the proof of Proposition 4.7.10, since f is an S-morphism. □

Corollary 4.7.13. *Let X and Y be S-schemes. Suppose that X is reduced and X, Y are spearated over S. Let $p : Y \to X$ be an S-morphism, U be an open dense subset of X, and f be a U-section of Y. Then the rational map $\bar{f}$ from X to Y extending f is a rational X-section of Y.*

Proof. We only need to prove that $p \circ \bar{f}$ is the identity on the defining domain of $\bar{f}$; since X is separated over S, this follows from the proof of Proposition 4.7.10. □

Corollary 4.7.14. *Let X be a reduced scheme and U be an open dense subset of X. There exists a canonical bijective correspondence between sections of $\mathscr{O}_X$ over U and rational functions f over X defined on each point of U.*

Proof. It suffices to remark that the X-scheme $X \otimes_{\mathbb{Z}} \mathbb{Z}[T]$ is separated over X by Proposition 4.5.25. □

Corollary 4.7.15. *Let Y be a reduced scheme, $f : X \to Y$ be a separated morphism, U be an open dense subset of Y, $g : U \to f^{-1}(U)$ be a U-section of $f^{-1}(U)$, and Z the reduced subscheme of X induced on $\overline{g(U)}$. For g to be the restriction of a Y-section of X, it is necessary and sufficient that the restriction of f to Z is an isomorphism from Z to Y.*

Proof. The restriction of f to $f^{-1}(U)$ is a separated morphism (Proposition 4.5.25(i)), so g is a closed immersion by Proposition 4.5.19, and therefore $g(U) = Z \cap f^{-1}(U)$ coincides with the subscheme induced by Z over the open subset $g(U)$ of Z. It is then clear that the given condition is sufficient, since if $f_Z : Z \to Y$ is an isomorphism and $\bar{g} : Y \to Z$ is the inverse morphism, then $\bar{g}$ extends g. Conversely, if g is the restriction to U of a Y-section h of X, h is then a closed immersion by Proposition 4.5.19, so $h(Y)$ is clossed, and as it contains $g(U)$ and we have (as h is continuous) $h(Y) = h(\overline{U}) \subseteq \overline{h(U)} = \overline{g(U)}$, we conclude that $h(Y) = Z$. It then follows from Proposition 4.4.40 that h is necessarily an isomorphism from Y to the closed subscheme Z of X, so $f|_Z$ is also an isomorphism. □

Let X and Y be two S-schemes, where X is reduced and Y is separated over S. Let f be a rational S-map from X to Y, and let x be a point of X. We can compose f with the canonical morphism $\operatorname{Spec}(\mathscr{O}_{X,x}) \to X$ provided that the trace on $\operatorname{Spec}(\mathscr{O}_{X,x})$ of the defining domain of f is dense in $\operatorname{Spec}(\mathscr{O}_{X,x})$ (identified with the set $z \in X$ such as $x \in \overline{\{z\}}$ (cf. Proposition 4.2.10). This happens if the family of irreducible components of X is *locally finite*:

Lemma 4.7.16. *Let X be a scheme such that the family of irreducible components of X is locally finite, and x be a point of X. The irreducible components of $\operatorname{Spec}(\mathscr{O}_{X,x})$ are then the traces over $\operatorname{Spec}(\mathscr{O}_{X,x})$ of the irreducible components of X containing x. For an open subset U of X to be such that $U \cap \operatorname{Spec}(\mathscr{O}_{X,x})$ is dense in $\operatorname{Spec}(\mathscr{O}_{X,x})$, it is necessary and sufficient that it meets the irreducible components of X containing x (and this is ture in particular if U is dense in X).*

Proof. The second assertion clearly follows from the first one. As $\operatorname{Spec}(\mathscr{O}_{X,x})$ is contained in any affine open U containing x and the irreducible components of U containing x are the traces of the irreducible components of X containing x on U (Proposition **??**), we can suppose that X is affine with ring A. Then the prime ideals of A_x correspond to prime ideals of A contained in $\mathfrak{p}_x$, so the minimal prime ideals of A_x correspond to minimal prime ideals of A contained in $\mathfrak{p}_x$, and the lemma follows from Proposition **??**. □

Suppose that we are under the assumption of Lemma 4.7.16. If U is the defining domain of definition of the rational S-map f, denote by f' the rational map from $\operatorname{Spec}(\mathscr{O}_{X,x})$ to Y which coincides with f over $U \cap \operatorname{Spec}(\mathscr{O}_{X,x})$; we will say that this rational map is **induced** by f.

Proposition 4.7.17. *Let S be a scheme, X be a reduced S-scheme, and Y be a separted S-scheme that is locally of finite presentation over S. Suppose that the family of irreducible components of X is locally finite. Let f be a rational S-map from X to Y and x be a point of X. For f to be defined at the point x, it is necessary and sufficient that the rational map f' from $\operatorname{Spec}(\mathscr{O}_{X,x})$ to Y induced by f is a morphism.*

Proof. The conditions is clearly necessary since $\operatorname{Spec}(\mathscr{O}_{X,x})$ is contained in any open subset containing x. We now prove the sufficiency, so suppose that f' is a morphism. In view of Proposition 4.6.49, there exists an open neighborhood U of x in X and an S-morphism $g : U \to Y$ inducing f' on $\operatorname{Spec}(\mathscr{O}_{X,x})$. The point is that U is not necessarily dense in X, so we want to extend g to a morphism defined on an open dense subset of X. Now by Lemma 4.7.16, there are finitely many irreducible components X_i of X containing x, and we can assume that these are the only ones meeting U, by replacing U with a smaller open subset. As the generic points of X_i belong to the defining domain of f and to U, we see that f and g coincide over a non-empty open dense subset of each of the X_i (Proposition 4.6.49). Consider the morphism f_1 defined on an open dense subset of $U \cup (X - \overline{U})$ which equals to g over U and to f over the intersection of $X - \overline{U}$ and the defining domain of f (we also note that each X_i is contained in $\overline{U}$). As $U \cup (X - \overline{U})$ is dense in X, f_1 and f coincide on an open dense subset of X, and f is an extension of f_1. Since f_1 is defined at x, this shows f is defined at x. □

4.7.3 Sheaf of rational functions

Let X be a scheme. For each open subset U of X, we denote by $K(U)$ the ring of rational functions over U, which is an $\Gamma(U, \mathscr{O}_X)$-algebra. Moreover, if $V \subseteq U$ is a second open subset of X, any section of $\mathscr{O}_X$ over a dense subset of U restricts to a section over a dense subset of V, and if two sections coincide over an open dense subset of U, their restriction also coincide over a smaller open dense subset of V. We then define a homomorphism of algebras $\mathrm{Res}_V^U : K(U) \to K(V)$, and it is clear that for $U \supseteq V \supseteq W$ open in X we have $\mathrm{Res}_W^U = \mathrm{Res}_W^V \circ \mathrm{Res}_V^U$. Therefore, we get a presheaf of algebras over X. The associated sheaf of $\mathscr{O}_X$-algebras over X is then called the **sheaf of rational functions** over the scheme X, and denoted by $\mathscr{K}(X)$. For any open subset U of X, it is clear that the restriction $\mathscr{K}(X)|_U$ is equal to $\mathscr{K}(U)$.

Proposition 4.7.18. *Let X be a scheme such that the family (X_λ) of irreducible components of X is locally finite. Then the $\mathscr{O}_X$-module $\mathscr{K}(X)$ is quasi-coherent and for any open subset U of X, $K(U)$ is equal to $\Gamma(U, \mathscr{K}(X))$ and is identified with the product of the local rings of the generic points x_λ of the irreducible components X_λ such that $X_\lambda \cap U \neq \varnothing$.*

Proof. The fact that $K(U)$ is identified with the product follows from Proposition 4.7.2. We now show that the presheaf $U \mapsto K(U)$ is a sheaf. Consider an open subset U of X and an open covering (V_α) of U. If $s_\alpha \in K(V_\alpha)$ are such that s_α and s_β coincide over $V_\alpha \cap V_\beta$ for each pair of indices, we then conclude that for any index λ such that $U \cap X_\lambda \neq \varnothing$, the component in $K(X_\lambda)$ of all s_α such that $V_\alpha \cap X_\lambda \neq \varnothing$ are the same. Denoting by t_λ this component, it is clear that the element of $K(U)$ with component t_λ in $K(X_\lambda)$ has restriction s_α on each V_α. Finally, to see the sheaf $\mathscr{K}(X)$ is quasi-coherent, we can limit ourselves to the case $X = \mathrm{Spec}(A)$ is affine with finitely many irreducible components; by taking for U the affine open sets of the form $D(f)$, where $f \in A$, it follows from the above argument that we have $\mathscr{K}(X) = \widetilde{M}$, where M is the direct sum of the A-modules A_{x_λ}. ◻

Corollary 4.7.19. *Let X be a reduced scheme with irreducible components $(X_i)_{1\leq i\leq n}$, endowed with the reduced subscheme structures. If $\iota_i : X_i \to X$ is the canonical injection, $\mathscr{K}(X)$ is the direct product of the $\mathscr{O}_X$-algebras $(\iota_i)_*(\mathscr{K}(X_i))$.*

Proof. This is a particular case of Proposition 4.7.18, in view of the conditions in that proposition. ◻

Corollary 4.7.20. *If X is irreducible, any quasi-coherent $\mathscr{K}(X)$-module $\mathscr{F}$ is a simple sheaf.*

Proof. It suffices to show that any $x \in X$ admits a neighborhood U such that $\mathscr{F}|_U$ is a simple sheaf, which means we can assume that X is affine. We can then suppose that $\mathscr{F}$ is the cokernel of a homomorphism $\mathscr{K}(X)^{\oplus I} \to \mathscr{K}(X)^{\oplus J}$, and it all boils down to seeing that $\mathscr{K}(X)$ is a simple sheaf. But this is evident since $\Gamma(U, \mathscr{K}(X)) = K(X)$ for any nonempty open subset U, since U contains the generic point of X. ◻

Corollary 4.7.21. *If X is irreducible, for any quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{K}(X)$ is a simple sheaf. If moreover X is reduced (hence integral), $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{K}(X)$ is isomorphic to a sheaf of the form $\mathscr{K}(X)^{\oplus I}$.*

Proof. The first claim follows from Corollary 4.7.20, and the second one follows from the fact that if X is integral then $K(X)$ is a field. ◻

Proposition 4.7.22. *Let X be a scheme such that the family (X_λ) of irreducible components of X is locally finite. Then $\mathscr{K}(X)$ is a quasi-coherent $\mathscr{O}_X$-algebra. If X is reduced, the canonical homomorphism $\mathscr{O}_X \to \mathscr{K}(X)$ is injective.*

Proof. Since the question is local, the first claim follows from Proposition 4.7.18. The second one follows from Corollary 4.7.14. □

Let X and Y be integral schemes, so that $\mathscr{K}(X)$ (resp. $\mathscr{K}(Y)$) is a quasi-coherent $\mathscr{O}_X$-module (resp. $\mathscr{O}_Y$-module). Let $f : X \to Y$ be a dominant morphism; then there exists a canonical homomorphism of $\mathscr{O}_X$-modules:

$$\tau : f^*(\mathscr{K}(Y)) \to \mathscr{K}(X).$$

Suppose that $X = \mathrm{Spec}(A)$ and $Y = \mathrm{Spec}(B)$ are affine with integral domains A and B, then f corresponds to the injective homomorphism $B \to A$ (Corollary **??**), which extens to a monomorphism $L \to K$ of fraction fields. The homomorphism τ then corresponds to the canonical homomorphism $L \otimes_B A \to K$.

In the general case, for any couple of affine opens $U \subseteq X$, $V \subseteq Y$ such that $f(U) \subseteq V$, we define similarly a homomorphism $\tau_{U,V}$ and note that if $U' \subseteq U$, $V' \subseteq V$ and $f(U') \subseteq V'$, then $\tau_{U,V}$ extends $\tau_{U',V'}$. If x and y are the generic points of X and Y, respectively, then $f(x) = y$ and

$$(f^*(\mathscr{K}(Y)))_x = \mathscr{O}_{Y,y} \otimes_{\mathscr{O}_{Y,y}} \mathscr{O}_{X,x} = \mathscr{O}_{X,x}$$

and τ_x is therefore an isomorphism. However, the homomorphism τ is usually not an isomorphism: for example, if $B = L$ is a field containing the integral domain A and A is not a field, the canonical homomorphism $L \otimes_B A \to K$ is then the canonical homomorphism $A \to K$, which is not bijective.

4.7.4 Torsion sheaves and torsion-free sheaves

Let X be a reduced scheme whose family of irreducible components is locally finite. For any $\mathscr{O}_X$-module $\mathscr{F}$, the canonical homomorphism $\mathscr{O}_X \to \mathscr{K}(X)$ is injective by Proposition 4.7.22, and defines a homomorphism $\mathscr{F} \to \mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{K}(X)$ which on each stalk, is none other than the homomorphism $z \mapsto z \otimes 1$ from $\mathscr{F}_x$ to $\mathscr{F}_x \otimes_{\mathscr{O}_{X,x}} \mathscr{K}(X)_x$. The kernel $\mathscr{T}$ (also denoted by $\mathscr{T}(\mathscr{F})$) of this homomorphism is a sub-$\mathscr{O}_X$-module $\mathscr{F}$ is called the **torsion sheaf** of $\mathscr{F}$, which is quasi-coherent if $\mathscr{F}$ is quasi-coherent (Proposition 4.7.18). The sheaf $\mathscr{F}$ is called **torsion-free** if $\mathscr{T} = 0$, and a **torsion sheaf** if $\mathscr{T} = \mathscr{F}$. For any $\mathscr{O}_X$-module $\mathscr{F}$, $\mathscr{F}/\mathscr{T}$ is torsion-free.

Proposition 4.7.23. *If X is an integral scheme, for a quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$ to be torsion-free, it is necessary and sufficient that it is isomorphic to a sub-$\mathscr{O}_X$-module $\mathscr{G}$ of a simple sheaf of the form $\mathscr{K}(X)^{\oplus I}$, generated (as a $\mathscr{K}(X)$-module) by $\mathscr{G}$.*

Proof. This follows from Corollary 4.7.21, since $\mathscr{F}$ is torsion-free if and only if the canonical homomorphism $\mathscr{F} \to \mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{K}(X)$ is injective. □

The cardinality of the set I is called the **rank** of $\mathscr{F}$. For any affine open U of X, since the generic point of x is contained in U, the rank of $\mathscr{F}$ is also equal to the rank of $\Gamma(U, \mathscr{F})$ as a $\Gamma(U, \mathscr{O}_X)$-module.

Corollary 4.7.24. *Over an integral scheme X, any torsion-free quasi-coherent $\mathscr{O}_X$-module of rank 1 (and in particular any invertible $\mathscr{O}_X$-module) is isomorphic to a sub-$\mathscr{O}_X$-module of $\mathscr{K}(X)$, and the converse is also true.*

Corollary 4.7.25. *Let X be an integral scheme, $\mathscr{L}$, $\mathscr{L}'$ be two torsion-free $\mathscr{O}_X$-module, s (resp. s') be two sections of $\mathscr{L}$ (resp. $\mathscr{L}'$) over X. For $s \otimes s' = 0$, it is necessary and sufficient that one of the sections s, s' is zero.*

Proof. Let x be the generic point of X. We have by hypothesis $(s \otimes s')_x = s_x \otimes s'_x = 0$. As $\mathscr{L}_x$ and $\mathscr{L}'_x$ are identified with sub-$\mathscr{O}_{X,x}$-modules of the field $\mathscr{O}_{X,x}$, the preceding relation implies $s_x = 0$ or $s'_x = 0$, and therefore $s = 0$ or $s' = 0$ since $\mathscr{L}$ and $\mathscr{L}'$ are torsion-free (Corollary 4.7.20). □

Proposition 4.7.26. *Let X and Y be two integral schemes and $f : X \to Y$ be a dominant morphism. For any torsion-free quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, $f_*(\mathscr{F})$ is a torsion-free $\mathscr{O}_Y$-module.*

Proof. As f_* is left exact, it suffices, in view of Proposition 4.7.23, to prove the proposition for $\mathscr{F} = \mathscr{K}(X)^{\oplus I}$. Now any open subset U of Y contains the generic point of Y, hence $f^{-1}(U)$ contains the generic point of X, so we have $\Gamma(U, f_*(\mathscr{F})) = \Gamma(f^{-1}(U), \mathscr{F}) = K(X)^{\oplus I}$. Therefore $f_*(\mathscr{F})$ is the simple sheaf with stalk $K(X)^{\oplus I}$, considered as a $\mathscr{K}(Y)$-module, and it is evidently torsion-free. □

Proposition 4.7.27. *Let X be reduced scheme whose family of irreducible components is locally finite. For any quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$ of finite type, the following conditions are equivalent:*

(i) *$\mathscr{F}$ is a torsion sheaf.*

(ii) *$\mathscr{F}_x = 0$ for every maximal point of x.*

(iii) *$\operatorname{supp}(\mathscr{F})$ contains no irreducible component of X.*

Proof. Since the question is local, we may assume that X has finitely many irreducible components $(X_i)_{1 \leq i \leq n}$, with generic points x_i. Endow each X_i the reduced subscheme structure of X, and let $\iota_i : X_i \to X$ be the canonical injection. If we put $\mathscr{F} = \iota_i^*(\mathscr{F})$, we see immediately that (Corollary 4.7.19) $\mathscr{F}$ is torsion-free if and only if each $\mathscr{F}_i$ is torsion-free. As $\mathscr{F}_{x_i} = (\mathscr{F}_i)_{x_i}$, to establish the equivalence of (i) and (ii), we can assume that X is integral. But then if x is the generic point of X, the relation $\mathscr{F}_x = 0$ and $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{K}(X) = 0$ are equivalent by Proposition 4.7.23 and Corollary 4.7.20. The equivalenct of (ii) and (iii) results from the fact that $\operatorname{supp}(\mathscr{F})$ is closed in X (since $\mathscr{F}$ is quasi-coherent) and that the conditions $\operatorname{supp}(\mathscr{F}) \cap X_i = \varnothing$ and $x_i \notin \operatorname{supp}(\mathscr{F})$ are then equivalent. □

4.7.5 Separation criterion for integral schemes

Let X be an integral scheme, K its function field, identified with the local ring at the generic point ξ of X. For any $x \in X$, we can identify $\mathscr{O}_{X,x}$ as a subring of K, formed by the rational functions defined at the point x. For any rational function $f \in K$, the defining domain $\delta(f)$ of f is then the open subset of $x \in X$ such that $f \in \mathscr{O}_{X,x}$, and in view of Corollary 4.7.14 we have, for each open subset $U \subseteq X$, that

$$\Gamma(U, \mathscr{O}_X) = \bigcap_{x \in U} \mathscr{O}_{X,x}. \tag{7.5.1}$$

Given a field K, for any subring A of K, we denote by $L(A)$ the set of localizations $A_{\mathfrak{p}}$, where $\mathfrak{p}$ runs through prime ideals of A; they are identified with local subrings of K containing A. As $\mathfrak{p} = (\mathfrak{p}A_{\mathfrak{p}}) \cap A$, the map $\mathfrak{p} \mapsto A_{\mathfrak{p}}$ from $\mathrm{Spec}(A)$ to $L(A)$ is bijective.

Lemma 4.7.28. *Let K be a field and A be a subring of K. For a local subring R to dominate a ring in $L(A)$, it is necessary and sufficient that $A \subseteq R$. In this case, the local ring $A_{\mathfrak{p}}$ dominated by R is then unique and corresponds to the prime ideal $\mathfrak{p} = \mathfrak{m}_R \cap A$, where $\mathfrak{m}_R$ is the maximal ideal of R.*

Proof. In fact, if R dominates $A_{\mathfrak{p}}$, then $\mathfrak{m}_R \cap A_{\mathfrak{p}} = \mathfrak{p}A_{\mathfrak{p}}$ by Proposition **??**, hence the uniqueness of $\mathfrak{p}$. On the other hand, if $A \subseteq R$, $\mathfrak{m}_R \cap A = \mathfrak{p}$ is a prime ideal of A, and as the elements of $A - \mathfrak{p}$ are then invertible in R, we have $A_{\mathfrak{p}} \subseteq R$, so $\mathfrak{p}A_{\mathfrak{p}} \subseteq \mathfrak{m}_R$ and R dominates $A_{\mathfrak{p}}$. □

Lemma 4.7.29. *Let K be a field, A, B be two local subrings of K, and C be the subring of K generated by $A \cup B$. The following conditions are equivalent:*

(i) There exists a prime ideal $\mathfrak{r}$ of C such that $\mathfrak{m}_A = \mathfrak{r} \cap A$ and $\mathfrak{m}_B = \mathfrak{r} \cap B$.

(ii) The ideal $\mathfrak{c}$ generated in C by $\mathfrak{m}_A \cup \mathfrak{m}_B$ is proper.

(iii) There exists a local subring R of K dominating both A and B.

Proof. It is clear that (i) implies (ii). Conversely, if $\mathfrak{c}$ is proper, it is contained in a maximal ideal $\mathfrak{n}$ of C, and $\mathfrak{n} \cap A$ contains $\mathfrak{m}_A$ and is proper, so $\mathfrak{n} \cap A = \mathfrak{m}_A$ and similarly $\mathfrak{n} \cap B = \mathfrak{m}_B$. Finally, it is clear that if R dominates A and B then $C \subseteq R$ and $\mathfrak{m}_A = \mathfrak{m}_R \cap A = (\mathfrak{m}_R \cap C) \cap A$, $\mathfrak{m}_B = \mathfrak{m}_R \cap B = (\mathfrak{m}_R \cap C) \cap B$, so (iii) implies (i). the converse is clear since we can take $R = C_{\mathfrak{r}}$. □

If the equivalent conditions in Lemma 4.7.29 hold, we say the two local subrings A and B are **related**.

Proposition 4.7.30. *Let A and B be subrings of a field K and C be the subring of K generated by $A \cup B$. The following conditions are equivalent:*

(i) For any local ring R containing A and B, we have $A_{\mathfrak{p}} = B_{\mathfrak{q}}$, where $\mathfrak{p} = \mathfrak{m}_R \cap A$ and $\mathfrak{q} = \mathfrak{m}_R \cap B$.

(ii) For any prime ideal $\mathfrak{r}$ of C, we have $A_{\mathfrak{p}} = B_{\mathfrak{q}}$, where $\mathfrak{p} = \mathfrak{r} \cap A$ and $\mathfrak{q} = \mathfrak{r} \cap B$.

(iii) If $P \in L(A)$ and $Q \in L(B)$ are related, they are identical.

(iv) We have $L(A) \cap L(B) = L(C)$.

Proof. It follows from Lemma 4.7.28 and Lemma 4.7.29 that (i) and (iii) are equivalent, and (i) implies (ii) by applying (i) to the ring $R = C_{\mathfrak{r}}$. Conversely, (ii) implies (i) because if R contains $A \cup B$, it contains C, and if $\mathfrak{r} = \mathfrak{m}_R \cap C$, we have $\mathfrak{p} = \mathfrak{r} \cap A$ and $\mathfrak{q} = \mathfrak{r} \cap B$, so $A_{\mathfrak{p}} = B_{\mathfrak{q}}$. We also see that (iv) implies (i), because if R contains $A \cup B$, it then dominates a local ring $C_{\mathfrak{r}} \in L(C)$ by Lemma 4.7.28; we have by hypothesis that $L(C) = L(A) \cap L(B)$, and as R dominates a unique ring in $L(A)$ (resp. $L(B)$), we conclude that $C_{\mathfrak{r}} = A_{\mathfrak{p}} = B_{\mathfrak{q}}$.

Finally, we show that (iii) implies (iv). Let $R \in L(C)$; R then dominates a ring $P \in L(A)$ and a ring $Q \in L(B)$ by Lemma 4.7.28, so P and Q are related, hence identical by hypothesis. As we then have $C \subseteq P$, P dominates a ring $R' \in L(C)$ (Lemma 4.7.28), so R dominates the ring R', and by Lemma 4.7.28 we necessarily have $R = R' = P$, so $R \in L(A) \cap L(B)$. Conversely, if $R \in L(A) \cap L(B)$, we have $C \subseteq R$, so R dominates a ring $R'' \in L(C)$ by Lemma 4.7.28. The two subrings R and R'' are clearly related, and as $L(C) \subseteq L(A) \cap L(B)$, we conclude from condition (iii) that $R = R''$, so $R \in L(C)$ and the proof is complete. □

Proposition 4.7.31. *Let X be an integral scheme and K be its field of rational functions. Then for X to be separated, it is necessary and sufficient that the relation "$\mathscr{O}_{X,x}$ and $\mathscr{O}_{X,y}$ are related" for two points $x, y \in X$ implies $x = y$.*

Proof. Suppose the given condition on X, we prove that X is separated. Let $U = \operatorname{Spec}(A)$ and $V = \operatorname{Spec}(B)$ be two distinct affine opens of X, with A, B identified as subrings of K. Then U (resp. V) is identified with the set $L(A)$ (resp. $L(B)$), and by Proposition 4.7.30 the hypothesis on X implies that, if C is the subring of K generated by $A \cup B$, $W = U \cap V$ is identified with $L(A) \cap L(B) = L(C)$. Moreover, we have seen from Proposition **??** that any subring R of K is equal to the intersection of the local rings belong to $L(R)$, so

$$C = \bigcap_{z \in W} \mathscr{O}_{X,z} = \Gamma(W, \mathscr{O}_X) \tag{7.5.2}$$

where we use formula (7.5.1). Consider then the subscheme induced by X over W. The identity homomorphism $\varphi : C \to \Gamma(W, \mathscr{O}_X)$ corresponds to a morphism $\psi : W \to \operatorname{Spec}(C)$. In view of (7.5.2) and the relation $L(C) = L(A) \cap L(B)$, any prime ideal $\mathfrak{r}$ of C is of the form $\mathfrak{r} = \mathfrak{m}_x \cap C$, where $x \in W$ is the point in $\operatorname{Spec}(C)$ corresponding to $\mathfrak{r}$, and the map ψ just sends x to $\mathfrak{r}$, so it is bijective. On the other hand, for any $x \in W$, $\psi_x^\#$ is the canonical injection $C_{\mathfrak{r}} \to \mathscr{O}_{X,x}$, where $\mathfrak{r} = \mathfrak{m}_x \cap C$. Now the local ring $\mathscr{O}_{X,x}$ dominates $A_{\mathfrak{p}}$, $B_{\mathfrak{q}}$ and $C_{\mathfrak{r}}$, where $\mathfrak{p} = \mathfrak{m}_x \cap A$ and $\mathfrak{q} = \mathfrak{m}_x \cap B$, and as $C_{\mathfrak{r}} \in L(C) = L(A) \cap L(B)$, by Lemma 4.7.28 we then conclude that $C_{\mathfrak{r}} = A_{\mathfrak{p}} = B_{\mathfrak{q}}$ (we already have seen this in the proof of Proposition 4.7.30). But the local rings $A_{\mathfrak{p}}$ and $B_{\mathfrak{q}}$ are both identified with the stalk $\mathscr{O}_{X,x}$, so we see that $C_{\mathfrak{r}} = \mathscr{O}_{X,x}$ and $\psi_x^\#$ is bijective. It then remains to show that ψ is a homeomorphism, which amounts to show that, for any closed subset $F \subseteq W$, the image $\psi(F)$ is closed in $\operatorname{Spec}(C)$. Now F is the intersection with W of a closed subset E of the form $V(\mathfrak{a})$, where $\mathfrak{a}$ is an ideal of A. We claim that $\psi(F) = V(\mathfrak{a}C)$: in fact, the prime ideal of C containing $\mathfrak{a}C$ are the prime ideals of C containing $\mathfrak{a}$, hence the ideals of the form $\psi(x) = \mathfrak{m}_x \cap C$, where $\mathfrak{a} \subseteq \mathfrak{m}_x$ and $x \in W$. As $\mathfrak{a} \subseteq \mathfrak{m}_x$ is equivalent to $x \in V(\mathfrak{a}) = W \cap E$ for $x \in U$, we then get $\psi(F) = V(\mathfrak{a}C)$. In view of Proposition 4.5.30, we then conclude that X is separated, because $U \cap V$ is affine and the ring C is generated by $A \cup B$.

Conversely, suppse that X is separated, and let x, y be two points of X such that $\mathscr{O}_{X,x}$ and $\mathscr{O}_{X,y}$ are related. Let U (resp. V) be an open affine containing x (resp. y), with ring A (resp. B). We then see $U \cap V$ is affine and its ring C is generated by $A \cup B$ (Proposition 4.5.30). If $\mathfrak{p} = \mathfrak{m}_x \cap A$ and $\mathfrak{q} = \mathfrak{m}_x \cap B$, we have $A_{\mathfrak{p}} = \mathscr{O}_{X,x}$ and $B_{\mathfrak{q}} = \mathscr{O}_{X,y}$, so $A_{\mathfrak{p}}$ and $B_{\mathfrak{q}}$ are related. Then by

Lemma 4.7.29 there exists a prime ideal $\mathfrak{r}$ of C such that $\mathfrak{p} = \mathfrak{r} \cap A$, $\mathfrak{q} = \mathfrak{r} \cap B$. But the prime ideal $\mathfrak{r}$ then corresponds to a point $z \in U \cap V$ since $U \cap V$ is affine, and we have $x = z$ and $y = z$, so $x = y$. □

Corollary 4.7.32. *Let X be a separated integral scheme and x, y be two points of X. For $x \in \overline{\{y\}}$, it is necessary and sufficient that $\mathscr{O}_{X,x} \subseteq \mathscr{O}_{Y,y}$, which means the rational functions defined at x are also defined at y.*

Proof. This condition is clearly necessary since the defining domain $\delta(f)$ of a rational function is open, hence stable under generalization. To see it is also sufficient, assume that $\mathscr{O}_{X,x} \subseteq \mathscr{O}_{X,y}$, so there exists a prime ideal $\mathfrak{p}$ of $\mathscr{O}_{X,x}$ such that $\underline{\mathscr{O}_{Y,y}}$ dominates $(\mathscr{O}_{X,x})_{\mathfrak{p}}$ (Lemma 4.7.28). By Proposition 4.2.10, there exists $z \in X$ such that $x \in \overline{\{z\}}$ and $\mathscr{O}_{X,z} = (\mathscr{O}_{X,x})_{\mathfrak{p}}$; as $\mathscr{O}_{X,z}$ and $\mathscr{O}_{X,y}$ are then related, we have $z = y$ by Proposition 4.7.31, whence the corollary. □

Corollary 4.7.33. *If X is a separated integral scheme, the map $x \mapsto \mathscr{O}_{X,x}$ is injective. In other words, if x, y are two distinct points of X, there exists a rational function defined at only one of these points.*

Proof. This follows from Corollary 4.7.32 and the T_0-axiom. □

Corollary 4.7.34. *Let X be a Noetherian separated integral scheme. The sets $\delta(f)$ for $f \in K(X)$ form a subbasis the topology of X.*

Proof. In fact, any closed subset of X is then a finite union of irreducible closed subsets, which are of the form $\overline{\{y\}}$. Now if $x \notin \overline{\{y\}}$, there exists a rational function f defined at x but not at y (Corollary 4.7.33), which means $x \in \delta(f)$ and $\delta(f) \cap \overline{\{y\}} = \varnothing$. The complement of $\overline{\{y\}}$ is then a union of sets of the form $\delta(f)$, and in view of the previous remark, any open subset of X is a union of finite intersections of sets of the form $\delta(f)$. □

Proposition 4.7.35. *Let X, Y be two integral schemes with rational function fields K and L, respectively. Suppose that Y is separated and let $f : X \to Y$ be a dominant morphism. Then L is identified with a subfield of K, and for every point $x \in X$, $\mathscr{O}_{Y,f(x)}$ is the unique local ring of Y dominated by $\mathscr{O}_{X,x}$.*

Proof. Ths first assertion is already proved in Proposition 4.4.21. Now for every $x \in X$, the local homomorphism $f_x^{\#} : \mathscr{O}_{Y,f(x)} \to \mathscr{O}_{X,x}$ is injective by Proposition 4.4.21, so, if we identify L as a subfield of K, $\mathscr{O}_{Y,f(x)}$ is dominated by $\mathscr{O}_{X,x}$. As Y is separated, two local rings of Y can not be dominated by the same local ring (Proposition 4.7.31), so our assertion follows. □

Proposition 4.7.36. *Let X be an irreducible scheme and $f : X \to Y$ be a local immersion (resp. a local isomorphism). Suppose that f is separated, then it is an immersion (resp. an open immersion).*

Proof. It suffices to prove that f is a homeomorphism from X to $f(X)$ (Proposition 4.4.9). By replacing f with f_{red}, we may assume that X and Y are reduced. If Y' is the reduced subscheme of Y with underlying space $\overline{f(X)}$, f then factors into

$$X \xrightarrow{f'} Y' \xrightarrow{j} Y$$

where j is the canonical injection. Then f' is separated by Proposition 4.5.25(v) and is a local immersion by Proposition 4.5.15(iii), so we may reduce to the case that f is dominant. But then Y is irreducible by Proposition **??**, and by Proposition 4.4.21, we see f is in fact a local isomorphism, so for each $x \in X$ the homomorphism $f_x^\#$ is an isomorphism. By Corollary 4.7.33, this implies that f is injective, so f is in fact a homeomorphism. □

4.8 Formal schemes

4.8.1 Formal affine schemes and morphisms

LEt A be a admissible topological ring, with a defining ideal $\mathfrak{I}$; $\mathrm{Spec}(A/\mathfrak{I})$ is then a closed subscheme of $\mathrm{Spec}(A)$, which is the set of open prime ideals of A. This topological space does not depend on the defining ideal of $\mathfrak{I}$, and we denote it by $\mathfrak{X}$. Let $(\mathfrak{I}_\lambda)$ be a system of fundamental neighborhood of 0 in A, formed by the defining ideals of A, and for any λ, let $\mathscr{O}_\lambda$ be the structural sheaf of $\mathrm{Spec}(A/\mathfrak{I}_\lambda)$. This sheaf is induced over $\mathfrak{X}$ by $\widetilde{A}/\widetilde{\mathscr{F}}_\lambda$ (which is zero outside $\mathfrak{X}$). For $\mathfrak{I}_\mu \subseteq \mathfrak{I}_\lambda$, the canonical homomorphism $A/\mathfrak{I}_\mu \to A/\mathfrak{I}_\lambda$ defines a homomorphism $u_{\lambda\mu} : \mathscr{O}_\mu \to \mathscr{O}_\lambda$ of sheaves of rings, and $(\mathscr{O}_\lambda)$ is a projective system of sheaves of rings for these homomorphisms. As the topology of $\mathfrak{X}$ admits a basis formed by quasi-compact open subsets, we can associate to any $\mathscr{O}_\lambda$ a sheaf of discrete topological rings with $\mathscr{O}_\lambda$ as underlying ring, still denoted by $\mathscr{O}_\lambda$; the $\mathscr{O}_\lambda$ then form a projective system of sheaves of topological rings. We denote by $\mathscr{O}_\mathfrak{X}$ the limit of this system $(\mathscr{O}_\lambda)$; by Proposition **??**, for any quasi-compact open subset U of $\mathfrak{X}$, $\Gamma(U, \mathscr{O}_\mathfrak{X})$ is then the limit topological ring of the discrete rings $\Gamma(U, \mathscr{O}_\lambda)$.

Given an admissible topological ring A, the closed subspace $\mathfrak{X}$ of $\mathrm{Spec}(A)$ formed by open prime ideals of A is called the **formal spectrum** of A and denoted by $\mathrm{Spf}(A)$. A topoloigcally ringed space is called a **formal affine scheme** if it is isomorphic to a formal spectrum $\mathrm{Spf}(A) = \mathfrak{X}$ endowed with the sheaf of topological rings $\mathscr{O}_\mathfrak{X}$, which is the limit of the sheaf of discrete rings $(\widetilde{A}/\widetilde{\mathfrak{I}_\lambda})|_\mathfrak{X}$, where $\mathfrak{I}_\lambda$ runs through the filtered set of defining ideals of A. When we speak of a formal spectrum $\mathfrak{X} = \mathrm{Spf}(A)$ as an formal affine scheme, it will always be understood that the topologically ringed space $(\mathfrak{X}, \mathscr{O}_\mathfrak{X})$ where $\mathscr{O}_\mathfrak{X}$ is defined as above. By an **adic** (resp. **Noetherian**) formal affine scheme, we mean a formal affine scheme which is isomorphic to a formal specturm $\mathrm{Spf}(A)$, where A is adic (resp. adic and Noetherian).

We note that any affine scheme $X = \mathrm{Spec}(A)$ can be considered as a formal affine scheme in a unique way: consider A as a discrete topological ring, the rings $\Gamma(U, \mathscr{O}_X)$ are then discrete if U is quasi-compact (but not true in general if U is any open set of X).

Proposition 4.8.1. *If $\mathfrak{X} = \mathrm{Spf}(A)$, where A is an admissible ring, then $\Gamma(X, \mathscr{O}_\mathfrak{X})$ is homeomorphic to A.*

Proof. In fact, as $\mathfrak{X}$ is closed in $\mathrm{Spec}(A)$, it is quasi-compact, and therefore $\Gamma(\mathfrak{X}, \mathscr{O}_\mathfrak{X})$ is homeomorphic to the limit of the discrete rings $\Gamma(\mathfrak{X}, \mathscr{O}_\lambda)$. But $\Gamma(\mathfrak{X}, \mathscr{O}_\lambda)$ is isomorphic to $A/\mathfrak{I}_\lambda$, and as A is separated and complete, this is homeomorphic to $\varinjlim A/\mathfrak{I}_\lambda$, whence the proposition. □

Proposition 4.8.2. *Let A be an admissible ring, $\mathfrak{X} = \mathrm{Spf}(A)$, and for $f \in A$, let $\mathfrak{D}(f) = D(f) \cap \mathfrak{X}$. Then the topologically ringed space $(\mathfrak{D}(f), \mathscr{O}_\mathfrak{X}|_{\mathfrak{D}(f)})$ is isomorphic to a formal affine spectrum $\mathrm{Spf}(A_{\{f\}})$.*

Proof. For any defining ideal $\mathfrak{I}$ of A, the discrete ring $A_f/\mathfrak{I}_f$ is canonically identified with $A_{\{f\}}/\mathfrak{I}_{\{f\}}$ (Proposition **??**), so the topological space $\mathrm{Spf}(A_{\{f\}})$ is canonically identified with

$\mathfrak{D}(f)$. Moreover, for any quasi-compact open U of $\mathfrak{X}$ contained in $\mathfrak{D}(f)$, $\Gamma(U, \mathscr{O}_\lambda)$ is identified with the module of sections of the structural sheaf of $\mathrm{Spec}(A_f/\mathfrak{I}_\lambda)$ over U, so, if we put $\mathfrak{Y} = \mathrm{Spf}(A_{\{f\}})$, $\Gamma(U, \mathscr{O}_{\mathfrak{X}})$ is identified with $\Gamma(U, \mathscr{O}_{\mathfrak{Y}})$, whence the proposition. □

As a sheaf of rings, the stalk of the structural sheaf $\mathscr{O}_{\mathfrak{X}}$ of $\mathrm{Spf}(A)$ for any $x \in X$ is, by Proposition 4.8.2, identified with the inductive limit $\varinjlim A_{\{f\}}$ for $f \notin \mathfrak{p}_x$. Therefore, by Proposition **??** and Proposition **??**, we have the following result:

Proposition 4.8.3. *For any $x \in \mathfrak{X} = \mathrm{Spf}(A)$, the stalk $\mathscr{O}_{\mathfrak{X},x}$ is a local ring whose residue field is isomorphic to $\kappa(x) = A_x/\mathfrak{p}_x A_x$. If A is adic and Noetherian, then $\mathscr{O}_{X,x}$ is a Noetherian ring.*

As the field $\kappa(x)$ is not reduced to 0, we conclude in particular that the support of $\mathscr{O}_{\mathfrak{X}}$ is equal to $\mathfrak{X}$, and $(\mathfrak{X}, \mathscr{O}_{\mathfrak{X}})$ is a locally topologically ringed space.

We now consider morphisms of formal affine schemes. Let A, B be admissible rings, and $\varphi : B \to A$ be a continuous homomorphism. The continuous map ${}^a\varphi : \mathrm{Spec}(A) \to \mathrm{Spec}(B)$ then maps $\mathfrak{X} = \mathrm{Spf}(A)$ into $\mathfrak{Y} = \mathrm{Spf}(B)$, because the inverse image of an open prime ideal of A is an open prime ideal of B. On the other hand, for any $g \in B$, φ defines a continuous homomorphism $\Gamma(\mathfrak{D}(g), \mathscr{O}_{\mathfrak{Y}}) \to \Gamma(\mathfrak{D}(\varphi(g), \mathscr{O}_{\mathfrak{X}}))$ in view of Proposition 4.8.1 and Proposition 4.8.2; as these homomorphisms are compatible with restrictions and $\mathfrak{D}(\varphi(g)) = ({}^a\varphi)^{-1}(\mathfrak{D}(g))$, we obtain a continuous homomorphism of sheaves of topological rings $\mathscr{O}_{\mathfrak{Y}} \to {}^a\varphi_*(\mathscr{O}_{\mathfrak{X}})$, which we denoted by $\tilde{\varphi}$. We then get a morphism $({}^a\varphi, \tilde{\varphi}) : (X, \mathscr{O}_{\mathfrak{X}}) \to (\mathfrak{Y}, \mathscr{O}_{\mathfrak{Y}})$ of topologically ringed spaces.

Proposition 4.8.4. *Let A, B be admissible topological rings, and $\mathfrak{X} = \mathrm{Spf}(A)$, $\mathfrak{Y} = \mathrm{Spf}(B)$. For a morphism $\psi : \mathfrak{X} \to \mathfrak{Y}$ of topologically ringed spaces to be of the form $({}^a\varphi, \tilde{\varphi}) : \mathfrak{X} \to \mathfrak{Y}$, it is necessary and sufficient that for each $x \in X$, $f_x^\# : \mathscr{O}_{\mathfrak{Y},\psi(x)} \to \mathscr{O}_{\mathfrak{X},x}$ is a local homomorphism.*

Proof. This conditions is necessary: in fact, let $\mathfrak{p} = \mathfrak{p}_x \in \mathrm{Spf}(A)$, and $\mathfrak{q} = \varphi^{-1}(\mathfrak{p}_x)$; if $g \notin \mathfrak{q}$, then $\varphi(g) \notin \mathfrak{p}$, and it is immediate that the homomorphism $B_{\{g\}} \to A_{\{\varphi(g)\}}$ induced from φ maps $\mathfrak{q}_{\{g\}}$ into $\mathfrak{p}_{\{\varphi(g)\}}$; by passing to inductive limit, we then see that $\tilde{\varphi}_x$ is a local homomorphism.

Conversely, let ψ be a morphism satisfying this condition. By Proposition 4.8.1, $\psi^\#$ defines a continuous homomorphism

$$\varphi = \Gamma(\psi^\#) : B = \Gamma(\mathfrak{Y}, \mathscr{O}_{\mathfrak{Y}}) \to \Gamma(\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}) = A.$$

By the hypothesis on $\psi^\#$, for the section $\varphi(g)$ of $\mathscr{O}_{\mathfrak{X}}$ over $\mathfrak{X}$ has invertible germ at a point x, it is necessary and sufficient that g has invertible germ at $\psi(x)$. But by Proposition **??**, the sections of $\mathscr{O}_{\mathfrak{X}}$ (resp. $\mathscr{O}_{\mathfrak{Y}}$) over $\mathfrak{X}$ (resp. $\mathfrak{Y}$) which have non-invertible germs at x (resp. $\psi(x)$) are exactly the elements of $\mathfrak{p}_x$ (resp $\mathfrak{p}_{\psi(x)}$), so we conclude that $\psi = {}^a\varphi$. Finally, for any $g \in B$, the diagram

$$\begin{tikzcd} B = \Gamma(\mathfrak{Y}, \mathscr{O}_{\mathfrak{Y}}) \arrow[r, "\varphi"] & \Gamma(\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}) = A \arrow[d] \\ B_{\{g\}} = \Gamma(\mathfrak{D}(g), \mathscr{O}_{\mathfrak{Y}}) \arrow[r, "\Gamma(\psi^\#_{\mathfrak{D}(g)})"] & \Gamma(\mathfrak{D}(\varphi(g), \mathscr{O}_{\mathfrak{X}})) = A_{\{\varphi(g)\}} \end{tikzcd}$$

is commutative. By the universal property of localization of complete rings (Proposition **??**), we conclude that $\psi^\#_{\mathfrak{D}(g)}$ is equal to $\tilde{\varphi}_{\mathfrak{D}(g)}$ for $g \in B$, so we have $\psi^\# = \tilde{\varphi}$. □

We say a morphism $\psi : \mathfrak{X} \to \mathfrak{Y}$ satisfying the condition in Proposition 4.8.4 is a **morphism of formal affine schemes**. Then by Proposition 4.8.4, the functor $A \mapsto \mathrm{Spf}(A)$ and $\mathfrak{X} \to \Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ define an equivalence from the category of admissible topological rings to the opposite of the category of formal affine schemes.

As a particular case of Proposition 4.8.4, note that for $f \in A$, the canonical injection of the formal affine scheme over $\mathfrak{D}(f)$ induced by $\mathfrak{X}$ corresponds to the canonical homomorphism $A \to A_{\{f\}}$. Under the hypothesis of Proposition 4.8.4, let h be an element of B and g be an element of A, which is a multiple of $\varphi(h)$. We then have $\psi(\mathfrak{D}(g)) \subseteq \mathfrak{D}(h)$; the restriction of ψ to $\mathfrak{D}(g)$, considered as a morphism $\mathfrak{D}(g) \to \mathfrak{D}(h)$, is the unique morphism η such that the diagram

$$\begin{tikzcd} \mathfrak{D}(g) \arrow[r, "\eta"] \arrow[d] & \mathfrak{D}(h) \arrow[d] \\ \mathfrak{X} \arrow[r, "\psi"] & \mathfrak{Y} \end{tikzcd}$$

This morphism corresponds to the unique continuous homomorphism $\varphi' : B_{\{h\}} \to A_{\{g\}}$ (Proposition **??**) such that the diagram

$$\begin{tikzcd} B \arrow[r, "\varphi"] \arrow[d] & A \arrow[d] \\ B_{\{h\}} \arrow[r, "\varphi'"] & A_{\{g\}} \end{tikzcd}$$

is commutative.

Let A be an admissible ring, $\mathfrak{I}$ be an open ideal of A, $\mathfrak{X}$ the formal affine scheme $\mathrm{Spf}(A)$. Let $(\mathfrak{I}_\lambda)$ be the set of defining ideals of A contained in $\mathfrak{I}$; then $\widetilde{\mathfrak{I}}/\widetilde{\mathfrak{I}}_\lambda$ is a sheaf of ideals of $\widetilde{A}/\widetilde{\mathfrak{I}}_\lambda$. Denote by $\mathfrak{I}^\Delta$ the projective limit of the sheaves induced by $\widetilde{\mathscr{I}}/\widetilde{\mathfrak{I}}_\lambda$ over $\mathfrak{X}$, which is identified as an ideal of $\mathcal{O}_{\mathfrak{X}}$. For any $f \in A$, $\Gamma(\mathfrak{D}(f), \mathfrak{I}^\Delta)$ is the projective limit of $\mathfrak{I}_f/(\mathscr{I}_\lambda)_f$, which is identified with the open ideal $\mathfrak{I}_{\{f\}}$ of the ring $A_{\{f\}}$ (Proposition **??**), and in particular $\Gamma(\mathfrak{X}, \mathfrak{I}^\Delta) = \mathfrak{I}$. We then conclude that (the $\mathfrak{D}(f)$ form a base of $\mathfrak{X}$) that we have

$$\mathfrak{I}^\Delta|_{\mathfrak{D}(f)} = (\mathfrak{I}_{\{f\}})^\Delta \tag{8.1.1}$$

With these notations, for $f \in A$ the canonical map of $A_{\{f\}} = \Gamma(\mathfrak{D}, \mathcal{O}_{\mathfrak{X}})$ in $\Gamma(\mathfrak{D}(f), (\widetilde{A}/\widetilde{\mathfrak{I}})|_{\mathfrak{X}}) = A_f/\mathfrak{I}_f$ is surjective with kernel $\Gamma(\mathfrak{D}(f), \mathfrak{I}^\Delta) = \mathfrak{I}_{\{f\}}$ (Proposition **??**). These maps define a canonical continuous epimorphism from the sheaf $\mathcal{O}_{\mathfrak{X}}$ to the sheaf of discrete rings $(\widetilde{A}/\widetilde{\mathfrak{I}})|_{\mathfrak{X}}$, whose kernel is $\mathfrak{I}^\Delta$; this homomorphism is none other than the homomorphism $\tilde{\varphi}$, where φ is the canonical continuous homomorphism $A \to A/\mathfrak{I}$. The morphism $({}^a\varphi, \tilde{\varphi}) : \mathrm{Spec}(A/\mathfrak{I}) \to \mathfrak{X}$ of the formal affine schemes is then called the canonical morphism. We then have a canonical isomorphism

$$\mathcal{O}_{\mathfrak{X}}/\mathfrak{I}^\Delta \xrightarrow{\sim} (\widetilde{A}/\widetilde{\mathfrak{I}})|_{\mathfrak{X}}. \tag{8.1.2}$$

It is clear (in view of $\Gamma(X, \mathfrak{I}^\Delta) = \mathfrak{I}$) that the map $\mathfrak{I} \mapsto \mathfrak{I}^\Delta$ is strictly increasing: in fact, for $\mathfrak{I} \subseteq \mathfrak{I}'$, the sheaf $\mathfrak{I}'^\Delta/\mathfrak{I}^\Delta$ is canonically isomorphic to $\widetilde{\mathfrak{I}}'/\widetilde{\mathfrak{I}} = \widetilde{\mathfrak{I}'/\mathfrak{I}}$.

We now say that an ideal $\mathscr{I}$ of $\mathcal{O}_{\mathfrak{X}}$ is a defining ideal of $\mathfrak{X}$ if, for any $x \in \mathfrak{X}$, there exists an open neighborhood of x in $\mathfrak{X}$ of the form $\mathfrak{D}(f)$, where $f \in A$, such that $\mathscr{I}|_{\mathfrak{D}(f)}$ is of the form $\mathfrak{I}^\Delta$ for a defining ideal $\mathfrak{I}$ of $A_{\{f\}}$. It is clear from our definiton that for any $f \in A$, any defining ideal of $\mathfrak{X}$ induces an defining ideal of $\mathfrak{D}(f)$.

Proposition 4.8.5. *If A is an admissible ring, any defining ideal of $\mathfrak{X} = \mathrm{Spf}(A)$ is of the form $\mathfrak{I}^{\Delta}$, where $\mathfrak{I}$ is a defining ideal of A.*

Proof. Let $\mathscr{I}$ be a defining ideal of $\mathfrak{X}$; by hypothesis, and since $\mathfrak{X}$ is quasi-compact, there exist finitely many elements $f_i \in A$ such that $\mathfrak{D}(f)$ cover $\mathfrak{X}$ and such that $\mathscr{I}|_{\mathfrak{D}(f)} = \mathfrak{J}_i$, where $\mathfrak{J}_i$ is a defining ideal of $A_{\{f_i\}}$. For any i, there then exists an open ideal $\mathfrak{I}_i$ of A such that $(\mathfrak{I}_i)_{\{f_i\}} = \mathfrak{J}_i$ (Proposition **??**); let $\mathfrak{R}$ be a defining ideal of A contained in each the $\mathfrak{I}_i$. The canonical image of $\mathscr{I}/\mathfrak{R}^{\Delta}$ in the structural sheaf $\widetilde{(A/\mathfrak{R})}$ of $\mathrm{Spec}(A/\mathfrak{R})$ is then such that its restriction to each $\mathfrak{D}(f_i)$ is equal to $\widetilde{\mathfrak{I}_i/\mathfrak{R}}$; we then conclude that this canonical image is a quasi-coherent ideal over $\mathrm{Spec}(A/\mathfrak{R})$, hence is of the form $\widetilde{\mathfrak{I}/\mathfrak{R}}$, where $\mathfrak{I}$ is an ideal of A containing $\mathfrak{R}$, and whence $\mathscr{I} = \mathfrak{I}^{\Delta}$ by (8.1.2). Moreover, as for each i there exists an integer n_i such that $\mathfrak{J}_i^{n_i} \subseteq \mathfrak{R}_{\{f_i\}}$, we have $(\mathscr{I}/\mathfrak{R}^{\Delta})^n = 0$ for n sufficiently large, and therefore $(\widetilde{\mathfrak{I}/\mathfrak{R}})^n = 0$, and finally $(\mathfrak{I}/\mathfrak{R})^n = 0$, which proves that $\mathfrak{I}$ is a defining ideal of A. □

Proposition 4.8.6. *Let A be an adic ring, $\mathfrak{I}$ be a defining ideal of A such that $\mathfrak{I}/\mathfrak{I}$ is an $A/\mathfrak{I}$ of finte type. For any integer $n > 0$, we then have $(\mathfrak{I}^{\Delta})^n = (\mathfrak{I}^n)^{\Delta}$.*

Proof. In fact, for any $f \in A$ we have (since $\mathfrak{I}^n$ is an open ideal)

$$(\Gamma(\mathfrak{D}(f), \mathfrak{I}^{\Delta}))^n = (\mathfrak{I}_{\{f\}})^n = (\mathfrak{I}^n)_{\{f\}} = \Gamma(\mathfrak{D}(f^n), (\mathfrak{I}^n)^{\Delta})$$

in view of (8.1.1) and Proposition **??**. As $(\mathfrak{I}^{\Delta})^n$ is associated with the presheaf $U \mapsto (\Gamma(U, \mathfrak{I}^{\Delta}))^n$, the corollary then follows since $\mathfrak{D}(f)$ form a basis for $\mathfrak{X}$. □

We say that a family $(\mathscr{I}_\lambda)$ of defining ideals of $\mathfrak{X}$ is a **fundamental system of defining ideals** if any defining ideal of $\mathscr{I}$ contains at least one of these $\mathscr{I}_\lambda$. As $\mathscr{I}_\lambda = \mathfrak{I}_\lambda^{\Delta}$, this is equivalent to saying that the $\mathfrak{I}_\lambda$ form a fundamental neighborhood of 0 in A, where $\mathscr{I}_\lambda = \mathscr{I}_\lambda^{\Delta}$. Let (f_α) be a family of elements of A such that the $\mathfrak{D}(f_\alpha)$ cover $\mathfrak{X}$. If $(\mathscr{I}_\lambda)$ is a filtered decreasing family of ideals of $\mathscr{O}_{\mathfrak{X}}$ such that for any α, the family $(\mathscr{I}_\lambda|_{\mathfrak{D}(f_\alpha)})$ is a fundamental system of defining ideals of $\mathfrak{D}(f_\alpha)$, then $(\mathscr{I}_\lambda)$ is a fundamental system of defining ideals of $\mathfrak{X}$. In fact, for any defining ideal of $\mathfrak{X}$, there exists a finite covering of $\mathfrak{X}$ by the $\mathfrak{D}(f_i)$ such that, for any i, $\mathscr{I}_{\lambda_i}|_{\mathfrak{D}(f_i)}$ is a defining ideal of $\mathfrak{D}(f_i)$ contained in $\mathfrak{I}|_{\mathfrak{D}(f_i)}$. If μ is an index such that $\mathscr{I}_\mu \subseteq \mathscr{I}_{\lambda_i}$ for all i, then $\mathscr{I}_\mu$ is a defining ideal of $\mathfrak{X}$ which is evidently contained in $\mathscr{I}$, whence the assertion.

4.8.2 Formal schemes and morphisms

Given a topologically ringed space $\mathfrak{X}$, we say an open subset $U \subseteq \mathfrak{X}$ is a **formal affine open** (resp. **an adic formal affine open**, resp. **a Noetherian formal affine open**) if the topologically ringed space $(U, \mathscr{O}_{\mathfrak{X}}|_U)$ is a formal affine scheme (resp. an adic formal affine scheme, resp. a Noetherian formal affine scheme). We way $(\mathfrak{X}, \mathscr{O}_{\mathfrak{X}})$ is a **formal scheme** (resp. adic formal scheme, resp. locally Noetherian formal scheme) if each of its point admits a formal affine open neighborhood (resp. an adic formal affine open, resp. a locally Noetherian formal affine open). We say that $\mathfrak{X}$ is Noetherian if it is locally Noetherian and the underlying space is quasi-compact (hence Noetherian). As any affine scheme can be considered as a formal affine scheme, any scheme can be considered as a formal scheme.

Proposition 4.8.7. *If $\mathfrak{X}$ is a formal scheme (resp. locally Noetherian formal scheme), the set of formal affine opens (resp. Noetherian formal affine opens) form a base for $\mathfrak{X}$.*

Proof. This follows from Proposition 4.8.2, and the fact that if A is a Noetherian adic ring, so is $A_{\{f\}}$ for any $f \in A$ (Proposition **??**). □

Corollary 4.8.8. *If $\mathfrak{X}$ is a formal scheme (resp. locally Noetherian formal scheme, resp. Noetherian formal scheme), the topological ringed space over any open subset of $\mathfrak{X}$ is a formal scheme (resp. locally Noetherian formal scheme, resp. Noetherian formal scheme).*

Given two formal schemes $\mathfrak{X}, \mathfrak{Y}$, we say that $f : \mathfrak{X} \to \mathfrak{Y}$ is a morphism of formal schemes if it is a morphism of the underlying locally ringed spaces. That is, if $(f, f^\#)$ is a morphism of ringed spaces and $f_x^\# : \mathscr{O}_{\mathfrak{Y},f(x)} \to \mathscr{O}_{\mathfrak{X},x}$ is a local homomorphism. The composition of two morphisms are defined as the same and clearly a morphism of formal schemes. The formal schemes then form a category, which we denoted by **Shf**, and we denote by $\mathrm{Hom}_{\mathsf{Shf}}(\mathfrak{X}, \mathfrak{Y})$ the set of morphisms of formal schemes $\mathfrak{X} \to \mathfrak{Y}$.

If U is an open subset of $\mathfrak{X}$, the canonical injection $U \to \mathfrak{X}$ is then a morphism of formal schemes, if we endow U the formal scheme structure induced by $\mathfrak{X}$. It is clear that this morphism is a monomorphism in the category **Shf**.

Proposition 4.8.9. *Let $\mathfrak{X}$ be a formal scheme, $\mathfrak{Y} = \mathrm{Spec}(A)$ be a formal affine scheme. Then there exists a canonical bijection*

$$\mathrm{Hom}_{\mathbf{Scf}}(\mathfrak{X}, \mathfrak{Y}) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{TopRing}}(A, \Gamma(\mathfrak{X}, \mathscr{O}_{\mathfrak{X}})).$$

Proof. We first note that, if $(\mathfrak{X}, \mathscr{O}_{\mathfrak{X}})$ and $(\mathfrak{Y}, \mathscr{O}_{\mathfrak{Y}})$ are two topologically ringed spaces, a morphism $f : (\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}) \to (\mathfrak{Y}, \mathscr{O}_{\mathfrak{Y}})$ defines canonically a continuous homomorphism of rings $\varphi : \Gamma(\mathfrak{Y}, \mathscr{O}_{\mathfrak{Y}}) \to \Gamma(\mathfrak{X}, \mathscr{O}_{\mathfrak{X}})$. In our case, we need to show that a continuous hommorphism $\varphi : A \to \Gamma(\mathfrak{X}, \mathscr{O}_{\mathfrak{X}})$ is of the form $\Gamma(\psi^\#)$ for a unique morphism $(\psi, \psi^\#)$. Now there exists by hypothesis a covering (V_α) of $\mathfrak{X}$ by formal affine opens; by composing φ with the restriction homomorphisms $\Gamma(\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}) \to \Gamma(V_\alpha, \mathscr{O}_{\mathfrak{X}}|_{V_\alpha})$, we obtain a continuous homomorphism $\varphi_\alpha : A \to \Gamma(V_\alpha, \mathscr{O}_{\mathfrak{X}}|_{V_\alpha})$, which corresponds to a unique morphism $\psi_\alpha : (V_\alpha, \mathscr{O}_{\mathfrak{X}}|_{V_\alpha}) \to (\mathfrak{Y}, \mathscr{O}_{\mathfrak{Y}})$, in view of Proposition 4.8.4. Moreover, for any couple (α, β) of indices, any point of $V_\alpha \cap V_\beta$ admits a formal affine open neighborhood W contained in $V_\alpha \cap V_\beta$ and it is clear that the compositions of φ_α and φ_β with the canonical restriction are the same continuous homomorphism $\Gamma(\mathfrak{Y}, \mathscr{O}_{\mathfrak{Y}}) \to \Gamma(W, \mathscr{O}_{\mathfrak{X}}|_W)$, so, in view of the relations $(\psi_\alpha^\#)_x = (\tilde{\varphi}_\alpha)_x$ for any $x \in V_\alpha$, the restrictions of ψ_α and ψ_β coincides on $V_\alpha \cap V_\beta$. We then conclude that there exists a unique morphism $\psi : (\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}) \to (\mathfrak{Y}, \mathscr{O}_{\mathfrak{Y}})$ whose restriction to V_α coincides with ψ_α, and it is clear that this is the unique morphism such that $\Gamma(\psi^\#) = \varphi$. □

Given a formal scheme $\mathfrak{S}$, a **formal $\mathfrak{S}$-scheme** is defined to be a formal scheme $\mathfrak{X}$ together with a morphism $\varphi : \mathfrak{X} \to \mathfrak{S}$, called the **structural morphism** of $\mathfrak{X}$. If $\mathfrak{S} = \mathrm{Spf}(A)$, where A is an admissible ring, we also say that the $\mathfrak{S}$-formal scheme $\mathfrak{X}$ is a formal A-scheme or a formal scheme over A. Any formal scheme can be clearly considered as a formal scheme over $\mathbb{Z}$ (endowed with the discrete topology).

If $\mathfrak{X}, \mathfrak{Y}$ are two formal $\mathfrak{S}$-schemes, we say a morphism $f : \mathfrak{X} \to \mathfrak{Y}$ is an **$\mathfrak{S}$-morphism** if the diagram

$$\begin{tikzcd} \mathfrak{X} \arrow[rr, "f"] \arrow[dr] & & \mathfrak{Y} \arrow[dl] \\ & \mathfrak{S} & \end{tikzcd}$$

where the vertical arrows are structural morphisms, is commutative. With this definition, the $\mathfrak{S}$-schemes form (for $\mathfrak{S}$ fixed) a category $\mathbf{Scf}_{\mathfrak{S}}$. We denote by $\mathrm{Hom}_{\mathfrak{S}}(\mathfrak{X}, \mathfrak{Y})$ the set of $\mathfrak{S}$-morphisms $\mathfrak{X} \to \mathfrak{Y}$. If $\mathfrak{S} = \mathrm{Spf}(A)$, we also say A-morphism for $\mathfrak{S}$-morphisms.

Let $\mathfrak{X}$ be a formal scheme; we say an ideal $\mathscr{I}$ of $\mathscr{O}_{\mathfrak{X}}$ is a defining ideal of $\mathfrak{X}$ if any $x \in \mathfrak{X}$ admits a formal affine open neighborhood U such that $\mathscr{I}|_U$ is a defining ideal of the formal scheme U induced by $\mathfrak{X}$. In view of Proposition 4.8.7, for any open $V \subseteq \mathfrak{X}$, $\mathscr{I}|_V$ is then a defining ideal of the formal scheme induced over V.

We say that a family $(\mathscr{I}_\lambda)$ of defining ideals of $\mathfrak{X}$ is a **fundamental system of defining ideals** if there exists a covering (U_α) of $\mathfrak{X}$ by formal affine opens such that, for any α, the family $(\mathscr{I}_\lambda|_{U_\alpha})$ form a fundamental system of defining ideals of U_α. For any open subset V of $\mathfrak{X}$, the family $(\mathscr{I}_\lambda|_V)$ then forms a fundamental system of defining ideals for V, in view of (8.1.1). If $\mathfrak{X}$ is locally Noetherian, and $\mathscr{I}$ is a defining ideal of $\mathfrak{X}$, it then follows from Proposition 4.8.6 that the powers of $\mathscr{I}^n$ form a fundamental system of defining ideals of $\mathfrak{X}$.

Let $\mathfrak{X}$ be a formal scheme, $\mathscr{I}$ be a defining ideal of $\mathfrak{X}$. Then the ringed space $(\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}/\mathscr{I})$ is a scheme, which is affine (resp. locally Noetherian, resp. Noetherian) if $\mathfrak{X}$ is a formal affine scheme (resp. a locally Noetherian formal scheme, resp. a Noetherian formal scheme). Moreover, if $\varphi : \mathscr{O}_{\mathfrak{X}} \to \mathscr{O}_{\mathfrak{X}}/\mathscr{I}$ is the cannical homomorphism, then $(1_{\mathfrak{X}}, \varphi)$ is a morphism (called canonical) of formal schemes $(\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}/\mathscr{I}) \to (\mathfrak{X}, \mathscr{O}_{\mathfrak{X}})$.

Proposition 4.8.10. *Let $\mathfrak{X}$ be a formal scheme, $(\mathscr{I}_\lambda)$ be a fundamental system of defining ideals of $\mathfrak{X}$. Then the sheaf $\mathscr{O}_{\mathfrak{X}}$ is the projective limit of the sheaf of discrete rings $\mathscr{O}_{\mathfrak{X}}/\mathscr{I}_\lambda$.*

Proof. As $\mathfrak{X}$ admits a basis by quasi-compact open sets, we are reduced to the affine case, where the proposition follows from Proposition 4.8.5 and the definition of $\mathscr{O}_{\mathfrak{X}}$. □

Proposition 4.8.11. *Let $\mathfrak{X}$ be a locally Noetherian formal scheme. Then there exists a largest defining ideal $\mathscr{T}$ of $\mathfrak{X}$, which is the unique defining ideal $\mathscr{I}$ such that the scheme $(\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}/\mathscr{I})$ is reduced. If $\mathscr{I}$ is a defining ideal of $\mathfrak{X}$, $\mathscr{T}$ is the inverse image of the nilradical of $\mathscr{O}_{\mathfrak{X}}/\mathscr{I}$ under the homomorphism $\mathscr{O}_{\mathfrak{X}} \to \mathscr{O}_{\mathfrak{X}}/\mathscr{I}$. The reduced (usual) scheme $(\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}/\mathscr{T})$ is denoted by $\mathfrak{X}_{\mathrm{red}}$.*

Proof. Suppose first that $\mathfrak{X} = \mathrm{Spf}(A)$, where A is a Noetherian adic ring. The existence of $\mathscr{T}$ and its properties then follows from Proposition 4.8.5, in view of Proposition **??** and Proposition **??** about the largest defining ideal of A. To prove the existence of $\mathscr{T}$ in the general case, it suffices to prove that if $V \subseteq U$ are two Noetherian formal affine opens of X, the largest defining ideal $\mathscr{T}_U$ of U induces the largest defining ideal $\mathscr{T}_V$ of V; but as $(V, (\mathscr{O}_{\mathfrak{X}}|_V)/(\mathscr{T}_U|_V))$ is reduced, this is immediate. □

Corollary 4.8.12. *Let $\mathfrak{X}$ be a locally Noetherian formal scheme, $\mathscr{T}$ be the largest defining ideal of $\mathfrak{X}$. Then for any open subset V of $\mathfrak{X}$, $\mathscr{T}|_V$ is the largest defining ideal of V.*

Proof. This is already shown in the proof of Proposition 4.8.11. □

Proposition 4.8.13. *Let $\mathfrak{X}, \mathfrak{Y}$ be formal schemes, $\mathscr{I}$ (resp. $\mathscr{J}$) be the defining ideal of $\mathfrak{X}$ (resp. $\mathfrak{Y}$), $f : \mathfrak{X} \to \mathfrak{Y}$ be a morphism of formal schemes.*

(i) If $f^*(\mathscr{J})\mathscr{O}_{\mathfrak{X}} \subseteq \mathscr{I}$, there exists a unique morphism

$$f_{\mathrm{red}} : (\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}/\mathscr{I}) \to (\mathfrak{Y}, \mathscr{O}_{\mathfrak{Y}}/\mathscr{K})$$

of schemes such that the following diagram

$$\begin{tikzcd}
(\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}/\mathscr{I}) \arrow[r, "f_{\mathrm{red}}"] \arrow[d] & (\mathfrak{Y}, \mathscr{O}_{\mathfrak{Y}}/\mathscr{J}) \arrow[d] \\
(\mathfrak{X}, \mathscr{O}_{X}) \arrow[r, "f"] & (\mathfrak{Y}, \mathscr{O}_{\mathfrak{Y}})
\end{tikzcd} \tag{8.2.1}$$

(ii) Suppose that $\mathfrak{X} = \mathrm{Spf}(A)$, $\mathfrak{Y} = \mathrm{Spf}(B)$ are formal affine schemes, $\mathscr{I} = \mathfrak{I}^{\Delta}$, $\mathscr{K} = \mathfrak{K}^{\Delta}$, where $\mathfrak{I}$ (resp. $\mathfrak{K}$) is a defining ideal of A (reps. B), and $f = ({}^a\varphi, \tilde{\varphi})$ where $\varphi : B \to A$ is a continuous homomorphism. For $f^*(\mathscr{K})\mathscr{O}_{\mathfrak{X}} \subseteq \mathscr{K}$, it is necessary and sufficient that $\varphi(\mathfrak{K}) \subseteq \mathfrak{I}$, and f_{red} is then the morphism $({}^a\varphi', \tilde{\varphi}')$, where $\varphi' : B/\mathfrak{K} \to A/\mathfrak{I}$ is the induced homomorphism by passing to quotients.

Proof. In case (a), the hypothesis implies that the image of the ideal $f^{-1}(\mathscr{K})$ of $f^{-1}(\mathscr{O}_{\mathfrak{Y}})$ under $f^{\#} : f^{-1}(\mathscr{O}_{\mathfrak{Y}}) \to \mathscr{O}_{\mathfrak{X}}$ is contained in $\mathscr{I}$. By passing to quotients, we then deduce that $f^{\#}$ is a homomorphism

$$\omega : f^{-1}(\mathscr{O}_{\mathfrak{Y}}/\mathscr{K}) = f^{-1}(\mathscr{O}_{\mathfrak{Y}})/f^{-1}(\mathscr{K}) \to \mathscr{O}_{\mathfrak{X}}/\mathscr{I};$$

moreover, as for any $x \in \mathfrak{X}$, $f_x^{\#}$ is a local homomorphism, so is ω_x. The morphism $(f, \omega^{\flat})$ is then the unique morphism of ringed spaces satisfying the requirements.

With the assumptions of (b), the canonical correspondence between morphisms of formal affine schemes and continuous homomorphisms of ringed spaces shows that the relation $f^*(\mathscr{K})\mathscr{O}_{\mathfrak{X}} \subseteq \mathscr{I}$ implies that $f' = ({}^a\varphi', \tilde{\varphi}')$, where $\varphi' : B/\mathfrak{K} \to A/\mathfrak{I}$ is the unique homomorphism such that the diagram

$$\begin{tikzcd}
B \arrow[r, "\varphi"] \arrow[d] & A \arrow[d] \\
B/\mathfrak{K} \arrow[r, "\varphi'"] & A/\mathfrak{I}
\end{tikzcd} \tag{8.2.2}$$

The existence of φ' implies then that $\varphi(\mathfrak{K}) \subseteq \mathfrak{I}$. Conversely, if this condition is verified, we have a canonical homomorphism $\varphi' : B/\mathfrak{K} \to A/\mathfrak{I}$, whence the induced morphism $f' = ({}^a\varphi', \tilde{\varphi}')$ satisfies the commutativity of (8.2.2). By considering the homomorphisms ${}^a\varphi^*(\mathscr{O}_{\mathfrak{Y}}) \to \mathscr{O}_{\mathfrak{X}}$ and ${}^a\varphi'^*(\mathscr{O}_{\mathfrak{Y}}/\mathscr{K}) \to \mathscr{O}_{\mathfrak{X}}/\mathscr{I}$, we then see that $f^*(\mathscr{K})\mathscr{O}_{\mathfrak{X}} \subseteq \mathscr{I}$. □

To conclude this paragraph, we consider product of formal schemes; let $\mathfrak{S}$ be a formal scheme.

Proposition 4.8.14. *Let $\mathfrak{X} = \mathrm{Spf}(B)$, $\mathfrak{Y} = \mathrm{Spf}(C)$ be two formal affine schemes over a formal affine scheme $\mathrm{Spf}(A)$. Let $\mathfrak{Z} = \mathrm{Spf}(B\widehat{\otimes}_A C)$ and p_1, p_2 be the $\mathfrak{S}$-morphisms corresponding to the A-homomorphisms $\rho_1 : B \to B\widehat{\otimes}_A C$ and $\rho_2 : C \to B\widehat{\otimes}_A C$. Then $(\mathfrak{Z}, p_1, p_2)$ forms a product in the category of the formal $\mathfrak{S}$-schemes $\mathfrak{X}$ and $\mathfrak{Y}$.*

Proof. In view of Proposition 4.8.4, we note that for any continuous A-homomorphism φ : $B\widehat{\otimes}_A C \to D$, where D is an admissible ring that is a topological A-algebra, we can associate the couple $(\varphi \circ \sigma_1, \varphi \circ \sigma_2)$, so that we define a bijection

$$\mathrm{Hom}_A(B\widehat{\otimes}_A C, D) \xrightarrow{\sim} \mathrm{Hom}_A(B, D) \times \mathrm{Hom}_A(C, D)$$

which follows from the universal property of the complete tensor product. □

Proposition 4.8.15. *For any formal $\mathfrak{S}$-schemes $\mathfrak{X}, \mathfrak{Y}$, the product $\mathfrak{X} \times_{\mathfrak{S}} \mathfrak{Y}$ exists.*

Proof. The demonstration is similar as the case for usual schemes, where we replace affine schemes by formal affine schemes and use Proposition 4.8.14. □

4.8.3 Formal schemes as inductive limits of schemes

Let $\mathfrak{X}$ be a formal scheme, $(\mathscr{I}_\lambda)$ be a fundamental system of defining ideals of $\mathfrak{X}$; for each λ, let $f_\lambda : (\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}/\mathscr{I}_\lambda) \to \mathfrak{X}$. For $\mathscr{I}_\mu \subseteq \mathscr{I}_\lambda$, the canoincal homomorphism $\mathscr{O}_{\mathfrak{X}}/\mathscr{I}_\mu \to \mathscr{O}_{\mathfrak{X}}/\mathscr{I}_\lambda$ defines a canonical morphism

$$f_{\mu\lambda} : (\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}/\mathscr{I}_\lambda) \to (\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}/\mathscr{I}_\mu)$$

of (usual) schemes such that we have $f_\lambda = f_\mu \circ f_{\mu\lambda}$. The scheme $X_\lambda = (\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}/\mathscr{I}_\lambda)$ is the morphisms $f_{\mu\lambda}$ then constitutes an inductive system in the category of formal schemes.

Proposition 4.8.16. *The formal scheme and the morphisms f_λ constitute an inductive limit of the system $(X_\lambda, f_{\mu\lambda})$ in the cateogry of formal schemes.*

Proof. Let $\mathfrak{Y}$ be a formal scheme, and for each index λ, let

$$g_\lambda : X_\lambda \to \mathfrak{Y}$$

be a morphism such that $g_\lambda = g_\mu \circ f_{\mu\lambda}$ for $\mathscr{I}_\mu \subseteq \mathscr{I}_\lambda$. This last condition and the definition of X_λ then imply that the g_λ are equal to the same continuous map $g : \mathfrak{X} \to \mathfrak{Y}$ on the underlying space; moreover, the homomorphisms $g_\lambda^\# : g^{-1}(\mathscr{O}_{\mathfrak{Y}}) \to \mathscr{O}_{X_\lambda} = \mathscr{O}_{\mathfrak{X}}/\mathscr{I}_\lambda$ form a projective system of homomorphisms of sheaves of rings. By passing to projective limit, we then deduce a homomorphism $\omega : g^{-1}(\mathscr{O}_{\mathfrak{Y}}) \to \lim \mathscr{O}_{\mathfrak{X}}/\mathscr{I}_\lambda = \mathscr{O}_{\mathfrak{X}}$, and it is clear that (g, ω) is a morphism of ringed spaces such that the diagram

$$\begin{tikzcd} X_\lambda \arrow[rr, "g_\lambda"] \arrow[dr, "f_\lambda"'] & & \mathfrak{Y} \\ & \mathfrak{X} \arrow[ur, "g"'] & \end{tikzcd} \tag{8.3.1}$$

It remains to prove that g is a morphism of formal schemes; the question is local on $\mathfrak{X}$ and $\mathfrak{Y}$, we can assume that $\mathfrak{X} = \mathrm{Spf}(A)$, $\mathfrak{Y} = \mathrm{Spf}(B)$, where A, B are admissible rings, and $\mathscr{I}_\lambda = \mathfrak{I}^\Delta$ where $(\mathfrak{I}_\lambda)$ is a fundamental system of defining ideals of A (Proposition 4.8.5). As $A = \varprojlim A/\mathfrak{I}_\lambda$, the existence of the morphism of formal affine schemes g fitting into the diagram (8.3.1) then follows from the one-to-one correspondence Proposition 4.8.4 between morphisms of formal affine schemes and continuous homomorphisms of rings, and of the definition of the projective limit. But the uniqueness of g as a morphism of ringed spaces shows that it coincides with the morphism at the beginning of the demonstration. □

Proposition 4.8.17. *Let $\mathfrak{X}$ be a topological space, $(\mathscr{O}_i, u_{ji})$ a projective system of sheaves of rings over $\mathfrak{X}$, with $\mathbb{N}$ the index set. Let $\mathscr{I}_i$ be the kernel of $u_{0,i} : \mathscr{O}_i \to \mathscr{O}_0$. Suppose that*

(a) For each i, the ringed space $X_i = (\mathfrak{X}, \mathscr{O}_i)$ is a scheme.

(b) For any $x \in \mathfrak{X}$ and any $i \in \mathbb{N}$, there exists an open neighborhood U_i of x in $\mathfrak{X}$ such that the restriction $\mathscr{I}_i|_{U_i}$ is nilpotent.

(c) The homomorphisms u_{ji} are surjective.

Let $\mathscr{O}_{\mathfrak{X}}$ be the sheaf of topological rings which is the projective limit of the sheaf of discrete rings $\mathscr{O}_i$, and $u_i : \mathscr{O}_{\mathfrak{X}} \to \mathscr{O}_i$ be the canonical homomorphism. Then the topologically ringed space $(\mathfrak{X}, \mathscr{O}_{\mathfrak{X}})$ is a formal scheme; the homomorphisms u_i are surjective whose kernel $\mathscr{I}^{(i)}$ form a fundamental system of defining ideals of $\mathfrak{X}$, and $\mathscr{I}^{(0)}$ is the projective limit of the sheaf of ideals $\mathscr{I}_i$.

Proof. We first note that at each stalk, u_{ji} is a surjective hommorphism and a fortiori a local homomorphism, so $v_{ij} = (1_{\mathfrak{X}}, u_{ji}) : X_j \to X_i$ is a morphism of schemes for $i \geq j$. Suppose that each X_i is an affine scheme with ring A_i. Then there exists a homomorphism $\varphi_{ji} : A_i \to A_j$ such that $u_{ji} = \tilde{\varphi}_{ji}$, so the sheaf $\mathscr{O}_j$ is a quasi-coherent $\mathscr{O}_i$-module over X_i, associated with A_j considered as an A_i-module via φ_{ji}. For each $f \in A_i$, let $f' = \varphi_{ji}(f)$; by hypothesis, the opens $D(f)$ and $D(f')$ are identical over $\mathfrak{X}$, and the homomorphism from $\Gamma(D(f), \mathscr{O}_i) = (A_i)_f$ to $\Gamma(D(f), \mathscr{O}_j) = (A_j)_{f'}$ corresponding to u_{ji} is none other than $(A_j)_{f'}$. But if we consider A_j as an A_i-module, $(A_j)_{f'}$ is the $(A_j)_f$-module $(A_j)_f$, so we have $u_{ji} = \tilde{\varphi}_{ji}$, if φ_{ji} is considered as a homomorphism of A_i-modules. Then, as u_{ji} is surjective, so is the φ_{ji} and if $\mathfrak{I}_{ji}$ is the kernel of φ_{ji}, the kernel of u_{ji} is the quasi-coherent $\mathscr{O}_i$-module $\widetilde{\mathfrak{I}}_{ji}$. In particular, we have $\mathscr{I}_i = \widetilde{\mathfrak{I}}_i$, where $\mathfrak{I}_i$ is the kernel of $\varphi_{0,i} : A_i \to A_0$. The hypothesis (b) implies that $\mathscr{I}_i$ is nilpotent: in fact, as $\mathfrak{X}$ is quasi-compact, we can cover X by finitely many opens U_k such that $(\mathscr{I}_i|_{U_k})^{n_k} = 0$ and by choosing n to be the largest n_k, we have $\mathscr{I}_i^n = 0$. We then conclude that each $\mathfrak{I}_i$ is nilpotent. Then the ring $A = \varprojlim A_i$ is admissible by Proposition **??**, the canonical homomorphisms $\varphi_i : A \to A_i$ is surjective and its kernel $\mathfrak{I}^{(i)}$ is equal to the projective limit of the $\mathfrak{I}_{ik}$ for $k \geq i$; the $\mathfrak{I}^{(i)}$ form a fundamental system of neighborhoods of 0 in A. The assertion of the proposition then follows from the fact that $(\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}) = \operatorname{Spf}(A)$. We also note that if $f = (f_i)$ is an element in the projective limit $A = \varprojlim A_i$, the open subsets $D(f_i)$ (affine open in X_i) is then identified with $\mathfrak{D}(f)$, and the scheme induced by X_i over $\mathfrak{D}(f)$ is then identified with the affine scheme $\operatorname{Spec}((A_i)_{f_i})$.

In the general case, we remark that for any quasi-compact open U of $\mathfrak{X}$, the $\mathscr{I}_i|_U$ is nilpotent as we have seen. We claim that for any $x \in \mathfrak{X}$, there is an open neighborhood U of x in $\mathfrak{X}$ which is an affine open for any X_i. In fact, let U be an affine open for X_0, and observe that $\mathscr{O}_{X_0} = \mathscr{O}_X/\mathscr{I}_i$. As $\mathscr{I}_i|_U$ is nilpotent in view of the preceding arguments, U is also an affine open for X_i in view of Proposition 4.4.6. Now for any U satisfying this property, it follows from the same arguments that $(U, \mathscr{O}_{\mathfrak{X}}|_U)$ is a formal scheme such that the $\mathscr{I}^{(i)}|_U$ form a fundamental system of defining ideals and $\mathscr{I}^{(0)}|_U$ is the projective limit of $\mathscr{I}_i|_U$, whence the conclusion. □

Corollary 4.8.18. *Suppose that for $i \geq j$, the kernel of u_{ji} is $\mathscr{I}_i^{j+1}$ and that $\mathscr{I}_1/\mathscr{I}_1^2$ is of finite type over $\mathscr{O}_0 = \mathscr{O}_1/\mathscr{I}_1$. Then $\mathfrak{X}$ is an adic formal scheme, and if $\mathscr{I}^{(n)}$ is the kernel of $\mathscr{O}_{\mathfrak{X}} \to \mathscr{O}_n$, we have, putting $\mathscr{I} = \mathscr{I}^{(0)}$, $\mathscr{I}^{(n)} = \mathscr{I}^{n+1}$ and $\mathscr{I}/\mathscr{I}^2$ is isomorphic to $\mathscr{I}_1$. If moreover X_0 is locally Noetherian (resp. Noetherian), then $\mathfrak{X}$ is locally Noetherian (resp. Noetherian).*

Proof. As the underlying space of $\mathfrak{X}$ and X_0 are the same, the question is local and we can assume that each X_i is affine. In view of the relations $\mathscr{I}_{ji} = \widetilde{\mathfrak{I}}_{ji}$ (With the notations of Proposition 4.8.17), we are then reduced to the case of Proposition **??**, and note that $\mathfrak{I}_1/\mathfrak{I}_1^2$ is then an A_0-module of finite type (Proposition 4.1.23). ◻

In particular, any locally Noetherian formal scheme $\mathfrak{X}$ is the inductive limit of a sequence (X_n) of locally Noetherian (usual) schemes verifying the conditions of Proposition 4.8.17 and Corollary 4.8.18: it suffices to consider a defining ideal $\mathfrak{I}$ of $\mathfrak{X}$ (Proposition 4.8.11) and put $X_n = (\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}/\mathscr{I}^{n+1})$ (Proposition 4.8.16). More generally, we the same properties if $\mathfrak{X}$ is an adic formal scheme having a defining ideal $\mathscr{I}$ such that $\mathscr{I}/\mathscr{I}^2$ is a finitely generated $(\mathscr{O}_{\mathfrak{X}}/\mathscr{I})$-module.

Corollary 4.8.19. *Let A be an admissible ring. For the formal affine scheme $\mathfrak{X} = \mathrm{Spf}(A)$ to be Noetherian, it is necessary and sufficient that A is adic and Noetherian.*

Proof. This condition is clearly sufficient. Conversely, suppose that $\mathfrak{X}$ is Noetherian, and let $\mathfrak{I}$ be a defining ideal of A, $\mathscr{I} = \mathfrak{I}^{\Delta}$ the defining ideal of $\mathfrak{X}$. The (usual) schemes $X_n = (\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}/\mathscr{I}^{n+1})$ is then affine and Noetherian, so the ring $A_n = A/\mathscr{I}^{n+1}$ is Noetherian (Proposition 4.2.26), and we conclude that $\mathfrak{I}/\mathfrak{I}^2$ is a finitely generated $(A/\mathfrak{I})$-module. As the $\mathfrak{I}^n$ form a fundamental system of defining ideals of $\mathfrak{X}$, we have $\mathscr{O}_{\mathfrak{X}} = \varprojlim(\mathscr{O}_{\mathfrak{X}}/\mathfrak{I}^n)$ (Proposition 4.8.10). ◻

Remark 4.8.1. With the notations of Proposition 4.8.17, let $\mathscr{F}_i$ be an $\mathscr{O}_i$-module, and suppose that we are given for $i \geq j$ a v_{ij}-morphism $\theta_{ji} : \mathscr{F}_i \to \mathscr{F}_j$, such that $\theta_{kj} \circ \theta_{ji} = \theta_{ki}$ for $k \leq j \leq i$. As the underlying continuous map of v_{ij} is the identity, θ_{ji} is a homomorphism of sheaves of abelian groups over $\mathfrak{X}$. Moreover, if $\mathscr{F}$ is the limit of the projective system $(\mathscr{F}_i)$ of sheaves of abelian groups, the fact that θ_{ji} are v_{ij}-morphisms permits us to define over $\mathscr{F}$ an $\mathscr{O}_{\mathfrak{X}}$-module structure by passing to projective limits. With this, we say that $\mathscr{F}$ is the projective limit (for the θ_{ji}) of the system of $\mathscr{O}_i$-modules $(\mathscr{F}_i)$. In the particular case where $v_{ij}^*(\mathscr{F}_i) = \mathscr{F}_j$ and there θ_{ji} is the identity, we then say that $\mathscr{F}$ is the limit of the system $(\mathscr{F}_i)$ such that $v_{ij}^*(\mathscr{F}_i) = \mathscr{F}_j$ for $j \leq i$ (without mension of θ_{ji}).

Let $\mathfrak{X}, \mathfrak{Y}$ be two formal schemes, $\mathscr{I}$ (resp. $\mathscr{K}$) be a defining ideal of $\mathfrak{X}$ (resp. $\mathfrak{Y}$), and $f : \mathfrak{X} \to \mathfrak{Y}$ be a morphism such that $f^*(\mathscr{K})\mathscr{O}_{\mathfrak{X}} \subseteq \mathscr{I}$. We then have for each integer $n > 0$ that $f^*(\mathscr{K}^n)\mathscr{O}_{\mathfrak{X}} = (f^*(\mathscr{K})\mathscr{O}_{\mathfrak{X}}) \subseteq \mathscr{I}^n$, so Proposition 4.8.13 deduce a morphism $f_n : X_n \to Y_n$ of (usual) schemes such that the diagram

$$\begin{tikzcd} X_m \arrow[r, "f_m"] \arrow[d] & Y_m \arrow[d] \\ X_n \arrow[r, "f_n"] & Y_n \end{tikzcd} \tag{8.3.2}$$

is commutative for $m \leq n$; in other words, (f_n) is a inductive system of morphisms.

Conversely, let (X_n) (resp. (Y_n)) be a inductive system of schemes satisfying the conditions (b), (c) of Proposition 4.8.17, and let $\mathfrak{X}$ (resp. $\mathfrak{Y}$) be the inductive limits (whose existence is proved by Proposition 4.8.17). By the definition of inductive limits, any sequence (f_n) of morphisms $f_n : X_n \to Y_n$ forming an inductive limit admits an inductive limit $f : \mathfrak{X} \to \mathfrak{Y}$, which is the unique morphism of formal schemes rendering the diagram

$$\begin{array}{ccc} X_n & \xrightarrow{f_n} & Y_n \\ \downarrow & & \downarrow \\ \mathfrak{X} & \xrightarrow{f} & \mathfrak{Y} \end{array}$$

Proposition 4.8.20. *Let $\mathfrak{X},\mathfrak{Y}$ be adic formal schemes, $\mathscr{J}$ (resp. $\mathscr{K}$) be a defining ideal of $\mathfrak{X}$ (resp. $\mathfrak{Y}$). The map $f \mapsto (f_n)$ is a bijection from the set of morphisms $f : \mathfrak{X} \to \mathfrak{Y}$ such that $f^*(\mathscr{K})\mathscr{O}_{\mathfrak{X}} \subseteq \mathscr{J}$ to the set of sequences (f_n) of morphisms rendering the diagram (8.3.2).*

Proof. If f is the inductive limit of this sequence, it is necessary to prove that $f^*(\mathscr{K})\mathscr{O}_{\mathfrak{X}} \subseteq \mathscr{J}$. This question is local over $\mathfrak{X}$ and $\mathfrak{Y}$, so we can assume that $\mathfrak{X} = \mathrm{Spf}(A)$, $\mathfrak{Y} = \mathrm{Spf}(B)$ are affine, where A, B are adic, $\mathscr{J} = \mathfrak{J}^\Delta$, $\mathscr{K} = \mathfrak{K}^\Delta$, where $\mathfrak{J}$ (resp. $\mathfrak{K}$) is a defining ideal of A (resp. B). We then have $X_n = \mathrm{Spec}(A_n)$, $Y_n = \mathrm{Spec}(B_n)$, where $A_n = A/\mathfrak{J}^{n+1}$ and $B_n = B/\mathfrak{K}^{n+1}$, in view of Proposition 4.8.6. Then $f_n = ({}^a\varphi_n, \tilde{\varphi}_n)$, where $\varphi_n : B_n \to A_n$ is the homomorphism forming a projective system, so $f = ({}^a\varphi, \tilde{\varphi})$, where $\varphi = \varprojlim \varphi_n$. The commutative diagram (8.3.2) shows that $\varphi_n(\mathfrak{K}/\mathfrak{K}^{n+1}) \subseteq \mathfrak{J}/\mathfrak{J}^{n+1}$ for each n (by take $m = 0$), so by passing to limit, $\varphi(\mathfrak{K}) \subseteq \mathfrak{J}$, and this implies $f^*(\mathscr{K})\mathscr{O}_{\mathfrak{X}} \subseteq \mathscr{J}$ (Proposition 4.8.13). □

Corollary 4.8.21. *Let $\mathfrak{X},\mathfrak{Y}$ be locally Noetherian formal schemes, $\mathscr{T}$ be the largest defining ideal of $\mathfrak{X}$.*

(i) For any defining ideal $mathscrK$ of $\mathfrak{Y}$ and any morphism $f : \mathfrak{X} \to \mathfrak{Y}$, we have $f^*(\mathscr{K})\mathscr{O}_{\mathfrak{X}} \subseteq \mathscr{T}$.

(ii) There exists a bijective correspondence between $\mathrm{Hom}(\mathfrak{X},\mathfrak{Y})$ and the set of sequences (f_n) of morphisms rendering the diagram (8.3.2), where $X_n = (\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}/\mathscr{T}^{n+1})$, $Y_n = (\mathfrak{Y}, \mathscr{O}_{\mathfrak{Y}}/\mathscr{K}^{n+1})$.

Proof. It is clear that (ii) follows from (i) and Proposition 4.8.20. To prove (i), we can assume that $\mathfrak{X} = \mathrm{Spf}(A)$, $\mathfrak{Y} = \mathrm{Spf}(B)$, A, B being Noetherian and adic, $\mathscr{T} = \mathfrak{T}^\Delta$, $\mathscr{K} = \mathfrak{K}^\Delta$, where $\mathfrak{T}$ is the largest defining ideal of A and $\mathfrak{K}$ is a defining ideal of B. Let $f = ({}^a\varphi, \tilde{\varphi})$ where $\varphi : B \to A$ is a continuous homomorphism; as the elements of $\mathfrak{K}$ are topologically nilpotent, so are those of $\varphi(\mathfrak{K})$, so $\varphi(\mathfrak{K}) \subseteq \mathfrak{T}$ since $\mathfrak{T}$ is the set of topologically nilpotent elements of A (Proposition **??**). The conclusion then follows from Proposition 4.8.13(ii). □

Corollary 4.8.22. *Let $\mathfrak{X},\mathfrak{Y},\mathfrak{S}$ be adic formal schemes, $f : \mathfrak{X} \to \mathfrak{S}$, $g : \mathfrak{Y} \to \mathfrak{S}$ be morphisms. Let $\mathscr{J}$ (resp. $\mathscr{K}$, $\mathscr{L}$) be a defining ideal of $\mathfrak{S}$ (resp. $\mathfrak{X}$, $\mathfrak{Y}$), and suppose that $f^*(\mathscr{J})\mathscr{O}_{\mathfrak{X}} \subseteq \mathscr{K}$, $g^*(\mathscr{J})\mathscr{O}_{\mathfrak{Y}} = \mathscr{L}$. Let $S_n = (\mathfrak{S}, \mathscr{O}_{\mathfrak{S}}/\mathscr{J}^{n+1})$, $X_n = (\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}/\mathscr{K}^{n+1})$, $Y_n = (\mathfrak{Y}, \mathscr{O}_{\mathfrak{Y}}/\mathscr{L}^{n+1})$. Then there exists a bijective correspondence between $\mathrm{Hom}_{\mathfrak{S}}(\mathfrak{X},\mathfrak{Y})$ and the set of sequences (u_n) of S_n-morphisms $u_n : X_n \to Y_n$ rendering the diagram (8.3.2).*

Proof. For any $\mathfrak{S}$-morphism $u : \mathfrak{X} \to \mathfrak{Y}$, we have $f = g \circ u$ by definition, so

$$u^*(\mathscr{L})\mathscr{O}_{\mathfrak{X}} = u^*(g^*(\mathscr{J})\mathscr{O}_{\mathfrak{Y}})\mathscr{O}_{\mathfrak{X}} = f^*(\mathscr{J})\mathscr{O}_{\mathfrak{X}} \subseteq \mathscr{K},$$

so the corollary follows from Proposition 4.8.20. □

We note that, for $m \leq n$, the datum of a morphism $f_n : X_n \to Y_n$ determines uniquely a morphism $f_m : X_m \to Y_m$ rendering the diagram (8.3.2), as we immediately see by reducing to the affine case; we have thus defined a map

$$\varphi_{mn} : \mathrm{Hom}_{S_n}(X_n, Y_n) \to \mathrm{hom}_{S_m}(X_m, Y_m)$$

and the $\mathrm{Hom}_{S_n}(X_n, Y_n)$ form for the φ_{mn} a projective system of sets; Corollary 4.8.22 then shows that there exists a canonical bijection

$$\mathrm{Hom}_{\mathfrak{S}}(\mathfrak{X}, \mathfrak{Y}) \xrightarrow{\sim} \varprojlim_n \mathrm{Hom}_{S_n}(X_n, Y_n).$$

Remark 4.8.2. Let $\mathfrak{S}$, $\mathfrak{X}, \mathfrak{Y}$ be formal schemes and $f : \mathfrak{X} \to \mathfrak{S}$, $g : \mathfrak{Y} \to \mathfrak{S}$ be morphisms. Suppose that there are fundamental system of defining ideals $(\mathscr{I}_\lambda)$, $(\mathscr{K}_\lambda)$, $(\mathscr{L}_\lambda)$ in $\mathfrak{S}$, $\mathfrak{X}, \mathfrak{Y}$, respectively, with the same index set I, such that $f^*(\mathscr{I}_\lambda)\mathscr{O}_{\mathfrak{X}} \subseteq \mathscr{K}_\lambda$ and $g^*(\mathscr{I}_\lambda)\mathscr{O}_{\mathfrak{Y}} \subseteq \mathscr{L}_\lambda$ for any λ. Put $S_\lambda = (\mathfrak{S}, \mathscr{O}_{\mathfrak{S}}/\mathscr{I}_\lambda)$, $X_\lambda = (\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}/\mathscr{K}_\lambda)$, $Y_\lambda = (\mathfrak{Y}, \mathscr{O}_{\mathfrak{Y}}/\mathscr{L}_\lambda)$; for $\mathscr{I}_\mu \subseteq \mathscr{I}_\lambda$, $\mathscr{K}_\mu \subseteq \mathscr{K}_\lambda$, $\mathscr{L}_\mu \subseteq \mathscr{L}_\lambda$, note that S_λ (resp. X_λ, Y_λ) is a closed subscheme of S_μ (X_μ, Y_μ) with the same underlying space. As $S_\lambda \to S_\mu$ is a monomorphism of schemes, we then see that the products $X_\lambda \times_{S_\lambda} Y_\lambda$ and $X_\lambda \times_{S_\mu} Y_\lambda$ are identical (Proposition 4.3.2), because $X_\lambda \times_{S_\mu} Y_\lambda$ is identified with a closed subscheme of $X_\mu \times_{S_\mu} Y_\mu$ with the same underlying space. Now the product is the inductive limit of the schemes $X_\lambda \times_{S_\lambda} Y_\lambda$: in fact, we see as in Proposition 4.8.16, we can assume that $\mathfrak{S}$, $\mathfrak{X}$, $\mathfrak{Y}$ are formal affine schemes. In view of Proposition 4.8.13 and our hypotheses, immediately see that our assertion follows from the definition of the completed tensor product of two algebras.

Moreover, let $\mathfrak{Z}$ be a formal $\mathfrak{S}$-scheme, $(\mathscr{M}_\lambda)$ be a fundamental system of defining ideals of $\mathfrak{Z}$ with index set I, $u : \mathfrak{Z} \to \mathfrak{X}$, $v : \mathfrak{Z} \to \mathfrak{Y}$ be morphisms such that $u^*(\mathscr{K}_\lambda)\mathscr{O}_{\mathfrak{Z}} \subseteq \mathscr{M}_\lambda$ and $v^*(\mathscr{L}_\lambda)\mathscr{O}_{\mathfrak{Z}} \subseteq \mathscr{M}_\lambda$. If we put $Z_\lambda = (\mathfrak{Z}, \mathscr{O}_{\mathfrak{Z}}/\mathscr{M}_\lambda)$, and if $u_\lambda : Z_\lambda \to X_\lambda$ and $v_\lambda : Z_\lambda \to Y_\lambda$ are the corresponding S_λ-morphisms, we then verify that $(u, v)_{\mathfrak{S}}$ is the inductive limits of the S_λ-morphisms $(u_\lambda, v_\lambda)_{S_\lambda}$.

4.8.4 Formal completion of schemes

Let X be a (usual) scheme, Z be a subscheme of X, U be an open subset of X containing Z and such that Z is a closed subscheme of U; then Z is defined by a quasi-coherent ideal $\mathscr{I}_U$ of $\mathscr{O}_U$. For any integer $n > 0$, and any quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, $(\mathscr{F}|_U) \otimes_{\mathscr{O}_U} (\mathscr{O}_U/\mathscr{I}_U^n)$ is then a quasi-coherent $\mathscr{O}_U$-module whose support is contained in Z, which is therefore often identified with its restriction to Z. The family $((\mathscr{F}|_U) \otimes_{\mathscr{O}_U} (\mathscr{O}_U/\mathscr{I}_U^n))_{n \geq 1}$ then forms a projective system of sheaves of abelian groups. The limit $\varprojlim((\mathscr{F}|_U) \otimes_{\mathscr{O}_U} (\mathscr{O}_U/\mathscr{I}_U^n))$ is called the **completion of $\mathscr{F}$ along the subscheme X' of X**, and denoted by $\widehat{\mathscr{F}}_{/Z}$ of simply $\widehat{\mathscr{F}}$ (if there is no confusion). The sections of $\widehat{\mathscr{F}}$ over Z are called the **formal sections** of $\mathscr{F}$ along Z.

This definition is justified by the fact that it obviously does not depend on the choice open subset U, because at every point x of $U - Z$, there is a neighborhood of x in which $\mathscr{O}_U/\mathscr{I}_U^n$ is zero for any integer n. We can therefore limit ourselves to the case where Z is a closed subscheme of X, and we will always assume this henceforth. Also, it is clear that for any open subset $U \subseteq X$, we have $(\mathscr{F}|_U)_{/(U \cap Z)} = (\mathscr{F}_{/Z})|_{U \cap Z}$.

By passing to projective limits, it is clear that $(\mathscr{O}_X)_{/Z}$ is a sheaf of rings, and that $\mathscr{F}_{/Z}$ can be considered as an $(\mathscr{O}_{/Z})$-module. Furthermore, as there existss a basis for Z formed by quasi-compact opens, we can consider $(\mathscr{O}_X)_{/Z}$ (resp. $\mathscr{F}_{/Z}$) as a sheaf of topological rings (resp. topological groups) which is the projective limit of the sheaf of discrete rings (resp. groups) $\mathscr{O}_X/\mathscr{I}^n$ (resp. $\mathscr{F} \otimes_{\mathscr{O}_X} (\mathscr{O}_X/\mathscr{I}^n) = \mathscr{F}/\mathscr{I}^n\mathscr{F}$), and by passing to projective limits, $\mathscr{F}_{/Z}$ is a topological $(\mathscr{O}_X)_{/Z}$-module. Note that for any quasi-compact open subset U of X, $\Gamma(U \cap Z, (\mathscr{O}_X)_{/Z})$ (resp. $\Gamma(U \cap Z, \mathscr{F}_{/Z})$) is then the projective limit of the discrete rings (resp. groups) $\Gamma(U, \mathscr{O}_X/\mathscr{I})$ (resp. $\Gamma(U, \mathscr{F}/\mathscr{I}\mathscr{F})$).

Now let $u : \mathscr{F} \to \mathscr{G}$ be a homomorphism of $\mathscr{O}_X$-modules, we then deduce a canonical homomorphism

$$u_{\mathscr{I}^n} : \mathscr{F} \otimes_{\mathscr{O}_X} (\mathscr{O}_X/\mathscr{I}^n) \to \mathscr{G} \otimes_{\mathscr{O}_X} (\mathscr{O}_X/\mathscr{I}^n)$$

for any $n \geq 1$, and these homomorphisms form a projective system. By passing to projective limits and restricting to Z, we obtain a continuous $(\mathscr{O}_X)_{/Z}$-homomorphism $\mathscr{F}_{/Z} \to \mathscr{G}_{/Z}$, denoted by $u_{/Z}$ of $\hat{u}$, and is called the completion of u along Z. It is clear that if $v : \mathscr{G} \to \mathscr{H}$ is a second homomorphism of $\mathscr{O}_X$-modules, then $(v \circ u)_{/Z} = (v_{/Z}) \circ (u_{/Z})$, so $\mathscr{F}_{/Z}$ is a covariant additive functor on $\mathscr{F}$ from the category of quasi-coherent $\mathscr{O}_X$-modules, with values in the category of $(\mathscr{O}_X)_{/Z}$-modules.

Proposition 4.8.23. *Let Z be a closed subscheme of X defined by a quasi-coherent ideal $\mathscr{I}$ of finite type. Then the support of $(\mathscr{O}_X)_{/Z}$ is Z, the topologically ringed space $(Z, (\mathscr{O}_X)_{/Z})$ is an adic formal scheme, and $\mathscr{I}_{/Z}$ is a defining ideal of this formal scheme. If $X = \operatorname{Spec}(A)$ is an affine scheme, $\mathscr{I} = \widetilde{\mathfrak{I}}$ where $\mathfrak{I}$ is an ideal of A, and $Z = V(\mathfrak{I})$, then $(Z, (\mathscr{O}_X)_{/Z})$ is canonically identified with $\operatorname{Spf}(\widehat{A})$ where $\widehat{A}$ is the Hausdorff completion of A for the $\mathfrak{I}$-adic topology.*

Proof. We can evidently assume that $X = \operatorname{Spec}(A)$ is affine. By Proposition **??** the the Hausdorff completion $\widehat{\mathfrak{I}}$ of $\mathfrak{I}$ for the $\mathfrak{I}$-adic topology is identified with the ideal $\mathfrak{I}\widehat{A}$ of $\widehat{A}$, and that $\widehat{A}$ is a $\widehat{A}$-adic ring such that $\widehat{A}/\widehat{\mathfrak{I}}^n = A/\mathfrak{I}^n$. This last relation (for $n = 1$) proves that the open prime ideals of $\widehat{A}$ are the ideals $\hat{\mathfrak{p}} = \mathfrak{p}\widehat{\mathfrak{I}}$, where $\mathfrak{p}$ is a prime ideal of A containing $\mathfrak{I}$, whence $\operatorname{Spf}(\widehat{A}) = Z$. As $\mathscr{O}_X/\mathscr{I}^n = \widetilde{A/\mathfrak{I}^n}$, the proposition then follows from the definition of $\operatorname{Spf}(\widehat{A})$. ∎

The formal scheme therefore defined is called the **completion** of X along Z and is denoted by $X_{/Z}$ of $\widehat{X}$. If $Z = X$, we can put $\mathscr{I} = 0$, and we then have $X_{/Z} = X$. It is clear that if U is an open subscheme of X, then $U_{/(U \cap Z)}$ is canonically identified with the formal subscheme of $X_{/Z}$ induced over the open subset $U \cap Z$ of Z.

Corollary 4.8.24. *Under the hypothesis of Proposition 4.8.23, assume that X is locally Noetherian; then the (usual) scheme $\widehat{X}_{\mathrm{red}}$ is the unique reduced subscheme of X with underlying space Z (Proposition 4.4.40). For $\widehat{X}$ to be Noetherian, it is necessary and sufficient that $\widehat{X}_{\mathrm{red}}$ is Noetherian, and it is sufficient that X is.*

Proof. The determination of $\widehat{X}_{\mathrm{red}}$ is local (Proposition 4.8.11), so we can assume that X is affine; with the notations of Proposition 4.8.23, the ideal $\mathfrak{T}$ of topological nilpotent elements of $\widehat{A}$ is the inverse image of the nilradical of $A/\mathfrak{T}$ under the canonical map $\widehat{A} \to \widehat{A}/\widehat{\mathfrak{I}} = A/\mathfrak{I}$, so $\widehat{A}/\mathfrak{T}$ is isomorphic to $(A/\mathfrak{I})_{\mathrm{red}}$. The first assertion then follows from Proposition 4.8.11 and Proposition 4.4.40. If $\widehat{X}_{\mathrm{red}}$ is Noetherian, its underlying space Z is also Noetherian, so the $Z_n = \operatorname{Spec}(\mathscr{O}_X/\mathscr{I}^n)$ are Noetherian and so is $\widehat{X}$ (Corollary 4.8.18); the converse of this is immeidate. ∎

The canonical homomorphisms $\mathscr{O}_X \to \mathscr{O}_X/\mathscr{I}^n$ form a projective system and therefore gives, by passing to limit, a homomorphism of sheaves of rings $\theta : \mathscr{O}_X \to i_*((\mathscr{O}_X)_{/Z}) = \varprojlim(\mathscr{O}_X/\mathscr{I}^n)$, where we denote by $i : Z \to X$ the canonical injection. We therefore obtain a morphism

$$(i, \theta) : X_{/Z} \to X$$

of ringed spaces, called the canonical morphism. By tensoring, for any coherent $\mathscr{O}_X$-module $\mathscr{F}$, the canonical homomorphisms $\mathscr{O}_X \to \mathscr{O}_X/\mathscr{I}^n$ give the homomorphisms $\mathscr{F} \to \mathscr{F} \otimes_{\mathscr{O}_X} (\mathscr{O}_X/\mathscr{I}^n)$ of $\mathscr{O}_X$-modules which form a projective system, so define a canonical functorial homomorphism $\gamma : \mathscr{F} \to i_*(\mathscr{F}_{/Z})$ of $\mathscr{O}_X$-modules.

Example 4.8.25. Let Z, Z' be closed subschemes of X, defined by quasi-coherent ideals $\mathscr{I}, \mathscr{I}'$ of $\mathscr{O}_X$. Suppose that for any affine open U of X, the ideals $\mathscr{I}|_U, \mathscr{I}'|_U$ of $\mathscr{O}_U$ are such that there exists an integer $m > 0$ such that $(\mathscr{I}|_U)' \subseteq \mathscr{I}'|_U$ and $(\mathscr{I}'|_U)^m \subseteq \mathscr{I}|_U$. It is clear that under this condition, for any quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, the sheaf of abelian groups $\varprojlim(\mathscr{F} \otimes_{\mathscr{O}_X} (\mathscr{O}_X/\mathscr{I}^n))$ and $\varprojlim(\mathscr{F} \otimes_{\mathscr{O}_X} (\mathscr{O}_X/\mathscr{I}'^n))$ are canonically isomorphic, which means $\mathscr{F}_{/Z} = \mathscr{F}_{/Z'}$. Note that this condition on Z and Z' implies that these two closed subschemes have the same underlying space, but is not in general equivalent to the latter property.

However, if the quasi-coherent ideals $\mathscr{I}$ and $\mathscr{I}'$ are of finite type, then it follows from Proposition 4.6.17 that if the subschemes Z and Z' have the same underlying space, the above condition is satisfied. In particular, if X is locally Noetherian, so that any quasi-coherent ideal of $\mathscr{O}_X$ is of finite type, then for any closed subset (or locally closed) Z of X, we can define $\mathscr{F}_{/Z}$ to be equal to $\mathscr{F}_{/Y}$ for any subscheme Y of X with underlying space Z.

Proposition 4.8.26. *Suppose that X is locally Noetherian and let Z be a closed subset of X, $\mathscr{F}$ be a coherent $\mathscr{O}_X$-module.*

(i) The functor $\mathscr{F}_{/Z}$ is exact on the category of coherent $\mathscr{O}_X$-modules.

(ii) The functorial homomorphism $\gamma^\sharp : i^*(\mathscr{F}) \to \mathscr{F}_{/Z}$ of $(\mathscr{O}_X)_{/Z}$-modules is an isomorphism.

Proof. To prove (i), it suffices to prove that if $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ is an exact sequence of coherent $\mathscr{O}_X$-modules and U is an affine open of X with Noetherian ring $A = \Gamma(U, \mathscr{O}_X)$, the sequence

$$0 \longrightarrow \Gamma(U \cap Z, \mathscr{F}'_{/Z}) \longrightarrow \Gamma(U \cap Z, \mathscr{F}_{/Z}) \longrightarrow \Gamma(U \cap Z, \mathscr{F}''_{/Z}) \to 0$$

is exact. Then $\mathscr{F}|_U = \widetilde{M}$, $\mathscr{F}'|_U = \widetilde{M}'$, $\mathscr{F}''|_U = \widetilde{M}''$, where M, M', M'' are A-modules of finite type such that the sequence $0 \to M' \to M \to M'' \to 0$ is exact. Let $\mathscr{I}$ be a quasi-coherent ideal of $\mathscr{O}_U$ defining a subscheme of U with underlying space $U \cap Z$, and $\mathfrak{I}$ be the ideal of A such that $\mathscr{I} = \widetilde{\mathfrak{I}}$. We have (Corollary 4.1.9)

$$\Gamma(U \cap Z, \mathscr{F} \otimes_{\mathscr{O}_X} (\mathscr{O}_X/\mathscr{I}^n)) = M \otimes_A (A/\mathfrak{I}^n);$$

so by the definition of projective limit,

$$\Gamma(U \cap Z, \mathscr{F}_{/Z}) = \varprojlim(M \otimes_A (A/\mathfrak{I}^n)) = \widehat{M}$$

where $\widehat{M}$ is the Huasdorff completion of M for the $\mathfrak{I}$-adic topology, and similarly

$$\Gamma(U\cap Z,\mathscr{F}'_{/Z})=\widehat{M}',\quad \Gamma(U\cap Z,\mathscr{F}''_{/Z})=\widehat{M}'';$$

our assertion then follows from the fact that A is Noetherian and the functor $\widehat{M}$ on M is exact on the category of finitely generated A-modules (Proposition **??**).

For assertion (i), the assertion is local, so we can assume that there exists an exact sequence $\mathscr{O}_X^m\to\mathscr{O}_X^n\to\mathscr{F}\to 0$; as $\gamma^\sharp$ is functorial, and the functors $i^*(\mathscr{F})$ and $\mathscr{F}_{/Z}$ are right exact, we have a commutative diagram

$$\begin{tikzcd} i^*(\mathscr{O}_X^m) \arrow[r] \arrow[d, "\gamma^\sharp"] & i^*(\mathscr{O}_X^n) \arrow[r] \arrow[d, "\gamma^\sharp"] & i^*(\mathscr{F}) \arrow[r] \arrow[d, "\gamma^\sharp"] & 0 \\ (\mathscr{O}_X^m)_{/Z} \arrow[r] & (\mathscr{O}_X^n)_{/Z} \arrow[r] & \mathscr{F}_{/Z} \arrow[r] & 0 \end{tikzcd} \tag{8.4.1}$$

with exact rows. Moreover, the functors $i^*(\mathscr{F})$ and $\mathscr{F}_{/Z}$ commutes with finite sums, so we are reduced to the case where $\mathscr{F}=\mathscr{O}_X$. We then have $i^*(\mathscr{O}_X)=(\mathscr{O}_X)_{/Z}=\mathscr{O}_{\widehat{X}}$, and $\gamma^\sharp$ is a homomorphism of $\mathscr{O}_{\widehat{X}}$-modules; it then suffices to verify that $\gamma^\sharp$ maps the unit section of $\mathscr{O}_{\widehat{X}}$ over an open subset of Z to itself, which is immediate and also shows that $\gamma^\sharp$ is the identity. ◻

Corollary 4.8.27. *Under the hypotheses of Proposition 4.8.26, the morphism $i:\widehat{X}\to X$ is flat.*

Corollary 4.8.28. *Let X be a locally Noetherian scheme. If $\mathscr{F}$ and $\mathscr{G}$ are coherent $\mathscr{O}_X$-modules, there exists a canonical functorial isomorphism*

$$(\mathscr{F}_{/Z})\otimes_{(\mathscr{O}_X)_{/Z}}(\mathscr{G}_{/Z})\xrightarrow{\sim}(\mathscr{F}\otimes_{\mathscr{O}_X}\mathscr{G})_{/Z}, \tag{8.4.2}$$

$$(\mathscr{H}om_{\mathscr{O}_X}(\mathscr{F},\mathscr{G}))_{/Z}\xrightarrow{\sim}\mathscr{H}om_{(\mathscr{O}_X)_{/Z}}(\mathscr{F}_{/Z},\mathscr{G}_{/Z}). \tag{8.4.3}$$

Proof. This follows from the canonical identification of $i^*(\mathscr{F})$ and $\mathscr{F}_{/Z}$; the existence of the first isomorphism is clear for any morphism of ringed spaces and the second is the homomorphism of (3.4.5), which is an isomorphism for any flat morphism. ◻

Proposition 4.8.29. *Let X be a locally Noetherian scheme. For any coherent $\mathscr{O}_X$-module $\mathscr{F}$, the kernel of the canonical homomorphism $\Gamma(X,\mathscr{F})\to\Gamma(Z,\mathscr{F}_{/Z})$ induced from $\mathscr{F}\to\mathscr{F}_{/Z}$ is formed by the sections which are zero on an open neighborhood of Z.*

Proof. It follows from the definition of $\mathscr{F}_{/Z}$ that the canonical image of such a section is zero. Conversely, if the image of $s\in\Gamma(X,\mathscr{F})$ is zero in $\Gamma(Z,\mathscr{F}_{/Z})$, it suffices to see that any $x\in Z$ admits an open neighborhood in X over which s is zero, and we can therefore assume that $X=\operatorname{Spec}(A)$ is affine, A is Noetherian, $Z=V(\mathfrak{I})$ where $\mathfrak{I}$ is an ideal of A, and $\mathscr{F}=\widetilde{M}$, where M is a finitely generated A-module. Then $\Gamma(Z,\mathscr{F}_{/Z})$ is the Huasdorff completion $\widehat{M}$ of M for the $\mathfrak{I}$-topology, and the homomorphism $\Gamma(X,\mathscr{F})\to\Gamma(Z,\mathscr{F}_{/Z})$ is the canonical homomorphism $M\to\widehat{M}$. We have seen that the kernel of this homomorphism (Proposition **??**) consists of elements $z\in M$ which is annihilated by an element of $1+\mathfrak{I}$. We then have $(1+f)s=0$ for $f\in\mathfrak{I}$, and for any $x\in Z$ we deduce that $(1_x+f_x)s_x=0$; as 1_x+f_x is invertible in $\mathscr{O}_{X,x}$ ($\mathfrak{I}_x\mathscr{O}_{X,x}$ is contained in $\mathfrak{m}_x$), we then have $s_x=0$, which proves the assertion. ◻

Corollary 4.8.30. *Let X be a locally Noetherian scheme and $u : \mathscr{F} \to \mathscr{G}$ be a homomorphism of coherent $\mathscr{O}_X$-modules. For the homomorphism $u_{/Z} : \mathscr{F}_{/Z} \to \mathscr{G}_{/Z}$ to be zero, it is necessary and sufficient that u is zero on an open neighborhood of Z.*

Proof. In fact, by Proposition 4.8.26, $u_{/Z}$ is identified with $i^*(u)$, so if we consider u as a section of $\mathscr{H} = \mathscr{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})$ over X, $u_{/Z}$ is the section of $i^*(\mathscr{H}) = \mathscr{H}_{/Z}$ over Z. It then suffices to apply Proposition 4.8.29 on $\mathscr{H}$. □

Corollary 4.8.31. *Let X be a locally Noetherian sceme and $u : \mathscr{F} \to \mathscr{G}$ be a homomorphism of coherent $\mathscr{O}_X$-modules. For the homomorphism $u_{/Z} : \mathscr{F}_{/Z} \to \mathscr{G}_{/Z}$ to be a monomorphism (resp. epimorphism), it is necessary and sufficient that u is a monomorphism (resp. epimorphism) on an open neighborhood of Z.*

Proof. Let $\mathscr{P}$ and $\mathscr{N}$ be the kernel and cokernel of u, so that we have an exact sequence

$$0 \longrightarrow \mathscr{P} \longrightarrow \mathscr{F} \xrightarrow{u} \mathscr{G} \longrightarrow \mathscr{N} \longrightarrow 0$$

By applying $(-)_{/Z}$, we then get an exact sequence

$$0 \longrightarrow \mathscr{P}_{/Z} \longrightarrow \mathscr{F}_{/Z} \xrightarrow{u_{/Z}} \mathscr{G}_{/Z} \longrightarrow \mathscr{N}_{/Z} \longrightarrow 0$$

That $u_{/Z}$ is a monomorphism (resp. epimorphism) is equivalent to $\mathscr{P}_{/Z} = 0$ (resp. $\mathscr{N}_{/Z} = 0$), so we can apply Proposition 4.8.29 to get the conclusion. □

Corollary 4.8.32. *Let X be a locally Noetherian scheme and $\mathscr{F}$ be a coherent $\mathscr{O}_X$-module. For $\mathscr{F}_{/Z}$ to be locally free (resp. locally free of rank n), it is necessary and sufficient that there exists an open neighborhood U of Z such that $\mathscr{F}|_Z$ is locally free (resp. locally free of rank n).*

Proof. To say that $\mathscr{F}_{/Z}$ is locally free signifies that any point $x \in Z$ admits an open neighborhood V in X such that there exists an isomorphism $v : (\mathscr{O}_X^n)_{/Z} \xrightarrow{\sim} \mathscr{F}_{/Z}|_{V\cap Z}$. We can evidently assume that $V = X$, and then it follows from (8.4.3) that v is of the form $u_{/Z}$, where u is a homomorphism $\mathscr{O}_X^n \to \mathscr{F}$; moreover, by replacing X with an open neigbourhood of Z, we can assume, in view of Corollary 4.8.31, that u is bijective, whence the corollary. □

Chapter 5

Global properties of morphisms of schemes

5.1 Affine morphisms

5.1.1 Schemes affine over a scheme

Let S be a scheme and X be an S-scheme. If $f : X \to S$ is the structural morphism, then the direct image $f_*(\mathscr{O}_X)$ is an $\mathscr{O}_S$-algebra, which we denote by $\mathscr{A}(X)$ if there is no confusion. If U is an open subset of S, we have

$$\mathscr{A}(f^{-1}(U)) = \mathscr{A}(X)|_U.$$

Similarly, for any $\mathscr{O}_X$-module $\mathscr{F}$ (resp. any $\mathscr{O}_X$-algebra $\mathscr{B}$), we denote by $\mathscr{A}(\mathscr{F})$ (resp. $\mathscr{A}(\mathscr{B})$) the direct image $f_*(\mathscr{F})$ (resp. $f_*(\mathscr{B})$) which is an $\mathscr{A}(X)$-module (resp. $\mathscr{A}(X)$-algebra), and also an $\mathscr{O}_S$-module (resp. $\mathscr{O}_S$-algebra).

Let Y be another S-scheme with $g : Y \to S$ the structural morphism, and $h : X \to Y$ be an S-morphism. We then have a commutative diagram

$$\begin{tikzcd} X \arrow[rr, "h"] \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & S & \end{tikzcd}$$

By definition we have a homomorphism $h^\# : \mathscr{O}_Y \to h_*(\mathscr{O}_X)$ of sheaves of rings, and we deduce from this a homomorphism of $\mathscr{O}_S$-algebras $g_*(h^\#) : g_*(\mathscr{O}_Y) \to g_*(h_*(\mathscr{O}_X)) = f_*(\mathscr{O}_X)$, which means, a homomorphism $\mathscr{A}(Y) \to \mathscr{A}(X)$ of $\mathscr{O}_S$-algebras, and we denote it by $\mathscr{A}(h)$. If $h' : Y \to Z$ is another S-morphism, it is immediate that $\mathscr{A}(h' \circ h) = \mathscr{A}(h) \circ \mathscr{A}(h')$. Therefore we have defined a contravariant functor $\mathscr{A}(X)$ from the category of S-schemes to the category of $\mathscr{O}_S$-algebras.

Now let $\mathscr{F}$ be an $\mathscr{O}_X$-module, $\mathscr{G}$ be an $\mathscr{O}_Y$-module, and $u : \mathscr{G} \to \mathscr{F}$ be an h-morphism, wihch is a homomorphism $\mathscr{G} \to h_*(\mathscr{F})$ of $\mathscr{O}_Y$-modules. Then $g_*(u) : g_*(\mathscr{G}) \to g_*(h_*(\mathscr{F})) = f_*(\mathscr{F})$ is a homomorphism $\mathscr{A}(\mathscr{G}) \to \mathscr{A}(\mathscr{F})$ of $\mathscr{O}_S$-modules, which we denote by $\mathscr{A}(u)$. The couple $(\mathscr{A}(h), \mathscr{A}(u))$ is then a bi-homomorphism of $\mathscr{A}(Y)$-modules $\mathscr{A}(\mathscr{G})$ ot the $\mathscr{A}(X)$-module $\mathscr{A}(\mathscr{F})$. If we fix S and consider the couples $(X, \mathscr{F})$, where X is an S-scheme and $\mathscr{F}$ is an $\mathscr{O}_X$-module, we then see that $(\mathscr{A}(X), \mathscr{A}(\mathscr{F}))$ defines a contravariant functor from the category of these couples to the category of couples of $\mathscr{O}_S$-algebras and modules of this algebra.

Consider now an S-scheme X and let $f : X \to S$ be a structural morphism. We say that X is **affine over S** if there exists a covering (S_α) of S by affine opens such that, for each α, the open subscheme $f^{-1}(S_\alpha)$ of X is affine. If this is true, we also say that X is an **affine S-scheme**, or the structural morphism f is affine.

Example 5.1.1. Any closed subscheme of S is an affine S-scheme. In fact, if Y is a closed subscheme of S, then for any affine open U of S, the intersection $U \cap Y$ is a closed subscheme of U, whence affine.

Remark 5.1.1. One should note that an affine S-scheme X is not necessary an affine scheme (for example S is affine over S, but note that this is ture if S itself is affine). On the other hand, if X is an S-scheme and is affine, it is not necessarily ture that X is an affine S-scheme (we will see this later). However, if S is a separated scheme, then any affine scheme is affine over S by Proposition 4.5.33.

Proposition 5.1.2. *Any affine S-scheme is separated over S.*

Proof. Recall that separatedness is local on target (Proposition 4.5.29), and if $f^{-1}(S_\alpha)$ is affine, then the restriction of f to $f^{-1}(S_\alpha)$ is a morphism between affine schemes, so is separated. □

Proposition 5.1.3. *Let X be an affine S-scheme and $f : X \to S$ be the structural morphism. Then for any open subset $U \subseteq S$, $f^{-1}(U)$ is affine over U. In particular, if U is affine, so is $f^{-1}(U)$.*

Proof. In view of the definition, we can reduce to the case $S = \mathrm{Spec}(A)$ and $X = \mathrm{Spec}(B)$, so that f corresponds to a homomorphism $\rho : A \to B$. As the standard opens $D(g)$ with $g \in A$ form a basis for S, we only need to prove the assertion for $U = D(g)$. But recall that $f^{-1}(D(g)) = D(\rho(g))$, so our assertion follows. □

Corollary 5.1.4. *Let S be an affine scheme. Then for an S-scheme X to be affine over S, it is necessary and sufficient that X is an affine scheme.*

Proposition 5.1.5. *Let X be an affine S-scheme and $f : X \to S$ be the structural morphism. For any quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, $f_*(\mathscr{F})$ is a quasi-coherent $\mathscr{O}_S$-module. In particular, the $\mathscr{O}_S$-algebra $\mathscr{A}(X)$ is quasi-coherent.*

Proof. The morphism f is separted by Proposition 5.1.2 and quasi-compact by Proposition 5.1.3 (since any quasi-compact open subset is a finite union of affine opens), so we can apply Proposition 4.6.54. □

Proposition 5.1.6. *Let X ba an affine S-scheme. For any S-scheme Y, the map $h \mapsto \mathscr{A}(h)$ from* $\mathrm{Hom}_S(Y, X)$ *to* $\mathrm{Hom}_{\mathscr{O}_S\text{-}\mathbf{alg}}(\mathscr{A}(X), \mathscr{A}(Y))$ *is bijective.*

Proof. Let $f : X \to S$, $g : Y \to S$ be the structural morphisms. Suppose first that $S = \mathrm{Spec}(A)$ and $X = \mathrm{Spec}(B)$ are affine; we must shows that for any homomorphism $\omega : f_*(\mathscr{O}_X) \to g_*(\mathscr{O}_Y)$ of $\mathscr{O}_S$-algebras, there exists a unique S-morphism $h : Y \to X$ such that $\mathscr{A}(h) = \omega$. By definition, for any open subset $U \subseteq S$, ω defines a homomorphism $\omega_U : \Gamma(f^{-1}(U), \mathscr{O}_X) \to \Gamma(g^{-1}(U), \mathscr{O}_Y)$ of $\Gamma(U, \mathscr{O}_S)$-algebras. In particular, for $U = S$, this gives a homomorphism $\varphi : \Gamma(X, \mathscr{O}_X) \to$

$\Gamma(Y, \mathscr{O}_Y)$, which by Proposition 4.2.4, since X is affine, corresponds to a morphism $h : Y \to X$. To see that $\mathscr{A}(h) = \omega$, we need to prove that for any open subset $U \subseteq S$, ω_U coincides with the algebra homomorphism φ_U, which corresponds to the S-morphism $h|_{g^{-1}(U)} : g^{-1}(U) \to f^{-1}(U)$. We may assume that $U = D(\lambda)$ where $\lambda \in A$; then, if $f : X \to S$ corresponds to the ring homomorphism $\rho : A \to B$, we have $f^{-1}(U) = D(\mu)$ where $\mu = \rho(\lambda)$, and $\Gamma(f^{-1}(U), \mathscr{O}_X)$ is the fraction ring B_μ. Now the following diagram commutes

$$\begin{tikzcd}
B = \Gamma(X, \mathscr{O}_X) \arrow[r, "\varphi"] \arrow[d] & \Gamma(Y, \mathscr{O}_Y) \arrow[d] \\
B_\mu = \Gamma(f^{-1}(U), \mathscr{O}_X) \arrow[r, bend left, "\varphi_U"] \arrow[r, bend right, "\omega_U"'] & \Gamma(g^{-1}(U), \mathscr{O}_Y)
\end{tikzcd}$$

By the universal property of localization, we then conclude that $\varphi_U = \omega_U$, whence the assertion in this case.

In the general case, let (S_α) be a covering of S by affine opens such that $f^{-1}(S_\alpha)$ are affine. Then any homomorphism $\omega : \mathscr{A}(X) \to \mathscr{A}(Y)$ of $\mathscr{O}_S$-algebras restricts to a family of homomorphisms

$$\omega_\alpha : \mathscr{A}(f^{-1}(S_\alpha)) \to \mathscr{A}(g^{-1}(S_\alpha))$$

of $\mathscr{O}_{S_\alpha}$-algebras, so there is a family of S_α-morphisms $h_\alpha : g^{-1}(S_\alpha) \to f^{-1}(S_\alpha)$ such that $\mathscr{A}(h_\alpha) = \omega_\alpha$. It all boils down to seeing that for any affine open U of the base $S_\alpha \cap S_\beta$, the restriction of h_α and h_β to $g^{-1}(U)$ coincide, which is immediate since these restrictions both correspond to the restriction homomorphism $\mathscr{A}(X)|_U \to \mathscr{A}(Y)|_U$ of ω. □

Corollary 5.1.7. *Let X and Y be affine S-schemes. For an S-morphism $h : Y \to X$ to be an isomorphism, it is necessary and sufficient that $\mathscr{A}(h) : \mathscr{A}(X) \to \mathscr{A}(Y)$ is an isomorphism.*

Proof. This follows from Proposition 5.1.6 and the functoriality of $\mathscr{A}(X)$. □

5.1.2 Affine S-scheme associated with an $\mathscr{O}_S$-algebra

Proposition 5.1.8. *Let S be a scheme. For any quasi-coherent $\mathscr{O}_S$-algebra $\mathscr{B}$, there exists an affine S-scheme X, defined up to S-isomorphisms, such that $\mathscr{A}(X) = \mathscr{B}$. The affine S-scheme X is said to be **associated with the $\mathscr{O}_S$-algebra** $\mathscr{B}$, and denoted by* $\mathrm{Spec}(\mathscr{B})$.

Proof. The uniqueness follows from Corollary 5.1.7, so we only need to construct the affine S-scheme X. For any affine open $U \subseteq S$, let X_U be the scheme $\mathrm{Spec}(\Gamma(U, \mathscr{B}))$; as $\Gamma(U, \mathscr{B})$ is an $\Gamma(U, \mathscr{O}_S)$-algebra, X_U is an S-scheme, and is affine over U since U and X_U are both affine. Moreover, as $\mathscr{B}$ is quasi-coherent, the $\mathscr{O}_S$-algebra $\mathscr{A}(X_U)$ is canonically idnetified with $\mathscr{B}|_U$ (Proposition 4.1.11). Let V be another affine open of S, and $X_{U,V}$ be the open subscheme of X_U over $\varphi_U^{-1}(U \cap V)$, where $\varphi_U : X_U \to S$ is the structural morphism. Then $X_{U,V}$ and $X_{V,U}$ are affine over $U \cap V$ (Proposition 5.1.3), and by definition $\mathscr{A}(X_{U,V})$ and $\mathscr{A}(X_{V,U})$ are canonically identified with $\mathscr{B}|_{U \cap V}$. There then exists (Corollary 5.1.7) a canonical S-isomorphism $\theta_{U,V} : X_{V,U} \to X_{U,V}$; furthermore, if W is a third affine open of S, and if $\theta'_{U,V}, \theta'_{V,W}, \theta'_{U,W}$ are the restrictions of $\theta_{U,V}, \theta_{V,W}, \theta_{U,W}$ over the inverse images of $U \cap V \cap W$ in X_V, X_W, X_W, then $\theta'_{U,V} \circ \theta'_{V,W} = \theta'_{U,W}$. By glueing the $X_{U,V}$, there then exists a scheme X and an affine open cover (T_U) of X such that for each U there is an isomorphism $\varphi_U : X_U \to T_U$ such that φ_U maps $\varphi_U^{-1}(U \cap V)$ to $T_U \cap T_V$ and we have $\theta_{U,V} = \varphi_U^{-1} \circ \varphi_V$. The morphism $g_U = \varphi_U \circ \varphi_U^{-1}$ then makes T_U an S-scheme, and the morphisms g_U and g_V coincide on $T_U \cap T_V$, so X is an S-scheme. It is clear by definition that X is affine over S and $\mathscr{A}(T_U) = \mathscr{B}|_U$, so $\mathscr{A}(X) = \mathscr{B}$. □

Corollary 5.1.9. *Let S be a scheme. The functor $\mathscr{A}(X)$ defines an equivalence of categories between the category of affine S-schemes and the category of quasi-coherent $\mathscr{O}_S$-algebras.*

Proof. By Proposition 5.1.6 we now that $\mathscr{A}(X)$ is fully faithful, and Proposition 5.1.8 proves that it is essentially surjective, whence the claim. □

Corollary 5.1.10. *Let S be a scheme. Then for any quasi-coherent $\mathscr{O}_S$-algebra $\mathscr{B}$, the contravariant functor*

$$Y \mapsto \mathrm{Hom}_{\mathscr{O}_S-\mathbf{alg}}(\mathscr{B}, \mathscr{A}(Y)) = \mathrm{Hom}_{\mathscr{O}_Y-\mathbf{alg}}(\mathscr{B} \otimes_{\mathscr{O}_S} \mathscr{O}_Y, \mathscr{O}_Y)$$

from the category of S-schemes to the category of sets, is represented by $\mathrm{Spec}(\mathscr{B})$.

Proof. Let $X = \mathrm{Spec}(\mathscr{B})$, then we know that $\mathscr{B} = \mathscr{A}(X)$, so the claim follows from Proposition 5.1.6. □

Corollary 5.1.11. *Let X be an affine S-scheme and $f : X \to S$ be the structural morphism. For any affine open $U \subseteq S$, the open subscheme $f^{-1}(U)$ of X is an affine scheme with ring $\Gamma(U, \mathscr{A}(X))$.*

Proof. We can suppose that X is associated with the $\mathscr{O}_S$-algebra $\mathscr{A}(X)$, the corollary then follows from the construction of X in Proposition 5.1.8. □

Example 5.1.12. Let S be the affine plane for a field K with the point 0 is doubled (Example 4.5.34). With the notations there, S is the union of two affine opens Y_1, Y_2. If f is the open immersion $Y_1 \to S$, then $f^{-1}(Y_2) = Y_1 \cap Y_2$ and we have already seen in Example 4.5.34 that this is not affine. So we obtain an example of an affine scheme not affine over a scheme S.

Remark 5.1.2. Let S be a scheme and $f : X \to S$ be a quasi-compact and quasi-separated morphism, so that $\mathscr{A}(X) = f_*(\mathscr{O}_X)$ is a quasi-coherent $\mathscr{O}_S$-algebra (Proposition 4.6.54). The affine S-scheme

$$X^0 = \mathrm{Aff}(X/S) = \mathrm{Spec}(f_*(\mathscr{O}_X)) = \mathrm{Spec}(\mathscr{A}(X))$$

is called the **affine envelope** of the S-scheme X. If $f^0 : X^0 \to S$ is the structural morphism, by Proposition 5.1.8 we then have

$$\mathscr{A}(X^0) = f^0_*(\mathscr{O}_{X^0}) = \mathscr{A}(X) = f_*(\mathscr{O}_X);$$

by Corollary 5.1.10, the identity homomorphism on $\mathscr{A}(X)$ therefore corresponds to a canonical S-morphism $\iota_X : X \to X^0$ such that f factors into

$$X \xrightarrow{\iota_X} X^0 \xrightarrow{f^0} S$$

This factorization for f is called the **Stein factorization** of f. For the morphism i_X to be an isomorphism, it is necessary and sufficient that the morphism f is affine. Moreover, for any

S-scheme Y affine over S, the map $u \mapsto u \circ i_X$ is then a bijection

$$\mathrm{Hom}_S(X^0, Y) \xrightarrow{\sim} \mathrm{Hom}_S(X, Y). \tag{1.2.1}$$

which is functorial on Y: this follows from the canonical bijections

$$\mathrm{Hom}_S(X, Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{O}_S-\mathbf{alg}}(\mathscr{A}(Y), \mathscr{A}(X)) = \mathrm{Hom}_{\mathscr{O}_S-\mathbf{alg}}(\mathscr{A}(Y), \mathscr{A}(X^0)) \xrightarrow{\sim} \mathrm{Hom}_S(X^0, Y).$$

That is, the S-affine scheme X^0 satisfies the universal property that any S-morphism $f : X \to Y$ such that Y is affine over S must factors through X^0, or equivalently that X^0 represents the covariant functor $Y \mapsto \mathrm{Hom}_S(X, Y)$ on the category of S-affine schemes. We also deduce that for S fixed, $X \mapsto \mathrm{Aff}(X/S)$ is a covariant functor from the category of S-schemes that are quasi-compact and quasi-separted over S to the cateogry of S-schemes affine over S. Consider a commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\iota_X"'] & X' \arrow[d, "\iota_{X'}"] \\ X^0 \arrow[r, "f^0"] & X'^0 \end{tikzcd}$$

The relation (1.2.1) can then be interpreted as the following: the functor $X \mapsto \mathrm{Aff}(X/S)$ is the left adjoint of the forgetful functor from the category of S-schemes affine over S to the category of S-schemes. We then conclude that this functor commutes with inductive limits, hence finite sums.

Corollary 5.1.13. *Let X be an affine S-scheme and Y be an X-scheme. For Y to be affine over S, it is necessary and sufficient that Y is affine over X.*

Proof. We can assume that S is affine, and then X is also affine by Corollary 5.1.4. Then Y is affine over S if and only if it is affine over X, if and only if it is affine, so our claim follows. □

Let X be an affine S-scheme. Then by Corollary 5.1.13, to define of a scheme Y affine over X is equivalent to giving a scheme Y affine over S and an S-morphism $g : Y \to X$. In view of Proposition 5.1.8 and Proposition 5.1.6, this is equivalent to giving a quasi-coherent $\mathscr{O}_S$-algebra $\mathscr{B}$ and a homomorphism $\mathscr{A}(X) \to \mathscr{B}$ of $\mathscr{O}_S$-algebras (which dfines over $\mathscr{B}$ an $\mathscr{A}(X)$-algebra structure). If $f : X \to S$ is the structural morphism, we then have $\mathscr{B} = f_*(g_*(\mathscr{O}_Y))$.

Corollary 5.1.14. *Let X be an affine S-scheme. For X to be of finite type over S, it is necessary and sufficient that the quasi-coherent $\mathscr{O}_X$-algebra $\mathscr{A}(X)$ is of finite type.*

Proof. By definition, we can assume that S is affine. Then X is an affine scheme, hence quasi-compact; if $S = \mathrm{Spec}(A)$, $X = \mathrm{Spec}(B)$, $\mathscr{A}(X)$ is the $\mathscr{O}_S$-algebra $\widetilde{B}$. As $\Gamma(X, \tilde{B}) = B$, the corollary follows from Proposition 4.6.39. □

Corollary 5.1.15. *Let X be an affine S-scheme. For X to be reduced, it is necessary and sufficient that the quasi-coherent $\mathscr{O}_X$-algebra is reduced.*

Proof. The question is local on S so we can assume that S is affine, and the corollary then follows from Proposition 4.4.27. □

5.1.3 Quasi-coherent sheaves over affine S-schemes

Proposition 5.1.16. *Let X be an affine S-scheme, Y be an S-scheme, and $\mathscr{F}$ (resp. $\mathscr{G}$) be a quasi-coherent $\mathscr{O}_X$-module (resp. $\mathscr{O}_Y$-module). Then the map $(h,u)\mapsto(\mathscr{A}(h),\mathscr{A}(u))$ from the set of morphisms $(Y,\mathscr{G})\to(X,\mathscr{F})$ to the set of bi-homomorphisms $(\mathscr{A}(X),\mathscr{A}(\mathscr{F}))\to(\mathscr{A}(Y),\mathscr{A}(\mathscr{G}))$ is bijective.*

Proof. The proof is the same as in Proposition 5.1.6, by using Proposition 4.2.5. □

Corollary 5.1.17. *Under the hypotheses of Proposition 5.1.16, suppose that Y is affine over S. Then for the couple (h,u) to be an isomorphism, it is necessary and sufficient that $(\mathscr{A}(h),\mathscr{A}(u))$ is a bi-isomorphism.*

Proposition 5.1.18. *For any couple $(\mathscr{B},\mathscr{M})$ formed by a quasi-coherent $\mathscr{O}_S$-algebra $\mathscr{B}$ and a quasi-coherent $\mathscr{B}$-module $\mathscr{M}$, there exists a couple $(X,\mathscr{F})$ formed by an affine S-scheme and a quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$ such that $\mathscr{A}(X)=\mathscr{B}$ and $\mathscr{A}(\mathscr{F})=\mathscr{M}$, and this couple is determined up to isomorphisms.*

Proof. The uniqueness part follows from Corollary 5.1.17. The existence for the scheme X follows from Proposition 5.1.8. To define $\mathscr{M}$, we can consider an affine open $U\subseteq S$ and set $\mathscr{F}|_{f^{-1}(U)}=\widetilde{\Gamma(U,\mathscr{M})}$, where $f:X\to S$ is the structural morphism. We will use $\widetilde{\mathscr{M}}$ to denote the quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$ associated with $\mathscr{M}$. □

Corollary 5.1.19. *In the category of quasi-coherent $\mathscr{B}$-modules, $\widetilde{\mathscr{M}}$ is an additive covariant functor which commutes with inductive limits and direct sums.*

Proof. We can in fact assume that S is affine, and the claim then reduces to the functor $\widetilde{M}$ for B-modules, where $B=\Gamma(S,\mathscr{B})$. □

Corollary 5.1.20. *Under the hypotheses of Proposition 5.1.18, assume that $\mathscr{B}$ is an $\mathscr{O}_X$-algebra of finite type. Then for $\widetilde{\mathscr{M}}$ to be an $\mathscr{O}_X$-module of finite type, it is necessary and sufficient that $\mathscr{M}$ is an $\mathscr{B}$-module of finite type.*

Proof. We can reduce to the case where $S=\operatorname{Spec}(A)$ is affine. Then $\mathscr{B}=\widetilde{B}$ where B is an A-algebra of finite type, and $\mathscr{M}=\widetilde{M}$ where M is an B-module. Over the scheme X, $\mathscr{O}_X$ is associated with the ring B and $\widetilde{\mathscr{M}}$ is associated with the B-module M. For $\widetilde{\mathscr{M}}$ to be of finite type, it is necessary and sufficient that M is of finite type, whence our claim. □

Proposition 5.1.21. *Let Y be an affine S-scheme and X,X' be two schemes affine over Y. Let $\mathscr{B}=\mathscr{A}(Y)$, $\mathscr{A}=\mathscr{A}(X)$, and $\mathscr{A}'=\mathscr{A}(X')$. Then $X\times_Y X'$ is affine over Y and $\mathscr{A}(X\times_Y X')$ is identified with $\mathscr{A}\otimes_{\mathscr{B}}\mathscr{A}'$.*

Proof. In fact, $\mathscr{A}\otimes_{\mathscr{B}}\mathscr{A}'$ is a quasi-coherent $\mathscr{B}$-algebra (Proposition 4.2.22), so is a quasi-coherent $\mathscr{O}_S$-algebra. Let Z be the spectral of $\mathscr{A}\otimes_{\mathscr{B}}\mathscr{A}'$. The canonical $\mathscr{B}$-homomorphisms $\mathscr{A}\to\mathscr{A}\otimes_{\mathscr{B}}\mathscr{A}'$ and $\mathscr{A}'\to\mathscr{A}\otimes_{\mathscr{B}}\mathscr{A}'$ corresponds to Y-morphisms $p:Z\to X$ and $p':Z\to X'$ (Proposition 5.1.6). To see that the triple (Z,p,p') is a product $X\times_Y X'$, we can reduce to the case $S=\operatorname{Spec}(C)$ is affine. But then Y,X,X' are all affine schemes with rings B,A,A', which are C-algebras such that $\mathscr{B}=\widetilde{B}$, $\mathscr{A}=\widetilde{A}$, $\mathscr{A}'=\widetilde{A}$. We then see that $\mathscr{A}\otimes_{\mathscr{B}}\mathscr{A}'$ is identified with the $\mathscr{O}_S$-algebra $\widetilde{A\otimes_B A'}$ (Proposition 4.1.9), so the ring of Z is identified with $A\otimes_B A'$, and the morphisms p,p' correspond to the canonical homomorphisms $A\to A\otimes_B A'$ and $A'\to A\otimes_B A'$. The proposition then follows from Proposition 4.3.1. □

Corollary 5.1.22. *Let $\mathscr{F}$ (resp. $\mathscr{F}'$) be a quasi-coherent $\mathscr{O}_X$-module (resp. $\mathscr{O}_{X'}$-module). Then $\mathscr{A}(\mathscr{F}\otimes_Y \mathscr{F}')$ is canonically identified with $\mathscr{A}(\mathscr{F})\otimes_{\mathscr{A}(Y)}\mathscr{A}(\mathscr{F}')$.*

Proof. The sheaf $\mathscr{F}\otimes_Y\mathscr{F}'$ is coherent over $X\times_Y X'$ by Proposition 1.4.1. Let $g: Y\to S$, $f: X\to Y$, $f': X'\to Y$ be the structural morphisms, so the structural morphism $h: Z\to S$ is equal to $g\circ f\circ p$ and to $g\circ f'\circ p'$. We define a canonical homomorphism

$$\mathscr{A}(\mathscr{F})\otimes_{\mathscr{A}(Y)}\mathscr{A}(\mathscr{F}')\to\mathscr{A}$$

by the following: for any open subset $U\subseteq S$, we have canonical homomorphisms

$$\Gamma(f^{-1}(g^{-1}(U)),\mathscr{F})\to\Gamma(h^{-1}(U),p^*(\mathscr{F})),\quad \Gamma(f'^{-1}(g^{-1}(U)),\mathscr{F}')\to\Gamma(h^{-1}(U),p'^*(\mathscr{F}')),$$

whence a canonical homomorphism

$$\begin{array}{c}\Gamma(f^{-1}(g^{-1}(U)),\mathscr{F})\otimes_{\Gamma(g^{-1}(U),\mathscr{O}_Y)}\Gamma(f'^{-1}(g^{-1}(U)),\mathscr{F}')\\ \big\downarrow\\ \Gamma(h^{-1}(U),p^*(\mathscr{F}))\otimes_{\Gamma(h^{-1}(U),\mathscr{O}_Z)}\Gamma(h^{-1}(U),p'^*(\mathscr{F}'))\end{array}$$

To see this is an isomorphism of $\mathscr{A}(Z)$-modules, we can assume that S is affine, and $\mathscr{F}=\widetilde{M}$, $\mathscr{F}'=\widetilde{M}'$, where M (resp. M') is an A-module (resp. A'-module). Then $\mathscr{F}\otimes_Y\mathscr{F}'$ is identified with the sheaf over $X\times_Y X'$ associated with the $(A\otimes_B A')$-module $M\otimes_B M'$ and the corollary follows from the canonical indentification $\widetilde{M\otimes_B M'}$ with $\widetilde{M}\otimes_{\widetilde{B}}\widetilde{M}'$. □

Corollary 5.1.23. *Let X and Y be affine S-schemes and $f: Y\to X$ be an S-morphism. Let $\mathscr{F}$ be a quasi-coherent $\mathscr{O}_X$-module, then $\mathscr{A}(f^*(\mathscr{F}))$ is identified with $\mathscr{A}(\mathscr{F})\otimes_{\mathscr{A}(X)}\mathscr{A}(Y)$.*

Proof. This is a special case of Proposition 5.1.22, by replacing X' with Y and Y with X. □

In particular, if $X=X'=Y$ (where X is an affine S-scheme), we see that if $\mathscr{F},\mathscr{G}$ are two quasi-coherent $\mathscr{O}_X$-modules, then

$$\mathscr{A}(\mathscr{F}\otimes_{\mathscr{O}_X}\mathscr{G})=\mathscr{A}(\mathscr{F})\otimes_{\mathscr{A}(X)}\mathscr{A}(\mathscr{G})$$

If moreover $\mathscr{F}$ is of finite presentation, then it follows from Proposition 4.1.11 and Proposition 4.1.9 that

$$\mathscr{A}(\mathscr{H}om_{\mathscr{O}_X}(\mathscr{F},\mathscr{G}))=\mathscr{H}om_{\mathscr{A}(X)}(\mathscr{A}(\mathscr{F}),\mathscr{A}(\mathscr{G})).$$

Proposition 5.1.24. *If X and X' are two affine S-schemes with $\mathscr{B}=\mathscr{A}(X)$ and $\mathscr{B}'=\mathscr{A}(X')$. Then the coproduct $X\amalg X'$ is affine over S with $\mathscr{A}(X\amalg X')=\mathscr{B}\times\mathscr{B}'$.*

Proof. The coproduct is affine over S since the product of two affine schemes is affine, and the second assertion also follows from this, and the fact that $\mathrm{Spec}(A)\amalg\mathrm{Spec}(A')=\mathrm{Spec}(A\times A')$ for two rings A, A'. □

Proposition 5.1.25. *Let S be a scheme, $\mathscr{B}$ be a quasi-coherent $\mathscr{O}_S$-algebra, and $X=\mathrm{Spec}(\mathscr{B})$. For any quasi-coherent ideal $\mathscr{I}$ of $\mathscr{B}$, $\widetilde{\mathscr{I}}$ is a quasi-coherent ideal of $\mathscr{O}_X$, and the closed subschem of X defined by $\widetilde{\mathscr{I}}$ is canonically isomorphic to* $\mathrm{Spec}(\mathscr{B}/\mathscr{I})$.

Proof. In fact, it follows from Proposition 4.4.6 that Y is affine over S, and in view of Proposition 5.1.8, we can then assume that S is affine, and the proposition follows from the corresponding result in affine schemes. □

We can also express the result of Proposition 5.1.25 by saying that if $h : \mathscr{B} \to \mathscr{B}'$ is a surjective homomorphism of quasi-coherent $\mathscr{O}_S$-algebras, then $\mathscr{A}(h) : \mathrm{Spec}(\mathscr{B}') \to \mathrm{Spec}(\mathscr{B})$ is a closed immersion.

Proposition 5.1.26. *Let S be a scheme, $\mathscr{B}$ be a quasi-coherent $\mathscr{O}_S$-algebra, and $X = \mathrm{Spec}(\mathscr{B})$. For any quasi-coherent ideal $\mathscr{K}$ of $\mathscr{O}_S$, we have (where $f : X \to S$ is the structural morphism) $f^*(\mathscr{K})\mathscr{O}_X = \widetilde{\mathscr{K}\mathscr{B}}$.*

Proof. The question is local over S, so we can assume that $S = \mathrm{Spec}(A)$ is affine, and the proposition then follows Proposition 4.1.13. □

5.1.4 Base change of affine S-schemes

Proposition 5.1.27. *Let X be an affine S-scheme. For any extension $g : S' \to S$ of base scheme, $X' = X_{(S')}$ is affine over S'.*

Proof. If $f' : X' \to S'$ is the projection, it suffices to prove that $f'^{-1}(U')$ is an affine open for any affine open subset U' of S' such that $g(U')$ is contained in an affine open U of S. We can then assume that S and S' are affine, so X is affine. But then X' is affine, so the claim follows. □

Corollary 5.1.28. *Let $f : X \to S$ be the structural morphism, $f' : X' \to S'$, $g' : X' \to X$ the projections such that the diagram*

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$

is commutative. For any quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, there exists a canonical isomorphism $u : g^*(f_*(\mathscr{F})) \to f'_*(g'^*(\mathscr{F}))$ of $\mathscr{O}_{S'}$-modules. In particular, there exists a canonical isomorphism from $\mathscr{A}(X')$ to $g^*(\mathscr{A}(X))$.

Proof. To define u, it suffices to define a homomorphism

$$v : f_*(\mathscr{F}) \to g_*(f'^*(\mathscr{F})) = f_*(g'_*(g'^*(\mathscr{F})))$$

and let u be the homomorphism corresponding to v (via the adjointness). We set $v = f_*(\rho)$, where $\rho : \mathscr{F} \to g'_*(g'^*(\mathscr{F}))$ is the canonical homomorphism. To prove that u is an isomorphism, we can assume that S and S', hence X and X', are affine. Let A, A', B, B' be the ring of X, X', S, S', then $\mathscr{F} = \widetilde{M}$ where M is an B-module. We then see that $g^*(f_*(\mathscr{F}))$ and $f'_*(g'^*(\mathscr{F}))$ are equal to the $\mathscr{O}_{S'}$-module associated with the A'-module $A' \otimes_A M$, and u is the homomorphism associated with the identity. □

Corollary 5.1.29. *For any affine S-scheme X and $s \in S$, the fiber X_s is an affine scheme.*

Proof. It suffices to apply Proposition 5.1.27 on $\mathrm{Spec}(\kappa(s)) \to S$. □

Corollary 5.1.30. *Let X be an S-scheme and S' be an affine S-scheme. Then $X' = X_{(S')}$ is affine over X. Moreover, if $f : X \to S$ is the structural homomorphism, there exists a canonical isomorphism of $\mathscr{O}_X$-algebras $\mathscr{A}(X') \cong f^*(\mathscr{A}(S'))$, and for any quasi-coherent $\mathscr{A}(S')$-module $\mathscr{M}$, a canonical bi-isomorphism $f^*(\mathscr{M}) \cong \mathscr{A}(f'^*(\widetilde{\mathscr{M}}))$, where $f' = f_{(S')}$ is the structural morphism $X' \to S'$.*

Proof. It suffices to apply Proposition 5.1.27 and Corollary 5.1.28, with the role of X and S' exchanged. □

Let S, S' be two schemes, $q : S' \to S$ be a morphism, $\mathscr{B}$ (resp. $\mathscr{B}'$) be a quasi-coherent $\mathscr{O}_S$-algebra (resp. $\mathscr{O}_{S'}$-algebra), and $u : \mathscr{B} \to \mathscr{B}'$ be a q-morphism (which means a homomorphism $\mathscr{B} \to q_*(\mathscr{B}')$ of $\mathscr{O}_S$-algebras). If $X = \mathrm{Spec}(\mathscr{B})$ and $X' = \mathrm{Spec}(\mathscr{B}')$, we deduce a canonical morphism $v = \mathrm{Spec}(u) : X' \to X$ such that the diagram

$$\begin{tikzcd} X' \arrow[r, "v"] \arrow[d] & X \arrow[d] \\ S' \arrow[r, "q"] & S \end{tikzcd} \tag{1.4.1}$$

is commutatie. In fact, the homomorphism u corresponds to a homomorphism $u^\sharp : q^*(\mathscr{B}) \to \mathscr{B}'$ by adjointness, and there then exists a canonical S'-morphism

$$w : \mathrm{Spec}(\mathscr{B}') \to \mathrm{Spec}(q^*(\mathscr{B}))$$

such that $\mathscr{A}(w) = u^\sharp$ (Proposition 5.1.6). On the other hand, it follows from Corollary 5.1.28 that $\mathrm{Spec}(q^*(\mathscr{B}))$ is canonically identified with $X \times_S S'$; the morphism v is the defined to be the composition

$$X' \xrightarrow{w} X \times_S S' \xrightarrow{p_1} X$$

where p_1 is the projection, and the commutativity of (1.4.1) is easily verified. Let U (resp. U') be an affine open of S (resp. S') such that $q(U') \subseteq U$, $A = \Gamma(U, \mathscr{O}_S)$, $A' = \Gamma(U', \mathscr{O}_{S'})$, $B = \Gamma(U, \mathscr{B})$, $B' = \Gamma(U'\mathscr{B}')$. The restriction of u is a $(q|_{U'})$-morphism $u|_U : \mathscr{B}|_U \to \mathscr{B}'|_{U'}$ corresponding to a bi-homomorphism $B \to B'$ of algebras. If V, V' are the inverse images of U, U' in X, X', respectively, the morphism $V' \to V$, which is the restriction of v, corresponds to the preceding bi-homomorphism.

Now let $\mathscr{M}$ be a quasi-coherent $\mathscr{B}$-module. There then exists a canonical isomorphism of $\mathscr{O}_{X'}$-modules

$$v^*(\widetilde{\mathscr{M}}) \cong (q^*(\mathscr{M}) \otimes_{q^*(\mathscr{B})} \mathscr{B}')^{\sim}. \tag{1.4.2}$$

In fact, the canonical isomorphism of Corollary 5.1.28 provides a canonical isomorphism of $p_1^*(\widetilde{\mathscr{M}})$ with the sheaf over $\mathrm{Spec}(q^*(\mathscr{B}))$ associated with $q^*(\mathscr{B})$-module $q^*(\mathscr{M})$, and it suffices to apply Corollary 5.1.22.

Recall that we say a morphism $f : X \to Y$ is affine if X is an affine scheme over Y. The properties of affine S-schemes then translate into properties of affine morphisms.

Proposition 5.1.31 (Properties of Affine Morphisms).

(i) A closed immersion is affine.

(ii) The composition of two affine morphisms is affine.

(iii) If $f : X \to Y$ is an affine S-morphism, then $f_{(S')}$ is affine for any base change $S' \to S$.

(iv) If $f : X \to Y$ and $f' : X' \to Y'$ are two affine S-morphisms, then $f \times_S f'$ is affine.

(v) If $f : X \to Y$ and $g : Y \to Z$ are two morphisms such that $g \circ f$ is affine and g is separated, then f is affine.

(vi) If f is affine, so is f_{red}.

Proof. In view of Proposition 4.5.22, it suffices to prove (i), (ii), and (iii). Now (i) follows from Proposition 4.4.6, (ii) follows from Corollary 5.1.4, and (iii) follows from Proposition 5.1.27. □

Corollary 5.1.32. *If X is an affine scheme and Y is a separated scheme, any morphism $f : X \to Y$ is affine.*

Proof. This is a direct concequence of Proposition 5.1.31(v). □

Proposition 5.1.33. *Let Y be a locally Noetherian scheme and $f : X \to Y$ be a morphism of finite type. Then for f to be affine, it is necessary and sufficient that f_{red} is affine.*

Proof. It suffcies to prove that f is affine if f_{red} is affine. For this, we can assume that Y is affine and Noetherian, and show that X is affine. Now Y_{red} is affine, and X_{red} is therefore affine by hypothesis. Since X is Noetherian, the assertion follows from Proposition 4.4.34. □

5.1.5 Vector bundles

Let A be a ring and E be an A-module. Recall that the symmetric algebra over E is the A-algebra $\boldsymbol{S}(E)$ (or $\boldsymbol{S}_A(E)$) which is the quotient of $\boldsymbol{T}(E)$ by the ideal generated by elements $x \otimes y - y \otimes x$, where x, y belongs to E. The algebra $\boldsymbol{S}(E)$ is characterized by the universal property that if $\sigma : E \to \boldsymbol{S}(E)$ is the canonical map, any A-linear map $E \to B$, where B is a commutative algebra, factors through $\boldsymbol{S}(E)$ and gives a homomorphism $\boldsymbol{S}(E) \to B$ of A-algebras. We deduce from this property that for two A-modules E, F, we have

$$\boldsymbol{S}(E \oplus F) = \boldsymbol{S}(E) \otimes \boldsymbol{S}(F).$$

Moreover, $\boldsymbol{S}(E)$ is a covariant functor on E from the category of A-modules to that of commutative A-algebras. Finally, the preceding characterization shows that if $E = \varinjlim E_\lambda$, then $\boldsymbol{S}(E) = \varinjlim \boldsymbol{S}(E_\lambda)$. By abuse of language, a product $\sigma(x_1) \cdots \sigma(x_n)$, where $x_i \in E$, is usually written as $x_1 \cdots x_n$. The algebra $\boldsymbol{S}(E)$ is graded, where $\boldsymbol{S}_n(E)$ is the set of linear combinations of products of n elements of E. In particular, the algebra $\boldsymbol{S}(A)$ is canonically isomorphic to the polynomial algebra $A[T]$ over A, and the algebra $\boldsymbol{S}(A^n)$ is the polynomial algebra $A[T_1, \dots, T_n]$ over A. More particularly, if E is free of rank 1, then $\boldsymbol{S}_n(E)$ is isomorphic to the tensor algebra $\boldsymbol{T}(E) = \bigoplus_{n \geq 0} E^{\otimes n}$.

Let $\varphi : A \to B$ be a ring homomorphism. If F is an B-module, the canonical map $F \to \boldsymbol{S}(F)$ then gives a canonical map $F_{(\varphi)} \to \boldsymbol{S}(F)_{(\varphi)}$, which factors into $F_{(\varphi)} \to \boldsymbol{S}(F_{(\varphi)}) \to \boldsymbol{S}(F)_{(\varphi)}$. The canonical homomorphism $\boldsymbol{S}(F_{(\varphi)}) \to \boldsymbol{S}(F)_{(\varphi)}$ is surjective, but not necessarily bijective.

If E is an A-module, any bi-homomorphism $E \to F$ (which is an A-homomorphism $E \to F_{(\varphi)}$) then gives an A-homomorphism $\boldsymbol{S}(E) \to \boldsymbol{S}(F_{(\varphi)}) \to \boldsymbol{S}(F)_{(\varphi)}$ of algebras, which is a bi-homomorphism $\boldsymbol{S}(E) \to \boldsymbol{S}(F)$. Also, for any A-module E, $\boldsymbol{S}(E \otimes_A B)$ is canonically identified with the algebra $\boldsymbol{S}(E) \otimes_A B$, which follows from the universal property of $\boldsymbol{S}(E)$.

Let R be a multiplicative subset of A. Then apply the previous arguments for the ring $B = R^{-1}A$, and recall that $R^{-1}E = E \otimes_A R^{-1}A$, we see that $\boldsymbol{S}(R^{-1}E) = R^{-1}\boldsymbol{S}(E)$. Moreover, if $R' \supseteq R$ is another multiplicative subset of A, the diagram

$$\begin{tikzcd} R^{-1}E \arrow[r] \arrow[d] & R'^{-1}E \arrow[d] \\ \boldsymbol{S}(R^{-1}E) \arrow[r] & \boldsymbol{S}(R'^{-1}E) \end{tikzcd}$$

is commutative.

Now let $(S, \mathscr{A})$ be a ringed space and $\mathscr{E}$ be an $\mathscr{A}$-module over S. If for each open subset $U \subseteq S$, we associate the $\Gamma(U, \mathscr{A})$-module $\boldsymbol{S}(\Gamma(U, \mathscr{E}))$, we then define a presheaf of algebras. The associated sheaf is called the **symmetric $\mathscr{A}$-algebra** of the $\mathscr{A}$-module $\mathscr{E}$ and denoted by $\boldsymbol{S}(\mathscr{E})$. It follows immediately that $\boldsymbol{S}(\mathscr{E})$ satisfies the following universal property: any homomorphism $\mathscr{E} \to \mathscr{B}$ of $\mathscr{A}$-modules, where $\mathscr{B}$ is an $\mathscr{A}$-algebra, factors through $\boldsymbol{S}(\mathscr{E})$ to give a homomorphism $\boldsymbol{S}(\mathscr{E}) \to \mathscr{B}$ of $\mathscr{A}$-algebras. In particular, any homomorphism $u : \mathscr{E} \to \mathscr{F}$ of $\mathscr{A}$-modules defines a homomorphism $\boldsymbol{S}(u) : \boldsymbol{S}(\mathscr{E}) \to \boldsymbol{S}(\mathscr{F})$ of $\mathscr{A}$-algebras and $\boldsymbol{S}(\mathscr{E})$ is a covariant functor $\mathscr{E}$.

Now since the functor $\boldsymbol{S}$ commutes with inductive limits, we have $\boldsymbol{S}(\mathscr{E})_s = \boldsymbol{S}(\mathscr{E}_s)$ for any point $s \in S$. If $\mathscr{E}, \mathscr{F}$ are two $\mathscr{A}$-module, $\boldsymbol{S}(\mathscr{E} \oplus \mathscr{F})$ is canonically identified with $\boldsymbol{S}(\mathscr{E}) \otimes_{\mathscr{A}} \boldsymbol{S}(\mathscr{F})$, as we can check this for the corresponding presheaves.

We see that $\boldsymbol{S}(\mathscr{E})$ is a graded $\mathscr{A}$-algebra, and $\boldsymbol{S}_n(\mathscr{E})$ is the $\mathscr{A}$-module associated with the presheaf $U \mapsto \boldsymbol{S}_n(\Gamma(U, \mathscr{E}))$. In particular, if $\mathscr{E} = \mathscr{A}$, then $\boldsymbol{S}(\mathscr{A})$ is identified with $\mathscr{A}[T] = \mathscr{A} \otimes_{\mathbb{Z}} \mathbb{Z}[T]$, and if $\mathscr{E}$ is an invertible sheaf, then $\boldsymbol{S}(\mathscr{E})$ is isomorphic to the tensor algebra $\boldsymbol{T}(\mathscr{E}) = \bigoplus_{n \geq 0} \mathscr{E}^{\otimes n}$.

Proposition 5.1.34. *Let $f : (S, \mathscr{A}) \to (T, \mathscr{B})$ be a morphism of ringed spaces. If $\mathscr{F}$ is an $\mathscr{B}$-module, then $\boldsymbol{S}(f^*(\mathscr{F}))$ is canonically identified with $f^*(\boldsymbol{S}(\mathscr{F}))$*

Proof. To see this, we may make use the universal property of $\boldsymbol{S}$. By definition, $\boldsymbol{S}(f^*(\mathscr{F}))$ is defined to be the unique $\mathscr{A}$-algebra satisfying the following equality

$$\mathrm{Hom}_{\mathscr{A}-\mathbf{alg}}(\boldsymbol{S}(f^*(\mathscr{F})), \mathscr{C}) = \mathrm{Hom}_{\mathscr{A}}(f^*(\mathscr{F}), \mathscr{C})$$

for any $\mathscr{A}$-algebra $\mathscr{C}$. On the other hand, by the adjointness property of f_* and f^*, we have

$$\begin{aligned} \mathrm{Hom}_{\mathscr{A}-\mathbf{alg}}(f^*(\boldsymbol{S}(\mathscr{F})), \mathscr{C}) &= \mathrm{Hom}_{\mathscr{B}-\mathbf{alg}}(\boldsymbol{S}(\mathscr{F}), f_*(\mathscr{C})) \\ &= \mathrm{Hom}_{\mathscr{B}}(\mathscr{F}, f_*(\mathscr{C})) = \mathrm{Hom}_{\mathscr{A}}(f^*(\mathscr{F}), \mathscr{C}). \end{aligned}$$

This implies the desired isomorphism. □

Proposition 5.1.35. *Let A be a ring, $S = \mathrm{Spec}(A)$ be the spectrum, and $\mathscr{E} = \widetilde{M}$ be the $\mathscr{O}_S$-module assocaited with an A-module M. Then the $\mathscr{O}_S$-algebra $\boldsymbol{S}(\mathscr{E})$ is associated with the A-algebra $\boldsymbol{S}(M)$.*

Proof. In fact, for any $f \in A$, $\boldsymbol{S}(M_f) = \boldsymbol{S}(M)_f$, so the proposition follows from the definition of $\widetilde{\boldsymbol{S}(M)}$. ◻

Corollary 5.1.36. *If S is a scheme and $\mathscr{E}$ is a quasi-coherent $\mathscr{O}_S$-module. Then the $\mathscr{O}_S$-algebra $\boldsymbol{S}(\mathscr{E})$ is quasi-coherent. If moreover $\mathscr{E}$ is of finite type (resp. of finite presentation), then each $\mathscr{O}_S$-module $\boldsymbol{S}_n(\mathscr{E})$ is of finite type (resp. finite presentation) and the $\mathscr{O}_S$-algebra $\boldsymbol{S}(\mathscr{E})$ is of finite type (resp. of finite presentation).*

Proof. The first assertion is immediate by Proposition 5.1.35. The second one follows from the fact that, if E is a finitely generated A-module, $\boldsymbol{S}_n(E)$ is also finitely generated. For the last assertion, we are reduced to the case $S = \operatorname{Spec}(A)$ and $\mathscr{E} = \widetilde{E}$ where E is an A-module of finite type (resp. of finite presentation). Now if we have an exact sequence

$$0 \longrightarrow N \longrightarrow A^n \longrightarrow E \longrightarrow 0$$

then we deduce an exact sequence

$$0 \longrightarrow \mathfrak{I} \longrightarrow \boldsymbol{S}(A^n) \longrightarrow \boldsymbol{S}(E) \longrightarrow 0$$

where $\mathfrak{I}$ is the ideal of $\boldsymbol{S}(A^n)$ generated by $N \subseteq \boldsymbol{S}_1(A^n)$, whence our conclusion. ◻

Let S be a scheme and $\mathscr{E}$ be a quasi-coherent $\mathscr{O}_S$-module. For any S-scheme T, with structural morphism $f : T \to S$, let $\mathscr{E}_{(T)} = f^*(\mathscr{E})$, which is a quasi-coherent $\mathscr{O}_T$-module. The map

$$T \mapsto F_{\mathscr{E}}(T) = \operatorname{Hom}_{\mathscr{O}_T}(\mathscr{E}_{(T)}, \mathscr{O}_T) = \Gamma(T, \mathscr{E}^*_{(T)})$$

then defines a contravariant functor from the category of S-schemes to that of sets if for any S-morphism $g : T' \to T$ we define $F_{\mathscr{E}}(g) : F_{\mathscr{E}}(T) \to F_{\mathscr{E}}(T')$ to be the map $g^* : u \mapsto g^*(u)$ (note that the structural morphism $T' \to S$ is $f \circ g$ and we have $\mathscr{E}_{(T')} = g^*(\mathscr{E}_{(T)})$ and $\mathscr{O}_{T'} = g^*(\mathscr{O}_T)$).

Proposition 5.1.37. *For any quasi-coherent $\mathscr{O}_S$-module $\mathscr{E}$, the contravariant functor $F_{\mathscr{E}}$ is represented by the couple formed by the affine S-scheme $\mathbb{V}(\mathscr{E}) = \operatorname{Spec}(\boldsymbol{S}(\mathscr{E}))$. The S-scheme $\mathbb{V}(\mathscr{E})$ is called the* ***vector bundle over S defined by*** *$\mathscr{E}$.*

Proof. This follows from the following canonical isomorphisms for any S-scheme T:

$$\begin{aligned}\operatorname{Hom}_S(T, \mathbb{V}(\mathscr{E})) &= \operatorname{Hom}_{\mathscr{O}_S-\mathbf{alg}}(\boldsymbol{S}(\mathscr{E}), \mathscr{A}(T)) = \operatorname{Hom}_{\mathscr{O}_S}(\mathscr{E}, f_*(\mathscr{O}_T)) \\ &= \operatorname{Hom}_{\mathscr{O}_T}(f^*(\mathscr{E}), \mathscr{O}_T) = \operatorname{Hom}_{\mathscr{O}_T}(\mathscr{E}_{(T)}, \mathscr{O}_T).\end{aligned}$$ ◻

The canonical $\boldsymbol{S}(\mathscr{E})$-homomorphism $\mathscr{E} \otimes_{\mathscr{O}_S} \boldsymbol{S}(\mathscr{E}) \to \boldsymbol{S}(\mathscr{E})$ induced by Proposition 5.1.6 an $\mathscr{O}_{\mathbb{V}(\mathscr{E})}$-homomorphism $\mathscr{E}_{(\mathbb{V}(\mathscr{E}))} \to \mathscr{O}_{\mathbb{V}(\mathscr{E})}$, which is a section over $\mathbb{V}(\mathscr{E})$ of dual sheaf $\mathscr{E}^*_{(\mathbb{V}(\mathscr{E}))}$ of $\mathscr{E}_{(V(\mathscr{E}))}$, called the **universal section** of this dual. If $U = \operatorname{Spec}(A)$ is an affine open of S, its inverse image in $\mathbb{V}(\mathscr{E})$ is identified with $\operatorname{Spec}(\boldsymbol{S}(M))$, if $\mathscr{E}|_U = \widetilde{M}$ where M is an A-module. Over the scheme $\operatorname{Spec}(\boldsymbol{S}(M))$, the universal section is identified with the homomorphism $m \otimes p \mapsto mp$ of $M \otimes_A \boldsymbol{S}(M)$ to $\boldsymbol{S}(M)$, where M is identified with the subset $\boldsymbol{S}_1(M)$ of $\boldsymbol{S}(M)$.

Consider in particular an open subset U of S. Then the S-morphisms $U \to \mathbb{V}(\mathscr{E})$ are the U-sections of the U-scheme induced by $\mathbb{V}(\mathscr{E})$ over $p^{-1}(U)$ (where $p : \mathbb{V}(\mathscr{E}) \to S$ is the structural morphism). By the definition of $\mathbb{V}(\mathscr{E})$, these U-sections correspond bijectively to sections of the dual $\mathscr{E}^*$ of $\mathscr{E}$ over U. The functorial of $\mathbb{V}$ shows that this interpretation is compatible with the restriction to an open subset $U' \subseteq U$, so we can say that the dual $\mathscr{E}^*$ of $\mathscr{E}$ is canonically identified with the sheaf of germs of S-sections of $\mathbb{V}(\mathscr{E})$. In particular, if $T = S$, the zero homomorphism $\mathscr{E} \to \mathscr{O}_S$ corresponds to an S-section of $\mathbb{V}(\mathscr{E})$, called the **zero section**.

If now we choose T to be the spectrum $\{\xi\}$ of a field K, the structural morphism $f : T \to S$ corresponds to a monomorphism $\kappa(s) \to K$, where $s = f(\xi)$ (Corollary 4.2.15), and the S-morphisms $\{\xi\} \to \mathbb{V}(\mathscr{E})$ are none other than points of $\mathbb{V}(\mathscr{E})$ with values in the extension K of $\kappa(s)$, which all locate at some points of $p^{-1}(s)$. The set of these points, which is called the **rational fiber** of $\mathbb{V}(\mathscr{E})$ over K lying over the point s, is then identified (by the definition of $\mathbb{V}(\mathscr{E})$) with the dual of the K-vector space $\mathscr{E}_s \otimes_{\mathscr{O}_{S,s}} K = \mathscr{E}^s \otimes_{\kappa(s)} K$ where we set $\mathscr{E}^s = \mathscr{E}_s/\mathfrak{m}_s\mathscr{E}_s$. If $\mathscr{E}^s$ and K are of finite rank over $\kappa(s)$, this dual is identified with $(\mathscr{E}^s)^* \otimes_{\kappa(s)} K$, where $(\mathscr{E}^s)^*$ is the dual space of the $\kappa(s)$-vector space $\mathscr{E}^s$.

These properties justify the terminology of "vector bundle" introduced above, but note that the definition we obtained is dual to the classical definition, since one would expect to obtain the space $\mathscr{E}^s \otimes_{\kappa(s)} K$ for the fiber of $\mathbb{V}(\mathscr{E})$, rather than its dual. This distinction is imposed for the need of defining $\mathbb{V}(\mathscr{E})$ for any quasi-coherent $\mathscr{O}_S$-module $\mathscr{E}$, not only for locally free $\mathscr{O}_S$-modules of finite rank. We can indeed show that the functor $T \mapsto \Gamma(T, \mathscr{E}_{(T)})$ is only representable if $\mathscr{E}$ is locally free of finite rank.

Proposition 5.1.38. *Let S be a scheme.*

- *(i) $\mathbb{V}$ is a contravariant functor on $\mathscr{E}$ from the category of quasi-coherent $\mathscr{O}_S$-modules to the category of affine S-schemes.*
- *(ii) If $\mathscr{E}$ is of finite type (resp. of finite presentation), $\mathbb{V}(\mathscr{E})$ is of finite type (resp. of finite presentation) over S.*
- *(iii) If $\mathscr{E}$ and $\mathscr{F}$ are two quasi-coherent $\mathscr{O}_S$-modules, $\mathbb{V}(\mathscr{E} \oplus \mathscr{F})$ is canonically identified with $\mathbb{V}(\mathscr{E}) \times_S \mathbb{V}(\mathscr{F})$.*
- *(iv) Let $g : S' \to S$ be a morphism. For any quasi-coherent $\mathscr{O}_S$-module $\mathscr{E}$, $\mathbb{V}(g^*(\mathscr{E}))$ is canonically identified with $\mathbb{V}(\mathscr{E})_{(S')} = \mathbb{V}(\mathscr{E}) \times_S S'$.*
- *(v) A surjective homomorphism $\mathscr{E} \to \mathscr{F}$ of quasi-coherent $\mathscr{O}_S$-modules corresponds to a closed immersion $\mathbb{V}(\mathscr{F}) \to \mathbb{V}(\mathscr{E})$.*

Proof. Assertion (i) follows from Proposition 5.1.6, since for any homomorphism $\mathscr{E} \to \mathscr{F}$ of $\mathscr{O}_S$-modules we have a homomorphism $\boldsymbol{S}(\mathscr{E}) \to \boldsymbol{S}(\mathscr{F})$ of $\mathscr{O}_S$-algebras. Assertion (ii) follows immediately from Corollary 5.1.20 and Corollary 5.1.36. To prove (iii), it suffices to recall the canonical isomorphism $\boldsymbol{S}(\mathscr{E} \oplus \mathscr{F}) \cong \boldsymbol{S}(\mathscr{E}) \otimes_{\mathscr{O}_S} \boldsymbol{S}(\mathscr{F})$ and apply Proposition 5.1.21. Similarly, to prove (iv), it suffices to remark that if the homomorphism $\mathscr{E} \to \mathscr{F}$ is surjective, so is the corresponding homomorphism $\boldsymbol{S}(\mathscr{E}) \to \boldsymbol{S}(\mathscr{F})$ of $\mathscr{O}_S$-algebras, and apply Proposition 5.1.25. □

Example 5.1.39. Consider in particular $\mathscr{E} = \mathscr{O}_S$. The scheme $\mathbb{V}(\mathscr{O}_S)$ is the spectrum of the $\mathscr{O}_S$-algebra $\boldsymbol{S}(\mathscr{O}_S)$, which is identified with $\mathscr{O}_S[T] = \mathscr{O}_S \otimes_{\mathbb{Z}} \mathbb{Z}[T]$. This is evident if $S = \operatorname{Spec}(\mathbb{Z})$, in view of Proposition 5.1.35, and we pass from this to the general case by considering the structural morphism $S \to \mathrm{Spec}(\mathbb{Z})$ and using Proposition 5.1.38(iv). Because of this result, we again set $\mathbb{V}(\mathscr{O}_S) = S[T]$, and we obtain the identification of the sheaf of germs of S-sections of $S[T]$ over $\mathscr{O}_S$ as a particular case.

For any S-scheme X, by the definition of $\mathbb{V}(\mathscr{O}_S)$, the set $\mathrm{Hom}_S(X, S[T])$ is canonically identified with $\mathrm{Hom}_{\mathscr{O}_S}(\mathscr{O}_S, \mathscr{A}(X))$, which is canonically isomorphic to $\Gamma(S, \mathscr{A}(X))$ and therefore has a ring structure. Moreover, any S-morphism $h : X \to Y$ corresponds to a homomorphism $\Gamma(\mathscr{A}(h)) : \Gamma(S, \mathscr{A}(Y)) \to \Gamma(S, \mathscr{A}(X))$, so we obtain a contravariant functor $\mathrm{Hom}_S(X, S[T])$ from the category of S-schemes to the cateogry of rings. On the other hand, $\mathrm{Hom}_S(X, \mathbb{V}(\mathscr{E}))$ is identified similarly with $\mathrm{Hom}_{\mathscr{O}_S}(\mathscr{E}, \mathscr{A}(X))$ (where $\mathscr{A}(X)$ is considered as an $\mathscr{O}_S$-module); we can then endow this set a $\mathrm{Hom}_S(X, S[T])$-module structure, and the couple

$$(\mathrm{Hom}_S(X, S[T]), \mathrm{Hom}_S(X, \mathbb{V}(\mathscr{E})))$$

is a contravariant functor on X with values in the category of couples (A, M) formed by a ring A and an A-module M, with morphisms being the bi-homomorphisms. In view of this, we say that $S[T]$ is the **S-scheme of ring** and $\mathbb{V}(\mathscr{E})$ is the **S-scheme of module** over the S-scheme of ring $S[T]$.

5.2 Homogeneous specturm of graded algebras

Let S be a graded ring and S_+ be the irrelevant ideal. We say a subset $\mathfrak{I}$ of S_+ is an **ideal of** S_+ if it is an ideal of S, and it is called a **graded prime ideal of** S_+ if it is the intersection with S_+ of a graded prime ideal of S not containing S_+ (in particular $\mathfrak{I} \neq S_+$, and this graded prime ideal of S is unique by Proposition **??**). If $\mathfrak{I}$ is an ideal of S_+, the radical of $\mathfrak{I}$ in S_+, denoted by $\mathfrak{r}_+(\mathfrak{I})$, is the set of elements of S_+ which have some power contained in $\mathfrak{I}$, or equivalently, $\mathfrak{r}_+(\mathfrak{I}) = \sqrt{\mathfrak{I}} \cap S_+$. In particularm the radical of 0 in S_+ is called the **nilradical** of S_+ and denoted by $\mathfrak{n}_+$: this is the subset of nilpotent elements of S_+. If $\mathfrak{I}$ is a graded ideal of S_+, its radical $\mathfrak{r}_+(\mathfrak{I})$ is also graded: by passing to $S/\mathfrak{I}$, we may assume that $\mathfrak{I} = 0$, and note that if $x = x_h + x_{h+1} + \cdots + x_k$ is nilpotent, then so is each $x_i \in S_i$; we can suppose that $x_k = 0$ and the top degree component of x^n is then x_k^n, so x_k is nilpotent, and we then preceed by induction on k. We say the graded ring S is essentially reduced if $\mathfrak{n}_+ = 0$, which means S_+ contains no nonzero nilpotent elements.

We note that in a graded ring S, if an element x is a zero divisor, so is its homogeneous component of top degree. We then say that the ring S is **essentially integral** if the ring S_+ (with the unit element) does not contain nonzero zero divisors; it suffices for this that a nonzero homogeneous element in S_+ is not divisor of 0 in this ring. It is clear that if $\mathfrak{p}$ is a graded prime ideal of S_+, $S/\mathfrak{p}$ is essentially integral. Let S be an essentially integral graded ring, and let $x_0 \in S_0$. It there exists a homogeneous element $f \neq 0$ in S_+ such that $x_0 f = 0$, we then have $x_0 S_+ = 0$, because $(x_0 g)f = (x_0 f)g = 0$ for any $g \in S_+$, and the hypothesis on S implies that $x_0 g = 0$. Therfore, for that S is integral, it is necessary and sufficient that S_0 is integral and the annihilator of S_+ in S_0 reduces to zero.

5.2.1 Localization of graded rings

Let S be a graded ring with positive degrees, f be a homogeneous element of S of degree $d > 0$. Then the fraction ring S_f is graded, where $(S_f)_n$ is the set of elements x/f^k, where $x \in S_{n+kd}$ with $k \geq 0$ (note that n can be an arbitrary integer). We denote by $S_{(f)}$ the subring $(S_f)_0$ of S_f formed by elements of degree 0.

If $f \in S_d$, the monomials $(f/1)^h$ in S_f (where h is an integer) form a lineraly independent system over the ring $S_{(f)}$, and the set of their linear combinations over $S_{(f)}$ is exactly the ring $(S^{(d)})_f$ (recall that $S^{(d)}$ is the direct sum of S_{nd}), and then we get an isomorphism

$$(S^{(d)})_f \cong S_{(f)}[T, T^{-1}] = S_{(f)} \otimes_{\mathbb{Z}} \mathbb{Z}[T, T^{-1}] \tag{2.1.1}$$

(where T is an indeterminate). In fact, if we have a relatin

$$\sum_{h=-a}^{a} z_h (f/1)^h = 0$$

where $z_h = x_h/f^m \in S_{(f)}$, then there exists an integer $k > -a$ such that

$$\sum_{h=-a}^{b} f^{h+k} x_h = 0,$$

and as the degrees of these terms are distinct, we have $f^{h+k}x_h = 0$ for all h, so $z_h = 0$ for all h. Similarly, if M is a graded S-module, the localization M_f is a graded S_f-module with $(M_f)_n$ being the set of elements z/f^k where $z \in M_{n+kd}$. We denote by $M_{(f)}$ the set of elements of degree 0 in M_f. It is immediate that $M_{(f)}$ is an $S_{(f)}$-module and we have $(M^{(d)})_f = M_{(f)} \otimes_{S_{(f)}} (S^{(d)})_f$.

Lemma 5.2.1. *Let $f \in S_d$ and $g \in S_e$ be two homogeneous elements of S with positive degrees. Then there exists a canonical isomorphism*

$$S_{(fg)} \cong (S_{(f)})_{g^d/f^e}.$$

If we identify these two rings, then for any S-module M, we have a canonical isomorphism

$$M_{(fg)} \cong (M_{(f)})_{g^d/f^e}.$$

Proof. Note that (fg) divides $f^e g^d$ and $f^e g^d$ divides $(fg)^{de}$, so the rings S_{fg} and $S_{f^e g^d}$ are canonically identified. On the other hand, $S_{f^e g^d}$ is also identified with $(S_{f^e})_{g^d/1}$, and as $f^e/1$ is invertible in S_{f^e}, $s_{f^e g^d}$ is also identified with $(S_{f^e})_{g^d/f^e}$. Now the element g^e/f^e is of degree zero in S_{f^e}, so we can conclude that the subring of $(S_{f^e})_{g^d/f^e}$ formed by elements of degree zero is $(S_{(f^e)})_{g^d/f^e}$, and as we have $S_{(f^e)} = S_{(f)}$, we see the assertion follows. ◻

With the hypotheses of Lemma 5.2.1, it is clear that the canonical homomorphism $S_f \to S_{fg}$, which maps x/f^k to $xg^k/(fg)^k$, is of degree 0 so restricts to a canonical homomorphism

$S_{(f)} \to S_{(fg)}$, such that the diagram

$$\begin{tikzcd} & S_{(f)} \arrow[dl] \arrow[dr] & \\ S_{(fg)} \arrow[rr, "\sim"] & & (S_{(f)})_{(g^d/f^e)} \end{tikzcd}$$

is commutative. We define similarly a canonical homomorphism $M_{(f)} \to M_{(fg)}$.

Lemma 5.2.2. *If f, g are two homogeneous elements of S_+, the ring $S_{(fg)}$ is generated by the union of the canonical images of $S_{(f)}$ and $S_{(g)}$.*

Proof. In view of Lemma 5.2.1, it suffices to show that $1/(g^d/f^e) = f^{d+e}/(fg)^d$ belongs to the canonical image of $S_{(g)}$ in $S_{(fg)}$, which is evident from the definition. □

Proposition 5.2.3. *Let $f \in S_d$ be a homogeneous element of positive degree. Then there exists a canonical isomorphism $S_{(f)} \cong S^{(d)}/(f-1)S^{(d)}$ of rings. If we identify these two rings, then for any S-module M, there exists a canonical isomorphism of modules $M_{(f)} \cong M^{(d)}/(f-1)M^{(d)}$*

Proof. The first isomorphism is defined by sending the element x/f^n, where $x \in S_{nd}$, to the element $\bar{x}$, the class of x mod $(f-1)S^{(d)}$. This map is well-defined, because we have the congruence $f^h x \equiv x$ mod $(f-1)S^{(d)}$ for any $x \in S^{(d)}$, so if $f^h x = 0$ for some $h > 0$ then $\bar{x} = 0$. On the other hand, if $x \in S_{nd}$ is such that $x = (f-1)y$ with $y = y_{hd} + y_{(h+1)d} \cdots + y_{kd}$, where $y_{jd} \in S_{jd}$ and $y_{hd} \neq 0$, we have necessarily $h = n$ and $x = -y_{hd}$, as well as the relations $y_{(j+1)d} = f y_{jd}$ for $h \le j \le k-1$ and $f_{ykd} = 0$; in particular, this implies $f^{k-n}x = 0$. We therefore have an inverse homomorphism from $S^{(d)}/(f-1)S^{(d)}$ to $S_{(f)}$ by corresponding a class $\bar{x}$ mod $(f-1)S^{(d)}$ (where $x \in S_{nd}$) the element x/f^n of $S_{(f)}$, since the preceding remark shows that this map is well-defined. The first assertion is therefore proves, and the second one can be done similarly. □

Corollary 5.2.4. *If S is Noetherian, so is $S_{(f)}$ for any homogeneous element f of positive degree.*

Proof. This follows from Proposition 5.2.3 and Corollary **??**. □

Let T be a multiplicative subset of S_+ formed by homogeneous elements; $T_0 = T \cup \{1\}$ is then a multiplicative subset of S. As the elements of T_0 are homogeneous, the ring $T_0^{-1}S$ is graded in a natrual way, and we denote by $S_{(T)}$ the subring of $T_0^{-1}S$ formed by elements of degree 0. We know that $T_0^{-1}S$ is identified with the inductive limit of the rings S_f, where $f \in T$ (with the canonical homomorphisms $S_f \to S_{fg}$). As this identification preserves the degrees, it identified $S_{(T)}$ as the inductive limit of $S_{(f)}$, where $f \in T$. For any graded S-module M, we define similarly the module $M_{(T)}$ (over the ring $S_{(T)}$) formed by degree zero elements of $T_0^{-1}M$, and we conclude that $M_{(T)}$ is the inductive limit of $M_{(f)}$ for $f \in T$.

If $\mathfrak{p}$ is a graded prime ideal of S_+, we denote by $S_{(\mathfrak{p})}$ and $M_{(\mathfrak{p})}$ the ring $S_{(T)}$ and the module $M_{(T)}$ respectively, where T is the homogeneous elements of S_+ not contained in $\mathfrak{p}$.

5.2.2 The homogeneous specturm of a graded ring

Given a graded ring S with positive degrees, we denote by $\mathrm{Proj}(S)$ the **homogeneous specturm** of S, which is the set of graded prime ideals of S_+, or, equivalently, the set of graded prime ideals of S not containing S_+. We will define a scheme structure on $\mathrm{Proj}(S)$, just as we have done for $\mathrm{Spec}(A)$ for a ring A.

For a subset E of S, let $V_+(E)$ be the set of graded prime ideals of S containing E and not containing S_+, which is also the subset $V(E) \cap \mathrm{Proj}(S)$ of $\mathrm{Spec}(S)$. We have immediately the following equalities:

$$V_+(0) = \mathrm{Proj}(S), \quad V_+(S) = V_+(S_+) = \varnothing,$$
$$V_+\Big(\bigcup_\lambda E_\lambda\Big) = \bigcap_\lambda V_+(E_\lambda),$$
$$V_+(EF) = V_+(E) \cup V_+(F).$$

Again, the set $V_+(E)$ remain unchanged if we replace E by the graded ideal it generates; moreover, if $\mathfrak{I}$ is a graded ideal of S, we have

$$V_+(\mathfrak{I}) = V_+\Big(\bigcup_{i\geq n}(\mathfrak{I} \cap S_i)\Big) \tag{2.2.1}$$

for any $n > 0$: in fact, if $\mathfrak{p} \in \mathrm{Proj}(S)$ contains the homogeneous elements of $\mathfrak{I}$ with degrees $\geq n$, as by hypothesis there exists a homogeneous element $f \in S_d$ not contained in $\mathfrak{p}$, for any $m \geq 0$ and any $x \in S_m \cap \mathfrak{I}$, we have $f^r x \in \mathfrak{I} \cap S_{m+rd}$ for r sufficiently large, hence $f^r x \in \mathfrak{p} \cap S_{m+rd}$, which implies $x \in \mathfrak{p} \cap S_m$. Finally, for any graded ideal $\mathfrak{I}$ of S, we have

$$V_+(\mathfrak{I}) = V_+(\mathfrak{r}_+(\mathfrak{I}))$$

where $\mathfrak{r}_+(\mathfrak{I})$ is the radical of $\mathfrak{I}$ in S_+.

By definition, $V_+(E)$ is a closed subset of $X = \mathrm{Proj}(S)$ for the topology induced by $\mathrm{Spec}(S)$. For each element $f \in S$, we set

$$D_+(f) = D(f) \cap \mathrm{Proj}(S) = \mathrm{Proj}(S) \setminus V_+(f).$$

Then for two elements $f, g \in S$, $D_+(fg) = D_+(f) \cap D_+(g)$, and the subsets $D_+(f)$, with $f \in S_+$, form a basis for the topology of $X = \mathrm{Proj}(S)$.

Let f be a homogeneous element of S_+ with degree $d > 0$. For any prime ideal $\mathfrak{p}$ of S not containing f, we see the set of x/f^n, where $x \in \mathfrak{p}$ and $n \geq 0$, is a prime ideal of the fraction ring S_f. Its trace on $S_{(f)}$ is then a prime ideal of this ring, which we denote by $\psi_f(\mathfrak{p})$: this is the set of elements x/f^n, for $n \geq 0$, $x \in \mathfrak{p}_{nd}$. We have therefore defined a map

$$\psi_f : D_+(f) \to \mathrm{Spec}(S_{(f)});$$

moreover, if g is another homogeneous element of S_+ with degree $e > 0$, we have a commutative diagram

$$\begin{tikzcd} D_+(fg) \arrow[r, "\psi_{fg}"] \arrow[d] & \mathrm{Spec}(S_{(fg)}) \arrow[d] \\ D_+(f) \arrow[r, "\psi_f"] & \mathrm{Spec}(S_{(f)}) \end{tikzcd} \tag{2.2.2}$$

where the left vertical maps are inclusions, and the right one is the map ${}^a\omega_{fg,f}$ induced from the canonical homomorphism $\omega_{fg,f}: S_{(f)} \to S_{(fg)}$. In fact, if $x/f^n \in \omega^{-1}(\psi_{fg}(\mathfrak{p}))$, where $fg \notin \mathfrak{p}$, we then have $g^n x/(fg)^n \in \psi_{fg}(\mathfrak{p})$, so $g^n x \in \mathfrak{p}$ and therefore $x \in \mathfrak{p}$, and the converse inclusion is evident.

Proposition 5.2.5. *The map ψ_f is a homeomorphism from $D_+(f)$ to* $\mathrm{Spec}(S_{(f)})$.

Proof. For $h \in S_{nd}$ is such that $h/f^n \in \psi_f(\mathfrak{p})$, by definition it is necessary and sufficient that $h \in \mathfrak{p}$, so $\psi^{-1}(D(h/f^n)) = D_+(fh) = D_+(h) \cap D_+(f)$ and the map ψ_f is therefore continuous. Moreover, the sets $D_+(hf)$, where h runs through the set S_{nd}, form a basis of the topology of $D_+(f)$, so the preceding argument proves, in view of the T_0-axiom for $D_+(f)$ and $\mathrm{Spec}(S_{(f)})$, that ψ_f is injective and the inverse map $\psi_f(D_+(f)) \to D_+(f)$ is continuous. Finally, to show that ψ_f is surjective, we remark that, if $\mathfrak{q}_0$ is a prime ideal of $S_{(f)}$ and if, for any $n > 0$, we denote by $\mathfrak{p}_n$ the set of elements $x \in_S n$ such that $x^d/f^n \in \mathfrak{q}_n$, the $\mathfrak{p}_n$ then verify the conditions Proposition **??**: if $x \in S_n$, $y \in S_n$ are such that $x^d/f^n \in \mathfrak{q}_0$ and $y^d/f^n \in \mathfrak{q}_0$, we have $(x+y)^{2d}/f^{2n} \in \mathfrak{q}_0$, whence $(x+y)^d/f^n \in \mathfrak{q}_0$ since $\mathfrak{q}_0$ is prime; this proves that $\mathfrak{p}_n$ is a subgroup of S_n, and the verification of other conditions of Proposition **??** is immediate. If $\mathfrak{p}$ is the corresponding graded ideal of S, then $\psi_f(\mathfrak{p}) = \mathfrak{q}_0$, since if $x \in S_{nd}$, the relations $x/f^n \in \mathfrak{q}_0$ and $x^d/f^{nd} \in \mathfrak{q}_0$ are equivalent. □

Corollary 5.2.6. *For $D_+(f) \neq \varnothing$, it is necessary and sufficient that f is nilpotent.*

Proof. For $\mathrm{Spec}(S_{(f)}) = \varnothing$, it is necessary and sufficient that $S_{(f)} = 0$, which means $1 = 0$ in S_f, and this is equivalent to that f is nilpotent. □

Corollary 5.2.7. *Let E be a subset of S_+. The following conditions are equivalent:*

(i) $V_+(E) = X = \mathrm{Proj}(S)$.

(ii) Every element of E is nilpotent.

(iii) The homogeneous components of every element of E are nilpotent.

Proof. It is clear that (iii)⇒(ii)⇒(i). If $\mathfrak{I}$ is the graded ideal of S generated by E, conditions (i) is equivalent to that $V_+(\mathfrak{I}) = X$, and a fortiori, (i) implies that any homogeneous element $f \in \mathfrak{I}$ is such that $V_+(f) = X$, so f is nilpotent by Corollary 5.2.6. □

Corollary 5.2.8. *If $\mathfrak{I}$ is a graded ideal of S_+, $\mathfrak{r}_+(\mathfrak{I})$ is the intersection of graded prime ideals in $V_+(\mathfrak{I})$.*

Proof. By considering the ring $S/\mathfrak{I}$, we may assume that $\mathfrak{I} = 0$. It then suffices to prove that if $f \in S_+$ is not nilpotent, then there exists a graded prime ideal of S not containing f. Now, since there exists at least homogeneous component of f that is not nilpotent, we may assume that f is homogeneous, the result then follows from Corollary 5.2.6. □

For any subset Y of $X = \operatorname{Proj}(S)$, we denote by $I_+(Y)$ the subset of $f \in S_+$ such that $Y \subseteq V_+(f)$, which is in other words $I(Y) \cap S_+$; the set $I_+(Y)$ is then a radical ideal of S_+.

Proposition 5.2.9. *Let E be a subset of S and Y be a subset of X.*

(a) The ideal $I_+(V_+(E))$ is the radical in S_+ of the graded ideal of S_+ generated by E.

(b) The set $V_+(I_+(Y))$ is the closure of Y in X.

Proof. If $\mathfrak{I}$ is the graded ideal of S_+ generated by E, we have $V_+(E) = V_+(\mathfrak{I})$ and the first assertion follows from Corollary 5.2.8. As for (b), since $V_+(\mathfrak{I}) = \bigcap_{f\in\mathfrak{I}} V_+(f)$, the relation $Y \subseteq V_+(\mathfrak{I})$ implies $Y \subseteq V_+(f)$ for any $f \in \mathfrak{I}$, and therefore $I_+(Y) \supseteq \mathfrak{I}$, so $V_+(I_+(Y)) \subseteq V_+(\mathfrak{I})$, which implies (b) by the definition of closure. □

Corollary 5.2.10. *The closed subsets Y of $X = \operatorname{Proj}(S)$ and the graded radical ideals of S_+ correspond bijectively via $Y \mapsto I_+(Y)$ and $\mathfrak{I} \mapsto V_+(\mathfrak{I})$. Also, the union $Y_1 \cup Y_2$ of two closed subsets of X corresponds to $I_+(Y_1) \cap I_+(Y_2)$, and the intersection of a family (Y_λ) of closed subsets corresponds to the radical of the sum of $I_+(Y_\lambda)$.*

Corollary 5.2.11. *Let (f_α) be a family of homogeneous elements of S_+ and f be an element of S_+. The following conditions are equivalent:*

(i) $D_+(f) \subseteq \bigcup_\alpha D_+(f_\alpha)$;

(ii) $V_+(f) \supseteq \bigcap_\alpha V_+(f_\alpha)$;

(ii) a power of f is contained in the ideal generated by the f_α.

In particular, if $\mathfrak{I}$ is a graded ideal of S_+, then $V_+(\mathfrak{I}) = \varnothing$ if and only if $\mathfrak{r}_+(\mathfrak{I}) = S_+$.

Corollary 5.2.12. *For $X = \operatorname{Proj}(S)$ to be empty, it is necessary and sufficient that every element of S_+ is nilpotent.*

Corollary 5.2.13. *The closed irreducible subset of $X = \operatorname{Proj}(S)$ correspond to graded prime ideals of S_+.*

Proof. In fact, if $Y = Y_1 \cup Y_2$, where Y_1, Y_2 are closed and distinct in Y, then

$$I_+(Y) = I_+(Y_1) \cap I_+(Y_2)$$

the ideals $I_+(Y_1)$ and $I_+(Y_2)$ are distinct from $I_+(Y)$, so $I_+(Y)$ can not be prime. Conversely, if $\mathfrak{I}$ is a graded non-prime ideal of S_+, there exist elements f, g of S_+ such that $fg \in \mathfrak{I}$ but $f, g \notin \mathfrak{I}$. Then $V_+(f) \not\subseteq V_+(\mathfrak{I})$, $V_+(g) \not\subseteq V_+(\mathfrak{I})$, but $V_+(\mathfrak{I}) \subseteq V_+(f) \cup V_+(g)$. We then conclude that $V_+(\mathfrak{I})$ is the union of the closed subsets $V_+(f) \cap V_+(\mathfrak{I})$ and $V_+(g) \cap V_+(\mathfrak{I})$, both are distinct from $V_+(\mathfrak{I})$. □

We now define the scheme structure on the homogeneous specturm $\operatorname{Proj}(S)$. Let f, g be two homogeneous elements of S_+ and consider the affine schemes $Y_f = \operatorname{Spec}(S_{(f)})$, $Y_g = \operatorname{Spec}(S_{(g)})$, and $Y_{fg} = \operatorname{Spec}(S_{(fg)})$. In view of Lemma 5.2.1, the morphism $w_{fg,f} : Y_{fg} \to Y_f$ corresponding to the canonical homomorphism $\omega_{fg,f} : S_{(f)} \to S_{(fg)}$, is an open immersion. By the homeomorphism $\psi_f : D_+(f) \to Y_f$ (Proposition 5.2.5), we can transport to $D_+(f)$ the affine scheme structure of Y_f; in view of the commutative diagram (2.2.2), the affine scheme $D_+(fg)$ is identified with the subscheme induced over the open subset $D_+(fg)$ by the affine scheme $D_+(f)$. It is then clear that $X = \operatorname{Proj}(S)$ is endowed with a unique scheme structure such that each $D_+(f)$ is an affine open subscheme of X. When we speak of the homogeneous spectrum $\operatorname{Proj}(S)$ as a scheme, it will always be the structure defined in this way.

Proposition 5.2.14. *The scheme* $\operatorname{Proj}(S)$ *is separated.*

Proof. By Proposition 4.5.30, it suffices to show that for any homogeneous elements f, g of S_+, $D_+(f) \cap D_+(g) = D_+(fg)$ is affine and the canonical images of the rings of $D_+(f)$ and $D_+(g)$ in $D_+(fg)$ generate the ring of $D_+(fg)$. The first one is clear by definition, and the second one follows from Lemma 5.2.2, □

Example 5.2.15. Let $S = K[T_1, T_2]$ where K is a field and T_1, T_2 are indeterminates. Then it follows from Corollary 5.2.11 that $\operatorname{Proj}(S)$ is the union of $D_+(T_1)$ and $D_+(T_2)$. We see that each of these affine subscheme is isomorphic to $K[T]$, and that $\operatorname{Proj}(S)$ is obtained by glueing these two schemes as described in Example 4.2.8.

Proposition 5.2.16. *Let S be a graded ring with positive degrees and $X = \operatorname{Proj}(S)$.*

(i) If $\mathfrak{n}_+$ is the nilradical of S_+, the scheme X_{red} is canonically isomorphic to $\operatorname{Proj}(S/\mathfrak{n}_+)$. *In particular, if S is essentially reduced, then* $\operatorname{Proj}(S)$ *is reduced.*

(ii) Suppose that S is essentially reduced, then for X to be integral, it is necessary and sufficient that S is essentially integral.

Proof. Let $\bar{S} = S/\mathfrak{n}_+$, and denote by $x \mapsto \bar{x}$ the canonical homomorphism $S \to \bar{S}$, of degree 0. For any $f \in S_d$ $(d > 0)$, the canonical homomorphism $S_f \to \bar{S}$ is surjective and of degree 0, hence restricts to a surjection $S_{(f)} \to \bar{S}_{(\bar{f})}$. If we suppose that $f \notin \mathfrak{n}_+$, then $\bar{S}_{(\bar{f})}$ is reduced and the kenrel of the preceding homomorphism is the nilradical of $S_{(f)}$, whence $\bar{S}_{(\bar{f})} = (S_{(f)})_{\mathrm{red}}$. This homomorphism then corresponds to a closed immersion $D_+(\bar{f}) \to D_+(f)$ which identifies $D_+(\bar{f})$ with $(D_+(f))_{\mathrm{red}}$ (Corollary 4.4.26), and in particular is a homeomorphism of affine scheme. Further, if $g \notin \mathfrak{n}_+$ is another homogeneous element of S_+, the diagram

$$\begin{tikzcd}
S_{(f)} \arrow[r] \arrow[d] & \bar{S}_{(\bar{f})} \arrow[d] \\
S_{(fg)} \arrow[r] & \bar{S}_{(\overline{fg})}
\end{tikzcd}$$

is commutative. As the sets $D_+(f)$ for f homogeneous in S_+ and $f \notin \mathfrak{n}_+$ form a covering for $X = \operatorname{Proj}(S)$, we conclude that the morphisms $D_+(\bar{f}) \to D_+(f)$ glue together to a closed immersion $\operatorname{Proj}(\bar{S}) \to \operatorname{Proj}(S)$ which is a homeomorphism on the underlying spaces, whence the conclusion of (i) by Corollary 4.4.26.

Suppose now that S is essentially integral, which means (0) is a graded ideal of S_+ distinct from S_+. Then X is reduced by (a) and irreducible by Corollary 5.2.13. Conversely, if S is essentially reduced and X is integral, then for any $f \neq 0$ homogeneous in S_+, we have $D_+(f) \neq \varnothing$ by Corollary 5.2.6; the hypothesis that X is irreducible implies that $D_+(f) \cap D_+(g) \neq \varnothing$ for any f, g homogeneous and nonzero in S_+, so in particular $fg \neq 0$, and we then conclude that S_+ is integral. □

Proposition 5.2.17. *Suppose that S is a graded A-algebra where A is a ring. Then over $X = \mathrm{Proj}(S)$ the structural sheaf $\mathcal{O}_X$ is an A-algebra, which means X is a scheme over $\mathrm{Spec}(A)$.*

Proof. It suffices to note that for any f homogeneous in S_+, $S_{(f)}$ is an A-algebra and the homomorphism $S_{(f)} \to S_{(fg)}$ is an A-algebra homomorphism for any f, g homogeneous in S_+. □

Proposition 5.2.18. *Let S be a graded ring with positive degrees.*

(a) For any integer $d > 0$, there exists a canonical isomorphism $\mathrm{Proj}(S) \cong \mathrm{Proj}(S^{(d)})$.

(b) Let S' be the graded ring such that $S'_0 = \mathbb{Z}$ and $S'_n = S_n$ for $n > 0$, then there exists a canonical isomorphism $\mathrm{Proj}(S) \cong \mathrm{Proj}(S')$.

Proof. We have already seen in Proposition **??** that the map $\mathfrak{p} \mapsto \mathfrak{p} \cap S^{(d)}$ is a bijection from $\mathrm{Proj}(S)$ to $\mathrm{Proj}(S^{(d)})$. As for any f homogeneous in S_+, we have $V_+(f) = V_+(f^d)$, this bijection is a homeomorphism of topological spaces. Finally, with the same notations, $S_{(f)}$ and $S_{(f^d)}$ are canonically identified by Lemma 5.2.1, so $\mathrm{Proj}(S)$ and $\mathrm{Proj}(S^{(d)})$ are canonically identified as schemes.

If to any $\mathfrak{p} \in \mathrm{Proj}(S)$, we correspond the unique prime ideal $\mathfrak{p}' \in \mathrm{Proj}(S')$ such that $\mathfrak{p}' \cap S_n = \mathfrak{p} \cap S_n$ for $n > 0$, then it is clear that this defines a homeomorphism $\mathrm{Proj}(S) \cong \mathrm{Proj}(S')$ of the underlying spaces, since $V_+(f)$ is the same set for S and S' if f is a homogeneous element of S_+. We also note that $S_{(f)} = S'_{(f)}$: to see this it suffices to note that if $x/1$ is an element of $S_{(f)}$ with $x \in S_0$, then $x/1 = xf/f \in S'_{(f)}$; we then conclude that $\mathrm{Proj}(S)$ and $\mathrm{Proj}(S')$ are identified as schemes. □

Corollary 5.2.19. *Let S be a graded A-algebra and S_A be the graded A-algebra such that $(S_A)_0 = A$ and $(S_A)_n = S_n$ for $n > 0$, then there exists a canonical isomorphism $\mathrm{Proj}(S) \cong \mathrm{Proj}(S_A)$.*

Proof. In fact, these two schemes are isomorphic to $\mathrm{Proj}(S')$, where $S'_0 = \mathbb{Z}$ and $S'_n = S_n$ for $n > 0$, in view of Proposition 5.2.18. □

5.2.3 Sheaf associated with a graded module

Let M be a graded S-module. For any homogeneous element f of S_+, $M_{(f)}$ is an $S_{(f)}$-module, and it therefore corresponds to a quasi-coherent sheaf $\widetilde{M_{(f)}}$ over the affine $\mathrm{Spec}(S_{(f)})$, identified with $D_+(f)$.

Proposition 5.2.20. *There existss a unique quasi-coherent $\mathcal{O}_X$-module $\widetilde{M}$ such that for any homogeneous element $f \in S_+$, we have $\Gamma(D_+(f), \widetilde{M}) = M_{(f)}$, and the restriction homomorphism $\Gamma(D_+(f), \widetilde{M}) \to \Gamma(D_+(fg), \widetilde{M})$ for f, g homogeneous in S_+ corresponds to the canonical homomorphism $M_{(f)} \to M_{(fg)}$.*

Proof. Suppose that $f \in S_d$, $g \in S_e$. As $D_+(fg)$ is identified with the prime specturm $(S_{(f)})_{g^d/f^e}$ by Lemma 5.2.1, the restriction of $\widetilde{M_{(f)}}$ to $D_+(fg)$ is canonically identified with the sheaf associated with the module $(M_{(f)})_{(g^d/f^e)}$, hence to $\widetilde{M_{(fg)}}$ (Lemma 5.2.1). We then conclude that there is a canonical isomorphism

$$\theta_{g,f} : \widetilde{M_{(f)}}|_{D_+(fg)} \to \widetilde{M_{(g)}}|_{D_+(fg)}$$

such that, if g is another homogeneous element of S_+, we have $\theta_{f,h} = \theta_{f,g} \circ \theta_{g,h}$ over $D_+(fgh)$. By glueing, there then exists a quasi-coherent sheaf $\mathscr{F}$ over X such that for any homogeneous element $f \in S_+$, we have an isomorphism $\eta_f : \mathscr{F}|_{D_+(f)} \cong \widetilde{M_{(f)}}$ and $\theta_{g,f} = \eta_g \circ \eta_f^{-1}$. Since over $D_+(f)$ we have $\Gamma(D_+(f), \widetilde{M})$, $\mathscr{F}$ can be identified with the sheaf extended from the presheaf $D_+(f) \mapsto M_{(f)}$ over the basis of standard open sets of X, whence the assertions of the proposition. In particular, we have $\widetilde{S} = \mathscr{O}_X$. □

We say the quasi-coherent $\mathscr{O}_X$-module $\widetilde{M}$ is **associted** with the graded S-module M. Recall that the graded S-modules form a category whose morphisms are graded homomorphisms of degrees. With this convention:

Proposition 5.2.21. *The functor $\widetilde{M}$ is a covariant exact functor from the category of graded S-modules to the category of quasi-coherent $\mathscr{O}_X$-modules, which commutes with inductive limits and direct sums.*

Proof. Since the properties are local, it suffices to verify over the sheaf $\widetilde{M}|_{D_+(f)} = \widetilde{M_{(f)}}$. Now the functor M_f on M, the functor N_0 on N, and the functor $\widetilde{P}$ on P all satisfy the stated properties, whence the claim. □

We denote by $\tilde{u} : \widetilde{M} \to \widetilde{N}$ the homomorphism corresponding to a graded homomorphism $u : M \to N$ of degree 0. We also deduce from Proposition 5.2.21 that the results of Proposition 4.1.6 and Corollary 4.1.7 are also true for graded S-modules and homomorphism of degree 0, via the same demonstration.

Proposition 5.2.22. *For any $\mathfrak{p} \in X = \mathrm{Proj}(S)$, we have $\widetilde{M}_{\mathfrak{p}} = M_{(\mathfrak{p})}$.*

Proof. By definition we have $\widetilde{M}_{\mathfrak{p}} = \varinjlim \Gamma(D_+(f), \widetilde{M})$, where f runs through homogeneous elements $f \in S_+$ such that $f \notin \mathfrak{p}$. As $\Gamma(D_+(f), \widetilde{M}) = M_{(f)}$, the proposition follows from the definition of $M_{(\mathfrak{p})}$. □

In particular, the local ring $\mathscr{O}_{X,\mathfrak{p}}$ is just the ring $S_{(\mathfrak{p})}$, the set of elements x/f where f is homogeneous in S_+ and not contained in $\mathfrak{p}$, and x is homogeneous with the same degree as f. If moreover S is essentially integral, then $\mathrm{Proj}(S) = X$ is integral (Proposition 5.2.16), and if $\xi = (0)$ is the generic point of X, the rational function field $K(X) = \mathscr{O}_{X,\xi}$, is the field formed by f/g where f, g are homogeneous elements of S_+ and $g \neq 0$.

Proposition 5.2.23. *If, for any $z \in M$ and any homogeneous element $f \in S_+$, there exists a power of f annihilating z, then $\widetilde{M} = 0$. This condition is also necessary if $S = S_0[S_1]$.*

Proof. The condition $\widetilde{M} = 0$ is equivalent to $M_{(f)} = 0$ for any homogeneous element of S_+. Now if $f \in S_d$, the condition $M_{(f)} = 0$ signifies that for any $z \in M$ homogeneous whose degree is a multiple of d, there exists power f^n such that $f^n z = 0$; this implies the first claim. Conversely, if moreover S is generated by S_1, then condition then implies that $f^n z = 0$ for any $z \in M$ and any $f \in S_+$, since any element $f \in S_+$ is a finite linear combination of elements of S_1. □

Proposition 5.2.24. *Let $f \in S_d$ with $d > 0$. Then for any $n \in \mathbb{Z}$, the $(\mathscr{O}_X|_{D_+(f)})$-module $\widetilde{S(nd)}|_{D_+(f)}$ is canonically isomorphic to $\mathscr{O}_X|_{D_+(f)}$.*

Proof. The multiplication by the invertible element $(f/1)^n$ of S_f defines a bijection from $S_{(f)} = (S_f)_0$ to the ring

$$(S_f)_{nd} = (S_f(nd))_0 = (S(nd)_f)_0 = S(nd)_{(f)},$$

whence the assertion. □

Corollary 5.2.25. *Over the open subset $U = \bigcup_{f \in S_d} D_+(f)$, the restriction of the $\mathcal{O}_X$-module $\widetilde{S(nd)}$ is an invertible $(\mathcal{O}_X|_U)$-module.*

Corollary 5.2.26. *If the ideal S_+ of S is generated by S_1, then the $\mathcal{O}_X$-module $\widetilde{S(n)}$ is invertible for any $n \in \mathbb{Z}$.*

Proof. It suffices to note that under the hypothesis we have $X = \bigcup_{f \in S_1} D_+(f)$ by Corollary 5.2.11. □

The quasi-coherent $\mathcal{O}_X$-modules $\widetilde{S(n)}$ is of particular interest in the theory of projective schemes, so for each $n \in \mathbb{Z}$, we put $\mathcal{O}_X(n) = \widetilde{S(n)}$ and for an open subset U of X and any $(\mathcal{O}_X|_U)$-module $\mathscr{F}$, set

$$\mathscr{F}(n) = \mathscr{F} \otimes_{\mathcal{O}_X|_U} (\mathcal{O}_X(n)|_U).$$

If the ideal S_+ is generated by S_1, the functor $\mathscr{F}(n)$ is exact on $\mathscr{F}$ for any $n \in \mathbb{Z}$, since $\mathcal{O}_X(n)$ is then an invertible $\mathcal{O}_X$-module.

Example 5.2.27. Let k be a field and consider the graded algebra $S = k[x_0, \dots, x_n]$; let $X = \mathrm{Proj}(S)$. Let d be an inter and consider the twist sheaf $\mathcal{O}_X(d)$. We compute the global sections for $\mathcal{O}_X(d)$: by definition, for each $x_i \in S$, the section of $\mathcal{O}_X(d)$ over $U_i = D_+(x_i)$ is given by

$$\Gamma(U_i, \mathcal{O}_X(d)) = S(d)_{(x_i)} = (S_{(x_i)})_d = \{f/x_i^n : f \in S_{n+d}\} = \{x_i^d f : f \in S^{(i)}\},$$

where $S^{(i)} = k[x_0/x_i, \dots, x_n/x_i]$. Therefore a section of $\mathcal{O}_X(d)$ is a family of rational polynomials (f_i) with $f_i \in S^{(i)}$ such that $x_i^d f_i = x_j^d f_j$ for each $i \neq j$; let f be this common rational polynomial. Then by construction, we have $f/x_i^d \in S^{(i)}$ for each i, which implies that f is a polynomial in S of degree d when $d \geq 0$. If on the other hand $d < 0$, then f can only have poles at x_i for each i, which is impossible, so there is no global sections for $\mathcal{O}_X(d)$ when $d < 0$.

Let M, N be graded S-modules. For any $f \in S_d$ we define a canonical homomorphism of $S_{(f)}$-modules

$$\lambda_f : M_{(f)} \otimes_{S_{(f)}} N_{(f)} \to (M \otimes_S N)_{(f)}$$

by composing the homomorphism $M_{(f)} \otimes_{S_{(f)}} N_{(f)} \to M_f \otimes_{S_f} N_f$ (induced from the canonical injections) with the canonical isomorphism $M_f \otimes_{S_f} N_f \cong (M \otimes_S N)_{(f)}$, and note that by the definition of the grading of tensor products, these isomorphisms preserve degrees. Unwinding the definitions, for $x \in M_{md}$, $y \in N_{nd}$, we have

$$\lambda_f((x/f^m) \otimes (y/f^n)) = (x \otimes y)/f^{m+n}.$$

It then follows that, if $g \in S_e$ is another homogeneous element, the diagram

$$\begin{tikzcd} M_{(f)} \otimes_{S_{(f)}} N_{(f)} \arrow[r, "\lambda_f"] \arrow[d] & (M \otimes_S N)_{(f)} \arrow[d] \\ M_{(fg)} \otimes_{S_{(fg)}} N_{(fg)} \arrow[r, "\lambda_{fg}"] & (M \otimes_S N)_{(fg)} \end{tikzcd}$$

(wher the vertical homomorphism are canonical) is commutative. We then deduce that λ is a canonical homomorphism of $\mathscr{O}_X$-modules

$$\lambda : \widetilde{M} \otimes_{\mathscr{O}_X} \widetilde{N} \to \widetilde{M \otimes_S N}.$$

Consider in particular two graded ideals $\mathfrak{I}, \mathfrak{K}$ of S. As $\widetilde{\mathfrak{I}}$ and $\widetilde{\mathfrak{K}}$ are two quasi-coherent ideals of $\mathscr{O}_X$, we have a canonical homomorphism $\widetilde{\mathfrak{I}} \otimes_{\mathscr{O}_X} \widetilde{\mathfrak{K}} \to \mathscr{O}_X$, and the diagram

$$\begin{tikzcd} \widetilde{\mathfrak{I}} \otimes_{\mathscr{O}_X} \widetilde{\mathfrak{K}} \arrow[rr, "\lambda"] \arrow[dr] & & \widetilde{\mathfrak{I} \otimes_S \mathfrak{K}} \arrow[dl] \\ & \mathscr{O}_X & \end{tikzcd} \tag{2.3.1}$$

is commutative. Finally, note that if M, N, P are graded S-modules, the diagram

$$\begin{tikzcd} \widetilde{M} \otimes_{\mathscr{O}_X} \widetilde{N} \otimes_{\mathscr{O}_X} \widetilde{P} \arrow[r, "\lambda\otimes 1"] \arrow[d, "1\otimes\lambda"'] & \widetilde{M \otimes_S N} \otimes_{\mathscr{O}_X} \widetilde{P} \arrow[d, "\lambda"] \\ \widetilde{M} \otimes_{\mathscr{O}_X} \widetilde{N \otimes_S P} \arrow[r, "\lambda"] & (M \otimes_S N \otimes_S P)^{\sim} \end{tikzcd} \tag{2.3.2}$$

is commutative. Simialrly, we define a canonical homomorphism of $S_{(f)}$-modules

$$\mu_f : \mathrm{Hom}_S(M, N)_{(f)} \to \mathrm{Hom}_{S_{(f)}}(M_{(f)}, N_{(f)})$$

which sends an element u/f^n, where u is a homomorphism of degre nd, the homomorphism $M_{(f)} \to N_{(f)}$ which sends x/f^m $(x \in M_{md})$ to $u(x)/f^{m+n}$. For $g \in S_e$, we have the following commutative diagram

$$\begin{tikzcd} \mathrm{Hom}_S(M, N)_{(f)} \arrow[r, "\mu_f"] \arrow[d] & \mathrm{Hom}_{S_{(f)}}(M_{(f)}, N_{(f)}) \arrow[d] \\ \mathrm{Hom}_S(M, N)_{(fg)} \arrow[r, "\mu_{fg}"] & \mathrm{Hom}_{S_{(fg)}}(M_{(fg)}, N_{(fg)}) \end{tikzcd} \tag{2.3.3}$$

We then conclude that the μ_f define a canonical homomorphism

$$\mu : (\mathrm{Hom}_S(M, N))^{\sim} \to \mathscr{H}\!om_{\mathscr{O}_X}(\widetilde{M}, \widetilde{N}).$$

Proposition 5.2.28. *Suppose that the ideal S_+ is generated by S_1. Then the homomorphism λ is an isomorphism; this holds for μ if the graded S-module M is of finite presentation.*

Proof. As X is the union of $D_+(f)$ for $f \in S_1$, we are reduced to prove tht λ_f and μ_f are isomorphisms for f homogeneous of degree 1. We then define a $\mathbb{Z}$-linear map $M_n \times N_n \to$

$M_{(f)}\otimes_{S_{(f)}} N_{(f)}$ that send a pair (x,y) to the element $(x/f^m)\otimes(y/f^n)$. This then defines a $\mathbb{Z}$-linear map $M\otimes_{\mathbb{Z}} N\to M_{(f)}\otimes_{S_{(f)}} N_{(f)}$, and if $s\in S_q$, this map send $(sx)\otimes y$ to $(s/f^q)((x/f^m)\otimes(y/f^n))$ (where $x\in M_m$, $y\in N_n$), so we get a bi-homomorphism $\gamma_f: M\otimes_S N\to M_{(f)}\otimes_{S_{(f)}} N_{(f)}$ relative to the canonical homomorphism $S\to S_{(f)}$ (sending $s\in S_q$ to s/f^q). Suppose that for an element $\sum_i(x_i\otimes y_i)$ of $M\otimes_S N$ (where x_i,y_i are homogeneous elements of degrees m_i,n_i, respectively) we have $f^r\sum_i(x_i\otimes y_i)=0$, which means $\sum_i(f^rx_i\otimes y_i)=0$. Then we deduce from the isomorphism $M_f\otimes_{S_f} N_f\cong(M\otimes_S N)_f$ that $\sum_i(f^rx^i/f^{m_i+r})\otimes(y_i/f^{n_i})=0$, which means $\gamma_i(\sum_i(x_i\otimes y_i))=0$. Therefore γ_f factors through $(M\otimes_S N)_f$ and give a homomorphism $\tilde{\gamma}_f:(M\otimes_S N)_f\to M_{(f)}\otimes_{S_{(f)}} N_{(f)}$; if $\tilde{\lambda}_f$ is the restriction of $\tilde{\gamma}_f$ to $(M\otimes_S N)_{(f)}$, we then verify that λ_f and $\tilde{\lambda}_f$ are inverses of each other, so the first assertion follows.

To demonstrate the second assertion, we now assume that M is of finite presentation, so is the cokernel of a homomorphism $P\to Q$ of graded S-module, P,Q being direct sums of finitely many modules of the form $S(n)$. By using the left exactness of $\mathrm{Hom}_S(-,N)$ and the exactness of $M_{(f)}$ on M, we are reduced to prove that μ_f is an isomorphism in the case $M=S(n)$. Now for any homogeneous $z\in N$, let u_z be the homomorphism from $S(n)$ to N such that $u_z(1)=z$; we then see that $\eta: z\mapsto u_z$ is an isomorphism of degree 0 from $N(-n)$ to $\mathrm{Hom}_S(S(n),N)$. It thus corresponds to an isomorphism

$$\eta_f: N(-n)_{(f)}\to\mathrm{Hom}_S(S(n),N)_{(f)}.$$

On the other hand, let $\tilde{\eta}_f$ be the isomorphism $N_{(f)}\to\mathrm{Hom}_{S_{(f)}}(S(n)_{(f)},N_{(f)})$ which send $z'\in N_{(f)}$ to the homomorphism $v_{z'}$ such that $v_{z'}(s/f^k)=sz'/f^{n+k}$ (for $s\in S_{n+k}=S(n)_k$). We consider the composition

$$N(-n)_{(f)}\xrightarrow{\eta_f}\mathrm{Hom}_S(S(n),N)_{(f)}\xrightarrow{\mu_f}\mathrm{Hom}_{S_{(f)}}(S(n)_{(f)},N_{(f)})\xrightarrow{\tilde{\eta}_f^{-1}}N_{(f)}$$

is the isomorphism $z/f^h\mapsto z/f^{h-n}$ from $N(-n)_{(f)}\to N_{(f)}$, so μ_f is an isomorphism. □

If the ideal S_+ is generated by S_1, we deduce from Proposition 5.2.28 that for any graded ideal $\mathfrak{I}$ of S and any graded S-module M, we have $\widetilde{\mathfrak{I}}\widetilde{M}=\widetilde{\mathfrak{I}M}$.

Corollary 5.2.29. *Suppose that the ideal S_+ is generated by S_1. Then for integers m,n, we have canonical isomorphisms*

$$\mathscr{O}_X(m)\otimes_{\mathscr{O}_X}\mathscr{O}_X(n)=\mathscr{O}_X(m+n),\quad \mathscr{O}_X(n)=(\mathscr{O}_X(1))^{\otimes n}.$$

Proof. The first formula follows from Proposition 5.2.28 and the existence of the canonical isomorphism $S(m)\otimes_S S(n)\cong S(m+n)$, which sends the element $1\otimes 1\in S(m)_{-m}\otimes S(n)_{-n}$ to the element $1\in S(m+n)_{-(m+n)}$. It then suffices to demonstrate the second formula for $n=-1$, and in view of Proposition 5.2.28, this follows from the fact that $\mathrm{Hom}_S(S(1),S)$ is canonically isomorphic to $S(-1)$. □

Corollary 5.2.30. *Suppose that the ideal S_+ is generated by S_1. For any graded S-module M and $n\in\mathbb{Z}$, we have a canonical isomorphism $\widetilde{M(n)}=\widetilde{M}(n)$.*

Proof. This follows from Proposition 5.2.28 and the canonical isomorphism $M(n)\cong M\otimes_S S(n)$ which send $z\in M(n)_h=M_{n+h}$ to $z\otimes 1\in M_{n+h}\otimes S(n)_{-n}\subseteq(M\otimes_S S(n))_h$. □

Example 5.2.31. Let S' be the graded ring such that $S'_0 = \mathbb{Z}$ and $S'_n = S_n$ for $n > 0$. Then if $f \in S_d$ $(d > 0)$, we have $S(n)_{(f)} = S'(n)_{(f)}$ for any $n \in \mathbb{Z}$, because an element of $S'(n)_{(f)}$ is of the form x/f^k where $x \in S'_{n+kd}$ $(k > 0)$, and we can always choose k such that $n + kd \neq 0$. As $X = \mathrm{Proj}(S)$ and $X' = \mathrm{Proj}(S')$ are canonically identified, we see that for any $n \in \mathbb{Z}$, $\mathscr{O}_X(n)$ and $\mathscr{O}_{X'}(n)$ are canonically isomorphic under this identification.

On the other hand, for any $d > 0$ and $n \in \mathbb{Z}$, we have

$$S^{(d)}(n)_h = S_{(n+h)d} = S(nd)_{hd};$$

so for any $f \in S_d$ we have $S^{(d)}(n)_{(f)} = S(nd)_{(f)}$. We have seen that the schemes $X = \mathrm{Proj}(S)$ and $X^{(d)} = \mathrm{Proj}(S^{(d)})$ are canonically identified, so under this identification, $\mathscr{O}_X(nd)$ and $\mathscr{O}_{X^{(d)}}(n)$ are canonically isomorphic, for any $n \in \mathbb{Z}$.

Proposition 5.2.32. *Let $d > 0$ be an integer and $U = \bigcup_{f \in S_d} D_+(f)$. Then the restriction to U of the canonical homomorphism $\mathscr{O}_X(nd) \otimes_{\mathscr{O}_X} \mathscr{O}_X(-nd) \to \mathscr{O}_X$ is an isomorphism for each $n \in \mathbb{Z}$.*

Proof. In view of Corollary 5.2.30, we can assume that $d = 1$, and the conclusion then follows the proof of Proposition 5.2.24. ◻

5.2.4 Graded S-module associated with a sheaf

In this part, for simplicity, we always assume that the ideal S_+ is generated by S_1, which also means that $S = S_0[S_1]$ by Proposition **??**. The $\mathscr{O}_X$-module $\mathscr{O}_X(1)$ is then inveritble by Proposition 5.2.26; we then put, for any $\mathscr{O}_X$-module $\mathscr{F}$, that

$$\Gamma_*(\mathscr{F}) = \Gamma_*(\mathscr{O}_X(1), \mathscr{F}) = \bigoplus_{n \in \mathbb{Z}} \Gamma(X, \mathscr{F}(n))$$

Recall that $\Gamma_*(\mathscr{O}_X)$ is endowed with a graded ring structure and $\Gamma_*(\mathscr{F})$ is a graded module over $\Gamma_*(\mathscr{O}_X)$. Since $\mathscr{O}_X(n)$ is locally free, $\Gamma_*(\mathscr{F})$ is a covariant left-exact functor on $\mathscr{F}$; in particular, if $\mathscr{I}$ is a sheaf of ideals of $\mathscr{O}_X$, $\Gamma_*(\mathscr{F})$ is canonically a graded ideal of $\Gamma_*(\mathscr{O}_X)$.

Suppose that M is a graded S-module. For any $f \in S_d$ with $d > 0$, $x \mapsto x/1$ is a homomorphism of abelian groups $M_0 \mapsto M_{(f)}$, and as $M_{(f)}$ is canonically identified with $\Gamma(D_+(f), \widetilde{M})$, we obtain a homomorphism $\alpha_0^f : M_0 \to \Gamma(D_+(f), \widetilde{M})$ of abelian groups. It is clear that, for any $g \in S_e$ with $e > 0$, the diagram

$$\begin{tikzcd} & \Gamma(D_+(f), \widetilde{M}) \arrow[d] \\ M_0 \arrow[ur, "\alpha_0^f"] \arrow[dr, "\alpha_0^{fg}"'] & \\ & \Gamma(D_+(fg), \widetilde{M}) \end{tikzcd}$$

is commutative, and this signifies that for any $x \in M_0$, the sections $\alpha_0^f(x)$ and $\alpha_0^{fg}(x)$ of M coincide over $D_+(fg)$, and therefore there exists a unique section $\alpha_0(x) \in \Gamma(X, \widetilde{M})$ whose restriction on $D_+(f)$ is $\alpha_0^f(x)$. We then define (under the hypothesis that S_+ is generated by S_1) a homomorphism

$$\alpha_0 : M_0 \to \Gamma(X, \widetilde{M}).$$

By applying this result on each graded S-module $M(n)$ (where $n \in \mathbb{Z}$), we then obtain homomorphisms of abelian groups

$$\alpha_n : M_n = M(n)_0 \to \Gamma(X, \widetilde{M}(n))$$

and therefore a homomorphism of graded abelian groups

$$\alpha : M \to \Gamma_*(\widetilde{M})$$

(also denoted by α_M) such that α_M coincides with α_n on each M_n.

If we consider in particular $M = S$, then it is easy to see that (by the definition of the multiplication of $\Gamma_*(\mathscr{O}_X)$) $\alpha : S \to \Gamma_*(\mathscr{O}_X)$ is a homomorphism of graded rings, therefore, for any S-module M, α is a bi-homomorphism of graded modules.

Proposition 5.2.33. *For any $f \in S_d$ with $d > 0$, $D_+(f)$ is identified with the subset of $\mathfrak{p} \in X$ such that the section $\alpha_d(f)$ of $\mathscr{O}_X(d)$ is nonzero at $\mathfrak{p}$.*

Proof. As $X = \bigcup_{g \in S_1} D_+(g)$, it suffices to prove that for any $g \in S_d$, the set $\mathfrak{p} \in D_+(g)$ where $\alpha_d(f)$ is nonzero is identified with $D_+(fg)$. Now the restriction of $\alpha_d(f)$ to $D_+(g)$ is by defnition the section corresponding to the element $f/1$ of $S(d)_{(g)}$; by the canonical isomorphism $S(d)_{(g)} \cong S_{(g)}$, this section of $\mathscr{O}_X(d)$ over $D_+(g)$ is identified with the section of $\mathscr{O}_X$ over $D_+(g)$ corresponding to the element f/g^d of $S_{(g)}$. To see that this section is zero on $\mathfrak{p} \in D_+(g)$ then signifies that $f/g^d \in \mathfrak{q}$, where $\mathfrak{q}$ is the prime ideal of $S_{(g)}$ corresponding to $\mathfrak{p}$; by definition this means $f \in \mathfrak{p}$, whence the proposition. □

Now let $\mathscr{F}$ be an $\mathscr{O}_X$-module and put $M = \Gamma_*(\mathscr{F})$. In view of the homomorphism $\alpha : S \to \Gamma_*(\mathscr{O}_X)$ of graded rings, M can also be considered as a graded S-module. For any $f \in S_d$ ($d > 0$), it follows from Proposition 5.2.33 that the restriction of the section $\alpha_d(f)$ of $\mathscr{O}_X(d)$ to $D_+(f)$ is invertible, and so is the restriction to $D_+(f)$ of $\alpha_d(f^n)$ for any $n > 0$. Let $z \in M_{nd} = \Gamma(X, \mathscr{F}(nd))$, if there existss an integer $k \geq 0$ such that the restriction to $D_+(f)$ of $f^k z$, which is the section $(\alpha_d(f^k)z)|_{D_+(f)}$ of $\mathscr{F}((n+k)d)$, is zero, then we conclude that $z|_{D_+(f)} = 0$. This shows that we can define an $S_{(f)}$-homomorphism $\beta_f : M_{(f)} \to \Gamma(D_+(f), \mathscr{F})$ which corresponds the element $z/f^n \in M_{(f)}$ the section $(z|_{D_+(f)})(\alpha_d(f^n)|_{D_+(f)})^{-1}$ of $\mathscr{F}$ over $D_+(f)$. We also verify that for $g \in S_e$ ($e > 0$), the diagram

$$\begin{tikzcd}
M_{(f)} \arrow[r, "\beta_f"] \arrow[d] & \Gamma(D_+(f), \mathscr{F}) \arrow[d] \\
M_{(fg)} \arrow[r, "\beta_{fg}"] & \Gamma(D_+(fg), \mathscr{F})
\end{tikzcd}$$

is commutative. Since $M_{(f)}$ is canonically identified with $\Gamma(D_+(f), \widetilde{M})$ and the $D_+(f)$ form a basis for the topological space X, the homomorphisms β_f glue together to a unique canonical homomorphism of $\mathscr{O}_X$-modules

$$\beta : \widetilde{\Gamma_*(\mathscr{F})} \to \mathscr{F}$$

(also denoted by $\beta_{\mathscr{F}}$) which is evidently functorial.

Proposition 5.2.34. *Let M be a graded S-module and $\mathscr{F}$ be an $\mathscr{O}_X$-module. Then the composition homomorphisms*

$$\widetilde{M} \xrightarrow{\tilde{\alpha}} (\Gamma_*(\widetilde{M}))^{\sim} \xrightarrow{\beta} \widetilde{M} \tag{2.4.1}$$

$$\Gamma_*(\mathscr{F}) \xrightarrow{\alpha} \Gamma_*(\widetilde{\Gamma_*(\mathscr{F})}) \xrightarrow{\Gamma_*(\beta)} \Gamma_*(\mathscr{F}) \tag{2.4.2}$$

are isomorphisms.

Proof. The verification of (2.4.1) is local: on an open subset $D_+(f)$, this result follows from the definition and the fact that the action of β is determined by its action of sections over $D_+(f)$ (Corollary 4.1.3). The verification of (2.4.2) can be done at each degree: if we put $M = \Gamma_*(\mathscr{F})$, we have $M_n = \Gamma(X, \mathscr{F}(n))$ and $\Gamma_*(\widetilde{M})_n = \Gamma(X, \widetilde{M}(n)) = \Gamma(X, \widetilde{M(n)})$. If $f \in S_1$ and $z \in M_n$, $\alpha_n^f(z)$ is the element $z/1$ of $M(n)_{(f)}$, and equals to $(f/1)^n(z/f^n)$; it then corresponds by β_f to the section

$$(\alpha_1(f)^n|_{D_+(f)})(z|_{D_+(f)})(\alpha_1(f)^n|_{D_+(f)})^{-1}$$

over $D_+(f)$, which is the restriction of z to $D_+(f)$. □

In general, the homomorphisms α and β are not isomorphisms (for example, a graded S-module M can be nonzero with $\widetilde{M}$ being zero). To obtain some nice results about these two homomorphisms, we need to impose further conditions on the graded ring S and the graded S-module M.

Proposition 5.2.35. *Let S be a graded ring and A be a ring.*

(a) If S is Noetherian, then $X = \mathrm{Proj}(S)$ is a Noetherian scheme.

(b) If S is a graded A-algebra of finite type, then $X = \mathrm{Proj}(S)$ is a scheme of finite type over $Y = \mathrm{Spec}(A)$.

Proof. If S is Noetherian, the ideal S_+ is generated by finitely many homogeneous elements $(f_i)_{1\leq i\leq p}$, so the space X is the union of $D_+(f_i) = \mathrm{Spec}(S_{(f_i)})$, and since each $S_{(f_i)}$ is Noetherian by corollary 5.2.4, we see X is Noetherian.

Now assume that S is an A-algebra of finite type, then S_0 is an A-algebra of finite type and S is an S_0-algebra of finite type, so S_+ is a finitely generated ideal by Corollary **??**. We are then reduced to prove as in (a) that for any $f \in S_d$, $S_{(f)}$ is an A-algebra of finite type. In view of Proposition 5.2.3, it suffices to show that $S^{(d)}$ is an A-algebra of finite type, which follows from Proposition **??**. □

Let M be a graded S-module. We say M is **eventually null** if there exists an integer n such that $M_i = 0$ for $i \neq n$, and is **eventually finite** if there exists an integer n such that $\bigoplus_{i\geq n} M_i$ is a finitely generated S-module, or equivalently, that there exists a finitely generated graded sub-S-module M' of M such that M/M' is eventually null. We also note that if M is eventually null, then $M_{(f)} = 0$ for any homogeneous element f in S_+, so $\widetilde{M} = 0$.

Let M, N be two graded S-modules. We say a homomorphism $u : M \to N$ of degree 0 is **eventually injective** (resp. **eventually surjective**, **eventually bijective**) if there exists an integer

n such that $u_i : M_i \to N_i$ is injective (resp. surjective, bijective) for $i \geq n$. Equivalently, the homomorphism u is eventually injective (resp. eventually surjective) if and only if $\ker u$ (resp. $\operatorname{coker} u$) is eventually null. If u is eventually bijective, we say it is an **eventual isomorphism**.

Proposition 5.2.36. *Let S be a graded ring such that S_+ is finitely generated and M be a graded S-module.*

(a) If M is eventually finite, the $\mathscr{O}_X$-module $\widetilde{M}$ is of finite type.

(b) Suppose that M is eventually finite, then for $\widetilde{M} = 0$, it is necessary and sufficient that M is eventually null.

Corollary 5.2.37. *Let S be a graded ring such that S_+ is finitely generated. For $X = \operatorname{Proj}(S) = \varnothing$, it is necessary and sufficient that S is eventually null.*

Proof. The condition $X = \varnothing$ is in fact equivalent to $\mathscr{O}_X = \widetilde{S} = 0$, and S is clearly a finite generated S-module. □

Example 5.2.38. To give a counterexample of Proposition 5.2.36, let k be a field and $S = k[x_1, \dots, x_n]$ be the polynomial ring with n variables. Consider the graded S-module M given by

$$M = k[x_1, x_2, \dots, x_n, \dots]/(x_1, x_2^2, \dots, x_n^n, \dots)$$

with the usual multiplication of polynomials. Then M is not finitely generated over S, and for every homogeneous polynomial f of degree 1, we see $f^N M = 0$ for sufficiently large N. Therefore $\widetilde{M} = 0$. However, note that $M_n \neq 0$ for every n.

Theorem 5.2.39. *Suppose that S is a graded ring such that the ideal S_+ is finitely generated by S_1, and let $X = \operatorname{Proj}(S)$. Then, for any quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, the canonical homomorphisms $\beta : \widetilde{\Gamma_*(\mathscr{F})} \to \mathscr{F}$ is an isomorphism.*

Proof. If S_+ is generated by finitely many $f_i \in S_1$, then X is the union of $\operatorname{Spec}(S_{(f_i)})$ which are quasi-compact, so X is quasi-compact. Also, $\mathscr{O}_X(n)$ is invertible for any $n \in \mathbb{Z}$ by Corollary 5.2.29, and since X is separated, by Corollary 4.6.15 and Proposition 5.2.33, we have for any $f \in S_d$ a canonical isomorphism $\Gamma_*(\mathscr{F})_{(\alpha_d(f))} \cong \Gamma(D_+(f), \mathscr{F})$ (the first module (considered as a $\Gamma_*(\mathscr{O}_X)$-module) is none other than $\Gamma_*(\mathscr{F})_{(f)}$ (considered as an S-module)). If we trace the definition of this isomorphism, we see that it coincides with β_f, whence our assertion. □

Corollary 5.2.40. *Under the hypotheses of Theorem 5.2.39, any quasi-coherent $\mathscr{O}_X$-module (of finite type) is isomorphic to an $\mathscr{O}_X$-module of the form $\widetilde{M}$, where M is a (finitely generated) graded S-module.*

Proof. Let $\mathscr{F}$ be a quasi-coherent $\mathscr{O}_X$-module, then $\mathscr{F} = \widetilde{M}$ for a graded S-module M by Theorem 5.2.39. Let $(f_\lambda)_{\lambda \in I}$ be a system of homogeneous generators of M; for each finite subset H of I, let M_H be the graded submodule of M generated by f_λ for $\lambda \in H$. It is clear that M is the inductive limit of the submodules M_H, so $\mathscr{F}$ is the inductive limit of the sub-$\mathscr{O}_X$-modules $\widetilde{M}_H$ (Proposition 5.2.21). If $\mathscr{F}$ is of finite type, we conclude from Proposition 1.4.10. □

Corollary 5.2.41. *Under the hypotheses of Theorem 5.2.39, for any quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$ of finite type, there exists an integer n_0 such that for any $n \geq n_0$, $\mathscr{F}(n)$ is isomorphic to a quotient of $\mathscr{O}_X^r$ (where $r > 0$ depends on n), and therefore is generated by finitely many global sections.*

Proof. By Corollary 5.2.40, we can assume that $\mathscr{F} = \widetilde{M}$ where M is a quotient of a finite dierct sum of $S(m_i)$. By Proposition 5.2.21, we are therefore reduced to the case where $M = S(m)$, so $\mathscr{F}(n) = (S(m+n))^\sim = \mathscr{O}_X(m+n)$. It then suffices to prove that for each $n \geq 0$ there exists r and a surjective homomorphism $\mathscr{O}_X^r \to \mathscr{O}_X(n)$. For this, it suffcies to prove that, for a suitable r, there exists an eventually surjective homomorphism $u : S^r \to S(n)$ of degree zero. Now we have $S(n)_0 = S_n$, and by hypothesis $S_h = S_1^h$ for any $h > 0$, so $SS_n = \bigoplus_{h \geq n} S_h$. As S_n is a finitely generated S_0-module (Corollary **??**), consider a system $(a_i)_{1 \leq i \leq r}$ of generators of this module, and let $u : S^r \to S(n)$ be the homomorphism that sends the i-th basis e_i of S^r to a_i. Then the image of u contains $\bigoplus_{h \geq 0} S(n)_h$, so u satisfies the requirement and the proof is complete. □

Corollary 5.2.42. *Under the hypotheses of Theorem 5.2.39, for any quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$ of finite type, there exists an integer n_0 such that for any $n \geq n_0$, $\mathscr{F}$ is isomorphic to a quotient of an $\mathscr{O}_X$-module of the form $\mathscr{O}_X(-n)^r$ (where $r > 0$ depends on n).*

Proof. This follows from Corollary 5.2.41 by tensoring with the invertible sheaf $\mathscr{O}_X(-n)$, which preserves the exactness. □

Proposition 5.2.43. *Assume the hypotheses of Theorem 5.2.39 and let M be a graded S-module.*

(a) The cannical homomorphism $\tilde{\alpha} : \widetilde{M} \to (\Gamma_*(\widetilde{M}))^\sim$ is an isomorphism.

(b) Let $\mathscr{G}$ be a quasi-coherent sub-$\mathscr{O}_X$-module of $\widetilde{M}$ and let N be the graded sub-S-module of M which is the inverse image of $\Gamma_*(\mathscr{G})$ under α. Then we have $\widetilde{N} = \mathscr{G}$.

Proof. As $\beta : (\Gamma_*(\widetilde{M}))^\sim \to \widetilde{M}$ is an isomorphism, $\tilde{\alpha}$ is its inverse isomorphism in view of (2.4.1), whence (a). Let P be the graded submodule $\alpha(M)$ of $\Gamma_*(\widetilde{M})$; as $\widetilde{M}$ is an exact functor, the image of $\widetilde{M}$ under $\tilde{\alpha}$ is equal to $\widetilde{P}$, so in view of (a), $\widetilde{P} = (\Gamma_*(\widetilde{M}))^\sim$. Put $Q = \Gamma_*(\mathscr{G}) \cap P$, so that $N = \alpha^{-1}(Q)$. Then by the preceding argument and Proposition 5.2.21, the image of $\widetilde{N}$ under $\tilde{\alpha}$ is $\widetilde{Q}$, and we have $\widetilde{Q} = \widetilde{\Gamma_*(\mathscr{G})}$. Since the image of $\widetilde{\Gamma_*(\mathscr{G})}$ under β is $\mathscr{G}$ and $\tilde{\alpha}$ is the inverse of β, we conclude that $\widetilde{N} = \mathscr{G}$. □

5.2.5 Functorial properties of Proj(S)

Let S, S' be two graded rings with positive degree and $\varphi : S' \to S$ be a homomorphisms of graded rings. We denote by $G(\varphi)$ the open subset of $X = \mathrm{Proj}(S)$ which is the complement of $V_+(\varphi(S'_+))$, or, the union of $D_+(\varphi(f'))$ where f' runs through homogeneous elements of S'_+. The restriction to $G(\varphi)$ of the continuous map ${}^a\varphi : \mathrm{Spec}(S') \to \mathrm{Spec}(S)$ is then a continuous map from $G(\varphi)$ to $\mathrm{Proj}(S')$, which is still denoted by ${}^a\varphi$. If $f' \in S'_+$ is homogeneous, we have

$$ {}^a\varphi^{-1}(D_+(f')) = D_+(\varphi(f')) \tag{2.5.1}$$

since ${}^a\varphi$ maps $G(\varphi)$ into $\mathrm{Proj}(S')$. On the other hand, the homomorphism φ defines canonically a homomorphism of graded rings $S'_{f'} \to S_f$ of degree 0 (where $f = \varphi(f')$), whence a homomorphism $S'_{(f')} \to S_{(f)}$, which we denote by $\varphi_{(f)}$. It then corresponds to a morphism $({}^a\varphi_{(f)}, \tilde{\varphi}_{(f)}) : \mathrm{Spec}(S_{(f)}) \to \mathrm{Spec}(S'_{(f')})$ of affine schemes. If we identify $\mathrm{Spec}(S_{(f)})$ with the open subscheme $D_+(f)$ of $\mathrm{Proj}(S)$, we then obtain a morphism $\Phi_f : D_+(f) \to D_+(f')$ and ${}^a\varphi_{(f)}$ is identified with the restriction of ${}^a\varphi$ to $D_+(f)$. If g' is another homogeneous element of S'_+ and $g = \varphi(g')$, it is immediate that the diagram

$$\begin{tikzcd} D_+(fg) \arrow[r, "\Phi_{fg}"] \arrow[d] & D_+(f'g') \arrow[d] \\ D_+(f) \arrow[r, "\Phi_f"] & D_+(f') \end{tikzcd}$$

is commutative.

Proposition 5.2.44. *Let $\varphi : S' \to S$ be a homomorphism of graded rings. There exists a unique morphism $({}^a\varphi, \tilde{\varphi}) : G(\varphi) \to \mathrm{Proj}(S')$ (called the morphism associated with φ and denoted by $\mathrm{Proj}(\varphi)$), such that for any homogeneous element $f' \in S'_+$, the restriction os this morphism to $D_+(\varphi(f'))$ coincides with the morphism associated with the homomorphism $\varphi_{(f)} : S'_{(f')} \to S_{(\varphi(f'))}$.*

Proof. The morphism $({}^a\varphi, \tilde{\varphi})$ is obtained form glueing the morphisms Φ_f over $D_+(f)$, and the claim property is immediate. □

Corollary 5.2.45. *Let $\varphi : S' \to S$ be a homomorphism of graded rings.*

(a) The morphism $\mathrm{Proj}(\varphi)$ *is affine.*

(b) If $\ker\varphi$ *is nilpotent (and in particular if φ is injective), the morphism* $\mathrm{Proj}(\varphi)$ *is dominant.*

(c) If φ is eventually surjective, then $G(\varphi) = \mathrm{Proj}(S)$.

Proof. The first assertion follows from Proposition 5.2.45 and the relation ${}^a\varphi^{-1}(D_+(f')) = D_+(\varphi(f'))$. On the other hand, if $\ker\varphi$ is nilpotent, for any f' homogeneous in S'_+, we verify that $\ker\varphi_f$ is also nilpotent, and so is $\ker\varphi_{(f)}$. The conclusion then follows from Proposition **??**. Finally, if φ is eventually surjective, then every homogeneous element $f \in S_+$ has some power contained in the image of φ, so by Corollary 5.2.11 we conclude that $G(\varphi) = \bigcup_{f' \in S'_+} D_+(\varphi(f')) = \mathrm{Proj}(S)$, whence the claim. □

Remark 5.2.1. Note that there are in general morphisms from $\mathrm{Proj}(S)$ to $\mathrm{Proj}(S')$ which are not affine, and therefore do not come from graded ring homomorphisms $S' \to S$; an example is the structural morphism $\mathrm{Proj}(S) \to \mathrm{Spec}(A)$, where A is field ($\mathrm{Spec}(A)$ being identified with $\mathrm{Proj}(A[T])$ (cf. Corollary 5.3.5)).

Let $\varphi' : S'' \to S'$ be another homomorphism of graded rings, and put $\varphi'' = \varphi \circ \varphi'$. Then by the formula ${}^a\varphi'' = {}^a\varphi' \circ {}^a\varphi$ and $G(\varphi'') \subseteq G(\varphi)$, if Φ, Φ', and Φ'' are the associated morphisms of φ, φ' and φ'', we have $\Phi'' = \Phi' \circ (\Phi|_{G(\varphi'')})$.

Suppose that S (resp. S') is a graded A-algebra (resp. a graded A'-algebra), and let $\psi : A' \to A$ be a homomorphism of rings such that the diagram

$$\begin{tikzcd} A' \arrow[r, "\psi"] \arrow[d] & A \arrow[d] \\ S' \arrow[r, "\varphi"] & S \end{tikzcd}$$

is commutative. We can then consider $G(\varphi)$ and $\mathrm{Proj}(S')$ as schemes over $\mathrm{Spec}(A)$ and $\mathrm{Spec}(A')$, respectively. If Φ and Ψ are the associated morphisms of φ and ψ, respectively, the diagram

$$\begin{tikzcd} G(\varphi) \arrow[r, "\Phi"] \arrow[d] & \mathrm{Proj}(S') \arrow[d] \\ \mathrm{Spec}(A) \arrow[r, "\Psi"] & \mathrm{Spec}(A') \end{tikzcd}$$

is commutative.

Now let M be a graded S-module and consider the S'-module $\varphi^*(M)$, which is clearly graded. Let f' be a homogeneous element in S'_+, and set $f = \varphi(f')$. We then have a canonical isomorphism $(\varphi^*(M))_{f'} \cong \varphi^*_f(M_f)$, and it is clear that this isomorphism preserves degrees, so induces an isomorphism $(\varphi^*(M))_{(f')} \cong \varphi^*_{(f)}(M_{(f)})$. There is then canonically an isomorphism of sheaves $\widetilde{\varphi^*(M)}|_{D_+(f')} \cong (\Phi_f)_*(\widetilde{M}|_{D_+(f)})$ (Proposition 4.1.11). Moreover, if g' is another homogeneous element of S'_+ and $g = \varphi(g')$, the diagram

$$\begin{tikzcd} (\varphi^*(M))_{(f')} \arrow[r, "\sim"] \arrow[d] & (M_{(f)})_{(\varphi_{(f)})} \arrow[d] \\ (\varphi^*(M))_{(f'g')} \arrow[r, "\sim"] & (M_{(fg)})_{(\varphi_{(fg)})} \end{tikzcd}$$

is commutative, whence we conclude that the isomorphism

$$\widetilde{\varphi^*(M)}|_{D_+(f'g')} \cong (\Phi_{fg})_*(\widetilde{M}|_{D_+(fg)})$$

is the restriction to $D_+(f'g')$ of the isomorphism $\widetilde{\varphi^*(M)}|_{D_+(f')} \cong (\Phi_f)_*(\widetilde{M}|_{D_+(f)})$. As Φ_f is the restriction of Φ on $D_+(f)$, we then obtain the following result:

Proposition 5.2.46. *There exists a canonical isomorphism $\widetilde{\varphi^*(M)} \cong \Phi_*(\widetilde{M}|_{G(\varphi)})$ of $\mathscr{O}_X$-modules.*

We also deduce a canonical functorial map from the set of φ-homomorphisms $M' \to M$ from a graded S'-module to a graded S-module M, to the set of Φ-morphisms $\widetilde{M}' \to \widetilde{M}|_{G(\varphi)}$. If $\varphi' : S'' \to S'$ is another ring homomorphism and M'' is a graded S''-module, the composition of a φ-morphism $M' \to M$ and a φ'-morphism $M'' \to M'$ canonically corresponds to the composition of $\widetilde{M}'|_{G(\varphi')} \to \widetilde{M}|_{G(\varphi'')}$ and $\widetilde{M}'' \to \widetilde{M}'|_{G(\varphi')}$.

Proposition 5.2.47. *Let $\varphi : S' \to S$ be a homomorphism of graded rings and M' be a graded S'-module. Then there exists a canonical homomorphism $\nu : \Phi^*(\widetilde{M}') \to \widetilde{\varphi_*(M)}|_{G(\varphi)}$. If the ideal S'_+ is generated by S'_1, then ν is an isomorphism.*

Proof. For $f' \in S'_d$ with $d > 0$, we define a canonical homomorphism of $S_{(f)}$-modules (where $f = \varphi(f')$)

$$\nu_f : M'_{(f)} \otimes_{S'_{(f)}} S_{(f)} \to (M' \otimes_{S'} S)_{(f)}$$

by composing the homomorphism $M'_{(f)} \otimes_{S'_{(f)}} S_{(f)} \to M_{f'} \otimes_{S'_{f'}} S_f$ with the canonical homomorphism $M'_{f'} \otimes_{S'_{f'}} S_f \cong (M' \otimes_{S'} S)_f$. It is immediate to verify that compatibility of ν_f with the restriction homomorphisms $D_+(f)$ to $_+(fg)$ (for $g' \in S'_+$ and $g = \varphi(g')$), so we obtain a homomorphism

$$\nu : \Phi^*(\widetilde{M}') \to \widetilde{\varphi_*(M)}|_{G(\varphi)}.$$

For the second assertion, it suffices to prove that ν_f is an isomorphism for each $f' \in S'_1$, since $G(\varphi)$ is the union of $D_+(\varphi(f'))$. We first define a $\mathbb{Z}$-bilinear map $M'_m \times S_n \to M'_{(f')} \otimes_{S'_{f'}} S_{(f)}$ which sends (x', s) to the element $(x'/f'^m) \otimes (s/f^n)$. As in the proof of Proposition 5.2.28, this map then induces a bi-homomorphism

$$\eta_f : M' \otimes_{S'} S \to M'_{(f')} \otimes_{S'_{(f')}} S_{(f)}.$$

Moreover, if, for $r > 0$, we have $f^r \sum_i (x'_i \otimes s_i) = 0$, then $\sum_i (f'^r x'_i \otimes s_i) = 0$, so $\sum_i (f'^r x_i / f'^{m_i + r}) \otimes (s_i / f^{n_i}) = 0$, which means $\eta_f(\sum_i x_i \otimes y_i) = 0$; the homomorphism then factors through $(M' \otimes_{S'} S)_f$ and gives a homomorphism $\tilde{\eta}_f : (M' \otimes_{S'} S)_f \to M'_{(f')} \otimes_{S'_{(f')}} S_{(f)}$. It is easy to verify that $\tilde{\eta}_f$ is the inverse of ν_f, whence our assertion. □

In particular, since $\varphi_*(S'(n)) = S(n)$ for each $n \in \mathbb{Z}$, it follows from Proposition 5.2.47 that we have a canonical homomorphism $\Phi^*(\mathscr{O}_{X'}(n)) \to \mathscr{O}_X(n)|G(\varphi)$, and this an isomorphism if S'_+ is generated by S'_1.

Remark 5.2.2. We note that it follows from Proposition 5.2.18 that the morphism Φ is unchanged if we replace S by $S^{(d)}$, S' by $S'^{(d)}$, and φ by $\varphi^{(d)}$. Also, it is also unchanged if we replace S_0 and S'_0 by $\mathbb{Z}$ and φ_0 be the identity map.

Let A, A' be two rings and $\psi : A' \to A$ be a homomorphism of rings, which defines a morphism $\Psi : \mathrm{Spec}(A) \to \mathrm{Spec}(A')$. Let S' be an A'-algebra with positive degrees, and put $S = S' \otimes_{A'} A$, which is a graded A-algebra by setting $S_n = S'_n \otimes_{A'} A$. The map $s' \mapsto s' \otimes 1$ is then a homomorphism of graded rings and also a bi-homomorphism. Since S_+ is the A-module generated by $\varphi(S'_+)$, we have $G(\varphi) = \mathrm{Proj}(S) = X$, so, if we put $X' = \mathrm{Proj}(S')$, we have a commutative diagram

$$\begin{tikzcd} X \arrow[r, "\Phi"] \arrow[d, "p"'] & X' \arrow[d] \\ Y \arrow[r, "\Psi"] & Y' \end{tikzcd} \tag{2.5.2}$$

Now let M' be a graded S'-module, and set $M = M' \otimes_{A'} A = M' \otimes_{S'} S$.

Proposition 5.2.48. *The commutative diagram (2.5.2) is cartesian and the canonical homomorphism $\nu : \Phi^*(\widetilde{M}') \to \widetilde{M}$ in Proposition 5.2.47 is an isomorphism.*

Proof. The first assertion follows if we can prove that for any f' homogeneous in S'_+ and $f = \varphi(f')$, the restriction of Φ and p to $D_+(f)$ identify this scheme as $D_+(f') \times_{Y'} Y$; in other words, it suffices to prove that $S_{(f)}$ is canonically identified with $S'_{(f')} \otimes_{A'} A$, which is immedaite from the fact that the canonical isomorphism $S_f \cong S'_{f'} \otimes_{A'} A$ preserves degrees. The second assertion follows from the isomorphism $M'_{(f')} \otimes_{S'_{(f')}} S_{(f)} \cong M'_{(f')} \otimes_{A'} A$, and the later one is isomorphic to $M_{(f)}$ since M_f is canonically identified with $M'_{f'} \otimes_{A'} A$. □

Corollary 5.2.49. *For any integer $n \in \mathbb{Z}$, $\widetilde{M}(n)$ is identified with $\Phi^*(\widetilde{M}'(n)) = \widetilde{M}'(n) \otimes_{Y'} \mathscr{O}_Y$. In particular, $\mathscr{O}_X(n) = \Phi^*(\mathscr{O}_{X'}(n)) = \mathscr{O}_{X'}(n) \otimes_{Y'} \mathscr{O}_Y$.*

Proof. This follows from Proposition 5.2.48 and Corollary 5.2.30. □

Now let $\mathscr{F}'$ be an $\mathscr{O}_{X'}$-module and $\mathscr{F} = \Phi^*(\mathscr{F}')$. Then we have for each $n \in \mathbb{Z}$ that $\mathscr{F}(n) = \Phi^*(\mathscr{F}'(n))$ in view of Corollary 5.2.49. Therefore, by the definition of Φ^*, we have a canonical homomorphism

$$\Gamma(\rho) : \Gamma(X', \mathscr{F}'(n)) \to \Gamma(X, \mathscr{F}(n))$$

which then gives a canonical bi-homomorphism $\Gamma_*(\mathscr{F}') \to \Gamma_*(\mathscr{F})$ of graded modules.

Suppose that the ideal S'_+ is generated by S'_1 and $\mathscr{F}' = \widetilde{M}'$, so $\mathscr{F} = \widetilde{M}$ where $M = M' \otimes_{A'} A$. If f' is homogeneous in S'_+ and $f = \varphi(f')$, we have $M_{(f)} = M'_{(f')} \otimes_{A'} A$ and the diagram

$$\begin{tikzcd}
M'_0 \arrow[r] \arrow[d] & M'_{(f')} = \Gamma(D_+(f), \widetilde{M}') \arrow[d] \\
M_0 \arrow[r] & M_{(f)} = \Gamma(D_+(f), \widetilde{M})
\end{tikzcd}$$

is commutative. We then conclude from the definition of the homomorphism $\alpha : M \to \Gamma_*(\widetilde{M})$ that the following diagram

$$\begin{tikzcd}
M' \arrow[r, "\alpha_{M'}"] \arrow[d] & \Gamma_*(\widetilde{M}') \arrow[d] \\
M \arrow[r, "\alpha_M"] & \Gamma_*(\widetilde{M})
\end{tikzcd} \tag{2.5.3}$$

is commutative. Similarly, the diagram

$$\begin{tikzcd}
\widetilde{\Gamma_*(\mathscr{F}')} \arrow[r, "\beta_{\mathscr{F}'}"] \arrow[d] & \mathscr{F}' \arrow[d] \\
\widetilde{\Gamma_*(\mathscr{F})} \arrow[r, "\beta_{\mathscr{F}}"] & \mathscr{F}
\end{tikzcd} \tag{2.5.4}$$

is commutative (where the vertical is the canonical Φ-morphism $\mathscr{F}' \to \Phi^*(\mathscr{F}') = \mathscr{F}$).

Now let N' be another graded S'-module and $N = N' \otimes_{A'} A$. It is immediate that the canonical bi-homomorphisms $M' \to M$, $N' \to N$ give a bi-homomorphism $M' \otimes_{S'} N' \to M \otimes_S N$, and therefore an S-homomorphism $(M' \otimes_{S'} N') \otimes_{A'} A \to M \otimes_S N$ of degree 0, which then corresponds to an $\mathscr{O}_X$-homomorphism

$$\Phi^*((M' \otimes_{S'} N')^\sim) \to (M \otimes_S N)^\sim.$$

Moreover, it is immedaite to verify that the following diagram

$$\begin{tikzcd}
\Phi^*(\widetilde{M}' \otimes_{\mathscr{O}_{X'}} \widetilde{N}') \arrow[r, "\sim"] \arrow[d, "\Phi^*(\lambda)"] & \widetilde{M} \otimes_{\mathscr{O}_X} \widetilde{N} = \Phi^*(\widetilde{M}') \otimes_{\mathscr{O}_X} \Phi^*(\widetilde{N}') \arrow[d, "\lambda"] \\
\Phi^*((M' \otimes_{S'} N')^\sim) \arrow[r] & (M \otimes_S N)^\sim
\end{tikzcd} \tag{2.5.5}$$

is commutative (where the first row is an isomorphism by (3.3.1)). If the ideal S'_+ is generated by S'_1, it is clear that S_+ is generated by S_1, so the two vertical homomorphisms are isomorphisms, so the second row is also an isomorphism.

We have similarly a cannical bi-homomorphism $\mathrm{Hom}_{S'}(M', N') \to \mathrm{Hom}_S(M, N)$, which sends a homomorphism u' of degree k the homomorphism $u' \otimes 1$, which is also of degree k. We then deduce an S-homomorphism of degree 0:

$$\mathrm{Hom}_{S'}(M', N') \otimes_{A'} A \to \mathrm{Hom}_S(M, N)$$

which corresponds to a homomorphism of $\mathscr{O}_X$-modules:

$$\Phi^*((\mathrm{Hom}_{S'}(M', N'))^{\sim}) \to (\mathrm{Hom}_S(M, N))^{\sim}.$$

Similarly, the diagram

$$\begin{tikzcd} \Phi^*((\mathrm{Hom}_{S'}(M', N'))^{\sim}) \arrow[r] \arrow[d, "\Phi^*(\mu)"] & (\mathrm{Hom}_S(M, N))^{\sim} \arrow[d, "\mu"] \\ \Phi^*(\mathscr{H}om_{\mathscr{O}_{X'}}(\widetilde{M}', \widetilde{N}')) \arrow[r] & \mathscr{H}om_{\mathscr{O}_X}(\widetilde{M}, \widetilde{N}) \end{tikzcd} \tag{2.5.6}$$

is commutative (where the second row is the canonical homomorphism of (3.4.5)).

5.2.6 Closed subschemes of Proj(S)

Recall that if $\varphi : S \to S'$ is a homomorphism of graded rings, we say that φ is eventually surjective (resp. eventually injective, eventually bijective) if $\varphi_i : S_i \to S'_i$ is surjective (resp. injective, bijective) for sufficiently large i. It follows from Proposition 5.2.18 that the study of Φ can be reduced to the case where φ is surjective (resp. injectij, bijective). Instead of saying that φ is eventually bijective, we also say that it is then an eventual isomorphism.

Proposition 5.2.50. *Let S, S' be graded rings with positive degrees and set $X = \mathrm{Proj}(S)$, $X' = \mathrm{Proj}(S')$.*

(a) If $\varphi : S \to S'$ is an eventually surjective homomorphism of graded rings, the corresponding morphism Φ is defined over $\mathrm{Proj}(S')$ and is a closed immersion. If $\mathfrak{I}$ is the kernel of φ, the closed subscheme of X associated with Φ is defined by the quasi-coherent ideal $\widetilde{\mathfrak{I}}$ of $\mathscr{O}_X$.

(b) Suppose moreover that the ideal S_+ is finitely generated by S_1. Let X' be a closed subscheme of X defined by a quasi-coherent ideal $\mathscr{I}$ of $\mathscr{O}_X$. Let $\mathfrak{I}$ be the graded ideal of S which is the inverse image of $\Gamma_*(\mathscr{I})$ under the canonical homomorphism $\alpha : S \to \Gamma_*(\mathscr{O}_X)$, and put $S' = S/\mathfrak{I}$. Then X' is the subscheme associated with the closed immersion $\mathrm{Proj}(S') \to X$ corresponding to the canonical homomorphism $S \to S'$ of graded rings.

Proof. Let $\varphi : S \to S'$ be an eventually surjective homomorphism of graded rings. We can suppose that φ is surjective, so $\varphi(S_+)$ is generated by S'_+, we have $G(\varphi) = \mathrm{Proj}(S')$. Now the second assertion in (a) can be verified locally over X; let f be a homogeneous element of S_+ and put $f' = \varphi(f)$. As φ is a surjective homomorphism of rings, $\varphi_{(f')} : S_{(f)} \to S'_{(f')}$ is surjective with kernel $\mathscr{I}_{(f)}$, so the corresponding morphism is closed.

We now consider the case of (b); in view of (a), we only need to verify that the homomorphism $\tilde{j} : \widetilde{\mathscr{I}} \to \mathscr{O}_X$ induced from the injection $j : \mathscr{I} \to S$ is an isomorphism from $\widetilde{\mathscr{I}}$ to $\mathscr{I}$, which follows from Proposition 5.2.43(b). □

Remark 5.2.3. Note that $\mathfrak{I}$ is the largest graded ideal $\mathfrak{I}'$ of S such that $\widetilde{\mathscr{I}'} = \mathscr{I}$ (where we identify $\widetilde{\mathscr{I}'}$ as a subsheaf of $\mathscr{O}_X$), since one immedaitely verify that this relation implies $\alpha(\mathfrak{I}') \subseteq \Gamma_*(\mathscr{I})$.

Corollary 5.2.51. *Assume the hypotheses of Proposition 5.2.50(a) and that S_+ is generated by S_1. Then $\Phi^*(\widetilde{S(n)})$ is canonically isomorphic to $\widetilde{S'(n)}$ for any $n \in \mathbb{Z}$, and therefore $\Phi^*(\mathscr{F}(n))$ is isomorphic to $\Phi^*(\mathscr{F})(n)$ for any $\mathscr{O}_X$-module $\mathscr{F}$.*

Proof. This is a partiuclar case of Proposition 5.2.47, in view of the definition of $\mathscr{F}(n)$ and Proposition 5.2.50(a). □

Corollary 5.2.52. *Assume the hypotheses of Proposition 5.2.50(a). Then for the closed subscheme X' of X to be integral, it is necessary and sufficient that the graded ideal $\mathfrak{I}$ is prime in S.*

Proof. As X' is isomorphic to $\mathrm{Proj}(S/\mathfrak{I})$, this condition is sufficient in view of Proposition 5.2.16. To see the necessity, assume that $\mathrm{Proj}(S')$ is integral and consider the exact sequence $0 \to \mathscr{I} \to \mathscr{O}_X \to \mathscr{O}_X/\mathscr{I} \to 0$, which gives an exact sequence

$$0 \longrightarrow \Gamma_*(\mathscr{I}) \longrightarrow \Gamma_*(\mathscr{O}_X) \longrightarrow \Gamma_*(\mathscr{O}_X/\mathscr{I})$$

by the left-exactness of the global section functor. In view of the canonical homomorphism $\alpha : S/\mathfrak{I} \to \Gamma_*(\mathscr{O}_X/\mathscr{I})$, it then suffices to prove that if $f \in S_m$, $g \in S_n$ are such that the image in $\Gamma_*(\mathscr{O}_X/\mathscr{I})$ of $\alpha_{n+m}(fg)$ is zero, then one of the images of $\alpha_m(f)$, $\alpha_n(g)$ is zero. Now by definition, these images are sectins of the invertible $(\mathscr{O}_X/\mathscr{I})$-modules $\mathscr{L} = (\mathscr{O}_X/\mathscr{I})(m)$ and $\mathscr{L}' = (\mathscr{O}_X/\mathscr{I})(n)$ over the integral scheme X'. The hypotheses implies that their product is zero in $\mathscr{L} \otimes \mathscr{L}'$ (Corollary 5.2.29), so one of them is zero by Corollary 4.7.25. □

Corollary 5.2.53. *Let A be a ring, M be an A-module, and S be a graded A-algebra generated by S_1. Let $u : M \to S_1$ be a surjective homomorphism of A-modules and $\bar{u} : \boldsymbol{S}(M) \to S$ be the unique homomorphism of A-algebras extending u. Then the morphism corresponding to $\bar{u}$ is a closed immersion from* $\mathrm{Proj}(S)$ *to* $\mathrm{Proj}(\boldsymbol{S}(M))$.

Proof. The homomorphism $\bar{u}$ is surjective by hypothesis, so it suffices to apply Proposition 5.2.50. □

5.3 Homogeneous spectrum of sheaves of graded algebras

Let Y be a scheme and $\mathscr{S}$ be an $\mathscr{O}_Y$-algebra. We say that $\mathscr{S}$ is **graded** if $\mathscr{S}$ is the direct sum of a family $(\mathscr{S}_n)$ of $\mathscr{O}_Y$-algebras such that $\mathscr{S}_m\mathscr{S}_n \subseteq \mathscr{S}_{m+n}$. If $\mathscr{S}$ is a graded $\mathscr{O}_Y$-algebra, by a **graded $\mathscr{S}$-module** $\mathscr{M}$ we mean an $\mathscr{S}$-module $\mathscr{M}$ which is the direct sum of a family $(\mathscr{M}_n)_{n\in\mathbb{Z}}$ such that $\mathscr{S}_m\mathscr{M}_n \subseteq \mathscr{M}_{m+n}$. We say $\mathscr{S}$ is a graded $\mathscr{O}_Y$-algebra with positive degrees if $\mathscr{S}_n = 0$ for $n < 0$, and $\mathscr{M}$ is a graded $\mathscr{S}$-module if $\mathscr{M}_n = 0$ for $n < 0$. In this section, without further specifications, we will only consider graded algebras with positive degree.

5.3.1 Homogeneous spectrum of a graded $\mathscr{O}_Y$-algebra

Let $\mathscr{S}$ be a graded $\mathscr{O}_Y$-algebra (with positive degrees) and $\mathscr{M}$ be a graded $\mathscr{S}$-module. If $\mathscr{S}$ is quasi-coherent, each homogeneous component $\mathscr{S}_n$ is also a quasi-coherent $\mathscr{O}_Y$-module, since it is the image of $\mathscr{S}$ under the projection of $\mathscr{S}$ onto $\mathscr{S}_n$. Similarly, if $\mathscr{M}$ is quasi-coherent as an $\mathscr{O}_Y$-module, so is each of its homogeneous components, and the converse also holds. If $d > 0$ is an integer, we denote by $\mathscr{S}^{(d)}$ the direct sum of the $\mathscr{O}_Y$-modules $\mathscr{S}_{nd}$, which is quasi-coherent if $\mathscr{S}$ is; for any integer k such that $0 \le k \le d-1$, we denote by $\mathscr{M}^{(d,k)}$ (or $\mathscr{M}^{(d)}$ if $k=0$) the direct sum of $\mathscr{M}_{nd+k}$ (for $n \in \mathbb{Z}$). If $\mathscr{S}$ and $\mathscr{M}$ are quasi-coherent sheaves, $\mathscr{M}(n)$ is a quasi-coherent $\mathscr{S}$-module by Proposition 4.2.22.

We say that $\mathscr{M}$ is a graded $\mathscr{S}$-module **of finite type** (resp. **of finite presentation**) if for any $y \in Y$, there exists an open neighborhood U of y and integers n_i (resp. integers m_i and n_i) such that there exists a surjective homomorphism $\bigoplus_{i=1}^r(\mathscr{S}(n_i)|_U) \to \mathscr{M}|_U$ of degree 0 (resp. such that $\mathscr{M}|_U$ is isomorphic to the cokernel of a homomorphism $\bigoplus_{i=1}^r \mathscr{S}(m_i)|_U \to \bigoplus_{j=1}^r \mathscr{S}(n_j)|_U$ of degree 0).

Let U be an affine open of Y and $A = \Gamma(U, \mathscr{O}_Y)$ by its ring. By hypothesis, the graded $(\mathscr{O}_Y|_U)$-algebra $\mathscr{S}|_U$ is isomorphic to $\widetilde{S}$ where $S = \Gamma(U, \mathscr{S})$ is a graded A-algebra; we put $X_U = \mathrm{Proj}(\Gamma(U, \mathscr{S}))$. Let $U' \subseteq U$ be another affine open subset of Y and $j : U' \to U$ the canonical injection, which corresponds to a homomorphism $A \to A'$, we have $\mathscr{S}|_{U'} = j^*(\mathscr{S}|_U)$, and therefore $S' = \Gamma(U', \mathscr{S})$ is identified with $S \otimes_A A'$ by Proposition 4.1.13. We then conclude from Proposition 5.2.48 that $X_{U'}$ is canonically identified with $X_U \times_U U'$, and therefore with $p_U^{-1}(U')$, where p_U is the structural morphism $X_U \to U$. Let $\sigma_{U',U}$ be the canonical isomorphism $p_U^{-1}(U') \cong X_{U'}$ thus defined, and $\rho_{U',U}$ be the open immersion $X_{U'} \to X_U$ obtained by composing $\sigma_{U',U}^{-1}$ with the canonical injectin $p_U^{-1}(U') \to X_U$. It is immediate that if $U'' \subseteq U'$ is a third affine open of Y, we have $\rho_{U'',U} = \rho_{U'',U'} \circ \rho_{U',U}$.

Proposition 5.3.1. *Let Y be a scheme and $\mathscr{S}$ be a quasi-coherent graded $\mathscr{O}_Y$-algebra. Then there exists a unique scheme X over Y such that, if $p : X \to Y$ is the structural morphism, for any affine open U of Y, there exists an isomorphism $\eta_U : p^{-1}(U) \xrightarrow{\sim} X_U = \mathrm{Proj}(\Gamma(U, \mathscr{S}))$ such that, if V is another affine open of Y contained in U, the following diagram*

$$\begin{tikzcd} p^{-1}(V) \arrow[r, "\eta_V"] \arrow[d] & X_V \arrow[d, "\rho_{V,U}"] \\ p^{-1}(U) \arrow[r, "\eta_U"] & X_U \end{tikzcd}$$

is commutative. The scheme X is called the ***homogeneous specturm*** *of the quasi-coherent $\mathscr{O}_Y$-algebra $\mathscr{S}$ and denoted by* $\mathrm{Proj}(\mathscr{S})$.

Proof. For two affine opens U, V of Y, let $X_{U,V}$ be the scheme induced over $p_U^{-1}(U \cap V)$ by X_U; we shall define a Y-isomorphism $\theta_{U,V} : X_{V,U} \to X_{U,V}$. For this, consider an affine open $W \subseteq U \cap V$; by composing the isomorphisms

$$p_U^{-1}(W) \xrightarrow{\sigma_{W,U}} X_W \xrightarrow{\sigma_{W,V}^{-1}} p_V^{-1}(W)$$

we obtain an isomorphism $\tau_W : p_U^{-1}(W) \to p_V^{-1}(W)$, and we can verify that if $W' \subseteq W$ is an affine open, $\tau_{W'}$ is the restriction of τ_W to $p_U^{-1}(W')$; the morphisms τ_W then glue together to a Y-isomorphism $\theta_{V,U}$, which is what we want. Moreover, if U,V,W are affine opens of Y and $\theta'_{U,V}$, $\theta'_{V,W}$, and $\theta'_{U,W}$ are restrictions of $\theta_{U,V}$, $\theta_{V,W}$, $\theta_{U,W}$ on the inverse images of $U\cap V\cap W$ in X_V, X_W, X_W, respectively, it follows from the preceding definition that $\theta'_{U,V}\circ\theta'_{V,W}=\theta'_{U,W}$. The existence of X then follows from glueing these schemes via the isomorphisms $\theta_{U,V}$, and the uniqueness is clear. □

It is clear that the Y-scheme $\mathrm{Proj}(\mathscr{S})$ is separated over Y since homogeneous specturms are separated. If $\mathscr{S}$ is an $\mathscr{O}_Y$-algebra of finite type, it follows from Proposition 5.2.35 and Proposition 4.6.33 that $\mathrm{Proj}(\mathscr{S})$ is of finite type over Y. If $p:X\to Y$ is the structural morphism, it is immediate that for any open subschem U of Y, $p^{-1}(U)$ is identified with the homogeneous specturm $\mathrm{Proj}(\mathscr{S}|_U)$.

Proposition 5.3.2. *Let $f\in\Gamma(Y,\mathscr{S}_d)$ with $d>0$ and $X=\mathrm{Proj}(\mathscr{S})$. Then there exists an open subset X_f of X such that, for any affine open subset U of Y, we have $X_f\cap p^{-1}(U)=D_+(f|_U)$ in $X_U=\mathrm{Proj}(\Gamma(U,\mathscr{S}))$ (where $p:X\to Y$ is the structural morphism). Moreover, the Y-scheme induced over X_f by X is canonically isomorphic to $\mathrm{Spec}(\mathscr{S}^{(d)}/(f-1)\mathscr{S}^{(d)})$.*

Proof. For any affine open U, we have $f|_U\in\Gamma(U,\mathscr{S}_d)=\Gamma(U,\mathscr{S})_d$ since U is quasi-compact. If U,U' are two affine opens of Y such that $U'\subseteq U$, $f|_{U'}$ is the image of $f|_U$ by the restriction homomorphism $\Gamma(U,\mathscr{S})\to\Gamma(U',\mathscr{S})$, so $D_+(f|_{U'})$ is equal to the open subscheme $\rho_{U',U}^{-1}(D_+(f|_U))$ of $X_{U'}$. The subset X_f can be then defined by glueing these subschemes, and the first assertion is then obvious. On the other hand, the open subscheme $D_+(f|_U)$ of X_U is canonically identified with $\mathrm{Spec}(\Gamma(U,\mathscr{S})_{(f|_U)})$, and this identification is clearly compatible with restrictions; the second assertion then follows from Proposition 5.2.3. □

Corollary 5.3.3. *If $f\in\Gamma(Y,\mathscr{S}_d)$ and $g\in\Gamma(Y,\mathscr{S}_e)$, we have $X_{fg}=X_f\cap X_g$.*

Proof. It suffices to consider the intersection of two members of $p^{-1}(U)$, where U is an affine open of Y, and the assertion follows from $D_+(fg)=D_+(f)\cap D_+(g)$ for a graded ring S. □

Corollary 5.3.4. *Let (f_α) be a family of sections of $\mathscr{S}$ over Y such that $f_\alpha\in\Gamma(Y,\mathscr{S}_{d_\alpha})$. If the sheaf of ideals of $\mathscr{S}$ generated by this family containes all the $\mathscr{S}_n$ for sufficiently large n, then the underlying space X is the union of X_{f_α}.*

Proof. In fact, for any affine open U of Y, $p^{-1}(U)$ is the union of $X_{f_\alpha}\cap p^{-1}(U)$ by Corollary 5.2.11, so the claim follows from the construction of X_{f_α}. □

Corollary 5.3.5. *Let $\mathscr{A}$ be a quasi-coherent $\mathscr{O}_Y$-algebra and put*

$$\mathscr{S}=\mathscr{A}[T]=\mathscr{A}\otimes_{\mathbb{Z}}\mathbb{Z}[T]$$

where T is an indeterminate. Then $X=\mathrm{Proj}(\mathscr{S})$ is canonically identified with $\mathrm{Spec}(\mathscr{A})$. In particular, $\mathrm{Proj}(\mathscr{O}_Y[T])$ is identified with Y.

Proof. By applying Corollary 5.3.4 to the unique section $f\in\Gamma(Y,\mathscr{S})$ equal to T on each point of Y, we see that $X_f=X$. Moreover, we have $f\in\mathscr{S}_1$, and $\mathscr{S}^{(1)}/(f-1)\mathscr{S}^{(1)}=\mathscr{S}/(f-1)\mathscr{S}$ is canonically isomorphic to $\mathscr{A}$, whence the corollary □

Let $g \in \Gamma(Y, \mathscr{O}_Y)$; if we put $\mathscr{S} = \mathscr{O}_Y[T]$, then $g \in \Gamma(Y, \mathscr{S}_0)$; let

$$h = gT \in \Gamma(Y, \mathscr{S}_1).$$

If $X = \mathrm{Proj}(\mathscr{S})$, the canonical identification of Corollary 5.3.5 identifies X_h with the open subset Y_g of Y: in fact, we can assume that $Y = \mathrm{Spec}(A)$ is affine, and this then follows from the fact that the ring A_g is canonically identified with $A[T]/(gT-1)A[T]$.

Proposition 5.3.6. *Let $\mathscr{S}$ be a quasi-coherent graded $\mathscr{O}_Y$-algebra.*

(a) For any $d > 0$, there exists a canonical Y-isomorphism from $\mathrm{Proj}(\mathscr{S})$ to $\mathrm{Proj}(\mathscr{S}^{(d)})$.

(b) Let $\mathscr{S}'$ be the graded $\mathscr{O}_Y$-algebra $\mathscr{O}_Y \oplus \bigoplus_{n>0} \mathscr{S}_n$, then the schemes $\mathrm{Proj}(\mathscr{S}')$ and $\mathrm{Proj}(\mathscr{S})$ are canonically Y-isomorphic.

(c) Let $\mathscr{L}$ be an invertible $\mathscr{O}_Y$-module and $\mathscr{S}_{(\mathscr{L})}$ be the graded $\mathscr{O}_Y$-algebra $\bigoplus_{d>0} \mathscr{S}_d \otimes \mathscr{L}^{\otimes d}$; then the schemes $\mathrm{Proj}(\mathscr{S})$ and $\mathrm{Proj}(\mathscr{S}_{(\mathscr{L})})$ are canonically Y-isomorphic.

Proof. In all three cases, it suffices to define an isomorphism locally over Y and verifying the compatibility of restriction morphisms is immediate. We can then assume that Y is affine, and assertions (a) and (b) then follow from Proposition 5.2.18. As for (c), if the invertible sheaf $\mathscr{L}$ is just isomorphic to $\mathscr{O}_Y$ then the isomorphism between $\mathrm{Proj}(\mathscr{S})$ and $\mathrm{Proj}(\mathscr{S}_{(\mathscr{L})})$ is evident. To define a canonical isomorphism, let $Y = \mathrm{Spec}(A)$, $\mathscr{S} = \widetilde{S}$, where S is a graded A-algebra, and let c be a generator of the free A-module L such that $\mathscr{L} = \widetilde{L}$. Then for any $n > 0$, $x_n \mapsto x_n \otimes c^{\otimes n}$ is an A-isomorphism from S_n to $S_n \otimes L^{\otimes n}$, and these A-isomorphisms define an A-isomorphism of graded algebras

$$p_c : S \to S_{(L)} = \bigoplus_{n \geq 0} S_n \otimes L^{\otimes n}.$$

Let $f \in S_+$ be homogeneous of degree d; for any $x \in S_{nd}$, we have $(x \otimes c^{nd})/(f \otimes c^d)^n = (x \otimes (\varepsilon c)^{nd})/(f \otimes (\varepsilon c)^d)^n$ for any invertible element $\varepsilon \in A$, which implies that the isomorphism $S_{(f)} \to (S_{(L)})_{(f \otimes c^d)}$ induced by p_c is independent from the generator c of L, whence the assertion. □

Recall that for the quasi-coherent graded $\mathscr{O}_Y$-algebra $\mathscr{L}$ to be *generated* by the $\mathscr{O}_Y$-module $\mathscr{S}_1$, it is necessary and sufficient that there exists a covering (U_α) of Y by affine opens such that the graded algebra $\Gamma(U_\alpha, \mathscr{S})$ over $\Gamma(U_\alpha, \mathscr{S}_0)$ is generated by $\Gamma(U_\alpha, \mathscr{S}_1)$. If this is true, then for any open subset V of Y, $\mathscr{S}|_V$ is then generated by $(\mathscr{O}_Y|_V)$-algebra $\mathscr{S}_1|_V$.

Proposition 5.3.7. *Suppose that there exists a finite affine open cover (U_i) of Y such that the graded algebra $\Gamma(U_i, \mathscr{S})$ is of finite type over $\Gamma(U_i, \mathscr{O}_Y)$. Then there exists $d > 0$ such that $\mathscr{S}^{(d)}$ is generated by $\mathscr{S}_d$, which is an $\mathscr{O}_Y$-module of finite type.*

Proof. In fact, it follows from Proposition **??** that for each i, there exist an integer m_i such that $\Gamma(U_i, \mathscr{S}_{nm_i}) = (\Gamma(U_i, \mathscr{S}_{m_i}))^n$ for all $n > 0$; it suffices to take d a common multiple of the m_i. □

Corollary 5.3.8. *Under the hypotheses of Proposition 5.3.7, $\mathrm{Proj}(\mathscr{S})$ is Y-isomorphic to a homogeneous specturm $\mathrm{Proj}(\mathscr{S}')$, where $\mathscr{S}'$ is a graded $\mathscr{O}_Y$-algebra generated by $\mathscr{S}'_1$, where $\mathscr{S}'$ is an $\mathscr{O}_Y$-algebra of finite type.*

Proof. It suffices to take $\mathscr{S}' = \mathscr{S}^{(d)}$, where d is determined by the properties of Proposition 5.3.7, and apply Proposition 5.3.6(a). $\square$

If $\mathscr{S}$ is a quasi-coherent graded $\mathscr{O}_Y$-algebras, we have seen in Proposition 4.4.25 that its nilradical is a quasi-coherent $\mathscr{O}_Y$-module. We say that $\mathscr{N}_+ = \mathscr{N} \cap \mathscr{S}_+$ is the nilradical of $\mathscr{S}_+$, which is a quasi-coherent graded $\mathscr{S}_0$-module, since this is the case if Y is affine. For any $y \in Y$, $(\mathscr{N}_+)_y$ is then the nilradical of $(\mathscr{S}_+)_y = (\mathscr{S}_y)_+$. Similar to the local case, we say the graded $\mathscr{O}_Y$-algebra $\mathscr{S}$ is **essentially reduced** if $\mathscr{N}_+ = 0$, which means $\mathscr{S}_y$ is an essentially reduced graded $\mathscr{O}_{Y,y}$-algebra for any $y \in Y$. It is clear that for any quasi-coherent graded $\mathscr{O}_Y$-algebra $\mathscr{S}$, $\mathscr{S}/\mathscr{N}_+$ is essentially reduced. Finally, we say $\mathscr{S}$ is **integral** if $\mathscr{S}_y$ is an integral ring for each $y \in Y$ and if $(\mathscr{S}_+)_y = (\mathscr{S}_y))_+ \neq 0$ for all $y \in Y$.

Proposition 5.3.9. *Let $\mathscr{S}$ be a graded $\mathscr{O}_Y$-algebra. If $X = \mathrm{Proj}(\mathscr{S})$, the Y-scheme X_{red} is canonically isomorphic to $\mathrm{Proj}(\mathscr{S}/\mathscr{N}_+)$. In particular, if $\mathscr{S}$ is essentially reduced, then X is reduced.*

Proof. The fact that $X' = \mathrm{Proj}(\mathscr{S}/\mathscr{N}_+)$ is reduced follows from Proposition 5.2.16, since the question is local. Moreover, for any affine open $U \subseteq Y$, $p'^{-1}(U)$ is equal to $(p^{-1}(U))_{\mathrm{red}}$ (where p and p' are the structural morphisms $X \to Y$, $X' \to Y$, respectively); we also verify t hat the canonical U-morphisms $p'^{-1}(U) \to p^{-1}(U)$ is compatible with restrictions and define therefore a closed immersion $X' \to X$, which is a homeomorphism on underlying spaces. Our assertion then follows from Corollary 4.4.26. $\square$

Proposition 5.3.10. *Let Y be an integral scheme and $\mathscr{S}$ be a quasi-coherent graded $\mathscr{O}_Y$-algebra such that $\mathscr{S}_0 = \mathscr{O}_Y$.*

(a) If $\mathscr{S}$ is integral then $X = \mathrm{Proj}(\mathscr{S})$ is integral and the structural morphism $p : X \to Y$ is dominant.

(b) Suppose moreover that $\mathscr{S}$ is essentially reduced. Then, conversely, if X is integral and p is dominant, then $\mathscr{S}$ is integral.

Proof. We first assume that $\mathscr{S}$ is integral. Then if (U_α) is a basis of Y formed by affine opens, it suffices to prove for Y being replaced by U_α and $\mathscr{S}$ by $\mathscr{S}|_{U_\alpha}$: in fact, is this is true, the underlying space $p^{-1}(U_\alpha)$ is an open irreducible subset of X such that $p^{-1}(U_\alpha) \cap p^{-1}(U_\beta) \neq \varnothing$ for any couple of indices α, β (since $U_\alpha \cap U_\beta$ contains an U_γ and $\mathscr{S}$ is integral), so X is irreducible by Proposition **??**; it is clear that X is reduced since $\mathscr{S}$ is reduced, so X is integral. It is clear that $p(X)$ is dense in Y since this holds for each U_α.

Suppose then that $Y = \mathrm{Spec}(A)$ where A is integral (Proposition 4.4.27) and $\mathscr{S} = \widetilde{S}$, where S is a graded A-algebra; the hypotheses on $\mathscr{S}$ is that for any $y \in Y$, $\widetilde{S}_y = S_y$ is an integral graded ring such that $(S_y)_+ \neq 0$. It then suffices to prove that S is an integral ring, since then $S_+ \neq 0$ and we can apply Proposition 5.2.16. Now, let f, g be two nonzero elements of S and suppose that $fg = 0$; for any $y \in Y$ we have $(f/1)(g/1) =$ in S_y, so $f/1 = 0$ or $g/1 = 0$ by hypothesis. Suppose for example that $f/1 = 0$ in S_y, so there exists $a \in A$ such that $a \notin \mathfrak{p}_y$ and $af = 0$. We then see that for each $z \in Y$, $(a/1)(f/1) = 0$ in the integral ring S_z, and as $a/1 \neq 0$ (since A is integral), $f/1 = 0$, which implies $f = 0$.

Now consider the hypothesis in (b) and assume that X is integral and p is dominant. Since the question is local over Y, we can assume that $Y = \mathrm{Spec}(A)$, where A is an integral ring, and

$\mathscr{S} = \widetilde{S}$. By hypothesis for any $y \in Y$, $(S_y)_+$ is reduced, and so is $(S_0)_y = A_y$ by hypothesis, so S_y is a reduced ring and we conclude that S is reduced. The hypothesis that X is integral implies that S is essentially integral (Proposition 5.2.16). The proposition then boils down to see that the annihilator $\mathfrak{I}$ of S_+ over $A = S_0$ is reduced to zero. In the contrary case, we would have $(S_h)_+ = 0$ for an $h \neq 0$ in $\mathfrak{I}$, which implies $p^{-1}(D(h)) = \varnothing$ by Proposition 5.3.1, and $p(X)$ is then not dense in Y, contradicting the hypothesis (since $D(h) \neq \varnothing$, h is not nilpotent). We then see that the ring S is integral, which conclude our assertion. ◻

5.3.2 Sheaves associated with a graded $\mathscr{S}$-module

Let Y be a scheme, $\mathscr{S}$ be a quasi-coherent $\mathscr{O}_Y$-algebra, and $\mathscr{M}$ be a quasi-coherent graded $\mathscr{S}$-module over the ringed space $(Y, \mathscr{S})$. With the notations of Proposition 5.3.1, we denote by $\widetilde{\mathscr{M}}_U$ the quasi-coherent $\mathscr{O}_{X_U}$-module $\widetilde{\Gamma(U, \mathscr{M})}$. For $U' \subseteq U$, $\Gamma(U', \mathscr{M})$ is canonically identified with $\Gamma(U, \mathscr{M}) \otimes_A A'$ by Proposition 4.1.13, so $\widetilde{M}_{U'} = \rho^*_{U',U}(\widetilde{\mathscr{M}}_U)$ by Proposition 5.2.48.

Proposition 5.3.11. *There exists over* $\mathrm{Proj}(\mathscr{S}) = X$ *a unique quasi-coherent* $\mathscr{O}_X$*-module* $\widetilde{\mathscr{M}}$ *such that, for any affine open* U *of* Y*, we have* $\eta^*_U(\widetilde{\mathscr{M}}_U) = \widetilde{\mathscr{M}}|_{p^{-1}(U)}$*, where* $p : X \to Y$ *is the structural morphism and* η_U *is the isomorphism* $p^{-1}(U) \cong \mathrm{Proj}(\Gamma(U, \mathscr{S}))$*. We say that* $\widetilde{\mathscr{M}}$ *is the* $\mathscr{O}_X$*-module associated with* $\mathscr{M}$*.*

Proof. As $\rho_{U',U}$ is identified with the injection morphism $p^{-1}(U') \to p^{-1}(U)$, the proposition follows from the relation $\widetilde{M}_{U'} = \rho^*_{U',U}(\widetilde{\mathscr{M}}_U)$ and glueing the $\widetilde{\mathscr{M}}_U$. ◻

Proposition 5.3.12. *Let* $\mathscr{M}$ *be a quasi-coherent* $\mathscr{S}$*-module and* $f \in \Gamma(Y, \mathscr{S}_d)$ *with* $d > 0$*. If* ξ_f *is the canonical isomorphism from* X_f *to the* Y*-scheme* $Z_f = \mathrm{Spec}(\mathscr{S}^{(d)}/(f-1)\mathscr{S}^{(d)})$ *in Proposition 5.3.2, then* $(\xi_f)_*(\widetilde{\mathscr{M}}|_{X_f})$ *is the* $\mathscr{O}_{Z_f}$*-module* $(\mathscr{M}^{(d)}/(f-1)\mathscr{M}^{(d)})^\sim$*.*

Proof. The question is local over Y we we are reduced to Proposition 5.2.3, and its compatibility with restrictions. ◻

Proposition 5.3.13. *The* $\mathscr{O}_X$*-module* $\widetilde{\mathscr{M}}$ *is an additive exact covariant functor from the category of quasi-coherent graded* $\mathscr{S}$*-modules to the category of quasi-coherent* $\mathscr{O}_X$*-modules, which commutes with direct sums and inductive limits.*

Proof. This follows from Proposition 4.1.6 and Proposition 5.2.34, since the question is local on Y. ◻

In particular, if $\mathscr{N}$ is a quasi-coherent graded sub-$\mathscr{S}$-module of $\mathscr{M}$, $\widetilde{\mathscr{N}}$ is cannically identified with a quasi-coherent sub-$\mathscr{O}_X$-module of $\widetilde{\mathscr{M}}$; if we take $\mathscr{M} = \mathscr{S}$, then for any quasi-coherent ideal $\mathscr{J}$ of $\mathscr{S}$, $\widetilde{\mathscr{J}}$ is a quasi-coherent ideal of $\mathscr{O}_X$.

If $\mathscr{M}$ is a quasi-coherent graded $\mathscr{S}$-module and $\mathscr{J}$ is a quasi-coherent ideal of $\mathscr{O}_Y$, then $\mathscr{J}\mathscr{M}$ is a quasi-coherent graded sub-$\mathscr{S}$-module of $\mathscr{M}$ and we have $\widetilde{\mathscr{J}\mathscr{M}} = \mathscr{J} \cdot \widetilde{\mathscr{M}}$: it suffices to verify this formula if $Y = \mathrm{Spec}(A)$, $\mathscr{S} = \widetilde{S}$, where S is a graded A-algebra, $\mathscr{M} = \widetilde{M}$ where M is a graded S-module, and $\mathscr{J} = \widetilde{\mathfrak{a}}$, where $\mathfrak{a}$ is an ideal of A. For any homogeneous element f of S_+, the restriction to $D_+(f) = \mathrm{Spec}(S_{(f)})$ of $\widetilde{\mathscr{J}\mathscr{M}}$ is the assocaited sheaf of $(\mathfrak{a}M)_{(f)} = \mathfrak{a} \cdot M_{(f)}$, and the identification is compatible with restrictions.

Proposition 5.3.14. *Let $f \in \Gamma(Y, \mathscr{S}_d)$ with $d > 0$. Over the open subset X_f, the $(\mathscr{O}_X|_{X_f})$-module $\widetilde{\mathscr{S}(nd)}|_{X_f}$ is canonically isomorphic to $\mathscr{O}_X|_{X_f}$ for any $n \in \mathbb{Z}$. In particular, if the $\mathscr{O}_Y$-algebra $\mathscr{S}$ is generated by $\mathscr{S}_1$, the $\mathscr{O}_X$-module $\widetilde{\mathscr{S}(n)}$ is invertible for all $n \in \mathbb{Z}$.*

Proof. For any affine open U of Y, by Proposition 5.2.24 we have a canonical isomorphism $\widetilde{\mathscr{S}(nd)}|_{X_f \cap p^{-1}(U)} \cong \mathscr{O}_X|_{X_f \cap p^{-1}(U)}$, in view of Proposition 5.3.2 (where $p: X \to Y$ is the structural morphism). It is immediate that this isomorphism is compatible with restrictions, whence the first assertion. For the second one, if $\mathscr{S}$ is generated by $\mathscr{S}_1$, there exists an affine open cover (U_α) of Y such that $\Gamma(U_\alpha, \mathscr{S})$ is generated by $\Gamma(U_\alpha, \mathscr{S}_1) = \Gamma(U_\alpha, \mathscr{S})_1$, and we can then use Proposition 5.2.26. □

Again, for any integer $n \in \mathbb{Z}$ and any $\mathscr{O}_X$-module $\mathscr{F}$, we set

$$\mathscr{O}_X(n) = \widetilde{\mathscr{S}(n)}, \quad \mathscr{F}(n) = \mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{O}_X(n).$$

It follows from this definition that, for any open subset U of Y,

$$\widetilde{\mathscr{S}|_U(n)} = \mathscr{O}_X|_{p^{-1}(U)},$$

where $p: X \to Y$ is the structural morphism.

Proposition 5.3.15. *Let $\mathscr{M}$ and $\mathscr{N}$ be quasi-coherent graded $\mathscr{S}$-modules. Then there exists cannical homomorphisms*

$$\lambda : \widetilde{M} \otimes_{\mathscr{O}_X} \widetilde{\mathscr{N}} \to (\mathscr{M} \otimes_{\mathscr{S}} \mathscr{N})^\sim$$
$$\mu : (\mathscr{H}om_{\mathscr{S}}(\mathscr{M}, \mathscr{N}))^\sim \to \mathscr{H}om_{\mathscr{O}_X}(\widetilde{\mathscr{M}}, \widetilde{\mathscr{N}}).$$

If $\mathscr{S}$ is generated by $\mathscr{S}_1$, then λ is an isomorphism; if moreover $\mathscr{M}$ is of finite presentation, μ is an isomorphism.

Proof. The isomorphisms λ and μ are defined in the arguments before Proposition 5.2.28 if Y is affine, and this definition is local and then glue together to define global morphisms, in view of the diagrams (2.5.5) and (2.5.6). □

Corollary 5.3.16. *If $\mathscr{S}$ is generated by $\mathscr{S}_1$, for any integers $m, n \in \mathbb{Z}$, we have*

$$\mathscr{O}_X(m) \otimes_{\mathscr{O}_X} \mathscr{O}_X(n) = \mathscr{O}_X(m+n), \quad \mathscr{O}_X(n) = \mathscr{O}_X^{\otimes 1}.$$

Corollary 5.3.17. *If $\mathscr{S}$ is generated by $\mathscr{S}_1$, for any quasi-coherent graded $\mathscr{S}$-module $\mathscr{M}$ and $n \in \mathbb{Z}$, we have*

$$\widetilde{\mathscr{M}(n)} = \widetilde{\mathscr{M}}(n).$$

Remark 5.3.1. If $\mathscr{S} = \mathscr{A}[T]$ where $\mathscr{A}$ is a quasi-coherent $\mathscr{O}_Y$-algebra, we verify immediately that the invertible $\mathscr{O}_X$-module $\mathscr{O}_X(n)$ is canonically isomorphic to $\mathscr{O}_X$. Moreover, let $\mathscr{N}$ be a quasi-coherent $\mathscr{A}$-module, and put $\mathscr{M} = \mathscr{N} \otimes_{\mathscr{A}} \mathscr{A}[T]$. It follows from Proposition 5.3.12 and Corollary 5.3.5 that under the canonical isomorphism of $X = \mathrm{Proj}(\mathscr{A}[T])$ and $X' = \mathrm{Spec}(\mathscr{A})$, the $\mathscr{O}_X$-module $\widetilde{\mathscr{M}}$ is identified with the $\mathscr{O}_X$-module $\widetilde{\mathscr{N}}$.

Remark 5.3.2. Let $\mathscr{S}$ be a quasi-coherent graded $\mathscr{O}_Y$-algebra and define $\mathscr{S}'$ to be the $\mathscr{O}_Y$-algebra such that $\mathscr{S}' = \mathscr{O}_Y$ and $\mathscr{S}'_n = \mathscr{S}_n$ for $n > 0$. Then the canonical isomorphism of $X = \mathrm{Proj}(\mathscr{S})$ and $\mathrm{Proj}(\mathscr{S}')$ identifies $\mathscr{O}_X(n)$ and $\mathscr{O}_{X'}(n)$ for any $n \in \mathbb{Z}$: this follows from the corresponding result in the affine case (Example 5.2.31) and the fact that this identification is compatible with restrictions. Similarly, let $X^{(d)} = \mathrm{Proj}(\mathscr{S}^{(d)})$; the canonical isomorphism of X and $X^{(d)}$ identifies $\mathscr{O}_X(nd)$ with $\mathscr{O}_{X^{(d)}}(n)$ for any $n \in \mathbb{Z}$.

Proposition 5.3.18. *Let $\mathscr{L}$ be an invertible $\mathscr{O}_Y$-module and ψ be the structural morphism from $X_{(\mathscr{L})} = \mathrm{Proj}(\mathscr{S}_{(\mathscr{L})})$ to $X = \mathrm{Proj}(\mathscr{S})$. Then for any integer $n \in \mathbb{Z}$, $\psi_*(\mathscr{O}_{X_{(\mathscr{L})}}(n))$ is canonically isomorphic to $\mathscr{O}_X(n) \otimes_Y \mathscr{L}^{\otimes n}$.*

Proof. Suppose first that Y is affine with ring A and $\mathscr{L} = \widetilde{L}$, where L is a free A-module of rank 1. With the notations of Proposition 5.3.6(c), we define for each $f \in S_d$ an isomorphism from $S(n)_{(f)} \otimes_A L^{\otimes n}$ to $S_{(L)}(n)_{(f\otimes c^d)}$ which sends $(x/f^k) \otimes c^n$, where $x \in S_{kd+n}$, to the element $(x \otimes c^{n+kd})/(f \otimes c^d)^k$. It is immedaite that this isomorphism is independent of the generator c of L, and is compatible with restrictions $D_+(f) \to D_+(fg)$. The general case then follows from glueing these isomorphisms. □

5.3.3 Graded $\mathscr{S}$-modules assocaited with a sheaf

For simplicity, in the following discussion, we always assume that the quasi-coherent $\mathscr{O}_Y$-algebra $\mathscr{S}$ is generated by $\mathscr{S}_1$, which by Proposition 5.3.6 is not at all essential if we impose the finiteness conditions of Proposition 5.3.7 on Y. Let $p : X \to Y$ be the structural morphism where $X = \mathrm{Proj}(\mathscr{S})$, which is separated by Proposition 5.2.14. For any $\mathscr{O}_X$-module $\mathscr{F}$, we define

$$\Gamma_*(\mathscr{F}) = \bigoplus_{n\in\mathbb{Z}} p_*(\mathscr{F}(n))$$

and in particular

$$\Gamma_*(\mathscr{O}_X) = \bigoplus_{n\in\mathbb{Z}} p_*(\mathscr{O}_X(n)).$$

We have seen in (3.2.1) that there exists a canonical homomorphism

$$p_*(\mathscr{F}) \otimes_{\mathscr{O}_Y} p_*(\mathscr{G}) \to p_*(\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G})$$

for any $\mathscr{O}_X$-modules $\mathscr{F}$ and $\mathscr{G}$, so we deduce from Corollary 5.3.16 that $\Gamma_*(\mathscr{O}_X)$ is endowed with a graded $\mathscr{O}_Y$-algebra structure and $\Gamma_*(\mathscr{F})$ is a graded module over $\Gamma_*(\mathscr{O}_X)$.

In view of Proposition 5.3.14 and the left-exactness of the functor f_*, $\Gamma_*(\mathscr{F})$ is an additive left-exact covariant functor from the category of $\mathscr{O}_X$-modules to the category of graded $\mathscr{O}_Y$-modules. In particular, if $\mathscr{J}$ is a sheaf of ideals of $\mathscr{O}_X$, $\Gamma_*(\mathscr{J})$ is identified with a sheaf of graded ideals of $\Gamma_*(\mathscr{O}_X)$.

Now let $\mathscr{M}$ be a quasi-coherent graded $\mathscr{S}$-module. For any affine open U of Y, we have defined a homomorphism of abelian groups

$$\alpha_{0,U} : \Gamma(U, \mathscr{M}_0) \to \Gamma(p^{-1}(U), \widetilde{\mathscr{M}}).$$

It is immediate that these homomorphisms commutes with restrictions and define (which do not use the hypothesis that $\mathscr{S}$ is generated by $\mathscr{S}_1$) a homomorphism of sheaf of abelian groups

$$\alpha_0 : \mathscr{M}_0 \to \widetilde{\mathscr{M}}.$$

Apply this result to $\mathscr{M}_n = (\mathscr{M}(n))_0$ and use Corollary 5.3.17, we define a homomorphism of abelian groups

$$\alpha_n : \mathscr{M}_n \to p_*(\widetilde{\mathscr{M}}(n)) \tag{3.3.1}$$

for each $n \in \mathbb{Z}$, whence a functorial homomorphism of graded sheaves of abelian groups

$$\alpha : \mathscr{M} \to \Gamma_*(\widetilde{\mathscr{M}}) \tag{3.3.2}$$

(we also denote it by $\alpha_{\mathscr{M}}$). In the particular case $\mathscr{M} = \mathscr{S}$, we verify that $\alpha : \mathscr{S} \to \Gamma_*(\mathscr{O}_X)$ is a homomorphism of graded $\mathscr{O}_Y$-algebra and is a bi-homomorphism of graded modules, relative to this homomorphism of graded homomorphism of algebras.

We also remark that the homomorphism α_n corresponds to a canonical homomorphism of $\mathscr{O}_X$-modules

$$\alpha_n^\sharp : p^*(\mathscr{M}_n) \to \widetilde{\mathscr{M}}(n).$$

Moreover, it is easy to verify that this homomorphism is none other than the associated homomorphism (by Proposition 5.3.13) of the canonical homomorphism $\mathscr{M}_n \otimes_{\mathscr{O}_Y} \mathscr{S} \to \mathscr{M}(n)$ of $\mathscr{O}_Y$-modules, where the $\mathscr{O}_Y$-module $\mathscr{M}_n \otimes_{\mathscr{O}_Y} \mathscr{S}$ is given the natrual graduation. To see this, we can in fact assume that $Y = \mathrm{Spec}(A)$ is affine, $\mathscr{M} = \widetilde{M}$ and $\mathscr{S} = \widetilde{S}$, where S is a graded A-algebra generated by S_1. Returning to the definition of α, we see that the restriction to $D_+(f)$ of the homomorphism $\alpha_n^\sharp$ corresponds to the homomorphism $M_n \otimes_A S_{(f)} \to M(n)_{(f)}$, where $x \otimes 1$ is maped to $x/1$.

Proposition 5.3.19. *For any section $f \in \Gamma(Y, \mathscr{S}_d)$ with $d > 0$, X_f is identified with the set of points of X on which $\alpha_d(f)$ is nonzero.*

Proof. The element $\alpha_d(f)$ is a section of $p_*(\mathscr{O}_X(d))$ over Y, and by definition is then a section of $\mathscr{O}_X(d)$ over X. The definition of X_f (Proposition 5.3.2) proves our claim in the affine case, in view of Proposition 5.2.33. □

We shall henceforth suppose, in addition to the hypothesis at the beginning of this part, that for every quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, $p_*(\mathscr{F}(n))$ is quasi-coherent over Y, and therefore $\Gamma_*(\mathscr{F}) = \bigoplus_{n\in\mathbb{Z}} p_*(\mathscr{F}(n))$ is also a quasi-coherent $\mathscr{O}_Y$-module; this circumstance will always occur if X is of finite type on Y (Proposition 4.6.54). We then conclude that $\widetilde{\Gamma_*(\mathscr{F})}$ is defined and is a quasi-coherent $\mathscr{O}_X$-module. For any affine open subset U of Y, we have (Proposition 4.1.6, Proposition 5.3.13, and note that U is quasi-compact)

$$\begin{aligned}(\Gamma(U, \bigoplus_{n\in\mathbb{Z}} p_*(\mathscr{F}(n))))^\sim &= \bigoplus_{n\in\mathbb{Z}} (\Gamma(U, p_*(\mathscr{F}(n))))^\sim = \bigoplus_{n\in\mathbb{Z}} (\Gamma(p^{-1}(U), \mathscr{F}(n)))^\sim \\ &= \Big(\bigoplus_{n\in\mathbb{Z}} \Gamma(p^{-1}(U), \mathscr{F}(n))\Big)^\sim = (\Gamma_*(\mathscr{F}|_{p^{-1}(U)}))^\sim\end{aligned}$$

and therefore a canonical homomorphism

$$\beta_U : \big(\Gamma(U, \bigoplus_{n\in\mathbb{Z}} p_*(\mathscr{F}(n)))\big)^{\sim} \to \mathscr{F}|_{p^{-1}(U)}.$$

Moreover, the diagram (2.5.4) shows that these homomorphisms are compatible with restrictions on Y, so we deduce a canonical homomorphism

$$\beta : \widetilde{\Gamma_*(\mathscr{F})} \to \mathscr{F}$$

(also denoted by $\beta_{\mathscr{F}}$) for the quasi-coherent $\mathscr{O}_X$-modules.

Proposition 5.3.20. *Let $\mathscr{M}$ be a quasi-coherent $\mathscr{S}$-module and $\mathscr{F}$ be a quasi-coherent $\mathscr{O}_X$-module. Then the composition homomorphisms*

$$\widetilde{\mathscr{M}} \xrightarrow{\tilde{\alpha}} (\Gamma_*(\widetilde{\mathscr{M}}))^{\sim} \xrightarrow{\beta} \widetilde{\mathscr{M}} \tag{3.3.3}$$

$$\Gamma_*(\mathscr{F}) \xrightarrow{\alpha} \Gamma_*(\widetilde{\Gamma_*(\mathscr{F})}) \xrightarrow{\Gamma_*(\beta)} \Gamma_*(\mathscr{F}) \tag{3.3.4}$$

Proof. The question is local over Y, so we can apply Proposition 5.2.34. □

Again, the homomorphisms α and β are in general not isomorphisms, and further finiteness conditions must be imposed. We note also that the homomorphism β is not always defined, unlike the affine case. However, we shall see that if $\mathscr{S}$ is of finite type and generated by $\mathscr{S}_1$, the homomorphisms α and β are well defined and the corresponding results of the affine cases carry over without difficulties.

Proposition 5.3.21. *Let Y be a scheme and $\mathscr{S}$ be a quasi-coherent $\mathscr{O}_Y$-algebra generated by $\mathscr{S}_1$. Suppose that $\mathscr{S}_1$ is of finite type, then $X = \mathrm{Proj}(\mathscr{S})$ is of finite type over Y.*

Proof. Again we can assume that Y is affine with ring A, so $\mathscr{S} = \widetilde{S}$ where S is a graded A-algebra generated by S_1, and S_1 is a finitely generated A-module by hypothesis. Then S is an A-algebra of finite type, and the proposition follows from Proposition 5.2.35. □

Let $\mathscr{S}$ be a quasi-coherent graded $\mathscr{O}_Y$-algebra and $\mathscr{M}$ be a quasi-coherent $\mathscr{S}$-module. We say that $\mathscr{M}$ is **eventually null** if there exists an integer n such that $\mathscr{M}_k = 0$ for $k \geq n$, and is **eventually finite** if there exists an integer n such that the $\mathscr{S}$-module $\bigoplus_{k\geq n} \mathscr{M}_k$ is of finite type. If $\mathscr{M}$ is eventually null, it is clear that $\widetilde{\mathscr{M}} = 0$, as in the affine case.

Let $\mathscr{M}$ and $\mathscr{N}$ be quasi-coherent graded $\mathscr{S}$-modules. We say a homomorphism $u : \mathscr{M} \to \mathscr{N}$ of degree 0 is eventually injective (resp. eventually sujective, eventually bijective) if there exists an integer n such that $u_k : \mathscr{M}_k \to \mathscr{N}_k$ is injective (resp. surjective, bijective) for $k \geq n$. It is clear that in this case, $\tilde{u} : \widetilde{\mathscr{M}} \to \widetilde{\mathscr{N}}$ is injective (resp. surjective, bijective), since this can be checked locally over Y and we can apply Proposition 5.3.13. If u is eventually bijective, we also say that it is an eventual isomorphism.

Proposition 5.3.22. *Let $\mathscr{Y}$ be a scheme and $\mathscr{S}$ be a quasi-coherent $\mathscr{O}_Y$-algebra generated by $\mathscr{S}_1$ and such that $\mathscr{S}_1$ is of finite type. Let $\mathscr{M}$ be a quasi-coherent graded $\mathscr{S}$-module.*

(a) *If $\mathscr{M}$ is eventually finite, $\widetilde{\mathscr{M}}$ is of finite type.*

(b) *If $\mathscr{M}$ is eventually finite, for $\widetilde{\mathscr{M}} = 0$, it is necessary and sufficient that $\mathscr{M}$ is eventually null.*

Proof. The question is local over Y, so we can assume that $Y = \mathrm{Spec}(A)$, and the proposition then follows from Proposition 5.2.36. ☐

Theorem 5.3.23. *Let Y be a scheme and $\mathscr{S}$ be a quasi-coherent $\mathscr{O}_Y$-algebra generated by $\mathscr{S}_1$ and such that $\mathscr{S}_1$ is of finite type. Let $X = \mathrm{Proj}(\mathscr{S})$, then for any quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, the canonical homomorphism $\beta : \widetilde{\Gamma(\mathscr{F})} \to \mathscr{F}$ is an isomorphism.*

Proof. We first note that the homomorphism β is defined because of Proposition 5.3.21. To see that β is an isomorphism, we can assume $Y = \mathrm{Spec}(A)$ is affine, and then apply Proposition 5.2.39. ☐

Corollary 5.3.24. *Under the hypotheses of Theorem 5.3.23, any quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$ is isomorphic to an $\mathscr{O}_X$-module of the form $\widetilde{\mathscr{M}}$, where $\mathscr{M}$ is a quasi-coherent $\mathscr{S}$-module. If moreover $\mathscr{F}$ is of finite type, and if we suppose that Y is a quasi-compact scheme, then we can choose $\mathscr{M}$ to be of finite type.*

Proof. The first assertion follows from Theorem 5.3.23 by take $\mathscr{M} = \Gamma_*(\mathscr{F})$. For the second one, it suffices to prove that $\mathscr{M}$ is the inductive limit of graded sub-$\mathscr{S}$-modules of finite type $\mathscr{N}_\lambda$: in fact, it then follows that $\widetilde{\mathscr{M}}$ is the inductive limit of the $\widetilde{\mathscr{N}_\lambda}$ (Proposition 5.3.13), hene $\mathscr{F}$ is the inductive limit of the $\beta(\mathscr{N}_\lambda)$. As X is quasi-compact (Proposition 5.3.21) and $\mathscr{F}$ is of finite type, $\mathscr{F}$ then necessarily equal to one of the $\beta(\widetilde{\mathscr{N}_\lambda})$ (Proposition 1.4.10).

To define the $\mathscr{N}_\lambda$, it suffices to consider for each $n \in \mathbb{Z}$ the quasi-coherent $\mathscr{O}_Y$-module $\mathscr{M}_n$, which is the inductive limit of its sub-$\mathscr{O}_Y$-modules $\mathscr{M}_n^{(\mu_n)}$ of finite type (by Corollary 4.6.62). It is immediate that $\mathscr{P}_{\mu_n} = \mathscr{S} \cdot \mathscr{M}_n^{(\mu_n)}$ is a graded $\mathscr{S}_n$-module of finite type, and $\mathscr{M}$ is then the inductive limit of finite direct sums of these $\mathscr{S}$-modules. ☐

Corollary 5.3.25. *Suppose the hypotheses of Theorem 5.3.23 and that Y is quasi-compact. Let $\mathscr{F}$ be a quasi-coherent $\mathscr{O}_X$-module of finite type. There exists an integer n_0 such that for $n \geq n_0$, the canonical homomorphism $\sigma : p^*(p_*(\mathscr{F}(n))) \to \mathscr{F}(n)$ is surjective.*

Proof. For any $y \in Y$, let U be an affine open neighborhood of y in Y. There then exists an inter $n_0(U)$ such that, for $n \geq n_0(U)$, $\mathscr{F}(n)|_{p^{-1}(U)}$ is generated by finitely many sections over $p^{-1}(U)$ (Corollary 5.2.41); but these are canonical images of sections of $p^*(p_*(\mathscr{F}(n)))$ over $p^{-1}(U)$, so $\mathscr{F}(n)|_{p^{-1}(U)}$ is equal to the canonical image of $p^*(p_*(\mathscr{F}(n)))|_{p^{-1}(U)}$. Finally, as Y is quasi-compact, there is a finite affine open cover (U_i) of Y, and we can choose n_0 to be the largest of the $n_0(U_i)$. ☐

Remark 5.3.3. If $p : X \to Y$ is a morphism of ringed spaces and $\mathscr{F}$ is an $\mathscr{O}_X$-module, the fact that the canonical homomorphism $\sigma : p^*(p_*(\mathscr{F})) \to \mathscr{F}$ is surjective is explained as follows: for any $x \in X$ and any sections of $\mathscr{F}$ over an open neighborhood V of x, there exists an open neighborhood U of $p(x)$ in Y, finitely many sections $(t_i)_{1 \leq i \leq m}$ of $\mathscr{F}$ over $p^{-1}(U)$, a neighborhood $W \subseteq V \cap p^{-1}(U)$ of x and sections $(a_i)_{1 \leq i \leq m}$ of $\mathscr{O}_X$ over W such that

$$s|_W = \sum_i a_i \cdot (t_i|_W).$$

If Y is an affine scheme and $p_*(\mathscr{F})$ is *quasi-coherent*, this condition is equivalent to the fact that $\mathscr{F}$ is generated by its sections over X: in fact, if $Y = \operatorname{Spec}(A)$, we can suppose that $U = D(f)$ with $f \in A$. Since $p_*(\mathscr{F})$ is quasi-coherent, by Proposition 4.1.20 there exists an integer $n > 0$ and sections s_i of $\mathscr{F}$ over X such that $g^n t_i$ is the restriction of s_i (where $g = \rho(f)$, and $\rho : A \to \Gamma(X, \mathscr{O}_X)$ is the homomorphism corresponding to p by Proposition 4.2.4) to $p^{-1}(U)$. As g is invertible over $p^{-1}(U)$, we then have

$$s|_W = \sum_i b_i \cdot (s_i|_W)$$

where $b_i = a_i \cdot (g|_W)^{-n}$, whence our assertion. Therefore, if Y is affine, Corollary 5.3.25 then recovers Corollary 5.2.41, in view of Corollary 1.4.11.

We finally conclude that if Y is a scheme, then the following three conditions are equivalent for a quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$ such that $p_*(\mathscr{F})$ is quasi-coherent:

(i) The canonical homomorphism $\sigma : p^*(p_*(\mathscr{F})) \to \mathscr{F}$ is surjective.

(ii) There is a quasi-coherent $\mathscr{O}_Y$-module $\mathscr{G}$ and a surjective homomorphism $p^*(\mathscr{G}) \to \mathscr{F}$.

(iii) For any affine open U of Y, $\mathscr{F}|_{p^{-1}(U)}$ is generated by its sections over $p^{-1}(U)$.

We have already established the equivalence of (i) and (iii), and (i) clearly implies (ii). conversely, any homomorphism $u : p^*(\mathscr{G}) \to \mathscr{F}$ factors into $p^*(\mathscr{G}) \to p^*(p_*(\mathscr{F})) \xrightarrow{\sigma} \mathscr{F}$ by (3.4.3), so if u is surjective, so is the canonical homomorphism $\sigma : p^*(p_*(\mathscr{F})) \to \mathscr{F}$.

Corollary 5.3.26. *Suppose the hypotheses of Theorem 5.3.23 and that Y is quasi-compact. Let $\mathscr{F}$ be a quasi-coherent $\mathscr{O}_X$-module of finite type. There then exists an integer n_0 such that for $n \geq n_0$, $\mathscr{F}$ is isomorphic to a quotient of an $\mathscr{O}_X$-module of the form $(p^*(\mathscr{G}))(-n)$, where $\mathscr{G}$ is a quasi-coherent $\mathscr{O}_Y$-module of finite type (dependent on n).*

Proof. As the structural morphism $X \to Y$ is separated and of finite type, $p_*(\mathscr{F}(n))$ is quasi-coherent by Proposition 4.6.54, and so is the inductive limits of its sub-$\mathscr{O}_Y$-modules of finite type, in view of Corollary 4.6.62. Since p^* commutes with inductive limits, we deduce from Corollary 5.3.25 and Proposition 1.4.10 that $\mathscr{F}(n)$ is the canonical image under $\sigma_{\mathscr{F}(n)}$ of an $\mathscr{O}_X$-module of the form $p^*(\mathscr{G})$, where $\mathscr{G}$ is a quasi-coherent sub-$\mathscr{O}_Y$-module of $p_*(\mathscr{F}(n))$ of finite type. The corollary then follows from Corollary 5.3.16 and Corollary 5.3.17. □

5.3.4 Functorial properties of $\operatorname{Proj}(\mathscr{S})$

Let Y be a scheme and $\mathscr{S}, \mathscr{S}'$ be two quasi-coherent graded O_Y-algebras with positive degrees. Let $X = \operatorname{Proj}(\mathscr{S})$, $X' = \operatorname{Proj}(\mathscr{S}')$, and p, p' be the structural morphisms of X and X', respectively. Let $\varphi : \mathscr{S}' \to \mathscr{S}$ be an $\mathscr{O}_Y$-homomorphism of graded algebras. For any affine open U of Y, let $S_U = \Gamma(U, \mathscr{S})$, $S'_U = \Gamma(U, \mathscr{S}')$; the homomorphism φ defines a homomorphism $\varphi_U : S'_U \to S_U$ of graded A_U-algebras, where $A_U = \Gamma(U, \mathscr{O}_Y)$. It then corresponds to an open subset $G(\varphi_U)$ of $p^{-1}(U)$ and a morphism $\Phi_U : G(\varphi_U) \to p'^{-1}(U)$. Moreover, if $V \subseteq U$ is another affine open subset, the diagram

$$\begin{tikzcd} S'_U \arrow[r, "\varphi_U"] \arrow[d] & S_U \arrow[d] \\ S'_V \arrow[r, "\varphi_V"] & S_V \end{tikzcd} \tag{3.4.1}$$

is commutative, and we also verify, by the definition of $G(\varphi_U)$, that

$$G(\varphi_V) = G(\varphi_U) \cap p^{-1}(V)$$

and that Φ_V is the restriction of Φ_U to $G(\varphi_V)$. We thus define an open subset $G(\varphi)$ of X such that $G(\varphi) \cap p^{-1}(U) = G(\varphi_U)$ for any affine open $U \subseteq Y$, and an affine Y-morphism $\Phi : G(\varphi) \to X'$, which is called the morphism associated with φ and denoted by $\mathrm{Proj}(\varphi)$. If for any $y \in Y$, there exists an affine open neighborhood U of Y such that $\Gamma(U, \mathscr{O}_Y)$-module $\Gamma(U, \mathscr{S}_+)$ is generated by $\varphi(\Gamma(U, \mathscr{S}'_+))$, we then have $G(\varphi_U) = p^{-1}U)$, and thus $G(\varphi) = X$.

Proposition 5.3.27. *Let $\mathscr{M}$ (resp. $\mathscr{M}'$) be a quasi-coherent graded $\mathscr{S}$-module (resp. $\mathscr{S}'$-module). Then there exist a canonical isomorphism $\widetilde{\varphi^*(\mathscr{M})} \xrightarrow{\sim} \Phi_*(\widetilde{\mathscr{M}}|_{G(\varphi)})$ of $\mathscr{O}_{X'}$-modules and a canonical homomorphism $\nu : \Phi^*(\widetilde{\mathscr{M}'}) \to \widetilde{\varphi_*(\mathscr{M}')}|_{G(\varphi)}$. If $\mathscr{S}'$ is generated by $\mathscr{S}'_1$, ν is an isomorphism.*

Proof. The homomorphisms considered are in fact already defined locally over Y (see Proposition 5.2.46 and Proposition 5.2.47), and the general case then follows from their compatibility with restrictions, and diagram (3.4.1). □

In particular, for any $n \in \mathbb{Z}$, we have a canonical homomorphism $\Phi^*(\mathscr{O}_{X'}(n)) \to \mathscr{O}_X(n)|_{G(\varphi)}$, and this is a homomorphism if $\mathscr{S}'$ is generated by $\mathscr{S}'_1$.

Proposition 5.3.28. *Let Y, Y' be schemes, $\psi : Y' \to Y$ be a morphism, $\mathscr{S}$ be a quasi-coherent $\mathscr{O}_Y$-algebra, and put $\mathscr{S}' = \psi^*(\mathscr{S})$. Then the Y'-scheme $X' = \mathrm{Proj}(\mathscr{S}')$ is canonically identified with $\mathrm{Proj}(\mathscr{S}) \times_Y Y'$. Moreover, if $\mathscr{M}$ is a quasi-coherent graded $\mathscr{S}$-module, the $\mathscr{O}_{X'}$-module $\widetilde{\psi^*(\mathscr{M})}$ is identified with $\widetilde{\mathscr{M}} \otimes_Y \mathscr{O}_{Y'}$.*

Proof. We first note that $\psi^*(\mathscr{S})$ and $\psi^*(\mathscr{M})$ are quasi-coherent $\mathscr{O}_{Y'}$-modules. Let U be an affine open of Y, $U' \subseteq \psi^{-1}(U)$ an affine open of Y', and A, A' the ring of U, U', respectively. We then have $\mathscr{S}|_U = \widetilde{S}$ where S is a graded A-algebra, and $\mathscr{S}'|_{U'}$ is identified with $\widetilde{S \otimes_A A'}$ by Proposition 4.1.13. The first assertion then follows from Proposition 5.2.48 and Corollary 4.3.2, since we can easily verify that the projection $\mathrm{Proj}(\mathscr{S}'|_{U'}) \to \mathrm{Proj}(\mathscr{S}|_U)$ defined by this identification is compatible with restrictions over U and U' and therefore define a morphism $\mathrm{Proj}(\mathscr{S}') \to \mathrm{Proj}(\mathscr{S})$. Now let $p : \mathrm{Proj}(\mathscr{S}) \to Y$ and $p' : \mathrm{Proj}(\mathscr{S}') \to Y'$ be the structural morphisms; $p'^{-1}(U')$ is identified with $p^{-1}(U) \times_U U'$, and the two sheaves $\widetilde{\psi^*(\mathscr{M})}|_{p'^{-1}(U')}$ and $\widetilde{\mathscr{M}} \otimes_Y \mathscr{O}_{Y'}|_{p'^{-1}(U')}$ are then canonically identified to $\widetilde{M \otimes_A A'}$, where $M = \Gamma(U, \mathscr{M})$, in view of Proposition 5.2.48 and Proposition 4.1.13; whence the second assertion, since these identifications are compatible with restrictions. □

Corollary 5.3.29. *With the notations of Proposition 5.3.28, $\mathscr{O}_{X'}(n)$ is canonically identified with $\mathscr{O}_X(n) \otimes_Y \mathscr{O}_{Y'}$ for any $n \in \mathbb{Z}$.*

Proof. With the notations of Proposition 5.3.28, it is clear that $\psi^*(\mathscr{S}(n)) = \mathscr{S}'(n)$ for any $n \in \mathbb{Z}$, whence the corollary. □

Retain the notations in Proposition 5.3.28, denote by $\Psi : X' \to X$ the canonical projection, and put $\mathscr{M}' = \psi^*(\mathscr{M})$. We suppose that $\mathscr{S}$ is generated by $\mathscr{S}_1$ and that X is of finite type over Y (for example if $\mathscr{S}_1$ is of finite type, cf. Proposition 5.3.21). Then $\mathscr{S}'$ is generated by $\mathscr{S}'_1$ (as can be checked locally on affine opens of Y) and X' is of finite type over Y by Proposition 4.6.35. Let $\mathscr{F}$ be an $\mathscr{O}_X$-module and set $\mathscr{F}' = \Psi^*(\mathscr{F})$; it then follows from Corollary 5.3.29 that we have $\mathscr{F}'(n) = \Psi^*(\mathscr{F})$ for each $n \in \mathbb{Z}$. We define a canonical Ψ-homomorphism $\theta_n : p_*(\mathscr{F}(n)) \to p'_*(\mathscr{F}'(n))$ as follows: from the commutative diagram

$$\begin{tikzcd} X' \arrow[r, "\Psi"] \arrow[d, "p'"'] & X \arrow[d, "p"] \\ Y' \arrow[r, "\psi"] & Y \end{tikzcd}$$

we see that it suffices to define a homomorphism

$$p_*(\mathscr{F}(n)) \to \psi_*(q'_*(\Psi^*(\mathscr{F}(n)))) = q_*(\Psi_*(\Psi^*(\mathscr{F}(n)))),$$

and we can take $\theta_n = p_*(\rho_n)$, where ρ_n is the canonical homomorphism $\rho_n : \mathscr{F}(n) \to \Psi_*(\Psi^*(\mathscr{F}(n)))$. It is immediate that for any affine open U of Y and any affine open U' of Y such that $U' \subseteq \psi^{-1}(U)$, the homomorphism θ_n thus defined gives a canonical homomorphism $\Gamma(p^{-1}(U), \mathscr{F}(n)) \to \Gamma(p'^{-1}(U'), \mathscr{F}'(n))$, and the commutative diagram (2.5.4) shows that if $\mathscr{F}$ is quasi-coherent, the diagram

$$\begin{tikzcd} \widetilde{\Gamma_*(\mathscr{F})} \arrow[r, "\tilde{\theta}"] \arrow[d, "\beta_{\mathscr{F}}"'] & \widetilde{\Gamma_*(\mathscr{F}')} \arrow[d, "\beta_{\mathscr{F}'}"] \\ \mathscr{F} \arrow[r, "\rho"] & \mathscr{F}' \end{tikzcd}$$

is commutative (where the send row is the canonical Ψ-morphism $\mathscr{F} \to \Phi^*(\mathscr{F})$).

Similarly, the commutative diagram (2.5.3) shows that the diagram

$$\begin{tikzcd} \mathscr{M} \arrow[r, "\rho"] \arrow[d, "\alpha_{\mathscr{M}}"] & \mathscr{M}' \arrow[d, "\alpha_{\mathscr{M}'}"] \\ \Gamma_*(\widetilde{\mathscr{M}}) \arrow[r, "\theta"] & \Gamma_*(\widetilde{\mathscr{M}'}) \end{tikzcd}$$

is commutative (where the first row is the canonical ψ-morphism $\mathscr{M} \to \psi^*(\mathscr{M})$).

Consider now a morphism $\psi : Y' \to Y$ of schemes, a quasi-coherent graded $\mathscr{O}_Y$-algebra (resp. $\mathscr{O}_{Y'}$-algebra) $\mathscr{S}$ (resp. $\mathscr{S}'$), and a ψ-morphism $u : \mathscr{S} \to \mathscr{S}'$ of graded algebras. This is equivalent to giving an $\mathscr{O}_{Y'}$-homomorphism of graded algebras $u^\sharp : \psi_*(\mathscr{S}) \to \mathscr{S}'$, and we deduce from $u^\sharp$ an Y'-morphism

$$w = \operatorname{Proj}(u^\sharp) : G(u^\sharp) \to \operatorname{Proj}(\psi^*(\mathscr{S})),$$

where $G(u^\sharp)$ is an open subset of $X' = \operatorname{Proj}(\mathscr{S}')$. On the other hand, $\operatorname{Proj}(\psi^*(\mathscr{S}))$ is canonically identified with $X \times_Y Y'$, where $X = \operatorname{Proj}(\mathscr{S})$ (Proposition 5.3.28). By composing the morphism $\operatorname{Proj}(u^\sharp)$ with the first projection $\pi : X \times_Y Y' \to X$, we then obtain a morphism $v = \operatorname{Proj}(u) :$

$G(u^\sharp) \to X$ such that the diagram

$$\begin{tikzcd} G(u^\sharp) \arrow[r, "v"] \arrow[d] & X \arrow[d] \\ Y' \arrow[r, "\psi"] & Y \end{tikzcd} \tag{3.4.2}$$

is commutative. Moreover, for any quasi-coherent $\mathscr{O}_Y$-module $\mathscr{M}$, we have a canonical v-morphism

$$\nu : \widetilde{\mathscr{M}} \to (\psi^*(\mathscr{M}) \otimes_{\psi^*(\mathscr{S})} \mathscr{S}')^{\sim}|_{G(u^\sharp)} \tag{3.4.3}$$

such that $\nu^\sharp$ is the composition

$$v^*(\widetilde{\mathscr{M}}) = w^*(\pi^*(\widetilde{\mathscr{M}})) \xrightarrow{\sim} w^*(\widetilde{\psi^*(\mathscr{M})}) \xrightarrow{\nu} (\psi^*(\mathscr{M}) \otimes_{\psi^*(\mathscr{M})} \mathscr{S}')^{\sim}|_{G(u^\sharp)} \tag{3.4.4}$$

where the first arrow is the isomorphism in Proposition 5.3.28 and the second one is the homomorphism ν of Proposition 5.3.27. If $\mathscr{S}$ is generated by $\mathscr{S}_1$, then it follows from Proposition 5.3.27 that $\nu^\sharp$ is an isomorphism. As a particular case, for any $n \in \mathbb{Z}$ we have a canonical v-homomorphism

$$\nu : \mathscr{O}_X(n) \to \mathscr{O}_{X'}(n)|_{G(u^\sharp)}. \tag{3.4.5}$$

5.3.5 Closed subschemes of $\mathrm{Proj}(\mathscr{S})$

Let Y be a scheme and $\varphi : \mathscr{S} \to \mathscr{S}'$ be a homomorphism of quasi-coherent graded $\mathscr{O}_Y$-algebras of degree 0. We say that φ is **eventually injective** (resp. **eventually surjective**, **eventually bijective**) if there exists an integer n such that, for $k \geq n$, $\varphi_k : \mathscr{S}_k \to \mathscr{S}'_k$ is surjective (resp. injective, bijective). If this is the case, we can then reduce the study of the morphism $\Phi : \mathrm{Proj}(\mathscr{S}') \to \mathrm{Proj}(\mathscr{S})$ to the case where φ is surjective (resp. injective, bijective) (this follows from Proposition 5.3.6). If φ is eventually bijective, we also say that φ is an **eventual isomorphism**.

Proposition 5.3.30. *Let Y be a scheme, $\mathscr{S}$ be a quasi-coherent $\mathscr{O}_Y$-algebra, and $X = \mathrm{Proj}(\mathscr{S})$.*

(a) If $\varphi : \mathscr{S} \to \mathscr{S}'$ is a eventually surjective homomorphism of graded $\mathscr{O}_Y$-algebra, the corresponding morphism $\Phi = \mathrm{Proj}(\varphi)$ is defined over $\mathrm{Proj}(\mathscr{S}')$ and is a closed immersion from $\mathrm{Proj}(\mathscr{S}')$ into X. If $\mathscr{I}$ is the kernel of φ, the closed subscheme of X associated with Φ is defined by the quasi-coherent ideal $\widetilde{\mathscr{I}}$ of $\mathscr{O}_X$.

(b) Suppose moreover that $\mathscr{S}_0 = \mathscr{O}_Y$ and $\mathscr{S}$ is generated by $\mathscr{S}_1$ where $\mathscr{S}_1$ is of finite type. Let X' be a closed subscheme of X defined by a quasi-coherent ideal $\mathscr{I}$ of $\mathscr{O}_X$. Let $\mathscr{J}$ be the inverse image of $\Gamma_*(\mathscr{I})$ under the canonical homomorphism $\alpha : \mathscr{S} \to \Gamma_*(\mathscr{O}_X)$ and put $\mathscr{S}' = \mathscr{S}/\mathscr{J}$. Then X' is the subscheme associated with the closed immersion $\mathrm{Proj}(\mathscr{S}') \to X$ corresponding to the canonical homomorphism $\mathscr{S} \to \mathscr{S}'$ of graded $\mathscr{O}_Y$-algebras.

Proof. For the assertion of (a), we can assume that φ is surjective. Then for any affine open U of Y, $\Gamma(U, \mathscr{S}) \to \Gamma(U, \mathscr{S}')$ is surjective by Proposition 4.1.6, so we have $G(\varphi) = X$. We are immediately reduced to the case where Y is affine, and the assertion follows from Proposition 5.2.50(a).

As for case (b), we are reduced to prove that the homomorphism $\widetilde{\mathscr{J}} \to \mathscr{O}_X$ induced from the canonical injection $\mathscr{J} \to \mathscr{S}$ is an isomorphism from $\widetilde{\mathscr{J}}$ to $\mathscr{J}$. As the question is local, we can assume that Y is affine with ring A, which implies $\mathscr{S} = \widetilde{S}$, where S is a graded A-algebra generated by S_1 and S_1 is of finite type over A. It then suffices to apply Proposition 5.2.50(b). ◻

Corollary 5.3.31. *Under the hypotheses of Proposition 5.3.30(a), suppose that $\mathscr{S}$ is generated by $\mathscr{S}_1$. Then $\Phi^*(\mathscr{O}_X(n))$ is canonically identified with $\mathscr{O}_{X'}(n)$ for any $n \in \mathbb{Z}$.*

Proof. We have defined such an isomorphism if Y is affine; in the general case, it suffices to verify that these isomorphisms are compatible with restrictions. ◻

Corollary 5.3.32. *Let Y be a scheme, $\mathscr{S}$ be a quasi-coherent $\mathscr{O}_Y$-algebra generated by $\mathscr{S}_1$, $\mathscr{M}$ be a quasi-coherent $\mathscr{O}_Y$-module, and $u : \mathscr{M} \to \mathscr{S}_1$ be a surjective $\mathscr{O}_Y$-homomorphism. If $\bar{u} : \boldsymbol{S}(\mathscr{M}) \to \mathscr{S}$ is the canonical homomorphism of $\mathscr{O}_Y$-algebras extending u, then the morphism corresponding to $\bar{u}$ is a closed immersion from* $\mathrm{Proj}(\mathscr{S})$ *into* $\mathrm{Proj}(\boldsymbol{S}(\mathscr{M}))$.

Proof. In fact, $\bar{u}$ is surjective by hypothesis, so we can apply Proposition 5.3.30(a). ◻

5.3.6 Morphisms into $\mathrm{Proj}(\mathscr{S})$

Let $q : X \to Y$ be a morphism of schemes, $\mathscr{L}$ be an invertible $\mathscr{O}_X$-module, and $\mathscr{S}$ be a quasi-coherent $\mathscr{O}_Y$-algebra with positive degrees. Then $q^*(\mathscr{S})$ is a quasi-coherent graded $\mathscr{O}_X$-algebra with positive degrees. Suppose that we are given a graded homomorphism of $\mathscr{O}_X$-algebras

$$\psi : q^*(\mathscr{S}) \to \boldsymbol{S}(\mathscr{L}) = \bigoplus_{n \geq 0} \mathscr{L}^{\otimes n}$$

(recall that for an invertible $\mathscr{O}_X$-module $\mathscr{L}$ we have $\boldsymbol{T}(\mathscr{L}) = \boldsymbol{S}(\mathscr{L})$) or equivalently, a $\mathscr{O}_Y$-homorphism of graded algebras

$$\psi^\flat : \mathscr{S} \to q_*(\boldsymbol{S}(\mathscr{L})).$$

Since $\mathscr{L}$ is invertible and $\boldsymbol{T}(\mathscr{O}_X) = \boldsymbol{S}(\mathscr{O}_X) = \mathscr{O}_X[T]$, by Proposition 5.3.6 and Corollary 5.3.5 we know that $\mathrm{Proj}(\boldsymbol{S}(\mathscr{L}))$ is canonically identified with X. We then conclude that the homomorphism ψ induces a Y-morphism

$$r_{\mathscr{L},\psi} : G(\psi) \to \mathrm{Proj}(\mathscr{S}) = P,$$

where $G(\psi)$ is an open subset of X. Recall that this morphism is by definition obtained by composing the first projection $\pi : \mathrm{Proj}(q^*(\mathscr{S})) = P \times_Y X \to P$ with the Y-morphism $\tau = \mathrm{Proj}(\psi) : G(\psi) \to \mathrm{Proj}(q^*(\mathscr{S}))$, which is shown in the following diagram:

$$\begin{tikzcd}
 & P \times_Y X \arrow[r, "\pi"] \arrow[d] & P \arrow[d, "p"] \\
G(\psi) \arrow[ur, "\tau=\mathrm{Proj}(\psi)"] \arrow[urr, bend left, "r_{\mathscr{L},\psi}"] \arrow[r] & X \arrow[r, "q"] & Y
\end{tikzcd}$$

Remark 5.3.4. Let us explain the morphism $r = r_{\mathscr{L},\psi}$ when $Y = \operatorname{Spec}(A)$ is affine, and therefore $\mathscr{S} = \widetilde{S}$, where S is a graded A-algebra with positive degrees. First suppose that $X = \operatorname{Spec}(B)$ is also affine and $\mathscr{L} = \widetilde{L}$, where L is a free B-module of rank 1. We then have $q^*(\mathscr{S}) = \widetilde{S \otimes_A B}$ by Proposition 4.1.13. If c is a generator of L, the homomorphism $\psi_n : q^*(\mathscr{S}_n) \to \mathscr{L}^{\otimes n}$ then corresponds to a B-homomorphism

$$w_n : S_n \otimes_A B \to L^{\otimes n}, \quad s \otimes b \mapsto b v_n(s) c^{\otimes n}, \tag{3.6.1}$$

where $v_n : S_n \to B$ is the n-th component of a homomorphism $v : S \to B$ of algebras. Let $f \in S_d$ with positive degree and put $g = v_d(f)$. We have $\pi^{-1}(D_+(f)) = D_+(f \otimes 1)$ in view of Proposition 5.2.48 and the identification of $D_+(f)$ with $\operatorname{Spec}(S_{(f)})$. On the other hand, the formula (2.5.1) and (3.6.1) shows that (using the canonical isomorphism of X and $\operatorname{Proj}(\boldsymbol{S}(\mathscr{L}))$)

$$\tau^{-1}(D_+(f \otimes 1)) = D(g)$$

whence $r^{-1}(D_+(f)) = D(g)$. Furthermore, the restriction of the morphism $\tau = \operatorname{Proj}(\psi)$ to $D(g)$ corresponds to the homomorphism $(S \otimes_A B)_{(f \otimes 1)} \to B_g$, which send $(s \otimes 1)/(f \otimes 1)^n$ (for $s \in S_{nd}$) to $v_{nd}(s)/g^n$, and the restriction of the projection π to $D_+(f \otimes 1)$ corresponds to the homomorphism $S_{(f)} \to (S \otimes_A B)_{(f \otimes 1)}$ given by $s/f^n \mapsto (s \otimes 1)/(f \otimes 1)^n$. We then conclude that the restriction of the morphism r to $D(g)$ corresponds to the homomorphisms $\omega : S_{(f)} \to B_g$ of A-algebras such that $\omega(s/f^n) = v_{nd}(s)/g^n$ for $s \in S_{nd}$.

Proposition 5.3.33. *Let $Y = \operatorname{Spec}(A)$ be affine and $\mathscr{S} = \widetilde{S}$, where S is a graded A-algebra. For any $f \in S_d = \Gamma(Y, \mathscr{S}_d)$, we have (where $\psi^\flat(f) \in \Gamma(X, \mathscr{L}^{\otimes d})$)*

$$r_{\mathscr{L},\psi}^{-1}(D_+(f)) = X_{\psi^\flat(f)}. \tag{3.6.2}$$

Moreover, under the canonical isomorphism of X and $\operatorname{Proj}(\boldsymbol{S}(\mathscr{L}))$, the restriction morphism $r_{\mathscr{L},\psi} : X_{\psi^\flat(f)} \to D_+(f) = \operatorname{Spec}(S_{(f)})$ corresponds to the homomorphism

$$\psi^\flat_{(f)} : S_{(f)} \to \Gamma(X_{\psi^\flat(f)}, \mathscr{O}_X)$$

such that, for any $s \in S_{nd} = \Gamma(Y, \mathscr{S}_{nd})$, we have

$$\psi^\flat_{(f)}(s/f^n) = (\psi^\flat(s)|_{X_{\psi^\flat(f)}})(\psi^\flat(f)|_{X_{\psi^\flat(f)}})^{-n}.$$

Proof. This follows from Remark 5.3.4 by passing to the general case. □

We say the morphism $r_{\mathscr{L},\psi}$ is **everywhere defined** if $G(\psi) = X$. For this to be the case, it is necessary and sufficient that $G(\psi) \cap q^{-1}(U) = q^{-1}(U)$ for any affine open $U \subseteq Y$, so this question is local over Y. If Y is affine, $G(\psi)$ is then the union of $r^{-1}(D_+(f))$ for $f \in S_+$, so by (3.6.2) the $X_{\psi^\flat(f)}$ then form a covering of X. In other words:

Corollary 5.3.34. *Under the hypotheses of Proposition 5.3.33, for the morphism $r_{\mathscr{L},\psi}$ to be everywhere defined, it is necessary and sufficient that for any $x \in X$, there exists an integer $n > 0$ and a section $s \in S_n$ such that $t = \psi^\flat(s) \in \Gamma(X, \mathscr{L}^{\otimes n})$ is nonzero at x. In particular, this is true if ψ is eventually surjective.*

Corollary 5.3.35. *Under the hypotheses of Proposition 5.3.33, for the morphism $r_{\mathscr{L},\psi}$ to be dominant, it is necessary and sufficient that for any integer $n > 0$, any section $s \in S_n$ such that $\psi^\flat(b) \in \Gamma(X, \mathscr{L}^{\otimes n})$ is locally nilpotent, is itself nilpotent.*

Proof. We must check that $r_{\mathscr{L},\psi}^{-1}(D_+(s))$ is nonempty if $D_+(s)$ is nonempty, and the corollary follows from (3.6.2) and Corollary 5.2.6. □

Proposition 5.3.36. *Let $q : X \to Y$ be a morphism, $\mathscr{L}$ be an invertible $\mathscr{O}_X$-module, and $\mathscr{S}$, $\mathscr{S}'$ be quasi-coherent graded $\mathscr{O}_Y$-algebras. Let $u : \mathscr{S}' \to \mathscr{S}$ be a homomorphism of graded algebras, $\psi : q^*(\mathscr{S}) \to \boldsymbol{S}(\mathscr{L})$ be a homomorphism of graded algebras, and $\psi' = \psi \circ q^*(u)$ be the composition.*

(i) If $r_{\mathscr{L},\psi'}$ is everywhere defined, then $r_{\mathscr{L},\psi}$ is everywhere defined;

(ii) If u is eventually surjective and $r_{\mathscr{L},\psi'}$ is dominant, then $r_{\mathscr{L},\psi}$ is dominant;

(iii) If u is eventually injective and $r_{\mathscr{L},\psi}$ is dominant, then $r_{\mathscr{L},\psi}$ is dominant.

Proof. We have $G(\psi') \subseteq G(\psi)$, whence the first assertion. If u is eventually surjective, $\mathrm{Proj}(u) : \mathrm{Proj}(S) \to \mathrm{Proj}(S')$ is everywhere defined and is a closed immersion; as $r_{\mathscr{L},\psi'}$ is the composition of $\mathrm{Proj}(u)$ and the restriction of $r_{\mathscr{L},\psi}$ to $G(\psi')$, we then conclude that if $r_{\mathscr{L},\psi'}$ is dominant, so is $r_{\mathscr{L},\psi}$. Finally, if u is eventually injective, then $\mathrm{Proj}(u)$ is a dominant morphism from $G(u)$ into $\mathrm{Proj}(\mathscr{S}')$ (Corollary 5.2.45); as $G(\psi')$ is the inverse image of $G(u)$ under $r_{\mathscr{L},\psi}$, we see that if $r_{\mathscr{L},\psi}$ is dominant, so is $r_{\mathscr{L},\psi'}$. □

Proposition 5.3.37. *Let Y be a quasi-compact scheme, $q : X \to Y$ be a quasi-compact morphism, $\mathscr{L}$ be an invertible $\mathscr{O}_X$-module, and $\mathscr{S}$ be a quasi-coherent $\mathscr{O}_Y$-algebra which is the filtered limit of a system $(\mathscr{S}^\lambda)$ of quasi-coherent $\mathscr{O}_Y$-algebras. Let $\varphi_\lambda : \mathscr{S}^\lambda \to \mathscr{S}$ be the canonical homomorphism, $\psi : q^*(\mathscr{S}) \to \boldsymbol{S}(\mathscr{L})$ be a homomorphism of graded algebras, and put $\psi_\lambda = \psi \circ q^*(\varphi_\lambda)$. Then for the morphism $r_{\mathscr{L},\psi}$ to be everywhere defined, it is necessary and sufficient that there exists an index λ such that $r_{\mathscr{L},\psi_\lambda}$ is everywhere defined; in this case, $r_{\mathscr{L},\psi_\mu}$ is everywhere defined for $\mu \geq \lambda$.*

Proof. The condition is sufficient in view of Proposition 5.3.36. Conversely, suppose that $r_{\mathscr{L},\psi}$ is everywhere defined; we can assume that Y is affine, because if for any affine open $U \subseteq Y$ there exists $\lambda(U)$ such that the restriction of $r_{\mathscr{L},\psi_{\lambda_{\lambda(U)}}}$ to $q^{-1}(U)$ is defined everywhere, it then suffices to cover Y by finitely many affine opens U_i (recall that Y is quasi-compact) and choose $\lambda \geq \lambda(U_i)$ for all i, by Proposition 5.3.36. If Y is affine (so $\mathscr{S} = \widetilde{S}$ where $S = \Gamma(Y, \mathscr{S})$) the hypotheses implies that for any $x \in X$, there exists a section $s^{(x)} \in S_n$ for some integer n such that, if $t^{(x)} = \psi^\flat(s^{(x)})$, then $t^{(x)}(x) \neq 0$ (where $t^{(x)}$ is a section of $\mathscr{L}^{\otimes n}$ over X), which implies $t^{(x)}(z) \neq 0$ for z in a neighborhood $V(x)$ of X. As the morphism $q : X \to Y$ is quasi-compact, we see X is quasi-compact, so we can cover X by finitely many $V(x_i)$ and let $s^{(i)}$ be the corresponding section of S. There is then an index λ such that $s^{(i)}$ is of the form $\varphi_\lambda(s_\lambda^{(i)})$, where $s_\lambda^{(i)} \in S^\lambda$ for all i, and it follows from Corollary 5.3.34 that $r_{\mathscr{L},\psi_\lambda}$ is everywhere defined. The second assertion is trival by Proposition 5.3.36. □

Corollary 5.3.38. *Under the hypotheses of Proposition 5.3.37, if the morphisms $r_{\mathscr{L},\psi_\lambda}$ are dominant, so is $r_{\mathscr{L},\psi}$. The converse is also ture if the homomorphisms φ_λ are eventually injective.*

Proof. The second assertion is a particular case of Proposition 5.3.36. To show that $r_{\mathscr{L},\psi}$ is dominant if each $r_{\mathscr{L},\psi_\lambda}$ is, we can assume that Y is affine and thus $\mathscr{S} = \widetilde{S}$ where $S = \Gamma(Y, \mathscr{S})$. If $s \in S$ is such that $\psi^\flat(s)$ is locally nilpotent, as we can write $s = \varphi_\lambda(s_\lambda)$ for some λ, from the definition of ψ_λ and by Corollary 5.3.35, we conclude that s_λ is nilpotent, so s is nilpotent, and the assertion follows by applying Corollary 5.3.35. □

Remark 5.3.5. With the hypotheses and notations of Proposition 5.3.33, for each $n \in \mathbb{Z}$ we have a homomorphism

$$\nu : r^*_{\mathscr{L},\psi}(\mathscr{O}_P(n)) \to \mathscr{L}^{\otimes n}|_{G(\psi)} \tag{3.6.3}$$

which is in fact the homomorphism ν defined in (3.4.3) on $\mathscr{O}_P(n)$. We also see that under the hypotheses of Proposition 5.3.33, the restriction of ν to $X_{\psi^\flat(f)}$ corresponds to the homomorphism sending the element s/f^k (with $s \in S_{n+kd}$) to the section

$$(\psi^\flat(s)|_{X_{\psi^\flat(f)}})(\psi^\flat(f)|_{X_{\psi^\flat(f)}})^{-k} \in \Gamma(X_{\psi^\flat(f)}, \mathscr{L}^{\otimes n}),$$

where we also use the notations of Proposition 5.3.33.

Remark 5.3.6. Let $\mathscr{F}$ be a quasi-coherent $\mathscr{O}_X$-module and suppose that q is quasi-compact and quasi-separated, so for each $n \geq 0$, $q_*(\mathscr{F} \otimes \mathscr{L}^{\otimes n})$ is a quasi-coherent $\mathscr{O}_Y$-module (Proposition 4.6.54). Let $\mathscr{M}' = \bigoplus_{n\geq 0} \mathscr{F} \otimes \mathscr{L}^{\otimes n}$, which is a quasi-coherent $\mathscr{O}_Y$-module, and consider the image $\mathscr{M} = q_*(\mathscr{M}') = \bigoplus_{n\geq 0} q_*(\mathscr{F} \otimes \mathscr{L}^{\otimes n})$ (which is a quasi-coherent $\mathscr{S}$-module via the homomorphism $\psi^\flat$). We shall see that there is a canonical homomorphism of $\mathscr{O}_X$-modules

$$\xi : r^*_{\mathscr{L},\psi}(\widetilde{\mathscr{M}}) \to \mathscr{F}|_{G(\psi)}. \tag{3.6.4}$$

For this, recall that we have defined a canonical homomorphism (3.4.3):

$$\nu : r^*_{\mathscr{L},\psi}(\widetilde{\mathscr{M}}) \to (q^*(\mathscr{M}) \otimes_{q^*(\mathscr{S})} \boldsymbol{S}(\mathscr{L}))^\sim|_{G(\psi)},$$

where the right hand side is considered as a quasi-coherent sheaf over X. On the other hand, for any quasi-coherent graded $\boldsymbol{S}(\mathscr{L})$-module $\mathscr{M}'$, we have a canonical homomorphism

$$q^*(q_*(\mathscr{M}')) \otimes_{q^*(\mathscr{S})} \boldsymbol{S}(\mathscr{L}) \to \mathscr{M}'$$

which, for any open subset U of X, any section t' of $q^*(q_*(\mathscr{M}'_h))$ over U and any section b' of $\mathscr{L}^{\otimes k}$ over U, sends the section $t' \otimes b'$ to the section $b'\sigma(t')$ of $\mathscr{M}'_{h+k}$, where σ is the canonical homomorphism $q^*(q_*(\mathscr{M}')) \to \mathscr{M}'$. We then conclude a canonical homomorphism

$$(q^*(q_*(\mathscr{M}')) \otimes_{q^*(\mathscr{S})} \boldsymbol{S}(\mathscr{L}))^\sim|_{G(\psi)} \to \widetilde{\mathscr{M}}'|_{G(\psi)}$$

and as $\widetilde{\mathscr{M}}'$ is canonically identified with $\mathscr{F}$ by Remark 5.3.1, we obtain the canonical homomorphism ξ.

Under the hypotheses and notations of Proposition 5.3.18, the restriction of this homomorphism to $X_{\psi^\flat(f)}$ is defined as follows: giving a section t_{nd} of $\mathscr{F} \otimes \mathscr{L}^{\otimes d}$ over X (which is also a section of $q_*(\mathscr{F}\otimes\mathscr{L}^{\otimes n})$ over Y), we send the element t_{nd}/f^n to the sectin $(t_{nd}|_{X_{\psi^\flat(f)}})(\psi^\flat(f)|_{X_{\psi^\flat(f)}})^{-n}$ of $\mathscr{F}$ over $X_{\psi^\flat(f)}$.

We now consider the important question that whether the induced morphism $r_{\mathscr{L},\psi}$ is an immersion (reps. an open immersion, a closed immersion). It is clear that this question is local over Y, and we shall give a criterion in this situation together with the condition that $r_{\mathscr{L},\psi}$ is defined everywhere.

Proposition 5.3.39. *Under the hypothesis and notations of Proposition 5.3.33, for the morphism $r_{\mathscr{L},\psi}$ be everywhere defined and an immersion, it is necessary and sufficient that there exists a family of sections $s_\alpha\in S_{n_\alpha}$ (with $n_\alpha>0$) such that, if $f_\alpha=\psi^\flat(s_\alpha)$, the following conditions are satisfied:*

(i) The X_{f_α} form a covering of X.

(ii) The X_{f_α} are affine open subset of X.

(iii) For any index α and any section $t\in\Gamma(X_{f_\alpha},\mathscr{O}_X)$, there exists an integer $n>0$ and $s\in S_{mn_\alpha}$ such that $t=(\psi^\flat(s)|_{X_{f_\alpha}})(f_\alpha|_{X_{f_\alpha}})^{-m}$.

For the morphism $r_{\mathscr{L},\psi}$ to be everywhere defined and an open immersion, it is necessary and sufficient that the family (s_α) satisfies the following additional condition:

(iv) For any integer $m>0$ and any $s\in S_{mn_\alpha}$ such that $\psi^\flat(s)|_{X_{f_\alpha}}=0$, there exists an integer $n>0$ such that $s_\alpha^n s=0$.

Similarly, for the morphism $r_{\mathscr{L},\psi}$ to be everywhere defined and a closed immersion, it is necessary and sufficient that the family (s_α) satisfies the following additional condition:

(v) The $D_+(s_\alpha)$ form a covering of $P=\mathrm{Proj}(S)$.

Proof. By Proposition 4.4.9, for the morphism $r=r_{\mathscr{L},\psi}$ to be an immersion (resp. a closed immersion), it is necessary and sufficent that there exists a covering of $r(G(\psi))$ (resp. of P) by the sets $D_+(s_\alpha)$ such that if $V_\alpha=r^{-1}(D_+(s_\alpha))$, the restriction of r on V_α is a closed immersion of V_α into $D_+(s_\alpha)$ (cf. Corollary 4.4.10). Now condition (i) just means that r is everywhere defined and that $D_+(s_\alpha)$ cover $r(X)$, by (3.6.2). As each $D_+(s_\alpha)$ is affine, condition (ii) and (iii) express that the restriction of r to X_{f_α} is a closed immersion into $D_+(s_\alpha)$ (Example 4.4.6). Finally, as (iii) and (iv) means the ring homomorphism $\psi^\flat_{(s_\alpha)}:S_{(s_\alpha)}\to\Gamma(X_{f_\alpha},\mathscr{O}_X)$ is an isomorphism, (ii), (iii), (iv) mean that the restriction of r to X_{f_α} is an isomorphism from X_{f_α} to $D_+(s_\alpha)$ for each α, so together with (i), they mean that r is an open immersion. ◻

Corollary 5.3.40. *Under the hypothesis and notations of Proposition 5.3.36, if $r_{\mathscr{L},\psi'}$ is everywhere defined and is an immersion, so is $r_{\mathscr{L},\psi}$. If we suppose that u is eventually surjective and if $r_{\mathscr{L},\psi'}$ is everywhere defined and is a closed immersion (resp. open), then so is $r_{\mathscr{L},\psi}$.*

Proof. We first suppose that $r_{\mathscr{L},\psi'}$ is everywhere defined and and is an immersion. Then by Proposition 5.3.39, there is a family $s'_\alpha\in S'_{n_\alpha}$ such that, if $f_\alpha=\psi'^\flat(s'_\alpha)$, the conditions (i), (ii), (iii) are satisfied. Set $s_\alpha=u(s'_\alpha)$, then $f_\alpha=\psi^\flat(s_\alpha)$, and we have a commutative diagram

$$\begin{tikzcd}
S'_{(s'_\alpha)} \arrow[r, "\psi'^\flat_{(s'_\alpha)}"] \arrow[d, "u_{(s'_\alpha)}"'] & \Gamma(X_{f_\alpha},\mathscr{O}_X) \\
S_{(s_\alpha)} \arrow[ur, "\psi^\flat_{(s_\alpha)}"'] &
\end{tikzcd}$$

The hypothesis then implies that $\psi'^{\flat}_{(s'_\alpha)}$ is surjective, so the homomorphism $\psi^{\flat}_{(s_\alpha)}$ is also surjective. This shows that $r_{\mathscr{L},\psi}$ is everywhere defined and is an immersion, in view of Proposition 5.3.39. If $r_{\mathscr{L},\psi'}$ is moreover an open immersion, then the homomorphism $\psi'^{\flat}_{(s'_\alpha)}$ is also injective, and this implies the homomorphism $\psi^{\flat}_{(s_\alpha)}$ is injective if u is eventually surjective, since in this case the homomorphism $u_{(s'_\alpha)}$ is just surjective.

Finally, if $r_{\mathscr{L},\psi'}$ is a closed immersion, then condition (v) is satisfied for (s'_α), and hence satisfied for (s_α) if u is eventually surjective (since in this case $\mathrm{Proj}(u)$ is a closed immersion from $\mathrm{Proj}(S)$ to $\mathrm{Proj}(S')$); this implies $r_{\mathscr{L},\psi}$ is a closed immersion. □

Proposition 5.3.41. *Suppose the hypotheses of Proposition 5.3.37 and moreover that $q : X \to Y$ is a morphism of finite type. Then, for the morphism $r_{\mathscr{L},\psi}$ to be everywhere defined and an immersion, it is necessary and sufficient that there exists an index λ such that $r_{\mathscr{L},\psi_\lambda}$ is everywhere defined and is an immersion; in this case, $r_{\mathscr{L},\psi_\lambda}$ is everywhere defined and an immersion for $\mu \geq \lambda$.*

Proof. By Corollary 5.3.40, it suffices to prove that if $r_{\mathscr{L},\psi}$ is defined everywhere and is an immersion, then there exists an index λ such that $r_{\mathscr{L},\psi_\lambda}$ is defined everywhere and is an immersion. By the same reasoning of Proposition 5.3.37 and using the quasi-compactness of Y, we are reduced to the case where $Y = \mathrm{Spec}(A)$ is affine. As X is then quasi-compact, Proposition 5.3.39 shows that there exists a finite family (s_i) of elements of S ($s_i \in S_{n_i}$) satisfying the conditions (i), (ii), and (iii). The morphism $X_{f_i} \to Y$ (where $f_i = \psi^\flat(s_i)$) is of finite type since X_{f_i} is affine and the morphism $q : X \to Y$ is locally of finite type. The ring B_i of X_{f_i} is therefore an A-algebra of finite type by Proposition 4.6.39, and we choose (t_{ij}) to be a family of generators of this algebra. There are then elements $s_{ij} \in S_{m_{ij}n_i}$ such that

$$t_{ij} = (\psi^\flat(s_{ij})|_{X_{f_i}})(\psi^\flat(s_i)|_{X_{f_i}})^{-m_{ij}}$$

We can choose an index λ and elements $s_{i\lambda} \in S^\lambda_{n_i}$, $s_{ij\lambda} \in S^\lambda_{m_{ij}n_i}$ such that their images under φ_λ is s_i and s_{ij}, respectively. It is then clear that the family $(s_{i\lambda})$ satisfies the conditions (i), (ii), and (iii), so $r_{\mathscr{L},\psi}$ is everywhere defined and an immersion. □

Proposition 5.3.42. *Let Y be a quasi-compact scheme, $q : X \to Y$ be a morphism of finite type, $\mathscr{L}$ be an invertible $\mathscr{O}_X$-module, $\mathscr{S}$ be a quasi-coherent graded $\mathscr{O}_Y$-algebra, and $\psi : q^*(\mathscr{S}) \to \boldsymbol{S}(\mathscr{L})$ be a homomorphism of graded algebras. For the morphism $r_{\mathscr{L},\psi}$ to be defined everywhere and an immersion, it is necessary and sufficient that there exists an integer $n > 0$ and a quasi-coherent sub-$\mathscr{O}_Y$-module $\mathscr{E}$ of $\mathscr{S}_n$ of finite type such that:*

(a) the homomorphism $\psi_n \circ q^*(j_n) : q^*(\mathscr{E}) \to \mathscr{L}^{\otimes n}$ (where $j_n : \mathscr{E} \to \mathscr{S}_n$ is the canonical injection) is surjective.

(b) if $\mathscr{S}'$ is the graded sub-$\mathscr{O}_Y$-algebra of $\mathscr{S}$ genrated by $\mathscr{E}$ and ψ' is the homomorphism $\psi \circ q^*(j')$, where $j' : \mathscr{S}' \to \mathscr{S}$ is the canonical injection, $r_{\mathscr{L},\psi'}$ is everywhere defined and an immersion.

If these are true, any quasi-coherent sub-$\mathscr{O}_Y$-module $\mathscr{E}'$ of $\mathscr{S}_n$ containing $\mathscr{E}$ possesses the same properties, and so does the sub-$\mathscr{O}_Y$-module $\mathscr{S}'_{kn}$ of $\mathscr{S}_{kn}$ for any $k > 0$.

Proof. The sufficiency of these conditions is a particular case of Corollary 5.3.40 in view of the isomorphism between $\mathrm{Proj}(\mathscr{S})$ and $\mathrm{Proj}(\mathscr{S}^{(d)})$ (Proposition 5.3.6). We now prove the necessity, so let (U_i) be a finite affine open covering of Y and set $A_i = \Gamma(U_i, \mathscr{O}_Y)$. As $q^{-1}(U_i)$ is compact, the hypotheses that $r_{\mathscr{L},\psi}$ is an immersion defined on X implies by Proposition 5.3.39 the existence of a finite family (s_{ij}) of elements of $S^{(i)} = \Gamma(U_i, \mathscr{S})$ (where $s_{ij} \in S^{(i)}_{n_{ij}}$) satisfying conditions (i), (ii), and (iii). Since $q : X \to Y$ is of finite type, the restricted homomorphism $X_{f_{ij}} \to U_i$ is of finite type (where $f_{ij} = \psi^\flat(s_{ij})$), so the ring B_{ij} of $X_{f_{ij}}$ is an A_i-algebra of finite type, and we choose $(\psi^\flat(t_{ijk})|_{X_{f_{ij}}})(f_{ij}|_{X_{f_{ij}}})^{-m_{ijk}}$ to be a system of generators of B_{ij}, where $t_{ijk} \in S^{(i)}_{m_{ijk}n_{ij}}$. Let n be a common multiple of all the $m_{ijk}n_{ij}$ and put $s'_{ij} = s_{ij}^{h_{ij}} \in S^{(i)}_n$, where $h_{ij} = n/n_{ij}$. For any given piar (i,j,k), the element $t'_{ij} = s_{ij}^{h-m_{ijk}}t_{ijk}$ belongs to $S^{(i)}_n$, and it is clear that the $(\psi^\flat(t'_{ijk})|_{X_{f'_{ij}}})(f'_{ij}|_{X_{f'_{ij}}})^{-1}$ also generate B_{ij} (where $f'_{ij} = \psi^\flat(s'_{ij})$, and we note that $X_{f'_{ij}} = X_{f_{ij}}$). Let E_i be the sub-A_i-module of $S^{(i)}$ generated by these s'_{ij} and t'_{ijk}; then there exists a quasi-coherent sub-$\mathscr{O}_Y$-module $\mathscr{E}_i$ of $\mathscr{S}_n$ of finite type such that $\mathscr{E}_i|_{U_i} = \widetilde{E}_i$ (Theorem 4.6.60). It is then clear that the sub-$\mathscr{O}_Y$-module $\mathscr{E}$ of $\mathscr{E}$, which is the sum of the $\mathscr{E}_i$, satisfies the required properties. □

Remark 5.3.7. The point of Proposition 5.3.42 is that, for a scheme X of finite type over a quasi-compact scheme Y, if X can be embedded into $\mathrm{Proj}(\mathscr{S})$ via a morphism $r_{\mathscr{L},\psi}$, then we can choose $\mathscr{S}$ so that it is generated by $\mathscr{S}_1$ and $\mathscr{S}_1$ of finite type (we already know that in this case the twisted sheaves over $\mathrm{Proj}(\mathscr{S})$ have nice properties).

5.4 Projective bundles and ample sheaves

5.4.1 Projective bundles

Let Y be a scheme, $\mathscr{E}$ be a quasi-coherent $\mathscr{O}_Y$-module, and $\boldsymbol{S}(\mathscr{E})$ be the symmetric $\mathscr{O}_Y$-algebra of $\mathscr{E}$, which is quasi-coherent by Proposition 5.1.36. The **projective bundle** over Y associated with $\mathscr{E}$ is defined to be the Y-scheme $P = \mathrm{Proj}(\boldsymbol{S}(\mathscr{E}))$. The $\mathscr{O}_P$-module $\mathscr{O}_P(1)$ is called the **fundamental sheaf** of P.

If $\mathscr{E} = \mathscr{O}_Y^n$, we then denote by $\mathbb{P}_Y^{n-1}$ instead of $\mathbb{P}(\mathscr{E})$; if moreover Y is affine with ring A, we then denote this scheme by $\mathbb{P}_A^{n-1}$. As $\boldsymbol{S}(\mathscr{O}_Y)$ is canonically isomorphic to $\mathscr{O}_Y[T]$, we see $\mathbb{P}_Y^0$ is canonically identified with Y.

If $Y = \mathrm{Spec}(A)$ and $\mathscr{E} = \widetilde{E}$ where E is an A-module, we also denote by $\mathbb{P}(E)$ the projective bundle $\mathbb{P}(E)$. The simplest example is $\mathbb{P}(E)$ where E is a vector space over a field k. In this case, we see $\mathbb{P}(E)$ is isomorphic to $\mathbb{P}_k^{n-1}$, where n is the dimension of E.

Let $\mathscr{E}, \mathscr{F}$ be two qusi-coherent $\mathscr{O}_Y$-modules and $u : \mathscr{E} \to \mathscr{F}$ be an $\mathscr{O}_Y$-homomorphism. Then u corresponds canonically to a homomorphism $\boldsymbol{S}(u) : \boldsymbol{S}(\mathscr{E}) \to \boldsymbol{S}(\mathscr{F})$ of graded $\mathscr{O}_Y$-algebras. If u is surjective, so is $\boldsymbol{S}(u)$, and therefore $\mathrm{Proj}(\boldsymbol{S}(u))$ is a closed immersion $\mathbb{P}(\mathscr{F}) \to \mathbb{P}(\mathscr{E})$, which we then denoted by $\mathbb{P}(u)$. We can then say that $\mathbb{P}(\mathscr{E})$ is a contravariant functor on the category of quasi-coherent $\mathscr{O}_Y$-modules with *surjective homomorphisms*. Suppose that u is surjective and put $P = \mathbb{P}(\mathscr{E})$, $Q = \mathbb{P}(\mathscr{F})$, and $j = \mathbb{P}(u)$. We then have an isomorphism

$$j^*(\mathscr{O}_P(n)) = \mathscr{O}_Q(n)$$

by Corollary 5.3.31.

If $\psi : Y' \to Y$ is a morphism and $\mathscr{E}' = \psi^*(\mathscr{E})$, we then have $\boldsymbol{S}_{\mathscr{O}_{Y'}}(\mathscr{E}') = \psi^*(\boldsymbol{S}_{\mathscr{O}_Y}(\mathscr{E}))$ by

Proposition 5.1.34, so from Proposition 5.3.28 we deduce that

$$\mathbb{P}(\psi^*(\mathscr{E})) = \mathbb{P}(\mathscr{E}) \times_Y Y'. \tag{4.1.1}$$

Moreover, it is clear that $\psi^*((\boldsymbol{S}_{\mathscr{O}_Y}(\mathscr{E}))(n)) = (\boldsymbol{S}_{\mathscr{O}_{Y'}}(\mathscr{E}'))(n)$ for each $n \in \mathbb{Z}$, so if $P = \mathbb{P}(\mathscr{E})$ and $P' = \mathbb{P}(\mathscr{E}')$, we have a canonical isomorphism

$$\mathscr{O}_{P'}(n) = \mathscr{O}_P(n) \otimes_Y \mathscr{O}_{Y'}. \tag{4.1.2}$$

Proposition 5.4.1. *Let $\mathscr{L}$ be an invertible $\mathscr{O}_Y$-module. For any quasi-coherent $\mathscr{O}_Y$-module, there exists a canonical Y-isomorphism $i_{\mathscr{L}} : Q = \mathbb{P}(\mathscr{E} \otimes \mathscr{L}) \xrightarrow{\sim} P = \mathbb{P}(\mathscr{E})$. Moreover, $(i_{\mathscr{L}})_*(\mathscr{O}_Q(n))$ is cannically isomorphic to $\mathscr{O}_P(n) \otimes_Y \mathscr{L}^{\otimes n}$ for any $n \in \mathbb{Z}$.*

Proof. If A is a ring, E is an A-module, L is an A-module free of rank 1, we can define a canonical homomorphism

$$\boldsymbol{S}_n(E \otimes L) \to \boldsymbol{S}_n(E) \otimes L^{\otimes n}$$

which maps an element $(x_1 \otimes y_1) \cdots (x_n \otimes y_n)$ to

$$(x_1 \cdots x_n) \otimes (y_1 \otimes \cdots \otimes y_n)$$

where $x_i \in E$ and $y_i \in L$. This is easily seen to be an isomorphism, so we get an isomorphism $\boldsymbol{S}(E \otimes L) \cong \bigoplus_{n \geq 0} \boldsymbol{S}_n(E) \otimes L^{\otimes n}$. In the situation of the proposition, the preceding remark allows us to define a canonical isomorphism of graded $\mathscr{O}_Y$-algebras

$$\boldsymbol{S}(\mathscr{E} \otimes_{\mathscr{O}_Y} \mathscr{L}) \xrightarrow{\sim} \bigoplus_{n \geq 0} \boldsymbol{S}_n(\mathscr{E}) \otimes_{\mathscr{O}_Y} \mathscr{L}^{\otimes n}$$

The proposition then follows from Proposition 5.3.6 and Proposition 5.3.18. ◻

Let $P = \mathbb{P}(\mathscr{E})$ and denote by $p : P \to Y$ the structural morphism. As by definition $\mathscr{E} = (\boldsymbol{S}(\mathscr{E}))_1$, we have a canonical homomorphism $\alpha_1 : \mathscr{E} \to p_*(\mathscr{O}_P(1))$, and therefore a canonical homomorphism

$$\alpha_1^\sharp : p^*(\mathscr{E}) \to \mathscr{O}_P(1).$$

Proposition 5.4.2. *The canonical homomorphism $\alpha_1^\sharp$ is surjective.*

Proof. We have seen that $\alpha_1^\sharp$ corresponds to the functorial homomorphism $\mathscr{E} \otimes_{\mathscr{O}_Y} \boldsymbol{S}(\mathscr{E}) \to (\boldsymbol{S}(\mathscr{E}))(1)$ (see the remark before Proposition 5.3.19). Since $\mathscr{E}$ generates $\boldsymbol{S}(\mathscr{E})$, this homomorphism is surjective, whence our assertion in view of Proposition 5.3.13. ◻

5.4.2 Morphisms into $\mathbb{P}(\mathscr{E})$

With the notations of the last subsection, we now let X be an Y-scheme, $q : X \to Y$ be the structural morphism, and $r : X \to P$ be an Y-morphism, which gives a commutative diagram

$$\begin{tikzcd} X \arrow[r, "r"] \arrow[dr, "q"'] & P \arrow[d, "p"] \\ & Y \end{tikzcd}$$

As the functor r^* is right-exact, we deduce from the surjective homomorphism $\alpha_1^\sharp$ in Proposition 5.4.2 a surjective homomorphism

$$r^*(\alpha_1^\sharp) : r^*(p^*(\mathscr{E})) \to r^*(\mathscr{O}_P(1)).$$

But $r^*(p^*(\mathscr{E})) = q^*(\mathscr{E})$ and $r^*(\mathscr{O}_P(1))$ is locally isomorphic to $r^*(\mathscr{O}_P) = \mathscr{O}_X$, which is then an invertible sheaf $\mathscr{L}_r$ over X, so we obtain a canonical surjective $\mathscr{O}_X$-homomorphism

$$\varphi_r : q^*(\mathscr{E}) \to \mathscr{L}_r.$$

If $Y = \operatorname{Spec}(A)$ is affine and $\mathscr{E} = \widetilde{E}$, we can explicitly explain this homomorphism: given $f \in E$, it follows from Proposition 5.2.33 that

$$r^{-1}(D_+(f)) = X_{\varphi_r^\flat(f)}.$$

Let V be an affine open of X contained in $r^{-1}(D_+(f))$, and let B be its ring, which is an A-algebra; put $S = \boldsymbol{S}_A(E)$. The restriction of r to V then corresponds to an A-homomorphism $\omega : S_{(f)} \to B$, and we have $q^*(\mathscr{E})|_V = \widetilde{E \otimes_A B}$ and $\mathscr{L}_r|_V = \widetilde{L_r}$, where by Proposition 4.1.13, $L_r = S(1)_{(f)} \otimes_{S_{(f)}} B$. In view of the definition of α_1, the restriction of φ_r to $q^*(\mathscr{E})|_V$ therefore corresponds to the B-homomorphism

$$u : E \otimes_A B \to L_r, \quad x \otimes 1 \mapsto (x/1) \otimes 1 = (f/1) \otimes \omega(x/f)$$

The canonical extension of φ_r to a homomorphism of $\mathscr{O}_X$-algebras (recall that $(\mathscr{O}_P(1))^{\otimes n} = \mathscr{O}_P(n)$ by Proposition 5.2.29)

$$\psi_r : q^*(\boldsymbol{S}(\mathscr{E})) = \boldsymbol{S}(q^*(\mathscr{E})) \to \boldsymbol{S}(\mathscr{L}_r) = \bigoplus_{n \geq 0} \mathscr{L}_r^{\otimes n} = \bigoplus_{n \geq 0} r^*(\mathscr{O}_P(n))$$

is then such that the restriction of ψ_r to $q^*(\boldsymbol{S}_n(\mathscr{E}))|_V$ corresponds to the B-homomorphism

$$\boldsymbol{S}_n(\mathscr{E} \otimes_A B) = \boldsymbol{S}_n(E) \otimes_A B \to L_r^{\otimes n} = (S(1)_{(f)})^{\otimes n} \otimes_{S_{(f)}} B$$

which send the element $s \otimes 1$ to $(f/1)^{\otimes n} \otimes \omega(s/f^n)$.

Conversely, given an invertible $\mathscr{O}_X$-module $\mathscr{L}$ and a quasi-coherent $\mathscr{O}_Y$-module $\mathscr{E}$, then any homomorphism $\varphi : q^*(\mathscr{E}) \to \mathscr{L}$ corresponds to a canonical homomorphism of quasi-coherent $\mathscr{O}_X$-algebras

$$\psi : \boldsymbol{S}(q^*(\mathscr{E})) = q^*(\boldsymbol{S}(\mathscr{E})) \to \boldsymbol{S}(\mathscr{L}) = \bigoplus_{n \geq 0} \mathscr{L}^{\otimes n}$$

which then gives an Y-morphism $r_{\mathscr{L},\psi} : G(\psi) \to \operatorname{Proj}(\boldsymbol{S}(\mathscr{E})) \to \mathbb{P}(\mathscr{E})$, which we also denoted by $r_{\mathscr{L},\varphi}$. If φ is surjective, then so is ψ and by Corollary 5.3.34 the morphism $r_{\mathscr{L},\psi}$ is everywhere defined.

Proposition 5.4.3. *Let $q : X \to Y$ be a morphism and $\mathscr{E}$ be a quasi-coherent $\mathscr{O}_Y$-module. Then the maps $r \mapsto (\mathscr{L}_r, \varphi_r)$ and $(\mathscr{L}, \varphi) \mapsto r_{\mathscr{L},\varphi}$ form a bijective correspondence between the set of Y-morphisms $r : X \to \mathbb{P}(\mathscr{E})$ to the set of equivalence classes of couples $(\mathscr{L}, \varphi)$ formed by an invertible $\mathscr{O}_X$-module and* *a surjective homomorphism $\varphi : q^*(\mathscr{E}) \to \mathscr{L}$, where two couples $(\mathscr{L}, \varphi)$, $(\mathscr{L}', \varphi')$ are equivalent if there exists an $\mathscr{O}_X$-isomorphism $\tau : \mathscr{L} \to \mathscr{L}'$ such that $\varphi' = \tau \circ \varphi$.*

Proof. Let us start with an Y-morphism $r : X \to \mathbb{P}(\mathscr{E})$, form $\mathscr{L}_r$ and φ_r, and put $r' = r_{\mathscr{L}_r, \varphi_r}$. To see the morphisms r and r' coincide, we may assume that $Y = \operatorname{Spec}(A)$ is affine, so $\mathscr{E} = \widetilde{E}$, and let $S = \boldsymbol{S}_A(E)$. Let $V = \operatorname{Spec}(B)$ be an affine open of X contained in $r^{-1}(D_+(f))$, where $f \in E$. Then as we have already seen, the restriction of r to V corresponds to an A-homomorphism $\omega : S_{(f)} \to B$, and the restriction of ψ_r to $q^*(\boldsymbol{S}_n(\mathscr{E}))|_V$ corresponds to the B-homomorphism $\boldsymbol{S}_n(E) \otimes_A B \to L_r^{\otimes n}$ which sends $s \otimes 1$ to $(f/1)^{\otimes n} \otimes \omega(s/f^n)$. The restriction of $\psi_r^\flat$ to $\boldsymbol{S}(\mathscr{E})|_V$ then corresponds to the homomorphism $\boldsymbol{S}_n(E) \to L_r^{\otimes n}$ which sends $s \in \boldsymbol{S}_n(E)$ to $(f/1)^{\otimes n} \otimes \omega(s/f^n)$, and by Proposition 5.3.33, the restriction of $r_{\mathscr{L}_r, \psi_r}$ to V corresponds to the homomorphism $(\psi_r^\flat)_{(f)}$, which send $s \in S_n$ to

$$(\psi_r^\flat(s))(\psi_r^\flat(f))^{-n} = [(f/1)^{\otimes n} \otimes \omega(s/f^n)][(f/1) \otimes 1]^{-n} = 1 \otimes \omega(s/f^n).$$

Therefore, under the identification of X with $\operatorname{Proj}(\boldsymbol{S}(\mathscr{L}))$, $r_{\mathscr{L},\psi_r}$ coincides with r over V, so they coincide on X.

Conversely. let $(\mathscr{L}, \varphi)$ be a couple and form $r = r_{\mathscr{L},\varphi}$, $\mathscr{L}_r$, and φ_r. We show that there is a canonical isomorphism $\tau : \mathscr{L}_r \to \mathscr{L}$ such that $\varphi = \tau \circ \varphi_r$. For this, we can assume that $Y = \operatorname{Spec}(A)$ and $X = \operatorname{Spec}(B)$ are affine, and (with the notations of Remark 5.3.4) define τ be sending an element $(x/1) \otimes 1$ of L_r (where $x \in E$) to the element $v_1(x)c$ of L. It is easy to verify that τ is independent of the choice of the generator c of L. As v_1 is surjective, to show that τ is an isomorphism, it suffices to prove that if $x/1 = 0$ in $S(1)_{(f)}$, then $v_1(x)/1 = 0$ in B_g. But the first condition means that $f^n x = 0$ in S_{n+1} for some n, and we then deduce that $v_{n+1}(f^n x) = g^n v_1(x) = 0$ in B, whence the conclusion. Finally, it is immediate that for two equivalent couples $(\mathscr{L}, \varphi)$, $(\mathscr{L}', \varphi')$, we have $r_{\mathscr{L},\varphi} = r_{\mathscr{L}',\varphi'}$. □

Theorem 5.4.4. *The set of Y-sections of $\mathbb{P}(\mathscr{E})$ is in bijective correspondence to the set of quasi-coherent sub-$\mathscr{O}_Y$-modules $\mathscr{F}$ of $\mathscr{E}$ such that $\mathscr{E}/\mathscr{F}$ is invertible.*

Proof. This is a particular case of Proposition 5.4.3 by taking $X = Y$ and note that if two pairs $(\varphi, \mathscr{L})$ and $(\varphi', \mathscr{L}')$ are equivalent, then $\ker \varphi$ and $\ker \varphi'$ are identical. □

Note that this property corresponds to the classical definition of the "projective space" as the set of hyperplanes of a vector space (the classical case corresponding to $Y = \operatorname{Spec}(k)$, where k is a field, and $\mathscr{E} = \widetilde{E}$, E being a finite dimensional k-vector. The sheaves $\mathscr{F}$ having the property stated in Theorem 5.4.4 corresponds then to the hyperplanes of E.

Remark 5.4.1. As there is a canonical correspondence between Y-morphisms from X to P and their graph morphisms, which are X-sections of $P \times_Y X$, we see conversely that Proposition 5.4.3 can be deduced from Theorem 5.4.4. Let $\operatorname{Hyp}_Y(X, \mathscr{E})$ be the set of quasi-coherent sub-$\mathscr{O}_X$-modules $\mathscr{F}$ of $\mathscr{E} \otimes_Y \mathscr{O}_X = q^*(\mathscr{E})$ such that $q^*(\mathscr{E})/\mathscr{F}$ is an invertible $\mathscr{O}_X$-module. If $g : X' \to X$ is an Y-morphism, then $g^*(q^*(\mathscr{E})/\mathscr{F}) = g^*(q^*(\mathscr{E}))/g^*(\mathscr{F})$ by the right exactness of g^*, so the second sheaf is invertible, and therefore $\operatorname{Hyp}_Y(X, \mathscr{E})$ is a covariant functor over the category of Y-schemes. We can then interprete Theorem 5.4.4 by saying that the Y-scheme $\mathbb{P}(\mathscr{E})$ representes the functor $\operatorname{Hyp}_Y(-, \mathscr{E})$. This also provides a characterization of the projective bundle $P = \mathbb{P}(\mathscr{E})$

by the following universal property, more close to the geometric intuition that the constructions of $r_{\mathscr{L},\psi}$: for any morphism $q : X \to Y$ and any invertible $\mathscr{O}_X$-module $\mathscr{L}$ which is a quotient of $q^*(\mathscr{E})$, there exists a unique Y-morphism $r : X \to P$ such that $\mathscr{L}$ is isomorphic to $r^*(\mathscr{O}_P(1))$.

Corollary 5.4.5. *Suppose that any invertible $\mathscr{O}_Y$-module is trivial. Let $E = \mathrm{Hom}_{\mathscr{O}_Y}(\mathscr{E}, \mathscr{O}_Y)$ considered as a module over the ring $A = \Gamma(Y, \mathscr{O}_Y)$, and let $E^\times$ be the subset of E formed by surjective homomorphisms. Then the set of Y-sections of $\mathbb{P}(\mathscr{E})$ is canonically identified with $E^\times/A^\times$, where $A^\times$ is the group of units of A.*

Example 5.4.6. Let Y be a scheme, y be a point of Y, and $Y' = \mathrm{Spec}(\kappa(y))$. The fiber $p^{-1}(y)$ of $\mathbb{P}(\mathscr{E})$ is, in view of (4.1.1), identified with $\mathbb{P}(\mathscr{E}^y)$, where $\mathscr{E}^y = \mathscr{E}_y \otimes_{\mathscr{O}_{Y,y}} \kappa(y) = \mathscr{E}_y/\mathfrak{m}_y\mathscr{E}_y$ is considered as a vector space over $\kappa(y)$. More generally, if K is an extension of $\kappa(y)$, $p^{-1}(y) \otimes_{\kappa(y)} K$ is identified with $\mathbb{P}(\mathscr{E}^y \otimes_{\kappa(y)} K)$. Since any invertible sheaves over a local scheme is trivial, Corollary 5.4.5 shows that the points of $\mathbb{P}(\mathscr{E})$ lying over y with values in K, which are called the **rational geometric fibers** of $\mathbb{P}(\mathscr{E})$ over K lying over y, is identified with the projective space of the dual of the vector K-space $\mathscr{E}^y \otimes_{\kappa(y)} K$.

Example 5.4.7. Suppose now that Y is affine with ring A, and any invertible sheaf on Y is trivial; we put $\mathscr{E} = \mathscr{O}_Y^n$. Then with the notations of Corollary 5.4.5, E is identified with A^n by Proposition 4.1.3 and $E^\times$ is identified with the set of systems $(f_i)_{1\le i\le n}$ of elements of A which generate the unit ideal of A. By Corollary 5.4.5, two such systems determine the same Y-section of $\mathbb{P}_Y^{n-1} = \mathbb{P}_A^{n-1}$, which means the same point of $\mathbb{P}_A^{n-1}$ with values in A, if and only if one is deduced from the other by multiplication by an invertible element of A.

Remark 5.4.2. Let $u : X' \to X$ be a morphism. If the Y-morphism $r : X \to P$ corresponds to the homomorphism $\varphi : q^*(\mathscr{E}) \to \mathscr{L}$, then by definition, the morphism $r \circ u$ corresponds to $u^*(\varphi) : u^*(q^*(\mathscr{E})) \to u^*(\mathscr{L})$. On the other hand, let $\mathscr{E}, \mathscr{F}$ be two quasi-coherent $\mathscr{O}_Y$-modules and $j = \mathbb{P}(v)$ be the closed immersion $\mathbb{P}(\mathscr{F}) \to \mathbb{P}(\mathscr{E})$ corresponding to a surjective homomorphism $v : \mathscr{E} \to \mathscr{F}$. If the Y-morphism $r : X \to \mathbb{P}(\mathscr{E})$ corresponds to the homomorphism $\varphi : q^*(\mathscr{E}) \to \mathscr{L}$, then $j \circ r$ corresponds to the composition

$$q^*(\mathscr{E}) \xrightarrow{q^*(v)} q^*(\mathscr{F}) \xrightarrow{\varphi} \mathscr{L}$$

Let $\psi : Y' \to Y$ be a morphism and put $\mathscr{E}' = \psi^*(\mathscr{E})$. If the Y-morphism $r : X \to P$ corresponds to the homomorphism $\varphi : q^*(\mathscr{E}) \to \mathscr{L}$, the Y'-morphism

$$r_{(Y')} : X_{(Y')} \to P' = \mathbb{P}(\mathscr{E}')$$

correspond to $\varphi_{(Y')} : q^*_{(Y')}(\mathscr{E}') = q^*(\mathscr{E}) \otimes_Y \mathscr{O}_{Y'} \to \mathscr{L} \otimes_Y \mathscr{O}_{Y'}$. In fact, by (4.1.1), we have the following commutative diagram

$$\begin{tikzcd}
X_{(Y')} \arrow[r, "r_{(Y')}"] \arrow[d, "v"] & P' = P_{(Y')} \arrow[r, "p_{(Y')}"] \arrow[d, "u"] & Y' \arrow[d, "\psi"] \\
X \arrow[r, "r"] & P \arrow[r, "p"] & Y
\end{tikzcd}$$

In view of (4.1.2), we have

$$(r_{(Y')})^*(\mathscr{O}_{P'}(1)) = (r_{(Y')})^*(u^*(\mathscr{O}_P(1))) = v^*(r^*(\mathscr{O}_P(1))) = v^*(\mathscr{L}) = \mathscr{L} \otimes_Y \mathscr{O}_{Y'}.$$

Also, $u^*(\alpha_1^\sharp)$ is equal to the canonical homomorphism $\alpha_1^\sharp : (p_{(Y')})^*(\mathscr{E}') \to \mathscr{O}_{P'}(1)$, in view of the definition of α_1, whence our assertion.

5.4.3 The Segre morphism

Let Y be a scheme and $\mathscr{E}, \mathscr{F}$ be two quasi-coherent $\mathscr{O}_Y$-modules. Put $P_1 = \mathbb{P}(\mathscr{E})$, $P_2 = \mathbb{P}(\mathscr{F})$, and denote by p_1, p_2 their morphisms; let $Q = P_1 \times_Y P_2$ and q_1, q_2 be the canonical projections. The $\mathscr{O}_Q$-module

$$\mathscr{L} = \mathscr{O}_{P_1}(1) \times_Y \mathscr{O}_{P_2}(1) = q_1^*(\mathscr{O}_{P_1}(1)) \times_{\mathscr{O}_Q} q_2^*(\mathscr{O}_{P_2}(1))$$

is invertible as a tensor product of invertible modules. On the other hand, if $r = p_1 \circ q_1 = p_2 \circ q_2$ is the structural morphism of Q, we have $r^*(\mathscr{E} \otimes_{\mathscr{O}_Y} \mathscr{F}) \otimes_{\mathscr{O}_Q} q_2^*(p_2^*(\mathscr{F}))$; the canonical surjective homomorphism $p_1^1(\mathscr{E}) \to \mathscr{O}_{P_1}(1)$ and $p_2^*(\mathscr{F}) \to \mathscr{O}_{P_2}(1)$ then give a canonical surjective homomorphism

$$s : r^*(\mathscr{E} \otimes_{\mathscr{O}_Y} \mathscr{F}) \to \mathscr{L} \tag{4.3.1}$$

we then deduce a canonical homomorphism, called the Segre morphism:

$$\zeta : \mathbb{P}(\mathscr{E}) \times_Y \mathbb{P}(\mathscr{F}) \to \mathbb{P}(\mathscr{E} \otimes_{\mathscr{O}_Y} \mathscr{F}). \tag{4.3.2}$$

To explain this morphism ζ, let us consider the case where $Y = \operatorname{Spec}(A)$ is affine, $\mathscr{E} = \widetilde{E}$, $\mathscr{F} = \widetilde{F}$, where E and F are two A-module; whence $\mathscr{E} \otimes_{\mathscr{O}_Y} \mathscr{F} = \widetilde{E \otimes_A F}$. Put $R = \boldsymbol{S}(E)$, $S = \boldsymbol{S}(F)$, and $T = \boldsymbol{S}(E \otimes_A F)$. Let $f \in E$, $g \in F$, and consider the affine open

$$D_+(f) \times_Y D_+(g) = \operatorname{Spec}(B)$$

of Q, where $B = R_{(f)} \otimes_A S_{(g)}$. The restriction of $\mathscr{L}$ on this affine open is $\widetilde{L}$, where

$$L = (R(1)_{(f)}) \otimes_A (S(1)_{(g)})$$

and the element $c = (f/1) \otimes (g/1)$ is a generator of L as a free B-module (Proposition 5.2.24). The homomorphism (4.3.1) then corresponds to the homomorphism

$$(x \otimes y) \otimes b \mapsto b((x/1) \otimes (y/1))$$

from $(E \otimes_A F) \otimes_A B$ to L. With the notations of Remark 5.3.4, we then have $v_1(x \otimes y) = (x/f) \otimes (y/g)$, so the restriction of the morphism ζ to $D_+(f) \times_Y D_+(g)$ is a morphism from this affine scheme to $D_+(f \otimes g)$, which corresponds to the ring homomorphism

$$\omega((x \otimes y)/(f \otimes g)) = (x/f) \otimes (y/g) \tag{4.3.3}$$

for $x \in E$ and $y \in F$.

From Proposition 5.4.3, there is a canonical isomorphism

$$\tau : \zeta^*(\mathscr{O}_P(1)) \xrightarrow{\sim} \mathscr{O}_{P_1}(1) \otimes_Y \mathscr{O}_{P_2}(1)$$

where we put $P = \mathbb{P}(\mathscr{E} \otimes_{\mathscr{O}_Y} \mathscr{F})$. Moreover, for $x \in \Gamma(Y, \mathscr{E})$ and $y \in \Gamma(Y, \mathscr{F})$, we have

$$\tau(\alpha_1(x \otimes y)) = \alpha_1(x) \otimes \alpha_1(y) \tag{4.3.4}$$

To see this, we can assume that Y is affine, so with the notations above and the definition of α_1, we have $\alpha_1^{f\otimes g}(x \otimes y) = (x \otimes y)/1$, $\alpha_1^f(x) = x/1$, and $\alpha_1^g(y) = y/1$. The definition of τ given in the proof of Proposition 5.4.3 says τ maps $(x/1) \otimes 1$ to $v_1(x)c$. Since we have seen that $v_1(x \otimes y) = (x/f) \otimes (y/g)$, this implies the assertion by a simple computation. From this, we then deduce the formula

$$\zeta^{-1}(P_{x\otimes y}) = (P_1)_x \times_Y (P_2)_y \tag{4.3.5}$$

where we need to use the following lemma:

Lemma 5.4.8. *Let B, B' be two A-algebras, and let $Y = \mathrm{Spec}(A)$, $Z = \mathrm{Spec}(B)$, $Z' = \mathrm{Spec}(B')$. Then for $t \in B$, $t' \in B'$, we have $D(t \otimes t') = D(t) \times_Y D(t')$.*

Proof. Let p, p' be the canonical projections of $Z \times_Y Z'$. Then it follows from Proposition **??** that $p^{-1}(D(t)) = D(t \otimes 1)$ and $p'^{-1}(D(t')) = D(1 \otimes t')$. Corollary 4.3.2 then implies the lemma, since $(t \otimes 1)(1 \otimes t') = t \otimes t'$. □

Proposition 5.4.9. *The Segre morphism is a closed immersion.*

Proof. Since the question is local on Y, we can assume that Y is affine. With the previous notations, the $D_+(f \otimes g)$ form a basis for P, since the elements $f \otimes g$ generate T for $f \in E$, $g \in F$. On the other hand, we have $\zeta^{-1}(D_+(f \otimes g)) = D_+(f) \times_Y D_+(g)$ in view of (4.3.5). It then suffices to use Proposition 4.4.10 to prove that the restriction of ζ to $D_+(f) \times_Y D_+(g)$ is a closed immersion into $D_+(f \otimes g)$. But this is a morphism between affine schemes whose corresponding ring homomorphism ω is surjective in view of the formula (4.3.3), so our assertion follows. □

The Segre morphism is functorial on $\mathscr{E}$ and $\mathscr{F}$ if we restrict ourselves to quasi-coherent $\mathscr{O}_Y$-modules with *surjective* homomorphisms. To see this, it suffices to consider a surjective $\mathscr{O}_Y$-homomorphism $\mathscr{E} \to \mathscr{E}'$ and prove that the following diagram is commutative

$$\begin{tikzcd}
\mathbb{P}(\mathscr{E}') \times \mathbb{P}(\mathscr{F}) \arrow[r, "j\times 1"] \arrow[d, "\zeta"'] & \mathbb{P}(\mathscr{E}') \times \mathbb{P}(\mathscr{F}) \arrow[d, "\zeta"] \\
\mathbb{P}(\mathscr{E}' \otimes \mathscr{F}) \arrow[r] & \mathbb{P}(\mathscr{E} \otimes \mathscr{F})
\end{tikzcd}$$

where j is the canonical closed immersion $\mathbb{P}(\mathscr{E}') \to \mathbb{P}(\mathscr{E})$. Put $P_1' = \mathbb{P}(\mathscr{E}')$ and retain the previous notations; $j \times 1$ is a closed immersion by Proposition 5.3.30 and we have

$$(j \times 1)^*(\mathscr{O}_{P_1}(1) \otimes \mathscr{O}_{P_2}(1)) = j^*(\mathscr{O}_{P_1}(1)) \otimes \mathscr{O}_{P_2}(1) = \mathscr{O}_{P_1'}(1) \otimes \mathscr{O}_{P_2}(1)$$

in view of (4.1.2) and Proposition 4.3.14. The assertion then follows from Remark 5.4.2.

Proposition 5.4.10. *Let $\psi : Y \to Y'$ be a morphism and put $\mathscr{E}' = \psi^*(\mathscr{E})$, $\mathscr{F}' = \psi^*(\mathscr{F})$. Then the Segre morphism $\mathbb{P}(\mathscr{E}') \times \mathbb{P}(\mathscr{F}') \to \mathbb{P}(\mathscr{E}' \otimes \mathscr{F}')$ is identified with $\zeta_{(Y')}$.*

Proof. Put $P_1' = \mathbb{P}(\mathscr{E}')$ and $P_2' = \mathbb{P}(\mathscr{F}')$. Then by Remark 5.4.2, P_i' is identified with $(P_i)_{(Y')}$ for $i = 1, 2$, so the structural morphism $P_1' \times_{Y'} P_2' \to Y'$ is identified with $r_{(Y')}$, where r is the structural morphism of $P_1 \times_Y P_2$. On the other hand, $\mathscr{E}' \otimes \mathscr{F}'$ is identified with $\psi^*(\mathscr{E} \otimes \mathscr{F})$, so $\mathbb{P}(\mathscr{E}' \otimes \mathscr{F}')$ is identified with $\mathbb{P}(\mathscr{E} \otimes \mathscr{F})_{(Y')}$ by Proposition 5.3.28. Finally, $\mathscr{O}_{P_1'}(1) \otimes_{Y'} \mathscr{O}_{P_2'}(1) = \mathscr{L}$ is identified with $\mathscr{L} \otimes_Y \mathscr{O}_{Y'}$ in view of (4.1.2) and Proposition 4.3.13. The canonical homomorphism $(r_{(Y')})^*(\mathscr{E}' \otimes \mathscr{F}') \to \mathscr{L}'$ is then identified with $s_{(Y')}$, and our assertion follows from Proposition 5.4.3. □

Remark 5.4.3. The coproduct of $\mathbb{P}(\mathscr{E})$ and $\mathbb{P}(\mathscr{F})$ is similarly canonical isomorphic to a closed subscheme of $\mathbb{P}(\mathscr{E} \oplus \mathscr{F})$. In fact, the surjective homomorphisms $\mathscr{E} \oplus \mathscr{F} \to \mathscr{E}$, $\mathscr{E} \oplus \mathscr{F} \to \mathscr{F}$ correspond to closed immersions $\mathbb{P}(\mathscr{E}) \to \mathbb{P}(\mathscr{E} \oplus \mathscr{F})$, $\mathbb{P}(\mathscr{E}) \to \mathbb{P}(\mathscr{E} \oplus \mathscr{F})$; it then boils down to showing that the underlying spaces of these closed subschemes of $\mathbb{P}(\mathscr{E} \oplus \mathscr{F})$ thus obtained have no common point. The question being local on Y, we can assume that Y is affine adapt our previous notations. Now $\boldsymbol{S}_n(E)$ and $\boldsymbol{S}_n(F)$ are identified with submodules of $S_n(E \oplus F)$ with intersection reduced to 0, and if $\mathfrak{p}$ is a graded prime ideal of $\boldsymbol{S}(E)$ such that $\mathfrak{p} \cap S_n(E) \neq S_n(E)$ for all $n \geq 0$, then it corresponds to a unique graded prime ideal in $\boldsymbol{S}(E \oplus F)$ whose trace on $\boldsymbol{S}_n(E)$ is $\mathfrak{p} \cap S_n(E)$, but which contains $S_+(F)$. Therefore, two distinct points of $\mathrm{Proj}(\boldsymbol{S}(E))$ and $\mathrm{Proj}(\boldsymbol{S}(F))$ can not have same image in $\mathrm{Proj}(\boldsymbol{S}(E \oplus F))$.

5.4.4 Very ample sheaves

Proposition 5.4.11. *Let Y be a quasi-compact scheme, $q : X \to Y$ a morphism of finite type, and $\mathscr{L}$ be an invertible $\mathscr{O}_X$-module.*

(a) Let $\mathscr{S}$ be a quasi-coherent graded $\mathscr{O}_Y$-algebra and $\psi : q^*(\mathscr{S}) \to \boldsymbol{S}(\mathscr{L})$ be a graded homomorphism of algebras. For the morphism $r_{\mathscr{L},\psi}$ to be everywhere defined and an immersion, it is necessary and sufficent that there exists an integer $n \geq 0$ and a quasi-coherent sub-$\mathscr{O}_Y$-module of finite type $\mathscr{E}$ of $\mathscr{S}_n$ such that the homomorphism $\varphi' = \psi_n \circ q^*(j) : q^*(\mathscr{E}) \to \boldsymbol{S}(\mathscr{L}) = \mathscr{L}'$ ($j : \mathscr{E} \to \mathscr{S}_n$ being the canonical injection) is surjective and the morphism $r_{\mathscr{L}',\varphi'} : X \to \mathbb{P}(\mathscr{E})$ is an immersion.

(b) Let $\mathscr{F}$ be a quasi-coherent $\mathscr{O}_Y$-module and $\varphi : q^*(\mathscr{F}) \to \mathscr{L}$ be a surjective homomorphism. For the morphism $r_{\mathscr{L},\varphi} : X \to \mathbb{P}(\mathscr{F})$ to be an immersion, it is necessary and sufficent that there exists a quasi-coherent sub-$\mathscr{O}_Y$-module of finite type $\mathscr{E}$ of $\mathscr{F}$ such that the homomorphism $\varphi' = \varphi \circ q(j) : q^*(\mathscr{E}) \to \mathscr{L}$ (where $j : \mathscr{E} \to \mathscr{F}$ is the canonical injection) is surjective and such that $r_{\mathscr{L},\varphi'} : X \to \mathbb{P}(\mathscr{E})$ is an immersion.

Proof. We first consider case (a). The fact that $r_{\mathscr{L},\psi}$ is everywhere defined and an immersion is equivalent by Proposition 5.3.42 to the existence of an integer $n > 0$ and $\mathscr{E}$ such that, if $\mathscr{S}'$ is the subalgebra of $\mathscr{S}$ generated by $\mathscr{E}$, the homomorphism $q^*(\mathscr{E}) \to \mathscr{L}^{\otimes n}$ is surjective and $r_{\mathscr{L},\psi'} : X \to \mathrm{Proj}(\mathscr{S}')$ is everywhere defined and an immersion. We also have a closed immersion corresponding to the surjective homomorphism $\boldsymbol{S}(\mathscr{E}) \to \mathscr{S}'$, so these the morphism $X \to \mathbb{P}(\mathscr{E})$ is an immersion.

Now consider the situation of (b). As $\mathscr{F}$ is the inductive limit of its quasi-coherent submodules of finite type $\mathscr{E}_\lambda$ (Proposition 4.6.62), $\boldsymbol{S}(\mathscr{F})$ is the inductive limit of the $\boldsymbol{S}(\mathscr{E}_\lambda)$, so by Proposition 5.3.41 there exists λ such that $r_{\mathscr{L},\varphi_\mu}$ is everywhere defined and an immersion for $\mu \geq \lambda$. Also, since the functor f^* is left-adjoint, it commutes with inductive limits and therefore $q^*(\mathscr{F})$ is the inductive limits of the $q^*(\mathscr{E}_\lambda)$. Since $\mathscr{L}$ is an $\mathscr{O}_X$-module of finite type and $q^*(\mathscr{F}) \to \mathscr{L}$ is surjective, it follows from Proposition 1.4.10 that there exists λ' such that $q^*(\mathscr{E}_\mu) \to \mathscr{L}$ is surjective for $\mu \geq \lambda'$. It then suffices to choose $\mathscr{E} = \mathscr{E}_\mu$ for $\mu \geq \lambda$ and $\mu \geq \lambda'$. ◻

Let Y be a scheme and $q : X \to Y$ be a morphism. We say an invertible $\mathscr{O}_X$-module $\mathscr{L}$ is **very ample for q** (or **very ample relative to q**, or simple **very ample**) if there exists a quasi-coherent $\mathscr{O}_Y$-module $\mathscr{E}$ and a Y-immersion $r : X \to \mathbb{P}(\mathscr{E})$ such that $\mathscr{L}$ is isomorphic to $r^*(\mathscr{O}_P(1))$. In view of Proposition 5.4.3, this is equivalent to the existence of a quasi-coherent $\mathscr{O}_Y$-module $\mathscr{E}$ and a surjective homomorphism $\varphi : q^*(\mathscr{E}) \to \mathscr{L}$ such that the associated morphism $r_{\mathscr{L},\varphi} : X \to \mathbb{P}(\mathscr{E})$ is an immersion. We also note that the existence of a very ample $\mathscr{O}_X$-module relative to Y implies that q is separated (Proposition 5.2.14 and Proposition 4.5.25).

Corollary 5.4.12. *Suppose that there exists a quasi-coherent $\mathscr{O}_Y$-algebra $\mathscr{S}$ genrated by $\mathscr{S}_1$ and a Y-immersion $r : X \to P = \mathrm{Proj}(\mathscr{S})$ such that $\mathscr{L}$ is isomorphic to $r^*(\mathscr{O}_P(1))$. Then $\mathscr{L}$ is very ample relative to q.*

Proof. If $\mathscr{F} = \mathscr{S}_1$, the canonical homomorphism $\boldsymbol{S}(\mathscr{F}) \to \mathscr{S}$ is surjective, so by composing r with the corresponding closed immersion $\mathrm{Proj}(\mathscr{S}) \to \mathbb{P}(\mathscr{F})$, we obtain an immersion $r' : X \to \mathbb{P}(\mathscr{F}) = P'$ such that $\mathscr{L}$ is isomorphic to $r'^*(\mathscr{O}_{P'}(1))$. ◻

Proposition 5.4.13. *Let $q : X \to Y$ be a quasi-compact morphism and $\mathscr{L}$ be an invertible $\mathscr{O}_X$-module. Then $\mathscr{L}$ is very ample relative to q if and only if $q_*(\mathscr{L})$ is quasi-coherent, the cannical homomorphism $\sigma : q^*(q_*(\mathscr{L})) \to \mathscr{L}$ is surjective, and the morphism $r_{\mathscr{L},\sigma} : X \to \mathbb{P}(q_*(\mathscr{L}))$ is an immersion.*

Proof. As q is quasi-compact, $q_*(\mathscr{L})$ is quasi-coherent if q is separated (Proposition 4.6.54). By Remark 5.3.3, the existence of a surjective homomorphism $\varphi : q^*(\mathscr{E}) \to \mathscr{L}$ ($\mathscr{E}$ being a quasi-coherent $\mathscr{O}_Y$-module) implies that σ is surjective. Moreover, the factorization $\varphi : q^*(\mathscr{E}) \to q^*(q_*(\mathscr{L})) \xrightarrow{\sigma} \mathscr{L}$ of (3.4.3) corresponds to a canonical factorization (recall that q^* commutes with $\boldsymbol{S}$)

$$q^*(\boldsymbol{S}(\mathscr{E})) \longrightarrow q^*(\boldsymbol{S}(q_*(\mathscr{L}))) \longrightarrow \boldsymbol{S}(\mathscr{L})$$

so by Corollary 5.3.40 the hypothesis that $r_{\mathscr{L},\varphi}$ is an immersion implies that $j = r_{\mathscr{L},\sigma}$ is an immersion. Moreover, by Proposition 5.4.3, $\mathscr{L}$ is isomorphic to $j^*(\mathscr{O}_{P'}(1))$ where $P' = \mathbb{P}(q_*(\mathscr{L}))$. The converse of this is clear by the definition of very ampleness. ◻

Corollary 5.4.14. *Let $q : X \to Y$ be a quasi-compact morphism and $\mathscr{L}$ be an invertible $\mathscr{O}_X$-module. For $\mathscr{L}$ to be very ample relative to Y, it is necessary and sufficent that there exists an open covering (U_α) of Y such that $\mathscr{L}|_{q^{-1}(U_\alpha)}$ is very ample relative to U_α for each α.*

Proof. This follows from the fact that the criterion of Proposition 5.4.13 is local over Y. ◻

Proposition 5.4.15. *Let Y be a quasi-compact scheme, $q : X \to Y$ a morphism of finite type, and $\mathscr{L}$ be an invertible $\mathscr{O}_X$-module. Then the following conditions are equivalent:*

(i) *$\mathscr{L}$ is very ample relative to Y.*

(ii) *There exists a quasi-coherent $\mathscr{O}_Y$-module $\mathscr{E}$ of finite type and a surjective homomorphism φ : $q^*(\mathscr{E}) \to \mathscr{L}$ such that $r_{\mathscr{L},\varphi}$ is an immersion.*

(iii) *There exists a coherent sub-$\mathscr{O}_Y$-module of finite type $\mathscr{E}$ of $q_*(\mathscr{L})$ and a surjective homomorphism $\varphi : q^*(\mathscr{E}) \to \mathscr{L}$ such that $r_{\mathscr{L},\varphi}$ is an immersion.*

Proof. It is clear that (ii) or (iii) implies (i); but (i) implies (ii) by Proposition 5.4.11, and similarly (i) implies (iii) in view of Proposition 5.4.13. □

Corollary 5.4.16. *Suppose that Y is a quasi-compact scheme. If $\mathscr{L}$ is very ample relative to Y, there exists a quasi-coherent graded $\mathscr{O}_Y$-algebra $\mathscr{S}$ generated by $\mathscr{S}_1$ and such that $\mathscr{S}_1$ is of finite type, and a dominant open Y-immersion $i : X \to P = \mathrm{Proj}(\mathscr{S})$ such that $\mathscr{L}$ is isomorphic to $i^*(\mathscr{O}_P(1))$.*

Proof. Since $\mathscr{L}$ is very ample, by Proposition 5.4.15 there exists a coherent sub-$\mathscr{O}_Y$-module of finite type $\mathscr{E}$ of $q_*(\mathscr{L})$ and a surjective homomorphism $\varphi : q^*(\mathscr{E}) \to \mathscr{L}$ such that $r_{\mathscr{L},\varphi}$ is an immersion. The structural morphism $p : P' = \mathbb{P}(\mathscr{E}) \to Y$ is then separated and of finite type (Proposition 5.3.21), so P' is a quasi-compact scheme if Y is quasi-compact. Let Z be scheme theoretic image of X in P', with underlying space $\overline{j(X)}$, where $j = r_{\mathscr{L},\varphi}$; then j factors through Z into a dominant open immersion $i : X \to Z$. But Z is identified with the scheme $\mathrm{Proj}(\mathscr{S})$, where $\mathscr{S}$ is the quotient graded $\mathscr{O}_Y$-algebra of $\boldsymbol{S}(\mathscr{E})$ by a quasi-coherent graded ideal (Proposition 5.3.30), and it is clear that $\mathscr{S}_1$ is generated by $\mathscr{S}$ (since $\boldsymbol{S}(\mathscr{E})$ satisfies this condition). Moreover, by Corollary 5.3.31, $\mathscr{O}_Z(1)$ is the inverse image of $\mathscr{O}_{P'}(1)$ under the canonical injection, so $\mathscr{L} = i^*(\mathscr{O}_Z(1))$. □

Proposition 5.4.17. *Let $q : X \to Y$ be a morphism, $\mathscr{L}$ be a very ample $\mathscr{O}_X$-module relative to q, and $\mathscr{L}'$ be an invertible $\mathscr{O}_X$-module such that there exists a quasi-coherent $\mathscr{O}_Y$-module $\mathscr{E}'$ and a surjective homomorphism $q^*(\mathscr{E}') \to \mathscr{L}'$. Then $\mathscr{L} \otimes_{\mathscr{O}_X} \mathscr{L}'$ is very ample relative to q.*

Proof. The hypothesis on $\mathscr{L}'$ implies the existence of an Y-morphism $r' : X \to P' = \mathbb{P}(\mathscr{E}')$ such that $\mathscr{L}'$ is isomorphic to $r'^*(\mathscr{O}_{P'}(1))$ (Proposition 5.4.3). There is by hypothesis a quasi-coherent $\mathscr{O}_Y$-module $\mathscr{E}$ and a Y-immersion $r : X \to P = \mathbb{P}(\mathscr{E})$ such that $\mathscr{L}$ is isomorphic to $r^*(\mathscr{O}_P(1))$. Consider the Segre morphism $\zeta : P \times_Y P' \to Q$ where $Q = \mathbb{P}(\mathscr{E} \otimes \mathscr{E}')$. As r is an immersions, so is the morphism $(r, r')_Y : X \to P \times_Y P'$ by Corollary 4.5.16, and therefore we get an immersion

$$r'' : X \xrightarrow{(r,r')_Y} P \times_Y P' \xrightarrow{\zeta} Q.$$

Since $\zeta^*(\mathscr{O}_Q(1))$ is isomorphic to $\mathscr{O}_P(1) \otimes_Y \mathscr{O}_{P'}(1)$, we conclude from Proposition 4.3.14 that $r''^*(\mathscr{O}_Q(1))$ is isomorphic to $\mathscr{L} \otimes \mathscr{L}'$, this proves the assertion. □

Remark 5.4.4. Note that $q^*(\mathscr{O}_Y^{\oplus I}) = \mathscr{O}_X^{\oplus I}$ and there exists a surjection $\mathscr{O}_X^{\oplus I} \to \mathscr{L}'$ if and only if $\mathscr{L}'$ is generated by global sections, so Proposition 5.4.17 is applicable if $\mathscr{L}'$ is generated by global sections.

Corollary 5.4.18. *Let $q : X \to Y$ be a morphism.*

(a) Let $\mathscr{L}$ be an invertible $\mathscr{O}_X$-module and $\mathscr{K}$ be an invertible $\mathscr{O}_Y$-module. For $\mathscr{L}$ to be very ample relative to q, it is necessary and sufficent that $\mathscr{L} \otimes q^*(\mathscr{K})$ is very ample relative to q.

(b) If $\mathscr{L}$ and $\mathscr{L}'$ are two invertible $\mathscr{O}_X$-modules that are very ample relative to q, then so is $\mathscr{L} \otimes \mathscr{L}'$. In particular, $\mathscr{L}^{\otimes n}$ is very ample relative to q for any $n > 0$.

Proof. The assertions in (b) is an immediate concequence of Proposition 5.4.17, so is the half implication of (a). Now assume that $\mathscr{L} \otimes q^*(\mathscr{K})$ is very ample; then so is $(\mathscr{L} \otimes q^*(\mathscr{K})) \otimes q^*(\mathscr{K}^{-1})$ by Proposition 5.4.17, which is isomorphic to $\mathscr{L}$. □

Proposition 5.4.19. *Let $f : X \to Y$ be a quasi-compact morphism where Y is quasi-compact, and $\mathscr{L}$ be a very ample $\mathscr{O}_X$-module relative to f. Let $\mathscr{F}$ be a quasi-coherent $\mathscr{O}_X$-module of finite type and for each $n \in \mathbb{Z}$, set $\mathscr{F}(n) = \mathscr{F} \otimes \mathscr{L}^{\otimes n}$. Then there exists an integer n_0 such that for $n \geq n_0$, the canonical homomorphism $\sigma : f^*(f_*(\mathscr{F}(n))) \to \mathscr{F}(n)$ is surjective.*

Proof. □

Proposition 5.4.20. *Let $f : X \to Y$ be a quasi-compact morphism where Y is quasi-compact, and $\mathscr{L}$ be a very ample $\mathscr{O}_X$-module relative to f. Let $\mathscr{F}$ be a quasi-coherent $\mathscr{O}_X$-module of finite type. Then there exists an integer n_0 such that, for $n \geq n_0$, $\mathscr{F}$ is isomorphic to a quotient of an $\mathscr{O}_X$-module $f^*(\mathscr{G}) \otimes_{\mathscr{O}_X} \mathscr{L}^{\otimes(-n)}$, where $\mathscr{G}$ is a quasi-coherent $\mathscr{O}_Y$-module of finite type (dependent on n).*

Proof. Since $\mathscr{L}$ is very ample, f is separated and by Proposition 5.4.19 the canonical homomorphism $\sigma : f^*(f_*(\mathscr{F}(n))) \to \mathscr{F}(n)$ is surjective for n sufficiently large. The proposition is then a generalization of Corollary 5.3.26, and can be proved similarly. □

Proposition 5.4.21 (Properties of Very Ample Sheaves).

(i) For any scheme Y, any invertible $\mathscr{O}_Y$-module $\mathscr{L}$ is very ample relative to the identify morphism 1_Y.

(ii) Let $f : X \to Y$ be a morphism and $j : X' \to X$ be an immersion. If $\mathscr{L}$ is a very ample $\mathscr{O}_X$-module relative to f, then $j^*(\mathscr{L})$ is very ample relative to $f \circ j$.

(iii) Let $f : X \to Y$ be a morphism of finite type and $g : Y \to Z$ be a quasi-compact morphism where Z is quasi-compact. Let $\mathscr{L}$ a very ample $\mathscr{O}_X$-module relative to f and $\mathscr{K}$ be a very ample $\mathscr{O}_Y$-module relative to g. Then there exists an integer $n_0 > 0$ such that for any $n \geq n_0$, $\mathscr{L} \otimes f^*(\mathscr{K}^{\otimes n})$ is very ample relative to $g \circ f$.

(iv) Let $f : X \to Y$, $g : Y' \to Y$ be two morphisms, and put $X' = X_{(Y')}$. If $\mathscr{L}$ is a very ample $\mathscr{O}_X$-module relative to f, $\mathscr{L}' = \mathscr{L} \otimes_Y \mathscr{O}_{Y'}$ is very ample relative to $f_{(Y')}$.

(v) Let $f_i : X_i \to Y_i$ $(i = 1, 2)$ be two S-morphisms. If $\mathscr{L}_i$ is a very ample $\mathscr{O}_{X_i}$-module relative to f_i, then $\mathscr{L}_1 \otimes_S \mathscr{L}_2$ is very ample relative to $f_1 \times_S f_2$.

(vi) Let $f : X \to Y$, $g : Y \to Z$ be two morphisms. If an $\mathscr{O}_X$-module $\mathscr{L}$ is very ample relative to $g \circ f$, then $\mathscr{L}$ is very ample relative to f.

(vii) Let $f : X \to Y$ be a morphism and $j : X_{\mathrm{red}} \to X$ be the canonical injection. If $\mathscr{L}$ is a very ample $\mathscr{O}_X$-module relative to f, then $j^*(\mathscr{L})$ is very ample relative to f_{red}.

Proof. The property (ii) follows from the definition and it is immedaite that (vii) is deduced from (ii) and (vi). To prove (vi), consider the factorization

$$X \xrightarrow{\Gamma_f} X \times_Z Y \xrightarrow{p_2} Y$$

and note that $p_2 = (g \circ f) \times 1_Y$. It follows from the hypothesis and from (i) and (v) that $\mathscr{L} \otimes_{\mathscr{O}_Z} \mathscr{O}_Y$ is very ample relative to p_2. On the other hand, we have $\mathscr{L} = \Gamma_f^*(\mathscr{L} \otimes_{\mathscr{O}_Z} \mathscr{O}_Y)$ by Proposition 4.3.14, and Γ_f is an immersion (Proposition 4.5.8), so we can apply (ii). As for (i), we can apply the definition with $\mathscr{E} = \mathscr{L}$, and note that $\mathbb{P}(\mathscr{L})$ is identified with Y (Proposition 5.3.6).

We now prove (iv). Under the hypothesis of (iv), there exists a quasi-coherent $\mathscr{O}_Y$-module $\mathscr{E}$ and a Y-immersion $i : X \to P = \mathbb{P}(\mathscr{E})$ such that $\mathscr{L} = i^*(\mathscr{O}_P(1))$. Then, if $\mathscr{E}' = g^*(\mathscr{E})$, $\mathscr{E}'$ is a quasi-coherent $\mathscr{O}_Y$-module and we have $P' = \mathbb{P}(\mathscr{E}') = P_{(Y')}$, $i_{(Y')}$ is an immersion from $X_{(Y')}$ to P', and $\mathscr{L}'$ is isomorphic to $(i_{(Y')})^*(\mathscr{O}_{P'}(1))$ (Remark 5.4.2).

To prove (v), remark that there exists by hypothesis a Y_i-immersion $r_i : X_i \to P_i = \mathbb{P}(\mathscr{E}_i)$, where $\mathscr{E}_i$ is a quasi-coherent $\mathscr{O}_{Y_i}$-module, and $\mathscr{L}_i = r_i^*(\mathscr{O}_{P_i}(1))$. Then $r_1 \times_S r_2$ is an S-immersion of $X_1 \times_S X_2$ to $P_1 \times_S P_2$ (Proposition 4.4.11) and the inverse image of $\mathscr{O}_{P_1}(1) \otimes_S \mathscr{O}_{P_2}(1)$ by this immersion is $\mathscr{L}_1 \otimes_S \mathscr{L}_2$. On the other hand, put $T = Y_1 \times_S Y_2$, and let p_1, p_2 be the projection of T, respectively. If $P_i' = \mathbb{P}(p_i^*(\mathscr{E}_i))$, we have $P_i' = P_i \times_{Y_i} T$, whence

$$P_1' \times_T P_2' = (P_1 \times_{Y_1} T) \times_T (P_2 \times_{Y_2} T) = P_1 \times_{Y_1} (T \times_{Y_2} P_2) = P_1 \times_{Y_1} (Y_1 \times_S P_2) = P_1 \times_S P_2.$$

Similarly, we have $\mathscr{O}_{P_i'}(1) = \mathscr{O}_{P_i'}(1) \otimes_{Y_i} \mathscr{O}_T$, and $\mathscr{O}_{P_1'}(1) \otimes_T \mathscr{O}_{P_2'}(1)$ is identified with $\mathscr{O}_{P_1}(1) \otimes_S \mathscr{O}_{P_2}(1)$. We can then consider $r_1 \times_S r_2$ as an T-immersion from $X_1 \times_S X_2$ to $P_1' \times_T P_2'$, the inverse image of $\mathscr{O}_{P_1'}(1) \otimes_T \mathscr{O}_{P_2'}(1)$ by this immersion being $\mathscr{L}_1 \otimes_S \mathscr{L}_2$. We can then conclude as in Proposition 5.4.17 that $\mathscr{L}_1 \otimes_S \mathscr{L}_2$ is very ample.

It remains to prove (iii). We can first restrict to the case where Z is an affine scheme, since there exists a finite covering (U_i) of Z by affine opens; if the property is proved for $\mathscr{K}|_{g^{-1}(U_i)}$, $\mathscr{L}|_{f^{-1}(g^{-1}(U_i))}$ and an integer n_i, it suffices to choose n_0 to be the largest n_i to prove the property for $\mathscr{K}$ and $\mathscr{L}$ (Corollary 5.4.14). The hypotheses imply that f, g are spearated morphisms, so X and Y are quasi-compact schemes. Since $\mathscr{L}$ is very ample relative to f, there exists an immersion $r : X \to P = \mathbb{P}(\mathscr{E})$, where $\mathscr{E}$ is a quasi-coherent $\mathscr{O}_Y$-module of finite type and $\mathscr{L} = r^*(\mathscr{O}_P(1))$, in view of Proposition 5.4.15. We claim that there exists an integer m_0 such that for any $m \geq m_0$, there is a very ample $\mathscr{O}_P$-module $\mathscr{M}$ relative to the composition morphism $j : P \to Y \xrightarrow{g} Z$ such that $\mathscr{O}_P(1)$ is isomorphic to $\mathscr{M} \otimes_Y \mathscr{K}^{\otimes(-m)}$. For $n \geq m+1$, $\mathscr{O}_P(1) \otimes_Y \mathscr{K}^{\otimes n}$ will then be very ample relative to Z in view of the hypothesis and applying (v) to the morphism $j : P \to Z$ and 1_Z; as r is an immersion and $\mathscr{L} \otimes f^*(\mathscr{K}^{\otimes n}) = r^*(\mathscr{O}_P(1) \otimes_Y \mathscr{K}^{\otimes n})$, the conclusion then follows from (ii).

To estabilsh the claim, we can use Proposition 5.4.20 to obtain a closed immersion j_1 of P to $P_1 = \mathbb{P}(g^*(\mathscr{F}) \otimes \mathscr{K}^{\otimes(-m)})$ such that $\mathscr{O}_P(1)$ is isomorphic to $j_1^*(\mathscr{O}_{P_1}(1))$ (Proposition 5.3.30). On the other hand, there is an isomorphism from P_1 to $P_2 = \mathbb{P}(g^*(\mathscr{F}))$, identifying $\mathscr{O}_{P_1}(1)$ with $\mathscr{O}_{P_2}(1) \otimes_Y \mathscr{K}^{\otimes(-m)}$ (Proposition 5.4.1); we then have a closed immersion $j_2 : P \to P_2$ such that $\mathscr{O}_P(1)$ is isomorphic to $j_2^*(\mathscr{O}_{P_2}(1)) \otimes_Y \mathscr{K}^{\otimes(-m)}$. Finally, P_2 is identified with $P_3 \times_Z Y$ where $P_3 = \mathbb{P}(\mathscr{F})$, and $\mathscr{O}_{P_2}(1)$ is identified with $\mathscr{O}_{P_3}(1) \otimes_Z \mathscr{O}_Y$ (4.1.2). By definition, $\mathscr{O}_{P_3}(1)$ is very ample for Z, and so is $\mathscr{K}$, so it follows from (v) that the $\mathscr{O}_{P_2}$-module $\mathscr{O}_{P_3}(1)\otimes_Y\mathscr{K}$ is very ample for Z. In view of (ii), $\mathscr{M}=j_2^*(\mathscr{O}_{P_3}(1)\otimes_Y\mathscr{K})$ is then very ample for Z, and $\mathscr{O}_P(1)$ is isomorphic to $\mathscr{M}\otimes_Y\mathscr{K}^{\otimes(-m-1)}$, whence the demonstration. □

Proposition 5.4.22. *Let $f:X\to Y$, $f':X'\to Y$ be two morphisms, $X''=X\amalg X'$, and $f''=f\amalg f'$. Let $\mathscr{L}$ (resp. $\mathscr{L}'$) be an invertible $\mathscr{O}_X$-module (resp. $\mathscr{O}_{X'}$-module), and let $\mathscr{L}''$ be the invertible $\mathscr{O}_{X''}$-module which coincides with $\mathscr{L}$ over X and with $\mathscr{L}'$ over X'. For $\mathscr{L}''$ to be very ample relative to f'', it is necessary and sufficent that $\mathscr{L}$ is very ample relative to f and $\mathscr{L}'$ is very ample relative to f'.*

Proof. We can assume that Y is affine. If $\mathscr{L}''$ is very ample then so is $\mathscr{L}$ and $\mathscr{L}'$ in view of Proposition 5.4.21(ii). Conversely, if $\mathscr{L}$ and $\mathscr{L}'$ are very ample, it follows from Remak 5.4.3 that $\mathscr{L}''$ is very ample. □

5.4.5 Ample sheaves

Let X be a scheme, $\mathscr{L}$ be an invertible $\mathscr{O}_X$-module, and $\mathscr{F}$ be an $\mathscr{O}_X$-module. For any $n\in\mathbb{Z}$, we set $\mathscr{F}(n)=\mathscr{F}\otimes_{\mathscr{O}_X}\mathscr{L}^{\otimes n}$ (if there is no confusion), and $S=\bigoplus_{n\geq 0}\Gamma(X,\mathscr{L}^{\otimes n})$. If we consider X as a $\mathbb{Z}$-scheme and let $p:X\to Y=\operatorname{Spec}(\mathbb{Z})$ be the structural morphism, there are bijections

$$\operatorname{Hom}_{\mathsf{Qcoh}(X)}(p^*(\widetilde{S}),\boldsymbol{S}(\mathscr{L}))\xrightarrow{\sim}\operatorname{Hom}_{\mathsf{Qcoh}(Y)}(\widetilde{S},p_*(\boldsymbol{S}(\mathscr{L})))\xrightarrow{\sim}\operatorname{Hom}_{\mathbb{Z}}(S,\Gamma(X,\boldsymbol{S}(\mathscr{L})))$$

where we use Proposition 4.2.5. The homomorphism $\varepsilon:p^*(\widetilde{S})\to\boldsymbol{S}(\mathscr{L})$ corresponding to the canonical injection of S into $\Gamma(X,\boldsymbol{S}(\mathscr{L}))$ is called the **canonical homomorphism associated with** $\mathscr{L}$. It then corresponds to a canonical morphism

$$r_{\mathscr{L},\varepsilon}:G(\varepsilon)\to\operatorname{Proj}(S).\tag{4.5.1}$$

Theorem 5.4.23. *Let X be a quasi-compact and quasi-separated scheme, $\mathscr{L}$ be an invertible $\mathscr{O}_X$-module and $S=\bigoplus_{n\geq 0}\Gamma(X,\mathscr{L}^{\otimes n})$. The following conditions are equivalent:*

- *(i) The subsets X_f, as f runs through the set of homogeneous elements of S_+, form a basis for X.*
- *(i′) The subsets X_f which are affine, as f runs through the set of homogeneous elements of S_+, form a basis for X.*
- *(ii) The canonical morphism $r_{\mathscr{L},\varepsilon}:G(\varepsilon)\to\operatorname{Proj}(S)$ is everywhere defined and a dominant open immersin.*
- *(ii′) The canonical morphism $r_{\mathscr{L},\varepsilon}:G(\varepsilon)\to\operatorname{Proj}(S)$ is everywhere defined and a homeomorphism from X onto a subspace of* $\operatorname{Proj}(S)$.
- *(iii) For any quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, if $\mathscr{F}_n$ is the sub-$\mathscr{O}_X$-module of $\mathscr{F}(n)$ generated by the sections of $\mathscr{F}(n)$ over X, then $\mathscr{F}$ is the direct sum of the sub-$\mathscr{O}_X$-modules $\mathscr{F}_n(-n)$ for $n>0$.*
- *(iii′) Property (iii) holds for any quasi-coherent ideal of $\mathscr{O}_X$.*

Moreover, in this case, if (f_α) is a family of homogeneous elements of S_+ such that X_{f_α} is affine, then the restriction to $\bigcup_\alpha X_{f_\alpha}$ of the canonical morphism $r_{\mathscr{L},\varepsilon}:X\to\operatorname{Proj}(S)$ is an isomorphism from $\bigcup_\alpha X_{f_\alpha}$ to $\bigcup_\alpha(\operatorname{Proj}(S))_{f_\alpha}$.

Proof. It is clear that (ii) implies (ii'), and (ii') implies (i) in view of the formula (3.6.2). Condition (i) implies (i'), because any $x \in X$ admits an affine neighborhood U such that $\mathscr{L}|_U$ is isomorphic to $\mathscr{O}_X|_U$; if $f \in \Gamma(X, \mathscr{L}^{\otimes n})$ is such that $x \in X_f \subseteq U$, X_f is also the set of $x' \in U$ such that $(f|_U)(x') \neq 0$, and this is an affine open subset of U (hence of X). To prove that (i') implies (ii), it suffices to show the last assertion of the statement, hence to check that if $X = \bigcup_\alpha X_{f_\alpha}$, the condition (iv) of Proposition 5.3.39 is satisfied; this follows immediately from Theorem 4.6.14(a). To see that $r_{\mathscr{L},\varepsilon}$ is dominant, we note that for $f \in S_+$ homogeneous, X_f is the inverse image of $D_+(f)$ by $r_{\mathscr{L},\varepsilon}$ and by Corollary 4.6.15 we have $\Gamma(X_f, \mathscr{O}_X) = S_{(f)}$ is nonzero if f is not nilpotent, so X_f is nonempty if $D_+(f)$ is not empty.

To prove that (i') implies (iii), note that if X_f is affine (where $f \in S_d$), $\mathscr{F}|_{X_f}$ is generated by its sections over X_f (Proposition 4.1.20); on the other hand, by Theorem 4.6.14 such a section s is of the form $(t|_{X_f}) \otimes (f|_{X_f})^{-m}$ where $t \in \Gamma(X, \mathscr{F}(md))$. By definition, t is also a section of $\mathscr{F}_{md}$, so s is a section of $\mathscr{F}_{md}(-md)$ over X_f, which proves (iii). It is clear that (iii) implies (iii'), and it rests to show that (iii') implies (i). Now let U be an open neighborhood of $x \in X$, and let $\mathscr{I}$ be a quasi-coherent ideal of $\mathscr{O}_X$ defining a closed subscheme of X with underlying subspace $X - U$. The hypothesis of (iii') implies that there exists an integer $n > 0$ and a section $\mathscr{I}(n)$ over X such that $f(x) \neq 0$. Then we have evidently $f \in S_n$ and $x \in X_f \subseteq U$, this proves (i). □

If X is a quasi-compact and quasi-separated scheme, the equivalent conditions of Theorem 5.4.23 implies that X is separated, since it is isomorphic to a subscheme of $\mathrm{Proj}(S)$. We say an invertible $\mathscr{O}_X$-module $\mathscr{L}$ is **ample** if X is a qausi-compact and quasi-separated scheme and the equivalent conditions of Theorem 5.4.23 are satisfied. It follows from Theorem 5.4.23(i) that if $\mathscr{L}$ is an ample $\mathscr{O}_X$-module, then for any open subset U of X, $\mathscr{L}|_U$ is an ample $(\mathscr{O}_X|_U)$-module.

Corollary 5.4.24. *Let $\mathscr{L}$ be an ample $\mathscr{O}_X$-module. For any finite subspace Z of X and any open neighborhood U of Z, there exists an integer $n > 0$ and a section $f \in \Gamma(X, \mathscr{L}^{\otimes n})$ such that X_f is an affine neighborhood of Z contained in U.*

Proof. In view of Theorem 5.4.23(ii), we can view X as a subscheme of $\mathrm{Proj}(S)$ and we only need to prove that for any finite subset Z' of $\mathrm{Proj}(S)$ and any open neighborhood U of Z', there exists a homogeneous element $f \in S_+$ such that $Z \subseteq D_+(f) \subseteq U$. Now by definition the closed set Y, which is the complement of U in $\mathrm{Proj}(S)$, is of the form $V_+(\mathfrak{I})$ where $\mathfrak{I}$ is a graded ideal of S, not containing S_+; on the other hand, the points of Z' are by definition graded prime ideals $\mathfrak{p}_i$ of S_+ not containing $\mathfrak{I}$. There then exists an element $f \in \mathfrak{I}$ not contained in each $\mathfrak{p}_i$ (Proposition **??**), and as the $\mathfrak{p}_i$ are graded, we can assume that f is homogeneous. This element f then satisfies the required. □

Proposition 5.4.25. *Suppose that X is a quasi-compact and quasi-separated scheme. Then the conditions of Theorem 5.4.23 are equivalent to the following conditions:*

- *(iv) For any quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$ of finite type, there exists an integer $n_0 > 0$ such that, for any $n \geq n_0$, $\mathscr{F}(n)$ is generated by its sections over X.*
- *(iv') For any quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$ of finite type, there exists integers $n > 0, k > 0$ such that $\mathscr{F}$ is isomorphic to a quotient of the $\mathscr{O}_X$-module $\mathscr{L}^{\otimes(-n)} \otimes \mathscr{O}_X^k$.*

(iv″) Property (iv′) holds for any quasi-coherent ideal of $\mathscr{O}_X$ of finite type.

Proof. As X is quasi-compact, if a quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$ of finite type is such that $\mathscr{F}(n)$ is generated by global sections, $\mathscr{F}(n)$ is then generated by finitely many global sections (Proposition 1.4.11), so (iv) impliess (iv′) and it is clear that (iv′) implies (iv″). As any quasi-coherent $\mathscr{O}_X$-module $\mathscr{G}$ is the inductive limit of its sub-$\mathscr{O}_X$-modules of finte type (Proposition 4.6.62), to verify conditin (iii′) of Theorem 5.4.23, it suffices to verify that for a quasi-coherent ideal of $\mathscr{O}_X$ that is of finite type, and this is clear if condition (iv″) holds. It remains to prove that if $\mathscr{L}$ is an ample $\mathscr{O}_X$-module, then condition (iv) holds. Consider a finite affine open covering (X_{f_i}) of X with $f_i \in S_{n_i}$; by changing f_i by its power, we can assume that the integers n_i equal to the same integer m. The sheaf $\mathscr{F}|_{X_{f_i}}$, being of finite type by hypotheses, is generated by a finitely number of sections h_{ij} over X_{f_i} (Proposition 4.1.23). By Theorem 4.6.14, there then exists an integer k_0 such that the section $h_{ij} \otimes f_i^{k_0}$ extend to a section of $\mathscr{F}(k_0 m)$ over X for any couple (i,j). A fortiori the $h_{ij} \otimes f_i^k$ extend to sections of $\mathscr{F}(km)$ over X for $k \geq k_0$, and for such values of k, $\mathscr{F}(km)$ is then generated by global sections. For any integer p such that $0 < p < m$, $\mathscr{F}(p)$ is also of finite type, so there exist an integer k_p such that $\mathscr{F}(p)(km) = \mathscr{F}(p+km)$ is generated by global sections for $k \geq k_p$. Let n_0 be the largest of the $k_p m$ for $0 < p < m$; we then conclude that $\mathscr{F}(n)$ is generated by global sections for $n \geq n_0$. □

Proposition 5.4.26. *Let X be a quasi-compact and quasi-separated scheme and $\mathscr{L}$ be an invertible $\mathscr{O}_X$-module.*

(a) *Let $n > 0$ be an integer. For $\mathscr{L}$ to be ample, it is necessary and sufficent that $\mathscr{L}^{\otimes n}$ is ample.*

(b) *Let $\mathscr{L}'$ be an inveritble $\mathscr{O}_X$-module such that, for any $x \in X$, there exists a section s' of $\mathscr{L}'^{\otimes n}$ over X such that $s'(x) \neq 0$. Then, if $\mathscr{L}$ is ample, so is $\mathscr{L} \otimes \mathscr{L}'$.*

Proof. Property (a) is a concequence of (i) of Theorem 5.4.23 since $X_{f^{\otimes n}} = X_f$. On the other hand, if $\mathscr{L}$ is ample, for any $x \in X$ and any neighborhood U of x, there exists $m > 0$ and $f \in \Gamma(X, f^{\otimes m})$ such that $x \in X_f \subseteq U$; if $f' \in \Gamma(X, \mathscr{L}'^{\otimes n})$ is such that $f'(x) \neq 0$, then $s(x) \neq 0$ for $s = f^{\otimes n} \otimes f'^{\otimes m} \in \Gamma(X, (\mathscr{L} \otimes \mathscr{L}')^{\otimes mn})$, so $x \in X_s \subseteq X_f \subseteq U$, and therefore $\mathscr{L} \otimes \mathscr{L}'$ is ample. □

Corollary 5.4.27. *The tensor product of two ample $\mathscr{O}_X$-modules is ample.*

Proof. An ample $\mathscr{O}_X$-module satisfies the condition of Proposition 5.4.26(b). □

Corollary 5.4.28. *Let $\mathscr{L}$ be an ample $\mathscr{O}_X$-module and $\mathscr{L}'$ be an invertible $\mathscr{O}_X$-module. There then exists an integer $n_0 > 0$ such that $\mathscr{L}^{\otimes n} \otimes \mathscr{L}'$ is ample and generated by global sections for $n \geq n_0$.*

Proof. It follows from Proposition 5.4.25 that there exists an integer m_0 such that for $m \geq m_0$, $\mathscr{L}^{\otimes m} \otimes \mathscr{L}'$ is generated by global sections, and therefore satisfies the condition of Proposition 5.4.26(b); we can then choose $n_0 = m_0 + 1$. □

Remark 5.4.5. Let $P = \mathrm{Pic}(X) = H^1(X, \mathscr{O}_X^\times)$ be the picard group of $\mathscr{O}_X$-modules, and let P^+ be the subset of P formed by ample sheaves. Suppose that P^+ is nonempty. Then it follows from Corollary 5.4.27 and Corollary 5.4.28 that we have

$$P^+ + P^+ \subseteq P^+, \quad P^+ - P^+ = P.$$

which means $P^+ \cup \{0\}$ is the set of positive elements of P for an order structure over P compatible with the group structure, which is archimedean in view of Corollary 5.4.28.

Proposition 5.4.29. *Let $q : X \to Y$ be a quasi-compact and quasi-separated morphism where Y is affine, and $\mathscr{L}$ be an inveritble $\mathscr{O}_X$-module.*

(a) If $\mathscr{L}$ is very ample relative to q then $\mathscr{L}$ is ample.

(b) Suppose that q is of finite type. Then for $\mathscr{L}$ to be ample, it is necessary and sufficent that it satisfies the following equivalent conditions:

(v) There exists $n_0 > 0$ such that for any integer $n \geq n_0$, $\mathscr{L}^{\otimes n}$ is very ample relative to q.

(v′) There exists $n > 0$ such that $\mathscr{L}^{\otimes n}$ is very ample relative to q.

Proof. The first assertion follows from the definition of very ample: if A is the ring of Y, there exists an A-module E and a surjective homomorphism

$$\psi : q^*(\widetilde{\boldsymbol{S}(E)}) \to \boldsymbol{S}(\mathscr{L})$$

such that $i = r_{\mathscr{L},\psi}$ is an immersion from X to $P = \mathbb{P}(\widetilde{E})$ such that $\mathscr{L} \cong i^*(\mathscr{O}_P(1))$. As the $D_+(f)$ for $f \in \boldsymbol{S}(E)_+$ homogeneous form a basis for P and $i^{-1}(D_+(f)) = X_{\psi^\flat(f)}$, we see that condition (i) of Theorem 5.4.23 holds, so $\mathscr{L}$ is ample.

Now assume that q is of finite type and $\mathscr{L}$ is ample. It follows from Theorem 5.4.23(ii) and Proposition 5.4.11(a) that there exists an integer $k_0 > 0$ such that $\mathscr{L}^{\otimes k_0}$ is very ample relative to q. On the other hand, in view of Proposition 5.4.25, there exists an integer m_0 such that, for $m \geq m_0$, $\mathscr{L}^{\otimes m}$ is generated by global sections. Put $n_0 = k_0 + m_0$; if $n \geq n_0$, we can write $n = k_0 + m$ where $m \geq m_0$, whence $\mathscr{L}^{\otimes n} = \mathscr{L}^{\otimes k_0} \otimes \mathscr{L}^{\otimes m}$. As $\mathscr{L}^{\otimes m}$ is generated by global sections, it follow from Proposition 5.4.17 and Remark 5.3.3 that $\mathscr{L}^{\otimes n}$ is very ample relative to q. Finally, it is clear that (v) implies (v′), and (v′) implies that $\mathscr{L}$ is ample in view of (a) and Proposition 5.4.26. □

Corollary 5.4.30. *Let $q : X \to Y$ be a quasi-compact and quasi-separated morphism of finite type where Y is affine, $\mathscr{L}$ be an ample $\mathscr{O}_X$-module, and $\mathscr{L}'$ be an invertible $\mathscr{O}_X$-module. Then there exists an integer $n_0 > 0$ such that for $n \geq n_0$, $\mathscr{L}^{\otimes n} \otimes \mathscr{L}'$ is very ample relative to q.*

Proof. In fact, there exists integer m_0 such that for $m \geq m_0$, $\mathscr{L}^{\otimes m} \otimes \mathscr{L}'$ is generated by global section (Corollary 5.4.28); on the other hand, there exists k_0 such that $\mathscr{L}^{\otimes k}$ is very ample relative to q for $k \geq k_0$. Thus $\mathscr{L}^{\otimes (k+m_0)} \otimes \mathscr{L}'$ is very ample for $k \geq k_0$ (Proposition 5.4.16). □

Proposition 5.4.31. *Let X be a quasi-compact scheme, Z be a closed subscheme of X defined by a qausi-coherent nilpotent ideal $\mathscr{J}$ of $\mathscr{O}_X$, and $j : Z \to X$ be the canonical injectin. For an inveritble $\mathscr{O}_X$-module $\mathscr{L}$ to be ample, it is necessary and sufficient that $\mathscr{L}' = j^*(\mathscr{L})$ is an ample $\mathscr{O}_Z$-module.*

Proof. This conditions is necessary. In fact, for any section f of $\mathscr{L}^{\otimes n}$ over X, let f' be the image $f \otimes 1$, which is a section of $\mathscr{L}'^{\otimes n} = \mathscr{L}^{\otimes n} \otimes_{\mathscr{O}_X} (\mathscr{O}_X/\mathscr{J})$ over the subspace Z (identified with X); it is clear that $Z_{f'} = X_f$, hence condition (i) of Theorem 5.4.23 shows that $\mathscr{L}'$ is ample.

To prove the sufficiency, note first that we can reduce to the case $\mathscr{J}^2 = 0$ by considering the finite sequence of schemes $X_k = (X, \mathscr{O}_X/\mathscr{J}^{k+1})$, which is a closed subscheme of the previous one and is defined by a square zero ideal. Now X is quasi-separated if X_{red} is quasi-separated (Proposition 4.6.7(vi)). Criterion (i) of Theorem 5.4.23 shows that it will suffice to prove that, if g is a section of $\mathscr{L}'^{\otimes n}$ over Z such that Z_g is affine, then there exists $m > 0$ such that $g^{\otimes m}$ is the canonical image of a section f of $\mathscr{L}^{\otimes nm}$ over X. For this, we consider the exact sequence

$$0 \longrightarrow \mathscr{I}(n) \longrightarrow \mathscr{O}_X(n) = \mathscr{L}^{\otimes n} \longrightarrow \mathscr{O}_Z(n) = \mathscr{L}'^{\otimes n} \longrightarrow 0$$

since $\mathscr{F}(n)$ is an exact functor on $\mathscr{F}$; whence an exact sequence on cohomology:

$$0 \longrightarrow \Gamma(X, \mathscr{I}(n)) \longrightarrow \Gamma(X, \mathscr{L}^{\otimes n}) \longrightarrow \Gamma(X, \mathscr{L}'^{\otimes n}) \xrightarrow{\delta} H^1(X, \mathscr{I}(n))$$

which associates in particular g to an element $\delta g \in H^1(X, \mathscr{I}(n))$.

Note that since $\mathscr{I}^2 = 0$, $\mathscr{I}$ can be considered as a quasi-coherent $\mathscr{O}_Z$-module and we have, for any k, $\mathscr{L}'^{\otimes k} \otimes_{\mathscr{O}_Z} \mathscr{I}(n) = \mathscr{I}(n+k)$. For any section $s \in \Gamma(X, \mathscr{L}'^{\otimes k})$, tensoring by s is then a homomorphism $\mathscr{I}(n) \to \mathscr{I}(n+k)$ of $\mathscr{O}_Z$-modules, which gives a homomorphism $H^i(X, \mathscr{I}(n)) \to H^i(X, \mathscr{I}(n+k))$ of cohomology groups. We claim that

$$g^{\otimes m} \otimes \delta g = 0 \tag{4.5.2}$$

for $m > 0$ sufficiently large. In fact, Z_g is an affine open of Z and we have $H^1(Z_g, \mathscr{I}(n)) = 0$ where $\mathscr{I}(n)$ is considered as an $\mathscr{O}_Z$-module. In particular, if we put $g' = g|_{Z_g}$, and if we consider its image under $\delta : \Gamma(Z_g, \mathscr{L}'^{\otimes n}) \to H^1(Z_g, \mathscr{I}(n))$, we have $\delta g' = 0$. To explain this relation, observe that the first cohomology group of a sheaf coincides with the Cech cohomology; to form δg, it is necessary to consider an open covering (U_α) of X, which we can assume that is finite and formed by affine opens, and choose for each α a section $g_\alpha \in \Gamma(U_\alpha, \mathscr{L}^{\otimes n})$ whose image in $\Gamma(U_\alpha, \mathscr{L}'^{\otimes n})$ is $g|_{U_\alpha}$, and consider the class of cocycle $(g_{\alpha\beta} - g_{\beta\alpha})$, where $g_{\alpha\beta}$ is the restriction of g_α to $U_\alpha \cap U_\beta$ (a cocycle with values in $\mathscr{I}(n)$). We can moreover suppose that $\delta g'$ is in the same manner using the covering formed by $U_\alpha \cap Z_g$ and the restrictions $g_\alpha|_{U_\alpha \cap Z_g}$; the relation $\delta g' = 0$ signifies then that there exists for each α a section $h_\alpha \in \Gamma(U_\alpha \cap Z_g, \mathscr{I}(n))$ such that $(g_{\alpha\beta} - g_{\beta\alpha})|_{U_\cap \cap U_\beta \cap Z_g} = h_{\alpha\beta} - h_{\eta\alpha}$, where $h_{\alpha\beta}$ denotes the restriction of h_α to $U_\alpha \cap U_\beta \cap Z_g$. Now by Theorem 4.6.14 there exists an integer $m > 0$ such that $g^{\otimes m} \otimes h_\alpha$ is the restriction to $U_\alpha \cap Z_g$ of a section $t_\alpha \in \Gamma(U_\alpha, \mathscr{I}(n+nm))$ for each α; we then have $g^{\otimes m} \otimes (g_{\alpha\beta} - g_{\beta\alpha}) = t_{\alpha\beta} - t_{\beta\alpha}$ for any couple of indices, which proves $g^{\otimes m} \otimes \delta g = 0$.

We remark on the other hand that if $s \in \Gamma(X, \mathscr{O}_Z(p))$, $t \in \Gamma(X, \mathscr{O}_Z(q))$, we have, in the group $H^1(X, \mathscr{I}(p+q))$, that

$$\delta(s \otimes t) = (\delta s) \otimes t + s \otimes (\delta t). \tag{4.5.3}$$

For this, we can still consider an open cover (U_α) of X, and for each α a section $s_\alpha \in \Gamma(U_\alpha, \mathscr{O}_X(p))$ (resp. a section $t_\alpha \in \Gamma(U_\alpha, \mathscr{O}_X(q))$) whose canonical image in $\Gamma(U_\alpha, \mathscr{O}_Z(p))$ (resp. in $\Gamma(U_\alpha, \mathscr{O}_Z(q))$) is $s|_{U_\alpha}$; the relation (4.5.3) then follows from

$$(s_{\alpha\beta} \otimes t_{\alpha\beta}) - (s_{\beta\alpha} \otimes t_{\beta\alpha}) = (s_{\alpha\beta} - s_{\beta\alpha}) \otimes t_{\alpha\beta} + s_{\beta\alpha} \otimes (t_{\alpha\beta} - t_{\beta\alpha})$$

with the same notations before. By recurrence on k, we then have

$$\delta(g^{\otimes k}) = (kg^{\otimes(k-1)}) \otimes (\delta g) \tag{4.5.4}$$

and in view of (4.5.2) and (4.5.4), we have $\delta(g^{\otimes(m+1)}) = 0$, so $g^{\otimes(m+1)}$ is the canonical image of a section f of $\mathscr{L}^{\otimes n(m+1)}$ over X, whence our demonstration. □

Corollary 5.4.32. *Let X be a Noetherian scheme and $j : X_{\mathrm{red}} \to X$ be the canonical injection. For an inveritble $\mathscr{O}_X$-module $\mathscr{L}$ to be ample, it is necessary and sufficient that $j^*(\mathscr{L})$ is an ample $\mathscr{O}_{X_{\mathrm{red}}}$-module.*

Proof. The nilradical $\mathscr{N}_X$ is nilpotent and we can apply Proposition 5.4.31. □

5.4.6 Relatively ample sheaves

Let $f : X \to Y$ be a quasi-compact morphism and $\mathscr{L}$ be an invertible $\mathscr{O}_X$-module. We say that $\mathscr{L}$ is **ample relative to f**, or relative to Y, or **f-ample**, or **Y-ample**, if there exists an affine open covering (U_α) of Y such that if $X_\alpha = f^{-1}(U_\alpha)$, $\mathscr{L}|_{X_\alpha}$ is an ample $\mathscr{O}_{X_\alpha}$-module for each α. Again, the existence of an f-ample $\mathscr{O}_X$-module implies that X is separated, so f is necessarily separated by Proposition 4.5.25.

Proposition 5.4.33. *Let $f : X \to Y$ be a quasi-compact morphism and $\mathscr{L}$ be an invertible $\mathscr{O}_X$-module. If $\mathscr{L}$ is very ample relative to f, then it is ample relative to f.*

Proof. If $\mathscr{L}$ is very ample relative to f then the morphism f is separated, so by Proposition 5.4.29(a) the restriction $\mathscr{L}|_{f^{-1}(U)}$ for any affine open U of Y is very ample, hence ample. □

Let $f : X \to Y$ be a quasi-compact morphism and $\mathscr{L}$ be an invertible $\mathscr{O}_X$-module. We consider the graded $\mathscr{O}_Y$-algebra $\mathscr{S} = \bigoplus_{n\geq 0} f_*(\mathscr{L}^{\otimes n})$. Then the canonical homomorphisms $\sigma_n : f^*(f_*(\mathscr{L}^{\otimes n})) \to \mathscr{L}^{\otimes n}$ induce a canonical homomorphism

$$\sigma : f^*(\mathscr{S}) \to \bigoplus_{n\geq 0} \mathscr{L}^{\otimes n} = \boldsymbol{S}(\mathscr{L}).$$

On the other hand, it is easy to see that $\sigma^\flat$ is the canonical injection from $\mathscr{S}$ into $f_*(\boldsymbol{S}(\mathscr{L}))$. The homomorphism σ gives an Y-morphism

$$r_{\mathscr{L},\sigma} : G(\sigma) \to \mathrm{Proj}(\mathscr{S}) = P.$$

Proposition 5.4.34. *Let $f : X \to Y$ be a quasi-compact morphism and $\mathscr{L}$ be an invertible $\mathscr{O}_X$-module. Then the following conditions are equivalent:*

- *(i) $\mathscr{L}$ is f-ample.*
- *(ii) $\mathscr{S}$ is quasi-coherent and the Y-morphism $r_{\mathscr{L},\sigma}$ is everywhere defined and a dominant open immersion.*
- *(ii') The morphism f is separated, the Y-morphism $r_{\mathscr{L},\sigma}$ is everywhere defined and is a homeomorphism from X onto a subspace of* Proj($\mathscr{S}$).

Moreover, if these are satisfied, for any $n \in \mathbb{Z}$ the canonical homomorphism of (3.6.3)

$$\nu : r^*_{\mathscr{L},\sigma}(\mathscr{O}_P(n)) \to \mathscr{L}^{\otimes n} \tag{4.6.1}$$

is an isomorphism. Finally, for any quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, if we put $\mathscr{M} = \bigoplus_{n\geq 0} f_*(\mathscr{F}\otimes\mathscr{L}^{\otimes n})$, the canonical homomorphism of (3.6.4)

$$\xi : r^*_{\mathscr{L},\sigma}(\widetilde{\mathscr{M}}) \to \mathscr{F} \tag{4.6.2}$$

is an isomorphism.

Proof. We have remarked that (i) implies that f is separated, so $\mathscr{S}$ is quasi-coherent by Proposition 4.6.54. As the fact that $r_{\mathscr{L},\sigma}$ is an open immersion everywhere defined is local over Y, to shows that (i) implies (ii), we can assume that Y is affine and $\mathscr{L}$ is ample; the assertion then follows from Proposition 5.4.23. It is clear that (ii) implies (ii'); finally, to show that (ii') implies (i), it suffices to consider an affine open cover (U_α) of Y and use Theorem 5.4.23(ii') to $\mathscr{L}|_{X_\alpha}$.

To prove the last two assertions, we use the fact that $\sigma^\flat$ is the canonical injection of $\mathscr{S}$ to $f_*(\boldsymbol{S}(\mathscr{L}))$ and the expression of the morphisms ν and ξ in Remark 5.3.5 and Remark 5.3.6. It then follows that ν and ξ are injective. As for the surjectivity, we can assume that Y is affine, so $\mathscr{L}$ is ample; the criterion of Theorem 5.4.23(iii) then shows that ν and ξ are surjective, whence the assertion. □

Remark 5.4.6. From Proposition 5.4.34 and its proof, we conclude that if $\mathscr{L}$ is f-ample then $f_*(\boldsymbol{S}(\mathscr{L}))$ is equal to $\mathscr{S}$, so the homomorphism $\sigma^\flat$ is the identity on $\mathscr{S}$. This can also be seen from Proposition 4.6.55 since in this case f is separated.

Corollary 5.4.35. *Let (U_α) be an open covering of Y. For $\mathscr{L}$ to be Y-ample, it is necessary and sufficient that $\mathscr{L}|_{f^{-1}(U_\alpha)}$ is U_α-ample for each α.*

Proof. This is ture since the condition (ii) of Proposition 5.4.34 is local over Y. □

Corollary 5.4.36. *Let $\mathscr{K}$ be an invertible $\mathscr{O}_Y$-module. For $\mathscr{L}$ to be Y-ample, it is necessary and sufficient that $\mathscr{L}\otimes f^*(\mathscr{K})$ is Y-ample.*

Proof. This is a concequence of Corollary 5.4.35 by taking U_α to be such that $\mathscr{K}|_{U_\alpha}$ is isomorphic to $\mathscr{O}_Y|_{U_\alpha}$ for each α. □

Corollary 5.4.37. *Suppose that Y is affine. For $\mathscr{L}$ to be Y-ample, it is necessary and sufficient that $\mathscr{L}$ is ample.*

Proof. This is immediate from the definition of Y-ample, and Proposition 5.4.34(ii) and Proposition 5.4.23(ii), since

$$\mathrm{Proj}(\mathscr{S}) = \mathrm{Proj}(\Gamma(Y,\mathscr{S})) = \mathrm{Proj}(\bigoplus_{n\geq 0}\Gamma(Y, f_*(\mathscr{L}^{\otimes n}))) = \mathrm{Proj}(S)$$

in this case (note that Y is quasi-compact). □

Corollary 5.4.38. *Let $f : X \to Y$ be a quasi-compact morphism. Suppose that there exists a quasi-coherent $\mathscr{O}_Y$-module $\mathscr{E}$ and a Y-morphism $i : X \to P = \mathbb{P}(\mathscr{E})$ which is a homeomorphism from X onto a subspace of P. Then $\mathscr{L} = i^*(\mathscr{O}_P(1))$ is Y-ample.*

Proof. We can assume that Y is affine, and the corollary then follows from the criterion (i) of Proposition 5.4.23 and the formula (3.6.2). □

Proposition 5.4.39. *Let X be a quasi-compact and quasi-separated scheme, $f : X \to Y$ be a quasi-compact and quasi-separated morphism. For an invertible $\mathscr{O}_X$-module $\mathscr{L}$ to be f-ample, it is necessary and sufficient that following equivalent conditions are satisfied:*

(iii) For any quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$ of finite type, there exists an integer $n_0 > 0$ such that, for any $n \geq n_0$, the canonical homomorphism $\sigma : f^*(f_*(\mathscr{F} \otimes \mathscr{L}^{\otimes n})) \to \mathscr{F} \otimes \mathscr{L}^{\otimes n}$ is surjective.

(iii′) For any ideal $\mathscr{I}$ of $\mathscr{O}_X$ of finite type, there exist an integer $n > 0$ such that the canonical homomorphism $\sigma : f^*(f_*(\mathscr{I} \otimes \mathscr{L}^{\otimes n})) \to \mathscr{I} \otimes \mathscr{L}^{\otimes n}$ is surjective.

Proof. □

Proposition 5.4.40. *Let $f : X \to Y$ be a quasi-compact morphism and $\mathscr{L}$ be an invertible $\mathscr{O}_X$-module.*

(a) Let $n > 0$ be an integer. For $\mathscr{L}$ to be f-ample, it is necessary and necessary that $\mathscr{L}^{\otimes n}$ is f-ample.

(b) Let $\mathscr{L}'$ be an invertible $\mathscr{O}_X$-module, and suppose that there exists an integer $n > 0$ such that the canonical homomorphism $\sigma : f^*(f_*(\mathscr{L}'^{\otimes n})) \to \mathscr{L}'^{\otimes n}$ is surjective. Then, if $\mathscr{L}$ is f-ample, so is $\mathscr{L} \otimes \mathscr{L}'$.

Corollary 5.4.41. *The tensor product of two f-ample $\mathscr{O}_X$-modules is f-ample.*

Proposition 5.4.42. *Let $f : X \to Y$ be a morphism of finite type where Y is quasi-compact, and $\mathscr{L}$ be an invertible $\mathscr{O}_X$-module. For $\mathscr{L}$ to be f-ample, it is necessary and sufficient that the following equivalent conditions are satisfied:*

(iv) There exists $n_0 > 0$ such that, for any integer $n \geq n_0$, $\mathscr{L}^{\otimes n}$ is very ample relative to f.

(iv′) There exist $n > 0$ such that $\mathscr{L}^{\otimes n}$ is very ample relative to f.

Lemma 5.4.43. *Let $u : Z \to S$ be a morphism, $\mathscr{L}$ be an invertible $\mathscr{O}_S$-module, s a section of $\mathscr{L}$ over S, and t be the inverse image of s under u. Then $Z_t = u^{-1}(S_s)$.*

Proof. □

Lemma 5.4.44. *Let Z, Z' be two S-schemes, p, p' be the projections of $T = Z \times_S Z'$, $\mathscr{L}$ (resp. $\mathscr{L}'$) be an invertible $\mathscr{O}_Z$-module (resp. $\mathscr{O}_{Z'}$-module), t (resp. t') be a section of $\mathscr{L}$ (resp. $\mathscr{L}'$) over Z (resp. Z'), s (resp. s') be the inverse image of t (resp. t') under p (resp. p'). Then we have $T_{s\otimes s'} = Z_t \times_S Z'_{t'}$.*

Proof. □

Proposition 5.4.45 (Properties of Relative Ample Sheaves).

(i) For any scheme Y, any invertible $\mathscr{O}_Y$-module $\mathscr{L}$ is relative ample relative to the identify morphism 1_Y.

(ii) *Let $f : X \to Y$ be a quasi-compact morphism and $j : X' \to X$ be a quasi-compact morphism that is a homeomorphism from X' onto a subspace of X. If $\mathscr{L}$ is an $\mathscr{O}_X$-module relative to f, then $j^*(\mathscr{L})$ is relative relative to $f \circ j$.*

(iii) *Let $f : X \to Y$ and $g : Y \to Z$ be quasi-compact morphisms where Z is quasi-compact. Let $\mathscr{L}$ an ample $\mathscr{O}_X$-module relative to f and $\mathscr{K}$ be an ample $\mathscr{O}_Y$-module relative to g. Then there exists an integer $n_0 > 0$ such that for any $n \geq n_0$, $\mathscr{L} \otimes f^*(\mathscr{K}^{\otimes n})$ is ample relative to $g \circ f$.*

(iv) *Let $f : X \to Y$ be a quasi-compact morphism, $g : Y' \to Y$ be a morphism, and put $X' = X_{(Y')}$. If $\mathscr{L}$ is an ample $\mathscr{O}_X$-module relative to f, $\mathscr{L}' = \mathscr{L} \otimes_Y \mathscr{O}_{Y'}$ is ample relative to $f_{(Y')}$.*

(v) *Let $f_i : X_i \to Y_i$ $(i = 1, 2)$ be two quasi-compact S-morphisms. If $\mathscr{L}_i$ is an ample $\mathscr{O}_{X_i}$-module relative to f_i, then $\mathscr{L}_1 \otimes_S \mathscr{L}_2$ is ample relative to $f_1 \times_S f_2$.*

(vi) *Let $f : X \to Y$, $g : Y \to Z$ be two morphisms such that $g \circ f$ is quasi-compact. If an $\mathscr{O}_X$-module $\mathscr{L}$ is ample relative to $g \circ f$ and if g is separated or X is locally Noetherian, then $\mathscr{L}$ is ample relative to f.*

(vii) *Let $f : X \to Y$ be a quasi-compact morphism and $j : X_{\mathrm{red}} \to X$ be the canonical injection. If $\mathscr{L}$ is an ample $\mathscr{O}_X$-module relative to f, then $j^*(\mathscr{L})$ is ample relative to f_{red}.*

Proof. □

Proposition 5.4.46. *Let $f : X \to Y$ be a quasi-compact morphism, $\mathscr{I}$ be a locally nilpotent ideal of $\mathscr{O}_X$, Z the closed subscheme of X defined by $\mathscr{I}$, and $j : Z \to X$ be the cannical injection. For an invertible $\mathscr{O}_X$-module to be ample relative to f, it is necessary and sufficient that $j^*(\mathscr{L})$ is ample relative to $f \circ j$.*

Corollary 5.4.47. *Let X be a locally Noetherian scheme and $f : X \to Y$ be a quasi-compact morphism. For an invertible $\mathscr{O}_X$-module $\mathscr{L}$ to be ample relative to f, it is necessary and sufficient that its inverse image $\mathscr{L}'$ under the canonical injection $X_{\mathrm{red}} \to X$ is ample relative to f_{red}.*

Proposition 5.4.48. *Let $f : X \to Y$, $f' : X' \to Y$ be two morphisms, $X'' = X \amalg X'$, and $f'' = f \amalg f'$. Let $\mathscr{L}$ (resp. $\mathscr{L}'$) be an invertible $\mathscr{O}_X$-module (resp. $\mathscr{O}_{X'}$-module), and let $\mathscr{L}''$ be the invertible $\mathscr{O}_{X''}$-module which coincides with $\mathscr{L}$ over X and with $\mathscr{L}'$ over X'. For $\mathscr{L}''$ to be ample relative to f'', it is necessary and sufficent that $\mathscr{L}$ is ample relative to f and $\mathscr{L}'$ is ample relative to f'.*

Proposition 5.4.49. *Let Y be a quasi-compact scheme, $\mathscr{S}$ be a quasi-coherent graded $\mathscr{O}_Y$-algebra of finite type, $X = \mathrm{Proj}(\mathscr{S})$, and $f : X \to Y$ be the structural morphism. Then f is of finite type and there exists an integer $n > 0$ such that $\mathscr{O}_X(d)$ is invertible and f-ample.*

5.5 Quasi-affine, proper, and projective morphisms

5.5.1 Quasi-affine morphisms

We say a scheme is **quasi-affine** if it is isomorphic to the subscheme induced over a quasi-compact open subset of an affine scheme. We say a morphism $f : X \to Y$ is quasi-affine, or that X is a quasi-affine Y-scheme, if there exists an affine open cover (U_α) of Y such that $f^{-1}(U_\alpha)$ is a quasi-affine scheme. Since any quasi-compact open subscheme of an affine scheme is separated, it is clear that quasi-affine morphisms are separated and quasi-compact, and any affine morphism is quasi-affine.

Recall that for any scheme X, if $A = \Gamma(X, \mathscr{O}_X)$, the identify homomorphism $A \to A$ induces a canonical morphism $q : X \to \operatorname{Spec}(A)$ by Proposition 4.2.4. This is also the morphism $r_{\mathscr{L},\varepsilon} : X \to \operatorname{Proj}(S)$ in (4.5.1) defined for $\mathscr{L} = \mathscr{O}_X$, since $\Gamma(X, -)$ commutes with taking tensor product with $\mathscr{O}_X$ (cf. Example **??**) and we have $S = \bigoplus_{n\geq 0} \Gamma(X, \mathscr{O}_X)^{\otimes n} = A[T]$, and $\operatorname{Proj}(A[T])$ is canonically identified with $\operatorname{Spec}(A)$.

Proposition 5.5.1. *Let X be a quasi-compact scheme and $A = \Gamma(X, \mathscr{O}_X)$. The following conditions are equivalent:*

(i) X is a quasi-affine scheme.

(ii) The canonical morphism $q : X \to \operatorname{Spec}(A)$ is an open immersion.

(ii′) The canonical morphism $q : X \to \operatorname{Spec}(A)$ is a homeomorphism from X onto a subspace of $\operatorname{Spec}(A)$.

(iii) The $\mathscr{O}_X$-module $\mathscr{O}_X$ is very ample relative to q.

(iii′) The $\mathscr{O}_X$-module $\mathscr{O}_X$ is ample.

(iv) The subsets X_f, as f runs through A, form a basis for X.

(iv′) The subsets X_f which are affine, as f runs through A, form a basis for X.

(v) Any quasi-coherent $\mathscr{O}_X$-module is generated by its global sections.

(v′) Any quasi-coherent ideal of finite type of $\mathscr{O}_X$ is generated by its global sections.

Proof. It is clear that (ii)⇒(i) by definition, and (iii)⇒(iii′) by Proposition 5.4.29. Since the canonical morphisms $q : X \to \operatorname{Spec}(A)$ and $r_{\mathscr{O}_X,\varepsilon} : X \to \operatorname{Proj}(S)$ are identified, we see that (iii′)⇔(ii)⇔(ii′)⇔(iv)⇔(iv′) by Theorem 5.4.23. Also, (iii′)⇔(v)⇔(v′) in view of Proposition 5.4.25.

We also note that if X is quasi-affine, then it can be identified as an open subscheme of an affine scheme $Y = \operatorname{Spec}(B)$. Let $\varphi : B \to A$ be the correspond homomorphism (Proposition 4.2.4). Since the affine opens $D(g)$, with $g \in B$, form a basis of Y, and we have $X_f = D(g) \cap X$ where $f = \varphi(g)$, it follows that the subsets X_f which are affine, with $f \in A$, form a basis for X, which proves (i)⇒(iv′).

Finally, it remains to show that (i)⇒(iii). For this we first note that if X is quasi-affine then it is quasi-compact and separated, so by Corollary 4.6.15, for $f \in A$ we have $\Gamma(X_f, \mathscr{O}_X) = A_f$. Since we have $q^{-1}(D_+(f)) = X_f$, we conclude that the canonical morphism $q : X \to \operatorname{Spec}(A)$ is of finite type, and by Proposition 5.4.29, since $\mathscr{O}_X^{\otimes n}$ is isomorphic to $\mathscr{O}_X$ for any integer $n > 0$, $\mathscr{O}_X$ is very ample relative to q. This completes the proof. ∎

Remark 5.5.1. Let X be a quasi-affine scheme and $A = \Gamma(X, \mathscr{O}_X)$. By Theorem 5.5.1 we know that $\mathscr{O}_X$ is very ample relative to $q : X \to \operatorname{Spec}(A)$. Since X is separated and quasi-compact, by Proposition 4.6.54 we know that $q_*(\mathscr{O}_X)$ is quasi-coherent, and from $\Gamma(\operatorname{Spec}(A), q_*(\mathscr{O}_X)) = A$ we conclude that $q_*(\mathscr{O}_X) = \widetilde{A}$, so $q^*(q_*(\mathscr{O}_X)) = \mathscr{O}_X$ and the canonical homomorphism σ :

$q^*(q_*(\mathscr{O}_X)) \to \mathscr{O}_X$ is identified with the identity on $\mathscr{O}_X$. This being so, the canonical morphism $r_{\mathscr{O}_X,\sigma} : X \to \mathbb{P}(q_*(\mathscr{O}_X))$ is then identified with $q : X \to \operatorname{Spec}(A)$, because we have

$$\mathbb{P}(q_*(\mathscr{O}_X)) = \operatorname{Proj}(\boldsymbol{S}(q_*(\mathscr{O}_X))) = \operatorname{Proj}(\boldsymbol{S}(A)) = \operatorname{Proj}(A[T]) = \operatorname{Spec}(A);$$

and we conclude from Theorem 5.5.1 that this is an open immersion, which justifies Proposition 5.4.13.

Corollary 5.5.2. *Let X be a quasi-compact scheme. If there exists a morphism $r : X \to Y$ from X into an affine scheme Y which is a homeomorphism onto an open subspace of Y, then X is quasi-affine.*

Proof. In fact, there then exists a family (g_α) of sections of $\mathscr{O}_Y$ over Y such that the $D(g_\alpha)$ form a basis for the topology of $r(X)$. If we put $f_\alpha = \theta(g_\alpha)$ where $\theta : \Gamma(Y, \mathscr{O}_Y) \to \Gamma(X, \mathscr{O}_X)$ is the corresponding ring homomorphism, then we have $X_{f_\alpha} = r^{-1}(D(g_\alpha))$, so the X_{f_α} form a basis for X, and by Proposition 5.5.1 X is then quasi-affine. □

Corollary 5.5.3. *If X is a quasi-affine scheme, any invertible $\mathscr{O}_X$-module is very ample (relative to the canonical morphism $q : X \to \operatorname{Spec}(A)$) and a fortiori ample.*

Proof. In fact any such module $\mathscr{L}$ is generated by its global sections (Proposition 5.5.1(v)), so $\mathscr{L} \otimes \mathscr{O}_X$ is very ample by Proposition 5.4.17. We also note that the morphism q is of finite type. □

Corollary 5.5.4. *Let X be a quasi-compact scheme. If there exists an invertible $\mathscr{O}_X$-module $\mathscr{L}$ such that $\mathscr{L}$ and $\mathscr{L}^{-1}$ are ample, then X is quasi-affine.*

Proof. In fact, $\mathscr{O}_X = \mathscr{L} \otimes \mathscr{L}^{-1}$ is then ample by Proposition 5.4.27. □

Proposition 5.5.5. *Let $f : x \to Y$ be a quasi-compact morphism. The following conditions are equivalent:*

- *(i) f is quasi-affine.*
- *(ii) The $\mathscr{O}_Y$-algebra $f_*(\mathscr{O}_X) = \mathscr{A}(X)$ is quasi-coherent and the canonical morphism $X \to \operatorname{Spec}(\mathscr{A}(X))$ corresponding to the identity homomorphism of $\mathscr{A}(X)$ is an open immersion.*
- *(ii') The $\mathscr{O}_Y$-algebra $\mathscr{A}(X)$ is quasi-coherent and the canonical morphism $X \to \operatorname{Spec}(\mathscr{A}(X))$ is a homeomorphism from X onto a subspace of $\operatorname{Spec}(\mathscr{A}(X))$.*
- *(iii) The $\mathscr{O}_X$-module $\mathscr{O}_X$ is very ample relative to f.*
- *(iii') The $\mathscr{O}_X$-module $\mathscr{O}_X$ is ample relative to f.*
- *(iv) The morphism f is separated and for any quasi-coherent $\mathscr{O}_X$-modle $\mathscr{F}$, the canonical homomorphism $\sigma : f^*(f_*(\mathscr{F})) \to \mathscr{F}$ is surjective.*

Morever, if f is quasi-affine, any invertible $\mathscr{O}_X$-module $\mathscr{L}$ is very ample relative to f.

Proof. The equivalence of these properties follows from the fact that they are all local over Y and the criteria of Proposition 5.5.1. Also, we note that $f_*(\mathscr{F})$ is quasi-coherent if f is separated (Proposition 4.6.54). The last assertion follows from Corollary 5.5.3. □

Corollary 5.5.6. *Let Y be an affine scheme and $f : X \to Y$ be a quasi-compact morphism. For f to be quasi-affine, it is necessary and sufficient that X is quasi-affine scheme.*

Proof. This is an immediate concequence of Proposition 5.5.5 and Corollary 5.4.37. □

Corollary 5.5.7. *Let Y be a quasi-compact and quasi-separated scheme, $f : X \to Y$ be a morphism of finite type. If f is quasi-affine, there exists a quasi-coherent sub-$\mathcal{O}_Y$-algebra $\mathscr{B}$ of $\mathscr{A}(X)$ of finite type such that the morphism $X \to \mathrm{Spec}(\mathscr{B})$ corresponding to the canonical injection $\mathscr{B} \to \mathscr{A}(X)$ is an immersion. Moreover, any quasi-coherent sub-$\mathcal{O}_Y$-algebra of finite type $\mathscr{B}'$ of $\mathscr{A}(X)$, containing $\mathscr{B}$, has the same property.*

Proof. In fact, $\mathscr{A}(X)$ is the inductive limit of its quasi-coherent sub-$\mathcal{O}_Y$-algebras of finite type (Corollary 4.6.62); the assertion is then a particular case of Proposition 5.3.41, in view of the identification of $\mathrm{Spec}(\mathscr{A}(X))$ and $\mathrm{Proj}(\mathscr{A}(X)[T])$ (Corollary 5.3.5) and the canonical morphisms from X into them (cf. Remark 5.5.1). □

Proposition 5.5.8 (Properties of Quasi-affine Morphisms).

(i) A quasi-compact morphism $f : X \to Y$ that is a homeomorphism from X onto a subspace of Y (and in particular a quasi-compact immersion) is quasi-affine.

(ii) The composition of two quasi-affine morphisms is quasi-affine.

(iii) If $f : x \to Y$ is a quasi-affine S-morphism, $f_{(S')} : X_{(S')} \to Y_{(S')}$ is a quasi-affine morphism for any base change $S' \to S$.

(iv) If $f : X \to Y$ and $g : X' \to Y'$ are two quasi-affine S-morphisms, so is $f \times_S g$.

(v) If $f : X \to Y$, $g : Y \to Z$ are two morphisms such that $g \circ f$ is quasi-affine and if g is separated or X is locally Noetherian, then f is quasi-affine.

(vi) Let $f : X \to Y$ be a quasi-compact morphism and $j : X_{\mathrm{red}} \to X$ the canonical injection. If an $\mathcal{O}_X$-module $\mathscr{L}$ is ample relative to f, then $j^*(\mathscr{L})$ is ample relative to f_{red}.

Proof. In view of the criterion (iii') of Proposition 5.5.5, (i), (iii), (iv), (v) and (vi) are concequences of Proposition 5.4.45. To prove (ii), we can assume that Z is affine, and the assertion then follows from Proposition 5.4.45(iii), applied to $\mathscr{L} = \mathcal{O}_X$ and $\mathscr{K} = \mathcal{O}_Y$. □

Remark 5.5.2. Let $f : X \to Y$, $g : Y \to Z$ be two morphisms such that $X \times_Z Y$ is locally Noetherian. Then the graph morphism $\Gamma_f : X \to X \times_Z Y$ is a quasi-compact immersion, hence quasi-affine, and the reasoning of Proposition 4.5.14 shows that the conclusion of (v) remains valid if we remove the hypothesis that g is separated.

Proposition 5.5.9. *Let $f : X \to Y$ be a quasi-compact morphism and $g : X' \to X$ be a quasi-affine morphism. If $\mathscr{L}$ is an f-ample $\mathcal{O}_X$-module, then $g^*(\mathscr{L})$ is ample relative to $f \circ g$.*

Proof. By hypothesis $\mathcal{O}_{X'}$ is very ample relative to f, and since the question is local over Y, it follows from Proposition 5.4.45(iii) that there exists (for Y affine) an integer n such that $g^*(\mathscr{L}^{\otimes n}) = (g^*(\mathscr{L}))^{\otimes n}$ is ample relative to $f \circ g$, whence $g^*(\mathscr{L})$ is ample relative to $f \circ g$. □

5.5.2 Serre's criterion on affineness

Theorem 5.5.10 (**Serre's criterion**). *For a quasi-compact and quasi-separated scheme X, then the following conditions are equivalent:*

(i) *X is an affine scheme.*

(ii) *There exists a family (f_α) of elements of $A = \Gamma(X, \mathcal{O}_X)$ such that X_{f_α} are affine and the ideal generated by the f_α equals to A.*

(iii) *The functor $\Gamma(X, -)$ is exact on the category of quasi-coherent $\mathcal{O}_X$-modules.*

(iii') *For any exact sequence of quasi-coherent $\mathcal{O}_X$-modules*

$$0 \longrightarrow \mathscr{F}' \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}'' \longrightarrow 0$$

where $\mathscr{F}$ is isomorphic to a sub-$\mathcal{O}_X$-module of a finite product $\mathcal{O}_X^n$, the induced sequence

$$0 \longrightarrow \Gamma(X, \mathscr{F}') \longrightarrow \Gamma(X, \mathscr{F}) \longrightarrow \Gamma(X, \mathscr{F}'') \longrightarrow 0$$

is exact.

(iv) *$H^1(X, \mathscr{F}) = 0$ for any quasi-coherent $\mathcal{O}_X$-module $\mathscr{F}$.*

(iv') *$H^1(X, \mathscr{I}) = 0$ for any quasi-coherent ideal $\mathscr{I}$ of $\mathcal{O}_X$.*

Proof. It is clear that (i) implies (ii); (ii) implies on the oter hand that the X_{f_α} cover X, since by hypothesis the unit section 1 is a linear combination of f_α, and that the $D(f_\alpha)$ cover $\mathrm{Spec}(A)$. The last assertion of Proposition 5.4.23 then implies that $X \to \mathrm{Spec}(A)$ is an isomorphism.

We have seen that (i) implies (iii), and it is trivial that (iii) implies (iii'). On the other hand, (iii') implies that, for any closed point $x \in X$ and any open neighborhood U of x, there exists $f \in A$ such that $x \in X_f \subseteq X - U$. To see this, let $\mathscr{I}$ (resp. $\mathscr{I}'$) be the quasi-coherent ideal of $\mathcal{O}_X$ defining the reduced closed subscheme of X with underlying space $X - U$ (resp. $(X-U) \cup \{x\}$). It is clear that $\mathscr{I}' \subseteq \mathscr{I}$, and the quotient $\mathscr{I}'' = \mathscr{I}/\mathscr{I}'$ is a quasi-coherent $\mathcal{O}_X$-module. By hypothesis, the stalk of $\mathcal{O}_X/\mathscr{I}$ and $\mathcal{O}_X/\mathscr{I}'$ are zero at any point $x \in U - \{x\}$. Moreover, since $\{x\}$ is closed in X, the subscheme $X - U$ is *open and closed* in $(X - U) \cup \{x\}$, so we conclude that $(\mathcal{O}_X/\mathscr{I})_z = (\mathcal{O}_X/\mathscr{I}')_z$ for $z \in X - U$, and therefore $\mathscr{I}''_z = 0$. At the point x, we have $\mathscr{I}_x = \mathcal{O}_X$, while $\mathscr{I}'_x = \mathfrak{m}_x$ (cf. Example 4.4.42), so $\mathscr{I}''$ is supported at $\{x\}$ and $\mathscr{I}''_x = \kappa(x)$. The hypothesis of (iii') applied to the exact sequence $0 \to \mathscr{I}' \to \mathscr{I} \to \mathscr{I}'' \to 0$ shows that $\Gamma(X, \mathscr{I}) \to \Gamma(X, \mathscr{I}'')$ is surjective, so the section of $\mathscr{I}''$ whose germ at x equals to 1_x is the image of a section $f \in \Gamma(X, \mathscr{I}) \subseteq \Gamma(X, \mathcal{O}_X)$, and we have by definition $f(x) = 1_x$ and $f(y) = 0$ over $X - U$, which proves the assertion. Moreover, if U is affine, so is X_f, and the union X' of these affine opens X_f (with $f \in A$) is then an open subset of X containing any closed point of X. As X is a quasi-compact Kolmogoroff space, we then have $X' = X$ (Proposition **??**). Since X is quasi-compact, there are finitely many elements $f_i \in A$ $(1 \le i \le n)$ such that (X_{f_i}) is an affine open cover of X. Consider the homomorphism $\mathcal{O}_X^n \to \mathcal{O}_X$ defined by the sections f_i; since for any $x \in X$ at least one of the $(f_i)_x$ is invertible, this homomorphism is surjective, and we then get an exact sequence

$$0 \longrightarrow \mathscr{R} \longrightarrow \mathscr{O}_X^n \longrightarrow \mathscr{O}_X \longrightarrow 0$$

where $\mathscr{R}$ is a quasi-coherent sub-$\mathscr{O}_X$-module of $\mathscr{O}_X$. It then follows from (iii′) that the corresponding homomorphism $\Gamma(X, \mathscr{O}_X^n) \to \Gamma(X, \mathscr{O}_X)$ is surjective, which proves (ii).

Finally, (i) implies (iv) and (iv) implies (iv′). We show that (iv′) implies (iii′). Now, if $\mathscr{F}'$ is a quasi-coherent sub-$\mathscr{O}_X$-module of $\mathscr{O}_X^n$, the filtration $0 \subseteq \mathscr{O}_X \subseteq \mathscr{O}_X^2 \cdots \subseteq \mathscr{O}_X^n$ defines over $\mathscr{F}'$ a filtration of the form $\mathscr{F}_k' = \mathscr{F} \cap \mathscr{O}_X^k$ $(0 \leq k \leq n)$, which are quasi-coherent $\mathscr{O}_X$-modules (Proposition 4.2.20(ii)), and $\mathscr{F}_{k+1}'/\mathscr{F}_k'$ is isomorphic to a quasi-coherent sub-$\mathscr{O}_X$-module of $\mathscr{O}_X^{k+1}/\mathscr{O}_X^k = \mathscr{O}_X$, which is thus a quasi-coherent ideal of $\mathscr{O}_X$. In the exact sequence

$$H^1(X, \mathscr{F}_k') \longrightarrow H^1(X, \mathscr{F}_{k+1}') \longrightarrow H^1(\mathscr{F}_{k+1}'/\mathscr{F}_k') = 0$$

by hypothesis of (iv′) we have $H^1(\mathscr{F}_{k+1}'/\mathscr{F}_k') = 0$; since $H^0(X, \mathscr{F}_0') = 0$, we conclude by recurrence on k that $H^1(X, \mathscr{F}_k') = 0$ for each k, whence the claim. □

Remark 5.5.3. Note that if X is a covering of (X_{f_i}) with X_{f_i} being affine, then X is automatically quasi-separated, since for any couple (i, j) of indices we have $X_{f_i} \cap X_{f_j} = D_{X_{f_i}}(f_j|_{X_{f_i}})$, which is an affine open of X_{f_i} and hence quasi-compact (Proposition 4.6.10).

Remark 5.5.4. If X is a Noetherian scheme, then in conditions (iii′) and (iv′) we can replace "quasi-coherent" by "coherent". In fact, in the demonstration that (iii′) implies (ii), $\mathscr{I}$ and $\mathscr{I}'$ are then coherent ideals, and moreover, any quasi-coherent submodule of a coherent module is coherent (Proposition 4.1.29), whence the assertion.

Corollary 5.5.11. *Let $f : X \to Y$ be a quasi-compact and quasi-separated morphism. Then the following conditions are equivalent:*

(i) f is an affine morphism.

(ii) The functor f_* is exact on the cateogry of quasi-coherent $\mathscr{O}_X$-modules.

(iii) For any quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, we have $R^1 f_*(\mathscr{F}) = 0$.

(iii′) For any quasi-coherent ideal $\mathscr{I}$ of $\mathscr{O}_X$, we have $R^1 f_*(\mathscr{I}) = 0$.

Proof. Any of these conditions are local over Y, by the definition of $R^1 f_*(\mathscr{F})$ (that is, the sheaf associated with the presheaf $U \mapsto H^1(f^{-1}(U), \mathscr{F})$), so we may assume that Y is affine. If f is affine, X is then affine and (ii) follows from Proposition 4.1.12. Conversely, we prove that (ii) implies (i): for any quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, $f_*(\mathscr{F})$ is a quasi-coherent $\mathscr{O}_Y$-module by Proposition 4.6.54. By hypothesis the fonctor f_* is exact; $\Gamma(Y, -)$ is exact since Y is affine, so we conclude that $\Gamma(Y, f_*(-)) = \Gamma(X, -)$ is exact, which proves that X is affine in view of Theorem 5.5.10.

If f is affine, $f^{-1}(U)$ is affine for any affine open U of Y, so $H^1(f^{-1}(U), \mathscr{F}) = 0$ by Theorem 5.5.10, which means $R^1 f_*(\mathscr{F}) = 0$. Finally, suppose that (iii′) is satisfied; the exact sequence of low-degree terms in the Leray spectral sequence gives

$$0 \longrightarrow H^1(Y, f_*(\mathscr{I})) \longrightarrow H^1(X, \mathscr{I}) \longrightarrow H^0(Y, R^1 f_*(\mathscr{I}))$$

As Y is affine and $f_*(\mathscr{I})$ is quasi-coherent (Proposition 4.6.54), we have $H^1(Y, f_*(\mathscr{I})) = 0$, so the hypothesis of (iii′) imlies that $H^1(X, \mathscr{I}) = 0$, and we conclude from Theorem 5.5.10 that X is an affine scheme. □

Corollary 5.5.12. *If $f : X \to Y$ is an affine morphism then for any quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, the canonical homomorphism $H^1(Y, f_*(\mathscr{F})) \to H^1(X, \mathscr{F})$ is bijective.*

Proof. In fact, we have an exact sequence

$$0 \longrightarrow H^1(Y, f_*(\mathscr{F})) \longrightarrow H^1(X, \mathscr{F}) \longrightarrow H^0(Y, R^1 f_*(\mathscr{F}))$$

which comes from the lower terms of the Leray spectral sequence, and the conclusion follows from Proposition 5.5.11. □

5.5.3 Quasi-projective morphisms

We say a morphism $f : X \to Y$ is **quasi-projective**, of that X is **quasi-projective over Y**, or that X is a **quasi-projective Y-scheme**, if f is of finite type and there exists an invertible $\mathscr{O}_X$-module that is f-ample. It is clear that a quasi-projective morphism is necessarily separated. If Y is quasi-compact, it is also equivalent to say that f is of finite type and there exists a very ample $\mathscr{O}_X$-module relative to f (Proposition 5.4.33).

Remark 5.5.5. It should be noted that this definition is not local over Y. There exist examples where X and Y are nonsingular algebraic schemes over an algebraically closed field such that any point of Y admits an affine neighborhood U such that $f^{-1}(U)$ is quasi-projective over U, but f is not quasi-projective.

Proposition 5.5.13. *Let Y be a quasi-compact and quasi-separated scheme and X be a Y-scheme. Then the following conditions are equivalent:*

(i) X is a quasi-projective Y-scheme.

(ii) X is of finite type over Y and there exists a quasi-coherent $\mathscr{O}_Y$-module $\mathscr{E}$ of finite type such that X is Y-isomorphic to a subscheme of $\mathbb{P}(\mathscr{E})$.

(iii) X is of finite type over Y and there exists a quasi-coherent graded $\mathscr{O}_Y$-algebra $\mathscr{S}$ that is generated by $\mathscr{S}_1$ and $\mathscr{S}_1$ is of finite type such that X is isomorphic to a dense open subscheme of $\mathrm{Proj}(\mathscr{S})$.

Proof. This follows from Corollary 5.4.12, Proposition 5.4.15 and Corollary 5.4.16. □

Corollary 5.5.14. *Let Y be a quasi-compact and quasi-separated scheme such that there exists an ample $\mathscr{O}_Y$-module $\mathscr{L}$. For a Y-scheme X to be quasi-projective, it is necessary and sufficient that X is of finite type over Y and is isomorphic to a sub-Y-scheme of a projective bundle of the form $\mathbb{P}^r_Y$.*

Proof. By the hypothesis on Y, if $\mathscr{E}$ is a quasi-coherent $\mathscr{O}_Y$-module of finite type, $\mathscr{E}$ is isomorphic to a quotient of an $\mathscr{O}_Y$-module $\mathscr{L}^{\otimes(-n)} \otimes_{\mathscr{O}_Y} \mathscr{O}_Y^k$ (Proposition 5.4.25), so $\mathbb{P}(\mathscr{E})$ is isomorphic to a closed subscheme of $\mathbb{P}^{k-1}_Y$ (Proposition 5.4.1). □

Proposition 5.5.15 (Properties of Quasi-projective Morphisms).

(i) *A quasi-affine morphism of finite type (in particular a quasi-compact immersion or an affine morphism of finite type) is quasi-projective.*

(ii) *If $f : X \to Y$ and $g : Y \to Z$ are quasi-projective and if Z is quasi-compact, $g \circ f$ is quasi-projective.*

(iii) *If $f : X \to Y$ is a quasi-projective S-morphism, $f_{(S')} : X_{(S')} \to Y_{(S')}$ is quasi-projective for any base change $S' \to S$.*

(iv) *If $f : X \to Y$ and $g : X' \to Y'$ are two quasi-projective S-morphisms, $f \times_S g$ is quasi-projective.*

(v) *If $f : X \to Y$, $g : y \to Z$ are two morphisms such that $g \circ f$ is quasi-projective and if g is separated or X is locally Noetherian, then f is quasi-projective.*

(vi) *If f is quasi-projective, so is f_{red}.*

Proof. Property (i) follows from Proposition 5.5.5 and Proposition 5.4.45(i). The other parts follows from the definition of quasi-projective morphism and Proposition 5.4.45, with the corresponding properties of morphisms of finite type (Proposition 4.6.35). □

Remark 5.5.6. Note that it may happen that f_{red} is quasi-projective without f being so, even we assume that Y is the spectrum of a finite dimensional algebra over C and f is proper.

Corollary 5.5.16. *If X and X' are two quasi-projective Y-schemes, $X \amalg X'$ is quasi-projective over Y.*

Proof. This follows from Proposition 5.4.48. □

5.5.4 Universally closed and proper morphisms

As the terminology indicates, we say a morphism $f : X \to Y$ is **universally closed** if the projection $f \times_Y 1_{Y'} : X \times_Y Y' \to Y'$ is closed for any base change $Y' \to Y$. By Proposition 4.4.12, we know that a closed immersion is universally closed. We say a morphism $f : X \to Y$ is **proper** if it is separated, of finite type, and universally closed, and in this case X is said to be **proper over Y**, or a **proper Y-scheme**. It is clear that all these notations are local over Y. We also note that, to verify that the image of a closed subset Z of $X \times_Y Y'$ under the projection $q : X \times_Y Y' \to Y'$ is closed in Y', it suffices to shows that $q(Z) \cap U'$ is closed in U' for any affine open subset U' of Y'. As $q(Z) \cap U' = q(Z \cap q^{-1}(U'))$ and $q^{-1}(U')$ is identified with $X \times_Y U'$ (Corollary 4.3.2), we see that to verify the universally closedness of f, it suffices to limit the case where Y' is affine. We will see later that if Y is locally Noetherian, we can even assume that Y' is of finite type over Y.

Proposition 5.5.17 (Properties of Proper Morphisms).

(i) *A closed immersion is proper.*

(ii) *The composition of two proper morphisms is proper.*

(iii) *If $f : X \to Y$ is a proper S-morphism, then $f_{(S')} : X_{(S')} \to Y_{(S')}$ is proper for any base change $S' \to S$.*

(iv) If $f : X \to Y$ and $g : X' \to Y'$ are two proper S-morphisms, then $f \times_S g$ is proper.

Proof. It suffices to prove the first three properties. In view of Proposition 4.5.25 and Proposition 4.6.35, it suffices to verify the universally closedness in each cases. This is trivial in (i) since closed immersions are universal. For (ii), consider two proper morphisms $X \to Y$, $Y \to Z$, and a morphism $Z' \to Z$. We have $X \times_Z Z' = X \times_Y (Y \times_Z Z')$ and therefore the projection $X \times_Z Z' \to Z'$ factors into $X \times_Y (Y \times_Z Z') \to Y \times_Z Z' \to Z'$. By hypothesis, this is a composition of two closed morphisms, hence closed. Finally, in (iii), for any morphism $S' \to S$, $X_{(S')}$ is identified with $X \times_Y Y_{(S')}$; for any morphism $Z \to Y_{(S')}$, we have

$$X_{(S')} \times_{Y_{(S')}} Z = (X \times_Y Y_{(S')}) \times_{Y_{(S')}} Z = X \times_Y Z$$

and $X \times_Y Z \to Z$ is closed by hypothesis, so (iii) follows. □

Corollary 5.5.18. *Let $f : X \to Y$, $g : Y \to Z$ be two morphisms such that $g \circ f$ is proper.*

(a) If g is separated, f is proper.

(b) If g is separated and of finite type and f is surjective, then g is proper.

Proof. The first claim follows from Proposition 4.5.22. To prove (b), we only need to verify that g is universally closed. For any morphism $Z' \to Z$, the diagram

$$\begin{tikzcd} X \times_Z Z' \arrow[r, "f \times 1_{Z'}"] \arrow[dr, "p"'] & Y \times_Z Z' \arrow[d, "p'"] \\ & Z' \end{tikzcd}$$

(where p and p' are projections) is commutative. Moreover, $f \times 1_{Z'}$ is surjective if f is (Proposition 4.3.26), and p is a closed immersion by hypothesis. Any closed subset F of $Y \times_Z Z'$ is then the image under $f \times 1_{Z'}$ of a closed subset E of $X \times_Z Z'$, so $p'(F) = p(E)$ is closed in Z' by hypothesis, whence the corollary. □

Corollary 5.5.19. *If X is a proper scheme over Y and $\mathscr{S}$ is a quasi-coherent $\mathscr{O}_Y$-algebra, any Y-morphism $f : X \to \mathrm{Proj}(\mathscr{S})$ is proper (and a fortiori closed).*

Proof. In fact, the structural morphism $p : \mathrm{Proj}(\mathscr{S}) \to Y$ is separated, and $p \circ f$ is proper by hypothesis. □

Corollary 5.5.20. *Let $f : X \to Y$ be a separated morphism of finite type. Let $(X_i)_{1 \le i \le n}$ (resp. $(Y_i)_{1 \le i \le n}$) be a family of closed subscheme of X (Resp. Y), $j_i : X_i \to X$ (resp. $h_i : Y_i \to Y$) be the canonical injection. Suppose that (X_i) forms a covering of X and for each i, let $f_i : X_i \to Y_i$ be a morphism such that the diagram*

$$\begin{tikzcd} X_i \arrow[r, "f_i"] \arrow[d, "j_i"'] & Y_i \arrow[d, "h_i"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

is commutative. Then, for f to be proper, it is necessary and sufficient that each f_i is proper.

Proof. If f is proper, so is each $f \circ j_i$, since j_i is a closed immersion; as each h_i is a closed immersion, hence separated, f_i is proper by Corollary 5.5.18. Suppose conversely that each f_i is proper, and consider the sum Z of X_i; let $u : Z \to X$ be the morphism that induces j_i on X_i. The restriction of $f \circ u$ to each X_i is equal to $f \circ j_i = h_i \circ f_i$, hence proper; it then follows that $f \circ u$ is proper. Since u is surjective by hypothesis, we conclude from Corollary 5.5.18 that f is proper. □

Corollary 5.5.21. *Let $f : X \to Y$ be a separated morphism of finite type. For f to be proper, it is necessary and sufficient that $f_{\mathrm{red}} : X_{\mathrm{red}} \to Y_{\mathrm{red}}$ is proper.*

Proof. This is a particular case where $n = 1$, $X_1 = X_{\mathrm{red}}$ and $Y_1 = Y_{\mathrm{red}}$. □

If X and Y and Noetherian schemes and $f : X \to Y$ is a separated morphism of finite type, to verify that f is proper, we can reduce to dominant morphisms of integral schemes. In fact, let X_i $(1 \leq i \leq n)$ be the irreducible components of X and consider for each i the unique reduced closed subscheme structure on X_i. Let Y_i be the reduced closed subscheme with underlying space $\overline{f(X_i)}$. If $j_i : X_i \to X$ (resp. $h_i : Y_i \to Y$) are the canonical injections, we then have $f \circ j_i = h_i \circ f_i$, where f_i is a dominant morphism $f_i : X_i \to Y_i$. We then see that the conditions of Corollary 5.5.20 are satisfied, and for f to be proper, it is necessary and sufficient that each f_i is.

Corollary 5.5.22. *Let X and Y be separated S-schemes of finite type and $f : X \to Y$ be an S-morphism. For f to be proper, it is necessary and sufficient that for any S-scheme S', the morphism $f_{(S')} : X_{(S')} \to Y_{(S')}$ is closed.*

Proof. We note that if $\varphi : X \to S$ and $\psi : Y \to S$ are the structural morphisms, we have $\varphi = \psi \circ f$, so f is separated and of finite type (Proposition 4.5.25 and Proposition 4.6.35). If f is proper, so is $f_{(S')}$, and is a fortiori closed. Conversely, assume this condition and let Y' be a Y-scheme; Y' can be considered as an S-scheme, and the morphism $Y \to S$ is separated. In the commutative diagram

$$\begin{tikzcd} X \times_Y Y' \arrow[r, "f \times_Y 1_{Y'}"] \arrow[d] & Y \times_Y Y' = Y' \arrow[d] \\ X \times_S Y' \arrow[r, "f_{(Y')}"] & Y \times_S Y' \end{tikzcd}$$

the vertical morphisms are a closed immersion by Proposition 4.5.18. It follows from the assumption that $f_{(Y')}$ is closed, and so is $f \times_Y 1_{Y'}$. □

Let $f : X \to Y$ be a morphism of finit type. We say that a closed subset Z of X is **proper over** Y (or Y-proper, or f-proper) if the restriction of f to a closed subscheme of X with underlying space Z is proper. As this restriction is then separated, it follows from Corollary 5.5.21 and Proposition 4.5.25(vi) this property is independent of the closed subscheme structural chosen for Z.

Let Z be a proper subset of X for f and let $g : X' \to X$ be a proper morphism. Then $g^{-1}(Z)$ is then a proper subset of X': if T is a subscheme of X with underlying space Z, it suffices to note that the restriction of g to the closed subscheme $g^{-1}(T)$ of X' is a proper morphism

$g^{-1}(T) \to T$ by Proposition 5.5.17(iii), and we can apply Proposition 5.5.17(ii) to conclude that $g^{-1}(T)$ is proper.

On the other hand, if X'' is a Y-scheme of finite type and $h : X \to X''$ is a Y-morphism, $h(Z)$ is then a proper subset of X'': in fact, for any reduced closed subscheme T of X with underlying space Z. The restriction of f to T is proper, and so is the restriction of h to T (Corollary 5.5.18(a)), so $h(Z)$ is closed in X''. Let T'' be a closed subscheme of X'' with underlying space $h(Z)$ so that the morphism $h|_T$ factors into (cf. Proposition 4.4.44)

$$T \xrightarrow{h|_T} T'' \xrightarrow{j} X''$$

where j is the canonical injection. Then $h|_T$ is proper by Proposition 5.5.20 and surjective. If $\psi : X'' \to Y$ is the structural morphism, $\psi|_{T''}$ is then separated of finite type (Proposition 4.5.25 and Proposition 4.6.35), and we have $f|_T = (\psi|_{T''}) \circ (h|_T)$; it then follows from Proposition 5.5.17(ii) that $\psi|_{T''}$ is proper, whence the assertion.

In particular, for a Y-proper subset of X, we have the following:

(a) For any closed subset X' of X, $Z \cap X'$ is a Y-proper subset of X'.

(b) If X is a subscheme of a Y-scheme of finite type X'', Z is also a Y-proper subset of X'' (and in particular is closed in X'').

5.5.5 Projective morphisms

Proposition 5.5.23. *Let X be a Y-scheme. The following conditins are equivalent:*

(a) X is Y-isomorphic to a closed subscheme of a projective bundle $\mathbb{P}(\mathscr{E})$, where $\mathscr{E}$ is a quasi-coherent $\mathscr{O}_Y$-module of finite type.

(b) There exists a quasi-coherent graded $\mathscr{O}_Y$-algebra such that $\mathscr{S}_1$ is of finite type and generates $\mathscr{S}$, and that X is Y-isomorphic to a $\mathrm{Proj}(\mathscr{S})$.

Proof. Condition (a) implies (b) by Proposition 5.3.30(b): if $\mathscr{I}$ is the quasi-coherent graded ideal of $\boldsymbol{S}(\mathscr{E})$, the quasi-coherent graded $\mathscr{O}_Y$-algebra $\mathscr{S} = \boldsymbol{S}(\mathscr{E})/\mathscr{I}$ is generateed by $\mathscr{S}_1$ and the later, which is the canonical image of $\mathscr{E}$, is an $\mathscr{O}_X$-module of finite type. Condition (b) implies (a) in view of Corollary 5.3.32 applied to the case where $\mathscr{M} \to \mathscr{S}_1$ is the identify homomorphism. □

We say a Y-scheme X is **projective over $\boldsymbol{Y}$** or a projective **$\boldsymbol{Y}$-scheme** if it satisfies the equivalent conditions of Proposition 5.5.23. We say a morphism $f : X \to Y$ is **projective** if X is a projective Y-scheme via this morphism. It is clear that if $f : X \to Y$ is projective, then there exists a very ample $\mathscr{O}_X$-module relative to f (Corollary 5.4.12).

Theorem 5.5.24. *Any projective morphism is quasi-projective and proper. Conversely, if Y is a quasi-compact and quasi-separated scheme, any quasi-projective and proper morphism $f : X \to Y$ is projective.*

Proof. It is clear that any projective morphism is of finite type and quasi-projective. On the other hand, it follows from Proposition 5.5.23(b) and Proposition 5.3.28 that if f is projective, so is $f \times_Y 1_{Y'} : X \times_Y Y' \to Y'$ for any morphism $Y' \to Y$. The proof that f is universally closed then boils down to show that a projective morphism f is closed. The question is local over Y, so we can suppose that $Y = \mathrm{Spec}(A)$ is affine, so by Proposition 5.5.23 $X = \mathrm{Proj}(S)$ where S is a graded A-algebra generated by finitely many elements of S_1. For any $y \in Y$, the fiber $f^{-1}(y)$ is identified with $\mathrm{Proj}(S) \times_Y \mathrm{Spec}(\kappa(y))$, hence to $\mathrm{Proj}(S \otimes_A \kappa(y))$ (Proposition 5.2.48). Therefore, $f^{-1}(y)$ is empty if and only if $S \otimes_A \kappa(y)$ is eventually zero, which meaens $S_n \otimes_A \kappa(y) = 0$ for n sufficiently large. Now as $(S_n)_y$ is an $\mathscr{O}_{Y,y}$-module of finite type, the preceding condition signifies that $(S_n)_y = 0$ for n sufficiently large, in view of Nakayama's lemma. If $\mathfrak{a}_n$ is the annihilator in A of the A-module S_n, the preceding condition is then equivalent to that $\mathfrak{a}_n \subseteq \mathfrak{p}_y$ for n sufficiently large. Now as $S_n S_1 = S_{n+1}$ by hypothesis, we have $\mathfrak{a}_n \subseteq \mathfrak{a}_{n+1}$, and if $\mathfrak{a}$ is the sum of $\mathfrak{a}_n$, we then have $f(X) = V(\mathfrak{a})$, so $f(X)$ is closed in Y. If now X' is a closed subset of X, there exists a closed subscheme of X with underlying space X' and it is clear (by Proposition 5.5.23(a)) that the composition morphism $X' \to X \to Y$ is projective, so $f(X')$ is closed in Y.

Now conversely, assume that Y is quasi-compact and quasi-separated. The hypothesis that f is quasi-projective implies the existence of a quasi-coherent $\mathscr{O}_Y$-module of finite type $\mathscr{E}$ and a Y-immersion $j : X \to \mathbb{P}(\mathscr{E})$ (Proposition 5.5.13). Since f is proper and the structural morphism $\mathbb{P}(\mathscr{E}) \to Y$ is separated, j is proper (hence closed) by Corollary 5.5.18(a), so f is projective. □

Remark 5.5.7. Let $f : X \to Y$ be a morphism such that

(i) f is proper

(ii) there exists a very ample $\mathscr{O}_X$-module $\mathscr{L}$ relative to f

(iii) the quasi-coherent $\mathscr{O}_Y$-module $\mathscr{E} = f_*(\mathscr{L})$ is of finite type

Then f is projective: there then exists a Y-immersion $j : X \to \mathbb{P}(\mathscr{E})$, and since f is proper, j is a closed immersion by Corollary 5.5.18(a). We will see that if Y is locally Noetherian, the last condition (iii) is a concequence of the others, and conditions (i) and (ii) characterize projective morphisms. If Y is Noetherian, we can further repalce in (ii) that there exists a *ample* $\mathscr{O}_X$-module relative to f (Proposition 5.4.42). We also note that there are proper morphisms that is not projective.

Remark 5.5.8. Let Y be a quasi-compact scheme such that there exists an ample $\mathscr{O}_Y$-module. For a Y-scheme X to be projective, it is necessary and sufficient that it is isomorphic to a closed subscheme of a projective bundle of the form $\mathbb{P}^r_Y$. This conditions is clearly sufficient. Conversely, if X is projective over Y, it is quasi-projective, so there exists a Y-immersion $j : X \to \mathbb{P}^r_Y$ by Corollary 5.5.14, which is closed by Corollary 5.5.18(a) and Theorem 5.5.24.

Remark 5.5.9. The reasoning of Theorem 5.5.24 shows that for any scheme Y, and any integer $r \geq 0$, the structural morphism $\mathbb{P}^r_Y \to Y$ is surjective, because if we put $\mathscr{S}_{\mathscr{O}_Y} = \boldsymbol{S}(\mathscr{O}_Y^{r+1})$, we have evidently $\mathscr{S}_y = \boldsymbol{S}_{\kappa(y)}(\kappa(y)^{r+1})$, so $(\mathscr{S}_n)_y \neq 0$ for any $y \in Y$ and $n \geq 0$.

Proposition 5.5.25 (Properties of Projective Morphisms).

(i) A closed immersion is projective.

(ii) *If $f : X \to Y$ and $g : Y \to Z$ are projective morphisms and if Z is quasi-compact and quasi-separated, then $g \circ f$ is projective.*

(iii) *If $f : X \to Y$ is a projective morphism, $f_{(S')} : X_{(S')} \to Y_{(S')}$ is projective for any base change $S' \to S$.*

(iv) *If $f : X \to Y$ and $g : X' \to Y'$ are projective S-morphisms, so is $f \times_S g$.*

(v) *If $g \circ f$ is a projective morphism and if g is separated, f is projective.*

(vi) *If f is projective, so is f_{red}.*

Proof. Property (i) follows from Corollary 5.3.5. It is necessary here to prove (iii) and (iv) separately, because of the restriction introduced on Z in (ii). To prove (iii), we can reduce to the case where $S = Y$ (Corollary 4.3.8) and the assertion then follows immediately from Proposition 5.5.23(b) and from Proposition 5.3.28. To prove (iv), we can assume that $X = \mathbb{P}(\mathscr{E})$, $X' = \mathbb{P}(\mathscr{E}')$, where $\mathscr{E}$ (resp. $\mathscr{E}'$) is a quasi-coherent $\mathscr{O}_Y$-module of finite type. Let p, p' be the projection of $T = Y \times_S Y'$ to Y and Y' respectively; by (4.1.1), we have $\mathbb{P}(p^*(\mathscr{E})) = \mathbb{P}(\mathscr{E}) \times_Y T$ and $\mathbb{P}(p'^*(\mathscr{E})) = \mathbb{P}(\mathscr{E}') \times_{Y'} T$, whence

$$\begin{aligned}\mathbb{P}(p^*(\mathscr{E})) \times_T \mathbb{P}(p'^*(\mathscr{E}')) &= (\mathbb{P}(\mathscr{E}) \times_Y T) \times_T (T \times_{Y'} \mathbb{P}(\mathscr{E}')) = \mathbb{P}(\mathscr{E}) \times_Y (T \times_{Y'} \mathbb{P}(\mathscr{E}')) \\ &= \mathbb{P}(\mathscr{E}) \times_Y ((Y \times_S Y') \times_{Y'} \mathbb{P}(\mathscr{E}')) = \mathbb{P}(\mathscr{E}) \times_S \mathbb{P}(\mathscr{E}').\end{aligned}$$

Now $p^*(\mathscr{E})$ and $p'^*(\mathscr{E}')$ are of finite type over T (Proposition 1.4.5), and so is $p^*(\mathscr{E}) \otimes_{\mathscr{O}_T} p'^*(\mathscr{E}')$; as $\mathbb{P}(p^*(\mathscr{E})) \times_T \mathbb{P}(p'^*(\mathscr{E}'))$ is identified with a closed subscheme of $\mathbb{P}(p^*(\mathscr{E}) \otimes_{\mathscr{O}_T} p'^*(\mathscr{E}'))$ (Proposition 5.4.10), this proves (iv). For (v) and (vi), we can apply Proposition 4.5.22, since any closed subscheme of a projective Y-scheme is projective by Proposition 5.5.23(a). □

Proposition 5.5.26. *If X and X' are two projective Y-schemes, so is $X \amalg X'$.*

Proof. This is evident from Remark 5.4.3. □

Proposition 5.5.27. *Let X be a projective Y-scheme, $\mathscr{L}$ be a Y-ample $\mathscr{O}_Y$-module. For any section f of $\mathscr{L}$ over X, X_f is affine over Y.*

Proof. Since the question is local over Y, we can assume that $Y = \operatorname{Spec}(A)$ is affine; since $X_{f^{\otimes n}} = X_f$, by replacing $\mathscr{L}$ by $\mathscr{L}^{\otimes n}$ we can assume that $\mathscr{L}$ is very ample for the structural morphism $q : X \to Y$ (Proposition 5.4.29). The canonical homomorphism $\sigma : q^*(q_*(\mathscr{L})) \to \mathscr{L}$ is then surjective and the corresponding morphism

$$r = r_{\mathscr{L},\sigma} : X \to P = \mathbb{P}(q_*(\mathscr{L}))$$

is an immersion such that $\mathscr{L} \cong r^*(\mathscr{O}_P(1))$ (Proposition 5.4.13). Moreover, as X is proper over Y, the immersion r is closed by Corollary 5.5.18. By definition $f \in \Gamma(Y, q_*(\mathscr{L}))$ and $\sigma^\flat$ is the identity of $q_*(\mathscr{L})$; it then follows from the formula (3.6.2) that we have $X_f = r^{-1}(D_+(f))$. Then X_f is a closed subscheme of the affine scheme $D_+(f)$, and therefore is affine. □

If we take $Y = X$ in Proposition 5.5.27, we obtain that, for any scheme X and any invertible $\mathscr{O}_X$-module $\mathscr{L}$, the open subset X_f is affine over X.

5.5.6 Chow's lemma

Theorem 5.5.28 (Chow's lemma). *Let X be a separated S-scheme of finite type and suppose that one of the following conditions is satisfied:*

(a) S is Noetherian.

(b) S is quasi-compact and X has finitely many irreducible components.

Then there exists a quasi-projective S-scheme X' and a surjective projective S-morphism $f : X' \to X$ that induces an isomorphism an isomorphism $f^{-1}(U) \cong U$ for some open dense subset of X. If X is reduced (resp. irreducible), we can also choose X' to be reduced (resp. irreducible).

Proof. The proof is divided into several steps. First of all, we can assume that X is irreducible. To see this, we note that in both cases the scheme X has finitely many irreducible components X_i. If the theorem is demonstrated for each reduced subscheme X_i, and if $f_i : X_i' : X_i$ is the corresponding hommorphism which induces an isomorphism $f_i^{-1}(U_i) \cong U_i$ with $U_i \subseteq X_i$, the sum $X' = \coprod_i X_i'$ is then quasi-projective over S (Proposition 5.5.16 and Proposition 5.5.15) and the morphism $f : X' \to X$ whose restriction on X_i' equals to $j_i \circ f_i$ (where $j_i : X_i \to X$ is the canonical injection), is then surjective and projective (Proposition 5.5.26); it is immediate to see that X' is reduced if each X_i' is. We now choose U to be the union of $U_i \cap (\bigcup_{j\neq i} X_j)^c$; since U_i is dense in X_i and X_i is maximal irreducible, we conclude that each $U_i \cap (\bigcup_{j\neq i} X_j)$ is nonempty. The open subset U is then dense in X and f clearly induces an isomorphism $f^{-1}(U) \cong U$.

So suppose now that X is irreducible. As the structural morphism $\eta : X \to S$ is of finite type, there exists a finite covering (S_i) of S by affine opens, and for each i there is a finite covering (T_{ij}) of $\eta^{-1}(S_i)$ by affine opens, with the morphism $T_{ij} \to S_i$ being affine and of finite type, hence quasi-projective (Proposition 5.5.15(i)). As in both hypotheses the immersion $S_i \to S$ is quasi-compact, it is quasi-projective by Proposition 5.5.15(i), so the restriction of η to T_{ij} is quasi-projective (Proposition 5.5.15(ii)). We relabel the T_{ij} by U_k with $1 \leq k \leq n$. There exists, for each index k, an open immersion $\varphi_k : U_k \to P_k$, where P_k is projective over S (Proposition 5.5.13). Let $U = \bigcap_k U_k$; as X is irreducible and the U_k is nonempty, U is nonempty, and therefore is dense in X; the restrictions of φ_k to U together define a morphism

$$\varphi : U \to P = P_1 \times_S P_2 \times_S \cdots \times_S P_n$$

which fits into the diagram

$$\begin{tikzcd} U \arrow[r, "\varphi"] \arrow[d, "j_k"'] & P \arrow[d, "p_k"] \\ U_k \arrow[r, "\varphi_k"] & P_k \end{tikzcd} \tag{5.6.1}$$

where j_k is the canonical injection and p_k is the canonical projection. If $j : U \to X$ is the canonical injection, the morphism $\psi = (j, \varphi)_S : U \to X \times_S P$ is then an immersion by Corollary 4.5.16. Under the hypotheses of (a), $X \times_S P$ is locally Noetherian (Proposition 5.3.21 and Corollary 4.6.22), and under the hypotheses of (b), $X \times_S P$ is quasi-compact. In both cases the scheme-theoretic image X' of ψ in $X \times_S P$ exists (which is the closure of $\psi(U)$ in $X \times_S P$) and ψ factors into

$$\psi : U \xrightarrow{\psi'} X' \xrightarrow{h} X \times_S P$$

where ψ' is a dominant open immersion and h is a closed immersion. Let $q_1 : X \times_S P \to X$ and $q_2 : X \times_P \to P$ be the canonical projections; we put

$$f : X' \xrightarrow{h} X \times_S P \xrightarrow{q_1} X, \quad g : X' \xrightarrow{h} X \times_S P \xrightarrow{q_2} P. \tag{5.6.2}$$

We shall verify that the scheme X' and the morphism f satisfy the requirements. First we show that f is projective and surjective, and that the restriction of $U' = f^{-1}(U)$ is an isomorphism from U' to U. As the P_k are projective over S, so is P (Proposition 5.5.25(iv)), and $X \times_S P$ is projective over X by Proposition 5.5.25(iii); then X' is also projective over X, since it is a closed subscheme of $X \times_S P$. On the other hand, we have $f \circ \psi' = q_1 \circ (h \circ \psi') = q_1 \circ \psi = j$, so $f(X')$ contains the dense open subset U of X; but f is proper by Theorem 5.5.24, so $f(X') = X$. Now $q_1^{-1}(U) = U \times_S P$ is an open subscheme of $X \times_S P$, and the the immersion ψ factors into

$$\psi : U \xrightarrow{\Gamma_\varphi} U \times_S P \xrightarrow{j \times 1} X \times_S P.$$

By Proposition 4.6.67, $U' = h^{-1}(U \times_S P)$ is the scheme-theoretic image of $\psi^{-1}(U \times_S P) = U$ under $\psi_U : U \to U \times_S P$, and therefore the closure of the image of Γ_φ in $U \times_S P$. As P is separted over S, Γ_φ is a closed immersion (Proposition 4.5.19), so we conclude that $U' = \psi(U)$. As ψ is an immersion, the restriction of f to U' is then an isomorphism, with inverse ψ'. Finally, by definition, $U' = \psi(U) = \psi'(U)$ is open and dense in X'.

We now show that g is an immersion, which implies that X' is quasi-projective over S, since P is projective over S. Let $V_k = \varphi_k(U_k)$ be the image of U_k in P_k, $W_k = p_k^{-1}(V_k)$ be the inverse image in P, and put $U'_k = f^{-1}(U_k)$, $U''_k = g^{-1}(W_k)$. Since the U_k cover X, it clear that the U'_k form an open covering of X'; we first shows that this is also ture for the U''_k, by proving that $U'_k \subseteq U''_k$. For this, it suffices to estabilsh the commutativity of the diagram

$$\begin{tikzcd} U'_k \arrow[r, "g|_{U'_k}"] \arrow[d, "f|_{U'_k}"'] & P \arrow[d, "p_k"] \\ U_k \arrow[r, "\varphi_k"] & P_k \end{tikzcd} \tag{5.6.3}$$

Since $U'_k = h^{-1}(U_k \times_S P)$ and $\psi^{-1}(U_k \times_S P) = U$, by Proposition 4.6.67 U'_k is the scheme-theoretic image of U in $U_k \times_S P$ under the morphism $\psi_k : U \to U_k \times_S P$ induced by ψ. It then suffices to prove the commutativity of the diagram obtained by composing (5.6.3) with the morphism ψ_k (Corollary 4.6.65), and this comes from the commutative diagram (5.6.1).

The W_k then form an open covering of $g(X')$, so to show that g is an immersion, it suffices to show the restriction $g|_{U''_k}$ is an immersion into W_k (Proposition 4.4.10). For this, consider the morphism

$$u_k : W_k \xrightarrow{p_k} V_k \xrightarrow{\varphi_k^{-1}} U_k \to X$$

which fits into the following commutative diagram

$$\begin{tikzcd}
U' \arrow[r, hook] \arrow[dr] & U'_k \arrow[r, hook] \arrow[dr] & U''_k \arrow[r, hook, "h|_{U''_k}"] \arrow[dr, "f|_{U''_k}"'] \arrow[rr, bend left, "g|_{U''_k}"] & X \times_S W_k \arrow[r, "q_2"] \arrow[d, "q_1"] & W_k \arrow[l, bend right, "\Gamma_{u_k}"'] \arrow[dl, "u_k"'] \arrow[d, "p_k"] \\
 & U \arrow[r, hook] & U_k \arrow[r, hook] \arrow[rr, bend right, "\cong", "\varphi_k"'] & X & V_k
\end{tikzcd}$$

By the definition of g (formula (5.6.3)), we have $U''_k = h^{-1}(X \times_S W_k) \subseteq X'$ and $\psi^{-1}(X \times_S W_k) = U$, so by Proposition 4.6.67, U''_k is the scheme-theoretic image of U under the morphism $U \to X \times_S W_k$ induced by ψ; since $U' = \psi(U)$, it is therefore dense in U''_k. On the other hand, as X is separted over S, the graph morphism $\Gamma_{u_k} : W_k \to X \times_S W_k$ is a closed immersion, so the graph $T_k = \Gamma_{u_k}(W_k)$ is a closed subscheme of $X \times_S W_k$. If we can prove that T_k dominates the canonical image of the open subscheme U' in $X \times_S W_k$, it will then dominates the subscheme U''_k. As the restriction of q_2 to T_k is an isomorphism onto W_k and h is a closed immersion, the restriction of g to X''_k will then be an immersion in W_k, and our assertion will be proved. For this, we let $v_k : U' \to X \times_S W_k$ be the canonical injection, and $w_k = q_2 \circ v_k$; then from the definition of Γ_{u_k} we have $v_k = \Gamma_{u_k} \circ w_k$, and the image of U' in $X \times_S W_k$ is therefore contained in T_k, verifying our claim.

It is clear that U, and therefore U', are irreducible, and so is X' by our construction, and that f is birational. If X is reduced, so is U', and X' is then reduced (Proposition 4.6.66). This completes the proof. □

Corollary 5.5.29. *Suppose the hypotheses of Theorem 5.5.28. For X to be proper over S, it is necessary and sufficient that there exists a projective S-scheme X' and a surjective S-morphism $f : X' \to X$ (which is projective by Proposition 5.5.25(v)). If this is the case, we can choose an open dense subset U of X such that f induces an isomorphism $f^{-1}(U) \cong U$ and that $f^{-1}(U)$ is dense in X'. If X is irreducible (resp. reduced), we can choose X' to be irreducible (resp. reduced). If X and X' are irreducible, f is then a birational morphism.*

Proof. The conditions is sufficient by Theorem 5.5.24 and Corollary 5.5.18(b). This is necessary because with the notations of Theorem 5.5.28, if X is proper over S, X' is then proper over S (since it is proper over X by Theorem 5.5.24), and our assertion follows from Proposition 5.5.17(ii). Moreover, as X' is quasi-projective over S, it is projective over S in view of Theorem 5.5.24. □

Corollary 5.5.30. *Let S be a locally Noetherian scheme, X be an S-scheme of finite type, and $\varphi : X \to S$ be the structural morphism. For X to be proper over S, it is necessary and sufficient that for any morphism of finite type $S' \to S$, the morphism $\varphi_{(S')} : X_{(S')} \to S'$ is closed. Moreover, it suffices to verify this condition for any S-scheme of the form $S' = S \otimes_{\mathbb{Z}} \mathbb{Z}[T_1, \dots, T_n]$.*

Proof. The conditions is clearly necessary, and we now prove the sufficiency. The question is local over S and S', so we may assume that S, S' are affine and Noetherian. By Chow's lemma, there exists a projective S-scheme P, an immersion $j : X' \to P$, and a projective surjective morphism $f : X' \to X$ such that the diagram

$$\begin{tikzcd}
X' \arrow[dr, "\psi"] \arrow[drr, bend left, "f"] \arrow[ddr, bend right, "j"'] & & \\
& X \times_S P \arrow[r, "q_1"] \arrow[d, "q_2"'] & X \arrow[d, "\varphi"] \\
& P \arrow[r, "r"] & S
\end{tikzcd}$$

is commutative; let $\psi = (f, j)_S$. As P is of finite type over S, the projection $q_2 : X \times_S P \to P$ is a closed morphism by hypotheses. On the other hand, since f is projective and the projection $q_1 : X \times_S P \to X$ is separated (since P is separated over S), we conclude from Proposition 5.5.25(v) that ψ is projective, hence closed. Since the immersion j is the composition of q_2 with ψ, it is therefore a closed immersion, whence proper. Moreover, the structural morphism $r : P \to S$ is projective, hence proper (Theorem 5.5.24), so $\varphi \circ f = r \circ j$ is proper. As f is surjective, we conclude by Corollary 5.5.18(b) that φ is proper.

To estabilsh the second assertion of the proposition, it suffices to prove that it implies that $\varphi_{(S')}$ is closed for any morphism $S' \to S$ of finite type. Now if S' is affine and of finite type over $S = \mathrm{Spec}(A)$, we have $S' = \mathrm{Spec}(A[x_1, \dots, x_n])$, and S' is then isomorphic to a closed subscheme of $S'' = \mathrm{Spec}(A[T_1, \dots, T_n])$ (where T_i are indeterminates). In the following commutative diagram

$$\begin{tikzcd}
X \times_S S' \arrow[r, "1_X \times j"] \arrow[d, "\varphi_{(S'')}"'] & X \times_S S'' \arrow[d, "\varphi_{(S'')}"] \\
S' \arrow[r, "j"] & S''
\end{tikzcd}$$

where j and $q_X \times j$ are closed immersions (Proposition 4.4.12) and $\varphi_{(S'')}$ is closed by hypothesis. We then conclude that $\varphi_{(S')}$ is closed, whence the claim. □

5.6 Integral morphisms and finite morphisms

5.6.1 Integral and finite morphisms

Let X be an S-scheme and $f : X \to S$ be the structural morphism. We say that X is **integral over S**, or that f is an **integral morphism**, if there exists an affine open covering (S_α) of S such that for each α, the open subscheme $f^{-1}(S_\alpha)$ of X is affine and its ring $B_\aleph$ is an integral algebra over the ring A_α of S_α. We say that X is **finite over** S, or that f is a **finite morphism**, if X is integral and of finite type over S. If S is affine with ring A, we also say that X is inegral or finite over A.

It is clear that any integral S-scheme is affine over S. Conversely, form the definition of integral morphisms, we see that for an affine S-scheme X to be integral over S (resp. finite), it is necessary and sufficient that the associated quasi-coherent $\mathscr{O}_S$-algebra $\mathscr{A}(X)$ is such that there exists an affine open covering (S_α) of S such that for each α, $\Gamma(S_\alpha, \mathscr{A}(X))$ is an integral algebra (resp. an integral algebra of finite type) over $\Gamma(S_\alpha, \mathscr{O}_S)$. A quasi-coherent $\mathscr{O}_S$-algebra satisfying this property is said to be **integral** (resp. **finite**) over $\mathscr{O}_S$. We note that a quasi-coherent $\mathscr{O}_S$-algebra $\mathscr{B}$ is finite if and only if it is an $\mathscr{O}_S$-module of finite type; it amounts to the same thing to say that $\mathscr{B}$ is an integral $\mathscr{O}_S$-algebra of finite type, because an integer algebra of finite type over a ring A is an A-module of finite type.

Proposition 5.6.1. *Let S be a locally Noetherian scheme. For an S-scheme X affine over S to be finite over S, it is necessary and sufficient that the $\mathscr{O}_S$-algebra $\mathscr{A}(X)$ is coherent.*

Proof. With the preceding remark, this follows from the fact that if S is locally Noetherian, then a quasi-coherent $\mathscr{O}_S$-module is of finite type if and only if it is coherent (Proposition 4.1.29). □

Proposition 5.6.2. *Let X be an integral (resp. finite) scheme over S with $f : X \to S$ the structural morphism. Then for any affine open subset $U \subseteq S$ with ring A, $f^{-1}(U)$ is affine and its ring B is an integral (resp. finite) algebra over A.*

Proof. To prove this proposition, we need Proposition **??**. We now that $f^{-1}(U)$ is affine by Proposition 5.1.3. If $\varphi : A \to B$ is the corresponding homomorphism, there exists a finite covering of U by open subsets $D(g_i)$ ($g_i \in A$) such that, if $h_i = \varphi(g_i)$, then B_{h_i} is an integral (resp. finite) algebra over A_{g_i}. In fact, by assumption, there is a covering of U by affine open subsets $V_\alpha \subseteq U$ such that if $A_\alpha = \Gamma(V_\alpha, \mathscr{O}_S)$ and $B_\alpha = \Gamma(f^{-1}(V_\alpha), \mathscr{O}_X)$, then B_α is an integral (resp. finite) algebra over A_α. Any $x \in U$ belongs to one V_α, so there exists $g \in A$ such that $x \in D(g) \subseteq V_\alpha$. If g_α is the image of g in A_α, we have $\Gamma(D(g), \mathscr{O}_S) = A_g = (A_\alpha)_{g_\alpha}$; let $h = \varphi(g)$, and let h_α be the image of g_α in B_α. We have

$$\Gamma(D(h), \mathscr{O}_S) = B_h = (B_\alpha)_{h_\alpha}$$

and as B_α is integral over A_α, $(B_\alpha)_{h_\alpha}$ is integral (resp. finite) over $(A_\alpha)_{g_\alpha}$. Since U is quasi-compact, we obtain a finite cover.

If we suppose first that each B_{h_i} is a finite algebra over A_{g_i}, then as an A_{g_i}-module, B_{h_i} is finitely generated, so Proposition **??** shows that B is a finitely generated A-module. Now assume that each B_{h_i} is integral over A_{g_i}; let $b \in B$, and let C be a sub-A-algebra of B generated by b. For each i, C_{h_i} is the A_{g_i}-algebra generated by $b/1$ over B_{h_i}. It then follows from the hypothesis that each C_{h_i} is finitely generated A_{g_i}-module, so by Proposition **??** C is a finitely generated A-module. This shows that B is integral over A. □

Proposition 5.6.3 (Properties of Integral and Finite morphisms).

(i) A closed immersion is finite (and a fortiori integral).

(ii) The composition of two integral morphisms (resp. finite) is integral (resp. finite).

(iii) If $f : X \to Y$ is an integral (resp. finite) S-morphism, $f_{(S')} : X_{(S')} \to Y_{(S')}$ is integral (resp. finite) for any base change $S' \to S$.

(iv) If $f : X \to Y$ and $g : X' \to Y'$ are two integral (resp. finite) S-morphisms, so is $f \times_S g$.

(v) If $f : X \to Y$ and $g : Y \to Z$ are two morphisms such that $g \circ f$ is integral (resp. finite) and g is separated, then f is integral (resp. finite).

(vi) If $f : X \to Y$ is an integral (resp. finite) morphism, so is f_{red}.

Proof. In view of Proposition 4.5.22, it suffices to prove (i), (ii), and (iii). To prove that a closed immersion $X \to S$ is finite, we can assume that $S = \mathrm{Spec}(A)$, and this then follows from the fact that a quotient ring $A/\mathfrak{a}$ is a finitely generated A-module. To prove the composition of two integral (resp. finite) morphisms $X \to Y$, $Y \to Z$ is integral (finite), we can assume that Z (and therefore X and Y) is affine, and the assertion is then equivalent to that if B is an integral (resp. finite) A-algebra and C is an integral (resp. finite) B-algebra, then C is an integral (resp. finite) A-algebra, which is immediate. Finally, to prove (iii), we can Simialrly assume that $S = \mathrm{Spec}(A)$, $S' = \mathrm{Spec}(A')$; then X is affine with ring B (Proposition 5.6.2), $X_{(S')}$ is affine with ring $A' \otimes_A B$, and it suffices to note that if B is an integral (resp. finite) A-algebra, then $A' \otimes_A B$ is an integral (resp. finite) A'-algebra. □

Note also that if X and Y are two integral (resp. finite) S-schemes, the sum $X \amalg Y$ is an integral (resp. finite) over S, because a product of two integral (resp. finite) A-algebras is still integral (resp. finite).

Corollary 5.6.4. *If X is an integral (resp. finite) scheme over S, then for any open subset $U \subseteq S$, $f^{-1}(U)$ is integral (resp. finite) over U.*

Proof. This is a particular case of Proposition 5.6.3(iii). □

Corollary 5.6.5. *Let $f : X \to Y$ be a finite morphism. Then for any $y \in Y$, the fiber $f^{-1}(y)$ is a finite algebraic scheme over $\kappa(y)$, and a fortiori with discrete and finite underlying space.*

Proof. The $\kappa(y)$-scheme $f^{-1}(y)$ is identified with $X \times_Y \mathrm{Spec}(\kappa(y))$, so is finite over $\kappa(y)$ by Proposition 5.6.3(iii). This is then an affine scheme whose ring is a finite dimensional $\kappa(y)$-algebra, so is Artinian. The proposition then follows from Proposition 4.2.30. □

Corollary 5.6.6. *Let X and S be integral schemes and $f : X \to S$ be a dominant morphism. If f is integral (resp. finite), then the rational function field $K(X)$ of X is an algebraic (resp. finite) extension of $K(S)$.*

Proof. Let s be the generic point of S; the $\kappa(s)$-scheme $f^{-1}(s)$ is integral (resp. finite) over $\mathrm{Spec}(\kappa(s))$ by Proposition 5.6.3(iii), and contains by hypothesis the generic point of x of X; the local ring of $f^{-1}(s)$ at x, equal to $\kappa(x)$ (Proposition 4.3.34), is a localization of an integral (resp. finite) algebra over $\kappa(s)$, whence the corollary. □

Proposition 5.6.7. *Any integral morphism is universally closed.*

Proof. Let $f : X \to Y$ be an integral morphism. In view of Proposition 5.6.3(iii), it suffices to prove that f is closed. Let Z be a closed subset of X. In view of Proposition 5.6.3(vi), we can suppose that X and Y are reduced; moreover, if T is the reduced closed subscheme of Y with underlying space $\overline{f(X)}$, we see that f factors into

$$f : X \xrightarrow{g} T \xrightarrow{j} Y,$$

where $j : T \to Y$ is the canonical injection, and as j is separated, it follows from Proposition 5.6.3(v) that g is an integral morphism. We can then assume that $f(X)$ is dense in Y, and prove that $f(X) = Y$. Since the question is local over Y, we can assume that $Y = \mathrm{Spec}(A)$ is affine, so $X = \mathrm{Spec}(B)$ where B is an integral algebra over A (Proposition 5.6.2); moreover A is reduced and the hypothesis that $f(X)$ is dense in Y implies that the corresponding homomorphism $\varphi : A \to B$ is injective (Corollary **??**). The condition that $f(X) = Y$ then follows from Theorem **??**. □

Remark 5.6.1. The hypothesis that g is separated is essential for the validity of Proposition 5.6.3(v): in fact, if Y is not separated over Z, the identity 1_Y is the composition morphism

$$Y \xrightarrow{\Delta_Y} Y \times_Z Y \xrightarrow{p_1} Y$$

but Δ_Y is not integral, since it is not closed (Proposition 5.6.7).

Corollary 5.6.8. *Any finite morphism $f : X \to Y$ is projective.*

Proof. As f is affine, $\mathscr{O}_X$ is a very ample $\mathscr{O}_X$-module relative to f; moreover $f_*(\mathscr{O}_X)$ is a quasi-coherent $\mathscr{O}_Y$-algebra of finite type by hypothesis. Finally, f is separated, of finite type, and universally closed (Proposition 5.6.7), and we then have the conditions of Remark 5.5.7. □

Lemma 5.6.9. *Let Y be a scheme, $\mathscr{E}$ be a locally free $\mathscr{O}_Y$-module of rank r, and Z be a finite subset of Y contained in an affine open subset V. Then there exists an open neighborhood $U \subseteq V$ of Z such that $\mathscr{E}|_U$ is isomorphic to $\mathscr{O}_Y^r|_U$.*

Proof. □

Proposition 5.6.10. *Let $f : X' \to X$ be a finite morphism, and let $\mathscr{B} = f_*(\mathscr{O}_{X'})$ (which is a quasi-coherent finite $\mathscr{O}_X$-algebra). Let $\mathscr{F}'$ be a quasi-coherent $\mathscr{O}_{X'}$-module; for $\mathscr{F}'$ to be locally free of rank r, it is necessary and sufficient that $f_*(\mathscr{F}')$ is a locally free $\mathscr{B}$-module of rank r.*

Proof. It is clear that if $f_*(\mathscr{F}')|_U$ is isomorphic to $\mathscr{B}^r|_U$ (where U is open in X), $\mathscr{F}'|_{f^{-1}(U)}$ is isomorphic to $\mathscr{O}_{X'}^r|_{f^{-1}(U)}$ (Corollary 5.1.17). Conversely, suppose that $\mathscr{F}'$ is locally free of rank r and we prove that $f_*(\mathscr{F}')$ is locally isomorphic to $\mathscr{B}^r$ as $\mathscr{B}$-modules. Let x be a point of X; if U runs through affine neighborhoods of x, $f^{-1}(U)$ form a fundamental system of affine neighborhoods (Proposition 5.1.3) of the finite subset $f^{-1}(x)$, since f is closed (Proposition 5.6.7). The proposition then follows from Lemma 5.6.9. □

Proposition 5.6.11. *Let $g : X' \to X$ be an integral morphism of schemes, Y be a locally integral and normal scheme, f be a rational map from Y to X' such that $g \circ f$ is a everywhere defined rational map; then f is everywhere defined.*

Proof. Recall that we say a scheme X is normal if it is normal as a ringed space, which means the stalk $\mathscr{O}_{X,x}$ is an integrally closed domain for every $x \in X$. If f_1 and f_2 are two morphisms (densely defined from Y to X') in the class of f, it is clear that $g \circ f_1$ and $g \circ f_2$ are equivalent morphisms, which justifies the notation $g \circ f$ for their equivalent class. We recall also that if Y is locally Noetherian, then the hypothesis on Y implies that Y is locally integral (Proposition 4.4.32).

To prove the proposition, we first note that the question is local over Y and we can assume that there exists a morphism $h : Y \to X$ in the class of $g \circ f$. Consider the inverse image $Y' = X'_{(h)} = X'_{(Y)}$, and note that the morphism $g' = g_{(Y)} : Y' \to Y$ is integral by Proposition 5.6.3(iii). Via the correspondence of rational maps from Y to X' with rational Y-sections of Y', we see that we are reduced to the case $X = Y$. □

Corollary 5.6.12. *Let X be a locally integral and normal scheme, $g : X' \to X$ be an integral morphism, and f be a rational X-section of X'. Then f is everywhere defined.*

Proof. □

Corollary 5.6.13. *Let X be a normal and integral scheme, X' be an integral scheme, and $g : X' \to X$ be an integral morphism. If there exists a rational X-section f of X', g is an isomorphism.*

5.6.2 Quasi-finite morphisms

Proposition 5.6.14. *Let $f : X \to Y$ be a morphism locally of finite type, x be a point of X and $y = f(x)$. The following conditions are equivalent:*

(i) The point x is isolated in the fiber X_y.

(ii) The point x is closed in X_y and there is no generalization of x in X_y.

(iii) The $\mathscr{O}_{Y,y}$-module $\mathscr{O}_{X,x}$ is quasi-finite.

Proof. The question is local over X and Y, so we can suppose that $X = \mathrm{Spec}(B)$ and $Y = \mathrm{Spec}(A)$ are affine, where B is finite A-algebra. Moreover, we can replace X by $X \times_Y \mathrm{Spec}(\mathscr{O}_{Y,y})$ without changing the fiber X_y and the local ring $\mathscr{O}_{X,x}$ (Proposition 4.3.34). We can then suppose that A is a local ring with maximal ideal $\mathfrak{m}$ (which equals to the local ring $\mathscr{O}_{Y,y}$). The fiber X_y is then the affine sheme of the ring $B/\mathfrak{m}B$, of finite type over $\kappa(y) = A/\mathfrak{m}$ (Proposition 4.6.47). Let $\mathfrak{P}$ be the prime ideal of B corresponding to x.

We note that the fiber $X_y = X \times_Y \mathrm{Spec}(\kappa(y)) = \mathrm{Spec}(B \otimes_A \kappa(y))$ is Jacobson. If (ii) is satisfied, then $\{x\}$ is an open subset of X_y, hence contains a closed point (by the Jacobson property) which must be x, and x is therefore closed in X_y. Also, since $\{x\}$ is open in X_y, it is clear that there is no further generalization x' of x (which means $x \in \overline{\{x'\}}$) in X_y; this proves (ii).

We now consider the condition in (ii). Consider the ring $\overline{B} = B \otimes_A \kappa(y) = B/\mathfrak{m}B$ and let $\overline{\mathfrak{P}}$ be the prime ideal corresponding to $\mathfrak{P}$. If x is closed in X_y, we see that $\overline{\mathfrak{P}}$ is maximal in $\overline{B}$ and by condition (ii) there is no prime ideal of $\overline{B}$ contained in $\overline{\mathfrak{P}}$. This shows that $\overline{B}_{\overline{\mathfrak{P}}}$ is Artinian, so we conclude that the ring homomorphism $A \to B$ is quasi-finite at $\mathfrak{P}$; that is, the $\mathscr{O}_{Y,y}$-module $\mathscr{O}_{X,x}$ is quasi-finite. Conversely, if $\overline{B}_{\overline{\mathfrak{P}}}$ is Artinian, then $\overline{\mathfrak{P}}$ is maximal and there is no prime ideal of $\overline{B}$ contained in $\overline{\mathfrak{P}}$; we then conclude that x is closed in X_y and there is no generalization of x in X_y. This shows that (ii)⇔(iii).

We finally prove that (ii) implies (i). If (ii) is satisfied, then the prime ideal $\mathfrak{P}$ is maximal and minimal in B since $\overline{B}_{\overline{\mathfrak{P}}}$ is Artinian. Let $\overline{\mathfrak{P}}_1 = \overline{\mathfrak{P}}, \overline{\mathfrak{P}}_2, \dots, \overline{\mathfrak{P}}_r$ be the minimal prime ideals of $\overline{B}$. Then the intersection $\bigcap_{i=1}^r \overline{\mathfrak{P}}_i$ is equal to the nilradical of $\overline{B}$, and $\bigcap_{i=2}^r \overline{\mathfrak{P}}_i$ is contained in this intersection in view of Proposition **??**, so there exists an element $\bar{b} \in \bigcap_{i=2}^r \overline{\mathfrak{P}}_i$ that is not nilpotent. The open subset $D_{\bar{B}}(\bar{b})$ of $X_y = \mathrm{Spec}(\bar{B})$ then reduces to $\{x\}$, which shows that x is isolated. □

Corollary 5.6.15. *Let $f : X \to Y$ be a morphism of finite type. Then the following conditions are equivalent:*

(i) Any point $x \in X$ is isolated in the fiber $X_{f(x)}$ (that is, $X_{f(x)}$ is discrete).

(ii) For any $x \in X$, $X_{f(x)}$ is a finite $\kappa(f(x))$-scheme.

(iii) For any $x \in X$, the $\mathscr{O}_{Y,y}$-module $\mathscr{O}_{X,x}$ is quasi-finite.

Proof. The equivalence of (i) and (iii) follows from Proposition 5.6.14. On the other hand, as $X_{f(x)}$ is an algebraic $\kappa(f(x))$-scheme, the equivalence of (i) and (ii) follows from Proposition 4.6.43. □

We say a morphism $f : X \to Y$ of finite type is **quasi-finite**, or X is quasi-finite over Y, if it satisfies the equivalent conditions of Corollary 5.6.15. From Corollary 5.6.5, it is clear that any finite morphism is quasi-finite.

Proposition 5.6.16 (Properties of Quasi-finite Morphisms).

(i) Any quasi-compact immersion (in particular any closed immersion) is quasi-finite.

(ii) If $f : X \to Y$ and $g : Y \to Z$ are quasi-finite morphisms, $g \circ f$ is quasi-finite.

(iii) If $f : X \to Y$ is a quasi-finite S-morphism, $f_{(S')} : X_{(S')} \to Y_{(S')}$ is quasi-finite for any base change $S' \to S$.

(iv) If $f : X \to Y$ and $g : X' \to Y'$ are quasi-finite S-morphisms, $f \times_S g$ is quasi-finite.

(v) Let $f : X \to Y$ and $g : Y \to Z$ be two morphisms such that $g \circ f$ is quasi-finite; then, if g is separated, or X is Noetherian, or $X \times_Z Y$ is locally Noetherian, then f is quasi-finite.

(vi) If f is quasi-finite, so is f_{red}.

Proof. If $f : X \to Y$ is an immersion, any fiber is reduced to a singleton, so (i) follows from Proposition 4.6.35(i). For (ii), we note that $h = g \circ f$ is of finite type by Proposition 4.6.35(ii); if $z = h(x)$ and $y = f(x)$, y is isolated in $g^{-1}(z)$, so there exists an open neighborhood V of y in Y not containing other points of $g^{-1}(z)$; $f^{-1}(V)$ is then an open neighborhood of x not containing other points of $f^{-1}(y')$, where $y' \neq y$ is in $g^{-1}(z)$, and therefore not containing points $x' \neq x$ in $h^{-1}(z)$ that is not in $f^{-1}(y)$. As x is isolated in $f^{-1}(y)$ by hypothesis, it is then isolated in $h^{-1}(z) = f^{-1}(g^{-1}(z))$. As for (iii), we can limit ourselves to the case where $Y = S$ (Corollary 4.3.8); we first note that $f' = f_{(S')}$ is of finite type (Proposition 4.6.35(iii)). On the other hand, if $x' \in X' = X_{(S')}$ and $y' = f'(x')$, $X'_{y'}$ is identified with $X_y \otimes_{\kappa(y)} \kappa(y')$ by Proposition 4.3.35. As X_y is of finite dimension over $\kappa(y)$ by hypothesis, $X'_{y'}$ is of finite dimension over $\kappa(y')$, hence discrete. The assertions (iv), (v), (vi) then follows from the first three assertions in view of the general principle Proposition 4.5.22, where in (v) we assume that g is separated. The other cases, we first remark that if x is isolated in $X_{g(f(x))}$, it is also isolated in $X_{f(x)}$; the fact that f is of finite type follows from Proposition 4.6.35. □

Proposition 5.6.17. *Let A be a complete Noetherian local ring, $Y = \operatorname{Spec}(A)$, X be a separated Y-scheme locally of finite type, x be a point over the closed point y of Y, and suppose that x is isolated in the fiber X_y. Then $\mathscr{O}_{X,x}$ is a finitely generated A-module and X is Y-isomorphic to the sum of $X' = \operatorname{Spec}(\mathscr{O}_{X,x})$ (which is a finite Y-scheme) and an A-scheme X''.*

Proof. It follows from Proposition 5.6.14 that $\mathscr{O}_{X,x}$ is a quasi-finite A-module. As $\mathscr{O}_{X,x}$ is Noetherian (Proposition 4.6.20) and the homomorphism $A \to \mathscr{O}_{X,x}$ is local, the hypothesis that A is complete implies that $\mathscr{O}_{X,x}$ is a finitely generated A-module (Proposition **??**). Let $X' = \operatorname{Spec}(\mathscr{O}_{X,x})$ be the local scheme of X at x and $g : X' \to X$ be the canonical morphism. The composition $f \circ g : X' \to Y$ is then finite, and since f is separated, g is finite by Proposition 5.6.3, so $g(X')$ is closed in X (Proposition 5.6.8). On the other hand, as g is of finite type and A is Noetherian, it is of finite presentation, and hence a local immersion at the closed point x' of X' (Proposition 4.6.52 and the definition of g). But X' is the only open neighborhood of x' in X', so it follows that $g(X')$ is open in X, which proves our assertion. □

Corollary 5.6.18. *Let A be a complete Noetherian local ring, $Y = \operatorname{Spec}(A)$, $f : X \to Y$ be a quasi-finite and separted morphism. Then X is Y-isomorphic to a sum $X' \amalg X''$, where X' is a finite Y-scheme and X'' is a quasi-finite Y-scheme such that, if y is the closed point of y, $X'' \cap f^{-1}(y) \neq \varnothing$.*

Proof. The fiber $f^{-1}(y)$ is finite and discrete by hypothesis, and the corollary then follows by recurrence on the number of points of $f^{-1}(y)$, using Proposition 5.6.17. □

5.6.3 Integral closure of a scheme

Proposition 5.6.19. *Let $(X, \mathscr{A})$ be a ringed space, $\mathscr{B}$ be an $\mathscr{A}$-algebra, and f be a section of $\mathscr{B}$ over X. The following properties are equivalent:*

- *(i) The sub-$\mathscr{A}$-algebra of $\mathscr{B}$ generated by f is finite (that is, of finite type as an $\mathscr{A}$-module).*
- *(ii) There exists a sub-$\mathscr{A}$-algebra $\mathscr{C}$ of $\mathscr{B}$, which is an $\mathscr{A}$-module of finite type, such that $f \in \Gamma(X, \mathscr{C})$.*
- *(iii) For any $x \in X$, f_x is integral over the fiber $\mathscr{A}_x$.*

*If these equivalent conditions are satisfied, the section f is said to be **integral** over $\mathscr{A}$.*

Proof. As the sub-$\mathscr{A}$-module of $\mathscr{B}$ genearted by f^n is an $\mathscr{A}$-algebra, it is clear that (i) implies (ii). On the other hand, (ii) implies that for any $x \in X$, the $\mathscr{A}_x$-module $\mathscr{C}_x$ is of finite type, which implies that any element of the algebra $\mathscr{C}_x$, and in particular f_x, is integral over $\mathscr{A}_x$. Finally, if for any point $x \in X$, we have a relation

$$f_x^n + (a_1)_x f_x^{n-1} + \cdots + (a_n)_x = 0$$

where a_i are sections of $\mathscr{A}$ over an open neighborhood U of x, the section $f^n|_U + a_1 \cdot f^{n-1}|_U + \cdots + a_n$ is zero over an open neighborhood $V \subseteq U$ of x, so $f^k|_V$ (for $k \geq 0$) is a linear combination over $\Gamma(V, \mathscr{A})$ of $f^j|_V$ with $0 \leq j \leq n-1$. We then conclude that (iii) implies (i). □

Corollary 5.6.20. *Under the hypothesis of Proposition 5.6.19, there exists a (unique) sub-$\mathscr{A}$-algebra $\mathscr{A}'$ of $\mathscr{B}$ such that for any $x \in X$, $\mathscr{A}'_x$ is the set of germs $f_x \in \mathscr{B}_x$ that is integral over $\mathscr{A}_x$. For any open* *subset $U \subseteq X$, the sections of $\mathscr{A}'$ over U is the sections of $\Gamma(U, \mathscr{B})$ that is integral over $\mathscr{A}|_U$. We say that $\mathscr{A}'$ is the **integral closure** of $\mathscr{A}$ in $\mathscr{B}$.*

Proof. The existence of $\mathscr{A}'$ is immediate, by setting $\Gamma(U, \mathscr{A}')$ to be the set of $f \in \Gamma(U, \mathscr{B})$ such that f_x is integral over $\mathscr{A}_x$ for any $x \in U$. It is clear that $\mathscr{A}'$ is an algebra, and the second assertion follows from Proposition 5.6.19. ◻

Let $(X, \mathscr{A})$, $(Y, \mathscr{B})$ be two ringed space and $f: X \to Y$ be a morphism. Let $\mathscr{C}$ (resp. $\mathscr{D}$) be an $\mathscr{A}$-algebra (resp. $\mathscr{B}$-algebra) and let $u: \mathscr{D} \to \mathscr{C}$ be a f-morphism. Then, if $\mathscr{A}'$ (resp. $\mathscr{B}'$) is the integral closure of $\mathscr{A}$ (resp. $\mathscr{B}$) in $\mathscr{C}$ (resp. $\mathscr{D}$), the restriction of u to $\mathscr{B}'$ is then a f-morphism $u': \mathscr{B}' \to \mathscr{A}'$. In fact, if j is the canonical injection $\mathscr{B}' \to \mathscr{D}$, it suffices to show that

$$v = u^\sharp \circ f^*(j) : f^*(\mathscr{B}') \to \mathscr{C}'$$

maps $f^*(\mathscr{B}')$ into $\mathscr{A}'$. Now an element of $(f^*(\mathscr{B}'))_x = \mathscr{B}'_{f(x)} \otimes_{\mathscr{B}_{(f(x))}} \mathscr{A}_x$ is integral over $\mathscr{A}_x$ by the definition of $\mathscr{B}'$, and hence so is its image under v_x, which proves our assertion.

Proposition 5.6.21. *Let X be a scheme and $\mathscr{A}$ be a quasi-coherent $\mathscr{O}_X$-algebra. The integral closure $\mathscr{O}'_X$ of $\mathscr{O}_X$ in $\mathscr{A}$ is then a quasi-coherent $\mathscr{O}_X$-algebra, and for any affine open U of X, $\Gamma(X, \mathscr{O}'_X)$ is the integral closure of $\Gamma(U, \mathscr{O}_X)$ in $\Gamma(U, \mathscr{A})$.*

Proof. We can assume that $X = \mathrm{Spec}(B)$ is affine and $\mathscr{A} = \widetilde{A}$, where A is an B-algebra. Let B' be the integral closure of B in A. It then boils down to seeing that for any $x \in X$, an element of A_x, integer over B_x, necessarily belongs to B'_x, which follows from the fact that taking integral closure commutes with localization (Proposition **??**). ◻

Under the hypothesis of Proposition 5.6.21, the X-scheme $X' = \mathrm{Spec}(\mathscr{O}'_X)$ is then called the **integral closure of X relative to** $\mathscr{A}$. We also deduce from Proposition 5.6.21 that if $f: X' \to X$ is the structural morphism, then for any open subset U of X, $f^{-1}(U)$ is the integral closure of the induced subscheme U by X, relative to $\mathscr{A}|_U$. In particular, we conclude that f is integral.

Let X and Y be schemes, $f: X \to Y$ be a morphism, $\mathscr{A}$ (resp. $\mathscr{B}$) be a quasi-coherent $\mathscr{O}_X$-algebra (resp. a $\mathscr{O}_Y$-algebra), and $u: \mathscr{B} \to \mathscr{A}$ be an f-morphism. We have seen that we have an induced f-morphism $u': \mathscr{O}'_Y \to \mathscr{O}'_X$, where $\mathscr{O}'_X$ (resp. $\mathscr{O}'_Y$) is the integral closure of $\mathscr{O}_X$ (resp. $\mathscr{O}_Y$) relative to $\mathscr{A}$ (resp. $\mathscr{B}$), we deduce a canonical morphism $f': \mathrm{Spec}(u'): X' \to Y'$ (Corollary 5.1.10) fitting into the following commutative diagram

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d] & Y' \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd} \tag{6.3.1}$$

Suppose that X has only finitely many irreducible components $(X_i)_{1 \le i \le r}$, with generic points $(\xi_i)_{1 \le i \le r}$, and consider in particular the integral closure of X relative to a quasi-coherent $\mathscr{K}(X)$-algebra $\mathscr{A}$. By Corollary 4.7.19 and Proposition 4.7.20, $\mathscr{A}$ is the direct product of r quasi-coherent $\mathscr{O}_X$-algebras $\mathscr{A}_i$, the support of $\mathscr{A}_i$ being contained in X_i, and the induced sheaf of $\mathscr{A}_i$ over X_i is the constant sheaf whose fiber A_i is an algebra over $\mathscr{O}_{X,\xi_i}$. It is clear that the integral closure $\mathscr{O}'_X$ of $\mathscr{O}_X$ is the direct product of the integral closures $\mathscr{O}_X^{(i)}$ of $\mathscr{O}_X$ in each $\mathscr{A}_i$, and therefore the integral closure $X' = \operatorname{Spec}(\mathscr{O}'_X)$ of X relative to $\mathscr{A}$ is an X-scheme which is the sum of $\operatorname{Spec}(\mathscr{O}_X^{(i)}) = X'_i$.

Now suppose that the $\mathscr{O}_X$-algebra $\mathscr{A}$ is reduced, or equivalently, each algebra A_i is reduced, and therefore can be considered as an algebra over the field $\kappa(\xi_i)$ (equal to the rational function field of the reduced subscheme X_i of X); then the X'_i is a reduced X-scheme and X' is also the integral closure of X_{red}. Suppose moreover that the algebras A_i is a direct product of finitely many field K_{ij} $(1 \le j \le s_i)$; if $\mathscr{K}_{ij}$ is the subalgebra of $\mathscr{A}_i$ corresponding to K_{ij}, it is clear that $\mathscr{O}_X^{(i)}$ is the direct product of integral closures $\mathscr{O}_X^{(ij)}$ of $\mathscr{O}_X$ in $\mathscr{K}_{ij}$. Therefore, X'_i is the sum of $X'_{ij} = \operatorname{Spec}(\mathscr{O}_X^{(ij)})$. Moreover, under this hypothesis, we have the following:

Proposition 5.6.22. *Each X'_{ij} is an integral and normal X-scheme, and its rational function field $K(X'_{ij})$ is canonically identified with the algebraic closure K'_{ij} of $\kappa(\xi_i)$ in K_{ij}.*

Proof. In view of the preceding remarks, we can assume that X is integral, so $r = 1$, $s_1 = 1$, so that the unique algebra A_1 is a field K; let ξ be the generic point of X, and let $f : X' \to X$ be the structural morphism. For any nonempty affine open U of X, $f^{-1}(U)$ is identified with the integral closure B'_U in the field K of the intagral ring $B_U = \Gamma(U, \mathscr{O}_X)$ (Proposition 5.6.21); as the ring B'_U is integrally closed, so is its localizations, and $f^{-1}(U)$ is by definition an integral and normal scheme. Moreover, as (0) is the unique prime ideal of B'_U lying over the prime ideal (0) of B_U, $f^{-1}(\xi)$ is reduced to a singleton ξ', and $\kappa(\xi')$ is the fraction field K' of B'_U, which is none other than the algebraic closure of $\kappa(\xi)$ in K. Finally, X' is irreducible, because if U runs through the nonempty affine open subsets of X, the $f^{-1}(U)$ constitute an open covering of X' formed by irreducible open subsets; moreover the intersection $f^{-1}(U \cap V)$ two two opens contains ξ', hence nonempty, and we conclude from Proposition **??** that X' is irreducible. ◻

Corollary 5.6.23. *Let X be a reduced scheme with finitely many irreducible components (X_i), and let ξ_i be the generic point of X_i. The integral closure X' of X relative to $\mathscr{K}(X)$ is the sum of r separated X-schemes X'_i which are integral and normal. If $f : X' \to X$ is the structural morphism, $f^{-1}(\xi_i)$ is reduced to the generic point ξ'_i of X'_i and we have $\kappa(\xi'_i) = \kappa(\xi_i)$, which means f is birational.*

Proof. This is a particular case of Proposition 5.6.22 by taking $K'_{ij} = \kappa(\xi_i)$. The rational function field of X'_i (which is $\kappa(\xi'_i)$) is then equal to $\kappa(\xi_i)$, whence our claim. ◻

The integral closure X' of X relative to $\mathscr{K}(X)$ is called that **normalization** of the reduced scheme X. We note that the morphism $f : X' \to X$, being birational and integral, is closed by Proposition 5.6.7, hence surjective (recall that a birational morphism is dominant). For $X' = X$, it is necessary and sufficient that X is normal. If X is an integral scheme, it follows from Corollary 5.6.23 that its normalization X' is integral.

Let X, Y be integral schemes, $f : X \to Y$ be a dominant morphism, $L = K(X)$, $K = K(Y)$ be the rational function field of X and Y. The morphism f corresponds to an injection $K \to L$, and if we identify K (resp. L) with the simple sheaf $\mathscr{K}(Y)$ (resp. $\mathscr{K}(X)$), this injection is an f-morphism. Let K_1 (resp. L_1) be an extension of K (resp. L) and suppose that we are given a monomorphism $K_1 \to L_1$ such that the diagram

$$\begin{tikzcd} K_1 \arrow[r] & L_1 \\ K \arrow[u] \arrow[r] & L \arrow[u] \end{tikzcd}$$

is commutative; if K_1 (resp. L_1) is considered as a simple sheaf over Y (resp. X), hence a $\mathscr{K}(Y)$-algebra (resp. a $\mathscr{K}(X)$-algebra), this signifies that $K_1 \to L_1$ is an f-morphism. Now if X' (resp. Y') is the integral closure of X (resp. Y) relative to L_1 (resp. K_1), X' (resp. Y') is a normal and integral scheme (Proposition 5.6.22) and its rational function field is canonically identified with the algebraic closure L' (resp. K') of L (resp. K) in L_1 (resp. K_1), and there exists a canonical morphism (necessarily dominant) $f' : X' \to Y'$ rendering the diagram (6.3.1). The important case is that $L_1 = L$, K_1 is an extension of K contained in L, and where we suppose that X is integral and normal, hence $X' = X$.

The preceding arguments then show that if X is a normal scheme and Y' is integrally closure of Y relative to a field $K_1 \subseteq L = K(X)$, any dominant morphism $f : X \to Y$ factors into

$$f : X \xrightarrow{f'} Y' \to Y$$

where f' is dominant; if the monomorphism $K_1 \to L$ is fixed, f' is necessarily unique (this can be verified when X and Y are both affine). We then say that given Y, L, and a K-monomorphism $K_1 \to L$, the integral closure Y' of Y relative to K_1 is a universal object.

Remark 5.6.2. Retain the hypothesis of Proposition 5.6.22 and suppose moreover that each algebra A_i is of finite dimension over $\kappa(\xi_i)$ (which implies that A_i is a direct product of finitely many fields); we can prove that the structural morphism $X' \to X$ is finite. For this, we can reduce to the case where X is reduced and affine with ring C, and that C has finitely many minimal prime ideals $\mathfrak{p}_i$ ($1 \le i \le r$) with $C_i = C/\mathfrak{p}_i$. Then by Proposition 5.6.21 X' is finite over X if the integral closure of each C_i in finite extension of its fraction field is a finitely generated C-module, or equivalently, if C_i is Japanese for each i. We know that this condition is true if C is an algebra of finite type over a field, or over $\mathbb{Z}$, or over a complete Noetherian local ring. We then conclude that $X' \to X$ is a finite morphism if X is a scheme of finite type over a field, or over $\mathbb{Z}$, or over a complete Noetherian local ring.

5.6.4 Determinant of an endomorphism of $\mathscr{O}_X$-modules

Let A be a ring, E be a free A-module of rank n, and $u : E \to E$ be an endomorphism of E; recall that in order to define the characteristic polynomial of u, we consider the endomorphism $u \otimes 1$ of free the $A[T]$-module $E \otimes_A A[T]$ (which is of rank n), and we put

$$P(u,T) = \det(T \cdot I - (u \otimes 1))$$

(I is the identity morphism on $E \otimes_A A[T]$). We have

$$P(u,T) = T^n - \sigma_1(u)T^{n-1} + \cdots + (-1)^n \sigma_n(u)$$

where $\sigma_i(u)$ is an element of A, equal to a homogeneous polynomial of degree i (with integer coefficients) with entries the elements of the matrix of u relative to any basis of E. We say that the $\sigma_i(u)$ are the **elementray symmetric functions** of u, and we have in particular $\sigma_1(u) = \mathrm{tr}(u)$ and $\sigma_n(u) = \det(u)$. By Hamilton-Cayley theorem, we have

$$P(u,u) = u^n - \sigma_1(u)u^{n-1} + \cdots + (-1)^n\sigma_n(u) = 0 \tag{6.4.1}$$

which can also be written as

$$(\det(u)) \cdot 1_E = uQ(u) \tag{6.4.2}$$

(1_E is the identity morphism on E), where

$$Q(u) = (-1)^{n+1}(u^{n-1} - \sigma_1(u)u^{n-2} + \cdots + (-1)^{n-1}\sigma_{n-1}(u)) \tag{6.4.3}$$

Let $\varphi : A \to B$ be a homomorphism of rings; consider the B-module $E_{(B)} = E \otimes_A B$ which is free of rank n, and the extension $u \otimes 1$ of u to an endomorphism on $E_{(B)}$. It is immediate that we have $\sigma_i(u \otimes 1) = \varphi(\sigma_i(u))$ for all i.

Suppose now that A is an integral domain, with fraction field K, and E is a finitely generated A-module (not necessarily free now). Let n be the rank of E, which equals to the dimension of $E \otimes_A K$ over K. Any endomorphism u of E corresponds canonically to the endomorphism $u \otimes 1$ of $E \otimes_A K$. By abuse of language, we call $P(u \otimes 1, T)$ the characteristic polynomial of u and denoted by $P(u,T)$, and the coefficients $\sigma_i(u \otimes 1)$ is called the elementray symmetric functions of u and denoted by $\sigma_i(u)$. In particular the determinant $\det(u) = \det(u \otimes 1)$ is defiend. With these notations, the formulas (6.4.1) and (6.4.2) are meaningful and still valid, if we interpret the u^i as the homomorphism $E \to E \otimes_A K$ whici is the composition of the endomorphism $u^j \otimes 1 = (u \otimes 1)^j$ of $E \otimes_A K$ and the canonical homomorphism $x \mapsto x \otimes 1$.

If F is the torsion module of R and $E_0 = E/F$, we have $u(F) \subseteq F$, hence, by taking quotient, u induces an endomorphism u_0 of E_0; moreover $E \otimes_A K$ is identified with $E_0 \otimes_A K$ and $u \otimes 1$ is identified with $u_0 \otimes 1$, hence $\sigma_i(u) = \sigma_i(u_0)$ for $1 \le i \le n$.

If E is torsion-free, E is identified with a sub-A-module of $E \otimes_A K$, and the relation $u \otimes 1 = 0$ is equivalent to $u = 0$. If E is a free A-module, the two definitions of $\sigma_i(u)$ given above coincide according to the preceding remarks, which justifies the notations adopted. We also note that if E is a torsion module then $E_0 = \{0\}$, the exterior algebra of E_0 is reduced to K and the determinant of the endomorphism u_0 of E_0 is equal to 1.

Proposition 5.6.24. *Let A be an integral domain, E be a finitely generated A-module, u be an endomorphism of u. Then the elementray symmetric functions $\sigma_i(u)$ of u (and in particular* $\det(u)$*) are integral elements of K over A.*

Proof. This is a particular case of Proposition **??**, where we set $B = K$ and note that condition (ii) is satisfied for $M = E$. ☐

Corollary 5.6.25. *Under the hypothesis of Proposition* **??***, if A is normal, the $\sigma_i(u)$ belong to A.*

Proposition 5.6.26. *Let A be an integral domain, E be a finitely generated A-module, of rank n, and u be an endomorphism of E such that the $\sigma_i(u)$ belong to A. For u to be an automorphism of E, it is necessary that* $\det(u)$ *is invertible in A; this condition is sufficient if E is torsion free.*

Proof. This conditions is sufficient by (6.4.2) and (6.4.3), if E is torsion free, since E is then a sub-A-module of $E \otimes_A K$, and $(\det(u))^{-1}Q(u)$ is the inverse of u. Conversely, this is necessary, because if u is invertible, it follows from Proposition 5.6.24 that $\det(u^{-1})$ belongs to the integral closure A' of A in K, and is clearly the inverse of $\det(u)$ in A'. If $\det(u)$ is not invertible in A, then it belongs to a maximal ideal $\mathfrak{m}$ of A, which is the contraction of a maximal ideal of A' (Theorem **??**), contradiction. □

We note a generalization of the preceding results. Consider a reduced Noetherian ring A and let $\mathfrak{p}_i$ $(1 \leq i \leq r)$ be the minimal prime ideals of A, and K_i be the fractional field of $A_i = A/\mathfrak{p}_i$. Then the total fraction field K of A is the direct product of the fields K_i (Proposition **??**). Let E be a finitely generated A-module, and suppose that $E \otimes_A K$ is a K-module of dimension n. Then each K_i-vector space $E_i = E \otimes_A K_i$ is of dimension n. If u is an endomorphism of E, we put $P(u,T) = P(u \otimes 1, T)$ and $\sigma_j(u) = \sigma_j(u \otimes 1)$, and in particular $\det(u) = \det(u \otimes 1)$; the $\sigma_j(u)$ are then elements of K. It is immediate that $E \otimes_A K$ is a direct sum of E_i and each of them is stable under $u \otimes 1$. The restriction of $u \otimes 1$ to E_i is just the extension of u to E_i, and we conclude that $\sigma_j(u)$ is the element of K with component in K_i being $\sigma_j(u_i)$. As the integral closure of A in K is the direct product of that of A in K_i (Proposition **??**), the $\sigma_j(u)$ are integral over A.

Lemma 5.6.27. *The sub-A-algebra of K genrated by the elements $\sigma_j(u)$ $(1 \leq j \leq n)$ for $u \in \mathrm{Hom}_A(E,E)$, is a finitely generated A-module.*

Let $(X, \mathscr{A})$ be a ringed space, $\mathscr{E}$ be a locally free $\mathscr{A}$-module (of finite rank). There is then by hypothesis a basis $\mathfrak{B}$ of X such that for any $V \in \mathfrak{B}$, $\mathscr{E}|_V$ is isomorphic to $\mathscr{A}^n|_V$ (the integer n may vary with V). Let u be an endomorphism of $\mathscr{E}$; for any $V \in \mathfrak{B}$, u_V is then an endomorphism of the $\Gamma(V,\mathscr{A})$-module $\Gamma(V,\mathscr{E})$, which is free by hypothesis; the determinant of u_V is then defined and belongs to $\Gamma(V,\mathscr{A})$. Moreover, if $e_1,\dots,e_n$ is a basis of $\Gamma(V,\mathscr{E})$, their restriction to any open subset $W \subseteq V$ form a basis of $\Gamma(W,\mathscr{E})$ over $\Gamma(W,\mathscr{A})$, so $\det(u_W)$ is the restriction of $\det(u_V)$ to W. There then exists a unique section of $\mathscr{A}$ over X, which we denote by $\det(u)$ and call the **determinant** of u, such that the restriction of $\det(u)$ to any $V \in \mathfrak{B}$ is $\det(u_V)$. It is clear that for any $x \in X$, we have $\det(u)_x = \det(u_x)$; for two endomorphisms u, v of $\mathscr{E}$, we have

$$\det(u \circ v) = (\det(u))(\det(v)), \quad \det(1_{\mathscr{E}}) = 1_{\mathscr{A}}.$$

If $\mathscr{E}$ is of rank n (for example if X is connected), we have

$$\det(s \cdot u) = s^n \det(u)$$

for any $s \in \Gamma(X,\mathscr{A})$ (we note that $\det(0) = 0_{\mathscr{A}}$ if $n \geq 1$, but $\det(0) = 1_{\mathscr{A}}$ if $n = 0$). Moreover, for u to be an automorphism of $\mathscr{E}$, it is necessary and sufficient that $\det(u)$ is invertible in $\Gamma(X,\mathscr{A})$ (Proposition 5.6.26).

If $\mathscr{E}$ is of rank n, we can similarly define the elementary symmetric functions $\sigma_i(u)$ for u, which are elements of $\Gamma(X,\mathscr{A})$, and we also have the relations (6.4.2) and (6.4.3).

We have then define a homomorphism $\det : \mathrm{Hom}_{\mathscr{A}}(\mathscr{E},\mathscr{E}) \to \Gamma(X,\mathscr{A})$ of multiplicative monoids. Note that $\mathrm{Hom}_{\mathscr{A}}(\mathscr{E},\mathscr{E}) = \Gamma(X,\mathscr{H}om_{\mathscr{A}}(\mathscr{E},\mathscr{E}))$ by definition, so we can replace X by any open subset U in this definition of det, and therefore obtain a homomorphism $\det : \mathscr{H}om_{\mathscr{A}}(\mathscr{E},\mathscr{E}) \to \mathscr{A}$ of sheaves of multiplicative monoids. If $\mathscr{E}$ is of constant rank, we can similarly define the homomorphisms $\sigma_i : \mathscr{H}om_{\mathscr{A}}(\mathscr{E},\mathscr{E}) \to \mathscr{A}$ of sheaves of sets; for $i = 1$, the homomorphism $\sigma_1 = \mathrm{tr}$ is a homomorphism of $\mathscr{A}$-modules.

Let $(Y,\mathscr{B})$ be a second ringed space and $f : (X,\mathscr{A}) \to (Y,\mathscr{B})$ be a morphism of ringed spaces; if $\mathscr{F}$ is a locally free $\mathscr{B}$-module, $f^*(\mathscr{F})$ is a locally free $\mathscr{A}$-module (with the same rank of $\mathscr{F}$). For any endomorphism v of $\mathscr{F}$, $f^*(v)$ is then an endomorphism of $f^*(\mathscr{F})$, and it follows from these definitions that $\det(f^*(v))$ is the section of $\mathscr{A} = f^*(\mathscr{B})$ over X which corresponds canonically to $\det(v) \in \Gamma(Y,\mathscr{B})$. We can then say that the homomorphism $f^*(\det) : f^*(\mathscr{H}om_{\mathscr{B}}(\mathscr{F},\mathscr{F})) \to f^*(\mathscr{B}) = \mathscr{A}$ is the composition

$$f^*(\mathscr{H}om_{\mathscr{B}}(\mathscr{F},\mathscr{F})) \xrightarrow{\gamma^\sharp} \mathscr{H}om_{\mathscr{A}}(f^*(\mathscr{F}),f^*(\mathscr{F})) \xrightarrow{\det} \mathscr{A} \tag{6.4.4}$$

(formula (3.4.5)). We have a similar result for σ_i.

Suppose now that X is a locally integral scheme, so its sheaf of rational function $\mathscr{K}(X)$ is locally simple over X (Corollary 4.7.19) and quasi-coherent as $\mathscr{O}_X$-module. If $\mathscr{E}$ is a qausi-coherent $\mathscr{O}_X$-module of finite type, $\mathscr{E}' = \mathscr{E} \otimes_{\mathscr{O}_X} \mathscr{K}(X)$ is then a locally free $\mathscr{K}(X)$-module (Corollary 4.7.21). For any endomorphism u of $\mathscr{E}$, $u \otimes 1_{\mathscr{K}(X)}$ is then an endomorphism of $\mathscr{E}'$, and $\det(u \otimes 1)$ is a section of $\mathscr{K}(X)$ over X, which is called the **determinant** of u and denoted by $\det(u)$. It follows from Proposition 5.6.24 that $\det(u)$ is a section of the integral closure of $\mathscr{O}_X$ in $\mathscr{K}(X)$; if X is also normal, $\det(u)$ is then a section of $\mathscr{O}_X$ over X, and if we suppose moreover that $\mathscr{E}$ is torsion free, for u to be an automorphism of $\mathscr{E}$, it is necessary and sufficient that $\det(u)$ is invertible (Proposition 5.6.26). The formulae (6.4.2) and (6.4.3) are still valid; the homomorphism $u \mapsto \det(u)$ then defines a homomorphism $\det : \mathscr{H}om_{\mathscr{O}_X}(\mathscr{E},\mathscr{E}) \to \mathscr{K}(X)$, which has values in $\mathscr{O}_X$ if X is normal. We have analogous results for the elementary symmetric function functions $\sigma_j(u)$, if $\mathscr{E}'$ has constant rank; if moreover X is normal, the $\sigma_j(u)$ are sections of $\mathscr{O}_X$ over X.

Finally, let X and Y be integral schemes, and $f : X \to Y$ be a dominant morphism. We see that there exists a canonical homomorphism $f^*(\mathscr{K}(Y)) \to \mathscr{K}(X)$, whence induces, for any quasi-coherent $\mathscr{O}_Y$-module $\mathscr{F}$ of finite type, a canonical homomorphism $\theta : f^*(\mathscr{F} \otimes_{\mathscr{O}_Y} \mathscr{K}(Y)) \to f^*(\mathscr{F}) \otimes_{\mathscr{O}_X} \mathscr{K}(X)$. If v is an endomorphism of $\mathscr{F}$, $f^*(v \otimes 1_{\mathscr{K}(Y)})$ is an endomorphism of $f^*(\mathscr{F} \otimes_{\mathscr{O}_Y} \mathscr{K}(Y))$, and we have a commutative diagram

$$\begin{tikzcd}
f^*(\mathscr{F} \otimes_{\mathscr{O}_Y} \mathscr{K}(Y)) \arrow[r, "f^*(v\otimes 1)"] \arrow[d, "\theta"'] & f^*(\mathscr{F} \otimes_{\mathscr{O}_Y} \mathscr{K}(Y)) \arrow[d, "\theta"] \\
f^*(\mathscr{F}) \otimes_{\mathscr{O}_X} \mathscr{K}(X) \arrow[r, "f^*(v)\otimes 1"] & f^*(\mathscr{F}) \otimes_{\mathscr{O}_X} \mathscr{K}(X)
\end{tikzcd}$$

We then conclude that $\det(f^*(v))$ is the canonical image of the section $\det(v)$ of $\mathscr{K}(Y)$ under the canonical homomorphism $f^*(\mathscr{K}(Y)) \to \mathscr{K}(X)$. In fact, it is immediate that we are reduced to the case where $X = \mathrm{Spec}(B)$ and $Y = \mathrm{Spec}(A)$ are affine, A, B being integral domains with fraction fields K, L respectively, the hommorphism $A \to B$ being injective and extends to a monomorphism $K \to L$. If $\mathscr{F} = \widetilde{M}$ where M is a finitely generated A-module, the dimension of

$M \otimes_A K$ is equal to that of $(M \otimes_A B) \otimes_B L$ over L, and $\det((u \otimes 1) \otimes 1)$ is the image of $\det(u \otimes 1)$ in L for any endomorphism u of M, whence our conclusion.

Finally, suppose that X is a reduced locally Noetherian scheme, whose sheaf of rational functions $\mathscr{K}(X)$ is quasi-coherent by Corollary 4.7.22. Let $\mathscr{E}$ be a coherent $\mathscr{O}_X$-module such that $\mathscr{E}' = \mathscr{E} \otimes_{\mathscr{O}_X} \mathscr{K}(X)$ is locally free of rank n. We can then define for each endomorphism u of $\mathscr{E}$ the elementary symmetric functions $\sigma_j(u)$, which are sections of $\mathscr{K}(X)$ over X.

5.6.5 Norm of an invertible sheaf

Let $(X, \mathscr{A})$ be a ringed space and $\mathscr{B}$ be an $\mathscr{A}$-algebra. The $\mathscr{A}$-module $\mathscr{B}$ is canonically identified with a sub-$\mathscr{A}$-module of $\mathscr{H}om_{\mathscr{A}}(\mathscr{B}, \mathscr{B})$, where a section f of $\mathscr{B}$ over an open subset U of X is identified with the multiplication by this section. Assume that $(X, \mathscr{A})$ and $\mathscr{B}$ satisfies the conditions given in the previous subsection, so that we can define $\det(f)$ (resp. $\sigma_j(f)$) to be a section of $\mathscr{K}(X)$ over U, which is called the **norm** of f (resp. the elementary symmetric functions) of f and denoted by $N_{\mathscr{B}/\mathscr{A}}(f)$. We suppose that one of the following conditions is satisfied:

(α) $\mathscr{B}$ is a locally free $\mathscr{A}$-module of finite rank n.

(β) $(X, \mathscr{A})$ is a reduced locally Noetherian scheme, $\mathscr{B}$ is a coherent $\mathscr{A}$-module such that $\mathscr{B} \otimes_{\mathscr{A}} \mathscr{K}(X)$ is a locally free $\mathscr{K}(X)$-module of rank n, and for any section $f \in \Gamma(U, \mathscr{B})$ over an open subset $U \subseteq X$, $\sigma_j(f)$ $(1 \leq j \leq n)$ is a section of $\mathscr{A}$ over U (this is ture for example if X is normal).

The hypothesis that $\mathscr{B} \otimes_{\mathscr{A}} \mathscr{K}(X)$ is locally free can be expressed by the following: denote by X_i the reduced closed subschemes of X with underlying space the irreducible components of X, which are then locally Noetherian integral schemes. Any $x \in X$ belongs to finitely many X_i, and $\mathscr{B} \otimes_{\mathscr{A}} \mathscr{K}(X_i)$ is a locally free $\mathscr{K}(X_i)$-module of constant rank k_i (Proposition 4.7.20); to say that $\mathscr{B} \otimes_{\mathscr{A}} \mathscr{K}(X)$ is locally free $\mathscr{K}(X)$-module signifies that, for any $x \in X$, the ranks k_i such that $x \in X_i$ are all equal. This question is in fact local, and we can assume that $X = \operatorname{Spec}(A)$, where A is a reduced Noetherian ring, and $\mathscr{B} = \widetilde{B}$ where B is a finite A-algebra. If $\mathfrak{p}_i$ $(1 \leq i \leq r)$ are the minimal prime ideals of A, the total fraction ring K of A is then the direct product of K_i, where K_i is the fraction field of $A_i = A/\mathfrak{p}_i$, and $B \otimes_A K$ is then the direct sum of $B \otimes_A K_i$, whence our conclusion.

It is clear that under the hypotheses (α) or (β), we then define a homomorphism $N_{\mathscr{B}/\mathscr{A}} : \mathscr{B} \to \mathscr{A}$ of sheaves of multiplicative monoids, which is also denoted by N if there is no confusion, and called the norm homomorphism. For two sections f, g of $\mathscr{B}$ over an open subset U, we then have

$$N_{\mathscr{B}/\mathscr{A}}(fg) = N_{\mathscr{B}/\mathscr{A}}(f) N_{\mathscr{B}/\mathscr{A}}(g), \quad N_{\mathscr{B}/\mathscr{A}}(1_{\mathscr{B}}) = 1_{\mathscr{A}} \tag{6.5.1}$$

for the corresponding sections of $\mathscr{A}$ over U. Also, for any section s of $\mathscr{A}$ over U, we have

$$N_{\mathscr{B}/\mathscr{A}}(s \cdot 1_{\mathscr{B}}) = s^n. \tag{6.5.2}$$

In case (α), for any $f \in \Gamma(U, \mathscr{B})$ to be invertible, it is necessary and sufficient that $N(f) \in \Gamma(U, \mathscr{A})$ is invertiblel; in case (β), this condition is necessary, and is sufficient if $\mathscr{B}$ is a torsion free $\mathscr{A}$-module.

Suppose the one of the hypotheses (α), (β) is satisfied, and let $\mathscr{L}'$ be an invertible $\mathscr{B}$-module. We can canonically associate an invertible $\mathscr{A}$-module by the following. Denote by $\mathscr{A}^\times$ (resp. $\mathscr{B}^\times$) the subsheaf of $\mathscr{A}$ (resp. $\mathscr{B}$) such that $\Gamma(U, \mathscr{A}^\times)$ (resp. $\Gamma(U, \mathscr{B}^\times)$) is the set of invertible elements of $\Gamma(U, \mathscr{A})$ (resp. $\Gamma(U, \mathscr{B})$) for any open subset $U \subseteq X$; this is a sheaf of multiplicative groups, and $N_{\mathscr{B}/\mathscr{A}}$, restricted to $\mathscr{B}^\times$, is a homomorphism $\mathscr{B}^\times \to \mathscr{A}^\times$ of sheaves of groups. Let $\mathfrak{L}$ be the set of couples $(U_\lambda, \eta_\lambda)$, with the following property: U_λ is an open subset of X and $\eta_\lambda : \mathscr{L}'|_{U_\lambda} \xrightarrow{\sim} \mathscr{B}|_{U_\lambda}$ is an isomorphism of $(B|_{U_\lambda})$-modules. By hypothesis, the U_λ for an open covering og X; for two indices λ, μ, we put $\omega_{\lambda\mu} = (\eta_\lambda|_{U_\lambda \cap U_\mu}) \circ (\eta_\mu|_{U_\lambda \cap U_\mu})^{-1}$, which is an automorphism of $\mathscr{B}|_{U_\lambda \cap U_\mu}$, and canonically identified with a section of $\mathscr{B}^\times$ over $U_\lambda \cap U_\mu$, and $(\omega_{\lambda\mu})$ is a 1-cocycle over the covering $\mathfrak{U} = (U_\lambda)$ with values in $\mathscr{B}^\times$. The fact that $N_{\mathscr{B}/\mathscr{A}} : \mathscr{B}^\times \to \mathscr{A}^\times$ is a homomorphism implies that $(N_{\mathscr{B}/\mathscr{A}}(\omega_{\lambda\mu}))$ is a 1-cocycle of $\mathfrak{U}$ with values in $\mathscr{A}^\times$, which then corresponds (up to isomorphism) to an invertible $\mathscr{A}$-module. This invertible $\mathscr{A}$-module is denoted by $N_{\mathscr{B}/\mathscr{A}}(\mathscr{L}')$ and is called the norm of the invertible $\mathscr{B}$-module $\mathscr{L}'$.

Let $\mathfrak{M}$ be a subset of $\mathfrak{L}$ such that the U_λ form an open covering of X, and let $\mathfrak{V}$ be a covering of X. The restriction of the cocycle $(\omega_{\lambda\mu})$ to $\mathfrak{V}$ defines a 1-cocycle $(N_{\mathscr{B}/\mathscr{A}}(\omega_{\lambda\mu}))$, which is the restriction of the 1-cocycle $(N_{\mathscr{B}/\mathscr{A}}(\omega_{\lambda\mu}))$ to $\mathfrak{U}$; it is clear that there is a canonical isomorphism of the invertible $\mathscr{A}$-modules thus defined, and we can therefore define $N_{\mathscr{B}/\mathscr{A}}(\mathscr{L}')$ by a refinement of the covering $\mathfrak{U}$. This shows that, if $\mathscr{L}'$ and $\mathscr{K}'$ are two invertible $\mathscr{B}$-modules, by (6.5.1) we have

$$N(\mathscr{L}' \otimes_{\mathscr{B}} \mathscr{K}') = N(\mathscr{L}') \otimes_{\mathscr{A}} N(\mathscr{K}'), \quad N_{\mathscr{B}/\mathscr{A}}(\mathscr{B}) = \mathscr{A}, \tag{6.5.3}$$

and therefore $N(\mathscr{L}'^{-1}) = N(\mathscr{L}')^{-1}$. Also, it follows from (6.5.2) that if $\mathscr{L}$ is an invertible $\mathscr{A}$-module, we have

$$N_{\mathscr{B}/\mathscr{A}}(\mathscr{L} \otimes_{\mathscr{A}} \mathscr{B}) = \mathscr{L}^{\otimes n}. \tag{6.5.4}$$

We show that $N_{\mathscr{B}/\mathscr{A}}$ is a covariant functor on the category of invertible $\mathscr{B}$-modules. Let $u' : \mathscr{L}' \to \mathscr{K}'$ be a homomorphism of invertible $\mathscr{B}$-modules, and let $\mathfrak{V} = (U_\lambda)$ be an open covering of X such that for any λ, we have an isomorphism $\eta_\lambda : \mathscr{L}'|_{U_\lambda} \xrightarrow{\sim} \mathscr{B}|_{U_\lambda}$ and $\tau_\lambda : \mathscr{K}'|_{U_\lambda} \xrightarrow{\sim} \mathscr{B}|_{U_\lambda}$; there is then for each λ an endomorphism u'_λ of $\mathscr{B}|_{U_\lambda}$ such that $u'_\lambda \circ \eta_\lambda = \tau_\lambda \circ (u'|_{U_\lambda})$, and we can evidently identify u'_λ with a section of $\mathscr{B}$ over U_λ. Hence, for any couple (λ, μ) of indices, the restriction of $(\tau_\lambda)^{-1} \circ u'_\lambda \circ \eta_\lambda$ and $(\tau_\mu)^{-1} \circ u'_\mu \circ \eta_\mu$ to $U_\lambda \cap U_\mu$ coincide. We then deduce for the 1-cocycle $(\omega_{\lambda\mu})$ corresponding to $\mathscr{L}'$ and the 1-cocycle $(\gamma_{\lambda\mu})$ corresponding to $\mathscr{K}'$ the relation

$$\gamma_{\lambda\mu} u'_\mu = u'_\lambda \omega_{\lambda\mu}.$$

If we put $u_\lambda = N(u'_\lambda)$, we then have the analogous relation

$$N(\gamma_{\lambda\mu}) u_\mu = u_\lambda N(\omega_{\lambda\mu})$$

and therefore the u_λ define a homomorphism $N(\mathscr{L}') \to N(\mathscr{K}')$, which is denoted by $N_{\mathscr{B}/\mathscr{A}}(u)$ or $N(u)$. In view of Proposition 5.6.26, it is clear that under the hypothesis (α), u' is an isomorphism if and only if u is, and this is ture under the hypothesis (β) if $\mathscr{B}$ is moreover torsion free. In particular, if consider the hommorphisms $\mathscr{B} \to \mathscr{L}'$, which correspond to global sections of $\mathscr{L}'$, since $N_{\mathscr{B}/\mathscr{A}}(\mathscr{B}) = \mathscr{A}$, we get a canonical homomorphism

$$N_{\mathscr{B}/\mathscr{A}} : \Gamma(X, \mathscr{L}') \to \Gamma(X, N_{\mathscr{B}/\mathscr{A}}(\mathscr{L}')).$$

It also follows from (6.5.1) that if $f' \in \Gamma(X, \mathscr{L}')$, $g' \in \Gamma(X, \mathscr{K}')$, we have

$$N(f' \otimes g') = N(f') \otimes N(g'). \tag{6.5.5}$$

Also, for any invertible $\mathscr{A}$-module $\mathscr{L}$ and any section $f \in \Gamma(X, \mathscr{L})$, we have

$$N_{\mathscr{B}/\mathscr{A}}(f \otimes 1_{\mathscr{B}}) = f^{\otimes n}. \tag{6.5.6}$$

Finally, for the homomorphism $\mathscr{B} \to \mathscr{L}'$ corresponding to a section f' of $\mathscr{L}'$ over X to be an isomorphism, it is necessary and sufficient that f'_x generates $\mathscr{L}'_x$ for any $x \in X$; under condition (α), this is equivalent to that $N(f')_x$ generates $(N(\mathscr{L}'))_x$ for any x, and this is ture for condition (β) if $\mathscr{B}$ is torsion free.

Let $(X, \mathscr{A})$, $(X', \mathscr{A}')$ be two ringed spaces and $\varphi : X' \to X$ be a morphism, $\mathscr{B}$ be an $\mathscr{A}$-algebra, and $\mathscr{B}' = \varphi^*(\mathscr{B})$. Suppose that one of the following conditions is satisfied:

(i) $\mathscr{B}$ satisfies condition (α).

(ii) $(X, \mathscr{A})$ and $\mathscr{B}$ satisfy condition (β), $(X', \mathscr{A}')$ is a reduced locally Noehterian scheme, and if we denote by X_α and X'_β the reduced closed subschemes of X and X' with underlying space the irreducible components of these spaces, the restriction of φ to X'_β is a dominant morphism from X'_β to X_α.

Under these conditions, we claim that $\mathscr{B}'$ verifies the conditions (α) or (β); the first case is clear, and to prove the second one, it suffices to prove that for any $x' \in X'$, the ranks of $\mathscr{B}' \otimes_{\mathscr{O}_{X'}} \mathscr{K}(X'_\beta)$ for the indices β such that $x' \in X'_\beta$ are the same. Now, if the restriction of φ to X'_β is a dominant morphism into X_α, the rank of $\mathscr{B}' \otimes_{\mathscr{O}_{X'}} \mathscr{K}(X'_\beta)$ is equal to that of $\mathscr{B} \otimes_{\mathscr{O}_X} \mathscr{K}(X_\alpha)$ (which can be seen from the affine case), whence our claim.

This being established, it follows that if f is a section of $\mathscr{B}$ over an open subset $U \subseteq X$, and f' is the inverse image of f under φ, $N_{\mathscr{B}'/\mathscr{A}'}(f')$ is the section of $\mathscr{A}'$ over $\varphi^{-1}(U)$ which is the inverse image of $N_{\mathscr{B}/\mathscr{A}}(f)$ under φ. If $\mathscr{L}$ is an invertible $\mathscr{B}$-module and if $\mathscr{L}' = \varphi^*(\mathscr{L})$ (which is an invertible $\mathscr{B}'$-module), we have

$$N_{\mathscr{B}'/\mathscr{A}'}(\mathscr{L}') = \varphi^*(N_{\mathscr{B}/\mathscr{A}}(\mathscr{L})). \tag{6.5.7}$$

Suppose now that $(X, \mathscr{A})$ is a scheme. Then giving a quasi-coherent finite $\mathscr{A}$-algebra $\mathscr{B}$ is equivalent to giving a finite morphism $\varphi : X' \to X$ such that $\varphi_*(\mathscr{O}_{X'}) = \mathscr{B}$, defined up to X-isomorphisms (Corollary 5.1.10), and in this case X' is isomorphic to the affine specturm $\operatorname{Spec}(\mathscr{B})$. Moreover, if this morphism $\varphi : X' \to X$ is fixed, then giving a quasi-coherent

$\mathscr{O}_{X'}$-module $\mathscr{F}'$ is equivalent to giving a quasi-coherent $\mathscr{B}$-module such that $\varphi_*(\mathscr{F}') = \mathscr{F}$ (Proposition 5.1.18), and for $\mathscr{F}'$ to be invertible, it is necessary and sufficient that $\mathscr{F}$ is (Proposition 5.6.10). To utilize the preceding results for the finite morphism φ, it is then necessary to assume that $\mathscr{B} = \varphi_*(\mathscr{O}_{X'})$ satisfies condition (α) or (β). For any invertible $\mathscr{O}_{X'}$-module $\mathscr{L}'$, we then set

$$N_{X'/X}(\mathscr{L}') := N_{\varphi_*(\mathscr{O}_{X'})/\mathscr{O}_X}(\varphi_*(\mathscr{L}')) \tag{6.5.8}$$

which is called the **norm** (relative to φ) of $\mathscr{L}'$. Similarly, if $u' : \mathscr{L}' \to \mathscr{K}'$ is a homomorphism of invertible $\mathscr{O}_{X'}$-modules, we put

$$N_{X'/X}(u') = N_{\varphi_*(\mathscr{O}_X)/\mathscr{O}_X}(\varphi_*(u')) : N_{X'/X}(\mathscr{L}') \to N_{X'/X}(\mathscr{K}'). \tag{6.5.9}$$

In particular, if we consider homomorphisms $\mathscr{O}_{X'} \to \mathscr{L}'$, we obtain a canonical homomorphism

$$N_{X'/X} : \Gamma(X', \mathscr{L}') \to \Gamma(X, N_{X'/X}(\mathscr{L}')). \tag{6.5.10}$$

Proposition 5.6.28. *Let $\varphi : X' \to X$ be a finite morphism and suppose that condition (i) or (ii) is satisfied. For a homomorphism $u' : \mathscr{L}' \to \mathscr{K}'$ of invertible $\mathscr{O}_{X'}$-modules to be an isomorphism, it is necessary and sufficient that, in the first case, that $N_{X'/X}(u')$ is an isomorphism; in the second case, this condition is necessary, and is sufficient if $\varphi_*(\mathscr{O}_{X'})$ is torsion free.*

Proof. This is a particular case of our previous discussions, where we put $\mathscr{B} = \varphi_*(\mathscr{O}_{X'})$, which is a quasi-coherent finite $\mathscr{O}_X$-algebra. We also note that by Corollary 5.1.17, for $\varphi_*(u')$ to be an isomorphism, it is necessary and sufficient that u' is an isomorphism. □

Corollary 5.6.29. *Retain the hypothesis of Proposition 5.6.28 and suppose that $\varphi_*(\mathscr{O}_{X'})$ is torsion free. Let $\mathscr{L}'$ be an invertible $\mathscr{O}_{X'}$-module, f' be a section of $\mathscr{L}'$ over X', and $f = N_{X'/X}(f')$ the section of $\mathscr{L} = N_{X'/X}(\mathscr{L}')$ over X corresponding to f'. Then we have $\varphi(X' - X'_{f'}) = X - X_f$ and X_f is the largest open subset of X such that $\varphi^{-1}(U) \subseteq X'_{f'}$.*

Proof. In fact, $\varphi(X' - X'_{f'})$ is closed in X by Proposition 5.6.7, and it then suffices to prove the second assertion. Now the relation $U \subseteq X_f$ is equivalent to that the hommorphism $\mathscr{O}_X|_U \to \mathscr{L}|_U$ defined by $f|_U$ is an isomorphism. In view of Proposition 5.6.28, this is equivalent to that the homomorphism $\mathscr{O}_{X'}|_{\varphi^{-1}(U)} \to \mathscr{L}'|_{\varphi^{-1}(U)}$ defined by $f'|_{\varphi^{-1}(U)}$ is an isomorphism, which means $\varphi^{-1}(U) \subseteq X'_{f'}$. □

Proposition 5.6.30. *Let $\varphi : X' \to X$ be a finite morphism, $\psi : Y \to X$ be a morphism; let $Y' = X'_{(Y)}$, $\varphi' = \varphi_{(Y)}$, $\psi' = \psi_{(X')}$ such that the following diagram is commutative*

$$\begin{tikzcd} Y' \arrow[r, "\psi'"] \arrow[d, "\varphi'"'] & X' \arrow[d, "\varphi"] \\ Y \arrow[r, "\psi"] & X \end{tikzcd}$$

Assume the hypotheses of Proposition 5.6.28. Then for any invertible $\mathscr{O}_{X'}$-module $\mathscr{L}'$, we have

$$N_{Y'/Y}(\psi'^*(\mathscr{L}')) = \psi^*(N_{X'/X}(\mathscr{L}')).$$

Proof. Note that we have $\psi^*(\varphi_*(\mathscr{L}')) = \varphi'_*(\psi'^*(\mathscr{L}'))$ in view of Corollary 5.1.28, and in particular $\varphi'_*(\mathscr{O}_{Y'}) = \psi^*(\varphi_*(\mathscr{O}_{X'}))$; if $\varphi_*(\mathscr{O}_{X'})$ is locally free, so is $\varphi'_*(\mathscr{O}_{Y'})$. The conclusion then follows from the definition of $N_{X'/X}$, $N_{Y'/Y}$, and (6.5.7). □

5.6.6 Application: a criterion of amplitude

Proposition 5.6.31. *Let $f : X \to Y$ be a quasi-compact morphism, $g : X' \to X$ be a finite and surjective morphism such that $(X, \mathscr{O}_X)$ and $g_*(\mathscr{O}_{X'})$ satisfy condition (β). Then, for an ample invertible $\mathscr{O}_{X'}$-module $\mathscr{L}'$ relative to $f \circ g$, $N_{X'/X}(\mathscr{L}') = \mathscr{L}$ is ample relative to f.*

Proof. We can suppose that Y is affine, and then, in view of Proposition 5.4.37, it suffices to prove that, if $\mathscr{L}'$ is ample, then $\mathscr{L} = N_{X'/X}(\mathscr{L}')$ is ample. For this, we can assume that $g_*(\mathscr{O}_{X'})$ is torsion free. In fact, let $\mathscr{T}$ be the kernel of the homomorphism $g_*(\mathscr{O}_{X'}) \to g_*(\mathscr{O}_{X'}) \otimes_{\mathscr{O}_X} \mathscr{K}(X)$, which is a coherent ideal of $\mathscr{B} = g_*(\mathscr{O}_{X'})$ by hypothesis, and put $X'' = \mathrm{Spec}(\mathscr{B}/\mathscr{T})$; we then have a commutative diagram

$$\begin{tikzcd} X'' \arrow[rr, "j"] \arrow[dr, "g'"'] & & X' \arrow[dl, "g"] \\ & X & \end{tikzcd}$$

where j is a closed immersion (Proposition 5.1.25). Moreover, since $\mathscr{T}$ is a torsion sheaf, by Proposition 4.7.27 and Proposition 1.4.6 we see that the support of $\mathscr{T}$ is a closed subset that is rare in X, so for the generic point x of an irreducible component of X, there exists an affine open neighborhood U of x such that $\mathscr{B}|_U = (\mathscr{B}/\mathscr{T})|_U$. As g is by hypothesis surjective, we then conclude that $x \in g'(X'')$; g' is then dominant, and hence surjective by Proposition 5.6.7 since it is a finite morphism. By definition we have

$$g'_*(\mathscr{O}_{X''}) \otimes \mathscr{K}(X) = (\mathscr{B}/\mathscr{T}) \otimes_{\mathscr{O}_X} \mathscr{K}(X) = g_*(\mathscr{O}_{X'}) \otimes_{\mathscr{O}_X} \mathscr{K}(X),$$

so $(X, \mathscr{O}_X)$ and $g'_*(\mathscr{O}_{X''})$ satisfies condition (β), and $g'_*(\mathscr{O}_{X''})$ is torsion free. Finally, $j^*(\mathscr{L}') = \mathscr{L}''$ is an ample $\mathscr{O}_{X''}$-module (Proposition 5.4.45(ii)), and $N_{X''/X}(\mathscr{L}'') = N_{X'/X}(\mathscr{L}')$. To see this, we note that to define these two invertible $\mathscr{O}_X$-modules we can utilize an affine open covering (U_λ) of X such that $g_*(\mathscr{L}')$ and $g'_*(\mathscr{L}'')$ to U_λ are respectively isomorphic to $\mathscr{B}|_{U_\lambda}$ and $(\mathscr{B}/\mathscr{T})|_{U_\lambda}$. By Corollary 5.1.23, we immediately see that for any isomorphism $\eta_\lambda : g_*(\mathscr{L}')|_{U_\lambda} \to \mathscr{B}|_{U_\lambda}$ corresponds canonically to an isomorphism

$$\eta'_\lambda : g'_*(\mathscr{L}'')|_{U_\lambda} \to (\mathscr{B}/\mathscr{T})|_{U_\lambda}$$

so that, if $(\omega_{\lambda\mu})$ and $(\omega'_{\lambda\mu})$ are the 1-cocycles corresponding to the isomorphisms (η_λ) and (η'_λ), $\omega'_{\lambda\mu}$ is the canonical image of $\omega_{\lambda\mu} \in \Gamma(U_\lambda \cap U_\mu, \mathscr{B})$ to $\Gamma(U_\lambda \cap U_\mu, \mathscr{B}/\mathscr{T})$. In view of the definition of $\mathscr{T}$, we conclude that

$$N_{\mathscr{B}/\mathscr{A}}(\omega_{\lambda\mu}) = N_{(\mathscr{B}/\mathscr{T})/\mathscr{A}}(\omega'_{\lambda\mu})$$

(where $\mathscr{A} = \mathscr{O}_X$), whence the equality.

Suppose then that $g_*(\mathscr{O}_{X'})$ is torsion free. It then suffices to prove that if f runs through the sections of $\mathscr{L}^{\otimes n}$ ($n > 0$) over X, the X_f form a basis of X (Proposition 5.4.23). Now, let $x \in X$, and let U be an open neighborhood of x; as $g^{-1}(x)$ is finite by Corollary 5.6.5 and $\mathscr{L}'$

is ample, there exists an integer $n > 0$ and a section f' of $\mathscr{L}'^{\otimes n}$ over X' such that $X'_{f'}$ is an open neighborhood of $g^{-1}(x)$ contained in $g^{-1}(U)$. As we have $\mathscr{L}^{\otimes n} = N_{X'/X}(\mathscr{L}'^{\otimes n})$, it then suffices to choose $f = N_{X'/X}(f')$: in fact, we have $X - X_f = g(X' - X'_{f'})$ by Corollary 5.6.29, so $x \in X_f \subseteq U$. □

Corollary 5.6.32. *Under the hypotheses of Proposition 5.6.31, for an invertible $\mathscr{O}_X$-module $\mathscr{L}$ to be ample relative to f, it is necessary and sufficient that $\mathscr{L}' = g^*(\mathscr{L})$ is ample relative to $f \circ g$.*

Proof. This condition is necessary, since g is affine (Proposition 5.5.9). To see the sufficiency, we can assume that Y is affine, so X and X' are quasi-compact and $\mathscr{L}'$ is ample (Proposition 5.4.37). Now the set of points $x \in X$ such that there is a neighborhood of x over which $g_*(\mathscr{O}_{X'})$ (resp. $g_*(\mathscr{O}_{X'}) \otimes_{\mathscr{O}_X} \mathscr{K}(X)$) is of rank n is open and closed in X by hypotheses, so X is a finite sum of such open subschemes (recall that X is quasi-compact), and we can therefore suppose that it is equal to one of them (Proposition 5.4.48). But we then have $N_{X'/X}(\mathscr{L}') = \mathscr{L}^{\otimes n}$, so $\mathscr{L}^{\otimes n}$ is ample in view of Proposition 5.6.31, and $\mathscr{L}$ is therefore ample. □

Corollary 5.6.33. *Under the hypotheses of Proposition 5.6.31, suppose moreover that $f : X \to Y$ is of finite type. Then, for f to be quasi-projective, it is necessary and sufficient that $f \circ g$ is quasi-projective. If we suppose that Y is quasi-compact and quasi-compact, then, for f to be projective, it is necessary and sufficient that $f \circ g$ is projective.*

Proof. The hypotheses implies that $f \circ g$ is of finite type. By the definition of quasi-projective morphisms, the first assertion then follows from Proposition 5.6.31 and Corollary 5.6.32. In view of this result and Theorem 5.5.24, it remains to prove that if f is quasi-projective, then for f to be proper, it is necessary and sufficient that $f \circ g$ is proper. But f is then separated and of fintie type, and as g is surjective, this follows from Corollary 5.5.18(ii). □

Corollary 5.6.34. *Let X be a scheme of finite type over a field K and K' be finite extension of K. For X to be projective (resp. quasi-projective) over K, it is necessary and sufficient that $X' \otimes_K K'$ is projective (resp. quasi-projective) over K'.*

Proof. This condition is necessary by Proposition 5.5.15(iii) and Proposition 5.5.25(iii). Conversely, suppose that X' is projective (resp. quasi-projective), and let $g : X' \to X$ be the canonical projective. Since K' is finite over K, it is clear that g is a finite morphism by Proposition 5.6.3 and is surjective (Proposition 4.3.26). Moreover, $g_*(\mathscr{O}_{X'})$ is a locally free $\mathscr{O}_X$-module, being isomorphic to $\mathscr{O}_X \otimes_K K'$ (Corollary 5.1.30). It then follows from the hypotheses and Corollary 5.6.8 and Proposition 5.5.25(ii) that X' is projective (resp. quasi-projective) over K. We then deduce from Corollary 5.6.33 that X is projective (resp. quasi-projective) over K. □

Remark 5.6.3. In fact, later we will see that the statement of Corollary 5.6.34 is valid for arbitrary extension K' of K.

The end of this subsection is devoted to the proof of the criterion in Proposition 5.6.39, which is a refinement of the techniques we have currently used.

Lemma 5.6.35. *Let X be a reduced Noetherian scheme and $\mathscr{E}$ be a coherent $\mathscr{O}_X$-module such that $\mathscr{E} \otimes_{\mathscr{O}_X} \mathscr{K}(X)$ is locally free of rank n. Then there exists a reduced Noetherian scheme Z and a birational* *and finite morphism $h : Z \to X$ such that the homomorphisms $\sigma_i : \mathscr{H}om_{\mathscr{O}_X}(\mathscr{E}, \mathscr{E}) \to \mathscr{K}(X)$ send $\mathscr{H}om_{\mathscr{O}_X}(\mathscr{E}, \mathscr{E})$ into the coherent $\mathscr{O}_X$-algebra $h_*(\mathscr{O}_Z)$.*

Corollary 5.6.36. *Under the hypotheses of Lemma 5.6.35, let W be an open subset of X such that for any $x \in W$, either X is normal at x or $\mathscr{E}_x$ is a free $\mathscr{O}_{X,x}$-module. Then we can choose h so that the restriction of h to $h^{-1}(W)$ is an isomorphism $h^{-1}(W) \cong W$.*

Proof. In fact, the hypotheses imply that if $U \subseteq W$ is an affine open subset, we have, in the notations of Lemma 5.6.35, that $(\sigma_i(u))_x \in A_x$ for any $x \in U$ (Proposition 5.6.24), hence $\sigma_i(u) \in A$, and the conclusion follows from the definition of h given in Lemma 5.6.35. □

Proof. Let X be a reduced Noetherian scheme, $g : X' \to X$ be a finite and surjective morphism such that $\mathscr{B} = g_*(\mathscr{O}_{X'})$ is a coherent $\mathscr{O}_X$-module. Suppose that $\mathscr{B} \otimes_{\mathscr{O}_X} \mathscr{K}(X)$ is locally free of rank n. Then we can apply Lemma 5.6.35 on $\mathscr{E} = \mathscr{B}$, with the same notations, to obtain a homomorphism $\sigma_n : \mathscr{H}om_{\mathscr{O}_X}(\mathscr{B}, \mathscr{B}) \to h_*(\mathscr{O}_Z)$, and by composing with the canonical injection $\mathscr{B} \to \mathscr{H}om_{\mathscr{O}_X}(\mathscr{B}, \mathscr{B})$, we obtain a homomorphism of sheaves of multiplicative monoids:

$$\widetilde{N} : \mathscr{B} = g_*(\mathscr{O}_{X'}) \to h_*(\mathscr{O}_Z) = \mathscr{C}. \tag{6.6.1}$$

For any invertible $\mathscr{O}_{X'}$-module $\mathscr{L}'$, $g_*(\mathscr{L}')$ is an invertible $\mathscr{B}$-module by Proposition 5.6.10, and using the same method, we can define an invertible $\mathscr{C}$-module $\widetilde{N}(g_*(\mathscr{L}'))$, which is functorial on $\mathscr{L}'$. □

Lemma 5.6.37. *Let X be a reduced Noetherian scheme, $g : X' \to X$ be a finite and surjective morphism such that $g * (\mathscr{O}_{X'}) \otimes_{\mathscr{O}_X} \mathscr{K}(X)$ is locally free of rank n. Then there exists a reduced Noetherian scheme Z and a finite and birational morphism $h : Z \to X$ such that for any ample $\mathscr{O}_X$-module $\mathscr{L}'$, the invertible $\mathscr{O}_Z$-module $\mathscr{M}$ such that $h_*(\mathscr{M}) = \widetilde{N}(g_*(\mathscr{L}'))$ is ample.*

Corollary 5.6.38. *Under the hypotheses of Lemma 5.6.37, for any invertible $\mathscr{O}_X$-module $\mathscr{L}$ such that $g^*(\mathscr{L})$ is ample, $h^*(\mathscr{L})$ is ample.*

Proposition 5.6.39. *Let Y be an affine scheme, X be a reduced Noetherian scheme, $f : X \to Y$ be a quasi-compact morphism, and $g : X' \to X$ be a finite and surjective morphism. Let W be an open subset of X such that, for any $x \in W$, either X is normal at x, or there exists an open neighborhood $T \subseteq W$ of x such that $(g_*(\mathscr{O}_{X'}))|_T$ is a locally free $(\mathscr{O}_X|_T)$-module. Then there exists a reduced Y-scheme Z and a finite and birational Y-morphism $h : Z \to X$ such that the restriction of h to $h^{-1}(W)$ is an isomorphism $h^{-1}(W) \cong W$ and satisfies the following property: for any invertible $\mathscr{O}_X$-module such that $g^*(\mathscr{L})$ is ample relative to $f \circ g$, $h^*(\mathscr{L})$ is ample relative to $f \circ h$.*

Corollary 5.6.40. *If in Proposition 5.6.39 we have $W = X$, then for an invertible $\mathscr{O}_X$-module $\mathscr{L}$ to be ample relative to f, it is necessary and sufficient that $g^*(\mathscr{L})$ is ample relative to $f \circ g$.*

Remark 5.6.4. We will see that if Y is Noetherian, f is of finite type, and if the restriction of f to the reduced closed subscheme of X having $X - W$ as underlying space is proper, then the conclusion of Corollary 5.6.40 is still valid. But we will also give examples of algebraic schemes X over a field K (the structural morphism $X \to \operatorname{Spec}(K)$ not being proper) whose normalize X' is quasi-affine, but which is not quasi-affine (so that $\mathscr{O}_X$ is not ample, although $\mathscr{O}_{X'}$ is, cf.

Proposition 5.5.1, and that the morphism $g : X' \to X$ is finite and surjective (Remark 5.6.2)). We will also see that this circumstance cannot occur when we replace "quasi-affine" by "affine" (by Chevalley's theorem).

5.6.7 Chevally's theorem

Lemma 5.6.41. *Let X, Y be integral Noetherian schemes, x (resp. y) be the generic point of X (resp. Y), and $f : X \to Y$ be a finite and surjective morphism. Let $\mathscr{L}$ be an invertible $\mathscr{O}_X$-module such that there exists an affine neighborhood U of y and a section $s \in \Gamma(X, \mathscr{L})$ such that $x \in X_s \subseteq f^{-1}(U)$. Then there exist integers $m, n > 0$, a homomorphism $u : \mathscr{O}_Y^m \to f_*(\mathscr{L}^{\otimes n})$ and an open neighborhood V of y such that the restriction $u|_V$ is an isomorphism $\mathscr{O}_Y^m|_V \xrightarrow{\sim} f_*(\mathscr{L}^{\otimes n})|_V$.*

Theorem 5.6.42 (Chevalley). *Let X be an affine scheme, Y be a Noetherian scheme, and $f : X \to Y$ be a finite and surjective morphism. Then Y is an affine scheme.*

Proof. It is clear that $f_{\text{red}} : X_{\text{red}} \to Y_{\text{red}}$ is fiite (Proposition 5.6.3(vi)); as X_{red} is an affine scheme, and Y is affine if and only if Y_{red} is (Corollary 4.4.34), we can assume that X, Y are reduced. For any closed subset Y' of Y, there is then a unique reduced subscheme subscheme structure on Y' whose inverse image $f^{-1}(Y')$, canonically isomorphic to $X \times_Y Y'$, is affine as a closed subscheme of X, and the restriction of f to $f^{-1}(Y')$, which is identified with $f \times_Y 1_{Y'}$, is a finite and surjective morphism (Proposition 4.3.26 and Proposition 5.6.3(iv)). In view of the Noetherian induction principle (Lemma **??**), we are then (in view of Corollary 4.4.34) reduced to prove the theorem under the hypothesis that for any closed subset $Y' \neq Y$, any closed subscheme of Y with underlying space Y' is affine. With this hypothesis, we first note that, for any coherent $\mathscr{O}_Y$-module $\mathscr{F}$ whose support (closed) Z is distinct from Y, we have $H^1(Y, \mathscr{F}) = 0$. In fact, there exists a closed subscheme structure on Z such that, if $j : Z \to Y$ is the canonical injection, we have $\mathscr{F} = j_*(j^*(\mathscr{F}))$ (Corollary 4.6.18 and Proposition 4.1.29), and therefore $H^1(Y, \mathscr{F}) = H^1(Z, j^*(\mathscr{F})) = 0$ (Corollary 5.5.12, since j is affine).

Suppose first that Y is not irreducible, and let Y' be an irreducible component of Y; we endow Y' with the reduced subscheme structure, and let $j : Y' \to Y$ be the canonical injection. Let $\mathscr{F}$ be a coherent $\mathscr{O}_Y$-module, and consider the canonical homomorphism

$$\rho : \mathscr{F} \to \mathscr{F}' = j_*(j^*(\mathscr{F}));$$

Since j is proper and Y', Y are Noetherian schemes, $\mathscr{F}'$ is a coherent $\mathscr{O}_Y$-module by Proposition 1.4.26 (since we have $j_*(\mathscr{O}_{Y'}) = \mathscr{O}_Y/\mathscr{I}$, where $\mathscr{I}$ is the quasi-coherent ideal of $\mathscr{O}_Y$ defining the subscheme Y'), so $\mathscr{G} = \ker \rho$ and $\mathscr{K} = \operatorname{im} \rho$ are coherent $\mathscr{O}_Y$-modules (Proposition 1.4.18). On the other hand, by definition the fiber $\mathscr{F}'_y$ of $\mathscr{F}'$ at the generic point y of Y' is equal to $\mathscr{F}_y/\mathscr{I}_y\mathscr{F}_y$, and hence to $\mathscr{F}_y$ (Example 4.4.43), so y is not contained in the support of $\mathscr{G}$ and we conclude that $H^1(Y, \mathscr{G}) = 0$. Since the support of $\mathscr{F}'$ (and a fortiori that of $\mathscr{K}$) is contained in Y', it is distinct from Y, and we also conclude that $H^1(Y, \mathscr{K}) = 0$. From the exact sequence $0 \to \mathscr{G} \to \mathscr{F} \to \mathscr{K} \to 0$ we then get $H^1(Y, \mathscr{F}) = 0$, so by Serre's criterion Y is then affine.

Suppose now that Y is irreducible, and therefore integral. We can also assume that X is integral: in fact, if we denote by X_i the reduced closed subschemes of X with underlying spaces the irreducible components of X and by $f_i : X_i \to Y$ the restriction of f to X_i, then one of f_i is dominant (Y is irreducible, so if its generic point is contained in the image of f_i, then f_i is dominant, and we note that f is surjective), and as there are finite morphisms (Proposition 5.6.3), it is surjective (Proposition 5.6.7); as X_i is an affine scheme, we see that we can replace X by X_i. In this case, we can apply Lemma 5.6.41 to $\mathscr{L} = \mathscr{O}_X$, since X is affine, to obtain a homomorphism $u : \mathscr{O}_Y^m \to f_*(\mathscr{L}) = \mathscr{B}$ and an open neighborhood V of y such that $u|_V$ is an isomorphism. In view of Serre's criterion, it suffices to prove that for any coherent $\mathscr{O}_Y$-module $\mathscr{F}$ such that $\mathscr{F} \subseteq \mathscr{O}_Y$, we have $H^1(Y, \mathscr{F}) = 0$. We note that then $\mathscr{F}$ is torsion free since Y is integral, and we only need to show that $H^1(Y, \mathscr{F}) = 0$ for any torsion free coherent $\mathscr{O}_Y$-module $\mathscr{F}$. Now the homomorphism u defines a homomorphism

$$v : \mathscr{G} = \mathscr{H}om_{\mathscr{O}_Y}(\mathscr{B}, \mathscr{F}) \to \mathscr{H}om_{\mathscr{O}_Y}(\mathscr{O}_Y^m, \mathscr{F}) = \mathscr{F}^m.$$

By hypotheses the support of $\mathscr{T} = \operatorname{coker} u$ does not meet V, so is a torsion $\mathscr{O}_Y$-module (Proposition 4.7.27). From the exact sequence $\mathscr{O}_Y^m \to \mathscr{B} \to \mathscr{T} \to 0$ induces, by the left exactness of $\mathscr{H}om_{\mathscr{O}_Y}$, an exact sequence

$$0 \longrightarrow \mathscr{H}om_{\mathscr{O}_Y}(\mathscr{T}, \mathscr{F}) \longrightarrow \mathscr{G} \xrightarrow{\ v\ } \mathscr{F}^m$$

But as $\mathscr{F}$ is torsion free and $\mathscr{T}$ is torsion, we have $\mathscr{H}om_{\mathscr{O}_Y}(\mathscr{T}, \mathscr{F}) = 0$, so v is injective. We then obtain an exact sequence

$$0 \longrightarrow \mathscr{G} \longrightarrow \mathscr{F}^m \longrightarrow \operatorname{coker} v \longrightarrow 0$$

where $\mathscr{G}$ and $\operatorname{coker} v$ are coherent $\mathscr{O}_Y$-modules (Proposition 1.4.18). In view of the exact sequence on cohomology, it suffices to prove that $H^1(Y, \mathscr{F}) = H^1(Y, \operatorname{coker} v) = 0$ since this implies $H^1(Y, \mathscr{F}^m) = (H^1(Y, \mathscr{F}))^m = 0$, so $H^1(Y, \mathscr{F}) = 0$. Now the restriction $v|_V$ is an isomorphism, so the support of $\operatorname{coker} v$ is distinct from Y, and we have $H^1(Y, \operatorname{coker} v) = 0$ by our hypothesis. On the other hand, $\mathscr{G}$ is a coherent $\mathscr{B}$-module by Corollary 4.2.28; as X is affine over Y, there exists a quasi-coherent $\mathscr{O}_X$-module $\mathscr{K}$ such that $\mathscr{G}$ is isomorphic to $f_*(\mathscr{K})$ (Proposition 5.1.18), and since $H^1(X, \mathscr{K}) = 0$ (X is affine), we then have $H^1(Y, \mathscr{G}) = 0$ by Corollary 4.6.18, which completes the proof. □

Corollary 5.6.43. *Let X be a Noetherian scheme and $(X_i)_{1 \le i \le n}$ be a finite covering of X by closed subsets. Then for X to be affine, it is necessary and sufficient that for each i, there exists a closed subscheme of X that is affine and has underlying space X_i.*

Proof. Let X' be the sum of the X_i, then it is clear that X' is affine if each X_i is affine, and we have a surjective morphism $f : X' \to X$ whose restriction to X_i is the canonical injection. To apply Proposition 5.6.42, it remains to verifying that f is finite, and this follows from Proposition 5.6.3(i). □

Corollary 5.6.44. *For a Noetherian scheme X to be affine, it is necessary and sufficient that any of its reduced closed subschemes, with underlying space an irreducible component of X, is affine.*

5.7 Valuative criterion

In this section we give the valuative criterion of sparation and properness of a morphis, which are criteria which involve a auxiliary scheme Spec(A), where A is a valuation ring. With a convenient "Noetherian" hypothesis, these criterion can be refined to the case where A is a discrete valuation ring, and this will probably be the only case that we will apply later.

5.7.1 Remainders for valuation rings

Among the vast properties that characterize valuation rings, we shall use the following one: a ring A is called a valuation ring if it is an integral domain which is not a field, and if in the set of proper local rinng contained in the fraction field K of A, A is maximal under the dominant relation. Recall that a valuation ring is integrally closed. If A is a valuation ring, then $A_{\mathfrak{p}}$ is also a valuation ring for any nonzero prime ideal $\mathfrak{p} \neq 0$.

Let K be a field, A be a proper local subring of K; then there exists a valuation ring of K dominating A (Proposition **??**). On the other hand, let B be a valuation ring, k be its residue field, and K be the fraction field, L be an extension of k. Then there exists a complete valuation ring C dominating B with residue field equals to L. In fat, L is an algebraic extension of a purely transcendental extension $L' = k(T_\mu)_{\mu\in M}$; we can extend the valuation of K corresponding to B to a valuation of $K' = K(T_\mu)_{\mu\in M}$ with residue field L'; replace B by this complete valuation ring C, we can assume then that B is complete that L is an algebraic closure of k. If $\overline{K}$ is an algebraic closure of K, we can then extend the defining valuation of B to $\overline{K}$, and the corresponding residue field is an algebraic closure of k, as can be seen by lifting the coefficients of a monic polynomial of $k[T]$ to $\overline{K}$. We are therefore finally reduced to the case where $L = k$ and it suffices then to take for C the completion of B to answer the question.

Let K be a field and A be a subring of K; the integral closure A' of A in K is the intersection of valuation rings of the fraction field of A containing A (Theorem **??**). The preceding argument then have the following geometric form:

Proposition 5.7.1. *Let Y be a scheme, $p : X \to Y$ be a morphism, x be a point of X, $y = p(x)$, and $y' \neq y$ be a specialization of y. Then there exists a local scheme Y', which is the specturm of a valuation ring, and a separated morphism $f : Y' \to Y$ such that, if a is the unique closed point of Y' and b is the generic point of Y', we have $f(a) = y'$ and $f(b) = y$. We can moreover suppose that one of the following additional conditions are satisfied:*

- *(i) Y' is the specturm of a complete valuation ring whose residue field is algebraically closed, and there exists a $\kappa(y)$-homomorphism $\kappa(x) \to \kappa(b)$.*
- *(ii) There exists a $\kappa(y)$-isomorphism $\kappa(x) \cong \kappa(b)$.*

Proof. Let Y_1 be the reduced closed subscheme of Y with $\overline{\{y\}}$ as underlying space, and X_1 be the closed subscheme $p^{-1}(Y_1)$; as $y' \in \overline{\{y\}}$ by hypothesis and $\kappa(x)$ is the same for X and X_1, by replacing Y with Y_1 and X with X_1, we can suppose that Y is integral with generic point y; $\mathscr{O}_{Y,y'}$ is then an integral local ring which is not a field, whose fraction field is $\mathscr{O}_{Y,y} = \kappa(y)$, and $\kappa(x)$ is an extension of $\kappa(y)$. To realize the conditions $f(a) = y'$ and $f(b) = y$ with the additional condition (i) (resp. (ii)), we choose $Y' = \mathrm{Spec}(A')$, where A' is a valuation ring dominating $\mathscr{O}_{Y,y'}$

and which is complete and with residue field an algebraically closed extension of $\kappa(x)$ (resp. a valuation ring dominating $\mathscr{O}_{Y,y'}$ with fraction field $\kappa(x)$); the existence of such rings are proved by the above remarks. □

Recall that a local ring $(A, \mathfrak{m})$ is of dimension if and only if any prime ideal of A distinct from $\mathfrak{m}$ is minimal; if A is integral, this means $\mathfrak{m}$ and (0) are the only prime ideals, and $\mathfrak{m} \neq (0)$; equivalent, $Y = \operatorname{Spec}(A)$ is reduced to two points a, b: a is the closed point, $\mathfrak{p}_a = \mathfrak{m}$, and $\kappa(a) = k$ is the residue field $A/\mathfrak{m}$; b is the generic point of Y, $\mathfrak{p}_b = (0)$, the set $\{b\}$ is the unique nontrivial open subset of Y, and $\kappa(b) = K$ is the fraction field of A. For an integral Noetherian local ring A of dimension 1, the following conditions are then equivalent:

(i) A is normal;

(ii) A is regular;

(iii) A is a valuation ring.

Moreover, if these are ture, A is then a discrete valuation ring.

Proposition 5.7.2. *Let A be a Noetherian local integral domain which is not a field, K be its fraction field, L be a extension of K of finite type. There then exists a discrete valuation ring of L dominating A.*

Proof. □

Corollary 5.7.3. *Let A be an integral Noetherian ring, K be its fraction field, and L be an extension of K of finite type. Then the integral closure of A in L is the intersection of discrete valuation rings of L containing A.*

Proposition 5.7.4. *Let Y be a locally Noetherian scheme, $p : X \to Y$ be a morphism locally of finite type, x be a point of X, $y = p(x)$, $y' \neq y$ be a specialization of y. Then there exists a local scheme Y', which is a specturm of a discrete valuation ring, and a separated morphism $f : Y' \to Y$ and rational Y-map $g : Y' \dashrightarrow X$ such that, if a is the closed point of Y' and b is the generic point, we have $f(a) = y'$, $f(b) = y$, $g(b) = x$, and in the following commutative diagram*

$$\begin{tikzcd} & \kappa(x) \arrow[dl, "\gamma"'] \\ \kappa(b) & \kappa(y) \arrow[l, "\varphi"] \arrow[u, "\pi"'] \end{tikzcd}$$

(whee π, φ, γ are the homomorphisms corresponding to p, f and g), γ is a bijection.

Proof. □

5.7.2 Valuative criterion of separation

Proposition 5.7.5 (Valuative Criterion of Separation). *Let Y be a scheme (resp. a locally Noetherian scheme), $f : X \to Y$ be a morphism (resp. a morphism locally of finite type). The following conditions are equivalent:*

(i) f is separated.

(ii) *f is quasi-separated and for any Y-scheme of the form $Y' = \mathrm{Spec}(A)$, where A is a valuation ring (resp. a discrete valuation ring), two Y-morphisms of Y' to X which coincide at the generic point of Y' are equal.*

(iii) *f is quasi-seaprated and for any Y-scheme of the form $Y' = \mathrm{Spec}(A)$, where A is a valuation ring (resp. a discrete valuation ring), two Y'-sections of $X' = X_{(Y')}$ which coincide at the generic point of Y' are equal.*

Proof. The equivalence of (ii) and (iii) follows from the correspondence of Y-morphisms $Y' \to X$ and Y'-sections of X'. If X is separated over Y, condition (ii) follows from the proof of Proposition 4.7.10, since Y' is integral. It then remains to prove that condition (ii) implies that the diagonal morphism $\Delta : X \to X \times_Y X$ is closed, and for this we can use Proposition 4.6.6. Now, let z be a point of the diagonal $\Delta(X)$, $z' \neq z$ be a specialization of z in $X \times_Y X$. Then there exists by Proposition 5.7.1 a valuation ring A and a morphism $g : Y' \to X \times_Y X$ such that $g(a) = z'$, $g(b) = z$ (with the notations of Proposition 5.7.1, a is the closed point of Y' and b is the generic point of Y'); this morphism makes Y' an $(X \times_Y X)$-scheme, and a fortiori a Y-scheme. If we compose g with the two projections of $X \times_Y X$, we obtain two Y-morphisms $g_1, g_2 : Y' \to X$, which by hypotheses send the point b to the same point in X; in view of (ii), these two morphisms coincide with a morphism $h : Y' \to X$, which signifies that g factors into $g = \Delta \circ h$, and therefore $z' \in \Delta(X)$. If we suppose that Y is locally Noetherian and f is of finite type, $X \times_Y X$ is locally Noetherian by Corollaries 4.6.22, and we can therefore replace Proposition 4.7.10 by Proposition 5.7.4. ◻

The condition (ii) of Proposition 5.7.5 signifies that if $Y' = \mathrm{Spec}(A)$ and $X' = \mathrm{Spec}(K)$ where K is the fraction field of A, the canonical map

$$\mathrm{Hom}_Y(Y', X) \to \mathrm{Hom}_Y(X', X)$$

is injective. Equivalently, this means in the following diagram

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ Y' \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

the dashed morphism $Y' \to X$, if exists, is necessarily unique.

Remark 5.7.1. In the criterion (ii) of Proposition 5.7.5, we can restrict ourselves to valuation rings A which is complete whose residue field is algebraically closed; this follows from the additional condition (i) of Proposition 4.7.10.

5.7.3 Valuative criterion of properness

Proposition 5.7.6. *Let A be a valuation ring, $Y = \mathrm{Spec}(A)$, b be the generic point of Y. Let X be an integral and separated scheme and $f : X \to Y$ be a closed morphism such that $f^{-1}(b)$ is reduced to a point x and the corresponding homomorphism $\kappa(b) \to \kappa(x)$ is bijective. Then f is an isomorphism.*

Proof. As f is closed and dominant, we have $f(X) = Y$; it then suffices to prove that for any $y' \neq b$ in Y, there exists a unique point x' such that $f(x') = y'$ and the corresponding homomorphism $\mathscr{O}_{Y,y'} \to \mathscr{O}_{X,x'}$ is bijective, because f is then a homeomorphism. Now, if $f(x') = y'$, $\mathscr{O}_{X,x'}$ is a local ring contained in $K = \kappa(x) = \kappa(y)$ and dominates $\mathscr{O}_{Y,y'}$; the later is the local ring $A_{y'}$, which a valuation ring for the fraction field K of A. But $\mathscr{O}_{X,x'} \neq K$ since x' is not the generic point of X, and we then conclude that $\mathscr{O}_{X,x'} = \mathscr{O}_{Y,y'}$ by maximality. As X is an integral scheme, the relation $\mathscr{O}_{X,x'} = \mathscr{O}_{X,x''}$ implies $x' = x''$ by Proposition 4.7.31, which proves our claim. □

Let A be a valuation ring, $Y = \operatorname{Spec}(A)$, b the generic point of Y, so that $\mathscr{O}_{Y,b} = \kappa(b)$ is equal to the fraction field K of A. Let $f : X \to Y$ be a morphism. We have seen that the rational Y-sections of X correspond to the germs of Y-sections (defined over a neighborhood of b) at b, whence a canonical map

$$\Gamma_{\mathrm{rat}}(X/Y) \to \Gamma(f^{-1}(b)/\operatorname{Spec}(K)) \tag{7.3.1}$$

where the elements of $\Gamma(f^{-1}(b)/\operatorname{Spec}(K))$ are identified with the rational points of $f^{-1}(b) = X \otimes_A K$ over K. If f is separated, it then follows from Corollary 4.5.21 that the map of (7.3.1) is injective, since Y is integral.

Composing (7.3.1) with the canonical map $\Gamma(X/Y) \to \Gamma_{\mathrm{rat}}(X/Y)$, we then obtain a canonical map

$$\Gamma(X/Y) \to \Gamma(f^{-1}(b)/\operatorname{Spec}(K)). \tag{7.3.2}$$

If f is separated, this map is injective by Corollary 4.5.21.

Proposition 5.7.7. *Let A be a valuation ring with fraction field K, $Y = \operatorname{Spec}(A)$, b be the generic point of Y, and $f : X \to Y$ be a separated and closed morphism. Then the canonical map (7.3.2) is bijective.*

Proof. Let x be a rational point of $f^{-1}(b)$ over K. As f is separated, so is the morphism $f^{-1}(b) \to \operatorname{Spec}(K)$ corresponding to f (Proposition 4.5.25(iv)), since and any section of $f^{-1}(b)$ a closed immersion by Proposition 4.5.19, $\{x\}$ is closed in $f^{-1}(b)$. Consider the reduced closed subscheme X' of X with underlying space $\overline{\{x\}}$ of $\{x\}$ in X. It is clear that the restriction of f to X' satisfies the conditions of Proposition 5.7.6 (note that since x is rational over K, we have $\kappa(x) = K$), hence an isomorphism from X' to Y, whose inverse isomorphism is the Y-section of X we want. □

Recall that if F is a subset of the scheme Y, the codimension of F in Y equal to the infimum of $\dim(\mathscr{O}_{Y,z})$ where $z \in F$ (this can be easily verified after reducing to affine case), and we denote this number by $\operatorname{codim}_Y(F)$.

Corollary 5.7.8. *Let Y be a reduced locally Noetherian scheme such that the subset N of $y \in Y$ where Y is not regular has codimension $\operatorname{codim}_Y(N) \geq 2$. Let $f : X \to Y$ be a separted and closed morphism of finite type and g be a rational Y-section of X. If Y' is the set of points of Y where g is not defined, then $\operatorname{codim}_Y(Y') \geq 2$.*

Proof. It suffices to prove that g is defined at any point $y \in Y$ such that $\dim(\mathscr{O}_{Y,y}) \leq 1$. If $\dim(\mathscr{O}_{Y,y}) = 0$, then y is the generic point of an irreducible component of Y. For any open dense subset V of Y, by restricting to an affine neighborhood of y and apply Proposition **??**, we conclude that V contains y. In particular, y belongs to the defining domain of g. Suppose now that $\dim(\mathscr{O}_{Y,y}) = 1$; then $\mathscr{O}_{Y,y}$ is a regular local ring, hence a discrete valuation ring. Let $Z = \mathrm{Spec}(\mathscr{O}_{Y,y})$; as $U = Y - Y'$ is open and dense, by Proposition 4.2.10 and our preceding arguments, $U \cap Z$ is nonempty (contains the generic of an irreducible component of Y containing y), so we can consider the rational map $g' : Z \dashrightarrow X$ induced by g. It then suffices to prove that g' is a morphism (Proposition 4.7.17). Now, g' can be considered as a rational Z-section of the Z-scheme $f^{-1}(Z) = X \times_Y Z$; it is clear that the morphism $f^{-1}(Z) \to Z$ corresponding to f is closed, and is separted by Proposition 4.5.25(i). We then conclude from Proposition 5.7.7 that g' is everywhere defined, and as Z is reduced and X is separated over Y, g' is a morphism (Proposition 4.7.10). □

Corollary 5.7.9. *Let S be a locally Noetherian scheme, X, Y be S-scheme, and assume that X is proper over X. Suppose that Y is reduced and the subset N of $y \in Y$ where Y is not regular has codimension $\mathrm{codim}_Y(N) \geq 2$. Let $f : Y \dashrightarrow X$ be a rational map and Y' be the set of points where f is not defined, then $\mathrm{codim}_Y(Y') \geq 2$.*

Proof. The rational S-maps $Y \dashrightarrow X$ correspond to rational Y-sections of $X \times_S Y$; as the structural morphism $X \times_S Y \to Y$ is closed by Proposition 5.5.17, we can apply Corollary 5.7.8, whence the corollary. □

Remark 5.7.2. The hypothesis on Y in Corollary 5.7.8 and Corollary 5.7.9 are satisfied in particular if Y is normal (by Serre's criterion for normality).

Theorem 5.7.10 (Valuative Criterion of Properness). *Let Y be a (resp. locally Noetherian) scheme and $f : X \to Y$ be a quasi-compact and separted morphism (resp. a quasi-compact morphism of finite type). The following conditions are equivalent:*

(i) f is universally closed (resp. proper)

(ii) For any Y-scheme $Y' = \mathrm{Spec}(A)$ where A is a valuation ring (resp. a discrete valuation ring) with fraction field K, the canonical map (where $X' = \mathrm{Spec}(K)$)

$$\mathrm{Hom}_Y(Y', X) \to \mathrm{Hom}_Y(\mathrm{Spec}(K), X)$$

corresponding to the cannical injection $A \to K$, is surjective (resp. bijective).

(iii) For any Y-scheme $Y' = \mathrm{Spec}(A)$ where A is a valuation ring (resp. a discrete valuation ring) with fraction field K, the canonical map of (7.3.2) relative to the Y'-scheme $X' = X_{(Y')}$ is surjective (resp. bijective).

Proof. The equivalence of (ii) and (iii) follows from the correspondence of Y-morphisms $Y' \to X$ and Y'-morphisms $Y' \to X'$. If f is universally closed then $f_{(Y')}$ is closed and separated, and it then suffices to apply Proposition 5.7.7. It remains to prove that (ii) implies (i). Consider first the case where Y is arbitrary, f is separated and quasi-compact. If the condition of (ii) is satisfied for f, it is also true for $f_{(Y'')} : X_{(Y'')} \to Y''$, where Y'' is an arbitrary Y-scheme, in view of the equivalent of (ii) and (iii), and the fact that $X_{(Y'')} \times_{Y''} Y' = X \times_Y Y'$ for any morphism $Y' \to Y''$; as $f_{(Y'')}$ is also quasi-compact and separated, we then conclude that we only need to prove (ii) implies f is closed, and for this we shall use Proposition 4.6.6. Let $x \in X$, $y' \neq y$ be a specialization of $y = f(x)$; in view of Proposition 5.7.1, there is a scheme $Y' = \operatorname{Spec}(A)$ where A is a valuation ring, and a separated morphism $g : Y' \to Y$ such that, if a is the closed point and b is the generic point of Y, we have $g(a) = y'$, $g(b) = y$, and there exists a $\kappa(y)$-hommorphism $\kappa(x) \to \kappa(b)$. This homomorphism corresponds to a canonical Y-morphism $\operatorname{Spec}(\kappa(b)) \to X$ (Corollary 4.2.15), and it then follows from condition (ii) that there exists a Y-morphism $h : Y' \to X$ which corresponds the previous morphism such that the following diagram commutes

$$\begin{tikzcd} \operatorname{Spec}(\kappa(b)) \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \operatorname{Spec}(A) \arrow[r, "g"] & Y \end{tikzcd}$$

We then have $h(b) = x$, and if we put $h(a) = x'$, x' is then a specialization of x, and we have $f(x') = f(h(a)) = g(a) = y'$.

If now Y is locally Noetherian and f is a quasi-compact morphism of finite type, then condition (ii) implies that f is separated (Proposition 5.7.5), so the diagonal morphism $\Delta : X \to X \times_Y X$ is quasi-compact. Moreover, to verify that f is proper, it suffices to show that $f_{(Y'')} : X_{(Y'')} \to Y''$ is closed for any Y-scheme Y'' of finite type (Corollary 5.5.30). As then Y'' is locally Noetherian, we can resume the reasoning given in the first case by taking for Y' the specturm of discrete valuation ring, and applying Proposition 5.7.1 instead of Proposition 5.7.4. □

Remark 5.7.3. We deduce from the criterion (iii) of Theorem 5.7.10 a new proof of the fact that a projective morphism $X \to Y$ is closed, which is closer to classical methods. We can in fact assume that Y is affine, and X is therefore a closed subscheme of a projective bundle $\mathbb{P}^n_Y$ (Proposition 5.5.14). To show that $X \to Y$ is closed, it suffices to verify that the structural morphism $\mathbb{P}^n_Y \to Y$ is closed. The criterion (iii) of Theorem 5.7.10, together with (4.1.1), show that we are reduced to proving the following fact: if Y is the spectrum of a valuation ring A with fraction field K, every point of $\mathbb{P}^n_Y$ with values in K comes (by restriction to the generic point of Y) from a point of $\mathbb{P}^n_Y$ with values in A. Now, any invertible $\mathscr{O}_Y$-module is trivial, so it follows from Example 5.4.7 that a point of $\mathbb{P}^n_Y$ with values in K is identified with a class of elements $(\xi c_0, \xi c_1, \dots, \xi c_n)$ of K^{n+1}, where $\xi \neq 0$ and the c_i are elements of K which generate the unit ideal of K. By multiplying the c_i with an element of A, we can suppose that the c_i belong to A, and generate the unit ideal of A. But then (Example 5.4.7) the system $(c_0, \dots, c_n)$ defines a point of $\mathbb{P}^n_Y$ with values in A, whence our assertion.

Remark 5.7.4. The criteria Proposition 5.7.5 and Theorem 5.7.10 are especially convenient when we consider a Y-scheme X as a functor

$$X(Y') = \operatorname{Hom}_Y(Y', X)$$

where Y' is a Y-scheme. These criteria will allow us, for example, to prove that under certain conditions the "Picard schemas" are proper.

Corollary 5.7.11. *Let Y be a separated integral scheme (resp. a separated integral locally Noetherian scheme) and $f : X \to Y$ be a dominant morphism.*

(a) If f is quasi-compact and universally closed, any valuation ring with fraction field the rational function field $K(X)$ and which dominates a local ring of Y, also dominates a local ring of X.

(b) Conversely, suppose that f is of finite type, and the property of (a) is satisfied for any valuation ring (resp. any discrete valuation ring) with fraction field $K(X)$. Then f is proper.

Proof. Assume the hypotheses of (a) and let $K = K(Y)$, $L = K(X)$, y be a point of Y, A be a valuation ring with L the fraction field and dominate $\mathcal{O}_{Y,y}$. The injection $\mathcal{O}_{Y,y} \to A$ is local, so it defines a morphism

$$h : Y' = \operatorname{Spec}(A) \to \operatorname{Spec}(\mathcal{O}_{Y,y}) \to Y$$

(Proposition 4.2.12) such that $h(a) = y$, where a is the closed point of Y'. Moreover, since $K \subseteq L$, the morphism $\operatorname{Spec}(A) \to \operatorname{Spec}(\mathcal{O}_{Y,y})$ is dominant, so if η is the generic point of Y (which is also that of $\operatorname{Spec}(\mathcal{O}_{Y,y})$), we have $h(b) = \eta$, where b is the generic point of Y'. If ξ is the generic point of X, we have $\kappa(\xi) = \kappa(b) = L$ by hypothesis, so there is a Y-morphism $g : \operatorname{Spec}(L) \to X$ such that $g(b) = \xi$. In view of Proposition 5.7.10, we obtain a Y-morphism $g' : Y' \to X$ such that $g(b) = \xi$. If we set $x = g'(a)$, then A dominates $\mathcal{O}_{X,x}$.

We now prove (b); since the question is local over Y, we can assume that Y is affine (resp. affine and Noetherian). As f is of finite type, we can apply Chow's lemma, so there exists a projective morphism $p : P \to Y$, an immersion $j : X' \to P$, and a projective and surjective birational morphism $g : X' \to X$ (where X' is integral) such that the diagram

$$\begin{tikzcd} X' \arrow[r, "j"] \arrow[d, "g"'] & P \arrow[d, "\pi"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

is commutative. It suffices to prove that j is a closed immersion, because then $f \circ g = p \circ j$ is projective, hence proper, and as g is surjective we conclude that f is proper by Corollary 5.5.18. Let Z be a reduced closed subscheme of P with underlying space $\overline{j(X')}$; as X' is integral, j factors into

$$j : X' \xrightarrow{h} Z \xrightarrow{i} P$$

where $i : Z \to P$ is the canonical injection and $h : X' \to Z$ is a dominant open immersion. Since Z is integral and projective over Y by Proposition 5.5.25, we are then reduced to the case where P is integral, j is dominant and birational, and prove that j is surjective. Now let $z \in P$; $\mathcal{O}_{Z,z}$ is an integral local ring (resp. Noetherian integral) whose fraction field is

$$L = K(P) = K(X') = K(X).$$

We can assume that z is not the generic point of P (since the later is contained in $j(Z)$ as j is dominant), so $\mathcal{O}_{Z,z} \neq L$ and by Proposition **??** and Proposition 5.7.2, there exists a valuation ring (resp. a discrete valuation ring) A with fraction field L that dominates $\mathcal{O}_{Z,z}$. A fortiori A dominates the local ring $\mathcal{O}_{Y,y}$ where $y = p(z)$, and by hypotheses there is then a point $x \in X$ such that A dominates $\mathcal{O}_{X,x}$. As the morphism g is proper, it satisfies the conditions of (a), so our previous arguments then prove that A also dominates $\mathcal{O}_{X,x'}$, for some $x' \in X'$. Then the local rings $\mathcal{O}_{Z,z}$ and $\mathcal{O}_{Z,j(x')} = \mathcal{O}_{X,x'}$ are related, and by Proposition 4.7.31, as P is separated, we have $z = j(x')$, which completes the proof. □

Corollary 5.7.12. *Let A be an integral domain, $Y = \mathrm{Spec}(A)$, and $f : X \to Y$ be a dominant morphism of integral schemes which is quasi-compact and universally closed. Then $\Gamma(X, \mathcal{O}_X)$ is canonically isomorphic to a subring of the integral closure of A in $K(X)$.*

Proof. Recall that by (7.5.1), $B = \Gamma(X, \mathcal{O}_X)$ is indeitified with the intersection of $\mathcal{O}_{X,x}$ for $x \in X$. If R is a valuation ring of $K(X)$ containing A, then it dominates the local ring $A_{\mathfrak{P}}$ where $\mathfrak{P} = \mathfrak{m}_R \cap B$, and therefore by Corollary 5.7.11 dominates a local ring of X. Then B is contained in R, and the conclusion follows from Theorem **??**. □

Remark 5.7.5. Under the hypothesis of Corollary 5.7.12, if we suppose that $K(X)$ is a finite extension of $K(Y)$, then we can in many cases conclude that $\Gamma(X, \mathcal{O}_X)$ is a finitely generated module over the ring $B = \Gamma(Y, \mathcal{O}_X)$. This is the case for example if B is a Japanese ring. In particular, if $X = \mathrm{Spec}(A)$ and $Y = \mathrm{Spec}(k)$ where k is a algebraically closed field, then the the corresponding homomorphism $k \to A$ is injective by Corollary **??** and since the integral closure of k in $K(X)$ is equal to k, we conclude that $\Gamma(X, \mathcal{O}_X) = k$.

5.7.4 Algebraic curves

Let k be a field. In this subsection, all schemes are considered to be separated k-schemes of finite type, and any morphism are k-morphism.

Proposition 5.7.13. *Let X be a scheme of finite type over k; let x_i $(1 \le i \le n)$ be the generic points of the irreducible components X_i of X, and $K_i = \kappa(x_i)$. Then the following conditions are equivalent:*

(i) For each i, the transcendence degre of K_i over k is equal to 1*.*

(ii) For any closed point x of X, the local ring $\mathcal{O}_{X,x}$ is of dimension 1*.*

(iii) The closed irreducible subsets of X distinct from the X_i are closed points of X.

Proof. As X is quasi-compact, any irreducible closed subset F of X contains a closed point (Proposition **??**). Let x be a closed point of X; in view of Proposition 4.2.10, there is a correspondence between prime ideals of $\mathcal{O}_{X,x}$ and the irreducible closed subsets of X containing x. The equivalence of (ii) and (iii) then follows. On the other hand, if $\mathfrak{p}_\alpha$ $(1 \le \alpha \le r)$ is the minimal prime ideals of the local Noetherian ring $\mathcal{O}_{X,x}$, the local ring $\mathcal{O}_{X,x}/\mathfrak{p}_\alpha$ are integral, whose fraction fields are the K_i such that $x \in X_i$. Moreover, the dimension of a k-algebra of finite type is equal to its transcendental degree over k. Finally, the dimension of $\mathcal{O}_{X,x}$ is the supremum of the $\mathcal{O}_{X,x}/\mathfrak{p}_\alpha$, and a k-algebra of finite type is catenary, so (i) and (ii) are equivalent. □

We note that under the conditions of Proposition 5.7.13, by Proposition 4.6.43 that set X is either empty of finite. We define an algebraic curve over k to be a nonempty scheme X over k satisfying the conditions of Proposition 5.7.13. Equivalently, we will see that this condition is equivalent to that the irreducible components of X has dimension 1. In particular, we note that if X is an algebraic curve over k, the reduced closed subschemes X_i of X with underlying spaces the irreducible components of X are also algebraic curves over k.

Corollary 5.7.14. *Let X be an irreducible algebraic curve. Then the only non-closed point of X is its generic point, the proper closed subsets of X are the finite subsets of X, which are also the non dense subsets of X.*

Proof. If a point $x \in X$ is not closed, then its closure is an irreducible closed subset of X, hence equal to X by Proposition 5.7.13, so x is the generic point of X. A proper closed subset F of X can not contain the generic point of X, so its points are all closed, hence T1, and by Proposition 4.2.30 we conclude that F is finite and discrete. The closure of an infinite subset of X is therefore necessarily equal to X, which proves the last assertion. □

If X is an arbitrary algebraic curve, then by applying Corollary 5.7.14, we conclude that the only non-closed points of X are the generic points of the irreducible components of X.

Corollary 5.7.15. *Let X and Y be irreducible algebraic curves over k and $f : X \to Y$ be a k-morphism. Then for f to be dominant, it is necessary and sufficient that $f^{-1}(y)$ is finite for any $y \in Y$.*

Proof. If f is not dominant, $f(X)$ is necessarily a finite subset of Y by Corollary 5.7.14, so it is not possible that $f^{-1}(y)$ is finite for any $y \in Y$ (since X is an infinite set). Conversely, if f is dominant, for any $y \in Y$ which is not the generic point η of Y, $f^{-1}(y)$ is closed in X since $\{y\}$ is closed in Y (Corollary 5.7.14); on the other hand, by hypotheses, $f^{-1}(y)$ does not contain the generic point of X, so is finite by Corollary 5.7.14. Finally, to see that $f^{-1}(\eta)$ is finite, we note that the morphism f is of finite type by Corollary 4.6.37, so the fiber $f^{-1}(\eta)$ is an irreducible scheme of finite type over $\kappa(\eta)$ with generic point ξ (Proposition 4.6.35). As $\kappa(\xi)$ and $\kappa(\eta)$ are extensions of k of finite type with transcendental degree 1, it follows that $\kappa(\xi)$ is a finite extension of $\kappa(\eta)$, so ξ is closed in $f^{-1}(\eta)$ by Corollary 5.6.5, and $f^{-1}(\eta)$ is therefore reduced to a point ξ. □

Remark 5.7.6. We will see later that a proper morphism $f : X \to Y$ of Noetherian schemes, such that $f^{-1}(y)$ is finite for any $y \in Y$, is necessarily finite. It then follows from Corollary 5.7.14 that such a dominant proper morphism of irreducible algebraic curves is finite.

Corollary 5.7.16. *Let X be an algebraic curve over k. For X to be regular, it is necessary and sufficient that X is normal, or the local ring of its closed points are discrete valuation rings.*

Proof. This comes from conditions (ii) of Proposition 5.7.13. □

Corollary 5.7.17. *Let X be a reduced algebraic curve, $\mathscr{A}$ be a reduced coherent $\mathscr{K}(X)$-algebra. Then the integral closure X' of X relative to $\mathscr{A}$ is a normal algebraic curve, and the canonical morphism $X' \to X$ is finite.*

Proof. The fact that $X' \to X$ is finite follows from Remark 5.6.2, and X' is then a normal algebraic scheme over k. Moreover, we note that if X is irreducible with generic point ξ and its integral closure X' has generic point ξ', then $\kappa(\xi') = \kappa(\xi)$ by Corollary 5.6.23, so X' is also an algebraic curve over k. □

Corollary 5.7.18. *For a reduced algebraic curve X to be proper over k (which is called* ***complete****), it is necessary and sufficient that the normalization X' of X is proper over k.*

Proof. The canonical morphism $f : X' \to X$ is finite by Corollary 5.7.17, hence proper (Corollary 5.6.8) and surjective (Corollary 5.6.23). If $g : X \to \mathrm{Spec}(k)$ is the structural morphism, g and $g \circ f$ are then simultaneously proper, in view of Proposition 5.5.17 and Corollary 5.5.18. □

Proposition 5.7.19. *Let X be a normal algebraic curve over k and Y be a proper algebraic scheme over k. Then any rational k-map $f : X \dashrightarrow Y$ is everywhere defined, hence a morphism.*

Proof. It follows from Corollary 5.7.9 that the set of points $x \in X$ where this rational map is not defined, the dimnesion of $\mathscr{O}_{X,x}$ is ≥ 2, hence is empty. The assertion then follows from Proposition 4.7.10. □

Corollary 5.7.20. *A normal algebraic curve over k is quasi-projective over k.*

Proof. As X is the sum of finitely many integral and normal algerbaic curves (Corollary 5.6.23), we can assume that X is integral (Corollary 5.5.16). As X is quasi-compact, it can be covered by finitely many affine opens U_i ($1 \leq i \leq n$), and as each of this is of finite type over k, there exists an integer n_i and a k-immersion $f_i : U_i \to \mathbb{P}_k^{n_i}$ (Corollary 5.5.14 and Proposition 5.5.15(i)). As U_i is dense in X (recall that X is integral by our assumption), it follows from Proposition 5.7.19 that f_i extends to a k-morphism $g_i : X \to \mathbb{P}_k^{n_i}$, and we obtain a k-morphism $g = (g_1, \dots, g_n)_k$ from X into the product P of the $\mathbb{P}_k^{n_i}$ over k. Moreover, for each index i, as the restriction of g_i to U_i is an immersion, so is the restriction of g to U_i (Corollary 4.5.16). As the U_i cover X and g is separated by Proposition 4.5.25(v), g is an immersion from X into P by Proposition 4.7.36. The Segre morphism then provides from g an immersion of X into a projective bundle $\mathbb{P}_k^n$, so X is quasi-projective. □

Corollary 5.7.21. *A normal algebraic curve X is isomorphic to a dense open subscheme of a normal and complete algebraic curve $\widehat{X}$, determined up to isomorphisms.*

Proof. If X_1, X_2 are two normal and complete algebraic curves containing X as open dense subscheme, it follows from Proposition 5.7.19 there is an isomorphism of X_1 and X_2, whence the uniqueness of $\widehat{X}$. To prove the existence of $\widehat{X}$, it suffices to remark that we can consider X as a subscheme of a projective bundle $\mathbb{P}_k^n$ (Corollary 5.7.20). Let $\overline{X}$ be the scheme-theoretic closure of X in $\mathbb{P}_k^n$ (Proposition 4.6.68); as X is an open and dense subscheme of $\overline{X}$, the generic points x_i of the irreducible components of X are those of $\overline{X}$, and the residue fields $\kappa(x_i)$ are the same for both schemes, so $\overline{X}$ is an algebraic curve over k, which is reduced (Proposition 4.6.66) and projective over k (Proposition 5.2.50), hence complete by Theorem 5.5.24. We then take $\widehat{X}$ to be the noramlization of $\overline{X}$, which is complete by Proposition 5.7.18. If $h : \widehat{X} \to \overline{X}$ is the canonical morphism, the restriction of h to $h^{-1}(X)$ is an isomorphism since X is normal (Proposition 5.6.21), and as $h^{-1}(X)$ contains the generic point of the irreducible components of $\widehat{X}$ (Corollary 5.6.23), it is therefore dense in $\widehat{X}$, which proves the assertion. □

Remark 5.7.7. We will later see that the conclusion of Corollary 5.7.21 is still valid without assuming the algebraic curve to be normal (or even reduced); we will also see that for an algebraic curve (reduced or not) to be affine, it is necessary and sufficient that its irreducible (reduced) components are not complete.

Corollary 5.7.22. *Let X be an irreducible normal algebraic curve with $L = K(X)$, Y be a complete and integral algebraic curve with $K = K(Y)$. Then there exists a canonical correspondence between dominant k-morphisms $X \to Y$ and k-monomorphisms $K \to L$.*

Proof. By Proposition 5.7.19, the rational k-maps $X \dashleftarrow Y$ are identified with k-morphisms $f : X \to Y$. The morphism f is dominante if and only if $f(x) = y$, where x and y are the generic points of X and Y, respectively. The corollary then follows from Corollary 4.7.6. □

Example 5.7.23. We can precise the result of corollary 5.7.22 if Y is the projective line $\mathbb{P}^1_k = \mathrm{Proj}(k[T_0, T_1])$, where T_0 and T_1 are indeterminates. This is an integral and separated k-scheme, and the induced subscheme $D_+(T_0)$ of Y is isomorphic to $\mathrm{Spec}(k[T])$, so the generic point of Y is the ideal (0) of $k[T]$ and its rational function field is $k(T)$, which shows that Y is a complete algebraic cuver over k. Moreover, the only graded ideal of $S = k[T_0, T_1]$ containing T_0 and distinct from S_+ is the principal ideal (T_0), so the complement of $D_+(T_0)$ in Y is reduced to a closed point, called the "infinite point" and denoted by ∞.

Corollary 5.7.24. *Let X be an irreducible normal algebraic curve with $K = K(X)$. Then there exists a canonical correspondence between K and the set of morphisms $u : X \to \mathbb{P}^1_k$ which is distinct from the constant morphism with value ∞. For u to be dominant, it is necessary and sufficient that the corresponding element in K is transcendental over k.*

Proof. By Corollary 4.7.6 and Example 5.7.23, the rational maps $X \dashrightarrow \mathbb{P}^1_k$ (hence morphisms $X \to \mathbb{P}^1_k$, in view of Proposition 5.7.19) correspond to points of $\mathbb{P}^1_k$ with values in K. For any element $\xi \in K$, we have a induced homomorphism $k[T_0] \to K$ which maps T_0 to ξ, and therefore a morphism $\mathrm{Spec}(K) \to D_+(T_0)$. By composing with the canonical injection $D_+(T_0) \to \mathbb{P}^1_k$, we obtain a point of $\mathbb{P}^1_k$ with values in K which is not located at ∞, which is completely determined by ξ in view of Proposition 4.2.4. On the other hand, by Corollary 4.2.15, any constant morphism $u : \mathrm{Spec}(K) \to \mathbb{P}^1_k$ with value y factors into

$$u : \mathrm{Spec}(K) \to \mathrm{Spec}(\kappa(y)) \to \mathbb{P}^1_k.$$

Since $y \in D_+(T_0)$, the morphism $\mathrm{Spec}(\kappa(y)) \to \mathbb{P}^1_k$ factors through the canonical injection $D_+(T_0) \to \mathbb{P}^1_k$, and u therefore is obtained by morphism $\mathrm{Spec}(K) \to D_+(T_0)$, which corresponds to an element $\xi \in K$; this proves the first part of the corollary. For the morphism u to be dominant, it is necessary that the morphism $\mathrm{Spec}(K) \to \mathrm{Spec}(k[T_0]) \cong D_+(T_0)$ is dominant, which means the homomorphism $k[T] \to K$ is injective (Proposition **??**), and this is true if and only if ξ is transcendental. □

Remark 5.7.8. With the notations of Corollary 5.7.24, we now determine the image of the morphism $\mathrm{Spec}(K) \to \mathbb{P}^1_k$ induced by an algerbaic element $\xi \in K$ over k. By definition, if $y \in \mathbb{P}^1_k$ is this image, the morphism factors into

$$\mathrm{Spec}(K) \to \mathrm{Spec}(\kappa(y)) \to \mathrm{Spec}(k[T]) \to \mathbb{P}^1_k$$

where we write $\mathrm{Spec}(k[T])$ for $D_+(T_0)$. We first note that any prime ideal in $k[T]$ is maximal, so $\kappa(y) = k[T]/\mathfrak{p}_y$, and the morphism $\mathrm{Spec}(K) \to \mathrm{Spec}(\kappa(y))$ is identified with the canonical injection $k[T]/\mathfrak{p}_y \to K$. By definition, this homomorphism is induced by the homomorphism $k[T] \to K, T \mapsto \xi$, so if $f(T)$ is the irreducible polynomial of ξ over k, we have $\mathfrak{p}_y = (f)$, and y is therefore the point of $\mathrm{Spec}(k[T])$ corresponding to (f). In particular, if $\xi \in k$, then $\mathfrak{p}_y = (T - \xi)$, and if k is an algebraically closed field, we conclude that any element $\xi \in k$ corresponds to a morphism $\mathrm{Spec}(K) \to \mathbb{P}^1_k$ with image ξ (identified with its corresponding maximal ideal $(T-\xi)$), and any element $\xi \in K - k$ corresponds to a dominant morphism $\mathrm{Spec}(K) \to \mathbb{P}^1_k$.

Corollary 5.7.25. *Let X and Y be normal, complete and irreducible algebraic curves, with $K = K(Y)$, $L = K(X)$. Then there exist a bijective correspondense between the set of k-isomorphisms $X \xrightarrow{\sim} Y$ and the set of k-isomorphisms $K \xrightarrow{\sim} L$.*

Corollary 5.7.25 shows that a normal, complete and irreducible algerbaic curves over k is determined by its rational function field K up to an isomorphism. By definition, K is a field of finite type over k with transcendental degree 1 (which is called a algerbaic function field of one variable).

Proposition 5.7.26. *For any extension K of k of finite type and transcendental degree 1, there exists a normal, complete and irreducible algebraic curve X such that $K(X) = K$. The set of local rings of X is identified with the set formed by K and the valuation rings containing k with fraction field K.*

Proof. In fact, K is a finite extension of a purely transcendental extension $k(T)$ of k, which is identified with the rational function field of $Y = \mathbb{P}^1_k$. Let X be the inegral closure of Y relative to K; X is then a normal algebraic curve with field K (Proposition 5.6.22), and it is complete since the morphism $X \to Y$ is finite (Corollary 5.7.17). For $x \in X$, the local ring $\mathscr{O}_{X,x}$ is equals to K if x is the generic of X, and otherwise it is a discrete valuation ring of K. Conversely, let A be a discrete valuation ring with fraction field K; as the morphism $X \to \mathrm{Spec}(k)$ is proper and A dominates k, it also dominates a local ring $\mathscr{O}_{X,x}$ of X by Proposition 5.7.11, and therefore equals to $\mathscr{O}_{X,x}$. □

Remark 5.7.9. It follows from Proposition 5.7.26 and Corollary 5.7.25 that giving a normal, complete and irreducible algebraic curve over k is essentially equivalent to giving a extension K of k of finite type and transcendental degree 1. We note that if k' is an extension of k, $X \otimes_k k'$ is also a complete algebraic curve over k' (Proposition 5.5.17(iii)), but in general it is neither reduced nor irreducible. However, this will be the case if K is a separable extension of k and if k is algebraically closed in K (which is expressed, in a classical terminology, that K is a "regular extension" of k). But even in this case, it may happen that $X \otimes_k k'$ is not not normal.

5.8 Blow up of schemes, projective cones and closures

5.8.1 Blow up of schemes

Let Y be a scheme and $(\mathscr{I}_n)_{n\geq 0}$ be a decreasing sequence of quasi-coherent ideal of $\mathscr{O}_Y$. Suppose that the following conditions are satisfied:

$$\mathscr{I}_0 = \mathscr{O}_Y, \quad \mathscr{I}_n\mathscr{I}_n \subseteq \mathscr{I}_{m+n}$$

where m, n are integers. If this is true, we say that the sequence $(\mathscr{I}_n)_{n\geq 0}$ is **filtered**, or that $(\mathscr{I}_n)_{n\geq 0}$ is a **filtered sequence of quasi-coherent ideals of** $\mathscr{O}_Y$. We note that this hypothesis implies that $\mathscr{I}_1^n \subseteq \mathscr{I}_n$. Put

$$\mathscr{S} = \bigoplus_{n\geq 0} \mathscr{I}_n.$$

It follows the assumption that $\mathscr{S}$ is a quasi-coherent $\mathscr{O}_Y$-algebra, hence defines a Y-scheme $X = \mathrm{Proj}(\mathscr{S})$. If $\mathscr{J}$ is an invertible ideal of $\mathscr{O}_Y$, $\mathscr{I}_n \otimes_{\mathscr{O}_Y} \mathscr{J}^{\otimes n}$ is canonically idealtified with $\mathscr{I}_n\mathscr{J}^n$, and if we replace $\mathscr{I}_n$ by $\mathscr{I}_n\mathscr{J}^n$, then the obtained $\mathscr{O}_Y$-algebra $\mathscr{S}_{(\mathscr{J})}$ satisfies that $X_{(\mathscr{J})} = \mathrm{Proj}(\mathscr{S}_{(\mathscr{J})})$ is canonically isomorphic to X (Proposition 5.3.6).

Suppose that Y is locally integral, so that $\mathscr{K}(Y)$ is a quasi-coherent $\mathscr{O}_Y$-algebra (Proposition 4.7.22). We say a sub-$\mathscr{O}_Y$-module $\mathscr{I}$ of $\mathscr{K}(Y)$ is a **fractional ideal** of $\mathscr{K}(Y)$ if it is of finite type. Given a filtered sequence $(\mathscr{I}_n)_{n\geq 0}$ of quasi-coherent fractional ideal of $\mathscr{K}(Y)$, we can then define the quasi-coherent graded $\mathscr{O}_Y$-algebra $\mathscr{S}$ and the corresponding Y-scheme $X = \mathrm{Proj}(\mathscr{S})$. We then see that for an invertible fractional ideal $\mathscr{J}$ of $\mathscr{K}(Y)$, there is a canonical isomorphism of X and $X_{(\mathscr{J})}$.

Let Y be a scheme (resp. a locally integral scheme), and $\mathscr{I}$ be a quasi-coherent ideal of $\mathscr{O}_Y$ (resp. a quasi-coherent fractional ideal of $\mathscr{K}(Y)$); put $\mathscr{S} = \bigoplus_{n\geq 0}\mathscr{I}^n$. The Y-scheme $X = \mathrm{Proj}(\mathscr{S})$ is said to be the scheme obtained by **blowing up along the ideal** $\mathscr{I}$, or the **blow up** of Y relative to $\mathscr{I}$. If $\mathscr{I}$ is a quasi-coherent ideal of $\mathscr{O}_Y$ and Y' is the closed subscheme defined by $\mathscr{I}$, we also say that X is the Y-scheme obtained by blowing up Y'. By definition, $\mathscr{S}$ is generated by $\mathscr{S}_1 = \mathscr{I}$; if $\mathscr{I}$ is a $\mathscr{O}_Y$-module of finite type, X is then projective over Y. By the hypotheses on $\mathscr{I}$, the $\mathscr{O}_X$-module $\mathscr{O}_X(1)$ is invertible (Proposition 5.3.14) and very ample in view of Corollary 5.4.12 for the structural morphism $j : X \to Y$. We also note that the restriction of f to $f^{-1}(Y - Y')$ is an isomorphism if $\mathscr{I}$ is the quasi-coherent ideal of $\mathscr{O}_Y$ defining Y': in fact, this question is local over Y, so it suffice to suppose $\mathscr{I} = \mathscr{O}_Y$, and this then follows from Corollary 5.3.1.

If we replace $\mathscr{I}$ by $\mathscr{I}^d$ for some $d > 0$, the blow up Y-scheme X is then replaced by a Y-scheme canonically isomorphic to X' (Proposition 5.3.6). Simialrly, for any invertible ideal (resp. fractional ideal) $\mathscr{J}$, the blow up scheme $X_{(\mathscr{J})}$ relative to $\mathscr{I}\mathscr{J}$ is canonically isomorphic to X. In particular, if $\mathscr{I}$ is an invertible ideal (resp. fractional ideal), the blow up Y-scheme relative to $\mathscr{I}$ is isomorphic to Y.

Proposition 5.8.1. *Let Y be an integral scheme.*

(i) For any filtered sequence $(\mathscr{I}_n)$ of quasi-coherent fractional ideals of $\mathscr{K}(Y)$, the Y-scheme $X = \mathrm{Proj}(\bigoplus_{n\geq 0}\mathscr{I}_n)$ is integral and the structural morphism $f : X \to Y$ is dominant.

(ii) Let $\mathscr{I}$ be a quasi-coherent fractional ideal of $\mathscr{K}(Y)$ and X be the blow up Y-scheme relative to $\mathscr{I}$. If $\mathscr{I} \neq 0$, the structural morphism $f : X \to Y$ is surjective and birational.

Proof. In case (i) the quasi-coherent graded $\mathscr{O}_Y$-algebra $\mathscr{S} = \bigoplus_{n\geq 0}\mathscr{I}_n$ is integral since for any $y \in Y$, $\mathscr{O}_{Y,y}$ is an integral domain, so the claim follows from Proposition 5.3.10. For (ii), we obtain from (i) that X is integral; if morever x and y are generic points of X and Y, we have $f(x) = y$, and it is necessary to prove that $\kappa(x) = \kappa(y)$. Now x is also the generic point of the fiber $f^{-1}(y)$; if $\psi : Z \to Y$ is the canonical morphism, where $Z = \mathrm{Spec}(\kappa(y))$, then the scheme $f^{-1}(y)$ is identified with $\mathrm{Proj}(\mathscr{S}')$, where $\mathscr{S}' = \psi^*(\mathscr{S})$ (Proposition 5.3.28). But it is clear that $\mathscr{S}' = \bigoplus_{n\geq 0}(\widetilde{\mathscr{I}_y})^n$, and as $\mathscr{I}$ is a nonzero quasi-coherent fractional ideal of $\mathscr{K}(Y)$, $\mathscr{I}_y \neq 0$ (Corollary 4.7.21), so $\mathscr{I}_y = \kappa(y)$ (since y is the generic point of Y, $\mathscr{I}_y$ is a $\kappa(y)$-vector space). The scheme $\mathrm{Proj}(\mathscr{S}')$ is then identified with $\mathrm{Spec}(\kappa(y))$ (Corollary 5.3.1), whence the assertion. □

Retain the notations of Proposition 5.8.1. By definition, the injection $\mathscr{I}_{n+1} \to \mathscr{I}_n$ defines for each $k \in \mathbb{Z}$ a injective homomorphism of degree 0 of graded $\mathscr{S}$-modules

$$u_k : \mathscr{S}_+(k+1) \to \mathscr{S}(k). \tag{8.1.1}$$

As $\mathscr{S}_+(k+1)$ and $\mathscr{S}(k+1)$ are eventually isomorphic $\mathscr{S}$-modules, the hommorphism u_k corresponds to a canonical injective homomorphism of $\mathscr{O}_X$-modules (Proposition 5.3.22):

$$\tilde{u}_k : \mathscr{O}_X(k+1) \to \mathscr{O}_X(k). \tag{8.1.2}$$

Recall on the other hand that we have defined a canonical homomorphism

$$\lambda : \mathscr{O}_X(d) \otimes_{\mathscr{O}_X} \mathscr{O}_X(k) \to \mathscr{O}_X(d+k)$$

and as the diagram

$$\begin{tikzcd}
\mathscr{S}(d) \otimes_{\mathscr{S}} \mathscr{S}(k) \otimes_{\mathscr{S}} \mathscr{S}(l) \arrow[r] \arrow[d] & \mathscr{S}(d+k) \otimes_{\mathscr{S}} \mathscr{S}(l) \arrow[d] \\
\mathscr{S}(d) \otimes_{\mathscr{S}} \mathscr{S}(k+l) \arrow[r] & \mathscr{S}(d+k+l)
\end{tikzcd}$$

is commutative, it follows from the functoriality of λ that the homomorphism λ define a quasi-coherent graded $\mathscr{O}_X$-algebra structure on $\mathscr{S}_X = \bigoplus_{n\in\mathbb{Z}} \mathscr{O}_X(n)$. Moreover, the diagram

$$\begin{tikzcd}
\mathscr{S}(d) \otimes_{\mathscr{S}} \mathscr{S}_+(k+1) \arrow[r] \arrow[d, "1\otimes u_k"'] & \mathscr{S}_+(d+k+1) \arrow[d, "u_{k+d}"] \\
\mathscr{S}(d) \otimes_{\mathscr{S}} \mathscr{S}(k) \arrow[r] & \mathscr{S}(d+k)
\end{tikzcd}$$

is commutative; the functoriality of λ shows that we have a commutative diagram

$$\begin{tikzcd}
\mathscr{O}_X(d) \otimes_{\mathscr{O}_X} \mathscr{O}_X(k+1) \arrow[r, "\lambda"] \arrow[d, "1\otimes\tilde{u}_k"'] & \mathscr{O}_X(d+k+1) \arrow[d, "\tilde{u}_{d+h}"] \\
\mathscr{O}_X(d) \otimes_{\mathscr{O}_X} \mathscr{O}_X(k) \arrow[r, "\lambda"] & \mathscr{O}_X(d+k)
\end{tikzcd} \tag{8.1.3}$$

where the horizontal arrows are the canonical homomorphisms. We can then say that the $\tilde{u}_k$ define an injective homomorphism (of degree 0) of graded $\mathscr{S}_X$-modules

$$\tilde{u} : \mathscr{S}_X(1) \to \mathscr{S}_X \tag{8.1.4}$$

Now we consider for each $n \geq 0$ the homomorphism $\tilde{v}_n = \tilde{u}_{n-1} \circ \cdots \tilde{u}_0$, which is an injective homomorphism $\mathscr{O}_X(n) \to \mathscr{O}_X$; we denote its image by $\mathscr{I}_{n,X}$, which is a quasi-coherent ideal $\mathscr{O}_X$

isomorphic to $\mathcal{O}_X(n)$. Moreover, the diagram

$$\begin{tikzcd}
\mathcal{O}_X(m) \otimes_{\mathcal{O}_X} \mathcal{O}_X(n) \arrow[r, "\lambda"] \arrow[d, "\tilde{v}_m \otimes \tilde{v}_n"'] & \mathcal{O}_X(m+n) \arrow[d, "\tilde{v}_{m+n}"] \\
\mathcal{O}_X \arrow[r, "\mathrm{id}"] & \mathcal{O}_X
\end{tikzcd}$$

is commutative for $m, n \geq 0$. We also conclude that $(\mathscr{I}_{n,X})_{n\geq 0}$ is a filtered sequence of quasi-coherent ideals of $\mathcal{O}_X$.

Proposition 5.8.2. *Let Y be a scheme, $\mathscr{I}$ be a quasi-coherent ideal of $\mathcal{O}_Y$, and X be the blow up Y-scheme relative to $\mathscr{I}$. Then for each $n > 0$ we have a canonical isomorphism*

$$\mathcal{O}_X(n) \xrightarrow{\sim} \mathscr{I}^n\mathcal{O}_X = \mathscr{I}_{n,X}$$

and therefore $\mathscr{I}^n\mathcal{O}_X$ is a very ample invertible $\mathcal{O}_X$-module relative to Y for $n > 0$.

Proof. The last assertion is immediate since $\mathcal{O}_X(1)$ is invertible (Proposition 5.3.14) and very ample relative to Y by definition. On the other hand, the image of the homomorphism $v_n : S_+(n) \to S$ is none other than $\mathscr{I}^n\mathscr{S}$, and the first assertion then follows from the exactness of the functor $\widetilde{\mathscr{M}}$ (Proposition 5.3.13) and the formula $\widetilde{\mathscr{I}\mathscr{M}} = \mathscr{I} \cdot \widetilde{\mathscr{M}}$. □

Corollary 5.8.3. *Under the hypotheses of Proposition 5.8.2, if $f : X \to Y$ is the structural morphism and Y' is the closed subscheme of Y defined by $\mathscr{I}$, the closed subscheme $X' = f^{-1}(Y')$ of X is defined by $\mathscr{I}\mathcal{O}_X$ (isomorphic to $\mathcal{O}_X(1)$), so we have an exact sequence*

$$0 \longrightarrow \mathcal{O}_X(1) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_{X'} \longrightarrow 0$$

Proof. This follows from Proposition 5.8.2 and Proposition 4.4.14(b). □

Under the hypotheses of Proposition 5.8.2, we can specify that structure of $\mathscr{I}_{n,X}$. Note that that the homomorphism

$$\tilde{u}_{-1} : \mathcal{O}_X \to \mathcal{O}_X(-1)$$

corresponds canonically to a section s of $\mathcal{O}_X(-1)$ over X, which is called the canonical section (relative to $\mathscr{I}$). In the diagram (8.1.3), the horizontal homomorphisms are isomorphisms (Corollary 5.3.16), so by replacing d by k and k by -1 in that diagram, we obtain $\tilde{u}_k = 1_k \otimes \tilde{u}_{-1}$ (where 1_k is the identity of $\mathcal{O}_X(k)$), which means the homomorphism $\tilde{u}_k$ is none other than the tensor product by the canonical section k (for any $k \in \mathbb{Z}$). The homomorphism $\tilde{u}$ of (8.1.4) can be interpreted in the same manner, and we then deduce that, for any $n \geq 0$, the homomorphism $\tilde{v}_n : \mathcal{O}_X(n) \to \mathcal{O}_X$ is the tensor product by $s^{\otimes n}$.

Corollary 5.8.4. *With the notations of Corollary 5.8.3, the underlying of X' is the set of $x \in X$ such that $s(x) = 0$, where s is the canonical section of $\mathcal{O}_X(-1)$.*

Proof. In fact, if c_x is a generator for the fiber $(\mathcal{O}_X(1))_x$ at a point x, $s_x \otimes c_x$ is canonically identified with a generator for the fiber $\mathscr{I}_{1,X}$ at the point x, and is therefore invertible if and only if $s_x \notin \mathfrak{m}_x(\mathcal{O}_X(-1))_x$, which means $s(x) \neq 0$. □

Proposition 5.8.5. *Let Y be an integral scheme, $\mathscr{I}$ be a quasi-coherent fractional ideal of $\mathscr{K}(Y)$, and X be the blow up Y-scheme relative to $\mathscr{I}$. Then there is an isomorphism $\mathscr{O}_X \xrightarrow{\sim} \mathscr{I}\mathscr{O}_X$, and in particular $\mathscr{I}\mathscr{O}_X$ is a very ample invertible $\mathscr{O}_X$-module relative to Y.*

Proof. The question is local over Y (Proposition 5.4.14), so we can assume that $Y = \mathrm{Spec}(A)$ is affine, where A is an integral domain with fraction field K and $\mathscr{I} = \tilde{\mathfrak{I}}$, where $\mathfrak{I}$ is a fractional ideal of K. Then there exists an element $a \neq 0$ such that $a\mathfrak{I} \subseteq A$. Put $S = \bigoplus_n \mathfrak{I}^n$; the map $x \mapsto ax$ is an A-isomorphism of $\mathfrak{I}^{n+1} = S(1)_n$ to $a\mathfrak{I}^{n+1} = a\mathfrak{I}S_n \subseteq \mathfrak{I}^n = S_n$, so defines a eventual isomorphism of degree 0 of graded S-modules $S_+(1) \to a\mathfrak{I}S$. But $x \mapsto a^{-1}x$ is an isomorphism of degree 0 of graded S-modules $a\mathfrak{I}S \xrightarrow{\sim} \mathfrak{I}S$, so we obtain an isomorphism $\mathscr{O}_X(1) \xrightarrow{\sim} \mathscr{I}\mathscr{O}_X$. As S is generated by $S_1 = \mathfrak{I}$, $\mathscr{O}_X(1)$ is invertible and very ample, whence the conclusion. □

Proposition 5.8.6. *Let Y be a locally Noetherian scheme, $\mathscr{I}$ ba a quasi-coherent ideal of $\mathscr{O}_Y$, and X be the blow up Y-scheme relative to $\mathscr{I}$. Let $f : X \to Y$ be the structural morphism. If $g : Z \to Y$ is any morphism such that $g^*(\mathscr{I})\mathscr{O}_Z$ is an invertible $\mathscr{O}_Z$-module, then there exists a unique morphism $\tilde{g} : Z \to X$ such that the following diagram is commutative:*

$$\begin{tikzcd} Z \arrow[r, "\tilde{g}"] \arrow[dr, "g"'] & X \arrow[d, "f"] \\ & Y \end{tikzcd}$$

Proof. The question is local over Y, so we can assume that $Y = \mathrm{Spec}(A)$ is affine, where A is Noetherian, and $\mathscr{I} = \widetilde{\mathfrak{I}}$ where $\mathfrak{I}$ is an ideal of A. Then $X = \mathrm{Proj}(S)$ where $S = \bigoplus_{n \geq 0} \mathfrak{I}^n$, and we note that since A is Noetherian, $\mathfrak{I}$ is finitely generated, so S is of finite type over A. Let $a_0, \dots, a_n \in \mathfrak{I}$ be a set of generators for $\mathfrak{I}$, so that we have a surjective homomorphism $\varphi : A[T_0, \dots, T_n] \to S$ which maps T_i to a_i, and this gives a closed immersion $i : X \to \mathbb{P}^n_A$, and we can identify X with its image. If $g : Z \to Y$ is a morphism such that $\mathscr{L} = g^*(\mathscr{I})\mathscr{O}_Z$ is invertible, then the inverse images of the a_i, which are global sections of $\mathscr{I}$, give global sections $s_0, \dots, s_n$ of $\mathscr{L}$ which generate $\mathscr{L}$. Then by Proposition 5.4.3 $r : Z \to P = \mathbb{P}^n_A$ such that $\mathscr{L} \cong r^*(\mathscr{O}_P(1))$ and $s_i = r^{-1}(T_i)$. We now claim that this morphism factors through the closed subscheme X of $\mathbb{P}^n_A$: this follows from the fact that if $F(T_0, \dots, T_n)$ is a homogeneous polynomial of degree d of $\ker \varphi$, then $F(a_0, \dots, a_n) = 0$ in A and so $F(s_0, \dots, s_n) = 0$ in $\Gamma(Z, \mathscr{L}^{\otimes d})$. This gives the desired morphism $\tilde{g} : Z \to X$, and for any such morphism we necessarily have

$$g^*(\mathscr{I})\mathscr{O}_Z = \tilde{g}^*(f^*(\mathscr{I})\mathscr{O}_X)\mathscr{O}_Z = \tilde{g}^*(\mathscr{O}_X(1))\mathscr{O}_Z$$

by Proposition 5.8.2, so we obtain a surjective hommorphism $\tilde{g}^*(\mathscr{O}_X(1)) \to g^*(\mathscr{I})\mathscr{O}_Z = \mathscr{L}$, hence an isomorphism by Corollary 1.4.38. Clearly the sections s_i of $\mathscr{L}$ are the inverse images of the sections T_i of $\mathscr{O}_P(1)$ on $\mathbb{P}^n_A$, so the uniqueness of $\tilde{g}$ follows from Proposition 5.4.3. □

Corollary 5.8.7. *Let $q : Y' \to Y$ be a morphism of locally Noetherian schemes and $\mathscr{I}$ be a quasi-coherent ideal of $\mathscr{O}_Y$. Let X be the blow up Y-scheme relative to $\mathscr{I}$ and X' be the blow up Y'-scheme* *relative to* $\mathcal{J} = q^*(\mathcal{I})\mathcal{O}_{Y}$. *Then there exists a unique morphism* $p : X' \to X$ *such that the diagram*

$$\begin{tikzcd} X' \arrow[r, "p"] \arrow[d] & X \arrow[d] \\ Y' \arrow[r, "q"] & Y \end{tikzcd} \tag{8.1.5}$$

is commutative. Moreover, if q *is a closed immersion, so is* p.

Proof. The existence and uniqueness of q follows from Proposition 5.8.6 and Proposition 5.8.2. To show that p is a closed immersion if q is, we trace the definition of the blow up: $X = \mathrm{Proj}(\mathcal{S})$ where $\mathcal{S} = \bigoplus_{n\geq 0}\mathcal{I}^n$ and $X' = \mathrm{Proj}(\mathcal{S}')$ where $\mathcal{S}' = \bigoplus_{n\geq 0}\mathcal{J}^n$. Since Y' is a closed subscheme of Y, we can consider $\mathcal{S}'$ as a sheaf of graded algebras over Y. Then there exists a natrual surjective homomorphism $\mathcal{S} \to \mathcal{S}'$, which gives rise to the closed immersion p. □

In the situation of Corollary 5.8.7, if Y' is a closed subscheme of Y, we call the closed subscheme X' of X the **strict transform** of Y' under the blowing-up $f : X \to Y$.

Example 5.8.8. Let $Y = \mathbb{A}_k^n$ be the affine space over a field k and we consider the blow up of Y at the origin y of Y. Then $Y = \mathrm{Spec}(A)$ where $A = k[X_1, \dots, X_n]$, y corresponds to the ideal $\mathfrak{I} = (X_1, \dots, X_n)$, and $X = \mathrm{Proj}(S)$ where $S = \bigoplus_{n\geq 0}\mathfrak{I}^n$. We can define a surjective homomorphism

$$\varphi : A[Y_0, \dots, Y_n] \to S$$

of graded rings by sending Y_i to X_i as an element of degree 1 in S, which gives a closed immersion of X into $\mathbb{P}_A^{n-1}$. It is not hard to see that the kernel of φ is generated by the homogeneous polynomials $X_iY_j - X_jY_i$, where $i, j = 1, \dots, n$, so this definition is compatible with the definition of the blow up of the affine variety $\mathbb{A}_k^n$.

Now if Y' is a closed subscheme of Y passing through y, then the strict transform X' of Y' is a closed subscheme of X. Hence, provided that Y' is not reduced to the point y, we can recover X' as the closure of $f^{-1}(Y' - \{y\})$, where $f : f^{-1}(Y' - \{y\}) \to Y' - \{y\}$ is an isomorphism. Again this definition is compatible with the definition of blow up of closed subvarieties of $\mathbb{A}_k^n$.

Example 5.8.9. As an example of the general concept of blowing up a coherent sheaf of ideals, we show how to eliminate the points of indeterminacy of a rational map determined by an invertible sheaf. So let A be a ring, X be a Noetherian scheme over A, $\mathcal{L}$ be an invertible sheaf on X, and $s_0, \dots, s_n$ be global sections of $\mathcal{L}$. Let U be the open subset of X where the s_i generate the sheaf $\mathcal{L}$ (that is, the subset where the corresponding homomorphism $\mathcal{O}_X^{n+1} \to \mathcal{L}$ is surjective, cf. Proposition 1.4.6(iii)). Then the invertible sheaf $\mathcal{L}|_U$ on U and the global sections $s_0, \dots, s_n$ determine an A-morphism $\varphi : U \to \mathbb{P}_A^n$, which is also a rational map $X \dashrightarrow \mathbb{P}_A^n$. We will now show how to blow up a certain sheaf of ideals $\mathcal{I}$ on X, whose corresponding closed subsheme Y has support equal to $X - U$ (i.e., the underlying topological space of Y is $X - U$), so that the morphism φ extends to a morphism $\tilde{\varphi} : \widetilde{X} \to \mathbb{P}^n_A$.

$$\begin{tikzcd}
\widetilde{X} \arrow[d, "\pi"'] \arrow[dr, dashed, "\tilde{\varphi}"] & \\
X & \mathbb{P}^n_A \\
U \arrow[u, hook] \arrow[ur, "\varphi"'] &
\end{tikzcd}$$

Let $\mathscr{F}$ be the quasi-coherent sub-$\mathscr{O}_X$-module of $\mathscr{L}$ generated by $s_0, \dots, s_n$. We define a coherent ideal $\mathscr{I}$ of $\mathscr{O}_X$ as follows: for any open subset $V \subseteq X$ and an isomorphism $\psi : \mathscr{L}|_V \cong \mathscr{O}_X|_V$, we take $\mathscr{I}|_V = \psi(\mathscr{F}|_V)$. It is clear that $\mathscr{I}_V$ is independent of the choice of ψ, so we get a well-defined coherent ideal $\mathscr{I}$ of $\mathscr{O}_X$. We also note that $\mathscr{I}_x = \mathscr{O}_{X,x}$ if and only if $x \in U$, so the corresponding closed subscheme Y has support $X - U$. Let $\pi : \widetilde{X} \to X$ be the corresponding blow up relative to $\mathscr{I}$. We claim that $\pi^*(\mathscr{I})$ is a coherent ideal of $\mathscr{O}_{\widetilde{X}}$, so is invertible by Proposition 5.8.2. This can be verified on each affine open X_{s_i}. Then the global sections $\pi^*(s_i)$ of $\pi^*(\mathscr{L})$ generate an invertible sub-$\mathscr{O}_{\widetilde{X}}$-module $\mathscr{L}'$ of $\pi^*(\mathscr{L})$. Now $\mathscr{L}'$ and the sections $\pi^*(s_i)$ define a morphism $\tilde{\varphi} : \widetilde{X} \to \mathbb{P}^n_A$ whose restriction on $\pi^{-1}(U)$ corresponds to the morphism φ under the isomorphism $\pi : \pi^{-1}(U) \xrightarrow{\sim} U$.

5.8.2 Homogenization of graded rings

Let S be a graded ring, which we do not suppose to be with positive degrees. We set

$$S^{\geqslant} = \bigoplus_{n \geq 0} S_n, \quad S^{\leqslant} = \bigoplus_{n \leq 0} S_n$$

which are subrings of S, with positive of negative degrees respectively. If f is a homogeneous element of degree d (positive of negative) of S, the localization $S_f = S'$ is endowed with a graded ring structure, where S'_n is the set of elements x/f^k, where $x \in S_{n+kd}$ $(k \geq 0)$; we also note that $S_{(f)} = S'_0$, and we denote $S_f^{\geqslant}$ and $S_f^{\geqslant}$ for $S'^{\geqslant}$ and $S'^{\leqslant}$, respectively. This notation is justified by the fact that if $d > 0$, then we have

$$(S^{\geqslant})_f = S_f; \tag{8.2.1}$$

in fact, if $x \in S_{n+kd}$ where $n + kd < 0$, we can write $x/f^k = xf^h/f^{h+k}$ so that $n + (h+k)d > 0$ for $h > 0$ large enough. We then conclude by definition that

$$(S^{\geqslant})_{(f)} = (S_f^{\geqslant})_0 = S_{(f)}. \tag{8.2.2}$$

If M is a graded S-module, we put similarly

$$M^{\geqslant} = \bigoplus_{n \geq 0} M_n, \quad M^{\leqslant} = \bigoplus_{n \leq 0} M_n$$

which are respectively $S^{\geqslant}$-module and $S^{\geqslant}$-module, with intersection the S_0-module M_0. If $f \in S_d$, we also define M_f as the graded S_f-module such that $(M_f)_n$ is the set of elements

z/f^k, where $z \in M_{n+kd}$. We denote by $M_{(f)}$ the set of degree 0 elements in M_f, which is an $S_{(f)}$-module, and we write $M_f^{\geqslant}$ and $M_f^{\leqslant}$ for $(M_f)^{\geqslant}$ and $(M_f)^{\leqslant}$ respectively. If $d > 0$, we also have

$$(M^{\geqslant})_f = M_f, \quad (M^{\geqslant})_{(f)} = (M_f^{\geqslant})_0 = M_{(f)}. \tag{8.2.3}$$

Let $\boldsymbol{z}$ be an indeterminate, which is called the **homogenization variable**. If S is a graded ring, the polynomial algebra

$$\widehat{S} = S[\boldsymbol{z}]$$

is a graded S-algebra, where for f homogeneous we put

$$\deg(f\boldsymbol{z}^n) = n + \deg(f).$$

Lemma 5.8.10. *Let $f \in S_d$ with $d > 0$. We have canonical isomorphisms*

$$\widehat{S}_{(\boldsymbol{z})} \cong \widehat{S}/(\boldsymbol{z}-1)\widehat{S} \cong S, \tag{8.2.4}$$
$$\widehat{S}_{(f)} \cong S_f^{\leqslant}. \tag{8.2.5}$$

Proof. The first isomorphism of (8.2.4) is already defined in Proposition 5.2.3 and the second one is trivial; the isomorphism $\widehat{S}_{(z)} \cong S$ thus defined then send an element $x\boldsymbol{z}^n/\boldsymbol{z}^{n+k}$ (where $\deg(x) = k$ for $k \geq -n$) to the element x. The homomorphism (8.2.5) is defined by sending $x\boldsymbol{z}^n/f^k$ (where $\deg(x) = kd - n$) to the element x/f^k, of degree $-k$ in $S_f^{\leqslant}$, and it is easy to verify that this is an isomorphism. □

Let M be a graded S-module. It is clear that the S-module

$$\widehat{M} = M \otimes_S \widehat{S} = M \otimes_S S[\boldsymbol{z}]$$

is the direct sum of the S-modules $M \otimes S\boldsymbol{z}^n$, whence the abelian groups $M_k \otimes S\boldsymbol{z}^n$. We define over $\widehat{M}$ an $\widehat{S}$-module structure by

$$\deg(x \otimes \boldsymbol{z}^n) = n + \deg(x)$$

for x homogeneous in M.

Lemma 5.8.11. *Let $f \in S_d$ with $d > 0$. We have a canonical isomorphism*

$$\widehat{M}_{(\boldsymbol{z})} \cong \widehat{M}/(\boldsymbol{z}-1)\widehat{M} \cong M, \tag{8.2.6}$$
$$\widehat{M}_{(f)} \cong M_f^{\leqslant}. \tag{8.2.7}$$

Proof. This can be proved as Lemma 5.8.10 by using the second part of Proposition 5.2.3. □

Let S be a graded ring with positive degrees. Then for each $n \geq 0$, we can consider $S(n) = \bigoplus_{m \geq n} S_m$ as a graded iedal of S (in particular $S(0) = S$ and $S(1) = S_+$). As it is clear that

$S(m)S(n) \subseteq S(m+n)$, we can then define a graded ring

$$S^{\natural} = \bigoplus_{n \geq 0} S(n)$$

whence $S^{\natural}_n = S(n)$. Then $S^{\natural}_0$ is equal to S considered as a nongraded ring, and $S^{\natural}$ is therefore an S-algebra. For any homogeneous element $f \in S_d$ with $d > 0$, we denote by $f^{\natural}$ the element f considered as an element of $S(d) = S^{\natural}_d$.

Lemma 5.8.12. *Let S be a graded ring with positive degrees, f be a homogeneous element with $d > 0$. We have canonical isomorphisms:*

$$S_f \cong \bigoplus_{n \in \mathbb{Z}} S(n)_{(f)}, \tag{8.2.8}$$

$$(S^{\geqslant}_f)_{f/1} \cong S_f, \tag{8.2.9}$$

$$S^{\natural}_{(f^{\natural})} \cong S^{\geqslant}_f. \tag{8.2.10}$$

the first two of which are bi-isomorphisms of graded rings.

Proof. It is immediate that we have $(S_f)_n = (S(n)_f)_0 = S(n)_f$, whence the first isomorphism. On the other hand, as $f/1$ is invertible in S_f, there is a canonical isomorphism $S_f \cong S^{\geqslant}_f = (S_f)_{f/1}$ in view of (8.2.1) applide to S_f. Finally, if $x = \sum_{m \geq n} y_n$ is an element of $S(n)$, where $n = kd$, we can correspond the element $x/(f^{\natural})^k$ to the element $\sum_m y_m/f^k$ of $S^{\geqslant}_f$, and we verify that this is an isomorphism. □

If M is a graded S-module, we can similarly put for $n \in \mathbb{Z}$

$$M^{\natural} = \bigoplus_{n \in \mathbb{Z}} M(n)$$

as $S(m)M(n) \subseteq M(m+n)$. Then $M^{\natural}$ is a graded $S^{\natural}$-module, and similarly we have the following:

Lemma 5.8.13. *Let $f \in S_d$ be a homogeneous element with $d > 0$. We have the following bi-homomorphisms*

$$M_f \cong \bigoplus_{n \in \mathbb{Z}} M(n)_{(f)}, \tag{8.2.11}$$

$$(M^{\geqslant}_f)_{f/1} \cong M_f, \tag{8.2.12}$$

$$M^{\natural}_{(f^{\natural})} \cong M^{\geqslant}_f. \tag{8.2.13}$$

the first two of which are bi-isomorphisms of graded modules.

Remark 5.8.1. We can think that $S^{\natural}$ is obtained from S by adding a "phantom" element $\boldsymbol{y}$ of degree -1. The component $S(n)$ can be then considered as the S-module $(S\boldsymbol{y}^n)_0$, which is the set of degree 0 elements in $S\boldsymbol{y}^n$. With this understanding, we can then relate the results of Lemma 5.8.10 and Lemma 5.8.12.

Lemma 5.8.14. *Let S be a graded ring with positive degrees.*

(i) For $S^\natural$ to be an $S_0^\natural$-algebra of finite type (resp. Noetherian), it is necessary and sufficient that S is an S_0-algebra of finite type (resp. Noetherian).

(ii) For $S_{n+1}^\natural = S_1^\natural S_n^\natural$ for $n \geq n_0$, it is necessary and sufficient that $S_{n+1} = S_1 S_n$ for $n \geq n_0$.

(iii) For $S_n^\natural = (S_1^\natural)^n$ for $n \geq n_0$, it is necessary and sufficient that $S_n = S_1^n$ for $n \geq n_0$.

(iv) If (f_α) is a set of homogeneous elements of S_+ such that the radical in S_+ of the ideal of S_+ generated by the f_α is equal to S_+, then $S_+^\natural$ is the radical in $S_+^\natural$ of the ideal of $S_+^\natural$ generated by the $f_\alpha^\natural$.

Proof. If $S^\natural$ is an $S_0^\natural$-algebra of finite type, $S_+ = S_1^\natural$ is a finitely generated module over $S = S_0^\natural$ by Proposition **??**, so S is an S_0-algebra of finite type by Corollary **??**. If $S^\natural$ is a Noetherian ring, so is the ring $S_0^\natural = S$ by Corollary **??**. Conversely, if S is an S_0-algebra of finite type, then by Proposition **??** there exists $d > 0$ and $n_0 > 0$ such that $S_{n+d} = S_h S_n$ for any $n \geq n_0$; we can clearly suppose that $n_0 \geq d$. Moreover, the S_n are finitely generated S_0-modules (Corollary **??**(c)), so if $n \geq n_0 + d$, we have $S_n^\natural = S_n S_{n-d}^\natural = S_d^\natural S_{n-d}^\natural$; and if $n < n_0 + d$, we have

$$S_n^\natural = S_n + \cdots + S_{n_0+d-1} + S_d E + S_d^2 E + \cdots$$

where $E = S_{n_0} + \cdots + S_{n_0+d-1}$. For $1 \leq n \leq n_0$, let G_n be the union of the generators of the S_0-module S_i where $n \leq i \leq n_0 + d - 1$, considered as subsets of $S(n)$. For $n_0 + 1 \leq n \leq n_0 + d - 1$, similarly let G_n be the union of the generators of the S_0-module S_i where $n \leq i \leq n_0 + d - 1$ and of $S_d E$, considered as subsets of $S(n)$. Then it is clear that we have $S_n^\natural = S_0^\natural G_n$ for $1 \leq n \leq n_0 + d - 1$, and therefore the union G of the G_n for $1 \leq n \leq n_0 + d - 1$ is a system of generators of the $S_0^\natural$-algebra $S^\natural$. We then conclude that if $S = S_0^\natural$ is Noehterian, then so is $S^\natural$.

It is clear that if $S_{n+1} = S_1 S_n$ for $n \geq n_0$, then we have $S_{n+1}^\natural = S_1 S_n^\natural$, and a fortiori $S_{n+1}^\natural = S_1^\natural S_n^\natural$ for $n \geq n_0$. Conversely, the latter relation means that

$$S_{n+1} + S_{n+2} + \cdots = (S_1 + S_2 + \cdots)(S_n + S_{n+1} + \cdots);$$

by comparing the degree $n + 1$ component of both sides, we conclude that $S_{n+1} = S_1 S_n$. This proves the assertion (ii).

If $S_n = S_1^n$ for $n \geq n_0$, we have $S_n^\natural = \bigoplus_{m \geq n} S_1^m$, and as $S_1^\natural$ contains $\bigoplus_{m \geq 1} S_1^m$, we then have $S_n^\natural \subseteq (S_1^\natural)^n$ for $n \geq n_0$. Conversely, the degree n component of $(S_1^\natural)^n = (S_1 + S_2 + \cdots)^n$ considered as elements of S is equal to S_1^n; so the relation $S_n^\natural = (S_1^\natural)^n$ implies $S_n = S_1^n$.

Finally, to prove (iv), it suffices to show that if an element $g \in S_{k+d}$ is considered as an element of $S_k^\natural$ ($k > 0$, $d \geq 0$), then there exists an integer $n > 0$ such that in $S_{kn}^\natural$, g^n is a linear combination of the $f_\alpha^\natural$ with coefficients in $S^\natural$. By hypothesis, there exists an integer n_0 such that for $n \geq n_0$, we have $g^n = \sum_\alpha c_{\alpha n} f_\alpha$ in S, where the indices α appearing in this formula are independent of n. Moreover, we can evidently suppose that the $c_{\alpha n}$ are homogeneous, with

$$\deg(c_{\alpha n}) = n(k + d) - \deg(f_\alpha)$$

in S. Let n_0 be large enough such that we have $kn_0 > deg(f_\alpha)$ for the f_α appearing in the formula of g^n; for any α, let $c'_{\alpha n}$ be the element $c_{\alpha n}$ considered as an element of degree $kn - \deg(f_\alpha)$ in $S^\natural$. We then have $g^n = \sum_\alpha c'_{\alpha m} f_\alpha^\natural$ in $S^\natural$, which proves our assertion. $\square$

Consider the graded S_0-algebra

$$S^\natural \otimes_S S_0 = S^\natural / S_+ S^\natural = \bigoplus_{n\geq 0} S(n)/S_+S(n).$$

As S_n is a quotient S_0-module of $S(n)/S_+S(n)$, we have a canonical hommorphism of graded S_0-algebras

$$S^\natural \otimes_S S_0 \to S \tag{8.2.14}$$

which is evidently surjective, and corresponds to a canonical closed immersion

$$\mathrm{Proj}(S) \to \mathrm{Proj}(S^\natural \otimes_S S_0) \tag{8.2.15}$$

Proposition 5.8.15. *The canonical morphism (8.2.15) is bijective. For the homomorphism (8.2.14) to be eventually bijective, it is necessary and sufficient that there exists an integer n_0 such that $S_{n+1} = S_1 S_n$ for $n \geq n_0$. If this is satisfied, then the morphism (8.2.15) is an isomorphism, and the converse of this is also true if S is Noetherian.*

Proof. To prove the first assertion, it suffices (Corollary 5.2.45) to prove that the kernel $\mathfrak{I}$ of the homomorphism (8.2.14) is formed by nilpotent elements. Now if $f \in S(n)$ is an element whose class mod $S_+S(n)$ belongs to this kernel, then $f \in S(n+1)$ (by the definition of (8.2.14)); the element f^{n+1}, considered as an element of $S(n(n+1))$, then belongs to $S_+S(n(n+1))$, if we write it as $f \cdot f^n$. Then the class of f^{n+1} mod $S_+S(n(n+1))$ is zero, which proves our assertion.

As the hypothesis $S_{n+1} = S_1 S_n$ for $n \geq n_0$ is equivalent to $S^\natural_{n+1} = S^\natural_1 S^\natural_n$ for $n \geq n_0$ (Lemma 5.8.14(ii)), these hypotheses are equivalent to that (8.2.14) is eventually injective, hence eventually bijective, and then (8.2.15) is an isomorphism by Proposition 5.3.6(a). Conversely, assume that S is Noetherian, hence so is $S^\natural$ and $S^\natural \otimes_S S_0$ (Lemma 5.8.14(i)). If (8.2.15) is an isomorphism, the sheaf $\widetilde{\mathscr{F}}$ over $\mathrm{Proj}(S^\natural \otimes_S S_0)$ is zero (Proposition 5.3.30(a)); as $S^\natural \otimes_S S_0$ is Noetherian, we then conclude from Proposition 5.2.36(b) that $\mathfrak{I}$ is eventually null, so $S^\natural_{n+1} = S^\natural_1 S^\natural_n$ for $n \geq n_0$. ◻

Consider now the canonical injection $(S_+)^n \to S(n)$, which defines an injective homomorphism of degree 0 of graded rings

$$\bigoplus_{n\geq 0} (S_+)^n \to S^\natural. \tag{8.2.16}$$

Proposition 5.8.16. *For the homomorphism (8.2.16) to be an eventual isomorphism, it is necessary and sufficient that there exists an integer n_0 such that $S_n = S_1^n$ for $n \geq n_0$. If this is the case, the corresponding morphism*

$$\mathrm{Proj}(S^\natural) \to \mathrm{Proj}\Big(\bigoplus_{n\geq 0} (S_+)^n\Big) \tag{8.2.17}$$

is everywhere defined and an isomorphism, and the converse is also true if S is Noetherian.

Proof. The first two conditions are equivalent in view of Lemma 5.8.14(iii). The third assertion follows from Lemma 5.8.14(i), (iii) and Lemma 5.8.17. □

Lemma 5.8.17. *Let S be a graded ring with positive degrees which is a S_0-algebra of finite type. If the morphism corresponding to the canonical injection $S' = \bigoplus_{n\geq 0} S_1^n \to S$ is everywhere defined and an isomorphism, then there exists $n_0 > 0$ such that $S_n = S_1^n$ for $n \geq n_0$.*

Proof. In fact, let f_i $(1 \leq i \leq r)$ be a system of generators for the S_0-module S_1. Then the hypotheses implies that the $D_+(f_i)$ cover $\mathrm{Proj}(S)$. Let $(g_j)_{1\leq j\leq n}$ be a system of homogeneous elements of S_+, with $n_j = \deg(g_j)$, which together with the f_i form a system of generators of the ideal S_+, or a system of generators of S as an S_0-algebra. The elements $g_j/f_i^{n_j}$ of the ring $S_{(f_i)}$ then by hypotheses belong to the subring $S'_{(f_i)}$, so there exists an integer k such that $S_1^k g_j \subseteq (S_1)^{k+n_j}$ for any j. We then conclude by recurrence on r that $S_1^k g_j^r \subseteq S'$ for any $r \geq 1$, and by the choice of the g_j, we then have $S_1^k S \subseteq S'$. On the other hand there exists for any j an integer m_j such that $g_j^{m_j}$ belongs to the ideal of S generated by the f_i (Corollary 5.2.11), so $g_j^{m_j} \in S_1 S$, and $g_j^{m_j k} \in S_1^k S \subseteq S'$. Therefore there exists an integer $m_0 \geq k$ such that $g_j^m \in S_1^{mn_j}$ for $m \geq m_0$. Now if d is the largest of the n_j, the number $n_0 = dm_0 + k$ then satisfies the requirement. In fact, an element of S_n, for $n \geq n_0$, is a sum of elements of $S_1^\alpha u$, where u is a product of powers of g_j; if $\alpha \geq k$, it follows from the choice of k that $S_1^\alpha u \subseteq S_1^n$; in the contrary case, at least one of the exponent of g_j is $\geq m_0$, so $u \in S_1^\beta v$ where $\beta \geq m_0 \geq k$ and v is a product of powers of g_j, so we are reduced to the previous case, and $S_1^\alpha u \subseteq S_1^n$. This completes the proof. □

Remark 5.8.2. The condition $S_n = S_1^n$ for $n \geq n_0$ clearly implies that $S_{n+1} = S_1 S_n$ for $n \geq n_0$, but the converse is not ture, even we assume that S is Noetherian. For example, let K be a field, $A = K[\boldsymbol{x}]$, $B = K[\boldsymbol{y}]/\boldsymbol{y}^2 K[\boldsymbol{y}]$, where $\boldsymbol{x}, \boldsymbol{y}$ are two indeterminates, with $\deg(\boldsymbol{x}) = 1$ and $\deg(\boldsymbol{y}) = 2$, and let $S = A \otimes_K B$, so that S is a graded algebra over K having a basis formed by the elements $\boldsymbol{x}^n$ and $\boldsymbol{x}^n\boldsymbol{y}$. It is immediate that $S_{n+1} = S_1 S_n$ for $n \geq 2$, but $S_1^n = K\boldsymbol{x}^n$ while $S_n = K\boldsymbol{x}^n + K\boldsymbol{x}^{n-2}\boldsymbol{y}$ for $n \geq 2$.

5.8.3 Projective cones

Let Y be a scheme; in this subsection, we only consider Y-schemes and Y-morphisms. Let $\mathscr{S}$ be a quasi-coherent graded $\mathscr{O}_Y$-algebra with positive degrees, which we suppose that $\mathscr{S}_0 = \mathscr{O}_Y$. According to the notations of the previous part, we put

$$\widehat{\mathscr{S}} = \mathscr{S}[\boldsymbol{z}] = \mathscr{S} \otimes_{\mathscr{O}_Y} \mathscr{O}_Y[\boldsymbol{z}] \tag{8.3.1}$$

which we consider as a graded $\mathscr{O}_Y$-algebra with positive degrees, so that for any affine open subset U of Y, we have

$$\Gamma(U, \mathscr{S}) = \Gamma(U, \mathscr{S})[\boldsymbol{z}].$$

In the following, we put

$$P = \mathrm{Proj}(\mathscr{S}), \quad C = \mathrm{Spec}(\mathscr{S}), \quad \widehat{C} = \mathrm{Proj}(\widehat{\mathscr{S}})$$

(where in the definition of C, $\mathscr{S}$ is considered as a nongraded $\mathscr{O}_Y$-algebra), and we say that C (resp. $\widehat{C}$) is the **affine cone** (resp. **projective cone**) defined by $\mathscr{S}$; we will also say "cone" instead of "cone affine". By abuse of language, we sat that C (resp. $\widehat{C}$) is the **affine projecting cone** of P (resp. the **projective projecting cone** of P), which P is understood to be given of the form $\mathrm{Proj}(\mathscr{S})$. Finally, we say that $\widehat{C}$ is the projective closure of C, where C is understood to be a scheme of the form $\mathrm{Spec}(\mathscr{S})$.

Lemma 5.8.18. *Let S be a graded ring with positive degrees, $X = \mathrm{Proj}(S)$, and f be a homogeneous element of S with degree $d > 0$. If f is not a divisor of zero in S, X is the smallest closed subscheme of X such that $X_f = D_+(f)$.*

Proof. This question is clearly local on X; for any homogeneous element $g \in S_h$ $(h > 0)$, it suffices to prove that X_g is the smallest closed subscheme of X_g which dominates X_{fg}. It follows from the definition and Proposition **??** that this condition is equivalent to the fact that the canonical homomorphism $S_{(g)} \to S_{(fg)}$ is injective. Now this homomorphism is identified canonically with the homomorphism $S_{(g)} \to (S_{(g)})_{f^h/g^d}$ (Lemma 5.2.1). But as f^h is not a zero divisor of S, f^h/g^e is not a zero divisor in S_g (and a fortiori in $S_{(g)}$), because the relation $(f^h/g^d)(t/g^m) = 0$ with $t \in S$ and $m > 0$ implies the existence of an integer $n > 0$ such that $g^n f^h t = 0$, whence $g^n t = 0$, and therefore $t/g^m = 0$ in S_g. This proves the claim. □

Proposition 5.8.19. *There is a commutative diagram*

$$\begin{tikzcd} & \widehat{C} \arrow[dd] & \\ P \arrow[ur, "j"] \arrow[dr] & & C \arrow[ul, "i"'] \arrow[dl] \\ & Y \arrow[ur, bend left, "\varepsilon"] & \end{tikzcd}$$

where ε and j are closed immersions and i is an affine morphism which is an open dominant immersion such that

$$i(C) = \widehat{C} - j(P). \tag{8.3.2}$$

Moreover $\widehat{C}$ is the smallest closed subscheme of $\widehat{C}$ dominating $i(C)$.

Proof. To define i, we consider the open subset $\widehat{C}_z = \mathrm{Spec}(\widehat{\mathscr{S}}/(z-1)\widehat{\mathscr{S}})$ of $\widehat{C}$ (by (8.2.4)), where z is canonically identified with a section of $\widehat{\mathscr{S}}$ over Y. The isomorphism $i : C \xrightarrow{\sim} \widehat{C}_z$ corresponds then to the canonical isomorphism

$$\widehat{\mathscr{S}}/(z-1)\widehat{\mathscr{S}} \cong \mathscr{S}$$

of (8.2.4). The morphism ε corresponds to the augmentation hommorphism $\mathscr{S} \to \mathscr{S}_0 = \mathscr{O}_Y$ with kernel $\mathscr{S}_+$, which is surjective so ε is a closed immersion (Proposition 5.1.25). Finally, j corresponds similar to the surjective homomorphism $\widehat{\mathscr{S}} \to \mathscr{S}$ of degree 0, which is the identity on $\mathscr{S}$ and zero on $z\widehat{\mathscr{S}}$. By Proposition 5.3.30 it is clear that j is everywhere defined and a closed immersion.

To prove the other assertions of the proposition, we can evidently assume that $Y = \mathrm{Spec}(A)$ is affine, $\mathscr{S} = \widetilde{S}$ where S is a graded A-algebra, whence $\widehat{\mathscr{S}} = (\widehat{S})^\sim$); the homogeneous elements f of S_+ are then identified with the sections of $\widehat{\mathscr{S}}$ over Y, and the open subset $D_+(f)$ of $\widehat{C}$ is identified with $\widehat{C}_f$ (Proposition 5.2.5); similarly the open subset $D(f)$ of C is identified with C_f. It then follows from Proposition 5.2.11 and the definition of $\widehat{S}$ that the open subset $\widehat{C}_{\boldsymbol{z}} = i(C)$ together with $\widehat{C}_f$ (where f is homogeneous in S_+) constitute an open covering of $\widehat{C}$. Moreover, we have

$$i^{-1}(\widehat{C}_f) = C_f. \tag{8.3.3}$$

In fact, if we identify $i(C)$ with $\widehat{C}_{\boldsymbol{z}}$, then

$$\widehat{C}_f \cap i(C) = \widehat{C}_f \cap \widehat{C}_{\boldsymbol{z}} = \widehat{C}_{f\boldsymbol{z}} = \operatorname{Spec}(\widehat{S}_{(f\boldsymbol{z})}).$$

Now if $d = \deg(f)$, $\widehat{S}_{(f\boldsymbol{z})}$ is canonically isomorphic to $(\widehat{S}_{(\boldsymbol{z})})_{(f/\boldsymbol{z}^d)}$ (Lemma 5.2.1), and it follows from the definition of the isomorphism (8.2.4) that the image of $(\widehat{S}_{(\boldsymbol{z})})_{(f/\boldsymbol{z}^d)}$ under the corresponding isomorphism is exactly S_f. As $C_f = \operatorname{Spec}(S_f)$, this proves (8.3.3) and also shows that the morphism i is affine. Moreover, the restriction of i to C_f, considered as a morphism into $\widehat{C}_f$, corresponds (Proposition 4.2.4) to the canonical homomorphism $\widehat{S}_{(f)} \to \widehat{S}_{(f\boldsymbol{z})} \cong S_f$. We also note that under the isomorphism (8.2.5), $\widehat{C}_f$ is canonically idnetified with $\operatorname{Spec}(S_f^{\leqslant})$ and the morphism restriction $i|_{C_f} : C_f \to \widehat{C}_f$ corresponds to the canonical injection $S_f^{\leqslant} \to S_f$. The complement of $\widehat{C}_{\boldsymbol{z}}$ in $\widehat{C} = \operatorname{Proj}(\widehat{S})$ is, by definition, the set of graded prime ideals of $\widehat{S}$ containing $\boldsymbol{z}$, which is $j(P)$ from the defintion of j, whence (8.3.2).

To prove the final assertion, we can still assume that Y is affine. With the preceding notations, we note that $\boldsymbol{z}$ is not a zero divisor in $\widehat{S}$, so we can apply Lemma 5.8.18. □

We now identify the affine cone C with the open subscheme $i(C)$ of the projective cone $\widehat{C}$, which is dense in $\widehat{C}$. The closed subscheme of C associated with the closed immersion ε is called the **sommet scheme** of C. We also say that ε, which is a Y-section of C, is the **sommet section** or the **zero section** of C; we can then identify Y with the sommet scheme of C via the morphism ε. The composition $i \circ \varepsilon$ is a Y-section of $\widehat{C}$, which is also a closed immersion (Corollary 4.5.19), corresponding to the canonical surjective homomorphism $\widehat{\mathscr{S}} = \mathscr{S}[\boldsymbol{z}] \to \mathscr{O}_Y[\boldsymbol{z}]$ (cf. Corollary 5.3.1), with kernel $\mathscr{S}_+[\boldsymbol{z}] = \widehat{\mathscr{S}}_+$. The closed subscheme of $\widehat{C}$ associated with this closed immersion is called the **sommet scheme** of $\widehat{C}$, which can be identified with Y via $i \circ \varepsilon$, and $i \circ \varepsilon$ is called the **sommet section** of $\widehat{C}$. Finally, the closed subscheme of $\widehat{C}$ associated with j is called the **place of infinity** of C, which is identified with P via j.

The subscheme of C (resp. $\widehat{C}$) induced respectively over the open subsets

$$E = C - \varepsilon(Y), \quad \widehat{E} = \widehat{C} - i(\varepsilon(Y)) \tag{8.3.4}$$

are called respectively (by abuse of language) the **blunt affine cone** (resp. **blunt projective cone**) defined by $\mathscr{S}$. We note that with this terminology, E is not necessarily affine over Y, nor is it projective over Y (cf. Example 5.8.25). If we identify C with $i(C)$, we then have

$$C \cup \widehat{E} = \widehat{C}, \quad C \cap \widehat{E} = E. \tag{8.3.5}$$

so that $\widehat{C}$ can be considered as obtaining by gluing the open subschemes C and $\widehat{E}$ along E;

moreover, in view of (8.3.2),

$$E = \widehat{E} - j(P). \tag{8.3.6}$$

If $Y = \mathrm{Spec}(A)$ is affine, we then have (with the notations of Proposition 5.8.19),

$$E = \bigcup C_f, \quad \widehat{E} = \bigcup \widehat{C}_f, \quad C_f = C \cap \widehat{C}_f \tag{8.3.7}$$

where f runs through homogeneous elements of S_+ (or a family of homogeneous elements of S_+ generating the ideal S_+). The glueing of C and the $\widehat{C}_f$ along the C_f is then determined by the injections $C_f \to C$, $C_f \to \widehat{C}_f$, which correspond to the cannical homomorphisms $S \to S_f$, $S_f^{\leqslant} \to S_f$. On the other hand, we note that $\bigcup \widehat{E}_f$ is the defining domain $G(\varphi)$ of the morphism associated with the canonical injection $\varphi : \mathscr{S} \to \widehat{\mathscr{S}} = \mathscr{S}[z]$, so we obtain a morphism $p : \widehat{E} \to P$.

Proposition 5.8.20. *The associated morphism $p : \widehat{E} \to P$ is an affine and surjective morphism (called the **canonical retraction**) such that*

$$p^{-1}(P_f) = \widehat{C}_f \tag{8.3.8}$$

and we have $p \circ j = 1_P$. Moreover, if Y is affine and $f \in S_1$, then $\widehat{C}_f$ is isomorphic to $P_f \otimes_{\mathbb{Z}} \mathbb{Z}[T]$ (T is an indeterminate).

Proof. To prove the proposition we may assume that Y is affine, so $\mathscr{S} = \widetilde{S}$. For any $f \in S_+$ homogeneous, by (2.5.1) we have (8.3.8) and the restriction $p : \widehat{C}_f \to P_f$ corresponds to the canonical injection $S_{(f)} \to S_f^{\leqslant}$. The formula $p \circ j = 1_P$ and the fact that p is surjective follows from the fact that the composition $\mathscr{S} \to \widehat{\mathscr{S}} \to \mathscr{S}$ is the identity on $\mathscr{S}$. Finally, the last assertion follows from the fact that $S_f^{\leqslant}$ is isomorphic to $S_{(f)}[T]$ (cf. (2.1.1)). □

Corollary 5.8.21. *The restriction $\pi : E \to P$ of p to E is a surjective and affine morphism. If Y is affine and $f \in S_+$ is homogeneous, we have*

$$\pi^{-1}(P_f) = C_f \tag{8.3.9}$$

and the restriction of $\pi|_{C_f} : C_f \to P_f$ corresponds to the cannical injection $S_{(f)} \to S_f$. If $f \in S_1$, then C_f is isomorphic to $P_f \otimes_{\mathbb{Z}} \mathbb{Z}[T, T^{-1}]$ (T is an indeterminant).

Proof. The formula (8.3.9) follows from (8.3.8) and (8.3.3), which also proves the surjectivity of π. We also have seen that the canonical injection $C_f \to \widehat{C}_f$ corresponds to $S_{(f)} \to S_f$, whence the second assertion. Finally, the last assertion is a concequence of the fact that for $f \in S_1$, S_f is isomorphic to $S_{(f)}[T, T^{-1}]$ (cf. (2.1.1)). □

Remark 5.8.3. If Y is affine, the elements of the underlying space of E are the prime ideals $\mathfrak{p}$ (not necessarily graded) of S not containing S_+, in ivew of the definition of the immersion ε. For such a prime ideal $\mathfrak{p}$, the intersections $\mathfrak{p} \cap S_n$ satisfy the conditions of Proposition **??**, so there exists a graded prime ideal $\mathfrak{q}$ of S such that $\mathfrak{q} \cap S_n = \mathfrak{p} \cap S_n$ for any n. The map $\pi : E \to P$ on the underlying topological space is then interpreted by the relation

$$\pi(\mathfrak{p}) = \mathfrak{q}.$$

In fact, to verify this relation, it suffices to consider a homogeneous element f of S_+ such that $\mathfrak{p} \in D(f)$, and we then observe that $\mathfrak{q}_{(f)}$ is the inverse image of $\mathfrak{p}_f$ under the canonical injection $S_{(f)} \to S_f$.

Corollary 5.8.22. *If $\mathscr{S}$ is generated by $\mathscr{S}_1$, the morphism p and π are of finite type. Moreover, for any $x \in P$, the fiber $p^{-1}(x)$ is isomorphic to* $\mathrm{Spec}(\kappa(x)[T])$ *and $\pi^{-1}(x)$ is isomorphic to* $\mathrm{Spec}(\kappa(x)[T, T^{-1}])$.

Proof. This follows from Proposition 5.8.20 and Corollary 5.8.21, since if Y is affine and S is genrated by S_1, then the P_f for $f \in S_1$ form an open covering of P. □

Remark 5.8.4. The blunt affine cone E corresponding to the graded $\mathscr{O}_Y$-algebra $\mathscr{O}_Y[T]$ (where T is an indeterminate) is identified with $G_m = \mathrm{Spec}(\mathscr{O}_Y[T, T^{-1}])$, since it is none other than C_T as we have seen in Proposition 5.8.19. This scheme is canonically endowed with an abelian group Y-scheme structure.

Example 5.8.23. Let k be a field, $k[T_0, \dots, T_n]$ be the polynomial ring, and $\mathfrak{p}$ be a graded prime ideal of $k[T_0, \dots, T_n]$ not containing the irrelevant ideal. Consider the quotient graded ring $S = k[T_0, \dots, T_n]/\mathfrak{p}$, and set

$$P = \mathrm{Proj}(S), \quad C = \mathrm{Spec}(S), \quad \widehat{C} = \mathrm{Spec}(\widehat{S}).$$

In the language of varieties, if $V \subseteq \mathbb{P}_k^n$ is the variety defined by S, C can be viewed as the affine cone obtained by considering the lines connecting the origin with points of V. Moreover, $\widehat{C}$ is the closure of C in $\mathbb{P}_k^{n+1}$ if we embed $\mathbb{A}_k^{n+1}$ into $\mathbb{P}_k^{n+1}$ via the map $(x_0, \dots, x_n) \to [x_0 : \cdots : x_n : 1]$. Also, the morphism $j : P \to \widehat{C}$ corresponds to the injection $[x_0 : \cdots : x_n] \mapsto [x_0 : \cdots : x_n : 0]$. With these, the projective closure $\widehat{C}$ of C in $\mathbb{P}^{n+1}$ is given by the equivalent classes $[x_0 : \cdots : x_n : x_{n+1}]$ in $\mathbb{P}^{n+1}$ such that $[x_0 : \cdots : x_n] \in P$, and we can divide into two cases depending on whether $x_{n+1} \neq 0$:

(a) $(x_0, \dots, x_n) \in C$, $x_{n+1} \neq 0$;

(b) $[x_0 : \cdots : x_n] \in P$, $x_{n+1} = 0$.

Thus we see that the variety $\widehat{C}$ can be viewed as a union of P with C, which justifies the formula (8.3.2). Also, by definition the blunt affine cone E is the subvariety of C obtained by removing the origin of $\mathbb{A}^{n+1}$, and $\widehat{E}$ can be considered as a union of E and P, which is also the projective cone $\widehat{C}$ removing the point $[0 : \cdots : 0 : 1]$ in $\mathbb{P}^{n+1}$. We also remark that if we base change C through the structural morphism $P \to Y$, then the projection $C_{(P)} \to C$ can be viewed as the projection from $\mathbb{A}^{n+1} \times \mathbb{P}^n$ to $\mathbb{P}_k^n$ which maps (x, ξ) to ξ (where the class of $x \in \mathbb{A}_k^{n+1}$ is equal to ξ), and this is the blow up map of $\mathbb{A}_k^{n+1}$ at the origin.

Let Y be a scheme, $\mathscr{E}$ be a quasi-coherent $\mathscr{O}_Y$-module. If $\mathscr{S}$ is the graded $\mathscr{O}_Y$-algebra $\boldsymbol{S}_{\mathscr{O}_Y}(\mathscr{E})$, then $\widehat{\mathscr{S}}$ is identified with $\boldsymbol{S}_{\mathscr{O}_Y}(\mathscr{E} \oplus \mathscr{O}_Y)$. The affine cone $\mathrm{Spec}(\mathscr{S})$ defined by $\mathscr{S}$ is by definition the vector bundle $\boldsymbol{V}(\mathscr{E})$, and $\mathrm{Proj}(\mathscr{S})$ is by definition $\mathbb{P}(\mathscr{E})$, so we see that:

Proposition 5.8.24. *The projective closure of a vector bundle $\boldsymbol{V}(\mathscr{E})$ over Y is canonically isomorphic to $\mathbb{P}(\mathscr{E} \oplus \mathscr{O}_Y)$, and the place of infinity of this is canonically isomorphic to $\mathbb{P}(\mathscr{E})$.*

Example 5.8.25. Put for example $\mathscr{E} = \mathscr{O}_Y^r$ where $r \geq 2$. Then the blunt cones $E, \widehat{E}$ defined by $\mathscr{S}$ are neither affine nor projective over Y if $Y \neq \varnothing$. The second assertion is immediate, since $\widehat{C} = \mathbb{P}(\mathscr{O}_Y^{r+1})$ is projective over Y and the underlying spaces of E and $\widehat{E}$ are not closed in $\widehat{C}$, so the canonical immersions $E \to \widehat{C}$ and $\widehat{E} \to \widehat{C}$ are not projective (Theorem 5.5.24 and Proposition 5.5.25(v)). On the other hand, suppose that $Y = \operatorname{Spec}(A)$ is affine and for example $r = 2$; we have $C = \operatorname{Spec}(A[T_1, T_2])$ and E is the open subscheme $D(T_1) \cup D(T_2)$ of C, and we have seen that this is not affine (Example 4.5.34); a fortiori $\widehat{E}$ is not affine, since E is the open subset of $\widehat{E}$ where the section $\boldsymbol{z}$ is nonzero (8.3.5).

Proposition 5.8.26. *Let $\mathscr{L}$ be an invertible $\mathscr{O}_Y$-module, we have canonical isomorphisms for the blunt cones corresponding to $C = V(\mathscr{L})$:*

$$\operatorname{Spec}(\bigoplus_{n\in\mathbb{Z}} \mathscr{L}^n) \xrightarrow{\sim} E, \quad \boldsymbol{V}(\mathscr{L}^{-1}) \xrightarrow{\sim} \widehat{E}. \tag{8.3.10}$$

Moreover, there exists a canonical isomorphism from the projective closue of $V(\mathscr{L})$ to that of $V(\mathscr{L}^{-1})$, which transform the sommet scheme (resp. the place of infinity) of the first one to the place of infinity (resp. the sommet scheme) of the second one.

Proof. Here we have $\mathscr{S} = \bigoplus_{n\geq 0} \mathscr{L}^{\otimes n}$; the canonical injection $\mathscr{S} \to \bigoplus_{n\in\mathbb{Z}} \mathscr{L}^{\otimes n}$ defines a cannical dominant morphism

$$\operatorname{Spec}(\bigoplus_{n\in\mathbb{Z}} \mathscr{L}^{\otimes n}) \to \boldsymbol{V}(\mathscr{L}) = \operatorname{Spec}(\bigoplus_{n\geq 0} \mathscr{L}^{\otimes n}) \tag{8.3.11}$$

and it suffices to prove that this morphism is an isomorphism from $\operatorname{Spec}(\bigoplus_{n\in\mathbb{Z}} \mathscr{L}^{\otimes n})$ to E. The question is local over Y, so we can suppose that $Y = \operatorname{Spec}(A)$ is affine and $\mathscr{L} = \mathscr{O}_Y$, so $\mathscr{S} = \widetilde{A[T]}$ and $\bigoplus_{n\in\mathbb{Z}} \mathscr{L}^{\otimes n} = \widetilde{A[T, T^{-1}]}$. Now $A[T, T^{-1}]$ is the fraction ring $A[T]_T$ of $A[T]$, so (8.3.11) identify $\operatorname{Spec}(\bigoplus_{n\in\mathbb{Z}} \mathscr{L}^{\otimes n})$ as the open subscheme $D(T)$ of $C = \boldsymbol{V}(\mathscr{L})$, which by definition is E.

The isomorphism $\boldsymbol{V}(\mathscr{L}^{-1}) \cong \widehat{E}$ will on the other hand be a consequence of the last assertion, since $\boldsymbol{V}(\mathscr{L}^{-1})$ is the complement of the place of infinity of its integral closure and $\widehat{E}$ is the complement of the sommet scheme of projective closure of $C = \boldsymbol{V}(\mathscr{E})$. Now these projective closures are respectively $\mathbb{P}(\mathscr{L}^{-1} \oplus \mathscr{O}_Y)$ and $\mathbb{P}(\mathscr{L} \oplus \mathscr{O}_Y)$; but we have

$$\mathscr{L} \oplus \mathscr{O}_Y = \mathscr{L} \oplus (\mathscr{L} \otimes \mathscr{L}^{-1}) = \mathscr{L} \otimes (\mathscr{L}^{-1} \oplus \mathscr{O}_Y),$$

so the existence of the isomorphism follows from Proposition 5.4.1, and it remains to see that this isomorphism exchanges the sommet scheme and the place of infinity. For this we can assume that $Y = \operatorname{Spec}(A)$ is affine, $\mathscr{L} = \widetilde{L}$, with $L = Ac$, $L^{-1} = Ac'$, and the canonical isomorphism $L \otimes L^{-1} \to A$ sends $c \otimes c'$ to 1. Then

$$\boldsymbol{S}(L \oplus A) = A[\boldsymbol{z}] \otimes \bigoplus_{n\geq 0} Ac^{\otimes n}, \quad \boldsymbol{S}(L^{-1} \oplus A) = A[\boldsymbol{z}] \otimes \bigoplus_{n\geq 0} Ac'^{\otimes n},$$

and the isomorphism defined in Proposition 5.4.1 sends $\boldsymbol{z}^h \otimes c'^{\otimes(n-h)}$ to the element $\boldsymbol{z}^{n-h} \otimes c^{\otimes h}$. Now, in $\mathbb{P}(\mathscr{L}^{-1} \oplus \mathscr{O}_Y)$ the place of infinity is the set of points where the section $\boldsymbol{z}$ vanishes, and the sommet section is the set of points where the section c' vanishes. As we have a similary description for $\mathbb{P}(\mathscr{L}\oplus\mathscr{O}_Y)$, our conclusion follows immediately from the preceding explanations. □

5.8.4 Functorial properties

Let Y, Y' be two schemes, $q : Y' \to Y$ be a morphism, $\mathscr{S}$ (resp. $\mathscr{S}'$) be a quasi-coherent $\mathscr{O}_Y$-algebra (resp. $\mathscr{O}_{Y'}$-algebra) with positive degrees. Consider a q-morphism of graded algebras

$$\varphi : \mathscr{S} \to \mathscr{S}'.$$

We have seen that this corresponds canonically to a morphism

$$\Phi = \mathrm{Spec}(\varphi) = \mathrm{Spec}(\mathscr{S}') \to \mathrm{Spec}(\mathscr{S})$$

such that the diagram

$$\begin{tikzcd} C' \arrow[r, "\Phi"] \arrow[d] & C \arrow[d] \\ Y' \arrow[r, "q"] & Y \end{tikzcd} \tag{8.4.1}$$

where $C = \mathrm{Spec}(\mathscr{S})$, $C' = \mathrm{Spec}(\mathscr{S}')$, is commutative. Suppose moreover that $\mathscr{S}_0 = \mathscr{O}_Y$ and $\mathscr{S}'_0 = \mathscr{O}_{Y'}$; let $\varepsilon : Y \to C$ and $\varepsilon' : Y' \to C'$ be the cannical immersions, we then have a commutative diagram

$$\begin{tikzcd} C' \arrow[r, "\Phi"] & C \\ Y' \arrow[r, "q"] \arrow[u, "\varepsilon'"] & Y \arrow[u, "\varepsilon"'] \end{tikzcd} \tag{8.4.2}$$

which corresponds to the diagram

$$\begin{tikzcd} \mathscr{S} \arrow[r, "\varphi"] \arrow[d] & \mathscr{S}' \arrow[d] \\ \mathscr{O}_Y \arrow[r] & \mathscr{O}_{Y'} \end{tikzcd}$$

where the vertical are augmentation homomorphisms, and the commutativity follows from the hypotheses that φ is a homomorphism of graded algebras.

Proposition 5.8.27. *If E (resp. E') is the blunt affine cone defined by $\mathscr{S}$ (resp. $\mathscr{S}'$), we have $\Phi^{-1}(E) \subseteq E'$. Moreover, the morphism $\mathrm{Proj}(\varphi) : G(\varphi) \to \mathrm{Proj}(\mathscr{S})$ is everywhere defined (in other words $G(\varphi) = \mathrm{Proj}(\mathscr{S}')$) if and only if $\Phi^{-1}(E) = E'$.*

Proof. The first assertion followsfrom (8.4.2), since $E = C - \varepsilon(Y)$ and $E' = C' - \varepsilon'(Y')$. To prove the second assertion, we can assume that $Y = \mathrm{Spec}(A)$ and $Y' = \mathrm{Spec}(A')$ are affine, $\mathscr{S} = \widetilde{S}$, $\mathscr{S}' = \widetilde{S}'$. For f homogeneous in S_+, if we put $f' = \varphi(f)$, we have $\Phi^{-1}(C_f) = C'_{f'}$ (2.5.1); to say that $G(\varphi) = \mathrm{Proj}(S')$ signifies that in S'_+ the radical of the ideal generated by the $f' = \varphi(f)$ is equal to S'_+ (Proposition 5.2.11), and this is equivalent to that the $C'_{f'}$ cover E' (8.3.7). □

The q-morphism φ extends canonically to a q-morphism of graded algebras

$$\hat{\varphi} : \widehat{\mathscr{S}} \to \widehat{\mathscr{S}}'$$

which satisfies $\hat{\varphi}(z) = z$. We then deduce a morphism

$$\hat{\Phi} = \mathrm{Proj}(\hat{\varphi}) : G(\hat{\varphi}) \to \widehat{C} = \mathrm{Proj}(\widehat{\mathscr{S}})$$

such that the diagram

$$\begin{tikzcd} G(\hat{\varphi}) \arrow[r, "\hat{\Phi}"] \arrow[d] & \widehat{C} \arrow[d] \\ Y' \arrow[r, "q"] & Y \end{tikzcd}$$

is commutative. It then follows from the definition that if we denote by $i : C \to \widehat{C}$ and $i' : C' \to \widehat{C}'$ are the canonical immersions, we have $i'(C') \subseteq G(\hat{\varphi})$ and the diagram

$$\begin{tikzcd} G(\hat{\varphi}) \arrow[r, "\hat{\Phi}"] & \widehat{C} \\ C' \arrow[u, "i'"] \arrow[r, "\Phi"] & C \arrow[u, "i"'] \end{tikzcd} \tag{8.4.3}$$

is commutative. Finally, if we put $P = \mathrm{Proj}(\mathscr{S})$, $P' = \mathrm{Proj}(\mathscr{S}')$, and if $j : P \to \widehat{C}$, $j' : P' \to \widehat{C}'$ are the canonical closed immersions, we have $j'(G(\varphi)) \subseteq G(\hat{\varphi})$ and the diagram

$$\begin{tikzcd} G(\hat{\varphi}) \arrow[r, "\hat{\Phi}"] & \widehat{C} \\ G(\varphi) \arrow[u, "j"] \arrow[r, "\mathrm{Proj}(\varphi)"] & P \arrow[u, "j"'] \end{tikzcd} \tag{8.4.4}$$

is commutative.

Proposition 5.8.28. *If $\widehat{E}$ (resp. $\widehat{E}'$) is the blunt projective cone defined by $\mathscr{S}$ (resp. $\mathscr{S}'$), we have $\hat{\Phi}^{-1}(\widehat{E}) \subseteq \widehat{E}'$. Moreover, if $p : \widehat{E} \to P$ and $p' : \widehat{E}' \to P'$ are the canonical retractions, we have $p'(\hat{\Phi}^{-1}(\widehat{E})) \subseteq G(\varphi)$, and the diagram*

$$\begin{tikzcd} \hat{\Phi}^{-1}(\widehat{E}) \arrow[r, "\hat{\Phi}"] \arrow[d, "p'"'] & E \arrow[d, "p"] \\ G(\varphi) \arrow[r, "\mathrm{Proj}(\varphi)"] & P \end{tikzcd}$$

is commutative. If $\mathrm{Proj}(\varphi)$ *is everywhere defined, so is $\hat{\Phi}$ and we have $\hat{\Phi}^{-1}(\widehat{E}) = \widehat{E}'$.*

Proof. The first assertion follows from the commutative diagrams (8.4.1) and (8.4.3), and the next two follow from the definition of the canonical retraction, the definition of $\hat{\varphi}$, and the fact that $\widehat{E}$ is the defining domain of the morphism induced by the canonical injection $\mathscr{S} \to \widehat{\mathscr{S}}$. On the other hand, to see that $\hat{\Phi}$ is everywhere defined if $\mathrm{Proj}(\varphi)$ is, we can assume that $Y = \mathrm{Spec}(A)$, $Y' = \mathrm{Spec}(A')$ are affine, $\mathscr{S} = \widetilde{S}$, $\mathscr{S}' = \widetilde{S}'$; the hypothesis is that if f runs through homogeneous elements of S_+, the ideal in S'_+ generated by the $\varphi(f)$ has radical in S'_+ equal to S'_+. We then conclude that the radical of the ideal genrated by z and the $\varphi(f)$ in $(S'[z])_+$ is equal to $(S'[z])_+$, whence our assertion. This proves similarly that $\widehat{E}'$ is the union of the $\widehat{C}'_{(\varphi(f))}$, which is equal to $\hat{\Phi}^{-1}(\widehat{E})$. ◻

Corollary 5.8.29. *If Φ is everywhere defined, the inverse image under $\hat{\Phi}$ of underlying space of the place of infinity (resp. the sommet scheme) of $\widehat{C}'$ is the underlying space of the place of infinity (resp. the sommet scheme) of $\widehat{C}$.*

Proof. This follows from Proposition 5.8.28 and Proposition 5.8.27, in ivew of the relations (8.3.4) and (8.3.4). □

5.8.5 Blunt cones over a homogeneous specturm

Let Y be a scheme, $\mathscr{S}$ be a quasi-coherent $\mathscr{O}_Y$-algebra with positive degrees such that $\mathscr{S}_0 = \mathscr{O}_Y$, and $X = \mathrm{Proj}(\mathscr{S})$. We now apply the previous results to the structure morphism $q : X \to Y$. Let

$$\mathscr{S}_X = \bigoplus_{n\in\mathbb{Z}} \mathscr{O}_X(n) \tag{8.5.1}$$

which is a quasi-coherent $\mathscr{O}_X$-algebra, the multiplication γ being defined by the canonical homomorphisms

$$\mathscr{O}_X(m) \otimes \mathscr{O}_X(n) \to \mathscr{O}_X(m+n)$$

which satisfies the associativity in view of Proposition 5.3.15. Let $\mathscr{S}'$ be the quasi-coherent sub-algebra

$$\mathscr{S}_X^{\geqslant} = \bigoplus_{n\geq 0} \mathscr{O}_X(n)$$

of $\mathscr{S}_X$, with positive degrees. For each $n \in \mathbb{Z}$, we have a canonical q-morphisms $\alpha_n : \mathscr{S}_n \to \mathscr{O}_X(n)$ defined in (3.3.1), which together give a homomorphism

$$\alpha : \mathscr{S} \to \bigoplus_{n\geq 0} q_*(\mathscr{O}_X(n)).$$

By composing with the canonical homomorphism $\bigoplus_{n\geq 0} q_*(\mathscr{O}_X(n)) \to q_*(\mathscr{S}_X^{\geqslant})$, this gives a q-homomorphism $\mathscr{S} \to \mathscr{S}_X^{\geqslant}$, still denoted by α. We set

$$C_X = \mathrm{Spec}(\mathscr{S}_X^{\geqslant}), \quad \widehat{C}_X = \mathrm{Proj}(\mathscr{S}_X^{\geqslant}[z]), \quad P_X = \mathrm{Proj}(\mathscr{S}_X^{\geqslant})$$

and denote by E_X and $\widehat{E}_X$ the corresponding blunt cones. We then have the canonical morphisms

$$\begin{tikzcd}
 & \widehat{C}_X \arrow[dd, "\varepsilon_X"] & \\
P_X \arrow[ur, "j_X"] \arrow[dr] & & C_X \arrow[ul, "i_X"'] \arrow[dl, bend left] \\
 & X \arrow[ur, bend left] &
\end{tikzcd}$$

and $p_X : \widehat{E}_X \to P_X$, $\pi_X : E_X \to P_X$.

Proposition 5.8.30. *The structural morphism $\psi : P_X \to X$ is an isomorphism, and the morphism* $\mathrm{Proj}(\alpha)$ *is everywhere defined and equals to ψ. The morphism* $\mathrm{Proj}(\hat{\alpha}) : \widehat{C}_X \to \widehat{C}$ *is everywhere defined and its restriction to $\widehat{E}_X$ and E_X are isomorphisms into $\widehat{E}$ and E, respectively. Finally, if we identify P_X*

and X via ψ, the morphisms p_X and π_X are identified with the structural morphisms of the X-schemes $\widehat{E}_X$ and E_X.

Proof. We can clearly assume that $Y = \mathrm{Spec}(A)$ is affine and $\mathscr{S} = \widetilde{S}$; then X is the union of the affine opens X_f, where $f \in S_+$ is homogeneous, the ring of X_f being $S_{(f)}$. It follows from the isomorphism (8.2.8) that

$$\Gamma(X_f, \mathscr{S}_X^{\geqslant}) = S_f^{\geqslant}. \tag{8.5.2}$$

We then have $\psi^{-1}(X_f) = \mathrm{Proj}(S_f^{\geqslant})$. But if $f \in S_d$ with $d > 0$, $\mathrm{Proj}(S_f^{\geqslant})$ is canonically isomorphic to $\mathrm{Proj}((S_f^{\geqslant})^{(d)})$ by Proposition 5.3.2, and $(S_f^{\geqslant})^{(d)} = (S^{(d)})_f^{\geqslant}$ is identified with $S_{(f)}[T]$ by the map $T \mapsto f/1$ (cf. (2.1.1)), so we conclude from Corollary 5.3.1 that the structural morphism $\psi^{-1}(X_f) \to X_f$ is an isomorphism, whence the first assertion. To prove the second one, we first note that $\mathrm{Proj}(\alpha)$ is everywhere defined by Lemma 5.8.14. Since $\psi^{-1}(X_f) = (\psi^{-1}(X_f))_{f/1}$, it follows from (2.5.1) that the image of $\psi^{-1}(X_f)$ under $\mathrm{Proj}(\alpha)$ is contained in X_f, and the restriction of $\mathrm{Proj}(\alpha)$ to $\psi^{-1}(X_f)$, considered as a morphism into $X_f = \mathrm{Spec}(S_{(f)})$, is identified with ψ. Finally, the formula (8.3.8) and (8.2.5) show that $p_X^{-1}(\psi^{-1}(X_f)) = \mathrm{Spec}((S_f^{\geqslant})_{f/1}^{\leqslant})$, and this open subset is, by Proposition 5.8.28 and formula (8.3.8), the inverse image of $p^{-1}(X_f) = \mathrm{Spec}(S_f^{\leqslant})$ under $\mathrm{Proj}(\hat{\alpha})$. By the isomorphism (8.2.5), the restriction of $\mathrm{Proj}(\hat{\alpha})$ to $p_X^{-1}(\psi^{-1}(X_f))$ corresponds to the isomorphism $S_f^{\leqslant} \cong (S_f^{\geqslant})_{f/1}^{\leqslant}$, whence the third assertion. The last assertion is clear by definition. ◻

We note that by (8.4.3) the restriction of $\mathrm{Proj}(\hat{\alpha})$ to C_X is equal to $\mathrm{Spec}(\alpha)$.

Corollary 5.8.31. *Considered as X-schemes, $\widehat{E}_X$ is canonically isomorphic to $\mathrm{Spec}(\mathscr{S}_X^{\leqslant})$, E_X is canonically isomorphic to $\mathrm{Spec}(\mathscr{S}_X)$, and C_X is canonically isomorphic to $\mathrm{Spec}(\mathscr{S}_X^{\geqslant})$.*

Proof. As we have seen that p_X and π_X are affine, it suffices to verify the corollary if $Y = \mathrm{Spec}(A)$ is affine and $\mathscr{S} = \widetilde{S}$. The first assertion follows from the canonical isomorphism $(S_f^{\geqslant})_{f/1}^{\leqslant} \cong S_f^{\leqslant}$, which are compatible with passage from f to fg (f, g homogeneous in S_+). Similarly, the formula (8.3.9), applied to π_X, shows that $\pi_X^{-1}(\psi^{-1}(X_f)) = \mathrm{Spec}((S_f^{\geqslant})_{f/1})$ for f homogeneous in S_+, and the second assertion then follows from the canonical isomorphism $(S_f^{\geqslant})_{f/1} \cong S_f$. ◻

We can then say that $\widehat{C}_X$, considered as an X-scheme, is obtained by glueing the affine X-schemes $C_X = \mathrm{Spec}(\mathscr{S}_X^{\geqslant})$ and $\widehat{E}_X = \mathrm{Spec}(\mathscr{S}_X^{\leqslant})$ along their intersection $E_X = \mathrm{Spec}(\mathscr{S}_X)$.

Corollary 5.8.32. *Suppose that $\mathscr{O}_X(1)$ is an invertible $\mathscr{O}_X$-module and that $\mathscr{S}_X \cong \bigoplus_{n\in\mathbb{Z}}(\mathscr{O}_X(1))^{\otimes n}$ (for example if $\mathscr{S}$ is generated by $\mathscr{S}_1$). Then the blunt projective cone $\widehat{E}$ is identified with the rank one vector bundle $\boldsymbol{V}(\mathscr{O}_X(-1))$ over X, and the bulnk affine cone E is isomorphic to the open subscheme induced over the complement of the zero section in this vector bundle. With these indentifications, the canonical retraction $\widehat{E} \to X$ is identified with the structural morphism of the X-scheme $\boldsymbol{V}(\mathscr{O}_X(-1))$. Finally, there exists a canonical Y-morphism $\boldsymbol{V}(\mathscr{O}_X(1)) \to C$, whose restriction to the complement of the zero section of $\boldsymbol{V}(\mathscr{O}_X(1))$ is an isomorphism from this complement to the blunt affine cone E.*

Proof. In fact, if $\mathscr{L} = \mathscr{O}_X(1)$, then $\mathscr{S}_X^{\geqslant}$ is identified with $\boldsymbol{S}_{\mathscr{O}_X}(\mathscr{L})$ and $\mathscr{S}_X^{\leqslant}$ is identified with $\boldsymbol{S}_{\mathscr{O}_X}(\mathscr{L}^{-1})$, so $\widehat{E}_X$ is identified with $\boldsymbol{V}(\mathscr{L}^{-1})$ in view of Proposition 5.8.31 and C_X is identified with $\boldsymbol{V}(\mathscr{L})$. The morphism $\boldsymbol{V}(\mathscr{L}) \to C$ is the restriction of $\mathrm{Proj}(\hat{\alpha})$, and the assertion of the corollary is a particular case of Proposition 5.8.30. ◻

We note that the inverse image of the sommet scheme of C under the morphism $\boldsymbol{V}(\mathscr{O}_X(1)) \to C$ is the zero section of $\boldsymbol{V}(\mathscr{O}_X(1))$ (Corollary 5.8.29). But in general the corresponding subschemes of C and of $\boldsymbol{V}(\mathscr{O}_X(1))$ are not isomorphic.

5.8.6 Blow up of projective cones

With the notations of the previous subsection, we have a commutative diagram

$$\begin{tikzcd} \widehat{C}_X \arrow[r, "r"] & \widehat{C} \\ X \arrow[u, "i_X\circ\varepsilon_X"] \arrow[r, "q"] & Y \arrow[u, "i\circ\varepsilon"'] \end{tikzcd}$$

where $r = \operatorname{Proj}(\hat{\alpha})$. Moreover, the restriction of r to the complement $\widehat{C}_X - i_X(\varepsilon_X(X))$ of the sommet section is an isomorphism to $\widehat{C} - i(\varepsilon(Y))$ of the sommet section in view of Proposition 5.8.30. If we suppose for simplicity that Y is affine, $\mathscr{S}$ is of finite type and generated by $\mathscr{S}_1$, X is projective over Y and $\widehat{C}_X$ is projective over X, so $\widehat{C}_X$ is projective over Y (Proposition 5.5.25(ii)), and a fortiori over $\widehat{C}$ (Proposition 5.5.25(v)). We thus have a projective Y-morphism $r : \widehat{C}_X \to \widehat{C}$ (hence restricts to a projective Y-morphism $C_X \to C$) which contract X to Y and induces an isomorphism when restricted to the complement of X and of Y. We therefore have a relation between C_X and C, analogous to that which takes place between a blow up scheme and its initial scheme. We will effectively show that we can identify C_X with the homogeneous spectrum of a graded $\mathscr{O}_C$-algebra.

For each $n \geq 0$, we consider the quasi-coherent ideal

$$\mathscr{S}(n) = \bigoplus_{m\geq n} \mathscr{S}_m$$

of the graded $\mathscr{O}_Y$-algebra of $\mathscr{S}$. It is clear that $(\mathscr{S}(n))_{n\geq 0}$ is a filtered sequence of ideals of $\mathscr{S}$. Consider the $\mathscr{O}_C$-module associated with $\mathscr{S}(n)$, which is a quasi-coherent ideal of $\mathscr{O}_C = \widetilde{\mathscr{S}}$:

$$\mathscr{I}_n = \widetilde{\mathscr{S}(n)}.$$

Then $(\mathscr{I}_n)$ is also a filtered sequence of quasi-coherent $\mathscr{O}_C$-ideals, so we can consider the quasi-coherent graded $\mathscr{O}_C$-algebra

$$\mathscr{S}^\natural = \bigoplus_{n\geq 0} \mathscr{I}_n = \Big(\bigoplus_{n\geq 0} \mathscr{S}(n)\Big)^{\sim}.$$

Proposition 5.8.33. *There exists a canonical C-isomorphism*

$$h : C_X \to \operatorname{Proj}(\mathscr{S}^\natural). \tag{8.6.1}$$

Proof. Suppose first that $Y = \operatorname{Spec}(A)$ is affine, so $\mathscr{S} = \widetilde{S}$ where S is a graded A-algebra with positive degrees and $C = \operatorname{Spec}(S)$. We then have $\mathscr{S}^\natural = \widetilde{(S^\natural)}$. To define the morphism h, consider an element $f \in S_d$ ($d > 0$) and the corresponding element $f^\natural \in S^\natural$; the S-isomorphism (8.2.10) defines a C-isomorphism

$$\operatorname{Spec}(S_f^{\geqslant}) \xrightarrow{\sim} \operatorname{Spec}(S^{\natural}_{(f^{\natural})}). \tag{8.6.2}$$

But with the notations of Proposition 5.8.30, if $\varphi : C_X \to X$ is the structural morphism, it follows from (8.5.2) that $\varphi^{-1}(X_f) = \operatorname{Spec}(S_f^{\geqslant})$. We have on the other hand $D_+(f^{\natural}) = \operatorname{Spec}(S^{\natural}_{(f^{\natural})})$, so that (8.6.3) define an isomorphism $v^{-1}(X_f) \to D_+(f^{\natural})$. Moreover, if $g \in S_e$ with $e > 0$, the diagram

$$\begin{tikzcd} \varphi^{-1}(X_{fg}) \arrow[r, "\sim"] \arrow[d] & D_+(f^{\natural}g^{\natural}) \arrow[d] \\ \varphi^{-1}(X_f) \arrow[r, "\sim"] & D_+(f^{\natural}) \end{tikzcd}$$

is commutative, which is clear from the definition of (8.2.10). By definition S_+ is generated by these homogeneous elements F, so it follows from Lemma 5.8.14(iv) that the $D_+(f^{\natural})$ form a covering of $\operatorname{Proj}(S^{\natural})$ and the $\varphi^{-1}(X_f)$ form a covering of C_X, if X_f form a covering of X. These together gives a isomorphism $h : C_X \to \operatorname{Proj}(\mathscr{S}^{\natural})$.

To prove the proposition in the general case, it suffices to see that if U, U' are two affine opens of Y such that $U' \subseteq U$, with rings A and A', and if $\mathscr{S}|_U = \widetilde{S}$, $\mathscr{S}|_{U'} = \widetilde{S}'$, the diagram

$$\begin{tikzcd} C_{U'} \arrow[r] \arrow[d] & \operatorname{Proj}(S^{\natural}) \arrow[d] \\ C_U \arrow[r] & \operatorname{Proj}(S^{\natural}) \end{tikzcd} \tag{8.6.3}$$

is commutative. But S' is canonically identified with $S \otimes_A A'$, so $S'^{\natural}$ is identified with $S^{\natural} \otimes_S S' = S^{\natural} \otimes_A A'$ and we then have $\operatorname{Proj}(S'^{\natural}) = \operatorname{Proj}(S^{\natural}) \times_U U'$ (Proposition 5.2.48). Similarly, if $X = \operatorname{Proj}(S)$ and $X' = \operatorname{Proj}(S')$, we have $X' = X \times_U U'$ and $\mathscr{S}_{X'} = \mathscr{S}_X \otimes_{\mathscr{O}_U} \mathscr{O}_{U'}$ (Corollary 5.3.29), which means $\mathscr{S}_{X'} = j^*(\mathscr{S}_X)$, where $j : X' \to X$ is the projection. By Corollary 5.1.28 we then have $C_{U'} = C_U \times_X X' = C_U \times_U U'$, and the commutativity of (8.6.3) is immediate. □

Remark 5.8.5. The end of the reasoning of Proposition 5.8.33 is immediately generalized in the following way.: let $g : Y' \to Y$ be a morphism, $\mathscr{S}' = g^*(\mathscr{S})$, $X' = \operatorname{Proj}(\mathscr{S}')$; we then have a commutative diagram

$$\begin{tikzcd} C_{X'} \arrow[r] \arrow[d] & \operatorname{Proj}(\mathscr{S}'^{\natural}) \arrow[d] \\ C_X \arrow[r] & \operatorname{Proj}(\mathscr{S}^{\natural}) \end{tikzcd} \tag{8.6.4}$$

On the other hand, let $\varphi : \mathscr{S}'' \to \mathscr{S}$ be a homomorphism of graded $\mathscr{O}_Y$-algebras such that the induced morphism $\Phi = \operatorname{Proj}(\varphi) : X \to X''$ is everywhere defined, where $X'' = \operatorname{Proj}(\mathscr{S}'')$. We have an Y-morphism $v : C \to C''$ (where $C'' = \operatorname{Spec}(\mathscr{S}'')$) such that $\mathscr{A}(v) = \varphi$, and as φ is a graded homomorphism, we deduce from φ a v-morphism $\psi : \mathscr{S}''^{\natural} \to \mathscr{S}^{\natural}$ (Proposition 5.1.16). Moreover, it follows from Lemma 5.8.14(iv) and the hypotheses on φ that $\Psi = \operatorname{Proj}(\psi)$ is everywhere defined. Finally, in view of (3.4.5), we have a canonical Φ-morphism $\mathscr{S}_{X''} \to \mathscr{S}_X$, whence a morphism $w : C_{X''} \to C_X$. The diagram

$$\begin{tikzcd} C_{X''} \arrow[r, "\sim"] \arrow[d, "w"] & \mathrm{Proj}(\mathscr{S}^{\natural}) \arrow[d, "\Psi"] \\ C_X \arrow[r, "\sim"] & \mathrm{Proj}(\mathscr{S}^{\natural}) \end{tikzcd}$$

is commutative, as can be verified in the case where Y is affine.

Remark 5.8.6. Recall that $(\mathscr{I}_n)_{n\geq 0}$ is a filtered sequence where $\mathscr{I}_n = \mathscr{S}(n)$, so we have $\mathscr{I}_1^n \subseteq \mathscr{I}_n \subseteq \mathscr{I}_1$ for any $n > 0$. Now by definition, $\mathscr{I}_1 = \widetilde{\mathscr{S}_+}$, so $\mathscr{I}_1$ defines in C the closed subscheme $\varepsilon(Y)$ (Proposition 5.1.25 and Proposition 5.8.19). We then conclude that for any $n > 0$, the support of $\mathscr{O}_C/\mathscr{I}_n$ is contained in the underlying space of tge sommet scheme $\varepsilon(Y)$. In the inverse image of the blunt affine cone E, the structural morphism $\mathrm{Proj}(\mathscr{S}^{\natural}) \to C$ reduces to an isomorphism (as it follows from Proposition 5.8.33 and Proposition 5.8.30). Moreover, if we canonically identify C as a dense open subset of $\widehat{C}$, we can evidently extend the ideal $\mathscr{I}_n$ of $\mathscr{O}_C$ to an ideal $\mathscr{J}_n$ of $\mathscr{O}_{\widehat{C}}$, such that it coincides with $\mathscr{O}_{\widehat{C}}$ on the open subset $\widehat{E}$ of $\widehat{C}$. If we put $\mathscr{T} = \bigoplus_{n\geq 0} \mathscr{J}_n$, which is a graded $\mathscr{O}_{\widehat{C}}$-algebra, we can then extend the isomorphism (8.6.1) into a $\widehat{C}$-isomorphism

$$\widehat{C}_X \xrightarrow{\sim} \mathrm{Proj}(\mathscr{T}). \tag{8.6.5}$$

In fact, over $\widehat{E}$, it follows from the definition of $\mathscr{J}$ that $\mathrm{Proj}(\mathscr{T})$ is identified with $\widehat{E}$, and we therefore define the isomorphism (8.6.5) so that it coincides with the canonical isomorphism $\widehat{E}_X \to \widehat{E}$ on $\widehat{E}$ (Proposition 5.8.30); it is then clear that this isomorphism and (8.6.1) coincides over $\widehat{E}$.

Corollary 5.8.34. *Suppose that there exists $n_0 > 0$ such that $\mathscr{S}_{n+1} = \mathscr{S}_1\mathscr{S}_n$ for $n \geq n_0$. Then the sommet scheme of C_X (isomorphic to X) is the inverse image of the sommet scheme of C (isomorphic to Y) under the canonical morphism $r = \mathrm{Proj}(\alpha) : C_X \to C$. The converse of this is true if moreover Y is Noetherian and $\mathscr{S}$ is of finite type.*

Proof. The first assertion is local over Y, so we can assume that $Y = \mathrm{Spec}(A)$ is affine, $\mathscr{S} = \widetilde{S}$ where S is a graded A-algebra with positive degrees. This then follows from Proposition 5.8.15, because we have

$$\mathrm{Proj}(S^{\natural} \otimes_S S_0) = \mathrm{Proj}(S^{\natural} \otimes_S (S/S_+)) = C_X \times_C \varepsilon(Y)$$

(in view of the identification (8.6.1) and Proposition 5.2.48), which is also the inverse image of $\varepsilon(Y)$ in C_X under the morphism $r : C_X \to C$. The converse of this also follows from Proposition 5.8.15 if Y is Noetherian and affine and S is of finite type. If Y is Noetherian (not necessarily affine) and $\mathscr{S}$ is of finite type, there exists a finite convering of Y by Noetherian affine covers U_i, and we then deduce that for each i, there is an integer n_i such that $\mathscr{S}_{n+1}|_{U_i} = (\mathscr{S}_1|_U)(\mathscr{S}_n|_U)$ for $n \geq n_i$; the largest integer n_0 of the n_i then satisfies the requirement. ☐

We now consider the C-scheme $\widetilde{C}$ with is obtained by blowing up the affine cone C along the sommet scheme $\varepsilon(Y)$. By definiton this is $\mathrm{Proj}(\bigoplus_{n\geq 0}(\mathscr{S}_+)^n)$; the canonical injection

$$\iota : \bigoplus_{n\geq 0}(\mathscr{S}_+)^n \to \mathscr{S}^\natural$$

defines (by the identification of (8.6.1)) a canonical dominant C-morphism $\mathrm{Proj}(\iota) : G(\iota) \to \widetilde{C}$, where $G(\iota)$ is an open subset of C_X. We note that it is possible that $G(\iota) \neq C_X$; for example, $Y = \mathrm{Spec}(k)$ where k is a field, $\mathscr{S} = \widetilde{S}$ where $S = k[\boldsymbol{y}]$ and $\boldsymbol{y}$ is an indeterminate of degree 2. If R_n is the set $(S_+)^n$, considered as a subset of $S(n) = S_n^\natural$, then $S_+^\natural$ is not equal to the radical in $S_+^\natural$ of the ideal generated by the R_n.

Corollary 5.8.35. *Suppose that there exists $n_0 > 0$ such that $\mathscr{S}_n = \mathscr{S}_1^n$ for $n \geq n_0$. Then the canonical morphism $\mathrm{Proj}(\iota) : G(\iota) \to \widetilde{C}$ is everywhere defined and an isomorphism from C_X to $\widetilde{C}$. The converse of this is also ture if moreover Y is Noetherian and $\mathscr{S}$ is of finite type.*

Proof. This assertion is local over Y, and therefore follows from Proposition 5.8.16. The converse of this is also true if Y is Noetherian and $\mathscr{S}$ is of finite type, as can be shown similarly to Corollary 5.8.34. □

Remark 5.8.7. As the condition of Corollary 5.8.35 implies that of Corollary 5.8.34, we see that if this condition is verified, not only C_X is identified with the scheme obtained by blowing up C along the sommet scheme (isomorphic to Y), but also the sommet scheme of C_X (isomorphic to X) is identified with the inverse image of the sommet scheme of C in C_X. Moreover, the hypothesis of Corollary 5.8.35 implies that over $X = \mathrm{Proj}(\mathscr{S})$, the $\mathscr{O}_X$-modules $\mathscr{O}_X(n)$ are invertible (Proposition 5.3.14) and we have $\mathscr{O}_X(n) = \mathscr{L}^{\otimes n}$, where $\mathscr{L} = \mathscr{O}_X(1)$ (Corollary 5.3.16). By definition C_X is then the vector bundle $\boldsymbol{V}(\mathscr{L})$ over X, and the sommet scheme is the zero section of this vector bundle.

5.8.7 Ample sheaves and contractions

Let Y be a scheme, $f : X \to Y$ be a quasi-compact and quasi-separated morphism $\mathscr{L}$ be an ample inveritble $\mathscr{O}_X$-module realtive to f. Consider the graded $\mathscr{O}_Y$-algebra with positive degrees

$$\mathscr{S} = \mathscr{O}_Y \oplus \bigoplus_{n\geq 1} f_*(\mathscr{L}^{\otimes n})$$

which is quasi-coherent by Proposition 4.6.54. We have a canonical homomorphism of graded $\mathscr{O}_X$-algebras

$$\sigma : f^*(\mathscr{S}) \to \bigoplus_{n\geq 0} \mathscr{L}^{\otimes n}$$

which, for each $n \geq 1$, coincides with the canonical homomorphism $\sigma_n : f^*(f_*(\mathscr{L}^{\otimes n})) \to \mathscr{L}^{\otimes n}$, and for $n = 0$ is the identify on $\mathscr{O}_X$. The hypothesis that $\mathscr{L}$ is f-ample implies that (Proposition 5.4.34 and Proposition 5.3.22) the corresponding Y-morphism

$$r = r_{\mathscr{L},\sigma} : X \to P = \mathrm{Proj}(\mathscr{S})$$

is everywhere defined and a dominant open immersion, and we have $\mathscr{L}^{\otimes n} = r^*(\mathscr{O}_P(n))$ for $n \in \mathbb{Z}$.

Proposition 5.8.36. *Let $C = \operatorname{Spec}(\mathscr{S})$ the affine cone defined by $\mathscr{S}$. If $\mathscr{L}$ is f-ample, there exists a canonical Y-morphism*

$$g : V = \mathbf{V}(\mathscr{L}) \to C \tag{8.7.1}$$

such that the diagram

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[d, "f"'] & \mathbf{V}(\mathscr{L}) \arrow[r, "\pi"] \arrow[d, "g"] & X \arrow[d, "f"] \\ Y \arrow[r, "\varepsilon"] & C \arrow[r, "\psi"] & Y \end{tikzcd} \tag{8.7.2}$$

is commutative, where ψ and π are structural morphisms, j and ε are the canonical immersions which maps X and Y respectively to the zero section of $\mathbf{V}(\mathscr{L})$ and the sommet scheme of C. Moreover, the restriction of g to $\mathbf{V}(\mathscr{L}) - j(X)$ is an open immersion

$$\mathbf{V}(\mathscr{L}) - j(X) \to E = C - \varepsilon(Y)$$

into the blunt affine cone E corresponding to $\mathscr{S}$.

5.8.8 Quasi-coherent sheaves over the projective cone

Let Y be a scheme, $\mathscr{S}$ be a quasi-coherent graded $\mathscr{O}_Y$-algebra, $X = \operatorname{Proj}(\mathscr{S})$, $C = \operatorname{Spec}(\mathscr{S})$ and $\widehat{C} = \operatorname{Proj}(\widehat{\mathscr{S}})$. Let $\mathscr{M}$ be a quasi-coherent graded $\mathscr{S}$-module; to avoid any possible confusion, we denote by $\widetilde{\mathscr{M}}$ the quasi-coherent $\mathscr{O}_C$-module associated with $\mathscr{M}$ if $\mathscr{M}$ is considered as a *non-graded* $\mathscr{S}$-module, and by $\mathscr{P}roj_0(\mathscr{M})$ the quasi-coherent $\mathscr{O}_X$-module associated with $\mathscr{M}$, where $\mathscr{M}$ is considered as a graded $\mathscr{S}$-module. We also set

$$\mathscr{M}_X = \mathscr{P}roj(\mathscr{M}) = \bigoplus_{n\in\mathbb{Z}} \mathscr{P}roj_0(\mathscr{M}(n));$$

with the quasi-coherent $\mathscr{O}_X$-algebra being defined by (8.5.1), $\mathscr{P}roj(\mathscr{M})$ is endowed a quasi-coherent graded $\mathscr{S}_X$-module structure, via the canonical homomorphisms

$$\mathscr{O}_X(m) \otimes_{\mathscr{O}_X} \mathscr{P}roj_0(\mathscr{M}(n)) \to \mathscr{P}roj_0(\mathscr{S}(m) \otimes_{\mathscr{S}} \mathscr{M}(n)) \to \operatorname{Proj}_0(\mathscr{M}(m+n))$$

which satisfies the axioms of modules in view of the commutative diagram (2.3.2). If $Y = \operatorname{Spec}(A)$ is affine, $\mathscr{S} = \widetilde{S}$ and $\mathscr{M} = \widetilde{M}$, where S is a graded A-algebra and M is a graded S-module, then for any homogeneous element $f \in S_+$, we have

$$\Gamma(X_f, \mathscr{P}roj(\mathscr{M})) = M_f$$

in view of the definition and (8.2.11).

Now consider the quasi-coherent graded $\widehat{\mathscr{S}}$-module

$$\widehat{\mathscr{M}} = \mathscr{M} \otimes_{\mathscr{S}} \widehat{\mathscr{S}}$$

(where $\widehat{\mathscr{S}} = \mathscr{S}[T] = \mathscr{S} \otimes_{\mathbb{Z}} \mathbb{Z}[T]$); we then deduce a quasi-coherent graded $\mathscr{O}_{\widehat{C}}$-module $\mathscr{P}roj_0(\widehat{\mathscr{M}})$ (recall that $\widehat{C} = \mathrm{Proj}(\widehat{\mathscr{S}})$), which we also denote by

$$\mathscr{M}^{\square} = \mathscr{P}roj_0(\widehat{\mathscr{M}}).$$

It is clear that $\mathscr{M}^{\square}$ is an exact functor on $\mathscr{M}$ and commutes with inductive limits and direct sums.

Proposition 5.8.37. *With the notations of Proposition 5.8.19 and Proposition 5.8.20, we have canonical homomorphisms*

$$i^*(\mathscr{M}^{\square}) \xrightarrow{\sim} \widetilde{\mathscr{M}}, \tag{8.8.1}$$

$$j^*(\mathscr{M}^{\square}) \to \mathscr{P}roj_0(\mathscr{M}), \tag{8.8.2}$$

$$p^*(\mathscr{P}roj_0(\mathscr{M})) \to \mathscr{M}^{\square}|_{\widehat{E}} \tag{8.8.3}$$

Moreover, the homomorphism (8.8.2) is an isomorphism if $\mathscr{S}$ is generated by $\mathscr{S}_1$.

Proof. In fact, $i^*(\mathscr{M}^{\square})$ is canonically identified with $(\widehat{\mathscr{M}}/(z-1)\widehat{\mathscr{M}})^{\sim}$ over $\mathrm{Spec}(\widehat{\mathscr{S}}/(z-1)\widehat{\mathscr{S}})$ in view of Proposition 5.3.12 and the definition of i. The first isomorphism of (8.8.1) is then deduced from Proposition 5.1.16 and the canonical isomorphism $\widehat{\mathscr{M}}/(z-1)\widehat{\mathscr{M}} \cong \mathscr{M}$. On the other hand, the canonical immersion $j : X \to \widehat{C}$ corresponds to the canonical homomorphism $\widehat{\mathscr{S}} \to \mathscr{S}$ with kernel $z\widehat{\mathscr{S}}$, so the second isomorphism is a particular case of the canonical homomorphism of Proposition 5.3.27, using the fact that $\widehat{\mathscr{M}} \otimes_{\widehat{\mathscr{S}}} \mathscr{S} = \mathscr{M}$. Finally, the homomorphism of (8.8.3) is a particular case of the homomorphisms $\nu^{\sharp}$ defined in (3.4.4). If $Y = \mathrm{Spec}(A)$ is affine, $\mathscr{S} = \widetilde{S}$, $\mathscr{M} = \widetilde{M}$, then we see from Proposition 5.2.47 that the restriction of (8.8.3) to $p^{-1}(X_f) = \widehat{C}_f$ (for $f \in S_+$ homogeneous) corresponds to the canonical homomorphism

$$M_{(f)} \otimes_{S_{(f)}} S_f^{\leqslant} \to M_f^{\leqslant}$$

in view of (8.2.5) and (8.2.7). The last assertion also follows from Proposition 5.3.27. □

By abuse of language, we say that $\mathscr{M}^{\square}$ is the projective closure of the $\mathscr{O}_C$-module $\widetilde{\mathscr{M}}$, where $\mathscr{M}$ is understood to be a graded $\mathscr{S}$-module.

Let us consider a morphism $q : Y' \to Y$ and a q-homomorphism $\varphi : \mathscr{S} \to \mathscr{S}'$. It then follows from (1.4.2) that for any quasi-coherent graded $\mathscr{S}$-module $\mathscr{M}$, we have a canonical isomorphism

$$\Phi^*(\widetilde{\mathscr{M}}) \xrightarrow{\sim} (q^*(\mathscr{M}) \otimes_{q^*(\mathscr{S})} \mathscr{S}')^{\sim}$$

of $\mathscr{O}_C$-modules, where $\Phi = \mathrm{Spec}(\varphi)$. On the other hand, if $w = \mathrm{Proj}(\varphi)$ and $\hat{\Phi} = \mathrm{Proj}(\hat{\varphi})$, (3.4.3) gives a canonical w-homomorphism

$$\mathscr{P}roj_0(\mathscr{M}) \to (\mathscr{P}roj_0(q^*(\mathscr{M}) \otimes_{q^*(\mathscr{S})} \mathscr{S}'))|_{G(\varphi)} \tag{8.8.4}$$

and also a canonical $\hat{\Phi}$-morphism

$$\mathscr{P}roj_0(\widehat{\mathscr{M}}) \to (\mathscr{P}roj_0(q^*(\widehat{\mathscr{M}}) \otimes_{q^*(\widehat{\mathscr{S}})} \widehat{\mathscr{S}}'))|_{G(\hat{\varphi})}. \tag{8.8.5}$$

Now we consider the situation of the structural morphism $q: X \to Y$, where $X = \mathrm{Proj}(\mathscr{S})$, with the canonical q-homomorphism $\alpha: \mathscr{S} \to \mathscr{S}_X^{\geqslant}$. We then have a canonical isomorphism

$$q^*(\mathscr{M}) \otimes_{q^*(\mathscr{S})} \mathscr{S}_X^{\geqslant} \xrightarrow{\sim} \mathscr{M}_X^{\geqslant} \tag{8.8.6}$$

where $\mathscr{M}_X^{\geqslant} = \bigoplus_{n\geq 0} \mathscr{P}roj_0(\mathscr{M}(n))$. To see this, we can assume that $Y = \mathrm{Spec}(A)$ is affine, $\mathscr{S} = \widetilde{S}$ and $\mathscr{M} = \widetilde{M}$, and define the isomorphism (8.8.6) in each affine open X_f (f is homogeneous in S_+), and verify the compatibility when passing to a homogeneous multiple of f. Now, the restriction of the left side of (8.8.6) to X_f is $\widetilde{M}' = ((M \otimes_A S_{(f)}) \otimes_{S \otimes_A S_{(f)}} S_f^{\geqslant})^{\sim}$ by (8.5.2). As we have a canonical isomorphism $M \otimes_A S_{(f)} \cong M \otimes_S (S \otimes_A S_{(f)})$, we conclude that $\widetilde{M}' \cong (M \otimes_S S_f^{\geqslant})^{\sim}$, and this is canonically isomorphic to the restriction of $\mathscr{M}_X^{\geqslant}$ by (8.2.8). The compatibility of this isomorphism with restrictions is clear.

By replace $\mathscr{M}$ by $\widehat{\mathscr{M}}$, $\mathscr{S}$ by $\widehat{\mathscr{S}}$ and $\mathscr{S}_X$ by $(\mathscr{S}_X^{\geqslant})^{\wedge}$ in the preceding arguments, we obtain similarly a canonical isomorphism

$$q^*(\widehat{\mathscr{M}}) \otimes_{q^*(\widehat{\mathscr{S}})} (\mathscr{S}_X^{\geqslant})^{\wedge} \xrightarrow{\sim} (\mathscr{M}_X^{\geqslant})^{\wedge} \tag{8.8.7}$$

If we recall Proposition 5.8.30 that the structural morphism $\psi: \mathrm{Proj}(\mathscr{S}_X^{\geqslant}) \to X$ is an isomorphism, we then deduce a canonical ψ-isomorphism

$$\mathscr{P}roj_0(\mathscr{M}) \xrightarrow{\sim} \mathscr{P}roj_0(\mathscr{M}_X^{\geqslant}) \tag{8.8.8}$$

as a particular case of (8.8.4). In fact, we observe that, in the notations of Proposition 5.8.30, that this reduces to the fact that canonical homomorphism $M_{(f)} \otimes_{S_{(f)}} (S_f^{\geqslant})^{(d)} \to (M_f^{\geqslant})^{(d)}$ is an isomorphism if $f \in S_d$ is homogeneous, which is immediate.

The isomorphism (8.8.7) permits us, by apply (8.8.5) to the canonical morphism $r = \mathrm{Proj}(\hat{\alpha}): \widehat{C}_X \to \widehat{C}$, to obtain a canonical r-homomorphism

$$\mathscr{M}^{\square} \to (\mathscr{M}_X^{\geqslant})^{\square}. \tag{8.8.9}$$

Now recall that the restrictions of r to the blunt cones $\widehat{E}_X$ and E_X are isomorphisms onto $\widehat{E}$ and E, respectively.

Proposition 5.8.38. *The restriction of the canonical r-homomorphism (8.8.9) to $\widehat{E}_X$ and to E_X are isomorphisms*

$$\mathscr{M}^{\square}|_{\widehat{E}} \xrightarrow{\sim} (\mathscr{M}_X^{\geqslant})^{\square}|_{\widehat{E}_X}, \tag{8.8.10}$$

$$\mathscr{M}^{\square}|_{E} \xrightarrow{\sim} (\mathscr{M}_X^{\geqslant})^{\square}|_{E_X}. \tag{8.8.11}$$

Proof. We can assume that $Y = \mathrm{Spec}(A)$ is affine as in the proof of Proposition 5.8.30; with the notations there, we must show that the canonical homomorphism

$$\widehat{M}_{(f)} \otimes_{\widehat{S}_{(f)}} \widehat{(S_f^{\geqslant})_{(f/1)}} \to \widehat{(M \otimes_S S_f^{\geqslant})_{(f/1)}}$$

is an isomorphism. But in view of (8.2.5) and (8.2.7), the left side is canonically identified with $M_f^{\geqslant} \otimes_{S_f^{\geqslant}} (S_f^{\geqslant})_{f/1}^{\leqslant}$, hence with $M_f^{\geqslant}$ in view of (8.2.9); the right side is identified with $(M_f^{\geqslant})_{f/1}^{\leqslant}$, hence also to $M_f^{\geqslant}$ by (8.2.12), whence our assertion about (8.8.10). The isomorphism (8.8.11) then follows from (8.8.10) and (8.8.1). $\square$

Corollary 5.8.39. *With the notations of Corollary 5.8.31, the restriction of $(\mathscr{M}_X^{\geqslant})^{\square}$ to $\widehat{E}_X$ is identified with $\widetilde{\mathscr{M}_X^{\leqslant}}$ and its restriction to E_X is identified with $\widetilde{\mathscr{M}_X}$.*

Proof. We can clearly reduce to the affine case, and this follows from the idnetification of $(M_f^{\geqslant})_{f/1}^{\leqslant}$ with $M_f^{\leqslant}$ and $(M_f^{\geqslant})_{f/1}$ with M_f (cf. (8.2.12)). $\square$

Proposition 5.8.40. *Under the hypotheses of Corollary 5.8.32, the canonical homomorphism (8.8.3) is an isomorphism.*

Proof. In view of the fact that the structural morphism $\operatorname{Proj}(\mathscr{S}_X^{\geqslant}) \to X$ is an isomorphism and the isomorphisms (8.8.8) and (8.8.10), we only need to prove that the canonical homomorphism $p_X^*(\mathscr{P}roj_0(\mathscr{M}_X^{\geqslant})) \to (\mathscr{M}_X^{\geqslant})^{\square}|_{E_X}$ is an isomorphism, which means that we can assume that $\mathscr{S}_1$ is an invertible $\mathscr{O}_Y$-module and $\mathscr{S}$ is generated by $\mathscr{S}_1$. With the notations of Proposition 5.8.37, we then have, for $f \in S_1$, $S_f^{\leqslant} = S_{(f)}[1/f]$ and the canonical isomorphism $M_{(f)} \otimes_{S_{(f)}} S_f^{\leqslant} \to M_f^{\leqslant}$ is an isomorphism by the definition of $M_f^{\geqslant}$. $\square$

We now consider the quasi-coherent $\mathscr{S}$-module $\mathscr{M}(n) = \bigoplus_{m \geq n} \mathscr{M}_m$ and the quasi-coherent graded $\mathscr{S}^{\natural}$-module

$$\mathscr{M}^{\natural} = \Big(\bigoplus_{n \geq 0} \mathscr{M}(n)\Big)^{\sim}.$$

By Proposition 5.8.33 we have a canonical C-isomorphism $h : C_X \xrightarrow{\sim} \operatorname{Proj}(\mathscr{S}^{\natural})$.

Proposition 5.8.41. *There exists a canonical h-isomorphism*

$$\mathscr{P}roj_0(\mathscr{M}^{\natural}) \xrightarrow{\sim} \widetilde{\mathscr{M}_X}. \tag{8.8.12}$$

Proof. This can be proved as Proposition 5.8.33, by using the bi-isomorphism (8.2.13) here instead of (8.2.10). $\square$

Chapter 6

Cohomological study of coherent sheaves

6.1 Cohomology of affine schemes

6.1.1 Čech cohomology and Koszul complex

Let X be a quasi-compact and quasi-separated scheme and $\mathscr{F}$ be a quasi-coherent $\mathscr{O}_X$-module. Let $A = \Gamma(X, \mathscr{O}_X)$, $M = \Gamma(X, \mathscr{F})$, $\boldsymbol{f} = (f_i)_{1\leq i\leq r}$ be a family of elements of A, and $U_i = X_{f_i}$ be the open subset of X. Let $U = \bigcup_{i=1}^r U_i$ and we denote by $\mathfrak{U} = (U_i)_{1\leq i\leq r}$ the covering of U. Then by Proposition 4.6.15 we see that $\Gamma(U_i, \mathscr{F}) = M_{f_i}$, where $M = \Gamma(X, \mathscr{F})$. For any sequence $(i_0, i_1, \dots, i_p) \in I^{p+1}$ with $I = \{1, \dots, r\}$, we set

$$U_{i_0\dots i_p} = \bigcap_{j=0}^{p} U_{i_j} = X_{f_{i_0}\dots f_{i_p}}.$$

Then by Proposition 4.6.15, we also have

$$\Gamma(U_{i_0\dots i_p}, \mathscr{F}) = M_{f_{i_0}\cdots f_{i_p}}.$$

Now $M_{f_{i_0}\cdots f_{i_p}}$ is identified with the inductive limit $\varinjlim M^{(n)}_{i_0\dots i_p}$, where the inductive system is formed by $M^{(n)}_{i_0\dots i_p} = M$ and the homomorphism $\varphi_{nm} : M^{(m)}_{i_0\dots i_p} \to M^{(n)}_{i_0\dots i_p}$ is given by multiplication by $(f_{i_0}\cdots f_{i_p})^{n-m}$ for $m \leq n$. Denote by $C^{p+1}_n(M)$ the set of alternating maps from I^{p+1} to M (for any n), and consider the inductive system formed by these A-modules and the homomorphisms φ_{nm}. If $C^p(\mathfrak{U}, \mathscr{F})$ is the group of alternating Čech p-cochains relative to the covering $\mathfrak{U}$ with coefficients in $\mathscr{F}$, then by the preceding arguments we see that

$$C^p(\mathfrak{U}, \mathscr{F}) = \varinjlim_{n} C^{p+1}_n(M).$$

On the other hand, from the definition of $C^{p+1}_n(M)$ it is easy to see that it is canonically identified with the Koszul complex $K^{p+1}(\boldsymbol{f}^n, M)$, and the homomorphism φ_{nm} is identified with the map

$$\varphi_{\boldsymbol{f}^{n-m}} : K^\bullet(\boldsymbol{f}^n, M) \to K^\bullet(\boldsymbol{f}^m, M)$$

induced by the map $(x_1, \dots, x_r) \mapsto (f_1^{n-m}x_1, \dots, f_r^{n-m}x_r)$ on A^r. We then have, for any $p \geq 0$, a functorial isomorphism

$$C^p(\mathfrak{U}, \mathscr{F}) \xrightarrow{\sim} C^{p+1}(\mathfrak{f}, M), \tag{1.1.1}$$

where $\mathfrak{f}$ is the ideal generated by $\boldsymbol{f}$. Moreover, the definition of the differentials of $C^p(\mathfrak{U}, \mathscr{F})$ and $C_n^p(M)$ show that the isomorphism (1.1.1) is in fact a morphism of complexes.

Proposition 6.1.1. *If X is a quasi-compact and quasi-separated scheme, there exists a canonical functorial isomorphism*

$$H^p(\mathfrak{U}, \mathscr{F}) \xrightarrow{\sim} H^{p+1}(\mathfrak{f}, M) \quad \textit{for} \quad p \geq 1, \tag{1.1.2}$$

where $\mathfrak{f}$ is the ideal generated by $\boldsymbol{f}$. Moreover, we have a functorial exact sequence

$$0 \longrightarrow H^0(\mathfrak{f}, M) \longrightarrow M \longrightarrow H^0(\mathfrak{U}, \mathscr{F}) \longrightarrow H^1(\mathfrak{f}, M) \longrightarrow 0 \tag{1.1.3}$$

Proof. The relation (1.1.2) is in fact a concequence of (1.1.1). On the other hand, we have $C^0(\mathfrak{U}, \mathscr{F}) = C^1(\mathfrak{f}, M)$, so $H^0(\mathfrak{U}, \mathscr{F})$ is identified with a subgroup of 1-cocycles of $C^1(\mathfrak{f}, M)$. As $C^0(\mathfrak{f}, M) = M$, the exact sequence (1.1.3) follows from the definition of $H^0(\mathfrak{f}, M)$ and $H^1(\mathfrak{f}, M)$. □

Corollary 6.1.2. *Suppose that the X_{f_i} are quasi-compact and there exists $g_i \in \Gamma(U, \mathscr{F})$ such that $\sum_i g_i(f_i|_U) = 1|_U$. Then for any quasi-coherent $(\mathscr{O}_X|_U)$-module $\mathscr{G}$, we have $H^p(\mathfrak{U}, \mathscr{G}) = 0$ for $p > 0$. If moreover $U = X$, then the canonical homomorphism $M \to H^0(\mathfrak{U}, \mathscr{F})$ in (1.1.3) is bijective.*

Proof. By hypothesis $U_i = X_{f_i}$ is quasi-compact, so U is quasi-compact, and we can assume that $U = X$. Then the hypothesis implies that $\mathfrak{f} = A$, so by Corollary **??** we have $H^p(\mathfrak{f}, M) = 0$ for $p \geq 1$, and the corollary follows from (1.1.2) and (1.1.3). □

Remark 6.1.1. Let X be an affine scheme, so that the $U_i = X_{f_i} = D(f_i)$ are affine opens, and so is each $U_{i_0 \dots i_p}$ (but U is not necessarily affine). In this case, the functors $\Gamma(X, \mathscr{F})$ and $\Gamma(U_{i_0 \dots i_p}, \mathscr{F})$ are exact by Proposition 5.5.10. If we have an exact sequence $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ of quasi-coherent $\mathscr{O}_X$-modules, the sequence of complexes

$$0 \longrightarrow C^\bullet(\mathfrak{U}, \mathscr{F}') \longrightarrow C^\bullet(\mathfrak{U}, \mathscr{F}) \longrightarrow C^\bullet(\mathfrak{U}, \mathscr{F}'') \longrightarrow 0$$

is exact, so we obtain a long exact sequence of coholomogy groups

$$\cdots \longrightarrow H^p(\mathfrak{U}, \mathscr{F}') \longrightarrow H^p(\mathfrak{U}, \mathscr{F}) \longrightarrow H^p(\mathfrak{U}, \mathscr{F}'') \xrightarrow{\delta} H^{p+1}(\mathfrak{U}, \mathscr{F}') \longrightarrow 0$$

On the other hand, if we put $M' = \Gamma(X, \mathscr{F}')$, $M'' = \Gamma(X, \mathscr{F}'')$, $M = \Gamma(X, \mathscr{F})$, then the sequence $0 \to M' \to M \to M' \to 0$ is exact; as $C^\bullet(\mathfrak{f}, M)$ is an exact functor on M, we then get a long exact sequence on cohomology

$$\cdots \longrightarrow H^p(\mathfrak{f},\mathscr{F}') \longrightarrow H^p(\mathfrak{f},\mathscr{F}) \longrightarrow H^p(\mathfrak{f},\mathscr{F}'') \xrightarrow{\delta} H^{p+1}(\mathfrak{f},\mathscr{F}') \longrightarrow 0$$

Now as the diagram

$$\begin{tikzcd} 0 \arrow[r] & C^\bullet(\mathfrak{U},\mathscr{F}') \arrow[r] \arrow[d] & C^\bullet(\mathfrak{U},\mathscr{F}) \arrow[r] \arrow[d] & C^\bullet(\mathfrak{U},\mathscr{F}'') \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & C^\bullet(\mathfrak{f},\mathscr{F}') \arrow[r] & C^\bullet(\mathfrak{f},\mathscr{F}) \arrow[r] & C^\bullet(\mathfrak{f},\mathscr{F}'') \arrow[r] & 0 \end{tikzcd}$$

is commutative, we conclude that the diagram

$$\begin{tikzcd} H^p(\mathfrak{U},\mathscr{F}'') \arrow[r, "\delta"] \arrow[d] & H^{p+1}(\mathfrak{U},\mathscr{F}') \arrow[d] \\ H^{p+1}(\mathfrak{f},M'') \arrow[r, "\delta"] & H^{p+2}(\mathfrak{f},M') \end{tikzcd} \tag{1.1.4}$$

is commutative for any $p > 0$.

6.1.2 Cohomology of affine schemes

Theorem 6.1.3. *Let X be an affine scheme. For any quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, we have $H^p(X,\mathscr{F}) = 0$ for $p > 0$.*

Proof. Let $\mathfrak{U}$ be a finite covering of X by affine opens $X_{f_i} = D(f_i)$ $(1 \le i \le r)$; then the ideal generated by f_i is equal to $A = \Gamma(X,\mathscr{O}_X)$. We then conclude from Corollary 6.1.2 that we have $H^p(\mathfrak{U},\mathscr{F}) = 0$ for any $p > 0$. As affine opens for a basis for X, we then conclude from the definition of the Čech cohomology that $\check{H}^p(X,\mathscr{F}) = 0$ for any $p > 0$. But this is also applicable on X_f for $f \in A$, so $\check{H}^p(X_f,\mathscr{F}) = 0$ for $p > 0$; as $X_f \cap X_g = X_{fg}$, we then conclude from Leray's vanishing theorem that $H^p(X,\mathscr{F}) = 0$ for $p > 0$. □

Corollary 6.1.4. *Let $f : X \to Y$ be an affine morphism. Then for any quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, we have $R^p f_*(\mathscr{F}) = 0$ for $p > 0$.*

Proof. By definition $R^p f_*(\mathscr{F})$ is defined to be the sheaf associated with the presheaf $U \mapsto H^p(f^{-1}(U),\mathscr{F})$, where U runs through open subsets of Y. Now the affine opens U form a basis for Y, and for such U, $f^{-1}(U)$ is affine, so $H^p(f^{-1}(U),\mathscr{F}) = 0$ by Proposition 6.1.3, so we conclude that $R^p f_*(\mathscr{F}) = 0$. □

Corollary 6.1.5. *Let $f : X \to Y$ be an affine morphism. Then for any quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, the canonical homomorphism $H^p(Y, f_*(\mathscr{F})) \to H^p(X,\mathscr{F})$ is bijective for any p.*

Proof. It suffices to prove that the edge-homomorphisms

$$E_2^{p,0} = H^p(Y, f_*(\mathscr{F})) \to H^p(X,\mathscr{F})$$

of the Leray spectral sequence is bijective. But the E_2 page of this sequence is given by

$$E_2^{p,q} = (R^p\Gamma \circ R^q f_*)(\mathscr{F}) = H^p(Y, R^q f_*(\mathscr{F})),$$

so it follows from Corollary 6.1.4 that $E_2^{p,q} = 0$ for $q > 0$, and this sequence collapses at E_2 page, whence our assertion. □

Corollary 6.1.6. *Let $f : X \to Y$ be an affine morphism, $g : Y \to Z$ be a morphism. Then for any quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, the canonical homomorphism $R^p g_*(f_*(\mathscr{F})) \to F^p(g \circ f)_*(\mathscr{F})$ is bijective for any p.*

Proof. It suffices to remark that, by Corollary 6.1.5, for any affine open W of Z, the canonical homomorphism $H^p(g^{-1}(W), f_*(\mathscr{F})) \to H^p(f^{-1}(g^{-1}(W)), \mathscr{F})$ is bijective; this homomorphism of presheaves then defines a canonical homomorphism $R^p g_*(f_*(\mathscr{F})) \to R^p(g \circ f)_*(\mathscr{F})$ which is bijective. □

6.1.3 Applications to cohomology of schemes

Proposition 6.1.7. *Let X be a separated scheme, $\mathfrak{U} = (U_\alpha)$ be a covering of X by affine opens. For any quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, the cohomology module $H^\bullet(X, \mathscr{F})$ and $H^\bullet(\mathfrak{U}, \mathscr{F})$ (over $\Gamma(X, \mathscr{O}_X)$) are canonically isomorphic.*

Proof. In fact, as X is a scheme, any finite intersection V of open sets in the covering $\mathfrak{U}$ is affine (Proposition 4.5.30), so $H^p(V, \mathscr{F}) = 0$ for $q > 0$ in view of Theorem 6.1.3. The proposition then follows from Leray's vanishing theorem. □

Remark 6.1.2. We note that the conclusion of Proposition 6.1.7 is still valid if the finite intersections of U_α are affine, even if X is not necessarily separated.

Corollary 6.1.8. *Let X be a quasi-compact and quasi-separated scheme, $A = \Gamma(X, \mathscr{O}_X)$, $\boldsymbol{f} = (f_i)_{1 \le i \le r}$ be a sequence of elements of A such that the X_{f_i} are affine. Then for any quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, we have a canonical functorial isomorphism*

$$H^p(U, \mathscr{F}) \xrightarrow{\sim} H^{p+1}(\mathfrak{f}, M) \quad \text{for} \quad p > 0 \tag{1.3.1}$$

where $\mathfrak{f}$ is the ideal generated by $\boldsymbol{f}$. Moreover, we have a functorial exact sequence

$$0 \longrightarrow H^0(\mathfrak{f}, M) \longrightarrow M \longrightarrow H^0(U, \mathscr{F}) \longrightarrow H^1(\mathfrak{f}, M) \longrightarrow 0 \tag{1.3.2}$$

If X is an affine scheme, it then follows from Remark 6.1.1 and Proposition 6.1.7 that for any $q \ge 0$, the diagram

$$\begin{tikzcd} H^p(U, \mathscr{F}'') \arrow[r, "\delta"] \arrow[d] & H^{p+1}(U, \mathscr{F}') \arrow[d] \\ H^{p+1}(\mathfrak{f}, M'') \arrow[r, "\delta"] & H^{p+2}(\mathfrak{f}, M') \end{tikzcd} \tag{1.3.3}$$

corresponding to an exact sequence $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ of quasi-coherent $\mathscr{O}_X$-modules, is commutative.

Proposition 6.1.9. *Let X be a quasi-compact and quasi-separated X scheme, $\mathscr{L}$ be an invertible $\mathscr{O}_X$-module, and consider the graded ring $A_* = \Gamma_*(\mathscr{L})$. Then $H^\bullet(\mathscr{F}, \mathscr{L}) = \bigoplus_{n \in \mathbb{Z}} H^\bullet(X, \mathscr{F} \otimes \mathscr{L}^{\otimes n})$ is a* *graded A_*-module, and for any $f \in (A_*)_n$, we have a canonical isomorphism*

$$H^\bullet(X_f, \mathscr{F}) \xrightarrow{\sim} (H^\bullet(\mathscr{F}, \mathscr{L}))_{(f)} \tag{1.3.4}$$

of $(A_*)_{(f)}$-modules.

Proof. □