Algebra

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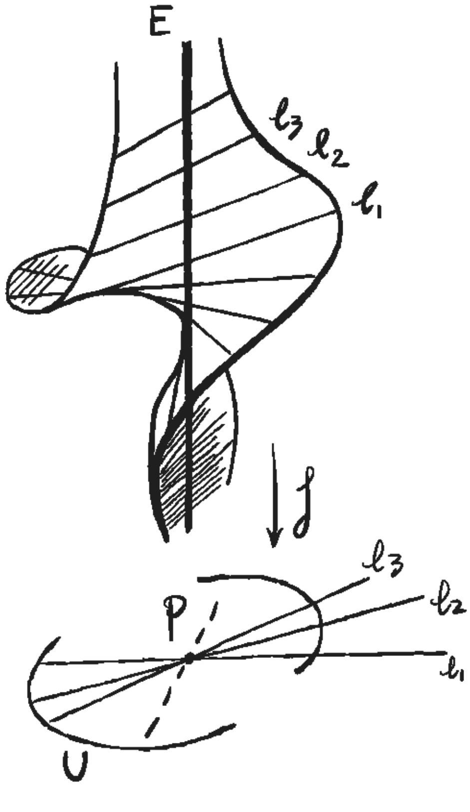

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1.1.1 Prime ideals

Definition 1.1.1. An ideal $\mathfrak{p}$ of a ring A is called **prime** if the ring $A/\mathfrak{p}$ is an integral domain.

A maximal ideal $\mathfrak{m}$ of A is prime since $A/\mathfrak{m}$ is a field; then it follows from Krull's theorem that every proper ideal of A is contained in at least one prime ideal. In particular, for prime ideals to exist in a ring A, it is necessary and sufficient that A be not reduced to 0.

Let $\rho : A \to B$ be a ring homomorphism and $\mathfrak{q}$ an ideal of B. Set $\mathfrak{p} = \rho^{-1}(\mathfrak{q})$. Then the homomorphism $\bar{\rho} : A/\mathfrak{p} \to B/\mathfrak{q}$ derived from ρ by taking quotients is injective. Suppose that $\mathfrak{q}$ is prime; as the ring $B/\mathfrak{q}$ is an integral domain, so is $A/\mathfrak{p}$, being isomorphic to a subring of $B/\mathfrak{q}$. Consequently the ideal $\mathfrak{p} = \rho^{-1}(\mathfrak{q})$ is prime. In particular, let A be a subring of B. For every ideal $\mathfrak{q}$ of B, $\mathfrak{q} \cap A$ is a prime ideal A.

If ρ is surjective, then $\bar{\rho}$ is an isomorphism, so the conditions "$\mathfrak{p}$ is prime" and "$\mathfrak{q}$ is prime" are then equivalent. Hence, if $\mathfrak{p}$ and $\mathfrak{a}$ are ideals of A such that $\mathfrak{a} \subseteq \mathfrak{p}$, a necessary and sufficient condition for $\mathfrak{p}$ to be prime is that $\mathfrak{p}/\mathfrak{a}$ be prime in $A/\mathfrak{a}$.

Proposition 1.1.2. *Let A be a ring, $\mathfrak{a}_1, \dots, \mathfrak{a}_n$ ideals of A and $\mathfrak{p}$ a prime ideal of A. If $\mathfrak{p}$ contains the product $\prod_{i=1}^n \mathfrak{a}_i$, it contains at least one of the $\mathfrak{a}_i$.*

Proof. Suppose in fact that $\mathfrak{p}$ contains none of the $\mathfrak{a}_i$. For $1 \leq i \leq n$l there exists then an element $s_i \in \mathfrak{a}_i - \mathfrak{p}$. Then $s = \prod_{i=1}^n s_i$ is contained in $\mathfrak{a}_1 \cdots \mathfrak{a}_n$ and is not contained in $\mathfrak{p}$, which is absurd. □

Corollary 1.1.3. *Let $\mathfrak{m}$ be a maximal ideal of A. Then for every integer $n > 0$, the only prime ideal containing $\mathfrak{m}^n$ is $\mathfrak{m}$.*

Proposition 1.1.4. *Let A be a ring, $\mathfrak{a}$ a non-empty set of A which is closed under addition and multiplication and $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ ideals of A. Suppose that $\mathfrak{a}$ is contained in the union of the $\mathfrak{p}_i$ and that at most two of the $\mathfrak{p}_i$ are not prime. Then $\mathfrak{a}$ is contained in one of the $\mathfrak{p}_i$.*

Proof. We argue by induction on n. The proposition is trivial if $n = 1$. Suppose that $n \geq 2$; if there exists an index j such that $\mathfrak{a} \cap \mathfrak{p}_j \subseteq \bigcup_{i \neq j} \mathfrak{p}_i$, then the set $\mathfrak{a}$, which is the union of the $\mathfrak{a} \cap \mathfrak{p}_i$ where $1 \leq i \leq n$, is contained in $\bigcup_{i \neq j} \mathfrak{p}_i$, and hence in one of the $\mathfrak{p}_i$, by the induction hypothesis. Suppose then that such an index does not exist.

For every $1 \leq j \leq n$ let y_j be an element of $\mathfrak{a} \cap \mathfrak{p}_j$ not belonging to any $\mathfrak{p}_i$ for $i \neq j$. Let k be chosen such that $\mathfrak{p}_k$ is prime if $n > 2$ and chosen arbitrarily if $n = 2$. Let $z = y_k + \prod_{j \neq k} y_j \in \mathfrak{a}$. If $i \neq k$ then $\prod_{j \neq k} y_j$ belongs to $\mathfrak{p}_i$, but $y_k \notin \mathfrak{p}_i$, whence $z \notin \mathfrak{p}_i$. On the other hand, $\prod_{j \neq k} y_j$ does not belong to $\mathfrak{p}_k$, as none of the factors y_j belongs to it and $\mathfrak{p}_k$ is prime if $n > 2$; as $y_k \in \mathfrak{p}_k$, z does not belong to $\mathfrak{p}_k$, and the proposition is established. □

Proposition 1.1.5. *Let A be a ring. Let $\mathfrak{a}_1, \ldots, \mathfrak{a}_n$ be ideals and let $\mathfrak{p}$ be a prime ideal containing $\bigcap_{i=1}^n \mathfrak{a}_i$. Then $\mathfrak{p} \supseteq \mathfrak{a}_i$ for some i. If $\mathfrak{p} = \bigcap_{i=1}^n \mathfrak{a}_i$, then $\mathfrak{p} = \mathfrak{a}_i$ for some i.*

Proof. Suppose that $\mathfrak{p} \not\supseteq \mathfrak{a}_i$ for all i. Then for each $1 \le i \le n$ there exist $x_i \in \mathfrak{a}_i$ such that $x_i \notin \mathfrak{p}$, and therefore $\prod x_i \in \prod \mathfrak{a}_i \subseteq \bigcap_i \mathfrak{a}_i$; but $\prod x_i \notin \mathfrak{p}$ (since $\mathfrak{p}$ is prime). Hence $\mathfrak{p} \not\supseteq \bigcap_i \mathfrak{a}_i$, contradiction. Finally, if $\mathfrak{p} = \bigcap_{i=1}^n \mathfrak{a}_i$, then $\mathfrak{p} \subseteq \mathfrak{a}_i$ for all i and hence $\mathfrak{p} = \mathfrak{a}_i$ for some i. □

1.1.2 Nilradical and Jacobson radical

An element $x \in A$ is **nilpotent** if $x^n = 0$ for some $n > 0$.

Proposition 1.1.6. *The set $\mathfrak{n}$ of all nilpotent elements in a ring A is an ideal, and $A/\mathfrak{n}$ has no nilpotent element except 0. This is called the **nilradical** of A.*

Proof. If $x \in \mathfrak{n}$, clearly $ax \in \mathfrak{n}$ for all $a \in A$. Let $x, y \in \mathfrak{n}$: say $x^m = 0$, $y^n = 0$. By the binomial theorem (which is valid in any commutative ring), $(x+y)^{m+n-1}$ is a sum of integer multiples of products $x^r y^s$. Since $r + s = m + n - 1$, we can not have $r < m, s < n$ simutanously. Hence $(x+y)^{m+n-1} = 0$, $x + y \in \mathfrak{n}$. □

The following proposition gives an alternative definition of the nilradical.

Proposition 1.1.7. *The nilradical of A is the intersection of all the prime ideals of A.*

Proof. We divide the proof into two parts. We first prove that $\mathfrak{n}$ is contained in every prime ideal of A. Suppose a is an element of the nilradical of A, so $a^n = 0$. And let $\mathfrak{p}$ be a prime ideal of A. If a is not contained in $\mathfrak{p}$, then $a + \mathfrak{p}$ is nonzero in $A/\mathfrak{p}$. But

$$(a + \mathfrak{p})^n = a^n + \mathfrak{p} = 0 + \mathfrak{p}$$

since $\mathfrak{p}$ is prime, $A/\mathfrak{p}$ is an integral domain, which is a contradiction.

Now consider the family $\mathscr{F}$ of ideals of A that intersect the set $\{r^n\}_{n\in\mathbb{N}}$ only at 0. Clearly $\mathscr{F}$ is nonempty and ordered by inclusion. So by Zorn's lemma there is a maximal element $\mathfrak{p}$ in $\mathscr{F}$.

Now we prove that $\mathfrak{p}$ is prime: Suppose $a, b \notin \mathfrak{p}$ but $ab \in \mathfrak{p}$. Then the ideals $\mathfrak{p} + (a)$ and $\mathfrak{p} + (b)$ properly contain $\mathfrak{p}$. Since $\mathfrak{p}$ is maximal, these two ideals are not in $\mathscr{F}$. So there are $m, n > 0$ such that

$$r^m \in \mathfrak{p} + (a), \quad r^n \in \mathfrak{p} + (b)$$

Now we find $r^{m+n} \in \mathfrak{p} + (ab) = \mathfrak{p}$, which is a contradiction. This shows at least one of a, b belongs to $\mathfrak{p}$, hence $\mathfrak{p}$ is prime.

Since we find a prime ideal $\mathfrak{p}$ such that $r \notin \mathfrak{p}$, we conclude that r is not in the intersection of all prime ideals. The contrapositive gives another inclusion, which finishes the proof. □

Proposition 1.1.8. *Let A be a commutative ring, and let $\mathfrak{n}$ be its nilradical. Then $A/\mathfrak{n}$ contains no nonzero nilpotent elements. (Such a ring is said to be **reduced**.)*

Definition 1.1.9. Define the **Jacobson radical** $\mathfrak{r}$ of A is defined to be the intersection of all the maximal ideals of A.

Proposition 1.1.10. *An element r of A is in the Jacobson radical $\mathfrak{r}$ if and only if $1 + rs$ is invertible for every $s \in A$.*

Proof. Consider the ideal $\mathfrak{a} := (1 + rs)$, if $1 + rs$ is not invertible, this is a proper ideal, hence is contained in some maximal ideal $\mathfrak{m}$. Since r is in the Jacobson radical, it is contained in $\mathfrak{m}$, and then we find $1 \in \mathfrak{m}$. But $\mathfrak{m}$ is proper, this is not possible.

Assume $1 + rs$ is invertible for every $s \in A$. Consider a maximal ideal $\mathfrak{m}$: If $r \notin \mathfrak{m}$, then $\mathfrak{m} + rA$ is an ideal properly contains $\mathfrak{m}$, then $A = \mathfrak{m} + rA$ since $\mathfrak{m}$ is maximal. In particular, $1 = m + rs$ for some $m \in \mathfrak{m}$ and $s \in A$. Then $m = 1 - rs$ is invertible by assumption, which contradicts that $\mathfrak{m}$ is a maximal ideal. □

1.1.3 Coprime ideals

Two ideals $\mathfrak{a}, \mathfrak{b}$ are said to be **coprime** (or **comaximal**) if $\mathfrak{a} + \mathfrak{b} = (1)$. Clearly two ideals $\mathfrak{a}, \mathfrak{b}$ are coprime if and only if there exist $x \in \mathfrak{a}$ and $y \in \mathfrak{b}$ such that $x + y = 1$.

Proposition 1.1.11. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be two relatively prime ideals of a ring A. Let $\mathfrak{a}'$ and $\mathfrak{b}'$ be two ideals of A such that every element of $\mathfrak{a}$ (resp. $\mathfrak{b}$) has a power in $\mathfrak{a}'$ (resp. $\mathfrak{b}'$). Then $\mathfrak{a}'$ and $\mathfrak{b}'$ are relatively prime.*

Proof. Under the given hypothesis, every prime ideal which contains $\mathfrak{a}'$ contains $\mathfrak{a}$ and every prime ideal which contains $\mathfrak{b}'$ contains $\mathfrak{b}$. If a prime ideal contains $\mathfrak{a}'$ and $\mathfrak{b}'$, then it contains $\mathfrak{a}$ and $\mathfrak{b}$, which is absurd, since $\mathfrak{a}$ and $\mathfrak{b}$ are relatively prime; hence $\mathfrak{a}'$ and $\mathfrak{b}'$ are relatively prime. □

Proposition 1.1.12. *Let $\mathfrak{a}$, $\mathfrak{b}_1, \dots, \mathfrak{b}_n$ be ideals of a ring A. If $\mathfrak{a}$ is relatively prime to each of the $\mathfrak{b}_i$, it is relatively prime to $\prod_{i=1}^n \mathfrak{b}_i$.*

Proof. Let $\mathfrak{p}$ be a prime ideal of A. If $\mathfrak{p}$ contains $\mathfrak{a}$ and $\prod_{i=1}^n \mathfrak{b}_i$ then it contains one of the $\mathfrak{b}_i$, which is absurd since $\mathfrak{a}$ and $\mathfrak{b}_i$ are relatively prime. □

Proposition 1.1.13. *Let A be a ring and $\mathfrak{a}_1, \dots, \mathfrak{a}_n$ ideals of A. Define a homomorphism*

$$\phi : A \to \prod_{i=1}^n (A/\mathfrak{a}_i), \quad \phi(x) = (x + \mathfrak{a}_1, \dots, x + \mathfrak{a}_n).$$

(a) If $\mathfrak{a}_i, \mathfrak{a}_j$ are coprime whenever $i \neq j$, then $\prod_i \mathfrak{a}_i = \bigcap_i \mathfrak{b}_i$.

(b) ϕ is surjective if and only if $\mathfrak{a}_i, \mathfrak{b}_j$ are coprime whenever $i \neq j$.

(c) ϕ is injective if and only if $\bigcap_i \mathfrak{a}_i = (0)$.

Proof. First we prove (a). The case $n = 2$ is dealt with above. Suppose $n > 2$ and the result true for $\mathfrak{a}_1, \dots, \mathfrak{a}_{n-1}$, and let $\mathfrak{b} = \mathfrak{a}_1 \cap \cdots \mathfrak{a}_{n-1}$. Since $\mathfrak{a}_i + \mathfrak{a}_j = (1)$ we have equations $x_i + y_i = 1$ $(x_i \in \mathfrak{a}_i, y_i \in \mathfrak{a}_n)$ and therefore

$$\prod_{i=1}^{n-1} x_i = \prod_{i=1}^{n-1} (1 - y_i) \equiv 1 \mod \mathfrak{a}_n.$$

This means $\mathfrak{a}_n + \mathfrak{b} = (1)$. And so

$$\prod_{i=1}^{n} \mathfrak{a}_i = \mathfrak{b}\mathfrak{a}_n = \mathfrak{b} \cap \mathfrak{a}_n = \bigcap_{i=1}^{n} \mathfrak{a}_i.$$

Now we turn to part (b). First assume that ϕ is surjective. Let us show for example that $\mathfrak{a}_1, \mathfrak{a}_2$ are coprime. There exists $x \in A$ such that $\phi(x) = (1, 0, \dots, 0)$; hence

$$x \equiv 1 \mod \mathfrak{a}_1 \quad \text{and} \quad x \equiv 0 \mod \mathfrak{a}_2,$$

so that $1 = (1 - x) + x \in \mathfrak{a}_1 + \mathfrak{a}_2$. The other case is done similarly.

Now suppose that $\mathfrak{a}_i$ and $\mathfrak{a}_j$ are coprime. It is enough to show, for example, that there is an element $x \in A$ such that $\phi(x) = (1, 0, \dots, 0)$. Since $\mathfrak{a}_1 + \mathfrak{b}_i = (1)$ for $i > 1$, we have equations

$$u_i + v_i = 1, \quad u_i \in \mathfrak{a}_1, v_i \in \mathfrak{a}_i, i = 2, \dots, n.$$

Take $x = \prod_{i=2}^n v_i$. then

$$x = \prod_{i=2}^{n} v_i = \prod_{i=2}^{n} (1 - u_i) = 1 \mod \mathfrak{a}_1.$$

Hence $\phi(x) = (1, 0, \dots, 0)$. The other case is done similarly. □

Corollary 1.1.14 (Chinese remainder theorem). *Let $\mathfrak{a}_1, \dots, \mathfrak{a}_n$ be ideals of A such that $\mathfrak{a}_i + \mathfrak{b}_j = (1)$ for all $i \neq j$. Then the natural homomorphism*

$$\phi : A \to \prod_{i=1}^{n} (A/\mathfrak{a}_i)$$

is surjective and induces an isomorphism $\tilde{\phi} : A/\prod_{i=1}^{n} \mathfrak{a}_i \to \prod_{i=1}^{n}(A/\mathfrak{a}_i)$.

1.1.4 Local rings and semilocal rings

A ring A with exactly one maximal ideal $\mathfrak{m}$ is called a **local ring**. The field $k = A/\mathfrak{m}$ is called the residue field of A.

Proposition 1.1.15. *Let $A \neq 0$ be a ring. The following are equivalent:*

(a) A has a proper ideal $\mathfrak{m}$ and every element of $A - \mathfrak{m}$ is a unit.

(b) A is not the zero ring and for every $x \in A$ either x or $1 - x$ is invertible or both.

Proof. Let $\mathfrak{m} \neq (1)$ be an ideal. If every element of $A - \mathfrak{m}$ is a unit then $\mathfrak{m}$ is clearly maximal and every ideal of A is contained in $\mathfrak{m}$. Thus A is local. The converse is clear.

If A is local and x is not a unit. Then $x \in \mathfrak{m}$, and $1 - x$ must be a unit since otherwise $1 = 1 - x + x \in \mathfrak{m}$. Conversely, the condition in (b) holds, and $\mathfrak{m}_1$ and $\mathfrak{m}_2$ are distince maximal ideal of A. Then by the Chinese remained theorem we can choose $x \in A$ such that $x \equiv 0 \bmod \mathfrak{m}_1$ and $x \equiv 1 \bmod \mathfrak{m}_2$. Then x is not invertible and neither is $1 - x$ which is a contradiction. □

Lemma 1.1.16. *Let $\varphi : A \to B$ be a ring map. Assume A and B are local rings. The following are equivalent:*

(a) $\varphi(\mathfrak{m}_A) \subseteq \mathfrak{m}_B$.

(b) $\varphi^{-1}(\mathfrak{m}_B) = \mathfrak{m}_A$.

(c) For any $x \in A$, if $\varphi(x)$ is invertible in B, then x is invertible in A.

If φ satisfies these conditions, we say φ is a ***local homomorphism****.*

Proof. For $(a) \Rightarrow (b)$, the condition $\varphi(\mathfrak{m}_A) \subseteq \mathfrak{m}_B$ clearly implies $\mathfrak{m}_A \subseteq \varphi^{-1}(\varphi(A)) \subseteq \varphi^{-1}(\mathfrak{m}_B)$, hence $\mathfrak{m}_A = \varphi^{-1}(\mathfrak{m}_B)$.

For $(b) \Rightarrow (c)$, assume that $\varphi^{-1}(\mathfrak{m}_B) = \mathfrak{m}_A$. If $\varphi(x)$ is invertible, then $\varphi(x) \notin \mathfrak{m}_B$, so $x \notin \mathfrak{m}_A$. This implies that x is invertible. The implication $(c) \Rightarrow (a)$ is clear. □

Now we consider semilocal rings. They are characterized by the following proposition.

Proposition 1.1.17. *Let A be a ring. The following properties are equivalent:*

(a) The set of maximal ideals of A is finite.

(b) The quotient of A by its Jacobson radical is the direct product of a finite number of fields.

Proof. Suppose that the quotient of A by its Jacobson radical $\mathfrak{r}(A)$ is a direct product of a finite number of fields. Then $A/\mathfrak{r}(A)$ possesses only a finite number of ideals and a fortiori only a finite number of maximal ideals. As every maximal ideal contains $\mathfrak{r}(A)$, the maximal ideals of A are the inverse images of the maximal ideals of $A/\mathfrak{r}(A))$ under the canonical homomorphism $A \to A/\mathfrak{r}(A)$; hence they are finite in number.

Conversely, suppose that A has a finite number of distinct maximal ideals $\mathfrak{m}_1, \dots, \mathfrak{m}_n$. The $A/\mathfrak{m}_i$ are fields and it follows from Proposition 1.1.13 that the canonical map $A \to \prod_{i=1}^{n} A/\mathfrak{m}_i$ is surjective; as its kernel $\bigcap_{i=1}^{n} \mathfrak{m}_i$ is the Jacobson radical A. Thus $A/\mathfrak{r}(A)$ is isomorphic to $\prod_{i=1}^{n} A/\mathfrak{m}_i$. □

Every local ring is semilocal. Every quotient of a semilocal ring is semilocal. Every finite product of semi-local rings is semi-local. Another example, generalizing the construction of the local rings $A_{\mathfrak{p}}$, is provided by the following proposition:

Proposition 1.1.18. *Let A be a ring and $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ prime ideals of A. We write $S = \bigcap_{i=1}^n (A - \mathfrak{p}_i) = A - \bigcup_{i=1}^n \mathfrak{p}_i$.*

(a) The ring $S^{-1}A$ is semilocal; if $\mathfrak{q}_1, \ldots, \mathfrak{q}_r$ are the distinct maximal elements (with respect to inclusion) of the set of $\mathfrak{p}_i$, the maximal ideals of $S^{-1}A$ are the $S^{-1}\mathfrak{q}_j$ and these ideals are distinct.

(b) The ring $A_{\mathfrak{p}_i}$ is canonically isomorphic to $(S^{-1}A)_{S^{-1}\mathfrak{p}_i}$.

(c) If A is an integral domain, then $S^{-1}A = \bigcap_{i=1}^n A_{\mathfrak{p}_i}$ in the field of fractions of A.

Proof. The ideals of A not meeting S are the ideals contained in the union of the $\mathfrak{p}_i$ and hence in at least one of the $\mathfrak{p}_i$ (by Proposition 1.1.4); the $\mathfrak{q}_i$ are therefore the maximal elements of the set of ideals not meeting S; consequently, the $S^{-1}\mathfrak{q}_i$ are the maximal ideals of $S^{-1}A$ by Proposition 1.2.37. Also, (b) follows from Corollary 1.2.23.

Suppose that A is an integral domain. If $\mathfrak{p}_i \subseteq \mathfrak{p}_j$ then $A_{\mathfrak{p}_i} \supseteq A_{\mathfrak{p}_j}$; to prove (c), we may therefore suppose that no two $\mathfrak{p}_i$ are comparable. Then it follows from (a) and Proposition 1.3.28 that $S^{-1}A = \bigcap_{i=1}^n (S^{-1}A)_{S^{-1}\mathfrak{p}_i}$; whence (c) in view of (b). □

Corollary 1.1.19. *Let A be an integral domain and $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ prime ideals of A, no two of which are comparable with respect to inclusion. If $A = \bigcap_{i=1}^n A_{\mathfrak{p}_i}$ in the field of fractions of A, then the maximal ideals of A are $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$.*

Proof. Setting $S = \bigcap_{i=1}^n (A - \mathfrak{p}_i)$. Then by Proposition 1.1.18 $S^{-1}A = A$; hence the elements of S are invertible in A and $S^{-1}\mathfrak{p}_i = \mathfrak{p}_i$ for all i. Our assertion then follows by virtue of Proposition 1.1.18(a). □

1.1.5 Operations on ideals

If $\mathfrak{a}, \mathfrak{b}$ are ideals in a ring A, their **ideal quotient** is

$$(\mathfrak{a} : \mathfrak{b}) = \{x \in A : x\mathfrak{b} \subseteq \mathfrak{a}\}$$

which is an ideal. In particular, $(0 : \mathfrak{b})$ is called the **annihilator** of $\mathfrak{b}$ and is also denoted by $\mathrm{Ann}(\mathfrak{b})$: it is the set of all $x \in A$ such that $x\mathfrak{b} = 0$. In this notation the set of all zero-divisors in A is

$$D = \bigcup_{x \neq 0} \mathrm{Ann}(x)$$

If $\mathfrak{b}$ is a principal ideal (x), we shall write $(\mathfrak{a} : x)$ in place of $(\mathfrak{a} : (x))$.

Example 1.1.20. If $A = \mathbb{Z}$, $\mathfrak{a} = (m)$, $\mathfrak{b} = (n)$, where $m = \prod_p p^{\mu_p}, \quad n = \prod_p p^{\nu_p}$, then $(\mathfrak{a} : \mathfrak{b}) = (q)$ where $q = \prod_p p^{\gamma_p}$ and

$$\gamma_p = \max(\mu_p - \nu_p, 0) = \mu_p - \min(\mu_p, \nu_p)$$

Hence $q = m/(m, n)$.

Proposition 1.1.21. *For ideals $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$ of A, we have the following properties.*

(a) $\mathfrak{a} \subseteq (\mathfrak{a} : \mathfrak{b})$.

(b) $(\mathfrak{a} : \mathfrak{b})\mathfrak{b} \subseteq \mathfrak{a}$.

(c) $((\mathfrak{a} : \mathfrak{b}) : \mathfrak{c}) = (\mathfrak{a} : \mathfrak{b}\mathfrak{c}) = ((\mathfrak{a} : \mathfrak{c}) : \mathfrak{b})$.

(d) $(\bigcap_i \mathfrak{a}_i : \mathfrak{b}) = \bigcap_i(\mathfrak{a}_i : \mathfrak{b})$.

(e) $(\mathfrak{a} : \sum_i \mathfrak{b}_i) = \bigcap_i(\mathfrak{a} : \mathfrak{b}_i)$.

Proof. Part (a), (b) and (e) follow from the defnition. As for (c), we have

$$((\mathfrak{a} : \mathfrak{b}) : \mathfrak{c}) = \{x \in A : x\mathfrak{c} \subseteq (\mathfrak{a} : \mathfrak{b})\} = \{x \in A : x\mathfrak{c}\mathfrak{b} \subseteq \mathfrak{a}\} = (\mathfrak{a} : \mathfrak{c}\mathfrak{b}),$$

and the second equality holds since $\mathfrak{b}\mathfrak{c} = \mathfrak{c}\mathfrak{b}$.

Now we prove (d). If $x\mathfrak{b} \subseteq \bigcap_i \mathfrak{a}_i$, then it is contained in $(\mathfrak{a}_i, \mathfrak{b})$ for all i, hence $(\bigcap_i \mathfrak{a}_i : \mathfrak{b}) \subseteq \bigcap_i(\mathfrak{a}_i : \mathfrak{b})$. Conversely, if $x \in (\mathfrak{a}_i, \mathfrak{b})$ for all i, then $X\mathfrak{b} \subseteq \mathfrak{a}_i$, hence in $\bigcap_i \mathfrak{a}_i$. □

Now if $\mathfrak{a}$ is any ideal of A, the **radical** of $\mathfrak{a}$ is defined to be

$$\sqrt{\mathfrak{a}} = \{x \in A : x^n \in \mathfrak{a} \text{ for some } n > 0\}.$$

If $\phi : A \to A/\mathfrak{a}$ is the standard homomorphism, then $\sqrt{\mathfrak{a}} = \phi^{-1}(\mathfrak{n}(A/\mathfrak{a}))$ and hence $\sqrt{\mathfrak{a}}$ is an ideal. Moreover, according to Proposition 1.1.7, we see the radical of an idea $\mathfrak{a}$ is the intersection of the prime ideals which contain $\mathfrak{a}$:

$$\sqrt{\mathfrak{a}} = \bigcap_{\mathfrak{p} \supseteq \mathfrak{a}} \mathfrak{p}.$$

Example 1.1.22. If $A = \mathbb{Z}$, $\mathfrak{a} = (m)$, let p_i $(1 \leq i \leq r)$ be the distinct prime divisors of m. Then $\sqrt{\mathfrak{a}} = (p_1 \cdots p_r) = \bigcap_i(p_i)$.

Proposition 1.1.23. *Let $\mathfrak{a}, \mathfrak{b}$ be ideals of A. Prove the following*

(a) $\mathfrak{a} \subseteq \sqrt{\mathfrak{a}}$.

(b) $\sqrt{\sqrt{\mathfrak{a}}} = \sqrt{\mathfrak{a}}$.

(c) $\sqrt{\mathfrak{a}\mathfrak{b}} = \sqrt{\mathfrak{a} \cap \mathfrak{b}} = \sqrt{\mathfrak{a}} \cap \sqrt{\mathfrak{b}}$.

(d) $\sqrt{\mathfrak{a}} = (1)$ *if and only if* $\mathfrak{a} = (1)$.

(e) $\sqrt{\mathfrak{a} + \mathfrak{b}} = \sqrt{\sqrt{\mathfrak{a}} + \sqrt{\mathfrak{b}}}$.

(f) If $\mathfrak{p}$ is prime, $\sqrt{\mathfrak{p}^n} = \mathfrak{p}$ for all $n > 0$.

Proof. Part (a), (d) are clear, and (b) holds since

$$x^n \in \sqrt{\mathfrak{a}} \Rightarrow (x^n)^m \in \mathfrak{a} \Rightarrow x^{nm} \in \mathfrak{a}.$$

For (c), since $\mathfrak{a}\mathfrak{b} \subseteq \mathfrak{a} \cap \mathfrak{b}$, we have $\sqrt{\mathfrak{a}\mathfrak{b}} \subseteq \sqrt{\mathfrak{a} \cap \mathfrak{b}}$. Conversely, if $x \in \sqrt{\mathfrak{a} \cap \mathfrak{b}}$, then $x^n \in \mathfrak{a} \cap \mathfrak{b}$ for some $n > 0$. This means $x^n \in \mathfrak{a}$ and $x^n \in \mathfrak{b}$, hence $x^{2n} = x^n \cdot x^n \subseteq \mathfrak{a}\mathfrak{b}$, which means $x \in \sqrt{\mathfrak{a}\mathfrak{b}}$.

The argument above also gives $x \in \sqrt{\mathfrak{a}}$ and $x \in \sqrt{\mathfrak{b}}$, hence $\sqrt{\mathfrak{a} \cap \mathfrak{b}} \subseteq \sqrt{\mathfrak{a}} \cap \sqrt{\mathfrak{b}}$. The other direction is also easy to verify.

Now we prove (e). Since $\mathfrak{a} + \mathfrak{b} \subseteq \sqrt{\mathfrak{a}} + \sqrt{\mathfrak{b}}$, we have $\sqrt{\mathfrak{a} + \mathfrak{b}} \subseteq \sqrt{\sqrt{\mathfrak{a}} + \sqrt{\mathfrak{b}}}$. Now assume that $x^n \in \sqrt{\mathfrak{a}} + \sqrt{\mathfrak{b}}$, then $x^n = m + n$ for $m^a \in \mathfrak{a}$ and $n^b \in \mathfrak{b}$. Then

$$(x^n)^{a+b} = (m+n)^{a+b} = \sum_{i=0}^{a+b} \binom{a+b}{i} m^i n^{a+b-i}$$

It is clear that the terms of the sum either belongs to $\mathfrak{a}$ or $\mathfrak{b}$. Hence this sum belongs to $\mathfrak{a} + \mathfrak{b}$. So $x \in \sqrt{\mathfrak{a} + \mathfrak{b}}$.

For (f), since $\mathfrak{p}^n \subseteq \mathfrak{p}$, we have $\sqrt{\mathfrak{p}^n} \subseteq \sqrt{\mathfrak{p}}$. But since $\mathfrak{p}$ is prime, we have $\sqrt{\mathfrak{p}} = \mathfrak{p}$, and therefore $\sqrt{\mathfrak{p}^n} \subseteq \mathfrak{p}$. Conversely, if $x \in \mathfrak{p}$, then $x^n \in \mathfrak{p}^n$, and therefore $x \in \sqrt{\mathfrak{p}^n}$. These together prove (f). □

Proposition 1.1.24. *Let $\mathfrak{a}$, $\mathfrak{b}$ be ideals in a ring A such that $\sqrt{\mathfrak{a}}$, $\sqrt{\mathfrak{b}}$ are coprime. Then $\mathfrak{a}$, $\mathfrak{b}$ are coprime.*

Proof. we have

$$\sqrt{\mathfrak{a}+\mathfrak{b}} = \sqrt{\sqrt{\mathfrak{a}}+\sqrt{\mathfrak{b}}} = \sqrt{(1)} = (1)$$

hence $\mathfrak{a}+\mathfrak{b} = (1)$, by Proposition 1.1.23. □

Let $\rho : A \to B$ be a ring homomorphism. If $\mathfrak{a}$ is an ideal in A, the set $\rho(\mathfrak{a})$ is not necessarily an ideal in B (e.g., let ρ be the embedding of $\mathbb{Z}$ in $\mathbb{Q}$, the field of rationals, and take $\mathfrak{a}$ to be any non-zero ideal in $\mathbb{Z}$.). We define the **extension** $\mathfrak{a}^e$ of $\mathfrak{a}$ to be the ideal generated by $\rho(\mathfrak{a})$ in B: explicitly,

$$\mathfrak{a}^e = \left\{ \sum y_i \rho(x_i) : x_i \in \mathfrak{a}, y_i \in B \right\}$$

If $\mathfrak{b}$ is an ideal of B, then $\rho^{-1}(\mathfrak{b})$ is always an ideal of A, called the **contraction** $\mathfrak{b}^c$ of $\mathfrak{b}$. If $\mathfrak{b}$ is prime, $\mathfrak{b}^c$ is prime. If $\mathfrak{a}$ is prime, $\mathfrak{a}^e$ need not to be prime: For example, if $\rho : \mathbb{Z} \to \mathbb{Q}$ is the inclusion and $\mathfrak{p}$ is any nonzero prime ideal of $\mathbb{Z}$, then $\mathfrak{p}^e = \mathbb{Q}$ is not prime.

We can factorize ρ as follows:

$$A \xrightarrow{\tilde{\rho}} \rho(B) \xrightarrow{j} B$$

where $\tilde{\rho}$ is surjective and j is injective. For ρ the situation is very simple: there is a one-to-one correspondence between ideals of $\rho(A)$ and ideals of A which contain $\ker f$, and prime ideals correspond to prime ideals. For j, on the other hand, the general situation is very complicated. The classical example is from algebraic number theory.

Example 1.1.25. Consider $\mathbb{Z} \to \mathbb{Z}[i]$, where $i = \sqrt{-1}$. A prime ideal (p) of $\mathbb{Z}$ may or may not stay prime when extended to $\mathbb{Z}[i]$. In fact $\mathbb{Z}[i]$ is a principal ideal domain (because it has a Euclidean algorithm) and the situation is as follows:

(a) $(2)^e = ((1+i)^2)$, the square of a prime ideal in $\mathbb{Z}[i]$.

(b) If $p \equiv 1 \bmod 4$ then $(p)^e$ is the product of two distinct prime ideals.

(c) If $p \equiv 3 \bmod 4$ then $(p)^e$ is prime in $\mathbb{Z}[i]$.

In fact the behavior of prime ideals under extensions of this sort is one of the central problems of algebraic number theory.

Proposition 1.1.26. *Let $\rho : A \to B$, $\mathfrak{a}$ and $\mathfrak{b}$ be as before. Then*

(a) $\mathfrak{a} \subseteq \mathfrak{a}^{ec}$, $\mathfrak{b} \supseteq \mathfrak{b}^{ce}$.

(b) $\mathfrak{b}^c = \mathfrak{b}^{cec}$, $\mathfrak{a}^e = \mathfrak{a}^{ece}$.

(c) If C is the set of contracted ideals in A and if E is the set of extended ideals in B, then

$$C = \{\mathfrak{a} \mid \mathfrak{a}^{ec} = \mathfrak{a}\}, \quad E = \{\mathfrak{b} \mid \mathfrak{b}^{ce} = \mathfrak{b}\},$$

and $\mathfrak{a} \mapsto \mathfrak{a}^e$ is a bijective map of C onto E, whose inverse is $\mathfrak{b} \mapsto \mathfrak{b}^c$.

Proof. The first part is clear, and (b) follows from (a):

$$\mathfrak{b}^c \subseteq (\mathfrak{b}^c)^{ec} = \mathfrak{b}^{cec} \quad \text{and} \quad \mathfrak{b} \supseteq \mathfrak{b}^{ce} \Rightarrow \mathfrak{b}^c \supseteq \mathfrak{b}^{cec}$$

Hence $\mathfrak{b}^c = \mathfrak{b}^{cec}$. Also, part (c) is a concequence of part (b). □

Proposition 1.1.27. *If $\mathfrak{a}_1, \mathfrak{a}_2$ are ideals of A and if $\mathfrak{b}_1, \mathfrak{b}_2$ are ideals of S, then we have* *(a)* $(\mathfrak{a}_1 + \mathfrak{a}_2)^e = \mathfrak{a}_1^e + \mathfrak{a}_2^e$ *and* $(\mathfrak{b}_1 + \mathfrak{b}_2)^c \supseteq \mathfrak{b}_1^c + \mathfrak{b}_2^c$.

(b) $(\mathfrak{a}_1 \cap \mathfrak{a}_2)^e \subseteq \mathfrak{a}_1^e \cap \mathfrak{a}_2^e$ *and* $(\mathfrak{b}_1 \cap \mathfrak{b}_2)^c = \mathfrak{b}_1^c \cap \mathfrak{b}_2^c$.

(c) $(\mathfrak{a}_1\mathfrak{a}_2)^e = \mathfrak{a}_1^e\mathfrak{a}_2^e$ *and* $(\mathfrak{b}_1\mathfrak{b}_2)^c \supseteq \mathfrak{b}_1^c\mathfrak{b}_2^c$.

(d) $(\mathfrak{a}_1 : \mathfrak{a}_2)^e \subseteq (\mathfrak{a}_1^e : \mathfrak{a}_2^e)$ *and* $(\mathfrak{b}_1 : \mathfrak{b}_2)^c \subseteq (\mathfrak{b}_1^c : \mathfrak{b}_2^c)$.

(e) $(\sqrt{\mathfrak{a}})^e \subseteq \sqrt{\mathfrak{a}^e}$ *and* $(\sqrt{\mathfrak{b}})^c = \sqrt{\mathfrak{b}^c}$.

Therefore, the set of ideals E is closed under sum and product, and C is closed under intersection and radical.

Proof. Note that the extension and contraction preserve inclusions. Since f is a ring homomorphism, part (a), (b), and (c) are easy to see. We first show part (d). If $x \in (\mathfrak{a}_1 : \mathfrak{a}_2)^e$, then $x = \sum y_i f(x_i)$ for $x_i \in (\mathfrak{a}_1 : \mathfrak{a}_2)$. Pick one term $y_i f(x_i)$, since $x_i\mathfrak{a}_2 \subseteq \mathfrak{a}_1$, we have $f(x_i)f(\mathfrak{a}_2) \subseteq f(\mathfrak{a}_1)$. So we have $y_i f(x_i) \in (\mathfrak{a}_1^e : \mathfrak{a}_2^e)$ and then $x \in (\mathfrak{a}_1^e : \mathfrak{a}_2^e)$. This gives $(\mathfrak{a}_1 : \mathfrak{a}_2)^e \subseteq (\mathfrak{a}_1^e : \mathfrak{a}_2^e)$. Similarly, if $x \in (\mathfrak{b}_1 : \mathfrak{b}_2)^c$, then $f(x)\mathfrak{b}_2 \subseteq \mathfrak{b}_1$. So $f(x\mathfrak{b}_2^c) \subseteq f(x)\mathfrak{b}_2 \subseteq \mathfrak{b}_1$. This shows $x\mathfrak{b}_2^c \subseteq \mathfrak{b}_1^c$, hence $(\mathfrak{b}_1 : \mathfrak{b}_2)^c \subseteq (\mathfrak{b}_1^c : \mathfrak{b}_2^c)$.

We turn to part (e). If $x = \sum y_i f(x_i)$ for $x_i^{n_i} \in \mathfrak{a}$. Then

$$(y_i f(x_i))^{n_i} = y_i^n f(x_i^{n_i}) \in \mathfrak{a}^e.$$

Since x is a finite sum, we get $x^N \in \mathfrak{a}^e$ for some N. Hence $x \in \sqrt{\mathfrak{a}^e}$, which shows $(\sqrt{\mathfrak{a}})^e \subseteq \sqrt{\mathfrak{a}^e}$. Finally, if $f(x) \in \sqrt{\mathfrak{b}}$, then $(f(x))^n = f(x^n) \in \mathfrak{b}$, so $x \in \sqrt{\mathfrak{b}^c}$. So $(\sqrt{\mathfrak{b}})^c \subseteq \sqrt{\mathfrak{b}^c}$. Conversely, if $x^n \in \mathfrak{b}^c$, then $f(x^n) = (f(x))^n \in \mathfrak{b}$. Hence $f(x) \in \sqrt{\mathfrak{b}}$, $x \in (\sqrt{\mathfrak{b}})^c$. □

1.1.6 Exercise

Exercise 1.1.1. Let x be a nilpotent element of a ring A. Show that $1 + x$ is a unit of A. Deduce that the sum of a nilpotent element and a unit is a unit.

Proof. Assume $x^n = 1$ for $n > 0$, then consider the polynomials

$$1 + x^n, \quad 1 - x^n$$

Either of them has a root -1: $1 + x^n$ for n odd, $1 - x^n$ for n even. Then we can insert x to get a decomposition of 1 since $x^n = 0$. Hence $1 + x$ is a unit.

For $u + x$ with u a unit, since there is $t \in A$ such that $ur = 1$, we have $r(u + x) = ur + xr = 1 + xr$. Since $(xr)^n = 0$, $1 + xr$ is a unit. Then $u + x$ is also a unit. □

Exercise 1.1.2. Let A be a ring and let $A[X]$ be the ring of polynomials in an indeterminate X, with coefficients in A. Let $f = a_0 + a_1X + \cdots + a_nX^n \in A[X]$. Prove that

(a) f is a unit in $A[X]$ if and only if a_0 is a unit in A and $a_1, \ldots, a_n$ are nilpotent.

(b) f is nilpotent if and only if $a_0, a_1, \ldots, a_n$ are nilpotent.

(c) f is a zero-divisor if and only if there exists $a \neq 0$ in A such that $af = 0$.

(d) f is said to be primitive if $(a_0, a_1, \ldots, a_n) = (1)$. Prove that if $f, g \in A[X]$, then fg is primitive if and only if f and g are primitive.

These results can be generized to a polynomial ring $A[X_1, \ldots, X_r]$ in several indeterminates.

Proof. Let f be a unit in $A[X]$. Then for any prime ideal $\mathfrak{p}$ of A, the image $\bar{f}$ in $(A/\mathfrak{p})[X]$ is also a unit. Since $A/\mathfrak{p}$ is an integral domain, this implies $\deg(\bar{f}) = 0$, hence $a_1, \ldots, a_n \in \mathfrak{p}$ and $a_0 \notin \mathfrak{p}$. Since this holds for any prime ideal of A, it follows that a_0 is a unit and $a_1, \ldots, a_n \in \bigcap \mathfrak{p} = \mathfrak{n}(A)$. That is, $a_1, \ldots, a_n$ are nilpotents. Conversely, if a_0 is a unit, then f has an inverse g in $A[[X]]$. If in addition $a_1, \ldots, a_n$ are nilpotent, this inverse g is in $A[X]$, hence f is a unit in $A[X]$.

Similarly, if f is nilpotent then it image in $(A/\mathfrak{p})[X]$ is nilpotent for any prime ideal $\mathfrak{p}$, hence is zero. This implies $a_0, \ldots, a_n \in \mathfrak{n}(A)$, so they are nilpotent. The converse can be proved similarly.

Let $g(X) = b_0 + b_1 X + \cdots + b_m X^m$ be such that $fg = 0$ and $g \neq 0$. If $f = 0$ then the claim follows, so we may assume that f is nonzero and in particular that $a_n \neq 0$. Then $a_n b_m = 0$ follows immediately. Note that $f \cdot (a_n g)$ is also zero, so by replacing g with $a_n g$ we elimate the degree of g by 1. This argument always holds except $\deg g = 0$, so there is $a \in A, a \neq 0$ such that $af = 0$. The converse is trivial.

Note that a polynomial is primitive just if no maximal ideal contains all its coefficients. Let $\mathfrak{m} \subseteq A$ be maximal. Since $A/\mathfrak{m}$ is a field, $A[X]/\mathfrak{m}[X] = (A/\mathfrak{m})[X]$ is an integral domain. Thus

$$f, g \notin \mathfrak{m}[X] \iff \bar{f}, \bar{g} \neq 0 \text{ in } (A/\mathfrak{m})[X] \iff \overline{fg} \neq 0 \text{ in } (A/\mathfrak{m})[X] \iff fg \notin \mathfrak{m}[X]$$

Therefore no maximal ideal contains all the coefficients of fg just if the same holds for f and g. $\square$

Exercise 1.1.3. In the ring $A[X]$, the Jacobson radical is equal to the nilradical.

Proof. Since maximal ideal is prime, $\mathfrak{n}(A) \subseteq \mathfrak{r}(A)$ holds for all ring A. Now assume $f(x) = a_0 + a_1 X + \cdots + a_n X^n$ is in $J(A[X])$, then $1 + f(X)g(X)$ is a unit for any $g(X) \in A[X]$. In particular, $1 + f(X)$ is a unit, then by Exercise 1.1.2, a_i are nilpotent for $i \geq 1$. Let $g(X) = 1 + X$, we obtain $a_0 + a_1$ is nilpotent (because it is the coefficient of X in $1 + fg$), hence a_0 is nilpotent. Agian by Exercise 1.1.2, $f(X)$ is nilpotent. $\square$

Exercise 1.1.4. Let A be a ring and let $A[[X]]$ be the ring of formal power series with coefficients in A. Show that

(a) f is a unit in $A[[X]]$ if and only if a_0 is a unit in A.

(b) If f is nilpotent, then a_n is nilpotent for all $n \geq 0$. Is the converse true?

(c) f belongs to the Jacobson radical of $A[[X]]$ if and only if a_0 belongs to the Jacobson radical of A.

(d) The contraction of a maximal ideal $\mathfrak{m}$ of $A[[X]]$ is a maximal ideal of A, and $\mathfrak{m}$ is generated by $\mathfrak{m}^c$ and X.

(e) Every prime ideal of A is the contraction of a prime ideal of $A[[X]]$.

Proof. If f is a unit, then a_0 is a unit. If a_0 is a unit, consider the expansion

$$\frac{1}{1 + a_0^{-1} X} \sum_{n=0}^{\infty} (-a_0^{-1} X)^n$$

Plug into $X = f - a_0$, we get an inverse of $1 + a_0^{-1}(f - a_0) = a_0^{-1} f$, hence $a_0^{-1} f$ is a unit, so is f.

If f is nilpotent, then a_0 is nilpotent, and $f - a_0$ is nilpotent. By repeting this we can show a_n are all nilpotent.

To see that the converse fails, let Y_n be a collection of indeterminates over $\mathbb{Z}$, indexed by the natural numbers. Let

$$A = \mathbb{Z}[Y_1, Y_2, \cdots]/(Y_1, Y_2^2, Y_3^3, \cdots)$$

Let

$$f(X) = \sum_n Y_n X^n \in A[[X]]$$

Then $a_n = Y_n$ is nilpotent for all n, but there is no N such that $f^N = 0$ (this would establish some non-trivial relation amongst the Y_n for $n > N$ other than $Y_n^n = 0$).

$f \in J(A[[X]])$ if and only if $1 + fg$ is a unit for all $g \in A[[X]]$. Since for $g = \sum b_n X^n$, the contant term of $1 + fg$ is $1 + a_0 b_0$, the condition is equivalent to that $1 + a_0 b_0$ is a unit for all b_0, and hence to $a_0 \in \mathfrak{r}(A)$.

Since x has constant term $0 \in \mathfrak{r}(A)$ in $A[[X]]$, by (c) above we get $X \in J(A[[X]])$, and hence $(X) \subseteq J(A[[X]])$. As $\mathfrak{m} - (X) = \mathfrak{m}^c$, we get $\mathfrak{m} = \mathfrak{m}^c + (X)$. Now $A/\mathfrak{m}^c \cong A[[X]]/\mathfrak{m}$ is a field, so $\mathfrak{m}^c$ is maximal.

To see $I = I^{ec}$, we only need to verify I^e is prime when I is prime. Let $f, g \in A[[X]]$ with contant term a_0, b_0. If $f, g \notin \mathfrak{p}^e$, then $a_0, b_0 \notin \mathfrak{p}$. Since fg has contant term $a_0 b_0$, which is not in $\mathfrak{p}$, $fg \notin \mathfrak{p}^e$. So $\mathfrak{p}^e$ is prime. □

Exercise 1.1.5. Let A be a ring, $\mathfrak{n}(A)$ its nilradical. Show that the following are equivalent:

(a) A has exactly one prime ideal.

(b) Every element of A is either a unit or nilpotent.

(c) $A/\mathfrak{n}(A)$ is a field.

Proof. Assume A has exactly one prime ideal $\mathfrak{p}$. Suppose $x \in A$ is not a unit, then $x \in \mathfrak{m}$ for some maximal ideal $\mathfrak{m}$. Since $\mathfrak{m}$ is also prime, we must have $\mathfrak{m} = \mathfrak{p}$. Hence $x \in \mathfrak{p}$. Since we also have $\mathfrak{n}(A) = \mathfrak{p}$, x is nilpotent.

Since $\mathfrak{n}(A)$ contains all nilpotent elements, $A - \mathfrak{n}(A)$ contains only units. Then every elements in $A/\mathfrak{n}(A)$ is a unit, so $A/\mathfrak{n}(A)$ is a field.

If $A/\mathfrak{n}(A)$ is a field, then $A - \mathfrak{n}(A)$ contains only units. Let $\mathfrak{p}$ be a prime ideal in A, then $\mathfrak{p} \in$
$n(A)$ since it contains no units, and then $\mathfrak{n}(A) = \mathfrak{p}$ since $\mathfrak{n}(A) \subseteq \mathfrak{p}$. So A contains only one prime ideal $\mathfrak{n}(A)$. □

Proof. Let $x \in A$ be idempotent, and let $\mathfrak{m}$ be the unique maximal ideal. Then in $A/\mathfrak{m}$ we see

$$\bar{x}(1 - \bar{x}) = 0$$

so $\bar{x} = 0$ or $\bar{x} = 1$, which means $x \in \mathfrak{m}$ or $x \in 1 + \mathfrak{m}$. Since $\mathfrak{m} = \mathfrak{r}(A)$, if $x \in \mathfrak{m}$ then $1 - x$ is a unit, and if $x \in 1 + \mathfrak{m}$ then x is a unit. Since $x(1 - x) = 0$, in the former case $x = 0$, and in the latter case $x = 1$. □

Exercise 1.1.6. In a ring A, let Σ be the set of all ideals in which every element is a zero-divisor. Show that the set Σ has maximal elements and that every maximal element of Σ is a prime ideal. Hence the set of zero-divisors in A is a union of prime ideals.

Proof. Clearly $(0) \in \Sigma$, so Σ is non-empty. Let $(I_i)_{i \in I}$ be a chain in Σ. The union a of the chain is still an ideal, and consists only of zero-divisors. Thus Σ, so by Zorn's Lemma Σ has maximal elements.

Let I be a maximal element of Σ, and suppose that $xy \in I$ with $y \notin I$. Then $I \subsetneq (I, a)$ so y is not a zero-divisor by the maximality of I. But xy is a zero-divisor, so there exists some $z \in A, z \neq 0$ such that $zxy = 0$. As y is not a zero-divisor we must have that $xz = 0$, so that x is a zero-divisor and thus is contained in I. □

1.2 Rings and modules of fractions

1.2.1 Rings of fractions

Definition 1.2.1. A subset S of a commutative ring A is a **multiplicative subset** (or **multiplicatively closed**) if

- $1 \in S$.
- if $s, t \in S$ then $st \in S$.

In other words, S is a subsemigroup of the multiplicative semigroup of A.

Example 1.2.2 (Example of multiplicative subsets).

(a) For every $a \in A$, the set of a^n, where $n \in \mathbb{N}$, is a multiplicative subset of A.

(b) Let A be a ring and $\mathfrak{a}$ an ideal of A, then $1 + \mathfrak{a}$ is multiplicative closed.

(c) Let $\mathfrak{p}$ be an ideal of A. For $A - \mathfrak{p}$ to be a multiplicative subset of A, it is necessary and sufficient that $\mathfrak{p}$ be prime.

(d) The set of elements of A which are not divisors of zero is a multiplicative subset of A.

(e) If S and T are multiplicative subsets of A, the set ST of products st, where $s \in S$ and $t \in T$, is a multiplicative subset.

(f) Let $\mathcal{S}$ be a directed set (with respect to inclusion) of multiplicative subsets of A. Then $T = \bigcup_{S \in \mathcal{S}} S$ is a multiplicative subset of A, as any two elements of T belong to some subset $S \in \mathcal{S}$, hence their product belongs to T.

Proposition 1.2.3. *Let A be a ring, S a multiplicative subset, and $\mathfrak{a}$ an ideal with $\mathfrak{a} \cap S = \varnothing$. Set $\mathscr{S} := \{\text{ideals } \mathfrak{b} \mid \mathfrak{b} \supseteq \mathfrak{a} \text{ and } \mathfrak{b} \cap S = \varnothing\}$. Then $\mathscr{S}$ has a maximal element $\mathfrak{p}$, and every such $\mathfrak{p}$ is prime.*

Proof. By Zorn's lemma, similar to Proposition 1.1.7. □

For every subset S of a ring A, there exist multiplicative subsets of A containing S, for example A itself. The intersection of all these subsets is the smallest multiplicative subset of A containing S; it is said to be **generated** by S. It follows immediately that it is the set consisting of all the finite products of elements of S.

Proposition 1.2.4. *Let A be a ring and S a subset of A. There exists a ring $\tilde{A}$ and a homomorphism $i : A \to \tilde{A}$ with the following properties:*

(a) the elements of $i(S)$ are invertible in $\tilde{A}$;

(b) for every homomorphism f of A to a ring B such that the elements of $f(S)$ are invertible in B, there exists a unique homomorphism $\tilde{f} : \tilde{A} \to B$ such that $f = \tilde{f} \circ i$.

Proof. Define a relation on the set of pairs $A \times S$ as follows:

$$(a, s) \sim (a', s') \iff (\exists t \in S)\, t(s'a - sa') = 0 \tag{1.2.1}$$

Now we denote by a/s the equivalence class of (a, s), and define the operations $+, \cdot$ on such fractions as

$$\frac{a}{s} + \frac{a'}{s'} = \frac{as' + a's}{ss'}, \qquad \frac{a}{s} \cdot \frac{a'}{s'} = \frac{aa'}{ss'}.$$

The set $S^{-1}A$ of fractions, endowed with the operations $+, \cdot$, is a commutative ring and the function $a \mapsto a/1$ defines a ring homomorphism $i_A^S : A \to S^{-1}A$. It is now clear that $i(s)$ is invertible for any $s \in S$. If $\rho : A \to B$ is a homomorphism such that $f(S)$ are invertible, then the map $\tilde{\rho} : S^{-1}A \to B$ defined by $\tilde{f}(a/s) = f(a)f(s)^{-1}$ is well-defined and clearly satisfies the requirement. □

Let A be a ring, S a subset of A and $\bar{S}$ the multiplicative subset generated by S. The ring of fractions of A defined by $\bar{S}$ and denoted by $S^{-1}A$ is the quotient set of $A \times \bar{S}$ under the equivalence relation (1.2.1) with the ring structure dfined by

$$a/s + b/t = (ta + bs)/st, \quad (a/s)(b/t) = ab/st.$$

for $a, b \in A$ and $s, t \in \bar{S}$. The canonical map i_A^S of A to $S^{-1}A$ is the homomorphism $a \mapsto a/1$, which makes $S^{-1}A$ into an A-algebra.

Corollary 1.2.5. *If $\rho : A \to B$ is a ring homomorphism such that*

(a) $\rho(S)$ is invertible in B.

(b) $\rho(a) = 0$ if and only if $as = 0$ for some $s \in S$.

(c) Every element of B is of the form $\rho(a)\rho(s)^{-1}$.

Then there is a unique isomorphism $h : S^{-1}A \to B$ such that $\rho = h \circ i$.

Proof. We only have to show that $h : S^{-1}A \to B$, defined by

$$h(a/s) = \rho(a)\rho(s)^{-1}$$

is an isomorphism. By (c), his surjective. To show h is injective, look at the kernel of h: if $h(a/s) = 0$, then $\rho(a) = 0$, hence by (b) we have $as = 0$ for some $s \in S$, hence $(a, s) = (0, 1)$, i.e., $a/s = 0$ in $S^{-1}A$. $\square$

Example 1.2.6 (Example of localizations).

(a) Let A be a commutative ring, and let $\mathfrak{p}$ be a prime ideal of A. Then the set $S = A - \mathfrak{p}$ is multiplicatively closed. The localizations $S^{-1}A$, $S^{-1}M$ are then denoted $A_\mathfrak{p}$, $M_\mathfrak{p}$. The process of passing from A to $A_\mathfrak{p}$ is called **localization at** $\mathfrak{p}$.

(b) Let $f \in A$ and let $S = \{f^n\}$, we write A_f for $S^{-1}A$ in this case.

(c) Let $A = \mathbb{Z}$, $\mathfrak{p} = (p)$ where p a prime number. Then $\mathbb{Z}_\mathfrak{p}$ is set of all rational numbers m/n where n is prime to p; if $f \in \mathbb{Z}$ and $f \neq 0$, then $\mathbb{Z}_f$ is the set of all rational numbers whose denominator is a power of f.

(d) Let $A = k[X_1, \dots, X_n]$, where k is a field and the X_i are independent indeterminates, $\mathfrak{p}$ a prime ideal in A. Then $A_\mathfrak{p}$ is the ring of all rational functions f/g, where $g \notin \mathfrak{p}$. If V is the variety defined by the ideal $\mathfrak{p}$, that is to say the set of all $x = (x_1, \dots, x_n) \in k^n$ such that $f(x) = 0$ whenever $f \in \mathfrak{p}$, then (provided k is infinite) $A_\mathfrak{p}$ can be identified with the ring of all rational functions on k^n which are defined at almost all points of V; it is the local ring of k^n **along the variety** V. This is the prototype of the local rings which arise in algebraic geometry.

Proposition 1.2.7. *Let A, B be two rings, S a subset of A, T a subset of B and $\rho : A \to B$ a homomorphism such that $\rho(S) \subseteq T$. There exists a unique homomorphism $\tilde{\rho}$ from S^{-1} to $T^{-1}B$ such that $\tilde{\rho}(a/1) = \rho(a)/1$ for all $a \in A$.*

Suppose further that T is contained in the multiplicative subset of B generated by $\rho(S)$. Then, if ρ is surjective (resp. injective) so is $\tilde{\rho}$.

Proof. The first assertion amounts to saying that there exists a unique homomorphism $\tilde{\rho} : S^{-1}A \to T^{-1}B$ giving a commutative diagram:

$$\begin{tikzcd} A \arrow[r, "\rho"] \arrow[d, "i_A^S"'] & B \arrow[d, "i_B^T"] \\ S^{-1}A \arrow[r, "\tilde{\rho}"] & T^{-1}B \end{tikzcd}$$

Now the relation $\rho(S) \subseteq T$ implies that $i_B^T(\rho(S))$ is invertible in $T^{-1}B$ for all $s \in S$ and it is sufficient to apply Proposition 1.2.4 to $i_B^T \circ \rho$. It follows easily from the definition that, for all $a \in A$ and $s \in \bar{S}$ (multiplicative subset of A generated by S),

$$\tilde{\rho}(a/s) = \rho(a)/\rho(s)$$

Suppose that T is contained in the multiplicative subset generated by $\rho(S)$, which is precisely $\rho(\bar{S})$. Then it follows from the formula above that, if ρ is surjective, so is $\tilde{\rho}$. Suppose now that ρ is injective. Let a/s be an element of the kernel of $\tilde{\rho}$. As the multiplicative subset generated by T is $\rho(\bar{S})$, there is an element $s_1 \in S$ such that $\rho(s_1)\rho(a) = 0$, whence $\rho(s_1 a) = 0$ and consequently $s_1 a = 0$ since ρ is injective; then $a/s = 0$, which proves that $\tilde{\rho}$ is injective. □

Corollary 1.2.8. *Let A be a ring, S a subset of A and ρ an injective homomorphism from A to a ring B such that the elements of $\rho(S)$ are invertible in B. The unique homomorphism $\tilde{\rho} : S^{-1}A \to B$ such that $\rho = \tilde{\rho} \circ i_A^S$ is then injective.*

Corollary 1.2.9. *Let A be a ring and S, T two subsets of A such that $S \subseteq T$. Then there exists a unique homomorphism $i_A^{T,S} : S^{-1}A \to T^{-1}A$ such that $i_A^T = i_A^{T,S} \circ i_A^S$.*

Proof. For all $a \in A$, $i_A^{T,S}$ then maps the element a/s in $S^{-1}A$ to the element a/s in $T^{-1}A$. □

Corollary 1.2.10. *Let A, B, C be three rings, S (resp. T, U) a multiplicative subset of A (resp. B, C), $\rho : A \to B$, $\nu : B \to C$ two homomorphism and $\eta = \nu \circ \rho$ the composite homomorphism. Suppose that $\rho(S) \subseteq T$ and $\nu(T) \subseteq U$. Let $\tilde{\rho} : S^{-1}A \to T^{-1}B$, $\tilde{\nu} : T^{-1}B \to U^{-1}C$, and $\tilde{\eta} : S^{-1}A \to U^{-1}C$ be the induced homomorphisms. Then $\tilde{\eta} = \tilde{\nu} \circ \tilde{\rho}$.*

In particular, if S, T, U are three multiplicative subsets of A such that $S \subseteq T \subseteq U$, then $i_A^{U,S} = i_A^{U,T} \circ i_A^{T,S}$.

Corollary 1.2.11. *Let S be a subset of a ring A, B a subring of $S^{-1}A$ containing the set $T := i_A^S(A)$. Let $j : B \to S^{-1}A$ be the canonical injection, then the unique homomorphism $\eta : T^{-1}B \to S^{-1}A$ such that $\eta \circ i_A^S = j$ is an isomorphism.*

Proof. The map η is injective by Corollary 1.2.8. The ring $\eta(T^{-1}B)$ contains T and the inverse of the elements of T. Hence it is equal to $S^{-1}A$. □

1.2.2 Modules of fractions

The canonical homomorphism $i_A^S : A \to S^{-1}A$ allows us to consider every $S^{-1}A$-module as an A-module.

Proposition 1.2.12. *Let A be a ring, S a subset of A, M an A-module and $\tilde{M}$ the A-module $M \otimes_A S^{-1}A$. If i_M^S is the canonical A-homomorphism $x \mapsto x \otimes 1$ of M to $\tilde{M}$, then:*

(a) For all $s \in S$, the homothety $z \mapsto sz$ of M' is bijective.

(b) For every A-module N such that, for all $s \in S$, the homothety $z \mapsto sz$ of N is bijective, and every homomorphism $\phi : M \to N$, there exists a unique homomorphism $\tilde{\phi} : \tilde{M} \to N$ such that $\phi = \tilde{\phi} \circ i_M^S$.

Proof. For every A-module N and all $a \in A$, denote by h_a the homothety $y \mapsto ay$ in N. Then $a \mapsto h_a$ is a ring homomorphism from A to $\mathrm{End}_A(N)$ To say that h_a is bijective means that h_a is an invertible element of $\mathrm{End}_A(N)$. Suppose that, for all $s \in S$, h_s is invertible in $\mathrm{End}_A(N)$. The elements h_a where $a \in A$ and the inverses of the elements h_s where $s \in S$ then generate in $\mathrm{End}_A(N)$ a commutative subring B and the homomorphism $a \mapsto h_a$ from A to B is such that the images of the elements of S are invertible. Then it follows that therc exists a unique homomorphism $\tilde{h}$ of $S^{-1}A$ to B such that

$$\tilde{h}(a/s) = h_a h_s^{-1}.$$

We know that such a homomorphism defines on N an $S^{-1}A$-module structure such that $(a/s) \cdot y = h_s^{-1}(a \cdot y)$. The A-module structure derived from this $S^{-1}A$-module structure by means of the homomorphism i_A^S is precisely the structure given initially.

Conversely, if N is an $S^{-1}A$-module and it is considered as an A-module by means of i_A^S, the homotheties $y \mapsto sy$, for $s \in S$, are bijective, for $y \mapsto (1/s)y$ is its inverse; and the $S^{-1}A$-module structure on N derived from its A-module structure by the process described above is the $S^{-1}A$-module structure given initially. Thus there is a canonical one-to-one correspondence between $S^{-1}A$-modules and A-modules in which the homotheties induced by the elements of S are bijective. Moreover, if N, N' are two A-modules with this property every A-module homomorphism $\phi : N \to N'$ is also a homomorphism of the $S^{-1}A$-module structures of N and N', as, for all $y \in N$ and all $s \in S$, we may write $\phi(y) = \phi(s \cdot ((1/s)y)) = s \cdot \phi((1/s)y)$, whence $\phi((1/s)y) = (1/s)\phi(y)$; the converse is obvious.

This being so, the statement is just the characterization of the module obtained from M by extending the scalars to $S^{-1}A$, taking account of the above interpretation. □

Let A be a ring, S a subset of A, $\bar{S}$ the multiplicative subset of A generated by S and M an A-module. Then the $S^{-1}A$-module $M \otimes_A S^{-1}A$ is called the **module of fractions of** M **defined by** S and denoted by $S^{-1}M$.

Proposition 1.2.13. *Let A be a ring. Let $S \subseteq A$ be a multiplicative subset. Let M be an A-module. Then*

$$S^{-1}M = \varinjlim_{s \in S} M_s$$

where the partial ordering on S is given by $s \geq s'$ if and only if $s = as'$ for some $a \in A$ in which case the map $M_{s'} \to M_s$ is given by $m/s^e = ma^e/(s')^e$.

Proposition 1.2.14. *Let S be a multiplicative subset of A and M an A-module. For $m/s = 0$ where $m \in M$ and $s \in S$, it is necessary and saflicient that there exist $s' \in S$ such that $s'm = 0$.*

Proof. If $s' \in S$ is such that $s'm = 0$, clearly $m/s = (s'm)/(ss') = 0$. Conversely, suppose that $m/s = 0$. As $1/s$ is invertible in $S^{-1}A$, $m/1 = 0$. For every sub-A-module P of $S^{-1}A$ containing 1, we denote by $\beta(P, m)$ the image of $(m, 1)$ under the canonical map of $M \times P$ to $M \otimes_A P$. Then $(MS^{-1}A, m) = 0$, so there exists a finitely generated submodule P of $S^{-1}A$ containing 1 and such that $\beta(P, m) = 0$. For all $t \in S$ we denote by A_t the set of a/t, where $a \in A$. As P is finitely generated, there exists $t \in S$ such that $P \subseteq A_t$, whence $\beta(A_t, m) = 0$. The map $a \mapsto a/t$ from A to A_t is surjective; let B be its kernel. Then we have a surjective map $h : M \otimes_A A \to M \otimes_A A_t$ whose kernel is $B \otimes_A M$. By definition $\beta(A_t, m) = h(tm)$ and consequently tm can be expressed in the form $\sum_i b_i m_i$, where $b_i \in B$ and $m_i \in M$. Since $b_i/t = 0$ for all i, there exists $t' \in S$ such that $t'tm = 0$, whence the claim. □

Corollary 1.2.15. *The module $S^{-1}M$ can be identified with the set $M \times S$ modulo the equivalence relation*

$$(m, s) \sim (m', s') \iff \exists t \in S \text{ such that } t(ms' - m's) = 0$$

and the $S^{-1}A$-module structure defined by $(a/s)(m, t) = (am, st)$.

Corollary 1.2.16. *Let M be a finitely generated A-module. For $S^{-1}M = 0$, it is necessary and suficient that there exists $s \in S$ such that $sM = 0$.*

Proof. Without any conditions on M, clearly the relation $sM = 0$ for some $s \in S$ implies $S^{-1}M = 0$. Conversely, suppose that $S^{-1}M = 0$ and let $x_1, \dots, x_n$ be a system of generators of M. The $m_i/1$ generate the $S^{-1}A$-module $S^{-1}M$, hence to say that $S^{-1}M = 0$ amounts to saying that $m_i/1 = 0$ for all i. By Corollary 1.2.15 there exist $s_i \in S$ such that $s_i m_i = 0$ and, taking $s = s_1 \cdots s_n$, we see $sm_i = 0$ for all i and hence $sM = 0$. □

Corollary 1.2.17. *Let M be a finitely generated A-module. For an ideal $\mathfrak{a}$ of A to be such that $\mathfrak{a}M = M$, it is necessary and sufiicient that there exist $a \in \mathfrak{a}$ such that $(1 + a)M = 0$.*

Proof. Clearly the relation $(1 + a)M = 0$ implies $M = aM$. To prove the converse, we use the following result: For every ideal $\mathfrak{a}$ of A, the set $S = 1 + \mathfrak{a}$ is a multiplieative subset of A and the set $S^{-1}\mathfrak{a}$ is an ideal contained in the Jacobson radical of $S^{-1}A$. The first assertion is obvious, as well as the fact that $S^{-1}\mathfrak{a}$ is an ideal of $S^{-1}A$. On the other hand, $(1/1) + (a/s) = (s + a)/s$ and $s + \mathfrak{a} \in S$ for all $s \in S$ and $a \in \mathfrak{a}$ by definition of S. Hence $(1/1) + (a/s)$ is invertible in $S^{-1}A$ for all $a/s \in S^{-1}\mathfrak{a}$, which completes the proof of this result. This being done, if we set $N = S^{-1}M$, clearly N is a finitely generated $S^{-1}A$-module. If $\mathfrak{a}M = M$, then $S^{-1}\mathfrak{a}N = N$ and it follows that $N = 0$ by Nakayama's Lemma. The corollary then follows from Corollary 1.2.16. □

Remark 1.2.18. Let M be an A-module, f be an element of A. Consider a sequence (M_n) of A-modules, where $M_n = M$ and for any integers $m \le n$, let $\varphi_{nm} : M_m \to M_n$ be the multiplication by f^{n-m}. Then it is immediate that (M_n, φ_{nm}) form an inductive system of A-modules, so let $N = \varinjlim M_n$ and $\varphi_n : N \to M_n$ be the canonical homomorphisms. We now define a functorial isomorphism from N to M_f: for each n, let $\theta_n : z \mapsto z/f^n$ be an A-homomorphism from $M_n = M$ to M_f, and since we clearly have $\theta_n \circ \varphi_{nm} = \theta_m$, let $\theta : N \to M_f$ be the induced homomorphism. Since any element of M_f is of the form z/f^n for some n, so θ is surjective. On the other hand, if $\theta(\varphi_n(z)) = 0$, which means $z/f^n = 0$, then there exists an integer $k > 0$ such that $f^k z = 0$, so $\varphi_{n+k,n}(z) = 0$, which means $\varphi_n(z) = 0$. We can then identify M_f with $\varinjlim M_n$ by the isomorphism θ.

Proposition 1.2.19. *Let A, B be two rings, S a multiplicative subset of A, T a multiplicative subset of B and f a homomorphism from A to B such that $f(S) \subseteq T$. Let M be an A-module, N a B-module and $\phi : M \to N$ an A-linear homomorphism. Then there exists a unique $S^{-1}A$-linear map $\tilde{\phi} : S^{-1}M \to T^{-1}N$ such that $\tilde{\phi}(m/1) = \phi(m)/1$ for all $m \in M$.*

Proof. Thc map $i_M^T \circ \phi$ from M to $T^{-1}N$ is A-linear. Moreover, if $s \in S$, then $f(s) \in T$, hence the homothety induced by s on $T^{-1}N$ is bijective. The existence and uniqueness of $\tilde{\phi}$ then follow from Proposition 1.2.12. Then, for $m \in M$ and $s \in S$,

$$\tilde{\phi}(m/s) = \phi(m)/f(s)$$

With the same notation, let C be a third ring, U a multiplicative subset of C, g a homomorphism from B to C such that $g(T) \subseteq U$, P a C-module, ψ a B-linear map from N to P and $\tilde{\phi}$ the $T^{-1}B$-linear map from B to $U^{-1}P$ associated with ψ. Then

$$\widetilde{(\psi \circ \phi)} = \tilde{\psi} \circ \tilde{\phi}$$

where the left hand side is the A-linear map $S^{-1}M \to U^{-1}P$ associated with $\psi \circ \phi$. Similarly, if ϕ' is a second A-linear map from M to N, then

$$\widetilde{(\phi + \phi')} = \tilde{\phi}' + \tilde{\phi}$$

the left-hand side being the A-linear map $S^{-1}M \to T^{-1}N$ associated with $\phi + \phi'$. □

If, in Proposition 1.2.19, we take $B = A$, $T = S$ and $\phi = \mathrm{id}_A$, it is easily seen that $\tilde{\phi}$ is just the map $\phi \otimes 1 : M \otimes_A S^{-1}A \to N \otimes_A S^{-1}A$. We shall henceforth denote it by $S^{-1}\phi$. If S is the complement of a prime ideal $\mathfrak{p}$ of A, we write $\phi_{\mathfrak{p}}$ instead of $S^{-1}\phi$.

Proposition 1.2.20. *Let f be a homomorphism from a ring A to a ring B and S a multiplicative subset of A. Then there exists a unique map $j : f(S)^{-1}B \to S^{-1}B$ (where B is considered as an A-module by means of f) such that $j(b/f(s)) = b/s$ for all $b \in B$, $s \in S$. If $\tilde{\phi} : S^{-1}A \to f(S)^{-1}B$ is the ring homomorphism associated with f, then $j \circ \tilde{f} = S^{-1}f$. The map j is an isomorphism of the $S^{-1}A$-module structure on $f(S)^{-1}B$ defined by $\tilde{f}$ onto that on $S^{-1}B$ and also of the B-module structure on $f(S)^{-1}B$ onto that on $S^{-1}B$ (resulting from the definition $S^{-1}B = S^{-1}A \otimes_A B$).*

Proof. If $b, b' \in B$ and $s, s' \in S$, the conditions $b/s = b'/s'$ and $b/f(s) = b'/f(s')$ are equivalent, as follows from Corollary 1.2.15, which establishes the existence of j and shows that j is bijective. The uniqueness of j is obvious. Clearly j is an additive group isomorphism. If $a \in A$, $b \in B$, $s, t \in S$ then

$$(a/s) \cdot (b/f(t)) = \tilde{f}(a/s)(b/f(t)) = f(a)/f(s)(b/f(t)) = (f(a)b)/f(st)$$

from which its follows that j is $S^{-1}A$-linear. Clearly $j \circ \tilde{f} = S^{-1}f$. Finally, if $b.b' \in B$, $s \in S$, then

$$j(b \cdot (b'/f(s)) = j(bb'/f(s)) = bb'/s = b \cdot (b'/s),$$

which proves the last assertion. □

The map j of Proposition 1.2.20 is called the canonical isomorphism of $f(S)^{-1}B$ onto $S^{-1}B$. These two sets are in general identified by means of f. Then $\tilde{f} = S^{-1}f$ and $i_B^S = i_B^{f(S)}$.

1.2.3 Change of multiplicative subset

Let A be a ring, S a multiplicative subset of A and M an A-module. If T is a multiplicative subset of A containing S, it follows from Proposition 1.2.19 that there exists a unique $S^{-1}A$-linear map $i_M^{T,S} : S^{-1}M \to T^{-1}M$ such that $i_M^T = i_M^{T,S} \circ i_M^T$. The map i_M^S maps the element m/s of $S^{-1}M$ to the element m/s of $T^{-1}M$. It is easily verified that $i_M^{T,S} = i_A^{T,S} \otimes 1_M$. If U is a third multiplicative subset of A such that $T \subseteq U$, then $i_M^{U,S} = i_M^{U,T} \circ i_M^{T,S}$. Moreover, if $\phi : M \to N$ is an A-module homomorphism, the diagram

$$\begin{array}{ccc} S^{-1}M & \xrightarrow{S^{-1}\phi} & S^{-1}N \\ \downarrow{\scriptstyle i_M^{T,S}} & & \downarrow{\scriptstyle i_N^{T,S}} \\ T^{-1}M & \xrightarrow{T^{-1}\phi} & T^{-1}N \end{array}$$

is commutative.

Proposition 1.2.21. *Let A be a ring and S, T two multiplicative subsets of A.*

(a) *There exists a unique isomorphism i_A from the ring $(ST)^{-1}A$ onto the ring $(S^{-1}T)^{-1}(S^{-1}A)$ such that the diagram*

$$\begin{array}{ccc} A & \xrightarrow{i_A^S} & S^{-1}A \\ {\scriptstyle i_A^{ST}}\downarrow & & \downarrow{\scriptstyle i_{S^{-1}A}^{S^{-1}T}} \\ (ST)^{-1}A & \xrightarrow{i} & (S^{-1}T)^{-1}(S^{-1}A) \end{array}$$

(b) *Let M be an A-module. There exists an $(ST)^{-1}A$-isomorphism i_M from the $(ST)^{-1}A$-module $(ST)^{-1}M$ onto the $(S^{-1}T)^{-1}(S^{-1}A)$-module $(S^{-1}T)^{-1}(S^{-1}M)$ such that the diagram*

$$\begin{array}{ccc} M & \xrightarrow{i_M^S} & S^{-1}M \\ {\scriptstyle i_M^{ST}}\downarrow & & \downarrow{\scriptstyle i_{S^{-1}M}^{S^{-1}T}} \\ (ST)^{-1}M & \xrightarrow{i_M} & (S^{-1}T)^{-1}(S^{-1}M) \end{array}$$

Proof. It suffices to prove (a). We use the definition of $(ST)^{-1}A$ as the solution of a universal map problem. Let B be a ring and f a homomorphism from A to B such that $f(ST)$ consists of invertible elements. As $f(S)$ consequently consists of invertible elements, there exists a unique homomorphism $f_1 : S^{-1}A \to B$ such that $f = f_1 \circ i_A^S$. For all $t \in T$, $f_1(i_A^S(t)) = f(t)$ is invertible in B by hypothesis, hence $f_1(S^{-1}T)$ consists of invertible elements; then there exists a unique homomorphism f_2 from $(S^{-1}T)^{-1}(S^{-1}A)$ to B such that $f_1 = f_2 \circ i_{S^{-1}A}^{S^{-1}T}$, whence $f = f_2 \circ i$, where $i = i_{S^{-1}A}^{S^{-1}T} \circ i_A^S$.

Moreover, if $f_2' : (S^{-1}T)^{-1}(S^{-1}A) \to B$ is a second homomorphism such that $f_2' \circ i = f$, then $(f_2' \circ i_{S^{-1}A}^{S^{-1}T}) \circ i_A^S = (f_2 \circ i_{S^{-1}A}^{S^{-1}T}) \circ i_A^S$, whence $f_2' \circ i_{S^{-1}A}^{S^{-1}T} = f_2 \circ i_{S^{-1}A}^{S^{-1}T}$ and consequently $f_2' = f_2$. As the images under i of the elements of ST are invertible, the ordered pair $((S^{-1}T)^{-1}(S^{-1}A), i)$ is a solution of the universal map problem (relative to A and ST). This shows the existence and uniqueness of i_A. $\square$

Corollary 1.2.22. *If S and T are two multiplicative subsets of A with $S \subseteq T$, then writing $S^{-1}T$ for the image of T in $S^{-1}A$, we have $(S^{-1}T)^{-1}(S^{-1}A) = T^{-1}A$.*

Corollary 1.2.23. *If $S \subseteq A$ is a multiplicative set and $\mathfrak{p}$ is a prime ideal of A disjoint from S then $(S^{-1}A)_{S^{-1}\mathfrak{p}} = A_{\mathfrak{p}}$. In particular if $\mathfrak{p}_1 \subseteq \mathfrak{p}_2$ are prime ideals of A, then $(A_{\mathfrak{p}_2})_{\mathfrak{p}_1} = A_{\mathfrak{p}_1}$.*

Proposition 1.2.24. *Let S, T be multiplicatively closed subsets of A, such that $S \subseteq T$. Then the following statements are equivalent:*

- *(a) The homomorphism $i_A^{T,S}$ is bijective.*
- *(b) For every A-module M, the homomorphism $i_M^{T,S} : S^{-1}M \to T^{-1}M$ is bijective.*
- *(c) For each $t \in T$, $t/1$ is a unit in $S^{-1}A$.*
- *(d) For each $t \in T$ there exists $x \in A$ such that $xt \in S$.*
- *(e) Every prime ideal which meets T also meets S.*

Proof. It has been seen above that $i_M^{T,S} = i_A^{T,S} \otimes 1_M$, which immediately proves the equivalence of (a) and (b). Also, the equivalence of (c) and (d) is easy to see. By Proposition 1.2.21, since $S \subseteq T$, $T^{-1}A$ is identified with $(S^{-1}T)^{-1}(S^{-1}A)$ and (a) is equivalent to saying that the elements of $S^{-1}T$ are invertible in $S^{-1}A$, so (a) is equivalent to (c). Now by definition, it is clear that (d) implies (e).

So assume (e). To show the injectivity of $i_A^{T,S}$, let $i_A^{T,S}(x/s) = 0$ in $T^{-1}A$, so that $xt = 0$ for some $t \in T$. Now assume that $x/s \neq 0$, which is to say $\mathrm{Ann}(x) \cap S = \varnothing$. Then by Proposition 1.2.3, there is a prime ideal $\mathfrak{p} \supseteq \mathrm{Ann}(x)$ such that $\mathfrak{p} \cap S = \varnothing$. By assumption, we also have $\mathfrak{p} \cap T = \varnothing$, which is a contradiction. This proves $x/s = 0$, so $i_A^{T,S}$ is injective.

For the surjectivity, let $t \in T$. We claim that there exists $x \in A$ such that $xt \in S$. Otherwise, $(t) \cap S = \varnothing$, and by Proposition 1.2.3, there is a prime ideal $\mathfrak{p} \supseteq (t)$ such that $\mathfrak{p} \cap S = \varnothing$. Then we have $\mathfrak{p} \cap T = \varnothing$, which is a contradiction. $\square$

By Proposition 1.2.24, we see that amongst the multiplicative subsets T of A containing S and satisfying the equivalent conditions of Proposition 1.2.24, there exists a greatest, consisting of all the elements of A which divide an element of S. This set can be characterized by another property, which we now define.

A multiplicative subset S of a ring A is called **saturated** if the relation $xy \in S$ implies $x \in S$ and $y \in S$. If this holds, then for any $x \in A \setminus S$ and any $s \in S$ we have $ax \notin S$, which shows $x/1$ is not a unit in $S^{-1}A$, hence there is a maximal ideal $\mathfrak{m}$ contains $x/1$. Now consider the contraction $\mathfrak{m}^c$ in A. It contains x, and is prime since $\mathfrak{m}$ is prime. Hence we see x is contained in a prime ideal $\mathfrak{p}$ disjoint from A, thus $A \setminus S$ is a union of prime ideals of A. In fact this condition is also sufficient: if $A \setminus S = \bigcup \mathfrak{p}_i$ for prime ideals $\mathfrak{p}_i$, then assume $xy \in S$. If $x \in A \setminus S$, then $x \in \mathfrak{p}_i$ for some $\mathfrak{p}_i$. Since $\mathfrak{p}_i$ is an ideal, $xy \in \mathfrak{p}_i$ for any $y \in A$, which means $xy \in A \setminus S$, a contradiction. So $x, y \in S$.

Proposition 1.2.25. *Let S be a multiplicative subset of A and define*

$$\bar{S} = \{x \in A : \textit{there exists } y \in S \textit{ such that } xy \in S\}.$$

Then S is the smallest saturated multiplicative subset containing S, called the ***saturation*** *of S, and $A \setminus \bar{S}$ is the union of the prime ideals of A not meeting S.*

Proof. It is easy to see $\bar{S}$ is a miltiplicative subset and saturated. In particular, $A \setminus \bar{S} = \bigcup_i \mathfrak{p}_i$, where $\mathfrak{p}_i$ are prime ideals. Now if $\mathfrak{p}$ is a prime ideal such that $\mathfrak{p} \cap S = \varnothing$, then for any $x \in \mathfrak{p}$, if there exists $y \in S$ such that $xy \in S$, then $xy \in \mathfrak{p} \cap S$, which is a contradiction. This shows $\mathfrak{p} \subseteq A \setminus S$, which finishes the proof. $\square$

Corollary 1.2.26. *Let S, T be multiplicatively closed subsets of A, such that $S \subseteq T$. Then $i_M^{T,S}$ is bijective if and only if T is contained in the saturation of S.*

Proposition 1.2.27. *Let A be a ring, S be a multiplicative subset of A. If B is a subring of $S^{-1}A$ containing $i_A^S(A)$, then we have*

$$S^{-1}A = S^{-1}B = T^{-1}B \quad \textit{where} \quad T = \{b \in B : b \textit{ is a unit of } S^{-1}A\}.$$

Proof. By definition of T we have $S \subseteq T$ and $T^{-1}B$ can be identified with a subring of $S^{-1}A$. We can write

$$\begin{tikzcd}
A \arrow[r, "i_A^S"] \arrow[d] & S^{-1}A \arrow[d, "i"] \\
B \arrow[r, "i_B^T"] \arrow[d, hook] & T^{-1}B \arrow[dl, "j"] \\
S^{-1}A &
\end{tikzcd}$$

where i and j the natural maps. Also, by definition we have $i \circ j = 1_{S^{-1}A}$. Also, by the definition of T and Proposition 1.2.24 we see $S^{-1}B = T^{-1}B$. Now, for any $b/t \in T^{-1}B$, we have $t^{-1} = a/s$ where $a \in A$, $s \in S$. Since $b \in B \subseteq S^{-1}A$, we then see

$$b/t = i(b) \cdot i(a/s) = i(b \cdot a/s)$$

whence i is surjective. Thus i and j are mutually inverse, giving an isomorphism $S^{-1}A \cong T^{-1}B$. $\square$

Proposition 1.2.28. *Let M be an A-module. For every submodule N' of the $S^{-1}A$-module $S^{-1}M$, let $\phi(N')$ be the inverse image of N' under i_M^S. Then:*

(a) $S^{-1}\phi(N') = N'$.

(b) *For every submodule N of M, the submodule $\phi(S^{-1}N)$ of M consists of those $m \in M$ for which there exists $s \in S$ such that $sm \in N$.*

(c) *ϕ is an isomorphism of the set of sub-$S^{-1}A$-modules of $S^{-1}M$ onto the set of submodules Q of M which satisfy the following condition:*

(MS) *If $sm \in Q$, where $s \in S$, $m \in M$, then $m \in Q$.*

Proof. Obviously $S^{-1}\phi(N') \subseteq N'$. Conversely, if $n' = m/s \in N'$, then $m/1 \in N'$, hence in $m \in \phi(N')$ and conscquently $n' \in S^{-1}(\phi(N'))$, whence (a). For an element $m \in M$ to be such that $m \in \phi(S^{-1}N)$, it is necessary and sufficient that $m/1 \in S^{-1}N$, that is, there exist $s \in S$ and $n \in N$ such that $m/1 = n/s$, which means there exists $s' \in S$ such that $s'sm = s'n \in N$, whence (b). Finally, the relation $sm \in \phi(N')$ is equivalent by definition to $sm/1 \in N'$ and as $s/1$ is invertible in $S^{-1}A$, this implies $m/1 \in N'$, or $m \in \phi(N')$, hence $\phi(N')$ satisfies condition (MS). On the other hand, it follows from (b) that, if N satisfies (MS), then $\phi(S^{-1}N) = N$, which completes the proof of (c). □

The submodule $\phi(S^{-1}N)$ is called the **saturation** of N in M with respect to S, and the submodules satifying condition (MS) (and hence equal to their saturations) are said to be **saturated** with respect to S. The submodule $\phi(S^{-1}N)$ is the kernel of the composite homomorphism

$$M \xrightarrow{\pi} M/N \xrightarrow{i^S_{M/N}} S^{-1}M/S^{-1}N$$

where π is the canonical homomorphism, as follows from the commutativity of the diagram

$$\begin{array}{ccc} M & \xrightarrow{\pi} & M/N \\ \downarrow{\scriptstyle i^S_M} & & \downarrow{\scriptstyle i^S_{M/N}} \\ S^{-1}M & \longrightarrow & S^{-1}M/S^{-1}N \end{array}$$

If S is the complement in A of a prime ideal $\mathfrak{p}$, $\phi(S^{-1}N)$ is also called the saturation of N in M with respect to $\mathfrak{p}$.

Corollary 1.2.29. *Let N_1, N_2 be two submodules of an A-modale M. For $S^{-1}N_1 \subseteq S^{-1}N_2$, it is necessary and sufficient that the saturation of N_1 with respect to S be contained in that of N_2.*

Corollary 1.2.30. *If M is a Noetherian (resp. Artinian) A-module, then $S^{-1}M$ is a Noetherian (resp. Artinian) $S^{-1}A$-module. In particular, if the ring A is Noetherian (resp. Artinian), so is the ring $S^{-1}A$.*

1.2.4 Properties of the localization operation

Proposition 1.2.31. *Let A be a ring and S a multiplicative subset of A. Then $S^{-1}A$ is A-flat.*

Proof. If $\phi : M \to N$ is an injective homomorphism of A-modules, it is necessary to establish that $S^{-1}\phi : S^{-1}M \to S^{-1}N$ is injective. Now, if m/s is such that $\phi(m)/s = 0$, this implies the existence of a $t \in S$ such that $t\phi(m) = 0$ or $\phi(tm) = 0$. As ϕ is injective, it follows that $tm = 0$, whence $m/s = 0$ in $S^{-1}M$. □

Corollary 1.2.32. *Formation of fractions commutes with formation of finite sums, finite intersections and quotients. Precisely, if N, P are submodules of an A-module M, then*

(a) $S^{-1}(N + P) = S^{-1}N + S^{-1}P$.

(b) $S^{-1}(N \cap P) = S^{-1}N \cap S^{-1}P$.

(c) The $S^{-1}A$-modules $S^{-1}(M/N)$ and $(S^{-1}M)/(S^{-1}N)$ are isomorphic.

Proof. Part (a) follows readily from the definitions and (b) is easy to verify: if $y/s = z/t$ ($y \in N$, $z \in P, s, t \in S$) then $u(ty - sz) = 0$ for some $u \in S$, hence $w = uty = usz \in N \cap P$ and therefore $y/s = w/stu \in S^{-1}(N \cap P)$. Consequently $S^{-1}(N \cap P) \subseteq S^{-1}(N) \cap S^{-1}P$, and the reverse inclusion is obvious.

Now to prove (c), we apply S^{-1} to the exact sequence

$$0 \longrightarrow N \longrightarrow M \longrightarrow M/N \longrightarrow 0$$

Since localization is exact, we get the claim. □

Proposition 1.2.33. *Let M be a finitely generated A-module, S a multiplicatively closed subset of A. Then $S^{-1}\mathrm{Ann}(M) = \mathrm{Ann}(S^{-1}M)$.*

Proof. If this is true for two A-modules, M, N, it is true for $M+N$:

$$\begin{aligned} S^{-1}(\mathrm{Ann}(M+N)) &= S^{-1}(\mathrm{Ann}(M) \cap \mathrm{Ann}(N)) = \mathrm{Ann}(S^{-1}M) \cap \mathrm{Ann}(S^{-1}N) \\ &= \mathrm{Ann}(S^{-1}M + S^{-1}N) = \mathrm{Ann}(S^{-1}(M+N)) \end{aligned}$$

where we use Corollary 1.2.32. Hence it is enough to prove for M generated by a single element: then $M = A/I$ as A-module, where $\mathfrak{a} = \mathrm{Ann}(M)$. We have

$$S^{-1}(M) = S^{-1}(A/\mathfrak{a}) \cong (S^{-1}A)/(S^{-1}\mathfrak{a})$$

by Corollary 1.2.32, which proves the claim. $\square$

Corollary 1.2.34. *If N, P are submodules of an A-module M and if P is finitely generated, then $S^{-1}(N : P) = (S^{-1}N : S^{-1}P)$.*

Proof. We have $(N : P) = \mathrm{Ann}((N+P)/P)$, so the claim follows by Proposition 1.2.33. $\square$

Proposition 1.2.35. *Let I be a directed set, $(S_i)_{i\in I}$ an inereasing family of multiplicative subsets of a ring A, and $S = \bigcup_i S_i$. We write $\rho_{\beta\alpha} = i_A^{S_\beta,S_\alpha}$. Then $(S_\alpha^{-1}A, \rho_{\beta,\alpha})$ is a direct system of rings and, if for $\alpha \in I$, ρ_α is the canonical map of $S_\alpha^{-1}A$ to $\varinjlim S_\alpha^{-1}A$, there exists a unique isomorphism $i : \varinjlim S_\alpha^{-1}A \to S^{-1}A$ such that $i \circ \rho_\alpha = i_A^{S,S_\alpha}$ for all $\alpha \in I$.*

$$\begin{tikzcd} S_\alpha^{-1}A \arrow[r, "\rho_{\beta\alpha}"] \arrow[drrr, "\rho_\alpha"'] & S_\beta^{-1}A \arrow[r, "\rho_{\gamma\beta}"] \arrow[drr, "\rho_\beta"'] & S_\gamma^{-1}A \arrow[r] \arrow[dr, "\rho_\gamma"] & \cdots \arrow[d] & \\ & & & \varinjlim S_\alpha^{-1}A \arrow[r, "i"] & S^{-1}A \end{tikzcd}$$

Proof. For $\alpha \preceq \beta \preceq \gamma$, $\rho_{\gamma\alpha} = \rho_{\gamma\beta} \circ \rho_{\beta\alpha}$, hence $(S_\alpha^{-1}A, \rho_{\beta\alpha})$ is a direct system of rings, and we write $B = \varinjlim S_\alpha^{-1}A$. As

$$i_A^{S,S_\alpha} = i_A^{S,S_\beta} \circ \rho_{\beta\alpha}$$

for $\alpha \preceq \beta$, (i_A^{S,S_α}) is a direct system of homomorphisms and $i = \varinjlim i_A^{S,S_\alpha}$ is the unique map such that $i \circ \rho_\alpha = i_A^{S,S_\alpha}$ for all $\alpha \in I$.

The homomorphisms $\rho_\alpha \circ i_A^{S_\alpha}$ are all equal since $\rho_{\alpha\beta} \circ i_A^{S_\alpha} = i_A^{S_\beta}$ for $\alpha \preceq \beta$, so let $\phi : A \to B$ be their common value. Clearly elements of $\phi(S)$ are invertible in B, which shows there exists a homomorphism $\tilde{\phi} : S^{-1}A \to B$ such that $\tilde{\phi} \circ i_A^S = \phi$. Then

$$i \circ \tilde{\phi} \circ i_A^S = i \circ \phi = i \circ \rho_\alpha \circ i_A^{S_\alpha} = i_A^{S,S_\alpha} \circ i_A^{S_\alpha} = i_A^S$$

for all $\alpha \in I$ and consequently $i \circ \tilde{\phi}$ is the identity map of $S^{-1}A$. On the other hand, for all $\alpha \in I$,

$$\tilde{\phi} \circ i \circ \rho_\alpha \circ i_A^{S_\alpha} = \tilde{\phi} \circ i_A^{S,S_\alpha} \circ i_A^{S_\alpha} = \tilde{\phi} \circ i_A^S = \phi = \rho_\alpha \circ i_A^{S_\alpha}$$

whence $\tilde{\phi} \circ i \circ \rho_\alpha = \rho_\alpha$ for all $\alpha \in I$; it follows that $\tilde{\phi} \circ i$ is the identity automorphism of B and consequently i is an isomorphism. $\square$

Corollary 1.2.36. *Under the hypotheses of Proposition 1.2.35, let M be an A-module. We write $\tau_{\beta\alpha} = i_M^{S_\beta,S_\alpha}$ for $\alpha \preceq \beta$. Then $(S_\alpha^{-1}M, \tau_{\beta,\alpha})$ is a direct system of rings and, if for $\alpha \in I$, τ_α is the canonical map of $S_\alpha^{-1}M$ to $\varinjlim S_\alpha^{-1}M$, there exists a unique isomorphism $i : \varinjlim S_\alpha^{-1}M \to S^{-1}M$ such that $i \circ \tau_\alpha = i_M^{S,S_\alpha}$ for all $\alpha \in I$.*

Proof. The corollary follows immediately from the definitions $S_\alpha^{-1}M = M \otimes_A S_\alpha^{-1}A$ and $S^{-1}M = M \otimes_A S^{-1}A$ and the fact that taking direct limits commutes with tensor products. $\square$

1.2.5 Ideals under localization

Proposition 1.2.37. *Let S be a multiplicative subset of a commutative ring A, and consider the localization operation. Let $\mathfrak{a}$ be an ideal in A, and $\mathfrak{b}$ an ideal in $S^{-1}A$.*

(a) $(\mathfrak{b}^c)^e = \mathfrak{b}$*, hence every ideal in $S^{-1}A$ is an extended ideal.*

(b) $(\mathfrak{a}^e)^c = \bigcup_{s \in S}(\mathfrak{a} : s)$*. Hence $S^{-1}\mathfrak{a}$ is proper if and only if $\mathfrak{a} \cap S \neq \varnothing$ and $\mathfrak{a}$ is a contracted ideal if and only if no element of S is a zero-divisor in $A/\mathfrak{a}$.*

(c) The assignment $\mathfrak{p} \to S^{-1}\mathfrak{p}$ gives an inclusion-preserving bijection between the set of prime ideals of A disjoint from S and the set of ideals of $S^{-1}A$.

(d) The operation S^{-1} commutes with formation of finite sums, products, intersections and radicals.

Proof. Recall that we always have $\mathfrak{b} \subseteq (\mathfrak{b}^c)^e$. Conversely, if $a/s \in \mathfrak{b}$ then $a/1 \in \mathfrak{b}$, so $a \in \mathfrak{b}^c$. It follows that $a/s \in S^{-1}(\mathfrak{b}^c) = (\mathfrak{b}^c)^e$, so $\mathfrak{b} = (\mathfrak{b}^c)^e$. For (b), by defninition we have

$$\begin{aligned}(\mathfrak{a}^e)^c &= \{a \in A : a/1 \in \mathfrak{a}^e\} = \{a \in A : (sa - r)u = 0 \text{ for some } r \in I \text{ and } u, s \in S\} \\ &= \{a \in A : (\exists s \in S)\ sx \in \mathfrak{a}\} = \bigcup_{s \in S}(\mathfrak{a} : s).\end{aligned}$$

From this, we see $\mathfrak{a}^e = (\bigcup_{s \in S}(\mathfrak{a} : s))^e$, and therefore $1 \in \mathfrak{a}^e$ if and only if $\mathfrak{a} \cap S \neq \varnothing$. From the description of $\mathfrak{a}^{ec}$, we also find that $\mathfrak{a}^{ec} = \mathfrak{a}$ if and only if $\bigcup_{s \in S}(\mathfrak{a} : s) \subseteq \mathfrak{a}$, if and only if no element of S is a zero-divisor in $A/\mathfrak{a}$. This proves (b).

If $\mathfrak{p}$ is prime in A such that $\mathfrak{p} \cap S = \varnothing$, then we see that $(\mathfrak{p} : s) = \mathfrak{p}$ for all $s \in S$, whence $(\mathfrak{p}^e)^c = \mathfrak{p}$. Thus $(\mathfrak{p})^e$ is a bijection on prime ideals of A. If $\mathfrak{p}$ is prime in $S^{-1}A$, then $\mathfrak{p}^c$ is prime in A. Further, by part (b) we have $\mathfrak{p}^c \cap S = \varnothing$.

If $\mathfrak{p}$ is prime in A such that $\mathfrak{p} \cap S = \varnothing$, then $S^{-1}\mathfrak{p}$ is a proper ideal, as we see in (b). Now we prove the extension $\mathfrak{p}^e = S^{-1}\mathfrak{p}$ is prime. Let $x/s, y/t \notin S^{-1}\mathfrak{p}$ but $xy/st \in S^{-1}\mathfrak{p}$. Then by definition for some $p \in \mathfrak{p}, u \in S$ we have

$$u(xy - pst) = 0.$$

That is, $uxy = ustp \in \mathfrak{p}$. Since $\mathfrak{p} \cap S = \varnothing$, we must have $xy \in \mathfrak{p}$. But since $x/s, y/t \notin S^{-1}\mathfrak{p}$, $x \notin \mathfrak{p}$ and $y \notin \mathfrak{p}$. This is a contradiction. Thus the extension and contraction corresponds the prime ideals in $S^{-1}A$ and prime ideals in A disjoint from S.

Finally, we deal with (e). For sums and products, this follows from Proposition 1.1.27; for intersections, from Corallary 1.2.32. As to radicals, we have $S^{-1}\sqrt{\mathfrak{a}} \subseteq \sqrt{S^{-1}\mathfrak{a}}$ from Proposition 1.1.27 since $S^{-1}\mathfrak{a} = \mathfrak{a}^e$. For the converse, let $x/s \in \sqrt{S^{-1}\mathfrak{a}}$, then $x^n/s^n \in S^{-1}\mathfrak{a}$ for some $n > 0$. This means $ux^n \in \mathfrak{a}$ for some $u \in S$. Then $u^n x^n$ also belongs to $\mathfrak{a}$, so $ux \in \sqrt{\mathfrak{a}}$. Moreover, since $x/s = (ux)/(us)$, we get $x/s \in S^{-1}\sqrt{\mathfrak{a}}$. □

Corollary 1.2.38. *There is an inclusion-preserving bijection between the prime ideals of $A_\mathfrak{p}$ and the prime ideals of A contained in $\mathfrak{p}$. Hence $A_\mathfrak{p}$ is a local ring.*

Proof. If $a, b \notin \mathfrak{p}$, then $ab \notin \mathfrak{p}$. So $A - \mathfrak{p}$ is a multiplicative set. The claim now comes from Proposition 1.2.37.

The elements a/s with $a \in \mathfrak{p}$ form an ideal $\mathfrak{m}$ in $A_\mathfrak{p}$. If $b/t \notin \mathfrak{m}$, then $b \notin \mathfrak{p}$, so $b \in A - \mathfrak{p}$ and is a unit in $A_\mathfrak{p}$. By Proposition 1.1.15, $A_\mathfrak{p}$ is a local ring.

Or we can use the correspondence to prove $S^{-1}\mathfrak{p}$ is the unique maximal ideal. □

Corollary 1.2.39. *If $\mathfrak{n}(A)$ is the nilradical of A, the nilradical of $S^{-1}A$ is $S^{-1}\mathfrak{n}(A)$.*

Remark 1.2.40. Thus the passage from A to $A_\mathfrak{p}$ cuts out all prime ideals except those contained in $\mathfrak{p}$. In the other direction, the passage from A to $A/\mathfrak{p}$ cuts out all prime ideals except those containing $\mathfrak{p}$. Hence if $\mathfrak{p}, \mathfrak{q}$ are prime ideals such that $\mathfrak{p} \supseteq \mathfrak{q}$, then by localizing with respect to $\mathfrak{p}$ and taking the quotient mod $\mathfrak{q}$ (in either order: these two operations commute, by Corallary 1.2.32), we restrict our attention to those prime ideals which lie between $\mathfrak{p}$ and $\mathfrak{q}$. In particular, if $\mathfrak{p} = \mathfrak{q}$ we end up with a field, called the **residue field at** $\mathfrak{p}$, which can be obtained either as the field of fractions of the integral domain $A/\mathfrak{p}$ or as the residue field of the local ring $A_\mathfrak{p}$.

Proposition 1.2.41. *Let A be a ring and S a multiplicative subset of A. For every ideal $\mathfrak{b}$ of $S^{-1}A$, let $\mathfrak{a} = (i_A^S)^{-1}(\mathfrak{b})$ be such that $S^{-1}\mathfrak{a} = \mathfrak{b}$.*

(a) Let π be the canonical homomorphism $A \to A/\mathfrak{a}$. Then the homomorphism from $S^{-1}A$ to $\pi(S)^{-1}(A/\mathfrak{a})$ canonically associated with π is surjective and its kernel is $\mathfrak{b}$, which defines, by taking quotients, a canonical isomorphism of $(S^{-1}A)/\mathfrak{b}$ onto $\pi(S)^{-1}(A/\mathfrak{a})$. Moreover, the canonical homomorphism from $A/\mathfrak{a}$ to $\pi(S)^{-1}(A/\mathfrak{a})$ is injective.

(b) If $\mathfrak{q}$ is a prime ideal of $S^{-1}A$ and $\mathfrak{p} = (i_A^S)^{-1}(\mathfrak{q})$, then there exists an isomorphism of the ring of fractions $A_\mathfrak{p}$ onto the ring $(S^{-1}A)_\mathfrak{q}$, which maps a/b to $(a/1)/(b/1)$, where $a \in A$, $b \in A - \mathfrak{p}$.

Proof. The ring $\pi(S)^{-1}(A/\mathfrak{a})$ can be identified with $S^{-1}(A/\mathfrak{a})$ by means of the canonical isomorphism between these two modules. The exact sequence $0 \to \mathfrak{a} \to A \to A/\mathfrak{a} \to 0$ then induces an exact sequence

$$0 \longrightarrow S^{-1}\mathfrak{a} \longrightarrow S^{-1}A \longrightarrow S^{-1}(A/\mathfrak{a}) \longrightarrow 0$$

whose existence proves the first assertion of (a), taking account of the fact that $\mathfrak{b} = S^{-1}\mathfrak{a}$. Since $\mathfrak{a}$ is saturated with respect S, the conditions $a \in A$, $s \in S$, $as \in \mathfrak{a}$ imply $a \in \mathfrak{a}$; the homothety of ratio s on $A/\mathfrak{a}$ is then injective, which proves the second assertion of (i).

Suppose that $\mathfrak{q}$ is prime and such that $\mathfrak{p}$ is also prime. The set $T = A - \mathfrak{p}$ is a multiplicative subset of A which contains S, whence $ST = T$. Then it follows from Corollary 1.2.22 that there exists a unique isomorphism i of $T^{-1}A = A_\mathfrak{p}$ onto $(S^{-1}T)^{-1}(S^{-1}A)$ such that

$$i(a/b) = (a/1)/(b/1)$$

where $a \in A$ and $b \in T$. On the other hand $S^{-1}T$ obviously does not meet $\mathfrak{q}$. Conversely, let $a/s \in S^{-1}A$; since $1/s$ is invertible in $S^{-1}A$, the condition $a/s \notin \mathfrak{q}$ is equivalent to $i_A^S(a) \notin \mathfrak{q}$ and hence to $a \notin \mathfrak{p}$. It follows that $S^{-1}A - \mathfrak{q} = S^{-1}T$ and hence $(S^{-1}T)^{-1}(S^{-1}A) = (S^{-1}A)_\mathfrak{q}$. □

Proposition 1.2.42. *Let $\rho : A \to B$ be a ring homomorphism and let $\mathfrak{p}$ be a prime ideal of A. Then $\mathfrak{p}$ is the contraction of a prime ideal of S if and only if $\mathfrak{p}^{ec} = \mathfrak{p}$.*

Proof. One direction is trivial. Conversely, suppose that $\mathfrak{p}^{ec} = \mathfrak{p}$ and consider the multiplicative subset $S = \rho(A - \mathfrak{p})$ of B. The hypothesis shows that $S \cap \mathfrak{p}^e = \varnothing$, so there exists a prime ideal $\mathfrak{q}$ of B disjoint with S such that $\mathfrak{p}^e \subseteq \mathfrak{q}$. Then $\mathfrak{q}^c$ contains $\mathfrak{p}$ and is disjoint with $A - \mathfrak{p}$, whence $\mathfrak{q}^c = \mathfrak{p}$. □

Corollary 1.2.43. *Let $\rho : A \to B$ be a ring homomorphism.*

(a) Suppose that there exists a B-module E such that $\rho^*(E)$ is a faithfully flat A-module. Then, for every prime ideal $\mathfrak{p}$ of A, there exists a prime ideal $\mathfrak{P}$ of B such that $\mathfrak{P}^c = \mathfrak{p}$.

(b) Conversely, suppose that B is a flat A-module. If for every prime ideal $\mathfrak{p}$ of A, there exists an ideal $\mathfrak{P}$ of B such that $\mathfrak{P}^c = \mathfrak{p}$, then B is a faithfully flat A-module.

Proof. The hypothesis in (a) implies that, for every ideal $\mathfrak{a}$ of A, we have $\mathfrak{a}^{ec} = \mathfrak{a}$ (**??**), and it is sufficient to apply Proposition 1.2.42. Now assume the conditions in (b). It is sufficient to show that, for every maximal ideal $\mathfrak{m}$ of A, there exists a maximal ideal $\mathfrak{M}$ of B such that $\mathfrak{M}^c = \mathfrak{m}$ (**??**). Now there exists by hypothesis an ideal $\mathfrak{Q}$ of B such that $\mathfrak{Q}^c = \mathfrak{m}$. As $\mathfrak{Q} \neq B$, there exists a maximal ideal $\mathfrak{M}$ of B containing $\mathfrak{Q}$ and consequently $\mathfrak{M}^c = \mathfrak{m}$ since $\mathfrak{M}^c$ cannot contain 1. □

Corollary 1.2.44. *Let $\rho : A \to B$ be a ring homomorphism such that B is a flat A-module. Let $\mathfrak{P}$ be a prime ideal of B and let $\mathfrak{p} = \mathfrak{P}^c$. Then ${}^a\rho_{\mathfrak{p}} : \operatorname{Spec}(B_{\mathfrak{P}}) \to \operatorname{Spec}(A_{\mathfrak{p}})$ is surjective.*

Proof. The map $\rho_{\mathfrak{p}} : A_{\mathfrak{p}} \to B_{\mathfrak{P}}$ is a local homomorphism since we have

$$(\mathfrak{p}A_{\mathfrak{p}})^e = \left(\frac{\mathfrak{p}}{A-\mathfrak{p}}\right)^e = \frac{\mathfrak{p}^e}{f(A-\mathfrak{p})} = \frac{\mathfrak{P}^{ce}}{f(A-\mathfrak{p})} \subseteq \mathfrak{P}B_{\mathfrak{P}} \neq B_{\mathfrak{P}},$$

so $B_{\mathfrak{P}}$ is a faithfully flat $A_{\mathfrak{p}}$-module by **??**, and the claim follows from Corollary 1.2.43. □

Proposition 1.2.45. *Let B be a ring and A a subring of B. For every minimal prime ideal $\mathfrak{p}$ of A there exists a minimal prime ideal $\mathfrak{P}$ of B such that $\mathfrak{P} \cap A = \mathfrak{p}$.*

Proof. Let $\mathfrak{p}$ be a minimal prime of A and $S = A - \mathfrak{p}$. Then the ring $A_{\mathfrak{p}}$ is identified with a subring of $S^{-1}B$ and it has a single prime ideal $\mathfrak{p}A_{\mathfrak{p}}$ since $\mathfrak{p}$ is minimal. As $S^{-1}B$ is not reduced to 0 (since it contains $A_{\mathfrak{p}}$), it has at least one prime ideal $\mathcal{P}$ and therefore $\mathcal{P} \cap A_{\mathfrak{p}} = \mathfrak{p}A_{\mathfrak{p}}$. If $\mathfrak{P}' = (i_B^S)^{-1}(\mathcal{P})$, we have

$$i_A^S(\mathcal{P} \cap A) \subseteq \mathcal{P} \cap A_{\mathfrak{p}} = \mathfrak{p}A_{\mathfrak{p}}.$$

hence $\mathfrak{P}' \cap A \subseteq \mathfrak{p}$, and as $\mathfrak{p}$ is minimal, $\mathfrak{P}' \cap A = \mathfrak{p}$. If $\mathfrak{P}$ is a minimal prime ideal of B contained in $\mathfrak{P}'$, then we also have $\mathfrak{P} \cap A = \mathfrak{p}$, whence the claim. □

1.2.6 Localization of tensor products and Hom sets

Proposition 1.2.46. *Let A be a ring and S a multiplicative subset of A.*

(a) If M and N are two A-modules, the there is a canonical isomorphism

$$S^{-1}M \otimes_{S^{-1}A} S^{-1}N \cong S^{-1}(M \otimes_A N).$$

(b) If M and N are two $S^{-1}A$-modules, the canonical homomorphism $M \otimes_A N \to M \otimes_{S^{-1}A} N$ derived from the A-bilinear map $(x, y) \mapsto x \otimes y$ is bijective.

Proof. Assertion (a) is an immediate consequence of the definition $S^{-1}M = M \otimes_A S^{-1}A$ and the associativity of tensor products. To prove (b), we note first that in M and N, considered as A-modules, the homotheties induced by the elements $s \in S$ are bijective, hence $M = S^{-1}M$ and $N = S^{-1}N$ and similarly $S^{-1}(M \otimes_A N) = M \otimes_A N$. Then (b) is then a special case of (a). □

Corollary 1.2.47. *Let M be an A-module and $\mathfrak{a}$ an ideal of A. The sub-$S^{-1}A$-modules*

$$(S^{-1}\mathfrak{a})(S^{-1}M), \quad \mathfrak{a}(S^{-1}M), \quad (S^{-1}\mathfrak{a})i_M^S(M), \quad S^{-1}(\mathfrak{a}M)$$

of $S^{-1}M$ are identical. In particular, if $\mathfrak{a}$ and $\mathfrak{b}$ are two ideals of A, then

$$(S^{-1}\mathfrak{a})(S^{-1}\mathfrak{b}) = \mathfrak{a}(S^{-1}\mathfrak{b}) = (S^{-1}\mathfrak{a})\mathfrak{b} = S^{-1}(\mathfrak{a}\mathfrak{b}).$$

Proposition 1.2.48. *Let A be a ring and S a multiplicative subset of A.*

(a) If M and N are two A-modules and M is finitely generated (resp. finitely presented), the canonical homomorphism

$$S^{-1}\mathrm{Hom}_A(M, N) \to \mathrm{Hom}_{S^{-1}A}(S^{-1}M, S^{-1}N)$$

is injective (resp. bijectioe).

(b) If M and N are two $S^{-1}A$-modules, the canonical bijection

$$\mathrm{Hom}_{S^{-1}A}(M, N) \to \mathrm{Hom}_A(M, N)$$

is bijective.

Proof. As $S^{-1}A$ is a flat A-module, (a) is a particular case of **??**. On the other hand, we note that every A-homomorphism of $S^{-1}A$-modules is necessarily $S^{-1}A$-linear, whence (b). □

Proposition 1.2.49. *Let A, B be two rings, $\rho : A \to B$ a homomorphism, S a multiplicative subset of A, and $\tilde{\rho} : S^{-1}A \to S^{-1}B$ the homomorphism corresponding to ρ.*

(a) For every B-module M there exists a unique $S^{-1}A$-isomorphism

$$i : S^{-1}\rho_*(M) \to \tilde{\rho}_*(S^{-1}M)$$

(b) For every A-module M, there exists a unique isomorphism

$$j : (S^{-1}M) \otimes_{S^{-1}A} (S^{-1}B) \to S^{-1}(M \otimes_A B).$$

Proof. If we consider $S^{-1}M$ as an A-module by means of the composite homomorphism $i_M^S \circ \rho$, the homotheties induced by the elements of S are bijective, hence there exists a unique homomorphism i. It is claer that i is surjective; moreover, if $m \in M$, $s \in S$ and $m/\rho(s) = 0$, then there exists $t \in S$ such that $\rho(t)m = 0$, and so $tm = 0$ in $\rho_*(M)$, hence $m/s = 0$ in $S^{-1}\rho_*(M)$.

For (b), as $(S^{-1}M) \otimes_{S^{-1}A} (S^{-1}B) = (M \otimes_A S^{-1}A) \otimes_{S^{-1}A} S^{-1}B$, we see

$$S^{-1}(M \otimes_A B) = (M \otimes_A B) \otimes_B (S^{-1}B)$$

the existence of j follows from the associativity of tensor products. □

1.3 Local rings and passing from local to global

1.3.1 Local rings

Proposition 1.3.1. *Let A be a ring and I the set of non-invertible elements of A. The set I is the union of the ideals of A which are distinct from A. Moreover, the following conditions are equivalent:*

(i) I is an ideal.

(ii) The set of ideals of A distinct from A has a greatest element.

(iii) A has a unique maximal ideal.

Proof. The relation $x \in I$ is equivalent to $1 \notin xA$ and hence $xA \neq A$. If $\mathfrak{a}$ is an ideal of A distinct from A and $x \in \mathfrak{a}$, then $xA \subseteq \mathfrak{a}$, hence $xA \neq A$ and $x \in I$. Hence every ideal of A distinct from A is contained in I and every element $x \in I$ belongs to a principal ideal $xA \neq A$. This proves the first assertion, which immediately implies the equivalence of (i), (ii) and (iii). □

A ring A is called a **local ring** if it satisfies the equivalent conditions of Proposition 1.3.1. The quotient of A by its Jacobson radical (which is then the unique maximal ideal of A) is called the **residue field** of A. Let A, B be two local rings and $\mathfrak{m}$, $\mathfrak{n}$ their respective maximal ideals. A homomorphism $\rho : A \to B$ is called local if $\rho(\mathfrak{m}) \subseteq \mathfrak{n}$. Note that this amounts to saying that $\rho^{-1}(\mathfrak{n}) = \mathfrak{n}$, for $\rho^{-1}(\mathfrak{n})$ is then an ideal containing $\mathfrak{m}$ and not containing 1 and hence equal to $\mathfrak{m}$. Taking quotients, we then derive canonically from ρ an injective homomorphism $A/\mathfrak{m} \to B/\mathfrak{n}$ from the residue field of A to that of B.

Example 1.3.2 (Example of local rings).

(a) A field is a local ring. A ring which is reduced to 0 is not a local ring.

(b) Let A be a local ring and κ its residue field. The ring of formal power series $B = A[[X_1, \dots, X_n]]$ is a local ring, for the non-invertible elements of B are the formal power series whose constant terms are not invertible in A. The canonical injection of A into B is a local homomorphism and the corresponding injection of residue fields is an isomorphism.

(c) Let $\mathfrak{a}$ be an ideal of a ring A which is only contained in a single maximal ideal $\mathfrak{m}$. Then $A/\mathfrak{a}$ is a local ring with maximal ideal $\mathfrak{m}/\mathfrak{a}$ and residue field canonically isomorphic to $A/\mathfrak{m}$. This applies in particular to the case $\mathfrak{a} = \mathfrak{m}^n$, where $\mathfrak{m}$ is any maximal ideal of A (Corollary 1.1.3). If A itself is a local ring with maximal ideal $\mathfrak{m}$, then for every proper ideal $\mathfrak{a}$ of A, $A/\mathfrak{a}$ is a local ring, the canonical homomorphism $A \to A/\mathfrak{a}$ a local homomorphism and the corresponding homomorphism of residue fields an isomorphism.

(d) Let X be a topological space, x_0 a point of X and A the ring of germs at the point x_0 of real-valued functions continuous in a neighbourhood of x_0. Clearly, for the germ at x_0 of a continuous function f to be invertible in A, it is necessary and sufficient that $f(x_0) \neq 0$, since this implies that $f(x) \neq 0$ in a neighbourhood of x_0. The ring A is therefore a local ring whose maximal ideal $\mathfrak{m}$ is the set of germs of functions which are zero at x_0. Taking quotients, the map $f \mapsto f(x_0)$ of A to $\mathbb{R}$ gives an isomorphism of the residue field $A/\mathfrak{m}$ onto $\mathbb{R}$.

Proposition 1.3.3. *Let A be a ring and $\mathfrak{p}$ a prime ideal of A. The ring $A_\mathfrak{p}$ is local, and its maximal ideal is the ideal $\mathfrak{p}A_\mathfrak{p} = \mathfrak{p}_\mathfrak{p}$, generated by the canonical image of $\mathfrak{p}$ in $A_\mathfrak{p}$. Its residue field is canonically isomorphic to the field of fractions of $A/\mathfrak{p}$.*

Proof. Let $S = A - \mathfrak{p}$ and $i_A^S : A \to A_\mathfrak{p}$ be the canonical homomorphism; the hypothesis that $\mathfrak{p}$ is prime implies that $\mathfrak{p}$ is saturated with respect to S, hence $(i_A^S)^{-1}(\mathfrak{p}A_\mathfrak{p}) = \mathfrak{p}$ and, as the ideals of A not meeting S are those contained in $\mathfrak{p}$, the first two assertions are special cases of Proposition 1.2.37. Moreover, if π is the canonical homomorphism $A \to A/\mathfrak{p}$, $\pi(S)$ is the set of nonzero elements of the integral domain $A/\mathfrak{p}$ and hence the last assertion is a special case of Proposition 1.2.41. □

Let A be a ring and $\mathfrak{p}$ a prime ideal of A. The ring $A_\mathfrak{p}$ is called the local ring of A at $\mathfrak{p}$, or the local ring of $\mathfrak{p}$, when there is no ambiguity. If A is a local ring and $\mathfrak{m}$ its maximal ideal, the elements of $A - \mathfrak{m}$ are invertible and hence $A_\mathfrak{m}$ is canonically identified with A.

Example 1.3.4.

(a) Let p be a prime number. The local ring $\mathbb{Z}_\mathfrak{p}$ is the set of rational numbers a/b, where a, b are rational integers with b prime to p. The residue field of $\mathbb{Z}_\mathfrak{p}$ is isomorphic to the prime field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$.

(b) Let V be an affine algebraic variety, A the ring of regular functions on V, W an irreducible subvariety of V and $\mathfrak{p}$ the (necessarily prime) ideal of A consisting of the functions which are zero at every point of W. The ring $A_\mathfrak{p}$ is called the local ring of W on V.

Proposition 1.3.5. *Let A, B be two rings, $\rho : A \to B$ a homomorphism, $\mathfrak{q}$ a prime ideal of B and $\mathfrak{p} = \mathfrak{q}^c$. Then there is a canonical homomorphism $\rho_{\mathfrak{q}} : A_{\mathfrak{p}} \to B_{\mathfrak{q}}$ which is local.*

Proof. As $\rho(A - \mathfrak{p}) \subseteq B - \mathfrak{q}$, a canonical homomorphism $\rho_{\mathfrak{q}} : A_{\mathfrak{p}} \to B_{\mathfrak{q}}$ is derived from ρ, and is immediate that $\rho_{\mathfrak{q}}(\mathfrak{p}A_{\mathfrak{p}}) \subseteq \mathfrak{q}B_{\mathfrak{q}}$, hence $\rho_{\mathfrak{q}}$ is local. □

1.3.2 Modules over a local ring

Proposition 1.3.6. *Let A be a ring, $\mathfrak{m}$ an ideal of A contained in the Jacobson radical of A and M an A-module. Suppose that one of the following conditions holds:*

(a) M is finitely generated;

(b) $\mathfrak{m}$ is nilpotent.

Then the relation $(A/\mathfrak{m}) \otimes_A M = 0$ implies $M = 0$.

Proof. The assertion with respect to hypothesis (a) is precisely Nakayama's Lemma. On the other hand, the relation $(A/\mathfrak{m}) \otimes_A M = 0$ is equivalent to $M = \mathfrak{m}M$ and hence implies $M = \mathfrak{m}^n M$ for every integer $n > 0$; whence the assertion with respect to hypothesis (b). □

Corollary 1.3.7. *Let A be a ring, $\mathfrak{m}$ an ideal of A contained in the Jacobson radical of A, M and N two A-modules and $\phi : M \to N$ an A-linear map. If N is finitely generated or $\mathfrak{m}$ is nilpotent and*

$$1 \otimes \phi : (A/\mathfrak{m}) \otimes_A M \to (A/\mathfrak{m}) \otimes_A N$$

is surjective, then ϕ is surjective.

Proof. Since $(A/\mathfrak{m}) \otimes_A (N/\phi(M))$ is canonically isomorphic to $((A/\mathfrak{m}) \otimes_A M)/\operatorname{im}(1 \otimes \phi)$, the hypothesis implies $(A/\mathfrak{m}) \otimes_A (N/\phi(M)) = 0$, hence $N/\phi(M) = 0$ by Proposition 1.3.6. □

Corollary 1.3.8. *Let A be a ring, $\mathfrak{m}$ an ideal of A contained in the Jacobson radical of A, M an A-module and $(x_i)_{i \in I}$ a family of elements of M. If M is finitely generated or $\mathfrak{m}$ is nilpotent and the elements $1 \otimes x_i$ generate the $(A/\mathfrak{m})$-module $M/\mathfrak{m}M$, then the x_i generate M.*

Proposition 1.3.9. *Let A be a ring, $\mathfrak{m}$ an ideal of A contained in the Jacobson radical of A and M an A-module. Suppose that one of the following conditions holds:*

(a) M is finitely presented;

(b) $\mathfrak{m}$ is nilpotent.

Then, if $M/\mathfrak{m}M$ is a free $(A/\mathfrak{m})$-module and the canonical homomorphism $\mathfrak{m} \otimes_A M \to M$ is injective, then M is a free A-module. More precisely, if $(x_i)_{i \in I}$ is a family of elements of M such that $(1 \otimes x_i)$ is a basis of the $(A/\mathfrak{m})$-module $M/\mathfrak{m}M$, then (x_i) is a basis of M.

Proof. We already know that the x_i generate M (Corollary 1.3.8). We shall see that they are linearly independent over A. To this end, let us consider the free A-module $L = A^I$. Let (e_i) be its canonical basis and $\phi : A^I \to M$ the A-linear map such that $\phi(e_i) = x_i$ for all $i \in I$. If R is the kernel of ϕ, we shall prove that that $\mathfrak{m}R = R$. Let j be the canonical injection $R \to L$; then there is a commutative diagram

$$\begin{tikzcd}
 & \mathfrak{m} \otimes R \arrow[r, "1\otimes j"] \arrow[d, "\alpha"] & \mathfrak{m} \otimes L \arrow[r, "1\otimes\phi"] \arrow[d, "\beta"] & \mathfrak{m} \otimes M \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & R \arrow[r, "j"] & L \arrow[r, "\phi"] & M \arrow[r] & 0
\end{tikzcd}$$

in which the two rows are exact. By hypothesis we have $\ker\gamma = 0$, hence by snake lemma, there is an exact sequence

$$0 \longrightarrow \operatorname{coker}\alpha \longrightarrow \operatorname{coker}\beta \longrightarrow \operatorname{coker}\gamma \longrightarrow 0$$

Now $\operatorname{coker}\beta = (A/\mathfrak{m}) \otimes L$ and $\operatorname{coker}\gamma = (A/\mathfrak{m}) \otimes M$. Since $(1 \otimes e_i)$ is a basis for $(A/\mathfrak{m}) \otimes L$ and its image in $(A/\mathfrak{m}) \otimes M$ is a basis for $(A/\mathfrak{m}) \otimes M$, we see $\operatorname{coker}\alpha = 0$, whence $\mathfrak{m}R = R$.

Under hypothesis (a), $(A/\mathfrak{m}) \otimes_A M$ is a finitely generated module, hence I is necessarily finite and R is a finitely generated A-module. Then by Proposition 1.3.6 we have $R = 0$. Similarly, under hypothesis (b) we also have $R = 0$, so the proof is completed. □

Corollary 1.3.10. *Let A be a local ring and $\mathfrak{m}$ its maximal ideal. For a family (y_i) of elements of M to be a basis of a direct factor of M, it is necessary and sufficient that the family $(1 \otimes y_i)$ be free in $M/\mathfrak{m}M$.*

Proof. If this condition holds, it can be assumed that (x_i) is a subfamily of a family (x_i) of elements of M such that $(1 \otimes x_i)$ is a basis of $M/\mathfrak{m}M$ and Proposition 1.3.9 then proves that (x_i) is a basis of M. □

Corollary 1.3.11. *Let A be a local ring, $\mathfrak{m}$ its maximal ideal and M an A-module. Suppose that one of the following conditions holds:*

(a) *M is finitely presented;*

(b) *$\mathfrak{m}$ is nilpotent.*

Then the following properties are equivalent:

(i) *M is free;*

(ii) *M is projective;*

(iii) *M is flat;*

(iv) *the canonical homomorphism $\mathfrak{m} \otimes_A M \to M$ is injective;*

(v) $\operatorname{Tor}_A^1(A/\mathfrak{m}, M) = 0$.

Proof. The implications (i)⇒(ii)⇒(iii)⇒(iv) are immediate, and we know that (iv) is equivalent to (v). As $A/\mathfrak{m}$ is a field, $(A/\mathfrak{m}) \otimes_A M$ is a free $(A/\mathfrak{m})$-module and Proposition 1.3.9 shows that (iv) implies (i). □

Proposition 1.3.12. *Let A be a local ring and $\mathfrak{m}$ its maximal radical. Let M and N be two finitely generated free A-modules and $\phi : M \to N$ a homomorphism. The following properties are equivalent:*

(i) *ϕ is an isomorphism of M onto a direct factor of N;*

(ii) *$1 \otimes \phi : (A/\mathfrak{m}) \otimes_A M \to (A/\mathfrak{m}) \otimes_A N$ is injective;*

(iii) *ϕ is injective and $\operatorname{coker}\phi$ is a free A-module;*

(iv) *the transpose homomorphism $\phi^t : N^* \to M^*$ is surjective.*

Proof. We know that, if $N/\phi(M)$ is free, then $\phi(M)$ is a direct factor of N, hence (iii) implies (i). Conversely, (i) implies that $\operatorname{coker}\phi$, isomorphic to a complement of $\phi(M)$ in N, is a finitely generated projective A-module and a fortiori finitely presented, hence this module is free by Corollary 1.3.11 and thus (i) implies (iii).

On the other hand, (i) obviously implies (ii). For simplicity we write $\widetilde{M} = (A/\mathfrak{m}) \otimes_A M$ and $\widetilde{N} = (A/\mathfrak{m}) \otimes_A N$. As M and N are finitely generated, the duals $\widetilde{M}^*$ and $\widetilde{N}^*$ of the

$(A/\mathfrak{m})$-modules $\widetilde{M}$ and $\widetilde{N}$ are canonically identified with $M^* \otimes_A (A/\mathfrak{m})$ and $N^* \otimes_A (A/\mathfrak{m})$ and $(1 \otimes \phi)^t$ with $1 \otimes \phi^t$. As $\widetilde{M}$ and $\widetilde{N}$ are vector spaces over the field $A/\mathfrak{m}$, the hypothesis that $1 \otimes \phi$ is injective implies that $(1 \otimes \phi)^t$ is surjective. Corollary 1.3.7 then shows that ϕ^t is surjective and we have thus proved that (ii) implies (iv).

Finally we show that (iv) implies (i). Suppose that ϕ^t is surjective. As M^* is free, there exists a homomorphism $\psi^t : M^* \to N^*$ such that $\phi^t \circ \phi^t = \mathrm{id}$. As M and N are free and finitely generated, there exists a homomorphism $\psi : N \to M$ such that the dual of ψ is ψ^t, whence

$$\mathrm{id}_M^t = \mathrm{id}_{M^*} = \phi^t \circ \psi^t = (\psi \circ \phi)^t$$

whence $\psi \circ \phi = \mathrm{id}$. This proves that ϕ is an isomorphism of M onto a submodule which is a direct factor of N. □

Corollary 1.3.13. *Under the hypotheses of Proposition 1.3.12 the following properties are equivalent:*

(i) ϕ *is an isomorphism of* M *onto* N*;*

(ii) M *and* N *have the same rank;*

(iii) $1 \otimes \phi : (A/\mathfrak{m}) \otimes_A M \to (A/\mathfrak{m}) \otimes_A N$ *is bijective.*

Proof. Clearly (i) implies (ii); (ii) implies that $1 \otimes \phi$ is surjective; moreover the hypothesis that M and N have the same rank implies that so do the vector spaces $(A/\mathfrak{m}) \otimes_A M$ and $(A/\mathfrak{m}) \otimes_A N$ over $A/\mathfrak{m}$, hence $1 \otimes \phi$ is bijective and (ii) implies (iii). Finally, condition (iii) implies, by Proposition 1.3.12, that N is the direct sum of $\phi(M)$ and a free submodule P and ϕ is an isomorphism of M onto $\phi(M)$. If $P \neq 0$, then $(A/\mathfrak{m}) \otimes_A P \neq 0$ and $1 \otimes \phi$ would not be sutjcctive; hence (iii) implies (i). □

Proposition 1.3.14. *Let* A *be a reduced local ring,* $\mathfrak{m}$ *its maximal ideal,* $(\mathfrak{p}_i)_{i \in I}$ *the family of minimal prime ideals of* A*,* K_i *the field of fractions of* $A/\mathfrak{p}_i$ *and* M *a finitely generated* A*-module. For* M *to be free it is necessary and sufficient that*

$$[(A/\mathfrak{m}) \otimes_A : (A/\mathfrak{m})] = [K_i \otimes_A M : K] \tag{1.3.1}$$

for all $i \in I$*.*

Proof. If M is free, clearly the two sides of (1.3.1) are equal to the rank of M for all $i \in I$. Suppose now that the condition is satisfied and denote by n the common value of the two sides of (1.3.1). By Corollary 1.3.8 M has a system of n generators $x_1, \ldots, x_n$. Suppose first that A is an integral domain, in which case $\mathfrak{p}_i = 0$ for all $i \in I$. The elements $(1 \otimes x_i)$ generate the vector space $K \otimes M$ over the field of fractions K of A. But as by hypothesis this space is of dimension n over K, the elements $1 \otimes x_i$ are linearly independent over K. It follows that the x_i are linearly independent over A and hence form a basis of M.

Passing to the general case, there exists a surjective homomorphism η from $L = A^n$ onto M. Consider the commutative diagram

$$\begin{tikzcd} L \arrow[r, "\eta"] \arrow[d, "\pi_L"] & M \arrow[d, "\pi_M"] \\ \prod_i((A/\mathfrak{p}_i) \otimes L) \arrow[r, "\tilde{\eta}"] & \prod_i((A/\mathfrak{p}_i) \otimes M) \end{tikzcd}$$

where ϕ_L (resp. π_M) is the map $x \mapsto (\phi_i(x))$ (resp. $y \mapsto (\psi_i(y))$), with $\phi_i : L \to (A/\mathfrak{p}_i) \otimes L$ (resp. $\psi_i : M \to (A/\mathfrak{p}_i) \otimes M$) being the canonical map, and $\tilde{\eta}$ is the product of the $1_{A/\mathfrak{p}_i} \otimes \eta$. Then as

$$((A/\mathfrak{p}_i)/(\mathfrak{m}/\mathfrak{p}_i)) \otimes_{A/\mathfrak{p}_i} ((A/\mathfrak{p}_i) \otimes_A M) = (A/m) \otimes_A M$$

and $A/\mathfrak{p}_i$ is a local integral domain, it follows from the first part of the argument that each of the M, $1_{A/\mathfrak{p}_i} \otimes \eta$ is an isomorphism; then so is $\tilde{\eta}$. On the other hand, as A is reduced, $\bigcap_i \mathfrak{p}_i = (0)$ whence $\bigcap_i \mathfrak{p}_i L = (0)$ since L is free. As this shows that π_L is injective, it follows that $\tilde{\eta} \circ \pi_L$ is injective, hence η is injective and, as η is sutjective by definition, this shows that M is free. □

1.3.3 Passing from local to global

Proposition 1.3.15. *Let A be a ring, $\mathfrak{m}$ a maximal ideal of A and M an A-module. If there exists an ideal $\mathfrak{a}$ of A such that $\mathfrak{m}$ is the only maximal ideal of A containing $\mathfrak{a}$ and $\mathfrak{a}M = 0$, then the canonical homomorphism $M \to M_{\mathfrak{m}}$ is bijective.*

Proof. By hypothesis $A/\mathfrak{a}$ is then a local ring with maximal ideal $\mathfrak{m}/\mathfrak{a}$; M can be considered as an $(A/\mathfrak{a})$-module. For all $s \in A - \mathfrak{m}$ the canonical image of s in $A/\mathfrak{a}$ is invertible, hence the homothety $x \mapsto sx$ of M is bijective from the definition of $M_{\mathfrak{m}}$ as the solution of a universal problem, whence the proposition. □

In particular, if there exists $k > 0$ such that $\mathfrak{m}^k M = 0$, the homomorphism $M \mapsto M_{\mathfrak{m}}$ is bijective.

Proposition 1.3.16. *Let A be a ring, $\mathfrak{m}$ a maximal ideal of A, M an A-module and $k > 0$ an integer. The canonical homomorphism $M \to M_{\mathfrak{m}}/\mathfrak{m}^k M_{\mathfrak{m}}$ is surjectice, has kernel $\mathfrak{m}^k M$ and defines an isomorphism of $M/\mathfrak{m}^k M$ onto $M_{\mathfrak{m}}/\mathfrak{m}^k M_{\mathfrak{m}}$.*

Proof. It follows from Proposition 1.3.15 the homomorphism $M/\mathfrak{m}^k M \to (M/\mathfrak{m}^k M)_{\mathfrak{m}}$ is bijective. On the other hand $(M/\mathfrak{m}^k M)_{\mathfrak{m}}$ is canonically identified with $M_{\mathfrak{m}}/(\mathfrak{m}^k M)_{\mathfrak{m}}$ and hence $(\mathfrak{m}^k M)_{\mathfrak{m}} = \mathfrak{m}^k M_{\mathfrak{m}}$, whence there is an isomorphism of $M/\mathfrak{m}^k M$ onto $M_{\mathfrak{m}}/\mathfrak{m}^k M_{\mathfrak{m}}$ which maps the class of an element $x \in M$ to the class of $x/1$. □

Corollary 1.3.17. *Let A be a ring, $\mathfrak{m}_1, \dots, \mathfrak{m}_n$ distinct maximal ideals of A, M an A-module and $k_1, \dots, k_n$ positive integers. The canonical homomorphism from M to $\bigoplus_{i=1}^n M_{\mathfrak{m}_i}/\mathfrak{m}_i^{k_i} M_{\mathfrak{m}_i}$ is surjective and its kernel is $(\bigcap_{i=1}^n \mathfrak{m}_i^{k_i})M$.*

Proposition 1.3.18. *Let A be a commutative ring, and let M be an A-module. The following are equivalent*

(a) $M = 0$.

(b) $M_{\mathfrak{p}} = 0$ *for every prime ideal* $\mathfrak{p}$.

(c) $M_{\mathfrak{m}} = 0$ *for every maximal ideal* $\mathfrak{m}$.

Proof. Clearly (a)⇒(b)⇒(c). So we only need to show $(c) \Rightarrow (a)$. Assume $M_{\mathfrak{m}} = 0$ for every maximal ideal $\mathfrak{m}$ but $M \neq 0$. Let x be a non-zero element of M, consider the ideal $\operatorname{Ann}(m)$: It is proper hence contained in a maxiaml ideal $\mathfrak{m}$. Consider the module $M_{\mathfrak{m}}$. We must have $x/1 = 0$ since $M_{\mathfrak{m}} = 0$, hence x is killed by some element in $A - \mathfrak{m}$. But this is impossible since $\operatorname{Ann}(x) \subseteq \mathfrak{m}$. □

Proposition 1.3.19. *Let $\phi : M \to N$ be an A-module homomorphism. Then the following are equivalent:*

(a) ϕ *is injective (resp. surjective).*

(b) $\phi_{\mathfrak{p}} : M_{\mathfrak{p}} \to N_{\mathfrak{p}}$ *is injective (resp. surjective) for each prime ideal* $\mathfrak{p}$ *of* A.

(c) $\phi_{\mathfrak{m}} : M_{\mathfrak{m}} \to N_{\mathfrak{m}}$ *is injective (resp. surjective) for each maximal ideal* $\mathfrak{m}$ *of* A.

Proof. Since localization is exact, we see (a)⇒(b)⇒(c). Now assume (c) and let $M' = \ker \phi$, then the sequence $0 \to M' \to M \to N$ is exact, hence $0 \to M'_\mathfrak{m} \to M_\mathfrak{m} \to N_\mathfrak{m}$ is exact and therefore $M'_\mathfrak{m} \cong \ker \phi_\mathfrak{m} = 0$ by assumption. Hence $M' = 0$ by Proposition 1.3.18, so ϕ is injective. For the other part of the proposition, just reverse all the arrows. □

Corollary 1.3.20. *The localization functor is faithfully exact in the following sense: let A be a commutative ring, and let*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0 \tag{1.3.2}$$

be a sequence of A-modules. Then (1.3.2) is exact if and only if the induced sequence of $A_\mathfrak{p}$-modules

$$0 \longrightarrow A_\mathfrak{p} \longrightarrow B_\mathfrak{p} \longrightarrow C_\mathfrak{p} \longrightarrow 0$$

is exact for every prime ideal $\mathfrak{p}$ of A, if and only if it is exact for every maximal ideal $\mathfrak{m}$.

Corollary 1.3.21. *Let M be an A-module, N a submodule of M and x an element of M. For $x \in N$, it is necessary and sufficient that, for any maximal ideal $\mathfrak{m}$, the canonical image of x in $M_\mathfrak{m}$ belong to $N_\mathfrak{m}$.*

Corollary 1.3.22. *Let M be an A-module and, for all maximal ideal $\mathfrak{m}$, let $\phi_\mathfrak{m} : M \to M_\mathfrak{m}$ be the canonical map. The homomorphism $x \mapsto (\phi_\mathfrak{m}(x))$ of M to $\bigoplus_\mathfrak{m} M_\mathfrak{m}$ is injective.*

Corollary 1.3.23. *Let A be an integral domain, K its field of fractions and M a torsion free A-module such that M is canonically identified with a sub-A-module of $K \otimes_A M$. Then, for any maximal ideal $\mathfrak{m}$, $M_\mathfrak{m}$ is canonically identified with a sub-A-module of $K \otimes_A M$ and $M = \bigcap_\mathfrak{m} M_\mathfrak{m}$.*

Proof. As M is identified with a submodule of $K \otimes_A M$, $M_\mathfrak{m}$ is identified with a sub-$A_\mathfrak{m}$-module of $(K \otimes_A M)_\mathfrak{m} = K_\mathfrak{m} \otimes_A M = K \otimes_A M$, so $M_\mathfrak{m}$ is torsion-free. Moreover, the commutativity of the diagram

$$\begin{tikzcd} M \arrow[r] \arrow[d] & K \otimes_A M \arrow[d] \\ M_\mathfrak{m} \arrow[r] & (K \otimes_A M) \end{tikzcd}$$

proves that the canonical map $M \mapsto M_\mathfrak{m}$ is injective. The corollary then follows from Corollary 1.3.21 applied to the A-module $K \otimes_A M$ and its submodule M. □

Proposition 1.3.24. *Let S be a multiplicative subset of a ring A and M be an A-module. If M is flat (resp. faithfully/flat), $S^{-1}M$ is a flat (resp. faithfully flat) $S^{-1}A$-module and a flat A-module.*

Proof. As $S^{-1}M = M \otimes_A S^{-1}A$, the first assertion follows from **??** (resp. **??**); moreover, $S^{-1}A$ is a flat A-module, hence if M is a flat A-module, so is $S^{-1}M$ by virtue of **??**. □

Proposition 1.3.25. *Let A be a ring, B an A-algebra and T a multiplicative subset of B. If N is a B-module which is flat as an A-module, $T^{-1}N$ is a flat A-module.*

Proof. We have $T^{-1}N = T^{-1}B \otimes_B N$, so the proposition follows from **??**. □

Proposition 1.3.26. *Let $\rho : A \to B$ be a homomorphism and N be a B-module. The following properties are equivalent:*

(a) N is a flat A-module;

(b) for every maximal ideal $\mathfrak{N}$ of B, $N_\mathfrak{N}$ is a flat A-module;

(c) for every maximal ideal $\mathfrak{N}$ of B, if $\mathfrak{m} = \mathfrak{N}^c$, $N_\mathfrak{N}$ is a flat $A_\mathfrak{m}$-module.

Proof. For all $a \notin \mathfrak{m}$, the homothety of $N_{\mathfrak{N}}$ induced by a is bijective, hence $N_{\mathfrak{N}}$ is canonically identified with $(N_{\mathfrak{N}})_{\mathfrak{m}}$ and the equivalence of (b) and (c) then follows. The fact that (a) implies (b) is a special case of Proposition 1.3.25. It remains to prove that (b) implies (a), that is, that, if (b) holds, for every injective A-module homomorphism $u : M \to M'$, the homomorphism $v = 1 \otimes u : N \otimes_A M \to N \otimes_A M'$ is injective. Now, v is also a B-module homomorphism and, for it to be injective, it is necessary and sufficient that $v_{\mathfrak{N}} : (N \otimes_A M)_{\mathfrak{N}} \to (N \otimes_A M')_{\mathfrak{N}}$ be so for every maximal ideal $\mathfrak{N}$ of B (Proposition 1.3.19). As we have

$$(N \otimes_A M)_{\mathfrak{N}} = B_{\mathfrak{N}} \otimes_B (N \otimes_A M) = N_{\mathfrak{N}} \otimes_A M$$

the homomorphism $v_{\mathfrak{N}}$ is identified with $1 \otimes u : N_{\mathfrak{N}} \otimes_A M \to N_{\mathfrak{N}} \otimes_A M'$, which is injective since $N_{\mathfrak{N}}$ is a flat A-module by hypothesis. □

Proposition 1.3.27. *For an A-module M to be flat (resp. faithfully flat), it is necessary and sufficient that, for every maximal ideal $\mathfrak{m}$ of A, $M_{\mathfrak{m}}$ be a flat (resp. faithfully flat) $A_{\mathfrak{m}}$-module.*

Proof. By Corollary 1.3.20, the sequence

$$0 \longrightarrow A \otimes M \longrightarrow B \otimes M \longrightarrow C \otimes M \longrightarrow 0$$

is exact if and only if

$$0 \longrightarrow (A \otimes M)_{\mathfrak{p}} \longrightarrow (B \otimes M)_{\mathfrak{p}} \longrightarrow (C \otimes M)_{\mathfrak{p}} \longrightarrow 0$$

is exact for any prime ideal $\mathfrak{p}$, iff for any maximal ideal $\mathfrak{m}$. From Proposition 1.2.46, we have $(A \otimes_A M)_{\mathfrak{p}} \simeq A_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} M_{\mathfrak{p}}$, so the claim follows. □

Proposition 1.3.28. *Let A be an integral domain with field of fractions K. We consider any ring of fractions of A as a subring of K. Then in this sense we have*

$$A = \bigcap_{\mathfrak{p} \in \operatorname{Spec}(A)} A_{\mathfrak{p}} = \bigcap_{\mathfrak{m} \in \operatorname{Max}(A)} A_{\mathfrak{m}}$$

Proof. For $x \in K$ the set $I = \{a \in A \mid ax \in A\}$ is an ideal of A. Now it is easy to see that $x \in A_{\mathfrak{p}}$ if and only if $I \cap (A - \mathfrak{p}) \neq \varnothing$, which is equivalenct to that $I \not\subseteq \mathfrak{p}$, so if $x \in A_{\mathfrak{m}}$ for every maximal ideal $\mathfrak{m}$ then I is not contained in any maximal ideal of A, whence $1 \in I$, that is, $x \in A$. □

Corollary 1.3.29. *Let A be an integral domain and $f \in A$, then $A_f = \bigcap_{f \notin \mathfrak{p}} A_{\mathfrak{p}}$.*

Proof. If A is an integral domain, then A_f is also a domain. Now $\operatorname{Spec}(A_f) = \{\mathfrak{p} \in \operatorname{Spec}(A) \mid f \notin \mathfrak{p}\}$, so by Proposition 1.3.28 we get the claim. □

1.4 The spectrum of a ring

1.4.1 The space $\operatorname{Spec}(A)$

Let A be a ring. The **spectrum of A** is the set of prime ideals of A. It is usually denoted $\operatorname{Spec}(A)$. To avoid confusion, we will sometimes write $[\mathfrak{p}]$ for the point of $\operatorname{Spec}(A)$ corresponding to the prime $\mathfrak{p}$ of A. Of course, the zero ideal (0) is a prime if A is a domain.

Example 1.4.1. Here are several examples.

(a) Prime ideals in $\mathbb{Z}$ is the form (p) with p prime, and every set $\{(p)\} = V(p)$ is closed. Hence $\operatorname{Spec}(\mathbb{Z})$ is a line, with one closed point for each prime number in $\mathbb{Z}$, and one non-closed point corresponding to the ideal (0).

(b) As $\mathbb{R}$ is a field, Spec($\mathbb{R}$) consists of a single point, corresponding to the ideal (0).

(c) The maximal ideals of the ring $\mathbb{C}[X]$ are all of the form $(x-a)$ for $a \in \mathbb{C}$. The only non-maximal prime ideal of $\mathbb{C}[X]$ is (0). Thus Spec($\mathbb{C}[X]$) is the complex plane together with a single non-closed point, corresponding to (0).

(d) There are two types of maximal ideals in $\mathbb{R}[X]$, those generated by a polynomial of degree one, and those generated by a polynomial of degree two. The degree one maximal ideals are all of the form $(x-a)$ for $a \in \mathbb{R}$. The maximal ideals of degree two correspond to complex conjugate pairs of elements of $\mathbb{C} - \mathbb{R}$. There is only one non-maximal prime ideal, (0). Therefore Spec($\mathbb{R}[X]$) is the upper-half plane, together with a single non-closed point.

Each element $f \in A$ defines a function, which we also write as f, on the space Spec(A): if $x = \mathfrak{p} \in X = \text{Spec}(A)$, we denote by $\kappa(x)$ or $\kappa(\mathfrak{p})$ the quotient field of the integral domain $A/\mathfrak{p}$, called the **residue field** of X at x, and we define $f(x) \in \kappa(x)$ to be the image of f via the canonical maps

$$A \to A/\mathfrak{p} \to \kappa(\mathfrak{p}).$$

The functions induced by elements of A are called the **regular functions** on Spec(A).

Example 1.4.2. Consider the ring of polynomials $\mathbb{C}[X]$, and let $p(x)$ be a polynomial. If $\alpha \in \mathbb{C}$ is a number, then $(x-\alpha)$ is a prime of $\mathbb{C}[X]$, and in fact is maximal. So $\kappa(x-\alpha)$ is just the field $\mathbb{C}$, and the value of $p(x)$ at the point $(x-\alpha) \in \text{Spec}(\mathbb{C}[X])$ is the number $p(\alpha)$.

More generally, if A is the coordinate ring of an affine variety V over an algebraically closed field K and $\mathfrak{m}$ is the maximal ideal corresponding to a point $x \in V$ in the usual sense, then $\kappa(x) = K$ and $f(x)$ is the value of f at x in the usual sense.

In general, the function f has values in fields that vary from point to point. Moreover, f is not necessarily determined by the values of this function. For example, if K is a field, the ring $A = K[X]/(x^2)$ has only one prime ideal, which is (x); and thus the element $x \in A$, albeit nonzero, induces a function whose value is 0 at every point of Spec(A).

By using regular functions, we make Spec(A) into a topological space. For each subset $S \subseteq A$, let

$$V(S) = \{x \in \text{Spec}(A) : f(x) = 0 \text{ for all } f \in S\} = \{\mathfrak{p} \in \text{Spec}(A) : \mathfrak{p} \supseteq S\}.$$

The impulse behind this definition is to make each $f \in A$ behave as much like a continuous function as possible. Of course the fields $\kappa(x)$ have no topology, and since they vary with x the usual notion of continuity makes no sense. But at least they all contain an element called zero, so one can speak of the locus of points in Spec(A) on which f is zero; and if f is to be like a continuous function, this locus should be closed. Since intersections of closed sets must be closed, we are led immediately to the definition above: $V(S)$ is just the intersection of the loci where the elements of S vanish.

Proposition 1.4.3. *Let A be a ring.*

(a) If $\mathfrak{a}$ is the ideal generated by E, then $V(E) = V(\mathfrak{a}) = V(\sqrt{\mathfrak{a}})$.

(b) Let $\mathfrak{a}, \mathfrak{b}$ be ideals of A, then $V(\mathfrak{a}) \subseteq V(\mathfrak{b})$ iff $\mathfrak{b} \subseteq \sqrt{\mathfrak{a}}$.

(c) If $(E_i)_{i\in I}$ is any family of subsets of A, then

$$V\Big(\bigcup_{i\in I} E_i\Big) = \bigcap_{i\in I} V(E_i).$$

(d) *For any ideals $\mathfrak{a}, \mathfrak{b}$ of A, we have $V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b})$.*

*In particular, the sets $V(E)$ satisfy the axioms for closed sets in a topological space. The resulting topology is called the **Zariski topology** on $\operatorname{Spec}(A)$.*

Proof. We only prove (d). By Proposition 1.1.23, we have

$$V(\mathfrak{a}\mathfrak{b}) = V(\sqrt{\mathfrak{a}\mathfrak{b}}) = V(\sqrt{\mathfrak{a} \cap \mathfrak{b}}) = V(\mathfrak{a} \cap \mathfrak{b}).$$

Now we only need to prove $V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b})$. Since $\mathfrak{a} \cap \mathfrak{b}$ is contained in $\mathfrak{a}$ and $\mathfrak{b}$, for either $\mathfrak{a} \subseteq \mathfrak{p}$ or $\mathfrak{b} \subseteq \mathfrak{p}$ we have $\mathfrak{a} \cap \mathfrak{b} \subseteq \mathfrak{p}$, whence $V(\mathfrak{a}) \cup V(\mathfrak{b}) \subseteq V(\mathfrak{a} \cap \mathfrak{b})$. Conversely, let $\mathfrak{p} \in V(\mathfrak{a} \cap \mathfrak{b})$ so that $\mathfrak{a} \cap \mathfrak{b} \subseteq \mathfrak{p}$. By Proposition 1.1.5, $\mathfrak{a} \subseteq \mathfrak{p}$ or $\mathfrak{a} \subseteq \mathfrak{p}$, so $\mathfrak{p} \in V(\mathfrak{a}) \cup V(\mathfrak{b})$. □

Let X be the prime spectrum of a ring A; for all $f \in A$, let us denote by X_f the set of prime ideals of A not containing f. Then $X_f = X - V(f)$ and X_f is therefore an open set, called the **distinguished** (or **standard**) **open subsets** of X. By Proposition 1.4.3, every closed subset of X is an intersection of closed sets of the form $V(f)$ and hence the X_f form a base of the spectral topology on X.

Proposition 1.4.4. *Ler A be a ring, $f, g \in A$ and $X = \operatorname{Spec}(A)$.*

(a) *$D(f) = \varnothing$ iff f is nilpotent and $D(f) = X$ iff f is a unit.*

(b) *$D(f) = D(g)$ iff $\sqrt{(f)} = \sqrt{(g)}$.*

(c) *$D(f) \cap D(g) = D(fg)$.*

(d) *If $\mathfrak{a} \subseteq A$ is an ideal and $\mathfrak{p}$ is a prime of A with $\mathfrak{p} \notin V(\mathfrak{a})$, then there exists an $f \in A$ such that $\mathfrak{p} \in D(f)$ and $D(f) \cap V(I) = \varnothing$.*

(e) *If $(f_i)_{i \in A}$ is a family of elements in A, then $\bigcup_i D(f_i) = X - V(\{f_i\})$.*

Proof. Part (a), (b) and (c) are immediate from the definition of $D(f)$ and Proposition 1.4.3. Now if $\mathfrak{p} \notin V(I)$, then there exists $f \in A$ such that $f \in \mathfrak{a} - \mathfrak{p}$. Then $f \notin \mathfrak{p}$ so that $\mathfrak{p} \in D(f)$. Also, if $\mathfrak{q} \in D(f)$ then $f \notin \mathfrak{q}$ and thus $\mathfrak{a}$ is not contained in $\mathfrak{q}$, which means $D(f) \cap V(\mathfrak{a}) = \varnothing$. This proves (d).

Finally, for (e) we note that

$$\bigcup_{i \in I} D(f_i) = \bigcup_{i \in I} (X - V(f_i)) = X - \bigcap_{i \in I} V(f_i) = X - V\Big(\bigcup_{i \in I} \{f_i\}\Big).$$

This completes the proof. □

For every subset Y of X, let us denote by $I(Y)$ the intersection of the prime ideals of A which belong to Y. Clearly $I(Y)$ is an ideal of A and the map $Y \mapsto I(Y)$ is decreasing with respect to inclusion in X and A, and we have

$$I\Big(\bigcup_{i \in I} Y_i\Big) = \bigcap_{i \in I} I(Y_i).$$

Proposition 1.4.5. *Let A be a ring, $\mathfrak{a}$ an ideal of A and Y a subset of $X = \operatorname{Spec}(A)$.*

(a) *$V(\mathfrak{a})$ is closed in X and $I(Y)$ is a radical ideal of A.*

(b) *$I(V(\mathfrak{a}))$ is the radical of $\mathfrak{a}$ and $V(I(Y))$ is the closure of Y in X.*

(c) *If E is any subset of A and Y is any subset of X, then*

$$V(E) = V(I(V(E))), \quad I(Y) = I(V(I(Y))).$$

(d) *The maps I and V define decreasing bijections, one of which is the inverse of the other, between the set of closed subsets of X and the set of radical ideals of A.*

Proof. Assertion (a) and the first assertion of (b) follow from the definitions. If a closed set $V(E)$ (for some $E \subseteq A$) contains Y, then $E \subseteq \mathfrak{p}$ for every prime ideal $\mathfrak{p} \in Y$, whence $E \subseteq I(Y)$ and consequently $V(E) \supseteq V(I(Y))$. As $Y \subseteq V(I(Y))$, $V(I(Y))$ is then the smallest closed set of X containing Y, which completes the proof of (b). Finally, it follows from (b) that, if $\mathfrak{a}$ is a prime ideal equal to its radical, then $I(V(\mathfrak{a})) = \mathfrak{a}$ and that, if Y is closed in X, then $V(I(Y)) = Y$. This proves (d). □

Corollary 1.4.6. *For every family $(Y_i)_{i\in I}$ of closed subsets of X, $I(\bigcap_{i\in I} Y_i)$ is the radical of the sum of the ideals $I(Y_i)$.*

Proof. It follows from Proposition 1.4.5 that $I(\bigcap_{i\in I} Y_i)$ is the smallest ideal which is equal to its radical and contains all the $I(Y_i)$; this ideal then contains $\sum_{i\in I} I(Y_i)$ and therefore also the radical of it, whence the corollary. □

Corollary 1.4.7. *The space $X = \operatorname{Spec}(A)$ is Noetherian if A is a Noetherian ring.*

Proof. If A is Noetherian, every ascending chain of prime ideals is stable, whence every descending chain of irreducible closed sets is stable. In other words, $\operatorname{Spec}(A)$ is Noetherian. □

Remark 1.4.8. Note that a ring A can be non-Noetherian even though its spectrum is Noetherian. For example, consider the ring

$$A = k[x_1, x_2, \dots]/(x_1, x_2^2, x_3^3, \dots)$$

where k is a field. Then A is not Noetherian since the ideal

$$\mathfrak{m} = (x_1, x_2, x_3, \dots)/(x_1, x_2^2, x_3^3, \dots)$$

is not finitely generated. However, this is the unique prime ideal of A (in fact maximal), so $\operatorname{Spec}(A)$ is a singleton and hence Noetherian.

Corollary 1.4.9. *Let A be a ring. For a subset Y of $X = \operatorname{Spec}(A)$ to be irreducible, it is necessary and suficient that the ideal $I(Y)$ be prime.*

Proof. Writing $\mathfrak{p} = I(Y)$, we note that, for an element $f \in A$, the relation $f \in \mathfrak{p}$ is equivalent to $Y \subseteq V(f)$. Suppose that Y is irreducible and let f, g be elements of A such that $fg \in \mathfrak{p}$. Then

$$Y \subseteq V(fg) = V(f) \cup V(g).$$

as Y is irreducible and $V(f)$ and $V(g)$ are closed, $Y \subseteq V(f)$ or $Y \subseteq V(g)$, hence $f \in \mathfrak{p}$ or $g \in \mathfrak{p}$, which proves that $\mathfrak{p}$ is prime.

Suppose now that $\mathfrak{p}$ is prime; then $\bar{Y} = V(\mathfrak{p})$ by Proposition 1.4.5 and, as $\mathfrak{p}$ is prime, $\mathfrak{p} = I(\{\mathfrak{p}\})$, whence $\bar{Y} = V(I(\{\mathfrak{p}\})) = \overline{\{\mathfrak{p}\}}$. As a set consisting of a single point is irreducible, Y is irreducible. □

Corollary 1.4.10. *For a ring A to be such that $X = \operatorname{Spec}(A)$ is irreducible, it is necessary and sufiicient that the quotient of A by its nilradical $\mathfrak{n}$ be an integral domain.*

Proof. By Proposition 1.4.5, $I(X)$ is the radical of the ideal (0), that is $\mathfrak{n}$. □

Remark 1.4.11. Note that a ring A can be non-Noetherian even through $\operatorname{Spec}(A)$ is Noetherian. For example, the ring $A = k[x_1, x_2, \dots]/(x_1^2, x_2^2, \dots)$ has a unique maximal ideal $\mathfrak{m} = (x_1, x_2, \dots)$, whence $\operatorname{Spec}(A)$ is a singleton and is Noetherian. But $\mathfrak{m}$ is not finitely generated, so A is not Noetherian.

Corollary 1.4.12. *The space $X = \operatorname{Spec}(A)$ is a Kolmogoroff space.*

Proof. If $\mathfrak{p}$ and $\mathfrak{q}$ are two different points of X, we have, either $\mathfrak{p} \not\subseteq \mathfrak{q}$ or $\mathfrak{q} \not\subseteq \mathfrak{p}$, hence one of the points $\mathfrak{p}$, $\mathfrak{q}$ does not belong to the closure of the other. $\square$

Proposition 1.4.13. *Let A be a ring. The topology on $X = \operatorname{Spec}(A)$ has the following properties:*

(a) X is quasi-compact.

(b) X has a basis for the topology consisting of quasi-compact opens.

(c) The intersection of any two quasi-compact opens is quasi-compact.

Proof. It suffices to prove that any covering of $\operatorname{Spec}(A)$ by standard opens can be refined by a finite covering. Thus suppose that $\operatorname{Spec}(A) = \bigcup_i D(f_i)$ for a set of elements f_i of A. This means that $\bigcap_i V(f_i) = \varnothing$. According to Proposition 1.4.3 this means that $V(\{f_i\}_{i\in I}) = \varnothing$. Then the ideal generated by the f_i is the unit ideal of A. This means that we can write

$$1 = \sum_{i=1}^{n} a_i f_i.$$

It follows that $\operatorname{Spec}(A) = \bigcup_{i=1}^{n} D(f_i)$. Since $D(f) \cong \operatorname{Spec}(A_f)$, each standard open is quasi-compact, and so their intersection $D(f) \cap D(g) = D(fg)$ is also quasi-compact. $\square$

Proposition 1.4.14. *Let A be a ring and $X = \operatorname{Spec}(A)$.*

(a) The irreducible closed subsets of X are exactly the subsets $V(\mathfrak{p})$, with $\mathfrak{p} \subseteq A$ a prime.

(b) The irreducible components of X are exactly the subsets $V(\mathfrak{p})$, with $\mathfrak{p} \subseteq A$ a minimal prime.

Proof. By Proposition 1.4.5 $V(\mathfrak{p}) = \overline{\{\mathfrak{p}\}}$ is the closure of a singleton and hence irreducible. conversely, let $V(\mathfrak{a}) \subseteq X$ with $\mathfrak{a}$ a radical ideal. If $\mathfrak{a}$ is not prime, then choose $a, b \in A$, $a, b \notin \mathfrak{a}$ with $ab \in \mathfrak{a}$. In this case $V(\mathfrak{a}, a) \cup V(\mathfrak{a}, b) = V(\mathfrak{a})$, but neither $V(\mathfrak{a}, b) = V(\mathfrak{a})$ nor $V(\mathfrak{a}, a) = V(\mathfrak{a})$ since $\sqrt{(\mathfrak{a}, a)} \neq \sqrt{\mathfrak{a}} = \mathfrak{a}$, $\sqrt{(\mathfrak{a}, b)} \neq \mathfrak{a}$. Hence $V(\mathfrak{a})$ is not irreducible. $\square$

Corollary 1.4.15. *Let A be a ring, $X = \operatorname{Spec}(A)$, and $\mathfrak{p} \in X$.*

(a) The set of irreducible closed subsets of X containing $\mathfrak{p}$ is in one-to-one correspondence with primes $\mathfrak{q} \supseteq \mathfrak{p}$.

(b) The set of irreducible component of X containing $\mathfrak{p}$ is in one-to-one correspondence with minimal primes $\mathfrak{q} \supseteq \mathfrak{p}$.

Corollary 1.4.16. *The set of minimal prime ideals of a Noetherian ring A is finite.*

Proof. The spectrum of a Noetherian is a Noetherian topological space, hence has finitely many irreducible components. $\square$

Proposition 1.4.17. *Let A be a ring and $\mathfrak{p}$ be a minimal prime of A. Let $V \subseteq \operatorname{Spec}(A)$ be a quasi-compact open not containing the point $\mathfrak{p}$. Then there exists an $f \in A$, $f \notin \mathfrak{p}$ such that $D(f) \cap V = \varnothing$.*

Proof. Since V is quasi-compact we may write $V = \bigcup_{i=1}^{n} D(g_i)$. Since $\mathfrak{p} \notin V$ we have $g_i \in \mathfrak{p}$ for all i. Since $\mathfrak{p}$ is minimal, we see $\mathfrak{n}(A_\mathfrak{p}) = \mathfrak{p}A_\mathfrak{p}$, so each g_i is nilpotent in $A_\mathfrak{p}$. Hence we can find an $f \in A$, $f \notin \mathfrak{p}$ such that $fg_i^{n_i} = 0$ for some $n_i > 0$. Then $D(f) \cap D(g_i) = D(fg_i^{n_i}) = \varnothing$, so this $D(f)$ works. $\square$

Proposition 1.4.18. *Let A be a ring and $X = \operatorname{Spec}(A)$. Let U be an open subset of X, then U is dense in X if and only if U meets every irreducible components of X. In particular, for an element $f \in A$ to be such that $D(f)$ is dense in X, it is necessary and sufficient that f does not belong to any minimal prime ideal of A. Moreover, every open dense subset of X contains an open subset of the form $D(f)$ with f not a zero divisor.*

Proof. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ be the minimal prime ideals of A and set $X_i = V(\mathfrak{p}_i)$. If $\mathfrak{p}_i \in U$ for all i, then $U \cap X_i \neq \varnothing$ so U is dense (in that for this direction we do not need to assume that X has finitely many irreducible components). Conversely, assume that U is dense and let $Z = \bigcup_{X_i \cap U \neq \varnothing} X_i$. Then Z is closed and contains U. But then $Z = X$ since U is dense, so there is no X_i such that $X_i \cap U = \varnothing$. In particular, since for an element $f \in A$, $D(f) \cap V(\mathfrak{p}_i) \neq \varnothing$ if and only if $f \notin \mathfrak{p}_i$, we see (f) is dense if and only if $f \notin \mathfrak{p}_i$ for all i.

On the other hand, if U is an open dense subset of X, the complement of U is of the form $V(\mathfrak{a})$, where $\mathfrak{a}$ is an ideal which is not contained in any of the $\mathfrak{p}_i$; it is therefore not contained in their union, and there then exists $f \in \mathfrak{a}$ contained in S; hence $D(f) \subseteq U$. ☐

Proposition 1.4.19. *Let A be a ring. Let $X = \operatorname{Spec}(A)$ as a topological space. The following are equivalent*

(a) X is Hausdorff.

(b) X is totally disconnected.

(c) Every prime ideal of A is maximal.

(d) Every quasi-compact open of X is closed.

(e) Every standard open $D(f) \subseteq X$ is closed.

Proof. It is clear that (d) and (e) are equivalent as every quasi-compact open is a finite union of standard opens. The implication (c)⇒(d) follows from Proposition 1.4.17. Assume (d) holds. Let $\mathfrak{p}, \mathfrak{p}'$ be distinct primes of A. Choose an $f \in \mathfrak{p}'$, $f \notin \mathfrak{p}$. Then $\mathfrak{p}' \notin D(f)$ and $\mathfrak{p} \in D(f)$. By (d) the open $D(f)$ is also closed. Hence $\mathfrak{p}$ and $\mathfrak{p}'$ are in disjoint open neighbourhoods whose union is X. Thus X is Hausdorff and totally disconnected. Thus (d) implies (a) and (b). Finally, if (b) holds, then the closure of $\{\mathfrak{p}\}$ must be itself, so we see (c) holds. If X is Hausdorff then every point is closed, so (a) implies (c). These together finish the proof. ☐

1.4.2 Functoriality of the spactrum

We will now show that $A \mapsto \operatorname{Spec}(A)$ defines a contravariant functor from the category of rings to the category of topological spaces. Let $\rho : A \to B$ be a homomorphism of rings. If $\mathfrak{q}$ is a prime ideal of B, $\rho^{-1}(\mathfrak{q})$ is a prime ideal of A. Therefore we obtain a map

$$^a\rho = \operatorname{Spec}(\rho) : \operatorname{Spec}(B) \to \operatorname{Spec}(A), \quad \mathfrak{q} \mapsto \rho^{-1}(\mathfrak{q}).$$

Proposition 1.4.20. *Let $\rho : A \to B$ be a ring homomorphism.*

(a) For every subset $S \subseteq A$, we have $(^a\rho)^{-1}(V(S)) = V(\rho(S))$, so $^a\rho$ is continuous.

(b) If $\mathfrak{b}$ is an ideal of B, then $\overline{^a\rho(V(\mathfrak{b}))} = V(\rho^{-1}(\mathfrak{b}))$.

Proof. A prime ideal $\mathfrak{q}$ of B contains $\rho(S)$ if and only if $\rho^{-1}(\mathfrak{q})$ contains S, so (a) holds. For part (b), we can rewrite the left hand side as $VI(^a\rho(V(\mathfrak{b})))$. But

$$I(^a\rho(V(\mathfrak{b}))) = \bigcap_{\mathfrak{q} \in V(\mathfrak{b})} \rho^{-1}(\mathfrak{q}) = \rho^{-1}\Big(\bigcap_{\mathfrak{q} \in V(\mathfrak{b})} \mathfrak{q}\Big) = \rho^{-1}(\sqrt{\mathfrak{b}}) = \sqrt{\rho^{-1}(\mathfrak{b})}.$$

and the claim follows by applying V. ☐

Corollary 1.4.21. *The map ${}^a\rho$ has dense image if and only if every element of* $\ker(\rho)$ *is nilpotent.*

Proof. By letting $\mathfrak{b} = 0$ in Proposition 1.4.20 we get $\overline{{}^a\rho(Y)} = V(\ker\rho)$, so the claim follows from the fact that $V(\mathfrak{a}) = \operatorname{Spec}(A)$ if and only if $\mathfrak{a} \subseteq \mathfrak{n}(A)$. $\square$

Proposition 1.4.22. *Suppose that for any $g \in B$ there exists $f \in A$ such that $g = u\rho(f)$, where u is invertible in B (this is the case if ρ is surjective). Then ${}^a\rho$ is a homeomorphism from X onto ${}^a\rho(X)$. In particular,* $\operatorname{Spec}(A)$ *and* $\operatorname{Spec}(A/\mathfrak{n}(A))$ *are naturally homeomorphic.*

Proof. We show that for any subset $F \subseteq B$, there exist $E \subseteq A$ such that $V(F) = V(\rho(E))$; by virtue of the T_0-axiom and the formula of Proposition 1.4.20(a), this will first imply that ${}^a\rho(X)$ is injective, then, still by virtue of Proposition 1.4.20(a), that ${}^a\rho(X)$ is a homeomorphism. Now, it suffices for each $g \in F$ to take $f \in A$ such that $u\rho(f) = g$ with u invertible in B; the set E of these elements then satisfies the requirement. $\square$

Corollary 1.4.23. *Let $\rho : A \to B$ be a homomorphism of rings. Let $\mathfrak{a}$ be an ideal of A and let $\mathfrak{b}$ be its extension in B. Let $\bar{\rho} : A/\mathfrak{a} \to B/\mathfrak{b}$ be the induced homomorphism. If* $\operatorname{Spec}(A/\mathfrak{a})$ *is identified with its canonical image $V(\mathfrak{a})$ in* $\operatorname{Spec}(A)$, *and* $\operatorname{Spec}(B/\mathfrak{b})$ *with its image $V(\mathfrak{b})$ in* $\operatorname{Spec}(B)$, *then ${}^a\bar{\rho}$ is the restriction of ${}^a\rho$ to $V(\mathfrak{b})$.*

Proof. Let $\mathfrak{q} \in V(\mathfrak{b})$, then $\mathfrak{a} \subseteq \mathfrak{a}^{ec} = \mathfrak{b}^c \subseteq \mathfrak{q}^c$, so ${}^a\rho(V(\mathfrak{b})) \subseteq V(\mathfrak{a})$. Let $\pi_A : A \to A/\mathfrak{a}$, $\pi_B : B \to B/\mathfrak{b}$ be the quotient maps. Then

$$\bar{\rho} \circ \pi_A(a) = \bar{\rho}(a + \mathfrak{a}) = \rho(a) + \mathfrak{b} = \pi_B \circ \rho(a).$$

Hence $\bar{\rho} \circ \pi_A = \pi_B \circ \rho$, which implies ${}^a(\pi_A) \circ {}^a\bar{\rho} = {}^a\rho \circ {}^a(\pi_B)$, so we have the followsing commutative diagram:

$$\begin{tikzcd}
\operatorname{Spec}(B/\mathfrak{b}) \arrow[r, "{}^a\bar{\rho}"] \arrow[d, "{}^a(\pi_B)"'] & \operatorname{Spec}(A/\mathfrak{a}) \arrow[d, "{}^a(\pi_A)"] \\
V(\mathfrak{b}) \arrow[r, "{}^a\rho"] & V(\mathfrak{a})
\end{tikzcd}$$

which is exactly the claim. $\square$

Proposition 1.4.24. *Let S be a multiplicative subset of A and let $i_A^S : A \to S^{-1}A$ be the canonical homomorphism. Then ${}^a(i_A^S)$ is a homeomorphism of* $\operatorname{Spec}(S^{-1}A)$ *onto the subspace of* $\operatorname{Spec}(A)$ *consisting of prime ideals $\mathfrak{p}$ with $S \cap \mathfrak{p} = \varnothing$.*

Proof. We already know that ${}^a(i_A^S)$ is injective by Proposition 1.2.37, and similar to Proposition 1.4.22, we see ${}^a(i_A^S)$ is closed, so the claim follows. $\square$

Due to Proposition 1.4.24, we may therefore identify $\operatorname{Spec}(S^{-1}A)$ with a subspace of $\operatorname{Spec}(A)$ and write $S^{-1}X$ for $\operatorname{Spec}(S^{-1}A)$, if $X = \operatorname{Spec}(A)$.

Corollary 1.4.25. *Let A be a ring and $f \in A$. Then the canonical map $i_A^{S_f} : A \to A_f$ induces a homeomorphism* $\operatorname{Spec}(A_f) \cong D(f) \subseteq \operatorname{Spec}(A)$, *with the inverse map given by $\mathfrak{p} \to \mathfrak{p}A_f$.*

Corollary 1.4.26. *Let A be a ring and $\mathfrak{p}$ a prime ideal of A. Then the canonical image of* $\operatorname{Spec}(A_\mathfrak{p})$ *in* $\operatorname{Spec}(A)$ *is equal to the intersection of all the open neighborhoods of $\mathfrak{p}$ in* $\operatorname{Spec}(A)$.

Proof. We have that $\operatorname{Spec}(A_f) = \{\mathfrak{q} \in \operatorname{Spec}(A) : f \notin \mathfrak{q}\}$ and $\operatorname{Spec}(A_\mathfrak{p}) = \{\mathfrak{q} \in \operatorname{Spec}(A) : \mathfrak{q} \subseteq \mathfrak{p}\}$, therefore

$$\operatorname{Spec}(A_\mathfrak{p}) = \bigcap_{f \in A - \mathfrak{p}} \operatorname{Spec}(A_f) = \bigcap_{\mathfrak{p} \in D(f)} D(f).$$

That is, $\operatorname{Spec}(A_\mathfrak{p})$ is the intersection of all basic open neighborhoods of the point $\mathfrak{p}$ in $\operatorname{Spec}(A)$. Since every open set is a union of principal open sets, we get the claim. $\square$

Proposition 1.4.27. *Let $\rho : A \to B$ be a homomorphism and S a multiplicative subset of A. Then ${}^a(S^{-1}\rho) : \mathrm{Spec}(S^{-1}B) \to \mathrm{Spec}(S^{-1}A)$ is the restriction of ${}^a\rho$ on $\mathrm{Spec}(S^{-1}B)$ and*

$$\mathrm{Spec}(S^{-1}B) = ({}^a\rho)^{-1}(\mathrm{Spec}(S^{-1}A)).$$

Proof. Since $S^{-1}B = \rho(S)^{-1}B$, we have the following commutative diagram:

$$\begin{tikzcd} A \arrow[r, "\rho"] \arrow[d, "i_A^S"'] & B \arrow[d, "i_B^S"] \\ S^{-1}A \arrow[r, "S^{-1}\rho"] & S^{-1}B \end{tikzcd}$$

whence the first claim. Now, by definition, for $\mathfrak{q} \in \mathrm{Spec}(B)$ we see $\mathfrak{q} \cap \rho(S) = \varnothing$ if and only if $\mathfrak{q}^c \cap S = \varnothing$, so the last claim follows. $\square$

Corollary 1.4.28. *Let $\rho : A \to B$ be a homomorphism and $\mathfrak{p}$ be a prime ideal of A. Then the continuous map ${}^a\nu : \mathrm{Spec}(B \otimes_A \kappa(\mathfrak{p})) \to \mathrm{Spec}(B)$, induced by the homomorphism $\nu : B \to B \otimes_A \kappa(\mathfrak{p})$, induces a homeomorphism from $\mathrm{Spec}(B \otimes_A \kappa(\mathfrak{p}))$ onto the fiber of $\mathfrak{p}$ under the map ${}^a\rho$.*

Proof. The homomorphism ν is the composition of the quotient homomorphism $B \to \mathfrak{p}^e$ and of the canonical homomorphism $B/\mathfrak{p}^e$ in its ring of fractions $(B/\mathfrak{p}^e)_{\mathfrak{p}}$. According to Proposition 1.4.22 and Corollary 1.4.26, ν^* thus induces a homeomorphism of $\mathrm{Spec}(B \otimes_A \kappa(\mathfrak{p}))$ on the subspace of $\mathrm{Spec}(B)$ consists of the prime ideals $\mathfrak{P}$ of B which contain $\mathfrak{p}^e$ and are disjoint of $\rho(A - \mathfrak{p})$. That is to say, which are lying above $\mathfrak{p}$. $\square$

Example 1.4.29. Let A be an integral domain with just one non-zero prime ideal $\mathfrak{p}$. and let K be the field of fractions of A. Let $B = (A/\mathfrak{p}) \times K$. Define $\rho : A \to B$ by $\rho(x) = (\bar{x}, x)$, where x is the image of $\bar{x}$ in $A/\mathfrak{p}$. Then ${}^a\rho$ is bijective but not a homeomorphism.

In fact, the idals in $A/\mathfrak{p}$ are $(0), A/\mathfrak{p}$, that in K is $(0), K$. So by the claim below the ideals in $(A/\mathfrak{p}) \times K$ are

$$(0) \times (0), (A/\mathfrak{p}) \times (0), (0) \times K, (A/\mathfrak{p}) \times K.$$

Note that $(0) \times (0)$ is not prime: $(a, 0) \cdot (0, b) = 0$ but neither of them is zero. So

$$\mathrm{Spec}(B) = \{(A/\mathfrak{p}) \times (0), (0) \times K\}$$

Since $\mathrm{Spec}(A) = \{0, \mathfrak{p}\}$, ${}^a\rho$ is bijective.

Now $\mathrm{Spec}(B)$ is a two point Hausdorff space, since the prime ideals in B are maximal. But $\mathrm{Spec}(A)$ is non-Hausdorff because it contains 0 and

$$\overline{\{0\}} = \{\mathfrak{p} \subseteq A : \mathfrak{p} \supseteq (0)\} = \{0, \mathfrak{p}\}$$

Therefore ${}^a\rho$ cannot be a homeomorphism.

Example 1.4.30. Let A be a principal ideal domain. In this case, the maximal ideals are of the form (p) for a prime element p of A, and all prime ideals are maximal or the zero ideal. Therefore the closed points of $\mathrm{Spec}(A)$ correspond to equivalence classes of prime elements. Let $\eta \in \mathrm{Spec}(A)$ be the point corresponding to the zero ideal. Then the closure of $\{\eta\}$ is $\mathrm{Spec}(A)$.

As A is a principal ideal domain, every closed subset of $\mathrm{Spec}(A)$ is of the form $V(f)$ for some $f \in A$. Assume $f \neq 0$ and let $f = p_1^{e_1} \cdots p_r^{e_r}$ with pairwise non-equivalent prime elements p_i and integers e_i. Then $V(f)$ consists of those closed points which correspond to the prime divisors of f, that is, $V(f) = \{(p_1), \ldots, (p_r)\}$. Therefore the proper closed subsets are the finite sets consisting of closed points.

If A is a local principal ideal domain, but not a field (i.e., A is a discrete valuation ring), $\mathrm{Spec}(A)$ consists only of two points η and x, where $\mathfrak{p}_x$ is the maximal ideal and $\mathfrak{p} = \{0\}$. The only nontrivial open subset of $\mathrm{Spec}(A)$ is then $\{\eta\}$.

Example 1.4.31. Let $A = R[X]$, where R is a PID. We assume that R is not a field and let K be its field of fractions. Let $X = \operatorname{Spec}(R[X])$. Then the prime elements of $R[X]$ are either of the form p, where p is a prime element of R, or of the form f, where $f \in R[X]$ is a primitive polynomial which is irreducible in $K[X]$

If $p \in R$ is a prime element, then the closure $V(p)$ is homeomorphic to $\operatorname{Spec}((R/p)[X])$. Since $(R/p)[X]$ is a PID but not be a field, we see that (p) is not a maximal ideal, and the prime ideals in $V(p)$ different from (p) are the maximal ideals generated by p and f where $f \in R[X]$ is a polynomial such that its image in $(R/p)[X]$ is irreducible.

The situation is more complicated for prime ideals of the form (f), where f is a primitive irreducible polynomial. If the leading coefficient of f is a unit in R, it is possible to divide in $R[X]$ by f with unique remainder, and therefore $R[X]/(f)$ is finitely generated as R-module. This implies that (f) is not a maximal ideal by Proposition 4.1.64, as otherwise R would be a field.

For other primitive irreducible polynomials f, (f) might be a maximal ideal, namely if R contains only finitely many prime elements (up to equivalence): If $0 \neq a \in R$ is an element which is divisible by all prime elements of R we have, with $f = ax - 1$,

$$R[X]/(f) \cong R[a^{-1}] = K$$

which shows that (f) is a maximal ideal.

1.4.3 The support of a module

Let A be a ring and M an A-module. The set of prime ideals $\mathfrak{p}$ of A such that $M_{\mathfrak{p}} \neq 0$ is called the **support** of M and is denoted by $\operatorname{supp}(M)$. As every maximal ideal of A is prime, it follows immediately from Proposition 1.3.18 that for A-module M to be equal to 0, it is necessary and suffieient that $\operatorname{supp}(M) = \varnothing$.

Example 1.4.32. Let $\mathfrak{a}$ be an ideal of A, then we have $\operatorname{supp}(A/\mathfrak{a}) = V(\mathfrak{a})$. In fact, if $\mathfrak{p}$ is a prime of A such that $\mathfrak{a} \not\subseteq \mathfrak{p}$, then $(A/\mathfrak{a})_{\mathfrak{p}} = 0$; if on the other hand $\mathfrak{a} \subseteq \mathfrak{p}$, then $\mathfrak{a}A_{\mathfrak{p}}$ is contained in the maximal ideal $\mathfrak{p}A_{\mathfrak{p}}$ of $A_{\mathfrak{p}}$ and $(A/\mathfrak{a})_{\mathfrak{p}}$ is isomorphic to $A_{\mathfrak{p}}/\mathfrak{a}A_{\mathfrak{p}}$ and hence is non-zero; whence our assertion.

Proposition 1.4.33. *Let A be a ring and M an A-module.*

(a) If N is a submodule of M, then

$$\operatorname{supp}(M) = \operatorname{supp}(N) \cup \operatorname{supp}(M/N).$$

(b) If M is the sum of a family $(M_i)_{i\in I}$ of submodules, then

$$\operatorname{supp}(M) = \bigcup_{I\in I} \operatorname{supp}(M_i).$$

Proof. From the exact sequence $0 \to N \to M \to N/N \to 0$, we derive, for every prime ideal $\mathfrak{p}$ of A, the exact sequence

$$0 \longrightarrow N_{\mathfrak{p}} \longrightarrow M_{\mathfrak{p}} \longrightarrow (M/N)_{\mathfrak{p}} \longrightarrow 0$$

For $M_{\mathfrak{p}}$ to be reduced to 0, it is necessary and sufficient that $N_{\mathfrak{p}}$ and $(M/N)_{\mathfrak{p}}$ be so. In other words, the relation $\mathfrak{p} \notin \operatorname{supp}(M)$ is equivalent tO $\mathfrak{p} \notin \operatorname{supp}(N)$ and $\mathfrak{p} \notin \operatorname{supp}(M/N)$, which proves (a). Part (b) can be proved similarly. $\square$

Corollary 1.4.34. *Let A be a ring, M an A-module, $(m_i)_{i\in I}$ a system of generators of M and $\mathfrak{a}_i$ the annihilator of m_i. Then $\operatorname{supp}(M) = \bigcup_{i\in I} V(\mathfrak{a}_i)$.*

Proof. We have $\operatorname{supp}(M) = \bigcup_{i \in I} \operatorname{supp}(Am_i)$ by Proposition 1.4.33. On the other hand, Am_i is isomorphic to the A-module $A/\mathfrak{a}_i$ and we have seen that $\operatorname{supp}(A/\mathfrak{a}_i) = V(\mathfrak{a}_i)$, hence the claim. □

Proposition 1.4.35. *Let A be a ring, M an A-module and $\mathfrak{a}$ its annihilator. If M is finitely generated, then $\operatorname{supp}(M) = V(\mathfrak{a})$ and $\operatorname{supp}(M)$ is therefore closed in $\operatorname{Spec}(A)$.*

Proof. Let $m_1, \dots, m_n$ be a system of generators of M and let $\mathfrak{a}_i$ be the annihilator of m_i for each i. Then $\mathfrak{a} = \bigcap_{i=1}^n \mathfrak{a}_i$, hence $V(\mathfrak{a}) = \bigcup_{i=1}^n V(\mathfrak{a}_i)$ and the proposition follows from the Corollary 1.4.34. □

Corollary 1.4.36. *Let A be a ring, M a finitely generated A-module and a an element of A. For a to belong to every prime ideal of the support of M, it is necessary and sufficient that the homothety of M with ratio a be nilpotent.*

Proof. It follows from Proposition 1.4.35 that the intersection of the prime ideals belonging to $\operatorname{supp}(M)$ is the radical of the annihilator $\mathfrak{a}$ of M. To say that a belongs to this radical is equ1valent to say that there exist a power a^n such that $a^n M = 0$. □

Corollary 1.4.37. *Let A be a Noetherian ring, M a finitely generated A-module and $\mathfrak{a}$ an ideal of A. For $\operatorname{supp}(M) \subseteq V(\mathfrak{a})$, it is necessary and sufficient that there exist an integer n such that $\mathfrak{a}^n M = 0$.*

Proof. If $\mathfrak{b}$ is the annihilator of M, the relation $\operatorname{supp}(M) \subseteq V(\mathfrak{a})$ is equivalent to $V(\mathfrak{b}) \subseteq V(\mathfrak{a})$ by Proposition 1.4.35 and hence to $\mathfrak{a} \subseteq \sqrt{\mathfrak{b}}$. Since A is Noetherian, this condition is also equivalent to the existence of an integer $n > 0$ such that $\mathfrak{a}^n \subseteq \mathfrak{b}$. □

Lemma 1.4.38. *Let A be a local ring and M and N two finitely generated A-modules. If $M \neq 0$ and $N \neq 0$, then $M \otimes N \neq 0$.*

Proof. Let κ be the residue field of A. By virtue of Proposition 1.3.6, $\kappa \otimes_A M \neq 0$, and $\kappa \otimes_A N \neq 0$, then we deduce that

$$(\kappa \otimes_A M) \otimes_\kappa (\kappa \otimes_A N) \neq 0$$

But, since the tensor product is associative, this tensor product is isomorphic to

$$M \otimes_A (\kappa \otimes_\kappa \kappa) \otimes_A N = M \otimes_A \kappa \otimes_A N$$

and therefore to $\kappa \otimes_A (M \otimes_A N)$, whence the lemma. □

Proposition 1.4.39. *Let M, N be two finitely generated modules over a ring A, then*

$$\operatorname{supp}(M \otimes_A N) = \operatorname{supp}(M) \cap \operatorname{supp}(N).$$

Proof. We need to prove that, if $\mathfrak{p}$ is a prime ideal of A, the relations "$(M \otimes_A N)_\mathfrak{p} \neq 0$" and "$M_\mathfrak{p} = 0$ and $N_\mathfrak{p} \neq 0$" are equivalent. As the $A_\mathfrak{p}$-modules $M_\mathfrak{p} \otimes_{A_\mathfrak{p}} N_\mathfrak{p}$ and $(M \otimes_A N)_\mathfrak{p}$ are isomorphic, our assertion follows from Lemma 1.4.38. □

Corollary 1.4.40. *Let M be a finitely generated A-module and $\mathfrak{n}$ its annihilator. For every ideal $\mathfrak{a}$ of A,* $\operatorname{supp}(M/\mathfrak{a}M) = V(\mathfrak{a}) \cap V(\mathfrak{n}) = V(\mathfrak{a} + \mathfrak{n})$.

Proof. We have $M/\mathfrak{a}M = M \otimes_A (A/\mathfrak{a})$ and $A/\mathfrak{a}$ is finitely generated, so the claim follows from Proposition 1.4.39. □

Lemma 1.4.41. *Let A, B be two local rings, $\rho : A \to B$ a local homomorphism and M a finitely generated A-module. If $M \neq 0$, then $\rho_*(M) \neq 0$.*

Proof. Let $\mathfrak{m}$ be the maximal ideal of A and $\kappa = A/\mathfrak{m}$ the residue field. The hypothesis implies that $B \otimes_A \kappa = B/\mathfrak{m}B \neq 0$ and $M \otimes_A \kappa = M/\mathfrak{m}M \neq 0$ by Proposition 1.3.6. Since the tensor product is associative, we have

$$(M \otimes_A B) \otimes_A \kappa = M \otimes_A (B \otimes_A \kappa) = M \otimes_A (\kappa \otimes_\kappa (B \otimes_A \kappa)) = (M \otimes_A \kappa) \otimes_\kappa (B \otimes_A \kappa)$$

therefore $M \otimes_A B \neq 0$. □

Proposition 1.4.42. *Let A, B be two rings, $\phi : A \to B$ a homomorphism and $\phi^* : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ the induced map. Then for every A-module M,*

$$\mathrm{supp}(M \otimes_A B) \subseteq \phi^{*-1}(\mathrm{supp}(M)).$$

If also M is finitely generated, then the equality holds.

Proof. Let $\mathfrak{q}$ be a prime ideal of B and $\mathfrak{p} = \phi^{-1}(\mathfrak{q})$. Suppose that $\mathfrak{q}$ belongs to $\mathrm{supp}(M \otimes_A B)$, then

$$(M \otimes_A B) \otimes_B B_\mathfrak{q} = M \otimes_A B_\mathfrak{q} = (M \otimes_A A_\mathfrak{p}) \otimes_A B_\mathfrak{q}.$$

since the homomorphism $A \to B \to B_\mathfrak{q}$ factors into $A \to A_\mathfrak{p} \to B_\mathfrak{q}$. The hypothesis $M \otimes_A B \otimes_B B_\mathfrak{q} \neq 0$ implies therefore $M \otimes_A A_\mathfrak{p} \neq 0$, whence the first assertion. As the homomorphism $\phi_\mathfrak{q} : A_\mathfrak{p} \to B_\mathfrak{q}$ is local, the second assertion follows from Lemma 1.4.41. □

Proposition 1.4.43. *Let A be a ring and M a finitely generated A-module. For every prime ideal $\mathfrak{p} \in \mathrm{supp}(M)$, there exists a non-zero A-homomorphism $\eta : M \to A/\mathfrak{p}$.*

Proof. Let $\mathfrak{p} \in \mathrm{supp}(M)$. As M is finitely generated and $M \neq 0$,

$$M_\mathfrak{p}/\mathfrak{p}M_\mathfrak{p} = M_\mathfrak{p} \otimes_A (A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}) \neq 0$$

Let $\kappa = A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$ be the field of fractions of the integral domain $A/\mathfrak{p}$. Since $M_\mathfrak{p}/\mathfrak{p}M_\mathfrak{p}$ is a vector space over κ which is not reduced to 0, there exists a non-zero linear form $u : M_\mathfrak{p}/\mathfrak{p}M_\mathfrak{p} \to \kappa$. If $x_1, \ldots, x_n$ is a system of generators of M, and $\bar{x}_i$ is the image of x_i in the $(A/\mathfrak{p})$-module $M_\mathfrak{p}/\mathfrak{p}M_\mathfrak{p}$. Let $\alpha \in A/\mathfrak{p}$ be such that $\alpha u(\bar{x}_i) \in A/\mathfrak{p}$ for $1 \leq i \leq n$, then $v = \alpha u$ is a non-zero $(A/\mathfrak{p})$-linear map from $M_\mathfrak{p}/\mathfrak{p}M_\mathfrak{p}$ to $A/\mathfrak{p}$. The composition

$$M \longrightarrow M_\mathfrak{p} \longrightarrow M_\mathfrak{p}/\mathfrak{p}M_\mathfrak{p} \xrightarrow{v} A/\mathfrak{p}$$

is therefore the required homomorphism. □

1.4.4 Clopen subsets of the spectrum

Lemma 1.4.44. *Let A be a ring. Let $e \in A$ be an idempotent. In this case*

$$\mathrm{Spec}(A) = D(e) \amalg D(1-e).$$

Proof. Note that an idempotent e of a domain is either 1 or 0. Hence we see that

$$D(e) = \{\mathfrak{p} \in \mathrm{Spec}(A) : e \neq 0 \text{ in } A/\mathfrak{p}\} = \{\mathfrak{p} \in \mathrm{Spec}(A) : e = 1 \text{ in } A/\mathfrak{p}\}.$$

Similarly we have

$$D(1-e) = \{\mathfrak{p} \in \mathrm{Spec}(A) : e \neq 1 \text{ in } A/\mathfrak{p}\} = \{\mathfrak{p} \in \mathrm{Spec}(A) : e = 0 \text{ in } A/\mathfrak{p}\}.$$

Since the image of e is either 1 or 0 we deduce that $D(e)$ and $D(1-e)$ cover all of $\mathrm{Spec}(A)$. □

Lemma 1.4.45. *Let A_1 and A_2 be rings. Let $A = A_1 \times A_2$. The maps $A \to A_1, (x, y) \mapsto x$ and $A \to A_2, (x, y) \mapsto y$ induce continuous maps $\operatorname{Spec}(A_1) \to \operatorname{Spec}(A)$ and $\operatorname{Spec}(A_2) \to \operatorname{Spec}(A)$. The induced map*

$$\operatorname{Spec}(A_1) \amalg \operatorname{Spec}(A_2) \to \operatorname{Spec}(A)$$

is a homeomorphism. In other words, the spectrum of $A = A_1 \times A_2$ is the disjoint union of the spectrum of A_1 and the spectrum of A_2.

Proof. Write $1 = e_1 + e_2$ with $e_1 = (1, 0)$ and $e_2 = (0, 1)$. Note that e_1 and $e_2 = 1 - e_1$ are idempotents. Thus $\operatorname{Spec}(A) = D(e_1) \amalg D(e_2)$ by the previous lemma. Now consider the localization A_{e_1}: $\{0\} \times A_2$ is mapped to zero in it, so $A_1 = A_{e_1}$. Similarly we have $A_{e_2} = A_2$. Thus by Corollary 1.4.25 $D(e_1) = \operatorname{Spec}(A_1), D(e_2) = \operatorname{Spec}(A_2)$. □

We already know that an idempotent element gives an open and closed subset of $\operatorname{Spec}(A)$. We now prove a converse.

Proposition 1.4.46. *Let A be a ring. For each $U \subseteq \operatorname{Spec}(A)$ which is open and closed there exists a unique idempotent $e \in A$ such that $U = D(e)$. This induces a one-to-one correspondence between open and closed subsets $U \subseteq \operatorname{Spec}(A)$ and idempotents $e \in A$.*

Proof. Let $U \subseteq \operatorname{Spec}(A)$ be open and closed, and V be its complement. We can write U and V as unions of standard opens such that

$$U = \bigcup_{i=1}^{n} D(f_i) \quad \text{and} \quad W = \bigcup_{j=1}^{m} D(g_j).$$

Since $\operatorname{Spec}(A) = U \amalg V$, we observe that the collection $\{f_i, g_j\}$ must generate the unit ideal in A by Proposition 1.4.4. So the following sequence is exact

$$0 \longrightarrow A \xrightarrow{\alpha} \bigoplus_{i=1}^{n} A_{f_i} \oplus \bigoplus_{j=1}^{m} A_{g_j} \xrightarrow{\beta} \bigoplus_{i_1,i_2} A_{f_{i_1} f_{i_2}} \oplus \bigoplus_{i,j} A_{f_i g_j} \oplus \bigoplus_{j_1,j_2} A_{g_{j_1} g_{j_2}}$$

However, notice that for any pair i, j, $D(f_i g_j) = D(f_i) \cap D(g_j) = \varnothing$ since $D(f_i) \subseteq U$ and $D(g_j) \subseteq V$. Therefore by Corollary 1.4.25 we see $\operatorname{Spec}(A_{f_i g_j}) = D(f_i g_j) = \varnothing$, which implies $A_{f_i g_j}$ is the zero ring for each pair i, j. Consider the element $(1, \ldots, 1, 0, \ldots, 0) \in \bigoplus_{i=1}^{n} A_{f_i} \oplus \bigoplus_{j=1}^{m} A_{g_j}$ whose coordinates are 1 in each A_{f_i} and 0 in each A_{g_j}. It is sent to 0 under the map

$$\beta : \bigoplus_{i=1}^{n} A_{f_i} \oplus \bigoplus_{j=1}^{m} A_{g_j} \to \bigoplus_{i_1,i_2} A_{f_{i_1} f_{i_2}} \oplus \bigoplus_{j_1.j_2} A_{g_{j_1} g_{j_2}}$$

so by the exactness of the sequence, there must be some element of A whose image under α is $(1, \ldots, 1, 0, \ldots, 0)$, which we denote by e. We see that $\alpha(e^2) = \alpha^2(e) = (1, \ldots, 1, 0, \ldots, 0) = \alpha(e)$. Since α is injective, $e = e^2$ in A and so e is an idempotent of A. We claim that $U = D(e)$. Notice that for any j, the map $A \mapsto A_{g_j}$ maps e to 0. Therefore there must be some positive integer k_j such that $g_j^{k_j} e = 0$ in A. Multiplying by e as necessary, we see that $(g_j e)^{k_j} = 0$, so $g_j e$ is nilpotent in A. By Proposition 1.4.4 $D(e) \cap D(g_j) = D(e g_j) = \varnothing$. So since $V = \bigcup_{j=1}^{m} D(g_j)$, we get $D(e) \cap V = \varnothing$; and then $D(e) \subseteq U$. Furthermore, for any i, the map $A \mapsto A_{f_i}$ maps e to 1, so there must be some l_i such that $f^{l_i}(e - 1) = 0$ in A. Hence $f_i^{l_i} e = f_i^{l_i}$. Suppose $\mathfrak{p} \in \operatorname{Spec}(A)$ contains e, then $\mathfrak{p}$ contains $f_i^{l_i} e = f_i^{l_i}$, and since $\mathfrak{p}$ is prime, $f_i \in \mathfrak{p}$. So $V(e) \subseteq V(f_i)$, implying that $D(f_i) \subseteq D(e)$. Therefore $U = \bigcup_{i=1}^{n} D(f_i) \subseteq D(e)$, and $U = D(e)$. Therefore any open and closed subset of $\operatorname{Spec}(A)$ is the standard open of an idempotent as desired. □

Corollary 1.4.47. *Let A be a nonzero ring. Then $\operatorname{Spec}(A)$ is connected if and only if A has no nontrivial idempotents.*

Proposition 1.4.48. *Let A be a ring. Let $\mathfrak{a}$ be a finitely generated ideal. Assume that $\mathfrak{a} = \mathfrak{a}^2$. Then $V(\mathfrak{a})$ is open and closed in* $\operatorname{Spec}(A)$*, and $A/\mathfrak{a} \cong A_e$ for some idempotent $e \in A$.*

Proof. By Nakayama's Lemma there exists an element $f = 1 + a$, $a \in \mathfrak{a}$ in A such that $f\mathfrak{a} = 0$. It follows that $V(\mathfrak{a}) = D(f)$ by a simple argument. Also, $0 = fa = a + a^2$, and hence

$$f^2 = (1+a)^2 = 1 + 2a + a^2 = 1 + a = f,$$

so f is an idempotent. Consider the canonical map $A \to A_f$. It is surjective since $x/f^n = x/f = xf/f^2 = x/1$. Any element of $\mathfrak{a}$ is in the kernel since $f\mathfrak{a} = 0$, and $x/1 = 0$ in A_f implies $f^n x = fx = (1+a)x = 0$, which means $x \in \mathfrak{a}$. □

1.4.5 Glueing properties of localization

In this part we show that given an open covering $\operatorname{Spec}(A) = \bigcup_{i=1}^n D(f_i)$ by standard opens, and given an element $h_i \in A_{f_i}$ for each i such that $h_i = h_j$ as elements of $A_{f_i f_j}$, then there exists a unique $h \in A$ such that the image of h in A_{f_i} is h_i. This result can be viewed as a sheaf property for $\operatorname{Spec}(A)$, and will play an important rule when we consider schemes.

Proposition 1.4.49. *Let A be a ring, and let $f_1, \dots, f_n \in A$ generate the unit ideal of A. Then the following sequence is exact:*

$$0 \longrightarrow A \xrightarrow{\alpha} \bigoplus_i A_{f_i} \xrightarrow{\beta} \bigoplus_{ij} A_{f_i f_j} \tag{1.4.1}$$

where the maps $\alpha : A \to \bigoplus_i A_{f_i}$ and $\beta : \bigoplus_i A_{f_i} \to \bigoplus_{i,j} A_{f_i f_j}$ are defined as

$$\alpha(x) = (x/1, \dots, x/1), \quad \beta(x_1/f_1^{r_1}, \dots, x_n/f_n^{r_n}) = x_i/f_i^{r_i} - x_j/f_j^{r_j}.$$

Proof. First let $x \in A$ such that $\alpha(x) = 0$. This means $x/1 = 0$ in A_{f_i} and so there exists n_i for each i such that

$$f_i^{n_i} x = 0.$$

Since $f_1, \dots, f_n$ generate A, we can find $a_i \in A$ such that $1 = \sum_{i=1}^n a_i f_i$. Then for all $N \geq \sum_i n_i$, we have

$$1^N = \Big(\sum_{i=1}^n a_i f_i\Big) = \sum \binom{N}{u_1, \dots, u_n} a_i^{u_1} \cdots a_n^{u_n} f_1^{u_1} \cdots f_n^{u_n}.$$

where each term has a factor of at least $f_i^{n_i}$ for some i. Therefore,

$$x = 1^N \cdot x = \sum \binom{N}{u_1, \dots, u_n} a_i^{u_1} \cdots a_n^{u_n} f_1^{u_1} \cdots f_n^{u_n} x = 0.$$

Thus, if $\alpha(x) = 0$, then $x = 0$ and so α is injective.

Now we check that the image of α equals the kernel of β. First, note that for $x \in A$ we have $\beta(\alpha(x)) = 0$, therefore the image of α is in the kernel of β. Now assume we have $x_1, \dots, x_n \in A$ such that

$$\beta\Big(\frac{x_1}{f_1^{r_1}}, \dots, \frac{x_n}{f_n^{r_n}}\Big) = 0.$$

Then, for all pairs i, j, there exists an n^{ij} such that

$$f_i^{n_{ij}} f_j^{n_{ij}} (x_i f_j^{r_j} - x_j f_i^{r_i}) = 0.$$

Choosing N so $N \geq n_{ij}$ for all i, j, we see that

$$f_i^N f_j^N (x_i f_j^{r_j} - x_j f_i^{r_i}) = 0.$$

Define elements $\tilde{x}_i$ and $\tilde{f}_i$ of A as follows:

$$\tilde{f}_i = f_i^{N+r_i}, \quad \tilde{x}_i = f_i^N x_i$$

so that $x_i/f_i^{r_i} = \tilde{x}_i/\tilde{f}_i$. Then we can use this to rewrite the above equation to get the following equality, for all i, j,

$$\tilde{f}_j \tilde{x}_i = \tilde{f}_i \tilde{x}_j.$$

Since $f_1, \ldots, f_n$ generate A, we clearly have that $\tilde{f}_1, \ldots, \tilde{f}_n$ also generate A. Therefore, there exist $a_1, \ldots, a_n$ in A so that

$$1 = \sum_{i=1}^{n} a_i \tilde{f}_i.$$

Therefore, we finally conclude that for all i,

$$\frac{x_i}{f_i^{r_i}} = \frac{\tilde{x}_i}{\tilde{f}_i} = \sum_{j=1}^{n} \frac{a_j \tilde{f}_j \tilde{x}_i}{\tilde{f}_i} = \sum_{j=1}^{n} \frac{a_j \tilde{f}_i \tilde{x}_j}{\tilde{f}_i} = \frac{\sum_{j=1}^{n} a_j \tilde{x}_j}{1}.$$

which means

$$\alpha\Big(\sum_{j=1}^{n} a_j \tilde{x}_j\Big) = \Big(\frac{x_1}{f_1^{r_1}}, \ldots, \frac{x_n}{f_n^{r_n}}\Big).$$

as required. □

Corollary 1.4.50. *Let A be a ring. Let $f_1, \ldots, f_n$ be elements of A generating the unit ideal. Let M be an A-module. Then the following sequence is exact:*

$$0 \longrightarrow M \xrightarrow{\alpha} \bigoplus_i M_{f_i} \xrightarrow{\beta} \bigoplus_{i,j} M_{f_i f_j} \tag{1.4.2}$$

Proof. This can be proved as Proposition 1.4.49, using the definition of M_f. □

Proposition 1.4.51. *Let A be a ring. Let $f_1, \ldots, f_n \in A$ and M be an A-module. Then $M \to \bigoplus M_{f_i}$ is injective if and only if the homomorphism*

$$\psi : M \to \bigoplus_{i=1}^{n} M, \quad m \mapsto (f_1 m, \ldots, f_n m)$$

is injective.

Proof. The map $M \to \bigoplus_{i=1}^{n} M_{f_i}$ is injective if and only if for all $m \in M$ and $e_1, \ldots, e_n \geq 1$ such that $f_i^{e_i} m = 0$, $1 \leq i \leq n$ we have $m = 0$. This clearly implies that ψ is injective. Conversely, suppose ψ is injective and $m \in M$ and $e_1, \ldots, e_n \geq 1$ are such that $f_i^{e_i} m = 0$. Let $N \geq e_i$ for all i, then $f_i^N m = 0$ for all i, which is $\psi^N(m) = 0$. Since ψ is injective, this implies $m = 0$. □

Proposition 1.4.52. *Let $(f_i)_{i\in I}$ be a finite family of elements of a ring A generating the unit ideal of A. Then the ring $B = \prod_{i\in I} A_{f_i}$ is a faifully flat A-module.*

Proof. We know each of the A_{f_i} is a flat A-module, hence so is B. On the other hand, if $\mathfrak{p}$ is a prime ideal of A, there exists an index i such that $f_i \notin \mathfrak{p}$ and $\mathfrak{p}_f = \mathfrak{p}A_f$ is therefore a prime ideal of A_f. Then $\mathfrak{p}B \subseteq \mathfrak{p}A_{f_i} \times \prod_{j\neq i} A_{f_j} \neq B$ since $\mathfrak{p}A_{f_i} \neq A_{f_i}$. This suffices to imply that B is a faithfully flat A-module. □

Proposition 1.4.53. *Let $(f_i)_{i\in I}$ be a finite family of elements of a ring A generating the unit ideal of A.*

(a) For an A-module M to be zero, it is necessary and sufficient that, for every index i, the A_{f_i}-module M_{f_i} is zero.

(b) *For an A-module M to be finitely generated (resp. finitely presented), it is necessary and sufficient that, for every index i, the A_{f_i}-module M_{f_i} be finitely generated (resp. finitely presented).*

(c) *Let $\phi : M \to N$ be a homomorphism of A-modules. Then ϕ is injective (resp. surjective) if and only if for every index i, the homomorphism $\phi_{f_i} : M_{f_i} \to N_{f_i}$ is injective (resp. surjective).*

Proof. If $M = 0$ then clearly $M_{f_i} = 0$ for all i. Conversely, if each M_{f_i} is zero, then the A-module $P = \bigoplus_{i\in I} M_{f_i}$ is zero. But $P = M \otimes_A \prod_{i\in I} A_{f_i}$, so the claim in (a) follows from Proposition 1.4.52. Also, (c) follows immediately from (a).

Now if M is finitely generated (resp. finitely presented) then clearly every M_{f_i} is finitely generated (resp. finitely presented). Conversely, if all the M_{f_i} are finitely generated (resp. finitely presented), then P is a finitely generated (resp. finitely presented) B-module (for we can obviously suppose that for each i there is an exact sequence $A_f^m \to A_f^n \to M_{f_i} \to 0$, where m and n are independent of i). Again by Proposition 1.4.52, we see M is finitely generated (resp. finitely presented). □

Corollary 1.4.54. *Let $(f_i)_{i\in I}$ be a finite family of elements of a ring A generating the unit ideal of A. Then A is Noetherian if and only if for every index $i \in I$, then ring A_{f_i} is Noetherian.*

Proof. This follows from Proposition 1.4.53 by applying to ideals of A. □

Corollary 1.4.55. *Let $(f_i)_{i\in I}$ be a finite family of elements of a ring A generating the unit ideal of A. For an A-algebra B to be of finite type (resp. finitely presented) over A, it is necessary and sufficient that, for every index i, the A_{f_i}-algebra B_{f_i} be of finite type (resp. finitely presented) over A_{f_i}.*

Proof. One direction is clear. Conversely, assume that each B_{f_i} is of finite type (finite presented) over A_{f_i}. Since A_{f_i} is finitely presented, we then see each B_{f_i} is of finite type (finite presented) over A. For each $i \in I$ take a finite generating set S_i of B_{f_i}. Without loss of generality, we may assume that the elements of S_i are in the image of the localization map $B \to B_{f_i}$, so we take a finite set T_i of preimages of the elements of T_i in B. Let T be the union of these sets. This is still a finite set. Consider the algebra homomorphism $A[X_t]_{t\in T} \to B$ induced by T. Since it is an algebra homomorphism, the image B' is an A-submodule of the A-module B, so we can consider the quotient module B/B'. Now it is clear that Proposition 1.4.53 implies both of the claims (note that $\{f_i\}$ also generates the unit ideal in $A[X_t]$). □

Corollary 1.4.56. *Let $\rho : A \to B$ be a ring map and $\{g_i\}_{i\in I}$ elements of B generating the unit ideal of B. Then for B to be of finite type (finite presented) over A, it is necessary and sufficient that for every index $i \in I$, the algebra B_{g_i} is of finite type (finite presented) over A.*

Proof. Choose h_i in B such that $1 = \sum_i g_i h_i$. We first prove that, if B_{g_i} is of finite type, then B is of finite type. For this, for each $i \in I$ choose a finite subset S_i which generate B_{g_i} as an A-algebra and let T_i be its preimage in B (as in the proof of Corollary 1.4.55). Consider the A-subalgebra $B' \subseteq B$ generated by $\{g_i\}$, $\{h_i\}$ and T_i for all $i \in I$. Since localization is exact, we see that $B'_{g_i} \to B_{g_i}$ is injective. On the other hand, it is surjective by our choice of T_i. The elements $\{g_i\}$ generate the unit ideal in B' as $h_i \in B'$, so $B' \to B$ viewed as an B'-module homomorphism is an isomorphism by Proposition 1.4.53.

Now we consider the case for finite presentation. Assume that each B_{g_i} is finitely presented. We already know that B is of finite type, so we may identify B with $A[x_1, \dots, x_n]/\mathfrak{a}$ for some ideal $\mathfrak{a}$. Moreover, we may choose elements $\tilde{g}_i, \tilde{h}_i \in A[x_1, \dots, x_n]$ whose image in B is g_i and h_i. Then we see

$$B_{g_i} = A[x_1, \dots, x_n, y_i]/(\mathfrak{a}_i + (1 - y_i\tilde{g}_i)),$$

where $\mathfrak{a}_i$ is the ideal of $A[x_1, \dots, x_n, y_i]$ generated by $\mathfrak{a}$. By **??**, we may choose a finite list of elements $f_{ij} \in \mathfrak{a}$ such that the image of f_{ij} in $\mathfrak{a}_i + (1 - y_i\tilde{g}_i)$ generate the ideal $\mathfrak{a}_i + (1 - y_i\tilde{g}_i)$. If we set

$$B' = A[x_1, \dots, x_n]/(\sum_i \tilde{g}_i \tilde{h}_i - 1, f_{ij}),$$

there is then a surjective A-algebra map $B' \to B$. The classes of the elements $\tilde{g}_i$ in B' generate the unit ideal and by construction the maps $B'_{g_i} \to B_{g_i}$ are injective, so we conclude that $B' \to B$ is injective, hence an isomorphism. □

Proposition 1.4.57. *Let $\mathfrak{a}$ be an ideal of A, $f \in A$ such that $A_f/\mathfrak{a}_f$ is an A-algebra of finite presentation. Then $\mathfrak{a}_f$ is finitely generated in A_f.*

Proof. By hypothesis, there exists a polynomial algebra $B = A[T_1, \dots, T_n]$ and a surjective A-homomorphism $\rho : B \to A_f/\mathfrak{a}_f$, whose kernel $\mathfrak{b}$ is finitely generated. Let $\pi : A_f \to A_f/\mathfrak{a}_f$ be the canonical homomorphism; for each $1 \leq i \leq n$, there exists $t_i \in A$ such that $\pi(t_i/s^k) = \rho(T_i)$ (we can choose an integer k for all index i). Consider then the A_f-algebra $C = A_f[T_1, \dots, T_n]$ and the homomorphism $\varphi : C \to A_f$ such that $\varphi(T_i) = t_i/s^k$. The homomorphism φ is clearly surjective, and so is the composition $\rho' = \pi \circ \varphi : C \to A_f/\mathfrak{a}_f$.

$$\begin{tikzcd}
& B = A[T_1, \dots, T_n] \arrow[d, "\rho"] \\
A_f \arrow[r, "\pi"] & A_f/\mathfrak{a}_f \\
& C = A_f[T_1, \dots, T_n] \arrow[u, "\rho'"'] \arrow[ul, "\varphi"]
\end{tikzcd}$$

On ther other hand, any polynomial in C is of the form P/s^m, where $P \in B$, and we have $\rho'(P/s^m) = (1/s^m)\rho(P)$. As $1/s$ is invertible in A_f, the relation $\rho'(P/s^m) = 0$ in $A_f/\mathfrak{a}_f$ is equivalent to $\rho(P) = 0$, and therefore the kernel $\mathfrak{b}'$ of ρ' is generated by the image of $\mathfrak{b}$ in C, and a fortiori is finitely generated in C. As $\mathfrak{b}' = \varphi^{-1}(\pi^{-1}(0)) = \varphi^{-1}(\mathfrak{a}_f)$ and φ is surjective, we see $\mathfrak{a}_f = \varphi(\mathfrak{b}')$, so $\mathfrak{a}_f$ is finitely generated in A_f. □

Remark 1.4.58. If B is an A-algebra of finite presentation, B' is an A-algebra of finite type, and $\rho : B' \to B$ is a surjective homomorphism, then the kernel $\mathfrak{b}$ of ρ is finitely generated in B'. In fact, there exists a surjective homomorphism $\psi : C \to B'$, where C is a polynomial algebra over A; as $\rho \circ \psi$ is surjective, $\ker(\rho \circ \psi)$ is a finitely generated ideal in C by Proposition 1.4.57. As we have $\mathfrak{b} = \psi(\ker(\rho \circ \psi))$, $\mathfrak{b}$ is then finitely generated in B'.

Conversely, if B' is an A-algebra of finite presentation, $\rho : B' \to B$ is a surjective homomorphism, and if $\mathfrak{b} = \ker \rho$ is finitely generated in B', then B is an A-algebra of finite presentation. In fact, we have a surjective homomorphism $\psi : C \to B'$ where C is a polynomial algebra and $\ker \psi$ is finitely generated in C. As $\ker(\rho \circ \psi)$ is generated by $\ker \psi$ and a system of elements whose image in B' generate $\mathfrak{b}$, it is finitely generated and therefore B is an A-algebra of finite presentation.

1.5 Finitely generated projective modules

1.5.1 Local characterization

Proposition 1.5.1. *Let A be a ring, $\phi : M \to N$ an A-module homomorphism and $\mathfrak{p}$ a prime ideal of A.*

(a) Suppose that $\phi_{\mathfrak{p}} : M_{\mathfrak{p}} \to N_{\mathfrak{p}}$ is suriective and that N is finitely generated. Then there exists $f \in A - \mathfrak{p}$ such that $\phi_f : M_f \to N_f$ is surjective.

(b) *Suppose that $\phi_{\mathfrak{p}} : M_{\mathfrak{p}} \to N_{\mathfrak{p}}$ is bijectioe, that M is finitely generated and that N is finitely presented. Then there exists $f \in A - \mathfrak{p}$ such that $\phi_f : M_f \to N_f$ is bijective.*

Proof. Let P and Q be the kernel and cokernel of ϕ. If $f \in A$, the kernel and cokernel of ϕ_f (resp. $\phi_{\mathfrak{p}}$) are P_f and Q_f (resp. $P_{\mathfrak{p}}$ and $Q_{\mathfrak{p}}$). In the case (a), $\phi_{\mathfrak{p}}$ is surjective so $Q_{\mathfrak{p}} = 0$; as N is finitely generated, so is Q and therefore there exists $f \in A - \mathfrak{p}$ such that $fQ = 0$, whence $Q_f = 0$.

Under the hypotheses of (b), the sequence $0 \to P_f \to M_f \to N_f \to 0$ is exact, hence P_f is finitely generated. Now, $(P_f)_{\mathfrak{p}P_f} = P_{\mathfrak{p}} = 0$, hence there exists $g/f^d \in A_f - \mathfrak{p}A_f$ (whence $g \in A - \mathfrak{p}$) such that $(g/f^d)P_f = 0$. Then as $1/f$ is invertible, we have $(g/1)P_f = 0$, whence $P_{fg} = (P_f)_{g/1} = 0$. Since $fg \in A - \mathfrak{p}$ and $Q_{fg} = 0$, we get the claim. □

Corollary 1.5.2. *If N is finitely presented and $N_{\mathfrak{p}}$ is a free $A_{\mathfrak{p}}$-module of rank p, there exists $f \in A - \mathfrak{p}$ such that N_f is a free A_f-module of rank p.*

Proof. There exist by hypothesis elements $x_1, \ldots, x_p$ such that the $x_i/1$ form a basis of the free $A_{\mathfrak{p}}$-modulc $N_{\mathfrak{p}}$. Consider the homomorphism $\phi : A^p \to N$ such that $\phi(e_i) = x_i$, (e_i) being the canonical basis of A^p. As $\phi_{\mathfrak{p}}$ is bijective by hypothesis, there exists $f \in A - \mathfrak{p}$ such that ϕ_f is bijective, by virtue of Proposition 1.5.1. □

Remark 1.5.3. In the language of algebraic geometry, Corallary 1.5.2 shows that if $\mathscr{F}$ is a coherent sheaf over a scheme X such that $\mathscr{F}|_x$ is free of rank n for some point $x \in X$, then there exist a neighborhood U of x such that $\mathscr{F}|_U$ is free of rank n.

Corollary 1.5.4. *Let N be a finitely generated A-module and consider the function ϕ on $X = \operatorname{Spec}(A)$ defined by*

$$\phi(\mathfrak{p}) = \dim_{\kappa(\mathfrak{p})}(N_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} \kappa(\mathfrak{p})).$$

Then ϕ is upper semi-continuous, i.e., for any $n \in \mathbb{Z}$, the set $\{\mathfrak{p} \in X : \phi(\mathfrak{p}) < n\}$ is open.

Proof. Note that $\phi(\mathfrak{p}) = [N_{\mathfrak{p}}/\mathfrak{p}N_{\mathfrak{p}} : \kappa(\mathfrak{p})]$. By Nakayama's Lemma, this number is equal to the minimal number of generators of $N_{\mathfrak{p}}$ as an $A_{\mathfrak{p}}$ module. Let $m_1, \ldots, m_r$ be such a minimal generating set with $r < n$. Write $m_i = x_i/f$ for $x_i \in N$ and $f \in A - \mathfrak{p}$, then we get a map $\phi : A_f^r \to N_f$ defined by $\phi(e_i/1) = x_i/f$, such that $\phi_{\mathfrak{p}}$ is surjective. By Proposition 1.5.1 there exist some $g \in A - \mathfrak{p}$ such that $\phi_g : A_{fg}^r \to N_{fg}$ is surjective. Then for any $\mathfrak{q} \in D(fg)$, by localizing we see $\phi_{\mathfrak{q}} : A_{\mathfrak{q}}^r \to N_{\mathfrak{q}}$ is also surjective, whence $\phi(\mathfrak{q}) \leq r < n$. Therefore the set $\{\mathfrak{p} \in X : \phi(\mathfrak{p}) < n\}$ is open. □

Theorem 1.5.5. *Let A be a ring and P an A-module. Then the following properties are equivalent:*

(i) *P is a finitely generated projective module.*

(ii) *P is a finitely presented module and, for every maximal ideal $\mathfrak{m}$ of A, $P_{\mathfrak{m}}$ is a free $A_{\mathfrak{m}}$-module.*

(iii) *P is a finitely generated module, for all $\mathfrak{p} \in \operatorname{Spec}(A)$, the $A_{\mathfrak{p}}$-module $P_{\mathfrak{p}}$ is free and, if we denote its rank by $\operatorname{rank}_{\mathfrak{p}}$, the function $\mathfrak{p} \mapsto \operatorname{rank}_{\mathfrak{p}}$ is locally constant in the topological space $\operatorname{Spec}(A)$.*

(iv) *There exists a finite family $(f_i)_{i \in I}$ of elements of A, generating the ideal A, such that for all $i \in I$, the A_{f_i}-module P_{f_i} is free of finite rank.*

(v) *For every maximal ideal $\mathfrak{m}$ of A, there exists $f \in A - \mathfrak{m}$ such that $P_{\mathfrak{m}}$ is free of finite rank.*

Proof. By **??**, we know that a finitely generated projective module is finitely presented. If P is a projective A-module, $P_{\mathfrak{m}} = P \otimes_A A_{\mathfrak{m}}$ is a projective $A_{\mathfrak{m}}$-module. Finally, as $A_{\mathfrak{m}}$ is a local ring, every finitely presented projective $A_{\mathfrak{m}}$-module is free (Corollary 1.3.11). This shows (i) implies (ii). Also, (ii) implies (v) by Corollary 1.5.2.

Now assume (v), and let E be the set of $f \in A$ such that P_f is a finitely generated free A_f-module. The hypothesis implies that E is contained in no maximal ideal of A, hence E generates the ideal A and there therefore exist a finite family $(f_i)_{i\in I}$ of elements of E that generates A, whence (iv).

Assume (iv) and consider the ring $B = \prod_{i\in I} A_{f_i}$ and the B-module $M = \bigoplus_{i\in I} P_{f_i} = P \otimes_A B$. For every index i, there exists a free A_{f_i}-module L_i such that P_i is a direct factor of L_i and it may be assumed that all the L_i have the same rank; then $L = \bigoplus_{i\in I} L_i$ is a free B-module of which M is a direct factor, in other words M is a finitely generated projective B-module. As B is a faithfully flat A-module, we conclude that P is a finitely generated projective A-module by **??**.

Finally, we show that (iv) implies (iii) and (iii) implies (v). If (iv) holds, then it follows from Proposition 1.4.53 that P is finitely generated. On the other hand, for every prime ideal $\mathfrak{p}$ of A, there exists an index i such that $f_i \notin \mathfrak{p}$; then $P_\mathfrak{p} = (P_{f_i})_{\mathfrak{p}_{f_i}}$ and hence by hypothesis $P_\mathfrak{p}$ is free and of the same rank as P_{f_i}, which proves (iii).

Assume (iii), let $\mathfrak{m}$ be a maximal ideal of A; write $\mathrm{rank}_\mathfrak{m} = n$ and let $x_1, \dots, x_n$ be a basis of $P_\mathfrak{m}$. We can assume that the x_i are canonical images of elements $p_i \in P$ to within multiplication by an invertible element of $A_\mathfrak{m}$. Let (e_i) be the canonical basis of A^n and $\phi : A^n \to P$ the homomorphism such that $\phi(e_i) = p_i$. As P is finitely generated, it follows from Proposition 1.5.1 that there exists $f \in A - \mathfrak{m}$ such that ϕ_f is surjective. We conclude that ϕ_{fg} is also surjective for all $g \in A - \mathfrak{m}$ and by hypothesis there exists $g \in A - \mathfrak{m}$ such that $\mathrm{rank}_\mathfrak{p} = n$ for $\mathfrak{p} \in D(g)$. Then, replacing f by fg, we may assume that $\mathrm{rank}_\mathfrak{p} = n$ for all $\mathfrak{p} \in D(f)$. Then $\phi_\mathfrak{p} : A^n_\mathfrak{p} \to P_\mathfrak{p}$ is a surjective homomorphism and $P_\mathfrak{p}$ and $A^n_\mathfrak{p}$ are both free $A_\mathfrak{p}$-modules of the same rank; hence by Corollary 1.3.13 $\phi_\mathfrak{p}$ is bijective for all $\mathfrak{p} \in D(f)$. Let $\mathfrak{p}'$ be a prime ideal of A_f and let $\mathfrak{p}$ be its inverse image in A under the canonical map; if $(A^n_f)_{\mathfrak{p}'}$ and $(P_f)_{\mathfrak{p}'}$ are identified with $A^n_\mathfrak{p}$ and $P_\mathfrak{p}$ under the canonical isomorphisms, $(\phi_f)_{\mathfrak{p}'}$ is then identified with $\phi_\mathfrak{p}$ and is consequently bijective. We conclude that ϕ is bijective, which establishes (v). □

Corollary 1.5.6. *Suppose that the equivalent properties of the statement of Theorem 1.5.5 hold. Let n be a positive integer such that, for every family $(x_1, \dots, x_n)$ of elements of P, there exists a family $(a_1, \dots, a_n)$ of elements of A, which are not all divisors of zero and for which $\sum_{i=1}^n a_i x_i = 0$. Then, for all $\mathfrak{p} \in \mathrm{Spec}(A)$, $\mathrm{rank}_\mathfrak{p} \leq n$.*

Proof. Let $\mathfrak{p}$ be a prime ideal of A, set $\mathrm{rank}_\mathfrak{p} = r$ and let $(y_1, \dots, y_r)$ be a basis of the free $A_\mathfrak{p}$-module $P_\mathfrak{p}$. There exist elements $(x_1, \dots, x_r)$ of P and $s \in A - \mathfrak{p}$ such that $y_i = x_i/s$ for all i. If $r > n$, then there exist a family $(a_1, \dots, a_r)$ of elements of A such that $\sum_{i=1}^r a_i x_i = 0$, which are not all divisors of zero (for example take $(a_1, \dots, a_n)$ as in the hypothesis and let $a_i = 0$ for $i > n$). But then we have $\sum_{i=1}^r (a_i/1) y_i = 0$ and therefore $a_i = 0$ in $A_\mathfrak{p}$, which implies a_i is a divisor of zero, contradiction. □

Corollary 1.5.7. *Every finitely presented flat module is projective.*

Proof. If P is a finitely presented flat A-module and $\mathfrak{m}$ a maximal ideal of A, the $A_\mathfrak{m}$-module $P_\mathfrak{m}$ is flat and finitely presented and hence free (Corollary 1.3.11). Condition (ii) of Theorem 1.5.5 therefore holds. □

1.5.2 The rank function

Definition 1.5.8. Let P be a finitely generated projective A-module. For every prime ideal $\mathfrak{p}$ of A, the rank of the free $A_\mathfrak{p}$-module P, is called the **rank** of P at $\mathfrak{p}$ and is denoted by $\mathrm{rank}_\mathfrak{p}(P)$.

By Theorem 1.5.5 the integer-valued function $\mathfrak{p} \mapsto \mathrm{rank}_\mathfrak{p}(P)$ is locally constant on $X = \mathrm{Spec}(A)$. It is therefore constant if X is connected and in particular if the ring A is an integral domain.

Definition 1.5.9. Let n be a positive integer. A projective A-module P is said to be **of rank n** if it is finitely generated and $\mathrm{rank}_{\mathfrak{p}}(P) = n$ for every prime ideal $\mathfrak{p}$ of A.

Clearly every finitely generated flee A-module L is of rank n in the sense of the above definition, n being equal to the dimension (or rank) of L. A projective module of rank 0 is zero. If A is not reduced to 0 and a projective A-module P is of rank n, the integer n is determined uniquely; it is then denoted by $\mathrm{rank}(P)$.

Theorem 1.5.10. *Let P be an A-module and n a psotive integer. The following properties are equivalent:*

(i) P is projective of rank n.

(ii) P is finitely generated and, for every maximal ideal $\mathfrak{m}$ of A, the $A_{\mathfrak{m}}$-module $P_{\mathfrak{m}}$ is free of rank n.

(iii) P is finitely generated and, for every prime ideal $\mathfrak{p}$ of A, the $A_{\mathfrak{p}}$-module $P_{\mathfrak{p}}$ is free of rank n.

(iv) For every maximal ideal $\mathfrak{m}$ of A, there exists $f \in A - \mathfrak{m}$ such that the $A_{\mathfrak{m}}$-module $P_{\mathfrak{m}}$ is free of rank n.

Proof. By definition and Theorem 1.5.5, (i) and (iii) are equivalent; (ii) implies (iii), as, for every prime ideal $\mathfrak{p}$ of A, there exists a maximal ideal $\mathfrak{m}$ containing $\mathfrak{p}$ and, writing $\mathfrak{p}' = \mathfrak{p}_{\mathfrak{m}}$, $P_{\mathfrak{p}}$ is isomorphic to $(P_{\mathfrak{m}})_{\mathfrak{p}'}$. If $P_{\mathfrak{m}}$ is free of rank n, so then is $P_{\mathfrak{p}}$. Property (iii) implies (i) by virtue of Theorem 1.5.5 and the fact that, iff $f \in A - \mathfrak{m}$ and $\mathfrak{m}' = \mathfrak{m}_f$, $P_{\mathfrak{m}}$ is isomorphic to $(P_f)_{\mathfrak{m}_f}$ and therefore the ranks of P_f and $P_{\mathfrak{m}}$ are equal. Finally, this last argument and Theorem 1.5.5 show that (iv) implies (ii). ☐

Corollary 1.5.11. *If A is an integral domain and P is a projective A-module, then* $\mathrm{rank}_{\mathfrak{p}}(P) = \mathrm{rank}(P)$ *for all prime ideal $\mathfrak{p}$ of A.*

Proof. It is sufficient to apply Theorem 1.5.10(iii) with $\mathfrak{p} = (0)$. ☐

Let E and F be two finitely generated projective A-modules. We know that $E \oplus F$, $E \otimes_A F$, $\mathrm{Hom}_A(E, F)$ and the dual E^* of E are projective and finitely generated, so is the exterior power $\bigwedge^p E$ for every integer $p > 0$. Also, it follows immediately from definition and the exactness of localization that, for every prime ideal $\mathfrak{p}$ of A,

$$\mathrm{rank}_{\mathfrak{p}}(E \oplus F) = \mathrm{rank}_{\mathfrak{p}}(E) + \mathrm{rank}_{\mathfrak{p}}(F), \quad \mathrm{rank}_{\mathfrak{p}}(E \times F) = \mathrm{rank}_{\mathfrak{p}}(E)\mathrm{rank}_{\mathfrak{p}}(F),$$
$$\mathrm{rank}_{\mathfrak{p}}(\mathrm{Hom}_A(E, F)) = \mathrm{rank}_{\mathfrak{p}}(E)\mathrm{rank}_{\mathfrak{p}}(F), \quad \mathrm{rank}_{\mathfrak{p}}(E^*) = \mathrm{rank}_{\mathfrak{p}}(E),$$
$$\mathrm{rank}_{\mathfrak{p}}(\bigwedge\nolimits^p E) = \binom{\mathrm{rank}_{\mathfrak{p}}(E)}{p}.$$

If the ranks of E and F are defined, so are those of $E \oplus F$, $E \otimes_A F$, $\mathrm{Hom}_A(E, F)$, E^* and $\bigwedge^p E$ and the above equations also hold with the index $\mathfrak{p}$ omittcd. Moreover:

Corollary 1.5.12. *For a finitely generated firqjective A-module P to be of rank n, it is necessary and sufficient that $\bigwedge^n P$ be of rank 1.*

Proposition 1.5.13. *Let B be a A-algebra and P a projective A-module of rank n. The B-module $B \otimes_A P$ is then projective of rank n.*

Proof. We know that $B \otimes_A P$ is projective and finitely generated. If $\mathfrak{q}$ is a prime ideal of B and $\mathfrak{p}$ its inverse image in A, then

$$(B \otimes_A P)_{\mathfrak{q}} = (P \otimes_A B) \otimes_B B_{\mathfrak{q}} = P \otimes_A B_{\mathfrak{q}} = (P \otimes_A A_{\mathfrak{p}}) \otimes_A B_{\mathfrak{q}}$$

and, as, by hypothesis, $P \otimes_A A_{\mathfrak{p}}$ is a free $A_{\mathfrak{p}}$-module of rank n, $(B \otimes_A P)_{\mathfrak{q}}$ is a free $B_{\mathfrak{q}}$-module of rank n. ☐

Proposition 1.5.14. *Let A be a semi-local ring and P a finitely generated projective A-module. If the rank of P is defined, P is a free A-module.*

Proof. Suppose first that $A = \prod_{i=1}^{n} K_i$ where K_i are fields. Then the K_i are then identified with the minimal ideals of A and, for all i, the sum $\mathfrak{p}_i = \sum_{j \neq i} K_j$ is a maximal ideal of A, and the $\mathfrak{p}_i$'s are the only prime ideals of A. Every finitely generated A-module P is therefore the direct sum of its isotypical components P_i, P_i being isomorphic to a direct sum of a finite number r, of A-modules isomorphic to K_i (A, VIII, §5, no.1, Proposition 1 and no.3, Proposition 11); the ring $A_{\mathfrak{p}_i}$ is identified with K_i and annihilates the P_j of index $j \neq i$, hence $n = \operatorname{rank}_{\mathfrak{p}_i}(P)$; if all the r_i are equal to the same number r, P is isomorphic to A^r, whence the proposition in this case. In the general case, let $\mathfrak{r}$ be the Jacobson radical of A and $B = A/\mathfrak{r}$; as B is a product of fields, the projective B-module $P \otimes_A B$ is free by the remarks preceding Proposition 1.5.13. Also P is a flat A-module and the proposition then follows from Proposition 1.3.9. □

1.5.3 Projective modules of rank 1

Theorem 1.5.15. *Let A be a ring and M a finitely generated A-module.*

(a) *If there exists an A-module N such that $M \otimes_A N$ is isomorphic to A, the module M is projective of rank* 1.

(b) *Conversely, if M is projective of rank* 1 *and M^* is the dual of M, the canonical homomorphism $v : M^* \otimes_A M \to A$ corresponding to the canonical bilinear form $(x^*, x) \mapsto \langle x^*, x \rangle$ on $M^* \times M$ is bijective.*

Proof. First assume that $M \otimes_A N \cong A$ for some A-module N. It is required to prove that, for every maximal ideal $\mathfrak{m}$ of A, the $A_{\mathfrak{m}}$-module $M_{\mathfrak{m}}$ is free of rank 1. We are free to replace A by $A_{\mathfrak{m}}$ and hence may assume that A is a local ring. Let $k = A/\mathfrak{m}$. The isomorphism $\eta : M \otimes_A N \to A$ defines an isomorphism

$$\eta \otimes 1_k : (M/\mathfrak{m}M) \otimes_k (N/\mathfrak{m}N) \to k$$

as the rank over k of $(M/\mathfrak{m}M) \otimes_k (N/\mathfrak{m}N)$ is the product of the ranks of $M/\mathfrak{m}M$ and $N/\mathfrak{m}N$, these latter are necessarily equal to 1, in other words $M/\mathfrak{m}M$ is monogenous. It follows that M is monogenous by Corollary 1.3.8. On the other hand, the annihilator of M also annihilates $M \otimes_A N$ and hence is zero, which proves that M is isomorphic to A.

For (b), assume that M is projective of rank 1. It is sufficient to prove that, for every maximal ideal $\mathfrak{m}$ of A, $v_{\mathfrak{m}}$ is an isomorphism. As M is finitely presented, $(M^*)_{\mathfrak{m}}$ is canonically identified with the dual $(M_{\mathfrak{m}})^*$ (Proposition 1.2.48) and, as $M_{\mathfrak{m}}$ is free of rank 1 like its dual $(M_{\mathfrak{m}})^*$, clearly the canonical homomorphism $v_{\mathfrak{m}} : (M_{\mathfrak{m}})^* \otimes_{A_{\mathfrak{m}}} (M_{\mathfrak{m}}) \to A_{\mathfrak{m}}$ is bijcctive, which completes the proof. □

Remark 1.5.16. If M is projective of rank 1 and N is such that $M \otimes_A N$ is isomorphic to A, then N is isomorphic to M^*, for there are isomorphisms

$$N \to N \otimes_A A \to N \otimes M \otimes M^* \to A \otimes M^* \to M^*.$$

Now let $A^n \to M$ be a surjective homomorphism. Then by applying the dual functor $\operatorname{Hom}(-, A)$ we get an injection $N \cong M^* \to (A^n)^* \cong A^n$. If we further assume that A is Noetherian, then N is a finitely generated A-module.

In the language of algebraic geometry, Theorem 1.5.15 and the argument above show that a coherent sheaf $\mathscr{F}$ on a Noetherian scheme X is invertible (locall of rank 1) if and only if there exist a coherent sheaf $\mathscr{G}$ such that $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G} \cong \mathscr{O}_X$. This justifies the terminology invertible: it means that $\mathscr{F}$ is an invertible element of the monoid of coherent sheaves under the operation $\otimes$.

Proposition 1.5.17. *Let M and N be projective A-modules of rank 1. Then $M \otimes_A N$, $\mathrm{Hom}_A(M, N)$ and the dual M^* of M are projective of rank 1.*

Let us now note that every finitely generated A-module is isomorphic to a quotient module of $L = A^n$. We may therefore speak of the set $F(A)$ of classes of finitely generated A-modules with respect to isomorphism. We denote by $P(A)$ the subset of $F(A)$ consisting of the classes of projective A-modules of rank 1 and by $\mathrm{cl}(M)$ the image in $P(A)$ of a projective A-module M of rank 1. It is immediate that, for two projective A-modules M, N of rank 1, $\mathrm{cl}(M \otimes_A N)$ depends only on $\mathrm{cl}(M)$ and $\mathrm{cl}(N)$. As definition we set

$$\mathrm{cl}(M) + \mathrm{cl}(N) = \mathrm{cl}(M \otimes_A N) \tag{1.5.1}$$

and an internal law of composition is thus defined on $P(A)$.

Proposition 1.5.18. *The set $P(A)$ of classes of projective A-modules of rank 1, with the law of composition (1.5.1), is a commutative group. If M is a projective A-module of rank 1 and M^* is its dual, then*

$$\mathrm{cl}(M^*) = -\mathrm{cl}(M), \quad \mathrm{cl}(A) = 0.$$

Proof. The associativity and commutativity of the tensor product show that the law of composition (6) is associative and commutative. The isomorphism between $A \otimes_A M$ and M prove that $\mathrm{cl}(A)$ is the identity element under this law and, by virtue of Theorem 1.5.15, $\mathrm{cl}(M) + \mathrm{cl}(M^*) = \mathrm{cl}(A)$, whence the proposition. □

Let B be an A-algebra and M a projective A-module of rank 1. Then $B \otimes_A M$ is a projective B-module of rank 1 by Proposition 1.5.13. Then there exists a map called canonical $\phi : P(A) \to P(B)$ such that

$$\phi(\mathrm{cl}(M)) = \mathrm{cl}(B \otimes_A M).$$

The equation $(M \otimes_A B) \otimes_B (N \otimes_A B) = (M \otimes_A N) \otimes_A B$ for two A-modules M, N proves that the map ϕ is a group homomorphism.

Remark 1.5.19. Condition (v) of Theorem 1.5.5 (equivalent to the fact that P is projective and finitely generated) may also be expressed by saying that the sheaf of modules P over $X = \mathrm{Spec}(A)$ associated with P is locally free and of finite type and may therefore be interpreted as the sheaf of sections of a vector bundle over X. Conversely, every vector bundle over X arises from a finitely generated projective module, which is determined to within a unique isomorphism. The projective modules of rank n thus correspond to the vector bundles all of whose fibres have dimension n. In particular, the vector bundles of rank 1 correspond to the projective modules of rank 1. If we denote by $\mathscr{O}_X$ the structure sheaf of X and by $\mathscr{O}_X^\times$ the sheaf of units of $\mathscr{O}_X$ (whose sections over an open set U of X are the invertible elements of the ring of sections of $\mathscr{O}_X$ over U), it follows that the group $P(A)$ is isomorphic to the first cohomology group $H^1(X, \mathscr{O}_X^\times)$.

1.5.4 Non-degenerate submodules and invertible submodules

In this part and the two following, A denotes a ring, S a multiplicative subset of A consisting of elements which are not divisors of zero in A, and B the ring $S^{-1}A$. The ring A is then canonically identified with a subring of B. The elements of S are therefore invertible in B. One of the most important special cases for applications is that where A is an integral domain and S is the set of nonzero elements of A; in this case B is the field of fractions of A.

Let M be a sub-A-module of B. Then M is called **non-degenerate** if $B \cdot M = B$. Note that if B is a field, this condition simply means that M is not reduced to 0.

Proposition 1.5.20. *Let M be a sub-A-module of B. The following conditions are equivalent:*

(i) M is non-degenerate;

(ii) M meets S;

(iii) If $i_M : M \to B$ is the canonical injection, then $S^{-1}i_M : S^{-1}M \to B$ is bijective.

Proof. Part (i) implies (ii), for if $B \cdot M = B$, there exists $a \in A$, $s \in S$ and $x \in M$ such that $(a/s)x = 1$, hence $ax = s$ belongs to $S \cap M$. To see that (ii) implies (iii), note that $S^{-1}i$ is already injective, since localization is a exact functor. Moreover, if $x \in M \cap S$, the image under $S^{-1}i$ of $x/x \in S^{-1}M$ in B is equal to 1 and $S^{-1}i$ is therefore surjective. Finally, (iii) clearly implies (i). □

Corollary 1.5.21. *If M and N are two non-degenerate sub-A-modules of B, the A-modules $M + N$, $M \cdot N$ and $M \cap N$ are non-degenerate.*

Proof. The assertion is trivial for $M + N$. On the other hand if $s \in S \cap M$ and $t \in S \cap N$, then $st \in S \cap (M \cdot N)$ and $st \in S \cap (M \cap N)$. □

Given two sub-A-modules M and N of B, let us denote by $(N : M)$ the sub-A-module of B consisting of those $b \in B$ such that $bM \subseteq N$. If every $b \in (N : M)$ is mapped to the homomorphism $h_b : x \mapsto bx$ of M to N, a canonical homomorphism $b \mapsto h_b$, is obtained from $(N : M)$ to $\mathrm{Hom}_A(M, N)$.

Proposition 1.5.22. *Let M, N be two sub-A-modules of B. If M is non-degenerate, the canonical homomorphism from $(N : M)$ to $\mathrm{Hom}_A(M, N)$ is bijective.*

Proof. Let $s \in S \cap M$. If $b \in (N : M)$ is such that $bx = 0$ for all $x \in M$, then $bs = 0$ whence $b = 0$ since s is not a divisor of 0 in B. On the other hand, let $f \in \mathrm{Hom}_A(M, N)$ and set $b = f(s)/s$; for all $x \in M$, there exists $t \in S$ such that $tx \in A$. Then

$$f(x) = f(stx)/(st) = (txf(s))/(st) = bx$$

whence $b \in (N : M)$ and $f = h_b$, which proves the proposition. □

In particular, $(A : M)$ is canonically identified with the dual M^* of M, the canonical bilinear form on $M^* \times M$ being identified with the restriction to $(A : M) \times M$ of the multiplication $B \times B \to B$.

Definition 1.5.23. A sub-A-module M of B is called **invertible** if there exists a sub-A-module N such that $M \cdot N = A$.

Example 1.5.24. If b is invertible element of B, the A-module Ab is invertible, as is seen by taking $N = Ab^{-1}$.

Proposition 1.5.25. *Let M be an invertible sub-A-module of B. Then:*

(a) There exists $s \in S$ such that $As \subseteq M \subseteq As^{-1}$ (and in particular M is nondegenerate).

(b) $(A : M)$ is the only sub-A-module N of B such that $M \cdot N = A$.

(c) M is finitely generated and $(A : M) = A$ if and only if $M = A$.

Proof. If there exists a sub-A-module N such that $M \cdot N = A$, then

$$B \cdot M = B \cdot (B \cdot M) \supseteq B \cdot (M \cdot N) = B \cdot A = B$$

hence M is non-degenerate. Similarly N is non-degenerate. If $t_1 \in S \cap M$ and $t_2 \in S \cap N$, the element $s = t_1 t_2$ belongs to $S \cap M \cap N$, whence $Ms \subseteq M \cdot N = A$ and therefore $As \subseteq M \subseteq As^{-1}$.

On the other hand, obviously $N \subseteq (A : M)$, whence

$$A = M \cdot N \subseteq M \cdot (A : M) \subseteq A$$

whence $A = M \cdot (A : M)$. Multiplying the two sides by N, we deduce then $N = (A : M)$, this prove (b). Now since $M \cdot N = A$, then there is an equation

$$\sum_i x_i y_i, \quad x_i \in M, y_i \in N,$$

so M is generated by (x_i) since $x = \sum_i (xy_i) x_i$ for each $x \in M$. Moreover, if $(A : M) = A$ then since M is invertible we have $M \cdot A = A$, which implies $M = A$. □

Theorem 1.5.26. *Let M be a non-degenerate sub-A-module of B. The following properties are equivalent:*

(i) *M is invertible.*

(ii) *M is projective.*

(iii) *M is projective of rank* 1.

(iv) *M is a finitely generated A-module and, for every maximal ideal $\mathfrak{m}$ of A, the $A_{\mathfrak{m}}$-module $M_{\mathfrak{m}}$ is principal.*

Proof. Let us show first the equivalence of properties (i), (ii) and (iii). If (i) holds and N is sub-A-module of B such that $M \cdot N = A$, then there is a relation

$$\sum_i m_i n_i = 1$$

For all $x \in M$, set $v_i(x) = n_i x$, the v_i are linear forms on M and by $x = \sum_i v_i(x) m_i$ for all $x \in M$. This proves that M is projective by(A, II, §2, no.6, Proposition 12) and generated by the x_i; hence M is a finitely generated projective module.

Let $\mathfrak{m}$ be a maximal ideal of A. We show that the integer $r = \mathrm{rank}_{\mathfrak{m}}(M)$ is equal to 1. Let $S_{\mathfrak{m}}$ be the image of S in $A_{\mathfrak{m}}$. As the elements of S are not divisors of 0 in A, those of $S_{\mathfrak{m}}$ are not divisors of 0 in $A_{\mathfrak{m}}$, since $A_{\mathfrak{m}}$ is a flat A-module. Then $(S_{\mathfrak{m}})^{-1} A_{\mathfrak{m}} \neq 0$ and, as $M_{\mathfrak{m}}$ is a free $A_{\mathfrak{m}}$-module of rank r, $(S_{\mathfrak{m}})^{-1} M_{\mathfrak{m}}$ is a free $(S_{\mathfrak{m}})^{-1} A_{\mathfrak{m}}$-module of rank r. Also note that $(S_{\mathfrak{m}})^{-1} A_{\mathfrak{m}}$ (resp. $(S_{\mathfrak{m}})^{-1} M_{\mathfrak{m}}$) is canonically identified with $(S^{-1}A)_{\mathfrak{m}}$ (resp. $(S^{-1}M)_{\mathfrak{m}}$). Now $S^{-1}M = B$ by Proposition 1.5.20(iii) and hence $(S^{-1}M)_{\mathfrak{m}}$ is a free $(S^{-1}A)_{\mathfrak{m}}$ of rank 1, which proves that $r = 1$ and shows the implication $(i) \Rightarrow (iii)$.

The implication $(iii) \Rightarrow (ii)$ is trivial. Let us show that $(ii) \Rightarrow (i)$. There exists by hypothesis a family (not necessarily finite) $(f_i)_{i \in I}$ of linear forms on M and a family $(m_i)_{i \in I}$ of elements of M such that, for all $x \in M$, the family $(f_i(x))$ has finite support and $x = \sum_i m_i f_i(x)$ (A, II, §2, no.6, Proposition 12). Since M is non-degenerate, $f_i(x) = n_i x$ for some $n_i \in (A : N)$ by virtue of Proposition 1.5.22. Taking x as an element of $M \cap S$, it is seen that of necessity $n_i = 0$ except for a finite number of indices and $\sum_i m_i n_i = 1$. This obviously implies $M \cdot (A : M) = A$, whence (i).

By definition we have (iii)⇒(iv), so let us show the converse. Since M is non-degenerate, its annihilator is zero by Proposition 1.5.20, then so is the annihilator of $M_{\mathfrak{m}}$. As $M_{\mathfrak{m}}$ is assumed to be a monogenous $A_{\mathfrak{m}}$-module, it is therefore free of rank 1 and it then follows from Theorem 1.5.10 that M is projective of rank 1. □

Corollary 1.5.27. *Every invertible sub-A-module of B is flat and finitely presented.*

Proof. This follows from Theorem 1.5.26(iii). □

Proposition 1.5.28. *Let M, N be two sub-A-modules of B. Suppose that M is invertible.*

(a) The canonical homomorphism $M \otimes_A N \to M \cdot N$ is bijective.

(b) $(N : M) = N \cdot (A : M)$ and $N = (N : M) \cdot M$.

Proof. Let i_N be the canonical injection from N to B. Since M is a flat A-module by Corollary 1.5.27, $1 \otimes i_N : M \otimes_A N \to M \otimes_A B$ is injective. But, as $B = S^{-1}A$, the B-module $M \otimes_A B$ is equal to $S^{-1}M$ and hence is identified with B since M is non-degenerate. If this identification is made, the image of $1 \otimes i_N$ is $M \cdot N$, whence (a).

Now write $M^{-1} := (A : M)$, then we have

$$M^{-1} \cdot N \subseteq (N : M) = M^{-1} \cdot M \cdot (N : M) \subseteq M^{-1} \cdot N$$

which proves $M^{-1} \cdot N = (N : M)$, whence $N = M \cdot (N : M)$. □

1.5.5 The class group of invertible submodules

Under multiplication, the sub-A-modules of B form a commutative monoid, with A as identity element. Then the invertible modules are the invertible elements and therefore form a commutative group $\mathfrak{I}$. We have seen that the inverse of $M \in \mathfrak{I}$ is $M^{-1} := (A : M)$.

Let $A^\times$ (resp. $B^\times$) be the multiplicative group of invertible elements of A (resp. B) and let i_A^S denote the canonical injection $A \to B$. For all $b \in B^\times$, $\theta(b) = bA$ is an invertible sub-A-module. The map $\theta : B^\times \to \mathfrak{I}$ is a homomorphism whose kernel is $i_A^S(A^\times)$. Its cokernel will be denoted by $\mathfrak{C}$ or $\mathfrak{C}(A)$. The group $\mathfrak{C}$ is called the **group of classes of invertible sub-A-modules** of B. The following exact sequence has been constructed

$$1 \longrightarrow A^\times \xrightarrow{i_A^S} B^\times \xrightarrow{\theta} \mathfrak{I} \xrightarrow{\pi} \mathfrak{C} \longrightarrow 1$$

where 1 denotcs the group consisting only of the identity element and π is the canonical map $\mathfrak{I} \to \mathfrak{C}$.

As every invertible sub-A-module M of B is projective of rank 1, the element $\mathrm{cl}(M) \in P(A)$ is defined.

Proposition 1.5.29. *The map $\mathrm{cl} : \mathfrak{I} \to P(A)$ defines, by taking quotients, an isomorphism from $\mathfrak{C}$ onto the kernel of the canonical homomorphism*

$$\phi : P(A) \to P(B).$$

In other words, there is an exact sequence

$$1 \longrightarrow A^\times \xrightarrow{i_A^S} B^\times \xrightarrow{\theta} \mathfrak{I} \xrightarrow{\mathrm{cl}} P(A) \xrightarrow{\phi} P(B)$$

Proof. It follows from Proposition 1.5.28 and the definition of addition in $P(A)$ that $\mathrm{cl}(M \cdot N) = \mathrm{cl}(M) + \mathrm{cl}(N)$ for M, N in $\mathfrak{I}$, which shows that cl is a homomorphism. If $M \in \mathfrak{I}$ is isomorphic to A, there exists $b \in B$ such that $M = Ab$ and, as M is invertible, there exists $b' \in B$ such that $bb' = 1$, in othe rwords $b \in B^\times$. The converse is immediate. Hence the kernel of cl is contained in $\theta(B^\times)$.

Let us now determine the image of cl. If $M \in \mathfrak{I}$, then $M \otimes_A B = S^{-1}M = B$, whence $\mathrm{cl}(M) \in \ker \phi$. Conversely, let P be a projective A-module of rank 1 such that $P(B) = P \otimes_A B$ is B-isomorphic to B. As P is a flat A-module, the injection $i : A \to B$ defines an injection $i \otimes i_B : P \to P \otimes_A B = B$ and P is thus identified with a sub-A-module of B. By virtue of Proposition 1.5.20(iii) P is non-degenerate and Theorem 1.5.26 shows that P is invertible. The kernel of ϕ is therefore equal to the image of cl. □

Corollary 1.5.30. *For two invertible sub-A-modules of B to have the same image in $\mathfrak{C}$, it is necessary and sufficient that they be isomorphic.*

Corollary 1.5.31. *If the ring B is semi-local, the group $\mathfrak{C}$ of classes of invertible sub-A-modules of B is canonically identified with the group $P(A)$ of classes of projective A-modules of rank* 1.

Proof. In this case $P(B) = 0$ by Proposition 1.5.14. □

Example 1.5.32. Let A be an integral domain and S is the set of nonzero elements of A, so that B is the field of fractions of A. The invertible sub-A-modules of B are also called in this case **invertible fractional ideals**. Those which are monogenous free A-modules are just the fractional principal ideals.

1.5.6 Exercise

Exercise 1.5.1. let A be a commutative ring. The following are equivalent:

(a) For every prime ideal $\mathfrak{p}$, the localization $A_\mathfrak{p}$ is an integral domain.

(b) For every maximal ideal $\mathfrak{m}$, the localization $A_\mathfrak{m}$ is an integral domain.

(c) For $x, y \in A$ such that $xy = 0$ we have $\mathrm{Ann}(x) + \mathrm{Ann}(y) = A$.

Proof. It is clear that (a) implies (b); for $(b) \Rightarrow (c)$, let $xy = 0$, assume $\mathrm{Ann}(x) + \mathrm{Ann}(y) \subseteq \mathfrak{m}$ for some maximal ideal $\mathfrak{m}$, then $xy/1 = 0$ in $A_\mathfrak{m}$. Then, since $A_\mathfrak{m}$ is an integral domain, we have $x/1 = 0$ or $y/1 = 0$. Assume $x/1 = 0$, then $xs = 0$ for some $s \in A - \mathfrak{m}$. But $\mathrm{Ann}(x) \subseteq \mathfrak{m}$, this is an contradiction. Similar for $y/1 = 0$.

We show that $(c) \Rightarrow (a)$. First we have the following observation: If $\mathfrak{p}$ is a prime ideal, then for any $x \in A$ such that $x/1 \neq 0 \in A_\mathfrak{p}$ then $\mathrm{Ann}(x) \subseteq \mathfrak{p}$. In fact, assume $tx = 0$ for $t \in A$. Then $t \notin A - \mathfrak{p}$, hence $t \in \mathfrak{p}$. Now, suppose that $x/s \cdot y/t = 0 \in A_\mathfrak{p}$. Then $xyz = 0$ for some $z \in A - \mathfrak{p}$. It will suffice to prove that $x/1 = 0$ or $y/1 = 0$. Suppose otherwise, namely that $x/1 \neq 0$ and $y/1 \neq 0$. Then, $xz/1 \neq 0$ since $z/1$ is invertible. By the observation, we know that $\mathrm{Ann}(xz) \subseteq \mathfrak{p}$ and $\mathrm{Ann}(y) \subseteq \mathfrak{p}$, thus $\mathrm{Ann}(x) + \mathrm{Ann}(y) \subseteq \mathfrak{p}$, which is a contradiction. □

Exercise 1.5.2. Let A be a ring. Suppose that, for each prime ideal $\mathfrak{p}$, the local ring $A_\mathfrak{p}$ has no nilpotent element $\neq 0$. Show that A has no nilpotent element $\neq 0$. If each $A_\mathfrak{p}$ is an integral domain, is A necessarily an integral domain?

Proof. By corollary 1.2.39, we have $\mathfrak{n}(A_\mathfrak{p}) = \mathfrak{n}(A)_\mathfrak{p}$ so $\mathfrak{n}(A)_\mathfrak{p} = 0$ for any prime ideal $\mathfrak{p}$. By Proposition 1.3.18, $\mathfrak{n}(A) = 0$.

The answer for next question is no. Suppose that $A = \prod_{i=1}^n k_i$, where $k_i = k$ an algebraic closed field. This is not integral domain, for example $(1, \dots, 0) \cdot (0, \dots, 1) = 0$. Then $\mathrm{Spec}(A) = \{\mathfrak{p}_i := 0 \times \prod_{i \neq j} k_j\}$. Consider the localization $A_{\mathfrak{p}_i}$:

$$A - \mathfrak{p}_i = \{(x_1, \dots, x_n) \in A : x_i \neq 0\}$$

We define a map $\varphi_i : A_{\mathfrak{p}_i} \to k_i$ by

$$\frac{x}{s} \mapsto \frac{x_i}{s_i}$$

This is surjective of course, since we can choose $s_i = 1$. To show the injectivity, let $x_i/s_i = y_i/t_i$, then since k_i is a field, we have

$$x_i t_i = y_i s_i.$$

Let $e_i = (0, \dots, 1, \dots, 0)$, then $e_i \in A - \mathfrak{p}_i$, and

$$e_i(xt - ys) = x_i t_i - y_i s_i = 0$$

so $x/s = y/t$. Hence φ_j is an isomorphism. In particular, $A_{\mathfrak{p}_i}$ is an integral domain. □

Exercise 1.5.3. Let A be a ring and let Σ be the set of all multiplicatively closed subsets S of A such that $0 \notin S$. Show that Σ has maximal elements, and that $S \in \Sigma$ is maximal if and only if $A - S$ is a minimal prime ideal of A.

Moreover, by replace 0 with any ideal I, we can show $S \in \Sigma$ is maximal if and only if $A - \mathfrak{p}$ is a minimal prime ideal among those containing I.

Proof. By Zorn's lemma, Σ has a maximal element. For any set $S \in \Sigma$, it is clear that $A - S$ has the prime property. When S is maximal in Σ, we claim that $A - S$ is an ideal. Let $a \in A$, consider the smallest multiplicatively closed subsets containing a and S: When $a \in S$, it is S, and when $a \notin S$, it is

$$S_a := \{sa^n : s \in S, n \in \mathbb{N}\}$$

Now assume $a, b \in A - s$, then S_a and S_b both contains S properly. By the maximality, $0 \in S_a, S_b$, so there are $s_a, s_b \in S, m, n \in \mathbb{N}$ such that

$$s_a a^n = s_b b^m = 0$$

Now assume $a + b \in S$, then $s_a s_b(a + b) \in S$. Consider the power

$$(s_a s_b(a + b))^{m+n} = (s_a s_b)^{m+n}(a + b)^{m+n} = 0$$

since every term of the sum is zero. This is a contradiction since $(s_a s_b(a + b))^{m+n} \in S$. So $a + b \in A - S$.

Let $r \in A$ and $a \notin S$. If $ra \in S$, then $rs_a a \in S$, and $(rs_a a)^n = 0 \in S$, contradiction. Hence $A - S$ is a prime ideal. Since for any prime ideal $\mathfrak{p}$, $A - \mathfrak{p}$ is multiplicatively closed and does not contain (0), it follows that $A - S$ is minimal.

Now assume $\mathfrak{p}$ is a minimal prime ideal, then $A - \mathfrak{p} \in \Sigma$. By Zorn's lemma, there is a maximal elements $T \in \Sigma$ such that $A - \mathfrak{p} \subseteq T$. Then $A - T$ is prime and contained in $\mathfrak{p}$, by assumption we have $T = A - \mathfrak{p}$. So $A - \mathfrak{p}$ is maximal. $\square$

Exercise 1.5.4. Let A be an integral domain and M an A-module. An element $x \in M$ is a torsion element of M if $\mathrm{Ann}(x) \neq 0$, that is if x is killed by some non-zero element of A.

Show that the torsion elements of M form a submodule of M. This submodule is called the **torsion submodule** of M and is denoted by $T(M)$. If $T(M) = 0$, the module M is said to be torsion-free. Show that

(a) If M is any A-module, then $M/T(M)$ is torsion-free.

(b) If $f : M \to N$ is a module homomorphism, then $f(T(M)) \subseteq T(N)$.

(c) The functor T is left-exact.

(d) If M is any A-module, then $T(M)$ is the kernel of the map $x \mapsto 1 \otimes x$ of M into $K \otimes_A M$, where K is the field of fractions of A.

Proof. Let

$$0 \longrightarrow L \xrightarrow{f} M \xrightarrow{g} N$$

be an exact sequence. Consider the sequence

$$0 \longrightarrow T(L) \xrightarrow{T(f)} T(M) \xrightarrow{T(g)} T(N)$$

where $T(f) = f|_{T(M)}$. Clearly $T(g) \circ T(f) = 0$. Since we have $\ker T(f) \subseteq \ker f$, we see $T(f)$ is injective.

Let $m \in \ker T(g)$, then m is also in $\ker g$, so there is $\ell \in L$ such that $f(\ell) = m$. m is torsion, so there is $r \in A$ such that $rm = 0$. Then $f(r\ell) = rf(\ell) = rm = 0$, and $r\ell = 0$ since f is injective. This implies $\ell \in T(L)$, so $T(f)(\ell) = m$.

Let $S = A - \{0\}$, then we have

$$(K \otimes_A M = S^{-1}A \otimes_A M = A \otimes_{S^{-1}A} S^{-1}M \cong S^{-1}M).$$

Now $1 \otimes m$ is $m/1$ in $S^{-1}M$, so $1 \otimes m = 0$ if and only if there is $a \in A$ such that $am = 0$. □

Exercise 1.5.5. Let S be a multiplicatively closed subset of an integral domain A. Show that $T(S^{-1}M) = S^{-1}T(M)$. Deduce that the following are equivalent:

(a) M is torsion-free.

(b) $M_\mathfrak{p}$ is torsion-free for all prime ideals $\mathfrak{p}$.

(c) $M_\mathfrak{m}$ is torsion-free for all maximal ideals $\mathfrak{m}$.

Proof. Let $m/s \in T(S^{-1}M)$, then there is $y/t \in S^{-1}A$ such that $ym/st = 0$. Hence $\exists u \in S$ such that

$$uym = 0$$

This implies $m \in T(m)$, so $m/s \in S^{-1}T(M)$.

Let $m/s \in S^{-1}T(M)$. Then there is $r \in A : rm = 0$. Then $r/1 \cdot m/s = 0$ so $m/s \in T(S^{-1}M)$.

M is torsion free if and only if $T(M) = 0$. Use Proposition 1.3.18. □

Chapter 2

Graded rings and filtrations

2.1 Graduation on rings and modules

2.1.1 Graded ring and graded module

Definition 2.1.1. Given an abelian group G and a set Δ, a **graduation** of type Δ on G is a family $(G_\lambda)_{\lambda\in\Delta}$, of which G is the direct sum. The set G with the structure defined by its group law and the graduation is called a **graded group** of type Δ.

The set Δ is called the **set ef degrees** of G. An element $x \in G$ is called **homogeneous** if it belongs to one of the G_λ, **homogeneous of degree $\boldsymbol{\lambda}$** if $x \in G_\lambda$. The element 0 is therefore homogeneous of all degrees; but if $x \neq 0$ is homogeneous, it belongs to only one of the G_λ; the index λ such that $x \in G_\lambda$ is then called the degree of x (or sometimes the weight of x) and is sometimes denoted by $\deg(x)$. Every $x \in G$ may be written uniquely as a sum $\sum_\lambda x_\lambda$ of homogeneous elements with $x_\lambda \in G_\lambda$; y_λ is called the **homogeneous component** of degree λ, or simply the component of degree λ of x.

Example 2.1.2. Given any commutative monoid Δ with identity element 0 and abelian group G, a graduation $(G_\lambda)_{\lambda\in\Delta}$ is defined on G by taking $G_0 = G$ and $G_\lambda = \{0\}$ for $\lambda \neq 0$; this graduation is called **trivial**.

Example 2.1.3. Let Δ and Δ' be two sets and ρ a map of Δ into Δ'. Let $(G_\lambda)_{\lambda\in\Delta}$ be a graduation of type Δ on a commutative group G; for $\mu \in \Delta'$, let G_μ be the sum of the G_λ's such that $\rho(\lambda) = \mu$; clearly $(G_\mu)_{\mu\in\Delta'}$ is a graduation of type Δ' on G, said to be **derived** from (G_λ) by means of the map ρ.

When Δ is a commutative group written additively and ρ the map $\lambda \mapsto -\lambda$ of Δ onto itself, (G_μ) is called the **opposite graduation** of (G_λ).

Example 2.1.4. If $\Delta = \Delta_1 \times \Delta_2$ is a product of two sets, a graduation of type Δ is called a **bigraduation** of types Δ_1, Δ_2. For all $\lambda \in \Delta_1$, let $G_\lambda = \bigoplus_{\mu\in\Delta_2} G_{\lambda\mu}$; for all $\mu \in \Delta_2$, let $G_\mu = \bigoplus_{\lambda\in\Delta_1} G_{\lambda\mu}$; clearly $(G_\lambda)_{\lambda\in\Delta_1}$ is a graduation of type Δ_1 and $(G_\mu)_{\mu\in\Delta_2}$ is a graduation of type Δ_2 on G; these graduations are called the **partial graduations** derived from the bigraduation $(G_{\lambda\mu})$. Note that $G_{\lambda\mu} = G_\lambda \cap G_\mu$. Conversely, if $(G_\lambda)_{\lambda\in\Delta_1}$ and $(G_\mu)_{\mu\in\Delta_2}$ are two graduations on G such that G is the direct sum of the $G_{\lambda\mu} = G_\lambda \cap G_\mu$, these subgroups form a bigraduation of types Δ_1, Δ_2 on G, of which $(G_\lambda)_{\lambda\in\Delta_1}$ and $(G_\mu)_{\mu\in\Delta_2}$ are the partial graduations.

Example 2.1.5. Let Δ_0 be a commutative monoid written additively, with identity element denoted by 0; let I be any set and $\Delta_0^{\oplus I}$ denote the submonoid of the product set Δ_0^I consisting of the families $(\lambda_i)_{i\in I}$ of finite support. Let $\rho : \Delta \to \Delta_0$ be the surjective (codiagonal) homomorphism of Δ into Δ_0 defined by $\rho((\lambda_i)) = \sum_i \lambda_i$. From every graduation of type Δ a graduation of type Δ_0 is derived by means of ρ; it is called the **total graduation** associated with the given "multigraduation" of type Δ.

Often the set Δ may also endow some algebraic structure and one want to have operations between different components of the graduation (G_λ). In the following, we may always assume that Δ is a commutative monoid, and consider graduations of type Δ on rings and modules.

Definition 2.1.6. Let Δ be a commutative moniod. Given a ring A and a graduation (A_λ) of type Δ on the additwe group A, this graduation is said to be **compatible** with the ring structure on A if

$$A_\lambda A_\mu \subseteq A_{\mu+\lambda} \quad \text{for all } \lambda, \mu \in \Delta.$$

The ring A with this graduation is called a **graded ring** of type Δ.

Proposition 2.1.7. *If the elements of Δ are cancellable and (A_λ) is a graduation of type Δ compatible with the structure of a ring A, then A_0 is a subring of A.*

Proof. As $A_0 A_0 \subseteq A_0$, by definition, it suffices to prove that $1 \in A_0$. Let $1 = \sum_{\lambda\in\Delta} e_\lambda$ be the decomposition of 1 into its homogeneous components. If $x \in A_\mu$ then $x = x \cdot 1 = \sum_\lambda x e_\lambda$; comparing the components of degree μ, (since every element in Δ is cancellable) we get $x = xe_0$. Since this relation is true for every homogeneous element of A, it is true for all $x \in A$; in particular $1 = 1 \cdot e_0 = e_0 \in A_0$. □

Definition 2.1.8. Let A be a graded ring of type Δ, (A_λ) its graduation and M a left A-module; a graduation (M_λ) of type Δ on the addition group M is compatible with the A-module structure on M if

$$A_\lambda M_\mu \subseteq M_{\lambda+\mu} \quad \text{for all } \lambda, \mu \in \Delta.$$

The module M with this graduation is then called a **graded module** of type Δ over the graded ring A.

When the elements of Δ are cancellable, it follows from Proposition 2.1.7 that the M_λ are A_0-modules. Clearly if A is a graded ring of type Δ, the A is itself a left graded A-module of type Δ.

Let A be a graded ring and M be a graded A-module. An element $x \in M$ is **homogeneous** if $x \in M_\lambda$ for some λ, and λ is then called the **degree** of x. A general element $x \in M$ can be written uniquely in the form $x = \sum_\lambda x_\lambda$ with $x_i \in M_\lambda$; x_λ is called the **homogeneous term of $\boldsymbol{x}$ of degree $\boldsymbol{\lambda}$**.

Example 2.1.9. On any ring A the trivial graduation of type Δ is compatible with the ring structure. If A is graded by the trivial graduation, for a graduation (M_λ) of type Δ on an A-module M to be compatible with the A-module structure, it is necessary and sufficient that the M_λ be submodules of M.

Example 2.1.10. Let A be a graded ring of type Δ, M a graded A-module of type Δ and ρ a homomorphism of Δ into a commutative monoid Δ' whose identity element is denoted by 0. Then A is a graded ring of type Δ' and M a graded module of type Δ' for the graduations of type Δ' derived from ρ and the graduations of type Δ on A and M: this follows immediately from the relation $\rho(\lambda + \mu) = \rho(\lambda) + \rho(\mu)$.

In particular, if $\Delta = A_1 \times A_2$ is a product of two commutative monoids, the projections π_1 and π_2 are homomorphisms and the corresponding graduations are just the parlial graduations derived from the graduations of type Δ; these partial graduations are thus compatible with the ring structure of A and the module structure of M. Similarly, if $\Delta = \Delta_0^{\oplus I}$ (where Δ_0 is a commutative monoid with identity element denoted by 0), the total graduation of type Δ_0 derived from the graduation of type Δ on A (resp. M) by means of the codiagonal homomorphism is compatible with the ring structure on A (resp. with the module structure on M).

Example 2.1.11. Let A be a graded ring of type Δ, M a graded A-module of type Δ and A_0 an element of Δ; for $\lambda \in \Delta$, let $M'_\lambda = M_{\lambda+\lambda_0}$, and let M' be the submodule $M' = \bigoplus_{\lambda\in\Delta} M'_\lambda$ is an A-module and the M'_λ form on M' a graduation of type A compatible with the A-module structure of M'; the graded A-module M' of type Δ thus defined is said to be obtained by shifting by λ_0 the graduation of M and it is denoted by $M(\lambda_0)$. When Δ is a group, the underlying A-module of the graded A-module M' is identified with M since in this case $\lambda \mapsto \lambda + \lambda_0$ is an isomorphism of Δ onto itself.

Example 2.1.12. Let A be a ring. The polynomial ring $A[X]$ in one indeterminate is graded of type $\mathbb{N}$ by the subgroups AX^n. Similarly, the polynomial ring $A[X_1, \dots, X_n]$ is graded of type $\mathbb{N}$ by the subgroups of homogeneous polynomials.

The graduations most often used are of type $\mathbb{Z}$ or of type $\mathbb{Z}^n$; when we speak of graded (resp. bigraded, trigraded, etc.) modules and rings without mentioning the type, it is understood that we mean graduations of type $\mathbb{Z}$ (resp. $\mathbb{Z}^2$, $\mathbb{Z}^3$ etc.); a graded ring (resp. module) of type $\mathbb{N}$ is also called a **graded ring (resp. module) with positive degrees**.

Now we introduce the morphisms between graded rings and modules.

Definition 2.1.13. Let A, B be two graded rings with the same type Δ. and (A_λ), (B_λ) their respective graduations. A ring homomorphism $\rho : A \to B$ is called **graded** if $\rho(A_\lambda) \subseteq B_\lambda$ for all $\lambda \in \Delta$.

Let M, N be two graded modules of type Δ over a graded ring A of type Δ. Let $\phi : M \to N$ be an A-homomorphism and δ an element of Δ; ϕ is called **graded of degree δ** if $\phi(M_\lambda) \subseteq N_{\lambda+\delta}$ for all $\lambda \in \Delta$.

An A-homomorphism $\phi : M \to N$ is called **graded** if there exists $\delta \in \Delta$ such that ϕ is graded of degree δ. If $\phi \neq 0$ and every element of Δ is cancellable, then the degree δ of ϕ is uniquely determined.

If $\rho : A \to B$ and $\nu : B \to C$ are two graded homomorphisms of graded rings of type Δ, so is $\nu \circ \rho : A \to B$; for a map $\rho : A \to B$ to be a graded ring isomorphism, it is necessary and sufficient that ρ be bijective and that ρ and the inverse map ρ^{-1} be graded homomorphisms. It also suffices for this that f be a bijective graded homomorphism. Thus it is seen that graded homomorphisms can be taken as the morphisms of the category of graded ring structure of type Δ.

Similarly, if $\phi : M \to N$ and $\psi : N \to P$ are two graded homomorphisms of graded A-modules of type Δ of respective degrees δ and γ, $\psi \circ \phi : M \to P$ is a graded homomorphism of degree $\delta + \gamma$. If δ admits an inverse $-\delta$ in Δ and $\phi : M \to N$ is a bijective graded homomorphism of degree δ, the inverse map $\phi^{-1} : N \to M$ is a bijective graded homomorphism of degree $-\delta$. It follows as above that the graded homomorphisms of degree 0 can be taken as the morphisms of the category of graded A-module of type Δ.

Example 2.1.14. If M is a graded A-module and $M(\lambda_0)$ is a graded A-module obtained by shifting, the $\mathbb{Z}$-linear map of $M(\lambda_0)$ into M which coincides with the canonical injection on each $M_{\lambda+\lambda_0}$ is a graded homomorphism of degree λ_0 (which is bijective when Δ is a group).

Example 2.1.15. If a is a homogeneous element of degree δ belonging to the centre of A, the homothety $x \mapsto ax$ of any graded A-module M is a graded homomorphism of degree δ.

Remark 2.1.16. A graded A-module M is called a graded free A-module if there exists a basis $(x_i)_{i\in I}$ of M consisting of homogeneous elements. Suppose it is and Δ is a commutative group; let λ_i be the degree of x_i and consider for each i the shifted A-module $A(-\lambda_i)$; if e_i denotes the element 1 of A considered as an element of degree λ_i in $A(-\lambda_i)$, the A-linear map $\phi : \bigoplus_{i\in I} A(-\lambda_i) \to M$ such that $\phi(e_i) = x_i$ for all i is a graded A-module isomorphism.

Example 2.1.17. A direct system $(A_\alpha, f_{\beta\alpha})$ of graded rings of type Δ is a direct system of rings such that each A_α is graded of type Δ and each $f_{\beta\alpha}$ is a homomorphism of graded ring. If (A_α^λ) is the graduation of A_α and we write

$$A = \varinjlim A_\alpha, \quad A^\lambda = \varinjlim A_\alpha^\lambda,$$

it follows that (A^λ) is a graduation of A and this graduation is compatible with the ring structure on A. The graded ring A is called the **direct limit** of the direct system of graded rings $(A_\alpha, f_{\beta\alpha})$. If $f_\alpha : A_\alpha \to A$ is the canonical map, then f_α is a homomorphism of graded rings.

Similarly, a direct system $(M_\alpha, \phi_{\beta\alpha})$ of graded A-modules of type Δ is a direct system of A-modules such that each M_α is graded of type Δ and each $\phi_{\beta\alpha}$ is a homomorphism of graded A-modules of degree 0. If (M_α^λ) is the graduation of M_α and

$$M = \varinjlim M_\alpha, \quad M^\lambda = \varinjlim M_\alpha^\lambda,$$

then (M^λ) is a graduation of M and this graduation is compatible with the module structure on M. The graded module M is called the **direct limit** of the direct system of graded A-modules $(M_\alpha, \phi_{\beta\alpha})$. If $\phi_\alpha : M_\alpha \to M$ is the canonical map, then ϕ_α is a homomorphism of graded modules of degree 0.

Proposition 2.1.18. *Let A be a graded ring of type Δ, M a graded A-module of type Δ, (M_λ) its graduation and N a sub-A-module of M. The following properties are equivalent:*

(i) $N = \bigoplus_\lambda (N \cap M_\lambda)$.

(ii) The homogeneous components of every element of N belongs to N

(iii) N can be generated by homogeneous elements.

Proof. Every element of N can be written uniquely as a sum of elements of the M_λ, and hence it is immediate that (i) and (ii) are equivalent and that (i) implies (iii). We show that (iii) implies (ii). Then let $(x_i)_{i\in I}$ be a family of homogeneous generators of N and let δ_i be the degree of x_i. Every element of N can be written as $\sum_i a_i x_i$ with $a_i \in A$; if $a_{i\mu}$ is the component of a_i of degree μ, the conclusion follows from the relation

$$\sum_{i\in I}\Big(\sum_{\mu\in\Delta} a_{i\mu}x_i\Big) = \sum_{\mu\in\Delta}\Big(\sum_{\mu+\delta_i=\lambda} a_{i\mu}x_i\Big).$$

This completes the proof. □

When a submodule N of M has the equivalent properties stated in Proposition 2.1.18, clearly the $N \cap M_\lambda$ form a graduation compatible with the A-module structure of N, called the graduatuin imluced by that on M; N with this graduation is called a graded submodule of M.

Corollary 2.1.19. *If N is a graded submodule of M and (x_i) is a generating system of N, then the homogeneous componmts of the (x_i) form a genetating system of N.*

Corollary 2.1.20. *If N is a finitely generated submodule of M, then N admits a finite generating system consists of homogeneous elements.*

A graded submodule of A is called a **graded ideal** of the graded ring A. For every subring B of A, if we set $B_\lambda = B \cap A_\lambda$ then $B_\lambda B_\mu \subseteq B_{\lambda+\mu}$, thus the graduation induced on B by that on A is compatible with the ring structure on B; B is then called a **graded subring** of A.

If N is a graded submodule of a graded A-module M and $(M_\lambda)_{\lambda\in\Delta}$ is the graduation of M, the submodules $(M_\lambda + N)/N$ of M/N form a gnoduation compatible with the structure of this quotient module. For, if $N_\lambda = M_\lambda \cap N$, then $(M_\lambda + N)/N$ is identified with M_λ/N_λ and it follows from Proposition 2.1.18 that M/N is their direct sum. Moreover,

$$A_\lambda(M_\mu + N) \subseteq A_\lambda M_\mu + N \subseteq M_{\lambda+\mu} + N$$

which establishes our assertion. The graduation $((M_\lambda + N)/N)_{\lambda\in\Delta}$ is called the **quotient graduation** of that on M by N and the quotient module M/N with this graduation is called the **graded quotient module** of M by the graded submodule N; the canonical homomorphism $\pi : M \to M/N$ is a graded homomorphism of degree 0 for this graduation.

If $\mathfrak{a}$ is a graded ideal of A, the quotient graduation on $A/\mathfrak{a}$ is compatible with the ring structure on $A/\mathfrak{a}$. The ring $A/\mathfrak{a}$ with this graduation is called the **quotient graded ring** of A by $\mathfrak{a}$. The canonical homomorphism $\pi : A \to A/\mathfrak{a}$ is a homomorphism of graded rings for this graduation.

Proposition 2.1.21. *Let A be a graded ring of type Δ, M and N two graded A-modules of type Δ and $\phi : M \to N$ a graded A-homomorphism of degree δ. Then:*

(a) $\operatorname{im}\phi$ *is a graded submodule of N.*

(b) If δ is a regular element of Δ, then $\ker\phi$ *is a graded submodule of M.*

(c) If $\delta = 0$, the bijection $M/\ker\phi \to \operatorname{im}\phi$ canonically associated with ϕ is an isomorphism of graded modules.

Proof. Assertion (a) follows immediately from the definitions and Proposition 2.1.18(iii). If x is an element of M such that $\phi(x) = 0$ and $x = \sum x_\lambda$ is its decomposition into homogeneous components (where x_λ is of degree λ), then

$$\sum_\lambda \phi(x_\lambda) = \phi(x) = 0$$

and $\phi(x_\lambda)$ is of degree $\lambda + \delta$; if δ is regular the relation $\lambda + \delta = \mu + \delta$ implies $\lambda = \mu$, hence the $\phi(x_\lambda)$ are the homogeneous components of $\phi(x)$ and necessarily $\phi(x_\lambda) = 0$ for all $\lambda \in \Delta$, which proves (b). The bijection $\pi : M/\ker\phi \to \operatorname{im}\phi$ canonically associated with ϕ is then a graded homomorphism of degree δ, as follows from the definition of the quotient graduation; whence (c) when $\delta = 0$. □

Corollary 2.1.22. *Let A, B be two graded rings of type Δ and $\rho : A \to B$ a graded homomorphism of graded rings. Then* $\operatorname{im}\rho$ *is a graded subring of B,* $\ker\rho$ *a graded ideal of type Δ and the bijection $A/\ker\rho \to \operatorname{im}\rho$ canonically associated with ρ is an isomorphism of graded rings.*

Proof. It suffices to apply Proposition 2.1.21 to ρ considered as a homomorphism of degree 0 of graded $\mathbb{Z}$-modules. □

Proposition 2.1.23. *Let A be a graded ring of type Δ and M a graded A-module of type Δ.*

(a) Every sum and every intersection of graded submodules of M is a graded submodule.

(b) If x is a homogeneous element of M of degree μ which is cancellable in Δ, then the annihilator of x is a graded ideal of A.

(c) If the elements of Δ are cancellable, the annihilator ef a graded submodule of M is a graded ideal of A.

Proof. If (N_i) is a family of graded submodules of M, property (iii) of Proposition 2.1.18 shows that the sum of the N_i is generated by homogeneous elements and property (ii) of Proposition 2.1.18 proves that the homogeneous components of every element of $\bigcap_i N_i$ belongs to $\bigcap_i N_i$; whence (a).

To prove (b), it suffices to note that $\mathrm{Ann}(x)$ is the kernel of the homomorphism $a \mapsto ax$ of the A-module A into M and that this homomorphism is graded of degree μ; the conclusion follows from Proposition 2.1.21(b). Finally (c) is a consequence of (a) and (b) for the annihilator of a graded submodule N of M is the intersection of the annihilators of the homogeneous elements of N, by virtue of Proposition 2.1.18. □

Example 2.1.24. Let M be a graded A-module and E a submodule of M; it follows from Proposition 2.1.23(a) that there exists a largest graded submodule N_1 of M contained in E and a smallest graded submodule N_2 of M containing E; N_1 is the set of $x \in E$ all of whose homogeneous components belong to E and N_2 is the submodule of M generated by the homogeneous components of a generating system of E.

Let Δ be a commutative monoid, A a graded ring of type Δ, $(A_\lambda)_{\lambda\in\Delta}$ its graduation and E an A-algebra. A graduation $(E_\lambda)_{\lambda\in\Delta}$ of type Δ on the additive group E is said to be **compatible** with the A-algebra structure on E if it is compatible both with the A-module and with the ring structure on E, in other words, if, for all λ, μ in Δ,

$$A_\lambda E_\mu \subseteq E_{\lambda+\mu}, \quad E_\lambda E_\mu \subseteq E_{\lambda+\mu}. \tag{2.1.1}$$

The A-algebra E, with this graduntion, is then called a **graded algebra** of type Δ over the graded ring A.

When the graduation on A is trivial (that is, $A_0 = A$, $A_\lambda = \{0\}$ for $\lambda \neq 0$), condition (2.1.1) means that the E_λ are sub-A-modules of E. This leads to the definition of the notion of graded algebra of type Δ over a non-graded commutative ring A: A is given the trivial graduation of type Δ and the above definition is applied.

When we consider graded A-algebras E with a unit element 1, it will always be understood that 1 is of degree 0. It follows that if an invertible element $x \in E$ is homogeneous and of degree p, its inverse x^{-1} is homogeneous and of degree $-p$: it suffice to decompose x^{-1} as a sum of homogeneous elements in the relation $x^{-1}x = xx^{-1} = 1$.

Let E and F be two graded algebras of type Δ over a graded ring A of type Δ. An A-algebra homomorphism $u : E \to F$ is called a **graded algebra homomorphism** is $u(E_\lambda) \subseteq F_\lambda$ for all $\lambda \in \Delta$ (where (E_λ) and (F_λ) denote the respective graduations of E and F); where E and F are associative and unital and u is unital, this condition means that u is a graded ring homomorphism.

Remark 2.1.25. Let E be a graded A-algebra of type $\mathbb{N}$. Then E often is identified with a graded A-algebra of type $\mathbb{Z}$ by writing $E_n = \{0\}$ for $n < 0$.

Remark 2.1.26. The definition can also be interpreted by saying that E is a graded A-module and that A-linear map

$$m : E \otimes_A E \to E$$

defining the multiplication on E, is homogeneous of degree 0 when $E \otimes_A E$ is given its graduation of type Δ.

To define a graded A-algebra structure of type Δ on the graded ring A, with E as underlying graded A-module, therefore amounts to defining for each ordered pair (λ, μ) of elements of Δ a $\mathbb{Z}$-bilinear map

$$m_{\lambda\mu} : E_\lambda \times E_\mu \to E_{\lambda+\mu}$$

such that for every triple of indices (λ, μ, ν) and for $\alpha \in A_\lambda$, $x \in E_\mu$, $y \in E_\mu$, we have $\alpha m_{\mu\nu}(x, y) = m_{\lambda+\mu,\nu}(\alpha x, y) = m_{\mu,\lambda+\nu}(x, \alpha y)$.

Example 2.1.27.

(a) Let B be a graded ring of type Δ; if B is given its canonical $\mathbb{Z}$-algebra structure, B is a graded A-algebra ($\mathbb{Z}$ being given the trivial graduation).

(b) Let M be a magma and $\phi : M \to \Delta$ a homomorphism. For all $\lambda \in \Delta$, we write $M_\lambda = \phi^{-1}(\lambda)$; then $M_\lambda M_\nu \subseteq M_{\lambda+\mu}$. Let A be a graded commutative ring of type Δ and $(A_\lambda)_{\lambda\in\Delta}$ its graduation; we shall define a graded A-algebra structure on the algebra $E = A^{\oplus M}$ of the magma M. To this end, let E_λ denote the additive subgroup of E generated by the elements of the form αm such that $\alpha \in A_\mu$, $m \in M_\nu$, and $\mu + \nu = \lambda$. As the M_λ are pairwise disjoint, E is the direct sum of the $A_\mu M_\nu$ and hence is the direct sum of E_λ and it is immediate that E_λ satisfy the condition (2.1.1). Therefore it defines on E the desired graded A-algebra structure. If M admits an identity element e, it may also be supposed that $\phi(e) = 0$. A particular case is the one where the graduation of the ring A is trivial; then E_λ is the sub-A-module of E generated by M_λ. More particularly, if we take $M = \mathbb{N}^{\oplus I}$, $\Delta = \mathbb{N}$ and ϕ the map such that $\phi((n_i)) = \sum_i n_i$, the ring A having the trivial graduation, a graduation is thus obtained on the polynomial algebra $A[X_i]_{i\in I}$, for which the degree of a homogeneous polynomial is the total degree.

We now take M to be the free monoid $\Delta(X)$ of a set X and ϕ the homomorphism $\Delta(X) \to \mathbb{N}$ which associates with each word its length. Thus a graded A-algebra structure is obtained on the free associated algebra of the set X.

Let E be a graded algebra of type Δ over a graded ring A of type Δ. If F is a sub-A-algebra of E which is a graded sub-A-module, then the graduation (F_λ) on F is compatible with its A-algebra structure, since $F_\lambda = F \cap E_\lambda$; in this case F is called a graded subalgebra of E and the canonical injection $F \to E$ is a graded algebra homomorphism.

Similarly, if $\mathfrak{a}$ is a left (resp. right) ideal of E which is a graded sub-A-module, then $E_\lambda\mathfrak{a}_\mu \subseteq \mathfrak{a}_{\lambda+\mu}$ (resp. $\mathfrak{a}_\lambda E_\mu \subseteq \mathfrak{a}_{\lambda+\mu}$), since $\mathfrak{a}_\lambda = \mathfrak{a} \cap E_\lambda$; then $\mathfrak{a}$ is called a graded ideal of the algebra E. If $\mathfrak{b}$ is a graded two-sided ideal of E the quotient graduation on the module $E/\mathfrak{b}$ is compatible with the algebra structure on $E/\mathfrak{b}$ and the canonical homomorphism $E \to E/\mathfrak{b}$ is a graded algebra homomorphism.

If $u : E \to F$ is a graded algebra homomorphism, then $\mathrm{im}(u)$ is a graded sub-algebra of F and $\ker u$ is a graded two-sided ideal of E and the bijection $E/\ker u \to \mathrm{im}\, u$ canonically associated with u is a graded algebra isomorphism.

Proposition 2.1.28. *Let A be a graded commutative ring of type Δ, E a graded A-algebra of type Δ and S a set of homogeneous elements of E. Then the sub-A-algebra (resp. left ideal, right ideal, two-sided ideal) generated by S is a graded subalgebra (resp. graded ideal).*

Proof. The subalgebra of E generated by S is the sub-A-module generated by the finite products of elements of S, which are homogeneous. Simialrly, the left (resp. right) ideal generated by S is the sub-A-module generated by the elements of the form $u_1(u_2(\cdots(u_n s))\cdots)$ (resp. $(\cdots((su_n)u_{n-1})\cdots)u_2)u_1$ where $s \in S$ and the $u_j \in E$ are homogeneous and these products are homogeneous, whence in the case the conclusion by Proposition 2.1.18. Finally, the two-sided ideal generated by S is the union of the sequence $(\mathfrak{J}_n)_{n\geq 1}$, where $\mathfrak{J}_1$ is the left ideal generated by S and $\mathfrak{J}_{2n}$ (resp. $\mathfrak{J}_{2n+1}$) is the right (resp. left) ideal generated by $\mathfrak{J}_{2n-1}$ (resp. $\mathfrak{J}_{2n}$), which completes the proof. □

Let $(A_\alpha, \phi_{\beta\alpha})$ is a directed direct system of graded commutative rings of type Δ and for each α let E_α be a graded A_α-algebra of type Δ; for $\alpha \leq \beta$ let $f_{\beta\alpha} : E_\alpha \to E_\beta$ be an A_α-homomorphism of graded algebras and suppose that $f_{\gamma\alpha} = f_{\gamma\beta} \circ f_{\beta\alpha}$ for $\alpha \leq \beta \leq \gamma$; then we shall write $(E_\alpha, f_{\beta\alpha})$ a directed direct system of graded algebras of type Δ over the directed direct system $(A_\alpha, \phi_{\beta\alpha})$ of graded commutative rings of type Δ. Then we know that $E = \varinjlim E_\alpha$ has canonically a graded module structure of type Δ over the graded ring $A = \varinjlim A_\alpha$ and a multiplication such that $E^\lambda E^\mu \subseteq E^{\lambda+\mu}$ (where (E^λ) denotes the graduation on E); then this multiplication and the graded A-module structure on E define on E a graded A-algebra structure of type Δ. The set E with this structure is called a direct limit of the direct system $(E_\alpha, f_{\beta\alpha})$ of graded algebras. Moreover, if F is a graded A-algebra of type Δ and (u_α) a direct system of A_α-homomorphism $u_\alpha : E_\alpha \to F$, $u = \varinjlim u_\alpha$ is an A-homomorphism of graded algebras.

2.1.2 Tensor products and Hom sets

Let Δ be a commutative monoid with its identity element denoted by 0, A a graded ring of type Δ and M, N graded A-modules of type Δ. Let (A_λ) (resp. (M_λ), (N_λ)) be the graduation of A (resp. M, N); the tensor product $M \otimes_{\mathbb{Z}} N$ of the $\mathbb{Z}$-modules M and N is the direct sum of the $M_\lambda \otimes_{\mathbb{Z}} N_\mu$ and hence the latter form a bigraduation of types Δ, Δ on this $\mathbb{Z}$-module. Consider on $M \otimes_{\mathbb{Z}} N$ the total graduation of type Δ associated with this bigraduation; it consists of the sub-$\mathbb{Z}$-modules $P_\lambda = \sum_{\mu+\nu=\lambda}(M_\mu \otimes_{\mathbb{Z}} N_\nu)$. It is known that the $\mathbb{Z}$-module $M \otimes_A N$ is the quotient of $M \otimes_{\mathbb{Z}} N$ by the sub-$\mathbb{Z}$-module Q generated by the elements $(ax) \otimes y - x \otimes (ay)$, where $x \in M$, $y \in N$ and $a \in A$; if, for all $\lambda \in \Delta$, x_λ, y_λ, a_λ are the homogeneous components of degree λ of x, y, a respectively, clearly $(ax) \otimes y - x \otimes (ay)$ is the sum of the homogeneous elements $(a_\nu x_\mu) \otimes y_\lambda - x_\mu \otimes (a_\nu y_\lambda)$, in other words Q is a graded sub-$\mathbb{Z}$-module of $M \otimes_{\mathbb{Z}} N$ and the quotient

$$M \otimes_A N = (M \otimes_{\mathbb{Z}} N)/Q$$

therefore has canonically a graded $\mathbb{Z}$-module structure of type Δ. Moreover, the graduation which we have just defined on $M \otimes_A N$ is compatible with its A-module structure. For if $x \in M_\lambda$, $y \in N_\mu$, $a \in A_\nu$, the element $a(x \otimes y)$ belong to $(M \otimes_{\mathbb{Z}} N)_{\lambda+\mu+\nu}$ and hence its image in $M \otimes_A N$ belongs to $(M \otimes_A N)_{\lambda+\mu+\nu}$, which establishes our assertion. When we speak of $M \otimes_A N$ as a graded A-module, we always mean with the structure thus defined, unless otherwise mentioned. Note that $(M \otimes_A N)$ can be defined as the additive group of $M \otimes_A N$ generated by the $x_\mu \otimes y_\nu$, where $x_\mu \in M_\mu$, $y_\nu \in N_\nu$ and $\mu + \nu = \lambda$.

Let M' (resp. N') be another graded A-module and $\phi : M \to M'$, $\psi : N \to N'$ graded homomorphisms of respective degrees α and β. Then it follows immediately from the above remark that $\phi \otimes \psi$ is a graded A-module homomorphism of degree $\alpha + \beta$.

Similarly, a graduation (compatible with the A-module structure) is similarly defined on the tensor product of any finite number of graded A-modules; it is moreover immediate that the associativity isomorphisms such as $(M \otimes N) \otimes P \to M \otimes (N \otimes P)$ are isomorphisms of graded modules.

Proposition 2.1.29. *Let M and N be graded A-modules of type Δ, P a graded A-module of type Δ and let f be a A-bilinear map of $M \times N$ to P such that*

$$f(x_\lambda, y_\mu) \in P_{\lambda+\mu} \quad \textit{for } x_\lambda \in M_\lambda,\, y_\nu \in N_\mu,\, \lambda, \mu \in \Delta.$$

Then $f(x, y) = \tilde{f}(x \otimes y)$ where $\tilde{f} : M \otimes_A N \to P$ is a graded A-module homomorphism of degree 0.

Proof. It is clear that $\tilde{f}$ is A-linear, and the given condition implies that it is a homomorphism of graded modules. □

Let B be another graded ring of type Δ and $\rho : A \to B$ a homomorphism of graded rings; then B is a graded A-module of type Δ via the map ρ. If E is a graded A-module of type Δ and $B \otimes_A E$ is given the graded A-module structure of type Δ defined above, the canonical B-module structure is compatible with the graduation of

$$\rho_!(E) = B \otimes_A E.$$

The graded B-module thus obtained is said to be obtained by extending the ring of scalars to B by means of f and when we speak of $\rho_!(E)$ as a graded B-module, we always mean this structure, unless otherwise mentioned.

Now we define graduations on the Hom sets. For simplicity we suppose that Δ is a group. Let A be a graded ring of type Δ and M, N two graded A-modules of type Δ. Let H_λ denote the additive group of graded homomorphisms of degree λ of M into N; in the additive group $\mathrm{Hom}_A(M,N)$ of all homomorphisms of M into N (with the un-graded A-module structures) the sum (for $\lambda \in \Delta$) of the H_λ is direct. For, if there is a relation $\sum_\lambda \phi_\lambda = 0$ with $\phi_\lambda \in H_\lambda$ for all λ, it follows that $\sum_\lambda \phi_\lambda(x_\mu) = 0$ for all μ and all $x_\mu \in M_\mu$. As the elements of Δ are cancellable, $\phi_\lambda(x_\mu)$ is the homogeneous component of $\sum_\lambda \phi_\lambda(x_\mu)$ of degree $\lambda + \mu$; hence $\phi_\lambda(x_\mu) = 0$ for every ordered pair (μ, λ) and every $x_\mu \in M_\mu$, which implies $\phi_\lambda = 0$ for all $\lambda \in \Delta$. We shall denote (in this paragraph) by HomHomgr$_A(M,N)$ the additive subgroup of $\mathrm{Hom}_A(M,N)$ the sum of the H_λ and we shall call it the additive group of graded A-module homomorphisms of M into N. For the canonical A-module structure on $\mathrm{Hom}_A(M,N)$, $\mathrm{Homgr}_A(M,N)$ is a submodule and the graduation (H_λ) is compatible with the A-module structure: for if $a_\nu \in A_\nu$, $x_\mu \in N_\nu$ and $\phi_\lambda \in H_\lambda$, then by definition $(a_\nu\phi_\lambda)(x_\mu) = a_\nu\phi_\lambda(x_\mu) \in N_{\lambda+\mu+\nu}$ and hence $a_\nu\phi_\lambda \in H_{\lambda+\nu}$.

Let M' and N' be two graded A-modules of type Δ. and $\phi : M' \to M$, $\psi : N \to N'$ graded homomorphisms of respective degrees α and β. Then it is immediate that $\mathrm{Hom}(\phi, \psi) : \eta \mapsto \psi \circ \eta \circ \phi$ maps $\mathrm{Homgr}_A(M,N)$ into $\mathrm{Homgr}_A(M',N')$ and that its restriction to $\mathrm{Homgr}_A(M,N)$ is a graded homomorphism into $\mathrm{Homgr}_A(M',N')$ of degree $\alpha + \beta$. In particular, $\mathrm{Homgr}_A(M,M)$ is a graded subring of $\mathrm{End}_A(M)$, which is denoted by $\mathrm{Homgr}_A(M)$.

If M and N are graded A-modules, the set $\mathrm{Homgr}_A(M,N)$ is in general distinct from $\mathrm{Hom}_A(M,N)$. However there is a special case where these tow sets equal.

Proposition 2.1.30. *Let A be a graded ring of type Δ and M a finitely generated graded A-module of type Δ. Then for any graded A-module N of type Δ, we have* $\mathrm{Homgr}_A(M,N) = \mathrm{Hom}_A(M,N)$.

Proof. Assume that M is generated by homogeneous elements $x_1 \dots, x_n$; let d_i be the degree of x_i. Let $\phi \in \mathrm{Hom}_A(M,N)$ and for all $\lambda \in \Delta$ let $z_{i,\lambda}$ denote the homogeneous component of $\phi(x_i)$ of degree $\lambda + d_i$. We show that there exists a homomorphism $\phi_\lambda : M \to N$ such that $\phi_\lambda(x_i) = z_{i,\lambda}$. for all i. For this, it suffices to prove that if $\sum_i a_i x_i = 0$ with $a_i \in A$ for $1 \le i \le n$, then $\sum_i a_i z_{i,\lambda} = 0$ for all $\lambda \in \Delta$, for then the formula $\sum_i a_i x_i \mapsto \sum_i a_i z_{i,\lambda}$ defines a map and satisfies the condition. To this end, it can be assumed that each a_i is homogeneous of degree δ_i such that $d_i + \delta_i = \mu$ for all i; then $\sum_i a_i \phi(x_i) = 0$; taking the homogeneous component of degree $\lambda + \mu$ on the left-hand sider, we obtain $\sum_i a_i z_{i,\lambda} = 0$, whence the existence of the homomorphism ϕ_λ; clearly ϕ_λ is graded of degree λ. Finally, $\phi_\lambda = 0$ except for a finite number of values of λ, and $\phi = \sum_\lambda \phi_\lambda$. by definition, which proves our assertion. □

In particular, $\mathrm{Homgr}_A(A,M) = \mathrm{Hom}_A(A,M)$ for every graded A-module M; moreover $\mathrm{Hom}_A(A,M)$ has a graded A-module structure, and it is immediate that with this structure the canonical map of M into $\mathrm{Hom}_A(A,M)$ is a graded A-module isomorphism.

Similarly, $\mathrm{Homgr}_A(M,A)$ has a graded A-module structure; it is called the **graded dual** of the graded A-module M and is denoted by $M^{*\mathrm{gr}}$, or simply M^* when no confusion results. If $\phi : M \to N$ is a graded homomorphism of degree δ, it follows from the above that the restriction to $N^{*\mathrm{gr}}$ of ϕ^* is a graded homomorphism of the graded dual $N^{*\mathrm{gr}}$ into the graded dual $M^{*\mathrm{gr}}$, of degree δ, called the graded dual of ϕ.

Let M, N, P, Q be graded A-modules of type Δ. Then there are canonical graded homomorphisms of degree 0:

$$\mathrm{Homgr}_A(M, \mathrm{Homgr}_A(N,P)) \to \mathrm{Homgr}_A(M \otimes_A N, P)$$

$$\mathrm{Homgr}_A(M,N) \otimes_A P \to \mathrm{Homgr}_A(M, N \otimes_A P)$$

$$\mathrm{Homgr}_A(M,P)\otimes\mathrm{Homgr}_A(N,Q)\to\mathrm{Homgr}_A(M\otimes_A N, P\otimes_A Q)$$

(the tensor products being given the graduations defined above) obtained by restricting the canonical homomorphisms for Hom sets; for, if $\phi : M \to \mathrm{Homgr}_A(N,P)$ is graded of degree δ, then, for all $x \in M_\lambda$, $\phi(x)$ is a graded homomorphism $N \to P$ of degree $\delta+\lambda$ and hence, for $y \in N_\mu$, $\phi(x)(y) \in P_{\delta+\lambda+\mu}$; if $\psi : M\otimes_A N \to P$ corresponds canonically to ϕ, it is then seen that ψ is a graded homomorphisms of degree δ, whence our assertion concerning the first map; moreover it is seen that this homomorphism is bijective. The argument is similar for the other maps.

2.1.3 Graduation by an ordered group

An order structure (denoted by $\leq$) on a commutative group Δ written additively is said to be compatible with the group structure if, for all $\lambda,\mu,\rho \in \Delta$, the relation $\lambda \leq \mu$ implies $\lambda+\rho \leq \mu+\rho$. The group Δ with this order structure is then called an ordered group.

Let Δ be an ordered commutative group, A a graded ring of type Δ and (A_λ) its graduation. We say A is a **graded ring with positive degrees** if $A_\lambda = \{0\}$ for $\lambda < 0$. In this case, it follows from definition that $A_+ = \bigoplus_{\lambda>0} A_\lambda$ is a graded ideal of A.

Proposition 2.1.31. *Let Δ be an ordered group, A a graded ring of type Δ with positive degrees, (A_λ) its gradualion, M a graded A-module of type Δ and (M_λ) its graduation. Suppose that there exists λ_0 such that $M_{\lambda_0} \neq \{0\}$ and $M_\lambda = \{0\}$ for $\lambda < \lambda_0$. Then $A_+M \neq M$.*

Proof. Let x be a non-zero element of M_{λ_0}; suppose that $x \in A_+M$. Then $x = \sum_i a_i x_i$ where the a_i are nonzero homogeneous elements of A_+ and the x_i nonzero homogeneous elements of M with $\deg(x) = \deg(a_i)+\deg(x_i)$ for all i. But, as $\deg(a_i) > 0$, $\lambda_0 = \deg(a_i)+\deg(x_i) > \deg(x_i)$, which contradicts the hypothesis. □

Corollary 2.1.32. *Let Δ be an ordered group, A a graded ring of type Δ with positive degrees. If M is a finitely generated graded A-module such that $A_+M = M$, then $M = \{0\}$.*

Proof. Suppose $M \neq \{0\}$. Let λ_0 be a minimal element of the set of degrees of a finite generating system of M consisting of nonzero homogeneous elements; then the hypotheses of Proposition 2.1.31 would be fulfilled, which implies a contradiction. □

Corollary 2.1.33. *Let Δ be an ordered group, A a graded ring of type Δ with positive degrees. If M is a finitely generated graded A-module and N is a graded submodule of M such that $N + A_+M = M$, then $N = M$.*

Proof. The quotient module M/N is a finitely generated graded A-module and the hypothesis implies that $A_+(M/N) = M/N$, henee $M/N = 0$. □

Corollary 2.1.34. *Let Δ be an ordered group, A a graded ring of type Δ with positive degrees. Let $\phi : M \to N$ be a graded homomorphism of graded A-modules, where N is assumed to be finitely generated. If the homomorphism*

$$\phi\otimes\mathrm{id} : M\otimes_A (A/A_+) \to N\otimes_A (A/A_+)$$

is surjective, then ϕ is surjective.

Proof. Note that $\phi(M)$ is a graded submodule of N and we have

$$(M/\phi(M))\otimes_A (A/A_+) \cong (N\otimes_A (A/A_+))/\operatorname{im}(\phi\otimes\mathrm{id})$$

as (A/A_+)-modules. The hypothesis therefore implies $(N/\phi(M))\otimes_A (A/A_+) = \{0\}$ and henee $N = \phi(M)$ by Corollary 2.1.32. □

Remark 2.1.35. It follows from the proof of Corollary 2.1.32 that Corollaries 2.1.32 and 2.1.33 (resp. Corollary 2.1.34) are still valid when, instead of assuming that M (resp. N) is finitely generated, the following hypothesis is made: there exists a subset Δ^+ of Δ satisfying the following conditions:

(a) $M_\lambda = \{0\}$ for all $\lambda \notin \Delta^+$.

(b) every non-empty subset of Δ^+ has a least element.

This will be the case if $\Delta = \mathbb{Z}$ and M (resp. N) is a graded module with positive degrees.

Proposition 2.1.36. *Suppose that $\Delta = \mathbb{Z}$. With the hypothesis of Proposition 2.1.31, consider the graded A_0-module $N = M/A_+M$ and suppose the following conditions hold:*

(a) each of the N_λ considered as an A_0-module admits a basis $(y_{i\lambda})_{i\in I_\lambda}$.

(b) the canonical homomorphism $A_+ \otimes_A M \to M$ is injective.

Then M is a graded free A-module and, to be precise, if $x_{i\lambda}$ is an element of M_λ whose image in N_λ is $y_{i\lambda}$, the family $(x_{i\lambda})_{(i,\lambda)\in I}$ (where I is the disjoint union of the I_λ) is a basis of M.

Proof. We know that there is a graded free A-module L (of graduation (L_λ)) and a surjective homomorphism $p : L \to M$ of degree 0 such that $p(e_{i\lambda}) = x_{i\lambda}$ for all $(i,\lambda) \in I$, where $(e_{i\lambda})_{(i,\lambda)\in I}$ is a basis of L consisting of homogeneous elements $e_{i\lambda} \in L_\lambda$. It follows from the above Remark that p is surjective. Consider the graded A-module $R = \ker p$ and note that $R_\lambda = \{0\}$ for $\lambda < \lambda_0$ by definition; we need to prove that $R = \{0\}$ and by Proposition 2.1.31 it suffices to show that $A_+R = R$. Consider the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & A_+ \otimes R \arrow[r] \arrow[d, "\phi"] & A_+ \otimes L \arrow[r, "1\otimes p"] \arrow[d, "\psi"] & A_+ \otimes M \arrow[r] \arrow[d, "\eta"] & 0 \\
0 \arrow[r] & R \arrow[r] & L \arrow[r, "p"] & M \arrow[r] & 0
\end{tikzcd}$$

where the vertical maps are induced from the canonical injection $A_+ \to A$. We need to show that ϕ is surjective. Note that, as L is free (hence flat), ψ is injective. Also, η is injective by hypothesis. Then by the snake lemma, we have an exact sequence

$$R/A_+R \longrightarrow L/A_+L \xrightarrow{\bar{p}} M/A_+M \longrightarrow 0$$

By hypothesis we see $\bar{p}$ is a bijection, hence the claim follows. □

2.1.4 Graded rings of type $\mathbb{Z}$

In this paragraph, all the graduations considered are assumed to be of type $\mathbb{Z}$. If A (resp. M) is a graded ring (resp. graded module), A_i (resp. M_i) will denote the set of homogeneous elements of degree i in A (resp. M). Recall that if $A_i = \{0\}$ (resp. $M_i = \{0\}$) for $i < 0$, A (resp. M) will, to abbreviate, be called a graded ring (resp. module) with positive degrees.

Proposition 2.1.37. *Let $A = \bigoplus_{i\geq 0} A_i$ be a graded ring with positive degrees and (x_λ) a family of homogeneous elements of A_+. The following conditions are equivalent:*

(i) The ideal of A generated by the family (x_λ) is equal to A_+.

(ii) The family (x_λ) is a system of generators of the A_0-algebra A.

(iii) For all $i \geq 0$, the A_0-module A_i is generated by the elements of the form $\prod_\lambda x_\lambda^{n_\lambda}$ which are of degree i in A.

Proof. Clearly conditions (ii) and (iii) are equivalent. If they hold, every element of A_+ is of the form $f(x_\lambda)$ where f is a polynomial of $A_0[X_\lambda]$ with no constant term; then $A_+ = \sum_\lambda Ax_\lambda$, which proves that (iii) implies (i). Conversely, suppose that condition (i) holds. Let $A' = A_0[x_\lambda]$ be the sub-A_0-algebra of A generated by the family (x_λ) and let us show that $A' = A$. For this, it is sufficicnt to show that $A_i \subseteq B$ for all $i \geq 0$. We proceed by induction on i, the property being obvious for $i = 0$. Then let $y \in A_i$ with $i \geq 1$. Sincey $y \in A_+$, which is the ideal generated by (x_λ), there exists a family (a_λ) of elements of A of finite support such that $y = \sum a_\lambda x_\lambda$ and we may assume that each of the a_λ is homogeneous of degree $i - \deg(x_\lambda)$ (by replacing it if need be by its homogeneous component of that degree); as $\deg(x_\lambda) > 0$, the induction hypothesis shows that $a_\lambda \in A'$ for all α, whence $y \in A'$ and $A_i \subseteq A'$, which proves that (i) implies (ii). $\square$

Proof. Part (a) follows from Proposition 2.1.37. As for (b), the condition in (b) is sufficient by Hilbert's basis theorem. If A is Noetherian, $A_0 = A/A_+$ is Noetherian and A_+ is finitely generated as an ideal. Thus A is an A_0-algebra of finite type by (a). $\square$

Corollary 2.1.38. *Let $A = \bigoplus_{i \geq 0} A_i$ be a graded ring with positive degrees.*

(a) The ideal A_+ is finitely generated if and only if A is an A_0-algebra of finite type.

(b) The ring A is Noetherian if and only if A_0 is Noetherian and A is an A_0-algebra of finite type.

(c) Suppose that the conditions in (a) hold and let $M = \bigoplus_{i \in \mathbb{Z}} M_i$ be a finitely generated graded A-module. Then, for all $i \in \mathbb{Z}$, M_i is a finitely generated A_0-module and there exists i_0 such that $M_i = \{0\}$ for $i < i_0$.

Proof. Part (a) follows from Proposition 2.1.37. As for (b), the condition in (b) is sufficient by Hilbert's basis theorem. If A is Noetherian, $A_0 = A/A_+$ is Noetherian and A_+ is finitely generated as an ideal. Thus A is an A_0-algebra of finite type by (a).

We prove (c). We may suppose that A is generated (as an A_0-algebra) by homogeneous elements $a_1, \ldots, a_r$ of degree $i \geq 1$ and M is generated (as an A-module) by homogeneous elements $x_1, \ldots, x_n$; let $h_i = \deg(a_i)$ and $k_j = \deg(x_j)$. Clearly M_n consists of the linear combinations with coefficients in A_0 of the elements $a_1^{\alpha_1} \cdots a_r^{\alpha_r} x_j$ such that the α_i are positive integers satisfying the relation $k_j + \sum_{i=1}^r h_i \alpha_i = n$; for each n there is only a finite number of families $(\alpha_i)_{i=1}^r$ satisfying these conditions, since $h_i \geq 0$ for all i; we conclude that M_n is a finitely generated A_0-module and moreover clearly $M_n = \{0\}$ when $n < \inf(k_i)$. $\square$

Let $A = \bigoplus_{i \in \mathbb{Z}} A_i$ be a graded ring and $M = \bigoplus_{i \in \mathbb{Z}} M_i$ a graded A-module; for each ordered pair (d, r) with $d \geq 1$ and $0 \leq r \leq d - 1$, set

$$A^{(d)} = \bigoplus_{i \in \mathbb{Z}} A_{id}, \quad M^{(d,r)} = \bigoplus_{i \in \mathbb{Z}} M_{id+r}.$$

Clearly $A^{(d)}$ is a graded subring of A and $M^{(d,r)}$ a graded $A^{(d)}$-module; moreover, if N is a graded submodule of M, $N^{(d,r)}$ is a graded sub-$A^{(d)}$-module of $M^{(d,r)}$. We shall write $M^{(d)}$ instead of $M^{(d,0)}$; for each $d \geq 1$, M is the direct sum of the $A^{(d)}$-modules $M^{(d,r)}$ for $0 \leq r \leq d - 1$.

Proposition 2.1.39. *Let A be a graded ring with positive degrees and $M = \bigoplus_{i \in \mathbb{Z}} M_i$ be a graded A-module. Suppose that A is an A_0-algebra of finite type and M a finitely generated A-module. Then*

(a) *For every ordered pair (d, r) of integers such that $d \geq 1$ and $0 \leq r \leq d - 1$, $M^{(d,r)}$ is a finitely generated $A^{(d)}$-module.*

(b) *$A_+^{(d)}$ is a finitely generated $A^{(d)}$-module and therefore $A^{(d)}$ is an A_0-algebra of finite type.*

Proof. Let us show that A is a finitely generated $A^{(d)}$-module. Let $a_1, \dots, a_s$ be a system of generators of the A_0-algebra A consisting of homogeneous elements. The elements of A (finite in number) of the form $a_1^{\alpha_1} \cdots a_s^{\alpha_s}$ such that $0 \le a_i \le d$ for $1 \le i \le s$ constitute a system of generators of the $A^{(d)}$-module A; for every system of integers $n_i \ge 0$, $1 \le i \le s$, there are positive integers q_i and r_i such that $n_i = q_i d + r_i$, where $0 \le r_i < d$ for $1 \le i \le s$; then

$$a_1^{n_1} \cdots a_s^{n_s} = (a_1^{q_1} \cdots a_s^{q_s})(a_1^{r_1} \cdots a_s^{r_s}).$$

which proves our assertion. Then, if M is a finitely generated A-module, it is also a finitely generated $A^{(d)}$-module; as M is the direct sum of the $M^{(d,r)}$ for $0 \le r \le d-1$, each of the $M^{(d,r)}$ is a finitely generated $A^{(d)}$-module, which proves (a).

Now by Corollary 2.1.38, A_+ is a finitely generated A-module, so (a) implies $A_+^{(d)}$ is a finitely-generated A_0-module, which again by Corollary 2.1.38 implies $A^{(d)}$ is a finite A_0-algebra. □

Proposition 2.1.40. *Let $A = \bigoplus_{i \ge 0} A_i$ be a graded ring with positive degrees which is a finitely generated A_0-algebra and $M = \bigoplus_{i \in \mathbb{Z}} M_i$ a graded A-module. Let $(x_\lambda)_{\lambda \in I}$ be a system of homogeneous generators of M such that $\sup_\lambda \deg(x_\lambda) < +\infty$. Then there exists $n_0 > 0$ and $d > 0$ such that, for any $n \ge n_0$ we have $M_{n+d} = A_d M_n$.*

Proof. Let A be generated as an A_0-algebra by homogeneous elements $a_1, \dots, a_s$, and set $d_i = \deg(a_i)$. Let d be a common multiple of all the d_i and set $b_i = a_i^{d/d_i}$ for $1 \le i \le s$, so that $\deg(b_i) = d$ for all i. Consider the set Z of elements of the form $a_1^{\alpha_1} \cdots a_s^{\alpha_s} x_\lambda$ with $0 \le \alpha_i \le d/d_i$ for $1 \le i \le s$ and $\lambda \in I$. Then by the hypothesis on (x_λ), we can choose $n_0 > 0$ be larger than the degree of every element of Z. If $n \ge n_0$ then any element of M_{n+d} can be written as an A_0-linear combination of elements $a_1^{n_1} \cdots a_s^{n_s} x_j$ as above. If $n_i = (d/d_i)\beta_i + \gamma_i$ with $\gamma_i < d/d_i$, then we can write

$$a_1^{n_1} \cdots a_s^{n_s} x_j = b_1^{\beta_1} \cdots b_s^{\beta_s} (a_1^{\gamma_1} \cdots a_s^{\gamma_s} x_j) = b_1^{\beta_1} \cdots b_s^{\beta_s} z$$

where every b_i is homogenous of degree d and $z \in Z$ is homogeneous of some degree t. Note that $n + d > n_0 \ge t$, so the last expression makes sense. Now it is clear that $b_1^{\beta_1} \cdots b_s^{\beta_s} z$ belongs to $A_d M_n$, so $M_{n+d} = A_d M_n$. □

Corollary 2.1.41. *Let A be a graded ring such that $A = A_0[A_1]$, M a graded A-module and $(x_\lambda)_{\lambda \in I}$ a system of homogeneous generators of M such that $\deg(x_\lambda) \le n_0$ for all $\lambda \in I$. Then for all $n \ge n_0$ and all $d > 0$, $M_{n+d} = A_d M_n$.*

Proof. In this case the integer d in Proposition 2.1.40 can be chosen to be 1, and the set Z has elements of degree smaller than n_0. Thus we have $M_{n+d} = A_1 M_n$ for $n \ge n_0$, and by induction $M_{n+d} = A_d M_n$ for $n \ge n_0$ and $d > 0$. □

Corollary 2.1.42. *Let A be a graded ring such that $A = A_0[A_1]$ and let $S = \bigoplus_{i \ge 0} S_i$ be a graded A-algebra with positive degrees which is a finitely generated A-module. Then there exists an integer $n_0 > 0$ such that:*

(a) *For $n \ge n_0$ and $d > 0$, $S_{n+d} = S_d S_n$.*

(b) *For $d \ge n_0$, $S^{(d)} = S_0[S_d]$.*

Proof. By Corollary 2.1.41 there exists an integer $n_0 \ge 0$ such that, for $n \ge n_0$ and $d > 0$, $S_{n+d} = A_d S_n$, whence a fortiori $S_{n+d} = S_d S_n$, which establishes (a). Then, for $d \ge n_0$ and $i > 0$, $S_{id} = (S_d)^i$ as follows by induction on i applying (a); this establishes (b). □

Corollary 2.1.43. *Let $A = \bigoplus_{i \ge 0} A_i$ be a graded ring with positive degrees which is a finitely generated A_0-algebra. There exists an integer $d > 0$ such that $A^{(md)} = A_0[A_{md}]$ for all $m \ge 1$.*

Proof. In Proposition 2.1.40 we can assume that $d \geq n_0$ (since the proof works with d any common multiple of the d_i). And therefore with $A = M$ we have $A_{d+d} = (A_d)^2$ and inductively $A_{id} = (A_d)^i$ for all $i > 0$. This shows the claim. □

Proposition 2.1.44. *Let Δ be an ordered group and A a graded ring of type Δ. If the product in A of two homogeneous nonzero elements is nonzero, then the ring A is integral.*

Proof. Let $x = \sum_\lambda x_\lambda$ and $y = \sum_\lambda y_\lambda$ be two non-zero elements of A, with x_λ, y_λ being homogeneous of degree λ for all $\lambda \in \Delta$. Let α (resp. β) be the greatest of the elements $\lambda \in \Delta$ such that $x_\lambda \neq 0$ (resp. $y_\lambda \neq 0$); it is immediate that if $\lambda \neq \alpha$ or $\mu \neq \beta$, either $x_\lambda y_\mu = 0$ or $\deg(x_\lambda y_\mu) < \lambda + \alpha$; the homogeneous component of xy of degree $\alpha + \beta$ is therefore $x_\lambda y_\beta$, which is non-zero by hypothesis; whence $xy \neq 0$. □

Proposition 2.1.45. *Let $\mathfrak{p} = \bigoplus_{i \geq 0} \mathfrak{p}_i$ be a graded ideal of A; for $\mathfrak{p}$ to be prime, it is necessary and sufficient that for homogeneous elements $x, y \in A$, $xy \notin \mathfrak{p}$ if and only if $x \notin \mathfrak{p}$ and $y \notin \mathfrak{p}$.*

Proof. The condition is obviously necessary. Conversely, if it is fulfilled, then in the graded ring $A/\mathfrak{p} = \bigoplus_{i \geq 0} A_i/\mathfrak{p}_i$ the product of two homogeneous nonzero elements is nonzero and hence $A/\mathfrak{p}$ is an integral domain. □

Let $\mathfrak{a}$ be an arbitrary ideal of A. We can associate to it a homogeneous ideal $\mathfrak{a}^h = \bigoplus_{i \geq 0}(\mathfrak{a} \cap A_i)$. It follows from the definition that $\mathfrak{a}^h \subseteq \mathfrak{a}$ and $\mathfrak{a}$ is homogeneous if and only if $\mathfrak{a} = \mathfrak{a}^h$.

Proposition 2.1.46. *Let A be a graded ring with positive degrees and $\mathfrak{p} \subseteq A$ be a prime ideal. Let $\mathfrak{p}^h$ be the homogeneous ideal generated by the homogeneous elements of $\mathfrak{p}$. Then $\mathfrak{p}^h$ is a prime ideal of A.*

Proof. For any homogeneous element $f, g \in A$ such that $fg \in \mathfrak{p}^h \subseteq \mathfrak{p}$, we have $f \in \mathfrak{p}$ or $g \in \mathfrak{p}$. Then by the definition of $\mathfrak{p}^h$, this implies $f \in \mathfrak{p}^h$ or $g \in \mathfrak{p}^h$. Thus $\mathfrak{p}^h$ is prime by Proposition 2.1.45. □

Corollary 2.1.47. *Let $A = \bigoplus_{i \geq 0} A_i$ be a graded ring with positive degrees.*

(a) Any minimal prime of A is a homogeneous ideal.

(b) Given a homogeneous ideal $\mathfrak{a} \subseteq A$ any minimal prime over $\mathfrak{a}$ is homogeneous.

Proof. The first claim is a direct application of Proposition 2.1.46, since the ideal $\mathfrak{p}^h$ is contained in $\mathfrak{p}$. The second follows by applying the result on $A/\mathfrak{a}$. □

Let A be a graded ring with positive degrees. Two graded ideals $\mathfrak{a} = \bigoplus_{i \geq 0} \mathfrak{a}_i$ and $\mathfrak{b} = \bigoplus_{i \geq 0} \mathfrak{b}_i$ of A are said to be **equivalent** if there exists an integer n_0 such that $\mathfrak{a}_n = \mathfrak{b}_n$ for $n \geq n_0$ (clearly it is an equivalence relation). A graded ideal is called **essential** if it is not equivalent to A_+. As we will see, this notation plays an important role when we study graded prime ideals.

Proposition 2.1.48. *Let A be a graded ring with positive degrees, $\mathfrak{a}$ a graded ideal, and $\mathfrak{p}$ a graded prime ideal of A with $\mathfrak{p} \not\supseteq A_+$. Then $\mathfrak{a} \subseteq \mathfrak{p}$ if and only if there exists an integer $n_0 \geq 0$ such that $\mathfrak{a}_n \subseteq \mathfrak{p}_n$ for all $n \geq n_0$.*

Proof. One implication is clear. Conversely, assume that $a_n \subseteq \mathfrak{p}_n$ for $n \geq n_0$. We now show that $a_{n_0-1} \subseteq \mathfrak{p}_{n_0-1}$. Since $\mathfrak{p} \not\supseteq A_+$, there exist $e > 0$ and $a \in A_e$ with $a \notin \mathfrak{p}$. If $b \in \mathfrak{a}_{n_0-1}$ then $ab \in \mathfrak{a}_{n_0-1+e} \subseteq \mathfrak{p}_{n_0-1+e}$, so $b \in \mathfrak{p}$ by primeness. This proves the claim. □

Corollary 2.1.49. *Let A be a graded ring with positive degrees and $\mathfrak{p}$, $\mathfrak{q}$ be graded prime ideals and $\mathfrak{p}$ with $\mathfrak{p} \not\supseteq A_+$ and $\mathfrak{q} \not\supseteq A_+$. Then $\mathfrak{p} = \mathfrak{q}$ if and only if they are equivalent.*

Proposition 2.1.50. *Let $\mathfrak{a}$ be a graded ideal of A and $n_0 > 0$ an integer. For there to exist a graded prime ideal $\mathfrak{p}$ such that $\mathfrak{p}_n = \mathfrak{a}_n$ for $n \geq n_0$, it is necessary and sufficient that, for any homogeneous elements x, y of degrees $\geq n_0$, the relation $xy \in \mathfrak{a}$ implies $x \in \mathfrak{a}$ or $y \in \mathfrak{a}$. If there exists $n \geq n_0$ such that $\mathfrak{a}_n \neq A_n$, then the prime ideal satisfying the above condition is unique.*

Proof. The condition of the statement is obviously necessary. If $\mathfrak{a}_n = A_n$ for all $n \geq n_0$, clearly every prime ideal containing A_+ is a solution to the problem; there may therefore be several prime ideals which solve the problem; however, any two of these ideals are obviously equivalent.

Now suppose that there exists a homogeneous element $a \in A_d$ with $d \geq n_0$ not belonging to $\mathfrak{a}_d$. Now we define

$$\mathfrak{p} = \{x \in A : ax \in \mathfrak{a}\}.$$

Clearly $\mathfrak{p}$ is an ideal of A; as the homogeneous components of ax are the products by a of those of x and a is a graded ideal, $\mathfrak{p}$ is a graded ideal; moreover, $1 \notin \mathfrak{p}$ and hence $\mathfrak{p} \neq A$. To prove that $\mathfrak{p}$ is prime, let $x \in A_m$ and $y \in A_n$ satisfy $x \notin \mathfrak{p}$ and $y \notin \mathfrak{p}$. Then $ax \notin \mathfrak{a}_{m+d}$ and $ay \notin \mathfrak{a}_{n+d}$, hence $a^2xy \notin \mathfrak{a}_{m+n+2d}$ by hypothesis. Since $\mathfrak{a}$ is an ideal, we then get $axy \notin \mathfrak{a}_{m+n+d}$, so $xy \in \mathfrak{p}$. Now if $n \geq n_0$ and $a \in A$, the conditions $x \in \mathfrak{a}_n$ and $ax \in \mathfrak{a}_{n+d}$ are equivalent by hypothesis and hence $\mathfrak{p} \cap A_n = \mathfrak{a}_n$, which completes the proof of the existence of the graded prime ideal $\mathfrak{p}$ which solves the problem. The uniqueness part now follows from Corollary 2.1.49. □

Proposition 2.1.51. *Let $A = \bigoplus_{i \geq 0} A_i$ be a graded ring with positive degrees and $d > 0$ an integer.*

- *(a) For every essential graded prime ideal $\mathfrak{p}$ of A, $\mathfrak{p} \cap A^{(d)}$ is an essential graded prime ideal of $A^{(d)}$.*
- *(b) Conversely, for every essential graded prime ideal $\mathfrak{p}'$ of $A^{(d)}$, there exists a unique (necessarily essential) graded prime ideal $\mathfrak{p}$ of A such that $\mathfrak{p} \cap A^{(d)} = \mathfrak{p}'$.*

Proof. If $a \in A_i$ does not belong to $\mathfrak{p}_i$, then a^d does not belong to $\mathfrak{p}_{id}$, and hence $\mathfrak{p} \cap A^{(d)}$ is essential. This proves (a). Now let $\mathfrak{p}'$ be an essential graded prime ideal of $A^{(d)}$. If $\mathfrak{p}$ is a graded prime ideal of A such that $\mathfrak{p} \cap A^{(d)} = \mathfrak{p}'$, then for all $n \geq 0$, the set $\mathfrak{p} \cap A_n$ must be equal to the set $\mathfrak{a}_n$ of $x \in A_n$ such that $x^d \in \mathfrak{p}'$. Let us show that $\mathfrak{a} = \bigoplus_{n \in \mathbb{N}} \mathfrak{a}_n$ is a graded prime ideal, which will proves (b). As $\mathfrak{p}'$ is prime, $\mathfrak{a}_n = \mathfrak{p}_n$ when n is a multiple of d. Now if $x \in \mathfrak{a}_n$, $y \in \mathfrak{a}_n$, then $(x-y)^{2d}$ is the sum of terms each of which is a product of x^d or y^d by a homogeneous element of degree nd and hence $(x-y)^{2d} \in \mathfrak{p}'$ and, since $\mathfrak{p}'$ is prime, $(x-y)^d \in \mathfrak{p}'$ and therefore $\mathfrak{a}_n$ is a subgroup of A. As $\mathfrak{p}'$ is an ideal of $A^{(d)}$, $\mathfrak{a}$ is a graded ideal of A; finally, the relation $(xy)^d \in \mathfrak{p}'$ implies $x^d \in \mathfrak{p}'$ or $y^d \in \mathfrak{p}'$, which completes the proof. □

Let A be a graded ring with positive degrees and $\mathfrak{p}$ an essential graded prime ideal of A. The set S of homogeneous elements of A not belonging to $\mathfrak{p}$ is multiplicative and the ring of fractions $S^{-1}A$ is therefore graded canonically (note that there will in general be homogeneous nonzero elements of negative degree in this graduation). We shall denote by $A_{\mathfrak{p}}$ the subring of $S^{-1}A$ consisting of the homogeneous elements of degree 0, in other words the set of fractions x/s, where x and s are homogeneous of the same degree in A and $s \notin \mathfrak{p}$. Similarly, for every graded A-module M, $S^{-1}M$ is graded canonically (loc. cit.) and we shall denote by $M_{\mathfrak{p}}$ the subgroup of homogeneous elements of degree 0, which is obviously an $A_{\mathfrak{p}}$-module.

Proposition 2.1.52. *Let $\mathfrak{p}$ be a graded prime ideal of A, $d > 0$ an integer, and $\mathfrak{p}'$ be the graded prime ideal $\mathfrak{p} \cap A^{(d)}$ of $A^{(d)}$. For every graded A-module M, the homomorphism $(M^{(d)})_{\mathfrak{p}'} \to M_{\mathfrak{p}}$ induced from the canonical injection $M^{(d)} \to M$ is bijective.*

Proof. If S is the set of homogeneous elements of A not belonging to $\mathfrak{p}$ and $S^{(d)} = S \cap A^{(d)}$, the canonical homomorphism $\phi : (S^{(d)})^{-1}M^{(d)} \to S^{-1}M$ is a homogeneous homomorphism of degree 0 and it is injective, for, if $x \in M_{nd}$ satisfies $sx = 0$ for $s \in A_m$, $s \notin \mathfrak{p}$, then also $s^d x = 0$ and $s^d \in A_{md}$, $s^d \notin \mathfrak{p}'$. It remains to show that the image under ϕ of $(M^{(d)})_{\mathfrak{p}'}$ is the whole of

$M_{\mathfrak{p}}$; but if $x \in M_n$, $s \in A_n$ and $s \notin \mathfrak{p}$, then also $x/s = (xs^{d-1})/s^d$ where $xs^{d-1} \in A_{nd}$, $s^d \in A_{md}$ and $s^d \notin \mathfrak{p}'$, whence our assertion. $\square$

Proposition 2.1.53. *Suppose A is a graded ring with positive degrees, $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ homogeneous prime ideals and $\mathfrak{a}$ a homogeneous ideal. Assume that $\mathfrak{a} \cap A_+ \nsubseteq \mathfrak{p}_i$ for all i, then there exists a homogeneous element $x \in \mathfrak{a} \cap A_+$ of positive degree such that $x \notin \mathfrak{p}_i$ for $1 \le i \le r$.*

Proof. We may assume that $\mathfrak{a} \subseteq A_+$ and there are no inclusions among the $\mathfrak{p}_i$. The result is true for $r = 1$. Suppose the result holds for $r - 1$. Pick $x \in \mathfrak{a}$ homogeneous of positive degree such that $x \notin \mathfrak{p}_i$ for all $i = 1, \dots, r-1$. If $x \notin \mathfrak{p}_r$ we are done, so assume that $x \in \mathfrak{p}_r$. Since $\mathfrak{a}\mathfrak{p}_1 \cdots \mathfrak{p}_{r-1}$ is a homogeneous ideal that is not contained in $\mathfrak{p}_r$, there is a homogeneous element $y \in \mathfrak{a}\mathfrak{p}_1 \cdots \mathfrak{p}_{r-1}$ of positive degree such that $y \notin \mathfrak{p}_r$. Then $x^{\deg(y)} + y^{\deg(x)}$ works. $\square$

2.2 Filtration and topologies

2.2.1 Filtrated rings and modules

Definition 2.2.1. An increasing (resp. decreasing) sequence $(G_n)_{n\in\mathbb{Z}}$ of subgroups of a commutative group G is called an **increasing (resp. decreasing) filtration** on G. A commutative group with a filtration is called a filtered group.

If $(G_n)_{n\in\mathbb{Z}}$ is an increasing (resp. decreasing) filtration on a commutative group G and we write $G'_n = G_{-n}$, clearly $(G'_n)_{n\in\mathbb{Z}}$ is a decreasing (resp. increasing) filtration on G. We may therefore restrict our study to decreasing filtrations and hence forth when we speak of a filtration, we shall mean a decreasing filtration, unless otherwise stated.

Given a decreasing filtration $(G_n)_{n\in\mathbb{Z}}$ on a commutative group G, clearly $\bigcap_{n\in\mathbb{Z}} G_n$ and $\bigcup_{n\in\mathbb{Z}} G_n$ are subgroups of G; the filtration is called **separated** if $\bigcap_{n\in\mathbb{Z}} G_n$ is reduced to the identity element and **exhaustive** if $\bigcup_{n\in\mathbb{Z}} G_n = G$.

Definition 2.2.2. Given a ring A, a filtration $(A_n)_{n\in\mathbb{Z}}$ over the additive group A is called compatible with the ring structure on A if

(a) $A_m A_n \subseteq A_{m+n}$ for all $m, n \in \mathbb{Z}$.

(b) $1 \in A_0$.

The ring A with this filtration is then called a **filtered ring**.

Conditions (a) and (b) show that A_0 is a subring of A and the A_n's are A_0-modules. The set $B = \bigcup_{n\in\mathbb{Z}} A_n$ is a subring of A and the set $\mathfrak{n} = \bigcap_{n\in\mathbb{Z}} A_n$ is an ideal of B; for if $x \in \mathfrak{n}$ and $a \in A_p$, then for all $k \in \mathbb{Z}$ we have $x \in A_{k-p}$, whence $ax \in A_k$ by (a); therefore $ax \in \mathfrak{n}$. An important particular case is that $A_0 = A$; then $A_n = A$ for all $n \le 0$ and all the A_n are ideals of A.

Definition 2.2.3. Let A be a filtered ring, $(A_n)_{n\in\mathbb{Z}}$ its filtration and M an A-module. A filtration $(M_n)_{n\in\mathbb{Z}}$ on M is called compatible with its module structure over the filtered ring A if

$$A_m M_n \subseteq M_{m+n} \quad \text{for all } m, n \in \mathbb{Z}. \tag{2.2.1}$$

The A-module M with this filtration is called a **filtered module**.

If M is a filtered module with filtration $(M_n)_{n\in\mathbb{Z}}$, then the M_n are all A_0-modules; if $B = \bigcup_{n\in\mathbb{Z}} A_n$, clearly $\bigcup_{n\in\mathbb{Z}} M_n$ is a B-module and so is $\bigcap_{n\in\mathbb{Z}} M_n$ by the same argument as above for $\mathfrak{n}$. If $A_0 = A$, all the M_n's are submodules of M.

Example 2.2.4. On a ring A the sets A_n such that $A_n = 0$ for $n > 0$, $A_n = A$ for $n \leq 0$ form what is called a **trivial filtration** associated with the trivial graduation on A; on an A-module M, every filtration (M_n) consisting of sub-A-modules is then compatible with the module structure on M over the filtered ring A. Then it is possible to say that every filtered commutative group G is a filtered $\mathbb{Z}$-module, if $\mathbb{Z}$ is given the trivial filtration.

Example 2.2.5. Let A be a graded ring of type $\mathbb{Z}$; for all $i \in \mathbb{Z}$, let A^n be the subgroup of homogeneous elements of degree n in A and set $A_n = \bigoplus_{i \geq n} A^i$. Then it is immediate that (A_n) is an exhaustive and separated decreasing filtration which is compatible with the ring structure on A; this filtration is said to be associated with the graduation (A^n) and the filtered ring A is said to be associated with the given graded ring A.

Now let M be a graded module of type $\mathbb{Z}$ over the graded ring A and for all $i \in \mathbb{Z}$ let M^n be the subgroup of homogeneous elements of degree n of M. If $M_n = \bigoplus_{i \geq n} M^i$, then (M_n) is an exhaustive and separated decreasing filtration which is compatible with the module structure on M over the filtered ring A; this filtration is said to be **associated** with the graduation (M^n) and the filtered module M is said to be **associated** with the given graded module M.

Example 2.2.6. Let A be a filtered ring, $(A_n)_{n \in \mathbb{Z}}$ its filtration and M an A-module. Let us write $M_n = A_n M$; it follows that

$$A_m M_n = A_m A_n M \subseteq A_{m+n} M = M_{m+n}$$

and that $M_0 = M$; therefore (M_n) is an exhaustive filtration which is compatible with the A-module structure on M. This filtration is said to be **derived** from the given filtration (A_n) on A; note that it is not necessarily separated, even if (A_n) is separated.

Example 2.2.7 (The $\mathfrak{a}$-adic filtration). Let A be a ring and $\mathfrak{a}$ an ideal of A. Let us write $A_n = \mathfrak{a}^n$ for $n > 0$ and $A_n = A$ for $n \leq 0$. It is immediate that (A_n) is an exhaustive filtration on A, called the **$\mathfrak{a}$-adic filtration**. Let M be an A-module; the filtration (M_n) derived from the $\mathfrak{a}$-adic filtration on A is called the $\mathfrak{a}$-adic filtration on M. If B is an A-algebra, then $\mathfrak{b} = \mathfrak{a}^e$ is an ideal of B and for every B-module N and $n \in \mathbb{Z}$ we have $\mathfrak{b}^n N = \mathfrak{a}^n N$, therefore the $\mathfrak{b}$-adic filtration on N coincides with the $\mathfrak{a}$-adic filtration (if N is considered as an A-module).

Let G be a filtered group and $(G_n)_{n \in \mathbb{Z}}$ its filtration; clearly, for every subgroup H of G, $(H \cap G_n)_{n \in \mathbb{Z}}$ is a filtration said to be **induced** by that on G. Similarly, if H is a normal subgroup of G, the family $((H + G_n)/H)_{n \in \mathbb{Z}}$ is a filtration on the group G/H, called the **quotient** under H of the filtration on G.

If G and G' are filtered groups and (G_n), (G'_n) are their filtrations. Then $(G_n \times G'_n)_{n \in \mathbb{Z}}$ is a filtration on $G \times G'$ called the **product** of the filtrations on G and G', which is exhaustive (resp. separated) if (G_n) and (G'_n) are.

Now let A be a filtered ring and (A_n) its filtration; on every subring B of A, clearly the filtration induced by that on A is compatible with the ring structure on B. If $\mathfrak{a}$ is an ideal of A, the quotient filtration on $A/\mathfrak{a}$ of that on A is compatible with the structure of this ring, for

$$(A_m + \mathfrak{a})(A_n + \mathfrak{a}) \subseteq A_m A_n + \mathfrak{a} \subseteq A_{m+n} + \mathfrak{a}.$$

If A' is another filtered ring, the product filtration on $A \times A'$ is compatible with the structure of this ring.

Simialrly, let M be a filtered A-module and (M_n) its filtration; on every submodule N of M, the filtration induced by that on M is compatible with the A-module structure on N and, on the quotient module M/N, the quotient filtration of that on M is compatible with the A-module structure, as

$$A_m(N + M_n) \subseteq N + A_m M_n \subseteq N + M_{m+n}.$$

Note that if the filtration on M is derived from that on A, so is the quotient filtration on M/N because we have $(N + A_n M)/N = A_n(M/N)$. However, this is not true in general for the filtration induced on N.

Let A be a filtered ring, M a filtered A-module and (M_n) the filtration of M. For all $x \in M$ let $v(x)$ denote the least upper bound in $\bar{\mathbb{R}}$ of the set on integers $n \in \mathbb{Z}$ such that $x \in M_n$. Then the following equivalences hold:

$$\begin{cases} v(x) = +\infty & x \in \bigcap_{n\in\mathbb{Z}} M_n \\ v(x) = p & x \in M_p \setminus M_{p+1} \\ v(x) = -\infty & x \notin \bigcup_{n\in\mathbb{Z}} M_n \end{cases}$$

Thé map $v : M \to \bar{\mathbb{R}}$ is called the **order function** of the filtered module M. If v is known then so are the M_n, for M_n is the set of $x \in M$ such that $v(x) \geq n$; the fact that the M_n are additive subgroups of M implies the relation

$$v(x + y) \geq \min\{v(x), v(y)\} \tag{2.2.2}$$

The above definition applies in particular to the filtered A-module A. If v_A is the order function of A and v_M the order function on M, it follows from (2.2.1) that for $a \in A$, and $x \in M$,

$$v_M(ax) \geq v_A(a) + v_M(x)$$

whenever the right-hand side is defined. In particular, for $a \in A$ and $b \in A$, we have $v_A(ab) \geq v_A(a) + v_A(b)$.

2.2.2 The associated graded module

Let G be a commutative group (written additively) and (G_n) a filtration on G. Let us write

$$\operatorname{gr}_n(G) = G_n/G_{n+1}, \quad \operatorname{gr}(G) = \bigoplus_{n\in\mathbb{Z}} \operatorname{gr}_n(G).$$

The commutative group $\operatorname{gr}(G)$ is then a graded group of type $\mathbb{Z}$, called the graded group associated with the filtered group G, the homogeneous elements of degree n of $\operatorname{gr}(G)$ being those of $\operatorname{gr}_n(G)$.

Now let A be a filtered ring, (A_n) its filtration, M a filtered A-module and (M_n) its filtration. For all $p \in \mathbb{Z}$, $q \in \mathbb{Z}$, we can define a map

$$\operatorname{gr}_p(A) \times \operatorname{gr}_q(M) \to \operatorname{gr}_{p+q}(M), \quad (\bar{a}, \bar{x}) \mapsto \overline{ax}. \tag{2.2.3}$$

This is well-defined in view of the equality $ax - by = (a - b)x + b(x - y)$ and the relations $A_{p+1}M_q \subseteq M_{p+q+1}$, $A_p M_{q+1} \subseteq M_{p+q+1}$. It is immediate that the map defined is $\mathbb{Z}$-bilinear; by linearity, we derive a $\mathbb{Z}$-bilinear map

$$\operatorname{gr}(A) \times \operatorname{gr}(M) \to \operatorname{gr}(M).$$

If this definition is first applied to the case $M = A$, the map (2.2.3) is an internal law of composition on $\operatorname{gr}(A)$, which it is immediately verified is associative and has an identity element which is the canonical image in $\operatorname{gr}_0(A)$ of the unit element of A; it therefore defines on $\operatorname{gr}(A)$ a ring structure and the graduation $(\operatorname{gr}_n(A))_{n\in\mathbb{Z}}$ is by definition compatible with this structure. The graded ring $\operatorname{gr}(A)$ (of type $\mathbb{Z}$) thus defined is called the **graded ring associated with the filtered ring** A. The map (2.2.3) is on the other hand a $\operatorname{gr}(A)$-module external law on $\operatorname{gr}(M)$, the module axioms being trivially satisfied, and the graduation $(\operatorname{gr}_n(M))_{n\in\mathbb{Z}}$ on $\operatorname{gr}(M)$ is obviously compatible with this module structure. The graded $\operatorname{gr}(A)$-module $\operatorname{gr}(M)$ (of type $\mathbb{Z}$) thus defined is called the graded module associated with the filtered A-module M.

Example 2.2.8. Let A be a ring and t an element of A which is not a divisor of 0. Let us give A the (t)-adic filtration. Then the associated graded ring $\mathrm{gr}(A)$ is canonically isomorphic to the polynomial ring $(A/(t))[X]$. For $\mathrm{gr}_n(A) = 0$ for $n < 0$ and by definition the ring $\mathrm{gr}_0(A)$ is the ring $A/(t)$. We now note that by virtue of the hypothesis on t the relation $at^n \equiv 0 \bmod (t^{n+1})$ is equivalent to $a \equiv 0 \bmod t$; if τ is the canonical image of t in $\mathrm{gr}_1(A)$, every element of $\mathrm{gr}_n(A)$ may then be written uniquely in the form $\alpha\tau^n$, where $\alpha \in \mathrm{gr}_0(A)$; whence our assertion.

Example 2.2.9. Let K be a ring and A the ring of formal power series

$$A = K[\![X_1, \dots, X_r]\!].$$

Let $\mathfrak{m}$ be the ideal of A whose elements are the formal power series with no constant term. Let us give A the $\mathfrak{m}$-adic filtration; if $M_1, \dots, M_s$ are the distinct monomials in $X_1, \dots, X_r$ of total degree $n-1$, clearly every formal power series u of total order $\deg(u) \geq n$ may be written as $\sum_{k=1}^{s} u_k M_k$, where the u_k belong to $\mathfrak{m}$; it is seen that $\mathfrak{m}^n$ is the set of formal power series u such that $\deg(u) \geq n$, which shows that deg is the order function for the $\mathfrak{m}$-adic filtration. Then clearly, for every formal power series $u \in \mathfrak{m}^n$, there exists a unique homogeneous polynomial of degree n in the X_i's which is congruent to u mod $\mathfrak{m}^{n+1}$, namely the sum of terms of degree n of u; we conclude that $\mathrm{gr}(A)$ is canonically isomorphic to the polynomial ring $K[X_1, \dots, X_r]$.

Example 2.2.10. More generally, let A be a ring and $\mathfrak{a}$ an ideal of A and A be given the $\mathfrak{a}$-adic filtration. Then the identity map of the $A/\mathfrak{a}$-module $\mathfrak{a}/\mathfrak{a}^2$ onto itself can be extended uniquely to a homomorphism ϕ from the symmetric algebra $\boldsymbol{S}(\mathfrak{a}/\mathfrak{a}^2)$ to the $A/\mathfrak{a}$-algebra $\mathrm{gr}(A)$; it follows from the definition of $\mathrm{gr}(A)$ that ϕ is a surjeelive homomorphism of graded algebras; for $n \geq 1$, every element of $\mathrm{gr}_n(A)$ is a sum of classes mod $\mathfrak{a}^{n+1}$ of elements of the form $y = x_1 \cdots x_n$, where $x_i \in \mathfrak{a}$ for $1 \leq i \leq n$; if ξ_i is the class of x_i mod $\mathfrak{a}^2$, clearly the class of y mod $\mathfrak{a}^{n+1}$ is the element $\phi(\xi_1) \cdots \phi(\xi_n)$, whence our assertion. In particular, every system of generators of the $A/\mathfrak{a}$-module $\mathfrak{a}/\mathfrak{a}^2$ is a system of generators of the $A/\mathfrak{a}$-algebra $\mathrm{gr}(A)$.

If now M is an A-module and M is given the $\mathfrak{a}$-adic filtration, it is seen similarly that the graded $\mathrm{gr}(A)$-module $\mathrm{gr}(M)$ is generated by $\mathrm{gr}_0(M) = M/\mathfrak{a}M$. To be precise, the restriction Γ to $\mathrm{gr}(A) \times \mathrm{gr}_0(M)$ of the external law on the $\mathrm{gr}(A)$-module $\mathrm{gr}(M)$ is a $\mathbb{Z}$-bilinear map of $\mathrm{gr}(A) \times \mathrm{gr}_0(M)$ to $\mathrm{gr}(M)$. Moreover, it is immediately verified that, for $\alpha \in \mathrm{gr}(A)$, $\beta \in \mathrm{gr}_0(A)$, $\xi \in \mathrm{gr}_0(M)$, we have $\Gamma(\alpha\beta, \xi) = \Gamma(\alpha, \beta\xi)$ and hence Γ defines a canonical surjective $\mathrm{gr}_0(A)$-linear map

$$\gamma_M : \mathrm{gr}(A) \otimes_{\mathrm{gr}_0(A)} \mathrm{gr}_0(M) \to \mathrm{gr}(M) \tag{2.2.4}$$

Example 2.2.11. Let A be a graded ring of type $\mathbb{Z}$ and M a graded A-module of type $\mathbb{Z}$; let A^i, (resp. M^i) be the subgroup of homogeneous elements of degree i A (resp. M). Let A and M be given the filtrations associated with their graduations. Then it is immediate that the $\mathbb{Z}$-linear map $A \to \mathrm{gr}(A)$ which maps an element of A^n to its canonical image in

$$\mathrm{gr}_n(A) = \bigoplus_{i \geq n} A^i / \bigoplus_{i \geq n+1} A^i$$

is a graded ring isomorphism. A canonical graded A-module isomorphism $E \to \mathrm{gr}(E)$ is defined similarly.

Proposition 2.2.12. *Let A be a filtered ring, $(A_n)_{n \in \mathbb{Z}}$ its filtration and v its order function. Suppose that* $\mathrm{gr}(A)$ *is an integral domain. Then, for every elements a, b of the ring $B = \bigcup_n A_n$, we have $v(ab) = v(a) + v(b)$.*

Proof. As $\mathfrak{n} = \bigcap_{n \in \mathbb{Z}} A_n$ is an ideal of the ring B, the formula holds if $v(a)$ or $v(b)$ is equal to $+\infty$. If not, $v(a) = r$ and $v(b) = s$ are integers. The classes $\bar{a}$ of a mod A_{r+1} and $\bar{b}$ of b mod A^{s+1} are nonzero by definition, whence by hypothesis $\bar{a}\bar{b} \neq 0$ in $\mathrm{gr}(A)$ and therefore $ab \notin A_{r+s+1}$; as $ab \in A_{r+s}$, we see $v(ab) = r + s$. $\square$

Corollary 2.2.13. *Let A be a filtered ring and $(A_n)_{n\in\mathbb{Z}}$ its filtration. Define $B = \bigcup_{n\in\mathbb{Z}} A_n$ and $\mathfrak{n} = \bigcap_{n\in\mathbb{Z}} A_n$. If the ring $\mathrm{gr}(A)$ is integral, so is $B/\mathfrak{n}$.*

Proof. If a and b are elements of B not belonging to $\mathfrak{n}$, then $v(a) \neq +\infty$ and $v(b) \neq +\infty$, whence $v(ab) \neq +\infty$ and therefore $ab \notin \mathfrak{n}$. □

2.2.3 Homomorphism compatible with filtrations

Let G, G' be two commutative groups (written additively), (G_n) a filtration on G and (G'_n) a filtration on G'. A homomorphism $\phi : G \to G'$ is called **compatible** with the filtrations on G and G' if $\phi(G_n) \subseteq G'_n$ for all $n \in \mathbb{Z}$. If this holds, then by taking quotients we get a homomorphism $\mathrm{gr}_n(\phi) : G_n/G_{n+1} \to G'_n/G'_{n+1}$ there is therefore a unique additive group homomorphism $\mathrm{gr}(\phi) : \mathrm{gr}(G) \to \mathrm{gr}(G')$ such that, for all $n \in \mathbb{Z}$, $\mathrm{gr}(\phi)$ coincides with $\mathrm{gr}_n(\phi)$ on $\mathrm{gr}_n(G)$. The map $\mathrm{gr}(\phi)$ is called the **graded group homomorphism associated with ϕ**. If G'' is a third filtered group and $\psi : G' \to G''$ is a homomorphism which is compatible with the filtrations, then $\psi \circ \phi$ is a homomorphism which is compatible with the filtrations and we have

$$\mathrm{gr}(\psi \circ \phi) = \mathrm{gr}(\psi) \circ \mathrm{gr}(\phi)$$

Proposition 2.2.14. *Let G be a filtered group and H a subgroup of G; let H be given the induced filtration and G/H the quotient filtration. If $\iota : H \to G$ is the canonical injection and $\pi : G \to G/H$ the canonical surjection, then ι and π are compatible with the filtrations and the sequence*

$$0 \longrightarrow \mathrm{gr}(H) \xrightarrow{\mathrm{gr}(\iota)} \mathrm{gr}(G) \xrightarrow{\mathrm{gr}(\pi)} \mathrm{gr}(G/H) \longrightarrow 0$$

is exact.

Proof. The first assertion is obvious: if (G_n) is the filtration on G, then

$$(H \cap G_n) \cap G_{n+1} = H \cap G_{n+1}.$$

and hence $\mathrm{gr}(\iota)$ is injective; moreover the canonical map $G_n \to (H + G_n)/H$ is surjective, hence so is $\mathrm{gr}(\pi)$ and $\mathrm{gr}(\pi) \circ \mathrm{gr}(\iota) = \mathrm{gr}(\pi \circ \iota) = 0$. Finally, let $\xi \in \mathrm{gr}_n(G)$ belong to the kernel of $\mathrm{gr}(\pi)$; then there exists $x \in \xi$ such that $x \in H + G_{n+1}$; but as $G_{n+1} \subseteq G_n$,

$$G_n \cap (H + G_{n+1}) = (H \cap G_n) + G_{n+1}$$

and hence $x = y + z$ where $y \in H \cap G_n$ and $z \in G_{n+1}$; this proves that ξ is the class mod G_{n+1} of $\iota(y)$, in other words it belongs to the image of $\mathrm{gr}(H)$ under $\mathrm{gr}(\iota)$. □

Example 2.2.15. Note that the functor $\mathrm{gr}(\cdot)$ is not exact. Let $A = A' = k[X]$ where k is a field and give A the trivial filtration while A' the (X)-adic filtration. Then the identity map η from A to A' is compatible with the filtrations and we get a sequence

$$0 \longrightarrow \mathrm{gr}(A) \xrightarrow{\mathrm{gr}(\eta)} \mathrm{gr}(A') \longrightarrow 0$$

It is easy to see the map $\mathrm{gr}(\eta)$ is neither injective nor surjective. The point is that the filtration on A differs from the filtration induced by A' via the map η.

If now A and B are two filtered rings and $\rho : A \to B$ a ring homomorphism which is compatible with the filtrations, it is immediately verified that the graded group homomorphism $\mathrm{gr}(\rho) : \mathrm{gr}(A) \to \mathrm{gr}(B)$ is also a ring homomorphism. In particular, if A' is a subring of A with the induced filtration, $\mathrm{gr}(A')$ is canonically identified with a graded subring of $\mathrm{gr}(A)$ (Proposition 2.2.14). If $\mathfrak{a}$ is an ideal of A and $A/\mathfrak{a}$ is given the quotient filtration, $\mathrm{gr}(A/\mathfrak{a})$ is canonically identified with the quotient graded ring $\mathrm{gr}(A)/gr(\mathfrak{a})$ (Proposition 2.2.14).

Finally, let A be a filtered ring, M, N two filtered A-modules and $\phi : M \to N$ an homomorphism compatible with the filtrations. Then it is immediate that $\mathrm{gr}(\phi) : \mathrm{gr}(M) \to \mathrm{gr}(N)$ is a $\mathrm{gr}(A)$-linear map and hence a graded homomorphism of degree 0 of graded $\mathrm{gr}(A)$-modules. Moreover, if $\psi : M \to N$ is another A-homomorphism compatible with the filtrations, so is $\phi + \psi$ and we gave $\mathrm{gr}(\phi + \psi) = \mathrm{gr}(\phi) + \mathrm{gr}(\psi)$.

2.3 Filtrations and completions

Let G be a commutative group filtered by a family $(G_n)_{n\in\mathbb{Z}}$ of subgroups of G. There exists a unique topology on G which is compatible with the group structure and for which the G_n constitute a fundamental system of neighbourhoods of the identity element 0 of G (**??**); it is called the topology on G defined by the filtration (G_n). When we use topological notions concerning a filtered group, we shall mean, unless otherwise stated, with the topology defined by the filtration. Note that the G_n, being subgroups of G, are both open and closed (**??**). As G is commutative, we deduce that G admits a Hausdorff completion group $\widehat{G}$ (**??**). For evcry subset E of G, the closure of E in G is equal to

$$\bar{E} = \bigcap_{n\in\mathbb{Z}} (E + G_n) = \bigcap_{n\in\mathbb{Z}} (G_n + E).$$

In particular $\bigcap_{n\in\mathbb{Z}} G_n$ is the closure of $\{0\}$; thus it is seen that for the topology on G to be Hausdorff it is necessary and suflicient that the filtration (G_n) be separated. On the other hand, for the topology on G to be discrete, it is necessary and sufficient that there exist $n \in \mathbb{Z}$ such that $G_n = \{0\}$; in this case the filtration (G_n) is called **discrete**.

Not let G' be another filtered group and $\phi : G \to G'$ a homomorphism compatible with the filtrations; the definition of the topologies on G and G' shows immediately that ϕ is continuous. If H is a subgroup of G, the topology induced on H by that on G (resp. the quotient topology with respect to H of that on G) is the topology on H (resp. G/H) defined by the filtration induced by that on G (resp. quotient topology of that on G). The product topology of those on G and G' is the topology defined by the product of filtrations on G and G'.

Proposition 2.3.1. *Let A be a filtered ring, (A_n) its filtration and $B = \bigcup_{n\in\mathbb{Z}} A_n$. If M is a filtered B-module, (M_n) its filtration and $N = \bigcup_{n\in\mathbb{Z}} M_n$ of M, then the map $(a, x) \mapsto ax$ from $B \times N$ to N is continuous.*

Proof. Let $a_0 \in B$, $x_0 \in N$; there exists by hypothesis integers r, s such that $a_0 \in A_r$ and $x_0 \in M_s$. The equality

$$ax - a_0x_0 = (a - a_0)x_0 + a_0(x - x_0) + (a - a_0)(x - x_0)$$

shows that if $a - a_0 \in A_i$ and $x - x_0 \in M_j$, then $ax - a_0x_0$ belongs to $M_{i+s} + M_{j+r} + M_{i+j}$. Then, given an integer n, $ax - a_0x_0 \in M_n$ provided $i \geq n - s$, $j \geq n - r$ and $i + j \geq n$; that is so long as i and j are sufficiently large. □

Corollary 2.3.2. *The ring B is a topological ring and the B-module N is a topological B-module.*

It is seen in particular that a filtered ring A whose filtration is exhaustive is a topological ring; if this is so every filtered A-module whose filtration is exhaustive is a topological A-module.

Example 2.3.3. Let A be a ring and $\mathfrak{a}$ an ideal of A; the topology defined on A by the $\mathfrak{a}$-adic filtration is called the $\mathfrak{a}$-adic topology; as the $\mathfrak{a}$-adic filtration is exhaustive, A is a topological ring with this topology. Similarly, for every A-module M, the topology defined by the $\mathfrak{a}$-adic filtration is called the $\mathfrak{a}$-adic topology on M; M is a topological A-module under this topology.

Proposition 2.3.4. *Let A be a ring filtered by an exhaustive filtration (A_n) and $\mathfrak{p}$ an ideal of A. Suppose that the ideal $\mathrm{gr}(\mathfrak{p})$ of the ring $\mathrm{gr}(A)$ is prime. Then the closure of $\mathfrak{p}$ in A is a prime ideal.*

Proof. We know that $\mathrm{gr}(A/\mathfrak{p})$ is isomorphic to $\mathrm{gr}(A)/\mathrm{gr}(\mathfrak{p})$ (Proposition 2.2.14) and hence an integral domain; we conclude that $A/\bigcap_{n\in\mathbb{Z}}(\mathfrak{p}+A_n)$ is an integral domain by Corollary 2.2.13. Thus the closure $\bar{\mathfrak{p}} = \bigcap_{n\in\mathbb{Z}}(\mathfrak{p}+A_n)$ is a prime ideal. □

2.3.1 Complete filtered groups

Proposition 2.3.5. *Let G be a filtered group with filtration (G_n). The following conditions are equivalent:*

(i) G is a complete topological group.

(ii) The associated Hausdorff group $G' = G/(\bigcap_n G_n)$ is complete.

(iii) Every Cauchy sequence in G is convergent.

(iv) Every family $(x_\lambda)_{\lambda\in I}$ of elements of G' which converges to 0 with respect to the filter $\mathfrak{F}$ complements of finite subsets of I is summable in G'.

Proof. For a filter on G to be a Cauchy filter (resp. a convergent filter), it is necessary and sufficient that its image under the canonical map $G \to G'$ be a Cauchy (resp. convergent) filter (**??**); whence first of all the equivalence of (i) and (ii); on the other hand, as G' is first countable, the equivalence of (i) and (iii) follows.

Suppose that G' is complete and let $(x_\lambda)_{\lambda\in I}$ be a family of elements of G' which converge to 0 with respect to G'. For every neighbourhood V' of 0 in G' which is a subgroup of G', there exists a finite subset J of I such that $x_\lambda \in V'$ whenever $\lambda \in I\setminus J$; then $\sum_{\lambda\in K} x_\lambda \in V'$ for every finite subset K of I not meeting J, which shows that the family $(x_\lambda)_{\lambda\in I}$ is summable. Conversely, suppose that condition (iv) holds and let (x_n) be a Cauchy sequence on G',the family $(x_{n+1}-x_n)$ is then summable and in particular the series with general term $x_{n+1}-x_n$ is convergent and hence the sequence (x_n) is convergent. □

Let G be a filtered group whose filtration (G_n); the quotient groups G/G_n are discrete and hence complete, since the G_n's are open in G. For each $n\in\mathbb{Z}$, we have a canonical map $\pi_n : G \to G/G_n$, and for $m<n$ there is a canonical map $\tau_{mn} : G/G_n \to G/G_m$ such that $(G/G_n, \pi_{mn})$ is an inverse system of discrete groups which is compatible with the maps (π_n). By the universal property of inductive limits, we then get a map $\pi : G \to \varprojlim G/G_n$. If $\iota : G \to \widehat{G}$ is the canonical map of G to its Hausdorff completion $\widehat{G}$, then as the G/G_n are complete, there exists a unique topological group isomorphism $\psi : \widehat{G} \to \varprojlim G/G_n$ such that $\pi = \psi\circ\iota$ (**??**), we shall call it the **canonical isomorphism** of $\widehat{G}$ onto $\varprojlim G/G_n$.

$$
\begin{tikzcd}
G/G_n \arrow[rr, "\tau_{mn}"] & & G/G_m \\
& \varprojlim G/G_n \arrow[ul, "\tau_n"'] \arrow[ur, "\tau_m"] & \\
& G \arrow[u, "\pi"] \arrow[uul, bend left, "\pi_n"] \arrow[uur, bend right, "\pi_m"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\widehat{G} \arrow[r, "\psi"] & \varprojlim G/G_n \\
G \arrow[u, "\iota"] \arrow[ur, "\pi"'] &
\end{tikzcd}
$$

With the result above, we can apply the prperties of inverse limit to completion. First let us observe that the inverse system $\{G/G_n\}$ has the special property that the transition maps $\tau_{mn} : G/G_n \to G/G_m$ are always surjective. Any inverse system with this property we shall call a **surjective system**. For such a system, we have the following result:

Proposition 2.3.6. *If $0 \to \{A_n\} \to \{B_n\} \to \{C_n\} \to 0$ is an exact sequence of inverse systems then we have an exact sequence*

$$0 \to \varprojlim A_n \to \varprojlim B_n \to \varprojlim C_n \to \varprojlim{}^1 A_n \to \varprojlim{}^1 B_n \to \varprojlim{}^1 C_n \to 0$$

Moreover, $\varprojlim^1 A_n = 0$ if $\{A_n\}$ is a surjective system.

Proof. For an inverse system $\{A_n\}$ let $A = \prod_n A_n$ and define $d_A : A \to A$ by

$$d_A((a_n)) = (a_n - \tau_{n,n+1}(a_{n+1}))$$

so that $\ker d_A = \varprojlim A_n$ and we set $\varprojlim^1 A_n := \operatorname{coker} d_A$. The exact sequence of inverse systems then defines a commutative diagram of exact sequences

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "d_A"] & B \arrow[r] \arrow[d, "d_B"] & C \arrow[r] \arrow[d, "d_C"] & 0 \\
0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0
\end{tikzcd}$$

and hence by snake lemma an exact sequence

$$0 \to \ker d_A \to \ker d_B \to \ker d_C \to \operatorname{coker} d_A \to \operatorname{coker} d_B \to \operatorname{coker} d_C \to 0$$

To complete the proof we have only to prove that d_A is surjective if $\{A_n\}$ is a surjective system. But this is clear because to show d_A surjective we have only to solve inductively the equations $x_n - \tau_{n,n+1}(x_{n+1}) = a_n$ for $x_n \in A_n$, given $a_n \in A_n$. □

Corollary 2.3.7. *Let G be a filtered group and H a subgroup of G; let H be given the induced filtration and G/H the quotient filtration. If $\iota : H \to G$ is the canonical injection and $\pi : G \to G/H$ the canonical surjection, then the sequence*

$$0 \longrightarrow \widehat{H} \longrightarrow \widehat{G} \longrightarrow \widehat{G/H} \longrightarrow 0$$

is exact.

Proof. We have an exact sequence of the correspond inverse systems so Proposition 2.3.6 gives the claim immediately. □

Corollary 2.3.8. *Let G be a filtered group with filtration (G_n). Then $\widehat{G}_n$ is a subgroup of $\widehat{G}$ and $\widehat{G}/\widehat{G}_n \cong G/G_n$. Therefore, we have $\operatorname{gr}(G) \cong \operatorname{gr}(\widehat{G})$.*

Proof. Just apply Corollary 2.3.7 with $H = G_n$, in which case G/G_n has the discrete topology so that $\widehat{G/G_n} = G/G_n$. □

Example 2.3.9. Let G be a complete filtered group. Every closed subgroup of G with the induced filtration is complete. Every quotient group of G with the quotient filtration is complete.

Example 2.3.10. Let A be a ring and $\mathfrak{a}$ an ideal. The completion $\widehat{A}$ of A under the $\mathfrak{a}$-adic topology is called the $\mathfrak{a}$-adic completion of A. The canonical map $\iota : A \to \widehat{A}$ is a continuous ring homomorphism, whose kernel is $\bigcap_n \mathfrak{a}^n$.

Let M be an A-module, the completion $\widehat{M}$ of M under the $\mathfrak{a}$-adic topology is a topological $\widehat{A}$-module. If $\phi : M \to N$ is any A-module homomorphism, then $\phi(\mathfrak{a}^n M) = \mathfrak{a}^n \phi(M) \subseteq \mathfrak{a}^n N$, and therefore ϕ is continuous (with respect to the $\mathfrak{a}$-topologies on M and N) and so defines $\widehat{\phi} : \widehat{M} \to \widehat{N}$.

Example 2.3.11. Let A be a filtered ring whose filtration we denote by $(A_n)_{n\in\mathbb{Z}}$; let $B = A[\![X_1,\dots,X_r]\!]$. For all $\alpha = (\alpha_1,\dots,\alpha_s) \in \mathbb{N}^s$, we write $|\alpha| = \sum_{i=1}^s e_i$ and $X^\alpha = X_1^{\alpha_1}\cdots X_s^{\alpha_s}$ so that every element $P \in B$ can be written uniquely $P = \sum_{\alpha\in\mathbb{N}^s} c_{\alpha,P}X^\alpha$, where $c_{\alpha,P} \in A$. For all $n \in \mathbb{Z}$, define

$$B_n = \{P = \sum_\alpha c_{\alpha,P}X^\alpha : c_{\alpha,P} \in A_{n-|\alpha|} \text{ for all } \alpha \in \mathbb{N}^s\}.$$

Clearly B_n is an additive subgroup of B; on the other hand, if $P \in B_n$ and $Q \in B$, then, for all $\alpha \in \mathbb{N}^s$, $c_{\gamma,PQ} = \sum_{\alpha+\beta=\gamma} c_{\alpha,P}c_{\beta,Q}$; as the relation $\alpha+\beta=\gamma$ implies $|\alpha| \le |\gamma|$, we have $PQ \in B_n$, so B_n is an ideal of B. Moreover, if $Q \in B_m$, then for $\alpha+\beta=\gamma$,

$$c_{\alpha,P}c_{\beta,Q} \in A_{n-|\alpha|}A_{n-|\beta|} \subseteq A_{m+n-|\gamma|},$$

which proves that $(B_n)_{n\in\mathbb{Z}}$ is a filtration compatible with the ring structure on B (for obviously $1 \in B_0$). When in future we speak of B as a filtered ring, we shall mean, unless otherwise stated, with the filtration (B_n). Clearly $\bigcap_{n\in\mathbb{Z}} B_n$ is the set of formal power series all of whose coefficients belong to $\bigcap_{n\in\mathbb{Z}} A_n$; then, if A is Hausdorff, so is B. If $A_0 = A$, then $B_0 = B$.

Proposition 2.3.12. *With the above notation, suppose that $A_0 = A$ and let h denote the map $P \mapsto (c_{\alpha,P})_{\alpha\in\mathbb{N}^s}$. Then h is an isomorphism of the additive topological group B onto the additive topological group $A^{\mathbb{N}^s}$. The polynomial ring $A[X_1,\dots,X_s]$ is dense in B; if A is complete, so is B.*

Proof. Clearly h is bijective; $V_n = h(B_n)$ is the set of $(c_\alpha)_{\alpha\in\mathbb{N}^s}$ such that $c_\alpha \in A_{n-|\alpha|}$ for all $\alpha \in \mathbb{N}^s$ such that $|\alpha| \le n$; as these elements α are finite in number, V_n is a neighbourhood of 0 in $A^{\mathbb{N}^s}$. Conversely, if V is a neighbourhood of 0 in $A^{\mathbb{N}^s}$, there is a finite subset E of $\mathbb{N}^s$ and an integer ν such that the conditions $c_\alpha \in A_\nu$ for all $\alpha \in E$ imply $(c_\alpha) \in V$; if then n is the greatest of the integers $\nu + |\alpha|$ for $\alpha \in E$, then $h(A_n) \subseteq V$, which proves the first assertion. Moreover, with n and E defined as above, $h(P - \sum_{\alpha\in E} c_{\alpha,P}X^\alpha) \in V$ for all $P \in B$, which shows that $A[X_1,\dots,X_s]$ is dense in B. The last assertion follows from the first and the fact that a product of complete spaces is complete. □

Let $\mathfrak{a}$ be an ideal of A and suppose that (A_n) is the $\mathfrak{a}$-adic filtration; then, if $\mathfrak{b}$ is the ideal of B generated by $\mathfrak{a}$ and $X_1,\dots,X_s$, the filtration (B_n) is the $\mathfrak{b}$-adic filtration. For clearly, for all $n \ge 0$, $\mathfrak{b}^n$ is generated by the elements cX^α such that $c \in \mathfrak{b}^{n-|\alpha|}$ for all $\alpha \in \mathbb{N}^s$ such that $|\alpha| \le n$, whence $\mathfrak{b}^n \subseteq B_n$. Conversely, for all $P \in B_n$, $P = P_1 + P_2$, where

$$P_1 = \sum_{|\alpha|<n} c_{\alpha,P}X^\alpha, \quad P_2 = \sum_{|\alpha|\ge n} c_{\alpha,P}X^\alpha$$

Clearly it is possible to write $P_2 = \sum_{|\alpha|=n} Q_\alpha X^\alpha$, where the Q_α are elements of B, whence $P_2 \in \mathfrak{b}^n$; on the other hand, since $P \in B_n$, we have $c_{\alpha,P}X^\alpha \in \mathfrak{b}^n$ for all $|\alpha| \le n$, whence $P_1 \in \mathfrak{b}^n$.

Corollary 2.3.13. *Let A be a ring and $B = A[\![X_1,\dots,X_r]\!]$ the ring of formal power series in s indeterminates over A and $\mathfrak{n}$ the ideal of B consisting of the formal power series with no constant term. The ring B is Hausdorff and complete with the $\mathfrak{n}$-adic topology and the polynomial ring $A[X_1,\dots,X_r]$ is everywhere dense in B.*

Proof. It is sufficient to apply what has just been said to the case $\mathfrak{m} = \{0\}$, so that the $\mathfrak{m}$-adic topology on A is trivial and A is thus complete. □

2.3.2 Completion of filtered rings and modules

Let A be a filtered ring, M a filtered A-module and (A_n) and (M_n) the respective filtrations of A and M; we shall assume that these filtrations are exhaustive so that for the corresponding topologies A is a topological ring and M a topological A-module (Proposition 2.3.1). Then we have defined $\widehat{A}$ as a topological ring and $\widehat{M}$ as a topological $\widehat{A}$-module. If $\iota : A \to \widehat{A}$

is the canonical homomorphism, then $\iota(A_m)\iota(A_n) \subseteq \iota(A_{m+n})$, whence by the continuity of multiplication in $\widehat{A}$,

$$\widehat{A}_n\widehat{A}_m \subseteq \widehat{A}_{m+n}$$

where $\widehat{A}_n$ is the closure of $\iota(A_n)$ in A. It can be similarly shown that $\widehat{A}_n\widehat{M}_m \subseteq \widehat{M}_{m+n}$, in other words,

Proposition 2.3.14. *Let A be a filtered ring and M a filtered A-module, the respective filtrations (A_n), (E_n) of A and M being exhaustive. Then $(\widehat{A}_n)$ is a filtration compatible with the ring structure on $\widehat{A}$ and $(\widehat{M}_n)$ is a filtration compatible with the module structure on $\widehat{A}$ over the filtered ring $\widehat{A}$. Moreover, these filtrations are exhaustive and define respectively the topologies on $\widehat{A}$ and $\widehat{M}$. Finally, the canonical maps $\mathrm{gr}(A) \to \mathrm{gr}(\widehat{A})$ and $\mathrm{gr}(M) \to \mathrm{gr}(\widehat{M})$ of graded $\mathbb{Z}$-modules are respectively a graded ring isomorphism and a graded $\mathrm{gr}(A)$ -module isomorphism.*

In what follows, for every uniform space X, ι_X will denote the canonical map from X to its Hausdorff completion $\widehat{X}$ and $X_0 = \iota_X(X)$ the uniform subspace of X, which is the Hausdorff space associated with X. Recall that the topology on X is the inverse image under ι_X of that on X_0, and for every uniformly continuous map $f : X \to Y$, $\hat{f}$ denotes the uniformly continuous map from $\widehat{X}$ to $\widehat{Y}$ such that $\hat{f} \circ \iota_X = \iota_Y \circ f$. If X is a uniform subspace of Y and i is the canonical injection, then $\widehat{X}$ is identified with a uniform subspace of $\widehat{Y}$ and $\hat{i}$ is the canonical injection of $\widehat{X}$ into $\widehat{Y}$ (**??**).

Lemma 2.3.15. *Let $0 \to X \to Y \to Z \to 0$ be an exact sequence of strict morphism of topological groups. Suppose that X, Y, Z admit Hausdorff completion groups and that the identity elements of X, Y, Z admit countable fundamental systems of neighbourhoods. Then*

$$0 \longrightarrow \widehat{X} \longrightarrow \widehat{Y} \longrightarrow \widehat{Z} \longrightarrow 0$$

is an exact sequence of strict morphisms.

Proof. Let $f : X \to Y$ and $g : Y \to Z$ be the given maps. We have already seen that the completion of a strict injection is strict injective, so it remains to show the following: if $Y = X/N$ (where N is a subgroup of X) and $f : X \to Y$ is the canonical map, then $\hat{f}$ is a surjective strict morphism with kernel $\widehat{N}$.

Let $X_0 = \iota_X(X)$ and $Y_0 = \iota_Y(Y)$; let $f_0 : X_0 \to Y_0$ be the map which coincides with $\hat{f}$ on X_0; as ι_X (resp. ι_Y) is a surjective strict morphism of X onto X_0 (resp. Y onto Y_0), f_0 is a surjective strict morphism by **??**. Now X_0 and Y_0 are metrizable (GT, IX, §3, no.1 Proposition 1), so it follows from (GT, IX, §3, no.1 Corollary to Proposition 4 and Lemma 1) that $\hat{f}_0 = \hat{f}$ is a surjective strict morphism and has kernel the closure $\widehat{N}_0'$ in $\widehat{X}$ of the kernel N_0' of f_0. Then it will be sufficient for us to prove that $\widehat{N}_0' = \widehat{N}$. Now N_0' obviously contains $N_0 = \iota_X(N)$, so $\widehat{N}_0' \supseteq \widehat{N}$; it will be sufficient to show that N_0' is contained in the closure $\bar{N}_0$ of N_0 in X_0. Now,

$$U = \iota_X^{-1}(X_0 \setminus \bar{N}_0) = X \setminus \iota_X^{-1}(\bar{N}_0)$$

is an open set in X which does not meet N; as f is a surjective strict morphism, $V = f(U)$ is an open set in Y not containing the identity element 0 of Y and hence not meeting the closure of $\{0\}$; then $\iota_Y(V)$ does not contain the identity element of Y_0. But $\iota_Y(V) = f_0(X_0 \setminus N_0)$ and hence $N_0' \subseteq \bar{N}_0$, which complete the proof. □

Proposition 2.3.16. *Let A be a filtered ring, (A_n) its filtration, M an A-module and (M_n) the filtration on M derived from that on A. Suppose that the filtration (A_n) is exhaustive and the module M is finitely generated. If $\iota_M : M \to \widehat{M}$ is the canonical map, then, for all $n \in \mathbb{Z}$,*

$$\widehat{M}_n = \widehat{A}_n\widehat{M} = \widehat{A}_n\iota_M(M), \quad \widehat{M} = \widehat{A}\widehat{M} = \widehat{A}\iota_M(M).$$

In particular $\widehat{M}$ is a finitely generated $\widehat{A}$-module.

Proof. By hypothesis there exists a surjective homomorphism $\phi : L \to M$, where $L = A^I$, I being a finite set; let L be given the product filtration, consisting of the $L_n = A_n^I$; then $\widehat{L} = \widehat{A}^I$ and $\widehat{L}_n = \widehat{A}_n^I$. Let $\iota_L : L \to \widehat{L}$ be the canonical map and $(e_i)_{i\in I}$ the canonical basis of L; for an element $\sum_i a_i\iota_L(e_i)$ of $\widehat{L}_n$ to belong to $\widehat{L}_n$, it is necessary and sufficient that $a_i \in \widehat{A}_n$ for all i, thus we see $\widehat{L}_n = \widehat{A}_n\widehat{L} = \widehat{A}_n\iota_L(L)$.

Now note that by definition $\phi(L_n) = M_n$ and hence ϕ is a strict morphism of L onto M. Lemma 2.3.15 then shows that $\hat{\phi} : \widehat{L} \to \widehat{M}$ is a surjective strict morphism. As L_n is an open subgroup of $\widehat{L}$, $\hat{\phi}(\widehat{L}_n)$ is an open (and therefore closed) subgroup of M; but by what we have just proved,

$$\hat{\phi}(\widehat{L}_n) = \hat{\phi}(\widehat{A}_n\iota_L(L)) = \widehat{A}_n\iota_M(M).$$

As $\iota_M(M_n) \subseteq \widehat{A}_n\iota_M(M)$, this then implies

$$\widehat{M}_n \subseteq \widehat{A}_n\iota_M(M) \subseteq \widehat{A}_n\widehat{M} \subseteq \widehat{M}_n$$

and therefore $\widehat{M}_n = \widehat{A}_n\widehat{M} = \widehat{A}_n\iota_M(M)$. Take the union of all $n \in \mathbb{Z}$, we obtain the second formula. □

Corollary 2.3.17. *Under the conditions of Proposition 2.3.16, if A is complete, so is M.*

Proof. As the canonical map $A \to \widehat{A}$ is then surjective, $M = \iota_M(M)$ by Proposition 2.3.5 and the conclusion follows from Proposition 2.3.5. □

Corollary 2.3.18. *Let A be a ring, $\mathfrak{a}$ a finitely generated ideal of A and $\widehat{A}$ the Hausdorff ompletion of A with respect to the $\mathfrak{a}$-adic topology. Then $\widehat{\mathfrak{a}^n} = (\hat{\mathfrak{a}})^n = \widehat{A}\mathfrak{a}^n$ for every integer $n > 0$ and the topology on $\widehat{A}$ is the $\hat{\mathfrak{a}}$-adic topology.*

Proof. Let us write $A_n = \mathfrak{a}^n$, which is a finitely generated ideal of A. The formula $\mathfrak{a}^pA_n = \mathfrak{a}^{n+p}$ shows that the topology induced on A_n by the $\mathfrak{a}$-adic topology coincides with the $\mathfrak{a}$-adic topology on the A-module A_n. By Proposition 2.3.16 applied to $M = A_n$, $\widehat{A}_n = \widehat{A}A_n$, in other words $\widehat{\mathfrak{a}^n} = \widehat{A}\mathfrak{a}^n$. In particular $\hat{\mathfrak{a}} = \widehat{A}\mathfrak{a}$, whence $(\hat{\mathfrak{a}})^n = \widehat{A}\mathfrak{a}^n$. □

Example 2.3.19. Let A be a graded ring of type $\mathbb{N}$ and let $(A^n)_{n\in\mathbb{N}}$ be its graduation; let it be given the associated filtration which is separated and exhaustive. The additive group A is canonically identified with the subspace $\bigoplus_{n\in\mathbb{N}} A^n$ in $B = \prod_{n\in\mathbb{N}} A^n$. If B is given the topology the product of the discrete topologies, the topology induced on A is the topology defined by the filtration on A; also B is a complete topological group and A is dense in B. The additive topological group B is then identified with the completion $\widehat{A}$ of the Hausdorff additive group A and it has a unique ring structure which makes it the completion of the topological ring A. To define multiplication in this ring, note that, if we write $A_n = \bigoplus_{i\geq n} A^i$, the closure in B of the ideal A_n is the set B_n of $x = (x_n)$ such that $x_n = 0$ for $i < n$. Let $x = (x_n)$, $y = (y_n)$ be two elements of B and $z = (z_n)$ their product. Then, for all $n > 0$, $x \equiv x^{(n)} \bmod B_{n+1}$ and $y \equiv y^{(n)} \bmod B_{n+1}$. But $x^{(n)}$ and $y^{(n)}$ belong to A and it is therefore seen that, for all $n \in \mathbb{N}$,

$$z_n = \sum_{i+j=n} x_iy_j.$$

In particular, we again obtain that, if A is a ring, the completion of $A[X_1, \dots, X_r]$ with the filtration associated with its usual graduation (by total degree) is canonically identified with the ring of formal power series $A[\![X_1, \dots, X_r]\!]$.

Example 2.3.20. Let α be a non-zero non-invertible element of a principal ideal domain; the (α)-adic topology on A is also called the α-adic topology; it is Hausdorff for the intersection of the ideals (α^n) reduces to 0. Note that the completion of A with respect to this topology is not necessarily an integral domain. The associated graded ring $\mathrm{gr}(A) = \mathrm{gr}(\widehat{A})$ is canonically isomorphic to $(A/\alpha)[X]$. If $A = \mathbb{Z}$, the completion of $\mathbb{Z}$ with respect to the α-adic topology $(n > 1)$ is denoted by $\mathbb{Z}_\alpha$, and its elements are called α-adic integers.

Example 2.3.21. Let k be a field, $A = k[(X_n)_{n\in\mathbb{N}}]$, and $\mathfrak{m} = (X_1, X_2, \dots)$. We will think of an element $f \in \widehat{A}$ as a (possibly) infinite sum

$$f(X) = \sum_I a_I X^I$$

(using multi-index notation) such that for each $d \geq 0$ there are only finitely many nonzero a_I for $|I| = d$. The maximal ideal $\hat{\mathfrak{m}}$ in $\widehat{A}$ is the collection of f with zero constant term. In particular, the element

$$f = X_1 + X_2^2 + X_3^2 + \cdots$$

is in $\hat{\mathfrak{m}}$ but not in $\mathfrak{m}\widehat{A}$. This shows the inclusion $\mathfrak{m}\widehat{A} \subseteq \hat{\mathfrak{m}}$ is proper.

Proposition 2.3.22. *Let A be a ring and $(\mathfrak{a}_\lambda)_{\lambda\in I}$ a family of ideals of A, distinct from A, such that $\mathfrak{a}_\lambda$ and $\mathfrak{a}_\mu$ are relatively prime for $\alpha, \mu \in I$. For every family $s = (s_\lambda)_{\lambda\in I}$ of positive integers of finite support, set $\mathfrak{a}_s = \bigcap_{\lambda\in I} \mathfrak{a}_\lambda^{s_\lambda}$, and let $\mathcal{T}$ be the topology compatible with the ring structure on A such that the $\mathfrak{a}_s$'s form a fundamental system of neighbourhoods of 0. Let $\widehat{A}$ be the Hausdorff completion of A with respect to this topology. On the other hand, for all $\lambda \in I$, let A_λ be the ring A with the $\mathfrak{a}_\lambda$-adic topology and let $\widehat{A}_\lambda$ be its Hausdorff completion. If $\phi : A \to \prod_{\lambda\in I} A_\lambda$ denotes the diagonal homomorphism, then ϕ is continuous and the corresponding homomorphism*

$$\hat{\phi} : \widehat{A} \to \prod_{\lambda\in I} \widehat{A}_\lambda$$

is a topological ring isomorphism.

Proof. Note that $\mathcal{T}$ is the least upper bound of all $\mathfrak{a}_\lambda$-adic topologies. The first assertion follows from **??**. Let us set $B = \prod_{\lambda\in I} A_\lambda$; as the topology on A is finer than each of the $\mathfrak{a}_\lambda$-adic topologies, the maps $\pi_\lambda \circ \phi$ are continuous and hence ϕ is continuous. Also, $\phi(\mathfrak{a}_s)$ is the intersection of the diagonal Δ of B and the open set $\bigcap_{\lambda\in I} \pi_\lambda^{-1}(\mathfrak{a}_\lambda^{s_\lambda})$, it follows that ϕ is a strict morphism from the additive group A to B with image A. Now A is dense in B. For let $b = (a_\lambda)$ be an element of B; every neighbourhood of b in B contains a set of the form $b + V$, where $V = \bigcap_{\lambda\in I} \pi_\lambda^{-1}(\mathfrak{a}_\lambda^{s_\lambda})$ for a family $s = (s_\lambda)$ of positive integers with finite support. As the $\mathfrak{a}_\lambda$'s are relatively prime in pairs, there exists $x \in A$ such that $x \equiv a_\lambda \bmod \mathfrak{a}_\lambda^{s_\lambda}$ for all $\lambda \in I$ (by CRT, note that (s_λ) has finite support) and hence $(b + V) \cap \Delta \neq \varnothing$. The Hausdorff completion of the group B/Δ is then $\{0\}$; applying Lemma 2.3.15 to the exact sequences $0 \to A \to B \to B/\Delta \to 0$, we see $\hat{\phi}$ is an isomorphism of $\widehat{A}$ onto $\widehat{B}$. □

Corollary 2.3.23. *Let A be a PID and P a representative system of elements of A such that the ideals πA with $\pi \in P$ run though all the maximal ideals of A. The topology on A with respect to which the nonzero ideals of A form a fundamental system of neighbourhoods of 0, which is compatible with the ring structure on A, is Hausdorff and the completion of A with this topology is canonically isomorphic to the product of the completions of A with respect to the π-adic topologies, where π runs through P.*

Proof. The principal ideals πA where $\pi \in P$ are maximal and distinct and hence relatively prime, we have already seen that the π-adic topologies are Hausdorff and hence so is the topology defined in the statement of Proposition 2.3.22, which is finer than each of the π-adic topologies. □

If Corollary 2.3.23 is applied when $A = \mathbb{Z}$, we denote by $\widehat{Z}$ the completion of $\mathbb{Z}$ with respect to the topology for which all the nonzero ideals of $\mathbb{Z}$ form a fundamental system of neighbourhoods of 0, the ring isomorphic to the product $\prod_{p\in P} \mathbb{Z}_p$ of the rings of p-adic integers (P being the set of prime numbers), and is usually denoted by $\widehat{\mathbb{Z}}$.

Proposition 2.3.24. *Let A be a ring, $(\mathfrak{m}_i)_{1\leq i\leq r}$ a finite family of distinct maximal ideals of A and set*

$$\mathfrak{t}=\prod_{i=1}^{r}\mathfrak{m}_i=\bigcap_{i=1}^{n}\mathfrak{m}_i,\quad S=\bigcap_{i=1}^{r}(A-\mathfrak{m}_i).$$

Endow A with the $\mathfrak{t}$-adic topology, $S^{-1}A$ the $S^{-1}\mathfrak{t}$-adic topology and each of the local ring $A_{\mathfrak{m}_i}$ the $(\mathfrak{m}_iA_{\mathfrak{m}_i})$-adic topology. Let $\phi:A\to S^{-1}A$ and $\psi:S^{-1}A\to\prod_{i=1}^{r}A_{\mathfrak{m}_i}$ be the canonical homomorphisms. Then ϕ and ψ are continuous and the corresponding homomorphisms $\hat{\phi}$ and $\hat{\psi}$ are topological ring isomorphism.

Proof. Since $\mathfrak{m}_i\cap S=\varnothing$ for each $i\in I$, the ideal $\mathfrak{m}_i'=\mathfrak{m}_iB$ of B is maximal and we have $S^{-1}\mathfrak{t}=\bigcap_{i=1}^{r}\mathfrak{m}_i'$. Also, $B_{\mathfrak{m}_i'}=A_{\mathfrak{m}_i}$ up to a canonical isomorphism. As $\phi^{-1}(\mathfrak{t}B)=\mathfrak{t}$ and $\psi_i^{-1}(\mathfrak{m}_iA_{\mathfrak{m}_i})\supseteq\mathfrak{t}B$, ϕ and ψ are continuous. Then it is sufficient to prove that, if

$$\omega=\psi\circ\phi:A\to\prod_{i=1}^{r}A_{\mathfrak{m}_i}$$

then $\hat{\omega}$ is an isomorphism of $\widehat{A}$ onto $\prod_{i=1}^{r}\widehat{A}_{\mathfrak{m}_i}$, for this result applied to B and the $\mathfrak{m}_i'$ will show that $\hat{\psi}$ is an isomorphism and therefore also $\hat{\phi}$. Note that every product of powers of the $\mathfrak{m}_i$ contains a power of $\mathfrak{t}$ and hence the $\mathfrak{t}$-adic topology is the least upper bound of the $\mathfrak{m}_i$-adic topologies; moreover, if A_i denotes the ring A with the $\mathfrak{m}_i$-adic topology and $\gamma:A\to\prod_{i=1}^{r}A_i$ is the diagonal map, then $\hat{\gamma}:\widehat{A}\to\prod_{i=1}^{r}\widehat{A}_i$ is an isomorphism (Proposition 2.3.22). Then it all amounts to proving that, if $v_i:A_i\to A_{\mathfrak{m}_i}$ is the canonical map, then $\hat{v}_i:\widehat{A}_i\to\widehat{A}_{\mathfrak{m}_i}$ is an isomorphism. Now, for all n, the map

$$v_{i,n}:A/\mathfrak{m}_i^n\to A_{\mathfrak{m}_i}/\mathfrak{m}_i^nA_{\mathfrak{m}_i}$$

derived from v_i by taking quotients is an isomorphism; our assertion follows from the fact that $\widehat{A}_i$ (resp. $\widehat{A}_{\mathfrak{m}_i}$) is the inverse limit of the discrete rings $A/\mathfrak{m}_i^n$ (resp. $A_{\mathfrak{m}_i}/\mathfrak{m}_iA_{\mathfrak{m}_i}$). □

As a particular example, we want to consider the $\mathfrak{m}$-adic completion of a maximal ideal $\mathfrak{m}$ of A. For this, we need the following useful lemma.

Lemma 2.3.25. *Let A be a complete Hausdorff topological ring, in which there exists a fundamental system of neighbourhoods of 0 consisting of additive subgroups of A.*

(a) For all $x\in A$ such that $\lim_n x^n=0$, the element $1-x$ is invertible in A and its inverse is equal to $\sum_n x^n$.

(b) Let $\mathfrak{a}$ be an ideal of A such that $\lim_n x^n=0$ for any $x\in\mathfrak{a}$. For an element y of A to be invertible, it is necessary and suflicient that its class mod $\mathfrak{a}$ be invertible in $A/\mathfrak{a}$. In particular, $\mathfrak{a}$ is contained in the Jacobson radical of A.

Proof. Note that

$$(1-x)(1+x+x^2+\cdots+x^n)=1-x^{n+1}$$

so for (a) it all amounts to proving that the series with general term x^n is convergent in A when $\lim_n x^n=0$; now, by hypothesis, for every symmetric neighbourhood V of 0 in A, there exists an integer $p>0$ such that $x^n\in V$ for all $n\geq p$. We conclude that

$$x^p+\cdots+x^q\in V$$

for all $q\geq p$ and our assertion then follows from Cauchy's criterion, since A is complete.

Suppose that there exists $y\in A$ such that $yz\equiv 1\bmod\mathfrak{a}$. The hypothesis on $\mathfrak{a}$ implies, by (a), that yz is invertible in A and hence y is invertible in A. In particular, every $x\in\mathfrak{a}$ is such that $1-x$ is invertible in A and, as $\mathfrak{a}$ is an ideal of A, it is contained in the Jacobson radical of A. □

Proposition 2.3.26. *Let A be a ring and $\mathfrak{m}$ a maximal ideal of A. The Hausdorff completion A of A with respect to the $\mathfrak{m}$-adic topology is a local ring whose maximal ideal is $\widehat{\mathfrak{m}}$.*

Proof. If $\mathfrak{n} = \bigcap_{n\geq 1}\mathfrak{m}^n$, then $\widehat{A}$ is the completion of the Hausdorff ring $A/\mathfrak{n}$ associated with A and, as $\mathfrak{m}/\mathfrak{n}$ is maximal in $A/\mathfrak{n}$, we may assume that A is Hausdorff with respect to the $\mathfrak{m}$-adic topology. As $A/\mathfrak{m}$ and $\widehat{A}/\widehat{\mathfrak{m}}$ are isomorphic rings, $\widehat{\mathfrak{m}}$ is maximal in $\widehat{A}$. As the topology on A is defined by the filtration $(\mathfrak{m}^n)$, the proposition will be a consequence of Lemma 2.3.25: apply it to the topological ring $\widehat{A}$ and the ideal $\widehat{\mathfrak{m}}$, as, for all $x \in \widehat{\mathfrak{m}}$, $x^n \in (\widehat{\mathfrak{m}})^n \subseteq \widehat{\mathfrak{m}^n}$ and the sequence (x^n) therefore tends to 0. □

Example 2.3.27. If we take $A = \mathbb{Z}$, every maximal ideal of $\mathbb{Z}$ is of the form $p\mathbb{Z}$ where p is prime. The ring of p-adic numbers $\mathbb{Z}_p$ is then a local ring of which $\mathfrak{p}\mathbb{Z}_p$ is the maximal ideal and whose residue field is isomorphic to $\mathbb{Z}/p\mathbb{Z} = \mathbb{F}_p$, and $\mathbb{Z}_{(p)}$ with the $p\mathbb{Z}_{(p)}$-adic topology is identified with a topological subring of $\mathbb{Z}_p$ containing $\mathbb{Z}$.

Corollary 2.3.28. *Let A be a semi-local ring, $\mathfrak{m}_1, \dots, \mathfrak{m}_r$ its distinct maximal ideals and*

$$\mathfrak{t} = \bigcap_{i=1}^{r} \mathfrak{m}_i = \prod_{i=1}^{r} \mathfrak{m}_i$$

its Jacobson radical. Then the Hausdorff completion $\widehat{A}$ of A with respect to the $\mathfrak{t}$-adic topology is a semi-local ring, canonically isomorphic to the product $\prod_{i=1}^{r} \widehat{A}_{\mathfrak{m}_i}$, where $\widehat{A}_{\mathfrak{m}_i}$ is the Hausdorff completion ring of the local ring $A_{\mathfrak{m}_i}$ with respect to the $(\mathfrak{m}_i A_{\mathfrak{m}_i})$-adic topology.

2.3.3 Lifting properties of the associated graduation

Theorem 2.3.29. *Let X, Y be two filtered groups with filtrations (X_n), (Y_n); let $\phi : X \to Y$ be a homomorphism compatible with the filtrations.*

(a) Suppose that the filtration (X_n) is exhaustive. For $\mathrm{gr}(\phi)$ to be injective, it is necessary and sufiicient that $\phi^{-1}(Y_n) = X_n$ for all $n \in \mathbb{Z}$.

(b) Suppose that one of the following hypotheses holds:

(α) X is complete and Y Hausdorff.

(β) Y is discrete.

Then, for $\mathrm{gr}(\phi)$ to be surjective, it is necessary and suficient that $Y_n = \phi(X_n)$ for all $n \in \mathbb{Z}$.

Proof. To say that the map $\mathrm{gr}_n(\phi)$ is injective means that

$$X_n \cap \phi^{-1}(Y_{n+1}) \subseteq X_{n+1}.$$

This is obviously the case if $\phi^{-1}(Y_{n+1}) = X_{n+1}$. Conversely, if this holds for all n, we deduce by induction on k that

$$X_m \cap \phi^{-1}(Y_{n+1}) \subseteq X_{n+1} \quad \text{for all } m, n \in \mathbb{Z} \text{ with } m < n+1.$$

As the filtration (X_n) is exhaustive, we see $\bigcup_{m<n} X_m = X$, hence $\phi^{-1}(Y_{n+1}) \subseteq X_{n+1}$ for all $n \in \mathbb{Z}$ and therefore $\phi^{-1}(Y_{n+1}) = X_{n+1}$, which completes the proof of (a).

To say that the map $\mathrm{gr}_n(\phi)$ is surjective means that

$$Y_n = \phi(X_n) + Y_{n+1}.$$

This is obviously the case if $Y_n = \phi(X_n)$. Conversely, suppose that $Y_n = \phi(X_n) + Y_{n+1}$ for all $n \in \mathbb{Z}$. Let n be an integer and y an element of Y_n; we now define a sequence (x_k) of elements of X_n such that for all $k \geq 0$,

$$x_{k+1} \equiv x_k \mod X_{n+k} \quad \text{and} \quad \phi(x_k) \equiv y \mod Y_{n+k}.$$

First we take x_0 equal to the identity element of X, which certainly gives $\phi(x_0) \equiv y \bmod Y_n$. Suppose that $x_k \in X_n$ has been constructed such that $\phi(x_k) \equiv y \bmod Y_{n+k}$; then $y - \phi(x_k) \in Y_{n+k} = \phi(X_{n+k}) + Y_{n+k+1}$ so there exists $t \in X_{n+k}$ such that $y - \phi(x_k) \equiv \phi(t) \bmod Y_{n+k+1}$ and hence $\phi(x_k + t) \equiv y \bmod Y_{n+k+1}$; it then suffices to take $x_{k+1} = x_k + t$ to carry out the induction.

This being done, if Y is discrete, there exists $k \geq 0$ such that $Y_{n+k} = \{0\}$, whence $\phi(x_k) = y$ and hence in this case it has been proved that $\phi(X_n) = Y_n$ for all n. Suppose now that X is complete and Y Hausdorff. As $x_k - x_j \in X_{n+k}$ for $j \geq k$, (x_k) is a Cauchy sequence in X; as X_n is closed in X and hence complete, this sequence has at least one limit x in X_n. By virtue of the continuity of ϕ, $\phi(x)$ is the unique limit of the sequence $(\phi(x_k))$ in Y, Y being Hausdorff. But the relations $\phi(x_k) \equiv y \bmod Y_{n+k+1}$ show that y is also a limit of this sequence, whence $\phi(x) = y$ and it has also been proved that $\phi(X_n) = Y_n$. □

Corollary 2.3.30. *Suppose that X is Hausdorff and its filtration exhaustive. Then, if $\mathrm{gr}(\phi)$ is injective, ϕ is injective.*

Proof. Since $\phi^{-1}(Y_n) = X_n$, we have $\ker \phi \subseteq \bigcap_n \phi^{-1}(Y_n) = \bigcap_n X_n$, whence the corollary. □

Corollary 2.3.31. *Suppose that one of the following hypotheses holds:*

(α) X is complete, Y is Hausdorff and its filtration is exhaustive;

(β) Y is discrete and its filtration is exhaustive.

Then, if $\mathrm{gr}(\phi)$ is surjective, ϕ is surjective.

Proof. In this case $Y = \bigcup_n Y_n = \bigcup_n \phi(X_n) \subseteq \phi(X)$ by Theorem 2.3.29. □

Corollary 2.3.32. *Suppose that X and Y are Hausdorff, the filtrations of X and Y exhaustive and X complete. Then, if $\mathrm{gr}(\phi)$ is bijective, ϕ is bijective.*

Proposition 2.3.33. *Let A be a filtered ring, M a filtered A-module and (A_n) and (M_n) the respective filtrations on A and M. Suppose that A is complete and the filtration (M_n) is exhaustive and separated. Let $(x_i)_{i \in I}$ be a finite family of elements of M and for $i \in I$ let n_i be an integer such that $x_i \in M_{n_i}$. Finally let ξ_i be the class of x_i in $\mathrm{gr}_{n_i}(M)$. Then, if (ξ_i) is a system of generators of the $\mathrm{gr}(A)$-module $\mathrm{gr}(M)$, then (x_i) is a system of generators of the A-module M.*

Proof. In the A-module $L = A^I$ let L_n denote the set (a_i) such that $a_i \in A_{n-n_i}$ for all $i \in I$; if p and q are the least and greatest of the n_i, then $A^I_{n-p} \subseteq L_n \subseteq A^I_{n-q}$, so the topology defined on L by the definition (L_n) is the same as the product topology; hence L is a complete filtered A-module. As L is free, there exists an A-linear map $\phi : L \to M$ such that $\phi((a_i)) = \sum_i a_i x_i$ and it is obviously compatible with the filtrations; we must prove that ϕ is surjective and for this it is sufficient, by virtue of Corollary 2.3.31, to show that $\mathrm{gr}(\phi)$ is surjective or also that, for all $x \in M_n$, there exist a family (a_i) such that $a_i \in A_{n-n_i}$ for all $i \in I$ and $x \equiv \sum_i a_i x_i \bmod M_{n+1}$. Let ξ be the class of x in $\mathrm{gr}_n(M)$; since the ξ_i generate the $\mathrm{gr}(A)$-module $\mathrm{gr}(M)$, there exist $\alpha_i \in \mathrm{gr}(A)$ such that $\xi = \sum_i a_i \xi_i$, and we may assume that $\alpha_i \in \mathrm{gr}_{n-n_i}(A)$ by replacing if need be α_i by its homogeneous component of degree $n - n_i$. Then α_i the image of an element $a_i \in A_{n-n_i}$ and the family (a_i) has the required property. □

Corollary 2.3.34. *Let A be a complete filtered ring and M a filtered A-module whose filtration is exhaustive and separated. If $\operatorname{gr}(M)$ is a finitely generated (resp. Noetherian) $\operatorname{gr}(A)$-module, then M is a finitely generated (resp. Noetherian) A-module.*

Proof. If $\operatorname{gr}(M)$ is finitely generated, there is a finite system of homogeneous generators and Proposition 2.3.33 shows that M is finitely generated. Suppose now that $\operatorname{gr}(M)$ is Noetherian and let N be a submodule of M; the filtration induced on N by that on M is exhaustive and separated and $\operatorname{gr}(N)$ is identified with a sub-$\operatorname{gr}(A)$-module of $\operatorname{gr}(M)$ by Proposition 2.2.14 and hence is finitely generated by hypothesis; we conclude that N is a finitely generated A-module and hence M is Noetherian. □

Corollary 2.3.35. *Let A be a complete Hausdorff filtered ring whose filtration is exhaustive. If $\operatorname{gr}(A)$ is Noetherian, so is A.*

Corollary 2.3.36. *Let A be a complete filtered ring, (A_n) its filtration, M a Hausdorff filtered A-module, (M_n) its filtration and N a finitely generated submodule of M; suppose that $A_0 = A$ and $M_0 = M$.*

- *(a) If for all $n \in \mathbb{Z}$ we have $M_n = M_{n+1} + A_nN$, then $M = N$.*
- *(b) If the filtration on M is derived from that on A, the relation $M = M_1 + N$ implies $M = N$.*

Proof. Let $\xi_1, \dots, \xi_s$ be the classes mod M_1 of a finite system of generators of N. It follows from the given hypothesis that for all $n \geq 0$ every element of $\operatorname{gr}_n(M)$ can be expressed in the form $\sum_{i=1}^{s} \alpha_i\xi_i$, where $\alpha_i \in \operatorname{gr}(A)$; the ξ_i therefore generate the $\operatorname{gr}(A)$-module $\operatorname{gr}(M)$, which proves (a) by virtue of Proposition 2.3.33. If the filtration on M is derived from that on A, the relation $M = M_1 + N$ implies

$$M_n = A_nM = A_nM_1 + A_nN = A_nA_1M + A_nN \subseteq A_{n+1}M + A_nN = M_{n+1} + A_nN \subseteq M_n$$

and assertion (b) then follows from this. □

Proposition 2.3.37. *Let A be a filtered ring and (A_n) its filtration. Suppose that there exist a subring C of A_0 such that $C \cap A_1 = \{0\}$ and C is identified with a subring of $\operatorname{gr}_0(A)$. Let $(x_1, \dots, x_r)$ be a family of element of A with $x_i \in A_{n_i}$ and let ξ_i be the class of x_i in $\operatorname{gr}_{n_i}(A)$.*

- *(a) If the family (ξ_i) of elements of $\operatorname{gr}(A)$ is algebraically independent over C, then the family (x_i) is algebraically independent over C.*
- *(b) If the filtration on A is exhaustive and discrete and (ξ_i) is a system of generators of the C-algebra $\operatorname{gr}(A)$, then (x_i) is a system of generators of the C-algebra A.*

Proof. Let B be the polynomial algebra $C[X_1, \dots, X_r]$ over C and give B the graduation (B_n) of type $\mathbb{Z}$ where B_n is the set of C-linear combinations of the monomials $X_1^{s_1}, \dots, X_r^{s_r}$ such that $\sum_i n_is_i = n$. Let ϕ be the homomorphism $f \mapsto f(x_1, \dots, x_r)$ from the C-algebra B to the C-algebra A. By definition, $\phi(B_n) \subseteq A_n$ for all $n \in \mathbb{Z}$ and hence ϕ is compatible with the filtrations. The hypothesis of (a) means that $\operatorname{gr}(\phi)$ is injective. As the filtration on B is exhaustive and separated, Corollary 2.3.30 may be applied and ϕ is injective, which proves the conclusion of (a). Similarly, the hypothesis (b) on the (ξ_i) means that $\operatorname{gr}(\phi)$ is surjective. As A is discrete and its filtration is exhaustive, Theorem 2.3.29 may be applied and ϕ is surjective, which proves the conclusion of (b). □

Proposition 2.3.38. *Let A be a complete Hausdorff filtered ring and (A_n) its filtration. Suppose that there exist a subring C of A_0 such that $C \cap A_1 = \{0\}$ and C is identified with a subring of $\operatorname{gr}_0(A)$. Let $(x_1, \dots, x_r)$ be a family of element of A with $x_i \in A_{n_i}$ and let ξ_i be the class of x_i in $\operatorname{gr}_{n_i}(A)$.*

(a) *There exists a unique C-homomorphism ρ from $B = C[\![X_1, \dots, X_r]\!]$ to A such that $\rho(X_i) = x_i$.*

(b) *If the family (x_i) is algebraically independent over C, the homomorphism ρ is injective.*

(c) *If the filtration on A is exhaustive and (ξ_i) is a system of generators of the C-algebra $\operatorname{gr}(A)$, then ρ is surjective.*

Proof. As $n_i \geq 1$ for all i, we have $\sum_i n_i s_i \geq \sum_i s_i$ for every monomial $X_1^{s_1}, \dots, X_r^{s_r}$ and on the other hand $\sum_i n_i s_i \leq N$ if N is the greatest of the n_i. If B_n denotes the set of formal power series whose non-zero terms $a_s X^s$ satisfy $\sum_i n_i s_i \leq n$, it follows from Corollary 2.3.13 that B is Hausdorff and complete with the exhaustive filtration (B_n) and that $B' = C[X_1, \dots, X_r]$ is dense in B. Moreover the homomorphism ϕ defined in the proof of Proposition 2.3.37 is continuum on B' and can be extended uniquely to a continuous homomorphism $\rho : B \to A$, since A is Hausdorff and complete, which proves (a). Also, $\operatorname{gr}(B) = \operatorname{gr}(B')$ and $\operatorname{gr}(v) = \operatorname{gr}(\phi)$, so (b) and (c) follow respectively from Corollary 2.3.30 and Theorem 2.3.29 in view Of the hypotheses on A. □

Let A be a local ring, $\mathfrak{m}$ its maximal ideal and M an A-module; let A and M be given the $\mathfrak{m}$-adic filtrations and let $\operatorname{gr}(A)$ and $\operatorname{gr}(M)$ be the graded ring and the graded $\operatorname{gr}(A)$-module associated with A and M. We have seen (Example 2.2.10) that the canonical map (2.2.4) is always surjective; we are going to consider the following property of M:

(GR) The canonical map $\gamma_M : \operatorname{gr}(A) \otimes_{\operatorname{gr}_0(A)} \operatorname{gr}_0(M) \to \operatorname{gr}(M)$ is bijective.

Proposition 2.3.39. *Let A be a local ring, $\mathfrak{m}$ its maximal ideal, M, N two A-modules and $\phi : N \to M$ an A-homomorphism. Suppose that M and N are given the $\mathfrak{m}$-adic filtrations and*

(a) *M satisfies property (GR);*

(b) *$\operatorname{gr}_0(\phi) : \operatorname{gr}_0(N) \to \operatorname{gr}_0(M)$ is injective.*

Then $\operatorname{gr}(\phi) : \operatorname{gr}(N) \to \operatorname{gr}(M)$ is injective and N and $P = \operatorname{coker}\phi$ satisfy property (GR).

Proof. It is immediately verified that the diagram

$$\begin{tikzcd}
\operatorname{gr}(A) \otimes_{\operatorname{gr}_0(A)} \operatorname{gr}_0(N) \arrow[r, "1\otimes\operatorname{gr}_0(\phi)"] \arrow[d, "\gamma_N"] & \operatorname{gr}(A) \otimes_{\operatorname{gr}_0(A)} \operatorname{gr}_0(M) \arrow[d, "\gamma_M"] \\
\operatorname{gr}(N) \arrow[r, "\operatorname{gr}(\phi)"] & \operatorname{gr}(M)
\end{tikzcd}$$

is commutative. As $\operatorname{gr}_0(A) = A/\mathfrak{m}$ is a field, the hypothesis implies that $1 \otimes \operatorname{gr}_0(\phi)$ is injective; as by hypothesis γ_M is injective, so is $\gamma_M \circ (1 \otimes \operatorname{gr}_0(\phi))$. This implies first that γ_N is injective and hence bijective and therefore that $\operatorname{gr}(\phi)$ is injective. The formula $\phi^{-1}(\mathfrak{m}^n M) = \mathfrak{m}^n N$ is then a consequence of Theorem 2.3.29(a).

Also, let us write $N' = \phi(N)$ and let $\iota : N' \to M$ be the canonical injection. If $\pi : M \to P = M/N'$ is the canonical homomorphism, then in the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{gr}(A) \otimes_{\operatorname{gr}_0(A)} \operatorname{gr}_0(N') \arrow[r] \arrow[d, "\gamma_{N'}"] & \operatorname{gr}(A) \otimes_{\operatorname{gr}_0(A)} \operatorname{gr}_0(M) \arrow[r] \arrow[d, "\gamma_M"] & \operatorname{gr}(A) \otimes_{\operatorname{gr}_0(A)} \operatorname{gr}(P) \arrow[r] \arrow[d, "\gamma_P"] & 0 \\
0 \arrow[r] & \operatorname{gr}(N) \arrow[r, "\operatorname{gr}(\iota)"] & \operatorname{gr}(M) \arrow[r, "\operatorname{gr}(\pi)"] & \operatorname{gr}(P) \arrow[r] & 0
\end{tikzcd}$$

the lower row is exact by Proposition 2.2.14 and so is the upper row by virtue of Proposition 2.2.14 and the fact that $\operatorname{gr}_0(A)$ is a field. The first part of the argument applied to ι shows that $\gamma_{N'}$ is bijective; as γ_M is also bijective by hypothesis, we conclude that γ_P is bijective by the snake lemma. □

Corollary 2.3.40. *Under the hypotheses of Proposition 2.3.39, if we assume also that N is Hausdorff with the $\mathfrak{m}$-adic filtration, then ϕ is injective.*

Proposition 2.3.41. *Let A be a ring, $\mathfrak{m}$ an ideal of A contained in the Jacobson radical of A and M an A-module. Let A and M be given the $\mathfrak{m}$-adic filtrations. Suppose that one of the following conditions holds:*

(a) M is a finitely generated A-module and A is Hausdorff.

(b) $\mathfrak{m}$ is nilpotent.

Then for M to be a free A-module, it is necessary and suffcient that $M/\mathfrak{m}M$ be a free $(A/\mathfrak{m})$-module and that M satisfies property (GR).

Proof. If M is a free A-module and (x_λ) a basis of M, $\mathfrak{m}^n M$ is the direct sum of the submodules $\mathfrak{m}^n x_\lambda$ of M for all $n \geq 0$; then $\mathfrak{m}^n M/\mathfrak{m}^{n+1}M$ is identified with the direct sum of the $\mathfrak{m}^n x_\lambda/\mathfrak{m}^{n+1}x_\lambda$. We deduce first (for $n = 0$) that the classes $1 \otimes x_\lambda$ of the x_λ in $M/\mathfrak{m}M = (A/\mathfrak{m}) \otimes_A M$ form a basis of the $(A/\mathfrak{m})$-module $M/\mathfrak{m}E$, since the canonical map

$$(\mathfrak{m}^n/\mathfrak{m}^{n+1}) \otimes_A (M/\mathfrak{m}M) \to \mathfrak{m}^n M/\mathfrak{m}^{n+1}M$$

is bijective for all $n \geq 0$; hence γ_M is bijective. Note that this part of the proof uses neither condition (a) nor condition (b).

Suppose conversely that the conditions of the statement hold and let $(x_\lambda)_{\lambda\in I}$ be a family of elements of M whose classes mod $\mathfrak{m}M$ form a basis of the $(A/\mathfrak{m})$-module $M/\mathfrak{m}M$; let L be the free A-module $A^{\oplus I}$ and (e_λ) a basis of L; let $\phi : L \to M$ the A-linear map such that $\phi(e_\lambda) = x_\lambda$ for all $\lambda \in I$. Now, each of the hypotheses (a) and (b) implies that A is Hausdorff and hence so is L with the $\mathfrak{m}$-adic filtration, since $\mathfrak{m}^n L = (\mathfrak{m}^n)^{\oplus I}$. Also, since $\mathrm{gr}(L)$ is identified with $\mathrm{gr}(A) \otimes_{A/\mathfrak{m}} (L/\mathfrak{m}L)$ from the first part of the proof, it is possible to write $\mathrm{gr}(\phi)$ as

$$\mathrm{gr}(L) \xrightarrow{\cong} \mathrm{gr}(A) \otimes_{A/\mathfrak{m}} (L/\mathfrak{m}L) \xrightarrow{\eta} \mathrm{gr}(A) \otimes_{A/\mathfrak{m}} (M/\mathfrak{m}M) \xrightarrow{\gamma_M} \mathrm{gr}(M)$$

where η is the bijection map the class $1 \otimes \bar{e}_\lambda$ onto $1 \otimes \bar{x}_\lambda$. The hypothesis then implies that $\mathrm{gr}(\phi)$ is bijective and the conclusion follows from Corollary 2.3.30, **??** and Corollary 2.3.31. □

We have already seen that if A is a ring and $\mathfrak{a}$ is an ideal of A such that A is Hausdorff and complete with the $\mathfrak{a}$-adic topology, then the topological ring A is cannically identified with the inverse limit of the discrete rings $A_i = A/\mathfrak{a}^{i+1}$ $(i \in \mathbb{N})$ with respect to the canonical homomorphisms

$$\pi_{ij} : A/\mathfrak{a}^{j+1} \to A/\mathfrak{a}^{i+1}$$

for $i \leq j$. Note that π_{ij} is surjective and if $\mathfrak{n}_{ij}$ is its kernel, then

$$\mathfrak{n}_{ij} = \mathfrak{a}^{i+1}/\mathfrak{a}^{j+1} = (\mathfrak{a}/\mathfrak{a}^{j+1})^{i+1} = (\mathfrak{n}_{0,j})^{i+1};$$

in particular $(\mathfrak{n}_{0,j})^{j+1} = 0$. Conversely, we have the following result:

Proposition 2.3.42. *Let (A_i, π_{ij}) be an inverse system of discrete rings indexed by $\mathbb{N}$ and let (M_i, u_{ij}) be an inverse system of modules over the inverse system of rings (A_i, π_{ij}). Let $\mathfrak{n}_j$ be the kernel of $\pi_{0,j} : A_j \to A_0$ and set $A = \varprojlim A_i$, $M = \varprojlim M_i$. Suppose that*

(a) for $i \leq j$, π_{ij} and u_{ij} are surjective;

(b) for $i \leq j$, the kernel of π_{ij} and u_{ij} are $\mathfrak{n}_j^{i+1}$ and $\mathfrak{n}_j^{i+1}M_j$, respectively.

Then the following assertions hold:

(i) *A is a complete Hausdorff topological ring, M is a complete topological A-module and the canonical homomorphisms $\pi_i : A \to A_i$, $u_i : M \to M_i$ are surjective.*

(ii) *If M_0 is a finitely generated A_0-module, then M is a finitely generated A-module. In fact, every finite subset E of M such that $u_0(E)$ generates M_0 is a system of generators of M.*

Proof. The first assertion follows from **??** and **??**. For each i, let $\mathfrak{m}_{i+1} = \ker \pi_i$ and $N_{i+1} = \ker u_i$; then by hypothesis (b),

$$\mathfrak{m}_{i+1} = \varprojlim_k (\ker \pi_{i,i+k}) = \varprojlim_k \mathfrak{n}_{i+1}^{i+k}, \quad N_{i+1} = \varprojlim_k (\ker u_{i,i+k}) = \varprojlim_k \mathfrak{n}_{i+k}^{i+1} M_{i+k},$$

and as π_{i+k} and u_{i+k} are surjective, we also have

$$\pi_{i+k}(\mathfrak{m}_{i+1}) = \mathfrak{n}_{i+k}^{i+1}, \quad u_{i+k}(N_{i+1}) = \mathfrak{n}_{i+k}^{i+1} M_{i+k}. \tag{2.3.1}$$

We first prove that $\mathfrak{m}_i N_j \subseteq N_{i+j}$, which amounts to saying that $u_{i+j-1}(\mathfrak{m}_i N_j) = 0$. For this, note that $\mathfrak{n}_k^{k+1} = 0$ as it is the kernel of $\pi_{kk} = \mathrm{id}$, so

$$u_{i+j-1}(\mathfrak{m}_i N_j) = \pi_{i+j-1}(\mathfrak{m}_i) u_{i+j-1}(N_j) = \mathfrak{n}_{i+j-1}^i (\mathfrak{n}_{i+j-1}^j M_{i+j-1}) = 0.$$

Similarly, we see that $\mathfrak{m}_i \mathfrak{m}_j \subseteq \mathfrak{m}_{i+j}$. If we set $\mathfrak{m}_i = A$ and $N_i = M$ for $i \leq 0$, then $(\mathfrak{m}_i)_{i\in\mathbb{Z}}$ is a filtration of A and $(N_i)_{i\in\mathbb{Z}}$ is a filtration of M compatible with that of A, and the topology on A and M are obviously those defined by these filtrations. Now let $\mathfrak{a}$ be an ideal of A such that $\pi_1(\mathfrak{a}) = \mathfrak{n}_1$ and M' be the submodule of M generated by a subset E of M such that $u_0(E)$ generates M_0. We are going to prove that for each $i \geq 0$,

$$N_i = \mathfrak{a}^i M' + N_{i+1}. \tag{2.3.2}$$

Let us write $\mathfrak{a}_i = \pi_i(\mathfrak{a})$ and $M'_i = u_i(M')$. From the definition of N_{i+1}, it then suffices to show that

$$u_i(N_i) = \mathfrak{a}_i^i M'_i.$$

This is true if $i = 0$, since $N_0 = M$ and $M'_0 = M_0$ by hypothesis. If $i \geq 1$, then $u_i(N_i) = \mathfrak{n}_i^i M_i$ by (2.3.1). As $\pi_{1,i}$ is surjective and $\pi_{0,i} = \pi_{0,1} \circ \pi_{1,i}$, the transition homomorphism $\pi_{1,i}$ maps the kernel $\mathfrak{n}_i$ of $\pi_{0,i}$ onto the kernel $\mathfrak{n}_1$ of $\pi_{0,1}$ and $\mathfrak{n}_i = \pi_{1,i}^{-1}(\mathfrak{n}_1)$, so

$$\pi_{1,i}(\mathfrak{a}_i) = \pi_1(\mathfrak{a}) = \mathfrak{n}_1 = \pi_{1,i}(\mathfrak{n}_i).$$

As the kernel of $\pi_{1,i}$ is $\mathfrak{n}_i^2$, we then conclude that $\mathfrak{a}_i \subseteq \mathfrak{n}_i \subseteq \mathfrak{a}_i + \mathfrak{n}_i^2$, whence $\mathfrak{n}_i = \mathfrak{a}_i + \mathfrak{n}_i^2$. On the other hand, we have $u_{0,i}(M'_i) = u_0(M') = M_0 = u_{0,i}(M_i)$, and as $\ker u_{0,i} = \mathfrak{n}_i M_i$ by hypothesis, this implies $M_i = M'_i + \mathfrak{n}_i M_i$, so

$$\mathfrak{n}_i^i M_i = (\mathfrak{a}_i + \mathfrak{n}_i^2)^i (M'_i + \mathfrak{n}_i M_i). \tag{2.3.3}$$

We note that $\mathfrak{a}_i^k \mathfrak{n}_i^{i+1-k} \subseteq \mathfrak{n}_i^{i+1} = 0$ for $0 \leq k \leq i$, so it follows from (2.3.1) and (2.3.3) that

$$u_i(N_i) = \mathfrak{n}_i^i M_i = \mathfrak{a}_i^i M'_i,$$

which proves (2.3.2). Now since $\mathfrak{m}_1 = \pi_1^{-1}(\mathfrak{n}_1)$, we have $\mathfrak{a} \subseteq \mathfrak{m}_1$ and then $\mathfrak{a}^i \subseteq \mathfrak{m}_1^i \subseteq \mathfrak{m}_i$, whence $N_i \subseteq \mathfrak{m}_i M' + N_{i+1}$ in view of (2.3.2). On the other hand, it is clear that $\mathfrak{m}_i M \subseteq N_i$, so $N_i = \mathfrak{m}_i M' + N_{i+1}$ for $i \geq 0$. It then follows from Corollary 2.3.36 that $M' = M$, which completes the proof. □

Corollary 2.3.43. *Under the hypotheses of Proposition 2.3.42, suppose that M_0 is a finitely generated A_0-module and that the ideal $\mathfrak{n}_1$ of A_1 is finitely generated. Let $\mathfrak{m}_{i+1}$ (resp. N_i) be the kernel of the canonical homomorphism $\pi_i : A \to A_i$ (resp. $u_i : M \to M_i$), and set $\mathfrak{m} = \mathfrak{m}_1$. Then*

(a) *For each $i > 0$, we have $\mathfrak{m}_i = \mathfrak{m}^i$ and $N_i = \mathfrak{m}^i M$. In other words, the topology of A and M are the $\mathfrak{m}$-adic topology.*

(b) $\mathfrak{m}/\mathfrak{m}^2$ *is a finitely generated A-module.*

Proof. We preserve the notation of the proof of Proposition 2.3.42. The hypotheses here allow us to assume that the ideal $\mathfrak{a}$ is *finitely generated*. Let i, j be positive integers, then by (2.3.2) we have

$$N_{i+j} = \mathfrak{a}^j(\mathfrak{a}^i M) + N_{i+j+1} \subseteq \mathfrak{m}_j(\mathfrak{a}^i M) + N_{i+j+1}.$$

Conversely, $\mathfrak{m}_j(\mathfrak{a}^i M) \subseteq \mathfrak{m}_j\mathfrak{m}_i M \subseteq \mathfrak{m}_{i+j}N \subseteq N_{i+j}$, so

$$N_{i+j} = \mathfrak{m}_i(\mathfrak{a}^i M) + N_{i+j+1}.$$

As $\mathfrak{a}$ and M are finitely generated A-modules, so is $\mathfrak{a}^i M$. Applying Corollary 2.3.36 to the module N_i with the filtration $(N_{ij})_{j\in\mathbb{Z}}$ defined by $N_{ij} = N_i$ if $j < 0$ and $N_{ij} = N_{i+j}$ if $j \geq 0$, we then obtain $N_i = \mathfrak{a}^i M$, whence $N_i \subseteq \mathfrak{m}^i M$. But we also have $\mathfrak{m}^i M \subseteq \mathfrak{m}_i M \subseteq N_i$, so $N_i = \mathfrak{m}^i M$. In particular, applying this to the case where $M_i = A_i$ and $u_{ij} = \pi_{ij}$, we obtain that $\mathfrak{m}_i = \mathfrak{m}^i$. Moreover, $\mathfrak{m} = \mathfrak{a} + \mathfrak{m}^2$ by (2.3.2), which proves the last assertion of the corollary. □

Corollary 2.3.44. *Under the hypotheses of Corollary 2.3.43, for A to be Noetherian, it is necessary and sufficient that A_0 is Noetherian.*

Proof. This condition is necessary since A_0 is isomorphic to a quotient of A, and is sufficient in view of Corollary 2.3.35. □

2.4 The adic topology on Noetherian rings

Let A be a filtered ring, M a filtered A-module and (A_n) and (M_n) be the corresponding filtrations. Since we only consider the $\mathfrak{a}$-adic filtration, we may assume that all filtrations are exhaustive and consist of submodules. The **Rees ring** of A is then defined by

$$\mathrm{Rees}(A) = \bigoplus_{n\in\mathbb{N}} A_n X^n \subseteq A[X]$$

and the **Rees mudule** of M is defined by

$$\mathrm{Rees}(M) = \bigoplus_{n\in\mathbb{N}} M_n \otimes_A AX^n \subseteq M \otimes_A A[X].$$

Note that $\mathrm{Rees}(M)$ is a graded $\mathrm{Rees}(A)$-module of type $\mathbb{N}$, since we have

$$A_m X^m (M_n \otimes_A AX^n) \subseteq (A_m M_n \otimes_A AX^{m+n}) \subseteq M_{m+n} \otimes_A AX^{m+n}.$$

The most important case is that $\mathfrak{a}$ is an ideal of A and A, M are given the $\mathfrak{a}$-adic filtration. In this case we write $\mathrm{Rees}_{\mathfrak{a}}(A)$ for the Rees ring $\bigoplus_{n\in\mathbb{N}} \mathfrak{a}^n X^n$ and $\mathrm{Rees}_{\mathfrak{a}}(M)$ for $\bigoplus_{n\in\mathbb{N}} \mathfrak{a}^n M \otimes_A AX^n$.

2.4.1 Good filtrations

Let A be a ring, $\mathfrak{a}$ an ideal of A, M an A-module and (M_n) a filtration on the additive group M consisting of submodules of M. The filtrations (M_n) is called **$\mathfrak{a}$-good** if:

(a) $\mathfrak{a}M_n \subseteq M_{n+1}$ for all $n \in \mathbb{Z}$.

(b) There exists an integer n_0 such that $\mathfrak{a}M_n = M_{n+1}$ for all $n \geq n_0$.

If (M_n) is $\mathfrak{a}$-good, then by induction on we see $\mathfrak{a}^p M_n = M_{n+p}$ for all $n \geq n_0$ and $p > 0$. Note that condition (a) means that the filtration (M_n) is compatible with the A-module structure on M if A is given the $\mathfrak{a}$-adic filtration. Clearly, on every A-module M, the $\mathfrak{a}$-adic filtration is $\mathfrak{a}$-good.

Theorem 2.4.1. *Let A be a ring, $\mathfrak{a}$ an ideal of A, and give A the $\mathfrak{a}$-adic filtration. Let M be an A-module and (M_n) a filtration of M consisting of finitely generated submodules. Suppose that $\mathfrak{a}M_n \subseteq M_{n+1}$ for all n. Then the two following conditions are equivalent:*

(a) The filtration (M_n) is $\mathfrak{a}$-good.

(b) $\mathrm{Rees}_{\mathfrak{a}}(M)$ is a finitely generated $\mathrm{Rees}_{\mathfrak{a}}(A)$-module.

Proof. Suppose that $\mathfrak{a}M_n = M_{n+1}$ for $n \geq n_0 \geq 0$. For $i \leq n_0$, let $(e_{ij})_{1 \leq j \leq r_i}$ be a finite system of generators of the A-module M_i. As the A-module $M_n \otimes_A AX^n$ is generated by the elements $e_{nj} \otimes X^n$ for $0 \leq n \leq n_0$ and is equal to

$$\mathfrak{a}^{n-n_0} M_{n_0} \otimes_A AX^n$$

for $n > n_0$, the $\mathrm{Rees}(A)$-module $\mathrm{Rees}(M)$ is generated by the elements $e_{nj} \otimes X^n$ for $0 \leq n \leq n_0$ and $1 \leq j \leq r_n$; then it is certainly finitely generated.

Conversely, if $\mathrm{Rees}(M)$ is a finitely generated $\mathrm{Rees}(A)$-module, it is generated by a finite family of elements of the form $e_k \otimes X^{n_k}$, where $e_k \in M_{n_k}$. Let n_0 be the greatest of the integers n_k. Then for $n \geq n_0$ and $x \in M_n$,

$$x \otimes X^n = \sum_k t_k (e_k \otimes X^{n_k})$$

where $t_k \in \mathrm{Rees}(A)$; replacing if need be to by its homogeneous component of degree $n - n_k$, we may assume that $t_k = a_k X^{n-n_k}$ where $a_k \in \mathfrak{a}^{n-n_k}$. As the unique element X^n forms a basis of the A-module AX^n, the equation $x \otimes X^n = (\sum_k a_k e_k) \otimes X^n$ implies $x = \sum_k a_k e_k$. Then $M_n \subseteq \mathfrak{a}^{n-n_0} M_{n_0}$; since the opposite inclusion is obvious by hypothesis, we see $M_n = \mathfrak{a}^{n-n_0} M_{n_0}$, whence $\mathfrak{a}M_n = M_{n+1}$ for $n \geq n_0$. □

Lemma 2.4.2. *Let A be a Noetherian ring and $\mathfrak{a}$ an ideal of A. Then the subring $\mathrm{Rees}_{\mathfrak{a}}(A)$ of $A[X]$ is Noetherian.*

Proof. Note that $\mathrm{Rees}_{\mathfrak{a}}(A)$ is an A-algebra generated by $\mathfrak{a}X$; as A is Noetherian, $\mathfrak{a}X$ is a finitely generated A-module, so the claim follows from Hilbert basis theorem. □

Proposition 2.4.3. *Let A be a filtered ring, $\mathfrak{a}$ an ideal of A, and M is an A-module with a $\mathfrak{a}$-good filtration. If N is a submodule of M then the quotient filtration on M/N is $\mathfrak{a}$-good. If further A is Noetherian, then the submodule filtration on N is $\mathfrak{a}$-good.*

Proof. If N and M/N are endowed with the submodule (resp. quotient) filtrations respectively, then $\mathrm{Rees}_{\mathfrak{a}}(N)$ is a $\mathrm{Rees}_{\mathfrak{a}}(A)$-submodule of $\mathrm{Rees}_{\mathfrak{a}}(M)$ and we have $\mathrm{Rees}_{\mathfrak{a}}(M/N) \cong \mathrm{Rees}_{\mathfrak{a}}(M)/\mathrm{Rees}_{\mathfrak{a}}(N)$ as $\mathrm{Rees}_{\mathfrak{a}}(A)$-modules. Since the filtration on M is $\mathfrak{a}$-good, $\mathrm{Rees}_{\mathfrak{a}}(M)$ is a finitely generated $\mathrm{Rees}_{\mathfrak{a}}(A)$-module and so $\mathrm{Rees}_{\mathfrak{a}}(M/N)$ is finitely generated, which implies the filtration on M/N is $\mathfrak{a}$-good. If further A is Noetherian then $\mathrm{Rees}_{\mathfrak{a}}(A)$ is Noetherian, and so $\mathrm{Rees}_{\mathfrak{a}}(N)$ is finitely generated. It then follows that the filtration on N is $\mathfrak{a}$-good. □

Corollary 2.4.4 (Artin-Rees Lemma). *Let A be a Noetherian ring, $\mathfrak{a}$ an ideal of A, M a finitely generated A-module and N a submodule of M. The filtration induced on N by the $\mathfrak{a}$-adic filtration on M is $\mathfrak{a}$-good. In other words, there exists an integer n_0 such that for $n \geq n_0$ we have*

$$\mathfrak{a}(\mathfrak{a}^n M \cap N) = \mathfrak{a}^{n+1} M \cap N.$$

Corollary 2.4.5. *Let A be a Noetherian ring and $\mathfrak{a}$, $\mathfrak{b}$ two ideals of A. Then there exists an integer $n > 0$ such that $\mathfrak{a}^n \cap \mathfrak{b} \subseteq \mathfrak{a}\mathfrak{b}$.*

Proof. By the Artin-Rees Lemma, there exist $n > 0$ such that $\mathfrak{a}^{n+1} \cap \mathfrak{b} = \mathfrak{a}(\mathfrak{a}^n \cap \mathfrak{b}) \subseteq \mathfrak{a}\mathfrak{b}$. □

Corollary 2.4.6. *Let A be a Noetherian ring, $\mathfrak{a}$ an ideal of A and x an element of A which is not a divisor of 0. There exists an integer $n_0 > 0$ such that, for all $n \geq n_0$, we have $(\mathfrak{a}^n : x) \subseteq \mathfrak{a}^{n-n_0}$.*

Proof. Corollary 2.4.4 applied to $M = A$, $N = Ax$ shows that there exists n_0 such that, for all $n \geq n_0$, $\mathfrak{a}^n \cap Ax = \mathfrak{a}^{n-n_0}(\mathfrak{a}^{n_0} \cap Ax)$. Then, if $xy \in \mathfrak{a}^n$,

$$xy \in \mathfrak{a}^n \cap Ax \subseteq \mathfrak{a}^{n-n_0}x$$

and, as x is not a divisor of 0, we deduce that $y \in \mathfrak{a}^{n-n_0}$. □

Corollary 2.4.7. *Let A be a Noetherian ring, $\mathfrak{a}$ an ideal of A, M a finitely generated A-module and (M_n) and (M'_n) two filtrations consisting of submodules of M. Suppose that the filtrations (M_n) and (M'_n) are compatible with the A-module structure on M when A is given the $\mathfrak{a}$-adic filtration. If the filtration (M_n) is $\mathfrak{a}$-good and $M'_n \subseteq M_n$ for all $n \in \mathbb{Z}$, then the filtration (M'_n) is $\mathfrak{a}$-good.*

Proof. This is a special case of Proposition 2.4.3. □

Lemma 2.4.8. *Let A, B be two Noetherian rings, $\rho : A \to B$ a ring homomorphism, M a finitely generated A-module and N a finitely generated B-module. Then $\mathrm{Hom}_A(M, N)$ is a finitely generated B-module.*

Proof. There exists by hypothesis a surjective A-homomorphism $\eta : A^n \to M$; the map $\phi \mapsto \phi \circ \eta$ of $\mathrm{Hom}_A(M, N)$ to $\mathrm{Hom}_A(A^n, N)$ is therefore injective and, as B is Noetherian, it is sufiicient to prove that $\mathrm{Hom}_A(A^n, N)$ is a finitely generated B-module; which 1s immediate since it is isomorphic to N^n. □

Proposition 2.4.9. *Let A be a Noetherian ring, $\mathfrak{a}$ an ideal of A and M, N two finitely generated A-modules. If (N_n) is an $\mathfrak{a}$-good filtration on N, the submodules $\mathrm{Hom}_A(E, N_n)$ form an $\mathfrak{a}$-good filtration on the A-module $\mathrm{Hom}_A(M, N)$.*

Proof. As $\mathfrak{a}^k N_n \subseteq N_{n+k}$ for $n \in \mathbb{Z}$ and $k \geq 0$, it is also true that

$$\mathfrak{a}^k \mathrm{Hom}_A(M, N_n) \subseteq \mathrm{Hom}_A(M, N_{n+k})$$

the family $(\mathrm{Hom}_A(M, N_n))$ is then a filtration on $\mathrm{Hom}_A(M, N)$ compatible with its module structure over the ring A filtered by the $\mathfrak{a}$-adic filtration. Since M is finitely generated, there exists an integer $r > 0$ and a surjective A-homomorphism $\eta : A^r \to M$ which defines an injective A-homomorphism

$$\eta^* : \mathrm{Hom}_A(M, N) \to \mathrm{Hom}_A(A^r, N)$$

clearly η^* is compatible with the filtrations $(\mathrm{Hom}_A(M, N))$ and $(\mathrm{Hom}_A(A^r, N))$. As the A-modules $\mathrm{Hom}_A(M, N)$ and $\mathrm{Hom}_A(A^r, N)$ are finitely generated (Lemma 2.4.8), it is sufficient by virtue of Proposition 2.4.3 to show that the filtration $(\mathrm{Hom}_A(A^r, N))$ is $\mathfrak{a}$-good; but this is immediate by virtue of the existence of the canonical isomorphism $\mathrm{Hom}_A(A^r, N) \to N^r$, and the fact that the relation $\mathfrak{a}N = N_{n+1}$ implies $\mathfrak{a}(N_n^r) = (\mathfrak{a}N_n)^r = N_{n+1}^r$. □

Proposition 2.4.10. *Let A be a Noetherian ring and $\mathfrak{a}$ an ideal of A such that A is Hausdorff and complete with respect to the $\mathfrak{a}$-adic topology. Let M be a filtered A-module over the filtered ring A, the filtration (M_n) of A being such that $M_0 = M$ and M is Hausdorff with respect to the topology defined by (M_n). Then the following conditions are equivalent:*

(i) M *is a finitely generated* A*-module and* (M_n) *is an* $\mathfrak{a}$*-good filtration.*

(ii) $\mathrm{gr}(M)$ *is a finitely generated* $\mathrm{gr}(A)$*-module.*

(iii) $\mathrm{gr}_n(M)$ *is a finitely generated* A*-module for all* n *and there exists* n_0 *such that for* $n \geq n_0$ *the canonical homomorphism* $\mathrm{gr}_1(A) \otimes_A \mathrm{gr}_n(M) \to \mathrm{gr}_{n+1}(M)$ *is surjective.*

Proof. It follows immediately from the definitions that (i) implies (iii). Also, (ii) is equivalent to (iii): the fact that (ii) implies (iii) is a consequence of Corollary 2.1.41; conversely, if (iii) holds, clearly $\mathrm{gr}(M)$ is generated as a $\mathrm{gr}(A)$-module by the sum of the $\mathrm{gr}_p(M)$ for $p \leq n_0$ and hence by hypothesis admits a finite system of generators. It remains to prove that (iii) implies (i).

Assume (iii), as the $\mathrm{gr}_n(M)$ are finitely generated and $M_0 = M$, clearly first, by induction on n, M/M_n is a finitely generated A-module for all n; it will therefore be sufficient to prove that, for $n > n_0$, M_n is a finitely generated A-module and that $\mathfrak{a}M_n = M_{n+1}$. Now, consider the A-module M_{n+1} with the exhaustive and separated filtration $(M_{n+k})_{k\geq 1}$. Now since $\mathrm{gr}_1(A) = \mathfrak{a}/\mathfrak{a}^2$, hypothesis (iii) implies that the image of $\mathfrak{a}M_n$ in $\mathrm{gr}_{n+1}(M)$ is equal to $\mathrm{gr}_{n+1}(M)$ and generates the graded $\mathrm{gr}(A)$-module $\mathrm{gr}(M_{n+1})$. As $\mathrm{gr}_{n+1}(M)$ is by hypothesis a finitely generated A-module, it follows from Proposition 2.3.33 that $\mathfrak{a}M_n = M_{n+1}$ and that M_{n+1} is a finitely generated A-module. □

Remark 2.4.11. Note that in the proof of Proposition 2.4.10, only the implication (iii)⇒(i) uses the hypothesis A is $\mathfrak{a}$-adic complete and Hausdorff. In particular, we see if A is Noetherian and the filtration (M_n) is $\mathfrak{a}$-good then $\mathrm{gr}(M)$ is a finitely generated $\mathrm{gr}(A)$-module.

2.4.2 The adic topology on Noetherian rings

Proposition 2.4.12. *Let* A *be a Noetherian ring,* $\mathfrak{a}$ *an ideal of* A *and* M *a finitely generated* A*-module. All the* $\mathfrak{a}$*-good filtrations on* M *define the same topology on* M*, namely the* $\mathfrak{a}$*-adic topology.*

Proof. Let (M_n) be an $\mathfrak{a}$-good filtration on M. As this filtration is exhaustive, every element of M belongs to one of the M_n and, as M is finitely generated (hence Noetherian) and the M_n are A-modules, there exists an integer n_1 such that $M_{n_1} = M$. On the other hand let n_0 be such that $\mathfrak{a}M_n = M_{n+1}$ for $n \geq n_0$; then, for $n > n_0 - n_1$,

$$\mathfrak{a}^n M \subseteq M_{n+n_1} = \mathfrak{a}^{n+n_1-n_0} M_{n_0} \subseteq \mathfrak{a}^{n+n_1-n_0} M,$$

which proves the proposition. □

Theorem 2.4.13 (Krull). *Let* A *be a Noetherian ring,* $\mathfrak{a}$ *an ideal of* A*,* M *a finitely generated* A*-module and* N *a submodule of* M*. Then the* $\mathfrak{a}$*-adic topology on* N *is induced by the* $\mathfrak{a}$*-adic topology on* M*.*

Proof. It follows from Artin-Rees Lemma that the filtration induced on N by the $\mathfrak{a}$-adic filtration on M is $\mathfrak{a}$-good and the conclusion then follows from Proposition 2.4.12. □

Corollary 2.4.14. *Let* A *be a Noetherian ring,* $\mathfrak{a}$ *an ideal of* A*,* M *an* A*-module and* N *a finitely generated* A*-module. Then every homomorphism* $\phi : M \to N$ *is a strict morphism for the* $\mathfrak{a}$*-adic topologies.*

Proof. As $\phi(\mathfrak{a}^n M) = \mathfrak{a}^n \phi(M)$, ϕ is a strict morphism of M onto $\phi(M)$ for the $\mathfrak{a}$-adic topologies on these two modules and the $\mathfrak{a}$-adic topology on $\phi(M)$ is induced by the $\mathfrak{a}$-adic topology on N by Theorem 2.4.13, hence the claim. □

Corollary 2.4.15. *Let* $0 \to M_1 \to M_2 \to M_3 \to 0$ *be an exact sequence of finitely-generated modules over a Noetherian ring* A*. Let* $\mathfrak{a}$ *be an ideal of* A*, then the sequence of* $\mathfrak{a}$*-adic completions*

$$0 \longrightarrow \widehat{M}_1 \longrightarrow \widehat{M}_2 \longrightarrow \widehat{M}_3 \longrightarrow 0$$

is exact.

Proposition 2.4.16. *Let A be a Noetherian ring, $\mathfrak{a}$ an ideal of A and M a finitely generated A-module. The closure $\bigcap_{n=0}^{\infty} \mathfrak{a}^n M$ of $\{0\}$ in M with respect to the $\mathfrak{a}$-adic topology is the set of $x \in M$ for which there exists an element $a \in \mathfrak{a}$ such that $(1-a)x = 0$.*

Proof. If $x = ax \bmod \mathfrak{a}$ where $a \in \mathfrak{a}$, then $x = a^n x$ for every integer $n > 0$ and hence $x \in \bigcap_n \mathfrak{a}^n M$. Conversely, if $x \in \bigcap_n \mathfrak{a}^n M$, Ax is contained in the intersection of the neighbourhoods of 0 in M; it then follows from Theorem 2.4.13 that the $\mathfrak{a}$-adic topology on Ax, which is induced by that on M, is the trivial topology; as $\mathfrak{a}x$ is by definition a neighbourhood of 0 with this topology, $\mathfrak{a}x = Ax$ and hence there exists $a \in \mathfrak{a}$ such that $x = ax$. □

Corollary 2.4.17 (Krull's Intersection Theorem). *Let A be a Noetherian ring and $\mathfrak{a}$ an ideal of A. Then the ideal $\bigcap_n \mathfrak{a}^n$ is the set of elements $x \in A$ for which there exists $a \in \mathfrak{a}$ such that $(1-a)x = 0$. In particular, for $\bigcap_n \mathfrak{a}^n = \{0\}$, it is necessary and sufficient that no element of $1 + \mathfrak{a}$ is a divisor of 0 in A.*

Remark 2.4.18. The hypothesis that A is Noetherian is essential in this corollary. For example, let A be the ring of infinitely differentiable maps from $\mathbb{R}$ to itself and let $\mathfrak{m}$ be the (maximal) ideal of A consisting of the functions f such that $f(0) = 0$. It is immediate that $\bigcap_n \mathfrak{m}^n$ is the set of functions f such that $f^{(n)}(0) = 0$ for all $n \geq 0$ and there exist such functions with $f(x) \neq 0$ for all $x \neq 0$, for example the function f defined by $f(x) = e^{-1/x^2}$ for $x \neq 0$ and $f(0) = 0$. Also, note that for such an f, the elment $1 + f$ is even invertible in A.

2.4.3 The adic completion of a Noetherian ring

Let A be a ring, $\mathfrak{a}$ an ideal of A and M an A-module; let $\widehat{A}$ and $\widehat{M}$ denote the respective Hausdorff completions of A and M with respect to the $\mathfrak{a}$-adic topology and $\iota_M : M \to \widehat{M}$ the canonical map. The A-bilinear map $(a, x) \mapsto a\iota_M(x)$ of $\widehat{A} \times M$ to $\widehat{M}$ defines a canonical A-linear map

$$\alpha_M : \widehat{A} \otimes_A M \to \widehat{M}.$$

Let $\phi : M \to N$ be an A-module homomorphism and let $\hat{\phi} : \widehat{M} \to \widehat{N}$ be the map obtained by passing to the Hausdorff completions; for $a \in \widehat{A}$ and $x \in M$,

$$\alpha_N(a \otimes \phi(x)) = a\iota_M(\phi(x)) = a\hat{\phi}(\iota_M(x)) = \hat{\phi}(\alpha_M(a \otimes x)).$$

In other words, the following diagram is commutative:

$$\begin{tikzcd} \widehat{A} \otimes_A M \arrow[r, "1\otimes\phi"] \arrow[d, "\alpha_M"] & \widehat{A} \otimes N \arrow[d, "\alpha_N"] \\ \widehat{M} \arrow[r, "\hat{\phi}"] & \widehat{N} \end{tikzcd}$$

Finally, recall that by Proposition 2.3.16, if M is finitely generated then the homomorphism α_M is surjective.

Theorem 2.4.19. *Let A be a Noetherian ring, $\mathfrak{a}$ an ideal of A and M, N, L be finitely generated A-modules. Then*

(a) If $0 \to M \to N \to L \to 0$ is an exact sequence of A-modules, then the sequence

$$0 \longrightarrow \widehat{M} \longrightarrow \widehat{N} \longrightarrow \widehat{L} \longrightarrow 0$$

obtained by passing to the Hausdorff completions (with respect to the $\mathfrak{a}$-adic topologies) is exact.

(b) The canonical $\widehat{A}$-linear map $\alpha_M : \widehat{A} \otimes M \to \widehat{M}$ is bijective.

(c) The A-module $\widehat{A}$ is flat.

Proof. We have seen that M and L carry the submodule filtration and quotient filtration, respectively, so claim (a) follows from Lemma 2.3.15. Assertion (b) is obvious when $M = A$ and the case where M is a free finitely generated A-module can be immediately reduced to that. In the general case, M admits a finite presentation

$$0 \longrightarrow N \longrightarrow A^n \longrightarrow M \longrightarrow 0$$

so we derive a commutative diagram

$$\begin{tikzcd} & \widehat{A}\otimes_A N \arrow[r] \arrow[d, "\alpha_N"] & \widehat{A}\otimes_A A^n \arrow[r] \arrow[d, "\alpha_{A^n}"] & \widehat{A}\otimes_A M \arrow[r] \arrow[d, "\alpha_M"] & 0 \\ 0 \arrow[r] & \widehat{N} \arrow[r] & \widehat{A}^n \arrow[r] & \widehat{M} \arrow[r] & 0 \end{tikzcd}$$

The first row is exact and so is the second by (a). Since A is Noetherian we know that N is also finitely generated, so α_M and α_N are both surjective. Since α_{A^n} is bijective, we then deduce that α_M is injective by the snake lemma.

Then it follows from (a) and (b) that, if $\mathfrak{a}$ is an ideal of A (necessarily finitely generated), the canonical map $\widehat{A}\otimes_A \mathfrak{a} \to \widehat{A}$ is injective, being the composition of $\hat{\mathfrak{a}} \to \widehat{A}$ and $\alpha_{\mathfrak{a}}$, which proves that A is a flat A-module. □

Under the conditions of Theorem, $\widehat{A}\otimes_A M$ is often identified with M by means of the canonical map α_M. If $\phi : M \to N$ is a homomorphism of finitely generated A-modules, $\hat{\phi}$: $\widehat{M} \to \widehat{N}$ is then identified with $1 \otimes \phi$ by virtue of the commutativity of diagram.

Corollary 2.4.20. *Let A be a Noetherian ring, $\mathfrak{a}$ an ideal of A and $\widehat{A}$ the Hausdorff completion of A with respect to the $\mathfrak{a}$-adic topology. If an element $a \in A$ is not a divisor of zero in A, its canonical image $\hat{a}$ in A is not a divisor of zero in $\widehat{A}$.*

Proof. This follows from the fact that $\widehat{A}$ is a flat A-module. □

Corollary 2.4.21. *Let A be a Noetherian ring, $\mathfrak{a}$ an ideal of A, M a finitely generated A-module and N and P two submodules of M. With the $\mathfrak{a}$-adic topologies and let ι_M be the canonical map from M to $\widehat{M}$, then*

$$\widehat{N} = \widehat{A}\iota_M(N), \quad \widehat{(N+P)} = \widehat{N} + \widehat{P}, \quad \widehat{N\cap P} = \widehat{N}\cap\widehat{P}, \quad \widehat{(N:P)} = (\widehat{N}:\widehat{P}).$$

Moreover, if $\mathfrak{a}$ and $\mathfrak{b}$ are two ideals of A, then $\widehat{\mathfrak{a}\mathfrak{b}} = \hat{\mathfrak{a}}\hat{\mathfrak{b}}$.

Proof. By Theorem 2.4.19, $\widehat{M}$, $\widehat{N}$, $\widehat{P}$ are canonically identified with $\widehat{A}\otimes_A M$, $\widehat{A}\otimes_A N$, and $\widehat{A}\otimes_A P$, which establishes the formulas. Finally, as $\hat{\mathfrak{a}} = \widehat{A}\mathfrak{a}$, $\hat{\mathfrak{b}} = \widehat{A}\mathfrak{b}$, we see that

$$\widehat{\mathfrak{a}\mathfrak{b}} = \widehat{A}\mathfrak{a}\mathfrak{b} = \widehat{A}\mathfrak{a}\widehat{A}\mathfrak{b} = \hat{\mathfrak{a}}\hat{\mathfrak{b}},$$

and this finishes the proof. □

Corollary 2.4.22. *Let A be a Noetherian ring, $\mathfrak{a}$ an ideal of A and $\widehat{A}$ the Hausdorff completion of A with respect to the $\mathfrak{a}$-adic topology. Then the topology on $\widehat{A}$ is the $\hat{\mathfrak{a}}$-adic topology.*

Proof. By Corollary 2.4.21 we have $\widehat{(\mathfrak{a}^n)} = (\hat{\mathfrak{a}})^n$. Since $\widehat{(\mathfrak{a}^n)}$ is a fundamental system of neighbourhoods in $\widehat{A}$ by **??**, the claim follows. □

Corollary 2.4.23. *If A is a Noetherian ring, the ring of normal power series $A[\![X_1, \dots, X_n]\!]$ is a flat A-module.*

Proof. The ring $A[\![X_1,\dots,X_n]\!]$ is the completion of $B = A[X_1,\dots,X_n]$ with respect to the $\mathfrak{m}$-adic topology, where $\mathfrak{m}$ is the set of polynomials with no constant terms. As B is Noetherian, $A[\![X_1,\dots,X_n]\!]$ is a flat B-module by Theorem 2.4.19 and, as B is free A-module, we deduce that $A[\![X_1,\dots,X_n]\!]$ is a flat A-module by **??**. □

Proposition 2.4.24. *Let A be a Noetherian ring, $\mathfrak{a}$ an ideal of A, $\widehat{A}$ the Hausdorff completion of A with respect to the $\mathfrak{a}$-adic topology and ι the canonical map from A to $\widehat{A}$. Then*

(a) The map $\mathfrak{m} \mapsto \hat{\mathfrak{m}}$ is a bijection of the set of maximal ideals of A containing $\mathfrak{a}$ onto the set of maximal ideals of $\widehat{A}$ and $\mathfrak{n} \mapsto \iota^{-1}(\mathfrak{n})$ is the inverse bijection.

(b) Let $\mathfrak{m}$ be a maximal ideal of A containing $\mathfrak{a}$. Then the homomorphism $\iota_{\mathfrak{m}} : A_{\mathfrak{m}} \to \widehat{A}_{\hat{\mathfrak{m}}}$ derived from ι is an embedding with dense image if $A_{\mathfrak{m}}$ is given the $\mathfrak{m}A_{\mathfrak{m}}$-adic topology and $\widehat{A}_{\hat{\mathfrak{m}}}$ the $\hat{\mathfrak{m}}\widehat{A}_{\hat{\mathfrak{m}}}$-adic topology.

Proof. The assertion of (a) follows immediately from the fact that the canonical homomorphism $A/\mathfrak{a} \to \widehat{A}/\hat{\mathfrak{a}}$ derived from ι is bijective and the fact that every maximal ideal of $\widehat{A}$ contains $\hat{\mathfrak{a}}$, since $\hat{\mathfrak{a}}$ is contained in the Jacobson radical of $\widehat{A}$ (Lemma 2.3.25).

Finally let us prove (b). As $\mathfrak{m} = \iota^{-1}(\hat{\mathfrak{m}})$, $\iota(A - \mathfrak{m}) \subseteq \widehat{A} - \hat{\mathfrak{m}}$ and ι certainly defines a homomorphism $\iota_{\mathfrak{m}} : A_{\mathfrak{m}} \to \widehat{A}_{\hat{\mathfrak{m}}}$. Let us show that $\iota_{\mathfrak{m}}$ is injective; let $a \in A$ and $s \in A - \mathfrak{m}$ be such that

$$\iota_{\mathfrak{m}}(a/s) = \iota(a)/\iota(s) = 0.$$

then there exists $t \in \widehat{A} - \hat{\mathfrak{m}}$ such that $t\iota(a) = 0$ and the annihilator of $\iota(a)$ in A is therefore not contained in $\hat{\mathfrak{m}}$. Now, if $\mathfrak{b}$ is the annihilator of a in A, then the annihilator of $\iota(a)$ in A is $\hat{\mathfrak{b}}$ (Corollary 2.4.20); hence $\mathfrak{b} \nsubseteq \mathfrak{m}$, which shows that $a/s = 0$.

Moreover, there is a commutative diagram

$$\begin{tikzcd}
A/\mathfrak{m}^k \arrow[r] \arrow[d, "\tau"] & A_{\mathfrak{m}}/(\mathfrak{m}A_{\mathfrak{m}})^k \arrow[d, "\tau_{\mathfrak{m}}"] \\
\widehat{A}/\hat{\mathfrak{m}}^k \arrow[r] & \widehat{A}_{\hat{\mathfrak{m}}}/(\hat{\mathfrak{m}}\widehat{A}_{\hat{\mathfrak{m}}})^k
\end{tikzcd}$$

where τ and $\tau_{\mathfrak{m}}$ are derived from ι and $\iota_{\mathfrak{m}}$ respectively and the horizontal arrows are the canonical isomorphisms. As $\mathfrak{m}^k$ is an open ideal of A (since it contains $\mathfrak{a}^k$), τ is bijective and hence so is $\tau_{\mathfrak{m}}$. This shows first that $(\hat{\mathfrak{m}}\widehat{A}_{\hat{\mathfrak{m}}})^k = \iota_{\mathfrak{m}}((\mathfrak{m}A_{\mathfrak{m}})^k)$ and hence the topology on $A_{\mathfrak{m}}$ is induced by that on $\widehat{A}_{\hat{\mathfrak{m}}}$. Moreover, $\widehat{A}_{\hat{\mathfrak{m}}} = A_{\mathfrak{m}} + (\hat{\mathfrak{m}}\widehat{A}_{\hat{\mathfrak{m}}})^k$ for all $k > 0$ and hence $A_{\mathfrak{m}}$ is dense in $\widehat{A}_{\hat{\mathfrak{m}}}$. □

Corollary 2.4.25. *Let A be a Noetherian local (resp. semi-local) ring and $\mathfrak{r}$ its Jacobson radical. Then $\widehat{A}$ is a Noetherian local (resp. semi-local) ring whose Jacobson radical is $\hat{\mathfrak{r}}$.*

Proof. As $\widehat{A}/\hat{\mathfrak{m}}$ is isomorphic to $A/\mathfrak{m}$, it is a Noetherian ring and $\hat{\mathfrak{m}} = \iota(\mathfrak{m})$ is a finitely generated A-module and therefore $\widehat{A}$ is Noetherian. The rest part follows from Proposition 2.4.24 and the third formula in Corollary 2.4.21. □

Corollary 2.4.26. *Let A be a Noetherian ring and $\mathfrak{m}$ an maximal ideal of A. Then the $\mathfrak{m}$-adic completion $\widehat{A}$ of A is a Noetherian local ring and is isomorphic to the $\mathfrak{m}A_{\mathfrak{m}}$-adic completion $\widehat{A_{\mathfrak{m}}}$ of $A_{\mathfrak{m}}$.*

Proof. Taking $\mathfrak{a}$ to be the maximal ideal $\mathfrak{m}$ in Proposition 2.4.24(a), we see the $\mathfrak{m}$-adic completion $\widehat{A}$ is Noetherian local with maximal ideal $\hat{\mathfrak{m}}$. Then by Proposition 2.4.24(b), since $\widehat{A}_{\hat{\mathfrak{m}}}$ is identified with $\widehat{A}$, we have an dense embedding $\iota_{\mathfrak{m}} : A_{\mathfrak{m}} \to \widehat{A}$, where $A_{\mathfrak{m}}$ is given the $\mathfrak{m}A_{\mathfrak{m}}$-adic topology and $\widehat{A}$ the $\hat{\mathfrak{m}}$-adic topology. Since $\widehat{A}$ is complete, this proves $\widehat{A} \cong \widehat{A_{\mathfrak{m}}}$. □

2.4.4 Zariski rings

For a topological ring A, if the given topology on A is the $\mathfrak{a}$-adic topology for an ideal $\mathfrak{a}$ of A, then $\mathfrak{a}$ is called a **defining ideal** of the topology on A.

Remark 2.4.27. Let A be a Noetherian ring and $\mathfrak{a}$ be an ideal of A. Note that if $\mathfrak{b}$ is a defining ideal of the $\mathfrak{a}$-adic topology on A, then there exists an integer $n > 0$ such that $\mathfrak{b}^n \subseteq \mathfrak{a}$ and hence $\mathfrak{b} \subseteq \sqrt{\mathfrak{a}}$. Conversely, since A is Noetherian, there exists an integer $k > 0$ such that $(\sqrt{\mathfrak{a}})^k \subseteq \mathfrak{a}$ (**??**) and hence $\sqrt{\mathfrak{a}}$ is the largest defining ideal of the $\mathfrak{a}$-adic topology.

Proposition 2.4.28. *Let A be a Noetherian ring and $\mathfrak{a}$ an ideal of A. The following properties are equivalent:*

(i) $\mathfrak{a}$ is contained in the Jacobson radical of A.

(ii) Every finitely generated A-module is Hausdorff with the $\mathfrak{a}$-adic topology.

(iii) For every finitely generated A-module M, every submodule of M is closed with respect to the $\mathfrak{a}$-adic topology on M.

(iv) Every maximal ideal of A is closed with respect to the $\mathfrak{a}$-adic topology.

(v) The Hausdorff completion $\widehat{A}$ is a faithfully flat A-module.

Proof. Let us show that (i) implies (ii). Suppose that $\mathfrak{a}$ is contained in the Jacobson radical of A and let M be a finitely generated A-module. If $x \in M$ and $a \in \mathfrak{a}$ are such that $(1-a)x = 0$, then $x = 0$, for $1 - a$ is invertible in A. Then M is Hausdorff with the $\mathfrak{a}$-adic topology.

Next we show that (ii) implies (iii). Suppose (ii) holds. Let M be a finitely generated A-module and N a submodule of M. Then M/N is Hausdorff with the $\mathfrak{a}$-adic topology, which is the quotient topology of the $\mathfrak{a}$-adic topology on M; thus N is closed in M.

Clearly (iii) implies (iv). Now assume (iv), then for every maximal ideal $\mathfrak{m}$ of A, the A-module $A/\mathfrak{m}$ is Hausdorff with the $\mathfrak{a}$-adic topology. This implies $\mathfrak{a}(A/\mathfrak{m}) \neq A/\mathfrak{m}$, unless the $\mathfrak{a}$-adic topology on $A/\mathfrak{m}$ were the trivial topology and $A/\mathfrak{m}$ were reduced to 0, which is absurd since $A/\mathfrak{m}$ is a field. The canonical image of $\mathfrak{a}$ in $A/\mathfrak{m}$ is therefore an ideal of $A/\mathfrak{m}$ distinct from $A/\mathfrak{m}$ and hence reduced to 0; then $\mathfrak{a} \subseteq \mathfrak{m}$, which proves that $\mathfrak{a}$ is contained in the Jacobson radical of A.

Finally, for every finitely generated A-module M, the canonical map $M \mapsto M \otimes_A \widehat{A}$ is identified with the canonical map $M \mapsto \widehat{M}$ from M to its Hausdorff completion with respect to the $\mathfrak{a}$-adic topology (by Theorem 2.4.19) and the kernel of this map is then the closure of $\{0\}$ in M with respect to this topology. As we already know that $\widehat{A}$ is a flat A-module, the equivalence of (v) and (ii) follows from the characterization of faithfully flat modules (**??**). This finishes the proof. □

A topological ring A is called a **Zariski ring** if it is Noetherian and there exists a defining ideal $\mathfrak{a}$ for the topology on A satisfiyng the equivalent conditions of Proposition 2.4.28. A Zariski ring A is necessarily Hausdorff (Proposition 2.4.28) and every defining ideal of its topology is contained in the Jacobson radical of A.

Example 2.4.29 (Examples of Zariski rings).

(a) Let A be a Noetherian ring and $\mathfrak{a}$ an ideal of A. If A is Hausdorff and complete with the $\mathfrak{a}$-adic topology, then A is a Zariski ring with this topology. In particular, $\widehat{A}$ is a Zariski ring.

(b) Every quotient ring $A/\mathfrak{b}$ of a Zariski ring A is a Zariski ring, for it is Noetherian and, if $\mathfrak{a}$ is a defining ideal of A, then $\mathfrak{a}(A/\mathfrak{b}) = (\mathfrak{a} + \mathfrak{b})/\mathfrak{b}$ is contained in the Jacobson radical of $A/\mathfrak{b}$.

(c) Let A be a Noetherian semi-local ring and $\mathfrak{r}$ its Jacobson radical. Then A with the $\mathfrak{r}$-adic topology is a Zariski ring. This will always be the topology in question (unless otherwise stated) when we consider a Noetherian semi-local ring as a topological ring.

If A is a Zariski ring and M is a finitely generated A-module, then since M is Hausdorff, we may (by virtue of Proposition 2.4.28) identify M with a subset of $\widehat{M}$ by means of the canonical map ι_M. With this identification, by Proposition 2.4.28 we have

Corollary 2.4.30. *Let A be a Zariski ring, M a finitely generated A-module and N a submodule of M. Then $N = \widehat{N} \cap M$.*

Corollary 2.4.31. *Let A be a Zariski ring and M a finitely generated A-madule. If $\widehat{M}$ is a free $\widehat{A}$-module, then M is a free A-madule.*

Proof. Let $\mathfrak{a}$ be a defining ideal of A, which is therefore contained in the Jacobson radical of A. We apply the criterion of Proposition 1.3.9: the canonical map $\iota_M : M \to \widehat{M}$ defines a bijection $i_M : M/\mathfrak{a}M \to \widehat{M}/\widehat{\mathfrak{a}M}$. Similarly the canonical map $\iota_M : A \to \widehat{A}$ defines a bijection $i_A : A/\mathfrak{a} \to \widehat{A}/\hat{\mathfrak{a}}$, which is a ring isomorphism. Then $\widehat{M}/\widehat{\mathfrak{a}M}$ is given an $(\widehat{A}/\hat{\mathfrak{a}})$-module structure and hence (by means of i_A) an $(A/\mathfrak{a})$-module structure. It is immediate that i_M is $(A/\mathfrak{a})$-linear, so that it is an $(A/\mathfrak{a})$-module isomorphism. As $\widehat{M}/\widehat{\mathfrak{a}M}$ is a free $(\widehat{A}/\hat{\mathfrak{a}})$-module, $M/\mathfrak{a}M$ is a free $(A/\mathfrak{a})$-module.

On the other hand, let $\eta : \mathfrak{a} \otimes_A M \to M$ be the canonical homomorphism; as $(\mathfrak{a} \otimes_A M) \otimes_A \widehat{A}$ is canonically identified with $\hat{\mathfrak{a}} \otimes_A \widehat{M}$ and $M \otimes_A \widehat{A}$ with $\widehat{M}$, the hypothesis that M is a free A-module implies that the homomorphism $\eta \otimes 1 : \hat{\mathfrak{a}} \otimes_A \widehat{M} \to \widehat{M}$ is injective. As $\widehat{A}$ is a faithfully flat A-module, we conclude that η is injective and the conditions for applying the above mentioned criterion are indeed fulfilled. □

Corollary 2.4.32. *Let A be a Zariski ring such that A is an integral domain and let $\mathfrak{a}$ be an ideal of A. If the ideal $\hat{\mathfrak{a}}$ of A is principal, then $\mathfrak{a}$ is principal.*

Corollary 2.4.33. *Let A be a Zariski ring such that $\widehat{A}$ is an integral domain, L the field of fractions of $\widehat{A}$ and $K \subseteq L$ the field of fractions of A. Then $\widehat{A} \cap K = A$.*

Proof. Clearly $A \subseteq \widehat{A} \cap K$; on the other hand, if $x \in \widehat{A} \cap K$, then $\widehat{A}x \subseteq \widehat{A}$ and hence, as $\widehat{A}x = \widehat{A} \otimes_A Ax$, we have

$$\widehat{A} \otimes_A ((Ax + A)/A) = (\widehat{A} \otimes_A Ax)/(\widehat{A} \otimes_A A) = 0.$$

As $\widehat{A}$ is a faithfully flat A-module, we deduce that $Ax \subseteq A$, whence $x \in A$. □

Corollary 2.4.34. *Let A be a Noetherian ring, M, N be finitely generated A-modules and $\phi : M \to N$ a homomorphism. For every maximal ideal $\mathfrak{m}$ of A, let $A(\mathfrak{m})$ (resp. $M(\mathfrak{m})$, $N(\mathfrak{m})$) denote the Hausdorff completion of A (resp. M, N) with respect to the $\mathfrak{m}$-adic topology and $\phi(\mathfrak{m}) : M(\mathfrak{m}) \to N(\mathfrak{m})$ the corresponding homomorphism to ϕ. For ϕ to be injective (resp. surjective, bijective, zero), it is necessary and sufiicient that $\phi(\mathfrak{m})$ be so for every maximal ideal $\mathfrak{a}$ of A.*

Proof. We know that for ϕ to be injective (resp. surjective, bijective, zero), it is necessary and sufficient that $\phi_{\mathfrak{m}} : M_{\mathfrak{m}} \to N_{\mathfrak{m}}$ be so for every maximal ideal $\mathfrak{m}$ of A. We now note that $A_{\mathfrak{m}}$ is a Noetherian local ring and hence a Zariski ring and there is a canonical A-algebra isomorphism $(A_{\mathfrak{m}})(\mathfrak{m}A_{\mathfrak{m}}) \to A(\mathfrak{m})$ (Corollary 2.3.28). On the other hand, there is a commutative diagram

$$\begin{tikzcd}
M_{\mathfrak{m}} \otimes_{A_{\mathfrak{m}}} A(\mathfrak{m}) \arrow[r] \arrow[d, "\phi_{\mathfrak{m}}\otimes 1"] & M \otimes_A A(\mathfrak{m}) \arrow[r] \arrow[d, "\phi\otimes 1"] & M(\mathfrak{m}) \arrow[d, "\phi(\mathfrak{m})"] \\
N_{\mathfrak{m}} \otimes_{A_{\mathfrak{m}}} A(\mathfrak{m}) \arrow[r] & N \otimes_A A(\mathfrak{m}) \arrow[r] & N(\mathfrak{m})
\end{tikzcd}$$

where the horizontal arrows on the left arise from the associativity of the tensor product and the isomorphisms $M_{\mathfrak{m}} \to M \otimes_A A_{\mathfrak{m}}$, $N_{\mathfrak{m}} \to N \otimes_A A_{\mathfrak{m}}$. As M and N are finitely generated A-modules, it follows from Theorem 2.4.19 that the rows of this diagram consist of isomorphisms; thus we are reduced to proving that $\phi_{\mathfrak{m}}$ being injective (resp. surjective, bijective, zero) is equivalent to $\phi_{\mathfrak{m}} \otimes 1$ being so. But this follows from the fact that $(A_{\mathfrak{m}})(\mathfrak{m}A_{\mathfrak{m}})$ (and hence also $A(\mathfrak{m})$) is a faithfully flat $A_{\mathfrak{m}}$-module. □

Proposition 2.4.35. *Let A, B be two rings and $\rho : A \to B$ be a ring homomorphism. Suppose that A is Noetherian and B is a finitely generated A-module (with the structure defined by f). Let $\mathfrak{a}$ be an ideal of A, then*

(a) For the $\mathfrak{a}^e$-adic topology on B to be Hausdorff, it is necessary and suficient that the elements of $1 + \rho(\mathfrak{a})$ be not divisors of 0 in B.

(b) If A with the $\mathfrak{a}$-adic topology is a Zariski ring, then B with the $\mathfrak{a}^e$-adic topology is a Zariski ring.

(c) If ρ is injective (thus identifying A with a subring of B), the $\mathfrak{a}^e$-adic topology on B induces on A the $\mathfrak{a}$-adic topology.

Proof. Recall that the $\mathfrak{a}^e$-adic filtration on B coincides with the $\mathfrak{a}$-adic filtration on the A-module B. Assertion (a) is thus a special case of Proposition 2.4.28 and assertion (c) a special case of Artin-Rees Lemma. Finally let us show (b). Suppose that A is a Zariski ring with the $\mathfrak{a}$-adic topology and let N be a finitely generated B-module; it is also a finitely generated A-module and the $\mathfrak{a}$-adic and $\mathfrak{a}^e$-adic filtrations on N coincide; then N is Hausdorff with the $\mathfrak{a}^e$-adic topology. Finally the A-module B is Noetherian and hence the ring B is Noetherian, which completes the proof that B is a Zariski ring. □

Proposition 2.4.36. *Let A, B be two Zariski rings, $\widehat{A}$, $\widehat{B}$ their completions, $\rho : A \to B$ a continuous ring homomorphism and $\hat{\rho} : \widehat{A} \to \widehat{B}$ the homomorphism obtained from f by passing to the completions. If $\hat{\rho}$ is bijective, then the A-module B is faithfully flat.*

Proof. As A and B are Hausdorff, the hypothesis that $\hat{\rho}$ is bijective implies first that ρ is injective. Identifying (algebraically) A with $\rho(A)$ by means of ρ and $\widehat{A}$ with $\widehat{B}$ by means of $\hat{\rho}$, we then obtain the inclusions $A \subseteq B \subseteq \widehat{A} = \widehat{B}$. Then we know that $\widehat{A}$ is a faithfully flat A-module and a faithfully flat B-module, hence B is a faithfully flat A-module. □

Proposition 2.4.37. *Let A be a Noetherian local ring, $\mathfrak{m}$ its maximal ideal, A its $\mathfrak{m}$-adic completion and B a ring such that $A \subseteq B \subseteq \widehat{A}$. Suppose that B is a Noetherian local ring whose maximal ideal $\mathfrak{n}$ satisfies the relation $\mathfrak{n} = \mathfrak{m}B$. Then for $k \geq 0$,*

$$\mathfrak{n}^k = \mathfrak{m}^k B = \hat{\mathfrak{m}}^k \cap B,$$

the $\mathfrak{n}$-adic topology on B is induced by the $\hat{\mathfrak{m}}$-adic topology on $\widehat{A}$, B is a faithfully flat A-module and there is an isomorphism of $\widehat{A}$ onto the $\mathfrak{n}$-adic completion $\widehat{B}$ of B, which extends the canonical injection $A \to B$.

Proof. It is sufficient to verify the relation $\mathfrak{n}^k = \hat{\mathfrak{m}}^k \cap B$, for, as B is dense in A and the $\mathfrak{n}$-adic topology is induced by the $\hat{\mathfrak{m}}$-adic topology, the last assertion will follow from **??** and the last but one from Proposition 2.4.36. The injections $A \to B \to \widehat{A} = \widehat{B}$ induce injective homomorphisms

$$A/(\hat{\mathfrak{m}} \cap A) \longrightarrow B/(\hat{\mathfrak{m}} \cap B) \longrightarrow \widehat{A}/\hat{\mathfrak{m}}$$

We know that $\hat{\mathfrak{m}} \cap A = \mathfrak{m}$ and that $A/\mathfrak{m} \cong \widehat{A}/\hat{\mathfrak{m}}$, hence $B/(\hat{\mathfrak{m}} \cap B) \cong \widehat{A}/\hat{\mathfrak{m}}$, which shows that $B/(\hat{\mathfrak{m}} \cap B)$ is a field, hence that $\hat{\mathfrak{m}} \cap B$ is a maximal ideal of B and therefore $\hat{\mathfrak{m}} \cap B = \mathfrak{n}$. As

$A/\mathfrak{m} \cong \widehat{A}/\widehat{\mathfrak{m}}$ we have $\widehat{A} = A + \widehat{\mathfrak{m}}$, thus $B = A + \mathfrak{n} = A + \mathfrak{m}B$. By induction on k we deduce that

$$B = A + \mathfrak{m}^k B = A + \mathfrak{n}^k$$

for all $k > 1$. As $\mathfrak{n}^k \subseteq \widehat{\mathfrak{m}}^k + B$, it is sufficient to show that $\widehat{\mathfrak{m}}^k \cap B \subseteq \mathfrak{n}^k$. If $b \in \widehat{\mathfrak{m}}^k \cap B$, we may write $b = a + z$ where $a \in A$, $z \in \mathfrak{n}^k$; whence

$$a = b - z \in \widehat{\mathfrak{m}}^k \cap A = \mathfrak{m}^k \subseteq \mathfrak{n}^k$$

and $b \in \mathfrak{n}^k$. □

Example 2.4.38. An important case where this applies is the following: B is the ring of integral series in n variables over a complete valued field K, which converge in the neighbourhood of 0, A is the local ring $K[X_1, \dots, X_n]_{\mathfrak{m}}$ where $\mathfrak{m}$ is the maximal ideal consisting of the polynomials with no constant term and $\widehat{A}$ is the ring of formal power series $K[\![X_1, \dots, X_n]\!]$.

Proposition 2.4.39. *Let A be a Noetherian ring, $\mathfrak{a}$ an ideal of A, S the multiplicative subset $1 + \mathfrak{a}$ of A and M a finitely generated A-module.*

(a) $S^{-1}A$ is a Zariski ring with the $(S^{-1}\mathfrak{a})$-adic topology.

(b) The canonical map $i : M \to S^{-1}M$ is continuous if M is given the $\mathfrak{a}$-adic topology and $S^{-1}M$ the $(S^{-1}\mathfrak{a})$-adic topology and $\hat{i} : \widehat{M} \to \widehat{S^{-1}M}$ is an isomorphism.

Proof. Every element of $1 + S^{-1}\mathfrak{a}$ is of the form

$$1 + (a/(1+b)) = (1 + a + b)/(1+b)$$

where $a, b \in \mathfrak{a}$; it is therefore invertible in $S^{-1}A$, which proves that $S^{-1}\mathfrak{a}$ is contained in the Jacobson radical of $S^{-1}A$. As $S^{-1}A$ is Noetherian, it is then a Zariski ring with the $(S^{-1}\mathfrak{a})$-adic topology, which proves (a). Let us show (b). For all $n > 0$, we have

$$i^{-1}((S^{-1}\mathfrak{a})^n M) = i^{-1}(S^{-1}\mathfrak{a}^n M) = \mathfrak{a}^n M$$

since by Proposition 1.2.37, if $x \in i^{-1}(S^{-1}\mathfrak{a}^n M)$ then $(1-a)x = y$, where $a \in \mathfrak{a}$, $y \in \mathfrak{a}^n M$, whence

$$x = (1 + a + a^2 + \cdots + a^{n-1})y + a^n x \in \mathfrak{a}^n M.$$

This proves that i is continuous and a strict morphism (Proposition 1.2.37). Moreover, the kernel of i, which is the set of $x \in M$ for which there exists some $s \in S$ such that $sx = 0$, is identical with the kernel of the canonical map $\iota : M \to \widehat{M}$. Then there exists a topological isomorphism $\iota_0 : i(M) \to \iota(M)$ such that the following diagram is commutative:

$$\begin{tikzcd}
 & \iota(M) \arrow[r, hook] \arrow[d, "\iota_0"] & \widehat{M} \arrow[d, "\hat{i}"] \\
M \arrow[ur, "\iota"] \arrow[dr, "i"'] & & \\
 & i(M) \arrow[r, hook] & \widehat{S^{-1}M}
\end{tikzcd}$$

Therefore the problem reduces to verify that $i(M)$ is dense in $S^{-1}M$. Now every element of $S^{-1}M$ may be written as $x/(1-a)$, where $a \in \mathfrak{a}$, and it is immediately verified that

$$x/(1-a) \equiv (1 + a + \cdots + a^{n-1})x \mod S^{-1}\mathfrak{a}^n M$$

which completes the proof. □

2.5 Lifting in complete rings

2.5.1 Strongly relatively prime polynomials

Let A be a ring. Two elements x, y of A are called **strongly relatively prime** if the principal ideals (x) and (y) are relatively prime, in other words if $Ax + Ay = A$; it amounts to the same to say that there exist two elements a, b of A such that $ax + by = 1$.

Lemma 2.5.1 (Euclid Lemma). *Let x, y be two strongly relatively prime elements of A; if $z \in A$ is such that x divides yz, then x divides z.*

Proof. If $1 = ax + by$, then $z = x(az) + (yz)b$, so x divides z. □

If x and y are strongly relatively prime in A, then $(xy) = (x) \cap (y)$. If A is an integral domain, two strongly relatively prime elements then have an lcm equal to their product and are therefore relatively prime in the sense that $\gcd(x, y) = 1$. Conversely, if A is a principal ideal domain, two relatively prime elements are also strongly relatively prime, as follows from Bezout's identity.

For polynomial rings there is the following result:

Proposition 2.5.2. *Let A be a ring and F, G two strongly relatively prime polynomials in $A[X]$. Suppose that F is monic and of degree s. Then every polynomial T in $A[X]$ may be written uniquely in the form*

$$T = PF + QG \tag{2.5.1}$$

where $P, Q \in A[X]$ and $\deg(Q) < s$. If further $\deg(T) \leq n$ and $\deg(G) \leq n - s$, then $\deg(P) \leq n - s$.

Proof. As F is monic, $FP \neq 0$ for every nonzero polynomial P of $A[X]$ and in this case $\deg(FP) = s + \deg(P)$. Let T be any polynomial in $A[X]$. As the ideal generated by F and G is the whole of $A[X]$, there exist polynomials P_1 and Q_1 such that

$$T = P_1F + Q_1G.$$

As F is monic of degree s, Euclidean division shows that there exist two polynomials Q' and Q'' such that $Q_1 = Q''F + Q'$ where $\deg(Q') < s$. Then we deduce that

$$T = P_1F + Q_1G = P_1F + Q''FG + Q'G = PF + Q'G$$

where $P = P_1 + Q''G$. To show the uniqueness of formula (2.5.1), it is sufficient to prove that the relations

$$0 = PF + QG, \quad \deg(Q) < s.$$

imply $P = Q = 0$. Now, if this holds, F divides QG and, as F and G are strongly relatively prime, F divides Q by Lemma 2.5.1, which is impossible unless $Q = 0$. This then implies $PF = 0$, and so $P = 0$ by the remark at the begining.

Finally, suppose that $\deg(T) \leq n$ and $\deg(G) \leq n - s$. With the polynomial T in the form (2.5.1),

$$\deg(QG) \leq \deg(Q) + \deg(G) < s + \deg(G) \leq n$$

and therefore

$$s + \deg(P) = \deg(PF) = \deg(T - QG) \leq n$$

whence $\deg(P) \leq n - s$. □

Example 2.5.3. For a polynomial $P \in A[X]$ to be strongly relatively prime to $X - a$ (where $a \in A$), it is necessary and sufiicient that $P(a)$ be invertible in A. For if P and $X - a$ are strongly relatively prime, it follows from Proposition 2.5.2 that there exist $c \in A$ and a polynomial $Q \in A[X]$ such that $1 = (X - a)Q + cP$, whence $cP(a) = 1$ and $P(a)$ is invertible. Conversely, by Euclidean division

$$P = (X - a)G + P(a)$$

and, if $P(a) = b^{-1}$ where $b \in A$, we deduce that $1 = bP - b(X - a)G$, which shows that P and $X - a$ are strongly relatively prime.

Let A and B be two rings and $\rho : A \to B$ a ring homomorphism. If $P = \sum_{n=0}^{\infty} a_n X^n$ is a formal power series in $A[\![X]\!]$, let $f(P)$ denote the formal power series $\sum_{n=0}^{\infty} \rho(a_n) X^n$ in $B[\![X]\!]$. If P is a polynomial, so is $\rho(P)$ and, if further P is monic, then $\rho(P)$ is monic of the same degree as P. Finally, $P \mapsto f(P)$ is clearly a homomorphism of $A[\![X]\!]$ to $B[\![X]\!]$ which extends f and maps X to X.

Proposition 2.5.4. *Let A and B be two rings, $\rho : A \to B$ a homomorphism and P, Q two polynomials in $A[X]$. If P and Q are strongly relatively prime in $A[X]$, then $\rho(P)$ and $\rho(Q)$ are strongly relatively prime in $B[X]$. The converse is true if ρ is surjective, if its kernel is contained in the Jacobson radical of A and if P is monic.*

Proof. Suppose that P and Q are strongly relatively prime; then there exist polynomials U, V in $A[X]$ such that $PU + QV = 1$; we deduce that

$$\rho(P)\rho(U) + \rho(Q)\rho(V) = 1$$

whence the first assertion. To show the second, let $\mathfrak{a}$ denote the kernel of ρ. Let $M = A[X]$ and N be the ideal of M generated by P and Q. As ρ is surjective and $\rho(P)$ is monic, Proposition 2.5.2 shows that for every polynomial $T \in A[X]$ there exist two polynomials U, V in $A[X]$ such that

$$\rho(T) = \rho(P)\rho(U) + \rho(Q)\rho(V)$$

whence the relation $M = N + \mathfrak{a}M$. Now, M/N is a finitely generated A-module, for every polynomial is congruent mod P to a polynomial of degree strictly smaller than $\deg(P)$, P being monic. As $\mathfrak{a}(M/N) = M/N$ and $\mathfrak{a}$ is contained in the Jacobson radical of A, Nakayama's Lemma shows that $M/N = 0$, which means that P and Q are strongly relatively prime. $\square$

2.5.2 Restricted power series and Hensel's lemma

Definition 2.5.5. A commutative topological ring A is said to be linearly topologized (and its topology is said to be **linear**) if there exists a fundamental system $\mathcal{B}$ of neighbourhoods of 0 consisting of ideals of A.

Note that in such a ring, the ideals $\mathfrak{a} \in \mathcal{B}$ are open and closed. For all $\mathfrak{a} \in \mathcal{B}$, the quotient topological ring $A/\mathfrak{a}$ is then discrete. For $\mathfrak{a}, \mathfrak{b} \in \mathcal{B}$ and $\mathfrak{b} \subseteq \mathfrak{a}$, let

$$\pi_{\mathfrak{a}\mathfrak{b}} : A/\mathfrak{b} \to A/\mathfrak{a}$$

be the canonical map. We know that $(A/\mathfrak{a}, \pi_{\mathfrak{a}\mathfrak{b}})$ is an inverse system of discrete rings (relative to the indexing set $\mathcal{B}$ which is ordered by inclusion and directed), whose inverse limit is a complete Hausdorff linearly topologized ring $\widetilde{A}$. Further, a strict morphism $\iota : A \to \widetilde{A}$ is defined, whose kernel is the closure of $\{0\}$ in A and whose image is dense in $\widetilde{A}$, so that $\widetilde{A}$ is canonically identified with the Hausdorff completion of A.

Definition 2.5.6. Given a commutative topological ring A, a formal power series

$$P = \sum_{\alpha \in \mathbb{N}^p} c_\alpha X^\alpha$$

in the ring $A[\![X_1, \ldots, X_p]\!]$ is called **restricted** if, for every neighbourhood V of 0 in A, there is only a finite number of coefiicients c_α not belonging to V (in other words, the family (c_α) tends to 0 in A with respect to the filter of complements of finite subsets of $\mathbb{N}^p$.

If A is linearly topologized, the restricted formal power series in $A[\![X_1, \ldots, X_p]\!]$ form a subring of $A[\![X_1, \ldots, Xp]\!]$, denoted by $A\{X_1, \ldots, X_p\}$, for if $P = \sum a_\alpha X^\alpha$ and $Q = \sum b_\alpha X^\alpha$ are two restricted formal power series and $\mathfrak{a}$ a neighbourhood of 0 in A which is an ideal of A, there exists an integer m such that $a_\alpha \in \mathfrak{a}$ and $b_\alpha \in \mathfrak{a}$ for every $\alpha \in \mathbb{N}^p$ such that $|\alpha| \geq m$. Now, if

$$PQ = \sum c_\alpha X^\alpha \quad \text{where} \quad c_\alpha = \sum_{\beta+\gamma=\alpha} a_\beta b_\gamma.$$

We conclude that if $|\alpha| \geq 2m$, then $|\beta| \geq m$ or $|\gamma| \geq m$ and hence, since $\mathfrak{a}$ is an ideal, $c_\alpha \in \mathfrak{a}$ so long as $|\alpha| \geq 2m$, which establishes our assertion. Moreover, every derivative $\partial(PQ)/\partial X_i$ $(1 \leq i \leq p)$ of a restricted formal power series is restricted, as follows immediately from the definition and the fact that the neighbourhoods $\mathfrak{a} \in \mathcal{B}$ are additive subgroups of A.

Let us always assume that A is linearly topologized and let $\mathcal{B}$ be a fundamental system of neighbourhoods of 0 in A consisting of ideals of A; for all $\mathfrak{a} \in \mathcal{B}$, let $\pi_\mathfrak{a} : A \to A/\mathfrak{a}$ be the canonical homomorphism. By definition, for every restricted formal power series $T \in A\{X_1, \ldots, X_n\}$,

$$\pi_\mathfrak{a}(T) \in (A/\mathfrak{a})[X_1, \ldots, X_p]$$

Clearly $((A/\mathfrak{a})[X_1, \ldots, X_p], \pi_{\mathfrak{a}\mathfrak{b}})$ is an inverse system of rings and $(\pi_\mathfrak{a})$ is an inverse system of homomorphisms $A\{X_1, \ldots, X_p\} \to (A/\mathfrak{a})[X_1, \ldots, X_p]$; as every polynomial is a restricted formal power series, $\pi_\mathfrak{a}$ is is surjective; its kernel $N_\mathfrak{a}$ is the ideal of $A\{X_1, \ldots, X_p\}$ consisting of the restricted formal power series all of whose coefficients belong to $\mathfrak{a}$; we shall give $A\{X_1, \ldots, X_p\}$ the (linear) topology for which the $N_\mathfrak{a}$ (for $\mathfrak{a} \in \mathcal{B}$) form a fundamental system of neighbourhoods of 0 (a topology which obviously depends only on that on A). Then it follows from **??** that

$$\pi = \varprojlim \pi_\mathfrak{a} : A\{X_1, \ldots, X_p\} \to \varprojlim_\mathfrak{a} (A/\mathfrak{a})[X_1, \ldots, X_p]$$

is a strict morphism whose kernel is the closure of $\{0\}$ in $A\{X_1, \ldots, X_p\}$ and whose image is dense in $B = \varprojlim_\mathfrak{a} (A/\mathfrak{a})[X_1, \ldots, X_p]$.

Proposition 2.5.7. *If the linearly topologized ring A is Hausdorff and complete, the canonical homomorphism π is a topological ring isomorphism.*

Proof. For all $\alpha \in \mathbb{N}^p$ and all $\mathfrak{a} \in \mathcal{B}$, let $\phi_\alpha^\mathfrak{a}$ be the map $(A/\mathfrak{a})[X_1, \ldots, X_p] \to A/\mathfrak{a}$ which maps every polynomial to the coefficient of X^α in this polynomial; clearly the $(\phi_\alpha^\mathfrak{a})_\mathfrak{a}$ form an inverse system of $(A/\mathfrak{a})$-module homomorphisms (relative to the ordered set $\mathcal{B}$) and, as A is canonically identified with $\varprojlim_\mathfrak{a} (A/\mathfrak{a})$ by hypothesis $\phi_\alpha = \varprojlim_\mathfrak{a} \phi_\alpha^\mathfrak{a}$ is a continuous A-homomorphisms from B to A. For every element $S = (S_\mathfrak{a})_{\mathfrak{a} \in \mathcal{B}}$ of B, we consider the formal power series $T = \sum_\alpha \phi_\alpha(S) X^\alpha$. For each $\mathfrak{a} \in \mathcal{B}$, since $S_\mathfrak{a}$ is a polynomial, there exist $n > 0$ such that $\phi_\alpha^\mathfrak{a}(S_\mathfrak{a}) = 0$ for $|\alpha| \geq n$, which implies $\phi_\alpha^\mathfrak{b}(S_\mathfrak{b}) \in \mathfrak{a}/\mathfrak{b}$ for all $\mathfrak{b} \in \mathcal{B}$, $\mathfrak{b} \subseteq \mathfrak{a}$. Taking limits for these $\mathfrak{b}$'s, we then see $\phi_\alpha(S) \in \mathfrak{a}$ for $|\alpha| \geq n$, which proves T is restricted. Also, it is immediate that $\pi(T) = S$. As A is Hausdorff, the intersection of the $N_\mathfrak{a}$ reduces to 0 and hence π is bijective, which completes the proof, since π is a strict morphism. $\square$

Proposition 2.5.8. *Let A, B be two linearly topologized rings, B being Hausdorff and complete, and $\rho : A \to B$ a continuous homomorphism. For every family $(b_i)_{1 \leq i \leq p}$ of elements of B, there exists a unique continuous homomorphism*

$$\tilde{\rho} : A\{X_1, \ldots, X_p\} \to B$$

such that $\tilde{\rho}(a) = f(a)$ for all $a \in A$ and $\tilde{\rho}(X_i) = b_i$ for $1 \leq i \leq p$.

Proof. There exists a unique homomorphism $v : A[X_1, \dots, X_p] \to B$ such that $v(a) = \rho(a)$ for $a \in A$ and $v(X_i) = b_i$, for $1 \leq i \leq p$. Moreover, if $\mathfrak{b}$ is a neighbourhood of 0 in B which is an ideal, $\rho^{-1}(\mathfrak{a})$ is an ideal of A which is a neighbourhood of 0 and, for every polynomial $P \in N_{\mathfrak{a}}$, clearly $v(P) \in \mathfrak{b}$ and hence v is continuous. As $A[X_1, \dots, X_p]$ is dense in $A\{X_1, \dots, X_p\}$, the existence and uniqueness of $\tilde{\rho}$ follow from **??** and the principle of extension of identities. □

Example 2.5.9. In the special case where $A = B$ and f is the identity map we shall write $P(b_1, \dots, b_p)$ for the value of $\tilde{f}(P)$ for every restricted formal power series $P \in A\{X_1, \dots, X_p\}$.

Now we turn to the most important application of restricted power series in this part. In a topological ring A, an element x is called **topologically nilpotent** if 0 is a limit of the sequence (x^n). If A is a linearly topologized ring, to say that x is topologically nilpotent means that for every open ideal $\mathfrak{a}$ of A the canonical image of x in $A/\mathfrak{a}$ is a nilpotent element of that ring. If $\mathfrak{r}_{\mathfrak{a}}$ is the nilradical of $A/\mathfrak{a}$, clearly $(\mathfrak{r}_{\mathfrak{a}})$ is an inverse system of subsets and the set $\mathfrak{t}$ of topological nilpotent elements of A is the inverse image of $\varprojlim_{\mathfrak{a}} \mathfrak{r}_{\mathfrak{a}}$ under the canonical homomorphism $A \to \varprojlim A/\mathfrak{a}$; it is therefore a closed ideal of A. If also A is Hausdorff and complete, this ideal is contained in the Jacobson radical of A and, for an element $x \in A$ to be invertible, it is necessary and sufficient that its class mod $\mathfrak{t}$ be invertible in $A/\mathfrak{t}$ (Lemma 2.3.25). Note that if A is a ring and $\mathfrak{a}$ an ideal of A, the elements of $\mathfrak{a}$ are topologically nilpotent with respect to the $\mathfrak{a}$-adic topology.

Theorem 2.5.10 (Hensel's Lemma). *Let A be a complete Hausdorff linearly topologized ring. Let $\mathfrak{m}$ be a closed ideal of A whose elements are topologically nilpotent. Let $A/\mathfrak{m}$ be the quotient topological ring and $\pi : A \to A/\mathfrak{m}$ the canonical map. Let T be a restricted formal power series in $A\{X\}$, $\bar{P}$ a monic polynomial in $(A/\mathfrak{m})[X]$ and $\bar{Q}$ a restricted formal power series in $(A/\mathfrak{m})\{X\}$. Suppose that $\pi(T) = \bar{P}\bar{Q}$ and that $\bar{P}$ and $\bar{Q}$ are strongly relatively prime in $(A/\mathfrak{m})\{X\}$. Then there exists a unique ordered pair (P, Q) consisting of a monic polynomial $P \in A[X]$ and a restricted formal power series $Q \in A\{X\}$ such that*

$$T = PQ, \quad \pi(P) = \bar{P}, \quad \pi(Q) = \bar{Q}. \tag{2.5.2}$$

Moreover, P and Q are strongly relatively prime in $A\{X\}$ and, if T is a polynomial, so is Q.

Proof. The proof is divided into several steps. In the first three we assume that A is discrete, in which case T and Q are *polynomials*. First assume that $\mathfrak{m}^2 = 0$. Let P_1, Q_1 be two polynomials of $A[X]$ such that P_1 is monic and $\pi(P_1) = \bar{P}$, $\pi(Q_1) = \bar{Q}$. Proposition 2.5.4 shows that P_1 and Q_1 are strongly relatively prime; hence (Proposition 2.5.2) there exists a unique ordered pair of polynomials (U, V) of $A[X]$ such that

$$T - P_1Q_1 = P_1U + Q_1V, \quad \deg(V) < \deg(P_1) = \deg(\bar{P}).$$

Applying π on this equation, we then see

$$\bar{P}\pi(U) + \bar{Q}\pi(V) = \pi(T) - \bar{P}\bar{Q} = 0.$$

As $\bar{P}$ is monic, $\bar{P}$ and $\bar{Q}$ strongly relatively prime and $\deg(\pi(V)) < \deg(\bar{P})$, the uniqueness part of Proposition 2.5.2 implies that $\pi(U) = \pi(V) = 0$, in other words the coefficients of P_1 and Q_1 belong to $\mathfrak{m}$ and the relation $\mathfrak{m}^2 = 0$ gives

$$T = P_1Q_1 + P_1U + Q_1V = P_1Q_1 + P_1U + Q_1V + UV = (P_1 + V)(Q_1 + U)$$

so the polynomials $P = P_1 + V$ and $Q = Q_1 + U$ are the solutions to the problem.

Now suppose that $\mathfrak{m}$ is nilpotent and let n be the smallest integer such that $\mathfrak{m}^n = 0$ and let us argue by induction on $n > 2$, the theorem having been shown for $n = 2$. Let

$$A' = A/\mathfrak{m}^{n-1}, \quad \mathfrak{m}' = \mathfrak{m}/\mathfrak{m}^{n-1}$$

As $(\mathfrak{m}')^{n-1} = 0$, by induction hypothesis there exists a unique ordered pair (P', Q') of polynomials in $A'[X]$ such that P' is monic and

$$T' = P'Q', \quad \pi'(P') = \bar{P}, \quad \pi'(Q') = \bar{Q},$$

where $\pi' : A' \to A'/\mathfrak{m}' = A/\mathfrak{m}$ and $\theta : A \to A'$ denote the canonical homomorphisms, and $T' = \theta(T)$. On the other hand, as $(\mathfrak{m}')^2 = 0$, there exists a unique ordered pair (P, Q) of polynomials in $A[X]$ such that P is monic and

$$T = PQ, \quad \theta(P) = P', \quad \theta(Q) = Q'.$$

As $\pi = \pi' \circ \theta$, this shows the existence and uniqueness of P and Q satisfying (2.5.2). Moreover P' and Q' are strongly relatively prime by the induction hypothesis and hence so are P and Q.

Next we turn to the case where A is discrete. In this case $\mathfrak{m}$ is no longer necessarily nilpotent, but it is always a nilideal (that is, every element in $\mathfrak{m}$ is nilpotent), since $\mathfrak{m}$ is topological nilpotent and $\{0\}$ is open in A. Let P_0, Q_0 be two polynomials of $A[X]$ such that $(P_0) = \bar{P}$, $\pi(Q_0) = \bar{Q}$ and P_0 is monic. Let us consider the ideal $\mathfrak{n}$ of A generated by the coefficients of $T - P_0Q_0$; it is finitely generated and contained in $\mathfrak{m}$, hence it is nilpotent and by definition, if $\psi : A \to A/\mathfrak{n}$ is the canonical map, then $\psi(T) = \psi(P_0)\psi(Q_0)$. Moreover, $\psi(P_0)$ and $\psi(Q_0)$ are strongly relatively prime, as follows from the hypothesis on P and Q and Proposition 2.5.4 applied to the canonical homomorphism $A/\mathfrak{n} \to A/\mathfrak{m}$. By virtue of the nilpotent case, there therefore exists an ordered pair (P, Q) of polynomials in $A[X]$ such that P is monic and relations (2.5.2) hold. The fact that P and Q are strongly relatively prime implies also here that P and Q are strongly relatively prime in $A[X]$ by virtue of Proposition 2.5.4, for $\mathfrak{m}$ is contained in the Jacobson radical of A. Suppose finally that P_1, Q_1 are two polynomials in $A[X]$ satisfying (2.5.2) and such that P_1 is monic and let $\mathfrak{n}_1$ be the finitely generated ideal of A generated by the coefficients of $P - P_1$ and the coefficients of $Q - Q_1$; as $\mathfrak{n}_1$ is contained in $\mathfrak{m}$, it is nilpotent and, if $\psi_1 : A \to A/\mathfrak{n}_1$ is the canonical map, then $\psi_1(P) = \psi_1(P_1)$ and $\psi_1(Q) = \psi_1(Q_1)$. The uniqueness property for nilpotent case therefore implies $P = P_1$ and $Q = Q_1$.

Finally, we deal with the general case. Let $\mathcal{B}$ be a fundamental system of neighbourhoods of 0 in A consisting of ideals of A. For all $\mathfrak{a} \in \mathcal{B}$, let $\pi_\mathfrak{a}$ be the canonical map $A \to A/\mathfrak{a}$, $\phi_\mathfrak{a}$ the canonical map

$$A/\mathfrak{a} \to (A/\mathfrak{a})/((\mathfrak{m} + \mathfrak{a})/\mathfrak{a}) = A/(\mathfrak{m} + \mathfrak{a}),$$

and $\psi_\mathfrak{a}$ the canonical map $A/\mathfrak{m} \to A/(\mathfrak{m} + \mathfrak{a})$ and write $T_\mathfrak{a} = \pi_\mathfrak{a}(T)$, $\bar{P}_\mathfrak{a} = \psi_\mathfrak{a}(\bar{P})$, $\bar{Q}_\mathfrak{a} = \psi_\mathfrak{a}(\bar{Q})$. As each ring $A/\mathfrak{a}$ is discrete, the preceeding argument can be applied to it and we see that there exists a unique ordered pair $(P_\mathfrak{a}, Q_\mathfrak{a})$ of polynomials in $(A/\mathfrak{a})[X]$ such that $P_\mathfrak{a}$ is monic and $T_\mathfrak{a} = P_\mathfrak{a}Q_\mathfrak{a}$, $\phi_\mathfrak{a}(P_\mathfrak{a}) = \bar{P}_\mathfrak{a}$, $\phi_\mathfrak{a}(Q_\mathfrak{a}) = \bar{Q}_\mathfrak{a}$. The uniqueness of this ordered pair implies that, if $\mathfrak{a}, \mathfrak{b} \in \mathcal{B}$, $\mathfrak{b} \subseteq \mathfrak{a}$ and $\pi_{\mathfrak{a}\mathfrak{b}} : A/\mathfrak{b} \to A/\mathfrak{a}$ is the canonical map, then $P_\mathfrak{a} = \pi_{\mathfrak{a}\mathfrak{b}}(P_\mathfrak{b})$ and $Q_\mathfrak{a} = \pi_{\mathfrak{a}\mathfrak{b}}(Q_\mathfrak{b})$. Then it follows from the canonical identification of $A\{X\}$ with $\varprojlim_\mathfrak{a}(A/\mathfrak{a})[X]$ that there exists $P \in A\{X\}$ and $Q \in A\{X\}$ such that $T = PQ$ and $\pi_\mathfrak{a}(P) = P_\mathfrak{a}$,$\pi_\mathfrak{a}(Q) = Q_\mathfrak{a}$ for all $\mathfrak{a} \in \mathcal{B}$. Moreover

$$\psi_\mathfrak{a}(\bar{P} - \pi(P)) = \psi_\mathfrak{a}(\bar{Q} - \pi(Q)) = 0,$$

for all $\mathfrak{a} \in \mathcal{B}$, which means that for all $\mathfrak{a} \in \mathcal{B}$ the coefficients of $\bar{P} - \pi(P)$ and $\bar{Q} - \pi(Q)$ all belong to $(\mathfrak{m} + \mathfrak{a})/\mathfrak{m}$. But, as $\mathfrak{m}$ is closed in A, $\bigcap_\mathfrak{a}(\mathfrak{m} + \mathfrak{a}) = \mathfrak{m}$, whence $\bar{P} = \pi(P)$, $\bar{Q} = \pi(Q)$ and P and Q certainly satisfy (2.5.2). Moreover, as the $P_\mathfrak{a}$ are monic and of the same degree, the restricted formal power series P is a monic polynomial. If (P', Q') were another ordered pair satisfying (2.5.2) and such that P' is a monic polynomial, we would deduce that

$$T_\mathfrak{a} = \pi_\mathfrak{a}(P')\pi_\mathfrak{a}(Q'), \quad \phi_\mathfrak{a}(\pi_\mathfrak{a}(P')) = \bar{P}_\mathfrak{a}, \quad \phi_\mathfrak{a}(\pi_\mathfrak{a}(Q')) = \bar{Q}_\mathfrak{a}.$$

By the uniqueness in the discrete case we have $\pi_{\mathfrak{a}}(P') = P_{\mathfrak{a}}$ and $\pi_{\mathfrak{a}}(Q') = Q_{\mathfrak{a}}$ for all $\mathfrak{a} \in \mathcal{B}$, which implies that $P = P'$ and $Q = Q'$. Let us show finally that P and Q are strongly relatively prime; by virtue of the discrete case and Proposition 2.5.2, for all $\mathfrak{a} \in \mathcal{B}$, there exists a unique ordered pair $(U_{\mathfrak{a}}, V_{\mathfrak{a}})$ of polynomials in $(A/\mathfrak{a})[X]$ such that

$$1 = P_{\mathfrak{a}}U_{\mathfrak{a}} + Q_{\mathfrak{a}}V_{\mathfrak{a}}, \quad \deg(V_{\mathfrak{a}}) < \deg(P_{\mathfrak{a}}) = \deg(\bar{P}).$$

The uniqueness of this ordered pair shows immediately that, for $\mathfrak{a}, \mathfrak{b} \in \mathcal{B}$ with $\mathfrak{b} \subseteq \mathfrak{a}$, $U_{\mathfrak{a}} = \pi_{\mathfrak{a}\mathfrak{b}}(U_{\mathfrak{b}})$ and $V_{\mathfrak{a}} = \pi_{\mathfrak{a}\mathfrak{b}}(V_{\mathfrak{b}})$. Taking account of Proposition 2.5.7, we conclude that there exist two restricted formal power series U, V of $A\{X\}$ such that $U_{\mathfrak{a}} = \pi_{\mathfrak{a}}(U)$, $V_{\mathfrak{a}} = \pi_{\mathfrak{a}}(V)$ and $1 = PU + QV$.

It remains to verify that, if T is a polynomial, so is Q. Now, the $Q_{\mathfrak{a}}$ are polynomials by construction and, as $P_{\mathfrak{a}}$ is monic, the relation $T_{\mathfrak{a}} = P_{\mathfrak{a}}Q_{\mathfrak{a}}$ implies

$$\deg(Q_{\mathfrak{a}}) \le \deg(T_{\mathfrak{a}}) \le \deg(T)$$

for all $\mathfrak{a} \in \mathcal{B}$, whence immediately the required result by definition of Q. □

2.5.3 System of equations in complete rings

Let A be a ring; we shall say that a system

$$\boldsymbol{f} = (f_1, \dots, f_p) \in (A[\![X_1, \dots, X_q]\!])^p$$

of formal power series in the $X_1, \dots, X_q$ with coefficients in A, is **without constant term** if this is true of all the f_j. For every systems of formal power series

$$\boldsymbol{f} = (f_1, \dots, f_p) \in (A[\![X_1, \dots, X_q]\!])^p, \quad \boldsymbol{g} = (g_1, \dots, g_q) \in (A[\![X_1, \dots, X_r]\!])^q$$

such that $\boldsymbol{g}$ is without constant term, we shall denote by $\boldsymbol{f} \circ \boldsymbol{g}$ (or $\boldsymbol{f}(\boldsymbol{g})$) the system of formal power series $f_j(g_1, \dots, g_q)$ in $(A[\![X_1, \dots, X_r]\!])^p$. If

$$\boldsymbol{h} = (h_1, \dots, h_r) \in (A[\![X_1, \dots, X_s]\!])^r$$

is a third system without constant term, then we have

$$(\boldsymbol{f} \circ \boldsymbol{g}) \circ \boldsymbol{h} = \boldsymbol{f} \circ (\boldsymbol{g} \circ \boldsymbol{h}).$$

For every system $\boldsymbol{f}$, we shall denote by $M_{\boldsymbol{f}}$ or $M_{\boldsymbol{f}}(X)$ the Jacobian matrix $(\partial f_i/\partial X_j)$, where i is the index of the rows and j that of the columns. For two systems $\boldsymbol{f}$ and $\boldsymbol{g}$, where $\boldsymbol{g}$ is without constant term, we have

$$M_{\boldsymbol{f} \circ \boldsymbol{g}} = M_{\boldsymbol{f}}(\boldsymbol{g}) \cdot M_{\boldsymbol{g}}. \tag{2.5.3}$$

where $M_{\boldsymbol{f}}(\boldsymbol{g})$ is the matrix whose elements are obtained by substituting g_j for X_j in each series element of $M_{\boldsymbol{f}}$. We shall denote by $M_{\boldsymbol{f}}(0)$ the matrix of constant terms of the elements of $M_{\boldsymbol{f}}$. Then we deduce from (2.5.3) that

$$M_{\boldsymbol{f} \circ \boldsymbol{g}}(0) = M_{\boldsymbol{f}}(0) \cdot M_{\boldsymbol{g}}(0).$$

Given an integer $n > 0$, we shall write

$$\mathbf{1}_n = \boldsymbol{X} = (X_1, \dots, X_n) \in (A[\![X_1, \dots, X_n]\!])^n$$

which will be considered as a matrix with a single column.

For every system $\boldsymbol{f} = (f_1, \dots, f_n) \in (A[\![X_1, \dots, X_n]\!])^n$, $M_{\boldsymbol{f}}$ is a square matrix of order n; we shall denote by $J_{\boldsymbol{f}}$ or $J_{\boldsymbol{f}}(X)$ its determinant and by $J_{\boldsymbol{f}}(0)$ the constant term of $J_{\boldsymbol{f}}$, equal to $\det M_{\boldsymbol{f}}(0)$. If $\boldsymbol{g} = (g_1, \dots, g_n)$ is a system without constant term in $(A[\![X_1, \dots, X_n]\!])^n$, then

$$J_{\boldsymbol{f} \circ \boldsymbol{g}} = J_{\boldsymbol{f}}(\boldsymbol{g}) J_{\boldsymbol{g}}, \quad J_{\boldsymbol{f} \circ \boldsymbol{g}}(0) = J_{\boldsymbol{f}}(0) J_{\boldsymbol{g}}(0).$$

Proposition 2.5.11. *Let A be a ring and $\boldsymbol{f} = (f_1, \dots, f_n)$ a system without constant term of n series in $A[\![X_1, \dots, X_n]\!]$. Suppose that $J_{\boldsymbol{f}}(0)$ is invertible in A. Then there exists a system without constant term $\boldsymbol{g} = (g_1, \dots, g_n)$ of n series in $A[\![X_1, \dots, X_n]\!]$ such that $\boldsymbol{f} \circ \boldsymbol{g} = \mathbf{1}_n$. This system is unique and $\boldsymbol{g} \circ \boldsymbol{f} = \mathbf{1}_n$.*

Proof. The existence and uniqueness of $\boldsymbol{g}$ are clear. It then follows that $J_{\boldsymbol{f}}(0)J_{\boldsymbol{g}}(0) = 1$ and hence $J_{\boldsymbol{g}}(0)$ is also invertible. We conclude that there exists a system $\boldsymbol{h} = (h_1, \dots, h_n)$ of n series without constant term in $A[\![X_1, \dots, X_n]\!]$ such that $\boldsymbol{g} \circ \boldsymbol{h} = \mathbf{1}_n$; from this relation it then follows that

$$\boldsymbol{h} = \mathbf{1}_n \circ \boldsymbol{h} = (\boldsymbol{f} \circ \boldsymbol{g}) \circ \boldsymbol{h} = \boldsymbol{f} \circ (\boldsymbol{g} \circ \boldsymbol{h}) = \boldsymbol{f} \circ \mathbf{1}_n = \boldsymbol{f}$$

so the second clam follows. □

To abbreviate, we shall say in what follows that a ring satisfies **Hensel's conditions** if it is linearly topologized, Hausdorff and complete; given an ideal $\mathfrak{m}$ in such a ring, $\mathfrak{a}$ (or the ordered pair $(A, \mathfrak{m})$) will be said to satisfy Hensel's conditions if $\mathfrak{m}$ is closed in A and its elements are topologically nilpotent. The ideal $\mathfrak{t}$ of A consisting of all the topologically nilpotent elements satisfies Hensel's conditions.

In particular, if A is a ring and $\mathfrak{m}$ an ideal of A and A is Hausdorff and complete with respect to the $\mathfrak{m}$-adic topology, the ordered pair $(A, \mathfrak{m})$ satisfies Hensel's conditions.

Proposition 2.5.12. *Let A be a commutative ring, B a ring satisfying Hensel's conditions and $\phi : A \to B$ a homomorphism. For every family $\boldsymbol{x} = (x_1, \dots, x_n)$ of topologically nilpotent elements of B, there exists a unique homomorphism $\tilde{\phi}$ from $A[\![X_1, \dots, X_n]\!]$ to B such that $\tilde{\phi}(a) = \phi(a)$ for all $a \in A$ and $\tilde{\phi}(X_i) = x_i$ for $1 \leq i \leq n$. Moreover, if $\mathfrak{m}$ denotes the ideal of series without constant term in $A[\![X_1, \dots, X_n]\!]$, then $\tilde{\phi}$ is continuous for the $\mathfrak{m}$-adic topology.*

Proof. Let $\mathfrak{a}$ be the finitely generated ideal generated in B by the x_i's; for every open ideal $\mathfrak{b}$ of B, the images of the x_i in $B/\mathfrak{b}$ are nilpotent, hence the ideal $(\mathfrak{a} + \mathfrak{b})/\mathfrak{b}$ is nilpotent in $B/\mathfrak{b}$ and there exists an integer k such that, for $\alpha \in \mathbb{N}^n$ with $|\alpha| \geq k$ we have $\boldsymbol{x}^\alpha \in \mathfrak{b}$. As every element of $\mathfrak{m}^k$ is a finite sum of formal power series of the form $X^\alpha g(X)$, where $|\alpha| \geq k$, it is seen that, if $\tilde{\phi}$ solves the problem, then $\tilde{\phi}(\mathfrak{m}^k) \subseteq \mathfrak{b}$, which proves the continuity of $\tilde{\phi}$. There obviously exists a unique homomorphism

$$v : A[X_1, \dots, X_n] \to B$$

such that $v(a) = \phi(a)$ for $a \in A$ and $v(X_i) = x_i$, for $1 \leq i \leq n$ and the above argument shows that v is continuous with respect to the topoloogy induced on $A[X_1, \dots, X_n]$ by the $\mathfrak{m}$-adic topology. As $A[X_1, \dots, X_n]$ is dense in $A[\![X_1, \dots, X_n]\!]$ with the $\mathfrak{m}$-adic topology and B is Hausdorff and complete, this completes the proof of the existence and uniqueness of $\tilde{\phi}$. □

If $B = A$ and ϕ is the identity map, we shall write $f(x_1, \dots, x_n)$ or $f(\boldsymbol{x})$ for the element $\tilde{\phi}(f)$ for every formal power seriesfe $A[\![X_1, \dots, X_n]\!]$. For every system $\boldsymbol{f} = (f_1, \dots, f_r)$ of formal power series of $A[\![X_1, \dots, X_n]\!]$, let $\boldsymbol{f}(\boldsymbol{x})$ denote the element $(f_1(\boldsymbol{x}), \dots, f_r(\boldsymbol{x}))$ of A^r, then it is said to be obtained by substituting the x_i for the X_i in $\boldsymbol{f}$. If $n \leq m$ and F is a formal power series of $A[\![X_1, \dots, X_m]\!]$, it is possible to consider F as a formal power series in $X_{n+1}, \dots, X_m$ with coefficients in $A[\![X_1, \dots, X_n]\!]$; let

$$F(x_1, \dots, x_n, X_{n+1}, \dots, X_m)$$

denote the formal power series in $A[\![X_{n+1}, \dots, X_m]\!]$ obtained by substituting the x_i for the X_i in the coefficients of F, for $1 \leq i \leq n$.

Corollary 2.5.13. *Let A be a ring satisfying Hensel's condition and $\boldsymbol{x} = (x_1, \dots, x_n)$ a family of topologically nilpotent elements of A. Let $\boldsymbol{g} = (g_1, \dots, g_q)$ be a system without constant term of series in $A[\![X_1, \dots, X_n]\!]$ and $\boldsymbol{f} = (f_1, \dots, f_p)$ a system of formal power series in $A[\![X_1, \dots, X_q]\!]$. Then $\boldsymbol{g}(\boldsymbol{x}) = (g_1(\boldsymbol{x}), \dots, g_q(\boldsymbol{x}))$ is a family of topologically nilpotent elements of A and*

$$(\boldsymbol{f} \circ \boldsymbol{g})(\boldsymbol{x}) = \boldsymbol{f}(\boldsymbol{g}(\boldsymbol{x})). \tag{2.5.4}$$

Proof. The fact that the $g_i(\boldsymbol{x})$ are topologically nilpotent follows immediately from Proposition 2.5.12 and the fact that in A the ideal of topologically nilpotent elements is closed. Relation (2.5.4) is obvious when the f, g are polynomials; on the other hand, if $\mathfrak{m}$ and $\mathfrak{m}'$ are the ideals of series without constant term in $A[\![X_1,\dots,X_q]\!]$ and $A[\![X_1,\dots,X_n]\!]$, respectively, clearly the relation $f \in \mathfrak{m}^k$ implies $f(g_1,\dots,g_q) \in (\mathfrak{m}')^k$. The two sides of (2.5.4) are therefore continuous functions of $\boldsymbol{f}$ to $(A[\![X_1,\dots,X_q]\!])^p$ if $A[\![X_1,\dots,X_q]\!]$ is given the $\mathfrak{m}$-adic topology, by virtue of the above remark and Proposition 2.5.12, whence the claim. □

In what follows, for a ring A and an ideal $\mathfrak{m}$ of A we shall denote by $\mathfrak{m}^{\times n}$ the product set $\prod_{i=1}^n \mathfrak{m}_i$ in A^n, where $\mathfrak{m}_i = \mathfrak{m}$ for all i, to avoid ambiguity.

Proposition 2.5.14. *Let A be a ring and $\mathfrak{m}$ an ideal of A such that the ordered pair $(A,\mathfrak{m})$ satisfies Hensel's conditions. Let $\boldsymbol{f} = (f_1,\dots,f_n)$ be a system without constant term of series in $A[\![X_1,\dots,X_n]\!]$ such that $J_{\boldsymbol{f}}(0)$ is invertible in A. Then, for all $\boldsymbol{x} \in \mathfrak{m}^{\times n}$, $\boldsymbol{f}(\boldsymbol{x}) \in \mathfrak{m}^{\times n}$ and $\boldsymbol{x} \mapsto \boldsymbol{f}(\boldsymbol{x})$ is a bijection of $\mathfrak{m}^{\times n}$ onto itself, the inverse bijection being $\boldsymbol{x} \mapsto \boldsymbol{g}(\boldsymbol{x})$, where $\boldsymbol{g}$ is given by relation $\boldsymbol{f} \circ \boldsymbol{g} = \mathbf{1}_n$.*

Proof. The fact that $\boldsymbol{f}(\boldsymbol{x}) \in \mathfrak{m}^{\times n}$ is obvious when the f_i are polynomials and follows in the general case from Proposition 2.5.12 and the fact that $\mathfrak{m}$ is closed in A. The other assertions of the proposition are then immediate consequences of (2.5.4). □

Corollary 2.5.15. *With the assumption of Proposition 2.5.14, let $\mathfrak{a}$ be a closed ideal of A contained in $\mathfrak{m}$. Then the relation $\boldsymbol{x} \equiv \boldsymbol{y} \bmod \mathfrak{a}^{\times n}$ is equivalent to $\boldsymbol{f}(\boldsymbol{x}) \equiv \boldsymbol{f}(\boldsymbol{y}) \bmod \mathfrak{a}^{\times n}$.*

Proof. For every formal power series $f \in A[\![X_1,\dots,X_n]\!]$,

$$f(X_1,\dots,X_n) - f(Y_1,\dots,Y_n) = \sum_{i=1}^n (X_i - Y_i) h_i(X_1,\dots,X_n,Y_1,\dots,Y_n)$$

where the h_i belong to $A[\![X_1,\dots,X_n,Y_1,\dots,Y_n]\!]$. It follows immediately that the relation $\boldsymbol{x} \equiv \boldsymbol{y} \bmod \mathfrak{a}^{\times n}$ implies $\boldsymbol{f}(\boldsymbol{x}) \equiv \boldsymbol{f}(\boldsymbol{y}) \bmod \mathfrak{a}^{\times n}$. The converse is obtained by replacing $\boldsymbol{f}$ by its "inverse" $\boldsymbol{g}$. □

Theorem 2.5.16. *Let A be a ring and $\mathfrak{m}$ an ideal of A such that the ordered pair $(A,\mathfrak{m})$ satisfies Hensel's conditions. Let $\boldsymbol{f} = (f_1,\dots,f_n)$ be a system of n elements of $A\{X_1,\dots,X_n\}$, let $\boldsymbol{a} \in A^n$ and write $e = J_{\boldsymbol{f}}(\boldsymbol{a})$. There exists a system $\boldsymbol{g} = (g_1,\dots,g_n)$ of restricted formal power series without constant term in $A\{X_1,\dots,X_n\}$ such that*

(a) $M_{\boldsymbol{g}}(0) = \mathfrak{a}_n$.

(b) For all $\boldsymbol{x} \in A^n$,

$$\boldsymbol{f}(\boldsymbol{a} + e\boldsymbol{x}) = \boldsymbol{f}(\boldsymbol{a}) + M_{\boldsymbol{f}}(\boldsymbol{a}) \cdot e\boldsymbol{g}(\boldsymbol{x}). \tag{2.5.5}$$

(c) Let $\boldsymbol{h} = (h_1,\dots,h_n)$ be the system offormal power series without constant term such that $\boldsymbol{g} \circ \boldsymbol{h} = \mathfrak{a}_n$. For all $\boldsymbol{y} \in \mathfrak{m}^{\times n}$,

$$\boldsymbol{f}(\boldsymbol{a} + e\boldsymbol{h}(\boldsymbol{y})) = \boldsymbol{f}(\boldsymbol{a}) + M_{\boldsymbol{f}}(\boldsymbol{a}) \cdot e\boldsymbol{y}. \tag{2.5.6}$$

Proof. For every formal power series $f \in A[\![X_1,\dots,X_n]\!]$,

$$f(\boldsymbol{X} + \boldsymbol{Y}) = f(\boldsymbol{X}) + M_f(\boldsymbol{X}) \cdot \boldsymbol{Y} + \sum_{1 \le i \le j \le n} G_{ij}(\boldsymbol{X},\boldsymbol{Y}) Y_i Y_j \tag{2.5.7}$$

where the G_{ij} are well determined formal power series in $A[\![X_1,\dots,X_n,Y_1,\dots,Y_n]\!]$. If f is restricted, so are the elements of M_f, and the G_{ij} for these formal power series are polynomials if f is a polynomial and it follows from their uniqueness that for every open ideal $\mathfrak{a}$ of A, denoting by $\pi_{\mathfrak{a}} : A \to A/\mathfrak{a}$ the canonical map, the image of G_{ij} under $\pi_{\mathfrak{a}}$ is the coefficient of Y_iY_j in $\pi_{\mathfrak{a}}(F)$ where F is the formal power series $f(\boldsymbol{X}+\boldsymbol{Y})$ in $A[\![X_1,\dots,X_n,Y_1,\dots,Y_n]\!]$; whence our assertion.

This being so, writing formula (2.5.7) for each series f_i, we obtain for all $x \in A^n$,

$$\boldsymbol{f}(\boldsymbol{a}+e\boldsymbol{x}) = \boldsymbol{f}(\boldsymbol{a}) + M_{\boldsymbol{f}}(\boldsymbol{a})\cdot e\boldsymbol{x} + e^2\boldsymbol{r}(\boldsymbol{x}) \tag{2.5.8}$$

where $\boldsymbol{r} = (r_1,\dots,r_n)$ is a system of restricted formal power series each of which is of total order ≥ 2. Now there exists a square matrix $N \in \mathcal{M}_n(A)$ such that $M_{\boldsymbol{f}}(\boldsymbol{a})N = e\mathfrak{a}_n$, whence using this in (2.5.8),

$$\boldsymbol{f}(\boldsymbol{a}+e\boldsymbol{x}) = \boldsymbol{f}(\boldsymbol{a}) + M_{\boldsymbol{f}}(\boldsymbol{a})\cdot e\boldsymbol{x} + M_{\boldsymbol{f}}(\boldsymbol{a})N\cdot e\boldsymbol{r}(\boldsymbol{x})$$

Writing $\boldsymbol{g} = \mathbf{1}_n + N\cdot\boldsymbol{r}$, we see that $\boldsymbol{g}$ satisfies conditions (a) and (b); then it is sufficient to replace $\boldsymbol{x}$ by $\boldsymbol{h}(\boldsymbol{y})$ to obtain (c). □

Corollary 2.5.17. *Let A be a ring and $\mathfrak{m}$ an ideal of A such that the ordered pair $(A,\mathfrak{m})$ satisfies Hensel's conditions. Let $f \in A\{X\}$, $a \in A$ and write $e = f'(a)$. If $f(a) \equiv 0 \bmod e^2\mathfrak{m}$, then there exists $b \in A$ such that $f(b) = 0$ and $b \equiv a \bmod e\mathfrak{m}$. If b' is another element such that $f(b') = 0$ and $b' \equiv a \bmod e\mathfrak{m}$, then $e(b-b') = 0$. In particular, b is unique if e is not a divisor of zero in A.*

Proof. Let $f(a) = e^2c$ where $c \in \mathfrak{m}$; formula (2.5.6) for $n = 1$ gives

$$f(a+eh(y)) = e^2(c+y)$$

and it is therefore sufficient to take $y = -c$, whence $b = a + eh(-y)$. Moreover if $b = a + ex$, $b' = a + ex'$ are such that $x, x' \in \mathfrak{m}$ and $f(b) = f(b') = 0$, then we deduce from (2.5.5) that $e^2(g(x) - g(x')) = 0$. As $g(X) - g(Y) = (X-Y)p(X,Y)$, where p is restricted and $p(0,0) = 1$, we see $g(x) - g(x') = (x-x')v$ where $v \in A$. Note that since $\mathfrak{m}$ is closed, p is restricted and 1 is the constant term of p, we have $p(x,x') - 1 \in \mathfrak{m}$, whence $v \in 1+\mathfrak{m}$ and so v is invertible. This proves the relation $e(b-b') = 0$. □

Example 2.5.18. Let p be a prime number $\neq 2$ and n an integer whose class mod p is a nonzero square in the prime field $\mathbb{F}_p$. If $\mathbb{Z}_p$ is the ring of p-adic integers, the application of Corollary 2.5.17 to the polynomial $X^2 - n$ shows that n is a square in $\mathbb{Z}_p$; for example 7 is a square in $\mathbb{Z}_3$.

Example 2.5.19. Let $A = K[\![Y]\!]$ be the ring of formal power series in one indeterminate with coefficients in a field K; with the (Y)-adic topology, the ring A is Hausdorff and complete and the map $f(Y) \mapsto f(0)$ defines by passing to the quotient ring an isomorphism of κ_A onto the field K. By Corollary 2.5.17, if $F(Y,X)$ is a polynomial in X with coefficients in A and a is a simple root of $F(0,X)$ in K, there exists a unique formal power series $f(Y)$ such that $f(0) = a$ and $F(Y, f(Y)) = 0$.

Corollary 2.5.20. *Let A be a ring and $\mathfrak{m}$ an ideal of A such that the ordered pair $(A,\mathfrak{m})$ satisfies Hensel's conditions. Let r, n be integers such that $0 \leq r < n$ and $\boldsymbol{f} = (f_{r+1},\dots,f_n)$ is a system of $n-r$ elements of $A\{X_1,\dots,X_n\}$. Let $J_{\boldsymbol{f}}^{(n-r)}$ denote the minor of $M_{\boldsymbol{f}}(X)$ consisting of the columns of index j such that $r+1 \leq j \leq n$. Let $\boldsymbol{a} \in A^n$ be such that $J_{\boldsymbol{f}}(\boldsymbol{a})$ is invertible in A and $\boldsymbol{f}(\boldsymbol{a}) \equiv 0 \bmod \mathfrak{m}^{\times(n-r)}$. Then there exists a unique $\boldsymbol{x} = (x_1,\dots,x_n) \in A^n$ such that $x_j = a_j$ for $1 \leq j \leq r$ and $x \equiv a \bmod \mathfrak{m}^{\times n}$ and $\boldsymbol{f}(\boldsymbol{x}) = 0$.*

Proof. Substituting a_j for X_j for $1 \leq j \leq r$ in the f_i, we see immediately that we may restrict our attention to the case where $r = 0$ to prove the corollary. Proposition 2.5.14 shows then that $\boldsymbol{f}$ defines a bijection on $\boldsymbol{a} + \mathfrak{m}^{\times n}$ onto $\boldsymbol{f}(\boldsymbol{a}) + \mathfrak{m}^{\times n} = \mathfrak{m}^{\times n}$; the corollary follows from the fact that $0 \in \mathfrak{m}^{\times n}$. □

2.6 Complements on complete rings

2.6.1 Admissible rings

Let A be a linearly topologized ring and $\mathfrak{I}$ be an open ideal of A. We say that $\mathfrak{I}$ is a **nilideal** if for any open neighborhood V of 0 in A, there exists an integer $n > 0$ such that $\mathfrak{I}^n \subseteq V$ (which means, by abusing language, that $(\mathfrak{I}^n)$ tends to 0). The linearly topologized A is called **preadmissible** if there exists a nilideal of A, and A is **admissible** if it is preadmissible and is complete and separated.

It is clear that if $\mathfrak{I}$ is a nilideal and $\mathfrak{K}$ is an open ideal of A, then $\mathfrak{I} \cap \mathfrak{K}$ is also a nilideal. The nilideals of a preadmissible ring A then form a fundamental system of open ideals (but note that the power $\mathfrak{I}^n$ of a nilideal if not necessarily open).

Lemma 2.6.1. *Let A be a linearly topologized ring.*

(a) For an element $x \in A$ to be topologically nilpotent, it is necessary and sufficient that for any open ideal $\mathfrak{I}$ of A, the canonical image of x in $A/\mathfrak{I}$ is nilpotent. The set $\mathfrak{T}$ of topological nilpotent elements of A is therefore an ideal.

(b) Suppose that A is preadmissible and let $\mathfrak{I}$ be a nilideal of A. For an element $x \in A$ to be topologically nilpotent, it is necessary and sufficient that the canonical image of x in $A/\mathfrak{I}$ is nilpotent. The ideal $\mathfrak{T}$ is then the inverse image of the nilradical of $A/\mathfrak{I}$ in A, and hence open.

Proof. By definition, x is topologically nilpotent if and only if x^n is contained in $\mathfrak{I}$ for sufficiently large n, whence the assertion of (a). As for assertion (b), it suffices to note that for any neighborhoof V of 0 in A, there exists $n > 0$ such that $\mathfrak{I}^n \subseteq V$. If $x \in A$ is such that $x^m \in \mathfrak{I}$, then $x^{mq} \in V$ for $q \geq n$, so x is topologically nilpotent. □

Proposition 2.6.2. *Let A be a preadmissible ring and $\mathfrak{I}$ be a nilideal of A.*

(a) For an ideal $\mathfrak{J}$ of A to be contained in a nilideal, it is necessary and sufficient that there exists an integer $n > 0$ such that $\mathfrak{J}^n \subseteq \mathfrak{I}$.

(b) For an element $x \in A$ to be contained in a nilideal, it is necessary and sufficient that it is topologically nilpotent.

Proof. If $\mathfrak{J}^n \subseteq \mathfrak{I}$, then $\mathfrak{I} + \mathfrak{J}$ is a nilideal, because it is open and $(\mathfrak{I} + \mathfrak{J})^n \subseteq \mathfrak{I}$; this proves assertion (a). For (b), the condition is evidently necessary, and it is sufficient because if this is satisfied, then there exists an integer $n > 0$ such that $x^n \in \mathfrak{I}$, so $\mathfrak{J} = \mathfrak{I} + Ax$ is a nilideal (it is open and $\mathfrak{J}^n \subseteq \mathfrak{I}$). □

Corollary 2.6.3. *In a preadmissible ring A, an open prime ideal contains any nilideal.*

Proof. If $x \in \mathfrak{I}$ is an element of a nilideal and $\mathfrak{p}$ is an open prime ideal, then x is topologically nilpotent by Proposition 2.6.2(b), so $x^n \in \mathfrak{p}$ for some integer $n > 0$, and hence $x \in \mathfrak{p}$. □

Corollary 2.6.4. *Let A be a preadmissible ring. Then the following properties for an ideal $\mathfrak{I}_0$ of A are equivalent:*

(i) $\mathfrak{I}_0$ is the largest nilideal of A;

(ii) $\mathfrak{I}_0$ is a maximal nilideal of A;

(iii) $\mathfrak{I}_0$ is a nilideal such that the ring $A/\mathfrak{I}_0$ is reduced.

For there exists an ideal $\mathfrak{I}_0$ satisfying these properties, it is necessary and sufficient that there exists a nilideal $\mathfrak{I}$ such that the nilradical $A/\mathfrak{I}$ is nilpotent. In this case $\mathfrak{I}_0$ is then equal to the ideal $\mathfrak{T}$ of topologically nilpotent elements of A, and we denote by A_{red} the reduced quotient ring $A/\mathfrak{T}$.

Proof. It is clear that (i) imples (ii), and (iii) implies (i) in view of Proposition 2.6.2(b) and Lemma 2.6.1(b). On the other hand, if $\mathfrak{I}_0$ is maximal among nilideals, then for any nilideal $\mathfrak{I}$, there exists an integer $n > 0$ such that $\mathfrak{I}_0^n \subseteq \mathfrak{I}$, so $(\mathfrak{I}_0 + \mathfrak{I})$ is a nilideal of A. By the maximality, we then conclude that $\mathfrak{I}_0 + \mathfrak{I} = \mathfrak{I}_0$, so $\mathfrak{I} \subseteq \mathfrak{I}_0$ and $\mathfrak{I}_0$ is the largest nilideal. The last assertion follows from Proposition 2.6.2(a) and Lemma 2.6.1(b). □

Corollary 2.6.5. *A Noetherian preadmissible ring admits a largest nilideal.*

Proof. If A is a Noetherian preadmissible ring, then $A/\mathfrak{I}$ is also Noetherian, so its nilradical is nilpotent. □

We note that any defining ideal $\mathfrak{I}$ of A is necessarily a nilideal, but the converse is not true. In view of this, we say that a ring A is **preadic** if there exists a nilideal $\mathfrak{I}$ of A such that $\mathfrak{I}^n$ form a fundamental system of open neighborhoods of 0 in A (which means each $\mathfrak{I}^n$ is open, or that $\mathfrak{I}$ is a defining ideal of A). The ring A is **adic** if it is preadic and separated and complete. If $\mathfrak{I}$ is a nilideal of a preadic ring A with defining ideal $\mathfrak{I}$, then it is easy to see that each $\mathfrak{I}^n$ is open in A, hence also a defining ideal of A. In other words, if A is a preadic ring, then any nilideal defines the topology of A, hence is a defining ideal of A.

Proposition 2.6.6. *Let A be a admissible ring and $\mathfrak{I}$ be a nilideal of A. Then $\mathfrak{I}$ is contained in the Jacobson radical of A. In particular, if A is adic, then it is a Zariski ring.*

Proof. In fact, since A is separated and complete, the conclusion follows from Lemma 2.3.25(b) as any element of $\mathfrak{I}$ is topologically nilpotent. □

We shall now provide a characterization for admissible rings via projective limits of discrete rings. For this, let us recall that if A is a linearly topologized ring and $(\mathfrak{I}_\lambda)$ is a fundamental system of open ideals of A, then the canonical homomorphisms $\varphi_\lambda : A \to A/\mathfrak{I}_\lambda$ form a projective system of discrete rings, hence define a continuous homomorphism $\varphi : A \to \varprojlim A/\mathfrak{I}_\lambda$ (the latter can be identified with the completion of A). This homomorphism is injective if and only if A is separated (and hence identified A with a dense subring of $\varprojlim A/\mathfrak{I}_\lambda$), and is an isomorphism if A is separated and complete.

Proposition 2.6.7. *For a linearly topologized ring be to admissible, it is necessary and sufficient that it is isomorphic to a projective limit $A = \varprojlim A_\lambda$, where $(A_\lambda, u_{\lambda\mu})$ is a projective system of discrete rings and the index set I has a smallest element 0 and satisfies the following conditions:*

- *(a) the homomorphisms $u_\lambda : A \to A_\lambda$ are surjective;*
- *(b) the kernel $\mathfrak{I}_\lambda$ of $u_{0,\lambda} : A_\lambda \to A_0$ is nilpotent.*

If these are satisfied, the kernel $\mathfrak{I}$ of $u_0 : A \to A_0$ is equal to $\varprojlim \mathfrak{I}_\lambda$.

Proof. This condition is necessary in view of the above remarks, since we can choose a fundamental system $(\mathfrak{I}_\lambda)$ of neighborhoods of 0 formed by nilideals contained in a fixed nilideal $\mathfrak{I}_0$, and note that $\mathfrak{I}_0^n \subseteq \mathfrak{I}_\lambda$ for sufficiently large n. The converse of this follows from the definition of projective limits, since $\mathfrak{I}$ is then a nilideal of A, and the last assertion is immediate. □

Let A be an admissible ring and $\mathfrak{I}$ be an ideal of A contained in a nilideal (which means $(\mathfrak{I}^n)$ tends to 0). We can then consider the $\mathfrak{I}$-adic topology on A, and the hypothesis on A implies that $\bigcap_n \mathfrak{I}^n = 0$, so the $\mathfrak{I}$-adic topology on A is separated.

Proposition 2.6.8. *If A is an admissible ring and $\mathfrak{I}$ be an ideal of A contained in a nilideal such that the $\mathfrak{I}^n$ are closed in A, then A is separated and complete for the $\mathfrak{I}$-adic topology.*

Proof. This follows from **??**, since the $\mathfrak{I}$-adic topology is finer than the original topology on A. □

Corollary 2.6.9. *Let A be an admissible ring, $\mathfrak{I}$ be a nilideal of A such that the $\mathfrak{I}^n$ are closed in A. For A to be Noetherian, it is necessary and sufficient that $A/\mathfrak{I}$ is Noetherian and that $\mathfrak{I}/\mathfrak{I}^2$ is a finitely generated $(A/\mathfrak{I})$-module.*

Proof. These conditions are clearly necessary. Conversely, if they are satisfied, then since $\mathrm{gr}(A)$ is generated by $\mathrm{gr}_1(A) = \mathfrak{I}/\mathfrak{I}^2$ over $\mathrm{gr}_0(A) = A/\mathfrak{I}$, it is Noetherian by Hilbert basis theorem, and the conclusion follows from Corollary 2.3.35. □

If A is an adic ring and $\mathfrak{I}$ is a defining ideal of A, then A is the projective limit of the discrete rings $A_i = A/\mathfrak{I}^{i+1}$, which is identified with the $\mathfrak{I}$-adic completion of A. Conversely, we can in fact use this to characterize adic rings such that there exists a defining ideal $\mathfrak{I}$ such that $\mathfrak{I}/\mathfrak{I}^2$ is finitely generated, as the following proposition shows:

Proposition 2.6.10. *Let A be a ring, $\mathfrak{I}$ be an ideal of A such that $\mathfrak{I}/\mathfrak{I}^2$ is a finitely generated $A/\mathfrak{I}$-module.*

(a) *The ring $\widehat{A} = \varprojlim(A/\mathfrak{I}^{n+1})$ is an adic ring. If $\widehat{\mathfrak{I}}$ is the closure of the canonical image of $\mathfrak{I}$ in A, then $\widehat{\mathfrak{I}}$ is a finitely generated nilideal of A, and $\widehat{\mathfrak{I}}^n$ is the closure of the canonical image of $\mathfrak{I}^n$ in A. Moreover, $\widehat{A}/\widehat{\mathfrak{I}}^n$ is isomorphic to $A/\mathfrak{I}^n$ and $\widehat{\mathfrak{I}}/\widehat{\mathfrak{I}}^2$ to $\mathfrak{I}/\mathfrak{I}^2$, and $\widehat{\mathfrak{I}}$ is isomorphic to $\varprojlim(\mathfrak{I}/\mathfrak{I}^{n+1})$.*

(b) *Let M be an A-module such that $M/\mathfrak{I}M$ is a finitely generated $A/\mathfrak{I}$-module. Then $\widehat{M} = \varprojlim(M/\mathfrak{I}^{n+1}M)$ is a finitely generated A-module, $\widehat{\mathfrak{I}}^n\widehat{M}$ is the closure of the canonical image of $\mathfrak{I}^n M$, and $\widehat{M}/\widehat{\mathfrak{I}}^n\widehat{M}$ is isomorphic to $M/\mathfrak{I}^n M$.*

Moreover, for A to be Noetherian, it is necessary and sufficient that A_0 is Noetherian.

Proof. It is clear that the conditions of Proposition 2.3.42 and Corollary 2.3.43 are satisfied, and the last assertion follows from Corollary 2.3.44. □

2.6.2 Complete localizations

Let A be a linearly topologized ring, $(\mathfrak{I}_\lambda)$ be a fundamental system of neighborhoods of 0 formed by ideals of A, and S be a multiplicative subset of A. Let $u_\lambda : A \to A_\lambda = A/\mathfrak{I}_\lambda$ be the canonical homomorphism, and for $\mathfrak{I}_\mu \subseteq \mathfrak{I}_\lambda$, let $u_{\lambda\mu} : A_\mu \to A_\lambda$ be the canonical homomorphism. Put $S_\lambda = u_\lambda(S)$, so that $u_{\lambda\mu}(S_\mu) = S_\lambda$. The homomorphisms $u_{\lambda\mu}$ then induce canonical surjections $S_\mu^{-1}A_\mu \to S_\lambda^{-1}A_\lambda$, which form a projective system and we denote by $A\{S^{-1}\}$ its projective limit. It is clear that this does not depend on the choice of the system $(\mathfrak{I}_\lambda)$.

Proposition 2.6.11. *The ring $A\{S^{-1}\}$ is homeomorphic to the completion of the localization $S^{-1}A$ for the topology induced by the closed ideals $S^{-1}\mathfrak{I}_\lambda$.*

Proof. In fact, if $v_\lambda : S^{-1}A \to S_\lambda^{-1}A_\lambda$ is the canonical homomorphism induced by u_λ, the kernel of v_λ is $S^{-1}\mathfrak{I}_\lambda$ and v_λ is surjective, and the completion of $S^{-1}A$ is identified with the limit $A\{S^{-1}\}$. □

Corollary 2.6.12. *If $\widehat{S}$ is the image of S in the completion $\widehat{A}$ of A, then $A\{S^{-1}\}$ is canonically identified with $\widehat{A}\{\widehat{S}^{-1}\}$.*

Remark 2.6.13. Note that even if A is separated and complete, the localization $S^{-1}A$ many be neither separated nor complete. For example, let S be the subset f^n for $n \geq 0$, where f is a topologically nilpotent element in A but not nilpotent. Then $S^{-1}A$ is nonzero, but for any λ there exists an integer n such that $f^n \in \mathfrak{I}_\lambda$, so $1 = f^n/f^n \in S^{-1}\mathfrak{I}_\lambda$ and $S^{-1}\mathfrak{I}_\lambda = S^{-1}A$.

Corollary 2.6.14. *If 0 is not contained in the closure of S, then the ring $A\{S^{-1}\}$ is nonzero.*

Proof. In fact, 0 is not contained in the closure of $\{1\}$ in the ring $S^{-1}A$, since otherwise $1 \in S^{-1}\mathfrak{I}_\lambda$ for any ideal $\mathfrak{I}_\lambda$, whence $\mathfrak{I}_\lambda \cap S \neq \varnothing$, a contradiction. □

The ring $A\{S^{-1}\}$ is called the **complete localization** of the ring A. It is clear that the inverse image of $S^{-1}\mathfrak{I}_\lambda$ in A contains $\mathfrak{I}_\lambda$, so the canonical homomorphism $A \to S^{-1}A$ is continuous. By composing with the canonical homomorphism $A \to A\{S^{-1}\}$, we then obtain a canonical continuous homomorphism $A \to A\{S^{-1}\}$, which is the projective limit of the homomorphisms $A \to S_\lambda^{-1}A_\lambda$. The ring $A\{S^{-1}\}$ satisfies a similar universal property as the localization rings:

Proposition 2.6.15. *Any continuous homomorphism u from A into a separated and complete linearly topologized ring B such that $u(S)$ is invertible in B factors uniquely through $A\{S^{-1}\}$:*

$$A \longrightarrow A\{S^{-1}\} \xrightarrow{\tilde{u}} B$$

where $\tilde{u}$ is continuous.

Proof. In fact, u factors through $S^{-1}A$:

$$A \longrightarrow S^{-1}A \xrightarrow{\tilde{v}} B$$

Now since for any open ideal $\mathfrak{K}$ of B, $u^{-1}(\mathfrak{K})$ contains an ideal $\mathfrak{I}_\lambda$, we see that $\tilde{v}^{-1}(\mathfrak{K})$ contains $S^{-1}\mathfrak{I}_\lambda$, so $\tilde{v}$ is continuous. Since B is complete and separated, the homomorphism $\tilde{v}$ then factors into the following

$$S^{-1}A \longrightarrow A\{S^{-1}\} \xrightarrow{\tilde{u}} B$$

where $\tilde{u}$ is continuous; this proves the proposition. □

Let B be a second linearly topologized ring, T be a multiplicative subset of B, and $\varphi : A \to B$ be a continuous homomorphism such that $\varphi(S) \subseteq T$. Then by Proposition 2.6.15, the continuous homomorphism

$$A \xrightarrow{\varphi} B \longrightarrow B\{T^{-1}\}$$

then factors into

$$A \longrightarrow A\{S^{-1}\} \xrightarrow{\tilde{\varphi}} B\{T^{-1}\}$$

where $\tilde{\varphi}$ is continuous. In particular, if $A = B$ and φ is the identity, we see that for any multiplicative subsets $S \subseteq T$ we have a continuous homomorphism $\rho^{T,S} : A\{S^{-1}\} \to A\{T^{-1}\}$, obtained by passing to completions from the canonical homomorphism $S^{-1}A \to T^{-1}A$. If U is a thrid multiplicative subset of A such that $S \subseteq T \subseteq U$, we have $\rho^{U,S} = \rho^{U,T} \circ \rho^{T,S}$.

Let S_1 and S_2 be multiplicative subset of A, and $\tilde{S}_2$ be the canonical image of S_2 in $A\{S_1^{-1}\}$. We then have a canonical isomorphism $A\{(S_1S_2)^{-1}\} \cong A\{S_1^{-1}\}\{\tilde{S}_2^{-1}\}$, which comes from the canonical isomorphism $(S_1S_2)^{-1}A \cong \bar{S}_2^{-1}(S_1^{-1}A)$, where $\bar{S}_2$ is the canonical image of S_2 in $S_1^{-1}A$. Let $\mathfrak{a}$ be an open ideal of A. We may suppose that $\mathfrak{I}_\lambda \subseteq \mathfrak{a}$ for any λ, and therefore $S^{-1}\mathfrak{I}_\lambda \subseteq S^{-1}\mathfrak{a}$ in the ring $S^{-1}A$, which means $S^{-1}\mathfrak{a}$ is an open ideal of $S^{-1}A$. We denote by $\mathfrak{a}\{S^{-1}\}$ the completion of $S^{-1}\mathfrak{a}$, which is identified with $\varprojlim(S^{-1}\mathfrak{a}/S^{-1}\mathfrak{I}_\lambda)$ and is an open ideal of $A\{S^{-1}\}$. Moreover, the discrete ring $A\{S^{-1}\}/\mathfrak{a}\{S^{-1}\}$ is canonically isomorphic to $S^{-1}A/S^{-1}\mathfrak{a} = S^{-1}(A/\mathfrak{a})$.

Proposition 2.6.16. *Let A be a linearly topologized ring and S be a multiplicative subset.*

(a) Any open ideal of $A\{S^{-1}\}$ is of the form $\mathfrak{a}\{S^{-1}\}$, where $\mathfrak{a}$ is an open ideal of A.

(b) The map $\mathfrak{p} \mapsto \mathfrak{p}\{S^{-1}\}$ is an increasing bijection from the set of open prime ideals $\mathfrak{p}$ in A such that $\mathfrak{p} \cap S = \varnothing$ to the set of open prime ideals of $A\{S^{-1}\}$. Moreover, the fraction field of $A\{S^{-1}\}/\mathfrak{p}\{S^{-1}\}$ is canonically isomorphic to $A/\mathfrak{p}$.

Proof. If $\mathfrak{a}'$ is an open ideal of $A\{S^{-1}\}$, then $\mathfrak{a}'$ contains an ideal of the form $\mathfrak{I}_\lambda\{S^{-1}\}$, hence is the inverse image of an ideal of $S^{-1}A/S^{-1}\mathfrak{I}_\lambda$, which is necessarily of the form $S^{-1}\mathfrak{a}$, where $\mathfrak{a} \supseteq \mathfrak{I}_\lambda$. The second assertion follows from Proposition 1.2.37 and the fact that taking completion does not change residue fields. □

Proposition 2.6.17. *Let A be a linearly topologized ring and S be a multiplicative subset of A.*

- *(a) If A is admissible, so is $A' = A\{S^{-1}\}$, and for any nilideal $\mathfrak{I}$ of A, $\mathfrak{I}' = \mathfrak{I}\{S^{-1}\}$ is a nilideal of $A\{S^{-1}\}$.*
- *(b) Suppose that A is adic and $\mathfrak{I}$ is a definig ideal of A such that $\mathfrak{I}/\mathfrak{I}^2$ is finitely generated over $A/\mathfrak{I}$. Then A' is an $\mathfrak{I}'$-adic ring and $\mathfrak{I}'/\mathfrak{I}'^2$ is finitely generated over $A'/\mathfrak{I}'$.*

Proof. If $\mathfrak{I}$ is a nilideal of A, then it is clear that $S^{-1}\mathfrak{I}$ is nilideal of $S^{-1}A$, because $(S^{-1}\mathfrak{I})^n = S^{-1}\mathfrak{I}^n$ for each n. Now let $\overline{A}$ be the separated ring associated with $S^{-1}A$, and $\bar{\mathfrak{I}}$ be the image of $S^{-1}\mathfrak{I}$ in $\overline{A}$. Then the image of $S^{-1}\mathfrak{I}^n$ is $\bar{\mathfrak{I}}^n$, so $\bar{\mathfrak{I}}^n$ tends to 0 in $\overline{A}$. As $\mathfrak{I}'$ is the closure of $\bar{\mathfrak{I}}$ in A', $\mathfrak{I}'^n$ is contained in the closure of $\bar{\mathfrak{I}}^n$, hence tends to 0 in A'.

For (b), put $A_i = A/\mathfrak{I}^{i+1}$, and let $u_{ij} : A_j \to A_i$ be the canonical homomorphism for $i \leq j$. Let S_i be the image of S in A_i, put $A'_i = S_i^{-1}A_i$, and let $u'_{ij} : A'_j \to A'_i$ be the induced homomorphism. We show that the projective system (A'_i, u'_{ij}) satisfies the conditions of Proposition 2.3.42. It is clear that each u'_{ij} is surjective, and by the flatness of localization, the kernel of u'_{ij} is equal to $S_j^{-1}(\mathfrak{I}^{i+1}/\mathfrak{I}^{j+1}) = \mathfrak{I}'^{i+1}_j$, where $\mathfrak{I}'_j = S_j^{-1}(\mathfrak{I}/\mathfrak{I}^{j+1})$. Finally, $\mathfrak{I}'_1/\mathfrak{I}'^2_1 = S_1^{-1}(\mathfrak{I}/\mathfrak{I}^2)$ is finitely generated over A'_1 since $\mathfrak{I}/\mathfrak{I}^2$ is finitely generated over $A/\mathfrak{I}$, and $A'_0 = S_0^{-1}(A/\mathfrak{I})$ is Noetherian if A is Noetherian. □

Corollary 2.6.18. *Under the hypothesis of Proposition 2.6.17(b), we have $(\mathfrak{I}\{S^{-1}\})^n = \mathfrak{I}^n\{S^{-1}\}$.*

Proof. This follows from Proposition 2.3.42, in view of the proof of Proposition 2.6.17. □

Proposition 2.6.19. *Let A be an adic Noetherian ring and S be a multiplicative subset of A. Then $A\{S^{-1}\}$ is a flat A-module.*

Proof. If $\mathfrak{I}$ is a defining ideal of A, $A\{S^{-1}\}$ is the completion of the Noetherian ring $S^{-1}A$ for the $S^{-1}\mathfrak{I}$-adic topology, so $A\{S^{-1}\}$ is a flat $S^{-1}A$-module (Theorem 2.4.19). As $S^{-1}A$ is flat over A, we conclude the proposition. □

Corollary 2.6.20. *Let A be an adic Noetherian ring and $T \subseteq S$ be multiplicative subsets of A. Then $A\{S^{-1}\}$ is a flat $A\{T^{-1}\}$-module.*

Proof. We have remarked that $A\{S^{-1}\}$ is canonically identified with $A\{T^{-1}\}\{S_0^{-1}\}$, where S_0 is the canonical image of S in $A\{T^{-1}\}$, and $A\{T^{-1}\}$ is Noetherian by Proposition 2.6.17. □

For any element $f \in A$, we denote by $A_{\{f\}}$ the complete localization $A\{S_f^{-1}\}$, where $S_f = \{f^n\}$. For any open ideal $\mathfrak{a}$ of A, we denote by $\mathfrak{a}_{\{f\}}$ the ideal $\mathfrak{a}\{S_f^{-1}\}$. If g is another element of A, we then have a continuous homomorphism $A_{\{f\}} \to A_{\{fg\}}$. As f runs through a multiplicative subset S of A, the rings $A_{\{f\}}$ then form an inductive system, and we set $A_{\{S\}} = \varinjlim_{f\in S} A_{\{f\}}$. For any $f \in S$, we have a canonical homomorphism $A_{\{f\}} \to A\{S^{-1}\}$, and they form an inductive system which induces a canoncal homomorphism $A_{\{S\}} \to A\{S^{-1}\}$.

Proposition 2.6.21. *If A is a Noetherian ring, $A\{S^{-1}\}$ is a flat module over $A_{\{S\}}$.*

Proof. By Corollary 2.6.20 the ring $A\{S^{-1}\}$ is flat over each $A_{\{f\}}$ for $f \in S$, so the conclusion follows from **??**. □

Proposition 2.6.22. *Let $\mathfrak{p}$ be an open prime ideal of an admissible ring A, and let $S = A - \mathfrak{p}$. Then the ring $A\{S^{-1}\}$ and $A_{\{S\}}$ are local, the canonical homomorphism $A_{\{S\}} \to A\{S^{-1}\}$ is local, and the residue fields of $A_{\{S\}}$ and $A\{S^{-1}\}$ are isomorphic to $\kappa(\mathfrak{p})$.*

Proof. Let $\mathfrak{I} \subseteq \mathfrak{p}$ be a nilideal of A. We then have $S^{-1}\mathfrak{I} \subseteq S^{-1}\mathfrak{p} = \mathfrak{p}A_{\mathfrak{p}}$, so $A_{\mathfrak{p}}/S^{-1}\mathfrak{I}$ is a local ring, and we conclude from Proposition 2.6.17(a) that $A\{S^{-1}\}$ is a local ring. Put $\mathfrak{m} = \varinjlim_{f\in S}\mathfrak{p}_{\{f\}}$, which is an ideal of $A_{\{S\}}$, we then see that any element of $A_{\{S\}} - \mathfrak{m}$ is invertible, so $A_{\{S\}}$ is a local ring with maximal ideal $\mathfrak{m}$. In fact, any such element is the image of an element $z \in A_{\{f\}} = \mathfrak{p}_{\{f\}}$ in $A_{\{S\}}$, for an element $f \in S$. Its canonical image z_0 in $A_{\{f\}}/\mathfrak{I}_{\{f\}} = S_f^{-1}(A/\mathfrak{I})$ is then not contained in $S_f^{-1}(\mathfrak{p}/\mathfrak{I})$, which means $z_0 = \bar{x}/\bar{f}^k$, where $x \notin \mathfrak{p}$ and $\bar{x}, \bar{f}$ are the classes of x, f mod $\mathfrak{I}$. As $x \in S$, we have $g = xf \in S$, so $y_0 = x^{k+1}/g^k$, the canonical image of $x/f^k \in S_f^{-1}A$ in the ring $S_g^{-1}A$, then admits an inverse $x^{k-1}f^{2k}/g^k$. This implies a fortiori that the image of y_0 in $S_g^{-1}A/S_g^{-1}\mathfrak{I}$ is invertible, so (Corollary 2.6.18) the canonical image y of z in $A_{\{g\}}$ is invertible, and so is its image in $A_{\{S\}}$. Now, the image of $\mathfrak{p}_{\{f\}}$ in $A\{S^{-1}\}$ is contained in the maximal ideal $\mathfrak{p}\{S^{-1}\}$ of this ring, so the image of $\mathfrak{m}$ in $A\{S^{-1}\}$ is contained in $\mathfrak{p}\{S^{-1}\}$, which means the homomorphism $A_{\{S\}} \to A\{S^{-1}\}$ is local. Finally, as any element $A\{S^{-1}\}/\mathfrak{p}\{S^{-1}\}$ is the image of an element of $S_f^{-1}A$ for some $f \in S$, the homomorphism $A_{\{S\}} \to A\{S^{-1}\}/\mathfrak{p}\{S^{-1}\}$ is surjective, and hence induces an isomorphism on residue fields. This residue field is isomorphic to $\kappa(\mathfrak{p})$ since taking completion does not change residue fields. □

Corollary 2.6.23. *Under the hypotheses of Proposition 2.6.22, if we suppose that A is an adic Noetherian ring, the local rings $A\{S^{-1}\}$ and $A_{\{S\}}$ are Noetherian, and $A\{S^{-1}\}$ is a faithfully flat $A_{\{S\}}$-module.*

Proof. By Proposition 2.6.17(b) and Corollary 2.6.20, the ring $A\{S^{-1}\}$ is Noetherian and flat over $A_{\{S\}}$. As the homomorphism $A_{\{S\}} \to A\{S^{-1}\}$ is local, we then conclude that $A\{S^{-1}\}$ is faithfully flat over $A_{\{S\}}$, and hence Noetherian (**??**). □

2.6.3 Complete tensor products

Let A be a linearly topologized ring and M, N be two linearly topologized A-modules. Let $\mathfrak{I}$, V, W be neighborhoods of 0 in A, M, N, respectively, which are A-modules and such that $\mathfrak{I}\cdot M \subseteq V$, $\mathfrak{I}\cdot N \subseteq W$. Then the quotient modules M/V and N/W can be considered as $(A/\mathfrak{I})$-modules. If $\mathfrak{I}$, V, W runs through the systems of open neighborhoods satisfying the preceding conditions, it is immediate that the modules $(M/V)\otimes_{A/\mathfrak{I}}(N/W)$ form a projective system of modules over the projective system $A/\mathfrak{I}$, so by passing to limit we obtain a module over the completion $\widehat{A}$ of A, which is called the complete tensor product of M and N, and denoted by $M\hat{\otimes}_A N$. Note that since M/V is canonically identified with $\widehat{M}/\widehat{V}$, where $\widehat{V}$ is the closure of V in $\widehat{M}$, the complete tensor product is canonically identified with the completion of $\widehat{M}\otimes_{\widehat{A}}\widehat{N}$, hence with $\widehat{M}\hat{\otimes}_{\widehat{A}}\widehat{N}$.

We also note that the tensor products $(M/V)\otimes_A(N/W)$ and $(M/V)\otimes_{A/\mathfrak{I}}(N/W)$ are both canonically idnetified with the quotient $(M\otimes_A N)/(\mathrm{im}(V\otimes_A N)+\mathrm{im}(M\otimes_A W))$, so the complete tensor product $M\hat{\otimes}_A N$ is the completion of the A-module $M\otimes_A N$ endowed with the topology induced by the submodules $\mathrm{im}(V\otimes_A N)+\mathrm{im}(M\otimes_A W)$. For simplicity, we say that this topology is the **tensor product** of the given topologies on M and N.

Let $u : M \to M'$ and $v : N \to N'$ be continuous homomorphisms of linearly topologized A-modules. It is immediate that $u\otimes v$ is continuous for the tensor product topologies on $M\otimes_A N$ and $M'\otimes_A N'$, respectively, so by passing to completion, we deduce a continuous homomorphism $M'\hat{\otimes}_A N' \to M\hat{\otimes}_A N$, which we denote by $u\hat{\otimes}v$. In this way, the construction $(M, N)\mapsto M\hat{\otimes}_A N$ is then a bifunctor on the category of linearly topologized A-modules.

Moreover, in the same way we can define the complete tensor product for finitely many linearly topologized A-modules, and it is clear that this construction satisfies the commutativity and associativity.

Let B and C be linearly topologized A-algebras. Then the tensor product of the topologies on B and C is a linear topology on $B \otimes_A C$, so $B\hat{\otimes}_A C$ is endowed with a topological $\widehat{A}$-algebra structure. This algebra is called the complete tensor product of the algebras B and C.

The homomorphisms $b \mapsto b \otimes 1$ and $c \mapsto 1 \otimes c$ of B and C into $B \otimes_A C$ are continuous for the tensor product topology, so by composing with the canonical homomorphism from $B \otimes_A C$ to its completion, we obtain canonical homomophisms $\rho : B \to B\hat{\otimes}_A C$ and $\sigma : C \to B\hat{\otimes}_A C$. The algebra $B\hat{\otimes}_A C$ and the homomorphisms ρ and σ then satisfies the following universal property:

Proposition 2.6.24. *For any complete and separated linearly topologized A-algebra D and any couple of continuous A-homomorphisms $u : B \to D$, $v : C \to D$, there exists a continuous A-homomorphism $w : B\hat{\otimes}_A C \to D$ such that the following diagram is commutative*

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[d, "\rho"'] \arrow[ddr, bend left, "u"] & \\
C \arrow[r, "\sigma"] \arrow[drr, bend right, "v"'] & B\hat{\otimes}_A C \arrow[dr, "w"] & \\
 & & D
\end{tikzcd}$$

Proof. In fact, there exists a unique homomorphism $w_0 : B \otimes_A C \to D$ such that $u(b) = w_0(b \otimes 1)$ and $v(c) = w_0(1 \otimes c)$, and it suffices to prove that w_0 is continuous, since it then induces a continuous homomorphism w by passing to completion. Now if $\mathfrak{M}$ is an open ideal of D, there exists by hypothesis open ideals $\mathfrak{K} \subseteq B$, $\mathfrak{N} \subseteq C$ such that $u(\mathfrak{K}) \subseteq \mathfrak{M}$, $v(\mathfrak{N}) \subseteq \mathfrak{M}$, so the image of $\mathrm{im}(\mathfrak{K} \otimes C) + \mathrm{im}(B \otimes \mathfrak{N})$ under w_0 is contained in $\mathfrak{M}$, whence the assertion. □

Proposition 2.6.25. *If B and C are two preadmissible A-algebras, $B\hat{\otimes}_A C$ is admissible, and if $\mathfrak{K}$ (resp. $\mathfrak{N}$) is a nilideal of B (resp. C), then the closure of $\mathfrak{L} = \mathrm{im}(\mathfrak{K} \otimes C) + \mathrm{im}(B \otimes \mathfrak{N})$ in $B\hat{\otimes}_A C$ is a nilideal.*

Proof. It suffices to show that $\mathfrak{L}^n$ tends to 0 in the tensor product topology, and this follows immediately from the inclusion $\mathfrak{L}^{2n} \subseteq \mathrm{im}(\mathfrak{K}^n \otimes C) + \mathrm{im}(B \otimes \mathfrak{N}^n)$. □

Proposition 2.6.26. *Let B, C be linearly topologized A-algebras, M be a linearly topologized B-module, whose topology is finer than the topology induced by B. Then the canonical isomorphism $M \otimes_B (B \otimes_A C) \cong M \otimes_A C$ is a topological isomorphism for the tensor product topologies. In particular, the comple tensor product $M\hat{\otimes}_B(B\hat{\otimes}_A C)$ is topologically isomorphic to $M\hat{\otimes}_A C$.*

Proof. An open neighborhood of 0 in the tensor product $M \otimes_B (B \otimes_A C)$ contains a neighborhood of the form

$$W = \mathrm{im}(V \otimes_B (B \otimes_A C)) + \mathrm{im}(M \otimes_B (\mathrm{im}(\mathfrak{K} \otimes_A C)) + \mathrm{im}(B \otimes_A \mathfrak{N}))$$

where V is an open submodule of M, $\mathfrak{K}$ (resp. $\mathfrak{N}$) is an open ideal of B (resp. C). This submodule can also be written as

$$\mathrm{im}(V \otimes_A C) + \mathrm{im}(M \otimes_A \mathfrak{N}) + \mathrm{im}(\mathfrak{K}M \otimes_A C)$$

and hence contains $W' = \mathrm{im}(V \otimes_A C) + \mathrm{im}(M \otimes_A \mathfrak{N})$, which is an open neighborhood of 0 in $M \otimes_A C$. Conversely, if $\mathfrak{K}$ is taken so that $\mathfrak{K}M \subseteq V$, which is possible by hypothesis, then $W = W'$ and the first assertion follows. The second assertion follows from the equality

$$M\hat{\otimes}_B(B\hat{\otimes}_A C) = \widehat{M}\hat{\otimes}_{\widehat{B}}(\widehat{B \otimes_A C}) = (\widehat{M \otimes_B (B \otimes_A C)}) = \widehat{M \otimes_A C} = M\hat{\otimes}_A C. \qquad \square$$

Proposition 2.6.27. *Let A be a preadic ring, $\mathfrak{I}$ be a defining ideal of A, and M be a finitely generated A-module. Then for any adic and Noetherian topological A-algebra B, $B \otimes_A M$ is identified with the complete tensor product $B\hat{\otimes}_A M$.*

Proof. If $\mathfrak{K}$ is a defining ideal of B, then there exists by hypothesis an integer m such that $\mathfrak{I}^m B \subseteq \mathfrak{K}$, so
$$\mathrm{im}(B \otimes_A \mathfrak{I}^{nm} M) = \mathrm{im}(\mathfrak{I}^{mn} B \otimes_A M) \subseteq \mathrm{im}(\mathfrak{K}^n B \otimes_A M) = \mathfrak{K}^n(B \otimes_A M).$$
We then conclude that over $B \otimes_A M$, the tensor product topology is equivalent to the $\mathfrak{K}$-adic topology. As $B \otimes_A M$ is a finitely generated B-module, it is equal to the $\mathfrak{K}$-adic completion (Theorem 2.4.19), since B is a Zariski ring (Proposition 2.4.28). □

Proposition 2.6.28. *Let A be a topological ring, B, C be topological Noetherian local A-algebras with maximal ideals $\mathfrak{m}, \mathfrak{n}$, respectively, endowed with the adic topologies. Suppose that C is complete and that the residue field $B/\mathfrak{m}$ is a finitely generated A-module. Let $E = B\hat{\otimes}_A C$ be the complete tensor product.*

(a) E is a complete Noetherian semi-local ring.

(b) The ideal $\mathfrak{m}E$ is contained in the radical of E, and for any $k > 0$, $E/\mathfrak{m}^k E$ is isomorphic to $(B/\mathfrak{m}^k) \otimes_A C$.

(c) If C is a flat A-module, then E is a flat B-module.

Proof. □

Corollary 2.6.29. *Under the hypotheses of Proposition 2.6.28, let M be a finitely generated B-module, endowed with the $\mathfrak{m}$-adic topology. Then $M\hat{\otimes}_A C$ is isomorphic to $M \otimes_B E$.*

Proof. In fact, as $\mathfrak{m}E \subseteq \mathfrak{r}E$, the tensor product topology on $M \otimes_B E$ is the $\mathfrak{r}$-adic topology, and as E is complete for this topology, so is $M \otimes_B E$. It then suffices to apply Proposition 2.6.26. □

Corollary 2.6.30. *Let k be a field, A, B be Noetherian local rings containing k, with maximal ideals $\mathfrak{m}, \mathfrak{n}$, respectively. Suppose that B is complete and that the residue field $A//\mathfrak{m}$ is a finite extension of k. Let C be the complete tensor product $A\hat{\otimes}_k B$.*

(a) C is a complete Noetherian semi-local ring.

(b) C is a flat A-module and a flat B-module.

(c) $\mathfrak{m}C$ is contained in the Jacobson radical of C and $C/\mathfrak{m}C$ is isomorphic to $(A/\mathfrak{m}) \otimes_k B$.

(d) The residue fields of C are finite extensions of that of B.

Moreover, if A' is a finite local A-algebra containing A, the canonical homomorphism $A\hat{\otimes}_k B \to A'\hat{\otimes}_k B$ is injective.

Proof. □

2.6.4 Flatness of graded modules

Let A be a ring and $\mathfrak{I}$ an ideal of A. An A-module M is called **ideally Hausdorff with respect to** $\mathfrak{I}$ (or simply ideally Hausdorff if there is no ambiguity) if, for every finitely generated ideal $\mathfrak{a}$ of A, the A-module $\mathfrak{a} \otimes_A M$ is Hausdorff with the $\mathfrak{I}$-adic topology.

Example 2.6.31 (Examples of ideally Hausdorff modules).

(a) If A is Noetherian and $\mathfrak{I}$ is contained in the Jacobson radical of A (in other words if A is a Zariski ring with the $\mathfrak{I}$-adic topology), every finitely generated A-module is ideally Hausdorff by Proposition 2.4.28.

(b) Every direct sum of ideally Hausdorff modules is an ideally Hausdorff module, by virtue of the observation

$$\mathfrak{I}^n(\mathfrak{a} \otimes_A \bigoplus_{i\in I} M_i) = \mathfrak{I}^n \bigoplus_{i\in I}(\mathfrak{a} \otimes_A M_i) = \bigoplus_{i\in I} \mathfrak{I}^n(\mathfrak{a} \otimes_A M_i).$$

(c) If an A-module M is flat and Hausdorff with the $\mathfrak{I}$-adic topology, it is ideally Hausdorff, for $\mathfrak{a} \otimes_A M$ is then identified with a submodule of M and the $\mathfrak{I}$-adic topology on $\mathfrak{a} \otimes_A M$ is finer than the topology induced on $\mathfrak{a} \otimes_A M$ by the $\mathfrak{I}$-adic topology on M, which is Hausdorff by hypothesis.

Let A be a ring, $\mathfrak{I}$ an ideal of A, M an A-module and $\mathrm{gr}(A)$ and $\mathrm{gr}(M)$ the graded ring and graded $\mathrm{gr}(A)$-module associated respectively with the ring A and with the module M with the $\mathfrak{I}$-adic filtrations. We have seen that for every nonngegative integer n there is a surjective $\mathbb{Z}$-module homomorphism (see (2.2.4))

$$\gamma_n : (\mathfrak{I}^n/\mathfrak{I}^{n+1}) \otimes_{A/\mathfrak{I}} (M/\mathfrak{I}M) \to \mathfrak{I}^n M/\mathfrak{I}^{n+1}M$$

and a graded homomorphism of degree 0 of graded $\mathrm{gr}(A)$-modules

$$\gamma_M : \mathrm{gr}(A) \otimes_{\mathrm{gr}_0(A)} \mathrm{gr}_0(M) \to \mathrm{gr}(M)$$

whose restriction to $\mathrm{gr}_n(A) \otimes \mathrm{gr}_0(M)$ is γ_n for all n and which is therefore surjective.

Proposition 2.6.32. *Let A be a ring, $\mathfrak{I}$ an ideal of A and M an A-module. The following conditions are equivalent:*

(i) For all $n > 0$, $\mathrm{Tor}_1^A(A/\mathfrak{I}^n, M) = 0$.

(ii) For all $n > 0$, the canonical homomorphism $\theta_n : \mathfrak{I}^n \otimes_A M \to \mathfrak{I}^n M$ is bijective.

Moreover these conditions imply:

(iii) The canonical homomorphism $\gamma_M : \mathrm{gr}(A) \otimes_{\mathrm{gr}_0(A)} \mathrm{gr}_0(M) \to \mathrm{gr}(M)$ bijective.

Conversely, if $\mathfrak{I}$ is nilpotent, then (iii) implies (i) and (ii).

Proof. The equivalence of (i) and (ii) follows from the exact sequence

$$\mathrm{Tor}_1^A(A, M) \longrightarrow \mathrm{Tor}_1^A(A/\mathfrak{I}^n, M) \longrightarrow \mathfrak{I}^n \otimes_A M \longrightarrow A \otimes_A M$$

Consider next the diagram

$$\begin{tikzcd} & \mathfrak{I}^{n+1} \otimes_A M \arrow[r] \arrow[d, "\theta_{n+1}"] & \mathfrak{I}^n \otimes_A M \arrow[r] \arrow[d, "\theta_n"] & (\mathfrak{I}^n/\mathfrak{I}^{n+1}) \otimes_A (M/\mathfrak{I}M) \arrow[r] \arrow[d, "\gamma_n"] & 0 \\ 0 \arrow[r] & \mathfrak{I}^{n+1}M \arrow[r] & \mathfrak{I}^n M \arrow[r] & \mathrm{gr}_n(M) \arrow[r] & 0 \end{tikzcd} \tag{2.6.1}$$

where we note that $(\mathfrak{I}^n/\mathfrak{I}^{n+1}) \otimes_A (M/\mathfrak{I}M)$ is canonically identified with $(\mathfrak{I}^n/\mathfrak{I}^{n+1}) \otimes_{A/\mathfrak{I}} (M/\mathfrak{I}M)$. This diagram is commutative by definition of γ_n, and its rows are exact. If (ii) holds, θ_n and θ_{n+1} are bijective and so therefore is γ_n, by definition of cokernel, hence (ii) implies (iii).

Conversely, assuming that $\mathfrak{I}$ is nilpotent, let us show that (iii) implies (ii). We shall argue by descending induction on n, since $\mathfrak{I}^n \otimes_A M = 0$ for n sufficiently large. Suppose then that in diagram (2.6.1), γ_n and θ_{n+1} are bijective. Then so is θ_n by the snake lemma, and the induction proves θ_n is bijective for all n. □

Theorem 2.6.33. *Let A be a ring, $\mathfrak{I}$ an ideal of A and M an A-module. Consider the following properties:*

(i) M is a flat A-module.

(ii) $\mathrm{Tor}_1(N,M)=0$ for every A-module N annihilated by $\mathfrak{I}$.

(iii) $M/\mathfrak{I}M$ is a flat $A/\mathfrak{I}$-module and the canonical map $\mathfrak{I}\otimes_A M\to\mathfrak{I}M$ is bijective.

(iv) $M/\mathfrak{I}M$ is a flat $A/\mathfrak{I}$-module and the canonical map $\gamma_M:\mathrm{gr}(A)\otimes_{\mathrm{gr}_0(A)}\mathrm{gr}_0(M)\to\mathrm{gr}(M)$ is bijective.

(v) For all $n>0$, $M/\mathfrak{I}^nM$ is a flat $(A/\mathfrak{I}^n)$-module.

Then (i)⇒(ii)⇔(iii)⇒(iv)⇔(v). If further $\mathfrak{I}$ is nilpotent or if A is Noetherian and M is ideally Hausdorff, then these properties are equivalent.

Proof. The implication (i)⇒(ii) is immediate. Also, condition (ii) is equivalent to the following:

(ii') $\mathrm{Tor}_1^A(N,M)=0$ for every A-module N annihilated by a power of $\mathfrak{I}$.

In fact, if (ii) holds, then then in particular $\mathrm{Tor}_1^A(\mathfrak{I}^n/\mathfrak{I}^{n+1},M)=0$ for all n. From the exact sequence

$$0\to\mathfrak{I}^{n+1}N\longrightarrow\mathfrak{I}^nN\longrightarrow\mathfrak{I}^nN/\mathfrak{I}^{n+1}N\longrightarrow 0$$

wc derive the exact sequence

$$\mathrm{Tor}_1^A(\mathfrak{I}^{n+1}N,M)\longrightarrow\mathrm{Tor}_1^A(\mathfrak{I}^nN,M)\longrightarrow\mathrm{Tor}_1^A(\mathfrak{I}^nN/\mathfrak{I}^{n+1}N,M)$$

and, as there exists an integer m such that $\mathfrak{I}^mN=0$, we deduce by descending induction on n that $\mathrm{Tor}_1^A(\mathfrak{I}^nN,M)=0$ for all $n<m$ and in particular for $n=0$. It follows from this that if $\mathfrak{I}$ is nilpotent, (ii) implies (i), for (ii') then means that $\mathrm{Tor}_1^A(N,M)=0$ for every A-module N and hence that M is flat. Note that the equivalence (ii)⇔(iii) is a special case of **??** applied to $R=A$, $S=A/\mathfrak{I}$, $F=M$, $E=N$, taking account of the fact that an $(A/\mathfrak{I})$-module N is the same as an A-module N annihilated by $\mathfrak{I}$.

Now if (ii) holds, so does (ii') and Proposition 2.6.32 shows that γ_M is an isomorphism. On the other hand, we already know that (ii) implies (iii) and hence $M/\mathfrak{I}M$ is a flat $(A/\mathfrak{I})$-module, which completes the proof that (ii) implies (iv). Also, Proposition 2.6.32 shows that, if $\mathfrak{I}$ is nilpotent, (iv) implies (iii). Taking account of the equivalence of (ii) and (iii), we have therefore proved in this case that (i), (ii), (iii) and (iv) are equivalent.

We prove the equivalence of (iv) and (v). For all $n>0$, $M/\mathfrak{I}^nM$ has a canonical $(A/\mathfrak{I}^n)$-module structure. If it is filtered by the $(\mathfrak{I}/\mathfrak{I}^n)$-adic filtration, it is immediate that

$$\mathrm{gr}_m(M/\mathfrak{I}^nM)=\begin{cases}\mathrm{gr}_m(M) & m<n\\ 0 & m\geq n.\end{cases}$$

For all $k>0$, let $A_k=A/\mathfrak{I}^k$, $\mathfrak{I}_k=\mathfrak{I}/\mathfrak{I}^k$, and $M_k=M/\mathfrak{I}^kM$. Let $(\mathrm{iv})_k$ (resp. $(\mathrm{v})_k$) denote the assertion derived from (iv) (resp. (v)) by replacing A, $\mathfrak{I}$, M by A_k, $\mathfrak{I}_k$, M_k. It follows from what has just been said that (iv) is equivalent to "for all $k>0$, $(\mathrm{iv})_k$" and obviously (v) is equivalent to "for all $k>0$, $(\mathrm{v})_k$". Then it will suffice to establish the equivalence $(\mathrm{iv})_k$ and $(\mathrm{v})_k$ for all k or also to show that (iv)⇔(v) when $\mathfrak{I}$ is nilpotent. Now we have seen that in that case (iv) is equivalent to (i), and (i) is clearly equivalent to (v). We have therefore shown the equivalence (iv)⇔(v) in all cases and also that of all the properties of the theorem in the case where $\mathfrak{I}$ is nilpotent.

Finaly we show the implication (v)⇒(i) when A is Noetherian and M ideally Hausdorff. It is sufficient to prove that for every ideal $\mathfrak{a}$ of A the canonical map $j : \mathfrak{a} \otimes_A M \to M$ is injective. As $\mathfrak{a} \otimes_A M$ is Hausdorff with the $\mathfrak{I}$-adic topology, it suffices to verify that $\ker j \subseteq \mathfrak{I}^n(\mathfrak{a} \otimes_A M)$ for every integer $n > 0$. By Theorem 2.4.13, there exists an integer k such that $\mathfrak{a}_k := \mathfrak{a} \cap \mathfrak{I}^k \subseteq \mathfrak{I}^n\mathfrak{a}$. Now, denoting by $\iota : \mathfrak{a}_k \to \mathfrak{a}$, $\pi : \mathfrak{a} \to \mathfrak{a}/\mathfrak{a}_k$ and $\psi : \mathfrak{a}/\mathfrak{a}_k \to A/\mathfrak{I}^k$ the canonical maps, there is a commutative diagram

$$\begin{tikzcd}
\mathfrak{a}_k \otimes_A M \arrow[r, "\iota\otimes 1_M"] & \mathfrak{a} \otimes_A M \arrow[r, "\pi\otimes 1_M"] \arrow[d] & (\mathfrak{a}/\mathfrak{a}_k) \otimes_A M \arrow[r] \arrow[d, "\psi\otimes 1_M"] & 0 \\
& M \arrow[r] & (A/\mathfrak{I}^k) \otimes_A M &
\end{tikzcd}$$

in which the first row is exact. The map $\psi \otimes 1_M$ is injective since it can also be written into

$$\psi \otimes 1_{M/\mathfrak{I}^k M} : (\mathfrak{a}/\mathfrak{a}_k) \otimes_{A/\mathfrak{I}^k} (M/\mathfrak{I}^k M) \to M/\mathfrak{I}^k M$$

and, as ψ is injective and by (v) $M/\mathfrak{I}^k M$ is a flat $(A/\mathfrak{I}^k)$-module. By a simple diagram chasing we see $\ker j \subseteq \operatorname{im}(\iota \otimes 1_M)$, which shows the claim and completes the proof. □

Proposition 2.6.34. *Let A be a ring, $\mathfrak{I}$ an ideal of A and B a Noetherian A-algebra such that $\mathfrak{I}$ is contained in the Jacobson radical of B. Then every finitely generated B-module M is an ideally Hausdorff A-module with respect to $\mathfrak{I}$.*

Proof. Wc shall see more generally that for every finitely generated A-module N, $N \otimes_A M$ is Hausdorff with the $\mathfrak{I}$-adic topology. Let $\mathfrak{r}$ be the Jacobson radical of B. As $\mathfrak{I}B$ is contained in $\mathfrak{r}$ and the B-module $N \otimes_A M$ is canonically identified with $N_{(B)} \otimes_B M$ by virtue of the associativity of the tensor product, the $\mathfrak{I}$-adic topology on $N \otimes_A M$ is therefore identified with a finer topology than the $\mathfrak{r}$-adic topology on $N_{(B)} \otimes_B M$. But this latter topology is Hausdorff since $N_{(B)} \otimes_B M$ is a finitely generated B-module, whence the conclusion. □

Proposition 2.6.35. *Let A be a ring, B an A-algebra, $\mathfrak{I}$ an ideal of A and M a B-module. Suppose that B is a Noetherian ring and a flat A-module and that M is ideally Hausdorff with respect to $\mathfrak{I}B$. The following conditions are equivalent:*

(i) M is a flat B-module.

(ii) M is a flat A-module and $M/\mathfrak{I}M$ is a flat $(B/\mathfrak{I}B)$-module.

If further the canonical homomorphism $A/\mathfrak{I} \to B/\mathfrak{I}B$ is bijective, then these are also equivalent to:

(iii) M is a flat A-module.

Proof. Condition (i) implies (ii) by **??** and **??** and the fact that $M/\mathfrak{I}M$ is isomorphic to $M \otimes_B B/\mathfrak{I}B$. Suppose condition (ii) holds. To show that M is a flat B-module, we shall apply Theorem 2.6.33 with A replaced by B and $\mathfrak{I}$ by $\mathfrak{I}B$. It will therefore be sufficient to show that the canonical map $\phi : \mathfrak{I}B \otimes_B M \to \mathfrak{I}M$ is injective. Let ϕ_1 be the canonical map $\mathfrak{I} \otimes_A B \to \mathfrak{I}B$ and ϕ_2 the canonical isomorphism $\mathfrak{I} \otimes_A M \to (\mathfrak{I} \otimes_A B) \otimes_B M$. Then

$$\mathfrak{I} \otimes_A M \xrightarrow{\phi_2} (\mathfrak{I} \otimes_A B) \otimes_B M \xrightarrow{\phi_1\otimes 1_M} \mathfrak{I}B \otimes_B M \xrightarrow{\phi} \mathfrak{I}M$$

Now $\psi : \mathfrak{I} \otimes_A M \to \mathfrak{I}M$ is an isomorphism since M is a flat A-module, whilst ϕ_1 is an isomorphism because B is flat over A. Therefore ϕ is then an isomorphism.

Let $\rho : A/\mathfrak{I} \to B/\mathfrak{I}B$ be the canonical homomorphism. The $(A/\mathfrak{I})$-module structure on $M/\mathfrak{I}M$ derived by means of ρ is isomorphic to that on $M \otimes_A (A/\mathfrak{I})$. Then it follows that, if M is a flat A-module, $M/\mathfrak{I}M$ is a flat $(A/\mathfrak{I})$-module and hence also a flat $(B/\mathfrak{I}B)$-module if ρ is an isomorphism. We have thus proved that (iii)⇒(ii) in that case. □

Corollary 2.6.36. *Let A be a Noetherian ring, $\mathfrak{I}$ an ideal of A, $\widehat{A}$ the Hausdorff completion of A with respect to the $\mathfrak{I}$-adic topology and M an ideally Hausdorff $\widehat{A}$-module with respect to $\widehat{\mathfrak{I}}$. For M ta be a flat A-module, it is necessary and sufficient that M be a flat $\widehat{A}$-module.*

Proof. We know in fact that A is a Noetherian ring and a flat A-module, that $\widehat{\mathfrak{I}} = \mathfrak{I}\widehat{A}$ and that the canonical homomorphism $A/\mathfrak{I} \to \widehat{A}/\widehat{\mathfrak{I}}$ is bijective. Proposition 2.6.35 can therefore be applied. □

Proposition 2.6.37. *Let A and B be two Noetherian rings, $\rho : A \to B$ a ring homomorphism, $\mathfrak{I}$ an ideal of A and $\mathfrak{K}$ an ideal of B containing $\mathfrak{I}B$ and contained in the Jacobson radical of B. Let $\widehat{A}$ be the Hausdorff completion of A with respect to the $\mathfrak{I}$-adic topology and B the Hausdorff completion of B with respect to the $\mathfrak{K}$-adic topology. Let M be a finitely generated B-module and $\widehat{M}$ its Hausdorff completion with respect to the $\mathfrak{K}$-adic topology. Then the following properties are equivalent:*

(i) M is a flat A-module.

(ii) $\widehat{M}$ is a flat A-module.

(iii) $\widehat{M}$ is a flat $\widehat{A}$-module.

Proof. As B with the $\mathfrak{K}$-adic topology is a Zariski ring, $\widehat{B}$ is a faithfully flat B-module and $\widehat{M}$ is canonically isomorphic to $M \otimes_B \widehat{B}$. It is immediately verified that this canonical isomorphism is an isomorphism of the A-module structure on $\widehat{M}$ onto the A-module structure on $M \otimes_B \widehat{B}$ derived from that on M. Applying **??**, we see that for M to be a flat A-module, it is necessary and sufficient that $\widehat{M}$ be a flat A-module. Moreover, $\widehat{M}$ is a finitely generated B-module and $\mathfrak{I}\widehat{B}$ is contained in $\widehat{\mathfrak{K}} = \mathfrak{K}\widehat{B}$ and hence in the Jacobson radical of $\widehat{B}$ (Proposition 2.4.24). Therefore $\widehat{M}$ is an ideally Hausdorff $\widehat{A}$-module with respect to $\widehat{\mathfrak{I}}$ by Proposition 2.6.34. Conditions (ii) and (iii) are therefore equivalent by the Corollary to Proposition 2.6.35. □

2.7 Exercise

Exercise 2.7.1. Show that p-adic completion is not a right-exact functor on the category of all $\mathbb{Z}$-modules.

Proof. Let $A = (\mathbb{Z}/p\mathbb{Z})^{\oplus\mathbb{N}}$ and $B = \bigoplus_n \mathbb{Z}/p^n\mathbb{Z}$. Consider the short exact sequence

$$0 \longrightarrow A \xhookrightarrow{\iota} B \xrightarrow{\pi} B/A \longrightarrow 0$$

If we give them the topology induced by B, namely $\{\iota^{-1}(p^nB)\}$ and $\{\pi(p^nB)\}$, then by Corollary 2.3.7 the induced conplition sequence is exact. Note that on $B/\alpha(A)$, the induced topology is just the same as the p-adic topology. So if we instead use the p-adic topology, the sequence is not exact. □

Remark 2.7.1. Though it does preserve surjectivity, because if $\rho : B \to C$ is a surjection, we have

$$\rho(p^nB) = p^n\rho(B) = p^nC,$$

so we have a surjective map of surjective inverse systems, and Corollary 2.3.7 tells us the map $\widehat{\rho} : \widehat{B} \to \widehat{C}$ is surjective.

The p-adic completion is not left-exact, by the same example; the essential reason is that given a general homomorphism $\alpha : A \to B$, we needn't have $\alpha(p^nA) = \alpha(A) \cap p^nB$, so the p-adic topology on A is not that induced from B and the hypotheses of Corollary 2.3.7 are not met.

Exercise 2.7.2. In Exercise 2.7.1, let $A_n = \iota^{-1}(p^n B)$, and consider the exact sequence

$$0 \longrightarrow A_n \longrightarrow A \longrightarrow A/A_n \longrightarrow 0$$

Show that $\varprojlim$ is not right exact, and compute $\varprojlim^1 A_n$.

Proof. From Exercise 2.7.1 we already have

$$A_n = \bigoplus_{j>n} G_j, \quad A/A_n = \bigoplus_{j=1}^{n} G_j$$

where $G_i = \mathbb{Z}/p\mathbb{Z}$ for all i. The maps

$$A_{n+1} \hookrightarrow A_n, \quad A \xrightarrow{\text{id}} A, \quad A/A_{n+1} \rightarrowtail A/A_n$$

give rise to an inverse system, hence we can form the inverse limits, and by Proposition 2.3.6 we get an exact sequence

$$0 \to \varprojlim A_n \to \varprojlim A \to \varprojlim A/A_n \to \varprojlim{}^1 A_n \to \varprojlim{}^1 A \to \varprojlim{}^1 A/A_n \to 0$$

Since $\{A\}$ and $\{A/A_n\}$ are surjective inverse systems, $\varprojlim^1 A = \varprojlim^1 A/A_n = 0$. Also, the inverse limits can be calculate to be

$$\varprojlim A_n = 0, \quad \varprojlim A = A = \bigoplus_{j=1}^{n} G_j, \quad \varprojlim A/A_n = \prod_{i=1}^{\infty} G_j$$

Hence we get a short exact sequence

$$0 \longrightarrow \bigoplus_{j=1}^{n} G_j \xrightarrow{\psi} \prod_{j=1}^{\infty} G_j \longrightarrow \varprojlim{}^1 A_n \longrightarrow 0$$

Then the map ψ is simply the inclusion, and we see $\varprojlim^1 A_n = \prod_{j=1}^{\infty} G_j / \bigoplus_{j=1}^{n} G_j$. $\square$

Exercise 2.7.3. Let A be a Noetherian ring and let $\mathfrak{a}, \mathfrak{b}$ be ideals in A. If M is any A-module, let $M^{\mathfrak{a}}, M^{\mathfrak{b}}$ denote its $\mathfrak{a}$-adic and $\mathfrak{b}$-adic completions respectively. If M is finitely generated, prove that $(M^{\mathfrak{a}})^{\mathfrak{b}} \cong M^{\mathfrak{a}+\mathfrak{b}}$.

Proof. Take the $\mathfrak{a}$-adic complition of $0 \to \mathfrak{b}^m M \to M \to M/\mathfrak{b}^m M \to 0$, we get

$$0 \longrightarrow (\mathfrak{b}^m M)^{\mathfrak{a}} \longrightarrow M^{\mathfrak{a}} \longrightarrow (M/\mathfrak{b}^m M)^{\mathfrak{a}} \longrightarrow 0$$

By Theorem 2.4.19 we have

$$\begin{aligned}(\mathfrak{b}^m M)^{\mathfrak{a}} &\cong A^{\mathfrak{a}} \otimes_A \mathfrak{b}^m M = A^{\mathfrak{a}} \otimes_A \mathfrak{b}^m \otimes_A M = A^{\mathfrak{a}} \otimes_A M \otimes_A \mathfrak{b}^m \\ &\cong \mathfrak{b}^m \otimes M^{\mathfrak{a}} = \mathfrak{b}^m M^{\mathfrak{a}}\end{aligned}$$

so from the sequence we get $(M/\mathfrak{b}^m M)^{\mathfrak{a}} \cong M^{\mathfrak{a}}/\mathfrak{b}^m M^{\mathfrak{a}}$. Now we have

$$(M^{\mathfrak{a}})^{\mathfrak{b}} = \varprojlim_m M^{\mathfrak{a}}/\mathfrak{b}^m M^{\mathfrak{a}} = \varprojlim_m (M/\mathfrak{b}^m M)^{\mathfrak{a}} = \varprojlim_m \varprojlim_n \frac{M/\mathfrak{b}^m M}{\mathfrak{a}^n (M/\mathfrak{b}^m M)}.$$

Since we have

$$\mathfrak{a}^n (M/\mathfrak{b}^m M) \cong \frac{\mathfrak{a}^n M + \mathfrak{b}^m M}{\mathfrak{b}^m M}.$$

we get
$$(M^{\mathfrak{a}})^{\mathfrak{b}} = \varprojlim_m \varprojlim_n M/(\mathfrak{a}^n M + \mathfrak{b}^m M) \cong \varprojlim_n M/(\mathfrak{a}^n M + \mathfrak{b}^n M)$$
Now note that
$$(\mathfrak{a}+\mathfrak{b})^{2n} \subseteq \mathfrak{a}^n + \mathfrak{b}^n \subseteq (\mathfrak{a}+\mathfrak{b})^n$$
so the topology induced by $\mathfrak{a}^n M + \mathfrak{b}^n M$ is the same as that by $(\mathfrak{a}+\mathfrak{b})^n$, hence we get the desired result. □

Exercise 2.7.4. Let A be the local ring of the origin in $\mathbb{C}^n$ (i.e., the ring of all rational functions $f/g \in \mathbb{C}(z_1, \ldots, z_n)$ with $g(0) \neq 0$), let B be the ring of power series in $z_1, \ldots, z_n$ which converge in some neighborhood of the origin, and let C be the ring of formal power series in $z_1, \ldots, z_n$, so that $A \subseteq B \subseteq C$. Show that B is a local ring and that its completion for the maximal ideal topology is C. Assuming that B is Noetherian, prove that B is A-flat.

Proof. The surjective ring homomorphism
$$\phi : B \to \mathbb{C}, \quad f \mapsto f(0)$$
shows the ideal $\mathfrak{m} = (z_1, \ldots, z_n)$ of power series with zero constant term is maximal. To show B is local, note that if $f \notin \mathfrak{m}$, then $f(0) \neq 0$ and f is a unit in C, and thus a unit in B. So $\mathfrak{m}$ is maximal.

We first compute the complition, note that
$$\mathbb{C}[z_1, \ldots, z_n] \subseteq A \subseteq B \subseteq C$$
and that C is the complition of $\mathbb{C}[z_1, \ldots, z_n]$ by the $(z_1, \ldots, z_n)$-adic topology. Thus we have
$$C = \widehat{\mathbb{C}[z_1, \ldots, z_n]} \subseteq \widehat{A} \subseteq \widehat{B} \subseteq \widehat{C} = C$$
So $\widehat{A} = \widehat{B} = C$. Since we know C is a faithfully flat B-algebra and A-algebra, we conclude B is a flat A-algebra. □

Chapter 3

Associated prime ideals and primary decomposition

3.1 Associated prime ideals of a module

3.1.1 Associated prime ideals

Let M be a module over a ring A. A prime ideal $\mathfrak{p}$ is said to be **associated with M** if there exists $x \in M$ such that $\mathfrak{p}$ is equal to the annihilator of x. The set of prime ideals associated with M is denoted by $\mathrm{Ass}_A(M)$, or simply $\mathrm{Ass}(M)$.

Example 3.1.1. Let $\mathfrak{a}$ be an ideal in the polynomial ring $A = \mathbb{C}[X_1, \ldots, X_n]$, V the corresponding affine algebraic variety and $V_1, \ldots, V_p$ the irreducible components of V. If M is taken to be the ring $A/\mathfrak{a}$ of functions which are regular on V, the set of prime ideals associated with M consists of the ideals of $V_1, \ldots, V_p$ and in general other prime ideals each of which contains one of the ideals of the V_i.

As the annihilator of 0 is A, an element $x \in M$ whose annihilator is a prime ideal is necessarily nonzero. To say that a prime idea $\mathfrak{p}$ is associated with M amounts to saying that M contains a submodule isomorphic to $A/\mathfrak{p}$ (namely Ax, for all $x \in M$ whose annihilator is $\mathfrak{p}$). If an A-module M is the union of a family $(M_i)_{i \in I}$ of submodules, then clearly

$$\mathrm{Ass}(M) = \bigcup_{i \in I} \mathrm{Ass}(M_i). \tag{3.1.1}$$

Proposition 3.1.2. *For every prime ideal $\mathfrak{p}$ of a ring A and every nonzero submodule M of $A/\mathfrak{p}$,* $\mathrm{Ass}(M) = \{\mathfrak{p}\}$.

Proof. As the ring $A/\mathfrak{p}$ is an integral domain, the annihilator of an nonzero element of $A/\mathfrak{p}$ is $\mathfrak{p}$. □

Proposition 3.1.3. *Let M be a module over a ring A. Every maximal element of the set of ideals* $\mathrm{Ann}(x)$ *of A, where x runs through the set of nonzero elements of M, belongs to* $\mathrm{Ass}(M)$.

Proof. Let $\mathfrak{p} = \mathrm{Ann}(x)$ (where $x \in M$ is nonzero) be such a maximal element; it is sufficient to show that $\mathfrak{p}$ is prime. As $x \neq 0$, $\mathfrak{p}$ is proper. Let a, b be elements of A such that $ab \in \mathfrak{p}$ but $a \notin \mathfrak{p}$. Then $ax \neq 0$ and $\mathfrak{p} \subseteq \mathrm{Ann}(ax)$. As $\mathfrak{p}$ is maximal, $\mathrm{Ann}(ax) = \mathfrak{p}$, whence $b \in \mathfrak{p}$, so that $\mathfrak{p}$ is prime. □

Corollary 3.1.4. *Let M be a module over a Noetherian ring A. Then the condition $M \neq \{0\}$ is equivalent to* $\mathrm{Ass}(M) \neq \varnothing$.

Proof. If $M = \{0\}$, clearly $\operatorname{Ass}(M)$ is empty (without any hypothesis on A). If $M \neq \{0\}$, the set of ideals of the form $\operatorname{Ann}(x)$, where $x \in M$ and $x \neq 0$, is non-empty and consists of ideals of A. As A is Noetherian, this set has a maximal element; then it suffices to apply Proposition 3.1.3. □

Corollary 3.1.5. *Let A be a Noetherian ring, M an A-module and $a \in A$. For the homothety on M with ratio a to be injective, it is necessary and sufficient that a belong to no prime ideal associated with M.*

Proof. If a belongs to a prime ideal $\mathfrak{p} \in \operatorname{Ass}(M)$, then $\mathfrak{p} = \operatorname{Ann}(x)$ where $x \in M$ and $x \neq 0$, whence $ax = 0$ and the homothety with ratio a is not injective. Conversely, if $ax = 0$ for some $x \in M$ such that $x \neq 0$, then $Ax \neq \{0\}$, whence $\operatorname{Ass}(Ax) \neq \varnothing$. Let $\mathfrak{p} \in \operatorname{Ass}(Ax)$, then obviously $\mathfrak{p} \in \operatorname{Ass}(M)$ and $\mathfrak{p} = \operatorname{Ann}(bx)$ for some $b \in A$; then $a \in \mathfrak{p}$, since $abx = 0$. □

Corollary 3.1.6. *The set of divisors of zero in a Noetherian ring A is the union of the ideals in* $\operatorname{Ass}(A)$.

Proposition 3.1.7. *If we have a short exact sequence of A-modules*

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

then $\operatorname{Ass}(L) \subseteq \operatorname{Ass}(M) \subseteq \operatorname{Ass}(L) \cup \operatorname{Ass}(N)$.

Proof. The inclusion $\operatorname{Ass}(L) \subseteq \operatorname{Ass}(M)$ is obvious. Let $\mathfrak{p} \in \operatorname{Ass}(M)$, E be a submodule of M isomorphic to $A/\mathfrak{p}$ and $F = E \cap L$. If $F = \{0\}$, E is isomorphic to a submodule of N, whence $\mathfrak{p} \in \operatorname{Ass}(N)$. If $F \neq \{0\}$, the annihilator of every nonzero element of F is $\mathfrak{p}$ (Proposition 3.1.3) and hence $\mathfrak{p} \in \operatorname{Ass}(F) \subseteq \operatorname{Ass}(L)$. □

Corollary 3.1.8. *If $M = \bigoplus_{i \in I} M_i$ then* $\operatorname{Ass}(M) = \bigcup_{i \in I} \operatorname{Ass}(M_i)$.

Proof. It may be reduced to the case where I is finite by means of (3.1.1), then to the case where $|I| = 2$ by induction. Then let $I = \{i, j\}$, where $i \neq j$. As M/M_i is isomorphic to M_j, we have $\operatorname{Ass}(M) \subseteq \operatorname{Ass}(M_i) \cup \operatorname{Ass}(M_j)$. Moreover, $\operatorname{Ass}(M_i)$ and $\operatorname{Ass}(M_j)$ are contained in $\operatorname{Ass}(M)$ by Proposition 3.1.7, whence the result. □

Corollary 3.1.9. *Let M be an A-module and $(Q_i)_{i \in I}$ a finite family of submodules of M. If $\bigcap_{i \in I} Q_i = \{0\}$ then*

$$\operatorname{Ass}(M) \subseteq \bigcup_{i \in I} \operatorname{Ass}(M/Q_i).$$

Proof. The canonical map $M \to \bigoplus_i (M/Q_i)$ is injective; then it suffices to apply Proposition 3.1.7 and Corollary 3.1.8. □

Proposition 3.1.10. *Let M be an A-module and Φ a subset of* $\operatorname{Ass}(M)$. *Then there exists a submodule N of M such that* $\operatorname{Ass}(N) = \operatorname{Ass}(M) - \Phi$ *and* $\operatorname{Ass}(M/N) = \Phi$.

Proof. Let $\mathcal{M}$ be the set of submodules P of M such that $\operatorname{Ass}(P) \subseteq \operatorname{Ass}(M) - \Phi$. Formula (3.1.1) shows that the set $\mathcal{M}$, ordered by inclusion, is inductive. Moreover, $\{0\} \in \mathcal{M}$ and hence $\mathcal{M} \neq \varnothing$. Let N be a maximal element of $\mathcal{M}$. Then $\operatorname{Ass}(N) \subseteq \operatorname{Ass}(M) - \Phi$. We shall see that $\operatorname{Ass}(M/N) \subseteq \Phi$, which, by Proposition 3.1.7, will complete the proof. Let $\mathfrak{p} \in \operatorname{Ass}(M/N)$; then M/N contains a submodule F/N isomorphic to $A/\mathfrak{p}$. By Proposition 3.1.7 and Proposition 3.1.2, $\operatorname{Ass}(F) \subseteq \operatorname{Ass}(N) \cup \{\mathfrak{p}\}$. Since N is maximal in $\mathcal{M}$, $F \notin \mathcal{M}$ and hence $\mathfrak{p} \in \Phi$. □

Proposition 3.1.11. *Let A be a ring, S a multiplicative subset of A, Φ the set of ideals of A which do not meet S and M an A-module. Then:*

(a) The map $\mathfrak{p} \mapsto S^{-1}\mathfrak{p}$ is a bijection of $\operatorname{Ass}_A(M) \cap \Phi$ *onto a subset of* $\operatorname{Ass}_{S^{-1}A}(S^{-1}M)$.

(b) *If $\mathfrak{p} \in \Phi$ is a finitely generated ideal and $S^{-1}\mathfrak{p} \in \mathrm{Ass}_{S^{-1}A}(S^{-1}M)$, then $\mathfrak{p} \in \mathrm{Ass}_A(M)$.*

Proof. Recall that the map $\mathfrak{p} \mapsto S^{-1}\mathfrak{p}$ is a bijection of Φ onto the set of prime ideals of $S^{-1}A$. If $\mathfrak{p} \in \mathrm{Ass}_A(M) \cap \Phi$, $\mathfrak{p}$ is the annihilator of a monogenous submodule N of M; then $S^{-1}\mathfrak{p}$ is the annihilator of the monogenous submodule $S^{-1}\mathbb{N}$ of $S^{-1}M$ (Proposition 1.2.33) and hence $S^{-1}\mathfrak{p} \in \mathrm{Ass}_{S^{-1}A}(S^{-1}M)$. Conversely, suppose that $\mathfrak{p} \in \Phi$ is finitely generated and such that $S^{-1}\mathfrak{p}$ is associated with $S^{-1}M$, then there exists $x \in M$ and $t \in S$ such that $S^{-1}\mathfrak{p}$ is the annihilator of x/t. Let $a_1, \dots, a_n$ be a system of generators of $\mathfrak{p}$; then $(a_i)(x/t) = 0$ and hence there exists $s_i \in S$ such that $s_i a_i x = 0$. Let us write $s = s_1 \cdots s_n$, then for all $a \in \mathfrak{p}$ we have $sax = 0$, whence $\mathfrak{p} \subseteq \mathrm{Ann}(sx)$. On the other hand, if $b \in A$ satisfies $bsx = 0$, then $b/1 \in S^{-1}\mathfrak{p}$ by definition, whence $b \in \mathfrak{p}$. Then $\mathfrak{p} = \mathrm{Ann}(sx)$ and $\mathfrak{p} \in \mathrm{Ass}_A(M)$. □

Corollary 3.1.12. *If the ring A is Noetherian, the map $\mathfrak{p} \mapsto S^{-1}\mathfrak{p}$ is a bijection of $\mathrm{Ass}_A(M) \cap \Phi$ onto $\mathrm{Ass}_{S^{-1}A}(S^{-1}M)$.*

Proposition 3.1.13. *Let A be a Noetherian ring, M an A-module, S a multiplicative subset of A and Φ the set of elements of $\mathrm{Ass}_A(M)$ which do not meet S. Then the kernel N of the canonical map $M \mapsto S^{-1}M$ is the unique submodule of M which satisfies the relations*

$$\mathrm{Ass}(N) = \mathrm{Ass}(M) - \Phi, \quad \mathrm{Ass}(M/N) = \Phi.$$

Proof. By Proposition 3.1.10, there exists a submodule N' of M which satisfies the relations $\mathrm{Ass}(N') = \mathrm{Ass}(M) - \Phi$ and $\mathrm{Ass}(M/N') = \Phi$. We need to prove $N' = N$. Consider the commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{\pi} & M/N' \\ {\scriptstyle i_M^S}\downarrow & & \downarrow{\scriptstyle i_{M/N'}^S} \\ S^{-1}M & \xrightarrow{S^{-1}\pi} & S^{-1}(M/N') \end{array}$$

where π, i_M^S, and $i_{M/N'}^S$ are the canonical homomorphisms. We shall show that $S^{-1}\pi$ and $i_{M/N'}^S$ are injective, which will prove that i_M^S and π have the same kernel and hence $N = N'$.

As $\mathrm{Ass}(N') \cap \Phi = \varnothing$, every element of $\mathrm{Ass}(N')$ meets S. Then by Proposition 3.1.11 we have $\mathrm{Ass}_{S^{-1}A}(S^{-1}N') = \varnothing$, whence $S^{-1}N' = \{0\}$ by Corollary 3.1.4. This proves that $S^{-1}\pi$ is injective. On the other hand, if x belongs to the kernel K of $i_{M/N'}^S$, then $\mathrm{Ann}(x) \cap S \neq \varnothing$, so $\mathrm{Ass}(K) \cap \Phi = \varnothing$. But $\mathrm{Ass}(K) \subseteq \mathrm{Ass}(M/N') = \Phi$, which means $\mathrm{Ass}(K) = \varnothing$ and $K = \{0\}$ by Corollary 3.1.4. This shows $i_{M/N'}^S$ is injective. □

Let M be a module over a ring A. Recall that the set of prime ideals $\mathfrak{p}$ of A such that $M_\mathfrak{p} \neq 0$ is called the *support* of M and is denoted by $\mathrm{supp}(M)$.

Proposition 3.1.14. *Let A be a ring and M an A-module.*

(a) *Every prime ideal $\mathfrak{p}$ of A containing an element of $\mathrm{Ass}(M)$ belongs to $\mathrm{supp}(M)$.*

(b) *Conversely, if A is Noetherian, every ideal $\mathfrak{p} \in \mathrm{supp}(M)$ contains an element of $\mathrm{Ass}(M)$.*

Proof. If $\mathfrak{p}$ contains an element $\mathfrak{q}$ of $\mathrm{Ass}(M)$, then $\mathfrak{q} \cap (A - \mathfrak{p}) = \varnothing$ and hence, if we write $S = A - \mathfrak{p}$, $S^{-1}\mathfrak{p}$ is a prime ideal associated with $S^{-1}M = M_\mathfrak{p}$, and a fortiori $M_\mathfrak{p} \neq \varnothing$, hence $\mathfrak{p} \in \mathrm{supp}(M)$. Conversely, if A is Noetherian, so is $A_\mathfrak{p}$. If $M_\mathfrak{p} \neq \varnothing$, then $\mathrm{Ass}_{A_\mathfrak{p}}(M_\mathfrak{p}) \neq \varnothing$ and hence there exists $\mathfrak{q} \in \mathrm{Ass}_A(M)$ such that $\mathfrak{q} \cap (A - \mathfrak{p}) = \varnothing$ by Proposition 3.1.11. □

Corollary 3.1.15. *If M is a module over a Noetherian ring, then $\mathrm{Ass}(M) \subseteq \mathrm{supp}(M)$ and these two sets have the same minimal elements.*

Corollary 3.1.16. *The nilradical of a Noetherian ring A is the intersection of the ideals $\mathfrak{p} \in \mathrm{Ass}(A)$.*

3.1.2 Associated prime ideals of Noetherian rings

Proposition 3.1.17. *Let A be a Noetherian ring and M a finitely generated A-module. Then there exists a chain $(M_i)_{0\leq i\leq n}$ of submodules of M such that M_i/M_{i+1} is isomorphic to $A/\mathfrak{p}_i$ where $\mathfrak{p}_i$ is a prime ideal of A.*

Proof. Let $\mathcal{M}$ be the set of submodules of M which have a chain with the property of the statement. As $\mathcal{M}$ is non-empty (for $\{0\}$ belongs to $\mathcal{M}$) and M is Noetherian, $\mathcal{M}$ has a maximal element N. If $M \subsetneq N$, then $M/N \neq \varnothing$ and hence $\mathrm{Ass}(M/N) \neq \varnothing$ by Corollary 3.1.4. The module M/N therefore contains a submodule N'/N isomorphic to an A-module of the form $A/\mathfrak{p}$, where $\mathfrak{p}$ is prime; then by definition $N' \in \mathcal{M}$, which contradicts the maximal character of N. Then necessarily $N = M$. □

Theorem 3.1.18. *Let M be a finitely generated module over a Noetherian ring A and $(M_i)_{0\leq i\leq n}$ a chain of submodules of M such that M_i/M_{i+1} is isomorphic to $A/\mathfrak{p}_i$ where $\mathfrak{p}_i$ is a prime ideal of A. Then*

$$\mathrm{Ass}(M) \subseteq \{\mathfrak{p}_0, \dots, \mathfrak{p}_{n-1}\} \subseteq \mathrm{supp}(M). \tag{3.1.2}$$

The minimal elements of these three sets are the same and coincide with the minimal elements of the set of prime ideals containing $\mathrm{Ann}(M)$.

Proof. The inclusion $\mathrm{Ass}(M) \subseteq \{\mathfrak{p}_0, \dots, \mathfrak{p}_{n-1}\}$ follows immediately from Proposition 3.1.7. For each i, we have

$$\mathfrak{p}_i \in \mathrm{supp}(A/\mathfrak{p}_i) = \mathrm{supp}(M_i/M_{i+1})$$

so $\mathfrak{p}_i \in \mathrm{supp}(M_i) \subseteq \mathrm{supp}(M)$ (Proposition 1.4.33), which shows $\{\mathfrak{p}_0, \dots, \mathfrak{p}_{n-1}\} \subseteq \mathrm{supp}(M)$. Proposition 3.1.14 shows that $\mathrm{Ass}(M)$ and $\mathrm{supp}(M)$ have the same minimal elements and (3.1.2) shows that these are just the minimal elements of $\{\mathfrak{p}_0, \dots, \mathfrak{p}_{n-1}\}$. The last assertion then follows from Proposition 1.4.35. □

Corollary 3.1.19. *If M is a finitely generated module over a Noetherian ring A,* $\mathrm{Ass}(M)$ *is finite.*

Proposition 3.1.20. *Let A be a Noetherian ring, $\mathfrak{a}$ an ideal of A and M a finitely generated A-module. The following conditions are equivalent:*

(i) there exists a nonzero element $x \in M$ such that $\mathfrak{a}x = 0$;

(ii) for all $a \in \mathfrak{a}$, there exists a nonzero element $x \in M$ such that $ax = 0$;

(iii) there exists $\mathfrak{p} \in \mathrm{Ass}(M)$ such that $\mathfrak{a} \subseteq \mathfrak{p}$.

Proof. Clearly (i) implies (ii). By virtue of Corollary 3.1.5, condition (ii) means that the ideal $\mathfrak{a}$ is contained in the union of the prime ideals associated with M and hence in one of them since $\mathrm{Ass}(M)$ is finite (Proposition 1.1.4), thus (ii) implies (iii). Finally, if there exists $\mathfrak{p} \in \mathrm{Ass}(M)$ such that $\mathfrak{a} \subseteq \mathfrak{p}$, then $\mathfrak{p}$ is the annihilator of a nonzero element $x \in M$ such that $\mathfrak{a}x = 0$, thus (iii) implies (i). □

Proposition 3.1.21. *Let A be a Noetherian ring, $\mathfrak{a}$ an ideal of A and M a finitely generated A-module. For there to exist an integer $n > 0$ such that $\mathfrak{a}^n M = 0$, it is necessary and sufficient that $\mathfrak{a}$ be contained in the intersection of the prime ideals associated with M.*

Proof. This intersection is also that of the minimal elements of $\mathrm{supp}(M)$ and to say that $\mathfrak{a}$ is contained in this intersection is equivalent to saying that $V(\mathfrak{a}) \supseteq \mathrm{supp}(M)$. The conclusion then follows from Corollary 1.4.37. □

Given an A-module M, an endomorphism ϕ of M is called **locally nilpotent** if, for all $x \in M$, there exists an integer $n(x) > 0$ such that $\phi(x)^{n(x)} = 0$. It is clear that, if M is finitely generated, every locally nilpotent endomorphism is nilpotent.

Corollary 3.1.22. *Let A be a Noetherian ring, M an A-module and a an element of A. For the homomorphism $h_a : x \mapsto ax$ of M to be locally nilpotent, it is necessary and suficient that a belong to every ideal of* $\operatorname{Ass}(M)$.

Proof. The condition of the statement is equivalent to saying that for all $x \in M$ there exists $n(x) > 0$ such that $(Aa)^{n(x)}(Ax) = 0$. By Proposition 3.1.21 this means also that a belongs to all the prime ideals associated with the submodule Ax of M. The corollary then follows from the fact that $\operatorname{Ass}(M)$ is the union of the $\operatorname{Ass}(Ax)$ where x runs through M. □

Proposition 3.1.23. *Let A be a Noetherian ring, M a finitely generated A-module and N an A-module. Then*

$$\operatorname{Ass}(\operatorname{Hom}_A(M, N)) = \operatorname{Ass}(N) \cap \operatorname{supp}(M).$$

Proof. By hypothesis, M is isomorphic to an A-module of the form A^n/R, hence $\operatorname{Hom}_A(M, N)$ is isomorphic to a submodule of $\operatorname{Hom}_A(A^n, N)$ and the latter is isomorphic to N^n. Now, $\operatorname{Ass}(N^n) = \operatorname{Ass}(N)$ by Corollary 3.1.8, thus $\operatorname{Ass}(\operatorname{Hom}_A(M, N)) \subseteq \operatorname{Ass}(N)$. On the other hand, for every prime ideal $\mathfrak{p}$ of A, $\operatorname{Hom}_{A_\mathfrak{p}}(M_\mathfrak{p}, N_\mathfrak{p})$ is isomorphic to $(\operatorname{Hom}_A(M, N))_\mathfrak{p}$ by Proposition 1.2.48

$$\operatorname{supp}(\operatorname{Hom}_A(M, N)) \subseteq \operatorname{supp}(M).$$

Then we conclude from Theorem 3.1.18 that $\operatorname{Ass}(\operatorname{Hom}_A(M, N)) \subseteq \operatorname{supp}(M)$.

Conversely, let $\mathfrak{p}$ be a prime ideal of A belonging to $\operatorname{Ass}(N) \cap \operatorname{supp}(M)$. By definition, N contains a submodule isomorphic to $A/\mathfrak{p}$. On the other hand, since M is finitely generated and $M_\mathfrak{p} \neq 0$, we know that there exists a nonzero homomorphism $\eta : M \to A/\mathfrak{p}$ (Proposition 1.4.43). As there exists an injective homomorphism $\iota : A/\mathfrak{p} \to N$, we have $\iota \circ \eta \in \operatorname{Hom}(M, N)$ and it is nonzero. On the other hand, since the annihilator of every nonzero element of $A/\mathfrak{p}$ is $\mathfrak{p}$, we see $\operatorname{Ann}(\iota \circ \eta) = \mathfrak{p}$, whence $\mathfrak{p} \in \operatorname{Ass}(\operatorname{Hom}_A(M, N))$. □

3.2 Primary decomposition

3.2.1 Primary submodules

Proposition 3.2.1. *Let A be a Noetherian ring and M an A-module. Then the following conditions are equivalent:*

(i) $\operatorname{Ass}(M)$ *is reduced to a single element.*

(ii) $M \neq 0$ *and every homothety of M is either injective or locally nilpotent.*

If these conditions are fulfilled and $\mathfrak{p}$ is the set of $a \in A$ such that the homothety of ratio a is locally nilpotent, then $\operatorname{Ass}(M) = \{\mathfrak{p}\}$.

Proof. This follows from Corollary 3.1.22 and Corollary 3.1.5. □

Let A be a Noetherian ring, M an A-module and N a submodule of N. If the module M/N satisfies the conditions of Proposition 3.2.1, N is called **$\mathfrak{p}$-primary** with respect to M (or in M).

In particular consider the case $M = A$, where the submodules of M are then the ideals of A and hence an ideal $\mathfrak{q}$ of A is called primary if $\operatorname{Ass}(A/\mathfrak{q})$ has a single element or, what amounts to the same, *if $A \neq \mathfrak{q}$ and every divisor of zero in the ring $A/\mathfrak{q}$ is nilpotent*. If $\mathfrak{q}$ is $\mathfrak{p}$-primary, then since $\operatorname{supp}(A/\mathfrak{q}) = V(\operatorname{Ann}(A/\mathfrak{q})) = V(\mathfrak{q})$, by Theorem 3.1.18 $\mathfrak{p}$ is the radical of the ideal $\mathfrak{q}$.

Example 3.2.2 (Example of primary submodules).

(a) If $\mathfrak{p}$ is a prime ideal of A, then $\mathfrak{p}$ is $\mathfrak{p}$-primary.

(b) Let $\mathfrak{q}$ be an ideal of A such that there exists a single prime ideal $\mathfrak{m}$ (necessarily maximal) containing $\mathfrak{q}$. Then, if M is an A-module such that $\mathfrak{q}M \neq M$, $\mathfrak{q}M$ is $\mathfrak{m}$-primary with respect to M, for every element of $\mathrm{Ass}(M/\mathfrak{q}M)$ contains $\mathfrak{q}$, hence is equal to $\mathfrak{m}$ and $\mathrm{Ass}(M/\mathfrak{q}M) \neq \varnothing$. In particular we see $\mathfrak{q}$ is an $\mathfrak{m}$-primary ideal in A.

(c) Let $\mathfrak{m}$ be a maximal ideal of a Noetherian ring A. Then the $\mathfrak{m}$-primary ideals are the ideals $\mathfrak{q}$ of A for which there exists an integer $n > 0$ such that $\mathfrak{m}^n \subseteq \mathfrak{q} \subseteq \mathfrak{m}$, for if $\mathfrak{m}^n \subseteq \mathfrak{q} \subseteq \mathfrak{m}$, then $\mathfrak{m}$ is the only prime ideal containing $\mathfrak{q}$ and the conclusion follows from (b). Conversely, if $\mathfrak{q}$ is $\mathfrak{m}$-primary, then $\mathfrak{m}$ is the radical of $\mathfrak{q}$ and there therefore exists $n > 1$ such that $\mathfrak{m}^n \subseteq \mathfrak{q}$ by **??**. In particular, if A is a PID then the primary ideals are (0) and the ideals of the form (p^n), where p is a prime element and $n > 0$.

Remark 3.2.3. The powers of any prime ideal are not necessarily primary ideals. For an example, let K be a field and A the quotient ring of $K[X,Y,Z]/(Z^2 - XY)$. Let x, y, z be the canonical images of X, Y, Z in A. Then the ideal $\mathfrak{p} = (x,z)$ is prime, since $A/\mathfrak{p} \cong K[X]$. Note that $\mathfrak{p}^2 = (x^2, z^2, xz) = (x^2, xy, xz)$ is not primary, since $x \notin \mathfrak{p}^2$, $y \notin \mathfrak{p}$ but $xy \in \mathfrak{p}^2$. Also, we have the following decomposition

$$\mathfrak{p}^2 = (x) \cap (x, y, z^2)$$

On the other hand, there exist primary ideals which are not powers of prime ideals. For example, consider the ring $A = \mathbb{Z}[X]$ and the maximal ideal $\mathfrak{m} = (2, X)$. The ideal $\mathfrak{q} = (4, X)$ is $\mathfrak{m}$-primary since $\sqrt{\mathfrak{q}} = \mathfrak{m}$, but it is not a power of $\mathfrak{m}$.

Proposition 3.2.4. *Let M be a module over a Noetherian ring A, $\mathfrak{p}$ a prime ideal of A and $(Q_i)_{i\in I}$ a non-empty finite family of submodules of M which are $\mathfrak{p}$-primary with respect to M. Then $\bigcap_{i\in I} Q_i$ is $\mathfrak{p}$-primary with respect to M.*

Proof. The quotient $M/(\bigcap_i Q_i)$ is isomorphic to a submodule of the direct sum $\bigoplus_i (M_i/Q_i)$. Now $\mathrm{Ass}(\bigoplus_i (M_i/Q_i)) = \bigcup_i \mathrm{Ass}(M_i/Q_i) = \{\mathfrak{p}\}$, whence $\mathrm{Ass}(M/(\bigcap_i Q_i)) = \{\mathfrak{p}\}$ and $\bigcap_i Q_i$ is $\mathfrak{p}$-primary. □

Proposition 3.2.5. *Let A be a Noetherian ring, S a multiplicative subset of A, $\mathfrak{p}$ a prime ideal of A, M an A-module, N a submodule of M and i_M^S the canonical map of M to $S^{-1}M$.*

(a) Suppose that $\mathfrak{p} \cap S \neq \varnothing$. If N is $\mathfrak{p}$-primary with respect to M, then $S^{-1}N = S^{-1}M$.

(b) Suppose that $\mathfrak{p} \cap S = \varnothing$. Then for N to be $\mathfrak{p}$-primary with respect to M, it is necessary and sufficient that N be of the form $(i_M^S)^{-1}(N')$, where N' is a sub-$S^{-1}A$-module of $S^{-1}M$ which is $(S^{-1}\mathfrak{p})$-primary with respect to $S^{-1}M$. In this case we have $N' = S^{-1}N$.

Proof. If $\mathfrak{p} \cap S \neq \varnothing$ and N is $\mathfrak{p}$-primary with respect to M, then

$$\mathrm{Ass}_{S^{-1}A}(S^{-1}(M/N)) = \varnothing$$

and hence $S^{-1}(M/N) = 0$, whence $S^{-1}N = S^{-1}M$.

Suppose that $\mathfrak{p} \cap S = \varnothing$. If N is $\mathfrak{p}$-primary with respect to M, then $\mathrm{Ass}_{S^{-1}A}(S^{-1}(M/N)) = \{S^{-1}\mathfrak{p}\}$ and hence the submodule $N' = S^{-1}N$ of $S^{-1}M$ is $(S^{-1}\mathfrak{p})$-primary. Moreover, if $s \in S$ and $m \in M$ are such that $sm \in N$, then $m \in N$, for the homothety with ratio s in M/N is injective, whence $N = (i_M^S)^{-1}(N')$.

Conversely, let N' be a submodule of $S^{-1}M$ which is $(S^{-1}\mathfrak{p})$-primary with respect to $S^{-1}M$. Let us write $N = (i_M^S)^{-1}(N')$, then $N' = S^{-1}N$ and

$$\mathrm{Ass}_{S^{-1}A}(S^{-1}(M/N)) = \mathrm{Ass}_{S^{-1}A}((S^{-1}M)/N') = \{S^{-1}\mathfrak{p}\}.$$

Since N is saturated, the canonical map $M/N \to S^{-1}(M/N)$ is injective, whence no prime ideal of A associated with M/N meets S (Proposition 3.1.13). It then follows that $\mathrm{Ass}(M/N) = \{\mathfrak{p}\}$, so that N is $\mathfrak{p}$-primary with respect to M. □

3.2.2 Primary decompositions

Let A be a Noetherian ring, M an A-module and N a submodule of M. A finite family $(Q_i)_{i\in I}$ of submodule of M which are primary with respect to M and such that $N = \bigcap_i Q_i$ is called a **primary decomposition** of N in M.

Example 3.2.6. Let us take $A = \mathbb{Z}$, $M = \mathbb{Z}$, $N = (n)$ for some integer $n > 0$. If $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ is the decomposition of n into prime factors, then

$$(n) = (p_1^{\alpha_1}) \cap \cdots \cap (p_k^{\alpha_k})$$

is a primary decomposition of (n) in $\mathbb{Z}$.

By an abuse of language, the relation $N = \bigcap_{i\in I} Q_i$ is called a primary decomposition of N in M. It amounts to the same to say that $\{0\} = \bigcap_{i\in I}(Q_i/N)$ is a primary decomposition of $\{0\}$ in M/N. If $(Q_i)_{i\in I}$ is a primary decomposition of N in M, the canonical map from M/N to $\bigcap_{i\in I}(M/Q_i)$ is injective. Conversely let N be a submodule of M and ϕ an injective homomorphism from M/N to a finite direct sum $P = \bigoplus_i P_i$, where each set $\mathrm{Ass}(P_i)$ is reduced to a single element $\mathfrak{p}_i$. Let ϕ_i be the homomorphism $M/N \to P_i$ obtained by taking the composition off with the projection $P \to P_i$, and let Q_i/N be the kernel of ϕ_i; then the Q_i distinct from M are primary with respect to M and $N = \bigcap_{i\in I} Q_i$. Moreover, $\mathrm{Ass}(M/N) \subseteq \bigcup_{i\in I}\{\mathfrak{p}_i\}$ by virtue of Proposition 3.1.7.

Theorem 3.2.7. *Let M be a finitely generated module over a Noetherian ring and let N be a submodule of M. Then there exists a primary decomposition of N of the form*

$$N = \bigcap_{\mathfrak{p}\in\mathrm{Ass}(M/N)} Q(\mathfrak{p}) \tag{3.2.1}$$

where for each $\mathfrak{p} \in \mathrm{Ass}(M/N)$, $Q(\mathfrak{p})$ is $\mathfrak{p}$-primary with respect to M.

Proof. We may replace M by M/N and therefore suppose that $N = 0$. By Corollary 3.1.19, $\mathrm{Ass}(M)$ is finite. By Proposition 3.1.10, there exists, for each $\mathfrak{p} \in \mathrm{Ass}(M)$, a submodule $Q(\mathfrak{p})$ of M such that $\mathrm{Ass}(M/Q(\mathfrak{p})) = \{\mathfrak{p}\}$ and $\mathrm{Ass}(Q(\mathfrak{p})) = \mathrm{Ass}(M) - \{\mathfrak{p}\}$. Let us write $P = \bigcap_{\mathfrak{p}\in\mathrm{Ass}(M)} Q(\mathfrak{p})$. For all $\mathfrak{p} \in \mathrm{Ass}(M)$, $\mathrm{Ass}(P) \subseteq \mathrm{Ass}(Q(\mathfrak{p}))$ and hence $\mathrm{Ass}(P) = \{\mathfrak{p}\}$, which implies $P = 0$ and therefore proves the theorem. $\square$

Let M be a module over a Noetherian ring and N a submodule of M. A primary decomposition $N = \bigcap_{i\in I} Q_i$ of N in M is called **reduced** if the following conditions are fulfilled:

(a) there exists no index $i \in I$ such that $\bigcap_{j\neq i} Q_j \subseteq Q_i$;

(b) if $\mathrm{Ass}(M/Q_i) = \{\mathfrak{p}_i\}$, the $\mathfrak{p}_i$'s are distinct.

From every primary decomposition $N = \bigcap_{i\in I} Q_i$ of N in M a reduced primary decomposition of M in N can be deduced as follows: let J be a minimal element of the set of subsets I' of I such that $N = \bigcap_{i\in I'} Q_i$. Clearly $(Q_i)_{i\in J}$ satisfies condition (a). Then let Φ be the set of $\mathfrak{p}_i$ for $i \in J$. For all $\mathfrak{p} \in \Phi$, let $H(\mathfrak{p})$ be the set of $i \in J$ such that $\mathfrak{p}_i = \mathfrak{p}$ and let $Q(\mathfrak{p}) = \bigcap_{i\in H(\mathfrak{p})} Q_i$. It follows from Proposition 3.2.4 that $Q(\mathfrak{p})$ is $\mathfrak{p}$-primary with respect to M; further $N = \bigcap_{\mathfrak{p}\in\Phi} Q(\mathfrak{p})$ and the family $(Q(\mathfrak{p}))_{\mathfrak{p}\in\Phi}$ is therefore a reduced primary decomposition of N in M.

Proposition 3.2.8. *Let M be a module over a Noetherian ring, N a submodule of M, $N = \bigcap_{i\in I} Q_i$ a primary decomposition of N in M and, for all $i \in I$, let $\{\mathfrak{p}_i\} = \mathrm{Ass}(M/Q_i)$. For this decomposition to be reduced, it is necessary and sufficient that the $\mathfrak{p}_i$ be distinct and belong to $\mathrm{Ass}(M/N)$. In this case, we have*

$$\mathrm{Ass}(M/N) = \bigcup_{i\in I} \mathfrak{p}_i, \quad \mathrm{Ass}(Q_i/N) = \bigcup_{j\neq i}\{\mathfrak{p}_i\} \quad \textit{for all } i \in I. \tag{3.2.2}$$

Proof. If the condition of the statement is fulfilled, $N = \bigcap_{j\neq i} Q_j$ cannot hold, for we would deduce that $\mathrm{Ass}(M/N) \subseteq \bigcup_{j\neq i}\{\mathfrak{p}_j\}$ by Corollary 3.1.9, contrary to the hypothesis. The primary decomposition $(Q_i)_{i\in I}$ of N is then certainly reduced.

Conversely, assume that $(Q_i)_{i\in I}$ is reduced. Note that $\mathrm{Ass}(M/N) \subseteq \bigcup_{i\in I}\{\mathfrak{p}_i\}$ always holds by Corollary 3.1.9. On the other hand, for all $i \in I$, let us write $P_i = \bigcap_{j\neq i} Q_j$. Then $P_i \cap Q_i = N$ and $P_i \neq N$ if $(Q_i)_{i\in I}$ is reduced, hence P_i/N is non-zero and is isomorphic to the submodule $(P_i + Q_i)/Q_i$ of M/Q_i, whence $\{\mathfrak{p}_i\} = \mathrm{Ass}(P_i/N)$ by Proposition 3.1.7. As $P_i/N \subseteq M/N$, we get $\mathfrak{p}_i \in \mathrm{Ass}(M/N)$, which completes the proof of the necessity of the condition in the statement and the first formula of (3.2.2). Finally, for each $i \in I$, as $N = \bigcap_{j\neq i}(Q_i \cap Q_j)$, by Corollary 3.1.9 we have

$$\mathrm{Ass}(Q_i/N) \subseteq \bigcup_{j\neq i} \mathrm{Ass}(Q_i/(Q_i \cap Q_j)) = \bigcup_{j\neq i} \mathrm{Ass}((Q_i + Q_j)/Q_j) \subseteq \bigcup_{j\neq i}\{\mathfrak{p}_j\}.$$

Taking account of the first formula of (3.2.2) and Proposition 3.1.7, the second formula of (3.2.2) follows easily. □

Corollary 3.2.9. *Let A be a Noetherian ring, M an A-module, N a submodule of M and $(Q_i)_{i\in I}$ a primary decomposition of N in M. Then $|I| \geq |\mathrm{Ass}(M/N)|$, and the equality holds if and only if $(Q_i)_{i\in I}$ is reduced.*

Proof. It follows from the remarks preceding Proposition 3.2.8 that there exists a reduced primary decomposition $(R_i)_{i\in J}$ of N in M such that $\mathrm{Card}(J) \subseteq \mathrm{Card}(I)$. The first assertion then follows from the second and the latter is a consequence of Proposition 3.2.8. □

Proposition 3.2.10. *Let A be a Noetherian ring, M an A-module and N a submodule of M. Let $N = \bigcap_{i\in I} Q_i$ be a reduced primary decomposition of N where Q_i is $\mathfrak{p}_i$-primary. If $\mathfrak{p}_i$ is a minimal element of* $\mathrm{Ass}(M/N)$*, then Q_i is equal to the saturation on N with respect to $\mathfrak{p}_i$.*

Proof. We can obviously restrict our attention to the case where $N = 0$, replacing if need be M by M/N. If $\mathfrak{p}_i$ is minimal in $\mathrm{Ass}(M)$, then the set of elements of $\mathrm{Ass}(M)$ which do not meet $A - \mathfrak{p}_i$ reduces to $\mathfrak{p}_i$. The proposition then follows from the second formula of (3.2.2) above and Proposition 3.1.13, since the kernel of the canonical map $M \mapsto M_{\mathfrak{p}_i}$ equals to the saturation of 0 with respect to $\mathfrak{p}_i$. □

The minimal elements of $\mathrm{Ass}(M/N)$ are called the **minimal** or **isolated prime ideals** associated with M/N, and the prime ideals $\mathfrak{p} \in \mathrm{Ass}(M/N)$ which are not minimal are called the **embedded prime ideals** associated with M/N. If M/N is finitely generated, then by Corollary 3.1.15 we know the isolated ideals of M/N are just the minimal prime ideals containing $\mathrm{supp}(M/N) = V(\mathrm{Ann}(M/N))$, so for $\mathfrak{p} \in \mathrm{Ass}(M/N)$ to be embedded, it is necessary and sufficient that $V(\mathfrak{p})$ be not an irreducible component of $\mathrm{supp}(M/N)$.

If $(Q(\mathfrak{p}))$ and $(R(\mathfrak{p}))$ are two reduced primary decompositions of N in M, then by Proposition 3.2.10 we see $Q(\mathfrak{p}) = R(\mathfrak{p})$ if $\mathfrak{p}$ is an isolated prime associated with M/N, being the saturation of N with respect to $\mathfrak{p}$. However, it may happen that $Q(\mathfrak{p}) \neq R(\mathfrak{p})$ for $\mathfrak{p}$ an embedded prime, as the following example shows.

Example 3.2.11. Consider the ring $A = K[X, Y]$ where K is a field. Then for every positive integer n, a primary decomposition of the ideal $\mathfrak{a} = (X^2, XY)$ is

$$(X^2, XY) = (X) \cap (X^2, XY, Y^n)$$

where (X) is prime and (X^2, XY, Y^n) is (X, Y)-primary. Note that (X) is isolated and (X, Y) is embedded.

Given a submodule N of a module M over a Noetherian ring A, to simplify we shall denote by $D_I(M/N)$, in this part, the set of reduced primary decompositions of N in M whose indexing set is I.

Proposition 3.2.12. *Let A be a Noetherian ring, M an A-module, N a submodule of M and $I = \mathrm{Ass}(M/N)$. Let S be a multiplicative subset of A and I_S the subset I consisting of the indices i such that $S \cap \mathfrak{p}_i = \varnothing$. Let $\phi(N)$ be the saturation of N with respect to S in M. Then:*

(a) If $(Q_i)_{i\in I}$ is an element of $D_I(M/N)$, the family $(Q_i)_{i\in I_S}$ is an element of $D_{I_S}(M/\phi(N))$ and the family $(S^{-1}Q_i)_{i\in I_S}$ is an element of $D_{I_S}(S^{-1}M/S^{-1}N)$.

(b) The map $(Q_i)_{i\in I_S} \mapsto (S^{-1}Q_i)_{i\in I_S}$ is a bijection from $D_{I_S}(M/\phi(N))$ to $D_{I_S}(S^{-1}M/S^{-1}N)$.

(c) If $(Q_i)_{i\in I_S}$ is an element of $D_{I_S}(M/\phi(N))$ and $(R_i)_{i\in I}$ is an element of $D_I(M/N)$, then the family $(T_i)_{i\in I}$ defined by $T_i = Q_i$ when $i \in I_S$ and $T_i = R_i$ for $i \in I - I_S$ is an element of $D_I(M/N)$.

Proof. We know that for $i \in I_S$, $S^{-1}Q_i$ is $S^{-1}\mathfrak{p}_i$-primary and that for $i \in I - I_S$, $S^{-1}Q_i = S^{-1}M$. As $S^{-1}N = \bigcap_{i\in I} S^{-1}Q_i$, we see $S^{-1}N = \bigcap_{i\in I_S} S^{-1}Q_i$. The ideals $S^{-1}\mathfrak{p}_i$ for $i \in I_S$ are distinct and their set is $\mathrm{Ass}(S^{-1}M/S^{-1}N)$ by Proposition 3.1.11. Then by Proposition 3.2.8 $(S^{-1}Q_i)_{i\in I_S}$ is a reduced primary decomposition of $S^{-1}N$. Moreover, Q_i is saturated for $i \in I_S$, hence $\phi(N) = (i_M^S)^{-1}(S^{-1}N) = \bigcap_{i\in I_S} Q_i$, and $(Q_i)_{i\in I_S}$ is obviously a reduced primary decomposition of $\phi(N)$ in M.

As $S^{-1}\phi(N) = S^{-1}N$, we may replace N by $\phi(N)$ and suppose that $I = I_S$. Let $(P_i)_{i\in I}$ be a reduced primary decomposition Of $S^{-1}N$ in $S^{-1}M$ and let us write $Q_i = (i_M^S)^{-1}(P_i)$. It follows from Proposition 3.2.5 that Q_i is $\mathfrak{p}_i$-primary and $(Q_i)_{i\in I}$ then a reduced primary decomposition of N of M by virtue of Corollaries 3.2.9. Finally, since $I = I_S$, by Proposition 3.2.5 we see every $\mathfrak{p}_i$-primary submodule of M is saturated, so the two maps

$$D_I(M/N) \to D_I(S^{-1}M/S^{-1}N) \quad \text{and} \quad D_I(S^{-1}M/S^{-1}N) \to D_I(M/N)$$

are inverse of each other, which proves (b).

Finally, we prove (c). From (a) we have $\phi(N) = \bigcap_{i\in I_S} R_i$, whence

$$N = \phi(N) \cap \bigcap_{i\in I-I_S} R_i = \Big(\bigcap_{i\in I_S} Q_i\Big) \cap \Big(\bigcap_{i\in I-I_S} R_i\Big)$$

and it follows immediately from Corollary 3.2.9 that this primary decomposition is reduced. ∎

Corollary 3.2.13. *The maps*

$$D_I(M/N) \to D_{I_S}(M/\phi(N)) \quad \textit{and} \quad D_I(M/N) \to D_{I_S}(S^{-1}M/S^{-1}N)$$

defined in Proposition 3.2.12 are surjective.

Proof. Proposition 3.2.12(c) shows that the map $D_I(M/N) \to D_{I_S}(M/\phi(N))$ is suijective and Proposition 3.2.12(b) then shows that $D_I(M/N) \to D_{I_S}(M/N) \to D_{I_S}(S^{-1}M/S^{-1}N)$ is surjective. ∎

3.2.3 Finite length modules

If an A-module M is of finite length, we shall denote this length by $\ell_A(M)$ or $\ell(M)$. Recall that every Artinian ring is Noetherian and that every finitely generated module over an Artinian ring is of finite length.

Proposition 3.2.14. *Let M be a finitely generated module over a Noetherian ring A. The following properties are equivalent:*

(i) M is of finite length.

(ii) Every ideal $\mathfrak{p} \in \operatorname{Ass}(M)$ is a maximal ideal of A.

(iii) Every ideal $\mathfrak{p} \in \operatorname{supp}(M)$ is a maximal ideal of A.

Proof. Let (M_i) be a chain of submodules of M such that M_i/M_{i+1} is isomorphic to $A/\mathfrak{p}_i$ where $\mathfrak{p}_i$ is prime. If M is of finite length, so is each of the A-modules $A/\mathfrak{p}_i$, which implies that each of the rings $A/\mathfrak{p}_i$ is Artinian. But as $A/\mathfrak{p}_i$ is an integral domain, it is therefore a field, in other words $\mathfrak{p}_i$ is maximal; we conclude that (i) implies (ii). Condition (ii) implies (iii) by Proposition 3.1.14. Finally, if all the ideals of $\operatorname{supp}(M)$ are maximal, so are the $\mathfrak{p}_i$, hence the $A/\mathfrak{p}_i$ are simple A-modules and M is of finite length, which completes the proof. □

Corollary 3.2.15. *For every module of finite length M over a Noetherian ring A, we have* $\operatorname{Ass}(M) = \operatorname{supp}(M)$.

Proof. Every element of $\operatorname{supp}(M)$ is then minimal in $\operatorname{supp}(M)$ and the conclusion follows from Corollary 3.1.15. □

Corollary 3.2.16. *Let M be a finitely generated module over a Noetherian ring A and $\mathfrak{p}$ a prime ideal of A. For $M_\mathfrak{p}$ to be a non-zero $A_\mathfrak{p}$-module of finite length, it is necessary and sufficient that $\mathfrak{p}$ be a minimal element of* $\operatorname{Ass}(M)$.

Proof. By Proposition 3.1.11, $\operatorname{Ass}_{A_\mathfrak{p}}(M_\mathfrak{p})$ is the set of ideals $\mathfrak{q}A_\mathfrak{p}$, where $\mathfrak{q}$ runs through the set of ideals of $\operatorname{Ass}(M)$ which are contained in $\mathfrak{p}$. Thus by Proposition 3.2.14, for $M_\mathfrak{p}$ to be an $A_\mathfrak{p}$-module of finite length, it is necessary and sufficient that no element of $\operatorname{Ass}(M)$ be strictly contained in $\mathfrak{p}$. On the other hand, for $M_\mathfrak{p} \neq 0$, it is necessary and sufficient by definition that $\mathfrak{p} \in \operatorname{supp}(M)$, that is that $\mathfrak{p}$ contain an element of $\operatorname{Ass}(M)$. This proves the corollary. □

Remark 3.2.17. Note that Proposition 3.2.14 is not equivalent to the condition that each prime ideal $\mathfrak{p} \in \operatorname{Ass}(M)$ is *maximal in* $\operatorname{Ass}(M)$. In fact, this condition signifies that each element of $\operatorname{Ass}(M)$ is minimal, so is a minimal prime ideal of $\operatorname{supp}(M)$ (in other words, M has no embedded prime ideals).

Remark 3.2.18. Let M be a finitely generated module over a Noetherian ring A and let (M_i) be a chain of M such that M_i/M_{i+1} is isomorphic to $A/\mathfrak{p}_i$ where $\mathfrak{p}_i$ is a prime ideal of A. If $\mathfrak{p}$ is a minimal element of $\operatorname{Ass}(M)$, the modules $(M_i)_\mathfrak{p}$ then form a chain of $M_\mathfrak{p}$ and $(M_i)_\mathfrak{p}/(M_{i+1})_\mathfrak{p}$ is isomorphic to $(A/\mathfrak{p}_i)_\mathfrak{p}$ and hence to $\{0\}$ if $\mathfrak{p}_i \neq \mathfrak{p}$, and to $(A/\mathfrak{p})_\mathfrak{p}$ which is a field, if $\mathfrak{p}_i = \mathfrak{p}$.

Proposition 3.2.19. *Let M be a module of finite length over a Noetherian ring A.*

(a) There exists a unique primary decomposition $\{0\} = \bigcap_{\mathfrak{p} \in \operatorname{Ass}(M)} Q(\mathfrak{p})$ of $\{0\}$ with respect to M indexed by $\operatorname{Ass}(M)$, where $Q(\mathfrak{p})$ is $\mathfrak{p}$-primary with respect to M.

(b) There exists an integer n_0 such that, for all $n \geq n_0$ and all $\mathfrak{p} \in \operatorname{Ass}(M)$, $Q(\mathfrak{p}) = \mathfrak{p}^n M$.

(c) For all $\mathfrak{p} \in \operatorname{Ass}(M)$, the canonical map of M to $M_\mathfrak{p}$ is surjective and its kernel is $Q(\mathfrak{p})$. Therefore $M_\mathfrak{p}$ can be identified with $M/Q(\mathfrak{p})$.

(d) The canonical injection of M into $\bigoplus_{\mathfrak{p} \in \operatorname{Ass}(M)} (M/Q(\mathfrak{p}))$ is bijective.

Proof. Since each ideal of M is maximal, part (a) follows from Proposition 3.2.10. As $M/Q(\mathfrak{p})$ is finitely generated, by Proposition 3.1.21 there exists $n_0 > 0$ such that $\mathfrak{p}^n M \subseteq Q(\mathfrak{p})$ for all $n \geq n_0$. But as $\mathfrak{p}$ is a maximal ideal, $\mathfrak{p}^n M$ is $\mathfrak{p}$-primary with respect to M and, as $\bigcap_{\mathfrak{p}\in \mathrm{Ass}(M)} \mathfrak{p}^n M = \{0\}$, it follows from (a) that necessarily $\mathfrak{p}^n M = Q(\mathfrak{p})$ for all $\mathfrak{p} \in \mathrm{Ass}(M)$, whence (b) follows. As the $\mathfrak{p}^n$, for $\mathfrak{p} \in \mathrm{Ass}(M)$, are relatively prime in pairs, the canonical map $M \mapsto \bigoplus_{\mathfrak{p}\in \mathrm{Ass}(M)}(M/\mathfrak{p}^n M)$ is suijective, whence (d).

Now $\mathrm{Ass}(Q(\mathfrak{p})) = \mathrm{Ass}(M) - \{\mathfrak{p}\}$ and $\mathrm{Ass}(M/Q(\mathfrak{p})) = \{\mathfrak{p}\}$. As the elements of Ass(M) are maximal ideals, $\mathfrak{p}$ is the only element of $\mathrm{Ass}(M)$ which does not meet $A - \mathfrak{p}$. Then $Q(\mathfrak{p})$ is the kernel of the canonical map $i_{\mathfrak{p}} : M \to M_{\mathfrak{p}}$ by Proposition 3.1.10. If $s \in A - \mathfrak{p}$, the homothety of $M/Q(\mathfrak{p})$ with ratio s isinjective by virtue of the relation $\mathrm{Ass}(M/Q(\mathfrak{p})) = \{\mathfrak{p}\}$. Since $M/Q(\mathfrak{p})$ is Artinian, this homothety is bijective. Therefore, if $m \in M$ and $s \in A - \mathfrak{p}$, then there exists $n \in M$ such that $\bar{m} = s\bar{n}$ in $M/Q(\mathfrak{p})$, whence $m - ns \in Q(\mathfrak{p})$. By the characterization of $M_{\mathfrak{p}}$, we can then find $t \in A - \mathfrak{p}$ such that $t(m - ns) = 0$, that is, $m/s = n/1$ in $M_{\mathfrak{p}}$. This proves the canonical map $M \to M_{\mathfrak{p}}$ is surjective and finishes the proof. □

Corollary 3.2.20. *If M is a module of finite length over a Noetherian ring A, then*

$$\ell_A(M) = \sum_{\mathfrak{p}\in \mathrm{Ass}(M)} \ell_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}).$$

Proof. This will follow from Proposition 3.2.19(d) if we prove that

$$\ell_A(M/Q(\mathfrak{p})) = \ell_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}).$$

Now, since $\mathrm{Ass}(M/Q(\mathfrak{p})) = \{\mathfrak{p}\}$, for all $s \in A - \mathfrak{p}$ the homothety with ratio s on $M/Q(\mathfrak{p})$ is injective; the homothety with ratio s on every submodule N of $M/Q(\mathfrak{p})$ is therefore injective and, as N is Artinian, it is bijective. We conclude that the sub-A-modules of $M/Q(\mathfrak{p})$ are the images under the bijection $M_{\mathfrak{p}} \to M/Q(\mathfrak{p})$ of the sub-$A_{\mathfrak{p}}$-modules of $M_{\mathfrak{p}}$, whence our assertion. □

Proposition 3.2.21. *Let A be a Noetherian ring. The following conditions are equivalent:*

(i) A is Artinian.

(ii) All the prime ideals of A are maximal ideals.

(iii) All the prime ideals of $\mathrm{Ass}(A)$ *are maximal ideals.*

If these conditions are fulfilled, A has only a finite number prime ideals, which are all maximal and associated with the A-module A. Further, A is a semi-local ring and its Jacobson radical is nilpotent.

Proof. To say that A is Artinian is equivalent to saying that A is an A-module of finite length, hence (i) and (ii) are equivalent by Proposition 3.2.14. We already know that (i) and (ii) are equivalent by **??**.

Suppose they hold. As every prime ideal of A belongs to $\mathrm{supp}(A)$ and every element of $\mathrm{supp}(A)$ contains an element of $\mathrm{Ass}(A)$, it follows from (iii) that $\mathrm{Ass}(A)$ is the set ofall prime ideals of A. Then A has only a finite number of prime ideals, all of them maximal and associated with the A-module A. Finally, we know that the Jacobson radical of an Artinian ring is nilpotent. □

Corollary 3.2.22. *Every Artinian ring A is isomorphic to the direct composition of a finite family of Artinian local rings.*

Proof. It follows from Proposition 3.2.21 and Proposition 3.2.19 (c) and (d) that, if $(\mathfrak{m}_i)$ is the family of maximal ideals of A, the canonical map $A \to \prod_i A_{\mathfrak{m}_i}$ is bijective. □

Corollary 3.2.23. *Let A be a Noetherian ring and $\mathfrak{a}$ an ideal of A. The following conditions are equivalent:*

(i) A is a semi-local ring and $\mathfrak{a}$ is a defining ideal of A.

(ii) A is a Zariski ring with the $\mathfrak{a}$-adic topology and $A/\mathfrak{a}$ is Artinian.

Proof. If (a) holds then by Example 2.4.29 we know A is a Zariski ring with the $\mathfrak{r}$-adic topology, where $\mathfrak{r}$ is the Jacobson radical of A. Moreover, as by hypothesis $\mathfrak{a}$ contains a power of $\mathfrak{r}$, every prime ideal of A which contains $\mathfrak{a}$ also contains $\mathfrak{r}$ and is therefore maximal, since $\mathfrak{r}$ is a finite intersection of maximal ideals. This shows that $A/\mathfrak{a}$ is Artinian.

Conversely, if (b) holds, then $\mathfrak{a} \subseteq \mathfrak{r}$, so every maximal ideal $\mathfrak{p}$ of A containing $\mathfrak{r}$ must contain $\mathfrak{a}$. As $A/\mathfrak{a}$ is Artinian, the ideals $\mathfrak{p}/\mathfrak{a}$ are finite in number (Proposition 3.2.21) and hence A has only a finite number of maximal ideals, which implies that it is semi-local. □

Corollary 3.2.24. *Let $\rho : A \to B$ be a ring homomorphism. Suppose that A is semi-local and Noetherian and that B is a finitely generated A-module. Then the ring B is semi-local and Noetherian. If $\mathfrak{a}$ is a defining ideal of A, then $\mathfrak{b} = \mathfrak{a}^e$ is a defining ideal of B.*

Proof. We know that B is a Zariski ring with the $\mathfrak{b}$-adic topology (Proposition 2.4.35). As $A/\mathfrak{a}$ is Artinian by Corollary 3.2.23 and $B/\mathfrak{b}$ is a finitely generated $(A/\mathfrak{a})$-module, it is an Artinian ring, hence B is semilocal and $\mathfrak{b}$ is a defining ideal of B by Corollary 3.2.23. □

Corollary 3.2.25. *Let A be a complete semi-local Noetherian ring, $\mathfrak{a}$ a defining ideal of A, M a finitely generated A-module and (N_n) a decreasing sequence of submodules of M such that $\bigcap_n N_n = 0$. Then, for all $p > 0$, there exists $n > 0$ such that $N_n \subseteq \mathfrak{m}^p M$.*

Proposition 3.2.26. *Let A be a Noetherian ring and $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ the prime ideals associated with the A-module A.*

(a) The set $S = \bigcap_{i=1}^n (A - \mathfrak{p}_i)$ is the set of elements which are not divisors of 0 in A.

(b) If all the $\mathfrak{p}_i$ are minimal elements of $\mathrm{Ass}(A)$, then the total ring of fractions $S^{-1}A$ of A is Artinian.

(c) If the ring A is reduced, then all the $\mathfrak{p}_i$ are minimal elements of $\mathrm{Ass}(A)$ (and therefore are the minimal prime ideals of A) and each of the $A_{\mathfrak{p}_i}$ is a field. For each index i, the canonical homomorphism $S^{-1}A \to A_{\mathfrak{p}_i}$ is surjective and its kernel is $S^{-1}\mathfrak{p}_i$. Finally, the canonical homomorphism $S^{-1}A \to \prod_{i=1}^n (S^{-1}A/S^{-1}\mathfrak{p}_i) \cong \prod_{i=1}^n A_{\mathfrak{p}_i}$ is bijective.

Proof. The fact that S is the set of elements which are not divisors of 0 in A has already been seen in Proposition 3.1.3. The prime ideals of $S^{-1}A$ are of the form $S^{-1}\mathfrak{p}$, where $\mathfrak{p}$ is a prime ideal of A contained in $\bigcup_i \mathfrak{p}_i$, and hence is contained in one of the $\mathfrak{p}_i$. If $\mathfrak{p}_i$ is a minimal element of $\mathrm{Ass}(A)$, it is then a minimal element of $\mathrm{Spec}(A) = \mathrm{supp}(A)$. If each of the $\mathfrak{p}_i$ is a minimal element of $\mathrm{Ass}(A)$, we then see that the prime ideals of $S^{-1}A$ are the $S^{-1}\mathfrak{p}_i$ and they are therefore all maximal, which proves that $S^{-1}A$ is Artinian.

Suppose finally that the ring A is reduced. Then $\bigcap_i \mathfrak{p}_i = \{0\}$, and we deduce that $\{0\} = \bigcap_i \mathfrak{p}_i$ is a reduced primary decomposition of the ideal $\{0\}$ by Corollary 3.2.9. In particular, none of the $\mathfrak{p}_i$ can contain a $\mathfrak{p}_i$ of index $j \neq i$ and therefore the $\mathfrak{p}_i$ are all minimal elements of $\mathrm{Ass}(A)$. The ring $S^{-1}A$ is then Artinian by (b). The $S^{-1}\mathfrak{p}$ are prime ideals associated with the $S^{-1}A$-module $S^{-1}A$ and $\{0\} = S^{-1}(\bigcap_{i=1}^n \mathfrak{p}_i) = \bigcap_{i=1}^n S^{-1}\mathfrak{p}_i$. As the $S^{-1}\mathfrak{p}_i$ are distinct, $(S^{-1}\mathfrak{p}_i)$ is a reduced primary decomposition of $\{0\}$ in $S^{-1}A$ by Corollary 3.2.9. Proposition 3.2.19 then shows that the canonical homomorphism $\phi_i : S^{-1}A \to (S^{-1}A)_{\mathfrak{p}_i}$ is surjective and has kernel $S^{-1}\mathfrak{p}_i$, and the canonical homomorphism

$$S^{-1}A \to \prod_{i=1}^n (S^{-1}A/S^{-1}\mathfrak{p}_i)$$

is bijective. We know moreover that the canonical homomorphism $S^{-1}A \to A_{\mathfrak{p}_i}$ is given by

$$S^{-1}A \xrightarrow{\phi_i} (S^{-1}A)_{\mathfrak{p}_i} \xrightarrow{\cong} (S^{-1}A)_{S^{-1}\mathfrak{p}_i} \xrightarrow{\cong} A_{\mathfrak{p}_i}$$

Finally, it follows from Proposition 3.2.19 that $(S^{-1}A)_{S^{-1}\mathfrak{p}_i}$ is isomorphic to $S^{-1}A/S^{-1}\mathfrak{p}_i$, and hence is a field since $S^{-1}\mathfrak{p}_i$ is a maximal ideal. □

3.2.4 Extension of scalars

In this part, A and B will denote two rings and we shall consider a ring homomorphism $\rho : A \to B$ which makes B into an A-algebra. Recall that, for every B-module P, $\rho^*(P)$ is the A-module whose structure is defined by $a \cdot y = \rho(a)y$ for all $a \in A$, $y \in P$.

Lemma 3.2.27. *Let A be a Noetherian ring, $\mathfrak{p}$ a prime ideal of A, M an A-module whose annihilator contains a power of $\mathfrak{p}$ and such that $\operatorname{Ass}_A(M) = \{\mathfrak{p}\}$. Let P be a B-module such that $\rho^*(P)$ is a flat A-module. The condition $\mathfrak{P} \in \operatorname{Ass}_B(M \otimes_A P)$ then implies $\mathfrak{P}^c = \mathfrak{p}$.*

Proof. If n is such that $\mathfrak{p}^n M = 0$, then $\mathfrak{p}^n B \subseteq \operatorname{Ann}(M \otimes_A P)$, whence $\mathfrak{p}^n B \subseteq \mathfrak{P}$, which implics $\mathfrak{p}^n \subseteq \mathfrak{P}^c$ and therefore $\mathfrak{p} \subseteq \mathfrak{P}$ sincen $\mathfrak{P}^c$ is prime. Moreover, if $a \in A - \mathfrak{p}$, the homothety h_a with ratio a on M is injective. As $h_a \otimes 1_P$ is the homothety $h_{\rho(a)}$ with ratio $\rho(a)$ on $M \otimes_A P$ and $\rho^*(P)$ is flat, we see $h_{\rho(a)}$ is injective. This proves that $\rho(a) \notin \mathfrak{P}$, whence $\mathfrak{P}^c = \mathfrak{p}$. □

Proposition 3.2.28. *Let $\rho : A \to B$ be a ring homomorphism, M an A-module and P a B-module such that $\rho^*(P)$ is a flat A-module. Then*

$$\operatorname{Ass}_B(M \otimes_A P) \supseteq \bigcup_{\mathfrak{p} \in \operatorname{Ass}_A(M)} \operatorname{Ass}_B(P/\mathfrak{p}P) \tag{3.2.3}$$

and the equality holds if A is Noetherian.

Proof. Let $\mathfrak{p} \in \operatorname{Ass}_A(M)$. By definition there exists an injection $A/\mathfrak{p} \to M$, and by tensoring with P we get an injection $P/\mathfrak{p}P \to M \otimes_A P$, whence $\operatorname{Ass}_B(P/\mathfrak{p}P) \subseteq \operatorname{Ass}_B(M \otimes_A P)$ and (3.2.3) follows. Suppose now that A is Noetherian and let us prove the opposite inclusion.

Suppose first that M is a finitely generated A-module and that $\operatorname{Ass}_A(M)$ is reduced to a single element $\mathfrak{p}$. By Theorem 3.1.18 there exists a chain (M_i) of M such that M_i/M_{i+1} is isomorphic to $A/\mathfrak{p}_i$ where $\mathfrak{p}_i$ is a prime ideal of A containing $\mathfrak{p}$. As P is a flat A-module, $(M_i \otimes_A P)$ is then a chain of $M \otimes_A P$ such that

$$(M_i \otimes_A P)/(M_{i+1} \otimes_A P) = (A/\mathfrak{p}_i) \otimes_A P = P/\mathfrak{p}_i P.$$

By Proposition 3.1.7, we get

$$\operatorname{Ass}_B(M \otimes_A P) \subseteq \bigcup_{i=0}^{n-1} \operatorname{Ass}_B(P/\mathfrak{p}_i P).$$

We know that M is annihilated by a power of $\mathfrak{p}$ by Proposition 3.1.21, so Lemma 3.2.27 shows that $\mathfrak{P}^c = \mathfrak{p}$ for all $\mathfrak{P} \in \operatorname{Ass}_B(M \otimes_A P)$. Also, since $P/\mathfrak{p}_i P$ is annihilated by $\mathfrak{p}_i$, by Lemma 3.2.27 we have $(\mathfrak{P}')^c = \mathfrak{p}_i$ for all $\mathfrak{P}' \in \operatorname{Ass}_B(P/\mathfrak{p}_i P)$. This then shows $\operatorname{Ass}_B(M \otimes_A P) \cap \operatorname{Ass}_B(P/\mathfrak{p}_i P) = \varnothing$ if $\mathfrak{p}_i \neq \mathfrak{p}$, which proves the claim in this case.

Next we drop the assumption $\operatorname{Ass}(M) = \{\mathfrak{p}\}$ and assume only that M is finitely generated. Let $\operatorname{Ass}_A(M) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$ and let $\{0\} = \bigcap_i Q_i$ be a corresponding reduced primary decomposition of $\{0\}$. Then M is isomorphic to a submodule of the direct sum of the submodules

$M_i = M/Q_i$ and, as $f^*(P)$ is a flat A-module, $M \otimes_A P$ is isomorphic to a submodule of the direct sum of the submodules $M_i \otimes_A P$. We deduce that

$$\mathrm{Ass}_B(M \otimes_A P) \subseteq \bigcup_{i=1}^n \mathrm{Ass}_B(M_i \otimes_A P).$$

But M_i is a finitely generated A-module such that $\mathrm{Ass}_A(M_i)$ is reduced to a single element $\mathfrak{p}_i$. By what we have proved, $\mathrm{Ass}_B(M_i \otimes_A P) = \mathrm{Ass}_B(P/\mathfrak{p}_i P)$, whence the claim in this case.

Now we turn to the general case. The B-module $M \otimes_A P$ is the union of the submodules $N \otimes_A P$, where N runs through the set of finitely generated submodules of the A-module M. If $\mathfrak{P}$ belongs to $\mathrm{Ass}_B(M \otimes_A P)$, then there exists a finitely generated submodule N of M such that $\mathfrak{P} \in \mathrm{Ass}_B(N \otimes_A P)$. By the preceeding argument, there exists $\mathfrak{p} \in \mathrm{Ass}_A(N)$ such that $\mathfrak{P} \in \mathrm{Ass}_B(P/\mathfrak{p}P)$. As $\mathrm{Ass}_A(N) \subseteq \mathrm{Ass}_A(M)$, this completes the proof. □

Corollary 3.2.29. *In the situation of Proposition 3.2.28, if A is Noetherian and $\mathfrak{P} \in \mathrm{Ass}_B(M \otimes_A P)$, then $\mathfrak{P}^c \in \mathrm{Ass}_A(M)$ and $\mathfrak{P}^c$ is the only prime ideal $\mathfrak{p}$ of A such that $\mathfrak{P} \in \mathrm{Ass}_B(P/\mathfrak{p}P)$.*

Proof. By Proposition 3.2.28 we see $\mathfrak{P} \in \mathrm{Ass}_B(P/\mathfrak{p}P) = \mathrm{Ass}_B((A/\mathfrak{p}) \otimes_A P)$ where $\mathfrak{p} \in \mathrm{Ass}_A(M)$. Then Lemma 3.2.27 says $\mathfrak{P}^c = \mathfrak{p}$. □

Corollary 3.2.30. *Suppose that A and B are Noetherian and that B is a flat A-module. Let $\mathfrak{p}$ be a prime ideal of A, Q a $\mathfrak{p}$-primary submodule of an A-module M and $\mathfrak{P}$ a prime ideal of B. For $Q \otimes_A B$ to be a $\mathfrak{P}$-primary submodule of $M \otimes_A B$, it is necessary and sufficient that $\mathfrak{p}B$ be a $\mathfrak{P}$-primary ideal of B.*

Proof. Let us apply Proposition 3.2.28 to the A-module M/Q and the B-module B. Then $\mathrm{Ass}_A(M/Q) = \{\mathfrak{p}\}$ and $(M/Q) \otimes_A B$ is isomorphic to $(M \otimes_A B)/(Q \otimes_A B)$ and hence $\mathrm{Ass}_B((E \otimes_A B)/(Q \otimes_A B)) = \mathrm{Ass}_B(B/\mathfrak{p}B)$. To say that $Q \otimes_A B$ is $\mathfrak{P}$-primary in $M \otimes_A B$ therefore means that $\mathrm{Ass}_B(B/\mathfrak{p}B)$ is reduced to $\{\mathfrak{P}\}$, whence the corollary. □

Proposition 3.2.31. *Suppose that A and B are Noetherian and that B is a flat A-module. Let M be an A-module and N a submodule of M such that, for every prime ideal $\mathfrak{p} \in \mathrm{Ass}_A(M/N)$, $\mathfrak{p}B$ is a prime ideal of B or equal to B. Let $N = \bigcap_{\mathfrak{p} \in \mathrm{Ass}_A(M/N)} Q(\mathfrak{p})$ be a reduced primary decomposition of N in M, $Q(\mathfrak{p})$ being $\mathfrak{p}$-primary for all $\mathfrak{p} \in \mathrm{Ass}(M/N)$.*

- *(a) If $\mathfrak{p} \in \mathrm{Ass}(M/N)$ and $\mathfrak{p}B = B$, then $Q(\mathfrak{p}) \otimes_A B = M \otimes_A B$.*
- *(b) If $\mathfrak{p} \in \mathrm{Ass}(M/N)$ and $\mathfrak{p}B$ is prime, then $Q(\mathfrak{p}) \otimes_A B$ is $\mathfrak{p}B$-primary in $M \otimes_A B$.*
- *(c) If Φ is the set of $\mathfrak{p} \in \mathrm{Ass}(M/N)$ such that $\mathfrak{p}B$ is prime, then*

$$N \otimes_A B = \bigcap_{\mathfrak{p} \in \Phi} (Q(\mathfrak{p}) \otimes_A B)$$

and this relation is a reduced primary decomposition of $N \otimes_A B$ in $M \otimes_A B$.

Proof. If $\mathfrak{p}B = B$, Proposition 3.2.28 applied to $M/Q(\mathfrak{p})$ and B shows that

$$\mathrm{Ass}_B((M/Q(\mathfrak{p})) \otimes_A B) = \varnothing$$

and, as B is Noetherian and is a flat A-module, we conclude that $Q(\mathfrak{p}) \otimes_A B = M \otimes_A B$. Assertion (b) follows from Corollary 3.2.30, taking $\mathfrak{P} = \mathfrak{p}B$. Finally the relation $N \otimes_A B = \bigcap_{\mathfrak{p} \in \Phi}(Q(\mathfrak{p}) \otimes_A B)$ follows from the fact that B is a flat A-module. Also, for each $\mathfrak{p} \in \Phi$ we have $\mathrm{Ass}_B((M/Q(\mathfrak{p})) \otimes B) = \{\mathfrak{p}B\}$, whence $\mathfrak{p} = (\mathfrak{p}B)^c$ by Lemma 3.2.27, and $\mathfrak{p}B \neq \mathfrak{q}B$ for two distinct ideals $\mathfrak{p}, \mathfrak{q}$ of Φ. On the other hand, by Proposition 3.2.28,

$$\mathrm{Ass}((M \otimes_A B)/(N \otimes_A B)) = \Phi$$

we conclude from Corollary 3.2.9 that

$$N \otimes_A B = \bigcap_{\mathfrak{p} \in \Phi} (Q(\mathfrak{p}) \otimes_A B)$$

is a reduced primary decomposition. □

Corollary 3.2.32. *In the situation of Proposition 3.2.31, suppose that $\mathfrak{p}B$ is prime for all $\mathfrak{p} \in \mathrm{Ass}_A(M/N)$. Then, if $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ are the minimal elements of $\mathrm{Ass}_A(M/N)$, the $\mathfrak{p}_i B$ are minimal elements of $\mathrm{Ass}_A((M \otimes_A B)/(N \otimes_A B))$.*

Proof. It is easy to see $\mathfrak{p}B$ is a minimal prime if $\mathfrak{p}$ is, and by Proposition 3.2.31 we see $\mathfrak{p}_i B \neq \mathfrak{p}_j B$ when $i \neq j$. □

Example 3.2.33.

(a) Let us take $B = S^{-1}A$, where S is a multiplicative subset of A. If A is Noetherian, the hypotheses of Proposition 3.2.31 are satisfied and we recover a part of Proposition 3.1.11.

(b) Let A be a Noetherian ring, $\mathfrak{a}$ an ideal of A and B the Hausdorff completion of A with respect to the $\mathfrak{a}$-adic topology. Then B is a flat A-module and Proposition 3.2.28 may be applied with $P = B$. But in general the hypotheses of Proposition 3.2.31 are not satisfied for the prime ideals of A.

(c) Let A be a Noetherian ring and B the polynomial algebra $A[X_1, \dots, X_n]$. Then B is Noetherian and is a free A-module and therefore flat. Also, if $\mathfrak{p}$ is a prime ideal of A, $B/\mathfrak{p}B$ is isomorphic to $(A/\mathfrak{p})[X_1, \dots, X_n]$, which is an integral domain, and hence $\mathfrak{p}B$ is prime. The hypotheses of Proposition 3.2.31 are therefore satisfied for every A-module M and every submodule N of M.

3.2.5 Primary decompositions of graded modules

Proposition 3.2.34. *Let Δ be a torsion free commutative group, A a graded ring of type Δ and M a graded A-module of type Δ. Every prime ideal associated with M is graded and is the annihilator of a homogeneous element of M.*

Proof. We know that Δ can be given a total order structure compatible with its group structure. Let $\mathfrak{p}$ be a prime ideal associated with M, the annihilator of an element $x \in M$, and let $(x_i)_{i \in \Delta}$ be the family of homogeneous components of x; let $i(1) < i(2) < \cdots < i(r)$ be the values of i for which $x_i \neq 0$. Consider an element $a \in \mathfrak{p}$ and let $(a_i)_{i \in \Delta}$ be the family of its homogeneous components; we shall prove that $a_i \in \mathfrak{p}$ for all $i \in \Delta$, which will show that $\mathfrak{p}$ is a homogeneous ideal.

We argue by induction on the number of indices i such that $a_i \neq 0$. Our assertion is obvious if this number is 0; if not, let m be the gratest of the indices i for which $a_i \neq 0$; if we prove that $a_m \in \mathfrak{p}$, the induction hypothesis applied to $a - a_m$ will give the conclusion. Now, $ax = 0$; for all $j \in \Delta$, using the fact that the homogeneous component of degree $m + j$ of ax is 0, we obtain $\sum_{i \in \Delta} a_{m-i} x_{j+i} = 0$; we conclude that $a_m x_j$ is a linear combination of the x_i of indices $i > j$. In particular, $a_m x_{i(r)} = 0$, whence, by descending induction on $k < r$, $a_m^{r-k+1} x_{i(k)} = 0$. Then $a_m^r x = 0$, whence $a_m^r \in \mathfrak{p}$ and, as $\mathfrak{p}$ is prime, $a_m \in \mathfrak{p}$.

We now show that $\mathfrak{p}$ is the annihilator of a homogeneous element of M. Let us write $\mathfrak{b}_n = \mathrm{Ann}(x_{i(n)})$ for $1 \leq n \leq r$. For every homogeneous element b of $\mathfrak{p}$ and all n the homogeneous component of bx of degree $i(n) + \deg(b)$ is $bx_{i(n)}$, hence $bx_{i(n)} = 0$ and therefore $b \in \mathfrak{b}_n$; as $\mathfrak{p}$ is generated by its homogeneous elements, $\mathfrak{p} \subseteq \mathfrak{b}_n$. On the other hand, clearly $\bigcap_{n=1}^r \mathfrak{b}_n = \mathfrak{p}$; as $\mathfrak{p}$ is prime, there exists an n such that $\mathfrak{b}_n \subseteq \mathfrak{p}$ (Proposition 1.1.4), whence $b_n = \mathfrak{p} = \mathrm{Ann}(x_{i(n)})$, which completes the proof. □

Corollary 3.2.35. *For every (necessarily homogeneous) prime ideal $\mathfrak{p}$ associated with a graded A-module M, there exists an index $k \in \Delta$ such that the shifted graded A-module $(A/\mathfrak{p})(k)$ is isomorphic to a graded submodule of M.*

Proof. With the notation of the proof of Proposition 3.2.34, consider the homomorphism obtained, by taking quotients, from the homomorphism $a \mapsto ax_{i(n)}$ of A to M; the latter is a graded homomorphism of degree $i(n)$ and hence it gives on taking quotients a graded bijective homomorphism of degree $i(n)$ of $A/\mathfrak{p}$ onto a graded submodule of M. □

Proposition 3.2.36. *Let Δ be a torsion-free commutative group, A a graded Noetherian ring of type Δ and M a graded finitely generated A-module of type Δ. There exists a composition series $(M_i)_{0 \le i \le n}$ consisting of graded submodules of M such that the graded module M_i/M_{i+1} is isomorphic to a shifted graded module $(A/\mathfrak{p}_i)(k_i)$, where $\mathfrak{p}_i$ is a homogeneous prime ideal of A and $k_i \in \Delta$.*

Proof. It is sufficient to retrace the argument of Theorem 3.1.18 taking on this occasion $\mathcal{M}$ to be the set of graded submodules of M with a composition series with the properties of the statement; we conclude using Corollary 3.2.35. □

Proposition 3.2.37. *Let Δ be a torsion-free commutative group, A a graded Noetherian ring of type Δ, $\mathfrak{p}$ a homogeneous ideal of A and M a graded A-module of type Δ not reduced to 0. Suppose that for every homogeneous element a of $\mathfrak{p}$ the homothety of ratio a on M is almost nilpotent and that for every homogeneous element b of $A - \mathfrak{p}$ the homothety of ratio b on M is infective. Then $\mathfrak{p}$ is prime and the submodule $\{0\}$ of M is $\mathfrak{p}$-primary.*

Proof. It suffices to show that $\mathrm{Ass}(M) = \{p\}$ (Proposition 3.2.1). Let $\mathfrak{q}$ be a prime ideal associated with M; it is a homogeneous ideal and it is the annihilator of a homogeneous element $x \neq 0$ of M by Proposition 3.2.34. For every homogcneous element a of $\mathfrak{q}$, $ax = 0$ and hence the homothety of ratio a on M is not injective, whence $a \in \mathfrak{p}$. Conversely, let b be a homogeneous element of $\mathfrak{p}$; there exists an integer $n > 0$ such that $b^n x = 0$, whence $b^n \in \mathrm{Ann}(x) = \mathfrak{q}$ and, as $\mathfrak{q}$ is prime, $b \in \mathfrak{q}$. As $\mathfrak{p}$ and $\mathfrak{q}$ are generated by their respective homogeneous element, $\mathfrak{p} = \mathfrak{q}$, which proves that $\mathrm{Ass}(M) \subseteq \{p\}$. As $M \neq \{0\}$, $\mathrm{Ass}(M) \neq \varnothing$ (Corollary 3.1.4), whence $\mathrm{Ass}(M) = \{\mathfrak{p}\}$. □

Proposition 3.2.38. *Let Δ be a torsion-free commutative group, A a graded Noetherian ring of type Δ and M a graded A-module of type Δ. Let $\mathfrak{p}$ be a prime ideal of A and N a submodule of M which is $\mathfrak{p}$-primary with respect to M.*

(a) The homogeneous ideal $\mathfrak{p}^h$ of A is prime.

(b) The graded submodule N^h of N is $\mathfrak{p}^h$-primary with respect to M.

Proof. We know that the homogeneous elements of $\mathfrak{p}^h$ (resp. N^h) are just the homogeneous elements of $\mathfrak{p}$ (resp. N). Let a be a homogeneous element of $\mathfrak{p}$; if x is a homogeneous element of M, there exists an integer $n > 0$ such that $a^n x \in N$; as $a^n x$ is homogeneous, $a^n x \in N^h$; as every $y \in M$ is the direct sum of a finite number of homogeneous elements, we conclude that there exists an integer $m > 0$ such that $a^m y \in N^h$, so that the homothety with ratio a in M/N^h is almost nilpotent.

Consider now a homogeneous element b of $A - \mathfrak{p}^h$; then $b \notin \mathfrak{p}$ since b is homogeneous. Let x be an element of M such that $bx \in N^h$ and let $(x_i)_{i \in \Delta}$ be the family of homogeneous components of x. As N^h is graded, $bx_i \in N^h$ for all i, hence $bx_i \in N$ and, as $b \notin \mathfrak{p}$, we conclude that $x_i \in N$; as x_i is homogeneous, $x_i \in N^h$, whence $x \in N^h$ and the homothety with ratio b on M/N^h is injective. The proposition then follows from Proposition 3.2.37 applied to $\mathfrak{p}^h$ and M/N^h. □

Proposition 3.2.39. *Let Δ be a torsion-free commutative group, A a graded Noetherian ring of type Δ, M a graded A-module of type M, N a graded submodule of M and $N = \bigcap_{i \in I} Q_i$ a primary decomposition of N in M.*

(a) Let Q_i^h be the graded submodule of Q_i, then the Q_i^h are primary and $N = \bigcap_{i \in I} Q_i^h$.

(b) If the primary decomposition $N = \bigcap_{i \in I} Q_i$ is reduced, so is the primary decomposition $N = \bigcap_{i \in I} Q_i^h$, and for all $i \in I$ the prime ideals corresponding to Q_i and Q_i^h are equal.

(c) If Q_i corresponds to a prime ideal $\mathfrak{p}_i$ which is a minimal element $\operatorname{Ass}(M/N)$, Q_i is a graded submodule of M.

Proof. We have seen (Proposition 3.2.38) that the Q_i^h are primary with respect to M and $N \subseteq Q_i^h \subseteq Q_i$, which proves (a). Proposition 3.2.38 also shows that the prime ideal $\mathfrak{p})i^h$ corresponding to Q_i^h is the largest homogeneous ideal contained in the prime ideal $\mathfrak{p}_i$ corresponding to Q_i. If the decomposition $N = \bigcap_{i \in I} Q_i$ is reduced, $\mathfrak{p}_i \in \operatorname{Ass}(M/N)$ for all i (Proposition 3.2.8), hence $\mathfrak{p}_i$ is a graded ideal (Proposition 3.2.34) and therefore $\mathfrak{p}_i = \mathfrak{p}_i^h$; then $\operatorname{Ass}(M/N) = \bigcup_{i \in I} \{\mathfrak{p}_i\}$ (Proposition 3.2.8), which proves that the decomposition $N = \bigcap_{i \in I} Q_i^h$ is reduced (Proposition 3.2.8). Finally, if $\mathfrak{p}_i$ is a minimal element of $\operatorname{Ass}(M/N)$, then $\mathfrak{p}_i^h = \mathfrak{p}_i$ since $\mathfrak{p}_i$ is homogeneous (Proposition 3.2.34), whence $Q_i^h = Q_i$, by virtue of Proposition 3.2.10. $\square$

Chapter 4

Integral dependence

4.1 Integral elements

Proposition 4.1.1. *Let A be a ring, R an algebra over A and x an element of R. The following properties are equivalent:*

(i) x is a root of a monic polynomial in the polynomial ring $A[X]$.

(ii) The subalgebra $A[x]$ of R is a finitely generated A-module.

(iii) There exists a faithful $A[x]$-module which is a finitely generated A-module.

Proof. The implications (i)⇒(ii)⇒(iii) are clear. Now assume that there exists a faithful $A[x]$-module M which is finitely generated as an A-module. Let ϕ to be multiplication by x and $I = A$ (we have $xM \subseteq M$ since M is an $A[x]$-module). Since M is faithful, we have $x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0$ for suitable $a_i \in A$ in view of **??**, whence (i) holds. □

An element $x \in R$ is called integral over A if it satisfies the equivalent properties of Proposition 4.1.1. A relation of the form $P(x) = 0$, where P is a monic polynomial in $A[X]$ is also called an equation of integral dependence with coefficients in A.

Example 4.1.2.

(a) Let K be a field and R a K-algebra; to say that an element $x \in R$ is integral over K is equivalent to saying that x is a root of a non-constant polynomial in the ring $K[X]$. Generalizing the terminology introduced when R is an extension of K, the elements $x \in R$ which are integral over K are also called the **algebraic elements** of R over K.

(b) The elements of $\mathbb{Q}(i)$ which are integral over $\mathbb{Z}$ are the elements of the form $a + ib$ where $a \in \mathbb{Z}$ and $b \in \mathbb{Z}$ ("Gaussian integers"); the elements of $\mathbb{Q}(\sqrt{5})$ which are integral over $\mathbb{Z}$ are the elements of the form $(a + b\sqrt{5})/2$ where a and b belong to $\mathbb{Z}$.

Proposition 4.1.3. *Let A be a ring, R an algebra over A and x an element of R. For x to be integral over A, it is necessary and sufficient that $A[x]$ be contained in a subalgebra R' of R which is a finitely generated A-module.*

Proof. The condition is obviously necessary by virtue of Proposition 4.1.1. It is also sufficient by virtue of Proposition 4.1.1, for R' is a faithful $A[x]$-module (since it contains the unit element of R). □

Corollary 4.1.4. *Let A be a Noetherian ring, R an A-algebra and x an element of R. For x to be integral over A, it is necessary and sufficient that there exist a finitely generated submodule of R containing $A[x]$.*

Proof. The condition is necessary by virtue of Proposition 4.1.1. It is sufficient for if $A[x]$ is a sub-A-module of a finitely generated A-module, it is itself a finitely generated A-module. □

Let A be a ring. An A-algebra R is called **integral over A** if every element of R is integral over A. Recall that R is called finite over A if R is a finitely generated A-*module*, and finite type if a finitely generated A-*algebra*. It follows from Proposition 4.1.3 that every finite A-algebra is integral.

Example 4.1.5. If M is a finitely generated A-module, the algebra $\mathrm{End}_A(M)$ of endomorphisms of M is integral over A by virtue of Proposition 4.1.1. In particular, for every integer n, the matrix algebra $\mathcal{M}_n(A) = \mathrm{End}_A(A^n)$ is integral (and even finite) over A.

Proposition 4.1.6. *Let A, B be two rings, R an A-algebra, S an B-algebra, and $\rho : A \to B$ and $\tau : R \to S$ two ring homomorphisms such that the diagram*

$$\begin{tikzcd} A \arrow[r, "\rho"] \arrow[d] & B \arrow[d] \\ R \arrow[r, "\tau"] & S \end{tikzcd}$$

is commutative. If an element $x \in R$ is integral over A, then $\tau(x)$ is integral over B.

Proof. If $P(x) = 0$ where $P \in A[X]$, then $\tau(x)$ is a zero of $\rho(P)$, hence $\tau(x)$ is integral over B. □

Corollary 4.1.7. *Let A be a ring, B an A-algebra and C a B-algebra. Then every element $x \in C$ which is integral over A is integral over B.*

Corollary 4.1.8. *Let K be a field, L an extension of K and x, y two elements of L which are conjugate over K. If A is a subring of K and x is integral over A, then y is also integral over A.*

Proof. There exists a K-isomorphism σ of $K(x)$ onto $K(y)$ such that $\sigma(x) = y$ and the elements of A are invariant under σ. □

Proposition 4.1.9. *Let $(R_i)_{1\leq i\leq n}$ be a finite family of A-algebras and let $R = \prod_{i=1}^n R_i$ be their product. For an element $x = (x_i)$ of R to be integral over A, it is necessary and sufficient that each of the x_i be integral over A. For R to be integral over A, it is necessary and sufficient that each of the R_i be integral over A.*

Proof. It is obviously sufficient to prove the first assertion. The condition is necessary by Proposition 4.1.6. Conversely, if each of the x_i is integral over A, the subalgebra $A[x_i]$ of R_i is a finitely generated A-module and hence so is the subalgebra $\prod_{i=1}^n A[x_i]$ of R. As $A[x]$ is contained in this subalgebra, x is integral over A. □

Proposition 4.1.10. *Let A be a ring, B an A-algebra and let $x_1, \dots, x_n$ be integral over A. Then the ring $A[x_1, \dots, x_n]$ is a finitely generated A-module.*

Proof. Proof by induction. For $n = 1$, this is a part of Proposition 4.1.1. Assume $n > 1$, let $A_{n-1} = A[x_1, \dots, x_{n-1}]$; then by the induction hypothesis A_{n-1} is a finitely generated A-module. Moreover, $A_n = A_{n-1}[x_n]$ is a finitely generated A_{n-1}-module by the case $n = 1$, since x_n is integral over A_{n-1}. Hence A_n is finitely generated as an A-module. □

Corollary 4.1.11. *Let A be a ring, R an A-algebra and E a set of elements of R which are integral over A. Then the sub-A-algebra of R generated by E is integral over A.*

Proof. Let B be the A-algebra generated by E in R. Then every element of B belongs to a sub-A-algebra of B generated by a finite subset of E, whence is integral over A by Proposition 4.1.10. □

Corollary 4.1.12. *The set $\overline{A}$ of elements of B which are integral over A is a subring of B containing A.*

Proof. If $x, y \in \overline{A}$ then $A[x, y]$ is a finitely generated A-module by Proposition 4.1.10. Hence $x \pm y$ and xy are integral over A, by (c) of Proposition 4.1.1. □

Proposition 4.1.13. *Let A be a ring and B and R two A-algebras. If R is integral over A, then $R \otimes_A B$ is integral over B.*

Proof. Consider any element $y = \sum_i \otimes b_i$ of $R \otimes_A B$, where the x, belong to R and the b_i to B. As $x_i \otimes a_i = (x_i \otimes 1)a_i$ and the $x_i \otimes 1$ are integral over B by Proposition 4.1.6, so is x. □

Corollary 4.1.14. *Let R be a ring and A, B, C subrings of R such that $A \subseteq B$. If B is integral over A, then $C[B]$ is integral over $C[A]$.*

Proof. The rng $B \otimes_A C[A]$ is integral over $C[A]$ by Proposition 4.1.13 and hence so is the canonical image $C[B]$ of $B \otimes_A C[A]$ in R (considered as an A-algebra) by Proposition 4.1.6. □

Proposition 4.1.15. *Let A be a ring and B an A-algebra. If B integral over A, $\mathfrak{b}$ is an ideal of B and $\mathfrak{a} = \mathfrak{b}^c$, then $B/\mathfrak{b}$ is integral over $A/\mathfrak{a}$.*

Proof. The follows by reducing an integral equation of an element of B mod $\mathfrak{b}$. □

Proposition 4.1.16. *Let A be a ring, B an A-algebra and C a B-algebra. If B is integral over A and C is integral over B, then C is integral over A.*

Proof. It is sufficient to verify that every $x \in C$ is integral over A. Then we have an equation

$$x^n + b_{n-1}x^{n-1} + \cdots + b_0, \quad b_i \in B.$$

The ring $B' = A[b_1, \ldots, b_n]$ is a finitely generated A-module by Proposition 4.1.10, and $B'[X]$ is a finitely generated B'-module (since x is integral over B'). Hence $B'[X]$ is a finitely generated A-module and therefore x is integral over A by (c) of Proposition 4.1.1. □

Corollary 4.1.17. *Let A be a ring and R, S two A-algebras integral over A. Then $R \otimes_A S$ is integral over A.*

Proof. The ring $R \otimes_A S$ is integral over S by Proposition 4.1.13 and hence the conclusion follows from Proposition 4.1.16. □

4.1.1 The integral closure of a ring

Let A be a ring and R a A-algebra. The sub-A-algebra $\overline{A}$ of R consisting of the elements of R integral over A is called the integral closure of A in R. If $\overline{A}$ is equal to the canonical image of A in R, A is called integrally closed in R.

Remark 4.1.18. If A is a field, the integral closure $\overline{A}$ of A in R consists of the elements of R which are algebraic over A. Generalizing the terminology used for field extensions, $\overline{A}$ is then also called the algebraic closure of the field A in the algebra R and A is called algebraically closed in R if $\overline{A} = A$.

If A is an integral domain, the integral closure of A in its field of fractions is called the **integral closure of A**. An integral domain is called **integrally closed** if it is equal to its integral closure.

Proposition 4.1.19. *Let A be a ring and R an A-algebra. The integral closure $\overline{A}$ of A in R is a subring integrally closed in R.*

Proof. The integral closure of $\overline{A}$ in R is integral over A. It is therefore equal to $\overline{A}$. □

Corollary 4.1.20. *The integral closure of an integral domain A is an integrally closed domain.*

Proof. Let K be the field of fractions of A and B the integral closure of A. Clearly K is the field of fractions of B and it is sufficient to apply Proposition 4.1.19 to $R = K$. □

Proposition 4.1.21. *Let R be a ring, $(B_i)_{i \in I}$ a family of subrings of R and for each $i \in I$ let A_i be a subring of B_i. If each A_i is integrally closed in B_i, then $A = \bigcap_i A_i$ is integrally closed in $B = \bigcap_i B_i$.*

Proof. This follows from Corollary 4.1.7. □

Corollary 4.1.22. *Every intersection of a non-empty family of integrally closed subdomains of an integral domain A is an integrally closed domain.*

Proof. It is sufficient to apply Proposition 4.1.21: taking R and the B_i equal to the field of fractions K of A and noting that a subring of K integrally closed in K is a fortiori an integrally closed domain since its field of fractions is contained in K. □

Proposition 4.1.23. *Let A be a ring, $(R_i)_{1 \leq i \leq n}$ a finite family of A-algebras and $\overline{A_i}$ the integral closure of A in R_i. Then the integral closure of A in $R = \prod_{i=1}^n R_i$ is $\prod_{i=1}^n \overline{A_i}$.*

Proof. This is an immediate consequence of Proposition 4.1.9. □

Corollary 4.1.24. *Let A be a reduced Noetherian ring, $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ its distinct minimal prime ideals, K_i the field of fractions of the integral domain $A/\mathfrak{p}_i$ and $\overline{A_i}$ the integral closure of A in K_i. Then the canonical isomorphism of the total ring of fractions $Q(A)$ of A onto $\prod_{i=1}^n K_i$ maps the integral closure of A in $Q(A)$ onto the product ring $\prod_{i=1}^n \overline{A_i}$.*

Proof. This follows from Proposition 3.2.26 and Proposition 4.1.23. □

Corollary 4.1.25. *For a reduced Noetherian ring to be integrally closed in its total ring of fractions, it is necessary and sufficient that it be a direct product of integrally closed (Noetherian) domains.*

4.1.2 Examples of integrally closed domains

Proposition 4.1.26. *A UFD is integrally closed.*

Proof. Let A be a UFD and K be its fraction field. Then an element $x = a/b \in K$ (where a, b are coprime) is integral over A if and only if $x \in A$. In fact, if there is an equation

$$a^n/b^n + c_{n-1}a^{n-1}/b^{n-1} + \cdots + c_1 a/b + c_0 = 0$$

then multiplying b^n on both sides we see $b \mid a$, which is a contradiction. □

Example 4.1.27. Let k be a field and X an indeterminate over k; set $A = k[X^2, X^3] \subseteq B = k[X]$. Then A and B both have the same field of fractions $K = k(X)$. Since B is a UFD, it is integrally closed; but X is integral over A, so that B is the integral closure of A in K.

Note that in this example $A \cong k[X, Y]/(Y^2 - X^3)$. Thus A is the coordinate ring of the plane curve $Y^2 = X^3$, which has a singularity at the origin. The fact that A is not integrally closed is related to the existence of this singularity.

Proposition 4.1.28. *Let A be a ring, R an A-algebra and P and Q monic polynomials in $R[X]$. If the cofficients of PQ are integral over A, then the coefficients of P and Q are integral over A.*

Proof. There exists a ring R' containing R and families of elements (a_i), (b_j) of R' such that in $R'[X]$ we have

$$P(X)=\prod_{i=1}^{n}(X-a_i),\quad Q(X)=\prod_{j=1}^{m}(X-b_j)$$

The coefficients of PQ are integral over A and so belong to the integral closure A' of A in R'. Thus the elements a_i and b_j are integral over A' and hence belong to A'. It the follows that the coefficients of P and Q are integral over A. □

Let A be an integral domain, K its field of fractions and L a K-algebra. Given an element $x\in L$ algebraic over K, the polynomials $P\in K[X]$ such that $P(x)=0$ form a nonzero ideal a of $K[X]$, necessarily principal. There exists a unique monic polynomial which generates $\mathfrak{a}$. Generalizing the tenninology, this monic polynomial will be called the **minimal polynomial** of x over K.

Corollary 4.1.29. *Let A be an integral domain, K its field of fractions and x an element of a K-algebra L. Then x is integral over A if and only if the coefficients of the minimal polynomial P of x over K are integral over A.*

Proof. If all coefficients of the minimal polynomial of x are integral over A, then by Proposition 4.1.16 we see x is integral over A. Conversely, assume that x is integral over A. Then there exists by hypothesis a monic polynomial $Q\in A[X]$ such that $Q(x)=0$. As P divides Q in $K[X]$, it follows from Proposition 4.1.28 that the coefficients of P are integral over A. □

Example 4.1.30. Let A be a UFD in which 2 is a unit and $a\in A$. Then $A[\sqrt{a}]$ is an integrally closed domain if and only if a is squre-free. In fact, let α be a square root of a. Let K be the field of fractions of A; then A is integrally closed in K, so that if $\alpha\in K$ we have $\alpha\in A$ and $A[\alpha]=A$, and the assertion is trivial. If $\alpha\notin K$ then the field of fractions of $A[\alpha]$ is $K(\alpha)$, and every element $\xi\in K(\alpha)$ can be written in a unique way as $\xi=x+y\alpha$ with $x,y\in K$. The minimal polynomial of ξ over K is

$$X^2-2xX+(x^2-y^2a)$$

Since α is integral over A, $\xi\in K(\alpha)$ is integral over $A[\alpha]$ if and only if it is integral over A, and by Corollary 4.1.29, if and only if $2x\in A$ and $x^2-y^2a\in A$.

If so, then by assumption, $2x\in A$ implies $x\in A$. Hence $y^2a\in A$. From this, if some prime p of A divides the denominator of y, we must have $p^2\mid a$, which is impossible if a is square-free. Thus $y\in A$ and $A[\alpha]$ is integrally closed in this case.

Note that if a is not squre-free then we can choose y such that $y^2a\in A$ and $y\notin A$. Thus, in this case $A[\alpha]$ is not integrally closed.

Example 4.1.31. If A is an integrally closed domain and $\mathfrak{p}$ is a prime ideal in A, then the residue ring $A/\mathfrak{p}$ is not integrally closed in general. In fact, any finite integral domain $k[x_1,\dots,x_n]$ over a field k is of the form $A/\mathfrak{p}$, where A is the polynomial ring $k[X_1,\dots,X_n]$. But finite integral domains are not in general integrally closed. In the case $n=2$ the simplest example is the one in which $\mathfrak{p}$ is the principal ideal $(X_1^2-X_2^3)$. In that case, x_1/x_2 does not belong to the ring $k[x_1,x_2]$, but x_1/x_2 is integral over that ring since $(x_1/x_2)^2=x_2$.

Let A be a ring and R an A-algebra. The homomorphism $A\to R$ defining the A-algebra structure on R can be extended uniquely to a homomorphism $A[X]\to R[X]$ of polynomial rings over A and R, leaving X invariant and hence $R[X]$ is given a canonical $A[X]$-algebra structure.

Proposition 4.1.32. *Let A be a ring, R an A-algebra and P a polynomial in $R[X_1,\dots,X_n]$. For P to be integral over $A[X_1,\dots,X_n]$, it is necessary and sufficient that the coefficients of P be integral over A.*

Proof. By considering the polynomials of $R[X_1, \ldots, X_n]$ as polynomials in X_n with coefficients in $R[X_1, \ldots, X_{n-1}]$, we see immediatelythat it is reduced to proving the proposition for $n = 1$. Then let P be a polynomial in $R[X]$. It follows immediately from Proposition 4.1.13 that, if the coefficients of P are in the integral closure B of A in R, the element P, which belongs to $B[X] = B \otimes_A A[X]$, is integral over $A[X]$. Conversely, suppose that P is integral over $A[X]$ and let

$$Q(Y) = Y^m + F_{m-1}Y^{m-1} + \cdots + F_0$$

be a monic polynomial with coefficients $F_i \in A[X]$ with P as a root. Let r be an integer strictly greater than all the degrees of the polynomials P and F_i and let us write $P_1(X) = P(X) - X^r$. Then P_1 is a root of the polynomial

$$Q_1(Y) = Q(Y + X^r) = Y^m + G_{m-1}Y^{m-1} + \cdots + G_0$$

with coefficients in $A[X]$. We may therefore write

$$G_0 = -P_1(P_1^{m-1} + G_{m-1}P_1^{m-2} + \cdots + G_1).$$

Now the choice of r implies that $-P_1$ is a manic polynomial of $R[X]$ and so is $G_0(X) = Q(X^r)$, the degrees of the polynomials $F_i(X)X^{r(m-i)}$ being all smaller than rm for $i \geq 1$. We conclude first of all that the polynomial

$$P_1^{m-1} + G_{m-1}P_1^{m-2} + \cdots + G_1$$

of $R[X]$ is also monic. Moreover, as the coefficients of G_0 belong to A, Proposition 4.1.28 shows that P_1 has coefficients integral over A and the coefficients of P are therefore certainly integral over A. □

Corollary 4.1.33. *Let A be a ring, R an A-algebra and $\overline{A}$ the integral closure of A in R. Then the integral closure of $A[X_1, \ldots, X_n]$ in $R[X_1, \ldots, X_n]$ is equal to $\overline{A}[X_1, \ldots, X_n]$.*

Corollary 4.1.34. *Let A be an integral domain and $\overline{A}$ its integral closure. Then the integral closure of the polynomial ring $A[X_1, \ldots, X_n]$ is $\overline{A}[X_1, \ldots, X_n]$.*

Proof. Arguing by induction on n, the problem is immediately reduced to the case $n = 1$. Let K be the field of fractions of A, which is also that of $\overline{A}$. If an element P of the field of fractions $K(X)$ of $A[X]$ is integral over $A[X]$, it belongs to the polynomial ring $K[X]$, for the latter is a principal ideal domain and hence integrally closed. The corollary then follows from Corollary 4.1.33 applied to $R = K$. □

Corollary 4.1.35. *Let A be an integral domain. For the polynomial ring $A[X_1, \ldots, X_n]$ to be integrally closed, it is necessary and sufiicient that A be integrally closed.*

4.1.3 Completely integrally closed domains

Proposition 4.1.36. *Let A be a ring, K the filed of fraction of A and x an element of K. The following properties are equivalent:*

(i) There exists nonzero $a \in A$ such that $ax^n \in A$.

(ii) All the powers x^n (with $n \geq 0$) are contained in a finitely generated sub-A-module of K.

(iii) The subalgebra $A[X]$ of K is a fractional ideal of K.

Proof. It is clear that (i) and (iii) are equivalent and (iii) implies (ii). Also, if all powers x^n is contained in a finitely generated sub-A-module M of K, then it can be easily seen that there exists $s \in A$ such that $sM \subseteq A$, whence (i) holds. □

An element x of K is called almost integral over A if it satisfies the conditions of Proposition 4.1.36. An integral domain A is called **completely integrally closed** if every almost integral element of K belongs to A. Clearly a completely integrally closed domain is integrally closed. Conversely, Corollary 4.1.4 shows that an integrally closed Noetherian domain is completely integrally closed. If (A_i) is a family of completely integrally closed domains with the same field of fractions K, then $A = \bigcap_i A_i$ is completely integrally closed. For if $x \in K$ is such that for somc non-zero d in A, $dA[x]$ belongs to A, the hypothesis implies that $x \in A_i$ for all i and hence $x \in A$.

Proposition 4.1.37. *Let A be a completely integrally closed domain. Then every polynomial ring $A[X_1, \dots, X_n]$ (resp. every ring of formal power series $A[\![X_1, \dots, X_n]\!]$) is completely integrally closed.*

Proof. By induction on n, it is sufficient to prove that $A[X]$ (resp. $A[\![X]\!]$) is completely integrally closed. Then let P be an element of the field of fractions of $A[X]$ (resp. $A[\![X]\!]$) and suppose that there exists a non-zero, element $Q \in A[X]$ (resp. $Q \in A[\![X]\!]$) such that, $QP^m \in A[X]$ (resp. $QP^m \in A[\![X]\!]$) for every integer $m \geq 0$. If K is the field of fractions of A, then $A[X]$ (resp. $A[\![X]\!]$) is a subring of $K[X]$ (resp. $K[\![X]\!]$) and $K[X]$ (resp. $K[\![X]\!]$) is a principal ideal domain and hence integrally closed and Noetherian and therefore completely integrally closed, therefore we have already seen that $P \in K[X]$ (resp. $P \in K[\![X]\!]$). Write

$$P = \sum_{k=0}^{\infty} a_k X^k, \quad Q = \sum_{k=0}^{\infty} b_k X^k$$

where $a_k \in K$ and $b_k \in A$, and we argue by reductio ad absurdum by supposing that the a_k do not all belong to A. Then there is a least index i such that $a_i \notin A$. If we write $P_1 = \sum_{k=0}^{i-1} a_k X^k \in A[X]$, it follows immediately from the hypothesis that also $Q(P - P_1)^m \in A[X]$ (resp. $Q(P - P_1)^m \in A[\![X]\!]$) for all $m \geq 0$. Let j be the least integer such that $b_j \neq 0$. Then in $Q(P - P_1)^m$ the term of least degree with a nonzero coefficient is $b_j a_i^m X^{j+mi}$ and hence $b_j a_i^m \in A$ for all $m \geq 0$. But as A is completely integrally closed this implies $a_i \in A$, contrary to the hypothesis. □

Proposition 4.1.38. *Let A be a filtered ring whose filtration is exhaustive and such that every principal ideal of A is closed under the topology defined by the filtration. If the associated graded ring $\mathrm{gr}(A)$ is a completely integrally closed domain, then A is a completely integrally closed domain.*

Proof. Let $(A_n)_{n \in \mathbb{Z}}$ be the filtration defined on A. As $\bigcap_n A_n$ is the closure of the ideal (0), the hypothesis implies first that the filtration (A_n) is separated and, as $\mathrm{gr}(A)$ is an integral domain, so then is A (Corollary 2.2.13). Let $x = a/b$ be an element of the field of fractions K of A for which there exists a nonzero element d of A such that $dx^n \in A$ for all n. We must prove that $a \in Ab$ and, as by hypothesis the ideal Ab is closed, it is sufficient to show that, for all $n \in \mathbb{Z}$, $a \in A_b + A_n$. As the filtration of A is exhaustive, there exists an integer $p \in \mathbb{Z}$ such that $a \in Ab + A_p$. It will therefore suffice to prove that the relation $a \in Ab + A_m$ implies $a \in Ab + A_{m+1}$ for any $m \in \mathbb{N}$.

Suppose then $a = by + z$ where $y \in A$ and $z \in A_m$. Since by hypothesis $dx^n \in A$ for all n, we have $d(x - y)^n \in A$ for all n. In other words, $dz^n = b^n t_n$ where $t_n \in A$ for all n. We may assume that $z \neq 0$. Let v denote the order function on A and let us write $v(z) = n_0$. If $\bar{x}$ denote the canonical image of an element $x \in A$ in $\mathrm{gr}(A)$, then since $b \neq 0$, we deduce that $\bar{d}(\bar{z}/\bar{b})^n \in \mathrm{gr}(A)$ for all $n \in \mathbb{N}$. Since $\mathrm{gr}(A)$ is completely integrally closed, there exists an element $\bar{s} \in \mathrm{gr}(A)$ such that $\bar{z} = \bar{b}\bar{s}$. Decomposing $\bar{s}$ into a sum of homogeneous elements, it is further seen (since $\bar{z}$ and $\bar{b}$ are homogeneous) that $\bar{s}$ may be assumed to be homogeneous and that is the image of an element $s \in A$. Then $v(bs) = v(z) = n_0$ and $z \equiv bs \bmod A_{n_0+1}$. As we have $n_0 \geq m$, a fortiori $z \equiv bs \bmod A_{m+1}$, hence $a \equiv b(y + s) \bmod A_{m+1}$ and the claim follows. □

4.1.4 The integral closure of a ring of fractions

Proposition 4.1.39. *Let A be a ring, R an A-algebra, $\overline{A}$ the integral closure of A in R and S a multiplicative subset of A. Then $S^{-1}\overline{A}$ is the integral closure of $S^{-1}A$ in $S^{-1}B$.*

Proof. Let $b/s \in S^{-1}\overline{A}$, where b is integral over A, say

$$b^n + a_{n-1}b^{n-1} + \cdots + a_0 = 0, \quad a_i \in A.$$

Then the equation of b gives an equation

$$(b/s)^n + (a_{n-1}/s)(b/s)^{n-1} + \cdots + a_0/s^n = 0$$

Hence $S^{-1}\overline{A}$ is integral over $S^{-1}A$. Conversely, if $b/s \in S^{-1}B$ is integral over $S^{-1}A$, then we have an equation of the form

$$(b/s)^n + (a_{n-1}/s_{n-1})(b/s)^{n-1} + \cdots + a_0/s_0 = 0$$

where $a_i \in A$, $s_i \in S$. Let $t = s_1 \cdots s_n$, and multiply this equation by $(st)^n$ throughout. Then it becomes an equation of integral dependence for bt over A. Hence $bt \in \overline{A}$ and therefore $b/s = bt/st \in S^{-1}\overline{A}$. □

Corollary 4.1.40. *Let A be an integral domain, $\overline{A}$ its integral closure and S a multiplicative subset such that $0 \notin S$. Then the integral closure of $S^{-1}A$ is $S^{-1}\overline{A}$.*

Proof. The field of fractions K of A is also the field of fractions of $S^{-1}A$ since $0 \notin S$, so Proposition 4.1.39 is then applied to K. □

Corollary 4.1.41. *Let A be an integral domain, K its field of fractions, R an algebra over K and B the integral closure of A in R. The elements of R which are algebraic over K are the elements of the form $a^{-1}b$ where $b \in B$ and $a \in A$. If L is the algebraic closure of K in R, then there exists a basis of L over K contained in B.*

Proof. The first assertion follows from Proposition 4.1.39 applied in the case $S = A - \{0\}$. If $(x_i)_{i\in I}$ is a basis of L over K, then there exists for all $i \in I$ an element a_i of A such that $a_ix_i \in B$. Then $(a_ix_i)_{i\in I}$ is also a basis of L over K. □

Proposition 4.1.42. *Let A be an integral domain. Then the following are equivalent:*

(i) *A is integrally closed.*

(ii) *$A_{\mathfrak{p}}$ is integrally closed for each prime ideal $\mathfrak{p}$.*

(iii) *$A_{\mathfrak{m}}$ is integrally closed for each maximal ideal $\mathfrak{m}$.*

Proof. It follows from Proposition 4.1.39 that the condition (iii) is necessary. The condition is sufficient, for $A = \bigcap_{\mathfrak{m}} A_{\mathfrak{m}}$ by Proposition 1.3.28 and it is sufficient to apply the Proposition 4.1.21. □

Corollary 4.1.43. *Let A be an integral domain, K its field of fractions and S a multiplicative subset of A such that $0 \notin S$.*

(a) *Let B be a subring of K which is integral over A and let $\mathfrak{f}$ be the annihilator of the A-module B/A (called the* ***conductor*** *of A in B). Then $S^{-1}\mathfrak{f}$ is contained in the conductor of $S^{-1}A$ in $S^{-1}B$ and is equal to it if B is a finitely generated A-module.*

(b) *Let $\overline{A}$ be the integral closure of A. For $S^{-1}A$ to be integrally closed, it is sufficient that the conductor $\mathfrak{f}$ of A in $\overline{A}$ meet S. This condition is also necessary if $\overline{A}$ is a finitely generated A-module.*

Proof. As $\mathfrak{f}B \subseteq A$, $(S^{-1}\mathfrak{f})(S^{-1}B) \subseteq S^{-1}A$ and hence $S^{-1}\mathfrak{f}$ is contained in $\operatorname{Ann}(S^{-1}B/S^{-1}A)$. If B is a finitely generated A-module, the equation $S^{-1}\mathfrak{f} = \operatorname{Ann}(S^{-1}B/S^{-1}A)$ is a special case of Proposition 1.2.33, $S^{-1}B/S^{-1}A$ being canonically identified with $S^{-1}(B/A)$. This proves (a).

By Proposition 4.1.39, $S^{-1}\overline{A}$ is the integral closure of $S^{-1}A$. As the relations $\mathfrak{f} \cap S \neq \varnothing$ and $S^{-1}\mathfrak{f} = S^{-1}A$ are equivalent, (b) is an immediate consequence of (a). □

Corollary 4.1.44. *Let A be an integral domain, $\overline{A}$ its integral closure and $\mathfrak{f}$ the conductor of A in $\overline{A}$. Suppose that $\overline{A}$ is a finitely generated A-module. Then the prime ideals $\mathfrak{p}$ of A such that $A_{\mathfrak{p}}$ is not integrally closed are those which contain $\mathfrak{f}$.*

4.1.5 Norms and traces of integral elements

Proposition 4.1.45. *Let A be a ring, B be an A-algebra, and T be a square matrix of order n over B. The following properties are equivalent:*

(i) T is integral over A;

(ii) There exists a finitely generated sub-A-module M of B^n such that $T(M) \subseteq M$ and M is a system of generators of the B-module B^n.

(iii) The coefficients of the characteristic polynomial of T are integral over A.

Proof. If $\chi(X)$ is the characteristic polynomial of X, the Cayley-Hamilton Theorem shows that $\chi(T) = 0$ and, as $\chi(X)$ is a monic polynomial, (iii) implies (i) by the transitivity of integrality. Suppose in the second place that (i) holds. If $e_1, \dots, e_n$ is the canonical basis of B^n, the sub-A-module M of B generated by the $T^k e_i$ is a finitely generated A-module, since the A-algebra $A[T]$ is a finitely generated A-module. As M contains the e_i, it is seen that (i) implies (ii). The converse is a consequence of Proposition 4.1.1.

Finally let us prove that (i) implies (iii). As T is integral over A and a fortiori over the polynomial ring $A[X]$, the polynomial $X - T$ is integral over $A[X]$ by Proposition 4.1.32. By Proposition 4.1.32, the problem is seen to reduce (by replacing T by $X - T$ and A by $A[X]$) to proving that, if T is integral over A, then $\det(T)$ is an element of B which is integral over A. Now, we have seen above that the endomorphism ϕ of B^n defined by the matrix T preserves a finitely generated sub-A-module M containing the e_i. Then $\bigwedge^n M$ is a finitely generated sub-A-module in $\bigwedge^n B^n$ containing $e_1 \wedge \cdots \wedge e_n$ and which is stable under $\wedge^n \phi$, in other words under the homothety of ratio $\det(T)$. As $e_1 \wedge \cdots \wedge e_n$ generates $\bigwedge^n B^n$, **??** then proves that $\det(T)$ is integral over A. □

Corollary 4.1.46. *Let A be an integral domain, K its field of fractions and L a finite dimensional K-algebra. If $x \in L$ is integral over A, then the coefieients of the characteristic polynomial of x over K are integral over A.*

Proof. Let h_x be the homothety of ratio x on L. Then by definition h_x is integral over A, and the characteristic polynomial of h_x is that of x over K. □

Corollary 4.1.47. *Let A be an integral domain, K its field of fractions and L a finite dimensional K-algebra. If $x \in L$ is integral over A, then $N_{L/K}(x)$ and $\operatorname{tr}_{L/K}(x)$ are integral over A.*

Proposition 4.1.48. *If L/K is a finite separable extension, then $(x, y) = \operatorname{tr}_{L/K}(xy)$ is a nondegenerate bilinear form on L.*

Proof. Let θ be a primitive element for L, so that $L = K(\theta)$. Then the bilinear form $(x, y) \mapsto \operatorname{tr}_{L/K}(xy)$ has matrix $G = (\operatorname{tr}_{L/K}(\theta^{i-1}\theta^{j-1}))$ under the basis $1, \theta, \dots, \theta^{n-1},$. It is nondegenerate because, if $\theta_1 = \theta, \dots, \theta_n$ are the K-conjugates of θ, then

$$\det(G) = \prod_{i<j} (\theta_i - \theta_j) \neq 0.$$

This proves the claim. □

Theorem 4.1.49. *Let A be an integrally closed domain, K its quotient field, L a finite-dimensional separable K-algebra, and B the integral closure of A in L. There exists a basis $\{x_1, \dots, x_n\}$ of L over K such that B is contained in the A-module $\sum_i Ax_i$.*

The proposition will follow from the following more precise lemma:

Lemma 4.1.50. *Under the hypotheses of Theorem 4.1.49, let $(w_1, \dots, w_n)$ be a basis of L over K contained in B (Corollary 4.1.41); then there is a unique dual basis $(w_1^*, \dots, w_n^*)$ of L over K with respect to the trace form $\mathrm{tr}_{L/K}$; if $d = D_{L/K}(w_1, \dots, w_n)$ is the discriminant of the basis $(w_1, \dots, w_n)$, then $d \neq 0$ and*

$$\sum_{i=1}^{n} Aw_i \subseteq B \subseteq \sum_{i=1}^{n} Aw_i^* \subseteq d^{-1}\Big(\sum_{i=1}^{n} Aw_i\Big). \tag{4.1.1}$$

In particular, if d is invertible in A, B is a free A-module with basis $(w_1, \dots, w_n)$.

Proof. As L is a separable K-algebra, $d \neq 0$ by (A, IX, §2, proposition 5) and the K-bilinear form $\mathrm{tr}_{L/K}$ is non-degenerate. This shows the existence and uniqueness of the dual basis (w_i^*). This being so, the first inclusion of (4.1.1) is obvious. Let x be an element of B; let us write $x = \sum_{i=1}^{n} \xi_i w_i^*$ where $\xi_i \in K$; for all i, we have $\xi_i = \mathrm{tr}_{L/K}(xw_i)$, so ξ_i is integral over A (Corollary 4.1.47) and, as A is integrally closed, $\xi_i \in A$ for all i; this shows the second inclusion in (4.1.1). Finally, let us write $w_j^* = \sum_{i=1}^{n} \alpha_{ji} w_i$ where $\alpha_{ji} \in K$; then $\sum_{i=1}^{n} \alpha_{ji}\mathrm{tr}_{L/K}(w_i w_k) = \delta_{jk}$ for all j and k; Cramer's formulae show that the α_{ji} belong to $d^{-1}A$, whence the third inclusion (4.1.1). The last assertion follows immediate from (4.1.1), which in this case gives $B = \sum_{i=1}^{n} Aw_i$. □

Proof. We first notice that $L = S^{-1}B$, where $S = A - \{0\}$; that is, given any element x of L there exists a non-zero element s of A such that $sx \in B$: in fact, if $p(X) = X^n + a_{n-1}X^{n-1} + \cdots + a_0$ is the minimal polynomial of x over K, and if we take a common denominator s in A such that $sc_i = a_i \in A$, then we have

$$(sx)^n + a_{n-1}(sx)^{n-1} + \cdots + s^{n-1}a_0 = 0$$

and sx is integral over A. It follows from this observation that there exists a basis $\{u_1, \dots, u_n\}$ of L over K such that $u \in B$ for every i. Let $\{v_1, \dots, v_n\}$ be the dual basis of $\{u_1, \dots, u_n\}$ under the bilinear form $\mathrm{tr}(xy)$. If an element $x = \sum_j a_j x_j$ is in B, we have $xu_i \in B$ for every i, whence $\mathrm{tr}(xu_i) \in A$ (A is integrally closed). Since $\mathrm{tr}(xu_i) = \sum_i a_i \mathrm{tr}(u_i v_j) = a_i$, we then have $B \subseteq \sum_j Av_j$. □

Corollary 4.1.51. *The assumptions being the same as in Theorem 4.1.49, let us furthermore assume that the ring A is Noetherian. Then B is a finite A-module and is a Noetherian ring.*

Proof. In fact, B is a submodule of the finite A-module $\sum_i Ax_i$, and is therefore a finite A-module. Thus B satisfies the a.c.c. as an A-module, and a fortiori satisfies the a.c.c. as an B-module. That is, B is Noetherian. □

Corollary 4.1.52. *The assumptions being the same as in Theorem 4.1.49, let us furthermore assume that A is a PID. Then there exists a basis $\{y_i\}$ of L over K such that $B = \sum_i Ay_i$.*

Proof. It was just shown that B is contained in an A-module generated by n elements x. Hence B has a basis consisting of n elements $\{y_i\}$, Since $L = S^{-1}B$, the set $\{y_i\}$ is necessarily also a basis of L over K. □

Corollary 4.1.52 is of particular importance for the case in which A is either the ring $\mathbb{Z}$ of rational integers, L being then an algebraic number field, or a polynomial ring $k[X]$ in one variable over a field k, L being then a field of algebraic functions of one variable. In the first case, the elements of L which are integral over $\mathbb{Z}$ are called the **algebraic integers** of the number field L; in the second case, the elements of L which are integral over $k[X]$ are called the **integral functions** of the function field L (with respect to the element X). Corollary 4.1.52 shows that these algebraic integers (or integral functions) are the linear combinations, with ordinary integral coefficients (or with coefficients in $k[X]$), of $n = [L : K]$ linearly independent algebraic integers y_i. Such a basis $\{y_i\}$ of L over the rational field (or over the rational function field $k(X)$) is called an **integral basis** of L.

Example 4.1.53. Let X be an indeterminate over a field k of characteristic $\neq 2$, and p an irreducible polynomial over k. Let $A = k[X]$ and $K = k(X)$, then the function field $L = k(X, Y)$ admits $\{1, Y\}$ as an integral basis (with respect to A). In fact, in the first place 1 and Y are integral over A. Furthermore, let f be an element of L which is integral over A. We write $f = a(X) + b(X)Y$ with $a, b \in k(X)$. Then the trace $2a(X)$ and the norm $a(X)^2 - b(X)^2 p(X)$ of f over $k(X)$ belong to $k[X]$, whence both $a(X)$ and $b(X)$ are polynomials, since otherwise $p(X)$ would be divisible by the square of the denominator of $b(X)$. Consequently, the integral closure of $A = k[X]$ in $L = k(X, Y)$ is the ring $k[X, Y]$.

Proposition 4.1.54. *Let k be a field, L a separable extension of k and A an integrally closed k-algebra. If the ring $L \otimes_k A$ is an integral domain, it is integrally closed.*

Proof. Let K be the field of fractions of A; as k is a field, $L \otimes_k A$ is canonically identified with a sub-k-algebra of $L \otimes_k K$ and L and A with sub-k-algebras of $L \otimes_k A$. Moreover, since an element $s \neq 0$ of A is not a divisor of 0 in A, $1 \otimes s$ is not a divisor of zero in $L \otimes_k A$ since L is flat over k; identifying s with $1 \otimes s$, it is therefore seen that, if $S = A \setminus \{0\}$, $L \otimes_k K$ is identified with $S^{-1}(L \otimes_k K)$; as $L \otimes_k A$ is assumed to be an integral domain, $L \otimes_k K$ is thus identified with a subring of the field of fractions Ω of $L \otimes_k A$.

First suppose that L is a *finite* extension of k; then $L \otimes_k K$ is an algebra of finite rank over K and by hypothesis has no divisor of 0; hence it is a field (it is sandwiched by K and Ω) and it is in this case the field of fraction Ω of $L \otimes A$. Let $(w_1, \dots, w_n)$ be a basis of L over k, which is therefore also a basis of $L \otimes_k K$ over K. There exists a basis $(w_1^*, \dots, w_n^*)$ of L such that $\mathrm{tr}_{L/k}(w_i w_j^*) = \delta_{ij}$ (Lemma 4.1.50); every $z \in L \otimes_k K$ may be written uniquely $z = \sum_{i=1}^n a_i w_i$ where $a_i \in K$; then

$$\mathrm{tr}_{(L\otimes K)/K}(z w_j^*) = \sum_{i=1}^n a_i \mathrm{tr}_{(L\otimes K)/K}(w_i w_j^*)$$

and as in L the traces $\mathrm{tr}_{(L\otimes K)/K}$ and $\mathrm{tr}_{L/K}$ coincide, finally $\mathrm{tr}_{(L\otimes K)/K}(z w_j^*) = a_j$ for each j. Note on the other hand that the elements of L are integral over k and hence also over A; therefore $L \otimes_k A$ is integral over A (Corollary 4.1.17). This being so, suppose that $z \in L \otimes_k K$ is integral over $L \otimes_k A$; then z is integral over A, hence so is $z w_j^*$ and therefore also $a_j = \mathrm{tr}_{L/K}(z w_j^*)$ for all j (Corollary 4.1.47). As A is integrally closed, $a_j \in A$ for all j and hence $z \in L \otimes_k A$, which proves the proposition in this case.

Suppose now that L is a finitely generated separable extension of k; then there exists a separating transcendence basis $(x_1, \dots, x_d)$ of L over k; as L and K are algebraically disjoint over k in the field Ω, the x_i are algebraically independent over K; hence $A[x_1, \dots, x_d]$ is integrally closed (Corollary 4.1.34). Let T be the set of nonzero elements of the ring $R = k[x_1, \dots, x_d] \subseteq L$, so that the field $k_1 = k(x_1, \dots, x_d) \subseteq L$ is equal to $T^{-1}R$; then

$$k_1 \otimes_k A = (T^{-1}R) \otimes_k A = T^{-1}R \otimes_R (R \otimes_k A) = T^{-1}(R \otimes_k A) = T^{-1}A[x_1, \dots, x_d]$$

by the associativity of the tensor product, hence this domain is integrally closed (Proposition 4.1.39). But $L \otimes_k A$ is identified with $L \otimes_{k_1} (k_1 \otimes_k A)$ and by definition L is a finite separable extension of k_1; it follows therefore from the preceding argument that $L \otimes_k A$ is integrally closed.

In the general case, if z is an element of Ω which is integral over $L \otimes_k A$, it satisfies a relation ofthe form $z^n + b_{n-1}z^{n-1} + \cdots + b_0 = 0$, where the b_i belong to $L \otimes_k A$; then there exists a finitely generated sub-extension L' of L over k such that the b_i belong to $L' \otimes_k A$ for all i and z to $L' \otimes_k K$. Then it follows from the previous argument that $z \in L' \otimes_k A$ and hence $L \otimes_k A$ is integrally closed. □

Remark 4.1.55. Let V be an affine irreducible algebraic variety over a field k and A the ring of functions regular on V defined over k; if A is integrally closed, V is called normal over k; Proposition 4.1.54 shows that, if V is normal over k, it remains normal over every separable extension L of k.

Corollary 4.1.56. *Let k be a field and A and B two integrally closed k-algebras. Suppose that the ring $A \otimes_k B$ is an integral domain and that the fields of fractions K and L of A and B respectively are separable over k. Then the ring $A \otimes_k B$ is an integrally closed domain.*

Proof. As A and B are identified with subalgebras of $A \otimes_k B$, K and L are identified with subfields of the field of fractions Ω of $A \otimes_k B$ which are linearly disjoint over k (A, V, §2, no.3, proposition 5). It then follows from Proposition 4.1.54 that $A \otimes_k L$ and $K \otimes_k B$ are integrally closed domains; as their intersection is $A \otimes_k B$ (**??**), $A \otimes_k B$ is an integrally closed domain (Proposition 4.1.21). □

4.1.6 Integers over a graded ring

All the graduations considered in this part are of type $\mathbb{Z}$; if A is a graded ring and $i \in \mathbb{Z}$, A_i denotes the set of homogeneous elements of degree i of the ring A. Let A be a graded ring and B a graded A-algebra. Let x be a homogeneous element of B which is integral over A; then there is a relation

$$x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0 \quad a_i \in A.$$

Let $m = \deg(x)$ and let a_i' be the homogeneous componenet of degree $m(n-i)$ of a_i for each i; then obviously

$$x^n + a_{n-1}'x^{n-1} + \cdots + a_0' = 0 \tag{4.1.2}$$

in other words x satisfies an equation of integral dependence with homogeneous coefficients.

Let $A[X, X^{-1}]$ denote the ring of fractions $S^{-1}A[X]$ of the polynomial ring $A[X]$ in one indeterminate, S being the multiplicative subset of $A[X]$ consisting of the powers X^n of X; as X is not a divisor of 0 in $A[X]$ it is immediate that the X^i's form a basis over A of the A-module $A[X, X^{-1}]$. For every element $a \in A$ with homogeneous components $(a_i)_{i\in\mathbb{Z}}$, we write

$$j_A(a) = \sum_{i\in\mathbb{Z}} a_i X^i \in A[X, X^{-1}]$$

it is immediate that $j_A : A \to A[X, X^{-1}]$ is an injective ring homomorphism.

Proposition 4.1.57. *Let $A = \bigoplus_{i\in\mathbb{Z}} A_i$ be a graded ring and B a graded A-algebra. Then the set R of elements of B integral over A is a graded subalgebra of B. Moreover, if $A_i = 0$ for $i < 0$ and R is a reduced ring, then $R_i = 0$ for $i < 0$.*

Proof. The diagram

$$\begin{tikzcd} A \arrow[r, "\rho"] \arrow[d, "j_A"] & B \arrow[d, "j_B"] \\ A[X, X^{-1}] \arrow[r, "\tilde{\rho}"] & B[X, X^{-1}] \end{tikzcd}$$

(where ρ is the homomorphism defining the A-algebra structure on B and $\tilde{\rho}$ the homomorphism canonically derived from it) is commutative, as is immediately verified from the definition. Let x be an element of B integral over A; then $j_B(x)$ is integral over $A[X, X^{-1}]$ (Proposition 4.1.6) and it therefore follows from Proposition 4.1.39 that there exists an integer $m > 0$ such that $X^m j_B(x)$ is an element of $B[X]$ integral over $A[X]$. We then deduce from Proposition 4.1.32 that the coefficients of the polynomial $X^m j_B(x)$ are integral over A; as these coefficients are by definition the homogeneous components of x, it is seen that these are integral over A, which proves that R is a graded subalgebra of B.

Suppose now that $x \in R_i$ where $i < 0$; the remark at the beginning of this part shows that x satisfies an equation of the form (4.1.2) where $a'_k \in A_{ki}$ for $1 \leq k \leq n$. If $A_j = 0$ for $j < 0$, then $x^n = 0$ and if B is a reduced ring we conclude that $x = 0$ and hence $R_i = 0$ for all $i < 0$ in this case. □

Recall that, if $A = \bigoplus_{i \in \mathbb{Z}} A_i$ is a graded ring and S is a multiplicative subset of A consisting of homogeneous elements, a graded ring structure is defined on $S^{-1}A$ by taking the set $(S^{-1}A)_i$ of homogeneous elements of degree i to be the set of elements of the form a/s, where $a \in A$ and $s \in S$ are homogeneous and such that $\deg(a) - \deg(s) = i$.

Lemma 4.1.58. *Let $A = \bigoplus_{i \in \mathbb{Z}} A_i$ be a graded integral domain and S the set of nonzero homogeneous elements of A.*

(a) Every nonzero homogeneous element of $S^{-1}A$ is invertible, the ring $K = (S^{-1}A)_0$ is a field and the set of $i \in \mathbb{Z}$ such that $(S^{-1}A)_i \neq 0$ is a subgroup $q\mathbb{Z}$ of $\mathbb{Z}$ (where $q \neq 0$).

(b) Suppose that $q \geq 1$ and let t be a non-zero element of $(S^{-1}A)_q$. Then the K-homomorphism f of the polynomial ring $K[X]$ to $S^{-1}A$ which maps X to t extends to an isomorphism of $K[X, X^{-1}]$ onto $S^{-1}A$ and $S^{-1}A$ is integrally closed.

Proof. The assertions in (a) follow immediately from the definitions and the hypothesis that A is an integral domain, for if a/s and b/t are two nonzero homogeneous elements of $S^{-1}A$ of degrees i and j, ab/st is a nonzero homogeneous element and of degree $i + j$. To show (b), we note that since t is invertible in $S^{-1}A$ the homomorphism f extends in a unique way to a homomorphism $\tilde{f} : K[X, X^{-1}] \to S^{-1}A$ and necessarily $\tilde{f}(X^{-1}) = t^{-1}$. On the other hand, by definition of q, every nonzero homogeneous element of $S^{-1}A$ is of degree qn ($n \in \mathbb{Z}$) and hence can be written uniquely in the form λt^n where $\lambda \in K$ (since $S^{-1}A$ is an integral domain); hence $\tilde{f}$ is bijective. Finally, we know that $K[X]$ is integrally closed and hence so is $K[X, X^{-1}]$ (Proposition 4.1.39), which completes the proof of the Lemma. □

Proposition 4.1.59. *Let $A = \bigoplus_{i \in \mathbb{Z}} A_i$ be a graded integral domain and S the set of nonzero homogeneous elements of A. The integral closure B of A is then a graded subring of $S^{-1}A$. If further $A_i = 0$ for $i < 0$, then $B_i = 0$ for $i < 0$.*

Proof. If $A = A_0$, the proposition is trivial. Otherwise we may apply Lemma 4.1.58; the ring $S^{-1}A$ is an integrally closed domain and therefore $B \subseteq S^{-1}A$; as $S^{-1}A$ is graded, so is B by Proposition 4.1.57; the latter assertion also follows from Proposition 4.1.57. □

Corollary 4.1.60. *With the hypotheses and notation of Proposition 4.1.59, if every homogeneous element of $S^{-1}A$ which is integral over A belongs to A, then A is integrally closed.*

Proof. Then $B_i \subseteq A$ for all $i \in \mathbb{Z}$, hence $B = A$. □

Corollary 4.1.61. *If $A = \bigoplus_{i \in \mathbb{Z}} A_i$ is an integrally closed graded domain, the domain A_0 is integrally closed.*

Proof. The field of fractions K_0 of A_0 is identified (in the notation of Proposition 4.1.59) with a subring of the ring of homogeneous elements of degree 0 of $S^{-1}A$; every element of K_0 integral over A_0 (and a fortiori over A) belongs therefore by hypothesis to A_0. $\square$

Corollary 4.1.62. *Let $A = \bigoplus_{i \in \mathbb{Z}} A_i$ be an integrally closed graded domain. Then, for every integer $d > 0$, the ring $A^{(d)}$ is an integrally closed domain.*

Proof. Let U be the set of nonzero homogeneous elements of $A^{(d)}$ and let x be a homogencous element of $U^{-1}A^{(d)}$ integral over $A^{(d)}$ and hence over A; as $x \in S^{-1}A$, x belongs to A by hypothesis; as its degree is divisible by d, it belongs to $A^{(d)}$ and it then follows from Corollary 4.1.60 that $A^{(d)}$ is integrally closed. $\square$

4.1.7 The lift of prime ideals

Let A, B be two rings and $\rho : A \to B$ a ring homomorphism. An ideal $\mathfrak{b}$ of B is said to **lie over** an ideal $\mathfrak{a}$ of A if $\mathfrak{a} = \rho^{-1}(\mathfrak{b})$.

To say that a prime ideal $\mathfrak{P}$ of B lies over an ideal $\mathfrak{p}$ of A therefore means that $\mathfrak{p}$ is the image of $\mathfrak{P}$ under the continuous map $f^* : \operatorname{Spec}(B) \to \operatorname{Spec}(A)$ associated with f. Note that for there to exist an ideal of B lying over the ideal (0) of A, it is necessary and sufficient that $\rho : A \to B$ be injective.

Let $\mathfrak{a}$ be an ideal of A. By taking quotients the homomorphism ρ gives a homomorphism $\bar{\rho} : A/\mathfrak{a} \to B/\mathfrak{b}$. To say that $\mathfrak{b}$ is an ideal of B lying over $\mathfrak{a}$ is equivalent to saying that $\mathfrak{a}B \subseteq \mathfrak{b}$ and that $\mathfrak{b}/\mathfrak{a}A$ is an ideal of $B/\mathfrak{a}B$ lying over (0).

Lemma 4.1.63. *Let $\rho : A \to B$ be a ring homomorphism, S a multiplicative subset of A, i_A^S and i_B^S the canonical homomorphisms, so that there is a commutative diagram*

$$\begin{tikzcd} A \arrow[r, "\rho"] \arrow[d, "i_A^S"] & B \arrow[d, "i_B^S"] \\ S^{-1}A \arrow[r, "S^{-1}\rho"] & S^{-1}B \end{tikzcd}$$

Let $\mathfrak{p}$ be a prime ideal of A such that $\mathfrak{p} \cap S = \varnothing$. Then $\mathfrak{b} \mapsto S^{-1}\mathfrak{b}$ is a bijective map of the set of ideals of B lying over $\mathfrak{p}$ and saturated with S onto the set of ideals of $S^{-1}B$ lying over $S^{-1}\mathfrak{p}$. In particular, $\mathfrak{P} \mapsto S^{-1}\mathfrak{P}$ is a bijection on the set of prime ideals of B lying over $\mathfrak{p}$ onto the set of prime ideals of $S^{-1}B$ lying over $S^{-1}\mathfrak{p}$.

Proof. This follows from Proposition 1.2.37 and Proposition 1.2.28. $\square$

Proposition 4.1.64. *Let $A \subseteq B$ be integral domains, B integral over A. Then B is a field if and only if A is a field.*

Proof. IfA is a field, then, for all nonzero y in B, $A[y]$ is by hypothesis a finitely generated A-module. As $A[y]$ is an integral domain, it is a field and a fortiori y is invertible in B and hence B is a field. Conversely, suppose that B is a field and let x be nonzero in A. As $x^{-1} \in B$ and B is integral over A, there is an equation of integral dependence

$$x^{-n} + a_{n-1}z^{-(n-1)} + \cdots + a_0 = 0$$

where the $a_i \in A$; now this relation shows that

$$-z^{-1} = a_{n-1} + a_{n-2}z + \cdots + a_0 z^{n-1} \in A,$$

hence A is a field. $\square$

Corollary 4.1.65. *Let $\rho : A \to B$ be a ring homomorphism such that B is integral over A, $\mathfrak{P}$ a prime ideal of B and $\mathfrak{p} = f^{-1}(\mathfrak{P})$. For $\mathfrak{p}$ to be maximal, it is necessary and sufficient that $\mathfrak{P}$ is maximal.*

Proof. Let $\bar{\rho} : A/\mathfrak{p} \to B/\mathfrak{P}$ be the homomorphism derived from f by taking quotients. Then $A/\mathfrak{p}$ and $B/\mathfrak{P}$ are integral domains and $B/\mathfrak{P}$ is integral over $A/\mathfrak{p}$. To say that $\mathfrak{p}$ (resp. $\mathfrak{P}$) is maximal means that $A/\mathfrak{p}$ (resp. $B/\mathfrak{P}$) is a field. The corollary then follows from Proposition 4.1.64. □

Corollary 4.1.66. *Let $\rho : A \to B$ a ring homomorphism such that B is integral over A, $\mathfrak{p}$ a prime ideal of A and $\mathfrak{P}_1$ and $\mathfrak{P}_2$ two ideals of B lying over $\mathfrak{p}$ such that $\mathfrak{P}_1 \subseteq \mathfrak{P}_2$. If $\mathfrak{P}_1$ and $\mathfrak{P}_2$ are prime, then $\mathfrak{P}_1 = \mathfrak{P}_2$.*

Proof. Let us write $S = A - \mathfrak{p}$. Then $S^{-1}B$ is integral over $S^{-1}A$ by Proposition 4.1.39, $S^{-1}\mathfrak{p}$ is a maximal ideal of $S^{-1}A$, $S^{-1}\mathfrak{P}_1$ and $S^{-1}\mathfrak{P}_2$ are ideals of $S^{-}B$ lying over $S^{-1}\mathfrak{p}$ and $S^{-1}\mathfrak{P}_1 \subseteq S^{-1}\mathfrak{P}_2$. As $S^{-1}\mathfrak{P}_1$ and $S^{-1}\mathfrak{P}_2$ are prime, they are maximal by Corollary 4.1.65 and hence $S^{-1}\mathfrak{P}_1 = S^{-1}\mathfrak{P}_2$. Therefore $\mathfrak{P}_1 = \mathfrak{P}_2$. □

Corollary 4.1.67. *Let B be an integral domain, A a subring of B such that B is integral over A and f a homomorphism from B to a ring C. If the restriction of f to A is injective, then f is injective.*

Proof. If $\mathfrak{b}$ is the kernel of f, the hypothesis means that $\mathfrak{b} \cap A = (0)$. As B is an integral domain, Corollary 4.1.66 may be applied taking $\mathfrak{p}$ and $\mathfrak{P}_1$ to be the ideal (0) of A and the ideal (0) of B respectively, whence $\mathfrak{b} = (0)$. □

Corollary 4.1.68. *Let $A \subseteq B$ be rings, B integral over A and $\mathfrak{m}$ a maximal ideal of A. Suppose that there are only finitely many distinct maximal ideals $\mathfrak{M}_1, \dots, \mathfrak{M}_s$ in B lying over $\mathfrak{m}$. Let $\mathfrak{Q}_i$ be the saturation of $\mathfrak{m}B$ with respect to $\mathfrak{M}_i$. Then:*

(a) In the ring $B/\mathfrak{Q}_i$, the zero divisors are the elements of $\mathfrak{M}_i/\mathfrak{Q}_i$, and they are nilpotent.

(b) $\mathfrak{m}B = \bigcap_{i=1}^s \mathfrak{Q}_i = \prod_{i=1}^s \mathfrak{Q}_i$.

(c) The canonical homomorphism $B/\mathfrak{m}B \to \prod_{i=1}^s B/\mathfrak{Q}_i$ is an isomorphism.

Proof. For a prime ideal of B to contain $\mathfrak{m}B$, it is necessary and sufficient that its intersection with A contains $\mathfrak{m}$, and hence that it lies over $\mathfrak{m}$, since $\mathfrak{m}$ is maximal in A. Therefore $\mathfrak{M}_i$ are therefore the only prime ideals of B containing $\mathfrak{m}B$ and therefore $\sqrt{\mathfrak{m}B} = \bigcap_{i=1}^s \mathfrak{M}_i$. By definition of $\mathfrak{Q}_i$, the image of an element of $B - \mathfrak{M}_i$ in $B/\mathfrak{Q}_i$ is not a divisor of 0. On the other hand, as the $\mathfrak{M}_i$ are distinct maximal ideals, for every index i there exists an element a_i belonging to $\bigcap_{j \neq i} \mathfrak{M}_j$ and not to $\mathfrak{M}_i$. Then for all $x \in \mathfrak{M}_i$ we have $a_i x \in \sqrt{\mathfrak{m}B}$, hence the image of $a_i x$ in $B/\mathfrak{Q}_i$ is nilpotent, and, as that of a_i is not a divisor of 0, we conclude that the image of x is nilpotent. In other words $\mathfrak{M}_i$ is the radical of $\mathfrak{Q}_i$ and proves (a).

It follows that $\mathfrak{Q}_i$ are relatively coprime in pair, and (c) will be a concequence of (b). To establish (b), we note that in the ring $B/\mathfrak{m}B$ the $\mathfrak{M}_j/\mathfrak{m}B$ are the only maximal ideals and $\mathfrak{Q}_j/\mathfrak{m}B$ is the saturation of (0) with respect to $\mathfrak{M}_j/\mathfrak{m}B$. We may therefore restrict our attention to the case $\mathfrak{m}B = (0)$. The assertion of (b) then follows from Proposition 1.3.19, by the definition of a saturation of (0). □

Theorem 4.1.69. *Let $\rho : A \to B$ be a ring homomorphism such that B is integral over A and let $\mathfrak{p}$ be a prime ideal of A containing* $\ker \rho$. *Then there exists a prime ideal $\mathfrak{P}$ of B such that $\mathfrak{P} \cap A = \mathfrak{p}$.*

Proof. By consider the image $\rho(A)$ in B, we may assume that ρ is injective. Suppose first that A is a local ring and $\mathfrak{p}$ the maximal ideal of A. Then, for every maximal ideal $\mathfrak{M}$ of B, $\mathfrak{M}^c$ is a maximal ideal of A and hence equal to $\mathfrak{p}$, which proves the theorem in this case (since B contains A by hypothesis and is therefore not reduced to 0). In the general case, let us write $S = A - \mathfrak{p}$, then $S^{-1}A$ is a local ring whose maximal ideal is $S^{-1}\mathfrak{p}$, $S^{-1}f : S^{-1}A \to S^{-1}A$ is injective and $S^{-1}B$ is integral over $S^{-1}A$. Then there exists a prime ideal $\mathfrak{P}'$ of $S^{-1}B$ lying over $S^{-1}\mathfrak{p}$ and we know that $\mathfrak{P}' = S^{-1}\mathfrak{P}$ where $\mathfrak{P}$ is a prime ideal of B lying over $\mathfrak{p}$ by Lemma 4.1.63. □

Corollary 4.1.70. *Let $\rho : A \to B$ be a ring homomorphism such that B is integral over A, $\mathfrak{a}$ and $\mathfrak{p}$ two ideals of A such that $\mathfrak{a} \subseteq \mathfrak{P}$ and $\mathfrak{b}$ an ideal of B lying over $\mathfrak{a}$. Suppose that $\mathfrak{p}$ is prime, then there exists a prime ideal $\mathfrak{P}$ of B lying over $\mathfrak{p}$ and containing $\mathfrak{b}$.*

Proof. If $\bar{\rho} : A/\mathfrak{a} \to B/\mathfrak{b}$ is the homomorphism derived from f by taking quotients, then $\bar{\rho}$ is injective by hypothesis and $B/\mathfrak{b}$ is integral over $A/\mathfrak{a}$, so the claim follows from Theorem 4.1.69. □

Corollary 4.1.71. *Let A be a ring and B a ring containing A and integral over A. Then there exists one-to-one correspondence between maximal ideals of A and maximal ideals of B.*

Proof. This follows from Corollary 4.1.65 and Theorem 4.1.69. □

Corollary 4.1.72. *Let A be a ring and B a ring containing A and integral over A. If $\mathfrak{R}$ is the Jacobson radical of B, then $\mathfrak{R} \cap A$ is the Jacobson radical of A.*

Corollary 4.1.73 (Going-up theorem). *Let A be a ring and B a ring containing A and integral over A. Let $\mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_n$ be a chain of prime ideals of A and $\mathfrak{P}_1 \subset \cdots \subset \mathfrak{P}_m$ a chain of prime ideals of B such that $\mathfrak{P}_i \cap A = \mathfrak{p}_i$ for $1 \leq i \leq m$ and $m < n$. Then the chain $\mathfrak{P}_1 \subset \cdots \subset \mathfrak{P}_m$ can be extended to a chain $\mathfrak{P}_1 \subset \cdots \subset \mathfrak{P}_n$ such that $\mathfrak{P}_i \cap A = \mathfrak{p}_i$ for $1 \leq i \leq n$.*

Proof. By induction we reduce immediately to the case $m = 1$, $n = 2$, and this follows from Corollary 4.1.70. □

Corollary 4.1.74. *Let $\rho : A \to B$ be a ring homomorphism such that B is integral over A. Then the associated continuous map $\rho^* : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is closed.*

Proof. For every ideal $\mathfrak{b}$ of B, $B/\mathfrak{b}$ is integral over B, hence also over A and $\mathrm{Spec}(B/\mathfrak{b})$ is identified with the closed subspace $V(\mathfrak{b})$ of $\mathrm{Spec}(B)$. To show that ρ^* is closed, we see then (replacing B by $B/\mathfrak{b}$) that it is sufficient to prove that the image of $\mathrm{Spec}(B)$ under ρ^* is a closed subset of $\mathrm{Spec}(A)$. Now it follows from Theorem 4.1.69 that this image is just the set of prime ideals of A containing the ideal $\ker \rho$ and this set is closed by definition of the topology on $\mathrm{Spec}(A)$. □

Corollary 4.1.75. *Let A be a ring, B a ring containing A and integral over A and ϕ a homomorphism from A to an algebraically closed field Ω. Then ϕ can be extended to a homomorphism from B to Ω.*

Proof. Let $\mathfrak{p}$ be the kernel of ϕ, which is a prime ideal since $\phi(A) \subseteq \Omega$ is an integral domain. Let $\mathfrak{P}$ be a prime ideal of B lying over $\mathfrak{p}$. Then $A/\mathfrak{p}$ is canonically identified with a subring of $B/\mathfrak{P}$ and $B/\mathfrak{P}$ is integral over $A/\mathfrak{p}$. The homomorphism ϕ defines, by taking the quotient, an isomorphism of $A/\mathfrak{p}$ onto the subring $\phi(A)$ of Ω, which can be extended to an isomorphism ψ of the field of fractions K of $A/\mathfrak{p}$ onto a subfield of Ω. As the field of fractions L of $B/\mathfrak{P}$ is algebraic over K, ψ can be extended to an isomorphism ψ' of L onto a subfield of Ω. If $\pi : B \to B/\mathfrak{P}$ is thc canonical homomorphism, $\psi' \circ \pi$ is a homomorphism from B to Ω extending ϕ. □

Proposition 4.1.76. *Let $\rho : A \to B$ be a ring homomorphism such that B is integral over A, $\mathfrak{p}$ a prime ideal of A, and $S = A - \mathfrak{p}$. If $(\mathfrak{P}_i)_{i\in I}$ is the family of all the prime ideals of B lying over $\mathfrak{p}$ and $T = \bigcap_{i\in I}(B - \mathfrak{P}_i)$, then $S^{-1}B = T^{-1}B$.*

Proof. By definition $\rho(S) \subseteq T$ and, as $\rho(S)^{-1}B = S^{-1}B$, it suffices to prove, by virtue of Proposition 1.2.24, that, if a prime ideal $\mathfrak{Q}$ of B does not meet $\rho(S)$, it does not meet T either. Now, suppose that $\mathfrak{Q} \cap \rho(S) = \varnothing$ and let $\mathfrak{q} = \rho^{-1}(\mathfrak{Q})$. Then $\mathfrak{q} \cap S = \varnothing$, in other words $\mathfrak{q} \subseteq \mathfrak{p}$. As $\mathfrak{Q}$ lies over $\mathfrak{q}$ by definition, it follows from Corollary 4.1.70 that there is an index i such that $\mathfrak{Q}_i \subseteq \mathfrak{p}_i$ and hence $\mathfrak{Q} \cap T = \varnothing$, which completes the proof. □

Corollary 4.1.77. *Let A be an integral domain, $\overline{A}$ its integral closure, and $\mathfrak{p}$ a prime ideal of A. If $(\mathfrak{P}_i)_{i\in I}$ is the family of all prime ideals of $\overline{A}$ lying over $\mathfrak{p}$ and $T = \bigcap_{i\in I}(\overline{A} - \mathfrak{P}_i)$, then $T^{-1}\overline{A}$ is the integral closure of $A_{\mathfrak{p}}$.*

Proof. By Proposition 4.1.39, the integral closure of $A_\mathfrak{p}$ is $\overline{A}_\mathfrak{p}$, which equals to $T^{-1}\overline{A}$ by Proposition 4.1.76. $\square$

Proposition 4.1.78. *Let $\rho : A \to B$ be a ring homomorphism sueh that B is a finitely generated A-module. Then, for every prime ideal $\mathfrak{p}$ of A, the set of prime ideals of B lying over $\mathfrak{p}$ is finite.*

Proof. By taking localization and quotient, we may reduce the problem to $\mathfrak{p} = (0)$ and A is a field. The ring B is then an A-algebra of finite rank and therefore Artinian and we know that in such an algebra there is only a finite number of prime ideals. $\square$

Proposition 4.1.79. *Let A be an integrally closed domain and B a ring containing A and integral over A. Suppose that B is a torsion-free A-module. Let $\mathfrak{p} \subseteq \mathfrak{q}$ be two prime ideals of A and $\mathfrak{Q}$ a prime ideal of A lying over $\mathfrak{q}$. Then there exists a prime ideal $\mathfrak{P}$ of B lying over $\mathfrak{p}$ and such that $\mathfrak{P} \subseteq \mathfrak{Q}$.*

Proof. Suppose first that B is an integral domain. Let K, L be the fields of fractions of A and B respectively. Let Ω be the algebraic closure of L and $\overline{A}$ the integral closure of A in Ω, so that $A \subseteq B \subseteq \overline{A}$. Let $\mathcal{P}$ be a prime ideal of $\overline{A}$ lying over $\mathfrak{p}$, then by Corollary 4.1.70 there is a prime ideal $\mathcal{Q}$ of $\overline{A}$ lying over $\mathfrak{q}$ and such that $\mathcal{P} \subseteq \mathcal{Q}$. Finally let $\mathcal{Q}_1$ be a prime ideal of $\overline{A}$ lying over $\mathfrak{Q}$. By Proposition 4.2.14, there exists a K-automorphism σ of Ω such that $\sigma(\mathcal{Q}) = \mathcal{Q}_1$. Then $\sigma(\mathcal{P})$ is a prime ideal of $\overline{A}$ lying over $\mathfrak{p}$ such that $\sigma(\mathcal{P}) \subseteq \mathcal{Q}_1$ and hence $\mathfrak{P} = B \cap \sigma(\mathcal{P})$ is a prime ideal of B lying over $\mathfrak{p}$ and contained in $\mathfrak{Q}$.

$$\begin{tikzcd}
\sigma(\mathcal{P}) \arrow[rr, hook] \arrow[d, dash] & & \mathcal{Q}_1 \arrow[d, dash] & \\
& \mathcal{P} \arrow[ul, "\sigma"'] \arrow[rr, hook] \arrow[ddd, dash] & & \mathcal{Q} \arrow[ul, "\sigma"'] \arrow[ddd, dash] \\
\mathfrak{P} \arrow[rr, hook] \arrow[d, dash] & & \mathfrak{Q} \arrow[d, dash] & \\
\mathfrak{p} \arrow[rr, hook] \arrow[dr, equal] & & \mathfrak{q} \arrow[dr, equal] & \\
& \mathfrak{p} \arrow[rr, hook] & & \mathfrak{q}
\end{tikzcd}$$

We pass to the general case. As A is an integral domain and $\mathfrak{Q}$ is prime, the subsets $A \setminus \{0\}$ and $B \setminus \mathfrak{Q}$ of B are multiplicative; then their product $S = (A \setminus \{0\})(B \setminus \mathfrak{Q})$ is a multiplicative subset of B which does not contain 0 since the non-zero elements of A are not divisors of 0 in B. Then there exists a prime ideal $\mathfrak{M}$ of B disjoint from S, in other words such that $\mathfrak{M} \subseteq \mathfrak{Q}$ and $\mathfrak{M} \cap A = 0$. Let π be the canonical homomorphism $B \to B/\mathfrak{M}$. Then the restriction of π to A is injective and hence $\pi(A)$ is integrally closed. Since $B/\mathfrak{M}$ is an integral domain, the first part of the proof proves that there exists a prime ideal $\mathfrak{N}$ of $B/\mathfrak{M}$ such that $\mathfrak{N} \cap \pi(A) = \pi(\mathfrak{p})$ and $\mathfrak{N} \subseteq \pi(\mathfrak{Q})$. The ideal $\mathfrak{P} = \pi^{-1}(\mathfrak{N})$ is a prime ideal of B and $\mathfrak{P} \subseteq \mathfrak{Q}$, since $\mathfrak{Q}$ contains the kernel of π. As π is injective on A, we see $\mathfrak{P} \cap A = \mathfrak{p}$. $\square$

Corollary 4.1.80 (Going-down theorem). *Let A be an integrally closed domain and B a ring containing A and integral over A. Suppose that B is a torsion-free A-module. Let $\mathfrak{p}_1 \supseteq \cdots \supseteq \mathfrak{p}_n$ be a chain of prime ideals of A and $\mathfrak{P}_1 \supseteq \cdots \supseteq \mathfrak{P}_m$ a chain of prime ideals of B such that $\mathfrak{P}_i \cap A = \mathfrak{p}_i$ for $1 \le i \le m$ and $m < n$. Then the chain $\mathfrak{P}_1 \supseteq \cdots \supseteq \mathfrak{P}_m$ can be extended to a chain $\mathfrak{P}_1 \supseteq \cdots \supseteq \mathfrak{P}_n$ such that $\mathfrak{P}_i \cap A = \mathfrak{p}_i$ for $1 \le i \le n$.*

Corollary 4.1.81. *let $\mathfrak{p}$ be a prime ideal of A. The prime ideals of B lying over $\mathfrak{p}$ are the minimal elements of the set $\mathcal{E}$ of prime ideals of B containing $\mathfrak{p}B$.*

Proof. A prime ideal of B lying over $\mathfrak{p}$ is minimal in $\mathcal{E}$ by virtue of Corollary 4.1.66. Conversely, let $\mathfrak{Q}$ be a minimal element of $\mathcal{E}$. As $\mathfrak{Q} \cap A \supseteq \mathfrak{p}$, the going down theorem shows that there exists a prime ideal $\mathfrak{P}$ of B lying over $\mathfrak{p}$ such that $\mathfrak{P} \subseteq \mathfrak{Q}$. As $\mathfrak{P}$ contains $\mathfrak{p}B$, the hypothesis made on $\mathfrak{Q}$ implies that $\mathfrak{P} = \mathfrak{Q}$ and hence $\mathfrak{Q}$ lies over $\mathfrak{p}$. $\square$

Proposition 4.1.82. *Let $\rho : A \to B$ be a flat homomorphism of rings. Then ρ has the going-down property.*

Proof. Let $\mathfrak{p}_1 \subseteq \mathfrak{p}$ be prime ideals in A, $\mathfrak{P}$ a prime ideal in B such that $\mathfrak{P} \cap A = \mathfrak{p}$. Consider the localization $A_\mathfrak{p}$ and $B_\mathfrak{P}$. From Corollary 1.2.44 we know the map $\rho^* : \mathrm{Spec}(B_\mathfrak{P}) \to \mathrm{Spec}(A_\mathfrak{p})$ is surjective, hence there is a prime ideal $\mathfrak{P}_1'$ lying over $\mathfrak{p}_1$. Contracting $\mathfrak{P}_1'$ to B gives the desired prime ideal. □

Theorem 4.1.83. *Let $A \subseteq B$ be integral domains such that A is integrally closed and B is integral over A. Let $\phi : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ be the canonical map. For $b \in B$, let*

$$f(X) = X^n + a_{n-1}X^{n-1} + \cdots + a_0$$

be a monic polynomial with coefficients in A having b as a root and of minimal degree. Then

$$\phi(D(b)) = \bigcup_{i=1}^{n} D(a_i).$$

In particular, ϕ is an open map.

Proof. By hypothesis, f is the minimal polynomial of b over the field of fraction of A. If we set $C = A[b]$ then $C \cong A[X]/(f(X))$ is a free A-module with basis $1, b, b^2, \dots, b^{n-1}$ and is hence faithfully flat over A. Suppose that $\mathfrak{P} \in D(b)$, so that $\mathfrak{P} \in \mathrm{Spec}(B)$ with $b \notin \mathfrak{P}$, and set $\mathfrak{p} = \mathfrak{P} \cap A$. Then $\mathfrak{p} \in \bigcup_i D(a_i)$, since otherwise $a_i \in \mathfrak{p}$ for all i, and so $b^n \in \mathfrak{P}$, hence $b \in \mathfrak{P}$, which is a contradiction. So we have $\phi(D(b)) \subseteq \bigcup_{i=1}^n D(a_i)$.

Conversely, let $\mathfrak{p} \in \bigcup_i D(a_i)$ and let $\mathcal{P}$ be a prime of C lying over $\mathfrak{p}$ and $\mathfrak{P}$ a prime of B lying over $\mathcal{P}$. Suppose that $b \in \sqrt{\mathfrak{p}C}$, then for sufficiently large m we have

$$b^m = \sum_{i=1}^{n} p_i b^i, \quad p_i \in \mathfrak{p}.$$

By the assumption on $f(X)$ we must have $m \geq n$. Then $X^m - \sum_{i=1}^n p_i X^i$ is divisible by $f(X)$ in $A[X]$, which implies that X^m is divisible by $\bar{f}(X) = X^n + \sum_{i=1}^{n-1} \bar{a}_i X^i$ in $(A/\mathfrak{p})[X]$; since at least one of the $\bar{a}_i$ is nonzero, this is a contradiction.

Thus $b \notin \sqrt{\mathfrak{p}C}$ and there exists some $\mathcal{P}_1 \in \mathrm{Spec}(C)$ with $b \notin \mathcal{P}_1$ and $\mathfrak{p}C \subset \mathcal{P}_1$. By Corollary 4.1.81 we have $\mathcal{P} \subseteq \mathcal{P}_1$, so $\mathfrak{P} \in D(b)$ since otherwise $b \in \mathfrak{P} \cap C = \mathcal{P} \subseteq \mathcal{P}_1$, which is a contradiction. This proves $\phi(D(b)) = \bigcup_{i=1}^n D(a_i)$. Since any open set of $\mathrm{Spec}(B)$ is a union of open sets of the form $D(a)$, it follows that ϕ is open. □

Proposition 4.1.84. *A ring homomorphism $\rho : A \to B$ is said to have the **going-up property** (resp. the **going-down property**) if the conclusion of the going-up theorem (resp. the going-down theorem) holds for B and its subring $\rho(A)$. Let $\rho^* : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ be the map associated with ρ.*

(i) Consider the following three statements:

- *(a) ρ^* is a closed map.*
- *(b) ρ has the going-up property.*
- *(c) Let $\mathfrak{P}$ be any prime ideal of B and let $\mathfrak{p} = \mathfrak{P}^c$. Then $\rho^* : \mathrm{Spec}(B/\mathfrak{P}) \to \mathrm{Spec}(A/\mathfrak{p})$ is surjective.*

Then we have (a)⇔(b)⇔(c).

(ii) Consider the following three statements:

- *(a′) ρ^* is an open map.*

(b′) *ρ has the going-down property.*

(c′) *Let $\mathfrak{P}$ be any prime ideal of B and let $\mathfrak{p} = \mathfrak{P}^c$. Then $\rho^* : \mathrm{Spec}(B_{\mathfrak{P}}) \to \mathrm{Spec}(A_{\mathfrak{p}})$ is surjective.*

Then we have (a′)⇒(b′)⇔(c′).

Proof. The equivalences (b)⇔(c) and (b′)⇔(c′) are clear. For (a)⇒(b), assume that ρ^* is closed, we claim that $\rho^*(V(\mathfrak{P})) = V(\mathfrak{p})$ if $\mathfrak{p} = \mathfrak{P}^c$. To see this, we note that since ρ^* is closed, $\rho^*(V(\mathfrak{P}))$ is closed and contains $\mathfrak{p}$, so $V(\mathfrak{p}) = \overline{\{\mathfrak{p}\}} \subseteq \rho^*(V(\mathfrak{P}))$. But $\mathfrak{P}' \supseteq \mathfrak{P}$ implies $(\mathfrak{P}')^c \supseteq \mathfrak{P}^c = \mathfrak{p}$, hence we also have $\rho^*(V(\mathfrak{P})) \subseteq V(\mathfrak{p}_1)$, and therefore $\rho^*(V(\mathfrak{P})) = V(\mathfrak{p})$. In particular, the map ρ^* is surjective on $V(\mathfrak{p})$, so if $\mathfrak{p} \subseteq \mathfrak{p}'$ then $\mathfrak{p}' \in V(\mathfrak{p})$ and there is a $\mathfrak{P}' \in V(\mathfrak{P})$ such that $(\mathfrak{P}')^c = \mathfrak{p}'$.

For (a′)⇒(c′), we first prove the following *downward property*: If $\mathfrak{p}' \subseteq \mathfrak{p}$ and $\mathfrak{p} \in U$ for open set U, then $\mathfrak{p}' \in U$. In fact, write $U = X \setminus C$ where C is closed, then $\mathfrak{p}' \in C$ would implies $V(\mathfrak{p}') = \overline{\{\mathfrak{p}'\}} \subseteq C$ since C is closed. But $\mathfrak{p} \in V(\mathfrak{p}')$ is not in C, contrdiction. By this property, if $\rho^*(U)$ is open, then for any $\mathfrak{p}' \supseteq \mathfrak{p}$ in A and $\mathfrak{P}$ in B such that $\mathfrak{P}^c = \mathfrak{p}$. Then let U be a neighborhood of $\mathfrak{P}$ in B, so $\rho^*(B)$ is a neighbrhood of $\mathfrak{p}_1$ in A. Now since $\mathfrak{p}' \subseteq \mathfrak{p}$, we conclude $\mathfrak{p}' \in f(U)$, so $\mathfrak{p}'$ has a preimage $\mathfrak{P}'$. This shows the map $\rho^* : \mathrm{Spec}(B_{\mathfrak{P}}) \to \mathrm{Spec}(A_{\mathfrak{p}})$ is surjective.

Now assume the going up property, we show that ρ^* is closed. To this end, let $V(\mathfrak{b})$ be a closed subset of $\mathrm{Spec}(B)$. Since $V(\mathfrak{b})$ is identified with $\mathrm{Spec}(B/\mathfrak{b})$ and the induced map $A/\mathfrak{a} \to B/\mathfrak{b}$ satisfies the going up property (where $\mathfrak{a} = \mathfrak{b}^c$), we only need to show the image of $\mathrm{Spec}(B)$ under ρ^* is closed. For this, let T be the image of ρ^*, and let $\mathfrak{p} \in \mathrm{Spec}(A)$ be in the closure of T. This means for every $f \in A - \mathfrak{p}$ we have $D(f) \cap T \neq \varnothing$. Note that $D(f) \cap T$ is the image of $\mathrm{Spec}(B_f)$ in $\mathrm{Spec}(A)$. Hence we conclude that $B_f \neq 0$. Since $B_{\mathfrak{p}}$ is the directed limit of the rings B_f with $f \in A - \mathfrak{p}$, we conclude that $B_{\mathfrak{p}} \neq 0$ by **??**. If $\mathcal{P}'$ is a prime ideal of $B_{\mathfrak{p}}$, then $\mathfrak{p}' = \mathcal{P}' \cap A$ is a prime ideal of A such that $\mathfrak{p}' \subseteq \mathfrak{p}$, and it is in the image of ρ^* since $\mathfrak{P}' \cap A = \mathfrak{p}'$, where $\mathfrak{P}' = (i_B^{\mathfrak{p}})^{-1}(\mathcal{P}')$. As we assumed that ρ satisfies the going up property, there exist a prime ideal $\mathfrak{P}$ of B lying over $\mathfrak{p}$, whence $\mathfrak{p} \in T$ and T is closed. □

4.2 Group action on algebras

4.2.1 Group acting on an algebra

Given a ring R, a R-algebra A and a group G acting on A, we shall say the action of G is ***R*-linear** or G is a R-action if for each $\sigma \in G$, the map $x \mapsto \sigma(x)$ is an endomorphism of the R-algebra A. We shall denote by A^G the set of elements of A which are invariant under G. Clearly it is a sub-R-algebra of A. We shall say that the **action is locally finite** if every orbit of G in A is finite.

Proposition 4.2.1. *Let A be a R-algebra and G a locally finite R-action on A. Then A is integral over the subalgebra A^G.*

Proof. For all $x \in A$, let $x_1, \dots, x_n$ be the distinct elements of the orbit of x under G. For all $\sigma \in G$, there exists a permutation $\pi_\sigma \in \mathfrak{S}_n$ such that $\sigma(x_i) = x_{\pi_\sigma(i)}$ for $1 \leq i \leq n$. Therefore the elementary symmetric functions of the x_i are elements of A which are invariant under G, in other words elements of A^G. As x is a root of the monic polynomial $\prod_i (X - x_i)$ and the coefficients of this polynomial belong to A^G, x is integral over A^G. □

Theorem 4.2.2. *Let A be a finitely generated R-algebra and G a locally finite R-action on A. Then A is a finitely generated A^G-module. If further R is Noetherian, A^G is a finitely generated R-algebra.*

Proof. Let $a_1, \ldots, a_n$ generate the R-algebra A. Then we see $A = A^G[a_1, \ldots, a_n]$ and the a_i are integral over A^G by Proposition 4.2.1, the first assertion follows. The second is a consequence of the following lemma:

$$\text{Let } A \text{ be a Noetherian ring, } B \text{ a finitely generated } A\text{-algebra and } C \text{ a sub-}A\text{-algebra of } B \text{ such that } B \text{ is integral over } C. \text{ Then } C \text{ is a finitely generated } A\text{-algebra.} \tag{4.2.1}$$

To prove the lemma, let $x_1, \ldots, x_n$ be a finite system of generators of the A-algebra B. For all i, there exists by hypothesis a monic polynomial $P_i \in C[X]$ such that $P_i(x_i) = 0$. Let C' be the sub-A-algebra of C generated by the coefficients of the P_i. Clearly the x_i are integral over C' and $B = C'[x_1, \ldots, x_n]$, hence B is a finitely generated C'-module. On the other hand C' is a Noetherian ring, hence C is a finitely generated C'-module, which proves that C is a finitely generated A-algebra. □

Let S be a multiplicative subset of a ring A and G a group acting on A and for which S is stable. Then, for all $\sigma \in G$, there exists a unique endomorphism $z \mapsto \sigma(z)$ of the ring $S^{-1}A$ such that $\sigma(a/1) = \sigma(a)/1$ for all $a \in A$, which is given by $\sigma(a/s) = \sigma(a)/\sigma(s)$. If τ is another element of G, then clearly $\sigma(\tau(z)) = \sigma\tau(z)$ for all $z \in S^{-1}A$, whence the group G operates on the ring $S^{-1}A$.

Proposition 4.2.3. *Let A be a R-algebra, G a locally finite R-action on A, S a multiplicative subset of A stable under G and S^G the set $S \cap A^G$. Then the canonical map $i_A^{S^G,S} : (S^G)^{-1}A \to S^{-1}A$ is an isomorphism which maps $(S^G)^{-1}A^G$ to $(S^{-1}A)^G$.*

Proof. For all $s \in S$, let $s_1 = s, s_2, \ldots, s_n$ be the distinct elements of the orbit of s under G. As $\prod_{i=1}^n s_i \in S^G$, the first assertion follows from Proposition 1.2.24. Identifying canonically $(S^G)^{-1}A$ with $S^{-1}A$, clearly every element of $(S^G)^{-1}A^G$ is invariant under G. Conversely, let a/s be an element of $(S^G)^{-1}A$ which is invariant under G. If $a_1, \ldots, a_m$ are the distinct elements of the orbit of a under G, then $a_j/s = a/s$ for $1 \leq j \leq m$ and therefore there exists $t \in S^G$ such that $t(a_j - a) = 0$ for $1 \leq j \leq m$. In other words, ta is invariant under G and, as $a/s = (ta)/(ts)$, certainly $a/s \in (S^G)^{-1}A^G$. □

Corollary 4.2.4. *Let A be an integral domain, K its field of fractions and G a locally finite R-action on A. Then G acts on K and K^G is the field of fractions of A^G.*

4.2.2 Decomposition group and inertial group

Let B be a ring and G a group acting on B. Given a prime ideal $\mathfrak{P}$ of B the subgroup of elements $\sigma \in G$ such that $\sigma(\mathfrak{P}) = \mathfrak{P}$ is called the **decomposition group** of $\mathfrak{P}$ (with respect to G) and is denoted by $G^Z(\mathfrak{P})$. The ring of elements of B invariant under $G^Z(\mathfrak{P})$ is called the **decomposition ring** of $\mathfrak{P}$ (with respect to G) and is denoted by $A^Z(\mathfrak{P})$.

We often write G^Z and A^Z instead of $G^Z(\mathfrak{P})$ and $A^Z(\mathfrak{P})$ respectively, when there is no ambiguity. For all $\sigma \in G^Z(\mathfrak{P})$, we also denote by $z \mapsto \sigma(z)$ the endomorphism of the ring $B/\mathfrak{P}$ derived from the endomorphism $x \mapsto \sigma(x)$ of B by taking quotients; clearly the group $G^Z(\mathfrak{P})$ operates in this way on the ring $B/\mathfrak{P}$. The subgroup of $G^Z(\mathfrak{P})$ consisting of those σ such that the endomorphism $z \mapsto \sigma(z)$ of $B/\mathfrak{P}$ is the identity is called the **inertia group** of $\mathfrak{P}$ (with respect to G) and denoted by $G^T(\mathfrak{P})$. Similarly, we write $A^T(\mathfrak{P})$ for the subring of B invariant under $G^T(\mathfrak{P})$.

If A is the subring of B consisting of the invariants of G (i.e., $A = B^G$), clearly we have

$$A \subseteq A^Z(\mathfrak{P}) \subseteq A^T(\mathfrak{P}) \subseteq B.$$

It follows from definition that, for $\rho \in G$,

$$G^Z(\rho(\mathfrak{P})) = \rho G^Z(\mathfrak{P})\rho^{-1}, \quad G^T(\rho(\mathfrak{P})) = \rho G^T(\mathfrak{P})\rho^{-1}. \tag{4.2.2}$$

We may also write $\mathfrak{P}^Z := \mathfrak{P} \cap A^Z$ and $\mathfrak{P}^T = \mathfrak{P} \cap A^T$.

If, for all $\sigma \in G^Z(\mathfrak{P})$, $\bar{\sigma}$ denotes the automorphism $z \mapsto \overline{\sigma(z)}$ of $B/\mathfrak{P}$, then the map $\sigma \mapsto \bar{\sigma}$ is a homomorphism (called canonical) of $G^Z(\mathfrak{P})$ to the group of automorphisms of $B/\mathfrak{P}$ leaving invariant the elements of $A^Z/(\mathfrak{P}^Z)$, and by definition $G^T(\mathfrak{P})$ is the kernel of this canonical homomorphism, so it is a normal subgroup of $G^Z(\mathfrak{P})$. If $\kappa(\mathfrak{P})$ is the field of fractions of $B/\mathfrak{P}$, every automorphism of $B/\mathfrak{P}$ can be extended uniquely to an automorphism of $\kappa(\mathfrak{P})$, so that $\sigma \mapsto \bar{\sigma}$ can be considered as a homomorphism from $G^Z(\mathfrak{P})$ to the group of automorphisms of $\kappa(\mathfrak{P})$. Note that, since $G^T(\mathfrak{P})$ is normal in $G^Z(\mathfrak{P})$, the ring $A^T(\mathfrak{P})$ is stable under G, in view of (4.2.2).

Proposition 4.2.5. *Let B be a ring, G a group acting on B, A the ring of invariants of G, $\mathfrak{P}$ a prime ideal of B and S a multiplicative subset of A not meeting $\mathfrak{P}$. Then*

$$G^Z(S^{-1}\mathfrak{P}) = G^Z(\mathfrak{P}), \quad G^T(S^{-1}\mathfrak{P}) = G^T(\mathfrak{P})$$

and, if the action of G is locally finite, then

$$S^{-1}A^Z(\mathfrak{P}) = A^Z(S^{-1}\mathfrak{P}), \quad S^{-1}A^T(\mathfrak{P}) = A^T(S^{-1}\mathfrak{P}).$$

Proof. As the elements of S are invariant under G, clearly, if $\sigma(\mathfrak{P}) = \mathfrak{P}$, also $\sigma(S^{-1}\mathfrak{P}) = S^{-1}\mathfrak{P}$. Conversely, suppose that $\sigma \in G$ is such that $\sigma(S^{-1}\mathfrak{P}) = S^{-1}\mathfrak{P}$, then, if $x \in \mathfrak{P}$, we have $\sigma(x/1) \in S^{-1}\mathfrak{P}$ and there therefore exists $s \in S$ such that $s\sigma(x) \in \mathfrak{P}$, whence $\sigma(x) \in \mathfrak{P}$ since $\mathfrak{P}$ is prime and $s \notin \mathfrak{P}$. This proves that $\sigma(\mathfrak{P}) \subseteq \mathfrak{P}$ and it can be similarly shown that $\sigma^{-1}(\mathfrak{P}) \subseteq \mathfrak{P}$, hence $\sigma(\mathfrak{P}) = \mathfrak{P}$ and $\sigma \in G^Z(\mathfrak{P})$. If $\sigma \in G^T(\mathfrak{P})$, then $\sigma(x) - x \in \mathfrak{P}$ for all $x \in B$, hence also, for all $s \in S$,

$$\sigma(x/s) - (x/s) = (\sigma(x) - x)/s \in S^{-1}\mathfrak{P}$$

and therefore $\sigma \in G^T(S^{-1}\mathfrak{P})$. Conversely, suppose that $\sigma \in G^T(S^{-1}\mathfrak{P})$. Then, for all $x \in B$, $\sigma(x/1) - (x/1) \in S^{-1}\mathfrak{P}$ and therefore there exists $s \in S$ such that $s(\sigma(x) - x) \in \mathfrak{P}$, whence as above $\sigma(x) - x \in \mathfrak{P}$, which proves that $\sigma \in G^T(\mathfrak{P})$. The last two assertions follow from Proposition 4.2.3. □

Proposition 4.2.6. *Let B be a ring, G a finite group acting on B and A the ring of invariants of G so that B is integral over A.*

(a) Given two prime ideals $\mathfrak{P}$, $\mathfrak{Q}$ of B lying over the same prime ideal $\mathfrak{p}$ of A, there exists $\sigma \in G$ such that $\sigma(\mathfrak{P}) = \mathfrak{Q}$. In other words, G operates transitively on the set of prime ideals of B lying over $\mathfrak{p}$.

(b) Let $\mathfrak{P}$ be a prime ideal of B and $\mathfrak{p} = \mathfrak{P} \cap A$. Then $\kappa(\mathfrak{P})$ is a normal extension of $\kappa(\mathfrak{p})$ and the canonical homomorphism $\sigma \mapsto \bar{\sigma}$ defines by taking the quotient an isomorphism of $G^Z(\mathfrak{P})/G^T(\mathfrak{P})$ onto $\mathrm{Gal}(\kappa(\mathfrak{P})/\kappa(\mathfrak{p}))$. In other words, we get the following exact sequence

$$1 \longrightarrow G^T(\mathfrak{P}) \longrightarrow G^Z(\mathfrak{P}) \longrightarrow \mathrm{Gal}(\kappa(\mathfrak{P})/\kappa(\mathfrak{p})) \longrightarrow 1$$

Proof. If $x \in \mathfrak{Q}$, then $\prod_{\sigma\in G}\sigma(x) \in \mathfrak{Q} \cap A = \mathfrak{p} \subseteq \mathfrak{P}$. Hence there exists $\sigma \in G$ such that $\sigma(x) \in \mathfrak{P}$, that is $x \in \sigma^{-1}(\mathfrak{P})$. We then conclude that $\mathfrak{Q} \subseteq \bigcup_{\sigma\in H}\sigma(\mathfrak{P})$, and hence ($G$ is finite) $\mathfrak{Q} \subseteq \sigma(\mathfrak{P})$ for some $\sigma \in G$. Since $\mathfrak{Q}$ and $\sigma(\mathfrak{P})$ both lie over $\mathfrak{p}$, we see $\mathfrak{Q} = \sigma(\mathfrak{P})$.

To see that $\kappa(\mathfrak{P})$ is a normal extension of $\kappa(\mathfrak{p})$, it suffices to prove that every element $\bar{x} \in B/\mathfrak{P}$ is a root of a polynomial P in $\kappa(\mathfrak{p})[X]$ all of whose roots are in $B/\mathfrak{P}$. Now, let $x \in B$ be a representative of the class $\bar{x}$. Then the polynomial $P(X) = \prod_{\sigma\in G}(X - \sigma(x))$ has all its coefficients in A. Let $\bar{P}(X)$ be the polynomial in $(A/\mathfrak{p})[X]$ which is the image of $P(X)$, then we see $\bar{P}(X) = \prod_{\sigma\in G}(X - \overline{\sigma(x)})$ and therefore solves the problem.

Clearly, for all $\sigma \in G^Z$, $\bar{\sigma}$ is a $\kappa(\mathfrak{p})$-automorphism of $\kappa(\mathfrak{P})$. It remains to verify that $\sigma \mapsto \bar{\sigma}$ maps G^Z onto the group of all $\kappa(\mathfrak{p})$-automorphisms of $\kappa(\mathfrak{P})$. Since $\kappa(\mathfrak{p})$ and $\kappa(\mathfrak{P})$ are not changed under localization and by Proposition 4.2.5 neither G^Z nor its operation on $\kappa(\mathfrak{P})$ is changed, we may therefore restrict our attention to the case where $\mathfrak{p}$ is maximal, in which case we know that so is $\mathfrak{P}$ and every element of $\kappa(\mathfrak{P})$ is therefore of the form $\bar{x}$ for some x in B. It has been seen above that such an element is a root of a polynomial in $\kappa(\mathfrak{p})[X]$ of degree smaller than $|G|$. As every finite separable extension of $\kappa(\mathfrak{p})$ admits a primitive element, it is seen that every finite separable extension of $\kappa(\mathfrak{p})$ contained in $\kappa(\mathfrak{P})$ is of degree smaller than $|G|$, whence the separable closure $\kappa(\mathfrak{p})^{\mathrm{sep}}$ in $\kappa(\mathfrak{P})$ is of degree smaller than $|G|$. Let $y \in B$ be an element such that $\bar{y}$ is a primitive element of $\kappa(\mathfrak{p})^{\mathrm{sep}}$. The ideals $\sigma(\mathfrak{P})$ for $\sigma \in G - G^Z$ are maximal and distinct from $\mathfrak{P}$ by definition. By Chinese remainder therorem there then exists $x \in B$ such that $x \equiv y \bmod \mathfrak{P}$ and $x \in \sigma(\mathfrak{P})$ for all $\sigma \in G - G^Z$. Now let θ be a $\kappa(\mathfrak{p})$-automorphism of $\kappa(\mathfrak{P})$ and let $P(X) = \prod_{\sigma\in G}(X - \overline{\sigma(x)})$. As $\bar{x}$ is a root of P and $P \in \kappa(\mathfrak{p})[X]$, $\theta(\bar{x})$ is also a root of P in $\kappa(\mathfrak{P})$ and hence there exists $\tau \in G$ such that

$$\theta(\bar{x}) = \tau(\bar{x}).$$

Since $\sigma(\bar{x}) = 0$ for all $\sigma \in G - G^Z$ and $\theta(\bar{x}) \neq 0$, we conclude that necessarily $\tau \in G^Z$. As θ and τ have the same value for the primitive element $\bar{x} = \bar{y}$ of $\kappa(\mathfrak{p})^{\mathrm{sep}}$, they coincide on $\kappa(\mathfrak{p})^{\mathrm{sep}}$ and, as $\kappa(\mathfrak{P})$ is an inseparable extension of $\kappa(\mathfrak{p})^{\mathrm{sep}}$, they coincide on $\kappa(\mathfrak{P})$. □

Corollary 4.2.7. *With the hypotheses and notation of Proposition 4.2.6, let f_1 and f_2 be two homomorphism of B to a field Ω with the same restriction to A. Then there exists a $\sigma \in G$ such that $f_2 = f_1 \circ \sigma$.*

Proof. Let $\mathfrak{P}_i$ be the kernel of f_i, which is a prime ideal of B. By hypothesis $\mathfrak{P}_1 \cap A = \mathfrak{P}_2 \cap A$ and this intersection is a prime ideal $\mathfrak{p}$ of A. There therefore exists $\tau \in G$ such that $\tau(\mathfrak{P}_2) = \mathfrak{P}_1$. Replacing f_1 by the homomorphism $f_1 \circ \tau$, we may then assume that $\mathfrak{P}_1 = \mathfrak{P}_2$ (an ideal which we shall denote by $\mathfrak{P}$). By taking the quotient and localization we then derive from f_1 and f_2 two injective homomorphisms $\tilde{f}_1$, $\tilde{f}_2$ from $\kappa(\mathfrak{P})$ to Ω. As $\kappa(\mathfrak{P})$ is a normal extension of $\kappa(\mathfrak{p})$, we see $\tilde{f}_2 \circ \tilde{f}_1^{-1}$ is a $\kappa(\mathfrak{p})$-automorphism of $\kappa(\mathfrak{P})$ (**??**) and it follows from Proposition 4.2.6 that $\tilde{f}_2 \circ \tilde{f}_1^{-1} = \bar{\sigma}$ for some $\sigma \in G$. In particular, for all $x \in B$ the elements $f_2(x)$ and $f_1(\sigma(x))$ are equal. □

Proposition 4.2.8. *Let B be a ring, G a finite group acting on B and H a subgroup of G. Let A^G and A^H be the rings of invariants of G and H, respectively, so that $A^G \subseteq A^H$. For a prime ideal $\mathfrak{P}$ of B, set $\mathfrak{p}^G = \mathfrak{P} \cap A^G$ and $\mathfrak{p}^H = \mathfrak{P} \cap A^H$.*

- *(a) For H to be contained in the decomposition group $G^Z(\mathfrak{P})$, it is necessary and sufficient that $\mathfrak{P}$ be the only prime ideal of A lying over $\mathfrak{p}^H$.*
- *(b) If H contains $G^Z(\mathfrak{P})$, then*
  - *(α) The rings $A^G/\mathfrak{p}^G$ and $A^H/\mathfrak{p}^H$ have the same field of fractions;*
  - *(β) The maximal ideal of the local ring $(A^H)_{\mathfrak{p}^H}$ is generated by the image of $\mathfrak{p}^G$.*
- *(c) Suppose further that B is an integral domain and that $\bigcap_{n=1}^{\infty}(\mathfrak{p}^G)^n B_{\mathfrak{P}} = 0$, then the conditions of (b) imply that $G^Z(\mathfrak{P})$ preserves the elements of A^H.*

Proof. It follows from Proposition 4.2.6(a) that the prime ideals of B lying over $\mathfrak{p}^H$ are the ideals of the form $\sigma(\mathfrak{P})$ where $\sigma \in H$; whence immediately (a). We now prove (b); by taking localization, we may assume that $\mathfrak{p}^G$ is maximal. To establish (α) it will be sufficient to prove that

$$A^H = A^G + \mathfrak{p}^H, \tag{4.2.3}$$

for this will show that the fields $(A^G/\mathfrak{p}^G)$ and $(A^H)/(\mathfrak{p}^H)$ are canonically isomorphic. By Proposition 4.2.6 there is only a finite number of primes of B lying over $\mathfrak{p}^G$, therefore there is only a finite number of prime ideals of A^H lying over $\mathfrak{p}^G$; let $\mathfrak{n}_1, \dots, \mathfrak{n}_r$ denote those of these ideals that are different from $\mathfrak{p}^H$. Let x be an element of A^H, as the ideals $\mathfrak{p}^H$ and $\mathfrak{n}_i$ are maximal, there exists $y \in A^H$ such that $y \equiv x \bmod \mathfrak{p}^H$ and $y \in \mathfrak{n}_i$ for $1 \leq j \leq r$. Let $y_1 = y, y_2, \dots, y_s$ be distinct elements of the orbit of y under G. Clearly

$$z = y_1 + y_2 + \cdots + y_s \in A^G$$

and to establish (4.2.3) it will be sufficient to show that $y_i \in \mathfrak{P}$ for $i \geq 2$, for then we shall deduce that $z - y \in \mathfrak{P} \cap A^H = \mathfrak{p}^H$, whence $x \in A^G + \mathfrak{p}^H$ since $x \equiv y \bmod \mathfrak{p}^H$. Now let $i \geq 2$ and $\sigma \in G$ be such that $\sigma(y) = y_i$. Then $\sigma^{-1}(\mathfrak{P})$ does not lie over $\mathfrak{p}^H$, for otherwise there would exist $\tau \in H$ such that $\sigma^{-1}(\mathfrak{P}) = \tau(\mathfrak{P})$, whence $\sigma\tau \in G^Z \subseteq H$ (Proposition 4.2.6(a)). But as $y \in A^H$ and $\sigma(\tau(y)) = \sigma(y) \neq y$, this is a contradiction. We conclude that $\sigma^{-1}(\mathfrak{P})$ lies over one of the ideals $\mathfrak{n}_j$ and as $y \in \mathfrak{n}_j$ by construction, we see $y \in \sigma^{-1}(\mathfrak{P})$ or $y_i = \sigma(y) \in \mathfrak{P}$.

To prove (β) we show that $\mathfrak{p}^H$ is contained in the saturation $\mathfrak{q}$ of the ideal $(\mathfrak{p}^G)A^H$ with respect to $\mathfrak{p}^H$. As $\mathfrak{p}^H$ is contained in none of the $\mathfrak{n}_i$, by Proposition 1.1.4 it suffices to prove that

$$\mathfrak{p}^H \subseteq \mathfrak{q} \cup \mathfrak{n}_1 \cup \cdots \cup \mathfrak{n}_r. \tag{4.2.4}$$

For this, we consider an element $x \in \mathfrak{p}^H$ belonging to none of the $\mathfrak{n}_i$. Let $x_1 = x, x_2, \dots, x_m$ be the distinct elements of the orbit of x under G. Write $u = x_1 x_2 \cdots x_m$ and $v = x_2 \cdots x_m$, then $u \in A^G$. On the other hand, if $\tau \in H$ then $\tau(x) = x$ and hence necessarily $\tau(x_i) \neq x$ for $i \geq 2$, which shows that $\tau(v) = v$ and hence $v \in A^H$. It can be shown as in the proof of (α) that, if $\sigma \in G$ is such that $\sigma(x) = x_i$ where $i \geq 2$, then $\sigma^{-1}(\mathfrak{P})$ lies over one of the $\mathfrak{n}_j$ and, as $x \notin \mathfrak{n}_j$, we have $x \notin \sigma^{-1}(\mathfrak{P})$ whence $x_i \notin \mathfrak{P}$. We conclude that $v \notin \mathfrak{P}$ and therefore $v \notin \mathfrak{p}^H$. On the other hand, clearly $u \in \mathfrak{P} \cap A^G = \mathfrak{p}^G$ and the relation $u = xv$ then shows that x is in the saturation of $\mathfrak{p}^G(A^H)$ with respect to $\mathfrak{p}^H$ and hence establishes (4.2.4).

Suppose that B is an integral domain, that $\bigcap_{n=0}^{\infty}(\mathfrak{p}^G)^n B_{\mathfrak{P}} = 0$ and that conditions (α) and (β) of (b) hold. With the same notation as in (b), by localization we may assume also that the ideal $\mathfrak{p}^G$ is maximal. The hypotheses (α) and (β) then imply that for all $n > 0$ we have (by induction)

$$(A^H)_{\mathfrak{p}^H} = A^G + (\mathfrak{p}^G)^n (A^H)_{\mathfrak{p}^H}.$$

Then let σ be an element of G^Z and x be an element of A^H. For all $n > 0$, there exists $a_n \in A^G$ such that $x - a_n \in (\mathfrak{p}^G)^n (A^H)_{\mathfrak{p}^H} \subseteq (\mathfrak{p}^G)^n B_{\mathfrak{P}}$. As $\sigma(a_n) = a_n$ and $\sigma(\mathfrak{P}) = \mathfrak{P}$, we deduce that $\sigma(x) - x \in (\mathfrak{p}^G)^n B_{\mathfrak{P}}$. Since this relation holds for all n, we conclude from the hypothesis that $\sigma(x) = x$. □

Corollary 4.2.9. *Under the hypotheses of Proposition 4.2.8 the rings $(A^G)/(\mathfrak{p}^G)$ and $A^Z/(\mathfrak{P}^Z)$ have the same field of fractions and the maximal ideal of the local ring $(A^Z)_{\mathfrak{P}^Z}$ is generated by $\mathfrak{p}^G$.*

Proof. This follows from Proposition 4.2.8 by taking $H = G^Z$. □

Corollary 4.2.10. *Let B be an integral domain, G a finite group acting on B, A the ring of invariants of G and $\mathfrak{P}$ a prime ideal of B. Let K, K^Z and L be the fields of fractions of A, A^Z and L respectively. Then B is a Galois extension of K with Galois group G and the subfields E of L containing K and such that $\mathfrak{P}$ is the only prime ideal of B lying over the ideal $\mathfrak{P} \cap E$ of $B \cap E$ are just those which contain K^Z.*

Proof. By Proposition 4.2.3 applied to $S = A - \{0\}$, we see K is the field of invariants of G in L and similarly K^Z is the field of invariants of G^Z. Therefore, L is a Galois extension of K with Galois group G. By Galois correspondence, we see the condition "E contains K^Z" is equivalent to "H is contained in G^Z", where H is the fixed group of E. As E is then the field of invariants of H in L, $B \cap E$ is the ring of invariants of H in B. The second assertion then follows from Proposition 4.2.8. □

Proposition 4.2.11. *Let B be an integral domain, G a finite group acting on B, A the ring of invariants of G, $\mathfrak{p}$ a prime ideal of A and $\mathfrak{P}$ a prime of B lying over A. Then*

(a) *The residue field of $\mathfrak{P}^T$ is the separable closure of $\kappa(\mathfrak{p})$ in $\kappa(\mathfrak{P})$.*

(b) *The extension K^T/K is Galois and $\mathrm{Gal}(K^T/K)$ is isomorphic to $\mathrm{Gal}(\kappa(\mathfrak{P}^T)/\kappa(\mathfrak{p}))$.*

(c) *There exists a correspondence between the set of all intermediate extensions of $K^T/E/K$ and the set of all separable extensions $\kappa(\mathfrak{P}^T)/\kappa/\kappa(\mathfrak{p})$. The extension $\kappa/\kappa(\mathfrak{p})$ is a Galois extension if and only if E/K is Galois. Moreover, κ is the residue field of $\mathfrak{P}\cap E$.*

Proof. By localization this may be reduced to the case where $\mathfrak{p}$ is a maximal ideal of A, which implies that $\mathfrak{P}$, $\mathfrak{P}^Z$ and $\mathfrak{P}^T$ are maximal in B, A^Z and A^T respectively. Since G^T is a normal subgroup of G, we know K^T/K is Galois with Galois group G/G^T. Since K^T/K is normal, by Proposition 4.2.6 we see $\kappa(\mathfrak{P}^T)$ is a normal extension of $\kappa(\mathfrak{p})$, and Proposition 4.2.6 shows that $\mathrm{Gal}(K^T/K)$ is isomorphic to $\mathrm{Gal}(\kappa(\mathfrak{P}^T)/\kappa(\mathfrak{p}))$, hence the correspondence between the sets of intermediate fields $K\subseteq E\subseteq K^T$ and $\kappa(\mathfrak{p})\subseteq\kappa\subseteq\kappa(\mathfrak{P}^T)$ follows from Galois correspondence as well as the statement concerning normality. To see the last statement of (c), observe that K^T is also the inertial field of $\mathfrak{P}\cap E$ for any intermediate extension $K\subseteq E\subseteq K^T$, therefore if $\kappa(E)$ is the residue field of $\mathfrak{P}\cap E$, then $\mathrm{Gal}(K^T/E)\cong\mathrm{Gal}(\kappa(\mathfrak{P}^T)/\kappa(E))$, by what we have just seen.

For all $x\in B$, the polynomial $P(X)=\prod_{\sigma\in G^T}(X-\sigma(x))$ has its coefficients in the inertia ring A^T and, by definition of G^T, all its roots in B are congruent mod $\mathfrak{P}$. Let $\pi: B\to B/\mathfrak{P}$ be the canonical homomorphism; the polynomial $\pi(P)(X)$ over $A^T/\mathfrak{P}^T$ therefore has all its roots in $B/\mathfrak{P}$ equal to $\pi(x)$, which shows that $\kappa(\mathfrak{P})$ is purely inseparable over $\kappa(\mathfrak{P}^T)$, so $\kappa(\mathfrak{p})^{\mathrm{sep}}\subseteq\kappa(\mathfrak{P}^T)$.

We know that $\kappa(\mathfrak{p})^{\mathrm{sep}}/\kappa(\mathfrak{p})$ is a Galois extension and it follows from Proposition 4.2.6 that its Galois group is isomorphic to $G'=G^Z/G^T$. As $\kappa(\mathfrak{P}^T)$ is a purely inseparable extension of $\kappa(\mathfrak{p})^{\mathrm{sep}}$, $\kappa(\mathfrak{P}^T)$ is a normal extension of $\kappa(\mathfrak{p})$ and the separable degree of $\kappa(\mathfrak{P}^T)$ over $\kappa(\mathfrak{p})$ is

$$[\kappa(\mathfrak{P}^T):\kappa(\mathfrak{p})]_s=[\kappa(\mathfrak{p})^{\mathrm{sep}}:\kappa(\mathfrak{p})]=[G^Z:G^T]=:q.$$

It remains to see that $\kappa(\mathfrak{P}^T)$ is a separable extension of $\kappa(\mathfrak{p})$. We have seen above that G' is identified with an automorphism group of A^T and that A^Z is the ring of invariants of G'. If $x\in A^T$, the polynomial $Q(X)=\prod_{\sigma'\in G'}(X-\sigma'(x))$ therefore has its coefficients in A^Z. The polynomial $\pi(Q)$ over $A^Z/\mathfrak{p}^Z$ whose coefficients are the images of those of Q under the homomorphism π is of degree q and has a root $\pi(x)\in A^T/\mathfrak{p}^T$. As $A^Z/\mathfrak{p}^Z=\kappa(\mathfrak{p})$ by Proposition 4.2.8(b), we see that every element of $\kappa(\mathfrak{P}^T)$ is of degree $\leq q$ over $\kappa(\mathfrak{p})$.

This being so, let κ be the field of invariants of the Galois group $\mathrm{Gal}(\kappa(\mathfrak{P}^T)/\kappa(\mathfrak{p}))$; then $[\kappa(\mathfrak{P}^T):\kappa]=q$ by **??**. Let ξ be a primitive element of $\kappa(\mathfrak{P}^T)$ over κ; as it is of degree q over κ and of degree $\leq q$ over $\kappa(\mathfrak{p})$, it is of degree q over $\kappa(\mathfrak{p})$ and its minimal polynomial over κ has coefficients in $\kappa(\mathfrak{p})$; this shows that ξ is separable over $\kappa(\mathfrak{p})$. On the other hand, for all $\zeta\in\kappa$, there exists a power p^f of the characteristic p such that $\zeta^{p^f}\in\kappa(\mathfrak{p})$. We conclude that $\kappa(\mathfrak{p})(\xi-\zeta)$, which contains

$$(\xi-\zeta)^{p^f}=\xi^{p^f}-\zeta^{p^f}$$

contains ξ^{p^f} and consequently $\kappa(\mathfrak{p})(\xi^{p^f})$. But as ξ is separable over $\kappa(\mathfrak{p})$, $\kappa(\mathfrak{p})(\xi)=\kappa(\mathfrak{p})(\xi^{p^f})$ by **??**, whence $\kappa(\mathfrak{p})(\xi)\subseteq\kappa(\mathfrak{p})(\xi-\zeta)$. As ξ is of degree q over $\kappa(\mathfrak{p})$ and $\xi-\zeta$ of degree $\leq q$, it follows that $\kappa(\mathfrak{p})(\xi)=\kappa(\mathfrak{p})(\xi-\zeta)$, whence $\zeta\in\kappa(\mathfrak{p})(\xi)$. This shows that ζ is separable over $\kappa(\mathfrak{p})$, hence $\kappa=\kappa(\mathfrak{p})$ and $\kappa(\mathfrak{P}^T)$ is separable over $\kappa(\mathfrak{p})$. □

Corollary 4.2.12. *If the order of the inertia group $G^T(\mathfrak{P})$ is relatively prime to the characteristic p of $\kappa(\mathfrak{p})$, then $\kappa(\mathfrak{P})/\kappa(\mathfrak{p})$ is a Galois extension.*

Proof. With the notation of the proof of Proposition 4.2.11, the polynomial $\pi(P)$ has coefficients in $\kappa(\mathfrak{P}^T) = \kappa(\mathfrak{p})^{\mathrm{sep}}$; and all its roots equal to the image $\bar{x}$ of x in $B/\mathfrak{P}$. We immediately deduce that $\pi(P)$ is a power of a minimal polynomial of $\bar{x}$ over $\kappa(\mathfrak{p})^{\mathrm{sep}}$. But the latter has degree equal to a power of p and hence, as the degree of $\pi(P)$ is equal to the order of G^T, the hypothesis implies that $\pi(P)$ has degree zero, in other words $\bar{x} \in \kappa(\mathfrak{p})^{\mathrm{sep}}$. Thus $\kappa(\mathfrak{P}^T) = \kappa(\mathfrak{p})^{\mathrm{sep}}$ and $\kappa(\mathfrak{P})/\kappa(\mathfrak{p})$ is a Galois extension. □

4.2.3 Applications for integrally closed domains

Lemma 4.2.13. *Let A be an integrally closed domain, K its field of fractions, p the characteristic of K, L a radicial extension of K and B a subring of L containing A and integral over A. For every prime ideal $\mathfrak{p}$ of A, there exists a unique prime ideal $\mathfrak{P}$ of B lying over $\mathfrak{p}$ and $\mathfrak{P}$ is the set of $x \in B$ such that there exists an integer $n > 0$ for which $x^{p^n} \in \mathfrak{p}$.*

Proof. The existence of $\mathfrak{P}$ follows from Theorem 4.1.69. If $x \in \mathfrak{P}$, there exists $n > 0$ such that $x^{p^n} \in K$, whence $x^{p^n} \in A$ since A is integrally closed, hence $x^{p^n} \in A \cap \mathfrak{P} = \mathfrak{p}$. Conversely, if $x \in B$ is such that $x^{p^n} \in \mathfrak{p} \subseteq \mathfrak{P}$ then $x \in \mathfrak{p}$ since $\mathfrak{P}$ is prime. □

Proposition 4.2.14. *Let A be an integrally closed domain, K its field of fractions, L a normal extension of K and B the integral closure of A in L. Then*

(a) For every prime ideal $\mathfrak{p}$ of A, the group G of K-automorphisms of L acts transitively on the set of prime ideals of B lying over $\mathfrak{p}$.

(b) For every prime ideal $\mathfrak{P}$ of B and $\mathfrak{p} = \mathfrak{P} \cap A$, the residue field $\kappa(\mathfrak{P})$ is a nromal extension of $\kappa(\mathfrak{p})$ and the canonical homomorphism $\sigma \mapsto \bar{\sigma}$ defines by taking the quotient an isomorphism of $G^Z(\mathfrak{P})/G^T(\mathfrak{P})$ onto $\mathrm{Gal}(\kappa(\mathfrak{P})/\kappa(\mathfrak{p}))$.

Proof. Suppose first that L is a finite Galois extension of K. Then $A = B \cap K$ since A is integrally closed and A is therefore the ring of invariants of G in B. As G is finite, the proposition follows in this case from Proposition 4.2.6.

Suppose secondly that L is any Galois extension of K. Then L is the union of a right directed family $(K_\alpha)_{\alpha\in I}$ of finite Galois extensions of K. To show (a), consider two prime ideals $\mathfrak{P}$, $\mathfrak{Q}$ of B lying over $\mathfrak{p}$. For all $\alpha \in I$, $\mathfrak{P} \cap K_\alpha$ and $\mathfrak{Q} \cap K_\alpha$ are two prime ideals of $B \cap K_\alpha$ lying over $\mathfrak{p}$. Since $B \cap K_\alpha$ is the integral closure of A in K_α and the restrictions to K_α of the elements of G form the group of K-automorphisms of K_α, it follows from the finite case that there exists $\sigma \in G$ such that $\sigma(\mathfrak{P} \cap K_\alpha) = \mathfrak{Q} \cap K_\alpha$. Let $\mathcal{E}_\alpha$ be the set of $\sigma \in G$ which have the above property. Let $\sigma \in G - \mathcal{E}_\alpha$, then for all $\tau \in G$ leaving invariant the elements of K_α, we have

$$(\sigma\tau)(\mathfrak{P} \cap K_\alpha) = \sigma(\mathfrak{P} \cap K_\alpha) \neq \mathfrak{Q} \cap K_\alpha$$

and hence $\sigma\tau \in G - \mathcal{E}_\alpha$. It follows that $\mathcal{E}_\alpha$ is closed in the topological Galois group G and clearly the family $\mathcal{E}_\alpha$ is directed. As G is compact and the $\mathcal{E}_\alpha$ are non-empty, the intersection $\mathcal{E}$ of the family $(\mathcal{E}_\alpha)$ is non-empty and $\sigma(\mathfrak{P}) = \mathfrak{Q}$ for all $\sigma \in \mathcal{E}$, whence (a).

To show (b), note that $\kappa(\mathfrak{P})$ is the union of the right directed family $(\kappa_\alpha)_{\alpha\in I}$ where κ_α is the residue field of $\mathfrak{P} \cap K_\alpha$. As each κ_α is a normal extension of $\kappa(\mathfrak{p})$ by Proposition 4.2.6, so is $\kappa(\mathfrak{P})$. On the other hand, let θ be a $\kappa(\mathfrak{p})$-automorphism of $\kappa(\mathfrak{P})$. By virtue of Proposition 4.2.6 applied to $B \cap K_\alpha$, there exists for all α a non-empty set $\mathcal{F}_\alpha$ of elements $\sigma \in G$ such that $\sigma(\mathfrak{P} \cap K_\alpha) = \mathfrak{P} \cap K_\alpha$ and $\theta(\bar{x}) = \overline{\sigma(x)}$ for all $x \in B \cap K_\alpha$. As above it is seen that $\mathcal{F}_\alpha$ is closed in G and, as $(\mathcal{F}_\alpha)$ is a left directed set, its intersection $\mathcal{F}$ is non-empty. Clearly for $\sigma \in G$ we have $\sigma \in G^Z$ and $\bar{\sigma} = \theta$, which completes the proof of (b) in this case.

Finally we deal with the general case. The field of invariants K_1 of G is a purely inseparable extension of K. There therefore exists a single prime ideal $\mathfrak{p}_1$ of $A_1 = B \cap K_1$ lying over $\mathfrak{p}$ (Lemma 4.2.13). If $\mathfrak{P}$ and $\mathfrak{Q}$ are two prime ideals of B lying over $\mathfrak{p}$, then they lie over $\mathfrak{p}_1$. As L

is a Galois extension of K_1 and $B \cap K_1$ is integrally closed by Proposition 4.1.21, it follows from the Galois case that there exists a $\sigma \in G$ such that $\sigma(\mathfrak{P}) = \mathfrak{Q}$, whence (a). On the other hand, clearly the residue field of $\mathfrak{p}_1$ is a purely inseparable extension of $\kappa(\mathfrak{p})$. As $\kappa(\mathfrak{P})$ is a normal extension of κ_1 by the Galois case, we then see $\kappa(\mathfrak{P})$ is a normal extension of $\kappa(\mathfrak{p})$ (since every $\kappa(\mathfrak{p})$-isomorphism of $\kappa(\mathfrak{P})$ is a κ_1-isomorphism). This last remark shows also, taking account of the Galois case, that every $\kappa(\mathfrak{p})$-automorphism of $\kappa(\mathfrak{P})$ is of the form $\bar{\sigma}$ where $\sigma \in G^Z(\mathfrak{P})$, which completes the proof of (b). □

Corollary 4.2.15. *Let A be an integrally closed domain, K its field of fractions, L a normal extension of K and B the integral closure of A in L. Let θ_1 and θ_2 be two homomorphisms of B to a field Ω with the same restriction on A. Then there exists a K-automorphism σ of L such that $\theta_1 = \theta_2 \circ \sigma$.*

Proof. This can be proved as Corollary 4.2.7. □

Proposition 4.2.16. *Let A be an integrally closed domain, K its field of fractions, L a finite algebraic extension of K and B a subring of L containing A and integral over A.*

(a) For every prime ideal $\mathfrak{p}$ of A the set of prime ideals of B lying over $\mathfrak{p}$ is at most $[L : K]_s$.

(b) If $\mathfrak{P}$ is a prime ideal of B lying over $\mathfrak{p}$, every element of $\kappa(\mathfrak{P})$ is of degree smaller than $[L : K]$ over $\kappa(\mathfrak{p})$.

Proof. We may first restrict our attention to the case where L is a separable extension of K, for in general L is a purely inseparable extension of the separable closure K_0 of K contained in L, and $[L : K]_s = [K_0 : K]$ by definition and, if $A_0 = B \cap K_0$, the prime ideals of A_0 and B are in one-to-one correspondence (Lemma 4.2.13).

Suppose therefore that L is separable over K and let N be the Galois extension of K generated by L in an algebraic closure of K, G its Galois group, $\overline{A}$ the integral closure of A in N and $\mathcal{P}$ a prime ideal of $\overline{A}$ lying over $\mathfrak{p}$. Let H be the Galois group of N over L and G^Z the decomposition group of $\mathcal{P}$. The prime ideals of $\overline{A}$ lying over $\mathfrak{p}$ are the $\sigma(\mathcal{P})$ where $\sigma \in G$ (Proposition 4.2.6) and the relation $\sigma(\mathcal{P}) = \sigma'(\mathcal{P})$ means that $\sigma' = \sigma\tau$ where $\tau \in G^Z$. On the other hand, in order that $\sigma(\mathcal{P}) \cap L = \sigma'(\mathcal{P}) \cap L$, it is necessary and sufficient that $\sigma(\mathcal{P}) = \theta\sigma(\mathcal{P})$, where $\theta \in H$ (Proposition 4.2.6), whence finally $\sigma' = \theta\sigma\tau$ where $\theta \in H$ and $\tau \in G^Z$. The number of prime ideals of B lying over $\mathfrak{p}$ is therefore equal to the cardinality of the double coset $H \backslash G / G^Z$, and this number is smaller than $[G : H] = [L : K]$ be Galois theory.

The coefficients of the minimal polynomial (over K) of any element $x \in B$ belong to A (Corollary 4.1.29). Applying the canonical homomorphism $B \to B/\mathfrak{P}$ to the coefficients of this polynomial, an equation of integral dependence with coefficients in $A/\mathfrak{p}$ and of degree $\leq [L : K]$ is obtained for the class mod $\mathfrak{P}$ of x, whence the conclusion. □

Proposition 4.2.17. *Let A be an integrally closed domain, K its field of fractions, L a Galois extension of K, B the integral closure of A in L, $\mathfrak{P}$ a prime ideal of B, $\mathfrak{p} = A \cap \mathfrak{P}$. For a subfield E of L containing K, set $A(E) = B \cap E$ and $\mathfrak{p}(E) = \mathfrak{P} \cap E$.*

(a) If E is contained in K^Z, then $\mathfrak{p}$ and $\mathfrak{p}(E)$ have the same residue field and the maximal ideal of $(A(E))_{\mathfrak{p}(E)}$ is generated by $\mathfrak{p}$.

(b) Conversely, if these two conditions in (a) hold and $\bigcap_{n=0}^{\infty} \mathfrak{p}^n B_{\mathfrak{P}} = 0$, then E is contained in K^Z.

Proof. Let G be the Galois group of L over K and H the Galois group of L over E. Then to say that $E \subseteq K^Z$ means that $G^Z \subseteq H$ and the assertions are therefore special cases of Proposition 4.2.8 when $[L : K]$ is finite. In the general case the argument is as in the proof of Proposition 4.2.14. □

Proposition 4.2.18. *Let A be an integrally closed domain, K its field of fractions, L a Galois extension of K, B the integral closure of A in L, $\mathfrak{P}$ a prime ideal of B, $\mathfrak{p} = A \cap \mathfrak{P}$.*

(a) *The residue field $\kappa(\mathfrak{P}^T)$ is the separable closure of $\kappa(\mathfrak{p})$ in $\kappa(\mathfrak{P})$.*

(b) *The extension K^T/K is Galois and $\mathrm{Gal}(K^T/K)$ is isomorphic to $\mathrm{Gal}(\kappa(\mathfrak{P}^T)/\kappa(\mathfrak{p}))$.*

(c) *There exists an inclusion-preserving bijective correspondence between the set of all intermediate extensions $K \subseteq E \subseteq K^T$ and the set of all separable extensions $\kappa(\mathfrak{p}) \subseteq \kappa \subseteq \kappa(\mathfrak{P}^T)$. The extension $\kappa/\kappa(\mathfrak{p})$ is a Galois extension if and only if E/K is Galois. Moreover, in this case κ is the residue field of $\mathfrak{P} \cap E$.*

Proof. The ring A is the ring of invariants in B of the Galois group of L over K. If L is of finite degree over K, the corollary then follows from Proposition 4.2.8 and Proposition 4.2.11. Consider now the general case, L therefore being the union of a right directed set (K_α) of finite Galois extensions of K.

Suppose now that $x \in A^T$. There exists α such that $x \in A^T(\mathfrak{P} \cap K_\alpha)$ and Proposition 4.2.11 shows that the class $\bar{x}$ of x mod $\mathfrak{P}^T \cap K_\alpha$ is algebraic and separable over $\kappa(\mathfrak{p})$. A fortiori the class mod $\mathfrak{P}^T$ of x is separable over $\kappa(\mathfrak{p})$. To complete the proof of the corollary, it is sufficient to show that $\kappa(\mathfrak{P})$ is a purely inseparable extension of $\kappa(\mathfrak{P}^T)$. Now, $\kappa(\mathfrak{P})$ is the union of the right directed family of residue fields κ_α of $\mathfrak{P} \cap K_\alpha$. It follows therefore from Proposition 4.2.11 that, if an element of $\kappa(\mathfrak{P})$ belongs to κ_α, it is purely inseparable over the residue field of $\mathfrak{P}^T \cap K_\alpha$, and a fortiori over $\kappa(\mathfrak{P}^T)$. □

Let A be an integrally closed domain, K its field of fractions, L a normal extension of K and B the integral closure of A in L. The field of invariants $K^Z(\mathfrak{P})$ (resp. $K^T(\mathfrak{P})$) of the group $G^Z(\mathfrak{P})$ (resp. $G^T(\mathfrak{P})$) in the field L is called the **decomposition field** (resp. inertia field) of $\mathfrak{P}$ with respect to K. We also write K^z (resp. K^T) in place of $K^Z(\mathfrak{P})$ (resp. $K^T(\mathfrak{P})$). It follows from Proposition 4.2.3 that K^Z (resp. K^T) is the field of fractions of the ring A^Z (resp. A^T), and by Proposition 4.1.21 A^Z (resp. A^T) is the integral closure of A in K^Z (resp. K^T).

Proposition 4.2.19. *Let A be an integrally closed domain, K its field of fractions, L a Galois extension of K, B the integral closure of A in L, $\mathfrak{P}$ a prime ideal of B. For a subfield E of L containing K, set $A(E) = B \cap E$ and $\mathfrak{p}(E) = \mathfrak{P} \cap E$. Then the decomposition field (resp. inertia field) of $\mathfrak{P}$ with respect to E is EK^Z (resp. EK^T). If further E is a Galois extension of K, the decomposition field of $\mathfrak{p}(E)$ with respect to K is $E \cap K^Z$.*

Proof. If H is the Galois group of L over E, clearly the decomposition group (resp. inertia group) of $\mathfrak{P}$ with respect to E is $H \cap G^Z$ (resp. $H \cap G^T$) and the first assertion follows from Galois theory if L is a finite Galois extension of K. In the general case it follows from the fact that A^Z (resp. A^T) is the union of the $A^Z(\mathfrak{P} \cap K_\alpha)$ (resp. $A^T(\mathfrak{P} \cap K_\alpha)$): every element $x \in L$ belongs to some K_α and if it is invariant under $G^Z(\mathfrak{P}) \cap H$ (resp. $G^T(\mathfrak{P}) \cap H$) it is also invariant under $G^Z(\mathfrak{P} \cap K_\beta) \cap H$ (resp. $G^T(\mathfrak{P} \cap K_\beta) \cap H$) for some suitable β; hence it belongs by the beginning of the argument to $EK^Z(\mathfrak{P} \cap K_\alpha) \subseteq EK^Z$ (resp. $EK^T(\mathfrak{P} \cap K_\alpha) \subseteq EK^T$).

Suppose now that E is a Galois extension of K. The restriction to E of every $\sigma \in G^Z$ then preserves $\mathfrak{p}(E)$ and hence belongs to the decomposition group of $\mathfrak{p}(E)$ with respect to K. Conversely, let τ be an automorphism of E belonging to this group and let σ be an extension of τ to a K-automorphism of L. We write $\mathfrak{Q} = \sigma(\mathfrak{P})$. As $\mathfrak{P}$ and $\mathfrak{Q}$ both lie over $\mathfrak{p}(E)$, there exists an automorphism $\rho \in H$ such that $\mathfrak{Q} = \rho(\mathfrak{P})$, whence $\rho^{-1}\sigma \in G^Z$ and τ is the restriction of $\rho^{-1}\sigma$ to E. In other words, the decomposition group of $\mathfrak{p}(E)$ with respect to K is identical with the group of restrictions to E of the automorphisms $\sigma \in G^Z$ which proves the second assertion. □

4.3 Finite generated algebras over a field

In this part, k will be a field. Recall that, if A is a ring, by a finite generated A-algebra we mean an A-algebra B that is isomorphic to a quotient of some polynomial algebra $A[X_1, \dots, X_n]$.

Theorem 4.3.1 (Normalization Lemma). *Let A be a finitely generated k-algebra and let $\mathfrak{a}_1 \subseteq \mathfrak{a}_2 \subset \cdots \subset \mathfrak{a}_p$ be an increasing finite sequence of peroper ideals of A. Then there exists a sequence $(x_i)_{1\le i\le n}$ of elements of A which are algebraically independent over k and such that:*

(a) A is integral over the ring $B = k[x_1, \dots, x_n]$.

(b) There exists an increasing sequence $(h_j)_{1\le j\le n}$ of integers such that for all j the ideal $\mathfrak{a}_j \cap B$ of B is generated $x_1, \dots, x_{h_j}$.

Proof. It is sufficient to prove the theorem when A is a polynomial algebra $k[Y_1, \dots, Y_m]$. In fact, assume the theorem for this case. In the general case, A is isomorphic to a quotient of such an algebra $\tilde{A}$ by an ideal $\tilde{\mathfrak{a}}_0$. Let $\tilde{\mathfrak{a}}_j$ denote the inverse image of $\mathfrak{a}_j$ in $\tilde{A}$ and let $(\tilde{x}_i)_{1\le i\le r}$ be elements of $\tilde{A}$ satisfying the conditions of the statement for the ring $\tilde{A}$ and the increasing sequence of ideals of $\tilde{\mathfrak{a}}_0 \subset \cdots \subset \tilde{\mathfrak{a}}_p$. Then the images x_i of the $\tilde{x}_i$ in A for $i > h_0$ satisfy the desired conditions. In fact, condition (b) is obvious and condition (a) follows from Proposition 4.1.6. Finally, if the x_i for $i > h_0$ were not algebraically independent over k, there would be a non-zero polynomial $Q \in k[X_{h_0+1}, \dots, X_r]$ such that $Q(x_{h_0+1}, \dots, x_r) \in \tilde{\mathfrak{a}}_0 \cap \tilde{B}$, where $\tilde{B} = k[\tilde{x}_1, \dots, \tilde{x}_r]$. But by hypothesis, the ideal $\mathfrak{a}_0 \cap \tilde{B}$ is generated by $x_1, \dots, x_{h_0}$, a contradiction which proves our assertion.

We shall therefore suppose in the rest of the proof that $A = k[Y_1, \dots, Y_m]$ and we shall argue by induction on p. First assume $p = 1$, and we may further assume that $\mathfrak{a}_i$ is a principal ideal generated by an element $x_1 \notin k$. Then $x_1 = P(Y_1, \dots, Y_m)$, where P is a non-constant polynomial. We shall see that, for a suitable choice of integers $r_i > 0$, the ring A is integral over $B = k[x_1, x_2, \dots, x_m]$, where

$$x_i = Y_i - Y_1^{r_1} \quad \text{for } 2 \le i \le m. \tag{4.3.1}$$

For this it is sufficient to choose the r_i such that Y_1 is integral over B (Proposition 4.1.10). Now, there is the relation

$$P(Y_1, x_2 + Y_1^{r_2}, \dots, x_m + Y_1^{r_m}) - x_1 = 0 \tag{4.3.2}$$

Write $P = \sum_\alpha a_\alpha Y^\alpha$, where $\alpha \in \mathbb{N}^m$ and $a_\alpha \in k$ are nonzero elements of k. Then (4.3.2) becomes

$$\sum_\alpha a_\alpha Y_1^{\alpha_1}(x_2 + Y_1^{r_2})^{\alpha_2} \cdots (x_m + Y_1^{r_m})^{\alpha_m} - x_1 = 0. \tag{4.3.3}$$

Let us write $f(\alpha) = \alpha_1 + r_2\alpha_2 + \cdots + r_m\alpha_m$ and suppose that the r_i are chosen so that the $f(\alpha)$ are distinct (it suffices for example to take $r_i = h^i$, where h is an integer strictly greater than all the α_i. Then there will be a unique system $\alpha = (\alpha_1, \dots, \alpha_m)$ such that $f(\alpha)$ is maximal and relation (4.3.3) may be written

$$a_\alpha Y_1^{f(\alpha)} + \sum_{j<f(\alpha)} Q_j(x_1, \dots, x_m) Y_1^j = 0. \tag{4.3.4}$$

where the Q_i are polynomials in $k[Y_1, \dots, Y_m]$. As $a_\alpha \neq 0$ is invertible in k, (4.3.4) is certainly an equation of integral dependence with coefficients in B, whence our assertion.

The field of fractions $k(Y_1, \dots, Y_m)$ of A is therefore algebraic over the field of fractions $k(x_1, \dots, x_m)$ of B, which proves that the x_i are algebraically independent. Moreover, every element $z \in \mathfrak{a}_1 \cap B$ may be written $z = x_1\tilde{z}$, where $\tilde{z} \in A \cap k(x_1, \dots, x_m)$. But $A \cap k(x_1, \dots, x_m) = k[x_1, \dots, x_m] = B$ since B is integrally closed by Corollary 4.1.34, whence $\mathfrak{a}_1 \cap B = Bx_1$.

We now do not assume that $\mathfrak{a}_1$ is principal and prove the theorem for $p = 1$ by induction on m. We may obviously suppose that $\mathfrak{a}_1 \neq 0$. Let x_1 be a non-zero element of $\mathfrak{a}_1$. By the argument above there exist $t_2, \dots, t_m$ such that $x_1, t_2 \dots, t_m$ are algebraically independent over k, A is integral over $C = k[x_1, t_2, \dots, t_m]$ and $x_1 A \cap C = x_1 C$. By the induction hypothesis there exist algebraically independent elements $x_2, \dots, x_m$ of $k[t_2, \dots, t_m]$ and an integer h such that

$k[t_2,\dots,t_m]$ is integral over $\tilde{B} = k[x_2,\dots,x_m]$, and the ideal $\mathfrak{a}_1 \cap \tilde{B}$ is generated by $x_2,\dots,x_h$. Then C is integral over $B = k[x_1,x_2,\dots,x_m]$ and hence so is A, therefore $x_1,\dots,x_m$ are algebraically independent over k. Finally, as $x_1 \in \mathfrak{a}_1$ and $B = \tilde{B}[x_1]$, $\mathfrak{a}_1 \cap B = Bx_1 + (\mathfrak{a}_1 \cap \tilde{B})$ and, as $\mathfrak{a}_1 \cap \tilde{B}$ is generated (in $\tilde{B}$) by $x_2,\dots,x_h$, we see $\mathfrak{a}_1 \cap B$ is generated (in B) by $x_1,x_2,\dots,x_h$.

Now assume the we have proved the theorem for $p-1$ idaels and let $t_1,\dots,t_m$ be elements of A satisfying the conditions of the theorem for the increasing sequence of ideals $\mathfrak{a}_1 \subset \cdots \subset \mathfrak{a}_{p-1}$ and let us write $r = h_{p-1}$. Apply the preceding proof for $\mathfrak{a}_p$, there exist algebraically independent elements $x_{r+1},\dots,x_m$ of $C = k[t_{r+1},\dots,t_m]$ and an integer s such that C is integral over $\tilde{B} = k[x_{r+1},\dots,x_m]$ and the ideal $\mathfrak{a}_p \cap \tilde{B}$ is generated by $x_{r+1},\dots,x_s$. Now write $x_i = t_i$ for $1 \le i \le r$ and $h_p = s$, then A is integral over $C[t_1,\dots,t_r]$ and hence also over $B = k[x_1,\dots,x_m] = \tilde{B}[t_1,\dots,t_r]$ since C is integral over $\tilde{B}$ (Corollary 4.1.14). In particular, we see the x_i are algebraically independent over k. On the other hand, for $j \le p-1$, the ideal

$$\mathfrak{a}_j \cap k[x_1,\dots,x_r,t_{r+1},\dots,t_m]$$

is by hypothesis the set of polynomials in $x_1,\dots,x_r,t_{r+1},\dots,t_m$ all of whose monomials contain one of the elements $x_1,\dots,x_{h_j}$. As $x_{r+1},\dots,x_m$ are polynomials in $t_{r+1},\dots,t_m$ with coeflficients in k, it is seen immediately that a polynomial in $x_1,\dots,x_m$ (with coefficients in k) can belong to $\mathfrak{a}_j$ only if all its monomials contain one of the elements $x_1,\dots,x_{h_j}$. Finally, as $x_1,\dots,x_r$ belong to $\mathfrak{a}_{p-1}$ and hence also to $\mathfrak{a}_p$, $\mathfrak{a}_p \cap \tilde{B}[x_1,\dots,x_r]$ consists of the polynomials in $x_1,\dots,x_r$ with coefficients in $\tilde{B}$ whose constant term belongs to $\mathfrak{a}_p \cap \tilde{B}$. This ideal is therefore generated by $x_1,\dots,x_r,x_{r+1},\dots,x_s$. Therefore we see $(x_i)_{1\le i\le m}$ satisfies the required conditions, which proves the theorem. □

Corollary 4.3.2. *Let A be an integral domain and B a finitely generated A-algebra containing A as a subring. Then there exist a nonzero element s of A and a subalgebra C of B isomorphic to a polynomial algebra $A[Y_1,\dots,Y_n]$ such that B_s is integral over C_s.*

Proof. Wewrite $S = A - \{0\}$ and let $k = S^{-1}A$ the field of fractions of A. Clearly $S^{-1}B$ is a finitely generated k-algebra and, as it contains k by hypothesis, it is not reduced to 0. By Theorem 4.3.1 applied to $\mathfrak{a}_1 = (0)$, there exists therefore a finite sequence $(x_i)_{1\le i\le n}$ of elements of $S^{-1}B$ which are algebraically independent over k and such that $S^{-1}B$ is integral over $k[x_1,\dots,x_n]$. Let $(z_j)_{1\le j\le m}$ be a system of generators of the A-algebra B. Then in $S^{-1}B$ each of the $z_j/1$ satisfies an equation of integral dependence

$$(z_j/1)^{q_j} + \sum_{k<q_j} P_{kj}(x_1,\dots,x_n)(z_j/1)^k = 0.$$

where the P_{kj} are polynomials in the x_i with coefficients in k. There exists an element $s \ne 0$ of A such that we may write $x_i = y_i/s$ where $y_i \in B$ and all the coefficients of the P_{kj} are of the form c/s where $c \in A$. Finally, replacing if need be, s by a product of elements of S, we may assume that in B,

$$sz_j^{q_j} + \sum_{k<q_j} Q_{kj}z_j^h = 0. \tag{4.3.5}$$

where the Q_{kj} are polynomials in $y_1,\dots,y_n$ with coeflicients in A. If we write $\tilde{z}_j = sz_j$, it is seen from (4.3.5) by multiplying s^{q_j-1} that $\tilde{z}_j$ is integral over $\tilde{B} = A[y_1,\dots,y_n]$. We show that the y_i are algebraically independent over A. If there is a relation of the form $\sum_\alpha a_\alpha y_1^{\alpha_1}\cdots y_n^{\alpha_n} = 0$ where $a_\alpha \in A$ for all α, we deduce that $\sum_\alpha b_\alpha x_1^{\alpha_1}\cdots x_n^{\alpha_n} = 0$ in $S^{-1}B$, where $b_\alpha = a_\alpha s^{|\alpha|}$. By hypothesis therefore $b_\alpha = 0$ for all α, whence $a_\alpha = 0$ for all α. Moreover, by Proposition 4.1.39 in the ring B_s each of the $\tilde{z}_j/1$ is integral over $\tilde{B}_s$ and, as $z_j/1 = (\tilde{z}_j/1)(1/s)$ in B_s, it is seen that the $z_j/1$ are integral over $\tilde{B}_s$, which completes the proof. □

Corollary 4.3.3. *Let L be a field, A a subring of L and K the field of fractions of A. If L is a finitely generated A-algebra, then $[L:K]$ is finite and there exists a nonzero element a in A such that $K = A_a$.*

Proof. It follows from Corollary 4.3.2 that there exist elements $x_1, \dots, x_n$ of L and an element $a \neq 0$ of A such that $x_1, \dots, x_n$ are algebraically independent over A (and therefore over K) and that L is integral over the subring $A[x_1, \dots, x_n]_a$. Then it follows from Proposition 4.1.64 that $A[x_1, \dots, x_n]_a$ is a field. But the only invertible elements of a polynomial ring $C[Y_1, \dots, Y_n]$ over an integral domain C are the invertible elements of C. Applying this remark to $C = A_a$, it is seen that necessarily $n = 0$ and that A_a is a field equal to K by definition of the latter. As L is integral over K and is a finitely generated K-algebra, the degree $[L : K]$ is finite. □

Corollary 4.3.4. *Let A be an integral domain, B a finitely generated A-algebra and b a nonzero element of B such that $z^n b \neq 0$ for all $z \neq 0$ in A and every integer $n > 0$. Let $\rho : A \to B$ be the canonical homomorphism. Then there exists a nonzero a in A such that, for every homomorphism ϕ from A to an algebraically closed field Ω such that $\phi(a) \neq 0$, there exists a homomorphism $\tilde{\phi}$ from B to Ω such that $\tilde{\phi}(b) \neq 0$ and $\phi = \tilde{\phi} \circ f$.*

Proof. The hypothesis on b implies that, if i is the canonical homomorphism $x \mapsto x/1$ of B to B_b, the homomorphism $i \circ f$ of A to B_b is injective. By Corollary 4.3.2 there therefore exist an element $a \neq 0$ of A and a subring $\tilde{B}$ of B_b such that $(B_b)_a$ is integral over $\tilde{B}_a$ and $\tilde{B}$ is isomorphic to a polynomial algebra $A[Y_1, \dots, Y_n]$. Let ϕ be a homomorphism from A to an algebraically closed field Ω such that $\phi(a) \neq 0$. There exists a homomorphism from $A[Y_1, \dots, Y_n]$ to Ω extending ϕ and hence there exists a homomorphism from $\tilde{B}$ to Ω extending ϕ, which we still denote by ϕ. As $\phi(a) \neq 0$ in Ω, there exists a homomorphism ψ from $\tilde{B}_a$ to Ω such that

$$\psi(x/a^n) = \phi(x)(\phi(a))^{-n}$$

for all $x \in \tilde{B}$ and all $n > 0$. Finally, as $(B_b)_a$ is integral over $\tilde{B}_a$, there exists a homomorphism $\tilde{\psi}$ from $(B_b)_a$ to Ω extending ψ (Corollary 4.1.75). If $j : x \mapsto x/1$ is the canonical homomorphism from B to $(B_b)_a$, then $\tilde{\phi} = \tilde{\psi} \circ j$ solves the problem for $j(b)$ is invertible in $(B_b)_a$ and hence $\tilde{\psi}(j(b)) \neq 0$ in Ω. □

Theorem 4.3.5. *Let A be a finitely generated integral k-algebra, K its field of fractions, L a finite algebraic extension of K, and $\overline{A}$ the integral closure of A in L. Then $\overline{A}$ is a finitely generated A-module and a finitely generated k-algebra.*

Proof. By Theorem 4.3.1 there exists a subalgebra C of A isomorphic to a polynomial algebra $k[X_1, \dots, X_n]$ and such that A is integral over C. Then $\overline{A}$ is obviously the integral closure of C in L (Proposition 4.1.16). We may therefore confine our attention to the case where $A = k[X_1, \dots, X_n]$.

Let N be the normal extension of L (in an algebraic closure of L) generated by L, which is a finite algebraic extension of K. It will suffice to prove that the integral closure of A in N is a finitely generated A-module, for $\overline{A}$ is a sub-A-module of that ring and A is a Noetherian ring. We may therefore confine our attention to the case where L is a finite normal extension of K. Then we know that L is a (finite) Galois extension of a (finite) purely inseparable extension K' of K. If A' is the integral closure of A in K', $\overline{A}$ is the integral closure of A' in L and it will suffice to prove that A' is a finitely generated A-module and $\overline{A}$ is a finitely generated A'-module. Now, if it has been proved that A' is a finitely generated A-module, it is a Noetherian domain, integrally closed by definition. The fact that $\overline{A}$ is a finitely generated A'-module will follow from Corollary 4.1.51.

We see that we may confine our attention to the case where $A = k[X_1, \dots, X_n]$ and where L is a finite purely inseparable extension of $K = k(X_1, \dots, X_n)$. Then L is generated by a finite family of elements $(y_i)_{1 \le i \le m}$ and there exists a power q of the characteristic of k such that $y_i^q \in k(X_1, \dots, X_n)$ for all i. Let $(c_i)_{1 \le i \le r}$ be the coefficients of the numerators and denominators of the rational functions in $X_1, \dots, X_n$ that equal to y_i^q. Then L is contained in the extension $L' = k'(X_1^{1/q}, \dots, X_n^{1/q})$, where $k' = k(c_1^{1/q}, \dots, c_r^{1/q})$ (we are in an algebraic closure of L) and $\overline{A}$

is contained in the algebraic closure A' of A in L'. Now, k' is algebraic over k and hence $C' = k'[X_1, \dots, X_n]$ is integral over A. As $k'[X_1^{1/q}, \dots, X_n^{1/q}]$ is integrally closed by Corollary 4.1.35, it is seen that this ring is the integral closure of C' in L' and hence also that of A, in other words, $A' = k'[X_1^{1/q}, \dots, X_n^{1/q}]$. Now clearly A' is a finitely generated C'-module and, as k' is a finite extension of k, C' is a finitely generated A-module and hence A' is a finitely generated A-module. Since A is Noetherian and $\overline{A} \subseteq A'$, we see $\overline{A}$ is a finitely generated A-module. □

4.3.1 The Nullstellensatz

Proposition 4.3.6. *Let A be a finitely generated algebra over a field k and Ω the algebraic closure of k.*

(a) If $A \neq \{0\}$, there exists a nonzero k-homomorphism from A to Ω.

(b) Let ϕ_1, ϕ_2 be two k-homomorphisms from A to Ω. For ϕ_1 and ϕ_2 to have the same kernel, it is necessary and sufficient that there exist a k-automorphism σ of Ω such that $\phi_2 = \sigma \circ \phi_1$.

(c) Let $\mathfrak{a}$ be an ideal of A. For $\mathfrak{a}$ to be maximal, it is necessary and suffieient that it be the kernel of a k-homomorphism from A to Ω.

(d) For an element x of A to be such that $\phi(x) = 0$ for every k-homomorphism from A to L, it is necessary and sufficient that x be nilpotent.

Proof. Assertion (a) follows Corollary 4.3.4 applied replacing A by k, B by A, b by the unit element of B and ϕ by the canonical injection of k into Ω.

If ϕ is a k-homomorphism from A to Ω, $\phi(A)$ is a subring of Ω containing k. As Ω is an algebraic extension of k, $\phi(A)$ is a field and, if $\mathfrak{a}$ is the kernel of ϕ, $A/\mathfrak{a}$ is isomorphic to $\phi(A)$, therefore is a field, which proves that $\mathfrak{a}$ is maximal. Conversely, if $\mathfrak{a}$ is a maximal ideal of A it follows from (a) that there exists a k-homomorphism from $A/\mathfrak{a}$ to Ω and hence a k-homomorphism of A to Ω whose kernel $\mathfrak{b}$ contains $\mathfrak{a}$. But as $\mathfrak{a}$ is maximal, $\mathfrak{b} = \mathfrak{a}$, this proves (c).

We now prove (b). If σ is a k-automorphism of Ω such that $\phi_2 = \sigma \circ \phi_1$, clearly ϕ_1 and ϕ_2 have the same kernel. Conversely, suppose that ϕ_1 and ϕ_2 have the same kernel. Then there exists a k-isomorphism σ_0 of the field $\phi_1(A)$ onto the field $\phi_2(A)$ such that $\phi_2 = \sigma \circ \phi_1$. But by **??**, σ_0 extends to a k-automorphism σ of Ω and hence $\phi_2 = \sigma \circ \phi_1$.

Finally, if $x \in A$ is such that $x^n = 0$, for every k-homomorphism ϕ from A to Ω, $(\phi(x))^n = \phi(x^n) = 0$ and hence $\phi(x) = 0$ since Ω is a field. Conversely, suppose that $x \in A$ is not nilpotent. Then A_x is a finitely generated A-algebra (and therefore a finitely generated k-algebra) not reduced to 0 and hence there exists a k-homomorphism ψ from A_x to Ω by (a). If $j : A \to A_x$ is the canonical homomorphism, $\phi = \psi \circ j$ a k-homomorphism from A to Ω and $\phi(x)\psi(1/x) = \psi(x/1)\psi(1/x) = \psi(1) = 1$, whence $\phi(x) \neq 0$. □

Lemma 4.3.7. *Let A be a finitely generated algebra over a field k, $(a_i)_{1 \leq i \leq n}$ a system of generators of this algebra and $\mathfrak{r}$ the ideal of algebraic relations between the a_i with eoeflieients in k. For every extension field L of k, the map $\phi \mapsto (\phi(a_i))$ is a bijection of the set of k-homomorphisms from A to L onto the set of zeros of $\mathfrak{r}$ in L^n.*

Proof. There exists a unique k-algebra homomorphism η of $k[X_1, \dots, X_n]$ onto A such that $\eta(X_i) = a_i$ for $1 \leq i \leq n$ and by definition $\mathfrak{r}$ is the kernel of η. The map $\phi \mapsto \phi \circ \eta$ is a bijection of the set of k-homomorphisms from A to Ω onto the set of k-homomorphisms from $k[X_1, \dots, X_n]$ to Ω which are zero on $\mathfrak{r}$. For every polynomial $P \in k[X_1, \dots, X_n]$ and every element $\boldsymbol{x} = (x_1, \dots, x_n) \in L^n$ we write $\eta_{\boldsymbol{x}}(P) = P(\boldsymbol{x})$. Then the map $\boldsymbol{x} \mapsto \eta_{\boldsymbol{x}}$ is a bijection of L^n onto the set of k-homomorphisms from $k[X_1, \dots, X_n]$ to L (such a homomorphism being determined by its values at the X_i). To say that $\eta_{\boldsymbol{x}}$ is zero on $\mathfrak{r}$ means that $\boldsymbol{x}$ is a zero of $\mathfrak{r}$ in L^n, whence the lemma. □

Proposition 4.3.8 (Hilbert's Nullstellensatz). *Let k be a field and Ω an algebraic closure of k.*

(a) Every proper ideal $\mathfrak{r}$ of $k[X_1, \dots, X_n]$ admits at least one zero in Ω^n.

(b) Let $\boldsymbol{x}, \boldsymbol{y}$ be two elements of Ω^n. For the set of polynomials of $k[X_1, \dots, X_n]$ zero at $\boldsymbol{x}$ to be identical with the set of polynomials of $k[X_1, \dots, X_n]$ zero at $\boldsymbol{y}$, it is necessary and sufficient that there exists a k-automorphism σ such that $\boldsymbol{y} = \sigma(\boldsymbol{x})$.

(c) For an ideal $\mathfrak{a}$ of $k[X_1, \dots, X_n]$ to be maximal, it is necessary and sufficient that there exist an $\boldsymbol{x}$ in Ω^n such that $\mathfrak{a}$ is the set of polynomials of $k[X_1, \dots, X_n]$ zero at $\boldsymbol{x}$.

(d) For a polynomial P of $k[X_1, \dots, X_n]$ to be zero on the set of zeros in Ω^n of an ideal $\mathfrak{r}$ of $k[X_1, \dots, X_n]$, it is necessary and sufficient that there exist an integer $n > 0$ such that $P^n \in \mathfrak{r}$.

Proof. Apply Proposition 4.3.6 to the algebra $A = k[X_1, \dots, X_n]/\mathfrak{r}$ and use Lemma 4.3.7, we obtain part (a) and (d). Also, part (b) and (c) follows by the same argument applied to $A = k[X_1, \dots, X_n]$. □

4.3.2 Jacobson rings

Proposition 4.3.9. *Let A be a ring. Then the following conditions are equivalent.*

(i) Every prime ideal of A is the intersection of a family of maximal ideals.

(ii) For every ideal $\mathfrak{a}$ of A, the Jacobson radical of $A/\mathfrak{a}$ be equal to its nilradical.

(iii) For every ideal $\mathfrak{a}$ of A, the radical of $\mathfrak{a}$ is the intersection of the maximal ideals of A containing $\mathfrak{a}$.

(iv) Every prime ideal in A which is not maximal is equal to the intersection of the prime ideals which contain it strictly.

Proof. The Jacobson radical (resp. nilradical) of $A/\mathfrak{a}$ is the intersection of the maximal (resp. prime) ideals of $A/\mathfrak{a}$. Thus (ii) means for every ideal $\mathfrak{a}$ of A the intersection of the prime ideals containing $\mathfrak{a}$ is equal to the intersection of the maximal ideals containing $\mathfrak{a}$. This condition obviously holds for every ideal $\mathfrak{a}$ of A if (i) holds, and it implies (i) by taking $\mathfrak{a}$ to be prime ideals of A. This proves the equivalence of (i) and (ii). Also, it is clear that (iii) is equivalent to (ii)

Now assume (i) and let $\mathfrak{p}$ be a prime ideal of A that is not maximal. Then by (i), since $\mathfrak{p}$ is non maximal, we have

$$\mathfrak{p} = \bigcap_{\mathfrak{m} \supsetneq \mathfrak{p}, \mathfrak{m} \in \mathrm{Max}(A)} \mathfrak{m} \supseteq \bigcap_{\mathfrak{q} \supsetneq \mathfrak{p}, \mathfrak{q} \in \mathrm{Spec}(A)} \mathfrak{q} \supseteq \mathfrak{p}$$

whence (iv) holds. Conversely, suppose there is a prime ideal which is not an intersection of maximal ideals. Passing to the quotient ring, we may then assume that A is an integral domain whose Jacobson radical $\mathfrak{r}$ is not zero. Let a be a non-zero element of $\mathfrak{r}$. Then $A_a \neq 0$ since a is not nilpotent, hence has a maximal ideal, whose contraction in A is a prime ideal $\mathfrak{p}$ such that $a \notin \mathfrak{p}$. Since $a \in \mathfrak{r}$ we see $\mathfrak{p}$ is not maximal. Moreover, by our choice of $\mathfrak{p}$, every prime ideal of A that strictly contains $\mathfrak{p}$ has nonempty with the multiplicative subset $S = \{a^n : n > 0\}$ and therefore contains a. Hence a is contained in the intersection of prime ideals containing $\mathfrak{p}$ strictly, which therefore is not equal to $\mathfrak{p}$. This shows the equivalence of (i) and (iv) and completes the proof. □

A ring A is called a **Jacobson ring** if it satisfies the equivalent conditions of Proposition 4.3.9.

Example 4.3.10 (Examples of Jacobson rings).

(a) Every field is a Jacobson ring.

(b) The ring $\mathbb{Z}$ is a Jacobson ring, the unique prime ideal which is not maximal (0) being the intersection of the maximal ideals (p) of $\mathbb{Z}$, where p runs through the set of prime numbers.

(c) Let A be a Jacobson ring and let $\mathfrak{a}$ be an ideal of A. Then $A/\mathfrak{a}$ is a Jacobson ring, for the ideals of $A/\mathfrak{a}$ are of the form $\mathfrak{b}/\mathfrak{a}$, where $\mathfrak{b}$ is an ideal of A containing $\mathfrak{a}$ and $\mathfrak{b}/\mathfrak{a}$ is prime (resp. maximal) if and only if $\mathfrak{b}$ is.

Proposition 4.3.11. *Let A be a PID and $(p_i)_{i\in I}$ a representative system of irreducible elements of A. For A to be a Jacobson ring, it is necessary and sufficient that I be infinite.*

Proof. The maximal ideals of A are the (p_i). If I is finite, their intersection is the ideal (x) where $x = \prod_i p_i$ and hence different from (0). On the other hand, if I is infinite, the intersection of the (p_i) is (0), since every nonzero element of A can be divisible by only a finite number of irreducible elements. The proposition then follows from the fact that (0) is the only prime ideal which is not maximal in A. □

Proposition 4.3.12. *Let A be a ring and B an A-algebra integral over A. If A is a Jacobson ring, so is B.*

Proof. Replacing A by its canonical image in B, we may assume that $A \subseteq B$. Let $\mathfrak{P}$ be a prime ideal of B and let $\mathfrak{p} = A \cap \mathfrak{P}$. There exists by hypothesis a family $(\mathfrak{m}_i)_{i\in I}$ of maximal ideals of A whose intersection is equal to $\mathfrak{p}$. For all $i \in I$ there exists a maximal ideal $\mathfrak{M}_i$ of B lying over $\mathfrak{m}_i$ and containing $\mathfrak{P}$ by the going up theorem. If we write $\mathfrak{Q} = \bigcap_i \mathfrak{M}_i$, then $\mathfrak{Q} \cap A = \bigcap_i \mathfrak{m}_i = \mathfrak{p}$ and $\mathfrak{Q} \supseteq \mathfrak{P}$, whence $\mathfrak{Q} = \mathfrak{P}$. □

Proposition 4.3.13. *Let A be a Jacobson ring, B a finitely generated A-algebra and $\rho : A \to B$ the canonical homomorphism. Then B is a Jacobson ring and for every maximal ideal $\mathfrak{M}$ of B, $\mathfrak{m} = \mathfrak{M}^c$ is a maximal ideal of A and $B/\mathfrak{M}$ is a finite algebraic extension of $A/\mathfrak{m}$.*

Proof. Let $\mathfrak{P}$ be a prime ideal of B and $\mathfrak{p} = \mathfrak{P}^c$. Let v be a nonzero element of $B/\mathfrak{P}$. As $B/\mathfrak{P}$ is a finitely generated integral $(A/\mathfrak{p})$-algebra and the canonical homomorphism $\eta : A/\mathfrak{p} \to B/\mathfrak{P}$ is injective, there exists a nonzero element u of $A/\mathfrak{p}$ such that, for every homomorphism ϕ from $A/\mathfrak{p}$ to an algebraically closed field Ω such that $\phi(u) \neq 0$, there exists a homomorphism ψ from $B/\mathfrak{P}$ to Ω such that $\psi(v) \neq 0$ and for which $\phi = \psi \circ \eta$ (Corollary 4.3.4). Since A is a Jacobson ring, there exists a maximal ideal $\mathfrak{m}$ of A containing $\mathfrak{p}$ and such that $u \notin \mathfrak{m}/\mathfrak{p}$. We take Ω to be an algebraic closure of $A/\mathfrak{m}$ and ϕ to be the canonical homomorphism $A/\mathfrak{p} \to \Omega$. Let $\psi : B/\mathfrak{P} \to \Omega$ be a homomorphism such that $\phi = \psi \circ \eta$ and $\psi(v) \neq 0$. Then

$$A/\mathfrak{m} \subseteq \phi(B/\mathfrak{P}) \subseteq \Omega$$

hence $\psi(B/\mathfrak{P})$ is a subfield of Ω and the kernel of ψ is therefore a maximal ideal of $B/\mathfrak{P}$ not containing v. Thus it is seen that the intersection of the maximal ideals of $B/\mathfrak{P}$ is reduced to 0, which proves that B is a Jacobson ring. Moreover, if $\mathfrak{P}$ is maximal, ψ is necessarily injective and hence $\mathfrak{p} = \mathfrak{m}$ is maximal. Finally $B/\mathfrak{P}$ is then a finitely generated algebra over the field $A/\mathfrak{m}$ and hence is a finite extension of $A/\mathfrak{m}$. □

Corollary 4.3.14. *Let A be a Jacobson ring. If there exists a finitely generated A-algebra B containing A and which is a field, then A is a field and B is an algebraic extension of A.*

Proof. It suffices to apply Proposition 4.3.13 with $\mathfrak{M} = (0)$. □

Corollary 4.3.15. *Every finitely generated algebra A over $\mathbb{Z}$ is a Jacobson ring. For a prime ideal $\mathfrak{p}$ of A to be maximal, it is necessary and sufficient that the ring $A/\mathfrak{p}$ be finite.*

Proof. If the integral domain $A/\mathfrak{p}$ is finite, it is a field, as, for every $u \neq 0$ in $A/\mathfrak{p}$, the map $v \mapsto uv$ of $A/\mathfrak{p}$ to itself is injective and hence bijective since $A/\mathfrak{p}$ is finite. Conversely, for every maximal ideal $\mathfrak{m}$ of A, the inverse image of $\mathfrak{m}$ in $\mathbb{Z}$ is a maximal ideal (p) and $A/\mathfrak{m}$ is finite over the prime field $\mathbb{Z}/(p) = \mathbb{F}_p$ by Proposition 4.3.13. □

Remark 4.3.16. Let A be a Jacobson ring. Then the ring $A[\![X]\!]$ is not Jacobson: in fact, let $\mathfrak{p}$ be a prime ideal of A, then $\mathfrak{p}A[\![X]\!]$ is not an intersection of maximal ideals of $A[\![X]\!]$ since any such ideal contains $X \cdot A[\![X]\!]$.

Theorem 4.3.17. *Let A be a ring. Then the following are equivalent.*

(i) A is a Jacobson ring.

(ii) Every finitely generated A-algebra B that is a field is finite over A.

Proof. First assume (ii). Let $\mathfrak{p}$ be a prime ideal of A which is not maximal, and let $B = A/\mathfrak{p}$. Let v be a non-zero element of B. Then $B_v = B[v^{-1}]$ is a finitely generated A-algebra. If it is a field then by condition (ii) it is finite over A, hence integral over B and therefore B is a field by Proposition 4.1.64. Since $\mathfrak{p}$ is not maximal, we then see B_v is not a field and therefore has a non-zero maximal ideal, whose contraction in B is a non-zero prime ideal $\mathfrak{q}$ such that $v \notin \mathfrak{q}$. By Proposition 4.3.9(iv), this shows A is a Jacobson ring.

Conversely, let A be a Jacobson ring and B be a finitely generated A-algebra that is a field. Since a quotient ring of a Jacobson ring is Jacobson, we can assume B contains A as a subring. Let $s \in A$ be such that for every homomorphism ϕ from A to an algebraically closed field Ω such that $\phi(s) \neq 0$, there exists a homomorphism $\tilde{\phi}$ from B to Ω extending ϕ (Corollary 4.3.4). Since A is then a integral domain and Jacobson, there exists a maximal ideal $\mathfrak{m}$ of A not containing s. We take Ω to be the algebraic closure of $A/\mathfrak{m}$ and ϕ the canonical homomorphism $A \to A/\mathfrak{m} \to \Omega$. Then ϕ extends to a homomorphism ψ of B into Ω. Since B is a field, ψ is injective, and so B is algebraic (thus finite algebraic) over $A/\mathfrak{m}$, whence finite over A. □

Since a field is Jacobson, we get immediately the following statement.

Corollary 4.3.18 (Zariski's Lemma). *Let k ba a field and K a finitely generated k-algebra. If K is a field then it is a finite field extension of k.*

4.4 Exercise

Exercise 4.4.1. Let A be a Noetherian ring and $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ be all the minimal prime ideals of A. Suppose that $A_\mathfrak{p}$ is an integral domain for all $\mathfrak{p} \in \operatorname{Spec}(A)$. Then

(a) $\operatorname{Ass}(A) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$.

(b) $\mathfrak{n}(A) = \bigcap_{i=1}^n \mathfrak{p}_i = (0)$.

(c) $\mathfrak{p}_i + \bigcap_{j\neq i} \mathfrak{p}_j = A$ for all i.

Deduce that $A \cong A/\mathfrak{p}_1 \times \cdots \times A/\mathfrak{p}_n$.

Proof. Recall $\mathfrak{n}(A_\mathfrak{p}) = (\mathfrak{n}(A))_\mathfrak{p}$ from Corollary 1.2.39, but $(\mathfrak{n}(A))_\mathfrak{p} = (0)$ for all $\mathfrak{p} \in \operatorname{Spec}(A)$, so by Proposition 1.3.18 $\mathfrak{n}(A) = \bigcap_{i=1}^n \mathfrak{p}_i = (0)$. By Corollary 3.1.9,we then have

$$\operatorname{Ass}(A) \subseteq \bigcup_{i=1}^n \operatorname{Ass}(A/\mathfrak{p}_i) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}.$$

Since $\operatorname{supp}(A) = \operatorname{Spec}(A)$ and $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ are the minimal element of $\operatorname{Spec}(A)$, we then get $\operatorname{Ass}(A) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$.

We first prove that $\mathfrak{p}_i + \mathfrak{p}_j = A$ for any $i \neq j$: Assume the converse, then $\mathfrak{p}_i + \mathfrak{p}_j \subseteq \mathfrak{m}$ for some maximal ideal $\mathfrak{m}$. Then, in the localization $A_\mathfrak{m}$ there are two distinct minimal ideals $\mathfrak{p}_i A_\mathfrak{m}$ and $\mathfrak{p}_j A_\mathfrak{m}$. Thus

$$\mathfrak{n}(A_\mathfrak{m}) = \mathfrak{p}_i A_\mathfrak{m} \cap \mathfrak{p}_j A_\mathfrak{m}$$

Since $A_\mathfrak{m}$ is an integral domain, and $\mathfrak{p}_i \cap \mathfrak{p}_j \neq 0$, this is a contradiction.

Now since $\mathfrak{p}_i + \mathfrak{p}_j = A$, for $j \neq i$ we have

$$x_j + y_j = 1, \quad x_j \in \mathfrak{q}_i, y_j \in \mathfrak{q}_j$$

Now

$$\prod_{j \neq i} (1 - x_i) = \prod_{j \neq i} y_j \in \bigcap_{j \neq i} \mathfrak{q}_j$$

and expand the term $\prod_{j \neq i}(1 - x_i)$ we get $1 + x$ for $x \in \mathfrak{q}_i$. Hence $\mathfrak{q}_i + \bigcap_{j \neq i} \mathfrak{p}_j = A$.

Finally, $\mathfrak{p}_i$ and $\mathfrak{p}_j$ are coprime, by the Chinese remainder theorem we have the claimed isomorphism. $\square$

Chapter 5

Valuation rings

5.1 Valuation rings of fields

Let A be an integral domain, K its field of fractions. A is a **valuation ring** if, for each $x \in K$ and $x \neq 0$, either $x \in A$ or $x^{-1} \in A$ (or both). The case $A = K$ is the trivial valuation ring.

Proposition 5.1.1. *Let A be an integral domain, K its field of fractions. Then A is a valuation ring of K if and only if idels of A is totally ordered by inclusion.*

Proof. Assume that A is a valuation ring of K and let I, J be two ideals such that there is $b \in J$, $b \notin I$. Now for $a \in I$, either $a/b \in A$ or $b/a \in A$. If $b/a \in A$, then $b = (b/a) \cdot a \in I$, contradiction. So $a/b \in A$, and $a = b(a/b) \in J$. Hence $I \subseteq J$. Conversely, assume that ideals of A are totally ordered. Let $a, b \in A$. Then $(a) \subseteq (b)$ or $(b) \subseteq (a)$. Hence it is clear $a/b \in A$ or $b/a \in A$. This shows A is a valuation ring. □

Corollary 5.1.2. *If A is a valuation ring and $\mathfrak{p}$ is a prime ideal of A, then $A/\mathfrak{p}$ and $A_{\mathfrak{p}}$ are valuation rings of their fields of fractions.*

Proposition 5.1.3. *If A is a valuation ring of K, then*

(a) A is a local ring and if we write $\mathfrak{m}$ for the maximal ideal, then

$$K - A = \{x \in K^{\times} : x^{-1} \in \mathfrak{m}\}$$

Thus A is determined by K and $\mathfrak{m}$.

(b) A is integrally closed.

Proof. By Proposition 5.1.1, A has a unique maximal ideal, so is a local ring. Moreover, if $x \notin A$, then $x^{-1} \in A$ and x^{-1} is not a unit in A, so $x^{-1} \in \mathfrak{m}$. conversely, if $x^{-1} \in \mathfrak{m}$, then $x \notin A$ since x^{-1} is not a unit. Thus

$$K - A = \{x \in K^{\times} : x^{-1} \in \mathfrak{m}\}$$

Now given K and $\mathfrak{m}$, the set $K - A$ is then determined, so $A = K - (K - A)$ is uniquely determined.

Let $x \in K$ be integral over A. Then we have, say,

$$x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0$$

with the $b_i \in A$. If $x \in A$ there is nothing to prove. If not, then $x^{-1} \in A$, hence $x = -(a_{n-1} + a_{n-2}x^{-1} + \cdots + a_0x^{-n}) \in A$. □

Proposition 5.1.4. *Let A be an integral domain, then the followings are equivalent:*

(i) A is a valuation ring.

(ii) A is a local domain and any finitely generated ideal is principal.

Proof. Let A be a valuation ring. By Proposition 5.1.3 A is a local ring. Let $I = (x_1, \ldots, x_n)$, since by Proposition 5.1.1 the set $\{(x_1), \ldots, (x_n)\}$ is totally ordered. Let's assume that (x_1) is the biggest element in it, then $(x_i) \subseteq (x_1)$ for all i, that is, $x_i = a_i x_1$. This implies I is generated by x_1.

Conversely, assume (b), and let $\mathfrak{m}$ be the unique maximal ideal of A. Write $\mathfrak{m}$ as the unqiue maximal ideal of A, so $J(A) = \mathfrak{m}$. Let $a, b \in A$, and write $(a, b) = (h)$. Then we have

$$a = uh,\ b = vh, \quad h = xa + yb, \quad u, v, x, y \in A.$$

Then $h = xuh + yvh$ are since A is a domain we get $1 = xu + yv$. If u is not a unit, then $u \in \mathfrak{m}$ and hence $1 - xu = yv$ is a unit. This implies v is a unit, and vice versa. Hence either $a/b \in A$ or $b/a \in A$, so A is a valuation ring. □

Proposition 5.1.5. *Let A be a valuation ring. Every finitely generated torsion-free A-madule is free. Every torsion-free A-module is flat.*

Proof. Let M be a finitely generated torsion-free A-module and let $x_1, \ldots, x_n$ be generators of E which are minimal in number; we show that they are linearly independent. If $\sum_{i=1}^n a_i x_i = 0$ is a non-trivial relation between the x_i, one of the a_i's, say a_1, divides all the others since the set of principal ideals of A is totally ordered by inclusion; then $a_1 \neq 0$ since the relation is non-trivial. As M is torsion-free, we can divide by a_1, which amounts to assuming that $a_1 = 1$. But then x_1 is a linear combination of $x_2, \ldots, x_n$, contrary to the minimal character of n. Hence M is free. The second claim follows since every finitely generated ideal is principal. □

Quite generally, we write $\mathfrak{m}_A$ for the maximal ideal of a local ring A. If A and B are local rings with $B \subseteq A$ and $\mathfrak{m}_B \cap A = \mathfrak{m}_A$, we say that B **dominates** A and write $B \geq A$. If $B \geq A$ and $B \neq A$, we write $B > A$.

Proposition 5.1.6. *Let A and B be local rings such that $A \subseteq B$. Then the following conditions are equivalent:*

(i) $\mathfrak{m}_A \subseteq \mathfrak{m}_B$;

(ii) B dominates A;

(iii) the ideal generated by $\mathfrak{m}_A$ in B is proper.

Proof. If $\mathfrak{m}_A \subseteq \mathfrak{m}_B$, then $\mathfrak{m}_B \cap A$ is a proper prime ideal containing the maximal ideal of A, so $\mathfrak{m}_A = \mathfrak{m}_B \cap A$ and B dominates A. If B dominates A, then the ideal $\mathfrak{m}_A B$ is contained in $\mathfrak{m}_B$ and so is proper, this shows (ii)⇒(iii). Finally, if (iii) holds, $\mathfrak{m}_A B$ is contained in the unique maximal ideal of $\mathfrak{m}_B$ of B, hence (i). □

Theorem 5.1.7. *Let K be a field, $A \subseteq K$ a subring, and $\mathfrak{p}$ a prime ideal of A. Then there exists a valuation ring B of K satisfying*

$$A \subseteq B, \quad \mathfrak{m}_B \cap A = \mathfrak{p}.$$

Moreover, valuation rings are maximal with respect to the domination relation.

Proof. Replacing A by $A_{\mathfrak{p}}$ we can assume that A is a local ring with $\mathfrak{p} = \mathfrak{m}_A$. Now write $\mathscr{F}$ for the set of all subrings of K containing A and to which $\mathfrak{p}$ extended to a nontrivial ideal. Now $A \in \mathscr{F}$, and if $\mathscr{L} \subseteq \mathscr{F}$ is a subset totally ordered by inclusion then the union of all the elements of $\mathscr{L}$ is again an element of $\mathscr{F}$, so that, by Zorn's lemma $\mathscr{F}$ has a maximal element B. Since $\mathfrak{p}^e \neq B$, there is a maximal ideal $\mathfrak{m}$ of B containing $\mathfrak{p}^e$. Then $B \subseteq B_{\mathfrak{m}} \in \mathscr{F}$, so that $B = B_{\mathfrak{m}}$, and B

is local. Also $\mathfrak{p} \subseteq \mathfrak{m} \cap A$ and $\mathfrak{p}$ is a maximal ideal of A, so that $\mathfrak{m} \cap A = \mathfrak{p}$. Thus it only remains to prove that B is a valuation ring of K.

Let $x \in K$. If $x \notin B$ then since $B[x] \notin \mathscr{F}$ and $B[x] \supseteq A$, we must have $1 \in \mathfrak{p}B[x]$, and there exists a relation of the form

$$1 = a_0 + a_1 x + \cdots + a_n x^n, \quad a_i \in \mathfrak{p}B$$

Since $a_0 \in \mathfrak{p}B \subseteq \mathfrak{m} = J(B)$, $1 - a_0$ is unit of B and we can modify this to get a relation

$$1 = b_1 x + \cdots + b_n x^n, \quad b_i \in \mathfrak{m} \tag{5.1.1}$$

Among all such relations, choose one for which n is as small as possible. If we also have $x^{-1} \notin B$, then by the same argument, there is a relation

$$1 = c_1 x^{-1} + \cdots + c_m x^{-m}, \quad c_i \in \mathfrak{m} \tag{5.1.2}$$

for which m is as small as possible. If $n \geq m$ we can multiply (5.1.2) by $b_n x^n$ and substract from (5.1.1) to obtain a relation of the form (5.1.1) but with a strictly smaller degree n, which is a contradiction; if $n < m$ then we get the same contradiction on interchanging the roles of x and x^{-1}. Thus if $x \notin B$ we must have $x^{-1} \in B$.

Finally, if $A \subseteq B$ are valuation rings of K, then by Proposition 5.3.2 we have $\mathfrak{m}_B \subseteq \mathfrak{m}_A$, and $\mathfrak{m}_A = \mathfrak{m}_B$ iff $A = B$. It follows that B dominates A iff $A = B$. □

Theorem 5.1.8. *Let K be a field, $A \subseteq K$ a subring, and let $\overline{A}$ be the integral closure of A in K. Then $\overline{A}$ is the intersection of all the valuation rings of K containing A. If A is local, $\overline{A}$ is the intersection of the valuation rings of K which dominate A.*

Proof. Every element $x \in \overline{A}$ is integral over A, so over any valuation ring $B \supseteq A$. But since B is integer closed, we must have $x \in B$, thus $\overline{A} \subseteq B$ for every valuation ring containing A. Conversely, let $x \notin \overline{A}$. Then $x \notin A[x^{-1}]$, so x^{-1} is not a unit of $A[x^{-1}]$. It follows that x^{-1} is contained in a maximal ideal $\mathfrak{M}$ of $A[x^{-1}]$, and by Theorem 5.1.7 there exists a valuation ring B of K such that $B \supseteq A[x^{-1}]$ and $\mathfrak{m}_B \cap A[x^{-1}] = \mathfrak{M}$. Now since $x^{-1} \in \mathfrak{M} \subseteq \mathfrak{m}_B$, we have $x \notin B$. Further $\mathfrak{M} \cap A$ is a maximal ideal of A, hence, if A is local, $\mathfrak{M} \cap A = \mathfrak{m}_A$ and B dominates A. □

Corollary 5.1.9. *For an integral domain to be integrally closed, it is necessary and sufficient that it be the intersection of a family of valuation rings of its field of fractions.*

Corollary 5.1.10. *Let K be a field, L an extension of K and A a valuation ring of K. The integral closure of A in L is the intersection of the valuation rings B of L such that $B \cap K = A$.*

Proof. If B is a valuation ring of L, then $B \cap K$ is a valuation ring of K and B dominates $B \cap K$. For B to dominate A, it is necessary and sufficient that $B \cap K$ dominate A and therefore be equal to it. □

5.2 Ordered groups and valuations

Let Γ be a commutative group. A order $\leq$ on Γ is said to be compatible with the group structure of Γ if for $\alpha, \beta, \gamma, \delta \in \Gamma$,

$$\alpha \geq \beta, \gamma \geq \delta \Rightarrow \alpha + \gamma \geq \beta + \delta.$$

In this case, the group Γ is called an **ordered group**, and a **totally ordered group** if its order is a total order. In this section we only consider totally ordered groups and if Γ is an ordered group, we usually make an ordered set $\Gamma \cup \{\infty\}$ by adding to Γ an element ∞ bigger than all the elements of Γ, and fix the conventions $\infty + \alpha = \infty$ for $\alpha \in \Gamma$ and $\infty + \alpha = \infty$.

Example 5.2.1 (Examples of totally ordered groups).

(a) The additive group of real numbers $\mathbb{R}$, or any subgroup of this.

(b) The direct product $\mathbb{Z}^n$ of n copies of $\mathbb{Z}$, with lexicographical order.

(c) The group $\mathbb{Q}$ of rational integers.

Theorem 5.2.2. *For a commutative group Γ to be such that there exists on Γ a total ordering compatible with the group struclure of Γ, it is necessary aad sufficient that Γ be torsion free.*

Proof. If there exists such an order structure on Γ and if $\alpha > 0$, then $\alpha + \mu > 0$ for all $\mu > 0$ and in particular, by induction we see $n\alpha \geq 0$ which proves that Γ is torsion free. Conversely, if Γ is torsion-free, it is a sub-$\mathbb{Z}$-module of a vector $\mathbb{Q}$-space, which may be assumed of the form $\mathbb{Q}^{\oplus I}$. If I is given a well ordering and $\mathbb{Q}^{\oplus I}$ its usual ordering, the set $\mathbb{Q}^{\oplus I}$ with the lexicographical ordering is totally ordered. It is immediate that this ordering is compatible with the additive group structure of $\mathbb{Q}^{\oplus I}$. $\square$

Let K be a field and Γ a totally ordered group. A map $v : K \to \Gamma \cup \{\infty\}$ is called an **additive valuation** or just a **valuation** of K if it satisfies the conditions

(a) $v(xy) = v(x) + v(y)$.

(b) $v(x + y) \geq \min\{v(x), v(y)\}$.

(c) $v(x) = \infty$ if and only if $x = 0$.

If we write $K^\times$ for the multiplicative group of K then v defines a homomorphism $K^\times \to \Gamma$ whose image is a subgroup of Γ, called the **value group** of v. The valuation v defined by $(x) = 0$ for any $x \in K^\times$ is called the **trivial valuation**.

Proposition 5.2.3. *Let v be a valuation on a field K. For any elements $x_1, \dots, x_n \in K$, we have*

$$v(\sum_{i=1}^{n} x_i) \geq \min\{v(x_1), \dots, v(x_n)\}.$$

Moreover, if there exists a single index k such that $v(x_k) = \min_i v(x_i)$, then the equality holds. In particular, if $v(x) \neq v(y)$, then $v(x + y) = \min\{v(x), v(y)\}$.

Proof. The first inequality follows by induction. Now if there exists a single index k such that $v(x_k) = \min_i v(x_i)$, then, writing $y = \sum_{i \neq k} x_i$ and $z = \sum_{i=1}^{n} x_i$, we have $v(y) > v(x_k)$ and $v(z) \geq v(x_k)$. If $v(z) > v(x_k)$, then the relation $x_k = z - y$ implies $v(x_k) \geq \min\{v(y), v(z)\} > v(x_k)$, which is absurd. Thus $v(z) = v(x_k)$ and the claim is proved. $\square$

Valuations are connected to valuation rings in the following way. If v is a valuation on K, we set

$$A_v = \{x \in K \mid v(x) \geq 0\}, \quad \mathfrak{m}_v = \{x \in K \mid v(x) > 0\}, \quad \kappa_v = A_v/\mathfrak{m}_v.$$

Then by the properties of v we see that A_v is a valuation ring and $\mathfrak{m}_v$ is a maximal ideal. We call A_v the **valuation ring** of v, $\mathfrak{m}_v$ the **valuation ideal** of v, and κ_v the **residue field** of v. Conversely, if A is a valuation ring of K, then the set $\Gamma = \{xA : x \in K^\times\}$ is a group with operation defined by $xA \cdot yA = xyA$. Moreover, if we define an order $\leq$ on Γ by

$$xA \leq yA \iff yA \subseteq xA \iff y/x \in A$$

then Γ becomes a totally ordered group with $\leq$ (since A is a valuation ring), and we obtain an additive valuation of K with value group Γ by setting $v(x) = xA$ and $v(0) = \infty$. It is easy to see the valuation ring of v is A.

The additive valuation corresponding to a valuation ring A is not quite unique, so we introduce the following notation. Two valuations v and w on a field K with value groups Γ_v and Γ_w are called equivalent if and only if there exists an order-isomorphism $\varphi : \Gamma_v \to \Gamma_w$ such that $w = \varphi \circ v$.

Theorem 5.2.4. *Let K be a field. Then valuation rings of K are in one-to-one correspondence to equivalent class of valuations on K.*

Proof. One direction is clear: If two valuations are equivalent, then they determine the same valuation ring. For the converse, assume that v and w are valuations on K with value groups Γ_v and Γ_w such that $A_v = A_w$. We define the homomorphism $\varphi : \Gamma_v \to \Gamma_w$ by

$$\varphi(v(x)) = w(x) \quad \text{for} \quad x \in K^\times.$$

Since $U(A_v) = U(A_w)$, it is easy to see φ is well defined and injective. Clearly φ satisfies the condition $w = \varphi \circ v$ and is surjective. Now, for $x, y \in K^\times$ we have

$$\varphi(v(x) + v(y)) = \varphi(v(xy)) = w(xy) = w(x) + w(y),$$

and

$$v(x) > 0 \iff x \in A_v = A_w \iff w(x) = \varphi(v(x)) > 0.$$

Thus φ is an order-isomorphism from Γ_v to Γ_w and v and w are equivalent. □

In view of this proposition, we think of valuation rings and additive valuations as being two aspects of the same thing, and we may assume that all valuations are *surjective*. In this part, we use this point of view to establish some results for valuation rings.

The first non-trivial examples of valuations are the ***p*-adic valuation** v_p on $\mathbb{Q}$ where p is a prime number, and similarly the p-adic valuation v_p on the rational function field $k(X)$ where p is any irreducible polynomial from $k[X]$, k being an arbitrary field. In the second case, v_p restricted to k is trivial.

There is one more interesting valuation on $k[X]$, trivial on k. This is the degree valuation v_∞ defined for non-zero polynomials $f, g \in k[X]$ by

$$v_\infty(f/g) = \deg g - \deg f.$$

One easily checks the axioms of a valuation. Moreover, f/g is a unit in the valuation ring A_{v_∞} if and only if $\deg f = \deg g$. Thus if

$$f = \sum_{i=0}^{n} a_i X^i$$

with $a_i \in k$ and $a_n \neq 0$, then we see that $f(X)/X^n$ is a unit that maps to a_n in the residue class field. Hence it follows that the residue class field is exactly k.

Proposition 5.2.5. *Every non-trivial valuation on $\mathbb{Q}$ is a p-adic valuation for some rational prime p. Every non-trivial valuation on $k(X)$, trivial on k, is either the degree valuation v_∞ or a p-adic valuation for some irreducible polynomial $p \in k[X]$.*

Proof. Let K be either $\mathbb{Q}$ or $k(X)$ and v some non-trivial valuation on K. Then the valuation ring A is different from K. In the second case v is assumed to be trivial on k. This means that $k \subseteq A$.

First consider the case $K = \mathbb{Q}$. Since $1 \in A$, we have $\mathbb{Z} \subseteq A$. As $A \neq \mathbb{Q}$, at least one prime p must lie in $\mathfrak{m}_A$. If q is a prime different from p, we have $ap + bq = 1$ for some $a, b \in \mathbb{Z}$, whence $q \notin \mathfrak{m}_A$. Hence all primes $q \neq p$ are units in A. Using the factorization of integers we therefore see that for $a, b \in \mathbb{Z}$, relatively prime, we have $a/b \in A$ iff $p \nmid b$, whence $A = \mathbb{Z}_{(p)}$, i.e., v is equivalent to the p-adic valuation v_p.

Let $K = k(X)$ and $\mathbb{Z} \subseteq A \neq K$. If $X \in A$, then $k[X] \subseteq O$ and we can argue as in the case of $\mathbb{Q}$ replacing $\mathbb{Z}$ by $k[X]$. If $X \notin A$, then $X^{-1} \in \mathfrak{m}_A$. In this case $v(X) < 0$ and $v(X^m) < v(X^n)$ whenever $0 < n < m$. Since $v(a) = 0$ for every $a \in k^\times$, we get

$$v(a_n X^n + \cdots + a_1 X + a_0) = v(a_n X^n) = nv(X)$$

in the case $a_n \neq 0$. Hence the value group of v is $v(X)\mathbb{Z}$. By sending $v(X)$ to -1, we therefore get an order-preserving isomorphism with $\mathbb{Z}$, showing that v is equivalent to the degree valuation. □

Definition 5.2.6. A subset Δ of a totally ordered group Γ is called a **segment** if Δ is non-empty and if for any element α of Γ which belongs to Δ, all the elements β of Γ which lie between α and $-\alpha$ also belong to Δ. A subgroup of Γ is called an **isolated subgroup** if it is a segment of Γ.

Isolated subgroups occurs naturally in the theory of totally ordered groups due to the following proposition.

Proposition 5.2.7. *A subgroup Δ of Γ is isolated if and only if it is the kernel of an order homomorphism from Γ to a totally ordered group.*

Proof. Let $\phi : \Gamma \to \Gamma'$ be an order-homomorphism between to tally ordered groups and $\Delta = \ker \phi$. we show that Δ is isolated. In fact, if $\alpha \in \Delta$ and β lie between α and $-\alpha$, then since $\phi(\alpha) = \varphi(-\alpha)$, we must have $0 \leq \phi(\beta) \leq 0$, whence $\beta \in \Delta$.

Conversely, let Δ be a isolated subgroup of Γ and Γ/Δ be the quotient group. We define an order on Γ/Δ by

$$\alpha + \Delta \leq \beta + \Delta \iff \alpha \leq \beta.$$

This definition makes sense since for $\alpha > 0$ and $\alpha \notin \Delta$ we have $\alpha > \beta$ for all $y \in \Delta$. It is easy to see Γ/Δ becoms a tally ordered group and the quotient map $\pi : \Gamma \to \Gamma/\Delta$ is an order homomorphism with kernel Δ. □

Note that the set of segements of a tally ordered group Γ is totally ordered by inclusion. Therefore, we defined the **rank** of Γ, denoted by $\mathrm{rank}(\Gamma)$, to be the maximal length of ascending chains in this set (just like the definition of the Krull dimension of a ring). The **rank of a valuation** is defined to be the rank of the corresponding value group.

Using the valuation map, there is a connection between idelas of a valuation ring and isolated subgroups of its value group, which we now explain.

Theorem 5.2.8. *Let $v : K^\times \to \Gamma$ be a valuation and A be its valuation ring. For any subset $S \subseteq A$, we define*

$$\Delta_S = \{\alpha \in \Gamma : \alpha \notin v(S) \text{ and } \alpha \notin -v(S)\}.$$

(a) If I is an ideal of A then Δ_I is a segement in Γ, and is an isolated subgroup if and only if I is prime.

(b) The map $I \mapsto \Delta_I$ is an order-reversing bijection from the set of all ideals of A onto the set of all segments of Γ.

Proof. Let I be an ideal of A. To show Δ_I is a segement, we only need to verify that, if $\alpha \notin \Delta_I$ and $\beta > \alpha > 0$, then $\beta \notin \Delta_I$. Now by definition, if $\alpha \notin \Delta_I$ and $\alpha > 0$, then $\alpha \in v(I)$. Since $\beta > \alpha$, we have $\beta - \alpha \in v(A)$, and hence $\beta = \alpha + \beta - \alpha \in v(A)v(I) \subseteq v(I)$. Furthermore, from the definition of Δ_I, it is easy to see Δ_I is a subgroup if and only if I is prime. For (b), we construct an inverse of the map $I \mapsto \Delta_I$. Let Δ be a segment in Γ, and define

$$I_\Delta = \{a \in A : v(a) > \alpha \text{ for all } \alpha \in \Delta\}.$$

It is easy to see I_Δ is an ideal of A, and we have $\Delta_{I_\Delta} = \Delta$, $I_{\Delta_I} = I$, so the claim is proved. □

Corollary 5.2.9. *The rank of a valuation equals the Krull dimension of its valuation ring.*

5.2.1 Valuations of rank one

As an important example, we consider the condition under which a valuation has rank 1. By Corollary 5.2.9, we know this happens if and only if $\dim A_v = 1$. Here we provides another characterization.

Proposition 5.2.10. *Let v be a nontrivial valuation, Γ be the value group of v, and A the valuation ring. Then the following are equivalent.*

(i) $\mathrm{rank}(v) = 1$, *or equivalently* $\dim A = 1$.

(ii) Γ *is order isomorphic to a subgroup of* $\mathbb{R}$.

(iii) Γ *is* ***Archimedean***, *that is, for any $\alpha, \beta \in \Gamma$ with $\alpha > 0$, there exist $n \in \mathbb{N}$ such that $nx > y$.*

Proof. We will show that (iii)⇒(ii)⇒(i)⇒(iii). First, assume that Γ is Archimedean, and we may suppose that $\Gamma \neq \{0\}$. Fix some $0 < \alpha \in \Gamma$. For any $\beta \in \Gamma$, there is a well-defined largest integer n such that $n\alpha \leq \beta$ (if $\beta \geq 0$ this is clear by assumption; if $\beta < 0$, let m be the smallest integer such that $-\beta \leq m\alpha$, and set $n = -m$), which we will denote by n_0. Now set $\beta_1 = \beta - n_0\alpha$ and let n_1 be the largest integer n such that $n\alpha < 10\beta_1$; we have $0 \leq n_1 < 10$. Set $\beta_2 = 10\beta_1 - n_1\alpha$ and let n_2 be the largest integer n such that $n\alpha < 10\beta_2$. Continuing in the same way, we find a sequence of integers $\{n_i\}$ and set $\phi(\beta)$ to be the real number given by the decimal expression $n_0.n_1n_2\dots$. Then it can easily be checked that $\phi : \Gamma \to \mathbb{R}$ is order-preserving and injective. Now let $\beta, \beta' \in \Gamma$. For $r \in \mathbb{N}$ arbitrary, let n and n' be the interger determined by the inequalities

$$n\alpha \leq 10^r\beta < (n+1)\alpha, \quad n'\alpha \leq 10^r\beta' \leq (n'+1)\alpha.$$

Then we see $(n+n')\alpha \leq 10^r(\beta+\beta') < (n+n'+2)\alpha$. If we use $\phi_r(\beta)$ to denote the number obtained by taking the first r decimal places of $\phi(\beta)$, then $n = \phi_r(\beta)$, $n' = \phi_r(\beta')$, and hence

$$\begin{aligned}|\phi(\beta+\beta') - \phi(\beta) - \phi(\beta')| = |\phi(\beta+\beta') - \phi_r(\beta+\beta')| + |\phi_r(\beta+\beta') - \phi_r(\beta) - \phi_r(\beta')| \\ + |\phi_r(\beta) - \phi(\beta)| + |\phi(\beta') - \phi_r(\beta')| \leq 4 \cdot 10^{-r}.\end{aligned}$$

Since r is arbitrary, we see ϕ is a homomorphism, hence Γ is order isomorphic to a subgroup of $\mathbb{R}$.

Now let Γ be a nonzero subgroup of $\mathbb{R}$. Since $\Gamma \neq 0$, A is not a field. Suppose that $\mathfrak{p}$ is a prime ideal of A distinct from $\mathfrak{m}_A$. Let $a \in \mathfrak{m}_A - \mathfrak{p}$ and set $v(a) = \alpha$. Suppose that $0 \neq b \in \mathfrak{p}$, and set $\beta = v(b)$; then $\beta \in \Gamma$ and $\alpha > 0$, so that $n\alpha > y$ for some sufficiently large natural number n. This means $a^n/b \in A$, so that $a^n \in bA \subseteq \mathfrak{p}$; then since $\mathfrak{p}$ is prime we have $a \in \mathfrak{p}$, which is a contradiction. Thus $\mathfrak{p} = (0)$ and the only prime ideals in A are (0) and $\mathfrak{m}_A$. That is, $\dim A = 1$.

Finally, assume that $\dim A = 1$. If $0 \neq b \in \mathfrak{m}_A$ then $\mathfrak{m}_A$ is the unique prime ideal containing b, and hence $\sqrt{bA} = \mathfrak{m}_A$. Thus for any $a \in \mathfrak{m}_R$ there exists a natural number n such that $a^n \in bA$. From this one sees easily that Γ satisfies the Archimedean axiom. □

Proposition 5.2.11. *Let K be a field, v a nontrivial valuation on K and A the ring of v. For A to be completely integrally closed, it is necessary and sufficient that v be of rank* 1.

Proof. Suppose v is of rank 1. Let $x \in K$ be such that the $A[x]$ are all contained in a finitely generated sub-A-module of K. There exists $d \in A - \{0\}$ such that $dx^n \in A$ for all $n \geq 0$. Then $v(d) + nv(x) \geq 0$, that is $n(-v(x)) \leq v(d)$ for all $n \geq 0$, whence $-v(x) \leq 0$ (since Γ_v is Archimedean) and $x \in A$. Thus A is completely integrally closed.

Suppose now that v is not of rank 1. Then there exist $y \in \mathfrak{m}_v$ and $t \in A$ such that $nv(y) < v(t)$ for all $n \geq 0$. Then $ty^{-n} \in A$ for all $n \geq 0$, but $y^{-1} \notin A$. Hence A is not completely integrally closed. □

Corollary 5.2.12. *Let K be a field, $(v_i)_{i\in I}$ a family of valuations of rank 1 on K and A the intersection of the rings of the v_i. Then A is a completely integrally closed domain.*

Proof. An intersection of completely integrally closed rings is completely integrally closed, hence the claim. □

5.2.2 Discrete valuation rings

A valuation ring whose value group is isomorphic to $\mathbb{Z}$ is called a **discrete valuation ring** (DVR). Discrete refers to the fact that the value group is a discrete subgroup of $\mathbb{R}$.

Example 5.2.13. The two standard examples are:

(a) $K = \mathbb{Q}$. Take a fixed prime p, then any non zero $x \in \mathbb{Q}$ can be written uniquely in the form $p^a y$, where $a \in \mathbb{Z}$ and both numerator and denominator of y are prime to p. Define $v_p(x)$ to be a. The valuation ring of v_p is the local ring $\mathbb{Z}_{(p)}$.

(b) $K = k(X)$, where k is a field and X an indeterminate. Take a fixed irreducible polynomial $f \in k[X]$ and define v_f just as in (a). The valuation ring of v_f is then the local ring $k[X]_{(f)}$.

(c) Let k be a field, and let $k((X))$ be the field of formal Laurent series in one variable over k. For every non-zero series
$$f(X) = \sum_{n\geq n_0} a_n X^n, \quad a_{n_0} \neq 0$$
one defines the order $v(f)$ of f to be the integer n_0. One obtains thereby a discrete valuation of $k((X))$, whose valuation ring is $k[\![X]\!]$, the set of formal series; its residue field is k.

Proposition 5.2.14. *Let A be a valuation ring. Then the following conditions are equivalent.*

(a) A is a DVR.

(b) A is a PID.

(c) A is Noetherian.

(d) The value group Γ of A is discrete under order topology and has rank 1.

Proof. Let K be the field of fractions of A and $\mathfrak{m}$ its maximal ideal. Since A is a valuation ring, we know finitely generated idels of A are principal, so A is a PID if and only if it is Noetherian. Now, we first show that if A_v is Noetherian then v must has rank 1. For suppose that $\mathrm{rank}(v) > 1$, there must exists a nontrivial isolated subgroup Δ. Fix a positive element $x \in \Delta$, then we get a chain of elements
$$x < 2x < \cdots < nx < \cdots .$$
Since Δ is proper, we can find $y \in G$ which does not belong to Δ. Then since Δ is isolated, y is bigger than any elements in Δ, and in particular bigger than any nx, $n \in \mathbb{N}$. Equivalently, we get a decreasing sequence of positive elements in G:
$$y - x > y - 2x > \cdots > y - nx > \cdots > 0$$
which gives an infinite strictly descending segements in G, hence an infinite ascending sequence of ideals in A_v (by Theorem 5.2.8). Hence R_v is not Noetherian.

Now since v has rank 1, G can be identified as a subgroup of $\mathbb{R}$. Moreover, the maximal ideal $\mathfrak{m}$ is principal, which means G has a smallest positive element. Then we see G is discrete and hence isomorphic to $\mathbb{Z}$. The converse is obvious, since the group $\mathbb{Z}$ has no infinite descending sequence of positive elements.

Finally, it is clear that $\mathbb{Z}$ is discrete and has rank 1. Conversely, if the value group G of A has rank 1 then it can be embedded into $\mathbb{R}$, and is isomorphic to $\mathbb{Z}$ if it is discrete. This completes the proof. □

If A is a DVR with maximal ideal $\mathfrak{m}$ then an element $t \in R$ such that $\mathfrak{m} = (t)$ is called a **uniformising element** (or **prime element**) of A.

Example 5.2.15. A valuation ring B whose maximal ideal $\mathfrak{m}_B$ is principal does not have to be a DVR. In fact, the maximal ideal of a valuation ring is principal if and only if its value group G has a smallest positive element, if and only if G is discrete under order topology.

To obtain a counter-example, let K be a field, and A a DVR of K. If $\bar{B}$ is a DVR for the residue field $k = A/\mathfrak{m}_A$, then their composite B will be a valuation ring of K with rank bigger than 2, hence not a DVR. However, let $s \in B$ be a preimage of a uniformising element $\bar{s}$ of $\bar{B}$. Then $s \notin \mathfrak{m}_A$ and we have

$$\mathfrak{m}_A \subseteq \mathfrak{m}_B \subseteq B \subseteq A, \quad \mathfrak{m}_B/\mathfrak{m}_A = \bar{s}\bar{B},$$

so $\mathfrak{m}_B = \mathfrak{m}_A + sB$. On the other hand, since $s \in A - \mathfrak{m}_A$, we have $s^{-1} \in A$ and thus $s^{-1}\mathfrak{m}_A \subseteq A \subseteq B$. This implies $\mathfrak{m}_A \subseteq sB$, so $\mathfrak{m}_B = sB$.

The previous theorem gives a characterisation of DVRs among valuation rings; now we consider characterisations among all rings.

Proposition 5.2.16. *Let A be a ring with fraction field K. Then the following conditions are equivalent:*

(a) A is a DVR.

(b) A is a local PID, and not a field.

(c) A is a Noetherian local ring with positive dimension and the maximal ideal $\mathfrak{m}_A$ is principal.

(d) A is a one-dimensional normal Noetherian local ring.

Proof. If A is a DVR, conditions (b), (c), (d) hold by Proposition 5.2.14, Proposition 5.1.3 and Proposition 4.1.39. Conversely, we now show that either one of these three conditions implies A is a DVR.

If A is a local PID and not a field, then A is Noetherian and the maximal ideal $\mathfrak{m}$ is nonzero and principal, so (b) implies (c). Now assume (c), and let $\mathfrak{m} = (t)$ be the maximal ideal of A. Then t can not by nilpotent since otherwise $\mathfrak{m}$ is the unique prime ideal of A, whence $\dim A = 0$. By the Krull intersection theorem we have $\bigcap_{n=1}^{\infty}(t^n) = (0)$, so that for $0 \neq x \in A$ there is a well-determined n such that $x \in (t^n)$ but $x \notin (t^{n+1})$. If $x = t^n u$, then since $u \notin (t)$ it must be a unit. Similarly, for $0 \neq y \in A$ we have $y = t^m v$, with v a unit. Therefore $yz = t^{m+n}uv \neq 0$, and so A is an integral domain. Finally, any element α of K can be written $\alpha = x^{\nu}u$, with u a unit of A and $\nu \in \mathbb{Z}$, and it is easy to see that setting $v(t) = \nu$ defines an additive valuation of K whose valuation ring is A.

Finally, assume that A is a one-dimensional noraml Noetherian local ring. Let x be a nonzero element of $\mathfrak{m}$. By hypothesis, $\mathfrak{m}$ is the only nonzero prime ideal, so $\sqrt{(x)} = \mathfrak{m}$. Then by **??** (x) contains a power of $\mathfrak{m}$, and we may assume that $(x) \neq \mathfrak{m}$. Then there exists $n > 1$ such that $\mathfrak{m}^n \subseteq (x)$ but $\mathfrak{m}^{n-1} \nsubseteq (x)$. If $y \in \mathfrak{m}^{n-1} - (x)$ and $\beta = x/y \in K$, then we have $\beta^{-1} = y/x \notin A$ since $y \notin (x)$. Since A is integrally closed, β^{-1} is not integral over A. But then $\beta^{-1}\mathfrak{m} \nsubseteq \mathfrak{m}$, in view of **??**. Now use $y \in \mathfrak{m}^{n-1}$ we have

$$\beta^{-1}\mathfrak{m} = (y/x)\mathfrak{m} \subseteq (1/x)\mathfrak{m}^n \subseteq (1/x)(x) = A.$$

Thus $\beta^{-1}\mathfrak{m}$ is an ideal of A, and it must be A since $\beta^{-1}\mathfrak{m} \nsubseteq \mathfrak{m}$. Hence $\mathfrak{m}$ is the principal ideal (β), and by the preceeding argument we can show that A is a DVR. □

To conclude this part, we introduce the concept of rational rank of a totally ordered group Γ, denoted by $\mathrm{rank}_{\mathbb{Q}}(\Gamma)$, which is defined to be the maximal number of rationally independent elements of Γ. In other words,

$$\mathrm{rank}_{\mathbb{Q}}(\Gamma) := \dim_{\mathbb{Q}}(\mathbb{Q} \otimes_{\mathbb{Z}} \Gamma).$$

Note that the rational rank of a group Γ is zero if and only if Γ is a torsion group. If Γ is a value group of a valuation, Γ is totally ordered, then its rational rank is zero if and only if $\Gamma = \{0\}$, i.e. if and only if the valuation is the trivial valuation.

Proposition 5.2.17. *Let Γ be an abelian group and Δ a subgroup of Γ. Then*

(a) $\mathrm{rank}_{\mathbb{Q}}(\Gamma) = \mathrm{rank}_{\mathbb{Q}}(\Gamma/\Delta) + \mathrm{rank}_{\mathbb{Q}}(\Delta)$.

(b) If Γ is ordered then $\mathrm{rank}(\Gamma) = \mathrm{rank}(\Gamma/\Delta) + \mathrm{rank}(\Delta)$.

(c) If Γ is ordered then $\mathrm{rank}(\Gamma) \le \mathrm{rank}_{\mathbb{Q}}(\Gamma)$.

Proof. Since $\mathbb{Q}$ is $\mathbb{Z}$-flat (it is torsion free), we see that

$$(\mathbb{Q} \otimes_{\mathbb{Z}} \Gamma)/(Q \otimes_{\mathbb{Z}} \Delta) = \mathbb{Q} \otimes_{\mathbb{Z}} (\Gamma/\Delta)$$

whence the first equality follows. Now it is easy to see the isolated groups in Γ/Δ are in one-to-one correspondence to isolated subgroups in Γ containing Δ, so the second equality follows. Finally, assume that Γ is ordered and let $\Delta_0 \le \Delta_1 \le \cdots \le \Delta_{s-1}$ be a strict chain of isolated subgroups of Γ. For each i, choose an element $\alpha_i \in \Delta_i \setminus \Delta_{i-1}$ (with $\Delta_s = \Gamma$), then we see $\alpha_1, \dots, \alpha_r$ are rationally independent: let $\sum_{i=1}^{t} r_i \alpha_i = 0$ with $r_t \neq 0$ and $t \le s$, then $r_t \alpha_t \in \Delta_{t-1}$, and since Δ_{r-1} is a segment and $r_t \neq 0$, we find $\alpha_t \in \Gamma_{t-1}$, contradiction. Thus $\mathrm{rank}(\Gamma) \le \mathrm{rank}_{\mathbb{Q}}(\Gamma)$. □

5.3 Compare valuation rings of a field

As an immediate observation, if $\mathfrak{p}$ is a prime ideal of A, then the localized ring $A_{\mathfrak{p}}$ is a local ring containing A. Therefore, $A_{\mathfrak{p}}$ is also a valuation of K. We have seen in Theorem 5.2.8 that such a prime ideal $\mathfrak{p}$ corresponds to a proper isolated subgroup Δ of G. Now we determine the value group of $A_{\mathfrak{p}}$.

Proposition 5.3.1. *The value group of $A_{\mathfrak{p}}$ is isomorphic to the quotient group Γ/Δ, where Δ is the isolated subgroup of Γ corresponds to $\mathfrak{p}$. In this way the valuation $v_{\mathfrak{p}}$ associated to $A_{\mathfrak{p}}$ is given by the composition of $v : K^\times \to \Gamma$ and $\pi : \Gamma \to \Gamma/\Delta$.*

Proof. We define $\tilde{v} = \pi \circ v : K^\times \to \Gamma/\Delta$, which is easily seen to be a valuation of K. If we embedd $A_{\mathfrak{p}}$ in K, then we have $v(A_{\mathfrak{p}}) = \Delta \cup \Gamma_+$. On the other hand, we observe that

$$A_{\tilde{v}} = \{x \in K : \pi(v(x)) \ge 0\} = \{x \in K : v(x) > \Delta \text{ or } v(x) \in \Delta\} = A_{\mathfrak{p}},$$

so the claim follows from Theorem 5.2.4. □

With Proposition 5.3.1, we can now classifies all valuation rings of K containing A: they are in one-to-one correspondence to prime ideals of A.

Proposition 5.3.2. *Let A be a valuation ring and $\mathfrak{m}_A$ be the maximal ideal of A.*

(a) Any localization $A_{\mathfrak{p}}$ of A at a prime ideal $\mathfrak{p}$ of A is a valuation ring of K containing A, and $\mathfrak{p}$ is the maximal ideal of $A_{\mathfrak{p}}$.

(b) *The map $\mathfrak{p} \mapsto A_{\mathfrak{p}}$ is a bijection from the set of prime ideals of A onto the set of rings B with $A \subseteq B \subseteq K$, which is orderreversing for the relation of inclusion. The inverse map is defined by $B \mapsto \mathfrak{m}_B$.*

Proof. We have already seen that $A_{\mathfrak{p}}$ is a valuation ring containing A. Now, from the definition of the valuation ideal and Proposition 5.3.1, we have

$$\mathfrak{m}_{A_{\mathfrak{p}}} = \{x \in K : v_{\mathfrak{p}}(x) > 0\} = \{x \in K : \pi(v(x)) > 0\} = \{x \in K : v(x) > \Delta\} = \mathfrak{p}.$$

Therefore $\mathfrak{p}$ is the maximal ideal of $A_{\mathfrak{p}}$.

Now let B be a ring such that $A \subseteq B \subseteq K$. Note that if $x \in \mathfrak{m}_B$ then $x^{-1} \notin B \supseteq A$, and hence $x \in \mathfrak{m}_A$. This implies $\mathfrak{m}_B \subseteq \mathfrak{m}_A \subseteq A \subseteq B$. Also, note that by (a), $\mathfrak{m}_B$ is the maximal ideal of both $A_{\mathfrak{m}_B}$ and B. Since a valuation ring is uniquely determined by its maximal ideal, we get $B = A_{\mathfrak{m}_B}$, hence the claim. □

Since $A_{\mathfrak{p}}$ is a valuation ring and $\mathfrak{p}$ is its maximal ideal, we can consider its residue field $\kappa_{\mathfrak{p}} = A_{\mathfrak{p}}/\mathfrak{p}$. Then since A is a valuation ring, it is easy to see $\bar{A} = A/\mathfrak{p}$ is a valuation ring of $\kappa_{\mathfrak{p}}$. Moreover, we can determine the value group of $\bar{A}$.

Proposition 5.3.3. *Let $\mathfrak{p}$ be a prime ideal of A with corresponding isolated subgroup Δ. Then $\bar{A}$ is a valuation ring of κ_v with value group Δ.*

Proof. Let $\pi : A_{\mathfrak{p}} \to A_{\mathfrak{p}}/\mathfrak{p}$ be the quotient map. Then since $A/\mathfrak{p}$ is local with maximal ideal $\mathfrak{m}_A/\mathfrak{p}$, we see $U(A/\mathfrak{p}) = \pi(U(A))$, hence there is an isomorphism ψ given by

$$\psi : U(A_{\mathfrak{p}})/U(A) \to U(A_{\mathfrak{p}}/\mathfrak{p})/U(A/\mathfrak{p}), \quad x + U(A) \mapsto \pi(x) + U(A/\mathfrak{p}).$$

Also, we note that $U(A_{\mathfrak{p}})/U(A) \cong \Delta$, with the isomorphism given by

$$\psi : U(A_{\mathfrak{p}})/U(A) \to \Delta, \quad x + U(A) \mapsto v(x).$$

With these observations, we now define a valuation on $\kappa_v^{\times} = U(A_{\mathfrak{p}}/\mathfrak{p})$ by

$$\bar{v} : U(A_{\mathfrak{p}}/\mathfrak{p}) \to \Delta, \quad \bar{v}(\pi(x)) = \psi(\phi^{-1}(\pi(x) + U(A/\mathfrak{p}))) = v(x).$$

With this, it is easy to check that

$$A_{\bar{v}} = \{\pi(x) : \bar{v}(\pi(x)) \geq 0\} = \{\pi(x) : v(x) \geq 0\} = A/\mathfrak{p}.$$

Thus the claim follows. □

It turns out that all valuation rings contained in a given valuation ring arise in this way: we have the following proposition.

Proposition 5.3.4. *Let A be a valuation ring of K and $\pi : A \to A/\mathfrak{m}_A$ be the canonical map.*

(a) *Let $B \subseteq A$ be a valuation ring of K, then the image $\bar{B}$ under π_A is a valuation ring of κ_v.*

(b) *Let $\bar{B}$ be a valuation ring of κ_v, then its preimage $B \subseteq A$ under π_A is a valuation ring of K.*

Proof. Let $B \subseteq A$ be a valuation ring of K and $\pi(x) \in \kappa_v - \bar{B}$. Then $x \notin B$, and hence $x^{-1} \in B$. Since $\pi(x^{-1}) = \pi(x)^{-1}$, we see $\pi(x)^{-1} \in \bar{B}$, hence $\bar{B}$ is a valuation ring of κ_v.

Conversely, let $\bar{B}$ be a valuation ring of K and B be its preimage under π. Let $x \in K - B$, then $\pi(x) \in \kappa_v - \bar{B}$ and hence $\pi(x^{-1}) \in \bar{B}$. Then it follows that $x^{-1} \in B$ so B is a valuation of K. □

The formulation of Proposition 5.3.4 can be defined on valuations as follows. Let $v : K^\times \to \Gamma$ be a valuation of K with residue field κ_v, and let $\bar{v} : \kappa_v \to \bar{\Gamma}$ be a valuation of κ_v. Then we can define a new valuation w of K by letting its valuation ring to be

$$A_w = \{a \in A : \bar{v}(\bar{a}) \geq 0\}.$$

The valuation w is called the composite of v and $\bar{v}$, and denoted by $w = v \circ \bar{v}$.

Let v and w be two valuations of K. We say w dominates v, or $w \geq v$, if $A_w \subseteq A_v$ or equivalently $\mathfrak{m}_w \supseteq \mathfrak{m}_v$. Then Proposition 5.3.4 just says all valuations $w \geq v$ are composite of v. By Proposition 5.3.1 and Proposition 5.3.3, the following result is immediate.

Proposition 5.3.5. *If w is the composite valuation $v \circ \bar{v}$ we have the equalities*

$$\operatorname{rank}(w) = \operatorname{rank}(v) + \operatorname{rank}(\bar{v}), \quad \operatorname{rank}_{\mathbb{Q}}(w) = \operatorname{rank}_{\mathbb{Q}}(v) + \operatorname{rank}_{\mathbb{Q}}(\bar{v}).$$

In fact, by Proposition 5.3.1 and Proposition 5.3.3, if we have the valuations v of K and $\bar{v}$ of κ_v, then the composite valuation $w = v \circ \bar{v}$ defines an extension of the value group Γ of v by the value group $\bar{\Gamma}$ of $\bar{v}$, i.e. an exact sequence of totally ordered groups:

$$0 \longrightarrow \bar{\Gamma} \longrightarrow \widetilde{\Gamma} \longrightarrow \Gamma \longrightarrow 0$$

If this exact sequence splits, the value group $\widetilde{\Gamma}$ is isomorphic to the group $\Gamma \times \bar{\Gamma}$ with the lexicographic order. If the valuation v is a discrete valuation of rank one, i.e. for $\Gamma = \mathbb{Z}$, the exact sequence always splits (this follows from the observation that $\mathbb{Z}$ can be embedded into any nonzero totally ordered group) and we can describe the composite valuation $w = v \circ \bar{v}$ in the following way. The maximal ideal of the valuation ring A_v associated to v is generated by an element t and we can associate to any non zero element x in K the nonzero element $\pi(xt^{-v(x)})$ in the residue field, where $\pi : A_v \to \kappa_v$ is the canonical map. The composite valuation w is then defined by $w(x) = (v(x), \bar{v}(\pi(xt^{-v(x)}))$.

5.4 Dependent valuations and induced topology

With the notation of composite, we say two valuations v, w of K are **dependent** if they are the composite of a common nontrivial valuation. If these do not hold, we say v and w are **independent**. Equivalently, by Proposition 5.3.4 v and w are independent if and only if the valuation rings A_v and A_w are not contained in a proper valuation ring of K, if and only if AB, the smallest ring containing A and B, is a proper subring of K.

we see from Proposition 5.3.2 that the dependence relation is an equivalence relation. Indeed, if A and B are dependent and also B and C are dependent, then AB and BC are both overrings of B. Thus there is some inclusion, say $AB \subseteq BC$. But then A and B are both contained in BC. Hence they are dependent. We shall now study the topology induced by a valuation on a field K. Let $v : K^\times \to \Gamma$ be a valuation of K. For each $\gamma \in \Gamma$ and each $x \in K$ we define the set

$$U_\gamma(x) = \{y \in K : v(y - x) > \gamma\}.$$

These sets form a basis of open neighborhoods of x, due to the following observations:

- $x \in U_\gamma(x)$,
- If $y \in U_\gamma(x) \cap U_{\gamma'}(x')$ and $\delta > \max\{\gamma, \gamma'\}$, then for $z \in U_\delta(y)$, we have

  $$v(z - x) = v(z - y + y - x) > \gamma, \quad v(z - x') = v(z - y + y - x') > \gamma'$$

  whence $U_\delta(y) \subseteq U_\gamma(x) \cap U_{\gamma'}(x')$.

Therefore, the sets $\{U_\gamma(x) : x \in K, \gamma \in \Gamma\}$ is a basis for a topology $\mathcal{T}_v$ on K, called the **topology induced by the valuation** v. Some properties of the topology $\mathcal{T}_v$ are gathered below.

Proposition 5.4.1. *Let v be a valuation on K and $\mathcal{T}_v$ the topology induced by v.*

(a) The topology $\mathcal{T}_v$ is Hausdorff and totally disconnected, and it is discrete if and only if v is trivial.

(b) The field operations are continuous with respect to $\mathcal{T}_v$.

(c) The valuation $v : K^\times \to \Gamma$ is continuous if Γ is given the discrete topology.

(d) The quotient topology induced on the residue field κ_v is discrete.

Proof. Let A be the valuation ring of v. Let $x, y \in K$ and $x \neq y$. If $\gamma = v(x-y)$, then $\gamma \neq \infty$ and we see $U_\gamma(x) \cap U_\delta(y) = \varnothing$ if $\gamma, \delta > v(x-y)$. Therefore $\mathcal{T}_v$ is Hausdorff. Also, it is easy to see $\mathcal{T}_v$ is discrete if and only if v is trivial. We next claim that the set $U_\gamma^c(x) = \{y \in K : v(y-x) \leq \gamma\}$ is also open in K. To prove this, we first observe that, if $v(z-y) > v(y-x)$ then we have $(z-y)/(y-x) \in \mathfrak{m}_A$, whence $1 + (z-y)/(y-x) = (z-x)/(y-x)$ is a unit in A. That is, we have

$$v(z-y) > v(y-x) \Rightarrow v(z-x) = v(y-x).$$

From this, it follows that $U_{v(y-x)} \subseteq U_\gamma(x)^c$ if $y \in U_\gamma(x)^c$, so $U_\gamma(x)^c$ is open. Since $U_\gamma(x)$ is a basis for $\mathcal{T}_v$, it follows that $\mathcal{T}_v$ is totally disconnected.

Since $v(x+y) \geq \min\{v(x), v(y)\}$, we have $U_\gamma(x) + U_\gamma(y) \subseteq U_\gamma(x+y)$. Moreover, from

$$ab - xy = (a-x)(b-y) + (a-x)y + (b-y)x$$

it follows that $U_\gamma(x)U_\gamma(y) \subseteq U_\delta(xy)$, where $\delta = \min\{2\gamma, \gamma + v(x), \gamma + v(y)\}$. These shows addition and multiplication are continuous under $\mathcal{T}_v$.

Let $x_0 \in K^\times$, if $x \in K^\times$ satisfies $v(x - x_0) > \max\{\gamma + 2v(x_0), v(x_0)\}$, then we have $v(x) = v(x_0)$ and

$$\begin{aligned} v(x^{-1} - x_0^{-1}) &= v(x^{-1}(x_0 - x)x_0^{-1}) = v(x_0 - x) - v(x) - v(x_0) \\ &> \gamma + 2v(x_0) - 2v(x_0) = \gamma \end{aligned} \tag{5.4.1}$$

hence the inverse map is also continuous. These proves (b). Also, the single condition $v(x - x_0) > v(x_0)$ implies $v(x) = v(x_0)$, hence the map $v : K^\times \to G$ is continuous if G is given the discrete topology. Finally, note that $\mathfrak{m}_v = U_0(0)$ is clopen in A and so the quotient topology is discrete. □

Theorem 5.4.2. *Two nontrivial valuations v and w of K are dependent if and only if they induce the same topology on K.*

Proof. Since two dependent valuation rings have a common non-trivial coarsening, to show that they induce the same topology it is enough to consider the particular case $A_v \subseteq A_w$.

Let $v : K^\times \to \Gamma$ be the valuation. Since $A_v \subseteq A_w$, by Proposition 5.3.2 there exists a isolated subgroup Δ of Γ such that w is given by $K^\times \to \Gamma \to \Gamma/\Delta$. Since w is nontrivial, $A_w \neq K$ and $\Delta \neq \Gamma$. Write

$$U_\gamma(0) = \{x \in K : v(x) > \gamma\}, \quad U_{\gamma+\Delta}(0) = \{x \in K : w(x) > \gamma + \Delta\},$$

then we see $U_{\gamma+\Delta}(0) \subseteq U_\gamma(0)$. On the other hand, if $v(x) > 2\gamma$ with $0 < \gamma \notin \Delta$ then $2\gamma > \gamma + \Delta$, hence $U_{2\gamma}(0) \subseteq U_{\gamma+\Delta}(0)$, whence v and w induces the same topology on K.

Conversely, let $\mathfrak{m}_v$ and $\mathfrak{m}_w$ be the valuation ideals of v and w. If v and w induce the same topology on K, then w is an open neighbourhood of 0 in $\mathcal{T}_v$, so there exists $a \in K^\times$ such that $a\mathfrak{m}_v \subseteq \mathfrak{m}_w$. As $\mathfrak{m}_w$ is the maximal ideal of the valuation ring A_w, the set $K - \mathfrak{m}_w$ is multiplicatively closed. Thus we can form the ring

$$A = \{x/y \in K : x \in A_v, y \in A_v - \mathfrak{m}_w\} = \{x/y \in K : v(x) \geq 0, v(y) \geq 0, w(y) \leq 0\}.$$

Since $A_v \subseteq A$, A is also a valuation ring. Moreover, A contins A_w since if $x \in A_w - A_v$ then $w(x) \geq 0$ and we have $x = 1/x^{-1} \in A$. Finally, $A \neq K$: let $z \in A_v \setminus \{0\}$, then $\frac{1}{az} \notin A$ since otherwise $1/(az) = x/y$ with $x \in A_v$ and $y \in A_v - \mathfrak{m}_w$, and we have $y = azx \in a\mathfrak{m}_v \subseteq \mathfrak{m}_w$, a contradiction. Hence v and w are dependent. □

Theorem 5.4.2 is another way to see that the dependence relation among the valuation rings of a field K is an equivalence relation, as we already saw above. Let A be a non-trivial valuation ring of K and take the dependence class $[A]$ of all non-trivial valuation rings of K dependent on A. Clearly $[A]$ is an upwardly directed set with respect to the partial order of inclusion.

Proposition 5.4.3. *Let A be any non-trivial valuation ring of K. Then we have the following case distinction:*

(a) $[A]$ has a maximal valuation ring B_0 which is a maximal non-trivial overring of A; moreover B_0 has dimension 1 and its maximal ideal is the intersection of all non-zero prime ideals of A, or

(b) there is no maximal non-trivial overring of A. Then the maximal ideals $\mathfrak{m}_B$ of valuation rings $B \in [A]$ form a neighborhood system of 0 for the topology induced by A. In this case the set of all non-zero prime ideals of A is also a neighborhood system of 0.

Proof. If $[A]$ has a maximal element B_0, then it is clear that B_0 contains A and is the maximal overring of A. Thus the first part follows from Proposition 5.3.2. Now assume that $[A]$ has no maximal element. Suppose we are given a positive $\delta \in \Gamma$. We seek a valuation ring $B \supseteq A$ such that $B \neq K$ and whose maximal ideal $\mathfrak{m}_B$ satisfies $\mathfrak{m}_B \subseteq U_\delta(0)$. Let Δ be the convex hull of the subgroup generated by δ in Γ, i.e.,

$$\Delta = \{\gamma \in \Gamma : \gamma, -\gamma < n\delta \text{ for some } n \in \mathbb{N}\}.$$

Then Δ defines a valuation ring $B \supseteq A$ with $\mathfrak{m}_B \subseteq U_\delta(0)$. It remains to show that $B \neq K$.

If $B = K$, then $\Delta = \Gamma$ and thus δ is an element of Γ such that Γ is the convex hull of $\delta\mathbb{Z}$. Let Δ^* be the largest isolated subgroup of Γ not containing δ. Then $\Gamma^* = \Gamma/\Delta^*$ is archimedean ordered and thus the overring A^* of A corresponding to Δ^* is maximal according to Proposition 5.3.2. This contradicts our assumption. □

We are now in a position to prove an approximation theorem for independent valuations. For this we need some preparations.

Lemma 5.4.4. *Let $v_1, \ldots, v_n$ be valuations on the field K and $x \in K^\times$. Then there exists a polynomial $f(X)$ of the form*

$$f(X) = 1 + n_1 X + \cdots + n_{k-1}X^{k-1} + X^k, \quad n_i \in \mathbb{Z}, k \geq 2 \tag{5.4.2}$$

such that $f(x) \neq 0$ and $z = f(x)^{-1}$ satisfies

$$\begin{aligned} v_i(z) &= 0 && \text{if } v_i(x) \geq 0, \\ v_i(z) + v_i(x) &> 0 && \text{if } v_i(x) < 0. \end{aligned}$$

Proof. Let I be the set of indices i such that $v_i(x) \geq 0$. For all $i \in I$, let x_i denote the canonical image of x in κ_{A_i}. For all $i \in I$ we construct a polynomial f_i as follows: if there exists a polynomial $g(X)$ of the form (5.4.2) such that $g(\bar{x}_i) = 0$ in κ_{A_i}, we take f_i to be such a polynomial; otherwise we take $f_i = 1$. Then we write $f(X) = 1 + X^2 \prod_{i \in I} f_i(X)$. It is obviously a polynomial of the form (5.4.2). If $i \in I$, then $f(x) \in A_i$ and also $f(\bar{x}_i) \neq 0$ by construction; hence $f(x) \notin \mathfrak{m}_i$, $v_i(f(x)) = 0$ and $v_i(z) = 0$. If $i \notin I$, then $v_i(x) < 0$, whence $v_i(f(x)) = kv_i(x)$ (by Proposition 5.2.3) and

$$v_i(x) + v(z) = (1-k)v_i(x) > 0,$$

so the lemma is proved. □

Proposition 5.4.5. *Let $A_1, \dots, A_n$ be valuation rings of a field K, $B = \bigcap_{i=1}^n A_i$ and $\mathfrak{p}_i = \mathfrak{m}_{A_i} \cap B$. Then $A_i = B_{\mathfrak{p}_i}$ and the fraction field of B is K.*

Proof. Clearly $B_{\mathfrak{p}_i} \subseteq A_i$ since $B \subseteq A_i$ and $B - \mathfrak{p}_i \subseteq A - \mathfrak{m}_i$ is a unit in A_i. Now let x be a nonzero element in A_i. We apply the lemma to x and valuations v_i associated with the A_i. Then $v_i(z) \geq 0$ and $v_i(zx) \geq 0$ for all i, hence $z, zx \in B$. As $v_i(x) \geq 0$ and $v_i(z) = 0$, we see $z \notin \mathfrak{p}_i$. Hence $x = xz/z \in B_{\mathfrak{p}_i}$. The field of fractions of B then contains A_i and hence is K. □

Proposition 5.4.6. *With the hypotheses of Proposition 5.4.5, suppose further that $A_i \nsubseteq A_j$ for $i \neq j$. Then the $\mathfrak{p}_i$'s are distinct maximal ideals of B and every maximal ideal of B is equal to one of the $\mathfrak{p}_i$.*

Proof. If $\mathfrak{p}_i \subseteq \mathfrak{p}_j$ for $i \neq j$ then $A_i = B_{\mathfrak{p}_i} \sup B_{\mathfrak{p}_j} = A_j$, which is a contradiction. Then the claim follows from Corollary 1.1.19. □

Corollary 5.4.7. *Suppose that $A_1, \dots, A_n$ are incomparable valuation rings. For every family of element $a_1 \in A_1, \dots, a_n \in A_n$, there exists $x \in B$ such that $x \equiv a_i \bmod \mathfrak{m}_i$ for each i.*

Proof. Since the $\mathfrak{p}_i$ are maximal ideals of B, $A/\mathfrak{m}_A = B_{\mathfrak{p}_i}/\mathfrak{p}_i B_{\mathfrak{p}_i} = B/\mathfrak{p}_i$ and it may therefore be assumed that $a_i \in B$ for all i. The corollary then follows from the fact that the canonical map from B to $\prod_{i=1}^n (B/\mathfrak{p}_i)$ is surjective. □

Corollary 5.4.8. *Suppose that $A_1, \dots, A_n$ are incomparable valuation rings. There exist element $x_1, \dots, x_n$ of K such that $v_i(x_i) = 0$ and $v_j(x_i) > 0$ for $j \neq i$.*

Proof. For each index i apply Corollary 5.4.7 to the family (a_i) such that $a_i = 1$ and $a_j = 0$ for $j \neq i$. □

Corollary 5.4.9. *Every valuation ring of K containing B contains one of the A_i.*

Proof. We may confine our attention to the case where A_i's are incomparable. Let A be a valuation ring of K containing B. We write $\mathfrak{p} = \mathfrak{m}_A \cap B$. There exists a maximal ideal $\mathfrak{p}_i$ of B containing $\mathfrak{p}$, whence $A_i = B_{\mathfrak{p}_i} \subseteq B_{\mathfrak{p}} \subseteq A$. □

Now for independent valuations, we have the following approximation theorem.

Theorem 5.4.10 (Approximation Theorem). *Suppose $A_1, \dots, A_n$ are pairwise independent valuation rings of K. For every i, let $v_i : K^\times \to \Gamma_i$ be the valuation corresponding to A_i. Then for any $a_1, \dots, a_n \in K$ and $\gamma_1 \in \Gamma_1, \dots, \gamma_n \in \Gamma_n$, there exists an $x \in K$ with*

$$v_i(x - a_i) > \gamma_i \quad \textit{for all } i \in \{1, \dots, n\}.$$

Proof. Let A_i be the valuation ring of v_i and set $B = \bigcap_{i=1}^n A_i$, $\mathfrak{p}_i = \mathfrak{m}_{A_i} \cap B$. By Proposition 5.4.5 we have $A_i = B_{\mathfrak{p}_i}$ and K is the fraction field of B, so the a_i may be written as $a_i = b_i/s$ with $b_i \in B$ and $s \in B \setminus \{0\}$. If we write $x = y/s$ and $\gamma_i' = \gamma_i + v(s)$, then

$$v_i(x - a_i) = v_i(y - b_i) - v(s) > \gamma_i \iff v_i(y - b_i) > \gamma_i'.$$

This shows that we may assume that $a_i \in B$ for all i; we may also assume that $\gamma_i > 0$ for all i. Define ideals of A_i and B by

$$I_i = \{z \in K : v_i(z) \geq \gamma_i\} \subseteq \mathfrak{m}_i, \quad J_i = I_i \cap B \subseteq \mathfrak{p}_i.$$

For $x \in B$, $v_i(x - a_i) \geq \gamma_i$ is equivalent to $x \equiv a_i \bmod I_i$. We therefore need to show that the canonical homomorphism $B \to \prod_{i=1}^n (B/J_i)$ is surjective, that is, J_i's are pairwise coprime. As the maximal ideals of B are the $\mathfrak{p}_i$, it will suffice for this to show that $J_i \nsubseteq \mathfrak{p}_j$ for $i \neq j$.

Suppose that there exists i, j such that $J_i \subseteq \mathfrak{p}_j$ and $i \neq j$. We shall see shortly that the radical of J_i is a prime ideal $\mathfrak{p}$ of B, so that $\mathfrak{p} \subseteq \mathfrak{p}_j$. Since $\gamma_i > 0$ we also have $J_i \subseteq \mathfrak{p}_i$, so $A_j = B_{\mathfrak{p}_j} \subseteq B_{\mathfrak{p}}$

and similarly $A_i \subseteq B_\mathfrak{p}$. Now, as $I_i \neq (0)$ and $I_i = B_{\mathfrak{p}_i} J_i$ (Proposition 1.2.37), $J_i \neq (0)$ whence $\mathfrak{p} \neq (0)$ and $B_\mathfrak{p} \neq K$. This contradicts the hypothesis that A_i and A_j are independent.

It remains to show that $\mathfrak{p} = \sqrt{J_i}$ is prime for each i. Suppose that $xy \in \mathfrak{p}$, so that $x^n y^n \in J_i$ for some i. Then, if for example $v(x) \geq v(y)$, we have

$$v(x^{2n}) = 2nv(x) \geq nv(x) + nv(y) = v(x^n y^n) \geq \gamma_i$$

whence $x^{2n} \in J_i$ and $x \in \mathfrak{p}$. This finishes the proof. □

Corollary 5.4.11. *For every family of elements $\gamma_1 \in \Gamma_1, \dots, \gamma_n \in \Gamma_n$, there exists $x \in K$ such that $v_i(x) = \gamma_i$.*

Proof. We may assume that $A_i \neq K$ for all i. Then, there exists for all i an $a_i \in K$ such that $v_i(a_i) = \gamma_i$, and an $\alpha_i \in \Gamma_i$ such that $\gamma_i < \alpha_i$. We apply Theorem 5.4.10 to these elements a_i there exists a $x \in K$ such that $v_i(x - a_i) > \gamma_i = v_i(a_i)$, whence $v_i(x) = v_i(a_i) = \gamma_i$. □

Here we introduce the concept of absolute values over a field, which has a deep connection with valuations of rank 1. Let K be a field. An **absolute value** on K is a map $|\cdot| : K \to \mathbb{R}$ satisfying the following axioms for all $x, y \in K$,

(a) $|x| > 0$ for all $x \neq 0$ and $|0| = 0$.

(b) $|xy| = |x||y|$.

(c) $|x + y| \leq |x| + |y|$.

The absolute value sending all $x \neq 0$ to 1 is called the **trivial absolute value** on K. Note that by (b) we have $|1| = 1$, $|-x| = |x|$, and $|x^{-1}| = |x|^{-1}$, thus $|\cdot|$ is a group homomorphism from $K^\times$ to $\mathbb{R}^\times$.

Proposition 5.4.12. *The set $\{|n \cdot 1| : n \in \mathbb{Z}\}$ is bounded if and only if $|\cdot|$ satisfies the **ultrametric inequality***

$$|x + y| \leq \max\{|x|, |y|\}$$

for all $x, y \in K$.

Proof. If $|\cdot|$ satisfies the inequality, then by induction, the set $\{|n \cdot 1| : n \in \mathbb{Z}\}$ is bounded by 1. Conversely, let $|n \cdot 1| \leq C$. Then

$$|x + y|^n = |(x + y)^n| \leq \sum_\nu \left| \binom{n}{\nu} x^\nu y^{n-\nu} \right| \leq (n + 1)C \max\{|x|, |y|\}^n.$$

Taking n-th roots and letting n go to infinity proves the assertion of the proposition. □

If an absolute value satisfies the ultrametric inequality, it is called **non-archimedean**; otherwise it is called **archimedean**. Clearly, if $\operatorname{char} K \neq 0$, K cannot carry any archimedean absolute value. A typical example of an archimedean absolute value is the ordinary absolute value on $\mathbb{R}$, which is given by $|x|_0 = x$ if $x \geq 0$ and $-x$ if $x < 0$. Note that if $v : K^\times \to \mathbb{R}$ is a rank 1 valuation on K then the absolute value defined by

$$|x| = e^{-v(x)}, \quad x \in K$$

is non-archimedean. Conversely, if $|\cdot|$ is a non-archimedean absolute value then $v(x) := -\ln |x|$ is a valuation on K. Thus we see non-archimedean absolute values corresponds to rank 1 valuations.

An absolute value $|\cdot|$ on K defines a metric by taking $|x - y|$ as distance, for $x, y \in K$. In particular, $|\cdot|$ induces a topology on K. If two absolute values induce the same topology on K, they are called dependent (otherwise independent).

Proposition 5.4.13. *Let $|\cdot|_1$ and $|\cdot|_2$ be two non-trivial absolute values on K. They are dependent if and only if for all $x \in K$,*

$$|x|_1 < 1 \Rightarrow |x|_2 < 1.$$

If they are dependent, then there exists a real number $\lambda > 0$ such that $|x|_1 = |x|_2^\lambda$ for all $x \in K$.

Proof. For $|\cdot|_1$ and $|\cdot|_2$ non-trivial and dependent absolute values on K there exists $\varepsilon > 0$ such that $\{x \in K : |x|_1 < \varepsilon\} \subseteq \{x \in K : |x|_2 < 1\}$. If $|x|_1 < 1$, there is $m \geq 1$ such that $(|x|_1)^m = |x^m| < \varepsilon$. Hence $(|x|_2)^m = |x^m|_2 < 1$ and consequently $|x|_2 < 1$, as required.

Conversely, by the non-triviality of $|\cdot|_1$, there exists $z \in K$ with $|z|_2 > 1$. Thus $|z^{-1}|_1 < 1$ and so $|z^{-1}|_1 < 1$, by assumption. Hence $|z|_2 > 1$, too. Now we claim that, for every $x \in K$, $x \neq 0$, we have

$$\frac{\log |x|_1}{\log |x|_2} = \frac{\log |z|_1}{\log |z|_2}.$$

In fact, for $m, n \in \mathbb{Z}$, $n > 0$, such that

$$\frac{m}{n} > \frac{\log |x|_1}{\log |z|_1},$$

it follows that $(|z|_1)^m > (|x|_1)^n$. Consequently, $|x^n z^{-m}|_1 < 1$ and then, by assumption, $|x^n z^{-m}|_2 < 1$. Walking back the steps of the last argument one gets

$$\frac{m}{n} > \frac{\log |x|_2}{\log |z|_2}.$$

It then follows that

$$\frac{\log |x|_1}{\log |z|_1} \geq \frac{\log |x|_2}{\log |z|_2}.$$

Similarly one proves the reverse inequality. So the claim follows.

Now set $\lambda = \log |z|_1 / \log |z|_2$, it follows that $|x|_1 = |x|_2^\lambda$, for every $x \in K$, as required. Finally, the last equation implies that $|\cdot|_1$ and $|\cdot|_2$ are dependent. □

Now we prove an approximation theorem for absolute values.

Theorem 5.4.14. *Let K be a field and $|\cdot|_1, \dots, |\cdot|_n$ be non-trivial pairwise-independent absolute values on K. Moreover let $x_1, \dots, x_n \in K$, and $0 < \varepsilon \in \mathbb{R}$. Then there exists $x \in K$ such that $|x - x_i| < \varepsilon$ for all i.*

Proof. We shall first prove that for every $1 \leq i \leq n$ there exists $a_i \in K$ such that $|a_i|_i > 1$ and $|a_i|_j < 1$, for all $j \neq i$. We may fix $i = 1$ without loss of generality, and write $a = a_1$. We proceed by induction on n. For $n = 2$, Proposition 5.4.13 implies the existence of $b, c \in K$ such that

$$|b|_1 < 1, |b|_2 \geq 1, \quad |c|_1 \geq 1, |c|_2 < 1.$$

Thus $a = b^{-1}c$ has the desired properties.

Assume next that there is $y \in K$ such that $|y|_1 > 1$ and $|y|_j < 1$ for all $j = 2, \dots, n-1$. Applying the first case to $|\cdot|_1$ and $|\cdot|_n$, one has $|z|_1 > 1$ and $|z|_n < 1$, for some $z \in K$. Therefore, if $|y|_n \leq 1$, then $|zy^\nu|_1 > 1$ and $|zy^\nu|_n < 1$ for every integer $\nu \geq 1$. On the other hand, for a sufficiently large integer $\nu \geq 1$, $|zy^\nu|_j < 1$ for every $j = 2, \dots, n-1$. For such a ν, $a = zy^\nu$ satisfies the requirements.

Consider now the case $|y|_n > 1$, and form the sequence

$$w_\nu = \frac{y^\nu}{1 + y^\nu}$$

The usual properties of sequences of ordinary real numbers imply that $\lim_\nu |w_\nu|_j = 0$ for $j = 2, \dots, n-1$ and $\lim_\nu |w_\nu - 1|_j = 0$ for $j = 1$ and n. Consequently, $\lim_\nu |zw_\nu|_j = 0$ for $j = 2, \dots, n-1$ and $\lim_\nu |zw_\nu|_j = |z|_j$ for $j = 1$ and n. Hence, for sufficiently large ν, $a = zw_\nu$ has the required properties.

Now we prove that for any real number $\varepsilon > 0$ and every i such that $1 \leq i \leq n$, there exists $c_i \in K$ such that $|c_i - 1|_i < \varepsilon$ and $|c_i|_j < \varepsilon$, for all $j \neq 1$. As before, it is enough to consider the case $i = 1$. Let $a \in K$ satisfy $|a|_1 > 1$ and $|a|_j < 1$ for $j > 1$. Then the sequence $|a^\nu/(1+a^\nu)|_j$ converges to 1 for $j = 1$, and converges to 0 if $j > 1$. Thus for sufficiently large ν,

$$c_1 = \frac{a^\nu}{1 + a^\nu}$$

has the required property.

According to the argument above, there exist elements $c_1, \dots, c_n$ in K such that c_i is close to 1 at $|\cdot|_i$, and for every $j \neq i$, c_i is close to 0 at $|\cdot|_j$. The element $x = c_1x_1 + \cdots + c_nx_n$ is then arbitrarily close to x_i at $|\cdot|_i$, for every $i = 1, \dots, n$, and therefore satisfies the requirements of the theorem. □

We now prove some result about topological vector spaces over a valued field, which will be used later.

Proposition 5.4.15. *Let (K, v) be a valued field with valued group Γ. Let X be a topological vector space over K which is Hausdorff and of dimension 1. Suppose that v is not trivial. For all $x_0 \in X$, the map $a \mapsto ax_0$ of K onto X is a topological isomorphism.*

Proof. This map is a continuous algebraic isomorphism. It is sufficient to show that it is bicontinuous. Let $\alpha \in \Gamma$. We need to show that there exists a neighbourhood V of 0 in X such that the relation $ax_0 \in V$ implies $v(a) > \alpha$. Let $a_0 \in K^\times$ be such that $v(a_0) = \alpha$. As X is Hausdorff, there exists a neighbourhood W of 0 in X such that $a_0x_0 \notin W$. Since v is not improper, there exist a neighbourhood W' of 0 in X and an element β of Γ such that the relations $y \in W'$ and $v(a) \geq \beta$ imply $ay \in W'$. Let $a_1 \in K^\times$ be such that $v(a_1) = -\beta$. The relations $ax_0 \in a_1^{-1}W'$ and $v(a) \leq \alpha$ imply $a_1ax_0 \in W'$ and

$$v(a_0a^{-1}a_1^{-1}) = v(a_0) - v(a) - v(a_1) = \alpha + \beta - v(a) \geq \beta$$

hence $a_0x_0 = (a_0a^{-1}a_1^{-1})a_1ax_0 \in W$, which is a contradiction. In other words, the relation $ax_0 \in a_1^{-1}W'$ implies $v(a) > \alpha$, so we are done. □

Proposition 5.4.16. *Suppose that v is not trivial. If M is a closed maximal subspace of a topological vector space X over K then any algebraic complement N of M is a topological complement.*

Proof. Let M be a closed maximal subspace of a topological vector space X. Since M is maximal, $\dim X/M = 1$; hence if N is an algebraic complement of M, $\dim N = 1$. Therefore there must be some $x \neq 0$ such that $N = Kx$. Consequently $X = M \oplus Kx$ and each vector y has a unique representation in the form $y = m + tx$, $m \in M$, $t \in K$. To show that Kx is a topological complement of M, we use the criterion of **??**: We show that the projection P on Kx along M, $tx + m \mapsto tx$, is continuous.

To this end note that $N(P) = M$. As M is closed and $x \notin M$, there exists a neighborhood U of 0 in X such that $(x + U) \cap M = \varnothing$ and $xU \subseteq U$ whenever $v(x) \geq -\gamma_0$, for some fixed $\gamma_0 \in \Gamma$. Now if $m + tx \in U$ then we must have $v(t) > \gamma_0$: otherwise we have $v(t) \leq \gamma_0$ and so $v(t^{-1}) \geq -\gamma_0$, thus $t^{-1}(m + tx) = m/t + x \in U$, which contradicts $(x + U) \cap M = \varnothing$. Hence if $\gamma > \gamma_0$ and $m + tx \in U_\gamma(0)U$, then $t > \gamma + \gamma_0$.

To establish the continuity of P at 0, suppose that the net $m_\alpha + t_\alpha x \to 0$. As such, for any $\gamma > 0$, $m_\alpha + t_\alpha x \in U_\gamma(0)U$ eventually. Therefore $t_\alpha > \gamma + \gamma_0$ eventually. In other words, $t_\alpha \to 0$ in K, which implies that $t_\alpha x = P(m_\alpha + t_\alpha x) \to 0$, and proves the continuity of P. □

Proposition 5.4.17. *Suppose that v is nontrivial and K is complete. Let X be a topological vector space over K, which is Hausdorff and of finite dimension n. For every basis $(e_i)_{i=1}^n$ of X over K, the map $(a_i) \mapsto \sum_{i=1}^n a_i e_i$ from K^n onto X is a topological vector space isomorphism.*

Proof. This follows from induction, using Proposition 5.4.15 and Proposition 5.4.16. □

Corollary 5.4.18. *Suppose that v is nontrivial and K is complete. Let X be a Hausdorff topological vector space over K and F a finite-dimensional vector subspace of X. Then F is closed.*

5.5 Completion of valued fields

Since a non-trivial absolute value v makes K into a topological field, we may consider the completion of K with respect to the additive uniform structure. The next theorem will show that every field K with a non-trivial valuation can be densely embedded into a field complete with respect to a valuation extending the given one on K.

Theorem 5.5.1. *Let K be a field, v a valuation on K and Γ the value group of v with the discrete topology.*

(a) The completion ring $\widehat{K}$ of K (with respect to $\mathcal{T}_v$) is a topological field.

(b) The map $v : K^\times \to \Gamma$ can be extended uniquely to a continuous map $\hat{v} : \widehat{K}^\times \to \Gamma$. The map $\hat{v}$ is a valuation on $\widehat{K}$ with value group Γ.

(c) The topology on $\widehat{K}$ is the topology defined by the valuation $\hat{v}$.

(d) If $U_\gamma(x), V_\gamma(x)$ and $\widehat{U}_\gamma(x), \widehat{V}_\gamma(x)$ are the basic neighborhood of x in (K, v) and $(\widehat{K}, \hat{v})$, then $\widehat{U}_\gamma(x)$ is the closure of $U_\gamma(x)$ in $\widehat{K}$ and $\widehat{V}_\gamma(x)$ is the closure of $V_\gamma(x)$ in $\widehat{K}$.

(e) The valuation ring of $\hat{v}$ is the completion $\widehat{A}$ of the valuation ring A of v; the maximal ideal $\widehat{\mathfrak{m}}$ is the completion of the maximal ideal $\mathfrak{m}$ of v.

(f) $\widehat{A} = A + \widehat{\mathfrak{m}}$ and the residue field of $\hat{v}$ is canonically identified with that of v.

Proof. By **??**, to prove (a), it suffices to show the following: let $\mathfrak{U}$ be a Cauchy filter (with respect to the additive uniform structure) on $K^\times$ for which 0 is not a cluster point; then the image of $\mathfrak{U}$ under the bijection $x \mapsto x^{-1}$ is a Cauchy filter (with respect to the additive uniform structure). For since 0 is not a clustcr point of $\mathfrak{U}$, there exists $V \in \mathfrak{U}$ and $\gamma \in \Gamma$ such that γ is an upper bound of $v(M)$. Let $\alpha \in \Gamma$. If V' is an element of $\mathfrak{U}$ contained in M and such that $v(x-y) > \max\{\alpha + 2\gamma, \gamma\}$ for $x, y \in V'$, then $v(x^{-1} - y^{-1}) > \gamma$ for $x, y \in V$ (see (5.4.1)). Whence (a) follows.

By Proposition 5.4.1, $v : K^\times \to \Gamma$ is a continuous homomorphism from $K^\times$ to Γ and hence can be extended uniquely to a continuous homomorphism $\hat{v}$ from $\widehat{K}^\times$ to Γ. The relation $\hat{v}(x+y) \geq \min\{\hat{v}(x), \hat{v}(y)\}$ holds in $K^\times$ and hence also holds in $\widehat{K}^\times$ by continuity. Thus $\hat{v}$ (extended by $\widehat{K}$ by $\hat{v}(0) = \infty$) is a valuation on $\widehat{K}$ and (b) is proved.

Wc now show (d). Let $\gamma \in \Gamma$ and $x \in \bar{U}_\alpha(0)$. For y in $U_\alpha(0)$ sufficiently close to x, $v(y) = \hat{v}(y) = \hat{v}(x)$ and hence $\hat{v}(x) > \gamma$. Conversely, let $x \in \widehat{K}^\times$ be such that $\hat{v}(x) > \alpha$; for y in $K^\times$ sufficiently close to x, $v(y) = \hat{v}(y) = \hat{v}(x)$ and therefore $y \in U_\gamma(0)$, whence $x \in \bar{U}_\gamma(0)$. Thus $\bar{U}_\gamma(0)$ is the set of $x \in \widehat{K}$ such that $\hat{v}(x) > \gamma$. The general case for $U_\gamma(x)$ now follows from homogeneity, and the result for $V_\gamma(x)$ can be proved similarly.

Taking account of **??**, assertion (c) is a consequence of (d). Assertion (e) is a special case of (d). Finally let $x \in \widehat{A}$; since $\widehat{A}$ is the closure of A in $\widehat{K}$, there exists $y \in A$ such that $\hat{v}(x-y) > 0$; then $z = x - y \in \widehat{\mathfrak{m}}$ and hence $x = y + z \in A + \widehat{\mathfrak{m}}$; thus $\widehat{A} = A + \widehat{\mathfrak{m}}$ which shows (f). □

5.5.1 Archimedean complete fields

Let K be a field complete with respect to an archimedean absolute value $|\cdot|$. Since the set $\{|n \cdot 1| : n \in \mathbb{Z}\}$ is not bounded, $\operatorname{char} K = 0$. Thus K contains the field $\mathbb{Q}$ of rationals. We shall first show that $|\cdot|$ restricted to $\mathbb{Q}$ is dependent on the usual absolute value of $\mathbb{Q}$. Thus the complete field K contains the completion of $\mathbb{Q}$ with respect to the ordinary absolute value, i.e., K contains $\mathbb{R}$ as a closed subfield. We shall then show that K must be equal to $\mathbb{R}$ or to $\mathbb{C}$. Consequently, every field K admitting an archimedean absolute value may be considered as a subfield of $\mathbb{C}$ or even $\mathbb{R}$ with the absolute value dependent on the induced one from $\mathbb{C}$ (or from $\mathbb{R}$).

Proposition 5.5.2. *Every archimedean absolute value on $\mathbb{Q}$ is dependent on the usual one.*

Proof. Let $|\cdot|$ be an archimedean absolute value on $\mathbb{Q}$. Denote by $|\cdot|_0$ the usual absolute value on $\mathbb{Q}$. Next, for integers $m, n \geq 2$ and $t \geq 1$, expand m^t in powers of n:

$$m^t = c_0 + c_1 n + \cdots + c_s n^s, \quad 0 \leq c_0, \ldots, c_s < n, c_s \neq 0.$$

Since each c_i is integer, we have $|c_i| \leq c_i < n$. It follows that

$$|m|^t \leq \sum_{i=0}^{s} |c_i||n|^i \leq n \sum_{i=0}^{s} |n|^i \leq n(s+1) \max\{1, |n|^s\}.$$

As $n^s \leq m^t$, $s \leq t(\log m)/\log n$. Thus

$$|m|^t \leq n\Big(\frac{t \log m}{\log n} + 1\Big) \max\{1, |n|\}^{t(\log m/\log n)}$$

or equivalently,

$$|m| \leq n^{1/t}\Big(\frac{t \log m}{\log n} + 1\Big)^{1/t} \max\{1, |n|\}^{\log m/\log n}.$$

Letting t go to infinity and taking limits, one gets

$$|m| \leq \max\{1, |n|\}^{\log m/\log n}.$$

Now, if $|n| < 1$ for some $n \in \mathbb{N}$, the above inequality implies $|m| < 1$ for every integer $m \geq 2$, contradicting the archimedeanness of $|\cdot|$. Therefore, $|n| > 1$ for all integers $n \geq 2$, and thus

$$|m| \leq |n|^{\log m/\log n}.$$

Interchanging the roles of m and n in the above inequality gives the reverse inequality. Hence

$$|m| = |m|^{\log m/\log n}$$

Therefore, if $m > n \geq 2$, then $\log m/\log n > 1$ and so $|m| > |n|$. Since $|-m| = |m|$ for all $m \in \mathbb{Z}$, it follows that $|m|_0 > |n|_0$ implies $|m| > |n|$, for non-zero $m, n \in \mathbb{Z}$. Consequently, if $m/n \in \mathbb{Q}$ satisfies $|m/n|_0 < 1$, then $|m/n| < 1$. By Proposition 5.4.13, $|\cdot|_0$ and $|\cdot|$ are dependent. $\square$

In case E is a field extension of K, every absolute value $|\cdot|'$ of E that restricts to $|\cdot|$ on K is a norm of E compatible with $|\cdot|$. If E has finite degree over K and K is complete with respect to $|\cdot|$, the next proposition will imply that up to equivalence of norms (and hence up to dependence as absolute values), E admits only one absolute value $|\cdot|'$ restricting to $|\cdot|$ on K. Moreover, E is complete with respect to $|\cdot|'$.

Proposition 5.5.3. *Let K be a field complete with respect to a non-trivial absolute value $|\cdot|$. Then every two norms (compatible with $|\cdot|$) of a finite dimensional K-vector space E are equivalent.*

Proof. We shall prove that any such norm on E is equivalent to the max-norm of E. This will be done by induction on the dimension n of the K-vector space E. For $n = 1$ the statement is obvious. Assume the proposition is proved for $n-1$, $n \geq 2$. One inequality is very simple to prove. Fix a basis $\omega_1, \ldots, \omega_n$ of E over K, and for $\xi = x_1\omega_1 + \cdots + x_n\omega_n$, denote $\|\xi\|_{\max} = \max_i |x_i|$. Then

$$\|\xi\| \leq \sum_{i=1}^{n} |x_i| \|\omega_i\| \leq C\|\xi\|_{\max} \quad \text{where } C = n \max_i \|\omega_i\|.$$

We must now prove that there exists a number $C' > 0$ such that for all $\xi \in E$,

$$\|\xi\|_{\max} \leq C'\|\xi\|.$$

Suppose no such number exists. Then, for every positive integer m there exists $\xi \in E$ such that $\|\xi\|_{\max} > m\|\xi\|$. Let j be such that $|x_j| = \max_{1\leq i\leq n} |x_i|$. Letting $\xi_m = x_j^{-1}\xi$ we get $\|\xi\|_{\max} = 1$ and thus $\|\xi_m\| < 1/m$. For every $m \geq 1$, one of the components of ξ_m equals 1. Thus there must be an infinite subset T of $\mathbb{N}$ and a fixed number j such that the j-th component of ξ_m equals 1 for all $m \in T$. We fix this number j from now until the end of the proof.

Consider the subspace E_1 of E consisting of all vectors whose j-th coordinate is equal to 0, equipped with the norm induced by $\|\cdot\|$. By induction, the restrictions of $\|\cdot\|$ and max-norm $\|\cdot\|_{\max}$ to E_1 are equivalent. In particular, a sequence of elements of E_1 converges to $\zeta \in E_1$ with respect to $\|\cdot\|_{\max}$ if and only if it converges to ζ with respect to $\|\cdot\|$.

For each $m \in T$ we can write $\xi_m = \omega_j + \zeta_m$, for some $\zeta_m \in E_1$. Now, for every $\varepsilon > 0$, take $N \in \mathbb{N}$ such that $2/N < \varepsilon$. If $m, n \geq N$, $m, n \in T$, then

$$\|\zeta_m - \zeta_n\| = \|\zeta_m + \omega_j - \omega_j - \zeta_n\| = \|\xi_m - \xi_n\| \leq \|\xi_m\| + \|\xi_n\| < \frac{1}{m} + \frac{1}{n} \leq \frac{2}{N} < \varepsilon.$$

Consequently, (ζ_m) is a Cauchy sequence with respect to the restriction of $\|\cdot\|$ to E_1. From the induction hypothesis, it follows that this sequence is also a Cauchy sequence with respect to the max-norm. Since E_1 is complete with respect to the max-norm, this sequence converges to some $\zeta \in E_1$ (with respect to the max-norm).

The choice of T implies $\|\omega_j + \zeta_m\| < 1/m$ for each $m \in T$. Since the restrictions of $\|\cdot\|$ and the max-norm to E_1 are equivalent,

$$\|\zeta - \zeta\| \leq C\|\zeta_m - \zeta\|_{\max}$$

for some number $C > 0$. Therefore,

$$\|\omega_j + \zeta\| \leq \|\omega_j + \zeta_m\| + \|\zeta - \zeta_m\| \leq \frac{1}{m} + C\|\zeta - \zeta_m\|_{\max}.$$

Letting $m \in T$ go to infinity, the right-hand side of the preceding inequality tends to 0. Hence $\omega_j + \zeta = 0$. But, this cannot occur, because $\zeta \in E_1$ has the j-th coordinate equal to 0 and $\omega_1, \ldots, \omega_n$ is a basis of E over K. This contradiction finishes the proof of the proposition. $\square$

Theorem 5.5.4. *Let K be a field containing $\mathbb{R}$ and having an absolute value that induces the ordinary one on $\mathbb{R}$. Then $K = \mathbb{R}$ or $K = \mathbb{C}$.*

In particular, the only fields complete with respect to an archimedean absolute value $|\cdot|$ are (up to isomorphism) $\mathbb{R}$ and $\mathbb{C}$ with $|\cdot|$ dependent on the ordinary absolute value.

This theorem is a consequence of the following proposition:

Proposition 5.5.5 (Gelfand-Mazur). *Let A be a commutative normed algebra with identity and assume that A contains an element j such that $j^2 = -1$, and let $\mathbb{C} = \mathbb{R} + j\mathbb{R} \subseteq A$ (identify $\mathbb{R} \cdot 1$ with $\mathbb{R}$). Then for every nonzero element $x_0 \in A$, there exists an element $c \in \mathbb{C}$ such that $x_0 - c$ is not invertible in A.*

Proof. Suppose that $x_0 - z$ is invertible for all $z \in \mathbb{C}$. The map $f : \mathbb{C} \to A$ defined by $f(z) = (x_0 - z)^{-1}$, is then well defined. Moreover, we shall see that taking inverses is a continuous operation on the group of units of A, from which it will follow that f is continuous. In order to show that $x \mapsto x^{-1}$ is continuous on the group of units of A, note that for any units a and x in A,

$$\|x^{-1} - a^{-1}\| = \|(a - x)a^{-1}x^{-1}\| \leq \|x - a\|\|a^{-1}\|\|x^{-1}\|.$$

Thus it remains to show that $\|x^{-1}\|$ is bounded as x varies through units near a. Let $\|a^{-1}\|\|x - a\| \leq 1/2$, and set $w = a^{-1}(x - a)$. Then clearly $\|w\| \leq 1/2$. Hence we get

$$\Big\|\frac{1}{1+w}\Big\| = \Big\|1 - \frac{w}{1+w}\Big\| \leq 1 + \frac{\|w\|}{1+\|w\|} \leq 1 + \frac{1}{2}\Big\|\frac{1}{1+w}\Big\|.$$

This implies $\|(1 + w)^{-1}\| \leq 1/2$. Thus finally we get

$$\|x^{-1}\| = \|a^{-1}(1 + w)^{-1}\| \leq 2\|a^{-1}\|.$$

Back to the proof of the proposition, observe that for $0 \neq z \in \mathbb{C}$,

$$f(z) = \frac{1}{x_0 - z} = \frac{1}{z}\Big(\frac{1}{x_0 z^{-1} - 1}\Big).$$

Since z^{-1} and $x_0 z^{-1}$ approach 0 when z goes to infinity in $\mathbb{C}$, it follows that $f(z) \to 0$ when $z \to \infty$. On the other hand, the map $z \mapsto \|f(z)\|$ is continuous, being the composition of two continuous maps. Consequently, $\|f(z)\| \to 0$ when $z \to \infty$. Hence this map may be considered as a real valued continuous map on the one-point compactification $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ of $\mathbb{C}$. Hence $\|f\|$ has a maximum $M \in \mathbb{R}$, $M > 0$. Let D be the set of elements $z \in \mathbb{C}$ such that $\|f(z)\| = M$. Then D is a non-empty, bounded, and closed subset of $\mathbb{C}$. We shall prove that it is also open, a contradiction.

For $z_0 \in D$, let us write $y = x_0 - z_0$ and let t be a nonzero complex number to be determined later. Consider the polynomial $h(X) = X^n - t^n = \prod_{j=1}^{n}(X - \zeta^j t)$, whence

$$\frac{h'(X)}{h(X)} = \frac{nX^{n-1}}{X^n - t^n} = \sum_{j=1}^{n-1} \frac{1}{X - \zeta_n^j t} + \frac{1}{X - t}.$$

Dividing by n and replacing every occurrence of X by y, taking account of the definitions of f and y, we obtain

$$f(z_0 + t) + \sum_{j=1}^{n-1} f(z_0 + \zeta_n^j t) - nf(z_0) = \frac{ny^{n-1}}{y^n - t^n} - \frac{n}{y} = \frac{n}{y} \cdot \frac{t^n y^{-n}}{1 - t^n y^{-n}}. \tag{5.5.1}$$

If we choose t such that $\|ty^{-1}\| < 1$, then the last expression in (5.5.1) tends to 0 as n tends to ∞; hence

$$\|f(z_0 + t)\| = \lim_{n\to\infty} \|nf(z_0) - \sum_{j=1}^{n-1} f(z_0 + \omega_n^j t)\|. \tag{5.5.2}$$

Now, $\|f(z_0)\| = M$ and $\|f(z_0 + \omega_n^j t)\| \leq M$ by definition of M, whence

$$\|nf(z_0) - \sum_{j=1}^{n-1} f(z_0 + \omega_n^j t)\| \geq n\|f(z_0)\| - \sum_{j=1}^{n-1} \|f(z_0 + \omega_n^j t)\| \geq nM - (n-1)M = M.$$

Therefore by (5.5.2), letting n tend to ∞, $\|f(z_0+t)\| \geq M$ and by definition of M this implies $\|f(z_0+t)\| = M$; in other words $z_0 + t \in D$. This proves that the set D is open in $\mathbb{C}$; as it is also closed and non-empty and $\mathbb{C}$ is connected, $D = \mathbb{C}$ and f is therefore constant on $\mathbb{C}$; as this function tends to 0 at the point at infinity, $\|f(z)\| = 0$ in $\mathbb{C}$ and in particular $\|f(0)\| = \|x^{-1}\| = 0$, which is absurd. $\square$

Proof of Theorem 5.5.4. We apply Proposition 5.5.5 to the $\mathbb{R}$-algebra K, and let the given absolute value serve as norm. If K contains $\mathbb{C}$, then $K = \mathbb{C}$, because every element of K is invertible. If K does not contain $\mathbb{C}$, let $L = K(j)$, where $j^2 = -1$. Define a norm on L by putting $\|x + yj\| = \|x\| + \|y\|$ for $x, y \in K$. This clearly makes L a normed $\mathbb{R}$-algebra. Moreover, by standard calculations one proves that $\|1\| = 1$ and $\|zw\| \leq \|z\|\|w\|$. Now, applying Proposition 5.5.5 to $A = L$, we obtain $L = \mathbb{C}$ as before. Thus K must be $\mathbb{R}$. $\square$

5.5.2 Non-archimedean complete fields

As we have already noticed, non-archimedean absolute values corresponds to rank 1 valuations. Therefore, for non-archimedean absolute values, we will use this additive notation and refer non-archimedean absolute values to valuations of rank 1.

Now we prove an important property of a field K complete with respect to a (non-trivial) valuation v. This theorem is widely known as Hensel's Lemma.

Theorem 5.5.6 (Hensel's Lemma). *Let K be a field complete with respect to a non-archimedean absolute value v. Let $f \in A_v[X]$ be a polynomial, and let $a_0 \in A_v$ be such that $v(f(a_0)) > 2v(f'(a_0))$. Then there exists some $a \in A_v$ with $f(a) = 0$ and $v(a_0 - a) > v(f'(a_0))$.*

Proof. The natural way to reach the conclusions of the theorem is to construct a suitable Cauchy sequence which must converge to a root a of f (recall that every polynomial f is a continuous map).

We use the Nowton approximation method: define a sequence (a_n) by

$$a_{n+1} = a_n - \frac{f(a_0)}{f'(a_0)}.$$

We will show that (a_n) has a limit a and this element a satisfies the requirements. In fact, set $r = v(f(a_0)) - 2v(f'(a_0)) > 0$, we will prove by induction that

(a) $v(a_n) \geq 0$.

(b) $v(a_{n+1} - a_n) = v(f(a_n)/f'(a_n)) \geq v(f(a_n)/f'(a_n)^2) \geq 2^n r$.

(c) $v(a_n - a_0) \geq v(f(a_0)/f'(a_0))$.

(d) $v(f'(a_n)) = v(f'(a_0))$.

By hypothesis, (a), (b), and (c) holds for $n = 0$. Now assume these conditions for n, then we can see (a) holds for $n+1$. Since a_n and $f(a_n)/f'(a_n)$ are in A_v, by Talor expansion we have

$$\begin{aligned} v(f(a_{n+1})) &= v\Big(f\Big(a_n - \frac{f(a_n)}{f'(a_n)}\Big)\Big) = v\Big(f(a_n) - f'(a_n)\cdot\frac{f(a_n)}{f'(a_n)} + M\Big(\frac{f(a_n)}{f'(a_n)}\Big)^2\Big) \\ &= v\Big(M\Big(\frac{f(a_n)}{f'(a_n)}\Big)^2\Big) \geq v\Big(\Big(\frac{f(a_n)}{f'(a_n)}\Big)^2\Big) \end{aligned} \tag{5.5.3}$$

where $M \in A_v$. Similarly,

$$\frac{f'(a_{n+1})}{f'(a_n)} = \frac{1}{f'(a_n)}\Big(f'(a_i) + M'\frac{f(a_n)}{f'(a_n)}\Big) = 1 + M'\frac{f(a_n)}{f'(a_n)^2}$$

for some $M' \in A_v$, whence

$$v\Big(\frac{f'(a_{n+1})}{f'(a_n)}\Big) = 0 \tag{5.5.4}$$

and $v(f'(a_{n+1})) = v(f'(a_n)) \neq \infty$. Now by (5.5.3) and (5.5.4),

$$\begin{aligned} v\Big(\frac{f(a_{n+1})}{f'(a_{n+1})^2}\Big) &\geq v\Big(\Big(\frac{f(a_n)}{f'(a_n)}\Big)^2\Big) - v(f'(a_{n+1})^2) = v\Big(\Big(\frac{f(a_n)}{f'(a_n)}\Big)^2\Big) - v(f'(a_n)^2) \\ &= 2v\Big(\frac{f(a_n)}{f'(a_n)^2}\Big) \geq 2^{n+1}r. \end{aligned} \tag{5.5.5}$$

This proves condition (b) for $n+1$. Moreover, by (5.5.5),

$$v\Big(\frac{f(a_{n+1})}{f'(a_{n+1})^2}\Big) \geq v\Big(\frac{f(a_n)}{f'(a_n)^2}\Big).$$

From (5.5.4), this implies

$$v\Big(\frac{f(a_{n+1})}{f'(a_{n+1})}\Big) \geq v\Big(\frac{f(a_n)}{f'(a_n)}\Big) \geq \cdots \geq v\Big(\frac{f(a_0)}{f'(a_0)}\Big)$$

which is to say

$$v(a_{n+1} - a_n) \geq v(a_n - a_{n-1}) \geq \cdots \geq v(a_1 - a_0),$$

and therefore

$$v(a_n - a_0) = v(\sum_{i=1}^{n}(a_n - a_{n-1})) = v(a_1 - a_0) = v\Big(\frac{f(a_0)}{f'(a_0)}\Big).$$

This proves (c), and completes the induction process.

From (a) and (b) we see (a_n) is a Cauchy sequence in K, so admits a limit $a \in K$. Also, since the valuation is continuous, we have $v(a) \geq 0$, whence $a \in A_v$. Taking limit in the recursion formula, we see $f(a) = 0$. Also, by (c), we have $v(a - a_0) \geq v(f(a_0)/f'(a_0)) > v(f'(a_0))$. □

Corollary 5.5.7. *Let K, v be as in Theorem 5.5.6. If $f \in A_v[X]$ has a simple zero a_0 in the residue field κ_v, i.e., $\bar{f}(\bar{a}_0) = 0$ and $\bar{f}'(a_0) \neq 0$, then f has a zero $a \in A_v$ such that $\bar{a} = \bar{a}_0$.*

The proof of Theorem 5.5.6 clearly shows that under the assumption of Theorem 5.5.6, the sequence $(f(a_n))$ converges to 0. At this point, essential use is made of the fact that $v(K^\times)$ is a subgroup of the additive reals. Thus even without the completeness of K, we could still notice:

Proposition 5.5.8. *Let v be a non-archimedean absolute value on K. Then for every $f \in A_v[X]$, if $\bar{f}$ has a simple zero in κ_v, then $f(K)$ approximates 0.*

Now consider the completion $(\widehat{K}, \hat{v})$ of the field with respect to a non-archimedean absolute value v, but now using the additive notation for absolute values. The density of K in $\widehat{K}$ has the following important consequence.

Theorem 5.5.9. *Denote by $\widehat{A}_v, \kappa_{\hat{v}}$ and A_v, κ_v the valuation ring and the residue field of $\hat{v}$ and v, respectively. Then the residue fields κ_v and $\kappa_{\hat{v}}$, as well as the groups Γ_v and $\Gamma_{\hat{v}}$, are canonically isomorphic.*

Proof. It follows from the constructions that $A_{\hat{v}} \cap K = A_v$ and $\mathfrak{m}_{\hat{v}} \cap K = \mathfrak{m}$, where $\mathfrak{m}_{\hat{v}}$ and $\mathfrak{m}_v$ are the respective maximal ideals. Thus the map that sends the residue class of $a \in A_v$ to the residue class $\bar{a} \in A_{\hat{v}}/\mathfrak{m}_{\hat{v}}$ is well defined; and it is clearly a ring homomorphism. It remains to be seen that it is surjective. For every $x \in A_{\hat{v}}$, the set $x + \mathfrak{m}_{\hat{v}}$ is an open neighbourhood of x. It consists of all elements z such that $\hat{v}(z - x) > 0$, or $|z - x| < 1$ in terms of the absolute value.

Thus the set $(x+\mathfrak{m}_v)\cap K$ is non-empty, by the density property. Hence the residue class of $y \in (x+\mathfrak{m}_{\hat{v}})\cap K$ is sent by the map above to $\bar{x}$, as required.

Similarly, the map $\Gamma_v \to \Gamma_{\hat{v}}$ sending $v(x)$ to $\hat{v}(x)$ for every $x \in K^\times$ is an order-preserving group monomorphism. In order to show surjectivity, let $x \in \widehat{K}^\times$ be given. By the density of K in $\widehat{K}$ there exists $z \in K$ with $\hat{v}(z-x) > v(x)$, or $|z-x|_{\widehat{K}} < |x|$ in terms of the absolute value. But then $\hat{v}(z) = v(x)$. □

Example 5.5.10. Now consider the p-adic valuation on $\mathbb{Q}$; the completion of $(\mathbb{Q}, v_p)$ is denoted by $\mathbb{Q}_p$, and is called the field of ***p*-adic numbers**. The valuation ring of $\mathbb{Q}_p$, denoted by $\mathbb{Z}_p$, is the ring of p-adic integers; it is the topological closure of $\mathbb{Z}$ in $\mathbb{Q}_p$, as we shall see below. According to our previous discussion of this example, Theorem 5.5.9 implies that the residue class field of $\mathbb{Z}_p$ is $\mathbb{F}_p$. Observe also that p is a local parameter for v_p in both fields, $\mathbb{Q}$ and $\mathbb{Q}_p$.

Example 5.5.11. If we fix X as the irreducible polynomial p, then the completion of $(k(X), v_X)$ is the field $k((X))$ of formal Laurent series over k, and the valuation ring is $k[\![X]\!]$, the ring of formal power series.

The last two examples above are special cases of the following more general result:

Proposition 5.5.12. *Let v be a discrete absolute value on the field K, with uniformizer π. Then every element $x \in K^\times$ can be written uniquely as a convergent series*

$$x = \sum_{i=\nu}^{\infty} r_i \pi^i$$

where $\nu = v(x)$, $r_\nu \neq 0$, and the coefficients r_i are taken from a set $R \subseteq A_v$ of representatives of the residue classes in the field κ_v (i.e., the canonical map $A_v \to \kappa_v$ induces a bijection of R onto κ_v). In particular, the valuation ring of $\widehat{K}$ is a complete discrete valuation ring with uniformizer π, and we have an isomorphism of topological rings $\widehat{A} \cong \varprojlim A/\pi^n A$.

Proof. We proceed by induction. As observed above, $u = x\pi^{-\nu}$ is a unit in A_v. Choose $r_\nu \in R$ such that $\bar{r}_\nu = \bar{u}$. Then clearly $v(x\pi^{-\nu} - r_\nu) > 0$ or, equivalently,

$$v(x - r_\nu \pi^\nu) > v(\pi^\nu) = \nu.$$

Let $x_1 = x - r_\nu\pi^\nu$ and $\nu_1 = v(x_1) > \nu$. Then by the same argument we get $r_{\nu_1} \in R$ such that

$$v(x - (r_\nu \pi^\nu + r_{\nu_1}\pi^{\mu_1})) = v(x_1 - r_{\mu_1}\pi^{\mu_1}) > \nu_1.$$

Repeating this argument and adding "zero coefficients" (i.e. a representative for zero in κ_v) if necessary, we obtain the existence of the "series"

$$r_\nu \pi^\nu + r_{\nu+1}\pi^{\nu+1} + \cdots$$

At the same time we see that it converges to x.

The uniqueness of the coefficients is clear. Indeed, otherwise 0 would have a representation

$$0 = (r_\nu - r'_\nu)\pi^\nu + (r_{\nu+1} - r'_{\nu+1})\pi^{\nu+1} + \cdots$$

with $r_\nu \neq r'_\nu \in R$, and hence $r_\nu - r'_\nu \neq 0$ in κ_v. But $v(0) = \infty$, a contradiction.

Let $\widehat{A}$ be the valuation ring of $\hat{v}$. Then $\widehat{A}$ is complete (it is closed in $\widehat{K}$) and contains A. For each $n \geq 1$ we define a ring homomorphism $\phi_n : \widehat{A} \to A/\pi^n A$ as follows: for each $x = \sum_{i=0}^{\infty} r_i\pi^i$ let $\phi_n(x)$ be the n-th truncation $\sum_{i=0}^{n-1} r_i\pi^i$. We thus obtain an infinite sequence of surjective maps $\phi_n : \widehat{A} \to A/\pi^n A$ that are compatible in that for all $n \geq m > 0$ and all $x \in \widehat{A}$ the image of $\phi_n(x)$ in $A/\pi^m A$ is $\phi_m(x)$. This defines a surjective ring homomorphism

$\phi : \widehat{A} \to \varprojlim A/\pi^n A$. Now note that $\ker \phi = \bigcap_n \pi^n \widehat{A} = \{0\}$ so ϕ is injective and therefore an isomorphism.

To show that ϕ is also a homeomorphism, it suffices to note that if $x + \pi^m A$ is a coset of $\pi^m A$ in A and U is the corresponding open set in $\varprojlim A/\pi^n A$, then $\phi^{-1}(U)$ is the closure of $x + \pi^m A$ in $\widehat{A}$, which is the coset $x + \pi^m \widehat{A}$, an open subset in $\widehat{A}$ (as explained in the discussion above, every open set in the inverse limit corresponds to a finite union of cosets $x + \pi^m A$ for some m). Conversely ϕ maps open sets $x + \pi^m \widehat{A}$ to open sets in $\varprojlim A/\pi^n A$. $\square$

Example 5.5.13. Returning once more to our typical examples, any p-adic number $z \in \mathbb{Q}_p^\times$ has a unique representation in the form

$$z = \sum_{i=\nu}^{\infty} a_i p^i$$

where $\nu = v_p(z)$, $0 \leq a_i < p$ for every i, and $a_\nu \neq 0$. The set of representatives chosen here is then $R = \{0, \ldots, p-1\}$. If $z \in \mathbb{Z}_p$, i.e., if $v(z) \geq 0$, then $z = \sum_{i=0}^{\infty} a_i p^i$. This shows, in particular, that $\mathbb{Z}$ is dense in $\mathbb{Z}_p$. The reader should be aware of the fact that addition of two "series" of the form $\sum_{i=\nu} a_i p^i$ is not coefficient-wise, as the set R is not closed under addition. As a simple example observe that (choosing $p = 7$)

$$5p^i + 4p^i = p^{i+1} + 2p^i$$

Example 5.5.14. For the X-adic valuation of $k(X)$, we can take the elements of k itself as representatives of the residue field. In this case every $z \in k(X))^\times$ has a unique representation in the form

$$z = \sum_{i=\nu}^{\infty} a_i X^i$$

where $v_X(z) = \nu \in \mathbb{Z}$ and $a_i \in k$ for every i. This time, addition of two such series is coefficient-wise. These series are called formal Laurent series. They form a field $k((X))$, the field of formal Laurent series. The canonical discrete absolute value on $k((X))$ is just given by

$$v\Big(\sum_{i=\nu}^{\infty} a_i X^i\Big) = \nu \quad \text{if } \nu \neq 0.$$

Clearly its valuation ring consists of the ring $k[\![X]\!]$ of formal power series, i.e., series of the type $\sum_{i=0}^{\infty} a_i X^i$.

5.6 Extensions of valuation rings

In this part we consider valuation rings in different fields. In particular, we will see how different valuation rings relate in a field extension, and we will prove a fundamental inequality in this situation.

5.6.1 Ramification index and inertial degree

Let $K \subseteq L$ be a field extension and w a valuation of L. Then it is easy to see $v = w|_K$ is a valuation of K and for maximal ideals we have

$$\mathfrak{m}_v = \mathfrak{m}_w \cap K.$$

That is, A_v is dominated by A_w. Conversely, if v is a valuation of K, then by Theorem 5.1.7 we can find a valuation ring of L dominating A_v. If w is the valuation of this ring, then the valuation v is given by the restriction of w on K and for value groups we have $\Gamma_v \subseteq \Gamma_w$. Also, since A_w dominates A_v, we also have an extension of residue fields : $\kappa_v \subseteq \kappa_w$. Now we make the following definition.

Definition 5.6.1. The index $[\Gamma_w : \Gamma_v]$ is called the **ramification index** of w over v and denoted by $e(w/v)$. If this index equals 1 then w is called **unramified** over v.

Definition 5.6.2. The index $[\kappa_w : \kappa_v]$ is called the **inertial degree** of w over v and denoted by $f(w/v)$.

In particular, if $e(w/v) = 1$ and $f(w/v) = 1$, the extension w/v is called **immediate**. We follow the usual convention that both e and f can be either finite or infinite, without distinguishing between different infinite cardinalities. With this convention, we have the following transitivity due to that of group index and field degree.

Proposition 5.6.3. *Let $K \subseteq L \subseteq L'$ be field extension, w' an extension of L' and w, v the restriction of w' to L' and L. Then*

$$e(w'/v) = e(w'/w)e(w/v), \quad f(w'/v) = f(w'/w)f(w/v).$$

Now, before we prove anything about the quantities e and f, we first provide some examples.

Example 5.6.4. Let (K, v) be a valued field with value group Γ_v and Γ a totally ordered group with $\Gamma_v \subseteq \Gamma$. Consider some t transcendental over K and any $\gamma \in \Gamma$. Define a valuation w on $K(t)$ with values in Γ via

$$w\Big(\sum_{i=0}^{n} a_i t^i\Big) = \min\{v(a_i) + i\gamma : 0 \le i \le n\}$$

for $f = \sum_{i=0}^{n} a_i t^i \in K[t]$ and extend w to all of $K(t)$. Then w is a well-defined valuation extending v with value group $\Gamma_w = \Gamma_v + \gamma\mathbb{Z}$.

The following inequality is fundamental for extension of valuations.

Proposition 5.6.5. *Let K be a field, L a extension of K of degree n, w a valuation on L and v its restriction to K. Choose $x_1, \dots, x_e \in L^\times$ and $y_1, \dots, y_f \in A_w$ such that*

(a) the values $w(x_1), \dots, v(x_e)$ are representatives of the distinct cosets of Γ_w/Γ_v.

(b) the residues $\bar{y}_1, \dots, \bar{y}_f \in \kappa_w$ are linearly independent over κ_v;

Then for all $a_{ij} \in K$, we have

$$w\Big(\sum_{ij} a_{ij} x_i y_j\Big) = \min_{ij}\{w(a_{ij} x_i y_j)\}$$

In particular, the products $x_i y_j$ are linearly independent over K.

Proof. Let $a_{ij} \in K$, not all zero, and $I \in \{1, \dots, e\}$ and $J \in \{1, \dots, f\}$ such that

$$w(a_{IJ} x_I) = \min_{ij}\{w(a_{ij} x_i)\}.$$

and observe first that $w(a_{IJ} x_I) < w(a_{ij} x_i)$ for all $i \neq I$ since otherwise

$$w(x_I) - w(x_i) = w(a_{ij}) - w(a_{IJ}) = v(a_{ij}/a_{IJ}) \in \Gamma_v,$$

which contradicts our assumption on x.

Next, write $z = \sum_{ij} a_{ij} x_i y_j$. Since $w(y_j) \ge 0$ for each j, for the sake of obtaining a contradiction, we assume that $w(z) > w(a_{IJ} x_I)$. Then $z(a_{IJ} x_I)^{-1} \in \mathfrak{m}_w$. Also, according to the previous paragraph we have $a_{ij} x_i (a_{IJ} x_I)^{-1} \in \mathfrak{m}_w$ for all $i \neq I$. Dividing everything by $a_{IJ} x_I$ one gets

$$\sum_j \frac{a_{Ij}}{a_{IJ}} y_j = \frac{z}{a_{IJ} x_I} - \sum_{j=1}^{f} \sum_{i \neq I} \frac{a_{ij} x_i}{a_{IJ} x_I} y_j \in \mathfrak{m}_w$$

which gives a relation $\sum_j \bar{a}_{Ij} (\bar{a}_{IJ})^{-1} \bar{y}_j = 0$, this contradicts the hypothesis made on y_j. ☐

Corollary 5.6.6. *Let K be a field, L a finite extension of K of degree n, w a valuation on L and v its restriction to K. Then the $e(w/v)f(w/v) \leq n$. In particular, $e(w/v)$ and $f(w/v)$ are finite.*

Theorem 5.6.7. *Let K be a field, L an algebraic extension of K, w a valuation on L, v its restriction to K. Then Γ_w/Γ_v is torsion and κ_w/κ_v is algebraic.*

Proof. Pick $x \in L$ such that $w(x) = \gamma \in \Gamma_w$. If $L' = K(x)$ and v' is the restriction of w on L' then by Corollary 5.6.6, since L'/K is finite degree, the group $\Gamma_{v'}/\Gamma_v$ is finite, whence torsion. It then follows that Γ_w/Γ_v is torsion.

Similarly, for $x \in A_w$ we take L' and v' as above. It follows from Corollary 5.6.6 that the residue class field $\kappa_{v'}$ is a finite extension of κ_v. Therefore $\bar{x} \in \kappa_{v'} \subseteq \kappa_w$ is algebraic over κ_v. □

Corollary 5.6.8. *Let K be a field, L an algebraic extension of K, w a valuation on L, v its restriction to K. Then the map $\Delta \mapsto \Delta \cap \Gamma_v$ is a one-to-one correspondence between isolated subgroups of Γ_w and Γ_v. In particular,* $\operatorname{rank}(w) = \operatorname{rank}(v)$.

Proof. It is clear that $\Delta \cap \Gamma_v$ is an isolated subgroup, if Δ is. Now let $\Delta \subseteq \Gamma_v$ be an isolated subgroup and Δ' denote the set of $\alpha \in \Gamma_w$ such that there exists $\beta \in \Delta$ satisfying $-\beta \leq \alpha \leq \beta$; it is immediately verified that Δ' is an isolated subgroup of Γ_v with $\Delta' \cap \Delta = \Delta$ (since Δ is isolated); hence the map $\Delta \mapsto \Delta \cap \Gamma_v$ is surjective. Finally, let Δ_1 and Δ_2 be two isolated subgroups of Γ_w such that $\Delta := \Delta_1 \cap \Gamma_v = \Delta_2 \cap \Gamma_v$ and assume, for example, $\Delta_1 \subseteq \Delta_2$; then Δ_2/Γ_1 is a totally ordered group and is isomorphic to a quotient group of Δ_2/Δ which itselfis identified with a subgroup of Γ_w/Γ_v. Hence Δ_2/Δ_1 is a torsion group and therefore reduces to 0. □

Corollary 5.6.9. *Suppose that L is a finite extension of K. For w to be discrete, it is necessary and sufficient that v be discrete.*

Proof. If w is discrete, then Γ_v is isomorphic to a non-zero subgroup of $\mathbb{Z}$ and hence to $\mathbb{Z}$. Conversely, if v is discrete, Γ_v is isomorphic to $\mathbb{Z}$ and Γ_w/Γ_v is a finite group. Hence Γ_w is a finitely generated commutative group of rank 1 and torsion-free. Consequently it is isomorphic to $\mathbb{Z}$. □

5.6.2 Prolongations of a valuation

Given a valuation ring $A \subseteq K$, and an extension L/K, any valuation ring B of L dominating A is called a **prolongation** of A to L. Equivalently, if v is the valuation corresponding to K, then prolongations of A corresponds to extensions of v to L. In this part, we consider the connection bewteen these prolongations.

The following lemma will be used throughout this part, and it states that two distinct prolongations of A to an algebraic extension L/K are incomparable.

Lemma 5.6.10. *Suppose L/K is an algebraic extension of fields, A is a valuation ring of K, and B_1, B_2 are two prolongations of A to L. If $B_1 \subseteq B_2$, then $B_1 = B_2$.*

Proof. The valuation ring B_1 maps to a valuation ring $\bar{B}_1 = B_1/\mathfrak{m}_{B_2}$ of the residue field $\kappa_{B_2} = B_2/\mathfrak{m}_{B_2}$. Since B_1 is an extension of A, it follows that $\bar{B}_1$ is also an extension of κ_v. According to Theorem 5.6.7, κ_{B_2} is an algebraic extension of κ_v, so it follows that $\bar{B}_1$ is also a field. But $\bar{B}_1$ is a valuation ring of κ_{B_2}, so it follows that $\bar{B}_1 = \kappa_{B_2}$ whence $B_1 = B_2$. □

There may exist infinitely many valuation rings of L lying over A, of course. But sometimes, their number has a natural bound. This is the content of the next result.

Theorem 5.6.11. *Let L be algebraic over K and $[L : K]_s < \infty$. Let A be a valuation ring of K. Then the number n of all prolongations of A to L is finite, and $n \leq [L : K]_s$.*

Proof. Let $B_1, \ldots, B_n$ be the distinct prolongations of A to L, with maximal ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$, respectively. Also, by Lemma 5.6.10 these prolongations are pairwise incomparable. Therefore, by Corollary 5.4.7 there exist $x_1, \ldots, x_n \in L$ such that for all $i, j \in \{1, \ldots, n\}$,

$$x_i - 1 \in \mathfrak{m}_i, \quad x_i \in \mathfrak{m}_j \text{ for } i \neq j.$$

If K has characteristic $p > 0$, pick k large enough to guarantee that $x_i^{p^k} \in K^s$, the separable closure of K in L. Then we claim that these n elements $x_i^{p^k}$ are lineraly independent over K. Consequently, $n \leq [K^s : K] = [L : K]_s$, and the result follows.

We shall prove the claim by contradiction. For $a_1, \ldots, a_n \in K$, not all zero, such that $\sum_{i=1}^n a_i x_i^{p^k} = 0$, pick j such that

$$v(a_j) = \min\{v(a_1), \ldots, v(a_n)\}.$$

Then $a_j \neq 0$ and we have

$$x_j^{p^k} = -\sum_{i \neq j} a_i x_i^{p^k} \in \mathfrak{m}_j.$$

Since this would imply $x_j \in \mathfrak{m}_j$ and hence $1 \in \mathfrak{m}_j$, we get the desired contradiction. □

Corollary 5.6.12. *If the extension L/K is purely inseparable, then every valuation ring of K has a unique prolongation to L.*

With Theorem 5.6.11 in hand, we now classifies all prolongations of a given valuation ring to L: they corresponds to maximal ideals of the integral closure of A in L.

Theorem 5.6.13. *Let L be an algebraic extension of a field K, A be a valuation ring of K, and R the integral closure of A in L.*

(a) If B is a prolongation of A to L then $\mathfrak{m} = \mathfrak{m}_B \cap R$ is a maximal ideal of R.

(b) If $\mathfrak{m}$ is a maximal ideal of R then $R_{\mathfrak{m}}$ is a prolongation of A to L.

(c) The map $B \mapsto \mathfrak{m}_B \cap R$ gives a one-to-one correspondence between maximal ideals of R and prolongations of A to L, and its inverse is given by $\mathfrak{m} \mapsto R_{\mathfrak{m}}$.

Proof. Note that any prolongation B of A to L must contains R, by Theorem 5.1.8. Since $\mathfrak{m}_B$ is maximal in B, $\mathfrak{m}$ is also maximal in R. Conversely, since a contraction of maximal ideal is maximal, we see $R_{\mathfrak{m}}$ is a prolongation of A for every maximal ideal $\mathfrak{m}$ in R.

In remains to prove that, for a prolongation B of A to L we have $B = R_{\mathfrak{m}}$, where $\mathfrak{m} = \mathfrak{m}_B \cap R$. The inclusion $R_{\mathfrak{m}} \subseteq B$ is clear. We first consider the case where the extension L/K is finite. In this case, there are finitely many prolongations of A to L by Theorem 5.6.11, say $B_1, \ldots, B_n$ with maximal ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$. Then by Proposition 5.4.5 we have $B_i = R_{\mathfrak{m}_i}$, where $\mathfrak{m}_i = \mathfrak{m}_{B_i} \cap R$, whence the claim holds in this case.

Now consider the general case. Let L' be a subfield of L such that L'/K is finite. Then we see $R' = R \cap L'$ is the integral closure of A in L' and $B' = B \cap L'$ is a prolongation of A to L' with maximal ideal $\mathfrak{m}_{B'} = \mathfrak{m}_B \cap L'$. By the argument above, we get $B' = R'_{\mathfrak{m}'}$, where $\mathfrak{m}' = \mathfrak{m} \cap L' = \mathfrak{m}_{B'} \cap R'$. Since the subfield L' is arbitrary, it follows that $B = R_{\mathfrak{m}}$. □

Next we shall consider the set of all prolongations of a fixed valuation ring of a field K to normal extensions of K. It turn out that the automorphism group $\mathrm{Aut}(L/K)$ plays an important rule.

Theorem 5.6.14. *Suppose L/K is a normal extension of fields, with $G = \mathrm{Aut}(L/K)$. Suppose A is a valuation ring of K, and B_1 and B_2 are valuation rings in L extending A. Then B_1 and B_2 are conjugate over K, i.e., there exists $\sigma \in G$ with $\sigma(B_1) = B_2$.*

Proof. First, split the extension L/K into the steps $K \subseteq K^s \subseteq L$. Corollary 5.6.12 implies that every extension of A to K^s has just one prolongation to L. Furthermore, $\mathrm{Aut}(K^s/K)$ and G can be canonically identified. Therefore, we see that it is enough to consider the case where L is separable over K.

We may assume $[L : K] < \infty$, and the general case follows by considering the compactness of Galois groups. In this case let

$$H_1 = \{\sigma \in G : \sigma(B_1) = B_1\}, \quad H_2 = \{\tau \in G : \tau(B_2) = B_2\}.$$

Then H_1 and H_2 are subgroups of G. Moreover, for every $\sigma \in H_1$, it follows that $\sigma(\mathfrak{m}_1) = \mathfrak{m}_1$, for the maximal ideal of B_1. Indeed, it is enough to observe that $\sigma(\mathfrak{m}_1)$ must be the maximal ideal of $\sigma(B_1)$. Analogously for the maximal ideal $\mathfrak{m}_2$ of B_2, it follows that $\tau(\mathfrak{m}_2) = \mathfrak{m}_2$ for all $\tau \in H_2$. Next write G as disjoint unions of cosets of H_1 and H_2, respectively:

$$G = \bigcup_{i=1}^{n} H_1\sigma_i^{-1}, \quad G = \bigcup_{k=1}^{m} H_2\tau_k^{-1}.$$

Suppose now, for the sake of contradiction, that $\sigma_i(B_1) \nsubseteq \tau_k(B_2)$ and $\tau_k(B_2) \nsubseteq B_1$ for all i, k. Since $\sigma_1^{-1}, \ldots, \sigma_n^{-1}$ is a complete set of representatives of cosets of H_1, for all $i \neq j$, $\sigma_i(B_1) \nsubseteq \sigma_j(B_1)$. Similarly, $\tau_k(H_2) \nsubseteq \tau_l(B_2)$ for every $k \neq l$. Now take

$$C = \bigcap_{i=1}^{n} \sigma_i(H_1) \cap \bigcap_{k=1}^{m} \tau_j(H_2).$$

According to Corollary 5.4.7, there exists an $x \in C$ such that, for each i, k,

$$x - 1 \in \sigma_i(\mathfrak{m}_1), \quad x \in \tau_k(\mathfrak{m}_2)$$

As a consequence, for $\sigma \in G$, writing $\sigma = \rho\sigma_i^{-1}$ and $\rho \in H_1$, it follows that

$$\sigma(x-1) \in \rho\sigma_i^{-1}(\sigma_i(\mathfrak{m}_1)) = \rho(\mathfrak{m}_1) = \mathfrak{m}_1.$$

Analogously, $\sigma(x) \in \mathfrak{m}_2$ for every $\sigma \in G$. Taking norms, it then follows

$$N_{L/K}(a) = \prod_{\sigma \in G} \sigma(a) \in (\mathfrak{m}_1 + 1) \cap K = \mathfrak{m} + 1$$

and

$$N_{L/K}(a) = \prod_{\sigma \in G} \sigma(a) \in \mathfrak{m}_2 \cap K = \mathfrak{m}.$$

This contradiction implies $\sigma_i(\mathfrak{m}_1) \subseteq \tau_j(\mathfrak{m}_2)$ or $\tau_j(\mathfrak{m}_2) \subseteq \sigma_i(\mathfrak{m}_1)$. Thus the claim holds in this case by Lemma 5.6.10. $\square$

Now, since we now all prolongations of A are conjugate under $\mathrm{Aut}(L/K)$, we are ready to prove the following theorem.

Theorem 5.6.15. *Let L be a normal extension of a field K, A a valuation ring of K, and B a prolongation of A to L. Let v and w be valuations corresponding to A and B.*

(a) For $\sigma \in \mathrm{Aut}(L/K)$, the map $w \circ \sigma$ is the unique valuation of L that corresponds to $\sigma^{-1}(B)$. In particular, if $\sigma(B) = B$, then $w \circ \sigma = w$.

(b) κ_B is a normal extension of κ_v.

(c) The map $x \mapsto \overline{\sigma(x)}$ induces a K-isomorphism from $\sigma^{-1}(B)/\sigma^{-1}(\mathfrak{m}_B)$ onto κ_B. In particular, if $\sigma(B) = B$ then $\bar{\sigma} \in \mathrm{Aut}(\kappa_B/\kappa_v)$.

(d) *For every* $\sigma \in \mathrm{Aut}(L/K)$ *we have* $e(\sigma^{-1}(B)/A) = e(B/A)$ *and* $f(\sigma^{-1}(B)/A) = f(B/A)$.

Proof. Part (a), (b), (d) are easily verified, so we concentrate on (b). Let $\bar{f} \in \kappa_v[X]$ be an irreducible polynomial with a root $\bar{x} \in \kappa_v$. Let R be the integral closure of A in L and write $\mathfrak{m} = \mathfrak{m}_B \cap R$. By Theorem 5.6.13, we have $B = R_{\mathfrak{m}}$. Observe next that R is fixed by $\mathrm{Aut}(L/K)$. Now let $x \in R$ be a preimage of $\bar{x}$, and let $g \in A[X]$ be the minimal polynomial of x over K. Since L/K is normal, g splits completely as $g(X) = \prod_{i=1}^{n}(X - x_i)$ with $x_i \in R$. From $\bar{g}(\bar{x}) = 0$, it follows that $\bar{f}$ divides $\bar{g}$. But $\bar{g} = \prod_{i=1}^{n}(X - \bar{x}_i)$ in κ_B, hence $\bar{f}$ has all its roots in κ_B, proving (b). □

5.6.3 Henselian fields

In this part we introduce the basic concepts of a Henselian field. These will be used when we establish the structure theorem about extension of valuations.

We have seen that every rank-one valuation v of a complete field K has a unique prolongation to each algebraic extension of K. By Corollary 5.6.12, also every valuation of a separably closed field K has this property. Valuation rings, or valuations, with this property are very important. They are the so-called "Henselian" valuation rings. We shall see that they are the suitable substitute for rank-one valuation rings of complete fields. In particular, they are the valuation rings for which Hensel's Lemma holds.

A valued field (K, v) is called **Henselian** if v has a unique extension to every algebraic extension L of K. We see from the definition that the property of being Henselian is hereditary. Let $\bar{K}$ be an algebraic closure and K^s be the separable closure of K in $\bar{K}$. We then have the following apparently easier characterization of Henselian valuations.

Proposition 5.6.16. *A valued field* (K, v) *is Henselian if and only if it extends uniquely to* K^s.

Proof. By definition, if (K, v) is Henselian then v extends uniquely to K^s. Conversely, take an algebraic extension L of K. By Corollary 5.6.12, every extension of v to $L \cap K^s$ has an extension to K^s. By assumption, v extends uniquely to $L \cap K^s$, so v has a unique prolongation to L. □

The next theorem will highlight the importance of Henselian valuation rings. Comparing with completions we see that rank-one complete valuation rings are a particular case of Henselian valuation rings. Before stating the theorem we need a little preparation.

Let $v : K^\times \to \Gamma$ be a valuation of the field K. The **Gauss extension** w of v to the rational function field $K(X)$ is given by

$$w\Big(\sum_{i=0}^{n} a_i X^i\Big) = \min_i\{v(a_i)\}.$$

We say a polynomial is primitive if $w(f) = 0$. The following properties of primitive polynomials hold, which can be seen as a generalization of the well known Gauss's lemma.

Lemma 5.6.17. *Let* $v : K^\times \to \Gamma$ *be a valuation with valuations ring* A.

(a) *If two polynomials* f *and* g *are primitive, then so is their product* fg.

(b) *Every* $f \in K[X]$ *admits a decomposition* $f = a\tilde{f}$ *with* $a \in K$ *and* $\tilde{f} \in K[X]$ *primitive.*

(c) *If* $f \in A[X]$ *decomposes as* $f = \prod_{i=1}^{m} g_i$ *with irreducible factors* $g_1, \dots, g_m \in K[X]$*, then there are* $h_1, \dots, h_m \in A[X]$*, irreducible in* $K[X]$*, such that* $f = \prod_{i=1}^{m} h_i$.

Proof. Part (a) and (b) are easy to prove. To see (c), write $f = a\tilde{f}$ and $g_i = b_i\tilde{g}_i$, where $a, b_1, \dots, b_m \in K$ and $\tilde{f}, \tilde{g}_1, \dots, \tilde{g}_m$ are primitive. Then

$$v(a) = w(f) = \sum_{i=1}^{n} w(g_i) = w(b_1 \cdots b_m).$$

Since $f \in A[X]$, we have $v(a) \geq 0$, whence $b_1 \cdots b_m \in A$. Define $h_1 = b_1 \cdots b_m \tilde{g}_1$ and $h_i = \tilde{g}_i$, we see the desired decomposition. $\square$

Theorem 5.6.18. *Let (K, v) be a valued field with valuation ring A and $\mathfrak{m}$ its maximal ideal. The following statements are equivalent:*

(i) *(K, v) is Henselian.*

(ii) *For each irreducible polynomial $f \in A[X]$ with $\bar{f} \notin (\kappa_v)[X]$, there exists $g \in A[X]$ such that $\bar{g}$ is irreducible in $(\kappa_v)[X]$ and $\bar{f} = \bar{g}^s$, for some $s \geq 1$.*

(iii) *Let $f, g, h \in A[X]$ satisfy $\bar{f} = \bar{g}\bar{h}$, with $\bar{g}, \bar{h}$ relatively prime in $(\kappa_v)[X]$. Then there exist $g_1, h_1 \in A[X]$ such that*

$$f = g_1 h_1, \quad \bar{g}_1 = \bar{g}, \bar{h}_1 = \bar{h}, \quad \deg g_1 = \deg g, \deg h_1 = \deg h.$$

(iv) *For each $f \in A[X]$ and $a \in A$ with $\bar{f}(\bar{a}) = 0$ and $\bar{f}'(\bar{a}) \neq 0$, there exists an $\alpha \in A$ with $f(\alpha) = 0$ and $\bar{\alpha} = \bar{a}$.*

(v) *For each $f \in A[X]$ and $a \in A$ with $v(f(a)) > 2v(f'(a))$, there exists an $\alpha \in A$ with $f(\alpha) = 0$ and $v(a - \alpha) > v(f'(\alpha))$.*

(vi) *Every polynomial $f(X) = X^n + a_{n-1}X^{n-1} + \cdots + a_1 X + a_0 \in A[X]$ with $a_{n-1} \in\notin \mathfrak{m}$ and $a_{n-2}, \ldots, a_0 \in \mathfrak{m}$ has a zero in K.*

(vii) *Every polynomial $f(X) = X^n + X^{n-1} + \cdots + a_1 X + a_0 \in A[X]$ with $a_{n-2}, \ldots, a_0 \in \mathfrak{m}$ has a zero in K.*

Proof. Let (K, v) be Henselian, denote by w the unique extension of v to $\bar{K}$, the algebraic closure of K. Write B the valuation ring of w and $\mathfrak{m}_B$ for the maximal ideal, and κ_B for the residue class field of B. Since w is the unique extension of v, for every K-automorphism σ of $\bar{K}$ we have $\sigma(B) = B$ and $\sigma(\mathfrak{m}_B) = \mathfrak{m}_B$. Let f an be irreducible polynomial $f \in A[X]$ with $\bar{f} \notin (\kappa_v)[X]$. Write

$$f(X) = \prod_{i=1}^{n} (aX - x_j)$$

where $a, x_1, \ldots, x_n \in K$ and a is an n-th root of the leading coefficient of f. Then $a \in B$ since $w(a) = v(a)/n \geq 0$. Also, $(-1)^n x_1 \ldots x_n = f(0) \in A$.

The roots $x_1/a, \ldots, x_n/a$ of f are all K-conjugate, and w is invariant under $\mathrm{Aut}(\bar{K}/K)$. Consequently, there exists $\gamma \in \Gamma_w$ such that $w(x_j/a) = \gamma$ for all j. So, for $\delta = \gamma + w(a)$, we have that $w(x_j) = \delta$ for each j. Therefore, $x_1, \ldots, x_n \in A$ implies $\delta \geq 0$, and thus $x_1, \ldots, x_n \in \mathfrak{m}_B$ or $x_1, \ldots, x_n \in B - \mathfrak{m}_B$.

In the first case, $\bar{f} = (\bar{a}X)^n$ and the claim (b) follows. In the second case, we obtain

$$\bar{f} = \prod_{i=1}^{n} (\bar{a}X - \bar{x}_i)$$

with $\bar{x}_i \neq 0$. Since $\bar{f} \notin (\kappa_v)[X]$, we also have $\bar{a} \neq 0$. For the sake of seeking a contradiction, let us assume that $\bar{f} = \bar{g}\bar{h}$ for some relatively prime polynomials $\bar{g}$ and $\bar{h}$ with $g, h \in A[X]$. Let $\overline{x_i/a}$ be a root of $\bar{g}$ and take some $x_j \neq x_i$ such that $\overline{x_j/a}$ is a root of $\bar{h}$. Hence $g(x_i/a) \in \mathfrak{m}_A$ and $h(x_j/a) \in \mathfrak{m}_A$. Take $\sigma \in \mathrm{Aut}(\bar{K}/K)$ such that $\sigma(x_i/a) = x_j/a$. Then

$$g(x_j/a) = g(\sigma(x_i/a)) = \sigma(g(x_i/a)) \in \sigma(\mathfrak{m}_B) = \mathfrak{m}_B$$

which means $\overline{x_j/a}$ is also a root of $\bar{g}$, contradiction. This proves $(i) \Rightarrow (ii)$.

Now assume (ii), and let $f = gh$ with $f, g, h \in A[X]$ and g, h coprime. By Lemma 5.6.17, let $f = \prod_{i=1}^m g_i$ be a factorization of f with irreducible factors $g_1, \ldots, g_m \in A[X]$. By condition (ii), for every i, either $g_i \in K$ or there exists $p_i \in A[X]$ such that p_i is irreducible and $g_i = p_i^{t_i}$ for some $t_i \geq 1$. Clearly, we may assume that p_i has no non-zero coefficient in $\mathfrak{m}_A$. Renumbering the polynomials $g_1, \ldots, g_m$, we may assume that

$$\bar{g} = \bar{a}\prod_{i=1}^{k}\bar{p}_i^{t_i}, \quad \bar{b} = \bar{h}\prod_{i=k+1}^{\ell}\bar{p}_i^{t_i}, \quad \bar{c} = \prod_{i=\ell+1}^{m}\bar{p}_i^{t_i}$$

for some $a, b, c \in A - \mathfrak{m}$, since $\bar{g}$ and $\bar{h}$ are coprime. Define now

$$g_1 = a\prod_{i=1}^{k}p_i^{t_i}, \quad h_1 = \Big(b\prod_{i=k+1}^{\ell}p_i^{t_i}\Big)\Big(c^{-1}\prod_{i=\ell+1}^{m}p_i^{t_i}\Big)$$

then the polynomials g_1 and h_1 satisfies the requirements in (iii), so we have shown $(ii) \Rightarrow (iii)$.

Next, we will deduce (iv) from (iii). If $f \in A[X]$ and $a \in A$ with $\bar{f}(\bar{a}) = 0$ and $\bar{f}'(\bar{a}) \neq 0$, set $g(X) = X - a$ and $\bar{h} = \bar{f}/\bar{g}$, then $\bar{f} = \bar{g}\bar{h}$. Since $\bar{f}'(\bar{a}) \neq 0$, $\bar{g}$ and $\bar{h}$ are coprime, so by condition (iii) there exist polynomials g_1 and h_2 such that

$$f = g_1h_1, \quad \bar{g}_1 = \bar{g} = X - \bar{a}, \bar{h}_1 = \bar{h}, \quad \deg g_1 = \deg g = 1, \deg h_1 = \deg h.$$

It then follows that $g_1 = u(X - b)$ for some $u \in A^\times$ and $b \in A$, with $\bar{u} = 1$ and $\bar{b} = \bar{a}$. Clearly $f(b) = 0$, so this proves (iv).

For each $f \in A[X]$ and $a \in A$ with $v(f(a)) > 2v(f'(a))$, we have $f'(a) \neq 0$. Now by a simple computation we know $f(a - X) = f(a) - f'(a)X + X^2g(X)$ for some $g \in A[X]$, and by a change of variable $X = f'(a)Y$ we can define

$$h(Y) := \frac{f(a - f'(a)Y)}{f'(a)^2} = \frac{f(a)}{f'(a)^2} - Y + Y^2g(f'(a)Y).$$

Since $v(f(a)) > 2v(f'(a))$ we have $h \in A[Y]$. Note that $f(a)/f'(a)^2 \in \mathfrak{m}_A$, so $\bar{h}$ has a simple root $\bar{0}$ in κ_v. If condition (iv) holds, this will give a root $b \in \mathfrak{m}$ of h. Then by the construction of h, $\alpha = a - f'(a)b$ will a root of f. Moreover, since $v(b) > 0$, we have $v(a - \alpha) > v(f'(a))$. This shows $(iv) \Rightarrow (v)$.

The implication $(v) \Rightarrow (vi)$ is immediate. In fact, let $f(X) = X^n + a_{n-1}X^{n-1} + \cdots + a_1X + a_0$ be as in (vi). Then

$$\bar{f}(X) = X^n + \bar{a}_{n-1}X^{n-1} = X^{n-1}(X + \bar{a}_{n-1}).$$

Hence $-\bar{a}_{n-1}$ is a nonzero simple zero of $\bar{f}$. In particular, $f(-a_{n-1}) > 0$ and $f'(-a_{n-1}) \in A - \mathfrak{m}$. This means

$$v(f(-a_{n-1})) > 0 = 2v(f'(-a_{n-1}))$$

thus f has a root in A, by (v).

It is clear that $(vi) \Rightarrow (vii)$. Conversely, if (vii) holds and $f(X) = X^n + a_{n-1}X^{n-1} + \cdots + a_1X + a_0$ is a polynomial as in (vi), then the changing $X = a_{n-1}Y$ and dividing by a_{n-1}^n, we obtain

$$g(Y) = Y^n + Y^{n-1} + \frac{a_{n-2}}{a_{n-1}^2}Y^{n-2} + \cdots + \frac{a_1}{a_{n-1}^n}Y + \frac{a_0}{a_{n-1}^n}.$$

It is clear that a root of $g(Y)$ gives a root of f, so (vi) and (vii) are equivalent.

Finally, we prove $(vii) \Rightarrow (i)$. Suppose (K, v) were not henselian. Then there would be a finite Galois extension L/K with Galois group $\mathrm{Gal}(L/K)$ in which v has more than one extension. Let B be a prolongation of A to L and set $H = \{\sigma \in \mathrm{Gal}(L/K) : \sigma(B) = B\}$. As B is not the only prolongation of A to L, the group H is a proper subgroup of $\mathrm{Gal}(L/K)$, and thus the fixed field $L' := L^H$ of H is a proper extension of K. Let $B_1, \ldots, B_n$ be all conjugates of B in L with maximal ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$ and consider the subring

$$R = \bigcap_{i=1}^{m} (B_i \cap L')$$

of L'. By Corollary 5.4.7, we find $\alpha \in R$ such that $\alpha - 1 \in \mathfrak{m}_1$ and $\alpha \in \mathfrak{m}_i$ for $i = 2, \ldots, n$. Since $n > 1$, α cannot lie in K, so its minimal polynomial

$$f = \min(\alpha, K) = X^n + a_{n-1}X^{n-1} + \cdots + a_1 X + a_0$$

cannot have a zero in K. However, if $\alpha_1 = \alpha, \alpha_2, \ldots, \alpha_n$ are conjugates of α in L, we have

$$a_{n-i} = (-1)^i e_i(\alpha_1, \ldots, \alpha_n), \tag{5.6.1}$$

where e_i is the i-th symmetric function. For each $i \geq 2$, we have $\alpha_i \neq \alpha$, so $\alpha_i = \tau(\alpha)$ for some $\tau \in \mathrm{Gal}(L/K) \setminus H$ (note that $\alpha \in L'$). Hence $\tau^{-1}(B) = B_j$ for some $j \geq 2$. Since by the approximation condition we have $\alpha \in \mathfrak{m}_j = \tau^{-1}(\mathfrak{m}_1)$, we then get $\alpha_i = \tau(\alpha) \in \mathfrak{m}_1$. By (5.6.1), this implies $1 + a_{n-1} \in \mathfrak{m}_1$ and $a_{n-2}, \ldots, a_1, a_0 \in \mathfrak{m}_1$. Thus by (vi) the polynomial f has a root in K, contradiction. □

Corollary 5.6.19. *Let (K, w) be a composition of valuations (K, v) and $(\kappa_v, \bar{v})$. Then (K, w) is Henselian if and only if both (K, v) and $(\kappa_v, \bar{v})$ are Henselian.*

Proof. Suppose (K, w) is Henselian. Then (K, v) is also Henselian, using $\mathfrak{m}_v \subseteq \mathfrak{m}_w \subseteq A_w \subseteq A_v$ and Theorem 5.6.18. To show that $(\kappa_v, \bar{v})$ is Henselian, let $\bar{f} = X^n + X^{n-1} + \cdots + \bar{a}_0$, with $a_i \in \mathfrak{m}_{\bar{v}}$ we must show that $\bar{f}$ has a zero in κ_v (again using Theorem 5.6.18). The polynomial

$$f = X^n + X^{n-1} + \cdots + a_0 \in A_w[X]$$

has a zero $x \in A_w$ (yet again by Theorem 5.6.18), since $a_i \in \mathfrak{m}_v \subseteq \mathfrak{m}_w$); therefore $\bar{x} \in A_{\bar{v}}$ is a zero of $\bar{f}$.

Now we prove the converse. Let $f = X_n + a_{n-1}X^{n-1} \cdots + a_0 \in A_w[X]$, and suppose that $\bar{f}$ has a simple zero in the residue field κ_w. As κ_w is also the residue field of $\bar{v}$, and $\bar{v}$ is Henselian, this simple zero of $\bar{f}$ lifts to κ_v. As the field (K, v) is also Henselian, the zero (which is again simple) can be lifted further to K. □

Let (K, v) be a valued field. Assume L/K is a Galois extension. We then say (K, v) is ***L*-Henselian** if v has exactly one extension to L. By taking $L = K^s$, the separable closure of K, we return to our usual definition of Henselian field. Just as Theorem 5.6.18, we can also obtain many equivalent characterizations of L-Henselian fields. One need only go through the proof of Theorem 5.6.18 and restrict consideration to those polynomials f that split in L. The corresponding condition for the Galois extension K^s/K is always satisfied, hence, of course, was not mentioned in Theorem 5.6.18.

5.6.4 The fundamental inequality

Definition 5.6.20. Let K be a field, v a valuation on K and L an extension of K. A family $(w_i)_{i \in I}$ of valuations on L which extend v and such that every valuation on L extending v is equivalent to a unique w_i is called a **complete system of extensions of v to L**.

We have already proved that the completion is an immediate extension of a given valued field. Now we go deeper to this result and consider the case of a extension of valued fields.

Proposition 5.6.21. *Let (K, v) be a valued field and $(\widehat{K}, \hat{v})$ its completion of K. Let L be a finite extension of K of degree n.*

(a) Let w be a valuation on L extending v and $(\widehat{L}_w, \hat{w})$ denote the completion of (L, w). Identity $\widehat{K}$ with the closure of K in $\widehat{L}_w$, then

$$e(\hat{w}/\hat{v}) = e(w/v), \quad f(\hat{w}/\hat{v}) = f(w/v), \quad e(w/v)f(w/v) \leq [\widehat{L}_w : \widehat{K}] \leq n.$$

(b) Every set of pairwise independent valuations on L extending a nontrivial valuation v is finite. Let $(w_i)_{1\leq i\leq s}$ denote pairwise independent valuations on L extending v such that every valuation on L extending v is dependent on one of the w_i; let L_i be the field L with the topology defined by w_i and $\widehat{L}_i$ its completion. Then the canonical map

$$\phi : \widehat{K} \otimes_K L \to \prod_{i=1}^{s} \widehat{L}_i$$

(extending by continuity the diagonal map $L \to \prod_{i=1}^{s} L_i$) is surjective, its kernel is the Jacobson radical of $\widehat{K} \otimes_K L$. In particular, $\sum_{i=1}^{s} [\widehat{L}_i : \widehat{K}] \leq n$.

Proof. Let us first prove (a). Suppose that v is nontrivial. As v and $\hat{v}$ (resp. w and $\hat{w}$) have the same order group and the same residue field, we see $e(\hat{w}/\hat{v}) = e(w/v)$ and $e(\hat{w}/\hat{v}) = e(w/v)$. In particular, $e(w/v)f(w/v) \leq [\widehat{L}_w : \widehat{K}]$ by Corollary 5.6.6. Finally the vector sub-$\widehat{K}$-space of $\widehat{L}_w$ generated by L is closed in $\widehat{L}_w$ (by Corollary 5.4.18) and also dense in $\widehat{L}_w$ so it equals to $\widehat{L}_w$. This shows $[\widehat{L}_w : \widehat{K}] \leq n$ and completes the proof of (a).

We now pass to (b). We may still assume that v is not trivial. Let $(w_1, \dots, w_r)$ be any finite family of pairwise independent valuations on L extending v. The image of L in $\prod_{i=1}^{r} L_i$ under the diagonal map is dense by the approximation theorem, and $\prod_i L_i$ is dense in $\prod_{i=1}^{r} \widehat{L}_i$. Hence the canonical Image of $\widehat{K} \otimes_K L$ in $\prod_{i=1}^{r} \widehat{L}_i$ is dense. On the other hand this image is a vector sub-$\widehat{K}$-space of $\prod_{i=1}^{r} \widehat{L}_i$; as $\prod_{i=1}^{r} \widehat{L}_i$ is of finite dimension over $\widehat{K}$ by (a), every subspace of $\prod_{i=1}^{r} \widehat{L}_i$ is closed. Thus the image of $\widehat{K} \otimes_K L$ is closed and equal to $\prod_{i=1}^{r} \widehat{L}_i$. As the dimension of $K \otimes_K L$ over K is n, we see $\sum_{i=1}^{s} [\widehat{L}_i : \widehat{K}] \leq n$. This shows in particular that the integer r is bounded above by n and shows the first assertion of (b).

We now take $(w_1, \dots, w_s)$ as in the statement. The fact that

$$\phi : \widehat{K} \otimes_K L \to \prod_{i=1}^{s} \widehat{L}_i$$

is suriective have already been shown. It remains to verify that the kernel of ϕ is the Jacobsen radical J of $K \otimes_K L$. As $\prod_{i=1}^{s} \widehat{L}_i$ is semi-simple, we see $J \subseteq \ker \phi$. On the other hand, for every maximal ideal $\mathfrak{m}$ of $\widehat{K} \otimes_K L$, the quotient field $L(\mathfrak{m}) = (\widehat{K} \otimes_K L)/\mathfrak{m}$ is a composite extension of $\widehat{K}$ and L over K (**??**). There exists a valuation μ on $L(\mathfrak{m})$ extending $\hat{v}$; the restriction w of μ to L then extends v. As $[L(\mathfrak{m}) : \widehat{K}]$ is finite, $L(\mathfrak{m})$ is complete with respect to μ (Proposition 5.4.17). Now the closure of L in $L(\mathfrak{m})$ is a field containing $\widehat{K}$ and L and hence is equal to $L(\mathfrak{m})$. Consequently $L(\mathfrak{m})$ is identified with the completion $\widehat{L}_w$ and $\mathfrak{m}$ is the kemel of the canonical map of $\widehat{K} \otimes_K L$ onto $\widehat{L}_w$. Now, by hypothesis, there exists an index i such that w and w_i are dependent; whence $\widehat{L}_w = \widehat{L}_i$. Thus $\ker \phi \subseteq \mathfrak{m}$, which proves that $\ker \phi \subseteq J$ and completes the proof. □

Corollary 5.6.22. *If K is complete with respect to v and v is nontrivial, two valuations on L extending v are dependent.*

Proof. In this case we have $\widehat{K} \otimes_K L = K \otimes_K L = L$, so the claim follows by Proposition 5.6.21(b). □

Corollary 5.6.23. *If the extension L/K or $\widehat{K}/K$ is separable then the map ϕ is an isomorphism.*

Proof. In this case the Jacobsen radical of $\widehat{K} \otimes_K L$ is zero. □

Theorem 5.6.24 (The Fundamental Inequality). *Let (K, v) be a valued field and L a finite extension of K of degree n. Then every complete system $(w_i)_{i\in I}$ of extensions of v to L is finite and we have*

$$\sum_{i\in I} e(w_i/v)f(w_i/v) \le n.$$

Proof. Since the theorem is trivial if v is trivial, we shall assume that v is nontrivial. Let $(w_1, \dots, w_s)$ be any finite family of valuations on L extending v, no two of which are equivalent. We shall show that $\sum_{i=1}^{s} e(w_i/v)f(w_i/v) \le n$. This will prove the theorem.

We argue by induction on s and suppose therefore that the inequality has been established for the case of valuations of number smaller than s. We distinguish two cases.

Suppose that there exist at least two independent valuations of w_i. Then there exists a partition $[1, s] = I_1 \cup \cdots \cup I_t$ of $[1, s]$ such that:

(i) for w_i and w_j to be dependent, it is necessary and suflicient that i and j belong to the same I_k;

(ii) $|I_k| < s$ for all k.

We choose in each I_k an index i_k. Let $\widehat{L}_k$ denote the completion of L with respect to v_{i_k} and $n_k = [\widehat{L}_k : \widehat{K}]$. For all $i \in I_k$, w_i defines on L the same topology as w_{i_k} and hence may be extended to a valuation $\hat{w}_i$ on $\widehat{L}_k$ whose restriction to $\widehat{K}$ is $\hat{v}$. Since no two of the w_i for $i \in I_k$ are equivalent, the same is true of the $\hat{w}_i$. The induction hypothesis applied to the ordered pair $(\widehat{K}, \widehat{L}_k)$ shows

$$\sum_{i\in I_k} e(w_i/v)f(w_i/v) \le n_k.$$

As $\sum_k n_k \le n$ by Proposition 5.6.21, we see the claim follows in this case.

We now pass to the case where any two of the w_i are dependent. In this case there is a nontrivial valuation w' of L such that $w_i = w \circ \bar{w}_i$, where $\bar{w}_i$ is a valuation of the residue field $\kappa_{w'}$. Denote by v' the restriction of w' to K, then v is also a composition of v', say $v = v' \circ \bar{v}$, where $\bar{v}$ is a valuation of $\kappa_{v'}$. If we choose w such that the valuation rings A_{w_i} generates $A_{w'}$, then the valuation rings of $\bar{w}_i$ generates $\kappa_{w'}$, so that they are not all dependent. From the first part of the proof, we then have

$$\sum_{i=1}^{s} e(\bar{w}_i/\bar{v})f(\bar{w}_i/\bar{v}) \le [\kappa_{w'} : \kappa_{v'}] = f(w'/v')$$

and hence

$$\sum_{i=1}^{s} e(w'/v')e(\bar{w}_i/\bar{v})f(\bar{w}_i/\bar{v}) \le e(w'/v')f(w'/v') \le n.$$

Now it suffices to prove that

$$f(w_i/v) = f(\bar{w}_i/\bar{v}), \quad e(w'/v')e(\bar{w}_i/\bar{v}) = e(w_i/v). \tag{5.6.2}$$

For this, we note that v and $\bar{v}$ (resp. w_i and $\bar{w}_i$) have the same residue field, so the first equality holds. For the second there is, by Proposition 5.3.3 and Proposition 5.3.1, the following commutative diagram, where the rows are exact sequences and the vertical arrows represent canonical injections:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \Gamma_{\bar{v}} \arrow[r] \arrow[d] & \Gamma_v \arrow[r] \arrow[d] & \Gamma_{v'} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Gamma_{\bar{w}_i} \arrow[r] \arrow[d] & \Gamma_{w_i} \arrow[r] \arrow[d] & \Gamma_{w'} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Gamma_{\bar{w}_i}/\Gamma_{\bar{v}} \arrow[r] \arrow[d] & \Gamma_{w_i}/\Gamma_v \arrow[r] \arrow[d] & \Gamma_{w'}/\Gamma_{v'} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

From this we see the second equality is valid, therefore the proof is completed. $\square$

Now we preceed a step further to investigate when the equality in Theorem 5.6.24 holds. For this, we need the following definition.

Definition 5.6.25. Let Γ be a totally ordered group. A subset of Γ is called **major** if the relations $\alpha \in M$, $\beta \in \Gamma$ and $\beta \geq \alpha$ imply $\beta \in M$.

Example 5.6.26. Let Δ be a isolated subgroup of a totally ordered group Γ. Then the set

$$M = \{\alpha \in \Gamma : \alpha \geq \gamma \text{ for all } \gamma \in \Delta\}$$

is a major subset of Γ.

Let K be a field, v a valuation on K, and A the ring of v and Γ the order group of v. For every major subset $M \subseteq G$, let $\mathfrak{a}(M)$ be the set of $x \in K$ such that $v(x) \in M \cup \{\infty\}$. Clearly $\mathfrak{a}(M)$ is a sub-A-module of K.

Proposition 5.6.27. *The map $M \mapsto \mathfrak{a}(M)$ is an increasing bijection of the set of major subsets of Γ onto the set of sub-A-modules of K.*

Proof. Let $\mathfrak{b}$ be a sub-A-module of K. The set of $v(x)$ for $x \in \mathfrak{b} \setminus \{0\}$ is a major subset $M(\mathfrak{b})$ of Γ. We now show the following equations are satisfied:

(a) $M(\mathfrak{a}(N)) = N$ for every major subset N of Γ;

(b) $\mathfrak{a}(M(\mathfrak{b})) = \mathfrak{b}$ for every sub-A-module $\mathfrak{b}$ of K.

Part (a) is easy, since, for all $\alpha \in N$, there exists $x \in K$ such that $v(x) = \alpha$. Then obviously $\mathfrak{b} \subseteq \mathfrak{a}(M(\mathfrak{b}))$; conversely, let $x \in \mathfrak{a}(M(\mathfrak{b}))$ and suppose $x \neq 0$, then $v(x) \in M(\mathfrak{b})$ and therefore there exists $y \in \mathfrak{b}$ such that $v(x) = v(y)$; whence $x = uy$ where $v(u) = 0$, which proves that $x \in \mathfrak{b}$ and completes the proof. $\square$

Corollary 5.6.28. *Let Γ_+ be the set of positive elements in Γ. The map $M \mapsto \mathfrak{a}(M)$ is a bijection of the set of major subsets of Γ_+ onto the set of ideals of A.*

Proof. As $A = \mathfrak{a}(\Gamma_+)$, $\mathfrak{a}(M) \subseteq A$ is equivalent to $M \subseteq \Gamma_+$. $\square$

Definition 5.6.29. Let Γ be a totally ordered group and Δ a subgroup of Γ of finite index. The number of major subsets of Γ consisting of strictly positive elements and containing all strict positive elements of Δ is called the **initial index** of Δ in Γ and denoted by $\varepsilon(\Gamma, \Delta)$.

This initial index is a natural number by virtue of the following proposition:

Proposition 5.6.30. *Let Γ be a totally ordered group and Δ a subgroup of Γ of finite index. If Γ has no least positive element, then $\varepsilon(\Gamma,\Delta) = 1$ for all Δ. If there exists a least positive element of Γ and Γ_0 denotes the subgroup it generates, then*

$$\varepsilon(\Gamma,\Delta) = [\Gamma_0 : \Gamma_0 \cap \Delta].$$

Proof. In the first case, let α be a strict positive element in Γ. The set of $\alpha \in \Gamma$ such that $0 < \beta < \alpha$ is infinite and hence there exist two elements of this set which are distinct and congruent modulo Δ; their difference is an element γ of Γ such that $0 < \gamma < \alpha$. Hence every major subset which contains all the strictly positive elements of Δ contains α and hence all strict positive elements of Γ.

In the second case, let δ be the least strict positive element of Γ and let n be the least positive integer such that $mx \in \Delta$. Clearly $n = [\Gamma_0 : \Gamma_0 \cap \Delta]$. On the other hand, writing $M(\alpha)$ for the set of $\beta \in \Gamma$ such that $\beta > \alpha$, it is immediately seen that the admissible major sets are just $M(\delta)$, $M(2\delta), \dots, M(n\delta)$. □

Corollary 5.6.31. *The initial index $\varepsilon(\Gamma,\Delta)$ divides the index $[\Gamma : \Delta]$ and is equal to it if Γ is isomorphic to $\mathbb{Z}$.*

Definition 5.6.32. Let K be a field, L a finite extension of K, w a valuation on L, v its restriction to K and Γ_v and Γ_w their order groups. The initial index of Γ_v in Γ_w is called the **initial ramification index** of w with respect to v (or with respect to K) and denoted by $\varepsilon(w/v)$.

From the above corollary, $\varepsilon(w/v)$ divides $e(w/v)$ with equality in the case of a discrete valuation.

Proposition 5.6.33. *Let K be a field, L a finite extension of K, w a valuation on L, v its restriction to K. Then*

$$\dim_{\kappa_v}(A_w/\mathfrak{m}_v A_w) = \varepsilon(w/v)f(w/v).$$

Proof. The ideals of A_w containing $\mathfrak{m}_v$ and distinct from A_w correspond to the major subsets of Γ_w, consisting of strict positive elements and containing the elements strict positive of Γ_v. They are therefore equal in number to $\varepsilon(w/v)$ and, as they form a totally ordered set under inclusion, this number is equal to the length of the quotient ring $A_w/\mathfrak{m}_v A_w$. Now a module of length 1 over A_w is a 1-dimensional vector space over κ_w and hence a module of length $f(w/v)$ over A; hence, as $A_w/\mathfrak{m}_v A_w$ is of length $\varepsilon(w/v)$ over A_w, it is of length $\varepsilon(w/v)f(w/v)$ over A_v, that is over κ_v. □

Theorem 5.6.34. *Let K be a field, v a valuation on K, A its ring, $\mathfrak{m}$ its maximal ideal, L a finite extension of K of degree n, B the integral closure of A in L and $(w_i)_{1\le i\le s}$ a complete system of extensions of v to L. Then*

$$\dim_{\kappa_v}(B/\mathfrak{m}B) = \sum_{i=1}^{s}\varepsilon(w_i/v)f(w_i/v) \le \sum_{i=1}^{s} e(w_i/v)f(w_i/v) \le n.$$

Proof. Let B_i be the ring of w_i; then $B_i = B_{\mathfrak{m}_i}$, where $\mathfrak{m}_i$ runs through the family of maximal ideals of B. Let $\mathfrak{q}_i$ be the saturation of $\mathfrak{m}B$ with respect to $\mathfrak{m}_i$. By Corollary 4.1.68, the canonical homomorphism $B/\mathfrak{m}B \to \prod_{i=1}^{s} B/\mathfrak{q}_i$ is an isomorphism and $\mathfrak{m}_i$ is the only maximal ideal containing $\mathfrak{q}_i$. Hence $B/\mathfrak{q}_i$ is canonically isomorphic to $(B/\mathfrak{q}_i)_{\mathfrak{m}_i}$. On the other hand, we note that

$$(B/\mathfrak{q}_i)_{\mathfrak{m}_i} = B_{\mathfrak{m}_i}/\mathfrak{m}B_{\mathfrak{m}_i} = B_i/\mathfrak{m}B_i,$$

therefore there is a canonical isomorphism $B/\mathfrak{m}B \to \prod_{i=1}^{s} B_i/\mathfrak{m}B_i$, whence the result in view of Proposition 5.6.33. □

Theorem 5.6.35. *With the hypotheses and notation of Theorem 5.6.34, the following conditions are equivalent:*

(a) B is a finitely generated A-module;

(b) B is a free A-module;

(c) $\dim_{\kappa_v}(B/\mathfrak{m}B) = n$;

(d) the equality in Theorem 5.6.34 holds and $\varepsilon(w_i/v) = e(w_i/v)$ for all i.

Proof. The equivalence of (a) and (b) follows from Proposition 5.1.5. Also, the equivalence of (c) and (d) follows from Theorem 5.6.34. It remains to show that (c) and (b) are equivalent.

First, assume that B is a free A-module. From the proof of Theorem 4.1.49, we see for any $x \in L$ there exists $s \in A$ such that sx is integral over K, hence in B. Since K is the fraction field of A, any independent set of B over A is an independent subset of L over K. These together show that any basis of B over A is also a basis of L over K, hence $\operatorname{rank}_A(B) = n$. Next, we prove that $\dim_{\kappa_v}(B/\mathfrak{m}B)$ also equals to n. Let $\{x_1, \dots, x_n\}$ be a basis of B over A. If $\bar{x}_i$ denotes the image of x_i in $B/\mathfrak{m}B$ then $\bar{x}_1, \dots, \bar{x}_n$ span the vector space $B/\mathfrak{m}B$ over κ_v. We assert that the n vectors are independent, hence the claim. For this, it suffices to show that if we have a relation of the form $\sum_{i=1}^n a_i x_i \in \mathfrak{m}B$, $a_i \in A$, then the a_i's necessarily belong to A. But this follows at once from the linear independence of the x_i over A, for we have, by assumption: $\sum_{i=1}^n a_i x_i = \sum_{i=1}^n b_i x_i$, where the y_i are suitable elements of $\mathfrak{m}$, and this relation implies $a_i = b_i$ for each i.

Conversely, assume that $\dim_{\kappa_v}(B/\mathfrak{m}B) = n$. Let $x_1, \dots, x_n$ be elements of B whose canonical images in $B/\mathfrak{m}B$ form a basis of $B/\mathfrak{m}B$ and let $M \subseteq B$ be the sub-Amodule which they generate. As M is torsion-free ($M \subseteq B$ and B is an integral domain) and finitely generated, it is free by Proposition 5.1.5. We shall see that $B = M$. Let $y \in B$; we write $N = M + Ay$; this is also a free A-module. The canonical injections $M \to N \to B$ give canonical homomorphisms

$$M/\mathfrak{m}M \to N/\mathfrak{m}N \to B/\mathfrak{m}B.$$

Now, by hypothesis, $M/\mathfrak{m}M \to B/\mathfrak{m}B$ is surjective and $B/\mathfrak{m}B$ is n-dimensional, hence $M/\mathfrak{m}M$ and $N/\mathfrak{m}N$ are n-dimensional and $M/\mathfrak{m}M \to N/\mathfrak{m}N$ is surjective. As N is finitely generated, $M \to N$ is then surjective by **??**, whence $M = N$ and $B = M$. Hence B is free. □

Corollary 5.6.36. *With the same hypotheses and notation, suppose further that v is discrete and L separable. Then*

$$\sum_{i=1}^{s} e(w_i/v) f(w_i/v) = n.$$

Proof. In this case B is a free A-module of rank n by Theorem 4.1.49, since A is a PID. □

Corollary 5.6.37. *Let (K, v) be a discrete valued complete field and L a finite extension of K of degree n. Then v admits a unique (up to equivalence) extension w to L, the ring B of w is a finitely generated free module over the ring A of v and $e(w/v)f(w/v) = n$.*

Proof. All the extensions of v to L are dependent by Corollary 5.6.22. Since they are discrete by Corollary 5.6.9, they are therefore equivalent. This shows the uniqueness of w. The integral closure of A in L is therefore B. As v is discrete, the topology induced on A by that on K is the $\mathfrak{m}$-adic topology (where $\mathfrak{m} = \mathfrak{m}_A$); the ring A is complete, for it is closed in K. We conclude that, since $B/\mathfrak{m}B$ is a finite-dimensional vector $(A/\mathfrak{m})$-space (Proposition 5.6.33), B is a finitely generated A-module (Corollary 2.3.36). It is therefore free and $e(v'/v)f(v'/v) = n$ by Theorem 5.6.35. □

Corollary 5.6.38. *Suppose that v is of rank 1 and that the equivalent conditions of Theorem 5.6.35 hold. If $\widehat{L}_i$ is the completion of L with respect to w_i, then the degree $n_i = [\widehat{L}_i : \widehat{K}]$ is equal to $e(w_i/v)f(w_i/v)$ for all i and the canonical homomorphism*

$$\phi : \widehat{K} \otimes_K L \to \prod_{i=1}^{s} \widehat{L}_i$$

is bijective. For all $x \in L$, the characteristic polynomial $\min_{L/K}(x)$ is equal to the product of the characteristic polynomials $\min_{\widehat{L}_i/\widehat{K}}(x)$. In particular, we have

$$\operatorname{tr}_{L/K}(x) = \sum_{i=1}^{s} \operatorname{tr}_{\widehat{L}_i/\widehat{K}}(x), \quad N_{L/K}(x) = \prod_{i=1}^{s} N_{\widehat{L}_i/\widehat{K}}(x), \quad v(N_{L/K}(x)) = \sum_{i=1}^{s} n_i w_i(x). \tag{5.6.3}$$

Proof. As no two of the w_i are equivalent and they are of rank 1 by Corollary 5.6.8, they are independent and Theorem 5.6.24 therefore shows that $e(w_i/v)f(w_i/v) \le n_i$ for all i and $\sum_i n_i \le n$. The first assertion therefore follows from these inequalities and the relation $\sum_i e(w_i/v)f(w_i/v) = n$. Under the isomorphism ϕ the endomorphism $z \mapsto z(1 \otimes x)$ of $\widehat{K} \otimes_K L$ (for $x \in L$) is transformed into the endomorphism of $\prod_{i=1}^{s} \widehat{L}_i$, leaving invariant each of the factors and reducing on each factor to multiplication by x (L_i being canonically imbedded in its completion $\widehat{L}_i$); whence the assertion relating to the characteristic polynomial of x and the first two formulae of (5.6.3). Finally, let E be a finite quasi-Galois extension of $\widehat{K}$ containing $\widehat{L}_i$. As $\widehat{K}$ is complete and v has rank 1, there exists only one valuation (up to equivalence) w on E extending $\hat{v}$ (Corollary 5.6.22). Then for every $\widehat{K}$-automorphism σ of E we have $w(\sigma(x)) = w(x)$ and therefore

$$\hat{v}(N_{\widehat{L}_i/\widehat{K}}(x)) = n_i w_i(x)$$

which proves the formula. □

5.7 Exercise

Exercise 5.7.1. If A is a valuation ring of dimension ≥ 2 then the formal power series ring $A[\![X]\!]$ is not integrally closed.

Proof. Let $0 \subset \mathfrak{p}_1 \subset \mathfrak{p}_2$ be a strict chain. Let $b \in \mathfrak{p}_1$, $a \in \mathfrak{p}_2 - \mathfrak{p}_1$, then $a^n/b \notin A$ for all n:

$$a^n/b \in A \Rightarrow a^n = (a^n/b) \cdot b \in \mathfrak{p}_1 \Rightarrow a \in \mathfrak{p}_1 (\mathfrak{p}_1 \text{ is prime})$$

Now A is a valuation ring so $ba^{-n} \in A$ for all n.

Let $f = \sum_{n=0}^{\infty} u_n x^n$ be a solution of

$$f^2 + af + x = 0$$

We shall take $u_0 = -a$, then from the equation

$$f^2 + af = \sum_{n=0}^{\infty} (u_0 u_n + \cdots + u_n u_0 + a u_n) x^n$$

we claim that $u_n \in a^{-2n+1}A$ for all $n \ge 1$. In fact, we can compute $u_1 = a^{-1}$. And assume this holds for $u_1, \dots, u_{n-1}$, for u_n we have an equation

$$u_0 u_n + u_1 u_{n-1} + \cdots + u_{n-1} u_1 + u_n u_0 + a u_n = 0$$

For $1 \le i \le n-1$, since $u_i \in a^{-2i+1}A$ we have

$$u_i u_{n-i} = a^{-2i+1} r_1 \cdot a^{-2(n-i)+1} r_2 = a^{-2n+2} r_1 r_2 \in a^{-2n+2} A$$

Hence
$$u_n = -(a+2u_0)^{-1}(u_1u_{n-1} + \cdots + u_{n-1}u_1) \in a^{-1} \cdot a^{-2n+2}A = a^{-2n+1}A.$$

Now we only need to prove that f is in the field of fractions of $A[\![X]\!]$. By our observation we have $bf \in A[\![X]\!]$, so $f \in b^{-1}A[\![X]\!] \subseteq \mathrm{Frac}(A[\![X]\!])$. But since $a \in \mathfrak{p}_2$, $a^{-1} \notin A$, which means $f \notin A[\![X]\!]$. So $A[\![X]\!]$ is not integrally closed. □

Exercise 5.7.2. If v is an additive valuation of a field K and $\alpha_1, \ldots, \alpha_n \in K$ are such that $\alpha_1 + \cdots + \alpha_n = 0$ then there exist two indices i, j such that $i \neq j$ and $v(\alpha_i) = v(\alpha_j)$.

Proof. If all $\alpha_i = 0$ then $v(\alpha_i) = \infty$ for all i, and the claim holds. Hence we may assume that $a_n \neq 0$, and from $\alpha_1 + \cdots + \alpha_n = 0$ we get

$$1 = -\Big(\frac{\alpha_1}{\alpha_n} + \cdots + \frac{\alpha_{n-1}}{\alpha_n}\Big).$$

Now, since $1 \notin \mathfrak{m}$, there exist an index $1 \le i \le n-1$ such that α_i/α_n is not in $\mathfrak{m}$, hence is a unit. Then by our definition $v(\alpha_i/\alpha_n) = 0$, so $v(\alpha_i) = v(\alpha_n)$. □

Exercise 5.7.3. Let k be a field, x and y indeterminates, and supposeth α is a positive irrational number. Then the map $v : k[x,y] \to \mathbb{R} \cup \{\infty\}$ defined by

$$\sum_{n,m} c_{n,m}x^n y^m \mapsto \min\{n + m\alpha \mid c_{n,m} \neq 0\}$$

determines a valuation of $k(x,y)$ with value group $\mathbb{Z} + \alpha\mathbb{Z}$.

Proof. For $f = \sum_{n,m} a_{n,m}x^n y^m$, $g = \sum_{n,m} b_{n,m}x^n y^m$, we have

$$v(fg) = \min\{n_1 + n_2 + (m_1 + m_2)\alpha \mid a_{n_1,m_1} \neq 0, b_{n_2,m_2} \neq 0\}$$

which is easily seen to be equal to $v(f) + v(g)$.

For $v(f+g)$, it is also easy to verify $v(f+g) \geq \min\{v(f), v(g)\}$. Also, $v(0) = \infty$. Thus by the previous exercise, v extends to a valuation $k(x,y) \to \mathbb{R} \cup \{\infty\}$. It is clear that $\Gamma_v = \mathbb{Z} + \alpha\mathbb{Z}$. □

Exercise 5.7.4. Let A be a DVR and $\mathfrak{m}$ its maximal ideal; then the $\mathfrak{m}$-adic completion $\widehat{A}$ of A is again a DVR.

Proof. Let $\mathfrak{m} = (x)$, then every element in A has a form rx^n for r a unit. Hence we can think the complition of A as the power series ring $(A - \mathfrak{m})[\![t]\!]$. Thus, we can define a valuation for $f = \sum_{n=0}^{\infty} u_n t^n$:
$$v : \widehat{A} \to \mathbb{Z}, \quad v(f) = \{n \mid u_n \neq 0\}$$

Since $f = \sum_{n=0}^{\infty} u_n t^n$ is a unit if and only if u_0 is a unit, this is well defined, and extending it to the field of fractions of $\widehat{A}$ gives makes $\widehat{A}$ a valuation ring. □

Chapter 6

Krull domains and divisors

6.1 Krull domains

6.1.1 Fractional ideals

Definition 6.1.1. Let A be an integral domain and K its field of fractions. A sub-A-module $\mathfrak{a}$ of K such that there exists a nonzero element $d \in A$ for which $d\mathfrak{a} \subseteq A$ is called a **fractional ideal** of A (or of K, by an abuse of language).

Every finitely generated sub-A-module $\mathfrak{a}$ of K is a fractional ideal: for if $a_1, \ldots, a_n$ is a system of generators of $\mathfrak{a}$, we may write $a_i = x_i/y_i$, where $x_i, y_i \in A$ and $y_i \neq 0$. If $d = y_1 \cdots y_n$ then clearly $d\mathfrak{a} \subseteq A$. In particular the principal sub-A-modules of K are fractional ideals. It is clear that every ideal of A is a fractional ideal. To avoid confusion, these will also be called the **integral ideals** of A. We denote by $\mathfrak{F}(A)$ the set of non-zero fractional ideals of A. Given two elements $\mathfrak{a}, \mathfrak{b}$ of $\mathfrak{F}(A)$, we shall write $\mathfrak{a} \preceq \mathfrak{b}$ for the relation "every fractional principal ideal containing $\mathfrak{a}$ also contains $\mathfrak{b}$. Clearly this relation is a **preordering** on $\mathfrak{F}(A)$ (it is reflexive and transitive). Consider the associated equivalence relation "$\mathfrak{a} \preceq \mathfrak{b}$ and $\mathfrak{b} \preceq \mathfrak{a}$" and let $\mathfrak{D}(A)$ be the equivalent class of this relation. We shall say that the elements of $\mathfrak{D}(A)$ are the **divisors** of A and, for every fractional ideal $\mathfrak{a} \in \mathfrak{F}(A)$, we shall denote by $\mathrm{div}(\mathfrak{a})$ (or $\mathrm{div}_A(\mathfrak{a})$) the canonical image of $\mathfrak{a}$ in $\mathfrak{D}(A)$ and we shall say that $\mathrm{div}(\mathfrak{a})$ is the **divisor of** $\mathfrak{a}$. If $a = Ax$ is a fractional principal ideal, we write $\mathrm{div}(x)$ instead of $\mathrm{div}(Ax)$ and $\mathrm{div}(x)$ is called the **divisor of x**. The elements of $\mathfrak{D}(A)$ of the form $\mathrm{div}(x)$ are called **principal divisors**. By taking the quotient, the preordering $\preceq$ on $\mathfrak{F}(A)$ defines on $\mathfrak{D}(A)$ an ordering which we shall denote by $\leq$.

For any $\mathfrak{a} \in \mathfrak{F}(A)$ there exists by hypothesis some $d \in A$ such that $\mathfrak{a} \subseteq Ad^{-1}$. The intersection $\tilde{\mathfrak{a}}$ of the fractional principal ideals containing $\mathfrak{a}$ is therefore an element of $\mathfrak{F}(A)$. Clearly the relation $\mathfrak{a} \preceq \mathfrak{b}$ is equivalent to the relation $\tilde{\mathfrak{a}} \supseteq \tilde{\mathfrak{b}}$, which is the case if $\mathfrak{a} \supseteq \mathfrak{b}$. For two elements $\mathfrak{a}, \mathfrak{b}$ of $\mathfrak{F}(A)$ to be equivalent modulo R, it is necessary and sufficient that $\tilde{\mathfrak{a}} = \tilde{\mathfrak{b}}$.

Definition 6.1.2. Every element $\mathfrak{a}$ of $\mathfrak{F}(A)$ such that $\mathfrak{a} = \tilde{\mathfrak{a}}$ is called a **divisorial fractional ideal** of A.

In other words a divisorial ideal is just a non-zero intersection of a non-empty family of fractional principal ideals. Every non-zero intersection of divisorial ideals is a divisorial ideal. If $\mathfrak{a}$ is divisorial, so is $\mathfrak{a}x$ for all $x \in K$, the map $\mathfrak{b} \mapsto \mathfrak{b}x$ being a bijection of the set of fractional principal ideals onto itself. By definition, for all $\mathfrak{a} \in \mathfrak{F}(A)$, $\tilde{\mathfrak{a}}$ is then the least divisorial ideal containing $\mathfrak{a}$ and is equivalent to $\mathfrak{a}$ modulo R. Moreover, if $\mathfrak{b}$ is a divisorial ideal equivalent to $\mathfrak{a}$ modulo R, then $\tilde{\mathfrak{a}} = \tilde{\mathfrak{b}} = \mathfrak{b}$. Hence $\tilde{\mathfrak{a}}$ is the unique divisorial ideal $\mathfrak{b}$ such that $\mathrm{div}(\mathfrak{a}) = \mathrm{div}(\mathfrak{b})$ (in other words, the restriction of the map $\mathfrak{a} \mapsto \mathrm{div}(\mathfrak{a})$ to the set of divisorial ideals is bijective).

Let $\mathfrak{a}$ and $\mathfrak{b}$ be two fractional ideals of K. Recall that $(\mathfrak{b} : \mathfrak{a})$ denotes the set of $x \in K$ such that $x\mathfrak{a} \subseteq \mathfrak{b}$. This is obviously an A-module. Note that $(\mathfrak{b} : \mathfrak{a})$ is also a fractional ideal, for if d is a non-zero element of A such that $d\mathfrak{b} \subseteq A$ and $d\mathfrak{a} \subseteq A$ and a is a non-zero element of $A \cap \mathfrak{a}$, then $da(\mathfrak{b} : \mathfrak{a}) \subseteq A$. On the other hand, if $b \neq 0$ belongs to $\mathfrak{b}$, then $bd\mathfrak{a} \subseteq \mathfrak{b}$, hence $bd \in (\mathfrak{b} : \mathfrak{a})$ and $(\mathfrak{b} : \mathfrak{a}) \neq 0$. Note that the definition of $(\mathfrak{b} : \mathfrak{a})$ can also be written as

$$(\mathfrak{b} : \mathfrak{a}) = \bigcap_{x \in \mathfrak{a}, x \neq 0} \mathfrak{b}x^{-1}. \tag{6.1.1}$$

Proposition 6.1.3. *Let $\mathfrak{a}, \mathfrak{b}$ be fractional ideals of A.*

(a) If $\mathfrak{b}$ is a divisorial ideal then $(\mathfrak{b} : \mathfrak{a})$ is divisorial.

(b) In order that $\mathrm{div}(\mathfrak{a}) = \mathrm{div}(\mathfrak{b})$, it is necessary and sufficient that $(A : \mathfrak{a}) = (A : \mathfrak{b})$.

(c) We have $\tilde{\mathfrak{a}} = (A : (A : \mathfrak{a}))$.

Proof. Assertion (a) follows immediately from equation (6.1.1) since, if $\mathfrak{b}$ is divisorial, so is $\mathfrak{b}x^{-1}$ for all $x \neq 0$. To show (b), let $P(\mathfrak{a})$ denote the set of fractional principal ideals containing $\mathfrak{a}$. The relation $Ax \in P(\mathfrak{a})$ is equivalent to $x^{-1}\mathfrak{a} \subseteq A$ and hence to $x^{-1} \in (A : \mathfrak{a})$. As the relation $\mathrm{div}(\mathfrak{a}) = \mathrm{div}(\mathfrak{b})$ is by definition equivalent to $P(\mathfrak{a}) = P(\mathfrak{b})$, it is also equivalent to $(A : \mathfrak{a}) = (A : \mathfrak{b})$.

Finally, as $\mathfrak{a}(A : a) \subseteq A$, we see $\mathfrak{a} \subseteq (A : (A : \mathfrak{a}))$. Replacing $\mathfrak{a}$ by $(A : \mathfrak{a})$ in this formula, it is seen that $(A : \mathfrak{a}) \subseteq (A : (A : (A : \mathfrak{a})))$. On the other hand, the relation $\mathfrak{a} \subseteq (A : (A : \mathfrak{a}))$ implies

$$(A : \mathfrak{a}) \supseteq (A : (A : (A : \mathfrak{a})))$$

Therefore $(A : \mathfrak{a}) = (A : (A : (A : \mathfrak{a})))$ and it follows from (b) that $\mathrm{div}(\mathfrak{a}) = \mathrm{div}(A : (A : \mathfrak{a}))$. As $(A : (A : \mathfrak{a}))$ is divisorial by (a), certainly $\tilde{\mathfrak{a}} = (A : (A : \mathfrak{a}))$, which proves (c). □

Proposition 6.1.4 (Lattice Structure in $\mathfrak{D}(A)$).

(a) In $\mathfrak{D}(A)$ every non-empty set bounded above admits a least upper bound. Moreprecisely, if $(\mathfrak{a}_i)$ is a non-empty family of elements of $\mathfrak{F}(A)$ which is bounded above, then

$$\sup_i(\mathrm{div}(\mathfrak{a}_i)) = \mathrm{div}(\bigcap_i \tilde{\mathfrak{a}}_i).$$

(b) In $\mathfrak{D}(A)$ every non-empty set bounded below admits a greatest lower bound. More precisely, if $(\mathfrak{a}_i)$ is a non-empty family of elements of $\mathfrak{F}(A)$ which is bounded below, then

$$\inf_i(\mathrm{div}(\mathfrak{a}_i)) = \mathrm{div}(\sum_i \tilde{\mathfrak{a}}_i).$$

(c) $\mathfrak{D}(A)$ is a lattice.

Proof. Let $(\mathfrak{a}_i)$ be a non-empty family of elements of $\mathfrak{F}(A)$ which is bounded above. To say that a divisorial ideal $\mathfrak{b}$ bounds this family above amounts to saying that it is contained in all the $\tilde{\mathfrak{a}}_i$. Hence $\bigcap_i \tilde{\mathfrak{a}}_i \neq 0$ and $\sup_i(\mathrm{div}(\mathfrak{a}_i)) = \mathrm{div}(\bigcap_i \tilde{\mathfrak{a}}_i)$.

Now let $(\mathfrak{a}_i)$ be a non-empty family of elements of $\mathfrak{F}(A)$ which is bounded below. To say that a divisorial ideal $\mathfrak{b}$ bounds this family below means that it contains all the $\tilde{\mathfrak{a}}_i$, so we see $\inf_i(\mathrm{div}(\mathfrak{a}_i)) = \mathrm{div}(\sum_i \tilde{\mathfrak{a}}_i)$.

Finally, it is sufficient to note that, if $\mathfrak{a}, \mathfrak{b}$ are in $\mathfrak{F}(A)$, then $\mathfrak{a} \cap \mathfrak{b}$ and $\mathfrak{a} + \mathfrak{b}$ are both nonzero fractional ideals, so the claim follows. □

Corollary 6.1.5. *If $x, y, x+y \in K^\times$ then $\mathrm{div}(x+y) \geq \inf\{\mathrm{div}(x), \mathrm{div}(y)\}$.*

Proof. We have $A(x+y) \subseteq Ax + Ay$ and hence $\mathrm{div}(x+y) \geq \mathrm{div}(Ax + Ay)$. □

6.1.2 The monoid structure on $\mathfrak{D}(A)$

Proposition 6.1.6. *Let* $\mathfrak{a}$, $\mathfrak{a}'$, $\mathfrak{b}$, $\mathfrak{b}'$ *be elements of* $\mathfrak{F}(A)$*. Then the relations* $\mathfrak{a} \succeq a'$ *and* $\mathfrak{b} \succeq \mathfrak{b}'$ *imply* $\mathfrak{a}\mathfrak{b} \succeq \mathfrak{a}'\mathfrak{b}'$.

Proof. We may restrict our attention to the case where $\mathfrak{b} = \mathfrak{b}'$, by transitivity. Then let Ax be a fractional principal ideal containing $\mathfrak{a}'\mathfrak{b}$. For every non-zero element y of $\mathfrak{b}$, $Ax \supseteq \mathfrak{a}'y$ and hence $Axy^{-1} \supseteq \mathfrak{a}'$, whence $Axy^{-1} \supseteq \mathfrak{a}$ and $Ax \supseteq \mathfrak{a}y$. Varying y, it is seen that $Ax \supseteq \mathfrak{a}\mathfrak{b}$, whence $\mathfrak{a}\mathfrak{b} \succeq \mathfrak{a}'\mathfrak{b}$. □

It follows from Proposition 6.1.6 that multiplication on $\mathfrak{F}(A)$ defines, by passing to the quotient, a law of composition on $\mathfrak{D}(A)$ which is obviously associative and commutative. It is written additively so that we may write:

$$\operatorname{div}(\mathfrak{a}\mathfrak{b}) = \operatorname{div}(\mathfrak{a}) + \operatorname{div}(\mathfrak{b})$$

for $\mathfrak{a}$, $\mathfrak{b}$ in $\mathfrak{F}(A)$. Clearly $\operatorname{div}(1)$ is an identity element for this addition. This element is denoted by 0. Proposition 6.1.6 proves further that the order structure on $\mathfrak{D}(A)$ is compatible with this addition:

$$\begin{aligned}\inf(\operatorname{div}(\mathfrak{a}) + \operatorname{div}(\mathfrak{b}), \operatorname{div}(\mathfrak{a}) + \operatorname{div}(\mathfrak{c})) &= \inf(\operatorname{div}(\mathfrak{a}\mathfrak{b}), \operatorname{div}(\mathfrak{a}\mathfrak{c})) = \operatorname{div}(\mathfrak{a}\mathfrak{b} + \mathfrak{a}\mathfrak{c}) \\ &= \operatorname{div}(\mathfrak{a}(\mathfrak{b} + \mathfrak{c})) = \operatorname{div}(\mathfrak{a}) + \operatorname{div}(\mathfrak{b} + \mathfrak{c}) \\ &= \operatorname{div}(\mathfrak{a}) + \inf(\operatorname{div}(\mathfrak{b}), \operatorname{div}(\mathfrak{c})).\end{aligned}$$

For a fractional ideal $\mathfrak{a}$ to be such that $\operatorname{div}(\mathfrak{a}) \geq 0$ in $\mathfrak{D}(A)$, it is necessary and sufficient that $\mathfrak{a} \subseteq A$ (in other words, that $\mathfrak{a}$ be an integral ideal of A).

For two elements x, y of $K^\times$, the relation $\operatorname{div}(x) = \operatorname{div}(y)$ is equivalent to $Ax = Ay$. Therefore the set of principal divisors of A with the order relation and the monoid law induced by that on $\mathfrak{D}(A)$ is an ordered group canonically isomorphic to the multiplicative group of fractional principal ideals ordered by the opposite order relation to inclusion. The relation S between two elements P, Q of $\mathfrak{D}(A)$:

$$\text{there exists } x \in K^\times \text{ such that } P = Q + \operatorname{div}(x)$$

is therefore an equivalence relation since the relation $P = Q + \operatorname{div}(x)$ is equivalent to $Q = P + \operatorname{div}(x^{-1})$. If P and Q are congruent modulo S, they are called **equivalent divisors** of A. Clearly moreover the relation S is compatible with the law of the monoid $\mathfrak{D}(A)$ and the latter therefore defines, by taking quotients, a monoid structure on $\mathfrak{D}(A)/S$. This monoid is called the **divisor class monoid** of A.

Proposition 6.1.7. *Let* $\mathfrak{a}$, $\mathfrak{b}$ *be two divisorial fractional ideals of* A*. Then* $\operatorname{div}(\mathfrak{a})$ *and* $\operatorname{div}(\mathfrak{b})$ *are equivalent divisors if and only if there exists* $x \in K^\times$ *such that* $\mathfrak{b} = x\mathfrak{a}$.

Proof. If $\operatorname{div}(\mathfrak{b}) = \operatorname{div}(\mathfrak{a}) + \operatorname{div}(x)$ for some $x \in K^\times$, then $\operatorname{div}(\mathfrak{b}) = \operatorname{div}(x\mathfrak{a})$ and, as $\mathfrak{b}$ and $x\mathfrak{a}$ are divisorial, $\mathfrak{b} = x\mathfrak{a}$, which proves the proposition. □

Let $\mathfrak{a}$ be an invertible fractional ideal; then $\mathfrak{a} = (A : (A : \mathfrak{a}))$ by Proposition 1.5.25(b) and hence $\mathfrak{a}$ is divisorial by Proposition 6.1.3. The group $\mathfrak{I}(A)$ of invertible fractional ideals is therefore identified with a subgroup of the monoid $\mathfrak{D}(A)$ and the canonical image of $\mathfrak{I}(A)$ in $\mathfrak{D}(A)/S$ with the group of classes of projective A-modules of rank 1 (Proposition 1.5.29).

Proposition 6.1.8. *Let* A *be an integral domain. For the monoid* $\mathfrak{D}(A)$ *of divisors of* A *to be a group, it it necessary and sufficient that* A *be completely integrally closed.*

Proof. Suppose that $\mathfrak{D}(A)$ is a group. Let $x \in K$ and suppose that $A[x]$ is contained in a finitely generated sub-A-module of K. Then we have seen that $\mathfrak{a} = A[x]$ is an element of $\mathfrak{F}(A)$. Then $x\mathfrak{a} \subseteq \mathfrak{a}$ and hence $\operatorname{div}(x) + \operatorname{div}(\mathfrak{a}) \geq \operatorname{div}\mathfrak{a}$. Since $\mathfrak{D}(A)$ is an ordered group, we conclude that $\operatorname{div}(x) \geq 0$, whence $x \in A$. Thus A is completely integrally closed.

Conversely, suppose that A is completely integrally closed. Let $\mathfrak{a}$ be a divisorial ideal. We shall show that $\operatorname{div}(\mathfrak{a}(A : \mathfrak{a})) = 0$, which will prove that $\mathfrak{D}(A)$ is a group. As $\mathfrak{a}(A : \mathfrak{a}) \subseteq A$, it suffices to verify that every fractional principal ideal Ax^{-1} which contains $\mathfrak{a}(A : \mathfrak{a})$ also contains A. Now, for $y \in K^\times$, the relation $Ay \supseteq \mathfrak{a}$ implies $y^{-1} \in (A : \mathfrak{a})$, whence $y^{-1}\mathfrak{a} \subseteq \mathfrak{a}(A : \mathfrak{a}) \subseteq Ax^{-1}$ and hence $x\mathfrak{a} \subseteq Ay$. As $\mathfrak{a}$ is divisorial, we then deduce that $x\mathfrak{a} \subseteq \mathfrak{a}$, whence $x^n\mathfrak{a} \subseteq \mathfrak{a}$ for all $n \in \mathbb{N}$. Since $\mathfrak{a}$ is divisorial, there exist elements x_0, x_1 of $K^\times$ such that $Ax_0 \subseteq \mathfrak{a} \subseteq Ax_1$; therefore $x^n x_0 \in Ax_1$, whence $x^n \in Ax_1x_0^{-1}$. As A is completely integrally closed, $x \in A$, that is $Ax^{-1} \supseteq A$, which completes the proof. □

Remark 6.1.9. Note that, if A is completely integrally closed (and even Noetherian), a divisorial ideal of A is not necessarily invertible, in other words, in general $\mathfrak{I}(A) \neq \mathfrak{D}(A)$.

Corollary 6.1.10. *Let A be a completely integrally closed domain and $\mathfrak{a}$ a divisorial fractional ideal of A. Then, for every nonzero fractional ideal $\mathfrak{b}$ of A,* $\operatorname{div}(\mathfrak{a} : \mathfrak{b}) = \operatorname{div}(\mathfrak{a}) - \operatorname{div}(\mathfrak{b})$.

Proof. By the formula (6.1.1),

$$\operatorname{div}(\mathfrak{a} : \mathfrak{b}) = \operatorname{div}\Big(\bigcap_{y \in \mathfrak{b}, y \neq 0} y^{-1}\mathfrak{a}\Big) = \sup_{y \in \mathfrak{b}, y \neq 0} \operatorname{div}(y^{-1}\mathfrak{a})$$

taking account of Proposition 6.1.4 and the fact that the fractional ideals $y^{-1}\mathfrak{a}$ are divisorial. But since $D(A)$ is an ordered group,

$$\sup_{y \in \mathfrak{b}, y \neq 0} \operatorname{div}(y^{-1}\mathfrak{a}) = \sup_{y \in \mathfrak{b}, y \neq 0} (\operatorname{div}(\mathfrak{a}) - \operatorname{div}(y)) = \operatorname{div}(\mathfrak{a}) - \inf_{y \in \mathfrak{b}, y \neq 0} \operatorname{div}(y) = \operatorname{div}(\mathfrak{a}) - \operatorname{div}(\mathfrak{b})$$

so the claim follows. □

6.1.3 Krull domains

Definition 6.1.11. An integral domain A is called a **Krull domain** if there exists a family $(v_i)_{i \in I}$ valuations on the field of fractions K of A with the following properties:

(K1) the valuations v_i are discrete;

(K2) the intersection of the rings of the v_i is A;

(K3) for all $x \in K^\times$, the set of indices $i \in I$ such that $v_i(x) \neq 0$ is finite.

Such a family $(v_i)_{i \in I}$ is called a **defining family of valuations**.

Example 6.1.12 (Examples of Krull domains).

(a) Every discrete valuation ring is a Krull domain.

(b) More generally, every PID A is a Krull domain. For let $(p_i)_{i \in I}$ be a representative system of irreducible elements of A and let v_i be the valuation on the field of fractions of A defined by p_i. It is immediately seen that the family $(v_i)_{i \in I}$ satisfies properties (K1), (K2) and (K3).

Let A be a Krull domain and let $(v_i)_{i\in I}$ be a family of valuations on the field of fractions K of A satisfying (K1), (K2) and (K3). The v_i may be assumed to be normed. For all $\mathfrak{a} \in I(A)$, we shall write:

$$v_i(\mathfrak{a}) = \sup_{\mathfrak{a}\subseteq Ax} v_i(x).$$

Then $v_i(\mathfrak{a}) \in \mathbb{Z}$, for, if a is a non-zero element of $\mathfrak{a}$, the relation $Ax \supseteq Aa$ implies that $v_i(x) \leq v_i(a)$ (by (K2)), which shows that the family $\{v_i(x) : \mathfrak{a} \subseteq Ax\}$ is bounded above. We establish the following properties.

Proposition 6.1.13. *Let A be a Krull domain, $(v_i)_{i\in I}$ a family of normed valuations on K satisfying (K1), (K2) and (K3), and $v_i(\mathfrak{a})$ be defined above.*

- *(a) If $x \in K^\times$ then $v_i(Ax) = v_i(x)$.*
- *(b) Let $\mathfrak{a}$ and $\mathfrak{b}$ be two divisorial fractional ideals of A. In order that $\mathfrak{a} \subseteq \mathfrak{b}$, it is necessary and sufiicient that $v_i(\mathfrak{a}) \geq v_i(\mathfrak{b})$ for all $i \in I$.*
- *(c) For all $\mathfrak{a} \in I(A)$, the indices $i \in I$ such that $v_i(\mathfrak{a}) \neq 0$ are finite in number.*

Proof. If $Ay \supseteq Ax$, then $v_i(y) \leq v_i(x)$ by (K2) and the minimum value of $v_i(y)$ is taken at $y = x$. This proves (a). For a divisorial ideal $\mathfrak{a}$ be, the relation $y \in \mathfrak{a}$ is equivalent to the relation "$\mathfrak{a} \subseteq Ax$ implies $y \in Ax$". Now, by (K2), the relation $y \in Ax$ is equivalent to $v_i(y) \geq v_i(x)$ for all $i \in I$, whence we see $y \in \mathfrak{a}$ if and only if $v_i(y) \geq v_i(\mathfrak{a})$ for all i, and (b) follows.

Finally, for $\mathfrak{a} \in I(A)$ there exist x, y in K^* such that $Ax \subseteq \mathfrak{a} \subseteq Ay$. By properties (a) and (b), $v_i(x) \geq v_i(\mathfrak{a}) \geq v_i(y)$ for all $i \in I$. It then follows from (K3) that (c) holds. □

Corollary 6.1.14. *If A is a Krull domain and $(v_i)_{i\in I}$ is a defining family of normed valuations on K, the map $\mathfrak{a} \mapsto (v_i(\mathfrak{a}))_{i\in I}$ is a decreasing injective map of the set of divisorial integer ideals of A (ordered by inclusion) to the set of positive elements of the ordered group the direct sum $\mathbb{Z}^{\oplus I}$. In particular every non-empty family of divisorial integral ideals of A admits a maximal element.*

The condition in Corollary 6.1.14 is also suffcient. In fact, we have the following characterization for Krull domains.

Theorem 6.1.15. *Let A be an integral domain. For A to be a Krull domain, it is necessary and sufficient that the two following conditions be satisfied:*

- *(a) A is completely integrally closed.*
- *(b) every non-empty family of divisorial integral ideals of A admits a maximal element (with respect to inclusion).*

Moreover, if $P(A)$ is the set of irreducible elements of $\mathfrak{D}(A)$, then $P(A)$ is a basis of the $\mathbb{Z}$-module $\mathfrak{D}(A)$ and the positive elements of $\mathfrak{D}(A)$ are the linear combinations of the elements of $P(A)$ with positive coefficients.

Proof. A Krull domain is completely integrally closed by Corollary 5.2.12, and it satisfies (b) by Corollary 6.1.14. Conversely, let A be an integral domain satisfying properties (a) and (b) of the statement. Since A is completely integrally closed, $\mathfrak{D}(A)$ is an ordered group and a lattice. By condition (b) of the statement, every non-empty family of positive elements of $\mathfrak{D}(A)$ has a minimal element. Let $P(A)$ be the set of irreducible elements of $\mathfrak{D}(A)$. Then by (A, VI, §1, no.13, Theorem 2) $P(A)$ is a basis of the $\mathbb{Z}$-module $\mathfrak{D}(A)$ and the positive elements of $\mathfrak{D}(A)$ are the linear combinations with positive integer coefficients of the elements of $P(A)$. Thus, for $x \in K^\times$, integers $v_P(x)$ are defined (for $P \in P(A)$) by writing:

$$\operatorname{div}(x) = \sum_{P\in P(A)} v_P(x)P. \tag{6.1.2}$$

From the relations $\operatorname{div}(xy) = \operatorname{div}(x) + \operatorname{div}(y)$ and $\operatorname{div}(x+y) \geq \inf\{\operatorname{div}(x), \operatorname{div}(y)\}$ for $x, y, x+y \in K^\times$, we deduce that the v_P are discrete valuations on K. In order that $x \in A$, it is necessary and sufficient that $\operatorname{div}(x) \geq 0$, that is that $v_P(x) \geq 0$ for all $P \in P(A)$. Thus the v_P satisfy conditions (K1) and (K2) and obviously also (K3). □

Corollary 6.1.16. *For a Noetherian ring to be a Krull domain, it is necessary and sufieient that it be an integrally closed domain.*

Proof. An integrally closed Noetherian domain is completely integrally closed. □

Note that there are non-Noetherian Krull domains, for example the polynominal ring $K[X_n]_{n\in\mathbb{N}}$ over a field K in an infinity of indeterminates.

6.1.4 Essential valuations for a Krull domain

Let A be a Krull domain and K its field of fractions. The valuations defined by formula (6.1.2) (for $x \in K^\times$) are called the **essential valuations** of K (or A). We have remarked in the course of the proof of Theorem 6.1.15 that the valuations (v_P) satisfy properties (K1), (K2) and (K3). Moreover, these discrete valuations v_P are normed: for every irreducible divisor $P \in P(A)$, $P < 2P$ and hence, if $\mathfrak{a}$ and $\mathfrak{b}$ are the divisorial ideals corresponding to P and $2P$, then $\mathfrak{a} \supsetneq \mathfrak{b}$. For $x \in \mathfrak{a} - \mathfrak{b}$, $\operatorname{div}(x) \geq P$ and $\operatorname{div}(x) \ngeq 2P$, whence $v_P(x) = 1$, which proves our assertion.

Proposition 6.1.17. *Let A be a Krull domain, K its field of fractions and $(v_P)_{P\in P(A)}$ the family of its essential valuations. Let $(n_P)_{P\in P(A)}$ be a family of integers with finite support. Then the set of $x \in K$ such that $v_P(x) \geq n_P$ for all $P \in P(A)$ is the divisorial ideal $\mathfrak{a}$ of A such that $\operatorname{div}(\mathfrak{a}) = \sum_P n_P P$.*

Proof. Let $x \in K^\times$ and $\mathfrak{a}$ the divisorial ideal such that $\operatorname{div}(\mathfrak{a}) = \sum_P n_P P$. In order that $x \in \mathfrak{a}$, it is necessary and sufficient that $Ax \subseteq \mathfrak{a}$, hence that $\operatorname{div}(x) \geq \operatorname{div}(\mathfrak{a})$ and hence, by (6.1.2), that $v_P(x) \geq n_P$ for all $P \in P(A)$. □

Proposition 6.1.18. *Let A be a Krull domain, K its field of fractions, $(v_i)_{i\in I}$ a defining family of valuations on K and A_i the ring of v_i. Let S be a multiplicative subset of A not containing 0, then*

$$S^{-1}A = \bigcap_{S^{-1}A\subseteq A_i} A_i.$$

In particular $S^{-1}A$ is a Krull domain.

Proof. Let $J = \{i \in I : S^{-1}A \subseteq A_i\}$ and write $B = \bigcap_{i\in J} A_i$, then we see $i \in J$ if and only if v_i is zero on S. Let $x \in B$ and $J(x)$ denote the finite set of indices $i \in I$ such that $v_i(x) < 0$. For each $i \in J(x)$ we have $x \notin A_i$, hence $i \notin J$ and so there exists $s_i \in S$ such that $v_i(s_i) > 0$. Let n_i be a positive integer such that $v_i(s_i^{n_i}x) \geq 0$ and we write $s = \prod_{i\in J} s_i^{n_i}$. Then $v_i(s_i x) \geq 0$ for all $i \in I$ and hence $sx \in A$ and $x \in s^{-1}A$. Thus $B = S^{-1}A$. □

Corollary 6.1.19. *Let P be an irreducible divisor of A and $\mathfrak{p}$ the corresponding divisorial ideal. Then $\mathfrak{p}$ is a prime ideal of height 1, the ring of v_P is $A_{\mathfrak{p}}$ and the residue field of v_P is identified with the field of fractions of $A/\mathfrak{p}$.*

Proof. By Proposition 6.1.17 we see v_P is zero on $S = A - \mathfrak{p}$ and positive on $\mathfrak{p}$, hence $\mathfrak{p} = \mathfrak{m}_{v_P} \cap A$ and $\mathfrak{p}$ is prime. On the other hand, since P is irreducible, for every divisor $Q \neq P$, $Q \ngeq P$ and hence the divisorial ideal $\mathfrak{q}$ corresponding to Q is not contained in $\mathfrak{p}$. This proves $\operatorname{ht}\mathfrak{p} = 1$ and the corollary then follows from Proposition 6.1.18 and Proposition 1.2.41 □

Corollary 6.1.20. *Let A be a Krull domain, K its field of fractions and $(v_i)_{i\in I}$ a defining family of valuations. Then every essential valuation of A is equivalent to one of the v_i.*

Proof. Let P be an irreducible divisor of A and $\mathfrak{p}$ the corresponding divisorial ideal. By Proposition 6.1.18 we have

$$A_{\mathfrak{p}} = \bigcap_{A_{\mathfrak{p}} \subseteq A_i} A_i = \bigcap_{\mathfrak{m}_i \cap A \subseteq \mathfrak{p}} A_i.$$

If $\mathfrak{m}_i \cap A = \{0\}$ then $A_i = K$ which is a contradiction, so $\mathfrak{m}_i \cap A \neq \{0\}$, and since $\mathfrak{p}$ is of height 1 we must have $\mathfrak{m}_i \cap A = \mathfrak{p}$. Since $\mathfrak{p}$ in nonzero, we then see there are only finitely many $i \in I$ such that $A_{\mathfrak{p}} \subseteq A_i$ by (K3). Now $A_{\mathfrak{p}}$ is an intersection of finitely many valuation rings and we can apply Proposition 5.4.6 Proposition 5.4.5 to claim that there is a unique $i \in I$ such that $A_{\mathfrak{p}} = A_i$, so v_P is equivalent to v_i. $\square$

Proposition 6.1.21. *Let A be a Krull domain, $(v_P)_{P \in P(A)}$ the family of its essential valuations and $\mathfrak{a} \in \mathfrak{F}(A)$. Then the coefficient of P in $\operatorname{div}(\mathfrak{a})$ is $\inf_{x \in \mathfrak{a}}(v_P(x))$. If $\mathfrak{p}$ is the divisorial prime ideal corresponding to the irreducible divisor P, then $\mathfrak{a}A_{\mathfrak{p}} = \tilde{\mathfrak{a}}A_{\mathfrak{p}}$.*

Proof. As $\mathfrak{a} = \sum_{x \in \mathfrak{a}} Ax$, Proposition 6.1.4 shows that $\operatorname{div}(\mathfrak{a}) = \inf_{x \in \mathfrak{a}} \operatorname{div}(Ax)$, whence our first assertion. The second follows immediately, since $\operatorname{div}(\tilde{\mathfrak{a}}) = \operatorname{div}(\mathfrak{a})$ and $A_{\mathfrak{p}}$ is the ring of the discrete valuation v_P. $\square$

Proposition 6.1.22. *Let A be an integrally closed Noetherian domain.*

*(a) Let P be an irreducible divisor of A and $\mathfrak{p}$ the corresponding divisorial prime ideal. Then the n-th **symbolic power** $\mathfrak{p}^{(n)}$ (that is, the saturation of $\mathfrak{p}^n$ with respect to $S = A - \mathfrak{p}$) is the set of $x \in A$ such that $v_P(x) \geq n$ and is a $\mathfrak{p}$-primary ideal.*

(b) Let $\mathfrak{a}$ be a divisorial integral ideal and $\operatorname{div}(\mathfrak{a}) = \sum_{i=1}^r n_i P_i$, with $\mathfrak{p}_i$ the corresponding prime ideal of P_i. Then

$$\mathfrak{a} = \bigcap_{i=1}^{r} \mathfrak{p}_i^{(n_i)}$$

is the unique reduced primary decomposition of $\mathfrak{a}$ and the $\mathfrak{p}_i$ are isolated primes of $\mathfrak{a}$.

Proof. By Corollary 6.1.19, the relation $x \in \mathfrak{p}^n A_{\mathfrak{p}} = (\mathfrak{p}A_{\mathfrak{p}})^n$ is equivalent to $v_P(x) \geq n$. On the other hand, as $A_{\mathfrak{p}}$ is a discrete valuation ring, $(\mathfrak{p}A_{\mathfrak{p}})^n$ is $(\mathfrak{p}A_{\mathfrak{p}})$-primary and hence $\mathfrak{p}^{(n)}$ is $\mathfrak{p}$-primary; this shows (a). Proposition 6.1.17 certainly shows that $\mathfrak{a} = \bigcap_{i=1}^r \mathfrak{p}_i^{(n_i)}$. As $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ are primes of height 1, this primary decomposition is reduced. The uniqueness follows from Proposition 3.2.10. $\square$

6.1.5 Approximation theorem

As the essential valuations of a Krull domain are discrete and normed, no two of them are equivalent and hence they are independent. Corollary 5.4.11 to the approximation theorem may therefore be applied to them: given some $n_i \in \mathbb{Z}$ and some essential valuations v_i finite in number and distinct, there exists $x \in K$ such that $v_i(x) = n_i$ for all i. But here there is a more precise result:

Proposition 6.1.23. *Let $v_1, \dots, v_r$ be essential valuations of a Krull domain A and $n_1, \dots, n_r$ integers. There exists an element x of the field of fractions K of A such $v_i(x) = n_i$ for all i and $v(x) \geq 0$ for every essential valuation v of A distinct from $v_1, \dots, v_r$.*

Proof. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ be the divisorial ideals of A corresponding to the valuations $v_1, \dots, v_r$. There exists $y \in K$ such that $v_i(y) = n_i$ for all i by Corollary 5.4.11. The essential valuations $w_1, \dots, w_s$ of A distinct from the v_i for which the integer $w_i(y) = -m_j < 0$ are finite in number. Let $\mathfrak{q}_1, \dots, \mathfrak{q}_s$ be the corresponding ideals. There exists no inclusion relation between $\mathfrak{p}_1, \dots, \mathfrak{p}_r, \mathfrak{q}_1, \dots, \mathfrak{q}_s$ since these ideals correspond to irreducible divisors and these ideals are prime. Hence the integral ideal $\mathfrak{a} = \mathfrak{q}_1^{m_1} \cdots \mathfrak{q}_s^{m_s}$ is contained in none of the $\mathfrak{p}_i$ and is therefore not contained in their union. Therefore there exists $z \in \mathfrak{a}$ such that $z \notin \mathfrak{p}_i$ for all i. Then $v_i(z) = 0$ for all i and $w_j(z) \geq m_j$ for all j; hence the element $x = yz$ solves the problem. □

Corollary 6.1.24. *Let A be a Krull domain, K its field of fractions and $\mathfrak{a}$, $\mathfrak{b}$ and $\mathfrak{c}$ three divisorial fractional ideals of A such that $\mathfrak{a} \subseteq \mathfrak{b}$. Then there exists $x \in K$ such that $\mathfrak{a} = \mathfrak{b} \cap x\mathfrak{c}$.*

Proof. Let $(v_i)_{i \in I}$ be the family of essential valuations of A and let (m_i) (resp. (n_i), (p_i)) be the family of integers (zero except for a finite number of indices) such that $\mathfrak{a}$ (resp. $\mathfrak{b}$, $\mathfrak{c}$) is the set of $x \in K$ for which $v_i(x_i) \geq m_i$ (resp. n_i, p_i) for all $i \in I$ (Proposition 6.1.17). The set J of $i \in I$ such that $m_i > n_i$ is finite. As $p_i = m_i = 0$ except for a finite number of indices, Proposition 6.1.23 shows that there exists $x \in K^\times$ such that $v_i(x^{-1}) + m_i = p_i$ for $i \in J$ and

$$v_i(x^{-1}) + m_i \geq p_i$$

for $i \in I - J$. Then, for all $i \in I$, $m_i = \sup(n_i, v_i(x) + p_i)$. Whence $\mathfrak{a} = \mathfrak{b} \cap x\mathfrak{c}$. □

Corollary 6.1.25. *Let A be a Krull domain. For a fractional ideal $\mathfrak{a}$ of A to be divisorial, it is necessary and sufficient that it be the intersection of two fractional principal ideals.*

Proof. The sufficiency is obvious. The necessity follows from Corollary 6.1.24 by taking $\mathfrak{b}$ and $\mathfrak{c}$ to be principal and such that $\mathfrak{b} \supseteq \mathfrak{a}$. □

6.1.6 Prime ideals of height 1

We have seen in Corollary 6.1.19 that irreducible divisors corresponds to prime ideals of height 1. Now we show that the converse also holds.

Theorem 6.1.26. *Let A be a Krull domain and $\mathfrak{p}$ an integral ideal of A. For $\mathfrak{p}$ to be the divisorial ideal corresponding to an irreducible divisor, it is necessary and suficient that $\mathfrak{p}$ be a prime ideal of height 1.*

Proof. If $\mathfrak{p}$ is the divisorial ideal corresponding to an irreducible divisor, we know that $\mathfrak{p}$ is prime of height 1. Conversely, let $\mathfrak{p}$ be a nonzero prime ideal. As $A_\mathfrak{p} \neq K$, by Proposition 6.1.18 $A_\mathfrak{p}$ is the intersection of a non-empty family (A_i) of essential valuation rings, each A_i being of the form $A_{\mathfrak{q}_i}$ (Corollary 6.1.19) and from $A_\mathfrak{p} \subseteq A_{\mathfrak{q}_i}$ we deduce that $\mathfrak{q}_i \subseteq \mathfrak{p}$. Thus, if $\mathfrak{p}$ is of height 1, then $\mathfrak{p} = \mathfrak{q}$, which shows that $\mathfrak{p}$ is the divisorial ideal corresponding to an irreducible divisor. □

Corollary 6.1.27. *In a Krull domain every non-zero prime ideal $\mathfrak{p}$ contains a prime ideal of height 1. If $\mathfrak{p}$ is not of height 1, then $\operatorname{div}(\mathfrak{p}) = 0$ and $(A : \mathfrak{p}) = A$.*

Proof. The first assertion is clear. If $\mathfrak{p}$ is not of height 1 and $\mathfrak{q}$ is a prime ideal of height 1 contained in $\mathfrak{p}$, then $\mathfrak{q} \subseteq \tilde{\mathfrak{p}}$ and $\mathfrak{q} \neq \tilde{\mathfrak{p}}$. As $\operatorname{div}(\mathfrak{q})$ is irreducible, necessarily $\operatorname{div}(\mathfrak{p}) = \operatorname{div}(\tilde{\mathfrak{p}}) = 0$; hence $\operatorname{div}(A : \mathfrak{p}) = 0$ and, as $(A : \mathfrak{p})$ is divisorial, we get $(A : \mathfrak{p}) = A$. □

Corollary 6.1.28. *Let A be a Krull domain, K its field of fractions, v a valuation on K which is positive on A and $\mathfrak{p}$ the set of $x \in A$ such that $v(x) > 0$. If the prime ideal $\mathfrak{p}$ is of height 1, v is equivalent to an essential valuation of A.*

Proof. Let B be the ring of v and $\mathfrak{m}$ its ideal. Then $\mathfrak{m} \cap A = \mathfrak{p}$ and hence $A_\mathfrak{p} \subseteq B$. Now $A_\mathfrak{p}$ is a DVR by Theorem 6.1.26 and $B \neq K$, thus $B = A_\mathfrak{p}$. □

Theorem 6.1.29. *Let A be an integral domain. For A to be a Krull domain, it is necessary and suficient that the following properties are satisfied:*

(a) For all prime ideal $\mathfrak{p}$ of A of height 1, $A_\mathfrak{p}$ is a DVR.

(b) $A = \bigcap_{\mathrm{ht}\mathfrak{p}=1} A_{\mathfrak{p}}$.

(c) *For all* $x \neq 0$ *in* A, *there exists only a finite number of prime ideals* $\mathfrak{p}$ *of height* 1 *such that* $x \in \mathfrak{p}$.

Moreover, in the admissible case, the valuations corresponding to the A *for prime ideals of height* 1 *are the essential valuations of* A.

Proof. The conditions are trivially sufficient. Their necessity follows immediately from Theorem 6.1.26, Corollary 6.1.19 and the fact that the essential valuations of A is a defining family. □

Proposition 6.1.30. *Let* A *be an integrally closed Noetherian domain and* $\mathfrak{a}$ *an integral ideal of* A. *Then* $\mathfrak{a}$ *is divisorial if and only if the prime ideals associated with* $A/\mathfrak{a}$ *are of height* 1.

Proof. Recall that, if $\mathfrak{a} = \bigcap_{i=1}^{n} \mathfrak{q}_i$ is a reduced primary decomposition of $\mathfrak{a}$ and $\mathfrak{p}_i$ denotes the prime ideal corresponding to $\mathfrak{q}_i$, the prime ideals associated with $A/\mathfrak{a}$ are just the $\mathfrak{p}_i$. The necessity then follows from Proposition 6.1.22. Conversely, if, in the above notation, the $\mathfrak{p}_i$ are of height 1, then $A_{\mathfrak{p}_i}$ is a DVR and each $\mathfrak{p}_i$ is isolated, so $\mathfrak{q}_i = \mathfrak{q}_i A_{\mathfrak{p}_i} \cap A$ (Proposition 3.2.5). Denoting by v_i the essential valuation corresponding to $\mathfrak{p}_i$, there therefore exists an integer n_i such that $\mathfrak{q}_i$ is the set of $x \in A$ such that $v_i(x) \geq n_i$ (n_i is just the integer such that $\mathfrak{q}_i A_{\mathfrak{p}_i} = (\mathfrak{p}_i A_{\mathfrak{p}_i})^{n_i}$ in $A_{\mathfrak{p}_i}$). This shows the $\mathfrak{q}_i$ are divisorial by Proposition 6.1.17, hence also is $\mathfrak{a}$. □

Proposition 6.1.31. *Let* A *be a local Krull domain (in particular an integrally closed local Noetherian domain) and* $\mathfrak{m}$ *its maximal ideal. The following conditions are equivalent:*

(i) A *is a DVR;*

(ii) $\mathfrak{m}$ *is invertible;*

(iii) $(A : \mathfrak{m}) \neq A$;

(iv) $\mathfrak{m}$ *is divisorial;*

(v) $\mathfrak{m}$ *is the only non-zero prime ideal of* A.

Proof. As every non-zero ideal of a DVR is principal, it is invertible and hence (i) implies (ii). If $\mathfrak{m}$ is invertible, its inverse is $(A : \mathfrak{m})$ and hence $(A : \mathfrak{m}) \neq A$, hence (ii) implies (iii). If $(A : \mathfrak{m}) \neq A$, then $(A : (A : \mathfrak{m})) \neq A$. Now $\mathfrak{m} \subseteq (A : (A : \mathfrak{m}))$, hence $\mathfrak{m} = (A : (A : \mathfrak{m}))$ since $\mathfrak{m}$ is maximal, so that $\mathfrak{m}$ is divisorial. Thus (iii) implies (iv). The fact that (iv) implies (v) follows from Theorem 6.1.26, Finally, if $\mathfrak{m}$ is the only non-zero prime ideal of A, it is of height 1 and hence $A_{\mathfrak{m}}$ is a DVR by Theorem 6.1.29. As A is local, $A_{\mathfrak{m}} = A$, which shows that (v) implies (i). □

6.1.7 Permanence properties

Proposition 6.1.32. *Let* A *be a Krull domain,* K *its field of fractions,* L *a finite extension of* K *and* R *the integral closure of* A *in* L. *Then* R *is a Krull domain. The essential valuations of* R *are the normed discrete valuations on* L *which are equivalent to the extensions of the essential valuations of* A.

Proof. Let $(v_i)_{i \in I}$ be a complete family of extensions to L of the essential valuations of A. Since the degree $n = [L : K]$ is finite, the v_i are discrete valuations on L by Corollary 5.6.9. Let B_i be the ring of v_i. Then $R \subseteq \bigcap_{i \in I} B_i$ by Theorem 5.1.8. Conversely, every element of $x \in \bigcap_i B_i$ is integral over each of the essential valuation rings of A by Corollary 5.1.10, hence the coefficients of the minimal polynomial of x over K belong to A (Corollary 4.1.29), so that $x \in R$; thus $R = \bigcap_i B_i$. Now let x be a non-zero element of R. It satisfies an equation of the form

$$x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$$

where $a_i \in A$ and $a_0 \neq 0$. If $v_i(x) > 0$, then $v_i(a_0) > 0$. Now the essential valuations v of A such that $v(a_0) > 0$ are finite in number and the valuations on L extending a given valuation on K are also finite in number, hence $v_i(x) = 0$ except for a finite number of indices $i \in I$. Thus it has been proved that R is a Krull domain.

It remains to show that the v_i are equivalent to essential valuations of R, that is, that the prime ideal $\mathfrak{p}_i$ consisting of the $x \in R$ such that $v_i(x) > 0$ is of height 1. If this were not so, there would exist a prime ideal $\mathfrak{q}$ of R such that $(0) \subseteq \mathfrak{q} \subseteq \mathfrak{p}_i$ distinct from (0) and $\mathfrak{p}_i$. Then $(0) \subseteq \mathfrak{q} \cap A \subseteq \mathfrak{p}_i \cap A$ and $\mathfrak{q} \cap A$ would be distinct from (0) and $\mathfrak{p}_i \cap A$ by Corollary 4.1.66. The prime ideal $\mathfrak{p}_i \cap A$ would therefore not be of height 1, which contradicts the fact that it corresponds to an essential valuation of A. □

Corollary 6.1.33. *Let $\mathfrak{p}$ (resp. $\mathfrak{P}$) be a prime ideal of A (resp. R) of height 1 and v (resp. w) the essential valuation of A (resp. R) corresponding to it. For $\mathfrak{P}$ to lie over $\mathfrak{p}$, it is necessary and sufficient that the restriction of w to K be equivalent to v.*

Proof. The valuation w is equivalent to the extension of an essential valuation v' of A. Let $\mathfrak{q} = \mathfrak{P} \cap A$, which is a prime ideal of A of height 1. For the restriction of w to K to be equivalent to v, it is necessary and suflicient that $v' = v$ and hence that $\mathfrak{q} = \mathfrak{p}$. □

Lemma 6.1.34. *Let A be an integral domain, K its field of fraction and L an extension of K. If $(A_i)_{i \in I}$ is a family of Krull domains contained in L satisfying the following two conditions:*

(a) $A = \bigcap_i A_i$.

(b) Each nonzero element of A is a nonunit in only finitely many A_i.

Then A is a Krull doamin.

Proof. For each $i \in I$ let $(v_{ij})_{j \in J_i}$ be a defining family of valuations of A_i. Let $x \in A$ be nonzero, then x is not a unit for only finitely many A_i's. Also, for each i, since A_i is a Krull domain, $v_{ij}(x) \neq 0$ for only finitely many $j \in J_i$, whence we see the family $(v_{ij})_{i \in I, j \in J_i}$ satisfies the condition (K3) and hence A is a Krull domain. □

Proposition 6.1.35. *Let A be a Krull domain, then the ring $A[X_1, \dots, X_n]$ is a Krull domain.*

Proof. Arguing by induction on n, it is sufficient to show that, if X is an indeterminate, $A[X]$ is a Krull domain. Let K be the field of fractions of A. The field of fractions of $A[X]$ is $K(X)$. The ring $K[X]$ is a PID and therefore a Krull ring. Moreover, for every prime ideal $\mathfrak{p}$ of A of height 1, let $\bar{v}_{\mathfrak{p}}$ be the extension of v to $K(X)$ defined by

$$\bar{v}_{\mathfrak{p}}\Big(\sum_j a_j X^j\Big) = \inf_j v_{\mathfrak{p}}(a_j)$$

for $\sum_j a_j X^j \in K[X]$. Then the valuation ring of $\bar{v}_{\mathfrak{p}}$ in $K(X)$ is $A[X]_{\mathfrak{p}A[X]}$. We also note that $K \cap A[X]_{\mathfrak{p}A[X]} = A_{\mathfrak{p}}[X]$, in fact, for $\sum_j a_j X^j \in K[X]$, the relation $\bar{v}_{\mathfrak{p}}(\sum_j a_j X^j) \geq 0$ is equivalent to $v_{\mathfrak{p}}(a_j) \geq 0$ for all j, hence equivalent to $\sum_j a_j X^j \in A_{\mathfrak{p}}[X]$. With this, we have

$$A[X] = K[X] \cap \bigcap_{\mathrm{ht}\mathfrak{p}=1} A[X]_{\mathfrak{p}A[X]}$$

Since A is Krull we see every nonzero element in $A[X]$ is a nonunit in only finitely many $A_{\mathfrak{p}}[X]$, so the condition of Lemma 6.1.34 is satisfied and we see $A[X]$ is a Krull domain. □

Corollary 6.1.36. *Let A be a Krull domain, then the prime ideals of $A[X]$ of height 1 are:*

(a) the prime ideals of the form $\mathfrak{p}A[X]$, where $\mathfrak{p}$ is a prime ideal of A of height 1;

(b) *the prime ideals of the form $\mathfrak{m} \cap A[X]$, where $\mathfrak{m}$ is a (necessarily principal) prime ideal of $K[X]$.*

Proof. This follows from the observation that that the prime ideals of height 1 in $K[X]$ are exactly maximal ideals. $\square$

Proposition 6.1.37. *Let A be a Krull domain, then the ring $A[\![X_1, \dots, X_n]\!]$ is a Krull domain.*

Proof. By induction we only need to show $A[\![X]\!]$ is Krull. Since A is Krull we have $A = \bigcap_{\mathrm{ht}\mathfrak{p}=1} A_{\mathfrak{p}}$, and therefore $A[\![X]\!] = \bigcap_{\mathrm{ht}\mathfrak{p}=1} A_{\mathfrak{p}}[\![X]\!]$. Also, each $A_{\mathfrak{p}}$ is a DVR and thus $A_{\mathfrak{p}}[\![X]\!]$ is integrally closed and Noetherian, hence a Krull domain. However, as X is a non-unit of all the rings $A_{\mathfrak{p}}[\![X]\!]$, we can not apply Lemma 6.1.34 directly. On the other hand, note that

$$A[\![X]\!] = K[\![X]\!] \cap \bigcap_{\mathrm{ht}\mathfrak{p}=1} (A_{\mathfrak{p}}[\![X]\!][X^{-1}]).$$

Now the hypothesis in Lemma 6.1.34 is easily verified for the rings $A_{\mathfrak{p}}[\![X]\!][X^{-1}]$. Indeed, an element $f(X) = \sum_{i=r}^{\infty} a_i X^i$ (with $a_r \neq 0$) in $A[\![X]\!]$ is a nonunit in $A_{\mathfrak{p}}[\![X]\!][X^{-1}]$ if and only if a_r is a nonunit of $A_{\mathfrak{p}}$, and there finitely many such $\mathfrak{p}$. Therefore $A[\![X]\!]$ is a Krull domain. $\square$

6.1.8 Divisor classes in a Krull domain

Let A be a Krull domain. Recall that the group $\mathfrak{D}(A)$ of divisors of A is the free commutative group generated by the set $P(A)$ of its irreducible elements and that $P(A)$ is identified with the set of prime ideals of A of height 1. For $\mathfrak{p} \in P(A)$ we shall denote by $v_{\mathfrak{p}}$ the normed essential valuation corresponding to $\mathfrak{p}$. Recall that the ring of $v_{\mathfrak{p}}$ is $A_{\mathfrak{p}}$. We shall denote by $\mathfrak{P}(A)$ the subgroup of $\mathfrak{D}(A)$ consisting of the principal divisors and by $\mathfrak{C}(A) = \mathfrak{D}(A)/\mathfrak{P}(A)$ the divisor class group of A.

Proposition 6.1.38. *Let A be a Krull domain and B a Krull domain containing A. Suppose that the following condition holds*

(PD) *For every prime ideal $\mathfrak{P}$ of B of height 1, the prime ideal $\mathfrak{P} \cap A$ is zero or of height 1.*

For $\mathfrak{p} \in P(A)$ the $\mathfrak{P} \in P(B)$ such that $\mathfrak{P} \cap A = \mathfrak{p}$ are finite in number. We write

$$i(\mathfrak{p}) = \sum_{\mathfrak{P} \in P(B), \mathfrak{P} \cap A = \mathfrak{p}} e(\mathfrak{P}/\mathfrak{p})\mathfrak{P},$$

where $e(\mathfrak{P}/\mathfrak{p})$ denotes the ramification index of $v_{\mathfrak{P}}$ over $v_{\mathfrak{p}}$. Then i defines, by linearity, an increasing homomorphism of $\mathfrak{D}(A)$ to $\mathfrak{D}(B)$, which enjoys the following properties:

(a) *for every non-zero element x of the field of fractions of A,*

$$i(\mathrm{div}_A(x)) = \mathrm{div}_B(x),$$

(b) *for all D_1, D_2 in $\mathfrak{D}(A)$,*

$$i(\sup\{D_1, D_2\}) = \sup\{i(D_1), i(D_2)\}.$$

Proof. Let $\mathfrak{p} \in P(A)$, consider a non-zero element a of $\mathfrak{p}$; the $\mathfrak{P} \in P(B)$ which contain a are finite in number by Theorem 6.1.29. A fortiori the $\mathfrak{P} \in P(B)$ such that $\mathfrak{P} \cap A = \mathfrak{p}$ are finite in number. We now show (a). By additivity, it may be assumed that $x \in A - \{0\}$. By definition, $\mathrm{div}_B(x) = \sum_{\mathfrak{P} \in P(B)} v_{\mathfrak{P}}(x)\mathfrak{P}$. For all $\mathfrak{P} \in P(B)$ such that $v_{\mathfrak{P}}(x) > 0$, $\mathfrak{P} \cap A$ is non-zero (for x is in it) and is therefore of height 1 by (PD). Setting $\mathfrak{p} = \mathfrak{P} \cap A$, by definition of the ramification index, $v_{\mathfrak{P}}(x) = e(\mathfrak{P}/\mathfrak{p})v_{\mathfrak{p}}(x)$ (since $v_{\mathfrak{P}}$ and $v_{\mathfrak{p}}$ are normed). As $\mathrm{div}_A(x) = \sum_{\mathfrak{p} \in P(A)} v_{\mathfrak{p}}(x)\mathfrak{p}$, and $i(\mathfrak{q}) = 0$ for all $\mathfrak{q} \in P(A)$ wich is not of the form $\mathfrak{Q} \cap A$ for some $\mathfrak{Q} \in P(B)$, we deduce (a).

To prove (b) we write

$$D_1 = \sum_{\mathfrak{p}\in P(A)} n_1(\mathfrak{p})\mathfrak{p}, \quad D_2 = \sum_{\mathfrak{p}\in P(A)} n_2(\mathfrak{p})\mathfrak{p}$$

the coefficient of $\mathfrak{p}$ in $\sup\{D_1, D_2\}$ is $\sup\{n_1(\mathfrak{p}), n_2(\mathfrak{p})\}$. Let $\mathfrak{P}$ be an element of $P(B)$. If $\mathfrak{P}\cap A = (0)$, the coefficients of $\mathfrak{P}$ in $i(D_1)$, $i(D_2)$, and hence also in $\sup\{i(D_1), i(D_2)\}$ are zero. Also the coefficient of $\mathfrak{P}$ in $i(\sup\{D_1, D_2\})$ is zero. If $\mathfrak{P}\cap A \neq (0)$, it is a prime ideal $\mathfrak{p}$ of height 1 (by (PD)); writing $e = e(\mathfrak{P}/\mathfrak{p})$, the coeflicients of $\mathfrak{P}$ in $i(D_1)$, $i(D_2)$ and $i(\sup\{D_1, D_2\})$ are respectively $en_1(\mathfrak{p}), en_2(\mathfrak{p})$ and $e\sup\{n_1(\mathfrak{p}), n_2(\mathfrak{p})\}$. That of $\sup\{i(D_1), i(D_2)\}$ is

$$\sup\{en_1(\mathfrak{p}), en_2(\mathfrak{p})\} = e\sup\{n_1(\mathfrak{p}), n_2(\mathfrak{p})\}.$$

This proves (b). □

Under the hypotheses of Proposition 6.1.38, it follows from (a) that i defines, by taking quotients, a canonical homomorphism $\bar{i}$ of $\mathfrak{C}(A)$ to $\mathfrak{C}(B)$.

Example 6.1.39. Let A, B be Krull domains and B be integral over A. In this case, for the prime ideal $\mathfrak{P}$ of B to be of height 1, it is necessary and sufficient that $\mathfrak{p} = \mathfrak{P}\cap A$ be of height 1 (Corollary 4.1.66), so the condition (PD) is satisfied.

Proposition 6.1.40. *Let A and B be Krull domains such that B contains A and is a flat A-module. Then:*

(a) the condition (PD) is satisfied;

(b) for every divisorial ideal $\mathfrak{a}$ of A, $B\mathfrak{a}$ is the divisorial ideal of B which corresponds to the divisor $i(\mathrm{div}_A(\mathfrak{a}))$.

Proof. To show (a), suppose that there exists a prime ideal $\mathfrak{P}$ of B of height 1 such that $\mathfrak{P}\cap A$ is neither 0 nor of height 1. Take an elemcnt $x \neq 0$ in $\mathfrak{P}\cap A$. The ideals $\mathfrak{p}_i$ of A of height 1 which contain x are finite in number and none contains $\mathfrak{P}\cap A$. There therefore exists an element y of $\mathfrak{P}\cap A$ such that $y \notin \mathfrak{p}_i$ for all i. Thus $\mathrm{div}_A(x)$ and $\mathrm{div}_A(y)$ are relatively prime elements of the ordered group $\mathfrak{D}(A)$, so that

$$\sup\{\mathrm{div}_A(x), \mathrm{div}_A(y)\} = \mathrm{div}_A(x) + \mathrm{div}_A(y) = \mathrm{div}_A(xy).$$

As $\sup\{\mathrm{div}_A(x), \mathrm{div}_A(y)\} = \mathrm{div}(Ax\cap Ay)$ and the ideals $Ax\cap Ay$ and Axy are divisorial, we deduce that $Ax\cap Ay = Axy$. Since B is a flat A-module, this implies $Bx\cap By = Bxy$ (**??**), and therefore

$$\sup\{v_{\mathfrak{P}}(x), v_{\mathfrak{P}}(y)\} = v_{\mathfrak{P}}(Bx\cap By) = v_{\mathfrak{P}}(xy) = v_{\mathfrak{P}}(x) + v_{\mathfrak{P}}(y).$$

Since $v_{\mathfrak{P}}(x)$ and $v_{\mathfrak{P}}(y)$ are positive integers (which hold since x and y are in $\mathfrak{P}$), this is a contradiction. Thus (a) has been proved by reductio ad absurdum.

We now show (b). If $\mathfrak{a}$ is a divisorial ideal of A, it is the intersection of two fractional principal ideals (Corollary 6.1.25), say

$$\mathfrak{a} = d^{-1}(Aa\cap Ab)$$

where a, b, d are in A and nonzero. As B is flat over A, $Ba = d^{-1}(Ba\cap Bb)$, which shows that $B\mathfrak{a}$ is divisorial. This shows also that $\mathrm{div}_B(B\mathfrak{a}) = \sup\{\mathrm{div}_B(a), \mathrm{div}_B(b)\} - \mathrm{div}_B(d)$, using Proposition 6.1.38 (a) and (b), it is seen that

$$\begin{aligned}\mathrm{div}_B(B\mathfrak{a}) &= \sup\{i(\mathrm{div}_A(a)), i(\mathrm{div}_A(b))\} - i(\mathrm{div}_A(d))\\ &= i(\sup\{\mathrm{div}_A(a), \mathrm{div}_A(b)\}) - i(\mathrm{div}_A(d))\\ &= i(\mathrm{div}_A(Aa\cap Ab)) - i(\mathrm{div}_A(d))\\ &= i(\mathrm{div}_A(d^{-1}(Aa\cap Ab))) = i(\mathrm{div}_A(\mathfrak{a}))\end{aligned}$$

which completes the proof. □

Corollary 6.1.41. *Let A be a local Krull domain and B a DVR such that B dominates A and is a flat A-module. Then A is a field or a DVR.*

Proof. Let $\mathfrak{M}$ be the maximal ideal of B. By (PD), $\mathfrak{M} \cap A$ is zero or of height 1. As it is, by hypothesis the maximal ideal of A, our assertion follows from Proposition 6.1.31. □

Remark 6.1.42. As the elements of $P(B)$ form a basis of $\mathfrak{D}(B)$ and two distinct ideals of $P(A)$ cannot be the traces on A of the same ideal of $P(B)$, for the injectivity of i it amounts to verifying that $i(\mathfrak{p}) \neq 0$ for all $\mathfrak{p} \in P(A)$. Now this is equivalent to say every $\mathfrak{p} \in P(A)$ is the contraction of an element in $P(B)$, and in particular this holds if B is a faithfully flat A-algebra or B is a superring of A and integral over A.

In what follows, we propose to study the canonical homomorphism i from $\mathfrak{C}(A)$ to $\mathfrak{C}(B)$ for certain ordered pairs of Krull domains A, B.

Proposition 6.1.43. *Let A be a Zariski ring such that its completion $\widehat{A}$ is a Krull domain. Then A is a Krull domain and the canonical homomorphism $\bar{i}$ from $\mathfrak{C}(A)$ to $\mathfrak{C}(\widehat{A})$ (which is defined since $\widehat{A}$ is a flat A-module) is injective.*

Proof. As $\widehat{A}$ is an integral domain and $A \subseteq \widehat{A}$, A is an integral domain. Let L be the field of fractions of $\widehat{A}$ and $K \subseteq L$ that of A. As $A = \widehat{A} \cap K$ (Corollary 2.4.33), A is a Krull domain. The fact that $\bar{i} : \mathfrak{C}(A) \to \mathfrak{C}(\widehat{A})$ is injective follows from Proposition 6.1.40 and the fact that, if $\mathfrak{b}\widehat{A}$ is principal, $\mathfrak{b}$ is principal (Corollary 2.4.32) □

Now let A be a Krull domain and S a multiplicative subset of A not containing 0. The group $\mathfrak{D}(A)$ (resp. $\mathfrak{D}(S^{-1}A)$ is the free commutative group with basis the set of $\operatorname{div}(\mathfrak{p})$ (resp. $\operatorname{div}(S^{-1}\mathfrak{p})$), where $\mathfrak{p}$ runs through the set of prime ideals of A of height 1 (resp. the set of prime ideals of A of height 1 such that $\mathfrak{p} \cap S = \varnothing$) and, if $\mathfrak{p} \cap S = \varnothing$, then $i(\operatorname{div}(\mathfrak{p})) = \operatorname{div}(S^{-1}\mathfrak{p})$. Thus $\mathfrak{D}(S^{-1}A)$ is identified with the direct factor of $\mathfrak{D}(A)$ generated by the elements $\operatorname{div}(\mathfrak{p})$ such that $\mathfrak{p} \cap S = \varnothing$ and admits as complement the free commutative subgroup of $\mathfrak{D}(A)$ with basis the set of $\operatorname{div}(\mathfrak{p})$ such that $\mathfrak{p} \cap S \neq \varnothing$. We shall denote this complement by $\mathfrak{G}(S)$. As the map $i : \mathfrak{D}(A) \to \mathfrak{D}(S^{-1}A)$ is suijective, so is the induced map $\bar{i} : \mathfrak{C}(A) \to \mathfrak{C}(S^{-1}A)$ and

$$\ker \bar{i} = (\mathfrak{G}(S) + \mathfrak{P}(A))/\mathfrak{P}(A) = \mathfrak{G}(S)/(\mathfrak{G}(S) \cap \mathfrak{P}(A)). \tag{6.1.3}$$

In fact, if an element of $\mathfrak{D}(S^{-1}A)$ is equal to $\operatorname{div}_{S^{-1}A}(x/s)$, where $x \in A$ and $s \in S$, it is then the image under i of the principal divisor $\operatorname{div}_A(x/s)$ (Proposition 6.1.38).

Proposition 6.1.44. *Let A be a Krull domain and S a multiplicative subset of A not containing 0. Then the canonical homomorphism $\bar{i}$ from $\mathfrak{C}(A)$ to $\mathfrak{C}(S^{-1}A)$ is suijective. If further S is generated by a family of elements p_i such that the principal ideals Ap_i are all prime, then $\bar{i}$ is bijective.*

Proof. Suppose now that S is generated by a family of elements $(p_i)_{i \in I}$ of A such that the principal ideals Ap_i are all prime. Then, if $\mathfrak{p}$ is a prime ideal of A of height 1 such that $p \cap S \neq \varnothing$, $\mathfrak{p}$ contains a product of powers of the p_i and therefore one of the p_i, say p_0. As Ap_0 is non-zero and prime and $\mathfrak{p}$ is of height 1, it follows that $\mathfrak{p} = Ap_0$. In the above notation, we then have $\mathfrak{G}(S) \subseteq \mathfrak{P}(A)$ and the kernel of $\bar{i}$ is zero. □

Proposition 6.1.45. *Let R be a Krull domain and consider the polynomial ring $R[X]$. The canonical homomorphism of $\mathfrak{C}(R)$ to $\mathfrak{C}(R[X])$ is bijective.*

Proof. Let R be a Krull domain; take A to be the polynomial ring $A = R[X]$ and S to be the set $R - (0)$ of non-zero constant polynomials of A. The prime ideals $\mathfrak{p}$ of A of height 1 such that $\mathfrak{p} \cap S \neq \varnothing$ are those of the form $\mathfrak{p}_0 A$, where $\mathfrak{p}_0$ is a prime ideal of R of height 1. Hence, in the notation introduced above, $\mathfrak{G}(S)$ is identified with $\mathfrak{D}(R)$ by identifying $\operatorname{div}_A(\mathfrak{p}_0 A)$ with $\operatorname{div}_R(\mathfrak{p}_0)$. On the other hand $\mathfrak{G}(S) \cap \mathfrak{P}(A)$ is identified with $\mathfrak{P}(R)$: for if an ideal $\mathfrak{a}_0$ of R

generates a principal ideal $f(X)A$ in $A = R[X]$, then $f(0) \in \mathfrak{a}_0 A$ since $\mathfrak{a}_0 A$ is a graded ideal of the ring A (graded by the usual degree of polynomials) and hence $f(0) \in \mathfrak{a}_0$. Further, for any $a \in \mathfrak{a}_0$ we have $a = f(X)g(X)$ where $g(X) \in R$, whence $a = f(0)g(0)$. It follows that $\mathfrak{a}_0$ is the principal ideal of R generated by $f(0)$. Finally, denoting by K the field of fractions of R, $S^{-1}A$ is identified with the polynomials ring $K[X]$, which is a principal ideal domain; hence $\mathfrak{C}(S^{-1}A) = (0)$. Thus $\mathfrak{C}(A) = \ker \bar{i}$ is identified with $\mathfrak{C}(R)$ and we have proved the proposition. □

6.1.9 Dedekind domains

Let A be an integral domain. Clearly the following conditions are equivalent:

(i) no two of the non-zero prime ideals of A are comparable with respect to inclusion;

(ii) the non-zero prime ideals of A are maximal;

(iii) the non-zero prime ideals of A are of height 1.

A Krull domain which satisfies the above equivalent conditions is called a **Dedekind domain**.

Example 6.1.46 (Examples of Dedekind domains).

(a) Every principal ideal domain is a Dedekind domain.

(b) Let K be a finite extension of $\mathbb{Q}$ and A the integral closure of $\mathbb{Z}$ over K. The ring A is a Krull domain by Proposition 6.1.32. Let $\mathfrak{p}$ be a non-zero prime ideal of A. Then by the going down theorem $\mathfrak{p} \cap Z$ is non-zero and hence is a maximal ideal of $\mathbb{Z}$. Hence $\mathfrak{p}$ is a maximal ideal of A. Therefore, A is a Dedekind domain. In general, A is not a principal ideal domain.

(c) Let V be an affine algebraic variety and A the ring of functions regular on V. Suppose that A is not a field (i.e. that V is not reduced to a point). For A to be a Dedekind domain, it is necessary and sufficient that V be an irreducible curve with no singular point: for to say that A is an integral domain amounts to saying that V is irreducible; to say that every non-zero prime ideal of A is maximal amounts to saying that $\dim A = 1$, so A is a curve; finally, as A is Noetherian, to say that it is a Krull domain amounts to saying that it is integrally closed, that is, V is a normal curve, or also that it has no singular point.

(d) A ring of fractions $S^{-1}A$ of a Dedekind domain A is a Dedekind domain if $0 \notin S$. For $S^{-1}A$ is a Krull domain and every nonzero prime ideal of $S^{-1}A$ is maximal.

Theorem 6.1.47. *Let A be an integral domain and K its field of fractions. Then the following conditions are equivalent:*

(i) A is a Dedekind domain;

(ii) A is a Krull domain and every nontrivial valuation on K which is positive on A is equivalent to an essential valuation of A;

(iii) A is a Krull domain and every nonzero fractional ideal $\mathfrak{a}$ of A is divisorial;

(iv) every nonzero fractional ideal $\mathfrak{a}$ of A is invertible;

(v) A is a Noetherian integrally closed domain and every non-zero prime ideal of A is maximal;

(vi) A is Noetherian and, for every maximal ideal $\mathfrak{m}$ of A, $A_{\mathfrak{m}}$ is either a field or a DVR;

(vii) A is Noetherian and, for every maximal ideal $\mathfrak{m}$ of A, $A_{\mathfrak{m}}$ is a PID.

Proof. We show first the equivalence of (i) and (ii). Corollary 6.1.28 shows immediately that (i) implies (ii). Conversely, (ii) implies (i), since, for every prime ideal $\mathfrak{p}$ of A, there exists a valuation ring of K which dominates $A_{\mathfrak{p}}$, and (ii) this implies that $\mathfrak{p}$ has height 1, so A is a Dedekind domain.

If A is a Dedekind domain and $\mathfrak{a}$ is a non-zero fractional ideal, then $\mathfrak{a}A_{\mathfrak{m}} = \tilde{\mathfrak{a}}A_{\mathfrak{m}}$ for every maximal ideal $\mathfrak{m}$ (by Proposition 6.1.21) and hence $\mathfrak{a} = \tilde{\mathfrak{a}}$; thus (i) implies (iii).

We now show that (iii) implies (iv). If (iii) holds, the map $\mathfrak{a} \mapsto \operatorname{div}(\mathfrak{a})$ is a bijection of $\mathfrak{F}(A)$ onto $\mathfrak{D}(A)$, as it is a homomorphism and $\mathfrak{D}(A)$ is a group, every element of $\mathfrak{F}(A)$ is invertible.

We show that (iv) implies (v). If (iv) holds, every integral ideal of A is finitely generated and hence A is Noetherian. As $\mathfrak{F}(A)$ is a group, $\mathfrak{D}(A)$ is a group and A is therefore completely integrally closed. Finally, if $\mathfrak{p}$ is a non-zero prime ideal of A and $\mathfrak{m}$ is a maximal ideal of A containing $\mathfrak{p}$, the ring $A_{\mathfrak{m}}$ is a PID by Theorem 1.5.26. As $\mathfrak{p}A_{\mathfrak{m}}$ is prime and non-zero, necessarily $\mathfrak{p}A_{\mathfrak{m}} = \mathfrak{m}A_{\mathfrak{m}}$, whence $\mathfrak{p} = \mathfrak{m}$ and $\mathfrak{p}$ is maximal.

We now show that (v) implies (vi). If $\mathfrak{m}$ is a maximal ideal of A and (v) holds, $A_{\mathfrak{m}}$ is an integrally closed Noetherian domain and its maximal ideal $\mathfrak{m}A_{\mathfrak{m}}$ is, either (0), or the only non-zero prime ideal of $A_{\mathfrak{m}}$. Hence $A_{\mathfrak{m}}$ is a field or a DVR by Proposition 6.1.31.

The fact (vi) implies (vii) is obvious. We show finally that (vii) implies (i). As A is the intersection of the $A_{\mathfrak{m}}$ where $\mathfrak{m}$ runs through the set of maximal ideals (Proposition 1.3.28), (vii) implies that A is integrally closed and Noetherian and hence that A is a Krull domain. On the other hand, it can be shown that every non-zero prime ideal of A is maximal as in the proof that (iv) implies (v). □

Proposition 6.1.48. *A semi-local Dedekind domain is a PID.*

Proof. Let A be a semi-local Dedekind domain, K its field of fractions, $\mathfrak{m}_1, \dots, \mathfrak{m}_n$ its maximal ideals and $v_1, \dots, v_n$ the corresponding essential valuations. These are the only essential valuations of A. Let $\mathfrak{a}$ be a non-zero integral ideal of A. Since it is divisorial, there exists integers $q_1, \dots, q_r$ such that $\mathfrak{a}$ is the set of $x \in K$ such that $v_i(x) \geq \mathfrak{q}_i$ for all i (Proposition 6.1.22). Let x_0 be an element of K such that $v_i(x_0) = q_i$ for all i (Proposition 6.1.23). Then $\mathfrak{a}$ is the set of $x \in K$ such that $v_i(xx_0^{-1}) \geq 0$ for all i and thus $\mathfrak{a} = Ax_0$. □

If A is a Dedekind domain, it has been seen, in Theorem 6.1.47, that the group $\mathfrak{D}(A)$ of divisors of A is identified with the group $\mathfrak{F}(A)$ of nonzero fractional ideals $\mathfrak{a}$ (as A is Noetherian, every non-zero fractional ideal is finitely generated). The divisor class group $\mathfrak{C}(A)$ of A is then identified with the group of class of nonzero ideals of A.

Let A be a Dedekind domain, $\mathfrak{F}(A)$ the ordered multiplicative group of nonzero fractional ideals of A and $\mathfrak{D}(A)$ the group of divisors of A. The isomorphism $\mathfrak{a} \mapsto \operatorname{div}(\mathfrak{a})$ of $\mathfrak{F}(A)$ onto $\mathfrak{D}(A)$ maps the non-zero prime ideals of A to irreducible divisors and hence the multiplicative group $\mathfrak{F}(A)$ admits as basis the set of non-zero prime ideals of A. In other words, every non-zero fractional ideal $\mathfrak{a}$ of A admits a unique decomposition of the form:

$$\mathfrak{a} = \prod \mathfrak{p}^{n_{\mathfrak{p}}} \tag{6.1.4}$$

where the product extends to the non-zero prime ideals of A, and the exponents $n_{\mathfrak{p}}$ equals $v_{\mathfrak{p}}(\mathfrak{a})$, where $v_{\mathfrak{p}}$ denotes the essential valuation corresponding to $\mathfrak{p}$. Further $\mathfrak{a}$ is integral if and only if the $n_{\mathfrak{p}}$ are all positive. The relation (6.1.4) is called the **decomposition of $\mathfrak{a}$ into prime factors**. For $\mathfrak{a}, \mathfrak{b} \in \mathfrak{F}(A)$, write

$$\mathfrak{a} = \prod \mathfrak{p}^{n_{\mathfrak{p}}}, \quad \mathfrak{b} = \prod \mathfrak{p}^{m_{\mathfrak{p}}}$$

then we have

$$\mathfrak{a}\mathfrak{b} = \prod \mathfrak{p}^{n_{\mathfrak{p}}+m_{\mathfrak{p}}}, \quad (\mathfrak{a} : \mathfrak{b}) = \mathfrak{a}\mathfrak{b}^{-1} = \prod \mathfrak{p}^{n_{\mathfrak{p}}-m_{\mathfrak{p}}},$$
$$\mathfrak{a} + \mathfrak{b} = \prod \mathfrak{p}^{\inf\{n_{\mathfrak{p}}, m_{\mathfrak{p}}\}}, \quad \mathfrak{a} \cap \mathfrak{b} = \prod \mathfrak{p}^{\sup\{n_{\mathfrak{p}}, m_{\mathfrak{p}}\}}.$$

Proposition 6.1.49. *Let A be a Dedekind domain, K its field of fractions and $\mathcal{P}$ the set of non-zero prime ideals of A. For $\mathfrak{p} \in \mathcal{P}$ let $v_{\mathfrak{p}}$ denote the corresponding essential valuation of A. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ be distinct elements of $\mathcal{P}$ and $n_1, \dots, n_r$ integers and $x_1, \dots, x_r$ elements of K. Then there exists $x \in K$ such that $v_{\mathfrak{p}_i}(x - x_i) \geq n_i$ for all i and $v_{\mathfrak{p}}(x) \geq 0$ for all $\mathfrak{p} \in \mathcal{P}$ distinct from the $\mathfrak{p}_i$.*

Proof. Replacing if need be the n_i by greater integers, they may be assumed all to be positive. We examine first the case where the x_i are in A. It obviously amounts to finding an $x \in A$ satisfying the congruences
$$x \equiv x_i \mod \mathfrak{p}_i^{n_i}$$
and the existence of x then follows from Chinese remainder theorem.

We pass now to the general case. We may write $x_i = y_i/s$, where y_i, s are in A. Writing $x = y/s$, it amounts to finding a $y \in A$ such that, on the one hand, $v_{\mathfrak{p}_i}(y - y_i) \geq n_i + v_{\mathfrak{p}_i}(s)$ and, on the other, $v_{\mathfrak{p}}(y) \geq v_{\mathfrak{p}}(s)$ for all $p \in \mathcal{P}$ distinct from the $\mathfrak{p}_i$. As $v_{\mathfrak{p}}(s) = 0$ except for a finite number of indices $\mathfrak{p}$, it is thus reduced to the above case, whence the proposition. □

Proposition 6.1.50. *Let A be a Dedekind domain and $\mathfrak{a}$ a fractional ideal of A. Then $\mathfrak{a}$ is generated by at most two elements.*

Proof. By multiplying by an element of A, we can reduce to the case that $\mathfrak{a}$ is an ideal of A. Write $\mathfrak{a} = \prod \mathfrak{p}_i^{n_i}$ as a finite product and choose $a \in \mathfrak{a}$ with $aA = \prod \mathfrak{q}_j^{m_j} \prod \mathfrak{p}_i^{f_i}$ and the $\mathfrak{q}_j$ different from $\mathfrak{p}_i$. Then since $aA \subseteq \mathfrak{a}$ we have $f_i \geq n_i$ for each i. Choose $b \in A$ such that $v_{\mathfrak{p}_i}(b) = n_i$ and $v_{\mathfrak{q}_j}(b) = 0$, then we see
$$aA + bA = \prod \mathfrak{p}_i^{\inf\{f_i, f_i+1\}} \mathfrak{q}_j^{\inf\{0, m_j\}} = \prod \mathfrak{p}_i^{n_i} = \mathfrak{a}.$$
This proves the claim. □

6.1.10 The Krull-Akizuki theorem

Lemma 6.1.51. *Let A be a one-dimensional Noetherian domain and M a finitely generated torsion A-module. Then the length $\ell_A(M)$ of M is finite.*

Proof. As M is a torsion module, every prime ideal associated with M is nonzero and therefore maximal. The lemma then follows from Proposition 3.2.14. □

Lemma 6.1.52. *Let A be a ring, M an A-module and $(M_i)_{i \in I}$ a directed family of submoduler of M with union M. Then $\ell_A(M) = \sup_i \ell_A(M_i)$.*

Proof. We have $\ell_A(M_i) \leq \ell_A(M)$ for all i. The lemma is obvious if no integer exceeds the $\ell_A(M_i)$, both sides then being infinite. Otherwise, let i_0 be an index for which $\ell_A(M_{i_0})$ takes its greatest value. Then $M = M_{i_0}$ since the family (M_i) is directed, whence our assertion in this case. □

Lemma 6.1.53. *Let A be a one-dimensional Noetherian domain and M a torsion-free A-module of rank $r < \infty$. Then for every nonzero $a \in A$, we have*
$$\ell_A(M/aM) \leq r \cdot \ell_A(A/aA). \tag{6.1.5}$$

Proof. Lemma 6.1.51 shows that $\ell_A(A/Aa)$ is finite. We show (6.1.5) first in the case where M is finitely generated. As M is torsion-free and of rank r, there exists a submodule L of M which is isomorphic to A^r and such that $Q = M/L$ is a finitely generated torsion A-module and hence of finite length (Lemma 6.1.51). For every integer $n \geq 1$, we have an exact sequence
$$0 \longrightarrow L/(a^n M \cap L) \longrightarrow M/a^n M \longrightarrow Q/a^n Q \longrightarrow 0$$

as $a^nL \subseteq a^nM \cap L$, this implies

$$\ell_A(M/a^nM) \leq \ell_A(L/a^nL) + \ell_A(Q/a^nQ) \leq \ell_A(L/a^nL) + \ell_A(Q). \tag{6.1.6}$$

Now, since M is torsion-free, multiplication by a defines an isomorphism of M/aM onto aM/a^2M and similarly for L; whence, by induction on n, the formulae:

$$\ell_A(M/a^nM) = n \cdot \ell_A(M/aM), \quad \ell_A(L/a^nL) = n \cdot \ell_A(L/aL) \tag{6.1.7}$$

Taking account of (6.1.6), we deduce that

$$\ell_A(M/aM) \leq \ell_A(L/aL) + n^{-1}\ell_A(Q) \tag{6.1.8}$$

for all $n > 0$. As L is isomorphic to A^r, we see $\ell_A(L/aL) = r\ell_A(A/aA)$, so we get (6.1.5) by letting n tend to infinity in (6.1.8).

We now pass to the general case. Let (M_i) be the family of finitely generated submodules of M. Then the module $T = M/aM$ is the union of the submodules $T_i = M_i/(M_i \cap aM)$. Now, T_i is isomorphic to a quotient of M_i/aM_i and hence

$$\ell_A(T_i) \leq r \cdot \ell_A(A/aA)$$

by what we have just proved. Whence $\ell_A(T) \leq r \cdot \ell_A(A/aA)$ by Lemma 6.1.52. □

Proposition 6.1.54 (Krull-Akizuki). *Let A be a one-dimensional Noetherian domain, K its field of fractions, L a finite extension of K and B a subring of L containing A. Then B is a one-dimensional Noetherian domain. Moreover, for every nonzero ideal $\mathfrak{b}$ of B, $B/\mathfrak{b}$ is an A-module of finite length.*

Proof. Let $\mathfrak{b}$ be a non-zero ideal of B. We shall show that $B/\mathfrak{b}$ is an A-module of finite length (hence, a fortiori, a B-module of finite length) and that $\mathfrak{b}$ is a finitely generated B-module. A non-zero element y of $\mathfrak{b}$ satisfies an equation of the form:

$$a_ny^n + a_{n-1}y^n + \cdots + a_1y + a_0 = 0, \quad a_i \in A, a_0 \neq 0.$$

This equation shows that $a_0 \in By \subseteq \mathfrak{b}$. Applying Lemma 6.1.53 to $M = B$, it is seen that B/a_0B is an A-module of finite length, and so is $B/\mathfrak{b}$ which is a quotient module of it. Further the B-module $\mathfrak{b}$ contains, as a submodule, a_0B which is finitely generated. As $\mathfrak{b}/a_0B$ is of finite length (as a submodule of B/a_0B) and hence finitely generated, $\mathfrak{b}$ is certainly a finitely generated B-module.

The above shows first that B is Noetherian. On the other hand, if $\mathfrak{P}$ is a nonzero prime ideal of B, the ring $B/\mathfrak{P}$ is an integral domain and of finite length and hence is a field, so that $\mathfrak{P}$ is maximal. □

Corollary 6.1.55. *Let A be a one-dimensional Noetherian domain, K its field of fractions, L a finite extension of K and B a subring of L containing A. Then for every prime ideal $\mathfrak{p}$ of A, the set of prime ideals of B lying over $\mathfrak{p}$ is finite.*

Proof. Suppose first that $\mathfrak{p} = (0)$. Then the only prime ideal $\mathfrak{P}$ of B such that $\mathfrak{P} \cap A = (0)$ is (0): otherwise, writing $S = A - \{0\}$, then $S^{-1}\mathfrak{P}$ would be a non-zero prime ideal of $S^{-1}B$ and $S^{-1}B$ is just the field of fractions of B, for it is a subring of L containing K, whence an absurd conclusion. If $\mathfrak{p}$ is nonzero, it follows from Proposition 6.1.54 that $B/\mathfrak{p}B$ is a finite-dimensional vector space over the field $A/\mathfrak{p}$, hence an Artinian ring and therefore has only a finite number of prime ideals, which proves that there is only a finite number of prime ideals of B containing $\mathfrak{p}$. □

Corollary 6.1.56. *Let A be a one-dimensional Noetherian domain, K its field of fractions, L a finite extension of K and B a subring of L containing A. Then the integral closure of A in L is a Dedekind domain. In particular, the integral closure of a Dedekind domain in a finite extension of its field of fractions is a Dedekind domain.*

Proof. This integral closure is a one-dimensional integrally closed Noetherian domain, hence Dedekind. □

Proposition 6.1.57. *Let A be a Dedekind domain, K its field of fractions, L a finite extension of K and B the integral closure of A in L. Let $\mathfrak{p}$ be a non-zero prime ideal of A, $v_{\mathfrak{p}}$ the corresponding essential valuation of K and*

$$B\mathfrak{p} = \prod_i \mathfrak{P}_i^{e_i}$$

the decomposition of the ideal $B\mathfrak{p}$ as a product of prime ideals. Then:

(a) the prime ideals of B lying over $\mathfrak{p}$ are the $\mathfrak{P}_i$ such that $e_i > 0$.

(b) the valuations v_i on L corresponding to these ideals $\mathfrak{P}_i$ are, up to equivalence, the valuations on L extending $v_{\mathfrak{p}}$.

(c) $[B/\mathfrak{P}_i : A/\mathfrak{p}] = f(v_i/v)$ and $e_i = e(v_i/v)$.

Proof. To say that a prime ideal $\mathfrak{P}$ of B lies over $\mathfrak{p}$ amounts to saying that $\mathfrak{P} \supseteq \mathfrak{p}$, hence that $\mathfrak{P} \supseteq B\mathfrak{p}$ and that $\mathfrak{P}$ contains one of the $\mathfrak{P}_i$, such that $e_i > 0$. Now part (b) follows from Corollary 6.1.28. Finally, the residue field of v is identified with $A/\mathfrak{p}$ and that of v_i with $B/\mathfrak{P}_i$, hence the first claim of (c). Let t (resp. t_i) be a uniformizer for v (resp. v_i). Then

$$tB_{\mathfrak{P}_i} = tA_{\mathfrak{p}}B_{\mathfrak{P}_i} = \mathfrak{p}B_{\mathfrak{P}_i} = \Big(\prod_i \mathfrak{P}_j^{e_j}\Big)B_{\mathfrak{P}_i} = \prod_j (\mathfrak{p}_j B_{\mathfrak{P}_i})^{e_j} = (\mathfrak{p}_i B_{\mathfrak{P}_i})^{e_i} = t_i^{e_i} B_{\mathfrak{P}_i}$$

since $\mathfrak{p}_j B_{\mathfrak{P}_i} = B_{\mathfrak{P}_i}$ for $j \neq i$, whence the second claim of (c), since $e(v_i/v) = v_i(t)$. □

6.1.11 Unique factorization domains

A Krull domain all of whose divisorial ideals are principal is called a **factorial** or **unique factorization domain**. In other words, A is factorial if and only if $\mathfrak{C}(A)$ is reduced to 0.

Example 6.1.58.

(a) Every PID is factorial (and, recall, is a Dedekind domain). Conversely, every factorial Dedekind domain is a PID by Theorem 6.1.47.

(b) In particular, if K is a field, the rings $K[X]$ and $K[\![X]\!]$ are factorial domains.

(c) The local ring of a simple point of an algebraic variety is a factorial domain. The ring of germs of functions analytic at the origin of $\mathbb{C}^n$ is a factorial domain.

Theorem 6.1.59. *Let A be an integral domain. Thefollowing conditions are equivalent*

(i) A is factorial;

(ii) the ordered group of non-zero fractional principal ideal of A is a direct sum of groups isomorphic to $\mathbb{Z}$ (ordered by the product order).

(iii) Every non-empty family of integral principal ideals of A has a maximal element and the intersection of two principal ideals of A is a principal ideal;

(iv) *Every non-empty family of integral principal ideals of A has a maximal element and every irreducible element of A is prime;*

(v) *A is a Krull domain and every prime ideal of height 1 is principal.*

Proof. We shall denote by K the field of fractions of A and by $\mathscr{P}$ (or $\mathscr{P}(A)$) the ordered group of non-zero fractional principal ideals of A. We show that (i) implies (ii). If A is factorial, $\mathscr{P}$ is isomorphic to the group of divisors of A and hence to a direct sum of groups $\mathbb{Z}$.

Note now that the relation "the intersection of two integral principal ideals of A is a principal ideal" means that every ordered pair ofelements of A admits a lcm, that is, $\mathscr{P}$ is a lattice-ordered group. The fact that (ii) implies (iii) (and even is equivalent to it) therefore follows from (A, VI, §1, no.13, Theorem 2). The fact that (iii) implies (iv) follows from (A, VI, §1, no.13, Proposition 14), and the fact that (iv) implies (ii) follows from (A, VI, §1, no.13, Theorem 2) applied to the group $\mathscr{P}$.

We show that (ii) implies (v). If (ii) holds, there is an isomorphism of $\mathscr{P}$ onto $\mathbb{Z}^{\oplus I}$. Let $(v_i(x))_{i\in I}$ denote the element of $\mathbb{Z}^{\oplus I}$ corresponding to the ideal Ax. It is seen immediately that each v_i is a discrete valuation on K, that A is the intersection of the rings of the vI and that, for $x \neq 0$ in A, $v_i(x) \neq 0$ for a finite number of indices i. Hence A is a Krull domain. On the other hand, let $\mathfrak{p}$ be a prime ideal of A of height 1; it contains a non-zero element a which is necessarily not invertible and hence also (by definition of a prime ideal) one of the irreducible elements p of A. As p is prime and non zero, $\mathfrak{p} = p$, which proves that $\mathfrak{p}$ is principal.

Finally we show that (v) implies (i). Let $\mathfrak{a}$ be a divisorial ideal of A. There exist prime ideals $\mathfrak{p}_i$ of A of height 1 such that $\operatorname{div}(\mathfrak{a}) = \sum_i n_i \operatorname{div}(\mathfrak{p}_i)$ where $n_i \in \mathbb{Z}$. If (v) holds, $\mathfrak{p}_i$ is of the form Ap_i whence $\operatorname{div}(\mathfrak{a}) = \operatorname{div}(Ap_i^{n_i})$ and hence $\mathfrak{a} = \prod_i Ap_i^{n_i}$ since $\mathfrak{a}$ is divisorial. □

Proposition 6.1.60. *Let A be a Krull domain. If every divisorial ideal of A is invertible, then, for every maximal ideal $\mathfrak{m}$ of A, $A_\mathfrak{m}$ is factorial. The converse is true if it is also assumed that every divisorial ideal of A is finitely generated (in particular if A is Noetherian).*

Proof. Suppose that every divisorial ideal of A is invertible. As $A_\mathfrak{m}$ is a Krull domain, every divisorial ideal $\mathfrak{a}$ of $A_\mathfrak{m}$ is the intersection of two principal fractional ideals (Corollary 6.1.25), hence $\mathfrak{a} = \mathfrak{b}A_\mathfrak{m}$, where $\mathfrak{b}$ is a divisorial ideal of A (Corollary 1.2.34). As $\mathfrak{b}$ is invertible by hypothesis, we deduce from Theorem 1.5.26 that $\mathfrak{a}$ is principal and hence $A_\mathfrak{m}$ is a factorial domain.

Conversely, if all the $A_\mathfrak{m}$ are factorial and $\mathfrak{a}$ is a finitely generated divisorial ideal of A, then $\mathfrak{a}A_\mathfrak{m}$ is a divisorial ideal of $A_\mathfrak{m}$, as follows from Corollary 6.1.25 and Corollary 1.2.32. By hypothesis $\mathfrak{a}A_\mathfrak{m}$ is principal and hence it follows from Theorem 1.5.26 that $\mathfrak{a}$ is invertible. □

Let A be an integral domain, K its field of fractions and $U(A)$ the multiplicative group of invertible elements of A. Recall that there is a canonical isomorphism of $K^\times/U$ onto the group $\mathscr{P}$ of non-zero fractional principal ideals of A. Condition (ii) of Theorem 6.1.59 may then be translated as follows:

Proposition 6.1.61. *Let A be an integral domain. For A to befactorial, it is necessary and sufficient that there exist a subset P of A such that every $a \in A$ may be written uniquely in the form $a = u\prod_{p\in P} p^{n_p}$, where $u \in U(A)$ and the n_p are positive integers which are zero exceptfor a finite number of them.*

If P satisfies this condition, clearly all its elements are irreducible and every irreducible element of A is associated with a unique element of P. Recall that P is then called a representative system of irreducible elements of A.

Suppose always that A is factorial. It has been seen that the group $\mathscr{P}$ is a lattice. In particular, every element of $K^\times$ may be written in an essentially unique way in the form of an irreducible fraction. Any two elements a, b of $K^\times$ have a gcd and a lcm. If $a = u\prod p^{n_p}$ and $b = v\prod p^{m_p}$ are decompositions of a and b as products of irreducible elements, then, then

$$\gcd(a,b) = w\prod p^{\inf\{n_p,m_p\}},\quad \operatorname{lcm}(a,b) = w\prod p^{\sup\{n_p,m_p\}}.$$

For all $p \in P$, the map $a \mapsto n_p$ is a discrete valuation v_p on K whose ring is obviously A_{Ap}. It follows from Theorem 6.1.59 that the v_p are just the essential valuations of A and that the ideals Ap are just the prime ideals of A of height 1.

Proposition 6.1.62. *Let A be a Krull domain and S a multiplicative subset of A not containing* 0*.*

(a) If A is factorial, $S^{-1}A$ is factorial;

(b) If S is generated by a family of elements p_i such that the principal ideals Ap_i are prime and $S^{-1}A$ isfactorial, then A is factorial.

Proof. This follows from the definition of factorial domains and Proposition 6.1.44. □

Let A be a factorial domain, K its field of fractions and f a non-zero element of $K[X]$. The **content** of f is defined to be the gcd of the coefficients of f (up to a unit of A) and denote by $\operatorname{cont}(f)$. Let v be a valuation on K which is essential for A and $\bar{v}$ its canonical extension to $K[X]$, then

$$\bar{v}(f) = v(\operatorname{cont}(f)).$$

Moreover, it is easy to see $\operatorname{cont}(f)$ is uniquely determined by this property.

Lemma 6.1.63 (Guass's Lemma). *Let f, g be non-zero elements of $K[X]$, then*

$$\operatorname{cont}(fg) = \operatorname{cont}(f)\operatorname{cont}(g).$$

Proof. For every valuation v on K which is essential for A, let $\bar{v}$ denote its canonical extension to $K[X]$. Then

$$v(\operatorname{cont}(fg)) = \bar{v}(fg) = \bar{v}(f) + \bar{v}(g) = v(\operatorname{cont}(f)) + v(\operatorname{cont}(g)) = v(\operatorname{cont}(fg))$$

which proves the claim. □

Note that for every $f \in K[X]$, $f/\operatorname{cont}(f)$ is a polynomial with content 1 (such a polynomial is called **primitive**) and we can always write

$$f = \operatorname{cont}(f)\tilde{f}$$

where $\tilde{f}$ is a primitive polynomial.

Theorem 6.1.64. *Let A be a factorial domain, K its field of fractions, (p_i) a representative system of irreducible elements of A and (P_α) a representative system of primitive irreducible polynomials of $K[X]$. Then:*

(a) $A[X]$ is a factorial domain;

(b) the set p_i and P_i is a representative system of irreducible elements of $A[X]$.

Proof. Let f be a non-zero element of $A[X]$. In the ring $K[X]$, f can be decomposed uniquely in the form:

$$f = a\prod_\alpha P_\alpha^{n_\alpha}$$

where $a \in K^\times$ and $n_\alpha \geq 0$. Lemma 6.1.63 proves that a is the content of f. Hence $a \in A$. As A is factorial, a can be decomposed uniquely in the form:

$$a = u \prod_i p_i^{m_i}$$

whence the existence and uniqueness of the decomposition

$$f = u \prod_i p_i^{m_i} \prod_\alpha P_\alpha^{n_\alpha}.$$

Note that this theorem proves that every element of A admits the same decomposition into irreducible elements in A and $A[X]$. The gcd of a family of elements of A is therefore the same in A and in $A[X]$. □

Corollary 6.1.65. *If A is a factorial domain, the domain $A[X_1, \dots, X_n]$ is factorial. Also the domain $A[X_n]_{n\in\mathbb{N}}$ is factorial.*

Proposition 6.1.66. *Let A be a Zariski ring and $\widehat{A}$ its completion. If $\widehat{A}$ is a factorial domain, A is a factorial domain.*

Proof. This follows from definition of factorial domain and Proposition 6.1.43. □

Proposition 6.1.67. *Let C be a ring which is either a field or a DVR. Then the domain of formal power series $C[\![X_1, \dots, X_n]\!]$ is factorial.*

Proof. Let $\mathfrak{m}$ be the maximal ideal of C and π a generator of $\mathfrak{m}$ (if C is a field, then $\pi = 0$). Let C be given the $\mathfrak{m}$-adic topology, which is Hausdorff. As C is a Noetherian local ring, $B = C[\![X_1, \dots, X_n]\!]$ is a Noetherian local ring and its completion is $\widehat{C}[\![X_1, \dots, X_n]\!]$. By (AC, VII, §3, Corollary to Proposition 4, no.6), it suffices to prove that $\widehat{C}[\![X_1, \dots, X_n]\!]$ is factorial. Now, if C is a field, then $\widehat{C} = C$; if C is a DVR, the same is true of $\widehat{C}$ (Theorem 5.5.1). We shall therefore assume in the remainder of the proof that C is complete.

Arguing by induction starting with the trivial case n = 0, we shall assume that it has been proved that $A = C[\![X_1, \dots, X_{n-1}]\!]$ is factorial. We shall identify B with $A[\![X_n]\!]$ and denote by $\mathfrak{M}$ the maximal ideal of A (generated by $\pi, X_1, \dots, X_{n-1}$). We shall prove that every non-zero element g of B is, in an essentially unique way, a product of irreducible elements.

Let K be the field $C/C\pi$. As $B/B\pi$ is identified with $K[\![X_1, \dots, X_n]\!]$, the ideal $B\pi$ is prime and π is irreducible. If $\pi \neq 0$, $B_{B\pi}$ is therefore the ring of a normed discrete valuation w; every nonzero element g of B may therefore be written as $g = \pi^{w(g)} f$, where $f \in B$ and f is not a multiple of π. It will therefore suffice to show that f is an essentially unique product of extremal elements. Now the canonical image of f in $K[\![X_1, \dots, X_n]\!]$ is not zero; (AC, VII, §3, Lemma 3, no.7) therefore shows that there exists an automorphism of B which maps f to an element $\tilde{f}$ such that the coefficients of $\tilde{f}(0, \dots, 0, X_n)$ are not all in $C\pi$; this means that the coefficients of the series $\tilde{f}$, considered as a formal power series in X_n, are not all in $\mathfrak{M}$. It will suffice to prove our assertion for $\tilde{f}$.

In what follows, all the elements of B will be considered as formal power series in X_n with coefficients in A. By (AC, VII, §3, Proposition 6, no.8) (applicable since C and therefore A are separable and complete and the reduced series of $\tilde{f}$ is nonzero), $\tilde{f}$ is associated, in B, with a unique distinguished polynomial F. By (AC, VII, §3, Proposition 7, no.8), every series which divides $\tilde{f}$ (or, what amounts to the same, which divides F) is associated with a distinguished polynomial which divides F and every decomposition of $\tilde{f}$ is, to within invertible factors, of the form $\tilde{f} = uF_1 \cdots F_r$, where u is invertible and the F_i are irreducible distinguished polynomials (in B) such that $F = F_1 \dots F_r$. By (AC, VII, §3, Corollary to Proposition 7, no.8), the F_i are also irreducible in $A[X_n]$. Now, as A is factorial by the induction hypothesis, so is $A[X_n]$; hence, since they are monic, the F_i are uniquely determined by F (up to a permutation). This shows the uniqueness of the decomposition $\tilde{f} = uF_1 \cdots F_r$; its existence follows from the fact that B is Noetherian, which completes the proof. □

6.2 Modules over integrally closed Noetherian domains

Throughout this section, A will be a integral domain with field of fractions K. Then $P(A)$, $\mathfrak{D}(A)$ and $\mathfrak{C}(A)$ will respectively denote the set of prime ideals of A of height 1, the divisor group of A and the divisor class group of A, these latter being written additively.

The general method of studying finite modules over an integrally closed Noetherian domain A consists of "localizing" the modules with respect to all the prime ideals $\mathfrak{p} \in P(A)$ of height 1 in A. As $A_\mathfrak{p}$ is then a DVR, the structure of finitely generated $A_\mathfrak{p}$-modules is well known and therefore gives information about the structure of finitely generated A-modules. In the particular case where A is a Dedekind domain, we can arrive at as complete a theory as when A is a principal ideal domain.

6.2.1 Lattices of a vector space

Let V be a finite-dimensional vector space over the field K. A **lattice of V with respect to A** (or simply a lattice of V) is defined to be any sub-A-module M of V such that there exist two free sub-A-modules L_1, L_2 of V such that $L_1 \subseteq M \subseteq L_2$ and $\operatorname{rank}_A(L_1) = \dim_K V$.

Example 6.2.1.

(a) If we take $V = K$, the lattices of K are just the nonzero fractional ideals of K.

(b) If $\dim_K V = n$, every free sub-A-module L of V has a basis containing at most n elements, every subset of V which is free over A being free over K. For L to be a lattice of V, it is necessary and sufficient that L have a basis of n elements (in other words, that $\operatorname{rank}_A(L) = n$).

(c) If A is a PID, every lattice M of V is a finitely generated A-module (since A is Noetherian) which is torsion-free and hence a free A-module.

Proposition 6.2.2. *For a sub-A-module M of V to be a lattice of V, it is necessary and sufficient that $KM = V$ and that M be contained in a finitely generated sub-A-module of V.*

Proof. The conditions are obviously necessary, for a free sub-A-module of V with the same rank as V generates V. Conversely, if $KM = V$, M contains a basis $a_1, \dots, a_n$ of V over K and hence it contains the free sub-A-modulc L_1 generated by the a_i. On the other hand, if $M \subseteq M_1$, where M_1 is a sub-A-module of V generated by a finite number of elements b_i and $e_1, \dots, e_n$ is a basis of V over K, there exists a nonzero element s of A such that each of the b_i is a linear combination of the $s^{-1}e_i$ with coefficients in A. If L_2 is the free sub-A-modules of V generated by the $s^{-1}e_i$, then $M \subseteq L_2$. □

Corollary 6.2.3. *Suppose that A is Noetherian. For a sub-A-module M of V to be a lattice of V, it is necessary and safiicient that $KM = V$ and M be finitely generated.*

Proposition 6.2.4. *Let M be a lattice of V and M_1 a sub-A-modale of V. If there exist two elements $x, y \in K^\times$ such that $xM \subseteq M_1 \subseteq yM$, then M_1 is a lattice of V. Conversely, if M_1 is a lattice of V, there exist two non-zero elements a, b of A such that $aM \subseteq M_1 \subseteq b^{-1}M$.*

Proof. If L_1, L_2 are two free lattices of V such that $L_1 \subseteq M \subseteq L_2$, the relations $xM \subseteq M_1 \subseteq yM$ imply $xL_1 \subseteq M_1 \subseteq yL_2$ and xL_1 and yL_2 are free lattices. Conversely, if M_1 is a lattice and $e_1, \dots, e_n$ is a basis of L_2 over A, the relation $KM_1 = V$ implies the existence of $x = a/s \in K^\times$ (where a and s are non-zero elements of A) such that $xe_i \in M_1$ for all i, whence $xM \subseteq xL_2 \subseteq M_1$ and a fortiori $aM \subseteq M_1$. Exchanging the roles of M and M_1 it can be similarly shown that there exists $b \neq 0$ in A such that $bM_1 \subseteq M$. □

Proposition 6.2.5 (Properties of Lattices).

(a) If M_1 and M_2 are lattices of V, so are $M_1 \cap M_2$ and $M_1 + M_2$.

(b) If W is a vector subspace of V and M is a lattice of V, then $M \cap W$ is a lattice of W.

(c) Let $V, V_1, \ldots, V_k$ be vector spaces of finite dimension over K and let $f : V_1 \times \cdots \times V_k \to V$ be a multilinear map whose image generates V. If M_i is a lattice of V_i for each i, then the sub-A-module of V generated by $f(M_1 \times \cdots \times M_k)$ is a lattice of V.

(d) Let V and W be two vector spaces of finite dimension over K, M a lattice of V and N a lattice of W. Then the sub-A-module $(N : M)$ of $\mathrm{Hom}_K(V, W)$, consisting of the K-linear maps f such that $f(M) \subseteq N$, is a lattice of $\mathrm{Hom}_K(V, W)$.

Proof. For (a), by virtue of Proposition 6.2.4, there exist non-zero a and b in A such that $aM_1 \subseteq M_2 \subseteq b^{-1}M_1$. We conclude that $M_1 \cap M_2$ and $M_1 + M_2$ lie betwcen aM_1 and $b^{-1}M_1$ and are therefore lattices by virtue of Proposition 6.2.4.

As for (b), let S be a complement of W in V, L_W a free lattice of W and L_S a free lattice of S, so that $L = L_S + L_W$ is a free lattice of V. Then there exist x, y in $K^\times$ such that $xL \subseteq M \subseteq yL$. We deduce that $xL_W \subseteq M \cap W \subseteq yL_W$, which shows that $M \cap W$ is a lattice of W.

Now we prove (c). As $KM_i = V_i$ clearly by linearity $f(M_1 \times \cdots \times M_k)$ generates the vector K-space V. On the other hand, for all i, there exists a finitely generated sub-A-module N_i of V_i such that $M_i \subseteq N_i$. The sub-A-module N of V generated by $f(N_1 \times \cdots \times N_k)$ is finitely generated and contains M and hence M is a lattice of V (Proposition 6.2.2).

Finally, consider (d). Let P (resp. Q) be a free lattice of V (resp. W) containing M (resp. contained in N). Obviously $(N : M) \subseteq (Q : P)$. Now it is immediate that $(Q : P)$ is isomorphic to $\mathrm{Hom}_A(P, Q)$, hence is a free A-module of rank $(\mathrm{rank}_A P)(\mathrm{rank} Q)$ and therefore a lattice of $\mathrm{Hom}_K(V, W)$. Similarly, if P' (resp. Q') is a free lattice of V (resp. W) contained in M, (resp. containing N), then $(Q' : P') \subseteq (N : M)$ and $(Q' : P')$ is a lattice of $\mathrm{Hom}_K(V, W)$, whence the conclusion. □

In the notation of Proposition 6.2.5(d), the canonical map $(N : M) \to \mathrm{Hom}_A(M, N)$ which maps every K-linear map $f \in (N : M)$ to the A-linear map from M to N which has the same graph as $f|_M$, is bijective: for every A-linear map $g : M \to N$ can be imbedded in a K-linear map

$$g \otimes 1 : M \otimes_A K \to N \otimes_A K$$

and by Proposition 6.2.2 $M \otimes_A K$ and $N \otimes_A K$ are respectively identified with V and W.

In particular, if we take $W = K$, $N = A$, $\mathrm{Hom}_K(V, W)$ is just the dual vector K-space V^* of V and $(A : M)$ is identified with the dual A-module M^* of M. We shall henceforth make this identification and we shall say that M^* is the dual lattice of M: it is therefore the set of $x^* \in V^*$ such that $\langle x^*, x\rangle \in A$ for all $x \in M$.

Corollary 6.2.6. *Let U, V, W be three vector spaces of finite dimension over K and $f : U \times V \to W$ a left non-degenerate K-bilinear map. If M is a lattice of V and N a lattice of W, then the set $(N :_f M)$ of $x \in U$ such that $f(x, y) \in N$ for all $y \in M$ is a lattice of U.*

Proof. Let $\phi_f : U \to \mathrm{Hom}_K(V, W)$ be the K-linear map left associated with f such that $\phi_f(x)$ is the linear mapy $y \mapsto f(x, y)$.Recall that to say that f is left non-degenerate means that ϕ_f is injective. By Proposition 6.2.5(d) $(N : M)$ is a lattice of $\mathrm{Hom}_K(V, W)$, as $(N :_f M) = \phi_f^{-1}(N : M)$ and ϕ_f is injective, the corollary follows from Proposition 6.2.5(b). □

Example 6.2.7. Let S be a (not necessarily associative) K-algebra of finite dimension with a unit element. Then the bilinear map $(x, y) \mapsto xy$ of $S \times S$ to S is (left and right) non-degenerate. If M and N are lattices of S with respect to A, so are $M \cdot N$ and the set of $x \in S$ such that $xM \subseteq N$. Note that there exists a sub-A-algebra of S containing the unit element of S which is a lattice of S: for consider a basis $e_1, \ldots, e_n$ of S such that e_1 is the unit element of S and let $e_i e_j = \sum_k c_{ij}^k e_k$

be the multiplication table of S, so that $c_{1j}^k = \delta_j^k$ and $c_{i1}^k = \delta_i^k$. Let $s \in A$ be a non-zero element such that $\tilde{c}_{ij}^k := s \cdot c_{ij}^k \in A$ for all triplets of indices (i, j, k). If we write $w_i = s^{-1} e_i$ for $i \geq 2$, then

$$w_i w_j = s\tilde{c}_{ij}^1 e_1 + \sum_{k \geq 2} \tilde{c}_{ij}^k w_k$$

for $i, j \geq 2$. The lattice of S with basis e_1 and the $w_2, \dots, w_n$ is a sub-A-algebra of S with unit element e_1.

Example 6.2.8. Let V be a finite-dimensional vector space over K and f a non-degenerate bilinear form on V. If M is a lattice of V, it follows from the Corollary 6.2.6 that the set M_f^* of $x \in V$ such that $f(x, y) \in A$ for all $y \in M$ is also a lattice of V. If $\phi_f : V \to V^*$ is the linear map left associated with f (which is bijective), $\phi_f(M_f^*)$ is just the dual lattice M^* of M.

Proposition 6.2.9. *Let B be an integral domain, A a subring of B and K and L the respective field of fractions of A and B. Let V be a finite-dimensional vector space over K.*

(a) For every lattice M of V with respect to A, the image BM of $M_{(B)} = M \otimes_A B$ in $V_{(L)} = V \otimes_K L$ is a lattice of $V_{(L)}$ with respect to B.

(b) Suppose further that B is a flat A-module. Then the canonical map $M_{(B)} \to BM$ is bijective. If further B is faithfully flat, the map which maps every lattice M of V with respect to A to the lattice BM of $V_{(L)}$ with respect to B is injective.

Proof. As $KM = V$, clearly $L(BM) = V_{(L)}$. On the other hand M is contained in a finitely generated sub-A-module M_1 of V and hence BM is contained in BM_1 which is a finitely generated B-module; whence assertion (a).

We see that $V_{(L)} = V \otimes_K L = V \otimes_A L$ and, as L is a flat B-module, it is also a flat A-module. If B is a flat A-module, then the canonical map $M \otimes_A B \to V \otimes_A B$ is injective. On the other hand, since V is a free K-module and K a flat A-module, V is a flat A-module and hence the canonical map $V \otimes_A B \to V \otimes_A L$ is injective, which establishes the first assertion. To see also that the relation $BM_1 = BM_2$ implies $M_1 = M_2$ for two lattices M_1, M_2 of V with respect to A when B is a faithfully flat A-module, note first that $BM_1 \cap BM_2 = B(M_1 \cap M_2)$. We may therefore confine our attention to the case where $M_1 \subseteq M_2$ and our assertion then follows from **??** applied to the canonical injection $M_1 \to M_2$. □

Corollary 6.2.10. *Suppose that A is a DVR. Let $\widehat{A}$ be its completion and let $\widehat{K}$ be the field of fractions of $\widehat{A}$. The map η which maps every lattice M of V to the lattice $\widehat{A}M$ of $\widehat{V} = V \otimes_K \widehat{K}$ with respect to A is bijective and its inverse maps every lattice $\widehat{M}$ of $\widehat{V}$ with respect to $\widehat{A}$ to its intersection $\widehat{M} \cap V$ (V being canonically identified with a vector sub-K-space of $\widehat{V}$).*

Proof. If L is a free lattice of V, the lattices aL (for $a \in A$ nonzero) form a fundamental system of neighbourhoods of 0 for a topology $\mathcal{T}$ on V (compatible with its A-module structure), which (when a basis of L over A is taken) is identified with the product topology on K^n. By virtue of Proposition 6.2.4, a fundamental system of neighbourhoods of 0 for $\mathcal{T}$ also consists of all the lattices of V with respect to A. Clearly $\widehat{V}$ is the completion of V with respect to $\mathcal{T}$. Moreover, if $\mathfrak{m}$ is the maximal ideal of A, the topology $\mathcal{T}$ induces on every lattice M of V with respect to A the $\mathfrak{m}$-adic topology since M is a finitely generated A-module (Theorem 2.4.13) and $\widehat{A}M$ is the completion of M with respect to this topology (Proposition 2.4.3). Moreover, as M is open (and therefore closed) in V, $\widehat{A}M \cap V = M$, which proves again the fact that η is injective (which follows directly from Proposition 6.2.9(b), since $\widehat{A}$ is a faithfully flat A-module).

Finally, if $\widehat{M}$ is a lattice of $\widehat{V}$ with respect to A and $M = \widehat{M} \cap V$, then since $\widehat{A}L$ is a free lattice of $\widehat{V}$ with respect to $\widehat{A}$ and every element of $\widehat{A}$ is the product of an element of A and an invertible element of $\widehat{A}$, there exists $a, b \in \widehat{A}$ such that $a\widehat{A}L \subseteq \widehat{M} \subseteq b^{-1}\widehat{A}L$, whence $aL \subseteq$

$\widehat{M} \cap V \subseteq b^{-1}L$, so $\widehat{M} \cap V$ is a lattice of V with respect to A. Moreovcr $\widehat{M}$ is open in V and, as V is dense in $\widehat{V}$, $\widehat{M}$ is the completion of $\widehat{M} \cap V = M$. This proves that η is surjective, whence the corollary $\square$

Let S be a multiplicative subset of A not containing 0. Then we can apply Proposition 6.2.9 to $B = S^{-1}A$ with $L = K$, $BM = S^{-1}M$. Hence $S^{-1}M$ is a lattice of V with respect to $S^{-1}A$. Moreover:

Proposition 6.2.11. *Let V, W be vector spaces of finite dimension over K, M a lattice of V and N a lattice of W. If M is finitely generated, then*

$$S^{-1}(N : M) = (S^{-1}N : S^{-1}M)$$

in $\mathrm{Hom}_K(V, W)$.

Proof. Clearly the left hand side is contained in the right hand side. Conversely, let $f \in (S^{-1}N : S^{-1}M)$ and let $x_1, \dots, x_n$ be a system of generators of M. There exists $s \in S$ such that $f(x_i) \in s^{-}1N$ for all i and hence $sf \in (N : M)$, which proves the proposition. $\square$

6.2.2 Duality and reflexive modules

From now on the domain A is assumed to be Noetherian and integrally closed and that $P(A)$ denotes the set of prime ideals of A of height 1. Every lattice with respect to A is a finitely generated A-module by Corollary 6.2.3.

Let V be a vector space of finite dimension over K, V^* its dual and V^{**} its bidual. We shall identify V and V^{**} by means of the canonical map J_M. Let M be a lattice of V, recall that the dual A-module M^* of M is canonically identified with the dual lattice of M, which is the set of $x^* \in V^*$ such that $\langle x^*, x\rangle \in A$ for all $x \in M$. The bidual A-module M^{**} of M is therefore a lattice of V which contains M. Moreover $M^{***} = M^*$, for the relation $M \subseteq M^{**}$ implies $(M^{**})^* \subseteq M^*$ and on the other hand $M^* \subseteq (M^*)^{**}$ by the above.

If $\mathfrak{p}$ is a prime ideal, Proposition 6.2.11 applied with $N = A$ gives the relation $(M^*)_{\mathfrak{p}} = (M_{\mathfrak{p}})^*$, which justifies the notation $M_{\mathfrak{p}}^*$ for both terms.

Theorem 6.2.12. *If M is a lattice of V, then $M^* = \bigcap_{\mathfrak{p} \in P(A)} M_{\mathfrak{p}}^*$.*

Proof. Clearly M^* is contained in each of the $M_{\mathfrak{p}}^*$. Conversely, suppose that $x^* \in \bigcap_{\mathfrak{p} \in P(A)} M_{\mathfrak{p}}^*$. If $x \in M$, then $\langle x^*, x\rangle \in \bigcap_{\mathfrak{p} \in P(A)} A_{\mathfrak{p}} = A$, and thus $x^* \in M^*$. $\square$

Corollary 6.2.13. *If M is a lattice of V, then $M^{**} = \bigcap_{\mathfrak{p} \in P(A)} M_{\mathfrak{p}}$.*

Proof. Theorcm 6.2.12 applied to M^* shows that $M^{**} = \bigcap_{\mathfrak{p} \in P(A)} M_{\mathfrak{p}}^{**}$. But as $A_{\mathfrak{p}}$ is a principal ideal domain, $M_{\mathfrak{p}}^*$ is a finitely generated free $A_{\mathfrak{p}}$-module and hence $M_{\mathfrak{p}}^{**}$ is canonically identified with $M_{\mathfrak{p}}$, whence the corollary. $\square$

For any lattice M with respect to A, the canonical map $J_M : M \to M^{**}$ identifies an element $x \in M$ with itself, for x is the unique element y of $V = V^{**}$ such that $\langle x^*, x\rangle = \langle x^*, y\rangle$ for all $x^* \in M^*$, since M^* generates V^*. We shall say that M is **reflexive** if $M^{**} = M$. As we have above $M^* = (M^*)^{**}$, it is seen that the dual of any lattice M is always reflxive.

Remark 6.2.14. Let M be a finitely generated A-module; it is immediate that the dual M^* of M, identified with a sub-A-module of $\mathrm{Hom}_A(M, K)$, is a lattice of the vector K-space $\mathrm{Hom}_A(M, K)$; in particular, every finitely generated reflexive A-module is isomorphic to a lattice of a suitable vector K-space.

Theorem 6.2.15. *If M is a lattice of V, the following conditions are equivalent:*

(i) M *is reflexive;*

(ii) $M = \bigcap_{\mathfrak{p} \in P(A)} M_{\mathfrak{p}}$*;*

(iii) $\operatorname{Ass}(V/M) \subseteq P(A)$.

Proof. The equivalence of (i) and (ii) follows from the Corollary 6.2.13. If (ii) holds, then $\bigcap_{\mathfrak{p} \in P(A)} (V/M_{\mathfrak{p}}) = \{0\}$ in V/M, hence by Corollary 3.1.9 we have

$$\operatorname{Ass}(V/M) \subseteq \bigcup_{p \in P(A)} \operatorname{Ass}(V/M_{\mathfrak{p}}).$$

As $V/M_{\mathfrak{p}}$ is an $A_{\mathfrak{p}}$-module, an element of $A - \mathfrak{p}$ cannot annihilate a nonzero element of $V/M_{\mathfrak{p}}$, since the elements of $A - \mathfrak{p}$ are invertible in $A_{\mathfrak{p}}$. The elements of $\operatorname{Ass}(V/M_{\mathfrak{p}})$ are therefore contained in $\mathfrak{p}$ and are nonempty, since $V/M_{\mathfrak{p}}$ is a torsion $A_{\mathfrak{p}}$-module. As $\mathfrak{p}$ is of height 1, necessarily $\operatorname{Ass}(V/M_{\mathfrak{p}}) = \{p\}$ if $V/M_{\mathfrak{p}} \neq \{0\}$ and $\operatorname{Ass}(V/M_{\mathfrak{p}}) = \varnothing$ if $V/M_{\mathfrak{p}} = \{0\}$; hence $\operatorname{Ass}(V/M) \subseteq P(A)$.

Finally, if condition (iii) holds, then

$$\operatorname{Ass}(M^{**}/M) \subseteq \operatorname{Ass}(V/M) \subseteq P(A)$$

On the other hand, if $\mathfrak{p} \in P(A)$, then it has been seen in the proof of the Corollary 6.2.13 that $^{**}_{\mathfrak{p}} = M_{\mathfrak{p}}$, whence $\mathfrak{p} \notin \operatorname{Ass}(M^{**}/M)$ by Proposition 3.1.14. We conclude that $\operatorname{Ass}(M^{**}/M) = \varnothing$, whence $M^{**} = M$. □

Corollary 6.2.16. *Let* M*,* N *be two lattices of* V *with respect to* A *such that* N *is reflexive. In order that* $M \subseteq N$*, it is necessary and sufficient that, for all* $\mathfrak{p} \in P(A)$*,* $M_{\mathfrak{p}} \subseteq N_{\mathfrak{p}}$*.*

Proof. The condition is obviously necessary and, if it is fulfilled, then

$$\bigcap_{\mathfrak{p} \in P(A)} M_{\mathfrak{p}} \subseteq \bigcap_{\mathfrak{p} \in P(A)} N_{\mathfrak{p}} = N$$

As $M \subseteq M^{**} = \bigcap_{\mathfrak{p} \in P(A)} M_{\mathfrak{p}}$, certainly $M \subseteq N$. □

Example 6.2.17 (Example of reflexive submodules).

(a) Every free lattice is reflexive.

(b) Take $V = K$. For a fractional ideal $\mathfrak{a}$ of K to be a reflexive lattice, it is necessary and sufficient that it be a divisorial ideal by the observation $\mathfrak{a}^* = (A : \mathfrak{a})$ and Proposition 6.1.3.

(c) Let M be a lattice with respect to A. If S is a multiplicative subset of A not containing 0, Proposition 6.2.11 shows that $S^{-1}(M^*) = (S^{-1}M)^*$. If M is reflexive, $S^{-1}M$ is therefore a reflexive lattice with respect to $S^{-1}A$.

(d) If M is a finitely generated A-module and T its torsion submodule, the dual M^* of M is the same as the dual of M/T, since, for every linear form f on M, the image $f(T)$ is a torsion submodule of A and hence zero. As M/T is isomorphic to a lattice of a vector space over K, it is seen that the dual of every finitely generated A-module is reflexive.

Proposition 6.2.18. *Let* V *be a vector space of finite dimension over* K*.*

(a) If M_1 *and* M_2 *are reflexive lattices of* V*, so is* $M_1 \cap M_2$*.*

(b) If W *is a vector subspace of* V *and* M *is a reflexive lattice of* V*, then* $M \cap W$ *is a reflexive lattice of* W*.*

(c) Let V, W be two vector spaces of finite dimension over K and M (resp. N) a lattice of V (resp. W). If N is reflexive, the lattice $(N : M)$ is reflexive.

Proof. For (a), note that $(M_1 \cap M_2)_\mathfrak{p} = (M_1)_\mathfrak{p} \cap (M_2)_\mathfrak{p}$ for all $\mathfrak{p} \in P(A)$. If $M_1 = \bigcap_{\mathfrak{p}\in P(A)}(M_1)_\mathfrak{p}$ and $M_2 = \bigcap_{\mathfrak{p}\in P(A)}(M_2)_\mathfrak{p}$, then

$$M_1 \cap M_2 = \bigcap_{\mathfrak{p}\in P(A)} (M_1 \cap M_2)_\mathfrak{p}$$

whence the conclusion by virtue of Theorem 6.2.15. Similalrly $(M \cap W)_\mathfrak{p} = M_\mathfrak{p} \cap W$, whence (b). Now for (c), as M is finitely generated, it follows from Proposition 6.2.11 that $(N : M)_\mathfrak{p} = (N_\mathfrak{p} : M_\mathfrak{p})$. Moreover, the relation $N = \bigcap_{\mathfrak{p}\in P(A)} N_\mathfrak{p}$ implies

$$(N : M) = \bigcap_{\mathfrak{p}\in P(A)} (N_\mathfrak{p} : M_\mathfrak{p})$$

For if $f \in \bigcap_{\mathfrak{p}\in P(A)}(N_\mathfrak{p} : M_\mathfrak{p})$ and $x \in M$, then $f(x) \in N_\mathfrak{p}$ for all $\mathfrak{p} \in P(A)$, whence $f \in (N : M)$. This shows $(N : M)$ is reflexive. □

Proposition 6.2.19. *Let $0 \to M \to N \to Q \to 0$ be an exact sequence of A-modules. Suppose that N is finitely generated and torsion-free.*

(a) If M is reflexive, then $\mathrm{Ass}(Q) \subseteq P(A) \cup \{(0)\}$ (in other words, every ideal associated with Q is either zero or of height 1).

(b) Conversely, if N is reflexive and $\mathrm{Ass}(Q) \subseteq P(A) \cup \{(0)\}$, then M is reflexive.

Proof. As A is Noetherian, M is finitely generated. If we write $V = M_{(K)}$ and $W = N_{(K)}$, then M (resp. N) is canonically identified with a lattice of V (resp. W) (Proposition 6.2.2). Consider the two exact sequences:

$$0 \to V/M \to W/M \to W/V \to 0, \quad 0 \to Q \to W/M \to W/N \to 0.$$

from which we deduce that

$$\mathrm{Ass}(Q) \subseteq \mathrm{Ass}(W/M) \subseteq \mathrm{Ass}(V/M) \cup \mathrm{Ass}(W/V)$$

If M is reflexive, then $\mathrm{Ass}(V/M) \subseteq P(A)$. On the other hand, clearly $\mathrm{Ass}(W/V)$ is, either empty, or reduced to $\{0\}$, whence (a). Similarly,

$$\mathrm{Ass}(V/M) \subseteq \mathrm{Ass}(W/M) \subseteq \mathrm{Ass}(Q) \cup \mathrm{Ass}(W/N)$$

The hypotheses N is reflexive then imply that $\mathrm{Ass}(V/M) \subseteq P(A) \cup \{(0)\}$. But V/M is a torsion A-module and hence $(0) \notin \mathrm{Ass}(V/M)$; Theorem 6.2.15 then shows that M is reflexive. □

Proposition 6.2.20. *Let A and B be two rings, $\rho : A \to B$ a ring homomorphism and M a finitely generated A-module. Suppose that A is Noetherian and that B is a flat A-module. Then, if M is reflexive, so is the B-module $M_{(B)} = M \otimes_A B$.*

Proof. We know that there exists a canonical isomorphism $\omega_M : (M^*)_{(B)} \to (M_{(B)})^*$, such that

$$\langle \omega_M(x^* \otimes 1), x \otimes 1\rangle = f(\langle x^*, x\rangle)$$

for $x \in M$, $x^* \in M^*$. As M is a quotient of a finitely generated free A-module L, M^* is isomorphic to a sub-A-module of the dual L^* and L^* is free and finitely generated; since A

is Noetherian, M^* is therefore also a finitely generated A-module, whence an isomorphism $\omega_{M^*} : (M^{**})_{(B)} \to ((M^*)_{(B)})^*$ such that

$$\langle \omega_{M^*}(x^{**} \otimes 1), x^* \otimes 1\rangle = f(\langle x^{**}, x^*\rangle)$$

for $x^* \in M$ and $x^{**} \in M^{**}$. On the other hand, there is an isomorphism $\omega^t_M : (M_{(B)})^{**} \to ((M^*)_{(B)})^*$, whence by composition a canonical isomorphism

$$\phi = (\omega^t_M)^{-1} \circ \omega_{M^*} : (M^{**})_{(B)} \to (M_{(B)})^{**}$$

such that, in the above notation:

$$\langle \phi(x^{**} \otimes 1), \omega_M(x^* \otimes 1)\rangle = f(\langle x^{**}, x^*\rangle). \tag{6.2.1}$$

We consider now the canonical homomorphism $J_M : M \to M^{**}$ and show that the composite homomorphism:

$$\psi = (J_M \otimes 1) \circ \phi : M_{(B)} \to (M^{**})_{(B)} \to (M_{(B)})^{**}$$

is just the canonical homomorphism $J_{M_{(B)}}$. This follows immediately from (6.2.1) which gives the relations:

$$\langle \psi(x \otimes 1), \omega_M(x^* \otimes 1)\rangle = f(\langle J_M(x), x^*\rangle) = f(\langle x^*, x\rangle) = \langle \omega_M(x^* \otimes 1), x \otimes 1\rangle.$$

and from the fact that the elements $\omega_M(x^* \otimes 1)$ generate $(M_{(B)})^*$. This being so, the hypothesis that M is reflexive means that J_M is bijective, hence so is $J_M \otimes 1$ and therefore $\psi = J_{M_{(B)}}$ is bijective, which shows the proposition. □

6.2.3 Local construction of reflexive modules

We keep the notation and hypotheses of the last paragraph. We shall say that a property holds "for almost all $\mathfrak{p} \in P(A)$" if the set of $\mathfrak{p} \in P(A)$ for which it is not true is finite.

Lemma 6.2.21. *Let $\mathfrak{p}$ and $\mathfrak{q}$ be two prime ideals of A such that (0) is the only prime ideal of A contained in $\mathfrak{p} \cap \mathfrak{q}$. Then, for every sub-$A$-module E of V, $(E_\mathfrak{p})_\mathfrak{q} = KE$.*

Proof. Let S be the multiplicative subset $(A - \mathfrak{p})(A - \mathfrak{q})$ of A; by Corollary 1.2.22, $(E_\mathfrak{p})_\mathfrak{q} = S^{-1}E$. The prime ideals of $S^{-1}A$ correspond to the prime ideals $\mathfrak{q}$ of A such that $\mathfrak{p}' \cap S = \varnothing$ and by hypothesis (0) is the only prime ideal of A not meeting S; hence $S^{-1}A = K$ and $S^{-1}E = KE$. □

Theorem 6.2.22. *Let V be a vector space of finite dimension over K and M a lattice of V with respect to A.*

(a) Let N be a lattice of V with respect to A. Then for every prime ideal $\mathfrak{p}$ of A, $N_\mathfrak{p}$ is a lattice of V with respect to $A_\mathfrak{p}$ and, for almost all $\mathfrak{p} \in P(A)$, $N_\mathfrak{p} = M_\mathfrak{p}$.

(b) Conversely, suppose given for all $\mathfrak{p} \in P$ a lattice $N(\mathfrak{p})$ of V with respect to $A_\mathfrak{p}$ such that $N(\mathfrak{p}) = M_\mathfrak{p}$ for almost all $\mathfrak{p} \in P(A)$. Then $N = \bigcap_{\mathfrak{p}\in P(A)} N(\mathfrak{p})$ is a reflexive lattice of V with respect to A and it is the only reflexive lattice of V with respect to A whose localization at $\mathfrak{p}$ is $N(\mathfrak{p})$ for all $\mathfrak{p} \in P(A)$.

Proof. The first assertion follows from Proposition 6.2.9. Moreover, there exist x, y in $K^\times$ such that $xN \subseteq M \subseteq yN$. We know that, for almost all $\mathfrak{p} \in P(A)$, $v_\mathfrak{p}(x) = v_\mathfrak{p}(y) = 0$, which shows that x and y are invertible in A and hence $M_\mathfrak{p} = N_\mathfrak{p}$.

Now we prove (b). We may replace M by $x^{-1}M$ where $x \neq 0$ in A and assume that $N(\mathfrak{p}) \subseteq M_\mathfrak{p}$ for all $\mathfrak{p} \in P(A)$. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ be the elements of $P(A)$ such that $N(\mathfrak{p}) =$ for $\mathfrak{p}$ distinct from the $\mathfrak{p}_i$, we write:

$$Q = M \cap N(\mathfrak{p}_1) \cap \cdots \cap N(\mathfrak{p}_n).$$

As each of the $N(\mathfrak{p}_i)$ contain a free lattice with respect to $A_{\mathfrak{p}_i}$, it contains a fortiori a lattice of V with respect to A, hence Q contains a lattice of V with respect to A (Proposition 6.2.5) and, as Q is contained in M, Q is a lattice with respect to A. If $\mathfrak{p} \in P$ is distinct from the $\mathfrak{p}_i$, Lemma 6.2.21 applied to $N(\mathfrak{p}_i)$ gives

$$(N(\mathfrak{p}_i))_{\mathfrak{p}} = ((N(\mathfrak{p}_i))_{\mathfrak{p}_i})_{\mathfrak{p}} = KN(\mathfrak{p}_i) = V$$

since the $\mathfrak{p}_i$ and $\mathfrak{p}$ are of height 1. Then

$$Q_{\mathfrak{p}} = M_{\mathfrak{p}} \cap (N(\mathfrak{p}_i))_{\mathfrak{p}} \cap \cdots \cap (N(\mathfrak{p}_n))_{\mathfrak{p}} = M_{\mathfrak{p}} = N(\mathfrak{p}).$$

On the other hand, if $\mathfrak{p}$ is equal to $\mathfrak{p}_i$, then $(N(\mathfrak{p}_i))_{\mathfrak{p}_j} = V$ for $i \neq j$ by the argument as above and $(N(\mathfrak{p}_i))_{\mathfrak{p}_i} = N(\mathfrak{p}_i)$, whence

$$Q_{\mathfrak{p}_i} = M_{\mathfrak{p}_i} \cap N(\mathfrak{p}_i) = N(\mathfrak{p}_i).$$

We have therefore proved that $Q_{\mathfrak{p}} = N(\mathfrak{p})$ for all $\mathfrak{p} \in P(A)$. Then $N = Q^{**} = \bigcap_{\mathfrak{p} \in P(A)} Q_{\mathfrak{p}}$ is reflexive and satisfies the relations $N_{\mathfrak{p}} = Q_{\mathfrak{p}} = N(\mathfrak{p})$ for all $\mathfrak{p} \in P(A)$. The uniqueness property follows immediately from Theorem 6.2.15. ☐

Proposition 6.2.23. *Let M be a finitely generated A-module. The following conditions are equivalent:*

(i) $M_{\mathfrak{p}} = 0$ for every prime ideal $\mathfrak{p}$ of height ≤ 1.

(ii) The annihilator $\mathfrak{a}$ of M is nonzero and $(A : \mathfrak{a}) = A$.

An A-module M is called ***pseudo-zero*** *if it is finitely generated and it satisfies the equivalent conditions above.*

Proof. We know (by Corollary 1.2.16) that the condition $M_{\mathfrak{p}} = 0$ is equivalent to $\mathfrak{a} \not\subseteq \mathfrak{p}$ and hence to $\mathfrak{a}A_{\mathfrak{p}} = A_{\mathfrak{p}}$. On the other hand, for every integral ideal $\mathfrak{b} \neq 0$ of A, the relation "$\mathfrak{b}A_{\mathfrak{p}} = A_{\mathfrak{p}}$ for all $\mathfrak{p} \in P(A)$" is equivalent to $\operatorname{div}(\mathfrak{b}) = \operatorname{div}(A) = 0$ in $\mathfrak{D}(A)$ (Proposition 6.1.17), or also to $\operatorname{div}(A : \mathfrak{b}) = 0$ and, as $(A : \mathfrak{b})$ is divisorial, this relation is also equivalent to $(A : \mathfrak{b}) = A$. The proposition then follows by noting that to say that $\mathfrak{a} \not\subseteq (0)$ means that $\mathfrak{a} \neq (0)$. ☐

Example 6.2.24 (Example of pseudo-zero modules).

(a) If A is a Dedekind domain, every prime ideal of A is of height 1, so to say that M is pseudo-zero means then that $\operatorname{supp}(M) = \varnothing$ and hence that $M = 0$.

(b) Let k be a field and $A = k[X, Y]$ the polynomial ring over k in two indeterminates. If $\mathfrak{m}$ is the maximal ideal $AX + AY$ of A, the A-module $A/\mathfrak{m}$ is pseudo-zero, for its annihilator $\mathfrak{m}$ is not of height ≤ 1 since it contains the principal prime ideals AX and AY and is distinct from them; therefore $(A : \mathfrak{m}) = A$.

Let $\phi : M \to N$ a homomorphism of A-modules. Then ϕ is called **pseudo-injective** (resp. **pseudo-surjective**, **pseudo-zero**) if $\ker\phi$ (resp. $\operatorname{coker}\phi$, $\operatorname{im}\phi$) is pseudo-zero. Also ϕ is called **pseudo-bifective** if it is both pseudo-injective and pseudo-surjective. A pseudo-bijective homomorphism is also called a **pseudo-isomorphism**.

Suppose that M and N are finitely generated. Then, for $\phi : M \to N$ to be pseudo-injective (resp. pseudo-surjective, pseudo-zero), it is necessary and sufficient that, for all $\mathfrak{p} \in P(A) \cup \{(0)\}$, $\phi_{\mathfrak{p}} : M_{\mathfrak{p}} \to N_{\mathfrak{p}}$ be injective (resp. surjective, zero) (this follows from the exactness of the A-module $A_{\mathfrak{p}}$).

Example 6.2.25. Let M be a torsion-free finitely generated A-module. Then M is identified with a lattice of $V = M \otimes_A K$. We have seen that $(M_{\mathfrak{p}})_{\mathfrak{q}} = V$ for $\mathfrak{p} \neq \mathfrak{q}$ in $P(A)$, therefore $M_{\mathfrak{p}} = M_{\mathfrak{p}}^{**}$ for all $\mathfrak{p} \in P(A)$. Also, for $\mathfrak{p} = 0$, $M_{\mathfrak{p}}$ and $M_{\mathfrak{p}}^{**}$ are both equal to V. Therefore the canonical map $J_M : M \to M^{**}$ of M to its bidual is a pseudo-isomorphism.

Example 6.2.26. For all $\mathfrak{p} \in P(A)$, the canonical map $\phi : A/\mathfrak{p}^n \to A/\mathfrak{p}^{(n)} = A/(A \cap \mathfrak{p}^n A_\mathfrak{p})$ is a pseudo-isomorphism, as, for all $\mathfrak{q} \in P(A)$ distinct from $\mathfrak{p}$, $A_\mathfrak{q}/\mathfrak{p}^n A_\mathfrak{q} = A_\mathfrak{q}/\mathfrak{p}^{(n)} A_\mathfrak{q} = 0$ and $A_\mathfrak{p}/\mathfrak{p}^n A_\mathfrak{p} = A_\mathfrak{p}/\mathfrak{p}^{(n)} A_\mathfrak{p}$.

Theorem 6.2.27. *Let E be a finitely generated A-module, T the torsion submodule of E and $M = E/T$. There exists a pseudo-isomorphism $\phi : E \to T \times M$.*

For this theorem, we shall first prove two lemmas.

Lemma 6.2.28. *Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_k$ be a family of prime ideals of A of height 1 and let $S = \bigcap_{i=1}^k (A - \mathfrak{p}_i)$. Then the ring $S^{-1}A$ is a principal ideal domain.*

Proof. The ring $S^{-1}A$ is a semi-local ring whose maximal ideals are the $\mathfrak{m}_i = S^{-1}\mathfrak{p}_i$ for $1 \leq i \leq k$, the local ring $(S^{-1}A)_{\mathfrak{m}_i}$ being isomorphic to $A_{\mathfrak{p}_i}$ and hence a discrete valuation ring. The ring $S^{-1}A$ is therefore a Dedekind domain and, as it is semi-local it is a principal ideal domain. □

Lemma 6.2.29. *There exists a homomorphism $\eta : E \to T$ whose restriction to T is both a homothety and a pseudo-isomorphism.*

Proof. Let $\mathfrak{a}$ be the annihilator of T. As T is a finitely generated torsion A-module, $\mathfrak{a} \neq 0$. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_k$ be the prime ideals of height 1 containing $\mathfrak{a}$ (which are finite in number). If this number is 0, T is pseudo-zero (Proposition 6.2.23) and we may take $\eta = 0$. Otherwise, let $S = \bigcap_{i=1}^k (A - \mathfrak{p}_i)$; by Lemma 6.2.28, $S^{-1}A$ is a PID and hence $S^{-1}M$, which is a torsion-free finitely generated $S^{-1}A$-module, is free and, as $S^{-1}M = (S^{-1}E)/(S^{-1}T)$, $S^{-1}T$ is a direct factor of $S^{-1}E$. Now,

$$\mathrm{Hom}_{S^{-1}A}(S^{-1}E, S^{-1}T) = S^{-1}\mathrm{Hom}_A(E, T)$$

hence there exist $s_0 \in S$ and $\eta_0 \in \mathrm{Hom}_A(E, T)$ such that $s_0^{-1}\eta_0$ is a projector of $S^{-1}E$ onto $S^{-1}T$. If $\psi_0 \in \mathrm{Hom}_A(T, T)$ denotes the restriction of η_0 to T, there therefore exists $s_1 \in S$ such that $s_1\psi_0(x) = s_1 s_0 x$ for all $x \in T$. Writing $s = s_1 s_0$, $\eta = s_1\eta_0$, $\psi = s_1\psi_0$, then ψ is the homothety of ratio s on T and is the restriction of η to T. It remains to verify that ψ is a pseudo-isomorphism.

Now, if $\mathfrak{p} = 0$ or if $\mathfrak{p} \in P(A)$ is distinct from the $\mathfrak{p}_i$, then $T_\mathfrak{p} = 0$ and $\psi_\mathfrak{p} : T_\mathfrak{p} \to T_\mathfrak{p}$ is an isomorphism. If on the contrary $\mathfrak{p}$ is equal to one of the $\mathfrak{p}_i$, then s is invertible in $A_{\mathfrak{p}_i}$ and $h_{\mathfrak{p}_i}$ is the homothety of ratio s on $T_{\mathfrak{p}_i}$ is also an isomorphism, which completes the proof of Lemma 6.2.29. □

Proof of Theorem 6.2.27. Let $\eta : E \to T$ be a homomorphism satisfying the properties of Lemma 6.2.29. Let ψ be the restriction of η to T and let π be the canonical projection of E onto M. We show that the homomorphism $\phi = (\eta, \pi) : E \to T \times M$ solves the problem. There is the commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T & \longrightarrow & E & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle 1_M} & & \\ 0 & \longrightarrow & T & \longrightarrow & T \times M & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

where the rows are exact. The snake diagram gives the exact sequence:

$$0 \longrightarrow \ker\psi \longrightarrow \ker\phi \longrightarrow 0 \longrightarrow \operatorname{coker}\psi \longrightarrow \operatorname{coker}\phi \longrightarrow 0$$

and hence $\ker\phi$ is isomorphic to $\ker\psi$ and $\operatorname{coker}\phi$ to $\operatorname{coker}\psi$. As ψ is a pseudo-isomorphism, so is ϕ. □

We can say that "to within a pseudo-isomorphism" Theorem 6.2.27 reduces the study of finitely generated A-modules to that of torsion-free modules on the one hand and to that of torsion modules on the other. Moreover, we have seen above that a torsion-free module is pseudo-isomorphic to its bidual and hence to a reflexive module. As for torsion modules, there is the following result, which determines them to within a pseudo-isomorphism:

Theorem 6.2.30. *Let T be a finitely generated torsion A-module. There exist two finite families $(n_i)_{i\in I}$ and $(\mathfrak{p}_i)_{i\in I}$ where the n_i are positive integers and the $\mathfrak{p}_i$ are prime ideals of A of height 1 such that there exists a pseudo-isomorphism of T to $\bigoplus_{i\in I} A/\mathfrak{p}_i^{n_i}$. Moreover, the families $(n_i)_{i\in I}$ and $(\mathfrak{p}_i)_{i\in I}$ with this property are unique to within a bijection of the indexing set and the $\mathfrak{p}_i$ containing the annihilator of T.*

Proof. Let $T' = \bigoplus_{i\in I} A/\mathfrak{p}_i^{n_i}$. If $\phi : T \to T'$ is a pseudo-isomorphism and $\mathfrak{p} \in P(A)$, then $\phi_{\mathfrak{p}} : T_{\mathfrak{p}} \to T_{\mathfrak{p}}$ is an isomorphism. Now, $T'_{\mathfrak{p}}$ is the direct sum of the $A_{\mathfrak{p}}/\mathfrak{p}^{n_i}A_{\mathfrak{p}}$ the sum being over the indices i such that $\mathfrak{p}_i = \mathfrak{p}$. The $\mathfrak{p}^{n_i}A_{\mathfrak{p}}$ are therefore the elementary divisors of the torsion $A_{\mathfrak{p}}$-module $T_{\mathfrak{p}}$ and are therefore unique.

We may confine our attention to the case where $T \neq 0$. Let $\mathfrak{a}$ be the annihilator (non-zero and distinct from A) of T and $\mathfrak{p}_1, \dots, \mathfrak{p}_k$ the prime ideals of A of height 1 containing $\mathfrak{a}$ (which are finite in number and $S = \bigcap_{i=1}^k (A - \mathfrak{p}_i)$. The semi-local ring $B = S^{-1}A$ is a principal ideal domain and has maximal ideals the $\mathfrak{m}_i = S^{-1}\mathfrak{p}_i$; as $S^{-1}T$ is a finitely generated torsion B-module, it is isomorphic to a finite direct sum $\bigoplus_{i=1}^k B/\mathfrak{m}_i^{n_i}$. As $B/\mathfrak{m}_i^{n_i}$ is isomorphic to $S^{-1}(A/\mathfrak{p}_i^{n_i})$, we have obtained a torsion A-module T' of the desired type and an isomorphism ϕ_0 of $S^{-1}T$ onto $S^{-1}T'$. As $\mathrm{Hom}_{S^{-1}A}(S^{-1}T, S^{-1}T')$ is equal to $S^{-1}\mathrm{Hom}_A(T, T')$, there exist $s \in S$ and a homomorphism $\phi : T \to T'$ such that $\phi_0 = s^{-1}f$. It remains to show that f is a pseudo-isomorphism. Now if $\mathfrak{p} = 0$ or $\mathfrak{p} \in P(A)$ is distinct from the $\mathfrak{p}_i$, then $T_{\mathfrak{p}} = T'_{\mathfrak{p}} = 0$. If on the contrary $\mathfrak{p}$ is one of the $\mathfrak{p}_i$, s is invertible in $A_{\mathfrak{p}_i}$ and, as $\phi_{\mathfrak{p}_i} = s(\phi_0)$ and $(\phi_0)_{\mathfrak{p}_i}$ is an isomorphism of $T_{\mathfrak{p}_i} = (S^{-1}T)_{\mathfrak{m}_i}$ onto $T'_{\mathfrak{p}_i} = (S^{-1}T')_{\mathfrak{m}_i}$, so is $\phi_{\mathfrak{p}_i}$. □

Given an exact sequence of A-modules $0 \to E \to F \to G \to 0$, if E and G are pseudo-zero, so is F, as follows from definition and the exactness of localization. In the language of categories, we may then say that, in the category $\mathcal{C}$ of A-modules, the sub-category $\mathcal{C}'$ of pseudo-zero modules is full and we may then define the quotient category $\mathcal{C}/\mathcal{C}'$. The objects in this category are also A-modules but the set of morphisms from E to F (for E, F in $\mathcal{C}$) is the direct limit of the set of commutative groups $\mathrm{Hom}_A(E', F')$, where E' (resp. F') runs through the set of submodules of E (resp. the set of quotient modules F/F'' of F) such that E/E' (resp. F'') is pseudo-zero. Of course, for every ordered pair of A-modules E, F, there is a canonical homomorphism $\mathrm{Hom}_{\mathcal{C}}(E, F) \to \mathrm{Hom}_{\mathcal{C}/\mathcal{C}'}(E, F)$. To say that a homomorphism $\phi \in \mathrm{Hom}_A(E, F)$ is pseudo-zero (resp. pseudo-injective, pseudo-surjective, pseudo-bijective) means that its canonical image in $\mathrm{Hom}_{\mathcal{C}/\mathcal{C}'}(E, F)$ is zero (resp. a monomorphism, an epimorphism, an isomorphism).

6.2.4 Divisors of finite generated torsion modules

We keep to assume that A is Noetherian and integrally closed. Recall that $\mathfrak{D}(A)$ denotes the divisor group of A, written additively. We know that $\mathfrak{D}(A)$ is the free $\mathbb{Z}$-module generated by the elements of $P(A)$.

Let T be a finitely generated torsion A-module. For all $\mathfrak{p} \in P(A)$, $T_{\mathfrak{p}}$ is a finitely generated torsion $A_{\mathfrak{p}}$-module and hence a module of finite length (Corollary 3.2.16). We shall denote this length by $\ell_{\mathfrak{p}}(T)$. Now $T_{\mathfrak{p}} = 0$ for all $\mathfrak{p}$ not containing the annihilator of T and hence for almost all $\mathfrak{p}$, which justifies the following definition:

Definition 6.2.31. If T is a finitely generated torsion A-module, the divisor:

$$\chi(T) = \sum_{\mathfrak{p}\in P(A)} \ell_{\mathfrak{p}}(T) \cdot \mathfrak{p}$$

is called the **content** of T.

Proposition 6.2.32. *Let χ be the content function defined above.*

(a) The function χ is additive.

(b) *If T_1 and T_2 are pseudo-isomorphic, then $\chi(T_1) = \chi(T_2)$.*

(c) *In order that $\chi(T) = 0$, it is necessary and sufficient that T be pseudo-zero.*

Proof. In view of the definition, it suffices to consider for each $\mathfrak{p} \in P(A)$ the values of $\ell_{\mathfrak{p}}$, for the torsion modules considered. The claimed properties then follows from that of length function. □

Corollary 6.2.33. *If there is a long exact sequence*

$$0 \longrightarrow T_n \longrightarrow T_{n-1} \longrightarrow \cdots \longrightarrow T_0 \longrightarrow 0$$

of finitely generated torsion A-modules, then $\sum_{i=0}^{n}(-1)^i\chi(T_i) = 0$.

Recall that we may speak of the set $F(A)$ of classes of finitely generated A-madules with respect to the relation of isomorphism; for every finitely generated A-module M, let $\mathrm{cl}(M)$ denote the corresponding element of $F(A)$. We shall denote by $T(A)$ the subset of $F(A)$ consisting of the classes of finitely generated torsion A-modules. Clearly χ defines a map of $T(A)$ to $\mathfrak{D}(A)$, also denoted by χ, such that $\chi(\mathrm{cl}(T)) = \chi(T)$.

Proposition 6.2.34. *Let G be a commutative group and $\delta : T(A) \to G$ a function. For every finitely generated torsion A-module T, we also write, by an abuse of language, $\delta(T) = \delta(\mathrm{cl}(T))$. Suppose that the following conditions are satisfied:*

(a) *The function δ is additive.*

(b) *If T is pseudo-zero, then $\delta(T) = 0$.*

Then there exists a unique homomorphism $\theta : \mathfrak{D}(A) \to G$ such that $\delta = \theta \circ \chi$.

Proof. As $\chi(A/\mathfrak{p}) = \mathfrak{p}$ for all $\mathfrak{p}$, necessarily $\theta(\mathfrak{p}) = \delta(A/\mathfrak{p})$ for all $\mathfrak{p} \in P(A)$, which proves the uniqueness of θ, since the elements of $P(A)$ form a basis of $\mathfrak{D}(A)$. Conversely, let θ be the homomorphism from $\mathfrak{D}(A)$ to G such that $\theta(p) = \delta(A/\mathfrak{p})$ for all $\mathfrak{p} \in P(A)$ and let us show that it solves the problem. For this, we write

$$\psi(T) = \delta(T) - \theta(\chi(T))$$

for every finitely generated torsion A-module T. Clearly conditions (a) and (b) are also satisfied if δ is replaced by ψ. On the other hand, $\psi(A/\mathfrak{p}) = 0$ if $\mathfrak{p} \in P(A)$. If $\mathfrak{p}$ is a nonzero prime ideal and not in P, the annihilator of $A/\mathfrak{p}$ is contained in no ideal of $P(A)$, hence $A/\mathfrak{p}$ is pseudo-zero and therefore $\psi(A/\mathfrak{p}) = 0$. This being so, every finitely generated torsion A-module T admits a chain og submodules whose factors are isomorphic to A-modules of the form $A/\mathfrak{p}$, where $\mathfrak{p} \in \mathrm{supp}(T)$ (Theorem 3.1.18), and hence $\mathfrak{p} \neq 0$ since T is a torsion module. By induction on the length of this decomposition series, we deduce (in view of property (a) for ψ) that $\psi(T) = 0$. □

Remark 6.2.35. We may consider the quotient category $\mathcal{A}/\mathcal{A}'$ of the category $\mathcal{A}$ of finitely generated torsion A-modules by the full sub-category $\mathcal{A}'$ of pseudo-zero finitely generated torsion A-modules. In the language of Abelian categories, Proposition 6.2.34 then expresses the fact that the Grothendieck group of the Abelian category $\mathcal{A}/\mathcal{A}'$ is canonically isomorphic to $\mathfrak{D}(A)$.

Proposition 6.2.36. *If $\mathfrak{a}$ is a nonzero ideal of A, then*

$$\chi(A/\mathfrak{a}) = \chi((A : \mathfrak{a})/A) = \mathrm{div}(\mathfrak{a}).$$

Proof. Let $\mathfrak{p} \in P(A)$. Then since $A_\mathfrak{p}$ is a DVR we have $\mathfrak{a}A_\mathfrak{p} = \mathfrak{p}^{n_p}A_\mathfrak{p}$ where some $n_\mathfrak{p} \neq 0$. As $(A/\mathfrak{a})_\mathfrak{p} = A_\mathfrak{p}/\mathfrak{a}A_\mathfrak{p} = (A/a)$, we then see $\ell_\mathfrak{p}(A/\mathfrak{a}) = n_\mathfrak{p}$, whence $\chi(A/\mathfrak{a}) = \sum_{\mathfrak{p}\in P(A)} n_\mathfrak{p}\mathfrak{p} = \operatorname{div}(\mathfrak{a})$ by Proposition 6.1.21. On the other hand, $(A : a)_\mathfrak{p} = (A_\mathfrak{p} : \mathfrak{a}A_\mathfrak{p}) = \mathfrak{p}^{-n_\mathfrak{p}}A_\mathfrak{p}$, hence $\ell_\mathfrak{p}((A : \mathfrak{a})/A) = n_\mathfrak{p}$ and we conclude in the same way. □

Corollary 6.2.37. *Let M be a finitely generated tosion A-module. If $\bigoplus_{i\in I} A/\mathfrak{p}_i^{n_i}$ is the canonical A-module to which M is pseudo-isomorphic to, then*

$$\chi(M) = \sum_{i\in I} n_i\mathfrak{p}_i.$$

Proof. By Proposition 6.2.36, we see $\chi(A/\mathfrak{p}_i^{n_i}) = \operatorname{div}(\mathfrak{p}_i^{n_i}) = n_i\mathfrak{p}_i$, whence the claim. □

Now let V be a vector space of dimension n over K and M a lattice of V with respect to A. Let W be the exterior power $\bigwedge^n V$, which is a one-dimensional vector space over K and let M_W denote the lattice of W generated by the image of M under the canonical map $V^n \to \bigwedge^n V$ (Proposition 6.2.5). If e is a basis of W over K, we may write $M_W = \mathfrak{a}e$, where $\mathfrak{a}$ is a nonzero fractional ideal of A.

Let M' be another lattice of V and let us write $M'_W = \mathfrak{a}'e$, where $\mathfrak{a}'$ is a nonzero fractional ideal of A. The divisor $\operatorname{div}(\mathfrak{a}) - \operatorname{div}(\mathfrak{a}')$ does not depend on the choice of basis e of W, $\mathfrak{a}$ and $\mathfrak{a}'$ being multiplied by the same element of $K^\times$ when the basis is changed. We shall write $\chi(M, M') = \operatorname{div}(\mathfrak{a}) - \operatorname{div}(\mathfrak{a}')$ and say that this divisor is the **relative invariant** of M' with respect to M. Clearly, if M, M', M'' are three lattices of V, then:

$$\chi(M, M') + \chi(M', M'') + \chi(M, M'') = 0, \quad \chi(M, M') + \chi(M', M) = 0.$$

For all $\mathfrak{p} \in P(A)$, it follows immediately from the definitions that $(M_W)_\mathfrak{p} = (M_\mathfrak{p})_W$. Moreover, since $M_\mathfrak{p}$ is then a free A-module since A is a principal ideal domain, a basis of M over A is a basis of V over K, hence $(M_\mathfrak{p})_W = \bigwedge^n(M_\mathfrak{p})$ and the fractional ideal $\mathfrak{a}_\mathfrak{p}$ equals $\mathfrak{a}A_\mathfrak{p}$. If we set $\mathfrak{a}_\mathfrak{p} = \mathfrak{p}^{n_\mathfrak{p}}A_\mathfrak{p}$ and $\mathfrak{a}'_\mathfrak{p} = \mathfrak{p}^{n'_\mathfrak{p}}A_\mathfrak{p}$, then

$$\chi(M, M') = \sum_{\mathfrak{p}\in P(A)} (n_\mathfrak{p} - n'_\mathfrak{p})\mathfrak{p},$$

which may also be written as:

$$\chi(M, M') = \sum_{\mathfrak{p}\in P(A)} \chi(M_\mathfrak{p}, M'_\mathfrak{p}).$$

identifying $\mathfrak{D}(A_\mathfrak{p})$ with the sub-$\mathbb{Z}$-module of $\mathfrak{D}(A)$ generated by $\mathfrak{p}$.

Proposition 6.2.38. *Let M be a lattice of V and ϕ a K-automorphism of V. Then:*

$$\chi(M, \phi(M)) = -\operatorname{div}(\det(\phi)).$$

Proof. For all $\mathfrak{p} \in P(A)$, we have $\bigwedge^n \phi(M_\mathfrak{p}) = \bigwedge^n \phi(M)_\mathfrak{p}$. If $e_1, \ldots, e_n$ is a basis of $M_\mathfrak{p}$, then

$$\bigwedge^n(M_\mathfrak{p}) = A_\mathfrak{p} \cdot e_1 \wedge \cdots \wedge e_n, \quad \bigwedge^n \phi(M_\mathfrak{p}) = A_\mathfrak{p}(\det(\phi)) \cdot e_1 \wedge \cdots \wedge e_n$$

whence the claim. □

Proposition 6.2.39. *If M, M' are two lattices of V such that $M' \subseteq M$, then M/M' is a finitely generated torsion A-module and $\chi(M, M') = -\chi(M/M')$.*

Proof. Clearly $M/M' \subseteq V/M'$ is a finitely generated torsion module. On the other hand, for all $\mathfrak{p} \in P(A)$, since $A_\mathfrak{p}$ is a PID, we know that there exist bases $e_1, \ldots, e_n$ of $M_\mathfrak{p}$ and $e'_1, \ldots, e'_n$ of $M'_\mathfrak{p}$ such that $e'_i = \pi^{\nu_i} e_i$ for all i and integers v, π being a uniformizer of $A_\mathfrak{p}$. Therefore (in the notation introduced above) $n_i - n'_i = \sum_{i=1}^n \nu_i$. Also, $(M/M')_\mathfrak{p} = M_\mathfrak{p}/M'_\mathfrak{p}$ is isomorphic to the torsion $A_\mathfrak{p}$-module $\bigoplus_{i=1}^n A_\mathfrak{p}/\mathfrak{p}_i^{\nu_i} A_\mathfrak{p}$, and hencc its length is $\sum_{i=1}^n \nu_i$, which proves the proposition. □

Corollary 6.2.40. *Let L_1, L_2 be two free A-modules of the same rank n and let $\phi : L_1 \to L_2$ be a homomorphism. For* $\operatorname{coker} \phi$ *to be a torsion A-module, it is necessary and sufficient that* $\det(\phi) \neq 0$ *and in this case,*

$$\chi(\operatorname{coker} \phi) = \operatorname{div}(\det(\phi)).$$

Proof. The modules L_1 and L_2 can be considered as lattices in $V_1 = L_1 \otimes_A K$ and $V_2 = L_2 \otimes_A K$ respectively, ϕ extending to a K-homomorphism $\phi_{(K)}$ from V_1 to V_2. Then $(\operatorname{coker} \phi)_{(K)} = \operatorname{coker} \phi_{(K)}$, and to say that $\operatorname{coker} \phi$ is a torsion A-module means that $\operatorname{coker} \phi_{(K)} = 0$. Now, it amounts to the same to say that $\phi_{(K)}$ is surjective or that $\det(\phi_{(K)}) = \det(\phi) \neq 0$, whence the first assertion. On the other hand, we may write $\phi(L_1) = \psi(L_2)$, where ϕ is an endomorphism of L_2 of determinant $\det(\phi)$. As $\operatorname{coker} \phi = L_2/\psi(L_2)$, the formula follows from Proposition 6.2.38. □

Example 6.2.41. If $A = \mathbb{Z}$, the divisor group of A is identified with the multiplicative group $\mathbb{Q}_+^\times$ of rational numbers > 0. For every finite commutative group T, $\chi(T)$ is the order of T; the above corollary shows that the order of the group $\operatorname{coker} \phi$ is equal to the absolute value of $\det(\phi)$.

6.2.5 Divisor classes of finite generated modules

Recall that $\mathfrak{C}(A)$ denotes the divisor class group of A, the quotient of $\mathfrak{D}(A)$ by the subgroup of principal divisors. For every divisor $D \in \mathfrak{D}(A)$, we shall denote by $c(D)$ its class in $\mathfrak{C}(A)$.

Proposition 6.2.42. *Let M be a finitely generated A-module. There exists a free submodule L of M such that M/L is a torsion module and the element $c(\chi(M/L))$ of $\mathfrak{C}$ does not depend on the free submodule L. The element $c(M) := -c(\chi(M/L))$ will be called the* ***divisor class attached to M****.*

Proof. We write $S = A - \{0\}$ and let $V = S^{-1}M = M \otimes_A K$. If n is the dimension of V over K, there exist n elements $e_1, \ldots, e_n$ of M whose canonical images in V form a basis of V. These elements are obviously linearly independent in M and hence generate a free submodule L of M such that $S^{-1}(M/L) = S^{-1}M/S^{-1}L = 0$, so that M/L is a torsion module.

Now let L_1 be another free submodule of M of rank n. Since $S^{-1}L = S^{-1}L_1$, there exists $s \in S$ such that $sL_1 \subseteq L$. We may therefore limit ourselves to proving that, if $L_1 \subseteq L_2$ are two free submodules of M of rank n, then

$$c(\chi(M/L_1)) = c(\chi(M/L_2)).$$

Now, $\chi(M/L_1) = \chi(M/L_2) + \chi(L_2/L_1)$ and it follows from Corollary 6.2.40 that $\chi(L_2/L_1)$ is a principal divisor and therefore the claim follows. □

Proposition 6.2.43 (Properties of the divisor class of modules).

(a) The function c is additive.

(b) If M_1 and M_2 are pseudo-isomorphic, then $c(M_1) = c(M_2)$.

(c) If T is a torsion A-module, then $c(T) = -c(\chi(T))$.

(d) If $\mathfrak{a}$ is a nonzero fractional ideal of A, then $c(\mathfrak{a}) = -c(\operatorname{div}(\mathfrak{a}))$.

(e) If L is a free A-module then $c(L) = 0$.

Proof. To prove (a), consider a exact sequence

$$0 \longrightarrow M_1 \xrightarrow{\phi} M_2 \xrightarrow{\psi} M_3 \longrightarrow 0$$

Then there are free sub-module L_1 (resp. L_3) of M_1 (resp. M_3) such that M_1/L_1 (resp. M_3/L_3) is a torsion module. Since L_3 is free and ψ is surjective, there exists in $\psi^{-1}(L_3)$ a free complement L_{23} of $\ker\psi$ which is isomorphic to L_3. But $\ker\psi = \operatorname{im}\phi$ contains $\phi(L_1) = L_{12}$ which is free since ϕ is injective. The sum $L_2 = L_{12} + L_{23}$ is direct and L_2 is therefore a free submodule of M_2. There is moreover the commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & L_1 \arrow[r] \arrow[d] & L_2 \arrow[r] \arrow[d] & L_3 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M_1 \arrow[r, "\phi"] & M_2 \arrow[r, "\psi"] & M_3 \arrow[r] & 0
\end{tikzcd}$$

where the rows are exact and the vertical arrows are injections. We therefore obtain from the snake diagram the exact sequence:

$$0 \longrightarrow M_1/L_1 \longrightarrow M_2/L_2 \longrightarrow M_3/L_3$$

As M_1/L_1 and M_3/L_3 are torsion modules, this exact sequence shows first that so is M_2/L_2 and then

$$\chi(M_2/L_2) = \chi(M_1/L_1) + \chi(M_3/L_3)$$

which proves (a).

Assertions (c) and (e) are obvious from the definition. We prove (b). Therefore let $\phi : M_1 \to M_2$ be a pseudo-isomorphism and let L_1 be a free submodule of M_1 such that M_1/L_1 is a torsion module. We set $L_2 = \phi(L_1)$; as $\ker\phi$ is pseudo-zero, it is a torsion module, hence $\ker\phi \cap L_1 = 0$ and therefore L_2 is free. Let $\bar\phi : M_1/L_1 \to M_2/L_2$ be the homomorphism derived from ϕ by taking quotients; $\ker\phi$ is isomorphic to $\ker\bar\phi$ and $\operatorname{coker}\phi$ to $\operatorname{coker}\bar\phi$ and hence $\bar\phi$ is a pseudo-isomorphism. Moreover $\operatorname{coker}\bar\phi = M_2/\bar\phi(M_1)$ is a torsion module and so is $\phi(M_1)/L_2 = \phi(M_1/L_1)$, hence M_2/L_2 is a torsion module and it follows that $\chi(M_1/L_1) = \chi(M_2/L_2)$.

Finally it remains to prove (d). Let $x \in K^\times$ be such that $\mathfrak{a} \subseteq xA$. By considering the exact sequence $0 \to \mathfrak{a} \to xA \to xA/\mathfrak{a} \to 0$, we obtain

$$c(\mathfrak{a}) = c(xA) - c(xA/\mathfrak{a}) = -c(xA/\mathfrak{a})$$

by (a) and (e). But $xA/\mathfrak{a}$ is isomorphic to $A/x^{-1}\mathfrak{a}$, whence, by virtue of (c),

$$c(xA/\mathfrak{a}) = -c(\chi(A/x^{-1}\mathfrak{a})) = -c(\operatorname{div}(x^{-1}\mathfrak{a})) = -c(\operatorname{div}(\mathfrak{a}))$$

This completes the proof. □

Corollary 6.2.44. *If there is a long exact sequence*

$$0 \longrightarrow M_n \longrightarrow M_{n-1} \longrightarrow \cdots \longrightarrow M_0 \longrightarrow 0$$

of finitely generated A-modules, then $\sum_{i=0}^{n}(-1)^i c(M_i) = 0$.

Proof. We argue by induction on n, the case $n = 2$ being Proposition 6.2.43(a). If $N_{n-1} = \operatorname{coker}(M_n \to M_{n-1})$, there are the two exact sequences:

$$0 \longrightarrow M_n \longrightarrow M_{n-1} \longrightarrow N_{n-1} \longrightarrow 0$$

$$0 \longrightarrow N_{n-1} \longrightarrow M_{n-2} \longrightarrow \cdots \longrightarrow M_0 \longrightarrow 0$$

The first shows that M_{n-1} is finitely generated and the induction hypothesis gives

$$(-1)^{n-1}c(N_{n-1}) + \sum_{i=0}^{n-2} c(M_i) = 0$$

and

$$c(N_{n-1}) = c(M_{n-1}) - c(M_n),$$

whence the corollary. □

Corollary 6.2.45. *If a nonzero divisorial fractional ideal $\mathfrak{a}$ of A admits a finite free resolution, it is principal.*

Proof. In fact we apply Corollary 6.2.44 to a finite free resolution of $\mathfrak{a}$:

$$0 \longrightarrow L_n \longrightarrow L_{n-1} \longrightarrow \cdots \longrightarrow L_0 \longrightarrow \mathfrak{a} \longrightarrow 0$$

By Proposition 6.2.43(e), we see $c(\mathfrak{a}) = 0$, so $\operatorname{div}(\mathfrak{a})$ is principal (Proposition 6.2.43(d)). As $\mathfrak{a}$ is assumed to be divisorial, it is principal. □

Corollary 6.2.46. *If every nonzero divisorial ideal of A admits a finite free resolution, A is factorial.*

If M is a finitely generated A-module, we shall denote its rank by $\operatorname{rank}(M)$ (recall that it is the dimension over K of $M \otimes_A K$). We know that $\operatorname{rank}(M)$ is a additive function. We write

$$\gamma(M) = (\operatorname{rank}(M), c(M)) \in \mathbb{Z} \times \mathfrak{C}(A).$$

Then γ is also additive and, if M is pseudo-zero, $\gamma(M) = 0$ (since M is a torsion module). There exists a unique map from $F(A)$ to $\mathbb{Z} \times \mathfrak{C}(A)$, also denoted by γ, such that $\gamma(M) = \gamma(\operatorname{cl}(M))$ for every finitely generated A-module M. We shall see that the above properties essentially characterize γ:

Proposition 6.2.47. *Let G be a commutative group and δ a map from the set $F(A)$ of classes of finitely generated A-modules to G. For every finitely generated A-module M we also write, by an abuse of language, $\delta(M) = \delta(\operatorname{cl}(M))$. Suppose the following conditions are satisfied:*

(a) δ is additive.

(b) If T is pseudo-zero, then $\delta(T) = 0$.

Then there exists a unique homomorphism $\delta : \mathbb{Z} \times \mathfrak{C}(A) \to G$ such that $\delta = \theta \circ \gamma$.

Proof. By Proposition 6.2.43, every element of $\mathbb{Z} \times \mathfrak{C}(A)$ is of the form $(\operatorname{rank}(M), c(M))$ for some suitable finitely generated A-module M; whence the uniqueness of θ. We apply Proposition 6.2.34 to the restriction of $-\delta$ to $T(A)$: then there exists a homomorphism $\theta_0 : \mathfrak{D}(A) \to G$ such that

$$-\delta(T) = \theta_0(\chi(T)).$$

for every finitely generated torsion A-module T. Let x be a non-zero element of A; applying property (a) to the exact sequence:

$$0 \longrightarrow A \xrightarrow{h_x} A \longrightarrow A/xA \longrightarrow 0$$

where h_x is multiplication by x, we obtain $\delta(A/xA) = 0$, whence $\theta_0(\mathrm{div}(x)) = 0$. Taking quotients, θ_0 therefore defines a homomorphism $\theta_1 : \mathfrak{C}(A) \to G$ and $\delta(T) = \theta_1(c(T))$ for every torsion A-module T. We show now that the homomorphism θ defined by $\theta(n, z) = n\delta(A) + \theta_0(z)$ solves the problem. For this, we write $\delta'(M) = \delta(M) - \theta(\gamma(M))$ for every finitely generated A-module M. Clearly condition (a) is still satisfied if δ is replaced by δ'. Moreover, $\delta'(M) = 0$ when M is a torsion module or a free module. But as for every finitely generated A-module M, there exists a free sub-module L of M such that M/L is a torsion module, property (a) shows that $\delta'(M) = 0$ for every finitely generated A-module M. □

Remark 6.2.48. In the language of Abelian categories, Proposition 6.2.47 shows that $\mathbb{Z} \times \mathfrak{C}(A)$ is canonically isomorphic to the Grothendieck group of the quotient category $\mathcal{F}/\mathcal{F}'$, where $\mathcal{F}$ is the category of finitely generated A-modules and $\mathcal{F}'$ the full sub-category of $\mathcal{F}$ consisting of the pseudo-zero modules.

6.2.6 Finite field extensions

In this part A and B denote two integrally closed Naetherian domains such that $A \subseteq B$ and B is a finitely generated A-module. Let K and L the fields of fractions of A and B respectively. We shall write instead of div_A, χ_A, c_A, χ_A respectively where A-modules are concerned and use analogous notation for B-modules.

We know that B is integral over A by Proposition 4.1.1, so for a prime ideal $\mathfrak{P}$ of B to be of height 1, it is necessary and sufficient that $\mathfrak{p} = \mathfrak{P} \cap A$ be of height 1. Moreover by Corollary 6.1.55, for $\mathfrak{p} \in P(A)$, there is only a finite number of prime ideals $\mathfrak{P} \in P(B)$ lying over $\mathfrak{p}$. To abbreviate, we shall denote by $\mathfrak{P}|\mathfrak{p}$ the relation "$\mathfrak{P}$ lies over $\mathfrak{p}$". We shall then denote by $e_{\mathfrak{P}/\mathfrak{p}}$ or $e(\mathfrak{P}/\mathfrak{p})$ the ramification index of the valuation $v_{\mathfrak{P}}$ over the valuation $v_{\mathfrak{p}}$ and by $f_{\mathfrak{P}/\mathfrak{p}}$ or $f(\mathfrak{P}/\mathfrak{p})$ the residue degree $f(v_{\mathfrak{P}}/v_{\mathfrak{p}})$. Recall that the discrete valuations $v_{\mathfrak{p}}$ and $v_{\mathfrak{P}}$ are normed and that $f_{\mathfrak{P}/\mathfrak{p}}$ is the degree of the field of fractions of $B/\mathfrak{P}$ over the field of fractions of $A/\mathfrak{p}$. We set $n = \mathrm{rank}_A(B)$, where B is considered as an A-module; hence by definition $n = [L : K]$ and, for all $\mathfrak{p} \in P(A)$, n is also the rank of the free $A_{\mathfrak{p}}$-module $B_{\mathfrak{P}}$ for all $\mathfrak{P}|\mathfrak{p}$. Then it follows from Theorem 5.6.35 that for all $\mathfrak{p} \in P(A)$:

$$\sum_{\mathfrak{P}|\mathfrak{p}} e_{\mathfrak{P}/\mathfrak{p}} f_{\mathfrak{P}/\mathfrak{p}} = n. \tag{6.2.2}$$

This being so, as $\mathfrak{D}(A)$ and $\mathfrak{D}(B)$ are free $\mathbb{Z}$-modules, we define an increasing homomorphism of ordered groups $N : \mathfrak{D}(B) \to \mathfrak{D}(A)$ (also denoted by $N_{B/A}$) by the condition:

$$N(\mathfrak{P}) = f_{\mathfrak{P}/\mathfrak{p}}\mathfrak{p} \quad \text{for } \mathfrak{P} \in P(B) \text{ and } \mathfrak{P}|\mathfrak{p}.$$

On the other hand we have defined an increasing homomorphism of ordered groups $i : \mathfrak{D}(A) \to \mathfrak{D}(B)$ (also denoted by $i_{B/A}$) by the condition:

$$i(\mathfrak{p}) = \sum_{\mathfrak{P}|\mathfrak{p}} e_{\mathfrak{P}/\mathfrak{p}}\mathfrak{P}$$

for $\mathfrak{p} \in P(A)$. Clearly for every family (D_i) (resp. (E_i)) of divisors of A (resp. B):

$$i(\sup_i D_i) = \sup_i i(D_i), \qquad i(\inf_i D_i) = \inf_i i(D_i), \tag{6.2.3}$$

$$N(\sup_i E_i) = \sup_i N(E_i), \qquad N(\inf_i E_i) = \inf_i N(E_i). \tag{6.2.4}$$

Also, the formula (6.2.2) implies $N \circ i = n \cdot 1_{\mathfrak{D}(A)}$.

Recall that in Proposition 6.1.38 we have proved that $i(\mathrm{div}_A(a)) = \mathrm{div}_B(a)$ for any $a \in A$. Now let $b \in B$ and consider $N(\mathrm{div}_B(b))$. We recall that the norm $N_{L/K}(b)$ of b is in A and, as $v_{\mathfrak{P}}$ and $v_{\mathfrak{p}}$ are normed, we have

$$(v_{\mathfrak{P}})|_K = e_{\mathfrak{P}/\mathfrak{p}} \cdot v_{\mathfrak{p}} \quad \text{for } \mathfrak{P}|\mathfrak{p}.$$

The formula in (5.6.3) then becomes

$$v_{\mathfrak{p}}(N_{L/K}(b)) = \sum_{\mathfrak{P}|\mathfrak{p}} f_{\mathfrak{P}/\mathfrak{p}}(b)$$

which implies $N(\mathrm{div}_B(b)) = \mathrm{div}_A(N_{L/K}(b))$. Therefore, by taking quotients, the homomorphisms N and i define homomorphisms which will also be denoted, by an abuse of language, by:

$$i : \mathfrak{C}(A) \to \mathfrak{C}(B), \quad N : \mathfrak{C}(B) \to \mathfrak{C}(A).$$

Proposition 6.2.49. *Let $A \subseteq B$ be Noetherian integrally closed domains and B a finitely generated A-module.*

(a) For E to be a pseudo-zero B-module, it is necessary and sufficient that the A-module $\mathrm{Res}_A^B E$ be pseudo-zero.

(b) For E to be finitely generated torsion B-module, it is necessary and sufieient that $\mathrm{Res}_A^B E$ is a finitely generated torsion A-module and in this case we have

$$\chi_A(\mathrm{Res}_A^B E) = N(\chi_B(E)).$$

(c) For E to be finitely generated B-module, it is necessary and suflicient that $\mathrm{Res}_A^B E$ be a finitely generated A-module and in this case we have

$$c_A(\mathrm{Res}_A^B E) = N(c_B(E)) + \mathrm{rank}_B(E) c_A(B), \quad \mathrm{rank}_A(\mathrm{Res}_A^B E) = n \cdot \mathrm{rank}_B(E).$$

Proof. As B is a finitely generated A-module, for E to be a finitely generated B-module, it is necessary and sufficient that $\mathrm{Res}_A^B E$ be a finitely generated A-module. Moreover, if $\mathfrak{b}$ is the annihilator of E, $\mathfrak{b} \cap A = \mathfrak{a}$ is the annihilator of $\mathrm{Res}_A^B E$. As B is integral over A, there is no ideal other than 0 lying over the ideal 0 of A and hence it amounts to the same to say that $\mathfrak{a} \neq 0$ or that $\mathfrak{b} \neq 0$.

By virtue of this last remark, we may confine our attention to the case where E is a torsion B-module. If $\mathfrak{b}$ is contained in a prime ideal $\mathfrak{P} \in P(B)$, $\mathfrak{a}$ is contained in $\mathfrak{P} \cap A = \mathfrak{p}$, which is of height 1. Conversely, if $\mathfrak{a}$ is contained in a prime ideal $\mathfrak{p} \in P(A)$, there exists a prime ideal $\mathfrak{P}$ of B which contains $\mathfrak{b}$ and lies over $\mathfrak{p}$. Assertion (a) follows from these remarks and Proposition 6.2.23.

For every finitely generated torsion B-module E, we write

$$\delta(E) = \chi_A(\mathrm{Res}_A^B E).$$

clearly δ satisfies that conditions Proposition 6.2.34 (taking account of part (a)). There therefore exists a homomorphism $\theta : \mathfrak{D}(B) \to \mathfrak{D}(A)$ such that $\delta(E) = \theta(\chi_B(E))$ for every finitely generated torsion B-module E. The homomorphism θ is determined by its value for every B-module of the form $B/\mathfrak{P}$ where $\mathfrak{P} \in P(B)$, since $\chi_B(B/\mathfrak{P}) = \mathfrak{P}$. Now, for every prime ideal $\mathfrak{q} \neq \mathfrak{p} = \mathfrak{P} \cap A$ in $P(A)$, $\mathfrak{p} \nsubseteq \mathfrak{q}$ and hence $(B/\mathfrak{P})_{\mathfrak{q}} = 0$. On the other hand, $\mathfrak{P}B_{\mathfrak{P}}$ is a maximal ideal of $B_{\mathfrak{P}}$ and $(B/\mathfrak{P})_{\mathfrak{p}} = B_{\mathfrak{P}}/\mathfrak{P}B_{\mathfrak{P}}$ is isomorphic to the field of fractions of $B/\mathfrak{P}$, that is to the residue field of $v_{\mathfrak{P}}$. Its length as an $A_{\mathfrak{p}}$-module is therefore $f_{\mathfrak{P}/\mathfrak{p}}$, which proves that $\theta = N$, by the definition of χ_A.

If T is the torsion submodule of E, then $\mathrm{Res}_A^B T$ is the torsion submodule of $\mathrm{Res}_A^B E$ and $\mathrm{Res}_A^B(E/T) = \mathrm{Res}_A^B E/\mathrm{Res}_A^B T$. To prove (c) we may therefore confine our attention to the case where E is torsion-free. Then E is identified with a sub-B-module of $E_{(L)}$ and contains a basis $e_1, \dots, e_m$ over L. If $b_1, \dots, b_s$ is a basis of L over K consisting ofelements of B, the $b_j e_i$ form a basis of $E_{(L)}$ over K consisting of elements of E, whence the second equality (c). On the other hand, let F be a free sub-B-module of E such that E/F is a torsion B-module; as $\mathrm{Res}_A^B F$ is a direct sum of $\mathrm{rank}_B(E)$ A-modules isomorphic to B, by Proposition 6.2.43,

$$c_A(\mathrm{Res}_A^B F) = \mathrm{rank}_B(E) \cdot c_A(B).$$

Moreover, by definiton of N and part (b), we have

$$c_A(\mathrm{Res}_A^B(E/F)) = -c_A(N(\chi_B(E/F))) = -N(c_B(\chi_B(E/F))) = N(c_B(E))$$

Then it suffices to apply Proposition 6.2.43 to finish the proof. □

Proposition 6.2.50. *Let E be a finitely generated B-module. For E to be reflexive, it is necessary and suficient that $\mathrm{Res}_A^B E$ be a reflexive A-module.*

Proof. We have remarked in the proof of Proposition 6.2.49 that for E to be a torsion-free B-module, it is necessary and sufficient that $\mathrm{Res}_A^B E$ be a torsion-free A-module. We may therefore assume that E is a lattice of $W = E \otimes_B L$ with respect to B. We shall use the following lemma:

Lemma 6.2.51. *Let W be a vector space of finite dimension over L and let E be a lattice of W with respect to B. Then, for all $\mathfrak{p} \in P(A)$, we have $(\mathrm{Res}_A^B E)_{\mathfrak{p}} = \bigcap_{\mathfrak{P}|\mathfrak{p}} E_{\mathfrak{P}}$.*

If $S = A - \mathfrak{p}$, the prime ideals of the ring $S^{-1}B$ are generated by the prime ideals of B not meeting S, in other words the ideals $\mathfrak{P}_i$ lying over $\mathfrak{p}$ and the ideal (0). This shows that $S^{-1}B$ is a semi-local ring whose maximal ideals are the $\mathfrak{m}_i = S^{-1}\mathfrak{P}_i$ for $1 \leq i \leq m$. Moreover the local ring $(S^{-1}B)_{\mathfrak{m}_i}$ in is isomorphic to $B_{\mathfrak{P}_i}$ and hence is a discrete valuation ring. The ring $S^{-1}B$ is therefore a Dedekind domain and, as it is semi-local, it is a principal ideal domain. This being so, $(\mathrm{Res}_A^B E)_{\mathfrak{p}}$ is equal to $S^{-1}E$ considered as an $A_{\mathfrak{p}}$-module; by the above, $S^{-1}E$ is a free lattice of W with respect to $S^{-1}B$ (hence reflexive) and Theorem 6.2.15 may therefore be applied to it, giving $S^{-1}E = \bigcap_i (S^{-1}E)_{\mathfrak{m}_i}$. But $(S^{-1}E)_{\mathfrak{m}_i} = E_{\mathfrak{P}_i}$, which proves the lemma.

Returning to the proof, by the above lemma we have

$$\bigcap_{\mathfrak{P} \in P(B)} E_{\mathfrak{P}} = \bigcap_{\mathfrak{p} \in P(A)} (\mathrm{Res}_A^B E)_{\mathfrak{p}}.$$

and the conclusion follows from Theorem 6.2.15. □

Corollary 6.2.52. *The ring B is a reflexive A-module.*

Proposition 6.2.53. *Let $A \subseteq B$ be Noetherian integrally closed domains and B a finitely generated A-module.*

(a) For a finitely generated A-module M to be pseudo-Zero, it is necessary and sufficient that $M \otimes_A B$ be a pseudo-zero B-module.

(b) If M is a finitely generated torsion A-module, then $M \otimes_A B$ is a finitely generated B-module and

$$\chi_B(M \otimes_A B) = i(\chi_A(M)).$$

(c) If M is a finitely generated A-module, $M \otimes_A B$ is a finitely generated B-module

$$c_B(M \otimes_A B) = i(c_A(M)), \quad \mathrm{rank}_B(M \otimes_A B) = \mathrm{rank}_A(M).$$

Proof. Let $\mathfrak{P}$ be a prime ideal of B and $\mathfrak{p} = \mathfrak{P} \cap A$, then $(M \otimes_A B)_{\mathfrak{P}} = M \otimes_A B_{\mathfrak{P}}$, and on the other hand

$$M \otimes_A B_{\mathfrak{P}} = (M \otimes_A A_{\mathfrak{p}}) \otimes_{A_{\mathfrak{p}}} B_{\mathfrak{P}} = M_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} B_{\mathfrak{P}}.$$

the relation $M_{\mathfrak{p}} = 0$ is therefore equivalent to $(M \otimes_A B)_{\mathfrak{P}} = 0$. It suffices to apply this remark to the ideal $\mathfrak{P} = (0)$ and the ideals $\mathfrak{P} \in P(B)$ to prove (a), taking account of the definition. To prove (b), we shall use the following lemma:

Lemma 6.2.54. *Let M_1, M_2 be two finitely generated A-modules, and $\phi : M_1 \to M_2$ an injective homomorphism. Then the kernel of $\phi \otimes 1 : M_1 \otimes_A B \to M_2 \otimes_A B$ is pseudo-zero.*

Let $\mathfrak{p}$ be a prime ideal of A of height ≤ 1. Then $(M_i \otimes_A B)_{\mathfrak{p}} = (M_i)_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} B_{\mathfrak{p}}$ and $(\phi \otimes 1_B)_{\mathfrak{p}} = \phi_{\mathfrak{p}} \otimes 1_{B_{\mathfrak{p}}}$ the hypothesisthat ϕ is injective implies that so is $\phi_{\mathfrak{p}}$. On the other hand, in view of the choice of $\mathfrak{p}$, $A_{\mathfrak{p}}$ is a principal ideal domain and $B_{\mathfrak{p}}$ a finitely generated torsion-free $A_{\mathfrak{p}}$ -module and hence free; we conclude that $\phi_{\mathfrak{p}} \otimes 1_{B_{\mathfrak{p}}}$ is itself injective. If $I = \ker(\phi \otimes 1)$, then $I_{\mathfrak{p}} = \ker((\phi \otimes 1)_{\mathfrak{p}})$, therefore $I_{\mathfrak{p}} = 0$, whence a fortiori $I_{\mathfrak{P}} = (I_{\mathfrak{p}})_{\mathfrak{P}} = 0$ for $\mathfrak{P}|\mathfrak{p}$, which proves the lemma.

We return now to the proof of (b). For every finitely generated torsion Amodule M, write $\delta(M) = \chi_B(M \otimes_A B)$. It follows from (a) that, if M is pseudo-zero, then $\delta(M) = 0$. On the other hand, consider an exact sequence of finitely generated torsion A-modules:

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$$

It follows from the above lemma that there is an exact sequence of B-modules:

$$0 \longrightarrow I \longrightarrow M_1 \otimes_A B \longrightarrow M_2 \otimes_A B \longrightarrow M_3 \otimes_A B \longrightarrow 0$$

where I is pseudo-zero. Using the additivity of χ we therefore see δ is additive. We therefore conclude from Proposition 6.2.34 that there exists a homomorphism $\delta : \mathfrak{D}(A) \to \mathfrak{D}(B)$ such that $\delta(M) = \theta(\chi_A(M))$ for every finitely generated torsion A-module M. To prove that $\theta = i$, it suffices to show that $\delta(A/\mathfrak{p}) = i(\mathfrak{p})$ for all $\mathfrak{p} \in P(A)$. Now $(A/\mathfrak{p}) \otimes_A B = B/\mathfrak{p}B$ and, for all $\mathfrak{P} \in P(B)$, $(B/\mathfrak{p}B)_{\mathfrak{P}} = B_{\mathfrak{P}}/\mathfrak{p}B_{\mathfrak{P}}$. The last module is 0 if $\mathfrak{P}$ does not lie over $\mathfrak{p}$. If on the contrary $\mathfrak{P}|\mathfrak{p}$, then $B_{\mathfrak{P}}/\mathfrak{p}B_{\mathfrak{P}}$ is a $B_{\mathfrak{P}}$-module of length $e(\mathfrak{P}/\mathfrak{p})$ by definition of the ramification index. Therefore $\chi_B(B/\mathfrak{p}B) = \sum_{\mathfrak{P}|\mathfrak{p}} e_{\mathfrak{P}/\mathfrak{p}}\mathfrak{P} = i(\mathfrak{p})$, which proves (b).

The second formula in (c) is immediate, for

$$(M \otimes_A B) \otimes_B L = M \otimes_A L = M \otimes_A K \otimes_K L$$

and the rank of $(M \otimes_A K) \otimes_K L$ over L is equal to the rank of $M \otimes_A K$ over K. To show the first, consider a free submodule H of M such that $Q = M/H$ is a torsion A-module. Applying the lemma as above, we obtain an exact sequence of B-modules:

$$0 \longrightarrow I \longrightarrow H \otimes_A B \longrightarrow M \otimes_A B \longrightarrow Q \otimes_A B \longrightarrow 0$$

where I is pseudo-zero. It therefore follows from Proposition 6.2.43(a), (b) and (e) that

$$c_B(M \otimes_A B) = c_B(Q \otimes_A B) - c_B(\chi_B(Q \otimes_A B)) = -c_B(i(\chi_A(Q))) = -i(c_A(\chi_A(Q))) = i(c_A(M))$$

by virtue of (b), which completes the proof. □

6.2.7 Modules over Dedekind domains

We now assume that A is a Dedekind domain. Then we know that the ideals $\mathfrak{p} \in P(A)$ are maximal and that they are the only nonzero prime ideals of A. The group $\mathfrak{D}(A)$ is identified with the group $\mathfrak{F}(A)$ of fractional nonzero ideals of A.

Proposition 6.2.55. *Let A be a Dedekind domain. Every pseudo-zero A-module is zero. Every pseudo-infective (resp. pseudo-surjeetive, pseudo-bijective, pseudo-zero) A-module homomorphism is injective (resp. smjeetive, bijective, zero).*

Proof. The first assertion has already been shown; the others follow from it immediately. ☐

Proposition 6.2.56. *Let A be a Dedekind domain and M a finitely generated A-module. The following properties are equivalent:*

(a) M is torsion-free;

(b) M is reflexive;

(c) M is projective.

Proof. We already know (with no hypothesis on the integral domain A) that (ii) implies (i) and that (iii) implies (ii). If M is torsion-free, it is identified with a lattice of $V = M \otimes_A K$ with respect to A. $M_\mathfrak{p}$ is therefore a free $A_\mathfrak{p}$-module for every maximal ideal $\mathfrak{p} \in P(A)$, since $A_\mathfrak{p}$ is a principal ideal domain. The conclusion then follows from Theorem 1.5.5. ☐

Corollary 6.2.57. *Let M be a finitely generated A-module and let T be its torsion submodule. Then T is a direct factor of M.*

Proof. As M/T is torsion-free and finitely generated, it is projective by Proposition 6.2.56 and the corollary therefore follows. ☐

Proposition 6.2.58. *Let A be a Dedekind domain and T a finitely generated torsion A-module. There exist two finite families $(n_i)_{i\in I}$ and $(\mathfrak{p}_i)_{i\in I}$ where the n_i are positive integers and the $\mathfrak{p}_i$ are elements of $P(A)$, such that T is isomorphic to the direct sum $\bigoplus_{i\in I}(A/\mathfrak{p}_i^{n_i})$. Further, the families $(n_i)_{i\in I}$ and $(\mathfrak{p}_i)_{i\in I}$ are unique to within a bijection of the indexing set.*

Proof. This follows from Theorem 6.2.30 since a pseudo-isomorphism is here an isomorphism. ☐

Proposition 6.2.59. *Let A be a Dedekind domain and M a finitely generated torsion-free A-module of rank $n \geq 1$. Then there exists a nonzero ideal $\mathfrak{a}$ of A such that M is isomorphic to the direct sum of the modules A^{n-1} and $\mathfrak{a}$. Moreover, the class of the ideal $\mathfrak{a}$ is determined uniquely by this condition.*

Proof. (CA, Theorem 6 of no.9) shows that there exists a free submodule L of M such that M/L is isomorphic to an ideal $\mathfrak{b}$ of A. If $\mathfrak{b} = 0$, we take $\mathfrak{a} = A$. Otherwise, $\mathfrak{b}$ is of rank 1, hence $L = A^{n-1}$ and $\mathfrak{b}$ is a projective module by Proposition 6.2.56. M is therefore isomorphic to the direct sum of L and $\mathfrak{b}$, which proves the first part of the proposition. Moreover, it follows from Proposition 6.2.43 that $c(M) = c(\mathfrak{a})$ whence the uniqueness of the class of $\mathfrak{a}$. ☐

Chapter 7

Dimension and Hilbert function

7.1 Dimension of rings

7.1.1 Dimension of rings and modules

Definition 7.1.1. Let A be a ring. Then **dimension** of A, denoted by $\dim(A)$, is defined to be the Krull dimension of the topological space $\mathrm{Spec}(A)$. For a prime ideal $\mathfrak{p}$ of A, the dimension of A at $\mathfrak{p}$, denoted by $\dim_{\mathfrak{p}}(A)$, is defined to be $\dim_{\mathfrak{p}}(\mathrm{Spec}(A))$.

Recall that the map $\mathfrak{p} \mapsto V(\mathfrak{p})$ is a bijection from prime ideals of A to irreducible closed subsets of $\mathrm{Spec}(A)$. The dimension of A is then the supremum of the length of chains of prime ideals of A. Since we do not consider the zero ring, $\dim(A)$ will always be a positive integer.

Let $\mathfrak{p} \in \mathrm{Spec}(A)$, the sets $D(f)$, where f runs through A, form a basis of the topology of $\mathrm{Spec}(A)$, and $\mathfrak{p}$ belongs to the open set $D(f)$ if and only if f does not belong to $\mathfrak{p}$. Therefore, $\dim_{\mathfrak{p}}(A)$ is the infimum of the numbers $\dim(A_f)$, where f runs through $A - \mathfrak{p}$ (Corollary 1.4.25).

Example 7.1.2 (Examples of dimension of rings).

(a) For $\dim(A) = 0$, it is necessary and sufficient that every prime ideal of A is maixmal. Thus a zero-dimensional integral domain is a field, and a zero-dimensional Noetherian ring is Artinian.

(b) A Dedekind domain is a Noetherian integrally closed domain of dimension one. More generally, a ring is a finite product of Dedekind's rings if and only if it is Noetherian, reduced, integrally closed in its total ring of fractions, and of dimension one.

(c) Let k be a field and A an k-algebra that is integral over k. Then since $\dim(k) = 0$ we have $\dim(A) = 0$.

(d) For a ring A we have
$$\dim(A[X]) \geq \dim(A) + 1.$$
In fact, if $\mathfrak{p}_0 \subset \cdots \subset \mathfrak{p}_n$ is a chain of prime ideals of A, then we have a chain $\mathfrak{P}_0 \subset \cdots \subset \mathfrak{P}_n \subset \mathfrak{P}_{n+1}$ of prime ideals of $A[X]$, where $\mathfrak{P}_i = \mathfrak{p}_i A[X]$ for $1 \leq i \leq n$ and $\mathfrak{P}_{n+1} = \mathfrak{p}_n A[X] + XA[X]$. By the same reasoning, we prove the inequality $\dim(A[\![X]\!]) \geq \dim(A) + 1$. We deduces by induction the inequalities
$$\dim(A[X_1, \dots, X_n]) \geq \dim(A) + n, \quad \dim(A[\![X_1, \dots, X_n]\!]) \geq \dim(A) + n.$$
Later we will see that the equality holds if A is a Noetherian ring.

Proposition 7.1.3. *Let A be a ring.*

(a) *For any ideal* $\mathfrak{a}$ *of* A*, we have* $\dim(A/\mathfrak{a}) \leq \dim(A)$.

(b) *Let* S *be a multiplicative subset of* A*, then* $\dim(S^{-1}A) \leq \dim(A)$.

(c) *We have* $\dim(A) = \sup_{\mathfrak{p}} \dim(A/\mathfrak{p}) = \sup_{\mathfrak{m}} \dim(A_{\mathfrak{m}})$*, where* $\mathfrak{p}$ *runs through minimal prime ideals of* A *and* $\mathfrak{m}$ *runs through maximal ideals of* A.

(d) *Assume that* A *has finitely many minimal prime ideals. Then for any prime ideal* $\mathfrak{p}$ *of* A*,*

$$\dim_{\mathfrak{p}}(A) = \sup_{\mathfrak{q}} \dim_{\mathfrak{p}/\mathfrak{q}}(A/\mathfrak{q})$$

where $\mathfrak{q}$ *runs through minimal prime ideals of* A *contained in* $\mathfrak{p}$.

(e) *Let* $\mathfrak{a}$ *be an ideal of* A *which is not contained in any minimal prime ideal of* A*. Then we have* $\dim(A) \geq \dim(A/\mathfrak{a}) + 1$*. In particular, if* A *is an integral domain, then* $\dim(A) \geq \dim(A/\mathfrak{a}) + 1$ *for any nonzero ideal* $\mathfrak{a}$ *of* A.

Proof. By Proposition 1.4.22, $\mathrm{Spec}(A/\mathfrak{a})$ is homeomorphic to the subspace $V(\mathfrak{a})$ of $\mathrm{Spec}(A)$, and we get $\dim(A/\mathfrak{a}) \leq \dim(A)$ therefore. Similarly, part (b) follows from Corollary 1.4.25. Also, by Proposition 1.4.14, the irreducible components of $\mathrm{Spec}(A)$ are homeomorphic to $\mathrm{Spec}(A/\mathfrak{p})$, where $\mathfrak{p}$ are minimal prime ideals of A. This proves (c), and (d) follows from **??**.

Finally, let $\mathfrak{a}$ be an ideal of A which is not contained in any minimal prime ideal of A. We prove that, for any chain $\mathfrak{p}_0 \subset \cdots \subset \mathfrak{p}_n$ of prime ideals of A such that $\mathfrak{a} \subseteq \mathfrak{p}_0$ we have $\dim(A) \geq n+1$. By the hypothesis on $\mathfrak{a}$ we see $\mathfrak{p}_0$ is not minimal, so there exists a prime ideal $\mathfrak{p}_{-1}$ of A contained in $\mathfrak{p}_0$, distinct from $\mathfrak{p}_0$, and $\mathfrak{p}_{-1} \subset \mathfrak{p}_0 \subset \cdots \subset \mathfrak{p}_0$ is a chain of prime ideals of A, whence $\dim(A) \geq n+1$. □

Corollary 7.1.4. *Let* $\rho : A \to B$ *be a ring homomorphism. Then* $\dim(B)$ *is the supremum of the numbers* $\dim(B/\mathfrak{p}^e)$*, where* $\mathfrak{p}$ *covers the set of minimal prime ideals of* A.

Proof. In fact, for each minimal prime ideal $\mathfrak{P}$ of B, there exists a minimal prime ideal $\mathfrak{p}$ of A contained in $\mathfrak{P}^c$, and therefore

$$\dim(B/\mathfrak{P}) \leq \dim(B/\mathfrak{p}^e) \leq \dim(B)$$

whence the claim by Proposition 7.1.3. □

For an ideal $\mathfrak{a}$ of A, we define the **height** (resp. **coheight**) $\mathrm{ht}(\mathfrak{a})$ (resp. $\mathrm{coht}(\mathfrak{a})$) of $\mathfrak{a}$ to be the codimension (resp. dimension) of $V(\mathfrak{a})$ in $\mathrm{Spec}(A)$. Also, the ring is called **catenary** if $\mathrm{Spec}(A)$ is catenary.

Proposition 7.1.5. *Let* A *be a ring.*

(a) *The height of a prime ideal* $\mathfrak{p}$ *is the supremum of the length of chains* $\mathfrak{p}_0 \subseteq \cdots \subseteq \mathfrak{p}_n$ *of prime ideals of* A *such that* $\mathfrak{p}_n = \mathfrak{p}$.

(b) *Let* $\mathfrak{p}$ *be a prime ideal of* A *and* $\mathfrak{a}$ *an ideal of* A*. Then* $\dim(A_{\mathfrak{p}}/\mathfrak{a}A_{\mathfrak{p}}) = -\infty$ *if* $\mathfrak{a}$ *is not contained in* $\mathfrak{p}$*, and* $\dim(A_{\mathfrak{p}}/\mathfrak{a}A_{\mathfrak{p}}) = \mathrm{codim}(V(\mathfrak{p}), V(\mathfrak{a}))$ *if* $\mathfrak{a}$ *is contained in* $\mathfrak{p}$*. In particular, for a prime ideal* $\mathfrak{p}$ *of* A*, we have* $\dim(A_{\mathfrak{p}}) = \mathrm{ht}(\mathfrak{p})$.

(c) *For any ideal* $\mathfrak{a}$ *of* A*, we have* $\mathrm{ht}(\mathfrak{a}) = \inf_{\mathfrak{p} \supseteq \mathfrak{a}} \mathrm{ht}(\mathfrak{p}) = \inf_{\mathfrak{p} \supseteq \mathfrak{a}} \dim(A_{\mathfrak{p}})$.

Proof. Assertion (a) follows from definition, and (b) follows from the fact that the map $\mathfrak{q} \mapsto \mathfrak{q}(A_{\mathfrak{p}}/\mathfrak{a}A_{\mathfrak{p}})$ is an increasing isomorphism of the set of prime ideals $\mathfrak{q}$ of A such that $\mathfrak{a} \subseteq \mathfrak{q} \subseteq \mathfrak{p}$ on the set of prime ideals of the local ring $A_{\mathfrak{p}}/\mathfrak{a}A_{\mathfrak{p}}$. Now let $\mathfrak{a}$ be an ideal of A. Then the irreducible closed subset of $V(\mathfrak{a})$ are the sets $V(\mathfrak{p})$, where $\mathfrak{p}$ is a prime ideal of A containing $\mathfrak{a}$. Part (c) then follows from the definition of codimension. □

Corollary 7.1.6. *Let $\mathfrak{p}$ be a prime ideal of A and S a multiplicative subset of A not meeting $\mathfrak{p}$. Then* $\mathrm{ht}(\mathfrak{p}) = \mathrm{ht}(S^{-1}\mathfrak{p})$.

Proof. This follows from Proposition 1.2.37. □

Proposition 7.1.7. *Let A be a ring. Let $\mathfrak{a}, \mathfrak{b}$ be proper ideals of A such that $\mathfrak{a} \subseteq \mathfrak{b}$. Then*

$$\mathrm{codim}(V(\mathfrak{b})), V(\mathfrak{a})) \leq \dim(A/\mathfrak{a}) - \dim(A/\mathfrak{b})$$

In particular, for any ideal $\mathfrak{a}$ of A, we have $\mathrm{ht}(\mathfrak{a}) + \dim(A/\mathfrak{a}) \leq \dim(A)$.

Proof. This from **??** and of the relations $\dim(A/\mathfrak{a}) = \dim(V(\mathfrak{a}))$, $\dim(A/\mathfrak{b}) = \dim(V(\mathfrak{b}))$. □

Proposition 7.1.8. *Let A be a ring.*

(a) For A to be catenary, it is necessary and sufficient that, for every triple $(\mathfrak{p}, \mathfrak{q}, \mathfrak{r})$ of prime ideals of A such that $\mathfrak{r} \subseteq \mathfrak{q} \subseteq \mathfrak{p}$, we have

$$\dim(A_\mathfrak{p}/\mathfrak{q}A_\mathfrak{p}) + \dim(A_\mathfrak{q}/\mathfrak{r}A_\mathfrak{q}) = \dim(A_\mathfrak{p}/\mathfrak{r}A_\mathfrak{p}).$$

(b) If A is an integral domain with finite dimension, then for A to be catenary, it is necessary and sufficient that, for every pair $(\mathfrak{p}, \mathfrak{q})$ of prime ideals of A such that $\subseteq \mathfrak{q} \subseteq \mathfrak{p}$, we have

$$\mathrm{ht}(\mathfrak{q}) + \dim(A_\mathfrak{p}/\mathfrak{q}A_\mathfrak{p}) = \mathrm{ht}(\mathfrak{p}).$$

Proof. This follows directly from **??** and **??**. □

Proposition 7.1.9. *Let A be a ring of finite dimension such that all maximal chains of prime ideals have the same length. Then A is catenary, and for any prime ideal $\mathfrak{p}$ of A we have*

$$\mathrm{ht}(\mathfrak{p}) = \dim(A) - \mathrm{coht}(\mathfrak{p}).$$

Moreover, for any pair $(\mathfrak{p}, \mathfrak{q})$ of prime ideals such that $\mathfrak{q} \subseteq \mathfrak{p}$, we have

$$\dim(A_\mathfrak{p}/\mathfrak{q}A_\mathfrak{p}) = \mathrm{coht}(\mathfrak{q}) - \mathrm{coht}(\mathfrak{p}).$$

Proof. This is a reformulation of **??**. □

Now let M be a finitely generated A-module. The **dimension** (resp. **codimension**) of M, denoted by $\dim_A(M)$ (resp. $\mathrm{codim}(M)$), is defined to be the dimension (resp. codimension) of the support of M. The support of the A-module A is $\mathrm{Spec}(A)$, so the dimension of the A-module A equals the dimension of the ring A. Also, since M is finitely generated, we have

$$\mathrm{supp}(M) = V(\mathfrak{a}) = \mathrm{supp}(A/\mathfrak{a}),$$

where $\mathfrak{a}$ is the annihilator of M. Consequently, the dimension of the A-mqdule M, the dimension of the A-module $A/\mathfrak{a}$, the dimension of the ring $A/\mathfrak{a}$ and the dimension of the $(A/\mathfrak{a})$-module M are all the supremum of the set of lengths of the chains $\mathfrak{p}_0 \subset \cdots \subset \mathfrak{p}_n$ of prime ideals of A such that $\mathfrak{a} \subseteq \mathfrak{p}_0$. By Proposition 7.1.3(c), the dimension of M is also the supremum of the dimensions of the rings (or of the A-modules) $A/\mathfrak{p}$, where $\mathfrak{p}$ runs through the set of prime ideals of A, minimal among those which contain $\mathfrak{a}$.

Example 7.1.10. If A is Noetherian, then by Proposition 3.2.14, we see $\dim_A(M) = 0$ if and only $\mathrm{supp}(M)$ consists of maximal ideals, or equivalently M is of finite length.

Example 7.1.11. If M is a finitely generated module over a Noetherian ring A, $\dim_A(M)$ is then the supremum of the numbers $\dim(A/\mathfrak{p})$, where $\mathfrak{p}$ runs through the set $\mathrm{Ass}_A(M)$ of prime ideals of A associated with M (Proposition 3.1.14).

Proposition 7.1.12. *Let A be a ring and M a finitely generated A-module.*

(a) For each $\mathfrak{p} \in \operatorname{supp}(M)$, we have $\dim_{A_\mathfrak{p}}(M_\mathfrak{p}) = \operatorname{codim}(V(\mathfrak{p}), \operatorname{supp}(M))$.

(b) We have $\dim_A(M) = \sup_\mathfrak{p} \dim_{A_\mathfrak{p}}(M_\mathfrak{p})$, where $\mathfrak{p}$ runs through prime ideals in $\operatorname{supp}(M)$ (resp. maximal ideal in $\operatorname{supp}(M)$).

(c) Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of finitely generated A-modules. Then $\dim_A(M) = \sup\{\dim_A(M'), \dim_A(M'')\}$.

Proof. Let $\mathfrak{a}$ be the annihilator of M, then the annihilator of $M_\mathfrak{p}$ is $\mathfrak{a}A_\mathfrak{p}$, whence $\dim_{A_\mathfrak{p}}(M_\mathfrak{p}) = \dim(A_\mathfrak{p}/\mathfrak{a}A_\mathfrak{p})$, and the claim follows from Proposition 7.1.5 and $\operatorname{supp}(M) = V(\mathfrak{a})$. Also, part (b) follows from (a), and (c) follows from Proposition 1.4.33. □

Now let A be a Noetherian ring. Let $Z(A)$ be the free $\mathbb{Z}$-module generated by the set of closed irreducible subsets of $\operatorname{Spec}(A)$. For any irreducible closed subset Y of $\operatorname{Spec}(A)$, we denote by $[Y]$ the element correspondent of $Z(A)$. The elements of $Z(A)$ are sometimes called **cycles**. Let M be a finitely generated A-module. For any prime ideal $\mathfrak{p}$ of A which is a minimal element of $\operatorname{supp}(M)$, we have $0 < \ell_{A_\mathfrak{p}}(M_\mathfrak{p}) < +\infty$ by Corollary 3.2.16. We set

$$z(M) = \sum_{\mathfrak{p}} \ell_{A_\mathfrak{p}}(M_\mathfrak{p}) \cdot [V(\mathfrak{p})]$$

where $\mathfrak{p}$ runs through the finite set of minimal element of $\operatorname{supp}(M)$.

Example 7.1.13. For any $\mathfrak{p} \in \operatorname{Spec}(A)$, we have $z(A/\mathfrak{p}) = [V(\mathfrak{p})]$. More generally, let M be a finitely generated A-module, and let $(M_i)_{1 \le i \le n}$ be a chain of M such that for each i, the modulus M_i/M_{i+1} is isomorphic to $A/\mathfrak{p}_i$, where $\mathfrak{p}_i$ is a prime ideal of A. Then we have $z(M) = \sum_{i \in J} \ell_i [V(\mathfrak{p}_i)]$, where J is the subset of I consists of i such that $\mathfrak{p}_i$ be a minimal element of $\{\mathfrak{p}_0, \dots, \mathfrak{p}_{n-1}\}$, and ℓ_i is the number of indices j such that $\mathfrak{p}_j = \mathfrak{p}_i$.

For any integer d, let $Z_{\le d}$ (resp. Z_d, resp. $Z^{\ge d}$, resp. Z^d) be the $\mathbb{Z}$-submodule of $Z(A)$ generated by the elements $[V(\mathfrak{p})]$ where $\mathfrak{p}$ is a prime ideal of A such that $\operatorname{coht}(\mathfrak{p}) \le d$ (resp. $\operatorname{coht}(\mathfrak{p}) = d$, resp. $\operatorname{ht}(\mathfrak{p}) \ge d$, resp. $\operatorname{ht}(\mathfrak{p}) = d$). An element of Z_d (resp. Z^d) is called a **cycle** of dimension d (resp. codimension d). We have

$$Z_{\le d} = Z_{\le d-1} \oplus Z_d, \quad Z^{\ge d} = Z^{\ge d-1} \oplus Z^d.$$

Let $\mathcal{C}$ be the set of classes of finitely generated A-modules, and for each integer d, let $C_{\le d}$ (resp. $\mathcal{C}^{\ge d}$) be the subgroup generated by finitely generated A-modules of dimension $\le d$ (resp. of codimension $\ge d$).

Lemma 7.1.14. *Let M be a finitely generated A-module.*

(a) For M be in $\mathcal{C}_{\le d}$, it is necessary and sufficient that $z(M) \in Z_{\le d}$. In this case, the projection $z_d(M)$ of $z(M)$ on Z_d parallel to $Z_{\le d-1}$ is given by

$$z_d(M) = \sum_{\operatorname{coht}(\mathfrak{p})=d} \ell_{A_\mathfrak{p}}(M_\mathfrak{p}) \cdot [V(\mathfrak{p})].$$

(b) For M be in $\mathcal{C}^{\ge d}$, it is necessary and sufficient that $z(M) \in Z^{\ge d}$. In this case, the projection $z^d(M)$ of $z(M)$ on Z^d parallel to $Z^{\ge d-1}$ is given by

$$z^d(M) = \sum_{\operatorname{ht}(\mathfrak{p})=d} \ell_{A_\mathfrak{p}}(M_\mathfrak{p}) \cdot [V(\mathfrak{p})].$$

Proof. For M to be dimension $\leq d$, it is necessary and sufficient that for any minimal prime $\mathfrak{p}$ of $\mathrm{supp}(M)$, we have $\mathrm{coht}(\mathfrak{p}) \leq d$, which means $z(M) \in Z_{\leq d}$. Suppose now $\dim(M) \leq d$, and let $\mathfrak{p} \in \mathrm{Spec}(A)$ be such that $\mathrm{coht}(\mathfrak{p}) = d$. Then, either $\mathfrak{p} \notin \mathrm{supp}(M)$ and $M_{\mathfrak{p}} = 0$, or $\mathfrak{p} \in \mathrm{supp}(M)$ and $\mathfrak{p}$ is a minimal element of $\mathrm{supp}(M)$. The coefficient of $[V(\mathfrak{p})]$ in $z(M)$ is in both cases $\ell_{A_{\mathfrak{p}}}(M_{\mathfrak{p}})$, whence (a). Part (b) is proved analogously, noting that a finitely generated module M is in $\mathcal{C}^{\geq d}$ if and only if we have $M_{\mathfrak{p}} = 0$ for any prime ideal $\mathfrak{p}$ of height $< d$. $\square$

By Proposition 1.4.33, the sets $\mathcal{C}_{\leq d}$ and $C^{\geq d}$ are hereditary, and we can consider the corresponding Grothendieck groups $K(\mathcal{C}_{\leq d})$ and $K(\mathcal{C}^{\geq d})$, which are subgroups of $K(\mathcal{C})$, the Grothendieck group of finitely generated A-modules. By Lemma 7.1.14, the functions z_d and z^d are additive, whence extends to homomorphisms

$$\zeta_d : K(\mathcal{C}_{\leq d}) \to Z_d, \quad \zeta^d : K(\mathcal{C}^{\geq d}) \to Z^d$$

Moreover, since $\mathcal{C}_{\leq d-1} \subseteq \mathcal{C}_{\leq d}$ and $\mathcal{C}^{\geq d+1} \subseteq \mathcal{C}^{\geq d}$, we have canonical homomorphisms

$$i_d : K(\mathcal{C}_{\leq d-1}) \to K(\mathcal{C}_{\leq d}), \quad i^d : K(\mathcal{C}^{\geq d+1}) \to K(\mathcal{C}^{\geq d}).$$

Proposition 7.1.15. *The following sequence of $\mathbb{Z}$-modules*

$$K(\mathcal{C}_{\leq d-1}) \xrightarrow{i_d} K(\mathcal{C}_{\leq d}) \xrightarrow{\zeta_d} Z_d \longrightarrow 0$$

$$K(\mathcal{C}^{\geq d+1}) \xrightarrow{i^d} K(\mathcal{C}^{\geq d}) \xrightarrow{\zeta^d} Z^d \longrightarrow 0$$

are exact.

Proof. We have $\zeta_d \circ i_d = 0$ by Lemma 7.1.14. For every $\mathfrak{p} \in \mathrm{Spec}(A)$ such that $\mathrm{coht}(\mathfrak{p}) = d$, we see $\zeta_d([A/\mathfrak{p}]) = z_d(Z/\mathfrak{p}) = [V(\mathfrak{p})]$, so the homomorphism ζ_d is surjective. Now by Proposition 3.1.17, $K(\mathcal{C}_{\leq d})$ is generated by the elements $[A/\mathfrak{p}]$, where $p \in \mathrm{Spec}(A)$ and $\mathrm{coht}(\mathfrak{p}) \leq d$. Therefore, every element ξ can be written into $\xi = i_d(\eta) + \sum_{i=1}^k n_i[A/\mathfrak{p}_i]$, where $\eta \in K(\mathcal{C}_{\leq d-1})$, $n_i \in \mathbb{Z}$ and $\mathrm{coht}(\mathfrak{p}_i) = d$ for each i. From this, we see $\zeta_d(\xi) = \sum_{i=1}^k n_i[V(\mathfrak{p}_i)]$, and $\zeta_d(\xi) = 0$ if and only if $\xi \in \mathrm{im}\, i_d$, which proves the claim. The proof for i^d and ζ^d can be done similarly. $\square$

Example 7.1.16. Let A be a Noetherian integral domain. Then $Z^0 = \mathbb{Z}\mathrm{Spec}(A)$ and $z^0(M) = \mathrm{rank}(M) \cdot \mathrm{Spec}(A)$. Therefore the modules in $\mathcal{C}^{\geq 1}$ are the torsion modules.

Example 7.1.17. Assume that A is Noetherian and integrally closed. Then Z^1 is identified with the group $\mathfrak{D}(A)$ of divisors of A. The modules in $\mathcal{C}^{\geq 2}$ are therefore pseudo-zero modules (Proposition 6.2.23). If M is a finitely generated torsion module, then $z^1(M) \in Z^1\mathfrak{D}(A)$ is the content $\chi(M)$ of M. Proposition 6.2.32 and Proposition 6.2.34 is then equivalent to the exactness of the sequence $K(\mathcal{C}^{\geq 2}) \longrightarrow K(\mathcal{C}^{\geq 1}) \longrightarrow Z^1 \longrightarrow 0$.

Example 7.1.18. The elements in $\mathcal{C}_0$ are finitely generated A-modules of dimension 0, that is, of finite length. Let $\epsilon : Z_0 \to \mathbb{Z}$ be the evaluation map, then $\ell_A(M) = \epsilon(z_0(M))$, in view of Corollary 3.2.20.

Example 7.1.19. Let A be a integral domain of finite dimension. Let $d = \dim(A)$, then $\mathcal{C}_{\leq d} = \mathcal{C}$ and $Z_d = \mathbb{Z}\mathrm{Spec}(A)$. Also, we have $z_d(M) = \mathrm{rank}(M)\mathrm{Spec}(A) = z^0(M)$, whence the modules in $\mathcal{C}_{\leq d-1}$ are precisely the torsion A-modules.

7.1.2 Dimension and going-down property

Let $\rho : A \to B$ be a ring homomorphism. Recall that we say ρ has the going-down property if it satisfies the going down theorem.

Proposition 7.1.20. *Let $\rho : A \to B$ be a ring homomorphism with the going-down property. Let F be a closed subset of $\mathrm{Spec}(A)$. If Y is an irreducible component of the inverse image of F under $\rho^* : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$, then the closure of $\rho^*(Y)$ is an irreducible component of F.*

Proof. Let $\mathfrak{a}$ be an ideal of A such that $F = V(\mathfrak{a})$. By Proposition 1.4.20, the inverse image by ρ^* of F is the subset $V(\mathfrak{a}^e)$ of $\mathrm{Spec}(B)$. Let $Y = V(\mathfrak{P})$ be an irreducible component of $V(\mathfrak{a}^e)$, then the closure of $\rho^*(Y)$ is the irreducible closed subset $V(\mathfrak{P}^c)$. So we have to prove that, if $\mathfrak{P}$ is a prime ideal minimal among the prime ideals of B containing $\mathfrak{a}^e$, then $\mathfrak{P}^c$ is minimal among the prime ideals of A containing $\mathfrak{a}$.

Suppose the contrary, then there would exist a prime ideal $\mathfrak{q}$ of A with $\mathfrak{a} \subseteq \mathfrak{q} \subset \mathfrak{P}^c$. According to the going-down property, there exist a prime ideal $\mathfrak{Q}$ of B lying over $\mathfrak{q}$ such that $\mathfrak{Q} \subset \mathfrak{P}$, whence $\mathfrak{a}^e \subseteq \mathfrak{Q} \subset \mathfrak{P}$, contrary to the hypothesis made on $\mathfrak{P}$. □

Proposition 7.1.21. *Let $\rho : A \to B$ be a ring homomorphism with the going-down property. Then we have the inequality*

$$\dim(B) \geq \dim(A) + \inf_{\mathfrak{m} \in \mathrm{Max}(A)} \dim(B/\mathfrak{m}B) \geq \dim(A) + \inf_{\mathfrak{p} \in \mathrm{Spec}(A)} \dim(B \otimes_A \kappa(\mathfrak{p})). \tag{7.1.1}$$

Proof. We have $\dim(A) = \sup_{\mathfrak{m}} \dim(A_{\mathfrak{m}})$, so it suffices to show the inequality

$$\dim(B) \geq \dim(A_{\mathfrak{m}}) + \dim(B/\mathfrak{m}B) \tag{7.1.2}$$

for each maximal ideal $\mathfrak{m}$ of A. In other words, we have to prove the inequality $\dim(B) \geq n + r$ if $\mathfrak{p}_0 \subset \cdots \subset \mathfrak{p}_n$ is a chain of prime ideals of A contained in $\mathfrak{m}$ and $\bar{\mathfrak{Q}}_0 \subset \cdots \subset \bar{\mathfrak{Q}}_r$ is a chain of prime ideals of $B/\mathfrak{m}B$. For $1 \leq i \leq r$, there exists a prime ideal $\mathfrak{Q}_{n+i}$ of B containing $\mathfrak{m}B$ such that $\bar{\mathfrak{Q}}_i = \mathfrak{Q}_{n+i}/\mathfrak{m}B$, and $\mathfrak{Q}_n \subset \cdots \subset \mathfrak{Q}_{n+r}$ is a chain of prime ideals of B. Let $\mathfrak{Q}$ be a prime ideal lying over $\mathfrak{m}$, then by the going down property, there exists prime ideals $\mathfrak{Q}_0 \subset \cdots \subset \mathfrak{Q}_{n-1} \subset \mathfrak{Q}$ such that $\mathfrak{Q}_i$ is lying over $\mathfrak{p}_i$ for $0 \leq i \leq n-1$. Then we see $\mathfrak{Q}_0 \subset \cdots \subset \mathfrak{Q}_{n-1} \subset \mathfrak{Q}_n \subset \cdots \subset \mathfrak{Q}_{n+r}$ is a chain of prime ideals of B, whence $\dim(B) \geq n + r$. □

Corollary 7.1.22. *Let $\rho : A \to B$ be a ring homomorphism with the going-down property. Then for any ideal $\mathfrak{a}$ of A we have $\mathrm{ht}(\mathfrak{a}) \leq \mathrm{ht}(\mathfrak{a}^e)$.*

Proof. Let $\mathfrak{P}$ be a prime ideal of B containing $\mathfrak{a}^e$, and $\mathfrak{p} = \mathfrak{P}^c$. By **??**, the homomorphism $\rho_{\mathfrak{p}} : A_{\mathfrak{p}} \to B_{\mathfrak{P}}$ satisfies condition (PM), whence $\dim(A_{\mathfrak{p}}) \leq \dim(B_{\mathfrak{P}})$ by Proposition 7.1.21. But $\mathrm{ht}(\mathfrak{a}) \leq \dim(A_{\mathfrak{p}})$ since $\mathfrak{p}$ contains $\mathfrak{a}$, so $\mathrm{ht}(\mathfrak{a}) \leq \dim(B_{\mathfrak{P}})$ for any prime ideal $\mathfrak{P}$ containing $\mathfrak{a}^e$, whence $\mathrm{ht}(\mathfrak{a}) \leq \mathrm{ht}(\mathfrak{a}^e)$. □

Remark 7.1.23. Let $\rho : A \to B$ be a local homomorphism of Noetherian local rings satisfying the going down property. Then (Corollary 7.2.27) the equality holds in (7.1.1). On the other hand, the inequality in (7.1.1) can be strict (c.f. Exercise 7.1.11).

7.1.3 Dimension of finite type algebras

Proposition 7.1.24. *Let $\rho : A \to B$ be a ring homomorphism. If $n = \sup_{\mathfrak{p}} \dim(B \otimes_A \kappa(\mathfrak{p}))$ with $\mathfrak{p}$ taking over prime ideals of A, then we have the inequality*

$$\dim(B) + 1 \leq (n+1)(\dim(A) + 1).$$

In particular, if $\dim(A)$ is finite and $\dim(B \otimes_A \kappa(\mathfrak{p}))$ is bounded from above, then $\dim(B)$ is finite.

Proof. We may suppose that $\dim(A) < +\infty$ and $n < +\infty$. Let $\mathfrak{P}_0 \subset \cdots \subset \mathfrak{P}_m$ be a chain of prime ideals of B and set $\mathfrak{p}_i = \mathfrak{P}_i^c$. Then the sequence $\mathfrak{p}_i$ is increasing, so the set of its values is of cardinal $\leq \dim(A) + 1$. For each $\mathfrak{p} \in \operatorname{Spec}(A)$, the set of $\mathfrak{P}_i$ such that $\mathfrak{P}_i = \mathfrak{p}$ is a chain of the subset $\rho^{*-1}(\mathfrak{p})$ of $\operatorname{Spec}(B)$, therefore has cardinality less than $\dim(B \otimes_A \kappa(\mathfrak{p})) + 1$, and concequent to $(n+1)$. It follows that $m + 1 \leq (n+1)(\dim(A) + 1)$, hence the proposition. □

Corollary 7.1.25. *Let A be a ring, then*

$$\dim(A) + 1 \leq \dim(A[X]) \leq 2\dim(A) + 1.$$

Proof. One direction is established. For the other, note that for any prime ideal $\mathfrak{p} \in \operatorname{Spec}(A)$, $A[X] \otimes_A \kappa(\mathfrak{p})$ is isomorphic to $\kappa(\mathfrak{p})[X]$, which is a PID hence of dimension 1. Thus the other side comes from Proposition 7.1.24. □

Corollary 7.1.26. *Let A be a ring and B a finitely generated A-algebra. If $\dim(A) < +\infty$, then we also have $\dim(B) < +\infty$.*

Proof. It follows from Corollary 7.1.25 that the polynomial ring $A[X_1, \dots, X_n]$ has finite dimension for all n, whence $\dim(B) < +\infty$. □

Lemma 7.1.27. *Let $\rho : A \to B$ be a ring homomorphism such that, for each prime ideal $\mathfrak{p}$ of A, the algebra $B \otimes_A \kappa(\mathfrak{p})$ is integral over $\kappa(\mathfrak{p})$. If $\mathfrak{P}$ and $\mathfrak{Q}$ are prime ideals of B such that $\mathfrak{P} \subset \mathfrak{Q}$, then $\mathfrak{P}^c \neq \mathfrak{Q}^c$.*

Proof. If $\mathfrak{P}^c = \mathfrak{Q}^c = \mathfrak{p}$, then $\mathfrak{P} \subset \mathfrak{Q}$ is a chain in $B \otimes_A \kappa(\mathfrak{p})$, whence $\dim(B \otimes_A \kappa(\mathfrak{p})) \geq 1$. But since $\kappa(\mathfrak{p})$ is a field and $B \otimes_A \kappa(\mathfrak{p})$ is integral over $\kappa(\mathfrak{p})$, we have $\dim(B \otimes_A \kappa(\mathfrak{p})) = 0$, a contradiction. □

Theorem 7.1.28. *Let $\rho : A \to B$ be a ring homomorphism such that B is integral over A, and $\rho^* : \operatorname{Spec}(B) \to \operatorname{Spec}(A)$ the induced map.*

(a) Let M be a finitely generated A-module. Then $\dim_B(M \otimes_A B) \leq \dim_A(M)$. If ρ^* is surjective, then $\dim_B(M \otimes_A B) = \dim_A(M)$.

(b) Let $\mathfrak{b}$ be an ideal of B. Then $\operatorname{ht}(\mathfrak{b}) \leq \operatorname{ht}(\mathfrak{b}^c)$ and $\dim(B/\mathfrak{b}) = \dim(A/\mathfrak{b}^c)$. If ρ^* is surjective, then $\operatorname{ht}(\mathfrak{a}^e) \leq \operatorname{ht}(\mathfrak{a})$ for any ideal $\mathfrak{a}$ of A.

(c) Suppose that B is a finitely generated A-module and let N be a finitely generated B-module. Then we have $\dim_B(N) = \dim_A(N)$. In particular, we have $\dim(B) = \dim_A(B)$.

Proof. By Corollary 4.1.17, the algebra $B \otimes_A \kappa(\mathfrak{p})$ is integral over $\kappa(\mathfrak{p})$. Let $\mathfrak{P}_0 \subset \cdots \subset \mathfrak{P}_m$ be a chain of prime ideals of B. Then by Lemma 7.1.27, the ideals $\mathfrak{p}_i = \mathfrak{P}_i^c$ are pairwise distinct, whence $\mathfrak{p}_0 \subset \cdots \subset \mathfrak{p}_m$ is a chain of prime ideals of A, and $m \leq \dim(A)$. This proves $\dim(B) \leq \dim(A)$.

Suppose further that ρ^* is surjective. Let $\mathfrak{p}_0 \subset \cdots \subset \mathfrak{p}_n$ be a chain of prime ideals of A. Then there exists a prime ideal $\mathfrak{P}_0$ of B lying over of $\mathfrak{p}_0$. According to Corollary 4.1.70, we can construct, by induction on n, a chain $\mathfrak{P}_0 \subset \cdots \subset \mathfrak{P}_n$ of prime ideals of B such that $\mathfrak{P}_i$ is lying over $\mathfrak{p}_i$ for each i. We therefore have $n \leq \dim(B)$ and consequently $\dim(A) = \dim(B)$.

This proves (a) in the case $M = A$. In the general case, let $\mathfrak{a}$ be the annihilator of M, so that the support of M is identified with $\operatorname{Spec}(A/\mathfrak{a})$, and we have $\dim_A(M) = \dim(A/\mathfrak{a})$. According to Proposition 1.4.42, the support of $M \otimes_A B$ is the inverse image by ρ^* of the support of M, therefore identifies with $\operatorname{Spec}(B/\mathfrak{a}^e)$, and we have $\dim_B(M \otimes_A B) = \dim(B/\mathfrak{a}^e)$. It remains to notice that the homomorphism $\bar{\rho} : A/\mathfrak{a} \to B/\mathfrak{a}^e$ deduced from ρ makes $B/\mathfrak{a}^e$ an integral $(A/\mathfrak{a})$-algebra, and that $\bar{\rho}^*$ is surjective when ρ^* is.

Now we prove (b). By the definition of height, it suffices to prove that $\operatorname{ht}(\mathfrak{b}) \leq \dim(A_{\mathfrak{p}})$ for any prime ideal $\mathfrak{p}$ of A containing $\mathfrak{b}^c$. Let $\mathfrak{p}$ be such an ideal, according to Corollary 4.1.70, there exists a prime ideal $\mathfrak{P}$ of B above $\mathfrak{p}$ and containing $\mathfrak{b}$, and we have $\mathrm{ht}(\mathfrak{b}) \leq \dim(B_{\mathfrak{P}})$. Now $B_{\mathfrak{P}}$ is identified with a ring of fractions of the integral $A_{\mathfrak{p}}$-algebra $B \otimes_A A_{\mathfrak{p}}$, from which we get

$$\dim(B_{\mathfrak{P}}) \leq \dim(B \otimes_A A_{\mathfrak{p}}) \leq \dim(A_{\mathfrak{p}})$$

by (a) and Proposition 7.1.3. We have thus proved the inequality $\mathrm{ht}(\mathfrak{b}) \leq \mathrm{ht}(\mathfrak{b}^c)$. Moreover, the homomorphism of $A/\mathfrak{a}$ in $B/\mathfrak{b}$ deduced from ρ is injective and makes $B/\mathfrak{b}$ an integral $(A/\mathfrak{a})$-algebra. We therefore have $\dim(B/\mathfrak{b}) = \dim(A/\mathfrak{a})$ according to (a). Assume now ρ^* surjective and let $\mathfrak{a}$ be an ideal of A and $\mathfrak{p}$ a prime ideal of A containing $\mathfrak{a}$. There exists by hypothesis a prime ideal $\mathfrak{P}$ of B above $\mathfrak{p}$. We have $\mathfrak{a}^e \subseteq \mathfrak{P}$, hence $\mathrm{ht}(\mathfrak{a}^e) \leq \mathrm{ht}(\mathfrak{P}) \leq \mathrm{ht}(\mathfrak{p})$ according to the above argument. Passing to the infimum, we obtain $\mathrm{ht}(\mathfrak{a}^e) \leq \mathrm{ht}(\mathfrak{a})$. Now part (c) follows from (b) by letting $\mathfrak{b}$ be the annihilator of N and noting that the annihilator of $\rho^*(N)$ is then $\mathfrak{b}^c$. $\square$

Theorem 7.1.29. *Let A be an integrally closed ring, B a ring containing A and integral over A. Suppose that B is a torsion-free A-module, then*

(a) For any ideal $\mathfrak{a}$ of A, we have $\mathrm{ht}(\mathfrak{a}) = \mathrm{ht}(\mathfrak{a}^e)$.

(b) For any ideal $\mathfrak{b}$ of B, we have $\mathrm{ht}(\mathfrak{b}) = \mathrm{ht}(\mathfrak{b}^c)$.

Proof. Let $\rho : A \to B$ be the canonical homomorphism. Let $\mathfrak{a}$ be an ideal of A. If $\mathfrak{a} = A$, then (a) is clear by Theorem 7.1.28. Suppose $\mathfrak{a} \neq A$, so that $\mathfrak{a}^e \neq B$. Since ρ is injective, ρ^* is surjective by Theorem 4.1.69. Let $\mathfrak{P}$ be a prime ideal of B containing $\mathfrak{a}^e$ and $\mathfrak{p} = \mathfrak{P} \cap A$. We have $\mathfrak{a} \subseteq \mathfrak{p}$, whence $\mathrm{ht}(\mathfrak{a}) \leq \mathrm{ht}(\mathfrak{p})$. Let $\mathfrak{p}_0 \subset \cdots \subset \mathfrak{p}_n$ be a chain of prime ideals of A with $\mathfrak{p}_n = \mathfrak{p}$. According to the going down theorem (Corollary 4.1.80), we have a chain $\mathfrak{P}_0 \subset \cdots \subset \mathfrak{P}_n$ of prime ideals of B such that $\mathfrak{P}_n = \mathfrak{P}$ and $\mathfrak{P}_i$ is above $\mathfrak{p}_i$. We then have $n \leq \mathrm{ht}(\mathfrak{P})$, whence $\mathrm{ht}(\mathfrak{a}) \leq \mathrm{ht}(\mathfrak{P})$. By taking infimum we get $\mathrm{ht}(\mathfrak{a}) \leq \mathrm{ht}(\mathfrak{a}^e)$. The inequality $\mathrm{ht}(\mathfrak{a}^e) \leq \mathrm{ht}(\mathfrak{a})$ follows from Theorem 7.1.28, whence the first equality.

Let $\mathfrak{b}$ be an ideal of B and $\mathfrak{a} = \mathfrak{b}^c$. We have $\mathfrak{a}^e \subseteq \mathfrak{b}$, hence $\mathrm{ht}(\mathfrak{a}) = \mathrm{ht}(\mathfrak{a}^e) \leq \mathrm{ht}(\mathfrak{b})$. The inequality $\mathrm{ht}(\mathfrak{b}) \leq \mathrm{ht}(\mathfrak{a})$ follows from Theorem 7.1.28, hence the theorem. $\square$

7.1.4 Dimension of finite type algebras over a field

As a special case of the materials discussed in the last paragraph, we now consider finite type algebras over a field k. As we shall see, the dimension theory for such algebras is extremely simple and beautiful as it connects with the transcendental degree of such algebras.

Lemma 7.1.30 (Noether's Normalization Theorem). *Let A be a finitely generated k-algebra and $\mathfrak{p}_0 \subset \cdots \subset \mathfrak{p}_m$ a maximal chain of prime ideals of A. Then there exists an integer $n \geq m$ and a sequence $(x_i)_{1 \leq i \leq n}$ of elements of A algebraically independent over k such that*

(a) A is integral over $B = k[x_1, \dots, x_n]$.

(b) The ideal $\mathfrak{p}_j \cap B$ is generated by $x_1, \dots, x_{n-m+j}$.

Proof. By Theorem 4.3.1, there exists an integer $n \geq 0$, a sequence $(x_i)_{1 \leq i \leq n}$ of elements of A algebraically independent over k and an increasing sequence $(h_j)_{1 \leq j \leq m}$ such that A is integral over $B = k[x_1, \dots, x_n]$ and $\mathfrak{q}_j = \mathfrak{p}_j \cap B$ is generated by $x_1, \dots, x_{h_j}$. By passing to the quotients, we deduce from the injective canonical homomorphism of B into A an injective homomorphism from $B/\mathfrak{q}_j$ into $A/\mathfrak{p}_j$ which makes $A/\mathfrak{p}_j$ an integral $(B/\mathfrak{q}_j)$-algebra. As the ring $B/\mathfrak{q}_j$ is isomorphic to a polynomial algebra in $n - h_j$ indeterminate over k, it is integrally closed. According to Theorem 7.1.29, we therefore have

$$1 = \mathrm{ht}(\mathfrak{p}_{j+1}/\mathfrak{p}_j) = \mathrm{ht}(\mathfrak{q}_{j+1}/\mathfrak{q}_j) \geq h_{j+1} - h_j.$$

whence $h_{j+1} \leq h_j + 1$. Since h_j is increasing and the ideals $\mathfrak{q}_j$ are pairwise distinct, we get $h_{j+1} = h_j + 1$. Also note that $h_m = n$ since $\mathfrak{q}_m$ is maximal ($\mathfrak{p}_m$ is maximal), so $h_j = n - m + j$ and the claim follows. $\square$

Theorem 7.1.31. *Let A be a finitely generated k-algebra.*

(a) For every minimal prime ideal $\mathfrak{p}$ of A, the maximal chains of prime ideals of A starting from $\mathfrak{p}$ have length $\operatorname{tr.deg}_k(\kappa(\mathfrak{p}))$.

(b) The ring A is catenary and its dimension is the supremum of the integers $\operatorname{tr.deg}_k(\kappa(\mathfrak{p}))$, *where $\mathfrak{p}$ runs through minimal prime ideals of A.*

(c) Suppose that A is an integral domain. Then the maximal chains of prime ideals of A have the same length, and the dimension of A equals to $\operatorname{tr.deg}_k(K)$, *where K is the field of fraction of A.*

Proof. Suppose that A is an integral domain and consider a maximal chain $\mathfrak{p}_0 \subset \cdots \subset \mathfrak{p}_m$ of prime ideals of A. We have $\mathfrak{p}_0 = 0$. We then deduce from Lemma 7.1.30 the existence of a injective homomorphism $\varphi : k[X_1, \dots, X_m] \to A$ of k-algebras which makes A an integral $k[X_1, \dots, X_m]$-algebra. Consequently, the transcendence degree over k of the field of fractions of A is equal to m, hence (c). Assertion (a) follows from (c) applied to the ring $A/\mathfrak{p}$ and assertion (b) is an immediate consequence of (a). □

Corollary 7.1.32. *Let n be a positive integer, then $\dim(k[X_1, \dots, X_n]) = n$. Moreover, for a finitely generated k-algebra A to be of dimension n, it is necessary and sufficient that there exists a injective k-homomorphism $\varphi : k[X_1, \dots, X_n] \to A$ such that A is finite over $k[X_1, \dots, X_n]$.*

Proof. This follows from Theorem 7.1.31, Theorem 4.3.1, and Theorem 7.1.28(a). □

Corollary 7.1.33. *Let A be a finitely generated k-algebra that is an integral domain. Then for any prime ideal $\mathfrak{p}$ of A, we have*

$$\operatorname{ht}(\mathfrak{p}) = \dim(A_{\mathfrak{p}}) = \dim(A) - \dim(A/\mathfrak{p}) = \dim(A) - \operatorname{tr.deg}_k(\kappa(\mathfrak{p})).$$

Proof. This is a concequence of Theorem 7.1.31 and Proposition 7.1.9. □

Corollary 7.1.34. *Let A be a finitely generated k-algebra and let f be an element of A which does not belong to any minimal prime ideal of A. Then $\dim(A) = \dim(A_f)$.*

Proof. The map $\mathfrak{p} \mapsto \mathfrak{p}A_f$ is a bijection between the set of minimal prime ideals of A and that of A_f. Since $A/\mathfrak{p}$ and $A_f/\mathfrak{p}A_f$ have the same field of fraction, it suffices to apply Theorem 7.1.31. □

Corollary 7.1.35. *Let A be a finitely generated k-algebra and $\mathfrak{p}$ a prime ideal of A.*

(a) For $\mathfrak{p}$ to be maximal, it is necessary and sufficient that $\kappa(\mathfrak{p})$ is a finite extension of k.

(b) Let $f \in A - \mathfrak{p}$. Then the ideal $\mathfrak{p}$ is a maximal ideal of A if and only if $\mathfrak{p}A_f$ is a maximal ideal of A_f.

Proof. If $\mathfrak{p}$ is a maixmal ideal of A, then $A/\mathfrak{p}$ is a field, whence a finitely generated extension of k of transcendence degree 0. It is therefore a finite extension of k (Theorem 7.1.31). Conversely, if the field $\kappa(\mathfrak{p})$ is a finite extension of k, we have $\dim(A/\mathfrak{p}) = 0$, so $\mathfrak{p}$ is maximal. The assertion (b) now follows from (a), taking into account that $A/\mathfrak{p}$ and $A_f/\mathfrak{p}A_f$ have the same field of fractions. □

Corollary 7.1.36. *Let A be a finitely generated k-algebra, $\mathfrak{p}$ a prime ideal of A and $(\mathfrak{p}_i)_{i \in I}$ the family of minimal prime ideals of A. Then we have*

$$\dim_{\mathfrak{p}}(A) = \sup_{i \in I_{\mathfrak{p}}} \dim(A/\mathfrak{p}_i) = \dim(A_{\mathfrak{p}}) + \dim(A/\mathfrak{p}) = \dim(A_{\mathfrak{p}}) + \operatorname{tr.deg}_k(\kappa(\mathfrak{p}))$$

where $I_{\mathfrak{p}}$ is the subset consists of $i \in I$ such that $\mathfrak{p}_i \subseteq \mathfrak{p}$.

Proof. We have $\dim_{\mathfrak{p}}(A) = \sup_{i \in I_{\mathfrak{p}}} \dim_{\mathfrak{p}/\mathfrak{p}_i}(A/\mathfrak{p}_i)$ by Proposition 7.1.3, since A is Noetherian. Also, from Corollary 7.1.34 we conclude that

$$\dim_{\mathfrak{p}/\mathfrak{p}_i}(A/\mathfrak{p}_i) = \inf_{f \notin \mathfrak{p}/\mathfrak{p}_i} \dim((A/\mathfrak{p}_i)_f) = \dim(A/\mathfrak{p}_i)$$

so the first equality follows. Now since $A/\mathfrak{p}_i$ is an integral domain, Corollary 7.1.33 shows $\dim(A/\mathfrak{p}_i) = \dim((A/\mathfrak{p}_i)_{\mathfrak{p}}) + \dim(A/\mathfrak{p})$. The other equalities then follows from the observation that $\dim(A_{\mathfrak{p}}) = \sup_{i \in I_{\mathfrak{p}}} \dim((A/\mathfrak{p}_i)_{\mathfrak{p}})$ and Theorem 7.1.31. □

Corollary 7.1.37. *Let A be a finitely generated k-algebra, n a positive integer. Then the following conditions are equivalent:*

(i) For every $\mathfrak{p} \in \operatorname{Ass}(A)$, we have $\dim(A/\mathfrak{p}) = n$.

(ii) Every prime ideal associated with A is minimal and all irreducible components of $\operatorname{Spec}(A)$ are of dimension n.

(iii) There exists an injective homomorphism $\varphi : k[X_1, \dots, X_n] \to A$ such that A is a finitely generated torsion-free $k[X_1, \dots, X_n]$-module.

Proof. The equivalence of (i) and (ii) is immediate. Now suppose that (ii) holds. Then $\dim(A) = n$ and by Corollary 7.1.32 there is an injective k-homomorphism

$$\varphi : k[X_1, \dots, X_n] \to A$$

such that A is finite over $k[X_1, \dots, X_n]$. For any prime ideal $\mathfrak{p} \in \operatorname{Ass}(A)$, the ring $A/\mathfrak{p}$ is then integral over $k[X_1, \dots, X_n]$, and we have so $n = \dim(A/\mathfrak{p}) = \dim(k[X_1, \dots, X_n]/\varphi^{-1}(\mathfrak{p}))$ according to Theorem 7.1.28(a), which proves $\varphi^{-1}(\mathfrak{p}) = 0$. By Corollary 3.1.6, this shows the image by the injective homomorphism φ of a nonzero element of $k[X_1, \dots, X_n]$ is not a divisor of 0 in A, whence (c).

Conversely, assume that (iii) holds. For each prime ideal $\mathfrak{p} \in \operatorname{Ass}(A)$, the homomorphism $\varphi : k[X_1, \dots, X_n] \to A/\mathfrak{p}$ induced by φ is injective by Corollary 3.1.6, whence $\dim(A/\mathfrak{p}) = n$ by Corollary 7.1.32. □

Corollary 7.1.38. *Under the condition of Corollary 7.1.37, $\dim_{\mathfrak{p}}(A) = \dim(A)$ for every prime ideal $\mathfrak{p}$ of A.*

Proof. This follows from Corollary 7.1.36 and Corollary 7.1.37. □

Proposition 7.1.39. *Let A and B be finitely generated k-algebras and $\rho : A \to B$ a k-algebra homomorphism. Suppose that A is an integral domain and B is a torsion-free A-module, and let K be the field of fraction of A. Then we have*

$$\dim(B) = \dim(A) + \dim(B \otimes_A K).$$

Proof. Suppose that B is an integral domain. The algebra $B \otimes_A K$ is then a ring of fractions of B defined by a multiplicative part not containing 0. It therefore has field of fractions the field of fractions L of B. According to Theorem 7.1.31, we have

$$\dim(B) = \operatorname{tr.deg}_k(L), \quad \dim(A) = \operatorname{tr.deg}_k(K), \quad \dim(B \otimes_A K) = \operatorname{tr.deg}_K(L).$$

By the transitivity of transcendental degree, we see the claim is true in this case.

Let's consider to the general case. Every minimal prime ideal $\mathfrak{P}$ of B is formed by divisors of zero in B, so is lying over the ideal 0 of A. By Proposition 1.2.37 it follows that the map $\mathfrak{P} \mapsto \mathfrak{P}(B \otimes_A K)$ is a bijection of the set of minimal prime ideals of B on the set of minimal prime ideals of $B \otimes_A K$. The proposition follows therefore from the first part of the proof and from Proposition 7.1.3. □

Corollary 7.1.40. *Let $\rho : A \to B$ be an injective homomorphism of finitely generated k-algebras. Then* $\dim(A) \leq \dim(B)$.

Proof. In fact, let $\mathfrak{p}$ be a minimal prime of A such that $\dim(A) = \dim(A/\mathfrak{p})$. Then there exists a prime ideal $\mathfrak{P}$ of A lying over $\mathfrak{p}$ by Proposition 1.2.45. According to Proposition 7.1.39 applied to $A/\mathfrak{p}$ and $B/\mathfrak{P}$, we have $\dim(A) = \dim(A/\mathfrak{p}) \leq \dim(B/\mathfrak{P}) \leq \dim(B)$, whence the corollary. □

Lemma 7.1.41. *Let A and B be k-algebras that are integral domains, M a torsion-free A-module, N a torsion-free B-module. Then the ring $A \otimes_k B$ is an integral domain, and $M \otimes_k N$ is a torsion-free $A \otimes_k B$ module.*

Proof. Let K (resp. L) be the field of fraction of A (resp. B). Then there exists a set I (resp. J) such that M (resp. N) is isomorphic to a submodule of $K^{\oplus I}$ (resp. $L^{\oplus J}$). The $(A \otimes_k B)$-module $M \otimes_k N$ is then isomorphic to a submodule of $K^{\oplus I} \otimes_k L^{\oplus J}$, which is isomorphic to $(K \otimes_k L)^{\oplus (I \times J)}$. As $K \otimes_k L$ is a ring of fractions of the ring $A \otimes_k B$, it is a torsion-free $A \otimes_k B$-module, hence the lemma. □

Proposition 7.1.42. *Let k' be an extension field of k, A a finitely generated k-algebra and B a finitely generated k'-algebra.*

(a) The k'-algebra $A \otimes_k B$ is finitely generated and we have

$$\dim(A \otimes_k B) = \dim(A) + \dim(B).$$

(b) Let $\mathfrak{r}$ be a prime ideal of $A \otimes_k B$. If $\mathfrak{p}$ (resp. $\mathfrak{q}$) is the contraction of $\mathfrak{r}$ to A (resp. B), then

$$\dim_{\mathfrak{r}}(A \otimes_k B) = \dim_{\mathfrak{p}}(A) + \dim_{\mathfrak{q}}(B).$$

Proof. Put $n = \dim(A)$ and $m = \dim(B)$. There exists according to Corollary 7.1.32 injective homomorphisms of algebras $\varphi : k[X_1, \dots, X_n] \to A$ and $\psi : k'[Y_1, \dots, Y_m] \to B$ making respectively of A and B finite algebras over $k[X_1, \dots, X_n]$ and $k'[Y_1, \dots, Y_m]$. The homomorphism $\varphi \otimes \psi$ is then injective and makes $A \otimes_k B$ a finite algebra over the k'-algebra $k[X_1, \dots, X_n] \otimes_k k'[Y_1, \dots, Y_m]$, which identifies with $k'[X_1, \dots, X_n, Y_1, \dots, Y_m]$ (to see that $\varphi \otimes \psi$ is injective, we may consider the following commutative diagram

$$\begin{tikzcd}
k[X] \otimes k'[Y] \arrow[r, "1\otimes\psi"] \arrow[d, "\varphi\otimes 1"'] & k[X] \otimes B \arrow[d, "\varphi\otimes 1"] \\
A \otimes k'[Y] \arrow[r, "1\otimes\psi"] & A \otimes B
\end{tikzcd}$$

where $\varphi \otimes \psi$ is the composition homomorphism, and note that all modules are flat since we are tensoring over k). We then have $\dim(A \otimes_k B) = n + m$ by Corollary 7.1.32, which proves (a).

Note that when A and B are integral domains, the $k'[X_1, \dots, X_n, Y_1, \dots, Y_m]$-module $A \otimes_k B$ is torsion free by Lemma 7.1.41 and therfore we have

$$\dim_{\mathfrak{r}}(A \otimes_k B) = \dim(A \otimes_k B) = \dim(A) + \dim(B)$$

for any prime ideal $\mathfrak{r}$ of $A \otimes_k B$ by Corollary 7.1.38.

Now let us show (b). Let $\mathfrak{r}_0$ be a minimal prime ideal of $A \otimes_k B$ contained in $\mathfrak{r}$, and let $\mathfrak{p}_0$ (resp. $\mathfrak{q}_0$) be the contraction of $\mathfrak{r}_0$ to A (resp. B). The ring $(A \otimes_k B)/\mathfrak{r}_0$ is isomorphic to a quotient of the ring $(A/\mathfrak{p}_0) \otimes_k (B/\mathfrak{q}_0)$. Thus we have, by (a),

$$\dim((A \otimes_k B)/\mathfrak{r}_0) \leq \dim((A/\mathfrak{p}_0) \otimes_k (B/\mathfrak{q}_0)) = \dim(A/\mathfrak{p}_0) + \dim(B/\mathfrak{q}_0).$$

According to Corollary 7.1.36, this shows

$$\dim_{\mathfrak{r}}(A \otimes_k B) \leq \dim_{\mathfrak{p}}(A) + \dim_{\mathfrak{q}}(B).$$

Conversely, let $\mathfrak{p}_0$ (resp. $\mathfrak{q}_0$) be a minimal prime ideal of A (resp. B) contained in $\mathfrak{p}$ (resp. $\mathfrak{q}$). By the observation above, we have

$$\dim(A/\mathfrak{p}_0) + \dim(B/\mathfrak{q}_0) = \dim_{\bar{\mathfrak{r}}}((A/\mathfrak{p}_0) \otimes_k (B/\mathfrak{q}_0)) \leq \dim_{\mathfrak{r}}(A \otimes_k B)$$

where $\bar{\mathfrak{r}}$ is the image of $\mathfrak{r}$ by the canonical surjection $A \otimes_k B \to (A/\mathfrak{p}_0) \otimes_k (B/\mathfrak{q}_0)$. Applying Corollary 7.1.36, we deduce the inequality

$$\dim_{\mathfrak{p}}(A) + \dim_{\mathfrak{q}}(B) \leq \dim_{\mathfrak{r}}(A \otimes_k B)$$

whence (b) follows. □

Corollary 7.1.43. *Let A be a finitely generated k-algebra, k' an extension field of k, and A' the k'-algebra $A \otimes_k k'$.*

(a) We have $\dim(A') = \dim(A)$.

(b) Let $\mathfrak{p}'$ be a prime ideal of A and $\mathfrak{p}$ its contraction to A. Then $\dim_{\mathfrak{p}'}(A') = \dim_{\mathfrak{p}}(A)$.

(c) Let $\mathfrak{p}'$ be a minimal prime ideal of A and $\mathfrak{p}$ be the contraction of $\mathfrak{p}'$. Then $\mathfrak{p}$ is minimal and $\dim(A'/\mathfrak{p}') = \dim(A/\mathfrak{p})$.

(d) If k' is a purely inseparable extension of k, then the canonical map ${}^a\rho : \mathrm{Spec}(A') \to \mathrm{Spec}(A)$ is a homeomorphism.

Proof. Assertions (a) and (b) follow from Proposition 7.1.42 by setting $B = k'$. For (c), note that the homomorphism $\nu : k \to k'$ satisfies condition (PM), so the ideal $\mathfrak{p}'$ is minimal by **??**(a) and (c). By corollary 7.1.36, we then have

$$\dim(A'/\mathfrak{p}') = \dim_{\mathfrak{p}'}(A') = \dim_{\mathfrak{p}}(A) = \dim(A/\mathfrak{p})$$

which completes the proof of (c).

Finally, assume that k'/k is purely inseparable and let $\rho : A \to A'$ be the canonical homomorphism. Let $\mathfrak{p} \in \mathrm{Spec}(A)$, recall that the fiber of $\mathfrak{p}$ under ${}^a\rho$ is homeomorphic to $\mathrm{Spec}(\kappa(\mathfrak{p}) \otimes_k k')$. Now the set $\mathfrak{n}$ of nilpotent elements of $\kappa(\mathfrak{p}) \otimes_k k'$ is a prime ideal (A, V, p.134, corollarie) and the quotient ring $(\kappa(\mathfrak{p}) \otimes_k k')/\mathfrak{n}$ is a field by (A, V, p.16, cor.1 et p.10, prop.1). Therefore $({}^a\rho)^{-1}(\mathfrak{p})$ is reduced to a singleton. It follows that the map ${}^a\rho$ is a bijection of $\mathrm{Spec}(A')$ on $\mathrm{Spec}(A)$. Since we have observed that ρ has the going up property (**??**), it follows from Proposition 4.1.84 that ${}^a\rho$ is a closed map, whence a homeomorphism. □

Theorem 7.1.44 (Grothendieck's Generic Freeness Lemma). *Let A be a Noetherian domain, B be an A-algebra of finite type, and M be a finitely generated B-module. Then there exists a nonzero element $f \in A$ such that M_f is a free A_f-module.*

Proof. Let K be the fraction field of A; then $B \otimes_A K$ is a finite type algebra over K and $M \otimes_A K$ is a finitely generated $(B \otimes_A K)$-module. We preceed by induction on the dimension $d = \dim(M \otimes_A K)$. If $d = -\infty$, we have $M \otimes_A K = 0$, which means M is annihilated by an nonzero element of $f \in A$, so $M_f = 0$ and the theorem is trivially satisfied.

Now assume that $d \geq 0$. By Proposition 3.1.17, there exists a filtration $(M_i)_{0 \leq i \leq n}$ of the B-module M such that $N_i = M_i/M_{i+1} \cong B/\mathfrak{P}_i$, where $\mathfrak{P}_i$ is a prime ideal of B. If the theorem is proved for each N_i, then there exists for each i an element $f_i \in A$ such that $(N_i)_{f_i}$ is free over A_{f_i}. If we set $f = f_1 \cdots f_{n-1}$, then $(N_i)_f$ is a free A_f-module for $0 \leq i \leq n-1$, and since $(N_i)_f = (M_i)_f/(M_{i+1})_f$, the A_f-module M_f then possesses a filtration by free modules, hence is free. By replacing B with $B/\mathfrak{P}$ (where $\mathfrak{P}$ is a prime ideal of B), which is of finite type over A, we see that we are reduced to the case where $M = B$ and B is an integral domain. By Corollary 4.3.2, there then exists a nonzero element $g \in A$ and algebraically indepedent elements $x_1, \dots, x_m$ of B such that B_g is integral over $A_g[x_1, \dots, x_m]$. By replacing A by A_g and B by B_g, we may assume that B is integral over $C = A[x_1, \dots, x_m]$, hence a finitely generated torsion-free C-module. By Corollary 7.1.32, we also note that the dimension of $B \otimes_A K$ is equal to m, so we have $m = d$.

Now if r is the rank of the torsion-free C-module B, there exists an exact sequence of C-modules

$$0 \longrightarrow C^r \longrightarrow B \longrightarrow M' \longrightarrow 0$$

where M' is a finitely generated torsion C-module. The support of M' then does not contain the generic point of $\mathrm{Spec}(C)$, and therefore the support of $M' \otimes_A K$ does not contain the generic point of $\mathrm{Spec}(C \otimes_A K)$ (Proposition 1.4.42). We then conclude that $\dim_C(M') < d$, so by the induction hypotheses, there exists a nonzero element $f \in A$ such that M'_f is free over A_f. Since C_f is also free over A_f (a basis of C_f is the image of the monomials in the x_i), we then conclude that B_f is a free A_f-module. □

7.1.5 Exercise

Exercise 7.1.1. In this exercise, we develop a way to measure the complexity of a partially ordered set.

(a) Prove that, for any partially ordered set E we can associate an element of $\mathbb{N} \cup \{\pm\infty\}$, denoted by $\mathrm{dev}(E)$, and called the **deviation** of E, which satisfies the following conditions:

  (α) A trivial poset (one in which no two elements are comparable) has deviation $-\infty$.

  (β) A poset E is said to have deviation at most n (for a positive integer n) if for every descending chain of elements $(a_k)_{k\in\mathbb{N}}$, all but a finite number of the posets (a_k, a_{k+1}) have deviation less than n. The deviation of E (if it exists) is the minimum value of α for which this is true.

(b) Show that for $\mathrm{dev}(E) \le 0$, it is necessary and sufficient that any decreasing sequence of E is stationary. We have $\mathrm{dev}(\mathbb{N}) = 0$, $\mathrm{dev}(\mathbb{Z}) = 1$, $\mathrm{dev}(\mathbb{Q}) = +\infty$.

(c) Let E and F be partially ordered sets. Show that if there exists an increasing map from E to F, then $\mathrm{dev}(E) \le \mathrm{dev}(F)$. Show that $\mathrm{dev}(E \times F) = \sup\{\mathrm{dev}(E), \mathrm{dev}(F)\}$.

Exercise 7.1.2. Let A be a ring (not necessarily commutative), M a left A-module, we denote by $\mathrm{dev}(M)$ the deviation of the set of submodules of M, orederd by inclusion. We put $\mathrm{dev}(A)$ to be the deviation of the left A-module A.

(a) If N is a sub-A-module of M, then

$$\mathrm{dev}(M) = \sup\{\mathrm{dev}(N), \mathrm{dev}(M/N)\}.$$

(b) Suppose that A is commutative. If $\mathfrak{p}$ and $\mathfrak{q}$ are distinct prime ideals of A and $\mathfrak{p} \subseteq \mathfrak{q}$, then $\mathrm{dev}(A/\mathfrak{p}) > \mathrm{dev}(A/\mathfrak{q})$. In particular, $\dim(A) \le \mathrm{dev}(A)$.

(c) Suppose that A is commutative and Noetherian. Let $\mathscr{P}_A$ be the set of prime ideals of A such that $\dim(A/\mathfrak{p}) = \dim(A)$ and S be complement of the union of prime ideals in $\mathscr{P}_A$. Let M be a finitely generated A-module. If $S^{-1}M = 0$ then

$$\mathrm{dev}(M) \le \sup_{\mathfrak{p}\notin\mathscr{P}_A} \mathrm{dev}(A/\mathfrak{p}).$$

(d) Suppose that A is commutative and Noetherian. Prove that in this case we have $\dim(A) = \operatorname{dev}(A)$ and $\dim(M) = \operatorname{dev}(M)$ for any finitely generated A-module M.

Proof. For a module E, we let $\mathcal{L}(E)$ denote the lattice of submodules of E. Then if N is a submodule of M, there are ordered homomorphisms $\mathcal{L}(N) \to \mathcal{L}(M)$ and $\mathcal{L}(M/N) \to \mathcal{L}(M)$, whence

$$\operatorname{dev}(M) \geq \sup\{\operatorname{dev}(N), \operatorname{dev}(M/N)\}.$$

On the other hand, we have an ordered homomorphism

$$\mathcal{L}(M) \to \mathcal{L}(N) \times \mathcal{L}(M/N), \quad M' \mapsto (M' \cap N, M' + N/N).$$

This proves the equality in (a).

Suppose that A is commutative and let $\mathfrak{p} \subseteq \mathfrak{q}$ be distinct prime ideals. By replacing A with $A/\mathfrak{p}$, we may assume that $\mathfrak{p} = 0$. For any nonzero element $x \in \mathfrak{q}$, consider the sequence $(x^n A)$ in A. We have

$$\operatorname{dev}(A) > \operatorname{dev}(x^n A/x^{n+1}A) = \operatorname{dev}(A/xA) \geq \operatorname{dev}(A/\mathfrak{q})$$

so the claim in (b) follows. Now let $\mathfrak{p}_0 \subset \cdots \subset \mathfrak{p}_n$ be a saturated chain in A. Then $\operatorname{dev}(A/\mathfrak{p}_0) > \cdots > \operatorname{dev}(A/\mathfrak{p}_n)$, which shows $\operatorname{dev}(A) \geq n = \dim(A)$.

Suppose that A is Noetherian and let M be a finitely generated A-module. Then there is a composition series $(M_i)_{0 \leq i \leq n}$ of M such that M/M_{i+1} is isomomorphic to $A/\mathfrak{p}_i$. If $S^{-1}M = 0$, we have $\mathfrak{p}_i \notin \mathscr{P}_A$ for all i. Since $\operatorname{dev}(M) = \sup_i \operatorname{dev}(M_i/M_{i+1})$, we conclude that

$$\operatorname{dev}(M) \leq \sup_{\mathfrak{p} \notin \mathscr{P}_A} \operatorname{dev}(A/\mathfrak{p}).$$

Finally, we show that $\operatorname{dev}(A) = \dim(A)$ by induction on $\dim(A)$. So assume that $\operatorname{dev}(B) = \dim(B)$ for every Noetherian commutative B such that $\dim(B) < \dim(A)$. Then in the notation of (c), since $\dim(A/\mathfrak{p}) < \dim(A)$, we have $\operatorname{dev}(A/\mathfrak{p}) = \dim(A/\mathfrak{p})$ for $\mathfrak{p} \notin \mathscr{P}_A$, whence

$$\operatorname{dev}(M) \leq \sup_{\mathfrak{p} \in \mathscr{P}_A} \dim(A/\mathfrak{p}) < \dim(A) \tag{7.1.3}$$

if M is finitely generated and $S^{-1}M = 0$. Now let $(\mathfrak{a}_k)$ be a sequence of decreasing ideals in A. Since $S^{-1}A$ is Artinian (it has dimension zero), there exists an integer N such that

$$S^{-1}\mathfrak{a}_n = S^{-1}\mathfrak{a}_{n+1} = \cdots$$

for $n \geq N$. In other words, $S^{-1}(\mathfrak{a}_n/\mathfrak{a}_{n+1}) = 0$ for $n \geq N$, so by (7.1.2) we have $\operatorname{dev}(\mathfrak{a}_n/\mathfrak{a}_{n+1}) < \dim(A)$ for $n \geq N$. From the definition of $\operatorname{dev}(A)$, this shows $\operatorname{dev}(A) \leq \dim(A)$, which gives $\operatorname{dev}(A) = \dim(A)$.

Now let M be a finitely generated A-module and let $(M_i)_{0 \leq i \leq n}$ be the composition series of M such that $M_i/M_{i+1} \cong A/\mathfrak{p}_i$, where $\mathfrak{p}_i$ is a prime ideal of M. Then

$$\operatorname{dev}(M) = \sup_i\{\operatorname{dev}(M_i/M_{i+1})\} = \sup_i\{\operatorname{dev}(A/\mathfrak{p}_i)\} = \sup_i\{\dim(A/\mathfrak{p}_i)\}.$$

We also note that, since the minimal elements in $\{\mathfrak{p}_0, \ldots, \mathfrak{p}_{n-1}\}$ are the embedded primes for M, which correspond to irreducible components of $\mathfrak{a} = \operatorname{Ann}(M)$. Since $\dim(M) = \dim(V(\mathfrak{a}))$ is the supremum of the dimension of these components, we conclude that $\operatorname{dev}(M) = \dim(M)$. □

Exercise 7.1.3. Let A be a Noetherian commutative ring, $\mathfrak{a}$ an ideal of A contained in the Jacobson radical of A, and $\operatorname{gr}(A)$ the graded ring associated with the $\mathfrak{a}$-adic filtration. Associate to each ideal of A an ideal of $\operatorname{gr}(A)$, and deduce that $\operatorname{dev}(A) \leq \operatorname{dev}(\operatorname{gr}(A))$. Concequently, deduce that $\dim(A) \leq \dim(\operatorname{gr}(A))$.

Exercise 7.1.4. Let A be a ring such that

(a) For every maximal ideal $\mathfrak{m}$ of A, $A_{\mathfrak{m}}$ is Noetherian.

(b) Every nonzero element $x \in A$ is contained in a finite number of maximal ideals of A.

Show that A is Noetherian.

Proof. Let $(\mathfrak{a}_n)$ be an increasing sequence of ideals in A, and assume that $\mathfrak{a}_1 \neq 0$. Then by condition (b), the ideal $\mathfrak{a}_1$ is contained in finitely many maximal ideals, say $\mathfrak{m}_1, \dots, \mathfrak{m}_r$. Since each ring $A_{\mathfrak{m}_i}$ is Noetherian, there exists an integer N such that

$$\mathfrak{a}_n A_{\mathfrak{m}_i} = \mathfrak{a}_{n+1} A_{\mathfrak{m}_i} = \cdots$$

for $n \geq N$ and $i = 1, \dots, r$. But if $\mathfrak{m}$ is a maximal ideal other than $\mathfrak{m}_i$, then $\mathfrak{a}_n \not\subseteq \mathfrak{m}$ for all n, so

$$\mathfrak{a}_n A_{\mathfrak{m}} = \mathfrak{a}_{n+1} A_{\mathfrak{m}} = \cdots = A_{\mathfrak{m}}$$

trivially holds. By Corollary 1.3.21, we conclude that $\mathfrak{a}_n = \mathfrak{a}_{n+1} = \cdots$ for $n \geq N$, so A is Noetherian. □

Exercise 7.1.5. Let K be a field, $(n_i)_{i \geq 1}$ a strict increasing sequence of positive integers; for each $i \geq 0$, put $\mathfrak{p}_i$ to be the ideal of the ring $R = K[(X_j)_{j \in \mathbb{N}}]$ generated by the elements X_j where $n_1 + \cdots + n_i \leq j < n_1 + \cdots + n_{i+1}$. Let S be the complement of the union of the $\mathfrak{p}_i$'s, and consider the ring $A = S^{-1}R$. Prove that $S^{-1}R$ is Noetherian but has infinite dimension.

Proof. We first note that an nonzero element in $S^{-1}R$ can only be contained in finitely many $S^{-1}\mathfrak{p}_i$ (since this is true in R for $\mathfrak{p}_i$). Since the ideals of $S^{-1}A$ correspond to ideals of R contained in $\bigcup_i \mathfrak{p}_i$, and any such ideal can only be contained in finitely many $\mathfrak{p}_i$, by prime avoidence we conclude that the $S^{-1}\mathfrak{p}_i$ are the only maximal ideals in $S^{-1}R$. Moreover, for each $S^{-1}\mathfrak{p}_i$, $(S^{-1}R)_{S^{-1}\mathfrak{p}_i}$ is isomomorphic to $R_{\mathfrak{p}_i}$, which is Noetherian since it is just $K(x_J)[x_I]_{(x_I)}$ where I is indices appears in the generators of $\mathfrak{p}_i$ and J its complement. By Exercise 7.1.4, we conclude that $S^{-1}R$ is Noetherian. Now it is easy to see each maximal ideal $S^{-1}\mathfrak{p}_i$ has height $\geq n_i$ in $S^{-1}R$, so $S^{-1}R$ has infinite dimension. □

Exercise 7.1.6. Let A be the ring of germs at 0 of smooth functions on $\mathbb{R}$. We propose to show that the ring A is of infinite dimension. We denote by C the algebra of smooth functions on $\mathbb{R}$, F the ideal of functions zero on a neighborhood of 0, so that $A = C/F$.

(a) Let $(x_n)_{n \in \mathbb{N}}$ be a strict decreasing sequence of positive numbers tending to 0, and let $\mathcal{U}$ be an ultrafilter on $\mathbb{N}$ finer than the filter of complements of finite subsets. Show that the set J_1 of $f \in C$ such that the set of $n \in \mathbb{N}$ such that $f(x_n) = 0$ belongs to $\mathcal{U}$, is a prime ideal of C.

(b) We denote by $\tau_n(f)$ the function $x \mapsto f(x + x_n)$. Show that if J is a prime ideal of C, the set $\varphi(J)$ of $f \in C$ such that the set of $n \in \mathbb{N}$ where $\tau_n(f) \in J$ belongs to $\mathcal{U}$ is a prime ideal of C.

(c) We define a sequence I_n of ideals of C by $I_0 = J_1$ and $I_{n+1} = \varphi(I_n)$. Show that $(I_n)_{n \in \mathbb{N}}$ is a strict decreasing sequence of prime ideals of C containing F.

Proof. Let (x_n) and $\mathcal{U}$ be as in (a), and define J_1 by

$$J_1 = \{f \in C : \{n \in \mathbb{N} : f(x_n) = 0\} \in \mathcal{U}\}.$$

Now let $f, g \in C$ and assume that $fg \in J_1$; put

$$E_f = \{n \in \mathbb{N} : f(x_n) = 0\}, \quad E_g = \{n \in \mathbb{N} : g(x_n) = 0\}.$$

Then by definition, the set $E_{fg} = \{n \in \mathbb{N} : f(x_n)g(x_n) = 0\}$ belongs to $\mathcal{U}$. Assume that $E_f, E_g \notin \mathcal{U}$, then since $\mathcal{U}$ is a ultrafilter, we have $E_f^c, E_g^c \in \mathcal{U}$. But then

$$E_f^c \cap E_g^c = \{n \in \mathbb{N} : f(x_n) \neq 0, g(x_n) \neq 0\} = \{n \in \mathbb{N} : f(x_n)g(x_n) \neq 0\} = E_{fg}^c \in \mathcal{U}$$

which is a contradiction since $E_{fg} \in \mathcal{U}$. This proves that either of E_f and E_g is in $\mathcal{U}$, so J_1 is a prime ideal.

Now for a prime ideal J of C, define $\varphi(J)$ by

$$\varphi(J) = \{f \in C : \{n \in \mathbb{N} : \tau_n(f) \in J\} \in \mathcal{U}\}.$$

Again, for an element $f \in C$ we put $E_f(J) = \{n \in \mathbb{N} : \tau_n(f) \in J\}$. By noting that, since J is a prime ideal,

$$E_f(J)^c \cap E_g(J)^c = \{n \in \mathbb{N} : \tau_n(f) \notin J, \tau_n(g) \notin J\} = \{n \in \mathbb{N} : \tau_n(f)\tau_n(g) \notin J\} = E_{fg}(J)^c,$$

it is easy to show that $\varphi(J)$ is a prime ideal, just as in (a).

Define the sequence (I_n) by $I_0 = J_1$ and $I_{n+1} = \varphi(I_n)$. To prove the first assertion in (c), we first note that, since $\mathcal{U}$ is finer than the finite complement filter on $\mathbb{N}$, every element in $\mathcal{U}$ is infinite (otherwise its complement belongs to $\mathcal{U}$). Let $f \in F$, then there exists a neighborhood U of 0 such that $f|_U = 0$. Then since $x_n \to 0$, there exists an integer N such that $x_n \in U$ when $n \geq N$, whence $f(x_n) = 0$ for $n \geq N$. Since $\mathcal{U}$ is finer than the finite complement filter on $\mathbb{N}$, this means $E_f = \{n \in \mathbb{N} : f(x_n) = 0\} \in \mathcal{U}$, so $f \in J_1$. This shows $F \subseteq J_1 = I_0$. Now let J be a prime ideal of C containing F, and consider $\varphi(J)$. Let $f \in F$, and let U be a neighborhood of 0 such that $f|_U \equiv 0$. Then since x_n is strict decreasing and tend to 0, there exists $N > 0$ such that $(x_n + U) \cap U$ contains a neighborhood of 0 for $n \geq N$, whence $\tau_n(f) \in F \subseteq J$ for $n \geq N$. This shows $f \in \varphi(J)$, so $F \subseteq \varphi(J)$. Thus we have shown that I_n contains F for each n.

It remains to prove that (I_n) is strict decreasing. We prove a more general fact: let J be a prime ideal of C such that, if f_n, f are elements in C and $f_n \to f$ pointwisely and $f_n \in J$ for all n, then $f \in J$ (we say J is *pointwise closed* if this is true). Then $\varphi(J) \subseteq J$ and $\varphi(J)$ is pointwise closed. For this, let $f \in \varphi(J)$, then $E_f(J)$ is infinite, hence contains a sequence (y_n) tends to 0; we have $\tau_{y_n}(f) \in J$ for each n, and letting n tends to infinite shows that $f \in J$, since $y_n \to 0$ by hypothesis and J is pointwise closed; this shows $\varphi(J) \subseteq J$. □

Exercise 7.1.7. Let R be a DVR and π a uniformizer for R. In the ring $R[T]$, the ideal $\mathfrak{m}_1 = (\pi T - 1)$ is maximal with height 1, the ideal $\mathfrak{m}_2 = (\pi) + (T)$ is maximal with height 2. The field $R[T]/\mathfrak{m}_1$ and $R[T]/\mathfrak{m}_2$ are isomomorphic to the field of fractions of R and to the residual field of R respectively. We have

$$\dim(R[T]) = 2, \quad \dim(R[T]/\mathfrak{m}_1) + \dim(R[T]_{\mathfrak{m}_1}) = 1, \quad \dim(\mathrm{gr}_{\mathfrak{m}_1}(R[T])) = 1. \tag{7.1.4}$$

Proof. Let K be the fraction field of R. Then it is easy to see $R[T]/\mathfrak{m}_1 \cong R[\pi^{-1}] = K$, so $\mathfrak{m}_1$ is maximal. Moreover, since $\mathfrak{m}_1$ is principal, it has height 1 by Corollary 7.2.19. Also, the quotient $R[T]/\mathfrak{m}_2$ is clearly isomomorphic to κ_R, so $\mathfrak{m}_2$ is maximal, and has height 2 by Corollary 7.2.19.

Clearly $\dim(R[T]) = \dim(R) + 1 = 2$ since R is Noetherian. The ring $R[T]_{\mathfrak{m}_1}$ has dimension equal to $\mathrm{ht}(\mathfrak{m}_1)$, which is 1. These prove the first two equalities in (7.1.4). To see $\dim(\mathrm{gr}_{\mathfrak{m}_1}(R[T])) = 1$, note that $\mathrm{gr}_{\mathfrak{m}_1}(R[T])$ is isomorphic to a polynomial ring over K. □

Exercise 7.1.8. With the notations of the previous exercise, suppose that the residue field of R and fraction field of R are isomomorphic (this is the case for example when $R = k[\![X]\!]$, the field k being the field of fractions of a ring of formal series with an infinite number of indeterminates with coefficients in a field). Let σ be an isomorphism of $R[T]/\mathfrak{m}_1$ to $R[T]/\mathfrak{m}_2$.

Let C be the subring of $R[T] = E$ formed of the elements of E whose classes modulo $\mathfrak{m}_1$ and $\mathfrak{m}_2$ are associated by σ: in other words, if $\pi_i : R[T] \to R[T]/\mathfrak{m}_i$ is the natural map, then

$$C = \{x \in R[T] : \sigma(\pi_1(x)) = \pi_2(x)\}.$$

Show that C is noetherian and that $\mathfrak{m}_1\mathfrak{m}_2 = \mathfrak{m}_1 \cap \mathfrak{m}_2$ is a maximal ideal of C. Show that E is the integral closure of C (note that we have $E = C + (\pi T - 1)C$ and that $(\pi T - 1)^2 + (\pi T - 1)$ belongs to C).

Let A be the local ring $C_{\mathfrak{m}_1\mathfrak{m}_2}$. Then A is integral with dimension 2, hence catenary; the integral closure $B = E \otimes_C A$ of A is a semi-local Noetherian ring with exactly two maximal ideals $\mathfrak{n}_1$ and $\mathfrak{n}_2$, and we have $\dim(B_{\mathfrak{n}_1}) = 1$, $\dim(B_{\mathfrak{n}_2}) = 2$.

Proof. □

Exercise 7.1.9. Retain the notations of the previous exercise, and identify B with a quotient ring of the polynomial ring $A[U]$ by the prime ideal generated by $U^2 + U - \pi T(\pi T - 1)$. Let $\mathfrak{q}_i$ be the ideal of $A[U]$ such that $\mathfrak{n}_i = \mathfrak{q}_i/\mathfrak{p}$. Then $\mathrm{ht}(\mathfrak{q}_1) = 3$, $\mathrm{ht}(\mathfrak{p}) = 1$ and the chain $\mathfrak{p} \subsetneq \mathfrak{q}_1$ is saturated. In particular, $A[U]$ and $A[U]_{\mathfrak{q}_1}$ are not catenary.

Exercise 7.1.10. Let A be a ring.

(a) Assume that A is an integral domain and let $f \in A[T]$, $f \neq 0$. Prove that there exists a maximal ideal $\mathfrak{m}$ of $A[T]$ such that $f \notin \mathfrak{m}$.

(b) Prove that $\dim(\mathrm{Max}(A[T])) \geq \dim(A) + 1$, so $\dim(\mathrm{Max}(A[T])) = \dim(A[T]) = \dim(A) + 1$ when A is Noetherian.

(c) Let k be a field, S the subset $1 + (X)$ of $k[X, Y]$, and $A = S^{-1}k[X, Y]$. Show that we have $\dim(\mathrm{Max}(A)) = 1$ and $\dim(\mathrm{Spec}(A)) = 2$. Generalize this to several indeterminates.

(d) Show that for any couple of integers $0 \leq n < m$, there exists a Noetherian ring A such that $\dim(\mathrm{Max}(A)) = n$ and $\dim(\mathrm{Max}(A[T])) = m$.

(e) Let A be the Noetherian ring defined in Exercise 7.1.5. Show that $\dim(\mathrm{Max}(A)) = 1$ and $\dim(\mathrm{Max}(A[T])) = \infty$.

Proof. Since the Jacobson radical of $A[T]$ is zero if A is integral, we have (a). Now let $\mathfrak{p}_0 \subset \cdots \subset \mathfrak{p}_n$ be a chain of prime ideals in A. Consider the chain

$$\mathfrak{p}_0[T] \subset \cdots \subset \mathfrak{p}_1[T] \subset \cdots \subset \mathfrak{p}_n[T] \subset (\mathfrak{p}_n, T)$$

of prime ideals of $A[T]$ and put

$$\begin{cases} U_i = V(\mathfrak{p}_i[T]) \cap \mathrm{Max}(A[T]), & 0 \leq i \leq n \\ U_{n+1} = V(\mathfrak{p}_n, T) \cap \mathrm{Max}(A[T]). \end{cases}$$

It is easy to see the chain $(U_i)_{0 \leq i \leq n+1}$ is decreasing and each U_i is closed; the fact it is strict follows immediately from (a). It remains to show that each U_i is irreducible. This shows the equality $\dim(\mathrm{Max}(A[T])) \geq \dim(A) + 1$. Recall that $\dim(\mathrm{Max}(A)) \leq \dim(A)$ for any ring and $\dim(A[T]) = \dim(A) + 1$ if A is Noetherian. So if A is Noetherian, we have

$$\dim(A) + 1 \leq \dim(\mathrm{Max}(A[T])) \leq \dim(A[T]) = \dim(A) + 1.$$

Let A be the ring in (c). Then the ring $A = S^{-1}k[X, Y]$ is just the ring $S^{-1}k[Y][X]$, and it is easy to see the chain $0 \subsetneq (X) \subsetneq (X, Y)$ corresponds to a chain of length 2 in A, whence $\dim(A) = 2$. For the space $\mathrm{Max}(A)$, note that a maximal ideal $(X - a, Y - b)$ is disjoint from

S if and only if $a = 0$, which indicates $\dim(\mathrm{Max}(A)) = 1$ at least when k is algebraically closed. In general, let $0 \leq n \leq m$ and consider the ring $k[X_1, \dots, X_m]$. Let S be the subset $1 + (X_1, \dots, X_{m-n})$, and $A = S^{-1}k[X_1, \dots, X_m]$. Then $\dim(A) = m$ and $\dim(\mathrm{Max}(A)) = n$. Note that in this case, since A is Noetherian, we have $\dim(\mathrm{Max}(A[T])) = \dim(A) + 1 = m + 1$, so (d) is proved.

Finally, consider the example of Exercise 7.1.5. Since A is Noetherian, we have

$$\dim(\mathrm{Max}(A[T])) = \dim(A[T]) = \infty.$$

But the maximal ideals of A are the $S^{-1}\mathfrak{p}_i$'s, and if $S^{-1}\mathfrak{p}$ is a prime ideal of A, it is contained in finitely many $S^{-1}\mathfrak{p}_i$'s. However, note that each singleton in $\mathrm{Max}(A)$ is closed, so the set $V(S^{-1}\mathfrak{p}) \cap \mathrm{Max}(A)$ is not irreducible unless it is a singleton or the whole space $\mathrm{Max}(A)$. This shows $\dim(\mathrm{Max}(A)) = 1$ (in fact, the topology on $\mathrm{Max}(A)$ is the cofinite topology, hence has Krull dimension 1). $\square$

Exercise 7.1.11. Let R be a DVR with fraction field K and n an integer. Let A be the subring of the polynomial ring $K[X_1, \dots, X_n]$ consists of polynomials whose constant terms are in R. Then the canonical homomorphism $\rho : R \to A$ satisfies the going down property and the canonical homomorphism $R/\mathfrak{m}_R \to A/\mathfrak{m}_R A$ is an isomorphism. On the other hand, we have $\dim(A) \geq n + 1$ and $\dim(R) = 1$. As long as $n \geq 1$, we have $\dim(A) > \dim(R) + \dim(A/\mathfrak{m}_R A)$. Show that A is not Noetherian. By using formal series, construct an similar example where A is local and not Noetherian.

Proof. It is clear that A is a subring of $K[X_1, \dots, X_n]$ and $A/\mathfrak{m}_R A$ is isomorphic to $R/\mathfrak{m}_R$; A is flat over R since it is torsion-free (Proposition 5.1.5). Since $\mathfrak{a}^{ec} = \mathfrak{a}$ for every ideal of R, by **??** we conclude that A is faithfully flat over R, whence the going down property is satisfied (**??**). Let $\mathfrak{m}_R[X_1, \dots, X_n]$ be the ideal consists of polynomials with coefficients in $\mathfrak{m}_R$. Then it is clear that $\mathfrak{m}_R[X_1, \dots, X_n]$ is prime in A and we have a chain

$$0 \subset \mathfrak{m}_R[X_1, \dots, X_n] \subset \mathfrak{m}_R[X_1, \dots, X_n] + (X_1) \subset \cdots \subset \mathfrak{m}_R[X_1, \dots, X_n] + (X_1, \dots, X_n)$$

in the ring $K[X_1, \dots, X_n]$. Contracting this to A, we get a chain of length $n + 1$ in A, so $\dim(A) \geq n + 1$. Since $A/\mathfrak{m}_R A \cong R/\mathfrak{m}_R$ is a field, this shows

$$\dim(A) \geq n + 1 > \dim(R) + \dim(A/\mathfrak{m}_R A)$$

as long as $n \geq 1$.

To see A is not Noetherian, we claim that the ideal $\mathfrak{p} = (X_1, \dots, X_n) \cap A$ in A is not finitely generated. In fact, if $f_1, \dots, f_n \in A$ generate $\mathfrak{p}$, then $\sum_i f_i(0)g_i(0)$ can take any element in K as g_i varies in A. But this is not possible since $f_i(0), g_i(0) \in R$.

Finally, replacing $K[X_1, \dots, X_n]$ with $K[\![X_1, \dots, X_n]\!]$, we see the resulting ring A is then local with maximal ideal $\mathfrak{m}_R[\![X_1, \dots, X_n]\!] + (X_1, \dots, X_n)$. $\square$

Exercise 7.1.12. Let R be a DVR with fraction field K and residue field k. Put $A = K \times k$, and let $\rho : R \to A$ be the homomorphism deduced from the canonical homomorphisms $R \to K$ and $R \to k$. Then the homomorphism ρ is injective, the induced map $\rho^* : \mathrm{Spec}(A) \to \mathrm{Spec}(R)$ is surjective, A is a finitely generated R-algebra, and we have $\dim(A) < \dim(R)$.

Proof. Since $R \to K$ is injective, it is clear that ρ is injective. Also, we have $\mathrm{Spec}(A) = \mathrm{Spec}(K) \amalg \mathrm{Spec}(k)$, which consists of two elements $(0)_K$ and $(0)_k$. The image of $(0)_K$ under ρ is $(0)_R$, and that of $(0)_k$ is $\mathfrak{m}_R$. Therefore ρ^* is surjective. Since K and k are finitely generated R-algebra, so is A, and we have $\dim(A) = 0 < \dim(R) = 1$. $\square$

Exercise 7.1.13. Let A be an integral domain and $\mathfrak{P}_0 \subset \mathfrak{P}_1 \subset \mathfrak{P}_2$ be a chain of prime ideals in $A[X]$. Show that $\mathfrak{P}_0, \mathfrak{P}_1, \mathfrak{P}_2$ can not have the same contraction in A. In particular, if A is one-dimensional and $\mathfrak{P}_i$ are nonzero, then $\mathfrak{P}_0 \cap A = \{0\}$, $\mathfrak{P}_2 = (\mathfrak{P}_1 \cap A) \cdot A[X]$, and $\mathfrak{P}_2$ is maximal.

Proof. Let $\mathfrak{P}_0 \cap A = \mathfrak{P}_1 \cap A = \mathfrak{P}_2 \cap A = \mathfrak{p}$. Then $\mathfrak{P}_i = \mathfrak{p}A[X] + (X)$ for $i = 0, 1, 2$, which contradicts that $\mathfrak{P}_i$ are distinct. Now assume that $\dim(A) = 1$ and $\mathfrak{P}_i$ are nonzero. Recall that we have $2 \leq \dim(A[X]) \leq 3$ (Corollary 7.1.25), and since 0 is a prime ideal of $A[X]$, the chain $\mathfrak{P}_0 \subset \mathfrak{P}_1 \subset \mathfrak{P}_2$ shows that $\dim(A[X]) = 3$. Therefore $\mathfrak{P}_2$ is a maximal ideal in $A[X]$ and it contains (X). Now consider the contracted chain

$$0 \subseteq \mathfrak{P}_0^c \subseteq \mathfrak{P}_1^c \subseteq \mathfrak{P}_2^c.$$

Since A has dimension 1 and $\mathfrak{P}_2$ is maximal in $A[X]$, $\mathfrak{P}_2^c$ must be a maximal ideal in A. The inclusion $0 \subseteq \mathfrak{P}_0^c$ can not be strict since otherwise we have $\mathfrak{P}_0^c = \mathfrak{P}_1^c = \mathfrak{P}_2^c$. This proves $\mathfrak{P}_0^c = \{0\}$. We also note that $\mathfrak{P}_1^c \neq \{0\}$ since otherwise $\mathfrak{P}_0 = \mathfrak{P}_1 = (X)$ (again by Example 7.5.5). Therefore $\mathfrak{P}_1^c = \mathfrak{P}_2^c$ is a maximal ideal of A, and since $\mathfrak{P}_1 \neq \mathfrak{P}_2$, we have $X \notin \mathfrak{P}_2$ and $\mathfrak{P}_2 = \mathfrak{P}_2^c \cdot A[X]$. □

Exercise 7.1.14. Let A be an integrally closed domain with K the fraction field. For any element $x \in K$, we denote by $A[x]$ the sub-A-algebra of K generated by x. Let x be nonzero in K and $x^{-1} \notin A$; let $A[X]$ be the polynomial ring over A, and $\varphi : A[X] \to A[x]$ the canonical map $P \mapsto P(x)$.

(a) The kernel of φ is the ideal $\mathfrak{Q}$ generated by the polynomials of the form $aX + b$, where $a, b \in A$ and $ax + b = 0$.

(b) Let A be local with maximal ideal $\mathfrak{m}_A$. Then the ideal $\mathfrak{m}_A A[x]$ of $A[x]$ is prime, and is maximal if and only if $x \in A$.

Proof. Let φ be the canonical map. Assume that $P(X) = \sum_{i=0}^{n} a_i X^i$ and $P(x) = 0$. Then by multiplying a_n^{n-1}, we get

$$(a_n x)^n + a_{n-1}(a_n x)^{n-1} + \cdots + a_n^{n-2} a_1 (a_n x) + a_n^{n-1} a_0 = 0,$$

so $-b := a_n x$ is integral over A, hence belongs to A. Consider the polynomial $a_n X + b$: it has x as a root and the same leading coefficient with P. Hence $a_n X + b \mid P(X)$ and assertion (a) is proved. We also note that, since $x^{-1} \notin A$, the element b can not be invertible, hence $b \in \mathfrak{m}_A$.

Now by (a), we have

$$A[x]/\mathfrak{m}_A A[x] \cong A[X]/(aX + b, \mathfrak{m}_A) = \kappa_A[X]/(\bar{a}X + \bar{b}) = \kappa_A[X]/(\bar{a}X)$$

where a, b are the generators of $\ker \varphi$, and we have remarked that $b \in \mathfrak{m}_A$. Since these are irreducible elements, we conclude that $\mathfrak{m}_A A[x]$ is prime, and maximal iff there exists $a \in A$ in these generators such that $\bar{a} \neq 0$. But this just means $a \notin \mathfrak{m}_A$, hence invertible, and we then have $x = a^{-1} b \in A$. □

Exercise 7.1.15. Let A be an integrally closed domain with fraction field K. Given a prime ideal $\mathfrak{p}$ of A, an element x of K such that $x \notin A_{\mathfrak{p}}$, and $x^{-1} \notin A_{\mathfrak{p}}$, the ideal $\mathfrak{p}A[x]$ of $A[x]$ is prime; we have $\mathfrak{p}A[x] \cap A = \mathfrak{p}$ and the canonical homomorphism $(A/\mathfrak{p})[X] \to A[x]/\mathfrak{p}A[x]$ is an isomorphism.

Proof. Since $x \notin A_{\mathfrak{p}}$, and $x^{-1} \notin A_{\mathfrak{p}}$, we have $x \notin A$, so we can prove that $\mathfrak{p}A[x]$ is prime. Since A is integrally closed, it is not hard to se $\mathfrak{p}A[x] \cap A = \mathfrak{p}$. Since $x \notin A_{\mathfrak{p}}$ and $x^{-1} \notin A_{\mathfrak{p}}$, by

Exericse 7.1.14, the kernel of the canonical map $\varphi : A[X] \to A[x]$ the ideal $\mathfrak{Q}$ generated by the polynomials of the form $aX + b$, where $a, b \in \mathfrak{p}$ and $ax + b = 0$. Since we have

$$A[x]/\mathfrak{p}A[x] = A[X]/(\mathfrak{p}, \mathfrak{Q}) = (A/\mathfrak{p})[X]/\bar{\mathfrak{Q}}$$

and the image of $\mathfrak{Q}$ in $A/\mathfrak{p}$ is zero, we see $(A/\mathfrak{p})[X]$ is isomorphic to $A[x]/\mathfrak{p}A[x]$. □

Exercise 7.1.16. We say an integral domain A is an ***F*-ring** if $\dim(A) = 1$ but $\dim(A[X]) > 2$.

(a) Using the previous exercise, show that if A is a one-dimensional integral domain then $\dim(A[X]) = 2$ if and only if the localization at any prime ideal of the integral closure of A is a valuation ring.

(b) Show that if we have $\dim(A) = n$ and $\dim(A[X]) > n + 1$, there exists a prime ideal $\mathfrak{p}$ of A such that $\mathrm{ht}(\mathfrak{p}) = 1$ and, either $\dim(A_{\mathfrak{p}}[X]) > 2$, or $\dim((A/\mathfrak{p})[X]) > \dim(A/\mathfrak{p}) + 1$.

Proof. Let A be as in (a) and by taking integral closure and note that if R is the integral closure of A, then $R[X]$ is the integral closure of $A[X]$, we may assume that A is integrally closed. Assume that there exists $0 \neq x \in K$ such that $x, x^{-1} \notin A_{\mathfrak{p}}$. Then by Exercise 7.1.14 applied to $A_{\mathfrak{p}}$, the ideal $\mathfrak{p}A_{\mathfrak{p}}[x]$ is not maximal in $A_{\mathfrak{p}}[x]$, whence $A_{\mathfrak{p}}[x]$ has dimension ≥ 2. But since $A_{\mathfrak{p}}[x]$ is a quotient of $A_{\mathfrak{p}}[X]$, this implies $\dim(A[X]) \geq \dim(A_{\mathfrak{p}}[X]) \geq 3$, so $\dim(A[X]) = 2$ iff there is not such x, or in other words $A_{\mathfrak{p}}$ is a valuation ring.

Now let A be n-dimensional and $\dim(A[X]) \neq n + 1$. Suppose that for some minimal prime ideal $\mathfrak{p}$ of A, $\mathfrak{p}A[X]$ is not minimal in $A[X]$; that is, there exists a prime ideal $\mathfrak{P}$ of $A[X]$ such that

$$0 \subset \mathfrak{P} \subset \mathfrak{p}A[X].$$

Then $0 \subset \mathfrak{P}A_{\mathfrak{p}}[X] \subset \mathfrak{p} \cdot A_{\mathfrak{p}}[X]$ is also a chain of prime ideals in $A_{\mathfrak{p}}[X]$, as one easily verifies. Since $\mathfrak{p} \cdot A_{\mathfrak{p}}[X]$ is not maximal (for example $\mathfrak{p} \cdot A_{\mathfrak{p}}[X] + (X)$ contains it), this shows that $A_{\mathfrak{p}}$ is an F-ring. We pass then to the case that $\mathfrak{p}A[X]$ is minimal for every minimal prime ideal $\mathfrak{p}$ of A. Let

$$0 \subset \mathfrak{P}_1 \subset \cdots \subset \mathfrak{P}_{n+2}$$

be a chain of prime ideals in $A[X]$ (recall that $\dim(A[X]) \geq n + 2$). If $\mathfrak{P}_1 \cap A = \mathfrak{p}$ is nonzero, then $A/\mathfrak{p}$ is at most $(n-1)$-dimensional, and $A[X]/\mathfrak{p}A[X]$ is a polynomial ring in one variable over $A/\mathfrak{p}$ and is at least $(n+1)$-dimensional, which proves the claim. So we may suppose $\mathfrak{P}_1 \cap A = 0$. But then $\mathfrak{P}_2 \cap A = \mathfrak{p}_2 \neq 0$ (by Exercise 7.1.13); let $\mathfrak{p}$ be a minimal prime ideal contained in $\mathfrak{p}_2$-such exists since A is finite dimensional; then $\mathfrak{p}A[x] \subseteq \mathfrak{P}_2$, properly, since $\mathfrak{p}A[X]$ is minimal but $\mathfrak{P}_2$ is not. Replacing $\mathfrak{P}_1$ by $\mathfrak{p}A[X]$, we come back to a previous case, and the proof is complete. □

Exercise 7.1.17. Let O be an integrally closed ring, with fraction field k. We put $d = \dim(O)$ and $t = \dim(O[X])$. Let k' be a non-trivial extension of k in which k is algebraically closed and let R be a discrete valuation ring (which is not a field) with residual field k'. We denote K the field of fractions of R and A the set of elements of R whose image in k' belongs to O.

(a) Show that the ring A is integrally closed with fraction field K.

(b) Let $\mathfrak{p}$ be the kernel of the surjective homomorphism $A \to O$. Show that every nonzero prime ideal of A contains $\mathfrak{p}$ and hence $\dim(A) = d + 1$.

(c) Considering an element x of R whose image in k' does not belong to k, show that the local ring $A_{\mathfrak{p}}$ is not not a valuation ring, and that $\mathfrak{p}A_{\mathfrak{p}}[x]$ is not minimal among non-zero prime ideals of $A_{\mathfrak{p}}[x]$. By using Exercise 7.1.16, deduce that we have $\dim(A[x]) \geq t + 2$. Conclude that we have $\dim(A[x]) = t + 2$ by a direct argument.

(d) Finally, show that for any pair (d, t) of integers with $d + 1 \leq t \leq 2d + 1$, there exists an integrally closed domain A of dimension d such that $\dim(A[X]) = t$.

Proof. Let v be the valuation of K associated with R. Note that any element in K can be written as a/b where $a, b \in R$. Moreover, we can assume that a, b has positive valuations, whence are in $\mathfrak{m}_R \subseteq A$. This shows the fraction field of A is K. Now let $x \in K$ be integral over A, and let

$$x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0$$

be an equation of integral dependence. Dividing this equation by x^s and supposing x^{-1} to have residue 0 in k', we get the contradiction $1 = 0$, so $x^{-1} \notin \mathfrak{m}_R$ and thus $x \in R$; we then have

$$\bar{x}^n + \bar{a}_{n-1}\bar{x}^{n-1} + \cdots + \bar{a}_0 = 0$$

where the bars indicate residues. Since each $\bar{a}_i$ is in O and k is algebraically closed in k' we deduce that $\bar{x} \in K$; whence $\bar{x} \in O$, since O is integrally closed. Hence A is integrally closed.

Let $\mathfrak{p}$ be the kernel of the canonical map $A \to O$. Then $\mathfrak{p}$ is a prime ideal. From the definitions one obtains $A/\mathfrak{p} \cong O$, whence A is at least $(n+1)$-dimensional. If $\mathfrak{q}$ is a nonzero prime ideal in A, then $\mathfrak{q}$ contains $\mathfrak{p}$: In fact, let $x \in \mathfrak{q}$; since x is $A \subseteq R$, we have $v(x) = e \geq 0$. Since any element in $\mathfrak{p}$ has valuation ≥ 1, its $(e+1)$-th power is divisible by x, whence $\mathfrak{p}^{e+1} \subseteq \mathfrak{q}$ and so $\mathfrak{p} \subseteq \mathfrak{q}$. From this it follows that A is at most $(d+1)$-dimensional, so $\dim(A) = d + 1$.

The quotient ring $A_\mathfrak{p}$ is integrally closed and has one nonzero prime ideal. Moreover it is not a valuation ring if $k \neq k'$. In fact, let $x \in R$ be an element having residue in k' but not in k (as in (c)). Note that x is in the quotient field of $A_\mathfrak{p}$, which is K; but neither x nor x^{-1} has residue in k, so neither x nor x^{-1} is in $A_\mathfrak{p}$. Thus $A_\mathfrak{p}$ is not a valuation ring, and hence is an F-ring (Exercise 7.1.16(a) applied to $A_\mathfrak{p}$). It follows that $\mathfrak{p}A[X]$ is not minimal among nonzero prime ideals in $A[X]$. Now

$$A[X]/\mathfrak{p}A[X] \cong (A/\mathfrak{p})[X] \cong O[X]$$

so $A[X]$ is at least $(t+2)$-dimensional (recall that is integral).

Finally, let $0 = \mathfrak{P}_0 \subset \mathfrak{P}_1 \subset \cdots \subset \mathfrak{P}_n$ be a chain of prime ideals in $A[X]$. Assume that the chain $0 \subset \mathfrak{P}_1$ is saturated; then $\mathfrak{P}_1 \cap A = 0$, as otherwise

$$\mathfrak{p} \subset \mathfrak{P}_1 \cap A \quad \text{and} \quad \mathfrak{p} \cdot A[X] \subset \mathfrak{P}_1.$$

Similarly one concludes that if the chain $0 \subset \mathfrak{P}_1 \subset \mathfrak{P}_2$ is saturated, then

$$\mathfrak{P}_2 \cap A = \mathfrak{p} \quad \text{and} \quad \mathfrak{P}_2 = \mathfrak{p}A[X]$$

(by Exercise 7.1.13, $\mathfrak{P}_2 \cap A$ can not be 0). From this it follows at once that $A[X]$ is at most $(t+2)$-dimensional.

We say an integral domain A is *of type* (d, t) if $\dim(A) = d$ and $\dim(A[X]) = t$. Now any field is of type $(0, 1)$, and from the above construction we get an integrally closed domain of type $(1, 3)$. Note that a Noetherian integrally closed domain is of type $(1, 2)$. We now prove (d) by induction on d. So suppose by induction that for some d and each t (where $d + 1 \leq t \leq 2d + 1$), we have an integrally closed ring of type (d, t). Note that if $d + 3 \leq t \leq 2d + 3$, then $d + 1 \leq t - 2 \leq 2d + 1$ and from an integrally closed ring of type $(d, t-2)$ we get an integrally closed ring of type (d, t). If on the other hand $t = d + 2$, then a Noetherian integrally closed domain of dimension $d + 1$ (for example the ring $k[X_1, \ldots, X_{d+1}]$ where k is a field) is of type $(d + 1, d + 2)$, so the claim is proved. □

Exercise 7.1.18.

(a) Let A be an integral domain, K its fraction field and n an integer. Let $t_1, \dots, t_n$ be elements of K and $\varphi : A[X_1, \dots, X_n] \to A[t_1, \dots, t_n]$ the homomorphism such that $\varphi(X_i) = t_i$. Show that the height of the kernel of φ equals to n.

(b) Deduce that for any n-tuple $(t_1, \dots, t_n)$ of elements of K, we have

$$\dim(A[t_1, \dots, t_n]) \leq \dim(A[X_1, \dots, X_n]) - n.$$

Exercise 7.1.19. Let A be an integral domain with fraction field K. Let $0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_n$ be a chain of prime ideals of A. Show that there exists a valuation ring R of K and a chain of prime ideals $0 \subset \mathfrak{q}_1 \subset \cdots \subset \mathfrak{q}_n$ of R such that $\mathfrak{q}_i \cap A = \mathfrak{p}_i$.

Exercise 7.1.20. Let A be an integral domain and K its fraction field. For any integer $k \geq 0$, the following properties are equivalent:

(i) Any subring B of K containing A has dimension $\leq k$.

(ii) Any valuation ring R of K containing A has dimension $\leq k$.

(iii) For any k elements $(t_1, \dots, t_k)$ of K, we have $\dim(A[t_1, \dots, t_k]) \leq k$.

(iv) We have $\dim(A[X_1, \dots, X_m]) \leq k + m$ for any integer $m \geq 0$.

7.2 Dimension of Noetherian rings

7.2.1 Dimension of a quotient ring

Proposition 7.2.1. *Let A be a Noetherian integral domain, x a nonzero element of A and $\mathfrak{p}$ a minimal prime belonging to x. Then $\mathfrak{p}$ has height 1.*

Proof. Let $\mathfrak{q} \subset \mathfrak{p}$ be a prime ideal distinct from $\mathfrak{p}$. Then $x \notin \mathfrak{q}$ by the hypothesis on $\mathfrak{p}$. Since A is integral, $A_{\mathfrak{p}}$ is identified with a subring of $A_{\mathfrak{q}}$. For any positive integer n, we denote by $\mathfrak{q}_n$ the ideal $\mathfrak{q}^n A_{\mathfrak{q}} \cap A_{\mathfrak{p}}$ of $A_{\mathfrak{p}}$. The minimal property of $\mathfrak{p}$ means that the local ring $A_{\mathfrak{p}}/xA_{\mathfrak{p}}$ is 0-dimensional. It is therefore Artinian and by **??** there is a positive integer n_0 such that

$$\mathfrak{q}_n + xA_{\mathfrak{p}} = \mathfrak{q}_{n+1} + xA_{\mathfrak{p}} \quad \text{for} \quad n \geq n_0.$$

Let's now fix an integer $n \geq n_0$. Given $y \in \mathfrak{q}_n$, there exists $a \in A_{\mathfrak{p}}$ such that $y - ax \in \mathfrak{q}_{n+1}$. We then have $ax \in \mathfrak{q}_n$, whence $a \in \mathfrak{q}_n$ since $x \notin \mathfrak{q}$, and finally we have $y \in \mathfrak{q}_{n+1} + x\mathfrak{q}_n$. So we have proved that

$$\mathfrak{q}_n = \mathfrak{q}_{n+1} + x\mathfrak{q}_n.$$

As x belongs to the maximal ideal of the Noetherian local ring $A_{\mathfrak{p}}$, the Nakayama lemma shows that we have $\mathfrak{q}_n = \mathfrak{q}_{n+1}$. As $\mathfrak{q}_n A_{\mathfrak{q}} = (\mathfrak{q}A_{\mathfrak{q}})^n$, we conclude that

$$(\mathfrak{q}A_{\mathfrak{q}})^n = (\mathfrak{q}A_{\mathfrak{q}})^{n+1} \quad \text{for} \quad n \geq n_0.$$

Since $A_{\mathfrak{q}}$ is a local Noetherian ring, we have $\bigcap_n (\mathfrak{q}A_{\mathfrak{q}})^n = \{0\}$ by Corollary 2.4.17, whence $(\mathfrak{q}A_{\mathfrak{q}})^{n_0} = 0$ and the prime ideal $\mathfrak{q}A_{\mathfrak{q}}$ of $A_{\mathfrak{q}}$ reduces to 0. Then we must have $\mathfrak{q} = 0$ and therefore $\mathfrak{p}$ has height 1. □

Lemma 7.2.2. *Let A be a Noetherian ring, $\mathfrak{p}_0 \subset \cdots \subset \mathfrak{p}_n$ a chain of prime ideals of A and x an element of $\mathfrak{p}_n$. Then there exists a chain $\mathfrak{p}'_0 \subset \cdots \subset \mathfrak{p}'_n$ of prime ideals of A such that $\mathfrak{p}'_0 = \mathfrak{p}_0$, $\mathfrak{p}'_n = \mathfrak{p}_n$, and $x \in \mathfrak{p}'_1$.*

Proof. We prove by induction. The case $n = 1$ is trivial, so suppose $n \geq 2$ and that x does not belong to $\mathfrak{p}_{n-1}$ (otherwise we can apply to $\mathfrak{p}_0 \subset \cdots \subset \mathfrak{p}_{n-1}$ the induction hypothesis). Let $\mathfrak{p}'_{n-1}$ be a minimal element of the set of prime ideals of A contained in $\mathfrak{p}'_n = \mathfrak{p}_n$ and containing $\mathfrak{p}_{n-2} + Ax$. According to Proposition 7.2.1, the ideal $\mathfrak{p}'_{n-1}/\mathfrak{p}_{n-2}$ of the ring $A/\mathfrak{p}_{n-2}$ is of height 1, and since $\mathfrak{p}_{n-2} \subset \mathfrak{p}_{n-1} \subset \mathfrak{p}_n$ is a chain of length 2, so is $\mathfrak{p}_{n-2} \subset \mathfrak{p}'_{n-1} \subset \mathfrak{p}'_n$. Since now $x \in \mathfrak{p}'_{n-1}$, the induction hypothesis applied to the chain $\mathfrak{p}_0 \subset \cdots \subset \mathfrak{p}'_{n-1}$ shows that there is a chain $\mathfrak{p}'_0 \subset \cdots \subset \mathfrak{p}'_{n-1}$ with $x \in \mathfrak{p}'_1$ and $\mathfrak{p}'_0 = \mathfrak{p}_0$. The chain

$$\mathfrak{p}'_0 \subset \cdots \subset \mathfrak{p}'_{n-1} \subset \mathfrak{p}'_n$$

then satisfies the requirements. $\square$

Proposition 7.2.3. *Let A be a Noetherian ring and $\mathfrak{a}$ an ideal contained in the Jacobson radical of A and generated by m elements. Then we have*

$$\dim(A/\mathfrak{a}) \leq \dim(A) \leq \dim(A/\mathfrak{a}) + m.$$

Proof. The inequality $\dim(A/\mathfrak{a}) \leq \dim(A)$ is clear. By induction it suffices to prove the inequality

$$\dim(A) \leq \dim(A/xA) + 1$$

for any element x contained in the Jacobson radical of A. That is, to show that $\dim(A/xA) \geq n - 1$ for any chain $\mathfrak{p}_0 \subset \cdots \subset \mathfrak{p}_n$ of prime ideals of A. Now since x is in the Jacobson radical of A, we may assume that $x \in \mathfrak{p}_n$. Then it suffices to construct a chain $\mathfrak{q}_1 \subset \cdots \subset \mathfrak{q}_n$ of prime ideals of A, with $x \in \mathfrak{q}_1$ and this follows from Lemma 7.2.2. $\square$

Corollary 7.2.4.

(a) Every Noetherian semilocal ring is of finite dimension.

(b) Let A be a Noetherian ring. Then every proper ideal of A has finite height.

(c) Any decreasing sequence of prime ideals of a Noetherian ring A is stationary.

Proof. Let A be a Noetherian semilocal ring and $\mathfrak{r}$ its Jacobson radical. The quotient ring $A/\mathfrak{r}$ is then of dimension 0 by Proposition 1.1.5. If the ideal $\mathfrak{r}$ is generated by m elements, then we have $\dim(A) \leq m$ by Proposition 7.2.3.

Let A be Noetherian and $\mathfrak{a}$ a proper ideal of A. Let $\mathfrak{m}$ be a maximal ideal of A containing A, then $0 \leq \operatorname{ht}(\mathfrak{a}) \leq \dim(A_{\mathfrak{m}})$. Since $A_{\mathfrak{m}}$ is a Noetherian local ring, we have $\dim(A_{\mathfrak{m}}) < +\infty$ by (a), whence (b) follows.

Finally, any finite strictly decreasing sequence $(\mathfrak{p}_i)_{1 \leq i \leq n}$ of prime ideals of a Noetherian ring A defines a chain $\mathfrak{p}_n \subset \cdots \subset \mathfrak{p}_0$, hence $n < \dim(A_{\mathfrak{p}_0}) < +\infty$ and (c) follows. $\square$

Corollary 7.2.5. *Let A be a Noetherian local ring.*

(a) Let $x \in \mathfrak{m}_A$. Then $\dim(A) - 1 \leq \dim(A/xA) \leq \dim(A)$, and $\dim(A/xA) = \dim(A) - 1$ if and only if x does not belong to any of the minimal prime ideals $\mathfrak{p}$ of A such that $\dim(A/\mathfrak{p}) = \dim(A)$, and it suffices that x is not a zero divisor in A.

(b) Let $\mathfrak{a}$ be a proper ideal of A such that $\dim(A/\mathfrak{a}) < \dim(A)$. Then there exists $x \in \mathfrak{a}$ such that $\dim(A/xA) = \dim(A) - 1$.

(c) If $\dim(A) \geq 1$, then there exists $x \in \mathfrak{m}_A$ such that $\dim(A/xA) = \dim(A) - 1$.

Proof. By Proposition 7.2.3, we see $\dim(A) - 1 \leq \dim(A/xA) \leq \dim(A)$. For $\dim(A/xA) = \dim(A) = n$, it is necessary and sufficient that there exists a chain $\mathfrak{p}_0 \subset \cdots \subset \mathfrak{p}_n$ such that $x \in \mathfrak{p}_0$, and this is equivalent to say there exists a prime ideal $\mathfrak{p}_0$ containing x such that $\dim(A/\mathfrak{p}_0) = \dim(A)$. But such an ideal prime $\mathfrak{p}_0$ is necessarily minimal, and any element of $\mathfrak{p}_0$ is therefore a divisor of 0 in A. This proves (a).

Let Φ be the set of minimal prime ideals of A, and Φ' the subset $\mathfrak{p} \in \Phi$ such that $\dim(A) = \dim(A/\mathfrak{p})$. Since A is Noetherian, Φ is finite. Let $\mathfrak{a}$ be a proper ideal such that $\dim(A/\mathfrak{a}) < \dim(A)$. Then for every $\mathfrak{p} \in \Phi'$ we have $\dim(A/\mathfrak{a}) < \dim(A/\mathfrak{p})$, whence $\mathfrak{a} \not\subseteq \mathfrak{p}$. Then by Proposition 1.1.4, there exists $x \in \mathfrak{a}$ such that $x \notin \mathfrak{p}$ for all $\mathfrak{p} \in \Phi'$, and therefore $\dim(A/xA) = \dim(A) - 1$ by (a), whence (b). Now (c) follows from (b) by taking $\mathfrak{a} = \mathfrak{m}_A$. □

Example 7.2.6. The claim in Proposition 7.2.1 fails if A is not Noetherian. For example, let A be a valuation ring of dimension $d \geq 2$. Then the prime ideals of A form a chain of length d:

$$0 = \mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_d = \mathfrak{m}_A.$$

Therefore, for each i, there exists $x_i \in \mathfrak{p}_i \setminus \mathfrak{p}_{i-1}$, and it is easy to see $\mathfrak{p}_i$ is the only minimal prime belonging to x_i.

7.2.2 Dimension and secant sequences

Let A be a Noetherian, M a finitely generated A-module, S a subset of A contained in the Jacobson radical of A. Let $\mathfrak{S}$ be the ideal generated by S and $\mathfrak{a}$ the annihilator of M. Then we have

$$\dim_A(M) = \dim(A/\mathfrak{a}) \leq \dim(A). \tag{7.2.1}$$

Moreover, the support of the A-module M/SM is equal to $V(\mathfrak{a} + \mathfrak{S})$ by Corollary 1.4.40, hence

$$\dim_A(M/SM) = \dim(A/(\mathfrak{a} + I)). \tag{7.2.2}$$

By Proposition 7.2.3, we then have the following inequality

$$\dim_A(M/SM) \leq \dim_A(M) \leq |S| + \dim_A(M/SM). \tag{7.2.3}$$

Definition 7.2.7. Let A be a Noetherian local ring and S a subset of $\mathfrak{m}_A$. Then S is said to be **secant for M** if

$$\dim_A(M) = \dim_A(M/SM) + |S|. \tag{7.2.4}$$

A family $(x_i)_{i \in I}$ of element of $\mathfrak{m}_A$ is said to be **secant for M** if

$$\dim_A(M) = |I| + \dim(M/\sum x_i M)$$

and an element $x \in \mathfrak{m}_A$ is called secant for M if $\{x\}$ is secant for M.

By definition, if $\mathfrak{a}$ is the annihilator of M, then a subset S of $\mathfrak{m}_A$ is secant for M if and only if it is secant for the A-module $A/\mathfrak{a}$. Also note that, since A is Noetherian and local, we have $\dim(A) < +\infty$, so if S is M-secant then S is finite.

Proposition 7.2.8. *Let A be a Noetherian local ring, M a finitely generated A-module, and S, S' two disjoint subsets of $\mathfrak{m}_A$. Then the set $S \cup S'$ is secant for M if and only if S is secant for M and S' is secant for $M' = M/SM$.*

Proof. For this, we first observe that

$$M/(SM+S'M) = (M/SM)/((SM+S'M)/SM) = M'/S'M'$$

and therefore

$$\dim_A(M/(SM+S'M)) = \dim_A(M'/S'M') \geq \dim_A(M') - |S'| \geq \dim_A(M) - |S| - |S'|.$$

This proves the claim by the definition of secanteness. □

Corollary 7.2.9. *Let A be a Noetherian local ring and $x_1, \dots, x_r$ be distinct elements of $\mathfrak{m}_A$. Define $M_1 = M$ and $M_{i+1} = M_i/x_iM_i$. Then the following conditions are equivalent.*

(i) The sequence $(x_1, \dots, x_r)$ is M-secant.

(ii) The element x_i is M_i-secant for each i.

(iii) The element x_i is $M/(x_1M + \cdots + x_{i-1}M)$-secant for each i.

Proof. The equivalence of (i) and (ii) follows by applying Proposition 7.2.8 on the subsets $\{x_1\}, \dots, \{x_n\}$, and that of (i) and (iii) follows similarly. □

Proposition 7.2.10. *Let A be a Noetherian local ring, M a finitely generated A-module, $x_1, \dots, x_r$ elements of $\mathfrak{m}_A$ and $n_1, \dots, n_r$ integers. Then the sequence $(x_1, \dots, x_r)$ is secant for M if and only if the sequence $(x_1^{n_1}, \dots, x_r^{n_r})$ is secant for M.*

Proof. We may reduce the case $r = 1$, so let $x \in \mathfrak{m}_A$ and n be a positive integer. If $\mathfrak{a}$ is the annihilator of M, then by Corollary 1.4.40,

$$\operatorname{supp}(M/xM) = V(\mathfrak{a}) \cap V(x), \quad \operatorname{supp}(M/x^nM) = V(\mathfrak{a}) \cap V(x^n).$$

By Proposition 1.4.3 we have $V(x^n) = V(x)$, whence the proposition. □

Proposition 7.2.11. *Let A be a Noetherian local ring and M a finitely generated A-module. For an element x in $\mathfrak{m}_A$ to be secant for M, it is necessary and sufficient that it does not belong to any minimal elements $\mathfrak{p}$ of $\operatorname{supp}(M)$ such that $\dim(A/\mathfrak{p}) = \dim_A(M)$, and it suffices that the homothety h_x with ratio x on M be injective.*

Proof. Let $\mathfrak{a}$ be the annihilator of M. Then x is secant for M means it is secant for $A/\mathfrak{a}$, and if h_x is injective on M, then x is not a zero divisor of $A/\mathfrak{a}$. The claim now follows from Corollary 7.2.5 applying to the ring $A/\mathfrak{a}$. □

Corollary 7.2.12. *Any seuqnece of elements of $\mathfrak{m}_A$ that is complete secant for M is secant for M.*

Proof. Let $(x_1, \dots, x_r)$ be a sequence of elements of $\mathfrak{m}_A$ that is complete secant for M. Put $M_0 = M$ and $M_i = M_{i-1}/x_iM_{i-1}$ for $1 \leq i \leq r$. Then by **??**, the homothety of ratio x_i is injective on M_{i-1}, so $\dim_A(M_i) = \dim_A(M_{i-1}) - 1$ by Proposition 7.2.11. We then conclude that

$$\dim_A(M) = r + \dim_A(M/(x_1M + \cdots + x_rM)),$$

so the sequence $(x_1, \dots, x_r)$ is secant for M. □

Remark 7.2.13. A sequence $(x_1, \dots, x_r)$ is said to be **regular for M** (or an **M-regular sequence**) if for each i, the element x_i is not a zerodivisor on $M/(x_1M + \cdots + x_{i-1}M)$. By Corollary 7.2.9 and Proposition 7.2.11, it follows that every regular sequence for M is secant for M.

Theorem 7.2.14. *Let A be a Noetherian local ring, M a finitely generated A-module, and S a subset of $\mathfrak{m}_A$.*

(a) If M/SM has finite length, then $|S| \geq \dim_A(M)$.

(b) If S is secant for M, then $|S| \leq \dim_A(M)$.

(c) Any secant subset for M is contained in a maximal secant subset for M.

(d) The following conditions are equivalent:

(i) S is a maximal secant sequence for M;
(ii) S is a secant sequence for M and $|S| = \dim_A(M)$;
(iii) M/SM has finite length and $|S| = \dim_A(M)$;
(iv) S is a secant sequence for M and M/SM has finite length.

Proof. Since $S \subseteq \mathfrak{m}_A$, by Nakayama lemma we have $M/SM \neq 0$, whence $\dim_A(M/SM) \geq 0$ with equality if and only if M/SM is of finite length. Assertions (a) and (b) then follows from (7.2.3) and (7.2.4), as well as the equivalences of (ii), (iii), and (iv) in (d).

By (a), any secant subset for M with cardinality $\dim_A(M)$ is maximal. It remains to prove that, if S is secant for M and if $|S| < \dim_A(M)$, then S is not not maximum. Let $\mathfrak{a}$ be the annihilator of M, and B the local Noetherian ring $A/(\mathfrak{a}+I)$, where I is the ideal generated by S. According to Corollary 7.2.5(c), there exists an element x of $\mathfrak{m}_A$ such that $\dim(B/xB) = \dim(B) - 1$, whence $x \notin S$. According to Proposition 7.2.8, the subst $S \cup \{x\}$ of $\mathfrak{m}_A$ is secant for $A/\mathfrak{a}$, hence for M. This proves assertion (c), and the equivalence of (i) and (ii) in (d). □

Corollary 7.2.15. *Let A be a Noetherian local ring, M a finitely generated A-module. Then the dimension of M is the smallest of the positive integers d for which it d there exists a sequence $(x_1, \dots, x_d)$ of elements of $\mathfrak{m}_A$ such that the A-module $M/\sum_i x_iM$ is of finite length.*

Proof. As $\varnothing$ is a secant subset for M, Theorem 7.2.14(c) shows the existence of a secant sequence for M, say $(x_1, \dots, x_d)$. But then $d = \dim_A(M)$ and the A-module $M/\sum_i x_iM$ is of finite length by property (iii) of Theorem 7.2.14(d). Conversely if $(x_1, \dots, x_{d'})$ is a sequence of elements of $\mathfrak{m}_A$ such that the A-module of $M/\sum_i x_iM$ is of finite length, we have $d' \geq \dim_A(M)$ according to Theorem 7.2.14(a). □

If $\dim_A(M) = d$, then a sequence $(x_1, \dots, x_d)$ of elements of $\mathfrak{m}_A$ such that $M/\sum_i x_iM$ is of finite length is called a **system of parameters** for M. Note that such a sequence must be secant for M, by (iii) of Theorem 7.2.14(d).

Let A be a Noetherian local ring and $\mathfrak{m}_A$ be its maximal ideal. Endow A with its $\mathfrak{m}_A$-adic topology, recall that an ideal $\mathfrak{a}$ of A is called a **defining ideal** of A if the $\mathfrak{a}$-adic topology coincides with its $\mathfrak{m}_A$-adic topology.

Proposition 7.2.16. *Let A be a Noetherian local ring and $\mathfrak{a}$ an ideal of A. Then the following conditions are equivalent.*

(i) $\mathfrak{a}$ is a defining ideal of A.

(ii) $\mathfrak{a}$ is $\mathfrak{m}_A$-primary.

(iii) There exist a positive integer n such that $\mathfrak{m}_A^n \subseteq \mathfrak{a} \subseteq \mathfrak{m}_A$.

(iv) $\mathfrak{a}$ is a proper ideal and $A/\mathfrak{a}$ is an A-module of finite legngth.

Proof. By Example 3.2.2, we see (ii) and (iii) are equivalent, also (i) is equivalent to (iii) by the definition of adic topologies and the fact that $\mathfrak{m}_A$ is a prime ideal. Finally, the equivalence of (i) and (iv) follows from Proposition 3.2.14 and $\operatorname{supp}(A/\mathfrak{a}) = V(\mathfrak{a})$. □

Corollary 7.2.17. *The dimension of a local Noetherian ring A is the smallest of the positive integers d for which there exists a defining ideal of A generated by d elements.*

Proof. This follows from Corollary 7.2.15 and Proposition 7.2.16. □

Let A be a ring and $X = \mathrm{Spec}(A)$ be its spectrum. A closed subset of the form $V(x)$ with $x \in A$ of X is called a **hypersurface** of X.

Proposition 7.2.18. *Let A be a Noetherian ring, Y be a closed subset of $X = \mathrm{Spec}(A)$, $H_1, \dots, H_m$ be hypersurfaces of X, and $\tilde{Y} = Y \cap H_1 \cap \cdots \cap H_m$.*

(a) For any closed subset V of Y contained in $\tilde{Y}$, we have $\mathrm{codim}(V, \tilde{Y}) \geq \mathrm{codim}(V, Y) - m$.

(b) For any irreducible componenet Z subset of $\tilde{Y}$, we have $\mathrm{codim}(Z, Y) \leq m$.

(c) If Z is an irreducible closed subset of X contained in Y such that $\mathrm{codim}(Z, Y) \leq m$, there are hypersurfaces $\tilde{H}_1, \dots, \tilde{H}_m$ such that Z is an irreducible component of $Y \cap \tilde{H}_1 \cdots \cap \tilde{H}_m$.

Proof. Let $\mathfrak{a}$ be an ideal of A and $x_1, \dots, x_m$ be elements of A such that $Y = V(\mathfrak{a})$ and $H_i = V(x_i)$ for each i. Let $V(\mathfrak{p})$ be an irreducible closed subset contained in Y, where $\mathfrak{p}$ is a prime ideal of A containing $\mathfrak{a}$. Suppose first that Z is contained in $\tilde{Y}$ and denote by ξ_i the image of x_i in the Noetherian local ring $B = A_{\mathfrak{p}}/\mathfrak{a}A_{\mathfrak{p}}$. According to Proposition 7.1.5, we have

$$\mathrm{codim}(Z, Y) = \dim(B), \quad \mathrm{codim}(Z, \tilde{Y}) = \dim(B/(\xi_1 B + \cdots + \xi_m B))$$

so by Proposition 7.2.3 we have

$$\mathrm{codim}(Z, \tilde{Y}) \geq \mathrm{codim}(Z, Y) - m. \tag{7.2.5}$$

If Z is an irreducible component of $\tilde{Y}$, we have $\mathrm{codim}(Z, \tilde{Y}) = 0$, whence $\mathrm{codim}(Z, Y) \leq m$. This proves (b), and (a) follows by taking in both sides of (7.2.5) the lower bound on the set of irreducible components Z of V.

Conversely, suppose that we have $\mathrm{codim}(Z, Y) \leq m$, ie, $\dim(B) \leq m$. As any element of $A_{\mathfrak{p}}$ is the product of an invertible element of $A_{\mathfrak{p}}$ by the image of an element of A, Corollary 7.2.17 demonstrates the existence of elements $\tilde{x}_1, \dots, \tilde{x}_m$ of A whose images in B generate a defining ideal of B. Set $\tilde{H}_i = V(\tilde{x}_i)$, it is then clear that Z is an irreducible component of $Y \cap \tilde{H}_1 \cap \cdots \cap \tilde{H}_m$. □

Corollary 7.2.19 (Krull's Hauptidealsatz). *Let A be a Noetherian ring and $x_1, \dots, x_m$ be elements of A.*

(a) For any prime ideal $\mathfrak{p}$ belonging to $Ax_1 + \cdots + Ax_m$, we have $\mathrm{ht}(\mathfrak{p}) \leq m$.

(b) If $\mathfrak{p}$ is a prime ideal of A such that $\mathrm{ht}(\mathfrak{p}) \leq m$, then there exist elements $\tilde{x}_1, \dots, \tilde{x}_m$ of A such that $\mathfrak{p}$ is a minimal prime belonging to $A\tilde{x}_1 + \cdots + A\tilde{x}_m$.

Proof. This is a reformulation of Proposition 7.2.18 in the case $Y = X$. □

Corollary 7.2.20. *Let A be a Noetherian ring and H be a hypersurface of $X = \mathrm{Spec}(A)$. Then the codimension of H in X is equal to 0 or 1. We have $\mathrm{codim}(H, X) = 1$ if and only if H does not contain any irreducible component of X. If so, all the irreducible components of H are of codimension 1 in X.*

Proof. By Proposition 7.2.18 we have $\mathrm{codim}(H, X) \leq 1$, and we have $\mathrm{codim}(H, X) = 0$ if and only if H contains an irreducible component of X. By definition,

$$\mathrm{codim}(H, X) = \inf_Z \mathrm{codim}(Z, X)$$

where Z runs through the set of irreducible components of H. The claim then follows from these remarks. □

Corollary 7.2.21. *Let A be a Noetherian ring, Y be an irreducible closed subset of $X = \mathrm{Spec}(A)$, and H a hypersurface in X. Then only three cases are possible:*

(a) $Y \subseteq H$;

(b) $Y \cap H$ *is nonempty and each of its irreducible components satisfies* $\mathrm{codim}(Z, X) = 1$;

(c) $Y \cap H$ *is empty.*

Proof. Suppose that $Y \cap H$ is nonempty and is not equal to Y. Then any irreducible component Z of $Y \cap H$ satisfies $\mathrm{codim}(Z, Y) \leq 1$ by Proposition 7.2.18. Since Z and Y are both irreducible closed subsets, $\mathrm{codim}(Z, Y) = 0$ iff $Y = Z$, which is impossible. □

Corollary 7.2.22. *If A is a Noetherian UFD, then the prime ideals of height 1 of A are the principal ideals generated by the irreducible elements of A. If moreover A is local, we have $\dim(A/\mathfrak{p}) = \dim(A) - 1$ for any prime ideal $\mathfrak{p}$ of height 1 of A.*

Proof. Let x be an irreducible element of A. Then Ax is a prime ideal because x is prime, of height 1 because A is integral. Let $\mathfrak{p}$ be a prime ideal of height 1 of A. Then $V(\mathfrak{p})$ is an irreducible component of a hypersurface $V(x)$ for some x (Proposition 7.2.18). Let $x = \prod_i p_i^{n_i}$ be a decomposition of x into products of irreducible elements. Then the irreducible components of $V(x)$ is the $V(p_i)$, so $\mathfrak{p} = Ap_i$ for some i. The last statement follows from Corollary 7.2.5. □

Proposition 7.2.23. *Let A be a Noetherian ring and $\mathfrak{p} \subset \mathfrak{q}$ be an unsaturated chain of prime ideals of A. Then the set E of prime ideals $\mathfrak{r}$ of A such that $\mathfrak{p} \subset \mathfrak{r} \subset \mathfrak{q}$ is infinite and we have*

$$\bigcap_{\mathfrak{r} \in E} \mathfrak{r} = \mathfrak{p}, \quad \bigcup_{\mathfrak{r} \in E} \mathfrak{r} = \mathfrak{q}.$$

Proof. By passing to the quotient ring, we may assume that $\mathfrak{p} = \{0\}$. Now by Lemma 7.2.2 we have $\bigcup_{\mathfrak{r} \in E} \mathfrak{r} = \mathfrak{q}$, and it follows from Proposition 1.1.4 that E is infinite.

Now let y be a nonzero element in $\bigcap_{\mathfrak{r} \in E} \mathfrak{r}$. Since the height of $\mathfrak{q}$ is finite by Corollary 7.2.4, by localization at a prime ideal of height 2, we may assume that $\mathrm{ht}(\mathfrak{q}) = 2$. Then the ring A/yA has dimension 1, hence each prime ideal $\mathfrak{r} \in E$ is minimal. But a Noetherian ring can have only finitely many minimal prime ideals, a contradiction. Therefore $\bigcap_{\mathfrak{r} \in E} \mathfrak{r} = \{0\}$, and the proof is finished. □

Proposition 7.2.24. *Let A be a Noetherian ring with dimension ≥ 2, and h an integer such that $0 < h < \dim(A)$.*

(a) The ring A has infinitely many prime ideals of A with height h.

(b) If A is of finite dimension, then there are infinitely many prime ideals $\mathfrak{p}$ such that $\mathrm{ht}(\mathfrak{p}) = h$, $\mathrm{coht}(\mathfrak{p}) = \dim(A) - h$.

Proof. Since a Noetherian local ring has finite dimension, each prime ideal of A has finite height. Since $h < \dim(A)$, there exist an integer $n > h$, a prime ideal $\mathfrak{p}$ with $\mathrm{ht}(\mathfrak{p}) = n$, and a chain $\mathfrak{p}_0 \subset \cdots \subset \mathfrak{p}_n = \mathfrak{p}$ of prime ideals of A. Then $\mathrm{ht}(\mathfrak{p}_i) = i$ for each i, so $\mathrm{ht}(\mathfrak{r}) = h$ for every prime ideal $\mathfrak{r}$ such that $\mathfrak{p}_{h-1} \subset \mathfrak{r} \subset \mathfrak{p}_{h+1}$. By Proposition 7.2.23 the set E of such prime ideals $\mathfrak{r}$ is infinite, whence (a).

If A is of finite dimension, then we may take $n = \dim(A)$ in the above proof. Then for each $\mathfrak{r} \in E$, we have $\mathrm{ht}(\mathfrak{r}) = h$ and $\mathrm{coht}(\mathfrak{r}) \leq n - h$. Since $\mathfrak{r} \subset \mathfrak{p}_{h+1} \subset \cdots \subset \mathfrak{p}_n$ is a chain of length $n - h$, we then get $\mathrm{coht}(\mathfrak{r}) = \dim(A) - h$. □

Example 7.2.25. There exist non-Noetherian integral domains of dimension 2 with only one prime ideal of height 1, for example the ring of a valuation of rank 2.

7.2.3 Extension of scalars

Proposition 7.2.26. *Let $\rho : A \to B$ be a local homomorphism of Noetherian local rings, M a finitely generated A-module and N a finitely generated B-module. If $\bar{B} = B \otimes_A \kappa_A$ and $\bar{N} = N \otimes_B \bar{B}$, then*

$$\dim_B(M \otimes_A N) \leq \dim_A(M) + \dim_{\bar{B}}(\bar{N})$$

and the equality holds if N is flat over A.

Proof. Let $S \subseteq \mathfrak{m}_A$ be a system of parameters for the A-module M and $T \subseteq \mathfrak{m}_B$ a system of parameters for the $\bar{B}$-module $\bar{N}$. Let E be the B-module $M \otimes_A N$. Since ρ is local, we have $\rho(S) \subseteq \mathfrak{m}_B$, and

$$\begin{aligned} E/(\rho(S)E + TE) &= E \otimes_B (B/(\rho(S)B + TB)) = M \otimes_A N \otimes_B (B/TB) \otimes_A (A/SA) \\ &= M \otimes_A (A/SA) \otimes_A N \otimes_B (B/TB) = (M/SM) \otimes_A (N/TN) \end{aligned}$$

Moreover, by Proposition 3.2.14, we have

$$\mathrm{supp}(M/SM) = \{\mathfrak{m}_A\}, \quad \mathrm{supp}(\bar{N}/T\bar{N}) = \{\mathfrak{m}_B\}$$

which implies, according to Proposition 1.4.39 and 1.4.42, that

$$\begin{aligned} \mathrm{supp}((M/SM) \otimes_A (N/TN)) &= \mathrm{supp}(N/TN) \cap \rho^{*-1}(\{\mathfrak{m}_A\}) \\ &= \mathrm{supp}((N/TN) \otimes_A \kappa_A) = \mathrm{supp}(\bar{N}/T\bar{N}) = \{\mathfrak{m}_B\}. \end{aligned}$$

Therefore the B-module $E/(\rho(S)E + TE)$ has finite length, which implies by Theorem 7.2.14 that

$$\dim_B(E) \leq |S| + |T| = \dim_A(B) + \dim_{\bar{B}}(\bar{N}).$$

Suppose now that N is flat over A. Let $\mathfrak{a}$ (resp. $\mathfrak{b}$) be the annihilator of M (resp. N). Then we have

$$\mathrm{supp}(E) = \mathrm{supp}(N) \cap \rho^{*-1}(\mathrm{supp}(N)) = V(\mathfrak{b}) \cap \rho^{*-1}(V(\mathfrak{a})) = V(\mathfrak{b} + \mathfrak{a}B),$$

and therefore $\dim_B(M \otimes_A N) = \dim(B/(\mathfrak{b} + \mathfrak{a}B))$. On the other hand,

$$\dim_A(M) + \dim_{\bar{B}}(\bar{N}) = \dim(A/\mathfrak{a}) + \dim(B/(\mathfrak{b} + \mathfrak{m}_A B)).$$

Let $A' = A/\mathfrak{a}$ and $B' = B/(\mathfrak{b} + \mathfrak{a}B)$ and let $\rho' : A' \to B'$ be the local homomorphism deduced from ρ by passing to quotients. Since the annihilator of N is $\mathfrak{b}$, N is a finitely generated $B/\mathfrak{b}$-module with support $\mathrm{Spec}(B/\mathfrak{b})$, and flat on A. By Proposition 1.4.39 and **??**, the local homomorphism $A \to B/\mathfrak{b}$ deduced from ρ therefore has the property (PM). By extension of the scalars, we deduce that ρ' has the property (PM). According to Proposition 7.1.21, we have

$$\dim(B') \geq \dim(A') + \dim(B'/\mathfrak{m}_{A'}B')$$

and as the ring $B'/\mathfrak{m}_{A'}B'$ is isomorphic to $B/(\mathfrak{b} + \mathfrak{m}_A B)$, our assertion follows. □

Corollary 7.2.27. *Let $\rho : A \to B$ be a local homomorphism of Noetherian local rings.*

(a) *We have $\dim(B) \leq \dim(A) + \dim(B \otimes_A \kappa_A)$, and the equality holds if B is flat over A.*

(b) *Suppose that B is flat over A, then a subset S of $\mathfrak{m}_A$ is secant for A if and only if $\rho(S)$ is secant for B.*

Proof. The assertion in (a) follows from Proposition 7.2.26, by setting $M = A$, $N = B$. For (b), if B is flat over A, then

$$\dim(B) = \dim(A) + \dim(\bar{B})$$

where $\bar{B} = B \otimes_A \kappa_A$. Since ρ is injective by **??**, we have $|\rho(S)| = |S|$. Finally $B' = B/\rho(S)B$ is a flat module on $A' = A/SA$, hence

$$\dim(B') = \dim(A') + \dim(\bar{B})$$

since $B'/\mathfrak{m}_{A'}B'$ is isomorphic to $\bar{B}$. The claim in (b) then follows from these equalities. $\square$

Corollary 7.2.28. *Let $\rho : A \to B$ be a ring homomorphism of Noetherian rings. Then*

$$\dim(B) \leq \dim(A) + \sup_{\mathfrak{p}\in\mathrm{Spec}(A)} \dim(B \otimes_A \kappa(\mathfrak{p})).$$

Proof. Let $\mathfrak{P}$ be a prime ideal of B and $\mathfrak{p} = \mathfrak{P}^c$. By Corollary 7.2.27, we have

$$\dim(B_{\mathfrak{P}}) \leq \dim(A_{\mathfrak{p}}) + \dim(B_{\mathfrak{P}} \otimes_A \kappa(\mathfrak{p})) \leq \dim(A) + \dim(B \otimes_A \kappa(\mathfrak{p}))$$

takign supremum on $\mathfrak{P}$, we get the claim. $\square$

Corollary 7.2.29. *Let $\rho : A \to B$ be a ring homomorphism of Noetherian rings and $\mathfrak{p} \subset \mathfrak{q}$ be prime ideals of A.*

(a) If the going-up property holds for ρ, then

$$\dim(B \otimes_A \kappa(\mathfrak{p})) \leq \dim(B \otimes_A \kappa(\mathfrak{q})).$$

(b) If the going-down property holds for ρ, then

$$\dim(B \otimes_A \kappa(\mathfrak{p})) \geq \dim(B \otimes_A \kappa(\mathfrak{q})).$$

Proof. First assume the going-up property for ρ. Let $\mathfrak{P}_0 \subset \cdots \subset \mathfrak{P}_r$ be a chain of prime ideals of B lying over $\mathfrak{p}$ and $\mathfrak{p} = \mathfrak{p}_0 \subset \cdots \subset \mathfrak{p}_s = \mathfrak{q}$ be a chain of prime ideals of A. Then by the going-up property, there is a chain $\mathfrak{P}_r \subset \cdots \subset \mathfrak{P}_{r+s}$ of prime ideals of B such that $\mathfrak{P}_{r+i}^c = \mathfrak{p}_i$. If $\mathfrak{Q} := \mathfrak{P}_{r+s}$, then $\mathfrak{Q}$ is lying over $\mathfrak{q}$ and $\mathrm{ht}(\mathfrak{Q}/\mathfrak{p}^e) \geq r + s$. Applying Corollary 7.2.28 to the homomorphism $(A/\mathfrak{p})_{\mathfrak{q}} \to (B/\mathfrak{p}^e)_{\mathfrak{Q}}$, we get

$$r + s \leq \mathrm{ht}(\mathfrak{Q}/\mathfrak{p}^e) \leq \mathrm{ht}(\mathfrak{q}/\mathfrak{p}) + \dim(B_{\mathfrak{Q}}/\mathfrak{q}B_{\mathfrak{Q}}) \leq s + \dim(B \otimes_A \kappa(\mathfrak{q}))$$

whence the claim in (a).

Now we prove (b), so assume the going down property for ρ. We may assume that $\mathrm{ht}(\mathfrak{q}/\mathfrak{p}) = 1$, and it is enough to prove that, given a chain $\mathfrak{Q}_0 \subset \cdots \subset \mathfrak{Q}_t$ of prime ideals of B lying over $\mathfrak{q}$ such that $\mathrm{ht}(\mathfrak{Q}_i/\mathfrak{Q}_{i-1}) = 1$, we can construct a chain of prime ideals $\mathfrak{P}_0 \subset \cdots \subset \mathfrak{P}_t$ of B lying over $\mathfrak{p}$ such that $\mathfrak{P}_i \subset \mathfrak{Q}_i$ and $\mathrm{ht}(\mathfrak{P}_i/\mathfrak{P}_{i-1}) = 1$ for all i.

The existence of $\mathfrak{P}_0$ is guaranteed by the going-down property, and if $\mathfrak{P}_0, \dots, \mathfrak{P}_i$ is constructed, take $x \in \mathfrak{q} \setminus \mathfrak{p}$ and let $\mathfrak{T}_1, \dots, \mathfrak{T}_s$ be the minimal prime ideals belonging to $\mathfrak{P}_i + xB$. Then by Proposition 7.2.1 we have $\mathrm{ht}(\mathfrak{T}_j/\mathfrak{P}_i) = 1$, while $\mathrm{ht}(\mathfrak{Q}_{i+1}/\mathfrak{P}_i) \geq 2$, so there exists an element $y \in \mathfrak{Q}_{i+1} \setminus \bigcup_{j=1}^{s} \mathfrak{T}_j$. Let $\mathfrak{P}_{i+1}$ be a minimal prime ideal of $\mathfrak{P}_i + yB$, then $\mathfrak{P}_{i+1} \neq \mathfrak{T}_j$ for all j and Proposition 7.2.1 shows that $\mathrm{ht}(\mathfrak{P}_{i+1}/\mathfrak{P}_i) = 1$, hence $\rho(x) \notin \mathfrak{P}_{i+1}$. Since $x \in \mathfrak{q}$ and $\mathrm{ht}(\mathfrak{q}/\mathfrak{p}) = 1$, we must have $\mathfrak{P}_{i+1}^c = \mathfrak{p}$, which finishes the induction process. $\square$

Corollary 7.2.30. *Let A be a Noetherian ring and n a positive integer. Then*

$$\dim(A[X_1, \dots, X_n]) = \dim(A) + n.$$

Proof. Let $B = A[X_1, \dots, X_n]$, then for any prime ideal $\mathfrak{p}$ of A, $B \otimes_A \kappa(\mathfrak{p})$ is identified with the polynomial ring on $\kappa(\mathfrak{p})$ with indeterminates $X_1, \dots, X_n$, whence of dimension n. By Corollary 7.2.28 we then have $\dim(B) \leq \dim(A) + n$, and the reverse inequality is already established. □

Corollary 7.2.31. *Let $\rho : A \to B$ be a homomorphism of Noetherian rings and $\rho^* : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ be the induced map.*

(a) If ρ^* is surjective, then $\mathrm{ht}(\mathfrak{a}^e) \leq \mathrm{ht}(\mathfrak{a})$ for any ideal $\mathfrak{a}$ of A.

(b) If B is a faithfully flat A-module, then $\mathrm{ht}(\mathfrak{a}^e) = \mathrm{ht}(\mathfrak{a})$ for any ideal $\mathfrak{a}$ of A.

Proof. If B is faithfully flat, then ρ^* is surjective by Corollary 1.2.43 and we have $\mathrm{ht}(\mathfrak{a}) \leq \mathrm{ht}(\mathfrak{a}^e)$ by **??**. Therefore it suffices to prove assertion (a). Assume that ρ^* is surjective and let $\mathfrak{p}$ be a prime ideal of A containing $\mathfrak{a}$ such that $\mathrm{ht}(\mathfrak{p}) = \mathrm{ht}(\mathfrak{a})$. Let $\mathfrak{P}$ be a prime ideal of B lying over $\mathfrak{p}$ whose image in $B \otimes \kappa(\mathfrak{p})$ is minimal. Then $\dim(B_{\mathfrak{P}} \otimes \kappa(\mathfrak{p})) = 0$ and Corollary 7.2.28 implies $\dim(B_{\mathfrak{P}}) \leq \dim(A_{\mathfrak{p}})$, whence

$$\mathrm{ht}(\mathfrak{a}^e) \leq \mathrm{ht}(\mathfrak{P}) = \dim(B_{\mathfrak{P}}) \leq \dim(A_{\mathfrak{p}}) = \mathrm{ht}(\mathfrak{p}) = \mathrm{ht}(\mathfrak{a}).$$

This proves the claim. □

Proposition 7.2.32. *Let A be a Noetherian ring, $\mathfrak{a}$ an ideal of A, and M a finitely generated A-module. Let $\widehat{A}$ (resp. $\widehat{M}$) be the Hausdorff completions of A (resp. M) under the $\mathfrak{a}$-adic topology.*

(a) Let $\mathfrak{p}$ be a prime ideal of A containing $\mathfrak{a}$. Then $\hat{\mathfrak{p}}$ is a prime ideal of $\widehat{A}$ and $\dim_{\widehat{A}_{\hat{\mathfrak{p}}}}(\widehat{M}_{\hat{\mathfrak{p}}}) = \dim_{A_{\mathfrak{p}}}(M_{\mathfrak{p}})$.

(b) We have $\dim_{\widehat{A}}(\widehat{M}) = \sup_{\mathfrak{m}} \dim_{\widehat{A}_{\hat{\mathfrak{p}}}}(\widehat{M}_{\hat{\mathfrak{p}}}) = \sup_{\mathfrak{m}} \dim_{A_{\mathfrak{m}}}(M_{\mathfrak{m}})$, where $\mathfrak{m}$ runs through prime (resp. maximal) ideals of A containing $\mathfrak{a}$. In particular, $\dim_{\widehat{A}}(\widehat{M}) \leq \dim_A(M)$.

Proof. Since the $\mathfrak{a}$-adic topology on $A/\mathfrak{p}$ is trivial, by Corollary 2.4.15 $\widehat{A}/\hat{\mathfrak{p}}$ is identified with $A/\mathfrak{p}$, so $\hat{\mathfrak{p}}$ is a prime ideal of $\widehat{A}$. Since $\widehat{A}$ is flat over A, $\widehat{A}_{\hat{\mathfrak{p}}}$ is flat over $A_{\mathfrak{p}}$. Moreover the canonical homomorphism of A in $\widehat{A}$ induces an isomorphism of $A/\mathfrak{a}$ on $\widehat{A}/\hat{\mathfrak{a}}$, therefore also an isomorphism of $A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$ on $\widehat{A}_{\hat{\mathfrak{p}}}/\mathfrak{p}\widehat{A}_{\hat{\mathfrak{p}}}$. We conclude assertion (a) by applying Proposition 7.2.26 to the homomorphism $A_{\mathfrak{p}} \to \widehat{A}_{\hat{\mathfrak{p}}}$ and to the modules $M_{\mathfrak{p}}$ and $\widehat{A}_{\hat{\mathfrak{p}}}$, noting that $M_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} \widehat{A}_{\hat{\mathfrak{p}}}$ is isomorphic to $\widehat{M}_{\hat{\mathfrak{p}}}$.

By Proposition 2.4.24, the map $\mathfrak{m} \mapsto \hat{\mathfrak{m}}$ is a bijection from the set of maximal ideals of A containing $\mathfrak{a}$ to the set of maximal ideals of $\widehat{A}$. Thus assertion (b) follows from Proposition 7.1.12. □

Corollary 7.2.33. *Let A be a Noetherian Zariski ring. Then for any finitely generated A-module M, we have $\dim_A(M) = \dim_{\widehat{A}}(\widehat{M})$.*

Proof. The topology on A is $\mathfrak{a}$-adic, where $\mathfrak{a}$ is an ideal contained in the Jacobson radical of A. Thus the claim follows from Proposition 7.2.32. □

Corollary 7.2.34. *Let A be a Noetherian ring, $\mathfrak{a}$ an ideal of A, and $\widehat{A}$ the Hausdorff completion with respect to the $\mathfrak{a}$-adic topology. Then $\dim(\widehat{A}) \leq \dim(A)$, and the equality holds if A is local and $\mathfrak{a}$ is proper.*

Corollary 7.2.35. *Let A be a Noetherian ring and n a positive integer. Then*

$$\dim(A[\![X_1, \dots, X_n]\!]) = \dim(A) + n.$$

Proof. The ring $A[\![X_1,\dots,X_n]\!]$ is the Hausdorff completion of $A[X_1,\dots,X_n]$ with respect to the $\mathfrak{m}$-adic topology, where $\mathfrak{m}$ is generated by $X_1,\dots,X_n$. We then have

$$\dim(A[\![X_1,\dots,X_n]\!]) \leq \dim(A[X_1,\dots,X_n]) = \dim(A) + n$$

and the reverse inequality is already established. □

Corollary 7.2.36. *Let A be a Noetherian ring and $\mathfrak{a}$ an ideal of A. Suppose that A is Hausdorff and complete under the $\mathfrak{a}$-adic topology. Then for any positive integer n, we have*

$$\dim(A\{X_1,\dots,X_n\}) = \dim(A) + n.$$

Proof. The ring $A\{X_1,\dots,X_n\}$ is the $\mathfrak{a}$-adic completion of $A[X_1,\dots,X_n]$, so

$$\dim(A\{X_1,\dots,X_n\}) \leq \dim(A) + n.$$

The reverse inequality can be established just like the power series ring. □

Corollary 7.2.37. *Let A be a Noetherian local ring, identified with a subring of its completion $\widehat{A}$, and B a subring of $\widehat{A}$ containing A. Suppose that B is a Noetherian local ring and that we have $\mathfrak{m}_A B = \mathfrak{m}_B$. Then $\dim(A) = \dim(B)$.*

Proof. By Proposition 2.4.37 the canonical injection of A into B extends to an isomorphism of $\widehat{A}$ on the completion $\widehat{B}$ of B, whence $\dim(B) = \dim(A)$. □

7.2.4 Exersise

Exercise 7.2.1. Let k be a field, and set

$$A = k[x,y], \quad B = [x,y,x/y], \quad \mathfrak{p} = (x,y)A, \quad \mathfrak{q} = (y,x/y)B$$

check that

$$\mathfrak{p} = \mathfrak{q} \cap A, \quad \mathrm{ht}\mathfrak{p} = \mathrm{ht}\mathfrak{q} = 2, \quad \dim B_{\mathfrak{q}}/\mathfrak{p}B_{\mathfrak{q}} = 1$$

so that

$$\mathrm{ht}\mathfrak{q} < \mathrm{ht}\mathfrak{p} + \dim B_{\mathfrak{q}}/\mathfrak{p}B_{\mathfrak{q}}$$

Show also by a concrete example that the going-down property does not hold between A and B.

Proof. Clealry we can see $(x,y)A \subseteq (y,x/y)B$ so that $\mathfrak{p} \subseteq \mathfrak{q} \cap A$. Also, $\mathfrak{p}$ is a maximal ideal in A, hence $\mathfrak{p} = \mathfrak{q} \cap A$.

We have $(y) = (y,x/y)$ in B, so

$$(y,x/y) \supset (y,x) \supset (0)$$

is a maximal chain of $\mathfrak{q}$. For $\mathfrak{p}$, it is easy to verify height $\mathfrak{p} = 2$.

For $\dim B_{\mathfrak{q}}/\mathfrak{p}B_{\mathfrak{q}}$, note that

$$(x,y)B \subset (y,x/y)B$$

Since (x,y) is maximal in A, if there is an ideal Q sandwiched between them, then Q contains a polynomial in x/y, hence contains x/y since Q is prime. This implies $\dim B_{\mathfrak{q}}/\mathfrak{p}B_{\mathfrak{q}} = 1$.

Consider the following chain in A:

$$P_1 = (y)A \subset (x,y)A = P_2$$

And choose a prime $Q_2 = (x,y)B$ in B such that $Q_2^c = P_2$. If Q_1 is a prime ideal in B such that $Q_1^c = P_1$, then $(y)A \subseteq Q_1$, in particular $y \in Q_1$. Note that since $x/y \in B$, this implies $x = (x/y) \cdot y \in Q_1$. But this means $Q_2 \subseteq Q_1$, hence there is no prime ideal contained in Q_2 satisfying the condition, therefore the going down property does not hold. □

Exercise 7.2.2. Let A be a ring, U an open subset of $\operatorname{Spec}(A)$.

(a) Show that the map $\mathfrak{p} \mapsto V(\mathfrak{p}) \cap U$ induce a bijection of U onto the set of irreducible closed subsets of U.

(b) Show that $\dim(U)$ is the supremum of the length of chains of prime ideals of A contained in U.

(c) Let Φ be the set of minimal prime ideals of A contained in U. Show that

$$\dim(U) = \sup_{\mathfrak{p}\in\Phi} \dim(V(\mathfrak{p}) \cap U)$$

7.3 Hilbert-Samuel series

Let A be an Artinian ring and consider the power series ring $A[\![T]\!]$ with one indeterminate on A. In this section, we denote by $A((T))$ the fraction ring $A[\![T]\!]_T$ of $A[\![T]\!]$ at the element T, which is identified with the subring of the Laurent series ring $A[\![T, T^{-1}]\!]$ consisting of elements $\sum_{n\in\mathbb{Z}} a_n T^n$ that is **bounded below**, that is, there exist $n_0 \in \mathbb{N}$ such that $a_n = 0$ for all $n < n_0$. The ring $\mathbb{Z}((T))$ will be our main focus in this section.

For any $n, p \in \mathbb{Z}$, we extend the definition of the binomial coefficient $\binom{n}{p}$ by defining $\binom{n}{p} = 0$ if $p < 0$ or $p > n$.

Lemma 7.3.1. *The element* $(1 - T)$ *is invertible in* $\mathbb{Z}((T))$*, and for any* $r > 0$ *we have*

$$(1-T)^{-r} = \sum_{n\in\mathbb{Z}} \binom{n+r-1}{r-1} T^n = \sum_{n\in\mathbb{N}} \binom{n+r-1}{r-1} T^n.$$

Proof. The element $1 - T$ is invertible with inverse $\sum_{n\in\mathbb{N}} T^n$, so

$$(1-T)^{-r} = \Big(\sum_{n\in\mathbb{N}} T^n\Big)^r = \sum_{n_1,\dots,n_r\in\mathbb{N}} T^{n_1+\cdots+n_r} = \sum_{n\in\mathbb{N}} \binom{n+r-1}{r-1} T^n$$

so the claim follows. □

Let $Q(T) \in \mathbb{Z}[T, T^{-1}]$ and r be a positive integer. If $P(T) = (1-T)^{-r}Q(T)$, then it is easy to see $P(T) \in \mathbb{Z}((T))$. In fact, if

$$Q(T) = \sum_{n\in\mathbb{Z}} a_n T^n, \quad P = \sum_{n\in\mathbb{Z}} b_n T^n$$

then

$$b_n = \sum_{i\in\mathbb{Z}} a_i \binom{n-i+r-1}{r-1} = \sum_{i\leq n} a_i \binom{n-i+r-1}{r-1}$$

where the summation is finite since $Q(T)$ is bounded below. If n_0 is the supremum of the integers $i \in \mathbb{Z}$ such that $a_i \neq 0$, then for $n \geq n_0$, we can write

$$b_n = \sum_{i\in\mathbb{Z}} a_i \binom{n-i+r-1}{r-1} = \frac{1}{(r-1)!} \sum_{i\in\mathbb{Z}} a_i \prod_{j=1}^{r-1} (n-i+j).$$

If $c = Q(1) = \sum_i a_i$, then we get

$$b_n = c\frac{n^{r-1}}{(r-1)!} + \rho_n n^{r-2} \tag{7.3.1}$$

where the rational number ρ_n tends to a limit as n increases indefinitely. Therefore we deduce the relationship

$$Q(1) = (r-1)! \lim_{n\to\infty} n^{1-r} b_n. \tag{7.3.2}$$

For two elements $F = \sum_n a_n T^n$ and $G = \sum_n b_n T^n$ in $\mathbb{Z}((T))$, we denote $F \leq G$ by the relation $a_n \leq b_n$ for all n, which is an order relation compatible with the ring structure of $\mathbb{Z}((T))$. We have $(1-T)^{-r} \geq 1$, and if an element Q is bigger than 0, then the integer $Q(1)$ is positive.

Lemma 7.3.2. *Let P be a nonzero element in $\mathbb{Z}((T))$ such that $(1-T)^r P \in \mathbb{Z}[T, T^{-1}]$ for some $r \in \mathbb{Z}$. Then there exists a unique $Q \in \mathbb{Z}[T, T^{-1}]$ such that $P = (1-T)^{-d} Q$ and $Q(1) \neq 0$, for some $d \in \mathbb{Z}$. If $P \geq 0$, then $Q(1) \geq 0$.*

Proof. Since $(1-T)^r P \in \mathbb{Z}[T, T^{-1}]$, we can write $P = (1-T)^{-1} T^n R(T)$ with $R(T) \in \mathbb{Z}[T]$, and by Euclidean division we may assume that $R(1) \neq 0$, which proves the existence of d and Q. On the other hand, if we have

$$(1-T)^r Q(T) = (1-T)^s R(T)$$

with $r > s$ and $Q, R \in \mathbb{Z}[T, T^{-1}]$, then $R(T) = (1-T)^{r-s} Q(T)$, whence $R(1) = 0$. This proves the uniqueness part.

Finally, assume that $P \geq 0$. If $P = (1-T)^{-d} Q$ with $d < 0$ then $P(1) = 0$ and we must have $P = 0$, which is a contradiction. Therefore $d \geq 0$, and if $d = 0$ then $P(1) = Q(1) \geq 0$. If $d > 0$, then $Q(1)$ is given by the formula (7.3.2), so $Q(1) \geq 0$. □

Lemma 7.3.3. *Let P, Q be elements of $\mathbb{Z}[T, T^{-1}]$, $p, q \in \mathbb{Z}$ and $P(1) > 0$. If*

$$(1-T)^{-p} P \leq (1-T)^{-q} Q$$

then, either $p < q$, or $p = q$ and $P(1) \leq Q(1)$.

Proof. Suppose $p \geq q$, then we have $(1-T)^{-p}[(1-T)^{p-q} Q - P] \geq 0$. Since $R = (1-T)^{p-q} Q - P \in \mathbb{Z}[T, T^{-1}]$, by Lemma 7.3.3 we have $R(1) \geq 0$. If $p > q$, then $R(1) = -P(1) \geq 0$, which is a contradiction. If $p = q$, then $R(1) = Q(1) - P(1)$, whence $P(1) \leq Q(1)$. □

7.3.1 Poincaré series of graded modules over polynomial rings

Let A_0 be an Artinian ring, I a finite set and consider the ring $A = A_0[(X_i)_{i\in I}]$. For each $i \in I$, let d_i be a positive integer. Endow A with the graded ring structure of type $\mathbb{Z}$ such that the elements of A_0 are homogeneous of degree 0 and each X_i is homogeneous of degree d_i. If $d_i = 1$ for each i, we then get the usual graduation on the polynomial ring A.

Let M be a finitely generated graded A-module such that all the homogeneous components of M are A_0-module of finite length. The **Poincaré series** of M, denoted by P_M, is the elemnt in $\mathbb{Z}((T))$ defined by

$$P_M = \sum_{n\in\mathbb{Z}} \ell_{A_0}(M_n) \cdot T^n.$$

Theorem 7.3.4. *The element $Q_M = P_M \prod_{i\in I}(1 - T^{d_i})$ is in $\mathbb{Z}[T, T^{-1}]$.*

Proof. If $I = \varnothing$, then $A = A_0$, and the family $(\ell_{A_0}(M_n))_{n\in\mathbb{Z}}$ has finite support since M is a finitely generated A_0-module, whence has finite length. The theorem is therefore proved in this case.

Now we prove by induction on the cardinality of I. Fix $j \in I$, $J = I \setminus \{j\}$, and denote by A' the subring of A generated by A_0 and the X_i with $i \in J$. Consider the homothety h_{X_j} with ratio X_j with kernel R and cokernel S. Then for each $n \in \mathbb{Z}$, we have an exact sequence

$$0 \longrightarrow R_{n-d_j} \longrightarrow M_{n-d_j} \xrightarrow{h_{X_j}} M_n \longrightarrow S_n \longrightarrow 0$$

whence R_n and S_n have finite legnth, and we have

$$\ell_{A_0}(M_n) - \ell_{A_0}(M_{n-d_j}) = \ell_{A_0}(S_n) - \ell_{A_0}(R_{n-d_j}). \tag{7.3.3}$$

Since M is a finitely generated module over the Noetherian A, it is Noetherian so the A-modules R and S are finitely generated. Since they are both annihilated by X_j, they are finitely generated as A'-modules. According to the induction hypothesis, the elements Q_R and Q_S are in $\mathbb{Z}[T, T^{-1}]$. Moreover, due to (7.3.3), we have

$$(1 - T^{d_j})P_M = P_M - T^{d_j}P_M = P_S - T^{d_j}P_R \tag{7.3.4}$$

which implies

$$Q_M = P_M \prod_{i\in I}(1 - T^{d_i}) = P_S \prod_{i\in J}(1 - T^{d_i}) - T^{d_j}P_R \prod_{i\in J}(1 - T^{d_i}).$$

This shows Q_M is in $\mathbb{Z}[T, T^{-1}]$, and our conclusion is then proved. □

Example 7.3.5. Let M_0 be an A_0-module and $M = A \otimes_{A_0} M_0$. Then with the notations above, we have $R = 0$ and $S = A' \otimes_{A_0} M_0$, so $Q_M = Q_S$ by (7.3.4). Since $Q_{M_0} = \ell_{A_0}(M_0)$, by induction $Q_M = \ell_{A_0}(M_0)$, whence

$$P_M = \ell_{A_0}(M_0) \prod_{i\in I}(1 - T^{d_i})^{-1}.$$

Corollary 7.3.6. *Assume that $d_i = 1$ for all $i \in I$ and set $r = |I|$, $c_M = Q_M(1)$.*

(a) If $r = 0$, then $\ell_{A_0}(M) = c_M$.

(b) If $r = 1$, then $\ell_{A_0}(M_n) = c_M$ for n large enough.

(c) If $r > 1$, then $\ell_{A_0}(M_n) = \dfrac{c_M}{(r-1)!}n^{r-1} + \rho_n n^{r-2}$, where ρ_n has a limit when $n \to +\infty$.

Proof. In this case we have $P_M = Q_M(1 - T)^{-r}$, so this follows from the formula (7.3.2). □

For simplicity, we may endow A with the usual graduation henceforth, so we have

$$P_M = Q_M(1 - T)^{-r}.$$

Example 7.3.7 (Example of Poincaré series for graded modules).

(a) Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of graded A-modules and of homomorphisms of degree 0 such that M is finitely generated over A and M_n of finite length over A_0 for each n. Then by the additivity of legnth, we have

$$P_M = P_{M'} + P_{M''}, \quad Q_M = Q_{M'} + Q_{M''}, \quad c_M = c_{M'} + c_{M''}.$$

(b) Let $M(p)$ be the modulu deduced from M by shifting p from the graduation of M. Since $M(p)_n = M_{p+n}$, we have

$$P_{M(p)} = T^{-p}P_M, \quad Q_{M(p)} = T^{-p}Q_M, \quad c_{M(p)} = c_M.$$

(c) Let M be a free graded A-module generated by lineraly independent homogeneous elements of degrees $\delta_1, \dots, \delta_s$. As M is isomorphic to $A(-\delta_1) \oplus \cdots \oplus A(-\delta_s)$, we have

$$P_M = \ell(A_0)\Big(\sum_{i=1}^{s} T^{\delta_s}\Big)(1 - T)^{-r}, \quad Q_M = \ell(A_0)\Big(\sum_{i=1}^{s} T^{\delta_s}\Big), \quad c_M = s \cdot \ell(A_0).$$

(d) Let M be a finitely generated A-module and suppose that there exists a long exact sequence of graded A-modules and of homomorphisms of degree 0

$$0 \longrightarrow L_n \longrightarrow L_{n-1} \longrightarrow \cdots \longrightarrow L_0 \longrightarrow M \longrightarrow 0$$

such that for each k, the module L_k is free and generated by linearly independent homogeneous elements of degrees $\delta_{k,1}, \ldots, \delta_{k,m_k}$. Then by (c),

$$Q_M = \ell(A_0) \sum_{k=0}^{n} \sum_{j=1}^{m_k} (-1)^k T^{\delta_{k,j}}, \quad c_M = \ell(A_0) \sum_{k=0}^{n} (-1)^k m_k.$$

Proposition 7.3.8. *Let M be a graded A-module. If M is generated by M_0 and M_0 is an A_0-module with finite length. Then we have*

$$P_M \leq (1-T)^{-r} \ell_{A_0}(M_0), \quad c_M \leq \ell_{A_0}(M_0).$$

Moreover, the following conditions are equivalent:

(i) $c_M = \ell_{A_0}(M_0)$.

(ii) $P_M = \ell_{A_0}(M_0)(1-T)^{-r}$.

(iii) The canonical homomorphism $\varphi : A \otimes_{A_0} M_0 \to M$ is bijective.

Proof. Let R be the kernel of φ. Then since φ is surjective, we have

$$P_M = P_{A \otimes_{A_0} M_0} - P_R = \ell_{A_0}(M_0)(1-T)^{-r} - P_R, \quad c_M = \ell_{A_0}(M_0) - c_R$$

which proves the first assertion. Now the conditions (i), (ii), and (iii) are equivalent to $c_R = 0$, $P_R = 0$, and $R = 0$, respectively, so we have (iii)$\Rightarrow$(ii)$\Rightarrow$(i). It remains to show that $c_R = 0$ implies $R = 0$. Now assume that $R \neq 0$ and let

$$0 = M^r \subset M^{r-1} \subset \cdots \subset M^0 = M_0$$

be a Jordan-Hölder series of the A_0-module M_0. Let R^k be the intersection of R and the image of $A \otimes_{A_0} M^k$ in $A \otimes_{A_0} M_0$. Then there exists an integer k such that $R^k \neq R^{k-1}$. Define $L = R^{k-1}/R^k$, we have $0 \leq c_L \leq c_R$, and it suffices to prove that $c_L \neq 0$. Now, if K is the quotient field of A_0 by the maximal ideal annihilating M^{k-1}/M^k, then L is identified with a nonzero graded submodule of $K[(X_i)_{i \in I}]$. So L contains a submodule isomorphic to a shifted module of $K[(X_i)_{i \in I}]$. As $c_{K[(X_i)_{i \in I}]} = 1$, we have $c_L \geq 1$, which proves the claim. □

Proposition 7.3.9. *Suppose that A_0 is a field and M is a finitely generated graded A-module. If K is the field of fraction of A, then c_M is equal to the rank of the A-module M, which is also the dimension of the K-vector space $M \otimes_A K$.*

Proof. This is clear if $M = A$, since $c_A = 1$. On the other hand, let $x \in A$, homogeneous of degree d, and not zero. We have $(A/xA) \otimes_A K = 0$. From the exact sequence

$$0 \longrightarrow A(-d) \xrightarrow{h_x} A \longrightarrow A/xA \longrightarrow 0$$

we get $c_{A/xA} = 0$. The proposition is therefore verified when M is generated by a homogeneous element. The general case follows, since any finite type graded A-module has a composition sequence whose quotients are of the previous form. □

Example 7.3.10. Let A_0 be a field, P a homogeneous polynomial in A of degree s, and M the module A/PA. Since the rank of M equals to the degree of P, by Proposition 7.3.9 we have $c_M = s$. On the other hand, it is not hard to compute that

$$\ell_{A_0}(M_n) = \binom{n+r-1}{r-1} - \binom{n-s+r-1}{r-1}$$

and therefore

$$c_M = (r-1)! \lim_{n\to\infty} n^{1-r}\left(\binom{n+r-1}{r-1} - \binom{n-s+r-1}{r-1}\right) = s$$

verifying our claim.

7.3.2 Hilbert-Samuel series of a good-filtered module

Let A be a Noetherian ring, $\mathfrak{I}$ an ideal of A, and M a finitely generated A-module. Recall that an exhaustive filtration $\mathcal{F} = (M_n)_{n\in\mathbb{Z}}$ on M consisting of submodules is called **$\mathfrak{I}$-good** if

(a) $\mathfrak{I}M_n \subseteq M_{n+1}$ for all $n \in \mathbb{Z}$.

(b) There exists an integer n_0 such that $\mathfrak{I}M_n = M_{n+1}$ for all $n \geq n_0$.

Similar to the case of series, we say the filtration $\mathcal{F}$ is **bounded below** if there exist $n_1 \in \mathbb{Z}$ such that $M_n = M$ for all $n \leq n_1$. Note that if the filtration $\mathcal{F}$ is exhaustive and M is finitely generated, then $\mathcal{F}$ is bounded below since M is then Noetherian.

Lemma 7.3.11. *Let (M_n) be an $\mathfrak{I}$-good filtration on M. If $M/\mathfrak{I}M$ has finite length, then M/M_{n+1} and M_n/M_{n+1} have finite length for all $n \in \mathbb{Z}$.*

Proof. Since $\mathcal{F}$ is bounded below, there exist n_1 such that $M_{n_1} = M$. Also, as (M_n) is $\mathfrak{I}$-good, there exist n_0 such that $\mathfrak{I}M_n = M_{n+1}$ for all $n \geq n_0$. Then for all $n \in \mathbb{Z}$,

$$\mathfrak{I}^{n-n_1}M \subseteq M_n \subseteq \mathfrak{I}^{n-n_0}M \tag{7.3.5}$$

where we define $\mathfrak{I}^n = A$ for $n \leq 0$. Then $\ell(M/M_n) \leq \ell(M/\mathfrak{I}^{n-n_1}M)$ and it suffices to prove that $\mathfrak{I}^nM/\mathfrak{I}^{n+1}M$ has finite length for each n. So we are reduced to the case of $\mathfrak{I}$-adic filtration. Let $(x_1,\dots,x_r)$ be a finite generating system of the A-module $\mathfrak{I}$, and let I be the finite set of monomials of total degree n in r variables $X_1,\dots,X_r$. The homomorphism from $(M/\mathfrak{I}M)^I$ to $\mathfrak{I}^nM/\mathfrak{I}^{n+1}M$ which maps $(u_m)_{m\in I}$ to $\sum_m m(x_1,\dots,x_r)u_m$ is surjective. Since $M/\mathfrak{I}M$ has finite length, so does $\mathfrak{I}^nM/\mathfrak{I}^{n+1}M$. ∎

Suppose henceforth that $M/\mathfrak{I}M$ has finite length. Let $\mathcal{F} = (M_n)$ be an $\mathfrak{I}$-good filtration on M. We define therefore the **Hilbert-Samuel series** H_M of M with respect to the $\mathfrak{I}$-good filtration (M_n) by

$$H_{M,\mathscr{F}} = \sum_{n\in\mathbb{Z}} \ell_{A/\mathfrak{I}}(M_n/M_{n+1})\cdot T^n$$

and the map $n \mapsto \ell_{A/\mathfrak{I}}(M_n/M_{n+1})$ is called the **Hilbert-Samuel function** of M with respect to (M_n).

Since there exists an integer n_1 such that $M_n = M$ for $n \leq n_1$, we see H_M is an element in $\mathbb{Z}((T))$. We denote by $H_{M,\mathfrak{I}}$ by the Hilbert-Samuel series of M with respect to the $\mathfrak{I}$-adic filtration, that is,

$$H_{M,\mathfrak{I}} = \sum_{n\in\mathbb{Z}} \ell_{A/\mathfrak{I}}(\mathfrak{I}^nM/\mathfrak{I}^{n+1}M)\cdot T^n.$$

If $P \in \mathbb{Z}((T))$ and $r \in \mathbb{N}$, we may use $P^{(1)}$ to denote the series $(1-T)^{-r}P$. In other words, if $P = \sum_{n\in\mathbb{Z}} a_nT^n$, then

$$P^{(1)} = \sum_{n\in\mathbb{Z}}\Big(\sum_{i\leq n} a_i\Big)T^n.$$

Proposition 7.3.12. *Let $\mathcal{F}$ be an $\mathfrak{I}$-good filtration on M. Then*

$$H^{(1)}_{M,\mathcal{F}} = \sum_{n\in\mathbb{Z}} \ell_A(M/M_{n+1}) \cdot T^n.$$

If $\mathcal{F}'$ is another $\mathfrak{I}$-good filtration on M, then there exists an integer m such that $H^{(1)}_{M,\mathcal{F}'} \geq T^m H^{(1)}_{M,\mathcal{F}}$.

Proof. Since $\ell_{A/\mathfrak{I}}(M_n/M_{n+1}) = \ell_A(M_n/M_{n+1})$, the first claim follows from the definition of $H_{M,\mathcal{F}}$ and the observation

$$\sum_{i\leq n} \ell_{A/\mathfrak{I}}(M_i/M_{i+1}) = \sum_{i\leq n} \ell_A(M_i/M_{i+1}) = \ell_A(M/M_{n+1}).$$

If $\mathcal{F} = (M'_n)$ is another filtration, then there exist an integer n_2 such that $M'_n \subseteq \mathfrak{I}^{n-n_2}M$ for all n, whence $M'_n \subseteq M_{n-(n_2-n_1)}$, and the second claim therefore follows. □

Theorem 7.3.13. *Let A be a Noetherian ring, $\mathfrak{I}$ an idela of A, M a finitely generated A-module such that $M/\mathfrak{I}M$ is nonzero and has finite length. Let $\mathcal{F}$ be an $\mathfrak{I}$-good filtration on M, then there exists a positive integer d and a uniquely determined element R in $\mathbb{Z}[T, T^{-1}]$ such that*

$$H_{M,\mathcal{F}} = (1-T)^{-d}R \quad \textit{and} \quad R(1) > 0.$$

Moreover, the integers d and $R(1)$ are independent of the filtration $\mathcal{F}$.

Proof. Consider the graded ring $\mathrm{gr}(A)$ and the graded module $\mathrm{gr}(M)$. Since $M_{n_1} = M$ and $\mathfrak{I}M_n = M_{n+1}$ for $n \geq n_0$, $\mathrm{gr}(M)$ is generated by $\bigoplus_{n_1\leq n\leq n_0} \mathrm{gr}_n(M)$, so is finitely generated. Furthermore, if $(x_1, \dots, x_r)$ is a finite generating system of the A-module $\mathfrak{I}$, then $\mathrm{gr}(A)$ is generated by $\mathrm{gr}_0(A)$ and the classes of x_i modulo $\mathfrak{I}^2$, so is isomorphic to a graded ring quotient of $B = (A/\mathfrak{I})[X_1, \dots, X_r]$. According to Theorem 7.3.4, we have

$$(1-T)^r H_{M,\mathcal{F}} \in \mathbb{Z}[T, T^{-1}].$$

Since $H_{M,\mathcal{F}} \neq 0$, by Lemma 7.3.2 there exist $d \in \mathbb{N}$ and $R \in \mathbb{Z}[T, T^{-1}]$ uniquely determined such that $R(1) > 0$ and $H_{M,\mathcal{F}} = (1-T)^{-d}R$.

If $\mathcal{F}'$ is another $\mathfrak{I}$-good filtration on M, then similarly

$$H_{M,\mathcal{F}'} = (1-T)^{-d'}R'.$$

By Proposition 7.3.12, there exists an integer m such that

$$(1-T)^{-d'-1}R' \geq T^m(1-T)^{-d-1}R$$

and due to Lemma 7.3.3, this implies $d' \geq d$, or $d' = d$ and $R'(1) \geq R(1)$. Exchange the role of $\mathcal{F}$ and $\mathcal{F}'$, we then get $d = d'$ and $R'(1) = R(1)$. □

Remark 7.3.14. In the notations of Theorem 7.3.13, if $R = \sum_{i\in\mathbb{Z}} a_i T^i$ and suppose that $d > 0$, then the relation $H_{M,\mathcal{F}} = (1-T)^{-d}R$ implies

$$\ell_{A/\mathfrak{I}}(M_n/M_{n+1}) = \sum_{i\in\mathbb{Z}} a_i \binom{n-i+d-1}{d-1} = \sum_{i\leq n} a_i \binom{n-i+d-1}{d-1}$$

Similarly, the relation $H^{(1)}_{M,\mathcal{F}} = (1-T)^{-d-1}R$ implies

$$\ell_A(M/M_{n+1}) = \sum_{i\in\mathbb{Z}} a_i \binom{n-i+d}{d} = \sum_{i\leq n} a_i \binom{n-i+d}{d}$$

Let A be a Noetherian ring, $\mathfrak{I}$ an idela of A, M a finitely generated A-module such that $M/\mathfrak{I}M$ has finite length. If $M \neq \mathfrak{I}M$, then by Theorem 7.3.13 there exists integers $d_{\mathfrak{I}}(M) \geq 0$ and $e_{\mathfrak{I}}(M) > 0$ such that, for any $\mathfrak{I}$-good filtration $\mathcal{F}$ on M,

$$H_{M,\mathcal{F}} = (1-T)^{-d_{\mathfrak{I}}(M)}R, \quad R(1) = e_{\mathfrak{I}}(M).$$

If $M = \mathfrak{I}M$, then we may set $d_{\mathfrak{I}}(M) = -\infty$ and $e_{\mathfrak{I}}(M) = 0$.

Corollary 7.3.15. *Let A be a Noetherian ring, $\mathfrak{I}$ an idela of A, M a finitely generated A-module such that $M/\mathfrak{I}M$ is nonzero and has finite length. Let $\mathcal{F}$ be an $\mathfrak{I}$-good filtration on M.*

(a) *For $d_{\mathfrak{I}}(M) \leq 0$, it is necessary and sufficient that the sequence $(\mathfrak{I}^n M)$ be stationary, or that the sequence (M_n) is stationary. In this case, we have, for all n large enough,*

$$\ell_A(M/M_{n+1}) = \ell_A(M/\mathfrak{I}^{n+1}M) = e_{\mathfrak{I}}(M).$$

(b) *Suppose that $d_{\mathfrak{I}}(M) > 0$, then*

$$\ell_{A/\mathfrak{I}}(M_n/M_{n+1}) = e_{\mathfrak{I}}(M)\frac{n^{d_{\mathfrak{I}}(M)-1}}{(d_{\mathfrak{I}}(M)-1)!} + \rho_n n^{d_{\mathfrak{I}}(M)-2} \tag{7.3.6}$$

and

$$\ell_A(M/M_{n+1}) = e_{\mathfrak{I}}(M)\frac{n^{d_{\mathfrak{I}}(M)}}{d_{\mathfrak{I}}(M)!} + \sigma_n n^{d_{\mathfrak{I}}(M)-1}. \tag{7.3.7}$$

where ρ_n and σ_n have limits when $n \to \infty$.

Proof. This follows from formulas (7.3.1) and (7.3.2), and the definitons above. □

Example 7.3.16. Let k be an algebraically closed field and $F \in k[X_1, \dots, X_r]$ be an irreducible polynomial F in A contained in the maximal ideal $\mathfrak{m} = (X_1, \dots, X_r)$, and let m be the largest integer such that $F \in \mathfrak{m}^m$. Set $A = (k[X_1, \dots, X_r]/(F))_{\mathfrak{m}}$ and let $\mathfrak{I} = (\mathfrak{m}/(F))_{\mathfrak{m}}$ be the maximal ideal of A; we want to determine $d_{\mathfrak{I}}(A)$ and $e_{\mathfrak{I}}(A)$. For this, we first note that $A/\mathfrak{I} = k$ by the Nullstellensatz and

$$0 \longrightarrow \mathfrak{I}^n/\mathfrak{I}^{n+1} \longrightarrow A/\mathfrak{I}^{n+1} \longrightarrow A/\mathfrak{I}^n \longrightarrow 0$$

Therefore we are reduced to compute $\dim_k(A/\mathfrak{I}^n)$. To this end, we first note that

$$A/\mathfrak{I}^n \cong (k[X_1, \dots, X_r]/(F))/(\mathfrak{m}^n/(F)) = k[X_1, \dots, X_n]/(\mathfrak{m}^n + (F)).$$

and we can use the following exact sequence to compute $\dim_k(k[X_1, \dots, X_r]/(\mathfrak{m}^n + (F)))$:

$$0 \to k[X_1, \dots, X_r]/\mathfrak{m}^{n-m} \xrightarrow{\psi} k[X_1, \dots, X_r]/\mathfrak{m}^n \to k[X_1, \dots, X_r]/(\mathfrak{m}^n + (F)) \to 0$$

where ψ is the homomorphism given by $\psi(\bar{G}) = \overline{FG}$. Since

$$\dim_k(k[X_1, \dots, X_r]/\mathfrak{m}^n) = \sum_{k=0}^{n-1}\binom{k+r-1}{r-1} \tag{7.3.8}$$

it follows that

$$\dim_k(k[X_1, \dots, X_r]/(\mathfrak{m}^n + (F))) = \sum_{k=n-m}^{n-1}\binom{k+r-1}{r-1}. \tag{7.3.9}$$

Since $\binom{n+r-1}{r-1} = \frac{n^{r-1}}{(r-1)!} + \cdots$, from (7.3.9) we deduce that

$$\dim_k(k[X_1,\dots,X_r]/(\mathfrak{m}^n + (F))) = m\frac{n^{r-1}}{(r-1)!} + \cdots .$$

This shows $d_{\mathfrak{J}}(A) = r-1$ and $e_{\mathfrak{J}}(A) = m$. In the language of algebraic geometry, the ring A is the coordinate ring of the variety $V = V(F)$, and the integer m is called the **multiplicity** of the point $p = (0,\dots,0)$ in V.

Example 7.3.17. Suppose that $\mathfrak{J}$ is contained in the Jacobson radical of A. Then from the Nakayama lemma, the sequence $(\mathfrak{J}^n M)$ is stationary if and only if we have $\mathfrak{J}^n M = 0$ for n large enough. It then follows from part (a) of Corollary 7.3.15 that we have $d_{\mathfrak{J}}(M) \leq 0$ if and only if M is of finite length and in this case $e_{\mathfrak{J}}(M) = \ell_A(M)$.

Consider now the special case A is local, $\mathfrak{J} = \mathfrak{m}$ is the maximal ideal of A, and $M = A$. Then by **??**, we see $\mathfrak{m}^n = 0$ if and only if A is Artinian. By Corollary 7.3.15, in this case we have $d_{\mathfrak{m}}(A) = 0$ and $e_{\mathfrak{m}}(A) = \ell_A(A)$.

Furthermore, to provide a concrete example, let consider $A = k[X]/(X^n)$ where k is an algebraically closed field. Then $\mathfrak{m} = (X)/(X^n)$ is the maximal ideal of A and $\mathfrak{m}^n = 0$, so $e_{\mathfrak{m}}(A) = n$.

Proposition 7.3.18. *Let A be a Noetherian ring, $x_1,\dots,x_r$ elements of A, $\mathfrak{J}$ the ideal they generate and M a finitely generated A-module such that $M/\mathfrak{J}M$ is non-zero and of finite length.*

(a) We have $d_{\mathfrak{J}}(M) \leq r$, and if $d_{\mathfrak{J}}(M) = r$ then $e_{\mathfrak{J}}(M) \leq \ell_A(M/\mathfrak{J}M)$.

(c) If $(x_1,\dots,x_r)$ is completely secant for M, then $d_{\mathfrak{J}}(M) = r$ and $e_{\mathfrak{J}}(M) = \ell_A(M/\mathfrak{J}M)$. The converse is also true if $\mathfrak{J}$ belongs to the Jacobson radical of A.

Proof. Let R be the ring $(A/\mathfrak{J})[X_1,\dots,X_r]$, and give $N = \bigoplus_n \mathfrak{J}^n M/\mathfrak{J}^{n+1}M$ be the graded R-module structure for which h_{X_i} is the multiplication by the class x_i mod $\mathfrak{J}^2$. Then we have

$$P_N = H_{M,\mathfrak{J}} = (1-T)^{-d_{\mathfrak{J}}(M)}R = (1-T)^{-r}Q_N$$

where $R(1) = e_{\mathfrak{J}}(M) > 0$ and $Q_N(1) = c_N$. By Lemma 7.3.3, we therefore have either $d_{\mathfrak{J}}(M) < r$ and $c_N = 0$, or $d_{\mathfrak{J}}(M) = r$ and $c_N = e_{\mathfrak{J}}(M)$. Furthermore, according to Proposition 7.3.8, we have $c_N \leq \ell_A(M/\mathfrak{J}M)$, and the equality holds if and only if canonical homomorphism

$$\varphi : (A/\mathfrak{J})[X_1,\dots,X_r] \otimes_{A/\mathfrak{J}} (M/\mathfrak{J}M) \to \bigoplus_n \mathfrak{J}^n M/I^{n+1}M$$

is bijective. This then proves the claim, in view of **??**. □

Proposition 7.3.19. *Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of finitely generated modules over an Noetherian ring A and $\mathfrak{J}$ be an ideal of A.*

(a) For $M/\mathfrak{J}M$ to have finite length, it is necessary and sufficient that $M'/\mathfrak{J}M'$ and $M''/\mathfrak{J}M''$ both have finite length.

(b) Suppose that $M/\mathfrak{J}M$ has finite length. Then there are exactly the following three cases:

(i) $d_{\mathfrak{J}}(M) = d_{\mathfrak{J}}(M') > d_{\mathfrak{J}}(M'')$ and $e_{\mathfrak{J}}(M) = e_{\mathfrak{J}}(M')$.

(ii) $d_{\mathfrak{J}}(M) = d_{\mathfrak{J}}(M'') > d_{\mathfrak{J}}(M')$ and $e_{\mathfrak{J}}(M) = e_{\mathfrak{J}}(M'')$.

(iii) $d_{\mathfrak{J}}(M) = d_{\mathfrak{J}}(M') = d_{\mathfrak{J}}(M'')$ and $e_{\mathfrak{J}}(M) = e_{\mathfrak{J}}(M') + e_{\mathfrak{J}}(M'')$.

Proof. By Proposition 3.2.14 and Corollary 1.4.40, for the module $M/\mathfrak{I}M$ to have finite length, it is necessary and sufficient that the element in

$$\operatorname{supp}(M/\mathfrak{I}M) = \operatorname{supp}(M) \cap V(\mathfrak{I})$$

is maximal. Since $\operatorname{supp}(M) = \operatorname{supp}(M') \cap \operatorname{supp}(M'')$, the assertion in (a) follows.

Now endow M with an $\mathfrak{I}$-good filtration $\mathcal{F}$ (for example the $\mathfrak{I}$-adic filtration), M' the submodule filtration $\mathcal{F}'$, and M'' the quotient filtration $\mathcal{F}''$. By Proposition 2.4.3, the filtrations $\mathcal{F}'$ and $\mathcal{F}''$ are $\mathfrak{I}$-good. Then we have for each n an exact sequence of A-modules

$$0 \longrightarrow M'_n/M'_{n+1} \longrightarrow M_n/M_{n+1} \longrightarrow M''_n/M''_{n+1} \longrightarrow 0$$

which implies $H_{M,\mathcal{F}} = H_{M',\mathcal{F}'} + H_{M'',\mathcal{F}''}$, and

$$(1-T)^{-d_{\mathfrak{q}}(M)}R = (1-T)^{-d_{\mathfrak{q}}(M')}R' + (1-T)^{-d_{\mathfrak{q}}(M'')}R''$$

with $R, R', R'' \in \mathbb{Z}[T, T^{-1}]$. Assertion (b) therefore follows. □

Theorem 7.3.20. *Let A be a Noetherian local ring, $\mathfrak{I}$ a proper ideal of A and M a finitely generated A-module such that $M/\mathfrak{I}M$ is non-zero and of finite length. Then the integer $d_{\mathfrak{I}}(M)$ equals to the dimension of the A-module M.*

Proof. We may suppose that $M \neq 0$. Let $(x_1, \dots, x_r)$ be a system of parameters for M and $\mathfrak{x}$ the ideal they generate. Then by Proposition 7.3.18 we have $d_{\mathfrak{x}}(M) \le r$. Since $\mathfrak{x} \subseteq \mathfrak{I}$, we have $H^{(1)}_{M,\mathfrak{I}} \le H^{(1)}_{M,I}$ and therefore (Lemma 7.3.3)

$$d_{\mathfrak{I}}(M) \le d_{\mathfrak{x}}(M) \le r = \dim_A(M).$$

We now prove the reverse inequality $\dim_A(M) \le d_{\mathfrak{I}}(M)$ by induction on $\dim_A(M)$. The claim is clear when $\dim_A(M) = 0$. Suppose then we have $\dim_A(M) > 0$, and $\dim_A(N) \le d_{\mathfrak{I}}(N)$ for any finitely generated A-module N such that $\dim_A(N) < \dim_A(M)$. If

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$$

is a composition sequence of M, we have

$$\dim_A(M) = \sup(\dim_A(M_i/M_{i-1})), \quad d_{\mathfrak{I}}(M) = \sup(d_{\mathfrak{I}}(M_i/M_{i-1}))$$

by Proposition 7.1.12 and 7.3.19. Due to Proposition 3.1.17, we are now reduced to the case that M is of the form $A/\mathfrak{p}$, where $\mathfrak{p}$ is a prime ideal of A such that $\mathfrak{p} \neq \mathfrak{m}_A$ (otherwise $\dim(A/\mathfrak{p}) = 0$). Let $x \in \mathfrak{m}_A - \mathfrak{p}$, then the homothety h_x with ratio x is injective on M, and we have an exact sequence

$$0 \longrightarrow M \xrightarrow{h_x} M \longrightarrow M/xM \longrightarrow 0$$

By Proposition 7.2.11, the element x is secant for M, so we have $\dim_A(M/xM) = \dim_A(M) - 1$. On the other hand, since $e_{\mathfrak{I}}(M/xM) > 0$, from Proposition 7.3.19 we conclude $d_{\mathfrak{I}}(M/xM) \le d_{\mathfrak{I}}(M) - 1$. Now by the induction hypothesis,

$$\dim_A(M) = \dim_A(M/xM) + 1 \le d_{\mathfrak{I}}(M/xM) + 1 \le d_{\mathfrak{I}}(M)$$

which finishes the induction process. □

Corollary 7.3.21. *Let A be a Noetherian ring, $\mathfrak{I}$ an idela of A, M a finitely generated A-module such that $M/\mathfrak{I}M$ is nonzero and has finite length. Then*

(a) $d_{\mathfrak{I}}(M)$ *is the supremum of the integers* $\dim_{A_{\mathfrak{m}}}(M_{\mathfrak{m}})$, *where* $\mathfrak{m}$ *runs through the set* $\mathrm{supp}(M) \cap V(\mathfrak{I})$.

(b) $e_{\mathfrak{I}}(M)$ *is the sum of the integers* $e_{\mathfrak{I}_{\mathfrak{m}}}(M_{\mathfrak{m}})$, *where* $\mathfrak{m}$ *runs through the elements of* $\mathrm{supp}(M) \cap V(\mathfrak{I})$ *such that* $\dim_{A_{\mathfrak{m}}}(M_{\mathfrak{m}}) = d_{\mathfrak{I}}(M)$.

Proof. By Corollary 3.2.20, the length of $M/\mathfrak{I}^n M$ is the sum of the $\ell_{A_{\mathfrak{m}}}(M_{\mathfrak{m}}/\mathfrak{I}_{\mathfrak{m}}^n M_{\mathfrak{m}})$ where $\mathfrak{m}$ runs through $\mathrm{Ass}(M/\mathfrak{I}^n M)$. Since $M/\mathfrak{I}^n M$ has finite length, the elements of $\mathrm{Ass}(M/\mathfrak{I}^n M)$ are maximal ideals and we have

$$\mathrm{Ass}(M/\mathfrak{I}^n M) = \mathrm{supp}(M/\mathfrak{I}^n M) = \mathrm{supp}(M) \cap V(\mathfrak{I}^n) = \mathrm{supp}(M) \cap V(\mathfrak{I})$$

(Proposition 3.2.14 and Corollary 1.4.40), so we have

$$H_{M,\mathfrak{I}} = \sum_{\mathfrak{m}} H_{M_{\mathfrak{m}},\mathfrak{I}_{\mathfrak{m}}}$$

where $\mathfrak{m}$ runs through prime ideals in $\mathrm{supp}(M) \cap V(\mathfrak{I})$. The claims then follow from this. □

Corollary 7.3.22. *Let* A *be a Noetherian ring,* $\mathfrak{I}$ *an ideal of* A*, and* M *a finitely generated* A*-module such that* $M/\mathfrak{I}M$ *has finite length. Let* $\widehat{A}$ *and* $\widehat{M}$ *be the Hausdorff completions of* A *and* M *under the* $\mathfrak{I}$*-adic topology. Then* $d_{\mathfrak{I}}(M) = \dim_{\widehat{A}}(\widehat{M})$*. If* $\mathfrak{I}$ *is contained in the Jacobson radical of* A *then* $d_{\mathfrak{I}}(M) = \dim_A(M)$*.*

Proof. By Proposition 7.2.32 and Corollary 7.3.21, we have

$$\dim_{\widehat{A}}(\widehat{M}) = \sup_{\mathfrak{m} \in V(\mathfrak{I})} \dim_{A_{\mathfrak{m}}}(M_{\mathfrak{m}}) = \sup_{\mathfrak{m} \in \mathrm{supp}(M) \cap V(\mathfrak{I})} \dim_{A_{\mathfrak{m}}}(M_{\mathfrak{m}}) = d_{\mathfrak{I}}(M).$$

If $\mathfrak{I}$ is contained in the Jacobson radical of A then we have $\dim_A(M) = \dim_{\widehat{A}}(\widehat{M})$, whence $d_{\mathfrak{I}}(M) = \dim_A(M)$. □

Lemma 7.3.23. *Let* A *be a ring,* M *an* A*-module, and* (P_n)*,* (Q_n) *two filtrations on* M *of submodules with* (Q_n) *bounded below. Suppose that* $Q_n \subseteq P_n$ *and* $\ell_A(P_n/Q_n) < +\infty$ *for each* $n \in \mathbb{Z}$*. Then*

$$\begin{aligned}\sum_{n\in\mathbb{Z}} \ell_A((P_{n+1} \cap Q_n)/Q_{n+1})T^n &\leq \sum_{n\in\mathbb{Z}} \ell_A(P_{n+1}/Q_{n+1})T^n \\ &\leq (1-T)^{-1} \sum_{n\in\mathbb{Z}} \ell_A((P_{n+1} \cap Q_n)/Q_{n+1})T^n.\end{aligned} \tag{7.3.10}$$

Proof. It suffices to prove the following inequalities:

$$\ell_A((P_{n+1} \cap Q_n)/Q_{n+1}) \leq \ell_A(P_{n+1}/Q_{n+1}) \leq \sum_{i\leq n} \ell_A((P_{n+1} \cap Q_n)/Q_{n+1}). \tag{7.3.11}$$

The first part is immediate. For the second part, since (Q_n) is bounded below, for i small enough we have $P_{n+1} \cap Q_i = P_{n+1}$, whence

$$\ell_A(P_{n+1}/Q_{n+1}) \leq \sum_{i\leq n} \ell_A((P_{n+1} \cap Q_i)/(P_{n+1} \cap Q_{i+1})).$$

On the other hand, the module $(P_{n+1} \cap Q_i)/(P_{n+1} \cap Q_{i+1})$ is isomorphic to $(P_{n+1} \cap Q_i + Q_{i+1})/Q_{i+1}$, which is a submodule of $(P_{i+1} \cap Q_i)/Q_{i+1}$ when $i \leq n$. From these remarks we see (7.3.11) follows. □

Proposition 7.3.24. *Let A be a ring, M an A-module and $\mathcal{F}$ a bounded below filtration on M consisting of submodules such that M_n/M_{n+1} has finite length for all $n \in \mathbb{Z}$. Let ϕ be an endomorphism of M with kernel M' and cokernel M''. Endow M' with the submodule filtration $\mathcal{F}'$ and M'' the quotient filtration $\mathcal{F}''$. If there exist an integer δ such that $\phi(M_n) \subseteq M_{n+\delta}$ and $\mathrm{gr}(\phi)$ is the associated homomorphism on $\mathrm{gr}(M)$, then we have*

$$H_{M',\mathcal{F}'} \le P_{\ker \mathrm{gr}(\phi)}, \tag{7.3.12}$$

$$(1-T^\delta)H^{(1)}_{M,\mathcal{F}} + T^\delta P_{\ker \mathrm{gr}(\phi)} \le H^{(1)}_{M'',\mathcal{F}''} \le (1-T^\delta)H^{(1)}_{M,\mathcal{F}} + T^\delta P^{(1)}_{\ker \mathrm{gr}(\phi)}. \tag{7.3.13}$$

Proof. The sequence $N_n = \phi^{-1}(M_{n+\delta})$ forms a filtration on M and $M_n \subseteq N_n$ for all $n \in \mathbb{Z}$. By definition, we have $\ker(\mathrm{gr}(\phi)_n) = (N_{n+1} \cap M_n)/M_{n+1}$, and

$$P_{\ker \mathrm{gr}(\phi)} = \sum_{n\in\mathbb{Z}} \ell_A((N_{n+1} \cap M_n)/M_{n+1})T^n. \tag{7.3.14}$$

For each n, the module $(M' \cap M_n)/(M' \cap M_{n+1})$ is identified with the submodule $(M' \cap M_n + M_{n+1})/M_{n+1}$ of $(N_{n+1} \cap M_n)/M_{n+1}$, so follows from (7.3.14) that $H_{M',\mathcal{F}'} \le P_{\ker \mathrm{gr}(\phi)}$. According to Lemma 7.3.23, we also have

$$P_{\ker \mathrm{gr}(\phi)} \le \sum_{n\in\mathbb{Z}} \ell_A(N_{n+1}/M_{n+1})T^n \le P^{(1)}_{\ker \mathrm{gr}(\phi)}. \tag{7.3.15}$$

For each $n \in \mathbb{Z}$, we have an exact sequence

$$0 \longrightarrow N_{n+1}/M_{n+1} \longrightarrow M/M_{n+1} \longrightarrow M/M_{n+\delta+1} \longrightarrow M''/M''_{n+\delta+1} \longrightarrow 0$$

This then implies, by the additivity of length, that

$$\ell_A(M''/M''_{n+\delta+1}) = \ell_A(M/M_{n+\delta+1}) - \ell_A(M/M_{n+1}) + \ell_A(N_{n+1}/M_{n+1}).$$

Multiply by $T^{n+\delta}$ on both sides and take summation on n, we then get

$$H^{(1)}_{M'',\mathcal{F}''} = (1-T^\delta)H^{(1)}_{M,\mathcal{F}} + T^\delta \sum_{n\in\mathbb{Z}} \ell_A(N_{n+1}/M_{n+1})T^n. \tag{7.3.16}$$

The inequality (7.3.13) then follows from (7.3.15) and (7.3.16). $\square$

Corollary 7.3.25. *Under the conditions of Proposition 7.3.24, we have $H_{M'',\mathcal{F}''} \ge (1-T^\delta)H^{(1)}_{M,\mathcal{F}}$, and the equality holds if and only if $\mathrm{gr}(\phi)$ is injective. In this case, we have $M' \subseteq \bigcap_n M_n$, and the following sequence is exact:*

$$0 \longrightarrow \mathrm{gr}(M) \xrightarrow{\mathrm{gr}(\phi)} \mathrm{gr}(M) \longrightarrow \mathrm{gr}(M'') \longrightarrow 0$$

Proof. The first assertion follows from (7.3.12). Suppose now $\mathrm{gr}(\phi)$ is injective. By Theorem 2.3.29 we have $\ker\phi \subseteq \phi^{-1}(M_{n+\delta}) = M_n$ for all n, whence $M' \subseteq \bigcap_n M_n$. Now consider the following exact sequence

$$0 \longrightarrow M/M' \xrightarrow{\bar{\phi}} M \longrightarrow M/\phi(M) \longrightarrow 0$$

Since $\mathrm{gr}(\phi)$ is injective, we have $\bar{\phi}^{-1}(M_n) = M_{n-\delta}/M'$. Hence the filtration on M/M' deduced by the filtration $\mathcal{F}$ on M is the filtration $(M_{n-\delta}/M')$. The associated graded module is $\mathrm{gr}(M)(-\delta)$ and we have an exact sequence of graded modules

$$0 \longrightarrow \mathrm{gr}(M)(-\delta) \longrightarrow \mathrm{gr}(M) \longrightarrow \mathrm{gr}(M'') \longrightarrow 0$$

which proves our claim. $\square$

Proposition 7.3.26. *Let A be a Noetherian ring, M a finitely generated A-module, $\mathfrak{I}$ an ideal of A such that $M/\mathfrak{I}M$ has finite length, and $\mathcal{F}$ an $\mathfrak{I}$-good filtration on M. Let $(x_1,\dots,x_s)$ be a sequence of elements of A, $(\delta_1,\dots,\delta_s)$ a sequence of positive integers such that $x_i\in\mathfrak{I}^{\delta_i}$ for each i, and let ξ_i be the image of x_i in $\mathrm{gr}_{\delta_i}(A)=\mathfrak{I}^{\delta_i}/\mathfrak{I}^{\delta_i+1}$.*

(a) Endow the A-module $\overline{M}=M/\sum_i x_iM$ with the $\mathfrak{I}$-good filtration $\overline{\mathcal{F}}$. Then we have

$$H^{(s)}_{\overline{M},\overline{\mathcal{F}}}\geq\prod_{i=1}^{s}(1-T^{\delta_i})H^{(s)}_{M,\mathcal{F}}. \tag{7.3.17}$$

(b) For the equality in (a) holds, it is necessary and sufficient that the sequence $(\xi_1,\dots,\xi_s)$ of elements of $\mathrm{gr}(A)$ is completely secant for $\mathrm{gr}(M)$. In this case, the canonical homomorphism

$$\theta:\mathrm{gr}(M)/\sum_i\xi_i\cdot\mathrm{gr}(M)\to\mathrm{gr}(\overline{M})$$

is bijective.

(c) Suppose the condition in (b) hold, and the modules $M_i=M/(x_1M+\cdots+x_iM)$ is Hausdorff in the $\mathfrak{I}$-adic topology for each i. Then the sequence $(x_1,\dots,x_s)$ is completely secant for M.

Proof. When $s=1$, we have $\bigcap_n M_n=\bigcap_n\mathfrak{I}^nM$ and the sequence $\{\xi_1\}$ is completely secant for $\mathrm{gr}(M)$ if and only if the homothety with ratio ξ_1 in $\mathrm{gr}(M)$ is injective. The claim then results immediately from Corollary 7.3.25 applied to the homothety $\phi=h_{x_1}$ in M.

Suppose then $s\geq 2$ and we prove by induction on s. By the induction hypothesis applied on $M_1=M/x_1M$ with the quoteint filtration $\mathcal{F}_1$ and the sequence $(x_2,\dots,x_s)$, we have

$$H^{(s-1)}_{\overline{M},\overline{F}}\geq\prod_{i=2}^{s}(1-T^{\delta_i})H^{(s-1)}_{M_1,\mathcal{F}_1} \tag{7.3.18}$$

and the equality holds if and only if the sequence $(\xi_2,\dots,\xi_s)$ is completely secant for the $\mathrm{gr}(A)$-module $\mathrm{gr}(M_1)$. Since the element $(1-T^{\delta_1})/(1-T)$ is positive, the case $s=1$ already treated and the formula (7.3.18) provide the inequalities

$$H^{(s)}_{\overline{M},\overline{\mathcal{F}}}\geq\prod_{i=2}^{s}(1-T^{\delta_i})H^{(s)}_{M_1,\mathcal{F}_1}\geq\prod_{i=1}^{s}(1-T^{\delta_i})H^{(s)}_{M_1,\mathcal{F}_1} \tag{7.3.19}$$

which proves (a).

We can have equality in (7.3.17) only if we have simultaneously equality in (7.3.18) and equality

$$H^{(1)}_{M_1,\mathcal{F}_1}=(1-T^{\delta_1})H^{(1)}_{M,\mathcal{F}}. \tag{7.3.20}$$

This last relation means that $\{\xi_1\}$ is completely secant for $\mathrm{gr}(M)$ and implies that the canonical homomorphism from $\mathrm{gr}(M)/\xi_1\mathrm{gr}(M)$ to $\mathrm{gr}(M_1)$ is a isomorphism. In other words, we have equality in (7.3.17) if and only if $\{\xi_1\}$ is completely secant for $\mathrm{gr}(M)$ and $\{\xi_2,\dots,\xi_s\}$ is completely secant for $\mathrm{gr}(M)$. This means $\{\xi_1,\dots,\xi_s\}$ is completely secant for $\mathrm{gr}(M)$. We have thus demonstrated the equivalence of the two conditions of (b).

Now suppose that $\{\xi_1,\dots,\xi_s\}$ is completely secant for $\mathrm{gr}(M)$ and M_i is Hausdorff for the $\mathfrak{I}$-adic topology for all i. According to the above argument and the induction hypothesis, the sequence $(x_2,\dots,x_s)$ is completely secant for M_1. As we have $M_1=M/x_1M$ and that $\{x_1\}$ is completely secant for M, the sequence $(x_1,x_2,\dots,x_s)$ is completely secant for M. ∎

7.4 Regular local rings

7.4.1 Regularity of local rings

Let $(A, \mathfrak{m})$ be a Noetherian local ring and $(x_i)_{i\in I}$ a family of elements of $\mathfrak{m}$. In view of Corollary 1.3.8, it amounts to the same thing to suppose that the family $(x_i)_{i\in I}$ generates the ideal $\mathfrak{m}$ of A, or that the classes of x_i modulo $\mathfrak{m}^2$ generate the κ_A-vector space $\mathfrak{m}/\mathfrak{m}^2$. In this case, we have $\dim(A) \leq |I|$ by Corollary 7.2.17, and therefore

$$\dim(A) \leq [\mathfrak{m}/\mathfrak{m}^2 : \kappa_A] \leq |I|$$

for every generating family $(x_i)_{i\in I}$ of the ideal $\mathfrak{m}$.

Definition 7.4.1. A Noetherian local ring $(A, \mathfrak{m})$ is called **regular** if $\dim(A) = [\mathfrak{m}/\mathfrak{m}^2, \kappa_A]$. A family of elements of $\mathfrak{m}$ whose calsses modulo $\mathfrak{m}^2$ form a basis of the κ_A-space $\mathfrak{m}/\mathfrak{m}^2$ is called a **system of parameters** for A.

A system of parameters in a regular local ring A is therefore a finite family $(x_i)_{i\in I}$ generating the ideal $\mathfrak{m}$ of A and such that $|I| = \dim(A)$. Conversely, if the maximal ideal $\mathfrak{m}$ of a Noetherian local ring A is generated by d elements with $d \leq \dim(A)$, then the ring A is regular.

Example 7.4.2 (Examples of regular local rings).

(a) The regular local rings of dimension 0 (resp. 1) are the fields (resp. the discrete valuation rings). In fact, the claim is clear for dimension 0, and a discrete valuation ring is clearly regular (since it is a PID). Conversely, assume that $(A, \mathfrak{m})$ is a regular local ring of dimension one, with t its generator of $\mathfrak{m}$. For all $n \geq 0$ we have $\dim_{\kappa_A}(\mathfrak{m}^n/\mathfrak{m}^{n+1}) = 1$ because it is generated by t^n (and it cannot be zero since otherwise $\mathfrak{m}$ will be the nilradical of A, which implies $\dim(A) = 0$, contradiction). In particular $\mathfrak{m}^n = (t^n)$ and the graded ring $\mathrm{gr}_{\mathfrak{m}}(A)$ is isomorphic to the polynomial ring $\kappa_A[T]$. For $x \in A \setminus \{0\}$, define $v(x)$ to be the largest integer n such that $x \in \mathfrak{m}^n$. Then we have $v(xy) = v(x) + v(y)$ for all $x, y \in A \setminus \{0\}$. We extend this to the field of fractions K of A. Then it is clear that A is the set of elements of K which have positive valuation. Hence we see that A is a discrete valuation ring. Note that if $(A, \mathfrak{m})$ is a discrete valuation ring, then an element t of $\mathfrak{m}$ is a uniformizer if and only if $\{t\}$ is a system of parameters of A.

(b) Let k be a field and n a positive integer. The ring $A = k[\![X_1, \dots, X_n]\!]$ is Noetherian and local with dimension n. Since its maximal ideal $\mathfrak{m}$ is generated by $X_1, \dots, X_n$, we see A is regular. Moreover, a sequence $(F_1, \dots, F_n)$ is a system of parameters if and only if their image in $\mathfrak{m}/\mathfrak{m}^2$ is lineraly independent, or equivalently the Jacobi matrix $(\partial F_i/\partial X_j)$ is nonsingular at the point 0.

More generally, let A be a regular local ring of dimension r. The ring $A[\![X_1, \dots, X_n]\!]$ is then a regular local ring of dimension $r + n$. If $(a_1, \dots, a_r)$ is a system of parameters for A, then $(a_1, \dots, a_r, X_1, \dots, X_n)$ is a system of parameters for $A[\![X_1, \dots, X_n]\!]$.

(c) Let A be a regular local ring of dimension r. The ring $A\{X_1, \dots, X_n\}$ of restricted formal power series is a regular local ring of dimension $n + r$. If $(a_1, \dots, a_r)$ is a system of parameters for A, then $(a_1, \dots, a_r, X_1, \dots, X_n)$ is a system of parameters for $A\{X_1, \dots, X_n\}$.

(d) Let k be a field, A a finitely generated k-algebra that is an integral domain and $\mathfrak{m}$ a maximal ideal of A. Then the Noetherian local ring $A_{\mathfrak{m}}$ is regular if and only if we have $\dim(A) = [\mathfrak{m}/\mathfrak{m}^2 : A/\mathfrak{m}]$. In fact, we have $\dim(A_{\mathfrak{m}}) = \dim(A)$ by Theorem 7.1.31 and the $A/\mathfrak{m}$-vector space $\mathfrak{m}A_{\mathfrak{m}}/(\mathfrak{m}A_{\mathfrak{m}})^2$ is isomorphic to $\mathfrak{m}/\mathfrak{m}^2$. In particular, if k is algebraically closed, this condition is equivalent to $\dim(A) = [\mathfrak{m}/\mathfrak{m}^2 : k]$ by the Nullstellensatz.

(e) Let A be a regular local ring. We will see later that the local ring $A_{\mathfrak{p}}$ is regular for any prime ideal $\mathfrak{p}$ of A.

Proposition 7.4.3. *Let $\rho : A \to B$ be a local homomorphism of Noetherian local rings such that B is flat over A. Suppose that $\mathfrak{m}_B = \mathfrak{m}_A^e$, then $\dim(B) = \dim(A)$ and B is regular if and only if A is regular.*

Proof. The first assertion follows from Proposition 7.2.26. Since B is flat over A, we can identify $\mathfrak{m}_B/\mathfrak{m}_B^2$ with $B \otimes_A (\mathfrak{m}_A/\mathfrak{m}_A^2)$ or with $\kappa_B \otimes_{\kappa_A} (\mathfrak{m}_A/\mathfrak{m}_A^2)$. We then have

$$[\mathfrak{m}_B/\mathfrak{m}_B^2 : \kappa_A] = [\mathfrak{m}_A/\mathfrak{m}_A^2 : \kappa_A]$$

so the second claim follows. □

Corollary 7.4.4. *A Noetherian local ring A is regular if and only if its completion $\widehat{A}$ is.*

Proof. The completion $\widehat{A}$ is flat over A and $\widehat{\mathfrak{m}}$ is generated by $\mathfrak{m}$ in $\widehat{A}$, so we can apply Proposition 7.4.3. □

Theorem 7.4.5. *Let A be a Noetherian local ring. The following conditions are equivalent:*

(i) A is regular.

(ii) The ideal $\mathfrak{m}_A$ is generated by a subset of $\mathfrak{m}_A$ secant for A.

(iii) The ideal $\mathfrak{m}_A$ is generated by a subset of $\mathfrak{m}_A$ completely secant for A.

(iv) The homomorphism $\gamma : \boldsymbol{S}_A(\mathfrak{m}_A/\mathfrak{m}_A^2) \to \mathrm{gr}(A)$ is bijective.

(v) There exists a positive integer r such that $H_{A,\mathfrak{m}_A} = (1-T)^{-r}$.

(vi) We have $H_{A,\mathfrak{m}_A} = (1-T)^{-d}$ where $d = \dim(A)$.

If these conditions are fulfilled, every system of parameters for A is a completely secant sequence for A.

Proof. Since every secant subset has cardinality smaller than $\dim(A)$, we see (iii)⇒(ii)⇒(i). To see (iv)⇒(iii), let $(x_1, \dots, x_r)$ be a sequence of elements of $\mathfrak{m}_A$ whose classes modulo $\mathfrak{m}_A^2$ form a basis $(\xi_1, \dots, \xi_r)$ for $\mathfrak{m}_A/\mathfrak{m}_A^2$. If property (iv) is satisfied, $\mathrm{gr}(A)$ is then the polynomial algebra $\kappa_A[\xi_1, \dots, \xi_r]$ and the sequence $(x_1, \dots, x_r)$ is completely secant by **??**. This also proves the last assertion of the theorem.

Let $r = [\mathfrak{m}_A/\mathfrak{m}_A^2 : \kappa_A]$. Then the Poincaré series of the graded module $S = \boldsymbol{S}_A(\mathfrak{m}_A/\mathfrak{m}_A^2)$ over κ_A is equal to

$$P_S = \sum_{n \in \mathbb{N}} [S_n : \kappa_A] T^n = (1-T)^{-r}.$$

Suppose that the homomorphism γ is not bijective. As γ is surjective, there exists a homogeneous element u of S of degree $d > 0$, canceled by γ. Then

$$H_{A,\mathfrak{m}_A} = P_S - P_{\ker\gamma} \leq P_S - P_{uS} = (1-T^d)(1-T)^{-r} = (1+T+\cdots+T^{d-1})(1-T)^{-(r-1)}.$$

By Theorem 7.3.20 and Lemma 7.3.3, we get $\dim(A) < r$, so A is not regular.

Finally, let us prove the equivalence of conditions (iv) and (vi). If we have $H_{A,\mathfrak{m}_A} = (1-T)^{-r}$ for some r, then by Theorem 7.3.20 we must have $r = \dim(A)$. Also, $[\mathfrak{m}_A/\mathfrak{m}_A^2 : \kappa_A] = r$ because $(1-T)^{-r} = 1 + rT + \cdots$. Therefore our claim follows. □

Corollary 7.4.6. *A regular local ring is integrally closed, and in particular an integral domain.*

Proof. If A is a regular local ring, then $\mathrm{gr}(A)$ is isomorphic to a polynomial algebra over a field, so it is integrally closed. Since A is a Zariski ring, by Proposition 4.1.38 we see A is integrally closed. □

Corollary 7.4.7. *Let $\rho : A \to B$ be a local homomorphism of Noetherian local rings. Suppose that A is complete and B is regular. Then for ρ to be bijective, it is necessary and sufficient that it induces bijections of κ_A on κ_B and of $\mathfrak{m}_A/\mathfrak{m}_A^2$ to $\mathfrak{m}_B/\mathfrak{m}_B^2$.*

Proof. These conditions are clearly necessary. Suppose now ρ induces bijections of κ_A on κ_B and of $\mathfrak{m}_A/\mathfrak{m}_A^2$ to $\mathfrak{m}_B/\mathfrak{m}_B^2$. Since the ring $\mathrm{gr}(A)$ is generated by $\mathrm{gr}_0(A)$ and $\mathrm{gr}_1(A)$, it follows that $\mathrm{gr}(\rho)$ is bijective. Therefore ρ is bijective by Theorem 2.3.29. □

Corollary 7.4.8. *Let k be a ring and A a Noetherian local k-algebra with residue field equal to k. Then A is regular if and only if the completion $\widehat{A}$ is isomorphic to a formal series ring $k[\![X_1, \dots, X_n]\!]$.*

Proof. This follows from Theorem 7.4.5 and Proposition 2.3.38. □

7.4.2 Quotients of regular local rings

Proposition 7.4.9. *Let A be a Noetherian local ring, $\boldsymbol{x} = (x_1, \dots, x_r)$ a sequence of elements of $\mathfrak{m}_A$ and $\mathfrak{x}$ the ideal it generates. The following conditions are equivalent.*

(i) A is regular and $\boldsymbol{x}$ can be extended to a system of parameters for A.

(ii) $A/\mathfrak{x}$ is regular and $\boldsymbol{x}$ is secant for A.

(iii) $A/\mathfrak{x}$ is regular and $\boldsymbol{x}$ is completely secant for A.

Moreover, if these conditions are satisfied, then $\mathfrak{x}$ is a prime ideal of A.

Proof. Clearly (iii) implies (ii). Suppose now that $\boldsymbol{x}$ is secant for A and the Noetherian local ring $A/\mathfrak{x}$ is regular. Let $(x_{r+1}, \dots, x_d)$ be a sequence of elements of A whose class modulo I form a system of parameters for $A/\mathfrak{x}$. Then the sequence $(x_1, \dots, x_d)$ generates the maixmal ideal $\mathfrak{m}_A$ of A, and we have

$$\dim(A) = r + \dim(A/\mathfrak{x}) = r + (d - r) = d.$$

Therefore, A is regular and $(x_1, \dots, x_d)$ is a system of parameters for A.

Finally, assume that (i) is satisfied. Then A is regular and $\boldsymbol{x}$ is completely secant for A, hence secant for A. Then we have

$$\dim(A/\mathfrak{x}) = \dim(A) - r$$

and the classes $x_1, \dots, x_r$ modulo $\mathfrak{m}_A^2$ is lineraly independent over κ_A, so

$$[\mathfrak{m}_A/(\mathfrak{m}_A^2 + \mathfrak{x}) : \kappa_A] = [\mathfrak{m}_A/\mathfrak{m}_A^2 : \kappa_A] - r.$$

Therefore we see $A/\mathfrak{x}$ is regular, and in particular $\mathfrak{x}$ is prime since $A/\mathfrak{x}$ is integral. □

Corollary 7.4.10. *Let A be a Noetherian local ring and x and element of $\mathfrak{m}_A$. The following conditions are equivalent.*

(i) A is regular, and x does not belong to $\mathfrak{m}_A$.

(ii) A/xA is regular and $\dim(A/xA) = \dim(A) - 1$.

(iii) A/xA is regular and x is not a divisor of 0 in A.

Proof. This follows by applying Proposition 7.4.9 on $\mathfrak{x} = xA$. □

Corollary 7.4.11. *Let A be a regular local ring and $\mathfrak{I}$ an ideal of A. Then $A/\mathfrak{I}$ is regular if and only if $\mathfrak{I}$ is generated by a subset of a system of parameters for A.*

Proof. The condition is sufficient by Proposition 7.4.9. Suppose now $A/\mathfrak{I}$ is regular, and $\boldsymbol{x} = (x_1, \dots, x_r)$ a sequence of elements of $\mathfrak{I}$ whose classes module $\mathfrak{m}_A^2$ form a basis for $(\mathfrak{I} + \mathfrak{m}_A^2)/\mathfrak{m}_A^2$ over κ_A. Let $\mathfrak{x}$ be the ideal generated by $\boldsymbol{x}$, so we have $\mathfrak{x} \subseteq \mathfrak{I}$ and $\boldsymbol{x}$ is part of a system of parameters for A, so the local Noetherian ring $A/\mathfrak{x}$ is regular. Moreover, the vector spaces $\mathfrak{m}_A/(\mathfrak{I} + \mathfrak{m}_A^2)$ and $\mathfrak{m}_A/(\mathfrak{x} + \mathfrak{m}_A^2)$ have the same dimension over κ_A. Consequently, the regular local rings $A/\mathfrak{I}$ and $A/\mathfrak{x}$ have the same dimension. As the ideals $\mathfrak{I}$ and $\mathfrak{x}$ are prime and we have $\mathfrak{x} \subseteq \mathfrak{I}$, we finally have $\mathfrak{I} = \mathfrak{x}$. $\square$

Example 7.4.12. Let k be a field. Then the ring $A = k[\![X_1, \dots, X_n]\!]$ of power series is local. Let $\mathfrak{I}$ be a proper ideal of A. By Corollary 7.4.11, for $A/\mathfrak{I}$ to be regular, it is necessary and sufficient that we can find a positive integer r and elements $F_1, \dots, F_r$ of A, generating $\mathfrak{I}$, and such that the matrix $(\partial F_i/\partial X_j(0))$ is of rank r (Jacobian criterion). We then have $\dim(A/\mathfrak{I}) = n - r$.

7.4.3 Eisenstein polynomials

Let A be a ring, $\mathfrak{p}$ a prime ideal of A, and P a polynomial in $A[T]$. We say P is an **Eisenstein polynomial** for $\mathfrak{p}$ if it satisfies the following conditions:

(a) P is monic with degree $d \geq 2$;

(b) $P(T) \equiv T^d \bmod \mathfrak{p}A[T]$;

(c) $P(0) \notin \mathfrak{p}^2$.

In other words, an Eisenstein polynomial for $\mathfrak{p}$ is of the form $P(T) = T^d + \sum_{i=0}^{d-1} a_i T^i$, with $d \geq 1$ and $a_1, \dots, a_{d-1}$ belong to $\mathfrak{p}$ with $a_0 \in \mathfrak{p} \setminus \mathfrak{p}^2$.

We say P is an Eisenstein polynomial for $\mathfrak{p}A_\mathfrak{p}$ if the canonical image of P in the polynomial ring $A_\mathfrak{p}[T]$ is an Eisenstein polynomial for the ideal $\mathfrak{p}A_\mathfrak{p}$. This means P is an Eisenstein polynomial for $\mathfrak{p}$ and that it also satisfies the following condition, stronger than (c):

(c′) any element a in A such that $aP(0) \in \mathfrak{p}^2$ belongs to $\mathfrak{p}$.

Proposition 7.4.13. *Let A be a ring, $\mathfrak{p}$ a prime ideal of A and $P \in A[T]$ an Eisenstein polynomial for $\mathfrak{p}$.*

(a) P has no decomposition of the form $P = P_1P_2$, where P_1, P_2 are two monic polynomials in $A[T]$ unequal to 1.

(b) Suppose that A is integrally closed with K its fraction field. Then P is irreducible in $K[T]$.

Proof. Let π be the canonical map from A to the residue field $\kappa(\mathfrak{p})$ of $\mathfrak{p}$ and $\tilde{\pi} : A[T] \to \kappa(\mathfrak{p})[T]$ be the extension of π such that $\tilde{\pi}(T) = T$. Suppose that $P = P_1P_2$ where P_1, P_2 are two monic polynomials unequal to 1. Then we have $\tilde{\pi}(P) = T^d = \tilde{\pi}(P_1)\tilde{\pi}(P_2)$ in $\kappa(\mathfrak{p})[T]$, with d the degree of P. If d_i is the degree of P_i, we get $\tilde{\pi}(P_i) = T^{d_i}$, which means $P_i(T) \equiv T^{d_i} \bmod \mathfrak{p}A[T]$, in particular $P_i(0) \in \mathfrak{p}$. BUt then $P(0) = P_1(0)P_2(0) \in \mathfrak{p}^2$, which is a contradiction. The assertion (b) now follows from (a). $\square$

Proposition 7.4.14. *Let A be a Noetherian local ring and $P_1, \dots, P_r$ be monic polynomials in $A[T]$ such that $P_i(T) \equiv T^{d_i} \bmod \mathfrak{m}_A A[T]$ and $\deg(P_i) \geq 2$ for each i. Let $\mathfrak{q}$ be the ideal in $A[T_1, \dots, T_r]$ generated by $P_1(T_1), \dots, P_r(T_r)$ and B the A-algebra $A[T_1, \dots, T_r]/\mathfrak{q}$. For each i, we denote by d_i the degree of P_i, ξ_i the class of T_i mod $\mathfrak{q}$, and γ_i the class of $c_i = P_i(0)$ mod $\mathfrak{m}_A^2$.*

(a) B is a Noetherian local ring with maximal ideal

$$\mathfrak{m}_B = B\mathfrak{m}_A + \sum_{i=1}^{r} B\xi_i.$$

We have $\dim(A) = \dim(B)$ and $[\kappa_B : \kappa_A] = 1$. Moreover, the monomials $\xi_1^{\alpha_1} \cdots \xi_r^{\alpha_r}$, with $0 \leq \alpha_i < d$ for $1 \leq i \leq r$, form a basis for the A-module B.

(b) *Let λ be the canonical homomorphism from $\mathfrak{m}_A/\mathfrak{m}_A^2$ to $\mathfrak{m}_B/\mathfrak{m}_B^2$. Then the kernel of λ is the κ_A-vector space generated by $\gamma_1,\dots,\gamma_r$ and the classes $\xi_1,\dots,\xi_r$ form a basis over κ_A for the cokernel of λ.*

(c) *For B to be regular, it is necessary and sufficient that A is regular and the $\gamma_1,\dots,\gamma_r$ is linearly independent in the κ_A-vector space $\mathfrak{m}_A/\mathfrak{m}_A^2$.*

Proof. The A-algebra B is isomorphic to the tensor product $B_1\otimes_A\otimes\cdots\otimes_A B_r$ where $B_i = A[T]/(P_i)$ for each i. It then follows that the monomials $\xi_1^{\alpha_1}\cdots\xi_r^{\alpha_r}$, with $0\le\alpha_i<d$ for $1\le i\le r$, form a basis for the A-module B. In particular, B is integral over A, so A and B has the same dimension.

By Corollary 3.2.24, B is Noetherian, and any maximal ideal of B contains $B\mathfrak{m}_A$. COnversely, from the hypothesis on $P_1,\dots,P_r$ and the relations $P_i(\xi_i)=0$, we get $\xi_i^{d_i}\in B\mathfrak{m}_A$ for each i. Therefore any maximal ideal of B containing $\xi_1,\dots,\xi_r$ must contains the ideal $\mathfrak{n}=B\mathfrak{m}_A+B\xi_1+\cdots+B\xi_r$. But we have $\mathfrak{m}_A=A\cap\mathfrak{n}$ and $B=A+\mathfrak{n}$, so $B/\mathfrak{n}$ is isomorphic to $A/\mathfrak{m}_A$ and therefore is maixmal in B. This shows B is local with maximal ideal $B\mathfrak{m}_A+\sum_{i=1}^r B\xi_i$, and we have $[\kappa_B:\kappa_A]=1$. This proves (a).

Let φ be the canonical homomorphism of $(A/\mathfrak{m}_A^2)[T_1,\dots,T_r]$ to $B/\mathfrak{m}_B^2$. By part (a), the kernel of φ is the ideal generated by the classes $\bar{P}_i(T_i)$ of $P_i(T_i)$ modulo $\mathfrak{m}_A^2A[T_1,\dots,T_r]$, the monomials T_iT_j, and xT_i with $1\le i,j\le r$ and $x\in\mathfrak{m}_A/\mathfrak{m}_A^2$. By hypothesis, we have $P_i(T)\equiv T^{d_i}$ mod $\mathfrak{m}_AA[T]$, so we can replace $\bar{P}_i(T_i)$ with γ_i in the description above. As a result, if we set $\mathfrak{r}=\mathfrak{m}_A^2+\sum_{i=1}^r Ac_i$, the ring $B/\mathfrak{m}_B^2$ is then isomorphic to $(A/\mathfrak{r})[T_1,\dots,T_r]$ quotient the ideal generated by the monomials T_iT_j and xT_i, with $1\le i,j\le r$ and $x\in\mathfrak{m}_A/\mathfrak{r}$. If we denote by τ_i the class of ξ_i modulo $\mathfrak{m}_B^2$, this means

$$B/\mathfrak{m}_B^2=(A/\mathfrak{r})\oplus\kappa_A\tau_1\oplus\cdots\oplus\kappa_A\tau_r. \tag{7.4.1}$$

and

$$\mathfrak{m}_B/\mathfrak{m}_B^2=(\mathfrak{m}_A/\mathfrak{r})\oplus\kappa_A\tau_1\oplus\cdots\oplus\kappa_A\tau_r. \tag{7.4.2}$$

Assertion (b) now follows from these.

According to formula (7.4.2) and the relation $[\kappa_B:\kappa_A]=1$, we get

$$[\mathfrak{m}_B/\mathfrak{m}_B^2:\kappa_B]=[\mathfrak{m}_A/\mathfrak{m}_A^2:\kappa_A]+(r-[\mathfrak{r}/\mathfrak{m}_A^2:\kappa_A]). \tag{7.4.3}$$

But the κ_A-vector space $\mathfrak{r}/\mathfrak{m}_A^2$ is generated by $\gamma_1,\dots,\gamma_r$, and we have

$$\dim(B)=\dim(A)\le[\mathfrak{m}_A/\mathfrak{m}_A^2:\kappa_A]. \tag{7.4.4}$$

Combine formula (7.4.3) and formula (7.4.4), we see assertion (c) follows. □

Corollary 7.4.15. *Let A be a regular Noetherian local ring and $P\in A[T]$ an Eisenstein polynomial for $\mathfrak{m}_A$. Then $B=A[T]/(P)$ is a Noetherian local regular ring, with the same dimension as A, and $[\kappa_B:\kappa_A]=1$. In fact, we have $\mathfrak{m}_B=B\mathfrak{m}_A+B\xi$, where ξ is the class of T modulo (P).*

Proof. By definition we have $\deg(P)\ge 2$ and $P(T)\equiv T^d$ mod $\mathfrak{m}_A$, so this is a special case of Proposition 7.4.14 with $r=1$. □

Proposition 7.4.16. *Let A be an integral domain, with fraction field K, and L an algebraic extension of K. Let B be the integral closure of A in L and $\mathfrak{p}$ a prime ideal of A. Suppose that the local ring $A_{\mathfrak{p}}$ is regular and Noetherian; let ξ be a primitive element for L over K (i.e., $L=K(\xi)$) and suppose that the minimal polynomial of ξ over K is an Eisenstein polynomial $P\in A[T]$ for $\mathfrak{p}A_{\mathfrak{p}}$.*

(a) *There exists in B a unique prime ideal $\mathfrak{P}$ lying over $\mathfrak{p}$.*

(b) *The local ring $B_{\mathfrak{P}}$ is Noetherian and regular, with the same dimension as $A_{\mathfrak{p}}$. Moreover, $B_{\mathfrak{P}} = A_{\mathfrak{p}}[\xi]$.*

(d) *The canonical homomorphism from $A/\mathfrak{p}$ to $B/\mathfrak{P}$ induces an isomorphism on the fraction fields.*

Proof. Set $C = A_{\mathfrak{p}}[\xi]$ and let d be the degree of P. By Proposition 7.4.14 applied to the ring $A_{\mathfrak{p}}$, the Eisenstein polynomial P is irreducible in $K[T]$ and $(1, \xi, \dots, \xi^{d-1})$ is a basis for L over K, and for C over $A_{\mathfrak{p}}$. Since P is monic, the kernel of the canonical homomorphism from $A_{\mathfrak{p}}[T]$ to C is equal to (P); in other words, $C = A_{\mathfrak{p}}[T]/(P)$, so we can apply Corollary 7.4.15: C is a regular Noetherian local with dimension equal to that of $A_{\mathfrak{p}}$, the maximal ideal $\mathfrak{m}_C$ of C is generated by $\mathfrak{p} \cup \{\xi\}$ and the residue field κ_C is a trivial extension of $\kappa(\mathfrak{p})$. To prove the proposition, it suffices to find a unique prime ideal $\mathfrak{P}$ of B lying over $\mathfrak{p}$, and such that $C = B_{\mathfrak{P}}$.

Let $S = A - \mathfrak{p}$, then the integral closure of $A_{\mathfrak{p}}$ in L is equal to $S^{-1}B$. But t is integral over $A_{\mathfrak{p}}$, and $C = A_{\mathfrak{p}}[\xi]$ is a regular local ring, hence integrally closed (Corollary 7.4.6). It then follows that $C = S^{-1}B$, so $S^{-1}B$ is local. By Theorem 4.1.69 and Corollary 4.1.65, there then exists a unique prime ideal of $S^{-1}B$ lying over $\mathfrak{p}A_{\mathfrak{p}}$, which corresponds to a unique prime ideal $\mathfrak{P}$ of B lying over $\mathfrak{p}$, and we have $B_{\mathfrak{P}} = S^{-1}B = C$ (Lemma 4.1.63). □

Corollary 7.4.17. *With the hypothesis in Proposition 7.4.16, suppose that $A_{\mathfrak{p}}$ is a DVR. Then $B_{\mathfrak{P}}$ is a DVR, ξ is a uniformizer for $B_{\mathfrak{P}}$, and we have*

$$f(B_{\mathfrak{P}}/A_{\mathfrak{p}}) = 1, \quad e(B_{\mathfrak{P}}/A_{\mathfrak{p}}) = [L:K]. \tag{7.4.5}$$

Proof. In fact, DVRs are regular Noetherian local rings with dimension 1; set $d = [L : K]$ and $P(T) = T^d + \sum_{i=0}^{d-1} a_i T^i$. Let $v_{\mathfrak{P}}$ be the valuation on $B_{\mathfrak{P}}$. Then since P is an Eisenstein polynomial, $a_0 \in \mathfrak{m}_{A_{\mathfrak{p}}} \setminus \mathfrak{m}_{A_{\mathfrak{p}}}^2$ and $a_i \in \mathfrak{m}_{A_{\mathfrak{p}}}$ for each i, whence

$$d = v_{\mathfrak{P}}(\xi^d) = v_{\mathfrak{P}}(-\sum_{i=0}^{d-1} a_i \xi^i) = \min_i \{v_{\mathfrak{P}}(a_i) + i\} = v_{\mathfrak{P}}(a_0).$$

This shows $d = e(B_{\mathfrak{P}}/A_{\mathfrak{p}})$. Since $[\kappa_B : \kappa_A] = 1$, we also have $f(B_{\mathfrak{P}}/A_{\mathfrak{p}}) = 1$. □

Example 7.4.18 (Examples of extensions by Eisenstein polynomials).

(a) Let $A = \mathbb{Z}$ and $L = \mathbb{Q}(p^{1/d})$, where p is a prime number and $d \geq 2$. Let B be the integral closure of $\mathbb{Z}$ in L. Then $P(T) = T^d - p$ is an Eisenstein polynomial in $\mathbb{Z}[T]$ for $p\mathbb{Z}_{(p)}$, and there exists a unique prime ideal $\mathfrak{P}$ of B lying over $p\mathbb{Z}$. Since $Z_{(p)}$ is a DVR, there exists on $Q(p^{1/d})$ a unique discrete valuation $v_{\mathfrak{P}}$ such that $v(p) > 0$. We have $[L : K] = v_{\mathfrak{P}}(p) = d$, and $B/\mathfrak{P}$ is a field with p elements. The valuation ring $B_{\mathfrak{P}}$ of $v_{\mathfrak{P}}$ is equal to $\mathbb{Z}_{(p)}[p^{1/d}]$.

(b) Let $A = \mathbb{Z}$ and $L = \mathbb{Q}(\zeta)$ where $\zeta = e^{2\pi i/p^f}$, with p a prime and $f \geq 1$. Let B be the integral closure of $\mathbb{Z}$ in L and P the polynomial in $\mathbb{Z}[T]$ defined by

$$P(T-1) = \frac{T^{p^f} - 1}{T^{p^{f-1}} - 1}.$$

Set $d = p^f - p^{f-1}$. We have $P(\zeta - 1) = 0$, $P(0) = p$, and

$$P(T-1) \equiv (T-1)^d \mod p\mathbb{Z}[T],$$

so $P(T) \equiv T^d$. Therefore P is an Eisenstein polynomial for $p\mathbb{Z}_{(p)}$. There exists a unique prime ideal $\mathfrak{P}$ of B lying over $\mathfrak{p}\mathbb{Z}$, and we have $B_{\mathfrak{P}} = \mathbb{Z}_{(p)}[\zeta]$. Moreover, $\zeta - 1$ is a uniformizer for $B_{\mathfrak{P}}$ and

$$[L:K] = d = p^f - p^{f-1}.$$

If $v_{\mathfrak{P}}$ is the unique normalized valuation on $\mathbb{Q}(\zeta)$ such that $v_{\mathfrak{P}}(p) > 0$, we have $v_{\mathfrak{P}}(p) = d$. Moreover, the field $B/\mathfrak{P}$ has p elements. We can prove that B is equal to $\mathbb{Z}[\zeta]$.

7.5 Dimension and graded rings

Let R be a graded ring of type $\mathbb{Z}$ and $(R_n)_{n\in\mathbb{Z}}$ its graduation; we assume that $R_n = 0$ for $n < 0$. For each $n \in \mathbb{Z}$, let $R_{\geq n} = \bigoplus_{i\geq n} R_i$. We have $R = R_{\geq 0}$ and each $R_{\geq n}$ is an homogeneous ideal of R. Denote by S the multiplicative subset $1 + R_{\geq n}$ consists of elements of R with degree zero term equals to 1, and consider the localization $S^{-1}R$. By identifying R with a subring of $\widehat{R} = \prod_n R_n$, we see S is invertible in the completion $\widehat{R}$, whence $S^{-1}R$ can be identified with a subring of $\widehat{R}$ containing R. For $s \in S$ and $a \in R_{\geq n}$, the element $s^{-1}a - a$ of $\widehat{R}$ belongs to $\prod_{i\geq n+1} R_i$; therefore we have $S^{-1}R_{\geq n} = (S^{-1}R) \cap \prod_{i\geq n} R_i$. From these we deduce the following results:

Proposition 7.5.1. *Let $R = \bigoplus_{n\in\mathbb{N}} R_n$ be a graded ring and S be defined as above.*

(a) The ideals $S^{-1}R_{\geq n}$ form an exhaustive and separated filtration of $S^{-1}R$.

(b) The homomorphism $i_S : R \to S^{-1}R$ induces an isomorphism u_n of R_n to $S^{-1}R_{\geq n}/S^{-1}R_{\geq n+1}$ for each n; the u_n are homogeneous components of an isomorphism of R onto the graded ring associated with $S^{-1}R$, filtered by $S^{-1}R_{\geq n}$.

Remark 7.5.2. Rn element a/s of $S^{-1}R$ with $a \in R$, $s \in S$, is invertible if and only if the degree 0 component of a is invertible in R_0. Therefore, if R_0 is a local ring, so is $S^{-1}R$ and the canonical injection $R_0 \to S^{-1}R$ induces an isomorphism on residue fields.

Remark 7.5.3. Suppose that R is generated by R_0 and R_1; then for each n we have $R_{n+1} = R_1 \cdot R_n$, so $R_{\geq n+1} = R_1 \cdot R_{\geq n}$ and $S^{-1}R_{\geq n+1} = R_1 \cdot S^{-1}R_{\geq n}$. As a result, the filtration $(S^{-1}R_{\geq n})$ of $S^{-1}R$ is $S^{-1}R_{\geq 1}$-adic.

Example 7.5.4.

(a) Let $\mathfrak{p}$ be a homogeneous prime ideal of $\mathbb{C}[X_0, \dots, X_n]$ different to the ideal generated by the X_i's (denoted by $\mathfrak{m}$). Let $X \subseteq \mathbb{CP}^n$ be the projective variety defined by $\mathfrak{p}$ and C the algebraic variety in $\mathbb{C}^{n+1}$ defined by $\mathfrak{p}$ (so that C the **cone** of X). Let $R = \mathbb{C}[X_0, \dots, X_n]/\mathfrak{p}$ be the coordinate ring of C and consider the localization $S^{-1}R$. We claim that $S^{-1}R$ is the local ring at the origin, that is, $S^{-1}R = R_{\mathfrak{m}_0}$. This follows from the fact that a prime ideal $\mathfrak{q}$ is disjoint from S if and only if every element of $\mathfrak{q}$ has zero constant term, or equivalently is contained in the maximal ideal $\mathfrak{m}$. (Therefore S and $R \setminus \mathfrak{m}/\mathfrak{p}$ have the same saturation.)

(b) Let A be a local ring and $\mathfrak{a}$ a proper ideal of A. Then $R = \bigoplus_n \mathfrak{a}^n/\mathfrak{a}^{n+1}$ is a graded ring and $R_0 = A/\mathfrak{a}$ is local; it is generated by R_0 and R_1. The ring $S^{-1}R$ is then local and the filtration $(S^{-1}R_{\geq 1})$ is the $S^{-1}R_{\geq 1}$-adic filtration. Note that in general the rings A and $S^{-1}R$ are not isomorphic.

7.5.1 Chains of homogeneous ideals

In this part, we will use $\mathrm{dimgr}(R)$ to denote the supremum of the length of chains homogeneous prime ideals in R; similarly, if $\mathfrak{p}$ is a homogeneous ideal of R, we use $\mathrm{htgr}(\mathfrak{p})$ to denote the supremum of the length of chains homogeneous prime ideals in R for which $\mathfrak{p}$ is the largest element. Note that if $\mathfrak{p}$ is a homogeneous prime ideal of R, we have $\mathfrak{p} \cap S = \varnothing$; in fact, if $\mathfrak{p}$ contains an element with degree zero componenet 1, then $1 \in \mathfrak{p}$. The map $\mathfrak{p} \mapsto S^{-1}\mathfrak{p}$ from the set of homogeneous prime ideals of R to the set of prime ideals of $S^{-1}R$ is then injective and increasing. Concequently, we have

$$\mathrm{dimgr}(R) \leq \dim(S^{-1}R) \leq \dim(R), \quad \mathrm{htgr}(\mathfrak{p}) \leq \mathrm{ht}(S^{-1}\mathfrak{p}) = \mathrm{ht}(\mathfrak{p}). \tag{7.5.1}$$

Example 7.5.5. Let A be an arbitrary ring and consider the polynomial ring $A[X]$. For a homogeneous prime ideal $\mathfrak{p}$ of $A[X]$, we set $\mathfrak{p}_0 = \mathfrak{p} \cap A$ to be the constant part of elements in $\mathfrak{p}$. We distinguish two cases:

(a) If $X \in \mathfrak{p}$, then it is easy to see $\mathfrak{p} = \mathfrak{p}_0 + X \cdot A[X]$ because any polynomial $f \in \mathfrak{p}$ can be written as $f = f(0) + X \cdot g$.

(b) If $X \notin \mathfrak{p}$, then $\mathfrak{p} = \mathfrak{p}_0 \cdot A[X]$. In fact, let $f \in \mathfrak{p}$ and write

$$f(X) = a_0 + a_1 X + \cdots + a_n X^n$$

with $a_i \in A$. Then $a_0 \in \mathfrak{p}_0$ by definition, and we have $g = a_1 X + \cdots + a_n X^n \in \mathfrak{p}$. Note that g is divisible by X, so $g/X \in \mathfrak{p}$ since $\mathfrak{p}$ is prime and $X \notin \mathfrak{p}$. Replacing this process we eventually conclude that $a_i \in \mathfrak{p}_0$ for all i, whence $f \in \mathfrak{p}_0 \cdot A[X]$.

We now compute $\operatorname{htgr}(\mathfrak{p})$. Let $\mathfrak{q}_0 \subset \mathfrak{q}_1 \subset \cdots \subset \mathfrak{q}_n = \mathfrak{p}_0$ be a saturated chain in A, so that $n = \operatorname{ht}(\mathfrak{p}_0)$. Then by extending them in $A[X]$, we get a chain

$$\mathfrak{q}_0 \cdot A[X] \subset \mathfrak{q}_1 \cdot A[X] \subset \cdots \subset \mathfrak{q}_n \cdot A[X] = \mathfrak{p}_0 \cdot A[X].$$

In case (b), we have $\mathfrak{p} = \mathfrak{p}_0 \cdot A[X]$, and any homogeneous prime ideal $\mathfrak{p}' \subseteq \mathfrak{p}$ of $A[X]$ satisfies $\mathfrak{p}' = \mathfrak{p}'_0 \cdot A[X]$ (using the claim in (b)). From this, it is easy to conclude that $\operatorname{htgr}(\mathfrak{p}) = \operatorname{ht}(\mathfrak{p})$ in this case.

On the other hand, in case (a) we claim that the chain $\mathfrak{p}_0 \cdot A[X] \subsetneq \mathfrak{p} = \mathfrak{p}_0 + X \cdot A[X]$ is saturated for homogeneous prime ideals of A, so that $\operatorname{htgr}(\mathfrak{p}) = \operatorname{ht}(\mathfrak{p}) + 1$. For this, let $\mathfrak{p}' \subseteq \mathfrak{p}$ be a homogeneous prime ideal containing $\mathfrak{p}_0 \cdot A[X]$ and choose $f \in \mathfrak{p}' \setminus (\mathfrak{p}_0 \cdot A[X])$; write $f = a_0 + a_1 X + \cdots + a_n X^n$ as usual, and recall that $a_0 \in \mathfrak{p}_0$ with $a_i \notin \mathfrak{p}_0$. Since $f \notin \mathfrak{p}_0 \cdot A[X]$, we may assume that $a_d \notin \mathfrak{p}_0$ with $1 \leq d \leq n$. As $\mathfrak{p}'$ is homogeneous, each homogeneous componenet of f belongs to $\mathfrak{p}'$, so $a_d X^d \in \mathfrak{p}' \subseteq \mathfrak{p}$. But every element of $\mathfrak{p}$ has constant term in $\mathfrak{p}_0$ by definition and $\mathfrak{p}'$ is prime, so $X^d \in \mathfrak{p}'$, and therefore $X \in \mathfrak{p}'$. This shows $\mathfrak{p}' = \mathfrak{p}$ and justifies our claim.

In summary, we have proved the following equality

$$\operatorname{htgr}(\mathfrak{p}) = \begin{cases} \operatorname{ht}(\mathfrak{p}_0) + 1 & \text{if } X \in \mathfrak{p}, \\ \operatorname{ht}(\mathfrak{p}_0) & \text{if } X \notin \mathfrak{p}. \end{cases}$$

Now by taking $\mathfrak{p}$ to be the maximal ideals of $A[X]$, we deduce that $\operatorname{dimgr}(A[X]) \leq \dim(A) + 1$. However, it is easy to see this equality actually holds, for example we can take a saturated chain $\mathfrak{q}_0 \subset \mathfrak{q}_1 \subset \cdots \subset \mathfrak{q}_n$ in A, extending it to a chain

$$\mathfrak{q}_0 \cdot A[X] \subset \mathfrak{q}_1 \cdot A[X] \subset \cdots \subset \mathfrak{q}_n \cdot A[X] \subset \mathfrak{q}_n \cdot A[X] + X \cdot A[X].$$

It is easy to see the ideals above are homogeneous and prime, which shows $\operatorname{dimgr}(A[X]) \geq \dim(A) + 1$.

Finally, recall that in Exercise 7.1.17 we have shown that for any positive integers n and d with $n + 1 \leq d \leq 2n + 1$, there exists a ring A such that $\dim(A) = n$ $\dim(A[X]) = d$. In view of our previous result, we see the inequality in (7.5.1) can be strict. But also recall that for a Noetherian ring A, we always have $\dim(A[X]) = \dim(A) + 1$ (Corollary 7.2.30).

We now turn to our main theorem of this part, which asserts that if the graded ring R is Noetherian then $\operatorname{dimgr}(R) = \dim(R)$. First, we recall that for an arbitrary ideal $\mathfrak{a}$ of R, we denote by $\mathfrak{a}^h$ the ideal generated by homogeneous elements in $\mathfrak{a}$: in other words $\mathfrak{a}^h = \sum_n (\mathfrak{a} \cap R_n)$. We note that $\mathfrak{a}^h$ is a homogeneous and contained in $\mathfrak{a}$. Moreover, by Proposition 2.1.46, $\mathfrak{p}^h$ is prime if $\mathfrak{p}$ is prime.

Lemma 7.5.6. *Any maximal element of the set of homogeneous prime ideals of R is a maximal ideal of R containing $R_{\geq 1}$.*

Proof. Let $\mathfrak{m}$ be a proper homogeneous ideal of R. Then we have

$$\mathfrak{m} \subseteq (\mathfrak{m} \cap R_0) + R_{\geq 1} \neq R.$$

If $\mathfrak{m}$ is maximal, then $\mathfrak{m} = \mathfrak{m}_0 + R_{\geq 1}$, where $\mathfrak{m}_0$ is a maximal ideal of R_0. □

Lemma 7.5.7. *Let $\mathfrak{p}$ and $\mathfrak{q}$ be distinct prime ideals of R such that $\mathfrak{q} \subset \mathfrak{p}$. If $\mathfrak{q}^h = \mathfrak{p}^h$, then $\mathfrak{q}$ is homogeneous, $\mathfrak{p}$ is not homogeneous, and* $\mathrm{ht}(\mathfrak{p}/\mathfrak{q}) = 1$.

Proof. By replacing R with $R/\mathfrak{q}^h$, we can assume that $\mathfrak{q}^h = 0$, so that R is integral (since $\mathfrak{q}^h$ is prime), $\mathfrak{p}^h = 0$. It suffices to prove that $\mathrm{ht}(\mathfrak{p}) \leq 1$, because $\mathrm{ht}(\mathfrak{p}) = 0$ implies $\mathfrak{p} = \{0\}$ when R is integral. Now since $\mathfrak{p}^h = \{0\}$, we have $\mathfrak{p} \cap R_n = \{0\}$ for each n, so $\mathfrak{p}$ is disjoint from the multiplicative subset $T = \bigcup_n (R_n \setminus \{0\})$. The ring $R_\mathfrak{p}$ is then isomomorphic to a subring of $T^{-1}R$, and we get

$$\mathrm{ht}(\mathfrak{p}) = \dim(R_\mathfrak{p}) \leq \dim(T^{-1}R).$$

However, by Lemma 4.1.58 the ring $T^{-1}R$ is isomomorphic to $K[X, X^{-1}]$, where K is a field. Therefore $\dim(T^{-1}R) \leq 1$ and $\mathrm{ht}(\mathfrak{p}) \leq 1$, which proves the claim. □

Proposition 7.5.8. *Let $\mathfrak{p}$ be a prime ideal of R. If $\mathfrak{p} \neq \mathfrak{p}^h$ then* $\mathrm{ht}(\mathfrak{p}^h) = \mathrm{ht}(\mathfrak{p}) - 1$.

Proof. By Proposition 2.1.46, $\mathfrak{p}^h$ is a prime ideal contained in $\mathfrak{p}$, so $\mathrm{ht}(\mathfrak{p}^h) \leq \mathrm{ht}(\mathfrak{p}) - 1$. The proposition is trivial if $\mathrm{ht}(\mathfrak{p}^h) = +\infty$, so we may assume that $\mathrm{ht}(\mathfrak{p}^h) < +\infty$. We now demonstrate the inequality $\mathrm{ht}(\mathfrak{p}) \leq \mathrm{ht}(\mathfrak{p}^h) + 1$ by proving that, for any prime ideal $\mathfrak{q}$ contained in $\mathfrak{p}$ and distinct from $\mathfrak{p}$, we have $\mathrm{ht}(\mathfrak{q}) \leq \mathrm{ht}(\mathfrak{p}^h)$.

We now proceed by induction on $\mathrm{ht}(\mathfrak{p})$. We distinguish two cases: if $\mathfrak{q}^h \neq \mathfrak{p}^h$, then we have $\mathrm{ht}(\mathfrak{q}^h) < \mathrm{ht}(\mathfrak{p}^h)$ since $\mathfrak{q}^h \subset \mathfrak{p}^h$; also, the equality

$$\mathrm{ht}(\mathfrak{q}) \leq \mathrm{ht}(\mathfrak{q}^h) + 1$$

holds by the induction hypothesis if $\mathfrak{q} \neq \mathfrak{q}^h$ and is trivial if $\mathfrak{q} = \mathfrak{q}^h$. Concequently, we get $\mathrm{ht}(\mathfrak{q}) \leq \mathrm{ht}(\mathfrak{q}^h) + 1 \leq \mathrm{ht}(\mathfrak{p}^h)$, which proves our claim. On the other hand, if $\mathfrak{q}^h = \mathfrak{p}^h$, then $\mathfrak{q} = \mathfrak{q}^h$ by Lemma 7.5.7, so $\mathrm{ht}(\mathfrak{q}) = \mathrm{ht}(\mathfrak{q}^h) = \mathrm{ht}(\mathfrak{p}^h)$. □

Theorem 7.5.9. *Suppose that the ring R is Noetherian.*

(a) Any chain of homogeneous prime ideals of R, saturated as a chain of homogeneous prime ideals, is saturated as a chain of prime ideals.

(b) For any homogeneous prime ideal $\mathfrak{p}$ of R, we have $\mathrm{htgr}(\mathfrak{p}) = \mathrm{ht}(S^{-1}\mathfrak{p}) = \mathrm{ht}(\mathfrak{p})$.

(c) We have $\mathrm{dimgr}(R) = \dim(S^{-1}R) = \dim(R)$.

Proof. To show (a), it suffices to prove that, if $\mathfrak{p}$ and $\mathfrak{q}$ are distinct homogeneous prime idelas of R such that $\mathfrak{q} \subset \mathfrak{p}$ and there exists no homogeneous prime ideals between $\mathfrak{q}$ and $\mathfrak{p}$, then $\mathrm{ht}(\mathfrak{p}/\mathfrak{q}) = 1$. By quotienting by $\mathfrak{q}$, we may assume that $\mathfrak{q} = \{0\}$ (the quotient ring is still Noetherian). In other words, it suffices to show that, if R is integral and Noetherian, and if $\mathfrak{p}$ is an homogeneous prime ideal of R minimal among nonzero homogeneous prime ideals, then $\mathrm{ht}(\mathfrak{p}) = 1$. Now let a be a nonzero homogeneous element of $\mathfrak{p}$ and $\mathfrak{r}$ a prime ideal of R such that $a \in \mathfrak{r} \subset \mathfrak{p}$ and is minimal with this property. Then $\mathfrak{r}^h$ is a nonzero prime ideal ($a \in \mathfrak{r}^h$) contained in $\mathfrak{r}$, so $\mathfrak{r}^h = \mathfrak{r}$, which means $\mathfrak{r}$ is homogeneous. By the hypothesis on $\mathfrak{p}$, this means $\mathfrak{p} = \mathfrak{r}$. Since R is Noetherian and integral, $\mathfrak{p}$ has height 1 (Proposition 7.2.1), which proves (a).

Let $\mathfrak{p}$ be a homogeneous prime ideal of R. We already have

$$\mathrm{htgr}(\mathfrak{p}) \leq \mathrm{ht}(S^{-1}\mathfrak{p}) \leq \mathrm{ht}(\mathfrak{p}).$$

We now prove the converse inequality by induction on $\mathrm{htgr}(\mathfrak{p})$. If $\mathrm{htgr}(\mathfrak{p}) = 0$, $\mathfrak{p}$ is minimal among the homogeneous prime ideals, so it is a minimal prime ideal of R (Corollary 2.1.47) and $\mathrm{ht}(\mathfrak{p}) = 0$. Suppose now $\mathrm{ht}(\mathfrak{p}) > 0$, we show the inequality $\mathrm{ht}(\mathfrak{q}) \leq \mathrm{htgr}(\mathfrak{p}) - 1$ for any prime ideal $\mathfrak{q}$ contained in $\mathfrak{p}$ and distinct from $\mathfrak{p}$. We distinguish two cases: If $\mathfrak{q}$ is homogeneous, this is true from the induction hypothesis:

$$\mathrm{ht}(\mathfrak{q}) \leq \mathrm{htgr}(\mathfrak{q}) \leq \mathrm{htgr}(\mathfrak{p}) - 1.$$

If $\mathfrak{q}$ is not homogeneous, we have $\mathfrak{q}^h \subset \mathfrak{q} \subset \mathfrak{p}$, so $\mathrm{ht}(\mathfrak{q}^h) \leq \mathrm{htgr}(\mathfrak{q}^h)$ by the induction hypothesis. Recall that $\mathrm{ht}(\mathfrak{q}) = \mathrm{ht}(\mathfrak{q}^h) + 1$ by Proposition 7.5.8, so $\mathrm{ht}(\mathfrak{q}) \leq \mathrm{htgr}(\mathfrak{q}^h) + 1$. The proof will now be completed if we can prove the inequality $\mathrm{htgr}(\mathfrak{q}^h) \leq \mathrm{htgr}(\mathfrak{p}) - 2$. Assume on the contrary that we have $\mathrm{htgr}(\mathfrak{q}^h) = \mathrm{htgr}(\mathfrak{p}) - 1$; then the chain $\mathfrak{q}^h \subset \mathfrak{p}$ is saturated by part (a), which contradicts the fact $\mathfrak{q}^h \subset \mathfrak{q} \subset \mathfrak{p}$.

Finally, we show that inequality $\dim(R) \leq \mathrm{dimgr}(R)$, or equivalently $\mathrm{ht}(\mathfrak{p}) \leq \mathrm{dimgr}(R)$ for any prime ideal $\mathfrak{p}$ of R. If $\mathfrak{p}$ is homogeneous, we have $\mathrm{ht}(\mathfrak{p}) = \mathrm{htgr}(\mathfrak{p}) \leq \mathrm{dimgr}(R)$ by (b). If $\mathfrak{p}$ is not homogeneous, we have $\mathrm{ht}(\mathfrak{p}) = \mathrm{htgr}(\mathfrak{p}^h) + 1$ by Proposition 7.5.8; let $\mathfrak{m}$ be a homogeneous maximal ideal of R containing $\mathfrak{p}^h$ (note that the homogeneous ideal is not maximal among homogeneous prime ideals by Lemma 7.5.6, since we have $\mathfrak{p}^h \subset \mathfrak{p}$). Then by Lemma 7.5.6, $\mathfrak{m}$ is maximal, so distinct from $\mathfrak{p}^h$, and we have

$$\mathrm{htgr}(\mathfrak{p}^h) + 1 \leq \mathrm{htgr}(\mathfrak{m}) \leq \mathrm{dimgr}(R).$$

This shows $\mathrm{ht}(\mathfrak{p}) \leq \mathrm{dimgr}(R)$, which finishes the proof. □

7.5.2 Dimension of graded modules

Again R denote a graded ring with positive degrees and $S = 1 + R_{\geq 1}$. We now let M be a graded R-module of type $\mathbb{Z}$. Then $S^{-1}M$ is an $S^{-1}R$-module, and if we write $M_{\geq n} = \bigoplus_{i\geq n} M_i$, the submodules $S^{-1}M_{\geq n}$ form an exhaustive and separated filtration of $S^{-1}M$ and the canonical map $M \mapsto S^{-1}M$ induces an isomorphism of M onto the graded module of $S^{-1}M$ associated to this filtration.

Lemma 7.5.10. *Suppose that R is generated by $R_0 \cup R_1$ and that M is generated by $\bigoplus_{i\leq n_0} M_i$ for an integer n_0. Then the filtration $(S^{-1}M_{\geq n})$ of $S^{-1}M$ is good for the ideal $S^{-1}R_{\geq 1}$ of $S^{-1}R$.*

Proof. For $n \geq n_0$, we have $M_{\geq n+1} = R_1 \cdot M_{\geq n}$, so

$$S^{-1}M_{\geq n+1} = R_1 \cdot S^{-1}M_{\geq n} = S^{-1}R_{\geq 1} \cdot S^{-1}M_{\geq n},$$

which shows the filtration $(S^{-1}M_{\geq n})$ is $(S^{-1}R_{\geq 1})$-good. □

Proposition 7.5.11. *Suppose that R is Noetherian and M is finitely generated. Then $\dim_R(M) = \dim_{S^{-1}R}(S^{-1}M)$.*

Proof. Let $\mathfrak{a}$ be the annihilator of the R-module M, which is a homogeneous ideal of R. Since M is finitely generated, the annihilator of $S^{-1}M$ is the ideal $S^{-1}\mathfrak{a}$. We have $\dim_R(M) = \dim(R/\mathfrak{a})$ and $\dim_{S^{-1}R}(S^{-1}M) = \dim(S^{-1}R/S^{-1}\mathfrak{a})$ by definition. Since $R/\mathfrak{a}$ is Noetherian, an application of Theorem 7.5.9 proves the claim. □

Proposition 7.5.12. *Suppose that R_0 is an Artinian local ring, R is generated by $R_0 \cup R_1$, and R_1 is a finitely generated R_0-module. If M is nonzero and finitely generated as an R-module, then M_n is an R_0-module with finite length for each n, and there exists $Q(T) \in \mathbb{Z}[T, T^{-1}]$ such that $Q(1) > 0$ and the following equality holds in the ring $\mathbb{Z}((T))$:*

$$\sum_{n\in\mathbb{Z}} \ell_{R_0}(M_n) \cdot T^n = (1 - T^{-d})Q(T),$$

where $d = \dim_R(M)$.

Proof. The ring $S^{-1}R$ is local and Noetherian by Remark 7.5.2, the $S^{-1}R$-module $S^{-1}M$ is finitely generated and nonzero, and $d = \dim_R(M)$ is nonzero. Also, by Lemma 7.5.10 the filtration $(S^{-1}M_{\geq n})$ is $S^{-1}R_{\geq 1}$-good and $S^{-1}R_{\geq 1}$ is a defining ideal of $S^{-1}R$ (Proposition 7.2.16). Finally, we have $\ell_{S^{-1}R}(S^{-1}M_{\geq n}/S^{-1}M_{\geq n+1}) = \ell_{R_0}(M_n)$ for each n, so it suffices to apply Theorem 7.3.13. □

Corollary 7.5.13. *If R is a Noetherian local ring and $\mathfrak{a}$ is a defining ideal of R, then* $\dim(R) = \dim(\mathrm{gr}_{\mathfrak{a}}(R))$.

Proof. Apply Proposition 7.5.12 on the module $M = B = \mathrm{gr}_{\mathfrak{a}}(R)$, we get

$$\sum_{n=0}^{\infty} \ell_{R/\mathfrak{a}}(\mathfrak{a}^n/\mathfrak{a}^{n+1}) \cdot T^n = (1-T)^{-d}Q(T)$$

where $d = \dim(\mathrm{gr}_{\mathfrak{a}}(R))$ and $Q(1) \neq 0$. We have $d = \dim(R)$ by Theorem 7.3.20, whence the corollary. □

Corollary 7.5.14. *Suppose that M is a finitely generated R-module. Let a be a homogeneous element of R with positive degree, and not belonging to any minimal element $\mathfrak{p}$ in* $\mathrm{supp}(M)$ *such that* $\dim(R/\mathfrak{p}) = \dim_R(M)$. *Then* $\dim_R(M/aM) = \dim_R(M) - 1$.

Proof. Put $d = \dim_R(M)$; by the hypothesis on a, we have $\dim_R(M/aM) < d$. Also, by Proposition 7.5.11, we have

$$\dim_R(M/aM) = \dim_{S^{-1}M}(S^{-1}M/(b/1) \cdot S^{-1}M)$$

and the formula (7.2.3) implies the following inequality:

$$\dim_{S^{-1}R}(S^{-1}M/(b/1) \cdot S^{-1}M) \geq \dim_{S^{-1}R}(S^{-1}M) - 1.$$

Finally, $\dim_{S^{-1}R}(S^{-1}M) = \dim_R(M) = d$ by Proposition 7.5.11. We then conclude that $\dim_R(M/bM) \geq d - 1$, which proves the claim. □

Proposition 7.5.15. *Suppose that R_0 is an Artinian local ring, R is an R_0-algebra of finite type and M is a finitely generated R-module.*

(a) *Let $a_1, \ldots, a_n$ be homogeneous elements of R with positive degrees and φ the homomorphism from $R_0[X_1, \ldots, X_n]$ to R map X_i to a_i for each i. Then $S^{-1}M/\sum_{i=1}^n (a_i/1) \cdot S^{-1}M$ has finite length over $S^{-1}R$ if and only if $\varphi^*(M)$ is a finitely generated module over $R_0[X_1, \ldots, X_n]$.*

(b) *There exists a family $(a_1, \ldots, a_d)$ of elements of R, homogeneous with the same positive degree, where $d = \dim_R(M)$, such that $(a_1/1, \ldots, a_d/1)$ is a maximal secant sequence for the $S^{-1}R$-module $S^{-1}M$. If moreover R is generated by R_1 as an R_0-algebra and the residue field of R_0 is infinite, each a_i can be taken to have degree* 1.

Proof. Put $N = M/\sum_{i=1}^n a_iM$, we have $\dim_R(N) = \dim_{S^{-1}R}(S^{-1}N)$ by Proposition 7.5.11. Also, the $S^{-1}R$-module $S^{-1}N$ is of finite length if and only if the R-module N has finite length, that is, N is a finitely generated R_0-module. If $\varphi^*(M)$ is the pullback module over $R_0[X_1, \ldots, X_n]$, we have $\varphi^*(N) = \varphi^*(M)/\sum_{i=1}^n X_i \cdot \varphi^*(M)$. Therefore $\varphi^*(M)$ is a finitely generated $R_0[X_1, \ldots, X_n]$-module if and only if N is a finitely generated R_0-module, which proves (a).

Let's assume the hypothesis in Proposition 7.5.15(b). We may assume that $\dim_R(M) > 0$. Note that any minimal element of $\mathrm{supp}(M)$ is homogeneous (Corollary 2.1.47). According to Proposition 2.1.53, there exists a homogeneous element a of R with positive degree, not belonging to any minimal element $\mathfrak{p}$ in $\mathrm{supp}(M)$ such that $\dim(R/\mathfrak{p}) = \dim_R(M)$. By Corollary 7.5.14, we have $\dim_R(M/aM) = \dim_R(M) - 1$. Suppose further that R is generated by R_1 as an R_0-algebra and the residue field k of R_0 is infinite. For any minimal element $\mathfrak{p}$ in

$\operatorname{supp}(M)$ such that $\dim(R/\mathfrak{p}) = \dim_R(M)$, consider the subspace $V_\mathfrak{p} = (\mathfrak{p} \cap R_1) \otimes_{R_0} k$ of the k-vector space $V = R_1 \otimes_{R_0} k$. These subspaces are proper: if $V_\mathfrak{p} = V$, then $\mathfrak{p} \cap H_1 = H_1$ by **??**, so $H_1 \subseteq \mathfrak{p}$ and $\dim_R(M) = \dim(R/\mathfrak{p}) \leq \dim(R/R_1) = 0$, which contradicts to our assumption. Since k is infinite, the union of $V_\mathfrak{p}$'s is distinct from V. If $a \in R_1$ is such that $a \otimes 1$ is not contained in any of the $V_\mathfrak{p}$, then a is not contained in the minimal element $\mathfrak{p}$ such that $\dim(R/\mathfrak{p}) = \dim_R(M)$, whence $\dim_R(M/aM) = \dim_R(M) - 1$.

By induction on the dimension $d = \dim_R(M)$, we can construct a sequence $(b_1, \dots, b_d)$ of elements in R, with $n_i = \deg(b_i) > 0$, wuch that $M/\sum_{i=1}^d b_i M$ is a R-module of finite length. Moreover, if R is generated by R_1 as an R_0-algebra and the residue field of R_0 is infinite, each b_i can be taken to have degree 1. Now by Proposition 7.5.11, $\dim_{S^{-1}R}(S^{-1}M) = d$ and

$$\dim_{S^{-1}R}(S^{-1}M/\sum_{i=1}^{d}(b_i/1) \cdot S^{-1}M) = 0.$$

Therefore $(b_1/1, \dots, b_d/1)$ is a maximal secant sequence for $S^{-1}M$. Put $a_i = b_i^{(n_1 \cdots n_d)/n_i}$, we see the a_i's have the same degree, and $(a_1/1, \dots, a_d/1)$ is a maximal secant sequence for $S^{-1}M$ by Proposition 7.2.10. □

Corollary 7.5.16. *Suppose that R_0 is a field and R is an R_0-algebra of finite type. Put $n = \dim(R)$. Then there exists homogeneous elements $a_1, \dots, a_n$ of R with the same positive degree such that the R_0-homomorphism $\varphi : R_0[X_1, \dots, X_n] \to R$, defined by $\varphi(X_i) = a_i$ for each i, is injective and makes R a finitely generated $R_0[X_1, \dots, x_n]$-module. If R is generated by R_1 as an R_0-algebra and R_0 is infinite, each a_i can be taken to have degree 1.*

Proof. There exists by Proposition 7.5.15 an R_0-homomorphism φ of the indicated form making R a finitely generated $R_0[X_1, \dots, X_n]$-module. By Theorem 7.1.28, we also have

$$\dim(R) = n = \dim(R_0[X_1, \dots, X_n]).$$

Since the ring $R_0[X_1, \dots, X_n]$ is integral and $R \cong R_0[X_1, \dots, X_n]/\ker\varphi$, we conclude that $\ker\varphi = 0$. □

Remark 7.5.17. Assume the hypothesis in Corollary 7.5.16. Let $(h_1, \dots, h_r)$ be a basis for the R_0-vector space R_1. For $\lambda = (\lambda_1, \dots, \lambda_r) \in R_0^r$, put $h_\lambda = \lambda_1 h_1 + \cdots + \lambda_r h_r$. Then Proposition 7.5.15 and Corollary 7.5.16 imply the following result: the set of elements $(\lambda^{(1)}, \dots, \lambda^{(n)})$ of $(R_0^r)^n$ such that the elements $a_i = h_{\lambda_i} \in R_1$ satisfies the conclusion of Corollary 7.5.16 contains the complement of the union of a finite number of vector subspaces of $(R_0^r)^n$ distinct from the entire space.

Corollary 7.5.18. *Let A be a Noetherian local ring and d a non-negative integer. For $\dim(A) \geq n$, it is necessary and sufficient that for any integer $r \geq 0$ we have*

$$[\mathfrak{m}_A^r/\mathfrak{m}_A^{r+1} : \kappa_A] \geq \binom{n+r-1}{n-1}.$$

Moreover, the equality holds if and only if A is regular of dimension n.

Proof. By Theorem 7.3.20 and Theorem 7.4.5, this condition is sufficient. We now demonstrate the necessity, so assume that $\dim(A) \geq n$. Consider the graded ring $\operatorname{gr}(A) = \operatorname{gr}_{\mathfrak{m}_A}(A)$; let k be a extension of the residue field κ_A such that k is infinite, and put $R = k \otimes_{\kappa_A} \operatorname{gr}(A)$. The ring R has dimension $\geq n$ by Corollary 7.5.13. We deduce by Corollary 7.5.16 the existence of an injective k-algebra homomorphism $\varphi : R_0[X_1, \dots, X_n] \to R$. Therefore for any integer $r \geq 0$,

$$[\mathfrak{m}_A^r/\mathfrak{m}_A^{r+1} : \kappa_A] = [R_r : R_0] \geq \binom{n+r-1}{n-1},$$

and the equality holds if and only if φ is bijective, whence A is regular (Theorem 7.4.5). □

Remark 7.5.19. Note that we have the equality

$$\binom{n+r}{n} = \sum_{i=0}^{r} \binom{n+i-1}{n-1}, \quad \ell_A(A/\mathfrak{m}_A^{r+1}) = \sum_{i=0}^{r} [\mathrm{gr}_i(A) : \kappa_A].$$

Therefore the condition in Corollary 7.5.18 is equivalent to $\ell_A(A/\mathfrak{m}_A^{r+1}) \geq \binom{n+r}{n}$.

7.5.3 Dimension and graded algebras

Lemma 7.5.20. *Let A be a ring, $\mathfrak{r}$ its Jacobson radical, $R = \bigoplus_{i\in\mathbb{Z}} R_i$ a graded A-algebra, and $M = \bigoplus_{i\in\mathbb{Z}} M_i$ a graded R-module. Suppose that each M_i is a finitely generated A-module and $M/\mathfrak{r}M$ is a finitely generated $R/\mathfrak{r}R$ module. Then M is a finitely generated R-module.*

Proof. Let $m_1, \dots, m_n$ be homogeneous elements of M whose images generate $M/\mathfrak{r}M$. Let N be the graded submodule of M generated by $m_1, \dots, m_n$. We have $M_i = N_i + \mathfrak{r}M_i$ for each i, whence $M_i = N_i$ (**??**); this shows $M = N$. □

Lemma 7.5.21. *Let $\rho : A \to B$ be a ring homomorphism and S be a multiplicative subset of A. Suppose that B is an A-algebra of finite type and $S^{-1}B$ is a finitely generated $S^{-1}A$-module. Then there exists $f \in S$ such that B_f is a finitely generated A_f-module.*

Proof. Let X be a generating set of the A-algebra B. For each $x \in X$, the image of x in $S^{-1}B$ is integral over $S^{-1}A$, so there exist an integer $n(x) \geq 0$, elements $b_1(x), \dots, b_{n(x)}(x)$ of A, and an element $f(x) \in S$, such that

$$f(x)x^{n(x)} + b_1(x)x^{n(x)-1} + \cdots + b_{n(x)} = 0.$$

Let $f = \prod_{x\in X} f(x)$; the image of $x \in X$ in B_f is then integral over A_f, so B_f is a finitely generated A_f-module. □

Proposition 7.5.22. *Suppose that R is an R_0-algebra of finite type. Then the function $\mathfrak{p} \mapsto \dim(R \otimes_{R_0} \kappa(\mathfrak{p}))$ is upper semi-continuous on $\mathrm{Spec}(A_0)$.*

Proof. Since R is an R_0-algebra of finite type, each R_i is finitely generated as an R_0-module (Corollary 2.1.38). Let $\mathfrak{p} \in \mathrm{Spec}(R_0)$ and suppose $\dim(R \otimes_{R_0} \kappa(\mathfrak{p})) = n \geq 0$. By Corollary 7.5.16, there exists $a_1, \dots, a_n$ in R, homogeneous of degree $d > 0$, such that the $\kappa(\mathfrak{p})$-homomorphism $\bar{\varphi} : \kappa(\mathfrak{p})[X_1, \dots, X_n] \to R \otimes_{R_0} \kappa(\mathfrak{p})$ map X_i to $a_i \otimes 1$ for each i, makes $R \otimes_{R_0} \kappa(\mathfrak{p})$ a finitely generated $\kappa(\mathfrak{p})[X_1, \dots, X_n]$-module. Let φ be the R_0-homomorphism from $R_0[X_1, \dots, X_n] = H$ to R map X_i to a_i for each i.

For $m \in \mathbb{Z}$, define

$$\tilde{R}_m = \bigoplus_{(m-1)d < i \leq md} R_i.$$

Then $(\tilde{R}_m)$ is a graduation of type $\mathbb{Z}$ on R which is compatible with the $R_0[X_1, \dots, X_n]$-module structure induced by φ; moreover, each $\tilde{R}_m$ is finitely generated over R_0. Now note that the Jacobson radical of $(R_0)_\mathfrak{p}$ is $\mathfrak{p}(R_0)_\mathfrak{p}$ and

$$R_\mathfrak{p}/\mathfrak{p}(R_0)_\mathfrak{p}R_\mathfrak{p} = R \otimes_{R_0} \kappa(\mathfrak{p}), \quad H_\mathfrak{p}/\mathfrak{p}(R_0)_\mathfrak{p}H_\mathfrak{p} = \kappa(\mathfrak{p})[X_1, \dots, X_n].$$

By using Lemma 7.5.20, we conclude that $R_\mathfrak{p}$ is a finitely generated $H_\mathfrak{p}$-module. Now apply Lemma 7.5.21, there then exists $f \in R_0 \setminus \mathfrak{p}$ such that R_f is finitely generated H_f-module. For any $\mathfrak{q} \in \mathrm{Spec}((R_0)_f)$, $R \otimes_{R_0} \kappa(\mathfrak{q})$ is then a finitely generated $\kappa(\mathfrak{q})[X_1, \dots, X_n]$-module, whence $\dim(R \otimes_{R_0} \kappa(\mathfrak{q})) \leq n$ by Theorem 7.1.28. This completes the proof. □

From now on, we assume that R_0 is a field and R is an R_0-algebra of finite type. In this case, we set $R_+ = R_{\geq 1}$, which is a maximal ideal of R. The ring $S^{-1}R$ can be then identified with the local ring R_{R_+}, whose maximal ideal is $(R_+)_{R_+} = S^{-1}R_+$ with residue field A_0.

Proposition 7.5.23. *Suppose that R_0 a field and R is a finite type R_0-algebra.*

(a) We have $\dim(R) \leq [R_+/R_+^2 : R_0]$.

(b) The following conditions are equivalent:

(i) $\dim(R) = [R_+/R_+^2 : R_0]$.

(ii) The Noetherian local ring $S^{-1}R$ is regular.

(iii) R is generated as an R_0-algebra by a family of homogeneous elements with positive degrees, algebraically independent over R_0.

(c) Suppose that the conditions in (b) holds, and let $a_1, \dots, a_n \in R$ be homogeneous elements with positive degrees. Then the following conditions are equivalent:

(i') The images of a_i's form a basis for the R_0-vector space R_+/R_+^2.

(ii') The images of a_i's form a system of parameters in the Noetherian local ring $S^{-1}R$.

(iii') The a_i's are algebraically independent over R_0 and generate R as an R_0-algbera.

Proof. Since R is Noetherian, we have $\dim(R) = \dim(S^{-1}R)$ by Theorem 7.5.9; also,

$$[R_+/R_+^2 : R_0] = [(S^{-1}R_+)/(S^{-1}R_+)^2 : R_0]$$

by Corollary 1.2.32. These together prove (a) and the equivalences (i)⇔(ii), (i')⇔(ii'). The implications (iii)⇒(i) and (iii')⇒(i') are trivial. To establish (i)⇒(iii), suppose that $\dim(R) = [R_+/R_+^2 : R_0]$ and let $a_1, \dots, a_n$ be homogeneous elements of R with positive degrees whose classes in R_+/R_+^2 form a basis. Consider the homomorphism $\varphi : R_0[X_1, \dots, X_n]$ to R map X_i to a_i. The ideal R_+ of R is generated by a_i, so the homomorphism φ is surjective. Thus we have

$$\dim(R_0[X_1, \dots, X_n]) = n = \dim(R) = \dim(R_0[X_1, \dots, X_n]/\ker\varphi)$$

which implies $\ker\varphi = 0$ (recall that $R_0[X_1, \dots, X_n]$ is an integra ldomain). This implies (iii), and the implication (i')⇒(iii') can be done similarly. □

If R satisfies the hypothesis in (b), we say R is an regular graded R_0-algebra, or an polynomial graded R_0-algebra. In this case, any family $(a_1, \dots, a_n)$ satisfying the hypothesis in (c) will be called a system of parameters of R.

Remark 7.5.24. In the notations of (c), let d_i be the degree of a_i for each i. Then the Poincaré series of H is given by

$$P_R = \sum_{n\in\mathbb{Z}} [R_n : R_0] \cdot T^n = \prod_{i=1}^{n} (1 - T^{d_i})^{-1}.$$

Conversely, if there exist integers $d_i > 0$ such that $P_R = \prod_i (1 - T^{d_i})^{-1}$, we may conclude that R is a polynomial graded algebra. For example, if R is generated by X with degree 1 and Y with degree 2 with the sole relation $X^2 = 0$, we have $P_R = (1 - T)^{-1}$, so R is regular.

7.6 Multiplicity

7.6.1 Multiplicity relative to an ideal

Let A be a Noetherian ring and M be a finitely generated A-module. Let $\mathfrak{a}$ be an ideal of A contained in the Jacobson radical of A and such that $M/\mathfrak{a}M$ is of finite length. Suppose that M is nonzero and put $d = \dim_A(M)$. Recall that (Corollary 7.3.15) there exists a unique integer $e_{\mathfrak{a}}(M) > 0$ such that, for $n \geq 1$,

$$\ell_A(M/\mathfrak{a}^{n+1}M) = e_{\mathfrak{a}}(M)\frac{n^d}{d!} + \beta_n n^{d-1} \tag{7.6.1}$$

where β_n has a limit when n goes to infinity. The integer $e_{\mathfrak{a}}(M)$ (or $e_{\mathfrak{a}}^A(M)$ if we want to emphasize the ring A) is called the **multiplicity** of M relative to $\mathfrak{a}$. If A is a local ring and $\mathfrak{m}$ is the maximal ideal of A, we write $e(M)$ or $e^A(M)$ for $e_{\mathfrak{m}}^A(M)$. Recall that equivalently, we have

$$\ell_A(\mathfrak{a}^n M/\mathfrak{a}^{n+1}M) = e_{\mathfrak{q}}(M)\frac{n^{d-1}}{(d-1)!} + \alpha_n n^{d-1}$$

where α_n has a limit when n goes to infinity.

Remark 7.6.1. Let $\mathfrak{b}$ be an ideal of A contained in the Jacobson radical of A and contains $\mathfrak{a}$. Then $M/\mathfrak{b}^n M$ is a quotient of $M/\mathfrak{a}^n M$, whence $e_{\mathfrak{b}}(M) \leq e_{\mathfrak{a}}(M)$ by the formula (7.6.1). Moreover, if the $\mathfrak{b}$-adic filtration on M is $\mathfrak{a}$-good, then $e_{\mathfrak{a}}(M) = e_{\mathfrak{b}}(M)$ by Theorem 7.3.13.

Recall that we can reduce the calculus of the multiplicities to the case where A is local since, according to Corollary 7.3.21, we have

$$e_{\mathfrak{a}}(M) = \sum_{\mathfrak{m}} e_{\mathfrak{a}_{\mathfrak{m}}}(M_{\mathfrak{m}})$$

where $\mathfrak{m}$ runs through the elements of $\mathrm{supp}(M) \cap V(\mathfrak{a})$ such that $\dim_{A_{\mathfrak{m}}}(M_{\mathfrak{m}}) = d_{\mathfrak{a}}(M)$.

Proposition 7.6.2. *Let $\widehat{A}$ (resp. $\widehat{M}$) be the $\mathfrak{a}$-adic completion of A (resp. M), then $e_{\mathfrak{a}}^A(M) = e_{\hat{\mathfrak{a}}}^{\widehat{A}}(\widehat{M})$.*

Proof. This follows from the fact that $\mathrm{gr}_{\mathfrak{a}}(M) \cong \mathrm{gr}_{\hat{\mathfrak{a}}}(\widehat{M})$. □

Proposition 7.6.3. *If A is regular, then $e(A) = 1$.*

Proof. If A is regular then $H_{A,\mathfrak{m}_A} = (1-T)^{-r}$ (Theorem 7.4.5), whence $e(A) = 1$. □

Example 7.6.4. By definition $e_{\mathfrak{a}^r}(M) = r^d e_{\mathfrak{a}}(M)$ where $d = \dim_A(M)$. Concequently, if A is a regular local ring, we have $e_{\mathfrak{m}_A^r}(A) = r^d$. For example, if A is a discrete valuation ring, then $e_{\mathfrak{m}_A^r}(A) = r$, whence $e_{\mathfrak{a}}(A) = \ell(A/\mathfrak{a})$ for any ideal $\mathfrak{a}$.

For an ideal $\mathfrak{a}$ of A contained in the Jacobson radical of A, we let $\mathcal{C}(\mathfrak{a})$ denote the set of classes of finitely generated A-modules M such that $M/\mathfrak{a}M$ is of finite length. For each $d \in \mathbb{N}$, let $\mathcal{C}(\mathfrak{a})_{\leq d}$ be the subset of $\mathcal{C}(\mathfrak{a})$ consists of classes of A-modules of dimension $\leq d$. We have a map $e_{\mathfrak{a},d} : \mathcal{C}(\mathfrak{a})_{\leq d} \to \mathbb{Z}$ such that $e_{\mathfrak{a},d}(M) = e_{\mathfrak{a}}(M)$ if $\dim_A(M) = d$ and $e_{\mathfrak{a},d}(M) = 0$ otherwise. This map is additive by Proposition 7.3.19, so we get an induced homomorphism, still denoted by $e_{\mathfrak{a},d}$, from the Grothendieck group $K(\mathcal{C}(\mathfrak{a})_{\leq d})$ to $\mathbb{Z}$, which is zero on $K(\mathcal{C}(\mathfrak{a})_{\leq d-1})$.

Proposition 7.6.5. *Let M be a finitely generated A-module with dimension $d \geq 0$. Let $\mathfrak{a}$ be an ideal of A contained in the Jacobson radical of A such that $M/\mathfrak{a}M$ has finite length. Then*

$$e_{\mathfrak{a}}(M) = \sum_{\mathrm{coht}(\mathfrak{p})=d} \ell_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}) \cdot e_{\mathfrak{a}}(A/\mathfrak{p})$$

Proof. By Theorem 3.1.18, M has a composition series $(M_i)_{0\leq i\leq n}$ such that M_i/M_{i+1} is isomorphic to $A/\mathfrak{p}_i$, where $\mathfrak{p}_i$ is a prime ideal of M. Also, by Remark 3.2.18, the number $\ell_{A_\mathfrak{p}}(M_\mathfrak{p})$ is the times of i such that $M_i/M_{i+1} \cong A/\mathfrak{p}$. The proposition then follows from Proposition 7.3.19. □

Corollary 7.6.6. *Suppose that A is Noetherian semi-local and let $\mathfrak{a}$ be a defining ideal for A.*

(a) We have $e_\mathfrak{a}(A) = \sum_\mathfrak{p} \ell_{A_\mathfrak{p}}(A_\mathfrak{p}) \cdot e_\mathfrak{a}(A/\mathfrak{p})$, where $\mathfrak{p}$ turns through minimal primes of A such that $\mathrm{coht}(\mathfrak{p}) = \dim(A)$.

(b) Suppose that A is integral and M is a finitely generated A-module such that $\dim_A(M) = \dim(A)$. Then $e_\mathfrak{a}(M) = \mathrm{rank}(M) \cdot e_\mathfrak{a}(A)$.

Proof. Part (a) follows directly from Proposition 7.6.5 by applying on $M = A$. Under the hypothesis of (b), the minimal prime ideal of A is (0), and $\ell_{A_{(0)}}(M_{(0)}) = \mathrm{rank}(M)$, so the claim follows. □

7.6.2 Extension of scalars

Proposition 7.6.7. *Let $\rho : A \to B$ be a local homomorphism of Noetherian local rings. Let N be a finitely generated B-module, flat over A, and such that $N \otimes_A \kappa_A$ is a B-module of finite length. If M is a nonzero finitely generated A-module and $\mathfrak{a}$ is a proper ideal of A such that $M/\mathfrak{a}M$ is of finite length, then $(M \otimes_A N)/\mathfrak{a}^e(M \otimes_A N)$ is a B-module of finite length, and we have*

$$e^B_{\mathfrak{a}^e}(M \otimes_A N) = \ell_B(N \otimes_A \kappa_A) \cdot e^A_\mathfrak{a}(M).$$

Proof. Let L be an A-module of finite length r. Then by Theorem 3.1.18 and Proposition 3.2.14, L has a composition series of length r, with quotients κ_A. Since N is flat over A, the B-module $L \otimes_A N$ possesses a composition series of length r, with quotients $N \otimes_A \kappa_A$. Therefore the length of $L \otimes_A N$ is $r \cdot \ell_B(N \otimes_A \kappa_A)$. The B-module $(M \otimes_A N)/(\mathfrak{a}^e)^n(M \otimes_A N)$ is isomorphic to $(M/\mathfrak{a}^n M) \otimes_A N$ for each $n \in \mathbb{N}$, the proposition then follows from the definition of multiplicity. □

Corollary 7.6.8. *Suppose that $\rho : A \to B$ be a local homomorphism of Noetherian local rings, B is flat over A and $\mathfrak{m}^e_A = \mathfrak{m}_B$. Then*

$$e^B_{\mathfrak{a}^e}(M \otimes_A B) = e^A_\mathfrak{a}(M).$$

Proof. In this case we have $B \otimes_A \kappa_A = B/\mathfrak{m}^e_A$, which has length 1 over B, so the claim follows from Proposition 7.6.7. □

Example 7.6.9. Let X be a complex algebraic variety, $\mathscr{O}_{X,x}$ the local ring of X at a rational point x, and X^{an} the analytic space associated with X. Let $\mathscr{O}_{X^{an},x}$ be the local ring of X^{an} at x. Then $e(\mathscr{O}_{X^{an},x}) = e(\mathscr{O}_{X,x})$.

Proposition 7.6.10. *Suppose that A is Noetherian semi-local and let $\rho : A \to B$ be a ring homomorphism making B a finitely generated A-module. Let N be a nonzero finitely generated B-module and $\mathfrak{a}$ an ideal of A contained in the Jacobson radical of A such that $N/\mathfrak{a}N$ has finite length. Let $\mathfrak{m}_1, \dots, \mathfrak{m}_r$ be the maximal ideals of B such that $\dim_{B_{\mathfrak{m}_i}}(N_{\mathfrak{m}_i}) = \dim_B(N)$ (by Corollary 3.2.24 B is semi-local). Put $B_i = B_{\mathfrak{m}_i}$ and $\mathfrak{a}_i = \mathfrak{a}B_i$ for each i. Then we have the equalities*

$$\dim_A(N) = \dim_B(N), \quad e^B_{\mathfrak{a}^e}(N) = \sum_{i=1}^r e^{B_i}_{\mathfrak{a}_i}(N_{\mathfrak{m}_i}), \quad e^A_\mathfrak{a}(N) = \sum_{i=1}^r [B/\mathfrak{m}_i : A/\mathfrak{m}^c_i] \cdot e^{B_i}_{\mathfrak{a}_i}(N_{\mathfrak{m}_i}).$$

Proof. Since B is integral over A, we have $\dim_A(N) = \dim_B(N)$. The second equality follows from Corollary 7.3.21 (note that $\mathfrak{m}_i$ is in $V(\mathfrak{a}^e)$ for each i since $\mathfrak{m}_i^c \supset \mathfrak{a}$ by Corollary 4.1.65). Now let E be a B-module of finite length; we have

$$\ell_A(E) = \sum_{\mathfrak{m}} [B/\mathfrak{m} : A/\mathfrak{m}^c] \cdot \ell_{B_\mathfrak{m}}(E_\mathfrak{m}),$$

where $\mathfrak{m}$ runs through maximal ideals of B: this is evident when E is one of the $B/\mathfrak{m}$, and the general case follows since E has a composition sequence with quotients isomomorphic to $B/\mathfrak{m}$, with $\mathfrak{m}$ a maximal ideal of B. Apply this formula to the B-module $N/\mathfrak{a}^{n+1}N$, we deduce the equality from the definition of multiplicity. □

Corollary 7.6.11. *If $[B/\mathfrak{m}_i : A/\mathfrak{m}_i^c] = 1$ for each i, then $e_\mathfrak{a}^A(N) = e_{\mathfrak{a}^e}^B(N)$.*

Lemma 7.6.12. *Let $\rho : A \to B$ be a ring homomorphism and $\mathfrak{p}$ a prime ideal of A. Consider the following conditions:*

(i) The canonical homomorphism $\tilde{\rho} : A_\mathfrak{p} \to A_\mathfrak{p} \otimes_A B$ is bijective.

(ii) There is only one prime ideal $\mathfrak{P}$ of B lying over $\mathfrak{p}$ and the canonical homomorphism $\rho_\mathfrak{p} : A_\mathfrak{p} \to B_\mathfrak{P}$ is bijective.

We have (i)⇒(ii). If $\mathfrak{p}$ is minimal, or if B is integral over A, then (i)⇔(ii).

Proof. The ring $A_\mathfrak{p} \otimes_A B$ can be identified with the localization $S^{-1}B$ by the multiplicative subset $S = \rho(A - \mathfrak{p})$. The prime ideals of $S^{-1}B$ are of the form $S^{-1}\mathfrak{P}$, wherer $\mathfrak{P}$ is a prime ideal of B such that $\mathfrak{P}^c \subseteq \mathfrak{p}$. If $\mathfrak{P}$ is such an ideal, $(S^{-1}B)_{S^{-1}\mathfrak{P}}$ is identified with $B_\mathfrak{P}$.

If condition (i) is satisfied, there exists (Lemma 4.1.63) a unique prime ideal $\mathfrak{P}$ of B such that $\mathfrak{P}^c = \mathfrak{p}$. Moreover, $B_\mathfrak{P}$ is identified with the ring $(S^{-1}B)_{S^{-1}\mathfrak{P}}$, hence with $(A_\mathfrak{p})_\mathfrak{q}$, where $\mathfrak{q}$ is the inverse image of $S^{-1}\mathfrak{P}$ under the isomorphism $\tilde{\rho} : A_\mathfrak{p} \to S^{-1}B$; then $\tilde{\rho}^{-1}(S^{-1}\mathfrak{P}) = (A - \mathfrak{p})^{-1}\mathfrak{p} = \mathfrak{p}A_\mathfrak{p}$, hence (ii).

Conversely, asssume (ii) and let $\mathfrak{P}$ be the unique prime ideal of B lying over $\mathfrak{p}$. Since $(S^{-1}B)_{S^{-1}\mathfrak{P}}$ is identified with $B_\mathfrak{P}$, it suffices to prove that $S^{-1}B$ is a local ring with maximal ideal $S^{-1}\mathfrak{P}$, which means any prime ideal $\mathfrak{Q}$ of B such that $\mathfrak{Q}^c \subseteq \mathfrak{p}$ is contained in $\mathfrak{P}$. If $\mathfrak{p}$ is minimal, we have $\mathfrak{Q}^c = \mathfrak{p}$, whence $\mathfrak{Q} = \mathfrak{P}$. If B is integral over A, then by the going up theorem, there exists a prime ideal $\mathfrak{P}'$ of B such that $\mathfrak{Q} \subseteq \mathfrak{P}'$ and $(\mathfrak{P}')^c = \mathfrak{p}$. Then $\mathfrak{P}' = \mathfrak{P}$, whence $\mathfrak{Q} \subseteq \mathfrak{P}$. □

Lemma 7.6.13. *Suppose that A is Noetherian and semi-local. Let $\mathfrak{a}$ be a defining ideal of A, and $\rho : A \to B$ a ring homomorphism making B a finitely generated A-module. Suppose that, for any prime ideal $\mathfrak{p}$ (necessarily minimal) of A such that $\dim(A/\mathfrak{p}) = \dim(A)$, there exists a unique prime ideal $\mathfrak{P}$ of B lying over $\mathfrak{p}$ such that the canonical homomorphism $\rho_\mathfrak{p} : A_\mathfrak{p} \to B_\mathfrak{P}$ is bijective. Then $\dim_A(B) = \dim(A)$ and $e_\mathfrak{a}^A(B) = e_\mathfrak{a}^A(A)$.*

Proof. Let $\mathscr{P}_A$ (resp. $\mathscr{P}_B$) be the set of prime ideals of A such that $\dim(A/\mathfrak{p}) = \dim(A)$ (resp. $\dim_A(B/\mathfrak{p}^e) = \dim_A(B)$); we have $\mathscr{P}_A \neq \varnothing$. Let $\mathfrak{p} \in \mathscr{P}_A$; there exists by hypothesis a unique ideal of B lying over $\mathfrak{p}$. By Proposition 1.2.42 we have $\mathfrak{p}^{ec} = \mathfrak{p}$, and Theorem 7.1.28 then shows that $\dim(A/\mathfrak{p}) = \dim(B/\mathfrak{p}^e) = \dim_A(B/\mathfrak{p}^e)$, whence

$$\dim_A(B) \geq \dim_A(B/\mathfrak{p}^e) = \dim(A/\mathfrak{p}) = \dim(A) \geq \dim_A(B).$$

This implies $\mathscr{P}_A \subseteq \mathscr{P}_B$ and $\dim(A) = \dim_A(B)$. Conversely, if $\mathfrak{p} \in \mathscr{P}_B$, we have the inequalities

$$\dim_A(B/\mathfrak{p}^e) = \dim_A(B) = \dim(A) \geq \dim(A/\mathfrak{p}) \geq \dim_A(B/\mathfrak{p}^e)$$

so $\mathfrak{p} \in \mathscr{P}_B$ and therefore $\mathscr{P}_A = \mathscr{P}_B$. Now by Proposition 7.6.5 and Corollary 7.6.6, we have

$$e_{\mathfrak{a}}^A(A) = \sum_{\mathfrak{p} \in \mathscr{P}_A} \ell_{A_\mathfrak{p}}(A_\mathfrak{p}) \cdot e_{\mathfrak{a}}^A(A/\mathfrak{p}), \quad e_{\mathfrak{a}}^A(B) = \sum_{\mathfrak{p} \in \mathscr{P}_B} \ell_{A_\mathfrak{p}}(A_\mathfrak{p} \otimes_A B) \cdot e_{\mathfrak{a}}^A(A/\mathfrak{p}).$$

Also, by Lemma 7.6.12, under the hypothesis on ρ we have $\ell_{A_\mathfrak{p}}(A_\mathfrak{p} \otimes_A B) = \ell_{A_\mathfrak{p}}(A_\mathfrak{p}) = 1$ for every $\mathfrak{p} \in \mathscr{P}_A$, so $e_{\mathfrak{a}}^A(A) = e_{\mathfrak{a}}^A(B)$. ☐

Proposition 7.6.14. *Suppose that A is a Noetherian semi-local ring and reduced; let $\mathfrak{a}$ be a defining ideal of A; let $Q(A)$ be the total ring of fraction of A, and B a sub A-algebra of $Q(A)$ that is finitely generated as an A-module. Then B is semi-local and $\mathfrak{a}^e$ is a defining ideal of B. Moreover, if for any maximal ideal $\mathfrak{m}$ of B such that $\dim(B_\mathfrak{m}) = \dim(B)$ we have $[B/\mathfrak{m} : A/\mathfrak{m}^c] = 1$, then $e_{\mathfrak{a}}^A(A) = e_{\mathfrak{a}^e}^B(B)$.*

Proof. By Corallary 3.2.24, B is semi-local with defining ideal $\mathfrak{a}^e$. Then $e_{\mathfrak{a}^e}^B(B) = e_{\mathfrak{a}}^A(B)$ by Corollary 7.6.11. Recall that $Q(A)$ can be identified with the ring $\prod_\mathfrak{p} A_\mathfrak{p}$, where $\mathfrak{p}$ runs through minimal prime ideals of A (Proposition 3.2.26), so the canonical map $A_\mathfrak{p} \to A_\mathfrak{p} \otimes_A B$ is bijective for any minimal prime $\mathfrak{p}$ of A. The proposition them follows from Lemma 7.6.12 and Lemma 7.6.13. ☐

Example 7.6.15. Let k be a field with $\mathrm{char}(k) \neq 2$ and let A be the local ring $k[\![X,Y]\!]/(X^2+Y^2)$, with residue field k. It is easy to compute that $e_{\mathfrak{m}_A}^A(A) = 2$. Put $B = k[\![X,T]\!]/(T^2+1)$ where $T = Y/X$, we distinguish two cases:

(a) If -1 is a squre of an element i in k, then B has two maximal ideals generated by $\{X, T+i\}$ and $\{X, T-i\}$, respectively, with residue field k. We have $e_{\mathfrak{m}_A B}^B(B) = 2$.

(b) If -1 is not a squre in k, then B has a unique maximal ideal (X) with residue field $k[T]/(T^2+1)$, and we have $e_{\mathfrak{m}_A B}^B(B) = 1$.

7.6.3 Multiplicity and secant sequences

Proposition 7.6.16. *Suppose that A is a Noetherian local ring and $s \geq 1$ be an integer. For $1 \leq i \leq s$, let δ_i be a positive integer, x_i an element of $\mathfrak{m}_A^{\delta_i}$, and ξ_i the its class in $\mathfrak{m}_A^{\delta_i}/\mathfrak{m}_A^{\delta_i+1}$. Suppose that $(x_1,\dots,x_s)$ is a secant sequence for A and let $\mathfrak{x}$ be the ideal of A generated by $(x_1,\dots,x_s)$. Then $e(A/\mathfrak{x}) \geq \delta_1 \cdots \delta_s e(A)$ and the equality holds if $(\xi_1,\dots,\xi_s)$ is a completely secant sequence for $\mathrm{gr}(A)$.*

Proof. Put $B = A/\mathfrak{x}$, and consider the formal series

$$H_A = \sum_{n=0}^\infty \ell_{\kappa_A}(\mathfrak{m}_A^n/\mathfrak{m}_A^{n+1}) \cdot T^n, \quad H_B = \sum_{n=0}^\infty \ell_{\kappa_B}(\mathfrak{m}_B^n/\mathfrak{m}_B^{n+1}) \cdot T^n.$$

Recall that, by Proposition 7.3.26, we have the following inequality in $\mathbb{Z}[\![T]\!]$:

$$H_B^{(r)} \geq \prod_{i=1}^s (1-T^{\delta_i}) H_A^{(s)} \tag{7.6.2}$$

and the equality holds if and only if $(\xi_1,\dots,\xi_s)$ is completely secant for $\mathrm{gr}(A)$. Put $d = \dim(A)$, so that $\dim(B) = d - s$ since $(x_1,\dots,x_s)$ is secant for A. By Theorem 7.3.13, there then exists elements R_A and R_B in $\mathbb{Z}[T,T^{-1}]$ such that

$$H_A = (1-T)^{-d} R_A, \quad H_B = (1-T)^{-d+s} R_B$$

and we have $R_A(1) = e(A)$, $R_B(1) = e(A/\mathfrak{x})$. Plug this into (7.6.2), we find that

$$(1-T)^{-d} R_B(T) = H_B^{(s)} \geq \prod_{i=1}^s (1-T^{\delta_i}) H_A^{(s)} = (1-T)^{-d} R(T) R_A(T)$$

where $R(T)$ is the polynomial in $\mathbb{Z}[T]$ given by

$$R(T)=\prod_{i=1}^{s}\frac{1-T^{\delta_i}}{1-T}.$$

Since $R(1)=\delta_1\cdots\delta_s$, we conclude that $e(B)\geq\delta_1\cdots\delta_s e(A)$, and the equality holds if and only if $(\xi_1,\dots,\xi_s)$ is completely secant for $\mathrm{gr}(A)$. □

Remark 7.6.17. We can show conversely that, if A is a regular local ring and $(x_1,\dots,x_s)$ is an arbitrary sequence of elements in $\mathfrak{m}$, then $e(A/\mathfrak{x})=\delta_1\cdots\delta_s$ if and only if the sequence $(\xi_1,\dots,\xi_s)$ is completely secant for $\mathrm{gr}(A)$. The point is that in this case we do not need to assume that $(x_1,\dots,x_s)$ is secant for A.

Example 7.6.18. Let A be the formal series ring $k[\![X_1,\dots,X_n]\!]$ over a field k; let $F_1,\dots,F_s$ be elements of A, $\mathfrak{a}$ the ideal they generate, and $B=A/\mathfrak{a}$. Let $P_1,\dots,P_s\in k[X_1,\dots,X_n]$ be the initial forms of the series $F_1,\dots,F_s$ and $\delta_1,\dots,\delta_s$ be their degrees, respectively. By Proposition 7.6.16, if the sequence $(F_1,\dots,F_s)$ is secant for A, we have $e(B)\geq\delta_1,\dots,\delta_s$, and the equality holds if and only if if the sequence $(P_1,\dots,P_s)$ is completely secant for the ring $k[X_1,\dots,X_n]$.

Consider for example the ring $B=k[\![X,Y]\!]/\mathfrak{a}$, where $\mathfrak{a}$ is generated by X^2+Y^3 and X^2+Y^4; the preceding inequality is $e(B)\geq 4$, and this inequality is strict because (X^2,X^2) is not completely secant. On the other hand, note that $\mathfrak{a}$ is also generated by X^2+Y^3 and Y^4-Y^3, for which the sequence of initial forms, namely (X^2,Y^3), is completely secant, so we obtain that $e(B)=6$.

From now on, let A be a Noetherian ring, $\mathfrak{a}$ an ideal of A contained in the Jacobson radical of A, and M a finitely generated A-module such that $M/\mathfrak{a}M$ has finite length. Recall that in this case we can talk about the multiplicity $e_{\mathfrak{a}}(M)$ of M relative to $\mathfrak{a}$.

Proposition 7.6.19. *Let δ be a positive integer, x an element of $\mathfrak{a}^\delta$, ξ its class in $\mathfrak{a}^\delta/\mathfrak{a}^{\delta+1}$, and φ the multiplication by ξ on $\mathrm{gr}(M)$.*

- *(a) The dimension of the A-module M/xM is equal to $\dim_A(M)$ or $\dim_A(M)-1$. In the second case, we have $e_{\mathfrak{a}}(M/xM)\geq\delta e_{\mathfrak{a}}(M)$.*
- *(b) Suppose that $\dim_A(M)\geq 1$ and the kernel of φ has finite length over $A/\mathfrak{a}$. Then we have $\dim_A(M/xM)=\dim_A(M)-1$. Moreover:*
  - *(i) If $\dim_A(M)>1$, then $e_{\mathfrak{a}}(M/xM)=\delta e_{\mathfrak{a}}(M)$.*
  - *(ii) If $\dim_A(M)=1$, then for any $n\geq 0$ we have*

$$n\delta e_{\mathfrak{a}}(M)\leq\ell_A(M/x^nM)\leq n\delta e_{\mathfrak{a}}(M)+\ell_{A/\mathfrak{a}}(\ker\varphi^n)$$

and $\delta e_{\mathfrak{a}}(M)=e_{xA}(M)\leq\ell_A(M/xM)$.

Proof. Put $M'=M/xM$; consider the Hilbert-Samuel series $H_M=H_{M,\mathfrak{a}}$ and $H_{M'}=H_{M',\mathfrak{a}}$, as well as the Poincare series $P(T)=\sum_{n=0}^{\infty}\ell_{A/\mathfrak{a}}(\ker\varphi_n)\cdot T^n$ (where φ_n is the restriction of φ on $\mathrm{gr}_n(M)$). By Theorem 7.3.13, there then exists elements R_M and $R_{M'}$ in $\mathbb{Z}[T,T^{-1}]$ such that

$$H_M=(1-T)^{-d}R_M,\quad H_{M'}=(1-T)^{-d'}R_{M'}$$

where $d=\dim_A(M)$, $d'=\dim_A(M')$, $R_M(1)=e_{\mathfrak{a}}(M)$, and $R_{M'}=e_{\mathfrak{a}}(M')$. By Proposition 7.3.24, we have the inequality

$$(1-T^\delta)H_M^{(1)}\leq H_{M'}^{(1)}\leq(1-T^\delta)H_M^{(1)}+T^\delta P^{(1)}$$

in the ring $\mathbb{Z}((T))$. Posing $R(T) = (1-T^\delta)/(1-T)$, this in equivalent to

$$(1-T)^{-d}R(T)R_M(T) \leq (1-T)^{-d'-1}R_{M'}(T) \leq (1-T)^{-d}R(T)R_M(T) + (1-T)^{-1}T^\delta P(T). \tag{7.6.3}$$

By Lemma 7.3.3, the first inequality implies that, either $d < d'+1$, or $d = d'+1$ and $R(1)R_M(1) \leq R_{M'}(1)$, which means $e_{\mathfrak{a}}(M') \geq \delta e_{\mathfrak{a}}(M)$. Since we always have $d' \leq d$, this proves (a).

Under the hypothesis of (b), we have $P(T) \in \mathbb{Z}[T]$ and $P(1) = \ell_A(\ker\varphi)$. The second inequality of (7.6.3) shows

$$(1-T)^{-d'-1}R_{M'}(T) \leq (1-T)^{-d}[R(T)R_M(T) + T^\delta(1-T)^{-d-1}P(T)].$$

Suppose that $d > 1$, again by Lemma 7.3.3, either $d'+1 < d$, or $d'+1 = d$ and $R_{M'}(1) = R(1)R(1)$, which means $e_{\mathfrak{a}}(M') = \delta e_{\mathfrak{a}}(M)$. This demonstrate the case (i) in (b), in view of part (a).

Suppose now $d = 1$. Then by the same reasoning, we have $d' = 0$ and

$$R_{M'}(1) \leq R(1)R_M(1) + P(1).$$

Recall that if $d' = 0$ then $R_{M'}(1) = \ell_A(M')$; therefore we conclude that

$$\delta e_{\mathfrak{a}}(M) \leq \ell_A(M/xM) \leq \delta e_{\mathfrak{a}}(M) + \ell_A(\ker\varphi). \tag{7.6.4}$$

Since $\ker\varphi$ has finite length, φ_m is injective for m large enough, so $(\varphi^n)_m$ is also injective for m large enough for each integer $n \geq 1$, which means $\ker\varphi^n$ has finite length. By replacing x with x^n in (7.6.4), we then get

$$n\delta e_{\mathfrak{a}}(M) \leq \ell_A(M/x^nM) \leq n\delta e_{\mathfrak{a}}(M) + \ell_A(\ker\varphi^n). \tag{7.6.5}$$

Now the submodules $\ker\varphi^n$ of the $\mathrm{gr}(A)$-Noetherian module $\mathrm{gr}(M)$ form an increasing sequence therefore stationary and that each of them is of finite length over $A/\mathfrak{a}$. Dividing by n in inequality (7.6.5) and letting n tend to $+\infty$, we find that $e_{xA}(M) = \delta e_{\mathfrak{a}}(M)$ by definition of $e_{xA}(M)$ (Corollary 7.3.15). □

Lemma 7.6.20. *Let R be a Noetherian graded ring with positive degrees, E a finitely generated graded R-module such that E_n has finite length over R_0 for each $n \in \mathbb{Z}$. The following conditions are equivalent:*

(i) E is an R-module of finite length;

(ii) there exists an integer n_0 such that $E_n = 0$ for $n \geq n_0$;

(iii) every associated prime of E contains R_+.

Proof. The equivalence of (i) and (ii) is clear. Let $\mathfrak{p}$ be an associated prime of E. If (iii) is satisfied, we have $\mathfrak{p} = \mathfrak{p}_0 + R_+$, where $\mathfrak{p}_0$ is a prime ideal of R_0, and the R-module $R/\mathfrak{p}$ is isomorphic to $R_0/\mathfrak{p}_0$. By Corollary 3.2.35, the R_0-module $R_0/\mathfrak{p}_0$ is isomorphic to a submodule of one of the E_k, hence has finite length. Concequently, $\mathfrak{p}_0$ is a maximal ideal of R_0, and $\mathfrak{p}$ is then maximal. Since $\mathfrak{p}$ is arbitrary, it follows from Proposition 3.2.14 that E has finite length.

Conversely, assume that E has finite length and let $\mathfrak{p}$ be an associated prime of E. Then $\mathfrak{p}$ is homogeneous by Proposition 3.2.34 and maximal by Proposition 3.2.14, hence contains R_+ (Lemma 7.5.6). □

Proposition 7.6.21. *Let $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ be the prime ideals of $\mathrm{gr}(A)$ associated with the graded module $\mathrm{gr}(M)$ and not containing $\mathrm{gr}_1(A)$. Let $\delta > 0$ be an integer, ξ an element of $\mathrm{gr}_\delta(A)$, and φ the homothety of $\mathrm{gr}(M)$ with ratio ξ. Then for φ_n to be injective for all n large enough, it is necessary and sufficient that ξ does not belong to any of the $\mathfrak{p}_i$.*

Proof. In fact, the prime ideals associated with the $\mathrm{gr}(A)$-module $\ker\varphi$ is the primes associated with $\mathrm{gr}(M)$ and contain ξ. By Lemma 7.6.20, $(\ker\varphi)_n$ is zero for n large enough, if and only if these prime ideals contain $\mathrm{gr}_+(A)$ (or equivalently they contain $\mathrm{gr}_1(A)$), whence the proposition. □

An element x of A is said to be **superficial** for M relative to $\mathfrak{a}$ if it is contained in $\mathfrak{a}$ and if, for n large enough, the map $\mathfrak{a}^nM/\mathfrak{a}^{n+1}M \to \mathfrak{a}^{n+1}M/\mathfrak{a}^{n+2}M$ induced by the multiplication by x is injective. If moreover $x \in \mathfrak{a}^\delta$, then x is said to be superficial of order δ for M. In the notation of Proposition 7.6.21, we see x is superficial of order δ for M if its class ξ in $\mathrm{gr}(A)$ belongs to $\mathrm{gr}_\delta(A)$ and is not contained in the $\mathfrak{p}_i$.

By Proposition 2.1.53, there exists a homogeneous element of $\mathrm{gr}(A)$ with positive degree that does not belong to any $\mathfrak{p}_i$. Concequently, there exists an integer $\delta > 0$ and an element of A that is superficial of order δ for M.

Remark 7.6.22. We note that, by Lemma 7.6.20, an element x of A satisfies the hypothesis in Proposition 7.6.19 if and only if it is superficial for M relative to $\mathfrak{a}$.

Example 7.6.23. Suppose that A is local with maximal ideal $\mathfrak{m}$ and residue field k, and consider the canonical surjection $\pi : \mathfrak{a} \to \mathfrak{a}\otimes_A k$. Since $\mathfrak{a}\otimes_A k = \mathfrak{a}/\mathfrak{a}\mathfrak{m}$ and $\mathfrak{a} \subseteq \mathfrak{m}$, we see π factors through the canonical map $\mathfrak{a} \to \mathfrak{a}/\mathfrak{a}^2$:

$$\mathfrak{a} \longrightarrow \mathfrak{a}/\mathfrak{a}^2 \xrightarrow{\bar{\pi}} \mathfrak{a}\otimes_A k$$

By Nakayama lemma, the subspace $V_i = \bar{\pi}(\mathfrak{p}_i \cap (\mathfrak{a}/\mathfrak{a}^2))$ of $\mathfrak{a}\otimes_A k$ is proper. If $\alpha \in \mathfrak{a}\otimes_A k$ is not contained in any of the V_i, then $\pi^{-1}(\alpha)$ consists of superficial elements for M by Proposition 7.6.21. In particular, if k is infinite, the union of V_i is distinct from $\mathfrak{a}\otimes_A k$ so there exists a superficial element for M.

Theorem 7.6.24. *Let A be a Noetherian ring, $\mathfrak{a}$ an ideal of A contained in the Jacobson radical of A and M a finitely generated A-module such that $M/\mathfrak{a}M$ is of finite length. Let $x_1,\dots,x_r$ be elements of $\mathfrak{a}$ and $\mathfrak{x}$ the ideal they generate.*

(a) We have $\dim_A(M/\mathfrak{x}M) \geq \dim_A(M) - r$.

(b) If $\dim_A(M/\mathfrak{x}M) = \dim_A(M) - r$, then $e_{\mathfrak{a}}(M/\mathfrak{x}M) \geq e_{\mathfrak{a}}(M)$.

(c) If $r < \dim_A(M)$ and for each i the element x_i is superficial for $M/(x_1M+\cdots+x_{i-1}M)$ relative to $\mathfrak{a}$, then
$$\dim_A(M/\mathfrak{x}M) = \dim_A(M) - 1, \quad e_{\mathfrak{a}}(M/\mathfrak{x}M) = e_{\mathfrak{a}}(M).$$

(d) If $r = \dim_A(M)$ and for each i the element x_i is superficial for $M/(x_1M+\cdots+x_{i-1}M)$ relative to $\mathfrak{a}$, then
$$e_{\mathfrak{a}}(M) = e_{\mathfrak{x}}(M) \leq \ell_A(M/\mathfrak{x}M) < +\infty.$$

Proof. The assertions in (a), (b), and (c) for $r = 1$ follows from Proposition 7.6.19, and the general case follows by induction. Now assume the hypothesis in (d) and put $\mathfrak{x}' = Ax_1 + \cdots + Ax_{r-1}$, $M' = M/\mathfrak{x}'M$; then $M/\mathfrak{x}M$ is identified with M'/x_rM'. By part (c), we have $\dim_A(M') = 1$ and $e_{\mathfrak{a}}(M) = e_{\mathfrak{a}}(M')$; Proposition 7.6.19 also implies that $M/\mathfrak{x}M$ has finite length and
$$e_{\mathfrak{a}}(M') = e_{x_rA}(M') \leq \ell_A(M'/\mathfrak{x}_rM') = \ell_A(M/\mathfrak{x}M).$$
On the other hand, since $x_r^nM' = \mathfrak{x}^nM'$ for each n, we have $e_{x_rA}(M') = e_{\mathfrak{x}}(M')$. By applying (b) to $x_1,\dots,x_{r-1}$, we also obtain that $e_{\mathfrak{x}}(M') \geq e_{\mathfrak{x}}(M)$, so
$$e_{\mathfrak{x}}(M) \leq e_{\mathfrak{x}}(M') = e_{x_rA}(M') = e_{\mathfrak{a}}(M') = e_{\mathfrak{a}}(M).$$
Since $\mathfrak{x}$ is contained in $\mathfrak{a}$, this implies $e_{\mathfrak{x}}(M) = e_{\mathfrak{a}}(M)$ and completes the proof. □

Corollary 7.6.25. *Suppose that A is local, with residue field infinite, and put $d = \dim_A(M)$. Then there exists a sequence $(x_1, \ldots, x_d)$ of elements in $\mathfrak{a}$ such that, if $\mathfrak{x}$ is the ideal they generated, then*

$$e_{\mathfrak{a}}(M) = e_{\mathfrak{x}}(M) \leq \ell_A(M/\mathfrak{x}M) < +\infty$$

Proof. This follows from Theorem 7.6.24 and Example 7.6.23. □

Remark 7.6.26. In the situation of Corollary 7.6.25, we have

$$e_{\mathfrak{a}}(M) = e_{\mathfrak{x}}(M) \leq \ell_A(M/\mathfrak{x}M), \quad \ell_A(M/\mathfrak{a}M) \leq \ell_A(M/\mathfrak{x}M).$$

The three cases

$$e_{\mathfrak{a}}(M) < \ell_A(M/\mathfrak{a}M), \quad e_{\mathfrak{a}}(M) = \ell_A(M/\mathfrak{a}M), \quad e_{\mathfrak{a}}(M) > \ell_A(M/\mathfrak{a}M)$$

are all possible.

Chapter 8

Complete Noetherian local rings

In this section, all rings are assumed to be commutative with unit. We denote by 1_A for the unit of a ring A. If $\mathfrak{p}$ a prime ideal of A, we denote by $\kappa(\mathfrak{p})$ the residue field of $A_{\mathfrak{p}}$. If A is local, $\mathfrak{m}_A$ will denote its maximal ideal and κ_A its residue field.

We say a ring homomorphism $\rho : A \to B$ is flat (resp. faithfully flat) if B is a flat A-module (resp. faithfully flat A-module). Recall that by **??**, if A and B are local rings, ρ is faithfully flat if and only if it is local and flat.

8.1 Witt vectors and Witt rings

8.1.1 Witt polynomials

In this part, we fix a prime number p. For a positive integer n, we define the **n-th Witt polynomial** Φ_n to be

$$\Phi_n(X_0, \dots, X_n) = \sum_{i=0}^{n} p^i X_i^{p^{n-i}} = X_0^{p^n} + pX_1^{p^{n-1}} + \cdots + p^n X_n. \tag{8.1.1}$$

Clearly we have $\Phi_0 = X_0$ and the following recursive formulae

$$\Phi_{n+1}(X_0, \dots, X_{n+1}) = \Phi_n(X_0^p, \dots, X_n^p) + p^{n+1} X_{n+1}, \tag{8.1.2}$$

$$\Phi_{n+1}(X_0, \dots, X_{n+1}) = X_0^{p^{n+1}} + p\Phi_n(X_1, \dots, X_{n+1}). \tag{8.1.3}$$

If we assign X_i with weight p^i, the polynomial Φ_n is isobare with weight p^n.

Lemma 8.1.1. *Let A be a filtered ring with $(A_n)_{n\in\mathbb{Z}}$ its filtration. Let x, y be elements of A congruent mod A_m, then*

$$x^{p^n} \equiv y^{p^n} \mod A_{m+n}. \tag{8.1.4}$$

Proof. Let $P = \sum_{i=0}^{p-1} X^i Y^{p-1-i}$ in $\mathbb{Z}[X, Y]$. From the hypothesis on x and y, we have

$$P(x, y) \equiv P(x, x) \equiv px^{p-1} \mod A_m.$$

But we have $A_m + pA \subseteq A_1$, from which $P(x, y) \in A_1$. Finally, $x^p - y^p = (x - y)P(x, y)$ is contained in $A_m A_1 \subseteq A_{m+1}$, which complete the induction process. □

Proposition 8.1.2. *Let A be a filtered ring with $(A_n)_{n\in\mathbb{Z}}$ its filtration. Suppose that $A_0 = A$ and $p \cdot 1 \in A_1$. Let $a_0, \dots, a_n$ and $b_0, \dots, b_n$ be elements of A, and $m \geq 1$ be an integer. If $a_i \equiv b_i$ mod A_m for each i, then*

$$\Phi_i(a_0, \dots, a_i) \equiv \Phi_i(b_0, \dots, b_i) \mod A_{m+i} \ \text{ for } \ 0 \leq i \leq n. \tag{8.1.5}$$

Moreover, the converse holds if for each integer $k \geq 1$ and $x \in A$, the relation $px \in A_{k+1}$ implies $x \in A_k$.

Proof. We prove (8.1.5) by induction on n. The case $n = 0$ is immediate, so suppose $n \geq 1$. Apply (8.1.4) on a_i and b_i, we get

$$a_i^p \equiv b_i^p \mod A_{m+1} \quad \text{for } 0 \leq i \leq n-1, \tag{8.1.6}$$

also, by the induction hypothesis on $(a_0, \dots, a_{n-1})$ and $(b_0, \dots, b_{n-1})$,

$$\Phi_{n-1}(a_0^p, \dots, a_{n-1}^p) \equiv \Phi_{n-1}(b_0^p, \dots, b_{n-1}^p) \mod A_{m+n}. \tag{8.1.7}$$

Therefore, from (8.1.2) we conclude that

$$\Phi_n(a_0, \dots, a_n) - p^n a_n \equiv \Phi_n(b_0, \dots, b_n) - p^n b_n \mod A_{m+n}. \tag{8.1.8}$$

Now $a_n - b_n$ is contained in A_m, the element $p^n a_n - p^n b_n$ then belongs to A_{m+n} and this implies the congruence

$$\Phi_n(a_0, \dots, a_n) \equiv \Phi_n(b_0, \dots, b_n) \mod A_{m+n}.$$

Conversely, assume that for each integer $k \geq 1$ and $x \in A$, the relation $px \in A_{k+1}$ implies $x \in A_k$. We prove the inverse direction by induction on n. The case $n = 0$ is clear, and suppose that $n \geq 1$. Since $a_i \equiv b_i \bmod A_m$ for $0 \leq i \leq n-1$, by the induction hypothesis, we deduce the congruence (8.1.6), (8.1.7), and (8.1.8). But by hypothesis $\Phi_n(a_0, \dots, a_n) \equiv \Phi_n(b_0, \dots, b_n) \bmod A_{m+n}$, so $p^n(a_n - b_n) \in A_{m+n}$. Since $px \in A_{k+1}$ implies $x \in A_k$, we conclude that $a_n \equiv b_n \bmod A_m$, which completes the proof. □

Let A be a ring, and give $A^{\mathbb{N}}$ the structure of product ring. We define two maps f_A and v_A by the following formulae

$$f_A(\boldsymbol{a}) = (a_1, a_2, \dots), \quad v_A(\boldsymbol{a}) = (0, pa_0, pa_1, \dots).$$

where $\boldsymbol{a} = (a_n)_{n \in \mathbb{N}}$. For any positive integer m, let Φ_m be the map on $A^{\mathbb{N}}$ defined by $\boldsymbol{a} = (a_n)_{n \in \mathbb{N}} \mapsto \Phi_m(a_0, \dots, a_m)$. We denote by Φ_A the map $\boldsymbol{a} \mapsto (\Phi_n(\boldsymbol{a}))_{n \in \mathbb{N}}$ on $A^{\mathbb{N}}$.

Lemma 8.1.3. *Suppose that A is endowed with an endomorphism σ satisfying $\sigma(a) = a^p \bmod pA$ for any $a \in A$. Let $a_0, \dots, a_{n-1}$ be elemnts of A. Put $u_i = \Phi_i(a_0, \dots, a_i)$ for each i, and let u_n be an element of A. Then the following conditions are equivalent:*

(i) There exists $a_n \in A$ such that $u_n = \Phi_n(a_0, \dots, a_n)$.

(ii) We have $\sigma(u_{n-1}) \equiv u_n \bmod p^n A$.

Proof. For $0 \leq i \leq n-1$, we have $\sigma(a_i) \equiv a_i^p \bmod pA$. Applying Proposition 8.1.2 on the filtration $(p^n A)_{n \in \mathbb{N}}$ and $m = 1$, we have a congruence

$$\sigma(u_{n-1}) = \Phi_{n-1}(\sigma(a_0), \dots, \sigma(a_{n-1})) \equiv \Phi_{n-1}(a_0^p, \dots, a_0^p) \mod p^n A.$$

But, from formula (8.1.2), the relation $u_n = \Phi_n(a_0, \dots, a_n)$ is equivalent to

$$u_n = \Phi_{n-1}(a_0^p, \dots, a_{n-1}^p) + p^n a_n.$$

Thus the lemma is proved. □

Proposition 8.1.4. *Let A be a ring and f_A, v_A, and Φ_A be the maps defined above.*

(a) If p is not a divisor of zero in A, the map Φ_A is injective.

(b) *If p is invertible in A, the map Φ_A is bijective.*

(c) *If σ is an endomorphism of A satisfying $\sigma(a) = a^p \bmod pA$ for any $a \in A$, the image of Φ_A is a subring of $A^{\mathbb{N}}$ invariant under f_A and v_A, which consists of elements $(u_n)_{n\in\mathbb{A}}$ such that $\sigma(u_n) \equiv u_{n+1} \bmod p^{n+1}A$ for all $n \in \mathbb{N}$.*

Proof. If $\boldsymbol{a} = (a_n)_{n\in\mathbb{N}}$ and $\boldsymbol{u} = (u_n)_{n\in\mathbb{N}}$ are elements of $A^{\mathbb{N}}$, the relation $\Phi_A(\boldsymbol{a}) = \boldsymbol{u}$ is equivalent, by formula (8.1.3), to

$$\begin{cases} u_0 = a_0, \\ u_n = \Phi_{n-1}(a_0^p, \dots, a_{n-1}^p) + p^n a_n \quad \text{for} \ \ n \geq 1. \end{cases} \tag{8.1.9}$$

Let $\boldsymbol{u} = (u_n)_{n\in\mathbb{N}}$ be in $A^{\mathbb{N}}$. If p is not a zero divisor of A (resp. if p is invertible in A), there then exists at most (resp. exactly) one sequence $(a_n)_{n\in\mathbb{N}}$ satisfying (8.1.9), whence (a) and (b).

Now by Lemma 8.1.3, the image of Φ_A consists of elements $\boldsymbol{u} = (u_n)_{n\in\mathbb{N}}$ of $A^{\mathbb{N}}$ such that $\sigma(u_n) = u_{n+1} \bmod p^{n+1}A$ for all $n \in \mathbb{N}$. Since σ is an endomorphism, it follows that this is a subring of $A^{\mathbb{N}}$ and invariant under l_A and v_A. □

Remark 8.1.5. Let $\boldsymbol{a} = (a_n)_{n\in\mathbb{N}}$ and $\boldsymbol{u} = (u_n)_{n\in\mathbb{N}}$ be elements of $A^{\mathbb{N}}$ such that $\Phi_A(\boldsymbol{a}) = \boldsymbol{u}$. Then by (8.1.9), we deduce the following assertions:

(a) If the u_n, for $0 \leq n \leq m$, is contained in a subring B of A and if, for any $x \in A$, the relation $px \in B$ implies $x \in B$, then a_n, for $0 \leq n \leq m$, are contained in B.

(b) If A is graded of type $\mathbb{N}$, p is not a zero divisor of A, and u_n is homogeneous of degree dp^n for $0 \leq n \leq m$ (where $d \in \mathbb{N}$), then a_n is homogeneous of degree dp^n for $0 \leq n \leq m$.

Now let A be the polynomial ring $\mathbb{Z}[\boldsymbol{X}, \boldsymbol{Y}]$ with indeterminates $\boldsymbol{X} = (X_n)_{n\in\mathbb{N}}$ and $\boldsymbol{Y} = (Y_n)_{n\in\mathbb{N}}$. Let θ be the endomorphism of A defined by $\theta(X_n) = X_n^p$ and $\theta(Y_n) = Y_n^p$ for $n \in \mathbb{N}$. Then p is not a divisor of zero in A and the set of $a \in A$ such that $\theta(a) = a^p \bmod pA$ is a subring of A containing X_n and Y_n, hence equal to A. By Proposition 8.1.4(a) and (c), there exist elements $\boldsymbol{S} = (S_n)_{n\in\mathbb{N}}$, $\boldsymbol{P} = (P_n)_{n\in\mathbb{N}}$, $\boldsymbol{I} = (I_n)_{n\in\mathbb{N}}$, and $\boldsymbol{F} = (F_n)_{n\in\mathbb{N}}$ in $A^{\mathbb{N}}$ characterized by the equalities

$$\begin{cases} \Phi_A(\boldsymbol{S}) = \Phi_A(\boldsymbol{X}) + \Phi_A(\boldsymbol{Y}) \\ \Phi_A(\boldsymbol{P}) = \Phi_A(\boldsymbol{X})\Phi_A(\boldsymbol{Y}) \\ \Phi_A(\boldsymbol{I}) = -\Phi_A(\boldsymbol{X}) \\ \Phi_A(\boldsymbol{F}) = f_A(\Phi_A(\boldsymbol{X})). \end{cases} \tag{8.1.10}$$

The elements S_n, P_n, I_n, and F_n of A are then characterized by the following formulae (where $n \in \mathbb{N}$):

$$\Phi_n(S_0, \dots, S_n) = \Phi_n(X_0, \dots, X_n) + \Phi_n(Y_0, \dots, Y_n), \tag{8.1.11}$$
$$\Phi_n(P_0, \dots, P_n) = \Phi_n(X_0, \dots, X_n)\Phi_n(Y_0, \dots, Y_n), \tag{8.1.12}$$
$$\Phi_n(I_0, \dots, I_n) = -\Phi_n(X_0, \dots, X_n), \tag{8.1.13}$$
$$\Phi_n(F_0, \dots, F_n) = \Phi_{n+1}(X_0, \dots, X_{n+1}). \tag{8.1.14}$$

Formula (8.1.3) makes it possible in practice to determine the polynomials S_n, P_n, I_n and F_n step by step.

Example 8.1.6 (First terms of the polynomials S_n, P_n, I_n, and F_n).

(a) We have

$$S_0 = X_0 + Y_0, \quad S_1 = X_1 + Y_1 - \sum_{i=1}^{p-1} \frac{1}{p}\binom{p}{i} X_0 Y_0^{p-i}.$$

Furthermore, $S_n - X_n - Y_n$ is in the ring $\mathbb{Z}[X_0, \dots, X_{n-1}, Y_0, \dots, Y_{n-1}]$.

(b) We have
$$P_0 = X_0Y_0, \quad P_1 = pX_1Y_1 + X_0^pY_1 + X_1Y_0^p.$$

(c) If $p \neq 2$, we have $I_n = -X_n$. If $p = 2$, then
$$I_0 = -X_0, \quad I_1 = -(X_0^2 + X_1), \quad I_2 = -X_0^4 - X_0^2X_1 - X_1^2 - X_2.$$

(d) We have
$$F_0 = X_0^p + pX_1, \quad F_1 = X_1^p + pX_2 - \sum_{i=0}^{p-1} \binom{p}{i} p^{p-i-1} X_0^{pi} X_1^{p-i}.$$
Note that $\Phi_n(F_0, \ldots, F_n) \equiv \Phi_n(X_0^p, \ldots, X_n^p) \bmod p^{n+1}A$ for each $n \in \mathbb{N}$ since $F_n \equiv X_n^p \bmod pA$ for each $n \in \mathbb{N}$.

Remark 8.1.7. A subset $J \subseteq \mathbb{N}_+$ is called **divisor-stable** provided that $J \neq \varnothing$, and if $n \in J$, then all proper divisors of n are also in J. If J is a divisor-stable set, we let $\mathfrak{p}(J)$ denote the set of prime numbers contained in J. For any element j in J, define the polynomial φ_j of $\mathbb{Z}[(X_j)_{j\in J}]$ by the formula
$$\varphi_j = \sum_{d|j} dX_d^{j/d}.$$
Note that we have, for each integer $n \geq 0$, that
$$\varphi_{p^n} = \Phi_n(X_{p^0}, \ldots, X_{p^n}).$$
For any ring A and any $n \in J$, we denote by φ_n the map from A^J to A given by $(a_j)_{j\in J} \mapsto \varphi_n((a_j)_{j\in J})$; we denote by φ_A, or simply φ, the map A^J to A defined by $\boldsymbol{a} = (a_j)_{j\in J} \mapsto (\varphi_n(\boldsymbol{a}))_{n\in J}$. Let $\mathcal{A} = \mathbb{Z}[(X_j)_{j\in J}, (Y_j)_{J\in J}]$ be the polynomial ring of indeterminates $\boldsymbol{X} = (X_j)_{j\in J}$ and $\boldsymbol{Y} = (Y_j)_{j\in J}$. We can show that there exists elements $\boldsymbol{s} = (s_j)_{j\in J}$, $\boldsymbol{p} = (p_j)_{j\in J}$ and $\boldsymbol{i} = (i_j)_{j\in J}$ characterized by the following equalities:
$$\begin{aligned} \varphi_{\mathcal{A}}(\boldsymbol{s}) &= \varphi_{\mathcal{A}}(\boldsymbol{X}) + \varphi_{\mathcal{A}}(\boldsymbol{Y}) \\ \varphi_{\mathcal{A}}(\boldsymbol{p}) &= \varphi_{\mathcal{A}}(\boldsymbol{X})\varphi_{\mathcal{A}}(\boldsymbol{Y}) \\ \varphi_{\mathcal{A}}(\boldsymbol{i}) &= -\varphi_{\mathcal{A}}(\boldsymbol{X}). \end{aligned}$$
In particular, this definition works, for example, if J is the set of integers $j \geq 1$.

8.1.2 The ring of Witt vectors

Let A be a ring. If $\boldsymbol{a} = (a_n)_{n\in\mathbb{N}}$ and $\boldsymbol{b} = (b_n)_{n\in\mathbb{N}}$ are elements of $A^{\mathbb{N}}$, we denote by $S_A(\boldsymbol{a}, \boldsymbol{b})$, or simply $S(\boldsymbol{a}, \boldsymbol{b})$, the sequence $(S_n(a_0, \ldots, a_n, b_0, \ldots, b_n))_{n\in\mathbb{N}}$. Simialrly, we denote by $P_A(\boldsymbol{a}, \boldsymbol{b})$ and $I_A(\boldsymbol{a})$ the resulting substitutions. By substituting X_n into a_n, Y_n into b_n, for each $n \in \mathbb{N}$, in the formulae (8.1.11), (8.1.12), and (8.1.13), we obtain the equalities
$$\Phi_A(S_A(\boldsymbol{a}, \boldsymbol{b})) = \Phi_A(\boldsymbol{a}) + \Phi_A(\boldsymbol{b}) \tag{8.1.15}$$
$$\Phi_A(P_A(\boldsymbol{a}, \boldsymbol{b})) = \Phi_A(\boldsymbol{a})\Phi_A(\boldsymbol{b}) \tag{8.1.16}$$
$$\Phi_A(I_A(\boldsymbol{a})) = -\Phi_A(\boldsymbol{a}). \tag{8.1.17}$$
We will denote by $W(A)$ the set $A^{\mathbb{N}}$ endowed with the laws of composition S_A and P_A.

Let $\rho : A \to B$ be a ring homomorphism. We denote $W(\rho)$ the map $\rho^{\mathbb{N}} : A^{\mathbb{N}} \to B^{\mathbb{N}}$ which sends an element $\boldsymbol{a} = (a_n)_{n\in\mathbb{N}}$ to $(\rho(a_n))_{n\in\mathbb{N}}$. From the definition, we have
$$W(\rho) \circ S_A = S_B \circ (W(\rho) \times W(\rho)) \tag{8.1.18}$$
$$W(\rho) \circ P_A = P_B \circ (W(\rho) \times W(\rho)) \tag{8.1.19}$$
$$W(\rho) \circ I_A = I_B \circ W(\rho) \tag{8.1.20}$$
$$W(\rho) \circ \Phi_A = \Phi_B \circ W(\rho) \tag{8.1.21}$$

Lemma 8.1.8. *Let A be a ring. Then there exists a surjective homomorphism $\rho : B \to A$, where B is a ring satisfying the following condition: p is not a divisor of zero in B, and there exists an endomorphism σ of B such that $\sigma(b) \equiv b^p \bmod pB$ for any $b \in B$.*

Proof. It suffices to take $B = \mathbb{Z}[(X_a)_{a\in A}]$, and let σ be the endomorphism on B defined by $\sigma(X_a) = X_a^p$ for any $a \in A$. The homomorphism $\rho : B \to A$ is defined to send X_a to a for any $a \in A$. □

Theorem 8.1.9. *Let A be a ring.*

(a) With the map S_A and P_A, the set $W(A)$ is a commutative ring. The addition identity is the sequence $\mathbf{0}_A$ with zero terms, and the multiplication identity is the sequence $\mathbf{1}_A$ of which all the terms are zero except that of index 0 which is equal to 1. The additive inverse of an element $\boldsymbol{a}$ in $W(A)$ is $I_A(\boldsymbol{a})$.

(b) Let $\rho : A \to B$ be a homomorphism of rings. Then the induced map $W(\rho) : W(A) \to W(B)$ is a homomorphism of rings.

(c) The map Φ_A is a homomorphism from $W(A)$ to the product ring $A^{\mathbb{N}}$. In particular, for each $n \in \mathbb{N}$, the map $\Phi_n : \boldsymbol{a} \mapsto \Phi_n(a_0, \ldots, a_n)$ is a homomorphism from $W(A)$ to A.

Proof. Concerning the formulae (8.1.15), (8.1.16), (8.1.18), and (8.1.21), it suffices to prove (a). Let $\rho : B \to A$ be a ring homomorphism satisfying the conditions in Lemma 8.1.8. Let B' be the image of Φ_B in $B^{\mathbb{N}}$, which by Proposition 8.1.4 consists of elements $(b_n)_{n\in\mathbb{N}}$ such that $\sigma(b_n) \equiv b_{n+1} \bmod p^{n+1}B$ for all $n \in \mathbb{N}$. By Proposition 8.1.4, Φ_B induces a bijection between $W(B)$ and B'. In view of the formulas (8.1.15) to (8.1.17) and the relations $\Phi_n(\mathbf{0}_B) = 0$ and $\Phi_n(\mathbf{1}_B) = 1$, we get a ring structure on $W(B)$ by transporting structure, with addition identity $\mathbf{0}_B$, multiplication identity $\mathbf{1}_B$, and additive inverse $I_B(\boldsymbol{b})$.

The map $W(\rho) : W(B) \to W(A)$ is surjective (since ρ is surjective). By the fomulae (8.1.18) and (8.1.19), the equivalence relation R on $W(B)$ associated with the map $W(\phi)$ is compatible with the ring structure on $W(B)$. Thus $W(\rho)$ is a bijection from the quotient ring $W(B)/R$ to $W(A)$, compatible with addition and multiplication. Assertion (a) then follows by transporting structure. □

For a ring A, the ring $W(A)$ is called the **Witt ring** with coefficients in A. For $\boldsymbol{a}$ in $W(A)$ and $n \in \mathbb{N}$, the element $\Phi_n(\boldsymbol{a}) = \Phi_n(a_0, \ldots, a_n)$ is called the **phantom components** of order n of $\boldsymbol{a}$.

Remark 8.1.10. Retain the notations in Remark 8.1.7. Let A be a ring. If $\boldsymbol{a}$ and $\boldsymbol{b}$ and elements of A^J and $\boldsymbol{r} = (r_j)_{j\in J}$ belongs to $\mathcal{A}^J$, we denote by $\boldsymbol{r}_A(\boldsymbol{a}, \boldsymbol{b})$ the element $(r_j(\boldsymbol{a}, \boldsymbol{b}))_{j\in J}$ of A^J. Let $U(A)$ be the set A^J endowed with the maps $\boldsymbol{s}_A$ and $\boldsymbol{p}_A$. WE can show that, with the addition $\boldsymbol{s}_A$ and multiplication $\boldsymbol{p}_A$, $U(A)$ is a commutative ring; we call it the **universal Witt ring** of A. The addition idnetity of $U(A)$ is the element with components all zero, and the multiplication identity is the element with components zero except 1_A at index 1; the additive inverse of $\boldsymbol{a}$ in $U(A)$ is $\boldsymbol{i}_A(\boldsymbol{a})$. The map φ_A is a homomorphism of $U(A)$ to the product ring A^J.

From now on, we use $+$ and $\times$ to denote the addition and multiplication in the ring $W(A)$; for simplicity, we write $\mathbf{0}$ for $\mathbf{0}_A$ and $\mathbf{1}$ for $\mathbf{1}_A$. Let us introduce[1] two maps F_A and V_A on $W(A)$ by the formulae

$$F_A(\boldsymbol{a}) = (F_n(a_0, \ldots, a_{n+1}))_{n\in\mathbb{N}}, \tag{8.1.22}$$
$$V_A(\boldsymbol{a}) = (0, a_0, a_1, \ldots). \tag{8.1.23}$$

[1]The letter F is the initial of the name of Frobenius, and the latter V is that of the German word *Verschiebung*, which means shift.

where $\boldsymbol{a} = (a_n)_{n\in\mathbb{N}}$. The map V_A is often called the **shift** operator. Note that by (8.1.14) we have the formula

$$\Phi_n(F_0(\boldsymbol{a}), \ldots, F_n(\boldsymbol{a})) = \Phi_{n+1}(a_0, \ldots, a_{n+1}).$$

We can also write it in the form

$$\Phi_A \circ F_A = f_A \circ \Phi_A. \tag{8.1.24}$$

From the recursive formula (8.1.3), we can also prove that

$$\Phi_A \circ V_A = v_A \circ \Phi_A. \tag{8.1.25}$$

Let $\rho : A \to B$ be a ring homomorphism. It is clear that we have

$$W(\rho) \circ F_A = F_B \circ W(\rho), \tag{8.1.26}$$
$$W(\rho) \circ V_A = V_B \circ W(\rho). \tag{8.1.27}$$

Proposition 8.1.11. *Let A be a ring and $W(A)$ be the Witt ring with coefficients A.*

(a) The map F_A is an endomorphism of the ring $W(A)$, and V_A is an endomorphism of the additive group of $W(A)$.

(b) For $\boldsymbol{a} \in W(A)$, we have $F_A(V_A(\boldsymbol{a})) = p \cdot \boldsymbol{a}$.

(c) Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be elements in $W(A)$. Then we have

$$V_A(\boldsymbol{a} \times F_A(\boldsymbol{b})) = V_A(\boldsymbol{a}) \times \boldsymbol{b}, \tag{8.1.28}$$
$$V_A(\boldsymbol{a}) \times V_A(\boldsymbol{b}) = p \cdot V_A(\boldsymbol{a} \times \boldsymbol{b}). \tag{8.1.29}$$

(d) Put $\boldsymbol{\mu} = V_A(\mathbf{1}) = (0, 1, 0, \ldots)$. Then for $\boldsymbol{b}$ in $W(A)$ we have

$$V_A(F_A(\boldsymbol{b})) = \boldsymbol{\mu} \times \boldsymbol{b}. \tag{8.1.30}$$

(e) For any element $\boldsymbol{a}$ in $W(A)$, denote by $\boldsymbol{a}^p$ the p-th power of $\boldsymbol{a}$ in $W(A)$. Then

$$F_A(\boldsymbol{a}) \equiv \boldsymbol{a}^p \mod pW(A). \tag{8.1.31}$$

Proof. Let $\rho : B \to A$ be a ring homomorphism satisfying the conditions of Lemma 8.1.8. Then $W(\rho) : W(B) \to W(A)$ is a surjective ring homomorphism, and $\Phi_B : W(B) \to B^{\mathbb{N}}$ is an injective homomorphism of rings. Furthermore, $f_B : B^{\mathbb{N}} \to B^{\mathbb{N}}$ is a ring homomorphism. By the formulae (8.1.24) and (8.1.26), we have

$$\Phi_B \circ F_B = f_B \circ \Phi_B, \quad W(\rho) \circ F_B = F_A \circ W(\rho),$$

which proves the first assertion of (a). The second one can be proved similarly, using (8.1.25) and (8.1.27).

Let $\boldsymbol{a}$ be an element of $W(A)$, and choose a element $\boldsymbol{x}$ in $W(B)$ whose image under $W(\rho)$ is $\boldsymbol{a}$. Put $\xi = \Phi_B(\boldsymbol{x})$. By definition of f_B and v_B, we have $f_B(v_B(\xi)) = p \cdot \xi$. By formulae (8.1.24) and (8.1.25), the elements $F_B(V_B(\boldsymbol{x}))$ and $p \cdot \boldsymbol{x}$ of $W(B)$ then have the same image under the injective map Φ_B, and therefore are equal. The formula $F_A(V_A(\boldsymbol{a})) = p \cdot \boldsymbol{a}$ follows then from (8.1.26) and (8.1.27). This proves (b).

Reasoning analogously, we reduce the demonstration of formula (8.1.28) to the relation

$$v_B(\xi f_B(\eta)) = v_B(\xi)\eta$$

where ξ, η are in $B^{\mathbb{N}}$. This follows from the equalities

$$\xi f_B(\eta) = (\xi_0\eta_1, \xi_1\eta_2, \dots), \quad v_B(\xi)\eta = (0, p\xi_0\eta_1, p\xi_1\eta_2, \dots).$$

Taking into account (a) and (b), the formula (8.1.29) results from the formula (8.1.28), where we replace $\boldsymbol{b}$ by $V_A(\boldsymbol{b})$. Formula (8.1.30) is the particular case $\boldsymbol{a} = \mathbf{1}$ of formula (8.1.28).

Simialrly, we deduce the formula (8.1.31) from the relation

$$f_B(\xi) \equiv \xi^p \mod p\Phi_B(B^{\mathbb{N}})$$

where ξ^p denote the product of $B^{\mathbb{N}}$. By Proposition 8.1.4(c), this is equivalent to the fact that for all $n \geq 0$, we have

$$\sigma(\xi_{n+1} - \xi_n^p) \equiv \xi_{n+2} - \xi_{n+1}^p \mod p^{n+2}B.$$

But, for $n \geq 0$, we have $\sigma(\xi_n) \equiv \xi_{n+1} \mod p^{n+1}B$, since $\xi = \Phi_B(\boldsymbol{x})$ (again by Proposition 8.1.4(c)); we then deduce from Lemma 8.1.1 that

$$\sigma(\xi_n)^p \equiv \xi_{n+1}^p \mod p^{n+2}B,$$

which completes the proof. □

8.1.3 Filtration and topology of the Witt ring

Lemma 8.1.12. *Let A be a ring and $m \geq 1$ an integer. Then*

$$\boldsymbol{a} = (a_0, \dots, a_{m-1}, 0, \dots) + (\underbrace{0, \dots, 0}_{m\text{-terms}}, a_m, a_{m+1}, \dots)$$

for any $\boldsymbol{a}$ in $W(A)$.

Proof. Let $\rho : B \to A$ be a ring homomorphism satisfying the conditions of Lemma 8.1.8. Then $W(\rho) : W(B) \to W(A)$ is a surjective ring homomorphism, and $\Phi_B : W(B) \to B^{\mathbb{N}}$ is an injective homomorphism of rings. It suffices to prove that

$$\Phi_n(\boldsymbol{b}) = \Phi_n(b_0, \dots, b_{m-1}, 0, \dots) + \Phi_n(0, \dots, 0, b_m, b_{m+1}, \dots) \tag{8.1.32}$$

for any $\boldsymbol{b}$ in $W(B)$ and $m \geq 1$. But we have

$$\Phi_n(b_0, \dots, b_{m-1}, 0, \dots) = \begin{cases} \Phi_n(b_0, \dots, b_n) & \text{if } 0 \leq n < m \\ \sum_{i=0}^{m-1} p^i b_i^{p^{n-i}} & \text{if } m \leq n \end{cases}$$

and

$$\Phi_n(0, b_m, b_{m+1}, \dots) = \begin{cases} 0 & \text{if } 0 \leq n < m \\ \sum_{i=m}^{m} p^i b_i^{p^{n-i}} & \text{if } m \leq n, \end{cases}$$

whence the formula (8.1.32). □

Let A be a ring. For each integer $m \geq 0$, we denote by $V_m(A)$ the set of Witt vectors $\boldsymbol{a} = (a_n)_{n\in\mathbb{N}}$ such that $a_n = 0$ for $0 \leq n < m$, which is also the m-fold image of V_A. From Proposition 8.1.11 and induction on m, we have the formulae

$$V^m(\boldsymbol{a} + \boldsymbol{b}) = V^m(\boldsymbol{a}) + V^m(\boldsymbol{b}), \tag{8.1.33}$$

$$V^m(\boldsymbol{a}) \times \boldsymbol{b} = V^m(\boldsymbol{a} \times F^m(\boldsymbol{b})). \tag{8.1.34}$$

We set $V_m(A) = W(A)$ if $m < 0$. The sequence $(V_m(A))_{m\in\mathbb{Z}}$ is then a decreasing filtration of additive subgroups of $W(A)$, and is compatible with the ring structure of $W(A)$ if and only if the ring A has characteristic p.

In the following, we denote by $\mathcal{T}$ the topology on $W(A)$ associated with the filtration $(V_m(A))_{m\in\mathbb{Z}}$. As $V_m(A)$ is an ideal of $W(A)$ for all $m \in \mathbb{Z}$, the topology $\mathcal{T}$ is compatible with the ring structure of $W(A)$ (**??**). Let $\boldsymbol{a} \in W(A)$; the sets $\boldsymbol{a} + V_m(A)$, for $m \in \mathbb{N}$, form a fundamental system of neighborhoods of $\boldsymbol{a}$ in $\mathcal{T}$. By Lemma 8.1.12, $\boldsymbol{a} + V_m(A)$ consists of Witt vectors $\boldsymbol{b}$ such that $a_i = b_i$ for $0 \le i < m$. Consequently, $\mathcal{T}$ is none other than the topology product on $A^{\mathbb{N}}$ of the discrete topology on each of the factors, and $W(A)$ is therefore a separated and complete topological ring.

Let τ_A be the map of A to $W(A)$ that sends an element a of A to $(a, 0, 0, \dots)$. We have $\Phi_n(\tau(a)) = a^{p^n}$ for each $n \in \mathbb{N}$. For any ring homomorphism $\rho : B \to A$, it is clear that $W(\rho) \circ \tau_B = \tau_A \circ \rho$.

Proposition 8.1.13. *Let a, b be elements of A and $\boldsymbol{x} = (x_n)_{n\in\mathbb{N}}$ an element of $W(A)$.*

(a) We have the formulas

$$\tau(ab) = \tau(b) \times \tau(b), \tag{8.1.35}$$
$$\tau(a) \times \boldsymbol{x} = (a^{p^n} x_n)_{n\in\mathbb{N}}. \tag{8.1.36}$$

(b) The series $\sum_n V^n(\tau(x_n))$ converges in $W(A)$ to $\boldsymbol{x}$.

Proof. Let n be a positive integer. The polynomial $P_n(X_0, \dots, X_n, Y_0, \dots, Y_n)$ is isobare with weight p^n is we assign X_i with weight p^i. We then have

$$P_n(X_0, 0, \dots, 0, Y_0, \dots, Y_n) = X_0^{p^n} P_n(1, 0, \dots, 0, Y_0, \dots, Y_n).$$

Since $\mathbf{1} = (1, 0, 0, \dots)$ is a unit in the Witt ring with coefficients in $\mathbb{Z}[(X_n)_{n\in\mathbb{N}}, (Y_n)_{n\in\mathbb{N}}]$, we have

$$P_n(1, 0, \dots, 0, Y_0, \dots, Y_n) = Y_n.$$

By Substituting X_0 with a and Y_i with x_i, we deduce that

$$P_n(a, 0, \dots, 0, x_0, \dots, x_n) = a^{p^n} x_n.$$

By the definition of product in $W(A)$, this proves (8.1.36), and (8.1.35) is a special case of (8.1.36).

We now prove (b). By definition, $V^n(\tau(x_n))$ is the sequence of which all the components are zero, except that of index n which is equal to x_n. It follows from Lemma 8.1.12, by induction on m, which we have

$$\sum_{n=0}^{m} V^n(\tau(x_n)) = (x_0, \dots, x_m, 0, 0, \dots)$$

for each $m \ge 0$; we then deduce (b) by passing to limit in the topology $\mathcal{T}$ of $W(A)$, which is the product of the discrete topology on A. □

Now for each integer $n \ge 1$, let $W_n(A)$ be the quotient ring $W(A)/V_n(A)$. For any elements $a_0, \dots, a_{n-1}$ of A, we denote by $[a_0, \dots, a_{n-1}]$ or $[a_i]_{0\le i<n}$ the class modulo $V_n(A)$ of the element $(a_0, \dots, a_{n-1}, 0, 0, \dots)$ of $W(A)$. By Lemma 8.1.12, the map $(a_0, \dots, a_{n-1}) \mapsto [a_0, \dots, a_{n-1}]$ from A^n to $W_n(A)$ is bijective. For this reasong, we say that the elements of $W_n(A)$ are the Witt vectors of length n; similarly, one sometimes qualifies as Witt vectors of infinite length the elements of $W(A)$.

We denote by π_n the canonical homomorphism from $W(A)$ to $W_n(A)$. According to Lemma 8.1.12, for $\boldsymbol{a} = (a_n)_{n\in\mathbb{N}}$ in $W(A)$, we have

$$\pi_n(\boldsymbol{a}) = [a_0, \ldots, a_{n-1}].$$

By the definition of the ring structure of $W(A)$, we have the following description of operations in $W_n(A)$:

$$\begin{aligned}
[a_0, \ldots, a_{n-1}] + [b_0, \ldots, b_{n-1}] &= [S_i(a_0, \ldots, a_i; b_0, \ldots, b_i)]_{0\le i<n} \\
[a_0, \ldots, a_{n-1}] \times [b_0, \ldots, b_{n-1}] &= [S_i(a_0, \ldots, a_i; b_0, \ldots, b_i)]_{0\le i<n} \\
-[a_0, \ldots, a_{n-1}] &= [I_i(a_0, \ldots, a_i)]_{0\le i<n}.
\end{aligned}$$

Moreover, the addition identity of $W_n(A)$ is $[0, \ldots, 0]$ and the multiplication identity of $W(A)$ is $[1, 0, \ldots, 0]$.

Let i be an integer such that $0 \le i \le n$. By passing to quotient, the homomorphism Φ_i from $W(A)$ to A defines a homomorphism from $W_n(A)$ to A, still denoted by Φ_i. It associates an Witt vector the element $\Phi_i(a_0, \ldots, a_i)$ (called the phantom component of index i of $[a_0, \ldots, a_{n-1}]$).

Let $\rho : A \to B$ be a ring hormormphism. By passing to quotient, the ring homomorphism $W(\rho)$ induces a homomorphism $W_n(\rho) : W_n(B) \to W_n(A)$. We have the following formula

$$W_n(\rho)[b_0, \ldots, b_{n-1}] = [\rho(b_0), \ldots, \rho(b_{n-1})] \tag{8.1.37}$$

for $[b_0, \ldots, b_{n-1}]$ in $W_n(B)$.

Let m and n be integers such that $1 \le n \le m$. We have $V_n(A) \supset V_m(A)$, whence a canonical homomorphism $W_m(A) = W(A)/V_m(A)$ to $W_n(A) = W(A)/V_n(A)$, denoted by $\pi_{n,m}$. More precisely,

$$\pi_{n,m}[a_0, \ldots, a_{m-1}] = [a_0, \ldots, a_{n-1}] \tag{8.1.38}$$

for $[a_0, \ldots, a_{m-1}]$ in $W_m(A)$. The family $(W_n(A), \pi_{n,m})$ is a inverse system of rings and the map $\pi : \boldsymbol{a} \mapsto (\pi_n(\boldsymbol{a}))_{n\ge1}$ is a ring homomorphism from $W(A)$ to $\varprojlim W_n(A)$. Since $W(A)$ is separated and complete for the filtration $(V_n(A))_{n\in\mathbb{Z}}$, the canonical homomorphism π is an isomorphism of topological rings, where $W_n(A)$ has the discrete topology for each $n \ge 1$.

Henceforth, the homomorphisms π_n and $\pi_{n,m}$ will be called the projection from $W(A)$ to $W_n(A)$ and $W_m(A)$ to $W_n(A)$, respectively.

Example 8.1.14.

(a) The homomorphism $\Phi_0 : W_1(A) \to A$ is an isomorphism.

(b) Explicitly, the operations on $W_2(A)$ are given by

$$\begin{aligned}
[a_0, a_1] + [b_0, b_1] &= [a_0 + b_0, a_1 + b_1 - \sum_{i=1}^{p-1} \frac{1}{p}\binom{p}{i} a_0^i b_0^{p-i}], \\
[a_0, a_1] \times [b_0, b_1] &= [a_0 b_0, a_0^p b_1 + a_1 b_0^p + p a_1 b_1].
\end{aligned}$$

where $[a_0, a_1]$ and $[b_0, b_1]$ are in $W_2(A)$. The phantom componenet of $[a_0, a_1]$ is a_0 and $a_0^p + p a_1$.

(c) Let $n \ge 1$ be an integer. If $a_0, \ldots, a_{n-1}$ and $b_0, \ldots, b_{n-1}$ are integers such that $a_i \equiv b_i \bmod p$ for each i, then (Proposition 8.1.2)

$$\Phi_{n-1}(a_0, \ldots, a_{n-1}) \equiv \Phi_{n-1}(b_0, \ldots, b_{n-1}) \mod p^n.$$

Therefore, by passing to quotient, Φ_{n-1} defines a homomorphism $\varphi_n : W_n(\mathbb{Z}/p\mathbb{Z}) \to \mathbb{Z}/p^n\mathbb{Z}$. The image of φ_n is a subgroup of $\mathbb{Z}/p^n\mathbb{Z}$ containing 1, hence the whole $\mathbb{Z}/p^n\mathbb{Z}$. Since $\mathbb{Z}/p^n\mathbb{Z}$ and $W_n(\mathbb{Z}/p\mathbb{Z})$ are both finite sets, φ_n is an isomorphism.

Let m and n be integers such that $1 \leq n \leq m$. There exists a ring homomorphism $\alpha_{n,m} : \mathbb{Z}/p^m\mathbb{Z} \to \mathbb{Z}/p^n\mathbb{Z}$; moreover, the diagram

$$\begin{tikzcd} \mathbb{Z}/p^m\mathbb{Z} \arrow[r, "\alpha_{n,m}"] \arrow[d] & \mathbb{Z}/p^n\mathbb{Z} \arrow[d] \\ W_m(\mathbb{Z}/p\mathbb{Z}) \arrow[r, "\pi_{n,m}"] & W_n(\mathbb{Z}/p\mathbb{Z}) \end{tikzcd}$$

is commutative. This then gives a map $\varphi = \varprojlim \varphi_n$ and an isomorphism of topological rings $W(\mathbb{Z}/p\mathbb{Z}) = \varprojlim W_n(\mathbb{Z}/p\mathbb{Z})$ and $\mathbb{Z}_\mathfrak{p} = \varprojlim \mathbb{Z}/p^n\mathbb{Z}$.

Let $m, n \geq 1$ be integers. By construction, we have an exact sequence of additive groups

$$0 \longrightarrow W(A) \xrightarrow{V^m} W(A) \xrightarrow{\pi_m} W_m(A) \longrightarrow 0 \tag{8.1.39}$$

By passing to quotient, the endomorphism V^n of the additive group of $W(A)$ defines a homomorphism V^n_m on the additive group $W_m(A)$ to $W_{m+n}(A)$. In other words, we have a commutative diagram

$$\begin{tikzcd} W(A) \arrow[r, "V^n"] \arrow[d, "\pi_n"] & W(A) \arrow[d, "\pi_{n+m}"] \\ W_m(A) \arrow[r, "V^n_m"] & W_{n+m}(A) \end{tikzcd}$$

Moreover, from (8.1.39) we deduce the following exact sequence

$$0 \longrightarrow W_m(A) \xrightarrow{V^n_m} W_{m+n}(A) \xrightarrow{\pi_{n,n+m}} W_n(A) \longrightarrow 0 \tag{8.1.40}$$

and for $[a_0, \ldots, a_{m-1}]$ in $W_m(A)$,

$$V^n_m[a_0, \ldots, a_{m-1}] = [\underbrace{0, \ldots, 0}_{n \text{ terms}}, a_0, \ldots, a_{m-1}] \tag{8.1.41}$$

From Eq. (8.1.30)(c), we have $FV^{m+1}(\boldsymbol{a}) = p \cdot V^m(\boldsymbol{a})$ for $\boldsymbol{a}$ in $W(A)$ and therefore $F(V_{m+1}(A)) \subseteq V_m(A)$. By induction on n, we then deduce that F^n maps $V_{n+m}(A)$ to $V_m(A)$, and therefore defines, by passing to quotient, a homomorphism $F^n_m : W_{n+m}(A) \to W_m(A)$. By construction, we have the following exact sequence

$$\begin{tikzcd} W(A) \arrow[r, "F^n"] \arrow[d, "\pi_{n+m}"] & W(A) \arrow[d, "\pi_m"] \\ W_{n+m}(A) \arrow[r, "F^n_m"] & W_m(A) \end{tikzcd}$$

Recall the polynomial F_i in $\mathbb{Z}[X_0, \ldots, X_{i+1}]$ for each $i \geq 0$. The homomorphism F^1_m of $W_{m+1}(A)$ to $W_m(A)$ has the following explicit expression

$$F^1_m[a_0, \ldots, a_m] = [F_i(a_0, \ldots, a_{i+1})]_{0 \leq i < m}. \tag{8.1.42}$$

Let $\boldsymbol{a}, \boldsymbol{a}' \in W_m(A)$ and $\boldsymbol{b} \in W_{m+1}(A)$. The following formulae then follows from Eq. (8.1.30):

$$F^1_m(V^1_m(\boldsymbol{a})) = p \cdot \boldsymbol{a}$$

$$V_m^1(\boldsymbol{a} \times F_m^1(\boldsymbol{b})) = V_m^1(\boldsymbol{a}) \times \boldsymbol{b}$$
$$V_m^1(\boldsymbol{a}) \times V_m^1(\boldsymbol{a}') = p \cdot V_m^1(\boldsymbol{a} \times \boldsymbol{a}')$$
$$V_m^1(F_m^1(\boldsymbol{b})) = \boldsymbol{\mu}_{m+1} \times \boldsymbol{b}$$

where $\boldsymbol{\mu}_{m+1} = [0, 1, 0 \dots, 0]$.

8.1.4 The Witt ring with characteristic p

Proposition 8.1.15. *Let A be a ring with characteristic p. Then for $\boldsymbol{a}, \boldsymbol{b}$ in $W(A)$ and positive integers m, n, we have*

$$F(\boldsymbol{a}) = (a_n^p)_{n \in \mathbb{N}} \tag{8.1.43}$$
$$p \cdot \boldsymbol{a} = VF(\boldsymbol{a}) = FV(\boldsymbol{a}) = (0, a_0^p, a_1^p, \dots) \tag{8.1.44}$$
$$V^m(\boldsymbol{a}) \times V^n(\boldsymbol{b}) = V^{m+n}(F^n(\boldsymbol{a}) \times F^m(\boldsymbol{b})). \tag{8.1.45}$$

Proof. The formula (8.1.43) follows from Example 8.1.6(d). We immediately deduce from this the equality
$$VF(\boldsymbol{a}) = FV(\boldsymbol{a}) = (0, a_0^p, a_1^p, \dots)$$
and the equality $p \cdot \boldsymbol{a} = FV(\boldsymbol{a})$ is already proved in Proposition 8.1.11(b), whence (8.1.44). Now recall that by (8.1.34), we have

$$V^m(\boldsymbol{a}) \times V^n(\boldsymbol{b}) = V^m(\boldsymbol{a} \times F^m(V^n(\boldsymbol{b}))).$$

By (8.1.33), we also deduce that

$$V^n(F^m(\boldsymbol{b})) \times \boldsymbol{a} = V^n(F^m(\boldsymbol{b}) \times F^n(\boldsymbol{a})).$$

Fomula (8.1.45) then follows from these two equalities and the relation $F^m \circ V^n = V^n \circ F^m$, which comes from (8.1.43). □

Corollary 8.1.16. *Let m and n be positive integers, then*

$$V_m(A) \times V_n(A) \subseteq V_{m+n}(A).$$

Proof. This follows from (8.1.45), since $V_m(A)$ is the image of V^m. □

Remark 8.1.17. Let A be a ring with characteristic p. By Proposition 8.1.15, we have the formulae

$$F_m^n[a_0, \dots, a_{n+m-1}] = [a_0^{p^n}, \dots, a_{m-1}^{p^n}], \quad p^n \cdot [a_0, \dots, a_{n+m-1}] = [\underbrace{0, \dots, 0}_{n \text{ terms}}, a_0^{p^n}, \dots, a_{m-1}^{p^n}]$$

for any Witt vector $[a_0, \dots, a_{n+m-1}]$ of length $n + m$.

Proposition 8.1.18. *Let A be a ring.*

(a) For each integer $k \geq 1$, we have $V_1(A)^k = p^{k-1} \cdot V_1(A)$.

(b) Suppose that A is a ring with characteristic p. Then the $V_1(A)$-adic topology and the p-adic topology coincide on $W(A)$, and is finer than the topology $\mathcal{T}$. The ring $W(A)$ is separated and complete for the p-adic topology.

Proof. We prove (a) by induction on k. The case $k = 1$ is evident, so suppose $k \geq 2$. By the induction hypothesis, we have $V_1(A)^{k-1} = p^{k-2} \cdot V_1(A)$ and Concequently $V_1(A)^k = p^{k-2} \cdot V_1(A)^2$. But by Eq. (8.1.29)(d), $V_1(A)^2 = p \cdot V_1(A)$, so (a) follows.

Suppose now that A has characteristic p. Then by (8.1.44),

$$pW(A) = VF(W(A)) \subseteq V_1(A)$$

combine this with (a), we deduce by induction that $p^k \cdot W(A) \subseteq V_1(A)^k \subseteq p^{k-1} \cdot W(A)$, and by Corollary 8.1.16 the inclusion $V_1(A)^k \subseteq V_k(A)$, for each $k \geq 1$. The first assertion in (b) then follow.

Let $k \geq 1$ be an integer. By (8.1.44), the ideal $p^k \cdot W(A)$ of $W(A)$ is the set of elements $\boldsymbol{a} = (a_n)_{n\in\mathbb{N}}$ of $W(A)$ such that $a_n = 0$ for $n < k$ and $a_n \in A^{p^k}$ for $n \geq k$. It is therefore closed in the topology $\mathcal{T}$. Since $W(A)$ is separated and complete for the topology $\mathcal{T}$ and that the ideals $p^k \cdot W(A)$ of $W(A)$, for $k \geq 1$, form a base of neighborhoods of $\mathbf{0}$ in $W(A)$ for the p-adic topology, the ring $W(A)$ is separated and complete for the p-adic topology (**??**). □

Proposition 8.1.19. *Let A be a perfect ring of characteristic p.*

(a) For any element $\boldsymbol{a} = (a_n)_{n\in\mathbb{N}}$ in $W(A)$, the series $\sum_n p^n \tau(a_n^{p^{-n}})$ converges in $W(A)$ to $\boldsymbol{a}$.

(b) In $W(A)$, the $V_1(A)$-adic topology, the p-adic topology, and the topology $\mathcal{T}$ coincide. Moreover precisely, we have $V_n(A) = p^n W(A) = V_1(A)^n$ for each integer $n \geq 0$. In particular, Φ_0 defines an isomorphism of $W(A)/pW(A)$ to A.

Proof. By definition, the map $a \mapsto a^p$ is an automorphism of A. Proposition 8.1.15 then show that F is an automorphism of $W(A)$, and for $n \in \mathbb{N}$, we have

$$p^n \cdot W(A) = V^n F^n(W(A)) = V^n(W(A)) = V_n(A).$$

In particular, $V_1(A)^n = (pW(A))^n = p^n \cdot W(A)$. Assertion (b) then follows. By Proposition 8.1.15, we have

$$p^n \cdot \tau(a_n^{p^{-n}}) = V^n F^n \tau(a_n^{p^{-n}}) = V^n \tau(a_n)$$

so (a) follows from Proposition 8.1.13. □

Proposition 8.1.20. *Let A be a field with characteristic p. Then $W(A)$ is a local integral domain, separated and complete, with maximal ideal $V_1(A)$ and residue field isomorphic to A. If A is perfect, the ring $W(A)$ is a DVR, and its maximal ideal is $pW(A)$.*

Proof. The homomorphism Φ_0 defines an isomorphic of $W(A)/V_1(A)$ to A. The ideal $V_1(A)$ of $W(A)$ is therefore maximal. The ring $W(A)$ is separated and complete for the $V_1(A)$-topology (Proposition 8.1.18), so is local with maximal ideal $V_1(A)$ by Proposition 2.3.26.

Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be two nonzero elements in $W(A)$. There exists integers $n, m \geq 0$ and elements $\tilde{\boldsymbol{a}} = (\tilde{a}_n)_{n\in\mathbb{N}}$ and $\tilde{\boldsymbol{b}} = (\tilde{b}_n)_{n\in\mathbb{N}}$ of $W(A)$ such that $\boldsymbol{a} = V^m(\tilde{\boldsymbol{a}})$, $\boldsymbol{b} = V^n(\tilde{\boldsymbol{b}})$ and the eleemnts $\tilde{a}_0$, $\tilde{b}_0$ are nonzero. Then the $m+n$-th component of $\boldsymbol{a} \times \boldsymbol{b}$ is equal to the 0-th component of $F^n(\tilde{\boldsymbol{a}}) \times F^m(\tilde{\boldsymbol{b}})$ (formula (8.1.45)), which is $\tilde{a}^{p^n} \tilde{b}^{p^m}$ (formula (8.1.43) and Example 8.1.6). Therefore $\boldsymbol{a} \times \boldsymbol{b}$ is nonzero and $W(A)$ is integral.

If the field A is perfect, the maximal ideal $V_1(a)$ of $W(A)$ is equal to $pW(A)$ by Proposition 8.1.19(b) and therefore $W(A)$ is a DVR by Proposition 5.2.16(c). □

8.2 Cohen rings

In this part, we fix a prime integer p.

8.2.1 p-rings

Let C be a ring. We say C is a ***p*-ring** if the ideal pC of C is maximal and if C is separated and complete for the pC-adic topology.

If $p1_C$ is nilpotent in C and the ideal pC is maximal, then C is a p-ring and the pC-adic topology is discrete. In particular, every field with characteristic p is a p-ring.

Proposition 8.2.1. *Let C be a p-ring.*

(a) The ring C is local with maximal ideal pC.

(b) Suppose that $p1_C$ is nilpotent and let d be the smallest positive integer such that $p^d 1_C = 0$. Then the ideals of C are of the form p^iC and $p^iC \neq p^jC$ if i, j are distinct integers such that $0 \leq i, j \leq d$. The C-module C has length d.

(c) Suppose that $p1_C$ is not nilpotent. Then C is a DVR whose residue field is of characteristic p, and the fraction field of characteristic 0. The ideals p^nC, where $n \in \mathbb{N}$, are distinct; they are the nonzero ideals of C. The C-module C has infinite length.

Proof. Assertion (a) follows from Proposition 2.3.26. By hypothesis we have $\bigcap_{n\in\mathbb{N}} p^nC = \{0\}$. Let $x \neq 0$ be in C; there exists an integer $n \geq 0$ such that $x \in p^nC$, $x \notin p^{n+1}C$. Write $x = p^n y$ where $y \in C$, then $y \notin pC$, so it is invertible.

Suppose that $p1_C$ is not nilpotent. If x_1 and x_1 are two nonzero elements in C, there exist integers $n_1, n_2 \geq 0$ and invertible elements y_1, y_2 of C such that $x_1 = p^{n_1}y_1$, $x_2 = p^{n_2}y_2$. We then have $x_1x_2 = p^{n_1+n_2}y_1y_2 \neq 0$, so C is integral. Since C is a local ring, not a field, and the maximal ideal $\mathfrak{m}_C = pC$ is principal, we conclude that C is a DVR (Proposition 5.2.16). The nonzero ideals of C are then of the form p^nC, and are all distinct. In particular, the ring C is not Artinian, so the C-module C has infinite length. The residue field C/pC of C clearly has characteristic p. Let q be the characteristic of the fraction field of C. We have $p1_C \neq 0$, so $q \neq p$. But if $q \neq 0$, we have $q1_C \neq 0$ and C/pC then has characteristic $q \neq p$, which is absurd. This proves (c).

Suppose that $p1_C$ is nilpotent and let d be the smallest positive integer such that $p^d1_C = 0$. We have a sequence of ideals

$$C \supsetneq pC \supsetneq \cdots \supsetneq p^{d-1}C \supsetneq p^dC = \{0\}. \tag{8.2.1}$$

If i is an integer such that $0 \leq i < d$ and $p^iC = p^{i+1}C$, then

$$p^{d-i-1}p^iC = p^{d-i-1}p^{i+1}C = p^dC = \{0\}$$

contradicting the hypothesis $p^{d-1}1_C \neq 0$. Therefore the ideals in the sequence above are all distinct. Let $\mathfrak{a}$ be an ideal of C and i the smallest integer such that $\mathfrak{a} \supseteq p^iC$. Let $x \neq 0$ be in $\mathfrak{a}$; we can write $x = p^m u$ where $m \geq 0$ and u is invertible. Then $p^m1_C \in \mathfrak{a}$, whence $p^mC \subseteq \mathfrak{a}$ and $m \geq i$. This shows $x \in p^iC$, so $\mathfrak{a} = p^iC$. We now conclude that (8.2.1) is a Jordan-Hölder sequence for the C-module C, so C has length d. □

Corollary 8.2.2. *If the p-ring C is integral, it is a DVR, or a field with characteristic p.*

Proof. Suppose that C is integral. If $p1_C$ is nilpotent, then $p1_C = 0$, and $\{0\}$ is a maximal ideal of C, whence C is a field with characteristic p. Otherwise, $p1_C$ is not nilpotent and C is a DVR by Proposition 8.2.1. □

Corollary 8.2.3. *Let C be a p-ring and $\mathfrak{a}$ a proper ideal of C. Then $C/\mathfrak{a}$ is a p-ring.*

Proof. We may suppose that $\mathfrak{a} \neq \{0\}$. Then there exists an integer $i \geq 1$ such that $\mathfrak{a} = p^iC$. The idela $pC/\mathfrak{a}$ of $C/\mathfrak{a}$ is maximal and we have $p^i1_{C/\mathfrak{a}} = 0$, so $C/\mathfrak{a}$ is a p-ring. □

Let C be a p-ring. The length of C, denoted by $l(C)$, is the supremum in $\bar{\mathbb{R}}$, of the set of integers $n \geq 1$ such that $p^{n-1}1_C \neq 0$. By Proposition 8.2.1, $l(C)$ is finite and equal to the length of the C-module C, or is $+\infty$, in which case the C-module C has infinite length.

Example 8.2.4 (Example of p-rings).

(a) For each integer $n \geq 1$, the ring $\mathbb{Z}/p^n\mathbb{Z}$ is a p-ring of length n. The ring $\mathbb{Z}_p$ of p-adic integers is a p-ring of length infinite.

(b) Let K be a perfect field of characteristic p. By Proposition 8.1.20, the Witt ring $W(K)$ is a p-ring of infinite length. The map $(a_n)_{n\in\mathbb{N}} \mapsto a_0$ induces by passing to quotient an isomorphism of $W(K)/pW(K)$ to the field K. For each integer $n \geq 1$, the ring

$$W_n(K) = W(K)/p^nW(K)$$

is a p-ring of length n. Note that $W(\mathbb{F}_p) = \mathbb{Z}_p$, so this generalize the example in (a).

Proposition 8.2.5. *Let C_1 and C_2 be p-rings and $\rho : C_1 \to C_2$ a ring homomorphism. Let $\bar{\rho}$ be the homomorphism of $\kappa_{C_1} = C_1/pC_1$ to $\kappa_{C_2} = C_2/pC_2$ induced by ρ on the residue field.*

(a) We have $l(C_1) \geq l(C_2)$, and ρ is injective if and only if $l(C_1) = l(C_2)$.

(b) For ρ to be surjective, it is necessary and sufficient that $\bar{\rho}$ being an isomorphism.

(c) For ρ to be an isomorphism, it is necessary and sufficient that $\bar{\rho}$ being an isomorphism and we have $l(C_1) = l(C_2)$.

Proof. Let $n \geq$ be an integer. We have $\rho(p^{n-1}1_{C_1}) = p^{n-1}1_{C_2}$, so the relation $p^{n-1}1_{C_2} \neq 0$ implies $p^{n-1}1_{C_1} \neq 0$ and they are equivalent if ρ is injective. We then have $l(C_2) \leq l(C_1)$ where the equality holds if ρ is injective. If ρ is not injective, there exists an integer $i < i(C_1)$ such that the kernel of ρ is the ideal p^iC_1 of C_1; we have $p^i1_{C_2} = 0$, so $l(C_2) \leq i$. This proves (a). Since κ_{C_1} and κ_{C_2} are fields, the homomorphism $\bar{\rho}$ is injective. If ρ is surjective, so is $\bar{\rho}$ and hence it is an isomorphism. Conversely, assume that $\bar{\rho}$ is surjective. Then for each integer $n \geq 0$, the map $\bar{\rho}_n : p^nC_1/p^{n+1}C_1 \to p^nC_2/p^{n+1}C_2$ induced by ρ is surjective. Since C is complete for the pC_1-adic topology, ρ is then surjective by Corollary 2.3.31. This proves (b). Finally, (a) and (b) imply (c). □

Proposition 8.2.6. *Let $(C_n, \pi_{n,m})$ be a inverse system of rings indexed by $\mathbb{N}$. Suppose that C_n is a p-ring for each $n \in \mathbb{N}$ and the homomorphism $\pi_{n,m}$ are surjective. Then $C = \varprojlim C_n$ is a p-ring, and for each $n \in \mathbb{N}$, the canonical homomorphism $\pi_n : C \to C_n$ is surjective and induces an isomorphism of κ_C to κ_{C_n}.*

Proof. Since the maps $\pi_{n,m}$ are surjective, π_n is surjective (Proposition 2.3.6). Let d_n be the length of C_n. By Proposition 8.2.5(a), the sequence d_n in $\mathbb{N} \cup \{+\infty\}$ is increasing; if it is stationary, there exists an integer n_0 such that $\pi_{n,m}$ is an isomorphism of C_m to C_n for $m \geq n \geq N_0$, hence C is isomorphic to C_{n_0}, a p-ring.

It then suffices to consider the case where d_n are all finite, and (d_n) tends to infinity. Let us endow the ring C with the trivial filtration. For each $n \in \mathbb{N}$, let I_n be the kernel of π_n, and put $I_n = C$ for $n < 0$. Endow C with the filtration $(I_n)_{n\in\mathbb{N}}$. It is then separated and complete, for the topology $\mathcal{T}$ defined bby the filtration is the inverse limit topology of the discrete topologis over C_n. Let $k \geq 1$ be an integer. We have $p^kC \subseteq \varprojlim_n (p^kC_n)$. Conversely, if $x = (x_n)_{n\in\mathbb{N}} \in \varprojlim_n (p^kC_n)$ and if we set $X_n = \{y \in C : \pi_n(p^ky) = x_n\}$, the sequence $(X_n)_{n\in\mathbb{N}}$ is a decreasing sequence of nonempty closed affine subsets of C. Since C/I_n is an Artinian C-module, the intersection of X_n is nonempty (AC, §2, n7, prop.7); for every $z \in \bigcap_{n\in\mathbb{N}} X_n$, we have $p^kz = x$. We have therefore proved that $p^kC = \varprojlim_n (p^kC_n)$ for each integer $k \geq 1$. In particular, the ideal p^kC of C is closed for the topology $\mathcal{T}$. For C, the p-adic topology is finer than the topology $\mathcal{T}$ since we have $p^{d_n}C \subseteq I_n$. It then follows from **??** that C is separated and complete for the pC-adic topology. In addition we have $pC = \varprojlim pC_n = \pi_0^{-1}(pC_0)$ and therefore the surjective homomorphism of C/pC to C_0/pC_0 induced by π_0 is an isomorphism. This shows that pC is maximal and therefore C is a p-ring. The final assertion follows from Proposition 8.2.5(b). $\square$

Now let A be a separated and complete local ring, with residue field of characteristic p. A Cohen subring of A is defined to be a subring C of A that is a p-ring and such that $A = \mathfrak{m}_A + C$ (i.e., $A/\mathfrak{m}_A = C/(\mathfrak{m}_A \cap C)$). In this case, the ideal $\mathfrak{m}_A \cap C$ is maximal in C, hence equal to pC. The canonical map $\kappa_C = C/pC$ to $\kappa_A = A/\mathfrak{m}_A$ is then an isomorphism.

Example 8.2.7. Let C be a p-ring. The formal series ring $A = C[\![X_1, \dots, X_n]\!]$ is then Noetherian, local, separated and complete, with maximal ideal generated by $(p, T_1, \dots, T_n)$. It is immediate that C is a Cohen subring of A. This applies in particular if C is equal to $\mathbb{Z}_p$, to $\mathbb{Z}/p^n\mathbb{Z}$, or a field with characteristic p.

Theorem 8.2.8. *Let A be a seaprated and complete local ring, with residue field k of characteristic p. Let $\pi : A \to k$ be the canonical map, and S a subset of A such that π induces a bijection of S to a p-basis for k.*

(a) There exists a unique Cohen subring C of A containing S.

(b) The subring C is closed in A, and the pC-adic topology of C is induced by the $\mathfrak{m}_A$-adic topology of A.

(c) Every closed subring B of A, containing S, and such that $A = B + \mathfrak{m}_A$, contains C.

Proof. We divide the proof into several parts. First, we assume that $\mathfrak{m}_A$ is nilpotent. Let n be a positive integer such that $\mathfrak{m}_A^{n+1} = \{0\}$. If Φ_n is the n-th Witt polynomial, the map $\rho : [a_0, \dots, a_n] \mapsto \Phi_n(a_0, \dots, a_n)$ is a homomorphism from $W_{n+1}(A)$ to A. Let B_n be the image of ρ_n and C_n the subring of A generated by $B_n \cup S$. We note that $pA \subseteq \mathfrak{m}_A$ and B_n consists of elements of the form $a_0^{p^n} + pa_1^{p^{n-1}} + \cdots + p^n a_n$, where $a_0, \dots, a_n \in A$. Concequently, we have $\pi(B_n) = k^{p^n}$, and $\pi(C_n) = k^{p^n}[\pi(S)]$. Since $\pi(S)$ is a p-basis for k, we have $k = k^{p^n}[\pi(S)]$ (A, V, p.96), whence $\pi(C_n) = k$, and $C_n + \mathfrak{m}_A = A$.

Let B be a subring of A containing S. We claim that, for B to contain C_n, it is necessary and sufficient that $A = B + \mathfrak{m}_A$. First, if B contains C_n, then

$$B + \mathfrak{m}_A \supseteq C_n + \mathfrak{m}_A = A$$

so $B + \mathfrak{m}_A = A$. Conversely, suppose that $B + \mathfrak{m}_A = A$. Let $a_0, \dots, a_n$ be elements of A; there exists by hypothesis elements $b_0, \dots, b_0$ of B such that $a_i \equiv b_i \bmod \mathfrak{m}_A$ for each i. By Proposition 8.1.2 and the hypothesis $\mathfrak{m}_A^{n+1} = 0$, we then have $\Phi_n(a_0, \dots, a_n) = \Phi_n(b_0, \dots, b_n) \in B$, whence $B_n \subseteq B$. Since C_n is the ring generated by $B_n \cup S$, we have $C_n \subseteq B$.

Let $\mathcal{S}$ be the set of subrings B of A containing S and such that $B + \mathfrak{m}_A = A$. There exists by the above claim a smallest element C in $\mathcal{S}$ (take the smallest integer n such that $\mathfrak{m}_A^{n+1} = \{0\}$, then C_n is the desired subring), and we have $C_n = C$ for any integer $n \geq 0$ such that $\mathfrak{m}_A^{n+1} = \{0\}$. We have $C + \mathfrak{m}_A = A$ by construction and $p1_C$ is nilpotent; clearly $pC \subseteq C \cap \mathfrak{m}_A$. We now prove that $C \cap \mathfrak{m}_A \subseteq pC$. Choose an integer $n \geq 1$ such that $\mathfrak{m}_A^n = \{0\}$, so $C = C_n = C_{n-1}$. Let Λ be the subset of $\mathbb{N}^{\oplus S}$ formed by elements $(\alpha_s)_{s \in S}$ with finite support such that $0 \leq \alpha_s < p^n$ for each $s \in S$. Since B_n contains $s^{p^n} = \Phi_n(s, 0, \dots, 0)$ for each $s \in S$, the monomials $z_\alpha = \prod_{s \in S} s^{\alpha_s}$, for $\alpha \in \Lambda$, generate the C_n as a B_n-module. Moreover, by the formula

$$\Phi_n(a_0, \dots, a_n) = a_0^{p^n} + p\Phi_{n-1}(a_1, \dots, a_n),$$

any element of B_n is of the form $a^{p^n} + pb$ where $a \in A$ and $b \in B_{n-1}$. Therefore any element of $C = C_n$ is of the form

$$x = \sum_{\alpha \in \Lambda} c_\alpha^{p^n} z_\alpha + py \tag{8.2.2}$$

where $c_\alpha \in A$ for $\alpha \in \Lambda$, and $y \in C_{n-1} = C$. If x is in $C \cap \mathfrak{m}_A$, we have $\pi(x) = 0$ whence $\sum_{\alpha \in \Lambda} \pi(c_\alpha)^{p^n} \pi(z_\alpha) = 0$. Since $\pi(S)$ is a p-basis for k, we have $\pi(c_\alpha) = 0$ for all $\alpha \in \Lambda$ (A, V, p.96). Then $c_\alpha \in \mathfrak{m}_A$, whence $c_\alpha^n = 0$ and a fortiori $c_\alpha^{p^n} = 0$. By (8.2.2), we then get $x = py$, whence $C \cap \mathfrak{m}_A \subseteq pC$. With this, it is now clear that C is a Cohen subring of A. We have $p^n C = \mathfrak{m}_A^n = \{0\}$ for n large enough and assertion (b) is then trivial. Assertion (c) follows from the construction of C.

We now turn to the general case. For each integer $n \geq 0$, let A_n be the local ring $A/\mathfrak{m}_A^{n+1}$, $\mathfrak{m}_n = \mathfrak{m}_A/\mathfrak{m}_A n+1$, and $\pi_n : A \to A_n$ the canonical homomorphism. By the previous case, there exists a unique Cohen subring C_n in A_n containing $\pi_n(S)$. For $0 \leq n \leq m$, we denote by $\pi_{n,m}$ the canonical map from A_m to A_n. By Corollary 8.2.3, $\pi_{n,m}(C_m)$ is a p-ring; we have $\pi_{n,m}(C_m) + \mathfrak{m}_n = A_n$, so $\pi_{n,m}(C_m)$ is equal to the Cohen subring C_n of A_n. By Proposition 8.2.6, the subring $\varprojlim C_n$ of $\varprojlim A_n$ is a p-ring. Put $C = \bigcap_{n \in \mathbb{N}} \pi_n^{-1}(C_n)$. Then C is the preimage of $\varprojlim C_n$ under the isomorphism $a \mapsto (\pi_n(a))_{n \in \mathbb{N}}$ of A to $\varprojlim A_n$, whence a closed subring of A, and a p-ring. We have $\pi_n(C) = C_n$ for each $n \in \mathbb{N}$ (Proposition 8.2.6) and in particular $\pi_0(C) = A_0$, which means $\pi(C) = k$. Therefore C is a Cohen subring of A.

For each integer $n \geq 0$, let $J_n = C \cap \mathfrak{m}_A^n$. Since the local ring A is separated, we have $\bigcap_{n \in \mathbb{N}} J_n = \{0\}$, and by the structure of ideals of p-rings, every ideal of C is of the form $p^i C$ and contains some J_n. Conversely, J_n contains $p^n C$. Therefore the pC-adic topology is induced by the $\mathfrak{m}_A$-topology on A. This proves (b). Now let B be a closed subring of A containing S and such that $B + \mathfrak{m}_A = A$. Since B is closed, we have $B = \bigcap_{n \in \mathbb{N}} \pi_n^{-1}(\pi_n(B))$. We have $\pi_n(B) \supset \pi_n(S)$ and $\pi_n(B) + \mathfrak{m}_n = A_n$, whence $\pi_n(B) \supseteq C_n$ and therefore $B \supseteq C$. This proves (c) and completes the proof. □

Remark 8.2.9. Suppose that $p1_A$ is not nilpotent (in particular if A is an integral domain with fraction field of characteristic 0). Then C is a DVR with fraction field of characteristic 0.

8.2.2 Existence and uniqueness of p-rings

Proposition 8.2.10. *Let C_1 and C_2 be p-rings such that $l(C) \geq l(C_2)$, π_1 (resp. π_2) the canonical homomorphism of C_1 (resp. C_2) to κ_{C_1} (resp. κ_{C_2}). Let $(x_\lambda)_{\lambda \in \Lambda}$ (resp. $(y_\lambda)_{\lambda \in \Lambda}$) be a family of elements of C_1 (resp. C_2) whose image under π_1 (resp. π_2) is a p-basis of κ_{C_1} (resp. κ_{C_2}). Let η be an isomorphism of κ_{C_1} to κ_{C_2} such that $\eta(\pi_1(x_\lambda)) = \pi_2(y_\lambda)$ for every $\lambda \in \Lambda$. There then exists a unique homomorphism $\rho : C_1 \to C_2$ such that $\eta \circ \pi_1 = \pi_2 \circ \rho$ and $\rho(x_\lambda) = y_\lambda$ for every $\lambda \in \Lambda$. It is surjective, and an isomorphism if $l(C_1) = l(C_2)$.*

Proof. Let A be the subring of $C_1 \times C_2$ formed by pairs (x, y) such that $\eta(\pi_1(x)) = \pi_2(y)$. The map $(x, y) \mapsto \pi_1(x)$ is a surjective homomorphism of A to κ_C. Its kernel $\mathfrak{m}$, equal to $pC_1 \times pC_2$, is then a maximal ideal of A. The subspace A of $C_1 \times C_2$ is closed, hence complete, and the topology induced on A from $C_1 \times C_2$ is the $\mathfrak{m}$-adic topology. Thus A is a separated and complete local ring with maximal ideal $\mathfrak{m}$. For each $\lambda \in \Lambda$, we have $(x_\lambda, y_\lambda) \in A$ by hypothesis; if ξ_λ is the calss of (x_λ, y_λ) modulo $\mathfrak{m}$, the family $(\xi_\lambda)_{\lambda \in \Lambda}$ is a p-basis of $A/\mathfrak{m}$. By Theorem 8.2.8, there exists a unique Cohen subring $\tilde{C}$ of A containing (x_λ, y_λ) for each $\lambda \in \Lambda$. By definition, we have $l(\tilde{C}) = l(C_1) \geq l(C_2)$. The restriction to $\tilde{C}$ of the projection of $C_1 \times C_2$ to C_1 is a homomorphism $h_1 : \tilde{C} \to C_1$ which induces an isomorphism of $\kappa_{\tilde{C}}$ to κ_{C_1}. By Proposition 8.2.5(c), h_1 is an isomorphism of $\tilde{C}$ to C_1. We also see that the restriction h_2 to $\tilde{C}$ of the projection from $C_1 \times C_2$ to C_2 is a surjective homomorphism. Therefore, $\tilde{C}$ is the graph of a surjective homomorphism $\rho = h_2 \circ h_1^{-1}$ of C_1 to C_2, and we have $\eta \circ \pi_1 = \pi_2 \circ \rho$, $\rho(x_\lambda) = y_\lambda$ for $\lambda \in \Lambda$. Moreover, if $l(C_1) = l(C_2)$, ρ is an isomorphism.

Let ρ_1 be a homomorphism of C_1 to C_2 such that $\eta \circ \pi_1 = \pi_2 \circ \rho$, $\rho(x_\lambda) = y_\lambda$ for $\lambda \in \Lambda$, and let $\tilde{C}'$ be the graph of ρ_1. It is immediate that $\tilde{C}'$ is a Cohen subring of A, containing (x_λ, y_λ) for $\lambda \in \Lambda$, whence $\tilde{C}' = \tilde{C}$ and $\rho_1 = \rho$. □

Proposition 8.2.11. *Let k be a field of characteristic p, and let $n \geq 1$ an integer, or $+\infty$. Then there exists a p-ring of length n with residue field isomorphic to k.*

Proof. The ring $W(k)$ of Witt vectors with coefficients in k is an integral seaprated and complete local ring, with residue field isomorphic to k (Proposition 8.1.20), and we have $p \cdot 1_{W(k)} \neq 0$. Let C be a Cohen subring of $W(k)$ (Theorem 8.2.8). Then C is a p-ring of length $+\infty$ with residue field isomorphic to k, and, if $n \geq 1$ is an integer, the quotient C/p^nC is a p-ring of length n with residue field isomomorphic to k. □

Remark 8.2.12. Let $n \geq 1$ be an integer and S a p-basis of k. We can show that the subring $W_n(k)$ generated by $W_n(k^{p^n})$ and the elements $[\xi, 0, \dots, 0]$ (where $\xi \in S$), is a p-ring of length n with residue field isomorphic to k.

Corollary 8.2.13. *Let C be a p-ring of finite length n. Then there exists a p-ring $\tilde{C}$ of infinite length such that C is isomorphic to $\tilde{C}/p^n\tilde{C}$.*

Proof. By Proposition 8.2.11, there exists a p-ring $\tilde{C}$ of infinite length such that $\kappa_{\tilde{C}}$ is isomorphic to κ_C. Then $\tilde{C}/p^n\tilde{C} = \tilde{C}_n$ is a p-ring of length n, and $\kappa_{\tilde{C}_n}$ is isomorphic to $\kappa_{\tilde{C}}$, hence to κ_C. By Proposition 8.2.10, the ring C and $\tilde{C}_n$ is isomorphic. □

Proposition 8.2.14. *Let C be a p-ring, with residue field k prefect. Suppose that C has finite length n (resp. infinite length). There exists a unique isomorphism $\rho_C : W_n(k) \to C$ (resp. $\rho_C : W(k) \to C$) which induces by passing to quotient the identity map on k.*

Proof. Since $W_n(k)$ (resp. $W(k)$) is a p-ring with residue field k and length n (infinite), and $\varnothing$ is a p-basis for k, Proposition 8.2.10 then implies the claim. □

8.2.3 Multiplicative representatives

Theorem 8.2.15. *Let A be a separated and complete local ring, k the residue field, and $\pi : A \to k$ the canonical map. Suppose that k is perfect with characteristic p.*

(a) There exists a unique homomorphism $\rho : W(k) \to A$ such that $\pi(\rho(\boldsymbol{a})) = a_0$ for each $\boldsymbol{a} = (a_n)_{n\in\mathbb{N}}$ in $W(k)$.

(b) The homomorphism ρ is continuous if we endow $W(k)$ with the $pW(k)$-adic topology, and its image is the unique Cohen subring of A.

Proof. By Theorem 8.2.8, there exists a unique Cohen subring C of A. Let $\rho : W(k) \to A$ be a ring homomorphism such that $\pi(\rho(\boldsymbol{a})) = a_0$ for $\boldsymbol{a} = (a_n)_{n\in\mathbb{N}}$ in $W(k)$; is is immediate that the image of ρ is a Cohen subring of A, hence equal to C. The existence and uniqueness of ρ then follows from Proposition 8.2.14. The pC-adic topology on C is induced by the $\mathfrak{m}_A$-adic topology (Theorem 8.2.8(b)), so ρ is continuous. □

Proposition 8.2.16. *Retain the hypothesis and notations in Therorem 8.2.15. There exist a multiplicative subset S of A such that π induces a bijection of S to k. For an element $a \in A$ to belong to S, it is necessary and sufficient that for each $n \in \mathbb{N}$, there exists an element a_n in A such that $a = a_n^{p^n}$. The set S consists of elements of the form $\rho(x, 0, 0, \dots,)$.*

Proof. Let S be a multiplicative subset of A such that π induces a bijection of S to k. Let T be the set of elements of A which are p^n-th powers for every $n \in \mathbb{N}$. Let $a \in S$ and $n \in \mathbb{N}$; since k is perfect, there exists x_n in k such that $x_n^{p^n} = \pi(\boldsymbol{a})$. But $\pi(S) = k$, so there exists $a_n \in S$ such that $x_n = \pi(a_n)$. Then we have $\pi(a_n^{p^n}) = \pi(a)$ whence $a_n^{p^n} = a$ since π is injective on S. This shows $S \subseteq T$. Conversely, we show that the restriction of π on T is injective, which then proves that $S = T$: let a and b be elements of T such that $\pi(a) = \pi(b)$. Let $n \in \mathbb{N}$; there exists two elements a_n and b_n of A such that $a = a_n^{p^n}$, $b = b_n^{p^n}$. Then $\pi(a_n)^{p^n} = \pi(b_n)^{p^n}$, hence $\pi(a_n) = \pi(b_n)$. This means $a_n \equiv b_n \bmod \mathfrak{m}_A$, so $a_n^{p^n} \equiv b_n^{p^n} \bmod \mathfrak{m}_A^{n+1}$ (Lemma 8.1.1). But then $a \equiv b \bmod \mathfrak{m}_A^{n+1}$. Since n is arbitrary and A is separated, we conclude that $a = b$.

In the notations of Therorem 8.2.15, let $\varphi = \rho \circ \tau_k$, which means

$$\varphi(x) = \rho(x, 0, 0, \dots)$$

for $x \in k$. Since τ_k is multiplicative (Proposition 8.1.13), so is φ and it is clear that $\pi \circ \varphi$ is the identity map on k. The image of φ then satisfies the requirements. $\square$

The elements of S are often called the **multiplicative (or Teichmüller) representatives** of A. Recall that by Proposition 8.1.19, we have

$$\boldsymbol{a} = \sum_{n=0}^{\infty} p^n \tau_k(a_n^{p^{-n}})$$

for $\boldsymbol{a} = (a_n)_{n \in \mathbb{N}}$ in $W(k)$. Therefore

$$\rho(\boldsymbol{a}) = \sum_{n=0}^{\infty} p^n \varphi(a_n^{p^{-n}})$$

because ρ is continuous. By this formula, the unique Cohen subring of A consists of elements of the form $\sum_{n=0}^{\infty} p^n s_n$ where $s_n \in S$ for each $n \in \mathbb{N}$.

Example 8.2.17 (Example of multiplicative representatives).

(a) Let k be a perfect field of characteristic p. The multiplicative representatives of the ring $W(k)$ is the Witt vectors $\tau(x) = (x, 0, 0, \dots)$ for $x \in k$.

(b) Let A be an integral separated and complete local ring. We suppose that the residue field k of A is finite, with $q = p^f$ elements, hence perfect with characteristic p. We have $x^q = x$ for each $x \in k$, hence $s^q = s$ for each multiplicative representative. It follows that the set of multiplicative representatives consists of 0 and the $q-1$-th roots of the unity in the field of fractions of A. If the field of fractions of A is locally compact, the existence of the multiplicative representatives follows also from (AC, VI, §9, n2, prop.3).

In particular, consider the case $A = \mathbb{Z}_p$. Then the multiplicative representatives are 0 and the $(p-1)$-th roots of unity in the field Q_p.

8.2.4 Structure of Noetherian complete local rings

We now use the results in this part to give a structural theorem for Noetherian complete local rings (such rings are separated by Corollary 2.4.17). Let A and C be Noetherian complete local rings and $\rho : C \to A$ a ring homomorphism which induces an isomorphism residue fields. Let $(p_1, \dots, p_m)$ be a sequence generating the maximal ideal $\mathfrak{m}_C$ of C, and let $(t_1, \dots, t_n)$ be a sequence of elements of A. Let S denote the sequence $(\rho(p_1), \dots, \rho(p_m), t_1, \dots, t_n)$.

Lemma 8.2.18. *Let $B = C[\![T_1, \dots, T_n]\!]$ and $\eta : B \to A$ be the unique homomorphism extending ρ that sends T_i to t_i for each i.*

(a) *For η to be surjective, it is necessary and suffcient that S generates the ideal $\mathfrak{m}_A$ of A, or its class modulo $\mathfrak{m}_A^2$ generates $\mathfrak{m}_A/\mathfrak{m}_A^2$ as a vector space over κ_A.*

(b) *For η to make A a finite B-algebra, it is necessary and sufficient that S generates a defining ideal of the $\mathfrak{m}_A$-adic topology on A.*

Proof. Let $\mathfrak{n}$ be the ideal of B generated by $T_1, \dots, T_n$. Any homomorphism η of B to A extending ρ and such that $\eta(T_i) = t_i$ maps $\mathfrak{n}$ to $\mathfrak{m}_A$, hence is continuous when B is endowed with the $\mathfrak{n}$-adic topology. The existence and uniqueness of η then follows from (A, IV, p.26, prop.4).

The ring $B = C[\![T_1, \dots, T_n]\!]$ is a Noetherian complete local ring, and $\mathfrak{m}_B$ is generated by $p_1, \dots, p_m, T_1, \dots, T_n$. We then have $\eta(\mathfrak{m}_B) \subseteq \mathfrak{m}_A$ and η is defines a homomorphism $\mathrm{gr}(\eta)$: $\mathrm{gr}(B) \to \mathrm{gr}(A)$. But the ring $\mathrm{gr}(A)$ is generated by $A/\mathfrak{m}_A = \kappa_A$ and $\mathfrak{m}_A/\mathfrak{m}_A$, $\mathrm{gr}(\eta)$ induces an isomorphism of $\kappa_B = \kappa_C$ to κ_A, and the classes modulo $\mathfrak{m}_B^2$ of the elements $p_1, \dots, p_m, T_1, \dots, T_n$ generate $\mathfrak{m}_B/\mathfrak{m}_B^2$ as a κ_B-vector space; moreover η is surjective if and only if $\mathrm{gr}(\eta)$ is surjective (Corollary 2.3.31). These prove (a).

The ideal $\mathfrak{a}$ of A generated by S is equal to $\eta(\mathfrak{m}_B)A$. Since $\mathfrak{m}_A$ contains $\eta(\mathfrak{m}_B)$, A is a Zariski ring for the $\mathfrak{a}$-adic topology. The ring $A/\mathfrak{a}$ is Artinian if and only if its length as an A-module is finite. But as any simple module of A is killed by $\mathfrak{m}_A$ and, by hypothesis, $A/\mathfrak{m}_A$ and $B/\mathfrak{m}_B$ are isomorphic, this happens exactly when the dimension of the vector space $A/\mathfrak{a}$ over $B/\mathfrak{m}_B$ is finite. By Corollary 3.2.23, we see $\mathfrak{a}$ is a defining ideal of A if and only if the dimension of $A/\mathfrak{a}$ over $B/\mathfrak{m}_B$ is finite. This is indeed the case if A is a finite B-algebra. Conversely, suppose then $\mathfrak{a}$ is a defining ideal of A. The $\mathfrak{m}_B$-adic topology on the B-module A then coincides with the $\mathfrak{m}_A$-adic topology on A, hence is separated. Since $A/\mathfrak{a}$ is a finitely generated $B/\mathfrak{m}_B$-module, A is a finitely generated B-module (Corollary 2.3.34). This proves (c). □

Lemma 8.2.19. *Suppose that C is regular and $(p_1, \dots, p_m)$ is a system of parameters for C.*

(a) *If the sequence S is secant for A, the homomorphism $\eta : B \to A$ is injective.*

(b) *For η to be inejctive and make A a finite B-algebra, it is necessary and sufficient that S is a maximal secant sequence for A. In this case A has dimension $m + n$.*

Proof. For the sequence S to be maximal secant for A, it is necessary and sufficient that it generates a defining ideal of A, and in that case A has dimension $m + n$ (Theorem 7.2.14 and Proposition 7.2.16). By Lemma 8.2.18(b), this is equivalent to that A is a finite B-algebra, and has dimension $m + n$. Now C is a Noetherian integral domain with dimension m, so $B = C[\![T_1, \dots, T_n]\!]$ is a Noetherian integral domain with dimension $m + n$. If A is a finite B-algebra, let $\mathfrak{a}$ be the kernel of η, then $\dim(A) = \dim(B/\mathfrak{a})$. Since B is a Noetherian integral domain with finite dimension, we have $\dim(B/\mathfrak{a}) < \dim(B)$ if $\mathfrak{a} \neq \{0\}$. Thus, if A is a finite B algebra, η is injective if and only if A has dimension $m + n$. This proves (b).

Suppose that S is secant for A. We may add elements $t_{n+1}, \dots, t_{n+r}$ of $\mathfrak{m}_A$ to form a maximal secant sequence. By the argument above, there exists an injective homomorphism $\tilde{\eta}$ from $C[\![T_1, \dots, T_n, T_{n+1}, \dots, T_{n+r}]\!] = B[\![T_{n+1}, \dots, T_{n+r}]\!]$ which extends η and sends T_{n+j} to t_{n+j} for $1 \leq j \leq r$. Hence η is injective. This proves (a). □

Theorem 8.2.20 (Cohen Structure Theorem). *Let A be a Noetherian complete local ring with residue field k of characteristic p. Let C be a p-ring of infinite length, whose residue field is isomorphic to k.*

(a) *If n is the dimension of $\mathfrak{m}_A/(\mathfrak{m}_A^2 + pA)$ over k, there exists an ideal $\mathfrak{a}$ of $C[\![T_1, \dots, T_n]\!]$ such that A is isomorphic to $C[\![T_1, \dots, T_n]\!]/\mathfrak{a}$.*

(b) *Let d be the dimension of A. Suppose that $p1_A$ is not a divisor of zero in A. Then there exists a subring B of A isomorphic to $C[\![T_1, \dots, T_{d-1}]\!]$ such that A is a finite B-algebra.*

Proof. Let $\tilde{C}$ be a Cohen subring of A. Since C has infnite length, there exists a homomorphism of C to $\tilde{C}$. Concequently, there exists a local homomorphism $\rho : C \to A$. Choose elements $t_1, \dots, t_n$ of $\mathfrak{m}_A$ whose classes form a basis for $\mathfrak{m}_A/(\mathfrak{m}_A^2 + pA)$ over k. We have $\rho(p1_C) = p1_A$, and Lemma 8.2.18(b) proves the existence of a surjective homomorphism of $C[\![T_1, \dots, T_m]\!]$ to A, which sends T_i to t_i for each i. This proves (a).

Suppose now that $p1_A$ is not a divisor of zero in A, hence secant for A. There exists by Proposition 7.4.9 elements $t_1, \dots, t_{d-1}$ of $\mathfrak{m}_A$ such that $(p1_A, t_1, \dots, t_{d-1})$ is a maximal secant sequence for A. The Noetherian local ring C is regular (it is a DVR), and $p1_C$ is a system of parameters of C. Assertion (b) now follows from Lemma 8.2.19(b). □

8.2.5 Coefficient fields

Let A be a ring. Recall that the characteristic of A is defined if A contains a subfield. It is equal to 0 if and only if A contains a subfield isomorphic to $\mathbb{Q}$, and equal to a prime number p if and only if $p1_A = 0$. If the characteristic of A is defined and $\rho : A \to B$ is a nonzero ring homomorphism, then the characteristic of B is defined and equal to A.

Now let A be a local ring with maximal ideal $\mathfrak{m}$ and residue field k. We distinguish two cases.

(a) Suppose that k has characteristic 0. Then A contains a field and the characteristic of A is 0. In fact, the canonical homomorphism of $\mathbb{Z}$ to A is injective since the canonical map of $\mathbb{Z}$ to k factors through it, and for any nonzero integer n, $n1_A$ is invertible in A since it is not contained in the maximal ideal $\mathfrak{m}$.

(b) Suppose that k has characteristic $p \neq 0$. Then A contains a field if and only if $p1_A = 0$ (since such a field must have characteristic p), and in this case A has characteristic p.

Now assume that A is an integral domain with fraction field K. We now the field K also has a characteristic, which is equal to that of A (if the characteristic of A is defined). Then A contains a subfield if and only if the characteristic of K and k are equal. In this case, we say A is a local ring with **equal characteristic**. We also note that, if the characteristic of k and K are unequal, then k must have characteristic $p \neq 0$ with K characteristic 0. In this case we say A is a local ring with **unequal characteristic**.

Proposition 8.2.21. *Let k_0 be a field, A a k_0-algebra and a separated and complete local ring, k a sub-k_0-extension of κ_A which has a separable transcendental basis $(\xi)_{\lambda\in\Lambda}$ over k_0. For any $\lambda \in \Lambda$, let x_λ be a representative of ξ_λ in A. Then there exists a unique subfield K of A, containing k_0 and the elements x_λ, such that the canonical homomorphism $\pi : A \to \kappa_A$ induces an isomorphism of K to k.*

Proof. Let φ be the k_0-homomorphism of the polynomial ring $k_0[(X_\lambda)_{\lambda\in\Lambda}]$ to A which sends X_λ to x_λ for each λ. Let u be a nonzero element in $k_0[(X_\lambda)_{\lambda\in\Lambda}]$; we have $\pi(\varphi(u)) \neq 0$ since the family $(\xi)_\lambda$ is algebraically independent over k_0 in κ_A, so $\varphi(u)$ is invertible in A. It follows that φ extens to a homomorphism ψ of the field $k_1 = k_0((X_\lambda)_{\lambda\in\Lambda})$ to A. Then A is a k_1- algebra, κ_A is an extension of k_1 and k a subextension of κ_A which is algebraic and separable over k_1. It suffices to prove that there exists a unique subfield K of A containing $\psi(k_1)$ and such that $\pi(K) = k$.

Let $\mathcal{S}$ be the set of subfield K of A, containing $\psi(k_1)$ and such that $\pi(K) \subseteq k$; this is inductive with respect to the inclusion relation. Let K be a maximal element in $\mathcal{S}$; we consider k as an extension (algebraic and separable) of K. Let $\xi \in k$ and $P \in K[X]$ its minimal polynomial over k. Since ξ is a simple root of P, the Hensel lemma (Corollary 2.5.17) then shows the existence of an element $x \in A$ such that $\pi(x) = \xi$ and $P(x) = 0$. The subfield $K(x)$ of A then belongs to $\mathcal{S}$, so $x \in K$ by the maximality of K. This shows $\xi \in \pi(K)$, so $\pi(K) = k$.

We now show the uniqueness of K. Let K and K' be subfields of A containing $\psi(k_1)$ and such that $\pi(K) = \pi(K') = k$. Let $\xi \in k$, and let $x \in K$, $x' \in K'$ be elements such that

$\pi(x) = \pi(x') = \xi$. If $P \in k_1[X]$ is the minimal polynomial of ξ over k_1, then ξ is a simple root of P, and we have $P(x) = P(x') = 0$. By Hensel lemma we then have $x = x'$, so $K = K'$. □

Remark 8.2.22. The previous proof applies more generally to the case where we only assume that A is a Henselian local ring. The uniqueness proof uses the assumption that the local ring A is separate, but not that it is complete.

Let A be a local ring. A **coefficient field** of A is defined to be a subfield K of A such that the canonical homomorphism of A to κ_A induces an isomorphism of K to κ_A (in other words, $A = K + \mathfrak{m}_A$). It is clear that such a field exists only when the characteristic of A is defined. This condition is also sufficient when A is separated and complete. More precisely, we have the following theorem:

Theorem 8.2.23. *Let A be a seperated and complete local ring of equal characteristic p.*

(a) Suppose $p = 0$ and let $(x_\lambda)_{\lambda \in \Lambda}$ be a family of elements of A whose class modulo $\mathfrak{m}_A$ forms a transcendental basis for κ_A over $\mathbb{Q}$. Then there exists a unique coefficient field of A containing the elements x_λ.

(b) Suppose $p \neq 0$ and let $(x_\lambda)_{\lambda \in \Lambda}$ be a family of elements of A whose class modulo $\mathfrak{m}_A$ forms a p-basis for κ_A. Then there exists a unique coefficient field of A containing the elements x_λ, which is a Cohen subring of A.

Proof. Suppose that $p = 0$, so A is a $\mathbb{Q}$-algebra. Since any transcendental basis for κ_A over $\mathbb{Q}$ is separable, the assertion in (a) follows from Proposition 8.2.21 applied to $k_0 = \mathbb{Q}$, $k = \kappa_A$.

Now suppose that $p \neq 0$. Then we have $p1_A = 0$, and any Cohen subring C of A satisfies $pC = 0$. Therefore, the notations of coefficient field and Cohen subring coincide on A. Assertion (b) therefore follows from Theorem 8.2.8. □

Corollary 8.2.24. *Let A be a separated and complete local ring, whose residue field is an algebraic extension of $\mathbb{Q}$. Then there exists a unique coefficient field of A.*

Proof. In fact, the ring A is of characteristic 0, and the empty set is a transcendental basis for κ_A over $\mathbb{Q}$. □

Corollary 8.2.25. *Let A be a separated and complete local ring, with residue field of characteristic $p \neq 0$. Suppose that the residue field κ_A is perfect. Then there exists a unique coefficient field of A, which coincide with the multiplicative representatives.*

Proof. This follows from Theorem 8.2.23(b) and Proposition 8.2.16. □

Theorem 8.2.26. *Let A be a Noetherian complete local ring with dimension d. Let K be a coefficient field of A, and m the dimension of $\mathfrak{m}_A/\mathfrak{m}_A^2$ over K.*

(a) There exists an ideal $\mathfrak{a}$ of $K[\![T_1, \dots, T_m]\!]$ such that A is isomorphic to $K[\![T_1, \dots, T_m]\!]/\mathfrak{a}$.

(b) There exists a sub-K-algebra B of A, isomorphic to $K[\![T_1, \dots, T_d]\!]$, such that A is a finite B-algebra.

(c) If A is regular then there exists an isomorphism of A to $K[\![T_1, \dots, T_d]\!]$.

Proof. Let $t_1, \dots, t_m$ be elements of $\mathfrak{m}_A$ whose classes in $\mathfrak{m}_A$ form a basis for $\mathfrak{m}_A/\mathfrak{m}_A^2$. By Lemma 8.2.18, there exists a surjective K-homomorphism of $K[\![T_1, \dots, T_m]\!]$ to A, sending T_i to t_i. This proves (a). Similarly, (b) follows from Lemma 8.2.19, and the existence of a maximal secant sequence for A. Finally, assertion (c) follows from Corollary 7.4.8. □

8.3 Integral closure of complete local rings

8.3.1 Japanese rings

Let A be a Noetherian integral domain. It is called an **N-1 ring** if the integral closure of A in its fraction field K is a finite A-algebra. It is called an **N-2 ring** (or **Japanese**) if the integral closure of A in any finite extension of K is a finite A-algebra. In other words, A is Japanese if it satisfies the following condition: any integral A-algebra B integral over A, contained in a finite type extension of the field of fractions K of A, is a finite A-algebra. Indeed, the field of fractions L of B is an algebraic extension of K, so is of finite degree over K. The A-algebra B is contained in the integral closure of A in L, and is therefore finite if the latter is finite.

We note that Japanese rings are stable under localization. That is, if A is a Noetherian integral Japanese ring and S a multiplicative subset of A non containing 0, then the fraction ring $S^{-1}A$ is Japanese. This follows from the observation that the integral closure of $S^{-1}A$ is the localization of that of A, and hence is finite over A if A is Japanese.

Example 8.3.1. We recall that, by Theorem 4.3.5, any integral k-algebra of finite type, where k is a field, is a Japanese ring. Note that this is true for the coordinate ring of an algebraic variety.

Proposition 8.3.2. *Let A be a Noetherian integral domain, with fraction field K. Suppose that for any finite purely inseparable extension L of K, the integral closure of A in L is a finite A-algebra. Then A is Japanese.*

Proof. Let E be a finite extension of K. Let N be the normal closure of E over K and L the invariant field of $\mathrm{Aut}_K(N)$. Then L is the purely inseparable closure of K in N and the extension L/N is separable (**??**). The integral closure B of A in L is then by hypothesis a finite A-algebra; the integral closure C of B in E is a finite B-algebra by Corollary 4.1.51, which contains the integral closure of A in E. Since A is Noetherian, this completes the proof. □

Corollary 8.3.3. *Suppose that K is perfect (for example of characteristic 0). Then A is Japanese if and only if it is integrally closed.*

Proposition 8.3.4. *Let B be a Noetherian integral domain and A a Noetherian subring of B, such that B is a finite A-algebra. For A to be Japanese, it is necessary and sufficient that B is Japanese.*

Proof. Let K (resp. L) be the fraction field of A (resp. B). Suppose that A is Japanese, and let E be a finite extension of L. Let C be the integral closure of B in E. By Proposition 4.1.6, C is also the integral closure of A in E, hence is a finite A-algebra (since E is a finite extension of K and A is Japanese). A fortiori, C is a finite B-algebra, this proves B is Japanese.

Conversely, suppose that B is Japanese and let N be a finite extension of K. Let D be the integral closure of A in N. Let E be the composed extension of L and N (over K); since B is Japanese, the integral closure D' of B in E is a finite B-algebra, hence a finite A-algebra; the A-module D is a sub-module of D', so is finite over A. This shows A is Japanese. □

Theorem 8.3.5 (Tate). *Let A be an integrally closed Noetherian ring, a an element of A. Suppose that the ideal aA is prime, the ring A/aA is Japanese and A is complete for the aA-adic topology. Then the ring A is Japanese.*

Proof. Let K be the fraction field of A. The assertion is trivial if K has characteristic 0. We may therefore assume that K has characteristic $p > 0$. Also, suppose that $a \neq 0$. Let L be a finite purely inseparable extension of K and $q = p^e$ a power of p such that $L \subseteq K^{1/q}$. By extending our field, we may assume that $a = x^q$ for some $x \in L$. It then suffices to prove that the integral closure B of A is a finite A-algebra.

We claim that xB is the unique ideal of B lying over aA (such an ideal exists by going up theorem). In fact, let $\mathfrak{P}$ be such an ideal. We have $x^q = a \in \mathfrak{P}$, hence $xB \subseteq \mathfrak{P}$ since $\mathfrak{P}$ is prime.

Conversely, let y be an element of $\mathfrak{P}$; the element y^q of K is integral over A, hence belongs to A (recall that A is integrally closed). Since $\mathfrak{P} \cap A = aA$, there exist $b \in A$ such that $y^q = ab = x^q b$. Then the lement y/x of L is integral over A, hence contained in B; this means $y \in xB$, hence $\mathfrak{P} = xB$.

Recall that the ring B_{xB} is the integral closure of A_{aA} in L. By Proposition 5.2.16, A_{aA} is a DVR; we then deduce from the Krull-Akizuki theorem that B_{xB} is Noetherian (the ring $\kappa(xB)$ is a finite extension of $\kappa(aA)$). The ring B/xB is integral over the Japanese ring A/aA and its fraction field is a finite extension of that of A. Concequently, B/xB is a finitely generated (A/aA)-module. For each integer $i \geq 0$, this is isomorphic to the module $x^i B/x^{i+1}B$; so the (A/aA)-module B/aB possesses a composition sequence of length q with quotients finitely generated (A/aA)-modules, hence is also finitely generated over (A/aA).

Endow the ring A with the aA-adic topology and B the aB-adic topology. Then A is complete by hypothesis; since B_{xB} is a Noetherian integral domain, the aB_{xB}-adic filtration on B_{xB} is separated (Krull intersection theorem); therefore we have $\bigcap_n a^n B \subseteq \bigcap_n a^n B_{xB} = \{0\}$, and the aB-adic filtration on B is separated. The $\mathrm{gr}_{aA}(A)$-module $\mathrm{gr}_{aB}(B)$ is generated by $\mathrm{gr}_0(B)$, hence is finitely generated. It then follows from Corollary 2.3.34 that B is a finitely generated A-module, which completes the proof. □

Corollary 8.3.6. *Let R be a Noetherian integral domain and n an integer. If R is Japanese, so is the ring $R[\![T_1, \dots, T_n]\!]$.*

Proof. Argue by induction, we may assume that $n = 1$. Let S be the integral closure of R. If R is Japanese, S is a finite R-algebra, hence a Japanese ring (Proposition 8.3.4). The ring $S[\![T]\!]$ is Noetherian and integrally closed, so by applying Theorem 8.3.5 on $A = S[\![T]\!]$ and $a = T$, we deduce that $S[\![T]\!]$ is Japanese. Thus $R[\![T]\!]$ is Japanese (Proposition 8.3.4). □

Theorem 8.3.7 (Nagata). *Every Noetherian integral complete local ring is Japanese.*

Proof. By Theorem 8.2.23, there exist an integer $n \geq 0$, a ring R that is a field of a DVR whose fraction field has characteristic 0, and a subring B of A, isomorphic to $R[\![T_1, \dots, T_n]\!]$, such that A is a finite B-algebra. Since R is Japanese, we see B is Japanese, hence so is A. □

A Noetherian semi-local ring A is called **analytically unramified** if its completion $\widehat{A}$ is reduced. A prime ideal $\mathfrak{p}$ of A is said to be **analytically unramified** if $\widehat{A}/\widehat{\mathfrak{p}} = \widehat{A/\mathfrak{p}}$ is reduced. Theorem 8.3.7 then imply the following important result:

Corollary 8.3.8. *Let A be a semi-local Noetherian ring which is analytically unramified. Then the integral closure $\overline{A}$ of A in the total fraction ring $Q(A)$ is a finite A-algebra.*

Proof. Suppose first that A is local and complete, with $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ the minimal prime ideals of A. For each $i = 1, \dots, n$, let K_i be the fraction field of $A/\mathfrak{p}_i$ and $\overline{A_i}$ the integral closure of $A/\mathfrak{p}_i$ in K_i. Since A is reduced, $Q(A)$ is the product of the K_i and $\overline{A}$ is the product of $\overline{A_i}$ (Corollary 4.1.24). Since the local ring $A/\mathfrak{p}_i$ is integral and complete, it is Japanese (Theorem 8.3.7), so $\overline{A_i}$ is a finite A_i-algebra, and therefore $\overline{A}$ is finite over A. If A is semi-local and complete, it is isomorphic to a finite product of complete local rings (Corollary 2.3.28), and we conclude the claim by the previous case.

In the general case, note that the completion $\widehat{A}$ of A is semi-local, complete, Noetherian and faithfully flat (A is a Zariski ring). Let S be the set of elements of A that are not zero divisors; we have $Q(A) = S^{-1}A$. Since $\widehat{A}$ is flat over A, the elements of S are not zero divisors in $\widehat{A}$, and $S^{-1}\widehat{A}$ is identified with a subring of $Q(\widehat{A})$. Moreover, the ring $\overline{A} \otimes_A \widehat{A}$ is identified as a subring of $Q(A) \otimes_A \widehat{A}$, hence a subring T integral over $\widehat{A}$. The preceding arguments then apply and show that $\overline{A} \otimes_A \widehat{A}$ is a finitely generated $\widehat{A}$-module. Therefore, $\overline{A}$ is a finitely generated A-module (**??**). □

Recall that an algebra E over a field K is called separable if the ring $L \otimes_K E$ is reduced for any extension L of K. The following proposition generalizes Theorem 8.3.7.

Proposition 8.3.9. *Let A be a semi-local Noetherian integral ring, K its fraction field. If the K-algebra $K \otimes_A \widehat{A}$ is separable, the ring A is Japanese.*

Proof. Let L be a finite extension of K and B the integral closure of A in L. Let F be a finite subset of B such that $L = K[F]$ (Theorem 4.1.49); let C be the A-algebra generated by F. Then L is the fraction field of C, the ring B is the integral closure of C, and it suffices to prove that B is a finite C-algebra. But C is a Noetherian semi-local ring, with completion identified with $C \otimes_A \widehat{A}$, which is a subring of the reduced ring $L \otimes_A \widehat{A} = L \otimes_K (K \otimes_A \widehat{A})$ and therefore reduced. The proposition then follows from Corollary 8.3.8. □

Finally, before considering Nagata ring, we prove some lemmas which will be used latter.

Lemma 8.3.10. *Let A be a Noetherian semi-local ring and B a finite A-algebra. Then B is Noetherian and semi-local; let $\mathfrak{m}_1, \dots, \mathfrak{m}_n$ be its maximal ideals. The canonical homomorphism of B to $\prod_{i=1}^n \widehat{B}_{\mathfrak{m}_i}$ then extends to an isomorphism of $\widehat{A} \otimes_A B$ to $\prod_{i=1}^n \widehat{B}_{\mathfrak{m}_i}$.*

Proof. By Corollary 3.2.24, the ring B is semi-local and $\mathfrak{m}_A B$ is a defining ideal. By Theorem 2.4.19(b), the ring $\widehat{A} \otimes_A B$ is the completion of B for the topology defined by its Jacobson radical. Then we can apply Corollary 2.3.28. □

Lemma 8.3.11. *Let A be a Noetherian ring and M an A-module. The canonical map of M into $\prod_{\mathfrak{p} \in \mathrm{Ass}(M)} M_{\mathfrak{p}}$ is injective.*

Proof. Let m be a nonzero element in M; then $\mathrm{Ann}(m)$ is contained in a prime ideal $\mathfrak{p}$ in $\mathrm{Ass}(M)$ (Proposition 3.1.3), and the image of m in $M_{\mathfrak{p}}$ is thus nonzero. □

Lemma 8.3.12. *Let A be a Noetherian local ring and $\mathfrak{p}$ a prime ideal of A. Assume that $A_{\mathfrak{p}}$ is a DVR and $\mathfrak{p}$ is analytically unramified, then for any associated prime $\hat{\mathfrak{q}}$ of $\widehat{A}/\mathfrak{p}\widehat{A}$, the local ring $\widehat{A}_{\hat{\mathfrak{q}}}$ is a DVR.*

Proof. Let π be a uniformizer for $A_{\mathfrak{p}}$. Since $\mathfrak{p}$ is analytically unramified, the associated primes of $\widehat{A}/\mathfrak{p}\widehat{A}$ are exactly the minimal primes of $\widehat{A}$ over $\mathfrak{p}\widehat{A}$, and in particular there is no embedded associated primel. Also, since $\widehat{A}/\mathfrak{p}\widehat{A} = \widehat{A/\mathfrak{p}}$ is reduced, we have $\{0\} = \bigcap \hat{\mathfrak{q}}/\mathfrak{p}\widehat{A}$ in $\widehat{A}/\mathfrak{p}\widehat{A}$, where $\hat{\mathfrak{q}}$ runs through minimal primes over $\mathfrak{p}\widehat{A}$. Therefore in $\widehat{A}$ we have

$$\mathfrak{p}\widehat{A} = \bigcap_{\hat{\mathfrak{q}} \in \mathrm{Ass}(\widehat{A}/\mathfrak{p}\widehat{A})} \hat{\mathfrak{q}}$$

which is then the unique primary decomposition of $\mathfrak{p}\widehat{A}$ in $\widehat{A}$. By Proposition 3.2.10, the $\hat{\mathfrak{q}}$-primary component of this decomposition is given by the saturation of $\mathfrak{p}\widehat{A}$ with respect to $\hat{\mathfrak{q}}$ (which is equal to $\hat{\mathfrak{q}}$), so in particular, we have

$$\hat{\mathfrak{q}}\widehat{A}_{\hat{\mathfrak{q}}} = \mathfrak{p}\widehat{A}_{\hat{\mathfrak{q}}} = \pi\widehat{A}_{\hat{\mathfrak{q}}}.$$

Since π is $\widehat{A}$-regular by the flatness of $\widehat{A}$ over A, the local ring $\widehat{A}_{\hat{\mathfrak{q}}}$ is regular of dimension 1, hence a DVR. □

Lemma 8.3.13. *Let A be a Noetherian semi-local integral domain with maximal ideal $\mathfrak{m}$. Let x be a nonzero element in $\mathfrak{m}$ such that*

(a) A/xA has no embedded primes;

(b) for each associated prime $\mathfrak{p}$ of A/xA, the local ring $A_{\mathfrak{p}}$ is regular and $\mathfrak{p}$ is analytically unramified.

Then A is analytically unramified.

Proof. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ be the associated primes of A/xA. Since A/xA has no embedded primes, we see that each $\mathfrak{p}_i$ has height 1, and is a minimal prime over xA. For each i, let $\hat{\mathfrak{q}}_{i,1}, \ldots, \hat{\mathfrak{q}}_{i,n_i}$ be the associated primeds of the $\widehat{A}$-module $\widehat{A}/\mathfrak{p}_i\widehat{A}$. By Lemma 8.3.12 we see that $\widehat{A}_{\hat{\mathfrak{q}}_{i,j}}$ is a DVR for each i, j. Also, since $\widehat{A}$ is flat over A, by Proposition 3.2.28 we have

$$\mathrm{Ass}_{\widehat{A}}(\widehat{A}/x\widehat{A}) = \bigcup_{\mathfrak{p}\in\mathrm{Ass}_A(A/xA)} \mathrm{Ass}_{\widehat{A}}(\widehat{A}/\mathfrak{p}\widehat{A}) = \{\hat{\mathfrak{q}}_{i,j}\}.$$

Let y be a nonzero nilpotent in $\widehat{A}$. Then since $\widehat{A}_{\hat{\mathfrak{q}}_{i,j}}$ is integral for each i, j, the element y is mapped to zero in $\widehat{A}_{\hat{\mathfrak{q}}_{i,j}}$. By Lemma 8.3.11, the canonical image of y in $\widehat{A}/x\widehat{A}$ is then zero, which means $y = xy'$ for some $y' \in \widehat{A}$. But since the homothety with ratio x is injective on A, it is also injective on $\widehat{A}$, which implies $y' \in \widehat{A}$ is also a nonzero nilpotent element. Repeating this process, we conclude that $y \in \bigcap_n x^n\widehat{A} = \{0\}$ (Corollary 2.4.17), which shows $\widehat{A}$ is reduced and A is therefore analytically unramified. $\square$

8.3.2 Nagata rings

A Noetherian ring is called Nagata if for any prime ideal $\mathfrak{p}$ of A, the Noetherian integral domain $A/\mathfrak{p}$ is Japanese. In this part, we give some of the properties of Nagata rings, and a criterion of Noetherian semi-local rings to be Nagata.

Example 8.3.14 (Example of Nagata rings).

(a) Any algebra of finite type over a field is a Nagata ring.

(b) Any Noetherian complete local ring is a Nagata ring by Theorem 8.3.7.

(c) The ring $\mathbb{Z}$ is a Nagata ring since any quotient $\mathbb{Z}/p\mathbb{Z}$ is a perfect field (hence Japanese).

(d) We will see that any algebra of finite type over a Nagata ring is Nagata.

Proposition 8.3.15. *Let A be a Nagata ring.*

(a) Any finite A-algebra is a Nagata ring.

(b) For any multiplicative subset S of A, the ring $S^{-1}A$ is a Nagata ring.

Proof. Let B be a finite A-algebra and $\rho : A \to B$ the homomorphism. For each prime ideal $\mathfrak{P}$ of B, the ring $B/\mathfrak{P}$ is a finite algebra of the Japanese ring $A/\mathfrak{p}^c$, hence is Japanese.

Let S be a multiplicative subset of A and $\mathfrak{q}$ a prime ideal of $S^{-1}A$; then there exist a prime ideal $\mathfrak{p}$ of A such that $\mathfrak{q} = S^{-1}\mathfrak{p}$. The ring $S^{-1}A/\mathfrak{q}$ is a fraction ring of the Japanese ring $A/\mathfrak{p}$, hence is Japanese. $\square$

Theorem 8.3.16 (Nagata). *Let A be a Noetherian semi-local integral domain. If A is a Nagata ring then it is analytically unramified.*

Proof. We use induction on $\dim(A)$. Let B be the integral closure of A in its fraction field K. Then B is finite over A, hence $\dim(A) = \dim(B)$ and for any $\mathfrak{P} \in \mathrm{Spec}(B)$, the domain $B/\mathfrak{P}$ is finite over $A/\mathfrak{p}$ (where $\mathfrak{p} = \mathfrak{P} \cap A$), which is assumed to be Japanese. Thus $B/\mathfrak{P}$ is Japanese and B is therefore a Nagata ring. Moreover, the defining ideal of B is given by the extension of that of A, which implies $\widehat{B} = B \otimes_A \widehat{A}$; since $\widehat{A}$ is a flat A-module, we can then identify $\widehat{A}$ as a subring of $\widehat{B}$. Thus we can replace A by its integral closure and assume that A is integrally closed. In this case, for any nonzero element $x \in A$, the A-module A/xA has no embedded primes by Proposition 6.1.30. If $\mathfrak{p} \in \mathrm{Ass}(A/xA)$, then $A/\mathfrak{p}$ is a Nagata domain and $\dim(A/\mathfrak{p}) < \dim(A)$, hence $\mathfrak{p}$ is analytically unramified by the induction hypothesis. Moreover, $A_\mathfrak{p}$ is a DVR because $\mathrm{ht}(\mathfrak{p}) = 1$ (Theorem 6.1.29 and Proposition 6.1.30). So the conditions of Lemma 8.3.13 are satisfied, and A is analytically unramified. $\square$

Theorem 8.3.17 (Zariski-Nagata). *Let A be a Noetherian semi-local ring. The following conditions are equivalent:*

(i) A is a Nagata ring;

(ii) for any prime ideal $\mathfrak{p}$ of A, the $\kappa(\mathfrak{p})$-algebra $\kappa(\mathfrak{p})\otimes_A\widehat{A}$ is separable;

(iii) for any reduced A-algebra R, the ring $R\otimes_A\widehat{A}$ is reduced.

Proof. We first demonstrate the equivalence of (ii) and (iii). The implication (iii)⇒(ii) is trivial; suppose conversely that A satisfies (ii). Then for any A-algebra K that is a field, the ring $K\otimes_A\widehat{A}$ is reduced since it is isomorphic to $K\otimes_{\kappa(\mathfrak{p})}\kappa(\mathfrak{p})\otimes_A\widehat{A}$. Now let C be a reduced A-algebra of finite type, then the ring C, being Noetherian, is isomorphic to a subring of a finite product $K_1\times\cdots\times K_n$ of fields (Proposition 3.2.26). Since $\widehat{A}$ is flat over A, the ring $C\otimes_A\widehat{A}$ is isomorphic to a subring of the reduced ring $\prod_i(K_i\otimes_A\widehat{A})$, hence is reduced. Finally, since any reduced A-algebra R is the union of a family of finite type A-algebras (C_α), we see $R\otimes_A\widehat{A}$ is reduced (it is the inductive limit of the family $(C_\alpha\otimes_A\widehat{A})$).

We now show that (ii)⇔(i). Let $\mathfrak{p}$ be a prime ideal of A. The fraction field K of $A/\mathfrak{p}$ is then identified with $\kappa(\mathfrak{p})$, and the K-algebra $K\otimes_{A/\mathfrak{p}}\widehat{A/\mathfrak{p}}$ is identified with $\kappa(\mathfrak{p})\otimes_{A/\mathfrak{p}}\widehat{A}/\mathfrak{p}\widehat{A}$, hence to $\kappa(\mathfrak{p})\otimes_A\widehat{A}$. If $\kappa(\mathfrak{p})\otimes_A\widehat{A}$ is a separable $\kappa(\mathfrak{p})$-algebra, then $A/\mathfrak{p}$ is Japanese by Proposition 8.3.9.

Conversely, we prove (i)⇒(ii). Let A be a semi-local Noetherian Nagata ring, let $\mathfrak{p}$ be an ideal of A, and L a finite extension of $\kappa(\mathfrak{p})$. It suffices to show that the ring $L\otimes_A\widehat{A}$ is reduced. Let B by the integral closure of $A/\mathfrak{p}$ in L; since $A/\mathfrak{p}$ is Japanese, B is a finite A-algebra hence a semi-local Nagata ring (Proposition 8.3.15). Let $\mathfrak{m}_1,\dots,\mathfrak{m}_r$ be the maximal ideal of B; the ring $L\otimes_A\widehat{A}$ is identified with a fraction ring of $B\otimes_A\widehat{A}$, and with that of a product of complete local rings $\widehat{B}_{\mathfrak{m}_i}$ (Lemma 8.3.10). It then suffices to prove that, for any maximal ideal $\mathfrak{m}$ of B, the ring $\widehat{B}_{\mathfrak{m}}$ is reduced (Corollary 1.2.39). But the ring $B_{\mathfrak{m}}$ is a Noetherian local ring that is Nagata (Proposition 8.3.15), so by Theorem 8.3.16 it is analytically unramified, which is what we want. □

Corollary 8.3.18 (Chevally). *Let A be a reduced algebra of finite type over a field, and $\mathfrak{p}$ a prime ideal of A. Then the local ring $A_{\mathfrak{p}}$ is analytically unramified.*

Proof. Since A is reduced, the local ring $A_{\mathfrak{p}}$ is reduced. The ring A is Nagata, hence $A_{\mathfrak{p}}$ is also Nagata, and Theorem 8.3.16 implies the claim. □

Corollary 8.3.19. *Let k be a field of characteristic 0 and A a Noetherian local k-algebra. Then A is a Nagata ring if and only if every prime ideal $\mathfrak{p}$ of A is analytically unramified.*

Proof. In fact, since the residue field $\kappa(\mathfrak{p})$ has characteristic 0, the $\kappa(\mathfrak{p})$-algebra $\kappa(\mathfrak{p})\otimes_A\widehat{A}=\kappa(\mathfrak{p})\otimes_{A/\mathfrak{p}}\widehat{A/\mathfrak{p}}$ is separable if and only if it is reduced, which shows that the stated condition is sufficient (Theorem 8.3.17, not that $\kappa(\mathfrak{p})\otimes_{A/\mathfrak{p}}\widehat{A/\mathfrak{p}}$ is a localization of $\widehat{A/\mathfrak{p}}$). It is also necessary by take $R=A/\mathfrak{p}$ in Theorem 8.3.17(iii). □

A Noetherian ring A is called **universally Japanese** if any integral A-algebra B of finite type over A is Japanese. It is clear that such a ring is Nagata, since $A/\mathfrak{p}$ is a finite type A-algebra for any prime ideal $\mathfrak{p}$ of A. We now prove the reverse direction, which is not at all obvious. For this, we first establish some lemmas.

Lemma 8.3.20. *Let A be a Noetherian integral domain and put $X=\operatorname{Spec}(A)$. Suppoe that there exist nonzero element f in A such that A_f is integrally closed, then the set*

$$\operatorname{Nor}(X)=\{\mathfrak{p}\in\operatorname{Spec}(A):A_{\mathfrak{p}}\ \textit{is integrally closed}\}$$

is open in X.

Proof. If $f \notin \mathfrak{p}$ for some $\mathfrak{p} \in X$, then $A_\mathfrak{p}$ is a localization of A_f, hence integrally closed. This shows $D(f) \subseteq \mathrm{Nor}(X)$. Now we put

$$E = \{\mathfrak{p} \in \mathrm{Ass}_A(A/fA) : \text{either } \mathrm{ht}(\mathfrak{p}) = 1 \text{ and } A_\mathfrak{p} \text{ is not regular, or } \mathrm{ht}(\mathfrak{p}) > 1\}.$$

Since $\mathrm{Ass}_A(A/fA)$ is finite, it is clear that E is a finite subset of X. On the other hand, by Serre's criterion for normality, if $\mathfrak{p} \notin \mathrm{Nor}(X)$ then there exist a prime $\mathfrak{q} \subseteq \mathfrak{p}$ such that either $\mathrm{ht}(\mathfrak{q}) \geq 2$ and $\mathrm{depth}(A_\mathfrak{q}) < 2$, or $\mathrm{ht}(\mathfrak{q}) = 1$ and $A_\mathfrak{q}$ is not regular. This in particular also means that $A_\mathfrak{q}$ is not integrally closed, and hence $f \in \mathfrak{q}$. In both cases we see $\mathfrak{q}$ is an associated prime of A/fA. It then follows that $\mathrm{Nor}(X) = X \setminus \bigcup_{\mathfrak{p} \in E} V(\mathfrak{p})$, so $\mathrm{Nor}(X)$ is open in X. □

Lemma 8.3.21. *Let A be a Noetherian domain with quotient field K. Then A is* N-1 *if and only if the following conditions hold:*

(a) there exists a nonzero element $f \in A$ such that A_f is integrally closed;

(b) $A_\mathfrak{m}$ is N-1 *for each maximal ideal $\mathfrak{m}$ of A.*

Proof. First assume that A is N-1. Let $\overline{A}$ be the integral closure of A in its fraction field K. By assumption we can find $x_1, \dots, x_n$ in $\overline{A}$ which generate it as an A-module. Since $\overline{A} \subseteq K$, we may further assume that $x_i = a_i/b$ for $a_i, b \in A$. Then $A_f \cong \overline{A}_f$, where the latter is the integral closure of the former. This shows A_f is integrally closed, hence (a) holds. It is clear that (b) follos, since the integral closure of $A_\mathfrak{m}$ is just $\overline{A}_\mathfrak{m}$.

Conversely, assume (a), (b) for A and let K be the fraction field of A. For a subring B of K that is finite over A, we set $X = \mathrm{Spec}(A)$, $Y = \mathrm{Spec}(B)$. Note that $A_f = B_f$ since A_f is integrally closed, so by Lemma 8.3.20 the set $\mathrm{Nor}(Y)$ is open in Y. Since the canonical map $\bar{X} \to X$ is closed (Proposition 4.1.84), the image of the complement of $\mathrm{Nor}(Y)$ is then closed in X, we denote it by Z_B for such a ring B.

Pick a maximal ideal $\mathfrak{m}$ of A and let $A_\mathfrak{m} \subseteq \overline{A_\mathfrak{m}}$ be the integral closure of the local ring in K. By assumption, this is a finite ring extension, so we can find finitely many elements $x_1, \dots, x_n$ in $\overline{A}$ such that $\overline{A}_\mathfrak{m}$ is generated by $x_1, \dots, x_n$ over $A_\mathfrak{m}$. Let $B(\mathfrak{m}) = A[x_1, \dots, x_n] \subseteq K$, then $B(\mathfrak{m})$ is finite over A, hence is Noetherian. Let $\mathfrak{M}$ be any prime ideal of $B(\mathfrak{m})$ lying over $\mathfrak{m}$, then

$$B(\mathfrak{m})_\mathfrak{M} \supset B(\mathfrak{m})_\mathfrak{m} \supset B(\mathfrak{m}),$$

and $B(\mathfrak{m})_\mathfrak{m} = \overline{A}_\mathfrak{m}$ is integrally closed. Thus $B(\mathfrak{m})_\mathfrak{M}$ is a localization of the integrally closed ring $\overline{A}_\mathfrak{m}$, so is integrally closed. This shows $\mathfrak{m} \notin Z_{B(\mathfrak{m})}$, so the intersection $\bigcap_{\mathfrak{m} \in \mathrm{Max}(A)} Z_{B(\mathfrak{m})}$ contains no closed point, and is therefore empty (since it is a closed subset of X). As X is quasi-compact, we can choose maximal ideals $\mathfrak{m}_1, \dots, \mathfrak{m}_r$ such that $\bigcap_i Z_{B(\mathfrak{m}_i)} = \varnothing$. Put $B_i = B(\mathfrak{m}_i)$ and let C be the ring generated by all of these; it is finite over A. We claim that $Z_C = \varnothing$, in other words, $C_\mathfrak{Q}$ is integrally closed for any $\mathfrak{Q} \in \mathrm{Spec}(C)$. In fact, the contraction of any prime $\mathfrak{Q}$ of C into A is not contained in some Z_{B_i}, so it lies over a prime $\mathfrak{P}_i$ of B_i such that $(B_i)_{\mathfrak{P}_i}$ is integrally closed. This implies $C_\mathfrak{Q} = (B_i)_{\mathfrak{P}_i}$ is integrally closed too, so C is integrally closed (Proposition 4.1.42). In other words, C is the integral closure of A in K, so A is N-1. □

Theorem 8.3.22 (Nagata). *Let A be a Noetherian ring. The following are equivalent:*

(i) A is a Nagata ring;

(ii) any finite type A-algebra B is Nagata;

(iii) A is universally Japanese.

Proof. It is clear that a Noetherian universally Japanese ring is universally Nagata (i.e., condition (ii) holds). Let A be a Noetherian Nagata ring. We will show that any finite type A-algebra B is Nagata. Note that the canonical image of A in B is also a Nagata ring, so we may assume that A is a subring of B. Then $B = A[x_1, \ldots, x_n]$ with $x_i \in B$, and by induction it suffices to consider the case $B = A[x]$.

Let $\mathfrak{P}$ be a prime ideal of B and $\mathfrak{p} = \mathfrak{P}^c$. Then $B/\mathfrak{P} = (A/\mathfrak{p})[\bar{x}]$, where $A/\mathfrak{p}$ is a Nagata domain (Proposition 8.3.15), and we must show that $B/\mathfrak{P}$ is Japanese. We are then reduced to prove the following assertion:

(A1) If A is a Noetherian Nagata domain and $B = A[x]$ is an integral domain generated by a single x over A, then B is Japanese.

Let K be the fraction field of A and $\overline{A}$ the integral closure of A in K. Let $\bar{B}$ be the subring of the fraction field of B generated by $\overline{A}$ and B. As $\overline{A}$ is finite over A (by the Nagata property, since A is an integral domain), $\bar{B}$ is finite over B. Since B is Noetherian, it then suffices to prove that $\bar{B}$ is Japanese (Proposition 8.3.4). Hence we can add the integrally closed assumption in (A1):

(A2) If A is an integrally closed Noetherian Nagata domain and $B = A[x]$ is an integral domain generated by a single x over A, then B is Japanese.

In this case, let L be the fraction field of B. We first note that, if x is transcendental over A, then $B = A[x]$ is isomorphic to the polynomial ring $A[X]$, hence is integrally closed. Thus if L has characteristic 0, we are done (Corollary 8.3.3). Suppose otherwise L has characteristic $p > 0$, and take a finite purely inseparable extension E of $L = K(x)$. Let $q = p^e$ be a power of p such that $E \subseteq L^{1/q}$. Then there exists a finite purely inseparable extension K' of K such that $E \subseteq K'(x^{1/q})$. If $\overline{A}$ (resp. $\bar{B}$) is the integral closure of A in K' (resp. of B in L), then $\overline{A}[x^{1/q}]$ is normal and we have $B = A[x] \subseteq \bar{B} \subseteq \overline{A}[x^{1/q}]$. Since $\overline{A}[x^{1/q}]$ is finite over B, we conclude that $\bar{B}$ is finite over B.

Now assume that x is algebraic over A, so that L/K is a finite extension. Let E be a finite extension of L (hence finite over K) and $\overline{A}$ be the integral closure of A in E. Then the integral closure $\bar{B}$ of B in E is equal to the integral closure of $\overline{A}[x]$ in E. Also the fraction field of $\overline{A}$ is E and $\overline{A}$ is a finite A-algebra (A is Japanese). This implies that $\overline{A}$ is a Nagata ring (Proposition 8.3.15). To show that $\bar{B}$ is finite over B is the same as showing that $\bar{B}$ is finite over $\overline{A}[x]$, so by replacing A by $\overline{A}$ and B by $\bar{B}[x]$, we are reduced to prove the following statement:

(A3) If A is an integrally closed Noetherian Nagata domain with fraction field K and x is an element of K, the ring $B = A[x]$ is N-1.

We now proceed with proving (A3). Since $x \in K$, we can write $x = a/b$ where $a, b \in A$. Then $B_a = B[1/a] = A[1/a]$ is integrally closed since it is a localization of A. By Lemma 8.3.21, it then suffices to prove that $B_{\mathfrak{M}}$ is N-1 for any maximal ideal $\mathfrak{M}$ of B. Now pick such a maximal ideal and set $\mathfrak{m} = \mathfrak{M} \cap A$. The residue field extension $\kappa(\mathfrak{m})/\kappa(\mathfrak{M})$ is finite and generated by the image of x. Hence there exists a monic polynomial $f(X) \in A[X]$ with $f(x) \in \mathfrak{M}$. Let E/K be a finite extension of fields such that $f(X)$ splits completely in $E[X]$. Let $\overline{A}$ be the integral closure of A in E, and $\bar{B} \subseteq E$ be the subring generated by $\overline{A}$ and x. As A is a Nagata ring, we see $\overline{A}$ is finite over A and hence Nagata (Proposition 8.3.15). Moreover, $\bar{B}$ is finite over B. If for every maximal ideal $\overline{\mathfrak{M}}$ of $\bar{B}$ the local ring $\bar{B}_{\overline{\mathfrak{M}}}$ is N-1, then $\bar{B}_{\overline{\mathfrak{M}}}$ is N-1 by Lemma 8.3.21, which in turn implies that $B_{\mathfrak{M}}$ is N-1 (by a proof similar to that of Proposition 8.3.4). Thus, after replacing A by $\overline{A}$ and B by $\bar{B}$, and $\mathfrak{M}$ by any of the maximal ideals $\overline{\mathfrak{M}}$ lying over $\mathfrak{M}$, we reach the situation where the polynomial $f(X)$ above split completely: $f(X) \prod_{i=1}^{d}(X - a_i)$ with $a_i \in A$. Since $f(x) \in \mathfrak{M}$ we see that $x - a_i \in \mathfrak{M}$ for some i. Finally, after replacing x by $x - a_i$ we may assume that $x \in \mathfrak{M}$.

We will show that Lemma 8.3.13 applies to the local ring $B_{\mathfrak{M}}$ and the element x. This will imply that $B_{\mathfrak{M}}$ is analytically unramified, whence it is N-1 by Corollary 8.3.8. Let Q be the kenrel of the canonical map $A[X] \to B$, where X is mapped to x. Then by Exercise 7.1.14, the ideal Q is generated by all linear forms $aX - b$, where $ax = b$ in K. Therefore, we see that

$$B/xB = A[X]/(x, \mathfrak{a}) = A/I$$

where I is the constant term of the elements in Q, or in other words $I = xA \cap A$. Now note that A is a Noetherian integrally closed domain and I is divisorial, so by Proposition 6.1.22, if $\operatorname{div}(xA \cap A) = \sum_i n_i P_i$ with $\mathfrak{p}_i$ the corresponding prime of P_i, then

$$I = xA \cap A = \bigcap_i \mathfrak{p}_i^{(n_i)}$$

is the unique reduced primary decomposition of I and the $\mathfrak{p}_i$ are isolated primes of I. This shows $A/I = B/xB$ has no embedded primes, so does $B_{\mathfrak{M}}$, in view of Proposition 3.1.11.

Now let $\mathcal{P} \in \operatorname{Spec}(B_{\mathfrak{M}})$ be an associated prime of $B_{\mathcal{P}}/xB_{\mathfrak{M}}$. Then $\mathfrak{p} = \mathcal{P} \cap A$ is an associated prime of $A/(xB_{\mathfrak{M}} \cap A) = A/I$ and $\operatorname{ht}(\mathfrak{p}) = 1$ by Proposition 6.1.30, so $(B_{\mathfrak{M}})_{\mathcal{P}} = A_{\mathfrak{p}}$ is a DVR (Theorem 6.1.29), hence regular. Finally, $B_{\mathfrak{M}}/\mathcal{P}$ is a localization of $B/(\mathcal{P} \cap B)$ (which is isomorphic to $A/\mathfrak{p}$ since $x \in \mathfrak{p}$), so $B_{\mathfrak{M}}/\mathcal{P}$ is a Noetherian Nagata local domain, hence analytically unramified. Thus the conditions in Lemma 8.3.13 are verified and our proof is completed. $\square$

Chapter 9

Depth, regularity, and duality

9.1 Depth of modules

9.1.1 Homological definition of depth

Let A be a ring, $\mathfrak{I}$ an ideal of A, and M an A-module. We define the **depth of M relative to $\mathfrak{I}$**, denoted by $\operatorname{depth}_A(\mathfrak{I}, M)$ or $\operatorname{depth}(\mathfrak{I}, M)$, to be the infimum in $\mathbb{N} \cup \{+\infty\}$ of the set of integers i such that $\operatorname{Ext}^i_A(A/\mathfrak{I}, M)$ is nonzero. If A is a local ring, the depth of M relative to the maximum ideal $\mathfrak{m}_A$ of A is simply called the **depth of M** and denoted by $\operatorname{depth}_A(M)$ or $\operatorname{depth}(M)$; the **depth of the local ring A** is defined to be the depth of the A-module A.

Example 9.1.1. Let M be a fnitely generated A-module and such that $M = \mathfrak{I}M$, which means $\operatorname{supp}(M) \cap V(\mathfrak{I}) = \varnothing$ (Corollary 1.4.40). In this case, $\operatorname{depth}_A(\mathfrak{I}, M)$ is equal to $+\infty$: in fact, the ideal $\operatorname{Ann}(M) + \mathfrak{I}$ is equal to A (since $V(\operatorname{Ann}(M) + \mathfrak{I}) = \varnothing$) and is contained in the annihilator of $\operatorname{Ext}^i_A(A/\mathfrak{I}, M)$ for each i. Conversely, we shall see that (Corollary 9.1.13) if $\mathfrak{I}$ is a finitely generated ideal, then $\operatorname{depth}_A(\mathfrak{I}, M) = +\infty$ implies $M = \mathfrak{I}M$.

Example 9.1.2. For $\operatorname{depth}_A(\mathfrak{I}, M)$ to be zero, it is necessary and sufficient that $\operatorname{Hom}_A(A/\mathfrak{I}, M)$ is zero, which signifies that M has a nonzero element annihilated by $\mathfrak{I}$. This in particular implies that $\operatorname{Ass}(M) \cap V(\mathfrak{I}) \neq \varnothing$ (Proposition 3.1.3). If A is Noetherian and the A-module M is finitely generated, the following conditions are equivalent (Corollary 3.1.19):

(i) $\operatorname{depth}_A(\mathfrak{I}, M) = 0$;

(ii) for any $x \in \mathfrak{I}$, the homothety x_M is not injective;

(iii) $\operatorname{Ass}(M) \cap V(\mathfrak{I}) \neq \varnothing$.

In particular, for a local ring to have zero depth, it is necessary and sufficient that there exist a nonzero element x of A such tha t$\mathfrak{m}_A x = 0$. If A is not a field, such an element is then not invertible, hence belongs to $\mathfrak{m}_A$ and satisfies $x^2 = 0$. Therefore, any reduced local ring of dimension ≥ 1 has depth ≥ 1.

Example 9.1.3. Let $\{M_i\}_{i \in I}$ be a family of A-module and $\mathfrak{I}$ be an ideal of A. Then in view of (A, X p.89, prop.7), we have $\operatorname{depth}_A(\mathfrak{I}) = \inf_{i \in I} \operatorname{depth}_A(\mathfrak{I}, M_i)$.

Proposition 9.1.4. *Let A be a ring, $\mathfrak{I}$ an ideal of A, and let*

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

be an exact sequence of A-modules. Then there are exactly three cases:

(i) $\operatorname{depth}(\mathfrak{I}, M) = \operatorname{depth}(\mathfrak{I}, M') \leq \operatorname{depth}(\mathfrak{I}, M'')$;

(ii) $\operatorname{depth}(\mathfrak{I}, M) = \operatorname{depth}(\mathfrak{I}, M'') \leq \operatorname{depth}(\mathfrak{I}, M')$;

(iii) $\operatorname{depth}(\mathfrak{I}, M'') = \operatorname{depth}(\mathfrak{I}, M') - 1 < \operatorname{depth}(\mathfrak{I}, M'')$;

Proof. Consider the long exact sequence associated with this short exact sequence:

$$0 \to \operatorname{Ext}^0_A(A/\mathfrak{I}, M') \to \operatorname{Ext}^0_A(A/\mathfrak{I}, M) \to \operatorname{Ext}^0_A(A/\mathfrak{I}, M'') \to \operatorname{Ext}^1_A(A/\mathfrak{I}, M') \to \cdots$$

We may exclude the case $\operatorname{depth}(\mathfrak{I}, M') = \operatorname{depth}(\mathfrak{I}, M) = \operatorname{depth}(\mathfrak{I}, M') = +\infty$. There then exists in this sequence a first non-zero module, and the next module is thus nonzero. The three cases then correspond to which module is the first to be nonzero. □

Remark 9.1.5. Suppose that we have $\operatorname{depth}(\mathfrak{I}, M') = \operatorname{depth}(\mathfrak{I}, M)$ and the injection $u : M' \to M$ belongs to $\mathfrak{I}\operatorname{Hom}_A(M', M)$ (which is equivalent to that $1_{A/\mathfrak{I}} \otimes u$ is zero). We then have $\operatorname{depth}(\mathfrak{I}, M'') = \operatorname{depth}(\mathfrak{I}, M) - 1$: in fact, the hypotheses implies that $\operatorname{Ext}^i_A(1_{A/\mathfrak{I}}, u) = 0$ for any integer i, which means the homomorphism $\operatorname{Ext}^i_A(A/\mathfrak{I}, M'') \to \operatorname{Ext}^i_A(A/\mathfrak{I}, M'')$ is surjective.

Proposition 9.1.6. *Let A be a ring, $\mathfrak{I}$ an ideal of A, M an A-module and N an A-module killed by a power of $\mathfrak{I}$. Then $\operatorname{Ext}^i_A(N, M) = 0$ for any integer $i < \operatorname{depth}_A(\mathfrak{I}, M)$.*

Proof. We first assume that $\mathfrak{I}N = 0$ and prove by induction on the integer $i < \operatorname{depth}_A(\mathfrak{I}, M)$. The assertion is clear for $i < 0$; now consider N as a $(A/\mathfrak{I})$-module and choose an exact sequence of $(A/\mathfrak{I})$-modules

$$0 \longrightarrow K \longrightarrow (A/\mathfrak{I})^{\oplus I} \longrightarrow N \longrightarrow 0$$

We then deduce a long exact sequence of extension modules

$$\cdots \to \operatorname{Ext}^{i-1}_A(K, M) \to \operatorname{Ext}^i_A(N, M) \to \operatorname{Ext}^i_A((A/\mathfrak{I})^{\oplus I}, M) \to \operatorname{Ext}^i_A(K, M) \to \cdots$$

The A-module $\operatorname{Ext}^{i-1}_A(K, M)$ is zero by induction hypothesis, and $\operatorname{Ext}^i_A((A/\mathfrak{I})^{\oplus I}, M)$ is isomorphic to $\operatorname{Ext}^i_A((A/\mathfrak{I}), M)^{\oplus I}$ (A, X, p.89, prop.7), which is zero by the definition of $\operatorname{depth}_A(\mathfrak{I}, M)$. We therefore conclude that $\operatorname{Ext}^i_A(N, M) = 0$.

Passing to the general case, we preceed by induction on the smallest number $m > 0$ such that $\mathfrak{I}^m N = 0$. We have proved the case $m = 1$, so assume that $m > 1$ and let $i < \operatorname{depth}_A(\mathfrak{I}, M)$ be an integer. Consider the sequence

$$\operatorname{Ext}^i_A(N/\mathfrak{I}N, M) \longrightarrow \operatorname{Ext}^i_A(N, M) \longrightarrow \operatorname{Ext}^i_A(\mathfrak{I}N, M)$$

The extremal modules are zero by induction hypothesis, since $N/\mathfrak{I}N$ and $\mathfrak{I}N$ are annihilated by $\mathfrak{I}^{m-1}$. We then deduce that $\operatorname{Ext}^i_A(N, M) = 0$, which completes the proof. □

Corollary 9.1.7. *Let $m > 0$ be an integer and $\mathfrak{I}'$ be an ideal of A contained in $\mathfrak{I}^m$. Then for any A-module M we have $\operatorname{depth}_A(\mathfrak{I}, M) \leq \operatorname{depth}_A(\mathfrak{I}', M)$.*

Proof. In fact, $\mathfrak{I}^m$ annihilates the A-module $A/\mathfrak{I}'$, so $\operatorname{Ext}^i_A(A/\mathfrak{I}', M) = 0$ for $i < \operatorname{depth}_A(\mathfrak{I}, M)$ by Proposition 9.1.6. □

Corollary 9.1.8. *Suppose that the ideal $\mathfrak{I}$ is finitely generated, and let $\mathfrak{I}'$ be an ideal of A such that $V(\mathfrak{I}') \subseteq V(\mathfrak{I})$. Then we have $\operatorname{depth}_A(\mathfrak{I}, M) \leq \operatorname{depth}_A(\mathfrak{I}', M)$. Moreover, if the ideal $\mathfrak{I}'$ is also finitely generated and $V(\mathfrak{I}) = V(\mathfrak{I}')$, then $\operatorname{depth}_A(\mathfrak{I}, M) \leq \operatorname{depth}_A(\mathfrak{I}', M)$.*

Proof. By Proposition 1.4.3, there exists an integer $m > 0$ such that $\mathfrak{I}^m \subseteq \mathfrak{I}'$. The first assertion therefore follows from Corollary 9.1.7, and the second one follows also from this. □

We conclude this paragraph by a cohomological application of depth. This result can also be servered as our motivation of defining the depth via extension module.

Proposition 9.1.9. *Let A be a ring, $C_\bullet$ be a right bounded complex of A-modules, and n_0 be an integer. Suppose that for any integers $m \geq n \geq n_0$, the depth of the A-module C_m relative to the annihilator of $H_n(C_\bullet)$ is strictly bigger than $m-n$, then we have $H_n(C_\bullet) = 0$ for $n \geq n_0$.*

Proof. Since $C_\bullet$ is right bounded, we have $H_n(C_\bullet) = 0$ for $n \gg 0$. If the conclusion is false, then there exists an integer $m \geq n_0$ such that $H_n(C_\bullet) = 0$ for $n > m$ and $H_m(C_\bullet) \neq 0$. We denote by $\mathfrak{I}$ the annihilator of $H_m(C_\bullet)$, so that $\operatorname{depth}_A(\mathfrak{I}, H_m(C_\bullet)) = 0$ by Example 9.1.2. Moreover, since $Z_m(C_\bullet)$ is a submodule of C_m, and by hypothesis we have $\operatorname{depth}_A(\mathfrak{I}, C_m) > m - m = 0$, we conclude that $\operatorname{depth}_A(\mathfrak{I}, Z_m(C_\bullet)) > 0$. From the exact sequence

$$0 \longrightarrow B_m(C_\bullet) \longrightarrow Z_m(C_\bullet) \longrightarrow H_m(C_\bullet) \longrightarrow 0$$

we then deduce that $\operatorname{depth}_A(\mathfrak{I}, B_m(C_\bullet)) = 1$ (Proposition 9.1.4). Now by the definition of m, we have $B_n(C_\bullet) = Z_n(C_\bullet) = 0$ for $n > m$, so from the canonical exact sequences

$$0 \longrightarrow B_n(C_\bullet) \longrightarrow C_n \longrightarrow B_{n-1}(C_\bullet) \longrightarrow 0$$

and the hypothesis $\operatorname{depth}_A(\mathfrak{I}, C_n) > n - m$, we derive by induction that $\operatorname{depth}_A(\mathfrak{I}, B_n(C_\bullet)) = n - m + 1$ for any $n \geq m$ (Proposition 9.1.4). But this is absurd since $B_n(C_\bullet) = 0$ for $n \gg 0$. □

Corollary 9.1.10. *Let A be a ring, $\mathfrak{I}$ be an ideal of A, $C_\bullet$ be a right bounded complex of A-modules, and n_0 be an integer. Suppose that $\mathfrak{I}H_n(C_\bullet) = 0$ and $\operatorname{depth}_A(\mathfrak{I}, C_n) > n - n_0$ for $n \geq n_0$. Then we have $H_n(C_\bullet) = 0$ for $n \geq n_0$.*

Proof. In fact, for $n \geq n_0$ the annihilator $\mathfrak{I}_n$ of $H_n(C_\bullet)$ contains $\mathfrak{I}$, so we have $\operatorname{depth}_A(\mathfrak{I}_n, C_m) \geq \operatorname{depth}_A(\mathfrak{I}, C_m)$ for $m \geq n \geq n_0$ (Corollary 9.1.7), and Proposition 9.1.9 then proves the corollary. □

Corollary 9.1.11. *Let A be a local ring with maximal $\mathfrak{m}$, $C_\bullet$ be a right bounded complex of A-modules, and n_0 be an integer. Suppose that for $n \geq n_0$, $H_n(C_\bullet)$ is of finite length and $\operatorname{depth}_A(\mathfrak{m}, C_n) > n - n_0$. Then we have $H_n(C_\bullet) = 0$ for $n \geq n_0$.*

Proof. The A-module $\bigoplus_{n \geq n_0} H_n(C_\bullet)$ is then of finite length. Let $\mathfrak{I}$ be its annihilator, then $A/\mathfrak{I}$ is Artinian, so $\mathfrak{I}$ is contained in a power of the maximal ideal $\mathfrak{m}$ of A. We then have $\operatorname{depth}_A(\mathfrak{I}, C_n) \geq \operatorname{depth}_A(\mathfrak{m}, C_n) > n - n_0$ (Corollary 9.1.7), so we can apply Proposition 9.1.9. □

9.1.2 Depth and Koszul complex

Let A be a ring, M be an A-module, and $\boldsymbol{x} = (x_i)_{i \in I}$ be a family of elements of A. Let $K^\bullet(\boldsymbol{x}, M)$ be the Koszul complex associated with $\boldsymbol{x}$ and M. By definition, we have $K^p(\boldsymbol{x}, M) = 0$ for $p < 0$, and for $p \geq 0$ the A-module $K^p(\boldsymbol{x}, M) = \operatorname{Hom}_A(\bigwedge^p A^{\oplus I}, M)$ is canonically identified with the A-module $C^p(M)$ formed by alternating maps from I^p to M, with differential $\partial^p : K^p(\boldsymbol{x}, M) \to K^{p+1}(\boldsymbol{x}, M)$ given by the formula

$$(\partial^p m)(\alpha_1, \dots, \alpha_{p+1}) = \sum_{j=1}^{p+1} (-1)^{j+1} x_{\alpha_j} \cdot m(\alpha_1, \dots, \hat{\alpha}_i, \dots, \alpha_{p+1})$$

where $m \in K^p(\boldsymbol{x}, M)$ and $(\alpha_1, \dots, \alpha_{p+1}) \in I^{p+1}$. In particular, the complex $K^\bullet(\boldsymbol{x}, M)$ only depends on the $\mathbb{Z}$-module structure on M and the endomorphisms $(x_i)_M$.

We denote by $H^\bullet(\boldsymbol{x}, M)$ the cohomology of the complex $K^\bullet(\boldsymbol{x}, M)$. By definition, the A-module $H^0(\boldsymbol{x}, M)$ is identified with $\operatorname{Hom}_A(A/\mathfrak{I}, M)$, where $\mathfrak{I}$ is the ideal generated by the x_i.

Theorem 9.1.12. *Let A be a ring, $\mathfrak{I}$ be an ideal of A, $\boldsymbol{x} = (x_i)_{i \in I}$ be a family of generator of $\mathfrak{I}$, and M be an A-module. Then the depth of M relative to $\mathfrak{I}$ is the infermum of integers i such that $H^i(\boldsymbol{x}, M) \neq 0$.*

Proof. We put $p = \operatorname{depth}_A(\mathfrak{I}, M)$, and consider the complex $K^\bullet(\boldsymbol{x}, M)$. Its cohomology is annihilated by $\mathfrak{I}$ (**??**), and the depth of $K^i(\boldsymbol{x}, M)$ relative to $\mathfrak{I}$ is equal to p of $+\infty$ since $K^i(\boldsymbol{x}, M)$ is either isomorphic to a product of M or zero (Example 9.1.3). By applying Corollary 9.1.10 to the relabeled complex $K_{-i}(\boldsymbol{x}, M)$ (defined by $K_{-i}(\boldsymbol{x}, M) = K^i(\boldsymbol{x}, M)$), we conclude that $H^i(\boldsymbol{x}, M) = 0$ for $i < p$. It then remains to prove that $H^p(\boldsymbol{x}, M)$ is nonzero if $p < +\infty$.

The case $p = 0$ is evident, so we suppose that $0 < p < +\infty$ and $H^p(\boldsymbol{x}, M) = 0$. Let $L^\bullet$ be a free resolution of the A-module $A/\mathfrak{I}$, and denote by $C^\bullet$ the Hom complex $\operatorname{Hom}(L^\bullet, M)$. The A-module $H^i(C^\bullet)$ is then isomorphic to $\operatorname{Ext}^i_A(A/\mathfrak{I}, M)$, and is hence zero for $i < p$. For each $i < p$, we then have a canonical exact sequence

$$0 \longrightarrow B^i(C^\bullet) \longrightarrow C^i \longrightarrow B^{i+1}(C^\bullet) \longrightarrow 0$$

Since each C^i is isomorphic to a product of M, by hypothesis we have $H^n(\boldsymbol{x}, C^i) = 0$ for $n \leq p$, and the connecting homomorphism $\partial^n : H^n(\boldsymbol{x}, B^{i+1}(C^\bullet)) \to H^{n+1}(\boldsymbol{x}, B^i(C^\bullet))$ in the induced long exact sequence is then injective for $n \leq p$ and $i < p$. As $B^0(C^\bullet) = 0$, it then follows that $H^{p-i}(\boldsymbol{x}, B^{i+1}(C^\bullet)) = 0$ for $i < p$, and in particular $H^1(\boldsymbol{x}, B^p(C^\bullet)) = 0$. Now from the exact sequence

$$0 \longrightarrow B^p(C^\bullet) \longrightarrow Z^p(C^\bullet) \longrightarrow H^p(C^\bullet) \longrightarrow 0$$

we then obtain a surjection $H^0(\boldsymbol{x}, Z^p(C^\bullet)) \to H^0(\boldsymbol{x}, H^p(C^\bullet))$. As the A-module $H^p(C^\bullet)$ is isomorphic to $\operatorname{Ext}^p_A(A/\mathfrak{I}, M)$, which is nonzero and annihilated by $\mathfrak{I}$, we finally conclude that $H^0(\boldsymbol{x}, H^p(C^\bullet)) \neq 0$, whence $H^0(\boldsymbol{x}, Z^p(C^\bullet)) \neq 0$ and then $H^0(\boldsymbol{x}, C^p) \neq 0$. But then $H^0(\boldsymbol{x}, M) \neq 0$, which is a contradiction and therefore compeltes the proof. □

Corollary 9.1.13. *Suppose that $\mathfrak{I}$ is finitely generated and $\mathfrak{I}M \neq M$. Then $\operatorname{depth}_A(\mathfrak{I}, M) \leq |I|$, and for the equality holds, it is necessary and sufficient that the family $\boldsymbol{x}$ is complete secant for M.*

Proof. Suppose that I is finite and $n = |I|$. Then the A-module $H^n(\boldsymbol{x}, M)$ is canonically isomorphic to $H_0(\boldsymbol{x}, M)$, hence to $M/\mathfrak{I}M$. The inequality $\operatorname{depth}_A(\mathfrak{I}, M) \leq n$ then follows from Theorem 9.1.12. For the equality holds, it is necesary and sufficient that the A-modules $H^i(\boldsymbol{x}, M) = 0$ for $i < n$, which means that $\boldsymbol{x}$ is complete secant for M. □

Corollary 9.1.14. *Let A be a local ring, $\mathfrak{I}$ be a finitely generated proper ideal of A, and M be a finitely generated nonzero A-module. Put $r = [\mathfrak{I}/\mathfrak{m}_A\mathfrak{I} : \kappa_A]$, then we have $\operatorname{depth}_A(\mathfrak{I}, M) \leq r$, and the equality holds if and only if $\mathfrak{I}$ is generated by a family that is complete secant for M. In this case, for a generating family of $\mathfrak{I}$ to be complete secant for M, it is necesary and sufficient that it consists of r elements.*

Proof. By Nakayama's lemma, we have $\mathfrak{I}M \neq M$, and r is the minimal number of generators of $\mathfrak{I}$. The assertion then follows from Corollary 9.1.13. □

Proposition 9.1.15. *Let A be a ring, $\mathfrak{I}$ be a finitely generated ideal of A, and M be an A-module. Denote by Ω the set of maximal ideals of $V(\mathfrak{I}) \cap \operatorname{supp}(M)$. Then we have*

$$\operatorname{depth}_A(\mathfrak{I}, M) = \inf_{\mathfrak{p} \in \operatorname{Spec}(A)} \operatorname{depth}_{A_\mathfrak{p}}(\mathfrak{I}_\mathfrak{p}, M_\mathfrak{p}) = \inf_{\mathfrak{m} \in \Omega} \operatorname{depth}_{A_\mathfrak{m}}(\mathfrak{I}_\mathfrak{m}, M_\mathfrak{m}).$$

Proof. Let $\boldsymbol{x} = (x_i)_{i \in I}$ be a finite generating family of $\mathfrak{I}$. Let $\mathfrak{p}$ be a prime ideal of A, then the ideal $\mathfrak{I}_\mathfrak{p}$ is generated by the image $\boldsymbol{x}_\mathfrak{p}$ of the family $\boldsymbol{x}$ in $A_\mathfrak{p}$. For any $p \geq 0$, the $A_\mathfrak{p}$-module $(H^p(\boldsymbol{x}, M))_\mathfrak{p}$ is isomorphic to $H^p(\boldsymbol{x}_\mathfrak{p}, M_\mathfrak{p})$, so by Theorem 9.1.12 we have

$$\operatorname{depth}_A(\mathfrak{I}, M) \leq \inf_{\mathfrak{p} \in \operatorname{Spec}(A)} \operatorname{depth}_{A_\mathfrak{p}}(\mathfrak{I}_\mathfrak{p}, M_\mathfrak{p}) = \inf_{\mathfrak{m} \in \Omega} \operatorname{depth}_{A_\mathfrak{m}}(\mathfrak{I}_\mathfrak{m}, M_\mathfrak{m}).$$

Let p be an integer that is strictly smaller than $\operatorname{depth}_{A_{\mathfrak{m}}}(\mathfrak{I}_{\mathfrak{m}}, M_{\mathfrak{m}})$ for any $\mathfrak{m} \in \Omega$. We then have $H^p(\boldsymbol{x}_{\mathfrak{m}}, M_{\mathfrak{m}}) = 0$ for any maximal ideal $\mathfrak{m}$ of A: this follows from Theorem 9.1.12 if $\mathfrak{m} \in \Omega$, and from that fact that $M_{\mathfrak{m}} = 0$ if $\mathfrak{m} \notin \operatorname{supp}(M)$, and that $\mathfrak{I}A_{\mathfrak{m}} = A_{\mathfrak{m}}$ if $\mathfrak{m} \notin V(\mathfrak{I})$ (which annihilates $H^p(\boldsymbol{x}_{\mathfrak{m}}, M_{\mathfrak{m}})$). We then conclude that $(H^p(\boldsymbol{x}, M_{\mathfrak{m}}))_m = 0$ for any maximal ideal $\mathfrak{m}$ of A, whence $H^p(\boldsymbol{x}, M) = 0$. The proposition then follows from Theorem 9.1.12. □

We now turn to another characterization of depth, which utilize the concept of regular sequences. Let A be a ring and M be an A-module. Recall that a sequence $(x_1, \dots, x_r)$ of elements of A is said to be **regular for M** or **M-regular** if, for any $1 \le i \le r$, the homothety with ratio x_i is injective on the A-module $M_i = M/(x_1M + \cdots + x_{i-1}M)$. If $(x_1, \dots, x_r)$ is an M-regular sequence, then for any flat A-module N, the sequence $(x_1, \dots, x_r)$ is $M \otimes N$-regular, and for any flat ring homomorphism $\rho : A \to B$, the image $(\rho(x_1), \dots, \rho(x_r))$ is regular for the B-module $B \otimes_A M$. In particular, for any prime ideal $\mathfrak{p}$ of A, the image of $(x_1, \dots, x_r)$ in $A_{\mathfrak{p}}$ is $M_{\mathfrak{p}}$-regular.

The imporance of regular sequence is that the depth of M relative to an ideal $\mathfrak{I}$ can also be characterized by the length of maximal regular M-sequences. Therefore, we can also define $\operatorname{depth}_A(\mathfrak{I}, M)$ by this maximal length. The connection of regular sequences with Ext modules then allows us to apply homological methods to commutative algebras.

Proposition 9.1.16. *Let A be a ring, $\mathfrak{I}$ be an ideal of A, M be an A-module, and $(x_1, \dots, x_r)$ be an M-regular sequence of elements of $\mathfrak{I}$. Then we have*

$$\operatorname{depth}_A(\mathfrak{I}, M) = r + \operatorname{depth}_A(\mathfrak{I}, M/(x_1M + \cdots + x_rM)),$$

and in particular $\operatorname{depth}_A(\mathfrak{I}, M) \le r$.

Proof. The case $n = 1$ follows from Proposition 9.1.4, since $M \xrightarrow{\cdot x_1} M$ is injective by hypothesis. The general case can then be proved by induction on n. □

Theorem 9.1.17. *Let A be a Noetherian ring, $\mathfrak{I}$ be an ideal of A, and M be a finitely generated A-module.*

(a) Suppose that $\operatorname{depth}_A(\mathfrak{I}, M)$ is finite, then any M-regular sequence of elements of $\mathfrak{I}$ can be completed into an M-regular sequence of length $\operatorname{depth}_A(\mathfrak{I}, M)$ of elements of $\mathfrak{I}$.

(b) The depth of M relative to $\mathfrak{I}$ is the maximal length of M-regular sequences of elements of $\mathfrak{I}$.

(c) For $\operatorname{depth}_A(\mathfrak{I}, M)$ to be finite, it is necessary and sufficient that the support of M meets $V(\mathfrak{I})$, or that we have $\mathfrak{I}M \neq M$.

Proof. Let $(x_1, \dots, x_r)$ be an M-regular sequence of elements of $\mathfrak{I}$. We have $n \le \operatorname{depth}_A(\mathfrak{I}, M)$ by Proposition 9.1.16; suppose that this inequality is strict, and denote by N the A-module $M/(x_1M + \cdots + x_rM)$. We then have $\operatorname{depth}_A(\mathfrak{I}, N) > 0$, so there exists an element x of $\mathfrak{I}$ such that the homothety x_N is injective (Example 9.1.2), which means $(x_1, \dots, x_r, x)$ is M-regular. It then follows by recurrence that for any integer s such that $r \le s \le \operatorname{depth}_A(\mathfrak{I}, M)$ the sequence $(x_1, \dots, x_r)$ can be extended into an M-regular sequence of length s, which proves the assertion (a) and (b). Assertion (c) then follows from Example 9.1.2 and Corollary 9.1.13. □

Corollary 9.1.18. *For an M-regular $(x_1, \dots, x_r)$ of elements of $\mathfrak{I}$, the following properties are equivalent:*

(i) $r = \operatorname{depth}_A(\mathfrak{I}, M)$;

(ii) the sequence $(x_1, \dots, x_r)$ is a maximal M-regular sequence of elements of $\mathfrak{I}$.

(iii) the A-module $M/(x_1M + \cdots + x_rM)$ possesses a nonzero element annihilated by $\mathfrak{I}$;

(iv) $\operatorname{Ass}(M/(x_1M+\cdots+x_rM))\cap V(\mathfrak{I})\neq\varnothing$.

Proof. The equivalence of (i) and (ii) follows from Theorem 9.1.17, and that of (ii), (iii) and (iv) follows from Example 9.1.2 applied to $M/(x_1M+\cdots+x_rM)$. □

Corollary 9.1.19. *Let A be a Noetherian local ring and M be a finitely generated A-module. Then* $\operatorname{depth}_A(M)\leq\dim_A(M)<+\infty$.

Proof. In fact, any M-regular sequence of elements of $\mathfrak{m}_A$ is complete secant for M (**??**), hence secant for M (Proposition 7.2.11). □

Proposition 9.1.20. *Let A be a Noetherian local ring, M be a finitely generated nonzero A-module, and $\mathfrak{I}$ be a proper ideal of A. Then we have the inequalities*

$$\operatorname{depth}_A(\mathfrak{I},M)\leq\operatorname{codim}(\operatorname{supp}(M)\cap V(\mathfrak{I}),\operatorname{supp}(M))\leq\dim(M)-\dim(M/\mathfrak{I}M)\leq[\mathfrak{I}/\mathfrak{m}_A\mathfrak{I}:\kappa_A].$$

Proof. For any element $\mathfrak{p}$ of $\operatorname{supp}(M)\cap V(\mathfrak{I})$, $\operatorname{depth}_A(\mathfrak{I},M)$ is smaller than $\dim_{A_{\mathfrak{p}}}(M_{\mathfrak{p}})$ (Proposition 9.1.15 and Corollary 9.1.19), which is equal to $\operatorname{codim}(V(\mathfrak{p}),\operatorname{supp}(M))$. If $\mathfrak{p}$ runs through $\operatorname{supp}(M)\cap V(\mathfrak{I})$, then $V(\mathfrak{p})$ runs through irreducible closed subsets of $\operatorname{supp}(M)\cap V(\mathfrak{I})$, whence the first inequality. The second one follows from **??**. Moreover, we can always find a generating set of $\mathfrak{I}$ with cardinality $[\mathfrak{I}/\mathfrak{m}_A\mathfrak{I}:\kappa_A]$, so the third inequality follows from (7.2.3). □

Remark 9.1.21. Consider the inequalities in Proposition 9.1.20.

(a) For that we have $\operatorname{depth}_A(\mathfrak{I},M)=[\mathfrak{I}/\mathfrak{m}_A\mathfrak{I}:\kappa_A]$, it is necessary and sufficient that $\mathfrak{I}$ can be generated by a M-regular sequence (Corollary 9.1.14).

(b) The equality $\dim(M)-\dim(M/\mathfrak{I}M)=[\mathfrak{I}/\mathfrak{m}_A\mathfrak{I}:\kappa_A]$ signifies that $\mathfrak{I}$ can be generated by a sequence that is secant for M (Theorem 7.2.14).

(c) If M is Cohen-Macaulay, then we have $\operatorname{depth}_A(\mathfrak{I},M)=\dim(M)-\dim(M/\mathfrak{I}M)$.

Lemma 9.1.22. *Let A be a Noetherian ring, $\mathfrak{p}\subseteq\mathfrak{p}_1\subseteq\cdots\subseteq\mathfrak{p}_{r-1}\subseteq\mathfrak{q}$ be a saturated chain of length r of prime ideals of A, M be a finitely generated A-module, and n be an integer. If the A-module $\operatorname{Ext}^n_{A_{\mathfrak{p}}}(\kappa(\mathfrak{p}),M_{\mathfrak{p}})$ is nonzero, then so is $\operatorname{Ext}^{n+r}_{A_{\mathfrak{q}}}(\kappa(\mathfrak{q}),M_{\mathfrak{q}})$.*

Proof. It evidently suffices to prove the case $r=1$; by replacing A, M, $\mathfrak{p}$ and $\mathfrak{q}$ with $A_{\mathfrak{q}}$, $M_{\mathfrak{q}}$, $\mathfrak{p}A_{\mathfrak{q}}$ and $\mathfrak{q}A_{\mathfrak{q}}$ respectively, we can then assume that A is local and $\mathfrak{q}=\mathfrak{m}_A$. Let x be an element of $\mathfrak{m}_A-\mathfrak{p}$. The $A_{\mathfrak{p}}$-module $\operatorname{Ext}^n_A(A/\mathfrak{p},M)\otimes_A A_{\mathfrak{p}}$ is isomorphic to $\operatorname{Ext}^n_{A_{\mathfrak{p}}}(\kappa(\mathfrak{p}),M_{\mathfrak{p}})$ (A, X, p.111, prop.10(b)), so is nonzero by hypothesis; a fortiori $\operatorname{Ext}^n_A(A/\mathfrak{p},M)$ is nonzero. The exact sequence

$$0\longrightarrow A/\mathfrak{p}\xrightarrow{x_{A/\mathfrak{p}}}A/\mathfrak{p}\longrightarrow A/(\mathfrak{p}+xA)\longrightarrow 0$$

induces an exact sequence

$$\operatorname{Ext}^n_A(A/\mathfrak{p},M)\xrightarrow{u}\operatorname{Ext}^n_A(A/\mathfrak{p},M)\longrightarrow\operatorname{Ext}^{n+1}_A(A/(\mathfrak{p}+xA),M)$$

where u is the homothety with ratio x. By Nakayama's lemma, this homomorphism is not surjective, so the A-module $\operatorname{Ext}^{n+1}_A(A/(\mathfrak{p}+xA),M)$ is nonzero. Now if $\operatorname{Ext}^{n+1}_A(\kappa_A,M)$ is zero, we then deduce, by induction on the length of N, that $\operatorname{Ext}^{n+1}_A(N,M)=0$ for any A-module N of finite length. But since $r=1$, the unique prime ideal of A containing $\mathfrak{p}+xA$ is $\mathfrak{m}_A$, so the A-module $A/(\mathfrak{p}+xA)$ is of finite length. This contradicts the fact that $\operatorname{Ext}^{n+1}_A(A/(\mathfrak{p}+xA),M)\neq 0$, which proves our claim. □

Proposition 9.1.23. *Let A be a Noetherian ring, M be a finitely generated A-module, $\mathfrak{p} \subseteq \mathfrak{q}$ be prime ideals of* $\mathrm{supp}(M)$. *Then we have*

$$\mathrm{depth}_{A_\mathfrak{q}}(M_\mathfrak{q}) \leq \mathrm{depth}_{A_\mathfrak{p}}(M_\mathfrak{p}) + \dim(A_\mathfrak{q}/\mathfrak{p}A_\mathfrak{q}).$$

More precisely, for any saturated chain of prime ideals $\mathfrak{p} \subseteq \mathfrak{p}_1 \subseteq \cdots \subseteq \mathfrak{p}_{r-1} \subseteq \mathfrak{q}$, we have

$$\mathrm{depth}_{A_\mathfrak{q}}(M_\mathfrak{q}) \leq \mathrm{depth}_{A_\mathfrak{p}}(M_\mathfrak{p}) + r.$$

Proof. Put $p = \mathrm{depth}_{A_\mathfrak{p}}(M_\mathfrak{p})$, it suffices to prove the second inequality. This is evident if $p = +\infty$; in the contrary case we have $\mathrm{Ext}^p_{A_\mathfrak{p}}(\kappa(\mathfrak{p}), M_\mathfrak{p}) \neq 0$, so $\mathrm{Ext}^{p+r}_{A_\mathfrak{q}}(\kappa(\mathfrak{q}), M_\mathfrak{q}) \neq 0$ by Lemma 9.1.22, which implies $\mathrm{depth}_{A_\mathfrak{q}}(M_\mathfrak{q}) \leq p + r$. As $\dim(A_\mathfrak{q} + \mathfrak{p}A_\mathfrak{q})$ is the supremum of the length of saturated chains of prime ideals with endpoints $\mathfrak{p}$ and $\mathfrak{q}$, the first assertion then follows. □

Corollary 9.1.24. *We have the inequality*

$$\dim(M_\mathfrak{q}) - \mathrm{depth}_{A_\mathfrak{q}}(M_\mathfrak{q}) \geq \dim(M_\mathfrak{p}) - \mathrm{depth}_{A_\mathfrak{p}}(M_\mathfrak{p}) \geq 0.$$

Proof. This follows from Proposition 9.1.23 and $\dim(M_\mathfrak{q}) \geq \dim(M_\mathfrak{p}) + \dim(A_\mathfrak{q}/\mathfrak{p}A_\mathfrak{q})$ (Proposition 7.1.12(a) and **??**(b)). □

Corollary 9.1.25. *Let A be a Noetherian local ring and M be a finitely generated A-module. Then we have the inequality*

$$\mathrm{depth}_A(M) \leq \inf_{\mathfrak{p} \in \mathrm{Ass}(M)} \dim(A/\mathfrak{p}).$$

Proof. Let $\mathfrak{p}$ be an associated prime of M; we have $\mathrm{depth}_{A_\mathfrak{p}}(M_\mathfrak{p}) = 0$ by Example 9.1.2. Proposition 9.1.23 applied to the ideals $\mathfrak{p} \subseteq \mathfrak{m}_A$ then implies the inequality $\mathrm{depth}_A(M) \leq \dim(A/\mathfrak{p})$, whence the corollary. □

Remark 9.1.26. We have $\sup_{\mathfrak{p} \in \mathrm{Ass}(M)} \dim(A/\mathfrak{p}) = \dim(M)$ in view of Example 7.1.11, so Corollary 9.1.25 implies that $\mathrm{depth}_A(M) \leq \dim(M)$ for $M \neq 0$, which is Corollary 9.1.19.

9.1.3 Extension of scalars

Proposition 9.1.27. *Let $\rho : A \to B$ be a ring homomorphism, $\mathfrak{I}$ be an ideal of A, and N be a B-module. Then we have* $\mathrm{depth}_A(\mathfrak{I}, N) = \mathrm{depth}_B(\mathfrak{I}B, N)$.

Proof. Let $\boldsymbol{x} = (x_i)_{i \in I}$ be a generating family of $\mathfrak{I}$; the family $\rho(\boldsymbol{x}) = (\rho(x_i))_{i \in I}$ then generates $\mathfrak{I}B$. By construction the complex $K^\bullet(\rho(\boldsymbol{x}), N)$ is equal to $K^\bullet(\boldsymbol{x}, N)$, so the proposition follows from Theorem 9.1.12. □

Proposition 9.1.28. *Let A be a ring, $\mathfrak{I}$ be a finitely generated ideal of A and M be an A-module. Let $\rho : A \to B$ be a flat ring homomorphism.*

- *(a)* *We have* $\mathrm{depth}_A(\mathfrak{I}, M) \leq \mathrm{depth}_B(\mathfrak{I}B, B \otimes_A M)$.
- *(b)* *Suppose moreover that any maximal ideal of* $\mathrm{supp}(M) \cap V(\mathfrak{I})$ *belongs to the image of the canonical map* $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$. *Then we have* $\mathrm{depth}_A(\mathfrak{I}, M) = \mathrm{depth}_B(\mathfrak{I}B, B \otimes_A M)$. *This is the case for example if the A-module B is faithfully flat.*

Proof. Let $\boldsymbol{x} = (x_i)_{i\in I}$ be a family of elements of A. For each integer $p \geq 0$, let $u^p : B \otimes_A C^p(M) \to C^p(B \otimes_A M)$ be the B-linear homomorphism give by

$$b \otimes m \mapsto ((\alpha_1, \dots, \alpha_p) \mapsto b \otimes m(\alpha_1, \dots, \alpha_p)).$$

The family (u^p) then defines an isomorphism of complexes $u : B \otimes_A K^\bullet(\boldsymbol{x}, M) \to K^\bullet(\boldsymbol{x}, B \otimes_A M)$. Now consider the canonical homomorphism

$$\gamma^p(B, K^\bullet(\boldsymbol{x}, M)) : B \otimes_A H^p(\boldsymbol{x}, M) \to H^p(B \otimes_A K^\bullet(\boldsymbol{x}, M))$$

By composing with $H^p(u)$, we obtain a homomorphism $v^p : B \otimes_A H^p(\boldsymbol{x}, M) \to H^p(\boldsymbol{x}, B \otimes_A M)$. It is clear from **??** that v^p is an isomorphism if B is flat over A, so assertion (a) follows from Theorem 9.1.12.

Suppose that p is an integer that is strictly smaller than $\operatorname{depth}_B(\mathfrak{J}B, B \otimes_A M)$, and let $\mathfrak{m}$ be a maximal ideal of A belonging to $\operatorname{supp}(M) \cap V(\mathfrak{J})$. Let $\boldsymbol{x}$ be a finite generating family of $\mathfrak{J}$. Under the hypothesis of (b), there exists a prime ideal $\mathfrak{P}$ of B lying over $\mathfrak{m}$, and we have a canonical isomorphism

$$B_{\mathfrak{P}} \otimes_{A_{\mathfrak{m}}} (A_{\mathfrak{m}} \otimes_A H^p(\boldsymbol{x}, M)) \to B_{\mathfrak{P}} \otimes_B (B \otimes_A H^p(\boldsymbol{x}, M)).$$

Now $B \otimes_A H^p(\boldsymbol{x}, M)$ is isomorphic to $H^p(\rho(\boldsymbol{x}), B \otimes_A M)$, hence is zero; moreover $B_{\mathfrak{P}}$ is faithfully flat over $A_{\mathfrak{m}}$ (Proposition 1.3.27 and **??**), so we conclude that $A_{\mathfrak{m}} \otimes_A H^p(\boldsymbol{x}, M) = 0$, and therefore $p < \operatorname{depth}_{A_{\mathfrak{m}}}(\mathfrak{J}_{\mathfrak{m}}, M_{\mathfrak{m}})$. The first assertion of (b) then follows from Proposition 9.1.15, and the second one follows from **??**. □

Corollary 9.1.29. *Let A be a Noetherian ring, $\mathfrak{J}$ be an ideal of A, M be a finitely generated A-module, $\widehat{A}$ and $\widehat{M}$ be the $\mathfrak{J}$-adic completion of A and M. Then we have* $\operatorname{depth}_A(\mathfrak{J}, M) = \operatorname{depth}_{\widehat{A}}(\mathfrak{J}\widehat{A}, \widehat{M})$.

Proof. In fact, the A-module $\widehat{A}$ is flat and $\widehat{M}$ is isomorphic to $\widehat{A} \otimes_A M$ (Proposition 2.3.16); moreover, any maximal ideal of A containing $\mathfrak{J}$ belongs to the image of the map $\operatorname{Spec}(\widehat{A}) \to \operatorname{Spec}(A)$ (Proposition 2.4.24). □

Lemma 9.1.30. *Let $\rho : A \to B$ be a homomorphism of Noetherian local rings, N be a finitely generated B-module, and y be an element of $\mathfrak{m}_B$. Then the following conditions are equivalent:*

(i) the A-module N/yN is flat and the homothety y_N is injective;

(ii) the A-module N is flat and the homothety $y_{\kappa_A \otimes N}$ is injective.

If these are satisfied, then the homothery $y_{M \otimes_A N}$ is injective for any A-module M.

Proof. Suppose that the hypotheses of (i) are satisfied, let us prove (ii) as well as the last assertion. Let M be an A-module; since the A-module N/yN is flat, we deduce from the exact sequence $0 \to N \xrightarrow{y_N} N \to N/yN \to 0$ the following exact sequences

$$0 \longrightarrow M \otimes_A N \xrightarrow{u} M \otimes_A N \longrightarrow M \otimes_A (N/yN) \longrightarrow 0$$

$$0 \longrightarrow \operatorname{Tor}_1^A(M, N) \xrightarrow{v} \operatorname{Tor}_1^A(M, N) \longrightarrow 0$$

where $u = 1_M \otimes y_N$ and $v = \operatorname{Tor}_1^A(1_M, y_N)$. It then follows that the homothety with ratio y is injective on $M \otimes_A N$, and bijective on $\operatorname{Tor}_1^A(M, N)$. Suppose moreover that M is finitely generated, then the B-module $\operatorname{Tor}_1^A(M, N)$ is also finitely generated, and hence is zero by Nakayama's lemma. This implies the flatness of the A-module N.

Conversely, assume that the hypotheses in (ii) are satisfied. Consider the exact sequences of B-modules

$$0 \longrightarrow \ker y_N \longrightarrow N \xrightarrow{p} \operatorname{im} y_N \longrightarrow 0 \tag{9.1.1}$$

$$0 \longrightarrow \operatorname{im} y_N \xrightarrow{i} N \longrightarrow N/yN \longrightarrow 0 \tag{9.1.2}$$

where p and i are canonical homomorphisms. We then deduce that the homomorphism $1 \otimes p : \kappa_A \otimes_A N \to \kappa_A \otimes \operatorname{im} y_N$ is surjective, and (since N is flat) that the kernel of the homomorphism $1 \otimes i : \kappa_A \otimes_A \operatorname{im} y_N \to \kappa_A \otimes_A N$ is isomorphic to $\operatorname{Tor}_1^A(\kappa_A, N/yN)$. But the map $(1 \otimes i) \circ (1 \otimes p)$, equal to $y_{\kappa_A \otimes_A N}$, is injective by hypothesis; we then deduce that $1 \otimes p$ is bijective and $1 \otimes i$ is injective, and therefore $\operatorname{Tor}_1^A(\kappa_A, N/yN) = 0$. It then follows that the A-module N/yN is flat (Theorem 2.6.33 and Proposition 2.6.34).

Since N and N/yN are flat over A, so is $\operatorname{im} y_N$ (by exace sequence (9.1.2)). We then deduce from (9.1.1) that $\kappa_A \otimes_A \ker y_N$ is isomorphic to the kernel of $1 \otimes p$, which is then zero. The homothety y_N is then injective by Nakayama's lemma. □

Proposition 9.1.31. *Let $\rho : A \to B$ be a homomorphism of Noetherian local rings, N be a finitely generated B-module, and $\boldsymbol{y} = (y_1, \dots, y_s)$ be a sequence of elements of $\mathfrak{m}_B$. Denote by $\mathfrak{Y}$ the ideal of B generated by this sequence, then the following conditions are equivalent:*

(i) the A-module $N/\mathfrak{Y}N$ is flat and the sequence $\boldsymbol{y}$ is N-regular.

(ii) the A-module N is flat and the sequence $\boldsymbol{y}$ is $(\kappa_A \otimes_A N)$-regular.

If these are satisfied, then for any A-module M, the sequence $\boldsymbol{y}$ is $(M \otimes_A N)$-regular.

Proof. We prove the equivalence by recurrence on s. The case $s = 0$ is evident, so suppose that $s \geq 1$. Denote by $\tilde{\boldsymbol{y}}$ the sequence $(y_1, \dots, y_{s-1})$ and $\widetilde{\mathfrak{Y}}$ the ideal of B it generates. By Lemma 9.1.30 applied to the B-module $N/\widetilde{\mathfrak{Y}}N$ and to the element y_s of B, we see that (i) is equivalent to the following condition:

(i′) the A-module $N/\widetilde{\mathfrak{Y}}N$ is flat, and sequence $\tilde{\boldsymbol{y}}$ is N-regular, and the homothery with ratio y_s is injective on $\kappa_A \otimes_A (N/\widetilde{\mathfrak{Y}}N) = (\kappa_A \otimes N)/\widetilde{\mathfrak{Y}}(\kappa_A \otimes_A N)$.

This condition is equivalent to (ii) by the recurrence hypotheses. Finally, the last assertion follows similarly by recurrence on s and utilize Lemma 9.1.30. □

Proposition 9.1.32. *Let $\rho : A \to B$ be a homomorphism of Noetherian local rings, M be a finitely generated A-module, and N be a finitely generated B-module. Suppose that the A-module N is flat.*

(a) Let $(x_1, \dots, x_r)$ be an M-regular sequence of elements of $\mathfrak{m}_A$ and $(y_1, \dots, y_s)$ be a $(\kappa_A \otimes_A N)$-regular sequence of elements of $\mathfrak{m}_B$. Then $(y_1, \dots, y_s, \rho(x_1), \dots, \rho(x_r))$ is an $(M \otimes_A N)$-regular sequence of elements of $\mathfrak{m}_B$.

(b) We have the equality

$$\operatorname{depth}_B(M \otimes_A N) = \operatorname{depth}_A(M) + \operatorname{depth}_B(\kappa_A \otimes_A N).$$

Proof. Denote by $\mathfrak{x}$ the ideal of A generated $\boldsymbol{x}$ and $\mathfrak{Y}$ the ideal of B generated by $\boldsymbol{y}$. By Proposition 9.1.31, the sequence $\boldsymbol{y}$ is $M \otimes_A N$-regular and $N/\mathfrak{Y}N$ is flat for A, so that the sequence $\rho(\boldsymbol{x}) = (\rho(x_1), \dots, \rho(x_r))$ is regular for $M \otimes_A (N/\mathfrak{Y}N) = (M \otimes_A N)/\mathfrak{Y}(M \otimes_A N)$. This proves the assertion in (a).

To prove (b), we can suppose that M and N are nonzero. By Nakayama's lemma, $\kappa_A \otimes_A N$ is also nonzero, so that $\operatorname{depth}_A(M)$ and $\operatorname{depth}_B(\kappa_A \otimes N)$ are finite (Corollary 9.1.19). Choose $\boldsymbol{x}$ and $\boldsymbol{y}$ to be maximal regular sequences, we then have $r = \operatorname{depth}_A(M)$, $s = \operatorname{depth}_B(\kappa_A \otimes N)$, and there exists an injective A-linear map $u : \kappa_A \to M/\mathfrak{x}M$ and an injective B-linear map $v : \kappa_B \to \kappa_A \otimes_A (N/\mathfrak{Y}N)$ (Corollary 9.1.18). Since $N/\mathfrak{Y}N$ is flat over A, the B-linear map

$$(u \otimes 1_{N/\mathfrak{Y}N}) \circ v : \kappa_B \to (M/\mathfrak{x}M) \otimes_A (N/\mathfrak{Y}N) = (M \otimes_A N)/(\rho(\mathfrak{x}) + \mathfrak{Y})(M \otimes_A N)$$

is injective. This implies the equality $\operatorname{depth}_B(M \otimes_A N) = r + s$, in view of Corollary 9.1.18. □

Remark 9.1.33. We note that under the hypotheses of Proposition 9.1.32, we have a similar equality for dimensions: (c.f. Proposition 7.2.26)

$$\dim_B(M \otimes_A N) = \dim_A(M) + \dim_B(\kappa_A \otimes_A N).$$

Corollary 9.1.34. *Let $\rho : A \to B$ be a flat homomorphism of Noetherian local rings. Then we have*

$$\begin{aligned} \operatorname{depth}(B) &= \operatorname{depth}(A) + \operatorname{depth}(\kappa_A \otimes_A B), \\ \dim(B) &= \dim(A) + \dim(\kappa_A \otimes_A B). \end{aligned}$$

Proof. In fact, the depth (resp. dimension) of the B-module $\kappa_A \otimes_A B$ is equal to the depth (resp. dimension) of the ring $\kappa_A \otimes_A B$ by Proposition 9.1.27. □

9.1.4 Depth along a closed subset

We now introduce the concept of depth along a closed subset. Let A be a Noetherian ring, F be a closed subset of $\operatorname{Spec}(A)$, and M be an A-module. By Corollary 9.1.8, the integer $\operatorname{depth}_A(\mathfrak{I}, M)$ does not depend on the ideal $\mathfrak{I}$ such that $F = V(\mathfrak{I})$, we then write $\operatorname{depth}_F(M)$ for this integer, and call it the **depth of** M **along** F. By Proposition 9.1.6 and Corollary 1.4.36, we see that the inequality $\operatorname{depth}_F(M) \geq r$ is equivalent to the property that for any finitely generated A-module N with support contained in F, we have $\operatorname{Ext}^i_A(N, M) = 0$ for $i < r$. Also, if M is finitely generated, then we have $\operatorname{depth}_F(M) = 0$ if and only if $\operatorname{Ass}(M) \cap F = \varnothing$, and $\operatorname{depth}_F(M) < +\infty$ if and only if $\operatorname{supp}(M) \cap F \neq \varnothing$.

Proposition 9.1.35. *Let A be a Noetherian ring, F be a closed subset of* $\operatorname{Spec}(A)$*, and M be a finitely generated A-module. Then we have*

$$\operatorname{depth}_F(M) = \inf_{\mathfrak{p} \in F} \operatorname{depth}_{A_\mathfrak{p}}(M_\mathfrak{p}) = \inf_{\mathfrak{p} \in \operatorname{supp}(M) \cap F} \operatorname{depth}_{A_\mathfrak{p}}(M_\mathfrak{p}).$$

Proof. This is clear if $\operatorname{depth}_F(M) = +\infty$. If $\operatorname{depth}_F(M) = 0$, there exists a prime ideal $\mathfrak{p} \in \operatorname{Ass}(M) \cap F$; we have $\mathfrak{p}A_\mathfrak{p} \in \operatorname{Ass}(M_\mathfrak{p})$ (Proposition 3.1.11), so $\operatorname{depth}_{A_\mathfrak{p}}(M_\mathfrak{p}) = 0$, whence the assertion in this case.

Suppose that $0 < \operatorname{depth}_F(M) < +\infty$; let $\mathfrak{I}$ be an ideal of A such that $V(\mathfrak{I}) = F$, and x be an element of $\mathfrak{I}$ such that the homothety x_M is injective (Example 9.1.2). For any prime ideal $\mathfrak{p}$, the homothety $x_{M_\mathfrak{p}}$ is injective, and by Proposition 9.1.16 we have

$$\operatorname{depth}_F(M/xM) = \operatorname{depth}_F(M) - 1, \quad \operatorname{depth}_{A_\mathfrak{p}}((M/xM)_\mathfrak{p}) = \operatorname{depth}_{A_\mathfrak{p}}(M_\mathfrak{p}) - 1.$$

The conclusion then follows by induction on $\operatorname{depth}_F(M)$. □

Remark 9.1.36. If $\mathfrak{q}$ is a point of $\operatorname{supp}(M)$, we then have $\operatorname{depth}_A(\mathfrak{q}, M) = \inf_{\mathfrak{p} \supseteq \mathfrak{q}} \operatorname{depth}_{A_\mathfrak{p}}(M_\mathfrak{p})$. In particular, we have $\operatorname{depth}_A(\mathfrak{q}, M) \leq \operatorname{depth}_{A_\mathfrak{q}}(M_\mathfrak{q})$, and the equality holds if $\mathfrak{q}$ is maximal. In the general case, we may have $\operatorname{depth}_A(\mathfrak{q}, M) < \operatorname{depth}_{A_\mathfrak{q}}(M_\mathfrak{q})$, or $\operatorname{depth}_A(\mathfrak{q}, M) < \inf \operatorname{depth}_{A_\mathfrak{m}}(M_\mathfrak{m})$ where $\mathfrak{m}$ runs through maximal ideals of A cotaining $\mathfrak{q}$. For example, let $\mathfrak{p}$ be a non maximal prime ideal of A, cotaining $\mathfrak{q}$ and distinct from $\mathfrak{q}$; put $M = A/\mathfrak{p}$, then we have $\operatorname{depth}_A(\mathfrak{q}, M) = 0$, $\operatorname{depth}_{A_\mathfrak{q}}(M_\mathfrak{q}) = +\infty$ and $\operatorname{depth}_{A_\mathfrak{m}}(M_\mathfrak{m}) > 0$ for any maximal ideal $\mathfrak{m}$ of A.

Proposition 9.1.37. *Let A be a Noetherian ring, M and N be finitely generated A-modules, and F be the support of N. Then* $\operatorname{depth}_F(M)$ *is the infermum of the integers i such that* $\operatorname{Ext}^i_A(N, M) \neq 0$.

Proof. By Proposition 9.1.6 and Corollary 1.4.36, we have $\operatorname{Ext}^i_A(N, M) = 0$ for $i < \operatorname{depth}_F(M)$. It remains to prove that if $\operatorname{depth}_F(M) = n <= \infty$, then $\operatorname{Ext}^n_A(N, M) \neq 0$. Let $\mathfrak{I}$ be the annihilator of N; we then have $F = V(\mathfrak{I})$, so $\operatorname{depth}_F(M) = \operatorname{depth}_A(\mathfrak{I}, M)$. By Theorem 9.1.17, there exists an M-regular sequence $(x_1, \dots, x_n)$ of length n formed by elements of $\mathfrak{I}$, and the depth of $\overline{M} = M/(x_1M + \cdots + x_nM)$ relative to $\mathfrak{I}$ is zero. By **??**, it then suffices to prove that $\mathrm{Hom}_A(N, \overline{M}) \neq 0$. Now by Proposition 9.1.35, there exists $\mathfrak{p} \in \mathrm{supp}(M) \cap \mathrm{supp}(N)$ such that $\mathrm{depth}_{A_\mathfrak{p}}(\overline{M}_\mathfrak{p}) = 0$, which means $\mathrm{Hom}_{A_\mathfrak{p}}(\kappa(\mathfrak{p}), \overline{M}_\mathfrak{p}) \neq 0$. Since $N_\mathfrak{p}$ is nonzero, the $\kappa(\mathfrak{p})$-vector space $N_\mathfrak{p} \otimes_{A_\mathfrak{p}} \kappa(\mathfrak{p})$ is nonzero (Nakayama's lemma), so there exists a surjective $A_\mathfrak{p}$-linear map from $N_\mathfrak{p}$ to $\kappa(\mathfrak{p})$. It follows that $\mathrm{Hom}_{A_\mathfrak{p}}(N_\mathfrak{p}, \overline{M}_\mathfrak{p}) \neq 0$, so $\mathrm{Hom}_A(N, \overline{M}) \neq 0$ (Proposition 1.2.48), which proves the assertion. □

Remark 9.1.38. Let A be a Noetherian ring and N be a fintely generated A-module. The **grade** of N, denoted by $\mathrm{grade}(N)$, is defined to be the infermum of the integers i such that $\mathrm{Ext}^i_A(N, A) \neq 0$. By Proposition 9.1.37, this is also the depth of A along the support of N, and is the maximal length of A-regular sequences of elements of the annihilator of N. As for any prime ideal of A, the annihilator of $N_\mathfrak{p}$ is equal to $\mathrm{Ann}(N)_\mathfrak{p}$, we deduce from Proposition 9.1.15 the equality

$$\mathrm{grade}(N) = \inf_{\mathfrak{p} \in \mathrm{Spec}(A)} \mathrm{grade}(N_\mathfrak{p}) = \inf_{\mathfrak{m} \in \Omega} \mathrm{grade}(N_\mathfrak{m})$$

where Ω denote the set of maximal ideals of A.

We recall that the localization operation is faithfully exact in the sense that a homomorphism $u : M \to N$ of A-modules is injective (resp. surjective) if and only if $u_\mathfrak{p} : M_\mathfrak{p} \to N_\mathfrak{p}$ is injective (resp. surjective) for any prime ideal $\mathfrak{p}$ of A. Using the notion of depth, we can show that under certain conditions, this injectivity (resp. surjectivity) of $u_\mathfrak{p}$ only need to be checked on an open subset of $\mathrm{Spec}(A)$.

Lemma 9.1.39. *Let A be a Noetherian ring, F be a closed subset of $\mathrm{Spec}(A)$, U be its complement, and $u : M \to N$ be a homomorphism of finitely generated A-module.*

(a) Suppose that $u_\mathfrak{p} : M_\mathfrak{p} \to N_\mathfrak{p}$ is injective for $\mathfrak{p} \in U$ and that $\mathrm{depth}_F(M) \geq 1$, then u is injective.

(b) Suppose that $u_\mathfrak{p} : M_\mathfrak{p} \to N_\mathfrak{p}$ is bijective for $\mathfrak{p} \in U$ and that $\mathrm{depth}_F(M) \geq 2$ and $\mathrm{depth}_F(N) \geq 1$, then u is bijective.

Proof. For the first assertion, we note that the hypotheses of (a) imply that $\mathrm{supp}(\ker u) \subseteq F$, so that $\mathrm{Hom}_A(\ker u, M) = 0$ by Proposition 9.1.6. We then conclude that $\ker u = 0$, since $\ker u$ is a submodule of M. As for the assertions in (b), we note that in this case u is injective by (a), and we have $\mathrm{supp}(\mathrm{coker}\, u) \subseteq F$. Again by Proposition 9.1.6, we have $\mathrm{Hom}_A(\mathrm{coker}\, u, N) = 0$ and $\mathrm{Ext}^1_A(\mathrm{coker}\, u, M) = 0$. From the exact sequence

$$\mathrm{Hom}_A(\mathrm{coker}\, u, N) \longrightarrow \mathrm{Hom}_A(\mathrm{coker}\, u, \mathrm{coker}\, u) \longrightarrow \mathrm{Ext}^1_A(\mathrm{coker}\, u, M)$$

we deduce that $\mathrm{Hom}_A(\mathrm{coker}\, u, \mathrm{coker}\, u) = 0$, whence $\mathrm{coker}\, u = 0$. □

Remark 9.1.40. Let A be a Noetherian ring, F be a closed subset of $\mathrm{Spec}(A)$, and U be its complement. For that we have $\mathrm{depth}_F(A) \geq 1$, it is necessary and sufficient that $\mathrm{Ass}(A) \subseteq U$ (Example 9.1.2). If this condition is satisfied, then any irreducible component of $\mathrm{Spec}(A)$ meets U, so that U is dense in $\mathrm{Spec}(A)$.

Theorem 9.1.41 (Hartshorne). *Let A be a Noetherian ring, F be a closed subset of A, and U be its complement. Suppose that $\mathrm{depth}_F(A) \geq 2$, then for any connected component Y of $\mathrm{Spec}(A)$, the set $Y \cap U$ is connected and dense in Y.*

Proof. Suppose first that $\mathrm{Spec}(A)$ is connected. By Remark 9.1.40, U is dense in $\mathrm{Spec}(A)$ and it then suffices to prove that it is connected. To this end, suppose that there are two nonempty open disjoint subsets U_0 and U_1 of $\mathrm{Spec}(A)$ whose union is U. As the set $\mathrm{Ass}(A)$ is contained in U by Remark 9.1.40, it is the union of the disjoint subsets $\mathrm{Ass}(A) \cap U_0$ and $\mathrm{Ass}(A) \cap U_1$. By Proposition 3.1.10, there exists ideals $\mathfrak{J}_0$ and $\mathfrak{J}_1$ of A such that

$$\mathrm{Ass}(\mathfrak{J}_i) = \mathrm{Ass}(A) \cap U_i, \quad \mathrm{Ass}(A/\mathfrak{J}_i) = \mathrm{Ass}(A) \cap U_{1-i} \quad (i = 0, 1).$$

The complement of U_i in $\mathrm{Spec}(A)$ contains $\mathrm{Ass}(A/\mathfrak{J}_i)$ and $\mathrm{Ass}(\mathfrak{J}_j)$; as it is closed, it also contains $\mathrm{supp}(A/\mathfrak{J}_i)$ and $\mathrm{supp}(\mathfrak{J}_{1-i})$. For $\mathfrak{p} \in U_i$, we then conclude that $(A/\mathfrak{J}_i)_\mathfrak{p} = 0$ and $(\mathfrak{J}_{1-i})_\mathfrak{p} = 0$, which also implies that $\mathfrak{J}_0$ and $\mathfrak{J}_1$ are proper ideals of A. Now let B be the A-module $A/\mathfrak{J}_0 \times A/\mathfrak{J}_1$ and $u : A \to B$ be the canonical homomorphism. By the preceding remarks, the homomorphism $u_\mathfrak{p}$ is bijective for $\mathfrak{p} \in U$; on the other hand, we have $\mathrm{Ass}(B) \subseteq U_0 \cup U_1 = U$ (Proposition 3.1.7), so $\mathrm{depth}_F(B) \geq 1$ in view of Remark 9.1.40. Lemma 9.1.39 then implies that u is bijective, which contradicts the connectedness of $\mathrm{Spec}(A)$.

In the general case, let $\mathfrak{J}$ be an ideal of A such that $F = V(\mathfrak{J})$ and let Y be a connected component of $\mathrm{Spec}(A)$. By Corollary 1.4.47, there exists an element f of A such that Y is identified with the subset $\mathrm{Spec}(A_f)$ of $\mathrm{Spec}(A)$ (for example, if $Y = V(e)$ for some idempotent e, then we can choose $f = 1 - e$). Then $Y \cap F$ is identified with $V(\mathfrak{J}_f)$, and we have $\mathrm{depth}_{A_f}(\mathfrak{J}_f, A_f) \geq \mathrm{depth}_A(\mathfrak{J}, A) \geq 2$ in view of Proposition 9.1.28. It then follows from our previous arguments that $Y \cap U = Y - (Y \cap F)$ is connected and dense in Y. □

Corollary 9.1.42. *The map which associates each connected component of U with its closure in* $\mathrm{Spec}(A)$ *is a bijection from the set of connected components of U onto the set of connected components of* $\mathrm{Spec}(A)$.

Corollary 9.1.43. *For any Noetherian local ring B with* $\mathrm{depth}(B) \geq 2$, *the space* $\mathrm{Spec}(B) - \{\mathfrak{m}_B\}$ *is connected.*

Proof. This follows from the observation that any local ring has no idempotents, so $\mathrm{Spec}(B)$ is connected and we can apply Theorem 9.1.41 to $F = V(\mathfrak{m}_B) = \{\mathfrak{m}_B\}$. □

Corollary 9.1.44. *Under the hypotheses of Theorem 9.1.41, suppose that* $\mathrm{Spec}(A_\mathfrak{p})$ *is irreducible (resp. $A_\mathfrak{p}$ is integral) for any $\mathfrak{p} \in U$. Then* $\mathrm{Spec}(A_\mathfrak{p})$ *is irreducible (resp. $A_\mathfrak{p}$ is integral) for any $\mathfrak{p} \in$* $\mathrm{Spec}(A)$.

Proof. Let $(Y_i)_{i \in I}$ be the (finite) family of irreducible components of $\mathrm{Spec}(A)$. Let $\mathfrak{p} \in U$; as $\mathrm{Spec}(A_\mathfrak{p})$ is irreducible, $\mathfrak{p}$ contains a unique minimal prime ideal of A, so is contained in a unique Y_i. The intersection $Y_i \cap U$ is then nonempty disjoint and closed in U, dense in Y_i, and irreducible by **??**, so they form the connected components of U. The closures Y_i are then the connected components of $\mathrm{Spec}(A)$ by Corollary 9.1.42. This proves that the connected components of $\mathrm{Spec}(A)$ are irreducible, so that $\mathrm{Spec}(A_\mathfrak{p})$ is irreducible for any $\mathfrak{p} \in \mathrm{Spec}(A_\mathfrak{p})$.

Now suppose that $A_\mathfrak{q}$ is integral for each $\mathfrak{q} \in U$. Let $\mathfrak{p} \in \mathrm{Spec}(A)$, since $\mathrm{Spec}(A_\mathfrak{p})$ is irreducible, the nilradical of $A_\mathfrak{p}$ is the unique prime ideal of $A_\mathfrak{p}$, and it therefore belongs to $\mathrm{Ass}(A_\mathfrak{p})$ (Proposition 3.1.14), and equal to $\mathfrak{q}A_\mathfrak{p}$, where $\mathfrak{q}$ is an associated prime of A (Proposition 3.1.11). We have $\mathfrak{q} \in U$ by Theorem 9.1.41 and $\mathfrak{q}A_\mathfrak{q} \in \mathrm{Ass}(A_\mathfrak{q})$ (Proposition 3.1.11); since $A_\mathfrak{q}$ is integral, $\mathfrak{q}$ is then zero, so $A_\mathfrak{p}$ is also integral. □

Corollary 9.1.45. *Let A be a Noetherian ring with* $\mathrm{Spec}(A)$ *connected. Suppose that there exists an integer $d \geq 1$ such that we have* $\mathrm{depth}(A_\mathfrak{p}) \geq 2$ *for any prime ideal of A with* $\mathrm{ht}(\mathfrak{p}) > d$.

(a) *For any closed subset Z of* $\mathrm{Spec}(A)$ *with* $\mathrm{codim}(Z) > d$, *the space* $\mathrm{Spec}(A) - Z$ *is connected.*

(b) *Let Y and Y' be irreducible components of* $\mathrm{Spec}(A)$. *Then there exists a sequence $(X_1, \dots, X_n)$ of irreducible components of* $\mathrm{Spec}(A)$ *such that $X_1 = Y$, $X_n = Y'$, and for each $i = 1, \dots, n-1$, we have* $\mathrm{codim}(X_i \cap X_{i+1}) \leq d$.

Proof. Let $Z \subseteq \operatorname{Spec}(A)$ be a closed subset with $\operatorname{codim}(Z) > d$. For any $\mathfrak{p} \in Z$, we have $\dim(A_{\mathfrak{p}}) > d$, so $\operatorname{depth}(A_{\mathfrak{p}}) \geq 2$, which implies that $\operatorname{depth}_Z(A) \geq 2$ (Proposition 9.1.35). Then $\operatorname{Spec}(A) - Z$ is connected by Theorem 9.1.41.

To prove (b), denote by Z the union of subsets $X' \cap X''$ where (X', X'') runs through subsets of couples of irreducible components of $\operatorname{Spec}(A)$ such that $\operatorname{codim}(X' \cap X'') > d$. In view of (a), the subset $\operatorname{Spec}(A) - Z$ is then connected. All irreducible components of $\operatorname{Spec}(A)$ meet $\operatorname{Spec}(A) - Z$, and their trace over $\operatorname{Spec}(A) - Z$ are the irreducible components of $\operatorname{Spec}(A) - Z$ (**??**). Since $\operatorname{Spec}(A) - Z$ is connected, there exists a sequence $(X_1, \dots, X_n)$ of irreducible components of $\operatorname{Spec}(A)$ such that $X_1 - Z = Y - Z$, $X_n - Z = Y' - Z$, and $(X_i - Z) \cap (X_{i+1} - Z) \neq \varnothing$ for $1 \leq i \leq n-1$. The construction of Z then implies that $X_1 = Y$, $X_n = Y'$, and $\operatorname{codim}(X_i \cap X_{i+1}) \leq d$. □

Example 9.1.46. Let k be a field and $S = k[T_1, T_2, T_3, T_4]$ be the polynomial ring. Recall that any maximal chain of prime ideals of S has length 4 (Theorem 7.1.31). Let $\mathfrak{m}$ be the maximal ideal of S generated by the T_i, and for $i \leq i < j \leq 4$, let $\mathfrak{p}_{ij} = (T_i, T_j)$. The ideals $\mathfrak{p}_{ij}$ are then prime with height 2, and their sum is the maximal ideal $\mathfrak{m}$.

(a) Let $\mathfrak{a}$ be the ideal of S defined by $\mathfrak{a} = (T_1T_2, T_3T_4)$, and $A = S/\mathfrak{a}$. Then we have $\mathfrak{a} = \mathfrak{p}_{13} \cap \mathfrak{p}_{14} \cap \mathfrak{p}_{23} \cap \mathfrak{p}_{24}$, and the ring $A = S/\mathfrak{a}$ is reduced. The irreducible components of $\operatorname{Spec}(A)$ are the subsets $X_{ij} = V(\mathfrak{p}_{ij}/\mathfrak{a})$ for $i = 1, 2$, $j = 3, 4$, which are all of dimension 2 and contain the closed point $\mathfrak{m}/\mathfrak{a}$. In particular, $\operatorname{Spec}(A)$ is connected of dimension 2. The intersection of two distinct components X_{ij} and X_{kl} is reduced to $\{\mathfrak{m}/\mathfrak{a}\}$ if $\{i, j\} \cap \{k, l\} = \varnothing$, and is of dimension 1 otherwise. It then follows that the conclusion of Corollary 9.1.45 is valid for $d = 1$ (in fact A is Cohen-Macaulay).

(b) Let $\mathfrak{b}$ be the ideal of S defined by $\mathfrak{b} = (T_1T_2, T_1T_3, T_2T_4, T_3T_4)$, and $B = S/\mathfrak{b}$. Then we have $\mathfrak{b} = \mathfrak{p}_{14}\mathfrak{p}_{23} = \mathfrak{p}_{14} \cap \mathfrak{p}_{23}$, and the ring B is reduced. The space $\operatorname{Spec}(B)$ is identified with the closed subset $X_{14} \cup X_{23}$ of $\operatorname{Spec}(A)$, it have two irreducible components (of dimension 2), whose intersection reduces to $\{\mathfrak{m}/\mathfrak{b}\}$. The depth of B along this closed point is strictly positive because B is reduced, and smaller than 1 in view of Theorem 9.1.41 (since $\operatorname{Spec}(B) - \{\mathfrak{m}/\mathfrak{b}\}$ is not connected), so it is equal to 1 (the ring B is not Cohen-Macaulay).

9.1.5 Serre's criterion for normality

Let A be a Noetherian ring. We denote by $(Y_i)_{i \in I}$ the finite family of connected components of $\operatorname{Spec}(A)$. By Corollary 1.4.47, for each i there exists a unique idempotent e_i of A such that $Y_i = V(e_i)$, and the canonical homomorphism $A \to \prod_i A/Ae_i$ is bijective. The quotient rings A/Ae_i are called the canonical components of A. Put $f_i = 1 - e_i$, then we have $\sum_i f_i = 1$ and $(f_i)_{i \in I}$ is an orthogonal family of nonzero idempotents of A. It then follows that the image of f_i in A/Ae_j is equal to 1 if $j = i$, and to 0 otherwise. The canonical homomorphism $A \to \prod_j A/Ae_j$ then induces a canonical isomorphism $A_{f_i} \to A/Ae_i$.

By Proposition 1.4.14, we see that the following conditions are equivalent:

(i) the connected components of $\operatorname{Spec}(A)$ are irreducible;

(ii) each prime (resp. maximal) ideal of A belongs to a unique irreducible component of $\operatorname{Spec}(A)$;

(iii) each prime (resp. maximal) ideal of A contains a unique minimal prime ideal;

(iv) for any prime (resp. maximal) ideal $\mathfrak{p}$ of A, the topological space $\operatorname{Spec}(A_{\mathfrak{p}})$ is irreducible;

(v) for any canonical component C of A, the topological space $\operatorname{Spec}(C)$ is irreducible.

Now note that if A is reduced, then each ring $A_{\mathfrak{p}}$ is reduced, and the converse is also true by Corollary 1.2.39. Applying Corollary 1.4.10, we then deduce that the following conditions are equivalent:

(i) A is reduced and the connected components of $\operatorname{Spec}(A)$ are irreducible;

(ii) for any prime (resp. maximal) ideal of A, the ring $A_{\mathfrak{p}}$ is integral;

(iii) the canonical components of A are integral.

The Noetherian ring A is called **locally integral** if it satisfies the above equivalent condition. Suppose that A is locally integral; let $u : A \to \prod_{j\in J} A_j$ be an isomorphism from A to a (finite) product of integral rings. Then there exists a bijection $\sigma : J \to I$ such that the map from $\operatorname{Spec}(\prod_{j\in J} A_j)$ to $\operatorname{Spec}(A)$ associated with u defines a homeomorphism from $\operatorname{Spec}(A_j)$ to the connected component $Y_{\sigma(j)}$ of $\operatorname{Spec}(A)$. We then deduce that u is an isomorphism from the canonical component $A/Ae_{\sigma(j)}$ to A_j.

Proposition 9.1.47. *Let A be a Noetherian ring. The following conditions are equivalent:*

(i) A is reduced and integrally closed in its total fraction ring;

(ii) A is isomorphic to the product of a finite family of integrally closed rings;

(iii) the canonical components of A are integrally closed;

(iv) for any prime (resp. maximal) ideal $\mathfrak{p}$ of A, the ring $A_{\mathfrak{p}}$ is integrally closed.

Proof. The equivalence of (i) and (ii) follows from Corollary 4.1.24, and that of (ii) and (iii) follows from the preceding remarks. Let $\mathfrak{p}$ be a prime ideal of A, then there exists a unique canonical component A' of A such that $\mathfrak{p}$ belongs to the closed subset $\operatorname{Spec}(A')$ of $\operatorname{Spec}(A)$ and we have a canonical isomorphism $A_{\mathfrak{p}} \to A'_{\mathfrak{p}A'}$. The equivalence of (iii) and (iv) then follows from Proposition 4.1.39 and Proposition 4.1.42. □

A ring A is called **normal** if it is Noetherian and it satisfies the equivalent conditions of Proposition 9.1.47. With this notion, a Noetherian ring is integrally closed if and only if it is integral and normal. It is clear that a local normal ring is integrally closed.

We now prresent Serre's criterion for the normality of a Noetherian ring. To beging with, we give the following characterization for a Noetherian ring to be reduced. Recall that for any prime ideal $\mathfrak{p}$ of A, we have $\operatorname{depth}(A_{\mathfrak{p}}) \le \operatorname{ht}(\mathfrak{p})$ by Corollary 9.1.19.

Proposition 9.1.48. *Let A be a Noetherian ring. Then A is reduced if and only if it satisfies the following conditions:*

(R0) for any minimal prime ideal, $A_{\mathfrak{p}}$ is a regular local ring;

(S1) for any prime ideal $\mathfrak{p}$ of A with $\operatorname{ht}(\mathfrak{p}) \ge 1$, we have $\operatorname{depth}(A_{\mathfrak{p}}) \ge 1$.

Proof. Denote by $\mathfrak{n}$ the nilradical of A. If A is reduced, then each local ring $A_{\mathfrak{p}}$ is reduced, so if $\operatorname{ht}(\mathfrak{p}) = 0$, then $A_{\mathfrak{p}}$ is a field, and if $\operatorname{ht}(\mathfrak{p}) \ge 1$, then $\operatorname{depth}(A_{\mathfrak{p}}) \ge 1$ (Example 9.1.2). Conversely, assume that A satisfies the above conditions. Then for any minimal prime ideal $\mathfrak{p}$ of A, we have $\mathfrak{n}_{\mathfrak{p}} = 0$ by condition (R0), which means $\mathfrak{p} \notin \operatorname{supp}(\mathfrak{n})$, and a fortiori $\mathfrak{p} \notin \operatorname{Ass}(\mathfrak{n})$. For any $\mathfrak{p} \in \operatorname{Spec}(A)$ with $\operatorname{ht}(\mathfrak{p}) \ge 1$, by condition (S1) and Example 9.1.2 we have $\mathfrak{p}A_{\mathfrak{p}} \notin \operatorname{Ass}_{A_{\mathfrak{p}}}(A_{\mathfrak{p}})$ and a fortiori $\mathfrak{p}A_{\mathfrak{p}} \notin \operatorname{Ass}_{A_{\mathfrak{p}}}(\mathfrak{n}_{\mathfrak{p}})$ (Proposition 3.1.7), so $\mathfrak{p} \notin \operatorname{Ass}_A(\mathfrak{n})$ (Proposition 3.1.11). We then conclude that $\operatorname{Ass}(\mathfrak{n}) = \varnothing$, so $\mathfrak{n} = 0$ and A is reduced. □

Proposition 9.1.49. *Let A be an integrally closed Noetherian ring, $\mathfrak{I}$ be an ideal of A with $\operatorname{ht}(\mathfrak{I}) \ge 2$, and M be a finitely generated reflexive A-module. Then we have $\operatorname{depth}_A(\mathfrak{I}, M) \ge 2$.*

Proof. Since M is reflexive, we can choose a finite dimensional vector space V over the fraction field of A such that M is a lattice in V (Remark 6.2.14). The associated primes of V/M, being of height 1 (Theorem 6.2.15), then do not belong to $V(\mathfrak{I})$. By Example 9.1.2, we then conclude that $\mathrm{depth}_A(\mathfrak{I}, V/M) \geq 1$. On the other hand, the A-module V is divisible and torsion free, hence injective, and this implies $\mathrm{depth}_A(\mathfrak{I}, V) = +\infty$ by Example 9.1.1. The inequality $\mathrm{depth}_A(\mathfrak{I}, M) \geq 2$ then follows from Proposition 9.1.4. □

Corollary 9.1.50. *An integrally closed Noetherian local ring of dimension ≥ 2 has depth ≥ 2.*

Proof. This follows from the fact that an integrally closed Noetherian local ring A is Krull (Theorem 6.1.15), hence reflexive (Theorem 6.1.29), so we can apply Proposition 9.1.49 to $\mathfrak{I} = \mathfrak{m}_A$. □

Theorem 9.1.51 (Serre's Normality Criterion). *Let A be a Noetherian ring. Then A is normal if and only if it satisfies the following properties:*

(R1) for any prime ideal $\mathfrak{p}$ of A with $\mathrm{ht}(\mathfrak{p}) \leq 1$, $A_\mathfrak{p}$ is a regular local ring.

(S2) for any prime ideal $\mathfrak{p}$ of A with $\mathrm{ht}(\mathfrak{p}) \geq 2$, we have $\mathrm{depth}(A_\mathfrak{p}) \geq 2$.

Proof. By definition, if A is normal, then $A_\mathfrak{p}$ is integrally closed for any prime ideal $\mathfrak{p}$ of A, so condition (S2) follows from Corollary 9.1.50. Also, it is clear that an integrally closed Noetherian local ring of dimension ≤ 1 must be regular. Conversely, assume that A satisfies condition (R1) and (S2). We show that $A_\mathfrak{p}$ is integrally closed by recurrence on $\mathrm{ht}(\mathfrak{p})$. For $\mathrm{ht}(\mathfrak{p}) \leq 1$, this follows directly from condition (R1). Suppose then that $\mathrm{ht}(\mathfrak{p}) \geq 2$ and that $A_\mathfrak{q}$ is integrally closed for any prime ideal $\mathfrak{q}$ with $\mathrm{ht}(\mathfrak{q}) < \mathrm{ht}(\mathfrak{p})$. By condition (S2), we have $\mathrm{depth}(A_\mathfrak{p}) \geq 2$. By the recurrence hypotheses and Corollary 9.1.44 applied to the ring $A_\mathfrak{p}$ and the closed subset $\{\mathfrak{p}A_\mathfrak{p}\}$ of $\mathrm{Spec}(A_\mathfrak{p})$, we then conclude that $A_\mathfrak{p}$ is integral. Let K be the fraction field of $A_\mathfrak{p}$ and B be a subring of K which is finite over $A_\mathfrak{p}$. It then suffices to prove that $B = A_\mathfrak{p}$, so let $i : A_\mathfrak{p} \to B$ be the canonical injection. As B is contained in K, it is a torsion-free $A_\mathfrak{p}$-module, so $\mathrm{depth}_{A_\mathfrak{p}}(B) \geq 1$. On the other hand, for any prime ideal $\mathfrak{q}$ of $A_\mathfrak{p}$ distinct from $\mathfrak{p}A_p$, the homomorphism $i_\mathfrak{q} : (A_\mathfrak{p})_\mathfrak{q} \to B_\mathfrak{q}$ is bijective since $A_\mathfrak{q}$ is integrally closed by hypotheses. By Lemma 9.1.39 applied to the closed subset $F = \{\mathfrak{p}A_\mathfrak{p}\}$ of $\mathrm{Spec}(A_\mathfrak{p})$, the homomorphism i is then bijective, which proves our assertion. □

Remark 9.1.52. A convenient form of Theorem 9.1.51 is the following: let A be a Noetherian ring such that for any prime ideal $\mathfrak{p}$ of A, the ring $A_\mathfrak{p}$ is either integrally closed, or $\mathrm{depth}(A_\mathfrak{p}) \geq 2$, then A is normal. In fact, in this case if $\mathrm{ht}(\mathfrak{p}) \leq 1$ then $\mathrm{depth}(A_\mathfrak{p}) \leq \mathrm{ht}(\mathfrak{p}) \leq 1$, so $A_\mathfrak{p}$ is integrally closed. If $\mathrm{ht}(\mathfrak{p}) \geq 2$, then Noe integral closed depth geq 2 if dim geq 2 implies in both cases that $\mathrm{depth}(A_\mathfrak{p}) \geq 2$, so we can apply Theorem 9.1.51. Similarly, we see if for any prime ideal $\mathfrak{p}$ of A, the ring $A_\mathfrak{p}$ is either regular of $\mathrm{depth}(A_\mathfrak{p}) \geq 1$, then A is reduced.

Corollary 9.1.53. *Let A be a Noetherian ring, F be a closed subset of $\mathrm{Spec}(A)$, and U be its complement. Suppose that $\mathrm{depth}_F(A) \geq 2$ (resp. $\mathrm{depth}_F(A) \geq 1$) and that for any $\mathfrak{p} \in U$, the ring $A_\mathfrak{p}$ is integrally closed (resp. reduced). Then A is normal (resp. reduced).*

Proof. For any $\mathfrak{p} \in F$, we have $\mathrm{depth}(A_\mathfrak{p}) \geq \mathrm{depth}_F(A)$ by Proposition 9.1.35, so it suffices to apply the preceding remark. □

Corollary 9.1.54. *Let $\rho : A \to B$ be a flat homomorphism of rings.*

(a) If B is normal and faithfully flat over A, then A is normal.

(b) Suppose that A is normal and the ring $\kappa(\mathfrak{p}) \otimes_A B$ is normal (resp. reduced) for any minimal prime ideal $\mathfrak{p}$ (resp. prime ideal $\mathfrak{p}$ of height one) of A. Then the ring B is normal.

Corollary 9.1.55. *Let $\rho : A \to B$ be a homomorphism of Noetherian rings. Suppose that B is a flat A-module, that A is normal, and that $\kappa(\mathfrak{p}) \otimes_A B$ is normal for any $\mathfrak{p} \in \mathrm{Spec}(A)$. Then B is normal.*

9.2 Cohen-Macaulay rings and modules

9.2.1 Cohen-Macaulay modules

Let A be a Noetherian ring, M be a finitely generated A-module, and $\mathfrak{p}$ be a prime ideal of A. If $\mathfrak{p} \notin \mathrm{supp}(M)$, then we have $M_{\mathfrak{p}} = 0$, so $\mathrm{depth}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}) = +\infty$. If $\mathfrak{p} \in \mathrm{supp}(M)$, then by Corollary 9.1.25,

$$0 \le \mathrm{depth}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}) \le \dim_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}) < +\infty.$$

The module M is called **Cohen-Macaulay** if for any maximal ideal $\mathfrak{m} \in \mathrm{supp}(M)$, we have $\mathrm{depth}_{A_{\mathfrak{m}}}(M_{\mathfrak{m}}) = \dim_{A_{\mathfrak{m}}}(M_{\mathfrak{m}})$. If A is local, then for a finitely generated nonzero A-module to be Cohen-Macaulay, it is necessary and sufficient that its depth is equal to its dimension.

Example 9.2.1. Any A-module of finite length is Cohen-Macaulay. In fact, if M is a module of finite length, then $M_{\mathfrak{m}}$ is of finite length for any maximal ideal $\mathfrak{m}$ of A (Corollary 3.2.16), and we have $\dim_{A_{\mathfrak{m}}}(M_{\mathfrak{m}}) = 0$ by Proposition 3.2.14, whence $\mathrm{depth}_{A_{\mathfrak{m}}}(M_{\mathfrak{m}}) = 0$.

Example 9.2.2. Let N be a direct factor of a finitely generated Cohen-Macaulay A-module M. Then N is Cohen-Macaulay; in factm for any maximal ideal $\mathfrak{m}$ of A, the $A_{\mathfrak{m}}$-module $N_{\mathfrak{m}}$ is a direct factor of $M_{\mathfrak{m}}$, and we therefore have

$$\mathrm{depth}_{A_{\mathfrak{m}}}(N_{\mathfrak{m}}) \ge \mathrm{depth}_{A_{\mathfrak{m}}}(M_{\mathfrak{m}}) \ge \dim_{A_{\mathfrak{m}}}(M_{\mathfrak{m}}) \ge \dim_{A_{\mathfrak{m}}}(N_{\mathfrak{m}}),$$

where we use Example 9.1.3 and Proposition 7.1.12.

Example 9.2.3. Let M be a finitely generated Cohen-Macaulay A-module and $(x_1, \dots, x_n)$ be an M-regular sequence for elements of A. Then the A-module $\overline{M} = M/(x_1M + \cdots + x_nM)$ is Cohen-Macaulay. To see this, let $\mathfrak{m}$ be a maximal ideal of A belonging to the support of $\overline{M}$; we have $x_i \in \mathfrak{m}$ for each i since x_i annihilates $\overline{M}$, and the canonical image of x_i in $A_{\mathfrak{m}}$ form an $M_{\mathfrak{m}}$-regular sequence of elements of $\mathfrak{m}A_{\mathfrak{m}}$. We then have the equalities (Proposition 9.1.16 and Corollary 7.2.12)

$$\mathrm{depth}_{A_{\mathfrak{m}}}(\overline{M}_{\mathfrak{m}}) = \mathrm{depth}_{A_{\mathfrak{m}}}(M_{\mathfrak{m}}) - n, \quad \dim_{A_{\mathfrak{m}}}(\overline{M}_{\mathfrak{m}}) = \dim_{A_{\mathfrak{m}}}(M_{\mathfrak{m}}) - n.$$

whence our assertion.

Example 9.2.4. Let M be a finitely generated A-module and $\mathfrak{a}$ be an ideal of A such that $\mathfrak{a}M = 0$. For that the A-module M to be Cohen-Macaulay, it is necessary and sufficient that it is Cohen-Macaulay as an $(A/\mathfrak{a})$-module. In fact, put $B = A/\mathfrak{a}$; let $\mathfrak{M}$ be the maximal ideal of B and $\mathfrak{m}$ be its inverse image in A. We then have $\dim_{A_{\mathfrak{m}}}(M_{\mathfrak{m}}) = \dim_{B_{\mathfrak{M}}}(M_{\mathfrak{M}})$ and $\mathrm{depth}_{A_{\mathfrak{m}}}(M_{\mathfrak{m}}) = \mathrm{depth}_{B_{\mathfrak{M}}}(M_{\mathfrak{M}})$ (Proposition 9.1.27).

Proof. This follows directly from Corollary 9.1.24 and depth of module prime ideal inclusion inequality. □

Proposition 9.2.5. *Let A be a Noetherian ring and M be a finitely generated A-module. Then the following conditions are equivalent:*

(i) the A-module M is Cohen-Macaulay;

(ii) $\mathrm{depth}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}) = \dim_{A_{\mathfrak{p}}}(M_{\mathfrak{p}})$ *for any* $\mathfrak{p} \in \mathrm{supp}(M)$*;*

(iii) $\mathrm{depth}_F(M) = \mathrm{codim}(\mathrm{supp}(M) \cap F, \mathrm{supp}(M))$ *for any closed subset F of* $\mathrm{Spec}(A)$.

(iv) $\mathrm{depth}_A(\mathfrak{p}, M) = \dim_{A_{\mathfrak{p}}}(M_{\mathfrak{p}})$ *for any* $\mathfrak{p} \in \mathrm{supp}(M)$.

Proof. We first note that (i)⇒(ii) in view of Corollary 9.1.24 and depth of module prime ideal inclusion inequality. For (ii)⇒(iii), note that by Proposition 9.1.35, $\operatorname{depth}_F(M)$ is the infermum of the integers $\operatorname{depth}_{A_\mathfrak{p}}(M_\mathfrak{p})$ for $\mathfrak{p} \in \operatorname{supp}(M) \cap F$. If M is Cohen-Macaulay, then for each such ideal $\mathfrak{p}$ we have the equality

$$\operatorname{depth}_{A_\mathfrak{p}}(M_\mathfrak{p}) = \dim_{A_\mathfrak{p}}(M_\mathfrak{p}) = \operatorname{codim}(V(\mathfrak{p}), \operatorname{supp}(M))$$

(Proposition 7.1.12), so (iii) follows from the definition of $\operatorname{codim}(\operatorname{supp}(M) \cap F, \operatorname{supp}(M))$. On the other hand, it is clear that (iii)⇒(iv) by taking $F = V(\mathfrak{p})$ (Proposition 7.1.12), and (iv)⇒(i) since we have the inequality $\operatorname{depth}_A(\mathfrak{p}, M) \le \operatorname{depth}_{A_\mathfrak{p}}(M_\mathfrak{p}) \le \dim(M_\mathfrak{p})$ for any $\mathfrak{p} \in \operatorname{supp}(M)$ (Remark 9.1.36). □

Corollary 9.2.6. *Let A be a Noetherian ring and M be a finitely generated A-module. Then for any prime ideals $\mathfrak{p} \subseteq \mathfrak{q}$ of $\operatorname{supp}(M)$, we have*

$$\operatorname{depth}_{A_\mathfrak{q}}(M_\mathfrak{q}) - \operatorname{depth}_{A_\mathfrak{p}}(M_\mathfrak{p}) = \dim(A_\mathfrak{q}/\mathfrak{p}A_\mathfrak{q}) = \dim_{A_\mathfrak{q}}(M_\mathfrak{q}) - \dim_{A_\mathfrak{p}}(M_\mathfrak{p}).$$

Proof. This follows from Proposition 9.1.23 and $\dim(M_\mathfrak{q}) \ge \dim(M_\mathfrak{p}) + \dim(A_\mathfrak{q}/\mathfrak{p}A_\mathfrak{q})$ (Proposition 7.1.12(a) and **??**(b)) □

Corollary 9.2.7. *Let S be a multiplicative subset of A and M be a finitely generated Cohen-Macaulay A-module. Then $S^{-1}M$ is a Cohen-Macaulay $S^{-1}A$-module.*

Proof. In fact, let $\mathfrak{q} \in \operatorname{Spec}(S^{-1}A)$, $i_A^S : A \to S^{-1}A$ be the canonical homomorpism, and $\mathfrak{p} = (i_A^S)^{-1}(\mathfrak{q})$. The ring $(S^{-1}A)_\mathfrak{q}$ is then identified with $A_\mathfrak{p}$, and the $A_\mathfrak{p}$-module $(S^{-1}M)_\mathfrak{q}$ is identified with the $A_\mathfrak{p}$-module $M_\mathfrak{p}$, whence the corolary. □

Proposition 9.2.8. *Let A be a Noetherian ring and M be a finitely generated Cohen-Macaulay A-module.*

(a) The A-module M has no embedded associated prime ideals.

(b) Let X be a closed irreducible subset of $\operatorname{supp}(M)$ and Y be a closed subset of X, then we have

$$\operatorname{codim}(Y, X) + \operatorname{codim}(X, \operatorname{supp}(M)) = \operatorname{codim}(Y, \operatorname{supp}(M)).$$

(c) The topological space $\operatorname{supp}(M)$ is catenary.

(d) Let X_1 and X_2 be irreducible components of $\operatorname{supp}(M)$ and Y be a closed subset of $X_1 \cap X_2$. Then we have $\operatorname{codim}(Y, X_1) = \operatorname{codim}(Y, X_2)$.

Proof. Let $\mathfrak{p} \in \operatorname{Ass}(M)$; we have $\operatorname{depth}_A(\mathfrak{p}, M) = 0$ (Example 9.1.2), so $\dim_{A_\mathfrak{p}}(M_\mathfrak{p}) = 0$ by Proposition 9.2.5. By Proposition 7.1.12, this implies that $\mathfrak{p}$ is a minimal element of $\operatorname{supp}(M)$, so the assertion in (a) follows.

To prove the assertion in (b), it suffices to let X be a closed irreducible subset of $\operatorname{supp}(M)$ and Y be a closed irreducible subset of X. Let $\mathfrak{p}$ and $\mathfrak{q}$ be prime ideals of $\operatorname{supp}(M)$ such that $Y = V(\mathfrak{q})$ and $X = V(\mathfrak{p})$, then it follows from Corollary 9.2.6 that

$$\begin{aligned}\operatorname{codim}(Y, X) &= \dim(A_\mathfrak{q}/\mathfrak{p}A_\mathfrak{q}) = \dim(M_\mathfrak{q}) - \dim(M_\mathfrak{p}) \\ &= \operatorname{codim}(Y, \operatorname{supp}(M)) - \operatorname{codim}(X, \operatorname{supp}(M)).\end{aligned}$$

Now if X, Y, Z are closed irreducible subsets of $\operatorname{supp}(M)$ such that $Z \subseteq Y \subseteq X$, then the codimensions of them in $\operatorname{supp}(M)$ are finite, and we deduce from (b) the equality

$$\operatorname{codim}(Z, Y) + \operatorname{codim}(Y, X) = \operatorname{codim}(Z, X)$$

so we conclude from **??** that $\operatorname{supp}(M)$ is catenary. Finally, if X_1 and X_2 are irreducible components of $\operatorname{supp}(M)$ and Y be a closed subset of $X_1 \cap X_2$, then

$$\operatorname{codim}(Y, X_1) = \operatorname{codim}(Y, \operatorname{supp}(M)) = \operatorname{codim}(Y, X_2).$$ □

In particular, if there exists a finitely generated A-module M such that $\operatorname{supp}(M) = \operatorname{Spec}(A)$, then the ring A is catenary and any fraction ring of quotient ring of A is catenary.

Remark 9.2.9. Under the hypotheses of Proposition 9.2.8, it may happen that two components irreducible X_1 and X_2 of $\operatorname{supp}(M)$ have an intersection Y reducing to a point and that $\dim(X_1) \neq \dim(X_2)$ and $\dim(X_2) \neq \operatorname{codim}(Y, X_2)$. However, this cannot happen if A is local, as shown in the next corollary.

Corollary 9.2.10. *Let A be a Noetherian local ring and M be a finitely generated nonzero Cohen-Macaulay ring.*

(a) Any maximal chain of irreducible closed subsets of $\operatorname{supp}(M)$ has length equal to $\dim(M)$.

(b) For any closed subset X of $\operatorname{supp}(M)$, we have

$$\operatorname{codim}(X, \operatorname{supp}(M)) = \dim(\operatorname{supp}(M)) - \dim(X).$$

(c) All irreducible components of $\operatorname{supp}(M)$ have the same dimension.

(d) For any ideal $\mathfrak{I}$ of A, we have

$$\operatorname{depth}_A(\mathfrak{I}, M) = \dim(M) - \dim(M/\mathfrak{I}M).$$

Proof. A maximal chain of irreducible closed subsets of $\operatorname{supp}(M)$ has smallest element $\{\mathfrak{m}_A\}$ and largest element a irreducible component X of $\operatorname{supp}(M)$. Its length is therefore equal to the codimension of $\{\mathfrak{m}_A\}$ in X (Proposition 9.2.8(c)); by Proposition 9.2.8(b), this is equal to $\operatorname{codim}(\{\mathfrak{m}_A\}, \operatorname{supp}(M))$, which is $\dim(M)$. Now (b) is a concequence of (a) if the subset X is irreducible, and the general case follows from (**??**), and (c) is a concequence of (b). Finally, for any ideal $\mathfrak{I}$, we have $\operatorname{depth}_A(\mathfrak{I}, M) = \operatorname{codim}(\operatorname{supp}(M) \cap V(\mathfrak{I}), \operatorname{supp}(M))$ by Proposition 9.2.5, so to prove (d) it suffices to apply (b) to $X = \operatorname{supp}(M) \cap V(\mathfrak{I}) = \operatorname{supp}(M/\mathfrak{I}M)$. □

We now consider Cohen-Macaulay modules over Noetherian local rings. In particular, we will consider the behaviour of these modules with regular sequence of $\mathfrak{m}_A$.

Proposition 9.2.11. *Let A be a Noetherian local ring, M be a finitely generated nonzero A-module of dimension d. The following conditions are equivalent:*

(i) the A-module M is Cohen-Macaulay;

(ii) $\operatorname{depth}(M) = d$;

(iii) $\operatorname{Ext}^i_A(\kappa_A, M) = 0$ for any integer $i < d$;

(iv) $\operatorname{Ext}^i_A(N, M) = 0$ for any A-module N of finite length and any integer $i < d$.

(v) $\operatorname{Ext}^i_A(N, M) = 0$ for any finitely generated A-module N and any integer $i < d - \dim(M \otimes_A N)$.

(vi) there exists an M-regular sequence of elements of $\mathfrak{m}_A$ of length d.

Proof. The equivalences of (i), (ii), (iii), (iv) and (vi) are clear from our definition. As for (v), note that it is clear that (v)⇒(iv), since if N is of finite length, then $\dim(M \otimes_A N) = 0$ for any finitely generated A-module M (Proposition 3.2.14 and Proposition 1.4.39). Conversely, suppose that M is Cohen-Macaulay and let N be a finitely generated A-module. Put $F = \operatorname{supp}(N)$, then by Proposition 1.4.39 we have $\operatorname{supp}(M) \cap F = \operatorname{supp}(M \otimes_A N)$, so that

$$\operatorname{depth}_F(M) = \operatorname{codim}(\operatorname{supp}(M) \cap F, \operatorname{supp}(M)) = \dim(M) - \dim(M \otimes_A N)$$

(Proposition 9.2.5 and Corollary 9.2.10). The implication (i)⇒(v) then follows from Proposition 9.1.37. □

A finitely generated module M over a Noetherian local ring is called **pure** if for any associated prime ideal $\mathfrak{p}$ of M, we have $\dim(A/\mathfrak{p}) = \dim(M)$. This signifies that M has no embedded associated primes and that the irreducible components of M have the same dimension. As we have seen (Corollary 9.2.10), every Cohen-Macaulay module over a Noetherian local ring is pure.

Lemma 9.2.12. *Let A be a Noetherian local ring, M be a finitely generated pure A-module, and x be an element of $\mathfrak{m}_A$. Then the following conditions are equivalent:*

(i) $\dim(M/xM) = \dim(M) - 1$;

(ii) the homothety x_M is injective.

Proof. We can suppose that M is nonzero. The assertion in (i) is equivalent to that fact that x does not belong to any minimal element $\mathfrak{p}$ of $\mathrm{supp}(M)$ such that $\dim(A/\mathfrak{p}) = \dim(M)$ (Proposition 7.2.11), while the assertion in (ii) is equivalent to the fact that x does not belong to any associated prime ideal of M (Corollary 3.1.5). Since M is pure, we see these two conditions are equivalent. □

Let A be a Noetherian local ring and M be a finitely generated nonzero A-module. Recall that a sequence $(x_1, \ldots, x_r)$ of elements of $\mathfrak{m}_A$ is said to be *secant* for M if we have

$$\dim(M/(x_1M + \cdots + x_rM)) = \dim(M) - r.$$

Proposition 9.2.13. *Let A be a Noetherian local ring, M be a finitely generated nonzero A-module, and $(x_1, \ldots, x_r)$ be a sequence of elements of $\mathfrak{m}_A$ that is secant for M. Then the following conditions are equivalent:*

(i) the A-module M is Cohen-Macaulay;

(ii) the sequence $(x_1, \ldots, x_r)$ is M-regular and the A-module $M/(x_1M + \cdots + x_rM)$ is Cohen-Macaulay.

Proof. Suppose that the sequence $(x_1, \ldots, x_r)$ is M-regular. Then by Proposition 9.1.16 we have

$$\dim(M) = r + \dim(M/(x_1M + \cdots + x_rM)), \quad \mathrm{depth}(M) = r + \mathrm{depth}(M/(x_1M + \cdots + x_rM))$$

whence the implication (ii)⇒(i). Suppose now that the A-module M is Cohen-Macaulay, we prove (ii) by recurrence on r. The assertion is evident if $r = 0$, so assume that $r \geq 1$; the A-module $N = M/(x_1M + \cdots + x_{r-1}M)$ is then Cohen-Macaulay by hypotheses and we have $\dim(N/x_rN) = \dim(N) - 1$ since the sequence $(x_1, \ldots, x_r)$ is secant. The hypothety $(x_r)_N$ is then injective by Lemma 9.2.12, and N/x_rN is Cohen-Macaulay by Example 9.2.3, whence (ii). □

Theorem 9.2.14. *Let A be a Noetherian local ring, M be a finitely generated nonzero A-module of dimension d, $\boldsymbol{x} = (x_1, \ldots, x_r)$ be a sequence of elements of $\mathfrak{m}_A$ that is secant for M, and $\mathfrak{I}$ be the ideal generated by $\boldsymbol{x}$. Then the following conditions are equivalent:*

(i) the A-module M is Cohen-Macaulay;

(ii) the sequence $\boldsymbol{x}$ is regular for M;

(iii) the sequence $\boldsymbol{x}$ is completely secant for M;

(iv) the multiplicity $e_{\mathfrak{I}}(M)$ of M relative to $\mathfrak{I}$ is equal to the length of the A-module $M/\mathfrak{I}M$;

(v) for each integer $1 \leq i \leq d$, the A-module $M/(x_1M + \cdots + x_{i-1}M)$ is pure.

Proof. The equivalence of (ii) and (iii) follows from **??**. The A-moduele $M.\mathfrak{I}M$ being of finite length (Theorem 7.2.14), the equivalence of (iii) and (iv) follows from Proposition 7.3.18.

It remains to see that equivalence of (i), (ii) and (v). If M is Cohen-Macaulay, then each $M/(x_1M+\cdots+x_{i-1}M)$ is Cohen-Macaulay by Example 9.2.3, hence pure. Conversely, by Noe local pure module secant element iff homothety injective, we see that (v)⇒(ii); the fact that (ii)⇒(i) follows from Proposition 9.2.13, since $M/\mathfrak{I}M$ is of finite length, hence Cohen-Macaulay. □

9.2.2 Strongly secant subsets

Let A be a Noetherian ring, M be a finitely generated A-module, and S be a subset of A. We denote by SM the submodule $\sum_{s\in S} sM$ of M, and $\mathfrak{S}$ the ideal generated by S.

Lemma 9.2.15. *Let $\bar{\mathfrak{S}}$ be the image of $\mathfrak{S}$ in $A/\mathrm{Ann}(M)$, then*

$$\mathrm{ht}(\bar{\mathfrak{S}}) = \mathrm{codim}(\mathrm{supp}(M/SM), \mathrm{supp}(M)).$$

If $M \neq SM$, then we have $\mathrm{ht}(\bar{\mathfrak{S}}) \leq |S|$.

Proof. We denote by $\mathfrak{a}$ the annihilator of M. By Corollary 1.4.40, the support of the A-module M/SM is equal to $V(\mathfrak{S}+\mathfrak{a})$, its codimension in $\mathrm{supp}(M)$ is therefore equal to the codimension of $V(\mathfrak{S}+\mathfrak{a})$ in $V(\mathfrak{a})$, which is also the codimension of $V((\mathfrak{S}+\mathfrak{a})/\mathfrak{a})$ in $\mathrm{Spec}(A/\mathfrak{a})$; this is exactly the height of $\bar{\mathfrak{S}}$. If $M \neq SM$, then the inequality $\mathrm{ht}(\bar{\mathfrak{S}}) \leq |S|$ is evident if S is infinite, and follows from Corollary 7.2.19 if S is finite. □

Let A be a Noetherian ring, M be a finitely generated A-module, and S be a subset of A. We say that the subset S is **strongly secant** for M if the reverse inequality of Lemma 9.2.15 is true, in other words, if we have

$$|S| \leq \mathrm{codim}(\mathrm{supp}(M/SM), \mathrm{supp}(M)).$$

From this definition, we see that any finite subset S of A such that $SM = M$ is strongly secant for M, and if $SM \neq M$, then for a subset S to be strongly secant for M, it is necessary and sufficient that

$$|S| = \mathrm{ht}(\bar{\mathfrak{S}}) = \mathrm{codim}(\mathrm{supp}(M/SM), \mathrm{supp}(M)).$$

Example 9.2.16. If A is a Noetherian local ring and M is a nonzero finitely generated A-module, then any subset S of $\mathfrak{m}_A$ that is strongly secant for M is secant for M. In fact, as the A-module M/SM is nonzero by Nayakama's lemma, we have

$$|S| \leq \mathrm{codim}(\mathrm{supp}(M/SM), \mathrm{supp}(M)) \leq \dim(M) - \dim(M/SM)$$

(**??**), whence our assertion.

Proposition 9.2.17. *Let A be a Noetherian ring, M be a finitely generated A-module, and S be a finite subset of A. Then the following conditions are equivalent:*

(i) S is strongly secant for M;

(ii) for any prime ideal $\mathfrak{p}$ of $\mathrm{supp}(M/SM)$, the canonical map $i_{\mathfrak{p}} : A \to A_{\mathfrak{p}}$ induces a bijection from S onto a subset of $\mathfrak{p}A_{\mathfrak{p}}$ that is secant for $M_{\mathfrak{p}}$.

Proof. Let $\mathfrak{p} \in \mathrm{supp}(M/SM)$ and $\tilde{S}$ be the image of S in $A_{\mathfrak{p}}$. The set $\tilde{S}$ is contained in the maximal ideal $\mathfrak{p}A_{\mathfrak{p}}$, and we have (Proposition 7.1.12)

$$\dim(M_{\mathfrak{p}}/\tilde{S}M_{\mathfrak{p}}) = \mathrm{codim}(V(\mathfrak{p}), \mathrm{supp}(M/SM)).$$

If S is strongly secant for M, the inequality $|S| \leq \operatorname{codim}(\operatorname{supp}(M/SM), \operatorname{supp}(M))$ and **??**(b) implies the relations

$$|S| + \dim(M_{\mathfrak{p}}/\tilde{S}M_{\mathfrak{p}}) \leq \operatorname{codim}(V(\mathfrak{p}), \operatorname{supp}(M)) = \dim(M_{\mathfrak{p}}).$$

As $M_{\mathfrak{p}}$ is nonzero, we also have $\dim(M_{\mathfrak{p}}) \leq |\tilde{S}| + \dim(M_{\mathfrak{p}}/\tilde{S}M_{\mathfrak{p}})$ by formula (7.2.3). Condition (ii) then follows from the inequality $|\tilde{S}| \leq |S|$.

Conversely, assume condition (ii); we may suppose that $M \neq SM$. For any prime ideal $\mathfrak{p} \in \operatorname{supp}(M/SM)$,

$$|S| = |\tilde{S}| \leq \dim(M_{\mathfrak{p}}) = \operatorname{codim}(V(\mathfrak{p}), \operatorname{supp}(M))$$

this implies (i) by passing to infermum, in view of the definition of codimensions. □

Corollary 9.2.18. *Let A be a Noetherian ring and M be a finitely generated A-module. Then any M-regular sequence is strongly secant for M.*

Proof. Let $\boldsymbol{x}$ be a M-regular sequence, and $\mathfrak{I}$ be the ideal it generates. For any prime ideal $\mathfrak{p} \in \operatorname{supp}(M/\mathfrak{I}M)$, the image of $\boldsymbol{x}$ in $A_{\mathfrak{p}}$ is an M-regular sequence of elements of $\mathfrak{p}A_{\mathfrak{p}}$, so it is secant for $M_{\mathfrak{p}}$. □

Proposition 9.2.19. *Let A be a Noetherian ring, M be a finitely generated Cohen-Macaulay A-module, and S be a finite subset of A that is strongly secant for M. Then the A-module M/SM is Cohen-Macaulay.*

Proof. For any maximal ideal $\mathfrak{m} \in \operatorname{supp}(M/SM)$, the image of S in $A_{\mathfrak{m}}$ is secant for $M_{\mathfrak{m}}$ (Proposition 9.2.17). Since $M_{\mathfrak{m}}$ is a Cohen-Macaulay $A_{\mathfrak{m}}$-module (Corollary 9.2.7), so is $(M/SM)_{\mathfrak{m}}$ by Proposition 9.2.13, whence the proposition. □

Theorem 9.2.20 (Cohen-Macaulay). *Let A be a Noetherian ring and M be a finitely generated A-module. Then the following conditions are equivalent:*

(i) the A-module is Cohen-Macaulay;

(ii) for any ideal $\mathfrak{I}$ of A generated by an M-regular sequence of elements of A, the A-module $M/\mathfrak{I}M$ has no embedded associated prime ideals;

(iii) for any finite subset S of A that is strongly secant for M, the A-module M/SM has no embedded associated prime ideals;

Proof. If M is Cohen-Macaulay and S is a finite subset that is strongly secant for M, the A-module M/SM is Cohen-Macaulay by Proposition 9.2.19, and in particular it has no embedded associated primes by Proposition 9.2.8. This proves (i)⇒(iii), and (iii)⇒(ii) follows from Corollary 9.2.18.

Finally, to prove that (ii)⇒(i), let $\mathfrak{p} \in \operatorname{supp}(M)$, we prove that the $A_{\mathfrak{p}}$-module $M_{\mathfrak{p}}$ is Cohen-Macaulay by recurrence on the integer $\dim(M_{\mathfrak{p}})$. If $\dim(M_{\mathfrak{p}}) = 0$, then $M_{\mathfrak{p}}$ is of finite length, so is Cohen-Macaulay (Example 9.2.1). Suppose that $\dim(M_{\mathfrak{p}}) > 0$, which means that $\mathfrak{p}$ is not a minimal element of $\operatorname{supp}(M)$. As M has no embedded associated primes, each element in $\operatorname{Ass}(M)$ is minimal, so $V(\mathfrak{p}) \cap \operatorname{Ass}(M) = \varnothing$ and there exists an element x of $\mathfrak{p}$ such that the homothety x_M is injective (Example 9.1.2). The homothety $x_{M_{\mathfrak{p}}}$ is then injective and we have $\dim(M_{\mathfrak{p}}/xM_{\mathfrak{p}}) < \dim(M_{\mathfrak{p}})$ by Proposition 7.2.11. By the recurrence hypotheses applied to the A-module M/xM and the prime ideal $\mathfrak{p}$ of $\operatorname{supp}(M/xM)$, the $A_{\mathfrak{p}}$-module $M_{\mathfrak{p}}/xM_{\mathfrak{p}}$ is Cohen-Macaulay, which implies that the $A_{\mathfrak{p}}$-module $M_{\mathfrak{p}}$ is Cohen-Macaulay (Proposition 9.2.13). □

9.2.3 Cohen-Macaulay rings

We say that a ring A is **Cohen-Macaulay** if it is Noetherian and the A-module A is Cohen-Macaulay.

Example 9.2.21 (Examples of Cohen-Macaulay rings).

(a) Any Artinian ring is Cohen-Macaulay: this follows directly from Example 9.2.1.

(b) A Cohen-Macaulay ring has no embedded associated primes (Proposition 9.2.8). Conversely, let A is Notherian ring of dimension ≤ 1 which has no embedded associated primes; for any nonempty finite strongly secant subset S of A, the A-module A/SA has dimension ≤ 0, hence is Cohen-Macaulay; therefore A is Cohen-Macaulay by Theorem 9.2.20. In particular, any reduced Noetherian ring of dimension ≤ 1 is Cohen-Macaulay (Corollary 9.2.18).

(c) A Noetherian normal ring of dimension ≤ 2 is Cohen-Macaulay by Theorem 9.1.51. Conversely, let A be a Cohen-Macaulay ring whose local ring $A_{\mathfrak{p}}$ is integrally closed for any prime ideal $\mathfrak{p}$ with $\operatorname{ht}(\mathfrak{p}) \leq 1$; then A is normal by Theorem 9.1.51.

(d) If A is Cohen-Macaulay, so is the localizatin $S^{-1}A$ for any multiplicative subset S of A (Corollary 9.2.7). Conversely, it follows from our definition that, if $A_{\mathfrak{m}}$ is Cohen-Macaulay for any maximal ideal $\mathfrak{m}$ of A, then A is Cohen-Macaulay.

(e) Let A be a Noetherian ring and $\mathfrak{I}$ be an ideal of A. For the ring $A/\mathfrak{I}$ to be Cohen-Macaulay, it is necessary and sufficient that it is a Cohen-Macaulay A-module (Example 9.2.4). On the other hand, if A is local and $\mathfrak{I}$ is generated by an A-regular sequence, then $A/\mathfrak{I}$ is Cohen-Macaulay if and only if A is (Proposition 9.2.13).

(f) For a Noetherian local ring to be Cohen-Macaulay, it is necessary and sufficient that it has a defining ideal that is generated by an A-regular sequence: this follows from Proposition 9.2.11 and the fact an A-regular sequence of $\mathfrak{m}_A$ generates a defining ideal if and only if its length is equal to $\dim(A)$ (Proposition 7.2.16). In particular, any Noetherian regular local ring is Cohen-Macaulay (Theorem 7.4.5). More generally, the quotient of a Noetherian regular local ring A by an ideal generated by an A-regular sequence is Cohen-Macaulay (Example 9.2.3)

Proposition 9.2.22. *For a Noetherian ring A, the following conditions are equivalent:*

(i) A is a Cohen-Macaulay ring;

(ii) for any closed subset F of $\operatorname{Spec}(A)$, we have $\operatorname{depth}_F(A) = \operatorname{codim}(F)$;

(iii) any ideal $\mathfrak{I}$ of A contains an A-regular sequence of length $\operatorname{ht}(\mathfrak{I})$;

(iii') any maximal ideal $\mathfrak{m}$ of A contains an A-regular sequence of length $\operatorname{ht}(\mathfrak{m})$;

(iv) for any ideal $\mathfrak{I}$ of A, we have $\operatorname{Ext}^i_A(A/\mathfrak{I}, A) = 0$ for $i < \operatorname{ht}(\mathfrak{I})$.

(iv') for any maximal ideal $\mathfrak{m}$ of A, we have $\operatorname{Ext}^i_A(A/\mathfrak{m}, A) = 0$ for $i < \operatorname{ht}(\mathfrak{m})$;

(v) for any prime ideal $\mathfrak{p}$ of A and any ideal $\mathfrak{I}$ of $A_{\mathfrak{p}}$ generated by a maximal secant sequence for $A_{\mathfrak{p}}$, we have $e_{\mathfrak{I}}(A_{\mathfrak{p}}) = \ell(A_{\mathfrak{p}}/\mathfrak{I}A_{\mathfrak{p}})$.

(v') for any maximal ideal $\mathfrak{m}$ of A, there exists an ideal $\mathfrak{I}$ of $A_{\mathfrak{m}}$, generated by a maximal secant sequence for $A_{\mathfrak{m}}$, satisfying $e_{\mathfrak{I}}(A_{\mathfrak{m}}) = \ell(A_{\mathfrak{m}}/\mathfrak{I}A_{\mathfrak{m}})$.

(vi) (***Cohen-Macaulay Criterion***) *for any finite subset S of A such that the ideal $\mathfrak{S}$ generated by S has height $|S|$, the A-module $A/\mathfrak{S}$ has no embedded associated primes.*

Proof. The equivalence of (i) and (ii) follows from Proposition 9.2.5. On the other hand, in view of Theorem 9.1.17 and the definition of depth, conditions (iii) and (iv) (resp. (iii') and (iv')) signifies that we have $\operatorname{depth}_A(\mathfrak{I}, A) \geq \operatorname{ht}(\mathfrak{I})$ for any ideal (resp. any maximal ideal) $\mathfrak{I}$ of A. We then have

$$\text{(i)}\Leftrightarrow\text{(ii)}\Rightarrow\text{(iii)}\Leftrightarrow\text{(iv)}\Rightarrow\text{(iii')}\Leftrightarrow\text{(iv')}.$$

But (iv') implies, for any maximal ideal $\mathfrak{m}$ of A, $\operatorname{Ext}^i_{A_\mathfrak{m}}(\kappa(\mathfrak{m}), A_\mathfrak{m}) = 0$ for $i < \dim(A_\mathfrak{m})$, whence $\operatorname{depth}(A_\mathfrak{m}) = \dim(A_\mathfrak{m})$, and A is then Cohen-Macaulay. Finally, the equivalence of (i), (v) and (v') follows from Theorem 9.2.14, and that of (i) and (vi) from Theorem 9.2.20. □

Remark 9.2.23. Let $\rho : A \to B$ be a ring homomorphism, and let $\mathfrak{p} \in \operatorname{Spec}(A)$. Denote by $\bar{B}$ the ring $\kappa(\mathfrak{p}) \otimes_A B$, which is identified with $S^{-1}B/\mathfrak{p}(S^{-1}B)$, where S is the multiplicative subset $\rho(A - \mathfrak{p})$ of B. The prime ideals of $\bar{B}$ are then of the form $\mathfrak{P}\bar{B}$, where $\mathfrak{P}$ is a prime ideal of B lying over $\mathfrak{p}$. For such a prime ideal $\mathfrak{P}$, we have $S \subseteq B - \mathfrak{P}$, so the local ring of $\bar{B}$ at $\mathfrak{P}\bar{B}$ is identified with $B_\mathfrak{P}/\mathfrak{p}B_\mathfrak{P}$, which is $\kappa(\mathfrak{p}) \otimes_A B_\mathfrak{P}$.

Similarly, if N is an B-module, the $\bar{B}_{\mathfrak{P}\bar{B}}$-module $(\kappa(\mathfrak{p}) \otimes_A N)_{\mathfrak{P}\bar{B}}$ is identified with $\kappa(\mathfrak{p}) \otimes_A N_\mathfrak{P}$. Suppose moreover that the B-module N is finitely generated, then by Nakayama's lemma, the condition $\kappa(\mathfrak{p}) \otimes_A N_\mathfrak{P} = 0$ is equivalent to $N_\mathfrak{P} = 0$. Therefore the support of $\bar{B}$-module $\kappa(\mathfrak{p}) \otimes_A N$ is formed by the prime ideals $\mathfrak{P}\bar{B}$, where $\mathfrak{P}$ runs through the prime ideals of $\operatorname{supp}_B(N)$ lying over $\mathfrak{p}$. In particular, for the module $\kappa(\mathfrak{p}) \otimes_A N$ to be nonzero, it is necessary and sufficient that there exists a prime ideal of $\operatorname{supp}_B(N)$ lying over $\mathfrak{p}$.

Proposition 9.2.24. *Let $\rho : A \to B$ be a homomorphism of Noetherian rings and N be a finitely generated B-module that is also a finitely generated A-module. For the A-module N to be Cohen-Macaulay, it is necessary and sufficient that the B-module N is Cohen-Macaulay and, for any couple $(\mathfrak{M}, \mathfrak{N})$ of maximal ideals of $\operatorname{supp}_B(N)$ such that $\mathfrak{M}^c = \mathfrak{N}^c$, we have $\dim_{B_\mathfrak{M}}(N_\mathfrak{M}) = \dim_{B_\mathfrak{N}}(N_\mathfrak{N})$.*

Proof. The A-module $B/\operatorname{Ann}_B(N)$ is isomorphic to a sub-A-module of the finitely generated A-module $\operatorname{End}_A(N)$, hence is finitely generated. By replacing A by $A/\operatorname{Ann}_A(N)$ and B by $B/\operatorname{Ann}_B(N)$, we may then reduce to the case where ρ is injective and B is a finite A-algebra, and where we have $\operatorname{supp}_A(N) = \operatorname{Spec}(A)$, $\operatorname{supp}_B(N) = \operatorname{Spec}(B)$. The map $f : \operatorname{Spec}(B) \to \operatorname{Spec}(A)$ induced by ρ is then surjective and a prime ideal $\mathfrak{P}$ is maximal if and only if $f(\mathfrak{P})$ is maxima in A (Corollary 4.1.65).

Let $\mathfrak{m}$ be a maximal ideal of A. By Remark 9.2.23, the prime ideals of the ring $B_\mathfrak{m}$ containing $\mathfrak{m}B_\mathfrak{m}$ are of the form $\mathfrak{P}B_\mathfrak{m}$ where $\mathfrak{P}$ is an ideal of B (necessarily maximal) such that $f(\mathfrak{P}) = \mathfrak{m}$. By Proposition 9.1.27 and Proposition 9.1.35, have

$$\operatorname{depth}_{A_\mathfrak{m}}(N_\mathfrak{m}) = \operatorname{depth}_{B_\mathfrak{m}}(\mathfrak{m}B_\mathfrak{m}, N_\mathfrak{m}) = \inf_{\mathfrak{P}\in f^{-1}(\mathfrak{m})} \operatorname{depth}_{B_\mathfrak{P}}(N_\mathfrak{P}),$$

while by Theorem 7.1.28 and Proposition 7.1.12,

$$\dim_{A_\mathfrak{m}}(N_\mathfrak{m}) = \dim_{B_\mathfrak{m}}(N_\mathfrak{m}) = \sup_{\mathfrak{P}\in f^{-1}(\mathfrak{m})} \dim_{B_\mathfrak{P}}(N_\mathfrak{P}).$$

As we have $\operatorname{depth}_{B_\mathfrak{P}}(N_\mathfrak{P}) \leq \dim_{B_\mathfrak{P}}(N_\mathfrak{P})$ for any $\mathfrak{P} \in f^{-1}(\mathfrak{m})$, the proposition then follows from these equalities. □

Corollary 9.2.25. *Let $\rho : A \to B$ be a homomorphism of Noetherian ring. If B is a finite A-algebra and a Cohen-Macaulay A-module, then it is a Cohen-Macaulay ring. If ρ is also injective, we have $\operatorname{ht}(\mathfrak{a}^e) = \operatorname{ht}(\mathfrak{a})$ for any ideal $\mathfrak{a}$ of A, and $\operatorname{ht}(\mathfrak{b}) = \operatorname{ht}(\mathfrak{b}^c)$ for any ideal $\mathfrak{b}$ of B.*

Proof. The first assertion follows from Proposition 9.2.24. Suppose that ρ is injective, and let $\mathfrak{a}$ be an ideal of A. We have $\mathrm{ht}(\mathfrak{a}) = \mathrm{depth}_A(\mathfrak{a}, B)$ and $\mathrm{ht}(\mathfrak{a}^e) = \mathrm{depth}_B(\mathfrak{a}^e, B)$ since the A-module B is Cohen-Macaulay with support equal to $\mathrm{Spec}(A)$ (Proposition 9.2.5), and $\mathrm{depth}_A(\mathfrak{a}, B) = \mathrm{depth}_B(\mathfrak{a}^e, B)$ by Proposition 9.1.27, whence $\mathrm{ht}(\mathfrak{a}^e) = \mathrm{ht}(\mathfrak{a})$. Now if $\mathfrak{b}$ is an ideal of B, then by the preceding arguments we have $\mathrm{ht}(\mathfrak{b}^c) = \mathrm{ht}(\mathfrak{b}^{ce})$. But $\mathfrak{b}^{ce}$ is contained in $\mathfrak{b}$, hence has height smaller than $\mathrm{ht}(\mathfrak{b})$ and we have $\mathrm{ht}(\mathfrak{b}) \leq \mathrm{ht}(\mathfrak{b}^c)$ by Theorem 7.1.28(b). □

Corollary 9.2.26. *Let A be a Noetherian integrally closed ring and B be a ring containing A. Suppose that B is a finitely generated torsion-free A-module. If B is a Cohen-Macaulay ring, the A-module B is Cohen-Macaulay.*

Proof. In fact, any prime ideals of B lying over the same ideal of A have the same height (Theorem 7.1.29). We can then apply Proposition 9.2.24 with $N = B$. □

Corollary 9.2.27. *Let A be an integrally closed ring, K be its fraction field, L be a finite K-algebra such that $[L : K]1_A$ is invertible in A, and B be a sub-A-algebra of L that is finite over A.*

(a) The sub-A-module $A1_B$ of B is a direct factor.

(b) For any ideal $\mathfrak{I}$ of A, we have $\mathrm{depth}_A(\mathfrak{I}, A) \geq \mathrm{depth}_B(\mathfrak{I}B, B)$.

(c) If B is a Cohen-Macaulay ring, so is A.

Proof. The K-linear map $\mathrm{tr}_{L/K} : L \to K$ sends B into A (Corollary 4.1.47), so defines an A-linear map $t : B \to A$. For any $x \in A$, we have $t(x1_B) = [L : K]x$, whence the assertion of (a). By Proposition 9.1.27, we have $\mathrm{depth}_A(\mathfrak{I}, B) = \mathrm{depth}_B(\mathfrak{I}B, B)$; but by (a) and Example 9.1.3, we have $\mathrm{depth}_A(\mathfrak{I}, A) \geq \mathrm{depth}_A(\mathfrak{I}, B)$, whence (b).

If the ring B is Noetherian, so is A: in fact, by (a) we have $\mathfrak{a}B \cap A = \mathfrak{a}$ for any ideal $\mathfrak{a}$ of A, so any increasing sequence $(\mathfrak{a}_n)$ of ideals of A is stationary if the sequence $(\mathfrak{a}_n B)$ is stationary. Under the hypothesis of (c), the A-module B is Cohen-Macaulay (Corollary 9.2.26), and so is the A-module A (Example 9.2.2). □

Example 9.2.28. The result of Corollary 9.2.27 is applicable in the following situations:

(a) Consider a Noetherian integrally closed ring A, a separable extension L of its fraction field, of finite degree n such that $n1_A$ is invertible in A, and let B be the integral closure of A in L (Corollary 4.1.51).

(b) Consider a Noetherian integrally closed ring B and a finite group G acting on B, such that $|G|$ is invertible in B. Let A be the ring of invariant elements of B under the action of G. Then we are in a particular case of (a): the group G acts on the fraction field L of B, and the invariant subfield of L for this action is the fraction field K of A (Corollary 4.2.4). The extension L/K is Galois, and a fortiori separable; its Galois group is isomorphic to G, so $[L : K]$ is equal to $|G|$. The inverse of $[L : K]1_B$ is invariant under G, so $[L : K]1_A$ is invertible in A. As B is integrally closed, the ring A, being equal to $K \cap B$, is then integrally closed and B is its integral closure in L (Proposition 4.2.1).

In particular, if the ring B is Cohen-Macaulay, then so is the ring A.

9.2.4 Flat base change

Proposition 9.2.29. *Let $\rho : A \to B$ be a homomorphism of Noetherian rings, M be a finitely generated A-module and N be a finitely generated B-module that is flat over A. Let $f : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ be the associated map. The following conditions are equivalent:*

(i) the B-module $M \otimes_A N$ is Cohen-Macaulay;

(ii) the $(\kappa(\mathfrak{p})\otimes_A B)$-module $\kappa(\mathfrak{p})\otimes_A N$ is Cohen-Macaulay for any $\mathfrak{p}\in\operatorname{supp}(M)$, and the $A_{\mathfrak{p}}$-module $M_{\mathfrak{p}}$ is Cohen-Macaulay for any $\mathfrak{p}\in f(\operatorname{supp}_B(N))$;

(iii) for any maximal ideal of $\operatorname{supp}_B(N)$ whose inverse image $\mathfrak{p}$ in A belongs to $\operatorname{supp}_A(M)$, the $A_{\mathfrak{p}}$-module $M_{\mathfrak{p}}$ and the $(\kappa(\mathfrak{p})\otimes_A B)$-module $\kappa(\mathfrak{p})\otimes_A N$ are Cohen-Macaulay.

If the B-module is faithfully flat, these conditions imply that the A-module M is Cohen-Macaulay.

Proof. Let $\mathfrak{P}$ be a prime ideal belonging to the support of $M\otimes_A N$, and put $\mathfrak{p}=\mathfrak{P}^c$. As the module $(M\otimes_A N)_{\mathfrak{P}}$ is identified with $M_{\mathfrak{p}}\otimes_{A_{\mathfrak{p}}}N_{\mathfrak{P}}$, the modules $M_{\mathfrak{p}}$ and $N_{\mathfrak{P}}$ are necessarily nonzero, and so is the module $\kappa(\mathfrak{p})\otimes_A N_{\mathfrak{P}}$ (Remark 9.2.23). The $A_{\mathfrak{p}}$-module $N_{\mathfrak{P}}$, being isomorphic to a fraction module of $N_{\mathfrak{p}}$, is flat and by Proposition 9.1.32(b) and Proposition 7.2.26 we have the equalities

$$\begin{aligned}\operatorname{depth}_{B_{\mathfrak{P}}}((M\otimes_A N)_{\mathfrak{P}})&=\operatorname{depth}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}})+\operatorname{depth}_{B_{\mathfrak{P}}}(\kappa(\mathfrak{p})\otimes_A N_{\mathfrak{P}}),\\ \dim_{B_{\mathfrak{P}}}((M\otimes_A N)_{\mathfrak{P}})&=\dim_{A_{\mathfrak{p}}}(M_{\mathfrak{p}})+\dim_{B_{\mathfrak{P}}}(\kappa(\mathfrak{p})\otimes_A N_{\mathfrak{P}}),\end{aligned}$$

where each term are nonnegative integers. In view of the fact that the $B_{\mathfrak{P}}$-module $\kappa(\mathfrak{p})\otimes_A N_{\mathfrak{P}}$ is Cohen-Macaulay if and only if so is it as a $(\kappa(\mathfrak{p})\otimes_A B_{\mathfrak{P}})$-module (Example 9.2.4), we then duce the equivalence of the following conditions:

(α) the $B_{\mathfrak{P}}$-module $(M\otimes_A N)_{\mathfrak{P}}$ is Cohen-Macaulay;

(β) the $A_{\mathfrak{p}}$-module $M_{\mathfrak{p}}$ and the $(\kappa(\mathfrak{p})\otimes_A B_{\mathfrak{P}})$-module $\kappa(\mathfrak{p})\otimes_A N_{\mathfrak{P}}$ are Cohen-Macaulay.

We now prove that (iii) implies (i). Let $\mathfrak{N}$ be a maximal ideal of B belonging to the support of $M\otimes_A N$, and put $\mathfrak{p}=\mathfrak{N}^c$. By the preceding remarks, we have $\mathfrak{p}\in\operatorname{supp}_A(M)$, so (iii) and Remark 9.2.23 imply that the condition (β) above is satisfied for $\mathfrak{P}=\mathfrak{N}$, so the $B_{\mathfrak{N}}$-module $(M\otimes_A N)_{\mathfrak{N}}$ is Cohen-Macaulay, whence (i).

The implication (ii)⇒(iii) is clear, so it remains to prove that (i)⇒(ii). Suppose that the B-module $M\otimes_A N$ is Cohen-Macaualy, and let $\mathfrak{p}$ be an element of $\operatorname{supp}_A(M)$. We can suppose that the $(\kappa(\mathfrak{p})\otimes_A B)$-module $\kappa(\mathfrak{p})\otimes_A N$ is nonzero, which means there exists a prime ideal $\mathfrak{P}$ of $\operatorname{supp}_B(N)$ lying over $\mathfrak{p}$ (Remark 9.2.23). The $B_{\mathfrak{P}}$-module $(M\otimes_A N)_{\mathfrak{P}}$ is Cohen-Macaulay by hypotheses, so it follows from the implication (α)⇒(β) and Remark 9.2.23 that the $A_{\mathfrak{p}}$-module $M_{\mathfrak{p}}$ and the $(\kappa(\mathfrak{p})\otimes_A B)$-module $(\kappa(\mathfrak{p})\otimes_A N)$ are Cohen-Macaulay, whence (ii).

If moreover N is faithfully flat over A, we have $\kappa(\mathfrak{p})\otimes_A N\neq 0$ for any $p\in\operatorname{Spec}(A)$, so $f(\operatorname{supp}_B(N))=\operatorname{Spec}(A)$ by Remark 9.2.23, and (ii) then implies that M is Cohen-Macaulay. □

Corollary 9.2.30. *Let $\rho: A\to B$ be a local homomorphism of Noetherian local rings, M be a finitely generated nonzero A-module and N be a finitely generated nonzero B-module which is flat over A. For the B-module $M\otimes_A N$ to be Cohen-Macaulay, it is necessary and sufficient that the A-module M and the $(\kappa_A\otimes_A B)$-module $\kappa_A\otimes_A N$ are Cohen-Macaulay.*

Proof. In fact, N is a faithfully flat A-module since $\kappa_A\otimes_A N$ is nonzero (**??**). □

Corollary 9.2.31. *Let A be a Noetherian ring, B be a finite and flat A-algebra, and M be a finitely generated Cohen-Macaulay A-module. Then the B-module $M\otimes_A B$ is Cohen-Macaulay.*

Proof. For any $\mathfrak{p}\in\operatorname{Spec}(A)$ the ring $\kappa(\mathfrak{p})\otimes_A B$ is a finite $\kappa(\mathfrak{p})$-algebra, hence Cohen-Macaulay, and we can apply Proposition 9.2.29. □

Corollary 9.2.32. *Let A be a Noetherian ring, $\mathfrak{J}$ be an ideal of A and M be a finitely generated A-module. Let $\widehat{A}$ and $\widehat{M}$ be the $\mathfrak{J}$-adic completion of A and M, and S be the multiplicative subset $1+\mathfrak{J}$ of A. Consider the following conditions:*

(i) *the A-module M is Cohen-Macaulay;*

(ii) *the $\widehat{A}$-module $\widehat{M}$ is Cohen-Macaulay;*

(iii) *the $S^{-1}A$-module $S^{-1}M$ is Cohen-Macaulay;*

(iv) *for any maximal ideal $\mathfrak{m} \in \operatorname{supp}(M) \cap V(\mathfrak{I})$, the $A_{\mathfrak{m}}$-module $M_{\mathfrak{m}}$ is Cohen-Macaulay;*

(v) *for any prime ideal $\mathfrak{p} \in \operatorname{supp}(M)$ not meeting S, the $A_{\mathfrak{p}}$-module $M_{\mathfrak{p}}$ is Cohen-Macaulay and the ring $\kappa(\mathfrak{p}) \otimes_A \widehat{A}$ is Cohen-Macaulay.*

Then conditions (ii) to (v) are equivalent, and are implied by (i). If the ideal $\mathfrak{I}$ is contained in the Jacobson radical of A, then they are all equivalent.

Proof. We have seen that (i) implies (iii) in Example 9.2.21(d), and (iii) is identical to (i) if the ideal $\mathfrak{I}$ is contained in the Jacobson radical of A (since the elements of S are then invertible). By Corollary 2.4.25, the ring $\widehat{A}$ is Noetherian, and it is identified with the $S^{-1}\mathfrak{I}$-adic completion of $S^{-1}A$ (Proposition 2.4.39); similarly, $\widehat{M}$ is identified with the $S^{-1}\mathfrak{I}$-adic completion of $S^{-1}M$. To prove the equivalence of (ii) to (v), we can then replace A by $S^{-1}A$, $\mathfrak{I}$ by $S^{-1}\mathfrak{I}$, and M by $S^{-1}M$. In other words, we can assume that $\mathfrak{I}$ is contained in the Jacobson radical of A. The A-module $\widehat{A}$ is then faithfully flat by Proposition 2.4.28.

Now it is clear that (v) implies (iv) and (iv) implies (i). Let $\mathfrak{m}$ be a maximal ideal of A; $\mathfrak{m}\widehat{A}$ is then a maximal ideal of $\widehat{A}$ lying over $\mathfrak{m}$, and any maximal ideal of $\widehat{A}$ is obtained in this way (Proposition 2.4.24). The ring $\kappa(\mathfrak{m}) \otimes_A \widehat{A}$ is the residue field of $\hat{\mathfrak{m}}$, hence Cohen-Macaulay. If the A-module M is Cohen-Macaulay, so is the $\widehat{A}$-module $\widehat{M}$ (Proposition 9.2.29); this porves (i)$\Rightarrow$(ii).

Finally, to see that (ii)$\Rightarrow$(v), note that $\widehat{A}$-module $\widehat{M}$ is isomorphic to $M \otimes_A \widehat{A}$ (Proposition 2.3.16); if it is Cohen-Macaulay, then by Proposition 9.2.29 we conclude that $\kappa(\mathfrak{p}) \otimes_A \widehat{A}$ is a Cohen-Macaulay ring for any $\mathfrak{p} \in \operatorname{supp}(M)$, and that the A-module M is Cohen-Macaulay. □

Proposition 9.2.33. *Let $\rho : A \to B$ be a flat homomorphism of Noetherian rings. Then the following conditions are equivalent:*

(i) *B is a Cohen-Macaulay ring;*

(ii) *for any prime ideal $\mathfrak{P}$ of B with $\mathfrak{p} = \mathfrak{P}^c$, the rings $A_{\mathfrak{p}}$ and $\kappa(\mathfrak{p}) \otimes_A B$ are Cohen-Macaulay;*

(iii) *for any maximal ideal $\mathfrak{N}$ of B with with $\mathfrak{p} = \mathfrak{N}^c$, the rings $A_{\mathfrak{p}}$ and $\kappa(\mathfrak{p}) \otimes_A B$ are Cohen-Macaulay;*

If B is faithfully flat over A, these conditions imply that A is a Cohen-Macaulay ring.

Proof. This is a particular case of Proposition 9.2.29, where $M = A$ and $N = B$. □

Corollary 9.2.34. *Any finite and flat algebra over a Cohen-Macaulay ring is Cohen-Macaulay.*

Corollary 9.2.35. *Let A be a Cohen-Macaulay ring and $n > 0$ be a positive integer. Then the rings $A[X_1, \dots, X_n]$ and $A[\![X_1, \dots, X_n]\!]$ are Cohen-Macaulay.*

Proof. It suffices to deal with the case where $n = 1$. The ring $A[T]$ is Noetherian by Hilbert basis theorem, and for any field k, the ring $k[T]$ is Cohen-Macaulay by Example 9.2.21. The ring $A[T]$ is then Cohen-Macaulay by Proposition 9.2.33 and $A[\![T]\!]$ is Cohen-Macaulay by Corollary 9.2.32. □

Corollary 9.2.36. *Any finitely generated algebra over a Cohen-Macaulay ring is catenary.*

Proof. In fact, any such algebra is a quotient of a polynomial algebra over a Cohen-Macaulay ring, hence a quotient of Cohen-Macaulay ring, and therefore catenary (Proposition 9.2.8). □

9.3 Depth and homological dimension

9.3.1 Projective dimension and injective dimension

Let A be a ring, M be a A-module. Recall that the **projective dimension** of M, denoted by $\mathrm{proj.dim}_A(M)$, is defined to be the infermum of the lengths of projective resolutions of M. We have $\mathrm{proj.dim}_A(0) = -\infty$ and $\mathrm{proj.dim}_A(M) \geq 0$ if M is nonzero. For the A-module M to be projective, it is necessary and sufficient that $\mathrm{proj.dim}_A(M) \leq 0$.

Example 9.3.1. Let $\mathfrak{I}$ be an ideal of A generated by an A-regular sequence $\boldsymbol{x} = (x_1, \dots, x_r)$. We have $\mathrm{proj.dim}_A(A/\mathfrak{I}) \leq r$: in fact, this is clear if $A = 0$, and in the contrary case, the Koszul complex $K_\bullet(\boldsymbol{x}, A)$ is a free resolution of $A/\mathfrak{I}$ of length r. Moreover, for any A-module N, the A-module $\mathrm{Ext}_A^r(A/\mathfrak{I}, N)$ and $N/\mathfrak{I}N$ are isomorphic, so for that $\mathrm{proj.dim}_A(A/\mathfrak{I}) = r$, it is necessary and sufficient that $\mathfrak{I}$ is a proper ideal of A (A, X, p.134, prop.1).

Similarly, we define the **injective dimension** of M, denoted by $\mathrm{inj.dim}_A(M)$, to be the infermum of the lengths of injective resolutions of M. It is clear that we have $\mathrm{inj.dim}_A(0) = -\infty$, and $\mathrm{inj.dim}_A(M) \geq 0$ if $M \neq 0$. For the A-module M to be injective, it is necessary and sufficient that $\mathrm{inj.dim}_A(M) \leq 0$.

Proposition 9.3.2. *Let A be a ring, M be an A-module and $n \geq 0$ be an integer. The following conditions are equivalent:*

(i) $\mathrm{inj.dim}_A(M) \leq n$;

(ii) for any A-module N and any integer $i > n$, we have $\mathrm{Ext}_A^i(N, M) = 0$;

(iii) for any ideal $\mathfrak{a}$ of A, we have $\mathrm{Ext}_A^{n+1}(A/\mathfrak{a}, M) = 0$;

(iv) for any exact sequence of A-modules

$$0 \longrightarrow M \longrightarrow I^0 \longrightarrow \cdots \longrightarrow I^{n-1} \longrightarrow Q \longrightarrow 0$$

where the I^i are injective, the A-module Q is injective.

Proof. The implications (i)⇒(ii)⇒(iii) are clear from definition. In the situation of (iv), we have for any A-module N an isomorphism $\mathrm{Ext}_A^1(N, Q) \cong \mathrm{Ext}_A^{n+1}(N, M)$, so the hypothesis of (iii) implies that $\mathrm{Ext}_A^1(A/\mathfrak{a}, Q) = 0$ for any ideal $\mathfrak{a}$ of A, and Q is therefore injective.

Finally, for (iv)⇒(i), consider the exact sequence

$$0 \longrightarrow M \longrightarrow I^0(M) \longrightarrow \cdots \longrightarrow I^{n-1}(M) \longrightarrow K^{n-1}(M) \longrightarrow 0$$

where $I^i(M)$ are free A-modules; if condition (iv) is satisfied, then the A-module $K^{n-1}(M)$ is injective, whence $\mathrm{inj.dim}_A(M) \leq n$. □

Recall that the **homological dimension** (or **global dimension**) of the ring A, denoted by $\mathrm{gl.dim}(A)$, is the supremum of integers n such that there exists two A-modules M and N such that $\mathrm{Ext}_A^n(M, N)$ is nonzero. This is then also the supremum of the projective (or injective) dimensions of A-modules.

Proposition 9.3.3. *Let A a ring, M and N be A-modules, i an integer and S be a multiplicative subset of A. Then we have a canonical isomorphism of $S^{-1}A$-modules*

$$S^{-1}\mathrm{Tor}_i^a(M, N) \to \mathrm{Tor}_i^{S^{-1}A}(S^{-1}M, S^{-1}N).$$

If the ring A is Noetherian and M is finitely generated, we have a canonical isomorphism

$$S^{-1}\mathrm{Ext}_A^i(M, N) \to \mathrm{Ext}_{S^{-1}A}^i(S^{-1}M, S^{-1}N).$$

Proof. As the A-module $S^{-1}A$ is flat, this follows from (A, X, p.110, prop.9) and (A, X, p.111, prop.10). $\square$

Corollary 9.3.4. *Let A be a ring, M and N be A-modules, and i be an integer.*

(a) The support of $\mathrm{Tor}_i^A(M,N)$ is contained in $\mathrm{supp}(M) \cap \mathrm{supp}(N)$, and so is the support of $\mathrm{Ext}_A^i(M,N)$ if A is Noetherian and M is finitely generated.

(b) Suppose that A is Noetherian and the module M and N are finitely generated. If the A-module $M \otimes_A N$ is of finite length, so are the A-modules $\mathrm{Tor}_i^A(M,N)$ and $\mathrm{Ext}_A^i(M,N)$.

Proof. If $\mathfrak{p}$ is a prime ideal of A not belonging to $\mathrm{supp}(M) \cap \mathrm{supp}(N)$, then the modules $M_\mathfrak{p}$ and $N_\mathfrak{p}$ are nonzero, and this implies (a) in view of Proposition 9.3.3. Now for a finitely generated module over a Noetherian ring to be of finite length, it is necessary and sufficient that its support consists of maximal ideals (Proposition 3.2.14). Under the hypotheses of (b), the A-modules $\mathrm{Tor}_i^A(M,N)$ and $\mathrm{Ext}_A^i(M,N)$ are finitely generated, so the assertions of (b) follow from (a) (Proposition 1.4.39). $\square$

Proposition 9.3.5. *Let A be a Noetherian ring, M be a finitely generated A-module and N be an A-module. Then we have*

$$\mathrm{proj.dim}_A(M) = \sup_\mathfrak{p} \mathrm{proj.dim}_{A_\mathfrak{p}}(M_\mathfrak{p}), \quad \mathrm{inj.dim}_A(M) = \sup_\mathfrak{p} \mathrm{inj.dim}_{A_\mathfrak{p}}(N_\mathfrak{p}) \tag{9.3.1}$$

where $\mathfrak{p}$ runs through prime (resp. maximal) ideals of A. Moreover, the map $\mathfrak{p} \mapsto \mathrm{proj.dim}_{A_\mathfrak{p}}(M_\mathfrak{p})$ on $\mathrm{Spec}(A)$ is upper semi-continuous.

Proof. Let $n \geq 0$ be an integer, and suppose that we have $\mathrm{proj.dim}_A(M) < n$. For any prime ideal $\mathfrak{p}$ of A and any $A_\mathfrak{p}$-module Q, the $A_\mathfrak{p}$-module $\mathrm{Ext}^n_{A_\mathfrak{p}}$ is isomorphic to $(\mathrm{Ext}^n_A(M,Q))_\mathfrak{p}$ (Proposition 9.3.3), hence is zero. In view of (A, X, p.134, prop.1), we then deduce the inequality $\mathrm{proj.dim}_{A_\mathfrak{p}}(M_\mathfrak{p}) \leq \mathrm{proj.dim}_A(M)$. Suppose conversely that we have $\mathrm{proj.dim}_{A_\mathfrak{m}}(M_\mathfrak{m}) < n$ for any maximal ideal $\mathfrak{m}$ of A, and let R be an A-module. We have $(\mathrm{Ext}^n_A(M,R))_\mathfrak{m} = 0$ for any $\mathfrak{m}$ (Proposition 9.3.3), so $\mathrm{Ext}^n_A(M,R) = 0$, which implies $\mathrm{proj.dim}_A(M) < n$. The first inequality of (9.3.1) then follows, and the second one can be proved similarly. As $\mathrm{gl.dim}(A)$ is the suporemum of injective dimensions of A-modules, the third one then follows.

Now let $\mathfrak{p}$ be a prime ideal of A and $n = \mathrm{proj.dim}_{A_\mathfrak{p}}(M_\mathfrak{p})$; we want to show that there exists an open neighborhood U of $\mathfrak{p}$ in $\mathrm{Spec}(A)$ such that $\mathrm{proj.dim}_{A_\mathfrak{q}}(M_\mathfrak{q}) \leq n$ for any $\mathfrak{q} \in U$. This is clear if $n = +\infty$, and if $n = -\infty$, this follows from the fact that $\mathrm{supp}(M)$ is closed. Suppose now that n is finite and choose an exact sequence of A-modules

$$P_{n-1} \xrightarrow{d_{n-1}} P_{n-2} \longrightarrow \cdots \longrightarrow P_0 \xrightarrow{d_0} M \longrightarrow 0$$

where the P_i are free of finite rank. Put $P = \ker d_{n-1}$, which is an A-module of finite presentation. The $A_\mathfrak{p}$-module $P_\mathfrak{p}$ is projective, so by Theorem 1.5.10, there exists an element f of $A - \mathfrak{p}$ such that A_f-module P_f is free; the $A_\mathfrak{q}$-module $P_\mathfrak{q}$ is then free for $\mathfrak{q} \in D(f)$, which proves the second assertion. $\square$

Corollary 9.3.6. *For any multiplicative subset S of A, we have*

$$\mathrm{proj.dim}_{S^{-1}A}(S^{-1}M) \leq \mathrm{proj.dim}_A(M), \quad \mathrm{inj.dim}_{S^{-1}A}(S^{-1}N) \leq \mathrm{inj.dim}_A(N).$$

Corollary 9.3.7. *If $\mathrm{proj.dim}_{A_\mathfrak{m}}(M_\mathfrak{m})$ is finite for any maximal ideal $\mathfrak{m}$ of $\mathrm{supp}(M)$, then we have $\mathrm{proj.dim}_A(M) < +\infty$.*

Proof. In fact, the subspace X of $\mathrm{supp}(M)$ formed by maximal ideals is quasi-compact (since $\mathrm{Spec}(A)$ is Noetherian); the map $\mathfrak{m} \mapsto \mathrm{proj.dim}_{A_\mathfrak{m}}(M_\mathfrak{m})$ from X to $\bar{\mathbb{R}}$ is upper semi-continuous, hence bounded. $\square$

9.3.2 Homological dimension of Noetherian rings

Let A be a Noetherian local ring and M be a finitely generated A-module. Recall that a resolution

$$\cdots \longrightarrow L_n \xrightarrow{d_n} L_{n-1} \longrightarrow \cdots \longrightarrow L_0 \xrightarrow{d_0} M \longrightarrow 0$$

of M is called a **minimal projective resolution** if each module L_i is free of finite rank, and if the complex $L \otimes_A \kappa_A$ has zero differential. For any integer $i \geq 0$, we then have (A, X p103, example 3)

$$[\mathrm{Ext}^i_A(M, \kappa_A) : \kappa_A] = [\mathrm{Tor}^A_i(M, \kappa_A) : \kappa_A] = \mathrm{rank}_A(L_i) \tag{9.3.2}$$

By (A, X p.56, prop.10), any finitely generated A-module admits a minimal projective resolution.

Proposition 9.3.8. *Let A be a Noetherian local ring, M be a finitely generated A-module and $n \geq 0$ be an integer. The following conditions are equivalent:*

(i) $\mathrm{proj.dim}_A(M) < n$;

(ii) $\mathrm{Tor}^A_n(M, \kappa_A) = 0$;

(iii) $\mathrm{Ext}^n_A(M, \kappa_A) = 0$;

(iv) any minimal projective resolution of M has length $< n$.

Proof. The assertions (i)⇒(ii) and (i)⇒(iii) are immediate. Let L be a minimal projective resolution of M; if (ii) or (iii) are satisfied, we have $L_n = 0$ by 9.3.2. As any minimal projective resolution of M is isomorphic to L (A, X, p.54, prop.8), we then deduce (iv). The implication (iv)⇒(i) is trivial. □

Corollary 9.3.9. *Let A be a Noetherian local ring and $n \geq 0$ be an integer. The following conditions are equivalent:*

(i) $\mathrm{gl.dim}(A) < n$;

(ii) $\mathrm{Ext}^i_A(M, N) = 0$ *and* $\mathrm{Tor}^A_i(M, N) = 0$ *for any couple (M, N) of A-modules and any integer $i \geq n$;*

(iii) $\mathrm{Tor}^A_n(\kappa_A, \kappa_A) = 0$;

(iv) $\mathrm{Ext}^n_A(\kappa_A, \kappa_A) = 0$;

(v) $\mathrm{proj.dim}_A(\kappa_A) < n$.

Proof. It is clear that (i) implies (ii) and (ii) implies (iii) and (iv). By Proposition 9.3.8 applied to the A-module κ_A, conditions (iii) and (iv) both implies (v). To see (v)⇒(i), we note that if $\mathrm{proj.dim}_A(\kappa_A) < n$, then $\mathrm{Tor}^A_n(M, \kappa_A) = 0$ for any A-module M; then any finitely generated A-module has projective dimension $< n$ (Proposition 9.3.8), so $\mathrm{gl.dim}(A) < n$. □

Corollary 9.3.10. *For a Noetherian local ring, we have* $\mathrm{gl.dim}(A) = \mathrm{proj.dim}_A(\kappa_A)$.

Example 9.3.11. Let A be a local ring. Then the A-module $\mathrm{Tor}^A_1(\kappa_A, \kappa_A)$ is isomorphic to $\mathfrak{m}_A/\mathfrak{m}_A^2$ (A, X, p.72, example), so if A is Noetherian, for that $\mathrm{Tor}^A_1(\kappa_A, \kappa_A) = 0$, it is necessary and sufficient that $\mathfrak{m}_A = 0$, which means A is a field. Corollary 9.3.9 then implies that a Noetherian local ring with homological dimension being zero is a field.

Example 9.3.12. Let A be a Noetherian local ring, M be a finitely generated A-module with finite projective dimension n, and N be a nonzero finitely generated A-module. The A-module $\mathrm{Ext}_A^n(M,N)$ is then nonzero: let L be a minimal projective resolution of M, and d be its differential. We then have an exact sequence

$$\mathrm{Hom}_A(L_{n-1},N)\xrightarrow{\mathrm{Hom}(d_n,1)}\mathrm{Hom}_A(L_n,N)\longrightarrow\mathrm{Ext}_A^n(M,N)\longrightarrow 0$$

As $d_n\otimes 1_{\kappa_A}$ is zero, we then deduce that tensoring with κ_A induces an isomorphism $\kappa_A\otimes_A\mathrm{Hom}_A(L_n,N)\to\kappa_A\otimes_A\mathrm{Ext}_A^n(M,N)$, whence in view of the formula (9.3.2),

$$[\kappa_A\otimes_A\mathrm{Ext}_A^n(M,N):\kappa_A]=[\mathrm{Ext}_A^n(M,\kappa_A):\kappa_A][\kappa_A\otimes_A N:\kappa_A];$$

which is nonzero by Proposition 9.3.8 and Nakayama's lemma. Therefore the projective dimension of M is the largest integer i such that $\mathrm{Ext}_A^i(M,N)$ is nonzero.

Example 9.3.13. Let A be a Noetherian ring, M be a finitely generated A-module with finite projective dimension, N be a finitely generated module with support equal to $\mathrm{Spec}(A)$. By Example 9.3.12 and Proposition 9.3.5, the projective dimension n of M is the largest integer i such that $\mathrm{Ext}_A^i(M,N)\neq 0$, and the support of the A-module $\mathrm{Ext}_A^n(M,N)$ is the set of elements $\mathfrak{p}\in\mathrm{Spec}(A)$ such that $\mathrm{proj.dim}_{A_\mathfrak{p}}(M_\mathfrak{p})=n$.

Proposition 9.3.14. *Let A be a Noetherian ring, M be a finitely generated A-module and $n\geq 0$ be an integer. Then the following conditions are equivalent:*

(i) $\mathrm{proj.dim}_A(M)<n$;

(ii) for any maximal ideal $\mathfrak{m}$ of A, we have $\mathrm{Ext}_A^n(M,A/\mathfrak{m})=0$ *(resp.* $\mathrm{Tor}_n^A(M,A/\mathfrak{m})=0$*);*

(iii) for any maximal ideal $\mathfrak{m}$ of A, we have $\mathrm{Ext}_{A_\mathfrak{m}}^n(M_\mathfrak{m},A/\mathfrak{m})=0$ *(resp.* $\mathrm{Tor}_n^{A_\mathfrak{m}}(M_\mathfrak{m},A/\mathfrak{m})=0$*).*

Proof. It is clear that (i)⇒(ii), and (ii)⇒(i) by Proposition 9.3.5. Finally, by Proposition 9.3.8, condition (iii) implies the inequality $\mathrm{proj.dim}_{A_\mathfrak{m}}(M_\mathfrak{m})<n$ for any maximal ideal $\mathfrak{m}$ of A, whence $\mathrm{proj.dim}_A(M)<n$ by Proposition 9.3.5. □

Remark 9.3.15. Let A be a Noetheiran ring and $n\geq 0$ be an integer. If $\mathfrak{m}$ and $\mathfrak{n}$ are two distinct maximal ideals of A, the A-modules $\mathrm{Ext}_A^n(A/\mathfrak{m},A/\mathfrak{n})$ and $\mathrm{Tor}_n^A(A/\mathfrak{m},A/\mathfrak{n})$ are then annihilated by $\mathfrak{m}+\mathfrak{n}$, whence is zero. By an argument similar to Corollary 9.3.9, we deduce from Proposition 9.3.14 the equivalence of the following conditions:

(i) $\mathrm{gl.dim}_A(M)<n$;

(ii) $\mathrm{Ext}_A^i(M,N)=0$ and $\mathrm{Tor}_i^A(M,N)=0$ for any couple (M,N) of A-modules and any integer $i\geq n$;

(iii) $\mathrm{Tor}_n^A(A/\mathfrak{m},A/\mathfrak{m})=0$ for any maximal ideal $\mathfrak{m}$ of A;

(iv) $\mathrm{Ext}_A^n(A/\mathfrak{m},A/\mathfrak{m})=0$ for any maximal ideal $\mathfrak{m}$ of A;

(v) $\mathrm{proj.dim}_A(A/\mathfrak{m})<n$ for any maximal ideal $\mathfrak{m}$ of A.

In particular, we have $\mathrm{gl.dim}(A)=\sup_\mathfrak{m}\mathrm{proj.dim}_A(A/\mathfrak{m})$, where $\mathfrak{m}$ runs through maximal ideals of A.

Proposition 9.3.16. *Let A be a Noetherian ring, N be an A-module, $n\geq 0$ be an integer. Then the following conditions are equivalent:*

(i) $\mathrm{inj.dim}_A(N)<n$;

(ii) for any prime ideal $\mathfrak{p}$ *of* A*, we have* $\mathrm{Ext}^n_A(A/\mathfrak{p}, N) = 0$*;*

(iii) for any prime ideal $\mathfrak{p}$ *of* A*, we have* $\mathrm{Ext}^n_{A_\mathfrak{p}}(\kappa(\mathfrak{p}), N_\mathfrak{p}) = 0$*.*

If the A*-module* N *is finitely generated, these conditions are equivalent to:*

(iv) for any maximal ideal $\mathfrak{m}$ *of* A*, we have* $\mathrm{Ext}^i_A(A/\mathfrak{m}, N) = 0$ *for* $n \le i \le n + \mathrm{ht}(\mathfrak{m})$*.*

Proof. □

Remark 9.3.17. Let N be a finitely generated A-module; the condition $\mathrm{Ext}^n_A(A/\mathfrak{m}, N) = 0$ for any maximal ideal $\mathfrak{m}$ of A does not imply necessarily $\mathrm{inj.dim}_A(N) < n$. For example, if A is local and not Gorenstein, we then have $\mathrm{Ext}^n_A(A/\mathfrak{m}, A) = 0$ for $n < \mathrm{depth}(A)$, but $\mathrm{inj.dim}_A(A) = +\infty$.

Proposition 9.3.18. *Let* A *be a Noetherian ring,* M *be a finitely generated* A*-module. Then we have* $\dim_A(M) \le \mathrm{inj.dim}_A(M)$*.*

Proof. Let $r \le \dim_A(M)$ be a positive integer. Then there exists a saturated chain of prime ideals $\mathfrak{p} \subseteq \mathfrak{p}_1 \subseteq \cdots \subseteq \mathfrak{p}_{r-1} \subseteq \mathfrak{q}$ such that $\mathfrak{p}$ is a minimal element of $\mathrm{supp}(M)$; the $A_\mathfrak{p}$-module $M_\mathfrak{p}$ is then of finite length, so we have $\mathrm{Hom}_{A_\mathfrak{p}}(\kappa(\mathfrak{p}), M_\mathfrak{p}) \ne 0$ (Example 9.1.2), whence $\mathrm{Ext}^r_{A_\mathfrak{q}}(\kappa(\mathfrak{q}), M_\mathfrak{q}) \ne 0$ (Lemma 9.1.22); this implies $\mathrm{inj.dim}_A(M) \ge r$ by Proposition 9.3.16. □

Proposition 9.3.19. *Let* A *be a Noetherian local ring and* M *be a finitely generated nonzero* A*-module with finite injective dimension. Then we have* $\mathrm{inj.dim}_A(M) = \mathrm{depth}(A)$*.*

Proof. Put $r = \mathrm{inj.dim}_A(M)$, we then have $\mathrm{Ext}^i_A(\kappa_A, M) = 0$ for $i > r$, so $\mathrm{Ext}^r_A(\kappa_A, M) \ne 0$ by Proposition 9.3.16(iv). Let $s = \mathrm{depth}(A)$ and $(x_1, \dots, x_s)$ be an A-regular sequence of element sof $\mathfrak{m}_A$; put $N = A/(x_1A + \cdots + x_sA)$. By Example 9.3.1, we have $\mathrm{proj.dim}_A(N) = s$ and $\mathrm{Ext}^s_A(N, M) \ne 0$, so $s \le \mathrm{inj.dim}_A(M) = r$. But N is of zero depth (Proposition 9.1.16), so there exists an exact sequence of A-modules

$$0 \longrightarrow \kappa_A \longrightarrow N \longrightarrow N' \longrightarrow 0$$

from which we deduce an exact sequence of extension modules

$$\mathrm{Ext}^r_A(N, M) \longrightarrow \mathrm{Ext}^r_A(\kappa_A, M) \longrightarrow \mathrm{Ext}^{r+1}_A(N', M)$$

As we have $\mathrm{Ext}^{r+1}_A(N', M) = 0$ and $\mathrm{Ext}^r_A(\kappa_A, M) \ne 0$, we obtain that $\mathrm{Ext}^r_A(N, M) \ne 0$, so $r \le \mathrm{proj.dim}_A(N) = s$; we then conclude that $r = s$, whence the proposition. □

9.3.3 Depth and projective dimension

Theorem 9.3.20 (Auslander-Buchsbaum). *Let* A *be a Noetherian local ring and* M *be a finitely generated* A*-module with finite projective dimension. Then we have*

$$\mathrm{proj.dim}_A(M) + \mathrm{depth}_A(M) = \mathrm{depth}(A).$$

Proof. We prove by induction on $\mathrm{proj.dim}_A(M)$. If $\mathrm{proj.dim}_A(M) = 0$, then M is free of finite rank, so $\mathrm{depth}_A(M) = \mathrm{depth}(A)$ by Example 9.1.3. Now suppose that $\mathrm{depth}_A(M) = 1$ and choose a minimal projective resolution of M:

$$0 \longrightarrow L_1 \xrightarrow{d_1} L_0 \xrightarrow{d_0} M \longrightarrow 0$$

The A-modules L_0 and L_1 are free of finite rank and nonzero, so their depths are equal to $\operatorname{depth}(A)$ (Example 9.1.3). The map $1_{\kappa_A} \otimes d_1 : \kappa_A \otimes_A L_1 \to \kappa_A \otimes_A L_0$ is zero, so d_1 belongs to $\mathfrak{m}_A \operatorname{Hom}_A(L_1, L_0)$. By Remark 9.1.5, we then have $\operatorname{depth}_A(M) = \operatorname{depth}(A) - 1$.

Finally, suppose that $\operatorname{proj.dim}_A(M) > 1$. We choose an exact sequence

$$0 \longrightarrow N \longrightarrow L \longrightarrow M \longrightarrow 0$$

where L is a free A-module of finite rank. By Example 9.1.3, we then have $\operatorname{depth}_A(L) = \operatorname{depth}(A)$, and $\operatorname{proj.dim}_A(N) = \operatorname{proj.dim}_A(M) - 1$ in view of (A, X, p.135, cor.2(c)); the induction hypotheses implies that $\operatorname{depth}_A(N) = \operatorname{depth}(A) - \operatorname{proj.dim}_A(N)$, and in particular $\operatorname{depth}_A(N) < \operatorname{depth}_A(L)$. Now Proposition 9.1.4 implies that $\operatorname{depth}(M) = \operatorname{depth}_A(N) - 1$, which proves the assertion. □

Remark 9.3.21. In view of Corollary 9.3.10, the fomula Theorem 9.3.20 applied to the A-module κ_A implies that we are in exactly one of the following two cases:

(i) $\operatorname{proj.dim}_A(\kappa_A) = \operatorname{gl.dim}(A) = +\infty$;

(ii) $\operatorname{proj.dim}_A(\kappa_A) = \operatorname{gl.dim}(A) = \operatorname{depth}(A) < +\infty$.

Later we shall see that case (ii) characterizes regular local rings.

Corollary 9.3.22. *Retain the hypotheses of 9.3.20.*

(a) We have $\operatorname{proj.dim}_A(M) \leq \operatorname{depth}(A)$, *and for the equality holds, it is necessary and sufficient that* $\mathfrak{m}_A \in \operatorname{Ass}(M)$.

(b) We have $\operatorname{depth}_A(M) \leq \operatorname{depth}(A)$, *and for the equality holds, it is necessary and sufficient that M is free.*

Proof. This follows from the fact that $\operatorname{depth}_A(M) = 0$ if and only if $\mathfrak{m}_A \in \operatorname{Ass}(A)$ by Example 9.1.2, and $\operatorname{proj.dim}_A(M) = 0$ if and only if M is projective, hence free. □

Corollary 9.3.23. *Retain the hypotheses of 9.3.20 and suppose that A is a Cohen-Macaulay ring. Then* $\operatorname{proj.dim}_A(M)$ *is the sum of the positive integers* $\dim(A) - \dim_A(M)$ *and* $\dim_A(M) - \operatorname{depth}(M)$.

In particular, we have $\operatorname{proj.dim}_A(M) \geq \dim(A) - \dim_A(M)$, and the equality holds if and only if M is Cohen-Macaulay.

Corollary 9.3.24. *Let A be a Noetherian ring, M be a finitely generated A-module with finite projective dimension, N be a finitely generated A-module, $i \geq 0$ be an integer, and F be the support of* $\operatorname{Ext}^i_A(M, N)$ *(resp.* $\operatorname{Tor}^A_i(M, N)$*). Then we have* $\operatorname{depth}_F(A) \geq i$.

Proof. For $\mathfrak{p} \in F$ we have $\operatorname{Ext}^i_{A_\mathfrak{p}}(M_\mathfrak{p}, N_\mathfrak{p}) \neq 0$ (resp. $\operatorname{Tor}^{A_\mathfrak{p}}_i(M_\mathfrak{p}, N_\mathfrak{p}) \neq 0$) by Proposition 9.3.3, so $i \leq \operatorname{proj.dim}_{A_\mathfrak{p}}(M_\mathfrak{p}) \leq \operatorname{proj.dim}_A(M) < +\infty$ in view of Proposition 9.3.5. The formula Theorem 9.3.20 then implies that $\operatorname{depth}(A_\mathfrak{p}) \geq i$, and therefore $\operatorname{depth}_F(A) = \inf_{\mathfrak{p} \in F} \operatorname{depth}(A_\mathfrak{p}) \geq i$ (Proposition 9.1.35). □

With the terminologies of Remark 9.1.38, the conclusion of Corollary 9.3.24 signifies that the modules $\operatorname{Ext}^i_A(M, N)$ and $\operatorname{Tor}^A_i(M, N)$ are of grade $\geq i$. This then implies that the codimension of their support in $\operatorname{Spec}(A)$ is $\geq i$ (Proposition 9.1.20).

Corollary 9.3.25. *Let A be a Noetherian Cohen-Macaulay ring and M be a finitely generated A-module with finite projective dimension.*

(a) *For* $\mathfrak{p} \in \operatorname{Spec}(A)$*, denote by* $\mathscr{C}(\mathfrak{p})$ *the set of irreducible components of* $\operatorname{supp}(M)$ *containing* $\mathfrak{p}$*. Then we have*

$$\dim_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}) - \operatorname{depth}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}) = \operatorname{proj.dim}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}) - \inf_{X \in \mathcal{C}(\mathfrak{p})} \operatorname{codim}(X, \operatorname{Spec}(A)).$$

(b) *The function* $\mathfrak{p} \mapsto \dim_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}) - \operatorname{depth}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}})$ *on* $\operatorname{Spec}(A)$ *is upper semi-continuous.*

(c) *The set of prime ideals* $\mathfrak{p}$ *of* A *such that the* $A_{\mathfrak{p}}$*-module* $M_{\mathfrak{p}}$ *is Cohen-Macaulay is open and dense in* $\operatorname{Spec}(A)$*. Its intersection with* $\operatorname{supp}(M)$ *is dense in* $\operatorname{supp}(M)$*.*

9.3.4 Gorenstein rings

We say that a ring A is **Gorenstein** if it is Noetherian and the $A_{\mathfrak{m}}$-module $A_{\mathfrak{m}}$ is of finite injective dimension for any maximal ideal $\mathfrak{m}$ of A. For a Noetherian local ring to be Gorenstein, it is then necessary and sufficient that $\operatorname{inj.dim}_A(A) < +\infty$.

Proposition 9.3.26. *A Gorenstein ring* A *is Cohen-Macaulay and we have* $\operatorname{inj.dim}_A(A) = \dim(A)$*.*

Proof. For any maximal ideal $\mathfrak{m}$ of A, we have (Proposition 9.3.18, Proposition 9.3.19 and Corollary 9.1.19)

$$\dim(A_{\mathfrak{m}}) \leq \operatorname{inj.dim}_{A_{\mathfrak{m}}}(A_{\mathfrak{m}}), \quad \operatorname{inj.dim}_{A_{\mathfrak{m}}}(A_{\mathfrak{m}}) = \operatorname{depth}(A_{\mathfrak{m}}), \quad \operatorname{depth}(A_{\mathfrak{m}}) \leq \dim(A_{\mathfrak{m}}).$$

It then follows that A is Cohen-Macaulay, and we have $\operatorname{inj.dim}_A(A) = \dim(A)$ by passing to supremum (Proposition 9.3.5). □

Thus the Noetherian rings A such that $\operatorname{inj.dim}_A(A)$ is finite are finite-dimensional Gorenstein rings (Proposition 9.3.5), and the Noetherian rings such that the A-module A is injective are the Artinian Gorenstein rings.

Example 9.3.27. For any multiplicative subset S of a Gorenstein ring A, the fraction ring $S^{-1}A$ is Gorenstein: in fact, any maximal ideal of $S^{-1}A$ is of the form $S^{-1}\mathfrak{p}$, where $\mathfrak{p}$ is a prime ideal of A not meeting S. Let $\mathfrak{m}$ be a maximal ideal of A containing $\mathfrak{p}$, then the ring $B = (S^{-1}A)_{S^{-1}\mathfrak{p}}$ is isomorphic to $A_{\mathfrak{p}}$, hence is a fraction ring of $A_{\mathfrak{m}}$, and therefore satisfies $\operatorname{inj.dim}_B(B) < +\infty$ (Corollary 9.3.6).

Example 9.3.28. Let A be a Gorenstein ring and $\mathfrak{J}$ be an ideal of A, generated by an A-regular sequence $\boldsymbol{x}$. The quotient ring $A/\mathfrak{J}$ is a Gorenstein ring: for any maximal ideal of A containing $\mathfrak{J}$, the image of $\boldsymbol{x}$ in $A_{\mathfrak{m}}$ is $A_{\mathfrak{m}}$-regular and generates the ideal $\mathfrak{J}_{\mathfrak{m}}$, so $A_{\mathfrak{m}}/\mathfrak{J}_{\mathfrak{m}}$ is Gorenstein by (CA, X, cor. de la prop.7 n4). On the other hand, if A is a Noetherian local ring and $\mathfrak{J}$ is an ideal generated by an A-regular sequence of elements of $\mathfrak{m}_A$, then A is Gorenstein if $A/\mathfrak{J}$ is Gorenstein.

Example 9.3.29. Let A be a regular local ring, then A is Gorenstein. In fact, let $\boldsymbol{x}$ be a system of parameters of A. Then $\boldsymbol{x}$ is A-regular and generates the ideal $\mathfrak{m}_A$ (Theorem 7.4.5), so we can apply Example 9.3.28.

9.4 Regular rings

9.4.1 Homological properties of regular local rings

Proposition 9.4.1. *Let* A *be a Noetherian regular local ring of dimension* n*. Then* $\operatorname{gl.dim}(A) = n$*, and for any integer* $i \geq 0$*,*

$$[\operatorname{Ext}^i_A(\kappa_A, \kappa_A) : \kappa_A] = [\operatorname{Tor}^A_i(\kappa_A, \kappa_A) : \kappa_A] = \binom{n}{i}$$

Proof. Let $\boldsymbol{x} = (x_1, \dots, x_n)$ be a system of parameters of A. The sequence $\boldsymbol{x}$ generates $\mathfrak{m}_A$ and is completely secant for A (Theorem 7.4.5). The Koszul complex $K_\bullet(\boldsymbol{x}, A)$ is then a free resolution of κ_A, whose differentials are zero modulo $\mathfrak{m}_A$. For any intger $i \geq 0$, we have (formula (9.3.2))

$$[\mathrm{Ext}^i_A(\kappa_A, \kappa_A) : \kappa_A] = [\mathrm{Tor}^A_i(\kappa_A, \kappa_A) : \kappa_A] = \mathrm{rank}_A(K_i(\boldsymbol{x}, A)) = \binom{n}{i}.$$

I then follows from Corollary 9.3.9 that we have $\mathrm{proj.dim}_A(\kappa_A) \geq n$, whence $\mathrm{proj.dim}_A(\kappa_A) = n$ and $\mathrm{gl.dim}(A) = n$ by Corollary 9.3.10. □

Proposition 9.4.2. *A Noetherian regular local ring is factorial.*

Proof. By Proposition 9.4.1, the minimal projective resolution of a finitely generated module over a Noetherian regular local ring has finite length. It then follows from Corollary 6.2.45 that this ring is factorial. □

Proposition 9.4.3. *Let A be a Noetherian regular local ring and M be a finitely generated nonzero A-module. Then the projective dimension of M is finite and we have*

$$\mathrm{proj.dim}_A(M) + \mathrm{depth}_A(M) = \dim(A).$$

Proof. In fact, M has finite projective dimension by Proposition 9.4.1, and we have $\mathrm{depth}(A) = \dim(A)$ since A is Cohen-Macaulay (Example 9.2.21). We can therefore apply the formula Theorem 9.3.20. □

Corollary 9.4.4. *We have $\mathrm{proj.dim}_A(M) \geq \dim(A) - \dim(M)$, and the equality holds if and only if M is Cohen-Macaulay.*

Corollary 9.4.5. *For the A-module M to be free, it is necessary and sufficient it is Cohen-Macaulay and of dimension $\dim(A)$, or that it has depth $\geq \dim(A)$.*

Corollary 9.4.6. *Any finitely generated reflexive module over a Noetherian regular local ring of dimension 2 is free.*

Proof. In fact, a Noetherian regular local ring is integrally closed (Corollary 7.4.6). The corollary then follows from Corollary 9.4.5 and Proposition 9.1.49. □

Corollary 9.4.7. *Let $\rho : A \to B$ be a local homomorphism of Noetherian local rings. Suppose that A is regular and that B is a finitely generated A-module. Then we have $\mathrm{proj.dim}_A(B) \geq \dim(A) - \dim(B)$, and the equality holds if and only if B is Cohen-Macaulay. In particular, for the ring B to be Cohen-Macaulay with dimension equal to $\dim(A)$, it is necessary and sufficient that the A-module B is free.*

Proof. In fact, we have $\dim(B) = \dim_A(B)$ by Theorem 7.1.28, so B is Cohen-Macaulay if and only if it is a Cohen-Macaulay A-module (Proposition 9.2.24). It then suffices to apply Corollary 9.4.4 and Corollary 9.4.5. □

9.4.2 Homological characterization for regularity

Theorem 9.4.8 (Serre). *For a Noetherian local ring to be regular, it is necessary and sufficient that it has finite homological dimension.*

Proof. We have seen that a Noetherian regular local ring has finite homological dimension. Conversely, let A be a Noetherian local ring with finite homological dimension n. By Corollary 9.3.10 and Theorem 9.3.20, we have

$$n = \mathrm{gl.dim}(A) = \mathrm{proj.dim}_A(\kappa_A) = \mathrm{depth}(A).$$

If $n = 0$, the A-module κ_A is free, so $\mathfrak{m}_A = 0$ and A is a field. Suppose that $n > 0$ and we preceed by induction on n. Since $\mathrm{depth}(A) > 0$, the ideal $\mathfrak{m}_A$ is not assocaited with A (Example 9.1.2), hence is not contained in the union of $\mathfrak{m}_A^2$ with the associated primes of A (Proposition 1.1.4). By Corollary 3.1.5, we can therefore choose an element x of $\mathfrak{m}_A - \mathfrak{m}_A^2$ such that the homothety x_A is injective. Let B be the Noetherian local ring A/xA and consider the exact sequence

$$0 \longrightarrow \kappa_A \xrightarrow{i} \mathfrak{m}_A/x\mathfrak{m}_A \xrightarrow{p} \mathfrak{m}_B \longrightarrow 0$$

where i is the map induced by the homothety x_A on $\mathfrak{m}_A$ and p is the canonical surjection. Since the class of x in the κ_A-vector space $\mathfrak{m}_A/\mathfrak{m}_A^2$ is nonzero, there exists an A-linear map $\phi : \mathfrak{m}_A \to \kappa_A$ with $\phi(x) = 1$; by passing to quotient, we then from ϕ a retraction of i, such that the preceding sequence splits. By (CA, X, cor.2 de la prop.7 du §3, n4) and (A, X, p.135, cor.1), this implies the relations

$$\mathrm{proj.dim}_B(\mathfrak{m}_B) \le \mathrm{proj.dim}_B(\mathfrak{m}_A/x\mathfrak{m}_A) = \mathrm{proj.dim}_A(\mathfrak{m}_A) < +\infty.$$

(CA, X, cor.2 de la prop.7 du §3, n4) applied to the exact sequence $0 \to \mathfrak{m}_B \to B \to \kappa_B \to 0$ of B-modules then implies that $\mathrm{proj.dim}_B(\kappa_B) < +\infty$. The ring B is then of finite homological dimension (Corollary 9.3.10), and of depth $n-1$ (Proposition 9.1.16). It then follows from the induction hypotheses that B is regular, so A is regular by Corollary 7.4.10. □

Therefore, if A is a Noetherian local ring, the following conditions are equivalent:

(i) A is regular;

(ii) the A-module κ_A has finite projective dimension;

(iii) any finitely generated A-module has finite projective dimension.

Now we say that a ring A is **regular** if it is Noetherian and the local ring $A_\mathfrak{m}$ is regular for any maximal ideal $\mathfrak{m}$ of A. From the above equivalence, it is not hard to derive the following characterization for regular rings.

Proposition 9.4.9. *Let A be a Noetherian ring. The following conditions are equivalent:*

(i) A is regular;

(ii) any finitely generated A-module has finite projective dimension;

(iii) for any maximal ideal $\mathfrak{m}$ of A, the projective dimension of $A/\mathfrak{m}$ is finite;

(iv) for any prime ideal $\mathfrak{p}$ of A, the local ring $A_\mathfrak{p}$ is regular.

Proof. Let $\mathfrak{p}$ be a prime ideal of A; if the A-module $A/\mathfrak{p}$ has finite projective dimension, so does the $A_\mathfrak{p}$-module $\kappa(\mathfrak{p})$ (Corollary 9.3.6), so the local ring $A_\mathfrak{p}$ is regular by Noe local regular iff finite gldim. We then reduce that (ii)⇒(iv) and (iii)⇒(i). The implications (iv)⇒(i) and (ii)⇒(iii) are clear.

To see that (i)⇒(ii), let M be a fintely generated A-module. Under the hypotheses of (i), we have $\mathrm{proj.dim}_{A_\mathfrak{m}}(M_\mathfrak{m}) \le \mathrm{gl.dim}(A_\mathfrak{m}) < +\infty$ for any maximal ideal $\mathfrak{m}$ of A (Proposition 9.4.1), so M has finite projective dimension (Proposition 9.3.5), whence (ii). □

Example 9.4.10. If a ring A is regular, then so is $S^{-1}A$ for any multiplicative subset S of A: this follows for example by the characterization (iii) of Proposition 9.4.9.

Example 9.4.11. For a ring to be regular, it is necessary and sufficient that it is isomorphic to the product of a finite family of regular integral domains. This follows from the fact that any regular ring is locally integral, since regular local rings are integral.

Corollary 9.4.12. *Let A be a Noetherian ring. Then the following conditions are equivalent:*

(i) $\mathrm{gl.dim}(A) < +\infty$*;*

(ii) *A is regular and* $\dim(A) < +\infty$*.*

If these conditions are satisfied, we have $\dim(A) = \mathrm{gl.dim}(A)$*.*

Proof. If A is regular, for any maximal ideal $\mathfrak{m}$ of A we have $\dim(A_{\mathfrak{m}}) = \mathrm{gl.dim}(A_{\mathfrak{m}})$ (Proposition 9.4.1), and then (Proposition 7.1.3 and Proposition 9.3.5)

$$\mathrm{gl.dim}(A) = \sup_{\mathfrak{m}} \mathrm{gl.dim}(A_{\mathfrak{m}}) = \sup_{\mathfrak{m}} \dim(A_{\mathfrak{m}}) = \dim(A).$$

On the other hand, if $\mathrm{gl.dim}(A) < +\infty$, the ring A is regular by Proposition 9.4.9, so the corollary follows. □

Corollary 9.4.13. *A regular ring is normal, Gorenstein and Cohen-Macaulay.*

Proof. In fact, a regular local ring is integrally closed (Corollary 7.4.6), Gorenstein (Example 9.3.29), and Cohen-Macaulay (Example 9.2.21(f)). □

Corollary 9.4.14. *Let A be a Noetherian ring, $\mathfrak{J}$ be an ideal of A and $\widehat{A}$ be the $\mathfrak{J}$-adic completion of A.*

(a) For the ring $\widehat{A}$ to be regular, it is necessary and sufficient that, for any maximal ideal $\mathfrak{m}$ of A containing $\mathfrak{J}$, the ring $A_{\mathfrak{m}}$ is regular.

(b) If the ring A is regular, then $\widehat{A}$ is regular. If the ring $\widehat{A}$ is regular and $\mathfrak{J}$ is contained in the Jacobson radical of A, then A is regular.

Proof. By Proposition 2.4.24, for the ring $\widehat{A}$ to be regular, it is necessary and sufficient that $\widehat{A}_{\widehat{\mathfrak{m}}}$ is regular for any maximal ideal $\mathfrak{m}$ of A containing $\mathfrak{J}$. As the local rings $\widehat{A}_{\widehat{\mathfrak{m}}}$ and $A_{\mathfrak{m}}$ are isomorphic, assertion of (a) then follows from Corollary 7.4.4, and (b) follows from (a). □

Corollary 9.4.15. *Let A be a regular ring and P be a finitely generated projective A-module. Then the symmetric algebra $\boldsymbol{S}_A(P)$ is regular.*

Proof. Let $\mathfrak{P}$ be a prime ideal of $\boldsymbol{S}_A(P)$ and $\mathfrak{p}$ be its contraction in A. The local ring $\boldsymbol{S}_A(P)_{\mathfrak{P}}$ is then a fraction field of $\boldsymbol{S}_A(P)_{\mathfrak{p}}$, which is isomorphic to $\boldsymbol{S}_{A_{\mathfrak{p}}}(P)_{\mathfrak{p}}$ by (A, III, p.72, prop.7); it then suffices to prove that the later is regular. This allow us to reduce to the case where A is local, and then P is free of finite rank. By Proposition 9.4.1 and (A, X, p.143, cor.1), we have

$$\mathrm{gl.dim}(\boldsymbol{S}_A(P)) = \mathrm{gl.dim}(A) + \mathrm{rank}_A(P) < +\infty$$

and $\boldsymbol{S}_A(P)$ is therefore regular by Proposition 9.4.9. □

Corollary 9.4.16. *Let A be a regular ring and $(T_i)_{i\in I}$ be a finite family of indeterminates. Then rings $A[(T_i)_{i\in I}]$ and $A[\![(T_i)_{i\in I}]\!]$ are regular.*

Proof. This follows from Corollary 9.4.16 and Corollary 9.4.14(b). □

Proposition 9.4.17. *Let $\rho : A \to B$ be a homomorphism of Noetherian rings and N be a B-module. Suppose that*

(a) the ring A is regular,

(b) N is a finitely generated A-module,

(c) the B-module N is Cohen-Macaulay,

(d) any minimal prime ideal of $\operatorname{supp}_B(N)$ *is lying over a minimal prime ideal of* A.

Then N *is a (finitely generated) projective* A*-module.*

Proof. It suffices to prove that, for any maximal ideal $\mathfrak{m}$ of A, the $A_{\mathfrak{m}}$-module $N_{\mathfrak{m}}$ is free (Theorem 1.5.5). The A-module $B/\mathrm{Ann}_B(N)$ is isomorphic to a sub-A-module of the finitely generated A-module $\mathrm{End}_A(N)$, hence is finitely generated. By replacing B by $B/\mathrm{Ann}_B(N)$, we may then reduce to the case where B is a finite A-algebra and where $\operatorname{supp}_B(N) = \operatorname{Spec}(B)$.

Let $\mathfrak{m}$ be a maximal ideal of $\operatorname{supp}_A(N)$, and put $n = \dim(A_{\mathfrak{m}})$. By Corollary 9.4.5, it suffices to prove that $N_{\mathfrak{m}}$ is a Cohen-Macaulay $A_{\mathfrak{m}}$-module of dimension n. By Corollary 4.1.65, any maximal ideal of $B_{\mathfrak{m}}$ is of the form $\mathfrak{N}B_{\mathfrak{m}}$, where $\mathfrak{N}$ is a maximal ideal of B lying over $\mathfrak{m}$. Let $\mathfrak{P}$ be a minimal prime ideal of $\operatorname{supp}_B(N)$ which is contained in $\mathfrak{N}$. The closed subset $V(\mathfrak{P}B_{\mathfrak{N}})$ of $\operatorname{supp}_{B_{\mathfrak{N}}}(N_{\mathfrak{N}})$ then has zero codimension, and since the $B_{\mathfrak{N}}$-module $N_{\mathfrak{N}}$ is Cohen-Macaulay, Corollary 9.2.10 implies that $\dim_{B_{\mathfrak{N}}}(N_{\mathfrak{N}}) = \dim(B_{\mathfrak{N}}/\mathfrak{P}B_{\mathfrak{N}})$. But $\mathfrak{P}$ is lying over a minimal prime of A which is contained in $\mathfrak{m}$, so $\mathfrak{P}B_{\mathfrak{N}}$ is lying over a minimal prime ideal of $A_{\mathfrak{m}}$, which is zero since the local ring $A_{\mathfrak{m}}$ is regular (hence integral). The canonical map $A_{\mathfrak{m}} \to B_{\mathfrak{N}}/\mathfrak{P}B_{\mathfrak{N}}$ is then injective and it follows from Theorem 7.1.28 that we have

$$\dim_{B_{\mathfrak{N}}}(N_{\mathfrak{N}}) = \dim(B_{\mathfrak{N}}/\mathfrak{P}B_{\mathfrak{N}}) = \dim(A_{\mathfrak{m}}) = n.$$

In view of this, Proposition 9.2.24 then implies that the $A_{\mathfrak{m}}$-module $N_{\mathfrak{m}}$ is Cohen-Macaulay, and the proposition then follows from the relations $\dim_{A_{\mathfrak{m}}}(N_{\mathfrak{m}}) = \dim_{B_{\mathfrak{m}}}(N_{\mathfrak{m}})$ (Theorem 7.1.28) and $\dim_{B_{\mathfrak{m}}}(N_{\mathfrak{m}}) \geq \dim_{B_{\mathfrak{N}}}(N_{\mathfrak{N}}) = n$. □

Corollary 9.4.18. *Let* B *be an integral Noetherian ring and* A *be a subring of* B *which is regular. Suppose that* B *is a finie* A*-algebra, then for the ring* B *to be Cohen-Macaulay, it is necessary and sufficient that the* A*-module* B *is projective.*

Proof. If the ring B is Cohen-Macaulay, the A-module B is projective by Proposition 9.4.17. Conversely, assume that the A-module B is projective. We know that the A-module A is Cohen-Macaulay (Corollary 9.4.13), and the A-module B is a direct factor of a free A-module of finite rank, hence Cohen-Macaulay (Example 9.2.2), so we can apply Proposition 9.2.24. □

Example 9.4.19. Let B be an integral algebra of finite type over a field k. By the normalization lemma (Theorem 4.3.1), there exists a subring A of B which is isomorphic to a polynomial algebra and such that B is finite over A. By Corollary 9.4.18, we see that B is a Cohen-Macaulay ring if and only if the A-module B is projective (hence free).

9.4.3 Flat base change

Proposition 9.4.20. *Let* $\rho : A \to B$ *be a homomorphism of Noetherian rings such that* B *is a faithfully flat* A*-module.*

(a) For any finitely generated A*-module* M*, we have* $\operatorname{proj.dim}_A(M) = \operatorname{proj.dim}_B(B \otimes_A M)$.

(b) If the ring B *is regular, so is* A.

Proof. The B-module $B \otimes_A M$ is finitely generated, and for it to be nonzero, it is necessary and sufficient that M is nonzero (**??**). For it to be projective, it is necessary and sufficient that M is a projective A-module (**??**). This proves (a) if $\operatorname{proj.dim}_A(M) \leq 0$, so suppose that $\operatorname{proj.dim}_A(M) \geq 1$ (whence $\operatorname{proj.dim}_B(B \otimes_A M) \geq 1$) and we proceed by induction on $\operatorname{proj.dim}_A(M)$. Choose an exact sequence of A-modules

$$0 \longrightarrow N \longrightarrow L \longrightarrow M \longrightarrow 0$$

where L is a free A-module of finite rank. We have $\text{proj.dim}_A(N) = \text{proj.dim}_A(M) - 1$ (A, X p.135, cor.2 de la prop.1). As B is flat over A, the sequence

$$0 \longrightarrow B \otimes_A N \longrightarrow B \otimes_A L \longrightarrow B \otimes_A M \longrightarrow 0$$

is exact and we have $\text{proj.dim}_B(B \otimes_A N) = \text{proj.dim}_B(B \otimes_A M) - 1$. Assertion (a) then follows from the induction hypotheses applied to N, and (b) follows from (a) and Proposition 9.4.9. □

Corollary 9.4.21. *Let B be a regular domain and A be a Noetherian subring of B such that B is finite over A. Then the following conditions are equivalent:*

(i) A is regular;

(ii) B is a projective A-module;

(iii) B is a flat A-module;

(iv) B is a faithfully flat A-module.

Proof. The implication (i)⇒(ii) follows from Corollary 9.4.18, and (ii)⇒(iii) is clear. For any prime ideal $\mathfrak{p}$ of A, we have $\mathfrak{p}^e \neq B$ (Theorem 4.1.69), so it suffices to apply (**??**) to see that (iii)⇒(iv). Finally, (iv) implies (i) by Proposition 9.4.20(b). □

For any Noetherian local ring A, we denote by $\delta(A)$ the integer

$$\delta(A) = [\mathfrak{m}_A/\mathfrak{m}_A^2 : \kappa_A] - \dim(A).$$

Recall that $\delta(A)$ is always positive, and its vanishing characterize regular local rings. Now let $\rho : A \to B$ be a local homomorphism of Noetherian local rings; we then have a κ_A-linear homomorphism $\mathfrak{m}_A/\mathfrak{m}_A^2 \to \mathfrak{m}_B/\mathfrak{m}_B^2$, whence a κ_B-linear homomorphism

$$d\rho : \kappa_B \otimes_{\kappa_A} (\mathfrak{m}_A/\mathfrak{m}_A^2) \to \mathfrak{m}_B/\mathfrak{m}_B^2.$$

Lemma 9.4.22. *Under the above hypotheses, we have*

$$\delta(B) + [\ker d\rho : \kappa_B] = \delta_A + \delta(\kappa_A \otimes_A B) + (\dim(A) - \dim(B) + \dim(\kappa_A \otimes_A B)).$$

Proof. Let $\bar{B}$ be the local ring $\kappa_A \otimes_A B$. Since $\mathfrak{m}_{\bar{B}} = \mathfrak{m}_B/\mathfrak{m}_A B$, we have the exact sequence of B-modules

$$B \otimes_A \mathfrak{m}_A \longrightarrow \mathfrak{m}_B \longrightarrow \mathfrak{m}_{\bar{B}} \longrightarrow 0$$

By tensoring with κ_B over B, we then obtain an exact sequence of κ_B-vector spaces

$$\kappa_B \otimes_{\kappa_A} (\mathfrak{m}_A/\mathfrak{m}_A^2) \xrightarrow{d\rho} \mathfrak{m}_B/\mathfrak{m}_B^2 \longrightarrow \mathfrak{m}_{\bar{B}}/\mathfrak{m}_{\bar{B}}^2 \longrightarrow 0 \tag{9.4.1}$$

where we use the equality

$$\kappa_B \otimes_B B \otimes_A \mathfrak{m}_A = \kappa_B \otimes_A \mathfrak{m}_A = \kappa_B \otimes_{\kappa_A} \kappa_A \otimes_A \mathfrak{m}_A = \kappa_B \otimes_{\kappa_A} (\mathfrak{m}_A/\mathfrak{m}_A^2).$$

Since $[\mathfrak{m}_{\bar{B}}/\mathfrak{m}_{\bar{B}}^2 : \kappa_B] = [\mathfrak{m}_{\bar{B}}/\mathfrak{m}_{\bar{B}}^2 : \kappa_{\bar{B}}]$ and $[\kappa_B \otimes_{\kappa_A} (\mathfrak{m}_A/\mathfrak{m}_A^2) : \kappa_B] = [\mathfrak{m}_A/\mathfrak{m}_A^2 : \kappa_A]$, from (9.4.1) we then deduce the equality

$$[\ker d\rho : \kappa_B] + [\mathfrak{m}_B/\mathfrak{m}_B^2 : \kappa_B] = [\mathfrak{m}_A/\mathfrak{m}_A^2 : \kappa_A] + [\mathfrak{m}_{\bar{B}}/\mathfrak{m}_{\bar{B}}^2 : \kappa_{\bar{B}}],$$

which proves the lemma. □

Proposition 9.4.23. *Let $\rho : A \to B$ be a local homomorphism of Noetherian local rings. The following conditions are equivalent:*

(i) B is regular and the κ_B-linear map $d\rho : \kappa_B \otimes_{\kappa_A} (\mathfrak{m}_A/\mathfrak{m}_A^2) \to \mathfrak{m}_B/\mathfrak{m}_B^2$ is injective;

(ii) B and $\kappa_A \otimes_A B$ are regular and the A-module B is flat;

(iii) A and $\kappa_A \otimes_A B$ are regular and the A-module B is flat.

(iv) A and $\kappa_A \otimes_A B$ are regular and we have $\dim(B) = \dim(A) + \dim(\kappa_A \otimes_A B)$.

Proof. We have $\dim(B) \leq \dim(A) + \dim(\kappa_A \otimes_A B)$ by Corollary 7.2.27; the equivalence of (i) and (iv) then follows from Lemma 9.4.22. Under the hypotheses of (ii), the A-module B is faithfully flat since ρ is a local homomorphism and this implies (iii) by Proposition 9.4.20. The implication (iii)⇒(iv) follows from Corollary 7.2.27.

Now it suffices to prove that if the equivalent conditions of (i) and (iv) are satisfied, then the A-module B is flat. To this end, let $\boldsymbol{x}$ be a system of parameters of A. Since $d\rho$ is injective, the image of $\boldsymbol{x}$ under ρ can be extended into a system of parameters of B. The sequence $\boldsymbol{x}$ is then completely secant for A and for B (Proposition 7.4.9), and generates the ideal $\mathfrak{m}_A$ of A. By **??**, the A-module $\mathrm{Tor}_1^A(\kappa_A, B)$ is then isomorphic to $H_1(\boldsymbol{x}, B)$, hence is zero. It then follows that B is flat over A (Theorem 2.6.33 and Proposition 2.6.34). □

Example 9.4.24. Let X and Y be finite dimensional complex analytic varieties, $f : X \to Y$ be a morphis, and x be a point of X. Consider the local homomorphism $\rho : \mathscr{O}_{Y,f(x)} \to \mathscr{O}_{X,x}$ induced by f. The map $d\rho$ is then the tangent map $d_x f : T_x X \to T_{f(x)} Y$. The equivalent conditions of Proposition 9.4.23 are therefore equivalent in this case to the fact that f is a submersion at x.

Corollary 9.4.25. *Let $\rho : A \to B$ be a flat homomorphism of Noetherian rings. If A is regular and $\kappa(\mathfrak{N}^c) \otimes_A B$ is regular for any maximal ideal $\mathfrak{N}$ of B, then the ring B is regular.*

Proof. In fact, for any maximal ideal $\mathfrak{N}$ of B, the $A_{\mathfrak{N}^c}$-module $B_{\mathfrak{N}}$ is flat (Proposition 1.3.27), so the ring $B_{\mathfrak{N}}$ is regular by Proposition 9.4.23. □

9.5 Complete intersections

9.5.1 Completely secant ideals

Let A be a ring and $\mathfrak{I}$ be an ideal of A. We say that the ideal $\mathfrak{I}$ is **completely secant at a point** $\mathfrak{p}$ of $V(\mathfrak{I})$ if the ideal $\mathfrak{I}_{\mathfrak{p}}$ of $A_{\mathfrak{p}}$ is generated by a completely secant sequence for $A_{\mathfrak{p}}$. We say that $\mathfrak{I}$ is **completely secant** if it is completely secant at every point of $V(\mathfrak{I})$.

If the ideal $\mathfrak{I}$ is completely secant, so is the ideal $S^{-1}\mathfrak{I}$ of $S^{-1}A$ for any multiplicative subset S of A. Any ideal generated by a completely secant sequence is completely secant. More precisely, we have the following:

Proposition 9.5.1. *Let A be a ring, $\mathfrak{I}$ be an ideal of A generated by a finite sequence $\boldsymbol{x} = (x_1, \ldots, x_r)$ of elements of A. Then the following conditions are equivalent:*

(i) the sequence $\boldsymbol{x}$ is completely secant for A;

(ii) for any prime (resp. maximal) ideal $\mathfrak{p} \in V(\mathfrak{I})$, the image of $\boldsymbol{x}$ in $A_{\mathfrak{p}}$ is completely secant for $A_{\mathfrak{p}}$;

(iii) for any prime (resp. maximal) ideal $\mathfrak{p} \in V(\mathfrak{I})$, the ideal $\mathfrak{I}$ is completely secant at $\mathfrak{p}$ and the image of $\boldsymbol{x}$ in the $\kappa(\mathfrak{p})$-vector space $\kappa(\mathfrak{p}) \otimes_A \mathfrak{I}$ form a basis.

If A is Noetherian, these conditions are equivalent to:

(iv) for any integer $n \geq 0$, the $A/\mathfrak{I}$-module $\mathfrak{I}^n/\mathfrak{I}^{n+1}$ is free and the image of the monomials $x_1^{\alpha_1}\cdots x_r^{\alpha_r}$ of total degree n form a basis.

Proof. Let $\mathfrak{p}$ be a prime ideal of A, and denote by $\boldsymbol{x}_\mathfrak{p}$ the image of $\boldsymbol{x}$ in $A_\mathfrak{p}$. For any integer $i \geq 0$, the $A_\mathfrak{p}$-module $H_i(\boldsymbol{x}_\mathfrak{p}, A_\mathfrak{p})$ is isomorphic to $(H_i(\boldsymbol{x}, A))_\mathfrak{p}$ (**??**(b)); it is then zero if $\mathfrak{p}$ does not contain $\mathfrak{I}$, since we then have $\mathfrak{I}_\mathfrak{p} = A_\mathfrak{p}$ (**??**). The equivalence of (ii) and (ii) then folows from Proposition 1.3.18. On the other hand, the equivalence of (ii) and (iii) follows from Corollary 9.1.14.

The implication (i)⇒(iv) is a concequence of **??**. Finally, for any $\mathfrak{p} \in V(\mathfrak{I})$, condition (iv) implies by localization the analogous results for the $(A_\mathfrak{p}/\mathfrak{I}_\mathfrak{p})$-module $\mathfrak{I}_\mathfrak{p}^n/\mathfrak{I}_\mathfrak{p}^{n+1}$, and this implies (i) by **??**. □

Remark 9.5.2. If A is Noetherian, we can replace "completely secant" by "regular" in condition (ii) of Proposition 9.5.1. However, this is not the case in condition (i): a completely secant sequence for A is not necessarily A-regular (exercise 1).

Remark 9.5.3. Let A be a ring, $\mathfrak{I}$ be a finitely generated ideal of A and $\mathfrak{p}$ be a prime ideal of A containing $\mathfrak{I}$. By Corollary 9.1.14, we have

$$\mathrm{depth}_{A_\mathfrak{p}}(\mathfrak{I}_\mathfrak{p}, A_\mathfrak{p}) \leq [\kappa(\mathfrak{p}) \otimes_A \mathfrak{I} : \kappa(\mathfrak{p})],$$

and the equality holds if and only if $\mathfrak{I}$ is completely secant at $\mathfrak{p}$. Suppose that $\mathfrak{I}$ is proper and generated by a completely secant sequence $(x_1, \dots, x_r)$. We then have (Proposition 9.1.27 and Proposition 9.5.1)

$$\mathrm{depth}_A(\mathfrak{I}, A) = \inf_{\mathfrak{p}\in V(\mathfrak{I})} \mathrm{depth}_{A_\mathfrak{p}}(\mathfrak{I}_\mathfrak{p}, A_\mathfrak{p}) = r.$$

If A is also Noetherian, then by Proposition 7.2.18 and Proposition 9.1.20 we have

$$\mathrm{depth}_A(\mathfrak{I}, A) \leq \mathrm{codim}(V(\mathfrak{I}), \mathrm{Spec}(A)) \leq r,$$

whence

$$\mathrm{depth}_A(\mathfrak{I}, A) = r = \mathrm{codim}(V(\mathfrak{I}), \mathrm{Spec}(A)) = \mathrm{ht}(\mathfrak{I}).$$

For an ideal $\mathfrak{I}$ of a ring A, the graded A-module $\bigoplus_{n\in\mathbb{N}} \mathfrak{I}^n$ possesses a natural graded A-algebra structure, induced from the multiplication of the ring A. The identity map on $\mathfrak{I}$ then extends to a canonical surjective homomorphism of graded A-algebras

$$\alpha_\mathfrak{I} : \boldsymbol{S}_A(\mathfrak{I}) \to \bigoplus_{n\in\mathbb{N}} \mathfrak{I}^n.$$

By extension of scalars to the ring $A/\mathfrak{I}$, we also deduce a canonical surjective homomorphism of graded $A/\mathfrak{I}$-algebras

$$\beta_\mathfrak{I} : \boldsymbol{S}_{A/\mathfrak{I}}(\mathfrak{I}/\mathfrak{I}^2) \to \mathrm{gr}_\mathfrak{I}(A).$$

As the case of **??**, the injectivity of the canonical homomorphisms $\alpha_\mathfrak{I}$ and $\beta_\mathfrak{I}$ characterizes completely secant ideals. More precisely, we have the following theorem:

Theorem 9.5.4. *Let A be a Noetherian ring and $\mathfrak{I}$ be an ideal of A. The following conditions are equivalent:*

(i) the ideal $\mathfrak{I}$ is completely secant;

(ii) the ideal $\mathfrak{I}$ is completely secant at every maximal ideal $\mathfrak{m} \in V(\mathfrak{I})$;

(iii) the $A/\mathfrak{I}$-module $\mathfrak{I}/\mathfrak{I}^2$ is projective and the canonical homomorphism $\alpha_\mathfrak{I} : \boldsymbol{S}_A(\mathfrak{I}) \to \bigoplus_{n\in\mathbb{N}} \mathfrak{I}^n$ is bijective;

(iv) the $A/\mathfrak{J}$-module $\mathfrak{J}/\mathfrak{J}^2$ is projective and the canonical homomorphism $\beta_{\mathfrak{J}} : \boldsymbol{S}_{A/\mathfrak{J}}(\mathfrak{J}/\mathfrak{J}^2) \to \mathrm{gr}_{\mathfrak{J}}(A)$ is bijective.

Proof. It is clear that (i)⇒(ii) and (iii)⇒(iv). Suppose that condition (ii) is satisfied; to see that (ii)⇒(iii), it suffices to prove that for any maximal ideal $\mathfrak{m}$ of A, the $A_{\mathfrak{m}}/\mathfrak{J}_{\mathfrak{m}}$-module $\mathfrak{J}_{\mathfrak{m}}/\mathfrak{J}_{\mathfrak{m}}^2$ is free and the homomorphism $\alpha_{\mathfrak{J}_{\mathfrak{m}}} : S_{A_{\mathfrak{m}}}(\mathfrak{J}_{\mathfrak{m}}) \to \bigoplus_{n\in\mathbb{N}} \mathfrak{J}_{\mathfrak{m}}^n$ is bijective (Theorem 1.5.5 and Proposition 1.3.19). But this assertion is trivial if $\mathfrak{m}$ does not belong to $V(\mathfrak{J})$ (since we then have $\mathfrak{J}_{\mathfrak{m}} = A_{\mathfrak{m}}$), and follows from **??** if $\mathfrak{m} \in V(\mathfrak{J})$.

Finally, suppose that condition (iv) is satisfied, and let $\mathfrak{p}$ be a prime ideal of A containing $\mathfrak{J}$. Then the $A_{\mathfrak{p}}/\mathfrak{J}_{\mathfrak{p}}$-module $\mathfrak{J}_{\mathfrak{p}}/\mathfrak{J}_{\mathfrak{p}}^2$ is free. Let $\boldsymbol{x} = (x_1,\dots,x_r)$ be a sequence of elements of $\mathfrak{J}_{\mathfrak{p}}$ whose image forms a basis for $\mathfrak{J}_{\mathfrak{p}}/\mathfrak{J}_{\mathfrak{p}}^2$. The sequence $\boldsymbol{x}$ generates $\mathfrak{J}_{\mathfrak{p}}$ by Nakayama's lemma, and satisfies condition (iv) of Proposition 9.5.1. Therefore the ideal $\mathfrak{J}_{\mathfrak{p}}$ of $A_{\mathfrak{p}}$ is completely secant, and $\mathfrak{J}$ is then completely secant at $\mathfrak{p}$. □

Remark 9.5.5. Suppose that the ideal $\mathfrak{J}$ is completely secant and let $\boldsymbol{x} = (x_1,\dots,x_r)$ be a sequence of elements of $\mathfrak{J}$ such that for any maximal ideal $\mathfrak{m} \in V(\mathfrak{J})$, the image of $\boldsymbol{x}$ in $\mathfrak{J}/\mathfrak{m}\mathfrak{J}$ forms a basis for the $A/\mathfrak{m}$-vector space $\mathfrak{J}/\mathfrak{m}\mathfrak{J}$. Then the $A/\mathfrak{J}$-module $\mathfrak{J}/\mathfrak{J}^2$ is free and the canonical image of $\boldsymbol{x}$ in $\mathfrak{J}/\mathfrak{J}^2$ forms a basis: in fact, it suffices to verify that for each $\mathfrak{m} \in V(\mathfrak{J})$, the image of $\boldsymbol{x}$ forms a basis for the $A_{\mathfrak{m}}/\mathfrak{J}_{\mathfrak{m}}$-module $\mathfrak{J}_{\mathfrak{m}}/\mathfrak{J}_{\mathfrak{m}}^2$, and this follows from Proposition 1.3.9 and Corollary 1.3.11 since the $A_{\mathfrak{m}}/\mathfrak{J}_{\mathfrak{m}}$-module $\mathfrak{J}_{\mathfrak{m}}/\mathfrak{J}_{\mathfrak{m}}^2$ is projective by Theorem 9.5.4.

Corollary 9.5.6. *Let $\rho : A \to B$ be a flat homomorphism of Noetherian rings, and $\mathfrak{J}$ be an ideal of A.*

(a) If $\mathfrak{J}$ is completely secant, the ideal $\mathfrak{J}B$ is completely secant.

(b) Suppose that the ideal $\mathfrak{J}B$ is completely secant and that any maximal ideal $\mathfrak{m} \in V(\mathfrak{J})$ is the inverse image of a maximal ideal of B. Then the ideal $\mathfrak{J}$ is completely secant. This is the case for example if B is faithfully flat over A.

Proof. Since the A-module B if flat, $\mathfrak{J}^n \otimes_A B$ is identified with $\mathfrak{J}^n B$ and $(\mathfrak{J}^n/\mathfrak{J}^{n+1}) \otimes_{A/\mathfrak{J}} (B/\mathfrak{J}B)$ is identified with $\mathfrak{J}^n B/\mathfrak{J}^{n+1}B$ for each integer $n \geq 0$. Assertion (a) then follows from criterion (iii) of Theorem 9.5.4. Under the hypotheses of (b), the $A/\mathfrak{J}$-module $B/\mathfrak{J}B$ is faithfully flat (**??** and **??**) and $\mathfrak{J}$ is completely secant by criterion (iv) of Theorem 9.5.4. The last assertion follows from **??**. □

Corollary 9.5.7. *Let A be a Noetherian ring, $\mathfrak{J}$ be an ideal of A, $\widehat{A}$ be the $\mathfrak{J}$-adic completion of A and $\widehat{\mathfrak{J}} = \mathfrak{J}\widehat{A}$ be the completion of $\mathfrak{J}$. For the ideal $\widehat{\mathfrak{J}}$ of $\widehat{A}$ to be completely secant, it is necessary and sufficient that the ideal $\mathfrak{J}$ of A is completely secant.*

Proof. In fact, by Proposition 2.4.24 the condition of Corollary 9.5.6(b) is satisfied. □

Corollary 9.5.8. *Let A be a Noetherian ring and $\mathfrak{J}$ be a completely secant ideal of A. If A is a Cohen-Macaulay (resp. Gorenstein) ring, so is $A/\mathfrak{J}$.*

Proof. Let $\mathfrak{m}$ be a maximal ideal of A containing $\mathfrak{J}$. The ideal $\mathfrak{J}_{\mathfrak{m}}$ of $A_{\mathfrak{m}}$ is generated by an $A_{\mathfrak{m}}$-regular sequence, so $(A/\mathfrak{J})_{\mathfrak{m}}$ is Cohen-Macaulay (resp. Corenstein) by Example 9.2.21(e) (resp. Example 9.3.28). □

Remark 9.5.9. A Noetherian ring A is said to be **complete intersection** if for any maximal ideal $\mathfrak{m}$ of A, the completion of the local ring $A_{\mathfrak{m}}$ is isomorphic to a quotient of a complete Noetherian regular local ring by a completely secant ideal. It then follows from Corollary 9.5.8 and Example 9.3.29 that such a ring is Gorenstein.

Proposition 9.5.10. *Let A be a Noetherian ring. Then the following conditions are equivalent:*

(i) A is regular;

(ii) any maximal ideal of A is completely secant;

(iii) any ideal $\mathfrak{I}$ of A such that $A/\mathfrak{I}$ is regular is completely secant.

Proof. Suppose that the ring A is regular; let $\mathfrak{I}$ be an ideal of A such that $A/\mathfrak{I}$ is regular and $\mathfrak{p}$ be a prime idela of A containing $\mathfrak{I}$. Then the local rings $A_\mathfrak{p}$ and $A_\mathfrak{p}/\mathfrak{I}_\mathfrak{p}$ are regular, so $\mathfrak{I}_\mathfrak{p}$ is generated by a completely secant sequence for $A_\mathfrak{p}$ (Proposition 7.4.9), and this signifies that $\mathfrak{I}$ is completely secant at $\mathfrak{p}$. This proves (i)⇒(iii), and (iii) ⇒(ii) since a field is regular. Finally, under the hypotheses of (ii), let $\mathfrak{m}$ be a maximal ideal of A. Then the maximal ideal $\mathfrak{m}A_\mathfrak{m}$ of $A_\mathfrak{m}$ is generated by a completely secant sequence for $A_\mathfrak{m}$, so $A_\mathfrak{m}$ is regular (Theorem 7.4.5), whence (i). □

Proposition 9.5.11. *Let A be a Noetherian ring and $\mathfrak{I}$ be an ideal of A such that $A/\mathfrak{I}$ is regular. Then the following conditions are equivalent:*

(i) the ideal $\mathfrak{I}$ is completely secant;

(ii) for any prime (resp. maximal) ideal $\mathfrak{p}$ of A containing $\mathfrak{I}$, the ring $A_\mathfrak{p}$ is regular;

(iii) the $\mathfrak{I}$-adic completion of A is regular.

Proof. By Theorem 9.5.4, condition (i) signifies that for any prime (resp. maximal) ideal $\mathfrak{p}$ of A containing $\mathfrak{I}$, the ideal $\mathfrak{I}_\mathfrak{p}$ of the local ring $A_\mathfrak{p}$ is generated by a completely secant sequence for $A_\mathfrak{p}$. Since $A_\mathfrak{p}/\mathfrak{I}_\mathfrak{p}$ is regular by hypothesis, this latter condition is equivalent to that $A_\mathfrak{p}$ is regular (Proposition 7.4.9). This proves the equivalence of (i) and (ii), and the equivalence of (ii) and (iii) follows from Corollary 9.4.14. □

Proposition 9.5.12. *Let A be a regular ring, $\mathfrak{I}$ be an ideal of A and A_0 be a subring of A such that the canonical homomorphism $A_0 \to A/\mathfrak{I}$ is bijective.*

(a) The ideal $\mathfrak{I}$ is completely secant, the A_0-module $\mathfrak{I}/\mathfrak{I}^2$ is finitely generated and projective, and the ring A_0 is regular.

(b) Let $\varphi : \mathfrak{I}/\mathfrak{I}^2 \to \mathfrak{I}$ be a A_0-linear section of the canonical surjection $\mathfrak{I} \to \mathfrak{I}/\mathfrak{I}^2$. Then the A_0-homomorphism $\boldsymbol{S}_{A_0}(\mathfrak{I}/\mathfrak{I}^2) \to A$ extending φ then extends to an isomorphism from the completion of the graded ring $\boldsymbol{S}_{A_0}(\mathfrak{I}/\mathfrak{I}^2)$ to the $\mathfrak{I}$-adic completion of A.

Proof. Let $\mathfrak{p}$ be a prime ideal of A containing $\mathfrak{I}$. We have $\mathfrak{p} = (\mathfrak{p} \cap A_0) \oplus \mathfrak{I}$ and therefore $\mathfrak{p}^2 = (\mathfrak{p} \cap A_0)^2 \oplus \mathfrak{p}\mathfrak{I}$, hence $\mathfrak{p}^2 \cap \mathfrak{I} = \mathfrak{p}\mathfrak{I}$. Denote by $i : \mathfrak{I} \to \mathfrak{p}$ be the canonical injection, then the map $i \otimes 1_{A/\mathfrak{p}} : \mathfrak{I} \otimes_A A/\mathfrak{p} \to \mathfrak{p} \otimes_A A/\mathfrak{p}$ is injective, and so is the map $i_\mathfrak{p} \otimes 1_{A/\mathfrak{p}} : \mathfrak{I}_\mathfrak{p} \otimes_{A_\mathfrak{p}} \kappa(\mathfrak{p}) \to \mathfrak{p} \otimes_{A_\mathfrak{p}} \kappa(\mathfrak{p})$. Nakayama's lemma then implies that the ideal $\mathfrak{I}_\mathfrak{p}$ of $A_\mathfrak{p}$ is generated by a subset of a system of parameters of the regular local ring $A_\mathfrak{p}$. By Corollary 7.4.11, the ring $A_\mathfrak{p}/\mathfrak{I}_\mathfrak{p}$ is regular and the ideal $\mathfrak{I}_\mathfrak{p}$ is completely secant. By Theorem 9.5.4, $\mathfrak{I}$ is then completely secant, the ring A_0, isomorphic to $A/\mathfrak{I}$, is regular, and the A_0-module $\mathfrak{I}/\mathfrak{I}^2$ is finitely generated and projective.

Now it follows from Theorem 9.5.4 that the canoncal homomorphism $\beta_\mathfrak{I} : \boldsymbol{S}_{A_0}(\mathfrak{I}/\mathfrak{I}^2)$ is bijective. Let $f : \boldsymbol{S}_{A_0}(\mathfrak{I}/\mathfrak{I}^2) \to A$ be the A_0-homomorphism extending the A_0-linear map $\varphi : \mathfrak{I}/\mathfrak{I}^2 \to \mathfrak{I}$. If we endow A with the $\mathfrak{I}$-adic filtration and $\boldsymbol{S}_{A_0}(\mathfrak{I}/\mathfrak{I}^2)$ the graduation filtration, $\beta_\mathfrak{I}$ is identified with the homomorphism induced from f by passing to graded algebras, and assertion (b) follows from Corollary 2.3.32. □

9.5.2 Graded regular rings

Let A_0 be a ring and P be a graded A_0-module of type $\mathbb{N}$. Denote by A the ring $\boldsymbol{S}_{A_0}(P)$, then there exists over A a unique graduation of type $\mathbb{N}$ for which A_0 is of degree 0 and P is a graded submodule of A. Let $A_+ = \bigoplus_{n>0} A_n$ be the irrelevant ideal of A, then the canonical map $P \to A_+/A_+^2$ is an isomorphism of graded A_0-modules (A, III, p76, prop.10).

If the graded A_0-module P is free and $(x_i)_{i\in I}$ is a basis for P formed by homogeneous elements, the graded A_0-algebra $\boldsymbol{S}_{A_0}(P)$ is then isomorphic to the polynomial algebra $A_0[(X_i)_{i\in I}]$, endowed with the graduation for which each X_i is homogeneous of degree $\deg(x_i)$. Any graded A_0-algbera of type $\mathbb{N}$, isomorphic to the graded A_0-algebra of this form, is said to be **polynomial**.

If A_0 is regular and the A_0-module P is projective and finitely generated, then the ring $\boldsymbol{S}_{A_0}(P)$ is regular by Corollary 9.4.15. Conversely, we have the following theorem which asserting that any regular graded A_0-algebra is of this form.

Theorem 9.5.13. *Let A be a regular graded ring of type $\mathbb{N}$. Then the ring A_0 formed by elements of degree 0 in A is regular, and there exists a finitely generated projective graded A_0-module P with positive degrees such that A is isomorphic to the graded A_0-algebra $\boldsymbol{S}_{A_0}(P)$.*

Proof. Denote by P the graded A_0-module A_+/A_+^2. By Proposition 9.5.12, the ring A_0 is regular and the A_0-module P is projective and finitely generated. The homogeneous components of P are then projective and there exists an A_0-linear section $\varphi : P \to A_+$ of the canonical surjection $A_+ \to P$, which is graded of degree 0. Let $f : \boldsymbol{S}_{A_0}(P) \to A$ be the induced homomorphism of graded A_0-algebras. Then by Proposition 9.5.12, $\hat{f} : \widehat{\boldsymbol{S}_{A_0}(P)} \to \widehat{A}$ is an isomorphism, so f is injective and its image is dense in A for the A_+-adic topology. But since the topologies induced over the homogeneous components of A are diecrete and the image of f is a graded submodule, this implies that f is bijective. □

Corollary 9.5.14. *Let B be a regular graded ring of positive degrees. Suppose that any finitely generated projective B_0-module is free.*

(a) The ring B_0 is integral and regular and the B_0-algebra B is a polynomial graded B_0-algebra of finite type.

(b) Let A be a graded subring of B such that $A_0 = B_0$ and that B is a finitely generated A-module. Then the following conditions are equivalent:

(i) the graded A-module B is free;

(ii) the A-module B is flat;

(iii) A is a polynomial graded A_0-algebra of finite type.

Corollary 9.5.15. *Let k be a field, B be a polynomial graded k-algebra of finite type, and A be a graded subring of B. Then the following conditions are equivalent:*

(i) B is a graded free A-module;

(ii) B is a flat A-module;

(iii) $\mathrm{Tor}_1^A(k, B) = 0$;

(iv) the algebra A is a polynomial graded k-algebra of finite type, and any algebraically free generating sequence of A formed by homogeneous elements is B-regular.

9.5.3 Extension of scalars

Proposition 9.5.16. *Let $\rho : A \to B$ be a homomorphism of Noetherian rings and $\mathfrak{I}$ be an ideal of B. Then the following conditions are equivalent:*

(i) the A-module $B/\mathfrak{I}$ is flat and the ideal $\mathfrak{I}$ is completely secant;

(ii) for any $\mathfrak{P} \in V(\mathfrak{I})$, the A-module $B_{\mathfrak{P}}$ is flat and, for any A-algebra A' such that the ring $A' \otimes_A B$ is Noetherian, the ideal $\mathfrak{I}(A' \otimes_A B)$ is completely secant;

(iii) for any maximal ideal $\mathfrak{N}$ of B containing $\mathfrak{I}$, the A-module $B_{\mathfrak{N}}$ is flat and the ideal $\mathfrak{I}(\kappa(\mathfrak{N}^c) \otimes_A B_{\mathfrak{N}})$ of $\kappa(\mathfrak{N}^c) \otimes_A B_{\mathfrak{N}}$ is completely secant.

9.6 Geometric regularity and normality

9.6.1 Algebras essentially of finite type

Let k be a ring, A be a k-algebra, and $\boldsymbol{x} = (x_i)_{i\in I}$ be a family of element of A; denote by A' the subalgebra of A generated by the x_i. We say that $\boldsymbol{x}$ is an **essentially generating family** of the k-algebra A if, for any element $a \in A$, there exists an element s of A', invertible in A, such that $sa \in A'$. Equivalently, this amounts to saying that, for any $a \in A$, there exist polynomials P, Q of $k[(X_i)_{i\in I}]$ such that $Q(\boldsymbol{x})$ is invertible in A and that $a = P(\boldsymbol{x})Q(\boldsymbol{x})^{-1}$.

A k-algebra A is said to be **essentially of finite type** if it admits a finite essentially generating family. Equivalently, this means there exists a k-algebra A' of finite type and a multiplicative subset S of A' such that A is isomorphic to the k-algebra $S^{-1}A'$.

Example 9.6.1. To say that a field extension L/K is essentially of finite type signifies that it is a finitely generated field extension (note that this is not equivalent to that K is a *k-algebra* of finite type). In fact, if $\boldsymbol{x} = (x_i)_{i\in I}$ is an essentially generating family for L, then we obtain a surjective homomorphism from $K((x_i)_{i\in I}) \to L$. Also note that by Zariski's lemma (4.3.18), L is of finite type over K if and only if it is a finite extension of K.

Example 9.6.2. For a local k-algebra to be essentially of finite type, it is necessary and sufficient that it is isomorphic to a k-algebra of the form $A_{\mathfrak{p}}$, where A is a k-algebra of finite type and $\mathfrak{p}$ is a prime ideal of A. In fact, if the local k-algebra is isomorphic to $S^{-1}A$, then by (Corollary 1.2.23), it is isomorphic to $A_{\mathfrak{p}}$, where $\mathfrak{p}$ is the inverse image of the maximal ideal of $S^{-1}A$ in A.

Proposition 9.6.3. *If the ring k is Noetherian, then any k-algebra essentially of finite type is a Noetherian ring.*

Proof. This follows from Hilbert basis theorem and Corollary 1.2.30. □

Proposition 9.6.4. *Let k be a ring.*

(a) Any quotient algebra of a k-algebra essentially of finite type is a k-algebra essentially of finite type.

(b) Any fraction ring of a k-algebra essentially of finite type is a k-algebra essentially of finite type.

(c) The product of a finite family of k-algebras essentially of finite type is a k-algebra essentially of finite type.

(d) Let $k \to k'$ be a ring homomorphism. For any k-algebra A essentially of finite type, the k'-algebra $A_{(k')} = k' \otimes_k A$ is essentially of finite type.

Proof. Since localization commutes with quotients, it is clear that assertion (a) is true, and assertion (b) follows from definition. Similarly, since localization commutes with base change, it is easy to conclude (d), and assertion (c) follows from the definition of essentially generating families. □

Corollary 9.6.5. *Let A be a k-algebra essentially of finite type and B be a Noetherian k-algebra. Then the ring $A \otimes_k B$ is Noetherian.*

Proof. In fact, this is a B-algebra essentially of finite type, so the conclusion follows from Proposition 9.6.3. □

Proposition 9.6.6. *Let k be a ring, A be a k-algebra and B be an A-algebra. If A is essentially of finite type over k and B is essentially of finite type over A, then B is essentially of finite type over k.*

Proof. Denote by $\rho : A \to B$ the canonical map. Let $\boldsymbol{x} = (x_i)_{i\in I}$ be an essentially generating family of the k-algebra A and A' the subalgebra it generates; let $\boldsymbol{y} = (y_j)_{j\in J}$ be an essentially generating family of the A-algebra B, and B' be the subalgebra generated by the $\rho(x_i)$ and y_j. Let $b \in B$, then by hypothesis there exist polynomials P, Q of $A[(Y_j)_{j\in J}]$ such that $Q(\boldsymbol{y})$ is invertible in B and that we have $Q(\boldsymbol{y})b = P(\boldsymbol{y})$. The nonzero coefficients of P and Q are finite in number, so there exists a polynomial $R \in k[(X_i)_{i\in I}]$ such that $R(\boldsymbol{x})$ is invertible in A and that $R(\boldsymbol{x})P$ and $R(\boldsymbol{x})Q$ belongs to $A'[(Y_j)_{j\in J}]$. Then $\rho(R(\boldsymbol{x}))Q(\boldsymbol{y})$ is invertible in B, and we have

$$\rho(R(\boldsymbol{x}))Q(\boldsymbol{y})b = \rho(R(\boldsymbol{x}))P(\boldsymbol{y}) \in B'.$$

In other words, the elements $\rho(x_i)$ and y_j for an essentially generating family for the k-algebra B. □

Corollary 9.6.7. *The tensor product of two k-algebras essentially of finite type is a k-algebra essentially of finite type.*

Proof. Let A and B be two k-algebras essentially of finite type. Then $A \otimes_k B$ is essentially of finite type over A (Proposition 9.6.4), and hence over k (Proposition 9.6.6). □

9.6.2 Tensor product of Cohen-Macaulay algebras

9.6.3 Geometrically regular and geometrically normal algebras

Recall that a separable algebra A over a field k is defined by the property that for any finite field extension k' of k, the ring $A_{(k')}$ is reduced (in other words, the k-algebra A is *geometrically reduced*). In this paragraph, we develop similar notions for a k-algebra A, concering the regularity (resp. normality) for any base change $A_{(k')}$ into a field extension k' of k. Such k-algebras are then called *geometrically regular* (resp. *geometrically normal*), and they will play an important role when we discuss smooth algebras.

Before giving the definition, we first recall if k' is a separable extension of k, then the base change $A_{(k')}$ of any reduced k-algebra is still reduced. We now prove a similar result for regular and normal algebras. However, for this to work, we need to impose some finite condition on the k-algebras. The technique points of our consideration are contained in the following two lemmas.

Lemma 9.6.8. *Let A be a Noetherian ring with an exhaustive filtration $(A_\alpha)_{\alpha\in I}$ of Noetherian subrings.*

(a) If the rings A_α are regular and A is a flat A_α-module for each $\alpha \in I$, then A is regular.

(b) If the rings A_α are normal, then A is normal.

Proof. □

Lemma 9.6.9. *Let k be a field and K, L be field extensions of k such that one of them is separable over k. Suppose that K is finitely generated over k, then the ring $K \otimes_k L$ is regular.*

Proof. □

Proposition 9.6.10. *Let k be a field, A be a k-algebra and K be a extension of k. Suppose that A is essentially of finite type or that the extension K is finitely generated.*

(a) If the ring $A_{(K)}$ is regular (resp. normal), then A is regular (resp. normal).

(b) If the ring A is regular (resp. normal) and the extension K/k is separable, then $A_{(K)}$ is regular (resp. normal).

Proof. Being the base change of the free k-module K by A, the A-module $A_{(K)}$ is free, hence faithfully flat. Assertion (a) then follows from **??** and Proposition 9.4.20 (resp. Lemma 9.6.9). Under the hypothesis of (b), for any prime ideal $\mathfrak{p}$ of A, the ring $\kappa(\mathfrak{p}) \otimes_k K$ is regular by Lemma 9.6.9, and a fortiori normal (Corollary 9.4.13). Assertion (b) then follows from Proposition 9.4.23 (resp. Corollary 9.1.55). □

Let k be a field and A be a k-algebra. We say that A is **geometrically regular** (resp. **geometrically normal**) if the ring $A_{(k')}$ is regular (resp. normal) for any finite extension k' of k. It is clear tha any geometrically regular (resp. geometrically normal) k-algebra is itself regular (resp. normal), as we can take $k' = k$ in the defintion. Also, in view of Proposition 9.6.10, to verify that a k-algebra is geometrically regular (resp. geometrically normal), it suffices to consider only finite purly separable extensions of k.

Example 9.6.11. If the field k is perfect, any regular (resp. normal) k-algebra is geometrically regular (resp. geometrically normal). This is an analogue for separable algebras, since in this case any reduced k-algebra is separable.

Example 9.6.12. Let A be an Artinian k-algebra. Then the following conditions are equivalent:

- A is separable;
- A is geometrically regular;
- A is geometrically normal.

In fact, if A is geometrically normal then it is separable, and if A is separable, for any finite extension k' of k, the ring $A_{(k')}$ is reduced and Artinian, hence isomorphic to a product of fields, and therefore regular (A, VIII, §8, n1, prop.2). Note that if the k-algebra A is finite dimensional, then these conditions are also equivalent to that A is étale (A, V, p.34, th.4).

Example 9.6.13. Let A be a regular local k-algebra. If the extension κ_A of k is separable, then A is geometrically regular. In fact, let k' be a finite extension of k. The A-module $A_{(k')}$ is then free and finitely generated, hence flat, and each maximal ideal of $A_{(k')}$ is lying over $\mathfrak{m}_A$ (Theorem 4.1.69). The ring $\kappa_A \otimes_A A_{(k')}$, being isomorphic to $\kappa_A \otimes_k k'$, is regular by Lemma 9.6.9. Corollary 9.4.25 then implies that the ring $A_{(k')}$ is regular, so the k-algebra A is geometrically regular.

Proposition 9.6.14. *Let k be a field and A be a Noetherian k-algebra.*

(a) If A is geometrically regular (resp. geometrically normal, resp. separable), so is the ring $S^{-1}A$ for any multiplicative subset S of A.

(b) If $A_{\mathfrak{m}}$ is geometrically regular (resp. geometrically normal, resp. separable) for any maximal ideal $\mathfrak{m}$ of A, then A is geometrically regular (resp. geometrically normal, resp. separable).

Proof. Assertion (a) follows from the fact that $(S^{-1}A)_{(k')}$ is isomorphic to a fraction ring of $A_{(k')}$ for any field extension k' of k. Now suppose that $A_{\mathfrak{m}}$ is geometrically regular (resp. geometrically normal, resp. separable) for any maximal ideal $\mathfrak{m}$ of A. Let k' be a finite extension of k, and $\mathfrak{m}'$ be a maximal ideal of $A_{(k')}$. It suffices to verify that the local ring $(A_{(k')})_{\mathfrak{m}'}$ is regular (resp. normal, resp. reduced).

The canonical homomorphism $A \to A_{(k')}$ is finite, so the maximal ideal $\mathfrak{m}'$ is lying over a maximal ideal $\mathfrak{m}$ of A (Theorem 4.1.69) and $(A_{(k')})_{\mathfrak{m}'}$ is isomorphic to a fraction ring of the regular (resp. normal, resp. reduced) ring $(A_{\mathfrak{m}})_{(k')}$, hence is regular (resp. normal, resp. reduced). □

Lemma 9.6.15. *Let k be a field and K be a finitely generated extension of k. Then there exists an finite extension L of K and a sub-k-extension k' of L which is finite and purely inseparable over k, such that the extension L/k' is separable.*

Proposition 9.6.16. *Let k be a field, A and B be k-algebras such that one of them is essentially of finite type. Suppose that A is geometrically regular (resp. geometrically normal). If the ring B is regular (resp. normal), so is the ring $A \otimes_k B$.*

Proof. □

Corollary 9.6.17. *Let k be a field. Then the tensor product of two geometrically regular (resp. geometrically normal) k-algebras, one of which is essentially of finite type, is a geometrically regular (resp. geometrically normal) k-algebra.*

Proof. Let A and B be k-algebras satisfying the hypotheses of the corollary, and k' be a finite extension of k. The ring $B_{(k')}$ is regular (resp. normal), so $A \otimes_k B_{(k')}$ is regular (resp. normal) by Proposition 9.6.16, which is isomorphic to $(A \otimes_k B)_{(k')}$. □

Corollary 9.6.18. *Let k be a field, A be a geometrically regular (resp. geometrically normal) k-algebra and K be an extension of k. Suppose that A is essentially of finite type or the extension K/k is finitely generated.*

(a) The ring $A_{(K)}$ is regular (resp. normal).

(b) If the extension K/k is separable, the k-algebra $A_{(K)}$ is geometrically regular (resp. geometrically normal).

Proof. Assertion (a) follows from Proposition 9.6.16, since a field is regular. Assertion (b) follows from Corollary 9.6.17 and Example 9.6.12. □

Corollary 9.6.19. *Let k be a field, A be a k-algebra and K be an extension of k. Suppose that the k-algebra A is essentially of finite type or the extension K/k is finitely generated. Fot the k-algebra A to be geometrically regular (resp. geometrically normal), it is necessary and sufficient that so is the K-algebra $A_{(K)}$.*

Proof. Suppose that A is geometrically regular (resp. geometrically normal) and let K' be a finite extension of K. The ring $K' \otimes_K A_{(K)}$, isomorphic to $K' \otimes_k A$, is regular (resp. normal) by Corollary 9.6.18.

Suppose conversely that the K-algebra $A_{(K)}$ is geometrically regular (resp. geometrically normal) and let k' be a finite extension of k. Let L be the composition extension of k' and K, then the ring $A_{(L)}$ is identified with $L \otimes_K A_{(K)}$, hence is regular (resp. normal); therefore the ring $A_{(k')}$ is regular (resp. normal) by Proposition 9.6.10. □

Corollary 9.6.20. *Let k be a field, A be a k-algebra essentially of finite type and K be a extension of k which is a perfect field. For the k-algebra A to be geometrically regular (resp. geometrically normal), it is necessary and sufficient that $A_{(K)}$ is regular (resp. normal).*

Proof. This follows from Corollary 9.6.19 and Example 9.6.11. □

9.6.4 Characterization of geometrically regular algebras

Proposition 9.6.21. *Let k be a field and A be a k-algebra essentially of finite type. Let $\mathfrak{I}$ be the kernel of the multiplication homomorpism $\mu : A \otimes_k A \to A$. Then the following conditions are equivalent:*

(i) the k-algebra A is geometrically regular;

(ii) for any regular k-algebra R, the ring $A \otimes_k R$ is regular;

(iii) the ring $A \otimes_k A$ is regular;

(iv) the ideal $\mathfrak{I}$ of $A \otimes_k A$ is completely secant.

Proof. We denote by B the ring $A \otimes_k A$, which is endowed with the A-algebra structure defined by the homomorphism $\rho : A \to A \otimes_k A$ such that $\rho(x) = x \otimes 1$. Then μ is a homomorphism of A-algebras and induces an isomorphism from $B/\mathfrak{I}$ to A. From Proposition 9.6.16, it is clear that (i) implies (ii), and (ii) implies (iii) by takeing $C = k$ and $C = A$. Also, to see that (iii)$\Rightarrow$(iv), note that the A-module B is faithfully flat as the base change of the free k-module A by A. If the ring B is regular, then A is regular by Proposition 9.4.20, and the ideal $\mathfrak{I}$ is then completely secant by Proposition 9.5.10.

Suppose that the ideal $\mathfrak{I}$ is completely secant, we prove that A is regular. Let $\mathfrak{m}$ be a maximal ideal of A and $\nu : (A/\mathfrak{m}) \otimes_k A \to A/\mathfrak{m}$ be the homomorphism induced by μ. The maximal ideal $\mathfrak{n} = \ker \nu$ is equal to $\mathfrak{I}((A/\mathfrak{m}) \otimes_k A)$, and by applying Proposition 9.5.16 to the ring homomorphism $B \to (A/\mathfrak{m}) \otimes_k A$ and use Proposition 9.1.31, we see that the ideal $\mathfrak{n}$ of $(A/\mathfrak{m}) \otimes_k A$ is completely secant. Therefore (Proposition 9.5.11) the local ring $((A/\mathfrak{m}) \otimes_k A)_{\mathfrak{n}}$ is regular. Denote by $j : A \to (A/\mathfrak{m}) \otimes_k A$ the homomorphism $x \mapsto 1 \otimes x$; since $\nu \circ j$ is the canonical homomorphism $A \to A/\mathfrak{m}$, we have $j^{-1}(\mathfrak{n}) = \mathfrak{m}$. Thus j extends to a local homomorphism $j_{\mathfrak{m}} : A_{\mathfrak{m}} \to ((A/\mathfrak{m}) \otimes_k A)_{\mathfrak{n}}$, which is faithfully flat since j is flat (the homomorphism $k \to A/\mathfrak{m}$ is free and j is its base change to A). By Proposition 9.4.20, the ring $A_{\mathfrak{m}}$ is then regular. This proves that the ring A is regular.

Now let k' be an extension of k. The kernel of the multiplication homomorphism $\mu' : A_{(k')} \otimes_{k'} A_{(k')} \to A_{(k')}$ is none other that $\mathfrak{I}B_{(k')}$, so it is completely secant in $A_{(k')}$ (Proposition 9.5.16). The k'-algebra $A_{(k')}$ then satisfies condition (iv), and by our preceding arguments, it is therefore regular. This proves (i) and completes the proof. □

Recall that (A, III, p.133-134) the quotient $\mathfrak{I}/\mathfrak{I}^2$ endowed with the A-module structure induced by ρ, is the differential module $\Omega_{A/k}$ of the k-algebra A. If the k-algebra A is essentially of finite type, the ring $A \otimes_k A$ is Noetherian (Proposition 9.6.3), so $\Omega_{A/k}$ is a finitely generated A-module. We denote by $d : A \to \Omega_{A/k}$ the k-linear map which sends an element $x \in A$ to the class of $x \otimes 1 - 1 \otimes x$ in $\Omega_{A/k}$. Then d is a k-derivation, and for any A-module M and any k-derivation $D : A \to M$, there exists a unique A-linear map $\delta : \Omega_{A/k} \to M$ such that $D = \delta \circ d$ (A, III, p.133-134 prop.18).

If S is a multiplicative subset of A, the canonical $S^{-1}A$-linear map (A, III, p.136) $S^{-1}\Omega_{A/k} \to \Omega_{S^{-1}A/k}$ is bijective: in fact, it suffices to verify that, for any $S^{-1}M$, any k-derivation $D : A \to M$ extends uniquely to a k-derivation $D : S^{-1}A \to M$, and this follows from (A, III, p.123, prop.4).

Now let k be a field and A be a local k-algebra essentially of finite type. Then by Example 9.6.2, A is isomorphic to a local ring $B_{\mathfrak{P}}$, where B is a finite type k-algebra and $\mathfrak{P}$ is an ideal of B. Then we have $\dim(A) = \mathrm{ht}(\mathfrak{P})$, and by Theorem 7.1.31(c), the dimension of the integral domain $B/\mathfrak{P}$ is given by

$$\dim(B/\mathfrak{P}) = \mathrm{tr.deg}_k(\kappa(\mathfrak{P})) = \mathrm{tr.deg}_k(\kappa_A).$$

It then follows from Corollary 7.1.33 that $n := \dim(A) + \mathrm{tr.deg}_k(\kappa_A)$ is equal to $\dim(B)$.

We now characterize geometrically regular local k-algebras by the differential module $\Omega_{A/k}$. We shall see that this property is equivalent to the freeness of $\Omega_{A/k}$ with rank equal to n. For simplicity, we single out the following two lemmas which will be use in our proof.

Lemma 9.6.22.

Lemma 9.6.23. *Let k*

Theorem 9.6.24. *Let k be a field and A be a local k-algebra essentially of finite type. Then we have*

$$[\kappa_A \otimes_A \Omega_{A/k} : \kappa_A] \geq n = \dim(A) + \operatorname{tr.deg}_k(\kappa_A),$$

and the following conditions are equivalent:

(i) the k-algebra A is geometrically regular;

(ii) the A-module $\Omega_{A/k}$ is free of rank n;

(iii) $[\kappa_A \otimes_A \Omega_{A/k} : \kappa_A] = n$.

Proof. It is clear that (ii) implies (iii), so to prove the theorem, it suffices to prove the implications (i)⇒(ii) and (iii)⇒(i). Choose a finite type k-algebra B such that A is isomorphic to $B_{\mathfrak{P}}$, where $\mathfrak{P}$ is a prime ideal of B. Then by the arguments above, we have $n = \dim(B)$, and by replacing B by a ring B_f, where $f \in B - \mathfrak{P}$, we can assume that B is connected of dimension n.

Suppose that the k-algebra A is □

Example 9.6.25. Let K be a finitely generated extension of k, Theorem 9.6.24 then proves that the extension K/k is separable if and only if $\Omega_{K/k}$ has dimension $\operatorname{tr.deg}_k(K)$, in view of Example 9.6.12.

Corollary 9.6.26. *Let k be a field and A be k-algebra essentially of finite type. Then the set of elements $\mathfrak{p}$ of $\operatorname{Spec}(A)$ such that the k-algebra $A_{\mathfrak{p}}$ is geometrically regular is open in $\operatorname{Spec}(A)$.*

Proof. We can suppose that the k-algebra A is of finite type. The considered subset is then formed by the prime ideals $\mathfrak{p}$ such that $[\kappa(\mathfrak{p}) \otimes_k \Omega_{A/k} : \kappa(\mathfrak{p})] \leq \dim_{\mathfrak{p}}(A)$. Now the function $\mathfrak{p} \mapsto \dim_{\mathfrak{p}}(A)$ is lower semi-continuous by definition, and the function $\mathfrak{p} \mapsto [\kappa(\mathfrak{p}) \otimes_k \Omega_{A/k} : \kappa(\mathfrak{p})]$ is upper semi-continuous (Nakayama's lemma and Proposition 1.5.1). □

Corollary 9.6.27. *Let k be a field and A be a k-algebra essentially of finite type. For A to be geometrically regular, it is necessary and sufficient that the A-module $\Omega_{A/k}$ is projective and for any minimal prime ideal $\mathfrak{p}$ of A, the k-algebra $A_{\mathfrak{p}}$ is separable.*

Proof. Suppose that A is geometrically regular. For any prime ideal of A, the k-algebra $A_{\mathfrak{p}}$ is geometrically regular, so the $A_{\mathfrak{p}}$-module $\Omega_{A_{\mathfrak{p}}/k}$ is free (Theorem 9.6.24). Moreover, if $\mathfrak{p}$ is minimal, the local k-algebra $A_{\mathfrak{p}}$ is also Artinian, hence is a separable field extension of k (Example 9.6.12).

Conversely, suppose that the A-module $\Omega_{A/k}$ is projective and $A_{\mathfrak{p}}$ is separable for any minimal prime $\mathfrak{p}$ of A. Now let $\mathfrak{p}$ be a prime ideal of A, and $\mathfrak{q}$ be a minimal prime contained in $\mathfrak{p}$. Since the $A_{\mathfrak{p}}$-module $(\Omega_{A/k})_{\mathfrak{p}}$ is free by Theorem 1.5.5, we have

$$[\kappa(\mathfrak{p}) \otimes_A \Omega_{A/k} : \kappa(\mathfrak{p})] = [\kappa(\mathfrak{q}) \otimes_A \Omega_{A/k} : \kappa(\mathfrak{q})].$$

The k-algebra $A_{\mathfrak{q}}$ is Artinian and separable, hence geometrically regular (Example 9.6.12). Theorem 9.6.24 then implies that

$$[\kappa(\mathfrak{q}) \otimes_A \Omega_{A/k} : \kappa(\mathfrak{q})] = \operatorname{tr.deg}_k(\kappa(\mathfrak{q})).$$

It then follows from Theorem 9.6.24 that the k-algebra $A_{\mathfrak{p}}$ is geometrically regular, whence the corollary. □

Remark 9.6.28. Suppose that the k-algebra A essentially of finite type is geometrically regular. Then the total fraction ring $Q(A)$ of A is identified with the product of $A_{\mathfrak{p}}$, where $\mathfrak{p}$ runs through minimal prime ideals of A (Proposition 3.2.26); it is then a separable k-algebra. Conversely, suppose that $Q(A)$ is a separable k-algebra; for any minimal prime ideal $\mathfrak{p}$ of A, the k-algebra $A_{\mathfrak{p}}$ is a fraction ring of $Q(A)$ (Proposition 3.2.26), hence a separable k-algebra (Proposition 9.6.14). If the A-module $\Omega_{A/k}$ is also projective, it then follows from Corollary 9.6.27 that the k-algebra A is geometrically regular.

9.7 Smooth algebras

9.7.1 Infinitesimal liftings

Let k be a ring, C be a k-algebra and N be a square zero ideal of C. We denote by $\pi : C \to C/N$ the canonical homomorphism; since $N^2 = 0$, the C-module N has a natrual C/N-module structure. Let A be an k-algebra and $\varphi : A \to C/N$ be a homomorphism of k-algebra. Endow N with the A-module structure induced by φ, we define a **lifting** of φ (to C) to be a homomorphism $\tilde{\varphi} : A \to C$ of k-algebras such that $\pi \circ \tilde{\varphi} = \varphi$.

$$\begin{tikzcd} k \arrow[r] \arrow[d] & C \arrow[d, "\pi"] \\ A \arrow[r, "\varphi"] \arrow[ur, dashed, "\tilde{\varphi}"] & C/N \end{tikzcd}$$

Proposition 9.7.1. *If φ admits a lefting, the map $(\delta, \tilde{\varphi}) \mapsto \delta + \tilde{\varphi}$ defines a simply transitive action of the group $\mathrm{Der}_k(A, N)$ of k-derivations from A to N on the set of leftings of φ.*

Proof. Let $\tilde{\varphi}_0 : A \to C$ be a lifting of φ. The map $\delta \mapsto \delta + \tilde{\varphi}_0$ induces a bijection from the set of maps $A \to N$ to the set of maps $\tilde{\varphi} : A \to C$ such that $\pi \circ \tilde{\varphi} = \varphi$. Fix δ, and put $\tilde{\varphi} = \delta + \tilde{\varphi}_0$. For $\tilde{\varphi}$ to be a homomorphism of k-algebras, it is necessary and sufficient that δ is a k-derivation: in fact, for $x, y \in A$ and $\lambda \in k$, we have the relations

$$\begin{aligned} \tilde{\varphi}(x+y) - \tilde{\varphi}(x) - \tilde{\varphi}(y) &= \delta(x+y) - \delta(x) - \delta(y) \\ \tilde{\varphi}(\lambda x) - \lambda\tilde{\varphi}(x) &= \delta(\lambda x) - \lambda\delta(x) \\ \tilde{\varphi}(xy) - \tilde{\varphi}(x)\tilde{\varphi}(y) &= \delta(xy) - \delta(x)\delta(y) - \delta(x)\tilde{\varphi}_0(y) - \tilde{\varphi}_0(x)\delta(y) \\ &= \delta(xy) - \varphi(x)\delta(y) - \varphi(y)\delta(x), \end{aligned}$$

the last equality resulting from the fact that N has zero square. □

Example 9.7.2. Let B be a k-algebra, N be a B-module. In this section, we often endow the k-module $B \oplus N$ with the k-algebra structure defined by $(b, x)(d, y) = (bd, by + dx)$; N is then a square zero ideal of $B \oplus N$. Let $\varphi : A \to B$ be a homomorphism of k-algebras, then the liftings of φ to $B \oplus N$ are the maps $x \mapsto (\varphi(x), \delta(x))$, where δ is a k-derivation of A to N.

Let $\Omega_{A/k}$ be the module of k-differentials of the ring A, and $d : A \to \Omega_{A/k}$ be the universal k-derivation. Recall that for any A-module M, the map $v \mapsto v \circ d$ is an A-linear isomorphism from $\mathrm{Hom}_A(\Omega_{A/k}, M)$ to $\mathrm{Der}_k(A, M)$. Let $\mathfrak{I}$ be an ideal of A. By (A, III, p.137), we have an exact sequence of $A/\mathfrak{I}$-modules

$$\mathfrak{I}/\mathfrak{I}^2 \xrightarrow{\bar{d}} (A/\mathfrak{I}) \otimes_A \Omega_{A/k} \longrightarrow \Omega_{(A/\mathfrak{I})/k} \longrightarrow 0$$

where $\bar{d}$ is the homomorphism induced by the restriction of d to $\mathfrak{I}$.

Let $\rho : A \to A/\mathfrak{I}^2$ and $\pi : A/\mathfrak{I}^2 \to A/\mathfrak{I}$ be the canonical homomorphisms. If

$$\alpha : (A/\mathfrak{I}) \otimes_A \Omega_{A/k} \to \mathfrak{I}/\mathfrak{I}^2$$

is a k-linear map, we then have an associated k-linear map $h_\alpha : A \to A/\mathfrak{I}^2$ defined by

$$h_\alpha(x) = \rho(x) - \alpha(1 \otimes dx).$$

If α is a *retraction* of $\bar{d}$ (that is, $\alpha \circ \bar{d} = \mathrm{id}_{\mathfrak{I}/\mathfrak{I}^2}$), then the map h_α factors into a k-linear map $h_\alpha : A/\mathfrak{I} \to A/\mathfrak{I}^2$. On the other hand, any k-linear map $h : A/\mathfrak{I} \to A/\mathfrak{I}^2$ induces a k-linear map $\psi_h : (A/\mathfrak{I}) \oplus (\mathfrak{I}/\mathfrak{I}^2) \to A/\mathfrak{I}^2$ given by $(x, y) \mapsto h(x) + y$.

Proposition 9.7.3. *Endow $(A/\mathfrak{I}) \oplus (\mathfrak{I}/\mathfrak{I}^2)$ with the k-algebra structure defined in Example 9.7.2. Then the maps $\alpha \mapsto h_\alpha$ and $h \mapsto \psi_h$ induce bijections between the following sets:*

(i) the set of $A/\mathfrak{I}$-linear retractions α of $\bar{d}$;

(ii) the set of k-algebra homomorphisms $h : A/\mathfrak{I} \to A/\mathfrak{I}^2$ such that $\pi \circ h = \mathrm{id}_{A/\mathfrak{I}}$.

(iii) the set of k-algebra isomorphisms $\psi : (A/\mathfrak{I}) \oplus (\mathfrak{I}/\mathfrak{I}^2) \to A/\mathfrak{I}^2$ such that $\pi \circ \psi = \mathrm{pr}_1$ and $\psi(0, z) = z$ for $z \in \mathfrak{I}/\mathfrak{I}^2$.

Proof. We apply Proposition 9.7.1 to $C = A/\mathfrak{I}^2$ and $N = \mathfrak{I}/\mathfrak{I}^2$. Let $\varphi : A \to A/\mathfrak{I}$ be the canonical surjection, then the homomorphism $\rho : A \to A/\mathfrak{I}^2$ is a lifting of φ to $A/\mathfrak{I}^2$.

$$\begin{tikzcd}
k \arrow[r] \arrow[d] & A/\mathfrak{I}^2 \arrow[d, "\pi"] \\
A \arrow[r, "\varphi"'] \arrow[ur, dashed, "\rho"] & A/\mathfrak{I}
\end{tikzcd}$$

The A-module $\mathrm{Hom}_{A/\mathfrak{I}}((A/\mathfrak{I}) \otimes_A \Omega_{A/k}, \mathfrak{I}/\mathfrak{I}^2)$ is identified with $\mathrm{Hom}_A(\Omega_{A/k}, \mathfrak{I}/\mathfrak{I}^2)$, so the map $\alpha \mapsto h_\alpha$ is a bijection from this set to the set of liftings of φ to $A/\mathfrak{I}^2$ (Proposition 9.7.1). Now for $x \in \mathfrak{I}$ we have $1 \otimes dx = \bar{d}(\rho(x))$, so for the map h_α to factor through $A/\mathfrak{I}$, it is necessary and sufficient that $\alpha \circ \bar{d}$ is the identity on $\mathfrak{I}/\mathfrak{I}^2$, which means that α is a retraction of $\bar{d}$. This proves the correspondence of the sets in (i) and (ii) under the map $\alpha \mapsto h_\alpha$.

The map $h \mapsto \psi_h$ is a bijection from the set of k-linear homomorphisms $A/\mathfrak{I} \to A/\mathfrak{I}^2$ to the set of k-linear homomorphisms $\psi : (A/\mathfrak{I}) \oplus (\mathfrak{I}/\mathfrak{I}^2) \to A/\mathfrak{I}^2$ such that $\psi(0, z) = z$ for $z \in \mathfrak{I}/\mathfrak{I}^2$. For the equality $\pi \circ \psi_h = \mathrm{pr}_1$ to be true, it is necessary and sufficient that $\pi \circ h = \mathrm{id}_{A/\mathfrak{I}}$, which means $h(\pi(z)) \equiv z \bmod \mathfrak{I}/\mathfrak{I}^2$ for any $z \in A/\mathfrak{I}^2$. Suppose that this condition is satisfied, then for h to be a ring homomorphism, it is necessary and sufficient that ψ_h is a ring homomorphism. Moreover, in this case ψ_h is bijective: the inverse map is given by $z \mapsto (\pi(z), z - h(\pi(z)))$, where $z \in A/\mathfrak{I}^2$. This proves the correspondence of the sets in (ii) and (iii). ☐

9.7.2 Formally smooth algebras

Let k be a ring and A be a linearly topologized k-algebra. We say that A is **formally smooth over k**, or that a **formally smooth k-algebra**, if it satisfies the following condition: for any k-algebra C and a square zero ideal N of C (endowed with the discrete topology), any continuous homomorphism from A into the k-algebra C/N can be lifted into a continuous homomorphism of A into the k-algebra C. We say that a k-algebra A is **formally smooth** if it is formally smooth endowed with the discrete topology, which is also the (0)-adic topology. In this case, it is then formally smooth for any $\mathfrak{I}$-adic topology.

Remark 9.7.4. Let k be a ring, A be a k-algebra and $\mathfrak{I}$ be an ideal of A. Endow A with the $\mathfrak{I}$-adic topology. Let C be an k-algebra, N be a square zero ideal of C, and endow C and C/N with the discrete topology. Let $\varphi : A \to C/N$ be a continuous homomorphism of k-algebras. Then any lifting $\tilde{\varphi} : A \to C$ of φ is continuous: in fact, there exists an integer $n > 0$ such that $\varphi(\mathfrak{I}^n) = 0$, and we have $\tilde{\varphi}(\mathfrak{I}^n) \subseteq N$, whence $\tilde{\varphi}(\mathfrak{I}^{2n}) \subseteq N^2 = 0$. From this, we conclude that if A is formally smooth for the $\mathfrak{I}$-adic topology, it is also formally smooth for the $\mathfrak{I}'$-adic topology for any ideal $\mathfrak{I}'$ containing $\mathfrak{I}$ (note that a homomorphism from A to a discrete k-algebra is continuous if and only if its kernel is open in A).

Example 9.7.5. Let k be a ring, A be a k-algebra and $\mathfrak{I}$ be an ideal of A. If the k-algebra $A/\mathfrak{I}$ is formally smooth (for the discrete topology), then the identity map on $A/\mathfrak{I}$ admits a lifting to $A/\mathfrak{I}^2$, so the sets of Proposition 9.7.3 are nonempty. In particular, the sequence

$$0 \to \mathfrak{I}/\mathfrak{I}^2 \xrightarrow{\bar{d}} (A/\mathfrak{I}) \otimes_A \Omega_{A/k} \longrightarrow \Omega_{(A/\mathfrak{I})/k} \longrightarrow 0$$

is then exact and split.

Remark 9.7.6. Let k be a ring, A be a linearly topologized formally smooth k-algebra, M be an A-module whose annihilator is open in A. Then *any derivation $\delta : k \to M$ extends to a derivation $\tilde{\delta} : A \to M$*. In fact, put $B = A/\mathrm{Ann}(M)$, the map $\lambda \mapsto (\lambda 1_B, \delta(\lambda))$ then defines an algebra homomorphism from k to $B \oplus M$, which gives a k-algebra structure on $B \oplus M$. The canonical surjection $\varphi : A \to B$ is continuous, hence admits a lifting $\tilde{\varphi} : A \to B \oplus M$. By Example 9.7.2, $\mathrm{pr}_2 \circ \tilde{\varphi}$ is then a derivation from A to M which extends δ.

Proposition 9.7.7. *Let k be a ring.*

(a) Let A and B be linearly topologized k-algebras and $\rho : A \to B$ be a continuous homomorphism of k-algebras. If A is formally smooth over k and B is formally smooth over A, then B is formally smooth over k.

(b) The product of a finite family of linearly topologized k-algebras is formally smooth over k if and only if each k-algebra is formally smooth over k.

(c) Let A be a linearly topologized k-algebra and $\widehat{A}$ be the completion of A. For A to be formally smooth over k, it is necessary and sufficient that $\widehat{A}$ is formally smooth over k.

Proof. Let C be a k-algebra, N be a square zero ideal of C, and $\pi : C \to C/N$ be the canonical surjection. Endow C and C/N the discrete topology. In the situation of (a), let $\psi : B \to C/N$ be a continuous homomorphism of k-algebras. Since A is formally smooth over k, there exists a continuous homomorphism of k-algebras $\tilde{\varphi} : A \to C$ such that $\pi \circ \tilde{\varphi} = \psi \circ \rho$.

$$\begin{tikzcd}
& & C \arrow[d, "\pi"] \\
A \arrow[r, "\rho"'] \arrow[urr, "\tilde{\varphi}"] & B \arrow[r, "\psi"'] \arrow[ur, "\tilde{\psi}"'] & C/N
\end{tikzcd}$$

Consider C and C/N as A-algebras via the homomorphism $\tilde{\varphi}$, so that ψ is a homomorphism of A-algebras. Since B is formally smooth over A, there exsits a continuous homomorphism $\tilde{\psi} : B \to C$ of A-algebras such that $\pi \circ \tilde{\psi} = \psi$, which proves (a). As for the assertion of (b), it suffices to note that giving a continuous k-homomorphism from a product $A = \prod_i A_i$ to C (resp. C/N) is equivalent to giving n continuous k-homomorphisms $A_i \to C$ (resp. $A_i \to C/N$) and that any continuous k-homomorphism $A_i \to C$ (resp. $A_i \to C/N$) gives by composition a continuous k-homomorphism $A \to A_i \to C$ (resp. $A \to A_i \to C/N$).

Finally, let $i : A \to \widehat{A}$ be the canonical homomorphism. For any ring D, endowed with the discrete topology, the map which assocaites a continuous homomorphism $f : \widehat{A} \to D$ the continuous homomorphism $f \circ i : A \to D$ is bijective, so the assertion of (c) follows. □

Example 9.7.8. Assertion (c) of Proposition 9.7.7 is applicable in particular if the topology of A is the $\mathfrak{J}$-adic topology, where $\mathfrak{J}$ is a finitely generated ideal; the closure $\widehat{\mathfrak{J}}$ of $\mathfrak{J}$ in $\widehat{A}$ is then equal to $\mathfrak{J}\widehat{A}$ and the topology of $\widehat{A}$ is the $\widehat{\mathfrak{J}}$-topology (Corollary 2.3.18). Therefore, it is equivalent to say that A is formally smooth for the $\mathfrak{J}$-adic topology or that the completion $\widehat{A}$ is formally smooth for the $\widehat{\mathfrak{J}}$-adic topology.

Proposition 9.7.9. *Let k be a ring, A and B be k-algebra, $\mathfrak{J}$ be an ideal of A and $\mathfrak{K}$ be an ideal of B.*

(a) Let S be a multiplicative subset of A and T be a multiplicative subset of k whose image in A is contained in S. If A is formally smooth over k for the $\mathfrak{J}$-adic topology, $S^{-1}A$ is formally smooth over $T^{-1}k$ for the $S^{-1}\mathfrak{J}$-adic topology.

(b) Let k' be a k-algebra. If A is formally smooth over k for the $\mathfrak{J}$-adic topology, the k'-algebra $A_{(k')}$ is formally smooth for the $\mathfrak{J}A_{(k')}$-algebra.

(c) *Let $\mathfrak{R}$ be the ideal of $A \otimes_k B$ generated by the image of $\mathfrak{I} \otimes_k B$ and $A \otimes_k \mathfrak{K}$. If A and B are formally smooth over k for the $\mathfrak{I}$-adic topology and $\mathfrak{K}$-adic topology, respectively, then the k-algebra $A \otimes_k B$ is formally smooth for the $\mathfrak{R}$-adic topology.*

Proof. Under the hypotheses of (a), let C be a $T^{-1}k$-algebra, N be a square zero ideal of C; endow C and C/N the discrete topology and let $\pi : C \to C/N$ be the canonical homomorphism. Let $\varphi : S^{-1}A \to C/N$ be a continuous homomorphism of $T^{-1}k$-algebras (for the $S^{-1}\mathfrak{I}$-topology). Let $i : A \to S^{-1}A$ be the canonical homomorphism, then $\varphi \circ i$ is a continuous homomorphism from A to C/N (for the $\mathfrak{I}$-adic topology), so it admits a lefting $\tilde{\varphi}_0 : A \to C$. The elements of $\tilde{\varphi}_0(S)$ are invertible modulo N, hence are invertible (N has zero square), therefore there exists a ring homomorphism $\tilde{\varphi} : S^{-1}A \to C$ such that $\tilde{\varphi} \circ i = \tilde{\varphi}_0$, and the homomorphism $\tilde{\varphi}$ is $T^{-1}k$-linear. We have $\pi \circ \tilde{\varphi} \circ i = \varphi \circ i$, whence $\pi \circ \tilde{\varphi} = \varphi$, and $\tilde{\varphi}$ is then a lifting of φ.

We now turn to the hypotheses of (b). Let C be a k'-algebra, N be a square zero ideal of C; endow C and C/N the discrete topology. Let $\varphi : A_{(k')} \to C/N$ be a homomorphism of k'-algebras, continuous for the $\mathfrak{I}A_{(k')}$-adic topology. Let $i : A \to A_{(k')}$ be the canonical homomorphism. The map $\varphi \circ i$ is then a homomorphism of k-algebras from A to C/N, continuous for the $\mathfrak{I}$-adic topology; if A is formally smooth over k for the $\mathfrak{I}$-adic toplogy, $\varphi \circ i$ admits a lifting $\tilde{\psi} : A \to C$. The homomorphism $\tilde{\varphi} : A_{(k')} \to C$ induced by $\tilde{\psi}$ is then a lifting of φ.

Finally, consider the hypotheses of (c). The B-algebra $A \otimes_k B$ is formally smooth for the $\mathfrak{I}(A \otimes_A B)$-adic topology by (b), hence for the $\mathfrak{R}$-adic topology; moreover the canonical homomorphism $B \to A \otimes_k B$ is continuous if we endow B with the $\mathfrak{K}$-adic topology and $A \otimes_k B$ with the $\mathfrak{R}$-adic topology. Assertion (c) then follows from Proposition 9.7.7(a). □

Example 9.7.10. Let A be a ring and $\mathfrak{p}$ be a prime ideal of A. Then the local ring $A_\mathfrak{p}$ is formally smooth over A. To see this, let C be an A-algebra, N be a square zero ideal of C, and $\pi : C \to C/N$ be the canonical homomorphism. If $\varphi : A_\mathfrak{p} \to C/N$ is a homomorphism of A-algebras, then the image of elements of $\varphi(A - \mathfrak{p})$ are invertible in C mod N, hence invertible. There then exists a unique lifting $\tilde{\varphi} : A_\mathfrak{p} \to C$ of φ, so $A_\mathfrak{p}$ is formally smooth over A (it is in fact formally étale over A).

Example 9.7.11. Let k be a ring, and P be a projective k-module. The symmetric k-algebra $\boldsymbol{S}_k(P)$ is formally smooth for the discrete topology, and a fortiori for that defined by its graduation. In fact, for any k-algebra C and any ideal N of C, any algebra homomorphism from $\boldsymbol{S}_k(P)$ to C (resp. resp. C/N) corresponds to k-linear maps from P to C (resp. C/N), and the canonical map $\mathrm{Hom}_k(P, C) \to \mathrm{Hom}_k(P, C/N)$ is surjective since P is projective. In particular, by Proposition 9.7.7 the k-algebra $\widehat{\boldsymbol{S}}_k(P) = \prod_{n\in\mathbb{N}} \boldsymbol{S}_k^n(P)$ is formally smooth (for the product topology of the discrete topology over $\boldsymbol{S}_k^n(P)$).

Example 9.7.12. For any family $\boldsymbol{T} = (T_i)_{i\in I}$ of indeterminates, the k-algebra $k[\boldsymbol{T}]$ and $k[\![\boldsymbol{T}]\!]$, endowded with the canonical topology, are formally smooth over k: this follows from Example 9.7.11. If k is a field, the extension $k(\boldsymbol{T})$ is formally smooth over k by Proposition 9.7.9.

Example 9.7.13. Let $f \in k[T]$ be a polynomial in one indeterminates. For the k-algebra $k[T]/(f)$ to be formally smooth over k, it is necessary and sufficient that the following property is satisfied: for any k-algebra C and any square zero ideal N of C, any root of f in C/N extends to a root of f in C. This is the case if f and f' generate the unit ideal of $k[T]$; to see this, let α be a root of f in C/N and a be an element of C whose image in C/N is equal to α. Then $f(a)$ belongs to N and therefore $f'(a)$ is invertible in C; the element $b = a - f'(a)^{-1}f(a)$ then has image α in C. Since $f'(a)^{-1}f(a)$ has zero square, we have

$$f(b) = f(a) - f'(a)f'(a)^{-1}f(a) = 0.$$

It is worth noting that in this case the lifting of any homomorphism $k[T]/(f) \to C$ is necessarily unique, which means $k[T]/(f)$ is always formally unramified.

Theorem 9.7.14 (Cohen). *Let k be a field anf K be a separable extension of k. Then K is a formally smooth k-algebra.*

Proof. Let C be a k-algebra, N be a square zero ideal of C, $\pi : C \to C/N$ be the canonical homomorphism, and $\varphi : K \to C/N$ be a homomorphism of k-algebras. In order to construct a lifting of φ, we distinguish into two cases.

Suppose first that k has characteristic zero. Consider the couple $(K', \tilde{\varphi}')$, where K' is a sub-extension of K and $\tilde{\varphi} : K' \to C$ is a lifting of the restriction of φ to K'. The set of such couples, endowed with the order defined by extension relation, is inductive, so by Zorn lemma there exists a maximal couple $(K', \tilde{\varphi}')$. We now prove that $K' = K$, so let $x \in K - K'$. If x is transcendantal over K', then the K'-algebra $K'(x)$ is formally smooth (Example 9.7.12). On the other hand, if x is algebraic over K', its minimal polynomial $f \in K'[T]$ is separable, so it is coprime to its derivative (**??**) and the field $K'(x)$, isomorphic to $K'[T]/(f)$, is then formally smooth (Example 9.7.13). In both cases, $K'(x)$ is formally smooth over K', and there exists an extension of $\tilde{\varphi}'$ to $K'(x)$ which lifts the restriction of φ to $K'(x)$; this contradicts the maximality of $(K', \tilde{\varphi}')$.

Now suppose that k has characteristic $p > 0$. Consider the Frobenius homomorphism $F : C \to C$ defined by $F(x) = x^p$. Since $F(x) = 0$ for $x \in N$, so there exists a unique homomorphism $\lambda : C/N \to C$ such that $\lambda \circ \pi = F$. We have $\pi(\lambda(\pi(x))) = \pi(x^p) = \pi(x)^p$, and since π is surjective, this implies $\pi(\lambda(z)) = z^p$ for any element $z \in C/N$. Moreover, let $f : K \to K^p$ be the isomorphic $y \mapsto y^p$ and $f^{-1} : K^p \to K$ be the inverse isomorphism. Let $g : K^p \to C$ be the composition homomorphism

$$K^p \xrightarrow{f^{-1}} K \xrightarrow{\varphi} C/N \xrightarrow{\lambda} C$$

For any $x \in K$, we have $g(x^p) = \lambda(\varphi(x))$, and the map g is k-linear because $\lambda(\alpha z) = \alpha^p \lambda(z)$ for $\alpha \in k$ and $z \in C/N$. Since the extension K/k is separable, $k(K^p)$ is identified with $k \otimes_{k^p} K^p$ (A, V, p.119, remark), so there exists a unique k-homomorphism $\tilde{g} : k(K^p) \to C$ which extends g. Let $(a_i)_{i \in I}$ be a p-basis for K over $k(K^p)$ (A, V, p.98, theorem 2); for any $i \in I$, choose an element $b_i \in C$ such that $\pi(b_i) = \varphi(a_i)$. We have

$$\tilde{g}(a_i^p) = g(a_i^p) = \lambda(\varphi(a_i)) = \lambda(\pi(b_i)) = b_i^p.$$

By (A, V, p.94, remark), there is then a unique homomorphism $\tilde{\varphi} : K \to C$ which extends $\tilde{g}$ and satisfies $\tilde{\varphi}(a_i) = b_i$ for each i. Note that $\pi(\tilde{\varphi}(a_i)) = \pi(b_i) = \varphi(a_i)$ and for each $x \in K$ we have

$$\pi(\tilde{\varphi}(x^p)) = \pi(h(x^p)) = \pi(g(x^p)) = \pi(\lambda(\varphi(x))) = \varphi(x^p);$$

so $\pi \circ \tilde{\varphi} = \varphi$ and this completes the proof of the theorem. □

Corollary 9.7.15. *Let k be a field, K be a separable extension of k and A be a linearly topologized K-algebra. If A is formally smooth over K, it is formally smooth over k.*

Proof. This follows from Theorem 9.7.14 and Proposition 9.7.7. □

Remark 9.7.16. Let k be a field, then any étale k-algebra is formally smooth (A, V, p.34, th.4(d)). Later we shall see that any field extension that is formally smooth is absolutely regular, hence separable.

We conclude this paragraph by establishing a stronger lifting property for formally smooth algebras. For this, we let k be a ring, C be a k-algebra, $(C_n)_{n \in \mathbb{Z}}$ be a filtration of C which is compatible with the k-algebra structure and such that $C_0 = C$. Suppose that C is separated and complete with respect to this filtration, so that the canonical homomorphism $C \to \varprojlim C/C_n$ is a homeomorphism. Let $m > 0$ be an integer and $\pi : C \to C/m$ be the canonical homomorphism.

Proposition 9.7.17. *Let A be a formally smooth lineraly topologized k-algebra. Then any continuous k-homomorphism $\varphi : A \to C/C_m$ admits a continuous lifting to C.*

Proof. For any integer $n > m$, let $\pi_n : C/C_n \to C/C_{n-1}$ be the canonical homomorphism. Since C is identified with the limit $\varprojlim C/C_n$, giving a continuous lifting of φ to C is equivalent to giving a family $(\varphi_n)_{n>m}$ of continuous k-homomorphisms $\varphi_n : A \to C/C_n$ satisfying $\pi_n \circ \varphi_n = \varphi_{n-1}$. By induction on m, we are then reduced to proving the statement when $C_{m+1} = 0$. The ideal C_m then has square zero (since $2m \geq m+1$), and we can use the hypothesis that A is formally smooth over k. □

Remark 9.7.18. The result of Proposition 9.7.17 if the filtration is defined by a nilpotent ideal N of C: that is, if $C_n = N^n$ and the ideal N is nilpotent. If A is a formally smooth lineraly topologized k-algebra, we then conclude that any k-homomorphism $\varphi : A \to C/N$ admits a lifting to C. This can also be considered as a slightly stronger (but in fact equaivalent) definition for formally smoothness, and justifies the name "infinitesimal lifting".

9.7.3 Criterion by associated graded algebras

Theorem 9.7.19. *Let k be a ring, A be a k-algebra and $\mathfrak{I}$ be an ideal of A such that the k-algebra $A/\mathfrak{I}$ is formally smooth. Endow A with the $\mathfrak{I}$-adic topology, the following conditions are equivalent:*

(i) the k-algebra A is formally smooth;

(ii) the $A/\mathfrak{I}$-module $\mathfrak{I}/\mathfrak{I}^2$ is projective and the canonical homomorphism

$$\beta_{\mathfrak{I}} : \boldsymbol{S}_{A/\mathfrak{I}}(\mathfrak{I}/\mathfrak{I}^2) \to \mathrm{gr}_{\mathfrak{I}}(A)$$

is bijective;

(iii) the $A/\mathfrak{I}$-module $\mathfrak{I}/\mathfrak{I}^2$ is projective and the completion of A is k-linearly homeomorphic to the completion of the graded algebra $\boldsymbol{S}_{A/\mathfrak{I}}(\mathfrak{I}/\mathfrak{I}^2)$.

If A is Noetherian, the these conditions are equivalent to

(iv) the ideal $\mathfrak{I}$ is completely secant.

Proof. We first show that (iii) implies (i): in fact, under the hypotheses of (iii), the algebra $\boldsymbol{S}_{A/\mathfrak{I}}(\mathfrak{I}/\mathfrak{I}^2)$, endowed with the topology given by its graduation, is formally smooth over $A/\mathfrak{I}$ (Example 9.7.11), hence over k (Proposition 9.7.7(a)); assertion (i) then follows from Proposition 9.7.7(c). Also, if A is Noetherian, then (ii) is equivalent to (iv) by Theorem 9.5.4.

Let $\widehat{A}$ be the completion of A and $\widehat{\mathfrak{I}}$ be the completion of $\mathfrak{I}$. The canonical homomorphism $i : A \to \widehat{A}$ induces an isomorphism $A/\mathfrak{I} \to \widehat{A}/\widehat{\mathfrak{I}}$, which admits a lifting $\varphi : A/\mathfrak{I} \to \widehat{A}$ (Proposition 9.7.17). Let $\lambda : \widehat{\mathfrak{I}} \to \mathfrak{I}/\mathfrak{I}^2$ be the surjection induced by the canonical isomorphism $\mathfrak{I}/\mathfrak{I}^2 \to \widehat{\mathfrak{I}}/\widehat{\mathfrak{I}}^2$. For $a \in A$ and $z \in \widehat{\mathfrak{I}}$, we then have $\varphi(\bar{a}) \equiv i(a) \bmod \widehat{\mathfrak{I}}$, whence $\varphi(\bar{a}z) \equiv i(a)z \bmod \widehat{\mathfrak{I}}^2$ and

$$\lambda(\varphi(\bar{a})z) = \lambda(i(a)z) = \bar{a}\lambda(z).$$

In other words, λ is $A/\mathfrak{I}$-linear if we endow $\widehat{\mathfrak{I}}$ the $A/\mathfrak{I}$-module structure induced by φ. Suppose that the homomorphism λ admits a $A/\mathfrak{I}$-linear *section* $\sigma : \mathfrak{I}/\mathfrak{I}^2 \to \widehat{\mathfrak{I}}$.

$$\begin{tikzcd}
\widehat{\mathfrak{I}} \arrow[rr, "\lambda"] \arrow[dr] & & \mathfrak{I}/\mathfrak{I}^2 \arrow[ll, bend right, "\sigma"'] \\
& \widehat{\mathfrak{I}}/\widehat{\mathfrak{I}}^2 \arrow[ur, "\sim"] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& \widehat{A} \arrow[d] \\
A/\mathfrak{I} \arrow[ur, "\varphi"] \arrow[r] & \widehat{A}/\widehat{\mathfrak{I}}
\end{tikzcd}$$

Let S be the symmetric algebra $\boldsymbol{S}_{A/\mathfrak{I}}(\mathfrak{I}/\mathfrak{I}^2)$ and $\widehat{S}$ be its completion, and $\theta : S \to \widehat{A}$ be the k-homomorphism defined by $\theta(x) = \varphi(x)$ for $x \in S^0 = A/\mathfrak{I}$ and $\theta(x) = \sigma(x)$ for $x \in S^1 = \mathfrak{I}/\mathfrak{I}^2$. Since θ sends S^1 into $\widehat{\mathfrak{I}}$, it sends S^n into $\widehat{\mathfrak{I}}^n$ and therefore extends to a continuous homomorphism $\hat{\theta} : \widehat{S} \to \widehat{A}$. The map $\mathrm{gr}_1(\theta) : \mathfrak{I}/\mathfrak{I}^2 \to \widehat{\mathfrak{I}}/\widehat{\mathfrak{I}}^2$ is the composition of σ with the surjection $\widehat{\mathfrak{I}} \to \widehat{\mathfrak{I}}/\widehat{\mathfrak{I}}$, and since σ is a section of λ, $\mathrm{gr}_1(\theta)$ then coincides with the canonical isomorphism $\mathfrak{I}/\mathfrak{I}^2 \to \widehat{\mathfrak{I}}/\widehat{\mathfrak{I}}$. The map $\mathrm{gr}(\theta) : S \to \mathrm{gr}_{\widehat{\mathfrak{I}}}(\widehat{A})$ is then the composition of the canonical surjection β with the canonical isomorphism $\mathrm{gr}_{\mathfrak{I}}(A) \to \mathrm{gr}_{\widehat{\mathfrak{I}}}(\widehat{A})$. Now under the hypothesis of (ii), the $A/\mathfrak{I}$-module $\mathfrak{I}/\mathfrak{I}^2$ is projective, so λ always admits a $A/\mathfrak{I}$-linear section. The induced homomorphism $\hat{\theta} : \widehat{S} \to \widehat{A}$ is then an isomorphism by the preceding arguments, whence (iii).

Suppose that the k-algebra A is formally smooth, we prove that (i)⇒(ii). We first show that $\mathfrak{I}/\mathfrak{I}^2$ is projective. Let M be a $A/\mathfrak{I}$-module and $f : M \to \mathfrak{I}/\mathfrak{I}^2$ be a surjective $A/\mathfrak{I}$-linear map. It suffices to prove that f admits a $A/\mathfrak{I}$-linear section. Let $\pi : A/\mathfrak{I}^2 \to A/\mathfrak{I}$ be the canonical surjection. By Proposition 9.7.3, there exists an isomorphism $\psi : A/\mathfrak{I} \oplus \mathfrak{I}/\mathfrak{I}^2 \to A/\mathfrak{I}^2$ such that $\pi \circ \psi = \mathrm{pr}_1$ and $\psi(0, z) = z$ for $z \in \mathfrak{I}/\mathfrak{I}^2$. Consider the k-algebra $(A/\mathfrak{I}) \oplus M$ and the map $u : (A/\mathfrak{I}) \oplus M \to A/\mathfrak{I}^2$ defined by $u(x, m) = \psi(x, f(m))$. This is a surjective homomorphism of k-algebras, whose kernel is the submodule $\ker f$ of M, hence has zero square. The canonical surjection $\rho : A \to A/\mathfrak{I}^2$ is continuous, and as the k-algebra A is formally smooth, there exists a k-homomorphism $\tilde{\rho} : A \to (A/\mathfrak{I}) \oplus M$ such that $u \circ \tilde{\rho} = \rho$. As $\mathrm{pr}_1 = \pi \circ \psi = \pi \circ u$, we have

$$\mathrm{pr}_1 \circ \tilde{\rho} = \pi \circ u \circ \tilde{\rho} = \pi \circ \rho,$$

so $\mathrm{pr}_1 \circ \tilde{\rho}$ is the canonical surjection $A \to A/\mathfrak{I}$. We then have $\tilde{\mathfrak{I}}(\mathfrak{I}) \subseteq M$ and therefore $\tilde{\rho}(\mathfrak{I}^2) = 0$, so $\tilde{\rho}$ induces a $A/\mathfrak{I}$-linear map $s : \mathfrak{I}/\mathfrak{I}^2 \to M$. We have $u \circ \tilde{\rho} = \rho$ and $\mathrm{pr}_2 \circ \psi^{-1} \circ u(y, m) = f(m)$ for $y \in A/\mathfrak{I}$ and $m \in M$. For $x \in \mathfrak{I}$, if $\bar{x}$ is the class of x in $\mathfrak{I}/\mathfrak{I}^2$, we have

$$f(s(\bar{x})) = f(\mathrm{pr}_2(\tilde{\rho}(x))) = \mathrm{pr}_2(\psi^{-1}(\rho(x))) = \bar{x},$$

so s is a section of f.

```tikzcd
A/\mathfrak{I} \arrow[r] \arrow[d, equal] & (A/\mathfrak{I}) \oplus M \arrow[r] \arrow[d] \arrow[dd, dashed, red, bend left, "u"] & M \arrow[d, "f"] \\
A/\mathfrak{I} \arrow[r] & (A/\mathfrak{I}) \oplus \mathfrak{I}/\mathfrak{I}^2 \arrow[r] \arrow[d, "\psi"] & \mathfrak{I}/\mathfrak{I}^2 \\
A \arrow[r, "\rho"] \arrow[dr] \arrow[uur, dashed, red, "\tilde{\rho}"] & A/\mathfrak{I}^2 \arrow[d, "\pi"] & \\
 & A/\mathfrak{I} &
```

Figure 9.1: The illustration of the proof of (i)⇒(ii) in Theorem 9.7.19.

Finally, it remains to prove that the homomorphism $\beta_{\mathfrak{I}}$ is injective. Since the $A/\mathfrak{I}$-module $\mathfrak{I}/\mathfrak{I}$ is injective, λ admits a $A/\mathfrak{I}$-linear section, and we have the induced homomorphism $\theta : S \to \widehat{A}$, the homomorphism $\mathrm{gr}(\theta)$ is identified with β by our previous arguments. For each integer $m \geq 0$, let $S_m = \sum_{i>m} S^i$ and $\theta_m : S/S_m \to A/\mathfrak{I}^{m+1}$ be the induced homomorphism. The composition of θ_m with the canonical surjection $A/\mathfrak{I}^{m+1} \to A/\mathfrak{I}$ is then the canonical projection of S/S_m fo $S^0 = A/\mathfrak{I}$, so its kernel is nilpotent. By Proposition 9.7.17, there exists a lifting $\psi_m : A \to S/S_m$ of the canonical surjection $A \to A/\mathfrak{I}^{m+1}$. As the composition of ψ_m with the projection $S/S_m \to A/\mathfrak{I}$ is the canonical surjection $A \to A/\mathfrak{I}$, $\psi_m(\mathfrak{I})$ is formed by elements of positive degrees. By passing to associated algebras, we then deduce a graded k-linear map $\mathrm{gr}(\psi_m) : \mathrm{gr}_{\mathfrak{I}}(A) \to S/S_m$ such that $\mathrm{gr}_m(\theta) \circ \mathrm{gr}_m(\psi_m) = \mathrm{id}_{\mathfrak{I}^m/\mathfrak{I}^{m+1}}$. It then follows that $\mathrm{gr}_m(\theta)$, and hence β_m, is injective. □

Corollary 9.7.20. *Let k be a field and A be a Noetherian local k-algebra such that the extension κ_A of k is separable. Then the following conditions are equivalent:*

(i) the k-algebra A is formally smooth for the $\mathfrak{m}_A$-adic topology;

(ii) the ring A is regular;

(iii) the k-algebra A is geometrically regular;

(iv) the k-algebra $\widehat{A}$ is isomorphic to $\kappa_A[[T_1,\dots,T_n]]$, where $n = \dim(A)$.

Proof. The conditions (ii) and (iii) are equivalent by (example 3 du §6, n4), and amounts to saying that the ideal $\mathfrak{m}_A$ is completely secant (Theorem 7.4.5). Moreover, any isomorphism from $\widehat{A}$ to $\kappa_A[[T_1,\dots,T_n]]$ is bicontinuous since they are local rings. As the k-algebra κ_A is formally smooth by Theorem 9.7.14, the corollary follows from Theorem 9.7.19 applied to $\mathfrak{I} = \mathfrak{m}_A$. □

Corollary 9.7.21. *Let k be a field, A be a Noetherian k-algebra and $\mathfrak{I}$ be an ideal contained in the Jacobson radical of A. Suppose that A is formally smooth over k for the $\mathfrak{I}$-adic topology, then it is geometrically regular.*

Proof. Let k' be a finite extension of k and $A' = A_{(k')}$. Then it suffices to prove that, for any maximal ideal $\mathfrak{m}'$ of A', the Noetherian local ring $A'_{\mathfrak{m}'}$ is regular. Now we have $\mathfrak{I}A' \subseteq \mathfrak{m}'$: in fact, the inverse image of $\mathfrak{m}'$ in A is a maximal ideal of A (Corollary 4.1.65), hence contains $\mathfrak{I}$. The k'-algebra A' is formally smooth for $\mathfrak{I}A'$-adic topology by Proposition 9.7.9(b), and the k'-algebra $A'_{\mathfrak{m}'}$ is then formally smooth for the $\mathfrak{I}A'_{\mathfrak{m}'}$-adic topology (Proposition 9.7.9(a)), hence also for the $\mathfrak{m}'A_{\mathfrak{m}'}$-adic topology (Remark 9.7.4). Let k_0 be the prime subfield of k', then $A'_{\mathfrak{m}'}$ is formally smooth over k_0 for the $\mathfrak{m}'A_{\mathfrak{m}'}$-adic topology (Corollary 9.7.15); as k_0 is perfect, $\kappa(\mathfrak{m}')$ is separable over k_0, so the ring $A'_{\mathfrak{m}'}$ is regular (Corollary 9.7.20). □

Corollary 9.7.22. *Let k be a ring and A be a formally smooth k-algebra.*

(a) The A-module $\Omega_{A/k}$ is projective.

(b) Suppose that the ring $A \otimes_k A$ is Noetherian and let $\mathfrak{I}$ be the kernel of the multiplication map $A \otimes_k A \to A$. Then the ideal $\mathfrak{I}$ is completely secant.

Proof. The k-algebras A and $A \otimes_k A$ are formally smooth (Proposition 9.7.9(c)), and A is isomorphic to a quotient algebra of $A \otimes_k A$ by the kernel $\mathfrak{I}$. By definition we have $\Omega_{A/k} = \mathfrak{I}$, so the corollary follows from Theorem 9.7.19. □

9.7.4 Formally smoothness for local algebras

Proposition 9.7.23. *Let k_0 be a ring, k be a k_0-algebra, A be a k-algebra, and $\mathfrak{m}$ be a maximal ideal of A. Suppose that k and $A/\mathfrak{m}$ are formally smooth over k_0. For A to be formally smooth over k for the $\mathfrak{m}$-adic topology, it is necessary and sufficient that the following two conditions are satisfied*

(i) the canonical homomorphism $\alpha_{\mathfrak{m}} : \boldsymbol{S}_{A/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2) \to \mathrm{gr}_{\mathfrak{m}}(A)$ is bijective;

(ii) the $A/\mathfrak{m}$-linear map

$$\omega : A/\mathfrak{m} \otimes_k \Omega_{k/k_0} \to A/\mathfrak{m} \otimes_A \Omega_{A/k_0}$$

induced by the canonical map $k \to A$ is injective.

Proof. Let $d_k : k \to \Omega_{k/k_0}$ and $d_A : A \to \Omega_{A/k_0}$ be the universal k_0-derivations, and endow A with the $\mathfrak{m}$-adic topology. Suppose first that A is formally smooth over k. Then A is formally smooth over k_0 (Proposition 9.7.7), which is equivalent to (I) in view of Theorem 9.7.19. Moreover, the k_0-derivation $\lambda \mapsto 1 \otimes d_k(\lambda)$ of k into $A/\mathfrak{m} \otimes_k \Omega_{k/k_0}$ can then be extended to a k_0-derivation from A into $A/\mathfrak{m} \otimes_k \Omega_{k/k_0}$ (Remark 9.7.6), so there exists an A-linear map $u : \Omega_{A/k_0} \to A/\mathfrak{m} \otimes_k \Omega_{k/k_0}$ such that $u(d_A(\lambda 1_A)) = 1 \otimes d_k(\lambda)$ for $\lambda \in k$. The $A/\mathfrak{m}$-linear map $\bar{u} : A/\mathfrak{m} \otimes_A \Omega_{A/k_0} \to A/\mathfrak{m} \otimes_k \Omega_{k/k_0}$ induced by u is then a retraction of ω, which proves (ii).

Conversely, assume that conditions (i) and (ii) are satisfied. Then A is formally smooth over k_0 (Theorem 9.7.19) and the A-module Ω_{A/k_0} is projective Corollary 9.7.22. Fix an integer $r \geq 0$ and consider the $A/\mathfrak{m}^r$-linear map

$$\omega_r : A/\mathfrak{m}^r \otimes \Omega_{k/k_0} \to A/\mathfrak{m}^r \otimes_A \Omega_{A/k_0}$$

induced by the canonical homomorphism $k \to A$. Let $(\lambda_i)_{i \in I}$ be a family of elements of k such that the $d_k(\lambda_i)$ form a basis for the k-vector space Ω_{k/k_0}. By condition (ii), the elements $1 \otimes d_A(\lambda_i 1_A)$ are linearly indepedent in $A/\mathfrak{m} \otimes_A \Omega_{A/k_0}$. By Corollary 1.3.10, the $1 \otimes d_A(\lambda_i 1_A)$ then form a basis for a direct summand of the $A/\mathfrak{m}^r$-module $A/\mathfrak{m}^r \otimes_A \Omega_{A/k_0}$. There then exists a $A/\mathfrak{m}^r$-linear map

$$u_r : A/\mathfrak{m}^r \otimes_A \Omega_{A/k_0} \to A/\mathfrak{m}^r \otimes_k \Omega_{k/k_0}$$

such that $u_r(1 \otimes d_A(\lambda_i 1_A)) = 1 \otimes d_k(\lambda_i)$ for each i, and hence $u_r \circ \omega_r = \mathrm{id}$.

We now verify that A is formally smooth over k. Let C be a k-algebra, N be a square zero ideal of C, and $\pi : C \to C/N$ be the canonical homomorphism. Endow C and C/N the discrete topology. Let $\varphi : A \to C/N$ be a continuous homomorphism of k-algebras. Since A is formally smooth over k_0, there exists a k_0-linear lifting $\tilde{\varphi}_0 : A \to C$ of φ. By Proposition 9.7.1, the k_0-linear liftings $\tilde{\varphi} : A \to C$ of φ are given by the maps $x \mapsto v(d_A(x)) + \tilde{\varphi}_0(x)$, where $v \in \mathrm{Hom}_A(\Omega_{A/k_0}, N)$, so we have to choose v so that $\tilde{\varphi}_0$ is a homomorphism of k-algebras. Now the map $\lambda \mapsto \lambda 1_C - \tilde{\varphi}_0(\lambda 1_A)$ is a k_0-derivation from k to N (Proposition 9.7.1), so can be written as $h \circ d_k$ where $h \in \mathrm{Hom}_k(\Omega_{k/k_0}, N)$. We choose an integer $r \geq 0$ such that the kernel of φ contains $\mathfrak{m}^r$ (note that $\ker \varphi$ is open in A). The A-module N is then annihilated by $\mathfrak{m}^r$ and it suffices to choose v to be the composition of the homomorphisms

$$\Omega_{A/k_0} \longrightarrow A/\mathfrak{m}^r \otimes_A \Omega_{A/k_0} \xrightarrow{u_r} A/\mathfrak{m}^r \otimes_k \Omega_{k/k_0} \xrightarrow{\tilde{h}} N$$

where $\tilde{h}$ is induced by h. In fact, in this case, for $\lambda \in k$ we have

$$v(d_A(\lambda 1_A)) = \tilde{h}(u_r(1 \otimes d_A(\lambda 1_A))) = \tilde{h}(1 \otimes d_k(\lambda 1_A)) = h(d_k(\lambda)) = \lambda 1_C - \tilde{\varphi}_0(\lambda 1_A). \qquad \square$$

Remark 9.7.24. If A is a Noetherian ring, condition (i) of Proposition 9.7.23 then signifies that the local ring $A_{\mathfrak{m}}$ is regular (Theorem 7.4.5).

Proposition 9.7.25. *Let k be a field anf A be a Noetherian local k-algebra. The following conditions are equivalent:*

(i) A is formally smooth for the $\mathfrak{m}_A$-adic topology;

(ii) A is regular and the κ_A-linear map

$$\omega : \kappa_A \otimes_k \Omega_k \to \kappa_A \otimes_A \Omega_A$$

induced by the canonical map $k \to A$ is injective;

(iii) A is geometrically regular.

Proof. To see that (i)⇔(ii), it suffices to apply Proposition 9.7.23 and Remark 9.7.24, where we choose k_0 to be the prime subfield of k. In fact, k and κ_A are then formally smooth over k_0 (Theorem 9.7.14). Now (i)⇒(iii) follows from Corollary 9.7.21.

If k has characteristic zero, then it follows from Corollary 9.7.20 that (iii) implies (i), whence the proposition in this case. Suppose now that k has characteristic $p > 0$, and we prove that (iii)⇒(ii). Let k'/k be a finite purely inseparable extension of height ≤ 1. If A and $A_{(k')}$ are regular, the canonical map $\kappa_A \otimes_{k'^p} \Omega_{k'^p/k^p} \to \kappa_A \otimes_A \Omega_A$ is injective by Proposition 9.5.16. By (A, V, p.97, th.1(b)) applied to the extension k/k^p, the k-vector space Ω_k, which coincides with Ω_{k/k^p}, is the union of the subspaces $k \otimes_{k'^p} \Omega_{k'^p/k^p}$, where k' runs through the set of finite purely inseparable extension of k of height ≤ 1 in a fixed algebraic closure of k. From this, we see that condition (ii) is then satisfied. □

9.7.5 Jacobian criterion

Let k be a ring, A be a k-algebra, $\mathfrak{I}$ be an ideal of A and $\bar{d} : \mathfrak{I}/\mathfrak{I}^2 \to A/\mathfrak{I} \otimes_A \Omega_{A/k}$ be the canonical map. For each $A/\mathfrak{I}$-algebra R, we denote by

$$\bar{d}_R : R \otimes_{A/\mathfrak{I}} \mathfrak{I}/\mathfrak{I}^2 \to R \otimes_A \Omega_{A/k}$$

the R-linear map induced by $\bar{d}$. If the k-algebra $A/\mathfrak{I}$ is formally smooth, $\bar{d}$ then possesses an A-linear retraction (Example 9.7.5) and $\bar{d}_R$ possesses an R-linear retraction for any R. More generally, we have the following lemma.

Lemma 9.7.26. *Let $\mathfrak{K}$ be an ideal of A containing $\mathfrak{I}$. Suppose that there exists an integer $m \geq 0$ such that $\mathfrak{I} \cap \mathfrak{K}^m$ is contained in $\mathfrak{I}\mathfrak{K}$ (this is satisfied if A is Noetherian). If $A/\mathfrak{I}$ is formally smooth over k for the $\mathfrak{K}/\mathfrak{I}$, the map $\bar{d}_{A/\mathfrak{K}} : A/\mathfrak{K} \otimes_{A/\mathfrak{I}} \mathfrak{I}/\mathfrak{I}^2 \to A/\mathfrak{K} \otimes_A \Omega_{A/k}$ possesses an A-linear retraction.*

Lemma 9.7.27. *Suppose that A is formally smooth over k for the $\mathfrak{I}$-adic topology. For $A/\mathfrak{I}$ to be formally smooth over k, it is necessary and sufficient that the canonical map $\bar{d} : \mathfrak{I}/\mathfrak{I}^2 \to A/\mathfrak{I} \otimes_A \Omega_{A/k}$ possesses an A-linear retraction.*

Theorem 9.7.28 (Jacobian Criterion). *Let k be a ring, A be a formally smooth k-algebra and $\mathfrak{I}$ be a finitely generated ideal of A; put $B = A/\mathfrak{I}$.*

(a) Let $\mathfrak{P}$ be a prime ideal of B and $\mathfrak{p}$ be the prime ideal of A such that $\mathfrak{P} = \mathfrak{p}/\mathfrak{I}$. Then the following conditions are equivalent:

(i) the k-algebra $B_{\mathfrak{P}}$ is formally smooth;

(ii) there exists $f \in B - \mathfrak{P}$ such that the k-algebra B_f is formally smooth;

(iii) the $\kappa(\mathfrak{P})$-linear map

$$\bar{d}_{\kappa(\mathfrak{P})} : \kappa(\mathfrak{P}) \otimes_B \mathfrak{I}/\mathfrak{I}^2 \to \kappa(\mathfrak{p}) \otimes_A \Omega_{A/k}$$

is injective;

(iv) there exist elements $f_1, \dots, f_m$ of $\mathfrak{I}$ whose images $(f_1)_{\mathfrak{p}}, \dots, (f_m)_{\mathfrak{p}}$ generate $\mathfrak{I}_{\mathfrak{p}}$ and k-derivations $D_1, \dots, D_m$ of A such that $\det(D_j(f_i)) \notin \mathfrak{p}$.

(b) The set of prime ideals $\mathfrak{P}$ of B satisfying the equivalent conditions of (a) is open in $\mathrm{Spec}(B)$. For B to be formally smooth over k, it is necessary and sufficient that any prime (resp. maximal) ideal of B satisfies these conditions.

(c) If A is Noetherian, the conditions of (a) are equivalent to the following:

(v) the k-algebra $B_{\mathfrak{P}}$ is formally smooth for the $\mathfrak{P}B_{\mathfrak{P}}$-adic topology.

Moreover, under condition (iv), the ideal $\mathfrak{I}_{\mathfrak{p}}$ is completely secant and $((f_1)_{\mathfrak{p}}, \dots, (f_m)_{\mathfrak{p}})$ is a completely secant sequence for $A_{\mathfrak{p}}$.

Proof. Put $M = \mathfrak{I}/\mathfrak{I}^2$ and $N = B \otimes_A \Omega_{A/k}$. The B-module M is finitely generated, and the B-module N is projective (Corollary 9.7.22). □

Corollary 9.7.29. *Let k_0 be a ring, k be a Noetherian formally smooth k_0-algebra, and A be a local k-algebra essentially of finite type. If the k_0-algebra A is formally smooth for the $\mathfrak{m}_A$-adic topology, it is formally smooth.*

Proof. By hypotheses, there exists an integer $n \geq 0$, a multiplicative subset S of $k[T_1, \dots, T_n]$, and a surjective k-homomorphism $S^{-1}k[T_1, \dots, T_n] \to A$. The k-algebra $S^{-1}k[T_1, \dots, T_n]$ is Noetherian and formally smooth by Proposition 9.7.9 and Example 9.7.12, and hence over k_0 (Proposition 9.7.7). The corollary then follows from Theorem 9.7.28(c). □

Corollary 9.7.30. *Let k_0 be a ring, k be a Noetherian formally smooth k_0-algebra, and A be a local k-algebra essentially of finite type. The set U of prime ideals $\mathfrak{p}$ of A such that the k_0-algebra $A_{\mathfrak{p}}$ is fomally smooth (for the discrete topology or the $\mathfrak{p}A_{\mathfrak{p}}$-adic topology) is open in $\mathrm{Spec}(A)$ and the following conditions are equivalent:*

(i) $U = \mathrm{Spec}(A)$;

(ii) U contains every maximal ideal of A;

(iii) the k_0-algebra A is formally smooth.

Proof. This is already contained in Theorem 9.7.28, in view of Corollary 9.7.29. □

Remark 9.7.31. The results of Corollary 9.7.29 and Corollary 9.7.30 are applicable if k_0 is a field and that we are in one of the following two cases:

(a) A is an algebra essentially of finite type over a separable extension of k.

(b) A is a complete Noetherian local algebra whose residue field κ_A is a separable extension of k_0 (in this case we choose k to be a formal series algebra over κ_A for which A is a quotient).

In each case, it follows from Corollary 9.7.30, in view of Proposition 9.7.25 and Proposition 9.6.14, that the k_0-algebra A is formally smooth if and only if it is gemetrically regular.

Corollary 9.7.32 (Zariski's Jacobian Criterion). *Let k be a field, A be a regular local k-algebra, and $\mathfrak{I}$ be a proper ideal of A. Suppose that the k-algebra A is essentially of finite type or complete. For the local ring $A/\mathfrak{I}$ to be regular, it is necessary and sufficient that there exists a generating family $f_1, \dots, f_m$ of $\mathfrak{I}$ and derivations $D_1, \dots, D_m$ of A such that $\det(D_j(f_i)) \notin \mathfrak{m}_A$. In this case, the elements $(f_1, \dots, f_m)$ is a subset of a system of parameters of A and the ideal $\mathfrak{I}$ is prime.*

Proof. Let k_0 be the prime subfield of k. The k_0-algebra A is geometrically regular by Example 9.6.13, hence formally smooth (Remark 9.7.31). By the same reasoning, the regularity of $A/\mathfrak{I}$ is equivalent to its formally smoothness over k_0. The first assertion then follows from Theorem 9.7.28, which also implies that the sequence $(f_1, \dots, f_m)$ is completely secant for A in this case. We can then apply Corollary 7.4.11 to conclude the second assertion (note that $A/\mathfrak{I}$ is then an integral domain). □

Remark 9.7.33. Under the hypotheses of Corollary 9.7.32, the A-module $\Omega_{A/k}$ is projective by Corollary 9.7.22, hence free. Any derivation of A in κ_A is therefore lifted into a derivation of A. The condition of the statement can therefore be expressed as follows: there exists a generating system $(f_1, \dots, f_m)$ of $\mathfrak{I}$ and derivations $D_1, \dots, D_m$ of A in κ_A such that $\det(D_j(f_i)) \neq 0$.

Corollary 9.7.34 (Zariski). *Let k be a field and A be a k-algebra essentially of finite type or a complete Noetherian local k-algebra. The set of prime ideals $\mathfrak{p}$ of A such that $A_{\mathfrak{p}}$ is regular is open in* $\mathrm{Spec}(A)$.

Proof. It suffices to apply Remark 9.7.31 and choose k_0 to be the prime field of k. $\square$

9.7.6 Smooth algebras

Lemma 9.7.35. *Let $\rho : A \to B$ be a local homomorphism of Noetherian local rings. Suppose that B is essentially of finite type over A. For the A-algebra to be formally smooth, it is necessary and sufficient that the A-module B is flat and the κ_A-algebra $\kappa_A \otimes_A B$ is geometrically regular.*

Proof. By hypotheses, there exists an integer $n \geq 0$, a prime ideal $\mathfrak{q}$ of $A[T_1, \dots, T_n]$ and a surjective homomorphism $h : A[T_1, \dots, T_n]_{\mathfrak{q}} \to B$. Denote by C be the local A-alebra $A[T_1, \dots, T_n]_{\mathfrak{q}}$; it is a formally smooth (Example 9.7.12) and flat over A, and we can identify B with the A-algebra $C/\mathfrak{J}$, where $\mathfrak{J} = \ker h$.

Put $\bar{C} = \kappa_A \otimes_A C$ and $\bar{B} = \kappa_A \otimes_A B$. Suppose that B is formally smooth over A. The κ_A-algebra $\bar{C}$ is then formally smooth (Proposition 9.7.9), hence geometrically regular (Corollary 9.7.21). Moreover, since $\bar{C}/\mathfrak{J}\bar{C}$ is identified with $\bar{B}$ and that the κ_A-algebra $\bar{C}$ is formally smooth, the ideal $\mathfrak{J}\bar{C}$ of $\bar{C}$ is completely secant (Theorem 9.7.19). It then follows from Proposition 9.5.16 that the A-module B is flat.

Conversely, suppose that B is flat over A and the κ_A-algebra $\bar{B}$ is geometrically regular. Then the local κ_A-algebra $\bar{B}$ is formally smooth (Proposition 9.7.25). Put $\bar{\mathfrak{J}} = \kappa_A \otimes_A \mathfrak{J}$; since B is a flat A-module, the canonical map $\bar{\mathfrak{J}} \to \mathfrak{J}\bar{C}$ is bijective and $\bar{B}$ is identified with $\bar{C}/\bar{\mathfrak{J}}$. It then follows from Example 9.7.5 that the canonical map

$$\bar{\mathfrak{J}}/\bar{\mathfrak{J}}^2 \to \bar{B} \otimes_{\bar{C}} \Omega_{\bar{C}/\kappa_A}$$

is injective and admits a retraction. Now $\bar{\mathfrak{J}}/\bar{\mathfrak{J}}^2$ is identified with $\kappa_A \otimes_A \mathfrak{J}/\mathfrak{J}^2$, hence with $\bar{B} \otimes_B \mathfrak{J}/\mathfrak{J}^2$. On the other hand the $\bar{C}$-module $\Omega_{\bar{C}/\kappa_A}$ is canonically isomorphic to $\bar{C} \otimes_C \Omega_{C/A}$ (A, III, p.136, prop.20), hence $\bar{B} \otimes_{\bar{C}} \Omega_{\bar{C}/\kappa_A}$ is canonically isomorphic to $\bar{B} \otimes_C \Omega_{C/A}$. Passing to quotient by the maximal ideal of $\bar{B}$, we then obtain an injective homomorphism (by the existence of retraction)

$$\kappa_B \otimes_B \mathfrak{J}/\mathfrak{J}^2 \to \kappa_B \otimes_C \Omega_{C/A}$$

which is none other than $\bar{d}_{\kappa_B}$, so B is formally smooth over A (Theorem 9.7.28). $\square$

Theorem 9.7.36. *Let A be a Noetherian ring and B be an A-algebra essentially of finite type. Then the following conditions are equivalent:*

(i) the A-algebra B is formally smooth;

(ii) for any prime ideal $\mathfrak{P} \in \mathrm{Spec}(B)$, the A-algebra $B_{\mathfrak{P}}$ is formally smooth (resp. formally smooth for the $\mathfrak{P}B_{\mathfrak{P}}$-adic topology);

(iii) the A-module B is flat and for any $\mathfrak{p} \in \mathrm{Spec}(A)$, the $\kappa(\mathfrak{p})$-algebra $\kappa(\mathfrak{p}) \otimes_A B$ is geometrically regular;

(iv) the A-module B is flat and for any regular A-algebra R, the ring $R \otimes_A B$ is geometrically regular;

(v) the A-module B is flat and the kernel of the multiplication homomorphism $\mu : B \otimes_A B \to B$ is completely secant.

Proof. The equivalence of (i) and (ii) folows from Corollary 9.7.30. To see that (i)⇒(v), suppose that B is formally smooth over A, let $\mathfrak{P}$ be a prime ideal of B and $\mathfrak{p}$ be its contraction into A. The $A_{\mathfrak{p}}$-algebra $B_{\mathfrak{P}}$ is formally smooth (Proposition 9.7.9), hence flat (Lemma 9.7.35), so the A-module B is flat (Proposition 1.3.27). On the other hand, the ring $B \otimes_A B$ is Noetherian (Corollary 9.6.5), so the ideal $\mathfrak{I}$ is completely secant in view of Corollary 9.7.22.

Suppose that condition (v) is satisfied and let $\mathfrak{p} \in \operatorname{Spec}(A)$; we show that (v)⇒(iii). The map

$$1 \otimes \mu : \kappa(\mathfrak{p}) \otimes_A (B \otimes_A B) \to \kappa(\mathfrak{p}) \otimes_A B$$

is identified with the multiplication map

$$\mu_{\mathfrak{p}} : (\kappa(\mathfrak{p}) \otimes_A B) \otimes_{\kappa(\mathfrak{p})} (\kappa(\mathfrak{p}) \otimes_A B) \to \kappa(\mathfrak{p}) \otimes_A B)$$

of the $\kappa(\mathfrak{p})$-algebra $\kappa(\mathfrak{p}) \otimes_A B$, so the kernel $\ker \mu_{\mathfrak{p}}$ is identified with $\mathfrak{I}(\kappa(\mathfrak{p}) \otimes_A (B \otimes_A B))$. It is completely secant since the A-module B is flat (Proposition 9.5.16). Condition (iii) then follows from Proposition 9.6.21.

Under the hypotheses of (iii), let $\mathfrak{P}$ be a prime ideal of B and $\mathfrak{p}$ be its contraction in A. The $A_{\mathfrak{p}}$-module $B_{\mathfrak{P}}$ is flat, and the $\kappa(\mathfrak{p})$-algebra $\kappa(\mathfrak{p}) \otimes_{A_{\mathfrak{p}}} B_{\mathfrak{P}}$, identified with a fraction ring of $\kappa(\mathfrak{p}) \otimes_A B$, is geometrically regular (Proposition 9.6.14). It then follows from Lemma 9.7.35 that $B_{\mathfrak{P}}$ is formally smooth over $A_{\mathfrak{p}}$, hence over A (Example 9.7.10); this proves (iii)⇒(ii).

It remains to prove the equivalence of (iii) and (iv). First consider the condition (iii). Let R be a regular A-algebra. The R-module $R \otimes_A B$ is flat by Proposition 1.3.25. Let $\mathfrak{r}$ be a prime ideal of R and $\mathfrak{p}$ be its contraction in A. The ring $\kappa(\mathfrak{r}) \otimes_R (R \otimes_A B)$, identified with $\kappa(\mathfrak{r}) \otimes_{\kappa(\mathfrak{p})} (\kappa(\mathfrak{p}) \otimes_A B)$, is regular by Corollary 9.6.18, so $R \otimes_A B$ is regular by Proposition 9.6.14. Conversely, if condition (iv) is satisfied, for any priem ideal $\mathfrak{p}$ of A and any extension k of $\kappa(\mathfrak{p})$, the ring $k \otimes_{\kappa(\mathfrak{p})} (\kappa(\mathfrak{p}) \otimes_A B)$, which is identified with $k \otimes_A B$, is regular, whence (iii). □

Let A be a Noetherian ring. We say that an A-algebra B is **smooth** if it is essentially of finite type are satisfies the equivalent conditions of Theorem 9.7.36.

Proposition 9.7.37. *Let A be a Noetherian ring.*

(a) Let A' be a Noetherian A-algebra and B be a smooth A-algebra. Then the A'-algebra $A' \otimes_A B$ is smooth.

(b) Let B be a smooth A-algebra and C be a smooth B-algebra. Then the A-algebra C is smooth.

(c) Let B and C be two smooth A-algebras. Then the A-algebra $B \otimes_A C$ is smooth.

Proof. This follows from the analogues results for algebras essentially of finite type and formally smooth. □

Example 9.7.38. The smooth algebras over a field k are the k-algebras essentially of finite type and geometrically regular. In particular, if k is a perfect field, then these are exactly regular k-algebras essentially of finite type.

Example 9.7.39. Let A be a Noetherian ring and $(T_i)_{i \in I}$ be a finite family of indeterminates. Then the polynomial algebra $A[(T_i)_{i\in I}]$ is smooth over A. More generally, let $F_1, \dots, F_m$ be elements of $A[(T_i)_{i\in I}]$ and $B = A[(T_i)_{i\in I}]/(F_1, \dots, F_m)$. If for any maximal ideal $\mathfrak{N}$ of B, the class of the matrix $(\partial F_j/\partial T_i)$ mod $\mathfrak{N}$ is of rank m, then the A-algebra B is smooth (Theorem 9.7.28).

9.8 Duality of finite length modules

9.8.1 Indecomposable injective modules

Chapter 10

The language of schemes

10.1 Affine schemes

Let A be a ring. Recall that we have associate with A a topological space $\mathrm{Spec}(A)$, called the spectrum of A. In this section we shall make $\mathrm{Spec}(A)$ a locally ringed space and consider sheaf of modules over it; such spaces will be called **affine schemes**.

10.1.1 Sheaves associated with a module

Let A be a ring and M an A-module. For any element $f \in A$, let S_f be the multiplicative subset consisting of powers of f. Recall that the localization of M with respect to S_f is then denoted by M_f, and that of A by A_f. Let $\bar{S}_f$ be the saturation of S_f, which is defined to be the complement of the union of prime ideals of A that are disjoint from S_f, or equivalently not contains f. By Proposition 1.2.25, the set $\bar{S}_f$ is also characterized by

$$\bar{S}_f = \{x \in A : \text{there exist } n, m \geq 0 \text{ such that } f^n x = f^m\}.$$

Also, by Proposition 1.2.24, we have $\bar{S}_f A = A_f$ and $\bar{S}_f M = M_f$.

Lemma 10.1.1. *Let f, g be elements of A. Then the following conditions are equivalent:*

(i) *$g \in \bar{S}_f$, or equivalently $\bar{S}_g \subseteq \bar{S}_f$;*

(ii) *$f \in \sqrt{(g)}$, or equivalently $\sqrt{(f)} \subseteq \sqrt{(g)}$;*

(iii) *$D(f) \subseteq D(g)$, or equivalently $V(g) \subseteq V(f)$.*

Proof. We first note that $g \in \bar{S}_f$ is equivalent to $S_g \subseteq \bar{S}_f$, so the equivalence in (i). Also, the equivalence of (ii) and (iii) follows from Proposition 1.4.3. Finally, if $g \in \bar{S}_f$, then there exist $n, m \geq 0$ such that $f^n g = f^m$, which is an element of (g), and thus $f \in \sqrt{(g)}$. Conversely, if $D(f) \subseteq D(g)$, then by the discriptions $S_f = \bigcup_{f \notin \mathfrak{p}} \mathfrak{p}$ and $S_g = \bigcup_{g \notin \mathfrak{p}} \mathfrak{p}$, we conclude that $\bar{S}_g \subseteq \bar{S}_f$, whence the lemma. □

If $D(g) \subseteq D(f)$ in $\mathrm{Spec}(A)$, then by Lemma 10.1.1, we have $\bar{S}_f \subseteq \bar{S}_g$, so there is a canonical homomorphism $\rho_{g,f} : M_f \to M_g$; moreover, if $D(f) \supseteq D(g) \supseteq D(h)$, we then have

$$\rho_{h,g} \circ \rho_{g,f} = \rho_{h,f}.$$

As f runs through $A - \mathfrak{p}$ (where $\mathfrak{p}$ is a point in $X = \mathrm{Spec}(A)$), the set S_f then constitute a filtered set indexed by $A - \mathfrak{p}$, since any two element f, g of $A - \mathfrak{p}$ contains S_{fg}; as the union of the S_f for $f \in A - \mathfrak{p}$ is $A - \mathfrak{p}$, we conclude from Proposition 1.2.35 that the $A_{\mathfrak{p}}$-module $M_{\mathfrak{p}}$ is canonically identified with the direct limit $\varinjlim M_f$, relative to the family $(\rho_{g,f})$ of homomorphisms. For each $f \in A - \mathfrak{p}$, we denote the canonical homomorphism from M_f to $M_\mathfrak{p}$ by

$$\rho_\mathfrak{p}^f : M_f \to M_\mathfrak{p}.$$

We now define the **structural sheaf** of the prime spectrum $X = \operatorname{Spec}(A)$, denoted by $\widetilde{A}$, to be the sheaf of rings associated with the presheaf $D(f) \mapsto A_f$ over the basis $\mathcal{B}$ of X, formed by $D(f)$ with $f \in A$. Simialrly, for an A-module M,we define the **associated sheaf** $\widetilde{M}$ to be the sheaf associated presheaf $D(f) \mapsto M_f$ over the basis $\mathcal{B}$ of X. By the property of sheafification, it is clear that the stalk $\widetilde{A}_\mathfrak{p}$ (resp. $\widetilde{M}_\mathfrak{p}$) is identified with the ring $A_\mathfrak{p}$ (resp. with $A_\mathfrak{p}$-module $M_\mathfrak{p}$).

Theorem 10.1.2. *For each A-module M, the presheaf $D(f) \to M_f$ is a sheaf on the basis $\mathcal{B}$ of X, so for each $f \in A$ we have a canonical isomorphism*

$$M_f \to \Gamma(D(f), \widetilde{M}).$$

In particular, M is canonically identified with $\Gamma(X, \widetilde{M})$.

Proof. To show that the presheaf $D(f) \mapsto M_f$ is a sheaf on the basis $\mathcal{B}$ of X, we need to check the sheaf condition for the standard open coverings. If $D(f) = \bigcup_{i=1}^n D(g_i)$, then the sheaf condition for this covering is equivalent with the exactness of the sequence

$$0 \longrightarrow M_f \longrightarrow \bigoplus_i M_{g_i} \longrightarrow \bigoplus_{i,j} M_{g_i g_j} \ .$$

Note that $D(g_i) = D(fg_i)$, and hence we can rewrite this sequence as the sequence

$$0 \longrightarrow M_f \longrightarrow \bigoplus_i M_{fg_i} \longrightarrow \bigoplus_{i,j} M_{fg_i g_j} \ .$$

Since the $D(g_i)$'s cover $D(f)$ (which is identified with $\operatorname{Spec}(A_f)$), the elements $g_1, \dots, g_n$ generate the unit ideal in A_f, so we may apply Corollary 1.4.50 to the module M_f over A_f and the elements $g_1, \dots, g_n$ to conclude that the sequence is exact. $\square$

Corollary 10.1.3. *Let M, N be A-modules. The canonical homomorphism*

$$\operatorname{Hom}_A(M, N) \to \operatorname{Hom}_{\widetilde{A}}(\widetilde{M}, \widetilde{N}), \quad \phi \mapsto \tilde{\phi}$$

is bijective. In particular, the relations $M = 0$ and $\widetilde{M} = 0$ are equivalent.

Proof. Consider the canonical homomorphism $\operatorname{Hom}_{\widetilde{A}}(\widetilde{M}, \widetilde{N}) \to \operatorname{Hom}_A(M, N)$, $\varphi \mapsto \Gamma(\varphi)$ (Theorem 10.1.2). It suffices to show that $\phi \mapsto \tilde{\phi}$ and $\varphi \mapsto \Gamma(\varphi)$ are inverses of each other. Now, it is evident that $\Gamma(\tilde{\phi}) = \phi$, by the definition of $\tilde{\phi}$. On the other hand, if we put $\phi = \Gamma(\varphi)$ for $\varphi \in \operatorname{Hom}_{\widetilde{A}}(\widetilde{M}, \widetilde{N})$, the map $\varphi_{D(f)} : \Gamma(D(f), \widetilde{M}) \to \Gamma(D(f), \widetilde{N})$ induced by φ is making the following diagram commutes:

$$\begin{tikzcd} M \arrow[r, "\phi"] \arrow[d, "\rho_{f,1}"'] & N \arrow[d, "\rho_{f,1}"] \\ M_f \arrow[r, "\varphi_{D(f)}"] & N_f \end{tikzcd}$$

We then have necessarily $\varphi_{D(f)} = \phi_f$ for each $f \in A$, which shows $\widetilde{\Gamma(\varphi)} = \varphi$. $\square$

Proposition 10.1.4. *For each $f \in A$, the open set $D(f) \subseteq X$ is canonically identified with the spectrum $\operatorname{Spec}(A_f)$, and the sheaf $\widetilde{M}_f$ associated with the A_f-module M_f is canonically identified with the restriction $\widetilde{M}_{D(f)}$.*

Proof. The first assertion is proved in Corollary 1.4.25. Now for $D(g) \subseteq D(f)$, then M_g is identified with the localization of M_f with respect to the canonical image of g in A_f, so the canonical identification of $\widetilde{M}_f$ and $\widetilde{M}_{D(f)}$ follows by definition. □

Proposition 10.1.5. *The functor $M \mapsto \widetilde{M}$ is an exact functor from the category of A-modules to the category of $\widetilde{A}$-modules.*

Proof. Let M, N be two A-modules and $\phi : M \to N$ a homomorphism; for any $f \in A$, we have a corresponding homomorphism $\phi_f : M_f \to N_f$, and for $D(g) \subseteq D(f)$ the diagram

$$\begin{tikzcd} M_f \arrow[r, "\phi_f"] \arrow[d, "\rho_{g,f}"'] & N_f \arrow[d, "\rho_{g,f}"] \\ M_g \arrow[r, "\phi_g"] & N_g \end{tikzcd}$$

is commutative. These then give a homomorphism of $\widetilde{A}$-modules $\tilde{\phi} : \widetilde{M} \to \widetilde{N}$. Moreover, for each $\mathfrak{p} \in X$, $\tilde{\phi}_x$ is the direct limit of ϕ_f for $f \in A - \mathfrak{p}$, and concequently identified with the canonical homomorphism $\phi_\mathfrak{p} : M_\mathfrak{p} \to N_\mathfrak{p}$. If P is another A-module, $\psi : N \to P$ a homomorphism and $\eta = \psi \circ \phi$, then it is immediate that $\eta_\mathfrak{p} = \psi_\mathfrak{p} \circ \phi_\mathfrak{p}$, hence $\tilde{\eta} = \tilde{\psi} \circ \tilde{\phi}$. We thus get a covariant functor $\widetilde{(-)}$ from the category of A-modules to the category of $\widetilde{A}$-modules. This functor is exact since for each $\mathfrak{p} \in X$, $M \mapsto M_\mathfrak{p}$ is an exact functor; furthermore, we have $\operatorname{supp}(M) = \operatorname{supp}(\widetilde{M})$ by the definitions of these two members. □

Corollary 10.1.6. *Let M and N be two A-modules.*

(a) *If $\phi : M \to N$ is a homomorphism, then the sheaves associated with $\ker\phi$, $\operatorname{im}\phi$, and $\operatorname{coker}\phi$ are $\ker\tilde{\phi}$, $\operatorname{im}\tilde{\phi}$, and $\operatorname{coker}\tilde{\phi}$, respectively. In particular, $\tilde{\phi}$ is injective (resp. surjective, bijective) if and only if ϕ is injective (resp. surjective, bijective).*

(b) *If M is a filtered limit (resp. direct sum) of a family $(M_i)_{i\in I}$ of A-modules, then $\widetilde{M}$ is a filtered limit (resp. direct sum) of the family $(\widetilde{M}_i)$.*

Proof. For (a), it suffices to apply the exact functor $M \mapsto \widetilde{M}$ to the following exact sequences:

$$0 \longrightarrow \ker\phi \longrightarrow M \longrightarrow \operatorname{im}\phi \longrightarrow 0$$

$$0 \longrightarrow \operatorname{im}\phi \longrightarrow N \longrightarrow \operatorname{coker}\phi \longrightarrow 0$$

Now let (M_i, ρ_{ji}) be a filtered system of A-modules, with limit M, and let $\rho_i : M_i \to M$ be the canonical homomorphism. Since we have $\tilde{\rho}_{kj} \circ \tilde{\rho}_{ji} = \tilde{\rho}_{ji}$ and $\tilde{\rho}_i = \tilde{\rho}_j \circ \tilde{\rho}_{ji}$ for $i \le j \le k$, we see $(\widetilde{M}, \tilde{\rho}_{ji})$ is a direct system of sheaves over X, and if we denote by $\eta_i : \widetilde{M}_i \to \varinjlim \widetilde{M}_i$ the canonical homomorphism, a unique homomorphism $\psi : \varinjlim \widetilde{M}_i \to \widetilde{M}$ such that $\psi \circ \eta_i = \tilde{\rho}_i$. For this ψ to be bijective, it suffices that for each $\mathfrak{p} \in X$, $\psi_\mathfrak{p}$ is a bijection from $(\varinjlim \widetilde{M}_i)_\mathfrak{p}$ to $\widetilde{M}_\mathfrak{p}$; but $\widetilde{M}_\mathfrak{p} = M_\mathfrak{p}$ and

$$(\varinjlim \widetilde{M}_i)_\mathfrak{p} = \varinjlim (\widetilde{M}_i)_\mathfrak{p} = \varinjlim (M_i)_\mathfrak{p} = M_\mathfrak{p}$$

Also, it follows by definition that $(\tilde{\rho}_i)_\mathfrak{p}$ and $(\eta_i)_\mathfrak{p}$ are equal to the canonical homomorphism from $(M_i)_\mathfrak{p}$ to $M_\mathfrak{p}$; since $(\tilde{\rho}_i)_\mathfrak{p} = \psi_\mathfrak{p} \circ (\eta_i)_\mathfrak{p}$, $\psi_\mathfrak{p}$ is therefore the identity.

Finally, if M is a direct sum of two modules N and P, it is immediate that $\widetilde{M} = \widetilde{M} \oplus \widetilde{P}$; by taking filtered limits, we then generalize this result for the direct sum of an arbitrary family. This completes the proof. □

Remark 10.1.7. By Proposition 10.1.5, we conclude that the sheaves which are isomorphic to the sheaves associated with A-modules form an abelian category. Note also that it follows from Corollary 10.1.6 that if M is a finitely generated A-module, that is, if there exists a surjective homomorphism $A^n \to M$, then there exists a homomorphism surjective $\widetilde{A}^n \to \widetilde{M}$, in other words, the $\widetilde{A}$-module $\widetilde{M}$ is generated by a finite family of sections over X, and vice versa.

Corollary 10.1.8. *Let N and P be submodules of M. The sheaves $\widetilde{N}$ and $\widetilde{P}$ can be identified with sub-$\widetilde{A}$-modules of $\widetilde{M}$, and we have*

$$\widetilde{N+P} = \widetilde{N} + \widetilde{P}, \quad \widetilde{N \cap P} = \widetilde{N} \cap \widetilde{P}.$$

In particular, if $\widetilde{N} = \widetilde{P}$, then $N = P$.

Proof. If N is a submodule of an A-module M, the canonical injection $N \to M$ induced an injective homomorphism $\widetilde{N} \to \widetilde{M}$, hence identifies $\widetilde{N}$ with a sub-$\widetilde{A}$-module of $\widetilde{M}$. Now note that $N + P$ is the image of the canonical homomorphism $\alpha : N \oplus P \to M$, so by Corollary 10.1.6 we have

$$\widetilde{N+P} = \widetilde{\operatorname{im}\alpha} = \operatorname{im}\tilde{\alpha} = \widetilde{N} + \widetilde{P}$$

since $\tilde{\alpha}$ is equal to the canonical homomorphism $\widetilde{N} \oplus \widetilde{P} \to \widetilde{M}$. Similarly, since $N \cap P$ is the kenrel of the canonical homomorphism $M \to (M/N) \oplus (M/P)$, we also have $\widetilde{N \cap P} = \widetilde{N} \cap \widetilde{P}$. □

Corollary 10.1.9. *Over the category of sheaves isomorphic to sheaves associated with A-modules, the global section functor Γ is exact.*

Proof. In fact, let $\widetilde{M} \xrightarrow{\tilde{\phi}} \widetilde{N} \xrightarrow{\tilde{\psi}} P$ be an exact sequence corresponding to two homomorphisms $\phi : M \to N$ and $\psi : N \to P$. If $Q = \operatorname{im}\phi$ and $R = \ker\psi$, we have

$$\widetilde{Q} = \operatorname{im}\tilde{\phi} = \ker\tilde{\psi} = \widetilde{R}$$

by Corollary 10.1.6, so $Q = R$ and the sequence is exact. □

Corollary 10.1.10. *Let M and N be two A-modules.*

(a) *The sheaf associated with $M \otimes_A N$ is canonically identified with $\widetilde{M} \otimes_{\widetilde{A}} \widetilde{N}$.*

(b) *If moreover M is finitely presented, the sheaf associated with $\operatorname{Hom}_A(M, N)$ is canonically identified with $\mathcal{H}om_{\widetilde{A}}(\widetilde{M}, \widetilde{N})$.*

Proof. The sheaf $\mathscr{F} = \widetilde{M} \otimes_{\widetilde{A}} \widetilde{N}$ is the sheafification of the presheaf

$$U \mapsto \mathscr{F}(U) = \Gamma(U, \widetilde{M}) \otimes_{\Gamma(U, \widetilde{A})} \Gamma(U, \widetilde{N})$$

where U runs through the basis $\mathcal{B}$ of X formed by $D(f)$, $f \in A$. Now, $\mathscr{F}(D(f))$ is canonically identified with $M_f \otimes_{A_f} N_f$ by Theorem 10.1.2, which is isomorphic to $\Gamma(D(f), \widetilde{M \otimes_A N})$. Moreover, it is immediately verified that the canonical isomorphisms

$$\mathscr{F}(D(f)) \cong \Gamma(D(f), \widetilde{M \otimes_A N})$$

is compatible with the restriction maps, so they define a canonical isomorphism $\widetilde{M} \otimes_{\widetilde{A}} \widetilde{N} \cong \widetilde{M \otimes_A N}$.

Now assume that M is finitely presented. The sheaf $\mathscr{G} = \mathcal{H}om_{\widetilde{A}}(\widetilde{M}, \widetilde{N})$ is the sheafification of the presheaf

$$U \mapsto \mathscr{G}(U) = \operatorname{Hom}_{\widetilde{A}|_U}(\widetilde{M}|_U, \widetilde{N}|_U)$$

where U runs through the basis $\mathcal{B}$ of X. By Proposition 10.1.4 and Corollary 10.1.3, the module $\mathscr{G}(D(f))$ is then identified with $\mathrm{Hom}_{A_f}(M_f, N_f)$, which is isomorphic to $\Gamma(D(f), \widetilde{\mathrm{Hom}_A(M,N)})$ by Proposition 1.2.48. It is clear that these isomorphisms are compatible with the restriction maps, so we conclude that $\mathcal{H}om_{\widetilde{A}}(\widetilde{M}, \widetilde{N}) \cong \widetilde{\mathrm{Hom}_A(M,N)}$. $\square$

Now consider an A-algebra B (commutative); this can be interpreted by saying that B is an A-module and that we are given an element $e \in B$ and an A-homomorphism $\varphi : B \otimes_A B \to B$ so that the diagrams

$$\begin{tikzcd} B \otimes_A B \otimes_A B \arrow[r, "\varphi\otimes 1"] \arrow[d, "1\otimes\varphi"'] & B \otimes_A B \arrow[d, "\varphi"] \\ B \otimes_A B \arrow[r, "\varphi"] & B \end{tikzcd} \qquad \begin{tikzcd} B \otimes_A B \arrow[rr, "\sigma"] \arrow[dr, "\varphi"'] & & B \otimes_A B \arrow[dl, "\varphi"] \\ & B & \end{tikzcd}$$

(where σ is the canonical symmetry) commute, and that $\varphi(e \otimes x) = \varphi(x \otimes e) = x$. In view of Corollary 10.1.10, the homomorphism $\tilde{\varphi} : \widetilde{B} \otimes_{\widetilde{A}} \widetilde{B} \to \widetilde{B}$ of $\widetilde{A}$-modules satisfies similar conditions, hence defines a $\widetilde{A}$-**algebra** structure on $\widetilde{B}$. In the same way, the data of a B-module N amounts to giving an A-module N and an A-homomorphism $\psi : B \otimes_A N \to N$ such that the diagram

$$\begin{tikzcd} B \otimes_A B \otimes_A N \arrow[r, "\varphi\otimes 1"] \arrow[d, "1\otimes\psi"'] & B \otimes_A N \arrow[d, "\psi"] \\ B \otimes_A N \arrow[r, "\psi"] & N \end{tikzcd}$$

commutes and $\psi(e \otimes n) = n$; the homomorphism $\tilde{\psi} : \widetilde{B} \otimes_{\widetilde{A}} \widetilde{N} \to \widetilde{N}$ then satisfies similar conditions, and defines on $\widetilde{N}$ a $\widetilde{B}$-module structure.

If $\rho : B \to B'$ (resp. $\phi : N \to N'$) is a homomorphism of A-algebras (resp. a homomorphisms of B-modules), then $\tilde{\rho}$ (resp. $\tilde{\phi}$) is a homomorphism of $\widetilde{A}$-algebras (resp. a homomorphisms of $\widetilde{B}$-modules), $\ker\tilde{\rho}$ is an ideal of $\widetilde{B}$ (resp. $\ker\tilde{\phi}$, $\operatorname{coker}\tilde{\phi}$, and $\operatorname{im}\tilde{\phi}$ are $\widetilde{B}$-modules). Moreover, by Proposition 1.4.53(b) if N is a B-module, then $\widetilde{N}$ is a finitely generated $\widetilde{B}$-module if and only if N is finitely generated over B.

If M and N are two B-modules, the $\widetilde{B}$-module $\widetilde{M} \otimes_{\widetilde{B}} \widetilde{N}$ is canonically identified with $\widetilde{M \otimes_B N}$; similarly, $\mathcal{H}om_{\widetilde{B}}(\widetilde{M}, \widetilde{N})$ is canonically identified with $\widetilde{\mathrm{Hom}_B(M,N)}$ if M is finitely presented. If $\mathfrak{b}$ is an ideal of B, then $\widetilde{\mathfrak{b}N} = \tilde{\mathfrak{b}}\widetilde{N}$.

Finally, if B is a graded A-algebra with (B_n) its graduation, the $\widetilde{A}$-algebra $\widetilde{B}$ is then the direct sum of the sub-$\widetilde{A}$-modules $\widetilde{B}_n$ (Corollary 10.1.6), so $(\widetilde{B}_n)$ is a graduation of $\widetilde{B}$. Similarly, if M is a graded B-module with graduation (M_n), then $\widetilde{M}$ is a graded $\widetilde{B}$-module with graduation $(\widetilde{M}_n)$.

10.1.2 Functorial properties of the associated sheaf

We now consider the functorial properties of the operation $M \mapsto \widetilde{M}$. Let A and B be rings and $\varphi : B \to A$ be a ring homomorphism. Then we have an associated map

$$^a\varphi : X = \mathrm{Spec}(A) \to Y = \mathrm{Spec}(B)$$

We will define a canonical homomorphism

$$\varphi^\# : \mathscr{O}_Y \to {}^a\varphi_*(\mathscr{O}_X)$$

of sheaf of rings. For any $g \in B$, we set $f = \varphi(g)$; we have $\varphi^{-1}(D(g)) = D(f)$ by Proposition 1.4.20. Now the sections $\Gamma(D(g), \widetilde{B})$ and $\Gamma(D(f), \widetilde{A})$ are canonically identified with B_g and

A_f, respectively, and we have an induced map $\varphi_g : B_g \to A_f$, which then gives a homomorphism of rings

$$\Gamma(D(g), \widetilde{B}) \to \Gamma(\varphi^{-1}(D(g)), \widetilde{A}) = \Gamma(D(g), {}^a\varphi_*(\widetilde{A})).$$

Moreover, these homomorphisms satisfy the following compatible conditions: for $D(g) \supseteq D(g')$ in Y, the diagram

$$\begin{tikzcd} \Gamma(D(g), \widetilde{A}) \arrow[r] \arrow[d] & \Gamma(D(g), {}^a\varphi_*(\widetilde{A})) \arrow[d] \\ \Gamma(D(g'), \widetilde{A}) \arrow[r] & \Gamma(D(g'), {}^a\varphi_*(\widetilde{A})) \end{tikzcd}$$

is commutative; we then get a morphism of $\mathscr{O}_Y$-algebras, since $D(g)$ form a basis for the topological space Y. The couple $({}^a\varphi, \varphi^\#)$ is called the **canonical morphism** of the locally ringed spaces induced by φ.

We also note that, if $y = {}^a\varphi(x)$, the homomorphism $\varphi^\#_x$ is no other that the homomorphism

$$\varphi_x : B_y \to A_x$$

induced by the homomorphism $\varphi : B \to A$. In fact, for $b/g \in B_y$, where $b, g \in B$ and $g \notin \mathfrak{p}_y$, $D(g)$ is then an open neighborhood of y in Y, and the homomorphism

$$\Gamma(D(g), \widetilde{B}) \to \Gamma(D(g), ({}^a\varphi)_*(\widetilde{A}))$$

induced by $\varphi^\#$ is just φ_g; by considering the section $\xi \in \Gamma(D(g), \widetilde{B})$ corresponding to b/g, we then obtain that $\varphi^\#_x(\xi) = \varphi(b)/\varphi(g)$ in A_x.

Example 10.1.11. Let S be a miltiplicative subset of A and $\varphi : A \to S^{-1}A$ the canonical homomorphism. We have seen in Proposition 1.4.24 that ${}^a\varphi$ is a homeomorphism from $Y = \mathrm{Spec}(S^{-1}A)$ to the subspace $X = \mathrm{Spec}(A)$ formed by x such that $\mathfrak{p}_x \cap S = \varnothing$. Moreover, for any x in this subspace, hence of the form ${}^a\varphi(y)$ where $y \in Y$, the homomorphism $\varphi^\#_y : \mathscr{O}_{Y,y} \to \mathscr{O}_{X,x}$ is bijective; therefore, $\mathscr{O}_Y$ is identified with the sheaf induced over Y by $\mathscr{O}_X$.

Proposition 10.1.12. *Let $\varphi : B \to A$ be a ring homomorphism and $\Phi : X \to Y$ the associated morphism. For any A-module M, there exists a canonical functorial isomorphism of the $\mathscr{O}_Y$-module $\widetilde{\varphi^*(M)}$ to its direct image $\Phi_*(\widetilde{M})$.*

Proof. For $g \in B$, put $f = \varphi(g)$; the modules $\Gamma(D(g), \widetilde{\varphi^*(M)})$ and $\Gamma(D(f), \widetilde{M})$ are identified with $(\varphi^*(M))_g$ and M_f, respectively; moreover, the B_g-module $\varphi^*_g(M_f)$ is canonically isomorphic to $(\varphi^*(M))_g$. We then have a functorial isomorphism of $\Gamma(D(g), \widetilde{B})$-modules:

$$\Gamma(D(g), \widetilde{\varphi^*(M)}) \cong \varphi^*(\Gamma(D(\varphi(g)), \widetilde{M}))$$

and this isomorphism satisfies the compatible conditions with restrictions, hence define an isomorphism of sheaves. $\square$

This proof also proves that for any A-algebra R, the canonical functorial isomorphism $\widetilde{\varphi^*(R)} \to \widetilde{\Phi_*(\widetilde{R})}$ is an isomorphism of $\mathscr{O}_Y$-algebras. If M is an R-module, the canonical isomorphism $\widetilde{\varphi^*(M)} \cong \Phi_*(\widetilde{M})$ is an isomorphism of $\Phi_*(\widetilde{R})$-modules.

Corollary 10.1.13. *The direct image functor Φ_* is exact on the category of quasi-coherent sheaves.*

Proof. Recall that the functor φ^* is exact and $M \mapsto \widetilde{M}$ is an exact functor. $\square$

Proposition 10.1.14. *Let $\varphi : B \to A$ be a ring homomorphism and $\Phi : X \to Y$ the associated morphism. Let N be a B-module and $\varphi_*(N)$ the A-module $N \otimes_B A$. Then there exist a canonical functorial isomorphism of the $\mathscr{O}_X$-module $\Phi^*(\widetilde{N})$ to $\widetilde{\varphi_*(N)}$.*

Proof. We first remark that $j : z \mapsto z \otimes 1$ is a B-homomorphism from N to $\varphi^*\varphi_*(N)$: this holds because for $g \in B$, we have

$$(gz) \otimes 1 = z \otimes \varphi(g) = \varphi(g)(z \otimes 1).$$

By Corollary 10.1.3, this corresponds to a homomorphism $\tilde{j} : \widetilde{N} \to \varphi^*(\widetilde{\varphi_*(N)})$ of $\mathscr{O}_Y$-modules, and via Proposition 10.1.12 we can think that $\tilde{j}$ maps $\widetilde{N}$ to $\Phi_*(\widetilde{\varphi_*(N)})$. From the adjointness of Φ^* and Φ_*, this canonically corresponds to a homomorphism

$$\theta : \Phi^*(\widetilde{N}) \to \widetilde{\varphi_*(N)}.$$

It then remains to show that θ is bijective, or equivalently that θ_x is bijective for every $x \in X$. For this, put $y = {}^a\varphi(x)$, choose $g \in B$ such that $y \in D(g)$, and let $f = \varphi(g)$. Then the ring $\Gamma(D(f), \widetilde{A})$ is identified with A_f, the module $\Gamma(D(f), \widetilde{\varphi_*(N)})$ is identified with $(\varphi_*(N))_f$, and $\Gamma(D(g), \widetilde{N})$ is identified with N_g. Let $s = n/g^p$ ($n \in N$) be a section of $\Gamma(D(g), \widetilde{N})$ and $t = a/f^q$ ($a \in A$) a section of $\Gamma(D(f), \widetilde{A})$. Then, since s is send to $(n \otimes 1)/f^p$ be $\tilde{j}$, by definition we have

$$\theta_x(s_x \otimes t_x) = t_x \cdot s_x.$$

Recall that we can identify $(\varphi_*(N))_f$ with $N_f \otimes_{B_g} \varphi^*(A_f)$, under which n/g^p is identified with $(n/g^p) \otimes 1$. So it is immediately seen that θ_x is none other than the canonical isomorphism

$$N_y \otimes_{B_y} \varphi_y^*(A_x) \cong (\varphi_*(N))_x = (N \otimes_B \varphi^*(A))_x.$$

Finally, let $v : N_1 \to N_2$ be a homomorphism of B-modules; since $\tilde{v}_y = v_y$ for any $y \in Y$, it follows immediately from the preceding argument that $\Phi^*(\tilde{v})$ is canonically identified to $\widetilde{v \otimes 1}$, which completes the proof. □

If S is an B-algebra, the canonical isomorphism of $\Phi^*(\widetilde{S})$ to $\widetilde{\varphi_*(S)}$ is an isomorphism of $\mathscr{O}_X$-algebras; if moreover N is a S-module, the canonical isomorphism of $\Phi^*(\widetilde{N})$ to $\widetilde{\varphi_*(N)}$ is an isomorphism of $\Phi^*(\widetilde{S})$-algebras.

Corollary 10.1.15. *The sections of $\Phi^*(\widetilde{N})$ which are cannical images of section of the B-module $\Gamma(\widetilde{N})$, generate the A-module $\Gamma(\Phi^*(\widetilde{N}))$.*

Proof. In fact, these images are identified with the elements $z \otimes 1$ of $\varphi_*(N)$, if we identify N and $\varphi_*(N)$ with $\Gamma(\widetilde{N})$ and $\Gamma(\widetilde{\varphi_*(N)})$. □

By the proof of Proposition 10.1.14, we see that the canonical map $\alpha : \widetilde{N} \to \Phi_*\Phi^*(\widetilde{N})$ is none other than the homomorphism $\tilde{j}$, where $j : N \to \varphi_*(N)$ is the canonical map $z \mapsto z \otimes 1$. Similarly, the canonical map $\beta : \Phi^*\Phi_*(\widetilde{M}) \to \widetilde{M}$ is none other than the homomorphism $\tilde{p}$, where $p : \varphi^*(M) \otimes_B \varphi^*(A) \to M$ is the canonical homomorphism that sends $m \otimes a$ to am.

Corollary 10.1.16. *Let N_1 and N_2 be B-modules and assume that N_1 is finitely presented. Then there is a canonical homomorphism*

$$\Phi^*(\mathcal{H}om_{\widetilde{B}}(\widetilde{N}_1, \widetilde{N}_2)) \to \mathcal{H}om_{\widetilde{A}}(\Phi^*(\widetilde{N}_1), \Phi^*(\widetilde{N}_2)).$$

This homomorphism is bijective if φ is a flat homomorphism.

Proof. By the above remarks and Corollary 10.1.10, this homomorphism is induced by the homomorphism
$$\mathrm{Hom}_B(N_1, N_2) \otimes_B A \to \mathrm{Hom}_A(N_1 \otimes_B A, N_2 \otimes_B A).$$
The last assertion follows from **??**. □

A locally ringed space $(X, \mathscr{O}_X)$ is called an **affine scheme** if it is isomorphic to the spectrum of a ring A. In this case, the ring $\Gamma(X, \mathscr{O}_X)$ is canonically identified with A. By absuing language, we often call $\mathrm{Spec}(A)$ an affine scheme, without mention the structural sheaf.

Let A and B be two rings and $(X, \mathscr{O}_X)$, $(Y, \mathscr{O}_Y)$ the corresponding affine schemes. Then any ring homomorphism $\varphi : B \to A$ corresponds to a morphism $({}^a\varphi, \varphi^\#) : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$. Note that the homomorphism φ is completely determined by $({}^a\varphi, \varphi^\#)$, since by definition we have $\varphi = \Gamma(\varphi^\#) : \Gamma(\widetilde{B}) \to \Gamma(\widetilde{A})$.

Theorem 10.1.17. *Let $(X, \mathscr{O}_X)$ and $(Y, \mathscr{O}_Y)$ be two affine schemes. Then any morphism $(\psi, \psi^\#) : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ of locally ringed spaces is of the form $({}^a\varphi, \varphi^\#)$, where $\varphi : \Gamma(Y, \mathscr{O}_Y) \to \Gamma(X, \mathscr{O}_X)$ is a ring homomorphism.*

Proof. Put $A = \Gamma(X, \mathscr{O}_X)$ and $B = \Gamma(Y, \mathscr{O}_Y)$. Let $(\psi, \psi^\#) : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ be a morphism of locally ringed spaces. By definition, $\psi^\#$ is a homomorphism from $\mathscr{O}_Y$ to $\psi_*\mathscr{O}_X$, and we then deduce a canonical homomorphism of rings
$$\varphi = \Gamma(\psi^\#) : B = \Gamma(Y, \mathscr{O}_Y) \to \Gamma(Y, \psi_*\mathscr{O}_X) = \Gamma(X, \mathscr{O}_X) = A.$$
Since $\psi_x^\#$ is a local homomorphism, by passing to quotients we deduce a monomorphism θ^x from the residue field $\kappa(\psi(x))$ into the residue field $\kappa(x)$ such that, for any section $f \in \Gamma(Y, \mathscr{O}_Y)$, we have $\theta^x(f(\psi(x))) = \varphi(f)(x)$ (we consider the elements of $\Gamma(Y, \mathscr{O}_Y)$ as functions on B). The relationship $f(\psi(x)) = 0$ is therefore equivalent to $\varphi(f)(x) = 0$, which means $\psi(x) = {}^a\varphi(x)$. Since this hold for any $x \in X$, we conclude that $\psi = {}^a\varphi$. We also know that the diagram
$$\begin{tikzcd}
B = \Gamma(Y, \mathscr{O}_Y) \arrow[r, "{}^a\varphi"] \arrow[d] & \Gamma(X, \mathscr{O}_X) = A \arrow[d] \\
B_{\psi(x)} \arrow[r, "\psi_x^\#"] & A_x
\end{tikzcd}$$
is commutative, so $\psi_x^\#$ is equal to the homomorphism $\varphi_x : B_{\psi(x)} \to A_x$ induced from φ. Since the morphism $\psi^\#$ is determined by $\psi_x^\#$, we obtain that $\psi^\# = \varphi^\#$. □

Corollary 10.1.18. *Let $(X, \mathscr{O}_X)$ and $(Y, \mathscr{O}_Y)$ be two affine schemes. Then there is a canonical bijection*
$$\mathrm{Mor}(X, Y) \to \mathrm{Hom}_{\mathbf{Ring}}(\Gamma(Y, \mathscr{O}_Y), \Gamma(X, \mathscr{O}_X))$$
which sends a morphism $(f, f^\#)$ to the global section of $f^\#$.

We can also say that the functors $(\mathrm{Spec}(A), \widetilde{A})$ in A and $\Gamma(X, \mathscr{O}_X)$ in $(X, \mathscr{O}_X)$ define an equivalence of the opposite category of commutative rings and of the category of the category of affine schemes.

Corollary 10.1.19. *If $\varphi : B \to A$ is surjective, then the corresponding morphism Φ is a monomorphism of locally ringed spaces.*

Proof. The map ${}^a\varphi$ is injective by Proposition 1.4.22, and since φ is surjective, for any $x \in x$ the map $\varphi_x^\# : B_{{}^a\varphi(x)} \to A_x$, obtained by passing to localization, is surjective; these prove the assertion. □

10.1.3 Quasi-coherent sheaves over affine schemes

Recall that we have defined the abstract notion of a quasi-coherent sheaf. In this paragraph we show that any quasi-coherent sheaf on an affine scheme $\operatorname{Spec}(A)$ corresponds to the sheaf $\widetilde{M}$ associated with an A-module M.

Lemma 10.1.20. *Let $X = \operatorname{Spec}(A)$ and $V = \bigcup_{i=1}^{n} D(g_i)$ be a union of finitely may standard opens. Let $\mathscr{F}$ be an $\mathscr{O}_X$-module satisfying the conditions:*

(a) *For any $D(f) \subseteq D(g_i)$ (resp. $D(g_ig_j)$) and any section $s \in \Gamma(D(f), \mathscr{F})$, there exists $n \geq 0$ such that $f^n s$ can be extended to a section of $\mathscr{F}$ on $D(g_i)$ (resp. on $D(g_ig_j)$).*

(b) *For any $D(f) \subseteq D(g_i)$ (resp. $D(g_ig_j)$) and any section $t \in \Gamma(V, \mathscr{F})$ such that $t|_{D(f)} = 0$, there exists $n \geq 0$ such that $f^n t = 0$.*

Then we have the stronger conditions:

(α) *For any $f \in A$ and any section $s \in \Gamma(D(f) \cap V, \mathscr{F})$, there exists $n \geq 0$ such that $f^n s$ can be extended to a section of $\mathscr{F}$ on V.*

(β) *For any $f \in A$ and any section $t \in \Gamma(V, \mathscr{F})$ such that $t|_{D(f)\cap V} = 0$, there exists $n \geq 0$ such that $f^n t = 0$.*

Proof. First we prove condition (β). Since $D(f) \cap D(g_i) = D(fg_i)$, for each i we have an integer $n_i \geq 0$ such that $(fg_i)^{n_i} t$ restricts to zero on $D(g_i)$. Since g_i is invertible in $A_{g_i} = \Gamma(D(g_i), \mathscr{O}_X)$, this implies $f^{n_i} t = 0$ on $D(g_i)$. Take n such that $n \geq n_i$, then $f^n t = 0$ on each $D(g_i)$, whence $f^n t = 0$ and we get (β).

To show (α), we apply (a) on $\mathscr{F}|_{D(g_i)}$ to get an integer integers $n_i \geq 0$ and $s_i' \in \Gamma(D(g_i), \mathscr{F})$ such that

$$s_i'|_{D(fg_i)} = (fg_i)^{n_i} s|_{D(fg_i)}.$$

By inverting g_i, this produces sections $s_i \in \Gamma(D(g_i), \mathscr{F})$ such that

$$s_i|_{D(fg_i)} = f^{n_i} s|_{D(fg_i)}.$$

We may assume that all n_i take the same value n. Then each $s_i - s_j$ restricts to zero on $D(f) \cap D(g_i) \cap D(g_j) = D(fg_ig_j)$, so by applying (b) on $\mathscr{F}|_{D(g_ig_j)}$ we get an integer m_{ij} such that

$$(fg_ig_j)^{m_{ij}}(s_i - s_j)|_{D(g_ig_j)} = 0.$$

Then similarly, since g_ig_j is invertible in $A_{g_ig_j}$, this implies

$$f^{m_{ij}}(s_i - s_j)|_{D(g_ig_j)} = 0.$$

We can also assume that all m_{ij}'s take the same value m, so that by the sheaf condition there exists a section $u \in \Gamma(V, \mathscr{F})$ such that $u|_{D(g_i)} = f^m s_i$. Then $f^n u$ extends $f^{m+n} s$, as desired. □

Theorem 10.1.21. *Let $X = \operatorname{Spec}(A)$ be an affine scheme. Let V be a quasi-compact open subset and $\mathscr{F}$ be an $\mathscr{O}_X|_V$-module. Then the following are equivalent:*

(i) *There is a A-module M such that $\mathscr{F}$ is isomorphic to $\widetilde{M}|_V$.*

(ii) *There exists a finite open covering (V_i) of V by sets of the form $D(f_i)$ ($f_i \in A$) contained in V, such that, for each i, $\mathscr{F}|_{V_i}$ is isomorphic to a sheaf of the form $\widetilde{M}_i$, where M_i is an A_{f_i}-module.*

(iii) *The sheaf $\mathscr{F}$ is quasi-coherent.*

(iv) ***(Serre's lifting criterion)*** *The following conditions are satisfied:*

(a) *For any $D(f) \subseteq V$ and any section $s \in \Gamma(D(f), \mathscr{F})$, there exists $n \geq 0$ such that $f^n s$ can be extended to a section of $\mathscr{F}$ on V.*

(b) *For any $D(f) \subseteq V$ and any section $t \in \Gamma(V, \mathscr{F})$ such that $t|_{D(f)} = 0$, there exists $n \geq 0$ such that $f^n t = 0$.*

Proof. The implication (i)⇒(ii) is immediate from Proposition 10.1.4 since X can be covered by standard opens. Also, since any A-module is isomorphic to the kernel of a homomorphism $A^{\oplus I} \to A^{\oplus J}$, Corollary 10.1.6 shows that (ii)⇒(iii). Conversely, if $\mathscr{F}$ is quasi-coherent, any point $x \in V$ possesses a neighborhood of the form $D(f) \subseteq V$ such that $\mathscr{F}|_{D(f)}$ is isomorphic to the cokernel of a homomorphism $(\widetilde{A}_f)^{\oplus I} \to (\widetilde{A}_f)^{\oplus J}$, hence to the sheaf associated with the cokernel of the corresponding homomorphism $A_f^{\oplus I} \to A_f^{\oplus J}$ (Corollary 10.1.3 and Corollary 10.1.6); since V is quasi-compact, it then follows that (iii) implies (ii).

Now we prove that (ii)⇒(iv). First assume that $V = D(g)$ for some $g \in A$, and $\mathscr{F}$ is isomorphic to $\widetilde{N}$ for some A_g-module N. Since $D(g)$ can be identified with $\mathrm{Spec}(A_g)$, we can assume that $g = 1$ and $V = X$. In this case, the set $\Gamma(D(f), \mathscr{F})$ and N_f are canonically identified (Theorem 10.1.2), and it is clear that conditions (a) and (b) in (iv) are satisfied. To prove the general case, since V is quasi-compact we can choose a finite covering by standard opens $D(g_i)$ with $\mathscr{F}|_{D(g_i)}$ isomorphic to $\widetilde{M}_i$ for some A_{g_i}-module M_i. Then $\mathscr{F}$ satisfies the conditions (a) and (b) in Lemma 10.1.20, so by Lemma 10.1.20, $\mathscr{F}$ also satisfies conditions (α) and (β), which is what we want.

Finally, we show that (iv)⇒(i). First we prove that, if (a) and (b) hold for $\mathscr{F}$, then they hold for $\mathscr{F}|_{D(g)}$ with $D(g) \subseteq V$. This is evident for condition (a); as for (b), if $t \in \Gamma(D(g), \mathscr{F})$ restricts to zero on $D(f) \subseteq D(g)$, then by condition (a) there is an integer $m \geq 0$ such that $g^m t$ can be extended to V. By applying condition (b) on the extension of $g^m t$, we get another integer $n \geq 0$ such that $f^n g^m t = 0$. Since g is invertible in A_g, this gives $f^n t = 0$ as desired.

This being done, since V is quasi-compact, by Lemma 10.1.20 we know that conditions (α) and (β) holds for $\mathscr{F}$. Now consider the module $M = \Gamma(V, \mathscr{F})$; we shall define a morphism $\varphi : \widetilde{M} \to j_*\mathscr{F}$, where $j : V \hookrightarrow X$ is the inclusion. For this, it suffices to define

$$\varphi_f : M_f \to \Gamma(D(f), j_*\mathscr{F}) = \Gamma(D(f) \cap V, \mathscr{F})$$

for each $f \in A$. Since f is invertible in A_f and $\Gamma(D(f) \cap V, \mathscr{F})$ is a A_f-module, the restriction $M = \Gamma(V, \mathscr{F}) \to \Gamma(D(f) \cap V, \mathscr{F})$ factors into

$$M \longrightarrow M_f \xrightarrow{\varphi_f} \Gamma(D(f) \cap V, \mathscr{F})$$

which gives the desired maps φ_f. We now claim that the conditions (a) and (b) in (iv) imply that φ is an isomorphism. In fact, if $s \in \Gamma(D(f) \cap V, \mathscr{F})$, then by condition (a) there exist an integer $n \geq 0$ and $z \in \Gamma(V, \mathscr{F}) = M$ such that $z|_{D(f) \cap V} = f^n s$; then $\varphi_f(z/f^n) = s$, showing that φ is surjective. Similarly, if there is $z \in M$ such that $\varphi_f(z/1) = 0$ in $D(f) \cap V$, then by condition (b) there is $n \geq 0$ such that $f^n z = 0$, so that $z/1 = 0$ in M_f. This means φ_f is injective, so we get an isomorphism $\widetilde{M} \cong j_*\mathscr{F}$. By restriction, we then conclude that $\mathscr{F} \cong \widetilde{M}|_V$. □

Corollary 10.1.22. *Let $X = \mathrm{Spec}(A)$ be an affine scheme. Then the functors $M \mapsto \widetilde{M}$ and $\mathscr{F} \mapsto \Gamma(X, \mathscr{F})$ define equivalences of categories between the category of quasi-coherent $\mathscr{O}_X$-modules and the category of A-modules.*

Proof. The space X it self is quasi-compact, so we can apply Theorem 10.1.21. □

Corollary 10.1.23. *Let $X = \mathrm{Spec}(A)$ be an affine scheme. Then kernels and cokernels of maps of quasi-coherent $\mathscr{O}_X$-modules are quasi-coherent.*

Proof. This follows from the exactness of the functor $\widetilde{M}$ and Corollary 10.1.6. □

Corollary 10.1.24. *For $\widetilde{M}$ to be a $\mathscr{O}_X$-module of finite type (resp. of finite presentation), it is necessary and sufficient that M is a finitely generated A-module (resp. of finite presentation).*

Proof. In view of the exactness of the functor $M \mapsto \widetilde{M}$, it is immediate that if M is of finite type (resp. finite presentation), so is $\widetilde{M}$. Conversely, if $\widetilde{M}$ is of finite type (resp. finite presentation), since X is quasi-compact, there exists finitely many $f_i \in A$ such that $D(f_i)$ cover X and M_{f_i} is of finite type (resp. finite presentation) over A_{f_i}. It then follows from Proposition 1.4.53 that M is of finite type (resp. finite presentation). □

Corollary 10.1.25. *For an A-module M, the $\mathscr{O}_X$-module is locally free of finite rank if and only if M is a finitely generated projective A-module.*

Proof. Since X is quasi-compact, this follows from Theorem 1.5.5. □

Corollary 10.1.26. *Let $X = \operatorname{Spec}(A)$ be an affine scheme. Then any quasi-coherent $\mathscr{O}_X$-algebra over X is isomorphic to an $\mathscr{O}_X$-algebra of the form $\widetilde{B}$, where B is an algebra over A. Moreover, any quasi-coherent $\widetilde{B}$-module is isomorphic to a $\widetilde{B}$-module of the form $\widetilde{N}$, where N is a B-module.*

Proof. In fact, a quasi-coherent $\mathscr{O}_X$-algebra is a quasi-coherent $\mathscr{O}_X$-module, hence of the form $\widetilde{B}$, where B is an A-module. The fact that B is an A-algebra follows from the structural morphism $\widetilde{B} \otimes_{\mathscr{O}_X} \widetilde{B} \to \widetilde{B}$ of $\mathscr{O}_X$-modules, which induces an A-algebra map $B \otimes_A B \to B$.

If $\mathscr{G}$ is a quasi-coherent $\widetilde{B}$-module, it suffices to show that $\mathscr{G}$ is also a quasi-coherent $\mathscr{O}_X$-module to then conclude in the same way. As the question is local, we can, by restricting ourselves to an open set of X of the form $D(f)$, over which $\mathscr{G}$ is the cokernel of a morphism $\widetilde{B}^{\oplus I} \to \widetilde{B}^{\oplus J}$ of $\widetilde{B}$-modules (and a fortiori $\mathscr{O}_X$-modules). The claim then follows from Corollary 10.1.3 and Corollary 10.1.6. □

Proposition 10.1.27. *Let $X = \operatorname{Spec}(A)$ be an affine scheme, Suppose that*

$$0 \longrightarrow \mathscr{F}_1 \longrightarrow \mathscr{F}_2 \longrightarrow \mathscr{F}_3 \longrightarrow 0$$

is a short exact sequence of $\mathscr{O}_X$-modules, and assume that $\mathscr{F}_1$ is quasi-coherent. Then the sequence

$$0 \longrightarrow \Gamma(X, \mathscr{F}_1) \longrightarrow \Gamma(X, \mathscr{F}_2) \longrightarrow \Gamma(X, \mathscr{F}_3) \longrightarrow 0$$

is exact.

Proof. We know already that Γ is a left-exact functor so we have only to show that the last map is surjective (which we denot by $\psi : \Gamma(X, \mathscr{F}_2) \to \Gamma(X, \mathscr{F}_3)$). Let $s \in \Gamma(X, \mathscr{F}_3)$ be a global section. Since the morphism $\mathscr{F}_2 \to \mathscr{F}_3$ is surjective, for any $x \in X$ there is an open neighborhood $D(f)$ of x such that

$$s|_{D(f)} = \psi(t)$$

where $t \in \mathscr{F}_2(D(f))$. We claim that for some $n > 0$, $f^n s = \psi(u)$ for some $u \in \Gamma(X, \mathscr{F}_2)$. Indeed, we can cover X with a finite number of open sets $D(g_i)$ such that for each i, $s|_{D(g_i)} = \psi(t_i)$ for a section $t_i \in \mathscr{F}_2(D(g_i))$. Then by the exactness of the original sequence, on $D(f) \cap D(g_i) = D(fg_i)$ we have

$$(t - t_i)|_{D(fg_i)} \in \mathscr{F}_1(D(fg_i))$$

where we identify $\mathscr{F}_1$ as the kernel of ψ. Since $\mathscr{F}_1$ is quasi-coherent, by Corollary 10.1.22, there is an integer $n \geq 0$ such that the $f^n(t - t_i)|_{D(fg_i)}$ can be extended to a section $u_i \in \mathscr{F}_1(D(g_i))$. Let

$$\tilde{t}_i = f^n t_i + u_i \in \mathscr{F}_2(D(g_i)).$$

Then $\tilde{t}_i|_{D(fg_i)} = f^n t_i|_{D(fg_i)} + f^n(t - t_i)|_{D(fg_i)} = f^n t|_{D(fg_i)}$ and we have

$$f^n s|_{D(g_i)} = f^n \psi(t_i) = \psi(\tilde{t}_i - u_i) = \psi(\tilde{t}_i). \tag{10.1.1}$$

Now on $D(g_ig_j)$ the two sections $\tilde{t}_i$ and $\tilde{t}_j$ of $\mathscr{F}_2$ are mapped to $f^n s|_{D(g_ig_j)}$ by ψ, so $\tilde{t}_i - \tilde{t}_j \in \mathscr{F}_1(D(g_ig_j))$. Furthermore, since $\tilde{t}_i$ and $\tilde{t}_j$ are both equal to $f^n t|_{D(fg_ig_j)}$ on $D(fg_ig_j)$, by Corollary 10.1.22 there exists $m \geq 0$ such that $f^m(\tilde{t}_i - \tilde{t}_j) = 0$ on $D(g_ig_j)$, which we may take to be independent of i and j. Then the sections $f^m\tilde{t}_i$ glue to give a global section of $\mathscr{F}_2$ over X, which lifts $f^{m+n}s$ by (10.1.1). This proves the claim.

Now cover X by a finite number of open sets $D(f_i)$ such that $s|_{D(f_i)}$ lifts to a section of $\mathscr{F}_2$ over $D(f_i)$ for each i. Then by the previous proof, we can find an integer $n \geq 0$ (one for all i) and global sections $t_i \in \Gamma(X, \mathscr{F}_2)$ such that $\psi(t_i) = f^n s$. Since the open sets $D(f_i)$ cover X, the ideal $(f_1^n, \dots, f_r^n)$ is the unit ideal of A, and we can write $1 = \sum a_i f_i^n$, with $a_i \in A$. Let $t = \sum a_i t_i$. Then t is a global section of $\mathscr{F}_2$ whose image under ψ is $\sum a_i f_i^n s = s$. □

Proposition 10.1.28. *Let $X = \operatorname{Spec}(A)$ be an affine scheme. Then the direct sum of an arbitrary collection of quasi-coherent sheaves on X is quasi-coherent. More generally, colimits of quasi-coherent sheaves are quasi-coherent.*

Proof. Let $\{\mathscr{F}_i\}_{i\in I}$ be a collection of quasi-coherent sheaves on X. By Theorem 10.1.21 we can write $\mathscr{F}_i = \widetilde{M}_i$ for A-modules M_i, so the assertion follows from Corollary 10.1.6. □

Proposition 10.1.29. *Let X be an affine scheme. Suppose that*

$$0 \longrightarrow \mathscr{F}_1 \longrightarrow \mathscr{F}_2 \longrightarrow \mathscr{F}_3 \longrightarrow 0$$

is a short exact sequence of sheaves $\mathscr{O}_X$-modules. If two out of three are quasi-coherent then so is the third.

Proof. The statement about kernels and cokernels follows from the fact that the functor $M \mapsto \widetilde{M}$ is exact and fully faithful from A-modules to quasi-coherent sheaves. Now let $\mathscr{F}_1$ and $\mathscr{F}_3$ be quasi-coherent. By Proposition 10.1.27, the corresponding sequence of global sections over X is exact, say $0 \to M_1 \to M_2 \to M_3 \to 0$. Applying the functor $\widetilde{M}$ we get an exact commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \widetilde{M}_1 \arrow[r] \arrow[d] & \widetilde{M}_2 \arrow[r] \arrow[d] & \widetilde{M}_3 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathscr{F}_1 \arrow[r] & \mathscr{F}_2 \arrow[r] & \mathscr{F}_3 \arrow[r] & 0
\end{tikzcd}$$

The two outside arrows are isomorphisms, since $\mathscr{F}_1$ and $\mathscr{F}_3$ are quasi-coherent. So by the five lemma, the middle one is also, showing that $\mathscr{F}_2$ is quasi-coherent. □

Theorem 10.1.30. *Let $X = \operatorname{Spec}(A)$ be an affine scheme where A is Noetherian. Let V be an open subset of X and $\mathscr{F}$ an $\mathscr{O}_X|_V$-module. Then the following conditions are equivalent:*

(i) *$\mathscr{F}$ is coherent.*

(ii) *$\mathscr{F}$ is of finite type and quasi-coherent.*

(iii) *There exists a finitely generated A-moduel M such that $\mathscr{F} \cong \widetilde{M}|_V$.*

Proof. It is clear that (i) implies (ii). To show (ii) implies (iii), we note that V is quasi-compact since X is Noetherian, so by Theorem 10.1.21, $\mathscr{F}$ is isomorphic to $\widetilde{M}|_V$, where M is an A-module. Now we have $M = \varinjlim M_\lambda$, where M_λ is the set of finitely generated sub-A-modules of M. Since the functor $\widetilde{(-)}$ is exact, this implies $\mathscr{F} = \widetilde{N}|_V = \varinjlim \widetilde{M_\lambda}|_V$. But $\mathscr{F}$ is of finite type and V is quasi-compact, so by **??** there exists an index λ such that $\mathscr{F} = \widetilde{M_\lambda}|_V$ (note that the canonical homomorphism $\widetilde{M}_\lambda \to \widetilde{M}$ is injective). This proves (iii).

It remains to show that $\widetilde{M}|_V$ is coherent if M is finitely generated. Since $\mathscr{F}$ is clearly of finite type, it suffices to show that for every open $U \subseteq X$ and $s_1, \dots, s_n \in \mathscr{F}(U)$, the associated map $\bigoplus_{i=1}^n \mathscr{O}_U \to \mathscr{F}|_U$ is of finite type. Since the question is local, we may assume $V = D(f)$ for $f \in A$. Then it suffices to show the kernel of a morphism $\bigoplus_{i=1}^n \widetilde{A}_f \to \widetilde{M}$ is of finite type. But this morphism corresponds to a homomorphism $A_f^n \to M$, whose kernel is finitely generated since A_f is Noetherian, so the claim follows. □

Corollary 10.1.31. *Let $X = \operatorname{Spec}(A)$ be an affine scheme where A is Noetherian. Then any quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$ is the inductive limit of coherent $\mathscr{O}_X$-modules.*

Proof. We have $\mathscr{F} = \widetilde{M}$ for an A-module M, and M is the inductive limit of its finitely generated submodules. □

Corollary 10.1.32. *Let $X = \operatorname{Spec}(A)$ be an affine scheme where A is Noetherian. Then the functor $M \mapsto \widetilde{M}$ gives an equivalence of categories between the category of finite generated A-modules and the category of coherent $\mathscr{O}_X$-modules.*

10.2 General schemes

10.2.1 Schemes and morphisms of schemes

Let $(X, \mathscr{O}_X)$ be a ringed space. An oepn subset V of X is said to be **affine** if the ringed space $(V, \mathscr{O}_X|_V)$ is an affine scheme (i.e. isomorphic to the spectrum of a ring). We say $(X, \mathscr{O}_X)$ is a **scheme** if every point of X admits an affine open neighborhood. If $(X, \mathscr{O}_X)$ is a scheme, then affine open subsets of X form a basis for X (because the standard opens form a basis for a spectrum $\operatorname{Spec}(A)$, and they are again affine), and in particular $(X, \mathscr{O}_X)$ is a locally ringed space. With this, for any open subset U of X, the ringed space $(U, \mathscr{O}_X|_U)$ is also a scheme, called the scheme **induced** on U by X, or the **restriction** of $(X, \mathscr{O}_X)$ on U.

Proposition 10.2.1. *The underlying space of a scheme is Kolmogoroff.*

Proof. In fact, if x and y are two points of a scheme X, then it is obvious that there exists an open neighborhood of one of these points not containing the other if x, y are not not in a same open affine; and if they are in the same open affine, this follows from the fact that the underlying spaces of affine schemes are Kolmogoroff (Corollary 1.4.12). □

Proposition 10.2.2. *If $(X, \mathscr{O}_X)$ is a scheme, any irreducible closed subset of X admits a unique generic point, and the map $x \mapsto \overline{\{x\}}$ is a bijection of X to the family of irreducible closed subsets of X.*

Proof. Let Y is an irreducible closed subset of X and $y \in Y$. If U is an affine open neighborhood of y in X, then $U \cap Y$ is dense in Y and is irreducible (**??**), so it is the closure in U of a point $x \in U$, and therefore $Y \subseteq \bar{U}$ is the closure of x in X. The uniqueness of the generic point of X follows from Proposition 10.2.1 and ([**?**] 0_{I}, 2.1.3). □

If Y is an irreducible closed subset of X and y its generic point, the local ring $\mathscr{O}_{X,y}$ is then denoted by $\mathscr{O}_{X,Y}$ and called the **local ring of X along Y**, or the **local ring of Y in X**. We say a scheme $(X, \mathscr{O}_X)$ is **irreducible** (resp. **connected**) if the underlying space X is irreducible (resp. connected), and **integral** if it is irreducible and reduced. We say the scheme $(X, \mathscr{O}_X)$ is **locally integral** if each point $x \in X$ admits an open neighborhoods U such that the scheme induced over U by $(X, \mathscr{O}_X)$ is integral. If X is an irreducible scheme and x is its generic point, the local ring $\mathscr{O}_{X,x}$ is called **the ring of rational functions on X**.

Let $(X, \mathscr{O}_X)$ and $(Y, \mathscr{O}_Y)$ be schemes. A **morphism** (of schemes) from $(X, \mathscr{O}_X)$ to $(Y, \mathscr{O}_Y)$ is defined to be that of locally ringed space. That is, a pair $(f, f^\#)$ such that for each $x \in X$, the homomorphism $f_x^\#$ is local. In this case, by passing to quotients, $f_x^\#$ induces a monomorphism $f^x : \kappa(f(x)) \to \kappa(x)$, so $\kappa(x)$ can be considered as an extension of the field $\kappa(f(x))$.

The composition of two morphisms of schemes is defined in the same way with that of locally ringed spaces, and we then see that schemes form a category, denoted by **Sch**. Following the general notation, we denote by $\mathrm{Hom}_{\mathbf{Sch}}(X,Y)$ the set of morphisms from a scheme X to a scheme Y.

Example 10.2.3. Let U be an open subset of X. Then the canonical injection of $(U, \mathscr{O}_X|_U)$ to $(X, \mathscr{O}_X)$ is a morphism of schemes; it is moreover a monomorphism of ringed spaces (and a fortiori a monomorphism of schemes).

Proposition 10.2.4. *Let $(X, \mathscr{O}_X)$ be a scheme and $(Y, \mathscr{O}_Y)$ be an affine scheme. Then there exists a canonical bijection*

$$\mathrm{Hom}_{\mathbf{Sch}}(X,Y) \to \mathrm{Hom}_{\mathbf{Ring}}(\Gamma(X,\mathscr{O}_X), \Gamma(X,\mathscr{O}_X)).$$

Proof. Let $A = \Gamma(Y, \mathscr{O}_Y)$. Note first that, if $(X, \mathscr{O}_X)$ and $(Y, \mathscr{O}_Y)$ are any two ringed spaces, a morphism $(\psi, \psi^\#) : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ canonically defines a homomorphism of rings

$$\Gamma(Y, \mathscr{O}_Y) \to \Gamma(Y, \psi_*(\mathscr{O}_X)) = \Gamma(X, \mathscr{O}_X).$$

It then remains to see that any homomorphism $\rho : A \to \Gamma(X, \mathscr{O}_X)$ is of the form $\Gamma(\psi^\#)$ for a unique morphism $(\psi, \psi^\#)$. However, there is by hypothesis a covering (V_α) of X by affine open sets. By considering the composition

$$A \xrightarrow{\rho} \Gamma(X, \mathscr{O}_X) \longrightarrow \Gamma(V_\alpha, \mathscr{O}_X|_{V_\alpha})$$

we obtain a homomorphism $\rho_\alpha : A \to \Gamma(V_\alpha, \mathscr{O}_X|_{V_\alpha})$, which corresponds to a morphism $(\psi_\alpha, \psi_\alpha^\#)$ from the scheme $(V_\alpha, \mathscr{O}_X|_{V_\alpha})$ to $(Y, \mathscr{O}_Y)$ (Theorem 10.1.17). Moreover, for each pair (α, β) of indices, any point of $V_\alpha \cap V_\beta$ admits an affine open neighborhood W contained in $V_\alpha \cap V_\beta$; it is clear that by composing ρ_α and ρ_β with the restriction homomorphism to W, we obtain the same homomorphism $A \to \Gamma(W, \mathscr{O}_X|_W)$, so, by virtue of the relation $(\psi_\alpha^\#)_x = (\rho_\alpha)_x$ for any $x \in V_\alpha$ and any α, the restrictions of $(\psi_\alpha, \psi_\alpha^\#)$ and $(\psi_\beta, \psi_\beta^\#)$ to W coincide. By gluing we then get a unique morphism $(\psi, \psi^\#) : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ which extending $(\psi_\alpha, \psi_\alpha^\#)$ on each V_α. It is clear that $(\psi, \psi^\#)$ is a morphism of schemes, and we have $\Gamma(\psi^\#) = \rho$. □

Remark 10.2.5. Let $\rho : A \to \Gamma(X, \mathscr{O}_X)$ be a ring homomorphism, and let $(\psi, \psi^\#)$ be the corresponding morphism $(X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$. For each $f \in A$, we have

$$\psi^{-1}(D(f)) = \{x \in X : f \notin \mathfrak{m}_{\psi(x)}\} = \{x \in X : (\rho(x))_x \notin \mathfrak{m}_x\} = X_{\rho(f)}.$$

Note that this can be viewed as a generalization of Proposition 1.4.20(a).

Proposition 10.2.6. *Under the hypothesis of Proposition 10.2.4, let $\rho : A \to \Gamma(X, \mathscr{O}_X)$ be a ring homomorphism and $f : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ the corresponding morphism of schemes. Let $\mathscr{F}$ (resp. $\mathscr{G}$) be a quasi-coherent $\mathscr{O}_X$-module (resp. $\mathscr{O}_Y$-module). Then there exist a canonical bijection*

$$\mathrm{Hom}_{\mathbf{Qcoh}(Y)}(\mathscr{G}, f_*(\mathscr{F})) \to \mathrm{Hom}_A(\Gamma(Y, \mathscr{G}), \rho^*(\Gamma(X, \mathscr{F}))).$$

Proof. Indeed, by reasoning as in Proposition 10.2.4, we are immediately reduced to the case where X is affine and the proposition then follows from Corollary 10.1.3 and Proposition 10.1.12. □

We say a morphism $f : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ is **open** (resp. **closed**) if for any open subset U of X (resp. any closed subset F of X), $f(U)$ is open in Y (resp. $f(F)$ is closed in Y). We say f is **dominant** if $f(X)$ is dense in Y, and **surjective** if f is surjective. It should be noted that these conditions only involve the continuous map f.

Proposition 10.2.7. *Let $f : (X, \mathscr{O}_X)$ and $g : (Y, \mathscr{O}_Y) \to (Z, \mathscr{O}_Z)$ be morphisms of schemes.*

(a) *If f and g are open (resp. closed, dominant, surjective), so is the composition $g \circ f$.*

(b) *If f is surjective and $g \circ f$ is closed, g is closed.*

(c) *If $g \circ f$ is surjective, g is surjective.*

Proof. The assertions (a) and (c) are evident. Put $h = g \circ f$. If F is closed in Y, then $f^{-1}(F)$ is closed in X, so $h(f^{-1}(F))$ is closed in Z. But since f is surjective, we have $f(f^{-1}(F)) = F$, so $h(f^{-1}(F)) = g(F)$, which shows g is closed. □

Proposition 10.2.8. *Let $f : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ be a morphism of schemes and (U_α) an open covering of Y. For f to be open (resp. closed, surjective, dominant), it is necessary and sufficient that for each U_α, the restrictions $f : f^{-1}(U_\alpha) \to U_\alpha$ is open (resp. closed, surjective, dominant).*

Proof. The proposition follows immediately from the definitions, taking into account the fact that a subset F of Y is closed (resp. open, dense) in Y if and only if each of the sets $F \cap U_\alpha$ is closed (resp. open, dense) in U_α. □

Let $(X, \mathscr{O}_X)$ and $(Y, \mathscr{O}_Y)$ be schemes; suppose that X and Y have a same finite number of irreducible components X_i (resp. Y_i) ($1 \leq i \leq n$); let ξ_i (resp. η_i) be the generic point of X_i (resp. Y_i). We say that a morphism

$$f : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$$

is **birational** if, for any i, we have $f^{-1}(\eta_i) = \{\xi_i\}$ and $f^{\#}_{\xi_i} : \mathscr{O}_{Y,\eta_i} \to \mathscr{O}_{X,\xi_i}$ is an isomorphism. It is clear that any birational morphism is dominant, hence surjective if it is closed.

Remark 10.2.9. Throughout the remainder of this chapter and when there is no risk of creating confusion, we will omit in the notation of a scheme (resp. of a morphism) the structural sheaf (resp. the morphism of structural sheaf). If U is an open subset of the underlying space of a scheme X, when we speak of U as of a scheme, it will always be the scheme induced on U by X.

With the morphisms of schemes defined, we can talk about glueing schemes as in the case of ringed spaces. It follows immediately from the definition that any ringed space obtained by glueing schemes is again a scheme. In particular, since any scheme admits a basis of affine open subsets, we see any scheme is obtained by glueing affine schemes.

Example 10.2.10. Cosnider a field K, $A = K[s]$, $B = K[t]$ be two rings of polynomials over K with one indeterminate, and $X_1 = \mathrm{Spec}(A)$, $X_2 = \mathrm{Spec}(B)$. In X_1 (resp. X_2), let U_{12} (resp. U_{21}) be the affine open set $D(s)$ (resp. $D(t)$), whose ring A_s (resp. B_t) is formed by the rational fractions of the form $f(s)/s^m$ (resp. $g(t)/t^n$) with $f \in A$ (resp. $g \in B$). Let φ_{12} be the isomorphism of of schemes $U_{21} \to U_{12}$ corresponding to the isomorphism of A and B such that, $f(s)/s^m$ is mapped to the rational fraction $f(1/t)/(1/t^m)$ (i.e. we map s to $1/t$). We can then glue X_1 and X_2 along U_{12} and U_{21} by the isomorphism u_{12}, which evidently satisfies the glueing condition. We will see later the schem X thus obtained is a particular case of a general method of construction. We only show here that X is not an affine scheme, which will result from the fact that the ring $\Gamma(X, \mathscr{O}_X)$ is isomorphic to K, therefore has a spectrum reduced to a point. Indeed, a section of $\mathscr{O}_X$ above X has a restriction over X_1 (resp. X_2), identified with an open affine of X, which is a polynomial $f(s)$ (resp. $g(t)$), and it follows from the definition of u_{12} that we must have $g(t) = f(1/t)$, which is not possible only if $f = g \in K$.

10.2.2 Local schemes

Let X be a scheme and $A = \Gamma(X, \mathscr{O}_X)$. We say X is a local scheme if X is affine and the ring X is local. In this case, there then exists a unique closed point ξ in X, and for any point $x \in X$ we have $\xi \in \overline{\{x\}}$.

Following this notation, for a general scheme Y and $y \in Y$, the scheme $\mathrm{Spec}(\mathscr{O}_{Y,y})$ is called the **local scheme of Y at y**. Let V be an affine open subset of Y containing y, and B the ring of V. The local ring $\mathscr{O}_{Y,y}$ is then canonically identified with B_y, and the canonical homomorphism $B \to B_y$ then induces a morphism $\mathrm{Spec}(\mathscr{O}_{Y,y}) \to V$ of schemes. If we compose this with the canonical injection of V into Y, we then get a morphism $\mathrm{Spec}(\mathscr{O}_{Y,y}) \to Y$, which is independent of the choice of the affine open V containing y: in fact, if U is another affine neighborhood of y, there exists an affine open neighborhood W of y contained in $U \cap V$; we can then limit ourselves to the case $U \subseteq V$, and if A is the ring of U, we have a commutative diagram

$$\begin{tikzcd} A \arrow[rr] \arrow[dr] & & B \arrow[dl] \\ & \mathscr{O}_{Y,y} & \end{tikzcd}$$

The morphism $\mathrm{Spec}(\mathscr{O}_{Y,y}) \to Y$ thus defined is said to be **canonical**.

Proposition 10.2.11. *Let Y be a scheme, $y \in Y$, and $f : \mathrm{Spec}(\mathscr{O}_{Y,y}) \to Y$ be the canonical morphism.*

(a) *The map f is a homeomorphism of $\mathrm{Spec}(\mathscr{O}_{Y,y})$ onto the subspace S_y of points $z \in Y$ such that $y \in \overline{\{z\}}$ (i.e. the set of generalizations of y).*

(b) *For each $\mathfrak{p} \in \mathrm{Spec}(\mathscr{O}_{Y,y})$, the homomorphism $f^\#_{\mathfrak{p}} : \mathscr{O}_{Y,f(\mathfrak{p})} \to (\mathscr{O}_{Y,y})_{\mathfrak{p}}$ is an isomorphism.*

In particular, f is a monomorphism of locally ringed spaces.

Proof. Since the unique closed point η of $\mathrm{Spec}(\mathscr{O}_{Y,y})$ belongs to the closure of any other point in this space, and $f(\eta) = y$, the image of $\mathrm{Spec}(\mathscr{O}_{Y,y})$ by the continuous map f is contained in S_y. As S_y is contained in any affine open neighborhood of y, we can reduce to the case where Y is an affine scheme; but in this case the proposition follows immediately. □

Corollary 10.2.12. *There is a bijective correspondence between $\mathrm{Spec}(\mathscr{O}_{Y,y})$ and irreducible closed subsets of Y containing y.*

Proof. This follows from Proposition 10.2.11 and the fact that every irreducible closed set in Y has a unique generic point. □

Corollary 10.2.13. *For a point $y \in Y$ to be the generic point of an irreducible component of Y, it is necessary and suffcient that $\mathscr{O}_{Y,y}$ is zero-dimensional.*

Proof. This follows from the observation that y is the generic point of an irreducible component if and only if it is a maximal element under generalization, which is then equivalent by Corollary 10.2.12 to that $\mathrm{Spec}(\mathscr{O}_{Y,y})$ is a singleton. □

Proposition 10.2.14. *Let $(X, \mathscr{O}_X)$ be a local scheme with $A = \Gamma(X, \mathscr{O}_X)$, ξ its unique closed point, and $(Y, \mathscr{O}_Y)$ a scheme. Then any morphism $f : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ factors through $\mathrm{Spec}(\mathscr{O}_{Y,f(\xi)})$:*

$$f : X \to \mathrm{Spec}(\mathscr{O}_{Y,f(\xi)}) \to Y$$

where the second one is the canonical morphism, and the first one corresponds to a local homomorphism $\mathscr{O}_{Y,f(\xi)} \to A$.

Proof. In fact, for any $x \in X$, we have $\xi \in \overline{\{x\}}$, hence $f(\xi) \in \overline{\{f(x)\}}$. It then follows that $f(X)$ is contained in any affine open neighborhood of $f(\xi)$ (in fact any open neighborhood of $f(\xi)$). We can then reduce to the case that $(Y, \mathscr{O}_Y)$ is an affine scheme with ring $B = \Gamma(Y, \mathscr{O}_Y)$, and the morphism f corresponds to a ring homomorphism $\rho : B \to A$. We have $\rho^{-1}(\mathfrak{p}_\xi) = \mathfrak{p}_{f(\xi)}$, so the image under ρ of an element of $B - \mathfrak{p}_{f(\xi)}$ is invertible in the local ring A, and we get a canonical homomorphism $\rho_\xi : B_{f(\xi)} \to A$. □

Corollary 10.2.15. *There is a canonical bijection between* $\mathrm{Hom}_{\mathbf{Sch}}(X, Y)$ *to the set of local homomorphisms* $\mathscr{O}_{Y,y} \to A$*, where* $y \in Y$*.*

Proof. It suffices to note that any local homomorphism $\mathscr{O}_{Y,y} \to A$ corresponds to a unique morphism $f : X \to \mathrm{Spec}(\mathscr{O}_{Y,y})$ such that $f(\xi) = y$, and by composing this with the canonical morphism $\mathrm{Spec}(\mathscr{O}_{Y,y}) \to Y$, we get a morphism $X \to Y$. □

Example 10.2.16. The affine scheme whose ring is a field K have an underlying space reduced to one point. If A is a local ring of maximal ideal $\mathfrak{m}$, any local homomorphism $A \to K$ has a kernel equal to $\mathfrak{m}$, so factors into $A \to A/\mathfrak{m} \to K$, where the second arrow is a monomorphism. The morphisms $\mathrm{Spec}(K) \to \mathrm{Spec}(A)$ correspond therefore bijectively to the field extensions $A/\mathfrak{m} \to K$.

Let $(Y, \mathscr{O}_Y)$ be a scheme; for any $y \in Y$ and any ideal $\mathfrak{a}_y$ of $\mathscr{O}_{Y,y}$, the canonical homomorphism $\mathscr{O}_{Y,y}/\mathfrak{a}_y$ defines a morphism $\mathrm{Spec}(\mathscr{O}_{Y,y}/\mathfrak{a}_y) \to \mathrm{Spec}(\mathscr{O}_{Y,y})$; by composing this with the canonical morphism $\mathrm{Spec}(\mathscr{O}_{Y,y}) \to Y$, we obtain a morphism $\mathrm{Spec}(\mathscr{O}_{Y,y}/\mathfrak{a}_y) \to Y$, also called canonical. If $\mathfrak{a}_y = \mathfrak{m}_y$ is the maximal ideal of $\mathscr{O}_{Y,y}$, then $\mathscr{O}_{Y,y} = \kappa(y)$ and Corollary 10.2.15 then imply the following result:

Corollary 10.2.17. *Let* $(X, \mathscr{O}_X)$ *be a local scheme with* $K = \Gamma(X, \mathscr{O}_X)$ *a field,* ξ *its unique point, and* $(Y, \mathscr{O}_Y)$ *be a scheme. Then any morphism* $f : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ *factors into*

$$f : X \to \mathrm{Spec}(\kappa(f(\xi))) \to Y$$

where the second arrow is the canonical morphism, and the first arrow corresponds to a field extension $\kappa(f(\xi)) \to K$*. This establishes a canonical bijection between* $\mathrm{Hom}_{\mathbf{Sch}}(X, Y)$ *to the set of field extensions* $\kappa(y) \to K$*, where* $y \in Y$*.*

Corollary 10.2.18. *For any* $y \in Y$*, the canonical morphism* $\mathrm{Spec}(\mathscr{O}_{Y,y}/\mathfrak{a}_y) \to Y$ *is a monomorphism of locally ringed spaces.*

Proof. This follows from Proposition 10.2.11 and Corollary 10.1.19. □

Remark 10.2.19. Let X be a local scheme, ξ its unique closed point. Since any affine open neighborhood of ξ is necessarily all of X, any invertible $\mathscr{O}_X$-module is necessarily isomorphic to $\mathscr{O}_X$ (in other words, is trivial). This property does not hold in general for any affine scheme $\mathrm{Spec}(A)$, but we will see that if A is a normal ring, this is true when A is factorial.

10.2.3 Shemes over a scheme

As in any category, for a scheme S we can define the category $\mathbf{Sch}/S$ of S-objects in the category of schemes, which will be a morphism $\varphi : X \to S$ where X is a scheme. In this case we also say that X is a **scheme over S**, or an **S-scheme**. We say that S is the **base scheme** of the S-scheme X and φ is called the structural morphism of the S-scheme X. When S is an affine scheme of the ring A, we also say that X is a **scheme over A** or an **A-scheme**.

It follows from Proposition 10.2.4 that giving a scheme over a ring A is equivalent to giving a scheme $(X, \mathscr{O}_X)$, where $\mathscr{O}_X$ is an A-algebra. In particular, any scheme can be considered as a scheme over $\mathbb{Z}$. In other words, the scheme $\mathrm{Spec}(\mathbb{Z})$ is a final object in the category of schemes (also a final object in the category of locally ringed spaces).

If $\varphi : X \to S$ is the structural morphism of an S-scheme X, we say a point $x \in X$ is **lying over** a point $s \in S$ if $\varphi(x) = s$. We say X **dominates** S if the morphism φ is dominant. Let X and Y be two S-schemes; a morphism $u : X \to Y$ is called a **morphism of schemes over S** (or **S-morphism**) if the diagram

$$\begin{tikzcd} X \arrow[rr, "u"] \arrow[dr] & & Y \arrow[dl] \\ & S & \end{tikzcd}$$

is commutative; in other words, if for any $s \in S$ and any $x \in X$ lying over s, the point $u(x)$ is lying over s. This definition immediately shows that the composite of two S-morphisms is an S-morphism, so the S-schemes thus form a category. We denote by $\mathrm{Hom}_S(X, Y)$ the set of S-morphisms from an S-scheme X to an S-scheme Y; the identity morphism of an S-scheme X is then denoted by 1_X or id_X. If S is an affine scheme S, we also say A-morphisms for S-morphisms.

If X is an S-scheme, $\varphi : X \to S$ the structural morphism, an **S-section** of X is defined to be an S-morphism of S to X, which is equivalently a morphism $\psi : S \to X$ of schemes such that $\varphi \circ \psi = \mathrm{id}_S$. We denote by $\Gamma(X/S)$ the set of S-sections of X.

Example 10.2.20. If X is an S-scheme and $v : X' \to X$ a morphism of schemes, the composition scheme

$$X' \xrightarrow{v} X \longrightarrow S$$

then defines X' as an S-scheme; in particular, any scheme induceed over an open subset U of X can be considered as an S-scheme by means of the canonical injection.

Example 10.2.21. Let $u : X \to Y$ be an S-morphism of S-schemes, the restriction of u on any open subset U of X is then an S-morphism $U \to Y$. Conversely, let (U_α) be a covering of X and for each α, let $u_\alpha : U_\alpha \to Y$ be an S-morphism; if for any pair (α, β) of indices, the restrictions of u_α and u_β on $U_\alpha \cap U_\beta$ coincide, then there exists a unique S-morphism $X \to Y$ whose restriction on U_α equals to u_α.

Let $S \to S'$ be a morphism of schemes; for any S'-scheme X, the composition morphism $X \to S' \to S$ then defines X as an S-scheme. Conversely, suppose that S' is the scheme induced over an open subset of S; let X be an S-scheme and suppose that the structural morphism $X \to S$ has image contained in S'; then we can consider X as an S'-scheme. In the latter case, if Y is an S-scheme whose structural morphism also maps the underlying space in S', any S-morphism from X in Y is also an S'-morphism.

10.2.4 Quasi-coherent sheaves on schemes

Proposition 10.2.22. *Let X be a scheme. For an $\mathscr{O}_X$-module $\mathscr{F}$ to be quasi-coherent, it is necessary and sufficient that, for any affine open subset V of X, the restriction $\mathscr{F}|_V$ is isomorphism to the sheaf associated with a $\Gamma(V, \mathscr{O}_X)$-module.*

Proof. We recall that being quasi-coherent is a local property, and affine opens form a basis for X. Also, by Theorem 10.1.21, a quasi-coherent sheaf on an affine open V is isomorphic to $\widetilde{M}$ for some $\Gamma(V, \mathscr{O}_X)$-module M. □

Corollary 10.2.23. *Let X be an arbitrary scheme.*

(i) *Let $0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{H} \to 0$ be an exact sequence of $\mathscr{O}_X$-modules; if two of them are quasi-coherent, then so is the third one.*

(ii) *The images, kernels and cokernels of homomorphisms of quasi-coherent $\mathscr{O}_X$-modules are quasi-coherent. The inductive limits and direct sums of quasi-coherent sheaves are quasi-coherent. If $\mathscr{G}$ and $\mathscr{H}$ are quasi-coherent $\mathscr{O}_X$-modules of a quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, then $\mathscr{G}+\mathscr{H}$ and $\mathscr{G}\cap\mathscr{H}$ are quasi-coherent.*

(iii) *Let $0\to\mathscr{F}_1\to\mathscr{F}_2\to\mathscr{F}_3\to\mathscr{F}_4\to\mathscr{F}_5\to 0$ be an exact sequence of $\mathscr{O}_X$-modules. If $\mathscr{F}_1,\mathscr{F}_2,\mathscr{F}_4,\mathscr{F}_5$ are quasi-coherent, so is $\mathscr{F}_3$.*

(iv) *If $\mathscr{F}$ and $\mathscr{G}$ are quasi-coherent $\mathscr{O}_X$-modules, $\mathscr{F}\otimes_{\mathscr{O}_X}\mathscr{G}$ is quasi-coherent. In particular, if $\mathscr{J}$ is a quasi-coherent ideal of $\mathscr{O}_X$, $\mathscr{J}\mathscr{F}$ is a quasi-coherent $\mathscr{O}_X$-module.*

(v) *Let $\mathscr{F}$ be a quasi-coherent $\mathscr{O}_X$-module with finite presentation. Then for any quasi-coherent $\mathscr{O}_X$-module $\mathscr{G}$, $\mathcal{H}om_{\mathscr{O}_X}(\mathscr{F},\mathscr{G})$ is quasi-coherent.*

(vi) *If $\mathscr{F}$ is a quasi-coherent $\mathscr{O}_X$-module of finite type, the annihilator $\mathscr{J}$ of $\mathscr{F}$ is a quasi-coherent ideal of $\mathscr{O}_X$.*

Proof. By Proposition 10.2.22, assertions (i) to (v) follow from Proposition 10.1.29, Corollary 10.1.6, and Corollary 10.1.10. To prove (vi), we can assume that $X=\mathrm{Spec}(A)$ is affine, $\mathscr{F}=\widetilde{M}$, where M is a finitely generated A-module, with generators $t_1,\dots,t_r$. The ideal $\mathscr{J}$ is then the intersection of the annihilators of t_i. But the annihilator of t_i is by definition the kernel of the canonical morphism $\mathscr{O}_X\to\mathscr{F}$ corresponding to $s\mapsto st_i$ from A to M, hence quasi-coherent. It then follows that $\mathscr{J}$ is quasi-coherent, as an intersection of quasi-coherent $\mathscr{O}_X$-modules. □

Corollary 10.2.24. *Let X be a scheme, $0\to\mathscr{F}\to\mathscr{G}\to\mathscr{H}\to 0$ an exact sequence of quasi-coherent $\mathscr{O}_X$-modules. If $\mathscr{H}$ is finitely presented and $\mathscr{G}$ is of finite type, then $\mathscr{F}$ is of finite type.*

Proof. Since this question is local, we may assume that X is affine, and the corresponding result is then **??**. □

Proposition 10.2.25. *Let X be a scheme and $\mathscr{B}$ be a quasi-coherent $\mathscr{O}_X$-algebra. For a $\mathscr{B}$-module $\mathscr{F}$ to be quasi-coherent, it is necessary and sufficient that $\mathscr{F}$ is a quasi-coherent $\mathscr{O}_X$-module. In particular, if $\mathscr{F}$ and $\mathscr{G}$ are quasi-coherent $\mathscr{B}$-modules, $\mathscr{F}\otimes_{\mathscr{B}}\mathscr{G}$ is a quasi-coherent $\mathscr{B}$-module; the same holds for $\mathcal{H}om_{\mathscr{B}}(\mathscr{F},\mathscr{G})$ if $\mathscr{F}$ is a finitely presented $\mathscr{B}$-module.*

Proof. Since the question is local, we can suppose that X is affine with ring A, and then $\mathscr{B}=\widetilde{B}$, where B is an A-algebra. If $\mathscr{F}$ is quasi-coherent over the space $(X,\mathscr{B})$, we can write $\mathscr{F}$ as the cokernel of $\mathscr{B}$-homomorphism $\mathscr{B}^{\oplus I}\to\mathscr{B}^{\oplus J}$; since this homomorphism is also an $\mathscr{O}_X$-homomorphism, and $\mathscr{B}^{\oplus I}$, $\mathscr{B}^{\oplus J}$ are quasi-coherent $\mathscr{O}_X$-modules, we conclude that $\mathscr{F}$ is also quasi-coherent.

Conversely, if $\mathscr{F}$ is a quasi-coherent $\mathscr{O}_X$-module, we have $\mathscr{F}=\widetilde{M}$ where M is a B-module (Corollary 10.1.26); M is isomorphic to the cokernel of a homomorphism $B^{\oplus I}\to B^{\oplus J}$, so $\mathscr{F}$ is a $\mathscr{B}$-module isomorphic to the cokernel of the corresponding homomorphism $\mathscr{B}^{\oplus I}\to\mathscr{B}^{\oplus J}$. This completes the proof. □

Let X be a scheme. A quasi-coherent $\mathscr{O}_X$-algebra $\mathscr{B}$ is **of finite type** (resp. **of finite presentation**) if for all $x\in X$, there exists an open affine neighborhood U of x such that $\Gamma(U,\mathscr{B})=B$ is an algebra of type finite (resp. of finite presentation[1]) over $\Gamma(U,\mathscr{O}_X)=A$. If this is the case, we have $\mathscr{B}|_U=\widetilde{B}$, and for all $f\in A$, the $(\mathscr{O}_X|_{D(f)})$-algebra $\mathscr{B}|_{D(f)}$ induced on $D(f)$ is of finite type (resp. of finite presentation), because it is isomorphic to $\widetilde{B\otimes_A A_f}$. As the $D(f)$ form a basis of the topology of X, we deduce that for any open set V of X, $\mathscr{B}|_V$ is a $(\mathscr{O}_X|_V)$-algebra of finite type (resp. of finite presentation).

[1] Recall that an algebra B is finitely presented over A if it is isomorphic to the quotient of a polynomial ring over A in finitely many variables by a finitely generated ideal.

Proposition 10.2.26. *Let X be a scheme, $\mathscr{E}$ a locally free $\mathscr{O}_X$-module of rank r, Z a finite subset of X contained in an affine open V. Then there exists an open neighborhood $U \subseteq V$ of Z such that $\mathscr{E}|_U$ is isomorphic to $\mathscr{O}_U^r$.*

Proof. By replacing X by V, we can assume that $X = \operatorname{Spec}(A)$ is affine. For each $z_i \in Z$ there exists a closed point z_i' in the closure $\overline{\{z_i\}}$ (that is, a maximal ideal containing $\mathfrak{p}_{z_i}$); if Z' is the set of the z_i', any neighborhood of Z' is a neighborhood of Z, and we can then suppose that Z is closed in X. Now, the subset Z of X is defined by an ideal $\mathfrak{a}$ of A; consider the scheme $\operatorname{Spec}(A/\mathfrak{a})$, with Z its underlying space, and the injection $\iota : Z \to X$ corresponds to the canonical homomorphism $A \to A/\mathfrak{a}$. Then $\iota^*(\mathscr{E}) = \mathscr{E} \otimes_{\mathscr{O}_X} \mathscr{O}_Z$ is locally free with rank r over the discrete scheme Z, so is isomorphic to $\mathscr{O}_Z^r$. In other words, there exist sections $s_1, \dots, s_r$ of $\mathscr{E} \otimes_{\mathscr{O}_X} \mathscr{O}_Z$ over Z such that the homomorphism $\mathscr{O}_Z^r \to \mathscr{E} \otimes_{\mathscr{O}_X} \mathscr{O}_Z$ defined by these sections is bijective. On the other hand, we have $\mathscr{E} = \widetilde{M}$ where M is an A-module; then each s_i belongs to $M \otimes_A (A/\mathfrak{a})$, and is then the image of an element $t_i \in M = \Gamma(X, \mathscr{E})$. For each $z_j \in Z$, by **??**, there then exists a neighborhood V_i of z_i in X such that the restrictions of t_i to V_j define an isomorphism $\mathscr{O}_X^r|_{V_j} \to \mathscr{E}|_{V_j}$; the union U of the V_j's then satisfies the requirement. □

Proposition 10.2.27. *Let X a scheme and $\mathscr{F}$ be a quasi-coherent $\mathscr{O}_X$-module of finite type. Then there exists a closed subscheme Y of X with underlying space* $\operatorname{supp}(\mathscr{F})$ *and a quasi-coherent $\mathscr{O}_Y$-module $\mathscr{G}$ of finite type supported on Y such that, if $j : Y \to X$ is the canonical injection, $\mathscr{F}$ is isomorphic to $j_*(\mathscr{G})$.*

Proof. It suffices to note that the annihilator $\mathscr{I}$ of $\mathscr{F}$ is a quasi-coherent ideal of $\mathscr{O}_X$ (Corollary 10.2.23), so if Y is the closed subscheme of X defined by $\mathscr{I}$, as $\mathscr{I}\mathscr{F} = 0$, $\mathscr{F}$ is an $(\mathscr{O}_X/\mathscr{I})$-module, and we can take $\mathscr{G} = j^*(\mathscr{F})$. □

10.2.5 Noetherian schemes and locally Noetherian schemes

We say a scheme X is Noetherian (resp. locally Noetherian) if there is a finite covering (resp. a covering) of open affines V_α such that each ring $\Gamma(V_\alpha, \mathscr{O}_X)$ is Noetherian. The underlying space of a Noetherian (resp. locally Noetherian) scheme is then a Noetherian space (resp. locally Noetherian). Moreover, if X is locally Noetherian. the structural sheaf $\mathscr{O}_X$ is coherent, any quasi-coherent $\mathscr{O}_X$-module of finite type is coherent (Theorem 10.1.30), and any local ring $\mathscr{O}_{X,x}$ is Noetherian. Any quasi-coherent sub-$\mathscr{O}_X$-module (resp. any quasi-coherent $\mathscr{O}_X$-quotient) of a coherent $\mathscr{O}_X$-module $\mathscr{F}$ is then coherent, because the question is local again, and we just apply Theorem 10.1.30, together with the fact that a sub-module (resp. quotient module) of a finitely generated module on a Noetherian ring is finitely generated. More particularly, any quasi-consistent ideal of $\mathscr{O}_X$ is consistent.

If a scheme X is a finite union (resp. a union) of open Noetherian (resp. locally Noetherian) subschemes W_λ, it is clear that X is then Noetherian (resp. locally Noetherian).

Proposition 10.2.28. *For a scheme X to be Noetherian, it is necessary and sufficient that it is locally Noetherian and its underlying space is quasi-compact.*

Proof. This follows from the definition, since a Noetherian space is quasi-compact. □

Proposition 10.2.29. *Let X be an affine scheme with ring A. Then the following conditions are equivalent:*

(i) *X is Noetherian;*

(ii) *X is locally Noetherian;*

(iii) *A is Noetherian.*

Proof. Since X is quasi-compact, it is clear that (i) and (ii) are equivalent. Also, (iii) implies (i) by definition. Now assume that X is Noetherian, then there is a finite covering (V_i) of X by affine opens where $A_i = \Gamma(V_i, \mathscr{O}_X)$ is Noetherian. Let $(\mathfrak{a}_n)$ be an increasing sequence of ideals of A; it corresponds to it canonically in a one-to-one way to an increasing sequence $(\tilde{\mathfrak{a}}_n)$ of ideals in $\widetilde{A} = \mathscr{O}_X$; to see that the sequence $(\mathfrak{a}_n)$ is stationary, it suffices to prove that the sequence $(\tilde{\mathfrak{a}}_n)$ is. However, the restriction $\tilde{\mathfrak{a}}_n|_{V_i}$ is a quasi-coherent ideal of $\mathscr{O}_X|_{V_i}$; $\tilde{\mathfrak{a}}_n|_{V_i}$ is then of the form $\tilde{\mathfrak{a}}_{n,i}$, where $\mathfrak{a}_{n,i}$ is an ideal of A_i. As A_i is Noetherian, the sequence $(\mathfrak{a}_{n,i})$ is stationary for all i, hence the proposition. □

Note that the above reasoning also proves that if X is a Noetherian scheme, any increasing sequence of coherent ideals of $\mathscr{O}_X$ is stationary.

Proposition 10.2.30. *Let X be a locally Noetherian scheme. Any quasi-coherent $\mathscr{O}_X$-algebra $\mathscr{B}$ of finite type is a coherent shaf of rings and an $\mathscr{O}_X$-algebra of finite presentation.*

Proof. We can assume that $X = \mathrm{Spec}(A)$ is affine, where A is a Noetherian ring, and $\mathscr{B} = \widetilde{B}$, where B is an A-algebra of finite type. It then follows that B is finitely presented over A, so $\mathscr{B}$ is of finite presentation. To show that $\mathscr{B}$ is coherent, we must prove that the kernel $\mathscr{N}$ of a $\mathscr{B}$-homomorphism $\mathscr{B}^m \to \mathscr{B}$ is a $\mathscr{B}$-module of finite type; but it is of the form $\widetilde{N}$, where N is the kernel of the corresponding homomorphism $B^m \to B$ of B-modules. Since B is also Noetherian, N is a finitely generated B-module. There then exists a surjective B-homomorphism $B^n \to N$, so a surjective homomorphism $\mathscr{B}^n \to \mathscr{N}$, which proves our assertion. □

Corollary 10.2.31. *Let X be a locally Noetherian scheme and $\mathscr{B}$ be a quasi-coherent $\mathscr{O}_X$-algebra of finite type. For a $\mathscr{B}$-module $\mathscr{F}$ to be coherent, it is necessary and sufficient that $\mathscr{F}$ is a quasi-coherent $\mathscr{O}_X$-module and a $\mathscr{B}$-module of finite type, and if $\mathscr{G}$ is a sub-$\mathscr{B}$-module or a quotient $\mathscr{B}$-module of $\mathscr{F}$, for $\mathscr{G}$ to be a coherent $\mathscr{B}$-module, it is necessary and sufficient that $\mathscr{G}$ is a quasi-coherent $\mathscr{O}_X$-module.*

Proof. Considering Proposition 10.2.25, the conditions on $\mathscr{F}$ is necessary. To prove the sufficiency, we can assume that $X = \mathrm{Spec}(A)$ is affine, where A is Noetherian, $\mathscr{B} = \widetilde{B}$, where B is an A-algebra of finite type, and $\mathscr{F} = \widetilde{M}$, where M is a B-module and there exists a surjective B-homomorphism $\mathscr{B}^m \to \mathscr{F}$. Then we get a corresponding homomorphism $B^m \to M$, so M is a finitely generated B-module; the kernel P of thie homomorphism is finitely generated since B is Noetherian, and $\mathscr{F}$ is therefore the cokernel of a morphism $\mathscr{B}^n \to \mathscr{B}^m$, so it is coherent (since $\mathscr{B}$ is a coherent sheaf of rings). The same reasoning shows that any quasi-coherent sub-$\mathscr{B}$-module (resp. quotient $\mathscr{B}$-module) of $\mathscr{F}$ is of finite type, whence the second part of the corollary. □

Proposition 10.2.32. *Let X be a locally Noetherian scheme and E be a subset of X. Any point $x \in E$ admits in E a maximal generalization y (i.e. y has no further generalization in Y). In particular, if $E \neq \varnothing$, there exists a maximal element $y \in E$ under generalization.*

Proof. The generalizations of x in X lie in the points of $\mathrm{Spec}(\mathscr{O}_{X,x})$ (Corollary 10.2.12), where $\mathscr{O}_{X,x}$ is a Noetherian local ring. We then know that the lengths of chains of prime ideals in this ring are bounded by $\dim(\mathscr{O}_{X,x})$, and to prove the proposition it suffices to consider a chain of prime ideals belonging to E and having the greatest possible length. □

Proposition 10.2.33. *Let X be a scheme. Then the following conditions are equivalent:*

(i) *$X = \mathrm{Spec}(A)$ is affine and A is Artinian;*

(ii) *X is Noetherian and has discrete underlying space;*

(iii) *X is Noetherian and every point in X is closed (in other words, X is T1).*

If these equivalent conditions hold, then X is finite and the ring A is a direct product of finitely many Artinian local rings.

Proof. We know that (i) implies the last asserion. It is clear that (i)⇒(ii)⇒(iii). To see that (iii) implies (i), let us first show that X is then finite; we can indeed reduce to case where X is affine, and we know that a Noetherian ring of which all the prime ideals are maximal is Artinian, hence our assertion. □

Note that a Noetherian scheme can have an underlying space finite without being artinian, as shown by the example of a spectrum discrete valuation ring.

10.3 Product of schemes

Let (X_α) be a family of schemes, and X be the topological space which is the **coproduct** of the underlying spaces of X. Then X is the union of its open subspaces U_α, and for each α we have a embedding $\iota_\alpha : X_\alpha \to X$ with image equal to U_α. If we endow each U_α the sheaf $(\iota_\alpha)_*(\mathscr{O}_{X_\alpha})$, it is clear that X becomes a scheme, which we will call the **coproduct** of the family (X_α), and denote by $\coprod_\alpha X_\alpha$. It is clear that the scheme X satisfies the universal property of coproducts of X_α's: for any scheme Y and morphisms $f_\alpha : X_\alpha \to Y$, there exists a unique morphism $f : X \to Y$ such that $f \circ \iota_\alpha = f_\alpha$. In other words, we have a functorial bijection

$$\mathrm{Hom}(\coprod_\alpha X_\alpha, Y) \to \prod_\alpha \mathrm{Hom}(X_\alpha, Y).$$

This fact can be also stated that $\coprod_\alpha X_\alpha$ represents the covariant functor $\prod_\alpha \mathrm{Hom}(X_\alpha, -)$ on the category of schemes. In particular, if X_α are S-schemes with structural morphisms ψ_α, then X is an S-scheme with structural morphism $\psi : X \to S$ such that $\psi \circ \iota_\alpha = \psi_\alpha$. We usually denote the coproduct of two schemes X and Y by $X \amalg Y$, and it is clear that if $X = \mathrm{Spec}(A)$ and $Y = \mathrm{Spec}(B)$, then $X \amalg Y$ is canonically identified with $\mathrm{Spec}(A \times B)$.

In this section, we shall consider product of schemes, which is far more complicated than coproducts. We will see that fiber products plays a central role of many construction and operations on schemes.

10.3.1 Product of schemes

Let X and Y be S-schemes. Recall that the object $X \times_S Y$ represents by definition the contravariant functor

$$T \mapsto F(T) = \mathrm{Hom}_S(T, X) \times \mathrm{Hom}_S(T, Y)$$

on the category of S-schemes. To prove the existence of $X \times_S Y$, we shall apply the methods used in **??**. We first verify condition (ii) in **??**, which means F is a sheaf over the category **Sch**: this is evident since the functors $T \mapsto \mathrm{Hom}_S(T, X)$ and $T \mapsto \mathrm{Hom}_S(T, Y)$ are sheaves, and a projective limit of sheaves over **Sch** is again a sheaf over **Sch**.

This already allows us to bring ourselves back to the case that the scheme S is affine. In fact, let (S_α) is a covering of S by affine open sets. In view of the above fact and of ([**?**] new, 0_{I}, 4.5.5), it suffices to show that each of the functors $F \times_{h_S} h_{S_\alpha}$ is representable. On the other hand, let $\varphi : X \to S$ and $\psi : Y \to S$ be the structural morphisms; it is immediate that when T is a locally ringed S_α-space (hence also an S-space), we have canonical identifies

$$\mathrm{Hom}_S(T, X) \stackrel{\sim}{\to} \mathrm{Hom}_{S_\alpha}(T, \varphi^{-1}(S_\alpha)), \quad \mathrm{Hom}_S(T, Y) \stackrel{\sim}{\to} \mathrm{Hom}_{S_\alpha}(T, \psi^{-1}(S_\alpha)).$$

Therefore, in view of ([**?**] new, 0_{I}, 4.5.5), we only need to show that F is representable when it is restricted to the subcategory of locally ringed S_α-spaces.

With these being done, assume that S is affine and consider a covering (X_λ) (resp. (Y_μ)) of X (resp. Y) by affine opens. We shall verify the conditions (i) and (iii) of **??** for the subfunctors $F_{\lambda\mu} : T \mapsto \mathrm{Hom}_S(T, X_\lambda) \times \mathrm{Hom}_S(T, Y_\mu)$ of F. Let Z be a locally ringed S-space and (p, q) be an element of $F(Z)$, i.e. $p : Z \to X$ and $q : Z \to Y$ are S-morphisms. These determine by Yoneda Lemma a natural transform $h_Z \to F$ which associates a locally ringed S-space T the map

$$\mathrm{Hom}_S(T, Z) \to F(T), \quad g \mapsto (p \circ g, q \circ g)$$

and every natural transform $h_Z \to F$ is of this form. We now show that the functor

$$T \mapsto F_{\lambda\mu}(T) \times_{F(T)} h_Z(T) \tag{10.3.1}$$

is representable by a locally ringed S-space induced by Z on an open subset of Z. In fact, an element of the right side of (10.3.1) (which is a fiber product of sets) is a triple (u_λ, v_μ, g), where $g : T \to Z$, $u_\lambda : T \to X_\lambda$, and $v_\mu : T \to Y_\mu$ are S-morphisms such that the following diagram commutes:

$$\begin{tikzcd}
T \arrow[dr, "g"] \arrow[drr, bend left, "v_\mu"] \arrow[ddr, bend right, "u_\lambda"'] & & \\
& Z \arrow[r, "q"] \arrow[d, "p"'] & Y \arrow[d] \\
& X \arrow[r] & S
\end{tikzcd}$$

Now this in particular implies that $g(T) \subseteq Z_{\lambda\mu} = p^{-1}(X_\lambda) \cap q^{-1}(Y_\mu)$, and conversely, any S-morphism $g : T \to Z$ verifying this condition corresponds to the unique triple $(p \circ g, q \circ g, g)$, since $p \circ g$ (resp $q \circ g$) can be viewed as a morphism from Z to X_λ (resp. Y_μ). In other word, we have a canonical bijection

$$F_{\lambda\mu}(T) \times_{F(T)} \mathrm{Hom}_S(T, Z) \xrightarrow{\sim} \mathrm{Hom}_S(T, Z_{\lambda\mu})$$

and the functor (10.3.1) is then represented by the couple $(Z_{\lambda\mu}, (p|_{Z_{\lambda\mu}}, q|_{Z_{\lambda,\mu}}), j_{\lambda\mu})$, where $j_{\lambda\mu} : Z_{\lambda\mu} \to Z$ is the cannical injection. Since the $Z_{\lambda\mu}$ form an open covering of Z, this proves both of the conditions (i) and (iii) of **??**.

It remains to show that the functors $F_{\lambda\mu}$ are representable, which means we need to construct $X \times_S Y$ when X, Y, and S are affine schemes. This is fairly easy, as we will now show.

Proposition 10.3.1. *Assume that $S = \mathrm{Spec}(A)$, $X = \mathrm{Spec}(B)$, and $Y = \mathrm{Spec}(C)$, where B and C are A-algebras. Then the scheme $Z = \mathrm{Spec}(B \otimes_A C)$, with p, q the S-morphisms corresponding to the canonical A-homomorphisms $B \to B \otimes_A C$ and $C \to B \otimes_A C$, representes the functor F in the category of locally rinfed S-spaces.*

Proof. In fact, in the category of rings, the tensor product $B \otimes_A C$ of two A-algebras B and C is a coproduct in the category of A-algebras, as can be easily verified. □

We then conclude that fiber products exist in the category of schemes. As always, the notation $X \times_S Y$ will be used to denote this product for two S-schemes X and Y. If $S = \mathrm{Spec}(A)$ is an affine scheme, we also write $X \times_A Y$. If $Y = \mathrm{Spec}(B)$ is an affine scheme, in view of Proposition 10.3.1, we use $X \otimes_S B$ to denote this product, and $X \otimes_A B$ if $S = \mathrm{Spec}(A)$ is also affine.

The general notations and results for fiber products in a category can be then used for the product of schemes. In particular, if $p_1 : X \times_X Y \to X$, $p_2 : X \times_S Y \to Y$ are the canonical projections, and $g : T \to X$, $h : T \to Y$ are two S-morphisms, we denote by $(g,h)_S$ the unique S-morphism fits into the following diagram:

$$\begin{tikzcd}
T \arrow[dr, "(g{,}h)_S"] \arrow[drr, bend left, "h"] \arrow[ddr, bend right, "g"'] & & \\
& X \times_S Y \arrow[r, "p_2"] \arrow[d, "p_1"] & Y \arrow[d] \\
& X \arrow[r] & S
\end{tikzcd}$$

If $S = \mathrm{Spec}(A)$, $X = \mathrm{Spec}(B)$, $Y = \mathrm{Spec}(C)$, and $T = \mathrm{Spec}(D)$ are all affine, and g,h correspond to homomorphisms $\rho : B \to D$, $\sigma : C \to D$ of A-algebras, then $(g,h)_S$ corresponds to the homomorphism $\tau : B \otimes_A C \to D$ such that

$$\tau(b \otimes c) = \rho(b)\sigma(c).$$

Again, if $S = \mathrm{Spec}(A)$ is affine, we also write $(g,h)_A$ instead of $(g,h)_S$.

Corollary 10.3.2. *Let $Z = X \times_S Y$ be the product of two S-schemes, $p : Z \to X$, $q : Z \to Y$ the canonical projections, φ (resp ψ) the structural morphisms of X (resp. Y). Let U, V be open subsets of X, Y respectively, and W be an open subset of S such that $p(U) \subseteq W$ and $p(V) \subseteq W$. Then the product $U \times_W V$ is canonically identified with the scheme induced by Z on the subset $p^{-1}(V) \cap q^{-1}(W)$ (considered as a U-scheme). Moreover, if $g : T \to X$, $h : T \to Y$ are S-morphisms such that $g(T) \subseteq V$, $h(T) \subseteq W$, the U-morphism $(g,h)_S$ is identified with $(g,h)_S$, considered as morphisms from T to $p^{-1}(V) \cap q^{-1}(W)$.*

Proof. We first note that, if U is an open set of S and $\varphi : X \to S$, $\psi : Y \to S$ are structural morphisms with images in U, then the fiber product $X \times_S Y$ is identified with $X \times_U Y$. Apply this to V and W, we conclude that $U \times_W V = U \times_W V$. So it suffices to prove that the subscheme $R = p^{-1}(U) \cap q^{-1}(V)$ with its restricted projections to U and V form a product of U and V. For this, we note that if T is an S-scheme, we can identify the S-morphisms $T \to R$ and the S-morphisms $T \to Z$ with image in R. If $g : T \to U$, $h : T \to V$ are two S-morphisms, we can consider them as S-morphisms of T in X and Y respectively, and by hypothesis there is therefore an S-morphism and there is a morphism $f : T \to Z$ such that $g = p \circ f$, $h = q \circ f$. Since $p(f(T)) \subseteq U$ and $q(f(T)) \subseteq V$, we have

$$f(T) \subseteq p^{-1}(U) \cap q^{-1}(V) = W$$

whence our claim. □

Corollary 10.3.3. *Let (X_λ) (resp. (Y_μ)) be a family of S-schemes and X (resp. Y) be their coproduct. Then $X \times_S Y$ is identified with the coproduct of the family $(X_\lambda \times_S Y_\mu)$.*

Proof. In fact, in the notations of Corollary 10.3.2, the underlying space of $X \times_S Y$ is the disjoint union of open sets $p^{-1}(X_\lambda) \cap q^{-1}(Y_\mu)$, and it suffices to apply Corollary 10.3.2. □

Remark 10.3.4. The product of two Noetherian S-schemes need not to be Noetherian, even if they are both spectrum of fields. For example, if k is a nonperfect field of characteristic $p > 0$, the tensor product $A = k^{p^{-\infty}} \otimes_k k^{p^{-\infty}}$ is not a Noetherian ring: in fact, for any integer $n > 0$, there exists $x_n \in k^{p^{-\infty}}$ such that $x_n^{p^n} \in k$ and $x_n^{p^{n-1}} \notin k$. If we consider the element $z_n = 1 \otimes x_n - x_n \otimes 1$ of A, we then have $z_n^{p^n} = 0$, and $z_n^{p^{n-1}} \neq 0$ since 1 and $x_n^{p^{n-1}}$ are linearly independent over k. We then conclude that the nilradical of A is not nilpotent, so A is not Noetherian.

Remark 10.3.5. We should note that the underlying topological space of the fiber product $X \times_S Y$ is not the fiber product of the underlying topological spaces. This can be seen from the tensor product of two fields, which can not be a field.

10.3.2 Base change of schemes

The functor $X \times_S Y$ is covariant in both of its variables, and this follows from the following commutative diagram:

$$\begin{tikzcd} X \times Y \arrow[r, "f\times 1"] \arrow[d] & X' \times Y \arrow[r, "f'\times 1"] \arrow[d] & X'' \times Y \arrow[d] \\ X \arrow[r, "f"] & X' \arrow[r, "f'"] & X'' \end{tikzcd}$$

Proposition 10.3.6. *For any S-scheme X, the first (resp. second) projection $X \times_S S$ (resp. $S \times_S X$) is a functorial isomorphism of $X \times_S S$ (resp. $S \times_S X$) to X, with inverse isomorphism $(1_X, \varphi)_S$ (resp. $(\varphi, 1_X)_S$), where $\varphi : X \to S$ is the structural morphism. We can therefore write*

$$X \times_S S = S \times_S X = X.$$

Proof. It suffices to prove that the triple $(X, 1_X, \varphi)$ form a product of X and S, which is immediate. □

Corollary 10.3.7. *Let X and Y be S-schemes, $\varphi : X \to S$, $\psi : Y \to S$ the structural morphisms. If we identify canonically X with $X \times_S S$ and Y with $S \times_S Y$, the projections $X \times_S Y \to X$ and $X \times_S Y \to Y$ is identified respectively to $1_X \times \psi$ and $\varphi \times 1_Y$.*

We can define similarly the fiber product of S-schemes $X_1, \dots, X_n$, whose existence can be proved by induction on n, which is isomorphic to $(X_1 \times_S \cdots \times_S X_{n-1}) \times_S X_n$. The uniqueness of the product entails, as in any category, its properties of commutativity and associativity. If, for example, p_1, p_2, p_3 denotes the projections of $X_1 \times_S X_2 \times_S X_3$, and if we identify this scheme with $(X_1 \times_S X_2) \times_S X_3$, the projection in $X_1 \times_S X_2$ is identified with $(p_1, p_2)_S$.

Let S, S' be two schemes, $\varphi : S \to S'$ an morphism, making S' an S-scheme. For any S-scheme X, consider the product $X \times_S S'$, and let p and π' the projections to X and S' respectively. Through the morphism π', this product is an S'-scheme, which we may denoted by $X_{(S')}$ or $X_{(\varphi)}$, and the obtained scheme is called the **base change** of X from S to S', or the inverse image of X via φ. We note that if π is the structural morphism of X and θ is the structural morphism of $X \times_S S'$, the following diagram is commutative:

$$\begin{tikzcd} X_{(S')} \arrow[r, "\pi'"] \arrow[d, "p"'] & S' \arrow[d, "\varphi"] \\ X \arrow[r, "\pi"] & S \end{tikzcd}$$

For any S-morphism $f : X \to Y$, we denote by $f_{(S')}$ the S'-morphism $f \times_S 1 : X_{(S')} \to Y_{(S')}$ and call it the inverse image of f by φ. The operation $X_{(S')}$ is clearly a covariant functor on X, from the category $\mathbf{Sch}/S$ to $\mathbf{Sch}/S'$.

Let S, S' be two affine schemes with rings A, A'; a morphism $S' \to S$ corresponds to a homomorphism $A \to A'$. If X is an S-scheme, we then denote by $X_{(A')}$ of $X \otimes_A A'$ by the S'-scheme $X_{(S')}$; if X is also affine with ring B, then $X_{(A')}$ is affine with ring $B_{(A')} = B \otimes_A A'$.

We point out that the scheme $X_{(S')}$ satisfies the following universal property: any S'-scheme T is an S-scheme via the morphism φ, and for any S-morphism $g : T \to X$ there exists a unique S'-morphism $f : T \to X_{(S')}$ such that $g = p \circ f$.

Proposition 10.3.8 (Transitivity). *Let $\varphi' : S'' \to S'$ and $\varphi : S' \to S$ be morphism of schemes. For any S-scheme X, there is a canonical functorial isomorphism of the S''-schemes $(X_{(\varphi)})_{(\varphi')}$ and $X_{(\varphi\circ\varphi')}$.*

Proof. In fact, let T be an S''-scheme, ψ its structural morphism, $g : T \to X$ an S-morphism (T is an S-scheme via the morphism $\varphi \circ \varphi' \circ \psi$). Since T is an S'-scheme with structural morphism $\varphi' \circ \psi$, we can write $g = p \circ g'$, where $g' : T \to X_{(\varphi)}$ is an S'-morphism. Then $g' = p' \circ g''$, where $g'' : T \to (X_{(\varphi)})_{\varphi'}$ is an S''-morphism:

$$\begin{tikzcd}
(X_{(\varphi)})_{\varphi'} \arrow[r, "p'"] \arrow[d, "\pi''"] & X_{(\varphi)} \arrow[r, "p"] \arrow[d, "\pi'"] & X \arrow[d, "\pi"] \\
S'' \arrow[r, "\varphi'"] & S' \arrow[r, "\varphi"] & S
\end{tikzcd}$$

The claim now follows from the definition of the universal property of $X_{(\varphi\circ\varphi')}$. □

The previous result can also be written as $(X_{(S')})_{(S'')} = X_{(S'')}$, if there is no risk of confusion. Moreover precisely, we have

$$(X \times_S S') \times_{S'} S'' = X \times_S S'';$$

the functorial of the isomorphism in Proposition 10.3.8 also shows the transitive of inverse image of morphisms:

$$(f_{(S')})_{(S'')} = f_{(S'')}$$

for any S-morphism $f : X \to Y$.

Corollary 10.3.9. *If X and Y are S-schemes, there exists a canonical functorial isomorphism of S'-schemes $X_{(S')} \times_{S'} Y_{(S')}$ and $(X \times_S Y)_{S'}$.*

Proof. In fact, we have, the following canonical isomorphisms:

$$(X \times_S S') \times_{S'} (Y \times_S S') = X \times_S (Y \times_S S') = (X \times_S Y) \times_S S'$$

where we use Proposition 10.3.8 and the associativity of fiber product. □

Again, the functorial isomorphism in Corollary 10.3.9 also gives the isomorphism

$$(u_{(S')}, v_{(S')})_{(S')} = ((u, v)_S)_{S'}$$

for any S-morphisms $u : T \to X$, $v : T \to Y$. In other words, the inverse image functor $X_{(S')}$ commutes on the formation of the products; note that it also commutes to the formation of coproducts.

Corollary 10.3.10. *Let Y be an S-scheme, $f : X \to Y$ a morphism making X a Y-scheme (and also an S-scheme). Then the scheme $X_{(S')}$ is canonically identified with the product $X \times_Y Y_{(S')}$, and the projection $X \times_Y Y_{(S')} \to Y_{(S')}$ is identified with $f_{(S')}$.*

Proof. Let $\psi : Y \to S$ be the structural morphism of Y; we have a commutative diagram

$$\begin{tikzcd}
X_{(S')} \arrow[r, "f_{(S')}"] \arrow[d] & Y_{(S')} \arrow[r, "\psi_{(S')}"] \arrow[d] & S' \arrow[d] \\
X \arrow[r, "f"] & Y \arrow[r, "\psi"] & S
\end{tikzcd}$$

Now $Y_{(S')}$ is identified with $S'_{(\psi)}$, and $X_{(S')}$ with $S'_{(\psi\circ f)}$, so by Proposition 10.3.8 and Proposition 10.3.6, we deduce the corollary. □

Example 10.3.11. Let A be a ring, X an A-scheme, and $\mathfrak{a}$ an ideal of A. Then $X_0 = X \otimes_A (A/\mathfrak{a})$ is an $(A/\mathfrak{a})$-scheme, called the scheme obtained from X by **reduction mod** $\mathfrak{a}$.

Proposition 10.3.12. *Let $f : X \to X'$, $g : Y \to Y'$ be two S-morphisms that are monomorphisms of schemes; then $f \times_S g$ is a monomorphism. In particular, for any extension $S' \to S$ of base scheme, the inverse image $f_{(S')} : X_{(S')} \to Y_{(S')}$ is a monomorphism.*

Proof. In fact, if p, q are the projections of $X \times_S Y$, and p', q' that of $X' \times_S Y'$:

$$\begin{tikzcd}
X \times_S Y \arrow[rr, "q"] \arrow[dr, "f \times_S g"] \arrow[dd, "p"'] & & Y \arrow[d, "g"] \\
& X' \times_S Y' \arrow[r, "q'"] \arrow[d, "p'"'] & Y' \arrow[d] \\
X \arrow[r, "f"] & X' \arrow[r] & S
\end{tikzcd}$$

then for any two morphisms $u, v : T \to X \times_S Y$, the relation $(f \times_S g) \circ u = (f \times_S g) \circ v$ implies

$$p' \circ (f \times_S g) \circ u = p' \circ (f \times_S g) \circ v$$

so $f \circ p \circ u = f \circ p \circ v$, and since f is a monomorphism, we conclude $p \circ u = p \circ v$. Similarly, since g is a monomorphism, we have $q \circ u = q \circ v$, whence $u = v$. □

For any S-morphism $f : S' \to X$, the morphism $f' = (f, 1_{S'})_S$ is then an S'-morphism from S' to $X' = X_{(S')}$ such that $p \circ f' = f$, $\pi' \circ f' = 1_{S'}$, which is called an ***S′*-section** of X':

$$\begin{tikzcd}
X' \arrow[r, "\pi'"] \arrow[d, "p"'] & S' \arrow[l, bend right, "f'"'] \arrow[dl, "f"] \arrow[d, "\varphi"] \\
X \arrow[r, "\pi"] & S
\end{tikzcd}$$

Conversely if f' is an S'-section, then $f = p \circ f'$ is an S-morphism $S' \to X$. We then deduce the following canonical correspondence

$$\mathrm{Hom}_S(S', X) \xrightarrow{\sim} \mathrm{Hom}_{S'}(S', X') \tag{10.3.2}$$

The morphism f' is called the **graph** of f, and denoted by Γ_f. A particularly important case is $S' = X$ and $f = 1_X$, where corresponding morphism $X \to X \times_S X$ is called the **diagonal morphism** of X, and denoted by Δ_X. Also, if $f : X \to Y$ is a morphism of schemes, we denote by Δ_f the diagonal map from X to $X \times_Y X$.

Example 10.3.13. Since any scheme X can be considered as a $\mathbb{Z}$-scheme, we can consider the X-sections of $X \otimes_{\mathbb{Z}} \mathbb{Z}[T]$ (where T is an indeterminate) corresponding to the $\mathbb{Z}$-morphisms $X \to \mathrm{Spec}(\mathbb{Z}[T])$.

$$\begin{tikzcd}
X \otimes_{\mathbb{Z}} \mathbb{Z}[T] \arrow[r] \arrow[d] & \mathrm{Spec}(\mathbb{Z}[T]) \arrow[d] \\
X \arrow[u, dashed, bend left] \arrow[ur, dashed] \arrow[r] & \mathrm{Spec}(\mathbb{Z})
\end{tikzcd}$$

We claim that such X-sections correspond to sections of the structural sheaf $\mathscr{O}_X$ of X. In fact, the morphisms $X \to \mathrm{Spec}(\mathbb{Z}[T])$ correspond to ring homomorphisms $\mathbb{Z}[T] \to \Gamma(X, \mathscr{O}_X)$, which in turn are entirely determined by the image of T, and can be an arbitrary element of $\Gamma(X, \mathscr{O}_X)$, whence our assertion.

10.3.3 Tensor product of quasi-coherent sheaves

Let S be a scheme, X,Y be two S-schemes, $Z = X \times_S Y$, and p,q be the projections of Z to X and Y, respectively. Let $\mathscr{F}$ (resp. $\mathscr{G}$) be an $\mathscr{O}_X$-module (resp. an $\mathscr{O}_Y$-module). Then the tensor product $p^*(\mathscr{F}) \otimes_{\mathscr{O}_Z} q^*(\mathscr{G})$ is called the **tensor product of $\mathscr{F}$ and $\mathscr{G}$ over $\mathscr{O}_S$** (or **over S**) and denoted by $\mathscr{F} \otimes_{\mathscr{O}_S} \mathscr{G}$ (or $\mathscr{F} \otimes_S \mathscr{G}$). More generaly, if $(X_i)_{1\le i\le n}$ is a finite family of S-schemes and for each i, $\mathscr{F}_i$ is an $\mathscr{O}_X$-module, we can define the tensor product $\mathscr{F}_1 \otimes_S \cdots \otimes_S \mathscr{F}_n$ over the scheme $Z = X_1 \times_S \cdots \times_S X_n$. This is a quasi-coherent $\mathscr{O}_Z$-module if each $\mathscr{F}_i$ is quasi-coherent (Corollary 10.2.23), and is coherent if each $\mathscr{F}_i$ is coherent and Z is locally Noetherian in view of **??**.

We note that if $X = Y = S$, the above definition coincide with the usual one of tensor product of $\mathscr{O}_S$-modules. Moreover, as $q^*(\mathscr{O}_Y) = \mathscr{O}_{X\times_S Y}$, the product $\mathscr{F} \otimes_S \mathscr{O}_Y$ is canonically identified with $p^*(\mathscr{F})$, and similarly $\mathscr{O}_X \otimes_S \mathscr{G}$ is identified with $q^*(\mathscr{G})$. In particular, if $Y = S$ and $f : X \to Y$ is the structural morphism, then $\mathscr{O}_X \otimes_Y \mathscr{G} = f^*(\mathscr{G})$: the ordinary tensor product and the inverse image therefore appears as a special case of the general tensor product. We also note that if X and Y are fixed, the operation $\mathscr{F} \otimes_S \mathscr{G}$ is a covariant bifunctor and is right exact on $\mathscr{F}$ and $\mathscr{G}$.

Proposition 10.3.14. *Let S,X,Y be affine schemes with rings A, B, C, respectively, where B,C are A-algebras. Let M (resp. B) be a B-module (resp. C-module) and $\mathscr{F} = \widetilde{M}$ (resp. $\mathscr{G} = \widetilde{N}$) the associated quasi-coherent sheaf. Then $\mathscr{F} \otimes_S \mathscr{G}$ is canonically isomorphic to the sheaf associated with the $(B \otimes_A C)$-module $M \otimes_A N$.*

Proof. In fact, in view of Proposition 10.1.14 and Corollary 10.1.10, $\mathscr{F} \otimes_S \mathscr{G}$ is canonically isomorphic to the sheaf associated with the $(B \otimes_A C)$-module

$$(M \otimes_B (B \otimes_A C)) \otimes_{B\otimes_A C} ((B \otimes_A C) \otimes_C N)$$

and due to the canonical isomorphisms between tensor products, the latter is isomorphic to $M \otimes_B (B \otimes_A C) \otimes_C N = (M \otimes_B B) \otimes_A (C \otimes_C N) = M \otimes_A N$. □

Proposition 10.3.15. *Let $f : T \to X$, $g : T \to Y$ be two S-morphisms, and $\mathscr{F}$ (resp. $\mathscr{G}$) be an $\mathscr{O}_X$-module (resp. $\mathscr{O}_Y$-module). Then we have $(f,g)_S^*(\mathscr{F} \otimes_S \mathscr{G}) = f^*(\mathscr{F}) \otimes_{\mathscr{O}_T} g^*(\mathscr{G})$.*

Proof. If p,q are the projections of $X \times_S Y$, the assertion follows from the relations $(f,g)_S^* \circ p^* = f^*$ and $(f,g)_S^* \circ q^* = g^*$, and the fact that the inverse image operation commutes with tensor products. □

Corollary 10.3.16. *Let $f : X \to X'$, $g : Y \to Y'$ be two S-schemes and $\mathscr{F}'$ (resp. $\mathscr{G}'$) be an $\mathscr{O}_{X'}$-module (resp. $\mathscr{O}_{Y'}$-module). Then $(f \times_S g)^*(\mathscr{F}' \otimes_S \mathscr{G}') = f^*(\mathscr{F}') \otimes_S g^*(\mathscr{G}')$.*

Proof. This follows from Proposition 10.3.15 and the fact that $f \times_S g = (f \circ p, g \circ q)_S$, where p,q are the projections of $X \times_S Y$. □

Corollary 10.3.17. *Let X,Y,Z be S-schemes and $\mathscr{F}$ (resp. $\mathscr{G}$, $\mathscr{H}$) be an $\mathscr{O}_X$-module (resp. $\mathscr{O}_Y$-module, $\mathscr{O}_Z$-module). Then the sheaf $\mathscr{F} \otimes_S \mathscr{G} \otimes_S \mathscr{H}$ is the inverse image of $(\mathscr{F} \otimes_S \mathscr{G} \otimes_S \mathscr{H})$ under the canonical isomorphism $X \times_S Y \times_S Z \to (X \times_S Y) \times_S Z$, and $\mathscr{G} \otimes_S \mathscr{F}$ is the inverse image of $\mathscr{F} \otimes_S \mathscr{G}$ under the canonical isomorphism $X \times_S \to Y \times_S X$.*

Proof. The first isomorphism is $(p_1, p_2)_S \times_S p_3$, where p_1, p_2, p_3 are the projections of $X \times_S Y \times_S Z$, and second one is similarly. □

Corollary 10.3.18. *If X is an S-scheme, any $\mathscr{O}_X$-module $\mathscr{F}$ is the inverse image of $\mathscr{F} \otimes_S \mathscr{O}_S$ under the canonical isomorphism from X to $X \times_S S$.*

Let X be an S-scheme, $\mathscr{F}$ be an $\mathscr{O}_X$-module, and $\varphi : S' \to S$ be an morphism. We denote by $\mathscr{F}_{(\varphi)}$ or $\mathscr{F}_{(S')}$ the sheaf $\mathscr{F} \otimes_S \mathscr{O}_{S'}$ over $X \times_S S' = X_{(\varphi)} = X_{(S')}$, so $\mathscr{F}_{(S')} = p^*(\mathscr{F})$, where p is the projection $X_{(S')} \to X$.

Proposition 10.3.19. *Let $\varphi' : S'' \to S'$ be a morphism. For any $\mathscr{O}_X$-module $\mathscr{F}$ over the S-scheme X, $(\mathscr{F}_{(\varphi)})_{(\varphi')}$ is the inverse image of $\mathscr{F}_{(\varphi\circ\varphi)}$ under the canonical isomorphism $(X_{(\varphi)})_{(\varphi')} \to X_{(\varphi\circ\varphi')}$.*

Proof. This follows from the definition and the associativity of base change, since $(\mathscr{F} \otimes_S \mathscr{O}_{S'}) \otimes_{S'} \mathscr{O}_{S''} = \mathscr{F} \otimes_S \mathscr{O}_{S''}$. ☐

Proposition 10.3.20. *Let Y be an S-scheme and $f : X \to Y$ be an S-morphism. For any $\mathscr{O}_Y$-module $\mathscr{G}$ and any morphism $S' \to S$, we have $(f_{(S')})^*(\mathscr{G}_{(S')}) = (f^*(\mathscr{G}))_{(S')}$.*

Proof. This follows from the diagram

$$\begin{tikzcd} X_{(S')} \arrow[r, "f_{(S')}"] \arrow[d] & Y_{(S')} \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

and the functoriality of inverse images. ☐

Corollary 10.3.21. *Let X, Y be S-schemes and $\mathscr{F}$ (resp. $\mathscr{G}$) be an $\mathscr{O}_X$-module (resp. $\mathscr{O}_Y$-module). For any morphism $S' \to S$, the inverse image of the sheaf $(\mathscr{F}_{(S')}) \otimes_{S'} (\mathscr{G}_{(S')})$ under the canonical isomorphism $(X \times_S Y)_{(S')} \cong (X_{(S')}) \times_{S'} (Y_{(S')})$ is equal to $(\mathscr{F} \otimes_S \mathscr{G})_{(S')}$.*

Proof. If p, q are the projections of $X \times_S Y$, the isomorphism is given by $(p_{(S')}, q_{(S')})'_{S'}$, so the corollary follows from Proposition 10.3.15 and Proposition 10.3.20. ☐

Proposition 10.3.22. *Let X, Y be two S-schemes and $\mathscr{F}$ (resp. $\mathscr{G}$) be an $\mathscr{O}_X$-module (resp. $\mathscr{O}_Y$-module). Let p, q be the projections of $Z = X \times_S Y$, z be a point of Z, and put $x = p(z)$, $y = q(z)$. Then the stalk $(\mathscr{F} \otimes_S \mathscr{G})_z$ is isomorphic to $(\mathscr{F}_x \otimes_{\mathscr{O}_{X,x}} \mathscr{O}_{Z,z}) \otimes_{\mathscr{O}_{Z,z}} (\mathscr{G}_y \otimes_{\mathscr{O}_{Y,y}} \mathscr{O}_{Z,z}) = \mathscr{F}_x \otimes_{\mathscr{O}_{X,x}} \mathscr{O}_{Z,z} \otimes_{\mathscr{O}_{Y,y}} \mathscr{G}_y$.*

Proof. Since the question is local, we can reduce to the affine case, and the assertion follows from Corollary 10.1.10. ☐

Corollary 10.3.23. *With the notations in Proposition 10.3.22, if $\mathscr{F}$ and $\mathscr{G}$ are of finite type, then*

$$\operatorname{supp}(\mathscr{F} \otimes_S \mathscr{G}) = p^{-1}(\operatorname{supp}(\mathscr{F})) \cap q^{-1}(\operatorname{supp}(\mathscr{G})).$$

Proof. As $p^*(\mathscr{F})$ and $q^*(\mathscr{G})$ are of finite type over $\mathscr{O}_Z$, in view of Proposition 10.3.22 and Proposition 1.4.39, we can reduce to the case $\mathscr{G} = \mathscr{O}_Y$, and the assertion then follows from the formula $\operatorname{supp}(p^{-1}(\mathscr{F})) = p^{-1}(\operatorname{supp}(\mathscr{F}))$. ☐

10.3.4 Scheme valued points

Let X be a scheme; for any scheme T, we denote by $X(T)$ the set $\operatorname{Hom}(T, X)$ of morphisms from T to X, and the elements of this set will be called **points of X with values in T**. The operation $T \mapsto X(T)$ is then a contravariant functor from the category of schemes to that of sets (in one word, we identify the scheme X with the induced functor h_X on **Sch**). Moreover, any morphism $g : X \to Y$ of schemes defines a natural transform $X(T) \to Y(T)$, which send $v \in X(T)$ to $g \circ v \in Y(T)$. The product of two S-schemes X and Y is then defined by the canonical isomorphism

$$(X \times_S Y)(T) \xrightarrow{\sim} X(T) \times_{S(T)} Y(T) \tag{10.3.3}$$

where the maps $X(T) \to S(T)$ and $Y(T) \to S(T)$ corresponds to the structural morphisms $X \to S$ and $Y \to S$.

If we are given a scheme S and consider S-schemes and S-morphisms, we denote by $X(T)_S$ the set $\mathrm{Hom}_S(T, X)$ of S-morphisms $T \to X$, and omit the index S if there is no risk of confusion. We also say the elements of $X(T)_S$ are the (S-)points of the S-scheme X with values in the S-scheme T. In particular, an S-section of X is none other than a point of X with values in S. The formula (10.3.3) is then written as

$$(X \times_S Y)(T)_S = X(T)_S \times Y(T)_S;$$

more generally, if Z is an S-scheme, X, Y, T are Z-schemes, we have

$$(X \times_Z Y)(T)_S = X(T)_S \times_{Z(T)_S} Y(T)_S.$$

We remark that for any morphism $S' \to S$, the set $X(S')_S = \mathrm{Hom}_S(S', X)$ is identified with the set $\mathrm{Hom}_{S'}(S', X')$, where $X' = X \times_S S'$, which is the set of S'-sections of X'.

In T (resp. S) is an affine scheme with ring B (resp. A), we replace T (resp. S) by B (resp. A) in the above notations, and we then refer points of X with values in the ring B, or points of the A-scheme X with values in the A-algebra B for the elements $X(B)$ of $X(B)_A$, respectively. We also call $X(T)_A$ the set of points of the A-scheme X with values in the A-scheme T.

Consider in particular the case where T is a local scheme $\mathrm{Spec}(A)$, where A is a local ring; the elements $X(A)$ corresponds to local homomorphisms $\mathscr{O}_{X,x} \to A$ for $x \in X$ (Corollary 10.2.15); we say the point x of the underlying topological space X is the **locality** of the point of X with values in A to which it corresponds (of course, several distinct points of X with values in A can have the same locality), or that the point of X with values in A which corresponds to x is **localizaed in x**.

Even more particularly, a point of X with values in a field K correspond to a point $x \in X$ and a field extension $\kappa(x) \to K$. If X is an S-scheme, saying that $S' = \mathrm{Spec}(K)$ is an S-scheme means K is an extension of the residue field $\kappa(s)$ for an point $s \in S$; an element of $X(K)_S$, which is called **a point of X lying over s with values in K**, corresponds then to a $\kappa(s)$-homomorphism $\kappa(x) \to K$, where x is a point of the topological space X lying over s (hence $\kappa(x)$ is an extension of $\kappa(s)$).

The points of X with values in an algebraically closed field K are called **geometric points** of the scheme X;[2] the field K is called the **value field** of the geometric point. If X is an S-scheme and s is an point of S, a **geometric point of X lying over s** is then a geometric point of X localizaed in a point of X lying over s. We then have a map $X(K) \to X$, which send a geometric point with values in K to the point it locates.

If $S = \mathrm{Spec}(k)$ is the spectrum of a field k and X is an S-scheme, the S-points of X with values in k is identified with the S-sections of X, or with the points x of X such that the canonical homomorphism $k \to \kappa(x)$ is an isomorphism since only at such a point there exists a homomorphism $\kappa(x) \to k$ such that the composition $k \to \kappa(x) \to k$ is the identity. Such points are called the **rational points** over k of the k-scheme X. Note that if k' is an extension of k, the points of X with values in k' correspond to the points of $X' = X_{(k')}$ rational over k' (cf. (10.3.2)).

The example $X = \mathrm{Spec}(K)$, where K is an nontrivial extension of k, shows that there do not necessarily exist in X rational points on k, even if X is nonempty. Still assuming that X is a k-scheme. For any point $x \in X$, there is always an extensions k' of k for which there is a point x' of $X' = X_{(k')}$ rational over k' and whose image by the canonical projection $X' \to X$ is x: it suffices to take for k' an extension of $\kappa(x)$, the k-monomorphism $\kappa(x) \to k'$ giving the sought point x'. When we thus passes from a point x to a rational point $x' \in X'$ over k' and above x, we say that we "make x rational."

[2] This terminology is also sometimes used when K is only separably closed, but at that time we will explicitly clarify which convention we adapt.

Proposition 10.3.24. *Let $S = \mathrm{Spec}(k)$ be the specturm of a field k, and X be an S-scheme. Then any k-rational point of X is closed in X.*

Proof. In fact, it suffices to show that the point x is closed in any open affine open set containing x, so we may assume that $X = \mathrm{Spec}(A)$ is affine. In this case, since the composition homomorphism $k \to A \to \kappa(x)$ is an isomorphism (we know that $\kappa(x) = k$), we conclude in particular that $A/\mathfrak{p}_x \to k$ is an integral extension, which implies that $A/\mathfrak{p}_x$ is a field (Proposition 4.1.64). □

Proposition 10.3.25. *Let $(X_i)_{1 \le i \le n}$ be S-schemes, s a point of X, and x_i a point of X_i lying over s for each i. For there to be a point y of the scheme $Y = X_1 \times_S \cdots \times_S X_n$ whose projections on X_i is x_i, it is necessary and sufficient that the x_i are over the same point s of S.*

Proof. This condition is clearly necessary. Now let s be an element of S and x_i a point of X_i lying over s. Then there exist $\kappa(s)$-homomorphisms $\kappa(x_i) \to K$ where K is a common field. The composition $\kappa(s) \to \kappa(x_i) \to K$ are all identical, so the morphisms $\mathrm{Spec}(K) \to X_i$ corresponding to $\kappa(x_i) \to K$ are S-morphisms, and we conclude that they define a unique morphism $\mathrm{Spec}(K) \to Y$. If y is the corresponding point of Y, it is clear that its projection into each of the X_i is x_i. □

In other words, if we denote by (X) the set underlying X, we see that we have a canonical surjective map $(X \times_S Y) \to (X) \times_{(S)} (Y)$; we have already pointed that this maps is not injective in general; that is, there can be multiple points distinct in $X \times_S Y$ having same projections to X and Y.

Corollary 10.3.26. *Let $f : X \to Y$ be an S-morphism, $f_{(S')} : X_{(S')} \to Y_{(S')}$ the S'-morphism induced by a base change $S' \to S$. Let p (resp. q) be the projection $X_{(S')} \to X$ (resp. $Y_{(S')} \to Y$); for any subset V of X, we have*

$$q^{-1}(f(M)) = f_{(S')}(p^{-1}(M)).$$

Proof. By Corollary 10.3.10, $X_{(S')}$ is identified with the product $X \times_Y Y_{(S')}$ and we have the following commutative diagram

$$\begin{tikzcd} X_{(S')} \arrow[r, "f_{(S')}"] \arrow[d, "p"'] & Y_{(S')} \arrow[d, "q"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

By Proposition 10.3.25, the relation $q(y') = f(x)$ for $x \in V$, $y' \in Y_{(S')}$ is equivalent to the existence of a point $x' \in X_{(S')}$ such that $p(x') = x$ and $f_{(S')}(x') = y'$, whence the corollary. □

Proposition 10.3.27. *Let X, Y be S-schemes and $x \in X$, $y \in Y$ two points lying over the same point $s \in S$. Then the set of points $X \times_S Y$ with projections x and y is in canonical correspondence with the set of types of the composition field extension of $\kappa(x)$ and $\kappa(y)$, considered as extensions of $\kappa(s)$.*

Proof. Let p (resp. q) be the projection of $X \times_S Y$ to X (resp. Y) and let E be the subspace $p^{-1}(x) \cap q^{-1}(y)$ of the underlying topological space of $X \times_S Y$. We first note that since x and y are lying over s, the morphisms $\mathrm{Spec}(\kappa(x)) \to S$ and $\mathrm{Spec}(\kappa(y)) \to S$ factor through $\mathrm{Spec}(\kappa(s))$:

$$\begin{tikzcd} & \mathrm{Spec}(\kappa(x)) \arrow[d] & \\ \mathrm{Spec}(\kappa(y)) \arrow[r] & \mathrm{Spec}(\kappa(s)) \arrow[r] & S \end{tikzcd}$$

since $\mathrm{Spec}(\kappa(s)) \to S$ is a monomorphism by Corollary 10.2.18, it follows immediately that we have

$$P = \mathrm{Spec}(\kappa(x)) \times_S \mathrm{Spec}(\kappa(y)) = \mathrm{Spec}(\kappa(x)) \times_{\mathrm{Spec}(\kappa(s))} \mathrm{Spec}(\kappa(y)) = \mathrm{Spec}(\kappa(x) \otimes_{\kappa(s)} \kappa(y)).$$

Let $i : \mathrm{Spec}(\kappa(x)) \to X$ and $j : \mathrm{Spec}(\kappa(y)) \to Y$ be the canonical morphisms, we put $\alpha = i \times_S j : P \to E$ to be the map on the underlying topological space. On the other hand, any point $z \in E$ defines two $\kappa(s)$-homomorphisms $\kappa(x) \to \kappa(z)$ and $\kappa(y) \to \kappa(z)$, hence a $\kappa(s)$-homomorphism $\kappa(x) \otimes_{\kappa(s)} \kappa(y) \to \kappa(z)$, which corresponds to a morphism $\mathrm{Spec}(\kappa(z)) \to P$; we take $\beta(z)$ to be the image of this morphism, which defines a map $\beta : E \to P$.

To verify that $\alpha \circ \beta$ and $\beta \circ \alpha$ are identity maps, we need the following commutative diagram

$$\begin{tikzcd}
\mathrm{Spec}(\kappa(z)) \arrow[dr] \arrow[drr, bend left] \arrow[ddr, bend right] & & & \\
& P \arrow[r] \arrow[d] \arrow[dr, "\alpha"] & \mathrm{Spec}(\kappa(y)) \arrow[dr, "j"] & \\
& \mathrm{Spec}(\kappa(x)) \arrow[dr, "i"'] & E \arrow[r] \arrow[d] & Y \arrow[d] \\
& & X \arrow[r] & S
\end{tikzcd}$$

for $z \in E$. By the uniqueness part of the universal property of fiber products, the canonical morphism $\mathrm{Spec}(\kappa(z)) \to E$ induced by $\mathrm{Spec}(\kappa(z)) \to X$ and $\mathrm{Spec}(\kappa(z)) \to Y$ is given by the composition $\mathrm{Spec}(\kappa(z)) \to P \to E$, and also equal to the canonical morphism $\mathrm{Spec}(\kappa(z)) \to E$ of the scheme $X \times_S Y$ at z (Corollary 10.2.17). This means the image of $\beta(z)$ under α is exactly the image of the canonical morphism $\mathrm{Spec}(\kappa(z)) \to E$, which is just z; this shows $\alpha \circ \beta = 1_E$. As for $\beta \circ \alpha$, we just note that if $z = \alpha(\mathfrak{p})$ for some $\mathfrak{p} \in P$ (a prime ideal), then the morphism α induces a field extensins $\kappa(z) \to \kappa(\mathfrak{p})$, which corresponds to morphism $\mathrm{Spec}(\kappa(\mathfrak{p})) \to \mathrm{Spec}(\kappa(z))$. Again by the uniqueness part of the fiber product P, we conclude that the canonical morphism $\mathrm{Spec}(\kappa(\mathfrak{p})) \to P$ factors through $\mathrm{Spec}(\kappa(z))$, which means $\beta(z) = \mathfrak{p}$, so $\beta \circ \alpha = 1_P$. Finally, we recall that the set P corresponds to composition fields of $\kappa(x)$ and $\kappa(y)$ over $\kappa(s)$. □

10.3.5 Surjective morphisms

Let $\mathcal{P}$ be a property for morphisms of schemes. We consider the following conditions:

(i) If $f : X \to X'$ and $g : Y \to Y'$ are S-morphisms satisfying $\mathcal{P}$, then $f \times_S g$ also satisfies $\mathcal{P}$.

(ii) If $f : X \to Y$ is an S-morphism satisfying $\mathcal{P}$ and $S' \to S$ is a morphism, then $f_{(S')} : X_{(S')} \to Y_{(S')}$ also satisfies $\mathcal{P}$.

Since $f_{(S')} = f \times_S 1_{S'}$, we see if any identity morphism satisfies $\mathcal{P}$, then (i) implies (ii). On the other hand, since $f \times_S g$ is the following composition

$$X \times_S Y \xrightarrow{f \times 1_Y} X' \times_S Y \xrightarrow{1_{X'} \times g} X' \times_S Y'$$

it is clear that if the composition of two morphisms satisfying $\mathcal{P}$ still satisfies $\mathcal{P}$, then (ii) implies (i) (in this case we say $\mathcal{P}$ is stable under composition). In general, a property $\mathcal{P}$ is called **stable under base change** if it satisfies the condition (ii). For example Proposition 10.3.12 just says that being a monomorphism is stable under base change. On the other hand, if $\mathcal{P}$ is an arbitrary property of morphisms, we say a morphism $f : X \to S$ **satisfies $\mathcal{P}$ universally** (or is **universally** $\mathcal{P}$), if for any morphism $S' \to S$ the inverse image $f_{(S')}$ satisfies $\mathcal{P}$.

Our first application of the above definition is that surjectivity is stable under base change:

Proposition 10.3.28. *Surjective morphisms of schemes are stable under base change.*

Proof. Note that it is clear that surjectivity is stable under composition, in fact we have the both conditions (i) and (ii) described above. But condition (ii) follows from Corollary 10.3.26 by setting $V = X$. $\square$

Proposition 10.3.29. *For a morphism $f : X \to Y$ of schemes to be surjective, it is necessary and sufficient that for any field K and any morphism $\mathrm{Spec}(K) \to Y$, there exists an extension K' of K and a morphism $\mathrm{Spec}(K') \to X$ fitting into the following diagram*

$$\begin{tikzcd} \mathrm{Spec}(K') \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \mathrm{Spec}(K) \arrow[r] & Y \end{tikzcd}$$

Proof. The condition is sufficient, since for any $y \in Y$, we can apply the canonical morphism $\mathrm{Spec}(\kappa(y)) \to Y$ to get a morphism $\mathrm{Spec}(K) \to X$, which gives a inverse image of y in X. Conversely, suppose that f is surjective, and let $y \in T$ be the image of $\mathrm{Spec}(K)$ in Y; there exists $x \in X$ such that $f(x) = y$. Consider the monomorphism $\kappa(y) \to \kappa(x)$ corresponding to f, and take an extension K' of $\kappa(y)$ containing $\kappa(x)$ and K; the morphism $\mathrm{Spec}(K') \to X$ corresponding to $\kappa(x) \to K'$ then satisfies the requirement. $\square$

Corollary 10.3.30. *For a morphism $f : X \to Y$ to be surjective, it is necessary and sufficient that, for any field K, there exist an algebraically closed extension K' of K such that the map $X(K') \to Y(K')$ corresponding to f is surjective.*

Proof. In view of Proposition 10.3.29, this condition is sufficient. Conversely, suppose that f is surjective and let K be a field. If p is the characteristic of K, let us take for K' an algebraically closed extension of K having over the prime field P a transcendence basis of strictly larger cardinality to the cardinals of all the transcendence bases on P of the residual fields of X and Y having characteristic p. It then remains to see, with the same notations as in Proposition 10.3.29, that any monomorphism $u : \kappa(y) \to K'$ factors into

$$\kappa(y) \xrightarrow{w} \kappa(x) \xrightarrow{v} K'$$

where $w = f^x$. Now, let L a purely transcendental extension of P contained in $\kappa(y)$ and over which $\kappa(y)$ is algebraic; if B is a transcendence basis of L over P, we can complete $w(B)$ into a transcendence basis B' of $\kappa(x)$ on P, and then (due to the assumption made on the trancendence bases of K') define a monomorphism $v_1 : P(B') \to K'$ such that $v_1 \circ (w|_L)$ coincides with $u|_L$. There is also an isomorphism $v_2 = u \circ w^{-1}$ from $w(\kappa(y))$ to $u(\kappa(y))$ such that v_2 and v_1 coincide in $w(L)$; as $w(\kappa(y))$ and $P(B' - w(B))$ are linearly disjoint on $w(L)$, we can extend v_1 and v_2 into a monomorphism v_0 of $M = P(B')(w(\kappa(y)))$ in K'; as K' is algebraically closed and $\kappa(x)$ is algebraic over M, we can finally extend v_0 into the monomorphism $v : \kappa(x) \to K'$, which completes the proof. $\square$

10.3.6 Radical morphisms

Proposition 10.3.31. *Let $f : X \to Y$ be a morphism of schemes. The following conditions are equivalent:*

(i) *f is universally injective.*

(ii) *The map f is injective and for any $x \in X$, the extension $f^x : \kappa(f(x)) \to \kappa(x)$ is purely inseparable.*

(iii) *For any field K, the map $X(K) \to Y(K)$ corresponding to f is injective.*

(iv) *For any field K, there exists an algebraically closed extension K' of K such that the map $X(K') \to X(K')$ corresponding to f is injective.*

(v) *The diagonal morphism $\Delta_f : X \to X \times_Y X$ is surjective.*

The morphism f is called ***radical*** *if it satisfies the above equivalent conditions.*

Proof. It is clear that (i) implies f is injective; on the other hand, if $\kappa(x)$ is not a purely inseparable extension of $\kappa(f(x))$, there exist two distinct $\kappa(f(x))$-monomorphisms $\kappa(x) \to K$ into an algebraically closed extension K of $\kappa(x)$; hence we get two distinct morphisms g_1, g_2 of $\mathrm{Spec}(K)$ to X, whose compositions $f \circ g_1$, $f \circ g_2$ equal to the same morphism $\mathrm{Spec}(K) \to Y$. If we set $Y' = \mathrm{Spec}(K)$, there then would be two distinct Y'-sections of $X_{(Y')}$; since K is a field, the Y'-sections of $X_{(Y')}$ correspond one-to-one to their images (the rational points of $X_{(Y')}$ over K), so $f_{(Y')} : X_{(Y')} \to Y'$ would not be injective, contrary to the assumption.

To show that (ii) implies (iii), we note that by Corollary 10.2.17, (iii) signifies that for any $y \in Y$ and a monomorphism $\kappa(y) \to K$ to a field K, there do not exist two distinct $\kappa(y)$-monomorphisms $\kappa(x_1) \to K$, $\kappa(x_2) \to K$, where x_1, x_2 are both lying over y. Now (ii) implis that if we have two such monomorphisms, they come from the same point x since f is injective; moreover, since $\kappa(x)$ is a purely inseparable extension of $\kappa(y)$, the two monomorphisms $\kappa(x) \to K$ are necessarily equal.

It is clear that (iii) implies (iv). Conversely, suppose that (iv) holds; let K be a field and K' be an algebraically closed extension of K; then the diagram

$$\begin{tikzcd} X(K) \arrow[r, "\alpha"] \arrow[d, "\varphi"] & Y(K) \arrow[d, "\varphi'"] \\ X(K') \arrow[r, "\alpha'"] & Y(K') \end{tikzcd} \tag{10.3.4}$$

is commutative. Since the homomorphism $K \to K'$ is injective, φ is injective by Corollary 10.2.17, and by hypothesis we can choose K' such that α' is also injective. Then α is injective, which shows (iii).

To see that (iv) and (v) are equivalent, we note that, for the morphism Δ_f to be surjective, it is necessary and sufficient that, in view of Corollary 10.3.30, for any field K, there exists an algebraically closed extension K' of K such that the diagonal map

$$X(K') \to (X \times_Y X)(K') = X(K') \times_{Y(K')} X(K')$$

corresponding to Δ_f is surjective. But by the definition of this fiber product, this signifies that the map $X(K') \to Y(K')$ is injective, whence our claim.

Finally, we prove that (iii) implies (i). If (iii) is satisfied, then for any base change $Y' \to Y$, the map

$$(X \times_Y Y')(K) \to Y'(K)$$

is still injective, as we immediately verify by noting that $(X \times_Y Y')(K) = X(K) \times_{Y(K)} Y'(K)$ and that $X(K) \to Y(K)$ is injective. Therefore, it suffices to prove that if $X(K) \to Y(K)$ is injective for any field K, then f is injective. Now if x_1 and x_2 are two points of X such that $f(x_1) = f(x_2) = y$, there then exists a field extension K of $\kappa(y)$ and $\kappa(y)$-monomorphisms $\kappa(x_1) \to K$, $\kappa(x_2) \to K$; the corresponding morphisms u_1, u_2 of $\mathrm{Spec}(K)$ to X are then such that $f \circ u_1 = f \circ u_2$, and by hypothesis this implies $u_1 = u_2$, so $x_1 = x_2$. □

Remark 10.3.32. We then obtain examples of injective morphisms (and even bijective) of schemes but not universally injective: it suffices to take a morphism $\mathrm{Spec}(K) \to \mathrm{Spec}(k)$, where K is a separable extension of k distinct from k.

Corollary 10.3.33. *A monomorphism of schemes $f : X \to Y$ is radical. In particular, if A is a ring, S is a multiplicative subset of A, then the canonical morphism* $\mathrm{Spec}(S^{-1}A) \to \mathrm{Spec}(A)$ *is radical.*

Proof. The first assertion follows from Proposition 10.3.31(iii), and the second from the fact that $\mathrm{Spec}(S^{-1}A) \to \mathrm{Spec}(A)$ is a monomorphism. □

Corollary 10.3.34. *Let $f : X \to Y$ be a radical morphism, $g : Y' \to Y$ a morphism, and $X' = X_{(Y')}$. Then the radical morphism $f_{(Y')}$ is a bijection from the underlying space X' to $g^{-1}(f(X))$. Moreover, for any field K, the set $X'(K)$ is identified with the inverse image in $Y'(K)$ under the map $Y'(K) \to Y(K)$ (corresponding to g) of the subset $X(K)$ of $Y(K)$.*

Proof. The first assertion follows from Proposition 10.3.31(ii) and Corollary 10.3.26; the second one follows from the commutative diagram (10.3.4). □

Proposition 10.3.35. *Let $f : X \to Y$ and $g : Y \to Z$ be two morphism of schemes.*

(a) *If f and g are radical, so is $g \circ f$.*

(b) *Conversely, if $g \circ f$ is radical, so is f.*

Proof. It suffices to apply the functors X, Y, Z on any field K, and use the characterization of Proposition 10.3.31(iii); the verification boils down to set-theoretic issues, which are straightforward. □

Proposition 10.3.36. *Radical morphisms are stable under base changes. In particular, if $f : X \to X'$ and $g : y \to Y'$ are radical S-morphisms, then so is $f \times_S g$.*

Proof. Since radical is equivalently to universally injective, the first assertion is clear. The second one follows from the first one since radical morphisms are stable under composition by Proposition 10.3.35. □

10.3.7 Fibers of morphisms

Proposition 10.3.37. *Let $f : X \to Y$ be a morphism, y be a point of Y, and $\mathfrak{a}_y$ be an ideal of $\mathscr{O}_{Y,y}$. Put $Y' = \mathrm{Spec}(\mathscr{O}_{Y,y}/\mathfrak{a}_y)$, $X' = X \times_Y Y'$, and let $p : X' \to X$ be the canonical projection. Then p is a homeomorphism from X' onto the subspace $f^{-1}(Y')$ of X (where we identify Y' are a subspace of Y, cf. Corollary 10.2.12). Moreover, for any $x' \in X'$, the homomorphism $p^\#_{x'} : \mathscr{O}_{X,p(x')} \to \mathscr{O}_{X',x'}$ is surjective with kernel $\mathfrak{a}_y\mathscr{O}_{X,x}$.*

Proof. The morphism $\mathrm{Spec}(\mathscr{O}_{Y,y}/\mathfrak{a}_y) \to Y$ is radical (Proposition 10.3.31), so we conclude from Corollary 10.3.34 that p identifies the space $X' = X \times_Y \mathrm{Spec}(\mathscr{O}_{Y,y}/\mathfrak{a}_y)$ with $f^{-1}(Y')$. It remains to show that p is a homeomorphism and identify its morphism on stalks. Since this question is local, we may assume that $X = \mathrm{Spec}(B)$, $Y = \mathrm{Spec}(A)$, where B is an A-algebra. Then the morphism p corresponds to the homomorphism $1 \otimes \rho : B \to B \otimes_A A'$, where $\rho : A \to A' = A_{\mathfrak{p}_y}/\mathfrak{a}_y$ is the canonical homomorphism. Now any element of $B \otimes_A A'$ is of the form

$$\sum_i b_i \otimes \rho(a_i)/\rho(s) = \rho\Big(\sum_i a_i b_i \otimes 1\Big)(1 \otimes \rho(s))^{-1}$$

where $s \notin \mathfrak{p}_y$, so we can apply Proposition 1.4.22. The assertion on the homomorphism $p^\#_{x'}$ also follows from the equality. □

We will mainly use Proposition 10.3.37 for the case $\mathfrak{a}_y = \mathfrak{m}_y$ is the maximal ideal of $\mathscr{O}_{Y,y}$. If there is no confusion, we denote by X_y the $\kappa(y)$-scheme obtained by transporting of scheme structure from $X' = X \otimes_S \kappa(s)$ to $f^{-1}(y)$ via the projection p, and this is always the scheme that will be involved when we speak of the **fiber** $f^{-1}(y)$ of the morphism f as a scheme.

Let X, Y be two S-schemes and $f : X \to Y$ an S-morphism. By the transitivity of base change, we have the canonical isomorphism

$$X_s = X \times_Y Y_s$$

for any $s \in S$; the morphism $f_s : X_s \to Y_s$ induced by f by the base change $Y_s \to Y$ is such that, for any $y \in Y_s$, the fiber $f_s^{-1}(y)$ is identified with the $\kappa(y)$-scheme $f^{-1}(y)$, since the residue field of Y_s at y is the same as that of Y at y, in view of Proposition 10.3.37

Proposition 10.3.38 (Transitivity of Fibers). *Let $f : X \to Y$, $g : Y' \to Y$ be two morphisms; put $X' = X_{(Y')}$ and $f' = f_{(Y')}$. For any $y' \in Y'$, if $y = g(y')$, then the scheme $X'_{y'}$ is canonically isomorphic to $X_y \otimes_{\kappa(y)} \kappa(y')$.*

Proof. In fact, by the transitivity of base change, we have canonical isomorphisms

$$(X \otimes_Y \kappa(y)) \otimes_{\kappa(y)} \kappa(y') \cong X \times_Y \operatorname{Spec}(\kappa(y')) \cong (X \times_Y Y') \otimes_{Y'} \kappa(y')$$

The left one is $X_y \otimes_{\kappa(y)} \kappa(y')$, and the right one is $X'_{y'} \otimes_{Y'} \kappa(y')$, so our assertion follows. □

Proposition 10.3.39. *Let $f : X \to Y$ be a monomorphism of schemes. Then for each $y \in Y$, the fiber X_y is a $\kappa(y)$-scheme which is either empty or $\kappa(y)$-isomorphic to $\operatorname{Spec}(\kappa(y))$.*

Proof. By Proposition 10.3.37, X_y is reduced to a point, and hence affine. By Proposition 10.3.12, the morphism $f_y : X_y \to \operatorname{Spec}(\kappa(y))$ induced by f under base change is still a monomorphism. If A is the ring of X_y, this signifies that the homomorphism $A \otimes_{\kappa(y)} A \to A$ which maps $a \times a'$ to aa' is bijective, and clearly implies that $A = \kappa(y)$, since otherwise there exist an element $a \in A$ not contained in $\kappa(y)$ and the image $a \otimes 1$ and $1 \otimes a$ are distinct, but both mapped to a. □

Proposition 10.3.40. *Let $f : X \to Y$ be an S-morphism of S-schemes, $g : S' \to S$ a surjective morphism, and $f' = f_{(S')} : X' = X_{(S')} \to Y' = Y_{(S')}$. Consider the following properties:*

(a) *surjective;*

(b) *injective;*

(c) *dominant;*

(d) *finite fiber (as sets);*

Then if $\mathcal{P}$ denotes one of the properties above and if f' satisfies $\mathcal{P}$, then so does f.

Proof. Since the projection $Y' \to Y$ is surjective by Proposition 10.3.28, we can, by virtue of Corollary 10.3.10, limiting ourselves to the case where $Y = S$, $Y' = S'$. For any $y' \in Y'$, let $y = g(y')$; we have the transitivity relation $X'_{y'} \cong X_y \otimes_{\kappa(y)} \kappa(y')$ (Proposition 10.3.38). Since the morphism $\operatorname{Spec}(\kappa(y')) \to \operatorname{Spec}(\kappa(y))$ is surjective, so is the projection $X'_{y'} \to X_y$ (Proposition 10.3.28). Thus, if $X'_{y'}$ is nonempty (resp. a singleton, resp. a finite set), the same holds for X_y. Since $S' \to S$ is surjective, this proves (a), (b), and (d). On the other hand, if f' is dominant, so is the composition $g \circ f' = f \circ g'$; but since $g' : X' \to X$ is surjective by Proposition 10.3.28, this implies f is dominant. □

10.3.8 Universally open and universally closed morphisms

Following the usual terminology, we say a morphism $f : X \to Y$ is **universally open** (resp. **universally closed**, resp. a **universal embedding,** resp. a **universal homeomorphism**) if for any base change $Y' \to Y$, the morphism $f_{(Y')} : X_{(Y')} \to Y'$ is open (resp. closed, resp. an embedding, resp. a homeomorphism).

Proposition 10.3.41.

(i) *The composition of two universally open morphisms (resp. universally closed morphisms, resp. two universal embeddings, resp. two universal homeomorphisms) is universally open (resp. universally closed, resp. a universal embedding, resp. a universal homeomorphism).*

(ii) *If $f : X \to X'$, $g : Y \to Y'$ are two universally open (resp. universally closed, resp. two universal embeddings, resp. two universal homeomorphisms) S-morphisms, so is the product $f \times_S g$.*

(iii) *If $f : X \to Y$ is a universally open (resp. universally closed, resp. a universal embedding, resp. a universal homeomorphism) S-morphism, so is $f_{(S')} : X_{(S')} \to Y_{(S')}$ for any base change $S' \to S$.*

(iv) *Let $f : X \to Y$, $g : Y \to Z$ be two morphism such that f is surjective; if $g \circ f$ is universally open (resp. universally closed, resp. a universal embedding, resp. a universal homeomorphism), so is g.*

(v) *Let (U_α) be an open cover of Y. For a morphism $f : X \to Y$ to be universally open (resp. universally closed, resp. a universal embedding, resp. a universal homeomorphism), it is necessary and sufficient that, for each α, the restriction $f_\alpha : f^{-1}(U_\alpha) \to U_\alpha$ is universally open (resp. closed).*

Proof. The assertion (i) follows from definiton, and so does (iii). We have already remareked that (i) and (iii) together imply (ii), since identity morphisms satisfies all the properties mentioned above. To prove (iv), we note that for any morphism $Z' \to Z$, the morphism $f_{(Z')} \to X_{(Z')}$ is surjective, so it suffices to prove that if $g \circ f$ is open (resp. closed, resp. an embedding, resp. a homeomorphism) and f is surjective, then so is g. For the case where $g \circ f$ is open or closed, the fact that g is open or closed result easily; for the other two cases, we can limit ourselves to the case $g(f(X)) = g(Y) = Z$, so that $g \circ f$ is a homeomorphism from X to Z. As f is surjective, g is necessarily bijective, and as it is open by the first two cases already shown, g is then a homeomorphism from Y to Z.

Finally, the necessity in (v) follows from (iii) and Corollary 10.3.2. Conversely, suppose the condition in (v) and let $g : Y' \to Y$ be a morphism; then $g^{-1}(U_\alpha) = U'_\alpha$ form an open cover of Y' and if $f' = f_{(Y')}$, the restriction $f'^{-1}(U'_\alpha) \to U'_\alpha$ of f' is none other than $(f_\alpha)_{(U'_\alpha)}$ (Corollary 10.3.2). We can then reduce to proving that f is open (resp. closed, resp. an embedding, resp. a homeomorphism) if each f_α is, which is immediate. □

We recall that openness of a map is a local property, i.e., a map $f : X \to Y$ is open if and only if it is open at every point of X. Simialrly, the property of universally open morphisms is also local. To justify this, we define a morphism $f : X \to Y$ is **universally open at a point** $x \in X$ if for any base change $Y' \to Y$ the morphism $f_{(Y')} : X_{(Y')} \to Y_{(Y')}$ is open at any point x' of X' lying over x.

Proposition 10.3.42. *Let $f : X \to Y$, $g : Y \to Z$ be two morphisms of schemes, x a point of X, and $y = f(x)$.*

(a) *If f is universally open at x and g is universally open at y, then $g \circ f$ is universally open at x. Conversely, if $g \circ f$ is universally open at x, then g is universally open at y.*

(b) *If $f : X \to Y$ is an S-morphism universally open at a point $x \in X$, then for any base change $S' \to S$, the morphism $f_{(S')} : X_{(S')} \to Y_{(S')}$ is universally open at any point of $X_{(S')}$ lying over x.*

Proof. Assertion (b) is an immediate concequence of the definition of universally openness at a point and transitivity of base change. Also, it follows from Corollary 10.3.26 that to prove (a), we may drop the "universally" condition and prove the assertion for openness, which then follows from ([**?**] new, 0_{I}, 2.8.2). ☐

Proposition 10.3.43. *Let X, Y be two schemes, $f : X \to Y$ be a morphism, and x be a point of X. Let (Y_i) be a locally finite covering of Y by closed subschemes, and suppose that for each i such that $f(x) \in Y_i$, the restriction $f_i : f^{-1}(Y_i) \to Y_i$ of f is an open morphism (resp. universally open) at the point x. Then f is open (resp. universally open) at the point x.*

Proof. The assertion about openness is immediate. For the universal part, consider a morphism $g : Y' \to Y$ and in Y' the closed subschemes $Y_i' = g^{-1}(Y_i)$ (Proposition 10.4.16), which underlying spaces form a locally finite covering of Y'. If $f' = f_{(Y')} : X_{(Y')} \to Y'$ is the base change of f, the restriction $f_i' : f'^{-1}(Y_i) \to Y_i'$ of f' equals to $(f_i)_{(Y')}$, so we can apply ([**?**] new, 0_{I}, 2.10.2 (ii)) to f'. ☐

Proposition 10.3.44. *Let $f : X \to Y$ be a morphism of schemes, $(X_i)_{1\leq i\leq n}$ (resp. $(Y_i)_{1\leq i\leq n}$) be a finite family of closed subschemes of X (resp. Y), and $j_i : X_i \to X$ (resp. $h_i : Y_i \to Y$) be the canonical injections. Suppose that $(X_i)_{1\leq i\leq n}$ covers X and for each i there exists a morphism $f_i : X_i \to Y_i$ fitting into the following diagram*

$$\begin{tikzcd} X_i \arrow[r, "f_i"] \arrow[d, "j_i"'] & Y_i \arrow[d, "h_i"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

Then for f to be closed (resp. universally closed), it is necessary and sufficient that each f_i is.

Proof. If f is closed, then each f_i is closed since Y_i are closed in Y. Conversely, if the f_i are closed, for any closed subset F of X, we have $f_i(F \cap X_i) = f(F \cap X_i)$ and it closed in Y_i, hence in Y, and as $f(F)$ is the union of $f(F \cap X_i)$, it is therefore closed in Y.

For the case of universally closed morphisms, the condition is necessary because j_i is a closed immersion (hence universally closed, cf. Corollary 10.4.14), and if f is universally closed, so is $f \circ j_i = h_i \circ f_i$. But h_i is a closed immersion, hence separated (Proposition 10.5.26), it then follows that f_i is universally closed (Proposition 10.5.23).

Conversely, suppose that each f_i is universally closed, and consider the scheme Z that is the coproduct of that X_i. Let $u : Z \to X$ be the induced morphism by the j_i's. The restriction of $f \circ u$ to X_i is equal to $f \circ j_i = h_i \circ f$, hence universally closed (Corollary 10.4.14 and Proposition 10.3.41(i)); we then deduce from Corollary 10.3.3 that $f \circ u$ is universally closed. But since u is surjective by hypotheses, we conclude that f is universally closed (Proposition 10.3.41(iv)). ☐

Remark 10.3.45. If X only have finitely many irreducible components, then we deduce from Proposition 10.3.44 that, to verify a morphism $f : X \to Y$ is closed (resp. universally closed), we can reduce ourselves to doing it for dominant morphisms of integral schemes. In fact, let $(X_i)_{1\leq i\leq n}$ be the reduced subschemes of X with underlying spaces the irreducible components of X (Proposition 10.4.44), which are then integral. Let Y_i be the unique reduced closed subscheme of Y with underlying space $\overline{f(X_i)}$ (Proposition 10.4.44), which iss irreducible (**??**). If $j_i : X_i \to X$ (resp. $h_i : Y_i \to Y$) is the canonical injection, there then exists a dominant morphism $f_i : X_i \to Y_i$ such that $f \circ g_i = h_i \circ f_i$ (Proposition 10.4.48); we are then in the case of Proposition 10.3.44, so f is closed (resp. universally closed) if and only if each f_i is.

Proposition 10.3.46. *Let X, Y be two S-schemes and $f : X \to Y$ be an S-morphism. For f to be universally closed, it suffices that, for any base change $S' \to S$ where $S' = S \otimes_{\mathbb{Z}} \mathbb{Z}[(T_\lambda)_{\lambda\in I}]$ (denoted by $S[(T_\lambda)_{\lambda\in I}]$), the morphism $f_{(S')} : X_{(S')} \to Y_{(S')}$ is closed.*

Proof. We note that if (S_α) is an open cover of S and Y_α is the inverse image of S_α in Y, it suffices to prove that the restricted morphism $f_\alpha : f^{-1}(Y_\alpha) \to Y_\alpha$ is closed (Proposition 10.3.41(v)). Now the inverse image of S_α in S' is $S_\alpha[(T_\lambda)_{\lambda\in I}] = S'_\alpha$ and $(f_\alpha)_{(S')}$ is the restriction $(f_{(S')})^{-1}(Y'_\alpha) \to Y'_\alpha$ of $f_{(S')}$, where Y'_α is the inverse image of S'_α in $Y_{(S')}$. If the proposition is proved for S_α and f_α, it is then true for S and f. We can then assume that S is affine.

Now let (U_β) be an open covering of Y; for f to be universally closed, it suffices to prove that $f_\beta : f^{-1}(U_\beta) \to U_\beta$ is universally closed for each β (Proposition 10.3.41(v)). Again, the morphism $(f_\beta)_{(S')}$ is the restriction $(f_{(S')})^{-1}(U'_\beta) \to U'_\beta$ of $f_{(S')}$, where $U'_\beta = U_\beta \times_S S'$ is the inverse image of U_β in $Y_{(S')}$. If the proposition is proved for U_β and f_β, it then holds for Y and f. Therefore, we can further assume that Y is affine.

Let us first show that if $f_{(S')}$ is closed for any base change $S' \to S$, then f is universally closed. In fact, any Y-scheme Y' can be considered as an S-scheme, and as the morphism $Y \to S$ is separated (recall that Y and S are assumed to be affine), $X \times_Y Y'$ (resp. $Y \times_Y Y' = Y'$) is identified with a closed subscheme of $X \times_S Y'$ (resp. $Y \times_S Y'$) (Proposition 10.4.13). In the following commutative diagram

$$\begin{tikzcd} X \times_Y Y' \arrow[r, "f\times 1_{Y'}"] \arrow[d] & Y \times_Y Y' = Y' \arrow[d] \\ X \times_S Y' \arrow[r, "f_{(Y')}"] & Y \times_S Y \end{tikzcd}$$

the vertical morphisms are closed immersions, so if $f_{(Y')}$ is a closed morphism, so is $f \times 1_{Y'}$.

It remains to prove that $f_{(S')}$ is closed for arbitrary base change $S' \to S$ if this is true for $S' = S[(T_\lambda)]$. Now by hypotheses S is affine, and if (S'_γ) is an open covering of S', we see in the same manner as before that for $f_{(S')}$ to be closed, it suffices to prove that $f_{(S'_\gamma)}$ is closed. We can then assume S' to be affine. If $S = \mathrm{Spec}(A)$, we have $S' = \mathrm{Spec}(A')$, where A' is an A-algebra. Let $(t_\lambda)_{\lambda\in I}$ be a generator for A', which means there is a surjective A-homomorphism $A[(T_\lambda)] \to A'$ identifying A' with $A[(T_\lambda)]/\mathfrak{b}$, where $\mathfrak{b}$ is an ideal. If $S'' = \mathrm{Spec}(A[(T_\lambda)])$, S' is then a closed subscheme of S'', and $X_{(S')}$ (resp. $Y_{(S')}$) is identified with a closed subscheme of $X_{(S'')}$ (resp. $Y_{(S'')}$). The morphism $f_{(S')}$ is the restriction of $f_{(S'')}$ on $X_{(S')}$, and since $f_{(S'')}$ is closed by hypotheses, we conclude that $f_{(S')}$ is closed. This completes the proof. □

10.4 Subschemes and immersions

10.4.1 Subschemes

Proposition 10.4.1. *Let X be a scheme and $\mathscr{I}$ be a quasi-coherent ideal of $\mathscr{O}_X$. The support Y of the sheaf $\mathscr{O}_X/\mathscr{I}$ is then closed, and if $\mathscr{O}_Y$ is the sheaf induced on Y by $\mathscr{O}_Y/\mathscr{I}$, $(Y, \mathscr{O}_Y)$ is a scheme.*

Proof. Since the problem is local, it suffices to consider the affine case and show that Y is closed and $(X, \mathscr{O}_Y)$ is an affine scheme. In fact, if $X = \mathrm{Spec}(A)$, we have $\mathscr{O}_X = \widetilde{A}$ and $\mathscr{I} = \tilde{\mathfrak{a}}$, where $\mathfrak{a}$ is an ideal of A (Theorem 10.1.21); Y is then equal to the closed subset $V(\mathfrak{a})$ of X and is identified with the spectrum of $B = A/\mathfrak{a}$ (Proposition 1.4.46). Moreover, if $\rho : A \to B = A/\mathfrak{a}$ is the canonical homomorphism, the direct image ${}^a\rho_*(\widetilde{B})$ is canonically identified with the sheaf $\widetilde{A}/\tilde{\mathfrak{a}} = \mathscr{O}_X/\mathscr{I}$ (Corollary 10.1.6 and Proposition 10.1.12). These complete the proof. □

We say $(Y, \mathscr{O}_Y)$ is the **subscheme of** $(X, \mathscr{O}_X)$ **defined by the quasi-coherent ideal** $\mathscr{I}$. More generally, we say a locally ringed space $(Y, \mathscr{O}_Y)$ is a **subscheme** of a scheme $(X, \mathscr{O}_X)$ if Y is a locally closed subspace of X and if U denote the largest open subset of X containing Y such that Y is open in U (in other words, the complement of $\bar{Y} - Y$, so $U = (X - \bar{Y}) \cup Y$), then $(Y, \mathscr{O}_Y)$ is a subscheme of $(U, \mathscr{O}_X|_U)$ defined by a quasi-coherent ideal of $\mathscr{O}_X|_U$. We say the subscheme

$(Y, \mathscr{O}_Y)$ of $(X, \mathscr{O}_X)$ is **closed** if Y is closed in X (in this case $U = X$). It follows from this definition and Proposition 10.4.1 that closed subschemes of X are in one-to-one correspondence with quasi-coherent ideals of $\mathscr{O}_X$, since if two such ideals $\mathscr{I}$, $\mathscr{J}$ have the same support (closed) Y and the sheaf induced by $\mathscr{O}_X/\mathscr{I}$ and $\mathscr{O}_X/\mathscr{J}$ on Y are identical, then $\mathscr{I} = \mathscr{J}$.

Let $(Y, \mathscr{O}_Y)$ be a subscheme of X, U the largest open subset of X containing Y such that Y is closed in U, V an open subset of X contained in U; then $V \cap Y$ is closed in V. Moreover, if Y is defined by the quasi-coherent ideal $\mathscr{I}$ of $\mathscr{O}_X|_U$, then $\mathscr{I}|_V$ is a quasi-coherent ideal of $\mathscr{O}_X|_V$, and it is immediate that the scheme induced by Y over $Y \cap V$ is the closed subscheme of V defined by the ideal $\mathscr{I}|_V$.

In particular, the scheme induced by X over an open subset of X is a subscheme of X; such schemes are called **open subschemes** of X. One should note that a subscheme of X can have the underlying space being an open set U of X without being induced on this open subset by X: it is induced over U by X only if it is defined by the ideal 0 of $\mathscr{O}_X|_U$, and there are in general quasi-coherent ideals $\mathscr{I}$ of $\mathscr{O}_X|_U$ such that $(\mathscr{O}_X|_U)/\mathscr{I}$ have support U but is nonzero.

Proposition 10.4.2. *Let $(Y, \mathscr{O}_Y)$ be a locally ringed space such that Y is a subspace of X and there exists a covering (V_α) of Y by open sets of X such that for each α, $Y \cap V_\alpha$ is closed in V_α and the locally ringed space $(Y \cap V_\alpha, \mathscr{O}_Y|_{Y\cap V_\alpha})$ is a closed subscheme of the scheme induced over V_α by X. Then $(Y, \mathscr{O}_Y)$ is a subscheme of X.*

Proof. The hypothesis implies that Y is locally closed in X and the largest open set U containing Y and in which Y is closed contains the V_α. We are then reduced to the case $U = X$ and Y is closed in X. We define a quasi-coherent ideal $\mathscr{I}$ of $\mathscr{O}_X$ by letting $\mathscr{I}_{V_\alpha}$ to be the sheaf of ideal of $\mathscr{O}_X|_{V_\alpha}$ that defines the closed subscheme $(Y \cap V_\alpha, \mathscr{O}_Y|_{Y\cap V_\alpha})$ and for any open set W of X not meeting Y, $\mathscr{I}_W = \mathscr{O}_X|_W$. It is immediately verified that there exists a unique sheaf of ideals $\mathscr{I}$ satisfying these conditions and that it defines the closed subscheme $(Y, \mathscr{O}_Y)$. ☐

Proposition 10.4.3. *A (closed) subscheme of a (closed) subscheme of X is canonically identified with a (closed) subscheme of X.*

Proof. Since a locally closed subset of a locally closed subset of X is still locally closed in X, it is clear by Proposition 10.4.2 that the question is local and we may assume that X is affine. The proposition then follows from the identification $A/\mathfrak{b}$ and $(A/\mathfrak{a})/(\mathfrak{b}/\mathfrak{a})$ where $\mathfrak{a}, \mathfrak{b}$ are ideals of the ring A such that $\mathfrak{a} \subseteq \mathfrak{b}$. ☐

Let Y be a subscheme of X and denote by $\iota : Y \to X$ the canonical injection of the underlying space; we know the inverse image $\iota^*(\mathscr{O}_X)$ is the restriction $\mathscr{O}_X|_Y$. If for any $y \in Y$, we denote by ι_y the canonical homomorphism $\mathscr{O}_{X,y} \to \mathscr{O}_{Y,y}$, these homomorphisms are then the restrictions to the stalks of $\mathscr{O}_X$ at the points of Y of a surjective homomorphism of sheaves of rings $\iota^\# : \mathscr{O}_X|_Y \to \mathscr{O}_Y$: it suffices indeed to check it locally on Y, so we can assume that X is affine and Y is a closed subscheme; in this case, if $\mathscr{I}$ is the quasi-coherent ideal of $\mathscr{O}_X$ which defines Y, the ι_y's are nothing but the restrictions to the stalks of the canonical homomorphism $\mathscr{O}_X|_Y \to (\mathscr{O}_X/\mathscr{I})|_Y$. We have therefore defined a monomorphism $j_Y = (\iota, \iota^\#)$ of locally ringed spaces, which is called the **canonical injection morphism**. If $f : X \to Z$ is another morphism of schemes, we say the composition

$$Y \xrightarrow{j_Y} X \xrightarrow{f} Z$$

is the **restriction** of f to the subscheme Y of X.

A subscheme Y of a scheme X is considered as an X-scheme via the canonical injection $j_Y : Y \to X$. Two subschemes Y, Z of X that are X-isomorphic are then necessary identical. In fact, if $u : Y \to Z$ is an X-isomorphism, the relation $j_Y = j_Z \circ u$ shows the underlying spaces of Y and Z are identical. Moreover if $U \supseteq Y$ is an open subset of X such that $Y = Z$ are closed in

U, and $\mathscr{I}$, $\mathscr{J}$ are the ideals of U defining respectively Y and Z, for each $x \in Y$ we then have a commutative diagram

$$\begin{tikzcd} \mathscr{O}_{X,x}/\mathscr{J}_x \arrow[rr, "u_x^\#"] & & \mathscr{O}_{X,x}/\mathscr{I}_x \\ & \mathscr{O}_{X,x} \arrow[ul] \arrow[ur] & \end{tikzcd}$$

Since u is an isomorphism, this implies $\mathscr{I}_x = \mathscr{J}_x$, so $Y = Z$ and $u = 1_Y$.

According to the general definitions, we say a morphism $f : Z \to X$ is **dominated by the canonical injection** $j_Y : Y \to X$ of a subscheme Y of X, if f factors through j_Y:

$$Z \xrightarrow{g} Y \xrightarrow{j_Y} X$$

where g is a morphism of schemes. Since j_Y is a monomorphism, the morphism g is unique.

Proposition 10.4.4. *Let Y be a subscheme of a scheme X and $j : Y \to X$ be the cannical injection. For a morphism $f : Z \to X$ to be dominated by the injection j, it is necessary and sufficient that $f(Z) \subseteq Y$ and for each $z \in Z$, the homomorphism $f_z^\# : \mathscr{O}_{X,f(z)} \to \mathscr{O}_{Z,z}$ factors through $\mathscr{O}_{Y,f(z)}$ (or equivalently, the kernel of $\mathscr{O}_{X,f(z)} \to \mathscr{O}_{Z,z}$ is contained in that of $\mathscr{O}_{X,f(z)} \to \mathscr{O}_{Y,f(z)}$).*

Proof. The condition is cleraly necessary. For the sufficiency, we may assume that Y is a closed subscheme of X, and replace X by an open subset U such that Y is closed in U. Y is then defined by a quasi-coherent ideal $\mathscr{I}$ of $\mathscr{O}_X$. Let $\mathscr{J}$ be the kernel of the homomorphism $f^*(\mathscr{O}_X) \to \mathscr{O}_Z$. In view of the properties of the functor f^*, the hypothesis implies tht for each $z \in Z$ we have $(f^*(\mathscr{I}))_z \subseteq \mathscr{J}_z$, and concequently $f^*(\mathscr{I}) \subseteq \mathscr{J}$. Therefore the homomorphism $f^*(\mathscr{O}_X) \to \mathscr{O}_Z$ factors into

$$f^*(\mathscr{O}_X) \longrightarrow f^*(\mathscr{O}_X)/f^*(\mathscr{I}) = f^*(\mathscr{O}_X/\mathscr{I}) \xrightarrow{\theta_z} \mathscr{O}_Z$$

the first arrow being the canonical homomorphism. Let g be the continuous map of Z in Y coincide with f; it is clear that $g^*(\mathscr{O}_Y) = f^*(\mathscr{O}_X/\mathscr{I})$; on the other hand, for any $z \in Z$, θ_z is obviously a local homomorphism, so $(g, \theta) : Z \to Y$ is a morphism of schemes which satisfies $f = j \circ g$, whence the proposition. □

Corollary 10.4.5. *Let Y and Z be subschemes of X. For the canonical injection $Z \to X$ to be dominated by the injection $Y \to X$, it is necessary and sufficient that Z is a subscheme of Y.*

Due to this corollary, for two subschemes Y, Z of X we write $Y \preceq Z$ if Y is a subscheme of Z. It is clear that this defines an order relation on the set of subschemes of X, since two subschemes Y and Z are identical if $Y \preceq Z$ and $Z \preceq Y$.

10.4.2 Immersions of schemes

We say a morphism $f : Y \to X$ is an **immersion** (resp. a **closed immersion**, resp. an **open immersion**) if it is factorized into

$$Y \xrightarrow{g} Z \xrightarrow{j} X$$

where g is an isomorphism, Z is a subscheme (resp. a closed subscheme, resp. an open subscheme) of X, and j is the canonical injection. The subscheme Z and the isomorphism g are then uniquely determined since two X-isomorphic subschemes are identical. We say $f = i \circ g$ is the **canonical factorization** of the immersion f, and the subscheme Z and the isomorphism g is called **associated** with f. It is clera that an immersion is a monomorphism of schemes (since j is a monomorphism), and a fortiori a radical morphism (Corollary 10.3.33). Also, it is clear from Proposition 10.4.3 that the composition of two immersions (resp. two open immersions, resp. two closed immersions) is an immersion (resp. an open immersion, resp. a closed immersion).

Again, one should note that an immersion $f : Y \to X$ such that $f(Y)$ is a open subset of X, in other words which is an open morphism, is not necessarily an open immersion.

Example 10.4.6. Let X be an affine scheme. Then from the definition of closed subschemes, we see that for a morphism $f : Y \to X$ to be a closed immersion, it is necessary and sufficient that Y is an affine scheme and $\Gamma(f) : \Gamma(X, \mathscr{O}_X) \to \Gamma(Y, \mathscr{O}_Y)$ is surjective.

Lemma 10.4.7. *Let $f : Y \to X$ be a morphism of schemes such that $f(Y)$ is closed and f is a homeomorphism onto $f(Y)$. Then for each point $x \in X$, there exists an affine open neighborhood U of x such that $f^{-1}(U)$ is an affine open of Y.*

Proof. Since $f(Y)$ is closed in X, the lemma is trivial if $x \notin f(Y)$, since it suffices to choose an affine open neighborhood of x disjoint from $f(Y)$. If $x \in f(Y)$, there exists a unique point $y \in Y$ such that $f(y) = x$. Let W be an affine open neighborhood of x in X and V an affine open neighborhood of y in Y such that $f(V) \subseteq W$. By hypothesis $f(V)$ is an open neighborhood of x in $f(Y)$, so there exists an open neighborhood $U' \subseteq W$ of x such that $U' \cap f(Y) = f(V)$. Let U be an open neighborhood of x contained in U' and is of the form $D(s)$, where $s \in A = \Gamma(W, \mathscr{O}_X)$ (recall that W is chosen to be affine); in view of Proposition 1.4.20(b), $f^{-1}(U) \subseteq V$ is of the form $D(t)$, where t is the image of s in $B = \Gamma(V, \mathscr{O}_Y|_V)$, hence proves the lemma. □

Lemma 10.4.8. *Let $f : Y \to X$ be a morphism of schemes and (U_λ) be an affine open covering of X such that for each λ, $f^{-1}(U_\lambda)$ is an affine open of Y. Then for any quasi-coherent $\mathscr{O}_Y$-module $\mathscr{F}$, the direct image $f_*(\mathscr{F})$ is a quasi-coherent $\mathscr{O}_X$-module.*

Proof. For each λ, put $V_\lambda = f^{-1}(U_\lambda)$, and let $f_\lambda : V_\lambda \to U_\lambda$ be the restriction of f to V_λ. Then the restriction $f_*(\mathscr{F})$ to U_λ is equal to $(f_\lambda)_*(\mathscr{F}_\lambda)$, where $\mathscr{F}_\lambda = \mathscr{F}|_{U_\lambda}$. But since U_λ and V_λ are affine by hypothesis, we see $(f_\lambda)_*(\mathscr{F}_\lambda)$ is quasi-coherent by Proposition 10.1.12. This proves the lemma. □

Proposition 10.4.9. *Let $f : Y \to X$ be a morphism of schemes.*

(a) *For f to be an open immersion, it is necessary and sufficient that f is a homeomorphism onto an open subset of X and for each $y \in Y$, the homomorphism $f_y^\# : \mathscr{O}_{X,f(y)} \to \mathscr{O}_{Y,y}$ is bijective.*

(b) *For f to be an immersion (resp. a closed immersion), it is necessary and sufficient that f is a homeomorphism onto a locally closed (resp. closed) subset of X and for each $y \in Y$, the homomorphism $f_y^\# : \mathscr{O}_{X,f(y)} \to \mathscr{O}_{Y,y}$ is surjective.*

Proof. In the two cases, the conditions are clearly necessary, so we only need to prove the sufficiency. If the conditions in (a) holds, it is clear that $f^\#$ induces an isomorphism of $\mathscr{O}_Y$ to $f^*(\mathscr{O}_X)$, and $f^*(\mathscr{O}_X)$ is the sheaf defined by the transport by structure by means of f^* from the induced sheaf $\mathscr{O}_X|_{f(Y)}$, hence the conclusion.

Suppose then the conditions in (b) holds. Let U_0 be the largest open set of X such that $Z = f(Y)$ is closed in U_0; by replacing X by the subscheme induced by X over U_0, we may assume that $Z = f(Y)$ is closed in X. By Lemma 10.4.7 and Lemma 10.4.8, the sheaf $f_*(\mathscr{O}_Y)$ is a quasi-coherent $\mathscr{O}_X$-module. We have an exact sequence

$$0 \longrightarrow \mathscr{I} \longrightarrow \mathscr{O}_X \xrightarrow{f^\#} f_*(\mathscr{O}_Y) \longrightarrow 0$$

where two terms are quasi-coherent $\mathscr{O}_X$-modules; we then deduce that (Corollary 10.2.23) $\mathscr{I}$ is a quasi-coherent ideal of $\mathscr{O}_X$, and $f^\#$ factors into

$$\mathscr{O}_X \longrightarrow \mathscr{O}_X/\mathscr{I} \xrightarrow{\theta} f_*(\mathscr{O}_Y)$$

where the first arrow is the canonical homomorphism and θ is an isomorphism. If $\mathscr{O}_Z = (\mathscr{O}_X/\mathscr{I})|_Z$, $(Z, \mathscr{O}_Z)$ is then a closed subscheme of $(X, \mathscr{O}_X)$ and f factors through the canonical injection $j_Z : Z \to X$. Since the corresponding morphism $Y \to Z$ is just (f_0, θ_0), where f_0 is the map f considered as a homeomorphism from Y to Z and θ_0 is the restriction of θ to $\mathscr{O}_Z$, it is clear that f is a closed immersion, which completes the proof. □

Remark 10.4.10. It may happen that $f : Y \to X$ is a closed immersion and for all $y \in Y$, $f_y^\# : \mathscr{O}_{X,f(y)} \to \mathscr{O}_{Y,y}$ is bijective, without f being an open immersion (that is, $f(Y)$ is not necessarily open in X). For example, let $X = \operatorname{Spec}(A)$ be an affine scheme and $x \in X$ be a closed point of X that is not isolated. Then if $Y = \operatorname{Spec}(A/\mathfrak{m}_x)$, the canonical morphism $Y \to X$ is a closed immersion satisfying the desired property, since the subspace $\{x\}$ is not open in X.

Corollary 10.4.11. *Let $f : Y \to X$ be a morphism of schemes.*

(a) *Let (V_λ) be a covering of $f(Y)$ by open subsets of X. Then for f to be an immersion (an open immersion), it is necessary and sufficient that for each λ, the restriction $f^{-1}(V_\lambda) \to V_\lambda$ of f is an immersion (an open immersion).*

(b) *Let (U_λ) be an open covering of X. Then for f to be a closed immersion, it is necessary and sufficient that for each λ, the restriction $f^{-1}(U_\lambda) \to U_\lambda$ of f is a closed immersion.*

Proof. In the case (a), $f_y^\#$ is surjective (resp. bijective) for every point $y \in Y$, and in case (b) it is surjective for every point $y \in Y$; it then suffices to verify that in case (a) f is a homeomorphism of Y onto a locally closed (resp. open) subset of X and in case (b), a homeomorphism onto a closed subset of X. Now, the hypothesis imply that f is clearly injective and maps each neighborhood of $y \in Y$ to a neighborhood of $f(y)$ in $f(Y)$. In case (a), $f(Y) \cap V_\lambda$ is locally closed (resp. open) in the union of the V_λ, and a fortiori in X; in case (b), $f(Y) \cap U_\lambda$ is closed in U_λ, hence closed in X since $X = \bigcup_\lambda U_\lambda$. □

Remark 10.4.12. We can generalize the notions of immersions to any ringed spaces. We define a **ringed subspace** of a ringed space $(X, \mathscr{O}_X)$ to be a ringed space of the form $(Y, (\mathscr{O}_U/\mathscr{I})|_Y)$, where U is an open of X, $\mathscr{I}$ an ideal of $\mathscr{O}_U$ and Y the support of the sheaf of rings $\mathscr{O}_U/\mathscr{I}$ (support which is no longer necessarily closed in U). We can define the canonical injection $Y \to X$, and the definitions and results of Proposition 10.4.4 are valid without modification. We then define the notion of immersion (resp. of closed immersion) in the same manner. The characterizations of open (resp. closed) immersions given in Proposition 10.4.9 still hold (observing that if $f(Y)$ is closed in X, $f_*(\mathscr{O}_Y)$ has support $f(Y)$). The result of Proposition 10.4.9 can therefore be stated by saying that if a scheme Y is a ringed subspace of a scheme X, then Y is a subscheme of X.

10.4.3 Inverse image of subschemes

Proposition 10.4.13. *Let $f : X' \to X$ and $g : Y' \to Y$ be two S-morphisms. Let p, q be the projections of $X \times_S Y$.*

(a) *If f and g are immersions (resp. open immersions, resp. closed immersions), then $f \times_S g$ is an immersion (resp. an open immersion, resp. a closed immersion).*

(b) *If X' (resp. Y') is identified with a subscheme of X (resp. Y) via the immersion f (resp. g), then $f \times_S g$ identifies the underlying space of $X' \times_S Y'$ with the subspace $p^{-1}(X') \cap q^{-1}(Y')$ of the underlying space of $X \times_S Y$.*

Proof. We can restrict ourselves to the case where X' and Y' are subschemas and f and g are the canonical injection morphisms. The proposition has already been established for the subschemes induced on the open sets (Corollary 10.3.2); as any subscheme is a closed subscheme of an open scheme, we are reduced in case X' and Y' are closed subschemes.

We can further assume that S is affine. In fact, let (S_λ) be an affine open cover of S, φ and ψ be the structural morphisms of X and Y, and let $X_\lambda = \varphi^{-1}(S_\lambda)$, $Y_\lambda = \psi^{-1}(S_\lambda)$. The restriction X'_λ (resp. Y'_λ) of X' (resp Y') to $X_\lambda \cap X'$ (resp. $Y_\lambda \cap Y'$) is a closed subscheme of X_λ (resp. Y_λ), the schemes X_λ, Y_λ, X'_λ, Y'_λ can then be considered as S_λ-schemes and the product $X_\lambda \times_S Y_\lambda$ and $X_\lambda \times_{S\lambda} Y_\lambda$ (resp. $X'_\lambda \times_S Y'_\lambda$ and $X'_\lambda \times_{S_\lambda} Y'_\lambda$) are identified (Corollary 10.3.2). If the proposition is true when S is affine, the restriction of $f \times_S g$ to the $X'_\lambda \times_S Y'_\lambda$ will therefore be an immersion. As the product $X'_\lambda \times_S Y'_\mu$ (resp. $X_\lambda \times_S Y_\mu$) is identified with $(X'_\lambda \cap X'_\mu) \times_S (Y'_\lambda \cap Y'_\mu)$ (resp. $(X_\lambda \cap X_\mu) \times_S (Y_\lambda \cap Y_\mu)$), the restriction of $f \times_S g$ to $X'_\lambda \times_S Y'_\mu$ is also an immersion; it follows from Corollary 10.4.11 that $f \times_S g$ is an immersion.

Secondly, let's prove that we can also assume that X and Y are affine. In fact, let (U_i) (resp. (V_j)) be an affine open cover of X (resp. Y), and let X'_i (resp. Y'_j) be the restriction of X' (resp. Y') to $X' \cap U_i$ (resp. $Y' \cap V_j$), which is a closed subscheme of U_i (resp. V_j). Then $U_i \times_S V_j$ is identified with the restriction of $X \times_S Y$ to $p^{-1}(U_i) \cap q^{-1}(V_j)$ by Corollary 10.3.2, and similarly, if $p' : X' \times_S Y' \to X'$ and $q' : X' \times_S Y' \to Y'$ are the canonical projections, the product $X'_i \times_S Y'_j$ is identified with the restriction of $X' \times_S Y'$ to $p'^{-1}(X'_i) \cap q'^{-1}(Y'_j)$. Put $h = f \times_S g$, then since $X'_i = f^{-1}(U_i)$, $Y'_j = g^{-1}(V_j)$, we have

$$p'^{-1}(X'_i) \cap q'^{-1}(Y'_j) = h^{-1}(p^{-1}(U_i) \cap q^{-1}(V_j)) = h^{-1}(U_i \times_S V_j).$$

Again, by the same reasoning and using Corollary 10.4.11, we can show that h is an immersion.

Suppose then that X, Y, and S are affine, with rings B, C, and A, respectively. Then B and C are A-algebras, X' and Y' are affine subschemes with rings quotients B', C' of B and C, respectively. Moreover, f and g are induced by ring homomorphisms $\rho : B \to B'$ and $\sigma : C \to C'$. With these, we see $X \times_S Y$ (resp. $X' \times_S Y'$) is the affine scheme with ring $B \otimes_A C$ (resp. $B' \otimes_A C'$), and $f \times_S g$ corresponds to the ring homomorphism $\rho \otimes \sigma : B \otimes_A C \to B' \otimes_A C'$. Since this homomorphism is surjective, $f \times_S g$ is an immersion. Moreover, if $\mathfrak{b}$ (resp. $\mathfrak{c}$) is the kernel of ρ (resp. σ), the kernel of $\rho \otimes \sigma$ is $u(\mathfrak{b}) + v(\mathfrak{c})$, where u (resp. v) is the homorphism $b \mapsto b \otimes 1$ (resp. $c \mapsto 1 \otimes c$). As p corresponds to the ring homomorphism u and q corresponds to v, this kernel corresponds, in the spectrum $\mathrm{Spec}(B \otimes_A C)$, to the closed subset $p^{-1}(X') \cap q^{-1}(Y')$, which proves the demonstration. □

Corollary 10.4.14. *If $f : X \to Y$ is an immersion (resp. an open immersion, resp. a closed immersion) and an S-morphism, then $f_{(S')}$ is an immersion (resp. an open immersion, resp. a closed immersion) for any extension $S' \to S$ of base schemes.*

Proof. This follows from the observation that the identity morphism is an immersion (resp. an open immersion, resp. a closed immersion). □

Corollary 10.4.15. *An immersion (resp. a closed immersion, resp. an open immersion) is a universally embedding (resp. universally closed, resp. universally open).*

Proposition 10.4.16. *Let $f : X \to Y$ be a morphism, Y' a subscheme (resp. a closed subscheme, resp. an open subscheme) of Y, and $j : Y' \to Y$ the canonical injection.*

(a) *The projection $j' : X \times_Y Y' \to X$ is an immersion (resp. a closed immersion, resp. an open immersion), and the subscheme of X associated with j' has underlying space $f^{-1}(Y')$.*

$$\begin{tikzcd} X \times_Y Y' \arrow[r] \arrow[d, hook, "p"'] & Y' \arrow[d, hook, "j"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

Moreover, for a morphism $h : Z \to X$ to be such that $f \circ h : Z \to Y$ is dominated by j, it is necessary and sufficient that h is dominated by j'.

(b) *If Z is a closed subscheme defined by a quasi-coherent ideal $\mathscr{K}$ of $\mathscr{O}_Y$, the inverse image of Z by f is defined by the quasi-coherent ideal $f^*(\mathscr{K})\mathscr{O}_X$.*

Proof. As $p = 1_X \times_Y j$, the first assertion in (a) follows from Proposition 10.4.13. The second one is a special case of Corollary 10.3.34. Finally, if we have $f \circ h = j \circ h'$, where $h' : Z \to Y'$ is a morphism, it follows from the universal property of product that we have $h = p \circ u$, where $u : Z \to X \times_Y Y'$ is a morphism, whence assertion (a).

To prove (b), since the question is local on X and Y, we may assume that X and Y are affine. It then suffices to note that if A is an B-algebra and $\mathfrak{b}$ is an ideal of B, we have $A \otimes_B (B/\mathfrak{b}) = A/\mathfrak{b}A$, and apply Proposition 10.1.14. □

We say the subscheme of X thus defined is the **inverse image** of the subscheme Y' of Y by the morphism f. We say the morphism $f \times 1_{Y'} : f^{-1}(Y') \to Y'$ is the restriction of f to $f^{-1}(Y')$. When we speak of $f^{-1}(Y')$ as a subscheme of X, it is always this subscheme that will be involved.

Example 10.4.17. If the scheme $f^{-1}(Y')$ and X are identical, $j' : f^{-1}(Y') \to X$ is then the identity and any morphism $h : Z \to X$ is dominated by j'; hence the morphism $f : X \to Y$ factors into

$$X \xrightarrow{g} Y' \xrightarrow{j} Y$$

Example 10.4.18. If y is a closed point of Y and $Y' = \operatorname{Spec}(\kappa(y))$ is the smallest closed subscheme of Y having $\{y\}$ as underlying space, the closed subscheme $f^{-1}(Y')$ is then canonically isomorphic to the **fiber** $f^{-1}(y)$.

Corollary 10.4.19. *Retain the hypotheses of Proposition 10.4.16(b). Let X' be a closed subscheme of X defined by a quasi-coherent ideal $\mathscr{I}$ of $\mathscr{O}_X$ and $i : X' \to X$ be the canonical injection. For the restriction $f \circ i$ of f to X' is dominated by the injection $j : Y' \to Y$, it is necessary and sufficient that $f^*(\mathscr{K})\mathscr{O}_X \subseteq \mathscr{I}$.*

Proof. This follows from Proposition 10.4.16(b) and (a). □

Corollary 10.4.20. *Let $f : X \to Y$, $g : Y \to Z$ be two morphisms, and $h = g \circ f$ be their composition. For any subscheme Z' of Z, the subscheme $f^{-1}(g^{-1}(Z'))$ and $h^{-1}(Z')$ of X are identical.*

Proof. This follows from the transitivity of products and Proposition 10.4.16. □

Corollary 10.4.21. *Let X', X'' be two subschemes of X and $j' : X' \to X$, $j'' : X'' \to X$ be the canonical injections. Then $j'^{-1}(X'')$ and $j''^{-1}(X')$ are both equal to the infimum $\inf(X', X'')$ of X' and X'' for the ordered relation on subschemes, and is canonically isomorphic to $X' \times_S X''$.*

Proof. This follows from Proposition 10.4.16, Proposition 10.4.3, and the universal property of products. □

Corollary 10.4.22. *Let $f : X \to Y$ be a morphism and Y', Y'' be two subschemes of Y. Then we have $f^{-1}(\inf(Y', Y'')) = \inf(f^{-1}(Y'), f^{-1}(Y''))$.*

Proof. In fact, we have the canonical isomorphism of $(X \times_Y Y') \times_X (X \times_Y Y'')$ and $X \times_Y (Y' \times_Y Y'')$. □

10.4.4 Local immersions and local isomorphisms

Let $f : X \to Y$ be a morphism of schemes. We say f is a **local immersion** (resp. a **local isomorphism**) **at a point** $x \in X$ if there exists an open neighborhood U of x in X and an open neighborhood V of $f(x)$ in Y such that the restriction of f to U is a closed immersion (resp. an open immersion) into V. We say f is a local immersion (resp. a local isomorphism) if f is a local immersion (resp. a local isomorphism) at every point of X.

An immersion (resp. closed immersion) $f : X \to Y$ can then be characterized as a local immersion such that f is a homeomorphism onto a subset of Y (resp. a closed subset of Y). An open immersion f can be characterized as an injective local isomorphism.

Proposition 10.4.23. *Let X be an irreducible scheme, $f : X \to Y$ be a injective dominant morphism. If f is a local immersion, then it is an immersion and $f(X)$ is open in Y.*

Proof. In fact, let $x \in X$ and U be an open neighborhood of x in X, V an open neighborhood of $f(x)$ in Y such that $f|_Y$ is a closed immersion into V. As U is dense in X, $f(U)$ is also dense in Y by hypothesis, hence $f(U) = V$ and f is a homeomorphism from U to V. The hypothesis f is injective implies $f^{-1}(V) = U$, whence the proposition. $\square$

Proposition 10.4.24. *Let $f : X \to X'$, $g : Y \to Y'$ be two S-schemes.*

(a) *The composition of two local immersions (resp. local isomorphisms) is a local immersion (resp. a local isomorphism).*

(b) *If f and g are local immersions (resp. local isomorphisms), so is the product $f \times_S g$.*

(c) *If f is a local immersion (resp. a local isomorphism), so is $f_{(S')}$ for any extension $S' \to S$ of base schemes.*

Proof. It suffices to prove (a) and (b). Now (a) follows from the transitivity of closed immersions (resp. open immersions) and the fact that if f is a homeomorphism of X to a closed subset Y, then for any open set $U \subseteq X$, $f(U)$ is open in $f(X)$, so there exists an open subset V of Y such that $f(U) = V \cap f(X)$, and $f(U)$ is therefore closed in V.

To prove (b), let p, q be the projections of $X \times_S Y$ and p', q' that of $X' \times_S Y'$. There exist by hypotheses open neighborhoods U, U', V, V' of $x = p(z)$, $x' = p'(z')$, $y = q(z)$, $y' = q'(z')$, respectively, such that $f|_U$ and $g|_V$ are closed immersions (resp. open immersions) onto U' and V', respectively. As the underlying space of $U \times_S V$ is $p^{-1}(U) \cap q^{-1}(V)$ and that of $U' \times_S V'$ is $p'^{-1}(U') \cap q'^{-1}(V')$, which are neighborhoods of z and z', respectively (Corollary 10.3.2), the claim follows by Corollary 10.4.14. $\square$

Remark 10.4.25. A local isomorphism is clearly flat and universally open, and therefore is universally generalizing.

Proposition 10.4.26. *Let X be an irreducible scheme, Y an integral scheme, and $f : X \to Y$ be a morphism.*

(a) *If f is dominant, then for any $x \in X$, the homomorphism $f_x^\# : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is injective.*

(b) *If f is dominant and a local immersion, then f is a local isomorphism (and therefore X is integral).*

Proof. Let ξ and η be the generic points of X and Y, respectively. If f is dominant, we then have $f(\xi) = \eta$; moreover $\mathcal{O}_{Y,\eta}$ is a field since Y is reduced, so $f_\xi^\# : \mathcal{O}_{Y,\eta} \to \mathcal{O}_{X,\xi}$ is a monomorphism. For any point $x \in X$, and any affine neighborhood U of $y = f(x)$, there exists an affine neighborhood V of x contained in $f^{-1}(U)$. The open set U (resp. V) contains η (resp. ξ), and the ring

$\Gamma(U,\mathscr{O}_Y)$ is integral with fraction field $\mathscr{O}_{Y,\eta}$. If $\rho:\Gamma(U,\mathscr{O}_Y)\to\Gamma(V,\mathscr{O}_X)$ is the homomorphism corresponding to f, the composition

$$\Gamma(U,\mathscr{O}_Y)\xrightarrow{\rho}\Gamma(V,\mathscr{O}_X)\longrightarrow\mathscr{O}_{X,\xi}$$

is then the restriction of $f_\xi^\#$ to $\Gamma(U,\mathscr{O}_Y)$, so the homomorphism ρ. We now deduce that $f_x^\#$ is injective: in fact the hypotheses that X is irreducible implies that $\Gamma(V,\mathscr{O}_X)$ has a unique minimal $\mathfrak{n}$, which is its nilradical; the homomorphism $f_x^\#$ just send an element u/s (where $u,s\in\Gamma(U,\mathscr{O}_Y)$ and $s\neq 0$) to the element $\rho(u)/\rho(s)\in\mathscr{O}_{X,x}$, which is zero only if there exists $t\notin\mathfrak{p}_x$ such that $t\rho(u)=0\in\mathfrak{n}$. But as $t\notin\mathfrak{n}$, this then implies $\rho(u)\in\mathfrak{n}$, so $\rho(u)$ is nilpotent and since ρ is injective, this shows that u is nilpotent, which means $u=0$ for $\Gamma(U,\mathscr{O}_Y)$ being integral.

To prove the second assertion, let f be dominant and a local immersion. We see $f(Y)$ is open in Y by Proposition 10.4.23. Since $f_x^\#$ is surjective for every point $x\in X$ (Proposition 10.4.9), it follows that $f_x^\#$ is an isomorphism by (a), and this shows f is a local isomorphism, again by Proposition 10.4.9 . □

Proposition 10.4.27. *Let Y be a reduced scheme such that the family of irreducible components of Y is locally finite. Let $j:X\to Y$ be an immersion. For j to be a local isomorphism at a point $x\in X$, it is necessary and sufficient that the homomorphism $j_x^\#:\mathscr{O}_{Y,j(x)}\to\mathscr{O}_{X,x}$ is flat.*

Proof. Since the question is local on Y, we may assume that Y is affine, that j is a closed immersion, and that all irreducible components of Y contain $j(x)$ (hence are finite in number), and we prove that j is an isomorphism in this case. If $Y=\mathrm{Spec}(A)$, and if $\mathfrak{p}$ is the prime ideal of A corresponding to $j(x)$, the morphism $\mathrm{Spec}(A_\mathfrak{p})\to\mathrm{Spec}(A)$ is then dominant since $\mathfrak{p}$ contains the minimal ideals of A (it is contaiend in every irreducible component of $\mathrm{Spec}(A)$). As A is reduced, the homomorphism $A\to A_\mathfrak{p}$ is injective (Corollary 1.4.21). If $\mathscr{O}$ is the ideal of $\mathscr{O}_Y$ defining the closed subscheme of Y associated with j, we have $\mathscr{J}=\widetilde{\mathfrak{a}}$, where $\mathfrak{a}$ is an ideal of A, and it is identified with a subset of $\mathfrak{a}_\mathfrak{p}$. If j is flat at x, then $(A/\mathfrak{a})_\mathfrak{p}$ is a flat $A_\mathfrak{p}$-module, and since the homomorphism $A_\mathfrak{p}\to(A/\mathfrak{a})_\mathfrak{p}$ is local, it is faithfully flat (**??**). By **??** this implies $\mathfrak{a}_\mathfrak{p}=0$, hence $\mathfrak{a}=0$ and the claim follows. □

10.4.5 Nilradical and associated reduced scheme

Proposition 10.4.28. *Let $(X,\mathscr{O}_X)$ be a scheme and $\mathscr{B}$ be a quasi-coherent $\mathscr{O}_X$-algebra. There exist a unique quasi-coherent ideal $\mathscr{N}$ of $\mathscr{B}$ such that for each point $x\in X$, the stalk $\mathscr{N}_x$ is the nilradical of the ring $\mathscr{B}_x$. If $X=\mathrm{Spec}(A)$ is affine and $\mathscr{B}=\widetilde{B}$, where B is an A-algebra, then $\mathscr{N}=\widetilde{\mathfrak{n}}$, where $\mathfrak{n}$ is the nilradical of B.*

Proof. Since the question is loca, we may assume that $X=\mathrm{Spec}(A)$ is affine and $\mathscr{B}=\widetilde{B}$. We know that $\widetilde{\mathfrak{n}}$ is a quasi-coherent $\mathscr{O}_X$-module and for each point $x\in X$, the stalk $\mathfrak{n}_x$ is an ideal of the fraction ring B_x. It suffices to show that the nilradical of B_x is contained in $\mathfrak{n}_x$, the opposite inclusion being evident. Now, let z/s be a nilpotent element in B_x, where $z\in B$ and $s\notin\mathfrak{p}_x$. By hypotheses, there exist $k\geq 0$ such that $(z/s)^k=0$, which means there exists $t\notin\mathfrak{p}_x$ such that $tz^k=0$. We then conclude that $(tz)^k=0$, so $z/s=(tz)/(ts)$ is indeed in $\mathfrak{n}_x$. □

The quasi-coherent ideal $\mathscr{N}$ is called the **nilradical** of the $\mathscr{O}_X$-algebra $\mathscr{B}$. In particular, we denote by $\mathscr{N}_X$ the nilradical of $\mathscr{O}_X$.

Corollary 10.4.29. *Let X be a scheme. Then the closed subscheme of X defined by the quasi-coherent ideal $\mathscr{N}_X$ is the unique reduced subscheme of X with underlying space X. It is also the smallest subscheme of X having X as underlying space.*

Proof. Since the structural sheaf of the closed subscheme Y defined by $\mathscr{N}_X$ is $\mathscr{O}_X/\mathscr{N}_X$, it is immediate that Y is reduced and has X as underlying space, since $\mathscr{N}_x \neq \mathscr{O}_{X,x}$ for each $x \in X$. To prove the second claim, let Z be a subscheme of X with X as underlying space. Then Z is closed in X, so let $\mathscr{I}$ be the ideal defining it. We can assume that X is affine, so $\mathscr{I} = \widetilde{\mathfrak{a}}$ where $\mathfrak{a}$ is an ideal of A. Then for each $x \in X$ we have $\mathfrak{a} \subseteq \mathfrak{p}_x$, so $\mathfrak{a} \subseteq \mathfrak{n}$, where $\mathfrak{n}$ is the nilradical of A. This shows Y is the smallest subscheme of X with underlying space X, and if Z is distinct from Y, we necessarily have $\mathscr{I}_x \neq \mathscr{N}_x$ for some $x \in X$, and concequently Z is not reduced. □

The reduced scheme defined by $\mathscr{N}_X$ on X is called the **reduced scheme associated with X**, and denoted by X_{red}. To say that a schema X is reduced therefore means that $X_{\mathrm{red}} = X$. Clearly, we have a canonical closed immersion $X_{\mathrm{red}} \to X$, which is also a universal homeomorphism.

Proposition 10.4.30. *For the spectrum of a ring A to be reduced (resp. integral), it is necessary and sufficient that A is reduced (resp. integral).*

Proof. In fact, the condition $\mathscr{N} = 0$ is necessary and sufficient for $\mathrm{Spec}(A)$ to be reduced, and the integral conditions follows from Corollary 1.4.9. □

Proposition 10.4.31. *A scheme X is integral if and only if for each open subset U of X, the ring $\Gamma(U, \mathscr{O}_X)$ is integral.*

Proof. We first assume that $\Gamma(U, \mathscr{O}_X)$ is integral for any open set U. It is clear that X is reduced. If X is not irreducible, then one can find two nonempty disjoint open subsets U_1 and U_2. Then $\Gamma(U_1 \cup U_2, \mathscr{O}_X) = \Gamma(U_1, \mathscr{O}_X) \times \Gamma(U_2, \mathscr{O}_X)$, which is not an integral domain. Thus X is an integral scheme.

Conversely, suppose that X is reduced and irreducible. Let $U \subseteq X$ be an open subset, and suppose that there are elements $f, g \in \Gamma(U, \mathscr{O}_X)$ with $fg = 0$. Let

$$Y = \{x \in U : f_x \in \mathfrak{m}_x\}, \quad Z = \{x \in U : g_x \in \mathfrak{m}_x\};$$

then Y and Z are closed subsets, and $Y \cup Z = U$. Since X is irreducible, so U is irreducible, and one of Y or Z is equal to U, say $Y = U$. But then the restriction of f to any open affine subset of U will be contained in every point of that subset, hence nilpotent and thus zero. This shows that $\Gamma(U, \mathscr{O}_X)$ is integral. □

Proposition 10.4.32. *Let X be a scheme and x a point of X.*

(a) *For x to belong to a unique irreducible component of X, it is necessary and sufficient that the nilradical of $\mathscr{O}_{X,x}$ is prime.*

(b) *If the nilradical of $\mathscr{O}_{X,x}$ is prime and the family of irreducible components of X is locally finite, there exists an open neighborhood U of x that is irreducible.*

(c) *For X to be the coproduct of its irreducible components, it is necessary and sufficient that the family of irreducible components of X is locally finite and for each $x \in X$, the nilradical of $\mathscr{O}_{X,x}$ is prime.*

Proof. To check that whether x belongs to distinct irreducible components of X, we may assume that $X = \mathrm{Spec}(A)$ is affine (**??**). Then this signifies that $\mathfrak{p}_x$ contains two distinct minimal prime ideals of A, and equivalently $\mathfrak{m}_x$ contains two distinct minimal prime ideals of $\mathscr{O}_{X,x}$, which is a contradiction if and only if the nilradical of $\mathscr{O}_{X,x}$ is prime.

Now assume the conditions in (a). As the family of irreducible components of X is locally finite, the union of those of these components which do not contain x is closed, so its complement U is open and contained in the unique irreducible component of X containing x, and therefore irreducible (**??**). The assertion in (c) follows from (b) and **??**. □

Proposition 10.4.33. *For a scheme X to be locally integral, it is necessary and sufficient that the family of irreducible components is locally finite and for each point $x \in X$, the local ring $\mathscr{O}_{X,x}$ is integral. In this case, X is the coproduct of its irreducible components, which are integral subschemes.*

Proof. As the localizations of integral domains are integral, for a locally integral scheme X, its local rings $\mathscr{O}_{X,x}$ are inegral. The set of the irreducible components of X is locally finite since each $x \in X$ admits an irreducible open neighborhoods; moreover, the irreducible components of X are all open and disjoint, so X is the coproduct of its irreducible components (**??**). Conversely, if X satisfies these conditions, then X is the coproduct of its irreducible components, which are open and integral. It follows immediately that X is locally integral. □

Corollary 10.4.34. *Let X be a scheme whose set of irreducible components is locally finite (for example if X is locally Noetherian). Then for X to be integral, it is necessary and sufficient that it is connected and for each $x \in X$, the local ring $\mathscr{O}_{X,x}$ is integral. In particular, a locally integral and connected scheme is integral.*

Proposition 10.4.35. *Let X be a locally Noetherian scheme and let $x \in X$ be a point such that the nilradical $\mathscr{N}_x$ of $\mathscr{O}_{X,x}$ is prime (resp. such that $\mathscr{O}_{X,x}$ is reduced, resp. such that $\mathscr{O}_{X,x}$ is integral). Then there exist an open neighborhood U of x that is irreducible (resp. reduced, resp. integral).*

Proof. It suffices to consider the case where $\mathscr{N}_x$ is prime and where $\mathscr{N}_x = 0$, the third one is the conjunction of the first two cases. If $\mathscr{N}_X$ is prime, the claim follows from Proposition 10.4.32. If $\mathscr{N}_x = 0$, we then have $\mathscr{N}_y = 0$ for y in a neighborhood of x, since $\mathscr{N}$ is quasi-coherent, hence coherent since X is locally Noetherian and $\mathscr{N}$ is of finite type, and the conclusion follows from **??**. □

Proposition 10.4.36. *For a Noetherian scheme X, the nilradical $\mathscr{N}_X$ of $\mathscr{O}_X$ is nilpotent.*

Proof. Since X is quasi-compact, We can cover X by a finite number of affine open sets U_i, and it suffices to prove that there exist integers n_i such that $(\mathscr{N}_X|_{U_i})^{n_i} = 0$. If n is the largest of the n_i, we will then have $\mathscr{N}_X^n = 0$. We are therefore reduced to the case where $X = \mathrm{Spec}(A)$ is affine, A being a Noetherian ring. It then suffices to observe that the nilradical of A is nilpotent. □

Corollary 10.4.37. *A Noetherian sheme X is affine if and only if X_{red} is affine.*

Proof. If X is affine it is clear that X_{red} is affine, regardless of X being Noetherian. Conversely, assume that X_{red} is affine and X is Noetherian. Then by Proposition 10.4.36, the nilradical $\mathscr{N}$ of $\mathscr{O}_X$ is nilpotent. For any quasi-coherent sheaf $\mathscr{F}$ on X, consider the follows exact sequence (where $k \geq 0$)

$$0 \longrightarrow \mathscr{N}^k\mathscr{F} \longrightarrow \mathscr{N}^{k-1}\mathscr{F} \longrightarrow \mathscr{N}^{k-1}\mathscr{F}/\mathscr{N}^k\mathscr{F} \longrightarrow 0$$

Since $\mathscr{N}^{k-1}\mathscr{F}/\mathscr{N}^k\mathscr{F}$ is an $\mathscr{O}_X/\mathscr{N}$-module and $(X, \mathscr{O}_X/\mathscr{N})$ is affine, by Serre's criterion we have $H^1(X, \mathscr{N}^{k-1}\mathscr{F}/\mathscr{N}^k\mathscr{F}) = 0$, so

$$H^1(X, \mathscr{N}^k\mathscr{F}) = 0 \Rightarrow H^1(X, \mathscr{N}^{k-1}\mathscr{F}) = 0.$$

Since $\mathscr{N}$ is nilpotent, this shows $H^1(X, \mathscr{F}) = 0$, so $(X, \mathscr{O}_X)$ is affine, by Serre's criterion again. □

Let $f : X \to Y$ be a morphism of schemes; let $i : X_{\mathrm{red}} \to X$ and $j : Y_{\mathrm{red}} \to Y$ be the canonical injections. The homomorphism $f_x^\# : \mathscr{O}_{Y,f(x)} \to \mathscr{O}_{X,x}$ maps nilpotent elements of $\mathscr{O}_{Y,f(x)}$ into nilpotent elements of $\mathscr{O}_{X,x}$, so $f^*(\mathscr{N}_Y)\mathscr{O}_X \subseteq \mathscr{N}_X$. It then follows from Proposition 10.4.4 that

$f \circ i$ factors through j, so we get an induced morphism $f_{\mathrm{red}} : X_{\mathrm{red}} \to Y_{\mathrm{red}}$ and a commutative diagram

$$\begin{tikzcd} X_{\mathrm{red}} \arrow[r, "f_{\mathrm{red}}"] \arrow[d, "i"'] & Y_{\mathrm{red}} \arrow[d, "j"] \\ X \arrow[r, "f"] & Y \end{tikzcd} \tag{10.4.1}$$

In particular, if X is reduced, then the morphism f factors into

$$X \xrightarrow{f_{\mathrm{red}}} Y_{\mathrm{red}} \xrightarrow{j} Y$$

or in other words, f is dominated by the canonical injection j. We also conclude that Y_{red} satisfies the universal property that any morphism from a reduced scheme to Y factors through Y_{red}.

For two morphisms $f : X \to Y$, $g : Y \to Z$, it follows from the uniqueness of factorization that we have

$$(g \circ f)_{\mathrm{red}} = g_{\mathrm{red}} \circ f_{\mathrm{red}};$$

we can equivalently say that the operation $X \mapsto X_{\mathrm{red}}$ is a covariant functor on the category of schemes.

Proposition 10.4.38. *If X and Y are S-schemes, the schemes $X_{\mathrm{red}} \times_{S_{\mathrm{red}}} Y_{\mathrm{red}}$ and $X_{\mathrm{red}} \times_S Y_{\mathrm{red}}$ are identical, and are canonically identified with a subscheme of $X \times_S Y$ having the same underlying space as this product.*

Proof. The fact that $X_{\mathrm{red}} \times_S Y_{\mathrm{red}}$ is identified with a subscheme of $X \times_S Y$ follows from the fact that if $f : X' \to X$ and $g : Y' \to Y$ are surjective S-immersions, then $f \times_S g : X' \times_S Y' \to X \times_S Y$ is a surjective immersion (Proposition 10.4.13 and Proposition 10.3.28). On the other hand, if $\varphi : X_{\mathrm{red}} \to S$ and $\psi : Y_{\mathrm{red}} \to S$ are the structural morphisms, it is clear that they factors through S_{red}, and as $S_{\mathrm{red}} \to S$ is a monomorphism, the second assertion follows. □

Corollary 10.4.39. *The schemes $(X \times_S Y)_{\mathrm{red}}$ and $(X_{\mathrm{red}} \times_{S_{\mathrm{red}}} Y_{\mathrm{red}})_{\mathrm{red}}$ are canonically identified.*

We note that if X and Y are reduced S-schemes, it is not necessarily that case that $X \times_S Y$ is reduced, since the tensor product of two reduced algebras (even two fields) may not be reduced.

Corollary 10.4.40. *For any morphism $f : X \to Y$ of schemes, the diagram (10.4.1) factors into*

$$\begin{tikzcd} X_{\mathrm{red}} = (X \times_Y Y_{\mathrm{red}})_{\mathrm{red}} \arrow[r] & X \times_Y Y_{\mathrm{red}} \arrow[r] \arrow[d] & Y_{\mathrm{red}} \arrow[d] \\ & X \arrow[r, "f"] & Y \end{tikzcd} \tag{10.4.2}$$

Proof. For this, we only need to note that $(X \times_Y Y_{\mathrm{red}})_{\mathrm{red}} = (X_{\mathrm{red}} \times_{Y_{\mathrm{red}}} Y_{\mathrm{red}})_{\mathrm{red}} = X_{\mathrm{red}}$. □

Proposition 10.4.41. *Let X and Y be two schemes. Consider the following properties for a morphism $f : X \to Y$:*

(i) *surjective;*

(ii) *injective;*

(iii) *open at the point x (resp. open);*

(iv) *closed;*

(v) *a homeomorphism onto its image;*

(vi) *universally open at a point x (resp. universally open);*

(vii) *universally closed;*

(viii) *a universal embedding;*

(ix) *a universal homeomorphism;*

(x) *radical;*

(xi) *generalizing at a point x (resp. generalizing);*

(xii) *universally generalizing at a point x (resp. universally generalizing).*

Then, if $\mathcal{P}$ denote one of the above properties, for f to possess the property $\mathcal{P}$, it is necessary and sufficient that f_{red} possess $\mathcal{P}$.

Proof. The proposition is evident for the properties (i), (ii), (iii), (iv), (v), (xi), which only depend on the map of the underlying spaces. For (x), the proposition follows from the fact that the fibers of f and f_{red} at a point $y \in Y$ have the same underlying space and the residue field at a point of X (resp. Y) is the same for X_{red} and Y_{red}. For the properties (vi), (vii), (viii), (ix) and (xii), if f possesses one of these properties, the same is true of f_{red} due to Corollary 10.4.40 and that the morphism $X_{\mathrm{red}} \to X \times_Y Y_{\mathrm{red}}$ is a universal homeomorphism. Conversely, if f_{red} possesses one of these properties, it suffices to note that for any morphism $Y' \to Y$ we have $(X_{\mathrm{red}} \times_{Y_{\mathrm{red}}} Y'_{\mathrm{red}})_{\mathrm{red}} = (X \times_Y Y')_{\mathrm{red}}$, so the morphism

$$(f_{(Y')})_{\mathrm{red}} : (X \times_Y Y')_{\mathrm{red}} \to Y'_{\mathrm{red}}$$

possesses the "nonuniversal" version of the same property, and by what we have already seen, $f_{(Y')}$ then has the corresponding property. □

Proposition 10.4.42. *Let X and Y be two schemes and x be a point of X. Consider the following properties for a morphism $f : X \to Y$:*

(i) *a monomorphism;*

(ii) *an immersion;*

(iii) *an open immersion;*

(iv) *a closed immersion;*

(v) *a local immersion at the point x;*

(vi) *a local isomorphism at the point x;*

(vii) *birational.*

Then, if f possesses one of the above properties, f_{red} also possesses that property.

Proof. For the properties (ii) to (vii), the result follows from the observation that $(f_{\mathrm{red}})^\#_x : \mathscr{O}_{Y,f(x)} \to \mathscr{O}_{X,x}$ is surjective (resp. bijective) if $f^\#_x : \mathscr{O}_{Y,f(x)} \to \mathscr{O}_{X,x}$ is surjective (resp. bijective) (Proposition 10.4.9). For (i) it suffices to note that a monomorphism is universal (Proposition 10.3.12), the diagram (10.4.2), and the fact that $X_{\mathrm{red}} \to X \times_Y Y_{\mathrm{red}}$ is a closed immersion, hence a monomorphism. □

Note that if f_{red} is an immersion, it is not necessarily true that f is. For example, let $Y = \operatorname{Spec}(k)$ where k is a field and $X = \operatorname{Spec}(A)$, where $A = k[T]/(T^2)$. Then the canonical injection $\rho : k \to k[T]/(T^2)$ corresponds to a morphism $f : X \to Y$. It is clear that f is not an immersion (in fact any nonzero immersion into Y is automatically closed); but $A_{\text{red}} = k$ so f_{red} is an isomorphism.

Remark 10.4.43. To say that an immersion $f : Y \to X$ is surjective means it is closed and that the subscheme of X associated with f is defined by an ideal $\mathscr{I}$ contained in the nilradical $\mathscr{N}_X$. In this case, we say f is a nilimmersion; f is then a homeomorphism from Y to X, and f_{red} is an isomorphism from Y_{red} to X_{red}. We say the nilimmersion f is **nilpotent** (resp. **locally nilpotent**) if the ideal $\mathscr{I}$ is nilpotent (resp. locally nilpotent, i.e. that every $x \in X$ has an open neighborhood U such that $\mathscr{I}|_U$ is nilpotent). More precisely, we say f is **nilpotent of order n** if $\mathscr{I}^{n+1} = 0$. If Y is a subscheme of X and f is the canonical immersion, we say X is an **infinitesimal neighborhood** (resp. an **infinitesimal neighborhood of order n**) of Y if f is nilpotent (resp. nilpotent of order n).

10.4.6 Reduced scheme structure on closed subsets

Proposition 10.4.44. *For any locally closed subspace Y of the underlying space of a scheme X, there exists a unique reduced subscheme of X with underlying space Y.*

Proof. The uniqueness is immediate from Corollary 10.4.29, so we only need to construct a reduced scheme structure on Y. If X is affine with ring A and Y is closed in X, the proposition is immediate: $I(Y)$ is the largest ideal $\mathfrak{a} \subseteq A$ such that $V(\mathfrak{a}) = Y$, and is radical, hence the ring $A/I(Y)$ is reduced, and we can take the scheme structure $(Y, \widetilde{A/I(Y)})$ on Y.

In the general case, for any affine open $U \subseteq X$ such that $U \cap Y$ is closed in U, consider the closed subscheme Y_U of U defined by the quasi-coherent ideal associated with the ideal $I(U \cap Y)$ of $\Gamma(U, \mathscr{O}_X|_U)$, which is reduced. If V is an open affine of X contained in U, then Y_V is induced by Y_U on $V \cap Y$ since this induced scheme is a closed subscheme of V which is reduced and has $V \cap Y$ as underlying space; the uniqueness of Y_V therefore entails our assertion. □

Corollary 10.4.45. *Let X be a scheme and Y be a locally closed subset of X. Then any point $x \in Y$ admits a maximal generalization y (i.e. y has no further generalization in Y). In particular, if $Y \neq \varnothing$, there exist a maximal element $y \in Y$ under generalization.*

Proof. It suffices to give Y a subscheme structure of X and take y to be the generic point of the irreducible components of Y containing x. □

Example 10.4.46. Let X be a scheme and x be a closed point of X. Let U be an open neighborhood of x. Then $Z = (X - U) \cup \{x\}$ is a closed subset of X, so we can consider the reduced scheme structure on it. Let $\mathscr{I}$ be the corresponding quasi-coherent ideal of $\mathscr{O}_X$, we want to determine the stalk $\mathscr{I}_x$. For this, we can assume that $X = \operatorname{Spec}(A)$ is affine and $X - U = V(\mathfrak{a})$, where $\mathfrak{a}$ is an ideal of A. Then the point x corresponds to a maximal ideal $\mathfrak{m}$ of A such that $\mathfrak{a} \not\subseteq \mathfrak{m}$, and $Z = V(\mathfrak{a}) \cup \{x\} = V(\mathfrak{a} \cap \mathfrak{m})$. By definition, $\mathscr{I}$ is the quasi-coherent ideal on X associated with $I(Z) = \sqrt{\mathfrak{a} \cap \mathfrak{m}}$, and therefore

$$\mathscr{I}_x = (\sqrt{\mathfrak{a} \cap \mathfrak{m}})_{\mathfrak{m}} = \sqrt{(\mathfrak{a} \cap \mathfrak{m})_{\mathfrak{m}}},$$

which is the intersection of prime ideals $\mathfrak{p}$ containing $\mathfrak{a} \cap \mathfrak{m}$ and contained in $\mathfrak{m}$. But if a prime ideal $\mathfrak{p} \subseteq \mathfrak{m}$ of A contains $\mathfrak{a} \cap \mathfrak{m}$, then by Proposition 1.1.5 we have $\mathfrak{p} \supset \mathfrak{a}$ or $\mathfrak{p} \supset \mathfrak{m}$. If $\mathfrak{p} \neq \mathfrak{m}$, then this implies $\mathfrak{p} \supset \mathfrak{a}$ and therefore $\mathfrak{m} \supset \mathfrak{p} \supset \mathfrak{a}$, which is a contradiction (since x is not contained in $X - U = V(\mathfrak{a})$). From this, we conclude that $\mathscr{I}_x = \mathfrak{m}_x$.

Example 10.4.47. Let X be a reduced locally Noetherian scheme and X' be a reduced closed subscheme of X with underlying space an irreducible component of X. Then X' is defined by a quasi-coherent ideal $\mathscr{I}$ of $\mathscr{O}_X$. Let x be the generic point of X'; we claim that $\mathscr{I}_x = 0$. For this, we can assume that $X = \operatorname{Spec}(A)$ is affine, where A is a reduced Noetherian ring, so $\mathscr{I} = \tilde{\mathfrak{p}}$ where $\mathfrak{p}$ is a minimal prime ideal of A. By definition the stalk of $\mathscr{I}$ at x is indeitified with $\mathfrak{p}A_{\mathfrak{p}}$, which is the maximal ideal of $A_{\mathfrak{p}} = \mathscr{O}_{X,x}$. But since A is reduced and $\mathfrak{p}$ is minimal, $A_{\mathfrak{p}}$ is a reduced Artinian local ring, whence a field, and we then conclude that $\mathfrak{p}A_p = 0$, so $\mathscr{I}_x = 0$.

Proposition 10.4.48. *Let X be a reduced scheme, $f : X \to Y$ be a morphism, and Z be a closed subscheme of Y containing $f(X)$. Then f factors into*

$$X \xrightarrow{g} Z \xrightarrow{j} Y$$

where j is the canonical injection.

Proof. The hypotheses implies that the closed subscheme $f^{-1}(Z)$ of X has underlying space X (Proposition 10.4.16). As X is reduced, this subscheme coincides with X by Corollary 10.4.29, and the proposition then follows from Proposition 10.4.16. □

Corollary 10.4.49. *Let X be a reduced subscheme of a scheme Y. If Z is the reduced closed subscheme of Y with underlying space $\bar{X}$, then X is an open subscheme of Z.*

Proof. Since X is locally closed, there is an open set U of Y such that $X = U \cap \bar{X}$. By Proposition 10.4.48, X is then a reduced subscheme of Z, with underlying space open in Z. Since the scheme structure induced by Z is also reduced, we conclude that X is induced by Z, in view of the uniqueness part of Proposition 10.4.44. □

Corollary 10.4.50. *Let $f : X \to Y$ be morphism and X' (resp. Y') be a closed subscheme of X (resp. Y) defined by a quasi-coherent ideal $\mathscr{I}$ (resp. $\mathscr{K}$) of $\mathscr{O}_X$ (resp. $\mathscr{O}_Y$). Suppose that X' is reduced and $f(X') \subseteq Y'$, then we have $f^*(\mathscr{K})\mathscr{O}_X \subseteq \mathscr{I}$.*

Proof. The restriction of f to X' factors into $X' \to Y' \to Y$ by Proposition 10.4.48, so it suffices to use Corollary 10.4.19. □

10.5 Separated schemes and morphisms

10.5.1 Diagonal and graph of a morphism

Let X be an S-scheme; recall that the diagonal morphism $X \to X \times_S X$, denoted by $\Delta_{X/S}$ or Δ_X, is the S-morphism $(1_X, 1_X)_S$, which means Δ_X is the unique S-morphism such that

$$p_1 \circ \Delta_X = p_2 \circ \Delta_X = 1_X,$$

where p_1, p_2 are the canonical projections of $X \times_S X$. If $f : T \to X$ and $g : T \to Y$ are two S-morphisms, we verify that

$$(f, g)_S = (f \times_S g) \circ \Delta_{T/S}.$$

If $\varphi : X \to S$ is the structural morphism of X, we also write Δ_φ for $\Delta_{X/S}$.

Proposition 10.5.1. *Let X, Y be S-schemes. If we identify $(X \times_S Y) \times_S (X \times_S Y)$ and $(X \times_S X) \times_S (Y \times_S Y)$, the morphism $\Delta_{X \times Y}$ is identified with $\Delta_X \times \Delta_Y$.*

Proof. In fact, if p_1, q_1 are the canonical projections $X \times_S X \to X$, $Y \times_S Y \to Y$, the projection $(X \times_S Y) \times_S (X \times_S Y) \to X \times_S Y$ is identified with $p_1 \times q_1$, and we have

$$(p_1 \times q_1) \circ (\Delta_X \times \Delta_Y) = (p_1 \circ \Delta_X) \times (q_1 \circ \Delta_Y) = 1_{X \times Y}$$

similar for the projection to the second factor. □

Corollary 10.5.2. *For any extension $S' \to S$ of base schemes, $\Delta_{X_{(S')}}$ is identified with $(\Delta_X)_{(S')}$.*

Proof. It suffices to remark that $(X \times_S X)_{(S')}$ is identified with $X_{(S')} \times_{S'} X_{(S')}$ canonically. □

Proposition 10.5.3. *Let X, Y be S-schemes and $S \to T$ be a morphism. Let $\varphi : X \to S$, $\psi : Y \to S$ be the structural morphisms, p, q the projection of $X \times_S Y$, and $\pi = \varphi \circ p = \psi \circ q$ the structural morphism $X \times_S Y \to S$. Then the diagram*

$$\begin{tikzcd}
X \times_S Y \arrow[r, "(p,q)_T"] \arrow[d, "\pi"'] & X \times_T Y \arrow[d, "\varphi \times_T \psi"] \\
S \arrow[r, "\Delta_{S/T}"] & S \times_T S
\end{tikzcd} \tag{10.5.1}$$

commutes and cartesian.

Proof. By the definition of products, we may prove the proposition in the category of sets, and replace X, Y, S by $X(Z)_T$, $Y(Z)_T$, $S(Z)_T$, where Z is an arbitrary T-scheme, and it is then immediate. □

Corollary 10.5.4. *If $f : X \to Y$ is an S-morphism, the diagram*

$$\begin{tikzcd}
X \arrow[r, "\Gamma_f"] \arrow[d, "f"'] & X \times_S Y \arrow[d, "f \times_S 1_Y"] \\
Y \arrow[r, "\Delta_Y"] & Y \times_S Y
\end{tikzcd} \tag{10.5.2}$$

is commutative and cartesian.

Proof. It suffcies to apply Proposition 10.5.3 with S replaced by Y and T by S, and note that $X \times_Y Y = X$. □

Proposition 10.5.5. *For a morphism $f : X \to Y$ of schemes to be a monomorphism, it is necessary and sufficient that $\Delta_{X/Y}$ is an isomorphism from X to $X \times_Y X$.*

Proof. In fact, f is monic means for any Y-scheme Z, the corresponding map $X(Z)_Y \to Y(Z)_Y$ is an injection, and as $Y(Z)_Y$ is reduced to a singleton, this means $X(Z)_Y$ is either empty or a singleton. But this is equivalent to saying that $X(Z)_Y \times X(Z)_Y$ is canonically isomorphic to $X(Z)_Y$ via the diagonal map, where the first set is $(X \times_Y X)(Z)_Y$, and this means $\Delta_{X/Y}$ is an isomorphism. □

Proposition 10.5.6. *The diagonal morphism is an immersion from X to $X \times_S X$, and the corresponding subscheme of $X \times_S X$ is called the **diagonal** of $X \times_S X$.*

Proof. Let p_1, p_2 be the projections of $X \times_S X$. As the continuous maps p_1 and Δ_X are such that $p_1 \circ \Delta_X = 1_X$, Δ_X is a homeomorphism from X onto $\Delta_X(X)$. Simialrly, the composition of the homomorphisms $\mathscr{O}_{X,x} \to \mathscr{O}_{\Delta_X(X),x} \to \mathscr{O}_{X,x}$ corresponding to p_1 and Δ_X is the identity, so the homomorphism corresponding to Δ_X on stalks are surjective. The proposition then follows from Proposition 10.4.9 . □

Corollary 10.5.7. *With the hypotheses of Proposition 10.5.3, the morphisms $(p,q)_T$ is an immersion.*

Proof. This follows from Proposition 10.5.3 and Corollary 10.4.14. □

Corollary 10.5.8. *Let X and Y be S-schemes and $f : X \to Y$ be an S-morphism. Then the graph morphism $\Gamma_f : X \to X \times_S Y$ of f is an immersion.*

Proof. This follows from the diagram (10.5.2) and Corollary 10.4.14. □

The subscheme of $X \times_S Y$ associated with the immersion Γ_f is called the **graph** of the morphism f; the subschemes of $X \times_S Y$ which are graphs of morphisms $X \to Y$ are characterized by the fact that the restriction of the projection $p_1 : X \times_S Y \to X$ to such a subscheme G is an isomorphism g from G to X: in fact, if this is the case, G is then the graph of the morphism $p_2 \circ g^{-1}$, where $p_2 : X \times_S Y \to Y$ is the second projection.

In particular, if $X = S$, the S-morphisms $S \to Y$, which are none other than the S-sections of Y, are equal to their graph morphisms; the subschemes of Y which are graphs of S-sections (in other words, those which are isomorphic to S by the restriction of the structural morphism $Y \to S$) are then called the **images of these sections**, or, by abuse of language, the S-sections of Y.

Proposition 10.5.9. *If $f : X \to Y$ is an S-morphism, the diagram*

$$\begin{tikzcd} X \arrow[r, "\Delta_X"] \arrow[d, "f"'] & X \times_S X \arrow[d, "f \times_S f"] \\ Y \arrow[r, "\Delta_Y"] & Y \times_S Y \end{tikzcd} \tag{10.5.3}$$

is commutative (in other words, Δ_X is a functorial morphism on the category of schemes).

Proof. The morphisms $\Delta_Y \circ f$ satisfies the condition that

$$p_1 \circ (\Delta_Y \circ f) = p_2 \circ (\Delta_Y \circ f) = f$$

where p_1, p_2 are the projections of $Y \times_S Y$. Similarly, if q_1, q_2 are the projections of $X \times_S X$,

$$p_1 \circ (f \times_S f) \circ \Delta_X = f \circ q_1 \circ \Delta_X = f,$$
$$p_2 \circ (f \times_S f) \circ \Delta_X = f \circ q_2 \circ \Delta_X = f.$$

It then follows from the universal property of products that $\Delta_Y \circ f = (f \times_S f) \circ \Delta_X$. □

Corollary 10.5.10. *If X is a subscheme of Y, the diagonal $\Delta_X(X)$ is identified with a subscheme of $\Delta_Y(Y)$ whose the underlying space is identified with*

$$\Delta_Y(Y) \cap p_1^{-1}(X) = \Delta_Y(Y) \cap p_2^{-1}(X)$$

where p_1, p_2 are the projections of $Y \times_S Y$.

Proof. Apply Proposition 10.5.9 to the immersion $f : X \to Y$, we see $(f \times_S f)$ is an immersion which identifies $X \times_S X$ with the subspace $p_1^{-1}(X) \cap p_2^{-1}(X)$ of $Y \times_S Y$ (Proposition 10.4.13). Moreover, if $z \in \Delta_Y \cap p_1^{-1}(X)$, we have $z = \Delta_Y(y)$ and $y = p_1(z) \in X$, so $y = f(y)$, and $z = \Delta_Y(f(y))$ belongs to $\Delta_X(X)$ in view of the diagram (10.5.3). □

Proposition 10.5.11. *Let $u_1 : X \to Y$, $u_2 : X \to Y$ be two S-morphisms. Then the kernel* $\ker(u_1, u_2)$ *is canonically isomorphic to the inverse image in X of the diagonal $\Delta_Y(Y)$ under the morphism $(u_1, u_2)_S : X \to Y \times_S Y$.*

Proof. Let $Z \to X$ be the inverse image of $\Delta_Y(Y)$ under the morphism $(u_1, u_2)_S$. Then if $f : T \to X$ is an S-morphism such that $u_1 \circ f = u_2 \circ f$, then

$$p_1 \circ (u_1, u_2)_S \circ f = u_1 \circ f = u_2 \circ f = p_1 \circ \Delta_Y \circ u_2 \circ f,$$
$$p_2 \circ (u_1, u_2)_S \circ f = u_2 \circ f = p_2 \circ \Delta_Y \circ u_2 \circ f$$

where p_1, p_2 are the projections of $Y \times_S Y$. We conclude that $(u_1, u_2)_S \circ f = \Delta_Y \circ u_2 \circ f$, and by the definition of Z, the morphism f factors uniquely through Z, which proves our claim. □

Corollary 10.5.12. *For a point $x \in X$ to belong to $\ker(u_1, u_2)$, it is necessary and sufficient that $u_1(x) = u_2(x) = y$ and the homomorphisms $\kappa(y) \to \kappa(x)$ corresponding to u_1 and u_2 are equal.*

Proof. In fact, if $j : Z \to X$ is the kernel of u_1 and u_2, to say $x \in Z$ signifies that the canonical morphism $h : \mathrm{Spec}(\kappa(x)) \to X$ factors into $h = j \circ g$, where g is a morphism from $\mathrm{Spec}(\kappa(x))$ to Z. This is equivalent to $u_1 \circ h = u_2 \circ h$, and by Corollary 10.2.17 to that $u_1(x) = u_2(x)$ and the field extensions $\kappa(y) \to \kappa(x)$ corresponding to u_1 and u_2 are equal. □

Proposition 10.5.13. *Let X and Y be S-schemes and $f : X \to Y$, $g : X \to Y$ be S-morphisms. Then we have the following commutative diagram*

$$\begin{tikzcd}
\ker(f,g) \arrow[r] \arrow[d] & X \arrow[r, "g"] \arrow[d, "\Gamma_g"] & Y \arrow[d, "\Delta_Y"] \\
X \arrow[r, "\Gamma_f"] \arrow[d, "f"'] & X \times_S Y \arrow[r, "g \times_S \mathbf{1}_Y"] \arrow[d, "f \times_S \mathbf{1}_Y"] & Y \times_S Y \\
Y \arrow[r, "\Delta_Y"] & Y \times_S Y &
\end{tikzcd}$$

where all squares are cartesian.

Proof. The fact that $\ker(f, g)$ is identified with the kernel of Γ_f and Γ_g can be deduced from by applying the projection $X \times_S Y \to Y$, or by Yoneda since this is clearly true for sets. The other two small squares are cartesian by Corollary 10.5.4, so the claim follows from the transitivity of products. □

Proposition 10.5.14. *Let $\mathcal{P}$ be a property for morphisms of schemes and consider the following conditions:*

(i) *If $j : X \to Y$ is an immersion and $g : Y \to Z$ is a morphism possessing the property $\mathcal{P}$, then $g \circ j$ possesses the property $\mathcal{P}$.*

(ii) *If $f : X \to X'$ and $g : y \to Y'$ are two S-morphisms possessing the property $\mathcal{P}$, then $f \times_S g$ possesses the property $\mathcal{P}$.*

(iii) *If the composition $g \circ f$ of two morphisms $f : X \to Y$, $g : Y \to Z$ possesses the property $\mathcal{P}$, then f possesses the property $\mathcal{P}$.*

Then (iii) is a concequence of (i) and (ii).

Proof. The morphism f factors into

$$X \xrightarrow{\Gamma_f} X \times_Z Y \xrightarrow{p_2} Y$$

where p_2 is the projection. On the other hand, p_2 is idnetified with $(g \circ f) \times_Z 1_Y$, and by (ii) it possesses the property $\mathcal{P}$; as Γ_f is an immersion, it then follows from (i) that f possesses $\mathcal{P}$. □

Proposition 10.5.15. *Let $f : X \to Y$ and $g : Y \to Z$ be two morphisms. Consider the following properties for a morphism:*

(i) *a monomorphism;*

(ii) *an immersion;*

(iii) *a local immersion;*

(iv) *a universal embedding;*

(v) *radical.*

Then, if $g \circ f$ possesses one of these properties, so does f.

Proof. The properties (i) and (v) have only been put for memory, because for (i) this is a property valid for any category, and for (v), the proposition has already been proven in Proposition 10.3.31.

An immersion has each of these properties, and the composition of two morphisms having one (determined) of these properties also possesses it; moreover, all the above properties are stable under products. Thus the claim follows from Proposition 10.5.14. $\square$

Corollary 10.5.16. *Let $j : X \to Y$ and $g : X \to Z$ be two S-morphisms. If j possesses one of the properties in Proposition 10.5.15, so does $(j, g)_S$.*

Proof. In fact, if $p : Y \times_S Z \to Y$ is the projection, we have $j = p \circ (j, g)_S$, and it suffices to apply Proposition 10.5.15. $\square$

10.5.2 Separated morphisms and schemes

A morphism $f : X \to Y$ of schemes is called **separated** if the diagonal morphism $\Delta_f : X \to X \times_Y X$ is a closed immersion. In this case, X is said to be a **separated Y-scheme**, or **separated over Y**. A scheme X is called **separated** if it is separated over $\mathbb{Z}$. In view of Proposition 10.5.6, for X to be separated over Y, it is necessary and sufficient that the diagonal is a closed subscheme of $X \times_Y X$.

Proposition 10.5.17. *Any morphism of affine schemes is separated. In particular, any affine scheme is separated.*

Proof. If $f : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is a morphism of affine schemes, where B is an A-algebra, the diagonal morphism corresponds to the ring homomorphism $B \otimes_A B \to B$ given by $b \otimes b' \mapsto bb'$. Since this is surjective, we conclude that Δ_f is a closed immersion, so f is separated. $\square$

Proposition 10.5.18. *Let X, Y be S-schemes and $S \to T$ be a separated morphism. Then the canonical immersion $X \times_S Y \to X \times_T Y$ in (10.5.1) is closed.*

Proof. In fact, in the diagram (10.5.1), the diagonal $\Delta_{S/T}$ is a closed immersion, so its base change $\varphi \times_T \psi : X \times_S Y \to X \times_T Y$ is also closed. $\square$

Corollary 10.5.19. *Let X, Y be S-schemes and $f : X \to Y$ be an S-morphism. If Y is spearated over S, the graph morphism $\Gamma_f : X \to X \times_S Y$ is a closed immersion. In particular, any S-section of Y is a closed immersion.*

Proof. This follows from (10.5.2) and Corollary 10.4.14. $\square$

Proposition 10.5.20. *Let Y be a separated S-scheme. Then for any S-morphisms $u_1 : X \to Y$, $u_2 : X \to Y$, the kernel of u_1 and u_2 is a closed subscheme of X.*

Proof. Recall that by Proposition 10.5.11 the kernel is the inverse image of $\Delta_Y(Y)$ under the morphism $(u_1, u_2)_S$. Since $\Delta_Y(Y)$ is closed, it follows from Corollary 10.4.14 that its inverse image is also closed. $\square$

Corollary 10.5.21. *Let S be an integral scheme, η its generic point, and X a separated S-scheme. If two S-sections u, v of X satisfy $u(\eta) = v(\eta)$, then $u = v$.*

Proof. In fact, if $x = u(\eta) = v(\eta)$, the corresponding homomorphisms $\kappa(x) \to \kappa(\eta)$ are necessarily identical, since their composition with the homomorphism $\kappa(\eta) \to \kappa(x)$ corresponding to the structural morphism $X \to S$ is the idnetity on $\kappa(\eta)$. We then deduce from Corollary 10.5.12 that $\eta \in \ker(u_1, u_2)$, and by hypothesis $\ker(u_1, u_2)$ is a closed subscheme of S (Proposition 10.5.20). As S is reduced and η is its generic point, the unique closed subscheme of S containing η is S (Corollary 10.4.29), so $u = v$. □

Proposition 10.5.22. *Let $\mathcal{P}$ be a property of morphisms of schemes, and consider the following properties:*

(i) *If $j : X \to Y$ is a closed immersion and $g : Y \to Z$ is a morphism possessing the property $\mathcal{P}$, then $g \circ j$ possesses the property $\mathcal{P}$.*

(ii) *If $f : X \to X'$ and $g : Y \to Y'$ are two S-morphisms possessing the property $\mathcal{P}$, then $f \times_S g$ possesses the property $\mathcal{P}$.*

(iii) *If the composition $g \circ f$ of two morphisms $f : X \to Y$, $g : Y \to Z$ possesses the property $\mathcal{P}$ and if g is separated, then f possesses the property $\mathcal{P}$.*

(iv) *If $f : X \to Y$ possesses the property $\mathcal{P}$, so does f_{red}.*

Then, (iii) and (iv) are concequences of (i) and (ii).

Proof. For the property (iii), the demonstration is similar to Proposition 10.5.14, with the fact that Γ_f is a closed immersion by Corollary 10.5.19. On the other hand, in the commutative diagram

$$\begin{tikzcd} X_{\text{red}} \arrow[r, "f_{\text{red}}"] \arrow[d] & Y_{\text{red}} \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

the vertical morphisms are closed immersions, so we see that (iv) is a consequence of (i) and (iii), observing that a closed immersion is separated in view of the definition and Proposition 10.5.5. □

Proposition 10.5.23. *Let $f : X \to Y$ be a morphism and $g : Y \to Z$ be a separated morphism. Suppose that $g \circ f$ possesses one of the following properties:*

(i) *universally closed;*

(ii) *a closed immersion;*

Then f possesses the same property.

Proof. In fact, these properties satisfy the conditions (i) and (ii) in Proposition 10.5.22. □

Corollary 10.5.24. *Let Z be a separated S-scheme and $f : X \to Y$, $g : X \to Z$ be two S-morphisms. If f is universally closed (resp. a closed immersion), so is $(f, g)_S : X \to Y \times_S Z$.*

Proof. The morphism j factors into

$$X \xrightarrow{(f,g)_S} Y \times_S Z \xrightarrow{p} Y$$

and the projection $p : Y \times_S Z \to Y$ is a separate morphism by Proposition 10.5.26 (which do not use Corollary 10.5.24), so it suffices to apply Proposition 10.5.23. □

Remark 10.5.25. From the diagram in Proposition 10.5.13, we conclude that a morphism $Y \to S$ is separated if and only if the following equivalent conditions holds:

(i) The diagonal morphism $\Delta_{Y/S}$ is a closed immersion.

(ii) For every S-scheme X and for any two S-morphisms $f, g : X \to Y$, the kernel $\ker(f, g)$ is a closed subscheme of X.

(iii) For every S-scheme X and for any S-morphism $f : X \to Y$, the graph morphism Γ_f is a closed immersion.

Also, if the conclusion in Proposition 10.5.23 holds for the morphisms $\Delta_Y : Y \to Y \times_S Y$ and $p_2 : Y \times_S Y \to Y$, then Δ_Y is a closed immersion so Y is separated over S.

10.5.3 Criterion of separated morphisms

Proposition 10.5.26 (Propoerties of Seperated Morphisms).

(i) *Any radical morphism (and in particular any monomorphism, hence any immersion) is a separated morphism.*

(ii) *The composition of two separated morphisms is separated.*

(iii) *If $f : X \to X'$ and $g : Y \to Y'$ are two separated S-morphisms, $f \times_S g$ is separated.*

(iv) *If $f : X \to Y$ is a separated S-morphism, the S'-morphism $f_{(S')}$ is separated for any base change $S' \to S$.*

(v) *If the composition $g \circ f$ of two morphisms is separated, then f is separated.*

(vi) *For a morphism f to be separated, it is necessary and sufficient that f_{red} is separated.*

Proof. A radical morphism, its diagonal morphism is surjective (Proposition 10.3.31), so it is separated. If $f : X \to Y$, $g : Y \to Z$ are two morphisms, the diagram

$$\begin{tikzcd} X \arrow[rr, "\Delta_{X/Z}"] \arrow[dr, "\Delta_{X/Y}"'] & & X \times_Z X \\ & X \times_Y X \arrow[ur, "j"'] & \end{tikzcd}$$

where j is the canonical immersion in (10.5.1), is commutative. If f and g are separated, $\Delta_{X/Y}$ is a closed immersion and j is a closed immersion by Proposition 10.5.18, so $\Delta_{X/Z}$ is closed, which proves (ii). With (i) and (ii), conditions (iii) and (iv) are equivalent, and it suffices to prove (iv). Now by transitivity, $X_{(S')} \times_{Y_{(S')}} X_{(S')}$ is identified with $(X \times_Y X) \times_Y Y_{(S')}$, and the diagonal morphism $\Delta_{X_{(S')}}$ is identified with $\Delta_X \times_Y 1_{Y_{(S')}}$. The assertion then follows from Corollary 10.4.14.

To prove (v), consider the factorization

$$X \xrightarrow{\Gamma_f} X \times_Z Y \xrightarrow{p_2} Y$$

of f, and note that $p_2 = (g \circ f) \times_Z 1_Y$; the hypothesis that $g \circ f$ is separated implies p_2 is separated by (iii), and as Γ_f is an immersion, Γ_f is separated by (i), hence f is separated by (ii). Finally, for (vi), we recall that the schemes $X_{\mathrm{red}} \times_{Y_{\mathrm{red}}} X_{\mathrm{red}}$ and $X_{\mathrm{red}} \times_Y X_{\mathrm{red}}$ is canonically identified (Proposition 10.4.38); if $j : X_{\mathrm{red}} \to X$ is the canonical injection, the diagram

$$\begin{tikzcd} X_{\mathrm{red}} \arrow[r, "\Delta_{X_{\mathrm{red}}}"] \arrow[d, "j"'] & X_{\mathrm{red}} \times_Y X_{\mathrm{red}} \arrow[d, "j\times_Y j"] \\ X \arrow[r, "\Delta_X"] & X \times_Y X \end{tikzcd}$$

is commutative, and the assertion follows from the fact that the vertical morphisms are homeomorphisms. $\square$

Corollary 10.5.27. *If $f : X \to Y$ is separated, the restriction of f to any subscheme of X is separated.*

Proof. This follows from Proposition 10.5.26(i) and (iii). $\square$

Corollary 10.5.28. *If X and Y are S-schemes and Y is separated over S, $X \times_S Y$ is separated over X.*

Proof. This is a particular case of Proposition 10.5.26(iv). $\square$

Proposition 10.5.29. *Let X be a scheme and suppose that $(X_i)_{1\le i\le n}$ is a finite covering of X by closed subsets. Let $f : X \to Y$ be a morphism and for each i let Y_i be a closed subset of Y such that $f(X_i) \subseteq Y_i$. Consider the reduced subscheme structures on each X_i and Y_i and let $f_i : X_i \to Y_i$ be the restriction of f on X_i. Then for f to be separated, it is necessary and sufficient that each f_i is separated.*

Proof. The necessity follows from Proposition 10.5.26(i), (ii) and (v). Conversely, if each f_i is separated, then the restriction $X_i \to Y$ of f is separated (Proposition 10.5.26). If p_1, p_2 are the projections of $X \times_Y X$, the subspace $\Delta_{X_i}(X_i)$ is identified with the subspace $\Delta_X(X) \cap p_1^{-1}(X_i)$ of $X \times_Y X$ (Corollary 10.5.10). This subspace is closed in $X \times_Y X$ by hypothesis, and their union is $\Delta_X(X)$, so Δ_X is closed and f is separated. $\square$

Suppose in particular that X_i are the irreducible components of X; then we can suppose that each Y_i is a irreducible closed subset of Y (**??**); Proposition 10.5.29 then enable us to reduce the separation problem to integral schemes.

Proposition 10.5.30. *Let (Y_λ) be an open covering of a scheme Y. For a morphism $f : X \to Y$ to be separated, it is necessary and sufficient that each of the restriction $f_\lambda : f^{-1}(Y_\lambda) \to Y_\lambda$ is separated.*

Proof. If we set $X_\lambda = f^{-1}(Y_\lambda)$ and identify $X_\lambda \times_Y X_\lambda$ and $X_\lambda \times_{Y_\lambda} X_\lambda$, the $X_\lambda \times_Y X_\lambda$ form an open covering of $X \times_Y X$. If $Y_{\lambda\mu} = Y_\lambda \cap Y_\mu$ and $X_{\lambda\mu} = X_\lambda \cap X_\mu = f^{-1}(Y_{\lambda\mu})$, then $X_\lambda \times_Y X_\mu$ is identified with $X_{\lambda\mu} \times_{Y_{\lambda\mu}} X_{\lambda\mu}$ by Corollary 10.3.2, hence with $X_{\lambda\mu} \times_Y X_{\lambda\mu}$, and finally to an open subset of $X_\lambda \times_Y X_\lambda$, which establishes our assertion (Corollary 10.4.11). $\square$

Proposition 10.5.30 allows, by taking a covering of Y byopen affines, to reduce the study of separated morphisms to separated morphisms with values in affine schemes.

Proposition 10.5.31. *Let Y be an affine scheme, X be a scheme, and (U_α) be an affine open covering of X. For a morphism $f : X \to Y$ to be separated, it ie necessary and sufficient that, for any couple of indices (α, β), the intersection $U_\alpha \cap U_\beta$ is affine, and the ring $\Gamma(U_\alpha \cap U_\beta, \mathscr{O}_X)$ is generated by the images of $\Gamma(U_\alpha, \mathscr{O}_X)$ and $\Gamma(U_\beta, \mathscr{O}_X)$.*

Proof. The open sets $U_\alpha \times_Y U_\beta$ form an open cover of $X \times_Y X$ (Corollary 10.3.2). Let p, q be the projections of $X \times_Y X$, we have

$$\Delta_X^{-1}(U_\alpha \times_Y U_\beta) = \Delta_X^{-1}(p^{-1}(U_\alpha) \cap q^{-1}(U_\beta)) = U_\alpha \cap U_\beta.$$

It therefore amounts to show that the restriction of Δ_X to $U_\alpha \cap U_\beta$ is a closed immersion into $U_\alpha \times_Y U_\beta$. But this restriction is just $(j_\alpha, j_\beta)_Y$, where j_α (resp. j_β) is the canonical injection of $U_\alpha \cap U_\beta$ to U_α (resp. to U_β). As $U_\alpha \times_Y U_\beta$ is an affine scheme with ring isomorphic to $\Gamma(U_\alpha, \mathscr{O}_X) \otimes_{\Gamma(Y,\mathscr{O}_Y)} \Gamma(U_\beta, \mathscr{O}_X)$, we see that $U_\alpha \cap U_\beta$ is a closed subscheme of $U_\alpha \times_Y U_\beta$ if and only if it is affine and the ring homomorphism

$$\Gamma(U_\alpha, \mathscr{O}_X) \otimes_{\Gamma(Y,\mathscr{O}_Y)} \Gamma(U_\beta, \mathscr{O}_X) \to \Gamma(U_\alpha \cap U_\beta, \mathscr{O}_X), \quad h_\alpha \otimes h_\beta \to h_\alpha h_\beta$$

is surjective (Example 10.4.6), which proves our assertion. □

Corollary 10.5.32. *Let Y be an affine scheme. For a morphism $f : X \to Y$ to be separated, it is necessary and sufficient that X is separated.*

Proof. In fact, the criterion in Proposition 10.5.31 does not depend on f. □

Corollary 10.5.33. *For a morphism $f : X \to Y$ to be separated, it is necessary that for any open affine subscheme U that is separated, the open subscheme $f^{-1}(U)$ is separated, and it suffices that this true for every affine open subset $U \subseteq Y$.*

Proof. The necessity follows from Proposition 10.5.29 and Proposition 10.5.26(ii). The sufficiency follows from Proposition 10.5.30 and Corollary 10.5.32. □

Proposition 10.5.34. *Let Y be a separated scheme and $f : X \to Y$ be a morphism. For any affine open U of X and any affine open V of Y, $U \cap f^{-1}(V)$ is affine.*

Proof. Let p_1, p_2 be the projections of $X \times_{\mathbb{Z}} Y$. Using the universal property of Γ_f, the subspace $U \cap f^{-1}(V)$ can be characterized by

$$\Gamma_f(U \cap f^{-1}(V)) = \Gamma_f(X) \cap p_1^{-1}(U) \cap p_2^{-1}(V)$$

Now $p_1^{-1}(U) \cap p_2^{-1}(V)$ is idnetified with the product $U \times_{\mathbb{Z}} V$, and therefore is affine; as $\Gamma_f(X)$ is closed in $X \times_{\mathbb{Z}} Y$ (Corollary 10.5.19), $\Gamma_f(X) \cap p_1^{-1}(U) \cap p_2^{-1}(V)$ is closed in $U \times_{\mathbb{Z}} V$, hence also affine. The assertion then follows from the fact that Γ_f is a closed immersion and Example 10.4.6. □

Example 10.5.35. The scheme in Example 10.2.10 is separated. In fact, for the covering (X_1, X_2) of X by affine opens, $X_1 \cap X_2 = U_{12}$ is affine and $\Gamma(U_{12}, \mathscr{O}_X)$, the fraction ring of the form $f(s)/s^m$ where $f \in K[s]$, is generated by $K[s]$ and $1/s$, so the conditions in Proposition 10.5.31 are satisfied.

With the same choice of X_1, X_2, U_{12} and U_{21} as in Example 10.2.10, take this time for u_{12} the isomorphism which sends $f(s)$ to $f(t)$; this time we obtain by gluing together a non-separated integral scheme X, because the first condition of Proposition 10.5.31 holds, but the second fails. It is immediate here that $\Gamma(X, \mathscr{O}_X) \to \Gamma(X_1, \mathscr{O}_X) = K[s]$ is an isomorphism; the inverse isomorphism defines a morphism $f : X \to \operatorname{Spec}(K[s])$ which is surjective, and for any $y \in \operatorname{Spec}(K[s])$ such that $\mathfrak{p}_y \neq (0)$, $f^{-1}(y)$ is reduced to a singleton, but for $\mathfrak{p}_y = (0)$, $f^{-1}(y)$ consists of two distinct points (we say that X is the "affine line on K, where the point 0 is doubled").

We can also give examples where neither of the two conditions of Proposition 10.5.31 does not hold. Note first that in the prime spectrum Y of the ring of polynomials $A = K[s,t]$ in two indeterminates over a field K, the open set $U = D(s) \cup D(t)$ is not an affine open set. Indeed, if z is a section of $\mathscr{O}_Y$ over U, there exist two integers $m, n \geq 0$ such that $s^m z$ and $t^n z$ are the restrictions to U of polynomials in s and t (Theorem 10.1.21), which is obviously only possible if the section z extends into a section over the entire space Y, identified with a polynomial in s and t. If U were affine, the canonical injection $U \to Y$ would therefore be an isomorphism by Theorem 10.1.17, which is absurd since $U \neq Y$.

This being so, let us take two affine schemes Y_1, Y_2, with rings $A_1 = K[x_1, t_1]$, $A_2 = K[s_2, t_2]$. Let $U_{12} = D(s_1) \cup D(t_1)$, $U_{21} = D(s_2) \cup D(t_2)$, and let u_{12} be the restriction to U_{21} of the isomorphism $Y_2 \to Y_1$ corresponding to the isomorphism of rints, which sends $f(s_1, t_1)$ to $f(s_2, t_2)$. We thus obtain an example where none of the conditions of Theorem 10.1.17 is satisfied (the integral scheme thus obtained is called "affine plane over K, where point 0 is doubled").

10.6 Finiteness conditions for morphisms

We study, in this section, various "finiteness conditions" for a morphism $f : X \to Y$ of schemes. There are basically two notions of "global finiteness" on X: quasi-compactness and quasi-separateness. On the other hand, there is two notions of "local finiteness" on X: locally of finite type and locally of finite presentation. By combining these local notions with the previous global notions, we obtain the notions of morphism of finite type and of morphism of finite presentation.

10.6.1 Quasi-compact and quasi-separated morphisms

We say a morphism $f : X \to Y$ is **quasi-compact** is for any quasi-compact open subset U of Y, the inverse image $f^{-1}(U)$ is quasi-compact. It is clear that this condition is purely topological, and if X is Noetherian, then any morphism $f : X \to Y$ is quasi-compact. We say a Y-scheme X is **quasi-compact over Y** if its structural morphism is quasi-compact.

If $\mathcal{B}$ is a base of Y formed by quasi-compact open sets (for example, affine opens), for a morphism f to be quasi-compact, it is necessary and sufficient that for any open set $V \in \mathcal{B}$, $f^{-1}(V)$, since any quasi-compact open set of Y is a finite union of open sets in $\mathcal{B}$.

If $f : X \to Y$ is a quasi-compact morphism, it is clear that for any open set V of Y, the restriction $f^{-1}(V) \to V$ of f is quasi-compact. Conversely, if (U_α) is an open covering of Y and $f : X \to Y$ is a morphism such that the restriction $f^{-1}(U_\alpha) \to U_\alpha$ is quasi-compact, then f is quasi-compact, since there exist a basis of quasi-compact open sets for Y, each set of which is contained in at least one of the U_α. We conclude that if $f : X \to Y$ is an S-morphism of S-schemes, and if there is an open covering (S_λ) of S such that the restrictions $\varphi^{-1}(S_\lambda) \to \psi^{-1}(S_\lambda)$ of f (where $\varphi : X \to S$ and $\psi : Y \to S$ are the structural morphisms) are quasi-compact morphisms, then f is quasi-compact.

Proposition 10.6.1. *Let Y be a separated scheme. Then for a morphism $f : X \to Y$ to be quasi-compact, it is necessary and sufficient that X is quasi-compact.*

Proof. If X is quasi-compact, it is a union of finitely many affine opens U_i, and for any affine open V of Y, $U_i \cap f^{-1}(V)$ is an affine open by Proposition 10.5.34, hence quasi-compact; therefore $f^{-1}(V)$ is quasi-compact. Conversely, if f is a quasi-compact morphism, then since Y is quasi-compact open in Y, we see $X = f^{-1}(Y)$ is also quasi-compact. □

Example 10.6.2. A closed immersion is quasi-compact since a closed subset of a quasi-compact set is again quasi-compact. However, open immersions are in general not quasi-compact: the standard example is the affine scheme $X = \mathrm{Spec}(k[x_1, x_2, \dots])$ and consider $U = X - \{0\}$, where 0 is the point of X corresponding to the maximal ideal $(x_1, x_2, \dots)$. The canonical injection $j : U \to X$ is not quasi-compact because U is not quasi-compact. To see this, consider the covering $(D(x_i))_{i\in\mathbb{N}}$ of U; for any finite subset J of $\mathbb{N}$, the family $(D(x_i))_{i\in J}$ can not cover U simply because the prime ideal $\mathfrak{p}_J$ generated by x_i with $i \in J$ is contained in U but not in the union of the $D(x_i)$ for $i \in J$.

We say a morphism $f : X \to Y$ of schemes is **quasi-separated** (of X is an **Y-scheme quasi-separated over Y**) if the diagonal morphism $\Delta_f : X \to X \times_Y X$ is quasi-compact. A scheme X is called **quasi-separated** if it is quasi-separated over $\mathbb{Z}$. Since a closed immersion is quasi-compact, we see any separated morphism is quasi-separated. In particular, any separated scheme is quasi-separated.

Proposition 10.6.3 (Properties of Quasi-Compact Morphisms).

(i) *An immersion $j : X \to Y$ is quasi-compact if it is closed, or Y is locally Noetherian, or X is Noetherian.*

(ii) *The composition of two quasi-compact morphisms is quasi-compact.*

(iii) *If $f : X \to Y$ is a quasi-compact S-morphism, $f_{(S')} : X_{(S')} \to Y_{(S')}$ is quasi-compact for any base change $g : S' \to S$.*

(iv) *If $f : X \to X'$ and $g : Y \to Y'$ is two quasi-compact S-morphisms, then $f \times_S g$ is quasi-compact.*

(v) *If the composition $g \circ f$ of two morphisms $f : X \to Y$, $g : Y \to Z$ is quasi-compact, and if f is quasi-separated or if X is locally Noetherian, then f is quasi-compact.*

(vi) *For a morphism f to be quasi-compact, it is necessary and sufficient that f_{red} is quasi-compact.*

Proof. Assertion (vi) is evident, since the quasi-compactness for a morphism only depends on the map of the underlying spaces. Also, (ii) follows from the definition of quasi-compactness.

The assertion in (i) is clear if j is closed; if j is an immersion and Y is locally Noetherian, any quasi-compact open V of Y is Noetherian, so $j^{-1}(V) = X \cap V \subseteq V$ is quasi-compact (here we identify X as a subscheme of Y). If X is Noetherian, then any morphism from X is quasi-compact.

To prove (iii), we can assume that $S = Y$ by the transitivity of products; put $f' = f_{(S')}$, and let U' be a quasi-compact open subset of S'. For any $s' \in U'$, let T be an affine open neighborhood of $g(s')$ in S, and let W be an affine open neighborhood of s' contained in $U' \cap g^{-1}(T)$; it suffices to show that $f'^{-1}(W)$ is quasi-compact, or in other words, we only need to show that if S and S' are affine, then $X \times_S S'$ is quasi-compact. This is true because by hypothesis X is a finite union of affine opens V_j, and $X \times_S S'$ is then the union of the affine schemes $V_j \times_S S'$, hence quasi-compact. With (ii) and (iii), asserion (iv) then follows.

We now prove (v) in the case where X is locally Noetherian. Put $h = g \circ f$ and let U be a quasi-compact open of Y; $g(U)$ is then quasi-compact in Z (not necessarily open), so it is contained in a finite union of quasi-compact opens V_j, and $f^{-1}(U)$ is contained in the union of the $h^{-1}(V_j)$, which are all quasi-compact by hypothesis. We then conclude that $f^{-1}(U)$ is a Noetherian space (**??**), and a fortiori quasi-compact.

To prove (v) in the case that g is quasi-separated, recall that f factors into

$$X \xrightarrow{\Gamma_f} X \times_Z Y \xrightarrow{p_2} Y$$

where p_2 is identified with $(g \circ f) \times_Z 1_Y$, and if $g \circ f$ is quasi-compact, so is p_2 by (iii). Finally, we have the following cartesian square (Corollary 10.5.4)

$$\begin{tikzcd}
X \arrow[r, "\Gamma_f"] \arrow[d, "f"'] & X \times_Z Y \arrow[d, "f \times 1_Y"] \\
Y \arrow[r, "\Delta_g"] & Y \times_Z Y
\end{tikzcd}$$

as by hypothesis Δ_g is quasi-compact, Γ_f is also quasi-compact, and by (ii) we conclude that f is quasi-compact. □

Proposition 10.6.4. *Let $f : X \to Y$, $g : Y \to Z$ be two morphisms. If $g \circ f$ is quasi-compact and f is surjective, then g is quasi-compact.*

Proof. If fact, if V is a quasi-compact open of Z, $f^{-1}(g^{-1}(V))$ is quasi-compact by hypothesis, and we have $g^{-1}(V) = f(f^{-1}(g^{-1}(V)))$ since f is surjective, so $g^{-1}(V)$ is quasi-compact. □

Proposition 10.6.5. *Let f be a quasi-compact morphism of schemes.*

(a) *The following conditions are equivalent:*

(i) *f is dominant;*

(ii) *for any maximal point $y \in Y$, $f^{-1}(y) \neq \varnothing$.*

(iii) *for any maximal point $y \in Y$, $f^{-1}(y)$ contains a maximal point of X.*

(b) *If f is dominant, for any generalizing morphism $g : Y' \to Y$, the morphism $f_{(Y')} : X_{(Y')} \to Y'$ is quasi-compact and dominant.*

Proof. It is clear that (iii)⇒(ii)⇒(i). Suppose that f is dominant and consider an affine open neighborhood U of y; $f^{-1}(U)$ is quasi-compact, hence a union of finitely many affine opens V_i, and by hypothesis y belongs to the closure of $f(V_i)$ in U. We can evidently suppose that X and Y are reduced. As the closure in X of an irreducible component of V_i is an irreducible component of X (**??**), we can replace X by V_i, Y by the closed reduced subscheme $\overline{f(V_i)} \cap U$ of U, and we are thus reduced to proving the proposition when $X = \operatorname{Spec}(B)$ and $Y = \operatorname{Spec}(A)$ are affine and reduced. Since f is dominant, A is then a subring of B (Proposition 1.4.20); the proposition then follows from the fact that any minimal prime ideal of A is the intersection with A of a minimal prime ideal of B (Proposition 1.2.45).

If $f : X \to Y$ is quasi-compact and dominant, then $f' = f_{(Y')}$ is quasi-compact by Proposition 10.6.3. On the other hand, a maximal point y' of Y' is hypothesis lying over a maximal point y of Y ([**?**] new, 3.9.5); as $f^{-1}(y)$ is nonempty by (i), the same holds for $f'^{-1}(y')$ (Proposition 10.3.38), whence the conclusion. □

Proposition 10.6.6. *For a quasi-compact morphism $f : X \to Y$, the following conditions are equivalent:*

(i) *The morphism f is closed.*

(ii) *For any $x \in X$ and any specialization y' of $y = f(x)$ distinct from y, there exists a specialization x' of x such that $f(x') = y'$.*

In particular, if $f : X \to Y$ is a quasi-compact immersion, for f to be a closed immersion, it is necessary and sufficient that X (considered as a subspace of Y) contains any specializations (in Y) of its points.

Proof. The condition (ii) expresses as $f(\overline{\{x\}}) = \overline{\{y\}}$, and is therefore a concequence of (i). To show that (ii) implies (i), consider a closed subset of X' of X; let $Y' = \overline{f(X')}$ and we prove that $Y' = f(X')$. Endow X' and Y' the reduced subscheme structure, there then exists a morphism $f' : X' \to Y'$ such that the diagram

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d] & Y' \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

commutes. As f is quasi-compact, so is f' (Proposition 10.6.3(i) and (v)). We are then reduced to proving that if f is a quasi-compact dominant morphism, then $f(X) = Y$. Now let y' be a point of y and let y be the generic point of a irreducible component of Y containing y'; by (ii), it suffices to note that $f^{-1}(y)$ is nonempty, which follows from Proposition 10.6.5. □

Proposition 10.6.7 (Properties of Quasi-Separated Morphisms).

(i) *Any radical morphism $f : X \to Y$ (in particular, any monomorphism and any immersion) is quasi-separated.*

(ii) *If $f : X \to Y$ and $g : Y \to Z$ are two quasi-separated morphisms, $g \circ f$ is quasi-separated.*

(iii) *Let X, Y be two S-schemes and $f : X \to Y$ be a quasi-separated S-morphism. Then, for any base change $g : S' \to S$, the morphism $f_{(S')} : X_{(S')} \to Y_{(S')}$ is quasi-separated.*

(iv) *If $f : X \to X'$, $g : Y \to Y'$ are two quasi-separated S-morphisms, $f \times_S g$ is quasi-separated.*

(v) *If the composition $g \circ f$ of two morphisms $f : X \to Y$, $g : Y \to Z$ is quasi-separated, then f is quasi-separated; if moreover f is quasi-compact and surjective, g is also quasi-separated.*

(vi) *For a morphism $f : X \to Y$ to be quasi-separated, it is necessary and sufficient that f_{red} is quasi-separated.*

Proof. The assertion (i) follows from Proposition 10.5.26(i). To prove (iii), we may reduce to the case $Y = S$, and the assertion then follows from $\Delta_{f_{(S')}} = (\Delta_f)_{(S')}$ (Corollary 10.5.2) and Proposition 10.6.3.

For assertion (ii), consider the projections p, q of $X \times_Y X$; if $\pi : X \times_Y X \to Y$ is the structural morphism and $j = (p, q)_Z$, we have the following cartesian square (Eq. (10.5.1))

$$\begin{tikzcd} X \times_Y X \arrow[r, "j"] \arrow[d, "\pi"'] & X \times_Z X \arrow[d, "f \times_Z f"] \\ Y \arrow[r, "\Delta_g"] & Y \times_Z Y \end{tikzcd}$$

If g is quasi-separated then Δ_g is quasi-compact, so j is also quasi-compact by Proposition 10.6.3(iii). If f is quasi-separated, Δ_f is quasi-compact and so is $j \circ \Delta_f$, which equals to $\Delta_{g \circ f}$. With these, assertion (iv) then follows from (ii) and (iii).

Suppose now that $g \circ f$ is quasi-separated. Then with the preceding notations, $\Delta_{g \circ f} = j \circ \Delta_f$ is quasi-compact, so Δ_f is quasi-compact by Proposition 10.6.3(v) and f is then quasi-separated. If moreover f is quasi-compact and surjective, $f \times_{\mathbb{Z}} f$ is also quasi-compact by Proposition 10.6.3(iv), and we conclude that $\Delta_g \circ \pi \circ \Delta_f$ is quasi-compact. Since $\pi \circ \Delta_f = f$ is surjective, it follows from Proposition 10.6.3(v) that Δ_g is quasi-compact, so g is quasi-separated.

Finally, for a morphism $f : X \to Y$, consider the following diagram

$$\begin{tikzcd} X_{\mathrm{red}} \arrow[r, "f_{\mathrm{red}}"] \arrow[d, "j_X"'] & Y_{\mathrm{red}} \arrow[d, "j_Y"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

where j_X and j_Y are surjective closed immersions, and so quasi-separated and quasi-compact. From the equality $f \circ j_X = j_Y \circ f_{\mathrm{red}}$ and (v), we see f is quasi-separated if and only if f_{red} is quasi-separated. □

Corollary 10.6.8. *Let X and Y be schemes.*

(i) *If f is quasi-separated, any morphism $f : X \to Y$ is quasi-separated.*

(ii) *If Y is quasi-separated, for a morphism $f : X \to Y$ to be quasi-separated, it is necessary and sufficient that the scheme X is quasi-separated.*

(iii) *Let X be a quasi-compact and Y be quasi-separated. Then any morphism $f : X \to Y$ is quasi-compact.*

Proof. To show (i) we only need to note that any morphism $f : X \to Y$ is a $\mathbb{Z}$-morphism, and if X is quasi-separated, then for any morphism $f : X \to Y$ the composition $X \to Y \to \operatorname{Spec}(\mathbb{Z})$ is quasi-separated, so f is quasi-separated by Proposition 10.6.7(v). Similarly, assertion (ii) follows from Proposition 10.6.7(ii) and (v). Assertion (iii) follows from Proposition 10.6.3(v). □

Proposition 10.6.9. *Let $f : X \to Y$ be a morphism and (U_α) be a covering of Y by open subschemes that are quasi-separated. For f to be quasi-separated, it is necessary and sufficient that the open subschemes $f^{-1}(U_\alpha)$ is quasi-separated.*

Proof. The inverse image of U_α in $X \times_Y X$ is $X_\alpha \times_{U_\alpha} X_\alpha$, where $X_\alpha = f^{-1}(U_\alpha)$, and the restriction $X_\alpha \to X_\alpha \times_{U_\alpha} X_\alpha$ of Δ_f is just Δ_{f_α}, where f_α is the restriction $X_\alpha \to U_\alpha$ of f. Since quasi-compactness is local on target, we see f is quasi-separated if and only if each f_α is. But by hypothesis U_α is separated, so the conclusion follows from Corollary 10.6.8(ii). □

By Proposition 10.6.9, to verify a morphism is quasi-separated, it suffices to verify the quasi-separateness of some subschemes. This can be done by the following simple criteria:

Proposition 10.6.10. *Let X be a scheme and (U_α) be a covering of X formed by quasi-compact open subsets. Then the following conditions are equivalent:*

(i) *X is a quasi-separated scheme.*

(ii) *For any quasi-compact open subset U of X, the canonical injection $U \to X$ is quasi-compact (that is, U is retrocompact in X).*

(iii) *The intersection of two quasi-compact open subsets of X is quasi-compact.*

(iv) *For any couple of indices (α, β), the intersection $U_\alpha \cap U_\beta$ is quasi-compact.*

Proof. Properties (ii) and (iii) are equivalent by the definition of quasi-compactness. As a quasi-compact open is a finie union of affine open sets, for two quasi-compact open subsets U, V of X, $U \times_{\mathbb{Z}} V$ is a quasi-compact open subset of $X \times_{\mathbb{Z}} X$ (Corollary 10.3.2), with inverse image $U \cap V$ under Δ_X, hence (i) implies (iii). It is clear that (iii) implies (iv); finally, if (iv) holds, the $U_\alpha \times_{\mathbb{Z}} U_\beta$ form a covering of $X \times_{\mathbb{Z}} X$ by quasi-compact open sets and the inverse image of $U_\alpha \times_{\mathbb{Z}} U_\beta$ under Δ_X is $U_\alpha \cap U_\beta$, hence quasi-compact. It then follows that Δ_X is quasi-compact, so (iv) implies (i). □

Corollary 10.6.11. *Any locally Noetherian scheme X is quasi-separated, and any morphism $f : X \to Y$ is then quasi-separated.*

Proof. It suffices to note that any quasi-compact open subset of X is Noetherian, so X is quasi-separated by Proposition 10.6.10 and any morphism $f : X \to Y$ is quasi-separated by Proposition 10.6.9, since any open subscheme of X is again locally Noetherian. □

Proposition 10.6.12. *Let $f : X \to Y$ be a morphism and $g : Y' \to Y$ be a base change that is surjective and quasi-compact. Put $f' = f_{(Y')}$ and consider the following properties:*

(i) *quasi-compact;*

(ii) *quasi-separated.*

Then if $\mathcal{P}$ denotes one of these properties and f' possesses the property $\mathcal{P}$, then f possesses the property $\mathcal{P}$.

Proof. Let $g' : X' \to X$ be the cannical projection, which is surjective and quasi-compact (Proposition 10.3.28 and Proposition 10.6.3(iii)). If f' is quasi-compact, so is $g \circ f'$ and since $f \circ g' = g \circ f'$ we conclude that f is quasi-compact by Proposition 10.6.3(v).

Now assume that f' is quasi-separated. We have $X' \times_{Y'} X' = (X \times_Y X)_{(Y')}$ and $\Delta_{f'} = (\Delta_f)_{(Y')}$. The projection $X' \times_{Y'} X' \to X \times_Y X$ is quasi-compact and surjective by the same reasoning, and we can apply (i) to the morphism Δ_f. Since by hypothesis $\Delta_{f'}$ is quasi-compact, we conclude that Δ_f is quasi-compact, so f is quasi-separated. □

Proposition 10.6.13. *Let $f_i : X_i \to Y$ be a finite family of morphisms and $f : X \to Y$ be their coproduct. Then for f to be quasi-compact (resp. quasi-separated), it is necessary and sufficient that each f_i is quasi-compact (resp. quasi-separated).*

Proof. The assertion about quasi-compactness follows from definition. We also note that $X \times_Y X$ is the coproduct of that $X_i \times_Y X_j$, and Δ_f is the morphism that coincides with Δ_{f_i} on each X_i, so the assertion for quasi-separatedness also follows. □

Theorem 10.6.14. *Let X be a quasi-compact and quasi-separated scheme, $\mathscr{L}$ be an invertible $\mathscr{O}_X$-module, s be a section over X, X_s the open subset of $x \in X$ such that $s(x) \neq 0$, and $\mathscr{F}$ be a quasi-coherent $\mathscr{O}_X$-module.*

(a) *If $t \in \Gamma(X, \mathscr{F})$ is such that $t|_{X_s} = 0$, there exists $n > 0$ such that $t \otimes s^{\otimes n} = 0$.*

(b) *For any section $t \in \Gamma(X_s, \mathscr{F})$, there exists an integer $n > 0$ such that $t \otimes s^{\otimes n}$ can be extended to a section $t' \in \Gamma(X, \mathscr{F} \otimes \mathscr{L}^{\otimes n})$.*

Proof. As the space X is a finite union of affine opens U_i such that $\mathscr{L}|_{U_i}$ is isomorphic to $\mathscr{O}_X|_{U_i}$, we can assume that X is affine and $\mathscr{L} = \mathscr{O}_X$. The assertion (a) then follows from Theorem 10.1.21(iv).

Now let t be a sectin of $\mathscr{F}$ over X_s. Since $\mathscr{L}|_{U_i}$ is isomorphic to $\mathscr{O}_X|_{U_i}$, the restriction $(t \otimes s^{\otimes n})|_{U_i \cap X_s}$ is identified with $(t|_{U_i \cap X_s})(s|_{U_i \cap X_s})^n$ under the isomorphism $(\mathscr{F} \otimes \mathscr{L}^{\otimes n})|_{U_i} \cong \mathscr{F}|_{U_i}$. We then conclude from Theorem 10.1.21(iv) that there exists an integer $n \geq 0$ such that for each i, $(t \otimes s^{\otimes n})|_{U_i \cap X_s}$ extends to a section t_i of $\mathscr{F} \otimes \mathscr{L}^{\otimes n})$ over U_i. Let t_{ij} be the restriction of t_i to $U_i \cap U_j$; by definition we have $(t_{ij} - t_{ji})|_{X_s \cap U_i \cap X_j} = 0$. Since X is quasi-separated, $U_i \cap U_j$ is quasi-compact, so by Theorem 10.1.21(iv) there exists an integer $m \geq 0$ such that $(t_{ij} - t_{ji}) \otimes s^{\otimes m} = 0$. The sections $t_i \otimes s^{\otimes m}$ then glue together to give a section $t' \in \Gamma(X, \mathscr{F} \otimes \mathscr{L}^{\otimes(n+m)})$, which induces $t_i \otimes s^{\otimes m}$ over each U_i, and hence induces $t \otimes s^{\otimes(n+m)}$ over X_s. □

Corollary 10.6.15. *With the hypotheses of Theorem 10.6.14, consider the ring $A = \Gamma_*(\mathscr{L})$ and the graded A-module $M = \Gamma_*(\mathscr{F}; \mathscr{L})$ of type $\mathbb{Z}$. Then for each $s \in A_n$, there exists a canonical isomorphism $\Gamma(X_s, \mathscr{F}) \cong M_{(s)}$, where $M_{(s)} = (M_s)_0$ is the degree zero part of the localization M_s.*

Proof. With the notations of Theorem 10.6.14(b), we see that any element $t \in \Gamma(X_s, \mathscr{F})$ corresponds to an element t'/s^n in $M_{(s)}$, which is independent of the integer n and the chosen extension t', in view of Theorem 10.6.14(a). It is immediate that this defines a homomorphism, and is bijective. □

Corollary 10.6.16. *Let X be a quasi-compact and quasi-separated scheme and $\mathscr{F}$ a quasi-coherent $\mathscr{O}_X$-module. Then if $A = \Gamma(X, \mathscr{O}_X)$ and $M = \Gamma(X, \mathscr{F})$, the A_s-module $\Gamma(X_s, \mathscr{F})$ is canonically isomorphic to M_s.*

Proof. This is a special case of Corollary 10.6.15, by taking $\mathscr{L} = \mathscr{O}_X$. □

Proposition 10.6.17. *Let X be a quasi-compact scheme, $\mathscr{F}$ be a quasi-coherent $\mathscr{O}_X$-module of finite type, and $\mathscr{I}$ be a quasi-coherent ideal of $\mathscr{O}_X$ of finite type such that* $\mathrm{supp}(\mathscr{F})$ *is contained in* $\mathrm{supp}(\mathscr{O}_X/\mathscr{I})$*. Then there exists an integer $n \geq 0$ such that $\mathscr{I}^n\mathscr{F} = 0$.*

Proof. As X is a finite union of affine open subsets, we may assume that $X = \mathrm{Spec}(A)$ is affine. Then $\mathscr{F} = \widetilde{M}$ and $\mathscr{I} = \tilde{\mathfrak{a}}$, where M is a finitely generated A-module and $\mathfrak{a}$ is a finitely generated ideal of A, and

$$\mathrm{supp}(\mathscr{F}) = \mathrm{supp}(M) = V(\mathrm{Ann}(M)), \quad \mathrm{supp}(\mathscr{O}_X/\mathscr{I}) = \mathrm{supp}(A/\mathfrak{a}) = V(\mathfrak{a}).$$

By hypothesis we have $V(\mathrm{Ann}(M)) \subseteq V(\mathfrak{a})$, so $\mathfrak{a} \subseteq \sqrt{\mathrm{Ann}(M)}$. Since $\mathfrak{a}$ is finitely generated, there exists an integer $n \geq 0$ such that $\mathfrak{a}^n \subseteq \mathrm{Ann}(M)$, and therefore $\mathscr{I}^n\mathscr{F} = \widetilde{\mathfrak{a}^n M} = 0$. □

Corollary 10.6.18. *Under the hypothesis of Proposition 10.6.17, there exists a closed subscheme Y of X with underlying space* $\operatorname{supp}(\mathscr{O}_X/\mathscr{J})$ *such that, if $j : Y \to X$ is the canonical injection, we have $\mathscr{F} = j_*(j^*(\mathscr{F}))$.*

Proof. Note that the support of $\mathscr{O}_X/\mathscr{J}$ and $\mathscr{O}_X/\mathscr{J}^n$ are the same because if $\mathscr{J}_x = \mathscr{O}_{X,x}$, we also have $\mathscr{J}_x^n = \mathscr{O}_{X,x}$, and on the other hand we have $\mathscr{J}_n^x \subseteq \mathscr{J}_x$ for each $x \in X$. In view of Proposition 10.6.17, we may then suppose that $\mathscr{J}\mathscr{F} = 0$, so $\mathscr{F}$ is also an $(\mathscr{O}_X/\mathscr{J})$-module. If Y is the subscheme defined by $\mathscr{J}$, the conclusion is immediate. □

10.6.2 Morphisms of finite type and of finite presentation

Let $f : X \to Y$ be a morphism of schemes, x be a point of X, and $y = f(x)$. We say f is **of finite type** (resp. **of finite presentation**) **at the point x** if there exist an affine neighborhood V of y and an affine neighborhood U of x such that $f(U) \subseteq V$ and $\Gamma(U, \mathscr{O}_X)$ is a $\Gamma(V, \mathscr{O}_Y)$-algebra of finite type (resp. of finite presentation). We say f is **locally of finite type** (resp. **locally of finite presentation**) if it is of finite type (resp. of finite presentation) at every point of X. In this case, we say the Y-scheme X is locally of finite type (resp. locally of finite presentation) over Y.

Lemma 10.6.19. *Let $f : X \to Y$ be a morphism of schemes, x be a point of X, and $y = f(x)$. If there exist an affine neighborhood V of y and an affine neighborhood U of x such that $f(U) \subseteq V$ and $\Gamma(U, \mathscr{O}_X)$ is a $\Gamma(V, \mathscr{O}_Y)$-algebra of finite type (resp. of finite presentation), then for any affine open neighborhoods U' of x and V' of y, there exist affine open neighborhoods $U_1 \subseteq U \cap U'$ of x and $V_1 \subseteq V \cap V'$ of y, respectively of the form* $\operatorname{Spec}(\Gamma(U', \mathscr{O}_X)_{s'})$ *and* $\operatorname{Spec}(\Gamma(V', \mathscr{O}_Y)_{t'})$*, such that $f(U_1) \subseteq V_1$ and $\Gamma(U_1, \mathscr{O}_X)$ is a $\Gamma(V_1, \mathscr{O}_Y)$-algebra of finite type (resp. of finite presentation).*

Proof. Let $t' \in \Gamma(V', \mathscr{O}_Y)$ such that $V_1 = \operatorname{Spec}(\Gamma(V', \mathscr{O}_Y)_t)$ is an affine neighborhood of y contained in $V \cap V'$ and choose $s_0' \in \Gamma(U', \mathscr{O}_X)$ such that $U'' = \operatorname{Spec}(\Gamma(U', \mathscr{O}_X)_{s_0'})$ is a neighborhood of x contained in $U \cap U' \cap f^{-1}(V_1)$. There then exists $s \in \Gamma(U, \mathscr{O}_X)$ such that $U_1 = \operatorname{Spec}(\Gamma(U, \mathscr{O}_X)_s)$ is a neighborhood of x contained in U''. If s'' is the image of s in $\Gamma(U'', \mathscr{O}_X)$, we then have $U_1 = \operatorname{Spec}(\Gamma(U'', \mathscr{O}_X)_{s''})$, so there exists $s' \in \Gamma(U', \mathscr{O}_X)$ such that $U_1 = \operatorname{Spec}(\Gamma(U', \mathscr{O}_X)_{s'})$. Now $\Gamma(U_1, \mathscr{O}_X) = \Gamma(U, \mathscr{O}_X)[1/s]$, so it is a $\Gamma(U, \mathscr{O}_X)$-algebra of finite presentation, and a fortiori a $\Gamma(V, \mathscr{O}_Y)$-algebra of finite type (resp. of finite presentation). The homomorphism $\Gamma(V, \mathscr{O}_Y) \to \Gamma(U_1, \mathscr{O}_X)$ factors into

$$\Gamma(V, \mathscr{O}_X) \longrightarrow \Gamma(V_1, \mathscr{O}_Y) \longrightarrow \Gamma(U_1, \mathscr{O}_X)$$

If $\Gamma(U_1, \mathscr{O}_X)$ is identified with a quotient algebra $\Gamma(V, \mathscr{O}_Y)[T_1, \dots, T_n]/\mathfrak{a}$, then it is also identified with the quotient algebra $\Gamma(V_1, \mathscr{O}_Y)[T_1, \dots, T_n]/\mathfrak{b}$, where $\mathfrak{b}$ is the ideal generated by $\mathfrak{a}$. It then follows that $\Gamma(U_1, \mathscr{O}_X)$ is a $\Gamma(V_1, \mathscr{O}_Y)$-algebra of finite type (resp. finite presentation). □

Proposition 10.6.20. *If Y is locally Noetherian, then $f : X \to Y$ is locally of finite type if and only if it is locally of finite presentation. Moreover, if this holds, then X is also locally Noetherian.*

Proof. The first assertion is clear since we can take $\Gamma(V, \mathscr{O}_Y)$ to be Noetherian. The second one follows because $\Gamma(U, \mathscr{O}_X)$ is then also Noetherian. □

Proposition 10.6.21 (Properties of Morphisms Locally of Finite Type).

(i) *Any local immersion is locally of finite type.*

(ii) *If $f : X \to Y$ and $g : Y \to Z$ are two morphisms locally of finite type, then $g \circ f$ is locally of finite type.*

(iii) *If $f : X \to Y$ is an S-morphism locally of finite type, $f_{(S')} : X_{(S')} \to Y_{(S')}$ is locally of finite type for any base change $g : S' \to S$.*

(iv) *If $f : X \to X'$ and $g : Y \to Y'$ are two S-morphisms locally of finite type, $f \times_S g$ is locally of finite type.*

(v) *If the composition $g \circ f$ of two morphisms is locally of finite type, then f is locally of finite type.*

(vi) *If a morphism f is locally of finite type, so is f_{red}.*

Proof. Assertion (vi) follows from the fact that if a ring homomorphism $A \to B$ is of finite type, then so is $A/\mathfrak{n}(A) \to B/\mathfrak{n}(B)$. Now in view of Proposition 10.5.14, it suffices to prove (i), (ii) and (iii). If $j : X \to Y$ is a local immersion, for any $x \in X$ there exists an affine open neighborhood V of $j(x)$ in Y and an affine open neighborhood U of x such that the restriction $U \to V$ of j is a closed immersion. Then $\Gamma(U, \mathscr{O}_X)$ is a quotient ring of $\Gamma(V, \mathscr{O}_Y)$, and is therefore of finite type.

To establish (iii), we may assume that $Y = S$; let $p : X_{(S')} \to X$ and $q : X_{(S')} \to S$ be the canonical projections, x' be a point of $X_{(S')}$, and $x = p(x')$, $s' = q(x')$, $s = f(p(x')) = g(q(x'))$. Let V be an affine neighborhood of s in S and U be an affine neighborhood of x in X such that $f(U) \subseteq V$ and $\Gamma(U, \mathscr{O}_X)$ is a $\Gamma(V, \mathscr{O}_S)$-algebra of finite type. Let V' be an affine open neighborhood of s' in S' contained in $g^{-1}(V)$; then $p^{-1}(U) \cap q^{-1}(V)$ is an affine neighborhood of x' and is identified with $U \times_V V'$ (Corollary 10.3.2). This is an affine scheme with ring $\Gamma(U, \mathscr{O}_X) \otimes_{\Gamma(V, \mathscr{O}_S)} \Gamma(V', \mathscr{O}_{S'})$; as this is a $\Gamma(V', \mathscr{O}_{S'})$-algebra of finite type, we see (iii) follows.

Finally, to prove (ii), consider a point $x \in X$; there exists by hypothesis an affine open neighborhood W of $g(f(x))$ in Z and an affine open neighborhood V of $f(x)$ in Y such that $g(V) \subseteq W$ and $\Gamma(V, \mathscr{O}_Y)$ is a $\Gamma(W, \mathscr{O}_Z)$-algebra of finite type. By Lemma 10.6.19 there exists an affine open neighborhood $V' \subseteq V$ of $f(x)$ and an affine open neighborhood $U \subseteq f^{-1}(V')$ of x such that $\Gamma(V', \mathscr{O}_Y)$ is a $\Gamma(V, \mathscr{O}_Y)$-algebra of finite type and $\Gamma(U, \mathscr{O}_X)$ is a $\Gamma(V', \mathscr{O}_Y)$-algebra of finite type. We then conclude that $\Gamma(U, \mathscr{O}_X)$ is a $\Gamma(W, \mathscr{O}_Z)$-algebra of finite type, so (ii) follows. □

Corollary 10.6.22. *Let $f : X \to Y$ be a morphism locally of finite type. For any morphism $Y' \to Y$ such that Y' is locally Noetherian, $X \times_Y Y'$ is locally Noetherian.*

Proof. This follows from Proposition 10.6.20, since $f_{(Y')} : X \times_Y Y' \to Y'$ is locally of finite type by Proposition 10.6.21. □

Proposition 10.6.23. *Let $\rho : A \to B$ be a homomorphism of rings. For the corresponding morphism $f : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ to be locally of finite type, it is necessary and sufficient that B is an A-algebra of finite type.*

Proof. This condition is clearly sufficient. Conversely, assume that f is locally of finite type. Then by Lemma 10.6.19 there exists a finite cover of $\mathrm{Spec}(B)$ by open sets $D(g_i)$ (where $g_i \in B$) such that B_{g_i} is an A-algebra of finite type. Since the $D(g_i)$'s cover $\mathrm{Spec}(B)$, we see g_i generate the ring B, and it follows from Corollary 1.4.56 that B is of finite type over A. □

Proposition 10.6.24 (Properties of Morphisms Locally of Finite Presentation).

(i) *Any local isomorphism is locally of finite presentation.*

(ii) *If two morphisms $f : X \to Y$, $g : Y \to Z$ are locally of finite presentation, so is $g \circ f$.*

(iii) *If $f : X \to Y$ is an S-morphism locally of finite presentation, $f_{(S')} : X_{(S')} \to Y_{(S')}$ is locally of finite presentation for any base change $g : S' \to S$.*

(iv) *If $f : X \to X'$ and $g : \to Y'$ are two S-morphisms locally of finite presentation, $f \times_S g$ is locally of finite presentation.*

(v) *If the compositin $g \circ f$ of two morphisms $f : X \to Y$, $g : Y \to Z$ is locally of finite presentation and if g is locally of finite type, then f is locally of finite presentation.*

Proof. The first assertion is trivial, and to prove (ii), (iii), it suffices to replace the "algebra of finite type" in the proof of Proposition 10.6.21 by "algebra of finite presentation", and use Lemma 10.6.19. Again, assertion (iv) then follows from these. For (v), consider the cartesian square

$$\begin{tikzcd} X \arrow[r, "\Gamma_f"] \arrow[d, "f"'] & X \times_Z Y \arrow[d, "f \times_Z 1_Y"] \\ Y \arrow[r, "\Delta_g"] & Y \times_Z Y \end{tikzcd}$$

If we can show that Δ_g is locally of finite presentation, then it follows from (iii) that Γ_f is also locally of finite presentation. But f factors into

$$X \xrightarrow{\Gamma_f} X \times_Z Y \xrightarrow{p_2} Y$$

and we have $p_2 = (g \circ f) \times_Z 1_Y$, which is locally of finite presentation by (iv) since $g \circ f$ is. We then deduce that f is locally of finite presentation.

It then suffices to prove that is $g : Y \to Z$ is a morphism locally of finite type, then the diagonal morphism $\Delta_g : Y \to Y \times_Z Y$ is locally of finite presentation. To do this we may assume that $Z = \mathrm{Spec}(A)$, $Y = \mathrm{Spec}(B)$, and B is an A-algebra of finite type. The diagonal Δ_g corresponds to the homomorphism $\pi : B \otimes_A B \to B$ such that $\pi(x \otimes y) = xy$. Let $\mathfrak{I}$ be the kernel of π. We claim that $\mathfrak{I}$ is generated by the elements $1 \otimes s - s \otimes 1$, where s runs through a system of generators for the A-algebra B (this then proves the claim since B is of finite type over A). Now, it is clear that for any $x \in B$, we have $x \otimes 1 - 1 \otimes x \in \mathfrak{I}$; on the other hand, for $x, y \in B$, we have

$$x \otimes y = xy \otimes 1 + (x \otimes 1)(1 \otimes y - y \otimes 1)$$

If $\sum_i (x_i \otimes y_i) \in \mathfrak{I}$, we have by definition that $\sum_i x_i y_i = 0$, so

$$\sum_i (x_i \otimes y_i) = \sum_i (x_i \otimes 1)(1 \otimes y_i - y_i \otimes 1).$$

Moreover, if $x = st$, then

$$x \otimes 1 - 1 \otimes x = (s \otimes 1)(t \otimes 1 - 1 \otimes t) + (s \otimes 1 - 1 \otimes t)(1 \otimes t).$$

The claim then follows by indcution on the number of factors of a product in B. □

Corollary 10.6.25. *Let $g : Y \to Z$ be a morphism locally of finite type. Then the diagonal morphism $\Delta_g : Y \to Y \times_Z Y$ is locally of finite presentation.*

Proof. This is contained in the proof of Proposition 10.6.24. □

Proposition 10.6.26. *Let A be a ring, B be an A-algebra, $B' = A[T_1, \dots, T_n]$, and $\rho : B' \to B$ be a surjective homomorphism of A-algebras. Then for B to be an A-algebra of finite presentation, it is necessary and sufficient that the kernel $\mathfrak{a}$ of ρ is finitely generated in B'.*

Proof. The condition is sufficient by definition. Conversely, we note that the morphism $g : \mathrm{Spec}(B') \to \mathrm{Spec}(A)$ is locally of finite type; if B is an A-algebra of finite presentation, the morphism $f : \mathrm{Spec}(B) \to \mathrm{Spec}(B')$ corresponding to ρ and $g \circ f : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ are locally of finite presentation, so it follows from Proposition 10.6.24(v) that f is locally of finite presentation. Now it suffices to apply Proposition 1.4.57. □

Corollary 10.6.27. *Let X, Y be two schemes, $j : X \to Y$ be an immersion, U an open subset of Y such that $j(X)$ is closed in U, and $\mathscr{I}$ the quasi-coherent ideal of $\mathscr{O}_U$ defining the closed subscheme of Y associated with j. For j to be locally of finite presentation, it is necessary and sufficient that $\mathscr{I}$ is a $\mathscr{O}_U$-module of finite type.*

Proof. Since the question is local, we can assume that X and Y are affine. The assertion then reduces to Proposition 10.6.26. □

Proposition 10.6.28. *Let $\rho : A \to B$ be a ring homomorphism. For the corresponding morphism $f : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ to be locally of finite presentation, it is necessary and sufficient that B is an A-algebra of finite presentation.*

Proof. The condition is clearly sufficient, so we only need to prove the necessity. If f is locally of finite presentation, then it follows from Proposition 10.6.23 that B is an A-algebra of finite type, so there exists a surjective homomorphism $\rho : C' = A[T_1, \dots, T_n] \to B$ of A-algebras. It then follows from Proposition 10.6.24(v) that the closed immersion $j : \mathrm{Spec}(B) \to \mathrm{Spec}(B')$ is locally of finite presentation, so, if $\mathfrak{b}$ is the kernel of ρ, the $\widetilde{B'}$-module $\tilde{\mathfrak{b}}$ is of finite type, and $\mathfrak{b}$ is therefore finitely generated in B' by Corollary 10.1.24. □

Proposition 10.6.29. *Let $\rho : A \to B$ be a homomorphism of rings such that B is a finite A-algebra. For B to be an A-algebra of finite presentation, it is necessary and sufficient that B is an A-module of finite presentation.*

Proof. There exists a finite A-algebra B' of finite presentation such that B' is a free A-module, and a surjective A-homomorphism of A-algebras $u : B' \to B$ (**??**); we have a surjective homomorphism $v : B'' = A[T_1, \dots, T_m] \to B'$ of A-algebras whose kernel is finitely generated. If $w = v \circ u : B'' \to B$ and $\mathfrak{b}$ (resp. $\mathfrak{a}$) is the kernel of w (resp. u), we have $\mathfrak{a} = v(\mathfrak{b})$ since v is surjective. If B is an A-algebra of finite presentation, $\mathfrak{b}$ is a finitely generated ideal of B'' by Proposition 10.6.26, so $\mathfrak{a}$ is a finitely generated ideal in B', hence a finitely generated A-module since B' is a finite A-algebra. As B' is a free A-module, B is then an A-module of finite presentation. The converse is proved in **??**. □

Proposition 10.6.30. *Let $f : X \to Y$ be a local immersion of finite type at a point of $y \in Y$. The following conditions are equivalent:*

(i) *f is an open map at y.*

(ii) *There exists an open neighborhood U of y in Y such that $f|_U$ is a nilimmersion over the open subscheme U.*

(iii) *There exists an open neighborhood U of y in Y such that $f|_U$ is a nilpotent immersion over the open subscheme U.*

Proof. It is clear that (iii) implies (ii) and (ii) implies (i). To show that (i) implies (iii), we can, by restricting f, suppose that f is a closed immersion from an affine open U of Y to an affine open V of X. Moreover, by choosing an irreducible component containing $f(y)$, we can further assume that V is irreducible. As f is a homeomorphism from U to $f(U)$, the hypothesis of (i) then implies that $f(U) = V$ since V is connected. If $V = \mathrm{Spec}(A)$, $U = \mathrm{Spec}(B)$, we have $B = A/\mathfrak{a}$, where $\mathfrak{a}$ is a nilideal of A. On the other hand, in view of Lemma 10.6.19, we can, by replacing A with a fraction field A_s, suppose that B is an A-algebra of finite presentation. But then $\mathfrak{a}$ is a finitely generated ideal of A by Proposition 10.6.26, so it is nilpotent. □

Proposition 10.6.31. *Let $f : X \to Y$ be a morphism locally of finite type. Suppose that f admits a Y-section g, and for every $y \in Y$, the fiber $X_y = f^{-1}(y)$ is a $\kappa(y)$-scheme $\kappa(y)$-isomorphic to $\mathrm{Spec}(\kappa(y))$ (and necessarily has underlying space $\{g(y)\}$). Then f is an isomorphism.*

Proof. In fact, g is a nilimmersion of Y to X (Corollary 10.5.8), so the image $g(Y)$ has underlying space X and is defined by a nilideal $\mathscr{I}$ of $\mathscr{O}_X$. As $f \circ g = 1_Y$ and f is locally of finite type, g is locally of finite presentation by Proposition 10.6.24(v), so $\mathscr{I}$ is an ideal of finite type of $\mathscr{O}_X$ (Corollary 10.6.27). For any $y \in Y$, put $x = g(y)$, and consider the following exact sequence:

$$0 \longrightarrow \mathscr{I}_x \longrightarrow \mathscr{O}_{X,x} \underset{f_x^\#}{\overset{g_y^\#}{\rightleftarrows}} \mathscr{O}_{Y,y} \longrightarrow 0$$

The relation $f \circ g = 1_Y$ implies $f_x^\# \circ g_y^\# = 1$, so the above exact sequence splits. By tensoring with $\kappa(y) = \mathscr{O}_{Y,y}/\mathfrak{m}_y$, we then get an isomorphism $\mathscr{O}_{X,x}/\mathfrak{m}_y\mathscr{O}_{X,x} \cong \kappa(y) \oplus (\mathscr{I}_x/\mathfrak{m}_y\mathscr{I}_x)$. But the hypothesis on X_y implies that $\kappa(y)$-isomorphic to $\kappa(y)$, so we deduce that $\mathfrak{m}_y\mathscr{I}_x = \mathscr{I}_x$ and a fortiori $\mathfrak{m}_x\mathscr{I}_x = \mathscr{I}_x$. As $\mathscr{I}_x$ is a finitely generated ideal of $\mathscr{O}_{X,x}$, by Nakayama lemma we conclude $\mathscr{I}_x = 0$, so $\mathscr{I} = 0$ and f is an isomorphism. □

Corollary 10.6.32. *Let X, Y be two S-schemes and $f : X \to Y$ be an S-morphism. Suppose that X is locally of finite type over S. For each $s \in S$, let X_s, Y_s be the fiber of X and Y at the point s, and $f_s : X_s \to Y_s$ be the morphism induced by f under the base change* $\mathrm{Spec}(\kappa(s)) \to S$*. Then if for each $s \in S$, f_s is a monomorphism, f is a monomorphism.*

Proof. If f_s is a monomorphism, so is $f_y : X_y \to \mathrm{Spec}(\kappa(y))$. By hypothesis f is locally of finite type (Proposition 10.6.21(v)), so we can limit ourselves to the case $Y = S$. To see that f is a monomorphism, it suffices to prove that the first projection $p : X \times_S X \to X$ is an isomorphism (Proposition 10.5.5). Now the hypothesis on f_s implies that the projections $p_s : X_s \otimes_{\kappa(s)} X_s \to X_s$ are isomorphisms for all $s \in S$. Since p admits an S-section, namely the diagonal Δ_f, it follows from Proposition 10.6.31 that p is an isomorphism. □

We now come to the definition of *morphisms of finite type*, which can be seen as a global version of morphisms locally of finite type. Briefly speaking, the notion of finite type concerns the "global finiteness" of a morphism: we have the following definition and proposition.

Proposition 10.6.33. *Let $f : X \to Y$ be a morphism and (U_α) be a covering of Y by affine opens. The following conditions are equivalent:*

(i) *f is locally of finite type and quasi-compact.*

(ii) *For each α, $f^{-1}(U_\alpha)$ is a finite union of affine opens $V_{\alpha,i}$ such that the ring $\Gamma(V_{\alpha,i}, \mathscr{O}_X)$ is a $\Gamma(U_\alpha, \mathscr{O}_Y)$-algebra of finite type.*

(iii) *For any affine open U of Y, $f^{-1}(U)$ is a finite union of affine opens V_j such that $\Gamma(V_j, \mathscr{O}_X)$ is a $\Gamma(U, \mathscr{O}_X)$-algebra of finite type.*

We say the morphism f is ***of finite type*** *if it satisfies the above equivalent conditions. In this case, we say X is of finite type over Y.*

Proof. It is clear that (iii)⇒(ii)⇒(i). To prove that (i) implies (iii), we may assume that $Y = U$ is affine; then X is quasi-compact, hence is a finte union of affine opens V_j such that the restriction $V_j \to U$ of f is locally of finite type. By Proposition 10.6.23, we see $\Gamma(V_j, \mathscr{O}_X)$ is a $\Gamma(U, \mathscr{O}_X)$-algebra of finite type. □

Proposition 10.6.34. *Let $f : X \to Y$ be a morphism of finite type. If Y is Noetherian (resp. locally Noetherian), so is X.*

Proof. This follows from Proposition 10.6.20 and **??**. □

Proposition 10.6.35 (Properties of Morphisms of Finite Type).

(i) *Any quasi-compact immersion is of finite type.*

(ii) *The composition of two morphisms of finite type is of finite type.*

(iii) *If $f : X \to Y$ is an S-morphism of finite type, $f_{(S')} : X_{(S')} \to Y_{(S')}$ is of finite type for any base change $S' \to S$.*

(iv) *If $f : X \to X'$ and $g : Y \to Y'$ are two S-morphisms of finite type, $f \times_S g$ is of finite type.*

(v) *If the composition $g \circ f$ of two morphisms $f : X \to Y$, $g : Y \to Z$ is of finite type and if g is quasi-separated or X is Noetherian, then f is of finite type.*

(vi) *If a morphism f is of finite type, so is f_{red}.*

Proof. This follows directly from Proposition 10.6.21 and Proposition 10.6.3. □

Corollary 10.6.36. *Let $f : X \to Y$ be a morphism of finite type. For any morphism $Y' \to Y$ such that Y' is Noetherian, $X \times_Y Y'$ is Noetherian.*

Proof. This follows from Proposition 10.6.35(iii) and Proposition 10.6.34. □

Corollary 10.6.37. *Let X be a scheme of finite type over a locally Noetherian scheme S. Then any S-morphism $f : X \to Y$ is of finite type.*

Proof. The morphism f is locally of finite type by Proposition 10.6.21(v). To see it is quasi-compact, we can suppose that S is Noetherian. If $\varphi : X \to S$ and $\psi : Y \to S$ are the structural morphism, we have $\varphi = \psi \circ f$ and X is Noetherian by Proposition 10.6.34, so f is of finite type, by Proposition 10.6.35(v). □

Let X and Y be two schemes. We say a morphism $f : X \to Y$ is **of finite presentation** if it satisfies the following conditions:

(i) f is locally of presentation;

(ii) f is quasi-compact;

(iii) f is quasi-separated.

In this case, we say X is **of finite presentation over Y**, or is an **Y-scheme of finite presentation**. It is clear that condition (iii) is automatic if f is separated, or if X is locally Noetherian. If Y is locally Noetherian, then again, f is of finite type if and only if it is of finite presentation, and in this case X is also locally Noetherian.

Proposition 10.6.38 (Properties of Morphisms of Finite Presentation).

(i) *Any quasi-compact immersion that is locally of finite presentation (in particular any quasi-compact open immersion) is of finite presentation.*

(ii) *The composition of two morphisms of finite presentation is of finite presentation.*

(iii) *If $f : X \to Y$ be an S-morphism of finite presentation, $f_{(S')} : X_{(S')} \to Y_{(S')}$ is of finite presentation for any base change $S' \to S$.*

(iv) *If $f : X \to X'$ and $g : Y \to Y'$ are two S-morphisms of finite presentation, $f \times_S g$ is of finite presentation.*

(v) *If the composition $g \circ f$ of two morphisms $f : X \to Y$, $g : Y \to Z$ is of finite presentation and if g is quasi-separated and locally of finite type, then f is of finite presentation.*

Proof. This follows from Proposition 10.6.3, Proposition 10.6.7, and Proposition 10.6.24. □

It follows from Proposition 10.6.38(iii) that if f is a morphism of finite presentation and U is an open subset of Y, the restriction $f^{-1}(U) \to U$ of f is also of finite presentation. conversely, let (U_α) be a covering of Y by affine opens and suppose that the restriction $f^{-1}(U_\alpha) \to U_\alpha$ of f is a morphism of finite presentation. Then it follows that f is of finite presentation, since f is clearly of finite presentation and quasi-compact and it is quasi-separated by Proposition 10.6.9.

If X is a quasi-separated scheme, any morphism $f : X \to Y$ is quasi-separated by Corollary 10.6.8. Therefore, if f is quasi-compact and locally of finite presentation, it is of finite presentation.

Corollary 10.6.39. *Let $\rho : A \to B$ be a ring homomorphism. For the corresponding morphism f : $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ to be of finite type (resp. of finite presentation), it is necessary and sufficient that B is an A-algebra of finite type (resp. of finite presentation).*

Proof. Since any morphism of affine schemes is quasi-compact and separated, this follows from Proposition 10.6.23 and Proposition 10.6.28. □

Remark 10.6.40. In the definition of morphisms of finite presentation, the condition (iii) is not a concequence of the other two conditions. For example, let Y be a non-Noetherian affine scheme and let Y be a non-quasi-compact open subset of Y (an example for this is $Y = \mathrm{Spec}(k[x_1, x_2, \dots])$ and $U = Y - \{0\}$, cf. Example 10.6.2). Let X be the scheme obtained by glueing two schemes Y_1, Y_2 isomorphic to Y along the open sets U_1, U_2 corresponding to U, so that X is the union of two affine opens isomorphic to Y_1, Y_2, respectively, and $Y_1 \cap Y_2 = U$. Let $f : X \to Y$ be the morphism which coincides with the canonical isomorphism $Y_i \to Y$ on each Y_i. Then it is clearly locally of finite presentation, and is quasi-compact since the inverse image of a quasi-compact open of Y is the union of its images in Y_1 and Y_2; but as $Y_1 \cap Y_2 = U$ is not quasi-compact, it is not quasi-separated by Proposition 10.6.10 and Corollary 10.6.8(ii).

Proposition 10.6.41. *Let $f_i : X_i \to Y$ be a finite family of morphisms and $f : X \to Y$ be their coproduct, where $X = \coprod_i X_i$. Then for f to be of finite type (resp. finite presentation), it s necessary and sufficient that each f_i is.*

Proof. In view of Proposition 10.6.13, it suffices to note that the same assertion holds for morphisms of locally finite type and of finite presentation. □

10.6.3 Algebraic schemes

We say a K-scheme is **algebraic** (resp. **locally algebraic**) if it is of finite type over K (resp. locally of finite type over K). The field K is called the **base field** of X.

Proposition 10.6.42. *Let K be a field. A locally algebraic (resp. algebraic) K-scheme is locally Noetherian (Noetherian). Moreover X is a Jacobson scheme and a point $x \in X$ is closed if and only if $\kappa(x)$ is a finite extension of K.*

Proof. The first assertion is clear, and X is Jacobson by **??**. To characterize close points in X, we note that for a point $x \in X$ to be closed, it is necessary and sufficient that for an open covering (U_α) of X, s is closed in the U_α containing it. As there is a covering of X by affine opens U_α such that $\Gamma(U_\alpha, \mathscr{O}_X)$ is a K-algebra of finite type, we can then assume that $X = \mathrm{Spec}(A)$ where A is a K-algebra of finite type. The closed points of X are then maximal ideals of A; but then $A/\mathfrak{p}_x = \kappa(x)$ is a finite extension by Proposition 4.3.13. Conversely, if $\kappa(x)$ is a finite K-algebra, so is the ring $A/\mathfrak{p}_x \subseteq \kappa(x)$, and as an integral K-algebra is also a field (Proposition 4.1.64), we have $A/\mathfrak{p}_x = \kappa(x)$, so x is closed. □

Corollary 10.6.43. *Let K be an algebraically closed field and X be a locally algebraic K-scheme. Then the closed points of X are exactly the rational points of X over K, which are identified with the K-points of X with values in K.*

Proposition 10.6.44. *Let K be a field and X be a locally algebraic scheme over K. Then the following conditions are equivalent:*

(i) *X is Artinian.*

(ii) *The underlying space of X has only finitely many closed points.*

(iii) *The underlying space of X is finite.*

(iv) *X is isomorphic to $\mathrm{Spec}(A)$ where A is K-algebra of finite dimension.*

If X is algebraic over K, then these conditions are equivalent to the following:

(v) *The underlying space of X is discrete.*

(vi) *The points of X are all closed.*

Proof. We see (i) implies any other conditions, and (v) or (vi) implies (i) if X is Noetherian. Moreover, it is clear that (iv) implies (i), since a finite dimensional K-algebra is Artianian. In the condition of (ii), the set X_0 of closed points of X is then finite, closed and very dense in X, whence equal to X and X is therefore Artinian, since it is then Noetherian. □

Corollary 10.6.45. *Let K be a field, X be a locally algebraic K-scheme, and x be a point of X. The following conditions are equivalent:*

(i) *x is isolated in X;*

(ii) *x is closed in X and $\mathscr{O}_{X,x}$ is Artinian;*

(iii) *$\mathscr{O}_{X,x}$ is a finite dimensional K-algebra.*

Proof. If $\mathscr{O}_{X,x}$ is a finite dimensional K-algebra, so is the residue field $\kappa(x) = \mathscr{O}_{X,x}/\mathfrak{m}_x$, so (iii) implies (ii) in view of Proposition 10.6.42. The local ring $\mathscr{O}_{X,x}$ is Artinian signifies that x is a maximal point of X, since a Noetherian local ring is Artinian if and only if it has a unique primei deal. If x is moreover closed, the set $\{x\}$ is closed and stable under generalization, hence open ([**?**] new, 0_{I}, 2.1.5), and this proves x is isolated in X. Finally, if x is isolated in X, there exists an affine open neighborhood U of x such that $U = \{x\}$ and $\Gamma(U, \mathscr{O}_X)$ is a finite type K-algebra. But then $\Gamma(U, \mathscr{O}_X) = \mathscr{O}_{X,x}$, so (i) implies (iii). □

If the conditions in Proposition 10.6.44, we say X is a finite scheme over K, or a finite K-scheme. For such a scheme, we denote by $\dim_K(X)$ the dimension of the ring $\Gamma(X, \mathscr{O}_X)$ over K. If X and Y are two finite schemes over K, we have

$$\dim_K(X \amalg Y) = \dim_K(X) + \dim_K(Y), \quad \dim_K(X \times_K Y) = \dim_K(X)\dim_K(Y).$$

Corollary 10.6.46. *Let X be a finite scheme over a field K. For any extension K' of K, $X \otimes_K K'$ is a finite scheme over K', with $\dim_{K'}(X') = \dim_K(X)$.*

Proof. In fact, if $X = \mathrm{Spec}(A)$, we have $[A \otimes_K K' : K'] = [A : K]$, whence the claim. □

Corollary 10.6.47. *Let X be a finite scheme over a field K. We put*

$$n = \sum_{x \in X} [\kappa(x) : K]_s$$

Then, for any algebraically closed extension Ω of K, the underlying space of $X \otimes_K \Omega$ has exactly n points, which are identified with the Ω-valued points of X.

Proof. By Proposition 10.2.33, we can assume that $A = \Gamma(X, \mathscr{O}_X)$ is local; let $\mathfrak{m}$ be the maximal ideal of A, $L = A/\mathfrak{m}$ the residue field, which is a finite algebraic extension of K by Proposition 10.6.44. The Ω-points of X correspond bijectively to Ω-sections of $X \otimes_K \Omega$, and to the closed points of $X \otimes_K \Omega$ by Corollary 10.6.43, and finally to the points of this Artinian scheme (Proposition 10.2.33). They also correspond to K-homomorphisms of L into Ω, and the assertion then follows from the definition of separable degree. □

The number n defined in Corollary 10.6.47 is called the **separable rank** of A (or X) over K, or the **geometric number of points** of X. This is also the number of elements in $X(\Omega)_K$. It follows from this definition that for any extension K' of K, $X \otimes_K K'$ has the same geometric number of points as X. If we denote this number by $n(X)$, it is clear that, if X and Y are two finite schemes over K, we have

$$n(X \amalg Y) = n(X) + n(Y), \quad n(X \times_K Y) = n(X)n(Y).$$

Proposition 10.6.48. *Let $f : X \to Y$ be a morphism locally of finite type (resp. of finite type). Then, for any $y \in Y$, the fiber $X_y = f^{-1}(y)$ is a locally algebraic (resp. algebraic) scheme over $\kappa(y)$, and for each $x \in X_y$, $\kappa(x)$ is a extension of $\kappa(y)$ of finite type.*

Proof. As $X_y = X \otimes_Y \kappa(y)$, this follows from Proposition 10.6.21(iii) and Proposition 10.6.35(iii). □

Proposition 10.6.49. *Let $f : X \to Y$, $g : Y' \to Y$ be two morphisms, and put $X' = X \times_Y Y'$, $f' = f_{(Y')} : X' \to Y'$. Let $y' \in Y'$ and $y = g(y')$. If the fiber $X_y = f^{-1}(y)$ is a finite scheme over $\kappa(y)$, then the fiber $X'_{y'} = f'^{-1}(y')$ is a finite scheme over $\kappa(y')$, and we have*

$$\dim_{\kappa(y')}(X'_{y'}) = \dim_{\kappa(y)}(X_y), \quad n(X'_{y'}) = n(X_y).$$

Proof. This follows from the observation $X'_{y'} = X_y \otimes_{\kappa(y)} \kappa(y')$. □

Proposition 10.6.48 shows that the morphisms of finite type (resp. locally of finite type) correspond intuitively to "algebraic families of algebraic varieties (resp. locally algebraic)", where Y plays the role of "parameters." Because of this, these morphisms are of significant geometric interests. The morphisms which are not locally of finite type will intervene them by the process of "base change", for example by localization and completion.

10.6.4 Local determination of morphisms

Proposition 10.6.50. *Let X and Y be S-schemes, $x \in X$, $y \in Y$ be points lying over the same point $s \in S$.*

(a) *Suppose that Y is locally of finite type over S at the point y. Then if two S-morphisms f, g from X to Y are such that $f(x) = g(x) = y$ and the $\mathscr{O}_{S,s}$-homomorphisms $f_x^\#$ and $g_x^\#$ from $\mathscr{O}_{Y,y}$ to $\mathscr{O}_{X,x}$ coincide, then f and g coincide in an open neighborhood of x.*

(b) *Suppose that Y is locally of finite presentation over S at the point y. Then, for any $\mathscr{O}_{X,x}$-homomorphism $\varphi : \mathscr{O}_{Y,y} \to \mathscr{O}_{X,x}$ there exists an open neighborhood U of x in X and an S-morphism f from U to Y such that $f(x) = y$ and $f_x^\# = \varphi$.*

Proof. We first consider case (a). The question is local over S, X and Y, so we can suppose that S, X, Y are affine with rings A, B, C, respectively. The morphisms f and g then correspond to A-homomorphisms ρ, σ from C to B such that $\rho^{-1}(\mathfrak{p}_x) = \sigma^{-1}(\mathfrak{p}_x) = \mathfrak{p}_y$, and the homomorphisms ρ_x and σ_x from C_y to B_x, deduced by ρ and σ, coincide. We can morphism suppose that C is an A-algebra of finite type. Let $(c_i)_{1 \le i \le n}$ be generators of the A-algebra C, and put $b_i = \rho(c_i)$, $b'_i = \sigma(c_i)$. By hypothesis, we have $b_i/1 = b'_i/1$ in the ring B_x. This means there exist elements

$s_i \in B - \mathfrak{p}_x$ such that $s_i(b_i - b_i') = 0$ for each i, and we can evidently choose one $s \in B - \mathfrak{p}_x$ for all i. We then conclude that $b_i/1 = b_i'/1$ for each i in the ring B_s; if $i_s : B \to B_s$ is the canonical homomorphism, we then have $i_s \circ \rho = i_s \circ \sigma$, so the restriction of f and g on $D(s)$ are identical.

We now come to case (b). Again we can suppose that S, X, Y are affine with rings A, B, C. Put $\mathfrak{p} = \mathfrak{p}_x$, $\mathfrak{q} = \mathfrak{p}_y$, and let $\varphi : C_\mathfrak{q} \to B_\mathfrak{p}$ be an A-homomorphism. We then get an A-homomorphism $\rho : C \to C_\mathfrak{q} \xrightarrow{\varphi} B_\mathfrak{p}$. Since we can consider $B_\mathfrak{p}$ as an inductive limit of the filtered system of A-algebras B_s, where s runs through elements of $B - \mathfrak{p}$, and C is by hypothesis an A-algebra of finite presentation, we deduce from **??** that there exists $s \notin \mathfrak{p}$ and an A-homomorphism $\sigma : C \to B_s$ whose canonical image is ρ, that is, the following diagram is commutative

$$\begin{tikzcd} C \arrow[r, "\sigma"] \arrow[d] & B \arrow[d] \\ C_\mathfrak{q} \arrow[r, "\varphi"] & B_\mathfrak{p} \end{tikzcd} \tag{10.6.1}$$

It then suffices to take $U = D(s)$ and let f be the morphism induced by σ. ☐

Corollary 10.6.51. *Under the hypotheses of Proposition 10.6.50(ii), if moreover X is locally of finite type over S at the point of x, we can choose f to be of finite type.*

Proof. To see this, we can assume that S, X, Y are affine, so that the structural morphisms $X \to S$ and $Y \to S$ are respectively of finite type and of finite presentation; then the results follows from Lemma 10.6.19 and Proposition 10.6.21(iv). ☐

Corollary 10.6.52. *Retain the hypotheses of Proposition 10.6.50(ii) and suppose that Y is integral and $\varphi : \mathscr{O}_{Y,y} \to \mathscr{O}_{X,x}$ is injective. Then we can suppose that U is affine and f factors into*

$$U \xrightarrow{g} V \longrightarrow Y$$

where V is an affine open containing y and $g : U \to V$ is a morphism corresponding to a injective homomorphism $\Gamma(V, \mathscr{O}_Y) \to \Gamma(U, \mathscr{O}_X)$.

Proof. In fact, with the notations of Proposition 10.6.50(ii), C is integral and the canonical homomorphism $C \to C_g$ is then injective; the result then follows from the diagram (10.6.1), since σ is injective. ☐

Proposition 10.6.53. *Let $f : X \to Y$ be a morphism, x be a point of X and $y = f(x)$.*

(a) *Suppose that f is locally of finite type at the point x. For f to be a local immersion at the point x, it is necessary and sufficient that $f_x^\# : \mathscr{O}_{Y,y} \to \mathscr{O}_{X,x}$ is surjective.*

(b) *Suppose that f is locally of finite presentation at the point x. For f to be a local isomorphism at the point x, it is necessary and sufficient that $f_x^\# : \mathscr{O}_{Y,y} \to \mathscr{O}_{X,x}$ is an isomorphism.*

Proof. We only need to prove the sufficiency, and we first consider case (b). Then X is locally of finite presentation over y at the point x, so by Proposition 10.6.50(i) and (ii), there exists an open neighborhood V of y and a morphism $g : V \to X$ such that $g \circ f$ (resp. $f \circ g$) is defined and coincide with the identity on an open neighborhood W of x (resp. an open neighborhood T of y). Put $T' = T \cap g^{-1}(w)$ and $W' = f^{-1}(T')$, we then verify that $g(T') \subseteq W'$, $f(W') \subseteq T'$ and $(g \circ f)|_{W'} = 1_{W'}$, whence f is a local isomorphism.

For (a), we can assume that X and Y are affine, with ring A and B. Then f corresponds to a homomorphism $\varphi : B \to A$ of finite type; we have $\varphi^{-1}(\mathfrak{p}_x) = \mathfrak{p}_y$, and the homomorphism

$\varphi_x : B_y \to A_x$ induced by φ is surjective. Let $(t_i)_{1\leq i\leq n}$ be a system of generators of the B-algebra A. The hypothesis on φ_x then implies $t_i/1 = \varphi(b_i)/\varphi(c)$ in the ring A_x, where $b_i \in B$ and $c \in B - \mathfrak{p}_y$, so we can find $a \in A - \mathfrak{p}_x$ such that

$$a(t_i\varphi(c) - \varphi(b_i)) = 0.$$

If we put $g = a\varphi(c)$, then $t_i/1 = a\varphi(b_i)/g$ in the ring A_g. Now there exists by hypothesis a polynomial $Q(X_1,\dots,X_n)$ with coefficients in $\varphi(B)$ such that $a = Q(t_1,\dots,t_n)$; write $Q(X_1/T,\dots,X_n/T) = P(X_1,\dots,X_n,T)/T^m$, where P is a polynomial of degree m. In the ring A_g, we then have

$$\begin{aligned} a/1 &= Q(t_1/1,\dots,t_n/1) = Q(a\varphi(b_1)/g,\dots,a\varphi(b_n)/g) \\ &= a^m P(\varphi(b_1),\dots,\varphi(b_n),\varphi(c))/g^m = a^m\varphi(d)/g^m \end{aligned}$$

where $d \in B$. Since $g/1 = (a/1)(\varphi(c)/1)$ is invertible in A_g by definition, so is $a/1$ and $\varphi(c)/1$, and we can then write $a/1 = (\varphi(d)/1)(\varphi(c)/1)^{-m}$. We conclude that $\varphi(d)/1$ is also invertible in A_g. Put $h = cd$, as $\varphi(h)/1$ is invertible in A_g, the composed homomorphism $B \to A \to A_g$ factors into

$$\begin{tikzcd} B \arrow[r, "\varphi"] \arrow[dr] & A \arrow[r] & A_g \\ & B_h \arrow[ur, "\gamma"'] & \end{tikzcd}$$

We claim that γ is surjective. For this, it suffices to verify that the image of B_h in A_g contains $t_i/1$ and $1/g$. Now we have

$$1/g = (\varphi(c)/1)^{m-1}(\varphi(d)/1)^{-1} = \gamma(c^m/h)$$

and $a/1 = \gamma(d^{m+1}/h^m)$, so $(a\varphi(b_i))/1 = \gamma(b_i d^{m+1}/h^m)$, and as $t_i/1 = (a\varphi(b_i)/1)(g/1)^{-1}$, we conclude our assertion. The choice of h implies $f(D(g)) \subseteq D(h)$, and the restriction of f to $D(g)$ is induced by γ. Since γ is surjective, this restriction is a closed immersion from $D(g)$ to $D(h)$, so f is a local immersion at x. □

Corollary 10.6.54. *With the notations of Proposition 10.6.53, suppose that f is a local immersion at the point x and is locally of finite presentation at x. For f to be open at x, it is necessary and sufficient that the kernel of $f_x^\#$ is nilpotent.*

Proof. In view of Proposition 10.6.30, it suffices to prove the sufficiency of the condition. We can suppose that $Y = \mathrm{Spec}(A)$ and $X = \mathrm{Spec}(A/\mathfrak{n})$, where $\mathfrak{n}$ is a finitely generated ideal of A (Corollary 10.6.27), and by hypothesis $\mathfrak{n}_x$ is nilpotent. If $(s_i)_{1\leq i\leq n}$ is a system of generators of $\mathfrak{n}$, we then have $s_i^m/1 = 0$ in A_x for an integer m and all i. Then there exists $t \in A - \mathfrak{p}_x$ such that $ts_i^m = 0$ for all i, so $(s_i/1)^m = 0$ in the ring A_t. This shows $\mathfrak{n}_t$ is nilpotent, whence the conclusion. □

10.6.5 Direct image of quasi-coherent sheaves

Proposition 10.6.55. *Let X, Y be two schemes and $f : X \to Y$ be a quasi-compact and quasi-separated morphism. Then for any quasi-coherent $\mathscr{O}_X$-module, $f_*(\mathscr{F})$ is quasi-coherent.*

Proof. Since the question is local over Y, we can assume that Y is affine. If f is quasi-compact, X is then a union of finitely many open affines X_i, and in view of Proposition 10.6.7(ii), X is a quasi-separated scheme, hence the intersections $X_i \cap X_j$ are qausi-compact (Proposition 10.6.10).

We first assume that each intersection $X_i \cap X_j$ is affine. Put $\mathscr{F}_i = \mathscr{F}|_{X_i}$, $\mathscr{F}_{ij} = \mathscr{F}|_{X_i \cap X_j}$ and let $\mathscr{F}_i'$ and $\mathscr{F}_{ij}'$ be the inverse image of $\mathscr{F}_i$ and $\mathscr{F}_j$ under the restriction of f to X_i and to $X_{i \cap X_j}$. We see that $\mathscr{F}_i'$ and $\mathscr{F}_{ij}'$ are quasi-coherent (**??**). We define a homomorphism

$$u : \bigoplus_i \mathscr{F}_i' \to \bigoplus_{i,j} \mathscr{F}_{ij}'$$

such that $f_*(\mathscr{F})$ is the kernel of u, and this then implies $f_*(\mathscr{F})$ is quasi-coherent by Corollary 10.1.6. For this, it suffices to define u as a homomorphism of presheaves, so for each open subset $W \subseteq Y$, we need a homomorphism

$$u_W : \bigoplus_i \Gamma(f^{-1}(W) \cap X_i, \mathscr{F}) \to \bigoplus_{ij} \Gamma(f^{-1}(W) \cap X_i \cap X_j, \mathscr{F})$$

so as to satisfy the compatibility for the restrictions to a smaller open subset. If for any section s_i of $\mathscr{F}$ over $f^{-1}(W) \cap X_i$, we denote by s_{ij} its restriction to $f^{-1}(W) \cap X_i \cap X_j$, we set

$$u_W((s_i)) = (s_{ij} - s_{ji})$$

and the compatibility is evident. To identity the kernel $\mathscr{R}$ of u, we define a homomorphism $v : f_*(\mathscr{F}) \to \mathscr{R}$ which sends a section s of $\mathscr{F}$ over $f^{-1}(W)$ to the family (s_i), where s_i is the restriction of s to $f^{-1}(W) \cap X_i$. By the sheaf axioms of $\mathscr{F}$, it is clear that v is bijective, which proves the assertion in this case.

In the general case, the same reasoning can be applied if we can show that each $\mathscr{F}_{ij}'$ is quasi-coherent. But by hypotheses, $X_i \cap X_j$ is a union of finitely many affine opens X_{ijk}, and since each X_{ijk} are affine open subschemes of the affine scheme X_i, their intersections are again affine (affine schemes are separated), so we can apply the previous arguments to conclude that $\mathscr{F}_{ij}'$ is quasi-coherent, and the proof is then complete. □

Remark 10.6.56. We should note that even if X and Y are Noetherian schemes and $f : X \to Y$ is a morphism of finite type, the direct image $f_*(\mathscr{F})$ of a coherent $\mathscr{O}_X$-module $\mathscr{F}$ is in general not coherent. For example, let Y be the spectrum of a field K, $X = \operatorname{Spec}(K[T])$, and choose $\mathscr{F} = \mathscr{O}_X$.

Proposition 10.6.57. *Let $f : X \to Y$ be a quasi-compact and quasi-separated morphism. Let $(\mathscr{F}_\lambda)$ be a inductive system of quasi-coherent $\mathscr{O}_X$-modules and $\mathscr{F} = \varinjlim \mathscr{F}_\lambda$ be the inductive limit. Then $\varinjlim f_*(\mathscr{F}_\lambda) \cong f_*(\mathscr{F})$.*

Proof. For each affine open subset W of Y and any λ, we have a canonical homomorphism

$$u_{W,\lambda} : (f_*(\mathscr{F}_\lambda))|_W \to f_*(\mathscr{F})|_W$$

whence a canonical homomorphism

$$u_W : (\varinjlim f_*(\mathscr{F}_\lambda))|_W \to f_*(\mathscr{F})|_W$$

and this hommorphism is compatible with restrictions. Since f is quasi-compact and quasi-separated, by Proposition 10.6.55 $\varinjlim f_*(\mathscr{f}_\lambda)$ and $f_*(\mathscr{F})$ are quasi-coherent. Moreover, the homomorphism u_W corresponds by taking global section over W to the canonical homomorphism

$$\varphi_W : \Gamma(W, \varinjlim f_*(\mathscr{F}_\lambda)) \to \Gamma(f^{-1}(W), \mathscr{F}).$$

Since f is quasi-compact and quasi-separated, by (Stack Project. Lemma 6.29.1) we have

$$\Gamma(W, \varinjlim f_*(\mathscr{F}_\lambda)) = \varinjlim \Gamma(W, f_*(\mathscr{F}_\lambda)) = \varinjlim \Gamma(f^{-1}(W), \mathscr{F}_\lambda) = \Gamma(f^{-1}(W), \mathscr{F})$$

and φ_W is therefore the identity homomorphism. By Corollary 10.1.22, it then follows that u_W is an isomorphism for each W, and the assertion then follows. □

10.6.6 Extension of quasi-coherent sheaves

Let X be a topological space and $\mathscr{F}$ be a sheaf of sets (resp. of groups, of rings) over X. Let U be an open subset of X with $j : U \to X$ the canonical injection, and let $\mathscr{G}$ be a subsheaf of $\mathscr{F}|_U = j^{-1}(\mathscr{F})$. As the functor j_* is left exact, $j_*(\mathscr{G})$ is then a subsheaf of $j_*(j^{-1}(\mathscr{F}))$. Let $\rho_{\mathscr{F}} : \mathscr{F} \to j_*(j^{-1}(\mathscr{F}))$ be the canonical homomorphism associated with $\mathscr{F}$ and consider the subsheaf $\bar{\mathscr{G}} = \rho_{\mathscr{F}}^{-1}(j_*(\mathscr{G}))$ of $\mathscr{F}$. It follows immediately from definition that, for any open subset V of X, $\Gamma(V, \bar{\mathscr{G}})$ is formed by sections $s \in \Gamma(V, \mathscr{F})$ whose restriction on $V \cap U$ is a section of $\mathscr{G}$ over $V \cap U$. In particular, we have $\bar{\mathscr{G}}|_U = j^{-1}(\bar{\mathscr{G}}) = \mathscr{G}$, and $\bar{\mathscr{G}}$ is the largest subsheaf of $\mathscr{F}$ inducing $\mathscr{G}$ on U. We say the subsheaf $\bar{\mathscr{G}}$ is the **canonical extension** of the subsheaf $\mathscr{G}$ of $\mathscr{F}|_U$ to a subsheaf of $\mathscr{F}$.

Proposition 10.6.58. *Let X be a scheme and U be an open subset of X such that the canonical injection $j : U \to X$ is quasi-compact (in other words, U is retrocompact in X).*

(a) *For any quasi-coherent $(\mathscr{O}_X|_U)$-module $\mathscr{G}$, $j_*(\mathscr{G})$ is a quasi-coherent $\mathscr{O}_X$-module and we have $j_*(\mathscr{G})|_U = j^*(j_*(\mathscr{G})) = \mathscr{G}$.*

(b) *For any quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$ and any quasi-coherent sub-$(\mathscr{O}_X|_U)$-module $\mathscr{G}$ of $\mathscr{F}|_U$, the canonical extension $\bar{\mathscr{G}}$ of $\mathscr{G}$ is a quasi-coherent sub-$\mathscr{O}_X$-module of $\mathscr{F}$.*

Proof. Assertion (a) is a special case of Proposition 10.6.55 since j is quasi-separated by Proposition 10.6.7(i), and the relation $j_*(\mathscr{G})|_U = j^*(j_*(\mathscr{G})) = \mathscr{G}$ can be checked directly. By the same reasoning, $j_*(j^*(\mathscr{F}))$ is quasi-coherent, and as $\bar{G}$ is the inverse image of $j_*(\mathscr{G})$ under the homomorphism $\rho_{\mathscr{F}} : \mathscr{F} \to j_*(j^*(\mathscr{F}))$, assertion (b) follows from Corollary 10.2.23. □

Corollary 10.6.59. *Let X be a scheme and U be a quasi-compact open subset of X such that the canonical injection $j : U \to X$ is quasi-compact. Suppose that any quasi-coherent $\mathscr{O}_X$-module is the inductive limit of its quasi-coherent sub-$\mathscr{O}_X$-modules of finite type (this is true if X is an affine scheme). Let $\mathscr{F}$ be a quasi-coherent $\mathscr{O}_X$-module and $\mathscr{G}$ be a quasi-coherent sub-$(\mathscr{O}_X|_U)$-module of finite type of $\mathscr{F}|_U$. Then there exists a quasi-coherent sub-$\mathscr{O}_X$-module $\mathscr{G}'$ of finite type such that $\mathscr{G}'|_U = \mathscr{G}$.*

Proof. We have $\mathscr{G} = \bar{\mathscr{G}}|_U$, and $\bar{\mathscr{G}}$ is quasi-coherent by Proposition 10.6.58, hence is the inductive limit of its quasi-coherent sub-$\mathscr{O}_X$-modules $\mathscr{H}_\lambda$ of finite type. Then $\mathscr{G}$ is the inductive limit of the $\mathscr{H}_\lambda|_U$, hence equals to one of $\mathscr{H}_\lambda|_U$ since they are of finite type (**??**). □

Remark 10.6.60. Suppose that for any affine open $U \subseteq X$ the injection $U \to X$ is quasi-compact. Then if the conclusion of Corollary 10.6.59 holds for any affine open U and any quasi-coherent sub-$(\mathscr{O}_X|_U)$-module of finite type $\mathscr{G}$ of $\mathscr{F}|_U$, it follows that $\mathscr{F}$ is the inductive limit of its quasi-coherent sub-$\mathscr{O}_X$-modules of finite type. In fact, for any affine open $U \subseteq X$, we have $\mathscr{F}|_U = \widetilde{M}$, where M is a $\Gamma(U, \mathscr{O}_X)$-module, and as the latter is the inductive limit of its finitely generated sub-modules, $\mathscr{F}|_U$ is the inductive limit of its quasi-coherent sub-$(\mathscr{O}_X|_U)$-modules of finite type. Now, by hypotheses, such a submodule is induced over U by a quasi-coherent sub-$\mathscr{O}_X$-module of finite type $\mathscr{G}_{\lambda,U}$ of $\mathscr{F}$. The finite sums of $\mathscr{G}_{\lambda,U}$ are then quasi-coherent of finite type, since the question is local and we can assume that X is affine, where the conclusion is trivial. It then follows that $\mathscr{F}$ is the inductive limit of these finite sums, whence our assertion.

Corollary 10.6.61. *Under the hypotheses of Corollary 10.6.59, if $\mathscr{G}$ is a quasi-coherent $(\mathscr{O}_X|_U)$-module of finite type, there exists a quasi-coherent $\mathscr{O}_X$-module $\mathscr{G}'$ such that $\mathscr{G}'|_U = \mathscr{G}$.*

Proof. As $\mathscr{F} = j_*(\mathscr{G})$ is quasi-coherent (Proposition 10.6.58) and $\mathscr{F}|_U = \mathscr{G}$, it suffices to apply Corollary 10.6.59 to $\mathscr{F}$. □

Lemma 10.6.62. *Let X be a scheme, $(V_\lambda)_{\lambda \in L}$ be a covering of X by affine opens where L is well-ordered, and U be an open subset of X. For each $\lambda \in L$, let $W_\lambda = \bigcup_{\mu < \lambda} V_\mu$. Suppose that*

(i) *for any $\lambda \in L$, $V_\lambda \cap W_\lambda$ is quasi-compact;*

(ii) *the canonical injection $j : U \to X$ is quasi-compact.*

Then for any quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$ and any quasi-coherent sub-$(\mathscr{O}_X|_U)$-module of finite type $\mathscr{G}$ of $\mathscr{F}|_U$, there exists a quasi-coherent sub-$\mathscr{O}_X$-module $\mathscr{G}'$ of $\mathscr{F}$ such that $\mathscr{G}'|_U = \mathscr{G}$.

Theorem 10.6.63. *Let X be a scheme and U be an open subset of X. Suppose that one of the following conditions is satisfied:*

(a) *X is locally Noetherian;*

(b) *X is quasi-compact and quasi-separated and U is quasi-compact.*

Then for any quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$ and any quasi-coherent sub-$(\mathscr{O}_X|_U)$-module of finite type $\mathscr{G}$ of $\mathscr{F}|_U$, there exists a quasi-coherent sub-$\mathscr{O}_X$-module $\mathscr{G}'$ of $\mathscr{F}$ such that $\mathscr{G}'|_U = \mathscr{G}$.

Corollary 10.6.64. *With the conditions of Theorem 10.6.63, for any quasi-coherent $(\mathscr{O}_X|_U)$-module $\mathscr{G}$ of finite type, there exists a quasi-coherent $\mathscr{O}_X$-module $\mathscr{G}'$ of finite type such that $\mathscr{G}'|_U = \mathscr{G}$.*

Corollary 10.6.65. *Let X be a locally Noetherian scheme or a quasi-compact and quasi-separated scheme. Then any quasi-coherent $\mathscr{O}_X$-module is the inductive limit of its quasi-coherent sub-$\mathscr{O}_X$-modules of finite type.*

Proof. This follows from Theorem 10.6.63 and Remark 10.6.60. □

Corollary 10.6.66. *Under the hypotheses of Corollary 10.6.65, if a quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$ is such that any quasi-coherent sub-$\mathscr{O}_X$-module of finite type of $\mathscr{F}$ is generated by its global sections, then $\mathscr{F}$ is generated by its global sections.*

Proof. Let U be an affine neighborhood of a point $x \in X$, and let s be a section of $\mathscr{F}$ over U. The sub-$\mathscr{O}_X$-module $\mathscr{G}$ of $\mathscr{F}|_U$ generated by s is quasi-coherent and of finite type, hence there exists a quasi-coherent sub-$\mathscr{O}_X$-module of finite type $\mathscr{G}'$ of $\mathscr{F}$ such that $\mathscr{G}'|_U = \mathscr{G}$ (Theorem 10.6.63). By hypotheses, there is then a finite number of sections t_i of $\mathscr{G}'$ over X and sections a_i of $\mathscr{O}_X$ over a neighborhood $V \subseteq U$ of x such that $s|_V = \sum_i a_i \cdot (t_i|_V)$, which proves the corollary. □

10.6.7 Scheme-theoretic image

Let $f : X \to Y$ be a morphism of schemes. If there exists a smallest closed subscheme Y' of Y such that the canonical injection $j : Y' \to Y$ dominates f (or equivalently, the inverse image $f^{-1}(Y')$ is equal to X), we then say that Y' is the **scheme-theoretic image** of X under f, or the **scheme-theoretic image of f**. If X is a subscheme of Y, the scheme-theoretic image of the canonical injection $j : X \to Y$ is called the **scheme-theoretic closure** of X.

Proposition 10.6.67 (Transitivity). *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms. Suppose that Y' is the scheme-theoretic image of X under f, and if g' is the restriction of g to Y', the scheme-theoretic image of Z' under g' exists. Then the scheme-theoretic image of X under $g \circ f$ is equal to Z'.*

Proof. To say that a closed subscheme Z_1 of Z is such that $(g \circ f)^{-1}(Z_1) = X$ signifies that $f^{-1}(g^{-1}(Z_1)) = X$, or that f is dominated by the canonical injection $g^{-1}(Z_1) \to Y$. Now, in view of the existence of the scheme-theoretic image Y', for any closed subscheme Z_1 of Z having this property, $g^{-1}(Z_1)$ dominates Y', which, if $j : Y' \to Y$ is the canonical injection, amounts to saying that $j^{-1}(g^{-1}(Z_1)) = g'^{-1}(Z_1) = Y'$. We then conclude that Z' is the smallest closed subscheme Z_1 having this property, whence our claim. □

Corollary 10.6.68. *Let $f : X \to Y$ be an S-morphism such that Y is the scheme-theoretic image of Y under f. Let Z be a separated S-scheme; if two S-morphisms $g_1, g_2 : Y \to Z$ are such that $g_1 \circ f = g_2 \circ f$, then $g_1 = g_2$.*

Proof. Let $h = (g_1, g_2)_S : Y \to Z \times_S Z$. As the diagonal $T = \Delta_Z(Z)$ is a closed subscheme of $Z \times_S Z$, $Y' = h^{-1}(T)$ is a closed subscheme of Y. Put $u = g_1 \circ f = g_2 \circ f$; we then have $h \circ f = (u, u)_S = \Delta_Z \circ u$. As $\Delta_Z^{-1}(T) = Z$, we have $(h \circ f)^{-1}(T) = u^{-1}(Z) = X$, so $f^{-1}(Y') = X$. We then conclude that the canonical injection $Y' \to Y$ dominates f, so $Y' = Y$ by hypothesis. Then by Proposition 10.4.16, h factors into $\Delta_Z \circ v$ where v is a morphism $Y \to Z$, which implies $g_1 = g_2 = v$. □

Let $f : X \to Y$ be a morphism and suppose that the scheme-theoretic image Y' of f exists. Then Y' is defined by a quasi-coherent ideal $\mathscr{J}'$ of $\mathscr{O}_Y$, and by definition, $\mathscr{J}'$ is the largest quasi-coherent ideal such that the homomorphism $f^\# : \mathscr{O}_Y \to f_*(\mathscr{O}_X)$ factors into $\mathscr{O}_Y \to \mathscr{O}_Y/\mathscr{J}' \to f_*(\mathscr{O}_X)$. This implies that $\mathscr{J}' \subseteq \ker f^\# = \mathscr{J}$; we therefore obtain a case where Y' exists, the one where $\mathscr{J}$ is quasi-coherent, and where $\mathscr{J}' = \mathscr{J}$.

Proposition 10.6.69. *Let $f : X \to Y$ be a morphism. Then the scheme-theoretic image of X under f exists if one of the following conditions is satisfied:*

(a) *$f_*(\mathscr{O}_X)$ is quasi-coherent (which is the case if f is quasi-compact and quasi-separated).*

(b) *X is reduced.*

In this case, the underlying space of Y' is equal to $\overline{f(X)}$, and if f factors into

$$X \xrightarrow{f'} Y' \xrightarrow{Y} Y$$

where j is the canonical injection, f' is scheme-theoretic dominant. Moreover, if X is reduced (resp. integral), so is Y'.

Proof. The case (a) is immediate by our previous argument; moreover, as $\mathscr{O}_Y/\mathscr{J} \to f_*(\mathscr{O}_X)$ is then injective, this shows that f' is scheme-theoretic dominant. We still need to verify that the closed subscheme of Y defined by $\mathscr{J} = \ker f^\#$ has underlying space $\overline{f(X)}$. Since the support of $f_*(\mathscr{O}_X)$ is contained in $\overline{f(X)}$, we have $\mathscr{J}_y = \mathscr{O}_y$ for $y \notin \overline{f(X)}$, so the support pf $\mathscr{O}_Y/\mathscr{J}$ is contained in $\overline{f(X)}$. Moreover, this support is closed and contains $f(X)$: if $y \in f(X)$, the identity element of the ring $(f_*(\mathscr{O}_X))_y$ is nonzero, being the germ at y of the section $1 \in \Gamma(X, \mathscr{O}_X) = \Gamma(Y, f_*(\mathscr{O}_X))$; as this is the image of the identity element under $f^\#$, it is not contained in $\mathscr{J}_y$, so $\mathscr{O}_y/\mathscr{J}_y \neq 0$; this proves our first claim. The case (b) follows from Proposition 10.4.48, because there is a smallest closed subscheme Z with underlying space $\overline{f(X)}$ such that $f(X) \subseteq Z$. □

Proposition 10.6.70. *Suppose the notations of Proposition 10.6.69 is satisfied, and let Y' be the scheme-theoretic image of X under f. For any open subset V of Y, let $f_V : f^{-1}(V) \to V$ be the restriction of f. Then the scheme-theoretic image of $f^{-1}(V)$ under f_V exists and is equal to the oepn subscheme $V \cap Y'$ of Y'.*

Proof. Put $X' = f^{-1}(V)$; as the direct image of $\mathscr{O}_{X'}$ is the restriction of $f_*(\mathscr{O}_X)$ to V, it is clear that the kernel of the homomorphism $\mathscr{O}_V \to (f_V)_*(\mathscr{O}_{X'})$ is the restriction of $\mathscr{J}$ to V, whence the assertion. □

Proposition 10.6.71. *Let Y be a subscheme of a scheme X, such that the canonical injection $j : Y \to X$ is quasi-compact. Then the scheme-theoretic closure of Y exists and has $\bar{Y}$ as underlying space.*

Proof. It suffices to apply Proposition 10.6.69 to the injection j, which is separated (Proposition 10.5.26) and quasi-compact by hypothesis. □

With these notations, let $\bar{Y}$ be the scheme-theoretic closure of Y in X. If the injection $\bar{Y} \to X$ is quasi-compact, and if $\mathscr{I}$ is the quasi-coherent ideal of $\mathscr{O}_X|_{\bar{Y}}$ defining the closed subscheme Y of $\bar{Y}$, then the quasi-coherent ideal of $\mathscr{O}_X$ defining $\bar{Y}$ is the canonical extension (Proposition 10.6.58) $\overline{\mathscr{I}}$ of $\mathscr{I}$, because it is evidently the largest quasi-coherent ideal of $\mathscr{O}_X$ inducing $\mathscr{I}$ over Y.

Corollary 10.6.72. *Under the hypothesis of Proposition 10.6.71, any section of $\mathscr{O}_{\bar{Y}}$ over an open subset V of $\bar{Y}$ that is zero on $V \cap Y$ is zero.*

Proof. In view of Proposition 10.6.70, we can assume that $V = \bar{Y}$. If we consider sections of $\mathscr{O}_{\bar{Y}}$ over $\bar{Y}$ as $\bar{Y}$-sections of $\bar{Y} \otimes_{\mathbb{Z}} \mathbb{Z}[T]$, which is separated over $\bar{Y}$, the assertion is then a particular case of Corollary 10.6.68. □

10.7 Rational maps over schemes

10.7.1 Rational maps and rational functions

Let X and Y be two schemes, U and V be open dense sets of X, and f (resp. g) be a morphism from U (resp. V) to Y. We say the morphisms f and g are equivalent if they coincide over an open subset dense in $U \cap V$. As the intersection of finitely many open dense subsets of X is an open dense subset of X, it is clear that this relation is an equivalence relation.

Given two schemes X and Y, a **rational map** from X to Y is defined to be an equivalent class of morphisms from an open dense subset of X to Y. If X and Y are S-schemes, this class is called an **rational S-map** is there exists an S-morphism in it. An rational S-map from S to X is called an **rational S-section** of X. The rational X-sections of the X-scheme $X \otimes_{\mathbb{Z}} \mathbb{Z}[T]$ (where T is an indeterminate) are called the **rational functions over** X (cf. Example 10.3.13). A rational map f from X to Y is usually denoted by $f : X \dashrightarrow Y$.

Let $f : X \dashrightarrow Y$ be a rational map and U be an open subset of X. If f_1, f_2 are morphisms belonging to the class f, defined respectively over the open dense sets V_1, V_2 of X, the restrictions $f_1|_{U \cap V_1}$ and $f_2|_{U \cap V_2}$ coincide on $U \cap V_1 \cap V_2$, which is dense in U; the class of morphisms f therefore defines a rational map $U \dashrightarrow Y$, called the **restriction** of f to U and denoted by $f|_U$.

It is clear that we have a canonical map from $\mathrm{Hom}_S(X, Y)$ to the set of rational S-maps from X to Y, which associates any S-morphism $f : X \to Y$ to the rational S-map it belongs to. If we denote by $\Gamma_{\mathrm{rat}}(X/Y)$ the set of rational Y-sections of X, we then have a canonical map $\Gamma(X/Y) \to \Gamma_{\mathrm{rat}}(X/Y)$. It is also clear that if X and Y are two S-schems, the set of rational S-maps from X to Y is canonically identified with $\Gamma_{\mathrm{rat}}((X \times_S Y)/X)$.

In view of Example 10.3.13, we see the rational functions over X are canonically identified with the equivalent classes of sections of the structural sheaf $\mathscr{O}_X$ over open dense sets of X, where two sections are equivalent if they coincide over an open dense subset of the intersection of their defining domain. In partiuclar, we see the rational functions over X form a ring $K(X)$.

If X is an irreducible scheme, any nonempty open subset of X is dense; we can also say that the non-empty open sets of X are the open neighborhoods of the generic point x of X. To say that two morphisms from nonempty open subsets of X to Y are equivalent therefore means in this case that they have the same germ at the point x. In other words, rational maps (resp. rational S-maps) $X \dashrightarrow Y$ are identified with the germs of morphisms (resp. of S-morphisms) of non-empty open subsets of X to Y at the generic point x of X. In particular:

Proposition 10.7.1. *If X is an irreducible scheme, the ring $K(X)$ of rational functions over X is canonically identified with the local ring $\mathscr{O}_{X,x}$ at the generic point x of X. This is a local ring of zero dimension, and therefore an Artinian local ring if X is Noetherian. It is a field if X is integral, and is identified with the fraction field of $\Gamma(X, \mathscr{O}_X)$ if X is moreover affine.*

Proof. Since we can identify rational functions with sections over X, the first assertion follows from the definiton of stalks. For the second one, we can assume that X is affine with ring A; then $\mathfrak{p}_x$ is the nilradical of A, and in particular $\mathscr{O}_{X,x}$ has zero dimension. If A is integral, $\mathfrak{p}_x = (0)$ and $\mathscr{O}_{X,x}$ is the fraction field of A. Finally, if A is Noetherian, then $\mathfrak{p}_x$ is nilpotent and $\mathscr{O}_{X,x} = A_x$ is Artinian. □

If X is integral, the ring $\mathscr{O}_{X,z}$ is integral for any $z \in X$. Any affine open U containing x must contain x as its generic point, and $\mathscr{O}_{X,z}$, equal to a fraction field of $\Gamma(U, \mathscr{O}_X)$, is identified with $K(X)$. We then conclude that $K(X)$ is identified with the fraction field of $\mathscr{O}_{X,z}$, and in this way, $\mathscr{O}_{X,z}$ is canonically identified with a subring of $K(X)$, so that a germ $s \in \mathscr{O}_{X,z}$ is canonically identified with a rational function over X.

Proposition 10.7.2. *Let X and Y be two S-schemes such that the family (X_λ) of irreducible components of X is locally finite. For each λ, let x_λ be the generic point of X_λ. If R_λ is the set of germs at x_λ of S-morphisms from open subsets of X to Y, the set of rational S-maps from X to Y is identified with the product of R_λ. In particular, the ring of rational functions over X is identified with the product of the local rings $\mathscr{O}_{X,x_\lambda}$.*

Proof. The set of the intersections $X_\lambda \cap X_\mu$ for $\lambda \neq \mu$ is then locally finite, so their union is closed and contains the maximal points of X. If we set $X'_\lambda = X_\lambda - \bigcup_{\mu \neq \lambda} X_\lambda \cap X_\mu$, then X'_λ is irreducible, with generic point equal to that of X_λ, and pairwise disjoint with union dense in X. For any open dense subset U of X, $U'_\lambda = U \cap X'_\lambda$ is a nonempty open dense subset of X'_λ, and U'_λ are pairwise disjoint with $U' = \bigcup_\lambda U'_\lambda$ closed in X. To give a morphism from U' to Y is then equivalent to giving (arbitrarily) a morphism from each of the U'_λ in Y, so the assertion follows. □

Corollary 10.7.3. *Let A be a Noetherian ring and $X = \operatorname{Spec}(A)$. The ring of rational function functions over X is identified with the total fraction ring $Q(A)$.*

Proof. Let S be the complement of the union of minimal prime ideals of A. Then by Proposition 1.4.18, the ring of sections $\Gamma(D(f), \mathscr{O}_X)$ is identified with A_f, so $D(f)$ with $f \in S$ form a cofinal subset of the open dense sets of X, and the ring of rational functions over X is then identified with the inductive limit of A_f, $f \in S$, which is exactly $Q(A)$. □

Suppose that X is irreducible with generic point x. As any nonempty open set U of X contains x, and therefore contains any generalization $z \in X$, any morphism $U \to Y$ can be composed with the canonical morphism $\operatorname{Spec}(\mathscr{O}_{X,x}) \to X$ (Corollary 10.2.12). Two morphisms from nonempty open subsets of X to Y which coincide on a smaller open subset then give the same morphism $\operatorname{Spec}(\mathscr{O}_{X,x}) \to Y$. In other words, to any rational map X to Y, there corresponds a well-defined morphism $\operatorname{Spec}(\mathscr{O}_{X,x}) \to Y$.

Proposition 10.7.4. *Let X and Y be S-schemes. Suppose that X is irreducible with generic point x, and Y is of finite type over S. Then two rational S-maps X to Y corresponding to the same S-morphism $\operatorname{Spec}(\mathscr{O}_{X,x}) \to Y$ are identical. If moreover S is locally of finite presentation over S, then any S-morphism $\operatorname{Spec}(\mathscr{O}_{X,x}) \to Y$ corresponds to a rational S-map from X to Y.*

Proof. Given that every non-empty open subset of X is dense, this result follows immediately from Proposition 10.6.50. □

Corollary 10.7.5. *Suppose that X is irreducible with generic point x and Y is localy of finite presentation. Then the rational S-maps from X to Y are identified with the points of Y with values in the S-scheme $\operatorname{Spec}(\mathscr{O}_{X,x})$.*

Proof. This is just a reformulation of Proposition 10.7.4. □

Corollary 10.7.6. *Suppose that X is irreducible with generic point x and Y is localy of finite presentation. Let s be the image of x in S. Then giving a rational S-map from X to Y is equivalent to giving a point y of Y lying over s and a $\mathscr{O}_{S,s}$-homomorphism $\mathscr{O}_{Y,y} \to \mathscr{O}_{X,x} = K(X)$.*

Proof. This follows from Proposition 10.7.4 and Proposition 10.2.14. □

Corollary 10.7.7. *Suppose that X is irreducible with generic point x and Y is localy of finite presentation, then the rational S-maps from X to Y (with Y given) only depends on the S-scheme* $\mathrm{Spec}(\mathscr{O}_{X,x})$ *and in particular remain the same if we replace X by* $\mathrm{Spec}(\mathscr{O}_{X,z})$*, $z \in X$.*

Proof. In fact, if $z \in \overline{\{x\}}$ then x is the generic point of $Z = \mathrm{Spec}(\mathscr{O}_{X,z})$ and $\mathscr{O}_{X,x} = \mathscr{O}_{Z,x}$. □

Corollary 10.7.8. *Suppose that X is integral with generic point x and Y is localy of finite presentation. Let s be the image of x in S. Then following dates are equivalent:*

(i) *a rational S-map from X to Y;*

(ii) *a point of $Y \otimes_S \kappa(s)$ with values in the extension $K(X)$ of $\kappa(s)$;*

(iii) *a point $y \in Y$ over s and an $\kappa(s)$-homomorphism $\kappa(y) \to \kappa(x) = K(X)$.*

Proof. The points of Y over s belong to $Y \otimes_S \kappa(s)$ and the $\mathscr{O}_{S,s}$-homomorphisms $\mathscr{O}_{Y,y} \to K(X)$ are $\kappa(s)$-homomorphisms $\kappa(y) \to K(X)$, since $K(X)$ is a field. □

Corollary 10.7.9. *Let k be a field and X, Y be two schemes locally algebraic over k. Suppose that X is integral, then the rational k-maps from X to Y are identified with the points of Y with values in the extension $K(X)$ of k.*

10.7.2 Defining domain of a rational map

Let X and Y be schemes, f a rational map from X to Y. We say f is **defined at a point** $x \in X$ if there exists an open dense subset U containing x and a morphism $U \to Y$ representing f. The set of points $x \in X$ where f is defined is called the **defining domain** of the rational map f. It is clearly an open dense subset of X.

Proposition 10.7.10. *Let X and Y be two S-schemes such that X is reduced and Y is separated over S. Let f be a rational S-map from X to Y, and U_0 be its domain. There then exists a unique S-morphism $U_0 \to Y$ belonging to the class f.*

Proof. For any morphism $U \to Y$ belonging to the class f, we necessarily have $U \subseteq U_0$, so we only need to prove that if U_1, U_2 are two dense subsets of X and $f_i : U_i \to Y$ $(i = 1, 2)$ are two S-morphisms that coincide on an open subset $V \subseteq U_1 \cap U_2$, then f_1 and f_2 coincide on $U_1 \cap U_2$. For this, we can clearly assume that $X = U_1 = U_2$. As X (hence V) is reduced, X is smallest closed subscheme of X dominating V (Proposition 10.4.48). Let $g = (f_1, f_2)_S : X \to Y \times_S Y$; as by hypothesis the diagonal $T = \Delta_Y(Y)$ is a closed subscheme of $Y \times_S Y$, $Z = g^{-1}(T)$ is a closed subscheme of X. If $h : V \to Y$ is the restriction of f_1 and f_2 to V, the restriction of g to V is $\tilde{g} = (h, h)_S$, which factors into $\tilde{g} = \Delta_Y \circ h$. As $\Delta_Y^{-1}(T) = Y$, we have $\tilde{g}^{-1}(T) = V$, and Z is therefore a closed subscheme of X inducing the subscheme structure on V, hence dominates V, and this implies $Z = X$. From the relation $g^{-1}(T) = X$, we deduce that g factors into $\Delta_Y \circ f$, where f is a morphism $X \to Y$ (Proposition 10.4.16), and we have $f_1 = f_2 = f$ from the definition of the diagonal morphism. □

It is clear that the morphism $U_0 \to Y$ defined in Proposition 10.7.10 is the unique morphism in the class f that admits no further extension to open dense subsets of X containing U_0. Under the conditions of Proposition 10.7.10, we can then identify the rational maps from X to Y with the morphisms unextendable to open dense subsets of X to Y.

Corollary 10.7.11. *Let X and Y be two S-schemes such that X is reduced and Y is separated over S. Let U be an open dense subset of X, then there is a canonical bijective correspondence between the S-morphisms from U to Y and the rational S-maps from X to Y defined at each point of U.*

Proof. In view of Proposition 10.7.10, for any S-morphism $f : U \to Y$, there exists a rational S-map $\bar{f}$ from X to Y which extends f. □

Corollary 10.7.12. *Let S be a separated scheme, X be a reduced S-scheme, Y be an S-scheme, and $f : U \to Y$ be an S-morphism from an open dense subset U of X to Y. If $\bar{f}$ is the rational $\mathbb{Z}$-map from X to Y extending f, $\bar{f}$ is an S-morphism (and therefore the rational S-map from X to Y extending f).*

Proof. In fact, if $\varphi : X \to S$ and $\psi : Y \to S$ are the structural morphisms, U_0 is the defining domain of $\bar{f}$, and $j : U_0 \to X$ is the injection, it suffices to prove that $\psi \circ \bar{f} = \varphi \circ j$, which follows from the proof of Proposition 10.7.10, since f is an S-morphism. □

Corollary 10.7.13. *Let X and Y be S-schemes. Suppose that X is reduced and X, Y are spearated over S. Let $p : Y \to X$ be an S-morphism, U be an open dense subset of X, and f be a U-section of Y. Then the rational map $\bar{f}$ from X to Y extending f is a rational X-section of Y.*

Proof. We only need to prove that $p \circ \bar{f}$ is the identity on the defining domain of $\bar{f}$; since X is separated over S, this follows from the proof of Proposition 10.7.10. □

Corollary 10.7.14. *Let X be a reduced scheme and U be an open dense subset of X. There exists a canonical bijective correspondence between sections of $\mathscr{O}_X$ over U and rational functions f over X defined on each point of U.*

Proof. It suffices to remark that the X-scheme $X \otimes_{\mathbb{Z}} \mathbb{Z}[T]$ is separated over X by Proposition 10.5.26. □

Corollary 10.7.15. *Let Y be a reduced scheme, $f : X \to Y$ be a separated morphism, U be an open dense subset of Y, $g : U \to f^{-1}(U)$ be a U-section of $f^{-1}(U)$, and Z the reduced subscheme of X induced on $\overline{g(U)}$. For g to be the restriction of a Y-section of X, it is necessary and sufficient that the restriction of f to Z is an isomorphism from Z to Y.*

Proof. The restriction of f to $f^{-1}(U)$ is a separated morphism (Proposition 10.5.26(i)), so g is a closed immersion by Corollary 10.5.19, and therefore $g(U) = Z \cap f^{-1}(U)$ coincides with the subscheme induced by Z over the open subset $g(U)$ of Z. It is then clear that the given condition is sufficient, since if $f_Z : Z \to Y$ is an isomorphism and $\bar{g} : Y \to Z$ is the inverse morphism, then $\bar{g}$ extends g. Conversely, if g is the restriction to U of a Y-section h of X, h is then a closed immersion by Corollary 10.5.19, so $h(Y)$ is clossed, and as it contains $g(U)$ and we have (as h is continuous) $h(Y) = h(\bar{U}) \subseteq \overline{h(U)} = \overline{g(U)}$, we conclude that $h(Y) = Z$. It then follows from Proposition 10.4.44 that h is necessarily an isomorphism from Y to the closed subscheme Z of X, so $f|_Z$ is also an isomorphism. □

Let X and Y be two S-schemes, where X is reduced and Y is separated over S. Let f be a rational S-map from X to Y, and let x be a point of X. We can compose f with the canonical morphism $\mathrm{Spec}(\mathscr{O}_{X,x}) \to X$ provided that the trace on $\mathrm{Spec}(\mathscr{O}_{X,x})$ of the defining domain of f is dense in $\mathrm{Spec}(\mathscr{O}_{X,x})$ (identified with the set $z \in X$ such as $x \in \overline{\{z\}}$ (cf. Corollary 10.2.12). This happens if the family of irreducible components of X is *locally finite*:

Lemma 10.7.16. *Let X be a scheme such that the family of irreducible components of X is locally finite, and x be a point of X. The irreducible components of $\mathrm{Spec}(\mathscr{O}_{X,x})$ are then the traces over $\mathrm{Spec}(\mathscr{O}_{X,x})$ of the irreducible components of X containing x. For an open subset U of X to be such that $U \cap \mathrm{Spec}(\mathscr{O}_{X,x})$ is dense in $\mathrm{Spec}(\mathscr{O}_{X,x})$, it is necessary and sufficient that it meets the irreducible components of X containing x (and this is true in particular if U is dense in X).*

Proof. The second assertion clearly follows from the first one. As $\mathrm{Spec}(\mathscr{O}_{X,x})$ is contained in any affine open U containing x and the irreducible components of U containing x are the traces of the irreducible components of X containing x on U (**??**), we can suppose that X is affine with ring A. Then the prime ideals of A_x correspond to prime ideals of A contained in $\mathfrak{p}_x$, so the minimal prime ideals of A_x correspond to minimal prime ideals of A contained in $\mathfrak{p}_x$, and the lemma follows from Proposition 1.4.18. □

Suppose that we are under the assumption of Lemma 10.7.16. If U is the defining domain of definition of the rational S-map f, denote by f' the rational map from $\mathrm{Spec}(\mathscr{O}_{X,x})$ to Y which coincides with f over $U \cap \mathrm{Spec}(\mathscr{O}_{X,x})$; we will say that this rational map is **induced** by f.

Proposition 10.7.17. *Let S be a scheme, X be a reduced S-scheme, and Y be a separted S-scheme that is locally of finite presentation over S. Suppose that the family of irreducible components of X is locally finite. Let f be a rational S-map from X to Y and x be a point of X. For f to be defined at the point x, it is necessary and sufficient that the rational map f' from* $\mathrm{Spec}(\mathscr{O}_{X,x})$ *to Y induced by f is a morphism.*

Proof. The conditions is clearly necessary since $\mathrm{Spec}(\mathscr{O}_{X,x})$ is contained in any open subset containing x. We now prove the sufficiency, so suppose that f' is a morphism. In view of Proposition 10.6.50, there exists an open neighborhood U of x in X and an S-morphism $g : U \to Y$ inducing f' on $\mathrm{Spec}(\mathscr{O}_{X,x})$. The point is that U is not necessarily dense in X, so we want to extend g to a morphism defined on an open dense subset of X. Now by Lemma 10.7.16, there are finitely many irreducible components X_i of X containing x, and we can assume that these are the only ones meeting U, by replacing U with a smaller open subset. As the generic points of X_i belong to the defining domain of f and to U, we see that f and g coincide over a non-empty open dense subset of each of the X_i (Proposition 10.6.50). Consider the morphism f_1 defined on an open dense subset of $U \cup (X - \bar{U})$ which equals to g over U and to f over the intersection of $X - \bar{U}$ and the defining domain of f (we also note that each X_i is contained in $\bar{U}$). As $U \cup (X - \bar{U})$ is dense in X, f_1 and f coincide on an open dense subset of X, and f is an extension of f_1. Since f_1 is defined at x, this shows f is defined at x. □

10.7.3 Sheaf of rational functions

Let X be a scheme. For each open subset U of X, we denote by $K(U)$ the ring of rational functions over U, which is an $\Gamma(U, \mathscr{O}_X)$-algebra. Moreover, if $V \subseteq U$ is a second open subset of X, any section of $\mathscr{O}_X$ over a dense subset of U restricts to a section over a dense subset of V, and if two sections coincide over an open dense subset of U, their restriction also coincide over a smaller open dense subset of V. We then define a homomorphism of algebras $\mathrm{Res}^U_V : K(U) \to K(V)$, and it is clear that for $U \supseteq V \supseteq W$ open in X we have $\mathrm{Res}^U_W = \mathrm{Res}^V_W \circ \mathrm{Res}^U_V$. Therefore, we get a presheaf of algebras over X. The associated sheaf of $\mathscr{O}_X$-algebras over X is then called the **sheaf of rational functions** over the scheme X, and denoted by $\mathscr{K}_X$. For any open subset U of X, it is clear that the restriction $\mathscr{K}_X|_U$ is equal to $\mathscr{K}_U$.

Proposition 10.7.18. *Let X be a scheme such that the family (X_λ) of irreducible components of X is locally finite. Then the $\mathscr{O}_X$-module $\mathscr{K}_X$ is quasi-coherent and for any open subset U of X, $K(U)$ is equal to $\Gamma(U, \mathscr{K}_X)$ and is identified with the product of the local rings of the generic points x_λ of the irreducible components X_λ such that $X_\lambda \cap U \neq \varnothing$.*

Proof. The fact that $K(U)$ is identified with the product follows from Proposition 10.7.2. We now show that the presheaf $U \mapsto K(U)$ is a sheaf. Consider an open subset U of X and an open covering (V_α) of U. If $s_\alpha \in K(V_\alpha)$ are such that s_α and s_β coincide over $V_\alpha \cap V_\beta$ for each pair of indices, we then conclude that for any index λ such that $U \cap X_\lambda \neq \varnothing$, the component in $K(X_\lambda)$ of all s_α such that $V_\alpha \cap X_\lambda \neq \varnothing$ are the same. Denoting by t_λ this component, it is clear that the element of $K(U)$ with component t_λ in $K(X_\lambda)$ has restriction s_α on each V_α. Finally, to see the sheaf $\mathscr{K}_X$ is quasi-coherent, we can limit ourselves to the case $X = \mathrm{Spec}(A)$ is affine with finitely many irreducible components; by taking for U the affine open sets of the form $D(f)$, where $f \in A$, it follows from the above argument that we have $\mathscr{K}_X = \widetilde{M}$, where M is the direct sum of the A-modules A_{x_λ}. □

Corollary 10.7.19. *Let X be a reduced scheme with irreducible components $(X_i)_{1 \leq i \leq n}$, endowed with the reduced subscheme structures. If $\iota_i : X_i \to X$ is the canonical injection, $\mathscr{K}_X$ is the direct product of the $\mathscr{O}_X$-algebras $(\iota_i)_*(\mathscr{K}_{X_i})$.*

Proof. This is a particular case of Proposition 10.7.18, in view of the conditions in that proposition. □

Corollary 10.7.20. *If X is irreducible, any quasi-coherent $\mathscr{K}_X$-module $\mathscr{F}$ is a simple sheaf.*

Proof. It suffices to show that any $x \in X$ admits a neighborhood U such that $\mathscr{F}|_U$ is a simple sheaf, which means we can assume that X is affine. We can then suppose that $\mathscr{F}$ is the cokernel of a homomorphism $\mathscr{K}_X^{\oplus I} \to \mathscr{K}_X^{\oplus J}$, and it all boils down to seeing that $\mathscr{K}_X$ is a simple sheaf. But this is evident since $\Gamma(U, \mathscr{K}_X) = K(X)$ for any nonempty open subset U, since U contains the generic point of X. □

Corollary 10.7.21. *If X is irreducible, for any quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{K}_X$ is a simple sheaf. If moreover X is reduced (hence integral), $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{K}_X$ is isomorphic to a sheaf of the form $\mathscr{K}_X^{\oplus I}$.*

Proof. The first claim follows from Corollary 10.7.20, and the second one follows from the fact that if X is integral then $K(X)$ is a field. □

Proposition 10.7.22. *Let X be a scheme such that the family (X_λ) of irreducible components of X is locally finite. Then $\mathscr{K}_X$ is a quasi-coherent $\mathscr{O}_X$-algebra. If X is reduced, the canonical homomorphism $\mathscr{O}_X \to \mathscr{K}_X$ is injective.*

Proof. Since the question is local, the first claim follows from Proposition 10.7.18. The second one follows from Corollary 10.7.14. □

Let X and Y be integral schemes, so that $\mathscr{K}_X$ (resp. $\mathscr{K}_Y$) is a quasi-coherent $\mathscr{O}_X$-module (resp. $\mathscr{O}_Y$-module). Let $f : X \to Y$ be a dominant morphism; then there exists a canonical homomorphism of $\mathscr{O}_X$-modules:

$$\tau : f^*(\mathscr{K}_Y) \to \mathscr{K}_X.$$

Suppose that $X = \mathrm{Spec}(A)$ and $Y = \mathrm{Spec}(B)$ are affine with integral domains A and B, then f corresponds to the injective homomorphism $B \to A$ (Corollary 1.4.21), which extens to a monomorphism $L \to K$ of fraction fields. The homomorphism τ then corresponds to the canonical homomorphism $L \otimes_B A \to K$.

In the general case, for any couple of affine opens $U \subseteq X$, $V \subseteq Y$ such that $f(U) \subseteq V$, we define similarly a homomorphism $\tau_{U,V}$ and note that if $U' \subseteq U$, $V' \subseteq V$ and $f(U') \subseteq V'$, then $\tau_{U,V}$ extends $\tau_{U',V'}$. If x and y are the generic points of X and Y, respectively, then $f(x) = y$ and

$$(f^*(\mathscr{K}_Y))_x = \mathscr{O}_{Y,y} \otimes_{\mathscr{O}_{Y,y}} \mathscr{O}_{X,x} = \mathscr{O}_{X,x}$$

and τ_x is therefore an isomorphism. However, the homomorphism τ is usually not an isomorphism: for example, if $B = L$ is a field containing the integral domain A and A is not a field, the canonical homomorphism $L \otimes_B A \to K$ is then the canonical homomorphism $A \to K$, which is not bijective.

10.7.4 Torsion sheaves and torsion-free sheaves

Let X be a reduced scheme whose family of irreducible components is locally finite. For any $\mathscr{O}_X$-module $\mathscr{F}$, the canonical homomorphism $\mathscr{O}_X \to \mathscr{K}_X$ is injective by Proposition 10.7.22, and defines a homomorphism $\mathscr{F} \to \mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{K}_X$ which on each stalk, is none other than the homomorphism $z \mapsto z \otimes 1$ from $\mathscr{F}_x$ to $\mathscr{F}_x \otimes_{\mathscr{O}_{X,x}} \mathscr{K}_{X,x}$. The kernel $\mathscr{T}$ (also denoted by $\mathscr{T}(\mathscr{F})$) of this homomorphism is a sub-$\mathscr{O}_X$-module $\mathscr{F}$ is called the **torsion sheaf** of $\mathscr{F}$, which is quasi-coherent if $\mathscr{F}$ is quasi-coherent (Proposition 10.7.18). The sheaf $\mathscr{F}$ is called **torsion-free** if $\mathscr{T} = 0$, and a **torsion sheaf** if $\mathscr{T} = \mathscr{F}$. For any $\mathscr{O}_X$-module $\mathscr{F}$, $\mathscr{F}/\mathscr{T}$ is torsion-free.

Proposition 10.7.23. *If X is an integral scheme, for a quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$ to be torsion-free, it is necessary and sufficient that it is isomorphic to a sub-$\mathscr{O}_X$-module $\mathscr{G}$ of a simple sheaf of the form $\mathscr{K}_X^{\oplus I}$, generated (as a $\mathscr{K}_X$-module) by $\mathscr{G}$.*

Proof. This follows from Corollary 10.7.21, since $\mathscr{F}$ is torsion-free if and only if the canonical homomorphism $\mathscr{F} \to \mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{K}_X$ is injective. □

The cardinality of the set I is called the **rank** of $\mathscr{F}$. For any affine open U of X, since the generic point of x is contained in U, the rank of $\mathscr{F}$ is also equal to the rank of $\Gamma(U, \mathscr{F})$ as a $\Gamma(U, \mathscr{O}_X)$-module.

Corollary 10.7.24. *Over an integral scheme X, any torsion-free quasi-coherent $\mathscr{O}_X$-module of rank 1 (and in particular any invertible $\mathscr{O}_X$-module) is isomorphic to a sub-$\mathscr{O}_X$-module of $\mathscr{K}_X$, and the converse is also true.*

Corollary 10.7.25. *Let X be an integral scheme, $\mathscr{L}$, $\mathscr{L}'$ be two torsion-free $\mathscr{O}_X$-module, s (resp. s') be two sections of $\mathscr{L}$ (resp. $\mathscr{L}'$) over X. For $s \otimes s' = 0$, it is necessary and sufficient that one of the sections s, s' is zero.*

Proof. Let x be the generic point of X. We have by hypothesis $(s \otimes s')_x = s_x \otimes s'_x = 0$. As $\mathscr{L}_x$ and $\mathscr{L}'_x$ are identified with sub-$\mathscr{O}_{X,x}$-modules of the field $\mathscr{O}_{X,x}$, the preceding relation implies $s_x = 0$ or $s'_x = 0$, and therefore $s = 0$ or $s' = 0$ since $\mathscr{L}$ and $\mathscr{L}'$ are torsion-free (Corollary 10.7.20). □

Proposition 10.7.26. *Let X and Y be two integral schemes and $f : X \to Y$ be a dominant morphism. For any torsion-free quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, $f_*(\mathscr{F})$ is a torsion-free $\mathscr{O}_Y$-module.*

Proof. As f_* is left exact, it suffices, in view of Proposition 10.7.23, to prove the proposition for $\mathscr{F} = \mathscr{K}_X^{\oplus I}$. Now any open subset U of Y contains the generic point of Y, hence $f^{-1}(U)$ contains the generic point of X, so we have $\Gamma(U, f_*(\mathscr{F})) = \Gamma(f^{-1}(U), \mathscr{F}) = K(X)^{\oplus I}$. Therefore $f_*(\mathscr{F})$ is the simple sheaf with stalk $K(X)^{\oplus I}$, considered as a $\mathscr{K}_Y$-module, and it is evidently torsion-free. □

Proposition 10.7.27. *Let X be reduced scheme whose family of irreducible components is locally finite. For any quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$ of finite type, the following conditions are equivalent:*

(i) *$\mathscr{F}$ is a torsion sheaf.*

(ii) *$\mathscr{F}_x = 0$ for every maximal point of x.*

(iii) *$\mathrm{supp}(\mathscr{F})$ contains no irreducible component of X.*

Proof. Since the question is local, we may assume that X has finitely many irreducible components $(X_i)_{1 \le i \le n}$, with generic points x_i. Endow each X_i the reduced subscheme structure of X, and let $\iota_i : X_i \to X$ be the canonical injection. If we put $\mathscr{F} = \iota_i^*(\mathscr{F})$, we see immediately that (Corollary 10.7.19) $\mathscr{F}$ is torsion-free if and only if each $\mathscr{F}_i$ is torsion-free. As $\mathscr{F}_{x_i} = (\mathscr{F}_i)_{x_i}$, to establish the equivalence of (i) and (ii), we can assume that X is integral. But then if x is the generic point of X, the relation $\mathscr{F}_x = 0$ and $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{K}_X = 0$ are equivalent by Proposition 10.7.23 and Corollary 10.7.20. The equivalenct of (ii) and (iii) results from the fact that $\mathrm{supp}(\mathscr{F})$ is closed in X (since $\mathscr{F}$ is quasi-coherent) and that the conditions $\mathrm{supp}(\mathscr{F}) \cap X_i = \varnothing$ and $x_i \notin \mathrm{supp}(\mathscr{F})$ are then equivalent. □

10.7.5 Separation criterion for integral schemes

Let X be an integral scheme, K its function field, identified with the local ring at the generic point ξ of X. For any $x \in X$, we can identify $\mathscr{O}_{X,x}$ as a subring of K, formed by the rational functions defined at the point x. For any rational function $f \in K$, the defining domain $\delta(f)$ of f is then the open subset of $x \in X$ such that $f \in \mathscr{O}_{X,x}$, and in view of Corollary 10.7.14 we have, for each open subset $U \subseteq X$, that

$$\Gamma(U, \mathscr{O}_X) = \bigcap_{x \in U} \mathscr{O}_{X,x}. \tag{10.7.1}$$

Given a field K, for any subring A of K, we denote by $L(A)$ the set of localizations $A_{\mathfrak{p}}$, where $\mathfrak{p}$ runs through prime ideals of A; they are identified with local subrings of K containing A. As $\mathfrak{p} = (\mathfrak{p}A_{\mathfrak{p}}) \cap A$, the map $\mathfrak{p} \mapsto A_{\mathfrak{p}}$ from $\operatorname{Spec}(A)$ to $L(A)$ is bijective.

Lemma 10.7.28. *Let K be a field and A be a subring of K. For a local subring R to dominate a ring in $L(A)$, it is necessary and sufficient that $A \subseteq R$. In this case, the local ring $A_{\mathfrak{p}}$ dominated by R is then unique and corresponds to the prime ideal $\mathfrak{p} = \mathfrak{m}_R \cap A$, where $\mathfrak{m}_R$ is the maximal ideal of R.*

Proof. In fact, if R dominates $A_{\mathfrak{p}}$, then $\mathfrak{m}_R \cap A_{\mathfrak{p}} = \mathfrak{p}A_{\mathfrak{p}}$ by Proposition 5.1.6, hence the uniqueness of $\mathfrak{p}$. On the other hand, if $A \subseteq R$, $\mathfrak{m}_R \cap A = \mathfrak{p}$ is a prime ideal of A, and as the elements of $A - \mathfrak{p}$ are then invertible in R, we have $A_{\mathfrak{p}} \subseteq R$, so $\mathfrak{p}A_{\mathfrak{p}} \subseteq \mathfrak{m}_R$ and R dominates $A_{\mathfrak{p}}$. □

Lemma 10.7.29. *Let K be a field, A, B be two local subrings of K, and C be the subring of K generated by $A \cup B$. The following conditions are equivalent:*

(i) *There exists a prime ideal $\mathfrak{r}$ of C such that $\mathfrak{m}_A = \mathfrak{r} \cap A$ and $\mathfrak{m}_B = \mathfrak{r} \cap B$.*

(ii) *The ideal $\mathfrak{c}$ generated in C by $\mathfrak{m}_A \cup \mathfrak{m}_B$ is proper.*

(iii) *There exists a local subring R of K dominating both A and B.*

Proof. It is clear that (i) implies (ii). Conversely, if $\mathfrak{c}$ is proper, it is contained in a maximal ideal $\mathfrak{n}$ of C, and $\mathfrak{n} \cap A$ contains $\mathfrak{m}_A$ and is proper, so $\mathfrak{n} \cap A = \mathfrak{m}_A$ and similarly $\mathfrak{n} \cap B = \mathfrak{m}_B$. Finally, it is clear that if R dominates A and B then $C \subseteq R$ and $\mathfrak{m}_A = \mathfrak{m}_R \cap A = (\mathfrak{m}_R \cap C) \cap A$, $\mathfrak{m}_B = \mathfrak{m}_R \cap B = (\mathfrak{m}_R \cap C) \cap B$, so (iii) implies (i). the converse is clear since we can take $R = C_{\mathfrak{r}}$. □

If the equivalent conditions in Lemma 10.7.29 hold, we say the two local subrings A and B are **related**.

Proposition 10.7.30. *Let A and B be subrings of a field K and C be the subring of K generated by $A \cup B$. The following conditions are equivalent:*

(i) *For any local ring R containing A and B, we have $A_{\mathfrak{p}} = B_{\mathfrak{q}}$, where $\mathfrak{p} = \mathfrak{m}_R \cap A$ and $\mathfrak{q} = \mathfrak{m}_R \cap B$.*

(ii) *For any prime ideal $\mathfrak{r}$ of C, we have $A_{\mathfrak{p}} = B_{\mathfrak{q}}$, where $\mathfrak{p} = \mathfrak{r} \cap A$ and $\mathfrak{q} = \mathfrak{r} \cap B$.*

(iii) *If $P \in L(A)$ and $Q \in L(B)$ are related, they are identical.*

(iv) *We have $L(A) \cap L(B) = L(C)$.*

Proof. It follows from Lemma 10.7.28 and Lemma 10.7.29 that (i) and (iii) are equivalent, and (i) implies (ii) by applying (i) to the ring $R = C_{\mathfrak{r}}$. Conversely, (ii) implies (i) because if R contains $A \cup B$, it contains C, and if $\mathfrak{r} = \mathfrak{m}_R \cap C$, we have $\mathfrak{p} = \mathfrak{r} \cap A$ and $\mathfrak{q} = \mathfrak{r} \cap B$, so $A_{\mathfrak{p}} = B_{\mathfrak{q}}$. We also see that (iv) implies (i), because if R contains $A \cup B$, it then dominates a local ring $C_{\mathfrak{r}} \in L(C)$ by Lemma 10.7.28; we have by hypothesis that $L(C) = L(A) \cap L(B)$, and as R dominates a unique ring in $L(A)$ (resp. $L(B)$), we conclude that $C_{\mathfrak{r}} = A_{\mathfrak{p}} = B_{\mathfrak{q}}$.

Finally, we show that (iii) implies (iv). Let $R \in L(C)$; R then dominates a ring $P \in L(A)$ and a ring $Q \in L(B)$ by Lemma 10.7.28, so P and Q are related, hence identical by hypothesis. As we then have $C \subseteq P$, P dominates a ring $R' \in L(C)$ (Lemma 10.7.28), so R dominates the ring R', and by Lemma 10.7.28 we necessarily have $R = R' = P$, so $R \in L(A) \cap L(B)$. Conversely, if $R \in L(A) \cap L(B)$, we have $C \subseteq R$, so R dominates a ring $R'' \in L(C)$ by Lemma 10.7.28. The two subrings R and R'' are clearly related, and as $L(C) \subseteq L(A) \cap L(B)$, we conclude from condition (iii) that $R = R''$, so $R \in L(C)$ and the proof is complete. □

Proposition 10.7.31. *Let X be an integral scheme and K be its field of rational functions. Then for X to be separated, it is necessary and sufficient that the relation "$\mathscr{O}_{X,x}$ and $\mathscr{O}_{X,y}$ are related" for two points $x, y \in X$ implies $x = y$.*

Proof. Suppose the given condition on X, we prove that X is separated. Let $U = \operatorname{Spec}(A)$ and $V = \operatorname{Spec}(B)$ be two distinct affine opens of X, with A, B identified as subrings of K. Then U (resp. V) is identified with the set $L(A)$ (resp. $L(B)$), and by Proposition 10.7.30 the hypothesis on X implies that, if C is the subring of K generated by $A \cup B$, $W = U \cap V$ is identified with $L(A) \cap L(B) = L(C)$. Moreover, we have seen from Proposition 1.3.28 that any subring R of K is equal to the intersection of the local rings belong to $L(R)$, so

$$C = \bigcap_{z \in W} \mathscr{O}_{X,z} = \Gamma(W, \mathscr{O}_X) \tag{10.7.2}$$

where we use formula (10.7.1). Consider then the subscheme induced by X over W. The identity homomorphism $\varphi : C \to \Gamma(W, \mathscr{O}_X)$ corresponds to a morphism $\psi : W \to \operatorname{Spec}(C)$. In view of (10.7.2) and the relation $L(C) = L(A) \cap L(B)$, any prime ideal $\mathfrak{r}$ of C is of the form $\mathfrak{r} = \mathfrak{m}_x \cap C$, where $x \in W$ is the point in $\operatorname{Spec}(C)$ corresponding to $\mathfrak{r}$, and the map ψ just sends x to $\mathfrak{r}$, so it is bijective. On the other hand, for any $x \in W$, $\psi_x^\#$ is the canonical injection $C_\mathfrak{r} \to \mathscr{O}_{X,x}$, where $\mathfrak{r} = \mathfrak{m}_x \cap C$. Now the local ring $\mathscr{O}_{X,x}$ dominates $A_\mathfrak{p}$, $B_\mathfrak{q}$ and $C_\mathfrak{r}$, where $\mathfrak{p} = \mathfrak{m}_x \cap A$ and $\mathfrak{q} = \mathfrak{m}_x \cap B$, and as $C_\mathfrak{r} \in L(C) = L(A) \cap L(B)$, by Lemma 10.7.28 we then conclude that $C_\mathfrak{r} = A_\mathfrak{p} = B_\mathfrak{q}$ (we already have seen this in the proof of Proposition 10.7.30). But the local rings $A_\mathfrak{p}$ and $B_\mathfrak{q}$ are both identified with the stalk $\mathscr{O}_{X,x}$, so we see that $C_\mathfrak{r} = \mathscr{O}_{X,x}$ and $\psi_x^\#$ is bijective. It then remains to show that ψ is a homeomorphism, which amounts to show that, for any closed subset $F \subseteq W$, the image $\psi(F)$ is closed in $\operatorname{Spec}(C)$. Now F is the intersection with W of a closed subset E of the form $V(\mathfrak{a})$, where $\mathfrak{a}$ is an ideal of A. We claim that $\psi(F) = V(\mathfrak{a}C)$: in fact, the prime ideal of C containing $\mathfrak{a}C$ are the prime ideals of C containing $\mathfrak{a}$, hence the ideals of the form $\psi(x) = \mathfrak{m}_x \cap C$, where $\mathfrak{a} \subseteq \mathfrak{m}_x$ and $x \in W$. As $\mathfrak{a} \subseteq \mathfrak{m}_x$ is equivalent to $x \in V(\mathfrak{a}) = W \cap E$ for $x \in U$, we then get $\psi(F) = V(\mathfrak{a}C)$. In view of Proposition 10.5.31, we then conclude that X is separated, because $U \cap V$ is affine and the ring C is generated by $A \cup B$.

Conversely, suppse that X is separated, and let x, y be two points of X such that $\mathscr{O}_{X,x}$ and $\mathscr{O}_{X,y}$ are related. Let U (resp. V) be an open affine containing x (resp. y), with ring A (resp. B). We then see $U \cap V$ is affine and its ring C is generated by $A \cup B$ (Proposition 10.5.31). If $\mathfrak{p} = \mathfrak{m}_x \cap A$ and $\mathfrak{q} = \mathfrak{m}_x \cap B$, we have $A_\mathfrak{p} = \mathscr{O}_{X,x}$ and $B_\mathfrak{q} = \mathscr{O}_{X,y}$, so $A_\mathfrak{p}$ and $B_\mathfrak{q}$ are related. Then by Lemma 10.7.29 there exists a prime ideal $\mathfrak{r}$ of C such that $\mathfrak{p} = \mathfrak{r} \cap A$, $\mathfrak{q} = \mathfrak{r} \cap B$. But the prime ideal $\mathfrak{r}$ then corresponds to a point $z \in U \cap V$ since $U \cap V$ is affine, and we have $x = z$ and $y = z$, so $x = y$. □

Corollary 10.7.32. *Let X be a separated integral scheme and x, y be two points of X. For $x \in \overline{\{y\}}$, it is necessary and sufficient that $\mathscr{O}_{X,x} \subseteq \mathscr{O}_{Y,y}$, which means the rational functions defined at x are also defined at y.*

Proof. This condition is clearly necessary since the defining domain $\delta(f)$ of a rational function is open, hence stable under generalization. To see it is also sufficient, assume that $\mathscr{O}_{X,x} \subseteq \mathscr{O}_{X,y}$, so there exists a prime ideal $\mathfrak{p}$ of $\mathscr{O}_{X,x}$ such that $\mathscr{O}_{Y,y}$ dominates $(\mathscr{O}_{X,x})_\mathfrak{p}$ (Lemma 10.7.28). By

Corollary 10.2.12, there exists $z \in X$ such that $x \in \overline{\{z\}}$ and $\mathscr{O}_{X,z} = (\mathscr{O}_{X,x})_{\mathfrak{p}}$; as $\mathscr{O}_{X,z}$ and $\mathscr{O}_{X,y}$ are then related, we have $z = y$ by Proposition 10.7.31, whence the corollary. □

Corollary 10.7.33. *If X is a separated integral scheme, the map $x \mapsto \mathscr{O}_{X,x}$ is injective. In other words, if x, y are two distinct points of X, there exists a rational function defined at only one of these points.*

Proof. This follows from Corollary 10.7.32 and the T_0-axiom. □

Corollary 10.7.34. *Let X be a Noetherian separated integral scheme. The sets $\delta(f)$ for $f \in K(X)$ form a subbasis the topology of X.*

Proof. In fact, any closed subset of X is then a finite union of irreducible closed subsets, which are of the form $\overline{\{y\}}$. Now if $x \notin \overline{\{y\}}$, there exists a rational function f defined at x but not at y (Corollary 10.7.33), which means $x \in \delta(f)$ and $\delta(f) \cap \overline{\{y\}} = \varnothing$. The complement of $\overline{\{y\}}$ is then a union of sets of the form $\delta(f)$, and in view of the previous remark, any open subset of X is a union of finite intersections of sets of the form $\delta(f)$. □

Proposition 10.7.35. *Let X, Y be two integral schemes with rational function fields K and L, respectively. Suppose that Y is separated and let $f : X \to Y$ be a dominant morphism. Then L is identified with a subfield of K, and for every point $x \in X$, $\mathscr{O}_{Y,f(x)}$ is the unique local ring of Y dominated by $\mathscr{O}_{X,x}$.*

Proof. Ths first assertion is already proved in Proposition 10.4.23. Now for every $x \in X$, the local homomorphism $f_x^\# : \mathscr{O}_{Y,f(x)} \to \mathscr{O}_{X,x}$ is injective by Proposition 10.4.23, so, if we identify L as a subfield of K, $\mathscr{O}_{Y,f(x)}$ is dominated by $\mathscr{O}_{X,x}$. As Y is separated, two local rings of Y can not be dominated by the same local ring (Proposition 10.7.31), so our assertion follows. □

Proposition 10.7.36. *Let X be an irreducible scheme and $f : X \to Y$ be a local immersion (resp. a local isomorphism). Suppose that f is separated, then it is an immersion (resp. an open immersion).*

Proof. It suffices to prove that f is a homeomorphism from X to $f(X)$ (Proposition 10.4.9). By replacing f with f_{red}, we may assume that X and Y are reduced. If Y' is the reduced subscheme of Y with underlying space $\overline{f(X)}$, f then factors into

$$X \xrightarrow{f'} Y' \xrightarrow{j} Y$$

where j is the canonical injection. Then f' is separated by Proposition 10.5.26(v) and is a local immersion by Proposition 10.5.15(iii), so we may reduce to the case that f is dominant. But then Y is irreducible by **??**, and by Proposition 10.4.23, we see f is in fact a local isomorphism, so for each $x \in X$ the homomorphism $f_x^\#$ is an isomorphism. By Corollary 10.7.33, this implies that f is injective, so f is in fact a homeomorphism. □

10.8 Formal schemes

10.8.1 Formal affine schemes and morphisms

Let A be a admissible topological ring, with a nilideal $\mathfrak{J}$ (recall that this means $\mathfrak{J}$ is open and $(\mathfrak{J}^n)$ tends to 0 in A). The specturm $\mathrm{Spec}(A/\mathfrak{J})$ is then a closed subscheme of $\mathrm{Spec}(A)$, which is the set of open prime ideals of A. This topological space does not depend on the nilideal of $\mathfrak{J}$, and we denote it by $\mathfrak{X}$. Let $(\mathfrak{J}_\lambda)$ be a system of fundamental neighborhood of 0 in A, formed by the nilideals of A, and for each λ, let $\mathscr{O}_\lambda$ be the structural sheaf of $\mathrm{Spec}(A/\mathfrak{J}_\lambda)$. This sheaf is induced over $\mathfrak{X}$ by $\widetilde{A}/\widetilde{\mathscr{I}}_\lambda$ (and is zero outside $\mathfrak{X}$). For $\mathfrak{J}_\mu \subseteq \mathfrak{J}_\lambda$, the canonical homomorphism $A/\mathfrak{J}_\mu \to A/\mathfrak{J}_\lambda$ defines a homomorphism $u_{\lambda\mu} : \mathscr{O}_\mu \to \mathscr{O}_\lambda$ of sheaves of rings, and $(\mathscr{O}_\lambda)$ is a projective system of sheaves of rings for these homomorphisms. As the topology of $\mathfrak{X}$ admits a basis formed by quasi-compact open subsets, if we view each $\mathscr{O}_\lambda$ as a sheaf of discrete rings, the $\mathscr{O}_\lambda$ then form a projective system of sheaves of topological rings, and we denote by $\mathscr{O}_\mathfrak{X}$ the limit of this system $(\mathscr{O}_\lambda)$. By ([?] 0_I 3.2.6), for any quasi-compact open subset U of $\mathfrak{X}$, $\Gamma(U, \mathscr{O}_\mathfrak{X})$ is then the limit topological ring of the discrete rings $\Gamma(U, \mathscr{O}_\lambda)$.

Given an admissible topological ring A, the closed subspace $\mathfrak{X}$ of $\mathrm{Spec}(A)$ formed by open prime ideals of A is called the **formal spectrum** of A and denoted by $\mathrm{Spf}(A)$. A topoloigcally ringed space is called a **formal affine scheme** if it is isomorphic to a formal spectrum $\mathrm{Spf}(A) = \mathfrak{X}$ endowed with the sheaf of topological rings $\mathscr{O}_\mathfrak{X}$, which is the limit of the sheaf of discrete rings $(\widetilde{A}/\widetilde{\mathfrak{J}}_\lambda)|_\mathfrak{X}$, where $\mathfrak{J}_\lambda$ runs through the filtered set of nilideals of A. When we speak of a formal spectrum $\mathfrak{X} = \mathrm{Spf}(A)$ as an formal affine scheme, it will always be understood that the topologically ringed space $(\mathfrak{X}, \mathscr{O}_\mathfrak{X})$ where $\mathscr{O}_\mathfrak{X}$ is defined as above. By an **adic** (resp. **Noetherian**) formal affine scheme, we mean a formal affine scheme which is isomorphic to a formal specturm $\mathrm{Spf}(A)$, where A is adic (resp. adic and Noetherian).

We note that any affine scheme $X = \mathrm{Spec}(A)$ can be considered as a formal affine scheme in a unique way: consider A as a discrete topological ring, the rings $\Gamma(U, \mathscr{O}_X)$ are then discrete if U is quasi-compact (but not true in general if U is any open set of X).

Proposition 10.8.1. *If $\mathfrak{X} = \mathrm{Spf}(A)$, where A is an admissible ring, then $\Gamma(X, \mathscr{O}_\mathfrak{X})$ is homeomorphic to A.*

Proof. In fact, as $\mathfrak{X}$ is closed in $\mathrm{Spec}(A)$, it is quasi-compact, and therefore $\Gamma(\mathfrak{X}, \mathscr{O}_\mathfrak{X})$ is homeomorphic to the limit of the discrete rings $\Gamma(\mathfrak{X}, \mathscr{O}_\lambda)$. But $\Gamma(\mathfrak{X}, \mathscr{O}_\lambda)$ is isomorphic to $A/\mathfrak{J}_\lambda$, and as A is separated and complete, this is homeomorphic to $\varinjlim A/\mathfrak{J}_\lambda$, whence the proposition. □

Proposition 10.8.2. *Let A be an admissible ring, $\mathfrak{X} = \mathrm{Spf}(A)$, and for $f \in A$, let $\mathfrak{D}(f) = D(f) \cap \mathfrak{X}$. Then the topologically ringed space $(\mathfrak{D}(f), \mathscr{O}_\mathfrak{X}|_{\mathfrak{D}(f)})$ is isomorphic to a formal affine spectrum $\mathrm{Spf}(A_{\{f\}})$.*

Proof. For any nilideal $\mathfrak{J}$ of A, the discrete ring $A_f/\mathfrak{J}_f$ is canonically identified with $A_{\{f\}}/\mathfrak{J}_{\{f\}}$, so the topological space $\mathrm{Spf}(A_{\{f\}})$ is canonically identified with $\mathfrak{D}(f)$. Moreover, for any quasi-compact open U of $\mathfrak{X}$ contained in $\mathfrak{D}(f)$, $\Gamma(U, \mathscr{O}_\lambda)$ is identified with the module of sections of the structural sheaf of $\mathrm{Spec}(A_f/\mathfrak{J}_\lambda)$ over U, so, if we put $\mathfrak{Y} = \mathrm{Spf}(A_{\{f\}})$, $\Gamma(U, \mathscr{O}_\mathfrak{X})$ is identified with $\Gamma(U, \mathscr{O}_\mathfrak{Y})$, whence the proposition. □

As a sheaf of rings, the stalk of the structural sheaf $\mathscr{O}_\mathfrak{X}$ of $\mathrm{Spf}(A)$ for any $x \in X$ is, by Proposition 10.8.2, identified with the inductive limit $\varinjlim A_{\{f\}}$ for $f \notin \mathfrak{p}_x$. Therefore, by Proposition 2.6.22 and Corollary 2.6.23, we have the following:

Proposition 10.8.3. *For any $x \in \mathfrak{X} = \mathrm{Spf}(A)$, the stalk $\mathscr{O}_{\mathfrak{X},x}$ is a local ring whose residue field is isomorphic to $\kappa(x)$. If A is adic and Noetherian, then $\mathscr{O}_{\mathfrak{X},x}$ is a Noetherian ring.*

As the field $\kappa(x)$ is not reduced to 0, we conclude in particular that the support of $\mathscr{O}_\mathfrak{X}$ is equal to $\mathfrak{X}$, and $(\mathfrak{X}, \mathscr{O}_\mathfrak{X})$ is a locally topologically ringed space.

We now consider morphisms of formal affine schemes. Let A, B be admissible rings, and $\varphi : B \to A$ be a continuous homomorphism. The continuous map ${}^a\varphi : \mathrm{Spec}(A) \to \mathrm{Spec}(B)$ then maps $\mathfrak{X} = \mathrm{Spf}(A)$ into $\mathfrak{Y} = \mathrm{Spf}(B)$, because the inverse image of an open prime ideal of A is an open prime ideal of B. On the other hand, for any $g \in B$, φ defines a continuous homomorphism $\Gamma(\mathfrak{D}(g), \mathscr{O}_\mathfrak{Y}) \to \Gamma(\mathfrak{D}(\varphi(g), \mathscr{O}_\mathfrak{X}))$ in view of Proposition 10.8.1 and Proposition 10.8.2; as these homomorphisms are compatible with restrictions and $\mathfrak{D}(\varphi(g)) = ({}^a\varphi)^{-1}(\mathfrak{D}(g))$, we obtain a continuous homomorphism of sheaves of topological rings $\mathscr{O}_\mathfrak{Y} \to {}^a\varphi_*(\mathscr{O}_\mathfrak{X})$, which we denoted by $\tilde{\varphi}$. We then get a morphism $({}^a\varphi, \tilde{\varphi}) : (X, \mathscr{O}_\mathfrak{X}) \to (\mathfrak{Y}, \mathscr{O}_\mathfrak{Y})$ of topologically ringed spaces.

Proposition 10.8.4. *Let A, B be admissible topological rings, and $\mathfrak{X} = \mathrm{Spf}(A)$, $\mathfrak{Y} = \mathrm{Spf}(B)$. For a morphism $\psi : \mathfrak{X} \to \mathfrak{Y}$ of topologically ringed spaces to be of the form $({}^a\varphi, \tilde{\varphi}) : \mathfrak{X} \to \mathfrak{Y}$, it is necessary and sufficient that for each $x \in X$, $f_x^\# : \mathscr{O}_{\mathfrak{Y},\psi(x)} \to \mathscr{O}_{\mathfrak{X},x}$ is a local homomorphism.*

Proof. This conditions is necessary: in fact, let $\mathfrak{p} = \mathfrak{p}_x \in \mathrm{Spf}(A)$, and $\mathfrak{q} = \varphi^{-1}(\mathfrak{p}_x)$; if $g \notin \mathfrak{q}$, then $\varphi(g) \notin \mathfrak{p}$, and it is immediate that the homomorphism $B_{\{g\}} \to A_{\{\varphi(g)\}}$ induced from φ maps $\mathfrak{q}_{\{g\}}$ into $\mathfrak{p}_{\{\varphi(g)\}}$; by passing to inductive limit, we then see that $\tilde{\varphi}_x$ is a local homomorphism.

Conversely, let ψ be a morphism satisfying this condition. By Proposition 10.8.1, $\psi^{\#}$ defines a continuous homomorphism

$$\varphi = \Gamma(\psi^{\#}) : B = \Gamma(\mathfrak{Y}, \mathscr{O}_{\mathfrak{Y}}) \to \Gamma(\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}) = A.$$

By the hypothesis on $\psi^{\#}$, for the section $\varphi(g)$ of $\mathscr{O}_{\mathfrak{X}}$ over $\mathfrak{X}$ has invertible germ at a point x, it is necessary and sufficient that g has invertible germ at $\psi(x)$. But by Corollary 2.6.23, the sections of $\mathscr{O}_{\mathfrak{X}}$ (resp. $\mathscr{O}_{\mathfrak{Y}}$) over $\mathfrak{X}$ (resp. $\mathfrak{Y}$) which have non-invertible germs at x (resp. $\psi(x)$) are exactly the elements of $\mathfrak{p}_x$ (resp $\mathfrak{p}_{\psi(x)}$), so we conclude that $\psi = {}^a\varphi$. Finally, for any $g \in B$, the diagram

$$\begin{tikzcd}
B = \Gamma(\mathfrak{Y}, \mathscr{O}_{\mathfrak{Y}}) \arrow[r, "\varphi"] \arrow[d] & \Gamma(\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}) = A \arrow[d] \\
B_{\{g\}} = \Gamma(\mathfrak{D}(g), \mathscr{O}_{\mathfrak{Y}}) \arrow[r, "\Gamma(\psi^{\#}_{\mathfrak{D}(g)})"] & \Gamma(\mathfrak{D}(\varphi(g), \mathscr{O}_{\mathfrak{X}})) = A_{\{\varphi(g)\}}
\end{tikzcd}$$

is commutative. By the universal property of localization of complete rings (Proposition 2.6.15), we conclude that $\psi^{\#}_{\mathfrak{D}(g)}$ is equal to $\tilde{\varphi}_{\mathfrak{D}(g)}$ for $g \in B$, so we have $\psi^{\#} = \tilde{\varphi}$. □

We say a morphism $\psi : \mathfrak{X} \to \mathfrak{Y}$ satisfying the condition in Proposition 10.8.4 is a **morphism of formal affine schemes**. Then by Proposition 10.8.4, the functor $A \mapsto \mathrm{Spf}(A)$ and $\mathfrak{X} \to \Gamma(\mathfrak{X}, \mathscr{O}_{\mathfrak{X}})$ define an equivalence from the category of admissible topological rings to the opposite of the category of formal affine schemes.

As a particular case of Proposition 10.8.4, note that for $f \in A$, the canonical injection of the formal affine scheme over $\mathfrak{D}(f)$ induced by $\mathfrak{X}$ corresponds to the canonical homomorphism $A \to A_{\{f\}}$. Under the hypothesis of Proposition 10.8.4, let h be an element of B and g be an element of A, which is a multiple of $\varphi(h)$. We then have $\psi(\mathfrak{D}(g)) \subseteq \mathfrak{D}(h)$; the restriction of ψ to $\mathfrak{D}(g)$, considered as a morphism $\mathfrak{D}(g) \to \mathfrak{D}(h)$, is the unique morphism η such that the diagram

$$\begin{tikzcd}
\mathfrak{D}(g) \arrow[r, "\eta"] \arrow[d] & \mathfrak{D}(h) \arrow[d] \\
\mathfrak{X} \arrow[r, "\psi"] & \mathfrak{Y}
\end{tikzcd}$$

This morphism corresponds to the unique continuous homomorphism $\tilde{\varphi} : B_{\{h\}} \to A_{\{g\}}$ such that the diagram

$$\begin{tikzcd}
B \arrow[r, "\varphi"] \arrow[d] & A \arrow[d] \\
B_{\{h\}} \arrow[r, "\tilde{\varphi}"] & A_{\{g\}}
\end{tikzcd}$$

is commutative.

Let A be an admissible ring, $\mathfrak{I}$ be an open ideal of A, and $\mathfrak{X}$ be the formal affine scheme $\mathrm{Spf}(A)$. Let $(\mathfrak{I}_\lambda)$ be the set of nilideals of A contained in $\mathfrak{I}$; then $\widetilde{\mathfrak{I}}/\widetilde{\mathfrak{I}}_\lambda$ is a sheaf of ideals of $\widetilde{A}/\widetilde{\mathfrak{I}}_\lambda$. We denote by $\mathfrak{I}^\Delta$ the projective limit of the sheaves induced by $\widetilde{\mathfrak{I}}/\widetilde{\mathfrak{I}}_\lambda$ over $\mathfrak{X}$, which is considered as an ideal of $\mathscr{O}_{\mathfrak{X}}$. For any $f \in A$, $\Gamma(\mathfrak{D}(f), \mathfrak{I}^\Delta)$ is the projective limit of $\mathfrak{I}_f/(\mathscr{I}_\lambda)_f$, which is identified with the open ideal $\mathfrak{I}_{\{f\}}$ of the ring $A_{\{f\}}$, and in particular $\Gamma(\mathfrak{X}, \mathfrak{I}^\Delta) = \mathfrak{I}$. We then conclude that (the $\mathfrak{D}(f)$ form a base of $\mathfrak{X}$) that we have

$$\mathfrak{I}^\Delta|_{\mathfrak{D}(f)} = (\mathfrak{I}_{\{f\}})^\Delta \tag{10.8.1}$$

With these notations, for $f \in A$ the canonical map of $A_{\{f\}} = \Gamma(\mathfrak{D}, \mathscr{O}_{\mathfrak{X}})$ in $\Gamma(\mathfrak{D}(f), (\widetilde{A}/\widetilde{\mathfrak{J}})|_{\mathfrak{X}}) = A_f/\mathfrak{J}_f$ is surjective with kernel $\Gamma(\mathfrak{D}(f), \mathfrak{J}^{\Delta}) = \mathfrak{J}_{\{f\}}$. These maps define a canonical continuous epimorphism from the sheaf $\mathscr{O}_{\mathfrak{X}}$ to the sheaf of discrete rings $(\widetilde{A}/\widetilde{\mathfrak{J}})|_{\mathfrak{X}}$, whose kernel is $\mathfrak{J}^{\Delta}$; this homomorphism is none other than the homomorphism $\tilde{\varphi}$, where φ is the canonical continuous homomorphism $A \to A/\mathfrak{J}$. The morphism $({}^a\varphi, \tilde{\varphi}) : \mathrm{Spec}(A/\mathfrak{J}) \to \mathfrak{X}$ of the formal affine schemes is then called the canonical morphism. We then have a canonical isomorphism

$$\mathscr{O}_{\mathfrak{X}}/\mathfrak{J}^{\Delta} \xrightarrow{\sim} (\widetilde{A}/\widetilde{\mathfrak{J}})|_{\mathfrak{X}}. \tag{10.8.2}$$

It is clear (in view of $\Gamma(X, \mathfrak{J}^{\Delta}) = \mathfrak{J}$) that the map $\mathfrak{J} \mapsto \mathfrak{J}^{\Delta}$ is strictly increasing: in fact, for $\mathfrak{J} \subseteq \mathfrak{J}'$, the sheaf $\mathfrak{J}'^{\Delta}/\mathfrak{J}^{\Delta}$ is canonically isomorphic to $\widetilde{\mathfrak{J}}'/\widetilde{\mathfrak{J}} = \widetilde{\mathfrak{J}'/\mathfrak{J}}$.

An ideal $\mathscr{I}$ of $\mathscr{O}_{\mathfrak{X}}$ is called a **nilideal** of $\mathfrak{X}$ if, for any $x \in \mathfrak{X}$, there exists an open neighborhood of x in $\mathfrak{X}$ of the form $\mathfrak{D}(f)$, where $f \in A$, such that $\mathscr{I}|_{\mathfrak{D}(f)}$ is of the form $\mathfrak{J}^{\Delta}$ for a nilideal $\mathfrak{J}$ of $A_{\{f\}}$. It is clear from our definiton that for any $f \in A$, any nilideal of $\mathfrak{X}$ induces a nilideal of $\mathfrak{D}(f)$.

Proposition 10.8.5. *If A is an admissible ring, any nilideal of $\mathfrak{X} = \mathrm{Spf}(A)$ is of the form $\mathfrak{J}^{\Delta}$, where $\mathfrak{J}$ is a nilideal of A.*

Proof. Let $\mathscr{I}$ be a nilideal of $\mathfrak{X}$; by hypothesis, and since $\mathfrak{X}$ is quasi-compact, there exist finitely many elements $f_i \in A$ such that $\mathfrak{D}(f)$ cover $\mathfrak{X}$ and such that $\mathscr{I}|_{\mathfrak{D}(f)} = \mathfrak{K}_i$, where $\mathfrak{K}_i$ is a nilideal of $A_{\{f_i\}}$. For any i, there then exists an open ideal $\mathfrak{N}_i$ of A such that $(\mathfrak{N}_i)_{\{f_i\}} = \mathfrak{K}_i$; let $\mathfrak{N}$ be a nilideal of A contained in each the $\mathfrak{N}_i$. The canonical image of $\mathscr{I}/\mathfrak{N}^{\Delta}$ in the structural sheaf $\widetilde{(A/\mathfrak{N})}$ of $\mathrm{Spec}(A/\mathfrak{N})$ is then such that its restriction to each $\mathfrak{D}(f_i)$ is equal to $\widetilde{\mathfrak{N}_i/\mathfrak{N}}$; we then conclude that this canonical image is a quasi-coherent ideal over $\mathrm{Spec}(A/\mathfrak{N})$, hence is of the form $\widetilde{\mathfrak{J}/\mathfrak{N}}$, where $\mathfrak{J}$ is an ideal of A containing $\mathfrak{N}$, and whence $\mathscr{I} = \mathfrak{J}^{\Delta}$ by (10.8.2). Moreover, as for each i there exists an integer n_i such that $\mathfrak{K}_i^{n_i} \subseteq \mathfrak{N}_{\{f_i\}}$, we have $(\mathscr{I}/\mathfrak{N}^{\Delta})^n = 0$ for n sufficiently large, and therefore $(\widetilde{\mathfrak{J}/\mathfrak{N}})^n = 0$, and finally $(\mathfrak{J}/\mathfrak{N})^n = 0$, which proves that $\mathfrak{J}$ is a nilideal of A. □

Proposition 10.8.6. *Let A be an adic ring, $\mathfrak{J}$ be a nilideal of A such that $\mathfrak{J}/\mathfrak{J}^2$ is an $A/\mathfrak{J}$ of finte type. For any integer $n > 0$, we then have $(\mathfrak{J}^{\Delta})^n = (\mathfrak{J}^n)^{\Delta}$.*

Proof. In fact, for any $f \in A$ we have (since $\mathfrak{J}^n$ is an open ideal)

$$(\Gamma(\mathfrak{D}(f), \mathfrak{J}^{\Delta}))^n = (\mathfrak{J}_{\{f\}})^n = (\mathfrak{J}^n)_{\{f\}} = \Gamma(\mathfrak{D}(f^n), (\mathfrak{J}^n)^{\Delta})$$

in view of (10.8.1) and Corollary 2.6.18. As $(\mathfrak{J}^{\Delta})^n$ is associated with the presheaf $U \mapsto (\Gamma(U, \mathfrak{J}^{\Delta}))^n$, the corollary then follows since $\mathfrak{D}(f)$ form a basis for $\mathfrak{X}$. □

A family $(\mathscr{I}_\lambda)$ of nilideals of $\mathfrak{X}$ is called a **fundamental system of nilideals** if any nilideal of $\mathscr{I}$ contains at least one of these $\mathscr{I}_\lambda$. As $\mathscr{I}_\lambda = \mathfrak{J}_\lambda^{\Delta}$, this is equivalent to saying that the $\mathfrak{J}_\lambda$ form a fundamental neighborhood of 0 in A, where $\mathscr{I}_\lambda = \mathscr{I}_\lambda^{\Delta}$. Let (f_α) be a family of elements of A such that the $\mathfrak{D}(f_\alpha)$ cover $\mathfrak{X}$. If $(\mathscr{I}_\lambda)$ is a filtered decreasing family of ideals of $\mathscr{O}_{\mathfrak{X}}$ such that for any α, the family $(\mathscr{I}_\lambda|_{\mathfrak{D}(f_\alpha)})$ is a fundamental system of nilideals of $\mathfrak{D}(f_\alpha)$, then $(\mathscr{I}_\lambda)$ is a fundamental system of nilideals of $\mathfrak{X}$. In fact, for any nilideal of $\mathfrak{X}$, there exists a finite covering of $\mathfrak{X}$ by the $\mathfrak{D}(f_i)$ such that, for any i, $\mathscr{I}_{\lambda_i}|_{\mathfrak{D}(f_i)}$ is a nilideal of $\mathfrak{D}(f_i)$ contained in $\mathfrak{J}|_{\mathfrak{D}(f_i)}$. If μ is an index such that $\mathscr{I}_\mu \subseteq \mathscr{I}_{\lambda_i}$ for all i, then $\mathscr{I}_\mu$ is a nilideal of $\mathfrak{X}$ which is evidently contained in $\mathscr{I}$, whence the assertion.

10.8.2 Formal schemes and morphisms

Given a topologically ringed space $\mathfrak{X}$, we say an open subset $U \subseteq \mathfrak{X}$ is a **formal affine open** (resp. **an adic formal affine open**, resp. **a Noetherian formal affine open**) if the topologically ringed space $(U, \mathscr{O}_{\mathfrak{X}}|_U)$ is a formal affine scheme (resp. an adic formal affine scheme, resp. a Noetherian formal affine scheme). We way $(\mathfrak{X}, \mathscr{O}_{\mathfrak{X}})$ is a **formal scheme** (resp. **adic formal scheme**, resp. **locally Noetherian formal scheme**) if each of its point admits a formal affine open neighborhood (resp. an adic formal affine open, resp. a locally Noetherian formal affine open). We say that $\mathfrak{X}$ is **Noetherian** if it is locally Noetherian and the underlying space is quasi-compact (hence Noetherian). As any affine scheme can be considered as a formal affine scheme, any scheme can be considered as a formal scheme.

Proposition 10.8.7. *If $\mathfrak{X}$ is a formal scheme (resp. locally Noetherian formal scheme), the set of formal affine opens (resp. Noetherian formal affine opens) form a base for $\mathfrak{X}$.*

Proof. This follows from Proposition 10.8.2, and the fact that if A is a Noetherian adic ring, so is $A_{\{f\}}$ for any $f \in A$ (Proposition 2.6.10). □

Corollary 10.8.8. *If $\mathfrak{X}$ is a formal scheme (resp. locally Noetherian formal scheme, resp. Noetherian formal scheme), the topological ringed space over any open subset of $\mathfrak{X}$ is a formal scheme (resp. locally Noetherian formal scheme, resp. Noetherian formal scheme).*

Given two formal schemes $\mathfrak{X}, \mathfrak{Y}$, we say that $f : \mathfrak{X} \to \mathfrak{Y}$ is a morphism of formal schemes if it is a morphism of the underlying locally ringed spaces. That is, if $(f, f^\#)$ is a morphism of ringed spaces and $f_x^\# : \mathscr{O}_{\mathfrak{Y},f(x)} \to \mathscr{O}_{\mathfrak{X},x}$ is a local homomorphism. The composition of two morphisms are defined as the same and clearly a morphism of formal schemes. The formal schemes then form a category, which we denoted by **Schf**, and we denote by $\mathrm{Hom}_{\mathsf{Schf}}(\mathfrak{X}, \mathfrak{Y})$ the set of morphisms of formal schemes $\mathfrak{X} \to \mathfrak{Y}$.

If U is an open subset of $\mathfrak{X}$, the canonical injection $U \to \mathfrak{X}$ is then a morphism of formal schemes, if we endow U the formal scheme structure induced by $\mathfrak{X}$. It is clear that this morphism is a monomorphism in the category **Schf**.

Proposition 10.8.9. *Let $\mathfrak{X}$ be a formal scheme, $\mathfrak{Y} = \mathrm{Spec}(A)$ be a formal affine scheme. Then there exists a canonical bijection*

$$\mathrm{Hom}_{\mathbf{Scf}}(\mathfrak{X}, \mathfrak{Y}) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{TopRing}}(A, \Gamma(\mathfrak{X}, \mathscr{O}_{\mathfrak{X}})).$$

Proof. We first note that, if $(\mathfrak{X}, \mathscr{O}_{\mathfrak{X}})$ and $(\mathfrak{Y}, \mathscr{O}_{\mathfrak{Y}})$ are two topologically ringed spaces, a morphism $f : (\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}) \to (\mathfrak{Y}, \mathscr{O}_{\mathfrak{Y}})$ defines canonically a continuous homomorphism of rings $\varphi : \Gamma(\mathfrak{Y}, \mathscr{O}_{\mathfrak{Y}}) \to \Gamma(\mathfrak{X}, \mathscr{O}_{\mathfrak{X}})$. In our case, we need to show that a continuous hommorphism $\varphi : A \to \Gamma(\mathfrak{X}, \mathscr{O}_{\mathfrak{X}})$ is of the form $\Gamma(\psi^\#)$ for a unique morphism $(\psi, \psi^\#)$. Now there exists by hypothesis a covering (V_α) of $\mathfrak{X}$ by formal affine opens; by composing φ with the restriction homomorphisms $\Gamma(\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}) \to \Gamma(V_\alpha, \mathscr{O}_{\mathfrak{X}}|_{V_\alpha})$, we obtain a continuous homomorphism $\varphi_\alpha : A \to \Gamma(V_\alpha, \mathscr{O}_{\mathfrak{X}}|_{V_\alpha})$, which corresponds to a unique morphism $\psi_\alpha : (V_\alpha, \mathscr{O}_{\mathfrak{X}}|_{V_\alpha}) \to (\mathfrak{Y}, \mathscr{O}_{\mathfrak{Y}})$, in view of Proposition 10.8.4. Moreover, for any couple (α, β) of indices, any point of $V_\alpha \cap V_\beta$ admits a formal affine open neighborhood W contained in $V_\alpha \cap V_\beta$ and it is clear that the compositions of φ_α and φ_β with the canonical restriction are the same continuous homomorphism $\Gamma(\mathfrak{Y}, \mathscr{O}_{\mathfrak{Y}}) \to \Gamma(W, \mathscr{O}_{\mathfrak{X}}|_W)$, so, in view of the relations $(\psi_\alpha^\#)_x = (\tilde{\varphi}_\alpha)_x$ for any $x \in V_\alpha$, the restrictions of ψ_α and ψ_β coincides on $V_\alpha \cap V_\beta$. We then conclude that there exists a unique morphism $\psi : (\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}) \to (\mathfrak{Y}, \mathscr{O}_{\mathfrak{Y}})$ whose restriction to V_α coincides with ψ_α, and it is clear that this is the unique morphism such that $\Gamma(\psi^\#) = \varphi$. □

Given a formal scheme $\mathfrak{S}$, a **formal $\mathfrak{S}$-scheme** is defined to be a formal scheme $\mathfrak{X}$ together with a morphism $\varphi : \mathfrak{X} \to \mathfrak{S}$, called the **structural morphism** of $\mathfrak{X}$. If $\mathfrak{S} = \mathrm{Spf}(A)$, where A is an admissible ring, we also say that the $\mathfrak{S}$-formal scheme $\mathfrak{X}$ is a formal A-scheme or a formal scheme over A. Any formal scheme can be clearly considered as a formal scheme over $\mathbb{Z}$ (endowed with the discrete topology).

If $\mathfrak{X}, \mathfrak{Y}$ are two formal $\mathfrak{S}$-schemes, we say a morphism $f : \mathfrak{X} \to \mathfrak{Y}$ is an **$\mathfrak{S}$-morphism** if the diagram

$$\begin{tikzcd} \mathfrak{X} \arrow[rr, "f"] \arrow[dr] & & \mathfrak{Y} \arrow[dl] \\ & \mathfrak{S} & \end{tikzcd}$$

where the vertical arrows are structural morphisms, is commutative. With this definition, the $\mathfrak{S}$-schemes form (for $\mathfrak{S}$ fixed) a category $\mathbf{Scf}_{\mathfrak{S}}$. We denote by $\mathrm{Hom}_{\mathfrak{S}}(\mathfrak{X}, \mathfrak{Y})$ the set of $\mathfrak{S}$-morphisms $\mathfrak{X} \to \mathfrak{Y}$. If $\mathfrak{S} = \mathrm{Spf}(A)$, we also say A-morphism for $\mathfrak{S}$-morphisms.

Let $\mathfrak{X}$ be a formal scheme; we say an ideal $\mathscr{I}$ of $\mathscr{O}_{\mathfrak{X}}$ is a nilideal of $\mathfrak{X}$ if any $x \in \mathfrak{X}$ admits a formal affine open neighborhood U such that $\mathscr{I}|_U$ is a nilideal of the formal scheme U induced by $\mathfrak{X}$. In view of Proposition 10.8.7, for any open $V \subseteq \mathfrak{X}$, $\mathscr{I}|_V$ is then a nilideal of the formal scheme induced over V.

A family $(\mathscr{I}_\lambda)$ of nilideals of $\mathfrak{X}$ is called a **fundamental system of nilideals** if there exists a covering (U_α) of $\mathfrak{X}$ by formal affine opens such that, for any α, the family $(\mathscr{I}_\lambda|_{U_\alpha})$ form a fundamental system of nilideals of U_α. For any open subset V of $\mathfrak{X}$, the family $(\mathscr{I}_\lambda|_V)$ then forms a fundamental system of nilideals for V, in view of (10.8.1). If $\mathfrak{X}$ is locally Noetherian, and $\mathscr{I}$ is a nilideal of $\mathfrak{X}$, it then follows from Proposition 10.8.6 that the powers of $\mathscr{I}^n$ form a fundamental system of nilideals of $\mathfrak{X}$.

Let $\mathfrak{X}$ be a formal scheme, $\mathscr{I}$ be a nilideal of $\mathfrak{X}$. Then the ringed space $(\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}/\mathscr{I})$ is a scheme, which is affine (resp. locally Noetherian, resp. Noetherian) if $\mathfrak{X}$ is a formal affine scheme (resp. a locally Noetherian formal scheme, resp. a Noetherian formal scheme). Moreover, if $\varphi : \mathscr{O}_{\mathfrak{X}} \to \mathscr{O}_{\mathfrak{X}}/\mathscr{I}$ is the cannical homomorphism, then $(1_{\mathfrak{X}}, \varphi)$ is a morphism (called canonical) of formal schemes $(\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}/\mathscr{I}) \to (\mathfrak{X}, \mathscr{O}_{\mathfrak{X}})$.

Proposition 10.8.10. *Let $\mathfrak{X}$ be a formal scheme, $(\mathscr{I}_\lambda)$ be a fundamental system of nilideals of $\mathfrak{X}$. Then the sheaf $\mathscr{O}_{\mathfrak{X}}$ is the projective limit of the sheaf of discrete rings $\mathscr{O}_{\mathfrak{X}}/\mathscr{I}_\lambda$.*

Proof. As $\mathfrak{X}$ admits a basis by quasi-compact open sets, we are reduced to the affine case, where the proposition follows from Proposition 10.8.5 and the definition of $\mathscr{O}_{\mathfrak{X}}$. □

Proposition 10.8.11. *Let $\mathfrak{X}$ be a locally Noetherian formal scheme. Then there exists a largest nilideal $\mathscr{T}$ of $\mathfrak{X}$, which is the unique nilideal $\mathscr{I}$ such that the scheme $(\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}/\mathscr{I})$ is reduced. If $\mathscr{I}$ is a nilideal of $\mathfrak{X}$, then $\mathscr{T}$ is the inverse image of the nilradical of $\mathscr{O}_{\mathfrak{X}}/\mathscr{I}$ under the homomorphism $\mathscr{O}_{\mathfrak{X}} \to \mathscr{O}_{\mathfrak{X}}/\mathscr{I}$. The reduced (usual) scheme $(\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}/\mathscr{T})$ is denoted by $\mathfrak{X}_{\mathrm{red}}$.*

Proof. Suppose first that $\mathfrak{X} = \mathrm{Spf}(A)$, where A is a Noetherian adic ring. The existence of $\mathscr{T}$ and its properties then follows from Proposition 10.8.5, in view of Corollary 2.6.4 about the largest nilideal of A. To prove the existence of $\mathscr{T}$ in the general case, it suffices to prove that if $V \subseteq U$ are two Noetherian formal affine opens of X, the largest nilideal $\mathscr{T}_U$ of U induces the largest nilideal $\mathscr{T}_V$ of V; but as $\Gamma(V, (\mathscr{O}_{\mathfrak{X}}|_V)/(\mathscr{T}_U|_V))$ is reduced, this is immediate. □

Corollary 10.8.12. *Let $\mathfrak{X}$ be a locally Noetherian formal scheme, $\mathscr{T}$ be the largest nilideal of $\mathfrak{X}$. Then for any open subset V of $\mathfrak{X}$, $\mathscr{T}|_V$ is the largest nilideal of V.*

Proof. This is already shown in the proof of Proposition 10.8.11. □

Proposition 10.8.13. *Let $\mathfrak{X}, \mathfrak{Y}$ be formal schemes, $\mathscr{I}$ (resp. $\mathscr{K}$) be the nilideal of $\mathfrak{X}$ (resp. $\mathfrak{Y}$), $f : \mathfrak{X} \to \mathfrak{Y}$ be a morphism of formal schemes.*

(i) *If $f^*(\mathscr{K})\mathscr{O}_{\mathfrak{X}} \subseteq \mathscr{I}$, there exists a unique morphism*

$$f_{\mathrm{red}} : (\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}/\mathscr{I}) \to (\mathfrak{Y}, \mathscr{O}_{\mathfrak{Y}}/\mathscr{K})$$

of schemes such that the following diagram

$$\begin{tikzcd} (\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}/\mathscr{I}) \arrow[r, "f_{\mathrm{red}}"] \arrow[d] & (\mathfrak{Y}, \mathscr{O}_{\mathfrak{Y}}/\mathscr{K}) \arrow[d] \\ (\mathfrak{X}, \mathscr{O}_X) \arrow[r, "f"] & (\mathfrak{Y}, \mathscr{O}_{\mathfrak{Y}}) \end{tikzcd} \tag{10.8.3}$$

(ii) *Suppose that $\mathfrak{X} = \mathrm{Spf}(A)$, $\mathfrak{Y} = \mathrm{Spf}(B)$ are formal affine schemes, $\mathscr{I} = \mathfrak{I}^\Delta$, $\mathscr{K} = \mathfrak{K}^\Delta$, where $\mathfrak{I}$ (resp. $\mathfrak{K}$) is a nilideal of A (reps. B), and $f = ({}^a\varphi, \tilde{\varphi})$ where $\varphi : B \to A$ is a continuous homomorphism. For $f^*(\mathscr{K})\mathscr{O}_{\mathfrak{X}} \subseteq \mathscr{K}$, it is necessary and sufficient that $\varphi(\mathfrak{K}) \subseteq \mathfrak{I}$, and f_{red} is then the morphism $({}^a\varphi', \tilde{\varphi}')$, where $\varphi' : B/\mathfrak{K} \to A/\mathfrak{I}$ is the induced homomorphism by passing to quotients.*

Proof. In case (a), the hypothesis implies that the image of the ideal $f^{-1}(\mathscr{K})$ of $f^{-1}(\mathscr{O}_{\mathfrak{Y}})$ under $f^\# : f^{-1}(\mathscr{O}_{\mathfrak{Y}}) \to \mathscr{O}_{\mathfrak{X}}$ is contained in $\mathscr{I}$. By passing to quotients, we then deduce that $f^\#$ is a homomorphism

$$\omega : f^{-1}(\mathscr{O}_{\mathfrak{Y}}/\mathscr{K}) = f^{-1}(\mathscr{O}_{\mathfrak{Y}})/f^{-1}(\mathscr{K}) \to \mathscr{O}_{\mathfrak{X}}/\mathscr{I};$$

moreover, as for any $x \in \mathfrak{X}$, $f_x^\#$ is a local homomorphism, so is ω_x. The morphism $(f, \omega^\flat)$ is then the unique morphism of ringed spaces satisfying the requirements.

With the assumptions of (b), the canonical correspondence between morphisms of formal affine schemes and continuous homomorphisms of ringed spaces shows that the relation $f^*(\mathscr{K})\mathscr{O}_{\mathfrak{X}} \subseteq \mathscr{I}$ implies that $f' = ({}^a\varphi', \tilde{\varphi}')$, where $\varphi' : B/\mathfrak{K} \to A/\mathfrak{I}$ is the unique homomorphism such that the diagram

$$\begin{tikzcd} B \arrow[r, "\varphi"] \arrow[d] & A \arrow[d] \\ B/\mathfrak{K} \arrow[r, "\varphi'"] & A/\mathfrak{I} \end{tikzcd} \tag{10.8.4}$$

The existence of φ' implies then that $\varphi(\mathfrak{K}) \subseteq \mathfrak{I}$. Conversely, if this condition is verified, we have a canonical homomorphism $\varphi' : B/\mathfrak{K} \to A/\mathfrak{I}$, whence the induced morphism $f' = ({}^a\varphi', \tilde{\varphi}')$ satisfies the commutativity of (10.8.4). By considering the homomorphisms ${}^a\varphi^*(\mathscr{O}_{\mathfrak{Y}}) \to \mathscr{O}_{\mathfrak{X}}$ and ${}^a\varphi'^*(\mathscr{O}_{\mathfrak{Y}}/\mathscr{K}) \to \mathscr{O}_{\mathfrak{X}}/\mathscr{I}$, we then see that $f^*(\mathscr{K})\mathscr{O}_{\mathfrak{X}} \subseteq \mathscr{I}$. □

To conclude this paragraph, we discuss the fiber product of formal schemes. Let $\mathfrak{S}$ be a formal scheme. It turns out that the fiber product of formal schemes has the same construction as that of usual schemes, provided that we replace tensor products by completed tensor products.

Proposition 10.8.14. *Let $\mathfrak{X} = \mathrm{Spf}(B)$, $\mathfrak{Y} = \mathrm{Spf}(C)$ be two formal affine schemes over a formal affine scheme $\mathrm{Spf}(A)$. Let $\mathfrak{Z} = \mathrm{Spf}(B\widehat{\otimes}_A C)$ and p_1, p_2 be the $\mathfrak{S}$-morphisms corresponding to the A-homomorphisms $\rho_1 : B \to B\widehat{\otimes}_A C$ and $\rho_2 : C \to B\widehat{\otimes}_A C$. Then $(\mathfrak{Z}, p_1, p_2)$ forms a product in the category of the formal $\mathfrak{S}$-schemes $\mathfrak{X}$ and $\mathfrak{Y}$.*

Proof. In view of Proposition 10.8.4, we note that for any continuous A-homomorphism $\varphi : B\widehat{\otimes}_A C \to D$, where D is an admissible ring that is a topological A-algebra, we can associate the couple $(\varphi \circ \sigma_1, \varphi \circ \sigma_2)$, so that we define a bijection

$$\mathrm{Hom}_A(B\widehat{\otimes}_A C, D) \xrightarrow{\sim} \mathrm{Hom}_A(B, D) \times \mathrm{Hom}_A(C, D)$$

which follows from the universal property of the complete tensor product. □

Proposition 10.8.15. *For any formal $\mathfrak{S}$-schemes $\mathfrak{X}, \mathfrak{Y}$, the product $\mathfrak{X} \times_{\mathfrak{S}} \mathfrak{Y}$ exists.*

Proof. The demonstration is similar as the case for usual schemes, where we replace affine schemes by formal affine schemes and use Proposition 10.8.14. $\square$

10.8.3 Inductive limits of schemes

Let $\mathfrak{X}$ be a formal scheme, $(\mathscr{I}_\lambda)$ be a fundamental system of nilideals of $\mathfrak{X}$; for each λ, let $f_\lambda : (\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}/\mathscr{I}_\lambda) \to \mathfrak{X}$. For $\mathscr{I}_\mu \subseteq \mathscr{I}_\lambda$, the canoincal homomorphism $\mathscr{O}_{\mathfrak{X}}/\mathscr{I}_\mu \to \mathscr{O}_{\mathfrak{X}}/\mathscr{I}_\lambda$ defines a canonical morphism

$$f_{\mu\lambda} : (\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}/\mathscr{I}_\lambda) \to (\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}/\mathscr{I}_\mu)$$

of (usual) schemes such that we have $f_\lambda = f_\mu \circ f_{\mu\lambda}$. The scheme $X_\lambda = (\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}/\mathscr{I}_\lambda)$ is the morphisms $f_{\mu\lambda}$ then constitutes an inductive system in the category of formal schemes.

Proposition 10.8.16. *The formal scheme and the morphisms f_λ constitute an inductive limit of the system $(X_\lambda, f_{\mu\lambda})$ in the cateogry of formal schemes.*

Proof. Let $\mathfrak{Y}$ be a formal scheme, and for each index λ, let

$$g_\lambda : X_\lambda \to \mathfrak{Y}$$

be a morphism such that $g_\lambda = g_\mu \circ f_{\mu\lambda}$ for $\mathscr{I}_\mu \subseteq \mathscr{I}_\lambda$. This last condition and the definition of X_λ then imply that the g_λ are equal to the same continuous map $g : \mathfrak{X} \to \mathfrak{Y}$ on the underlying space; moreover, the homomorphisms $g_\lambda^\# : g^{-1}(\mathscr{O}_{\mathfrak{Y}}) \to \mathscr{O}_{X_\lambda} = \mathscr{O}_{\mathfrak{X}}/\mathscr{I}_\lambda$ form a projective system of homomorphisms of sheaves of rings. By passing to projective limit, we then deduce a homomorphism $\omega : g^{-1}(\mathscr{O}_{\mathfrak{Y}}) \to \lim \mathscr{O}_{\mathfrak{X}}/\mathscr{I}_\lambda = \mathscr{O}_{\mathfrak{X}}$, and it is clear that (g, ω) is a morphism of ringed spaces such that the diagram

$$\begin{tikzcd} X_\lambda \arrow[rr, "g_\lambda"] \arrow[dr, "f_\lambda"'] & & \mathfrak{Y} \\ & \mathfrak{X} \arrow[ur, "g"'] & \end{tikzcd} \tag{10.8.5}$$

It remains to prove that g is a morphism of formal schemes; the question is local on $\mathfrak{X}$ and $\mathfrak{Y}$, we can assume that $\mathfrak{X} = \mathrm{Spf}(A)$, $\mathfrak{Y} = \mathrm{Spf}(B)$, where A, B are admissible rings, and $\mathscr{I}_\lambda = \mathfrak{I}^\Delta$ where $(\mathfrak{I}_\lambda)$ is a fundamental system of nilideals of A (Proposition 10.8.5). As $A = \varprojlim A/\mathfrak{I}_\lambda$, the existence of the morphism of formal affine schemes g fitting into the diagram (10.8.5) then follows from the one-to-one correspondence Proposition 10.8.4 between morphisms of formal affine schemes and continuous homomorphisms of rings, and of the definition of the projective limit. But the uniqueness of g as a morphism of ringed spaces shows that it coincides with the morphism at the beginning of the demonstration. $\square$

Proposition 10.8.17. *Let $\mathfrak{X}$ be a topological space, $(\mathscr{O}_i, u_{ji})$ a projective system of sheaves of rings over $\mathfrak{X}$ indexed by $\mathbb{N}$. Let $\mathscr{I}_i$ be the kernel of $u_{0,i} : \mathscr{O}_i \to \mathscr{O}_0$ and suppose that*

(a) *For each i, the ringed space $X_i = (\mathfrak{X}, \mathscr{O}_i)$ is a scheme.*

(b) *For any $x \in \mathfrak{X}$ and any $i \in \mathbb{N}$, there exists an open neighborhood U_i of x in $\mathfrak{X}$ such that the restriction $\mathscr{I}_i|_{U_i}$ is nilpotent.*

(c) *The homomorphisms u_{ji} are surjective.*

Let $\mathscr{O}_{\mathfrak{X}}$ be the sheaf of topological rings which is the projective limit of the sheaf of discrete rings $\mathscr{O}_i$, and $u_i : \mathscr{O}_{\mathfrak{X}} \to \mathscr{O}_i$ be the canonical homomorphism. Then the topologically ringed space $(\mathfrak{X}, \mathscr{O}_{\mathfrak{X}})$ is a formal scheme and the homomorphisms u_i are surjective. If $\mathscr{I}^{(i+1)}$ is the kernel of u_i, then $(\mathscr{I}^{(i)})$ form a fundamental system of nilideals of $\mathfrak{X}$, and $\mathscr{I}^{(1)}$ is the projective limit of the sheaf of ideals $\mathscr{I}_i$.

Proof. We first note that at each stalk, u_{ji} is a surjective hommorphism and a fortiori a local homomorphism, so $v_{ij} = (1_{\mathfrak{X}}, u_{ji}) : X_j \to X_i$ is a morphism of schemes for $i \geq j$. Suppose that each X_i is an affine scheme with ring A_i. Then there exists a homomorphism $\varphi_{ji} : A_i \to A_j$ such that $u_{ji} = \tilde{\varphi}_{ji}$, so the sheaf $\mathscr{O}_j$ is a quasi-coherent $\mathscr{O}_i$-module over X_i, associated with A_j considered as an A_i-module via φ_{ji}. For each $f \in A_i$, let $f' = \varphi_{ji}(f)$; by hypothesis, the opens $D(f)$ and $D(f')$ are identical over $\mathfrak{X}$, and the homomorphism from $\Gamma(D(f), \mathscr{O}_i) = (A_i)_f$ to $\Gamma(D(f), \mathscr{O}_j) = (A_j)_{f'}$ corresponding to u_{ji} is none other than $(A_j)_{f'}$. But if we consider A_j as an A_i-module, $(A_j)_{f'}$ is the $(A_j)_f$-module $(A_j)_f$, so we have $u_{ji} = \tilde{\varphi}_{ji}$, if φ_{ji} is considered as a homomorphism of A_i-modules. Then, as u_{ji} is surjective, so is the φ_{ji} and if ji is the kernel of φ_{ji}, the kernel of u_{ji} is the quasi-coherent $\mathscr{O}_i$-module $\widetilde{\mathfrak{I}}_{ji}$. In particular, we have $\mathscr{I}_i = \widetilde{\mathfrak{I}}_i$, where $\mathfrak{I}_i$ is the kernel of $\varphi_{0,i} : A_i \to A_0$. The hypothesis (b) implies that $\mathscr{I}_i$ is nilpotent: in fact, as $\mathfrak{X}$ is quasi-compact, we can cover X by finitely many opens U_k such that $(\mathscr{I}_i|_{U_k})^{n_k} = 0$ and by choosing n to be the largest n_k, we have $\mathscr{I}_i^n = 0$; we then conclude that each $\mathfrak{I}_i$ is nilpotent. Then the ring $A = \varprojlim A_i$ is admissible by Proposition 2.6.7, the canonical homomorphisms $\varphi_i : A \to A_i$ is surjective and its kernel $\mathfrak{I}^{(i+1)}$ is equal to the projective limit of the $\mathfrak{I}_{ik}$ for $k \geq i$; the $\mathfrak{I}^{(i+1)}$ form a fundamental system of neighborhoods of 0 in A. The assertion of the proposition then follows from the fact that $(\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}) = \mathrm{Spf}(A)$. We also note that if $f = (f_i)$ is an element in the projective limit $A = \varprojlim A_i$, the open subsets $D(f_i)$ (affine open in X_i) is then identified with $\mathfrak{D}(f)$, and the scheme induced by X_i over $\mathfrak{D}(f)$ is then identified with the affine scheme $\mathrm{Spec}((A_i)_{f_i})$.

In the general case, we remark that for any quasi-compact open U of $\mathfrak{X}$, the $\mathscr{I}_i|_U$ is nilpotent as we have seen. We claim that for any $x \in \mathfrak{X}$, there is an open neighborhood U of x in $\mathfrak{X}$ which is an affine open for any X_i. In fact, let U be an affine open for X_0, and observe that $\mathscr{O}_{X_0} = \mathscr{O}_X/\mathscr{I}_i$. As $\mathscr{I}_i|_U$ is nilpotent in view of the preceding arguments, U is also an affine open for X_i in view of Example 10.4.6. Now for any U satisfying this property, it follows from the same arguments that $(U, \mathscr{O}_{\mathfrak{X}}|_U)$ is a formal scheme such that the $\mathscr{I}^{(i)}|_U$ form a fundamental system of nilideals and $\mathscr{I}^{(1)}|_U$ is the projective limit of $\mathscr{I}_i|_U$, whence the conclusion. □

Corollary 10.8.18. *Suppose that for $i \geq j$, the kernel of u_{ji} is $\mathscr{I}_i^{j+1}$ and that $\mathscr{I}_1/\mathscr{I}_1^2$ is of finite type over $\mathscr{O}_0 = \mathscr{O}_1/\mathscr{I}_1$. Then $\mathfrak{X}$ is an adic formal scheme, and we have $\mathscr{I}^{(i)} = \mathscr{I}^{i+1}$ and $\mathscr{I}/\mathscr{I}^2$ is isomorphic to $\mathscr{I}_1$, where we put $\mathscr{I} = \mathscr{I}^{(1)}$. If moreover X_0 is locally Noetherian (resp. Noetherian), then $\mathfrak{X}$ is locally Noetherian (resp. Noetherian).*

Proof. As the underlying space of $\mathfrak{X}$ and X_0 are the same, the question is local and we can assume that each X_i is affine. In view of the relations $\mathscr{I}_{ji} = \widetilde{\mathfrak{I}}_{ji}$ (With the notations of Proposition 10.8.17), we are then reduced to the case of Proposition 2.3.42, and note that $\mathfrak{I}_1/\mathfrak{I}_1^2$ is then a finitely generated A_0-module (Corollary 10.1.24). □

In particular, any locally Noetherian formal scheme $\mathfrak{X}$ is the inductive limit of a sequence (X_n) of locally Noetherian (usual) schemes verifying the conditions of Proposition 10.8.17 and Corollary 10.8.18: it suffices to consider a nilideal $\mathscr{I}$ of $\mathfrak{X}$ (Proposition 10.8.11) and put $X_n = (\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}/\mathscr{I}^{n+1})$ (Proposition 10.8.16). More generally, the same is true if $\mathfrak{X}$ is an adic formal scheme having a nilideal $\mathscr{I}$ such that $\mathscr{I}/\mathscr{I}^2$ is a finitely generated $(\mathscr{O}_{\mathfrak{X}}/\mathscr{I})$-module.

Corollary 10.8.19. *Let A be an admissible ring. For the formal affine scheme $\mathfrak{X} = \mathrm{Spf}(A)$ to be Noetherian, it is necessary and sufficient that A is adic and Noetherian.*

Proof. This condition is clearly sufficient. Conversely, suppose that $\mathfrak{X}$ is Noetherian, and let $\mathfrak{I}$ be a nilideal of A, $\mathscr{I} = \mathfrak{I}^{\Delta}$ the nilideal of $\mathfrak{X}$. The (usual) schemes $X_n = (\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}/\mathscr{I}^{n+1})$ is then affine and Noetherian, so the ring $A_n = A/\mathscr{I}^{n+1}$ is Noetherian (Proposition 10.2.29), and we conclude that $\mathfrak{I}/\mathfrak{I}^2$ is a finitely generated $(A/\mathfrak{I})$-module. As the $\mathfrak{I}^n$ form a fundamental system of nilideals of $\mathfrak{X}$, we have $\mathscr{O}_{\mathfrak{X}} = \varprojlim(\mathscr{O}_{\mathfrak{X}}/\mathfrak{I}^n)$ (Proposition 10.8.10). □

Remark 10.8.20. With the notations of Proposition 10.8.17, let $\mathscr{F}_i$ be an $\mathscr{O}_i$-module, and suppose that for $i \geq j$ we are given a v_{ij}-morphism $\theta_{ji} : \mathscr{F}_i \to \mathscr{F}_j$, such that $\theta_{kj} \circ \theta_{ji} = \theta_{ki}$ for $k \leq j \leq i$. As the underlying continuous map of v_{ij} is the identity, θ_{ji} is a homomorphism of sheaves of abelian groups over $\mathfrak{X}$. Moreover, if $\mathscr{F}$ is the limit of the projective system $(\mathscr{F}_i)$ of sheaves of abelian groups, the fact that each θ_{ji} is a v_{ij}-morphism permits us to define over $\mathscr{F}$ an $\mathscr{O}_{\mathfrak{X}}$-module structure by passing to projective limits. With this, we say that $\mathscr{F}$ is the **projective limit** (for the θ_{ji}) of the system of $\mathscr{O}_i$-modules $(\mathscr{F}_i)$. In the particular case where $v_{ij}^*(\mathscr{F}_i) = \mathscr{F}_j$ and there θ_{ji} is the identity, we then say that $\mathscr{F}$ is the limit of the system $(\mathscr{F}_i)$ such that $v_{ij}^*(\mathscr{F}_i) = \mathscr{F}_j$ for $j \leq i$ (without mension of θ_{ji}).

Let $\mathfrak{X}, \mathfrak{Y}$ be two formal schemes, $\mathscr{J}$ (resp. $\mathscr{K}$) be a nilideal of $\mathfrak{X}$ (resp. $\mathfrak{Y}$), and $f : \mathfrak{X} \to \mathfrak{Y}$ be a morphism such that $f^*(\mathscr{K})\mathscr{O}_{\mathfrak{X}} \subseteq \mathscr{J}$. We then have for each integer $n > 0$ that $f^*(\mathscr{K}^n)\mathscr{O}_{\mathfrak{X}} = (f^*(\mathscr{K})\mathscr{O}_{\mathfrak{X}})^n \subseteq \mathscr{J}^n$, so Proposition 10.8.13 deduce a morphism $f_n : X_n \to Y_n$ of (usual) schemes such that the diagram

$$\begin{tikzcd} X_m \arrow[r, "f_m"] \arrow[d] & Y_m \arrow[d] \\ X_n \arrow[r, "f_n"] & Y_n \end{tikzcd} \tag{10.8.6}$$

is commutative for $m \leq n$; in other words, (f_n) is a inductive system of morphisms.

Conversely, let (X_n) (resp. (Y_n)) be a inductive system of schemes satisfying the conditions (b), (c) of Proposition 10.8.17, and let $\mathfrak{X}$ (resp. $\mathfrak{Y}$) be the inductive limits (whose existence is proved by Proposition 10.8.17). By the definition of inductive limits, any inductive system (f_n) of morphisms $f_n : X_n \to Y_n$ admits an inductive limit $f : \mathfrak{X} \to \mathfrak{Y}$, which is the unique morphism of formal schemes rendering the diagram

$$\begin{tikzcd} X_n \arrow[r, "f_n"] \arrow[d] & Y_n \arrow[d] \\ \mathfrak{X} \arrow[r, "f"] & \mathfrak{Y} \end{tikzcd}$$

Proposition 10.8.21. *Let $\mathfrak{X}, \mathfrak{Y}$ be adic formal schemes, $\mathscr{J}$ (resp. $\mathscr{K}$) be a nilideal of $\mathfrak{X}$ (resp. $\mathfrak{Y}$). The map $f \mapsto (f_n)$ is a bijection from the set of morphisms $f : \mathfrak{X} \to \mathfrak{Y}$ such that $f^*(\mathscr{K})\mathscr{O}_{\mathfrak{X}} \subseteq \mathscr{J}$ to the set of sequences (f_n) of morphisms rendering the diagram (10.8.6).*

Proof. If f is the inductive limit of this sequence, it is necessary to prove that $f^*(\mathscr{K})\mathscr{O}_{\mathfrak{X}} \subseteq \mathscr{J}$. This question is local over $\mathfrak{X}$ and $\mathfrak{Y}$, so we can assume that $\mathfrak{X} = \mathrm{Spf}(A)$, $\mathfrak{Y} = \mathrm{Spf}(B)$ are affine, where A, B are adic, $\mathscr{J} = \mathfrak{J}^\Delta$, $\mathscr{K} = \mathfrak{K}^\Delta$, where $\mathfrak{J}$ (resp. $\mathfrak{K}$) is a nilideal of A (resp. B). We then have $X_n = \mathrm{Spec}(A_n)$, $Y_n = \mathrm{Spec}(B_n)$, where $A_n = A/\mathfrak{J}^{n+1}$ and $B_n = B/\mathfrak{K}^{n+1}$, in view of Proposition 10.8.6. Then $f_n = ({}^a\varphi_n, \tilde{\varphi}_n)$, where $\varphi_n : B_n \to A_n$ is the homomorphism forming a projective system, so $f = ({}^a\varphi, \tilde{\varphi})$, where $\varphi = \varprojlim \varphi_n$. The commutative diagram (10.8.6) shows that $\varphi_n(\mathfrak{K}/\mathfrak{K}^{n+1}) \subseteq \mathfrak{J}/\mathfrak{J}^{n+1}$ for each n (by take $m = 0$), so by passing to limit, $\varphi(\mathfrak{K}) \subseteq \mathfrak{J}$, and this implies $f^*(\mathscr{K})\mathscr{O}_{\mathfrak{X}} \subseteq \mathscr{J}$ (Proposition 10.8.13). □

Corollary 10.8.22. *Let $\mathfrak{X}, \mathfrak{Y}$ be locally Noetherian formal schemes, $\mathscr{T}$ be the largest nilideal of $\mathfrak{X}$.*

(i) *For any nilideal $\mathscr{K}$ of $\mathfrak{Y}$ and any morphism $f : \mathfrak{X} \to \mathfrak{Y}$, we have $f^*(\mathscr{K})\mathscr{O}_{\mathfrak{X}} \subseteq \mathscr{T}$.*

(ii) *There exists a bijective correspondence between $\mathrm{Hom}(\mathfrak{X}, \mathfrak{Y})$ and the set of sequences (f_n) of morphisms rendering the diagram (10.8.6), where $X_n = (\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}/\mathscr{T}^{n+1})$, $Y_n = (\mathfrak{Y}, \mathscr{O}_{\mathfrak{Y}}/\mathscr{K}^{n+1})$.*

Proof. It is clear that (ii) follows from (i) and Proposition 10.8.21. To prove (i), we can assume that $\mathfrak{X} = \mathrm{Spf}(A)$, $\mathfrak{Y} = \mathrm{Spf}(B)$, A, B being Noetherian and adic, $\mathscr{T} = \mathfrak{T}^\Delta$, $\mathscr{K} = \mathfrak{K}^\Delta$, where

$\mathfrak{T}$ is the largest nilideal of A and $\mathfrak{K}$ is a nilideal of B. Let $f = ({}^a\varphi, \tilde{\varphi})$ where $\varphi : B \to A$ is a continuous homomorphism; as the elements of $\mathfrak{K}$ are topologically nilpotent, so are those of $\varphi(\mathfrak{K})$, so $\varphi(\mathfrak{K}) \subseteq \mathfrak{T}$ since $\mathfrak{T}$ is the set of topologically nilpotent elements of A (Proposition 2.6.2). The conclusion then follows from Proposition 10.8.13(ii). □

Corollary 10.8.23. *Let $f : \mathfrak{X} \to \mathfrak{S}$ and $g : \mathfrak{Y} \to \mathfrak{S}$ be morphisms of adic formal schemes. Let $\mathscr{J}$ (resp. $\mathscr{K}$, $\mathscr{L}$) be a nilideal of $\mathfrak{S}$ (resp. $\mathfrak{X}$, $\mathfrak{Y}$), and suppose that $f^*(\mathscr{J})\mathscr{O}_{\mathfrak{X}} \subseteq \mathscr{K}$, $g^*(\mathscr{J})\mathscr{O}_{\mathfrak{Y}} = \mathscr{L}$. Let $S_n = (\mathfrak{S}, \mathscr{O}_{\mathfrak{S}}/\mathscr{J}^{n+1})$, $X_n = (\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}/\mathscr{K}^{n+1})$, $Y_n = (\mathfrak{Y}, \mathscr{O}_{\mathfrak{Y}}/\mathscr{L}^{n+1})$. Then there exists a bijective correspondence between $\mathrm{Hom}_{\mathfrak{S}}(\mathfrak{X}, \mathfrak{Y})$ and the set of sequences (u_n) of S_n-morphisms $u_n : X_n \to Y_n$ rendering the diagram (10.8.6).*

Proof. For any $\mathfrak{S}$-morphism $u : \mathfrak{X} \to \mathfrak{Y}$, we have $f = g \circ u$ by definition, so

$$u^*(\mathscr{L})\mathscr{O}_{\mathfrak{X}} = u^*(g^*(\mathscr{J})\mathscr{O}_{\mathfrak{Y}})\mathscr{O}_{\mathfrak{X}} = f^*(\mathscr{J})\mathscr{O}_{\mathfrak{X}} \subseteq \mathscr{K},$$

so the corollary follows from Proposition 10.8.21. □

We note that, for $m \leq n$, the datum of a morphism $f_n : X_n \to Y_n$ determines uniquely a morphism $f_m : X_m \to Y_m$ rendering the diagram (10.8.6), as we immediately see by reducing to the affine case; we have thus defined a map

$$\varphi_{mn} : \mathrm{Hom}_{S_n}(X_n, Y_n) \to \mathrm{hom}_{S_m}(X_m, Y_m)$$

and the $\mathrm{Hom}_{S_n}(X_n, Y_n)$ form for the φ_{mn} a projective system of sets; Corollary 10.8.23 then shows that there exists a canonical bijection

$$\mathrm{Hom}_{\mathfrak{S}}(\mathfrak{X}, \mathfrak{Y}) \xrightarrow{\sim} \varprojlim_n \mathrm{Hom}_{S_n}(X_n, Y_n).$$

Remark 10.8.24. Let $\mathfrak{S}$, $\mathfrak{X}, \mathfrak{Y}$ be formal schemes and $f : \mathfrak{X} \to \mathfrak{S}$, $g : \mathfrak{Y} \to \mathfrak{S}$ be morphisms. Suppose that there are fundamental system of nilideals $(\mathscr{J}_\lambda)$, $(\mathscr{K}_\lambda)$, $(\mathscr{L}_\lambda)$ in $\mathfrak{S}$, $\mathfrak{X}, \mathfrak{Y}$, respectively, with the same index set I, such that $f^*(\mathscr{J}_\lambda)\mathscr{O}_{\mathfrak{X}} \subseteq \mathscr{K}_\lambda$ and $g^*(\mathscr{J}_\lambda)\mathscr{O}_{\mathfrak{Y}} \subseteq \mathscr{L}_\lambda$ for any λ. Put $S_\lambda = (\mathfrak{S}, \mathscr{O}_{\mathfrak{S}}/\mathscr{J}_\lambda)$, $X_\lambda = (\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}/\mathscr{K}_\lambda)$, $Y_\lambda = (\mathfrak{Y}, \mathscr{O}_{\mathfrak{Y}}/\mathscr{L}_\lambda)$; for $\mathscr{J}_\mu \subseteq \mathscr{J}_\lambda$, $\mathscr{K}_\mu \subseteq \mathscr{K}_\lambda$, $\mathscr{L}_\mu \subseteq \mathscr{L}_\lambda$, note that S_λ (resp. X_λ, Y_λ) is a closed subscheme of S_μ (X_μ, Y_μ) with the same underlying space. As $S_\lambda \to S_\mu$ is a monomorphism of schemes, we then see that the products $X_\lambda \times_{S_\lambda} Y_\lambda$ and $X_\lambda \times_{S_\mu} Y_\lambda$ are identical (Corollary 10.3.2), because $X_\lambda \times_{S_\mu} Y_\lambda$ is identified with a closed subscheme of $X_\mu \times_{S_\mu} Y_\mu$ with the same underlying space. Now the product is the inductive limit of the schemes $X_\lambda \times_{S_\lambda} Y_\lambda$: in fact, we see as in Proposition 10.8.16, we can assume that $\mathfrak{S}$, $\mathfrak{X}$, $\mathfrak{Y}$ are formal affine schemes. In view of Proposition 10.8.13 and our hypotheses, immediately see that our assertion follows from the definition of the completed tensor product of two algebras.

Moreover, let $\mathfrak{Z}$ be a formal $\mathfrak{S}$-scheme, $(\mathscr{M}_\lambda)$ be a fundamental system of nilideals of $\mathfrak{Z}$ with index set I, $u : \mathfrak{Z} \to \mathfrak{X}$, $v : \mathfrak{Z} \to \mathfrak{Y}$ be morphisms such that $u^*(\mathscr{K}_\lambda)\mathscr{O}_{\mathfrak{Z}} \subseteq \mathscr{M}_\lambda$ and $v^*(\mathscr{L}_\lambda)\mathscr{O}_{\mathfrak{Z}} \subseteq \mathscr{M}_\lambda$. If we put $Z_\lambda = (\mathfrak{Z}, \mathscr{O}_{\mathfrak{Z}}/\mathscr{M}_\lambda)$, and if $u_\lambda : Z_\lambda \to X_\lambda$ and $v_\lambda : Z_\lambda \to Y_\lambda$ are the corresponding S_λ-morphisms, we then verify that $(u, v)_{\mathfrak{S}}$ is the inductive limits of the S_λ-morphisms $(u_\lambda, v_\lambda)_{S_\lambda}$.

10.8.4 Formal completion of schemes

Let X be a (usual) scheme, X' be a subscheme of X, U be an open subset of X containing X' and such that X' is a closed subscheme of U; then X' is defined by a quasi-coherent ideal $\mathscr{J}_U$ of $\mathscr{O}_U$. For any integer $n > 0$, and any quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, $(\mathscr{F}|_U) \otimes_{\mathscr{O}_U} (\mathscr{O}_U/\mathscr{J}_U^n)$ is then a quasi-coherent $\mathscr{O}_U$-module whose support is contained in X', which is therefore often identified with its restriction to X'. The family $\{(\mathscr{F}|_U) \otimes_{\mathscr{O}_U} (\mathscr{O}_U/\mathscr{J}_U^n)\}_{n \geq 1}$ then forms a projective system of sheaves of abelian groups. The limit $\varprojlim((\mathscr{F}|_U) \otimes_{\mathscr{O}_U} (\mathscr{O}_U/\mathscr{J}_U^n))$ is called the

completion of $\mathscr{F}$ along the subscheme X' of X, and denoted by $\widehat{\mathscr{F}}_{/X'}$ of simply $\widehat{\mathscr{F}}$ (if there is no confusion). The sections of $\widehat{\mathscr{F}}$ over X' are called the **formal sections** of $\mathscr{F}$ along X'.

This definition is justified by the fact that it obviously does not depend on the choice open subset U, because at every point x of $U - X'$, there is a neighborhood of x in which $\mathscr{O}_U/\mathscr{I}_U^n$ is zero for any integer n. We can therefore limit ourselves to the case where X' is a closed subscheme of X, and we will always assume this henceforth. Also, it is clear that for any open subset $U \subseteq X$, we have $(\mathscr{F}|_U)_{/(U\cap X')} = (\mathscr{F}_{/X'})|_{U\cap X'}$.

By passing to projective limits, it is clear that $(\mathscr{O}_X)_{/X'}$ is a sheaf of rings, and that $\mathscr{F}_{/X'}$ can be considered as an $(\mathscr{O}_{/X'})$-module. Furthermore, as there existss a basis for X' formed by quasi-compact opens, we can consider $(\mathscr{O}_X)_{/X'}$ (resp. $\mathscr{F}_{/X'}$) as a sheaf of topological rings (resp. topological groups) which is the projective limit of the sheaf of discrete rings $\mathscr{O}_X/\mathscr{I}^n$ (resp. the sheaf of groups $\mathscr{F} \otimes_{\mathscr{O}_X} (\mathscr{O}_X/\mathscr{I}^n) = \mathscr{F}/\mathscr{I}^n\mathscr{F}$), and by passing to projective limits, $\mathscr{F}_{/X'}$ is then a topological $(\mathscr{O}_X)_{/X'}$-module. Note that for any quasi-compact open subset U of X, $\Gamma(U \cap X', (\mathscr{O}_X)_{/X'})$ (resp. $\Gamma(U \cap X', \mathscr{F}_{/X'})$) is then the projective limit of the discrete rings (resp. groups) $\Gamma(U, \mathscr{O}_X/\mathscr{I})$ (resp. $\Gamma(U, \mathscr{F}/\mathscr{I}\mathscr{F})$).

Now let $u : \mathscr{F} \to \mathscr{G}$ be a homomorphism of $\mathscr{O}_X$-modules, we then deduce a canonical homomorphism

$$u_{\mathscr{I}^n} : \mathscr{F} \otimes_{\mathscr{O}_X} (\mathscr{O}_X/\mathscr{I}^n) \to \mathscr{G} \otimes_{\mathscr{O}_X} (\mathscr{O}_X/\mathscr{I}^n)$$

for any $n \geq 1$, and these homomorphisms form a projective system. By passing to projective limits and restricting to X', we obtain a continuous $(\mathscr{O}_X)_{/X'}$-homomorphism $\mathscr{F}_{/X'} \to \mathscr{G}_{/X'}$, denoted by $u_{/X'}$ or $\hat{u}$, and is called the **completion of u along X'**. It is clear that if $v : \mathscr{G} \to \mathscr{H}$ is a second homomorphism of $\mathscr{O}_X$-modules, then $(v \circ u)_{/X'} = (v_{/X'}) \circ (u_{/X'})$, so $\mathscr{F}_{/X'}$ is a covariant additive functor on $\mathscr{F}$ from the category of quasi-coherent $\mathscr{O}_X$-modules, with values in the category of $(\mathscr{O}_X)_{/X'}$-modules.

Proposition 10.8.25. *Let X' be a closed subscheme of X defined by a quasi-coherent ideal $\mathscr{I}$ of finite type. Then the support of $(\mathscr{O}_X)_{/X'}$ is X', the topologically ringed space $(X', (\mathscr{O}_X)_{/X'})$ is an adic formal scheme, and $\mathscr{I}_{/X'}$ is a nilideal of this formal scheme. If $X = \mathrm{Spec}(A)$ is an affine scheme, $\mathscr{I} = \widetilde{\mathfrak{I}}$ where $\mathfrak{I}$ is an ideal of A, and $X' = V(\mathfrak{I})$, then $(X', (\mathscr{O}_X)_{/X'})$ is canonically identified with $\mathrm{Spf}(\widehat{A})$, where $\widehat{A}$ is the Hausdorff completion of A for the $\mathfrak{I}$-adic topology.*

Proof. We can evidently assume that $X = \mathrm{Spec}(A)$ is affine. By Corollary 2.3.18 the the Hausdorff completion $\widehat{\mathfrak{I}}$ of $\mathfrak{I}$ for the $\mathfrak{I}$-adic topology is identified with the ideal $\mathfrak{I}\widehat{A}$ of $\widehat{A}$, and that $\widehat{A}$ is a $\widehat{A}$-adic ring such that $\widehat{A}/\widehat{\mathfrak{I}}^n = A/\mathfrak{I}^n$. This last relation (for $n = 1$) proves that the open prime ideals of $\widehat{A}$ are the ideals $\hat{\mathfrak{p}} = \mathfrak{p}\widehat{\mathfrak{I}}$, where $\mathfrak{p}$ is a prime ideal of A containing $\mathfrak{I}$, whence $\mathrm{Spf}(\widehat{A}) = X'$. As $\mathscr{O}_X/\mathscr{I}^n = \widetilde{A/\mathfrak{I}^n}$, the proposition then follows from the definition of $\mathrm{Spf}(\widehat{A})$. □

The formal scheme therefore defined is called the **completion** of X along X' and denoted by $X_{/X'}$ or $\widehat{X}$. If $X' = X$, we can set $\mathscr{I} = 0$, and then $X_{/X'} = X$. It is clear that if U is an open subscheme of X, then $U_{/(U\cap X')}$ is canonically identified with the formal subscheme of $X_{/X'}$ induced over the open subset $U \cap X'$ of X'.

Corollary 10.8.26. *Under the hypothesis of Proposition 10.8.25, assume that X is locally Noetherian. Then the (usual) scheme $\widehat{X}_{\mathrm{red}}$ is the unique reduced subscheme of X with underlying space X' (Proposition 10.4.44). For $\widehat{X}$ to be Noetherian, it is necessary and sufficient that $\widehat{X}_{\mathrm{red}}$ is Noetherian, and it is sufficient that X is Noetherian.*

Proof. The determination of $\widehat{X}_{\mathrm{red}}$ is local (Proposition 10.8.11), so we can assume that X is affine; with the notations of Proposition 10.8.25, the ideal $\mathfrak{T}$ of topological nilpotent elements of $\widehat{A}$ is the inverse image of the nilradical of $A/\mathfrak{T}$ under the canonical map $\widehat{A} \to \widehat{A}/\widehat{\mathfrak{I}} = A/\mathfrak{I}$, so $\widehat{A}/\mathfrak{T}$ is isomorphic to $(A/\mathfrak{I})_{\mathrm{red}}$. The first assertion then follows from Proposition 10.8.11 and Proposition 10.4.44. If $\widehat{X}_{\mathrm{red}}$ is Noetherian, its underlying space X' is also Noetherian, so the $X'_n = \operatorname{Spec}(\mathscr{O}_X/\mathscr{I}^n)$ are Noetherian and so is $\widehat{X}$ (Corollary 10.8.18); the converse of this is immeidate. $\square$

The canonical homomorphisms $\mathscr{O}_X \to \mathscr{O}_X/\mathscr{I}^n$ form a projective system and therefore gives, by passing to limit, a homomorphism of sheaves of rings $\theta : \mathscr{O}_X \to i_*((\mathscr{O}_X)_{/X'}) = \varprojlim(\mathscr{O}_X/\mathscr{I}^n)$, where $i : X' \to X$ is the canonical injection. We therefore obtain a morphism

$$(i,\theta) : X_{/X'} \to X$$

of ringed spaces, called the **canonical morphism**. By tensoring, for any coherent $\mathscr{O}_X$-module $\mathscr{F}$, the canonical homomorphisms $\mathscr{O}_X \to \mathscr{O}_X/\mathscr{I}^n$ gives homomorphisms $\mathscr{F} \to \mathscr{F} \otimes_{\mathscr{O}_X} (\mathscr{O}_X/\mathscr{I}^n)$ of $\mathscr{O}_X$-modules which form a projective system, and hence defines a canonical functorial homomorphism $\gamma : \mathscr{F} \to i_*(\mathscr{F}_{/X'})$ of $\mathscr{O}_X$-modules.

Example 10.8.27. Let X', X'' be closed subschemes of X, defined by quasi-coherent ideals $\mathscr{I}$, $\mathscr{I}'$ of $\mathscr{O}_X$. Suppose that for any affine open U of X, the ideals $\mathscr{I}|_U$, $\mathscr{I}'|_U$ of $\mathscr{O}_U$ are such that there exists integers $m, m' > 0$ such that $(\mathscr{I}|_U)^m \subseteq \mathscr{I}'|_U$ and $(\mathscr{I}'|_U)^{m'} \subseteq \mathscr{I}|_U$. It is clear that under this condition, for any quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, the sheaf of abelian groups $\varprojlim(\mathscr{F} \otimes_{\mathscr{O}_X} (\mathscr{O}_X/\mathscr{I}^n))$ and $\varprojlim(\mathscr{F} \otimes_{\mathscr{O}_X} (\mathscr{O}_X/\mathscr{I}'^n))$ are canonically isomorphic, which means $\mathscr{F}_{/X'} = \mathscr{F}_{/X''}$. Note that this condition on X' and X'' implies that these two closed subschemes have the same underlying space, but is not in general equivalent to the latter property.

However, if the quasi-coherent ideals $\mathscr{I}$ and $\mathscr{I}'$ are of finite type, then it follows from Proposition 10.6.17 that if the subschemes X' and X'' have the same underlying space, the above condition is satisfied. In particular, if X is locally Noetherian, so that any quasi-coherent ideal of $\mathscr{O}_X$ is of finite type, then for any closed subset (or locally closed) X' of X, we can define $\mathscr{F}_{/X'}$ to be equal to $\mathscr{F}_{/Y}$ for any subscheme Y of X with underlying space X'.

Proposition 10.8.28. *Suppose that X is locally Noetherian and let X' be a closed subset of X, $\mathscr{F}$ be a coherent $\mathscr{O}_X$-module.*

(i) *The functor $\mathscr{F}_{/X'}$ is exact on the category of coherent $\mathscr{O}_X$-modules.*

(ii) *The functorial homomorphism $\gamma^\sharp : i^*(\mathscr{F}) \to \mathscr{F}_{/X'}$ of $(\mathscr{O}_X)_{/X'}$-modules is an isomorphism.*

Proof. To prove (i), it suffices to prove that if $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ is an exact sequence of coherent $\mathscr{O}_X$-modules and U is an affine open of X with Noetherian ring $A = \Gamma(U, \mathscr{O}_X)$, the sequence

$$0 \longrightarrow \Gamma(U \cap X', \mathscr{F}'_{/X'}) \longrightarrow \Gamma(U \cap X', \mathscr{F}_{/X'}) \longrightarrow \Gamma(U \cap X', \mathscr{F}''_{/X'}) \to 0$$

is exact. Then $\mathscr{F}|_U = \widetilde{M}$, $\mathscr{F}'|_U = \widetilde{M}'$, $\mathscr{F}''|_U = \widetilde{M}''$, where M, M', M'' are A-modules of finite type such that the sequence $0 \to M' \to M \to M'' \to 0$ is exact. Let $\mathscr{I}$ be a quasi-coherent ideal of $\mathscr{O}_U$ defining a subscheme of U with underlying space $U \cap X'$, and $\mathfrak{I}$ be the ideal of A such that $\mathscr{I} = \widetilde{\mathfrak{I}}$. We have (Corollary 10.1.10)

$$\Gamma(U \cap X', \mathscr{F} \otimes_{\mathscr{O}_X} (\mathscr{O}_X/\mathscr{I}^n)) = M \otimes_A (A/\mathfrak{I}^n);$$

so by the definition of projective limit,

$$\Gamma(U \cap X', \mathscr{F}_{/X'}) = \varprojlim(M \otimes_A (A/\mathfrak{I}^n)) = \widehat{M}$$

where $\widehat{M}$ is the Huasdorff completion of M for the $\mathfrak{I}$-adic topology, and similarly

$$\Gamma(U \cap X', \mathscr{F}'_{/X'}) = \widehat{M}', \quad \Gamma(U \cap X', \mathscr{F}''_{/X'}) = \widehat{M}'';$$

our assertion then follows from the fact that A is Noetherian and the functor $\widehat{M}$ on M is exact on the category of finitely generated A-modules (Theorem 2.4.19).

For assertion (i), the assertion is local, so we can assume that there exists an exact sequence $\mathscr{O}_X^m \to \mathscr{O}_X^n \to \mathscr{F} \to 0$; as $\gamma^\sharp$ is functorial, and the functors $i^*(\mathscr{F})$ and $\mathscr{F}_{/X'}$ are right exact, we have a commutative diagram

$$\begin{tikzcd} i^*(\mathscr{O}_X^m) \arrow[r] \arrow[d, "\gamma^\sharp"] & i^*(\mathscr{O}_X^n) \arrow[r] \arrow[d, "\gamma^\sharp"] & i^*(\mathscr{F}) \arrow[r] \arrow[d, "\gamma^\sharp"] & 0 \\ (\mathscr{O}_X^m)_{/X'} \arrow[r] & (\mathscr{O}_X^n)_{/X'} \arrow[r] & \mathscr{F}_{/X'} \arrow[r] & 0 \end{tikzcd} \tag{10.8.7}$$

with exact rows. Moreover, the functors $i^*(\mathscr{F})$ and $\mathscr{F}_{/X'}$ commutes with finite sums, so we are reduced to the case where $\mathscr{F} = \mathscr{O}_X$. We then have $i^*(\mathscr{O}_X) = (\mathscr{O}_X)_{/X'} = \mathscr{O}_{\widehat{X}}$, and $\gamma^\sharp$ is a homomorphism of $\mathscr{O}_{\widehat{X}}$-modules; it then suffices to verify that $\gamma^\sharp$ maps the unit section of $\mathscr{O}_{\widehat{X}}$ over an open subset of X' to itself, which is immediate and also shows that $\gamma^\sharp$ is the identity. □

Corollary 10.8.29. *Under the hypotheses of Proposition 10.8.28, the morphism $i : \widehat{X} \to X$ is flat.*

Corollary 10.8.30. *Let X be a locally Noetherian scheme. If $\mathscr{F}$ and $\mathscr{G}$ are coherent $\mathscr{O}_X$-modules, there exists a canonical functorial isomorphism*

$$(\mathscr{F}_{/X'}) \otimes_{(\mathscr{O}_X)_{/X'}} (\mathscr{G}_{/X'}) \xrightarrow{\sim} (\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G})_{/X'}, \tag{10.8.8}$$

$$(\mathcal{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G}))_{/X'} \xrightarrow{\sim} \mathcal{H}om_{(\mathscr{O}_X)_{/X'}}(\mathscr{F}_{/X'}, \mathscr{G}_{/X'}). \tag{10.8.9}$$

Proof. This follows from the canonical identification of $i^*(\mathscr{F})$ and $\mathscr{F}_{/X'}$; the existence of the first isomorphism is clear for any morphism of ringed spaces and the second is the homomorphism of (**??**), which is an isomorphism for any flat morphism. □

Proposition 10.8.31. *Let X be a locally Noetherian scheme. For any coherent $\mathscr{O}_X$-module $\mathscr{F}$, the kernel of the canonical homomorphism $\Gamma(X, \mathscr{F}) \to \Gamma(X', \mathscr{F}_{/X'})$ induced from $\mathscr{F} \to \mathscr{F}_{/X'}$ is formed by the sections which are zero on an open neighborhood of X'.*

Proof. It follows from the definition of $\mathscr{F}_{/X'}$ that the canonical image of such a section is zero. Conversely, if the image of $s \in \Gamma(X, \mathscr{F})$ is zero in $\Gamma(X', \mathscr{F}_{/X'})$, it suffices to see that any $x \in X'$ admits an open neighborhood in X over which s is zero, and we can therefore assume that $X = \operatorname{Spec}(A)$ is affine, A is Noetherian, $X' = V(\mathfrak{I})$ where $\mathfrak{I}$ is an ideal of A, and $\mathscr{F} = \widetilde{M}$, where M is a finitely generated A-module. Then $\Gamma(X', \mathscr{F}_{/X'})$ is the Huasdorff completion $\widehat{M}$ of M for the $\mathfrak{I}$-topology, and the homomorphism $\Gamma(X, \mathscr{F}) \to \Gamma(X', \mathscr{F}_{/X'})$ is the canonical homomorphism $M \to \widehat{M}$. We have seen that the kernel of this homomorphism (Corollary 2.4.17) consists of elements $z \in M$ which is annihilated by an element of $1 + \mathfrak{I}$. We then have $(1 + f)s = 0$ for $f \in \mathfrak{I}$, and for any $x \in X'$ we deduce that $(1_x + f_x)s_x = 0$; as $1_x + f_x$ is invertible in $\mathscr{O}_{X,x}$ ($\mathfrak{I}_x \mathscr{O}_{X,x}$ is contained in $\mathfrak{m}_x$), we then have $s_x = 0$, which proves the assertion. □

Corollary 10.8.32. *Let X be a locally Noetherian scheme and $u : \mathscr{F} \to \mathscr{G}$ be a homomorphism of coherent $\mathscr{O}_X$-modules. For the homomorphism $u_{/X'} : \mathscr{F}_{/X'} \to \mathscr{G}_{/X'}$ to be zero, it is necessary and sufficient that u is zero on an open neighborhood of X'.*

Proof. In fact, by Proposition 10.8.28, $u_{/X'}$ is identified with $i^*(u)$, so if we consider u as a section of $\mathscr{H} = \mathcal{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})$ over X, $u_{/X'}$ is the section of $i^*(\mathscr{H}) = \mathscr{H}_{/X'}$ over X'. It then suffices to apply Proposition 10.8.31 on $\mathscr{H}$. □

Corollary 10.8.33. *Let X be a locally Noetherian sceme and $u : \mathscr{F} \to \mathscr{G}$ be a homomorphism of coherent $\mathscr{O}_X$-modules. For the homomorphism $u_{/X'} : \mathscr{F}_{/X'} \to \mathscr{G}_{/X'}$ to be a monomorphism (resp. epimorphism), it is necessary and sufficient that u is a monomorphism (resp. epimorphism) on an open neighborhood of X'.*

Proof. Let $\mathscr{P}$ and $\mathscr{N}$ be the kernel and cokernel of u, so that we have an exact sequence

$$0 \longrightarrow \mathscr{P} \longrightarrow \mathscr{F} \xrightarrow{u} \mathscr{G} \longrightarrow \mathscr{N} \longrightarrow 0$$

By applying $(-)_{/X'}$, we then get an exact sequence

$$0 \longrightarrow \mathscr{P}_{/X'} \longrightarrow \mathscr{F}_{/X'} \xrightarrow{u_{/X'}} \mathscr{G}_{/X'} \longrightarrow \mathscr{N}_{/X'} \longrightarrow 0$$

That $u_{/X'}$ is a monomorphism (resp. epimorphism) is equivalent to $\mathscr{P}_{/X'} = 0$ (resp. $\mathscr{N}_{/X'} = 0$), so we can apply Proposition 10.8.31 to get the conclusion. □

Corollary 10.8.34. *Let X be a locally Noetherian scheme and $\mathscr{F}$ be a coherent $\mathscr{O}_X$-module. For $\mathscr{F}_{/X'}$ to be locally free (resp. locally free of rank n), it is necessary and sufficient that there exists an open neighborhood U of X' such that $\mathscr{F}|'_X$ is locally free (resp. locally free of rank n).*

Proof. To say that $\mathscr{F}_{/X'}$ is locally free signifies that any point $x \in X'$ admits an open neighborhood V in X such that there exists an isomorphism $v : (\mathscr{O}_X^n)_{/X'} \xrightarrow{\sim} \mathscr{F}_{/X'}|_{V\cap X'}$. We can evidently assume that $V = X$, and then it follows from (10.8.9) that v is of the form $u_{/Z}$, where u is a homomorphism $\mathscr{O}_X^n \to \mathscr{F}$; moreover, by replacing X with an open neigbourhood of Z, we can assume, in view of Corollary 10.8.33, that u is bijective, whence the corollary. □

We now consider the induced morphisms between formal completions. Let X, Y be schemes, $f : X \to Y$ be a morphism, X' (resp. Y') be a closed subscheme of X (resp. Y), and $i : X' \to X$, $j : Y' \to Y$ be the canonical injections. Suppose that the composition morphism $f \circ i$ dominates j, so that we have commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ X' \arrow[u, "i"] \arrow[r, "f'"] & Y' \arrow[u, "j"'] \end{tikzcd}$$

where f' is the restriction of f to X'. If $\mathscr{J}$ (resp. $\mathscr{K}$) is the quasi-coherent ideal defining X' (resp. Y'), then this means $f^*(\mathscr{K})\mathscr{O}_X \subseteq \mathscr{J}$ (Proposition 10.4.16). We then have, for each integer $n > 0$, $f^*(\mathscr{K}^n)\mathscr{O}_X \subseteq \mathscr{J}^n$, so if we put $X'_n = (X', \mathscr{O}_X/\mathscr{J}^{n+1})$, $Y'_n = (Y', \mathscr{O}_Y/\mathscr{K}^{n+1})$, the morphism f induces a morphism $f_n : X'_n \to Y'_n$, and it is immediate that the f_n form a inductive system. The inductive limit of this system (Proposition 10.8.21) is denote by $\hat{f} : X_{/X'} \to Y_{/Y'}$, and called the **completion of f along the subschemes X' and Y'**. It is clear from definition that the following diagram is commutative

$$\begin{tikzcd} X_{X'} \arrow[r, "\hat{f}"] \arrow[d, "i_X"'] & Y_{/Y'} \arrow[d, "i_Y"] \\ X \arrow[r, "f"] & Y \end{tikzcd} \tag{10.8.10}$$

where the vertical morphism are canonical morphism.

If $X = \mathrm{Spec}(B)$ and $Y = \mathrm{Spec}(A)$ are affine schemes, then $\mathscr{J} = \widetilde{\mathfrak{I}}$, $\mathscr{K} = \widetilde{\mathfrak{K}}$, where $\mathfrak{I}$ (resp. $\mathfrak{K}$) is an ideal of B (resp. A), and f corresponds to a homomorphism $\varphi : A \to B$ such that $\varphi(\mathfrak{K}) \subseteq \mathfrak{I}$. The morphism $\hat{f}$ then corresponds to the continuous homomorphism $\hat{\varphi} : \widehat{A} \to \widehat{B}$

(Proposition 10.8.9), where $\widehat{A}$ (resp. $\widehat{B}$) is the $\mathfrak{K}$-adic completion of A (resp. $\mathfrak{I}$-adic completion of B).

Let Z be a third scheme, $g : Y \to Z$ is a morphism, Z' is a closed subscheme of Z defined by a quasi-coherent ideal $\mathscr{R}$ of $\mathscr{O}_Z$, and suppose that we have $g^*(\mathscr{R})\mathscr{O}_Y \subseteq \mathscr{K}$. Then, if $\hat{g}$ is the completion of g along Y' and Z', then it follows from our definition that $\widehat{(g \circ f)} = \hat{g} \circ \hat{f}$.

Now suppose that X and Y are locally Noetherian schemes, X', Y' are closed subsets of X, Y, respectively, and $f : X \to Y$ is a morphism such that $f(X') \subseteq Y'$. Then there exists a coherent ideal $\mathscr{I}$ of $\mathscr{O}_X$ (resp. $\mathscr{K}$ of $\mathscr{O}_Y$) such that $\operatorname{supp}(\mathscr{O}_X/\mathscr{I}) = X'$ (resp. $\operatorname{supp}(\mathscr{O}_Y/\mathscr{K}) = Y'$) and that $f^*(\mathscr{K})\mathscr{O}_X \subseteq \mathscr{I}$: in fact, it suffices to choose $\mathscr{I}$ to be the ideal defining the reduced subscheme structure of X' and $\mathscr{K}$ to be the ideal defining that of Y'. The relation $f^*(\mathscr{K})\mathscr{O}_X \subseteq \mathscr{I}$ then follows from Corollary 10.4.50. We can then define the morphism $\hat{f} : X_{/X'} \to Y_{/Y'}$, and it follows from Example 10.8.27 that $\hat{f}$ does not depend on the choice of the idelas $\mathscr{I}$ and $\mathscr{K}$.

Proposition 10.8.35. *Let X and Y be locally Noetherian S-schemes and suppose that Y is of finite type over S. Let X', Y' be closed subsets of X, Y, respectively, and f, g be two S-morphisms from X to Y such that $f(X') \subseteq Y'$, $g(X') \subseteq Y'$. For that $\hat{f} = \hat{g}$, it is necessary and sufficient that f and g coincides in an open neighborhood of X'.*

Proof. This conditions is clearly sufficient (without the finiteness conditio on Y). To see that it is necessary, we first note that $\hat{f} = \hat{g}$ implies $f(x) = g(x)$ for any $x \in X'$. On the other hand, since the question is local, we can assume that $X = \operatorname{Spec}(B)$ and $Y = \operatorname{Spec}(A)$ are affine neighborhoods of x and $y = f(x) = g(x)$, respectively, with Noetherian rings, and that $S = \operatorname{Spec}(R)$ is affine. Then A is an R-algebra of finite type (Corollary 10.6.39), and f, g correspond to R-homomorphisms ρ, σ from A into B. By hypothesis, the induced homomorphisms $\hat{\rho}, \hat{\sigma}$ on completions are equal. We then conclude from Proposition 10.8.31 that for any section $s \in A$, the sections $\rho(s)$ and $\sigma(s)$ coincides in an open neighborhood of X' (dependent of s); as A is a finite type algebra over R, we then deduce that there exists an open neighborhood V of X' such that $\rho(s)$ and $\sigma(s)$ coincides over V for any section $s \in A$. If $h \in A$ is such that $D(h)$ is an open neighborhood of X' contained in V, we then conclude so f and g coincides on $D(h)$. □

Proposition 10.8.36. *Let X and Y be locally Noetherian schemes, $f : X \to Y$ be a morphism, X', Y' be closed subsets of X, Y, respectively, such that $f(X') \subseteq Y'$. Then, for any coherent $\mathscr{O}_Y$-module $\mathscr{G}$, there exists a canonical isomorphism of $(\mathscr{O}_X)_{/X'}$-modules*

$$(f^*(\mathscr{G}))_{/X'} \xrightarrow{\sim} \hat{f}^*(\mathscr{G}_{/Y'}). \tag{10.8.11}$$

Proof. If we identify canonically $(f^*(\mathscr{G}))_{/X'}$ with $i_X^*(f^*(\mathscr{G}))$ and $\hat{f}(\mathscr{G}_{/Y'})$ with $\hat{f}^*(i_Y^*(\mathscr{G}))$ (Proposition 10.8.28), the proposition then follows from the commutative diagram (10.8.10). □

Remark 10.8.37. Retain the hypotheses of Proposition 10.8.36, and let $\mathscr{F}$ be a coherent $\mathscr{O}_X$-module, $\mathscr{G}$ be a coherent $\mathscr{O}_Y$-module. If $u : \mathscr{G} \to \mathscr{F}$ is an f-morphism, then it correponds to a homomorphism $u^\sharp : f^*(\mathscr{G}) \to \mathscr{F}$, hence by completion a continuous $(\mathscr{O}_X)_{/X'}$-homomorphism

$$(u^\sharp)_{/X'} : (f^*(\mathscr{G}))_{/X'} \to \mathscr{F}_{/X'}.$$

In view of (10.8.11), there exists a unique $\hat{f}$-morphism $v : \mathscr{G}_{/Y'} \to \mathscr{F}_{/X'}$ such that $v^\sharp = (u^\sharp)_{/X'}$. If we consider the triple $(\mathscr{F}, X, X')$ (where $\mathscr{F}$ is quasi-coherent $\mathscr{O}_X$-module and X' is a closed subset of X) as a category $\mathcal{C}$, with morphisms $(\mathscr{F}, X, X') \to (\mathscr{G}, Y, Y')$ consisting of a morphism $f : X \to Y$ of schemes such that $f(X') \subseteq Y'$ and an f-morphism $u : \mathscr{G} \to \mathscr{F}$, we can then say that $(X_{/X'}, \mathscr{F}_{/X'})$ is a functor from $\mathcal{C}$ to the category of couples $(\mathfrak{Z}, \mathscr{H})$ formed by a locally Noetherian formal scheme $\mathfrak{Z}$ and an $\mathscr{O}_{\mathfrak{Z}}$-module $\mathscr{H}$, with morphisms given by morphisms g of formal schemes and g-morphisms of sheaves.

Proposition 10.8.38. *Let X and Y be S-schemes, S' be a closed subscheme of S and X', Y' be closed subschemes of X, Y, respectively, such that, if $\mathscr{I}, \mathscr{K}, \mathscr{L}$ are the nilideals of S', X', Y', then $\varphi^*(\mathscr{I})\mathscr{O}_X \subseteq \mathscr{K}$ and $\psi^*(\mathscr{I})\mathscr{O}_Y \subseteq \mathscr{L}$ (where $\varphi : X \to S$ and $\psi : Y \to S$ are the structural morphisms). Let $Z = X \times_S Y$ and $Z' = p^{-1}(X') \cap q^{-1}(Y')$, where p, q are the canonical projections.*

(a) *Then the completion $Z_{/Z'}$ is identified with the product of the formal $S_{S'}$-schemes $(X_{/X'}) \times_{S_{S'}} (Y_{/Y'})$, the structural morphisms with $\hat{\varphi}, \hat{\psi}$, and the projections with $\hat{p}, \hat{q}$.*

(b) *If T is an S-scheme, $u : T \to X$, $v : T \to Y$ are S-morphisms, and T' is a closed subscheme of T such that, if $\mathscr{M}$ is the ideal defining T', then $u^*(\mathscr{K})\mathscr{O}_T \subseteq \mathscr{M}$ and $v^*(\mathscr{L})\mathscr{O}_T \subseteq \mathscr{M}$. Then the completion of $(u, v)_S$ along T' and Z' is canonically identified with $(\hat{u}, \hat{v})_{S_{S'}}$.*

Proof. It is immediate that the question is local for S, X, Y, so we can assume that $S = \operatorname{Spec}(A)$, $X = \operatorname{Spec}(B)$, $Y = \operatorname{Spec}(C)$, $S' = V(\mathfrak{I})$, $X' = V(\mathfrak{K})$, $Y' = V(\mathfrak{R})$, where $\mathfrak{I}, \mathfrak{K}, \mathfrak{R}$ are ideals such that $\rho(\mathfrak{I}) \subseteq \mathfrak{K}$ and $\sigma(\mathfrak{I}) \subseteq \mathfrak{R}$, where $\sigma : A \to B$ and $\sigma : A \to C$ are the corresponding homomorphisms, Then we see that $Z = \operatorname{Spec}(B \otimes_A C)$ and that $Z' = V(\mathfrak{L})$, where $\mathfrak{L}$ is the ideal generated by $\operatorname{im}(\mathfrak{K} \otimes_A C) + \operatorname{im}(B \otimes_A \mathfrak{R})$. The conclusion then follows from Proposition 10.8.14 and the fact that the complete tensor product $\widehat{B} \hat{\otimes}_{\widehat{A}} \widehat{C}$ is the completion of $B \otimes_A C$ for the $\mathfrak{L}$-adic topology. □

Corollary 10.8.39. *With the hypotheses of Proposition 10.8.38, for any S-morphism $f : X \to Y$ satisfying $f^*(\mathscr{L})\mathscr{O}_X \subseteq \mathscr{K}$, the graph diagram $\Gamma_{\hat{f}}$ is identified with the completion $\widehat{(\Gamma_f)}$ of the diagram morphism of f.*

Corollary 10.8.40. *Let X, Y be schemes, $f : X \to Y$ be a morphism, Y' be a closed subscheme of Y, and $X' = f^{-1}(Y')$. Then following commutative diagram is cartesian:*

$$\begin{tikzcd} X_{/X'} \arrow[r, "\hat{f}"] \arrow[d, "i_X"'] & Y_{/Y'} \arrow[d, "i_Y"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

Proof. It suffices to apply Proposition 10.8.38(a), where we take $X' = X$ and $S = S' = Y$. □

10.8.5 Coherent sheaves over formal schemes

In this paragraph, we discuss coherent sheaves over locally Noetherian formal schemes, and characterize them as projective limits of coherent modules over (usual) schemes. For this, we shall first consider the affine case, so let A be an adic ring and $\mathfrak{I}$ be a nilideal of A. Let $X = \operatorname{Spec}(A)$, $\mathfrak{X} = \operatorname{Spf}(A)$, whose underlying space is identified with the closed subset $V(\mathfrak{I})$ of $\operatorname{Spec}(A)$. If $X' = \operatorname{Spec}(A/\mathfrak{I})$ is the closed subscheme of X defined by $\widetilde{\mathfrak{I}}$, it then follows from definigion that $\mathfrak{X}$ is identified with $X_{/X'}$. For any A-module M, the sheaf $M^\Delta = (\widetilde{M})_{/X'}$ is then an $\mathscr{O}_\mathfrak{X}$-module. Moreover, if $u : M \to N$ is a homomorphism of A-modules, it then corresponds to a homomorphism $\tilde{u} : \widetilde{M} \to \widetilde{N}$, and hence to a continuous homomorphism $\tilde{u}_{/X'} : (\widetilde{M})_{/X'} \to (\widetilde{N})_{/X'}$, which we denote by u^Δ. It is evident that $(v \circ u)^\Delta = v^\Delta \circ u^\Delta$, so we obtain an additive covariant functor $M \mapsto M^\Delta$ from the category of A-module to the category of $\mathscr{O}_\mathfrak{X}$-modules.

As A is an adic ring, the ideals $\mathfrak{I}^n$ are open in A, hence separated and complete. The ideal $(\mathfrak{I}^n)^\Delta$ of $\mathscr{O}_\mathfrak{X}$, with the preceding definition, is then equal to the ideal defined in Eq. (10.8.1), and if we put $\mathscr{I} = \mathfrak{I}^\Delta$, then $(\mathfrak{I}^n)^\Delta = \mathscr{I}^n$ if $\mathfrak{I}/\mathfrak{I}^2$ is a finitely generated A-module (Proposition 10.8.6). Under this hypothesis, let $A_n = A/\mathfrak{I}^{n+1}$ and $X_n = \operatorname{Spec}(A_n) = (\mathfrak{X}, \mathscr{O}_\mathfrak{X}/\mathscr{I}^{n+1})$. If $u_{nm} : X_m \to X_n$ is the canonical morphism induced by $A_n \to A_m$ for $m \leq n$, the formal scheme $\mathfrak{X}$ is then the inductive limit of X_n for the u_{nm} (Proposition 10.8.17).

Proposition 10.8.41. *Let A be an adic Noetherian ring. Then the functor $M \mapsto M^\Delta$ is exact on the category of finitely generated A-modules, and we have a canonical isomorphism*

$$\Gamma(\mathfrak{X}, M^\Delta) = M.$$

Proof. The exactness of the functor $M \mapsto M^\Delta$ follows from that of $M \mapsto \widetilde{M}$ (Proposition 10.1.5) and $\mathscr{F} \to \mathscr{F}_{/X'}$ (Proposition 10.8.28). By definition, $\Gamma(\mathfrak{X}, M^\Delta)$ is the $\mathfrak{J}$-adic completion of the A-module $\Gamma(X, \widetilde{M}) = M$ ($\mathfrak{J}$ is a nilideal of A). But as A is complete and M is finitely generated, we see that M is complete and separated (Theorem 2.4.19), and this proves the proposition. □

Proposition 10.8.42. *Let A be an adic Noetherian ring.*

(a) *If M and N are finitely generated A-modules, there exists a canonical isomorphism*

$$(M \otimes_A N)^\Delta \xrightarrow{\sim} M^\Delta \otimes_{\mathscr{O}_\mathfrak{X}} N^\Delta, \tag{10.8.12}$$
$$(\mathrm{Hom}_A(M, N))^\Delta \xrightarrow{\sim} \mathcal{H}om_{\mathscr{O}_\mathfrak{X}}(M^\Delta, N^\Delta). \tag{10.8.13}$$

(b) *The map $u \mapsto u^\Delta$ is a functorial isomorphism*

$$\mathrm{Hom}_A(M, N) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{O}_\mathfrak{X}}(M^\Delta, N^\Delta). \tag{10.8.14}$$

Proof. The first two isomorphisms follow from Corollary 10.1.10(a) and Corollary 10.8.30. Now as $\mathrm{Hom}_A(M, N)$ is a finitely generated A-module, we can apply Proposition 10.8.41 to identify $\Gamma(\mathfrak{X}, (\mathrm{Hom}_A(M, N))^\Delta)$ with $\mathrm{Hom}_A(M, N)$, and by (10.8.13), we see that (10.8.14) is an isomorphism. □

Proposition 10.8.43. *If A is an adic Noetherian ring, $\mathscr{O}_\mathfrak{X}$ is a coherent sheaf of rings.*

Proof. If $f \in A$, we see that $A_{\{f\}}$ is an adic Noetherian ring (Proposition 2.6.10) and as the question is local (Proposition 10.8.2), we are reduced to prove that the kernel of a homomorphism $v : \mathscr{O}_\mathfrak{X}^n \to \mathscr{O}_\mathfrak{X}$ is an $\mathscr{O}_\mathfrak{X}$-module of finite type. We then have $v = u^\Delta$, where $u : A^n \to A$ is a homomorphism (10.8.13). As A is Noetherian, the kernel of u is finitely generated, which means we have a homomorphism $w : A^m \to A^n$ such that the following sequence is exact:

$$A^m \xrightarrow{w} A^n \xrightarrow{u} A$$

We then conclude from Proposition 10.8.41 that the sequence

$$\mathscr{O}_\mathfrak{X}^m \xrightarrow{w^\Delta} \mathscr{O}_\mathfrak{X}^n \xrightarrow{v} \mathscr{O}_\mathfrak{X}$$

is exact, which means the kernel of v is of finite type. □

With the preceding notations, let $A_n = A/\mathfrak{J}^{n+1}$ and X_n be the affine scheme $\mathrm{Spec}(A_n) = (\mathfrak{X}, \mathscr{O}_\mathfrak{X}/\mathscr{J}^{n+1})$, where $\mathscr{J} = \mathfrak{J}^\Delta$ is the nilideal of $\mathscr{O}_\mathfrak{X}$ corresponding to $\mathfrak{J}$. Let $u_{nm} : X_m \to X_n$ be the morphism induced by the ring homomorphism $A_n \to A_m$ for $m \leq n$. As we have remarked, $\mathfrak{X}$ is then the inductive limit of the X_n (Proposition 10.8.17).

Proposition 10.8.44. *Suppose that A is an adic Noetherian ring and let $\mathscr{F}$ be an $\mathscr{O}_\mathfrak{X}$-module. Then the following conditions are equivalent:*

(i) *$\mathscr{F}$ is a coherent $\mathscr{O}_\mathfrak{X}$-module;*

(ii) *there exists a finitely generated A-module M (uniquely determined up to isomorphism) such that $\mathscr{F}$ is isomorphic to M^Δ.*

(iii) *$\mathscr{F}$ is isomorphic to the projective limit of a sequence $(\mathscr{F}_n)$ of coherent $\mathscr{O}_{X_n}$-modules such that $u_{nm}^*(\mathscr{F}_n) = \mathscr{F}_m$ for $m \leq n$, and the projective system $(\mathscr{F}_n)$ is then isomorphic to the system $(\mathscr{F} \otimes_{\mathscr{O}_{\mathfrak{X}}} \mathscr{O}_{X_n})$*

Proof. By definition, we have $\mathscr{O}_{\mathfrak{X}} = A^{\Delta}$. If condition (ii) is satisfied, then M is the cokernel of a homomorphism $A^m \to A^n$, so it follows from Proposition 10.8.41 that M^{Δ} is the cokernel of a homomorphism $\mathscr{O}_{\mathfrak{X}}^m \to \mathscr{O}_{\mathfrak{X}}^n$. As the sheaf $\mathscr{O}_{\mathfrak{X}}$ is coherent (Proposition 10.8.43), so is M^{Δ} (**??**) and this proves (ii)⇒(i).

Now assume the conditions in (iii). Then since each X_n is an affine (usual) scheme, we have $\mathscr{F}_n = \widetilde{M}_n$, where M_n is a finitely generated A_n-module. Since $u_{mn}^*(\mathscr{F}_n) = \mathscr{F}_m$ for $m \leq n$, we have $M_m = M_n \otimes_{A_n} A_m$ (Proposition 10.1.14). The modules M_n then form a projective system for the canonical bi-homomorphisms $M_n \to M_m$, and it follows from the definition of A_n that this projective system satisfies the conditions of Proposition 2.3.42, so its projective limit M is a finitely generated A-module such that $M_n = M \otimes_A A_n$ for each n. We then deduce that $\mathscr{F}_n$ is induced over X_n by $\widetilde{M} \otimes_{\mathscr{O}_{\mathfrak{X}}} (\mathscr{O}_{\mathfrak{X}}/\mathscr{J}^{n+1})$, so $\mathscr{F} = M^{\Delta}$ by definition. Conversely, if $\mathscr{F} = M^{\Delta}$ for a finitely generated A-module M, then by definition, $\mathscr{F}$ is the projective limit of the system $\widetilde{M}_n \otimes_{\mathscr{O}_{\mathfrak{X}}} (\mathscr{O}_{\mathfrak{X}}/\mathscr{J}^{n+1})$, and we have $u_{nm}^*(\mathscr{F}_n) = \mathscr{F}_m$ for $m \leq n$.

Finally, assume that $\mathscr{F}$ is a coherent $\mathscr{O}_{\mathfrak{X}}$-module. Considered as an $\mathscr{O}_{\mathfrak{X}}$-module, we have $\mathscr{O}_{X_n} = \mathscr{O}_{\mathfrak{X}}/\mathscr{J}^{n+1} = \widetilde{A}_n$, so $\mathscr{F}_n = \mathscr{F} \otimes_{\mathscr{O}_{\mathfrak{X}}} \mathscr{O}_{X_n}$ is a coherent $\mathscr{O}_{\mathfrak{X}}$-module (**??**), and since it is also a $\mathscr{O}_{X_n}$-module and $\mathscr{J}^{n+1}$ is coherent ($\mathfrak{J}$ is finitely generated), we conclude that $\mathscr{F}_n$ is a coherent $\mathscr{O}_{X_n}$-module (**??**), and it is immediate that $u_{mn}^*(\mathscr{F}_n) = \mathscr{F}_m$ for $m \leq n$. If $\mathscr{G} = \varprojlim \mathscr{F}_n$ is the projective limit, then it is a coherent $\mathscr{O}_{\mathfrak{X}}$-module in view of the equivalence of (ii) and (iii), and it remains to prove that $\mathscr{F}$ is isomorphic to $\mathscr{G}$. Now the canonical homomorphisms $\mathscr{F} \to \mathscr{F}_n$ form a projective system, hence induces a canonical homomorphism $w : \mathscr{F} \to \mathscr{G}$, so it suffices to prove that w is an isomorphism. Since this question is local, we can assume that $\mathscr{F}$ is the cokernel of a homomorphism $\mathscr{O}_{\mathfrak{X}}^m \to \mathscr{O}_{\mathfrak{X}}^n$, which is of the form v^{Δ}, where v is a homomorphism $v : A^m \to A^n$ of A-modules (Proposition 10.8.42), and $\mathscr{F}$ is then isomorphic to M^{Δ}, where $M = \operatorname{coker} v$. In view of Proposition 10.8.42, we then have

$$\mathscr{F}_n = M^{\Delta} \otimes_{\mathscr{O}_{\mathfrak{X}}} A_n^{\Delta} = (M \otimes_A A_n)^{\Delta} = (M \otimes_A A_n)^{\Delta} = \widetilde{M \otimes_A A_n}.$$

since the $\mathfrak{J}$-adic topology on $M \otimes_A A_n$ is discrete. This then implies $M^{\Delta} = \varprojlim \mathscr{F}_n = \mathscr{G}$, so w is an isomorphism. □

Theorem 10.8.45. *Let A be an adic Noetherian ring. Then the functor $M \mapsto M^{\Delta}$ is an equivalence from the category of finitely generated A-modules to the category of coherent $\mathscr{O}_{\mathfrak{X}}$-module.*

Proof. This follows from Proposition 10.8.41, Proposition 10.8.42 and Proposition 10.8.44. □

Now let A, B be two adic Noetherian rings and $\varphi : B \to A$ be a continuous homomorphism. We denote by $\mathfrak{J}$ (resp. $\mathfrak{K}$) the nilideal of A (resp. B), so that $\varphi(\mathfrak{K}) \subseteq \mathfrak{J}$, and we put $X = \operatorname{Spec}(A)$, $Y = \operatorname{Spec}(B)$, $\mathfrak{X} = \operatorname{Spf}(A)$, $Y = \operatorname{Spf}(B)$. Let $f : X \to Y$ be the corresponding morphism and $\hat{f} : \mathfrak{X} \to \mathfrak{Y}$ be the completion of f, which is also the morphism of formal schemes corresponding to φ. We then have a commutative diagram

$$\begin{tikzcd} \mathfrak{X} \arrow[r, "\hat{f}"] \arrow[d, "i_X"'] & \mathfrak{Y} \arrow[d, "i_Y"] \\ X \arrow[r, "f"] & Y \end{tikzcd} \tag{10.8.15}$$

Proposition 10.8.46. *For any finitely generated B-module N, there exists a canonical homomorphism*

$$\hat{f}^*(N^{\Delta}) \xrightarrow{\sim} (N \otimes_B A)^{\Delta}.$$

Proof. By Proposition 10.8.28, we have a canonical isomorphism $N^\Delta = i_Y^*(\widetilde{N})$, so by Proposition 10.1.14

$$(N \otimes_B A)^\Delta = i_X^*(\widetilde{N \otimes_B A}) = i_X^*(f^*(\widetilde{N})).$$

The proposition then follows from the commutative diagram Eq. (10.8.15). □

Corollary 10.8.47. *For any ideal $\mathfrak{b}$ of B, we have*

$$\hat{f}^*(\mathfrak{b}^\Delta)\mathcal{O}_{\mathfrak{X}} = (\mathfrak{b}A)^\Delta.$$

Proof. Let $j : \mathfrak{b} \to B$ be the canonical injection, which corresponds to the canonical injection $j^\Delta : b^\Delta \to \mathcal{O}_{\mathfrak{Y}}$. By definition, $\hat{f}^*(\mathfrak{b}^\Delta)\mathcal{O}_{\mathfrak{X}}$ is the image of the homomorphism $f^*(æ^\Delta) : \hat{f}^*(\mathfrak{b}^\Delta) \to \mathcal{O}_{\mathfrak{X}} = \hat{f}^*(\mathcal{O}_{\mathfrak{Y}})$. But this homomorphism is identified with $(j \otimes 1)^\Delta : (\mathfrak{b} \otimes_B A)^\Delta \to \mathcal{O}_{\mathfrak{X}} = (B \otimes_B A)^\Delta$ in view of Proposition 10.8.46. Since the image of $j \otimes 1$ is the ideal $\mathfrak{b}A$ of A, the image of $(j \otimes 1)^\Delta$ is equal to $(\mathfrak{b}A)^\Delta$, whence the corollary. □

Proposition 10.8.48. *If $\mathfrak{X}$ is a locally Noetherian formal scheme, the sheaf of rings $\mathcal{O}_{\mathfrak{X}}$ is coherent and any nilideal of $\mathfrak{X}$ is coherent.*

Proof. In fact, this question is local, so we can assume that $\mathfrak{X}$ is affine, and the proposition then follows from Proposition 10.8.43 and Proposition 10.8.44. □

Let $\mathfrak{X}$ be a locally Noetherian formal scheme, $\mathscr{J}$ be a nilideal of $\mathfrak{X}$, and X_n be the (usual) scheme $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}/\mathscr{J}^{n+1})$, so that $\mathfrak{X}$ is the inductive limit of (X_n) for the morphisms $u_{mn} : X_m \to X_n$ (Proposition 10.8.17). With these notations, we have the following theorem:

Theorem 10.8.49. *For an $\mathcal{O}_{\mathfrak{X}}$-module $\mathscr{F}$ to be coherent, it is necessary and sufficient that it is isomorphic to the projective limit of a sequence $(\mathscr{F}_n)$ of coherent $\mathcal{O}_{X_n}$-modules such that $u_{nm}^*(\mathscr{F}_n) = \mathscr{F}_m$ for $m \leq n$. In this case, the projective system $(\mathscr{F}_n)$ is then isomorphic to the system $u_n^*(\mathscr{F}) = \mathscr{F} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}_{X_n}$, where $u_n : X_n \to \mathfrak{X}$ is the canonical morphism.*

Proof. This question is local, so we can assume that $\mathfrak{X}$ is an affine formal scheme, and the theorem then follows from Proposition 10.8.44. □

In view of Theorem 10.8.49, we can then say that giving a coherent $\mathcal{O}_{\mathfrak{X}}$-module is equivalent to giving a projective system $(\mathscr{F}_n)$ of coherent $\mathcal{O}_{X_n}$-modules such that $u_{mn}^*(\mathscr{F}_n) = \mathscr{F}_m$ for $m \leq n$.

Corollary 10.8.50. *Under the hypotheses of Theorem 10.8.49, if $\mathscr{F}$ and $\mathscr{G}$ are coherent $\mathcal{O}_{\mathfrak{X}}$-modules, we have a canonical isomorphism*

$$\mathrm{Hom}_{\mathcal{O}_{\mathfrak{X}}}(\mathscr{F}, \mathscr{G}) \xrightarrow{\sim} \varprojlim \mathrm{Hom}_{\mathcal{O}_{X_n}}(\mathscr{F}_n, \mathscr{G}_n). \tag{10.8.16}$$

Proof. The transition homomorphism $\mathrm{Hom}_{\mathcal{O}_{X_n}}(\mathscr{F}_n, \mathscr{G}_n) \to \mathrm{Hom}_{\mathcal{O}_{X_m}}(\mathscr{F}_m, \mathscr{G}_m)$ is given by $\theta_n \mapsto u_{mn}^*(\theta_n)$ ($m \leq n$), and the homomorphism Eq. (10.8.16) sends an element $\theta \in \mathrm{Hom}_{\mathcal{O}_{\mathfrak{X}}}(\mathscr{F}, \mathscr{G})$ to the sequence $(u_n^*(\theta))$. In view of Theorem 10.8.49, we see that the inverse homomorphism is given by sending a projective system $(\theta_n) \in \varprojlim \mathrm{Hom}_{\mathcal{O}_{X_n}}(\mathscr{F}_n, \mathscr{G}_n)$ to its projective limit. □

Corollary 10.8.51. *Under the hypotheses of Theorem 10.8.49, for a homomorphism $\theta : \mathscr{F} \to \mathscr{G}$ to be surjective, it is necessary and sufficient that the corresponding homomorphism $\theta_0 = u_0^*(\theta) : \mathscr{F}_0 \to \mathscr{G}_0$ is surjective.*

Proof. The question is local, so we can assume that $\mathfrak{X} = \mathrm{Spf}(A)$, where A is an adic Noetherian ring, $\mathscr{F} = M^\Delta$, $\mathscr{G} = N^\Delta$, and $\theta = u^\Delta$, where M, N are finitely generated A-modules and $u : M \to N$ is a homomorphism. We then have $\theta_0 = \tilde{u}_0$, where u_0 is the induced homomorphism

$$u \otimes 1 : M \otimes_A A/\mathfrak{I} \to N \otimes_A A/\mathfrak{I}.$$

The conclusion then follows if we can prove that u is surjective if and only if u_0 is surjective. To this end, recall that $\mathfrak{I}$ is contained in the Jacobson radical of A (Lemma 2.3.25(b)), so the assertion follows from Nakayama's lemma. □

Remark 10.8.52. In view of Theorem 10.8.49, we see that any coherent $\mathscr{O}_{\mathfrak{X}}$-module $\mathscr{F}$ is canonically endowed with a structure of topological $\mathscr{O}_{\mathfrak{X}}$-module, which is the projective limit of the sheaf of discrete groups $\mathscr{F}_n$. It then follows from Corollary 10.8.50 that any homomorphism $u : \mathscr{F} \to \mathscr{G}$ of coherent $\mathscr{O}_{\mathfrak{X}}$-modules is automatically continuous. Moreover, if $\mathscr{H}$ is a coherent sub-$\mathscr{O}_{\mathfrak{X}}$-module of a coherent $\mathscr{O}_{\mathfrak{X}}$-module $\mathscr{F}$, then for any open subset $U \subseteq \mathfrak{X}$, $\Gamma(U, \mathscr{H})$ is the kernel of the (continuous) homomorphism $\Gamma(U, \mathscr{F}) \to \Gamma(U, \mathscr{F}/\mathscr{H})$ (because the functor Γ is left exact). Since $\Gamma(U, \mathscr{F}/\mathscr{H})$ is a separated topological group, we conclude that $\Gamma(U, \mathscr{H})$ is a *closed* subgroup of $\Gamma(U, \mathscr{F})$.

Proposition 10.8.53. *Let X be a locally Noetherian formal scheme, $\mathscr{F}$ and $\mathscr{G}$ be coherent $\mathscr{O}_{\mathfrak{X}}$-modules. Then there are canonical isomorphism of topological $\mathscr{O}_{\mathfrak{X}}$-modules*

$$\mathscr{F} \otimes_{\mathscr{O}_{\mathfrak{X}}} \mathscr{G} \xrightarrow{\sim} \varprojlim(\mathscr{F}_n \otimes_{\mathscr{O}_{X_n}} \mathscr{G}_n), \tag{10.8.17}$$

$$\mathcal{H}om_{\mathscr{O}_{\mathfrak{X}}}(\mathscr{F}, \mathscr{G}) \xrightarrow{\sim} \mathcal{H}om_{\mathscr{O}_{X_n}}(\mathscr{F}_n, \mathscr{G}_n). \tag{10.8.18}$$

As $\mathrm{Hom}_{\mathscr{O}_{\mathfrak{X}}}(\mathscr{F}, \mathscr{G})$ is the global section of the topological $\mathscr{O}_{\mathfrak{X}}$-module $\mathcal{H}om_{\mathscr{O}_{\mathfrak{X}}}(\mathscr{F}, \mathscr{G})$, it is endowed with a canonical topology. If $\mathfrak{X}$ is Noetherian, then it follows from Eq. (10.8.18) that a fundamental system of neighborhoods of 0 is given by the subgroups $\mathrm{Hom}_{\mathscr{O}_{\mathfrak{X}}}(\mathscr{F}, \mathscr{J}^n\mathscr{G})$ (cf. [**?**], 0, 7.8.2).

Proposition 10.8.54. *Let $\mathfrak{X}$ be a Noetherian formal scheme and $\mathscr{F}, \mathscr{G}$ be coherent $\mathscr{O}_{\mathfrak{X}}$-modules. Then in the topological group* $\mathrm{Hom}_{\mathscr{O}_{\mathfrak{X}}}(\mathscr{F}, \mathscr{G})$*, the set of surjective (resp. injective) homomorphisms is open.*

Proof. In view of Corollary 10.8.51, the set of surjective homomorphisms in $\mathrm{Hom}_{\mathscr{O}_{\mathfrak{X}}}(\mathscr{F}, \mathscr{G})$ is the inverse image of the set of surjective homomorphisms in $\mathrm{Hom}_{\mathscr{O}_{X_0}}(\mathscr{F}_0, \mathscr{G}_0)$ under the continuous map $\mathrm{Hom}_{\mathscr{O}_{\mathfrak{X}}}(\mathscr{F}, \mathscr{G}) \to \mathrm{Hom}_{\mathscr{O}_{X_0}}(\mathscr{F}_0, \mathscr{G}_0)$. For the second assertion, we may cover $\mathfrak{X}$ by finitely many open Noetherian affine formal schemes U_i. For an element $\theta \in \mathrm{Hom}_{\mathscr{O}_{\mathfrak{X}}}(\mathscr{F}, \mathscr{G})$, it is necessary and sufficient that its restriction in $\mathrm{Hom}_{\mathscr{O}_{\mathfrak{X}}|_{U_i}}(\mathscr{F}|_{U_i}, \mathscr{G}|_{U_i})$ is injective for each i. Since there are only finitely many U_i, we can then reduce to the affine case, and the assertion follows from ([**?**], 0_{I}, 7.8.3). ☐

Chapter 11

Global properties of morphisms of schemes

11.1 Affine morphisms

11.1.1 Schemes affine over a scheme

Let S be a scheme and X be an S-scheme. If $f : X \to S$ is the structural morphism, then the direct image $f_*(\mathscr{O}_X)$ is an $\mathscr{O}_S$-algebra, which we denote by $\mathscr{A}(X)$ if there is no confusion. If U is an open subset of S, we have

$$\mathscr{A}(f^{-1}(U)) = \mathscr{A}(X)|_U.$$

Similarly, for any $\mathscr{O}_X$-module $\mathscr{F}$ (resp. any $\mathscr{O}_X$-algebra $\mathscr{B}$), we denote by $\mathscr{A}(\mathscr{F})$ (resp. $\mathscr{A}(\mathscr{B})$) the direct image $f_*(\mathscr{F})$ (resp. $f_*(\mathscr{B})$) which is an $\mathscr{A}(X)$-module (resp. $\mathscr{A}(X)$-algebra), and also an $\mathscr{O}_S$-module (resp. $\mathscr{O}_S$-algebra).

Let Y be another S-scheme with $g : Y \to S$ the structural morphism, and $h : X \to Y$ be an S-morphism. We then have a commutative diagram

$$\begin{tikzcd} X \arrow[rr, "h"] \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & S & \end{tikzcd}$$

By definition we have a homomorphism $h^\# : \mathscr{O}_Y \to h_*(\mathscr{O}_X)$ of sheaves of rings, and we deduce from this a homomorphism of $\mathscr{O}_S$-algebras $g_*(h^\#) : g_*(\mathscr{O}_Y) \to g_*(h_*(\mathscr{O}_X)) = f_*(\mathscr{O}_X)$, which means, a homomorphism $\mathscr{A}(Y) \to \mathscr{A}(X)$ of $\mathscr{O}_S$-algebras, and we denote it by $\mathscr{A}(h)$. If $h' : Y \to Z$ is another S-morphism, it is immediate that $\mathscr{A}(h' \circ h) = \mathscr{A}(h) \circ \mathscr{A}(h')$. Therefore we have defined a contravariant functor $\mathscr{A}(X)$ from the category of S-schemes to the category of $\mathscr{O}_S$-algebras.

Now let $\mathscr{F}$ be an $\mathscr{O}_X$-module, $\mathscr{G}$ be an $\mathscr{O}_Y$-module, and $u : \mathscr{G} \to \mathscr{F}$ be an h-morphism, wihch is a homomorphism $\mathscr{G} \to h_*(\mathscr{F})$ of $\mathscr{O}_Y$-modules. Then $g_*(u) : g_*(\mathscr{G}) \to g_*(h_*(\mathscr{F})) = f_*(\mathscr{F})$ is a homomorphism $\mathscr{A}(\mathscr{G}) \to \mathscr{A}(\mathscr{F})$ of $\mathscr{O}_S$-modules, which we denote by $\mathscr{A}(u)$. The couple $(\mathscr{A}(h), \mathscr{A}(u))$ is then a bi-homomorphism of $\mathscr{A}(Y)$-modules $\mathscr{A}(\mathscr{G})$ ot the $\mathscr{A}(X)$-module $\mathscr{A}(\mathscr{F})$. If we fix S and consider the couples $(X, \mathscr{F})$, where X is an S-scheme and $\mathscr{F}$ is an $\mathscr{O}_X$-module, we then see that $(\mathscr{A}(X), \mathscr{A}(\mathscr{F}))$ defines a contravariant functor from the category of these couples to the category of couples of $\mathscr{O}_S$-algebras and modules of this algebra.

Consider now an S-scheme X and let $f : X \to S$ be a structural morphism. We say that X is **affine over S** if there exists a covering (S_α) of S by affine opens such that, for each α, the open subscheme $f^{-1}(S_\alpha)$ of X is affine. If this is true, we also say that X is an **affine S-scheme**, or the structural morphism f is affine.

Example 11.1.1. Any closed subscheme of S is an affine S-scheme. In fact, if Y is a closed subscheme of S, then for any affine open U of S, the intersection $U \cap Y$ is a closed subscheme of U, whence affine.

Remark 11.1.2. One should note that an affine S-scheme X is not necessary an affine scheme (for example S is affine over S, but note that this is true if S itself is affine). On the other hand, if X is an S-scheme and is affine, it is not necessarily true that X is an affine S-scheme (we will see this later). However, if S is a separated scheme, then any affine scheme is affine over S by Proposition 10.5.34.

Proposition 11.1.3. *Any affine S-scheme is separated over S.*

Proof. Recall that separatedness is local on target (Proposition 10.5.30), and if $f^{-1}(S_\alpha)$ is affine, then the restriction of f to $f^{-1}(S_\alpha)$ is a morphism between affine schemes, so is separated. $\square$

Proposition 11.1.4. *Let X be an affine S-scheme and $f : X \to S$ be the structural morphism. Then for any open subset $U \subseteq S$, $f^{-1}(U)$ is affine over U. In particular, if U is affine, so is $f^{-1}(U)$.*

Proof. In view of the definition, we can reduce to the case $S = \operatorname{Spec}(A)$ and $X = \operatorname{Spec}(B)$, so that f corresponds to a homomorphism $\rho : A \to B$. As the standard opens $D(g)$ with $g \in A$ form a basis for S, we only need to prove the assertion for $U = D(g)$. But recall that $f^{-1}(D(g)) = D(\rho(g))$, so our assertion follows. $\square$

Corollary 11.1.5. *Let S be an affine scheme. Then for an S-scheme X to be affine over S, it is necessary and sufficient that X is an affine scheme.*

Proposition 11.1.6. *Let X be an affine S-scheme and $f : X \to S$ be the structural morphism. For any quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, $f_*(\mathscr{F})$ is a quasi-coherent $\mathscr{O}_S$-module. In particular, the $\mathscr{O}_S$-algebra $\mathscr{A}(X)$ is quasi-coherent.*

Proof. The morphism f is separted by Proposition 11.1.3 and quasi-compact by Proposition 11.1.4 (since any quasi-compact open subset is a finite union of affine opens), so we can apply Proposition 10.6.55. $\square$

Proposition 11.1.7. *Let X ba an affine S-scheme. For any S-scheme Y, the map $h \mapsto \mathscr{A}(h)$ from* $\operatorname{Hom}_S(Y, X)$ *to* $\operatorname{Hom}_{\mathscr{O}_S\textbf{-alg}}(\mathscr{A}(X), \mathscr{A}(Y))$ *is bijective.*

Proof. Let $f : X \to S$, $g : Y \to S$ be the structural morphisms. Suppose first that $S = \operatorname{Spec}(A)$ and $X = \operatorname{Spec}(B)$ are affine; we must shows that for any homomorphism $\omega : f_*(\mathscr{O}_X) \to g_*(\mathscr{O}_Y)$ of $\mathscr{O}_S$-algebras, there exists a unique S-morphism $h : Y \to X$ such that $\mathscr{A}(h) = \omega$. By definition, for any open subset $U \subseteq S$, ω defines a homomorphism $\omega_U : \Gamma(f^{-1}(U), \mathscr{O}_X) \to \Gamma(g^{-1}(U), \mathscr{O}_Y)$ of $\Gamma(U, \mathscr{O}_S)$-algebras. In particular, for $U = S$, this gives a homomorphism $\varphi : \Gamma(X, \mathscr{O}_X) \to \Gamma(Y, \mathscr{O}_Y)$, which by Proposition 10.2.4, since X is affine, corresponds to a morphism $h : Y \to X$. To see that $\mathscr{A}(h) = \omega$, we need to prove that for any open subset $U \subseteq S$, ω_U coincides with the algebra homomorphism φ_U, which corresponds to the S-morphism $h|_{g^{-1}(U)} : g^{-1}(U) \to f^{-1}(U)$. We may assume that $U = D(\lambda)$ where $\lambda \in A$; then, if $f : X \to S$ corresponds to the ring homomorphism $\rho : A \to B$, we have $f^{-1}(U) = D(\mu)$ where $\mu = \rho(\lambda)$, and $\Gamma(f^{-1}(U), \mathscr{O}_X)$ is the fraction ring B_μ. Now the following diagram commutes

$$\begin{tikzcd}
B = \Gamma(X, \mathscr{O}_X) \arrow[r, "\varphi"] \arrow[d] & \Gamma(Y, \mathscr{O}_Y) \arrow[d] \\
B_\mu = \Gamma(f^{-1}(U), \mathscr{O}_X) \arrow[r, bend left, "\varphi_U"] \arrow[r, bend right, "\omega_U"'] & \Gamma(g^{-1}(U), \mathscr{O}_Y)
\end{tikzcd}$$

By the universal property of localization, we then conclude that $\varphi_U = \omega_U$, whence the assertion in this case.

In the general case, let (S_α) be a covering of S by affine opens such that $f^{-1}(S_\alpha)$ are affine. Then any homomorphism $\omega : \mathscr{A}(X) \to \mathscr{A}(Y)$ of $\mathscr{O}_S$-algebras restricts to a family of homomorphisms

$$\omega_\alpha : \mathscr{A}(f^{-1}(S_\alpha)) \to \mathscr{A}(g^{-1}(S_\alpha))$$

of $\mathscr{O}_{S_\alpha}$-algebras, so there is a family of S_α-morphisms $h_\alpha : g^{-1}(S_\alpha) \to f^{-1}(S_\alpha)$ such that $\mathscr{A}(h_\alpha) = \omega_\alpha$. It all boils down to seeing that for any affine open U of the base $S_\alpha \cap S_\beta$, the restriction of h_α and h_β to $g^{-1}(U)$ coincide, which is immediate since these restrictions both correspond to the restriction homomorphism $\mathscr{A}(X)|_U \to \mathscr{A}(Y)|_U$ of ω. □

Corollary 11.1.8. *Let X and Y be affine S-schemes. For an S-morphism $h : Y \to X$ to be an isomorphism, it is necessary and sufficient that $\mathscr{A}(h) : \mathscr{A}(X) \to \mathscr{A}(Y)$ is an isomorphism.*

Proof. This follows from Proposition 11.1.7 and the functoriality of $\mathscr{A}(X)$. □

11.1.2 Affine S-scheme associated with an $\mathscr{O}_S$-algebra

Proposition 11.1.9. *Let S be a scheme. For any quasi-coherent $\mathscr{O}_S$-algebra $\mathscr{B}$, there exists an affine S-scheme X, defined up to S-isomorphisms, such that $\mathscr{A}(X) = \mathscr{B}$. The affine S-scheme X is said to be* ***associated with the $\mathscr{O}_S$-algebra*** *$\mathscr{B}$, and denoted by* $\mathrm{Spec}(\mathscr{B})$.

Proof. The uniqueness follows from Corollary 11.1.8, so we only need to construct the affine S-scheme X. For any affine open $U \subseteq S$, let X_U be the scheme $\mathrm{Spec}(\Gamma(U, \mathscr{B}))$; as $\Gamma(U, \mathscr{B})$ is an $\Gamma(U, \mathscr{O}_S)$-algebra, X_U is an S-scheme, and is affine over U since U and X_U are both affine. Moreover, as $\mathscr{B}$ is quasi-coherent, the $\mathscr{O}_S$-algebra $\mathscr{A}(X_U)$ is canonically idnetified with $\mathscr{B}|_U$ (Proposition 10.1.12). Let V be another affine open of S, and $X_{U,V}$ be the open subscheme of X_U over $\varphi_U^{-1}(U \cap V)$, where $\varphi_U : X_U \to S$ is the structural morphism. Then $X_{U,V}$ and $X_{V,U}$ are affine over $U \cap V$ (Proposition 11.1.4), and by definition $\mathscr{A}(X_{U,V})$ and $\mathscr{A}(X_{V,U})$ are canonically identified with $\mathscr{B}|_{U \cap V}$. There then exists (Corollary 11.1.8) a canonical S-isomorphism $\theta_{U,V} : X_{V,U} \to X_{U,V}$; furthermore, if W is a third affine open of S, and if $\theta'_{U,V}, \theta'_{V,W}, \theta'_{U,W}$ are the restrictions of $\theta_{U,V}, \theta_{V,W}, \theta_{U,W}$ over the inverse images of $U \cap V \cap W$ in X_V, X_W, X_W, then $\theta'_{U,V} \circ \theta'_{V,W} = \theta'_{U,W}$. By glueing the $X_{U,V}$, there then exists a scheme X and an affine open cover (T_U) of X such that for each U there is an isomorphism $\varphi_U : X_U \to T_U$ such that φ_U maps $\varphi_U^{-1}(U \cap V)$ to $T_U \cap T_V$ and we have $\theta_{U,V} = \varphi_U^{-1} \circ \varphi_V$. The morphism $g_U = \varphi_U \circ \varphi_U^{-1}$ then makes T_U an S-scheme, and the morphisms g_U and g_V coincide on $T_U \cap T_V$, so X is an S-scheme. It is clear by definition that X is affine over S and $\mathscr{A}(T_U) = \mathscr{B}|_U$, so $\mathscr{A}(X) = \mathscr{B}$. □

Corollary 11.1.10. *Let S be a scheme. The functor $\mathscr{A}(X)$ defines an equivalence of categories between the category of affine S-schemes and the category of quasi-coherent $\mathscr{O}_S$-algebras.*

Proof. By Proposition 11.1.7 we now that $\mathscr{A}(X)$ is fully faithful, and Proposition 11.1.9 proves that it is essentially surjective, whence the claim. □

Corollary 11.1.11. *Let S be a scheme. Then for any quasi-coherent $\mathscr{O}_S$-algebra $\mathscr{B}$, the contravariant functor*

$$Y \mapsto \mathrm{Hom}_{\mathscr{O}_S\text{-}\mathbf{alg}}(\mathscr{B}, \mathscr{A}(Y)) = \mathrm{Hom}_{\mathscr{O}_Y\text{-}\mathbf{alg}}(\mathscr{B} \otimes_{\mathscr{O}_S} \mathscr{O}_Y, \mathscr{O}_Y)$$

from the category of S-schemes to the category of sets, is represented by $\mathrm{Spec}(\mathscr{B})$.

Proof. Let $X = \mathrm{Spec}(\mathscr{B})$, then we know that $\mathscr{B} = \mathscr{A}(X)$, so the claim follows from Proposition 11.1.7. □

Corollary 11.1.12. *Let X be an affine S-scheme and $f : X \to S$ be the structural morphism. For any affine open $U \subseteq S$, the open subscheme $f^{-1}(U)$ of X is an affine scheme with ring $\Gamma(U, \mathscr{A}(X))$.*

Proof. We can suppose that X is associated with the $\mathscr{O}_S$-algebra $\mathscr{A}(X)$, the corollary then follows from the construction of X in Proposition 11.1.9. □

Example 11.1.13. Let S be the affine plane for a field K with the point 0 is doubled (Example 10.5.35). With the notations there, S is the union of two affine opens Y_1, Y_2. If f is the open immersion $Y_1 \to S$, then $f^{-1}(Y_2) = Y_1 \cap Y_2$ and we have already seen in Example 10.5.35 that this is not affine. So we obtain an example of an affine scheme not affine over a scheme S.

Remark 11.1.14. Let S be a scheme and $f : X \to S$ be a quasi-compact and quasi-separated morphism, so that $\mathscr{A}(X) = f_*(\mathscr{O}_X)$ is a quasi-coherent $\mathscr{O}_S$-algebra (Proposition 10.6.55). The affine S-scheme

$$X^0 = \mathrm{Aff}(X/S) = \mathrm{Spec}(f_*(\mathscr{O}_X)) = \mathrm{Spec}(\mathscr{A}(X))$$

is called the **affine envelope** of the S-scheme X. If $f^0 : X^0 \to S$ is the structural morphism, by Proposition 11.1.9 we then have

$$\mathscr{A}(X^0) = f^0_*(\mathscr{O}_{X^0}) = \mathscr{A}(X) = f_*(\mathscr{O}_X);$$

by Corollary 11.1.11, the identity homomorphism on $\mathscr{A}(X)$ therefore corresponds to a canonical S-morphism $\iota_X : X \to X^0$ such that f factors into

$$X \xrightarrow{\iota_X} X^0 \xrightarrow{f^0} S$$

This factorization for f is called the **Stein factorization** of f. For the morphism i_X to be an isomorphism, it is necessary and sufficient that the morphism f is affine. Moreover, for any S-scheme Y affine over S, the map $u \mapsto u \circ i_X$ is then a bijection

$$\mathrm{Hom}_S(X^0, Y) \xrightarrow{\sim} \mathrm{Hom}_S(X, Y). \tag{11.1.1}$$

which is functorial on Y: this follows from the canonical bijections

$$\mathrm{Hom}_S(X, Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{O}_S\text{-}\mathbf{alg}}(\mathscr{A}(Y), \mathscr{A}(X)) = \mathrm{Hom}_{\mathscr{O}_S\text{-}\mathbf{alg}}(\mathscr{A}(Y), \mathscr{A}(X^0)) \xrightarrow{\sim} \mathrm{Hom}_S(X^0, Y).$$

That is, the S-affine scheme X^0 satisfies the universal property that any S-morphism $f : X \to Y$ such that Y is affine over S must factors through X^0, or equivalently that X^0 represents the covariant functor $Y \mapsto \mathrm{Hom}_S(X, Y)$ on the category of S-affine schemes. We also deduce that for S fixed, $X \mapsto \mathrm{Aff}(X/S)$ is a covariant functor from the category of S-schemes that are quasi-compact and quasi-separted over S to the cateogry of S-schemes affine over S. Consider a commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\iota_X"'] & X' \arrow[d, "\iota_{X'}"] \\ X^0 \arrow[r, "f^0"] & X'^0 \end{tikzcd}$$

The relation (11.1.1) can then be interpreted as the following: the functor $X \mapsto \mathrm{Aff}(X/S)$ is the left adjoint of the forgetful functor from the category of S-schemes affine over S to the category of S-schemes. We then conclude that this functor commutes with inductive limits, hence finite sums.

Corollary 11.1.15. *Let X be an affine S-scheme and Y be an X-scheme. For Y to be affine over S, it is necessary and sufficient that Y is affine over X.*

Proof. We can assume that S is affine, and then X is also affine by Corollary 11.1.5. Then Y is affine over S if and only if it is affine over X, if and only if it is affine, so our claim follows. □

Let X be an affine S-scheme. Then by Corollary 11.1.15, to define of a scheme Y affine over X is equivalent to giving a scheme Y affine over S and an S-morphism $g : Y \to X$. In view of Proposition 11.1.9 and Proposition 11.1.7, this is equivalent to giving a quasi-coherent $\mathscr{O}_S$-algebra $\mathscr{B}$ and a homomorphism $\mathscr{A}(X) \to \mathscr{B}$ of $\mathscr{O}_S$-algebras (which dfines over $\mathscr{B}$ an $\mathscr{A}(X)$-algebra structure). If $f : X \to S$ is the structural morphism, we then have $\mathscr{B} = f_*(g_*(\mathscr{O}_Y))$.

Corollary 11.1.16. *Let X be an affine S-scheme. For X to be of finite type over S, it is necessary and sufficient that the quasi-coherent $\mathscr{O}_X$-algebra $\mathscr{A}(X)$ is of finite type.*

Proof. By definition, we can assume that S is affine. Then X is an affine scheme, hence quasi-compact; if $S = \operatorname{Spec}(A)$, $X = \operatorname{Spec}(B)$, $\mathscr{A}(X)$ is the $\mathscr{O}_S$-algebra $\widetilde{B}$. As $\Gamma(X, \tilde{B}) = B$, the corollary follows from Corollary 10.6.39. □

Corollary 11.1.17. *Let X be an affine S-scheme. For X to be reduced, it is necessary and sufficient that the quasi-coherent $\mathscr{O}_X$-algebra is reduced.*

Proof. The question is local on S so we can assume that S is affine, and the corollary then follows from Proposition 10.4.30. □

11.1.3 Quasi-coherent sheaves over affine S-schemes

Proposition 11.1.18. *Let X be an affine S-scheme, Y be an S-scheme, and $\mathscr{F}$ (resp. $\mathscr{G}$) be a quasi-coherent $\mathscr{O}_X$-module (resp. $\mathscr{O}_Y$-module). Then the map $(h,u) \mapsto (\mathscr{A}(h), \mathscr{A}(u))$ from the set of morphisms $(Y,\mathscr{G}) \to (X,\mathscr{F})$ to the set of bi-homomorphisms $(\mathscr{A}(X), \mathscr{A}(\mathscr{F})) \to (\mathscr{A}(Y), \mathscr{A}(\mathscr{G}))$ is bijective.*

Proof. The proof is the same as in Proposition 11.1.7, by using Proposition 10.2.6. □

Corollary 11.1.19. *Under the hypotheses of Proposition 11.1.18, suppose that Y is affine over S. Then for the couple (h,u) to be an isomorphism, it is necessary and sufficient that $(\mathscr{A}(h), \mathscr{A}(u))$ is a bi-isomorphism.*

Proposition 11.1.20. *For any couple $(\mathscr{B}, \mathscr{M})$ formed by a quasi-coherent $\mathscr{O}_S$-algebra $\mathscr{B}$ and a quasi-coherent $\mathscr{B}$-module $\mathscr{M}$, there exists a couple $(X, \mathscr{F})$ formed by an affine S-scheme and a quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$ such that $\mathscr{A}(X) = \mathscr{B}$ and $\mathscr{A}(\mathscr{F}) = \mathscr{M}$, and this couple is determined up to isomorphisms.*

Proof. The uniqueness part follows from Corollary 11.1.19. The existence for the scheme X follows from Proposition 11.1.9. To define $\mathscr{M}$, we can consider an affine open $U \subseteq S$ and set $\mathscr{F}|_{f^{-1}(U)} = \widetilde{\Gamma(U, \mathscr{M})}$, where $f : X \to S$ is the structural morphism. We will use $\widetilde{\mathscr{M}}$ to denote the quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$ associated with $\mathscr{M}$. □

Corollary 11.1.21. *In the category of quasi-coherent $\mathscr{B}$-modules, $\widetilde{\mathscr{M}}$ is an additive covariant functor which commutes with inductive limits and direct sums.*

Proof. We can in fact assume that S is affine, and the claim then reduces to the functor $\widetilde{M}$ for B-modules, where $B = \Gamma(S, \mathscr{B})$. □

Corollary 11.1.22. *Under the hypotheses of Proposition 11.1.20, assume that $\mathscr{B}$ is an $\mathscr{O}_X$-algebra of finite type. Then for $\widetilde{\mathscr{M}}$ to be an $\mathscr{O}_X$-module of finite type, it is necessary and sufficient that $\mathscr{M}$ is an $\mathscr{B}$-module of finite type.*

Proof. We can reduce to the case where $S = \operatorname{Spec}(A)$ is affine. Then $\mathscr{B} = \widetilde{B}$ where B is an A-algebra of finite type, and $\mathscr{M} = \widetilde{M}$ where M is an B-module. Over the scheme X, $\mathscr{O}_X$ is associated with the ring B and $\widetilde{\mathscr{M}}$ is associated with the B-module M. For $\widetilde{\mathscr{M}}$ to be of finite type, it is necessary and sufficient that M is of finite type, whence our claim. □

Proposition 11.1.23. *Let Y be an affine S-scheme and X, X' be two schemes affine over Y. Let $\mathscr{B} = \mathscr{A}(Y)$, $\mathscr{A} = \mathscr{A}(X)$, and $\mathscr{A}' = \mathscr{A}(X')$. Then $X \times_Y X'$ is affine over Y and $\mathscr{A}(X \times_Y X')$ is identified with $\mathscr{A} \otimes_{\mathscr{B}} \mathscr{A}'$.*

Proof. In fact, $\mathscr{A} \otimes_{\mathscr{B}} \mathscr{A}'$ is a quasi-coherent $\mathscr{B}$-algebra (Proposition 10.2.25), so is a quasi-coherent $\mathscr{O}_S$-algebra. Let Z be the spectral of $\mathscr{A} \otimes_{\mathscr{B}} \mathscr{A}'$. The canonical $\mathscr{B}$-homomorphisms $\mathscr{A} \to \mathscr{A} \otimes_{\mathscr{B}} \mathscr{A}'$ and $\mathscr{A}' \to \mathscr{A} \otimes_{\mathscr{B}} \mathscr{A}'$ corresponds to Y-morphisms $p : Z \to X$ and $p' : Z \to X'$ (Proposition 11.1.7). To see that the triple (Z, p, p') is a product $X \times_Y X'$, we can reduce to the case $S = \operatorname{Spec}(C)$ is affine. But then Y, X, X' are all affine schemes with rings B, A, A', which are C-algebras such that $\mathscr{B} = \widetilde{B}$, $\mathscr{A} = \widetilde{A}$, $\mathscr{A}' = \widetilde{A}$. We then see that $\mathscr{A} \otimes_{\mathscr{B}} \mathscr{A}'$ is identified with the $\mathscr{O}_S$-algebra $\widetilde{A \otimes_B A'}$ (Corollary 10.1.10), so the ring of Z is identified with $A \otimes_B A'$, and the morphisms p, p' correspond to the canonical homomorphisms $A \to A \otimes_B A'$ and $A' \to A \otimes_B A'$. The proposition then follows from Proposition 10.3.1. □

Corollary 11.1.24. *Let $\mathscr{F}$ (resp. $\mathscr{F}'$) be a quasi-coherent $\mathscr{O}_X$-module (resp. $\mathscr{O}_{X'}$-module). Then $\mathscr{A}(\mathscr{F} \otimes_Y \mathscr{F}')$ is canonically identified with $\mathscr{A}(\mathscr{F}) \otimes_{\mathscr{A}(Y)} \mathscr{A}(\mathscr{F}')$.*

Proof. The sheaf $\mathscr{F} \otimes_Y \mathscr{F}'$ is coherent over $X \times_Y X'$ by **??**. Let $g : Y \to S$, $f : X \to Y$, $f' : X' \to Y$ be the structural morphisms, so the structural morphism $h : Z \to S$ is equal to $g \circ f \circ p$ and to $g \circ f' \circ p'$. We define a canonical homomorphism

$$\mathscr{A}(\mathscr{F}) \otimes_{\mathscr{A}(Y)} \mathscr{A}(\mathscr{F}') \to \mathscr{A}$$

by the following: for any open subset $U \subseteq S$, we have canonical homomorphisms

$$\Gamma(f^{-1}(g^{-1}(U)), \mathscr{F}) \to \Gamma(h^{-1}(U), p^*(\mathscr{F})), \quad \Gamma(f'^{-1}(g^{-1}(U)), \mathscr{F}') \to \Gamma(h^{-1}(U), p'^*(\mathscr{F}')),$$

whence a canonical homomorphism

$$\begin{array}{c} \Gamma(f^{-1}(g^{-1}(U)), \mathscr{F}) \otimes_{\Gamma(g^{-1}(U), \mathscr{O}_Y)} \Gamma(f'^{-1}(g^{-1}(U)), \mathscr{F}') \\ \big\downarrow \\ \Gamma(h^{-1}(U), p^*(\mathscr{F})) \otimes_{\Gamma(h^{-1}(U), \mathscr{O}_Z)} \Gamma(h^{-1}(U), p'^*(\mathscr{F}')) \end{array}$$

To see this is an isomorphism of $\mathscr{A}(Z)$-modules, we can assume that S is affine, and $\mathscr{F} = \widetilde{M}$, $\mathscr{F}' = \widetilde{M}'$, where M (resp. M') is an A-module (resp. A'-module). Then $\mathscr{F} \otimes_Y \mathscr{F}'$ is identified with the sheaf over $X \times_Y X'$ associated with the $(A \otimes_B A')$-module $M \otimes_B M'$ and the corollary follows from the canonical indentification $\widetilde{M \otimes_B M'}$ with $\widetilde{M} \otimes_{\widetilde{B}} \widetilde{M}'$. □

Corollary 11.1.25. *Let X and Y be affine S-schemes and $f : Y \to X$ be an S-morphism. Let $\mathscr{F}$ be a quasi-coherent $\mathscr{O}_X$-module, then $\mathscr{A}(f^*(\mathscr{F}))$ is identified with $\mathscr{A}(\mathscr{F}) \otimes_{\mathscr{A}(X)} \mathscr{A}(Y)$.*

Proof. This is a special case of Corollary 11.1.24, by replacing X' with Y and Y with X. □

In particular, if $X = X' = Y$ (where X is an affine S-scheme), we see that if $\mathscr{F}, \mathscr{G}$ are two quasi-coherent $\mathscr{O}_X$-modules, then

$$\mathscr{A}(\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G}) = \mathscr{A}(\mathscr{F}) \otimes_{\mathscr{A}(X)} \mathscr{A}(\mathscr{G})$$

If moreover $\mathscr{F}$ is of finite presentation, then it follows from Proposition 10.1.12 and Corollary 10.1.10 that

$$\mathscr{A}(\mathcal{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})) = \mathcal{H}om_{\mathscr{A}(X)}(\mathscr{A}(\mathscr{F}), \mathscr{A}(\mathscr{G})).$$

Proposition 11.1.26. *If X and X' are two affine S-schemes with $\mathscr{B} = \mathscr{A}(X)$ and $\mathscr{B}' = \mathscr{A}(X')$. Then the coproduct $X \amalg X'$ is affine over S with $\mathscr{A}(X \amalg X') = \mathscr{B} \times \mathscr{B}'$.*

Proof. The coproduct is affine over S since the product of two affine schemes is affine, and the second assertion also follows from this, and the fact that $\operatorname{Spec}(A) \amalg \operatorname{Spec}(A') = \operatorname{Spec}(A \times A')$ for two rings A, A'. $\square$

Proposition 11.1.27. *Let S be a scheme, $\mathscr{B}$ be a quasi-coherent $\mathscr{O}_S$-algebra, and $X = \operatorname{Spec}(\mathscr{B})$. For any quasi-coherent ideal $\mathscr{I}$ of $\mathscr{B}$, $\widetilde{\mathscr{I}}$ is a quasi-coherent ideal of $\mathscr{O}_X$, and the closed subschem of X defined by $\widetilde{\mathscr{I}}$ is canonically isomorphic to $\operatorname{Spec}(\mathscr{B}/\mathscr{I})$.*

Proof. In fact, it follows from Example 10.4.6 that Y is affine over S, and in view of Proposition 11.1.9, we can then assume that S is affine, and the proposition follows from the corresponding result in affine schemes. $\square$

We can also express the result of Proposition 11.1.27 by saying that if $h : \mathscr{B} \to \mathscr{B}'$ is a surjective homomorphism of quasi-coherent $\mathscr{O}_S$-algebras, then $\mathscr{A}(h) : \operatorname{Spec}(\mathscr{B}') \to \operatorname{Spec}(\mathscr{B})$ is a closed immersion.

Proposition 11.1.28. *Let S be a scheme, $\mathscr{B}$ be a quasi-coherent $\mathscr{O}_S$-algebra, and $X = \operatorname{Spec}(\mathscr{B})$. For any quasi-coherent ideal $\mathscr{K}$ of $\mathscr{O}_S$, we have (where $f : X \to S$ is the structural morphism) $f^*(\mathscr{K})\mathscr{O}_X = \widetilde{\mathscr{K}\mathscr{B}}$.*

Proof. The question is local over S, so we can assume that $S = \operatorname{Spec}(A)$ is affine, and the proposition then follows Proposition 10.1.14. $\square$

11.1.4 Base change of affine S-schemes

Proposition 11.1.29. *Let X be an affine S-scheme. For any extension $g : S' \to S$ of base scheme, $X' = X_{(S')}$ is affine over S'.*

Proof. If $f' : X' \to S'$ is the projection, it suffices to prove that $f'^{-1}(U')$ is an affine open for any affine open subset U' of S' such that $g(U')$ is contained in an affine open U of S. We can then assume that S and S' are affine, so X is affine. But then X' is affine, so the claim follows. $\square$

Corollary 11.1.30. *Let $f : X \to S$ be the structural morphism, $f' : X' \to S'$, $g' : X' \to X$ the projections such that the diagram*

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$

is commutative. For any quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, there exists a canonical isomorphism $u : g^*(f_*(\mathscr{F})) \to f'_*(g'^*(\mathscr{F}))$ of $\mathscr{O}_{S'}$-modules. In particular, there exists a canonical isomorphism from $\mathscr{A}(X')$ to $g^*(\mathscr{A}(X))$.

Proof. To define u, it suffices to define a homomorphism

$$v : f_*(\mathscr{F}) \to g_*(f'^*(\mathscr{F})) = f_*(g'_*(g'^*(\mathscr{F})))$$

and let u be the homomorphism corresponding to v (via the adjointness). We set $v = f_*(\rho)$, where $\rho : \mathscr{F} \to g'_*(g'^*(\mathscr{F}))$ is the canonical homomorphism. To prove that u is an isomorphism, we can assume that S and S', hence X and X', are affine. Let A, A', B, B' be the ring of X, X', S, S', then $\mathscr{F} = \widetilde{M}$ where M is an B-module. We then see that $g^*(f_*(\mathscr{F}))$ and $f'_*(g'^*(\mathscr{F}))$ are equal to the $\mathscr{O}_{S'}$-module associated with the A'-module $A' \otimes_A M$, and u is the homomorphism associated with the identity. $\square$

Corollary 11.1.31. *For any affine S-scheme X and $s \in S$, the fiber X_s is an affine scheme.*

Proof. It suffices to apply Proposition 11.1.29 on $\mathrm{Spec}(\kappa(s)) \to S$. $\square$

Corollary 11.1.32. *Let X be an S-scheme and S' be an affine S-scheme. Then $X' = X_{(S')}$ is affine over X. Moreover, if $f : X \to S$ is the structural homomorphism, there exists a canonical isomorphism of $\mathscr{O}_X$-algebras $\mathscr{A}(X') \cong f^*(\mathscr{A}(S'))$, and for any quasi-coherent $\mathscr{A}(S')$-module $\mathscr{M}$, a canonical bi-isomorphism $f^*(\mathscr{M}) \cong \mathscr{A}(f'^*(\widetilde{\mathscr{M}}))$, where $f' = f_{(S')}$ is the structural morphism $X' \to S'$.*

Proof. It suffices to apply Proposition 11.1.29 and Corollary 11.1.30, with the role of X and S' exchanged. $\square$

Let S, S' be two schemes, $q : S' \to S$ be a morphism, $\mathscr{B}$ (resp. $\mathscr{B}'$) be a quasi-coherent $\mathscr{O}_S$-algebra (resp. $\mathscr{O}_{S'}$-algebra), and $u : \mathscr{B} \to \mathscr{B}'$ be a q-morphism (which means a homomorphism $\mathscr{B} \to q_*(\mathscr{B}')$ of $\mathscr{O}_S$-algebras). If $X = \mathrm{Spec}(\mathscr{B})$ and $X' = \mathrm{Spec}(\mathscr{B}')$, we deduce a canonical morphism $v = \mathrm{Spec}(u) : X' \to X$ such that the diagram

$$\begin{CD} X' @>{v}>> X \\ @VVV @VVV \\ S' @>{q}>> S \end{CD} \tag{11.1.2}$$

is commutatie. In fact, the homomorphism u corresponds to a homomorphism $u^\sharp : q^*(\mathscr{B}) \to \mathscr{B}'$ by adjointness, and there then exists a canonical S'-morphism

$$w : \mathrm{Spec}(\mathscr{B}') \to \mathrm{Spec}(q^*(\mathscr{B}))$$

such that $\mathscr{A}(w) = u^\sharp$ (Proposition 11.1.7). On the other hand, it follows from Corollary 11.1.30 that $\mathrm{Spec}(q^*(\mathscr{B}))$ is canonically identified with $X \times_S S'$; the morphism v is the defined to be the composition

$$X' \xrightarrow{w} X \times_S S' \xrightarrow{p_1} X$$

where p_1 is the projection, and the commutativity of (11.1.2) is easily verified. Let U (resp. U') be an affine open of S (resp. S') such that $q(U') \subseteq U$, $A = \Gamma(U, \mathscr{O}_S)$, $A' = \Gamma(U', \mathscr{O}_{S'})$, $B = \Gamma(U, \mathscr{B})$, $B' = \Gamma(U'\mathscr{B}')$. The restriction of u is a $(q|_{U'})$-morphism $u|_U : \mathscr{B}|_U \to \mathscr{B}'|_{U'}$ corresponding to a bi-homomorphism $B \to B'$ of algebras. If V, V' are the inverse images of U, U' in X, X', respectively, the morphism $V' \to V$, which is the restriction of v, corresponds to the preceding bi-homomorphism.

Now let $\mathscr{M}$ be a quasi-coherent $\mathscr{B}$-module. There then exists a canonical isomorphism of $\mathscr{O}_{X'}$-modules

$$v^*(\widetilde{\mathscr{M}}) \cong (q^*(\mathscr{M}) \otimes_{q^*(\mathscr{B})} \mathscr{B}')^{\sim}. \tag{11.1.3}$$

In fact, the canonical isomorphism of Corollary 11.1.30 provides a canonical isomorphism of $p_1^*(\widetilde{\mathscr{M}})$ with the sheaf over $\mathrm{Spec}(q^*(\mathscr{B}))$ associated with $q^*(\mathscr{B})$-module $q^*(\mathscr{M})$, and it suffices to apply Corollary 11.1.24.

Recall that we say a morphism $f : X \to Y$ is affine if X is an affine scheme over Y. The properties of affine S-schemes then translate into properties of affine morphisms.

Proposition 11.1.33 (Properties of Affine Morphisms).

(i) *A closed immersion is affine.*

(ii) *The composition of two affine morphisms is affine.*

(iii) *If $f : X \to Y$ is an affine S-morphism, then $f_{(S')}$ is affine for any base change $S' \to S$.*

(iv) *If $f : X \to Y$ and $f' : X' \to Y'$ are two affine S-morphisms, then $f \times_S f'$ is affine.*

(v) *If $f : X \to Y$ and $g : Y \to Z$ are two morphisms such that $g \circ f$ is affine and g is separated, then f is affine.*

(vi) *If f is affine, so is f_{red}.*

Proof. In view of Proposition 10.5.22, it suffices to prove (i), (ii), and (iii). Now (i) follows from Example 10.4.6, (ii) follows from Corollary 11.1.5, and (iii) follows from Proposition 11.1.29. □

Corollary 11.1.34. *If X is an affine scheme and Y is a separated scheme, any morphism $f : X \to Y$ is affine.*

Proof. This is a direct concequence of Proposition 11.1.33(v), since the canonical morphism $X \to \mathrm{Spec}(\mathbb{Z})$ is affine. □

Proposition 11.1.35. *Let Y be a locally Noetherian scheme and $f : X \to Y$ be a morphism of finite type. Then for f to be affine, it is necessary and sufficient that f_{red} is affine.*

Proof. It suffcies to prove that f is affine if f_{red} is affine. For this, we can assume that Y is affine and Noetherian, and show that X is affine. Now Y_{red} is affine, and X_{red} is therefore affine by hypothesis. Since X is Noetherian, the assertion follows from Corollary 10.4.37. □

11.1.5 Vector bundles

Let A be a ring and E be an A-module. Recall that the symmetric algebra over E is the A-algebra $\boldsymbol{S}(E)$ (or $\boldsymbol{S}_A(E)$) which is the quotient of $\boldsymbol{T}(E)$ by the ideal generated by elements $x \otimes y - y \otimes x$, where x, y belongs to E. The algebra $\boldsymbol{S}(E)$ is characterized by the universal property that if $\sigma : E \to \boldsymbol{S}(E)$ is the canonical map, any A-linear map $E \to B$, where B is a commutative algebra, factors through $\boldsymbol{S}(E)$ and gives a homomorphism $\boldsymbol{S}(E) \to B$ of A-algebras. We deduce from this property that for two A-modules E, F, we have

$$\boldsymbol{S}(E \oplus F) = \boldsymbol{S}(E) \otimes \boldsymbol{S}(F).$$

Moreover, $\boldsymbol{S}(E)$ is a covariant functor on E from the category of A-modules to that of commutative A-algebras. Finally, the preceding characterization shows that if $E = \varinjlim E_\lambda$, then $\boldsymbol{S}(E) = \varinjlim \boldsymbol{S}(E_\lambda)$. By abuse of language, a product $\sigma(x_1) \cdots \sigma(x_n)$, where $x_i \in E$, is usually written as $x_1 \cdots x_n$. The algebra $\boldsymbol{S}(E)$ is graded, where $\boldsymbol{S}_n(E)$ is the set of linear combinations of products of n elements of E. In particular, the algebra $\boldsymbol{S}(A)$ is canonically isomorphic to the polynomial algebra $A[T]$ over A, and the algebra $\boldsymbol{S}(A^n)$ is the polynomial algebra $A[T_1, \dots, T_n]$ over A. More particularly, if E is free of rank 1, then $\boldsymbol{S}_n(E)$ is isomorphic to the tensor algebra $\boldsymbol{T}(E) = \bigoplus_{n \geq 0} E^{\otimes n}$.

Let $\varphi : A \to B$ be a ring homomorphism. If F is an B-module, the canonical map $F \to \boldsymbol{S}(F)$ then gives a canonical map $F_{(\varphi)} \to \boldsymbol{S}(F)_{(\varphi)}$, which factors into $F_{(\varphi)} \to \boldsymbol{S}(F_{(\varphi)}) \to \boldsymbol{S}(F)_{(\varphi)}$. The canonical homomorphism $\boldsymbol{S}(F_{(\varphi)}) \to \boldsymbol{S}(F)_{(\varphi)}$ is surjective, but not necessarily bijective. If E is an A-module, any bi-homomorphism $E \to F$ (which is an A-homomorphism $E \to F_{(\varphi)}$) then gives an A-homomorphism $\boldsymbol{S}(E) \to \boldsymbol{S}(F_{(\varphi)}) \to \boldsymbol{S}(F)_{(\varphi)}$ of algebras, which is a bi-homomorphism $\boldsymbol{S}(E) \to \boldsymbol{S}(F)$. Also, for any A-module E, $\boldsymbol{S}(E \otimes_A B)$ is canonically identified with the algebra $\boldsymbol{S}(E) \otimes_A B$, which follows from the universal property of $\boldsymbol{S}(E)$.

Let R be a multiplicative subset of A. Then apply the previous arguments for the ring $B = R^{-1}A$, and recall that $R^{-1}E = E \otimes_A R^{-1}A$, we see that $\boldsymbol{S}(R^{-1}E) = R^{-1}\boldsymbol{S}(E)$. Moreover, if $R' \supseteq R$ is another multiplicative subset of A, the diagram

$$\begin{tikzcd}
R^{-1}E \arrow[r] \arrow[d] & R'^{-1}E \arrow[d] \\
\boldsymbol{S}(R^{-1}E) \arrow[r] & \boldsymbol{S}(R'^{-1}E)
\end{tikzcd}$$

is commutative.

Now let $(S, \mathscr{A})$ be a ringed space and $\mathscr{E}$ be an $\mathscr{A}$-module over S. If for each open subset $U \subseteq S$, we associate the $\Gamma(U, \mathscr{A})$-module $\boldsymbol{S}(\Gamma(U, \mathscr{E}))$, we then define a presheaf of algebras. The associated sheaf is called the **symmetric $\mathscr{A}$-algebra** of the $\mathscr{A}$-module $\mathscr{E}$ and denoted by $\boldsymbol{S}(\mathscr{E})$. It follows immediately that $\boldsymbol{S}(\mathscr{E})$ satisfies the following universal property: any homomorphism $\mathscr{E} \to \mathscr{B}$ of $\mathscr{A}$-modules, where $\mathscr{B}$ is an $\mathscr{A}$-algebra, factors through $\boldsymbol{S}(\mathscr{E})$ to give a homomorphism $\boldsymbol{S}(\mathscr{E}) \to \mathscr{B}$ of $\mathscr{A}$-algebras. In particular, any homomorphism $u : \mathscr{E} \to \mathscr{F}$ of $\mathscr{A}$-modules defines a homomorphism $\boldsymbol{S}(u) : \boldsymbol{S}(\mathscr{E}) \to \boldsymbol{S}(\mathscr{F})$ of $\mathscr{A}$-algebras and $\boldsymbol{S}(\mathscr{E})$ is a covariant functor $\mathscr{E}$.

Now since the functor $\boldsymbol{S}$ commutes with inductive limits, we have $\boldsymbol{S}(\mathscr{E})_s = \boldsymbol{S}(\mathscr{E}_s)$ for any point $s \in S$. If $\mathscr{E}, \mathscr{F}$ are two $\mathscr{A}$-module, $\boldsymbol{S}(\mathscr{E} \oplus \mathscr{F})$ is canonically identified with $\boldsymbol{S}(\mathscr{E}) \otimes_{\mathscr{A}} \boldsymbol{S}(\mathscr{F})$, as we can check this for the corresponding presheaves.

We see that $\boldsymbol{S}(\mathscr{E})$ is a graded $\mathscr{A}$-algebra, and $\boldsymbol{S}_n(\mathscr{E})$ is the $\mathscr{A}$-module associated with the presheaf $U \mapsto \boldsymbol{S}_n(\Gamma(U, \mathscr{E}))$. In particular, the algebra $\boldsymbol{S}(\mathscr{A})$ is identified with $\mathscr{A}[T] = \mathscr{A} \otimes_{\mathbb{Z}} \mathbb{Z}[T]$, and if $\mathscr{E}$ is an invertible sheaf, then $\boldsymbol{S}(\mathscr{E})$ is isomorphic to the tensor algebra $\boldsymbol{T}(\mathscr{E}) = \bigoplus_{n \geq 0} \mathscr{E}^{\otimes n}$.

Proposition 11.1.36. *Let $f : (S, \mathscr{A}) \to (T, \mathscr{B})$ be a morphism of ringed spaces. If $\mathscr{F}$ is an $\mathscr{B}$-module, then $\boldsymbol{S}(f^*(\mathscr{F}))$ is canonically identified with $f^*(\boldsymbol{S}(\mathscr{F}))$*

Proof. To see this, we may make use the universal property of $\boldsymbol{S}$. By definition, $\boldsymbol{S}(f^*(\mathscr{F}))$ is defined to be the unique $\mathscr{A}$-algebra satisfying the following equality

$$\mathrm{Hom}_{\mathscr{A}\text{-}\mathbf{alg}}(\boldsymbol{S}(f^*(\mathscr{F})), \mathscr{C}) = \mathrm{Hom}_{\mathscr{A}}(f^*(\mathscr{F}), \mathscr{C})$$

for any $\mathscr{A}$-algebra $\mathscr{C}$. On the other hand, by the adjointness property of f_* and f^*, we have

$$\begin{aligned}\mathrm{Hom}_{\mathscr{A}\text{-}\mathbf{alg}}(f^*(\boldsymbol{S}(\mathscr{F})), \mathscr{C}) &= \mathrm{Hom}_{\mathscr{B}\text{-}\mathbf{alg}}(\boldsymbol{S}(\mathscr{F}), f_*(\mathscr{C})) \\ &= \mathrm{Hom}_{\mathscr{B}}(\mathscr{F}, f_*(\mathscr{C})) = \mathrm{Hom}_{\mathscr{A}}(f^*(\mathscr{F}), \mathscr{C}).\end{aligned}$$

This implies the desired isomorphism. □

Proposition 11.1.37. *Let A be a ring, $S = \mathrm{Spec}(A)$ be the spectrum, and $\mathscr{E} = \widetilde{M}$ be the $\mathscr{O}_S$-module assocaited with an A-module M. Then the $\mathscr{O}_S$-algebra $\boldsymbol{S}(\mathscr{E})$ is associated with the A-algebra $\boldsymbol{S}(M)$.*

Proof. In fact, for any $f \in A$, $\boldsymbol{S}(M_f) = \boldsymbol{S}(M)_f$, so the proposition follows from the definition of $\widetilde{\boldsymbol{S}(M)}$. □

Corollary 11.1.38. *If S is a scheme and $\mathscr{E}$ is a quasi-coherent $\mathscr{O}_S$-module. Then the $\mathscr{O}_S$-algebra $\boldsymbol{S}(\mathscr{E})$ is quasi-coherent. If moreover $\mathscr{E}$ is of finite type (resp. of finite presentation), then each $\mathscr{O}_S$-module $\boldsymbol{S}_n(\mathscr{E})$ is of finite type (resp. finite presentation) and the $\mathscr{O}_S$-algebra $\boldsymbol{S}(\mathscr{E})$ is of finite type (resp. of finite presentation).*

Proof. The first assertion is immediate by Proposition 11.1.37. The second one follows from the fact that, if E is a finitely generated A-module, $\boldsymbol{S}_n(E)$ is also finitely generated. For the last assertion, we are reduced to the case $S = \mathrm{Spec}(A)$ and $\mathscr{E} = \widetilde{E}$ where E is an A-module of finite type (resp. of finite presentation). Now if we have an exact sequence

$$0 \longrightarrow N \longrightarrow A^n \longrightarrow E \longrightarrow 0$$

then we deduce an exact sequence

$$0 \longrightarrow \mathfrak{I} \longrightarrow \boldsymbol{S}(A^n) \longrightarrow \boldsymbol{S}(E) \longrightarrow 0$$

where $\mathfrak{I}$ is the ideal of $\boldsymbol{S}(A^n)$ generated by $N \subseteq \boldsymbol{S}_1(A^n)$, whence our conclusion. □

Let S be a scheme and $\mathscr{E}$ be a quasi-coherent $\mathscr{O}_S$-module. For any S-scheme T, with structural morphism $f : T \to S$, let $\mathscr{E}_{(T)} = f^*(\mathscr{E})$, which is a quasi-coherent $\mathscr{O}_T$-module. The map

$$T \mapsto F_{\mathscr{E}}(T) = \mathrm{Hom}_{\mathscr{O}_T}(\mathscr{E}_{(T)}, \mathscr{O}_T) = \Gamma(T, \mathscr{E}^{\vee}_{(T)})$$

then defines a contravariant functor from the category of S-schemes to that of sets if for any S-morphism $g : T' \to T$ we define $F_{\mathscr{E}}(g) : F_{\mathscr{E}}(T) \to F_{\mathscr{E}}(T')$ to be the map $g^* : u \mapsto g^*(u)$ (note that the structural morphism $T' \to S$ is $f \circ g$ and we have $\mathscr{E}_{(T')} = g^*(\mathscr{E}_{(T)})$ and $\mathscr{O}_{T'} = g^*(\mathscr{O}_T)$).

Proposition 11.1.39. *For any quasi-coherent $\mathscr{O}_S$-module $\mathscr{E}$, the contravariant functor $F_{\mathscr{E}}$ is represented by the couple formed by the affine S-scheme $\mathbb{V}(\mathscr{E}) = \mathrm{Spec}(\boldsymbol{S}(\mathscr{E}))$. The S-scheme $\mathbb{V}(\mathscr{E})$ is called the* ***vector bundle over S defined by*** $\mathscr{E}$.

Proof. This follows from the following canonical isomorphisms for any S-scheme T:

$$\begin{aligned}\mathrm{Hom}_S(T, \mathbb{V}(\mathscr{E})) &= \mathrm{Hom}_{\mathscr{O}_S\text{-}\mathbf{alg}}(\boldsymbol{S}(\mathscr{E}), \mathscr{A}(T)) = \mathrm{Hom}_{\mathscr{O}_S}(\mathscr{E}, f_*(\mathscr{O}_T)) \\ &= \mathrm{Hom}_{\mathscr{O}_T}(f^*(\mathscr{E}), \mathscr{O}_T) = \mathrm{Hom}_{\mathscr{O}_T}(\mathscr{E}_{(T)}, \mathscr{O}_T).\end{aligned}$$

□

The canonical $\boldsymbol{S}(\mathscr{E})$-homomorphism $\mathscr{E} \otimes_{\mathscr{O}_S} \boldsymbol{S}(\mathscr{E}) \to \boldsymbol{S}(\mathscr{E})$ induced by Proposition 11.1.7 an $\mathscr{O}_{\mathbb{V}(\mathscr{E})}$-homomorphism $\mathscr{E}_{(\mathbb{V}(\mathscr{E}))} \to \mathscr{O}_{\mathbb{V}(\mathscr{E})}$, which is a section over $\mathbb{V}(\mathscr{E})$ of dual sheaf $\mathscr{E}^*_{(\mathbb{V}(\mathscr{E}))}$ of $\mathscr{E}_{(V(\mathscr{E}))}$, called the **universal section** of this dual. If $U = \mathrm{Spec}(A)$ is an affine open of S, its inverse image in $\mathbb{V}(\mathscr{E})$ is identified with $\mathrm{Spec}(\boldsymbol{S}(M))$, if $\mathscr{E}|_U = \widetilde{M}$ where M is an A-module. Over the scheme $\mathrm{Spec}(\boldsymbol{S}(M))$, the universal section is identified with the homomorphism $m \otimes p \mapsto mp$ of $M \otimes_A \boldsymbol{S}(M)$ to $\boldsymbol{S}(M)$, where M is identified with the subset $\boldsymbol{S}_1(M)$ of $\boldsymbol{S}(M)$.

Consider in particular an open subset U of S. Then the S-morphisms $U \to \mathbb{V}(\mathscr{E})$ are the U-sections of the U-scheme induced by $\mathbb{V}(\mathscr{E})$ over $p^{-1}(U)$ (where $p : \mathbb{V}(\mathscr{E}) \to S$ is the structural morphism). By the definition of $\mathbb{V}(\mathscr{E})$, these U-sections correspond bijectively to sections of the dual $\mathscr{E}^*$ of $\mathscr{E}$ over U. The functorial of $\mathbb{V}$ shows that this interpretation is compatible with the restriction to an open subset $U' \subseteq U$, so we can say that the dual $\mathscr{E}^*$ of $\mathscr{E}$ is canonically identified with the sheaf of germs of S-sections of $\mathbb{V}(\mathscr{E})$. In particular, if $T = S$, the zero homomorphism $\mathscr{E} \to \mathscr{O}_S$ corresponds to an S-section of $\mathbb{V}(\mathscr{E})$, called the **zero section**.

If now we choose T to be the spectrum $\{\xi\}$ of a field K, the structural morphism $f : T \to S$ corresponds to a monomorphism $\kappa(s) \to K$, where $s = f(\xi)$ (Corollary 10.2.17), and the S-morphisms $\{\xi\} \to \mathbb{V}(\mathscr{E})$ are none other than points of $\mathbb{V}(\mathscr{E})$ with values in the extension K of $\kappa(s)$, which all locate at some points of $p^{-1}(s)$. The set of these points, which is called the **rational fiber** of $\mathbb{V}(\mathscr{E})$ over K lying over the point s, is then identified (by the definition of $\mathbb{V}(\mathscr{E})$) with the dual of the K-vector space $\mathscr{E}_s \otimes_{\mathscr{O}_{S,s}} K = \mathscr{E}^s \otimes_{\kappa(s)} K$ where we set $\mathscr{E}^s = \mathscr{E}_s/\mathfrak{m}_s\mathscr{E}_s$. If $\mathscr{E}^s$ and K are of finite rank over $\kappa(s)$, this dual is identified with $(\mathscr{E}^s)^* \otimes_{\kappa(s)} K$, where $(\mathscr{E}^s)^*$ is the dual space of the $\kappa(s)$-vector space $\mathscr{E}^s$.

These properties justify the terminology of "vector bundle" introduced above, but note that the definition we obtained is dual to the classical definition, since one would expect to obtain the space $\mathscr{E}^s \otimes_{\kappa(s)} K$ for the fiber of $\mathbb{V}(\mathscr{E})$, rather than its dual. This distinction is imposed for the need of defining $\mathbb{V}(\mathscr{E})$ for any quasi-coherent $\mathscr{O}_S$-module $\mathscr{E}$, not only for locally free $\mathscr{O}_S$-modules of finite rank. We can indeed show that the functor $T \mapsto \Gamma(T, \mathscr{E}_{(T)})$ is only representable if $\mathscr{E}$ is locally free of finite rank.

Proposition 11.1.40. *Let S be a scheme.*

(i) *$\mathbb{V}$ is a contravariant functor on $\mathscr{E}$ from the category of quasi-coherent $\mathscr{O}_S$-modules to the category of affine S-schemes.*

(ii) *If $\mathscr{E}$ is of finite type (resp. of finite presentation), $\mathbb{V}(\mathscr{E})$ is of finite type (resp. of finite presentation) over S.*

(iii) *If $\mathscr{E}$ and $\mathscr{F}$ are two quasi-coherent $\mathscr{O}_S$-modules, $\mathbb{V}(\mathscr{E}\oplus\mathscr{F})$ is canonically identified with $\mathbb{V}(\mathscr{E})\times_S\mathbb{V}(\mathscr{F})$.*

(iv) *Let $g: S'\to S$ be a morphism. For any quasi-coherent $\mathscr{O}_S$-module $\mathscr{E}$, $\mathbb{V}(g^*(\mathscr{E}))$ is canonically identified with $\mathbb{V}(\mathscr{E})_{(S')}=\mathbb{V}(\mathscr{E})\times_S S'$.*

(v) *A surjective homomorphism $\mathscr{E}\to\mathscr{F}$ of quasi-coherent $\mathscr{O}_S$-modules corresponds to a closed immersion $\mathbb{V}(\mathscr{F})\to\mathbb{V}(\mathscr{E})$.*

Proof. Assertion (i) follows from Proposition 11.1.7, since for any homomorphism $\mathscr{E}\to\mathscr{F}$ of $\mathscr{O}_S$-modules we have a homomorphism $\boldsymbol{S}(\mathscr{E})\to\boldsymbol{S}(\mathscr{F})$ of $\mathscr{O}_S$-algebras. Assertion (ii) follows immediately from Corollary 11.1.22 and Corollary 11.1.38. To prove (iii), it suffices to recall the canonical isomorphism $\boldsymbol{S}(\mathscr{E}\oplus\mathscr{F})\cong\boldsymbol{S}(\mathscr{E})\otimes_{\mathscr{O}_S}\boldsymbol{S}(\mathscr{F})$ and apply Proposition 11.1.23. Similarly, to prove (iv), it suffices to remark that if the homomorphism $\mathscr{E}\to\mathscr{F}$ is surjective, so is the corresponding homomorphism $\boldsymbol{S}(\mathscr{E})\to\boldsymbol{S}(\mathscr{F})$ of $\mathscr{O}_S$-algebras, and apply Proposition 11.1.27. □

Example 11.1.41. Consider in particular $\mathscr{E}=\mathscr{O}_S$. The scheme $\mathbb{V}(\mathscr{O}_S)$ is the specturm of the $\mathscr{O}_S$-algebra $\boldsymbol{S}(\mathscr{O}_S)$, which is identified with $\mathscr{O}_S[T]=\mathscr{O}_S\otimes_{\mathbb{Z}}\mathbb{Z}[T]$. This is evident if $S=\mathrm{Spec}(\mathbb{Z})$, in view of Proposition 11.1.37, and we pass from this to the general case by considering the structural morphism $S\to\mathrm{Spec}(\mathbb{Z})$ and using Proposition 11.1.40(iv). Because of this result, we again set $\mathbb{V}(\mathscr{O}_S)=S[T]$, and we obtain the identification of the sheaf of germs of S-sections of $S[T]$ over $\mathscr{O}_S$ as a particular case.

For any S-scheme X, by the definition of $\mathbb{V}(\mathscr{O}_S)$, the set $\mathrm{Hom}_S(X,S[T])$ is canonically identified with $\mathrm{Hom}_{\mathscr{O}_S}(\mathscr{O}_S,\mathscr{A}(X))$, which is canonically isomorphic to $\Gamma(S,\mathscr{A}(X))$ and therefore has a ring structure. Moreover, any S-morphism $h: X\to Y$ corresponds to a homomorphism $\Gamma(\mathscr{A}(h)):\Gamma(S,\mathscr{A}(Y))\to\Gamma(S,\mathscr{A}(X))$, so we obtain a contravariant functor $\mathrm{Hom}_S(X,S[T])$ from the category of S-schemes to the cateogry of rings. On the other hand, $\mathrm{Hom}_S(X,\mathbb{V}(\mathscr{E}))$ is identified similarly with $\mathrm{Hom}_{\mathscr{O}_S}(\mathscr{E},\mathscr{A}(X))$ (where $\mathscr{A}(X)$ is considered as an $\mathscr{O}_S$-module); we can then endow this set a $\mathrm{Hom}_S(X,S[T])$-module structure, and the couple

$$(\mathrm{Hom}_S(X,S[T]),\mathrm{Hom}_S(X,\mathbb{V}(\mathscr{E})))$$

is a contravariant functor on X with values in the category of couples (A,M) formed by a ring A and an A-module M, with morphisms being the bi-homomorphisms. In view of this, we say that $S[T]$ is the S**-scheme of ring** and $\mathbb{V}(\mathscr{E})$ is the S**-scheme of module** over the S-scheme of ring $S[T]$.

11.2 Homogeneous specturm of graded algebras

Let S be a graded ring and S_+ be the irrelevant ideal. We say a subset $\mathfrak{I}$ of S_+ is an **ideal of** S_+ if it is an ideal of S, and it is called a **graded prime ideal of** S_+ if it is the intersection with S_+ of a graded prime ideal of S not containing S_+ (in particular $\mathfrak{I}\neq S_+$, and this graded prime ideal of S is unique by Proposition 2.1.50). If $\mathfrak{I}$ is an ideal of S_+, the radical of $\mathfrak{I}$ in S_+, denoted by $\mathfrak{r}_+(\mathfrak{I})$, is the set of elements of S_+ which have some power contained in $\mathfrak{I}$, or equivalently, $\mathfrak{r}_+(\mathfrak{I})=\sqrt{\mathfrak{I}}\cap S_+$. In particularm the radical of 0 in S_+ is called the **nilradical** of S_+ and denoted by $\mathfrak{n}_+$: this is the subset of nilpotent elements of S_+. If $\mathfrak{I}$ is a graded ideal of S_+, its radical $\mathfrak{r}_+(\mathfrak{I})$ is also graded: by passing to $S/\mathfrak{I}$, we may assume that $\mathfrak{I}=0$, and note that if $x=x_h+x_{h+1}+\cdots+x_k$ is nilpotent, then so is each $x_i\in S_i$; we can suppose that $x_k=0$ and the top degree component of x^n is then x_k^n, so x_k is nilpotent, and we then preceed by induction on k. We say the graded ring S is essentially reduced if $\mathfrak{n}_+=0$, which means S_+ contains no nonzero nilpotent elements.

We note that in a graded ring S, if an element x is a zero divisor, so is its homogeneous component of top degree. We then say that the ring S is **essentially integral** if the ring S_+ (with the unit element) does not contain nonzero zero divisors; it suffices for this that a nonzero homogeneous element in S_+ is not divisor of 0 in this ring. It is clear that if $\mathfrak{p}$ is a graded prime ideal of S_+, $S/\mathfrak{p}$ is essentially integral. Let S be an essentially integral graded ring, and let $x_0 \in S_0$. It there exists a homogeneous element $f \neq 0$ in S_+ such that $x_0 f = 0$, we then have $x_0 S_+ = 0$, because $(x_0 g)f = (x_0 f)g = 0$ for any $g \in S_+$, and the hypothesis on S implies that $x_0 g = 0$. Therfore, for that S is integral, it is necessary and sufficient that S_0 is integral and the annihilator of S_+ in S_0 reduces to zero.

11.2.1 Localization of graded rings

Let S be a graded ring with positive degrees, f be a homogeneous element of S of degree $d > 0$. Then the fraction ring S_f is graded, where $(S_f)_n$ is the set of elements x/f^k, where $x \in S_{n+kd}$ with $k \geq 0$ (note that n can be an arbitrary integer). We denote by $S_{(f)}$ the subring $(S_f)_0$ of S_f formed by elements of degree 0.

If $f \in S_d$, the monomials $(f/1)^h$ in S_f (where h is an integer) form a lineraly independent system over the ring $S_{(f)}$, and the set of their linear combinations over $S_{(f)}$ is exactly the ring $(S^{(d)})_f$ (recall that $S^{(d)}$ is the direct sum of S_{nd}), and then we get an isomorphism

$$(S^{(d)})_f \cong S_{(f)}[T, T^{-1}] = S_{(f)} \otimes_{\mathbb{Z}} \mathbb{Z}[T, T^{-1}] \tag{11.2.1}$$

(where T is an indeterminate). In fact, if we have a relatin

$$\sum_{h=-a}^{a} z_h (f/1)^h = 0$$

where $z_h = x_h/f^m \in S_{(f)}$, then there exists an integer $k > -a$ such that

$$\sum_{h=-a}^{b} f^{h+k} x_h = 0,$$

and as the degrees of these terms are distinct, we have $f^{h+k}x_h = 0$ for all h, so $z_h = 0$ for all h. Similarly, if M is a graded S-module, the localization M_f is a graded S_f-module with $(M_f)_n$ being the set of elements z/f^k where $z \in M_{n+kd}$. We denote by $M_{(f)}$ the set of elements of degree 0 in M_f. It is immediate that $M_{(f)}$ is an $S_{(f)}$-module and we have $(M^{(d)})_f = M_{(f)} \otimes_{S_{(f)}} (S^{(d)})_f$.

Lemma 11.2.1. *Let $f \in S_d$ and $g \in S_e$ be two homogeneous elements of S with positive degrees. Then there exists a canonical isomorphism*

$$S_{(fg)} \cong (S_{(f)})_{g^d/f^e}.$$

If we identify these two rings, then for any S-module M, we have a canonical isomorphism

$$M_{(fg)} \cong (M_{(f)})_{g^d/f^e}.$$

Proof. Note that (fg) divides $f^e g^d$ and $f^e g^d$ divides $(fg)^{de}$, so the rings S_{fg} and $S_{f^e g^d}$ are canonically identified. On the other hand, $S_{f^e g^d}$ is also identified with $(S_{f^e})_{g^d/1}$, and as $f^e/1$ is invertible in S_{f^e}, $s_{f^e g^d}$ is also identified with $(S_{f^e})_{g^d/f^e}$. Now the element g^e/f^e is of degree zero in S_{f^e}, so we can conclude that the subring of $(S_{f^e})_{g^d/f^e}$ formed by elements of degree zero is $(S_{(f^e)})_{g^d/f^e}$, and as we have $S_{(f^e)} = S_{(f)}$, we see the assertion follows. □

With the hypotheses of Lemma 11.2.1, it is clear that the canonical homomorphism $S_f \to S_{fg}$, which maps x/f^k to $xg^k/(fg)^k$, is of degree 0 so restricts to a canonical homomorphism $S_{(f)} \to S_{(fg)}$, such that the diagram

$$\begin{tikzcd} & S_{(f)} \arrow[dl] \arrow[dr] & \\ S_{(fg)} \arrow[rr, "\sim"] & & (S_{(f)})_{(g^d/f^e)} \end{tikzcd}$$

is commutative. We define similarly a canonical homomorphism $M_{(f)} \to M_{(fg)}$.

Lemma 11.2.2. *If f, g are two homogeneous elements of S_+, the ring $S_{(fg)}$ is generated by the union of the canonical images of $S_{(f)}$ and $S_{(g)}$.*

Proof. In view of Lemma 11.2.1, it suffices to show that $1/(g^d/f^e) = f^{d+e}/(fg)^d$ belongs to the canonical image of $S_{(g)}$ in $S_{(fg)}$, which is evident from the definition. □

Proposition 11.2.3. *Let $f \in S_d$ be a homogeneous element of positive degree. Then there exists a canonical isomorphism $S_{(f)} \cong S^{(d)}/(f-1)S^{(d)}$ of rings. If we identify these two rings, then for any S-module M, there exists a canonical isomorphism of modules $M_{(f)} \cong M^{(d)}/(f-1)M^{(d)}$*

Proof. The first isomorphism is defined by sending the element x/f^n, where $x \in S_{nd}$, to the element $\bar{x}$, the class of x mod $(f-1)S^{(d)}$. This map is well-defined, because we have the congruence $f^h x \equiv x$ mod $(f-1)S^{(d)}$ for any $x \in S^{(d)}$, so if $f^h x = 0$ for some $h > 0$ then $\bar{x} = 0$. On the other hand, if $x \in S_{nd}$ is such that $x = (f-1)y$ with $y = y_{hd} + y_{(h+1)d} \cdots + y_{kd}$, where $y_{jd} \in S_{jd}$ and $y_{hd} \neq 0$, we have necessarily $h = n$ and $x = -y_{hd}$, as well as the relations $y_{(j+1)d} = fy_{jd}$ for $h \leq j \leq k-1$ and $f_{ykd} = 0$; in particular, this implies $f^{k-n}x = 0$. We therefore have an inverse homomorphism from $S^{(d)}/(f-1)S^{(d)}$ to $S_{(f)}$ by corresponding a class $\bar{x}$ mod $(f-1)S^{(d)}$ (where $x \in S_{nd}$) the element x/f^n of $S_{(f)}$, since the preceding remark shows that this map is well-defined. The first assertion is therefore proves, and the second one can be done similarly. □

Corollary 11.2.4. *If S is Noetherian, so is $S_{(f)}$ for any homogeneous element f of positive degree.*

Proof. This follows from Proposition 11.2.3 and Corollary 2.1.38. □

Let T be a multiplicative subset of S_+ formed by homogeneous elements; $T_0 = T \cup \{1\}$ is then a multiplicative subset of S. As the elements of T_0 are homogeneous, the ring $T_0^{-1}S$ is graded in a natrual way, and we denote by $S_{(T)}$ the subring of $T_0^{-1}S$ formed by elements of degree 0. We know that $T_0^{-1}S$ is identified with the inductive limit of the rings S_f, where $f \in T$ (with the canonical homomorphisms $S_f \to S_{fg}$). As this identification preserves the degrees, it identified $S_{(T)}$ as the inductive limit of $S_{(f)}$, where $f \in T$. For any graded S-module M, we define similarly the module $M_{(T)}$ (over the ring $S_{(T)}$) formed by degree zero elements of $T_0^{-1}M$, and we conclude that $M_{(T)}$ is the inductive limit of $M_{(f)}$ for $f \in T$.

If $\mathfrak{p}$ is a graded prime ideal of S_+, we denote by $S_{(\mathfrak{p})}$ and $M_{(\mathfrak{p})}$ the ring $S_{(T)}$ and the module $M_{(T)}$ respectively, where T is the homogeneous elements of S_+ not contained in $\mathfrak{p}$.

11.2.2 The homogeneous specturm of a graded ring

Given a graded ring S with positive degrees, we denote by $\mathrm{Proj}(S)$ the **homogeneous spectrum** of S, which is the set of graded prime ideals of S_+, or, equivalently, the set of graded prime ideals of S not containing S_+. We will define a scheme structure on $\mathrm{Proj}(S)$, just as we have done for $\mathrm{Spec}(A)$ for a ring A.

For a subset E of S, let $V_+(E)$ be the set of graded prime ideals of S containing E and not containing S_+, which is also the subset $V(E)\cap\mathrm{Proj}(S)$ of $\mathrm{Spec}(S)$. We have immediately the following equalities:

$$\begin{gathered}V_+(0)=\mathrm{Proj}(S),\quad V_+(S)=V_+(S_+)=\varnothing,\\ V_+\Big(\bigcup_\lambda E_\lambda\Big)=\bigcap_\lambda V_+(E_\lambda),\\ V_+(EF)=V_+(E)\cup V_+(F).\end{gathered}$$

Again, the set $V_+(E)$ remain unchanged if we replace E by the graded ideal it generates; moreover, if $\mathfrak{J}$ is a graded ideal of S, we have

$$V_+(\mathfrak{J})=V_+\Big(\bigcup_{i\geq n}(\mathfrak{J}\cap S_i)\Big)\tag{11.2.2}$$

for any $n>0$: in fact, if $\mathfrak{p}\in\mathrm{Proj}(S)$ contains the homogeneous elements of $\mathfrak{J}$ with degrees $\geq n$, as by hypothesis there exists a homogeneous element $f\in S_d$ not contained in $\mathfrak{p}$, for any $m\geq 0$ and any $x\in S_m\cap\mathfrak{J}$, we have $f^rx\in\mathfrak{J}\cap S_{m+rd}$ for r sufficiently large, hence $f^rx\in\mathfrak{p}\cap S_{m+rd}$, which implies $x\in\mathfrak{p}\cap S_m$. Finally, for any graded ideal $\mathfrak{J}$ of S, we have

$$V_+(\mathfrak{J})=V_+(\mathfrak{r}_+(\mathfrak{J}))$$

where $\mathfrak{r}_+(\mathfrak{J})$ is the radical of $\mathfrak{J}$ in S_+.

By definition, $V_+(E)$ is a closed subset of $X=\mathrm{Proj}(S)$ for the topology induced by $\mathrm{Spec}(S)$. For each element $f\in S$, we set

$$D_+(f)=D(f)\cap\mathrm{Proj}(S)=\mathrm{Proj}(S)\setminus V_+(f).$$

Then for two elements $f,g\in S$, $D_+(fg)=D_+(f)\cap D_+(g)$, and the subsets $D_+(f)$, with $f\in S_+$, form a basis for the topology of $X=\mathrm{Proj}(S)$.

Let f be a homogeneous element of S_+ with degree $d>0$. For any prime ideal $\mathfrak{p}$ of S not containing f, we see the set of x/f^n, where $x\in\mathfrak{p}$ and $n\geq 0$, is a prime ideal of the fraction ring S_f. Its trace on $S_{(f)}$ is then a prime ideal of this ring, which we denote by $\psi_f(\mathfrak{p})$: this is the set of elements x/f^n, for $n\geq 0$, $x\in\mathfrak{p}_{nd}$. We have therefore defined a map

$$\psi_f:D_+(f)\to\mathrm{Spec}(S_{(f)});$$

moreover, if g is another homogeneous element of S_+ with degree $e>0$, we have a commutative diagram

$$\begin{tikzcd} D_+(fg) \arrow[r, "\psi_{fg}"] \arrow[d] & \mathrm{Spec}(S_{(fg)}) \arrow[d] \\ D_+(f) \arrow[r, "\psi_f"] & \mathrm{Spec}(S_{(f)}) \end{tikzcd}\tag{11.2.3}$$

where the left vertical maps are inclusions, and the right one is the map ${}^a\omega_{fg,f}$ induced from the canonical homomorphism $\omega_{fg,f}:S_{(f)}\to S_{(fg)}$. In fact, if $x/f^n\in\omega^{-1}(\psi_{fg}(\mathfrak{p}))$, where $fg\notin\mathfrak{p}$, we then have $g^nx/(fg)^n\in\psi_{fg}(\mathfrak{p})$, so $g^nx\in\mathfrak{p}$ and therefore $x\in\mathfrak{p}$, and the converse inclusion is evident.

Proposition 11.2.5. *The map ψ_f is a homeomorphism from $D_+(f)$ to* $\mathrm{Spec}(S_{(f)})$.

Proof. For $h\in S_{nd}$ is such that $h/f^n\in\psi_f(\mathfrak{p})$, by definition it is necessary and sufficient that $h\in\mathfrak{p}$, so $\psi^{-1}(D(h/f^n))=D_+(fh)=D_+(h)\cap D_+(f)$ and the map ψ_f is therefore continuous.

Moreover, the sets $D_+(hf)$, where h runs through the set S_{nd}, form a basis of the topology of $D_+(f)$, so the preceding argument proves, in view of the T_0-axiom for $D_+(f)$ and $\mathrm{Spec}(S_{(f)})$, that ψ_f is injective and the inverse map $\psi_f(D_+(f)) \to D_+(f)$ is continuous. Finally, to show that ψ_f is surjective, we remark that, if $\mathfrak{q}_0$ is a prime ideal of $S_{(f)}$ and if, for any $n > 0$, we denote by $\mathfrak{p}_n$ the set of elements $x \in_S n$ such that $x^d/f^n \in \mathfrak{q}_n$, the $\mathfrak{p}_n$ then verify the conditions Proposition 2.1.50: if $x \in S_n$, $y \in S_n$ are such that $x^d/f^n \in \mathfrak{q}_0$ and $y^d/f^n \in \mathfrak{q}_0$, we have $(x+y)^{2d}/f^{2n} \in \mathfrak{q}_0$, whence $(x+y)^d/f^n \in \mathfrak{q}_0$ since $\mathfrak{q}_0$ is prime; this proves that $\mathfrak{p}_n$ is a subgroup of S_n, and the verification of other conditions of Proposition 2.1.50 is immediate. If $\mathfrak{p}$ is the corresponding graded ideal of S, then $\psi_f(\mathfrak{p}) = \mathfrak{q}_0$, since if $x \in S_{nd}$, the relations $x/f^n \in \mathfrak{q}_0$ and $x^d/f^{nd} \in \mathfrak{q}_0$ are equivalent. □

Corollary 11.2.6. *For $D_+(f) \neq \varnothing$, it is necessary and sufficient that f is nilpotent.*

Proof. For $\mathrm{Spec}(S_{(f)}) = \varnothing$, it is necessary and sufficient that $S_{(f)} = 0$, which means $1 = 0$ in S_f, and this is equivalent to that f is nilpotent. □

Corollary 11.2.7. *Let E be a subset of S_+. The following conditions are equivalent:*

(i) $V_+(E) = X = \mathrm{Proj}(S)$.

(ii) *Every element of E is nilpotent.*

(iii) *The homogeneous components of every element of E are nilpotent.*

Proof. It is clear that (iii)⇒(ii)⇒(i). If $\mathfrak{I}$ is the graded ideal of S generated by E, conditions (i) is equivalent to that $V_+(\mathfrak{I}) = X$, and a fortiori, (i) implies that any homogeneous element $f \in \mathfrak{I}$ is such that $V_+(f) = X$, so f is nilpotent by Corollary 11.2.6. □

Corollary 11.2.8. *If $\mathfrak{I}$ is a graded ideal of S_+, $\mathfrak{r}_+(\mathfrak{I})$ is the intersection of graded prime ideals in $V_+(\mathfrak{I})$.*

Proof. By considering the ring $S/\mathfrak{I}$, we may assume that $\mathfrak{I} = 0$. It then suffices to prove that if $f \in S_+$ is not nilpotent, then there exists a graded prime ideal of S not containing f. Now, since there exists at least homogeneous component of f that is not nilpotent, we may assume that f is homogeneous, the result then follows from Corollary 11.2.6. □

For any subset Y of $X = \mathrm{Proj}(S)$, we denote by $I_+(Y)$ the subset of $f \in S_+$ such that $Y \subseteq V_+(f)$, which is in other words $I(Y) \cap S_+$; the set $I_+(Y)$ is then a radical ideal of S_+.

Proposition 11.2.9. *Let E be a subset of S and Y be a subset of X.*

(a) *The ideal $I_+(V_+(E))$ is the radical in S_+ of the graded ideal of S_+ generated by E.*

(b) *The set $V_+(I_+(Y))$ is the closure of Y in X.*

Proof. If $\mathfrak{I}$ is the graded ideal of S_+ generated by E, we have $V_+(E) = V_+(\mathfrak{I})$ and the first assertion follows from Corollary 11.2.8. As for (b), since $V_+(\mathfrak{I}) = \bigcap_{f\in\mathfrak{I}} V_+(f)$, the relation $Y \subseteq V_+(\mathfrak{I})$ implies $Y \subseteq V_+(f)$ for any $f \in \mathfrak{I}$, and therefore $I_+(Y) \supseteq \mathfrak{I}$, so $V_+(I_+(Y)) \subseteq V_+(\mathfrak{I})$, which implies (b) by the definition of closure. □

Corollary 11.2.10. *The closed subsets Y of $X = \mathrm{Proj}(S)$ and the graded radical ideals of S_+ correspond bijectively via $Y \mapsto I_+(Y)$ and $\mathfrak{I} \mapsto V_+(\mathfrak{I})$. Also, the union $Y_1 \cup Y_2$ of two closed subsets of X corresponds to $I_+(Y_1) \cap I_+(Y_2)$, and the intersection of a family (Y_λ) of closed subsets corresponds to the radical of the sum of $I_+(Y_\lambda)$.*

Corollary 11.2.11. *Let (f_α) be a family of homogeneous elements of S_+ and f be an element of S_+. The following conditions are equivalent:*

(i) $D_+(f) \subseteq \bigcup_\alpha D_+(f_\alpha)$;

(ii) $V_+(f) \supseteq \bigcap_\alpha V_+(f_\alpha)$;

(ii) *a power of f is contained in the ideal generated by the f_α.*

In particular, if $\mathfrak{I}$ is a graded ideal of S_+, then $V_+(\mathfrak{I}) = \varnothing$ if and only if $\mathfrak{r}_+(\mathfrak{I}) = S_+$.

Corollary 11.2.12. *For $X = \mathrm{Proj}(S)$ to be empty, it is necessary and sufficient that every element of S_+ is nilpotent.*

Corollary 11.2.13. *The closed irreducible subset of $X = \mathrm{Proj}(S)$ correspond to graded prime ideals of S_+.*

Proof. In fact, if $Y = Y_1 \cup Y_2$, where Y_1, Y_2 are closed and distinct in Y, then

$$I_+(Y) = I_+(Y_1) \cap I_+(Y_2)$$

the ideals $I_+(Y_1)$ and $I_+(Y_2)$ are distinct from $I_+(Y)$, so $I_+(Y)$ can not be prime. Conversely, if $\mathfrak{I}$ is a graded non-prime ideal of S_+, there exist elements f, g of S_+ such that $fg \in \mathfrak{I}$ but $f, g \notin \mathfrak{I}$. Then $V_+(f) \nsubseteq V_+(\mathfrak{I})$, $V_+(g) \nsubseteq V_+(\mathfrak{I})$, but $V_+(\mathfrak{I}) \subseteq V_+(f) \cup V_+(g)$. We then conclude that $V_+(\mathfrak{I})$ is the union of the closed subsets $V_+(f) \cap V_+(\mathfrak{I})$ and $V_+(g) \cap V_+(\mathfrak{I})$, both are distinct from $V_+(\mathfrak{I})$. □

We now define the scheme structure on the homogeneous spectrum $\mathrm{Proj}(S)$. Let f, g be two homogeneous elements of S_+ and consider the affine schemes $Y_f = \mathrm{Spec}(S_{(f)})$, $Y_g = \mathrm{Spec}(S_{(g)})$, and $Y_{fg} = \mathrm{Spec}(S_{(fg)})$. In view of Lemma 11.2.1, the morphism $w_{fg,f} : Y_{fg} \to Y_f$ corresponding to the canonical homomorphism $\omega_{fg,f} : S_{(f)} \to S_{(fg)}$, is an open immersion. By the homeomorphism $\psi_f : D_+(f) \to Y_f$ (Proposition 11.2.5), we can transport to $D_+(f)$ the affine scheme structure of Y_f; in view of the commutative diagram (11.2.3), the affine scheme $D_+(fg)$ is identified with the subscheme induced over the open subset $D_+(fg)$ by the affine scheme $D_+(f)$. It is then clear that $X = \mathrm{Proj}(S)$ is endowed with a unique scheme structure such that each $D_+(f)$ is an affine open subscheme of X. When we speak of the homogeneous spectrum $\mathrm{Proj}(S)$ as a scheme, it will always be the structure defined in this way.

Proposition 11.2.14. *The scheme $\mathrm{Proj}(S)$ is separated.*

Proof. By Proposition 10.5.31, it suffices to show that for any homogeneous elements f, g of S_+, $D_+(f) \cap D_+(g) = D_+(fg)$ is affine and the canonical images of the rings of $D_+(f)$ and $D_+(g)$ in $D_+(fg)$ generate the ring of $D_+(fg)$. The first one is clear by definition, and the second one follows from Lemma 11.2.2, □

Example 11.2.15. Let $S = K[T_1, T_2]$ where K is a field and T_1, T_2 are indeterminates. Then it follows from Corollary 11.2.11 that $\mathrm{Proj}(S)$ is the union of $D_+(T_1)$ and $D_+(T_2)$. We see that each of these affine subscheme is isomorphic to $K[T]$, and that $\mathrm{Proj}(S)$ is obtained by glueing these two schemes as described in Example 10.2.10.

Proposition 11.2.16. *Let S be a graded ring with positive degrees and $X = \mathrm{Proj}(S)$.*

(i) *If $\mathfrak{n}_+$ is the nilradical of S_+, the scheme X_{red} is canonically isomorphic to $\mathrm{Proj}(S/\mathfrak{n}_+)$. In particular, if S is essentially reduced, then $\mathrm{Proj}(S)$ is reduced.*

(ii) *Suppose that S is essentially reduced, then for X to be integral, it is necessary and sufficient that S is essentially integral.*

Proof. Let $\bar{S} = S/\mathfrak{n}_+$, and denote by $x \mapsto \bar{x}$ the canonical homomorphism $S \to \bar{S}$, of degree 0. For any $f \in S_d$ ($d > 0$), the canonical homomorphism $S_f \to \bar{S}$ is surjective and of degree 0, hence restricts to a surjection $S_{(f)} \to \bar{S}_{(\bar{f})}$. If we suppose that $f \notin \mathfrak{n}_+$, then $\bar{S}_{(\bar{f})}$ is reduced and the kenrel of the preceding homomorphism is the nilradical of $S_{(f)}$, whence $\bar{S}_{(\bar{f})} = (S_{(f)})_{\mathrm{red}}$. This homomorphism then corresponds to a closed immersion $D_+(\bar{f}) \to D_+(f)$ which identifies $D_+(\bar{f})$ with $(D_+(f))_{\mathrm{red}}$ (Corollary 10.4.29), and in particular is a homeomorphism of affine scheme. Further, if $g \notin \mathfrak{n}_+$ is another homogeneous element of S_+, the diagram

$$\begin{array}{ccc} S_{(f)} & \longrightarrow & \bar{S}_{(\bar{f})} \\ \downarrow & & \downarrow \\ S_{(fg)} & \longrightarrow & \bar{S}_{(\overline{fg})} \end{array}$$

is commutative. As the sets $D_+(f)$ for f homogeneous in S_+ and $f \notin \mathfrak{n}_+$ form a covering for $X = \mathrm{Proj}(S)$, we conclude that the morphisms $D_+(\bar{f}) \to D_+(f)$ glue together to a closed immersion $\mathrm{Proj}(\bar{S}) \to \mathrm{Proj}(S)$ which is a homeomorphism on the underlying spaces, whence the conclusion of (i) by Corollary 10.4.29.

Suppose now that S is essentially integral, which means (0) is a graded ideal of S_+ distinct from S_+. Then X is reduced by (a) and irreducible by Corollary 11.2.13. Conversely, if S is essentially reduced and X is integral, then for any $f \neq 0$ homogeneous in S_+, we have $D_+(f) \neq \varnothing$ by Corollary 11.2.6; the hypothesis that X is irreducible implies that $D_+(f) \cap D_+(g) \neq \varnothing$ for any f, g homogeneous and nonzero in S_+, so in particular $fg \neq 0$, and we then conclude that S_+ is integral. □

Proposition 11.2.17. *Suppose that S is a graded A-algebra where A is a ring. Then over $X = \mathrm{Proj}(S)$ the structural sheaf $\mathscr{O}_X$ is an A-algebra, which means X is a scheme over* $\mathrm{Spec}(A)$.

Proof. It suffices to note that for any f homogeneous in S_+, $S_{(f)}$ is an A-algebra and the homomorphism $S_{(f)} \to S_{(fg)}$ is an A-algebra homomorphism for any f, g homogeneous in S_+. □

Proposition 11.2.18. *Let S be a graded ring with positive degrees.*

(a) *For any integer $d > 0$, there exists a canonical isomorphism* $\mathrm{Proj}(S) \cong \mathrm{Proj}(S^{(d)})$.

(b) *Let S' be the graded ring such that $S'_0 = \mathbb{Z}$ and $S'_n = S_n$ for $n > 0$, then there exists a canonical isomorphism* $\mathrm{Proj}(S) \cong \mathrm{Proj}(S')$.

Proof. We have already seen in Proposition 2.1.51 that the map $\mathfrak{p} \mapsto \mathfrak{p} \cap S^{(d)}$ is a bijection from $\mathrm{Proj}(S)$ to $\mathrm{Proj}(S^{(d)})$. As for any f homogeneous in S_+, we have $V_+(f) = V_+(f^d)$, this bijection is a homeomorphism of topological spaces. Finally, with the same notations, $S_{(f)}$ and $S_{(f^d)}$ are canonically identified by Lemma 11.2.1, so $\mathrm{Proj}(S)$ and $\mathrm{Proj}(S^{(d)})$ are canonically identified as schemes.

If to any $\mathfrak{p} \in \mathrm{Proj}(S)$, we correspond the unique prime ideal $\mathfrak{p}' \in \mathrm{Proj}(S')$ such that $\mathfrak{p}' \cap S_n = \mathfrak{p} \cap S_n$ for $n > 0$, then it is clear that this defines a homeomorphism $\mathrm{Proj}(S) \cong \mathrm{Proj}(S')$ of the underlying spaces, since $V_+(f)$ is the same set for S and S' if f is a homogeneous element of S_+. We also note that $S_{(f)} = S'_{(f)}$: to see this it suffices to note that if $x/1$ is an element of $S_{(f)}$ with $x \in S_0$, then $x/1 = xf/f \in S'_{(f)}$; we then conclude that $\mathrm{Proj}(S)$ and $\mathrm{Proj}(S')$ are identified as schemes. □

Corollary 11.2.19. *Let S be a graded A-algebra and S_A be the graded A-algebra such that $(S_A)_0 = A$ and $(S_A)_n = S_n$ for $n > 0$, then there exists a canonical isomorphism* $\mathrm{Proj}(S) \cong \mathrm{Proj}(S_A)$.

Proof. In fact, these two schemes are isomorphic to $\mathrm{Proj}(S')$, where $S'_0 = \mathbb{Z}$ and $S'_n = S_n$ for $n > 0$, in view of Proposition 11.2.18. □

11.2.3 Sheaf associated with a graded module

Let M be a graded S-module. For any homogeneous element f of S_+, $M_{(f)}$ is an $S_{(f)}$-module, and it therefore corresponds to a quasi-coherent sheaf $\widetilde{M_{(f)}}$ over the affine $\mathrm{Spec}(S_{(f)})$, identified with $D_+(f)$.

Proposition 11.2.20. *There existss a unique quasi-coherent $\mathcal{O}_X$-module $\widetilde{M}$ such that for any homogeneous element $f \in S_+$, we have $\Gamma(D_+(f), \widetilde{M}) = M_{(f)}$, and the restriction homomorphism $\Gamma(D_+(f), \widetilde{M}) \to \Gamma(D_+(fg), \widetilde{M})$ for f, g homogeneous in S_+ corresponds to the canonical homomorphism $M_{(f)} \to M_{(fg)}$.*

Proof. Suppose that $f \in S_d$, $g \in S_e$. As $D_+(fg)$ is identified with the prime specturm $(S_{(f)})_{g^d/f^e}$ by Lemma 11.2.1, the restriction of $\widetilde{M_{(f)}}$ to $D_+(fg)$ is canonically identified with the sheaf associated with the module $(M_{(f)})_{(g^d/f^e)}$, hence to $\widetilde{M_{(fg)}}$ (Lemma 11.2.1). We then conclude that there is a canonical isomorphism

$$\theta_{g,f} : \widetilde{M_{(f)}}|_{D_+(fg)} \to \widetilde{M_{(g)}}|_{D_+(fg)}$$

such that, if g is another homogeneous element of S_+, we have $\theta_{f,h} = \theta_{f,g} \circ \theta_{g,h}$ over $D_+(fgh)$. By glueing, there then exists a quasi-coherent sheaf $\mathscr{F}$ over X such that for any homogeneous element $f \in S_+$, we have an isomorphism $\eta_f : \mathscr{F}|_{D_+(f)} \cong \widetilde{M_{(f)}}$ and $\theta_{g,f} = \eta_g \circ \eta_f^{-1}$. Since over $D_+(f)$ we have $\Gamma(D_+(f), \widetilde{M})$, $\mathscr{F}$ can be identified with the sheaf extended from the presheaf $D_+(f) \mapsto M_{(f)}$ over the basis of standard open sets of X, whence the assertions of the proposition. In particular, we have $\widetilde{S} = \mathcal{O}_X$. □

We say the quasi-coherent $\mathcal{O}_X$-module $\widetilde{M}$ is **associted** with the graded S-module M. Recall that the graded S-modules form a category whose morphisms are graded homomorphisms of degrees. With this convention:

Proposition 11.2.21. *The functor $\widetilde{M}$ is a covariant exact functor from the category of graded S-modules to the category of quasi-coherent $\mathcal{O}_X$-modules, which commutes with inductive limits and direct sums.*

Proof. Since the properties are local, it suffices to verify over the sheaf $\widetilde{M}|_{D_+(f)} = \widetilde{M_{(f)}}$. Now the functor M_f on M, the functor N_0 on N, and the functor $\widetilde{P}$ on P all satisfy the stated properties, whence the claim. □

We denote by $\tilde{u} : \widetilde{M} \to \widetilde{N}$ the homomorphism corresponding to a graded homomorphism $u : M \to N$ of degree 0. We also deduce from Proposition 11.2.21 that the results of Corollary 10.1.6 and Corollary 10.1.8 are also true for graded S-modules and homomorphism of degree 0, via the same demonstration.

Proposition 11.2.22. *For any $\mathfrak{p} \in X = \mathrm{Proj}(S)$, we have $\widetilde{M}_{\mathfrak{p}} = M_{(\mathfrak{p})}$.*

Proof. By definition we have $\widetilde{M}_{\mathfrak{p}} = \varinjlim \Gamma(D_+(f), \widetilde{M})$, where f runs through homogeneous elements $f \in S_+$ such that $f \notin \mathfrak{p}$. As $\Gamma(D_+(f), \widetilde{M}) = M_{(f)}$, the proposition follows from the definition of $M_{(\mathfrak{p})}$. □

In particular, the local ring $\mathcal{O}_{X,\mathfrak{p}}$ is just the ring $S_{(\mathfrak{p})}$, the set of elements x/f where f is homogeneous in S_+ and not contained in $\mathfrak{p}$, and x is homogeneous with the same degree as f. If moreover S is essentially integral, then $\mathrm{Proj}(S) = X$ is integral (Proposition 11.2.16), and if $\xi = (0)$ is the generic point of X, the rational function field $K(X) = \mathcal{O}_{X,\xi}$, is the field formed by f/g where f, g are homogeneous elements of S_+ and $g \neq 0$.

Proposition 11.2.23. *If, for any $z \in M$ and any homogeneous element $f \in S_+$, there exists a power of f annihilating z, then $\widetilde{M} = 0$. This condition is also necessary if $S = S_0[S_1]$.*

Proof. The condition $\widetilde{M} = 0$ is equivalent to $M_{(f)} = 0$ for any homogeneous element of S_+. Now if $f \in S_d$, the condition $M_{(f)} = 0$ signifies that for any $z \in M$ homogeneous whose degree is a multiple of d, there exists power f^n such that $f^n z = 0$; this implies the first claim. Conversely, if moreover S is generated by S_1, then condition then implies that $f^n z = 0$ for any $z \in M$ and any $f \in S_+$, since any element $f \in S_+$ is a finite linear combination of elements of S_1. □

Proposition 11.2.24. *Let $f \in S_d$ with $d > 0$. Then for any $n \in \mathbb{Z}$, the $(\mathscr{O}_X|_{D_+(f)})$-module $\widetilde{S(nd)}|_{D_+(f)}$ is canonically isomorphic to $\mathscr{O}_X|_{D_+(f)}$.*

Proof. The multiplication by the invertible element $(f/1)^n$ of S_f defines a bijection from $S_{(f)} = (S_f)_0$ to the ring

$$(S_f)_{nd} = (S_f(nd))_0 = (S(nd)_f)_0 = S(nd)_{(f)},$$

whence the assertion. □

Corollary 11.2.25. *Over the open subset $U = \bigcup_{f \in S_d} D_+(f)$, the restriction of the $\mathscr{O}_X$-module $\widetilde{S(nd)}$ is an invertible $(\mathscr{O}_X|_U)$-module.*

Corollary 11.2.26. *If the ideal S_+ of S is generated by S_1, then the $\mathscr{O}_X$-module $\widetilde{S(n)}$ is invertible for any $n \in \mathbb{Z}$.*

Proof. It suffices to note that under the hypothesis we have $X = \bigcup_{f \in S_1} D_+(f)$ by Corollary 11.2.11. □

The quasi-coherent $\mathscr{O}_X$-modules $\widetilde{S(n)}$ is of particular interest in the theory of projective schemes, so for each $n \in \mathbb{Z}$, we put $\mathscr{O}_X(n) = \widetilde{S(n)}$ and for an open subset U of X and any $(\mathscr{O}_X|_U)$-module $\mathscr{F}$, set

$$\mathscr{F}(n) = \mathscr{F} \otimes_{\mathscr{O}_X|_U} (\mathscr{O}_X(n)|_U).$$

If the ideal S_+ is generated by S_1, the functor $\mathscr{F}(n)$ is exact on $\mathscr{F}$ for any $n \in \mathbb{Z}$, since $\mathscr{O}_X(n)$ is then an invertible $\mathscr{O}_X$-module.

Example 11.2.27. Let k be a field and consider the graded algebra $S = k[x_0, \dots, x_n]$; let $X = \mathrm{Proj}(S)$. Let d be an inter and consider the twist sheaf $\mathscr{O}_X(d)$. We compute the global sections for $\mathscr{O}_X(d)$: by definition, for each $x_i \in S$, the section of $\mathscr{O}_X(d)$ over $U_i = D_+(x_i)$ is given by

$$\Gamma(U_i, \mathscr{O}_X(d)) = S(d)_{(x_i)} = (S_{(x_i)})_d = \{f/x_i^n : f \in S_{n+d}\} = \{x_i^d f : f \in S^{(i)}\},$$

where $S^{(i)} = k[x_0/x_i, \dots, x_n/x_i]$. Therefore a section of $\mathscr{O}_X(d)$ is a family of rational polynomials (f_i) with $f_i \in S^{(i)}$ such that $x_i^d f_i = x_j^d f_j$ for each $i \neq j$; let f be this common rational polynomial. Then by construction, we have $f/x_i^d \in S^{(i)}$ for each i, which implies that f is a polynomial in S of degree d when $d \geq 0$. If on the other hand $d < 0$, then f can only have poles at x_i for each i, which is impossible, so there is no global sections for $\mathscr{O}_X(d)$ when $d < 0$.

Let M, N be graded S-modules. For any $f \in S_d$ we define a canonical homomorphism of $S_{(f)}$-modules

$$\lambda_f : M_{(f)} \otimes_{S_{(f)}} N_{(f)} \to (M \otimes_S N)_{(f)}$$

by composing the homomorphism $M_{(f)} \otimes_{S_{(f)}} N_{(f)} \to M_f \otimes_{S_f} N_f$ (induced from the canonical injections) with the canonical isomorphism $M_f \otimes_{S_f} N_f \cong (M \otimes_S N)_{(f)}$, and note that by the definition of the grading of tensor products, these isomorphisms preserve degrees. Unwinding the definitions, for $x \in M_{md}$, $y \in N_{nd}$, we have

$$\lambda_f((x/f^m) \otimes (y/f^n)) = (x \otimes y)/f^{m+n}.$$

It then follows that, if $g \in S_e$ is another homogeneous element, the diagram

$$\begin{tikzcd} M_{(f)} \otimes_{S_{(f)}} N_{(f)} \arrow[r, "\lambda_f"] \arrow[d] & (M \otimes_S N)_{(f)} \arrow[d] \\ M_{(fg)} \otimes_{S_{(fg)}} N_{(fg)} \arrow[r, "\lambda_{fg}"] & (M \otimes_S N)_{(fg)} \end{tikzcd}$$

(wher the vertical homomorphism are canonical) is commutative. We then deduce that λ is a canonical homomorphism of $\mathscr{O}_X$-modules

$$\lambda : \widetilde{M} \otimes_{\mathscr{O}_X} \widetilde{N} \to \widetilde{M \otimes_S N}.$$

Consider in particular two graded ideals $\mathfrak{J}, \mathfrak{K}$ of S. As $\widetilde{\mathfrak{J}}$ and $\widetilde{\mathfrak{K}}$ are two quasi-coherent ideals of $\mathscr{O}_X$, we have a canonical homomorphism $\widetilde{\mathfrak{J}} \otimes_{\mathscr{O}_X} \widetilde{\mathfrak{K}} \to \mathscr{O}_X$, and the diagram

$$\begin{tikzcd} \widetilde{\mathfrak{J}} \otimes_{\mathscr{O}_X} \widetilde{\mathfrak{K}} \arrow[rr, "\lambda"] \arrow[dr] & & \widetilde{\mathfrak{J} \otimes_S \mathfrak{K}} \arrow[dl] \\ & \mathscr{O}_X & \end{tikzcd} \tag{11.2.4}$$

is commutative. Finally, note that if M, N, P are graded S-modules, the diagram

$$\begin{tikzcd} \widetilde{M} \otimes_{\mathscr{O}_X} \widetilde{N} \otimes_{\mathscr{O}_X} \widetilde{P} \arrow[r, "\lambda \otimes 1"] \arrow[d, "1 \otimes \lambda"'] & \widetilde{M \otimes_S N} \otimes_{\mathscr{O}_X} \widetilde{P} \arrow[d, "\lambda"] \\ \widetilde{M} \otimes_{\mathscr{O}_X} \widetilde{N \otimes_S P} \arrow[r, "\lambda"] & (M \otimes_S N \otimes_S P)^{\sim} \end{tikzcd} \tag{11.2.5}$$

is commutative. Simialrly, we define a canonical homomorphism of $S_{(f)}$-modules

$$\mu_f : \mathrm{Hom}_S(M,N)_{(f)} \to \mathrm{Hom}_{S_{(f)}}(M_{(f)}, N_{(f)})$$

which sends an element u/f^n, where u is a homomorphism of degre nd, the homomorphism $M_{(f)} \to N_{(f)}$ which sends x/f^m ($x \in M_{md}$) to $u(x)/f^{m+n}$. For $g \in S_e$, we have the following commutative diagram

$$\begin{tikzcd} \mathrm{Hom}_S(M,N)_{(f)} \arrow[r, "\mu_f"] \arrow[d] & \mathrm{Hom}_{S_{(f)}}(M_{(f)}, N_{(f)}) \arrow[d] \\ \mathrm{Hom}_S(M,N)_{(fg)} \arrow[r, "\mu_{fg}"] & \mathrm{Hom}_{S_{(fg)}}(M_{(fg)}, N_{(fg)}) \end{tikzcd} \tag{11.2.6}$$

We then conclude that the μ_f define a canonical homomorphism

$$\mu : (\mathrm{Hom}_S(M,N))^{\sim} \to \mathcal{H}om_{\mathscr{O}_X}(\widetilde{M}, \widetilde{N}).$$

Proposition 11.2.28. *Suppose that the ideal S_+ is generated by S_1. Then the homomorphism λ is an isomorphism; this holds for μ if the graded S-module M is of finite presentation.*

Proof. As X is the union of $D_+(f)$ for $f \in S_1$, we are reduced to prove tht λ_f and μ_f are isomorphisms for f homogeneous of degree 1. We then define a $\mathbb{Z}$-linear map $M_n \times N_n \to M_{(f)} \otimes_{S_{(f)}} N_{(f)}$ that send a pair (x, y) to the element $(x/f^m) \otimes (y/f^n)$. This then defines a $\mathbb{Z}$-linear map $M \otimes_{\mathbb{Z}} N \to M_{(f)} \otimes_{S_{(f)}} N_{(f)}$, and if $s \in S_q$, this map send $(sx) \otimes y$ to $(s/f^q)((x/f^m) \otimes (y/f^n))$ (where $x \in M_m$, $y \in N_n$), so we get a bi-homomorphism $\gamma_f : M \otimes_S N \to M_{(f)} \otimes_{S_{(f)}} N_{(f)}$ relative to the canonical homomorphism $S \to S_{(f)}$ (sending $s \in S_q$ to s/f^q). Suppose that for an element $\sum_i (x_i \otimes y_i)$ of $M \otimes_S N$ (where x_i, y_i are homogeneous elements of degrees m_i, n_i, respectively) we have $f^r \sum_i (x_i \otimes y_i) = 0$, which means $\sum_i (f^r x_i \otimes y_i) = 0$. Then we deduce from the isomorphism $M_f \otimes_{S_f} N_f \cong (M \otimes_S N)_f$ that $\sum_i (f^r x^i / f^{m_i + r}) \otimes (y_i / f^{n_i}) = 0$, which means $\gamma_i(\sum_i (x_i \otimes y_i)) = 0$. Therefore γ_f factors through $(M \otimes_S N)_f$ and give a homomorphism $\tilde{\gamma}_f : (M \otimes_S N)_f \to M_{(f)} \otimes_{S_{(f)}} N_{(f)}$; if $\tilde{\lambda}_f$ is the restriction of $\tilde{\gamma}_f$ to $(M \otimes_S N)_{(f)}$, we then verify that λ_f and $\tilde{\lambda}_f$ are inverses of each other, so the first assertion follows.

To demonstrate the second assertion, we now assume that M is of finite presentation, so is the cokernel of a homomorphism $P \to Q$ of graded S-module, P, Q being direct sums of finitely many modules of the form $S(n)$. By using the left exactness of $\mathrm{Hom}_S(-, N)$ and the exactness of $M_{(f)}$ on M, we are reduced to prove that μ_f is an isomorphism in the case $M = S(n)$. Now for any homogeneous $z \in N$, let u_z be the homomorphism from $S(n)$ to N such that $u_z(1) = z$; we then see that $\eta : z \mapsto u_z$ is an isomorphism of degree 0 from $N(-n)$ to $\mathrm{Hom}_S(S(n), N)$. It thus corresponds to an isomorphism

$$\eta_f : N(-n)_{(f)} \to \mathrm{Hom}_S(S(n), N)_{(f)}.$$

On the other hand, let $\tilde{\eta}_f$ be the isomorphism $N_{(f)} \to \mathrm{Hom}_{S_{(f)}}(S(n)_{(f)}, N_{(f)})$ which send $z' \in N_{(f)}$ to the homomorphism $v_{z'}$ such that $v_{z'}(s/f^k) = sz'/f^{n+k}$ (for $s \in S_{n+k} = S(n)_k$). We consider the composition

$$N(-n)_{(f)} \xrightarrow{\eta_f} \mathrm{Hom}_S(S(n), N)_{(f)} \xrightarrow{\mu_f} \mathrm{Hom}_{S_{(f)}}(S(n)_{(f)}, N_{(f)}) \xrightarrow{\tilde{\eta}_f^{-1}} N_{(f)}$$

is the isomorphism $z/f^h \mapsto z/f^{h-n}$ from $N(-n)_{(f)} \to N_{(f)}$, so μ_f is an isomorphism. $\square$

If the ideal S_+ is generated by S_1, we deduce from Proposition 11.2.28 that for any graded ideal $\mathfrak{I}$ of S and any graded S-module M, we have $\widetilde{\mathfrak{I}}\widetilde{M} = \widetilde{\mathfrak{I}M}$.

Corollary 11.2.29. *Suppose that the ideal S_+ is generated by S_1. Then for integers m, n, we have canonical isomorphisms*

$$\mathcal{O}_X(m) \otimes_{\mathcal{O}_X} \mathcal{O}_X(n) = \mathcal{O}_X(m+n), \quad \mathcal{O}_X(n) = (\mathcal{O}_X(1))^{\otimes n}.$$

Proof. The first formula follows from Proposition 11.2.28 and the existence of the canonical isomorphism $S(m) \otimes_S S(n) \cong S(m+n)$, which sends the element $1 \otimes 1 \in S(m)_{-m} \otimes S(n)_{-n}$ to the element $1 \in S(m+n)_{-(m+n)}$. It then suffices to demonstrate the second formula for $n = -1$, and in view of Proposition 11.2.28, this follows from the fact that $\mathrm{Hom}_S(S(1), S)$ is canonically isomorphic to $S(-1)$. $\square$

Corollary 11.2.30. *Suppose that the ideal S_+ is generated by S_1. For any graded S-module M and $n \in \mathbb{Z}$, we have a canonical isomorphism $\widetilde{M(n)} = \widetilde{M}(n)$.*

Proof. This follows from Proposition 11.2.28 and the canonical isomorphism $M(n) \cong M \otimes_S S(n)$ which send $z \in M(n)_h = M_{n+h}$ to $z \otimes 1 \in M_{n+h} \otimes S(n)_{-n} \subseteq (M \otimes_S S(n))_h$. $\square$

Example 11.2.31. Let S' be the graded ring such that $S'_0 = \mathbb{Z}$ and $S'_n = S_n$ for $n > 0$. Then if $f \in S_d$ ($d > 0$), we have $S(n)_{(f)} = S'(n)_{(f)}$ for any $n \in \mathbb{Z}$, because an element of $S'(n)_{(f)}$ is of the form x/f^k where $x \in S'_{n+kd}$ ($k > 0$), and we can always choose k such that $n + kd \neq 0$. As $X = \mathrm{Proj}(S)$ and $X' = \mathrm{Proj}(S')$ are canonically identified, we see that for any $n \in \mathbb{Z}$, $\mathscr{O}_X(n)$ and $\mathscr{O}_{X'}(n)$ are canonically isomorphic under this identification.

On the other hand, for any $d > 0$ and $n \in \mathbb{Z}$, we have

$$S^{(d)}(n)_h = S_{(n+h)d} = S(nd)_{hd};$$

so for any $f \in S_d$ we have $S^{(d)}(n)_{(f)} = S(nd)_{(f)}$. We have seen that the schemes $X = \mathrm{Proj}(S)$ and $X^{(d)} = \mathrm{Proj}(S^{(d)})$ are canonically identified, so under this identification, $\mathscr{O}_X(nd)$ and $\mathscr{O}_{X^{(d)}}(n)$ are canonically isomorphic, for any $n \in \mathbb{Z}$.

Proposition 11.2.32. *Let $d > 0$ be an integer and $U = \bigcup_{f \in S_d} D_+(f)$. Then the restriction to U of the canonical homomorphism $\mathscr{O}_X(nd) \otimes_{\mathscr{O}_X} \mathscr{O}_X(-nd) \to \mathscr{O}_X$ is an isomorphism for each $n \in \mathbb{Z}$.*

Proof. In view of Corollary 11.2.30, we can assume that $d = 1$, and the conclusion then follows the proof of Proposition 11.2.24. □

11.2.4 Graded S-module associated with a sheaf

In this paragraph, for simplicity, we always assume that the ideal S_+ is generated by S_1, which also means that $S = S_0[S_1]$ by Proposition 2.1.37. The $\mathscr{O}_X$-module $\mathscr{O}_X(1)$ is then inveritble by Corollary 11.2.26; we then put, for any $\mathscr{O}_X$-module $\mathscr{F}$, that

$$\Gamma_*(\mathscr{F}) = \Gamma_*(\mathscr{O}_X(1), \mathscr{F}) = \bigoplus_{n \in \mathbb{Z}} \Gamma(X, \mathscr{F}(n))$$

Recall that $\Gamma_*(\mathscr{O}_X)$ is endowed with a graded ring structure and $\Gamma_*(\mathscr{F})$ is a graded module over $\Gamma_*(\mathscr{O}_X)$. Since $\mathscr{O}_X(n)$ is locally free, $\Gamma_*(\mathscr{F})$ is a covariant left-exact functor on $\mathscr{F}$; in particular, if $\mathscr{J}$ is a sheaf of ideals of $\mathscr{O}_X$, $\Gamma_*(\mathscr{F})$ is canonically a graded ideal of $\Gamma_*(\mathscr{O}_X)$.

Suppose that M is a graded S-module. For any $f \in S_d$ with $d > 0$, $x \mapsto x/1$ is a homomorphism of abelian groups $M_0 \mapsto M_{(f)}$, and as $M_{(f)}$ is canonically identified with $\Gamma(D_+(f), \widetilde{M})$, we obtain a homomorphism $\alpha_0^f : M_0 \to \Gamma(D_+(f), \widetilde{M})$ of abelian groups. It is clear that, for any $g \in S_e$ with $e > 0$, the diagram

$$\begin{tikzcd} & \Gamma(D_+(f), \widetilde{M}) \arrow[d] \\ M_0 \arrow[ur, "\alpha_0^f"] \arrow[dr, "\alpha_0^{fg}"'] & \\ & \Gamma(D_+(fg), \widetilde{M}) \end{tikzcd}$$

is commutative, and this signifies that for any $x \in M_0$, the sections $\alpha_0^f(x)$ and $\alpha_0^{fg}(x)$ of M coincide over $D_+(fg)$, and therefore there exists a unique section $\alpha_0(x) \in \Gamma(X, \widetilde{M})$ whose restriction on $D_+(f)$ is $\alpha_0^f(x)$. We then define (under the hypothesis that S_+ is generated by S_1) a homomorphism

$$\alpha_0 : M_0 \to \Gamma(X, \widetilde{M}).$$

By applying this result on each graded S-module $M(n)$ (where $n \in \mathbb{Z}$), we then obtain homomorphisms of abelian groups

$$\alpha_n : M_n = M(n)_0 \to \Gamma(X, \widetilde{M}(n))$$

and therefore a homomorphism of graded abelian groups

$$\alpha : M \to \Gamma_*(\widetilde{M})$$

(also denoted by α_M) such that α_M coincides with α_n on each M_n.

If we consider in particular $M = S$, then it is easy to see that (by the definition of the multiplication of $\Gamma_*(\mathscr{O}_X)$) $\alpha : S \to \Gamma_*(\mathscr{O}_X)$ is a homomorphism of graded rings, therefore, for any S-module M, α is a bi-homomorphism of graded modules.

Proposition 11.2.33. *For any $f \in S_d$ with $d > 0$, $D_+(f)$ is identified with the subset of $\mathfrak{p} \in X$ such that the section $\alpha_d(f)$ of $\mathscr{O}_X(d)$ is nonzero at $\mathfrak{p}$.*

Proof. As $X = \bigcup_{g\in S_1} D_+(g)$, it suffices to prove that for any $g \in S_d$, the set $\mathfrak{p} \in D_+(g)$ where $\alpha_d(f)$ is nonzero is identified with $D_+(fg)$. Now the restriction of $\alpha_d(f)$ to $D_+(g)$ is by definition the section corresponding to the element $f/1$ of $S(d)_{(g)}$; by the canonical isomorphism $S(d)_{(g)} \cong S_{(g)}$, this section of $\mathscr{O}_X(d)$ over $D_+(g)$ is identified with the section of $\mathscr{O}_X$ over $D_+(g)$ corresponding to the element f/g^d of $S_{(g)}$. To see that this section is zero on $\mathfrak{p} \in D_+(g)$ then signifies that $f/g^d \in \mathfrak{q}$, where $\mathfrak{q}$ is the prime ideal of $S_{(g)}$ corresponding to $\mathfrak{p}$; by definition this means $f \in \mathfrak{p}$, whence the proposition. □

Now let $\mathscr{F}$ be an $\mathscr{O}_X$-module and put $M = \Gamma_*(\mathscr{F})$. In view of the homomorphism $\alpha : S \to \Gamma_*(\mathscr{O}_X)$ of graded rings, M can also be considered as a graded S-module. For any $f \in S_d$ ($d > 0$), it follows from Proposition 11.2.33 that the restriction of the section $\alpha_d(f)$ of $\mathscr{O}_X(d)$ to $D_+(f)$ is invertible, and so is the restriction to $D_+(f)$ of $\alpha_d(f^n)$ for any $n > 0$. Let $z \in M_{nd} = \Gamma(X, \mathscr{F}(nd))$, if there existss an integer $k \geq 0$ such that the restriction to $D_+(f)$ of $f^k z$, which is the section $(\alpha_d(f^k)z)|_{D_+(f)}$ of $\mathscr{F}((n+k)d)$, is zero, then we conclude that $z|_{D_+(f)} = 0$. This shows that we can define an $S_{(f)}$-homomorphism $\beta_f : M_{(f)} \to \Gamma(D_+(f), \mathscr{F})$ which corresponds the element $z/f^n \in M_{(f)}$ the section $(z|_{D_+(f)})(\alpha_d(f^n)|_{D_+(f)})^{-1}$ of $\mathscr{F}$ over $D_+(f)$. We also verify that for $g \in S_e$ ($e > 0$), the diagram

$$\begin{tikzcd} M_{(f)} \arrow[r, "\beta_f"] \arrow[d] & \Gamma(D_+(f), \mathscr{F}) \arrow[d] \\ M_{(fg)} \arrow[r, "\beta_{fg}"] & \Gamma(D_+(fg), \mathscr{F}) \end{tikzcd}$$

is commutative. Since $M_{(f)}$ is canonically identified with $\Gamma(D_+(f), \widetilde{M})$ and the $D_+(f)$ form a basis for the topological space X, the homomorphisms β_f glue together to a unique canonical homomorphism of $\mathscr{O}_X$-modules

$$\beta : \widetilde{\Gamma_*(\mathscr{F})} \to \mathscr{F}$$

(also denoted by $\beta_{\mathscr{F}}$) which is evidently functorial.

Proposition 11.2.34. *Let M be a graded S-module and $\mathscr{F}$ be an $\mathscr{O}_X$-module. Then the composition homomorphisms*

$$\widetilde{M} \xrightarrow{\tilde{\alpha}} (\Gamma_*(\widetilde{M}))^\sim \xrightarrow{\beta} \widetilde{M} \tag{11.2.7}$$

$$\Gamma_*(\mathscr{F}) \xrightarrow{\alpha} \Gamma_*(\widetilde{\Gamma_*(\mathscr{F})}) \xrightarrow{\Gamma_*(\beta)} \Gamma_*(\mathscr{F}) \tag{11.2.8}$$

are isomorphisms.

Proof. The verification of (11.2.7) is local: on an open subset $D_+(f)$, this result follows from the definition and the fact that the action of β is determined by its action of sections over $D_+(f)$ (Corollary 10.1.3). The verification of (11.2.8) can be done at each degree: if we put $M = \Gamma_*(\mathscr{F})$, we have $M_n = \Gamma(X, \mathscr{F}(n))$ and $\Gamma_*(\widetilde{M})_n = \Gamma(X, \widetilde{M}(n)) = \Gamma(X, \widetilde{M(n)})$. If $f \in S_1$ and $z \in M_n$,

$\alpha_n^f(z)$ is the element $z/1$ of $M(n)_{(f)}$, and equals to $(f/1)^n(z/f^n)$; it then corresponds by β_f to the section

$$(\alpha_1(f)^n|_{D_+(f)})(z|_{D_+(f)})(\alpha_1(f)^n|_{D_+(f)})^{-1}$$

over $D_+(f)$, which is the restriction of z to $D_+(f)$. □

In general, the homomorphisms α and β are not isomorphisms (for example, a graded S-module M can be nonzero with $\widetilde{M}$ being zero). To obtain some nice results about these two homomorphisms, we need to impose further conditions on the graded ring S and the graded S-module M.

Proposition 11.2.35. *Let S be a graded ring and A be a ring.*

(a) *If S is Noetherian, then $X = \mathrm{Proj}(S)$ is a Noetherian scheme.*

(b) *If S is a graded A-algebra of finite type, then $X = \mathrm{Proj}(S)$ is a scheme of finite type over $Y = \mathrm{Spec}(A)$.*

Proof. If S is Noetherian, the ideal S_+ is generated by finitely many homogeneous elements $(f_i)_{1\leq i\leq p}$, so the space X is the union of $D_+(f_i) = \mathrm{Spec}(S_{(f_i)})$, and since each $S_{(f_i)}$ is Noetherian by Corollary 11.2.4, we see X is Noetherian.

Now assume that S is an A-algebra of finite type, then S_0 is an A-algebra of finite type and S is an S_0-algebra of finite type, so S_+ is a finitely generated ideal by Corollary 2.1.38. We are then reduced to prove as in (a) that for any $f \in S_d$, $S_{(f)}$ is an A-algebra of finite type. In view of Proposition 11.2.3, it suffices to show that $S^{(d)}$ is an A-algebra of finite type, which follows from Proposition 2.1.39. □

Let M be a graded S-module. We say M is **eventually null** if there exists an integer n such that $M_i = 0$ for $i \neq n$, and is **eventually finite** if there exists an integer n such that $\bigoplus_{i\geq n} M_i$ is a finitely generated S-module, or equivalently, that there exists a finitely generated graded sub-S-module M' of M such that M/M' is eventually null. We also note that if M is eventually null, then $M_{(f)} = 0$ for any homogeneous element f in S_+, so $\widetilde{M} = 0$.

Let M, N be two graded S-modules. We say a homomorphism $u : M \to N$ of degree 0 is **eventually injective** (resp. **eventually surjective**, **eventually bijective**) if there exists an integer n such that $u_i : M_i \to N_i$ is injective (resp. surjective, bijective) for $i \geq n$. Equivalently, the homomorphism u is eventually injective (resp. eventually surjective) if and only if $\ker u$ (resp. $\mathrm{coker}\, u$) is eventually null. If u is eventually bijective, we say it is an **eventual isomorphism**.

Proposition 11.2.36. *Let S be a graded ring such that S_+ is finitely generated and M be a graded S-module.*

(a) *If M is eventually finite, the $\mathscr{O}_X$-module $\widetilde{M}$ is of finite type.*

(b) *Suppose that M is eventually finite, then for $\widetilde{M} = 0$, it is necessary and sufficient that M is eventually null.*

Proof. If $M_n = 0$ for $n \geq n_0$ then $M_{(f)} = 0$ for any homogeneous element $f \in S_+$, so if M is eventually null, then $\widetilde{M} = 0$. On the other hand, if M is eventually finite, then for $n \gg 0$ the graded submodule $M' = \bigoplus_{k\geq n} M_k$ is finitely generated by hypothesis, and M/M' is eventually null, so $\widetilde{(M/M')} = 0$ and therefore $\widetilde{M} = \widetilde{M}'$ by the exactness of the functor $\widetilde{M}$ (Proposition 11.2.21). Therefore, to prove that $\widetilde{M}$ is of finite type, we may assume that M is finitely generated. Now since this question is local, we only need to show that $M_{(f)}$ is finitely genrated over $S_{(f)}$ for any homogeneous $f \in S_d$ with $d > 0$. But $M^{(d)}$ is a finitely generated $S^{(d)}$ module by Proposition 2.1.39, and the assertion follows from Proposition 11.2.3.

Suppose now that M is eventually finite and $\widetilde{M} = 0$; then we have $\widetilde{M}' = 0$, so the condition that M' is eventually null is equivalent to that of M. We may therefore assume that M is finitely generated over S by homogeneous elements x_i $(1 \leq i \leq p)$; let $(f_j)_{1\leq j\leq q}$ be a system of generators of the ideal S_+. We have by hypothesis $M_{(f_j)} = 0$ for any j, so there exists an integer n such that $f_j^n x_i = 0$ for any i, j. Let $n_j = \deg(f_j)$ and m be the supremum of $\sum_j r_j n_j$ for any finite system of integers (r_j) such that $\sum_j r_j \leq nq$. It is then clear that if $k > m$, we have $S_k x_i = 0$ for any i; if d is the supremum of the degrees of x_i, we then conclude that $M_k = 0$ for $k > d + m$, which proves our assertion. □

Corollary 11.2.37. *Let S be a graded ring such that S_+ is finitely generated. For $X = \mathrm{Proj}(S) = \varnothing$, it is necessary and sufficient that S is eventually null.*

Proof. The condition $X = \varnothing$ is in fact equivalent to $\mathscr{O}_X = \widetilde{S} = 0$, and S is clearly a finite generated S-module. □

Example 11.2.38. To give a counterexample of Proposition 11.2.36, let k be a field and $S = k[X_1, \dots, X_n]$ be the polynomial ring with n variables. Consider the graded S-module M given by

$$M = k[X_1, X_2, \dots, X_n, \dots]/(X_1, X_2^2, \dots, X_n^n, \dots)$$

with the usual multiplication of polynomials. Then M is not finitely generated over S, and for every homogeneous polynomial f of degree 1, we see $f^N M = 0$ for sufficiently large N. Therefore $\widetilde{M} = 0$. However, note that $M_n \neq 0$ for every n.

Theorem 11.2.39. *Suppose that S is a graded ring such that the ideal S_+ is finitely generated by S_1, and let $X = \mathrm{Proj}(S)$. Then, for any quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, the canonical homomorphisms $\beta : \widetilde{\Gamma_*(\mathscr{F})} \to \mathscr{F}$ is an isomorphism.*

Proof. If S_+ is generated by finitely many $f_i \in S_1$, then X is the union of $\mathrm{Spec}(S_{(f_i)})$ which are quasi-compact, so X is quasi-compact. Also, $\mathscr{O}_X(n)$ is invertible for any $n \in \mathbb{Z}$ by Corollary 11.2.29, and since X is separated, by Corollary 10.6.15 and Proposition 11.2.33, we have for any $f \in S_d$ a canonical isomorphism $\Gamma_*(\mathscr{F})_{(\alpha_d(f))} \cong \Gamma(D_+(f), \mathscr{F})$ (the first module (considered as a $\Gamma_*(\mathscr{O}_X)$-module) is none other than $\Gamma_*(\mathscr{F})_{(f)}$ (considered as an S-module)). If we trace the definition of this isomorphism, we see that it coincides with β_f, whence our assertion. □

Corollary 11.2.40. *Under the hypotheses of Theorem 11.2.39, any quasi-coherent $\mathscr{O}_X$-module (of finite type) is isomorphic to an $\mathscr{O}_X$-module of the form $\widetilde{M}$, where M is a (finitely generated) graded S-module.*

Proof. Let $\mathscr{F}$ be a quasi-coherent $\mathscr{O}_X$-module, then $\mathscr{F} = \widetilde{M}$ for a graded S-module M by Theorem 11.2.39. Let $(f_\lambda)_{\lambda\in I}$ be a system of homogeneous generators of M; for each finite subset H of I, let M_H be the graded submodule of M generated by f_λ for $\lambda \in H$. It is clear that M is the inductive limit of the submodules M_H, so $\mathscr{F}$ is the inductive limit of the sub-$\mathscr{O}_X$-modules $\widetilde{M}_H$ (Proposition 11.2.21). If $\mathscr{F}$ is of finite type, we conclude from **??**. □

Corollary 11.2.41. *Under the hypotheses of Theorem 11.2.39, for any quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$ of finite type, there exists an integer n_0 such that for any $n \geq n_0$, $\mathscr{F}(n)$ is isomorphic to a quotient of $\mathscr{O}_X^r$ (where $r > 0$ depends on n), and therefore is generated by finitely many global sections.*

Proof. By Corollary 11.2.40, we can assume that $\mathscr{F} = \widetilde{M}$ where M is a quotient of a finite dierct sum of $S(m_i)$. By Proposition 11.2.21, we are therefore reduced to the case where $M = S(m)$, so $\mathscr{F}(n) = (S(m+n))^\sim = \mathscr{O}_X(m+n)$. It then suffices to prove that for each $n \geq 0$ there exists r and a surjective homomorphism $\mathscr{O}_X^r \to \mathscr{O}_X(n)$. For this, it suffcies to prove that, for a suitable r, there exists an eventually surjective homomorphism $u : S^r \to S(n)$ of degree zero. Now we have $S(n)_0 = S_n$, and by hypothesis $S_h = S_1^h$ for any $h > 0$, so $SS_n = \bigoplus_{h\geq n} S_h$. As S_n is a finitely generated S_0-module (Corollary 2.1.38), consider a system $(a_i)_{1\leq i\leq r}$ of generators of this module, and let $u : S^r \to S(n)$ be the homomorphism that sends the i-th basis e_i of S^r to a_i. Then the image of u contains $\bigoplus_{h\geq 0} S(n)_h$, so u satisfies the requirement and the proof is complete. $\square$

Corollary 11.2.42. *Under the hypotheses of Theorem 11.2.39, for any quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$ of finite type, there exists an integer n_0 such that for any $n \geq n_0$, $\mathscr{F}$ is isomorphic to a quotient of an $\mathscr{O}_X$-module of the form $\mathscr{O}_X(-n)^r$ (where $r > 0$ depends on n).*

Proof. This follows from Corollary 11.2.41 by tensoring with the invertible sheaf $\mathscr{O}_X(-n)$, which preserves the exactness. $\square$

Proposition 11.2.43. *Assume the hypotheses of Theorem 11.2.39 and let M be a graded S-module.*

(a) *The cannical homomorphism $\tilde{\alpha} : \widetilde{M} \to (\Gamma_*(\widetilde{M}))^\sim$ is an isomorphism.*

(b) *Let $\mathscr{G}$ be a quasi-coherent sub-$\mathscr{O}_X$-module of $\widetilde{M}$ and let N be the graded sub-S-module of M which is the inverse image of $\Gamma_*(\mathscr{G})$ under α. Then we have $\widetilde{N} = \mathscr{G}$.*

Proof. As $\beta : (\Gamma_*(\widetilde{M}))^\sim \to \widetilde{M}$ is an isomorphism, $\tilde{\alpha}$ is its inverse isomorphism in view of (11.2.7), whence (a). Let P be the graded submodule $\alpha(M)$ of $\Gamma_*(\widetilde{M})$; as $\widetilde{M}$ is an exact functor, the image of $\widetilde{M}$ under $\tilde{\alpha}$ is equal to $\widetilde{P}$, so in view of (a), $\widetilde{P} = (\Gamma_*(\widetilde{M}))^\sim$. Put $Q = \Gamma_*(\mathscr{G}) \cap P$, so that $N = \alpha^{-1}(Q)$. Then by the preceding argument and Proposition 11.2.21, the image of $\widetilde{N}$ under $\tilde{\alpha}$ is $\widetilde{Q}$, and we have $\widetilde{Q} = \widetilde{\Gamma_*(\mathscr{G})}$. Since the image of $\widetilde{\Gamma_*(\mathscr{G})}$ under β is $\mathscr{G}$ and $\tilde{\alpha}$ is the inverse of β, we conclude that $\widetilde{N} = \mathscr{G}$. $\square$

11.2.5 Functorial properties of Proj(S)

Let S, S' be two graded rings with positive degree and $\varphi : S' \to S$ be a homomorphisms of graded rings. We denote by $G(\varphi)$ the open subset of $X = \mathrm{Proj}(S)$ which is the complement of $V_+(\varphi(S'_+))$, or, the union of $D_+(\varphi(f'))$ where f' runs through homogeneous elements of S'_+. The restriction to $G(\varphi)$ of the continuous map ${}^a\varphi : \mathrm{Spec}(S') \to \mathrm{Spec}(S)$ is then a continuous map from $G(\varphi)$ to $\mathrm{Proj}(S')$, which is still denoted by ${}^a\varphi$. If $f' \in S'_+$ is homogeneous, we have

$$ {}^a\varphi^{-1}(D_+(f')) = D_+(\varphi(f')) \tag{11.2.9} $$

since ${}^a\varphi$ maps $G(\varphi)$ into $\mathrm{Proj}(S')$. On the other hand, the homomorphism φ defines canonically a homomorphism of graded rings $S'_{f'} \to S_f$ of degree 0 (where $f = \varphi(f')$), whence a homomorphism $S'_{(f')} \to S_{(f)}$, which we denote by $\varphi_{(f)}$. It then corresponds to a morphism $({}^a\varphi_{(f)}, \tilde{\varphi}_{(f)}) : \mathrm{Spec}(S_{(f)}) \to \mathrm{Spec}(S'_{(f')})$ of affine schemes. If we identify $\mathrm{Spec}(S_{(f)})$ with the open subscheme $D_+(f)$ of $\mathrm{Proj}(S)$, we then obtain a morphism $\Phi_f : D_+(f) \to D_+(f')$ and ${}^a\varphi_{(f)}$ is identified with the restriction of ${}^a\varphi$ to $D_+(f)$. If g' is another homogeneous element of S'_+ and $g = \varphi(g')$, it is immediate that the diagram

$$\begin{tikzcd} D_+(fg) \arrow[r, "\Phi_{fg}"] \arrow[d] & D_+(f'g') \arrow[d] \\ D_+(f) \arrow[r, "\Phi_f"] & D_+(f') \end{tikzcd}$$

is commutative.

Proposition 11.2.44. *Let $\varphi : S' \to S$ be a homomorphism of graded rings. There exists a unique morphism $({}^a\varphi, \tilde{\varphi}) : G(\varphi) \to \mathrm{Proj}(S')$ (called the morphism associated with φ and denoted by $\mathrm{Proj}(\varphi)$), such that for any homogeneous element $f' \in S'_+$, the restriction of this morphism to $D_+(\varphi(f'))$ coincides with the morphism associated with the homomorphism $\varphi_{(f)} : S'_{(f')} \to S_{(\varphi(f'))}$.*

Proof. The morphism $({}^a\varphi, \tilde{\varphi})$ is obtained form glueing the morphisms Φ_f over $D_+(f)$, and the claim property is immediate. $\square$

Corollary 11.2.45. *Let $\varphi : S' \to S$ be a homomorphism of graded rings.*

(a) *The morphism* $\mathrm{Proj}(\varphi)$ *is affine.*

(b) *If* $\ker\varphi$ *is nilpotent (and in particular if φ is injective), the morphism* $\mathrm{Proj}(\varphi)$ *is dominant.*

(c) *If φ is eventually surjective, then $G(\varphi) = \mathrm{Proj}(S)$.*

Proof. The first assertion follows from Corollary 11.2.45 and the relation ${}^a\varphi^{-1}(D_+(f')) = D_+(\varphi(f'))$. On the other hand, if $\ker\varphi$ is nilpotent, for any f' homogeneous in S'_+, we verify that $\ker\varphi_f$ is also nilpotent, and so is $\ker\varphi_{(f)}$. The conclusion then follows from Corollary 1.4.21. Finally, if φ is eventually surjective, then every homogeneous element $f \in S_+$ has some power contained in the image of φ, so by Corollary 11.2.11 we conclude that $G(\varphi) = \bigcup_{f' \in S'_+} D_+(\varphi(f')) = \mathrm{Proj}(S)$, whence the claim. $\square$

Remark 11.2.46. Note that there are in general morphisms from $\mathrm{Proj}(S)$ to $\mathrm{Proj}(S')$ which are not affine, and therefore do not come from graded ring homomorphisms $S' \to S$; an example is the structural morphism $\mathrm{Proj}(S) \to \mathrm{Spec}(A)$, where A is field ($\mathrm{Spec}(A)$ being identified with $\mathrm{Proj}(A[T])$ (cf. Corollary 11.3.5)).

Let $\varphi' : S'' \to S'$ be another homomorphism of graded rings, and put $\varphi'' = \varphi \circ \varphi'$. Then by the formula ${}^a\varphi'' = {}^a\varphi' \circ {}^a\varphi$ and $G(\varphi'') \subseteq G(\varphi)$, if Φ, Φ', and Φ'' are the associated morphisms of φ, φ' and φ'', we have $\Phi'' = \Phi' \circ (\Phi|_{G(\varphi'')})$.

Suppose that S (resp. S') is a graded A-algebra (resp. a graded A'-algebra), and let $\psi : A' \to A$ be a homomorphism of rings such that the diagram

$$\begin{tikzcd} A' \arrow[r, "\psi"] \arrow[d] & A \arrow[d] \\ S' \arrow[r, "\varphi"] & S \end{tikzcd}$$

is commutative. We can then consider $G(\varphi)$ and $\mathrm{Proj}(S')$ as schemes over $\mathrm{Spec}(A)$ and $\mathrm{Spec}(A')$, respectively. If Φ and Ψ are the associated morphisms of φ and ψ, respectively, the diagram

$$\begin{tikzcd} G(\varphi) \arrow[r, "\Phi"] \arrow[d] & \mathrm{Proj}(S') \arrow[d] \\ \mathrm{Spec}(A) \arrow[r, "\Psi"] & \mathrm{Spec}(A') \end{tikzcd}$$

is commutative.

Now let M be a graded S-module and consider the S'-module $\varphi^*(M)$, which is clearly graded. Let f' be a homogeneous element in S'_+, and set $f = \varphi(f')$. We then have a canonical isomorphism $(\varphi^*(M))_{f'} \cong \varphi_f^*(M_f)$, and it is clear that this isomorphism preserves degrees, so induces an isomorphism $(\varphi^*(M))_{(f')} \cong \varphi_{(f)}^*(M_{(f)})$. There is then canonically an isomorphism of sheaves $\widetilde{\varphi^*(M)}|_{D_+(f')} \cong (\Phi_f)_*(\widetilde{M}|_{D_+(f)})$ (Proposition 10.1.12). Moreover, if g' is another homogeneous element of S'_+ and $g = \varphi(g')$, the diagram

$$\begin{tikzcd} (\varphi^*(M))_{(f')} \arrow[r, "\sim"] \arrow[d] & (M_{(f)})_{(\varphi_{(f)})} \arrow[d] \\ (\varphi^*(M))_{(f'g')} \arrow[r, "\sim"] & (M_{(fg)})_{(\varphi_{(fg)})} \end{tikzcd}$$

is commutative, whence we conclude that the isomorphism

$$\widetilde{\varphi^*(M)}|_{D_+(f'g')} \cong (\Phi_{fg})_*(\widetilde{M}|_{D_+(fg)})$$

is the restriction to $D_+(f'g')$ of the isomorphism $\widetilde{\varphi^*(M)}|_{D_+(f')} \cong (\Phi_f)_*(\widetilde{M}|_{D_+(f)})$. As Φ_f is the restriction of Φ on $D_+(f)$, we then obtain the following result:

Proposition 11.2.47. *There exists a canonical isomorphism $\widetilde{\varphi^*(M)} \cong \Phi_*(\widetilde{M}|_{G(\varphi)})$ of $\mathscr{O}_X$-modules.*

We also deduce a canonical functorial map from the set of φ-homomorphisms $M' \to M$ from a graded S'-module to a graded S-module M, to the set of Φ-morphisms $\widetilde{M}' \to \widetilde{M}|_{G(\varphi)}$. If $\varphi' : S'' \to S'$ is another ring homomorphism and M'' is a graded S''-module, the composition of a φ-morphism $M' \to M$ and a φ'-morphism $M'' \to M'$ canonically corresponds to the composition of $\widetilde{M}'|_{G(\varphi')} \to \widetilde{M}|_{G(\varphi'')}$ and $\widetilde{M}'' \to \widetilde{M}'|_{G(\varphi')}$.

Proposition 11.2.48. *Let $\varphi : S' \to S$ be a homomorphism of graded rings and M' be a graded S'-module. Then there exists a canonical homomorphism $\nu : \Phi^*(\widetilde{M}') \to \widetilde{\varphi_*(M)}|_{G(\varphi)}$. If the ideal S'_+ is generated by S'_1, then ν is an isomorphism.*

Proof. For $f' \in S'_d$ with $d > 0$, we define a canonical homomorphism of $S_{(f)}$-modules (where $f = \varphi(f')$)

$$\nu_f : M'_{(f)} \otimes_{S'_{(f)}} S_{(f)} \to (M' \otimes_{S'} S)_{(f)}$$

by composing the homomorphism $M'_{(f)} \otimes_{S'_{(f)}} S_{(f)} \to M_{f'} \otimes_{S'_{f'}} S_f$ with the canonical homomorphism $M'_{f'} \otimes_{S'_{f'}} S_f \cong (M' \otimes_{S'} S)_f$. It is immediate to verify that compatibility of ν_f with the restriction homomorphisms $D_+(f)$ to $_+(fg)$ (for $g' \in S'_+$ and $g = \varphi(g')$), so we obtain a homomorphism

$$\nu : \Phi^*(\widetilde{M}') \to \widetilde{\varphi_*(M)}|_{G(\varphi)}.$$

For the second assertion, it suffices to prove that ν_f is an isomorphism for each $f' \in S'_1$, since $G(\varphi)$ is the union of $D_+(\varphi(f'))$. We first define a $\mathbb{Z}$-bilinear map $M'_m \times S_n \to M'_{(f')} \otimes_{S'_{f'}} S_{(f)}$ which sends (x', s) to the element $(x'/f'^m) \otimes (s/f^n)$. As in the proof of Proposition 11.2.28, this map then induces a bi-homomorphism

$$\eta_f : M' \otimes_{S'} S \to M'_{(f')} \otimes_{S'_{(f')}} S_{(f)}.$$

Moreover, if, for $r > 0$, we have $f^r \sum_i (x'_i \otimes s_i) = 0$, then $\sum_i (f'^r x'_i \otimes s_i) = 0$, so $\sum_i (f'^r x_i / f'^{m_i+r}) \otimes (s_i/f^{n_i}) = 0$, which means $\eta_f(\sum_i x_i \otimes y_i) = 0$; the homomorphism then factors through $(M' \otimes_{S'} S)_f$ and gives a homomorphism $\tilde{\eta}_f : (M' \otimes_{S'} S)_f \to M'_{(f')} \otimes_{S'_{(f')}} S_{(f)}$. It is easy to verify that $\tilde{\eta}_f$ is the inverse of ν_f, whence our assertion. □

In particular, since $\varphi_*(S'(n)) = S(n)$ for each $n \in \mathbb{Z}$, it follows from Proposition 11.2.48 that we have a canonical homomorphism $\Phi^*(\mathscr{O}_{X'}(n)) \to \mathscr{O}_X(n)|G(\varphi)$, and this an isomorphism if S'_+ is generated by S'_1.

Remark 11.2.49. We note that it follows from Proposition 11.2.18 that the morphism Φ is unchanged if we replace S by $S^{(d)}$, S' by $S'^{(d)}$, and φ by $\varphi^{(d)}$. Also, it is also unchanged if we replace S_0 and S'_0 by $\mathbb{Z}$ and φ_0 be the identity map.

Let A, A' be two rings and $\psi : A' \to A$ be a homomorphism of rings, which defines a morphism $\Psi : \mathrm{Spec}(A) \to \mathrm{Spec}(A')$. Let S' be an A'-algebra with positive degrees, and put $S = S' \otimes_{A'} A$, which is a graded A-algebra by setting $S_n = S'_n \otimes_{A'} A$. The map $s' \mapsto s' \otimes 1$

is then a homomorphism of graded rings and also a bi-homomorphism. Since S_+ is the A-module generated by $\varphi(S'_+)$, we have $G(\varphi) = \mathrm{Proj}(S) = X$, so, if we put $X' = \mathrm{Proj}(S')$, we have a commutative diagram

$$\begin{tikzcd} X \arrow[r, "\Phi"] \arrow[d, "p"'] & X' \arrow[d] \\ Y \arrow[r, "\Psi"] & Y' \end{tikzcd} \tag{11.2.10}$$

Now let M' be a graded S'-module, and set $M = M' \otimes_{A'} A = M' \otimes_{S'} S$.

Proposition 11.2.50. *The commutative diagram (11.2.10) is cartesian and the canonical homomorphism $\nu : \Phi^*(\widetilde{M}') \to \widetilde{M}$ in Proposition 11.2.48 is an isomorphism.*

Proof. The first assertion follows if we can prove that for any f' homogeneous in S'_+ and $f = \varphi(f')$, the restriction of Φ and p to $D_+(f)$ identify this scheme as $D_+(f') \times_{Y'} Y$; in other words, it suffices to prove that $S_{(f)}$ is canonically identified with $S'_{(f')} \otimes_{A'} A$, which is immedaite from the fact that the canonical isomorphism $S_f \cong S'_{f'} \otimes_{A'} A$ preserves degrees. The second assertion follows from the isomorphism $M'_{(f')} \otimes_{S'_{(f')}} S_{(f)} \cong M'_{(f')} \otimes_{A'} A$, and the later one is isomorphic to $M_{(f)}$ since M_f is canonically identified with $M'_{f'} \otimes_{A'} A$. □

Corollary 11.2.51. *For any integer $n \in \mathbb{Z}$, $\widetilde{M}(n)$ is identified with $\Phi^*(\widetilde{M}'(n)) = \widetilde{M}'(n) \otimes_{Y'} \mathscr{O}_Y$. In particular, $\mathscr{O}_X(n) = \Phi^*(\mathscr{O}_{X'}(n)) = \mathscr{O}_{X'}(n) \otimes_{Y'} \mathscr{O}_Y$.*

Proof. This follows from Proposition 11.2.50 and Corollary 11.2.30. □

Now let $\mathscr{F}'$ be an $\mathscr{O}_{X'}$-module and $\mathscr{F} = \Phi^*(\mathscr{F}')$. Then we have for each $n \in \mathbb{Z}$ that $\mathscr{F}(n) = \Phi^*(\mathscr{F}'(n))$ in view of Corollary 11.2.51. Therefore, by the definition of Φ^*, we have a canonical homomorphism

$$\Gamma(\rho) : \Gamma(X', \mathscr{F}'(n)) \to \Gamma(X, \mathscr{F}(n))$$

which then gives a canonical bi-homomorphism $\Gamma_*(\mathscr{F}') \to \Gamma_*(\mathscr{F})$ of graded modules.

Suppose that the ideal S'_+ is generated by S'_1 and $\mathscr{F}' = \widetilde{M}'$, so $\mathscr{F} = \widetilde{M}$ where $M = M' \otimes_{A'} A$. If f' is homogeneous in S'_+ and $f = \varphi(f')$, we have $M_{(f)} = M'_{(f')} \otimes_{A'} A$ and the diagram

$$\begin{tikzcd} M'_0 \arrow[r] \arrow[d] & M'_{(f')} = \Gamma(D_+(f), \widetilde{M}') \arrow[d] \\ M_0 \arrow[r] & M_{(f)} = \Gamma(D_+(f), \widetilde{M}) \end{tikzcd}$$

is commutative. We then conclude from the definition of the homomorphism $\alpha : M \to \Gamma_*(\widetilde{M})$ that the following diagram

$$\begin{tikzcd} M' \arrow[r, "\alpha_{M'}"] \arrow[d] & \Gamma_*(\widetilde{M}') \arrow[d] \\ M \arrow[r, "\alpha_M"] & \Gamma_*(\widetilde{M}) \end{tikzcd} \tag{11.2.11}$$

is commutative. Similarly, the diagram

$$\begin{tikzcd} \widetilde{\Gamma_*(\mathscr{F}')} \arrow[r, "\beta_{\mathscr{F}'}"] \arrow[d] & \mathscr{F}' \arrow[d] \\ \widetilde{\Gamma_*(\mathscr{F})} \arrow[r, "\beta_{\mathscr{F}}"] & \mathscr{F} \end{tikzcd} \tag{11.2.12}$$

is commutative (where the vertical is the canonical Φ-morphism $\mathscr{F}' \to \Phi^*(\mathscr{F}') = \mathscr{F}$).

Now let N' be another graded S'-module and $N = N' \otimes_{A'} A$. It is immediate that the canonical bi-homomorphisms $M' \to M$, $N' \to N$ give a bi-homomorphism $M' \otimes_{S'} N' \to M \otimes_S N$, and therefore an S-homomorphism $(M' \otimes_{S'} N') \otimes_{A'} A \to M \otimes_S N$ of degree 0, which then corresponds to an $\mathscr{O}_X$-homomorphism

$$\Phi^*((M' \otimes_{S'} N')^{\sim}) \to (M \otimes_S N)^{\sim}.$$

Moreover, it is immedaite to verify that the following diagram

$$\begin{tikzcd} \Phi^*(\widetilde{M}' \otimes_{\mathscr{O}_{X'}} \widetilde{N}') \arrow[r, "\sim"] \arrow[d, "\Phi^*(\lambda)"] & \widetilde{M} \otimes_{\mathscr{O}_X} \widetilde{N} = \Phi^*(\widetilde{M}') \otimes_{\mathscr{O}_X} \Phi^*(\widetilde{N}') \arrow[d, "\lambda"] \\ \Phi^*((M' \otimes_{S'} N')^{\sim}) \arrow[r] & (M \otimes_S N)^{\sim} \end{tikzcd} \tag{11.2.13}$$

is commutative (where the first row is an isomorphism by (**??**)). If the ideal S'_+ is generated by S'_1, it is clear that S_+ is generated by S_1, so the two vertical homomorphisms are isomorphisms, so the second row is also an isomorphism.

We have similarly a cannical bi-homomorphism $\mathrm{Hom}_{S'}(M', N') \to \mathrm{Hom}_S(M, N)$, which sends a homomorphism u' of degree k the homomorphism $u' \otimes 1$, which is also of degree k. We then deduce an S-homomorphism of degree 0:

$$\mathrm{Hom}_{S'}(M', N') \otimes_{A'} A \to \mathrm{Hom}_S(M, N)$$

which corresponds to a homomorphism of $\mathscr{O}_X$-modules:

$$\Phi^*((\mathrm{Hom}_{S'}(M', N'))^{\sim}) \to (\mathrm{Hom}_S(M, N))^{\sim}.$$

Similarly, the diagram

$$\begin{tikzcd} \Phi^*((\mathrm{Hom}_{S'}(M', N'))^{\sim}) \arrow[r] \arrow[d, "\Phi^*(\mu)"] & (\mathrm{Hom}_S(M, N))^{\sim} \arrow[d, "\mu"] \\ \Phi^*(\mathcal{H}om_{\mathscr{O}_{X'}}(\widetilde{M}', \widetilde{N}')) \arrow[r] & \mathcal{H}om_{\mathscr{O}_X}(\widetilde{M}, \widetilde{N}) \end{tikzcd} \tag{11.2.14}$$

is commutative (where the second row is the canonical homomorphism of (**??**)).

11.2.6 Closed subschemes of $\mathrm{Proj}(S)$

Recall that if $\varphi : S \to S'$ is a homomorphism of graded rings, we say that φ is eventually surjective (resp. eventually injective, eventually bijective) if $\varphi_i : S_i \to S'_i$ is surjective (resp. injective, bijective) for sufficiently large i. It follows from Proposition 11.2.18 that the study of Φ can be reduced to the case where φ is surjective (resp. injectij, bijective). Instead of saying that φ is eventually bijective, we also say that it is then an eventual isomorphism.

Proposition 11.2.52. *Let S, S' be graded rings with positive degrees and set $X = \mathrm{Proj}(S)$, $X' = \mathrm{Proj}(S')$.*

(a) *If $\varphi : S \to S'$ is an eventually surjective homomorphism of graded rings, the corresponding morphism Φ is defined over $\mathrm{Proj}(S')$ and is a closed immersion. If $\mathfrak{I}$ is the kernel of φ, the closed subscheme of X associated with Φ is defined by the quasi-coherent ideal $\widetilde{\mathfrak{I}}$ of $\mathscr{O}_X$.*

(b) *Suppose moreover that the ideal S_+ is finitely generated by S_1. Let X' be a closed subscheme of X defined by a quasi-coherent ideal $\mathscr{J}$ of $\mathscr{O}_X$. Let $\mathfrak{J}$ be the graded ideal of S which is the inverse image of $\Gamma_*(\mathscr{J})$ under the canonical homomorphism $\alpha : S \to \Gamma_*(\mathscr{O}_X)$, and put $S' = S/\mathfrak{J}$. Then X' is the subscheme associated with the closed immersion* $\mathrm{Proj}(S') \to X$ *corresponding to the canonical homomorphism $S \to S'$ of graded rings.*

Proof. Let $\varphi : S \to S'$ be an eventually surjective homomorphism of graded rings. We can suppose that φ is surjective, so $\varphi(S_+)$ is generated by S'_+, we have $G(\varphi) = \mathrm{Proj}(S')$. Now the second assertion in (a) can be verified locally over X; let f be a homogeneous element of S_+ and put $f' = \varphi(f)$. As φ is a surjective homomorphism of rings, $\varphi_{(f')} : S_{(f)} \to S'_{(f')}$ is surjective with kernel $\mathscr{J}_{(f)}$, so the corresponding morphism is closed.

We now consider the case of (b); in view of (a), we only need to verify that the homomorphism $\tilde{j} : \widetilde{\mathscr{J}} \to \mathscr{O}_X$ induced from the injection $j : \mathscr{J} \to S$ is an isomorphism from $\widetilde{\mathscr{J}}$ to $\mathscr{J}$, which follows from Proposition 11.2.43(b). □

Remark 11.2.53. Note that $\mathfrak{J}$ is the largest graded ideal $\mathfrak{J}'$ of S such that $\widetilde{\mathscr{J}'} = \mathscr{J}$ (where we identify $\widetilde{\mathscr{J}'}$ as a subsheaf of $\mathscr{O}_X$), since one immedaitely verify that this relation implies $\alpha(\mathfrak{J}') \subseteq \Gamma_*(\mathscr{J})$.

Corollary 11.2.54. *Assume the hypotheses of Proposition 11.2.52(a) and that S_+ is generated by S_1. Then $\Phi^*(\widetilde{S(n)})$ is canonically isomorphic to $\widetilde{S'(n)}$ for any $n \in \mathbb{Z}$, and therefore $\Phi^*(\mathscr{F}(n))$ is isomorphic to $\Phi^*(\mathscr{F})(n)$ for any $\mathscr{O}_X$-module $\mathscr{F}$.*

Proof. This is a partiuclar case of Proposition 11.2.48, in view of the definition of $\mathscr{F}(n)$ and Proposition 11.2.52(a). □

Corollary 11.2.55. *Assume the hypotheses of Proposition 11.2.52(a). Then for the closed subscheme X' of X to be integral, it is necessary and sufficient that the graded ideal $\mathfrak{J}$ is prime in S.*

Proof. As X' is isomorphic to $\mathrm{Proj}(S/\mathfrak{J})$, this condition is sufficient in view of Proposition 11.2.16. To see the necessity, assume that $\mathrm{Proj}(S')$ is integral and consider the exact sequence $0 \to \mathscr{J} \to \mathscr{O}_X \to \mathscr{O}_X/\mathscr{J} \to 0$, which gives an exact sequence

$$0 \longrightarrow \Gamma_*(\mathscr{J}) \longrightarrow \Gamma_*(\mathscr{O}_X) \longrightarrow \Gamma_*(\mathscr{O}_X/\mathscr{J})$$

by the left-exactness of the global section functor. In view of the canonical homomorphism $\alpha : S/\mathfrak{J} \to \Gamma_*(\mathscr{O}_X/\mathscr{J})$, it then suffices to prove that if $f \in S_m$, $g \in S_n$ are such that the image in $\Gamma_*(\mathscr{O}_X/\mathscr{J})$ of $\alpha_{n+m}(fg)$ is zero, then one of the images of $\alpha_m(f)$, $\alpha_n(g)$ is zero. Now by definition, these images are sectins of the invertible $(\mathscr{O}_X/\mathscr{J})$-modules $\mathscr{L} = (\mathscr{O}_X/\mathscr{J})(m)$ and $\mathscr{L}' = (\mathscr{O}_X/\mathscr{J})(n)$ over the integral scheme X'. The hypotheses implies that their product is zero in $\mathscr{L} \otimes \mathscr{L}'$ (Corollary 11.2.29), so one of them is zero by Corollary 10.7.25. □

Corollary 11.2.56. *Let A be a ring, M be an A-module, and S be a graded A-algebra generated by S_1. Let $u : M \to S_1$ be a surjective homomorphism of A-modules and $\bar{u} : \boldsymbol{S}(M) \to S$ be the unique homomorphism of A-algebras extending u. Then the morphism corresponding to $\bar{u}$ is a closed immersion from* $\mathrm{Proj}(S)$ *to* $\mathrm{Proj}(\boldsymbol{S}(M))$.

Proof. The homomorphism $\bar{u}$ is surjective by hypothesis, so it suffices to apply Proposition 11.2.52. □

11.3 Homogeneous spectrum of sheaves of graded algebras

Let Y be a scheme and $\mathscr{S}$ be an $\mathscr{O}_Y$-algebra. We say that $\mathscr{S}$ is **graded** if $\mathscr{S}$ is the direct sum of a family $(\mathscr{S}_n)$ of $\mathscr{O}_Y$-algebras such that $\mathscr{S}_m\mathscr{S}_n \subseteq \mathscr{S}_{m+n}$. If $\mathscr{S}$ is a graded $\mathscr{O}_Y$-algebra, by a **graded $\mathscr{S}$-module** $\mathscr{M}$ we mean an $\mathscr{S}$-module $\mathscr{M}$ which is the direct sum of a family $(\mathscr{M}_n)_{n\in\mathbb{Z}}$ such that $\mathscr{S}_m\mathscr{M}_n \subseteq \mathscr{M}_{m+n}$. We say $\mathscr{S}$ is a graded $\mathscr{O}_Y$-algebra with positive degrees if $\mathscr{S}_n = 0$ for $n < 0$, and $\mathscr{M}$ is a graded $\mathscr{S}$-module if $\mathscr{M}_n = 0$ for $n < 0$. In this section, without further specifications, we will only consider graded algebras with positive degree.

11.3.1 Homogeneous spectrum of a graded $\mathscr{O}_Y$-algebra

Let $\mathscr{S}$ be a graded $\mathscr{O}_Y$-algebra (with positive degrees) and $\mathscr{M}$ be a graded $\mathscr{S}$-module. If $\mathscr{S}$ is quasi-coherent, each homogeneous component $\mathscr{S}_n$ is also a quasi-coherent $\mathscr{O}_Y$-module, since it is the image of $\mathscr{S}$ under the projection of $\mathscr{S}$ onto $\mathscr{S}_n$. Similarly, if $\mathscr{M}$ is quasi-coherent as an $\mathscr{O}_Y$-module, so is each of its homogeneous components, and the converse also holds. If $d > 0$ is an integer, we denote by $\mathscr{S}^{(d)}$ the direct sum of the $\mathscr{O}_Y$-modules $\mathscr{S}_{nd}$, which is quasi-coherent if $\mathscr{S}$ is; for any integer k such that $0 \leq k \leq d-1$, we denote by $\mathscr{M}^{(d,k)}$ (or $\mathscr{M}^{(d)}$ if $k = 0$) the direct sum of $\mathscr{M}_{nd+k}$ (for $n \in \mathbb{Z}$). If $\mathscr{S}$ and $\mathscr{M}$ are quasi-coherent sheaves, $\mathscr{M}(n)$ is a quasi-coherent $\mathscr{S}$-module by Proposition 10.2.25.

We say that $\mathscr{M}$ is a graded $\mathscr{S}$-module **of finite type** (resp. **of finite presentation**) if for any $y \in Y$, there exists an open neighborhood U of y and integers n_i (resp. integers m_i and n_i) such that there exists a surjective homomorphism $\bigoplus_{i=1}^r(\mathscr{S}(n_i)|_U) \to \mathscr{M}|_U$ of degree 0 (resp. such that $\mathscr{M}|_U$ is isomorphic to the cokernel of a homomorphism $\bigoplus_{i=1}^r \mathscr{S}(m_i)|_U \to \bigoplus_{j=1}^r \mathscr{S}(n_j)|_U$ of degree 0).

Let U be an affine open of Y and $A = \Gamma(U, \mathscr{O}_Y)$ by its ring. By hypothesis, the graded $(\mathscr{O}_Y|_U)$-algebra $\mathscr{S}|_U$ is isomorphic to $\widetilde{S}$ where $S = \Gamma(U, \mathscr{S})$ is a graded A-algebra; we put $X_U = \mathrm{Proj}(\Gamma(U, \mathscr{S}))$. Let $U' \subseteq U$ be another affine open subset of Y and $j : U' \to U$ the canonical injection, which corresponds to a homomorphism $A \to A'$, we have $\mathscr{S}|_{U'} = j^*(\mathscr{S}|_U)$, and therefore $S' = \Gamma(U', \mathscr{S})$ is identified with $S \otimes_A A'$ by Proposition 10.1.14. We then conclude from Proposition 11.2.50 that $X_{U'}$ is canonically identified with $X_U \times_U U'$, and therefore with $p_U^{-1}(U')$, where p_U is the structural morphism $X_U \to U$. Let $\sigma_{U',U}$ be the canonical isomorphism $p_U^{-1}(U') \cong X_{U'}$ thus defined, and $\rho_{U',U}$ be the open immersion $X_{U'} \to X_U$ obtained by composing $\sigma_{U',U}^{-1}$ with the canonical injectin $p_U^{-1}(U') \to X_U$. It is immediate that if $U'' \subseteq U'$ is a third affine open of Y, we have $\rho_{U'',U} = \rho_{U'',U'} \circ \rho_{U',U}$.

Proposition 11.3.1. *Let Y be a scheme and $\mathscr{S}$ be a quasi-coherent graded $\mathscr{O}_Y$-algebra. Then there exists a unique scheme X over Y such that, if $p : X \to Y$ is the structural morphism, for any affine open U of Y, there exists an isomorphism $\eta_U : p^{-1}(U) \xrightarrow{\sim} X_U = \mathrm{Proj}(\Gamma(U, \mathscr{S}))$ such that, if V is another affine open of Y contained in U, the following diagram*

$$\begin{tikzcd}
p^{-1}(V) \arrow[r, "\eta_V"] \arrow[d] & X_V \arrow[d, "\rho_{V,U}"] \\
p^{-1}(U) \arrow[r, "\eta_U"] & X_U
\end{tikzcd}$$

*is commutative. The scheme X is called the **homogeneous specturm** of the quasi-coherent $\mathscr{O}_Y$-algebra $\mathscr{S}$ and denoted by* $\mathrm{Proj}(\mathscr{S})$.

Proof. For two affine opens U, V of Y, let $X_{U,V}$ be the scheme induced over $p_U^{-1}(U \cap V)$ by X_U; we shall define a Y-isomorphism $\theta_{U,V} : X_{V,U} \to X_{U,V}$. For this, consider an affine open $W \subseteq U \cap V$; by composing the isomorphisms

$$p_U^{-1}(W) \xrightarrow{\sigma_{W,U}} X_W \xrightarrow{\sigma_{W,V}^{-1}} p_V^{-1}(W)$$

we obtain an isomorphism $\tau_W : p_U^{-1}(W) \to p_V^{-1}(W)$, and we can verify that if $W' \subseteq W$ is an affine open, $\tau_{W'}$ is the restriction of τ_W to $p_U^{-1}(W')$; the morphisms τ_W then glue together to a Y-isomorphism $\theta_{V,U}$, which is what we want. Moreover, if U, V, W are affine opens of Y and $\theta'_{U,V}$, $\theta'_{V,W}$, and $\theta'_{U,W}$ are restrictions of $\theta_{U,V}, \theta_{V,W}, \theta_{U,W}$ on the inverse images of $U \cap V \cap W$ in X_V, X_W, X_W, respectively, it follows from the preceding definition that $\theta'_{U,V} \circ \theta'_{V,W} = \theta'_{U,W}$. The existence of X then follows from glueing these schemes via the isomorphisms $\theta_{U,V}$, and the uniqueness is clear. □

It is clear that the Y-scheme $\mathrm{Proj}(\mathscr{S})$ is separated over Y since homogeneous spectrums are separated. If $\mathscr{S}$ is an $\mathscr{O}_Y$-algebra of finite type, it follows from Proposition 11.2.35 and Proposition 10.6.33 that $\mathrm{Proj}(\mathscr{S})$ is of finite type over Y. If $p : X \to Y$ is the structural morphism, it is immediate that for any open subschem U of Y, $p^{-1}(U)$ is identified with the homogeneous spectrum $\mathrm{Proj}(\mathscr{S}|_U)$.

Proposition 11.3.2. *Let $f \in \Gamma(Y, \mathscr{S}_d)$ with $d > 0$ and $X = \mathrm{Proj}(\mathscr{S})$. Then there exists an open subset X_f of X such that, for any affine open subset U of Y, we have $X_f \cap p^{-1}(U) = D_+(f|_U)$ in $X_U = \mathrm{Proj}(\Gamma(U, \mathscr{S}))$ (where $p : X \to Y$ is the structural morphism). Moreover, the Y-scheme induced over X_f by X is canonically isomorphic to $\mathrm{Spec}(\mathscr{S}^{(d)}/(f-1)\mathscr{S}^{(d)})$.*

Proof. For any affine open U, we have $f|_U \in \Gamma(U, \mathscr{S}_d) = \Gamma(U, \mathscr{S})_d$ since U is quasi-compact. If U, U' are two affine opens of Y such that $U' \subseteq U$, $f|_{U'}$ is the image of $f|_U$ by the restriction homomorphism $\Gamma(U, \mathscr{S}) \to \Gamma(U', \mathscr{S})$, so $D_+(f|_{U'})$ is equal to the open subscheme $\rho_{U',U}^{-1}(D_+(f|_U))$ of $X_{U'}$. The subset X_f can be then defined by glueing these subschemes, and the first assertion is then obvious. On the other hand, the open subscheme $D_+(f|_U)$ of X_U is canonically identified with $\mathrm{Spec}(\Gamma(U, \mathscr{S})_{(f|_U)})$, and this identification is clearly compatible with restrictions; the second assertion then follows from Proposition 11.2.3. □

Corollary 11.3.3. *If $f \in \Gamma(Y, \mathscr{S}_d)$ and $g \in \Gamma(Y, \mathscr{S}_e)$, we have $X_{fg} = X_f \cap X_g$.*

Proof. It suffices to consider the intersection of two members of $p^{-1}(U)$, where U is an affine open of Y, and the assertion follows from $D_+(fg) = D_+(f) \cap D_+(g)$ for a graded ring S. □

Corollary 11.3.4. *Let (f_α) be a family of sections of $\mathscr{S}$ over Y such that $f_\alpha \in \Gamma(Y, \mathscr{S}_{d_\alpha})$. If the sheaf of ideals of $\mathscr{S}$ generated by this family containes all the $\mathscr{S}_n$ for sufficiently large n, then the underlying space X is the union of X_{f_α}.*

Proof. In fact, for any affine open U of Y, $p^{-1}(U)$ is the union of $X_{f_\alpha} \cap p^{-1}(U)$ by Corollary 11.2.11, so the claim follows from the construction of X_{f_α}. □

Corollary 11.3.5. *Let $\mathscr{A}$ be a quasi-coherent $\mathscr{O}_Y$-algebra and put*

$$\mathscr{S} = \mathscr{A}[T] = \mathscr{A} \otimes_{\mathbb{Z}} \mathbb{Z}[T]$$

where T is an indeterminate. Then $X = \mathrm{Proj}(\mathscr{S})$ is canonically identified with $\mathrm{Spec}(\mathscr{A})$. In particular, $\mathrm{Proj}(\mathscr{O}_Y[T])$ is identified with Y.

Proof. By applying Corollary 11.3.4 to the unique section $f \in \Gamma(Y, \mathscr{S})$ equal to T on each point of Y, we see that $X_f = X$. Moreover, we have $f \in \mathscr{S}_1$, and $\mathscr{S}^{(1)}/(f-1)\mathscr{S}^{(1)} = \mathscr{S}/(f-1)\mathscr{S}$ is canonically isomorphic to $\mathscr{A}$, whence the corollary □

Let $g \in \Gamma(Y, \mathscr{O}_Y)$; if we put $\mathscr{S} = \mathscr{O}_Y[T]$, then $g \in \Gamma(Y, \mathscr{S}_0)$; let

$$h = gT \in \Gamma(Y, \mathscr{S}_1).$$

If $X = \mathrm{Proj}(\mathscr{S})$, the canonical identification of Corollary 11.3.5 identifies X_h with the open subset Y_g of Y: in fact, we can assume that $Y = \mathrm{Spec}(A)$ is affine, and this then follows from the fact that the ring A_g is canonically identified with $A[T]/(gT-1)A[T]$.

Proposition 11.3.6. *Let $\mathscr{S}$ be a quasi-coherent graded $\mathscr{O}_Y$-algebra.*

(a) *For any $d > 0$, there exists a canonical Y-isomorphism from* $\operatorname{Proj}(\mathscr{S})$ *to* $\operatorname{Proj}(\mathscr{S}^{(d)})$.

(b) *Let $\mathscr{S}'$ be the graded $\mathscr{O}_Y$-algebra $\mathscr{O}_Y \oplus \bigoplus_{n>0} \mathscr{S}_n$, then the schemes* $\operatorname{Proj}(\mathscr{S}')$ *and* $\operatorname{Proj}(\mathscr{S})$ *are canonically Y-isomorphic.*

(c) *Let $\mathscr{L}$ be an invertible $\mathscr{O}_Y$-module and $\mathscr{S}_{(\mathscr{L})}$ be the graded $\mathscr{O}_Y$-algebra $\bigoplus_{d>0} \mathscr{S}_d \otimes \mathscr{L}^{\otimes d}$; then the schemes* $\operatorname{Proj}(\mathscr{S})$ *and* $\operatorname{Proj}(\mathscr{S}_{(\mathscr{L})})$ *are canonically Y-isomorphic.*

Proof. In all three cases, it suffices to define an isomorphism locally over Y and verifying the compatibility of restriction morphisms is immediate. We can then assume that Y is affine, and assertions (a) and (b) then follow from Proposition 11.2.18. As for (c), if the invertible sheaf $\mathscr{L}$ is just isomorphic to $\mathscr{O}_Y$ then the isomorphism between $\operatorname{Proj}(\mathscr{S})$ and $\operatorname{Proj}(\mathscr{S}_{(\mathscr{L})})$ is evident. To define a canonical isomorphism, let $Y = \operatorname{Spec}(A)$, $\mathscr{S} = \widetilde{S}$, where S is a graded A-algebra, and let c be a generator of the free A-module L such that $\mathscr{L} = \widetilde{L}$. Then for any $n > 0$, $x_n \mapsto x_n \otimes c^{\otimes n}$ is an A-isomorphism from S_n to $S_n \otimes L^{\otimes n}$, and these A-isomorphisms define an A-isomorphism of graded algebras

$$p_c : S \to S_{(L)} = \bigoplus_{n\geq 0} S_n \otimes L^{\otimes n}.$$

Let $f \in S_+$ be homogeneous of degree d; for any $x \in S_{nd}$, we have $(x \otimes c^{nd})/(f \otimes c^d)^n = (x \otimes (\varepsilon c)^{nd})/(f \otimes (\varepsilon c)^d)^n$ for any invertible element $\varepsilon \in A$, which implies that the isomorphism $S_{(f)} \to (S_{(L)})_{(f\otimes c^d)}$ induced by p_c is independent from the generator c of L, whence the assertion. □

Recall that for the quasi-coherent graded $\mathscr{O}_Y$-algebra $\mathscr{L}$ to be *generated* by the $\mathscr{O}_Y$-module $\mathscr{S}_1$, it is necessary and sufficient that there exists a covering (U_α) of Y by affine opens such that the graded algebra $\Gamma(U_\alpha, \mathscr{S})$ over $\Gamma(U_\alpha, \mathscr{S}_0)$ is generated by $\Gamma(U_\alpha, \mathscr{S}_1)$. If this is true, then for any open subset V of Y, $\mathscr{S}|_V$ is then generated by $(\mathscr{O}_Y|_V)$-algebra $\mathscr{S}_1|_V$.

Proposition 11.3.7. *Suppose that there exists a finite affine open cover (U_i) of Y such that the graded algebra $\Gamma(U_i, \mathscr{S})$ is of finite type over $\Gamma(U_i, \mathscr{O}_Y)$. Then there exists $d > 0$ such that $\mathscr{S}^{(d)}$ is generated by $\mathscr{S}_d$, which is an $\mathscr{O}_Y$-module of finite type.*

Proof. In fact, it follows from Corollary 2.1.42 that for each i, there exist an integer m_i such that $\Gamma(U_i, \mathscr{S}_{nm_i}) = (\Gamma(U_i, \mathscr{S}_{m_i}))^n$ for all $n > 0$; it suffices to take d a common multiple of the m_i. □

Corollary 11.3.8. *Under the hypotheses of Proposition 11.3.7,* $\operatorname{Proj}(\mathscr{S})$ *is Y-isomorphic to a homogeneous specturm* $\operatorname{Proj}(\mathscr{S}')$, *where $\mathscr{S}'$ is a graded $\mathscr{O}_Y$-algebra generated by $\mathscr{S}'_1$, where $\mathscr{S}'$ is an $\mathscr{O}_Y$-algebra of finite type.*

Proof. It suffices to take $\mathscr{S}' = \mathscr{S}^{(d)}$, where d is determined by the properties of Proposition 11.3.7, and apply Proposition 11.3.6(a). □

If $\mathscr{S}$ is a quasi-coherent graded $\mathscr{O}_Y$-algebras, we have seen in Proposition 10.4.28 that its nilradical is a quasi-coherent $\mathscr{O}_Y$-module. We say that $\mathscr{N}_+ = \mathscr{N} \cap \mathscr{S}_+$ is the nilradical of $\mathscr{S}_+$, which is a quasi-coherent graded $\mathscr{S}_0$-module, since this is the case if Y is affine. For any $y \in Y$, $(\mathscr{N}_+)_y$ is then the nilradical of $(\mathscr{S}_+)_y = (\mathscr{S}_y)_+$. Similar to the local case, we say the graded $\mathscr{O}_Y$-algebra $\mathscr{S}$ is **essentially reduced** if $\mathscr{N}_+ = 0$, which means $\mathscr{S}_y$ is an essentially reduced graded $\mathscr{O}_{Y,y}$-algebra for any $y \in Y$. It is clear that for any quasi-coherent graded $\mathscr{O}_Y$-algebra $\mathscr{S}$, $\mathscr{S}/\mathscr{N}_+$ is essentially reduced. Finally, we say $\mathscr{S}$ is **integral** if $\mathscr{S}_y$ is an integral ring for each $y \in Y$ and if $(\mathscr{S}_+)_y = (\mathscr{S}_y))_+ \neq 0$ for all $y \in Y$.

Proposition 11.3.9. *Let $\mathscr{S}$ be a graded $\mathscr{O}_Y$-algebra. If $X = \mathrm{Proj}(\mathscr{S})$, the Y-scheme X_{red} is canonically isomorphic to $\mathrm{Proj}(\mathscr{S}/\mathscr{N}_+)$. In particular, if $\mathscr{S}$ is essentially reduced, then X is reduced.*

Proof. The fact that $X' = \mathrm{Proj}(\mathscr{S}/\mathscr{N}_+)$ is reduced follows from Proposition 11.2.16, since the question is local. Moreover, for any affine open $U \subseteq Y$, $p'^{-1}(U)$ is equal to $(p^{-1}(U))_{\mathrm{red}}$ (where p and p' are the structural morphisms $X \to Y$, $X' \to Y$, respectively); we also verify t hat the canonical U-morphisms $p'^{-1}(U) \to p^{-1}(U)$ is compatible with restrictions and define therefore a closed immersion $X' \to X$, which is a homeomorphism on underlying spaces. Our assertion then follows from Corollary 10.4.29. □

Proposition 11.3.10. *Let Y be an integral scheme and $\mathscr{S}$ be a quasi-coherent graded $\mathscr{O}_Y$-algebra such that $\mathscr{S}_0 = \mathscr{O}_Y$.*

(a) *If $\mathscr{S}$ is integral then $X = \mathrm{Proj}(\mathscr{S})$ is integral and the structural morphism $p : X \to Y$ is dominant.*

(b) *Suppose moreover that $\mathscr{S}$ is essentially reduced. Then, conversely, if X is integral and p is dominant, then $\mathscr{S}$ is integral.*

Proof. We first assume that $\mathscr{S}$ is integral. Then if (U_α) is a basis of Y formed by affine opens, it suffices to prove for Y being replaced by U_α and $\mathscr{S}$ by $\mathscr{S}|_{U_\alpha}$: in fact, is this is true, the underlying space $p^{-1}(U_\alpha)$ is an open irreducible subset of X such that $p^{-1}(U_\alpha) \cap p^{-1}(U_\beta) \neq \varnothing$ for any couple of indices α, β (since $U_\alpha \cap U_\beta$ contains an U_γ and $\mathscr{S}$ is integral), so X is irreducible by **??**; it is clear that X is reduced since $\mathscr{S}$ is reduced, so X is integral. It is clear that $p(X)$ is dense in Y since this holds for each U_α.

Suppose then that $Y = \mathrm{Spec}(A)$ where A is integral (Proposition 10.4.30) and $\mathscr{S} = \widetilde{S}$, where S is a graded A-algebra; the hypotheses on $\mathscr{S}$ is that for any $y \in Y$, $\widetilde{S}_y = S_y$ is an integral graded ring such that $(S_y)_+ \neq 0$. It then suffices to prove that S is an integral ring, since then $S_+ \neq 0$ and we can apply Proposition 11.2.16. Now, let f, g be two nonzero elements of S and suppose that $fg = 0$; for any $y \in Y$ we have $(f/1)(g/1) =$ in S_y, so $f/1 = 0$ or $g/1 = 0$ by hypothesis. Suppose for example that $f/1 = 0$ in S_y, so there exists $a \in A$ such that $a \notin \mathfrak{p}_y$ and $af = 0$. We then see that for each $z \in Y$, $(a/1)(f/1) = 0$ in the integral ring S_z, and as $a/1 \neq 0$ (since A is integral), $f/1 = 0$, which implies $f = 0$.

Now consider the hypothesis in (b) and assume that X is integral and p is dominant. Since the question is local over Y, we can assume that $Y = \mathrm{Spec}(A)$, where A is an integral ring, and $\mathscr{S} = \widetilde{S}$. By hypothesis for any $y \in Y$, $(S_y)_+$ is reduced, and so is $(S_0)_y = A_y$ by hypothesis, so S_y is a reduced ring and we conclude that S is reduced. The hypothesis that X is integral implies that S is essentially integral (Proposition 11.2.16). The proposition then boils down to see that the annihilator $\mathfrak{I}$ of S_+ over $A = S_0$ is reduced to zero. In the contrary case, we would have $(S_h)_+ = 0$ for an $h \neq 0$ in $\mathfrak{I}$, which implies $p^{-1}(D(h)) = \varnothing$ by Proposition 11.3.1, and $p(X)$ is then not dense in Y, contradicting the hypothesis (since $D(h) \neq \varnothing$, h is not nilpotent). We then see that the ring S is integral, which conclude our assertion. □

11.3.2 Sheaves associated with a graded $\mathscr{S}$-module

Let Y be a scheme, $\mathscr{S}$ be a quasi-coherent $\mathscr{O}_Y$-algebra, and $\mathscr{M}$ be a quasi-coherent graded $\mathscr{S}$-module over the ringed space $(Y, \mathscr{S})$. With the notations of Proposition 11.3.1, we denote by $\widetilde{\mathscr{M}}_U$ the quasi-coherent $\mathscr{O}_{X_U}$-module $\widetilde{\Gamma(U, \mathscr{M})}$. For $U' \subseteq U$, $\Gamma(U', \mathscr{M})$ is canonically identified with $\Gamma(U, \mathscr{M}) \otimes_A A'$ by Proposition 10.1.14, so $\widetilde{M}_{U'} = \rho^*_{U',U}(\widetilde{\mathscr{M}}_U)$ by Proposition 11.2.50.

Proposition 11.3.11. *There exists over $\mathrm{Proj}(\mathscr{S}) = X$ a unique quasi-coherent $\mathscr{O}_X$-module $\widetilde{\mathscr{M}}$ such that, for any affine open U of Y, we have $\eta_U^*(\widetilde{\mathscr{M}}_U) = \widetilde{\mathscr{M}}|_{p^{-1}(U)}$, where $p : X \to Y$ is the structural morphism and η_U is the isomorphism $p^{-1}(U) \cong \mathrm{Proj}(\Gamma(U, \mathscr{S}))$. We say that $\widetilde{\mathscr{M}}$ is the $\mathscr{O}_X$-module associated with $\mathscr{M}$.*

Proof. As $\rho_{U',U}$ is identified with the injection morphism $p^{-1}(U') \to p^{-1}(U)$, the proposition follows from the relation $\widetilde{M}_{U'} = \rho^*_{U',U}(\widetilde{\mathscr{M}}_U)$ and glueing the $\widetilde{\mathscr{M}}_U$. □

Proposition 11.3.12. *Let $\mathscr{M}$ be a quasi-coherent $\mathscr{S}$-module and $f \in \Gamma(Y, \mathscr{S}_d)$ with $d > 0$. If ξ_f is the canonical isomorphism from X_f to the Y-scheme $Z_f = \operatorname{Spec}(\mathscr{S}^{(d)}/(f-1)\mathscr{S}^{(d)})$ in Proposition 11.3.2, then $(\xi_f)_*(\widetilde{\mathscr{M}}|_{X_f})$ is the $\mathscr{O}_{Z_f}$-module $(\mathscr{M}^{(d)}/(f-1)\mathscr{M}^{(d)})^{\sim}$.*

Proof. The question is local over Y we we are reduced to Proposition 11.2.3, and its compatibility with restrictions. □

Proposition 11.3.13. *The $\mathscr{O}_X$-module $\widetilde{\mathscr{M}}$ is an additive exact covariant functor from the category of quasi-coherent graded $\mathscr{S}$-modules to the category of quasi-coherent $\mathscr{O}_X$-modules, which commutes with direct sums and inductive limits.*

Proof. This follows from Corollary 10.1.6 and Proposition 11.2.34, since the question is local on Y. □

In particular, if $\mathscr{N}$ is a quasi-coherent graded sub-$\mathscr{S}$-module of $\mathscr{M}$, $\widetilde{\mathscr{N}}$ is cannically identified with a quasi-coherent sub-$\mathscr{O}_X$-module of $\widetilde{\mathscr{M}}$; if we take $\mathscr{M} = \mathscr{S}$, then for any quasi-coherent ideal $\mathscr{I}$ of $\mathscr{S}$, $\widetilde{\mathscr{I}}$ is a quasi-coherent ideal of $\mathscr{O}_X$.

If $\mathscr{M}$ is a quasi-coherent graded $\mathscr{S}$-module and $\mathscr{I}$ is a quasi-coherent ideal of $\mathscr{O}_Y$, then $\mathscr{I}\mathscr{M}$ is a quasi-coherent graded sub-$\mathscr{S}$-module of $\mathscr{M}$ and we have $\widetilde{\mathscr{I}\mathscr{M}} = \mathscr{I} \cdot \widetilde{\mathscr{M}}$: it suffices to verify this formula if $Y = \operatorname{Spec}(A)$, $\mathscr{S} = \widetilde{S}$, where S is a graded A-algebra, $\mathscr{M} = \widetilde{M}$ where M is a graded S-module, and $\mathscr{I} = \widetilde{\mathfrak{a}}$, where $\mathfrak{a}$ is an ideal of A. For any homogeneous element f of S_+, the restriction to $D_+(f) = \operatorname{Spec}(S_{(f)})$ of $\widetilde{\mathscr{I}\mathscr{M}}$ is the assocaited sheaf of $(\mathfrak{a}M)_{(f)} = \mathfrak{a} \cdot M_{(f)}$, and the identification is compatible with restrictions.

Proposition 11.3.14. *Let $f \in \Gamma(Y, \mathscr{S}_d)$ with $d > 0$. Over the open subset X_f, the $(\mathscr{O}_X|_{X_f})$-module $\widetilde{\mathscr{S}(nd)}|_{X_f}$ is canonically isomorphic to $\mathscr{O}_X|_{X_f}$ for any $n \in \mathbb{Z}$. In particular, if the $\mathscr{O}_Y$-algebra $\mathscr{S}$ is generated by $\mathscr{S}_1$, the $\mathscr{O}_X$-module $\widetilde{\mathscr{S}(n)}$ is invertible for all $n \in \mathbb{Z}$.*

Proof. For any affine open U of Y, by Proposition 11.2.24 we have a canonical isomorphism $\widetilde{\mathscr{S}(nd)}|_{X_f \cap p^{-1}(U)} \cong \mathscr{O}_X|_{X_f \cap p^{-1}(U)}$, in view of Proposition 11.3.2 (where $p : X \to Y$ is the structural morphism). It is immediate that this isomorphism is compatible with restrictions, whence the first assertion. For the second one, if $\mathscr{S}$ is generated by $\mathscr{S}_1$, there exists an affine open cover (U_α) of Y such that $\Gamma(U_\alpha, \mathscr{S})$ is generated by $\Gamma(U_\alpha, \mathscr{S}_1) = \Gamma(U_\alpha, \mathscr{S})_1$, and we can then use Corollary 11.2.26. □

Again, for any integer $n \in \mathbb{Z}$ and any $\mathscr{O}_X$-module $\mathscr{F}$, we set

$$\mathscr{O}_X(n) = \widetilde{\mathscr{S}(n)}, \quad \mathscr{F}(n) = \mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{O}_X(n).$$

It follows from this definition that, for any open subset U of Y,

$$\widetilde{\mathscr{S}|_U(n)} = \mathscr{O}_X|_{p^{-1}(U)},$$

where $p : X \to Y$ is the structural morphism.

Proposition 11.3.15. *Let $\mathscr{M}$ and $\mathscr{N}$ be quasi-coherent graded $\mathscr{S}$-modules. Then there exists cannical homomorphisms*

$$\lambda : \widetilde{M} \otimes_{\mathscr{O}_X} \widetilde{\mathscr{N}} \to (\mathscr{M} \otimes_{\mathscr{S}} \mathscr{N})^{\sim}$$
$$\mu : (\mathcal{H}om_{\mathscr{S}}(\mathscr{M}, \mathscr{N}))^{\sim} \to \mathcal{H}om_{\mathscr{O}_X}(\widetilde{\mathscr{M}}, \widetilde{\mathscr{N}}).$$

If $\mathscr{S}$ is generated by $\mathscr{S}_1$, then λ is an isomorphism; if moreover $\mathscr{M}$ is of finite presentation, μ is an isomorphism.

Proof. The isomorphisms λ and μ are defined in the arguments before Proposition 11.2.28 if Y is affine, and this definition is local and then glue together to define global morphisms, in view of the diagrams (11.2.13) and (11.2.14). $\square$

Corollary 11.3.16. *If $\mathscr{S}$ is generated by $\mathscr{S}_1$, for any integers $m, n \in \mathbb{Z}$, we have*

$$\mathscr{O}_X(m) \otimes_{\mathscr{O}_X} \mathscr{O}_X(n) = \mathscr{O}_X(m+n), \quad \mathscr{O}_X(n) = \mathscr{O}_X^{\otimes 1}.$$

Corollary 11.3.17. *If $\mathscr{S}$ is generated by $\mathscr{S}_1$, for any quasi-coherent graded $\mathscr{S}$-module $\mathscr{M}$ and $n \in \mathbb{Z}$, we have*

$$\widetilde{\mathscr{M}(n)} = \widetilde{\mathscr{M}}(n).$$

Remark 11.3.18. If $\mathscr{S} = \mathscr{A}[T]$ where $\mathscr{A}$ is a quasi-coherent $\mathscr{O}_Y$-algebra, we verify immediately that the invertible $\mathscr{O}_X$-module $\mathscr{O}_X(n)$ is canonically isomorphic to $\mathscr{O}_X$. Moreover, let $\mathscr{N}$ be a quasi-coherent $\mathscr{A}$-module, and put $\mathscr{M} = \mathscr{N} \otimes_{\mathscr{A}} \mathscr{A}[T]$. It follows from Proposition 11.3.12 and Corollary 11.3.5 that under the canonical isomorphism of $X = \mathrm{Proj}(\mathscr{A}[T])$ and $X' = \mathrm{Spec}(\mathscr{A})$, the $\mathscr{O}_X$-module $\widetilde{\mathscr{M}}$ is identified with the $\mathscr{O}_X$-module $\widetilde{\mathscr{N}}$.

Remark 11.3.19. Let $\mathscr{S}$ be a quasi-coherent graded $\mathscr{O}_Y$-algebra and define $\mathscr{S}'$ to be the $\mathscr{O}_Y$-algebra such that $\mathscr{S}' = \mathscr{O}_Y$ and $\mathscr{S}'_n = \mathscr{S}_n$ for $n > 0$. Then the canonical isomorphism of $X = \mathrm{Proj}(\mathscr{S})$ and $\mathrm{Proj}(\mathscr{S}')$ identifies $\mathscr{O}_X(n)$ and $\mathscr{O}_{X'}(n)$ for any $n \in \mathbb{Z}$: this follows from the corresponding result in the affine case (Example 11.2.31) and the fact that this identification is compatible with restrictions. Similarly, let $X^{(d)} = \mathrm{Proj}(\mathscr{S}^{(d)})$; the canonical isomorphism of X and $X^{(d)}$ identifies $\mathscr{O}_X(nd)$ with $\mathscr{O}_{X^{(d)}}(n)$ for any $n \in \mathbb{Z}$.

Proposition 11.3.20. *Let $\mathscr{L}$ be an invertible $\mathscr{O}_Y$-module and ψ be the structural morphism from $X_{(\mathscr{L})} = \mathrm{Proj}(\mathscr{S}_{(\mathscr{L})})$ to $X = \mathrm{Proj}(\mathscr{S})$. Then for any integer $n \in \mathbb{Z}$, $\psi_*(\mathscr{O}_{X_{(\mathscr{L})}}(n))$ is canonically isomorphic to $\mathscr{O}_X(n) \otimes_Y \mathscr{L}^{\otimes n}$.*

Proof. Suppose first that Y is affine with ring A and $\mathscr{L} = \widetilde{L}$, where L is a free A-module of rank 1. With the notations of Proposition 11.3.6(c), we define for each $f \in S_d$ an isomorphism from $S(n)_{(f)} \otimes_A L^{\otimes n}$ to $S_{(L)}(n)_{(f \otimes c^d)}$ which sends $(x/f^k) \otimes c^n$, where $x \in S_{kd+n}$, to the element $(x \otimes c^{n+kd})/(f \otimes c^d)^k$. It is immedaite that this isomorphism is independent of the generator c of L, and is compatible with restrictions $D_+(f) \to D_+(fg)$. The general case then follows from glueing these isomorphisms. $\square$

11.3.3 Graded $\mathscr{S}$-module assocaited with a sheaf

For simplicity, in the following discussion, we always assume that the quasi-coherent $\mathscr{O}_Y$-algebra $\mathscr{S}$ is generated by $\mathscr{S}_1$, which by Proposition 11.3.6 is not at all essential if we impose the finiteness conditions of Proposition 11.3.7 on Y. Let $p: X \to Y$ be the structural morphism where $X = \mathrm{Proj}(\mathscr{S})$, which is separated by Proposition 11.2.14. For any $\mathscr{O}_X$-module $\mathscr{F}$, we define

$$\Gamma_*(\mathscr{F}) = \bigoplus_{n \in \mathbb{Z}} p_*(\mathscr{F}(n))$$

and in particular

$$\Gamma_*(\mathscr{O}_X) = \bigoplus_{n \in \mathbb{Z}} p_*(\mathscr{O}_X(n)).$$

We have seen in (**??**) that there exists a canonical homomorphism

$$p_*(\mathscr{F}) \otimes_{\mathscr{O}_Y} p_*(\mathscr{G}) \to p_*(\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G})$$

for any $\mathscr{O}_X$-modules $\mathscr{F}$ and $\mathscr{G}$, so we deduce from Corollary 11.3.16 that $\Gamma_*(\mathscr{O}_X)$ is endowed with a graded $\mathscr{O}_Y$-algebra structure and $\Gamma_*(\mathscr{F})$ is a graded module over $\Gamma_*(\mathscr{O}_X)$.

In view of Proposition 11.3.14 and the left-exactness of the functor f_*, $\Gamma_*(\mathscr{F})$ is an additive left-exact covariant functor from the category of $\mathscr{O}_X$-modules to the category of graded $\mathscr{O}_Y$-modules. In particular, if $\mathscr{I}$ is a sheaf of ideals of $\mathscr{O}_X$, $\Gamma_*(\mathscr{I})$ is identified with a sheaf of graded ideals of $\Gamma_*(\mathscr{O}_X)$.

Now let $\mathscr{M}$ be a quasi-coherent graded $\mathscr{S}$-module. For any affine open U of Y, we have defined a homomorphism of abelian groups

$$\alpha_{0,U} : \Gamma(U, \mathscr{M}_0) \to \Gamma(p^{-1}(U), \widetilde{\mathscr{M}}).$$

It is immediate that these homomorphisms commutes with restrictions and define (which do not use the hypothesis that $\mathscr{S}$ is generated by $\mathscr{S}_1$) a homomorphism of sheaf of abelian groups

$$\alpha_0 : \mathscr{M}_0 \to \widetilde{\mathscr{M}}.$$

Apply this result to $\mathscr{M}_n = (\mathscr{M}(n))_0$ and use Corollary 11.3.17, we define a homomorphism of abelian groups

$$\alpha_n : \mathscr{M}_n \to p_*(\widetilde{\mathscr{M}}(n)) \tag{11.3.1}$$

for each $n \in \mathbb{Z}$, whence a functorial homomorphism of graded sheaves of abelian groups

$$\alpha : \mathscr{M} \to \Gamma_*(\widetilde{\mathscr{M}}) \tag{11.3.2}$$

(we also denote it by $\alpha_{\mathscr{M}}$). In the particular case $\mathscr{M} = \mathscr{S}$, we verify that $\alpha : \mathscr{S} \to \Gamma_*(\mathscr{O}_X)$ is a homomorphism of graded $\mathscr{O}_Y$-algebra and is a bi-homomorphism of graded modules, relative to this homomorphism of graded homomorphism of algebras.

We also remark that the homomorphism α_n corresponds to a canonical homomorphism of $\mathscr{O}_X$-modules

$$\alpha_n^\sharp : p^*(\mathscr{M}_n) \to \widetilde{\mathscr{M}}(n).$$

Moreover, it is easy to verify that this homomorphism is none other than the associated homomorphism (by Proposition 11.3.13) of the canonical homomorphism $\mathscr{M}_n \otimes_{\mathscr{O}_Y} \mathscr{S} \to \mathscr{M}(n)$ of $\mathscr{O}_Y$-modules, where the $\mathscr{O}_Y$-module $\mathscr{M}_n \otimes_{\mathscr{O}_Y} \mathscr{S}$ is given the natrual graduation. To see this, we can in fact assume that $Y = \operatorname{Spec}(A)$ is affine, $\mathscr{M} = \widetilde{M}$ and $\mathscr{S} = \widetilde{S}$, where S is a graded A-algebra generated by S_1. Returning to the definition of α, we see that the restriction to $D_+(f)$ of the homomorphism $\alpha_n^\sharp$ corresponds to the homomorphism $M_n \otimes_A S_{(f)} \to M(n)_{(f)}$, where $x \otimes 1$ is maped to $x/1$.

Proposition 11.3.21. *For any section $f \in \Gamma(Y, \mathscr{S}_d)$ with $d > 0$, X_f is identified with the set of points of X on which $\alpha_d(f)$ is nonzero.*

Proof. The element $\alpha_d(f)$ is a section of $p_*(\mathscr{O}_X(d))$ over Y, and by definition is then a section of $\mathscr{O}_X(d)$ over X. The definition of X_f (Proposition 11.3.2) proves our claim in the affine case, in view of Proposition 11.2.33. □

We shall henceforth suppose, in addition to the hypothesis at the beginning of this paragraph, that for every quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, $p_*(\mathscr{F}(n))$ is quasi-coherent over Y, and therefore $\Gamma_*(\mathscr{F}) = \bigoplus_{n\in\mathbb{Z}} p_*(\mathscr{F}(n))$ is also a quasi-coherent $\mathscr{O}_Y$-module; this circumstance will always occur if X is of finite type on Y (Proposition 10.6.55). We then conclude that $\widetilde{\Gamma_*(\mathscr{F})}$ is defined and is a quasi-coherent $\mathscr{O}_X$-module. For any affine open subset U of Y, we have (Corollary 10.1.6, Proposition 11.3.13, and note that U is quasi-compact)

$$(\Gamma(U, \bigoplus_{n\in\mathbb{Z}} p_*(\mathscr{F}(n))))^\sim = \bigoplus_{n\in\mathbb{Z}} (\Gamma(U, p_*(\mathscr{F}(n))))^\sim = \bigoplus_{n\in\mathbb{Z}} (\Gamma(p^{-1}(U), \mathscr{F}(n)))^\sim$$

$$= \Big(\bigoplus_{n\in\mathbb{Z}}\Gamma(p^{-1}(U),\mathscr{F}(n))\Big)^{\sim} = (\Gamma_*(\mathscr{F}|_{p^{-1}(U)}))^{\sim}$$

and therefore a canonical homomorphism

$$\beta_U : (\Gamma(U,\bigoplus_{n\in\mathbb{Z}} p_*(\mathscr{F}(n))))^{\sim} \to \mathscr{F}|_{p^{-1}(U)}.$$

Moreover, the diagram (11.2.12) shows that these homomorphisms are compatible with restrictions on Y, so we deduce a canonical homomorphism

$$\beta : \widetilde{\Gamma_*(\mathscr{F})} \to \mathscr{F}$$

(also denoted by $\beta_{\mathscr{F}}$) for the quasi-coherent $\mathscr{O}_X$-modules.

Proposition 11.3.22. *Let $\mathscr{M}$ be a quasi-coherent $\mathscr{S}$-module and $\mathscr{F}$ be a quasi-coherent $\mathscr{O}_X$-module. Then the composition homomorphisms*

$$\widetilde{\mathscr{M}} \xrightarrow{\tilde{\alpha}} (\Gamma_*(\widetilde{\mathscr{M}}))^{\sim} \xrightarrow{\beta} \widetilde{\mathscr{M}} \tag{11.3.3}$$

$$\Gamma_*(\mathscr{F}) \xrightarrow{\alpha} \Gamma_*(\widetilde{\Gamma_*(\mathscr{F})}) \xrightarrow{\Gamma_*(\beta)} \Gamma_*(\mathscr{F}) \tag{11.3.4}$$

Proof. The question is local over Y, so we can apply Proposition 11.2.34. □

Again, the homomorphisms α and β are in general not isomorphisms, and further finiteness conditions must be imposed. We note also that the homomorphism β is not always defined, unlike the affine case. However, we shall see that if $\mathscr{S}$ is of finite type and generated by $\mathscr{S}_1$, the homomorphisms α and β are well defined and the corresponding results of the affine cases carry over without difficulties.

Proposition 11.3.23. *Let Y be a scheme and $\mathscr{S}$ be a quasi-coherent $\mathscr{O}_Y$-algebra generated by $\mathscr{S}_1$. Suppose that $\mathscr{S}_1$ is of finite type, then $X = \mathrm{Proj}(\mathscr{S})$ is of finite type over Y.*

Proof. Again we can assume that Y is affine with ring A, so $\mathscr{S} = \widetilde{S}$ where S is a graded A-algebra generated by S_1, and S_1 is a finitely generated A-module by hypothesis. Then S is an A-algebra of finite type, and the proposition follows from Proposition 11.2.35. □

Let $\mathscr{S}$ be a quasi-coherent graded $\mathscr{O}_Y$-algebra and $\mathscr{M}$ be a quasi-coherent $\mathscr{S}$-module. We say that $\mathscr{M}$ is **eventually null** if there exists an integer n such that $\mathscr{M}_k = 0$ for $k \geq n$, and is **eventually finite** if there exists an integer n such that the $\mathscr{S}$-module $\bigoplus_{k\geq n}\mathscr{M}_k$ is of finite type. If $\mathscr{M}$ is eventually null, it is clear that $\widetilde{\mathscr{M}} = 0$, as in the affine case.

Let $\mathscr{M}$ and $\mathscr{N}$ be quasi-coherent graded $\mathscr{S}$-modules. We say a homomorphism $u : \mathscr{M} \to \mathscr{N}$ of degree 0 is eventually injective (resp. eventually sujective, eventually bijective) if there exists an integer n such that $u_k : \mathscr{M}_k \to \mathscr{N}_k$ is injective (resp. surjective, bijective) for $k \geq n$. It is clear that in this case, $\tilde{u} : \widetilde{\mathscr{M}} \to \widetilde{\mathscr{N}}$ is injective (resp. surjective, bijective), since this can be checked locally over Y and we can apply Proposition 11.3.13. If u is eventually bijective, we also say that it is an eventual isomorphism.

Proposition 11.3.24. *Let Y be a scheme and $\mathscr{S}$ be a quasi-coherent $\mathscr{O}_Y$-algebra generated by $\mathscr{S}_1$ and such that $\mathscr{S}_1$ is of finite type. Let $\mathscr{M}$ be a quasi-coherent graded $\mathscr{S}$-module.*

(a) *If $\mathscr{M}$ is eventually finite, $\widetilde{\mathscr{M}}$ is of finite type.*

(b) *If $\mathscr{M}$ is eventually finite, for $\widetilde{\mathscr{M}} = 0$, it is necessary and sufficient that $\mathscr{M}$ is eventually null.*

Proof. The question is local over Y, so we can assume that $Y = \mathrm{Spec}(A)$, and the proposition then follows from Proposition 11.2.36. □

Theorem 11.3.25. *Let Y be a scheme and $\mathscr{S}$ be a quasi-coherent $\mathscr{O}_Y$-algebra generated by $\mathscr{S}_1$ and such that $\mathscr{S}_1$ is of finite type. Let $X = \mathrm{Proj}(\mathscr{S})$, then for any quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, the canonical homomorphism $\beta : \widetilde{\Gamma(\mathscr{F})} \to \mathscr{F}$ is an isomorphism.*

Proof. We first note that the homomorphism β is defined because of Proposition 11.3.23. To see that β is an isomorphism, we can assume $Y = \mathrm{Spec}(A)$ is affine, and then apply Theorem 11.2.39. □

Corollary 11.3.26. *Under the hypotheses of Theorem 11.3.25, any quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$ is isomorphic to an $\mathscr{O}_X$-module of the form $\widetilde{\mathscr{M}}$, where $\mathscr{M}$ is a quasi-coherent $\mathscr{S}$-module. If moreover $\mathscr{F}$ is of finite type, and if we suppose that Y is a quasi-compact scheme, then we can choose $\mathscr{M}$ to be of finite type.*

Proof. The first assertion follows from Theorem 11.3.25 by take $\mathscr{M} = \Gamma_*(\mathscr{F})$. For the second one, it suffices to prove that $\mathscr{M}$ is the inductive limit of graded sub-$\mathscr{S}$-modules of finite type $\mathscr{N}_\lambda$: in fact, it then follows that $\widetilde{\mathscr{M}}$ is the inductive limit of the $\widetilde{\mathscr{N}_\lambda}$ (Proposition 11.3.13), hene $\mathscr{F}$ is the inductive limit of the $\beta(\mathscr{N}_\lambda)$. As X is quasi-compact (Proposition 11.3.23) and $\mathscr{F}$ is of finite type, $\mathscr{F}$ then necessarily equal to one of the $\beta(\widetilde{\mathscr{N}_\lambda})$ (**??**).

To define the $\mathscr{N}_\lambda$, it suffices to consider for each $n \in \mathbb{Z}$ the quasi-coherent $\mathscr{O}_Y$-module $\mathscr{M}_n$, which is the inductive limit of its sub-$\mathscr{O}_Y$-modules $\mathscr{M}_n^{(\mu_n)}$ of finite type (by Corollary 10.6.65). It is immediate that $\mathscr{P}_{\mu_n} = \mathscr{S} \cdot \mathscr{M}_n^{(\mu_n)}$ is a graded $\mathscr{S}_n$-module of finite type, and $\mathscr{M}$ is then the inductive limit of finite direct sums of these $\mathscr{S}$-modules. □

Corollary 11.3.27. *Suppose the hypotheses of Theorem 11.3.25 and that Y is quasi-compact. Let $\mathscr{F}$ be a quasi-coherent $\mathscr{O}_X$-module of finite type. There exists an integer n_0 such that for $n \geq n_0$, the canonical homomorphism $\sigma : p^*(p_*(\mathscr{F}(n))) \to \mathscr{F}(n)$ is surjective.*

Proof. For any $y \in Y$, let U be an affine open neighborhood of y in Y. There then exists an inter $n_0(U)$ such that, for $n \geq n_0(U)$, $\mathscr{F}(n)|_{p^{-1}(U)}$ is generated by finitely many sections over $p^{-1}(U)$ (Corollary 11.2.41); but these are canonical images of sections of $p^*(p_*(\mathscr{F}(n)))$ over $p^{-1}(U)$, so $\mathscr{F}(n)|_{p^{-1}(U)}$ is equal to the canonical image of $p^*(p_*(\mathscr{F}(n)))|_{p^{-1}(U)}$. Finally, as Y is quasi-compact, there is a finite affine open cover (U_i) of Y, and we can choose n_0 to be the largest of the $n_0(U_i)$. □

Remark 11.3.28. If $p : X \to Y$ is a morphism of ringed spaces and $\mathscr{F}$ is an $\mathscr{O}_X$-module, the fact that the canonical homomorphism $\sigma : p^*(p_*(\mathscr{F})) \to \mathscr{F}$ is surjective is explained as follows: for any $x \in X$ and any sections of $\mathscr{F}$ over an open neighborhood V of x, there exists an open neighborhood U of $p(x)$ in Y, finitely many sections $(t_i)_{1\leq i\leq m}$ of $\mathscr{F}$ over $p^{-1}(U)$, a neighborhood $W \subseteq V \cap p^{-1}(U)$ of x and sections $(a_i)_{1\leq i\leq m}$ of $\mathscr{O}_X$ over W such that

$$s|_W = \sum_i a_i \cdot (t_i|_W).$$

If Y is an affine scheme and $p_*(\mathscr{F})$ is *quasi-coherent*, this condition is equivalent to the fact that $\mathscr{F}$ is generated by its sections over X: in fact, if $Y = \mathrm{Spec}(A)$, we can suppose that $U = D(f)$ with $f \in A$. Since $p_*(\mathscr{F})$ is quasi-coherent, by Theorem 10.1.21 there exists an integer $n > 0$ and sections s_i of $\mathscr{F}$ over X such that $g^n t_i$ is the restriction of s_i (where $g = \rho(f)$, and $\rho : A \to \Gamma(X, \mathscr{O}_X)$ is the homomorphism corresponding to p by Proposition 10.2.4) to $p^{-1}(U)$. As g is invertible over $p^{-1}(U)$, we then have

$$s|_W = \sum_i b_i \cdot (s_i|_W)$$

where $b_i = a_i \cdot (g|_W)^{-n}$, whence our assertion. Therefore, if Y is affine, Corollary 11.3.27 then recovers Corollary 11.2.41, in view of **??**.

We finally conclude that if Y is a scheme, then the following three conditions are equivalent for a quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$ such that $p_*(\mathscr{F})$ is quasi-coherent:

(i) The canonical homomorphism $\sigma : p^*(p_*(\mathscr{F})) \to \mathscr{F}$ is surjective.

(ii) There is a quasi-coherent $\mathscr{O}_Y$-module $\mathscr{G}$ and a surjective homomorphism $p^*(\mathscr{G}) \to \mathscr{F}$.

(iii) For any affine open U of Y, $\mathscr{F}|_{p^{-1}(U)}$ is generated by its sections over $p^{-1}(U)$.

We have already established the equivalence of (i) and (iii), and (i) clearly implies (ii). conversely, any homomorphism $u : p^*(\mathscr{G}) \to \mathscr{F}$ factors into $p^*(\mathscr{G}) \to p^*(p_*(\mathscr{F})) \xrightarrow{\sigma} \mathscr{F}$ by (**??**), so if u is surjective, so is the canonical homomorphism $\sigma : p^*(p_*(\mathscr{F})) \to \mathscr{F}$.

Corollary 11.3.29. *Suppose the hypotheses of Theorem 11.3.25 and that Y is quasi-compact. Let $\mathscr{F}$ be a quasi-coherent $\mathscr{O}_X$-module of finite type. There then exists an integer n_0 such that for $n \geq n_0$, $\mathscr{F}$ is isomorphic to a quotient of an $\mathscr{O}_X$-module of the form $(p^*(\mathscr{G}))(-n)$, where $\mathscr{G}$ is a quasi-coherent $\mathscr{O}_Y$-module of finite type (dependent on n).*

Proof. As the structural morphism $X \to Y$ is separated and of finite type, $p_*(\mathscr{F}(n))$ is quasi-coherent by Proposition 10.6.55, and so is the inductive limits of its sub-$\mathscr{O}_Y$-modules of finite type, in view of Corollary 10.6.65. Since p^* commutes with inductive limits, we deduce from Corollary 11.3.27 and **??** that $\mathscr{F}(n)$ is the canonical image under $\sigma_{\mathscr{F}(n)}$ of an $\mathscr{O}_X$-module of the form $p^*(\mathscr{G})$, where $\mathscr{G}$ is a quasi-coherent sub-$\mathscr{O}_Y$-module of $p_*(\mathscr{F}(n))$ of finite type. The corollary then follows from Corollary 11.3.16 and Corollary 11.3.17. □

11.3.4 Functorial properties of $\mathrm{Proj}(\mathscr{S})$

Let Y be a scheme and $\mathscr{S}, \mathscr{S}'$ be two quasi-coherent graded $\mathscr{O}_Y$-algebras with positive degrees. Let $X = \mathrm{Proj}(\mathscr{S})$, $X' = \mathrm{Proj}(\mathscr{S}')$, and p, p' be the structural morphisms of X and X', respectively. Let $\varphi : \mathscr{S}' \to \mathscr{S}$ be an $\mathscr{O}_Y$-homomorphism of graded algebras. For any affine open U of Y, let $S_U = \Gamma(U, \mathscr{S})$, $S'_U = \Gamma(U, \mathscr{S}')$; the homomorphism φ defines a homomorphism $\varphi_U : S'_U \to S_U$ of graded A_U-algebras, where $A_U = \Gamma(U, \mathscr{O}_Y)$. It then corresponds to an open subset $G(\varphi_U)$ of $p^{-1}(U)$ and a morphism $\Phi_U : G(\varphi_U) \to p'^{-1}(U)$. Moreover, if $V \subseteq U$ is another affine open subset, the diagram

$$\begin{tikzcd} S'_U \arrow[r, "\varphi_U"] \arrow[d] & S_U \arrow[d] \\ S'_V \arrow[r, "\varphi_V"] & S_V \end{tikzcd} \tag{11.3.5}$$

is commutative, and we also verify, by the definition of $G(\varphi_U)$, that

$$G(\varphi_V) = G(\varphi_U) \cap p^{-1}(V)$$

and that Φ_V is the restriction of Φ_U to $G(\varphi_V)$. We thus define an open subset $G(\varphi)$ of X such that $G(\varphi) \cap p^{-1}(U) = G(\varphi_U)$ for any affine open $U \subseteq Y$, and an affine Y-morphism $\Phi : G(\varphi) \to X'$, which is called the morphism associated with φ and denoted by $\mathrm{Proj}(\varphi)$. If for any $y \in Y$, there exists an affine open neighborhood U of Y such that $\Gamma(U, \mathscr{O}_Y)$-module $\Gamma(U, \mathscr{S}_+)$ is generated by $\varphi(\Gamma(U, \mathscr{S}'_+))$, we then have $G(\varphi_U) = p^{-1}(U)$, and thus $G(\varphi) = X$.

Proposition 11.3.30. *Let $\mathscr{M}$ (resp. $\mathscr{M}'$) be a quasi-coherent graded $\mathscr{S}$-module (resp. $\mathscr{S}'$-module). Then there exist a canonical isomorphism $\widetilde{\varphi^*(\mathscr{M})} \xrightarrow{\sim} \Phi_*(\widetilde{\mathscr{M}}|_{G(\varphi)})$ of $\mathscr{O}_{X'}$-modules and a canonical homomorphism $\nu : \Phi^*(\widetilde{\mathscr{M}}') \to \widetilde{\varphi_*(\mathscr{M}')}|_{G(\varphi)}$. If $\mathscr{S}'$ is generated by $\mathscr{S}'_1$, ν is an isomorphism.*

Proof. The homomorphisms considered are in fact already defined locally over Y (see Proposition 11.2.47 and Proposition 11.2.48), and the general case then follows from their compatibility with restrictions, and diagram (11.3.5). ☐

In particular, for any $n \in \mathbb{Z}$, we have a canonical homomorphism $\Phi^*(\mathscr{O}_{X'}(n)) \to \mathscr{O}_X(n)|_{G(\varphi)}$, and this is a homomorphism if $\mathscr{S}'$ is generated by $\mathscr{S}'_1$.

Proposition 11.3.31. *Let Y, Y' be schemes, $\psi : Y' \to Y$ be a morphism, $\mathscr{S}$ be a quasi-coherent $\mathscr{O}_Y$-algebra, and put $\mathscr{S}' = \psi^*(\mathscr{S})$. Then the Y'-scheme $X' = \mathrm{Proj}(\mathscr{S}')$ is canonically identified with $\mathrm{Proj}(\mathscr{S}) \times_Y Y'$. Moreover, if $\mathscr{M}$ is a quasi-coherent graded $\mathscr{S}$-module, the $\mathscr{O}_{X'}$-module $\widetilde{\psi^*(\mathscr{M})}$ is identified with $\widetilde{\mathscr{M}} \otimes_Y \mathscr{O}_{Y'}$.*

Proof. We first note that $\psi^*(\mathscr{S})$ and $\psi^*(\mathscr{M})$ are quasi-coherent $\mathscr{O}_{Y'}$-modules. Let U be an affine open of Y, $U' \subseteq \psi^{-1}(U)$ an affine open of Y', and A, A' the ring of U, U', respectively. We then have $\mathscr{S}|_U = \widetilde{S}$ where S is a graded A-algebra, and $\mathscr{S}'|_{U'}$ is identified with $\widetilde{S \otimes_A A'}$ by Proposition 10.1.14. The first assertion then follows from Proposition 11.2.50 and Corollary 10.3.2, since we can easily verify that the projection $\mathrm{Proj}(\mathscr{S}'|_{U'}) \to \mathrm{Proj}(\mathscr{S}|_U)$ defined by this identification is compatible with restrictions over U and U' and therefore define a morphism $\mathrm{Proj}(\mathscr{S}') \to \mathrm{Proj}(\mathscr{S})$. Now let $p : \mathrm{Proj}(\mathscr{S}) \to Y$ and $p' : \mathrm{Proj}(\mathscr{S}') \to Y'$ be the structural morphisms; $p'^{-1}(U')$ is identified with $p^{-1}(U) \times_U U'$, and the two sheaves $\widetilde{\psi^*(\mathscr{M})}|_{p'^{-1}(U')}$ and $\widetilde{\mathscr{M}} \otimes_Y \mathscr{O}_{Y'}|_{p'^{-1}(U')}$ are then canonically identified to $\widetilde{M \otimes_A A'}$, where $M = \Gamma(U, \mathscr{M})$, in view of Proposition 11.2.50 and Proposition 10.1.14; whence the second assertion, since these identifications are compatible with restrictions. ☐

Corollary 11.3.32. *With the notations of Proposition 11.3.31, $\mathscr{O}_{X'}(n)$ is canonically identified with $\mathscr{O}_X(n) \otimes_Y \mathscr{O}_{Y'}$ for any $n \in \mathbb{Z}$.*

Proof. With the notations of Proposition 11.3.31, it is clear that $\psi^*(\mathscr{S}(n)) = \mathscr{S}'(n)$ for any $n \in \mathbb{Z}$, whence the corollary. ☐

Retain the notations in Proposition 11.3.31, denote by $\Psi : X' \to X$ the canonical projection, and put $\mathscr{M}' = \psi^*(\mathscr{M})$. We suppose that $\mathscr{S}$ is generated by $\mathscr{S}_1$ and that X is of finite type over Y (for example if $\mathscr{S}_1$ is of finite type, cf. Proposition 11.3.23). Then $\mathscr{S}'$ is generated by $\mathscr{S}'_1$ (as can be checked locally on affine opens of Y) and X' is of finite type over Y by Proposition 10.6.35. Let $\mathscr{F}$ be an $\mathscr{O}_X$-module and set $\mathscr{F}' = \Psi^*(\mathscr{F})$; it then follows from Corollary 11.3.32 that we have $\mathscr{F}'(n) = \Psi^*(\mathscr{F})$ for each $n \in \mathbb{Z}$. We define a canonical Ψ-homomorphism $\theta_n : p_*(\mathscr{F}(n)) \to p'_*(\mathscr{F}'(n))$ as follows: from the commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{\Psi} & X \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ Y' & \xrightarrow{\psi} & Y \end{array}$$

we see that it suffices to define a homomorphism

$$p_*(\mathscr{F}(n)) \to \psi_*(q'_*(\Psi^*(\mathscr{F}(n)))) = q_*(\Psi_*(\Psi^*(\mathscr{F}(n)))),$$

and we can take $\theta_n = p_*(\rho_n)$, where ρ_n is the canonical homomorphism $\rho_n : \mathscr{F}(n) \to \Psi_*(\Psi^*(\mathscr{F}(n)))$. It is immediate that for any affine open U of Y and any affine open U' of Y such that $U' \subseteq \psi^{-1}(U)$, the homomorphism θ_n thus defined gives a canonical homomorphism $\Gamma(p^{-1}(U), \mathscr{F}(n)) \to$

$\Gamma(p'^{-1}(U'), \mathscr{F}'(n))$, and the commutative diagram (11.2.12) shows that if $\mathscr{F}$ is quasi-coherent, the diagram

$$\begin{tikzcd}
\widetilde{\Gamma_*(\mathscr{F})} \arrow[r, "\tilde{\theta}"] \arrow[d, "\beta_{\mathscr{F}}"'] & \widetilde{\Gamma_*(\mathscr{F}')} \arrow[d, "\beta_{\mathscr{F}'}"] \\
\mathscr{F} \arrow[r, "\rho"] & \mathscr{F}'
\end{tikzcd}$$

is commutative (where the send row is the canonical Ψ-morphism $\mathscr{F} \to \Phi^*(\mathscr{F})$).

Similarly, the commutative diagram (11.2.11) shows that the diagram

$$\begin{tikzcd}
\mathscr{M} \arrow[r, "\rho"] \arrow[d, "\alpha_{\mathscr{M}}"] & \mathscr{M}' \arrow[d, "\alpha_{\mathscr{M}'}"] \\
\Gamma_*(\widetilde{\mathscr{M}}) \arrow[r, "\theta"] & \Gamma_*(\widetilde{\mathscr{M}'})
\end{tikzcd}$$

is commutative (where the first row is the canonical ψ-morphism $\mathscr{M} \to \psi^*(\mathscr{M})$).

Consider now a morphism $\psi : Y' \to Y$ of schemes, a quasi-coherent graded $\mathscr{O}_Y$-algebra (resp. $\mathscr{O}_{Y'}$-algebra) $\mathscr{S}$ (resp. $\mathscr{S}'$), and a ψ-morphism $u : \mathscr{S} \to \mathscr{S}'$ of graded algebras. This is equivalent to giving an $\mathscr{O}_{Y'}$-homomorphism of graded algebras $u^\sharp : \psi_*(\mathscr{S}) \to \mathscr{S}'$, and we deduce from $u^\sharp$ an Y'-morphism

$$w = \operatorname{Proj}(u^\sharp) : G(u^\sharp) \to \operatorname{Proj}(\psi^*(\mathscr{S})),$$

where $G(u^\sharp)$ is an open subset of $X' = \operatorname{Proj}(\mathscr{S}')$. On the other hand, $\operatorname{Proj}(\psi^*(\mathscr{S}))$ is canonically identified with $X \times_Y Y'$, where $X = \operatorname{Proj}(\mathscr{S})$ (Proposition 11.3.31). By composing the morphism $\operatorname{Proj}(u^\sharp)$ with the first projection $\pi : X \times_Y Y' \to X$, we then obtain a morphism $v = \operatorname{Proj}(u) : G(u^\sharp) \to X$ such that the diagram

$$\begin{tikzcd}
G(u^\sharp) \arrow[r, "v"] \arrow[d] & X \arrow[d] \\
Y' \arrow[r, "\psi"] & Y
\end{tikzcd} \tag{11.3.6}$$

is commutative. Moreover, for any quasi-coherent $\mathscr{O}_Y$-module $\mathscr{M}$, we have a canonical v-morphism

$$\nu : \widetilde{\mathscr{M}} \to (\psi^*(\mathscr{M}) \otimes_{\psi^*(\mathscr{S})} \mathscr{S}')^\sim|_{G(u^\sharp)} \tag{11.3.7}$$

such that $\nu^\sharp$ is the composition

$$v^*(\widetilde{\mathscr{M}}) = w^*(\pi^*(\widetilde{\mathscr{M}})) \xrightarrow{\sim} w^*(\widetilde{\psi^*(\mathscr{M})}) \xrightarrow{\nu} (\psi^*(\mathscr{M}) \otimes_{\psi^*(\mathscr{M})} \mathscr{S}')^\sim|_{G(u^\sharp)} \tag{11.3.8}$$

where the first arrow is the isomorphism in Proposition 11.3.31 and the second one is the homomorphism ν of Proposition 11.3.30. If $\mathscr{S}$ is generated by $\mathscr{S}_1$, then it follows from Proposition 11.3.30 that $\nu^\sharp$ is an isomorphism. As a particular case, for any $n \in \mathbb{Z}$ we have a canonical v-homomorphism

$$\nu : \mathscr{O}_X(n) \to \mathscr{O}_{X'}(n)|_{G(u^\sharp)}. \tag{11.3.9}$$

11.3.5 Closed subschemes of $\operatorname{Proj}(\mathscr{S})$

Let Y be a scheme and $\varphi : \mathscr{S} \to \mathscr{S}'$ be a homomorphism of quasi-coherent graded $\mathscr{O}_Y$-algebras of degree 0. We say that φ is **eventually injective** (resp. **eventually surjective**, **eventually bijective**) if there exists an integer n such that, for $k \geq n$, $\varphi_k : \mathscr{S}_k \to \mathscr{S}'_k$ is surjective (resp. injective, bijective). If this is the case, we can then reduce the study of the morphism $\Phi : \mathrm{Proj}(\mathscr{S}') \to \mathrm{Proj}(\mathscr{S})$ to the case where φ is surjective (resp. injective, bijective) (this follows from Proposition 11.3.6). If φ is eventually bijective, we also say that φ is an **eventual isomorphism**.

Proposition 11.3.33. *Let Y be a scheme, $\mathscr{S}$ be a quasi-coherent $\mathscr{O}_Y$-algebra, and $X = \mathrm{Proj}(\mathscr{S})$.*

(a) *If $\varphi : \mathscr{S} \to \mathscr{S}'$ is a eventually surjective homomorphism of graded $\mathscr{O}_Y$-algebra, the corresponding morphism $\Phi = \mathrm{Proj}(\varphi)$ is defined over $\mathrm{Proj}(\mathscr{S}')$ and is a closed immersion from $\mathrm{Proj}(\mathscr{S}')$ into X. If $\mathscr{I}$ is the kernel of φ, the closed subscheme of X associated with Φ is defined by the quasi-coherent ideal $\widetilde{\mathscr{I}}$ of $\mathscr{O}_X$.*

(b) *Suppose moreover that $\mathscr{S}_0 = \mathscr{O}_Y$ and $\mathscr{S}$ is generated by $\mathscr{S}_1$ where $\mathscr{S}_1$ is of finite type. Let X' be a closed subscheme of X defined by a quasi-coherent ideal $\mathscr{I}$ of $\mathscr{O}_X$. Let $\mathscr{J}$ be the inverse image of $\Gamma_*(\mathscr{I})$ under the canonical homomorphism $\alpha : \mathscr{S} \to \Gamma_*(\mathscr{O}_X)$ and put $\mathscr{S}' = \mathscr{S}/\mathscr{J}$. Then X' is the subscheme associated with the closed immersion $\mathrm{Proj}(\mathscr{S}') \to X$ corresponding to the canonical homomorphism $\mathscr{S} \to \mathscr{S}'$ of graded $\mathscr{O}_Y$-algebras.*

Proof. For the assertion of (a), we can assume that φ is surjective. Then for any affine open U of Y, $\Gamma(U,\mathscr{S}) \to \Gamma(U,\mathscr{S}')$ is surjective by Corollary 10.1.6, so we have $G(\varphi) = X$. We are immediately reduced to the case where Y is affine, and the assertion follows from Proposition 11.2.52(a).

As for case (b), we are reduced to prove that the homomorphism $\widetilde{\mathscr{J}} \to \mathscr{O}_X$ induced from the canonical injection $\mathscr{J} \to \mathscr{S}$ is an isomorphism from $\widetilde{\mathscr{J}}$ to $\mathscr{I}$. As the question is local, we can assume that Y is affine with ring A, which implies $\mathscr{S} = \widetilde{S}$, where S is a graded A-algebra generated by S_1 and S_1 is of finite type over A. It then suffices to apply Proposition 11.2.52(b). ◻

Corollary 11.3.34. *Under the hypotheses of Proposition 11.3.33(a), suppose that $\mathscr{S}$ is generated by $\mathscr{S}_1$. Then $\Phi^*(\mathscr{O}_X(n))$ is canonically identified with $\mathscr{O}_{X'}(n)$ for any $n \in \mathbb{Z}$.*

Proof. We have defined such an isomorphism if Y is affine; in the general case, it suffices to verify that these isomorphisms are compatible with restrictions. ◻

Corollary 11.3.35. *Let Y be a scheme, $\mathscr{S}$ be a quasi-coherent $\mathscr{O}_Y$-algebra generated by $\mathscr{S}_1$, $\mathscr{M}$ be a quasi-coherent $\mathscr{O}_Y$-module, and $u : \mathscr{M} \to \mathscr{S}_1$ be a surjective $\mathscr{O}_Y$-homomorphism. If $\bar{u} : \boldsymbol{S}(\mathscr{M}) \to \mathscr{S}$ is the canonical homomorphism of $\mathscr{O}_Y$-algebras extending u, then the morphism corresponding to $\bar{u}$ is a closed immersion from $\mathrm{Proj}(\mathscr{S})$ into $\mathrm{Proj}(\boldsymbol{S}(\mathscr{M}))$.*

Proof. In fact, $\bar{u}$ is surjective by hypothesis, so we can apply Proposition 11.3.33(a). ◻

11.3.6 Morphisms into $\mathrm{Proj}(\mathscr{S})$

Let $q : X \to Y$ be a morphism of schemes, $\mathscr{L}$ be an invertible $\mathscr{O}_X$-module, and $\mathscr{S}$ be a quasi-coherent $\mathscr{O}_Y$-algebra with positive degrees. Then $q^*(\mathscr{S})$ is a quasi-coherent graded $\mathscr{O}_X$-algebra with positive degrees. Suppose that we are given a graded homomorphism of $\mathscr{O}_X$-algebras

$$\psi : q^*(\mathscr{S}) \to \boldsymbol{S}(\mathscr{L}) = \bigoplus_{n\geq 0} \mathscr{L}^{\otimes n}$$

(recall that for an invertible $\mathscr{O}_X$-module $\mathscr{L}$ we have $\boldsymbol{T}(\mathscr{L}) = \boldsymbol{S}(\mathscr{L})$) or equivalently, a $\mathscr{O}_Y$-homorphism of graded algebras

$$\psi^\flat : \mathscr{S} \to q_*(\boldsymbol{S}(\mathscr{L})).$$

Since $\mathscr{L}$ is invertible and $\boldsymbol{T}(\mathscr{O}_X) = \boldsymbol{S}(\mathscr{O}_X) = \mathscr{O}_X[T]$, by Proposition 11.3.6 and Corollary 11.3.5 we know that $\mathrm{Proj}(\boldsymbol{S}(\mathscr{L}))$ is canonically identified with X. We then conclude that the homomorphism ψ induces a Y-morphism

$$r_{\mathscr{L},\psi} : G(\psi) \to \mathrm{Proj}(\mathscr{S}) = P,$$

where $G(\psi)$ is an open subset of X. Recall that this morphism is by definition obtained by composing the first projection $\pi : \mathrm{Proj}(q^*(\mathscr{S})) = P \times_Y X \to P$ with the Y-morphism $\tau = \mathrm{Proj}(\psi) : G(\psi) \to \mathrm{Proj}(q^*(\mathscr{S}))$, which is shown in the following diagram:

$$\begin{tikzcd}
 & P \times_Y X \arrow[r, "\pi"] \arrow[d] & P \arrow[d, "p"] \\
G(\psi) \arrow[ur, "\tau=\mathrm{Proj}(\psi)"] \arrow[urr, bend left, "r_{\mathscr{L},\psi}"] \arrow[r] & X \arrow[r, "q"] & Y
\end{tikzcd}$$

Remark 11.3.36. Let us explain the morphism $r = r_{\mathscr{L},\psi}$ when $Y = \mathrm{Spec}(A)$ is affine, and therefore $\mathscr{S} = \widetilde{S}$, where S is a graded A-algebra with positive degrees. First suppose that $X = \mathrm{Spec}(B)$ is also affine and $\mathscr{L} = \widetilde{L}$, where L is a free B-module of rank 1. We then have $q^*(\mathscr{S}) = \widetilde{S \otimes_A B}$ by Proposition 10.1.14. If c is a generator of L, the homomorphism $\psi_n : q^*(\mathscr{S}_n) \to \mathscr{L}^{\otimes n}$ then corresponds to a B-homomorphism

$$w_n : S_n \otimes_A B \to L^{\otimes n}, \quad s \otimes b \mapsto bv_n(s)c^{\otimes n}, \tag{11.3.10}$$

where $v_n : S_n \to B$ is the n-th component of a homomorphism $v : S \to B$ of algebras. Let $f \in S_d$ with positive degree and put $g = v_d(f)$. We have $\pi^{-1}(D_+(f)) = D_+(f \otimes 1)$ in view of Proposition 11.2.50 and the identification of $D_+(f)$ with $\mathrm{Spec}(S_{(f)})$. On the other hand, the formula (11.2.9) and (11.3.10) shows that (using the canonical isomorphism of X and $\mathrm{Proj}(\boldsymbol{S}(\mathscr{L}))$)

$$\tau^{-1}(D_+(f \otimes 1)) = D(g)$$

whence $r^{-1}(D_+(f)) = D(g)$. Furthermore, the restriction of the morphism $\tau = \mathrm{Proj}(\psi)$ to $D(g)$ corresponds to the homomorphism $(S \otimes_A B)_{(f\otimes 1)} \to B_g$, which send $(s \otimes 1)/(f \otimes 1)^n$ (for $s \in S_{nd}$) to $v_{nd}(s)/g^n$, and the restriction of the projection π to $D_+(f \otimes 1)$ corresponds to the homomorphism $S_{(f)} \to (S \otimes_A B)_{(f\otimes 1)}$ given by $s/f^n \mapsto (s \otimes 1)/(f \otimes 1)^n$. We then conclude that the restriction of the morphism r to $D(g)$ corresponds to the homomorphisms $\omega : S_{(f)} \to B_g$ of A-algebras such that $\omega(s/f^n) = v_{nd}(s)/g^n$ for $s \in S_{nd}$.

Proposition 11.3.37. *Let $Y = \mathrm{Spec}(A)$ be affine and $\mathscr{S} = \widetilde{S}$, where S is a graded A-algebra. For any $f \in S_d = \Gamma(Y, \mathscr{S}_d)$, we have (where $\psi^\flat(f) \in \Gamma(X, \mathscr{L}^{\otimes d})$)*

$$r_{\mathscr{L},\psi}^{-1}(D_+(f)) = X_{\psi^\flat(f)}. \tag{11.3.11}$$

Moreover, under the canonical isomorphism of X and $\mathrm{Proj}(\boldsymbol{S}(\mathscr{L}))$, the restriction morphism $r_{\mathscr{L},\psi} : X_{\psi^\flat(f)} \to D_+(f) = \mathrm{Spec}(S_{(f)})$ corresponds to the homomorphism

$$\psi^\flat_{(f)} : S_{(f)} \to \Gamma(X_{\psi^\flat(f)}, \mathscr{O}_X)$$

such that, for any $s \in S_{nd} = \Gamma(Y, \mathscr{S}_{nd})$, we have

$$\psi^\flat_{(f)}(s/f^n) = (\psi^\flat(s)|_{X_{\psi^\flat(f)}})(\psi^\flat(f)|_{X_{\psi^\flat(f)}})^{-n}.$$

Proof. This follows from Remark 11.3.36 by passing to the general case. □

We say the morphism $r_{\mathscr{L},\psi}$ is **everywhere defined** if $G(\psi) = X$. For this to be the case, it is necessary and sufficient that $G(\psi) \cap q^{-1}(U) = q^{-1}(U)$ for any affine open $U \subseteq Y$, so this question is local over Y. If Y is affine, $G(\psi)$ is then the union of $r^{-1}(D_+(f))$ for $f \in S_+$, so by (11.3.11) the $X_{\psi^\flat(f)}$ then form a covering of X. In other words:

Corollary 11.3.38. *Under the hypotheses of Proposition 11.3.37, for the morphism $r_{\mathscr{L},\psi}$ to be everywhere defined, it is necessary and sufficient that for any $x \in X$, there exists an integer $n > 0$ and a section $s \in S_n$ such that $t = \psi^\flat(s) \in \Gamma(X, \mathscr{L}^{\otimes n})$ is nonzero at x. In particular, this is true if ψ is eventually surjective.*

Corollary 11.3.39. *Under the hypotheses of Proposition 11.3.37, for the morphism $r_{\mathscr{L},\psi}$ to be dominant, it is necessary and sufficient that for any integer $n > 0$, any section $s \in S_n$ such that $\psi^\flat(b) \in \Gamma(X, \mathscr{L}^{\otimes n})$ is locally nilpotent, is itself nilpotent.*

Proof. We must check that $r_{\mathscr{L},\psi}^{-1}(D_+(s))$ is nonempty if $D_+(s)$ is nonempty, and the corollary follows from (11.3.11) and Corollary 11.2.6. □

Proposition 11.3.40. *Let $q : X \to Y$ be a morphism, $\mathscr{L}$ be an invertible $\mathscr{O}_X$-module, and $\mathscr{S}$, $\mathscr{S}'$ be quasi-coherent graded $\mathscr{O}_Y$-algebras. Let $u : \mathscr{S}' \to \mathscr{S}$ be a homomorphism of graded algebras, $\psi : q^*(\mathscr{S}) \to \boldsymbol{S}(\mathscr{L})$ be a homomorphism of graded algebras, and $\psi' = \psi \circ q^*(u)$ be the composition.*

(i) *If $r_{\mathscr{L},\psi'}$ is everywhere defined, then $r_{\mathscr{L},\psi}$ is everywhere defined;*

(ii) *If u is eventually surjective and $r_{\mathscr{L},\psi'}$ is dominant, then $r_{\mathscr{L},\psi}$ is dominant;*

(iii) *If u is eventually injective and $r_{\mathscr{L},\psi}$ is dominant, then $r_{\mathscr{L},\psi}$ is dominant.*

Proof. We have $G(\psi') \subseteq G(\psi)$, whence the first assertion. If u is eventually surjective, $\mathrm{Proj}(u) : \mathrm{Proj}(S) \to \mathrm{Proj}(S')$ is everywhere defined and is a closed immersion; as $r_{\mathscr{L},\psi'}$ is the composition of $\mathrm{Proj}(u)$ and the restriction of $r_{\mathscr{L},\psi}$ to $G(\psi')$, we then conclude that if $r_{\mathscr{L},\psi'}$ is dominant, so is $r_{\mathscr{L},\psi}$. Finally, if u is eventually injective, then $\mathrm{Proj}(u)$ is a dominant morphism from $G(u)$ into $\mathrm{Proj}(\mathscr{S}')$ (Corollary 11.2.45); as $G(\psi')$ is the inverse image of $G(u)$ under $r_{\mathscr{L},\psi}$, we see that if $r_{\mathscr{L},\psi}$ is dominant, so is $r_{\mathscr{L},\psi'}$. □

Proposition 11.3.41. *Let Y be a quasi-compact scheme, $q : X \to Y$ be a quasi-compact morphism, $\mathscr{L}$ be an invertible $\mathscr{O}_X$-module, and $\mathscr{S}$ be a quasi-coherent $\mathscr{O}_Y$-algebra which is the filtered limit of a system $(\mathscr{S}^\lambda)$ of quasi-coherent $\mathscr{O}_Y$-algebras. Let $\varphi_\lambda : \mathscr{S}^\lambda \to \mathscr{S}$ be the canonical homomorphism, $\psi : q^*(\mathscr{S}) \to \boldsymbol{S}(\mathscr{L})$ be a homomorphism of graded algebras, and put $\psi_\lambda = \psi \circ q^*(\varphi_\lambda)$. Then for the morphism $r_{\mathscr{L},\psi}$ to be everywhere defined, it is necessary and sufficient that there exists an index λ such that $r_{\mathscr{L},\psi_\lambda}$ is everywhere defined; in this case, $r_{\mathscr{L},\psi_\mu}$ is everywhere defined for $\mu \geq \lambda$.*

Proof. The condition is sufficient in view of Proposition 11.3.40. Conversely, suppose that $r_{\mathscr{L},\psi}$ is everywhere defined; we can assume that Y is affine, because if for any affine open $U \subseteq Y$ there exists $\lambda(U)$ such that the restriction of $r_{\mathscr{L},\psi_{\lambda_{\lambda(U)}}}$ to $q^{-1}(U)$ is defined everywhere, it then suffices to cover Y by finitely many affine opens U_i (recall that Y is quasi-compact) and choose $\lambda \geq \lambda(U_i)$ for all i, by Proposition 11.3.40. If Y is affine (so $\mathscr{S} = \widetilde{S}$ where $S = \Gamma(Y, \mathscr{S})$) the hypotheses implies that for any $x \in X$, there exists a section $s^{(x)} \in S_n$ for some integer n such that, if $t^{(x)} = \psi^\flat(s^{(x)})$, then $t^{(x)}(x) \neq 0$ (where $t^{(x)}$ is a section of $\mathscr{L}^{\otimes n}$ over X), which implies $t^{(x)}(z) \neq 0$ for z in a neighborhood $V(x)$ of X. As the morphism $q : X \to Y$ is quasi-compact, we see X is quasi-compact, so we can cover X by finitely many $V(x_i)$ and let $s^{(i)}$ be the corresponding section of S. There is then an index λ such that $s^{(i)}$ is of the form $\varphi_\lambda(s_\lambda^{(i)})$, where $s_\lambda^{(i)} \in S^\lambda$ for all i, and it follows from Corollary 11.3.38 that $r_{\mathscr{L},\psi_\lambda}$ is everywhere defined. The second assertion is trival by Proposition 11.3.40. □

Corollary 11.3.42. *Under the hypotheses of Proposition 11.3.41, if the morphisms $r_{\mathscr{L},\psi_\lambda}$ are dominant, so is $r_{\mathscr{L},\psi}$. The converse is also true if the homomorphisms φ_λ are eventually injective.*

Proof. The second assertion is a particular case of Proposition 11.3.40. To show that $r_{\mathscr{L},\psi}$ is dominant if each $r_{\mathscr{L},\psi_\lambda}$ is, we can assume that Y is affine and thus $\mathscr{S} = \widetilde{S}$ where $S = \Gamma(Y, \mathscr{S})$. If $s \in S$ is such that $\psi^\flat(s)$ is locally nilpotent, as we can write $s = \varphi_\lambda(s_\lambda)$ for some λ, from the definition of ψ_λ and by Corollary 11.3.39, we conclude that s_λ is nilpotent, so s is nilpotent, and the assertion follows by applying Corollary 11.3.39. □

Remark 11.3.43. With the hypotheses and notations of Proposition 11.3.37, for each $n \in \mathbb{Z}$ we have a homomorphism

$$\nu : r^*_{\mathscr{L},\psi}(\mathscr{O}_P(n)) \to \mathscr{L}^{\otimes n}|_{G(\psi)} \tag{11.3.12}$$

which is in fact the homomorphism ν defined in (11.3.7) on $\mathscr{O}_P(n)$. We also see that under the hypotheses of Proposition 11.3.37, the restriction of ν to $X_{\psi^\flat(f)}$ corresponds to the homomorphism sending the element s/f^k (with $s \in S_{n+kd}$) to the section

$$(\psi^\flat(s)|_{X_{\psi^\flat(f)}})(\psi^\flat(f)|_{X_{\psi^\flat(f)}})^{-k} \in \Gamma(X_{\psi^\flat(f)}, \mathscr{L}^{\otimes n}),$$

where we also use the notations of Proposition 11.3.37.

Remark 11.3.44. Let $\mathscr{F}$ be a quasi-coherent $\mathscr{O}_X$-module and suppose that q is quasi-compact and quasi-separated, so for each $n \geq 0$, $q_*(\mathscr{F} \otimes \mathscr{L}^{\otimes n})$ is a quasi-coherent $\mathscr{O}_Y$-module (Proposition 10.6.55). Let $\mathscr{M}' = \bigoplus_{n \geq 0} \mathscr{F} \otimes \mathscr{L}^{\otimes n}$, which is a quasi-coherent $\mathscr{O}_Y$-module, and consider the image $\mathscr{M} = q_*(\mathscr{M}') = \bigoplus_{n \geq 0} q_*(\mathscr{F} \otimes \mathscr{L}^{\otimes n})$ (which is a quasi-coherent $\mathscr{S}$-module via the homomorphism $\psi^\flat$). We shall see that there is a canonical homomorphism of $\mathscr{O}_X$-modules

$$\xi : r^*_{\mathscr{L},\psi}(\widetilde{\mathscr{M}}) \to \mathscr{F}|_{G(\psi)}. \tag{11.3.13}$$

For this, recall that we have defined a canonical homomorphism (11.3.7):

$$\nu : r^*_{\mathscr{L},\psi}(\widetilde{\mathscr{M}}) \to (q^*(\mathscr{M}) \otimes_{q^*(\mathscr{S})} \boldsymbol{S}(\mathscr{L}))^{\sim}|_{G(\psi)},$$

where the right hand side is considered as a quasi-coherent sheaf over X. On the other hand, for any quasi-coherent graded $\boldsymbol{S}(\mathscr{L})$-module $\mathscr{M}'$, we have a canonical homomorphism

$$q^*(q_*(\mathscr{M}')) \otimes_{q^*(\mathscr{S})} \boldsymbol{S}(\mathscr{L}) \to \mathscr{M}'$$

which, for any open subset U of X, any section t' of $q^*(q_*(\mathscr{M}'_h))$ over U and any section b' of $\mathscr{L}^{\otimes k}$ over U, sends the section $t' \otimes b'$ to the section $b'\sigma(t')$ of $\mathscr{M}'_{h+k}$, where σ is the canonical homomorphism $q^*(q_*(\mathscr{M}')) \to \mathscr{M}'$. We then conclude a canonical homomorphism

$$(q^*(q_*(\mathscr{M}')) \otimes_{q^*(\mathscr{S})} \boldsymbol{S}(\mathscr{L}))^{\sim}|_{G(\psi)} \to \widetilde{\mathscr{M}}'|_{G(\psi)}$$

and as $\widetilde{\mathscr{M}}'$ is canonically identified with $\mathscr{F}$ by Remark 11.3.18, we obtain the canonical homomorphism ξ.

Under the hypotheses and notations of Proposition 11.3.20, the restriction of this homomorphism to $X_{\psi^\flat(f)}$ is defined as follows: giving a section t_{nd} of $\mathscr{F} \otimes \mathscr{L}^{\otimes d}$ over X (which is also a section of $q_*(\mathscr{F} \otimes \mathscr{L}^{\otimes n})$ over Y), we send the element t_{nd}/f^n to the sectin $(t_{nd}|_{X_{\psi^\flat(f)}})(\psi^\flat(f)|_{X_{\psi^\flat(f)}})^{-n}$ of $\mathscr{F}$ over $X_{\psi^\flat(f)}$.

We now consider the important question that whether the induced morphism $r_{\mathscr{L},\psi}$ is an immersion (reps. an open immersion, a closed immersion). It is clear that this question is local over Y, and we shall give a criterion in this situation together with the condition that $r_{\mathscr{L},\psi}$ is defined everywhere.

Proposition 11.3.45. *Under the hypothesis and notations of Proposition 11.3.37, for the morphism $r_{\mathscr{L},\psi}$ be everywhere defined and an immersion, it is necessary and sufficient that there exists a family of sections $s_\alpha \in S_{n_\alpha}$ (with $n_\alpha > 0$) such that, if $f_\alpha = \psi^\flat(s_\alpha)$, the following conditions are satisfied:*

(i) *The X_{f_α} form a covering of X.*

(ii) *The X_{f_α} are affine open subset of X.*

(iii) *For any index α and any section $t \in \Gamma(X_{f_\alpha}, \mathscr{O}_X)$, there exists an integer $n > 0$ and $s \in S_{mn_\alpha}$ such that $t = (\psi^\flat(s)|_{X_{f_\alpha}})(f_\alpha|_{X_{f_\alpha}})^{-m}$.*

For the morphism $r_{\mathscr{L},\psi}$ to be everywhere defined and an open immersion, it is necessary and sufficient that the family (s_α) satisfies the following additional condition:

(iv) *For any integer $m > 0$ and any $s \in S_{mn_\alpha}$ such that $\psi^\flat(s)|_{X_{f_\alpha}} = 0$, there exists an integer $n > 0$ such that $s_\alpha^n s = 0$.*

Similarly, for the morphism $r_{\mathscr{L},\psi}$ to be everywhere defined and a closed immersion, it is necessary and sufficient that the family (s_α) satisfies the following additional condition:

(v) *The $D_+(s_\alpha)$ form a covering of $P = \mathrm{Proj}(S)$.*

Proof. By Proposition 10.4.9, for the morphism $r = r_{\mathscr{L},\psi}$ to be an immersion (resp. a closed immersion), it is necessary and suffcient that there exists a covering of $r(G(\psi))$ (resp. of P) by the sets $D_+(s_\alpha)$ such that if $V_\alpha = r^{-1}(D_+(s_\alpha))$, the restriction of r on V_α is a closed immersion of V_α into $D_+(s_\alpha)$ (cf. Corollary 10.4.11). Now condition (i) just means that r is everywhere defined and that $D_+(s_\alpha)$ cover $r(X)$, by (11.3.11). As each $D_+(s_\alpha)$ is affine, condition (ii) and (iii) express that the restriction of r to X_{f_α} is a closed immersion into $D_+(s_\alpha)$ (Example 10.4.6). Finally, as (iii) and (iv) means the ring homomorphism $\psi^\flat_{(s_\alpha)}: S_{(s_\alpha)} \to \Gamma(X_{f_\alpha}, \mathscr{O}_X)$ is an isomorphism, (ii), (iii), (iv) mean that the restriction of r to X_{f_α} is an isomorphism from X_{f_α} to $D_+(s_\alpha)$ for each α, so together with (i), they mean that r is an open immersion. □

Corollary 11.3.46. *Under the hypothesis and notations of Proposition 11.3.40, if $r_{\mathscr{L},\psi'}$ is everywhere defined and is an immersion, so is $r_{\mathscr{L},\psi}$. If we suppose that u is eventually surjective and if $r_{\mathscr{L},\psi'}$ is everywhere defined and is a closed immersion (resp. open), then so is $r_{\mathscr{L},\psi}$.*

Proof. We first suppose that $r_{\mathscr{L},\psi'}$ is everywhere defined and and is an immersion. Then by Proposition 11.3.45, there is a family $s'_\alpha \in S'_{n_\alpha}$ such that, if $f_\alpha = \psi'^\flat(s'_\alpha)$, the conditions (i), (ii), (iii) are satisfied. Set $s_\alpha = u(s'_\alpha)$, then $f_\alpha = \psi^\flat(s_\alpha)$, and we have a commutative diagram

$$\begin{tikzcd} S'_{(s'_\alpha)} \arrow[r, "\psi'^\flat_{(s'_\alpha)}"] \arrow[d, "u_{(s'_\alpha)}"'] & \Gamma(X_{f_\alpha}, \mathscr{O}_X) \\ S_{(s_\alpha)} \arrow[ur, "\psi^\flat_{(s_\alpha)}"'] & \end{tikzcd}$$

The hypothesis then implies that $\psi'^\flat_{(s'_\alpha)}$ is surjective, so the homomorphism $\psi^\flat_{(s_\alpha)}$ is also surjective. This shows that $r_{\mathscr{L},\psi}$ is everywhere defined and is an immersion, in view of Proposition 11.3.45. If $r_{\mathscr{L},\psi'}$ is moreover an open immersion, then the homomorphism $\psi'^\flat_{(s'_\alpha)}$ is also injective, and this implies the homomorphism $\psi^\flat_{(s_\alpha)}$ is injective if u is eventually surjective, since in this case the homomorphism $u_{(s'_\alpha)}$ is just surjective.

Finally, if $r_{\mathscr{L},\psi'}$ is a closed immersion, then condition (v) is satisfied for (s'_α), and hence satisfied for (s_α) if u is eventually surjective (since in this case $\mathrm{Proj}(u)$ is a closed immersion from $\mathrm{Proj}(S)$ to $\mathrm{Proj}(S')$); this implies $r_{\mathscr{L},\psi}$ is a closed immersion. □

Proposition 11.3.47. *Suppose the hypotheses of Proposition 11.3.41 and moreover that $q : X \to Y$ is a morphism of finite type. Then, for the morphism $r_{\mathscr{L},\psi}$ to be everywhere defined and an immersion, it is necessary and sufficient that there exists an index λ such that $r_{\mathscr{L},\psi_\lambda}$ is everywhere defined and is an immersion; in this case, $r_{\mathscr{L},\psi_\lambda}$ is everywhere defined and an immersion for $\mu \geq \lambda$.*

Proof. By Corollary 11.3.46, it suffices to prove that if $r_{\mathscr{L},\psi}$ is defined everywhere and is an immersion, then there exists an index λ such that $r_{\mathscr{L},\psi_\lambda}$ is defined everywhere and is an immersion. By the same reasoning of Proposition 11.3.41 and using the quasi-compactness of Y, we are reduced to the case where $Y = \mathrm{Spec}(A)$ is affine. As X is then quasi-compact, Proposition 11.3.45 shows that there exists a finite family (s_i) of elements of S ($s_i \in S_{n_i}$) satisfying the conditions (i), (ii), and (iii). The morphism $X_{f_i} \to Y$ (where $f_i = \psi^\flat(s_i)$) is of finite type since X_{f_i} is affine and the morphism $q : X \to Y$ is locally of finite type. The ring B_i of X_{f_i} is therefore an A-algebra of finite type by Corollary 10.6.39, and we choose (t_{ij}) to be a family of generators of this algebra. There are then elements $s_{ij} \in S_{m_{ij}n_i}$ such that

$$t_{ij} = (\psi^\flat(s_{ij})|_{X_{f_i}})(\psi^\flat(s_i)|_{X_{f_i}})^{-m_{ij}}$$

We can choose an index λ and elements $s_{i\lambda} \in S^\lambda_{n_i}$, $s_{ij\lambda} \in S^\lambda_{m_{ij}n_i}$ such that their images under φ_λ is s_i and s_{ij}, respectively. It is then clear that the family $(s_{i\lambda})$ satisfies the conditions (i), (ii), and (iii), so $r_{\mathscr{L},\psi}$ is everywhere defined and an immersion. □

Proposition 11.3.48. *Let Y be a quasi-compact scheme, $q : X \to Y$ be a morphism of finite type, $\mathscr{L}$ be an invertible $\mathscr{O}_X$-module, $\mathscr{S}$ be a quasi-coherent graded $\mathscr{O}_Y$-algebra, and $\psi : q^*(\mathscr{S}) \to \boldsymbol{S}(\mathscr{L})$ be a homomorphism of graded algebras. For the morphism $r_{\mathscr{L},\psi}$ to be defined everywhere and an immersion, it is necessary and sufficient that there exists an integer $n > 0$ and a quasi-coherent sub-$\mathscr{O}_Y$-module $\mathscr{E}$ of $\mathscr{S}_n$ of finite type such that:*

(a) *the homomorphism $\psi_n \circ q^*(j_n) : q^*(\mathscr{E}) \to \mathscr{L}^{\otimes n}$ (where $j_n : \mathscr{E} \to \mathscr{S}_n$ is the canonical injection) is surjective.*

(b) *if $\mathscr{S}'$ is the graded sub-$\mathscr{O}_Y$-algebra of $\mathscr{S}$ genrated by $\mathscr{E}$ and ψ' is the homomorphism $\psi \circ q^*(j')$, where $j' : \mathscr{S}' \to \mathscr{S}$ is the canonical injection, $r_{\mathscr{L},\psi'}$ is everywhere defined and an immersion.*

If these are true, any quasi-coherent sub-$\mathscr{O}_Y$-module $\mathscr{E}'$ of $\mathscr{S}_n$ containing $\mathscr{E}$ possesses the same properties, and so does the sub-$\mathscr{O}_Y$-module $\mathscr{S}'_{kn}$ of $\mathscr{S}_{kn}$ for any $k > 0$.

Proof. The sufficiency of these conditions is a particular case of Corollary 11.3.46 in view of the isomorphism between $\mathrm{Proj}(\mathscr{S})$ and $\mathrm{Proj}(\mathscr{S}^{(d)})$ (Proposition 11.3.6). We now prove the necessity, so let (U_i) be a finite affine open covering of Y and set $A_i = \Gamma(U_i, \mathscr{O}_Y)$. As $q^{-1}(U_i)$ is compact, the hypotheses that $r_{\mathscr{L},\psi}$ is an immersion defined on X implies by Proposition 11.3.45 the existence of a finite family (s_{ij}) of elements of $S^{(i)} = \Gamma(U_i, \mathscr{S})$ (where $s_{ij} \in S^{(i)}_{n_{ij}}$) satisfying conditions (i), (ii), and (iii). Since $q : X \to Y$ is of finite type, the restricted homomorphism $X_{f_{ij}} \to U_i$ is of finite type (where $f_{ij} = \psi^\flat(s_{ij})$), so the ring B_{ij} of $X_{f_{ij}}$ is an A_i-algebra of finite type, and we choose $(\psi^\flat(t_{ijk})|_{X_{f_{ij}}})(f_{ij}|_{X_{f_{ij}}})^{-m_{ijk}}$ to be a system of generators of B_{ij}, where $t_{ijk} \in S^{(i)}_{m_{ijk}n_{ij}}$. Let n be a common multiple of all the $m_{ijk}n_{ij}$ and put $s'_{ij} = s_{ij}^{h_{ij}} \in S^{(i)}_n$, where $h_{ij} = n/n_{ij}$. For any given piar (i,j,k), the element $t'_{ij} = s_{ij}^{h-m_{ijk}}t_{ijk}$ belongs to $S^{(i)}_n$, and it is clear that the $(\psi^\flat(t'_{ijk})|_{X_{f'_{ij}}})(f'_{ij}|_{X_{f'_{ij}}})^{-1}$ also generate B_{ij} (where $f'_{ij} = \psi^\flat(s'_{ij})$, and we note that $X_{f'_{ij}} = X_{f_{ij}}$). Let E_i be the sub-A_i-module of $S^{(i)}$ generated by these s'_{ij} and t'_{ijk}; then there exists a quasi-coherent sub-$\mathscr{O}_Y$-module $\mathscr{E}_i$ of $\mathscr{S}_n$ of finite type such that $\mathscr{E}_i|_{U_i} = \widetilde{E}_i$ (Theorem 10.6.63). It is then clear that the sub-$\mathscr{O}_Y$-module $\mathscr{E}$ of $\mathscr{E}$, which is the sum of the $\mathscr{E}_i$, satisfies the required properties. □

Remark 11.3.49. The point of Proposition 11.3.48 is that, for a scheme X of finite type over a quasi-compact scheme Y, if X can be embedded into $\mathrm{Proj}(\mathscr{S})$ via a morphism $r_{\mathscr{L},\psi}$, then we can choose $\mathscr{S}$ so that it is generated by $\mathscr{S}_1$ and $\mathscr{S}_1$ of finite type (we already know that in this case the twisted sheaves over $\mathrm{Proj}(\mathscr{S})$ have nice properties).

11.4 Projective bundles and ample sheaves

11.4.1 Projective bundles

Let Y be a scheme, $\mathscr{E}$ be a quasi-coherent $\mathscr{O}_Y$-module, and $\boldsymbol{S}(\mathscr{E})$ be the symmetric $\mathscr{O}_Y$-algebra of $\mathscr{E}$, which is quasi-coherent by Corollary 11.1.38. The **projective bundle** over Y associated with $\mathscr{E}$ is defined to be the Y-scheme $P = \mathrm{Proj}(\boldsymbol{S}(\mathscr{E}))$. The $\mathscr{O}_P$-module $\mathscr{O}_P(1)$ is called the **fundamental sheaf** of P.

If $\mathscr{E} = \mathscr{O}_Y^n$, we then denote by $\mathbb{P}_Y^{n-1}$ instead of $\mathbb{P}(\mathscr{E})$; if moreover Y is affine with ring A, we then denote this scheme by $\mathbb{P}_A^{n-1}$. As $\boldsymbol{S}(\mathscr{O}_Y)$ is canonically isomorphic to $\mathscr{O}_Y[T]$, we see $\mathbb{P}_Y^0$ is canonically identified with Y.

If $Y = \mathrm{Spec}(A)$ and $\mathscr{E} = \widetilde{E}$ where E is an A-module, we also denote by $\mathbb{P}(E)$ the projective bundle $\mathbb{P}(E)$. The simplest example is $\mathbb{P}(E)$ where E is a vector space over a field k. In this case, we see $\mathbb{P}(E)$ is isomorphic to $\mathbb{P}_k^{n-1}$, where n is the dimension of E.

Let $\mathscr{E},\mathscr{F}$ be two qusi-coherent $\mathscr{O}_Y$-modules and $u:\mathscr{E}\to\mathscr{F}$ be an $\mathscr{O}_Y$-homomorphism. Then u corresponds canonically to a homomorphism $\boldsymbol{S}(u):\boldsymbol{S}(\mathscr{E})\to\boldsymbol{S}(\mathscr{F})$ of graded $\mathscr{O}_Y$-algebras. If u is surjective, so is $\boldsymbol{S}(u)$, and therefore $\mathrm{Proj}(\boldsymbol{S}(u))$ is a closed immersion $\mathbb{P}(\mathscr{F})\to\mathbb{P}(\mathscr{E})$, which we then denoted by $\mathbb{P}(u)$. We can then say that $\mathbb{P}(\mathscr{E})$ is a contravariant functor on the category of quasi-coherent $\mathscr{O}_Y$-modules with *surjective homomorphisms*. Suppose that u is surjective and put $P = \mathbb{P}(\mathscr{E})$, $Q = \mathbb{P}(\mathscr{F})$, and $j = \mathbb{P}(u)$. We then have an isomorphism

$$j^*(\mathscr{O}_P(n)) = \mathscr{O}_Q(n)$$

by Corollary 11.3.34.

If $\psi: Y'\to Y$ is a morphism and $\mathscr{E}' = \psi^*(\mathscr{E})$, we then have $\boldsymbol{S}_{\mathscr{O}_{Y'}}(\mathscr{E}') = \psi^*(\boldsymbol{S}_{\mathscr{O}_Y}(\mathscr{E}))$ by Proposition 11.1.36, so from Proposition 11.3.31 we deduce that

$$\mathbb{P}(\psi^*(\mathscr{E})) = \mathbb{P}(\mathscr{E})\times_Y Y'. \tag{11.4.1}$$

Moreover, it is clear that $\psi^*((\boldsymbol{S}_{\mathscr{O}_Y}(\mathscr{E}))(n)) = (\boldsymbol{S}_{\mathscr{O}_{Y'}}(\mathscr{E}'))(n)$ for each $n\in\mathbb{Z}$, so if $P = \mathbb{P}(\mathscr{E})$ and $P' = \mathbb{P}(\mathscr{E}')$, we have a canonical isomorphism

$$\mathscr{O}_{P'}(n) = \mathscr{O}_P(n)\otimes_Y\mathscr{O}_{Y'}. \tag{11.4.2}$$

Proposition 11.4.1. *Let $\mathscr{L}$ be an invertible $\mathscr{O}_Y$-module. For any quasi-coherent $\mathscr{O}_Y$-module, there exists a canonical Y-isomorphism $i_{\mathscr{L}}: Q = \mathbb{P}(\mathscr{E}\otimes\mathscr{L}) \xrightarrow{\sim} P = \mathbb{P}(\mathscr{E})$. Moreover, $(i_{\mathscr{L}})_*(\mathscr{O}_Q(n))$ is cannically isomorphic to $\mathscr{O}_P(n)\otimes_Y\mathscr{L}^{\otimes n}$ for any $n\in\mathbb{Z}$.*

Proof. If A is a ring, E is an A-module, L is an A-module free of rank 1, we can define a canonical homomorphism

$$\boldsymbol{S}_n(E\otimes L)\to\boldsymbol{S}_n(E)\otimes L^{\otimes n}$$

which maps an element $(x_1\otimes y_1)\cdots(x_n\otimes y_n)$ to

$$(x_1\cdots x_n)\otimes(y_1\otimes\cdots\otimes y_n)$$

where $x_i\in E$ and $y_i\in L$. This is easily seen to be an isomorphism, so we get an isomorphism $\boldsymbol{S}(E\otimes L)\cong\bigoplus_{n\geq 0}\boldsymbol{S}_n(E)\otimes L^{\otimes n}$. In the situation of the proposition, the preceding remark allows us to define a canonical isomorphism of graded $\mathscr{O}_Y$-algebras

$$\boldsymbol{S}(\mathscr{E}\otimes_{\mathscr{O}_Y}\mathscr{L}) \xrightarrow{\sim} \bigoplus_{n\geq 0}\boldsymbol{S}_n(\mathscr{E})\otimes_{\mathscr{O}_Y}\mathscr{L}^{\otimes n}$$

The proposition then follows from Proposition 11.3.6 and Proposition 11.3.20. □

Let $P = \mathbb{P}(\mathscr{E})$ and denote by $p : P \to Y$ the structural morphism. As by definition $\mathscr{E} = (\boldsymbol{S}(\mathscr{E}))_1$, we have a canonical homomorphism $\alpha_1 : \mathscr{E} \to p_*(\mathscr{O}_P(1))$, and therefore a canonical homomorphism

$$\alpha_1^\sharp : p^*(\mathscr{E}) \to \mathscr{O}_P(1).$$

Proposition 11.4.2. *The canonical homomorphism $\alpha_1^\sharp$ is surjective.*

Proof. We have seen that $\alpha_1^\sharp$ corresponds to the functorial homomorphism $\mathscr{E} \otimes_{\mathscr{O}_Y} \boldsymbol{S}(\mathscr{E}) \to (\boldsymbol{S}(\mathscr{E}))(1)$ (see the remark before Proposition 11.3.21). Since $\mathscr{E}$ generates $\boldsymbol{S}(\mathscr{E})$, this homomorphism is surjective, whence our assertion in view of Proposition 11.3.13. □

11.4.2 Morphisms into $\mathbb{P}(\mathscr{E})$

With the notations of the last subsection, we now let X be an Y-scheme, $q : X \to Y$ be the structural morphism, and $r : X \to P$ be an Y-morphism, which gives a commutative diagram

$$\begin{tikzcd} X \arrow[r, "r"] \arrow[dr, "q"'] & P \arrow[d, "p"] \\ & Y \end{tikzcd}$$

As the functor r^* is right-exact, we deduce from the surjective homomorphism $\alpha_1^\sharp$ in Proposition 11.4.2 a surjective homomorphism

$$r^*(\alpha_1^\sharp) : r^*(p^*(\mathscr{E})) \to r^*(\mathscr{O}_P(1)).$$

But $r^*(p^*(\mathscr{E})) = q^*(\mathscr{E})$ and $r^*(\mathscr{O}_P(1))$ is locally isomorphic to $r^*(\mathscr{O}_P) = \mathscr{O}_X$, which is then an invertible sheaf $\mathscr{L}_r$ over X, so we obtain a canonical surjective $\mathscr{O}_X$-homomorphism

$$\varphi_r : q^*(\mathscr{E}) \to \mathscr{L}_r.$$

If $Y = \operatorname{Spec}(A)$ is affine and $\mathscr{E} = \widetilde{E}$, we can explicitly explain this homomorphism: given $f \in E$, it follows from Proposition 11.2.33 that

$$r^{-1}(D_+(f)) = X_{\varphi_r^\flat(f)}.$$

Let V be an affine open of X contained in $r^{-1}(D_+(f))$, and let B be its ring, which is an A-algebra; put $S = \boldsymbol{S}_A(E)$. The restriction of r to V then corresponds to an A-homomorphism $\omega : S_{(f)} \to B$, and we have $q^*(\mathscr{E})|_V = \widetilde{E \otimes_A B}$ and $\mathscr{L}_r|_V = \widetilde{L}_r$, where by Proposition 10.1.14, $L_r = S(1)_{(f)} \otimes_{S_{(f)}} B$. In view of the definition of α_1, the restriction of φ_r to $q^*(\mathscr{E})|_V$ therefore corresponds to the B-homomorphism

$$u : E \otimes_A B \to L_r, \quad x \otimes 1 \mapsto (x/1) \otimes 1 = (f/1) \otimes \omega(x/f)$$

The canonical extension of φ_r to a homomorphism of $\mathscr{O}_X$-algebras (recall that $(\mathscr{O}_P(1))^{\otimes n} = \mathscr{O}_P(n)$ by Corollary 11.2.29)

$$\psi_r : q^*(\boldsymbol{S}(\mathscr{E})) = \boldsymbol{S}(q^*(\mathscr{E})) \to \boldsymbol{S}(\mathscr{L}_r) = \bigoplus_{n \geq 0} \mathscr{L}_r^{\otimes n} = \bigoplus_{n \geq 0} r^*(\mathscr{O}_P(n))$$

is then such that the restriction of ψ_r to $q^*(\boldsymbol{S}_n(\mathscr{E}))|_V$ corresponds to the B-homomorphism

$$\boldsymbol{S}_n(\mathscr{E} \otimes_A B) = \boldsymbol{S}_n(E) \otimes_A B \to L_r^{\otimes n} = (S(1)_{(f)})^{\otimes n} \otimes_{S_{(f)}} B$$

which send the element $s \otimes 1$ to $(f/1)^{\otimes n} \otimes \omega(s/f^n)$.

Conversely, given an invertible $\mathscr{O}_X$-module $\mathscr{L}$ and a quasi-coherent $\mathscr{O}_Y$-module $\mathscr{E}$, then any homomorphism $\varphi : q^*(\mathscr{E}) \to \mathscr{L}$ corresponds to a canonical homomorphism of quasi-coherent $\mathscr{O}_X$-algebras

$$\psi : \boldsymbol{S}(q^*(\mathscr{E})) = q^*(\boldsymbol{S}(\mathscr{E})) \to \boldsymbol{S}(\mathscr{L}) = \bigoplus_{n \geq 0} \mathscr{L}^{\otimes n}$$

which then gives an Y-morphism $r_{\mathscr{L},\psi} : G(\psi) \to \operatorname{Proj}(\boldsymbol{S}(\mathscr{E})) \to \mathbb{P}(\mathscr{E})$, which we also denoted by $r_{\mathscr{L},\varphi}$. If φ is surjective, then so is ψ and by Corollary 11.3.38 the morphism $r_{\mathscr{L},\psi}$ is everywhere defined.

Proposition 11.4.3. *Let $q : X \to Y$ be a morphism and $\mathscr{E}$ be a quasi-coherent $\mathscr{O}_Y$-module. Then the maps $r \mapsto (\mathscr{L}_r, \varphi_r)$ and $(\mathscr{L}, \varphi) \mapsto r_{\mathscr{L},\varphi}$ form a bijective correspondence between the set of Y-morphisms $r : X \to \mathbb{P}(\mathscr{E})$ to the set of equivalence classes of couples $(\mathscr{L}, \varphi)$ formed by an invertible $\mathscr{O}_X$-module and a surjective homomorphism $\varphi : q^*(\mathscr{E}) \to \mathscr{L}$, where two couples $(\mathscr{L}, \varphi)$, $(\mathscr{L}', \varphi')$ are equivalent if there exists an $\mathscr{O}_X$-isomorphism $\tau : \mathscr{L} \to \mathscr{L}'$ such that $\varphi' = \tau \circ \varphi$.*

Proof. Let us start with an Y-morphism $r : X \to \mathbb{P}(\mathscr{E})$, form $\mathscr{L}_r$ and φ_r, and put $r' = r_{\mathscr{L}_r,\varphi_r}$. To see the morphisms r and r' coincide, we may assume that $Y = \operatorname{Spec}(A)$ is affine, so $\mathscr{E} = \widetilde{E}$, and let $S = \boldsymbol{S}_A(E)$. Let $V = \operatorname{Spec}(B)$ be an affine open of X contained in $r^{-1}(D_+(f))$, where $f \in E$. Then as we have already seen, the restriction of r to V corresponds to an A-homomorphism $\omega : S_{(f)} \to B$, and the restriction of ψ_r to $q^*(\boldsymbol{S}_n(\mathscr{E}))|_V$ corresponds to the B-homomorphism $\boldsymbol{S}_n(E) \otimes_A B \to L_r^{\otimes n}$ which sends $s \otimes 1$ to $(f/1)^{\otimes n} \otimes \omega(s/f^n)$. The restriction of $\psi_r^\flat$ to $\boldsymbol{S}(\mathscr{E})|_V$ then corresponds to the homomorphism $\boldsymbol{S}_n(E) \to L_r^{\otimes n}$ which sends $s \in \boldsymbol{S}_n(E)$ to $(f/1)^{\otimes n} \otimes \omega(s/f^n)$, and by Proposition 11.3.37, the restriction of $r_{\mathscr{L}_r,\psi_r}$ to V corresponds to the homomorphism $(\psi_r^\flat)_{(f)}$, which send $s \in S_n$ to

$$(\psi_r^\flat(s))(\psi_r^\flat(f))^{-n} = [(f/1)^{\otimes n} \otimes \omega(s/f^n)][(f/1) \otimes 1]^{-n} = 1 \otimes \omega(s/f^n).$$

Therefore, under the identification of X with $\operatorname{Proj}(\boldsymbol{S}(\mathscr{L}))$, $r_{\mathscr{L},\psi_r}$ coincides with r over V, so they coincide on X.

Conversely. let $(\mathscr{L}, \varphi)$ be a couple and form $r = r_{\mathscr{L},\varphi}$, $\mathscr{L}_r$, and φ_r. We show that there is a canonical isomorphism $\tau : \mathscr{L}_r \to \mathscr{L}$ such that $\varphi = \tau \circ \varphi_r$. For this, we can assume that $Y = \operatorname{Spec}(A)$ and $X = \operatorname{Spec}(B)$ are affine, and (with the notations of Remark 11.3.36) define τ be sending an element $(x/1) \otimes 1$ of L_r (where $x \in E$) to the element $v_1(x)c$ of L. It is easy to verify that τ is independent of the choice of the generator c of L. As v_1 is surjective, to show that τ is an isomorphism, it suffices to prove that if $x/1 = 0$ in $S(1)_{(f)}$, then $v_1(x)/1 = 0$ in B_g. But the first condition means that $f^n x = 0$ in S_{n+1} for some n, and we then deduce that $v_{n+1}(f^n x) = g^n v_1(x) = 0$ in B, whence the conclusion. Finally, it is immediate that for two equivalent couples $(\mathscr{L}, \varphi)$, $(\mathscr{L}', \varphi')$, we have $r_{\mathscr{L},\varphi} = r_{\mathscr{L}',\varphi'}$. □

Theorem 11.4.4. *The set of Y-sections of $\mathbb{P}(\mathscr{E})$ is in bijective correspondence to the set of quasi-coherent sub-$\mathscr{O}_Y$-modules $\mathscr{F}$ of $\mathscr{E}$ such that $\mathscr{E}/\mathscr{F}$ is invertible.*

Proof. This is a particular case of Proposition 11.4.3 by taking $X = Y$ and note that if two pairs $(\varphi, \mathscr{L})$ and $(\varphi', \mathscr{L}')$ are equivalent, then $\ker \varphi$ and $\ker \varphi'$ are identical. □

Note that this property corresponds to the classical definition of the "projective space" as the set of hyperplanes of a vector space (the classical case corresponding to $Y = \operatorname{Spec}(k)$, where k is a field, and $\mathscr{E} = \widetilde{E}$, E being a finite dimensional k-vector. The sheaves $\mathscr{F}$ having the property stated in Theorem 11.4.4 corresponds then to the hyperplanes of E.

Remark 11.4.5. As there is a canonical correspondence between Y-morphisms from X to P and their graph morphisms, which are X-sections of $P \times_Y X$, we see conversely that Proposition 11.4.3 can be deduced from Theorem 11.4.4. Let $\operatorname{Hyp}_Y(X, \mathscr{E})$ be the set of quasi-coherent sub-$\mathscr{O}_X$-modules $\mathscr{F}$ of $\mathscr{E} \otimes_Y \mathscr{O}_X = q^*(\mathscr{E})$ such that $q^*(\mathscr{E})/\mathscr{F}$ is an invertible $\mathscr{O}_X$-module. If $g : X' \to X$ is an Y-morphism, then $g^*(q^*(\mathscr{E})/\mathscr{F}) = g^*(q^*(\mathscr{E}))/g^*(\mathscr{F})$ by the right exactness of g^*, so the second sheaf is invertible, and therefore $\mathrm{Hyp}_Y(X, \mathscr{E})$ is a covariant functor over the category of Y-schemes. We can then interprete Theorem 11.4.4 by saying that the Y-scheme $\mathbb{P}(\mathscr{E})$ representes the functor $\mathrm{Hyp}_Y(-, \mathscr{E})$. This also provides a characterization of the projective bundle $P = \mathbb{P}(\mathscr{E})$ by the following universal property, more close to the geometric intuition that the constructions of $r_{\mathscr{L},\psi}$: for any morphism $q : X \to Y$ and any invertible $\mathscr{O}_X$-module $\mathscr{L}$ which is a quotient of $q^*(\mathscr{E})$, there exists a unique Y-morphism $r : X \to P$ such that $\mathscr{L}$ is isomorphic to $r^*(\mathscr{O}_P(1))$.

Corollary 11.4.6. *Suppose that any invertible $\mathscr{O}_Y$-module is trivial. Let $E = \mathrm{Hom}_{\mathscr{O}_Y}(\mathscr{E}, \mathscr{O}_Y)$ considered as a module over the ring $A = \Gamma(Y, \mathscr{O}_Y)$, and let $E^\times$ be the subset of E formed by surjective homomorphisms. Then the set of Y-sections of $\mathbb{P}(\mathscr{E})$ is canonically identified with $E^\times / A^\times$, where $A^\times$ is the group of units of A.*

Example 11.4.7. Let Y be a scheme, y be a point of Y, and $Y' = \mathrm{Spec}(\kappa(y))$. The fiber $p^{-1}(y)$ of $\mathbb{P}(\mathscr{E})$ is, in view of (11.4.1), identified with $\mathbb{P}(\mathscr{E}^y)$, where $\mathscr{E}^y = \mathscr{E}_y \otimes_{\mathscr{O}_{Y,y}} \kappa(y) = \mathscr{E}_y/\mathfrak{m}_y\mathscr{E}_y$ is considered as a vector space over $\kappa(y)$. More generally, if K is an extension of $\kappa(y)$, $p^{-1}(y) \otimes_{\kappa(y)} K$ is identified with $\mathbb{P}(\mathscr{E}^y \otimes_{\kappa(y)} K)$. Since any invertible sheaves over a local scheme is trivial, Corollary 11.4.6 shows that the points of $\mathbb{P}(\mathscr{E})$ lying over y with values in K, which are called the **rational geometric fibers** of $\mathbb{P}(\mathscr{E})$ over K lying over y, is identified with the projective space of the dual of the vector K-space $\mathscr{E}^y \otimes_{\kappa(y)} K$.

Example 11.4.8. Suppose now that Y is affine with ring A, and any invertible sheaf on Y is trivial; we put $\mathscr{E} = \mathscr{O}_Y^n$. Then with the notations of Corollary 11.4.6, E is identified with A^n by Corollary 10.1.3 and $E^\times$ is identified with the set of systems $(f_i)_{1 \le i \le n}$ of elements of A which generate the unit ideal of A. By Corollary 11.4.6, two such systems determine the same Y-section of $\mathbb{P}_Y^{n-1} = \mathbb{P}_A^{n-1}$, which means the same point of $\mathbb{P}_A^{n-1}$ with values in A, if and only if one is deduced from the other by multiplication by an invertible element of A.

Remark 11.4.9. Let $u : X' \to X$ be a morphism. If the Y-morphism $r : X \to P$ corresponds to the homomorphism $\varphi : q^*(\mathscr{E}) \to \mathscr{L}$, then by definition, the morphism $r \circ u$ corresponds to $u^*(\varphi) : u^*(q^*(\mathscr{E})) \to u^*(\mathscr{L})$. On the other hand, let $\mathscr{E}, \mathscr{F}$ be two quasi-coherent $\mathscr{O}_Y$-modules and $j = \mathbb{P}(v)$ be the closed immersion $\mathbb{P}(\mathscr{F}) \to \mathbb{P}(\mathscr{E})$ corresponding to a surjective homomorphism $v : \mathscr{E} \to \mathscr{F}$. If the Y-morphism $r : X \to \mathbb{P}(\mathscr{E})$ corresponds to the homomorphism $\varphi : q^*(\mathscr{E}) \to \mathscr{L}$, then $j \circ r$ corresponds to the composition

$$q^*(\mathscr{E}) \xrightarrow{q^*(v)} q^*(\mathscr{F}) \xrightarrow{\varphi} \mathscr{L}$$

Let $\psi : Y' \to Y$ be a morphism and put $\mathscr{E}' = \psi^*(\mathscr{E})$. If the Y-morphism $r : X \to P$ corresponds to the homomorphism $\varphi : q^*(\mathscr{E}) \to \mathscr{L}$, the Y'-morphism

$$r_{(Y')} : X_{(Y')} \to P' = \mathbb{P}(\mathscr{E}')$$

correspond to $\varphi_{(Y')} : q^*_{(Y')}(\mathscr{E}') = q^*(\mathscr{E}) \otimes_Y \mathscr{O}_{Y'} \to \mathscr{L} \otimes_Y \mathscr{O}_{Y'}$. In fact, by (11.4.1), we have the following commutative diagram

$$\begin{tikzcd}
X_{(Y')} \arrow[r, "r_{(Y')}"] \arrow[d, "v"] & P' = P_{(Y')} \arrow[r, "p_{(Y')}"] \arrow[d, "u"] & Y' \arrow[d, "\psi"] \\
X \arrow[r, "r"] & P \arrow[r, "p"] & Y
\end{tikzcd}$$

In view of (11.4.2), we have

$$(r_{(Y')})^*(\mathscr{O}_{P'}(1)) = (r_{(Y')})^*(u^*(\mathscr{O}_P(1))) = v^*(r^*(\mathscr{O}_P(1))) = v^*(\mathscr{L}) = \mathscr{L} \otimes_Y \mathscr{O}_{Y'}.$$

Also, $u^*(\alpha_1^\sharp)$ is equal to the canonical homomorphism $\alpha_1^\sharp : (p_{(Y')})^*(\mathscr{E}') \to \mathscr{O}_{P'}(1)$, in view of the definition of α_1, whence our assertion.

11.4.3 The Segre morphism

Let Y be a scheme and $\mathscr{E}, \mathscr{F}$ be two quasi-coherent $\mathscr{O}_Y$-modules. Put $P_1 = \mathbb{P}(\mathscr{E})$, $P_2 = \mathbb{P}(\mathscr{F})$, and denote by p_1, p_2 their morphisms; let $Q = P_1 \times_Y P_2$ and q_1, q_2 be the canonical projections. The $\mathscr{O}_Q$-module

$$\mathscr{L} = \mathscr{O}_{P_1}(1) \times_Y \mathscr{O}_{P_2}(1) = q_1^*(\mathscr{O}_{P_1}(1)) \times_{\mathscr{O}_Q} q_2^*(\mathscr{O}_{P_2}(1))$$

is invertible as a tensor product of invertible modules. On the other hand, if $r = p_1 \circ q_1 = p_2 \circ q_2$ is the structural morphism of Q, we have $r^*(\mathscr{E} \otimes_{\mathscr{O}_Y} \mathscr{F}) \otimes_{\mathscr{O}_Q} q_2^*(p_2^*(\mathscr{F}))$; the canonical surjective homomorphism $p_1^1(\mathscr{E}) \to \mathscr{O}_{P_1}(1)$ and $p_2^*(\mathscr{F}) \to \mathscr{O}_{P_2}(1)$ then give a canonical surjective homomorphism

$$s : r^*(\mathscr{E} \otimes_{\mathscr{O}_Y} \mathscr{F}) \to \mathscr{L} \tag{11.4.3}$$

we then deduce a canonical homomorphism, called the Segre morphism:

$$\zeta : \mathbb{P}(\mathscr{E}) \times_Y \mathbb{P}(\mathscr{F}) \to \mathbb{P}(\mathscr{E} \otimes_{\mathscr{O}_Y} \mathscr{F}). \tag{11.4.4}$$

To explain this morphism ζ, let us consider the case where $Y = \operatorname{Spec}(A)$ is affine, $\mathscr{E} = \widetilde{E}$, $\mathscr{F} = \widetilde{F}$, where E and F are two A-module; whence $\mathscr{E} \otimes_{\mathscr{O}_Y} \mathscr{F} = \widetilde{E \otimes_A F}$. Put $R = \boldsymbol{S}(E)$, $S = \boldsymbol{S}(F)$, and $T = \boldsymbol{S}(E \otimes_A F)$. Let $f \in E$, $g \in F$, and consider the affine open

$$D_+(f) \times_Y D_+(g) = \operatorname{Spec}(B)$$

of Q, where $B = R_{(f)} \otimes_A S_{(g)}$. The restriction of $\mathscr{L}$ on this affine open is $\widetilde{L}$, where

$$L = (R(1)_{(f)}) \otimes_A (S(1)_{(g)})$$

and the element $c = (f/1) \otimes (g/1)$ is a generator of L as a free B-module (Proposition 11.2.24). The homomorphism (11.4.3) then corresponds to the homomorphism

$$(x \otimes y) \otimes b \mapsto b((x/1) \otimes (y/1))$$

from $(E \otimes_A F) \otimes_A B$ to L. With the notations of Remark 11.3.36, we then have $v_1(x \otimes y) = (x/f) \otimes (y/g)$, so the restriction of the morphism ζ to $D_+(f) \times_Y D_+(g)$ is a morphism from this affine scheme to $D_+(f \otimes g)$, which corresponds to the ring homomorphism

$$\omega((x \otimes y)/(f \otimes g)) = (x/f) \otimes (y/g) \tag{11.4.5}$$

for $x \in E$ and $y \in F$.

From Proposition 11.4.3, there is a canonical isomorphism

$$\tau : \zeta^*(\mathscr{O}_P(1)) \xrightarrow{\sim} \mathscr{O}_{P_1}(1) \otimes_Y \mathscr{O}_{P_2}(1)$$

where we put $P = \mathbb{P}(\mathscr{E} \otimes_{\mathscr{O}_Y} \mathscr{F})$. Moreover, for $x \in \Gamma(Y, \mathscr{E})$ and $y \in \Gamma(Y, \mathscr{F})$, we have

$$\tau(\alpha_1(x \otimes y)) = \alpha_1(x) \otimes \alpha_1(y) \tag{11.4.6}$$

To see this, we can assume that Y is affine, so with the notations above and the definition of α_1, we have $\alpha_1^{f\otimes g}(x \otimes y) = (x \otimes y)/1$, $\alpha_1^f(x) = x/1$, and $\alpha_1^g(y) = y/1$. The definition of τ given in the proof of Proposition 11.4.3 says τ maps $(x/1) \otimes 1$ to $v_1(x)c$. Since we have seen that $v_1(x \otimes y) = (x/f) \otimes (y/g)$, this implies the assertion by a simple computation. From this, we then deduce the formula

$$\zeta^{-1}(P_{x \otimes y}) = (P_1)_x \times_Y (P_2)_y \tag{11.4.7}$$

where we need to use the following lemma:

Lemma 11.4.10. *Let B, B' be two A-algebras, and let $Y = \mathrm{Spec}(A)$, $Z = \mathrm{Spec}(B)$, $Z' = \mathrm{Spec}(B')$. Then for $t \in B$, $t' \in B'$, we have $D(t \otimes t') = D(t) \times_Y D(t')$.*

Proof. Let p, p' be the canonical projections of $Z \times_Y Z'$. Then it follows from Proposition 1.4.20 that $p^{-1}(D(t)) = D(t \otimes 1)$ and $p'^{-1}(D(t')) = D(1 \otimes t')$. Corollary 10.3.2 then implies the lemma, since $(t \otimes 1)(1 \otimes t') = t \otimes t'$. □

Proposition 11.4.11. *The Segre morphism is a closed immersion.*

Proof. Since the question is local on Y, we can assume that Y is affine. With the previous notations, the $D_+(f \otimes g)$ form a basis for P, since the elements $f \otimes g$ generate T for $f \in E$, $g \in F$. On the other hand, we have $\zeta^{-1}(D_+(f \otimes g)) = D_+(f) \times_Y D_+(g)$ in view of (11.4.7). It then suffices to use Corollary 10.4.11 to prove that the restriction of ζ to $D_+(f) \times_Y D_+(g)$ is a closed immersion into $D_+(f \otimes g)$. But this is a morphism between affine schemes whose corresponding ring homomorphism ω is surjective in view of the formula (11.4.5), so our assertion follows. □

The Segre morphism is functorial on $\mathscr{E}$ and $\mathscr{F}$ if we restrict ourselves to quasi-coherent $\mathscr{O}_Y$-modules with *surjective* homomorphisms. To see this, it suffices to consider a surjective $\mathscr{O}_Y$-homomorphism $\mathscr{E} \to \mathscr{E}'$ and prove that the following diagram is commutative

$$\begin{array}{ccc} \mathbb{P}(\mathscr{E}') \times \mathbb{P}(\mathscr{F}) & \xrightarrow{j \times 1} & \mathbb{P}(\mathscr{E}') \times \mathbb{P}(\mathscr{F}) \\ {\scriptstyle \zeta}\big\downarrow & & \big\downarrow{\scriptstyle \zeta} \\ \mathbb{P}(\mathscr{E}' \otimes \mathscr{F}) & \longrightarrow & \mathbb{P}(\mathscr{E} \otimes \mathscr{F}) \end{array}$$

where j is the canonical closed immersion $\mathbb{P}(\mathscr{E}') \to \mathbb{P}(\mathscr{E})$. Put $P_1' = \mathbb{P}(\mathscr{E}')$ and retain the previous notations; $j \times 1$ is a closed immersion by Proposition 11.3.33 and we have

$$(j \times 1)^*(\mathscr{O}_{P_1}(1) \otimes \mathscr{O}_{P_2}(1)) = j^*(\mathscr{O}_{P_1}(1)) \otimes \mathscr{O}_{P_2}(1) = \mathscr{O}_{P_1'}(1) \otimes \mathscr{O}_{P_2}(1)$$

in view of (11.4.2) and Corollary 10.3.16. The assertion then follows from Remark 11.4.9.

Proposition 11.4.12. *Let $\psi : Y \to Y'$ be a morphism and put $\mathscr{E}' = \psi^*(\mathscr{E})$, $\mathscr{F}' = \psi^*(\mathscr{F})$. Then the Segre morphism $\mathbb{P}(\mathscr{E}') \times \mathbb{P}(\mathscr{F}') \to \mathbb{P}(\mathscr{E}' \otimes \mathscr{F}')$ is identified with $\zeta_{(Y')}$.*

Proof. Put $P_1' = \mathbb{P}(\mathscr{E}')$ and $P_2' = \mathbb{P}(\mathscr{F}')$. Then by Remark 11.4.9, P_i' is identified with $(P_i)_{(Y')}$ for $i = 1, 2$, so the structural morphism $P_1' \times_{Y'} P_2' \to Y'$ is identified with $r_{(Y')}$, where r is the structural morphism of $P_1 \times_Y P_2$. On the other hand, $\mathscr{E}' \otimes \mathscr{F}'$ is identified with $\psi^*(\mathscr{E} \otimes \mathscr{F})$, so $\mathbb{P}(\mathscr{E}' \otimes \mathscr{F}')$ is identified with $\mathbb{P}(\mathscr{E} \otimes \mathscr{F})_{(Y')}$ by Proposition 11.3.31. Finally, $\mathscr{O}_{P_1'}(1) \otimes_{Y'} \mathscr{O}_{P_2'}(1) = \mathscr{L}$ is identified with $\mathscr{L} \otimes_Y \mathscr{O}_{Y'}$ in view of (11.4.2) and Proposition 10.3.15. The canonical homomorphism $(r_{(Y')})^*(\mathscr{E}' \otimes \mathscr{F}') \to \mathscr{L}'$ is then identified with $s_{(Y')}$, and our assertion follows from Proposition 11.4.3. □

Remark 11.4.13. The coproduct of $\mathbb{P}(\mathscr{E})$ and $\mathbb{P}(\mathscr{F})$ is similarly canonical isomorphic to a closed subscheme of $\mathbb{P}(\mathscr{E} \oplus \mathscr{F})$. In fact, the surjective homomorphisms $\mathscr{E} \oplus \mathscr{F} \to \mathscr{E}$, $\mathscr{E} \oplus \mathscr{F} \to \mathscr{F}$ correspond to closed immersions $\mathbb{P}(\mathscr{E}) \to \mathbb{P}(\mathscr{E} \oplus \mathscr{F})$, $\mathbb{P}(\mathscr{E}) \to \mathbb{P}(\mathscr{E} \oplus \mathscr{F})$; it then boils down to showing that the underlying spaces of these closed subschemes of $\mathbb{P}(\mathscr{E} \oplus \mathscr{F})$ thus obtained have no common point. The question being local on Y, we can assume that Y is affine adapt our previous notations. Now $\boldsymbol{S}_n(E)$ and $\boldsymbol{S}_n(F)$ are identified with submodules of $S_n(E \oplus F)$ with intersection reduced to 0, and if $\mathfrak{p}$ is a graded prime ideal of $\boldsymbol{S}(E)$ such that $\mathfrak{p} \cap S_n(E) \neq S_n(E)$ for all $n \geq 0$, then it corresponds to a unique graded prime ideal in $\boldsymbol{S}(E \oplus F)$ whose trace on $\boldsymbol{S}_n(E)$ is $\mathfrak{p} \cap S_n(E)$, but which contains $S_+(F)$. Therefore, two distinct points of $\mathrm{Proj}(\boldsymbol{S}(E))$ and $\mathrm{Proj}(\boldsymbol{S}(F))$ can not have same image in $\mathrm{Proj}(\boldsymbol{S}(E \oplus F))$.

11.4.4 Very ample sheaves

Proposition 11.4.14. *Let Y be a quasi-compact scheme, $q : X \to Y$ a morphism of finite type, and $\mathscr{L}$ be an invertible $\mathscr{O}_X$-module.*

(a) *Let $\mathscr{S}$ be a quasi-coherent graded $\mathscr{O}_Y$-algebra and $\psi : q^*(\mathscr{S}) \to \boldsymbol{S}(\mathscr{L})$ be a graded homomorphism of algebras. For the morphism $r_{\mathscr{L},\psi}$ to be everywhere defined and an immersion, it is necessary and sufficent that there exists an integer $n \geq 0$ and a quasi-coherent sub-$\mathscr{O}_Y$-module of finite type $\mathscr{E}$ of $\mathscr{S}_n$ such that the homomorphism $\varphi' = \psi_n \circ q^*(j) : q^*(\mathscr{E}) \to \boldsymbol{S}(\mathscr{L}) = \mathscr{L}'$ ($j : \mathscr{E} \to \mathscr{S}_n$ being the canonical injection) is surjective and the morphism $r_{\mathscr{L}',\varphi'} : X \to \mathbb{P}(\mathscr{E})$ is an immersion.*

(b) *Let $\mathscr{F}$ be a quasi-coherent $\mathscr{O}_Y$-module and $\varphi : q^*(\mathscr{F}) \to \mathscr{L}$ be a surjective homomorphism. For the morphism $r_{\mathscr{L},\varphi} : X \to \mathbb{P}(\mathscr{F})$ to be an immersion, it is necessary and sufficent that there exists a quasi-coherent sub-$\mathscr{O}_Y$-module of finite type $\mathscr{E}$ of $\mathscr{F}$ such that the homomorphism $\varphi' = \varphi \circ q(j) : q^*(\mathscr{E}) \to \mathscr{L}$ (where $j : \mathscr{E} \to \mathscr{F}$ is the canonical injection) is surjective and such that $r_{\mathscr{L},\varphi'} : X \to \mathbb{P}(\mathscr{E})$ is an immersion.*

Proof. We first consider case (a). The fact that $r_{\mathscr{L},\psi}$ is everywhere defined and an immersion is equivalent by Proposition 11.3.48 to the existence of an integer $n > 0$ and $\mathscr{E}$ such that, if $\mathscr{S}'$ is the subalgebra of $\mathscr{S}$ generated by $\mathscr{E}$, the homomorphism $q^*(\mathscr{E}) \to \mathscr{L}^{\otimes n}$ is surjective and $r_{\mathscr{L},\psi'} : X \to \operatorname{Proj}(\mathscr{S}')$ is everywhere defined and an immersion. We also have a closed immersion corresponding to the surjective homomorphism $\boldsymbol{S}(\mathscr{E}) \to \mathscr{S}'$, so these the morphism $X \to \mathbb{P}(\mathscr{E})$ is an immersion.

Now consider the situation of (b). As $\mathscr{F}$ is the inductive limit of its quasi-coherent submodules of finite type $\mathscr{E}_\lambda$ (Corollary 10.6.65), $\boldsymbol{S}(\mathscr{F})$ is the inductive limit of the $\boldsymbol{S}(\mathscr{E}_\lambda)$, so by Proposition 11.3.47 there exists λ such that $r_{\mathscr{L},\varphi_\mu}$ is everywhere defined and an immersion for $\mu \geq \lambda$. Also, since the functor f^* is left-adjoint, it commutes with inductive limits and therefore $q^*(\mathscr{F})$ is the inductive limits of the $q^*(\mathscr{E}_\lambda)$. Since $\mathscr{L}$ is an $\mathscr{O}_X$-module of finite type and $q^*(\mathscr{F}) \to \mathscr{L}$ is surjective, it follows from **??** that there exists λ' such that $q^*(\mathscr{E}_\mu) \to \mathscr{L}$ is surjective for $\mu \geq \lambda'$. It then suffices to choose $\mathscr{E} = \mathscr{E}_\mu$ for $\mu \geq \lambda$ and $\mu \geq \lambda'$. □

Let Y be a scheme and $q : X \to Y$ be a morphism. We say an invertible $\mathscr{O}_X$-module $\mathscr{L}$ is **very ample for q** (or **very ample relative to q**, or simple **very ample**) if there exists a quasi-coherent $\mathscr{O}_Y$-module $\mathscr{E}$ and a Y-immersion $r : X \to \mathbb{P}(\mathscr{E})$ such that $\mathscr{L}$ is isomorphic to $r^*(\mathscr{O}_P(1))$. In view of Proposition 11.4.3, this is equivalent to the existence of a quasi-coherent $\mathscr{O}_Y$-module $\mathscr{E}$ and a surjective homomorphism $\varphi : q^*(\mathscr{E}) \to \mathscr{L}$ such that the associated morphism $r_{\mathscr{L},\varphi} : X \to \mathbb{P}(\mathscr{E})$ is an immersion. We also note that the existence of a very ample $\mathscr{O}_X$-module relative to Y implies that q is separated (Proposition 11.2.14 and Proposition 10.5.26).

Corollary 11.4.15. *Suppose that there exists a quasi-coherent $\mathscr{O}_Y$-algebra $\mathscr{S}$ genrated by $\mathscr{S}_1$ and a Y-immersion $r : X \to P = \operatorname{Proj}(\mathscr{S})$ such that $\mathscr{L} \cong r^*(\mathscr{O}_P(1))$. Then $\mathscr{L}$ is very ample relative to q.*

Proof. If $\mathscr{F} = \mathscr{S}_1$, the canonical homomorphism $\boldsymbol{S}(\mathscr{F}) \to \mathscr{S}$ is surjective, so by composing r with the corresponding closed immersion $\operatorname{Proj}(\mathscr{S}) \to \mathbb{P}(\mathscr{F})$, we obtain an immersion $r' : X \to \mathbb{P}(\mathscr{F}) = P'$ such that $\mathscr{L}$ is isomorphic to $r'^*(\mathscr{O}_{P'}(1))$. □

Proposition 11.4.16. *Let $q : X \to Y$ be a quasi-compact morphism and $\mathscr{L}$ be an invertible $\mathscr{O}_X$-module. Then $\mathscr{L}$ is very ample relative to q if and only if $q_*(\mathscr{L})$ is quasi-coherent, the cannical homomorphism $\sigma : q^*(q_*(\mathscr{L})) \to \mathscr{L}$ is surjective, and the morphism $r_{\mathscr{L},\sigma} : X \to \mathbb{P}(q_*(\mathscr{L}))$ is an immersion.*

Proof. As q is quasi-compact, $q_*(\mathscr{L})$ is quasi-coherent if q is separated (Proposition 10.6.55). By Remark 11.3.28, the existence of a surjective homomorphism $\varphi : q^*(\mathscr{E}) \to \mathscr{L}$ ($\mathscr{E}$ being a quasi-coherent $\mathscr{O}_Y$-module) implies that σ is surjective. Moreover, the factorization $\varphi : q^*(\mathscr{E}) \to$

$q^*(q_*(\mathscr{L})) \xrightarrow{\sigma} \mathscr{L}$ of (**??**) corresponds to a canonical factorization (recall that q^* commutes with $\boldsymbol{S}$)

$$q^*(\boldsymbol{S}(\mathscr{E})) \longrightarrow q^*(\boldsymbol{S}(q_*(\mathscr{L}))) \longrightarrow \boldsymbol{S}(\mathscr{L})$$

so by Corollary 11.3.46 the hypothesis that $r_{\mathscr{L},\varphi}$ is an immersion implies that $j = r_{\mathscr{L},\sigma}$ is an immersion. Moreover, by Proposition 11.4.3, $\mathscr{L}$ is isomorphic to $j^*(\mathscr{O}_{P'}(1))$ where $P' = \mathbb{P}(q_*(\mathscr{L}))$. The converse of this is clear by the definition of very ampleness. □

Corollary 11.4.17. *Let $q : X \to Y$ be a quasi-compact morphism and $\mathscr{L}$ be an invertible $\mathscr{O}_X$-module. For $\mathscr{L}$ to be very ample relative to Y, it is necessary and sufficent that there exists an open covering (U_α) of Y such that $\mathscr{L}|_{q^{-1}(U_\alpha)}$ is very ample relative to U_α for each α.*

Proof. This follows from the fact that the criterion of Proposition 11.4.16 is local over Y. □

Proposition 11.4.18. *Let Y be a quasi-compact scheme, $q : X \to Y$ a morphism of finite type, and $\mathscr{L}$ be an invertible $\mathscr{O}_X$-module. Then the following conditions are equivalent:*

(i) *$\mathscr{L}$ is very ample relative to Y.*

(ii) *There exists a quasi-coherent $\mathscr{O}_Y$-module $\mathscr{E}$ of finite type and a surjective homomorphism $\varphi : q^*(\mathscr{E}) \to \mathscr{L}$ such that $r_{\mathscr{L},\varphi}$ is an immersion.*

(iii) *There exists a coherent sub-$\mathscr{O}_Y$-module of finite type $\mathscr{E}$ of $q_*(\mathscr{L})$ and a surjective homomorphism $\varphi : q^*(\mathscr{E}) \to \mathscr{L}$ such that $r_{\mathscr{L},\varphi}$ is an immersion.*

Proof. It is clear that (ii) or (iii) implies (i); but (i) implies (ii) by Proposition 11.4.14, and similarly (i) implies (iii) in view of Proposition 11.4.16. □

Corollary 11.4.19. *Suppose that Y is a quasi-compact scheme. If $\mathscr{L}$ is very ample relative to Y, there exists a quasi-coherent graded $\mathscr{O}_Y$-algebra $\mathscr{S}$ generated by $\mathscr{S}_1$ and such that $\mathscr{S}_1$ is of finite type, and a dominant open Y-immersion $i : X \to P = \mathrm{Proj}(\mathscr{S})$ such that $\mathscr{L}$ is isomorphic to $i^*(\mathscr{O}_P(1))$.*

Proof. Since $\mathscr{L}$ is very ample, by Proposition 11.4.18 there exists a coherent sub-$\mathscr{O}_Y$-module of finite type $\mathscr{E}$ of $q_*(\mathscr{L})$ and a surjective homomorphism $\varphi : q^*(\mathscr{E}) \to \mathscr{L}$ such that $r_{\mathscr{L},\varphi}$ is an immersion. The structural morphism $p : P' = \mathbb{P}(\mathscr{E}) \to Y$ is then separated and of finite type (Proposition 11.3.23), so P' is a quasi-compact scheme if Y is quasi-compact. Let Z be scheme theoretic image of X in P', with underlying space $\overline{j(X)}$, where $j = r_{\mathscr{L},\varphi}$; then j factors through Z into a dominant open immersion $i : X \to Z$. But Z is identified with the scheme $\mathrm{Proj}(\mathscr{S})$, where $\mathscr{S}$ is the quotient graded $\mathscr{O}_Y$-algebra of $\boldsymbol{S}(\mathscr{E})$ by a quasi-coherent graded ideal (Proposition 11.3.33), and it is clear that $\mathscr{S}_1$ is generated by $\mathscr{S}$ (since $\boldsymbol{S}(\mathscr{E})$ satisfies this condition). Moreover, by Corollary 11.3.34, $\mathscr{O}_Z(1)$ is the inverse image of $\mathscr{O}_{P'}(1)$ under the canonical injection, so $\mathscr{L} = i^*(\mathscr{O}_Z(1))$. □

Proposition 11.4.20. *Let $q : X \to Y$ be a morphism, $\mathscr{L}$ be a very ample $\mathscr{O}_X$-module relative to q, and $\mathscr{L}'$ be an invertible $\mathscr{O}_X$-module such that there exists a quasi-coherent $\mathscr{O}_Y$-module $\mathscr{E}'$ and a surjective homomorphism $q^*(\mathscr{E}') \to \mathscr{L}'$. Then $\mathscr{L} \otimes_{\mathscr{O}_X} \mathscr{L}'$ is very ample relative to q.*

Proof. The hypothesis on $\mathscr{L}'$ implies the existence of an Y-morphism $r' : X \to P' = \mathbb{P}(\mathscr{E}')$ such that $\mathscr{L}'$ is isomorphic to $r'^*(\mathscr{O}_{P'}(1))$ (Proposition 11.4.3). There is by hypothesis a quasi-coherent $\mathscr{O}_Y$-module $\mathscr{E}$ and a Y-immersion $r : X \to P = \mathbb{P}(\mathscr{E})$ such that $\mathscr{L}$ is isomorphic to $r^*(\mathscr{O}_P(1))$. Consider the Segre morphism $\zeta : P \times_Y P' \to Q$ where $Q = \mathbb{P}(\mathscr{E} \otimes \mathscr{E}')$. As r is an immersions, so is the morphism $(r, r')_Y : X \to P \times_Y P'$ by Corollary 10.5.16, and therefore we get an immersion

$$r'' : X \xrightarrow{(r,r')_Y} P \times_Y P' \xrightarrow{\zeta} Q.$$

Since $\zeta^*(\mathscr{O}_Q(1))$ is isomorphic to $\mathscr{O}_P(1) \otimes_Y \mathscr{O}_{P'}(1)$, we conclude from Corollary 10.3.16 that $r''^*(\mathscr{O}_Q(1))$ is isomorphic to $\mathscr{L} \otimes \mathscr{L}'$, this proves the assertion. □

Remark 11.4.21. Note that $q^*(\mathscr{O}_Y^{\oplus I}) = \mathscr{O}_X^{\oplus I}$ and there exists a surjection $\mathscr{O}_X^{\oplus I} \to \mathscr{L}'$ if and only if $\mathscr{L}'$ is generated by global sections, so Proposition 11.4.20 is applicable if $\mathscr{L}'$ is generated by global sections.

Corollary 11.4.22. *Let $q : X \to Y$ be a morphism.*

(a) *Let $\mathscr{L}$ be an invertible $\mathscr{O}_X$-module and $\mathscr{K}$ be an invertible $\mathscr{O}_Y$-module. For $\mathscr{L}$ to be very ample relative to q, it is necessary and sufficent that $\mathscr{L} \otimes q^*(\mathscr{K})$ is very ample relative to q.*

(b) *If $\mathscr{L}$ and $\mathscr{L}'$ are two invertible $\mathscr{O}_X$-modules that are very ample relative to q, then so is $\mathscr{L} \otimes \mathscr{L}'$. In particular, $\mathscr{L}^{\otimes n}$ is very ample relative to q for any $n > 0$.*

Proof. The assertions in (b) is an immediate concequence of Proposition 11.4.20, so is the half implication of (a). Now assume that $\mathscr{L} \otimes q^*(\mathscr{K})$ is very ample; then so is $(\mathscr{L} \otimes q^*(\mathscr{K})) \otimes q^*(\mathscr{K}^{-1})$ by Proposition 11.4.20, which is isomorphic to $\mathscr{L}$. □

Proposition 11.4.23. *Let $f : X \to Y$ be a quasi-compact morphism where Y is quasi-compact, and $\mathscr{L}$ be a very ample $\mathscr{O}_X$-module relative to f. Let $\mathscr{F}$ be a quasi-coherent $\mathscr{O}_X$-module of finite type and for each $n \in \mathbb{Z}$, set $\mathscr{F}(n) = \mathscr{F} \otimes \mathscr{L}^{\otimes n}$. Then there exists an integer n_0 such that for $n \geq n_0$, the canonical homomorphism $\sigma : f^*(f_*(\mathscr{F}(n))) \to \mathscr{F}(n)$ is surjective.*

Proof. □

Proposition 11.4.24. *Let $f : X \to Y$ be a quasi-compact morphism where Y is quasi-compact, and $\mathscr{L}$ be a very ample $\mathscr{O}_X$-module relative to f. Let $\mathscr{F}$ be a quasi-coherent $\mathscr{O}_X$-module of finite type. Then there exists an integer n_0 such that, for $n \geq n_0$, $\mathscr{F}$ is isomorphic to a quotient of an $\mathscr{O}_X$-module $f^*(\mathscr{G}) \otimes_{\mathscr{O}_X} \mathscr{L}^{\otimes(-n)}$, where $\mathscr{G}$ is a quasi-coherent $\mathscr{O}_Y$-module of finite type (dependent on n).*

Proof. Since $\mathscr{L}$ is very ample, f is separated and by Proposition 11.4.23 the canonical homomorphism $\sigma : f^*(f_*(\mathscr{F}(n))) \to \mathscr{F}(n)$ is surjective for n sufficiently large. The proposition is then a generalization of Corollary 11.3.29, and can be proved similarly. □

Proposition 11.4.25 (Properties of Very Ample Sheaves).

(i) *For any scheme Y, any invertible $\mathscr{O}_Y$-module $\mathscr{L}$ is very ample relative to the identify morphism 1_Y.*

(ii) *Let $f : X \to Y$ be a morphism and $j : X' \to X$ be an immersion. If $\mathscr{L}$ is a very ample $\mathscr{O}_X$-module relative to f, then $j^*(\mathscr{L})$ is very ample relative to $f \circ j$.*

(iii) *Let $f : X \to Y$ be a morphism of finite type and $g : Y \to Z$ be a quasi-compact morphism where Z is quasi-compact. Let $\mathscr{L}$ a very ample $\mathscr{O}_X$-module relative to f and $\mathscr{K}$ be a very ample $\mathscr{O}_Y$-module relative to g. Then there exists an integer $n_0 > 0$ such that for any $n \geq n_0$, $\mathscr{L} \otimes f^*(\mathscr{K}^{\otimes n})$ is very ample relative to $g \circ f$.*

(iv) *Let $f : X \to Y$, $g : Y' \to Y$ be two morphisms, and put $X' = X_{(Y')}$. If $\mathscr{L}$ is a very ample $\mathscr{O}_X$-module relative to f, $\mathscr{L}' = \mathscr{L} \otimes_Y \mathscr{O}_{Y'}$ is very ample relative to $f_{(Y')}$.*

(v) *Let $f_i : X_i \to Y_i$ $(i = 1, 2)$ be two S-morphisms. If $\mathscr{L}_i$ is a very ample $\mathscr{O}_{X_i}$-module relative to f_i, then $\mathscr{L}_1 \otimes_S \mathscr{L}_2$ is very ample relative to $f_1 \times_S f_2$.*

(vi) *Let $f : X \to Y$, $g : Y \to Z$ be two morphisms. If an $\mathscr{O}_X$-module $\mathscr{L}$ is very ample relative to $g \circ f$, then $\mathscr{L}$ is very ample relative to f.*

(vii) *Let $f : X \to Y$ be a morphism and $j : X_{\mathrm{red}} \to X$ be the canonical injection. If $\mathscr{L}$ is a very ample $\mathscr{O}_X$-module relative to f, then $j^*(\mathscr{L})$ is very ample relative to f_{red}.*

Proof. The property (ii) follows from the definition and it is immedaite that (vii) is deduced from (ii) and (vi). To prove (vi), consider the factorization

$$X \xrightarrow{\Gamma_f} X \times_Z Y \xrightarrow{p_2} Y$$

and note that $p_2 = (g \circ f) \times 1_Y$. It follows from the hypothesis and from (i) and (v) that $\mathscr{L} \otimes_{\mathscr{O}_Z} \mathscr{O}_Y$ is very ample relative to p_2. On the other hand, we have $\mathscr{L} = \Gamma_f^*(\mathscr{L} \otimes_{\mathscr{O}_Z} \mathscr{O}_Y)$ by Corollary 10.3.16, and Γ_f is an immersion (Corollary 10.5.8), so we can apply (ii). As for (i), we can apply the definition with $\mathscr{E} = \mathscr{L}$, and note that $\mathbb{P}(\mathscr{L})$ is identified with Y (Proposition 11.3.6).

We now prove (iv). Under the hypothesis of (iv), there exists a quasi-coherent $\mathscr{O}_Y$-module $\mathscr{E}$ and a Y-immersion $i : X \to P = \mathbb{P}(\mathscr{E})$ such that $\mathscr{L} = i^*(\mathscr{O}_P(1))$. Then, if $\mathscr{E}' = g^*(\mathscr{E})$, $\mathscr{E}'$ is a quasi-coherent $\mathscr{O}_Y$-module and we have $P' = \mathbb{P}(\mathscr{E}') = P_{(Y')}$, $i_{(Y')}$ is an immersion from $X_{(Y')}$ to P', and $\mathscr{L}'$ is isomorphic to $(i_{(Y')})^*(\mathscr{O}_{P'}(1))$ (Remark 11.4.9).

To prove (v), remark that there exists by hypothesis a Y_i-immersion $r_i : X_i \to P_i = \mathbb{P}(\mathscr{E}_i)$, where $\mathscr{E}_i$ is a quasi-coherent $\mathscr{O}_{Y_i}$-module, and $\mathscr{L}_i = r_i^*(\mathscr{O}_{P_i}(1))$. Then $r_1 \times_S r_2$ is an S-immersion of $X_1 \times_S X_2$ to $P_1 \times_S P_2$ (Proposition 10.4.13) and the inverse image of $\mathscr{O}_{P_1}(1) \otimes_S \mathscr{O}_{P_2}(1)$ by this immersion is $\mathscr{L}_1 \otimes_S \mathscr{L}_2$. On the other hand, put $T = Y_1 \times_S Y_2$, and let p_1, p_2 be the projection of T, respectively. If $P_i' = \mathbb{P}(p_i^*(\mathscr{E}_i))$, we have $P_i' = P_i \times_{Y_i} T$, whence

$$P_1' \times_T P_2' = (P_1 \times_{Y_1} T) \times_T (P_2 \times_{Y_2} T) = P_1 \times_{Y_1} (T \times_{Y_2} P_2) = P_1 \times_{Y_1} (Y_1 \times_S P_2) = P_1 \times_S P_2.$$

Similarly, we have $\mathscr{O}_{P_i'}(1) = \mathscr{O}_{P_i'}(1) \otimes_{Y_i} \mathscr{O}_T$, and $\mathscr{O}_{P_1'}(1) \otimes_T \mathscr{O}_{P_2'}(1)$ is identified with $\mathscr{O}_{P_1}(1) \otimes_S \mathscr{O}_{P_2}(1)$. We can then consider $r_1 \times_S r_2$ as an T-immersion from $X_1 \times_S X_2$ to $P_1' \times_T P_2'$, the inverse image of $\mathscr{O}_{P_1'}(1) \otimes_T \mathscr{O}_{P_2'}(1)$ by this immersion being $\mathscr{L}_1 \otimes_S \mathscr{L}_2$. We can then conclude as in Proposition 11.4.20 that $\mathscr{L}_1 \otimes_S \mathscr{L}_2$ is very ample.

It remains to prove (iii). We can first restrict to the case where Z is an affine scheme, since there exists a finite covering (U_i) of Z by affine opens; if the property is proved for $\mathscr{K}|_{g^{-1}(U_i)}$, $\mathscr{L}|_{f^{-1}(g^{-1}(U_i))}$ and an integer n_i, it suffices to choose n_0 to be the largest n_i to prove the property for $\mathscr{K}$ and $\mathscr{L}$ (Corollary 11.4.17). The hypotheses imply that f, g are spearated morphisms, so X and Y are quasi-compact schemes. Since $\mathscr{L}$ is very ample relative to f, there exists an immersion $r : X \to P = \mathbb{P}(\mathscr{E})$, where $\mathscr{E}$ is a quasi-coherent $\mathscr{O}_Y$-module of finite type and $\mathscr{L} = r^*(\mathscr{O}_P(1))$, in view of Proposition 11.4.18. We claim that there exists an integer m_0 such that for any $m \geq m_0$, there is a very ample $\mathscr{O}_P$-module $\mathscr{M}$ relative to the composition morphism $j : P \to Y \xrightarrow{g} Z$ such that $\mathscr{O}_P(1)$ is isomorphic to $\mathscr{M} \otimes_Y \mathscr{K}^{\otimes(-m)}$. For $n \geq m+1$, $\mathscr{O}_P(1) \otimes_Y \mathscr{K}^{\otimes n}$ will then be very ample relative to Z in view of the hypothesis and applying (v) to the morphism $j : P \to Z$ and 1_Z; as r is an immersion and $\mathscr{L} \otimes f^*(\mathscr{K}^{\otimes n}) = r^*(\mathscr{O}_P(1) \otimes_Y \mathscr{K}^{\otimes n})$, the conclusion then follows from (ii).

To estabilsh the claim, we can use Proposition 11.4.24 to obtain a closed immersion j_1 of P to $P_1 = \mathbb{P}(g^*(\mathscr{F}) \otimes \mathscr{K}^{\otimes(-m)})$ such that $\mathscr{O}_P(1)$ is isomorphic to $j_1^*(\mathscr{O}_{P_1}(1))$ (Proposition 11.3.33). On the other hand, there is an isomorphism from P_1 to $P_2 = \mathbb{P}(g^*(\mathscr{F}))$, identifying $\mathscr{O}_{P_1}(1)$ with $\mathscr{O}_{P_2}(1) \otimes_Y \mathscr{K}^{\otimes(-m)}$ (Proposition 11.4.1); we then have a closed immersion $j_2 : P \to P_2$ such that $\mathscr{O}_P(1)$ is isomorphic to $j_2^*(\mathscr{O}_{P_2}(1)) \otimes_Y \mathscr{K}^{\otimes(-m)}$. Finally, P_2 is identified with $P_3 \times_Z Y$ where $P_3 = \mathbb{P}(\mathscr{F})$, and $\mathscr{O}_{P_2}(1)$ is identified with $\mathscr{O}_{P_3}(1) \otimes_Z \mathscr{O}_Y$ (11.4.2). By definition, $\mathscr{O}_{P_3}(1)$ is very ample for Z, and so is $\mathscr{K}$, so it follows from (v) that the $\mathscr{O}_{P_2}$-module $\mathscr{O}_{P_3}(1) \otimes_Y \mathscr{K}$ is very ample for Z. In view of (ii), $\mathscr{M} = j_2^*(\mathscr{O}_{P_3}(1) \otimes_Y \mathscr{K})$ is then very ample for Z, and $\mathscr{O}_P(1)$ is isomorphic to $\mathscr{M} \otimes_Y \mathscr{K}^{\otimes(-m-1)}$, whence the demonstration. □

Proposition 11.4.26. *Let $f : X \to Y$, $f' : X' \to Y$ be two morphisms, $X'' = X \amalg X'$, and $f'' = f \amalg f'$. Let $\mathscr{L}$ (resp. $\mathscr{L}'$) be an invertible $\mathscr{O}_X$-module (resp. $\mathscr{O}_{X'}$-module), and let $\mathscr{L}''$ be the invertible $\mathscr{O}_{X''}$-module which coincides with $\mathscr{L}$ over X and with $\mathscr{L}'$ over X'. For $\mathscr{L}''$ to be very ample relative to f'', it is necessary and sufficent that $\mathscr{L}$ is very ample relative to f and $\mathscr{L}'$ is very ample relative to f'.*

Proof. We can assume that Y is affine. If $\mathscr{L}''$ is very ample then so is $\mathscr{L}$ and $\mathscr{L}'$ in view of Proposition 11.4.25(ii). Conversely, if $\mathscr{L}$ and $\mathscr{L}'$ are very ample, it follows from Remark 11.4.13 that $\mathscr{L}''$ is very ample. $\square$

11.4.5 Ample sheaves

Let X be a scheme, $\mathscr{L}$ be an invertible $\mathscr{O}_X$-module, and $\mathscr{F}$ be an $\mathscr{O}_X$-module. For any $n \in \mathbb{Z}$, we set $\mathscr{F}(n) = \mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{L}^{\otimes n}$ (if there is no confusion), and $S = \bigoplus_{n \geq 0} \Gamma(X, \mathscr{L}^{\otimes n})$. If we consider X as a $\mathbb{Z}$-scheme and let $p : X \to Y = \operatorname{Spec}(\mathbb{Z})$ be the structural morphism, there are bijections

$$\operatorname{Hom}_{\mathsf{Qcoh}(X)}(p^*(\widetilde{S}), \boldsymbol{S}(\mathscr{L})) \xrightarrow{\sim} \operatorname{Hom}_{\mathsf{Qcoh}(Y)}(\widetilde{S}, p_*(\boldsymbol{S}(\mathscr{L}))) \xrightarrow{\sim} \operatorname{Hom}_{\mathbb{Z}}(S, \Gamma(X, \boldsymbol{S}(\mathscr{L})))$$

where we use Proposition 10.2.6. The homomorphism $\varepsilon : p^*(\widetilde{S}) \to \boldsymbol{S}(\mathscr{L})$ corresponding to the canonical injection of S into $\Gamma(X, \boldsymbol{S}(\mathscr{L}))$ is called the **canonical homomorphism associated with** $\mathscr{L}$. It then corresponds to a canonical morphism

$$r_{\mathscr{L},\varepsilon} : G(\varepsilon) \to \operatorname{Proj}(S). \tag{11.4.8}$$

Theorem 11.4.27. *Let X be a quasi-compact and quasi-separated scheme, $\mathscr{L}$ be an invertible $\mathscr{O}_X$-module and $S = \bigoplus_{n \geq 0} \Gamma(X, \mathscr{L}^{\otimes n})$. The following conditions are equivalent:*

(i) *The subsets X_f, as f runs through the set of homogeneous elements of S_+, form a basis for X.*

(i') *The subsets X_f which are affine, as f runs through the set of homogeneous elements of S_+, form a basis for X.*

(ii) *The canonical morphism $r_{\mathscr{L},\varepsilon} : G(\varepsilon) \to \operatorname{Proj}(S)$ is everywhere defined and a dominant open immersin.*

(ii') *The canonical morphism $r_{\mathscr{L},\varepsilon} : G(\varepsilon) \to \operatorname{Proj}(S)$ is everywhere defined and a homeomorphism from X onto a subspace of* $\operatorname{Proj}(S)$.

(iii) *For any quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, if $\mathscr{F}_n$ is the sub-$\mathscr{O}_X$-module of $\mathscr{F}(n)$ generated by the sections of $\mathscr{F}(n)$ over X, then $\mathscr{F}$ is the direct sum of the sub-$\mathscr{O}_X$-modules $\mathscr{F}_n(-n)$ for $n > 0$.*

(iii') *Property (iii) holds for any quasi-coherent ideal of $\mathscr{O}_X$.*

Moreover, in this case, if (f_α) is a family of homogeneous elements of S_+ such that X_{f_α} is affine, then the restriction to $\bigcup_\alpha X_{f_\alpha}$ of the canonical morphism $r_{\mathscr{L},\varepsilon} : X \to \operatorname{Proj}(S)$ is an isomorphism from $\bigcup_\alpha X_{f_\alpha}$ to $\bigcup_\alpha (\operatorname{Proj}(S))_{f_\alpha}$.

Proof. It is clear that (ii) implies (ii'), and (ii') implies (i) in view of the formula (11.3.11). Condition (i) implies (i'), because any $x \in X$ admits an affine neighborhood U such that $\mathscr{L}|_U$ is isomorphic to $\mathscr{O}_X|_U$; if $f \in \Gamma(X, \mathscr{L}^{\otimes n})$ is such that $x \in X_f \subseteq U$, X_f is also the set of $x' \in U$ such that $(f|_U)(x') \neq 0$, and this is an affine open subset of U (hence of X). To prove that (i') implies (ii), it suffices to show the last assertion of the statement, hence to check that if $X = \bigcup_\alpha X_{f_\alpha}$, the condition (iv) of Proposition 11.3.45 is satisfied; this follows immediately from Theorem 10.6.14(a). To see that $r_{\mathscr{L},\varepsilon}$ is dominant, we note that for $f \in S_+$ homogeneous, X_f is the inverse image of $D_+(f)$ by $r_{\mathscr{L},\varepsilon}$ and by Corollary 10.6.15 we have $\Gamma(X_f, \mathscr{O}_X) = S_{(f)}$ is nonzero if f is not nilpotent, so X_f is nonempty if $D_+(f)$ is not empty.

To prove that (i') implies (iii), note that if X_f is affine (where $f \in S_d$), $\mathscr{F}|_{X_f}$ is generated by its sections over X_f (Theorem 10.1.21); on the other hand, by Theorem 10.6.14 such a section s is of the form $(t|_{X_f}) \otimes (f|_{X_f})^{-m}$ where $t \in \Gamma(X, \mathscr{F}(md))$. By definition, t is also a section of $\mathscr{F}_{md}$, so s is a section of $\mathscr{F}_{md}(-md)$ over X_f, which proves (iii). It is clear that (iii) implies (iii'), and it rests to show that (iii') implies (i). Now let U be an open neighborhood of $x \in X$, and let $\mathscr{I}$ be a quasi-coherent ideal of $\mathscr{O}_X$ defining a closed subscheme of X with underlying subspace $X - U$. The hypothesis of (iii') implies that there exists an integer $n > 0$ and a section $\mathscr{I}(n)$ over X such that $f(x) \neq 0$. Then we have evidently $f \in S_n$ and $x \in X_f \subseteq U$, this proves (i). $\square$

If X is a quasi-compact and quasi-separated scheme, the equivalent conditions of Theorem 11.4.27 implies that X is separated, since it is isomorphic to a subscheme of $\mathrm{Proj}(S)$. We say an invertible $\mathscr{O}_X$-module $\mathscr{L}$ is **ample** if X is a qausi-compact and quasi-separated scheme and the equivalent conditions of Theorem 11.4.27 are satisfied. It follows from Theorem 11.4.27(i) that if $\mathscr{L}$ is an ample $\mathscr{O}_X$-module, then for any open subset U of X, $\mathscr{L}|_U$ is an ample $(\mathscr{O}_X|_U)$-module.

Corollary 11.4.28. *Let $\mathscr{L}$ be an ample $\mathscr{O}_X$-module. For any finite subspace Z of X and any open neighborhood U of Z, there exists an integer $n > 0$ and a section $f \in \Gamma(X, \mathscr{L}^{\otimes n})$ such that X_f is an affine neighborhood of Z contained in U.*

Proof. In view of Theorem 11.4.27(ii), we can view X as a subscheme of $\mathrm{Proj}(S)$ and we only need to prove that for any finite subset Z' of $\mathrm{Proj}(S)$ and any open neighborhood U of Z', there exists a homogeneous element $f \in S_+$ such that $Z \subseteq D_+(f) \subseteq U$. Now by definition the closed set Y, which is the complement of U in $\mathrm{Proj}(S)$, is of the form $V_+(\mathfrak{I})$ where $\mathfrak{I}$ is a graded ideal of S, not containing S_+; on the other hand, the points of Z' are by definition graded prime ideals $\mathfrak{p}_i$ of S_+ not containing $\mathfrak{I}$. There then exists an element $f \in \mathfrak{I}$ not contained in each $\mathfrak{p}_i$ (Proposition 1.4.17), and as the $\mathfrak{p}_i$ are graded, we can assume that f is homogeneous. This element f then satisfies the required. □

Proposition 11.4.29. *Suppose that X is a quasi-compact and quasi-separated scheme. Then the conditions of Theorem 11.4.27 are equivalent to the following conditions:*

(iv) *For any quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$ of finite type, there exists an integer $n_0 > 0$ such that, for any $n \geq n_0$, $\mathscr{F}(n)$ is generated by its sections over X.*

(iv′) *For any quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$ of finite type, there exists integers $n > 0, k > 0$ such that $\mathscr{F}$ is isomorphic to a quotient of the $\mathscr{O}_X$-module $\mathscr{L}^{\otimes(-n)} \otimes \mathscr{O}_X^k$.*

(iv″) *Property (iv′) holds for any quasi-coherent ideal of $\mathscr{O}_X$ of finite type.*

Proof. As X is quasi-compact, if a quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$ of finite type is such that $\mathscr{F}(n)$ is generated by global sections, $\mathscr{F}(n)$ is then generated by finitely many global sections (**??**), so (iv) impliess (iv′) and it is clear that (iv′) implies (iv″). As any quasi-coherent $\mathscr{O}_X$-module $\mathscr{G}$ is the inductive limit of its sub-$\mathscr{O}_X$-modules of finte type (Corollary 10.6.65), to verify conditin (iii′) of Theorem 11.4.27, it suffices to verify that for a quasi-coherent ideal of $\mathscr{O}_X$ that is of finite type, and this is clear if condition (iv″) holds. It remains to prove that if $\mathscr{L}$ is an ample $\mathscr{O}_X$-module, then condition (iv) holds. Consider a finite affine open covering (X_{f_i}) of X with $f_i \in S_{n_i}$; by changing f_i by its power, we can assume that the integers n_i equal to the same integer m. The sheaf $\mathscr{F}|_{X_{f_i}}$, being of finite type by hypotheses, is generated by a finitely number of sections h_{ij} over X_{f_i} (Corollary 10.1.24). By Theorem 10.6.14, there then exists an integer k_0 such that the section $h_{ij} \otimes f_i^{k_0}$ extend to a section of $\mathscr{F}(k_0 m)$ over X for any couple (i, j). A fortiori the $h_{ij} \otimes f_i^k$ extend to sections of $\mathscr{F}(km)$ over X for $k \geq k_0$, and for such values of k, $\mathscr{F}(km)$ is then generated by global sections. For any integer p such that $0 < p < m$, $\mathscr{F}(p)$ is also of finite type, so there exist an integer k_p such that $\mathscr{F}(p)(km) = \mathscr{F}(p + km)$ is generated by global sections for $k \geq k_p$. Let n_0 be the largest of the $k_p m$ for $0 < p < m$; we then conclude that $\mathscr{F}(n)$ is generated by global sections for $n \geq n_0$. □

Proposition 11.4.30. *Let X be a quasi-compact and quasi-separated scheme and $\mathscr{L}$ be an invertible $\mathscr{O}_X$-module.*

(a) *Let $n > 0$ be an integer. For $\mathscr{L}$ to be ample, it is necessary and sufficent that $\mathscr{L}^{\otimes n}$ is ample.*

(b) *Let $\mathscr{L}'$ be an inveritble $\mathscr{O}_X$-module such that, for any $x \in X$, there exists a section s' of $\mathscr{L}'^{\otimes n}$ over X such that $s'(x) \neq 0$. Then, if $\mathscr{L}$ is ample, so is $\mathscr{L} \otimes \mathscr{L}'$.*

Proof. Property (a) is a concequence of (i) of Theorem 11.4.27 since $X_{f^{\otimes n}} = X_f$. On the other hand, if $\mathscr{L}$ is ample, for any $x \in X$ and any neighborhood U of x, there exists $m > 0$ and $f \in \Gamma(X, f^{\otimes m})$ such that $x \in X_f \subseteq U$; if $f' \in \Gamma(X, \mathscr{L}'^{\otimes n})$ is such that $f'(x) \neq 0$, then $s(x) \neq 0$ for $s = f^{\otimes n} \otimes f'^{\otimes m} \in \Gamma(X, (\mathscr{L} \otimes \mathscr{L}')^{\otimes mn})$, so $x \in X_s \subseteq X_f \subseteq U$, and therefore $\mathscr{L} \otimes \mathscr{L}'$ is ample. □

Corollary 11.4.31. *The tensor product of two ample $\mathscr{O}_X$-modules is ample.*

Proof. An ample $\mathscr{O}_X$-module satisfies the condition of Proposition 11.4.30(b). □

Corollary 11.4.32. *Let $\mathscr{L}$ be an ample $\mathscr{O}_X$-module and $\mathscr{L}'$ be an invertible $\mathscr{O}_X$-module. There then exists an integer $n_0 > 0$ such that $\mathscr{L}^{\otimes n} \otimes \mathscr{L}'$ is ample and generated by global sections for $n \geq n_0$.*

Proof. It follows from Proposition 11.4.29 that there exists an integer m_0 such that for $m \geq m_0$, $\mathscr{L}^{\otimes m} \otimes \mathscr{L}'$ is generated by global sections, and therefore satisfies the condition of Proposition 11.4.30(b); we can then choose $n_0 = m_0 + 1$. □

Remark 11.4.33. Let $P = \mathrm{Pic}(X) = H^1(X, \mathscr{O}_X^\times)$ be the picard group of $\mathscr{O}_X$-modules, and let P^+ be the subset of P formed by ample sheaves. Suppose that P^+ is nonempty. Then it follows from Corollary 11.4.31 and Corollary 11.4.32 that we have

$$P^+ + P^+ \subseteq P^+, \quad P^+ - P^+ = P.$$

which means $P^+ \cup \{0\}$ is the set of positive elements of P for an order structure over P compatible with the group structure, which is archimedean in view of Corollary 11.4.32.

Proposition 11.4.34. *Let $q : X \to Y$ be a quasi-compact and quasi-separated morphism where Y is affine, and $\mathscr{L}$ be an inveritble $\mathscr{O}_X$-module.*

(a) *If $\mathscr{L}$ is very ample relative to q then $\mathscr{L}$ is ample.*

(b) *Suppose that q is of finite type. Then for $\mathscr{L}$ to be ample, it is necessary and sufficent that it satisfies the following equivalent conditions:*

  (v) *There exists $n_0 > 0$ such that for any integer $n \geq n_0$, $\mathscr{L}^{\otimes n}$ is very ample relative to q.*

  (v′) *There exists $n > 0$ such that $\mathscr{L}^{\otimes n}$ is very ample relative to q.*

Proof. The first assertion follows from the definition of very ample: if A is the ring of Y, there exists an A-module E and a surjective homomorphism

$$\psi : q^*(\widetilde{\boldsymbol{S}(E)}) \to \boldsymbol{S}(\mathscr{L})$$

such that $i = r_{\mathscr{L},\psi}$ is an immersion from X to $P = \mathbb{P}(\widetilde{E})$ such that $\mathscr{L} \cong i^*(\mathscr{O}_P(1))$. As the $D_+(f)$ for $f \in \boldsymbol{S}(E)_+$ homogeneous form a basis for P and $i^{-1}(D_+(f)) = X_{\psi^\flat(f)}$, we see that condition (i) of Theorem 11.4.27 holds, so $\mathscr{L}$ is ample.

Now assume that q is of finite type and $\mathscr{L}$ is ample. It follows from Theorem 11.4.27(ii) and Proposition 11.4.14(a) that there exists an integer $k_0 > 0$ such that $\mathscr{L}^{\otimes k_0}$ is very ample relative to q. On the other hand, in view of Proposition 11.4.29, there exists an integer m_0 such that, for $m \geq m_0$, $\mathscr{L}^{\otimes m}$ is generated by global sections. Put $n_0 = k_0 + m_0$; if $n \geq n_0$, we can write $n = k_0 + m$ where $m \geq m_0$, whence $\mathscr{L}^{\otimes n} = \mathscr{L}^{\otimes k_0} \otimes \mathscr{L}^{\otimes m}$. As $\mathscr{L}^{\otimes m}$ is generated by global sections, it follow from Proposition 11.4.20 and Remark 11.3.28 that $\mathscr{L}^{\otimes n}$ is very ample relative to q. Finally, it is clear that (v) implies (v′), and (v′) implies that $\mathscr{L}$ is ample in view of (a) and Proposition 11.4.30. □

Corollary 11.4.35. *Let $q : X \to Y$ be a quasi-compact and quasi-separated morphism of finite type where Y is affine, $\mathscr{L}$ be an ample $\mathscr{O}_X$-module, and $\mathscr{L}'$ be an invertible $\mathscr{O}_X$-module. Then there exists an integer $n_0 > 0$ such that for $n \geq n_0$, $\mathscr{L}^{\otimes n} \otimes \mathscr{L}'$ is very ample relative to q.*

Proof. In fact, there exists integer m_0 such that for $m \geq m_0$, $\mathscr{L}^{\otimes m} \otimes \mathscr{L}'$ is generated by global section (Corollary 11.4.32); on the other hand, there exists k_0 such that $\mathscr{L}^{\otimes k}$ is very ample relative to q for $k \geq k_0$. Thus $\mathscr{L}^{\otimes(k+m_0)} \otimes \mathscr{L}'$ is very ample for $k \geq k_0$ (Corollary 11.4.19). □

Proposition 11.4.36. *Let X be a quasi-compact scheme, Z be a closed subscheme of X defined by a qausi-coherent nilpotent ideal $\mathscr{J}$ of $\mathscr{O}_X$, and $j: Z \to X$ be the canonical injectin. For an inveritble $\mathscr{O}_X$-module $\mathscr{L}$ to be ample, it is necessary and sufficient that $\mathscr{L}' = j^*(\mathscr{L})$ is an ample $\mathscr{O}_Z$-module.*

Proof. This conditions is necessary. In fact, for any section f of $\mathscr{L}^{\otimes n}$ over X, let f' be the image $f \otimes 1$, which is a section of $\mathscr{L}'^{\otimes n} = \mathscr{L}^{\otimes n} \otimes_{\mathscr{O}_X} (\mathscr{O}_X/\mathscr{J})$ over the subspace Z (identified with X); it is clear that $Z_{f'} = X_f$, hence condition (i) of Theorem 11.4.27 shows that $\mathscr{L}'$ is ample.

To prove the sufficiency, note first that we can reduce to the case $\mathscr{J}^2 = 0$ by considering the finite sequence of schemes $X_k = (X, \mathscr{O}_X/\mathscr{J}^{k+1})$, which is a closed subscheme of the previous one and is defined by a square zero ideal. Now X is quasi-separated if X_{red} is quasi-separated (Proposition 10.6.7(vi)). Criterion (i) of Theorem 11.4.27 shows that it will suffice to prove that, if g is a section of $\mathscr{L}'^{\otimes n}$ over Z such that Z_g is affine, then there exists $m > 0$ such that $g^{\otimes m}$ is the canonical image of a section f of $\mathscr{L}^{\otimes nm}$ over X. For this, we consider the exact sequence

$$0 \longrightarrow \mathscr{J}(n) \longrightarrow \mathscr{O}_X(n) = \mathscr{L}^{\otimes n} \longrightarrow \mathscr{O}_Z(n) = \mathscr{L}'^{\otimes n} \longrightarrow 0$$

since $\mathscr{F}(n)$ is an exact functor on $\mathscr{F}$; whence an exact sequence on cohomology:

$$0 \longrightarrow \Gamma(X, \mathscr{J}(n)) \longrightarrow \Gamma(X, \mathscr{L}^{\otimes n}) \longrightarrow \Gamma(X, \mathscr{L}'^{\otimes n}) \xrightarrow{\delta} H^1(X, \mathscr{J}(n))$$

which associates in particular g to an element $\delta g \in H^1(X, \mathscr{J}(n))$.

Note that since $\mathscr{J}^2 = 0$, $\mathscr{J}$ can be considered as a quasi-coherent $\mathscr{O}_Z$-module and we have, for any k, $\mathscr{L}'^{\otimes k} \otimes_{\mathscr{O}_Z} \mathscr{J}(n) = \mathscr{J}(n+k)$. For any section $s \in \Gamma(X, \mathscr{L}'^{\otimes k})$, tensoring by s is then a homomorphism $\mathscr{J}(n) \to \mathscr{J}(n+k)$ of $\mathscr{O}_Z$-modules, which gives a homomorphism $H^i(X, \mathscr{J}(n)) \to H^i(X, \mathscr{J}(n+k))$ of cohomology groups. We claim that

$$g^{\otimes m} \otimes \delta g = 0 \tag{11.4.9}$$

for $m > 0$ sufficiently large. In fact, Z_g is an affine open of Z and we have $H^1(Z_g, \mathscr{J}(n)) = 0$ where $\mathscr{J}(n)$ is considered as an $\mathscr{O}_Z$-module. In particular, if we put $g' = g|_{Z_g}$, and if we consider its image under $\delta: \Gamma(Z_g, \mathscr{L}'^{\otimes n}) \to H^1(Z_g, \mathscr{J}(n))$, we have $\delta g' = 0$. To explain this relation, observe that the first cohomology group of a sheaf coincides with the Cech cohomology; to form δg, it is necessary to consider an open covering (U_α) of X, which we can assume that is finite and formed by affine opens, and choose for each α a section $g_\alpha \in \Gamma(U_\alpha, \mathscr{L}^{\otimes n})$ whose image in $\Gamma(U_\alpha, \mathscr{L}'^{\otimes n})$ is $g|_{U_\alpha}$, and consider the class of cocycle $(g_{\alpha\beta} - g_{\beta\alpha})$, where $g_{\alpha\beta}$ is the restriction of g_α to $U_\alpha \cap U_\beta$ (a cocycle with values in $\mathscr{J}(n)$). We can moreover suppose that $\delta g'$ is in the same manner using the covering formed by $U_\alpha \cap Z_g$ and the restrictions $g_\alpha|_{U_\alpha \cap Z_g}$; the relation $\delta g' = 0$ signifies then that there exists for each α a section $h_\alpha \in \Gamma(U_\alpha \cap Z_g, \mathscr{J}(n))$ such that $(g_{\alpha\beta} - g_{\beta\alpha})|_{U_\cap \cap U_\beta \cap Z_g} = h_{\alpha\beta} - h_{\eta\alpha}$, where $h_{\alpha\beta}$ denotes the restriction of h_α to $U_\alpha \cap U_\beta \cap Z_g$. Now by Theorem 10.6.14 there exists an integer $m > 0$ such that $g^{\otimes m} \otimes h_\alpha$ is the restriction to $U_\alpha \cap Z_g$ of a section $t_\alpha \in \Gamma(U_\alpha, \mathscr{J}(n+nm))$ for each α; we then have $g^{\otimes m} \otimes (g_{\alpha\beta} - g_{\beta\alpha}) = t_{\alpha\beta} - t_{\beta\alpha}$ for any couple of indices, which proves $g^{\otimes m} \otimes \delta g = 0$.

We remark on the other hand that if $s \in \Gamma(X, \mathscr{O}_Z(p))$, $t \in \Gamma(X, \mathscr{O}_Z(q))$, we have, in the group $H^1(X, \mathscr{J}(p+q))$, that

$$\delta(s \otimes t) = (\delta s) \otimes t + s \otimes (\delta t). \tag{11.4.10}$$

For this, we can still consider an open cover (U_α) of X, and for each α a section $s_\alpha\in\Gamma(U_\alpha,\mathscr{O}_X(p))$ (resp. a section $t_\alpha\in\Gamma(U_\alpha,\mathscr{O}_X(q))$) whose canonical image in $\Gamma(U_\alpha,\mathscr{O}_Z(p))$ (resp. in $\Gamma(U_\alpha,\mathscr{O}_Z(q))$) is $s|_{U_\alpha}$; the relation (11.4.10) then follows from

$$(s_{\alpha\beta}\otimes t_{\alpha\beta})-(s_{\beta\alpha}\otimes t_{\beta\alpha})=(s_{\alpha\beta}-s_{\beta\alpha})\otimes t_{\alpha\beta}+s_{\beta\alpha}\otimes(t_{\alpha\beta}-t_{\beta\alpha})$$

with the same notations before. By recurrence on k, we then have

$$\delta(g^{\otimes k})=(kg^{\otimes(k-1)})\otimes(\delta g) \tag{11.4.11}$$

and in view of (11.4.9) and (11.4.11), we have $\delta(g^{\otimes(m+1)})=0$, so $g^{\otimes(m+1)}$ is the canonical image of a section f of $\mathscr{L}^{\otimes n(m+1)}$ over X, whence our demonstration. □

Corollary 11.4.37. *Let X be a Noetherian scheme and $j:X_{\mathrm{red}}\to X$ be the canonical injection. For an inveritble $\mathscr{O}_X$-module $\mathscr{L}$ to be ample, it is necessary and sufficient that $j^*(\mathscr{L})$ is an ample $\mathscr{O}_{X_{\mathrm{red}}}$-module.*

Proof. The nilradical $\mathscr{N}_X$ is nilpotent and we can apply Proposition 11.4.36. □

11.4.6 Relatively ample sheaves

Let $f:X\to Y$ be a quasi-compact morphism and $\mathscr{L}$ be an invertible $\mathscr{O}_X$-module. We say that $\mathscr{L}$ is **ample relative to f**, or relative to Y, or **f-ample**, or **Y-ample**, if there exists an affine open covering (U_α) of Y such that if $X_\alpha=f^{-1}(U_\alpha)$, $\mathscr{L}|_{X_\alpha}$ is an ample $\mathscr{O}_{X_\alpha}$-module for each α. Again, the existence of an f-ample $\mathscr{O}_X$-module implies that X is separated, so f is necessarily separated by Proposition 10.5.26.

Proposition 11.4.38. *Let $f:X\to Y$ be a quasi-compact morphism and $\mathscr{L}$ be an invertible $\mathscr{O}_X$-module. If $\mathscr{L}$ is very ample relative to f, then it is ample relative to f.*

Proof. If $\mathscr{L}$ is very ample relative to f then the morphism f is separated, so by Proposition 11.4.34(a) the restriction $\mathscr{L}|_{f^{-1}(U)}$ for any affine open U of Y is very ample, hence ample. □

Let $f:X\to Y$ be a quasi-compact morphism and $\mathscr{L}$ be an invertible $\mathscr{O}_X$-module. We consider the graded $\mathscr{O}_Y$-algebra $\mathscr{S}=\bigoplus_{n\geq 0}f_*(\mathscr{L}^{\otimes n})$. Then the canonical homomorphisms $\sigma_n:f^*(f_*(\mathscr{L}^{\otimes n}))\to\mathscr{L}^{\otimes n}$ induce a canonical homomorphism

$$\sigma:f^*(\mathscr{S})\to\bigoplus_{n\geq 0}\mathscr{L}^{\otimes n}=\boldsymbol{S}(\mathscr{L}).$$

On the other hand, it is easy to see that $\sigma^\flat$ is the canonical injection from $\mathscr{S}$ into $f_*(\boldsymbol{S}(\mathscr{L}))$. The homomorphism σ gives an Y-morphism

$$r_{\mathscr{L},\sigma}:G(\sigma)\to\mathrm{Proj}(\mathscr{S})=P.$$

Proposition 11.4.39. *Let $f:X\to Y$ be a quasi-compact morphism and $\mathscr{L}$ be an invertible $\mathscr{O}_X$-module. Then the following conditions are equivalent:*

(i) *$\mathscr{L}$ is f-ample.*

(ii) *$\mathscr{S}$ is quasi-coherent and the Y-morphism $r_{\mathscr{L},\sigma}$ is everywhere defined and a dominant open immersion.*

(ii') *The morphism f is separated, the Y-morphism $r_{\mathscr{L},\sigma}$ is everywhere defined and is a homeomorphism from X onto a subspace of* $\mathrm{Proj}(\mathscr{S})$.

Moreover, if these are satisfied, for any $n \in \mathbb{Z}$ the canonical homomorphism of (11.3.12)

$$\nu : r^*_{\mathscr{L},\sigma}(\mathscr{O}_P(n)) \to \mathscr{L}^{\otimes n} \tag{11.4.12}$$

is an isomorphism. Finally, for any quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, if we put $\mathscr{M} = \bigoplus_{n\geq 0} f_*(\mathscr{F} \otimes \mathscr{L}^{\otimes n})$, the canonical homomorphism of (11.3.13)

$$\xi : r^*_{\mathscr{L},\sigma}(\widetilde{\mathscr{M}}) \to \mathscr{F} \tag{11.4.13}$$

is an isomorphism.

Proof. We have remarked that (i) implies that f is separated, so $\mathscr{S}$ is quasi-coherent by Proposition 10.6.55. As the fact that $r_{\mathscr{L},\sigma}$ is an open immersion everywhere defined is local over Y, to shows that (i) implies (ii), we can assume that Y is affine and $\mathscr{L}$ is ample; the assertion then follows from Theorem 11.4.27. It is clear that (ii) implies (ii'); finally, to show that (ii') implies (i), it suffices to consider an affine open cover (U_α) of Y and use Theorem 11.4.27(ii') to $\mathscr{L}|_{X_\alpha}$.

To prove the last two assertions, we use the fact that $\sigma^\flat$ is the canonical injection of $\mathscr{S}$ to $f_*(\boldsymbol{S}(\mathscr{L}))$ and the expression of the morphisms ν and ξ in Remark 11.3.43 and Remark 11.3.44. It then follows that ν and ξ are injective. As for the surjectivity, we can assume that Y is affine, so $\mathscr{L}$ is ample; the criterion of Theorem 11.4.27(iii) then shows that ν and ξ are surjective, whence the assertion. □

Remark 11.4.40. From Proposition 11.4.39 and its proof, we conclude that if $\mathscr{L}$ is f-ample then $f_*(\boldsymbol{S}(\mathscr{L}))$ is equal to $\mathscr{S}$, so the homomorphism $\sigma^\flat$ is the identity on $\mathscr{S}$. This can also be seen from Proposition 10.6.57 since in this case f is separated.

Corollary 11.4.41. *Let (U_α) be an open covering of Y. For $\mathscr{L}$ to be Y-ample, it is necessary and sufficient that $\mathscr{L}|_{f^{-1}(U_\alpha)}$ is U_α-ample for each α.*

Proof. This is true since the condition (ii) of Proposition 11.4.39 is local over Y. □

Corollary 11.4.42. *Let $\mathscr{K}$ be an invertible $\mathscr{O}_Y$-module. For $\mathscr{L}$ to be Y-ample, it is necessary and sufficient that $\mathscr{L} \otimes f^*(\mathscr{K})$ is Y-ample.*

Proof. This is a concequence of Corollary 11.4.41 by taking U_α to be such that $\mathscr{K}|_{U_\alpha}$ is isomorphic to $\mathscr{O}_Y|_{U_\alpha}$ for each α. □

Corollary 11.4.43. *Suppose that Y is affine. For $\mathscr{L}$ to be Y-ample, it is necessary and sufficient that $\mathscr{L}$ is ample.*

Proof. This is immediate from the definition of Y-ample, and Proposition 11.4.39(ii) and Theorem 11.4.27(ii), since

$$\mathrm{Proj}(\mathscr{S}) = \mathrm{Proj}(\Gamma(Y, \mathscr{S})) = \mathrm{Proj}(\bigoplus_{n\geq 0} \Gamma(Y, f_*(\mathscr{L}^{\otimes n}))) = \mathrm{Proj}(S)$$

in this case (note that Y is quasi-compact). □

Corollary 11.4.44. *Let $f : X \to Y$ be a quasi-compact morphism. Suppose that there exists a quasi-coherent $\mathscr{O}_Y$-module $\mathscr{E}$ and a Y-morphism $i : X \to P = \mathbb{P}(\mathscr{E})$ which is a homeomorphism from X onto a subspace of P. Then $\mathscr{L} = i^*(\mathscr{O}_P(1))$ is Y-ample.*

Proof. We can assume that Y is affine, and the corollary then follows from the criterion (i) of Theorem 11.4.27 and the formula (11.3.11). □

Proposition 11.4.45. *Let X be a quasi-compact and quasi-separated scheme and $f : X \to Y$ be a quasi-compact and quasi-separated morphism. For an invertible $\mathscr{O}_X$-module $\mathscr{L}$ to be f-ample, it is necessary and sufficient that following equivalent conditions are satisfied:*

(iii) *For any quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$ of finite type, there exists an integer $n_0 > 0$ such that, for any $n \geq n_0$, the canonical homomorphism $\sigma : f^*(f_*(\mathscr{F} \otimes \mathscr{L}^{\otimes n})) \to \mathscr{F} \otimes \mathscr{L}^{\otimes n}$ is surjective.*

(iii′) *For any ideal $\mathscr{I}$ of $\mathscr{O}_X$ of finite type, there exist an integer $n > 0$ such that the canonical homomorphism $\sigma : f^*(f_*(\mathscr{I} \otimes \mathscr{L}^{\otimes n})) \to \mathscr{I} \otimes \mathscr{L}^{\otimes n}$ is surjective.*

Proof. □

Proposition 11.4.46. *Let $f : X \to Y$ be a quasi-compact morphism and $\mathscr{L}$ be an invertible $\mathscr{O}_X$-module.*

(a) *Let $n > 0$ be an integer. For $\mathscr{L}$ to be f-ample, it is necessary and necessary that $\mathscr{L}^{\otimes n}$ is f-ample.*

(b) *Let $\mathscr{L}'$ be an invertible $\mathscr{O}_X$-module, and suppose that there exists an integer $n > 0$ such that the canonical homomorphism $\sigma : f^*(f_*(\mathscr{L}'^{\otimes n})) \to \mathscr{L}'^{\otimes n}$ is surjective. Then, if $\mathscr{L}$ is f-ample, so is $\mathscr{L} \otimes \mathscr{L}'$.*

Corollary 11.4.47. *The tensor product of two f-ample $\mathscr{O}_X$-modules is f-ample.*

Proposition 11.4.48. *Let $f : X \to Y$ be a morphism of finite type where Y is quasi-compact, and $\mathscr{L}$ be an invertible $\mathscr{O}_X$-module. For $\mathscr{L}$ to be f-ample, it is necessary and sufficient that the following equivalent conditions are satisfied:*

(iv) *There exists $n_0 > 0$ such that, for any integer $n \geq n_0$, $\mathscr{L}^{\otimes n}$ is very ample relative to f.*

(iv′) *There exist $n > 0$ such that $\mathscr{L}^{\otimes n}$ is very ample relative to f.*

Lemma 11.4.49. *Let $u : Z \to S$ be a morphism, $\mathscr{L}$ be an invertible $\mathscr{O}_S$-module, s a section of $\mathscr{L}$ over S, and t be the inverse image of s under u. Then $Z_t = u^{-1}(S_s)$.*

Proof. □

Lemma 11.4.50. *Let Z, Z' be two S-schemes, p, p' be the projections of $T = Z \times_S Z'$, $\mathscr{L}$ (resp. $\mathscr{L}'$) be an invertible $\mathscr{O}_Z$-module (resp. $\mathscr{O}_{Z'}$-module), t (resp. t') be a section of $\mathscr{L}$ (resp. $\mathscr{L}'$) over Z (resp. Z'), s (resp. s') be the inverse image of t (resp. t') under p (resp. p'). Then we have $T_{s\otimes s'} = Z_t \times_S Z'_{t'}$.*

Proof. □

Proposition 11.4.51 (Properties of Relative Ample Sheaves).

(i) *For any scheme Y, any invertible $\mathscr{O}_Y$-module $\mathscr{L}$ is relative ample relative to the identify morphism 1_Y.*

(ii) *Let $f : X \to Y$ be a quasi-compact morphism and $j : X' \to X$ be a quasi-compact morphism that is a homeomorphism from X' onto a subspace of X. If $\mathscr{L}$ is an $\mathscr{O}_X$-module relative to f, then $j^*(\mathscr{L})$ is relative relative to $f \circ j$.*

(iii) *Let $f : X \to Y$ and $g : Y \to Z$ be quasi-compact morphisms where Z is quasi-compact. Let $\mathscr{L}$ an ample $\mathscr{O}_X$-module relative to f and $\mathscr{K}$ be an ample $\mathscr{O}_Y$-module relative to g. Then there exists an integer $n_0 > 0$ such that for any $n \geq n_0$, $\mathscr{L} \otimes f^*(\mathscr{K}^{\otimes n})$ is ample relative to $g \circ f$.*

(iv) *Let $f : X \to Y$ be a quasi-compact morphism, $g : Y' \to Y$ be a morphism, and put $X' = X_{(Y')}$. If $\mathscr{L}$ is an ample $\mathscr{O}_X$-module relative to f, $\mathscr{L}' = \mathscr{L} \otimes_Y \mathscr{O}_{Y'}$ is ample relative to $f_{(Y')}$.*

(v) *Let $f_i : X_i \to Y_i$ ($i = 1, 2$) be two quasi-compact S-morphisms. If $\mathscr{L}_i$ is an ample $\mathscr{O}_{X_i}$-module relative to f_i, then $\mathscr{L}_1 \otimes_S \mathscr{L}_2$ is ample relative to $f_1 \times_S f_2$.*

(vi) *Let $f : X \to Y$, $g : Y \to Z$ be two morphisms such that $g \circ f$ is quasi-compact. If an $\mathscr{O}_X$-module $\mathscr{L}$ is ample relative to $g \circ f$ and if g is separated or X is locally Noetherian, then $\mathscr{L}$ is ample relative to f.*

(vii) *Let $f : X \to Y$ be a quasi-compact morphism and $j : X_{\mathrm{red}} \to X$ be the canonical injection. If $\mathscr{L}$ is an ample $\mathscr{O}_X$-module relative to f, then $j^*(\mathscr{L})$ is ample relative to f_{red}.*

Proof. □

Proposition 11.4.52. *Let $f : X \to Y$ be a quasi-compact morphism, $\mathscr{I}$ be a locally nilpotent ideal of $\mathscr{O}_X$, Z the closed subscheme of X defined by $\mathscr{I}$, and $j : Z \to X$ be the cannical injection. For an invertible $\mathscr{O}_X$-module to be ample relative to f, it is necessary and sufficient that $j^*(\mathscr{L})$ is ample relative to $f \circ j$.*

Corollary 11.4.53. *Let X be a locally Noetherian scheme and $f : X \to Y$ be a quasi-compact morphism. For an invertible $\mathscr{O}_X$-module $\mathscr{L}$ to be ample relative to f, it is necessary and sufficient that its inverse image $\mathscr{L}'$ under the canonical injection $X_{\mathrm{red}} \to X$ is ample relative to f_{red}.*

Proposition 11.4.54. *Let $f : X \to Y$, $f' : X' \to Y$ be two morphisms, $X'' = X \amalg X'$, and $f'' = f \amalg f'$. Let $\mathscr{L}$ (resp. $\mathscr{L}'$) be an invertible $\mathscr{O}_X$-module (resp. $\mathscr{O}_{X'}$-module), and let $\mathscr{L}''$ be the invertible $\mathscr{O}_{X''}$-module which coincides with $\mathscr{L}$ over X and with $\mathscr{L}'$ over X'. For $\mathscr{L}''$ to be ample relative to f'', it is necessary and sufficent that $\mathscr{L}$ is ample relative to f and $\mathscr{L}'$ is ample relative to f'.*

Proposition 11.4.55. *Let Y be a quasi-compact scheme, $\mathscr{S}$ be a quasi-coherent graded $\mathscr{O}_Y$-algebra of finite type, $X = \mathrm{Proj}(\mathscr{S})$, and $f : X \to Y$ be the structural morphism. Then f is of finite type and there exists an integer $n > 0$ such that $\mathscr{O}_X(d)$ is invertible and f-ample.*

11.5 Projective morphisms and Chow's lemma

11.5.1 Quasi-affine morphisms

We say a scheme is **quasi-affine** if it is isomorphic to the subscheme induced over a quasi-compact open subset of an affine scheme. We say a morphism $f : X \to Y$ is quasi-affine, or that X is a quasi-affine Y-scheme, if there exists an affine open cover (U_α) of Y such that $f^{-1}(U_\alpha)$ is a quasi-affine scheme. Since any quasi-compact open subscheme of an affine scheme is separated, it is clear that quasi-affine morphisms are separated and quasi-compact, and any affine morphism is quasi-affine.

Recall that for any scheme X, if $A = \Gamma(X, \mathscr{O}_X)$, the identify homomorphism $A \to A$ induces a canonical morphism $q : X \to \mathrm{Spec}(A)$ by Proposition 10.2.4. This is also the morphism $r_{\mathscr{L},\varepsilon} : X \to \mathrm{Proj}(S)$ in (11.4.8) defined for $\mathscr{L} = \mathscr{O}_X$, since $\Gamma(X, -)$ commutes with taking tensor product with $\mathscr{O}_X$ and we have $S = \bigoplus_{n \geq 0} \Gamma(X, \mathscr{O}_X)^{\otimes n} = A[T]$, and $\mathrm{Proj}(A[T])$ is canonically identified with $\mathrm{Spec}(A)$.

Proposition 11.5.1. *Let X be a quasi-compact scheme and $A = \Gamma(X, \mathscr{O}_X)$. The following conditions are equivalent:*

(i) *X is a quasi-affine scheme.*

(ii) *The canonical morphism $q : X \to \mathrm{Spec}(A)$ is an open immersion.*

(ii') *The canonical morphism $q : X \to \mathrm{Spec}(A)$ is a homeomorphism from X onto a subspace of $\mathrm{Spec}(A)$.*

(iii) *The $\mathscr{O}_X$-module $\mathscr{O}_X$ is very ample relative to q.*

(iii') *The $\mathscr{O}_X$-module $\mathscr{O}_X$ is ample.*

(iv) *The subsets X_f, as f runs through A, form a basis for X.*

(iv') *The subsets X_f which are affine, as f runs through A, form a basis for X.*

(v) *Any quasi-coherent $\mathscr{O}_X$-module is generated by its global sections.*

(v′) *Any quasi-coherent ideal of finite type of $\mathscr{O}_X$ is generated by its global sections.*

Proof. It is clear that (ii)⇒(i) by definition, and (iii)⇒(iii′) by Proposition 11.4.34. Since the canonical morphisms $q : X \to \operatorname{Spec}(A)$ and $r_{\mathscr{O}_X,\varepsilon} : X \to \operatorname{Proj}(S)$ are identified, we see that (iii′)⇔(ii)⇔(ii′)⇔(iv)⇔(iv′) by Theorem 11.4.27. Also, (iii′)⇔(v)⇔(v′) in view of Proposition 11.4.29.

We also note that if X is quasi-affine, then it can be identified as an open subscheme of an affine scheme $Y = \operatorname{Spec}(B)$. Let $\varphi : B \to A$ be the correspond homomorphism (Proposition 10.2.4). Since the affine opens $D(g)$, with $g \in B$, form a basis of Y, and we have $X_f = D(g) \cap X$ where $f = \varphi(g)$, it follows that the subsets X_f which are affine, with $f \in A$, form a basis for X, which proves (i)⇒(iv′).

Finally, it remains to show that (i)⇒(iii). For this we first note that if X is quasi-affine then it is quasi-compact and separated, so by Corollary 10.6.15, for $f \in A$ we have $\Gamma(X_f, \mathscr{O}_X) = A_f$. Since we have $q^{-1}(D_+(f)) = X_f$, we conclude that the canonical morphism $q : X \to \operatorname{Spec}(A)$ is of finite type, and by Proposition 11.4.34, since $\mathscr{O}_X^{\otimes n}$ is isomorphic to $\mathscr{O}_X$ for any integer $n > 0$, $\mathscr{O}_X$ is very ample relative to q. This completes the proof. □

Remark 11.5.2. Let X be a quasi-affine scheme and $A = \Gamma(X, \mathscr{O}_X)$. By Proposition 11.5.1 we know that $\mathscr{O}_X$ is very ample relative to $q : X \to \operatorname{Spec}(A)$. Since X is separated and quasi-compact, $q_*(\mathscr{O}_X)$ is quasi-coherent by Proposition 10.6.55, and from $\Gamma(\operatorname{Spec}(A), q_*(\mathscr{O}_X)) = A$ we conclude that $q_*(\mathscr{O}_X) = \widetilde{A}$, so $q^*(q_*(\mathscr{O}_X)) = \mathscr{O}_X$ and the canonical homomorphism $\sigma : q^*(q_*(\mathscr{O}_X)) \to \mathscr{O}_X$ is identified with the identity on $\mathscr{O}_X$. This being so, the canonical morphism $r_{\mathscr{O}_X,\sigma} : X \to \mathbb{P}(q_*(\mathscr{O}_X))$ is then identified with $q : X \to \operatorname{Spec}(A)$, because we have

$$\mathbb{P}(q_*(\mathscr{O}_X)) = \operatorname{Proj}(\boldsymbol{S}(q_*(\mathscr{O}_X))) = \operatorname{Proj}(\boldsymbol{S}(A)) = \operatorname{Proj}(A[T]) = \operatorname{Spec}(A);$$

and we conclude from Proposition 11.5.1 that this is an open immersion, which justifies Proposition 11.4.16.

Corollary 11.5.3. *Let X be a quasi-compact scheme. If there exists a morphism $r : X \to Y$ from X into an affine scheme Y which is a homeomorphism onto an open subspace of Y, then X is quasi-affine.*

Proof. In fact, there then exists a family (g_α) of sections of $\mathscr{O}_Y$ over Y such that the $D(g_\alpha)$ form a basis for the topology of $r(X)$. If we put $f_\alpha = \theta(g_\alpha)$ where $\theta : \Gamma(Y, \mathscr{O}_Y) \to \Gamma(X, \mathscr{O}_X)$ is the corresponding ring homomorphism, then we have $X_{f_\alpha} = r^{-1}(D(g_\alpha))$, so the X_{f_α} form a basis for X, and by Proposition 11.5.1 X is then quasi-affine. □

Corollary 11.5.4. *If X is a quasi-affine scheme, any invertible $\mathscr{O}_X$-module is very ample (relative to the canonical morphism $q : X \to \operatorname{Spec}(A)$) and a fortiori ample.*

Proof. In fact any such module $\mathscr{L}$ is generated by its global sections (Proposition 11.5.1(v)), so $\mathscr{L} \otimes \mathscr{O}_X$ is very ample by Proposition 11.4.20. We also note that the morphism q is of finite type. □

Corollary 11.5.5. *Let X be a quasi-compact scheme. If there exists an invertible $\mathscr{O}_X$-module $\mathscr{L}$ such that $\mathscr{L}$ and $\mathscr{L}^{-1}$ are ample, then X is quasi-affine.*

Proof. In fact, $\mathscr{O}_X = \mathscr{L} \otimes \mathscr{L}^{-1}$ is then ample by Corollary 11.4.31. □

Proposition 11.5.6. *Let $f : x \to Y$ be a quasi-compact morphism. The following conditions are equivalent:*

(i) *f is quasi-affine.*

(ii) *The $\mathscr{O}_Y$-algebra $f_*(\mathscr{O}_X) = \mathscr{A}(X)$ is quasi-coherent and the canonical morphism $X \to \mathrm{Spec}(\mathscr{A}(X))$ corresponding to the identity homomorphism of $\mathscr{A}(X)$ is an open immersion.*

(ii') *The $\mathscr{O}_Y$-algebra $\mathscr{A}(X)$ is quasi-coherent and the canonical morphism $X \to \mathrm{Spec}(\mathscr{A}(X))$ is a homeomorphism from X onto a subspace of $\mathrm{Spec}(\mathscr{A}(X))$.*

(iii) *The $\mathscr{O}_X$-module $\mathscr{O}_X$ is very ample relative to f.*

(iii') *The $\mathscr{O}_X$-module $\mathscr{O}_X$ is ample relative to f.*

(iv) *The morphism f is separated and for any quasi-coherent $\mathscr{O}_X$-modle $\mathscr{F}$, the canonical homomorphism $\sigma : f^*(f_*(\mathscr{F})) \to \mathscr{F}$ is surjective.*

Morever, if f is quasi-affine, any invertible $\mathscr{O}_X$-module $\mathscr{L}$ is very ample relative to f.

Proof. The equivalence of these properties follows from the fact that they are all local over Y and the criteria of Proposition 11.5.1. Also, we note that $f_*(\mathscr{F})$ is quasi-coherent if f is separated (Proposition 10.6.55). The last assertion follows from Corollary 11.5.4. □

Corollary 11.5.7. *Let Y be an affine scheme and $f : X \to Y$ be a quasi-compact morphism. For f to be quasi-affine, it is necessary and sufficient that X is quasi-affine scheme.*

Proof. This is an immediate concequence of Proposition 11.5.6 and Corollary 11.4.43. □

Corollary 11.5.8. *Let Y be a quasi-compact and quasi-separated scheme, $f : X \to Y$ be a morphism of finite type. If f is quasi-affine, there exists a quasi-coherent sub-$\mathscr{O}_Y$-algebra $\mathscr{B}$ of $\mathscr{A}(X)$ of finite type such that the morphism $X \to \mathrm{Spec}(\mathscr{B})$ corresponding to the canonical injection $\mathscr{B} \to \mathscr{A}(X)$ is an immersion. Moreover, any quasi-coherent sub-$\mathscr{O}_Y$-algebra of finite type $\mathscr{B}'$ of $\mathscr{A}(X)$, containing $\mathscr{B}$, has the same property.*

Proof. In fact, $\mathscr{A}(X)$ is the inductive limit of its quasi-coherent sub-$\mathscr{O}_Y$-algebras of finite type (Corollary 10.6.65); the assertion is then a particular case of Proposition 11.3.47, in view of the identification of $\mathrm{Spec}(\mathscr{A}(X))$ and $\mathrm{Proj}(\mathscr{A}(X)[T])$ (Corollary 11.3.5) and the canonical morphisms from X into them (cf. Remark 11.5.2). □

Proposition 11.5.9 (Properties of Quasi-affine Morphisms).

(i) *A quasi-compact morphism $f : X \to Y$ that is a homeomorphism from X onto a subspace of Y (and in particular a quasi-compact immersion) is quasi-affine.*

(ii) *The composition of two quasi-affine morphisms is quasi-affine.*

(iii) *If $f : x \to Y$ is a quasi-affine S-morphism, $f_{(S')} : X_{(S')} \to Y_{(S')}$ is a quasi-affine morphism for any base change $S' \to S$.*

(iv) *If $f : X \to Y$ and $g : X' \to Y'$ are two quasi-affine S-morphisms, so is $f \times_S g$.*

(v) *If $f : X \to Y$, $g : Y \to Z$ are two morphisms such that $g \circ f$ is quasi-affine and if g is separated or X is locally Noetherian, then f is quasi-affine.*

(vi) *Let $f : X \to Y$ be a quasi-compact morphism and $j : X_{\mathrm{red}} \to X$ the canonical injection. If an $\mathscr{O}_X$-module $\mathscr{L}$ is ample relative to f, then $j^*(\mathscr{L})$ is ample relative to f_{red}.*

Proof. In view of the criterion (iii') of Proposition 11.5.6, (i), (iii), (iv), (v) and (vi) are concequences of Proposition 11.4.51. To prove (ii), we can assume that Z is affine, and the assertion then follows from Proposition 11.4.51(iii), applied to $\mathscr{L} = \mathscr{O}_X$ and $\mathscr{K} = \mathscr{O}_Y$. □

Remark 11.5.10. Let $f : X \to Y$, $g : Y \to Z$ be two morphisms such that $X \times_Z Y$ is locally Noetherian. Then the graph morphism $\Gamma_f : X \to X \times_Z Y$ is a quasi-compact immersion, hence quasi-affine, and the reasoning of Proposition 10.5.14 shows that the conclusion of (v) remains valid if we remove the hypothesis that g is separated.

Proposition 11.5.11. *Let $f : X \to Y$ be a quasi-compact morphism and $g : X' \to X$ be a quasi-affine morphism. If $\mathscr{L}$ is an f-ample $\mathscr{O}_X$-module, then $g^*(\mathscr{L})$ is ample relative to $f \circ g$.*

Proof. By hypothesis $\mathscr{O}_{X'}$ is very ample relative to f, and since the question is local over Y, it follows from Proposition 11.4.51(iii) that there exists (for Y affine) an integer n such that $g^*(\mathscr{L}^{\otimes n}) = (g^*(\mathscr{L}))^{\otimes n}$ is ample relative to $f \circ g$, whence $g^*(\mathscr{L})$ is ample relative to $f \circ g$. □

11.5.2 Serre's criterion on affineness

Theorem 11.5.12 (Serre's criterion). *For a quasi-compact and quasi-separated scheme X, then the following conditions are equivalent:*

(i) *X is an affine scheme.*

(ii) *There exists a family (f_α) of elements of $A = \Gamma(X, \mathscr{O}_X)$ such that X_{f_α} are affine and the ideal generated by the f_α equals to A.*

(iii) *The functor $\Gamma(X, -)$ is exact on the category of quasi-coherent $\mathscr{O}_X$-modules.*

(iii') *For any exact sequence of quasi-coherent $\mathscr{O}_X$-modules*

$$0 \longrightarrow \mathscr{F}' \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}'' \longrightarrow 0$$

where $\mathscr{F}$ is isomorphic to a sub-$\mathscr{O}_X$-module of a finite product $\mathscr{O}_X^n$, the induced sequence

$$0 \longrightarrow \Gamma(X, \mathscr{F}') \longrightarrow \Gamma(X, \mathscr{F}) \longrightarrow \Gamma(X, \mathscr{F}'') \longrightarrow 0$$

is exact.

(iv) *$H^1(X, \mathscr{F}) = 0$ for any quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$.*

(iv') *$H^1(X, \mathscr{I}) = 0$ for any quasi-coherent ideal $\mathscr{I}$ of $\mathscr{O}_X$.*

Proof. It is clear that (i) implies (ii); (ii) implies on the oter hand that the X_{f_α} cover X, since by hypothesis the unit section 1 is a linear combination of f_α, and that the $D(f_\alpha)$ cover $\mathrm{Spec}(A)$. The last assertion of Theorem 11.4.27 then implies that $X \to \mathrm{Spec}(A)$ is an isomorphism.

We have seen that (i) implies (iii), and it is trivial that (iii) implies (iii'). On the other hand, (iii') implies that, for any closed point $x \in X$ and any open neighborhood U of x, there exists $f \in A$ such that $x \in X_f \subseteq X - U$. To see this, let $\mathscr{I}$ (resp. $\mathscr{I}'$) be the quasi-coherent ideal of $\mathscr{O}_X$ defining the reduced closed subscheme of X with underlying space $X - U$ (resp. $(X - U) \cup \{x\}$). It is clear that $\mathscr{I}' \subseteq \mathscr{I}$, and the quotient $\mathscr{I}'' = \mathscr{I}/\mathscr{I}'$ is a quasi-coherent $\mathscr{O}_X$-module. By hypothesis, the stalk of $\mathscr{O}_X/\mathscr{I}$ and $\mathscr{O}_X/\mathscr{I}'$ are zero at any point $x \in U - \{x\}$. Moreover, since $\{x\}$ is closed in X, the subscheme $X - U$ is *open and closed* in $(X - U) \cup \{x\}$, so we conclude that $(\mathscr{O}_X/\mathscr{I})_z = (\mathscr{O}_X/\mathscr{I}')_z$ for $z \in X - U$, and therefore $\mathscr{I}''_z = 0$. At the point x, we have $\mathscr{I}_x = \mathscr{O}_X$, while $\mathscr{I}'_x = \mathfrak{m}_x$ (cf. Example 10.4.46), so $\mathscr{I}''$ is supported at $\{x\}$ and $\mathscr{I}''_x = \kappa(x)$. The hypothesis of (iii') applied to the exact sequence $0 \to \mathscr{I}' \to \mathscr{I} \to \mathscr{I}'' \to 0$ shows that $\Gamma(X, \mathscr{I}) \to \Gamma(X, \mathscr{I}'')$ is surjective, so the section of $\mathscr{I}''$ whose germ at x equals to 1_x is the image of a section $f \in \Gamma(X, \mathscr{I}) \subseteq \Gamma(X, \mathscr{O}_X)$, and we have by definition $f(x) = 1_x$ and $f(y) = 0$ over $X - U$, which proves the assertion. Moreover, if U is affine, so is X_f, and the union X' of these affine opens X_f (with $f \in A$) is then an open subset of X containing any closed point of X. As X is a quasi-compact Kolmogoroff space, we then have $X' = X$ ([**?**] new, 0_{I}, 2.1.3). Since X is quasi-compact, there are finitely many elements $f_i \in A$ $(1 \le i \le n)$ such that (X_{f_i}) is an affine open cover of X. Consider the homomorphism $\mathscr{O}_X^n \to \mathscr{O}_X$ defined by the sections f_i; since for any $x \in X$ at least one of the $(f_i)_x$ is invertible, this homomorphism is surjective, and we then get an exact sequence

$$0 \longrightarrow \mathscr{R} \longrightarrow \mathscr{O}_X^n \longrightarrow \mathscr{O}_X \longrightarrow 0$$

where $\mathscr{R}$ is a quasi-coherent sub-$\mathscr{O}_X$-module of $\mathscr{O}_X$. It then follows from (iii') that the corresponding homomorphism $\Gamma(X, \mathscr{O}_X^n) \to \Gamma(X, \mathscr{O}_X)$ is surjective, which proves (ii).

Finally, (i) implies (iv) and (iv) implies (iv'). We show that (iv') implies (iii'). Now, if $\mathscr{F}'$ is a quasi-coherent sub-$\mathscr{O}_X$-module of $\mathscr{O}_X^n$, the filtration $0 \subseteq \mathscr{O}_X \subseteq \mathscr{O}_X^2 \cdots \subseteq \mathscr{O}_X^n$ defines over $\mathscr{F}'$ a filtration of the form $\mathscr{F}'_k = \mathscr{F} \cap \mathscr{O}_X^k$ $(0 \le k \le n)$, which are quasi-coherent $\mathscr{O}_X$-modules (Corollary 10.2.23(ii)), and $\mathscr{F}'_{k+1}/\mathscr{F}'_k$ is isomorphic to a quasi-coherent sub-$\mathscr{O}_X$-module of $\mathscr{O}_X^{k+1}/\mathscr{O}_X^k = \mathscr{O}_X$, which is thus a quasi-coherent ideal of $\mathscr{O}_X$. In the exact sequence

$$H^1(X, \mathscr{F}'_k) \longrightarrow H^1(X, \mathscr{F}'_{k+1}) \longrightarrow H^1(\mathscr{F}'_{k+1}/\mathscr{F}'_k) = 0$$

by hypothesis of (iv') we have $H^1(\mathscr{F}'_{k+1}/\mathscr{F}'_k) = 0$; since $H^0(X, \mathscr{F}'_0) = 0$, we conclude by recurrence on k that $H^1(X, \mathscr{F}'_k) = 0$ for each k, whence the claim. □

Remark 11.5.13. Note that if X is a covering of (X_{f_i}) with X_{f_i} being affine, then X is automatically quasi-separated, since for any couple (i, j) of indices we have $X_{f_i} \cap X_{f_j} = D_{X_{f_i}}(f_j|_{X_{f_i}})$, which is an affine open of X_{f_i} and hence quasi-compact (Proposition 10.6.10).

Remark 11.5.14. If X is a Noetherian scheme, then in conditions (iii') and (iv') we can replace "quasi-coherent" by "coherent." In fact, in the demonstration that (iii') implies (ii), $\mathscr{J}$ and $\mathscr{J}'$ are then coherent ideals, and moreover, any quasi-coherent submodule of a coherent module is coherent (Theorem 10.1.30), whence the assertion.

Corollary 11.5.15. *Let $f : X \to Y$ be a quasi-compact and quasi-separated morphism. Then the following conditions are equivalent:*

(i) *f is an affine morphism.*

(ii) *The functor f_* is exact on the cateogry of quasi-coherent $\mathscr{O}_X$-modules.*

(iii) *For any quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, we have $R^1 f_*(\mathscr{F}) = 0$.*

(iii') *For any quasi-coherent ideal $\mathscr{J}$ of $\mathscr{O}_X$, we have $R^1 f_*(\mathscr{J}) = 0$.*

Proof. Any of these conditions are local over Y, by the definition of $R^1 f_*(\mathscr{F})$ (that is, the sheaf associated with the presheaf $U \mapsto H^1(f^{-1}(U), \mathscr{F})$), so we may assume that Y is affine. If f is affine, X is then affine and (ii) follows from Corollary 10.1.13. Conversely, we prove that (ii) implies (i): for any quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, $f_*(\mathscr{F})$ is a quasi-coherent $\mathscr{O}_Y$-module by Proposition 10.6.55. By hypothesis the fonctor f_* is exact; $\Gamma(Y, -)$ is exact since Y is affine, so we conclude that $\Gamma(Y, f_*(-)) = \Gamma(X, -)$ is exact, which proves that X is affine in view of Theorem 11.5.12.

If f is affine, $f^{-1}(U)$ is affine for any affine open U of Y, so $H^1(f^{-1}(U), \mathscr{F}) = 0$ by Theorem 11.5.12, which means $R^1 f_*(\mathscr{F}) = 0$. Finally, suppose that (iii') is satisfied; the exact sequence of low-degree terms in the Leray spectral sequence gives

$$0 \longrightarrow H^1(Y, f_*(\mathscr{J})) \longrightarrow H^1(X, \mathscr{J}) \longrightarrow H^0(Y, R^1 f_*(\mathscr{J}))$$

As Y is affine and $f_*(\mathscr{J})$ is quasi-coherent (Proposition 10.6.55), we have $H^1(Y, f_*(\mathscr{J})) = 0$, so the hypothesis of (iii') imlies that $H^1(X, \mathscr{J}) = 0$, and we conclude from Theorem 11.5.12 that X is an affine scheme. □

Corollary 11.5.16. *If $f : X \to Y$ is an affine morphism then for any quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, the canonical homomorphism $H^1(Y, f_*(\mathscr{F})) \to H^1(X, \mathscr{F})$ is bijective.*

Proof. In fact, we have an exact sequence

$$0 \longrightarrow H^1(Y, f_*(\mathscr{F})) \longrightarrow H^1(X, \mathscr{F}) \longrightarrow H^0(Y, R^1 f_*(\mathscr{F}))$$

which comes from the lower terms of the Leray spectral sequence, and the conclusion follows from Corollary 11.5.15. □

11.5.3 Quasi-projective morphisms

We say a morphism $f : X \to Y$ is **quasi-projective**, of that X is **quasi-projective over Y**, or that X is a **quasi-projective Y-scheme**, if f is of finite type and there exists an invertible $\mathscr{O}_X$-module that is f-ample. It is clear that a quasi-projective morphism is necessarily separated. If Y is quasi-compact, it is also equivalent to say that f is of finite type and there exists a very ample $\mathscr{O}_X$-module relative to f (Proposition 11.4.38).

Remark 11.5.17. It should be noted that this definition is not local over Y. There exist examples where X and Y are nonsingular algebraic schemes over an algebraically closed field such that any point of Y admits an affine neighborhood U such that $f^{-1}(U)$ is quasi-projective over U, but f is not quasi-projective.

Proposition 11.5.18. *Let Y be a quasi-compact and quasi-separated scheme and X be a Y-scheme. Then the following conditions are equivalent:*

(i) *X is a quasi-projective Y-scheme.*

(ii) *X is of finite type over Y and there exists a quasi-coherent $\mathscr{O}_Y$-module $\mathscr{E}$ of finite type such that X is Y-isomorphic to a subscheme of $\mathbb{P}(\mathscr{E})$.*

(iii) *X is of finite type over Y and there exists a quasi-coherent graded $\mathscr{O}_Y$-algebra $\mathscr{S}$ that is generated by $\mathscr{S}_1$ and $\mathscr{S}_1$ is of finite type such that X is isomorphic to a dense open subscheme of* $\mathrm{Proj}(\mathscr{S})$.

Proof. This follows from Corollary 11.4.15, Proposition 11.4.18 and Corollary 11.4.19. □

Corollary 11.5.19. *Let Y be a quasi-compact and quasi-separated scheme such that there exists an ample $\mathscr{O}_Y$-module $\mathscr{L}$. For a Y-scheme X to be quasi-projective, it is necessary and sufficient that X is of finite type over Y and is isomorphic to a sub-Y-scheme of a projective bundle of the form $\mathbb{P}^r_Y$.*

Proof. By the hypothesis on Y, if $\mathscr{E}$ is a quasi-coherent $\mathscr{O}_Y$-module of finite type, $\mathscr{E}$ is isomorphic to a quotient of an $\mathscr{O}_Y$-module $\mathscr{L}^{\otimes(-n)} \otimes_{\mathscr{O}_Y} \mathscr{O}_Y^k$ (Proposition 11.4.29), so $\mathbb{P}(\mathscr{E})$ is isomorphic to a closed subscheme of $\mathbb{P}^{k-1}_Y$ (Proposition 11.4.1). □

Proposition 11.5.20 (Properties of Quasi-projective Morphisms).

(i) *A quasi-affine morphism of finite type (in particular a quasi-compact immersion or an affine morphism of finite type) is quasi-projective.*

(ii) *If $f : X \to Y$ and $g : Y \to Z$ are quasi-projective and if Z is quasi-compact, $g \circ f$ is quasi-projective.*

(iii) *If $f : X \to Y$ is a quasi-projective S-morphism, $f_{(S')} : X_{(S')} \to Y_{(S')}$ is quasi-projective for any base change $S' \to S$.*

(iv) *If $f : X \to Y$ and $g : X' \to Y'$ are two quasi-projective S-morphisms, $f \times_S g$ is quasi-projective.*

(v) *If $f : X \to Y$, $g : y \to Z$ are two morphisms such that $g \circ f$ is quasi-projective and if g is separated or X is locally Noetherian, then f is quasi-projective.*

(vi) *If f is quasi-projective, so is f_{red}.*

Proof. Property (i) follows from Proposition 11.5.6 and Proposition 11.4.51(i). The other parts follows from the definition of quasi-projective morphism and Proposition 11.4.51, with the corresponding properties of morphisms of finite type (Proposition 10.6.35). □

Remark 11.5.21. Note that it may happen that f_{red} is quasi-projective without f being so, even we assume that Y is the spectrum of a finite dimensional algebra over C and f is proper.

Corollary 11.5.22. *If X and X' are two quasi-projective Y-schemes, $X \amalg X'$ is quasi-projective over Y.*

Proof. This follows from Proposition 11.4.54. □

11.5.4 Universally closed and proper morphisms

As the terminology indicates, we say a morphism $f : X \to Y$ is **universally closed** if the projection $f \times_Y 1_{Y'} : X \times_Y Y' \to Y'$ is closed for any base change $Y' \to Y$. By Corollary 10.4.14, we know that a closed immersion is universally closed. We say a morphism $f : X \to Y$ is **proper** if it is separated, of finite type, and universally closed, and in this case X is said to be **proper over Y**, or a **proper Y-scheme**. It is clear that all these notations are local over Y. We also note that, to verify that the image of a closed subset Z of $X \times_Y Y'$ under the projection $q : X \times_Y Y' \to Y'$ is closed in Y', it suffices to shows that $q(Z) \cap U'$ is closed in U' for any affine open subset U' of Y'. As $q(Z) \cap U' = q(Z \cap q^{-1}(U'))$ and $q^{-1}(U')$ is identified with $X \times_Y U'$ (Corollary 10.3.2), we see that to verify the universally closedness of f, it suffices to limit the case where Y' is affine. We will see later that if Y is locally Noetherian, we can even assume that Y' is of finite type over Y.

Proposition 11.5.23 (Properties of Proper Morphisms).

(i) *A closed immersion is proper.*

(ii) *The composition of two proper morphisms is proper.*

(iii) *If $f : X \to Y$ is a proper S-morphism, then $f_{(S')} : X_{(S')} \to Y_{(S')}$ is proper for any base change $S' \to S$.*

(iv) *If $f : X \to Y$ and $g : X' \to Y'$ are two proper S-morphisms, then $f \times_S g$ is proper.*

Proof. It suffices to prove the first three properties. In view of Proposition 10.5.26 and Proposition 10.6.35, it suffices to verify the universally closedness in each cases. This is trivial in (i) since closed immersions are universal. For (ii), consider two proper morphisms $X \to Y$, $Y \to Z$, and a morphism $Z' \to Z$. We have $X \times_Z Z' = X \times_Y (Y \times_Z Z')$ and therefore the projection $X \times_Z Z' \to Z'$ factors into $X \times_Y (Y \times_Z Z') \to Y \times_Z Z' \to Z'$. By hypothesis, this is a composition of two closed morphisms, hence closed. Finally, in (iii), for any morphism $S' \to S$, $X_{(S')}$ is identified with $X \times_Y Y_{(S')}$; for any morphism $Z \to Y_{(S')}$, we have

$$X_{(S')} \times_{Y_{(S')}} Z = (X \times_Y Y_{(S')}) \times_{Y_{(S')}} Z = X \times_Y Z$$

and $X \times_Y Z \to Z$ is closed by hypothesis, so (iii) follows. □

Corollary 11.5.24. *Let $f : X \to Y$, $g : Y \to Z$ be two morphisms such that $g \circ f$ is proper.*

(a) *If g is separated, f is proper.*

(b) *If g is separated and of finite type and f is surjective, then g is proper.*

Proof. The first claim follows from Proposition 10.5.22. To prove (b), we only need to verify that g is universally closed. For any morphism $Z' \to Z$, the diagram

$$\begin{tikzcd} X \times_Z Z' \arrow[r, "f \times 1_{Z'}"] \arrow[dr, "p"'] & Y \times_Z Z' \arrow[d, "p'"] \\ & Z' \end{tikzcd}$$

(where p and p' are projections) is commutative. Moreover, $f \times 1_{Z'}$ is surjective if f is (Proposition 10.3.28), and p is a closed immersion by hypothesis. Any closed subset F of $Y \times_Z Z'$ is then the image under $f \times 1_{Z'}$ of a closed subset E of $X \times_Z Z'$, so $p'(F) = p(E)$ is closed in Z' by hypothesis, whence the corollary. ☐

Corollary 11.5.25. *If X is a proper scheme over Y and $\mathscr{S}$ is a quasi-coherent $\mathscr{O}_Y$-algebra, any Y-morphism $f : X \to \mathrm{Proj}(\mathscr{S})$ is proper (and a fortiori closed).*

Proof. In fact, the structural morphism $p : \mathrm{Proj}(\mathscr{S}) \to Y$ is separated, and $p \circ f$ is proper by hypothesis. ☐

Corollary 11.5.26. *Let $f : X \to Y$ be a separated morphism of finite type. Let $(X_i)_{1 \le i \le n}$ (resp. $(Y_i)_{1 \le i \le n}$) be a family of closed subscheme of X (Resp. Y), $j_i : X_i \to X$ (resp. $h_i : Y_i \to Y$) be the canonical injection. Suppose that (X_i) forms a covering of X and for each i, let $f_i : X_i \to Y_i$ be a morphism such that the diagram*

$$\begin{tikzcd} X_i \arrow[r, "f_i"] \arrow[d, "j_i"'] & Y_i \arrow[d, "h_i"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

is commutative. Then, for f to be proper, it is necessary and sufficient that each f_i is proper.

Proof. If f is proper, so is each $f \circ j_i$, since j_i is a closed immersion; as each h_i is a closed immersion, hence separated, f_i is proper by Corollary 11.5.24. Suppose conversely that each f_i is proper, and consider the sum Z of X_i; let $u : Z \to X$ be the morphism that induces j_i on X_i. The restriction of $f \circ u$ to each X_i is equal to $f \circ j_i = h_i \circ f_i$, hence proper; it then follows that $f \circ u$ is proper. Since u is surjective by hypothesis, we conclude from Corollary 11.5.24 that f is proper. ☐

Corollary 11.5.27. *Let $f : X \to Y$ be a separated morphism of finite type. For f to be proper, it is necessary and sufficient that $f_{\mathrm{red}} : X_{\mathrm{red}} \to Y_{\mathrm{red}}$ is proper.*

Proof. This is a particular case where $n = 1$, $X_1 = X_{\mathrm{red}}$ and $Y_1 = Y_{\mathrm{red}}$. ☐

If X and Y and Noetherian schemes and $f : X \to Y$ is a separated morphism of finite type, to verify that f is proper, we can reduce to dominant morphisms of integral schemes. In fact, let X_i $(1 \le i \le n)$ be the irreducible components of X and consider for each i the unique reduced closed subscheme structure on X_i. Let Y_i be the reduced closed subscheme with underlying space $\overline{f(X_i)}$. If $j_i : X_i \to X$ (resp. $h_i : Y_i \to Y$) are the canonical injections, we then have $f \circ j_i = h_i \circ f_i$, where f_i is a dominant morphism $f_i : X_i \to Y_i$. We then see that the conditions of Corollary 11.5.26 are satisfied, and for f to be proper, it is necessary and sufficient that each f_i is.

Corollary 11.5.28. *Let X and Y be separated S-schemes of finite type and $f : X \to Y$ be an S-morphism. For f to be proper, it is necessary and sufficient that for any S-scheme S', the morphism $f_{(S')} : X_{(S')} \to Y_{(S')}$ is closed.*

Proof. We note that if $\varphi : X \to S$ and $\psi : Y \to S$ are the structural morphisms, we have $\varphi = \psi \circ f$, so f is separated and of finite type (Proposition 10.5.26 and Proposition 10.6.35). If f is proper, so is $f_{(S')}$, and is a fortiori closed. Conversely, assume this condition and let Y' be a Y-scheme; Y' can be considered as an S-scheme, and the morphism $Y \to S$ is separated. In the commutative diagram

$$\begin{array}{ccc} X \times_Y Y' & \xrightarrow{f \times_Y 1_{Y'}} & Y \times_Y Y' = Y' \\ \downarrow & & \downarrow \\ X \times_S Y' & \xrightarrow{f_{(Y')}} & Y \times_S Y' \end{array}$$

the vertical morphisms are a closed immersion by Proposition 10.5.18. It follows from the assumption that $f_{(Y')}$ is closed, and so is $f \times_Y 1_{Y'}$. □

Let $f : X \to Y$ be a morphism of finit type. We say that a closed subset Z of X is **proper over** Y (or $\boldsymbol{Y}$**-proper**, or $\boldsymbol{f}$**-proper**) if the restriction of f to a closed subscheme of X with underlying space Z is proper. As this restriction is then separated, it follows from Corollary 11.5.27 and Proposition 10.5.26(vi) the property that Z is proper over Y is independent of the closed subscheme structural chosen for Z.

Let Z be a proper subset of X for f and let $g : X' \to X$ be a proper morphism. Then $g^{-1}(Z)$ is then a proper subset of X': if T is a subscheme of X with underlying space Z, it suffices to note that the restriction of g to the closed subscheme $g^{-1}(T)$ of X' is a proper morphism $g^{-1}(T) \to T$ by Proposition 11.5.23(iii), and we can apply Proposition 11.5.23(ii) to conclude that $g^{-1}(T)$ is proper.

On the other hand, if X'' is a Y-scheme of finite type and $h : X \to X''$ is a Y-morphism, $h(Z)$ is also a proper subset of X'': in fact, for any reduced closed subscheme T of X with underlying space Z. The restriction of f to T is proper, and so is the restriction of h to T (Corollary 11.5.24(a)), so $h(Z)$ is closed in X''. Let T'' be a closed subscheme of X'' with underlying space $h(Z)$ so that the morphism $h|_T$ factors into (cf. Proposition 10.4.48)

$$T \xrightarrow{h|_T} T'' \xrightarrow{j} X''$$

where j is the canonical injection. Then $h|_T$ is proper by Corollary 11.5.26 and surjective. If $\psi : X'' \to Y$ is the structural morphism, $\psi|_{T''}$ is then separated of finite type (Proposition 10.5.26 and Proposition 10.6.35), and we have $f|_T = (\psi|_{T''}) \circ (h|_T)$; it then follows from Proposition 11.5.23(ii) that $\psi|_{T''}$ is proper, whence the assertion.

In particular, for a Y-proper subset of X, we have the following:

(a) For any closed subset X' of X, $Z \cap X'$ is a Y-proper subset of X'.

(b) If X is a subscheme of a Y-scheme of finite type X'', Z is also a Y-proper subset of X'' (and in particular is closed in X'').

11.5.5 Projective morphisms

Proposition 11.5.29. *Let X be a Y-scheme. The following conditins are equivalent:*

(a) *X is Y-isomorphic to a closed subscheme of a projective bundle $\mathbb{P}(\mathscr{E})$, where $\mathscr{E}$ is a quasi-coherent $\mathscr{O}_Y$-module of finite type.*

(b) *There exists a quasi-coherent graded $\mathscr{O}_Y$-algebra such that $\mathscr{S}_1$ is of finite type and generates $\mathscr{S}$, and that X is Y-isomorphic to a $\mathrm{Proj}(\mathscr{S})$.*

Proof. Condition (a) implies (b) by Proposition 11.3.33(b): if $\mathscr{J}$ is the quasi-coherent graded ideal of $\boldsymbol{S}(\mathscr{E})$, the quasi-coherent graded $\mathscr{O}_Y$-algebra $\mathscr{S} = \boldsymbol{S}(\mathscr{E})/\mathscr{J}$ is generated by $\mathscr{S}_1$ and the later, which is the canonical image of $\mathscr{E}$, is an $\mathscr{O}_X$-module of finite type. Condition (b) implies (a) in view of Corollary 11.3.35 applied to the case where $\mathscr{M} \to \mathscr{S}_1$ is the identify homomorphism. □

We say a Y-scheme X is **projective over $\boldsymbol{Y}$** or a projective **$\boldsymbol{Y}$-scheme** if it satisfies the equivalent conditions of Proposition 11.5.29. We say a morphism $f : X \to Y$ is **projective** if X is a projective Y-scheme via this morphism. It is clear that if $f : X \to Y$ is projective, then there exists a very ample $\mathscr{O}_X$-module relative to f (Corollary 11.4.15).

Theorem 11.5.30. *Any projective morphism is quasi-projective and proper. Conversely, if Y is a quasi-compact and quasi-separated scheme, any quasi-projective and proper morphism $f : X \to Y$ is projective.*

Proof. It is clear that any projective morphism is of finite type and quasi-projective. On the other hand, it follows from Proposition 11.5.29(b) and Proposition 11.3.31 that if f is projective, so is $f \times_Y 1_{Y'} : X \times_Y Y' \to Y'$ for any morphism $Y' \to Y$. The proof that f is universally closed then boils down to show that a projective morphism f is closed. The question is local over Y, so we can suppose that $Y = \mathrm{Spec}(A)$ is affine, so by Proposition 11.5.29 $X = \mathrm{Proj}(S)$ where S is a graded A-algebra generated by finitely many elements of S_1. For any $y \in Y$, the fiber $f^{-1}(y)$ is identified with $\mathrm{Proj}(S) \times_Y \mathrm{Spec}(\kappa(y))$, hence to $\mathrm{Proj}(S \otimes_A \kappa(y))$ (Proposition 11.2.50). Therefore, $f^{-1}(y)$ is empty if and only if $S \otimes_A \kappa(y)$ is eventually zero, which meaens $S_n \otimes_A \kappa(y) = 0$ for n sufficiently large. Now as $(S_n)_y$ is an $\mathscr{O}_{Y,y}$-module of finite type, the preceding condition signifies that $(S_n)_y = 0$ for n sufficiently large, in view of Nakayama lemma. If $\mathfrak{a}_n$ is the annihilator in A of the A-module S_n, the preceding condition is then equivalent to that $\mathfrak{a}_n \subseteq \mathfrak{p}_y$ for n sufficiently large. Now as $S_n S_1 = S_{n+1}$ by hypothesis, we have $\mathfrak{a}_n \subseteq \mathfrak{a}_{n+1}$, and if $\mathfrak{a}$ is the sum of $\mathfrak{a}_n$, we then have $f(X) = V(\mathfrak{a})$, so $f(X)$ is closed in Y. If now X' is a closed subset of X, there exists a closed subscheme of X with underlying space X' and it is clear (by Proposition 11.5.29(a)) that the composition morphism $X' \to X \to Y$ is projective, so $f(X')$ is closed in Y.

Now conversely, assume that Y is quasi-compact and quasi-separated. The hypothesis that f is quasi-projective implies the existence of a quasi-coherent $\mathscr{O}_Y$-module of finite type $\mathscr{E}$ and a Y-immersion $j : X \to \mathbb{P}(\mathscr{E})$ (Proposition 11.5.18). Since f is proper and the structural morphism $\mathbb{P}(\mathscr{E}) \to Y$ is separated, j is proper (hence closed) by Corollary 11.5.24(a), so f is projective. □

Remark 11.5.31. Let $f : X \to Y$ be a morphism such that

(i) f is proper;

(ii) there exists a very ample $\mathscr{O}_X$-module $\mathscr{L}$ relative to f;

(iii) the quasi-coherent $\mathscr{O}_Y$-module $\mathscr{E} = f_*(\mathscr{L})$ is of finite type.

Then f is projective: there then exists a Y-immersion $j : X \to \mathbb{P}(\mathscr{E})$, and since f is proper, j is a closed immersion by Corollary 11.5.24(a). We will see that if Y is locally Noetherian, the last condition (iii) is a concequence of the others, and conditions (i) and (ii) characterize projective morphisms. If Y is Noetherian, we can further repalce in (ii) that there exists a *ample* $\mathscr{O}_X$-module relative to f (Proposition 11.4.48). We also note that there are proper morphisms that is not projective.

Remark 11.5.32. Let Y be a quasi-compact scheme such that there exists an ample $\mathscr{O}_Y$-module. For a Y-scheme X to be projective, it is necessary and sufficient that it is isomorphic to a closed subscheme of a projective bundle of the form $\mathbb{P}^r_Y$. This conditions is clearly sufficient. Conversely, if X is projective over Y, it is quasi-projective, so there exists a Y-immersion $j : X \to \mathbb{P}^r_Y$ by Corollary 11.5.19, which is closed by Corollary 11.5.24(a) and Theorem 11.5.30.

Remark 11.5.33. The reasoning of Theorem 11.5.30 shows that for any scheme Y, and any integer $r \geq 0$, the structural morphism $\mathbb{P}^r_Y \to Y$ is surjective, because if we put $\mathscr{S}_{\mathscr{O}_Y} = \boldsymbol{S}(\mathscr{O}_Y^{r+1})$, we have evidently $\mathscr{S}_y = \boldsymbol{S}_{\kappa(y)}(\kappa(y)^{r+1})$, so $(\mathscr{S}_n)_y \neq 0$ for any $y \in Y$ and $n \geq 0$.

Proposition 11.5.34 (Properties of Projective Morphisms).

(i) *A closed immersion is projective.*

(ii) *If $f : X \to Y$ and $g : Y \to Z$ are projective morphisms and if Z is quasi-compact and quasi-separated, then $g \circ f$ is projective.*

(iii) *If $f : X \to Y$ is a projective morphism, $f_{(S')} : X_{(S')} \to Y_{(S')}$ is projective for any base change $S' \to S$.*

(iv) *If $f : X \to Y$ and $g : X' \to Y'$ are projective S-morphisms, so is $f \times_S g$.*

(v) *If $g \circ f$ is a projective morphism and if g is separated, f is projective.*

(vi) *If f is projective, so is f_{red}.*

Proof. Property (i) follows from Corollary 11.3.5. It is necessary here to prove (iii) and (iv) separately, because of the restriction introduced on Z in (ii). To prove (iii), we can reduce to the case where $S = Y$ (Corollary 10.3.10) and the assertion then follows immediately from Proposition 11.5.29(b) and from Proposition 11.3.31. To prove (iv), we can assume that $X = \mathbb{P}(\mathscr{E})$, $X' = \mathbb{P}(\mathscr{E}')$, where $\mathscr{E}$ (resp. $\mathscr{E}'$) is a quasi-coherent $\mathscr{O}_Y$-module of finite type. Let p, p' be the projection of $T = Y \times_S Y'$ to Y and Y' respectively; by (11.4.1), we have $\mathbb{P}(p^*(\mathscr{E})) = \mathbb{P}(\mathscr{E}) \times_Y T$ and $\mathbb{P}(p'^*(\mathscr{E})) = \mathbb{P}(\mathscr{E}') \times_{Y'} T$, whence

$$\begin{aligned}\mathbb{P}(p^*(\mathscr{E})) \times_T \mathbb{P}(p'^*(\mathscr{E}')) &= (\mathbb{P}(\mathscr{E}) \times_Y T) \times_T (T \times_{Y'} \mathbb{P}(\mathscr{E}')) = \mathbb{P}(\mathscr{E}) \times_Y (T \times_{Y'} \mathbb{P}(\mathscr{E}')) \\ &= \mathbb{P}(\mathscr{E}) \times_Y ((Y \times_S Y') \times_{Y'} \mathbb{P}(\mathscr{E}')) = \mathbb{P}(\mathscr{E}) \times_S \mathbb{P}(\mathscr{E}').\end{aligned}$$

Now $p^*(\mathscr{E})$ and $p'^*(\mathscr{E}')$ are of finite type over T (**??**), and so is $p^*(\mathscr{E}) \otimes_{\mathscr{O}_T} p'^*(\mathscr{E}')$; as $\mathbb{P}(p^*(\mathscr{E})) \times_T \mathbb{P}(p'^*(\mathscr{E}'))$ is identified with a closed subscheme of $\mathbb{P}(p^*(\mathscr{E}) \otimes_{\mathscr{O}_T} p'^*(\mathscr{E}'))$ (Proposition 11.4.12), this proves (iv). For (v) and (vi), we can apply Proposition 10.5.22, since any closed subscheme of a projective Y-scheme is projective by Proposition 11.5.29(a). □

Proposition 11.5.35. *If X and X' are two projective Y-schemes, so is $X \amalg X'$.*

Proof. This is clear from Remark 11.4.13. □

Proposition 11.5.36. *Let X be a projective Y-scheme, $\mathscr{L}$ be a Y-ample $\mathscr{O}_Y$-module. For any section f of $\mathscr{L}$ over X, X_f is affine over Y.*

Proof. Since the question is local over Y, we can assume that $Y = \operatorname{Spec}(A)$ is affine; since $X_{f^{\otimes n}} = X_f$, by replacing $\mathscr{L}$ by $\mathscr{L}^{\otimes n}$ we can assume that $\mathscr{L}$ is very ample for the structural morphism $q : X \to Y$ (Proposition 11.4.34). The canonical homomorphism $\sigma : q^*(q_*(\mathscr{L})) \to \mathscr{L}$ is then surjective and the corresponding morphism

$$r = r_{\mathscr{L},\sigma} : X \to P = \mathbb{P}(q_*(\mathscr{L}))$$

is an immersion such that $\mathscr{L} \cong r^*(\mathscr{O}_P(1))$ (Proposition 11.4.16). Moreover, as X is proper over Y, the immersion r is closed by Corollary 11.5.24. By definition $f \in \Gamma(Y, q_*(\mathscr{L}))$ and $\sigma^\flat$ is the identity of $q_*(\mathscr{L})$; it then follows from the formula (11.3.11) that we have $X_f = r^{-1}(D_+(f))$. Then X_f is a closed subscheme of the affine scheme $D_+(f)$, and therefore is affine. □

Remark 11.5.37. If we take $Y = X$ in Proposition 11.5.36, we obtain that, for any scheme X and any invertible $\mathscr{O}_X$-module $\mathscr{L}$, the open subset X_f is affine over X.

Proposition 11.5.38. *Let X be a projective Y-scheme and $\mathscr{L}$ be a Y-ample $\mathscr{O}_Y$-module. Then X is Y-isomorphic to* $\mathrm{Proj}(\mathscr{S})$*, where $\mathscr{S} = \bigoplus_{n\geq 0} f_*(\mathscr{L}^{\otimes n})$.*

Proof. By Proposition 11.4.39 we see that the canonical morphism $r_{\mathscr{L},\sigma} : X \to \mathrm{Proj}(\mathscr{S})$ is a dominant open immersion. Since X is proper over Y and $\mathrm{Proj}(\mathscr{S})$ is separated over Y, the image of $r_{\mathscr{L},\sigma}$ is closed by Corollary 11.5.24, so $r_{\mathscr{L},\sigma} : X \to \mathrm{Proj}(\mathscr{S})$ is an isomorphism. □

11.5.6 Chow's lemma

Theorem 11.5.39 (Chow's lemma). *Let X be a separated S-scheme of finite type and suppose that one of the following conditions is satisfied:*

(a) *S is Noetherian.*

(b) *S is quasi-compact and X has finitely many irreducible components.*

Then there exists a quasi-projective S-scheme X' and a surjective projective S-morphism $f : X' \to X$ that induces an isomorphism an isomorphism $f^{-1}(U) \cong U$ for some open dense subset of X. If X is reduced (resp. irreducible), we can also choose X' to be reduced (resp. irreducible).

Proof. The proof is divided into several steps. First of all, we can assume that X is irreducible. To see this, we note that in both cases the scheme X has finitely many irreducible components X_i. If the theorem is demonstrated for each reduced subscheme X_i, and if $f_i : X_i' : X_i$ is the corresponding hommorphism which induces an isomorphism $f_i^{-1}(U_i) \cong U_i$ with $U_i \subseteq X_i$, the sum $X' = \coprod_i X_i'$ is then quasi-projective over S (Corollary 11.5.22 and Proposition 11.5.20) and the morphism $f : X' \to X$ whose restriction on X_i' equals to $j_i \circ f_i$ (where $j_i : X_i \to X$ is the canonical injection), is then surjective and projective (Proposition 11.5.35); it is immediate to see that X' is reduced if each X_i' is. We now choose U to be the union of $U_i \cap (\bigcup_{j\neq i} X_j)^c$; since U_i is dense in X_i and X_i is maximal irreducible, we conclude that each $U_i \cap (\bigcup_{j\neq i} X_j)$ is nonempty. The open subset U is then dense in X and f clearly induces an isomorphism $f^{-1}(U) \cong U$.

So suppose now that X is irreducible. As the structural morphism $\eta : X \to S$ is of finite type, there exists a finite covering (S_i) of S by affine opens, and for each i there is a finite covering (T_{ij}) of $\eta^{-1}(S_i)$ by affine opens, with the morphism $T_{ij} \to S_i$ being affine and of finite type, hence quasi-projective (Proposition 11.5.20(i)). As in both hypotheses the immersion $S_i \to S$ is quasi-compact, it is quasi-projective by Proposition 11.5.20(i), so the restriction of η to T_{ij} is quasi-projective (Proposition 11.5.20(ii)). We relabel the T_{ij} by U_k with $1 \leq k \leq n$. There exists, for each index k, an open immersion $\varphi_k : U_k \to P_k$, where P_k is projective over S (Proposition 11.5.18). Let $U = \bigcap_k U_k$; as X is irreducible and the U_k is nonempty, U is nonempty, and therefore is dense in X; the restrictions of φ_k to U together define a morphism

$$\varphi : U \to P = P_1 \times_S P_2 \times_S \cdots \times_S P_n$$

which fits into the diagram

$$\begin{tikzcd} U \arrow[r, "\varphi"] \arrow[d, "j_k"'] & P \arrow[d, "p_k"] \\ U_k \arrow[r, "\varphi_k"] & P_k \end{tikzcd} \tag{11.5.1}$$

where j_k is the canonical injection and p_k is the canonical projection. If $j : U \to X$ is the canonical injection, the morphism $\psi = (j, \varphi)_S : U \to X \times_S P$ is then an immersion by Corollary 10.5.16. Under the hypotheses of (a), $X \times_S P$ is locally Noetherian (Proposition 11.3.23 and

Corollary 10.6.22), and under the hypotheses of (b), $X \times_S P$ is quasi-compact. In both cases the scheme-theoretic image X' of ψ in $X \times_S P$ exists (which is the closure of $\psi(U)$ in $X \times_S P$) and ψ factors into

$$\psi : U \xrightarrow{\psi'} X' \xrightarrow{h} X \times_S P$$

where ψ' is a dominant open immersion and h is a closed immersion. Let $q_1 : X \times_S P \to X$ and $q_2 : X \times_P \to P$ be the canonical projections; we put

$$f : X' \xrightarrow{h} X \times_S P \xrightarrow{q_1} X, \quad g : X' \xrightarrow{h} X \times_S P \xrightarrow{q_2} P. \tag{11.5.2}$$

We shall verify that the scheme X' and the morphism f satisfy the requirements. First we show that f is projective and surjective, and that the restriction of $U' = f^{-1}(U)$ is an isomorphism from U' to U. As the P_k are projective over S, so is P (Proposition 11.5.34(iv)), and $X \times_S P$ is projective over X by Proposition 11.5.34(iii); then X' is also projective over X, since it is a closed subscheme of $X \times_S P$. On the other hand, we have $f \circ \psi' = q_1 \circ (h \circ \psi') = q_1 \circ \psi = j$, so $f(X')$ contains the dense open subset U of X; but f is proper by Theorem 11.5.30, so $f(X') = X$. Now $q_1^{-1}(U) = U \times_S P$ is an open subscheme of $X \times_S P$, and the the immersion ψ factors into

$$\psi : U \xrightarrow{\Gamma_\varphi} U \times_S P \xrightarrow{j \times 1} X \times_S P.$$

By Proposition 10.6.70, $U' = h^{-1}(U \times_S P)$ is the scheme-theoretic image of $\psi^{-1}(U \times_S P) = U$ under $\psi_U : U \to U \times_S P$, and therefore the closure of the image of Γ_φ in $U \times_S P$. As P is separted over S, Γ_φ is a closed immersion (Corollary 10.5.19), so we conclude that $U' = \psi(U)$. As ψ is an immersion, the restriction of f to U' is then an isomorphism, with inverse ψ'. Finally, by definition, $U' = \psi(U) = \psi'(U)$ is open and dense in X'.

We now show that g is an immersion, which implies that X' is quasi-projective over S, since P is projective over S. Let $V_k = \varphi_k(U_k)$ be the image of U_k in P_k, $W_k = p_k^{-1}(V_k)$ be the inverse image in P, and put $U'_k = f^{-1}(U_k)$, $U''_k = g^{-1}(W_k)$. Since the U_k cover X, it clear that the U'_k form an open covering of X'; we first shows that this is also true for the U''_k, by proving that $U'_k \subseteq U''_k$. For this, it suffices to estabilsh the commutativity of the diagram

$$\begin{tikzcd} U'_k \arrow[r, "g|_{U'_k}"] \arrow[d, "f|_{U'_k}"'] & P \arrow[d, "p_k"] \\ U_k \arrow[r, "\varphi_k"] & P_k \end{tikzcd} \tag{11.5.3}$$

Since $U'_k = h^{-1}(U_k \times_S P)$ and $\psi^{-1}(U_k \times_S P) = U$, by Proposition 10.6.70 U'_k is the scheme-theoretic image of U in $U_k \times_S P$ under the morphism $\psi_k : U \to U_k \times_S P$ induced by ψ. It then suffices to prove the commutativity of the diagram obtained by composing (11.5.3) with the morphism ψ_k (Corollary 10.6.68), and this comes from the commutative diagram (11.5.1).

The W_k then form an open covering of $g(X')$, so to show that g is an immersion, it suffices to show the restriction $g|_{U''_k}$ is an immersion into W_k (Corollary 10.4.11). For this, consider the morphism

$$u_k : W_k \xrightarrow{p_k} V_k \xrightarrow{\varphi_k^{-1}} U_k \to X$$

which fits into the following commutative diagram

$$\begin{tikzcd}
U' \arrow[r, hook] \arrow[dr] & U'_k \arrow[r, hook] \arrow[dr] & U''_k \arrow[r, hook, "h|_{U''_k}"] \arrow[dr, "f|_{U''_k}"'] \arrow[rr, bend left, "g|_{U''_k}"] & X \times_S W_k \arrow[r, "q_2"] \arrow[d, "q_1"] & W_k \arrow[l, bend right, "\Gamma_{u_k}"'] \arrow[dl, "u_k"'] \arrow[d, "p_k"] \\
 & U \arrow[r, hook] & U_k \arrow[r, hook] \arrow[rr, bend right, "\cong"', "\varphi_k"'] & X & V_k
\end{tikzcd}$$

By the definition of g (formula (11.5.2)), we have $U''_k = h^{-1}(X \times_S W_k) \subseteq X'$ and $\psi^{-1}(X \times_S W_k) = U$, so by Proposition 10.6.70, U''_k is the scheme-theoretic image of U under the morphism $U \to X \times_S W_k$ induced by ψ; since $U' = \psi(U)$, it is therefore dense in U''_k. On the other hand, as X is separted over S, the graph morphism $\Gamma_{u_k} : W_k \to X \times_S W_k$ is a closed immersion, so the graph $T_k = \Gamma_{u_k}(W_k)$ is a closed subscheme of $X \times_S W_k$. If we can prove that T_k dominates the canonical image of the open subscheme U' in $X \times_S W_k$, it will then dominates the subscheme U''_k. As the restriction of q_2 to T_k is an isomorphism onto W_k and h is a closed immersion, the restriction of g to X''_k will then be an immersion in W_k, and our assertion will be proved. For this, we let $v_k : U' \to X \times_S W_k$ be the canonical injection, and $w_k = q_2 \circ v_k$; then from the definition of Γ_{u_k} we have $v_k = \Gamma_{u_k} \circ w_k$, and the image of U' in $X \times_S W_k$ is therefore contained in T_k, verifying our claim.

It is clear that U, and therefore U', are irreducible, and so is X' by our construction, and that f is birational. If X is reduced, so is U', and X' is then reduced (Proposition 10.6.69). This completes the proof. □

Corollary 11.5.40. *Suppose the hypotheses of Theorem 11.5.39. For X to be proper over S, it is necessary and sufficient that there exists a projective S-scheme X' and a surjective S-morphism $f : X' \to X$ (which is projective by Proposition 11.5.34(v)). If this is the case, we can choose an open dense subset U of X such that f induces an isomorphism $f^{-1}(U) \cong U$ and that $f^{-1}(U)$ is dense in X'. If X is irreducible (resp. reduced), we can choose X' to be irreducible (resp. reduced). If X and X' are irreducible, f is then a birational morphism.*

Proof. The conditions is sufficient by Theorem 11.5.30 and Corollary 11.5.24(b). This is necessary because with the notations of Theorem 11.5.39, if X is proper over S, X' is then proper over S (since it is proper over X by Theorem 11.5.30), and our assertion follows from Proposition 11.5.23(ii). Moreover, as X' is quasi-projective over S, it is projective over S in view of Theorem 11.5.30. □

Corollary 11.5.41. *Let S be a locally Noetherian scheme, X be an S-scheme of finite type, and $\varphi : X \to S$ be the structural morphism. For X to be proper over S, it is necessary and sufficient that for any morphism of finite type $S' \to S$, the morphism $\varphi_{(S')} : X_{(S')} \to S'$ is closed. Moreover, it suffices to verify this condition for any S-scheme of the form $S' = S \otimes_{\mathbb{Z}} \mathbb{Z}[T_1, \dots, T_n]$.*

Proof. The conditions is clearly necessary, and we now prove the sufficiency. The question is local over S and S', so we may assume that S, S' are affine and Noetherian. By Chow's lemma, there exists a projective S-scheme P, an immersion $j : X' \to P$, and a projective surjective morphism $f : X' \to X$ such that the diagram

$$\begin{tikzcd}
X' \arrow[dr, "\psi"] \arrow[drr, bend left, "f"] \arrow[ddr, bend right, "j"'] & & \\
& X \times_S P \arrow[r, "q_1"] \arrow[d, "q_2"'] & X \arrow[d, "\varphi"] \\
& P \arrow[r, "r"] & S
\end{tikzcd}$$

is commutative; let $\psi = (f, j)_S$. As P is of finite type over S, the projection $q_2 : X \times_S P \to P$ is a closed morphism by hypotheses. On the other hand, since f is projective and the projection $q_1 : X \times_S P \to X$ is separated (since P is separated over S), we conclude from Proposition 11.5.34(v) that ψ is projective, hence closed. Since the immersion j is the composition of q_2 with ψ, it is therefore a closed immersion, whence proper. Moreover, the structural morphism $r : P \to S$ is projective, hence proper (Theorem 11.5.30), so $\varphi \circ f = r \circ j$ is proper. As f is surjective, we conclude by Corollary 11.5.24(b) that φ is proper.

To estabilsh the second assertion of the proposition, it suffices to prove that it implies that $\varphi_{(S')}$ is closed for any morphism $S' \to S$ of finite type. Now if S' is affine and of finite type over $S = \mathrm{Spec}(A)$, we have $S' = \mathrm{Spec}(A[x_1, \dots, x_n])$, and S' is then isomorphic to a closed subscheme of $S'' = \mathrm{Spec}(A[T_1, \dots, T_n])$ (where T_i are indeterminates). In the following commutative diagram

$$\begin{tikzcd}
X \times_S S' \arrow[r, "1_X \times j"] \arrow[d, "\varphi_{(S'')}"'] & X \times_S S'' \arrow[d, "\varphi_{(S'')}"] \\
S' \arrow[r, "j"] & S''
\end{tikzcd}$$

where j and $q_X \times j$ are closed immersions (Corollary 10.4.14) and $\varphi_{(S'')}$ is closed by hypothesis. We then conclude that $\varphi_{(S')}$ is closed, whence the claim. □

11.6 Integral morphisms and finite morphisms

11.6.1 Integral and finite morphisms

Let X be an S-scheme and $f : X \to S$ be the structural morphism. We say that X is **integral over S**, or that f is an **integral morphism**, if there exists an affine open covering (S_α) of S such that for each α, the open subscheme $f^{-1}(S_\alpha)$ of X is affine and its ring $B_\aleph$ is an integral algebra over the ring A_α of S_α. We say that X is **finite over** S, or that f is a **finite morphism**, if X is integral and of finite type over S. If S is affine with ring A, we also say that X is inegral or finite over A.

It is clear that any integral S-scheme is affine over S. Conversely, form the definition of integral morphisms, we see that for an affine S-scheme X to be integral over S (resp. finite), it is necessary and sufficient that the associated quasi-coherent $\mathscr{O}_S$-algebra $\mathscr{A}(X)$ is such that there exists an affine open covering (S_α) of S such that for each α, $\Gamma(S_\alpha, \mathscr{A}(X))$ is an integral algebra (resp. an integral algebra of finite type) over $\Gamma(S_\alpha, \mathscr{O}_S)$. A quasi-coherent $\mathscr{O}_S$-algebra satisfying this property is said to be **integral** (resp. **finite**) over $\mathscr{O}_S$. We note that a quasi-coherent $\mathscr{O}_S$-algebra $\mathscr{B}$ is finite if and only if it is an $\mathscr{O}_S$-module of finite type; it amounts to the same thing to say that $\mathscr{B}$ is an integral $\mathscr{O}_S$-algebra of finite type, because an integer algebra of finite type over a ring A is an A-module of finite type.

Proposition 11.6.1. *Let S be a locally Noetherian scheme. For an S-scheme X affine over S to be finite over S, it is necessary and sufficient that the $\mathscr{O}_S$-algebra $\mathscr{A}(X)$ is coherent.*

Proof. With the preceding remark, this follows from the fact that if S is locally Noetherian, then a quasi-coherent $\mathscr{O}_S$-module is of finite type if and only if it is coherent (Theorem 10.1.30). □

Proposition 11.6.2. *Let X be an integral (resp. finite) scheme over S with $f : X \to S$ the structural morphism. Then for any affine open subset $U \subseteq S$ with ring A, $f^{-1}(U)$ is affine and its ring B is an integral (resp. finite) algebra over A.*

Proof. To prove this proposition, we need Proposition 1.4.53. We now that $f^{-1}(U)$ is affine by Proposition 11.1.4. If $\varphi : A \to B$ is the corresponding homomorphism, there exists a finite covering of U by open subsets $D(g_i)$ ($g_i \in A$) such that, if $h_i = \varphi(g_i)$, then B_{h_i} is an integral (resp. finite) algebra over A_{g_i}. In fact, by assumption, there is a covering of U by affine open subsets $V_\alpha \subseteq U$ such that if $A_\alpha = \Gamma(V_\alpha, \mathscr{O}_S)$ and $B_\alpha = \Gamma(f^{-1}(V_\alpha), \mathscr{O}_X)$, then B_α is an integral (resp. finite) algebra over A_α. Any $x \in U$ belongs to one V_α, so there exists $g \in A$ such that $x \in D(g) \subseteq V_\alpha$. If g_α is the image of g in A_α, we have $\Gamma(D(g), \mathscr{O}_S) = A_g = (A_\alpha)_{g_\alpha}$; let $h = \varphi(g)$, and let h_α be the image of g_α in B_α. We have

$$\Gamma(D(h), \mathscr{O}_S) = B_h = (B_\alpha)_{h_\alpha}$$

and as B_α is integral over A_α, $(B_\alpha)_{h_\alpha}$ is integral (resp. finite) over $(A_\alpha)_{g_\alpha}$. Since U is quasi-compact, we obtain a finite cover.

If we suppose first that each B_{h_i} is a finite algebra over A_{g_i}, then as an A_{g_i}-module, B_{h_i} is finitely generated, so Proposition 1.4.53 shows that B is a finitely generated A-module. Now assume that each B_{h_i} is integral over A_{g_i}; let $b \in B$, and let C be a sub-A-algebra of B generated by b. For each i, C_{h_i} is the A_{g_i}-algebra generated by $b/1$ over B_{h_i}. It then follows from the hypothesis that each C_{h_i} is finitely generated A_{g_i}-module, so by Proposition 1.4.53 C is a finitely generated A-module. This shows that B is integral over A. □

Proposition 11.6.3 (Properties of Integral and Finite morphisms).

(i) *A closed immersion is finite (and a fortiori integral).*

(ii) *The composition of two integral morphisms (resp. finite) is integral (resp. finite).*

(iii) *If $f : X \to Y$ is an integral (resp. finite) S-morphism, $f_{(S')} : X_{(S')} \to Y_{(S')}$ is integral (resp. finite) for any base change $S' \to S$.*

(iv) *If $f : X \to Y$ and $g : X' \to Y'$ are two integral (resp. finite) S-morphisms, so is $f \times_S g$.*

(v) *If $f : X \to Y$ and $g : Y \to Z$ are two morphisms such that $g \circ f$ is integral (resp. finite) and g is separated, then f is integral (resp. finite).*

(vi) *If $f : X \to Y$ is an integral (resp. finite) morphism, so is f_{red}.*

Proof. In view of Proposition 10.5.22, it suffices to prove (i), (ii), and (iii). To prove that a closed immersion $X \to S$ is finite, we can assume that $S = \operatorname{Spec}(A)$, and this then follows from the fact that a quotient ring $A/\mathfrak{a}$ is a finitely generated A-module. To prove the composition of two integral (resp. finite) morphisms $X \to Y$, $Y \to Z$ is integral (finite), we can assume that Z (and therefore X and Y) is affine, and the assertion is then equivalent to that if B is an integral (resp. finite) A-algebra and C is an integral (resp. finite) B-algebra, then C is an integral (resp. finite) A-algebra, which is immediate. Finally, to prove (iii), we can Simialrly assume that $S = \operatorname{Spec}(A)$, $S' = \operatorname{Spec}(A')$; then X is affine with ring B (Proposition 11.6.2), $X_{(S')}$ is affine with ring $A' \otimes_A B$, and it suffices to note that if B is an integral (resp. finite) A-algebra, then $A' \otimes_A B$ is an integral (resp. finite) A'-algebra. □

Note also that if X and Y are two integral (resp. finite) S-schemes, the sum $X \amalg Y$ is an integral (resp. finite) over S, because a product of two integral (resp. finite) A-algebras is still integral (resp. finite).

Corollary 11.6.4. *If X is an integral (resp. finite) scheme over S, then for any open subset $U \subseteq S$, $f^{-1}(U)$ is integral (resp. finite) over U.*

Proof. This is a particular case of Proposition 11.6.3(iii). $\square$

Corollary 11.6.5. *Let $f : X \to Y$ be a finite morphism. Then for any $y \in Y$, the fiber $f^{-1}(y)$ is a finite algebraic scheme over $\kappa(y)$, and a fortiori with discrete and finite underlying space.*

Proof. The $\kappa(y)$-scheme $f^{-1}(y)$ is identified with $X \times_Y \mathrm{Spec}(\kappa(y))$, so is finite over $\kappa(y)$ by Proposition 11.6.3(iii). This is then an affine scheme whose ring is a finite dimensional $\kappa(y)$-algebra, so is Artinian. The proposition then follows from Proposition 10.2.33. $\square$

Corollary 11.6.6. *Let X and S be integral schemes and $f : X \to S$ be a dominant morphism. If f is integral (resp. finite), then the rational function field $K(X)$ of X is an algebraic (resp. finite) extension of $K(S)$.*

Proof. Let s be the generic point of S; the $\kappa(s)$-scheme $f^{-1}(s)$ is integral (resp. finite) over $\mathrm{Spec}(\kappa(s))$ by Proposition 11.6.3(iii), and contains by hypothesis the generic point of x of X; the local ring of $f^{-1}(s)$ at x, equal to $\kappa(x)$ (Proposition 10.3.37), is a localization of an integral (resp. finite) algebra over $\kappa(s)$, whence the corollary. $\square$

Proposition 11.6.7. *Any integral morphism is universally closed.*

Proof. Let $f : X \to Y$ be an integral morphism. In view of Proposition 11.6.3(iii), it suffices to prove that f is closed. Let Z be a closed subset of X. In view of Proposition 11.6.3(vi), we can suppose that X and Y are reduced; moreover, if T is the reduced closed subscheme of Y with underlying space $\overline{f(X)}$, we see that f factors into

$$f : X \xrightarrow{g} T \xrightarrow{j} Y,$$

where $j : T \to Y$ is the canonical injection, and as j is separated, it follows from Proposition 11.6.3(v) that g is an integral morphism. We can then assume that $f(X)$ is dense in Y, and prove that $f(X) = Y$. Since the question is local over Y, we can assume that $Y = \mathrm{Spec}(A)$ is affine, so $X = \mathrm{Spec}(B)$ where B is an integral algebra over A (Proposition 11.6.2); moreover A is reduced and the hypothesis that $f(X)$ is dense in Y implies that the corresponding homomorphism $\varphi : A \to B$ is injective (Corollary 1.4.21). The condition that $f(X) = Y$ then follows from **??**. $\square$

Remark 11.6.8. The hypothesis that g is separated is essential for the validity of Proposition 11.6.3(v): in fact, if Y is not separated over Z, the identity 1_Y is the composition morphism

$$Y \xrightarrow{\Delta_Y} Y \times_Z Y \xrightarrow{p_1} Y$$

but Δ_Y is not integral, since it is not closed (Proposition 11.6.7).

Corollary 11.6.9. *Any finite morphism $f : X \to Y$ is projective.*

Proof. As f is affine, $\mathscr{O}_X$ is a very ample $\mathscr{O}_X$-module relative to f; moreover $f_*(\mathscr{O}_X)$ is a quasi-coherent $\mathscr{O}_Y$-algebra of finite type by hypothesis. Finally, f is separated, of finite type, and universally closed (Proposition 11.6.7), and we then have the conditions of Remark 11.5.31. $\square$

Lemma 11.6.10. *Let Y be a scheme, $\mathscr{E}$ be a locally free $\mathscr{O}_Y$-module of rank r, and Z be a finite subset of Y contained in an affine open subset V. Then there exists an open neighborhood $U \subseteq V$ of Z such that $\mathscr{E}|_U$ is isomorphic to $\mathscr{O}_Y^r|_U$.*

Proof. $\square$

Proposition 11.6.11. *Let $f : X' \to X$ be a finite morphism, and let $\mathscr{B} = f_*(\mathscr{O}_{X'})$ (which is a quasi-coherent finite $\mathscr{O}_X$-algebra). Let $\mathscr{F}'$ be a quasi-coherent $\mathscr{O}_{X'}$-module; for $\mathscr{F}'$ to be locally free of rank r, it is necessary and sufficient that $f_*(\mathscr{F}')$ is a locally free $\mathscr{B}$-module of rank r.*

Proof. It is clear that if $f_*(\mathscr{F}')|_U$ is isomorphic to $\mathscr{B}^r|_U$ (where U is open in X), $\mathscr{F}'|_{f^{-1}(U)}$ is isomorphic to $\mathscr{O}_{X'}^r|_{f^{-1}(U)}$ (Corollary 11.1.19). Conversely, suppose that $\mathscr{F}'$ is locally free of rank r and we prove that $f_*(\mathscr{F}')$ is locally isomorphic to $\mathscr{B}^r$ as $\mathscr{B}$-modules. Let x be a point of X; if U runs through affine neighborhoods of x, $f^{-1}(U)$ form a fundamental system of affine neighborhoods (Proposition 11.1.4) of the finite subset $f^{-1}(x)$, since f is closed (Proposition 11.6.7). The proposition then follows from Lemma 11.6.10. □

Proposition 11.6.12. *Let $g : X' \to X$ be an integral morphism of schemes, Y be a locally integral and normal scheme, f be a rational map from Y to X' such that $g \circ f$ is a everywhere defined rational map; then f is everywhere defined.*

Proof. Recall that we say a scheme X is normal if it is normal as a ringed space, which means the stalk $\mathscr{O}_{X,x}$ is an integrally closed domain for every $x \in X$. If f_1 and f_2 are two morphisms (densely defined from Y to X') in the class of f, it is clear that $g \circ f_1$ and $g \circ f_2$ are equivalent morphisms, which justifies the notation $g \circ f$ for their equivalent class. We recall also that if Y is locally Noetherian, then the hypothesis on Y implies that Y is locally integral (Proposition 10.4.35).

To prove the proposition, we first note that the question is local over Y and we can assume that there exists a morphism $h : Y \to X$ in the class of $g \circ f$. Consider the inverse image $Y' = X'_{(h)} = X'_{(Y)}$, and note that the morphism $g' = g_{(Y)} : Y' \to Y$ is integral by Proposition 11.6.3(iii). Via the correspondence of rational maps from Y to X' with rational Y-sections of Y', we see that we are reduced to the case $X = Y$. □

Corollary 11.6.13. *Let X be a locally integral and normal scheme, $g : X' \to X$ be an integral morphism, and f be a rational X-section of X'. Then f is everywhere defined.*

Corollary 11.6.14. *Let X be a normal and integral scheme, X' be an integral scheme, and $g : X' \to X$ be an integral morphism. If there exists a rational X-section f of X', g is an isomorphism.*

11.6.2 Quasi-finite morphisms

Proposition 11.6.15. *Let $f : X \to Y$ be a morphism locally of finite type, x be a point of X and $y = f(x)$. The following conditions are equivalent:*

(i) *The point x is isolated in the fiber X_y.*

(ii) *The point x is closed in X_y and there is no generalization of x in X_y.*

(iii) *The $\mathscr{O}_{Y,y}$-module $\mathscr{O}_{X,x}$ is quasi-finite.*

Proof. The question is local over X and Y, so we can suppose that $X = \operatorname{Spec}(B)$ and $Y = \operatorname{Spec}(A)$ are affine, where B is finite A-algebra. Moreover, replacing X by $X \times_Y \operatorname{Spec}(\mathscr{O}_{Y,y})$ does not change the fiber X_y and the local ring $\mathscr{O}_{X,x}$ (Proposition 10.3.37), so we can suppose that A is a local ring with maximal ideal $\mathfrak{m}$ (which equals to the local ring $\mathscr{O}_{Y,y}$). The fiber X_y is then the affine sheme of the ring $B/\mathfrak{m}B$, of finite type over $\kappa(y) = A/\mathfrak{m}$ (Proposition 10.6.48). Let $\mathfrak{P}$ be the prime ideal of B corresponding to x.

We note that the fiber $X_y = X \times_Y \operatorname{Spec}(\kappa(y)) = \operatorname{Spec}(B \otimes_A \kappa(y))$ is Jacobson. If (i) is satisfied, then $\{x\}$ is an open subset of X_y, hence contains a closed point (by the Jacobson property) which must be x, and x is therefore closed in X_y. Also, since $\{x\}$ is open in X_y, it is clear that there is no further generalization x' of x (which means $x \in \overline{\{x'\}}$) in X_y; this proves (i)⇒(ii).

We now consider the conditions in (ii) and (iii). Consider the ring $\bar{B} = B \otimes_A \kappa(y) = B/\mathfrak{m}B$ and let $\bar{\mathfrak{P}}$ be the prime ideal corresponding to $\mathfrak{P}$. If x is closed in X_y, we see that $\bar{\mathfrak{P}}$ is maximal in $\bar{B}$ and by condition (ii) there is no prime ideal of $\bar{B}$ contained in $\bar{\mathfrak{P}}$. This shows that $\bar{B}_{\bar{\mathfrak{P}}}$ is Artinian, so we conclude that the ring homomorphism $A \to B$ is quasi-finite at $\mathfrak{P}$; that is, the $\mathscr{O}_{Y,y}$-module $\mathscr{O}_{X,x}$ is quasi-finite. Conversely, if $\bar{B}_{\bar{\mathfrak{P}}}$ is Artinian, then $\bar{\mathfrak{P}}$ is maximal and there is no prime ideal of $\bar{B}$ contained in $\bar{\mathfrak{P}}$; we then conclude that x is closed in X_y and there is no generalization of x in X_y. This shows that (ii)⇔(iii).

We finally prove that (ii) implies (i). If (ii) is satisfied, then the prime ideal $\mathfrak{P}$ is both maximal and minimal in B since $\bar{B}_{\bar{\mathfrak{P}}}$ is Artinian. Let $\bar{\mathfrak{P}}_1 = \bar{\mathfrak{P}}, \bar{\mathfrak{P}}_2, \dots, \bar{\mathfrak{P}}_r$ be the minimal prime ideals of $\bar{B}$. Then the intersection $\bigcap_{i=1}^r \bar{\mathfrak{P}}_i$ is equal to the nilradical of $\bar{B}$, and $\bigcap_{i=2}^r \bar{\mathfrak{P}}_i$ is contained in this intersection in view of Proposition 1.1.5, so there exists an element $\bar{b} \in \bigcap_{i=2}^r \bar{\mathfrak{P}}_i$ that is not nilpotent. The open subset $D_{\bar{B}}(\bar{b})$ of $X_y = \operatorname{Spec}(\bar{B})$ then reduces to $\{x\}$, which shows that x is isolated. □

Corollary 11.6.16. *Let $f : X \to Y$ be a morphism of finite type. Then the following conditions are equivalent:*

(i) *Any point $x \in X$ is isolated in the fiber $X_{f(x)}$ (that is, $X_{f(x)}$ is discrete).*

(ii) *For any $x \in X$, $X_{f(x)}$ is a finite $\kappa(f(x))$-scheme.*

(iii) *For any $x \in X$, the $\mathscr{O}_{Y,y}$-module $\mathscr{O}_{X,x}$ is quasi-finite.*

Proof. The equivalence of (i) and (iii) follows from Proposition 11.6.15. On the other hand, as $X_{f(x)}$ is an algebraic $\kappa(f(x))$-scheme, the equivalence of (i) and (ii) follows from Proposition 10.6.44. □

We say a morphism $f : X \to Y$ of finite type is **quasi-finite**, or X is **quasi-finite** over Y, if it satisfies the equivalent conditions of Corollary 11.6.16. We say a morphism $f : X \to Y$ is quasi-finite at a point $x \in X$ if there exists an affine open neighborhood V of $y = f(x)$ and an affine open neighborhood U of x such that $f(U) \subseteq V$ and the restriction $f|_U : U \to V$ is quasi-finite. We say that $f : X \to Y$ is locally quasi-finite if it is quasi-finite at every point of X. From Corollary 11.6.5, it is clear that any finite morphism is quasi-finite.

Proposition 11.6.17 (Properties of Quasi-finite Morphisms).

(i) *Any quasi-compact immersion (in particular any closed immersion) is quasi-finite.*

(ii) *If $f : X \to Y$ and $g : Y \to Z$ are quasi-finite morphisms, $g \circ f$ is quasi-finite.*

(iii) *If $f : X \to Y$ is a quasi-finite S-morphism, $f_{(S')} : X_{(S')} \to Y_{(S')}$ is quasi-finite for any base change $S' \to S$.*

(iv) *If $f : X \to Y$ and $g : X' \to Y'$ are quasi-finite S-morphisms, $f \times_S g$ is quasi-finite.*

(v) *Let $f : X \to Y$ and $g : Y \to Z$ be two morphisms such that $g \circ f$ is quasi-finite; if g is separated, or X is Noetherian, or $X \times_Z Y$ is locally Noetherian, then f is quasi-finite.*

(vi) *If f is quasi-finite, so is f_{red}.*

Proof. If $f : X \to Y$ is an immersion, any fiber is reduced to a singleton, so (i) follows from Proposition 10.6.35(i). For (ii), we note that $h = g \circ f$ is of finite type by Proposition 10.6.35(ii); if $z = h(x)$ and $y = f(x)$, y is isolated in $g^{-1}(z)$, so there exists an open neighborhood V of y in Y not containing other points of $g^{-1}(z)$; $f^{-1}(V)$ is then an open neighborhood of x not containing other points of $f^{-1}(y')$, where $y' \neq y$ is in $g^{-1}(z)$, and therefore not containing points $x' \neq x$ in $h^{-1}(z)$ that is not in $f^{-1}(y)$. As x is isolated in $f^{-1}(y)$ by hypothesis, it is then isolated in $h^{-1}(z) = f^{-1}(g^{-1}(z))$. As for (iii), we can limit ourselves to the case where $Y = S$ (Corollary 10.3.10); we first note that $f' = f_{(S')}$ is of finite type (Proposition 10.6.35(iii)). On the other hand, if $x' \in X' = X_{(S')}$ and $y' = f'(x')$, $X'_{y'}$ is identified with $X_y \otimes_{\kappa(y)} \kappa(y')$ by Proposition 10.3.38. As X_y is of finite dimension over $\kappa(y)$ by hypothesis, $X'_{y'}$ is of finite dimension over $\kappa(y')$, hence discrete. The assertions (iv), (v), (vi) then follows from the first three assertions in view of the general principle Proposition 10.5.22, where in (v) we assume that g is separated. The other cases, we first remark that if x is isolated in $X_{g(f(x))}$, it is also isolated in $X_{f(x)}$; the fact that f is of finite type follows from Proposition 10.6.35. □

Proposition 11.6.18. *Let A be a complete Noetherian local ring, $Y = \operatorname{Spec}(A)$, X be a separated Y-scheme locally of finite type, x be a point over the closed point y of Y, and suppose that x is isolated in the fiber X_y. Then $\mathscr{O}_{X,x}$ is a finitely generated A-module and X is Y-isomorphic to the sum of $X' = \operatorname{Spec}(\mathscr{O}_{X,x})$ (which is a finite Y-scheme) and an A-scheme X''.*

Proof. It follows from Proposition 11.6.15 that $\mathscr{O}_{X,x}$ is a quasi-finite A-module. As $\mathscr{O}_{X,x}$ is Noetherian (Proposition 10.6.20) and the homomorphism $A \to \mathscr{O}_{X,x}$ is local, the hypothesis that A is complete implies that $\mathscr{O}_{X,x}$ is a finitely generated A-module ([?] 0_{I}, 7.4.3). Let $X' = \operatorname{Spec}(\mathscr{O}_{X,x})$ be the local scheme of X at x and $g : X' \to X$ be the canonical morphism. The composition $f \circ g : X' \to Y$ is then finite, and since f is separated, g is finite by Proposition 11.6.3, so $g(X')$ is closed in X (Corollary 11.6.9). On the other hand, as g is of finite type and A is Noetherian, it is of finite presentation, and hence a local immersion at the closed point x' of X' (Proposition 10.6.53 and the definition of g). But X' is the only open neighborhood of x' in X', so it follows that $g(X')$ is open in X, which proves our assertion. □

Corollary 11.6.19. *Let A be a complete Noetherian local ring, $Y = \operatorname{Spec}(A)$, $f : X \to Y$ be a quasi-finite and separted morphism. Then X is Y-isomorphic to a sum $X' \amalg X''$, where X' is a finite Y-scheme and X'' is a quasi-finite Y-scheme such that, if y is the closed point of y, $X'' \cap f^{-1}(y) \neq \varnothing$.*

Proof. The fiber $f^{-1}(y)$ is finite and discrete by hypothesis, and the corollary then follows by recurrence on the number of points of $f^{-1}(y)$, using Proposition 11.6.18. □

11.6.3 Integral closure of a scheme

Proposition 11.6.20. *Let $(X, \mathscr{A})$ be a ringed space, $\mathscr{B}$ be an $\mathscr{A}$-algebra, and f be a section of $\mathscr{B}$ over X. The following properties are equivalent:*

(i) *The sub-$\mathscr{A}$-algebra of $\mathscr{B}$ generated by f is finite (that is, of finite type as an $\mathscr{A}$-module).*

(ii) *There exists a sub-$\mathscr{A}$-algebra $\mathscr{C}$ of $\mathscr{B}$, which is an $\mathscr{A}$-module of finite type, such that $f \in \Gamma(X, \mathscr{C})$.*

(iii) *For any $x \in X$, f_x is integral over the fiber $\mathscr{A}_x$.*

*If these equivalent conditions are satisfied, the section f is said to be **integral** over $\mathscr{A}$.*

Proof. As the sub-$\mathscr{A}$-module of $\mathscr{B}$ genearted by f^n is an $\mathscr{A}$-algebra, it is clear that (i) implies (ii). On the other hand, (ii) implies that for any $x \in X$, the $\mathscr{A}_x$-module $\mathscr{C}_x$ is of finite type, which implies that any element of the algebra $\mathscr{C}_x$, and in particular f_x, is integral over $\mathscr{A}_x$. Finally, if for any point $x \in X$, we have a relation

$$f_x^n + (a_1)_x f_x^{n-1} + \cdots + (a_n)_x = 0$$

where a_i are sections of $\mathscr{A}$ over an open neighborhood U of x, the section $f^n|_U + a_1 \cdot f^{n-1}|_U + \cdots + a_n$ is zero over an open neighborhood $V \subseteq U$ of x, so $f^k|_V$ (for $k \geq 0$) is a linear combination over $\Gamma(V, \mathscr{A})$ of $f^j|_V$ with $0 \leq j \leq n-1$. We then conclude that (iii) implies (i). □

Corollary 11.6.21. *Under the hypothesis of Proposition 11.6.20, there exists a (unique) sub-$\mathscr{A}$-algebra $\mathscr{A}'$ of $\mathscr{B}$ such that for any $x \in X$, $\mathscr{A}'_x$ is the set of germs $f_x \in \mathscr{B}_x$ that is integral over $\mathscr{A}_x$. For any open subset $U \subseteq X$, the sections of $\mathscr{A}'$ over U is the sections of $\Gamma(U, \mathscr{B})$ that is integral over $\mathscr{A}|_U$. We say that $\mathscr{A}'$ is the **integral closure** of $\mathscr{A}$ in $\mathscr{B}$.*

Proof. The existence of $\mathscr{A}'$ is immediate, by setting $\Gamma(U, \mathscr{A}')$ to be the set of $f \in \Gamma(U, \mathscr{B})$ such that f_x is integral over $\mathscr{A}_x$ for any $x \in U$. It is clear that $\mathscr{A}'$ is an algebra, and the second assertion follows from Proposition 11.6.20. □

Let $(X, \mathscr{A})$, $(Y, \mathscr{B})$ be two ringed space and $f : X \to Y$ be a morphism. Let $\mathscr{C}$ (resp. $\mathscr{D}$) be an $\mathscr{A}$-algebra (resp. $\mathscr{B}$-algebra) and let $u : \mathscr{D} \to \mathscr{C}$ be a f-morphism. Then, if $\mathscr{A}'$ (resp. $\mathscr{B}'$) is the integral closure of $\mathscr{A}$ (resp. $\mathscr{B}$) in $\mathscr{C}$ (resp. $\mathscr{D}$), the restriction of u to $\mathscr{B}'$ is then a f-morphism $u' : \mathscr{B}' \to \mathscr{A}'$. In fact, if j is the canonical injection $\mathscr{B}' \to \mathscr{D}$, it suffices to show that

$$v = u^\sharp \circ f^*(j) : f^*(\mathscr{B}') \to \mathscr{C}'$$

maps $f^*(\mathscr{B}')$ into $\mathscr{A}'$. Now an element of $(f^*(\mathscr{B}'))_x = \mathscr{B}'_{f(x)} \otimes_{\mathscr{B}_{(f(x))}} \mathscr{A}_x$ is integral over $\mathscr{A}_x$ by the definition of $\mathscr{B}'$, and hence so is its image under v_x, which proves our assertion.

Proposition 11.6.22. *Let X be a scheme and $\mathscr{A}$ be a quasi-coherent $\mathscr{O}_X$-algebra. The integral closure $\mathscr{O}'_X$ of $\mathscr{O}_X$ in $\mathscr{A}$ is then a quasi-coherent $\mathscr{O}_X$-algebra, and for any affine open U of X, $\Gamma(X, \mathscr{O}'_X)$ is the integral closure of $\Gamma(U, \mathscr{O}_X)$ in $\Gamma(U, \mathscr{A})$.*

Proof. We can assume that $X = \mathrm{Spec}(B)$ is affine and $\mathscr{A} = \widetilde{A}$, where A is an B-algebra. Let B' be the integral closure of B in A. It then boils down to seeing that for any $x \in X$, an element of A_x, integer over B_x, necessarily belongs to B'_x, which follows from the fact that taking integral closure commutes with localization (Proposition 4.1.39). □

Under the hypothesis of Proposition 11.6.22, the X-scheme $X' = \mathrm{Spec}(\mathscr{O}'_X)$ is then called the **integral closure of X relative to** $\mathscr{A}$. We also deduce from Proposition 11.6.22 that if $f : X' \to X$ is the structural morphism, then for any open subset U of X, $f^{-1}(U)$ is the integral closure of the induced subscheme U by X, relative to $\mathscr{A}|_U$. In particular, we conclude that f is integral.

Let X and Y be schemes, $f : X \to Y$ be a morphism, $\mathscr{A}$ (resp. $\mathscr{B}$) be a quasi-coherent $\mathscr{O}_X$-algebra (resp. a $\mathscr{O}_Y$-algebra), and $u : \mathscr{B} \to \mathscr{A}$ be an f-morphism. We have seen that we have an induced f-morphism $u' : \mathscr{O}'_Y \to \mathscr{O}'_X$, where $\mathscr{O}'_X$ (resp. $\mathscr{O}'_Y$) is the integral closure of $\mathscr{O}_X$ (resp. $\mathscr{O}_Y$) relative to $\mathscr{A}$ (resp. $\mathscr{B}$), we deduce a canonical morphism $f' : \mathrm{Spec}(u') : X' \to Y'$ (Corollary 11.1.11) fitting into the following commutative diagram

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d] & Y' \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd} \tag{11.6.1}$$

Suppose that X has only finitely many irreducible components $(X_i)_{1 \le i \le r}$, with generic points $(\xi_i)_{1 \le i \le r}$, and consider in particular the integral closure of X relative to a quasi-coherent $\mathscr{K}_X$-algebra $\mathscr{A}$. By Corollary 10.7.19 and Corollary 10.7.20, $\mathscr{A}$ is the direct product of r quasi-coherent $\mathscr{O}_X$-algebras $\mathscr{A}_i$, the support of $\mathscr{A}_i$ being contained in X_i, and the induced sheaf of $\mathscr{A}_i$ over X_i is the constant sheaf whose fiber A_i is an algebra over $\mathscr{O}_{X,\xi_i}$. It is clear that the integral closure $\mathscr{O}'_X$ of $\mathscr{O}_X$ is the direct product of the integral closures $\mathscr{O}_X^{(i)}$ of $\mathscr{O}_X$ in each $\mathscr{A}_i$, and therefore the integral closure $X' = \mathrm{Spec}(\mathscr{O}'_X)$ of X relative to $\mathscr{A}$ is an X-scheme which is the sum of $\mathrm{Spec}(\mathscr{O}_X^{(i)}) = X'_i$.

Now suppose that the $\mathscr{O}_X$-algebra $\mathscr{A}$ is reduced, or equivalently, each algebra A_i is reduced, and therefore can be considered as an algebra over the field $\kappa(\xi_i)$ (equal to the rational function field of the reduced subscheme X_i of X); then the X_i' is a reduced X-scheme and X' is also the integral closure of X_{red}. Suppose moreover that the algebras A_i is a direct product of finitely many field K_{ij} $(1 \leq j \leq s_i)$; if $\mathscr{K}_{ij}$ is the subalgebra of $\mathscr{A}_i$ corresponding to K_{ij}, it is clear that $\mathscr{O}_X^{(i)}$ is the direct product of integral closures $\mathscr{O}_X^{(ij)}$ of $\mathscr{O}_X$ in $\mathscr{K}_{ij}$. Therefore, X_i' is the sum of $X_{ij}' = \mathrm{Spec}(\mathscr{O}_X^{(ij)})$. Moreover, under this hypothesis, we have the following:

Proposition 11.6.23. *Each X_{ij}' is an integral and normal X-scheme, and its rational function field $K(X_{ij}')$ is canonically identified with the algebraic closure K_{ij}' of $\kappa(\xi_i)$ in K_{ij}.*

Proof. In view of the preceding remarks, we can assume that X is integral, so $r = 1$, $s_1 = 1$, so that the unique algebra A_1 is a field K; let ξ be the generic point of X, and let $f : X' \to X$ be the structural morphism. For any nonempty affine open U of X, $f^{-1}(U)$ is identified with the integral closure B_U' in the field K of the intagral ring $B_U = \Gamma(U, \mathscr{O}_X)$ (Proposition 11.6.22); as the ring B_U' is integrally closed, so is its localizations, and $f^{-1}(U)$ is by definition an integral and normal scheme. Moreover, as (0) is the unique prime ideal of B_U' lying over the prime ideal (0) of B_U, $f^{-1}(\xi)$ is reduced to a singleton ξ', and $\kappa(\xi')$ is the fraction field K' of B_U', which is none other than the algebraic closure of $\kappa(\xi)$ in K. Finally, X' is irreducible, because if U runs through the nonempty affine open subsets of X, the $f^{-1}(U)$ constitute an open covering of X' formed by irreducible open subsets; moreover the intersection $f^{-1}(U \cap V)$ two two opens contains ξ', hence nonempty, and we conclude from **??** that X' is irreducible. □

Corollary 11.6.24. *Let X be a reduced scheme with finitely many irreducible components (X_i), and let ξ_i be the generic point of X_i. The integral closure X' of X relative to $\mathscr{K}_X$ is the sum of r separated X-schemes X_i' which are integral and normal. If $f : X' \to X$ is the structural morphism, $f^{-1}(\xi_i)$ is reduced to the generic point ξ_i' of X_i' and we have $\kappa(\xi_i') = \kappa(\xi_i)$, which means f is birational.*

Proof. This is a particular case of Proposition 11.6.23 by taking $K_{ij}' = \kappa(\xi_i)$. The rational function field of X_i' (which is $\kappa(\xi_i')$) is then equal to $\kappa(\xi_i)$, whence our claim. □

The integral closure X' of X relative to $\mathscr{K}_X$ is called that **normalization** of the reduced scheme X. We note that the morphism $f : X' \to X$, being birational and integral, is closed by Proposition 11.6.7, hence surjective (recall that a birational morphism is dominant). For $X' = X$, it is necessary and sufficient that X is normal. If X is an integral scheme, it follows from Corollary 11.6.24 that its normalization X' is integral.

Let X, Y be integral schemes, $f : X \to Y$ be a dominant morphism, $L = K(X)$, $K = K(Y)$ be the rational function field of X and Y. The morphism f corresponds to an injection $K \to L$, and if we identify K (resp. L) with the simple sheaf $\mathscr{K}_Y$ (resp. $\mathscr{K}_X$), this injection is an f-morphism. Let K_1 (resp. L_1) be an extension of K (resp. L) and suppose that we are given a monomorphism $K_1 \to L_1$ such that the diagram

$$\begin{array}{ccc} K_1 & \longrightarrow & L_1 \\ \uparrow & & \uparrow \\ K & \longrightarrow & L \end{array}$$

is commutative; if K_1 (resp. L_1) is considered as a simple sheaf over Y (resp. X), hence a $\mathscr{K}_Y$-algebra (resp. a $\mathscr{K}_X$-algebra), this signifies that $K_1 \to L_1$ is an f-morphism. Now if X' (resp. Y') is the integral closure of X (resp. Y) relative to L_1 (resp. K_1), X' (resp. Y') is a normal and integral scheme (Proposition 11.6.23) and its rational function field is canonically identified with the algebraic closure L' (resp. K') of L (resp. K) in L_1 (resp. K_1), and there exists a canonical morphism (necessarily dominant) $f' : X' \to Y'$ rendering the diagram (11.6.1). The important case is that $L_1 = L$, K_1 is an extension of K contained in L, and where we suppose that X is integral and normal, hence $X' = X$.

The preceding arguments then show that if X is a normal scheme and Y' is integrally closure of Y relative to a field $K_1 \subseteq L = K(X)$, any dominant morphism $f : X \to Y$ factors into

$$f : X \xrightarrow{f'} Y' \to Y$$

where f' is dominant; if the monomorphism $K_1 \to L$ is fixed, f' is necessarily unique (this can be verified when X and Y are both affine). We then say that given Y, L, and a K-monomorphism $K_1 \to L$, the integral closure Y' of Y relative to K_1 is a universal object.

Remark 11.6.25. Retain the hypothesis of Proposition 11.6.23 and suppose moreover that each algebra A_i is of finite dimension over $\kappa(\xi_i)$ (which implies that A_i is a direct product of finitely many fields); we can prove that the structural morphism $X' \to X$ is finite. For this, we can reduce to the case where X is reduced and affine with ring C, and that C has finitely many minimal prime ideals $\mathfrak{p}_i$ ($1 \leq i \leq r$) with $C_i = C/\mathfrak{p}_i$. Then by Proposition 11.6.22 X' is finite over X if the integral closure of each C_i in finite extension of its fraction field is a finitely generated C-module, or equivalently, if C_i is Japanese for each i. We know that this condition is true if C is an algebra of finite type over a field, or over $\mathbb{Z}$, or over a complete Noetherian local ring. We then conclude that $X' \to X$ is a finite morphism if X is a scheme of finite type over a field, or over $\mathbb{Z}$, or over a complete Noetherian local ring.

11.6.4 Determinant of an endomorphism of $\mathscr{O}_X$-modules

Let A be a ring, E be a free A-module of rank n, and $u : E \to E$ be an endomorphism of E; recall that in order to define the characteristic polynomial of u, we consider the endomorphism $u \otimes 1$ of free the $A[T]$-module $E \otimes_A A[T]$ (which is of rank n), and we put

$$P(u, T) = \det(T \cdot I - (u \otimes 1))$$

(I is the identity morphism on $E \otimes_A A[T]$). We have

$$P(u, T) = T^n - \sigma_1(u)T^{n-1} + \cdots + (-1)^n \sigma_n(u)$$

where $\sigma_i(u)$ is an element of A, equal to a homogeneous polynomial of degree i (with integer coefficients) with entries the elements of the matrix of u relative to any basis of E. We say that the $\sigma_i(u)$ are the **elementray symmetric functions** of u, and we have in particular $\sigma_1(u) = \mathrm{tr}(u)$ and $\sigma_n(u) = \det(u)$. By Hamilton-Cayley theorem, we have

$$P(u, u) = u^n - \sigma_1(u)u^{n-1} + \cdots + (-1)^n \sigma_n(u) = 0 \tag{11.6.2}$$

which can also be written as

$$(\det(u)) \cdot 1_E = uQ(u) \tag{11.6.3}$$

(1_E is the identity morphism on E), where

$$Q(u) = (-1)^{n+1}(u^{n-1} - \sigma_1(u)u^{n-2} + \cdots + (-1)^{n-1}\sigma_{n-1}(u)) \tag{11.6.4}$$

Let $\varphi : A \to B$ be a homomorphism of rings; consider the B-module $E_{(B)} = E \otimes_A B$ which is free of rank n, and the extension $u \otimes 1$ of u to an endomorphism on $E_{(B)}$. It is immediate that we have $\sigma_i(u \otimes 1) = \varphi(\sigma_i(u))$ for all i.

Suppose now that A is an integral domain, with fraction field K, and E is a finitely generated A-module (not necessarily free now). Let n be the rank of E, which equals to the dimension of $E \otimes_A K$ over K. Any endomorphism u of E corresponds canonically to the endomorphism $u \otimes 1$ of $E \otimes_A K$. By abuse of language, we call $P(u \otimes 1, T)$ the characteristic polynomial of u and denoted by $P(u,T)$, and the coefficients $\sigma_i(u\otimes 1)$ is called the elementray symmetric functions of u and denoted by $\sigma_i(u)$. In particular the determinant $\det(u) = \det(u\otimes 1)$ is defiend. With these notations, the formulas (11.6.2) and (11.6.3) are meaningful and still valid, if we interpret the u^i as the homomorphism $E\to E\otimes_A K$ whici is the composition of the endomorphism $u^j\otimes 1 = (u\otimes 1)^j$ of $E\otimes_A K$ and the canonical homomorphism $x\mapsto x\otimes 1$.

If F is the torsion module of R and $E_0 = E/F$, we have $u(F)\subseteq F$, hence, by taking quotient, u induces an endomorphism u_0 of E_0; moreover $E\otimes_A K$ is identified with $E_0\otimes_A K$ and $u\otimes 1$ is identified with $u_0\otimes 1$, hence $\sigma_i(u)=\sigma_i(u_0)$ for $1\leq i\leq n$.

If E is torsion-free, E is identified with a sub-A-module of $E\otimes_A K$, and the relation $u\otimes 1 = 0$ is equivalent to $u=0$. If E is a free A-module, the two definitions of $\sigma_i(u)$ given above coincide according to the preceding remarks, which justifies the notations adopted. We also note that if E is a torsion module then $E_0 = \{0\}$, the exterior algebra of E_0 is reduced to K and the determinant of the endomorphism u_0 of E_0 is equal to 1.

Proposition 11.6.26. *Let A be an integral domain, E be a finitely generated A-module, u be an endomorphism of u. Then the elementray symmetric functions $\sigma_i(u)$ of u (and in particular $\det(u)$) are integral elements of K over A.*

Proof. This is a particular case of Proposition 4.1.45, where we set $B=K$ and note that condition (ii) is satisfied for $M=E$. □

Corollary 11.6.27. *Under the hypothesis of Proposition 11.6.26, if A is normal, the $\sigma_i(u)$ belong to A.*

Proposition 11.6.28. *Let A be an integral domain, E be a finitely generated A-module, of rank n, and u be an endomorphism of E such that the $\sigma_i(u)$ belong to A for each i. For u to be an automorphism of E, it is necessary that $\det(u)$ is invertible in A; this condition is sufficient if E is torsion free.*

Proof. This conditions is sufficient by (11.6.3) and (11.6.4), if E is torsion free, since E is then a sub-A-module of $E\otimes_A K$, and $(\det(u))^{-1}Q(u)$ is the inverse of u. Conversely, this is necessary, because if u is invertible, it follows from Proposition 11.6.26 that $\det(u^{-1})$ belongs to the integral closure A' of A in K, and is clearly the inverse of $\det(u)$ in A'. If $\det(u)$ is not invertible in A, then it belongs to a maximal ideal $\mathfrak{m}$ of A, which is the contraction of a maximal ideal of A' (**??**), contradiction. □

We note a generalization of the preceding results. Consider a reduced Noetherian ring A and let $\mathfrak{p}_i$ $(1\leq i\leq r)$ be the minimal prime ideals of A, and K_i be the fractional field of $A_i = A/\mathfrak{p}_i$. Then the total fraction field K of A is the direct product of the fields K_i (Proposition 3.2.26). Let E be a finitely generated A-module, and suppose that $E\otimes_A K$ is a K-module of dimension n. Then each K_i-vector space $E_i = E\otimes_A K_i$ is of dimension n. If u is an endomorphism of E, we put $P(u,T) = P(u\otimes 1,T)$ and $\sigma_j(u) = \sigma_j(u\otimes 1)$, and in particular $\det(u) = \det(u\otimes 1)$; the $\sigma_j(u)$ are then elements of K. It is immediate that $E\otimes_A K$ is a direct sum of E_i and each of them is stable under $u\otimes 1$. The restriction of $u\otimes 1$ to E_i is just the extension of u to E_i, and we conclude that $\sigma_j(u)$ is the element of K with component in K_i being $\sigma_j(u_i)$. As the integral closure of A in K is the direct product of that of A in K_i (Proposition 4.1.23), the $\sigma_j(u)$ are integral over A.

Lemma 11.6.29. *The sub-A-algebra of K genrated by the elements $\sigma_j(u)$ $(1\leq j\leq n)$ for $u\in \mathrm{Hom}_A(E,E)$, is a finitely generated A-module.*

Let $(X,\mathscr{A})$ be a ringed space, $\mathscr{E}$ be a locally free $\mathscr{A}$-module (of finite rank). There is then by hypothesis a basis $\mathfrak{B}$ of X such that for any $V\in\mathfrak{B}$, $\mathscr{E}|_V$ is isomorphic to $\mathscr{A}^n|_V$ (the integer n may vary with V). Let u be an endomorphism of $\mathscr{E}$; for any $V\in\mathfrak{B}$, u_V is then an endomorphism of the $\Gamma(V,\mathscr{A})$-module $\Gamma(V,\mathscr{E})$, which is free by hypothesis; the determinant of u_V is then defined and belongs to $\Gamma(V,\mathscr{A})$. Moreover, if $e_1,\dots,e_n$ is a basis of $\Gamma(V,\mathscr{E})$, their restriction to any open subset $W\subseteq V$ form a basis of $\Gamma(W,\mathscr{E})$ over $\Gamma(W,\mathscr{A})$, so $\det(u_W)$ is the restriction of $\det(u_V)$ to W. There then exists a unique section of $\mathscr{A}$ over X, which we denote by $\det(u)$ and call the **determinant** of u, such that the restriction of $\det(u)$ to any $V \in \mathfrak{B}$ is $\det(u_V)$. It is clear that for any $x \in X$, we have $\det(u)_x = \det(u_x)$; for two endomorphisms u, v of $\mathscr{E}$, we have

$$\det(u \circ v) = (\det(u))(\det(v)), \quad \det(1_{\mathscr{E}}) = 1_{\mathscr{A}}.$$

If $\mathscr{E}$ is of rank n (for example if X is connected), we have

$$\det(s \cdot u) = s^n \det(u)$$

for any $s \in \Gamma(X, \mathscr{A})$ (we note that $\det(0) = 0_{\mathscr{A}}$ if $n \geq 1$, but $\det(0) = 1_{\mathscr{A}}$ if $n = 0$). Moreover, for u to be an automorphism of $\mathscr{E}$, it is necessary and sufficient that $\det(u)$ is invertible in $\Gamma(X, \mathscr{A})$ (Proposition 11.6.28).

If $\mathscr{E}$ is of rank n, we can similarly define the elementary symmetric functions $\sigma_i(u)$ for u, which are elements of $\Gamma(X, \mathscr{A})$, and we also have the relations (11.6.3) and (11.6.4).

We have then define a homomorphism $\det : \operatorname{Hom}_{\mathscr{A}}(\mathscr{E}, \mathscr{E}) \to \Gamma(X, \mathscr{A})$ of multiplicative monoids. Note that $\operatorname{Hom}_{\mathscr{A}}(\mathscr{E}, \mathscr{E}) = \Gamma(X, \mathcal{H}om_{\mathscr{A}}(\mathscr{E}, \mathscr{E}))$ by definition, so we can replace X by any open subset U in this definition of det, and therefore obtain a homomorphism $\det : \mathcal{H}om_{\mathscr{A}}(\mathscr{E}, \mathscr{E}) \to \mathscr{A}$ of sheaves of multiplicative monoids. If $\mathscr{E}$ is of constant rank, we can similarly define the homomorphisms $\sigma_i : \mathcal{H}om_{\mathscr{A}}(\mathscr{E}, \mathscr{E}) \to \mathscr{A}$ of sheaves of sets; for $i = 1$, the homomorphism $\sigma_1 = \operatorname{tr}$ is a homomorphism of $\mathscr{A}$-modules.

Let $(Y, \mathscr{B})$ be a second ringed space and $f : (X, \mathscr{A}) \to (Y, \mathscr{B})$ be a morphism of ringed spaces; if $\mathscr{F}$ is a locally free $\mathscr{B}$-module, $f^*(\mathscr{F})$ is a locally free $\mathscr{A}$-module (with the same rank of $\mathscr{F}$). For any endomorphism v of $\mathscr{F}$, $f^*(v)$ is then an endomorphism of $f^*(\mathscr{F})$, and it follows from these definitions that $\det(f^*(v))$ is the section of $\mathscr{A} = f^*(\mathscr{B})$ over X which corresponds canonically to $\det(v) \in \Gamma(Y, \mathscr{B})$. We can then say that the homomorphism $f^*(\det) : f^*(\mathcal{H}om_{\mathscr{B}}(\mathscr{F}, \mathscr{F})) \to f^*(\mathscr{B}) = \mathscr{A}$ is the composition

$$f^*(\mathcal{H}om_{\mathscr{B}}(\mathscr{F}, \mathscr{F})) \xrightarrow{\gamma^\sharp} \mathcal{H}om_{\mathscr{A}}(f^*(\mathscr{F}), f^*(\mathscr{F})) \xrightarrow{\det} \mathscr{A} \tag{11.6.5}$$

(formula (**??**)). We have a similar result for σ_i.

Suppose now that X is a locally integral scheme, so its sheaf of rational function $\mathscr{K}_X$ is locally simple over X (Corollary 10.7.19) and quasi-coherent as $\mathscr{O}_X$-module. If $\mathscr{E}$ is a qausi-coherent $\mathscr{O}_X$-module of finite type, $\mathscr{E}' = \mathscr{E} \otimes_{\mathscr{O}_X} \mathscr{K}_X$ is then a locally free $\mathscr{K}_X$-module (Corollary 10.7.21). For any endomorphism u of $\mathscr{E}$, $u \otimes 1_{\mathscr{K}_X}$ is then an endomorphism of $\mathscr{E}'$, and $\det(u \otimes 1)$ is a section of $\mathscr{K}_X$ over X, which is called the **determinant** of u and denoted by $\det(u)$. It follows from Proposition 11.6.26 that $\det(u)$ is a section of the integral closure of $\mathscr{O}_X$ in $\mathscr{K}_X$; if X is also normal, $\det(u)$ is then a section of $\mathscr{O}_X$ over X, and if we suppose moreover that $\mathscr{E}$ is torsion free, for u to be an automorphism of $\mathscr{E}$, it is necessary and sufficient that $\det(u)$ is invertible (Proposition 11.6.28). The formulae (11.6.3) and (11.6.4) are still valid; the homomorphism $u \mapsto \det(u)$ then defines a homomorphism $\det : \mathcal{H}om_{\mathscr{O}_X}(\mathscr{E}, \mathscr{E}) \to \mathscr{K}_X$, which has values in $\mathscr{O}_X$ if X is normal. We have analogous results for the elementary symmetric function functions $\sigma_j(u)$, if $\mathscr{E}'$ has constant rank; if moreover X is normal, the $\sigma_j(u)$ are sections of $\mathscr{O}_X$ over X.

Finally, let X and Y be integral schemes, and $f : X \to Y$ be a dominant morphism. We see that there exists a canonical homomorphism $f^*(\mathscr{K}_Y) \to \mathscr{K}_X$, whence induces, for any quasi-coherent $\mathscr{O}_Y$-module $\mathscr{F}$ of finite type, a canonical homomorphism $\theta : f^*(\mathscr{F} \otimes_{\mathscr{O}_Y} \mathscr{K}_Y) \to f^*(\mathscr{F}) \otimes_{\mathscr{O}_X} \mathscr{K}_X$. If v is an endomorphism of $\mathscr{F}$, $f^*(v \otimes 1_{\mathscr{K}_Y})$ is an endomorphism of $f^*(\mathscr{F} \otimes_{\mathscr{O}_Y} \mathscr{K}_Y)$, and we have a commutative diagram

$$\begin{tikzcd}
f^*(\mathscr{F} \otimes_{\mathscr{O}_Y} \mathscr{K}_Y) \arrow[r, "f^*(v\otimes 1)"] \arrow[d, "\theta"'] & f^*(\mathscr{F} \otimes_{\mathscr{O}_Y} \mathscr{K}_Y) \arrow[d, "\theta"] \\
f^*(\mathscr{F}) \otimes_{\mathscr{O}_X} \mathscr{K}_X \arrow[r, "f^*(v)\otimes 1"] & f^*(\mathscr{F}) \otimes_{\mathscr{O}_X} \mathscr{K}_X
\end{tikzcd}$$

We then conclude that $\det(f^*(v))$ is the canonical image of the section $\det(v)$ of $\mathscr{K}_Y$ under the canonical homomorphism $f^*(\mathscr{K}_Y) \to \mathscr{K}_X$. In fact, it is immediate that we are reduced to the case where $X = \operatorname{Spec}(B)$ and $Y = \operatorname{Spec}(A)$ are affine, A, B being integral domains with fraction fields K, L respectively, the hommorphism $A \to B$ being injective and extends to a monomorphism $K \to L$. If $\mathscr{F} = \widetilde{M}$ where M is a finitely generated A-module, the dimension of $M \otimes_A K$ is equal to that of $(M \otimes_A B) \otimes_B L$ over L, and $\det((u \otimes 1) \otimes 1)$ is the image of $\det(u \otimes 1)$ in L for any endomorphism u of M, whence our conclusion.

Finally, suppose that X is a reduced locally Noetherian scheme, whose sheaf of rational functions $\mathscr{K}_X$ is quasi-coherent by Proposition 10.7.22. Let $\mathscr{E}$ be a coherent $\mathscr{O}_X$-module such that $\mathscr{E}' = \mathscr{E} \otimes_{\mathscr{O}_X} \mathscr{K}_X$ is locally free of rank n. We can then define for each endomorphism u of $\mathscr{E}$ the elementary symmetric functions $\sigma_j(u)$, which are sections of $\mathscr{K}_X$ over X.

11.6.5 Norm of invertible sheafs

Let $(X, \mathscr{A})$ be a ringed space and $\mathscr{B}$ be an $\mathscr{A}$-algebra. The $\mathscr{A}$-module $\mathscr{B}$ is canonically identified with a sub-$\mathscr{A}$-module of $\mathcal{H}om_{\mathscr{A}}(\mathscr{B}, \mathscr{B})$, where a section f of $\mathscr{B}$ over an open subset U of X is identified with the multiplication by this section. Assume that $(X, \mathscr{A})$ and $\mathscr{B}$ satisfies the conditions given in the previous subsection, so that we can define $\det(f)$ (resp. $\sigma_j(f)$) to be a section of $\mathscr{K}_X$ over U, which is called the **norm** of f (resp. the elementary symmetric functions) of f and denoted by $N_{\mathscr{B}/\mathscr{A}}(f)$. We suppose that one of the following conditions is satisfied:

- (α) $\mathscr{B}$ is a locally free $\mathscr{A}$-module of finite rank n.
- (β) $(X, \mathscr{A})$ is a reduced locally Noetherian scheme, $\mathscr{B}$ is a coherent $\mathscr{A}$-module such that $\mathscr{B} \otimes_{\mathscr{A}} \mathscr{K}_X$ is a locally free $\mathscr{K}_X$-module of rank n, and for any section $f \in \Gamma(U, \mathscr{B})$ over an open subset $U \subseteq X$, $\sigma_j(f)$ $(1 \le j \le n)$ is a section of $\mathscr{A}$ over U (this is true for example if X is normal).

The hypothesis that $\mathscr{B} \otimes_{\mathscr{A}} \mathscr{K}_X$ is locally free can be expressed by the following: denote by X_i the reduced closed subschemes of X with underlying space the irreducible components of X, which are then locally Noetherian integral schemes. Any $x \in X$ belongs to finitely many X_i, and $\mathscr{B} \otimes_{\mathscr{A}} \mathscr{K}_{X_i}$ is a locally free $\mathscr{K}_{X_i}$-module of constant rank k_i (Corollary 10.7.20); to say that $\mathscr{B} \otimes_{\mathscr{A}} \mathscr{K}_X$ is locally free $\mathscr{K}_X$-module signifies that, for any $x \in X$, the ranks k_i such that $x \in X_i$ are all equal. This question is in fact local, and we can assume that $X = \operatorname{Spec}(A)$, where A is a reduced Noetherian ring, and $\mathscr{B} = \widetilde{B}$ where B is a finite A-algebra. If $\mathfrak{p}_i$ $(1 \le i \le r)$ are the minimal prime ideals of A, the total fraction ring K of A is then the direct product of K_i, where K_i is the fraction field of $A_i = A/\mathfrak{p}_i$, and $B \otimes_A K$ is then the direct sum of $B \otimes_A K_i$, whence our conclusion.

It is clear that under the hypotheses (α) or (β), we then define a homomorphism $N_{\mathscr{B}/\mathscr{A}} : \mathscr{B} \to \mathscr{A}$ of sheaves of multiplicative monoids, which is also denoted by N if there is no confusion, and called the norm homomorphism. For two sections f, g of $\mathscr{B}$ over an open subset U, we then have

$$N_{\mathscr{B}/\mathscr{A}}(fg) = N_{\mathscr{B}/\mathscr{A}}(f) N_{\mathscr{B}/\mathscr{A}}(g), \quad N_{\mathscr{B}/\mathscr{A}}(1_{\mathscr{B}}) = 1_{\mathscr{A}} \tag{11.6.6}$$

for the corresponding sections of $\mathscr{A}$ over U. Also, for any section s of $\mathscr{A}$ over U, we have

$$N_{\mathscr{B}/\mathscr{A}}(s \cdot 1_{\mathscr{B}}) = s^n. \tag{11.6.7}$$

In case (α), for any $f \in \Gamma(U, \mathscr{B})$ to be invertible, it is necessary and sufficient that $N(f) \in \Gamma(U, \mathscr{A})$ is invertiblel; in case (β), this condition is necessary, and is sufficient if $\mathscr{B}$ is a torsion free $\mathscr{A}$-module.

Suppose the one of the hypotheses (α), (β) is satisfied, and let $\mathscr{L}'$ be an invertible $\mathscr{B}$-module. We can canonically associate an invertible $\mathscr{A}$-module by the following. Denote by $\mathscr{A}^\times$ (resp. $\mathscr{B}^\times$) the subsheaf of $\mathscr{A}$ (resp. $\mathscr{B}$) such that $\Gamma(U,\mathscr{A}^\times)$ (resp. $\Gamma(U,\mathscr{B}^\times)$) is the set of invertible elements of $\Gamma(U,\mathscr{A})$ (resp. $\Gamma(U,\mathscr{B})$) for any open subset $U\subseteq X$; this is a sheaf of multiplicative groups, and $N_{\mathscr{B}/\mathscr{A}}$, restricted to $\mathscr{B}^\times$, is a homomorphism $\mathscr{B}^\times\to\mathscr{A}^\times$ of sheaves of groups. Let $\mathfrak{L}$ be the set of couples (U_λ,η_λ), with the following property: U_λ is an open subset of X and $\eta_\lambda:\mathscr{L}'|_{U_\lambda}\xrightarrow{\sim}\mathscr{B}|_{U_\lambda}$ is an isomorphism of $(B|_{U_\lambda})$-modules. By hypothesis, the U_λ for an open covering og X; for two indices λ,μ, we put $\omega_{\lambda\mu}=(\eta_\lambda|_{U_\lambda\cap U_\mu})\circ(\eta_\mu|_{U_\lambda\cap U_\mu})^{-1}$, which is an automorphism of $\mathscr{B}|_{U_\lambda\cap U_\mu}$, and canonically identified with a section of $\mathscr{B}^\times$ over $U_\lambda\cap U_\mu$, and $(\omega_{\lambda\mu})$ is a 1-cocycle over the covering $\mathfrak{U}=(U_\lambda)$ with values in $\mathscr{B}^\times$. The fact that $N_{\mathscr{B}/\mathscr{A}}:\mathscr{B}^\times\to\mathscr{A}^\times$ is a homomorphism implies that $(N_{\mathscr{B}/\mathscr{A}}(\omega_{\lambda\mu}))$ is a 1-cocycle of $\mathfrak{U}$ with values in $\mathscr{A}^\times$, which then corresponds (up to isomorphism) to an invertible $\mathscr{A}$-module. This invertible $\mathscr{A}$-module is denoted by $N_{\mathscr{B}/\mathscr{A}}(\mathscr{L}')$ and is called the norm of the invertible $\mathscr{B}$-module $\mathscr{L}'$.

Let $\mathfrak{M}$ be a subset of $\mathfrak{L}$ such that the U_λ form an open covering of X, and let $\mathfrak{V}$ be a covering of X. The restriction of the cocycle $(\omega_{\lambda\mu})$ to $\mathfrak{V}$ defines a 1-cocycle $(N_{\mathscr{B}/\mathscr{A}}(\omega_{\lambda\mu}))$, which is the restriction of the 1-cocycle $(N_{\mathscr{B}/\mathscr{A}}(\omega_{\lambda\mu}))$ to $\mathfrak{U}$; it is clear that there is a canonical isomorphism of the invertible $\mathscr{A}$-modules thus defined, and we can therefore define $N_{\mathscr{B}/\mathscr{A}}(\mathscr{L}')$ by a refinement of the covering $\mathfrak{U}$. This shows that, if $\mathscr{L}'$ and $\mathscr{K}'$ are two invertible $\mathscr{B}$-modules, by (11.6.6) we have

$$N(\mathscr{L}'\otimes_{\mathscr{B}}\mathscr{K}')=N(\mathscr{L}')\otimes_{\mathscr{A}}N(\mathscr{K}'),\quad N_{\mathscr{B}/\mathscr{A}}(\mathscr{B})=\mathscr{A},\tag{11.6.8}$$

and therefore $N(\mathscr{L}'^{-1})=N(\mathscr{L}')^{-1}$. Also, it follows from (11.6.7) that if $\mathscr{L}$ is an invertible $\mathscr{A}$-module, we have

$$N_{\mathscr{B}/\mathscr{A}}(\mathscr{L}\otimes_{\mathscr{A}}\mathscr{B})=\mathscr{L}^{\otimes n}.\tag{11.6.9}$$

We show that $N_{\mathscr{B}/\mathscr{A}}$ is a covariant functor on the category of invertible $\mathscr{B}$-modules. Let $u':\mathscr{L}'\to\mathscr{K}'$ be a homomorphism of invertible $\mathscr{B}$-modules, and let $\mathfrak{V}=(U_\lambda)$ be an open covering of X such that for any λ, we have an isomorphism $\eta_\lambda:\mathscr{L}'|_{U_\lambda}\xrightarrow{\sim}\mathscr{B}|_{U_\lambda}$ and $\tau_\lambda:\mathscr{K}'|_{U_\lambda}\xrightarrow{\sim}\mathscr{B}|_{U_\lambda}$; there is then for each λ an endomorphism u'_λ of $\mathscr{B}|_{U_\lambda}$ such that $u'_\lambda\circ\eta_\lambda=\tau_\lambda\circ(u'|_{U_\lambda})$, and we can evidently identify u'_λ with a section of $\mathscr{B}$ over U_λ. Hence, for any couple (λ,μ) of indices, the restriction of $(\tau_\lambda)^{-1}\circ u'_\lambda\circ\eta_\lambda$ and $(\tau_\mu)^{-1}\circ u'_\mu\circ\eta_\mu$ to $U_\lambda\cap U_\mu$ coincide. We then deduce for the 1-cocycle $(\omega_{\lambda\mu})$ corresponding to $\mathscr{L}'$ and the 1-cocycle $(\gamma_{\lambda\mu})$ corresponding to $\mathscr{K}'$ the relation

$$\gamma_{\lambda\mu}u'_\mu=u'_\lambda\omega_{\lambda\mu}.$$

If we put $u_\lambda=N(u'_\lambda)$, we then have the analogous relation

$$N(\gamma_{\lambda\mu})u_\mu=u_\lambda N(\omega_{\lambda\mu})$$

and therefore the u_λ define a homomorphism $N(\mathscr{L}')\to N(\mathscr{K}')$, which is denoted by $N_{\mathscr{B}/\mathscr{A}}(u)$ or $N(u)$. In view of Proposition 11.6.28, it is clear that under the hypothesis (α), u' is an isomorphism if and only if u is, and this is true under the hypothesis (β) if $\mathscr{B}$ is moreover torsion free. In particular, if consider the hommorphisms $\mathscr{B}\to\mathscr{L}'$, which correspond to global sections of $\mathscr{L}'$, since $N_{\mathscr{B}/\mathscr{A}}(\mathscr{B})=\mathscr{A}$, we get a canonical homomorphism

$$N_{\mathscr{B}/\mathscr{A}}:\Gamma(X,\mathscr{L}')\to\Gamma(X,N_{\mathscr{B}/\mathscr{A}}(\mathscr{L}')).$$

It also follows from (11.6.6) that if $f'\in\Gamma(X,\mathscr{L}')$, $g'\in\Gamma(X,\mathscr{K}')$, we have

$$N(f'\otimes g')=N(f')\otimes N(g').\tag{11.6.10}$$

Also, for any invertible $\mathscr{A}$-module $\mathscr{L}$ and any section $f \in \Gamma(X, \mathscr{L})$, we have

$$N_{\mathscr{B}/\mathscr{A}}(f \otimes 1_{\mathscr{B}}) = f^{\otimes n}. \tag{11.6.11}$$

Finally, for the homomorphism $\mathscr{B} \to \mathscr{L}'$ corresponding to a section f' of $\mathscr{L}'$ over X to be an isomorphism, it is necessary and sufficient that f'_x generates $\mathscr{L}'_x$ for any $x \in X$; under condition (α), this is equivalent to that $N(f')_x$ generates $(N(\mathscr{L}'))_x$ for any x, and this is true for condition (β) if $\mathscr{B}$ is torsion free.

Let $(X, \mathscr{A})$, $(X', \mathscr{A}')$ be two ringed spaces and $\varphi : X' \to X$ be a morphism, $\mathscr{B}$ be an $\mathscr{A}$-algebra, and $\mathscr{B}' = \varphi^*(\mathscr{B})$. Suppose that one of the following conditions is satisfied:

(i) $\mathscr{B}$ satisfies condition (α).

(ii) $(X, \mathscr{A})$ and $\mathscr{B}$ satisfy condition (β), $(X', \mathscr{A}')$ is a reduced locally Noehterian scheme, and if we denote by X_α and X'_β the reduced closed subschemes of X and X' with underlying space the irreducible components of these spaces, the restriction of φ to X'_β is a dominant morphism from X'_β to X_α.

Under these conditions, we claim that $\mathscr{B}'$ verifies the conditions (α) or (β); the first case is clear, and to prove the second one, it suffices to prove that for any $x' \in X'$, the ranks of $\mathscr{B}' \otimes_{\mathscr{O}_{X'}} \mathscr{K}_{X'_\beta}$ for the indices β such that $x' \in X'_\beta$ are the same. Now, if the restriction of φ to X'_β is a dominant morphism into X_α, the rank of $\mathscr{B}' \otimes_{\mathscr{O}_{X'}} \mathscr{K}_{X'_\beta}$ is equal to that of $\mathscr{B} \otimes_{\mathscr{O}_X} \mathscr{K}_{X_\alpha}$ (which can be seen from the affine case), whence our claim.

This being established, it follows that if f is a section of $\mathscr{B}$ over an open subset $U \subseteq X$, and f' is the inverse image of f under φ, $N_{\mathscr{B}'/\mathscr{A}'}(f')$ is the section of $\mathscr{A}'$ over $\varphi^{-1}(U)$ which is the inverse image of $N_{\mathscr{B}/\mathscr{A}}(f)$ under φ. If $\mathscr{L}$ is an invertible $\mathscr{B}$-module and if $\mathscr{L}' = \varphi^*(\mathscr{L})$ (which is an invertible $\mathscr{B}'$-module), we have

$$N_{\mathscr{B}'/\mathscr{A}'}(\mathscr{L}') = \varphi^*(N_{\mathscr{B}/\mathscr{A}}(\mathscr{L})). \tag{11.6.12}$$

Suppose now that $(X, \mathscr{A})$ is a scheme. Then giving a quasi-coherent finite $\mathscr{A}$-algebra $\mathscr{B}$ is equivalent to giving a finite morphism $\varphi : X' \to X$ such that $\varphi_*(\mathscr{O}_{X'}) = \mathscr{B}$, defined up to X-isomorphisms (Corollary 11.1.11), and in this case X' is isomorphic to the affine specturm $\operatorname{Spec}(\mathscr{B})$. Moreover, if this morphism $\varphi : X' \to X$ is fixed, then giving a quasi-coherent $\mathscr{O}_{X'}$-module $\mathscr{F}'$ is equivalent to giving a quasi-coherent $\mathscr{B}$-module such that $\varphi_*(\mathscr{F}') = \mathscr{F}$ (Proposition 11.1.20), and for $\mathscr{F}'$ to be invertible, it is necessary and sufficient that $\mathscr{F}$ is (Proposition 11.6.11). To utilize the preceding results for the finite morphism φ, it is then necessary to assume that $\mathscr{B} = \varphi_*(\mathscr{O}_{X'})$ satisfies condition (α) or (β). For any invertible $\mathscr{O}_{X'}$-module $\mathscr{L}'$, we then set

$$N_{X'/X}(\mathscr{L}') := N_{\varphi_*(\mathscr{O}_{X'})/\mathscr{O}_X}(\varphi_*(\mathscr{L}')) \tag{11.6.13}$$

which is called the **norm** (relative to φ) of $\mathscr{L}'$. Similarly, if $u' : \mathscr{L}' \to \mathscr{K}'$ is a homomorphism of invertible $\mathscr{O}_{X'}$-modules, we put

$$N_{X'/X}(u') = N_{\varphi_*(\mathscr{O}_X)/\mathscr{O}_X}(\varphi_*(u')) : N_{X'/X}(\mathscr{L}') \to N_{X'/X}(\mathscr{K}'). \tag{11.6.14}$$

In particular, if we consider homomorphisms $\mathscr{O}_{X'} \to \mathscr{L}'$, we obtain a canonical homomorphism

$$N_{X'/X} : \Gamma(X', \mathscr{L}') \to \Gamma(X, N_{X'/X}(\mathscr{L}')). \tag{11.6.15}$$

Proposition 11.6.30. *Let $\varphi : X' \to X$ be a finite morphism and suppose that condition (i) or (ii) is satisfied. For a homomorphism $u' : \mathscr{L}' \to \mathscr{K}'$ of invertible $\mathscr{O}_{X'}$-modules to be an isomorphism, it is necessary and sufficient that, in the first case, that $N_{X'/X}(u')$ is an isomorphism; in the second case, this condition is necessary, and is sufficient if $\varphi_*(\mathscr{O}_{X'})$ is torsion free.*

Proof. This is a particular case of our previous discussions, where we put $\mathscr{B} = \varphi_*(\mathscr{O}_{X'})$, which is a quasi-coherent finite $\mathscr{O}_X$-algebra. We also note that by Corollary 11.1.19, for $\varphi_*(u')$ to be an isomorphism, it is necessary and sufficient that u' is an isomorphism. □

Corollary 11.6.31. *Retain the hypothesis of Proposition 11.6.30 and suppose that $\varphi_*(\mathscr{O}_{X'})$ is torsion free. Let $\mathscr{L}'$ be an invertible $\mathscr{O}_{X'}$-module, f' be a section of $\mathscr{L}'$ over X', and $f = N_{X'/X}(f')$ the section of $\mathscr{L} = N_{X'/X}(\mathscr{L}')$ over X corresponding to f'. Then we have $\varphi(X' - X'_{f'}) = X - X_f$ and X_f is the largest open subset of X such that $\varphi^{-1}(U) \subseteq X'_{f'}$.*

Proof. In fact, $\varphi(X' - X'_{f'})$ is closed in X by Proposition 11.6.7, and it then suffices to prove the second assertion. Now the relation $U \subseteq X_f$ is equivalent to that the hommorphism $\mathscr{O}_X|_U \to \mathscr{L}|_U$ defined by $f|_U$ is an isomorphism. In view of Proposition 11.6.30, this is equivalent to that the homomorphism $\mathscr{O}_{X'}|_{\varphi^{-1}(U)} \to \mathscr{L}'|_{\varphi^{-1}(U)}$ defined by $f'|_{\varphi^{-1}(U)}$ is an isomorphism, which means $\varphi^{-1}(U) \subseteq X'_{f'}$. □

Proposition 11.6.32. *Let $\varphi : X' \to X$ be a finite morphism, $\psi : Y \to X$ be a morphism; let $Y' = X'_{(Y)}$, $\varphi' = \varphi_{(Y)}$, $\psi' = \psi_{(X')}$ such that the following diagram is commutative*

$$\begin{tikzcd} Y' \arrow[r, "\psi'"] \arrow[d, "\varphi'"'] & X' \arrow[d, "\varphi"] \\ Y \arrow[r, "\psi"] & X \end{tikzcd}$$

Assume the hypotheses of Proposition 11.6.30. Then for any invertible $\mathscr{O}_{X'}$-module $\mathscr{L}'$, we have

$$N_{Y'/Y}(\psi'^*(\mathscr{L}')) = \psi^*(N_{X'/X}(\mathscr{L}')).$$

Proof. Note that we have $\psi^*(\varphi_*(\mathscr{L}')) = \varphi'_*(\psi'^*(\mathscr{L}'))$ in view of Corollary 11.1.30, and in particular $\varphi'_*(\mathscr{O}_{Y'}) = \psi^*(\varphi_*(\mathscr{O}_{X'}))$; if $\varphi_*(\mathscr{O}_{X'})$ is locally free, so is $\varphi'_*(\mathscr{O}_{Y'})$. The conclusion then follows from the definition of $N_{X'/X}$, $N_{Y'/Y}$, and (11.6.12). □

11.6.6 A criterion for ample sheaves

Proposition 11.6.33. *Let $f : X \to Y$ be a quasi-compact morphism, $g : X' \to X$ be a finite and surjective morphism such that $(X, \mathscr{O}_X)$ and $g_*(\mathscr{O}_{X'})$ satisfy condition (β). Then, for an ample invertible $\mathscr{O}_{X'}$-module $\mathscr{L}'$ relative to $f \circ g$, $N_{X'/X}(\mathscr{L}') = \mathscr{L}$ is ample relative to f.*

Proof. We can suppose that Y is affine, and then, in view of Corollary 11.4.43, it suffices to prove that, if $\mathscr{L}'$ is ample, then $\mathscr{L} = N_{X'/X}(\mathscr{L}')$ is ample. For this, we can assume that $g_*(\mathscr{O}_{X'})$ is torsion free. In fact, let $\mathscr{T}$ be the kernel of the homomorphism $g_*(\mathscr{O}_{X'}) \to g_*(\mathscr{O}_{X'}) \otimes_{\mathscr{O}_X} \mathscr{K}_X$, which is a coherent ideal of $\mathscr{B} = g_*(\mathscr{O}_{X'})$ by hypothesis, and put $X'' = \operatorname{Spec}(\mathscr{B}/\mathscr{T})$; we then have a commutative diagram

$$\begin{tikzcd} X'' \arrow[rr, "j"] \arrow[dr, "g'"'] & & X' \arrow[dl, "g"] \\ & X & \end{tikzcd}$$

where j is a closed immersion (Proposition 11.1.27). Moreover, since $\mathscr{T}$ is a torsion sheaf, by Proposition 10.7.27 and **??** we see that the support of $\mathscr{T}$ is a closed subset that is rare in X, so for the generic point x of an irreducible component of X, there exists an affine open neighborhood U of x such that $\mathscr{B}|_U = (\mathscr{B}/\mathscr{T})|_U$. As g is by hypothesis surjective, we then conclude that

$x \in g'(X'')$; g' is then dominant, and hence surjective by Proposition 11.6.7 since it is a finite morphism. By definition we have

$$g'_*(\mathscr{O}_{X''}) \otimes \mathscr{K}_X = (\mathscr{B}/\mathscr{T}) \otimes_{\mathscr{O}_X} \mathscr{K}_X = g_*(\mathscr{O}_{X'}) \otimes_{\mathscr{O}_X} \mathscr{K}_X,$$

so $(X, \mathscr{O}_X)$ and $g'_*(\mathscr{O}_{X''})$ satisfies condition (β), and $g'_*(\mathscr{O}_{X''})$ is torsion free. Finally, $j^*(\mathscr{L}') = \mathscr{L}''$ is an ample $\mathscr{O}_{X''}$-module (Proposition 11.4.51(ii)), and $N_{X''/X}(\mathscr{L}'') = N_{X'/X}(\mathscr{L}')$. To see this, we note that to define these two invertible $\mathscr{O}_X$-modules we can utilize an affine open covering (U_λ) of X such that $g_*(\mathscr{L}')$ and $g'_*(\mathscr{L}'')$ to U_λ are respectively isomorphic to $\mathscr{B}|_{U_\lambda}$ and $(\mathscr{B}/\mathscr{T})|_{U_\lambda}$. By Corollary 11.1.25, we immediately see that for any isomorphism $\eta_\lambda : g_*(\mathscr{L}')|_{U_\lambda} \to \mathscr{B}|_{U_\lambda}$ corresponds canonically to an isomorphism

$$\eta'_\lambda : g'_*(\mathscr{L}'')|_{U_\lambda} \to (\mathscr{B}/\mathscr{T})|_{U_\lambda}$$

so that, if $(\omega_{\lambda\mu})$ and $(\omega'_{\lambda\mu})$ are the 1-cocycles corresponding to the isomorphisms (η_λ) and (η'_λ), $\omega'_{\lambda\mu}$ is the canonical image of $\omega_{\lambda\mu} \in \Gamma(U_\lambda \cap U_\mu, \mathscr{B})$ to $\Gamma(U_\lambda \cap U_\mu, \mathscr{B}/\mathscr{T})$. In view of the definition of $\mathscr{T}$, we conclude that

$$N_{\mathscr{B}/\mathscr{A}}(\omega_{\lambda\mu}) = N_{(\mathscr{B}/\mathscr{T})/\mathscr{A}}(\omega'_{\lambda\mu})$$

(where $\mathscr{A} = \mathscr{O}_X$), whence the equality.

Suppose then that $g_*(\mathscr{O}_{X'})$ is torsion free. It then suffices to prove that if f runs through the sections of $\mathscr{L}^{\otimes n}$ $(n > 0)$ over X, the X_f form a basis of X (Theorem 11.4.27). Now, let $x \in X$, and let U be an open neighborhood of x; as $g^{-1}(x)$ is finite by Corollary 11.6.5 and $\mathscr{L}'$ is ample, there exists an integer $n > 0$ and a section f' of $\mathscr{L}'^{\otimes n}$ over X' such that $X'_{f'}$ is an open neighborhood of $g^{-1}(x)$ contained in $g^{-1}(U)$. As we have $\mathscr{L}^{\otimes n} = N_{X'/X}(\mathscr{L}'^{\otimes n})$, it then suffices to choose $f = N_{X'/X}(f')$: in fact, we have $X - X_f = g(X' - X'_{f'})$ by Corollary 11.6.31, so $x \in X_f \subseteq U$. □

Corollary 11.6.34. *Under the hypotheses of Proposition 11.6.33, for an invertible $\mathscr{O}_X$-module $\mathscr{L}$ to be ample relative to f, it is necessary and sufficient that $\mathscr{L}' = g^*(\mathscr{L})$ is ample relative to $f \circ g$.*

Proof. This condition is necessary, since g is affine (Proposition 11.5.11). To see the sufficiency, we can assume that Y is affine, so X and X' are quasi-compact and $\mathscr{L}'$ is ample (Corollary 11.4.43). Now the set of points $x \in X$ such that there is a neighborhood of x over which $g_*(\mathscr{O}_{X'})$ (resp. $g_*(\mathscr{O}_{X'}) \otimes_{\mathscr{O}_X} \mathscr{K}_X$) is of rank n is open and closed in X by hypotheses, so X is a finite sum of such open subschemes (recall that X is quasi-compact), and we can therefore suppose that it is equal to one of them (Proposition 11.4.54). But we then have $N_{X'/X}(\mathscr{L}') = \mathscr{L}^{\otimes n}$, so $\mathscr{L}^{\otimes n}$ is ample in view of Proposition 11.6.33, and $\mathscr{L}$ is therefore ample. □

Corollary 11.6.35. *Under the hypotheses of Proposition 11.6.33, suppose moreover that $f : X \to Y$ is of finite type. Then, for f to be quasi-projective, it is necessary and sufficient that $f \circ g$ is quasi-projective. If we suppose that Y is quasi-compact and quasi-compact, then, for f to be projective, it is necessary and sufficient that $f \circ g$ is projective.*

Proof. The hypotheses implies that $f \circ g$ is of finite type. By the definition of quasi-projective morphisms, the first assertion then follows from Proposition 11.6.33 and Corollary 11.6.34. In view of this result and Theorem 11.5.30, it remains to prove that if f is quasi-projective, then for f to be proper, it is necessary and sufficient that $f \circ g$ is proper. But f is then separated and of fintie type, and as g is surjective, this follows from Corollary 11.5.24(ii). □

Corollary 11.6.36. *Let X be a scheme of finite type over a field K and K' be finite extension of K. For X to be projective (resp. quasi-projective) over K, it is necessary and sufficient that $X' \otimes_K K'$ is projective (resp. quasi-projective) over K'.*

Proof. This condition is necessary by Proposition 11.5.20(iii) and Proposition 11.5.34(iii). Conversely, suppose that X' is projective (resp. quasi-projective), and let $g : X' \to X$ be the canonical projective. Since K' is finite over K, it is clear that g is a finite morphism by Proposition 11.6.3 and is surjective (Proposition 10.3.28). Moreover, $g_*(\mathscr{O}_{X'})$ is a locally free $\mathscr{O}_X$-module, being isomorphic to $\mathscr{O}_X \otimes_K K'$ (Corollary 11.1.32). It then follows from the hypotheses and Corollary 11.6.9 and Proposition 11.5.34(ii) that X' is projective (resp. quasi-projective) over K. We then deduce from Corollary 11.6.35 that X is projective (resp. quasi-projective) over K. □

Remark 11.6.37. In fact, later we will see that the statement of Corollary 11.6.36 is valid for arbitrary extension K' of K.

The end of this subsection is devoted to the proof of the criterion in Proposition 11.6.42, which is a refinement of the techniques we have currently used.

Lemma 11.6.38. *Let X be a reduced Noetherian scheme and $\mathscr{E}$ be a coherent $\mathscr{O}_X$-module such that $\mathscr{E} \otimes_{\mathscr{O}_X} \mathscr{K}_X$ is locally free of rank n. Then there exists a reduced Noetherian scheme Z and a birational and finite morphism $h : Z \to X$ such that the homomorphisms $\sigma_i : \mathcal{H}om_{\mathscr{O}_X}(\mathscr{E}, \mathscr{E}) \to \mathscr{K}_X$ send $\mathcal{H}om_{\mathscr{O}_X}(\mathscr{E}, \mathscr{E})$ into the coherent $\mathscr{O}_X$-algebra $h_*(\mathscr{O}_Z)$.*

Corollary 11.6.39. *Under the hypotheses of Lemma 11.6.38, let W be an open subset of X such that for any $x \in W$, either X is normal at x or $\mathscr{E}_x$ is a free $\mathscr{O}_{X,x}$-module. Then we can choose h so that the restriction of h to $h^{-1}(W)$ is an isomorphism $h^{-1}(W) \cong W$.*

Proof. In fact, the hypotheses imply that if $U \subseteq W$ is an affine open subset, we have, in the notations of Lemma 11.6.38, that $(\sigma_i(u))_x \in A_x$ for any $x \in U$ (Proposition 11.6.26), hence $\sigma_i(u) \in A$, and the conclusion follows from the definition of h given in Lemma 11.6.38. □

Proof. Let X be a reduced Noetherian scheme, $g : X' \to X$ be a finite and surjective morphism such that $\mathscr{B} = g_*(\mathscr{O}_{X'})$ is a coherent $\mathscr{O}_X$-module. Suppose that $\mathscr{B} \otimes_{\mathscr{O}_X} \mathscr{K}_X$ is locally free of rank n. Then we can apply Lemma 11.6.38 on $\mathscr{E} = \mathscr{B}$, with the same notations, to obtain a homomorphism $\sigma_n : \mathcal{H}om_{\mathscr{O}_X}(\mathscr{B}, \mathscr{B}) \to h_*(\mathscr{O}_Z)$, and by composing with the canonical injection $\mathscr{B} \to \mathcal{H}om_{\mathscr{O}_X}(\mathscr{B}, \mathscr{B})$, we obtain a homomorphism of sheaves of multiplicative monoids:

$$\widetilde{\mathrm{N}} : \mathscr{B} = g_*(\mathscr{O}_{X'}) \to h_*(\mathscr{O}_Z) = \mathscr{C}. \tag{11.6.16}$$

For any invertible $\mathscr{O}_{X'}$-module $\mathscr{L}'$, $g_*(\mathscr{L}')$ is an invertible $\mathscr{B}$-module by Proposition 11.6.11, and using the same method, we can define an invertible $\mathscr{C}$-module $\widetilde{\mathrm{N}}(g_*(\mathscr{L}'))$, which is functorial on $\mathscr{L}'$. □

Lemma 11.6.40. *Let X be a reduced Noetherian scheme, $g : X' \to X$ be a finite and surjective morphism such that $g * (\mathscr{O}_{X'}) \otimes_{\mathscr{O}_X} \mathscr{K}_X$ is locally free of rank n. Then there exists a reduced Noetherian scheme Z and a finite and birational morphism $h : Z \to X$ such that for any ample $\mathscr{O}_X$-module $\mathscr{L}'$, the invertible $\mathscr{O}_Z$-module $\mathscr{M}$ such that $h_*(\mathscr{M}) = \widetilde{\mathrm{N}}(g_*(\mathscr{L}'))$ is ample.*

Corollary 11.6.41. *Under the hypotheses of Lemma 11.6.40, for any invertible $\mathscr{O}_X$-module $\mathscr{L}$ such that $g^*(\mathscr{L})$ is ample, $h^*(\mathscr{L})$ is ample.*

Proposition 11.6.42. *Let Y be an affine scheme, X be a reduced Noetherian scheme, $f : X \to Y$ be a quasi-compact morphism, and $g : X' \to X$ be a finite and surjective morphism. Let W be an open subset of X such that, for any $x \in W$, either X is normal at x, or there exists an open neighborhood $T \subseteq W$ of x such that $(g_*(\mathscr{O}_{X'}))|_T$ is a locally free $(\mathscr{O}_X|_T)$-module. Then there exists a reduced Y-scheme Z and a finite and birational Y-morphism $h : Z \to X$ such that the restriction of h to $h^{-1}(W)$ is an isomorphism $h^{-1}(W) \cong W$ and satisfies the following property: for any invertible $\mathscr{O}_X$-module such that $g^*(\mathscr{L})$ is ample relative to $f \circ g$, $h^*(\mathscr{L})$ is ample relative to $f \circ h$.*

Corollary 11.6.43. *If in Proposition 11.6.42 we have $W = X$, then for an invertible $\mathscr{O}_X$-module $\mathscr{L}$ to be ample relative to f, it is necessary and sufficient that $g^*(\mathscr{L})$ is ample relative to $f \circ g$.*

Remark 11.6.44. We shall see in Chapter 12 that if Y is Noetherian, f is of finite type, and if the restriction of f to the reduced closed subscheme of X having $X - W$ as underlying space is proper, then the conclusion of Corollary 11.6.43 is still valid. But we will also give examples of algebraic schemes X over a field K (the structural morphism $X \to \operatorname{Spec}(K)$ not being proper) whose normalize X' is quasi-affine, but which is not quasi-affine (so that $\mathscr{O}_X$ is not ample, although $\mathscr{O}_{X'}$ is, cf. Proposition 11.5.1, and that the morphism $g : X' \to X$ is finite and surjective (cf. Remark 11.6.25)). We will also see that this circumstance cannot occur when we replace "quasi-affine" by "affine" (by Chevalley's theorem).

11.6.7 Chevally's theorem

Lemma 11.6.45. *Let X, Y be integral Noetherian schemes, x (resp. y) be the generic point of X (resp. Y), and $f : X \to Y$ be a finite and surjective morphism. Let $\mathscr{L}$ be an invertible $\mathscr{O}_X$-module such that there exists an affine neighborhood U of y and a section $s \in \Gamma(X, \mathscr{L})$ such that $x \in X_s \subseteq f^{-1}(U)$. Then there exist integers $m, n > 0$, a homomorphism $u : \mathscr{O}_Y^m \to f_*(\mathscr{L}^{\otimes n})$ and an open neighborhood V of y such that the restriction $u|_V$ is an isomorphism $\mathscr{O}_Y^m|_V \xrightarrow{\sim} f_*(\mathscr{L}^{\otimes n})|_V$.*

Theorem 11.6.46 (Chevalley). *Let X be an affine scheme, Y be a Noetherian scheme, and $f : X \to Y$ be a finite and surjective morphism. Then Y is an affine scheme.*

Proof. It is clear that $f_{\text{red}} : X_{\text{red}} \to Y_{\text{red}}$ is fiite (Proposition 11.6.3(vi)); as X_{red} is an affine scheme, and Y is affine if and only if Y_{red} is (Corollary 10.4.37), we can assume that X, Y are reduced. For any closed subset Y' of Y, there is then a unique reduced subscheme subscheme structure on Y' whose inverse image $f^{-1}(Y')$, canonically isomorphic to $X \times_Y Y'$, is affine as a closed subscheme of X, and the restriction of f to $f^{-1}(Y')$, which is identified with $f \times_Y 1_{Y'}$, is a finite and surjective morphism (Proposition 10.3.28 and Proposition 11.6.3(iv)). In view of the Noetherian induction principle (**??**), we are then (in view of Corollary 10.4.37) reduced to prove the theorem under the hypothesis that for any closed subset $Y' \neq Y$, any closed subscheme of Y with underlying space Y' is affine. With this hypothesis, we first note that, for any coherent $\mathscr{O}_Y$-module $\mathscr{F}$ whose support (closed) Z is distinct from Y, we have $H^1(Y, \mathscr{F}) = 0$. In fact, there exists a closed subscheme structure on Z such that, if $j : Z \to Y$ is the canonical injection, we have $\mathscr{F} = j_*(j^*(\mathscr{F}))$ (Corollary 10.6.18 and Theorem 10.1.30), and therefore $H^1(Y, \mathscr{F}) = H^1(Z, j^*(\mathscr{F})) = 0$ (Corollary 11.5.16, since j is affine).

Suppose first that Y is not irreducible, and let Y' be an irreducible component of Y; we endow Y' with the reduced subscheme structure, and let $j : Y' \to Y$ be the canonical injection. Let $\mathscr{F}$ be a coherent $\mathscr{O}_Y$-module, and consider the canonical homomorphism

$$\rho : \mathscr{F} \to \mathscr{F}' = j_*(j^*(\mathscr{F}));$$

Since j is proper and Y', Y are Noetherian schemes, $\mathscr{F}'$ is a coherent $\mathscr{O}_Y$-module by **??** (since we have $j_*(\mathscr{O}_{Y'}) = \mathscr{O}_Y/\mathscr{J}$, where $\mathscr{J}$ is the quasi-coherent ideal of $\mathscr{O}_Y$ defining the subscheme Y'), so $\mathscr{G} = \ker \rho$ and $\mathscr{K} = \operatorname{im} \rho$ are coherent $\mathscr{O}_Y$-modules (**??**). On the other hand, by definition the fiber $\mathscr{F}'_y$ of $\mathscr{F}'$ at the generic point y of Y' is equal to $\mathscr{F}_y/\mathscr{J}_y\mathscr{F}_y$, and hence to $\mathscr{F}_y$ (Example 10.4.47), so y is not contained in the support of $\mathscr{G}$ and we conclude that $H^1(Y, \mathscr{G}) = 0$. Since the support of $\mathscr{F}'$ (and a fortiori that of $\mathscr{K}$) is contained in Y', it is distinct from Y, and we also conclude that $H^1(Y, \mathscr{K}) = 0$. From the exact sequence $0 \to \mathscr{G} \to \mathscr{F} \to \mathscr{K} \to 0$ we then get $H^1(Y, \mathscr{F}) = 0$, so by Serre's criterion Y is then affine.

Suppose now that Y is irreducible, and therefore integral. We can also assume that X is integral: in fact, if we denote by X_i the reduced closed subschemes of X with underlying spaces the irreducible components of X and by $f_i : X_i \to Y$ the restriction of f to X_i, then one of f_i is dominant (Y is irreducible, so if its generic point is contained in the image of f_i, then f_i is dominant, and we note that f is surjective), and as there are finite morphisms (Proposition 11.6.3), it is surjective (Proposition 11.6.7); as X_i is an affine scheme, we see that we can replace X by X_i. In this case, we can apply Lemma 11.6.45 to $\mathscr{L} = \mathscr{O}_X$, since X is affine, to obtain a homomorphism $u : \mathscr{O}_Y^m \to f_*(\mathscr{L}) = \mathscr{B}$ and an open neighborhood V of y such that $u|_V$ is an isomorphism. In view of Serre's criterion, it suffices to prove that for any coherent $\mathscr{O}_Y$-module $\mathscr{F}$ such that $\mathscr{F} \subseteq \mathscr{O}_Y$, we have $H^1(Y, \mathscr{F}) = 0$. We note that then $\mathscr{F}$ is torsion free since Y is integral, and we only need to show that $H^1(Y, \mathscr{F}) = 0$ for any torsion free coherent $\mathscr{O}_Y$-module $\mathscr{F}$. Now the homomorphism u defines a homomorphism

$$v : \mathscr{G} = \mathcal{H}om_{\mathscr{O}_Y}(\mathscr{B}, \mathscr{F}) \to \mathcal{H}om_{\mathscr{O}_Y}(\mathscr{O}_Y^m, \mathscr{F}) = \mathscr{F}^m.$$

By hypotheses the support of $\mathscr{T} = \operatorname{coker} u$ does not meet V, so is a torsion $\mathscr{O}_Y$-module (Proposition 10.7.27). From the exact sequence $\mathscr{O}_Y^m \to \mathscr{B} \to \mathscr{T} \to 0$ induces, by the left exactness of $\mathcal{H}om_{\mathscr{O}_Y}$, an exact sequence

$$0 \longrightarrow \mathcal{H}om_{\mathscr{O}_Y}(\mathscr{T}, \mathscr{F}) \longrightarrow \mathscr{G} \xrightarrow{v} \mathscr{F}^m$$

But as $\mathscr{F}$ is torsion free and $\mathscr{T}$ is torsion, we have $\mathcal{H}om_{\mathscr{O}_Y}(\mathscr{T}, \mathscr{F}) = 0$, so v is injective. We then obtain an exact sequence

$$0 \longrightarrow \mathscr{G} \longrightarrow \mathscr{F}^m \longrightarrow \operatorname{coker} v \longrightarrow 0$$

where $\mathscr{G}$ and $\operatorname{coker} v$ are coherent $\mathscr{O}_Y$-modules (**??**). In view of the exact sequence on cohomology, it suffices to prove that $H^1(Y, \mathscr{F}) = H^1(Y, \operatorname{coker} v) = 0$ since this implies $H^1(Y, \mathscr{F}^m) = (H^1(Y, \mathscr{F}))^m = 0$, so $H^1(Y, \mathscr{F}) = 0$. Now the restriction $v|_V$ is an isomorphism, so the support of $\operatorname{coker} v$ is distinct from Y, and we have $H^1(Y, \operatorname{coker} v) = 0$ by our hypothesis. On the other hand, $\mathscr{G}$ is a coherent $\mathscr{B}$-module by Corollary 10.2.31; as X is affine over Y, there exists a quasi-coherent $\mathscr{O}_X$-module $\mathscr{K}$ such that $\mathscr{G}$ is isomorphic to $f_*(\mathscr{K})$ (Proposition 11.1.20), and since $H^1(X, \mathscr{K}) = 0$ (X is affine), we then have $H^1(Y, \mathscr{G}) = 0$ by Corollary 10.6.18, which completes the proof. □

Corollary 11.6.47. *Let X be a Noetherian scheme and $(X_i)_{1 \leq i \leq n}$ be a finite covering of X by closed subsets. Then for X to be affine, it is necessary and sufficient that for each i, there exists a closed subscheme of X that is affine and has underlying space X_i.*

Proof. Let X' be the sum of the X_i, then it is clear that X' is affine if each X_i is affine, and we have a surjective morphism $f : X' \to X$ whose restriction to X_i is the canonical injection. To apply Theorem 11.6.46, it remains to verifying that f is finite, and this follows from Proposition 11.6.3(i). □

11.7 Valuative criterion

In this section we give the valuative criterion of sparation and properness of a morphis, which are criteria which involve a auxiliary scheme $\operatorname{Spec}(A)$, where A is a valuation ring. With a convenient "Noetherian" hypothesis, these criterion can be refined to the case where A is a discrete valuation ring, and this will probably be the only case that we will apply later.

11.7.1 Remainders for valuation rings

Among the vast properties that characterize valuation rings, we shall use the following one: a ring A is called a valuation ring if it is an integral domain which is not a field, and if in the set of proper local rinng contained in the fraction field K of A, A is maximal under the dominant relation. Recall that a valuation ring is integrally closed. If A is a valuation ring, then $A_\mathfrak{p}$ is also a valuation ring for any nonzero prime ideal $\mathfrak{p} \neq 0$.

Let K be a field, A be a proper local subring of K; then there exists a valuation ring of K dominating A (Theorem 5.1.7). On the other hand, let B be a valuation ring, k be its residue field, and K be the fraction field, L be an extension of k. Then there exists a complete valuation ring C dominating B with residue field equals to L. In fat, L is an algebraic extension of a purely transcendental extension $L' = k(T_\mu)_{\mu \in M}$; we can extend the valuation of K corresponding to B to a valuation of $K' = K(T_\mu)_{\mu \in M}$ with residue field L'; replace B by this complete valuation ring C, we can assume then that B is complete that L is an algebraic closure of k. If $\bar{K}$ is an algebraic closure of K, we can then extend the defining valuation of B to $\bar{K}$, and the corresponding residue field is an algebraic closure of k, as can be seen by lifting the coefficients of a monic polynomial of $k[T]$ to $\bar{K}$. We are therefore finally reduced to the case where $L = k$ and it suffices then to take for C the completion of B to answer the question.

Let K be a field and A be a subring of K; the integral closure A' of A in K is the intersection of valuation rings of the fraction field of A containing A (Theorem 5.1.8). The preceding argument then have the following geometric form:

Proposition 11.7.1. *Let Y be a scheme, $p : X \to Y$ be a morphism, x be a point of X, $y = p(x)$, and $y' \neq y$ be a specialization of y. Then there exists a local scheme Y', which is the specturm of a valuation ring, and a separated morphism $f : Y' \to Y$ such that, if a is the unique closed point of Y' and b is the generic point of Y', we have $f(a) = y'$ and $f(b) = y$. We can moreover suppose that one of the following additional conditions are satisfied:*

- (i) *Y' is the specturm of a complete valuation ring whose residue field is algebraically closed, and there exists a $\kappa(y)$-homomorphism $\kappa(x) \to \kappa(b)$.*
- (ii) *There exists a $\kappa(y)$-isomorphism $\kappa(x) \cong \kappa(b)$.*

Proof. Let Y_1 be the reduced closed subscheme of Y with $\overline{\{y\}}$ as underlying space, and X_1 be the closed subscheme $p^{-1}(Y_1)$; as $y' \in \overline{\{y\}}$ by hypothesis and $\kappa(x)$ is the same for X and X_1, by replacing Y with Y_1 and X with X_1, we can suppose that Y is integral with generic point y; $\mathscr{O}_{Y,y'}$ is then an integral local ring which is not a field, whose fraction field is $\mathscr{O}_{Y,y} = \kappa(y)$, and $\kappa(x)$ is an extension of $\kappa(y)$. To realize the conditions $f(a) = y'$ and $f(b) = y$ with the additional condition (i) (resp. (ii)), we choose $Y' = \operatorname{Spec}(A')$, where A' is a valuation ring dominating $\mathscr{O}_{Y,y'}$ and which is complete and with residue field an algebraically closed extension of $\kappa(x)$ (resp. a valuation ring dominating $\mathscr{O}_{Y,y'}$ with fraction field $\kappa(x)$); the existence of such rings are proved by the above remarks. □

Recall that a local ring $(A, \mathfrak{m})$ is of dimension if and only if any prime ideal of A distinct from $\mathfrak{m}$ is minimal; if A is integral, this means $\mathfrak{m}$ and (0) are the only prime ideals, and $\mathfrak{m} \neq (0)$; equivalent, $Y = \operatorname{Spec}(A)$ is reduced to two points a, b: a is the closed point, $\mathfrak{p}_a = \mathfrak{m}$, and $\kappa(a) = k$ is the residue field $A/\mathfrak{m}$; b is the generic point of Y, $\mathfrak{p}_b = (0)$, the set $\{b\}$ is the unique nontrivial open subset of Y, and $\kappa(b) = K$ is the fraction field of A. For an integral Noetherian local ring A of dimension 1, the following conditions are then equivalent:

- (i) A is normal;
- (ii) A is regular;
- (iii) A is a valuation ring.

Moreover, if these are ture, A is then a discrete valuation ring.

Proposition 11.7.2. *Let A be a Noetherian local integral domain which is not a field, K be its fraction field, L be a extension of K of finite type. There then exists a discrete valuation ring of L dominating A.*

Proof. □

Corollary 11.7.3. *Let A be an integral Noetherian ring, K be its fraction field, and L be an extension of K of finite type. Then the integral closure of A in L is the intersection of discrete valuation rings of L containing A.*

Proposition 11.7.4. *Let Y be a locally Noetherian scheme, $p : X \to Y$ be a morphism locally of finite type, x be a point of X, $y = p(x)$, $y' \neq y$ be a specialization of y. Then there exists a local scheme Y', which is a specturm of a discrete valuation ring, and a separated morphism $f : Y' \to Y$ and rational Y-map $g : Y' \dashrightarrow X$ such that, if a is the closed point of Y' and b is the generic point, we have $f(a) = y'$, $f(b) = y$, $g(b) = x$, and in the following commutative diagram*

$$\begin{tikzcd}
& \kappa(x) \arrow[dl, "\gamma"'] \\
\kappa(b) & \kappa(y) \arrow[l, "\varphi"] \arrow[u, "\pi"']
\end{tikzcd}$$

(whee π, φ, γ are the homomorphisms corresponding to p, f and g), γ is a bijection.

Proof. □

11.7.2 Valuative criterion of separation

Proposition 11.7.5 (Valuative Criterion of Separation). *Let Y be a scheme (resp. a locally Noetherian scheme), $f : X \to Y$ be a morphism (resp. a morphism locally of finite type). The following conditions are equivalent:*

(i) *f is separated.*

(ii) *f is quasi-separated and for any Y-scheme of the form $Y' = \operatorname{Spec}(A)$, where A is a valuation ring (resp. a discrete valuation ring), two Y-morphisms of Y' to X which coincide at the generic point of Y' are equal.*

(iii) *f is quasi-seaprated and for any Y-scheme of the form $Y' = \operatorname{Spec}(A)$, where A is a valuation ring (resp. a discrete valuation ring), two Y'-sections of $X' = X_{(Y')}$ which coincide at the generic point of Y' are equal.*

Proof. The equivalence of (ii) and (iii) follows from the correspondence of Y-morphisms $Y' \to X$ and Y'-sections of X'. If X is separated over Y, condition (ii) follows from the proof of Proposition 10.7.10, since Y' is integral. It then remains to prove that condition (ii) implies that the diagonal morphism $\Delta : X \to X \times_Y X$ is closed, and for this we can use Proposition 10.6.6. Now, let z be a point of the diagonal $\Delta(X)$, $z' \neq z$ be a specialization of z in $X \times_Y X$. Then there exists by Proposition 11.7.1 a valuation ring A and a morphism $g : Y' \to X \times_Y X$ such that $g(a) = z'$, $g(b) = z$ (with the notations of Proposition 11.7.1, a is the closed point of Y' and b is the generic point of Y'); this morphism makes Y' an $(X \times_Y X)$-scheme, and a fortiori a Y-scheme. If we compose g with the two projections of $X \times_Y X$, we obtain two Y-morphisms $g_1, g_2 : Y' \to X$, which by hypotheses send the point b to the same point in X; in view of (ii), these two morphisms coincide with a morphism $h : Y' \to X$, which signifies that g factors into $g = \Delta \circ h$, and therefore $z' \in \Delta(X)$. If we suppose that Y is locally Noetherian and f is of finite type, $X \times_Y X$ is locally Noetherian by Corollary 10.6.22, and we can therefore replace Proposition 10.7.10 by Proposition 11.7.4. □

The condition (ii) of Proposition 11.7.5 signifies that if $Y' = \operatorname{Spec}(A)$ and $X' = \operatorname{Spec}(K)$ where K is the fraction field of A, the canonical map

$$\operatorname{Hom}_Y(Y', X) \to \operatorname{Hom}_Y(X', X)$$

is injective. Equivalently, this means in the following diagram

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ Y' \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

the dashed morphism $Y' \to X$, if exists, is necessarily unique.

Remark 11.7.6. In the criterion (ii) of Proposition 11.7.5, we can restrict ourselves to valuation rings A which is complete whose residue field is algebraically closed; this follows from the additional condition (i) of Proposition 10.7.10.

11.7.3 Valuative criterion of properness

Proposition 11.7.7. *Let A be a valuation ring, $Y = \operatorname{Spec}(A)$, b be the generic point of Y. Let X be an integral and separated scheme and $f : X \to Y$ be a closed morphism such that $f^{-1}(b)$ is reduced to a point x and the corresponding homomorphism $\kappa(b) \to \kappa(x)$ is bijective. Then f is an isomorphism.*

Proof. As f is closed and dominant, we have $f(X) = Y$; it then suffices to prove that for any $y' \neq b$ in Y, there exists a unique point x' such that $f(x') = y'$ and the corresponding homomorphism $\mathscr{O}_{Y,y'} \to \mathscr{O}_{X,x'}$ is bijective, because f is then a homeomorphism. Now, if $f(x') = y'$, $\mathscr{O}_{X,x'}$ is a local ring contained in $K = \kappa(x) = \kappa(y)$ and dominates $\mathscr{O}_{Y,y'}$; the later is the local ring $A_{y'}$, which a valuation ring for the fraction field K of A. But $\mathscr{O}_{X,x'} \neq K$ since x' is not the generic point of X, and we then conclude that $\mathscr{O}_{X,x'} = \mathscr{O}_{Y,y'}$ by maximality. As X is an integral scheme, the relation $\mathscr{O}_{X,x'} = \mathscr{O}_{X,x''}$ implies $x' = x''$ by Proposition 10.7.31, which proves our claim. □

Let A be a valuation ring, $Y = \operatorname{Spec}(A)$, b the generic point of Y, so that $\mathscr{O}_{Y,b} = \kappa(b)$ is equal to the fraction field K of A. Let $f : X \to Y$ be a morphism. We have seen that the rational Y-sections of X correspond to the germs of Y-sections (defined over a neighborhood of b) at b, whence a canonical map

$$\Gamma_{\mathrm{rat}}(X/Y) \to \Gamma(f^{-1}(b)/\operatorname{Spec}(K)) \tag{11.7.1}$$

where the elements of $\Gamma(f^{-1}(b)/\operatorname{Spec}(K))$ are identified with the rational points of $f^{-1}(b) = X \otimes_A K$ over K. If f is separted, it then follows from Corollary 10.5.21 that the map of (11.7.1) is injective, since Y is integral.

Composing (11.7.1) with the canonical map $\Gamma(X/Y) \to \Gamma_{\mathrm{rat}}(X/Y)$, we then obtain a canonical map

$$\Gamma(X/Y) \to \Gamma(f^{-1}(b)/\operatorname{Spec}(K)). \tag{11.7.2}$$

If f is separated, this map is injective by Corollary 10.5.21.

Proposition 11.7.8. *Let A be a valuation ring with fraction field K, $Y = \operatorname{Spec}(A)$, b be the generic point of Y, and $f : X \to Y$ be a separated and closed morphism. Then the canonical map (11.7.2) is bijective.*

Proof. Let x be a rational point of $f^{-1}(b)$ over K. As f is separated, so is the morphism $f^{-1}(b) \to \operatorname{Spec}(K)$ corresponding to f (Proposition 10.5.26(iv)), since and any section of $f^{-1}(b)$ a closed immersion by Corollary 10.5.19, $\{x\}$ is closed in $f^{-1}(b)$. Consider the reduced closed subscheme X' of X with underlying space $\overline{\{x\}}$ of $\{x\}$ in X. It is clear that the restriction of f to X' satisfies the conditions of Proposition 11.7.7 (note that since x is rational over K, we have $\kappa(x) = K$), hence an isomorphism from X' to Y, whose inverse isomorphism is the Y-section of X we want. □

Recall that if F is a subset of the scheme Y, the codimension of F in Y is equal to the infimum of $\dim(\mathscr{O}_{Y,z})$ where $z \in F$ (this can be easily verified after reducing to affine case), and we denote this number by $\operatorname{codim}_Y(F)$.

Corollary 11.7.9. *Let Y be a reduced locally Noetherian scheme such that the subset N of $y \in Y$ where Y is not regular has codimension ≥ 2. Let $f : X \to Y$ be a separted and closed morphism of finite type and g be a rational Y-section of X. If Y' is the set of points of Y where g is not defined, then* $\operatorname{codim}_Y(Y') \geq 2$.

Proof. It suffices to prove that g is defined at any point $y \in Y$ such that $\dim(\mathscr{O}_{Y,y}) \leq 1$. If $\dim(\mathscr{O}_{Y,y}) = 0$, then y is the generic point of an irreducible component of Y. For any open dense subset V of Y, by restricting to an affine neighborhood of y and apply **??**, we conclude that V contains y. In particular, y belongs to the defining domain of g. Suppose now that $\dim(\mathscr{O}_{Y,y}) = 1$; then $\mathscr{O}_{Y,y}$ is a regular local ring, hence a discrete valuation ring. Let $Z = \operatorname{Spec}(\mathscr{O}_{Y,y})$; as $U = Y - Y'$ is open and dense, by Corollary 10.2.12 and our preceding arguments, $U \cap Z$ is nonempty (contains the generic of an irreducible component of Y containing y), so we can consider the rational map $g' : Z \dashrightarrow X$ induced by g. It then suffices to prove that g' is a morphism (Proposition 10.7.17). Now, g' can be considered as a rational Z-section of the Z-scheme $f^{-1}(Z) = X \times_Y Z$; it is clear that the morphism $f^{-1}(Z) \to Z$ corresponding to f is closed, and is separted by Proposition 10.5.26(i). We then conclude from Proposition 11.7.8 that g' is everywhere defined, and as Z is reduced and X is separated over Y, g' is a morphism (Proposition 10.7.10). □

Corollary 11.7.10. *Let S be a locally Noetherian scheme, X, Y be S-scheme, and assume that X is proper over X. Suppose that Y is reduced and the subset N of $y \in Y$ where Y is not regular has codimension ≥ 2. Let $f : Y \dashrightarrow X$ be a rational map and Y' be the set of points where f is not defined, then* $\operatorname{codim}_Y(Y') \geq 2$.

Proof. The rational S-maps $Y \dashrightarrow X$ correspond to rational Y-sections of $X \times_S Y$; as the structural morphism $X \times_S Y \to Y$ is closed by Proposition 11.5.23, we can apply Corollary 11.7.9, whence the corollary. □

Remark 11.7.11. The hypothesis on Y in Corollary 11.7.9 and Corollary 11.7.10 are satisfied in particular if Y is normal (by Serre's criterion for normality).

Theorem 11.7.12 (Valuative Criterion of Properness). *Let Y be a (resp. locally Noetherian) scheme and $f : X \to Y$ be a quasi-compact and separted morphism (resp. a quasi-compact morphism of finite type). The following conditions are equivalent:*

(i) *f is universally closed (resp. proper)*

(ii) *For any Y-scheme $Y' = \operatorname{Spec}(A)$ where A is a valuation ring (resp. a discrete valuation ring) with fraction field K, the canonical map (where $X' = \operatorname{Spec}(K)$)*

$$\operatorname{Hom}_Y(Y', X) \to \operatorname{Hom}_Y(\operatorname{Spec}(K), X)$$

corresponding to the cannical injection $A \to K$, is surjective (resp. bijective).

(iii) *For any Y-scheme $Y' = \operatorname{Spec}(A)$ where A is a valuation ring (resp. a discrete valuation ring) with fraction field K, the canonical map of (11.7.2) relative to the Y'-scheme $X' = X_{(Y')}$ is surjective (resp. bijective).*

Proof. The equivalence of (ii) and (iii) follows from the correspondence of Y-morphisms $Y' \to X$ and Y'-morphisms $Y' \to X'$. If f is universally closed then $f_{(Y')}$ is closed and separated, and it then suffices to apply Proposition 11.7.8. It remains to prove that (ii) implies (i). Consider first the case where Y is arbitrary, f is separated and quasi-compact. If the condition of (ii) is satisfied for f, it is also true for $f_{(Y'')} : X_{(Y'')} \to Y''$, where Y'' is an arbitrary Y-scheme, in view of the equivalent of (ii) and (iii), and the fact that $X_{(Y'')} \times_{Y''} Y' = X \times_Y Y'$ for any morphism $Y' \to Y''$; as $f_{(Y'')}$ is also quasi-compact and separated, we then conclude that we only need to prove (ii) implies f is closed, and for this we shall use Proposition 10.6.6. Let $x \in X$, $y' \neq y$ be a specialization of $y = f(x)$; in view of Proposition 11.7.1, there is a scheme $Y' = \operatorname{Spec}(A)$ where A is a valuation ring, and a separated morphism $g : Y' \to Y$ such that, if a is the closed point and b is the generic point of Y, we have $g(a) = y'$, $g(b) = y$, and there exists a $\kappa(y)$-hommorphism $\kappa(x) \to \kappa(b)$. This homomorphism corresponds to a canonical Y-morphism $\operatorname{Spec}(\kappa(b)) \to X$ (Corollary 10.2.17), and it then follows from condition (ii) that there exists a Y-morphism $h : Y' \to X$ which corresponds the previous morphism such that the following diagram commutes

$$\begin{tikzcd} \operatorname{Spec}(\kappa(b)) \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \operatorname{Spec}(A) \arrow[r, "g"] & Y \end{tikzcd}$$

We then have $h(b) = x$, and if we put $h(a) = x'$, x' is then a specialization of x, and we have $f(x') = f(h(a)) = g(a) = y'$.

If now Y is locally Noetherian and f is a quasi-compact morphism of finite type, then condition (ii) implies that f is separated (Proposition 11.7.5), so the diagonal morphism $\Delta : X \to X \times_Y X$ is quasi-compact. Moreover, to verify that f is proper, it suffices to show that $f_{(Y'')} : X_{(Y'')} \to Y''$ is closed for any Y-scheme Y'' of finite type (Corollary 11.5.41). As then Y'' is locally Noetherian, we can resume the reasoning given in the first case by taking for Y' the specturm of discrete valuation ring, and applying Proposition 11.7.1 instead of Proposition 11.7.4. □

Remark 11.7.13. We deduce from the criterion (iii) of Theorem 11.7.12 a new proof of the fact that a projective morphism $X \to Y$ is closed, which is closer to classical methods. We can in fact assume that Y is affine, and X is therefore a closed subscheme of a projective bundle $\mathbb{P}^n_Y$ (Corollary 11.5.19). To show that $X \to Y$ is closed, it suffices to verify that the structural morphism $\mathbb{P}^n_Y \to Y$ is closed. The criterion (iii) of Theorem 11.7.12, together with (11.4.1), show that we are reduced to proving the following fact: if Y is the spectrum of a valuation ring A with fraction field K, every point of $\mathbb{P}^n_Y$ with values in K comes (by restriction to the generic point of Y) from a point of $\mathbb{P}^n_Y$ with values in A. Now, any invertible $\mathscr{O}_Y$-module is trivial, so it follows from Example 11.4.8 that a point of $\mathbb{P}^n_Y$ with values in K is identified with a class of elements $(\xi c_0, \xi c_1, \dots, \xi c_n)$ of K^{n+1}, where $\xi \neq 0$ and the c_i are elements of K which generate the unit ideal of K. By multiplying the c_i with an element of A, we can suppose that the c_i belong to A, and generate the unit ideal of A. But then (Example 11.4.8) the system $(c_0, \dots, c_n)$ defines a point of $\mathbb{P}^n_Y$ with values in A, whence our assertion.

Remark 11.7.14. The criteria Proposition 11.7.5 and Theorem 11.7.12 are especially convenient when we consider a Y-scheme X as a functor

$$X(Y') = \operatorname{Hom}_Y(Y', X)$$

where Y' is a Y-scheme. These criteria will allow us, for example, to prove that under certain conditions the "Picard schemas" are proper.

Corollary 11.7.15. *Let Y be a separated integral scheme (resp. a separated integral locally Noetherian scheme) and $f : X \to Y$ be a dominant morphism.*

(a) *If f is quasi-compact and universally closed, any valuation ring with fraction field the rational function field $K(X)$ and which dominates a local ring of Y, also dominates a local ring of X.*

(b) *Conversely, suppose that f is of finite type, and the property of (a) is satisfied for any valuation ring (resp. any discrete valuation ring) with fraction field $K(X)$. Then f is proper.*

Proof. Assume the hypotheses of (a) and let $K = K(Y)$, $L = K(X)$, y be a point of Y, A be a valuation ring with L the fraction field and dominate $\mathscr{O}_{Y,y}$. The injection $\mathscr{O}_{Y,y} \to A$ is local, so it defines a morphism

$$h : Y' = \operatorname{Spec}(A) \to \operatorname{Spec}(\mathscr{O}_{Y,y}) \to Y$$

(Proposition 10.2.14) such that $h(a) = y$, where a is the closed point of Y'. Moreover, since $K \subseteq L$, the morphism $\operatorname{Spec}(A) \to \operatorname{Spec}(\mathscr{O}_{Y,y})$ is dominant, so if η is the generic point of Y (which is also that of $\operatorname{Spec}(\mathscr{O}_{Y,y})$), we have $h(b) = \eta$, where b is the generic point of Y'. If ξ is the generic point of X, we have $\kappa(\xi) = \kappa(b) = L$ by hypothesis, so there is a Y-morphism $g : \operatorname{Spec}(L) \to X$ such that $g(b) = \xi$. In view of Theorem 11.7.12, we obtain a Y-morphism $g' : Y' \to X$ such that $g(b) = \xi$. If we set $x = g'(a)$, then A dominates $\mathscr{O}_{X,x}$.

We now prove (b); since the question is local over Y, we can assume that Y is affine (resp. affine and Noetherian). As f is of finite type, we can apply Chow's lemma, so there exists a projective morphism $p : P \to Y$, an immersion $j : X' \to P$, and a projective and surjective birational morphism $g : X' \to X$ (where X' is integral) such that the diagram

$$\begin{tikzcd}
X' \arrow[r, "j"] \arrow[d, "g"'] & P \arrow[d, "\pi"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

is commutative. It suffices to prove that j is a closed immersion, because then $f \circ g = p \circ j$ is projective, hence proper, and as g is surjective we conclude that f is proper by Corollary 11.5.24. Let Z be a reduced closed subscheme of P with underlying space $\overline{j(X')}$; as X' is integral, j factors into

$$j : X' \xrightarrow{h} Z \xrightarrow{i} P$$

where $i : Z \to P$ is the canonical injection and $h : X' \to Z$ is a dominant open immersion. Since Z is integral and projective over Y by Proposition 11.5.34, we are then reduced to the case where P is integral, j is dominant and birational, and prove that j is surjective. Now let $z \in P$; $\mathscr{O}_{Z,z}$ is an integral local ring (resp. Noetherian integral) whose fraction field is

$$L = K(P) = K(X') = K(X).$$

We can assume that z is not the generic point of P (since the later is contained in $j(Z)$ as j is dominant), so $\mathscr{O}_{Z,z} \neq L$ and by Theorem 5.1.7 and Proposition 11.7.2, there exists a valuation ring (resp. a discrete valuation ring) A with fraction field L that dominates $\mathscr{O}_{Z,z}$. A fortiori A dominates the local ring $\mathscr{O}_{Y,y}$ where $y = p(z)$, and by hypotheses there is then a point $x \in X$ such that A dominates $\mathscr{O}_{X,x}$. As the morphism g is proper, it satisfies the conditions of (a), so our previous arguments then prove that A also dominates $\mathscr{O}_{X,x'}$, for some $x' \in X'$. Then the local rings $\mathscr{O}_{Z,z}$ and $\mathscr{O}_{Z,j(x')} = \mathscr{O}_{X,x'}$ are related, and by Proposition 10.7.31, as P is separated, we have $z = j(x')$, which completes the proof. $\square$

Corollary 11.7.16. *Let A be an integral domain, $Y = \mathrm{Spec}(A)$, and $f : X \to Y$ be a dominant morphism of integral schemes which is quasi-compact and universally closed. Then $\Gamma(X, \mathscr{O}_X)$ is canonically isomorphic to a subring of the integral closure of A in $K(X)$.*

Proof. Recall that by (10.7.1), $B = \Gamma(X, \mathscr{O}_X)$ is indeitified with the intersection of $\mathscr{O}_{X,x}$ for $x \in X$. If R is a valuation ring of $K(X)$ containing A, then it dominates the local ring $A_{\mathfrak{P}}$ where $\mathfrak{P} = \mathfrak{m}_R \cap B$, and therefore by Corollary 11.7.15 dominates a local ring of X. Then B is contained in R, and the conclusion follows from Theorem 5.1.8. □

Remark 11.7.17. Under the hypothesis of Corollary 11.7.16, if we suppose that $K(X)$ is a finite extension of $K(Y)$, then we can in many cases conclude that $\Gamma(X, \mathscr{O}_X)$ is a finitely generated module over the ring $B = \Gamma(Y, \mathscr{O}_X)$. This is the case for example if B is a Japanese ring. In particular, if $X = \mathrm{Spec}(A)$ and $Y = \mathrm{Spec}(k)$ where k is a algebraically closed field, then the the corresponding homomorphism $k \to A$ is injective by Corollary 1.4.21 and since the integral closure of k in $K(X)$ is equal to k, we conclude that $\Gamma(X, \mathscr{O}_X) = k$.

11.7.4 Algebraic curves

Let k be a field. In this paragraph, all schemes are considered to be separated k-schemes of finite type, and any morphism are k-morphism.

Proposition 11.7.18. *Let X be a scheme of finite type over k; let x_i ($1 \le i \le n$) be the generic points of the irreducible components X_i of X, and $K_i = \kappa(x_i)$. Then the following conditions are equivalent:*

(i) *For each i, the transcendence degre of K_i over k is equal to* 1.

(ii) *For any closed point x of X, the local ring $\mathscr{O}_{X,x}$ is of dimension* 1.

(iii) *The closed irreducible subsets of X distinct from the X_i are closed points of X.*

Proof. As X is quasi-compact, any irreducible closed subset F of X contains a closed point ([**?**] 0_{I}, 2.1.3). Let x be a closed point of X; in view of Corollary 10.2.12, there is a correspondence between prime ideals of $\mathscr{O}_{X,x}$ and the irreducible closed subsets of X containing x. The equivalence of (ii) and (iii) then follows. On the other hand, if $\mathfrak{p}_\alpha$ ($1 \le \alpha \le r$) is the minimal prime ideals of the local Noetherian ring $\mathscr{O}_{X,x}$, the local ring $\mathscr{O}_{X,x}/\mathfrak{p}_\alpha$ are integral, whose fraction fields are the K_i such that $x \in X_i$. Moreover, the dimension of a k-algebra of finite type is equal to its transcendental degree over k. Finally, the dimension of $\mathscr{O}_{X,x}$ is the supremum of the $\mathscr{O}_{X,x}/\mathfrak{p}_\alpha$, and a k-algebra of finite type is catenary, so (i) and (ii) are equivalent. □

We note that under the conditions of Proposition 11.7.18, by Proposition 10.6.44 that set X is either empty of finite. We define an algebraic curve over k to be a nonempty scheme X over k satisfying the conditions of Proposition 11.7.18. Equivalently, we will see that this condition is equivalent to that the irreducible components of X has dimension 1. In particular, we note that if X is an algebraic curve over k, the reduced closed subschemes X_i of X with underlying spaces the irreducible components of X are also algebraic curves over k.

Corollary 11.7.19. *Let X be an irreducible algebraic curve. Then the only non-closed point of X is its generic point, the proper closed subsets of X are the finite subsets of X, which are also the non dense subsets of X.*

Proof. If a point $x \in X$ is not closed, then its closure is an irreducible closed subset of X, hence equal to X by Proposition 11.7.18, so x is the generic point of X. A proper closed subset F of X can not contain the generic point of X, so its points are all closed, hence T1, and by Proposition 10.2.33 we conclude that F is finite and discrete. The closure of an infinite subset of X is therefore necessarily equal to X, which proves the last assertion. □

If X is an arbitrary algebraic curve, then by applying Corollary 11.7.19, we conclude that the only non-closed points of X are the generic points of the irreducible components of X.

Corollary 11.7.20. *Let X and Y be irreducible algebraic curves over k and $f : X \to Y$ be a k-morphism. Then for f to be dominant, it is necessary and sufficient that $f^{-1}(y)$ is finite for any $y \in Y$.*

Proof. If f is not dominant, $f(X)$ is necessarily a finite subset of Y by Corollary 11.7.19, so it is not possible that $f^{-1}(y)$ is finite for any $y \in Y$ (since X is an infinite set). Conversely, if f is dominant, for any $y \in Y$ which is not the generic point η of Y, $f^{-1}(y)$ is closed in X since $\{y\}$ is closed in Y (Corollary 11.7.19); on the other hand, by hypotheses, $f^{-1}(y)$ does not contain the generic point of X, so is finite by Corollary 11.7.19. Finally, to see that $f^{-1}(\eta)$ is finite, we note that the morphism f is of finite type by Corollary 10.6.37, so the fiber $f^{-1}(\eta)$ is an irreducible scheme of finite type over $\kappa(\eta)$ with generic point ξ (Proposition 10.6.35). As $\kappa(\xi)$ and $\kappa(\eta)$ are extensions of k of finite type with transcendental degree 1, it follows that $\kappa(\xi)$ is a finite extension of $\kappa(\eta)$, so ξ is closed in $f^{-1}(\eta)$ by Corollary 11.6.5, and $f^{-1}(\eta)$ is therefore reduced to a point ξ. □

Remark 11.7.21. We will see later that a proper morphism $f : X \to Y$ of Noetherian schemes, such that $f^{-1}(y)$ is finite for any $y \in Y$, is necessarily finite. It then follows from Corollary 11.7.19 that such a dominant proper morphism of irreducible algebraic curves is finite.

Corollary 11.7.22. *Let X be an algebraic curve over k. For X to be regular, it is necessary and sufficient that X is normal, or the local ring of its closed points are discrete valuation rings.*

Proof. This comes from conditions (ii) of Proposition 11.7.18. □

Corollary 11.7.23. *Let X be a reduced algebraic curve, $\mathscr{A}$ be a reduced coherent $\mathscr{K}_X$-algebra. Then the integral closure X' of X relative to $\mathscr{A}$ is a normal algebraic curve, and the canonical morphism $X' \to X$ is finite.*

Proof. The fact that $X' \to X$ is finite follows from Remark 11.6.25, and X' is then a normal algebraic scheme over k. Moreover, we note that if X is irreducible with generic point ξ and its integral closure X' has generic point ξ', then $\kappa(\xi') = \kappa(\xi)$ by Corollary 11.6.24, so X' is also an algebraic curve over k. □

Corollary 11.7.24. *For a reduced algebraic curve X to be proper over k (which is called **complete**), it is necessary and sufficient that the normalization X' of X is proper over k.*

Proof. The canonical morphism $f : X' \to X$ is finite by Corollary 11.7.23, hence proper (Corollary 11.6.9) and surjective (Corollary 11.6.24). If $g : X \to \mathrm{Spec}(k)$ is the structural morphism, g and $g \circ f$ are then simultaneously proper, in view of Proposition 11.5.23 and Corollary 11.5.24. □

Proposition 11.7.25. *Let X be a normal algebraic curve over k and Y be a proper algebraic scheme over k. Then any rational k-map $f : X \dashrightarrow Y$ is everywhere defined, hence a morphism.*

Proof. It follows from Corollary 11.7.10 that the set of points $x \in X$ where this rational map is not defined, the dimnesion of $\mathscr{O}_{X,x}$ is ≥ 2, hence is empty. The assertion then follows from Proposition 10.7.10. □

Corollary 11.7.26. *A normal algebraic curve over k is quasi-projective over k.*

Proof. As X is the sum of finitely many integral and normal algerbaic curves (Corollary 11.6.24), we can assume that X is integral (Corollary 11.5.22). As X is quasi-compact, it can be covered by finitely many affine opens U_i ($1 \leq i \leq n$), and as each of this is of finite type over k, there exists an integer n_i and a k-immersion $f_i : U_i \to \mathbb{P}_k^{n_i}$ (Corollary 11.5.19 and Proposition 11.5.20(i)). As U_i is dense in X (recall that X is integral by our assumption), it follows from Proposition 11.7.25 that f_i extends to a k-morphism $g_i : X \to \mathbb{P}_k^{n_i}$, and we obtain a k-morphism $g = (g_1, \dots, g_n)_k$ from X into the product P of the $\mathbb{P}_k^{n_i}$ over k. Moreover, for each index i, as the restriction of g_i to U_i is an immersion, so is the restriction of g to U_i (Corollary 10.5.16). As the U_i cover X and g is separated by Proposition 10.5.26(v), g is an immersion from X into P by Proposition 10.7.36. The Segre morphism then provides from g an immersion of X into a projective bundle $\mathbb{P}_k^n$, so X is quasi-projective. □

Corollary 11.7.27. *A normal algebraic curve X is isomorphic to a dense open subscheme of a normal and complete algebraic curve $\widehat{X}$, determined up to isomorphisms.*

Proof. If X_1, X_2 are two normal and complete algebraic curves containing X as open dense subscheme, it follows from Proposition 11.7.25 there is an isomorphism of X_1 and X_2, whence the uniqueness of $\widehat{X}$. To prove the existence of $\widehat{X}$, it suffices to remark that we can consider X as a subscheme of a projective bundle $\mathbb{P}_k^n$ (Corollary 11.7.26). Let $\bar{X}$ be the scheme-theoretic closure of X in $\mathbb{P}_k^n$ (Proposition 10.6.71); as X is an open and dense subscheme of $\bar{X}$, the generic points x_i of the irreducible components of X are those of $\bar{X}$, and the residue fields $\kappa(x_i)$ are the same for both schemes, so $\bar{X}$ is an algebraic curve over k, which is reduced (Proposition 10.6.69) and projective over k (Proposition 11.2.52), hence complete by Theorem 11.5.30. We then take $\widehat{X}$ to be the noramlization of $\bar{X}$, which is complete by Corollary 11.7.24. If $h : \widehat{X} \to \bar{X}$ is the canonical morphism, the restriction of h to $h^{-1}(X)$ is an isomorphism since X is normal (Proposition 11.6.22), and as $h^{-1}(X)$ contains the generic point of the irreducible components of $\widehat{X}$ (Corollary 11.6.24), it is therefore dense in $\widehat{X}$, which proves the assertion. □

Remark 11.7.28. We will later see that the conclusion of Corollary 11.7.27 is still valid without assuming the algebraic curve to be normal (or even reduced); we will also see that for an algebraic curve (reduced or not) to be affine, it is necessary and sufficient that its irreducible (reduced) components are not complete.

Corollary 11.7.29. *Let X be an irreducible normal algebraic curve with $L = K(X)$, Y be a complete and integral algebraic curve with $K = K(Y)$. Then there exists a canonical correspondence between dominant k-morphisms $X \to Y$ and k-monomorphisms $K \to L$.*

Proof. By Proposition 11.7.25, the rational k-maps $X \dashleftarrow Y$ are identified with k-morphisms $f : X \to Y$. The morphism f is dominante if and only if $f(x) = y$, where x and y are the generic points of X and Y, respectively. The corollary then follows from Corollary 10.7.6. □

Example 11.7.30. We can precise the result of Corollary 11.7.29 if Y is the projective line $\mathbb{P}_k^1 = \mathrm{Proj}(k[T_0, T_1])$, where T_0 and T_1 are indeterminates. This is an integral and separated k-scheme, and the induced subscheme $D_+(T_0)$ of Y is isomorphic to $\mathrm{Spec}(k[T])$, so the generic point of Y is the ideal (0) of $k[T]$ and its rational function field is $k(T)$, which shows that Y is a complete algebraic cuver over k. Moreover, the only graded ideal of $S = k[T_0, T_1]$ containing T_0 and distinct from S_+ is the principal ideal (T_0), so the complement of $D_+(T_0)$ in Y is reduced to a closed point, called the "infinite point" and denoted by ∞.

Corollary 11.7.31. *Let X be an irreducible normal algebraic curve with $K = K(X)$. Then there exists a canonical correspondence between K and the set of morphisms $u : X \to \mathbb{P}_k^1$ which is distinct from the constant morphism with value ∞. For u to be dominant, it is necessary and sufficient that the corresponding element in K is transcendental over k.*

Proof. By Corollary 10.7.6 and Example 11.7.30, the rational maps $X \dashrightarrow \mathbb{P}_k^1$ (hence morphisms $X \to \mathbb{P}_k^1$, in view of Proposition 11.7.25) correspond to points of $\mathbb{P}_k^1$ with values in K. For any element $\xi \in K$, we have a induced homomorphism $k[T_0] \to K$ which maps T_0 to ξ, and therefore a morphism $\mathrm{Spec}(K) \to D_+(T_0)$. By composing with the canonical injection $D_+(T_0) \to \mathbb{P}_k^1$, we obtain a point of $\mathbb{P}^1_k$ with values in K which is not located at ∞, which is completely determined by ξ in view of Proposition 10.2.4. On the other hand, by Corollary 10.2.17, any constant morphism $u : \mathrm{Spec}(K) \to \mathbb{P}^1_k$ with value y factors into

$$u : \mathrm{Spec}(K) \to \mathrm{Spec}(\kappa(y)) \to \mathbb{P}^1_k.$$

Since $y \in D_+(T_0)$, the morphism $\mathrm{Spec}(\kappa(y)) \to \mathbb{P}^1_k$ factors through the canonical injection $D_+(T_0) \to \mathbb{P}^1_k$, and u therefore is obtained by morphism $\mathrm{Spec}(K) \to D_+(T_0)$, which corresponds to an element $\xi \in K$; this proves the first part of the corollary. For the morphism u to be dominant, it is necessary that the morphism $\mathrm{Spec}(K) \to \mathrm{Spec}(k[T_0]) \cong D_+(T_0)$ is dominant, which means the homomorphism $k[T] \to K$ is injective (Corollary 1.4.21), and this is true if and only if ξ is transcendental. □

Remark 11.7.32. With the notations of Corollary 11.7.31, we now determine the image of the morphism $\mathrm{Spec}(K) \to \mathbb{P}^1_k$ induced by an algerbaic element $\xi \in K$ over k. By definition, if $y \in \mathbb{P}^1_k$ is this image, the morphism factors into

$$\mathrm{Spec}(K) \to \mathrm{Spec}(\kappa(y)) \to \mathrm{Spec}(k[T]) \to \mathbb{P}^1_k$$

where we write $\mathrm{Spec}(k[T])$ for $D_+(T_0)$. We first note that any prime ideal in $k[T]$ is maximal, so $\kappa(y) = k[T]/\mathfrak{p}_y$, and the morphism $\mathrm{Spec}(K) \to \mathrm{Spec}(\kappa(y))$ is identified with the canonical injection $k[T]/\mathfrak{p}_y \to K$. By definition, this homomorphism is induced by the homomorphism $k[T] \to K, T \mapsto \xi$, so if $f(T)$ is the irreducible polynomial of ξ over k, we have $\mathfrak{p}_y = (f)$, and y is therefore the point of $\mathrm{Spec}(k[T])$ corresponding to (f). In particular, if $\xi \in k$, then $\mathfrak{p}_y = (T - \xi)$, and if k is an algebraically closed field, we conclude that any element $\xi \in k$ corresponds to a morphism $\mathrm{Spec}(K) \to \mathbb{P}^1_k$ with image ξ (identified with its corresponding maximal ideal $(T - \xi)$), and any element $\xi \in K - k$ corresponds to a dominant morphism $\mathrm{Spec}(K) \to \mathbb{P}^1_k$.

Corollary 11.7.33. *Let X and Y be normal, complete and irreducible algebraic curves, with $K = K(Y)$, $L = K(X)$. Then there exist a bijective correspondense between the set of k-isomorphisms $X \xrightarrow{\sim} Y$ and the set of k-isomorphisms $K \xrightarrow{\sim} L$.*

Corollary 11.7.33 shows that a normal, complete and irreducible algerbaic curves over k is determined by its rational function field K up to an isomorphism. By definition, K is a field of finite type over k with transcendental degree 1 (which is called a algerbaic function field of one variable).

Proposition 11.7.34. *For any extension K of k of finite type and transcendental degree 1, there exists a normal, complete and irreducible algebraic curve X such that $K(X) = K$. The set of local rings of X is identified with the set formed by K and the valuation rings containing k with fraction field K.*

Proof. In fact, K is a finite extension of a purely transcendental extension $k(T)$ of k, which is identified with the rational function field of $Y = \mathbb{P}^1_k$. Let X be the inegral closure of Y relative to K; X is then a normal algebraic curve with field K (Proposition 11.6.23), and it is complete since the morphism $X \to Y$ is finite (Corollary 11.7.23). For $x \in X$, the local ring $\mathscr{O}_{X,x}$ is equals to K if x is the generic of X, and otherwise it is a discrete valuation ring of K. Conversely, let A be a discrete valuation ring with fraction field K; as the morphism $X \to \mathrm{Spec}(k)$ is proper and A dominates k, it also dominates a local ring $\mathscr{O}_{X,x}$ of X by Corollary 11.7.15, and therefore equals to $\mathscr{O}_{X,x}$. □

Remark 11.7.35. It follows from Proposition 11.7.34 and Corollary 11.7.33 that giving a normal, complete and irreducible algebraic curve over k is essentially equivalent to giving a extension K of k of finite type and transcendental degree 1. We note that if k' is an extension of k, $X \otimes_k k'$

is also a complete algebraic curve over k' (Proposition 11.5.23(iii)), but in general it is neither reduced nor irreducible. However, this will be the case if K is a separable extension of k and if k is algebraically closed in K (which is expressed, in a classical terminology, that K is a "regular extension" of k). But even in this case, it may happen that $X \otimes_k k'$ is not normal.

11.8 Blow up of schemes, projective cones and closures

11.8.1 Blow up of schemes

Let Y be a scheme and $(\mathscr{I}_n)_{n\geq 0}$ be a decreasing sequence of quasi-coherent ideal of $\mathscr{O}_Y$. Suppose that the following conditions are satisfied:

$$\mathscr{I}_0 = \mathscr{O}_Y, \quad \mathscr{I}_n\mathscr{I}_n \subseteq \mathscr{I}_{m+n}$$

where m, n are integers. If this is true, we say that the sequence $(\mathscr{I}_n)_{n\geq 0}$ is **filtered**, or that $(\mathscr{I}_n)_{n\geq 0}$ is a **filtered sequence of quasi-coherent ideals of** $\mathscr{O}_Y$. We note that this hypothesis implies that $\mathscr{I}_1^n \subseteq \mathscr{I}_n$. Put

$$\mathscr{S} = \bigoplus_{n\geq 0} \mathscr{I}_n.$$

It follows the assumption that $\mathscr{S}$ is a quasi-coherent $\mathscr{O}_Y$-algebra, hence defines a Y-scheme $X = \mathrm{Proj}(\mathscr{S})$. If $\mathscr{J}$ is an invertible ideal of $\mathscr{O}_Y$, $\mathscr{I}_n \otimes_{\mathscr{O}_Y} \mathscr{J}^{\otimes n}$ is canonically idealtified with $\mathscr{I}_n\mathscr{J}^n$, and if we replace $\mathscr{I}_n$ by $\mathscr{I}_n\mathscr{J}^n$, then the obtained $\mathscr{O}_Y$-algebra $\mathscr{S}_{(\mathscr{J})}$ satisfies that $X_{(\mathscr{J})} = \mathrm{Proj}(\mathscr{S}_{(\mathscr{J})})$ is canonically isomorphic to X (Proposition 11.3.6).

Suppose that Y is locally integral, so that $\mathscr{K}_Y$ is a quasi-coherent $\mathscr{O}_Y$-algebra (Proposition 10.7.22). We say a sub-$\mathscr{O}_Y$-module $\mathscr{I}$ of $\mathscr{K}_Y$ is a **fractional ideal** of $\mathscr{K}_Y$ if it is of finite type. Given a filtered sequence $(\mathscr{I}_n)_{n\geq 0}$ of quasi-coherent fractional ideal of $\mathscr{K}_Y$, we can then define the quasi-coherent graded $\mathscr{O}_Y$-algebra $\mathscr{S}$ and the corresponding Y-scheme $X = \mathrm{Proj}(\mathscr{S})$. We then see that for an invertible fractional ideal $\mathscr{J}$ of $\mathscr{K}_Y$, there is a canonical isomorphism of X and $X_{(\mathscr{J})}$.

Let Y be a scheme (resp. a locally integral scheme), and $\mathscr{I}$ be a quasi-coherent ideal of $\mathscr{O}_Y$ (resp. a quasi-coherent fractional ideal of $\mathscr{K}_Y$); put $\mathscr{S} = \bigoplus_{n\geq 0}\mathscr{I}^n$. The Y-scheme $X = \mathrm{Proj}(\mathscr{S})$ is said to be the scheme obtained by **blowing up along the ideal** $\mathscr{I}$, or the **blow up** of Y relative to $\mathscr{I}$. If $\mathscr{I}$ is a quasi-coherent ideal of $\mathscr{O}_Y$ and Y' is the closed subscheme defined by $\mathscr{I}$, we also say that X is the Y-scheme obtained by blowing up Y'. By definition, $\mathscr{S}$ is generated by $\mathscr{S}_1 = \mathscr{I}$; if $\mathscr{I}$ is a $\mathscr{O}_Y$-module of finite type, X is then projective over Y. By the hypotheses on $\mathscr{I}$, the $\mathscr{O}_X$-module $\mathscr{O}_X(1)$ is invertible (Proposition 11.3.14) and very ample in view of Corollary 11.4.15 for the structural morphism $j : X \to Y$. We also note that the restriction of f to $f^{-1}(Y - Y')$ is an isomorphism if $\mathscr{I}$ is the quasi-coherent ideal of $\mathscr{O}_Y$ defining Y': in fact, this question is local over Y, so it suffice to suppose $\mathscr{I} = \mathscr{O}_Y$, and this then follows from Remark 11.3.18.

If we replace $\mathscr{I}$ by $\mathscr{I}^d$ for some $d > 0$, the blow up Y-scheme X is then replaced by a Y-scheme canonically isomorphic to X' (Proposition 11.3.6). Simialrly, for any invertible ideal (resp. fractional ideal) $\mathscr{J}$, the blow up scheme $X_{(\mathscr{J})}$ relative to $\mathscr{I}\mathscr{J}$ is canonically isomorphic to X. In particular, if $\mathscr{I}$ is an invertible ideal (resp. fractional ideal), the blow up Y-scheme relative to $\mathscr{I}$ is isomorphic to Y.

Proposition 11.8.1. *Let Y be an integral scheme.*

(i) *For any filtered sequence $(\mathscr{I}_n)$ of quasi-coherent fractional ideals of $\mathscr{K}_Y$, the Y-scheme $X = \mathrm{Proj}(\bigoplus_{n\geq 0}\mathscr{I}_n)$ is integral and the structural morphism $f : X \to Y$ is dominant.*

(ii) *Let $\mathscr{I}$ be a quasi-coherent fractional ideal of $\mathscr{K}_Y$ and X be the blow up Y-scheme relative to $\mathscr{I}$. If $\mathscr{I} \neq 0$, the structural morphism $f : X \to Y$ is surjective and birational.*

Proof. In case (i) the quasi-coherent graded $\mathscr{O}_Y$-algebra $\mathscr{S} = \bigoplus_{n\geq 0}\mathscr{I}_n$ is integral since for any $y \in Y$, $\mathscr{O}_{Y,y}$ is an integral domain, so the claim follows from Proposition 11.3.10. For (ii), we obtain from (i) that X is integral; if morever x and y are generic points of X and Y, we have $f(x) = y$, and it is necessary to prove that $\kappa(x) = \kappa(y)$. Now x is also the generic point of the fiber $f^{-1}(y)$; if $\psi : Z \to Y$ is the canonical morphism, where $Z = \mathrm{Spec}(\kappa(y))$, then the scheme $f^{-1}(y)$ is identified with $\mathrm{Proj}(\mathscr{S}')$, where $\mathscr{S}' = \psi^*(\mathscr{S})$ (Proposition 11.3.31). But it is clear that $\mathscr{S}' = \bigoplus_{n\geq 0}(\widetilde{\mathscr{I}_y})^n$, and as $\mathscr{I}$ is a nonzero quasi-coherent fractional ideal of $\mathscr{K}_Y$, $\mathscr{I}_y \neq 0$ (Corollary 10.7.21), so $\mathscr{I}_y = \kappa(y)$ (since y is the generic point of Y, $\mathscr{I}_y$ is a $\kappa(y)$-vector space). The scheme $\mathrm{Proj}(\mathscr{S}')$ is then identified with $\mathrm{Spec}(\kappa(y))$ (Remark 11.3.18), whence the assertion. □

Retain the notations of Proposition 11.8.1. By definition, the injection $\mathscr{I}_{n+1} \to \mathscr{I}_n$ defines for each $k \in \mathbb{Z}$ a injective homomorphism of degree 0 of graded $\mathscr{S}$-modules

$$u_k : \mathscr{S}_+(k+1) \to \mathscr{S}(k). \tag{11.8.1}$$

As $\mathscr{S}_+(k+1)$ and $\mathscr{S}(k+1)$ are eventually isomorphic $\mathscr{S}$-modules, the hommorphism u_k corresponds to a canonical injective homomorphism of $\mathscr{O}_X$-modules (Proposition 11.3.24):

$$\tilde{u}_k : \mathscr{O}_X(k+1) \to \mathscr{O}_X(k). \tag{11.8.2}$$

Recall on the other hand that we have defined a canonical homomorphism

$$\lambda : \mathscr{O}_X(d) \otimes_{\mathscr{O}_X} \mathscr{O}_X(k) \to \mathscr{O}_X(d+k)$$

and as the diagram

$$\begin{tikzcd}
\mathscr{S}(d)\otimes_{\mathscr{S}}\mathscr{S}(k)\otimes_{\mathscr{S}}\mathscr{S}(l) \arrow[r] \arrow[d] & \mathscr{S}(d+k)\otimes_{\mathscr{S}}\mathscr{S}(l) \arrow[d] \\
\mathscr{S}(d)\otimes_{\mathscr{S}}\mathscr{S}(k+l) \arrow[r] & \mathscr{S}(d+k+l)
\end{tikzcd}$$

is commutative, it follows from the functoriality of λ that the homomorphism λ define a quasi-coherent graded $\mathscr{O}_X$-algebra structure on $\mathscr{S}_X = \bigoplus_{n\in\mathbb{Z}}\mathscr{O}_X(n)$. Moreover, the diagram

$$\begin{tikzcd}
\mathscr{S}(d)\otimes_{\mathscr{S}}\mathscr{S}_+(k+1) \arrow[r] \arrow[d, "1\otimes u_k"'] & \mathscr{S}_+(d+k+1) \arrow[d, "u_{k+d}"] \\
\mathscr{S}(d)\otimes_{\mathscr{S}}\mathscr{S}(k) \arrow[r] & \mathscr{S}(d+k)
\end{tikzcd}$$

is commutative; the functoriality of λ shows that we have a commutative diagram

$$\begin{tikzcd}
\mathscr{O}_X(d)\otimes_{\mathscr{O}_X}\mathscr{O}_X(k+1) \arrow[r, "\lambda"] \arrow[d, "1\otimes\tilde{u}_k"'] & \mathscr{O}_X(d+k+1) \arrow[d, "\tilde{u}_{d+h}"] \\
\mathscr{O}_X(d)\otimes_{\mathscr{O}_X}\mathscr{O}_X(k) \arrow[r, "\lambda"] & \mathscr{O}_X(d+k)
\end{tikzcd} \tag{11.8.3}$$

where the horizontal arrows are the canonical homomorphisms. We can then say that the $\tilde{u}_k$ define an injective homomorphism (of degree 0) of graded $\mathscr{S}_X$-modules

$$\tilde{u} : \mathscr{S}_X(1) \to \mathscr{S}_X \tag{11.8.4}$$

Now we consider for each $n \geq 0$ the homomorphism $\tilde{v}_n = \tilde{u}_{n-1} \circ \cdots \tilde{u}_0$, which is an injective homomorphism $\mathscr{O}_X(n) \to \mathscr{O}_X$; we denote its image by $\mathscr{I}_{n,X}$, which is a quasi-coherent ideal $\mathscr{O}_X$ isomorphic to $\mathscr{O}_X(n)$. Moreover, the diagram

$$\begin{tikzcd} \mathscr{O}_X(m) \otimes_{\mathscr{O}_X} \mathscr{O}_X(n) \arrow[r, "\lambda"] \arrow[d, "\tilde{v}_m \otimes \tilde{v}_n"'] & \mathscr{O}_X(m+n) \arrow[d, "\tilde{v}_{m+n}"] \\ \mathscr{O}_X \arrow[r, "\mathrm{id}"] & \mathscr{O}_X \end{tikzcd}$$

is commutative for $m, n \geq 0$. We also conclude that $(\mathscr{I}_{n,X})_{n \geq 0}$ is a filtered sequence of quasi-coherent ideals of $\mathscr{O}_X$.

Proposition 11.8.2. *Let Y be a scheme, $\mathscr{I}$ be a quasi-coherent ideal of $\mathscr{O}_Y$, and X be the blow up Y-scheme relative to $\mathscr{I}$. Then for each $n > 0$ we have a canonical isomorphism*

$$\mathscr{O}_X(n) \xrightarrow{\sim} \mathscr{I}^n \mathscr{O}_X = \mathscr{I}_{n,X}$$

and therefore $\mathscr{I}^n \mathscr{O}_X$ is a very ample invertible $\mathscr{O}_X$-module relative to Y for $n > 0$.

Proof. The last assertion is immediate since $\mathscr{O}_X(1)$ is invertible (Proposition 11.3.14) and very ample relative to Y by definition. On the other hand, the image of the homomorphism $v_n : S_+(n) \to S$ is none other than $\mathscr{I}^n \mathscr{S}$, and the first assertion then follows from the exactness of the functor $\widetilde{\mathscr{M}}$ (Proposition 11.3.13) and the formula $\widetilde{\mathscr{I}\mathscr{M}} = \mathscr{I} \cdot \widetilde{\mathscr{M}}$. □

Corollary 11.8.3. *Under the hypotheses of Proposition 11.8.2, if $f : X \to Y$ is the structural morphism and Y' is the closed subscheme of Y defined by $\mathscr{I}$, the closed subscheme $X' = f^{-1}(Y')$ of X is defined by $\mathscr{I}\mathscr{O}_X$ (isomorphic to $\mathscr{O}_X(1)$), so we have an exact sequence*

$$0 \longrightarrow \mathscr{O}_X(1) \longrightarrow \mathscr{O}_X \longrightarrow \mathscr{O}_{X'} \longrightarrow 0$$

Proof. This follows from Proposition 11.8.2 and Proposition 10.4.16(b). □

Under the hypotheses of Proposition 11.8.2, we can specify that structure of $\mathscr{I}_{n,X}$. Note that that the homomorphism

$$\tilde{u}_{-1} : \mathscr{O}_X \to \mathscr{O}_X(-1)$$

corresponds canonically to a section s of $\mathscr{O}_X(-1)$ over X, which is called the canonical section (relative to $\mathscr{I}$). In the diagram (11.8.3), the horizontal homomorphisms are isomorphisms (Corollary 11.3.16), so by replacing d by k and k by -1 in that diagram, we obtain $\tilde{u}_k = 1_k \otimes \tilde{u}_{-1}$ (where 1_k is the identity of $\mathscr{O}_X(k)$), which means the homomorphism $\tilde{u}_k$ is none other than the tensor product by the canonical section k (for any $k \in \mathbb{Z}$). The homomorphism $\tilde{u}$ of (11.8.4) can be interpreted in the same manner, and we then deduce that, for any $n \geq 0$, the homomorphism $\tilde{v}_n : \mathscr{O}_X(n) \to \mathscr{O}_X$ is the tensor product by $s^{\otimes n}$.

Corollary 11.8.4. *With the notations of Corollary 11.8.3, the underlying of X' is the set of $x \in X$ such that $s(x) = 0$, where s is the canonical section of $\mathscr{O}_X(-1)$.*

Proof. In fact, if c_x is a generator for the fiber $(\mathscr{O}_X(1))_x$ at a point x, $s_x \otimes c_x$ is canonically identified with a generator for the fiber $\mathscr{I}_{1,X}$ at the point x, and is therefore invertible if and only if $s_x \notin \mathfrak{m}_x(\mathscr{O}_X(-1))_x$, which means $s(x) \neq 0$. □

Proposition 11.8.5. *Let Y be an integral scheme, $\mathscr{I}$ be a quasi-coherent fractional ideal of $\mathscr{K}_Y$, and X be the blow up Y-scheme relative to $\mathscr{I}$. Then there is an isomorphism $\mathscr{O}_X \xrightarrow{\sim} \mathscr{I}\mathscr{O}_X$, and in particular $\mathscr{I}\mathscr{O}_X$ is a very ample invertible $\mathscr{O}_X$-module relative to Y.*

Proof. The question is local over Y (Corollary 11.4.17), so we can assume that $Y = \mathrm{Spec}(A)$ is affine, where A is an integral domain with fraction field K and $\mathscr{I} = \widetilde{\mathfrak{I}}$, where $\mathfrak{I}$ is a fractional ideal of K. Then there exists an element $a \neq 0$ such that $a\mathfrak{I} \subseteq A$. Put $S = \bigoplus_n \mathfrak{I}^n$; the map $x \mapsto ax$ is an A-isomorphism of $\mathfrak{I}^{n+1} = S(1)_n$ to $a\mathfrak{I}^{n+1} = a\mathfrak{I}S_n \subseteq \mathfrak{I}^n = S_n$, so defines a eventual isomorphism of degree 0 of graded S-modules $S_+(1) \to a\mathfrak{I}S$. But $x \mapsto a^{-1}x$ is an isomorphism of degree 0 of graded S-modules $a\mathfrak{I}S \xrightarrow{\sim} \mathfrak{I}S$, so we obtain an isomorphism $\mathscr{O}_X(1) \xrightarrow{\sim} \mathscr{I}\mathscr{O}_X$. As S is generated by $S_1 = \mathfrak{I}$, $\mathscr{O}_X(1)$ is invertible and very ample, whence the conclusion. □

Proposition 11.8.6. *Let Y be a locally Noetherian scheme, $\mathscr{I}$ ba a quasi-coherent ideal of $\mathscr{O}_Y$, and X be the blow up Y-scheme relative to $\mathscr{I}$. Let $f : X \to Y$ be the structural morphism. If $g : Z \to Y$ is any morphism such that $g^*(\mathscr{I})\mathscr{O}_Z$ is an invertible $\mathscr{O}_Z$-module, then there exists a unique morphism $\tilde{g} : Z \to X$ such that the following diagram is commutative:*

$$\begin{tikzcd} Z \arrow[r, dash, "\tilde{g}"] \arrow[dr, "g"'] & X \arrow[d, "f"] \\ & Y \end{tikzcd}$$

Proof. The question is local over Y, so we can assume that $Y = \mathrm{Spec}(A)$ is affine, where A is Noetherian, and $\mathscr{I} = \widetilde{\mathfrak{I}}$ where $\mathfrak{I}$ is an ideal of A. Then $X = \mathrm{Proj}(S)$ where $S = \bigoplus_{n\geq 0}\mathfrak{I}^n$, and we note that since A is Noetherian, $\mathfrak{I}$ is finitely generated, so S is of finite type over A. Let $a_0, \dots, a_n \in \mathfrak{I}$ be a set of generators for $\mathfrak{I}$, so that we have a surjective homomorphism $\varphi : A[T_0, \dots, T_n] \to S$ which maps T_i to a_i, and this gives a closed immersion $i : X \to \mathbb{P}^n_A$, and we can identify X with its image. If $g : Z \to Y$ is a morphism such that $\mathscr{L} = g^*(\mathscr{I})\mathscr{O}_Z$ is invertible, then the inverse images of the a_i, which are global sections of $\mathscr{I}$, give global sections $s_0, \dots, s_n$ of $\mathscr{L}$ which generate $\mathscr{L}$. Then by Proposition 11.4.3 $r : Z \to P = \mathbb{P}^n_A$ such that $\mathscr{L} \cong r^*(\mathscr{O}_P(1))$ and $s_i = r^{-1}(T_i)$. We now claim that this morphism factors through the closed subscheme X of $\mathbb{P}^n_A$: this follows from the fact that if $F(T_0, \dots, T_n)$ is a homogeneous polynomial of degree d of $\ker\varphi$, then $F(a_0, \dots, a_n) = 0$ in A and so $F(s_0, \dots, s_n) = 0$ in $\Gamma(Z, \mathscr{L}^{\otimes d})$. This gives the desired morphism $\tilde{g} : Z \to X$, and for any such morphism we necessarily have

$$g^*(\mathscr{I})\mathscr{O}_Z = \tilde{g}^*(f^*(\mathscr{I})\mathscr{O}_X)\mathscr{O}_Z = \tilde{g}^*(\mathscr{O}_X(1))\mathscr{O}_Z$$

by Proposition 11.8.2, so we obtain a surjective hommorphism $\tilde{g}^*(\mathscr{O}_X(1)) \to g^*(\mathscr{I})\mathscr{O}_Z = \mathscr{L}$, hence an isomorphism by **??**. Clearly the sections s_i of $\mathscr{L}$ are the inverse images of the sections T_i of $\mathscr{O}_P(1)$ on $\mathbb{P}^n_A$, so the uniqueness of $\tilde{g}$ follows from Proposition 11.4.3. □

Corollary 11.8.7. *Let $q : Y' \to Y$ be a morphism of locally Noetherian schemes and $\mathscr{I}$ be a quasi-coherent ideal of $\mathscr{O}_Y$. Let X be the blow up Y-scheme relative to $\mathscr{I}$ and X' be the blow up Y'-scheme relative to $\mathscr{J} = q^*(\mathscr{I})\mathscr{O}_Y$. Then there exists a unique morphism $p : X' \to X$ such that the diagram*

$$\begin{tikzcd} X' \arrow[r, "p"] \arrow[d] & X \arrow[d] \\ Y' \arrow[r, "q"] & Y \end{tikzcd} \tag{11.8.5}$$

is commutative. Moreover, if q is a closed immersion, so is p.

Proof. The existence and uniqueness of q follows from Proposition 11.8.6 and Proposition 11.8.2. To show that p is a closed immersion if q is, we trace the definition of the blow up: $X = \mathrm{Proj}(\mathscr{S})$ where $\mathscr{S} = \bigoplus_{n\geq 0}\mathscr{I}^n$ and $X' = \mathrm{Proj}(\mathscr{S}')$ where $\mathscr{S}' = \bigoplus_{n\geq 0}\mathscr{J}^n$. Since Y' is a closed subscheme of Y, we can consider $\mathscr{S}'$ as a sheaf of graded algebras over Y. Then there exists a natrual surjective homomorphism $\mathscr{S} \to \mathscr{S}'$, which gives rise to the closed immersion p. □

In the situation of Corollary 11.8.7, if Y' is a closed subscheme of Y, we call the closed subscheme X' of X the **strict transform** of Y' under the blowing-up $f : X \to Y$.

Example 11.8.8. Let $Y = \mathbb{A}^n_k$ be the affine space over a field k and we consider the blow up of Y at the origin y of Y. Then $Y = \mathrm{Spec}(A)$ where $A = k[X_1, \ldots, X_n]$, y corresponds to the ideal $\mathfrak{I} = (X_1, \ldots, X_n)$, and $X = \mathrm{Proj}(S)$ where $S = \bigoplus_{n \geq 0} \mathfrak{I}^n$. We can define a surjective homomorphism

$$\varphi : A[Y_0, \ldots, Y_n] \to S$$

of graded rings by sending Y_i to X_i as an element of degree 1 in S, which gives a closed immersion of X into $\mathbb{P}^{n-1}_A$. It is not hard to see that the kernel of φ is generated by the homogeneous polynomials $X_iY_j - X_jY_i$, where $i, j = 1, \ldots, n$, so this definition is compatible with the definition of the blow up of the affine variety $\mathbb{A}^n_k$.

Now if Y' is a closed subscheme of Y passing through y, then the strict transform X' of Y' is a closed subscheme of X. Hence, provided that Y' is not reduced to the point y, we can recover X' as the closure of $f^{-1}(Y' - \{y\})$, where $f : f^{-1}(Y' - \{y\}) \to Y' - \{y\}$ is an isomorphism. Again this definition is compatible with the definition of blow up of closed subvarieties of $\mathbb{A}^n_k$.

Example 11.8.9. As an example of the general concept of blowing up a coherent sheaf of ideals, we show how to eliminate the points of indeterminacy of a rational map determined by an invertible sheaf. So let A be a ring, X be a Noetherian scheme over A, $\mathscr{L}$ be an invertible sheaf on X, and $s_0, \ldots, s_n$ be global sections of $\mathscr{L}$. Let U be the open subset of X where the s_i generate the sheaf $\mathscr{L}$ (that is, the subset where the corresponding homomorphism $\mathscr{O}_X^{n+1} \to \mathscr{L}$ is surjective, cf. **??**(iii)). Then the invertible sheaf $\mathscr{L}|_U$ on U and the global sections $s_0, \ldots, s_n$ determine an A-morphism $\varphi : U \to \mathbb{P}^n_A$, which is also a rational map $X \dashrightarrow \mathbb{P}^n_A$. We will now show how to blow up a certain sheaf of ideals $\mathscr{J}$ on X, whose corresponding closed subsheme Y has support equal to $X - U$ (i.e., the underlying topological space of Y is $X - U$), so that the morphism φ extends to a morphism $\tilde{\varphi} : \widetilde{X} \to \mathbb{P}^n_A$.

$$\begin{tikzcd}
\widetilde{X} \arrow[d, "\pi"'] \arrow[dr, dashed, "\tilde{\varphi}"] & \\
X & \mathbb{P}^n_A \\
U \arrow[u, hook] \arrow[ur, "\varphi"'] &
\end{tikzcd}$$

Let $\mathscr{F}$ be the quasi-coherent sub-$\mathscr{O}_X$-module of $\mathscr{L}$ generated by $s_0, \ldots, s_n$. We define a coherent ideal $\mathscr{J}$ of $\mathscr{O}_X$ as follows: for any open subset $V \subseteq X$ and an isomorphism $\psi : \mathscr{L}|_V \cong \mathscr{O}_X|_V$, we take $\mathscr{J}|_V = \psi(\mathscr{F}|_V)$. It is clear that $\mathscr{J}_V$ is independent of the choice of ψ, so we get a well-defined coherent ideal $\mathscr{J}$ of $\mathscr{O}_X$. We also note that $\mathscr{J}_x = \mathscr{O}_{X,x}$ if and only if $x \in U$, so the corresponding closed subscheme Y has support $X - U$. Let $\pi : \widetilde{X} \to X$ be the corresponding blow up relative to $\mathscr{J}$. We claim that $\pi^*(\mathscr{J})$ is a coherent ideal of $\mathscr{O}_{\widetilde{X}}$, so is invertible by Proposition 11.8.2. This can be verified on each affine open X_{s_i}. Then the global sections $\pi^*(s_i)$ of $\pi^*(\mathscr{L})$ generate an invertible sub-$\mathscr{O}_{\widetilde{X}}$-module $\mathscr{L}'$ of $\pi^*(\mathscr{L})$. Now $\mathscr{L}'$ and the sections $\pi^*(s_i)$ define a morphism $\tilde{\varphi} : \widetilde{X} \to \mathbb{P}^n_A$ whose restriction on $\pi^{-1}(U)$ corresponds to the morphism φ under the isomorphism $\pi : \pi^{-1}(U) \xrightarrow{\sim} U$.

11.8.2 Homogenization of graded rings

Let S be a graded ring, which we do not suppose to be with positive degrees. We set

$$S^{\geqslant} = \bigoplus_{n \geq 0} S_n, \quad S^{\leqslant} = \bigoplus_{n \leq 0} S_n$$

which are subrings of S, with positive of negative degrees respectively. If f is a homogeneous element of degree d (positive of negative) of S, the localization $S_f = S'$ is endowed with a graded ring structure, where S'_n is the set of elements x/f^k, where $x \in S_{n+kd}$ ($k \geq 0$); we also note that $S_{(f)} = S'_0$, and we denote $S_f^{\geqslant}$ and $S_f^{\geqslant}$ for $S'^{\geqslant}$ and $S'^{\leqslant}$, respectively. This notation is justified by the fact that if $d > 0$, then we have

$$(S^{\geqslant})_f = S_f; \tag{11.8.6}$$

in fact, if $x \in S_{n+kd}$ where $n + kd < 0$, we can write $x/f^k = xf^h/f^{h+k}$ so that $n + (h+k)d > 0$ for $h > 0$ large enough. We then conclude by definition that

$$(S^{\geqslant})_{(f)} = (S_f^{\geqslant})_0 = S_{(f)}. \tag{11.8.7}$$

If M is a graded S-module, we put similarly

$$M^{\geqslant} = \bigoplus_{n \geq 0} M_n, \quad M^{\leqslant} = \bigoplus_{n \leq 0} M_n$$

which are respectively $S^{\geqslant}$-module and $S^{\geqslant}$-module, with intersection the S_0-module M_0. If $f \in S_d$, we also define M_f as the graded S_f-module such that $(M_f)_n$ is the set of elements z/f^k, where $z \in M_{n+kd}$. We denote by $M_{(f)}$ the set of degree 0 elements in M_f, which is an $S_{(f)}$-module, and we write $M_f^{\geqslant}$ and $M_f^{\leqslant}$ for $(M_f)^{\geqslant}$ and $(M_f)^{\leqslant}$ respectively. If $d > 0$, we also have

$$(M^{\geqslant})_f = M_f, \quad (M^{\geqslant})_{(f)} = (M_f^{\geqslant})_0 = M_{(f)}. \tag{11.8.8}$$

Let $\boldsymbol{z}$ be an indeterminate, which is called the **homogenization variable**. If S is a graded ring, the polynomial algebra

$$\widehat{S} = S[\boldsymbol{z}]$$

is a graded S-algebra, where for f homogeneous we put

$$\deg(f\boldsymbol{z}^n) = n + \deg(f).$$

Lemma 11.8.10. *Let $f \in S_d$ with $d > 0$. We have canonical isomorphisms*

$$\widehat{S}_{(\boldsymbol{z})} \cong \widehat{S}/(\boldsymbol{z}-1)\widehat{S} \cong S, \tag{11.8.9}$$

$$\widehat{S}_{(f)} \cong S_f^{\leqslant}. \tag{11.8.10}$$

Proof. The first isomorphism of (11.8.9) is already defined in Proposition 11.2.3 and the second one is trivial; the isomorphism $\widehat{S}_{(\boldsymbol{z})} \cong S$ thus defined then send an element $x\boldsymbol{z}^n/\boldsymbol{z}^{n+k}$ (where $\deg(x) = k$ for $k \geq -n$) to the element x. The homomorphism (11.8.10) is defined by sending $x\boldsymbol{z}^n/f^k$ (where $\deg(x) = kd - n$) to the element x/f^k, of degree $-k$ in $S_f^{\leqslant}$, and it is easy to verify that this is an isomorphism. □

Let M be a graded S-module. It is clear that the S-module

$$\widehat{M} = M \otimes_S \widehat{S} = M \otimes_S S[\boldsymbol{z}]$$

is the direct sum of the S-modules $M \otimes S\boldsymbol{z}^n$, whence the abelian groups $M_k \otimes S\boldsymbol{z}^n$. We define over $\widehat{M}$ an $\widehat{S}$-module structure by

$$\deg(x \otimes \boldsymbol{z}^n) = n + \deg(x)$$

for x homogeneous in M.

Lemma 11.8.11. *Let $f \in S_d$ with $d > 0$. We have a canonical isomorphism*

$$\widehat{M}_{(\boldsymbol{z})} \cong \widehat{M}/(\boldsymbol{z}-1)\widehat{M} \cong M, \tag{11.8.11}$$

$$\widehat{M}_{(f)} \cong M_f^{\leqslant}. \tag{11.8.12}$$

Proof. This can be proved as Lemma 11.8.10 by using the second part of Proposition 11.2.3. □

Let S be a graded ring with positive degrees. Then for each $n \geq 0$, we can consider $S(n) = \bigoplus_{m \geq n} S_m$ as a graded iedal of S (in particular $S(0) = S$ and $S(1) = S_+$). As it is clear that $S(m)S(n) \subseteq S(m+n)$, we can then define a graded ring

$$S^\natural = \bigoplus_{n \geq 0} S(n)$$

whence $S_n^\natural = S(n)$. Then $S_0^\natural$ is equal to S considered as a nongraded ring, and $S^\natural$ is therefore an S-algebra. For any homogeneous element $f \in S_d$ with $d > 0$, we denote by $f^\natural$ the element f considered as an element of $S(d) = S_d^\natural$.

Lemma 11.8.12. *Let S be a graded ring with positive degrees, f be a homogeneous element with $d > 0$. We have canonical isomorphisms:*

$$S_f \cong \bigoplus_{n \in \mathbb{Z}} S(n)_{(f)}, \tag{11.8.13}$$

$$(S_f^{\geqslant})_{f/1} \cong S_f, \tag{11.8.14}$$

$$S^\natural_{(f^\natural)} \cong S_f^{\geqslant}. \tag{11.8.15}$$

the first two of which are bi-isomorphisms of graded rings.

Proof. It is immediate that we have $(S_f)_n = (S(n)_f)_0 = S(n)_f$, whence the first isomorphism. On the other hand, as $f/1$ is invertible in S_f, there is a canonical isomorphism $S_f \cong S_f^{\geqslant} = (S_f)_{f/1}$ in view of (11.8.6) applide to S_f. Finally, if $x = \sum_{m \geq n} y_n$ is an element of $S(n)$, where $n = kd$, we can correspond the element $x/(f^\natural)^k$ to the element $\sum_m y_m/f^k$ of $S_f^{\geqslant}$, and we verify that this is an isomorphism. □

If M is a graded S-module, we can similarly put for $n \in \mathbb{Z}$

$$M^\natural = \bigoplus_{n \in \mathbb{Z}} M(n)$$

as $S(m)M(n) \subseteq M(m+n)$. Then $M^\natural$ is a graded $S^\natural$-module, and similarly we have the following:

Lemma 11.8.13. *Let $f \in S_d$ be a homogeneous element with $d > 0$. We have the following bi-homomorphisms*

$$M_f \cong \bigoplus_{n \in \mathbb{Z}} M(n)_{(f)}, \tag{11.8.16}$$

$$(M_f^{\geqslant})_{f/1} \cong M_f, \tag{11.8.17}$$

$$M^\natural_{(f^\natural)} \cong M_f^{\geqslant}. \tag{11.8.18}$$

the first two of which are bi-isomorphisms of graded modules.

Remark 11.8.14. We can think that $S^\natural$ is obtained from S by adding a "phantom" element $\boldsymbol{y}$ of degree -1. The component $S(n)$ can be then considered as the S-module $(S\boldsymbol{y}^n)_0$, which is the set of degree 0 elements in $S\boldsymbol{y}^n$. With this understanding, we can then relate the results of Lemma 11.8.10 and Lemma 11.8.12.

Lemma 11.8.15. *Let S be a graded ring with positive degrees.*

(i) *For $S^\natural$ to be an $S_0^\natural$-algebra of finite type (resp. Noetherian), it is necessary and sufficient that S is an S_0-algebra of finite type (resp. Noetherian).*

(ii) *For $S_{n+1}^\natural = S_1^\natural S_n^\natural$ for $n \geq n_0$, it is necessary and sufficient that $S_{n+1} = S_1 S_n$ for $n \geq n_0$.*

(iii) *For $S_n^\natural = (S_1^\natural)^n$ for $n \geq n_0$, it is necessary and sufficient that $S_n = S_1^n$ for $n \geq n_0$.*

(iv) *If (f_α) is a set of homogeneous elements of S_+ such that the radical in S_+ of the ideal of S_+ generated by the f_α is equal to S_+, then $S_+^\natural$ is the radical in $S_+^\natural$ of the ideal of $S_+^\natural$ generated by the $f_\alpha^\natural$.*

Proof. If $S^\natural$ is an $S_0^\natural$-algebra of finite type, $S_+ = S_1^\natural$ is a finitely generated module over $S = S_0^\natural$ by Proposition 2.1.37, so S is an S_0-algebra of finite type by Corollary 2.1.38. If $S^\natural$ is a Noetherian ring, so is the ring $S_0^\natural = S$ by Corollary 2.1.38. Conversely, if S is an S_0-algebra of finite type, then by Proposition 2.1.40 there exists $d > 0$ and $n_0 > 0$ such that $S_{n+d} = S_h S_n$ for any $n \geq n_0$; we can clearly suppose that $n_0 \geq d$. Moreover, the S_n are finitely generated S_0-modules (Corollary 2.1.38(c)), so if $n \geq n_0 + d$, we have $S_n^\natural = S_n S_{n-d}^\natural = S_d^\natural S_{n-d}^\natural$; and if $n < n_0 + d$, we have

$$S_n^\natural = S_n + \cdots + S_{n_0+d-1} + S_d E + S_d^2 E + \cdots$$

where $E = S_{n_0} + \cdots + S_{n_0+d-1}$. For $1 \leq n \leq n_0$, let G_n be the union of the generators of the S_0-module S_i where $n \leq i \leq n_0 + d - 1$, considered as subsets of $S(n)$. For $n_0 + 1 \leq n \leq n_0 + d - 1$, similarly let G_n be the union of the generators of the S_0-module S_i where $n \leq i \leq n_0 + d - 1$ and of $S_d E$, considered as subsets of $S(n)$. Then it is clear that we have $S_n^\natural = S_0^\natural G_n$ for $1 \leq n \leq n_0 + d - 1$, and therefore the union G of the G_n for $1 \leq n \leq n_0 + d - 1$ is a system of generators of the $S_0^\natural$-algebra $S^\natural$. We then conclude that if $S = S_0^\natural$ is Noehterian, then so is $S^\natural$.

It is clear that if $S_{n+1} = S_1 S_n$ for $n \geq n_0$, then we have $S_{n+1}^\natural = S_1 S_n^\natural$, and a fortiori $S_{n+1}^\natural = S_1^\natural S_n^\natural$ for $n \geq n_0$. Conversely, the latter relation means that

$$S_{n+1} + S_{n+2} + \cdots = (S_1 + S_2 + \cdots)(S_n + S_{n+1} + \cdots);$$

by comparing the degree $n+1$ component of both sides, we conclude that $S_{n+1} = S_1 S_n$. This proves the assertion (ii).

If $S_n = S_1^n$ for $n \geq n_0$, we have $S_n^\natural = \bigoplus_{m \geq n} S_1^m$, and as $S_1^\natural$ contains $\bigoplus_{m \geq 1} S_1^m$, we then have $S_n^\natural \subseteq (S_1^\natural)^n$ for $n \geq n_0$. Conversely, the degree n component of $(S_1^\natural)^n = (S_1 + S_2 + \cdots)^n$ considered as elements of S is equal to S_1^n; so the relation $S_n^\natural = (S_1^\natural)^n$ implies $S_n = S_1^n$.

Finally, to prove (iv), it suffices to show that if an element $g \in S_{k+d}$ is considered as an element of $S_k^\natural$ ($k > 0$, $d \geq 0$), then there exists an integer $n > 0$ such that in $S_{kn}^\natural$, g^n is a linear combination of the $f_\alpha^\natural$ with coefficients in $S^\natural$. By hypothesis, there exists an integer n_0 such that for $n \geq n_0$, we have $g^n = \sum_\alpha c_{\alpha n} f_\alpha$ in S, where the indices α appearing in this formula are independent of n. Moreover, we can evidently suppose that the $c_{\alpha n}$ are homogeneous, with

$$\deg(c_{\alpha n}) = n(k+d) - \deg(f_\alpha)$$

in S. Let n_0 be large enough such that we have $kn_0 > deg(f_\alpha)$ for the f_α appearing in the formula of g^n; for any α, let $c'_{\alpha n}$ be the element $c_{\alpha n}$ considered as an element of degree $kn - \deg(f_\alpha)$ in $S^\natural$. We then have $g^n = \sum_\alpha c'_{\alpha m} f_\alpha^\natural$ in $S^\natural$, which proves our assertion. □

Consider the graded S_0-algebra

$$S^\natural \otimes_S S_0 = S^\natural / S_+ S^\natural = \bigoplus_{n \geq 0} S(n)/S_+ S(n).$$

As S_n is a quotient S_0-module of $S(n)/S_+S(n)$, we have a canonical hommorphism of graded S_0-algebras

$$S^\natural \otimes_S S_0 \to S \tag{11.8.19}$$

which is evidently surjective, and corresponds to a canonical closed immersion

$$\mathrm{Proj}(S) \to \mathrm{Proj}(S^\natural \otimes_S S_0) \tag{11.8.20}$$

Proposition 11.8.16. *The canonical morphism (11.8.20) is bijective. For the homomorphism (11.8.19) to be eventually bijective, it is necessary and sufficient that there exists an integer n_0 such that $S_{n+1} = S_1 S_n$ for $n \geq n_0$. If this is satisfied, then the morphism (11.8.20) is an isomorphism, and the converse of this is also true if S is Noetherian.*

Proof. To prove the first assertion, it suffices (Corollary 11.2.45) to prove that the kernel $\mathfrak{I}$ of the homomorphism (11.8.19) is formed by nilpotent elements. Now if $f \in S(n)$ is an element whose class mod $S_+S(n)$ belongs to this kernel, then $f \in S(n+1)$ (by the definition of (11.8.19)); the element f^{n+1}, considered as an element of $S(n(n+1))$, then belongs to $S_+S(n(n+1))$, if we write it as $f \cdot f^n$. Then the class of f^{n+1} mod $S_+S(n(n+1))$ is zero, which proves our assertion.

As the hypothesis $S_{n+1} = S_1 S_n$ for $n \geq n_0$ is equivalent to $S^\natural_{n+1} = S^\natural_1 S^\natural_n$ for $n \geq n_0$ (Lemma 11.8.15(ii)), these hypotheses are equivalent to that (11.8.19) is eventually injective, hence eventually bijective, and then (11.8.20) is an isomorphism by Proposition 11.3.6(a). Conversely, assume that S is Noetherian, hence so is $S^\natural$ and $S^\natural \otimes_S S_0$ (Lemma 11.8.15(i)). If (11.8.20) is an isomorphism, the sheaf $\widetilde{\mathscr{I}}$ over $\mathrm{Proj}(S^\natural \otimes_S S_0)$ is zero (Proposition 11.3.33(a)); as $S^\natural \otimes_S S_0$ is Noetherian, we then conclude from Proposition 11.2.36(b) that $\mathfrak{I}$ is eventually null, so $S^\natural_{n+1} = S^\natural_1 S^\natural_n$ for $n \geq n_0$. □

Consider now the canonical injection $(S_+)^n \to S(n)$, which defines an injective homomorphism of degree 0 of graded rings

$$\bigoplus_{n \geq 0} (S_+)^n \to S^\natural. \tag{11.8.21}$$

Proposition 11.8.17. *For the homomorphism (11.8.21) to be an eventual isomorphism, it is necessary and sufficient that there exists an integer n_0 such that $S_n = S_1^n$ for $n \geq n_0$. If this is the case, the corresponding morphism*

$$\mathrm{Proj}(S^\natural) \to \mathrm{Proj}\Big(\bigoplus_{n \geq 0} (S_+)^n\Big) \tag{11.8.22}$$

is everywhere defined and an isomorphism, and the converse is also true if S is Noetherian.

Proof. The first two conditions are equivalent in view of Lemma 11.8.15(iii). The third assertion follows from Lemma 11.8.15(i), (iii) and Lemma 11.8.18. □

Lemma 11.8.18. *Let S be a graded ring with positive degrees which is a S_0-algebra of finite type. If the morphism corresponding to the canonical injection $S' = \bigoplus_{n \geq 0} S_1^n \to S$ is everywhere defined and an isomorphism, then there exists $n_0 > 0$ such that $S_n = S_1^n$ for $n \geq n_0$.*

Proof. In fact, let f_i ($1 \leq i \leq r$) be a system of generators for the S_0-module S_1. Then the hypotheses implies that the $D_+(f_i)$ cover $\mathrm{Proj}(S)$. Let $(g_j)_{1\leq j\leq n}$ be a system of homogeneous elements of S_+, with $n_j = \deg(g_j)$, which together with the f_i form a system of generators of the ideal S_+, or a system of generators of S as an S_0-algebra. The elements $g_j/f_i^{n_j}$ of the ring $S_{(f_i)}$ then by hypotheses belong to the subring $S'_{(f_i)}$, so there exists an integer k such that $S_1^k g_j \subseteq (S_1)^{k+n_j}$ for any j. We then conclude by recurrence on r that $S_1^k g_j^r \subseteq S'$ for any $r \geq 1$, and by the choice of the g_j, we then have $S_1^k S \subseteq S'$. On the other hand there exists for any j an integer m_j such that $g_j^{m_j}$ belongs to the ideal of S generated by the f_i (Corollary 11.2.11), so $g_j^{m_j} \in S_1 S$, and $g_j^{m_j k} \in S_1^k S \subseteq S'$. Therefore there exists an integer $m_0 \geq k$ such that $g_j^m \in S_1^{mn_j}$ for $m \geq m_0$. Now if d is the largest of the n_j, the number $n_0 = dm_0 + k$ then satisfies the requirement. In fact, an element of S_n, for $n \geq n_0$, is a sum of elements of $S_1^\alpha u$, where u is a product of powers of g_j; if $\alpha \geq k$, it follows from the choice of k that $S_1^\alpha u \subseteq S_1^n$; in the contrary case, at least one of the exponent of g_j is $\geq m_0$, so $u \in S_1^\beta v$ where $\beta \geq m_0 \geq k$ and v is a product of powers of g_j, so we are reduced to the previous case, and $S_1^\alpha u \subseteq S_1^n$. This completes the proof. □

Remark 11.8.19. The condition $S_n = S_1^n$ for $n \geq n_0$ clearly implies that $S_{n+1} = S_1 S_n$ for $n \geq n_0$, but the converse is not ture, even we assume that S is Noetherian. For example, let K be a field, $A = K[\boldsymbol{x}]$, $B = K[\boldsymbol{y}]/\boldsymbol{y}^2 K[\boldsymbol{y}]$, where $\boldsymbol{x}, \boldsymbol{y}$ are two indeterminates, with $\deg(\boldsymbol{x}) = 1$ and $\deg(\boldsymbol{y}) = 2$, and let $S = A \otimes_K B$, so that S is a graded algebra over K having a basis formed by the elements $\boldsymbol{x}^n$ and $\boldsymbol{x}^n\boldsymbol{y}$. It is immediate that $S_{n+1} = S_1 S_n$ for $n \geq 2$, but $S_1^n = K\boldsymbol{x}^n$ while $S_n = K\boldsymbol{x}^n + K\boldsymbol{x}^{n-2}\boldsymbol{y}$ for $n \geq 2$.

11.8.3 Projective cones

Let Y be a scheme; in this subsection, we only consider Y-schemes and Y-morphisms. Let $\mathscr{S}$ be a quasi-coherent graded $\mathscr{O}_Y$-algebra with positive degrees, which we suppose that $\mathscr{S}_0 = \mathscr{O}_Y$. According to the notations of the previous part, we put

$$\widehat{\mathscr{S}} = \mathscr{S}[\boldsymbol{z}] = \mathscr{S} \otimes_{\mathscr{O}_Y} \mathscr{O}_Y[\boldsymbol{z}] \tag{11.8.23}$$

which we consider as a graded $\mathscr{O}_Y$-algebra with positive degrees, so that for any affine open subset U of Y, we have

$$\Gamma(U, \mathscr{S}) = \Gamma(U, \mathscr{S})[\boldsymbol{z}].$$

In the following, we put

$$P = \mathrm{Proj}(\mathscr{S}), \quad C = \mathrm{Spec}(\mathscr{S}), \quad \widehat{C} = \mathrm{Proj}(\widehat{\mathscr{S}})$$

(where in the definition of C, $\mathscr{S}$ is considered as a nongraded $\mathscr{O}_Y$-algebra), and we say that C (resp. $\widehat{C}$) is the **affine cone** (resp. **projective cone**) defined by $\mathscr{S}$; we will also say "cone" instead of "cone affine." By abuse of language, we sat that C (resp. $\widehat{C}$) is the **affine projecting cone** of P (resp. the **projective projecting cone** of P), which P is understood to be given of the form $\mathrm{Proj}(\mathscr{S})$. Finally, we say that $\widehat{C}$ is the projective closure of C, where C is understood to be a scheme of the form $\mathrm{Spec}(\mathscr{S})$.

Lemma 11.8.20. *Let S be a graded ring with positive degrees, $X = \mathrm{Proj}(S)$, and f be a homogeneous element of S with degree $d > 0$. If f is not a divisor of zero in S, X is the smallest closed subscheme of X such that $X_f = D_+(f)$.*

Proof. This question is clearly local on X; for any homogeneous element $g \in S_h$ ($h > 0$), it suffices to prove that X_g is the smallest closed subscheme of X_g which dominates X_{fg}. It follows from the definition and Corollary 1.4.21 that this condition is equivalent to the fact that the canonical homomorphism $S_{(g)} \to S_{(fg)}$ is injective. Now this homomorphism is identified canonically with the homomorphism $S_{(g)} \to (S_{(g)})_{f^h/g^d}$ (Lemma 11.2.1). But as f^h is not a zero divisor of S, f^h/g^e is not a zero divisor in S_g (and a fortiori in $S_{(g)}$), because the relation $(f^h/g^d)(t/g^m) = 0$ with $t \in S$ and $m > 0$ implies the existence of an integer $n > 0$ such that $g^n f^h t = 0$, whence $g^n t = 0$, and therefore $t/g^m = 0$ in S_g. This proves the claim. $\square$

Proposition 11.8.21. *There is a commutative diagram*

$$\begin{tikzcd}
& \widehat{C} \arrow[dd] & \\
P \arrow[ur, "j"] \arrow[dr] & & C \arrow[ul, "i"'] \arrow[dl, bend left] \\
& Y \arrow[ur, bend left, "\varepsilon"] &
\end{tikzcd}$$

where ε and j are closed immersions and i is an affine morphism which is an open dominant immersion such that

$$i(C) = \widehat{C} - j(P). \tag{11.8.24}$$

Moreover $\widehat{C}$ is the smallest closed subscheme of $\widehat{C}$ dominating $i(C)$.

Proof. To define i, we consider the open subset $\widehat{C}_{\boldsymbol{z}} = \operatorname{Spec}(\widehat{\mathscr{S}}/(\boldsymbol{z}-1)\widehat{\mathscr{S}})$ of $\widehat{C}$ (by (11.8.9)), where $\boldsymbol{z}$ is canonically identified with a section of $\widehat{\mathscr{S}}$ over Y. The isomorphism $i : C \xrightarrow{\sim} \widehat{C}_{\boldsymbol{z}}$ corresponds then to the canonical isomorphism

$$\widehat{\mathscr{S}}/(\boldsymbol{z}-1)\widehat{\mathscr{S}} \cong \mathscr{S}$$

of (11.8.9). The morphism ε corresponds to the augmentation hommorphism $\mathscr{S} \to \mathscr{S}_0 = \mathscr{O}_Y$ with kernel $\mathscr{S}_+$, which is surjective so ε is a closed immersion (Proposition 11.1.27). Finally, j corresponds similar to the surjective homomorphism $\widehat{\mathscr{S}} \to \mathscr{S}$ of degree 0, which is the identity on $\mathscr{S}$ and zero on $\boldsymbol{z}\widehat{\mathscr{S}}$. By Proposition 11.3.33 it is clear that j is everywhere defined and a closed immersion.

To prove the other assertions of the proposition, we can evidently assume that $Y = \operatorname{Spec}(A)$ is affine, $\mathscr{S} = \widetilde{S}$ where S is a graded A-algebra, whence $\widehat{\mathscr{S}} = (\widehat{S})^{\sim}$); the homogeneous elements f of S_+ are then identified with the sections of $\widehat{\mathscr{S}}$ over Y, and the open subset $D_+(f)$ of $\widehat{C}$ is identified with $\widehat{C}_f$ (Proposition 11.2.5); similarly the open subset $D(f)$ of C is identified with C_f. It then follows from Corollary 11.2.11 and the definition of $\widehat{S}$ that the open subset $\widehat{C}_{\boldsymbol{z}} = i(C)$ together with $\widehat{C}_f$ (where f is homogeneous in S_+) constitute an open covering of $\widehat{C}$. Moreover, we have

$$i^{-1}(\widehat{C}_f) = C_f. \tag{11.8.25}$$

In fact, if we identify $i(C)$ with $\widehat{C}_{\boldsymbol{z}}$, then

$$\widehat{C}_f \cap i(C) = \widehat{C}_f \cap \widehat{C}_{\boldsymbol{z}} = \widehat{C}_{f\boldsymbol{z}} = \operatorname{Spec}(\widehat{S}_{(f\boldsymbol{z})}).$$

Now if $d = \deg(f)$, $\widehat{S}_{(f\boldsymbol{z})}$ is canonically isomorphic to $(\widehat{S}_{(\boldsymbol{z})})_{(f/\boldsymbol{z}^d)}$ (Lemma 11.2.1), and it follows from the definition of the isomorphism (11.8.9) that the image of $(\widehat{S}_{(\boldsymbol{z})})_{(f/\boldsymbol{z}^d)}$ under the corresponding isomorphism is exactly S_f. As $C_f = \operatorname{Spec}(S_f)$, this proves (11.8.25) and also shows that the morphism i is affine. Moreover, the restriction of i to C_f, considered as a morphism into $\widehat{C}_f$, corresponds (Proposition 10.2.4) to the canonical homomorphism $\widehat{S}_{(f)} \to \widehat{S}_{(f\boldsymbol{z})} \cong S_f$.

We also note that under the isomorphism (11.8.10), $\widehat{C}_f$ is canonically idnetified with $\operatorname{Spec}(S_f^{\leqslant})$ and the morphism restriction $i|_{C_f} : C_f \to \widehat{C}_f$ corresponds to the canonical injection $S_f^{\leqslant} \to S_f$. The complement of $\widehat{C}_{\boldsymbol{z}}$ in $\widehat{C} = \operatorname{Proj}(\widehat{S})$ is, by definition, the set of graded prime ideals of $\widehat{S}$ containing $\boldsymbol{z}$, which is $j(P)$ from the defintion of j, whence (11.8.24).

To prove the final assertion, we can still assume that Y is affine. With the preceding notations, we note that $\boldsymbol{z}$ is not a zero divisor in $\widehat{S}$, so we can apply Lemma 11.8.20. □

We now identify the affine cone C with the open subscheme $i(C)$ of the projective cone $\widehat{C}$, which is dense in $\widehat{C}$. The closed subscheme of C associated with the closed immersion ε is called the **sommet scheme** of C. We also say that ε, which is a Y-section of C, is the **sommet section** or the **zero section** of C; we can then identify Y with the sommet scheme of C via the morphism ε. The composition $i \circ \varepsilon$ is a Y-section of $\widehat{C}$, which is also a closed immersion (Corollary 10.5.19), corresponding to the canonical surjective homomorphism $\widehat{\mathscr{S}} = \mathscr{S}[\boldsymbol{z}] \to \mathscr{O}_Y[\boldsymbol{z}]$ (cf. Remark 11.3.18), with kernel $\mathscr{S}_+[\boldsymbol{z}] = \widehat{\mathscr{S}}_+$. The closed subscheme of $\widehat{C}$ associated with this closed immersion is called the **sommet scheme** of $\widehat{C}$, which can be identified with Y via $i \circ \varepsilon$, and $i \circ \varepsilon$ is called the **sommet section** of $\widehat{C}$. Finally, the closed subscheme of $\widehat{C}$ associated with j is called the **place of infinity** of C, which is identified with P via j.

The subscheme of C (resp. $\widehat{C}$) induced respectively over the open subsets

$$E = C - \varepsilon(Y), \quad \widehat{E} = \widehat{C} - i(\varepsilon(Y)) \tag{11.8.26}$$

are called respectively (by abuse of language) the **blunt affine cone** (resp. **blunt projective cone**) defined by $\mathscr{S}$. We note that with this terminology, E is not necessarily affine over Y, nor is it projective over Y (cf. Example 11.8.29). If we identify C with $i(C)$, we then have

$$C \cup \widehat{E} = \widehat{C}, \quad C \cap \widehat{E} = E. \tag{11.8.27}$$

so that $\widehat{C}$ can be considered as obtaining by gluing the open subschemes C and $\widehat{E}$ along E; moreover, in view of (11.8.24),

$$E = \widehat{E} - j(P). \tag{11.8.28}$$

If $Y = \operatorname{Spec}(A)$ is affine, we then have (with the notations of Proposition 11.8.21),

$$E = \bigcup C_f, \quad \widehat{E} = \bigcup \widehat{C}_f, \quad C_f = C \cap \widehat{C}_f \tag{11.8.29}$$

where f runs through homogeneous elements of S_+ (or a family of homogeneous elements of S_+ generating the ideal S_+). The glueing of C and the $\widehat{C}_f$ along the C_f is then determined by the injections $C_f \to C$, $C_f \to \widehat{C}_f$, which correspond to the cannical homomorphisms $S \to S_f$, $S_f^{\leqslant} \to S_f$. On the other hand, we note that $\bigcup \widehat{E}_f$ is the defining domain $G(\varphi)$ of the morphism associated with the canonical injection $\varphi : \mathscr{S} \to \widehat{\mathscr{S}} = \mathscr{S}[\boldsymbol{z}]$, so we obtain a morphism $p : \widehat{E} \to P$.

Proposition 11.8.22. *The associated morphism $p : \widehat{E} \to P$ is an affine and surjective morphism (called the **canonical retraction**) such that*

$$p^{-1}(P_f) = \widehat{C}_f \tag{11.8.30}$$

and we have $p \circ j = 1_P$. Moreover, if Y is affine and $f \in S_1$, then $\widehat{C}_f$ is isomorphic to $P_f \otimes_{\mathbb{Z}} \mathbb{Z}[T]$ (T is an indeterminate).

Proof. To prove the proposition we may assume that Y is affine, so $\mathscr{S} = \widetilde{S}$. For any $f \in S_+$ homogeneous, by (11.2.9) we have (11.8.30) and the restriction $p : \widehat{C}_f \to P_f$ corresponds to the canonical injection $S_{(f)} \to S_f^{\leqslant}$. The formula $p \circ j = 1_P$ and the fact that p is surjective follows from the fact that the composition $\mathscr{S} \to \widehat{\mathscr{S}} \to \mathscr{S}$ is the identity on $\mathscr{S}$. Finally, the last assertion follows from the fact that $S_f^{\leqslant}$ is isomorphic to $S_{(f)}[T]$ (cf. (11.2.1)). □

Corollary 11.8.23. *The restriction $\pi : E \to P$ of p to E is a surjective and affine morphism. If Y is affine and $f \in S_+$ is homogeneous, we have*

$$\pi^{-1}(P_f) = C_f \tag{11.8.31}$$

and the restriction of $\pi|_{C_f} : C_f \to P_f$ corresponds to the cannical injection $S_{(f)} \to S_f$. If $f \in S_1$, then C_f is isomorphic to $P_f \otimes_{\mathbb{Z}} \mathbb{Z}[T, T^{-1}]$ (T is an indeterminant).

Proof. The formula (11.8.31) follows from (11.8.30) and (11.8.25), which also proves the surjectivity of π. We also have seen that the canonical injection $C_f \to \widehat{C}_f$ corresponds to $S_{(f)} \to S_f$, whence the second assertion. Finally, the last assertion is a concequence of the fact that for $f \in S_1$, S_f is isomorphic to $S_{(f)}[T, T^{-1}]$ (cf. (11.2.1)). □

Remark 11.8.24. If Y is affine, the elements of the underlying space of E are the prime ideals $\mathfrak{p}$ (not necessarily graded) of S not containing S_+, in ivew of the definition of the immersion ε. For such a prime ideal $\mathfrak{p}$, the intersections $\mathfrak{p} \cap S_n$ satisfy the conditions of Proposition 2.1.50, so there exists a graded prime ideal $\mathfrak{q}$ of S such that $\mathfrak{q} \cap S_n = \mathfrak{p} \cap S_n$ for any n. The map $\pi : E \to P$ on the underlying topological space is then interpreted by the relation

$$\pi(\mathfrak{p}) = \mathfrak{q}.$$

In fact, to verify this relation, it suffices to consider a homogeneous element f of S_+ such that $\mathfrak{p} \in D(f)$, and we then observe that $\mathfrak{q}_{(f)}$ is the inverse image of $\mathfrak{p}_f$ under the canonical injection $S_{(f)} \to S_f$.

Corollary 11.8.25. *If $\mathscr{S}$ is generated by $\mathscr{S}_1$, the morphism p and π are of finite type. Moreover, for any $x \in P$, the fiber $p^{-1}(x)$ is isomorphic to $\operatorname{Spec}(\kappa(x)[T])$ and $\pi^{-1}(x)$ is isomorphic to $\operatorname{Spec}(\kappa(x)[T, T^{-1}])$.*

Proof. This follows from Proposition 11.8.22 and Corollary 11.8.23, since if Y is affine and S is genrated by S_1, then the P_f for $f \in S_1$ form an open covering of P. □

Remark 11.8.26. The blunt affine cone E corresponding to the graded $\mathscr{O}_Y$-algebra $\mathscr{O}_Y[T]$ (where T is an indeterminate) is identified with $G_m = \operatorname{Spec}(\mathscr{O}_Y[T, T^{-1}])$, since it is none other than C_T as we have seen in Proposition 11.8.21. This scheme is canonically endowed with an abelian group Y-scheme structure.

Example 11.8.27. Let k be a field, $k[T_0, \dots, T_n]$ be the polynomial ring, and $\mathfrak{p}$ be a graded prime ideal of $k[T_0, \dots, T_n]$ not containing the irrelevant ideal. Consider the quotient graded ring $S = k[T_0, \dots, T_n]/\mathfrak{p}$, and set

$$P = \operatorname{Proj}(S), \quad C = \operatorname{Spec}(S), \quad \widehat{C} = \operatorname{Spec}(\widehat{S}).$$

In the language of varieties, if $V \subseteq \mathbb{P}^n_k$ is the variety defined by S, C can be viewed as the affine cone obtained by considering the lines connecting the origin with points of V. Moreover, $\widehat{C}$ is the closure of C in $\mathbb{P}^{n+1}_k$ if we embed $\mathbb{A}^{n+1}_k$ into $\mathbb{P}^{n+1}_k$ via the map $(x_0, \dots, x_n) \to [x_0 : \cdots : x_n : 1]$. Also, the morphism $j : P \to \widehat{C}$ corresponds to the injection $[x_0 : \cdots : x_n] \mapsto [x_0 : \cdots : x_n : 0]$. With these, the projective closure $\widehat{C}$ of C in $\mathbb{P}^{n+1}$ is given by the equivalent classes $[x_0 : \cdots : x_n : x_{n+1}]$ in $\mathbb{P}^{n+1}$ such that $[x_0 : \cdots : x_n] \in P$, and we can divide into two cases depending on whether $x_{n+1} \neq 0$:

(a) $(x_0, \dots, x_n) \in C$, $x_{n+1} \neq 0$;

(b) $[x_0 : \cdots : x_n] \in P$, $x_{n+1} = 0$.

Thus we see that the variety $\widehat{C}$ can be viewed as a union of P with C, which justifies the formula (11.8.24). Also, by definition the blunt affine cone E is the subvariety of C obtained by removing the origin of $\mathbb{A}^{n+1}$, and $\widehat{E}$ can be considered as a union of E and P, which is also the projective cone $\widehat{C}$ removing the point $[0:\cdots:0:1]$ in $\mathbb{P}^{n+1}$. We also remark that if we base change C through the structural morphism $P\to Y$, then the projection $C_{(P)}\to C$ can be viewed as the projection from $\mathbb{A}^{n+1}\times\mathbb{P}^n$ to $\mathbb{P}^n_k$ which maps (x,ξ) to ξ (where the class of $x\in\mathbb{A}^{n+1}_k$ is equal to ξ), and this is the blow up map of $\mathbb{A}^{n+1}_k$ at the origin.

Let Y be a scheme, $\mathscr{E}$ be a quasi-coherent $\mathscr{O}_Y$-module. If $\mathscr{S}$ is the graded $\mathscr{O}_Y$-algebra $\boldsymbol{S}_{\mathscr{O}_Y}(\mathscr{E})$, then $\widehat{\mathscr{S}}$ is identified with $\boldsymbol{S}_{\mathscr{O}_Y}(\mathscr{E}\oplus\mathscr{O}_Y)$. The affine cone $\mathrm{Spec}(\mathscr{S})$ defined by $\mathscr{S}$ is by definition the vector bundle $\boldsymbol{V}(\mathscr{E})$, and $\mathrm{Proj}(\mathscr{S})$ is by definition $\mathbb{P}(\mathscr{E})$, so we see that:

Proposition 11.8.28. *The projective closure of a vector bundle $\boldsymbol{V}(\mathscr{E})$ over Y is canonically isomorphic to $\mathbb{P}(\mathscr{E}\oplus\mathscr{O}_Y)$, and the place of infinity of this is canonically isomorphic to $\mathbb{P}(\mathscr{E})$.*

Example 11.8.29. Put for example $\mathscr{E}=\mathscr{O}_Y^r$ where $r\geq 2$. Then the blunt cones $E,\widehat{E}$ defined by $\mathscr{S}$ are neither affine nor projective over Y if $Y\neq\varnothing$. The second assertion is immediate, since $\widehat{C}=\mathbb{P}(\mathscr{O}_Y^{r+1})$ is projective over Y and the underlying spaces of E and $\widehat{E}$ are not closed in $\widehat{C}$, so the canonical immersions $E\to\widehat{C}$ and $\widehat{E}\to\widehat{C}$ are not projective (Theorem 11.5.30 and Proposition 11.5.34(v)). On the other hand, suppose that $Y=\mathrm{Spec}(A)$ is affine and for example $r=2$; we have $C=\mathrm{Spec}(A[T_1,T_2])$ and E is the open subscheme $D(T_1)\cup D(T_2)$ of C, and we have seen that this is not affine (Example 10.5.35); a fortiori $\widehat{E}$ is not affine, since E is the open subset of $\widehat{E}$ where the section $\boldsymbol{z}$ is nonzero (11.8.27).

Proposition 11.8.30. *Let $\mathscr{L}$ be an invertible $\mathscr{O}_Y$-module, we have canonical isomorphisms for the blunt cones corresponding to $C=V(\mathscr{L})$:*

$$\mathrm{Spec}(\bigoplus_{n\in\mathbb{Z}}\mathscr{L}^n)\xrightarrow{\sim}E,\quad \boldsymbol{V}(\mathscr{L}^{-1})\xrightarrow{\sim}\widehat{E}. \tag{11.8.32}$$

Moreover, there exists a canonical isomorphism from the projective closue of $V(\mathscr{L})$ to that of $V(\mathscr{L}^{-1})$, which transform the sommet scheme (resp. the place of infinity) of the first one to the place of infinity (resp. the sommet scheme) of the second one.

Proof. Here we have $\mathscr{S}=\bigoplus_{n\geq 0}\mathscr{L}^{\otimes n}$; the canonical injection $\mathscr{S}\to\bigoplus_{n\in\mathbb{Z}}\mathscr{L}^{\otimes n}$ defines a cannical dominant morphism

$$\mathrm{Spec}(\bigoplus_{n\in\mathbb{Z}}\mathscr{L}^{\otimes n})\to\boldsymbol{V}(\mathscr{L})=\mathrm{Spec}(\bigoplus_{n\geq 0}\mathscr{L}^{\otimes n}) \tag{11.8.33}$$

and it suffices to prove that this morphism is an isomorphism from $\mathrm{Spec}(\bigoplus_{n\in\mathbb{Z}}\mathscr{L}^{\otimes n})$ to E. The question is local over Y, so we can suppose that $Y=\mathrm{Spec}(A)$ is affine and $\mathscr{L}=\mathscr{O}_Y$, so $\mathscr{S}=\widetilde{A[T]}$ and $\bigoplus_{n\in\mathbb{Z}}\mathscr{L}^{\otimes n}=\widetilde{A[T,T^{-1}]}$. Now $A[T,T^{-1}]$ is the fraction ring $A[T]_T$ of $A[T]$, so (11.8.33) identify $\mathrm{Spec}(\bigoplus_{n\in\mathbb{Z}}\mathscr{L}^{\otimes n})$ as the open subscheme $D(T)$ of $C=\boldsymbol{V}(\mathscr{L})$, which by definition is E.

The isomorphism $\boldsymbol{V}(\mathscr{L}^{-1})\cong\widehat{E}$ will on the other hand be a consequence of the last assertion, since $\boldsymbol{V}(\mathscr{L}^{-1})$ is the complement of the place of infinity of its integral closure and $\widehat{E}$ is the complement of the sommet scheme of projective closure of $C=\boldsymbol{V}(\mathscr{E})$. Now these projective closures are respectively $\mathbb{P}(\mathscr{L}^{-1}\oplus\mathscr{O}_Y)$ and $\mathbb{P}(\mathscr{L}\oplus\mathscr{O}_Y)$; but we have

$$\mathscr{L}\oplus\mathscr{O}_Y=\mathscr{L}\oplus(\mathscr{L}\otimes\mathscr{L}^{-1})=\mathscr{L}\otimes(\mathscr{L}^{-1}\oplus\mathscr{O}_Y),$$

so the existence of the isomorphism follows from Proposition 11.4.1, and it remains to see that this isomorphism exchanges the sommet scheme and the place of infinity. For this we can assume that $Y = \operatorname{Spec}(A)$ is affine, $\mathscr{L} = \widetilde{L}$, with $L = Ac$, $L^{-1} = Ac'$, and the canonical isomorphism $L \otimes L^{-1} \to A$ sends $c \otimes c'$ to 1. Then

$$\boldsymbol{S}(L \oplus A) = A[\boldsymbol{z}] \otimes \bigoplus_{n \geq 0} Ac^{\otimes n}, \quad \boldsymbol{S}(L^{-1} \oplus A) = A[\boldsymbol{z}] \otimes \bigoplus_{n \geq 0} Ac'^{\otimes n},$$

and the isomorphism defined in Proposition 11.4.1 sends $\boldsymbol{z}^h \otimes c'^{\otimes(n-h)}$ to the element $\boldsymbol{z}^{n-h} \otimes c^{\otimes h}$. Now, in $\mathbb{P}(\mathscr{L}^{-1} \oplus \mathscr{O}_Y)$ the place of infinity is the set of points where the section $\boldsymbol{z}$ vanishes, and the sommet section is the set of points where the section c' vanishes. As we have a similary description for $\mathbb{P}(\mathscr{L} \oplus \mathscr{O}_Y)$, our conclusion follows immediately from the preceding explanations. □

11.8.4 Functorial properties

Let Y, Y' be two schemes, $q : Y' \to Y$ be a morphism, $\mathscr{S}$ (resp. $\mathscr{S}'$) be a quasi-coherent $\mathscr{O}_Y$-algebra (resp. $\mathscr{O}_{Y'}$-algebra) with positive degrees. Consider a q-morphism of graded algebras

$$\varphi : \mathscr{S} \to \mathscr{S}'.$$

We have seen that this corresponds canonically to a morphism

$$\Phi = \operatorname{Spec}(\varphi) = \operatorname{Spec}(\mathscr{S}') \to \operatorname{Spec}(\mathscr{S})$$

such that the diagram

$$\begin{tikzcd} C' \arrow[r, "\Phi"] \arrow[d] & C \arrow[d] \\ Y' \arrow[r, "q"] & Y \end{tikzcd} \tag{11.8.34}$$

where $C = \operatorname{Spec}(\mathscr{S})$, $C' = \operatorname{Spec}(\mathscr{S}')$, is commutative. Suppose moreover that $\mathscr{S}_0 = \mathscr{O}_Y$ and $\mathscr{S}'_0 = \mathscr{O}_{Y'}$; let $\varepsilon : Y \to C$ and $\varepsilon' : Y' \to C'$ be the cannical immersions, we then have a commutative diagram

$$\begin{tikzcd} C' \arrow[r, "\Phi"] & C \\ Y' \arrow[r, "q"] \arrow[u, "\varepsilon'"] & Y \arrow[u, "\varepsilon"'] \end{tikzcd} \tag{11.8.35}$$

which corresponds to the diagram

$$\begin{tikzcd} \mathscr{S} \arrow[r, "\varphi"] \arrow[d] & \mathscr{S}' \arrow[d] \\ \mathscr{O}_Y \arrow[r] & \mathscr{O}_{Y'} \end{tikzcd}$$

where the vertical are augmentation homomorphisms, and the commutativity follows from the hypotheses that φ is a homomorphism of graded algebras.

Proposition 11.8.31. *If E (resp. E') is the blunt affine cone defined by $\mathscr{S}$ (resp. $\mathscr{S}'$), we have $\Phi^{-1}(E) \subseteq E'$. Moreover, the morphism $\operatorname{Proj}(\varphi) : G(\varphi) \to \operatorname{Proj}(\mathscr{S})$ is everywhere defined (in other words $G(\varphi) = \operatorname{Proj}(\mathscr{S}')$) if and only if $\Phi^{-1}(E) = E'$.*

Proof. The first assertion followsfrom (11.8.35), since $E = C - \varepsilon(Y)$ and $E' = C' - \varepsilon'(Y')$. To prove the second assertion, we can assume that $Y = \operatorname{Spec}(A)$ and $Y' = \operatorname{Spec}(A')$ are affine, $\mathscr{S} = \widetilde{S}$, $\mathscr{S}' = \widetilde{S}'$. For f homogeneous in S_+, if we put $f' = \varphi(f)$, we have $\Phi^{-1}(C_f) = C'_{f'}$ (11.2.9); to say that $G(\varphi) = \operatorname{Proj}(S')$ signifies that in S'_+ the radical of the ideal generated by the $f' = \varphi(f)$ is equal to S'_+ (Corollary 11.2.11), and this is equivalent to that the $C'_{f'}$ cover E' (11.8.29). □

The q-morphism φ extends canonically to a q-morphism of graded algebras

$$\hat{\varphi} : \widehat{\mathcal{S}} \to \widehat{\mathcal{S}}'$$

which satisfies $\hat{\varphi}(z) = z$. We then deduce a morphism

$$\hat{\Phi} = \mathrm{Proj}(\hat{\varphi}) : G(\hat{\varphi}) \to \widehat{C} = \mathrm{Proj}(\widehat{\mathcal{S}})$$

such that the diagram

$$\begin{tikzcd} G(\hat{\varphi}) \arrow[r, "\hat{\Phi}"] \arrow[d] & \widehat{C} \arrow[d] \\ Y' \arrow[r, "q"] & Y \end{tikzcd}$$

is commutative. It then follows from the definition that if we denote by $i : C \to \widehat{C}$ and $i' : C' \to \widehat{C}'$ are the canonical immersions, we have $i'(C') \subseteq G(\hat{\varphi})$ and the diagram

$$\begin{tikzcd} G(\hat{\varphi}) \arrow[r, "\hat{\Phi}"] & \widehat{C} \\ C' \arrow[u, "i'"] \arrow[r, "\Phi"] & C \arrow[u, "i"'] \end{tikzcd} \tag{11.8.36}$$

is commutative. Finally, if we put $P = \mathrm{Proj}(\mathcal{S})$, $P' = \mathrm{Proj}(\mathcal{S}')$, and if $j : P \to \widehat{C}$, $j' : P' \to \widehat{C}'$ are the canonical closed immersions, we have $j'(G(\varphi)) \subseteq G(\hat{\varphi})$ and the diagram

$$\begin{tikzcd} G(\hat{\varphi}) \arrow[r, "\hat{\Phi}"] & \widehat{C} \\ G(\varphi) \arrow[u, "j"] \arrow[r, "\mathrm{Proj}(\varphi)"] & P \arrow[u, "j"'] \end{tikzcd} \tag{11.8.37}$$

is commutative.

Proposition 11.8.32. *If $\widehat{E}$ (resp. $\widehat{E}'$) is the blunt projective cone defined by $\mathcal{S}$ (resp. $\mathcal{S}'$), we have $\hat{\Phi}^{-1}(\widehat{E}) \subseteq \widehat{E}'$. Moreover, if $p : \widehat{E} \to P$ and $p' : \widehat{E}' \to P'$ are the canonical retractions, we have $p'(\hat{\Phi}^{-1}(\widehat{E})) \subseteq G(\varphi)$, and the diagram*

$$\begin{tikzcd} \hat{\Phi}^{-1}(\widehat{E}) \arrow[r, "\hat{\Phi}"] \arrow[d, "p'"'] & E \arrow[d, "p"] \\ G(\varphi) \arrow[r, "\mathrm{Proj}(\varphi)"] & P \end{tikzcd}$$

is commutative. If $\mathrm{Proj}(\varphi)$ *is everywhere defined, so is $\hat{\Phi}$ and we have $\hat{\Phi}^{-1}(\widehat{E}) = \widehat{E}'$.*

Proof. The first assertion follows from the commutative diagrams (11.8.34) and (11.8.36), and the next two follow from the definition of the canonical retraction, the definition of $\hat{\varphi}$, and the fact that $\widehat{E}$ is the defining domain of the morphism induced by the canonical injection $\mathcal{S} \to \widehat{\mathcal{S}}$. On the other hand, to see that $\hat{\Phi}$ is everywhere defined if $\mathrm{Proj}(\varphi)$ is, we can assume that $Y = \mathrm{Spec}(A)$, $Y' = \mathrm{Spec}(A')$ are affine, $\mathcal{S} = \widetilde{S}$, $\mathcal{S}' = \widetilde{S}'$; the hypothesis is that if f runs through homogeneous elements of S_+, the ideal in S'_+ generated by the $\varphi(f)$ has radical in S'_+ equal to S'_+. We then conclude that the radical of the ideal genrated by z and the $\varphi(f)$ in $(S'[z])_+$ is equal to $(S'[z])_+$, whence our assertion. This proves similarly that $\widehat{E}'$ is the union of the $\widehat{C}'_{(\varphi(f))}$, which is equal to $\hat{\Phi}^{-1}(\widehat{E})$. □

Corollary 11.8.33. *If Φ is everywhere defined, the inverse image under $\hat{\Phi}$ of underlying space of the place of infinity (resp. the sommet scheme) of $\widehat{C}'$ is the underlying space of the place of infinity (resp. the sommet scheme) of $\widehat{C}$.*

Proof. This follows from Proposition 11.8.32 and Proposition 11.8.31, in ivew of the relations (11.8.26) and (11.8.26). □

11.8.5 Blunt cones over a homogeneous specturm

Let Y be a scheme, $\mathscr{S}$ be a quasi-coherent $\mathscr{O}_Y$-algebra with positive degrees such that $\mathscr{S}_0 = \mathscr{O}_Y$, and $X = \mathrm{Proj}(\mathscr{S})$. We now apply the previous results to the structure morphism $q : X \to Y$. Let

$$\mathscr{S}_X = \bigoplus_{n\in\mathbb{Z}} \mathscr{O}_X(n) \tag{11.8.38}$$

which is a quasi-coherent $\mathscr{O}_X$-algebra, the multiplication γ being defined by the canonical homomorphisms

$$\mathscr{O}_X(m) \otimes \mathscr{O}_X(n) \to \mathscr{O}_X(m+n)$$

which satisfies the associativity in view of Proposition 11.3.15. Let $\mathscr{S}'$ be the quasi-coherent sub-algebra

$$\mathscr{S}_X^{\geqslant} = \bigoplus_{n\geq 0} \mathscr{O}_X(n)$$

of $\mathscr{S}_X$, with positive degrees. For each $n \in \mathbb{Z}$, we have a canonical q-morphisms $\alpha_n : \mathscr{S}_n \to \mathscr{O}_X(n)$ defined in (11.3.1), which together give a homomorphism

$$\alpha : \mathscr{S} \to \bigoplus_{n\geq 0} q_*(\mathscr{O}_X(n)).$$

By composing with the canonical homomorphism $\bigoplus_{n\geq 0} q_*(\mathscr{O}_X(n)) \to q_*(\mathscr{S}_X^{\geqslant})$, this gives a q-homomorphism $\mathscr{S} \to \mathscr{S}_X^{\geqslant}$, still denoted by α. We set

$$C_X = \mathrm{Spec}(\mathscr{S}_X^{\geqslant}), \quad \widehat{C}_X = \mathrm{Proj}(\mathscr{S}_X^{\geqslant}[\boldsymbol{z}]), \quad P_X = \mathrm{Proj}(\mathscr{S}_X^{\geqslant})$$

and denote by E_X and $\widehat{E}_X$ the corresponding blunt cones. We then have the canonical morphisms

$$\begin{tikzcd}
& \widehat{C}_X \arrow[dd] & \\
P_X \arrow[ur, "j_X"] \arrow[dr] & & C_X \arrow[ul, "i_X"'] \arrow[dl] \\
& X \arrow[ur, bend left, "\varepsilon_X"] &
\end{tikzcd}$$

and $p_X : \widehat{E}_X \to P_X$, $\pi_X : E_X \to P_X$.

Proposition 11.8.34. *The structural morphism $\psi : P_X \to X$ is an isomorphism, and the morphism* $\mathrm{Proj}(\alpha)$ *is everywhere defined and equals to ψ. The morphism* $\mathrm{Proj}(\hat{\alpha}) : \widehat{C}_X \to \widehat{C}$ *is everywhere defined and its restriction to $\widehat{E}_X$ and E_X are isomorphisms into $\widehat{E}$ and E, respectively. Finally, if we identify P_X and X via ψ, the morphisms p_X and π_X are identified with the structural morphisms of the X-schemes $\widehat{E}_X$ and E_X.*

Proof. We can clearly assume that $Y = \mathrm{Spec}(A)$ is affine and $\mathscr{S} = \widetilde{S}$; then X is the union of the affine opens X_f, where $f \in S_+$ is homogeneous, the ring of X_f being $S_{(f)}$. It follows from the isomorphism (11.8.13) that

$$\Gamma(X_f, \mathscr{S}_X^{\geqslant}) = S_f^{\geqslant}. \tag{11.8.39}$$

We then have $\psi^{-1}(X_f) = \mathrm{Proj}(S_f^{\geqslant})$. But if $f \in S_d$ with $d > 0$, $\mathrm{Proj}(S_f^{\geqslant})$ is canonically isomorphic to $\mathrm{Proj}((S_f^{\geqslant})^{(d)})$ by Remark 11.3.19, and $(S_f^{\geqslant})^{(d)} = (S^{(d)})_f^{\geqslant}$ is identified with $S_{(f)}[T]$ by the map $T \mapsto f/1$ (cf. (11.2.1)), so we conclude from Remark 11.3.18 that the structural morphism $\psi^{-1}(X_f) \to X_f$ is an isomorphism, whence the first assertion. To prove the second one, we first note that $\mathrm{Proj}(\alpha)$ is everywhere defined by Lemma 11.8.15. Since $\psi^{-1}(X_f) = (\psi^{-1}(X_f))_{f/1}$, it follows from (11.2.9) that the image of $\psi^{-1}(X_f)$ under $\mathrm{Proj}(\alpha)$ is contained in X_f, and the restriction of $\mathrm{Proj}(\alpha)$ to $\psi^{-1}(X_f)$, considered as a morphism into $X_f = \mathrm{Spec}(S_{(f)})$, is identified with ψ. Finally, the formula (11.8.30) and (11.8.10) show that $p_X^{-1}(\psi^{-1}(X_f)) = \mathrm{Spec}((S_f^{\geqslant})_{f/1}^{\leqslant})$, and this open subset is, by Proposition 11.8.32 and formula (11.8.30), the inverse image of $p^{-1}(X_f) = \mathrm{Spec}(S_f^{\leqslant})$ under $\mathrm{Proj}(\hat{\alpha})$. By the isomorphism (11.8.10), the restriction of $\mathrm{Proj}(\hat{\alpha})$ to $p_X^{-1}(\psi^{-1}(X_f))$ corresponds to the isomorphism $S_f^{\leqslant} \cong (S_f^{\geqslant})_{f/1}^{\leqslant}$, whence the third assertion. The last assertion is clear by definition. □

We note that by (11.8.36) the restriction of $\mathrm{Proj}(\hat{\alpha})$ to C_X is equal to $\mathrm{Spec}(\alpha)$.

Corollary 11.8.35. *Considered as X-schemes, $\widehat{E}_X$ is canonically isomorphic to* $\mathrm{Spec}(\mathscr{S}_X^{\leqslant})$*, E_X is canonically isomorphic to* $\mathrm{Spec}(\mathscr{S}_X)$*, and C_X is canonically isomorphic to* $\mathrm{Spec}(\mathscr{S}_X^{\geqslant})$*.*

Proof. As we have seen that p_X and π_X are affine, it suffices to verify the corollary if $Y = \mathrm{Spec}(A)$ is affine and $\mathscr{S} = \widetilde{S}$. The first assertion follows from the canonical isomorphism $(S_f^{\geqslant})_{f/1}^{\leqslant} \cong S_f^{\leqslant}$, which are compatible with passage from f to fg (f, g homogeneous in S_+). Similarly, the formula (11.8.31), applied to π_X, shows that $\pi_X^{-1}(\psi^{-1}(X_f)) = \mathrm{Spec}((S_f^{\geqslant})_{f/1})$ for f homogeneous in S_+, and the second assertion then follows from the canonical isomorphism $(S_f^{\geqslant})_{f/1} \cong S_f$. □

We can then say that $\widehat{C}_X$, considered as an X-scheme, is obtained by glueing the affine X-schemes $C_X = \mathrm{Spec}(\mathscr{S}_X^{\geqslant})$ and $\widehat{E}_X = \mathrm{Spec}(\mathscr{S}_X^{\leqslant})$ along their intersection $E_X = \mathrm{Spec}(\mathscr{S}_X)$.

Corollary 11.8.36. *Suppose that $\mathscr{O}_X(1)$ is an invertible $\mathscr{O}_X$-module and that $\mathscr{S}_X \cong \bigoplus_{n\in\mathbb{Z}}(\mathscr{O}_X(1))^{\otimes n}$ (for example if $\mathscr{S}$ is generated by $\mathscr{S}_1$). Then the blunt projective cone $\widehat{E}$ is identified with the rank one vector bundle $\boldsymbol{V}(\mathscr{O}_X(-1))$ over X, and the bulnk affine cone E is isomorphic to the open subscheme induced over the complement of the zero section in this vector bundle. With these indentifications, the canonical retraction $\widehat{E} \to X$ is identified with the structural morphism of the X-scheme $\boldsymbol{V}(\mathscr{O}_X(-1))$. Finally, there exists a canonical Y-morphism $\boldsymbol{V}(\mathscr{O}_X(1)) \to C$, whose restriction to the complement of the zero section of $\boldsymbol{V}(\mathscr{O}_X(1))$ is an isomorphism from this complement to the blunt affine cone E.*

Proof. In fact, if $\mathscr{L} = \mathscr{O}_X(1)$, then $\mathscr{S}_X^{\geqslant}$ is identified with $\boldsymbol{S}_{\mathscr{O}_X}(\mathscr{L})$ and $\mathscr{S}_X^{\leqslant}$ is identified with $\boldsymbol{S}_{\mathscr{O}_X}(\mathscr{L}^{-1})$, so $\widehat{E}_X$ is identified with $\boldsymbol{V}(\mathscr{L}^{-1})$ in view of Corollary 11.8.35 and C_X is identified with $\boldsymbol{V}(\mathscr{L})$. The morphism $\boldsymbol{V}(\mathscr{L}) \to C$ is the restriction of $\mathrm{Proj}(\hat{\alpha})$, and the assertion of the corollary is a particular case of Proposition 11.8.34. □

We note that the inverse image of the sommet scheme of C under the morphism $\boldsymbol{V}(\mathscr{O}_X(1)) \to C$ is the zero section of $\boldsymbol{V}(\mathscr{O}_X(1))$ (Corollary 11.8.33). But in general the corresponding subschemes of C and of $\boldsymbol{V}(\mathscr{O}_X(1))$ are not isomorphic.

11.8.6 Blow up of projective cones

With the notations of the previous subsection, we have a commutative diagram

$$\begin{tikzcd} \widehat{C}_X \arrow[r, "r"] & \widehat{C} \\ X \arrow[u, "i_X \circ \varepsilon_X"] \arrow[r, "q"] & Y \arrow[u, "i \circ \varepsilon"'] \end{tikzcd}$$

where $r = \mathrm{Proj}(\hat{\alpha})$. Moreover, the restriction of r to the complement $\widehat{C}_X - i_X(\varepsilon_X(X))$ of the sommet section is an isomorphism to $\widehat{C} - i(\varepsilon(Y))$ of the sommet section in view of Proposition 11.8.34. If we suppose for simplicity that Y is affine, $\mathscr{S}$ is of finite type and generated by $\mathscr{S}_1$, X is projective over Y and $\widehat{C}_X$ is projective over X, so $\widehat{C}_X$ is projective over Y (Proposition 11.5.34(ii)), and a fortiori over $\widehat{C}$ (Proposition 11.5.34(v)). We thus have a projective Y-morphism $r : \widehat{C}_X \to \widehat{C}$ (hence restricts to a projective Y-morphism $C_X \to C$) which contract X to Y and induces an isomorphism when restricted to the complement of X and of Y. We therefore have a relation between C_X and C, analogous to that which takes place between a blow up scheme and its initial scheme. We will effectively show that we can identify C_X with the homogeneous spectrum of a graded $\mathscr{O}_C$-algebra.

For each $n \geq 0$, we consider the quasi-coherent ideal

$$\mathscr{S}(n) = \bigoplus_{m \geq n} \mathscr{S}_m$$

of the graded $\mathscr{O}_Y$-algebra of $\mathscr{S}$. It is clear that $(\mathscr{S}(n))_{n \geq 0}$ is a filtered sequence of ideals of $\mathscr{S}$. Consider the $\mathscr{O}_C$-module associated with $\mathscr{S}(n)$, which is a quasi-coherent ideal of $\mathscr{O}_C = \widetilde{\mathscr{S}}$:

$$\mathscr{I}_n = \widetilde{\mathscr{S}(n)}.$$

Then $(\mathscr{I}_n)$ is also a filtered sequence of quasi-coherent $\mathscr{O}_C$-ideals, so we can consider the quasi-coherent graded $\mathscr{O}_C$-algebra

$$\mathscr{S}^\natural = \bigoplus_{n \geq 0} \mathscr{I}_n = \Big(\bigoplus_{n \geq 0} \mathscr{S}(n)\Big)^{\sim}.$$

Proposition 11.8.37. *There exists a canonical C-isomorphism*

$$h : C_X \to \mathrm{Proj}(\mathscr{S}^\natural). \tag{11.8.40}$$

Proof. Suppose first that $Y = \mathrm{Spec}(A)$ is affine, so $\mathscr{S} = \widetilde{S}$ where S is a graded A-algebra with positive degrees and $C = \mathrm{Spec}(S)$. We then have $\mathscr{S}^\natural = \widetilde{(S^\natural)}$. To define the morphism h, consider an element $f \in S_d$ ($d > 0$) and the corresponding element $f^\natural \in S^\natural$; the S-isomorphism (11.8.15) defines a C-isomorphism

$$\mathrm{Spec}(S_f^{\geqslant}) \xrightarrow{\sim} \mathrm{Spec}(S^\natural_{(f^\natural)}). \tag{11.8.41}$$

But with the notations of Proposition 11.8.34, if $\varphi : C_X \to X$ is the structural morphism, it follows from (11.8.39) that $\varphi^{-1}(X_f) = \mathrm{Spec}(S_f^{\geqslant})$. We have on the other hand $D_+(f^\natural) = \mathrm{Spec}(S^\natural_{(f^\natural)})$, so that (11.8.41) define an isomorphism $v^{-1}(X_f) \to D_+(f^\natural)$. Moreover, if $g \in S_e$ with $e > 0$, the diagram

$$\begin{tikzcd} \varphi^{-1}(X_{fg}) \arrow[r, "\sim"] \arrow[d] & D_+(f^\natural g^\natural) \arrow[d] \\ \varphi^{-1}(X_f) \arrow[r, "\sim"] & D_+(f^\natural) \end{tikzcd}$$

is commutative, which is clear from the definition of (11.8.15). By definition S_+ is generated by these homogeneous elements F, so it follows from Lemma 11.8.15(iv) that the $D_+(f^\natural)$ form a covering of $\mathrm{Proj}(S^\natural)$ and the $\varphi^{-1}(X_f)$ form a covering of C_X, if X_f form a covering of X. These together gives a isomorphism $h : C_X \to \mathrm{Proj}(\mathscr{S}^\natural)$.

To prove the proposition in the general case, it suffices to see that if U, U' are two affine opens of Y such that $U' \subseteq U$, with rings A and A', and if $\mathscr{S}|_U = \widetilde{S}$, $\mathscr{S}|_{U'} = \widetilde{S}'$, the diagram

$$\begin{tikzcd} C_{U'} \arrow[r] \arrow[d] & \mathrm{Proj}(S^\natural) \arrow[d] \\ C_U \arrow[r] & \mathrm{Proj}(S^\natural) \end{tikzcd} \tag{11.8.42}$$

is commutative. But S' is canonically identified with $S \otimes_A A'$, so $S'^\natural$ is identified with $S^\natural \otimes_S S' = S^\natural \otimes_A A'$ and we then have $\mathrm{Proj}(S'^\natural) = \mathrm{Proj}(S^\natural) \times_U U'$ (Proposition 11.2.50). Similarly, if $X = \mathrm{Proj}(S)$ and $X' = \mathrm{Proj}(S')$, we have $X' = X \times_U U'$ and $\mathscr{S}_{X'} = \mathscr{S}_X \otimes_{\mathscr{O}_U} \mathscr{O}_{U'}$ (Corollary 11.3.32), which means $\mathscr{S}_{X'} = j^*(\mathscr{S}_X)$, where $j : X' \to X$ is the projection. By Corollary 11.1.30 we then have $C_{U'} = C_U \times_X X' = C_U \times_U U'$, and the commutativity of (11.8.42) is immediate. □

Remark 11.8.38. The end of the reasoning of Proposition 11.8.37 is immediately generalized in the following way.: let $g : Y' \to Y$ be a morphism, $\mathscr{S}' = g^*(\mathscr{S})$, $X' = \mathrm{Proj}(\mathscr{S}')$; we then have a commutative diagram

$$\begin{tikzcd} C_{X'} \arrow[r] \arrow[d] & \mathrm{Proj}(\mathscr{S}'^\natural) \arrow[d] \\ C_X \arrow[r] & \mathrm{Proj}(\mathscr{S}^\natural) \end{tikzcd} \tag{11.8.43}$$

On the other hand, let $\varphi : \mathscr{S}'' \to \mathscr{S}$ be a homomorphism of graded $\mathscr{O}_Y$-algebras such that the induced morphism $\Phi = \mathrm{Proj}(\varphi) : X \to X''$ is everywhere defined, where $X'' = \mathrm{Proj}(\mathscr{S}'')$. We have an Y-morphism $v : C \to C''$ (where $C'' = \mathrm{Spec}(\mathscr{S}''))$ such that $\mathscr{A}(v) = \varphi$, and as φ is a graded homomorphism, we deduce from φ a v-morphism $\psi : \mathscr{S}''^\natural \to \mathscr{S}^\natural$ (Proposition 11.1.18). Moreover, it follows from Lemma 11.8.15(iv) and the hypotheses on φ that $\Psi = \mathrm{Proj}(\psi)$ is everywhere defined. Finally, in view of (11.3.9), we have a canonical Φ-morphism $\mathscr{S}_{X''} \to \mathscr{S}_X$, whence a morphism $w : C_{X''} \to C_X$. The diagram

$$\begin{tikzcd} C_{X''} \arrow[r, "\sim"] \arrow[d, "w"] & \mathrm{Proj}(\mathscr{S}^\natural) \arrow[d, "\Psi"] \\ C_X \arrow[r, "\sim"] & \mathrm{Proj}(\mathscr{S}^\natural) \end{tikzcd}$$

is commutative, as can be verified in the case where Y is affine.

Remark 11.8.39. Recall that $(\mathscr{I}_n)_{n\geq 0}$ is a filtered sequence where $\mathscr{I}_n = \mathscr{S}(n)$, so we have $\mathscr{I}_1^n \subseteq \mathscr{I}_n \subseteq \mathscr{I}_1$ for any $n > 0$. Now by definition, $\mathscr{I}_1 = \widetilde{\mathscr{S}_+}$, so $\mathscr{I}_1$ defines in C the closed subscheme $\varepsilon(Y)$ (Proposition 11.1.27 and Proposition 11.8.21). We then conclude that for any $n > 0$, the support of $\mathscr{O}_C/\mathscr{I}_n$ is contained in the underlying space of tge sommet scheme $\varepsilon(Y)$. In the inverse image of the blunt affine cone E, the structural morphism $\mathrm{Proj}(\mathscr{S}^\natural) \to C$ reduces to an isomorphism (as it follows from Proposition 11.8.37 and Proposition 11.8.34). Moreover, if we canonically identify C as a dense open subset of $\widehat{C}$, we can evidently extend the ideal $\mathscr{I}_n$ of $\mathscr{O}_C$ to an ideal $\mathscr{J}_n$ of $\mathscr{O}_{\widehat{C}}$, such that it coincides with $\mathscr{O}_{\widehat{C}}$ on the open subset $\widehat{E}$ of $\widehat{C}$. If we put $\mathscr{T} = \bigoplus_{n\geq 0} \mathscr{J}_n$, which is a graded $\mathscr{O}_{\widehat{C}}$-algebra, we can then extend the isomorphism (11.8.40) into a $\widehat{C}$-isomorphism

$$\widehat{C}_X \xrightarrow{\sim} \mathrm{Proj}(\mathscr{T}). \tag{11.8.44}$$

In fact, over $\widehat{E}$, it follows from the definition of $\mathscr{J}$ that $\operatorname{Proj}(\mathscr{T})$ is identified with $\widehat{E}$, and we therefore define the isomorphism (11.8.44) so that it coincides with the canonical isomorphism $\widehat{E}_X \to \widehat{E}$ on $\widehat{E}$ (Proposition 11.8.34); it is then clear that this isomorphism and (11.8.40) coincides over $\widehat{E}$.

Corollary 11.8.40. *Suppose that there exists $n_0 > 0$ such that $\mathscr{S}_{n+1} = \mathscr{S}_1\mathscr{S}_n$ for $n \geq n_0$. Then the sommet scheme of C_X (isomorphic to X) is the inverse image of the sommet scheme of C (isomorphic to Y) under the canonical morphism $r = \operatorname{Proj}(\alpha) : C_X \to C$. The converse of this is true if moreover Y is Noetherian and $\mathscr{S}$ is of finite type.*

Proof. The first assertion is local over Y, so we can assume that $Y = \operatorname{Spec}(A)$ is affine, $\mathscr{S} = \widetilde{S}$ where S is a graded A-algebra with positive degrees. This then follows from Proposition 11.8.16, because we have

$$\operatorname{Proj}(S^\natural \otimes_S S_0) = \operatorname{Proj}(S^\natural \otimes_S (S/S_+)) = C_X \times_C \varepsilon(Y)$$

(in view of the identification (11.8.40) and Proposition 11.2.50), which is also the inverse image of $\varepsilon(Y)$ in C_X under the morphism $r : C_X \to C$. The converse of this also follows from Proposition 11.8.16 if Y is Noetherian and affine and S is of finite type. If Y is Noetherian (not necessarily affine) and $\mathscr{S}$ is of finite type, there exists a finite convering of Y by Noetherian affine covers U_i, and we then deduce that for each i, there is an integer n_i such that $\mathscr{S}_{n+1}|_{U_i} = (\mathscr{S}_1|_U)(\mathscr{S}_n|_U)$ for $n \geq n_i$; the largest integer n_0 of the n_i then satisfies the requirement. ◻

We now consider the C-scheme $\widetilde{C}$ with is obtained by blowing up the affine cone C along the sommet scheme $\varepsilon(Y)$. By definiton this is $\operatorname{Proj}(\bigoplus_{n\geq 0}(\mathscr{S}_+)^n)$; the canonical injection

$$\iota : \bigoplus_{n\geq 0}(\mathscr{S}_+)^n \to \mathscr{S}^\natural$$

defines (by the identification of (11.8.40)) a canonical dominant C-morphism $\operatorname{Proj}(\iota) : G(\iota) \to \widetilde{C}$, where $G(\iota)$ is an open subset of C_X. We note that it is possible that $G(\iota) \neq C_X$; for example, $Y = \operatorname{Spec}(k)$ where k is a field, $\mathscr{S} = \widetilde{S}$ where $S = k[\boldsymbol{y}]$ and $\boldsymbol{y}$ is an indeterminate of degree 2. If R_n is the set $(S_+)^n$, considered as a subset of $S(n) = S_n^\natural$, then $S_+^\natural$ is not equal to the radical in $S_+^\natural$ of the ideal generated by the R_n.

Corollary 11.8.41. *Suppose that there exists $n_0 > 0$ such that $\mathscr{S}_n = \mathscr{S}_1^n$ for $n \geq n_0$. Then the canonical morphism $\operatorname{Proj}(\iota) : G(\iota) \to \widetilde{C}$ is everywhere defined and an isomorphism from C_X to $\widetilde{C}$. The converse of this is also true if moreover Y is Noetherian and $\mathscr{S}$ is of finite type.*

Proof. This assertion is local over Y, and therefore follows from Proposition 11.8.17. The converse of this is also true if Y is Noetherian and $\mathscr{S}$ is of finite type, as can be shown similarly to Corollary 11.8.40. ◻

Remark 11.8.42. As the condition of Corollary 11.8.41 implies that of Corollary 11.8.40, we see that if this condition is verified, not only C_X is identified with the scheme obtained by blowing up C along the sommet scheme (isomorphic to Y), but also the sommet scheme of C_X (isomorphic to X) is identified with the inverse image of the sommet scheme of C in C_X. Moreover, the hypothesis of Corollary 11.8.41 implies that over $X = \operatorname{Proj}(\mathscr{S})$, the $\mathscr{O}_X$-modules $\mathscr{O}_X(n)$ are invertible (Proposition 11.3.14) and we have $\mathscr{O}_X(n) = \mathscr{L}^{\otimes n}$, where $\mathscr{L} = \mathscr{O}_X(1)$ (Corollary 11.3.16). By definition C_X is then the vector bundle $\boldsymbol{V}(\mathscr{L})$ over X, and the sommet scheme is the zero section of this vector bundle.

11.8.7 Ample sheaves and contractions

Let Y be a scheme, $f : X \to Y$ be a quasi-compact and quasi-separated morphism $\mathscr{L}$ be an ample inveritble $\mathscr{O}_X$-module realtive to f. Consider the graded $\mathscr{O}_Y$-algebra with positive degrees

$$\mathscr{S} = \mathscr{O}_Y \oplus \bigoplus_{n\geq 1} f_*(\mathscr{L}^{\otimes n})$$

which is quasi-coherent by Proposition 10.6.55. We have a canonical homomorphism of graded $\mathscr{O}_X$-algebras

$$\sigma : f^*(\mathscr{S}) \to \bigoplus_{n\geq 0} \mathscr{L}^{\otimes n}$$

which, for each $n \geq 1$, coincides with the canonical homomorphism $\sigma_n : f^*(f_*(\mathscr{L}^{\otimes n})) \to \mathscr{L}^{\otimes n}$, and for $n = 0$ is the identify on $\mathscr{O}_X$. The hypothesis that $\mathscr{L}$ is f-ample implies that (Proposition 11.4.39 and Proposition 11.3.24) the corresponding Y-morphism

$$r = r_{\mathscr{L},\sigma} : X \to P = \mathrm{Proj}(\mathscr{S})$$

is everywhere defined and a dominant open immersion, and we have $\mathscr{L}^{\otimes n} = r^*(\mathscr{O}_P(n))$ for $n \in \mathbb{Z}$.

Proposition 11.8.43. *Let $C = \mathrm{Spec}(\mathscr{S})$ the affine cone defined by $\mathscr{S}$. If $\mathscr{L}$ is f-ample, there exists a canonical Y-morphism*

$$g : V = \mathbf{V}(\mathscr{L}) \to C \tag{11.8.45}$$

such that the diagram

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[d, "f"'] & \mathbf{V}(\mathscr{L}) \arrow[r, "\pi"] \arrow[d, "g"] & X \arrow[d, "f"] \\ Y \arrow[r, "\varepsilon"] & C \arrow[r, "\psi"] & Y \end{tikzcd} \tag{11.8.46}$$

is commutative, where ψ and π are structural morphisms, j and ε are the canonical immersions which maps X and Y respectively to the zero section of $\mathbf{V}(\mathscr{L})$ and the sommet scheme of C. Moreover, the restriction of g to $\mathbf{V}(\mathscr{L}) - j(X)$ is an open immersion

$$\mathbf{V}(\mathscr{L}) - j(X) \to E = C - \varepsilon(Y)$$

into the blunt affine cone E corresponding to $\mathscr{S}$.

11.8.8 Quasi-coherent sheaves over the projective cone

Let Y be a scheme, $\mathscr{S}$ be a quasi-coherent graded $\mathscr{O}_Y$-algebra, $X = \mathrm{Proj}(\mathscr{S})$, $C = \mathrm{Spec}(\mathscr{S})$ and $\widehat{C} = \mathrm{Proj}(\widehat{\mathscr{S}})$. Let $\mathscr{M}$ be a quasi-coherent graded $\mathscr{S}$-module; to avoid any possible confusion, we denote by $\widetilde{\mathscr{M}}$ the quasi-coherent $\mathscr{O}_C$-module associated with $\mathscr{M}$ if $\mathscr{M}$ is considered as a *nongraded* $\mathscr{S}$-module, and by $\mathscr{P}roj_0(\mathscr{M})$ the quasi-coherent $\mathscr{O}_X$-module associated with $\mathscr{M}$, where $\mathscr{M}$ is considered as a graded $\mathscr{S}$-module. We also set

$$\mathscr{M}_X = \mathscr{P}roj(\mathscr{M}) = \bigoplus_{n\in\mathbb{Z}} \mathscr{P}roj_0(\mathscr{M}(n));$$

with the quasi-coherent $\mathscr{O}_X$-algebra being defined by (11.8.38), $\mathscr{P}roj(\mathscr{M})$ is endowed a quasi-coherent graded $\mathscr{S}_X$-module structure, via the canonical homomorphisms

$$\mathscr{O}_X(m) \otimes_{\mathscr{O}_X} \mathscr{P}roj_0(\mathscr{M}(n)) \to \mathscr{P}roj_0(\mathscr{S}(m) \otimes_{\mathscr{S}} \mathscr{M}(n)) \to \mathrm{Proj}_0(\mathscr{M}(m+n))$$

which satisfies the axioms of modules in view of the commutative diagram (11.2.5). If $Y = \mathrm{Spec}(A)$ is affine, $\mathscr{S} = \widetilde{S}$ and $\mathscr{M} = \widetilde{M}$, where S is a graded A-algebra and M is a graded S-module, then for any homogeneous element $f \in S_+$, we have

$$\Gamma(X_f, \mathscr{P}roj(\mathscr{M})) = M_f$$

in view of the definition and (11.8.16).

Now consider the quasi-coherent graded $\widehat{\mathscr{S}}$-module

$$\widehat{\mathscr{M}} = \mathscr{M} \otimes_{\mathscr{S}} \widehat{\mathscr{S}}$$

(where $\widehat{\mathscr{S}} = \mathscr{S}[T] = \mathscr{S} \otimes_{\mathbb{Z}} \mathbb{Z}[T]$); we then deduce a quasi-coherent graded $\mathscr{O}_{\widehat{C}}$-module $\mathscr{P}roj_0(\widehat{\mathscr{M}})$ (recall that $\widehat{C} = \mathrm{Proj}(\widehat{\mathscr{S}})$), which we also denote by

$$\mathscr{M}^{\square} = \mathscr{P}roj_0(\widehat{\mathscr{M}}).$$

It is clear that $\mathscr{M}^{\square}$ is an exact functor on $\mathscr{M}$ and commutes with inductive limits and direct sums.

Proposition 11.8.44. *With the notations of Proposition 11.8.21 and Proposition 11.8.22, we have canonical homomorphisms*

$$i^*(\mathscr{M}^{\square}) \xrightarrow{\sim} \widetilde{\mathscr{M}}, \tag{11.8.47}$$

$$j^*(\mathscr{M}^{\square}) \to \mathscr{P}roj_0(\mathscr{M}), \tag{11.8.48}$$

$$p^*(\mathscr{P}roj_0(\mathscr{M})) \to \mathscr{M}^{\square}|_{\widehat{E}} \tag{11.8.49}$$

Moreover, the homomorphism (11.8.48) is an isomorphism if $\mathscr{S}$ is generated by $\mathscr{S}_1$.

Proof. In fact, $i^*(\mathscr{M}^{\square})$ is canonically identified with $(\widehat{\mathscr{M}}/(z-1)\widehat{\mathscr{M}})^{\sim}$ over $\mathrm{Spec}(\widehat{\mathscr{S}}/(z-1)\widehat{\mathscr{S}})$ in view of Proposition 11.3.12 and the definition of i. The first isomorphism of (11.8.47) is then deduced from Proposition 11.1.18 and the canonical isomorphism $\widehat{\mathscr{M}}/(z-1)\widehat{\mathscr{M}} \cong \mathscr{M}$. On the other hand, the canonical immersion $j : X \to \widehat{C}$ corresponds to the canonical homomorphism $\widehat{\mathscr{S}} \to \mathscr{S}$ with kernel $z\widehat{\mathscr{S}}$, so the second isomorphism is a particular case of the canonical homomorphism of Proposition 11.3.30, using the fact that $\widehat{\mathscr{M}} \otimes_{\widehat{\mathscr{S}}} \mathscr{S} = \mathscr{M}$. Finally, the homomorphism of (11.8.49) is a particular case of the homomorphisms $\nu^{\sharp}$ defined in (11.3.8). If $Y = \mathrm{Spec}(A)$ is affine, $\mathscr{S} = \widetilde{S}$, $\mathscr{M} = \widetilde{M}$, then we see from Proposition 11.2.48 that the restriction of (11.8.49) to $p^{-1}(X_f) = \widehat{C}_f$ (for $f \in S_+$ homogeneous) corresponds to the canonical homomorphism

$$M_{(f)} \otimes_{S_{(f)}} S_f^{\leqslant} \to M_f^{\leqslant}$$

in view of (11.8.10) and (11.8.12). The last assertion also follows from Proposition 11.3.30. $\square$

By abuse of language, we say that $\mathscr{M}^{\square}$ is the projective closure of the $\mathscr{O}_C$-module $\widetilde{\mathscr{M}}$, where $\mathscr{M}$ is understood to be a graded $\mathscr{S}$-module.

Let us consider a morphism $q : Y' \to Y$ and a q-homomorphism $\varphi : \mathscr{S} \to \mathscr{S}'$. It then follows from (11.1.3) that for any quasi-coherent graded $\mathscr{S}$-module $\mathscr{M}$, we have a canonical isomorphism

$$\Phi^*(\widetilde{\mathscr{M}}) \xrightarrow{\sim} (q^*(\mathscr{M}) \otimes_{q^*(\mathscr{S})} \mathscr{S}')^{\sim}$$

of $\mathscr{O}_C$-modules, where $\Phi = \mathrm{Spec}(\varphi)$. On the other hand, if $w = \mathrm{Proj}(\varphi)$ and $\hat{\Phi} = \mathrm{Proj}(\hat{\varphi})$, (11.3.7) gives a canonical w-homomorphism

$$\mathscr{P}roj_0(\mathscr{M}) \to (\mathscr{P}roj_0(q^*(\mathscr{M}) \otimes_{q^*(\mathscr{S})} \mathscr{S}'))|_{G(\varphi)} \tag{11.8.50}$$

and also a canonical $\hat{\Phi}$-morphism

$$\mathscr{P}roj_0(\widehat{\mathscr{M}}) \to (\mathscr{P}roj_0(q^*(\widehat{\mathscr{M}}) \otimes_{q^*(\widehat{\mathscr{S}})} \widehat{\mathscr{S}}'))|_{G(\hat{\varphi})}. \tag{11.8.51}$$

Now we consider the situation of the structural morphism $q : X \to Y$, where $X = \mathrm{Proj}(\mathscr{S})$, with the canonical q-homomorphism $\alpha : \mathscr{S} \to \mathscr{S}_X^{\geqslant}$. We then have a canonical isomorphism

$$q^*(\mathscr{M}) \otimes_{q^*(\mathscr{S})} \mathscr{S}_X^{\geqslant} \xrightarrow{\sim} \mathscr{M}_X^{\geqslant} \tag{11.8.52}$$

where $\mathscr{M}_X^{\geqslant} = \bigoplus_{n\geq 0} \mathscr{P}roj_0(\mathscr{M}(n))$. To see this, we can assume that $Y = \mathrm{Spec}(A)$ is affine, $\mathscr{S} = \widetilde{S}$ and $\mathscr{M} = \widetilde{M}$, and define the isomorphism (11.8.52) in each affine open X_f (f is homogeneous in S_+), and verify the compatibility when passing to a homogeneous multiple of f. Now, the restriction of the left side of (11.8.52) to X_f is $\widetilde{M}' = ((M \otimes_A S_{(f)}) \otimes_{S \otimes_A S_{(f)}} S_f^{\geqslant})^\sim$ by (11.8.39). As we have a canonical isomorphism $M \otimes_A S_{(f)} \cong M \otimes_S (S \otimes_A S_{(f)})$, we conclude that $\widetilde{M}' \cong (M \otimes_S S_f^{\geqslant})^\sim$, and this is canonically isomorphic to the restriction of $\mathscr{M}_X^{\geqslant}$ by (11.8.13). The compatibility of this isomorphism with restrictions is clear.

By replace $\mathscr{M}$ by $\widehat{\mathscr{M}}$, $\mathscr{S}$ by $\widehat{\mathscr{S}}$ and $\mathscr{S}_X$ by $(\mathscr{S}_X^{\geqslant})^\wedge$ in the preceding arguments, we obtain similarly a canonical isomorphism

$$q^*(\widehat{\mathscr{M}}) \otimes_{q^*(\widehat{\mathscr{S}})} (\mathscr{S}_X^{\geqslant})^\wedge \xrightarrow{\sim} (\mathscr{M}_X^{\geqslant})^\wedge \tag{11.8.53}$$

If we recall Proposition 11.8.34 that the structural morphism $\psi : \mathrm{Proj}(\mathscr{S}_X^{\geqslant}) \to X$ is an isomorphism, we then deduce a canonical ψ-isomorphism

$$\mathscr{P}roj_0(\mathscr{M}) \xrightarrow{\sim} \mathscr{P}roj_0(\mathscr{M}_X^{\geqslant}) \tag{11.8.54}$$

as a particular case of (11.8.50). In fact, we observe that, in the notations of Proposition 11.8.34, that this reduces to the fact that canonical homomorphism $M_{(f)} \otimes_{S_{(f)}} (S_f^{\geqslant})^{(d)} \to (M_f^{\geqslant})^{(d)}$ is an isomorphism if $f \in S_d$ is homogeneous, which is immediate.

The isomorphism (11.8.53) permits us, by apply (11.8.51) to the canonical morphism $r = \mathrm{Proj}(\hat{\alpha}) : \widehat{C}_X \to \widehat{C}$, to obtain a canonical r-homomorphism

$$\mathscr{M}^\square \to (\mathscr{M}_X^{\geqslant})^\square. \tag{11.8.55}$$

Now recall that the restrictions of r to the blunt cones $\widehat{E}_X$ and E_X are isomorphisms onto $\widehat{E}$ and E, respectively.

Proposition 11.8.45. *The restriction of the canonical r-homomorphism (11.8.55) to $\widehat{E}_X$ and to E_X are isomorphisms*

$$\mathscr{M}^\square|_{\widehat{E}} \xrightarrow{\sim} (\mathscr{M}_X^{\geqslant})^\square|_{\widehat{E}_X}, \tag{11.8.56}$$
$$\mathscr{M}^\square|_{E} \xrightarrow{\sim} (\mathscr{M}_X^{\geqslant})^\square|_{E_X}. \tag{11.8.57}$$

Proof. We can assume that $Y = \mathrm{Spec}(A)$ is affine as in the proof of Proposition 11.8.34; with the notations there, we must show that the canonical homomorphism

$$\widehat{M}_{(f)} \otimes_{\widehat{S}_{(f)}} (S_f^{\geqslant})\widehat{\ }_{(f/1)} \to (M \otimes_S S_f^{\geqslant})\widehat{\ }_{(f/1)}$$

is an isomorphism. But in view of (11.8.10) and (11.8.12), the left side is canonically identified with $M_f^{\geqslant} \otimes_{S_f^{\geqslant}} (S_f^{\geqslant})_{f/1}^{\leqslant}$, hence with $M_f^{\geqslant}$ in view of (11.8.14); the right side is identified with $(M_f^{\geqslant})_{f/1}^{\leqslant}$, hence also to $M_f^{\geqslant}$ by (11.8.17), whence our assertion about (11.8.56). The isomorphism (11.8.57) then follows from (11.8.56) and (11.8.47). ∎

Corollary 11.8.46. *With the notations of Corollary 11.8.35, the restriction of $(\mathscr{M}_X^{\geqslant})^{\square}$ to $\widehat{E}_X$ is identified with $\widetilde{\mathscr{M}_X^{\leqslant}}$ and its restriction to E_X is identified with $\widetilde{\mathscr{M}_X}$.*

Proof. We can clearly reduce to the affine case, and this follows from the idnetification of $(M_f^{\geqslant})_{f/1}^{\leqslant}$ with $M_f^{\leqslant}$ and $(M_f^{\geqslant})_{f/1}$ with M_f (cf. (11.8.17)). $\square$

Proposition 11.8.47. *Under the hypotheses of Corollary 11.8.36, the canonical homomorphism (11.8.49) is an isomorphism.*

Proof. In view of the fact that the structural morphism $\mathrm{Proj}(\mathscr{S}_X^{\geqslant}) \to X$ is an isomorphism and the isomorphisms (11.8.54) and (11.8.56), we only need to prove that the canonical homomorphism $p_X^*(\mathscr{P}roj_0(\mathscr{M}_X^{\geqslant})) \to (\mathscr{M}_X^{\geqslant})^{\square}|_{E_X}$ is an isomorphism, which means that we can assume that $\mathscr{S}_1$ is an invertible $\mathscr{O}_Y$-module and $\mathscr{S}$ is generated by $\mathscr{S}_1$. With the notations of Proposition 11.8.44, we then have, for $f \in S_1$, $S_f^{\leqslant} = S_{(f)}[1/f]$ and the canonical isomorphism $M_{(f)} \otimes_{S_{(f)}} S_f^{\leqslant} \to M_f^{\leqslant}$ is an isomorphism by the definition of $M_f^{\geqslant}$. $\square$

We now consider the quasi-coherent $\mathscr{S}$-module $\mathscr{M}(n) = \bigoplus_{m \geq n} \mathscr{M}_m$ and the quasi-coherent graded $\mathscr{S}^{\natural}$-module

$$\mathscr{M}^{\natural} = \widetilde{\Big(\bigoplus_{n \geq 0} \mathscr{M}(n)\Big)}.$$

By Proposition 11.8.37 we have a canonical C-isomorphism $h : C_X \xrightarrow{\sim} \mathrm{Proj}(\mathscr{S}^{\natural})$.

Proposition 11.8.48. *There exists a canonical h-isomorphism*

$$\mathscr{P}roj_0(\mathscr{M}^{\natural}) \xrightarrow{\sim} \widetilde{\mathscr{M}_X}. \tag{11.8.58}$$

Proof. This can be proved as Proposition 11.8.37, by using the bi-isomorphism (11.8.18) here instead of (11.8.15). $\square$

Chapter 12

Cohomology of coherent sheaves over schemes

12.1 Cohomology of affine schemes

12.1.1 Čech cohomology and Koszul complex

Let X be a quasi-compact and quasi-separated scheme and $\mathscr{F}$ be a quasi-coherent $\mathscr{O}_X$-module. Let $A = \Gamma(X, \mathscr{O}_X)$, $M = \Gamma(X, \mathscr{F})$, $\boldsymbol{f} = (f_i)_{1 \le i \le r}$ be a family of elements of A, and $U_i = X_{f_i}$ be the open subset of X. Let $U = \bigcup_{i=1}^{r} U_i$ and $\mathfrak{U} = (U_i)$ be the covering of U. For any sequence $(i_0, i_1, \dots, i_p) \in I^{p+1}$ with $I = \{1, \dots, r\}$, we set

$$U_{i_0 \dots i_p} = \bigcap_{j=0}^{p} U_{i_j} = X_{f_{i_0} \cdots f_{i_p}}.$$

By Corollary 10.6.15, we have $\Gamma(U_{i_0 \dots i_p}, \mathscr{F}) = M_{f_{i_0} \cdots f_{i_p}}$, where $M = \Gamma(X, \mathscr{F})$. Note that the localization module $M_{f_{i_0} \cdots f_{i_p}}$ is identified with the limit of the inductive system $\{M^n_{i_0 \dots i_p}\}_{n \ge 0}$, where $M^n_{i_0 \dots i_p} = M$ and the homomorphism $\varphi_{nm} : M^m_{i_0 \dots i_p} \to M^n_{i_0 \dots i_p}$ is given by multiplication by $(f_{i_0} \dots f_{i_p})^{n-m}$ for $m \le n$.

For any $n \ge 0$, let $C_n^{p+1}(M)$ be the set of alternating maps from I^{p+1} to M, and consider the inductive system formed by these A-modules and the homomorphisms induced by φ_{nm}. If $C^p(\mathfrak{U}, \mathscr{F})$ is the group of alternating Čech p-cochains relative to the covering $\mathfrak{U}$ with coefficients in $\mathscr{F}$, then we see that

$$C^p(\mathfrak{U}, \mathscr{F}) = \varinjlim_n C_n^{p+1}(M).$$

On the other hand, from the definition of $C_n^{p+1}(M)$ it is easy to see that it is canonically identified with the Koszul complex $K^{p+1}(\boldsymbol{f}^n, M)$, and the homomorphism φ_{nm} is identified with the map

$$\varphi_{\boldsymbol{f}^{n-m}} : K^\bullet(\boldsymbol{f}^n, M) \to K^\bullet(\boldsymbol{f}^m, M)$$

induced by the map $(x_1, \dots, x_r) \mapsto (f_1^{n-m} x_1, \dots, f_r^{n-m} x_r)$ on A^r. We then have, for any $p \ge 0$, a functorial isomorphism

$$C^p(\mathfrak{U}, \mathscr{F}) \xrightarrow{\sim} C^{p+1}(\mathfrak{f}, M), \tag{12.1.1}$$

where $\mathfrak{f}$ is the ideal generated by $\boldsymbol{f}$. Moreover, the definition of the differentials of $C^p(\mathfrak{U}, \mathscr{F})$ and $C_n^p(M)$ show that the isomorphism (12.1.1) is in fact a morphism of complexes.

Proposition 12.1.1. *If X is a quasi-compact and quasi-separated scheme, there exists a canonical functorial isomorphism*

$$H^p(\mathfrak{U},\mathscr{F}) \xrightarrow{\sim} H^{p+1}(\mathfrak{f},M) \quad \text{for} \quad p \geq 1, \tag{12.1.2}$$

where $\mathfrak{f}$ is the ideal generated by $\boldsymbol{f}$. Moreover, we have a functorial exact sequence

$$0 \longrightarrow H^0(\mathfrak{f},M) \longrightarrow M \longrightarrow H^0(\mathfrak{U},\mathscr{F}) \longrightarrow H^1(\mathfrak{f},M) \longrightarrow 0 \tag{12.1.3}$$

Proof. The relation (12.1.2) is in fact a concequence of (12.1.1). On the other hand, we have $C^0(\mathfrak{U},\mathscr{F}) = C^1(\mathfrak{f},M)$, so $H^0(\mathfrak{U},\mathscr{F})$ is identified with a subgroup of 1-cocycles of $C^1(\mathfrak{f},M)$. As $C^0(\mathfrak{f},M) = M$, the exact sequence (12.1.3) follows from the definition of $H^0(\mathfrak{f},M)$ and $H^1(\mathfrak{f},M)$. □

Corollary 12.1.2. *Suppose that the X_{f_i} are quasi-compact and there exists $g_i \in \Gamma(U,\mathscr{F})$ such that $\sum_i g_i(f_i|_U) = 1|_U$. Then for any quasi-coherent $(\mathscr{O}_X|_U)$-module $\mathscr{G}$, we have $H^p(\mathfrak{U},\mathscr{G}) = 0$ for $p > 0$. If moreover $U = X$, then the canonical homomorphism $M \to H^0(\mathfrak{U},\mathscr{F})$ in (12.1.3) is bijective.*

Proof. By hypothesis $U_i = X_{f_i}$ is quasi-compact, so U is quasi-compact, and we can assume that $U = X$. Then the hypothesis implies that $\mathfrak{f} = A$, so by **??** we have $H^p(\mathfrak{f},M) = 0$ for $p \geq 1$, and the corollary follows from (12.1.2) and (12.1.3). □

Remark 12.1.3. Let X be an affine scheme, so that the $U_i = X_{f_i} = D(f_i)$ are affine opens, and so is each $U_{i_0\dots i_p}$ (but U is not necessarily affine). In this case, the functors $\Gamma(X,\mathscr{F})$ and $\Gamma(U_{i_0\dots i_p},\mathscr{F})$ are exact by Theorem 11.5.12. If we have an exact sequence $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ of quasi-coherent $\mathscr{O}_X$-modules, the sequence of complexes

$$0 \longrightarrow C^\bullet(\mathfrak{U},\mathscr{F}') \longrightarrow C^\bullet(\mathfrak{U},\mathscr{F}) \longrightarrow C^\bullet(\mathfrak{U},\mathscr{F}'') \longrightarrow 0$$

is exact, so we obtain a long exact sequence of coholomogy groups

$$\cdots \longrightarrow H^p(\mathfrak{U},\mathscr{F}') \longrightarrow H^p(\mathfrak{U},\mathscr{F}) \longrightarrow H^p(\mathfrak{U},\mathscr{F}'') \xrightarrow{\delta} H^{p+1}(\mathfrak{U},\mathscr{F}') \longrightarrow 0$$

On the other hand, if we put $M' = \Gamma(X,\mathscr{F}')$, $M'' = \Gamma(X,\mathscr{F}'')$, $M = \Gamma(X,\mathscr{F})$, then the sequence $0 \to M' \to M \to M' \to 0$ is exact; as $C^\bullet(\mathfrak{f},M)$ is an exact functor on M, we then get a long exact sequence on cohomology

$$\cdots \longrightarrow H^p(\mathfrak{f},M') \longrightarrow H^p(\mathfrak{f},M) \longrightarrow H^p(\mathfrak{f},M'') \xrightarrow{\delta} H^{p+1}(\mathfrak{f},M') \longrightarrow 0$$

Now as the diagram

$$\begin{tikzcd}
0 \arrow[r] & C^\bullet(\mathfrak{U},\mathscr{F}') \arrow[r] \arrow[d] & C^\bullet(\mathfrak{U},\mathscr{F}) \arrow[r] \arrow[d] & C^\bullet(\mathfrak{U},\mathscr{F}'') \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C^\bullet(\mathfrak{f},M') \arrow[r] & C^\bullet(\mathfrak{f},M) \arrow[r] & C^\bullet(\mathfrak{f},M'') \arrow[r] & 0
\end{tikzcd}$$

is commutative, we conclude that the diagram

$$\begin{tikzcd}
H^p(\mathfrak{U},\mathscr{F}'') \arrow[r, "\delta"] \arrow[d] & H^{p+1}(\mathfrak{U},\mathscr{F}') \arrow[d] \\
H^{p+1}(\mathfrak{f},M'') \arrow[r, "\delta"] & H^{p+2}(\mathfrak{f},M')
\end{tikzcd} \tag{12.1.4}$$

is commutative for any $p > 0$.

12.1.2 Cohomology of affine schemes

Theorem 12.1.4. *Let X be an affine scheme. For any quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, we have $H^p(X,\mathscr{F}) = 0$ for $p > 0$.*

Proof. Let $\mathfrak{U}$ be a finite covering of X by affine opens $X_{f_i} = D(f_i)$ $(1 \le i \le r)$; then the ideal generated by f_i is equal to $A = \Gamma(X,\mathscr{O}_X)$. We then conclude from Corollary 12.1.2 that we have $H^p(\mathfrak{U},\mathscr{F}) = 0$ for any $p > 0$. As affine opens for a basis for X, we then conclude from the definition of the Čech cohomology that $\check{H}^p(X,\mathscr{F}) = 0$ for any $p > 0$. But this is also applicable on X_f for $f \in A$, so $\check{H}^p(X_f,\mathscr{F}) = 0$ for $p > 0$; as $X_f \cap X_g = X_{fg}$, we then conclude from Leray's vanishing theorem that $H^p(X,\mathscr{F}) = 0$ for $p > 0$. □

Corollary 12.1.5. *Let $f : X \to Y$ be an affine morphism. Then for any quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, we have $R^p f_*(\mathscr{F}) = 0$ for $p > 0$.*

Proof. By definition $R^p f_*(\mathscr{F})$ is defined to be the sheaf associated with the presheaf $U \mapsto H^p(f^{-1}(U),\mathscr{F})$, where U runs through open subsets of Y. Now the affine opens U form a basis for Y, and for such U, $f^{-1}(U)$ is affine, so $H^p(f^{-1}(U),\mathscr{F}) = 0$ by Theorem 12.1.4, so we conclude that $R^p f_*(\mathscr{F}) = 0$. □

Corollary 12.1.6. *Let $f : X \to Y$ be an affine morphism. Then for any quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, the canonical homomorphism $H^p(Y, f_*(\mathscr{F})) \to H^p(X,\mathscr{F})$ is bijective for any p.*

Proof. Consider the Leray spectral sequence

$$E_2^{p,q} = (R^p\Gamma \circ R^q f_*)(\mathscr{F}) = H^p(Y, R^q f_*(\mathscr{F})),$$

it follows from Corollary 12.1.5 that $E_2^{p,q} = 0$ for $q > 0$, so this sequence collapses at E_2 page, whence our assertion. □

Corollary 12.1.7. *Let $f : X \to Y$ be an affine morphism and $g : Y \to Z$ be a morphism. Then for any quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, the canonical homomorphism $R^p g_*(f_*(\mathscr{F})) \to R^p(g\circ f)_*(\mathscr{F})$ is bijective for any p.*

Proof. It suffices to remark that, by Corollary 12.1.6, for any affine open W of Z, the canonical homomorphism $H^p(g^{-1}(W), f_*(\mathscr{F})) \to H^p(f^{-1}(g^{-1}(W)),\mathscr{F})$ is bijective; this homomorphism of presheaves then defines a canonical homomorphism $R^p g_*(f_*(\mathscr{F})) \to R^p(g\circ f)_*(\mathscr{F})$ which is bijective. □

12.1.3 Applications to cohomology of schemes

Proposition 12.1.8. *Let X be a separated scheme, $\mathfrak{U} = (U_\alpha)$ be a covering of X by affine opens. For any quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, the cohomology module $H^\bullet(X,\mathscr{F})$ and $H^\bullet(\mathfrak{U},\mathscr{F})$ (over $\Gamma(X,\mathscr{O}_X)$) are canonically isomorphic.*

Proof. In fact, as X is separated, any finite intersection V of open sets in the covering $\mathfrak{U}$ is affine (Proposition 10.5.31), so $H^p(V,\mathscr{F}) = 0$ for $q > 0$ in view of Theorem 12.1.4. The proposition then follows from Leray's vanishing theorem. □

Remark 12.1.9. We note that the conclusion of Proposition 12.1.8 is still valid if the finite intersections of U_α are affine, even if X is not necessarily separated.

Corollary 12.1.10. *Let X be a quasi-compact and separated scheme, $A = \Gamma(X,\mathscr{O}_X)$, $\boldsymbol{f} = (f_i)_{1\le i\le r}$ be a sequence of elements of A such that the X_{f_i} are affine. Then for any quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, we have a canonical functorial isomorphism*

$$H^p(U,\mathscr{F}) \xrightarrow{\sim} H^{p+1}(\mathfrak{f}, M) \quad \textit{for} \quad p > 0 \tag{12.1.5}$$

where $\mathfrak{f}$ is the ideal generated by $\boldsymbol{f}$. Moreover, we have a functorial exact sequence

$$0 \longrightarrow H^0(\mathfrak{f},M) \longrightarrow M \longrightarrow H^0(U,\mathscr{F}) \longrightarrow H^1(\mathfrak{f},M) \longrightarrow 0 \tag{12.1.6}$$

If X is an affine scheme, it then follows from Remark 12.1.3 and Proposition 12.1.8 that for any $q \geq 0$, the diagram

$$\begin{tikzcd} H^p(U,\mathscr{F}'') \arrow[r, "\delta"] \arrow[d] & H^{p+1}(U,\mathscr{F}') \arrow[d] \\ H^{p+1}(\mathfrak{f},M'') \arrow[r, "\delta"] & H^{p+2}(\mathfrak{f},M') \end{tikzcd} \tag{12.1.7}$$

corresponding to an exact sequence $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ of quasi-coherent $\mathscr{O}_X$-modules, is commutative.

Proposition 12.1.11. *Let X be a quasi-compact and separated X scheme, $\mathscr{L}$ be an invertible $\mathscr{O}_X$-module, and consider the graded ring $A_* = \Gamma_*(\mathscr{L})$. Then $H^\bullet(\mathscr{F},\mathscr{L}) = \bigoplus_{n\in\mathbb{Z}} H^\bullet(X,\mathscr{F}\otimes\mathscr{L}^{\otimes n})$ is a graded A_*-module, and for any $f \in (A_*)_n$, we have a canonical isomorphism*

$$H^\bullet(X_f,\mathscr{F}) \xrightarrow{\sim} (H^\bullet(\mathscr{F},\mathscr{L}))_{(f)} \tag{12.1.8}$$

of $(A_*)_{(f)}$-modules.

Proof. As X is quasi-compact and separated, we can compute the cohomology of $\mathscr{O}_X$-module $\mathscr{F}\otimes\mathscr{L}^{\otimes n}$ using the same finite covering $\mathfrak{U} = (U_i)$ by open affine subsets such that $\mathscr{L}|_{U_i}$ is isomorphic to $\mathscr{O}_X|_{U_i}$ for each i (Proposition 12.1.8). Also, since each $U_i \cap X_f$ is open and affine, we can also compute $H^\bullet(X_f,\mathscr{F}\otimes\mathscr{L}^{\otimes n})$ using the covering $\mathfrak{U}|_{X_f} = (U_i\cap X_f)$. Now for any $f \in A_n$, it is immediate that the multiplication by f defines a homomorphism $C^\bullet(\mathfrak{U},\mathscr{F}\otimes\mathscr{L}^{\otimes m}) \to C^\bullet(\mathfrak{U},\mathscr{F}\otimes\mathscr{L}^{\otimes(m+n)})$, whence a homomorphism $H^\bullet(\mathfrak{U},\mathscr{F}\otimes\mathscr{L}^{\otimes m}) \to H^\bullet(\mathfrak{U},\mathscr{F}\otimes\mathscr{L}^{\otimes(m+n)})$. On the other hand, by Corollary 10.6.16 and Corollary 10.1.6 (ii), any $f \in A_n$ gives an isomorphism of $(A_*)_{(f)}$-modules

$$C^\bullet(\mathfrak{U}|_{X_f},\mathscr{F}) \xrightarrow{\sim} (C^\bullet(\mathfrak{U},\bigoplus_{n\in\mathbb{Z}}\mathscr{F}\otimes\mathscr{L}^{\otimes n}))_{(f)}.$$

By passing to cohomologies, we then deduce the desired isomorphism (12.1.8), using the fact that $M \mapsto M_{(f)}$ is an exact functor on the category of graded modules. □

Corollary 12.1.12. *If $A = \Gamma(X,\mathscr{O}_X)$, then for any $f \in A$ we have a canonical isomorphism $H^\bullet(X_f,\mathscr{F}) \xrightarrow{\sim} (H^\bullet(X,\mathscr{F}))_f$ of A_f-module.*

Corollary 12.1.13. *Let X be a quasi-compact and separated scheme and $f \in \Gamma(X,\mathscr{O}_X)$.*

(a) *If X_f is affine, then for any quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$ and any $\xi \in H^i(X,\mathscr{F})$ with $i > 0$, there exists an integer $n > 0$ such that $f^n\xi = 0$.*

(b) *Conversely, suppose that X_f is quasi-compact and for any quasi-coherent ideal $\mathscr{I}$ and any $\zeta \in H^1(X,\mathscr{I})$, there exists $n > 0$ such that $f^n\zeta = 0$. Then X_f is affine.*

Proof. First, if X_f is affine then $H^i(X_f,\mathscr{F}) = 0$ for $i > 0$, so (a) follows from the isomorphism of Corollary 12.1.12. Conversely, in case (b), in view of Serre's criterion, it suffices to prove that for any quasi-coherent ideal $\mathscr{K}$ of $\mathscr{O}_X|_{X_f}$, we have $H^1(X_f,\mathscr{K}) = 0$. Now as X_f is a quasi-compact open subset of the quasi-compact scheme X, by Theorem 10.6.63 there exists a quasi-coherent ideal $\mathscr{I}$ of $\mathscr{O}_X$ such that $\mathscr{K} = \mathscr{I}|_{X_f}$. By Corollary 12.1.12 we have $H^1(X_f,\mathscr{K}) = (H^1(X,\mathscr{I}))_f$, and the hypothesis then implies our claim. □

Lemma 12.1.14 (Induction principle). *Let X be a quasi-compact and quasi-separated scheme and $\mathcal{P}$ be a property for quasi-compact open subsets of X. Assume that the following conditions are satisfied:*

(a) *$\mathcal{P}$ holds for affine opens of X,*

(b) *if U is a quasi-compact open subset of X, V is an affine open of X, and $\mathcal{P}$ holds for U, V, $U \cap V$, then $\mathcal{P}$ holds for $U \cup V$.*

Then $\mathcal{P}$ holds for every quasi-compact open subset of X, and in particular holds for X.

Proof. We first prove that $\mathcal{P}$ holds for separated quasi-compact open subset $W \subseteq X$. For this, note that W can be written as a union $W = U_1 \cup \cdots \cup U_n$ of affine opens and we can applying induction on n with $U = U_1 \cup \cdots \cup U_n$ and $V = U_n$. This is allowed because $U \cap V = (U_1 \cap U_n) \cup \cdots \cup (U_{n-1} \cap U_n)$ is again a union of $n-1$ affine open subschemes. Now for any quasi-compact open subset $W \subseteq X$, we can induct on the number of affine opens needed to cover W using the same trick as before and using that the quasi-compact open $U_i \cap U_n$ is separated as an open subscheme of the affine scheme U_n. □

Proposition 12.1.15. *Let $f : X \to Y$ be a quasi-compact and quasi-separated morphism. Then for any quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, $R^p f_*(\mathscr{F})$ is quasi-coherent for $p \geq 0$.*

Proof. Since taking restriction commutes with higher direct images, we may assume that Y is affine. Then X is quasi-compact and quasi-separated. For a quasi-compact open subset $U \subseteq X$ and $f_U = f|_U$, we let $\mathcal{P}(U)$ be the property that $R^p(f_U)_*(\mathscr{F})$ is quasi-coherent for all quasi-coherent modules $\mathscr{F}$ on U and $o \geq 0$. It then suffices to prove that the conditions of Lemma 12.1.14 hold. If U is affine, then the morphism f_U is affine, so by Corollary 11.5.15 we have $R^p(f_U)_*(\mathscr{F}) = 0$ for $p > 0$, and $f_*(\mathscr{F})$ is quasi-coherent by Proposition 10.6.55. Now let $U \subseteq X$ be a quasi-compact open subset, $V \subseteq X$ be an affine open subset, and assume that property $\mathcal{P}$ holds for U, V and $U \cap V$. Then for any quasi-coherent $\mathscr{O}_X|_{U \cup V}$-module $\mathscr{F}$, we have the relative Mayer-Vietoris sequence

$$0 \to (f_{U\cup V})_*(\mathscr{F}) \to (f_U)_*(\mathscr{F}|_U) \oplus (f_V)_*(\mathscr{F}|_V) \to (f_{U\cap V})_*(\mathscr{F}|_{U\cap V}) \to R^1(f_{U\cup V})_*(\mathscr{F}) \to \cdots$$

It then follows from our assumption and Corollary 10.1.6 that $R^p(f_{U\cup V})_*(\mathscr{F})$ is quasi-coherent for $p \geq 0$, so the assertion follows by applying Lemma 12.1.14. □

Corollary 12.1.16. *Let $f : X \to Y$ be a quasi-compact and quasi-separated morphism. Then for any quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$ and any affine open $V \subseteq Y$, the canonical homomorphism*

$$H^p(f^{-1}(V), \mathscr{F}) \to H^0(V, R^p f_*(\mathscr{F}))$$

is an isomorphism for $p \geq 0$.

Proof. Since this question is local, we may assume that Y is affine. Consider the Leray spectral sequence

$$E_2^{p,q} = H^p(Y, R^q f_*(\mathscr{F})) \Rightarrow H^{p+q}(X, \mathscr{F}).$$

By Proposition 12.1.15, we see that the $R^q f_*(\mathscr{F})$ are quasi-coherent $\mathscr{O}_Y$-module, so we have $H^p(Y, R^q f_*(\mathscr{F})) = 0$ for $p > 0$ (Theorem 11.5.12). The spectral sequence therefore collapses at E_2 page and we obtain an isomorphism $H^q(X, \mathscr{F}) \cong H^0(Y, R^q f_*(\mathscr{F}))$. □

Corollary 12.1.17. *Let $f : X \to Y$ be a quasi-compact and separated morphism, and suppose that Y is quasi-compact. Then there exists an integer $r > 0$ such that for any quasi-coherent $\mathscr{O}_X$-module, we have $R^p f_*(\mathscr{F}) = 0$ for $p > r$. If Y is affine, then we can choose r so that there exists a covering of X by r affine opens.*

Proof. Since Y is a union of affine opens, it suffices to prove the second assertion, in view of Corollary 12.1.16. Now if $\mathfrak{U}$ is a covering of X by r affine opens, then $H^p(\mathfrak{U}, \mathscr{F}) = 0$ for $p > r$, since the cochains can be taken to be alternating. The conclusion then follows from Proposition 12.1.8. □

Corollary 12.1.18. *Under the hypothesis of Proposition 12.1.15, suppose that $Y = \mathrm{Spec}(A)$ is affine. Then for any quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$ and any $f \in A$, we have*

$$\Gamma(Y_f, R^p f_*(\mathscr{F})) = (\Gamma(Y, R^p f_*(\mathscr{F})))_f.$$

Proof. This follows from Proposition 12.1.15 and Corollary 10.6.15. □

Proposition 12.1.19. *Let Y be a quasi-compact scheme, $f : X \to Y$ be a quasi-compact and quasi-separated morphism, $g : Y \to Z$ be an affine morphism. For any quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, the canonical homomorphism $R^p(g \circ f)_*(\mathscr{F}) \to g_*(R^p f_*(\mathscr{F}))$ is bijective for $p \geq 0$.*

Proof. Consider the Grothendieck spectral sequence

$$E_2^{p,q} = R^p g_*(R^q f_*(\mathscr{F})) \Rightarrow R^{p+q}(g \circ f)_*(\mathscr{F}).$$

Since each $R^q f_*(\mathscr{F})$ is quasi-coherent by Proposition 12.1.15, it suffices to prove that $R^p g_*(\mathscr{G}) = 0$ for any quasi-coherent $\mathscr{O}_Y$-module $\mathscr{G}$ and $p > 0$. Since this question is local, we may assume that Z is affine; but then Y is also affine so the assertion follows from Corollary 12.1.16. □

Proposition 12.1.20. *Let $f : X \to Y$ be a quasi-compact and separated morphism and $g : Y' \to Y$ be a morphism. Let $f' = f_{(Y')} : X' = X_{(Y')} \to Y'$, and consider the commutative diagram*

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "g'"'] & Y' \arrow[d, "g"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

Let $\mathscr{F}$ (resp. $\mathscr{G}$) be a quasi-coherent $\mathscr{O}_X$-module (resp. $\mathscr{O}_{Y'}$-module) and assume that $\mathscr{F}$ and $\mathscr{G}$ are tor-independent, i.e., that for any points $x \in X$, $y' \in Y'$ such that $g(y') = f(x) = y$, we have

$$\mathrm{Tor}_p^{\mathscr{O}_{Y,y}}(\mathscr{G}_{y'}, \mathscr{F}_x) = 0 \quad \textit{for} \quad p > 0.$$

Then there exists a natural isomorphism in the derived category $D(Y')$:

$$\mathscr{G} \otimes_Y^L Rf_*(\mathscr{F}) \xrightarrow{\sim} Rf'_*(\mathscr{G} \otimes_Y \mathscr{F}), \tag{12.1.9}$$

where $G \otimes_Y^L Rf_*(\mathscr{F}) := \mathscr{G} \otimes_{\mathscr{O}_{Y'}}^L Lg^*(Rf_*(\mathscr{F}))$.

When $Y = Y'$ (resp. $\mathscr{G} = \mathscr{O}_{Y'}$), the isomorphism (12.1.9) is called the **projection isomorphism** (resp. **base change isomorphism**). When $\mathscr{G} = \mathscr{O}_{Y'}$, we have $\mathscr{G} \otimes_Y \mathscr{F} = g'^*(\mathscr{F})$, so we deduce from (12.1.9) a canonical map

$$g^*(R^p f_*(\mathscr{F})) \to R^p f'_*(g'^*(\mathscr{F})) \tag{12.1.10}$$

This is the composition of the canonical map $g^*(R^p f_*(\mathscr{F})) \to H^p(Lg^*(Rf_*(\mathscr{F})))$ and the isomorphism $H^p(Lg^*(Rf_*(\mathscr{F}))) \xrightarrow{\sim} R^p f'_*(g'^*(\mathscr{F}))$ obtained from (12.1.9) by applying H^p. However, this is not an isomorphism in general.

Proof. First, we consider the case where $X = \operatorname{Spec}(B)$, $Y = \operatorname{Spec}(A)$, $Y' = \operatorname{Spec}(A')$ are affine, so that X' is affine with ring $B' = A' \otimes_A B$, and $\mathscr{F} = \widetilde{M}$, $\mathscr{G} = \widetilde{N}$ for some B-module M and A'-module N. Then $Rf_*(\mathscr{F})$ is represented by the underlying A-module $M_{[A]}$ of M, and $Rf'_*(\mathscr{G} \otimes_Y \mathscr{F})$ by the underlying A'-module $(N \otimes_A M)_{[A']}$ of $(N \otimes_{A'} B') \otimes_{B'} (B' \otimes_A M)$. On the other hand, the derived tensor product $\mathscr{G} \otimes_Y^L Rf_*(\mathscr{F})$ is represented by

$$N \otimes_A^L M_{[A]} := N \otimes_{A'}^L (A' \otimes_A^L M_{[A]}),$$

which can be calculated as $N \otimes_A P$, where P is a flat resolution of $M_{[A]}$. The tor-independence hypothesis implies that $\operatorname{Tor}_p^A(N, M_{[A]}) = 0$ for $p > 0$, i.e., the natural map

$$N \otimes_A^L M_{[A]} \to N \otimes_A M_{[A]} \tag{12.1.11}$$

is an isomorphism (in $D(A')$). The isomorphism (12.1.9) is then composition of (12.1.11) and the (trivial) isomorphism

$$N \otimes_A M_{[A]} \xrightarrow{\sim} (N \otimes_A M)_{[A']}.$$

In the general case, as f and Y are quasi-compact, X has a finite affine open covering $\mathfrak{U} = (U_i)$. Since f is separated, any finite intersection $U_{i_0 \dots i_p}$ is separated over Y, so it follows from Proposition 12.1.8 that $Rf_*(\mathscr{F}) = f_*(\check{\mathscr{C}}(\mathfrak{U}, \mathscr{F}))$, where $\check{\mathscr{C}}(\mathfrak{U}, \mathscr{F})$ is the alternating Čech complex of $\mathfrak{U}$ with values in $\mathscr{F}$. The preceding discussion, applied to affine open subsets of Y' above affine open subsets of Y, then shows that we have natural identifications

$$\mathscr{G} \otimes_Y^L Rf_*(\mathscr{F}) \xrightarrow{\sim} \mathscr{G} \otimes_{\mathscr{O}_{Y'}} g^*(f_*(\check{\mathscr{C}}(\mathfrak{U}, \mathscr{F}))) \xrightarrow{\sim} f'_*(\check{\mathscr{C}}(\mathfrak{U}', \mathscr{G} \otimes_Y \mathscr{F})) \xrightarrow{\sim} Rf'_*(\mathscr{G} \otimes_Y \mathscr{F}),$$

where $\mathfrak{U}'$ is the covering of X' formed by the inverse images of the U_i's. It is easy to check that the above composition does not depend on the choice of $\mathfrak{U}$, so we take this as the definition of the isomorphism (12.1.9). □

Corollary 12.1.21. *Let Y be a quasi-compact scheme, $f : X \to Y$ be a quasi-compact and separated morphism, and $g : Y' \to Y$ be a flat morphism. Then for any quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, we have a base change isomorphism*

$$g^* Rf_*(\mathscr{F}) \xrightarrow{\sim} Rf'_*(g'^*(\mathscr{F})),$$

and the induced base change maps (12.1.10) are isomorphisms.

Proof. If g is flat, then we can take $\mathscr{G} = \mathscr{O}_{Y'}$ in Proposition 12.1.20, so that we obtain a base change isomorphism

$$g^* Rf_*(\mathscr{F}) \xrightarrow{\sim} Rf'_*(g'^*(\mathscr{F})).$$

Moreover, since g is flat, the canonical map $g^*(R^p f_*(\mathscr{F})) \to H^p(Lg^*(Rf_*(\mathscr{F})))$ is an isomorphism, so the induced base change maps (12.1.10) are isomorphisms. □

Corollary 12.1.22. *Let Y be a quasi-compact scheme and $f : X \to Y$ be a quasi-compact and separated morphism. Let $y \in Y$ be a point and X_y be the fiber of f at y. Then for any quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$ which is flat over Y, we have a natural isomorphism (in the derived category of $\kappa(y)$-vector spaces)*

$$\kappa(y) \otimes_{\mathscr{O}_Y}^L Rf_*(\mathscr{F}) \xrightarrow{\sim} R\Gamma(X_y, \mathscr{F}_y) \tag{12.1.12}$$

where $\mathscr{F}_y = \mathscr{O}_{X_y} \otimes_{\mathscr{O}_X} \mathscr{F}$.

Proof. By hypothesis $\mathscr{F}$ is tor-independent of $\kappa(y)$, so we can apply Proposition 12.1.20. □

Corollary 12.1.23. *Let A be a ring, X be an A-scheme of finite type, and B be a faithfully flat A-algebra. Then for X to be affine, it is necesary and sufficient that $X \otimes_A B$ is affine.*

Proof. The condition is clearly necessary. Conversely, as X is separated over A and the morphism $\operatorname{Spec}(B) \to \operatorname{Spec}(A)$ is flat, it follows from Corollary 12.1.21 that we have

$$H^i(X \otimes_A B, \mathscr{F} \otimes_A B) = H^i(X, \mathscr{F}) \otimes_A B$$

for $i \geq 0$ and any quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$. As $X \otimes_A B$ is affine, we see that $H^i(X \otimes_A B, \mathscr{F} \otimes_A B) = 0$ for $i > 0$, so $H^i(X, \mathscr{F}) = 0$ for $i > 0$ since B is faithfully flat over A. As X is quasi-compact, the conclusion then follows from Serre's criterion. □

Remark 12.1.24. The projection isomorphism and the base change isomorphism together give a proof of the Künneth formula. To see this, let $f : X \to S$, $g : Y \to S$ be two quasi-compact and separated morphisms, and form the fiber product $X \times_S Y$:

$$\begin{tikzcd} X \times_S Y \arrow[r, "q"] \arrow[d, "p"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & S \end{tikzcd}$$

Let $\mathscr{F}$ (resp. $\mathscr{G}$) be a quasi-coherent $\mathscr{O}_X$-module (resp. $\mathscr{O}_Y$-module) and $h = f \circ p = g \circ q$. Then (assuming the projection isomorphism and the base change isomorphism) we have the following isomorphisms:

$$\begin{aligned} Rh_*(\mathscr{F} \otimes_S^L \mathscr{G}) = Rf_*(Rp_*(Lp^*(\mathscr{F}) \otimes_S^L Lq^*(\mathscr{G}))) &\cong Rf_*(\mathscr{F} \otimes_S^L Rp_*(Lq^*(\mathscr{G}))) \\ &\cong Rf_*(\mathscr{F} \otimes_S^L Lf_*(Rg_*(\mathscr{G}))) \cong Rf_*(\mathscr{F}) \otimes_S^L Rg_*(\mathscr{G}). \end{aligned}$$

12.2 Coholomogical properties of projective morphisms

12.2.1 Cohomology associated with an invertible sheaf

Let X be a scheme, $\mathscr{L}$ be an invertible $\mathscr{O}_X$-module, and consider the graded ring

$$S = \Gamma_*(X, \mathscr{L}) = \bigoplus_{n \in \mathbb{Z}} \Gamma(X, \mathscr{L}^{\otimes n}).$$

Let $\boldsymbol{f} = (f_i)_{1 \leq i \leq r}$ be a family of homogeneous elements of S, where $f_i \in S_{d_i}$. Put $U_i = X_{f_i}$, $U = \bigcup_i U_i$, and let $\mathfrak{U} = (U_i)$ be the covering of U. For any quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$ of X, we set

$$H^\bullet(\mathfrak{U}, \mathscr{F}; \mathscr{L}) := \bigoplus_{n \in \mathbb{Z}} H^\bullet(\mathfrak{U}, \mathscr{F} \otimes \mathscr{L}^{\otimes n}), \quad H^\bullet(U, \mathscr{F}; \mathscr{L}) := \bigoplus_{n \in \mathbb{Z}} H^\bullet(U, \mathscr{F} \otimes \mathscr{L}^{\otimes n}). \tag{12.2.1}$$

We note that the abelian groups in (12.2.1) are bigraded: for $m, n \in \mathbb{Z}$ we set

$$(H^\bullet(\mathfrak{U}, \mathscr{F}; \mathscr{L}))_{mn} = H^m(\mathfrak{U}, \mathscr{F} \otimes \mathscr{L}^n), \quad (H^\bullet(U, \mathscr{F}; \mathscr{L}))_{mn} = H^m(U, \mathscr{F} \otimes \mathscr{L}^n).$$

For any fixed $m \in \mathbb{Z}$, it is clear that $H^m(\mathfrak{U}, \mathscr{F}; \mathscr{L})$ and $H^m(U, \mathscr{F}; \mathscr{L})$ are graded S-modules. We now consider the graded S-module

$$M = \Gamma_*(\mathscr{F}; \mathscr{L}) = H^0(X, \mathscr{F}; \mathscr{L}) = \bigoplus_{n \in \mathbb{Z}} \Gamma(X, \mathscr{F} \otimes \mathscr{L}^{\otimes n}).$$

If X is quasi-compact and quasi-separated, then it follows from Theorem 10.6.14 that for any sequence $(i_0, i_1, \dots, i_p)$, we have a canonical isomorphism

$$\Gamma(U_{i_0 \dots i_p}, \mathscr{F}; \mathscr{L}) = H^0(U_{i_0 \dots i_p}, \mathscr{F}; \mathscr{L}) = M_{f_{i_0} \cdots f_{i_p}}.$$

Recall that $M_{f_{i_0}\cdots f_{i_p}}$ is idnetified with $\varinjlim_n M^n_{i_0\dots i_p}$, and this identification is an isomorphism of graded S-modules, if we define the degree of an element $z \in \varinjlim_n M^n_{i_0\dots i_p}$ to be the number $m - n(d_{i_0} + \cdots + d_{i_p})$ if z is the image of a homogeneous element $x \in M^n_{i_0\dots i_p} = M$ of degree m (from the definition of the transition homomorphism, it follows that this definition does not depend on the choice of x). Let $C^p_n(M)$ be the set of alternating maps $I^{p+1} \to M$ (for any n), then we can similarly define a graded S-module structure on $\varinjlim_n C^p_n(M)$. Now the canonical isomorphism

$$C^p(\mathfrak{U}, \mathscr{F}; \mathscr{L}) = \varinjlim_n C^p_n(M)$$

is then an isomorphism of graded S-modules. By Proposition 12.1.1, we have isomorphism of graded S-modules

$$C^p(\mathfrak{U}, \mathscr{F}; \mathscr{L}) = C^{p+1}(\mathfrak{f}, M) = \varinjlim_n K^{p+1}(\boldsymbol{f}^n, M)$$

where the degree of an element in $\varinjlim K^{p+1}(\boldsymbol{f}^n, M)$ is defined similarly. It is easy to see that the isomorphisms above are compatible with differential maps, so from Proposition 12.1.1 we conclude the following:

Proposition 12.2.1. *Let X be a quasi-compact and quasi-separated scheme. Then there exists a canonical isomorphism of graded S-modules*

$$H^p(\mathfrak{U}, \mathscr{F}; \mathscr{L}) \xrightarrow{\sim} H^{p+1}(\mathfrak{f}, M) \quad \text{for} \quad p \geq 1, \tag{12.2.2}$$

where $\mathfrak{f}$ is the ideal generated by $\boldsymbol{f}$. Moreover, we have a functorial exact sequence

$$0 \longrightarrow H^0(\mathfrak{f}, M) \longrightarrow M \longrightarrow H^0(\mathfrak{U}, \mathscr{F}; \mathscr{L}) \longrightarrow H^1(\mathfrak{f}, M) \longrightarrow 0 \tag{12.2.3}$$

of homomorphisms of degree 0.

Corollary 12.2.2. *If X is quasi-compact and separated and $U_i = X_{f_i}$ are affine, then there exists a canonical isomorphism of degree* 0*:*

$$H^p(U, \mathscr{F}; \mathscr{L}) \xrightarrow{\sim} H^{p+1}(\mathfrak{f}, M) \quad \text{for} \quad p > 0.$$

and we have a functorial exact sequence

$$0 \longrightarrow H^0(\mathfrak{f}, M) \longrightarrow M \longrightarrow H^0(\mathfrak{U}, \mathscr{F}; \mathscr{L}) \longrightarrow H^1(\mathfrak{f}, M) \longrightarrow 0$$

of homomorphisms of degree 0.

Corollary 12.2.3. *Let S be a graded ring with positive degrees, $(f_i)_{1 \leq i \leq r}$ be homogeneous elements of S_+ with $f_i \in S_{d_i}$, and M be a graded S-module. Let $X = \operatorname{Proj}(S)$, $U_i = D_+(f_i)$, and $\mathscr{L} = \mathscr{O}_X(1)$, then there exists a canonical isomorphism of degree* 0*:*

$$H^p(U, \widetilde{M}; \mathscr{L}) \xrightarrow{\sim} H^{p+1}(\mathfrak{f}, M) \quad \text{for} \quad p > 0.$$

and we have a functorial exact sequence

$$0 \longrightarrow H^0(\mathfrak{f}, M) \longrightarrow M \longrightarrow H^0(\mathfrak{U}, \widetilde{M}; \mathscr{L}) \longrightarrow H^1(\mathfrak{f}, M) \longrightarrow 0$$

of homomorphisms of degree 0.

Proof. In fact, we have $\Gamma(U_{i_0\dots i_p}, \widetilde{M(n)}) = (M_{f_{i_0}\cdots f_{i_p}})_n$, so $\Gamma(U_{i_0\dots i_p}, \widetilde{M}; \mathscr{L}) = M_{f_{i_0}\cdots f_{i_p}}$. The conclusion then follows from Corollary 12.2.2 since X is quasi-compact and separated. □

Remark 12.2.4. Corollary 12.2.3 is intersting if S is an A-algebra generated by S_1 where A is Noetherian. In fact, in this case any quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$ is of the form $\widetilde{M}$ by Theorem 11.2.39.

Remark 12.2.5. Under the hypothesis of Corollary 12.2.3, the functor $\Gamma(U_{i_0\dots i_p}, \widetilde{M};\mathscr{L}) = M_{f_{i_0}\cdots f_{i_p}}$ is exact on M, and as in Remark 12.1.3, if $0 \to M' \to M \to M'' \to 0$ is an exact sequence of graded S-modules, we have a commutative diagram for $p \geq 0$:

$$\begin{tikzcd} H^p(U, \widetilde{M}'';\mathscr{L}) \arrow[r, "\partial"] \arrow[d] & H^{p+1}(U, \widetilde{M}';\mathscr{L}) \arrow[d] \\ H^{p+1}(\mathfrak{f}, M'') \arrow[r, "\partial"] & H^{p+2}(\mathfrak{f}, M') \end{tikzcd}$$

We now apply Corollary 12.2.3 to the polynomial ring $S = A[T_0, \dots, T_r]$, where A is a ring and T_i are indeterminantes. Let $M = S$ and $f_i = T_i$, we are then reduced to compute $H^\bullet(\mathfrak{m}, S)$, where $\boldsymbol{T} = (T_i)_{0\leq i\leq r}$ and $\mathfrak{m}$ is the maximal ideal of S generated by $T_0, \dots, T_r$.

Lemma 12.2.6. *If $S = A[T_0, \dots, T_r]$ and $\boldsymbol{T} = (T_i)_{0\leq i\leq r}$, then*

$$H^i(\boldsymbol{T}^n, S) = \begin{cases} 0 & i \neq r+1, \\ S/\mathfrak{m}^n & i = r+1. \end{cases} \tag{12.2.4}$$

Proof. This is an immediate from the fact that the sequence $\boldsymbol{T}$ is regular. □

By passing to inductive limits over n, we see that $H^i(\mathfrak{m}, S) = 0$ for $i \neq r+1$. If $i = r+1$, then the inductive system is formed by $S/\mathfrak{m}^n$ and $\varphi_{nm} : S/\mathfrak{m}^n \to S/\mathfrak{m}^m$ for $0 \leq n \leq m$ is the multiplication by $(T_0 \cdots T_r)^{n-m}$. For any sequence $\alpha = (\alpha_0, \dots, \alpha_r)$ and integer $n \geq \sup_i\{\alpha_i\}$, we define

$$\xi_\alpha^n = T_0^{n-\alpha_0} \cdots T_r^{n-\alpha_r} \mod \mathfrak{m}^n.$$

Then $\varphi_{nm}(\xi_\alpha^n) = \xi_\alpha^m$, so the ξ_α^n form an element ξ_α in the inductive limit $H^{r+1}(\mathfrak{m}, S)$.

Corollary 12.2.7. *With the notations of Lemma 12.2.6, we have $H^i(\mathfrak{m}, S) = 0$ for $i \neq r+1$, and $H^{r+1}(\mathfrak{m}, S)$ is a free A-module with basis formed by the elements ξ_α with $\alpha_i > 0$ for each i.*

Proof. In fact, for each $n \geq 0$, the elements ξ_α^n for $0 < \alpha_i \leq n$ form a basis of the A-module $S/\mathfrak{m}^n$, so the corollary follows from (12.2.4). □

Proposition 12.2.8. *Let A be a ring, $r > 0$ be an integer, and $X = \mathbb{P}^r_A$.*

(a) *We have $H^i(X, \mathscr{O}_X; \mathscr{O}_X(1)) = 0$ for $i \neq 0, r$.*

(b) *The canonical homomorphism $\alpha : S \to H^0(X, \mathscr{O}_X; \mathscr{O}_X(1))$ is an isomorphism.*

(c) *The A-module $H^r(X, \mathscr{O}_X; \mathscr{O}_X(1))$ is free with a basis formed by the elements ξ_α with $\alpha_i > 0$ for each i. Moreover, ξ_α is of degree $-|\alpha| = -(\alpha_0 + \cdots + \alpha_r)$ and $T_i \cdot \xi_\alpha = \xi_{\alpha_0,\dots,\alpha_i-1,\dots,\alpha_r}$.*

Proof. By Corollary 12.2.7 we have $H^i(\mathfrak{m}, S) = 0$ for $i \neq r+1$, so the assertion in (a) follows from (12.2.2). From the exact sequence (12.2.3), it is easy to see that $S \cong H^0(X, \mathscr{O}_X; \mathscr{O}_X(1))$, and the isomorphism is given by the canonical homomorphism α. The last assertion also follows from (12.2.2) and Corollary 12.2.7. □

Corollary 12.2.9. *The values (i, n) such that $H^i(X, \mathscr{O}_X(n)) \neq 0$ are the following: $i = 0$ and $n \geq 0$, or $i = r$ and $n \leq -(r+1)$.*

Proof. We note that if $A \neq 0$ then $H^i(X, \mathscr{O}_X(n)) \neq 0$ by the listed values of (i, n). □

Corollary 12.2.10. *The A-modules $H^i(X, \mathscr{O}_X(n))$ are free of finite rank. If $i > 0$, then they are zero for $n > 0$.*

Proposition 12.2.11. *Let Y be a scheme, $\mathscr{E}$ be a locally free $\mathscr{O}_Y$-module of rank $r+1$, and $X = \mathbb{P}(\mathscr{E})$ be the projective bundle defined by $\mathscr{E}$. Let $f : X \to Y$ be the structural morphism, then the values (n, i) such that $R^i f_*(\mathscr{O}_X(n)) \neq 0$ are the following: $i = 0$ and $n \geq 0$, or $i = r$ and $n \leq -(r+1)$. Moreover, the canonical homomorphism*

$$\alpha : \boldsymbol{S}_{\mathscr{O}_Y}(\mathscr{E}) \to \Gamma_*(\mathscr{O}_X) = \bigoplus_{n\in\mathbb{Z}} f_*(\mathscr{O}_X(n))$$

is an isomorphism.

Proof. This question is local over Y, so we can assume that $Y = \operatorname{Spec}(A)$ is affine and $\mathscr{E} = \widetilde{E}$, where $E = A^{r+1}$. We can then apply Proposition 12.2.8, in view of Corollary 12.1.16. ☐

12.2.2 The coherence theorem for projective morphisms

Proposition 12.2.12. *Let A be a Noetherian ring and S be a graded A-algebra with positive degrees that is generated by $r+1$ elements of S_1. Let $X = \operatorname{Proj}(S)$, and consider a coherent $\mathscr{O}_X$-module $\mathscr{F}$.*

(a) *The A-module $H^p(X, \mathscr{F})$ is finitely generated.*

(b) *We have $H^p(X, \mathscr{F}) = 0$ for $p > r$.*

(c) *There exists an integer n_0 such that for $n \geq n_0$, we have $H^p(X, \mathscr{F}(n))$ for $p > 0$.*

(d) *There exists an integer n_0 such that for $n \geq n_0$, $\mathscr{F}(n)$ is generated by global sections.*

Proof. Note that X can be identified with a closed subscheme of $P = \mathbb{P}^r_A$ (Proposition 11.2.52). Moreover, if $j : X \to P$ is the canonical injection, $j_*(\mathscr{F})$ is a coherent $\mathscr{O}_P$-module and we have $j_*(\mathscr{F}(n)) = (j_*(\mathscr{F}))(n)$ (Corollary 11.3.26 and Proposition 11.3.30). In view of (G, II, cor. du th.4.9.1), we only need to consider the case where $X = \mathbb{P}^r_A$ and $S = A[T_0, \dots, T_r]$. Now X can be covered by $r+1$ affine opens $D_+(T_i)$, so (b) follows from Corollary 12.1.16 and Corollary 12.1.17. We also note that (d) is proved in Corollary 11.3.27.

We now prove (a) and (c). By Proposition 12.2.8, these assertions hold for $\mathscr{F} = \mathscr{O}_X(m)$, hence for direct sums of finitely many $\mathscr{O}_X$-module of the form $\mathscr{O}_X(m_j)$. On the other hand, (a) and (c) are trivial for $p > r$ in view of (b). We now proceed by descendent induction on p. Since $\mathscr{F}$ is coherent, it is a quotient of direct sume of finitely many sheaves $\mathscr{O}_X(m_j)$ (Corollary 11.2.42). That is, we have an exact sequence $0 \to \mathscr{R} \to \mathscr{E} \to \mathscr{F} \to 0$ where $\mathscr{R}$ is coherent and $\mathscr{E}$ satisfies (a) and (c). Since tensoring with $\mathscr{O}_X(n)$ is exact, we obtain an exact sequence

$$H^{p-1}(X, \mathscr{E}(n)) \longrightarrow H^{p-1}(X, \mathscr{F}(n)) \longrightarrow H^p(X, \mathscr{R}(n)).$$

As $\mathscr{E}(n)$ is a direct sum of $\mathscr{O}_X(m_j + n)$, we see $H^{p-1}(X, \mathscr{E}(n))$ is finitely generated by Corollary 12.2.10, and so is $H^p(X, \mathscr{R}(n))$ by induction hypothesis. As A is Noetherian, we then conclude that $H^{p-1}(X, \mathscr{F}(n))$ is finitely generated for any $n \in \mathbb{Z}$, and in particular for $n = 0$. On the other hand, by induction hypothesis there exists an integer n_0 such that for $n \geq n_0$ we have $H^p(X, \mathscr{R}(n)) = 0$, and we can choose n_0 such that $H^{p-1}(X, \mathscr{E}(n)) = 0$ for $n \geq n_0$, since $\mathscr{E}$ satisfies (c). Therefore we see that $H^{p-1}(X, \mathscr{F}(n)) = 0$ for $n \geq n_0$, which completes the proof. ☐

Theorem 12.2.13 (Serre). *Let Y be a Noetherian scheme, $f : X \to Y$ be a projective morphism, and $\mathscr{L}$ be an invertible $\mathscr{O}_X$-module that is ample for f. For any coherent $\mathscr{O}_X$-module $\mathscr{F}$, set $\mathscr{F}(n) = \mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{L}^{\otimes n}$.*

(a) *The $\mathscr{O}_Y$-module $R^p f_*(\mathscr{F})$ is coherent for $p \geq 0$.*

(b) *There exists an integer n_0 such that for $n \geq n_0$, we have $R^p f_*(\mathscr{F}(n)) = 0$ for $p > 0$.*

(c) *There exists an integer n_0 such that for $n \geq n_0$, the canonical homomorphism $f^*(f_*(\mathscr{F}(n))) \to \mathscr{F}(n)$ is surjective.*

Proof. We note that the theorem is unchanged if we replace $\mathscr{L}$ by $\mathscr{L}^{\otimes d}$ for $d > 0$. In fact, we then have $\mathscr{F}(n) = (\mathscr{F} \otimes \mathscr{L}^{\otimes r}) \otimes \mathscr{L}^{\otimes kd}$ for $k > 0$ and $0 \leq r < d$, and by hypothesis for any r there is an integer n_r such that for $k \geq n_r$, the properties (b) and (c) holds for $\mathscr{F} \otimes \mathscr{L}^{\otimes r}$. Let n_0 be the supremum of dn_r, then assertion (b) and (c) hold for $n \geq n_0$. In view of Proposition 11.4.48, we may assume that $\mathscr{L}$ is very ample relative to f, so there exists a dominant open immersion $i : X \to P$, where $P = \mathrm{Proj}(\mathscr{S})$ for a quasi-coherent graded $\mathscr{O}_Y$-algebra $\mathscr{S}$ with positive degrees that is finitely generated by $\mathscr{S}_1$; moreover, $\mathscr{L} \cong i^*(\mathscr{O}_P(1))$. Since f is proper, the morphism i is also proper by Corollary 11.5.24, so it is an isomorphism $X \cong P$. We can therefore assme that $X = \mathrm{Proj}(\mathscr{S})$ and $\mathscr{L} = \mathscr{O}_X(1)$, and the theorem then follows from Proposition 12.2.12. □

Corollary 12.2.14. *Under the hypothesis of Theorem 12.2.13, let $0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{H} \to 0$ be an exact sequence of coherent $\mathscr{O}_X$-modules. Then there exists an integer n_0 such that for $n \geq n_0$, the sequence*

$$0 \longrightarrow f_*(\mathscr{F}(n)) \longrightarrow f_*(\mathscr{G}(n)) \longrightarrow f_*(\mathscr{H}(n))$$

is exact.

Proof. This follows from the long exact sequence of f_* and property (b) of Theorem 12.2.13. □

Corollary 12.2.15. *Let Y be a Noetherian scheme, $f : X \to Y$ be a morphism of finite type, and $\mathscr{L}$ be an invertible $\mathscr{O}_X$-module that is ample for f. Let $0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{H} \to 0$ be an exact sequence of coherent $\mathscr{O}_X$-modules such that the supports of $\mathscr{F}$ and $\mathscr{H}$ are proper over Y. Then there exists an integer n_0 such that for $n \geq n_0$, the sequence*

$$0 \longrightarrow f_*(\mathscr{F}(n)) \longrightarrow f_*(\mathscr{G}(n)) \longrightarrow f_*(\mathscr{H}(n))$$

is exact.

Proof. The same reasoning of Theorem 12.2.13 show that we can assume that $\mathscr{L}$ is very ample relative to f, so we can identify X as an open subscheme of $Z = \mathrm{Proj}(\mathscr{S})$, where $\mathscr{S}$ is a quasi-coherent graded $\mathscr{O}_Y$-algebra with positive degrees, such that $\mathscr{S}$ is finitely generated by $\mathscr{S}_1$, and $\mathscr{L} = i^*(\mathscr{O}_Z(1))$, where $i : X \to P$ is the caonical immersion. Now as $\mathrm{supp}(\mathscr{G})$ is closed in X and contained in $\mathrm{supp}(\mathscr{F}) \cap \mathrm{supp}(\mathscr{H})$, it is proper over Y; the supports of $\mathscr{F}, \mathscr{G}, \mathscr{H}$ are then closed in Z. The sheaves $\mathscr{F}' = i_*(\mathscr{F})$, $\mathscr{G}' = i_*(\mathscr{G})$, and $\mathscr{H}' = i_*(\mathscr{H})$ are then coherent $\mathscr{O}_Z$-modules, and the sequence $0 \to \mathscr{F}' \to \mathscr{G}' \to \mathscr{H}' \to 0$ is exact. Moreover, if $g : Z \to Y$ is the structral morphism, then $f = g \circ i$, and it is clear that $\mathscr{F}'(n) = i_*(\mathscr{F}(n))$ and similarly for $\mathscr{G}'$, $\mathscr{H}'$. The conclusion then follows from Corollary 12.2.14. □

Remark 12.2.16. The assertion (a) of Theorem 12.2.13 is still valid if we only assume that Y is locally Noetherian. In fact, this property is local over Y, and the hypotheses inTheorem 12.2.13 imply that for any affine open $U \subseteq Y$, the restriction f to $f^{-1}(U)$ is projective and $\mathscr{L}|_{f^{-1}(U)}$ is ample for this restriction.

Remark 12.2.17. The assertion (a) of Theorem 12.2.13 is still valid, as we have seen, when we only assumes that X is a quasi-compact and quasi-separated scheme and $f : X \to Y$ is a quasi-compact and quasi-separated morphism (Proposition 11.4.45). However, note that assertion (b) is not true if we suppose that Y is the spectrum of a field k and that f is quasi-projective. For example, let $X' = \mathrm{Spec}(k[T_0, \dots, T_r])$ and X be the union of the affine opens $D(T_i)$ (so that

X can be considered as the space $\mathbb{A}^r_k$ with the origin removed). As the immersion $X \to X'$ is quasi-compact, the structural morphism $f : X \to Y$ is quasi-affine, so $\mathscr{O}_X$ is very ample relative to f (Proposition 11.5.1). But the ring $\Gamma(X, \mathscr{O}_X)$ is identified with the intersection of $K[T_0, \dots, T_r]_{T_i}$ for $0 \le i \le r$ (Eq. (10.7.1)), which is $K[T_0, \dots, T_r]$, so it follows from (12.1.5) that we have $H^r(X, \mathscr{O}_X^{\otimes n}) = H^r(X, \mathscr{O}_X) = A \neq 0$ for any $n \in \mathbb{Z}$.

Theorem 12.2.18. *Let Y be a Noetherian scheme and $\mathscr{S}$ be a quasi-coherent $\mathscr{O}_Y$-algebra of finite type. Let $f : X \to Y$ be a projective morphism, $\mathscr{S}' = f^*(\mathscr{S})$, $\mathscr{M}$ be a quasi-coherent $\mathscr{S}'$-module of finite type.*

(a) *For any $p \ge 0$, $R^p f_*(\mathscr{M})$ is an $\mathscr{S}$-module of finite type.*

(b) *Let $\mathscr{L}$ be an f-ample $\mathscr{O}_X$-module, and put $\mathscr{M}(n) = \mathscr{M} \otimes \mathscr{L}^{\otimes n}$. Then there exists an integer n_0 such that for $n \ge n_0$, we have $R^p f_*(\mathscr{M}(n)) = 0$ for $p > 0$, and the canonical homomorphism $f^*(f_*(\mathscr{M}(n))) \to \mathscr{M}(n)$ is surjective.*

12.2.3 Applications to associated sheaves of graded modules

Theorem 12.2.19. *Let Y be a Noetherian scheme, $\mathscr{S}$ be a quasi-coherent $\mathscr{O}_Y$-algebra of finite type with positive degrees, $X = \mathrm{Proj}(\mathscr{S})$, and $q : X \to Y$ be the structural morphism. Let $\mathscr{M}$ be a quasi-coherent graded $\mathscr{S}$-module that is eventually finite. Then there exists an integer n_0 such that for $n \ge n_0$, the canonical homomorphism of ([?], 8.14.5.1)*

$$\alpha_n : \mathscr{M}_n \to q_*(\mathscr{P}roj_0(\mathscr{M}(n))) = q_*((\mathscr{P}roj(\mathscr{M}))_n)$$

is bijective. In other words, the canonical homomorphism $\alpha : \mathscr{M} \to \Gamma_*(\mathscr{P}roj(\mathscr{M}))$ is an eventual isomorphisms.

Proof. By Proposition 11.2.36, we can assume that $\mathscr{M}$ is an $\mathscr{S}$-module of finite type. As Y is quasi-compact, by Proposition 11.3.7 there exists an integer $d > 0$ such that $\mathscr{S}^{(d)}$ is generated by $\mathscr{S}_d$, which is of finite type. Now $\mathscr{M}$ is the direct sum of $\mathscr{M}^{(d,k)}$ for $0 \le k < d$, and each $\mathscr{M}^{(d,k)}$ is quasi-coherent $\mathscr{S}^{(d)}$-module of finite type (Proposition 2.1.39), so it suffices to prove that the canonical homomorphism $\alpha : \mathscr{M}^{(d,k)} \to \Gamma_*((\mathscr{P}roj(\mathscr{M}))^{(d,k)})$ is an eventual isomorphism. In view of ([?], 8.14.13) (and the diagram ([?], 8.14.13.4)), we can therefore assume that $\mathscr{S}$ is finitely generated by $\mathscr{S}_1$. As Y is Noetherian, the same reasoning as in Proposition 12.2.12 shows that we can further assume that $Y = \mathrm{Spec}(A)$, $\mathscr{S} = \widetilde{\mathscr{S}}$, $\mathscr{M} = \widetilde{\mathscr{M}}$, where A is a Noetherian ring, S_1 is a finitely generated A-module, and M is a finitely generated S-module.

We also note that it suffices to prove the assertion for $M = S$. In fact, in the general case, we have an exact sequence $R \to L \to M \to 0$, where L and R are direct sums of graded modules of the form $S(m)$. If the assertion is proved for $M = S$, then it also holds for $M = S(m)$, hence for L and R. Consider the commutative diagram

$$\begin{tikzcd}
\widetilde{R}_n \arrow[r] \arrow[d, "\alpha_n"] & \widetilde{L}_n \arrow[r] \arrow[d, "\alpha_n"] & \widetilde{M}_n \arrow[r] \arrow[d, "\alpha_n"] & 0 \\
q_*(\widetilde{R}(n)) \arrow[r] & q_*(\widetilde{L}(n)) \arrow[r] & q_*(\widetilde{M}(n)) \arrow[r] & 0
\end{tikzcd}$$

The first and second vertical arrows are isomorphisms for $n \gg 0$, so by five lemma we conclude that the middle one is also an isomorphism for sufficiently large n, whence our assertion.

This being done, we are left to prove the theorem for $M = S$; for this, we first suppose that $S = A_0[T_0, \dots, T_r]$ (where T_i are indeterminates). In this case, the assertion follows from Proposition 12.2.8(b). In the general case, S is identified with a quotient of a ring $S' = A[T_0, \dots, T_n]$ by a graded ideal, so X is a closed subscheme of $X' = \mathbb{P}^r_A$. If $j : X \to X'$ is the canonical injection, then $j_*(\widetilde{S}(n))$ is equal to the $\mathscr{O}_{X'}$-module $(\mathscr{P}roj(\widetilde{S}))(n)$ where S is considered as a graded

S'-module (Proposition 11.2.47). As $j_*(\widetilde{S}(n))$ is a eventually finite $\mathcal{O}_{X'}$-module, the canonical homomorphism $\alpha_n : S_n \to \Gamma(X', j_*(\widetilde{S}(n)))$ is bijective for $n \gg 0$, and this proves our assertion since $\Gamma(X', j_*(\widetilde{S}(n))) = \Gamma(X, \widetilde{S}(n))$. □

Corollary 12.2.20. *Under the hypotheses of Theorem 12.2.19, let $\mathcal{S}_X = \bigoplus_{n\in\mathbb{Z}} \mathcal{O}_X(n)$, and $\mathcal{F}$ be a quasi-coherent graded $\mathcal{S}_X$-module of finite type. Then $\Gamma_*(\mathcal{F})$ is eventually finite.*

Proof. We see that in the proof of Theorem 12.2.19, X is isomorphic to $\mathrm{Proj}(\mathcal{S}^{(d)})$ which is of finite type over Y. It then follows from ([**?**], 8.14.9) that $\mathcal{F}$ is isomorphic to a graded $\mathcal{S}_X$-module of the form $\mathscr{P}roj_0(\mathcal{M})$, where $\mathcal{M}$ is a quasi-coherent $\mathcal{S}$-module of finite type. In view of Theorem 12.2.19, we see $\Gamma_*(\mathcal{F})$ is eventually isomorphic to $\mathcal{M}$, so is eventually finite. □

Remark 12.2.21. Let Y be a Noetherian scheme, $\mathcal{S}$ be a graded $\mathcal{O}_Y$-alegbra satisfying the conditions of Theorem 12.2.19, and $X = \mathrm{Proj}(\mathcal{S})$. Let $\mathcal{K}_{\mathcal{S}}$ be the abelian category of quais-coherent graded $\mathcal{S}$-modules that are eventually finite, and $\mathcal{K}'_{\mathcal{S}}$ be the subcategory of $\mathcal{K}_{\mathcal{S}}$ consists of quais-coherent graded $\mathcal{S}$-modules that are eventually null. Finally, let $\mathcal{K}_X$ be the category of quasi-coherent graded $\mathcal{S}_X$-module of finite type (which amounts to saying, since $\mathcal{S}_X$ is periodic by ([**?**], 8.14.4) and ([**?**], 8.14.12), that each $\mathcal{F}_i$ is a coherent $\mathscr{P}_X$-module). Then in view of ([**?**], 8.14.8), ([**?**], 8.14.10) and Theorem 12.2.19, the functors $\mathcal{M} \mapsto \mathscr{P}roj(\mathcal{M})$ and $\mathcal{F} \mapsto \Gamma_*(\mathcal{F})$ define an equivalence from the quotient category $\mathcal{K}_{\mathcal{S}}/\mathcal{K}'_{\mathcal{S}}$ to the category $\mathcal{K}_X$. If $\mathcal{S}$ is generated by $\mathcal{S}_1$, we can also replace $\mathcal{K}_X$ by the category of coherent $\mathcal{O}_X$-modules.

Proposition 12.2.22. *Let Y be a Noetherian scheme.*

(a) *Let $\mathcal{S}$ be a quasi-coherent graded $\mathcal{O}_Y$-algebra of finite type with positive degrees, $X = \mathrm{Proj}(\mathcal{S})$, and $\mathcal{S}_X = \mathscr{P}roj(\mathcal{S}) = \bigoplus_{n\in\mathbb{Z}} \mathcal{O}_X(n)$. Then $\mathcal{S}_X$ is a periodic graded $\mathcal{O}_X$-algebra whose homogeneous components $(\mathcal{S}_X)_n = \mathcal{O}_X(n)$ are coherent $\mathcal{O}_X$-modules. If $d > 0$ is a period of $\mathcal{S}_X$, then $(\mathcal{S}_X)_d = \mathcal{O}_X(d)$ is an invertible $\mathcal{O}_X$-module that is Y-ample. Moreover, the canonical homomorphism $\alpha : \mathcal{S} \to \Gamma_*(\mathcal{S}_X)$ is a eventual isomorphism.*

(b) *Conversely, let $q : X \to Y$ be a projective morphism, and $\mathcal{S}'$ be a graded $\mathcal{O}_X$-algebra whose homogeneous components $\mathcal{S}'_n$ are coherent $\mathcal{O}_X$-modules, and that admits a period $d > 0$ such that $\mathcal{S}'_d$ is an invertible $\mathcal{O}_X$-module that is Y-ample. Then $\mathcal{S} = \bigoplus_{n\geq 0} q_*(\mathcal{S}'_n)$ is a quasi-coherent graded $\mathcal{O}_Y$-algebra of finite type with positive degrees, and there exists a Y-isomorphism $i : X \cong \mathrm{Proj}(\mathcal{S})$ such that $i^*(\mathscr{P}roj(\mathcal{S}))$ is isomorphic to $\mathcal{S}'$.*

Proposition 12.2.23. *Let Y be a Noetherian scheme, $q : X \to Y$ be a projective morphism, and $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module that is very ample for q. Then $\mathcal{S} = \bigoplus_{n\geq 0} q_*(\mathcal{L}^{\otimes n})$ is a quasi-coherent $\mathcal{O}_Y$-algebra of finite type such that $\mathcal{S}_n = \mathcal{S}_1^n$ for $n \gg 0$, and there exists an Y-isomorphism $r : X \cong P = \mathrm{Proj}(\mathcal{S})$ such that $\mathcal{S} \cong r^*(\mathcal{O}_P(1))$.*

Proposition 12.2.24. *Let Y be a Noetherian integral scheme, X be an integral scheme, and $f : X \to Y$ be a projective birational morphism. Then there exists a coherent fractional ideal $\mathcal{I} \subseteq \mathcal{K}_Y$ such that X is Y-isomorphic to blow up Y-scheme relative to $\mathcal{I}$. Moreover, there exists an open subset $U \subseteq Y$ such that the restriction $f : f^{-1}(U) \to U$ is an isomorphic, and $\mathcal{I}|_U$ is invertible.*

Proof. By Proposition 11.5.18, there exists an invertible $\mathcal{O}_X$-module $\mathcal{L}$ that is very ample for f, so we can apply Proposition 12.2.23, hence identify X with $\mathrm{Proj}(\mathcal{S})$, where $\mathcal{S} = \bigoplus_{n\geq 0} f_*(\mathcal{L}^{\otimes n})$. We also see that each $f_*(\mathcal{L}^{\otimes n})$ is a torsion-free $\mathcal{O}_Y$-module (Proposition 10.7.26), so $\mathcal{S}$ is canonically identified with a sub-$\mathcal{O}_Y$-module of $\mathcal{S} \otimes_{\mathcal{O}_Y} \mathcal{K}_Y$. By Corollary 10.7.20 this sheaf is simple, so is completely determined when we know its restriction to any non-empty open set, for example to a open nonempty $U' \subseteq U$ (here $U \subseteq Y$ is an open subset such that f is an isomorphic on $f^{-1}(U)$) such that $\mathcal{L}|_{f^{-1}(U')}$ is isomorphic to $\mathcal{O}_X|_{f^{-1}(U')}$. As by hypotheses $f_*(\mathcal{L}^{\otimes n})|_{U'}$ is then isomorphic to $\mathcal{O}_Y|_{U'}$, we conclude that $\mathcal{S} \otimes \mathcal{K}_Y$ is an $\mathcal{K}_Y$-module isomorphic to $\mathcal{K}_Y[T]$, where T is an indeterminate, and $\mathcal{S}$ is eventually isomorphic to the sub-$\mathcal{O}_Y$-module generated by the canonical image of $f_*(\mathscr{L})$ in $\mathscr{S} \otimes_{\mathscr{O}_Y} \mathscr{K}_Y$ (Proposition 12.2.23). But if we identify $\mathscr{S} \otimes_{\mathscr{O}_Y} \mathscr{K}_Y$ with $\mathscr{K}_Y[T]$, then the image of $f_*(\mathscr{L})$ is identified with $\mathscr{I} \cdot T$, where $\mathscr{I}$ is a coherent sub-$\mathscr{O}_Y$-module of $\mathscr{K}_Y$ (Theorem 12.2.13), and its restriction to U' is isomorphic to $\mathscr{O}_Y|_{U'}$. We therefore conclude that $\mathscr{I}|_U$ is invertible and $\mathscr{S}$ is eventually isomorphic to $\bigoplus_{n\geq 0} \mathscr{I}^n$, whence our assertion. □

Corollary 12.2.25. *Under the hypothesis of Proposition 12.2.24, suppose that for any nontrivial coherent sub-$\mathscr{O}_Y$-module $\mathscr{I}$ of $\mathscr{K}_Y$, there exists an invertible $\mathscr{O}_Y$-module $\mathscr{L}$ such that*

$$\Gamma(Y, \mathscr{L} \otimes_{\mathscr{O}_Y} \mathscr{H}om_{\mathscr{O}_Y}(\mathscr{I}, \mathscr{O}_Y)) \neq 0.$$

Then, in the situation of Proposition 12.2.24, we can suppose that $\mathscr{I}$ is an ideal of $\mathscr{O}_Y$. This additional condition is always verified if there exists an ample $\mathscr{O}_Y$-module.

Proof. We first note that

$$\mathscr{L} \otimes_{\mathscr{O}_Y} \mathscr{H}om_{\mathscr{O}_Y}(\mathscr{I}, \mathscr{O}_Y) = \mathscr{H}om_{\mathscr{O}_Y}(\mathscr{L}^{-1}, \mathscr{H}om_{\mathscr{O}_Y}(\mathscr{I}, \mathscr{O}_Y)) = \mathscr{H}om_{\mathscr{O}_Y}(\mathscr{I} \otimes_{\mathscr{O}_Y} \mathscr{L}^{-1}, \mathscr{O}_Y)$$

so the hypothesis signifies that there is a nonzero homomorphism $u : \mathscr{I} \otimes \mathscr{L}^{-1} \to \mathscr{O}_Y$. Now for any $y \in Y$, $(\mathscr{I} \otimes \mathscr{L}^{-1})_y$ is identified with a sub-$\mathscr{O}_{Y,y}$-module of the fraction field $\mathscr{K}_{Y,y}$ of $\mathscr{O}_{Y,y}$, so u_y is necessarily injective, and u is therefore an isomorphism from $\mathscr{I} \otimes \mathscr{L}^{-1}$ onto an ideal $\mathscr{I}'$ of $\mathscr{O}_Y$. As the blow up Y-scheme relative to $\mathscr{I}$ and $\mathscr{I} \otimes \mathscr{L}^{-1}$ are isomorphic (Remark 11.3.19), this proves the corollary. The last remark is a direct concequence of Proposition 11.4.29. □

Corollary 12.2.26. *Let X and Y be integral schemes that are projective over a field k, and $f : X \to Y$ be a birational k-morphism. Then X is k-isomorphic to a blow up Y-scheme relative to a closed subscheme Y' (not necessarily reduced) of Y.*

Proof. In fact, f is projective by Proposition 11.5.34, and as Y is projective over k, the condition of Corollary 12.2.25 is satisfied. It then suffices to consider the closed subscheme of Y defined by the coherent ideal $\mathscr{I}$. □

12.2.4 Euler characteristic and Hilbert polynomial

Let A be an Artinian ring, X be a projective scheme over $Y = \operatorname{Spec}(A)$. For any coherent $\mathscr{O}_X$-module $\mathscr{F}$, the cohomology module $H^i(X, \mathscr{F})$ is finitely generated (Theorem 12.2.13), hence is of finite length. We also see that $H^i(X, \mathscr{F}) = 0$ for $i \gg 0$, so the integer

$$\chi_A(\mathscr{F}) = \sum_{i=0}^{\infty} (-1)^i \ell_A(H^i(X, \mathscr{F}))$$

is defined for any coherent $\mathscr{O}_X$-module. If A is local Artinian, then we say that $\chi_A(\mathscr{F})$ is the Euler characteristic of $\mathscr{F}$ (over the ring A). For $\mathscr{F} = \mathscr{O}_X$, the integer $\chi_A(\mathscr{O}_X)$ is called the **arithmetic genus** of X (over A).

It is clear that the map χ_A is additive on the category of coherent $\mathscr{O}_X$-modules. That is, for an exact sequence $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$, we have

$$\chi_A(\mathscr{F}) = \chi_A(\mathscr{F}') + \chi_A(\mathscr{F}'').$$

Theorem 12.2.27. *Let A be a local Artinian ring, X be a projective scheme over $Y = \operatorname{Spec}(A)$, $\mathscr{L}$ be a very ample $\mathscr{O}_X$-module relative to Y, and $\mathscr{F}$ be a coherent $\mathscr{O}_X$-module. For $n \in \mathbb{Z}$, we set $\mathscr{F}(n) = \mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{L}^{\otimes n}$.*

(a) *There exists a polynomial $P \in \mathbb{Q}[T]$ such that $\chi_A(\mathscr{F}(n)) = P(n)$ for $n \in \mathbb{Z}$ (the polynomial P is called the Hilbert polynomial of $\mathscr{F}$ over A).*

(b) *For $n \gg 0$, we have $\chi_A(\mathscr{F}(n)) = \ell_A(\Gamma(X, \mathscr{F}(n)))$.*

(c) *The leading coefficient of* $\chi_A(\mathscr{F}(n))$ *is positive.*

Example 12.2.28. Let k be a field, $r > 0$ be an integer, and $X = \mathbb{P}^r_k$. Then we have $\chi_A(\mathscr{O}_X(n)) = \binom{n+r}{r}$ for $n \in \mathbb{Z}$. To see this, we divide into three cases.

- For $n > 0$, we have $\chi_A(\mathscr{O}_X(n)) = \dim_k(H^0(X, \mathscr{O}_X(n)))$, which is the number of homogeneous polynomials of degree n and is equal to $\binom{n+r}{r}$.
- For $n < -r$, we have $\chi_A(\mathscr{O}_X(n)) = (-1)^r \dim_k(H^r(X, \mathscr{O}_X(n)))$. If $n = -r - d$, the dimension of $H^r(X, \mathscr{O}_X(n))$ over k is is the number of sequences of integers $(\alpha_0, \dots, \alpha_r)$ with $\alpha_i > 0$ and $|\alpha| = r + d$ (Proposition 12.2.8), which is equal to the number $\binom{d+r-1}{r} = (-1)^r \binom{n+r}{r}$.
- For $-r \le n \le 0$, we have $\binom{n+r}{r} = 0$ and also $H^i(X, \mathscr{O}_X(n)) = 0$ for $i \ge 0$.

Corollary 12.2.29. *Let A be a local Artinian ring, S be a graded A-algebra of finite type generated by S_1, M be a graded S-module, and $X = \mathrm{Proj}(S)$. Then we have $\chi_A(\widetilde{M}(n)) = \ell_A(M_n)$ for $n \gg 0$.*

Proof. This follows from Theorem 12.2.19, since $H^i(X, \widetilde{M}(n)) = 0$ for $i > 0$ if $n \gg 0$. □

12.2.5 Cohomological criterion for ampleness

Proposition 12.2.30. *Let Y be a Noetherian scheme, $f : X \to Y$ be a proper morphism, and $\mathscr{L}$ be an invertible $\mathscr{O}_X$-module. For any coherent $\mathscr{O}_X$-module $\mathscr{F}$, we set $\mathscr{F}(n) = \mathscr{F} \otimes \mathscr{L}^{\otimes n}$ for $n \in \mathbb{Z}$.*

(i) *$\mathscr{L}$ is ample for f.*

(ii) *For any coherent $\mathscr{O}_X$-module $\mathscr{F}$, there exists an integer n_0 such that for $n \ge n_0$, we have $R^p f_*(\mathscr{F}(n)) = 0$ for $p > 0$.*

(iii) *For any coherent ideal $\mathscr{I}$ of $\mathscr{O}_X$, there exists an integer n_0 such that for $n \ge n_0$, we have $R^1 f_*(\mathscr{I}(n)) = 0$.*

12.3 The finiteness theorem for proper morphisms

12.3.1 The dévissage lemma

Let $\mathcal{A}$ be an abelian category. We say that a subset $\mathfrak{E}$ of objects of $\mathcal{A}$ is exact if $0 \in \mathcal{E}$, and for any exact sequence $0 \to A' \to A \to A'' \to 0$ in $\mathcal{A}$ such that two of A, A', A'' are in $\mathfrak{E}$, then the third one is in $\mathfrak{E}$. The following technique, called the *dévissage lemma*, is introduced by Alexander Grothendieck to prove statements about coherent sheaves on Noetherian schemes. One can think this method as an adaption of Noetherian induction.

Theorem 12.3.1 (Dévissage Lemma). *Let X be a Noetherian scheme, $\mathfrak{E}$ be an exact subset of the category $\mathcal{A}$ of coherent $\mathscr{O}_X$-modules, and X' be a closed subset of X. Suppose that for any irreducible subset Y of X', with generic point y, there exists a coherent $\mathscr{O}_X$-module $\mathscr{G} \in \mathfrak{E}$ such that $\mathscr{G}_y$ is a $\kappa(y)$-vector space of dimension 1. Then any coherent $\mathscr{O}_X$-module with support in X' belongs to $\mathfrak{E}$.*

Proof. Consider the following property $\mathcal{P}(Y)$ on a closed subset Y of X: any coherent $\mathscr{O}_X$-module with support contained in Y belongs to $\mathfrak{E}$. In view of the Noetherian induction principle (**??**), we are reduced to prove the following: if Y is a closed subset of X' such that $\mathcal{P}(Y')$ is valid for any proper closed subset $Y' \subsetneq Y$, then $\mathcal{P}(Y)$ is valid.

Now let $\mathscr{F}$ be a coherent $\mathscr{O}_X$-module with support contained in Y, we show that $\mathscr{F} \in \mathfrak{E}$. In this case, we endow Y with the reduced subscheme structure of X, and let $\mathscr{I}$ be the ideal of $\mathscr{O}_X$

defining it. By Proposition 10.6.17, we see there exists an integer $n > 0$ such that $\mathscr{I}^n\mathscr{F} = 0$; for $1 \leq k \leq n$, we then have an exact sequence

$$0 \longrightarrow \mathscr{I}^{k-1}\mathscr{F} / \mathscr{I}^k\mathscr{F} \longrightarrow \mathscr{F} / \mathscr{I}^k\mathscr{F} \longrightarrow \mathscr{F} / \mathscr{I}^{k-1}\mathscr{F} \longrightarrow 0$$

of coherent $\mathscr{O}_X$-modules. As $\mathfrak{C}$ is exact, by recurrence on k, it suffices to prove that $\mathscr{F}_k = \mathscr{I}^{k-1}\mathscr{F} / \mathscr{I}^k\mathscr{F}$ belongs to $\mathfrak{C}$; in other words, we may also assume that $\mathscr{I}\mathscr{F} = 0$, which means $\mathscr{F} = j_*(j^*(\mathscr{F}))$, where $j : Y \to X$ is the canonical injection.

First suppose that Y is reducible, and let $Y = Y' \cup Y''$, where Y', Y'' are proper closed subsets of Y. We endow Y', Y'' with the reduced subscheme structure, and let $\mathscr{I}', \mathscr{I}''$ be the defining ideals of $\mathscr{O}_X$. Put $\mathscr{F}' = \mathscr{F} \otimes_{\mathscr{O}_X} (\mathscr{O}_X/\mathscr{I}')$ and $\mathscr{F}'' = \mathscr{F} \otimes_{\mathscr{O}_X} (\mathscr{O}_X/\mathscr{I}'')$. The canonical homomorphisms $\mathscr{F} \to \mathscr{F}', \mathscr{F} \to \mathscr{F}''$ then define a homomorphism $u : \mathscr{F} \to \mathscr{F} \oplus \mathscr{F}''$ such that any point $z \notin Y' \cap Y''$, the induced homomorphism $u_z : \mathscr{F} \to \mathscr{F}'_z \oplus \mathscr{F}''_z$ is bijective. In fact, we have $\mathscr{I}' \cap \mathscr{I}'' = \mathscr{I}$; if $z \notin Y''$ then $\mathscr{I}'_z = \mathscr{I}_z$, so $\mathscr{F}'_z = \mathscr{F}_z$ and $\mathscr{F}''_z = 0$; and similarly if $z \notin Y'$. The kernel and cokernel of u, which belong to $\mathcal{A}$, are then supported in $Y' \cap Y''$, and hence belong to $\mathfrak{C}$ by hypothesis. By the same reasoning, $\mathscr{F}'$ and $\mathscr{F}''$ are in $\mathfrak{C}$, hence so is $\mathscr{F}' \oplus \mathscr{F}''$. We now conclude from the following exact sequences

$$0 \longrightarrow \ker u \longrightarrow \mathscr{F} \longrightarrow \operatorname{im} u \longrightarrow 0$$
$$0 \longrightarrow \operatorname{im} u \longrightarrow \mathscr{F}' \oplus \mathscr{F}'' \longrightarrow \operatorname{coker} u \longrightarrow 0$$

that $\mathscr{F}$ belongs to $\mathfrak{C}$.

If on the other hand Y is irreducible, then the subscheme Y of X is integral. If y is the generic point of Y, we have $\mathscr{O}_{Y,y} = \kappa(y)$, and as $j^*(\mathscr{F})$ is a coherent $\mathscr{O}_Y$-module, $\mathscr{F}_y = (j^*(\mathscr{F}))_y$ is a $\kappa(y)$-vector space of finite dimension m. By hypothesis, there exists a coherent $\mathscr{O}_X$-module $\mathscr{G} \in \mathfrak{C}$ (necessarily supported in Y) such that $\mathscr{G}_y$ is a $\kappa(y)$-vector space of dimension 1, so there is an $\kappa(y)$-isomorphism $(\mathscr{G}_y)^m \cong \mathscr{F}_y$, which is also an $\mathscr{O}_Y$-isomorphism, and as $\mathscr{G}^m$ and $\mathscr{F}$ are coherent, there exists an open neighborhood W of y in X and an isomorphism $\mathscr{G}^m|_W \cong \mathscr{F}|_W$ (**??**). Let $\mathscr{H}$ be the graph of this isomorphism, which is a coherent sub-$(\mathscr{O}_X|_W)$-module of $(\mathscr{G}^m \oplus \mathscr{F})|_W$. Then there is a sub-$\mathscr{O}_X$-module $\mathscr{H}_0$ of $\mathscr{G}^m \oplus \mathscr{F}$, inducing $\mathscr{H}$ over W and zero over $X - Y$ (**??**). The restrictions $v : \mathscr{H}_0 \to \mathscr{G}^m$ and $w : \mathscr{H}_0 \to \mathscr{F}$ of the canonical projections of $\mathscr{G}^m \oplus \mathscr{F}$ are then homomorphisms of coherent $\mathscr{O}_X$-modules which are isomorphic over W and over $X - Y$. The kernel and cokernels of these homomorphisms are then supported in the proper closed subset $Y - (Y \cap W)$ of Y, so by hypotheses they belong to $\mathfrak{C}$. On the other hand, we have $\mathscr{G}^m \in \mathfrak{C}$ since $\mathscr{G} \in \mathfrak{C}$, so by the exactness of $\mathfrak{C}$ we conclude that $\mathscr{H}_0 \in \mathfrak{C}$, hence $\mathscr{F} \in \mathfrak{C}$. □

Corollary 12.3.2. *Suppose that the exact subset $\mathfrak{C}$ satisfies the additional property that any direct factor in $\mathcal{A}$ of a coherent $\mathscr{O}_X$-module $\mathscr{F} \in \mathfrak{C}$ belongs to $\mathfrak{C}$. Then the conclusion of Theorem 12.3.1 is still valid if we replace "$\mathscr{G}_y$ is a $\kappa(y)$-vector space of dimension 1" by the condition that $\mathscr{G}_y \neq 0$ (or equivalently $\operatorname{supp}(\mathscr{G}) = Y$).*

Proof. In fact, in this case the proof of Theorem 12.3.1 when Y is irreducible can be modified as follows: $\mathscr{G}_y$ is a $\kappa(y)$-vector space of dimension $q > 0$, and we then have an isomorphism $(\mathscr{G}_y)^m \cong (\mathscr{F}_y)^q$. By the same reasoning, we obtain that $\mathscr{F}^q \in \mathfrak{C}$, so $\mathscr{F} \in \mathfrak{C}$ by our additional assumption on $\mathfrak{C}$. □

12.3.2 The finiteness theorem for proper morphisms

Theorem 12.3.3. *Let Y be a locally Noetherian scheme and $f : X \to Y$ be a proper morphism. Then for any coherent $\mathscr{O}_X$-module $\mathscr{F}$, the $\mathscr{O}_Y$-modules $R^p f_*(\mathscr{F})$ are coherent for $p \geq 0$.*

Proof. Since this question is local over Y, we can suppose that Y is Noetherian, and hence X is Noetherian (Proposition 10.6.20). The coherent $\mathscr{O}_X$-module $\mathscr{F}$ satisfying the conclusion of Theorem 12.3.3 is easily seen to form an exact subset $\mathfrak{C}$ of the category $\mathcal{A}$ of coherent $\mathscr{O}_X$-modules. In fact, let $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ be an exact sequence of coherect $\mathscr{O}_X$-modules, and suppose for example that $\mathscr{F}'$, $\mathscr{F}$ belong to $\mathfrak{C}$. Then we have an exact sequence

$$R^{p-1}f_*(\mathscr{F}'') \xrightarrow{\partial} R^pf_*(\mathscr{F}) \longrightarrow R^pf_*(\mathscr{F}) \longrightarrow R^pf_*(\mathscr{F}'') \xrightarrow{\partial} R^{p+1}f_*(\mathscr{F}')$$

in which the four outer terms are coherent. It then follows from **??** that $R^pf_*(\mathscr{F})$ is coherent. We also note that any direct factor $\mathscr{F}' \in \mathcal{A}$ of a coherent $\mathscr{O}_X$-module $\mathscr{F} \in \mathfrak{C}$ belongs to $\mathfrak{C}$: in fact, $R^pf_*(\mathscr{F}')$ is then a direct factor of $R^pf_*(\mathscr{F})$ (G, II 4.4.4), so is of finite type, and since it is quasi-coherent (Proposition 12.1.15), it is coherent (note that Y is Noetherian). In view of Corollary 12.3.2, we only need to prove that if X is irreducible with generic point x, then there exists a coherent $\mathscr{O}_X$-module $\mathscr{F} \in \mathfrak{C}$ such that $\mathscr{F}_x \neq 0$. In fact, for any irreducible closed subscheme Y of X with canonical injection $j : Y \to X$, the composition $f \circ j$ is proper (Proposition 11.5.23), and if $\mathscr{G}$ is a coherent $\mathscr{O}_Y$-module supported in Y, then $j_*(\mathscr{G})$ is a coherent $\mathscr{O}_X$-module such that $R^p(f \circ j)_*(\mathscr{G}) = R^pf_*(j_*(\mathscr{G}))$ (G, II, 4.9.1), so we can apply our result on Y.

By Chow's lemma (Theorem 11.5.39), there exists an irreducible scheme X' and a surjective projective morphism $g : X' \to X$ such that $f \circ g : X' \to Y$ is projective. By Proposition 11.5.18, there exists a g-ample $\mathscr{O}_{X'}$-module $\mathscr{L}$, so by Theorem 12.2.13 we see that there exists an integer n_0 such that $\mathscr{F} = g_*(\mathscr{O}_{X'}(n))$ is a coherent $\mathscr{O}_X$-module and $R^pg_*(\mathscr{O}_{X'}(n)) = 0$ for $p > 0$ and $n \geq n_0$. Moreover, if x (resp. x') is the generic point of X (resp. X'), there exists an open subset U of x such that g is an isomorphism from $g^{-1}(U)$ onto U, so $\mathscr{F}_x \cong (\mathscr{O}_{X'}(n))_{x'} \neq 0$. On the other hand, since $f \circ g$ is projective and Y is Noetherian, the $R^p(f \circ g)(\mathscr{O}_{X'}(n))$ are coherent by Theorem 12.2.13. Consider the Grothendieck's spectral sequence:

$$E_2^{p,q} = R^pf_*(R^qg_*(\mathscr{O}_{X'}(n))) \Rightarrow R^*(f \circ g)_*(\mathscr{O}_{X'}(n)).$$

We have already remarked that for $n \gg 0$ we have $R^qg_*(\mathscr{O}_{X'}(n)) = 0$ for $q > 0$, so this sequence collapse at E_2 and we obtain an isomorphism $E_2^{p,0} = R^pf_*(\mathscr{F}) \cong R^p(f \circ g)_*(\mathscr{O}_{X'}(n))$, which implies $\mathscr{F} \in \mathfrak{C}$ and completes the proof. □

Corollary 12.3.4. *Let A be a Noetherian ring, X be a proper scheme over A. For any coherent $\mathscr{O}_X$-module $\mathscr{F}$, the $H^p(X, \mathscr{F})$ are finitely generated A-modules, and there exists an integer r such that $H^p(X, \mathscr{F}) = 0$ for any coherent $\mathscr{O}_X$-module $\mathscr{F}$ and $p > r$.*

Proof. The second assertion is proved in Proposition 12.2.12, and the first one follows from Theorem 12.3.3, in view of Corollary 12.1.16. □

Corollary 12.3.5. *Let Y be a locally Noetherian scheme, $f : X \to Y$ be a morphism of finite type. For any coherent $\mathscr{O}_X$-module with support proper over Y, the $\mathscr{O}_Y$-modules $R^pf_*(\mathscr{F})$ are coherent for $p \geq 0$.*

Proof. Since this question is local over Y, we may assume that Y is Noetherian, and then so is X. By hypothesis, any closed subscheme Z of X with underlying space $\mathrm{supp}(\mathscr{F})$ is proper over Y, so if $j : Y \to X$ is the canonical injection, $f \circ j$ is proper. Now we may choose Z such that $\mathscr{F} = j_*(\mathscr{G})$, where $\mathscr{G} = j^*(\mathscr{F})$ is a coherent $\mathscr{O}_Z$-module (Corollary 10.6.18). Since we have $R^pf_*(\mathscr{F}) = R^p(f \circ j)_*(\mathscr{G})$ by Corollary 12.1.7, the conclusion follows from Theorem 12.3.3. □

Proposition 12.3.6. *Let Y be a Noetherian scheme, $\mathscr{S}$ be a quasi-coherent graded $\mathscr{O}_Y$-algebra of finite type with positive degrees, $Y' = \mathrm{Proj}(\mathscr{S})$, and $g : Y' \to Y$ be the structural morphism. Let $f : X \to Y$ be a proper morphism, $\mathscr{S}' = f^*(\mathscr{S})$, and $\mathscr{M}$ be a quasi-coherent graded $\mathscr{S}'$-module of finite type. Then the $R^pf_*(\mathscr{M})$ are graded $\mathscr{S}$-modules of finite type for $p \geq 0$. Suppose moreover that $\mathscr{S}$ is generated by $\mathscr{S}_1$, then for any $p \in \mathbb{N}$, there exists an integer k_p such that for $k \geq k_p$ and $r \geq 0$, we have*

$$R^pf_*(\mathscr{M}_{k+r}) = \mathscr{S}_rR^pf_*(\mathscr{M}_k). \tag{12.3.1}$$

Proof. The first assertion follows from Theorem 12.2.18 since in its proof the condition on f is only used to derive the coherence of the $\mathscr{O}_{Y'}$-modules $R^p f'_*(\widetilde{\mathscr{M}})$. With the hypotheses of Proposition 12.3.6, f' is proper (Proposition 11.5.23(iii)), so we can now utilize Theorem 12.3.3 to complete the proof. □

Corollary 12.3.7. *Let A be a Noetherian ring, $\mathfrak{I}$ be an ideal of A, X be a proper A-scheme, and $\mathscr{F}$ be a coherent $\mathscr{O}_X$-module. Then for any $p \geq 0$, the direct sum $\bigoplus_{k\geq 0} H^p(X, \mathfrak{I}^k\mathscr{F})$ is a finitely generated module over $S = \bigoplus_{k\geq 0} \mathfrak{I}^k$. In particular, there exists an integer $k_p \geq 0$ such that for $k \geq k_p$, $r \geq 0$, we have*

$$H^p(X, \mathfrak{I}^{k+r}\mathscr{F}) = \mathfrak{I}^r H^p(X, \mathfrak{I}^k\mathscr{F}). \tag{12.3.2}$$

Proof. It suffices to apply 12.3.6 to $Y = \operatorname{Spec}(A)$, $\mathscr{S} = \widetilde{S}$, and $\mathscr{M}_k = \mathfrak{I}^k\mathscr{F}$. □

12.3.3 Generalization to formal schemes

12.4 Zariski's main theorem and applications

12.4.1 Grothendieck's comparison theorem

Let X, Y be Noetherian schemes, $f : X \to Y$ be a proper morohism, Y' be a closed subscheme of Y, and X' be the inverse image $f^{-1}(Y')$. We denote by $\widehat{X}$ and $\widehat{Y}$ the formal completion $X_{/X'}$ and $Y_{/Y'}$ of X and Y along X' and Y', respectively, and let $\hat{f} : \widehat{X} \to \widehat{Y}$ be the extension of f to completions. For any coherent $\mathscr{O}_X$-module $\mathscr{F}$, let $\widehat{\mathscr{F}}$ be the completion $\mathscr{F}_{/X'}$ of $\mathscr{F}$ along X', which is coherent by Proposition 10.8.28.

Let $\mathscr{J}$ be a coherent ideal of $\mathscr{O}_Y$ defining Y', then by Proposition 10.4.16(b) the coherent ideal $\mathscr{K} = f^*(\mathscr{J})\mathscr{O}_X$ defines the closed subscheme X' of X. For each $k \geq 0$, we consider the coherent $\mathscr{O}_X$-module

$$\mathscr{F}_k = \mathscr{F} \otimes_{\mathscr{O}_Y} (\mathscr{O}_Y/\mathscr{J}^{k+1}) = \mathscr{F}/\mathscr{K}^{k+1}\mathscr{F}.$$

The $\mathscr{O}_Y$-modules $R^p f_*(\mathscr{F})$ and $R^p f_*(\mathscr{F}_k)$ are coherent for $p \geq 0$ (Theorem 12.3.3). For any $k \geq 0$ and $p \geq 0$, the canonical homomorphism $\mathscr{F} \to \mathscr{F}_k$ defines by functoriality a homomorphism

$$R^p f_*(\mathscr{F}) \to R^p f_*(\mathscr{F}_k) \tag{12.4.1}$$

Moreover, as $\mathscr{F}_k$ is an $\mathscr{O}_X/\mathscr{K}^{k+1}$-module, $R^p f_*(\mathscr{F}_k)$ is an $\mathscr{O}_Y/\mathscr{J}^{k+1}$-module ([?] 0_{III}, 12.2.1) and we then deduce from (12.4.1) a homomorphism

$$R^p f_*(\mathscr{F}) \otimes_{\mathscr{O}_Y} (\mathscr{O}_Y/\mathscr{J}^{k+1}) \to R^p f_*(\mathscr{F}_k). \tag{12.4.2}$$

The two sides of (12.4.2) form two projective systems, and the projective limit of the left side is just the completion $(R^p f_*(\mathscr{F}))_{/Y'}$, which we also denote by $\widehat{R^p f_*(\mathscr{F})}$. Moreover, it is immediate that the homomorphisms in (12.4.2) form a projective system, so by passing to projective limit we obtain a canonocal homomorphism

$$\varphi_p : \widehat{R^p f_*(\mathscr{F})} \to \varprojlim_k R^p f_*(\mathscr{F}_k). \tag{12.4.3}$$

Since (12.4.2) is a homomorphism of $(\mathscr{O}_Y/\mathscr{J}^{k+1})$-modules, and can be considered as a continuous homomorphism of discrete $\mathscr{O}_{\widehat{Y}}$-modules, we see that the homomorphism φ_p is a continuous homomorphism of topological $\mathscr{O}_Y$-modules.

On the other hand, let $i_X : \widehat{X} \to X$ be the canonical morphism, which fits into the commutative diagram

$$\begin{tikzcd} X_k \arrow[r, "h_k"] \arrow[dr, "i_k"'] & \widehat{X} \arrow[d, "i_X"] \\ & X \end{tikzcd} \tag{12.4.4}$$

where X_k is the closed subscheme of X defined by the ideal $\mathscr{K}^{k+1}$, $i_k : X_k \to X$ is the canonical injection, and h_k is given the canonical homomorphism $\mathscr{O}_{\widehat{X}} \to \mathscr{O}_X/\mathscr{K}^{k+1}$. By Proposition 10.8.28, we have $\widehat{\mathscr{F}} = i_X^*(\mathscr{F})$. Since $\mathscr{F}_k = (i_k)_*(i_k^*(\mathscr{F}_k))$ (G, II, 4.9.1), we see that

$$H^p(X_k, i_k^*(\mathscr{F}_k)) = H^p(X, \mathscr{F}_k). \tag{12.4.5}$$

The canonical homomorphism $H^p(\widehat{X}, \widehat{\mathscr{F}}) \to H^p(X_k, h_k^*(\widehat{\mathscr{F}}))$ can then by identified as the following homomorphism:

$$H^p(\widehat{X}, \widehat{\mathscr{F}}) \to H^p(X, \mathscr{F}_k). \tag{12.4.6}$$

These homomorphisms evidently form a projective system, so by passing to projective limit we obtain a canonical homomorphism

$$\psi_{p,X} : H^p(\widehat{X}, \widehat{\mathscr{F}}) \to \varprojlim_k H^p(X, \mathscr{F}_k). \tag{12.4.7}$$

Now in view of Corollary 12.1.16, by replacing X with open subsets of the form $f^{-1}(V)$, where V is an affine open of Y, we also obtain homomorphisms

$$\psi_{p,V} : H^p(\widehat{X} \cap f^{-1}(V), \widehat{\mathscr{F}}) \to \varprojlim_k \Gamma(V, R^p f_*(\mathscr{F}_k)).$$

It is clear that the homomorphisms $\psi_{p,V}$ are compatible with restrictions, so by shefification, we obtain an induced canonical homomorphism

$$\psi_p : R^p \hat{f}_*(\widehat{\mathscr{F}}) \to \varprojlim_k R^p f_*(\mathscr{F}_k). \tag{12.4.8}$$

Finally, let $i_Y : \widehat{Y} \to Y$ be the canonical morphism; as $R^p f_*(\mathscr{F})$ is a coherent $\mathscr{O}_Y$-module by Theorem 12.3.3, we have $i_Y^*(R^p f_*(\mathscr{F})) = \widehat{R^p f_*(\mathscr{F})}$ (Proposition 10.8.28), and therefore a canonical homomorphism

$$\rho_p : \widehat{R^p f_*(\mathscr{F})} = i_Y^*(R^p f_*(\mathscr{F})) \to R^p \hat{f}_*(i_X^*(\mathscr{F})) = R^p \hat{f}_*(\widehat{\mathscr{F}}), \tag{12.4.9}$$

which is defined in the same way as the canonical homomorphism (**??**). From the commutative diagram (12.4.4), we then obtain a commutative diagram

$$\begin{tikzcd} \widehat{R^p f_*(\mathscr{F})} \arrow[rr, "\rho_p"] \arrow[dr, "\varphi_p"'] & & R^p \hat{f}_*(\widehat{\mathscr{F}}) \arrow[dl, "\psi_p"] \\ & \varprojlim_k R^p f_*(\mathscr{F}_k) & \end{tikzcd} \tag{12.4.10}$$

Theorem 12.4.1 (Grothendieck's Comparison Theorem). *Let $f : X \to Y$ be a proper morphism between Noetherian schemes, Y' be a closed subset of Y, and $X' = f^{-1}(Y')$. Then for any coherent $\mathscr{O}_X$-module $\mathscr{F}$, $R^p \hat{f}_*(\widehat{\mathscr{F}})$ is a coherent $\mathscr{O}_{\widehat{X}}$-module and the homomorphisms in (12.4.10) are homeomorphisms for $p \geq 0$.*

The fact that ρ_p is an isomorphism signifies that the formation of $R^p f_*$ commutes with that of completion, so Theorem 12.4.1 gives a comparision result between formal geometry and algebraic geometry. We being its proof by establishing the following affine case:

Corollary 12.4.2. *Under the hypotheses of Theorem 12.4.1, assume that $Y = \mathrm{Spec}(A)$, where A is Noetherian, and $\mathscr{J} = \widetilde{\mathfrak{J}}$ is an ideal of A, so that $\mathscr{F}_k = \mathscr{F}/\mathfrak{J}^{k+1}\mathscr{F}$. Then for each $p \geq 0$, the projective system $(H^p(X, \mathscr{F}_k))_{k\geq 0}$ satisfies the Mittag-Leffler condition, and the canonical homomorphism*

$$\psi_p : H^p(\widehat{X}, \widehat{\mathscr{F}}) \to \varprojlim_k H^p(X, \mathscr{F}_k) \tag{12.4.11}$$

is an isomorphism. Moreover, the filtration on $H^p(X, \mathscr{F})$ defined by the kernel of the canonical homomorphisms $H^p(X, \mathscr{F}) \to H^p(X, \mathscr{F}_k)$ is $\mathfrak{J}$-good and the canonical homomorphisms

$$\varphi_p : \widehat{H^p(X, \mathscr{F})} \to \varprojlim_k H^p(X, \mathscr{F}_k) \tag{12.4.12}$$

is a homeomorphism (where the left side is the $\mathfrak{J}$-adic completion of $H^p(X, \mathscr{F})$).

Proof. Fix an integer $p \geq 0$, and we simplfy the notation by setting

$$H = H^p(X, \mathscr{F}), \quad H_k = H^p(X, \mathscr{F}_k), \quad R_k = \ker(H \to H_k) \subseteq H.$$

The exact sequence on cohomology

$$H^p(X, \mathfrak{J}^{k+1}\mathscr{F}) \longrightarrow H^p(X, \mathscr{F}) \longrightarrow H^p(X, \mathscr{F}_k) \xrightarrow{\partial} H^{p+1}(X, \mathfrak{J}^{k+1}\mathscr{F}) \longrightarrow H^{+1}(X, \mathscr{F})$$

shows that R_k is also the image of the homomorphism $H^p(X, \mathfrak{J}^{k+1}\mathscr{F}) \to H^p(X, \mathscr{F})$; we set

$$Q_k = \ker(H^{p+1}(X, \mathfrak{J}^{k+1}\mathscr{F}) \to H^{p+1}(X, \mathscr{F})) = \mathrm{im}(H^p(X, \mathscr{F}_k) \to H^{p+1}(X, \mathfrak{J}^{k+1}\mathscr{F})) \tag{12.4.13}$$

so that there is an exact sequence

$$0 \longrightarrow R_k \longrightarrow H \longrightarrow H_k \longrightarrow Q_k \longrightarrow 0$$

Let x be an element of $\mathfrak{J}^m$ for $m \geq 0$; the multiplication by x on $\mathfrak{J}^k\mathscr{F}$ is a homomorphism $\mathfrak{J}^k\mathscr{F} \to \mathfrak{J}^{k+m}\mathscr{F}$, and therefore gives a homomorphism

$$\mu_{x,m} : H^p(X, \mathfrak{J}^k\mathscr{F}) \to H^p(X, \mathfrak{J}^{k+m}\mathscr{F}).$$

If we denote by S the graded A-algebra $\bigoplus_{k\geq 0} \mathfrak{J}^k$, then the multiplications $\mu_{x,m}$ define over $E = \bigoplus_{k\geq 0} H^p(X, \mathfrak{J}^k\mathscr{F})$ a finitely generated graded S-module structure (Corollary 12.3.7), which is Noetherian since S is Noetherian (Corollary 2.1.38).

We begin by showing that the submodules (R_k) define a $\mathfrak{J}$-good filtration on H. First, for any $x \in \mathfrak{J}^m$, the diagram

$$\begin{tikzcd}
\mathfrak{J}^{k+1}\mathscr{F} \arrow[r, "x"] \arrow[d] & \mathfrak{J}^{k+m+1}\mathscr{F} \arrow[d] \\
\mathscr{F} \arrow[r, "x"] & \mathscr{F}
\end{tikzcd}$$

is commutative, so the correponding diagram

$$\begin{tikzcd}
H^p(X, \mathfrak{J}^{k+1}\mathscr{F}) \arrow[r, "\mu_{x,m}"] \arrow[d] & H^p(X, \mathfrak{J}^{k+m+1}\mathscr{F}) \arrow[d] \\
H^p(X, \mathscr{F}) \arrow[r, "\mu_{x,0}"] & H^p(X, \mathscr{F})
\end{tikzcd} \tag{12.4.14}$$

is commutative, which proves, in view of the interpretation of R_k as the image of $H^p(X, \mathfrak{I}^{k+1}\mathscr{F})$ in $H^p(X, \mathfrak{F})$, that $\mathfrak{I}^m R_k \subseteq R_{k+m}$ and that the graded S-module $R = \bigoplus_{k\geq 0} R_k$ is a quotient of the sub-S-module $M = \bigoplus_{k\geq 0} H^p(X, \mathfrak{I}^{k+1}\mathscr{F})$ of E. Since M is also a finitely generated S-module by Corollary 12.3.7, the S-module R is then finitely generated, which is equivalent to the condition that (R_k) is $\mathfrak{I}$-good (Theorem 2.4.1).

Consider now the graded S-module $N = \bigoplus_{k\geq 0} H^{p+1}(X, \mathfrak{I}^{k+1}\mathscr{F})$, which is a finitely generated S-module in view of Corollary 12.3.7. For each $k \geq 0$, by (12.4.13) we have $Q_k \subseteq N_k$, and by replacing p by $p+1$ in the diagram (12.4.14) we see that $S_m Q_k = \mathfrak{I}^m Q_k \subseteq Q_{k+m}$. In other words, $Q = \bigoplus_{k\geq 0} Q_k$ is a graded sub-S-module of N, and is therefore finitely generated. We denote by $\alpha_m : \mathfrak{I}^m \to A$ the canonical injection, which can also be written as $S_m \to S_0$. Since $\mathfrak{I}^{k+1}\mathscr{F}_k = 0$, the A-module $H^p(X, \mathscr{F}_k)$ is annihilated by $\mathfrak{I}^{k+1}$, so Q_k, as the image of the A-homomorphism $H^p(X, \mathscr{F}_k) \to H^{p+1}(X, \mathfrak{I}^{k+1}\mathscr{F})$, is also annihilated by $\mathfrak{I}^{k+1}$. This signifies that, in the S-module Q, we have

$$\alpha_{k+1}(S_{k+1})Q_k = 0. \tag{12.4.15}$$

As Q is a finitely generated S-module, there exists an integers k_0 and d such that $Q_{k+d} = S_d Q_k$ for $k \geq k_0$ (Proposition 2.1.40); we then deduce from this relation and (12.4.15) that there exists an integer $r > 0$ such that

$$\alpha_r(S_r)Q = 0. \tag{12.4.16}$$

Now note that the the canonical injection $\mathfrak{I}^{k+m}\mathscr{F} \to \mathfrak{I}^k\mathscr{F}$ gives by passing to cohomology an A-homomorphism

$$\nu_m : H^{p+1}(X, \mathfrak{I}^{k+m}\mathscr{F}) \to H^{p+1}(X, \mathfrak{I}^k\mathscr{F})$$

and, for any $x \in \mathfrak{I}^m$, we have evidently a factorization

$$\mu_{x,0} : H^{p+1}(X, \mathfrak{I}^k\mathscr{F}) \xrightarrow{\mu_{x,m}} H^{p+1}(X, \mathfrak{I}^{k+m}\mathscr{F}) \xrightarrow{\nu_m} H^{p+1}(X, \mathfrak{I}^k\mathscr{F})$$

from which we conclude that, for any sub-A-module P of $H^{p+1}(X, \mathfrak{I}^k\mathscr{F})$, we have, in the S-module N,

$$\nu_m(S_m P) = \alpha_m(S_m)P. \tag{12.4.17}$$

If we choose $m \geq r$ to be a multiple of d, then as $Q_{k+m} = S_m Q_k$ for $k \geq k_0$ (by our choice of d), we derive from (12.4.17) and (12.4.16) that $\nu_m(Q_{k+m}) = \alpha_m(S_m)Q_k \subseteq \alpha_r(S_r)Q_k = 0$ for $k \geq k_0$.

Consider now the commutative diagram

$$\begin{tikzcd}
H^p(X,\mathscr{F}) \arrow[r] \arrow[d, equal] & H^p(X,\mathscr{F}_{k+m}) \arrow[r, "\partial"] \arrow[d] & H^{p+1}(X,\mathfrak{I}^{k+1}\mathscr{F}) \arrow[r] \arrow[d] & H^{p+1}(X,\mathscr{F}) \arrow[d] \\
H^p(X,\mathscr{F}) \arrow[r] & H^p(X,\mathscr{F}_k) \arrow[r, "\partial"] & H^{p+1}(X,\mathfrak{I}^{k+1}\mathscr{F}) \arrow[r] & H^{p+1}(X,\mathscr{F})
\end{tikzcd}$$

induced from the homomorphisms $\mathfrak{I}^{k+m+1}\mathscr{F} \to \mathfrak{I}^k\mathscr{F}$ and $\mathscr{F}_{k+m} \to \mathscr{F}_k$. From this, we conclude the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & R_{k+m} \arrow[r] \arrow[d] & H \arrow[r] \arrow[d, equal] & H_{k+m} \arrow[r] \arrow[d] & Q_{k+m} \arrow[r] \arrow[d, "\nu_m"] & 0 \\
0 \arrow[r] & R_k \arrow[r] & H \arrow[r] & H_k \arrow[r] & Q_k \arrow[r] & 0
\end{tikzcd}$$

whose rows are exact. As the last vertical arrow is zero for $k \geq k_0$, the image of H_{k+m} in H_k is contained in $\ker(H_k \to Q_k) = \operatorname{im}(H \to H_k)$, whence equals to $\operatorname{im}(H \to H_k)$ by the commutativity of the diagram. This image then equals to the image of $H_{k'}$ in H_k for $k' \geq k+m$, so the projective system $(H_k)_{k\geq 0}$ satisfies the Mittag-Leffler condition. Moreover, for any affine open U of X, we have $H^i(U, \mathscr{F}_k) = 0$ for $i > 0$, and the map $H^0(U, \mathscr{F}_{k+m}) \to H^0(U, \mathscr{F}_k)$ is surjective for $m \geq 0$ (Proposition 10.1.5). We can then apply ([?] 0_{III}, 13.3.1) to conclude that the canonical homomorphism $H^p(\widehat{X}, \widehat{\mathscr{F}}) \to \varprojlim_k H^p(X, \mathscr{F}_k)$ is bijective for each $p \geq 0$.

Now since the projective system (H/R_k) also satisfies the Mittag-Leffler condition, taking projective limit preserves the exactness of the sequence

$$0 \longrightarrow H/R_k \longrightarrow H_k \longrightarrow Q_k \longrightarrow 0$$

As $\nu_m(Q_{k+m}) = 0$, we have $\varprojlim_k Q_k = 0$, so we obtain an isomorphism $\varprojlim_k (H/R_k) \cong \varprojlim_k H_k$. But the filtration (R_k) of H is $\mathfrak{I}$-good, so it defines the $\mathfrak{I}$-adic topology on H, and $\varprojlim_k (H/R_k)$ is therefore the $\mathfrak{I}$-adic completion of H. □

Proof of Theorem 12.4.1. We now return to the proof of Theorem 12.4.1. For any affine open V of Y, since $R^p f_*(\mathscr{F})$ is coherent, we see that $\Gamma(V, \widehat{R^p f_*(\mathscr{F})})$ is equal to the $\mathfrak{I}$-adic completion of $\Gamma(V, R^p f_*(\mathscr{F}))$, and $\Gamma(V, \varprojlim_k R^p f_*(\mathscr{F}_k))$ is equal to $\varprojlim_k \Gamma(V, R^p f_*(\mathscr{F}_k))$ ([?] 0_{I}, 3.2.6). The fact that φ_p is a homeomorphism then follows from Corollary 12.4.2 and Corollary 12.1.16. Similarly, we see that each $\psi_{p,V}$ is an isomorphism, so ψ_p is an isomorphism by the definition of $R^p \hat{f}_*(\widehat{\mathscr{F}})$, and hence a homeomorphism by . □

Corollary 12.4.3. *Under the hypotheses of Theorem 12.4.1, for any affine open V of Y, the canonical homomorphism*

$$H^p(\widehat{X} \cap f^{-1}(V), \widehat{\mathscr{F}}) \to \Gamma(\widehat{Y} \cap V, R^p \hat{f}_*(\widehat{\mathscr{F}}))$$

is bijective.

Proof. This follows from the isomorphism ψ_p in Theorem 12.4.1 and Corollary 12.1.16. □

Remark 12.4.4. Let $f : X \to Y$ be a morphism of finite type between Noetherian schemes, and $\mathscr{F}$ be a coherent $\mathscr{O}_Y$-module with support proper over Y. Then we see from Corollary 12.3.5 that $R^p f_*(\mathscr{F})$ is coherent for $p \geq 0$. Moreover, we can also suppose that $\mathscr{F} = j_*(\mathscr{G})$, where $\mathscr{G} = j^*(\mathscr{F})$ is a coherent $\mathscr{O}_Z$-module, Z being a closed subscheme of X with underlying space $\operatorname{supp}(\mathscr{F})$, and $j : Z \to X$ is the canonical injection (Corollary 10.6.18). If we set $\mathscr{G}_k = \mathscr{G} \otimes_{\mathscr{O}_Y} (\mathscr{O}_Y/\mathscr{J}^{k+1})$, then

$$\mathscr{G}_k = j^*(\mathscr{F}_k), \quad R^p f_*(\mathscr{F}_k) = R^p(f \circ j)_*(\mathscr{G}_k), \quad R^p f_*(\mathscr{F}) = R^p(f \circ j)_*(\mathscr{G})$$

by Corollary 12.1.7. On the other hand, in view of Proposition 10.8.36, we also have

$$R^p \hat{f}(\widehat{\mathscr{F}}) = R^p \widehat{(f \circ j)}_*(\mathscr{G}).$$

We can then apply Theorem 12.4.1 to conclude that the result of Theorem 12.4.1 is also valid for $\mathscr{F}$ and f.

Proposition 12.4.5 (Formal Function Theorem). *Let Y be a locally Noetherian scheme, $f : X \to Y$ be a proper morphism, and $\mathscr{F}$ be a coherent $\mathscr{O}_X$-module. Then for any $y \in Y$ and $p \geq 0$, the $\mathscr{O}_{Y,y}$-module $(R^p f_*(\mathscr{F}))_y$ is finitely generated, separated for the $\mathfrak{m}_y$-adic topology, and we have a homeomorphism*

$$(\widehat{R^p f_*(\mathscr{F})})_y \xrightarrow{\sim} \varprojlim_k H^p(f^{-1}(y), \mathscr{F} \otimes_{\mathscr{O}_Y} (\mathscr{O}_{Y,y}/\mathfrak{m}_y^k))$$

where the left side is the $\mathfrak{m}_y$-adic completion of $(R^p f_*(\mathscr{F}))_y$ and $f^{-1}(y)$ is considered as the underlying space of $X \times_Y \operatorname{Spec}(\mathscr{O}_{Y,y}/\mathfrak{m}_y^k)$ (Proposition 10.4.16).

Proof. As $\mathscr{O}_{Y,y}$ is a Noetherian local ring and $(R^p f_*(\mathscr{F}))_y$ is finitely generated by Theorem 12.3.3, the $\mathfrak{m}_y$-adic topology on $(R^p f_*(\mathscr{F}))_y$ is separated (Proposition 2.4.28). The assertions therefore result from Corollary 12.4.2 if Y is Noetherian and the poinr y is closed, since we can then replace Y be an affine neighborhood of y and put $Y' = \{y\}$, in view of (G, II, 4.9.1). In the general case, we set

$$Y_1 = \mathrm{Spec}(\mathscr{O}_{Y,y}), \quad X_1 = X \times_Y Y_1, \quad \mathscr{F}_1 = \mathscr{F} \otimes_{\mathscr{O}_Y} \mathscr{O}_{Y_1}, \quad f_1 = f \times 1_{Y_1} : X_1 \to Y_1.$$

Then Y_1 is Noetherian, f_1 is proper, and $\mathscr{F}_1$ is coherent. Let y_1 be the unique closed point in Y_1; the proposition is then valid for f_1, $\mathscr{F}_1$ and y_1. We have $\mathscr{O}_{Y_1,y_1} = \mathscr{O}_{Y,y}$, $f_1^{-1}(y_1) = f^{-1}(y)$ (Proposition 10.3.37), and the schemes $X \times_Y \mathrm{Spec}(\mathscr{O}_{Y,y}/\mathfrak{m}_y^k)$ and $X_1 \times_{Y_1} \mathrm{Spec}(\mathscr{O}_{Y_1,y_1}/\mathfrak{m}_{y_1}^k)$ are canonically idnetified (Proposition 10.3.8). Moreover, $\mathscr{F}_1 \otimes_{\mathscr{O}_{Y_1}} (\mathscr{O}_{Y_1,y_1}/\mathfrak{m}_{y_1}^k)$ is identified with $\mathscr{F} \otimes_{\mathscr{O}_Y} (\mathscr{O}_{Y,y}/\mathfrak{m}_y^k)$ (Proposition 10.3.19). It then remains to see that $R^p(f_1)_*(\mathscr{F}_1)$ is canonically isomorphic to $R^p f_*(\mathscr{F}) \otimes_{\mathscr{O}_Y} \mathscr{O}_{Y_1}$, which follows from Corollary 12.1.21, since the local morphism $\mathrm{Spec}(\mathscr{O}_{Y,y}) \to Y$ is flat. □

Corollary 12.4.6. *Let Y be a locally Noetherian scheme. $f : X \to Y$ be a proper morphism, y be a point of Y, and r be the dimension of $f^{-1}(y)$. Then for any coherent $\mathscr{O}_X$-module $\mathscr{F}$, the sheaf $R^p f_*(\mathscr{F})$ vanishes in a neighborhood of y for $p > r$.*

Proof. In fact, if $p > r$, we then have $H^p(f^{-1}(y), \mathscr{F} \otimes (\mathscr{O}_{Y,y}/\mathfrak{m}_y^k)) = 0$ (by Leray's vanishing theorem) for any k, so the $\mathfrak{m}_y$-adic completion of $(R^p f_*(\mathscr{F}))_y$ is zero. As this topology is separated by Proposition 12.4.5, we then have $(R^p f_*(\mathscr{F}))_y = 0$, so the conclusion follows from **??**. □

Corollary 12.4.7. *Under the hypothesis of Proposition 12.4.5, we have a canonical homeomorphism*

$$(\widehat{f_*(\mathscr{F})})_y \xrightarrow{\sim} \varprojlim_k \Gamma(f^{-1}(y), \mathscr{F}_y/\mathfrak{m}_y^k \mathscr{F}_y).$$

Remark 12.4.8. Many applications of Proposition 12.4.5 use only the case $p = 0$, in which case the right-hand side is equal to $\Gamma(\widehat{X}, \widehat{\mathscr{F}})$, where $\widehat{X}$ is the formal completion of X along X_y, and $\widehat{\mathscr{F}} = \mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{O}_{\widehat{X}}$. In particular, if $\mathscr{F} = \mathscr{O}_X$, we have $\Gamma(\widehat{X}, \mathscr{O}_{\widehat{X}})$, which is the ring of **formal-regular functions** (also called **holomorphic functions**) on X along X_y.

12.4.2 Zariski's connectedness theorem

Theorem 12.4.9 (Zariski's Connectednes Theorem). *Let Y be a locally Noetherian scheme, $f : X \to Y$ be a proper morphism, and*

$$X \xrightarrow{f'} Y' \xrightarrow{g} Y$$

be the Stein factorization of f. Then g is finite, f' is proper, $f'_*(\mathscr{O}_X)$ is isomorphic to $\mathscr{O}_{Y'}$, and the fibers $f'^{-1}(y')$ are nonempty and connected for any $y' \in Y'$.

Proof. Let $\theta : \mathscr{O}_{Y'} \to f'_*(\mathscr{O}_X)$ be the morphism induced by f'. Then since $g_*(\mathscr{O}_{Y'}) = \mathscr{A}(Y')$ and $f_*(\mathscr{O}_X) = \mathscr{A}(X)$, the homomorphism $g_*(\theta) : g_*(\mathscr{O}_{Y'}) \to g_*(f'_*(\mathscr{O}_X)) = f_*(\mathscr{O}_X)$ is an isomorphism, so θ is an isomorphism by Corollary 11.1.8. Since $\mathscr{A}(X)$ is coherent by Theorem 12.3.3, we see f' is finite, and it is proper by Proposition 11.5.23. It then remains to prove the following assertion: if $f : X \to Y$ is a proper morphism with $f_*(\mathscr{O}_X) \cong \mathscr{O}_Y$ and Y being locally Noetherian, then the fibers $f^{-1}(y)$ are nonempty and connected for any $y \in Y$. To this end, we note that the hypothesis $f_*(\mathscr{O}_X) \cong \mathscr{O}_Y$ implies $(f_*(\mathscr{O}_X))_y \cong \mathscr{O}_{Y,y} \neq 0$ for any $y \in Y$, so f is dominant and hence is surjective since f is closed. We may then, as in 12.4.5, reduce to the case where y is closed in Y. Then $f^{-1}(y)$ is a Noetherian space with finitely many connected components, and is equal to the underlying space of the completion $\widehat{X}$ of X along $f^{-1}(y)$. If $(Z_i)_{1 \le i \le n}$ is its connected components, it is clear that $\Gamma(\widehat{X}, \mathscr{O}_{\widehat{X}})$ is the direct product of the rings $\Gamma(Z_i, \mathscr{O}_{\widehat{X}})$, and each of them is nonzero since the unit section is nozero at any point of $\widehat{X}$. Now if we apply Theorem 12.4.1 to $\mathscr{F} = \mathscr{O}_X$, whose completion along $f^{-1}(y)$ is $\mathscr{O}_{\widehat{X}}$, then we see that $\Gamma(\widehat{X}, \mathscr{O}_{\widehat{X}})$ is isomorphic to the $\mathfrak{m}_y$-adic completion $\widehat{\mathscr{O}_{Y,y}}$ of the local ring $\mathscr{O}_{\widehat{X}}$; this is a local ring which can not be a direct product of nonzero proper rings, since otherwise there are elements e_1, e_1 such that $e_1 + e_1 = 1$ and $e_1e_2 = 0$. But then e_1, e_2 are nonunits so are contained in the maximal ideal, and their sum cannot be 1. We therefore have $n = 1$, which proves our assertion. □

Corollary 12.4.10. *Under the hypothesis of Theorem 12.4.9, if $f_*(\mathscr{O}_X) \cong \mathscr{O}_Y$, then the fibers $f^{-1}(y)$ are connected and nonempty for any $y \in Y$.*

Corollary 12.4.11. *Under the hypothesis of Theorem 12.4.9, for any $y \in Y$, the set of connected components of the fiber $f^{-1}(y)$ is in one-to-one correspondence to the fiber $g^{-1}(y)$ (in other words, the set of maximal ideals of $(f_*(\mathscr{O}_X))_y$).*

Proof. Since Y' is finite over Y, it has finite fiber at y (Corollary 11.6.5). As $f^{-1}(y) = f'^{-1}(g^{-1}(y))$, the corollary then follows from Theorem 12.4.9. □

Remark 12.4.12. Let k be an extension field of $\kappa(y)$. If the scheme $f^{-1}(y) \otimes_{\kappa(y)} k = X \times_Y \mathrm{Spec}(k)$ is connected, so is $f^{-1}(y)$, since it is the image under a projection. For a morphism $f : X \to Y$ of scheme and a point $y \in Y$, we say that the fiber $f^{-1}(y)$ is **geometrically connected** if for any field extension k of $\kappa(y)$, the scheme $f^{-1}(y) \otimes_{\kappa(y)} k = X \times_Y \mathrm{Spec}(k)$ is connected. Under the hypothesis of Corollary 12.4.10, we can then conclude that the fibers $f^{-1}(y)$ are in fact geometrically connected. To see this, observe that for any extension k of $\kappa(y)$, there exists a Noetherian local ring A and a local hommorphism $\varphi : \mathscr{O}_{Y,y} \to A$ which is flat and such that the residue field of A is $\kappa(y)$-isomorphic to k (**[?]** 0_{III}, 10.3.1). Let $Y_1 = \mathrm{Spec}(A)$, $h : Y_1 \to Y$ be the local morphism corresponding to φ, sending the unique closed point y_1 of Y_1 to y, and put $X_1 = X \times_Y Y_1$ and $f_1 = f \times 1_{Y_1}$. Then f_1 is proper and $f_1^{-1}(y_1)$ is $\kappa(y_1)$-isomorphic to $X \times_Y \mathrm{Spec}(k)$. It then remains to show that $(f_1)_*(\mathscr{O}_{X_1}) = \mathscr{O}_{Y_1}$. Now since g is flat, we have $(f_1)_*(\mathscr{O}_{X_1}) = h^*(f_*(\mathscr{O}_X)) = h^*(\mathscr{O}_Y) = \mathscr{O}_Y$, in view of Corollary 12.1.21 applied to $p = 0$.

In the general case of Theorem 12.4.9, the same reasoning shows that we have $(f_1)_*(\mathscr{O}_{X_1}) = h^*(g_*(\mathscr{O}_{Y'}))$ (Corollary 11.1.30), and the Stein factorization $f_1 = g_1 \circ f_1'$ of f_1 is such that we have the Cartesian square

$$\begin{tikzcd} X_1 \arrow[r, "f_1'"] \arrow[d] & Y_1' \arrow[r, "g_1"] \arrow[d] & Y_1 \arrow[d, "h"] \\ X \arrow[r, "f'"] & Y' \arrow[r, "g"] & Y \end{tikzcd}$$

By the transitivity of fibers, we then see that the number of connected components of $f_1^{-1}(y_1)$ is, in view of Corollary 12.4.11, equal to the number of elements of $g_1^{-1}(y_1) = g^{-1}(y) \otimes_{\kappa(y)} k$. If we choose k to be an algebraic closure of $\kappa(y)$, this is also the number of geometric points of $g^{-1}(y)$, or the sum of the separable degrees $[\kappa(y_i') : \kappa(y)]_s$, where y_i' runs through the finite set $g^{-1}(y)$ (Corollary 10.6.47). Note that the $\kappa(y_i')$ are none other than the residual fields of the semi-local ring $(f_*(\mathscr{O}_X))_y$.

Proposition 12.4.13. *Let X and Y be two locally Noetherian integral schemes and $f : X \to Y$ be a proper and dominante morphism. For any $y \in Y$, the number of connected components of $f^{-1}(y)$ is also equal to the number of maximal ideals of the integral closure $\mathscr{O}_{Y,y}'$ of $\mathscr{O}_{Y,y}$ in the rational function field $K(X)$.*

Proof. Recall that for any open subset U of Y, $\Gamma(U, f_*(\mathscr{O}_X)) = \Gamma(f^{-1}(U), \mathscr{O}_X)$ is the intersection of the local rings $\mathscr{O}_{X,x}$ for $x \in f^{-1}(U)$ (formula (10.7.1)). We then conclude that the fiber $(f_*(\mathscr{O}_X))_y$ is a subring of $K(X)$ containing $\mathscr{O}_{Y,y}$. Moreover, as $f_*(\mathscr{O}_X)$ is a coherent $\mathscr{O}_X$-module,

$(f_*(\mathscr{O}_X))_y$ is a finitely generated $\mathscr{O}_{Y,y}$-module, hence contained in $\mathscr{O}'_{Y,y}$. By Corollary 4.1.73, any maximal ideal of the ring A is the intersection of A of a maximal ideal of $\mathscr{O}'_{Y,y}$, whence the proposition. □

A local ring A is called **unibranch** if A_{red} is an integral ring and the integral closure of A_{red} is a local ring. We say a point y of an integral scheme Y is unibrach if the local ring $\mathscr{O}_{Y,y}$ is **unibranch** (this is the case if Y is normal at y). Let A be an integral local ring, and K be its fractional field. For A to be unibranch, it is then necessary and sufficient that any subring R of K, containing A and is a finite A-algebra, is a local ring. In fact, if A' is the integral closure A, then any such ring is contaiend in A' and any maximal ideal of R is trace of a maximal ideal of A' along R, so if A' is local, so is R. Conversely, A' is the inductive limit of the filtered family of such finite A-algebras A_α of A', and if each A_α is a local ring, then for $A_\alpha \subseteq A_\beta$, the maximal ideal of A_α is the trace over A_α of the maximal ideal of A_β, so A' is a local ring.

We also note that if the completion of a Noetherian local ring A is integral (in this case A is called **analytically integral**), then A is unibranch. In fact, let $\mathfrak{m}$ be the maximal ideal of A, K be its fraction field, and L be that of $\widehat{A}$; we then have $L = K \otimes_A \widehat{A}$. Let B be a finite sub-A-algebra of K, then the subring R of L generated by $\widehat{A}$ and B is isomorphic to $B \otimes_A \widehat{A}$; this is a finitely generated $\widehat{A}$-module, and is the $\mathfrak{m}$-adic completion of B. As B is a semi-local ring (Corollary 3.2.24) and this completion is integral, we conclude from 2.4.25 that B has a unique maximal ideal, whence our assertion.

Corollary 12.4.14. *Under the hypotheses of 12.4.13, suppose that $[K(Y) : K(X)]_s = n$ and that $y \in Y$ is unibranch. Then the fiber $f^{-1}(y)$ has at most n connected components. In particular, if $K(X)$ is purely inseparable over $K(X)$, then $f^{-1}(y)$ is connected.*

Proof. Let $\mathscr{O}'_{Y,y}$ be the integral closure of $\mathscr{O}_{Y,y}$, then the integral closure R of $\mathscr{O}_{Y,y}$ in $K(X)$ is also that of $\mathscr{O}'_{Y,y}$. If $\mathscr{O}'_{Y,y}$ is a local ring, then R is a semi-local ring with at most n maximal ideals (Proposition 4.2.16). □

Remark 12.4.15. Corollary 12.4.14 is essentially the "connectedness theorem" proved by Zariski for algebraic schemes. Note that in Corollary 12.4.14 if we suppose that Y is normal at y, then the fiber $f^{-1}(y)$ is geometrically connected, since (with the notation of Remark 12.4.12) $g^{-1}(y)$ then reduces to a point y' and $\kappa(y')$ is purely inseparable over $\kappa(y)$.

Corollary 12.4.16. *Under the hypothesis of Proposition 12.4.13, suppose moreover that $K(Y)$ is algebraically closed in $K(X)$, and let y be a normal point of Y. Then $f^{-1}(y)$ is geometrically connected, and there exists an open neighborhood U of y such that $f_*(\mathscr{O}_X|_{f^{-1}(U)})$ is isomorphic to $\mathscr{O}_Y|_U$. In particular, if we suppose that Y is normal (and $K(Y)$ is algebraically closed in $K(X)$), then $f_*(\mathscr{O}_X)$ is isomorphic to $\mathscr{O}_Y$.*

Proof. The first assertion concerning $f^{-1}(y)$ is a particular case of Remark 12.4.15. If $f : X \xrightarrow{f'} Y' \xrightarrow{g} Y$ is the Stein factorization, then $g^{-1}(y)$ is reduced to a single point y'. Moreover, we have $\mathscr{O}_{Y,y} \subseteq \mathscr{O}_{Y',y'} = (f_*(\mathscr{O}_X))_y \subseteq K(X)$, and as $\mathscr{O}_{Y',y'}$ is finite over $\mathscr{O}_{Y,y}$ (and a fortiori over $K(Y)$), it is contained in $K(Y)$ by our hypothesis; as y is normal, we necessarily have $\mathscr{O}_{Y',y'} = \mathscr{O}_{Y,y}$. We then conclude that g is a local isomorphism at the point y' (Proposition 10.6.53), which completes the proof of the first part of the corollary. There seconds one follows from the first, because the additional assumption entails that g is an isomorphism in the neighborhood of any point of Y', whence an isomorphism. □

Proposition 12.4.17. *Let A be a unibranch Noethreian local ring, $\mathfrak{a}$ be a defining ideal of A, $A_0 = A/\mathfrak{a}$, and $S = \mathrm{gr}_{\mathfrak{a}}(A)$. Then $\mathrm{Proj}(S)$ is a connected A_0-scheme.*

Proof. Let $\mathfrak{m}$ be the maximal ideal of A; $Y = \operatorname{Spec}(A)$ is an integral scheme with unique closed point y. By hypothesis, we have $\mathfrak{m}^k \subseteq \mathfrak{a} \subseteq \mathfrak{m}$ for an integer k, so $V(\mathfrak{a}) = \{\mathfrak{m}\}$. Let $S' = \bigoplus_{n \geq 0} \mathfrak{a}^n$, and $X = \operatorname{Proj}(S')$, which is the blow up Y-scheme relative to $\mathfrak{a}$. Then X is integral and the structural morphism $f : X \to Y$ is birational and projective (Proposition 11.8.1). Corollary 12.4.14 is then applicable and shows that $f^{-1}(y)$ is connected. But $f^{-1}(y)$ is the underlying space of $\operatorname{Proj}(S' \otimes_A A_0)$ (Proposition 11.2.50 and Proposition 10.3.37), and as $S' \otimes_A A_0 = S$ by definition, the proposition follows. □

12.4.3 Zariski's "Main Theorem"

Proposition 12.4.18. *Let Y be a locally Noetherian scheme and $f : X \to Y$ be a proper morphism. Let X' be the set of points $x \in X$ which is isolated in the fiber $f^{-1}(f(x))$. Then X' is open in X, and if $f = g \circ f'$ is the Satein factorization of f, the restriction of f' to X' is an isomorphism from X' onto an open subscheme of Y', and we have $X' = f'^{-1}(U)$.*

Proof. As $g^{-1}(f(x))$ is finite and discrete (Corollary 11.6.5), for x to be isolated in $f^{-1}(f(x))$, it is necessary and sufficient that it is isolated in $f'^{-1}(f'(x))$. We may then assume that $f' = f$, so $f_*(\mathscr{O}_X) = \mathscr{O}_Y$. Then, if $x \in X'$, the fiber $f^{-1}(f(x))$ is connected by Corollary 12.4.10, and is necessarily reduced to the point x. As f is closed, for any open neighborhood V of x, $f(X - V)$ is closed in Y and does not contain $y = f(x)$, since $f^{-1}(y) = \{x\}$; if U is the complement of $f(X - V)$ in Y, then we have $f^{-1}(U) \subseteq V$, so we conclude that the inverse images under f of a fundamental system of open neighborhoods of y form a fundamental system of open neighborhoods of x. The hypothesis $f_*(\mathscr{O}_X) = \mathscr{O}_Y$ then implies that the homomorphism $f_y^\# : \mathscr{O}_{Y,y} \to \mathscr{O}_{X,x}$ is an isomorphism, so by Proposition 10.6.53 there exists an open neighborhood V of x and an open neighborhood U of y such that $f^{-1}(U) \subseteq V$ and the restriction $f|_V : V \to U$ is an isomorphism. Moreover, by the remarks above, we may also assume that $V = f^{-1}(U)$, so that $V \subseteq X'$, which completes the proof. □

Proposition 12.4.19. *Let Y be a locally Noetherian scheme, $f : X \to Y$ be a morphism. Then the following conditions are equivalent:*

(i) *f is finite.*

(ii) *f is affine and proper.*

(iii) *f is proper, and for any $y \in Y$, the fiber $f^{-1}(y)$ is a finite set.*

Proof. We see that (i)⇒(ii) by Corollary 11.6.9. If f is proper and affine, then so is the induced morphism $f^{-1}(y) \to \operatorname{Spec}(\kappa(y))$ (Proposition 11.5.23 and Proposition 11.1.33), and the finiteness theorem Theorem 12.3.3, applied to the structural sheaf of $f^{-1}(y)$, shows that $f^{-1}(y) = \operatorname{Spec}(A)$, where A is a finite dimensional $\kappa(y)$-algebra. Then $f^{-1}(y)$ is a finite $\kappa(y)$-scheme, so is a finite set (Proposition 10.6.44), and we see that (ii)⇒(iii). Finally, as $f^{-1}(y)$ is an algebraic scheme, the hypothesis that $f^{-1}(y)$ is a finite set implies that it is discrete (Proposition 10.6.44). With the notations of Proposition 12.4.18, we then have $X' = X$, and $f' : X' \to Y'$ is an isomorphism; as g is a finite morphism, we then conclude that (iii)⇒(i). □

Theorem 12.4.20 (Zariski's "Main Theorem"). *Let Y be a Noetherian scheme, $f : X \to Y$ be a quasi-projective morphism, and X' be the set of points $x \in X$ that is isolated in the fiber $f^{-1}(f(x))$. Then X' is an open subset of X, and the open subscheme X' of X is isomorphic via f to an open subscheme Y' of Y that is finite over Y.*

Proof. The hypotheses implies that there exists a projective Y-scheme P such that X is Y-isomorphic to an open subscheme of P. We then reduce to prove the theorem if f is a projective morphism, hence proper (Theorem 11.5.30), and this follows from Proposition 12.4.18. □

Remark 12.4.21. If X is reduced (resp. irreducible and X' is nonempty), we can suppose, in the situation of Theorem 12.4.20, that Y' is reduced (resp. irreducible). In fact, we can replace Y' by the scheme-theoretic closure $\overline{X'}$ of X' in Y', and it is reduced if X' is (Proposition 10.6.69). If X' is nonempty, then it is irreducible if X is, and then $\overline{X'}$ is also irreducible.

Corollary 12.4.22. *Let Y be a locally Noetherian and separated scheme, $f : X \to Y$ be morphism of finite type, and $x \in X$ be a point that is isolated in $f^{-1}(f(x))$. then there exists an open neighborhood U of x that is isomorphic to an open subscheme of Y that is finite over Y.*

Proof. Let $y = f(x)$, U be an affine open neighborhood of Y, V an affine open neighborhood of x contained in $f^{-1}(U)$. As Y is separated, the injection $U \to Y$ is affine (Corollary 11.1.34), and as V is affine over U (again by Corollary 11.1.34), the restriction of f to V is an affine morphism $V \to Y$ (Proposition 11.1.33(ii)); a fortiori, this restriction is quasi-projective since it is of finite type (Proposition 10.6.35(i) and Proposition 11.5.20). It then suffices to apply Theorem 12.4.20. □

Corollary 12.4.23. *Let A be a Noetherian ring, B be an A-algebra of finite type, $\mathfrak{P}$ be a prime ideal of B, $\mathfrak{p}$ be its contraction in A. Suppose that $\mathfrak{P}$ is both maximal and minimal among prime ideals of B with contraction $\mathfrak{p}$, then there exists $g \in B - \mathfrak{P}$, a finite A-algebra A' and an element $f' \in A'$ such that the A-algebra B_g and $A'_{f'}$ are isomorphic.*

Proof. It suffices to apply Corollary 12.4.22 to $Y = \operatorname{Spec}(A)$ and $X = \operatorname{Spec}(B)$, since the hypothesis on $\mathfrak{P}$ signified that it is isolated in the fiber. □

Corollary 12.4.24. *Let A be a Noetherian local ring, B be an A-algebra of finite type, $\mathfrak{M}$ be a maximal ideal of B whose contraction in A is the maximal ideal $\mathfrak{m}$ of A. Suppose that $\mathfrak{M}$ and minimal among prime ideals of B with contraction $\mathfrak{m}$. Then there exists a finite A-algebra A' and a maximal ideal $\mathfrak{m}'$ of A' (whose contraction is $\mathfrak{m}$) such that $B_{\mathfrak{M}}$ is isomorphic to the A-algebra $A'_{\mathfrak{m}'}$.*

Proof. This follows from Corollary 12.4.23 by taking stalks at $\mathfrak{m}$ and $\mathfrak{M}$. □

Corollary 12.4.25. *Under the hypothesis of Corollary 12.4.24, suppose that A and B are integral with the same fraction field K. Then, if A is integrally closed, we have $B = A$.*

Proof. By Remark 12.4.21 we can suppose that A' is integral with fraction field K. The hypothesis over A implies that $A' = A$, so $B_{\mathfrak{M}} = A$. As $A \subseteq B \subseteq B_{\mathfrak{M}}$, we conclude that $A = B$. □

Corollary 12.4.26. *Let Y be a locally Noetherian integral scheme, $f : X \to Y$ be a separated morphism that is of finite type and birational. Suppose that Y is normal and the fibers $f^{-1}(y)$ are finite for $y \in Y$. Then f is an open immersion; if moreover f is closed, then it is an isomorphic.*

Proof. Let $x \in X$ and put $y = f(x)$. As $f^{-1}(y)$ is an algebraic scheme over $\kappa(y)$, the hypothesis that it is finite implies that it is discrete (Proposition 10.6.44); moreover $\mathscr{O}_{Y,y}$ is integrally closed and $\mathscr{O}_{X,x}$ and $\mathscr{O}_{Y,y}$ have the same fraction field. We can then apply Corollary 12.4.25, and the homomorphism $f_y^\sharp : \mathscr{O}_{Y,y} \to \mathscr{O}_{X,x}$ is bijective. We then conclude that f is a local isomorphism. □

Proposition 12.4.27. *Let Y be a locally Noetherian scheme and $f : X \to Y$ be a morphism locally of finite type. Then the set X' of $x \in X$ isolated in the fiber $f^{-1}(f(x))$ is open in X.*

Proof. This question is local over X and Y, so we can assume that X and Y are affine Noetherian; f is then affine and of finite type, hence quasi-projective (Proposition 11.5.20(i)), and it suffices to apply Theorem 12.4.20. □

Corollary 12.4.28. *Let Y be a locally Noetherian scheme, $f : X \to Y$ be a proper morphism. Then the set U of points $y \in Y$ such that $f^{-1}(y)$ is discrete is open in Y, and the restriction morphism $f|_{f^{-1}(U)} : f^{-1}(U) \to U$ is finite. In particular, a proper and quasi-finite morphism is finite.*

Proof. In fact, the complement of U in Y is the image of $X - X'$ under f, which is closed in X by Proposition 12.4.27. As f is a closed map, U is then open in Y. Moreover, it follows from Proposition 10.6.44 that $f^{-1}(y)$ is finite for any $y \in U$; as the restriction morphism $f|_{f^{-1}(U)} : f^{-1}(U) \to U$ is proper (Proposition 11.5.23), it is finite in view of Proposition 12.4.19. □

12.5 Covariant functors on $\mathbf{Mod}(A)$ and base change

Let $f : X \to Y$ be a morphism of schemes and $\mathscr{F}$ be a quasi-coherent $\mathscr{O}_X$-module. In the study of the higher direct images $R^p f_*(\mathscr{F})$, the following problem is proposed: given a base change morphism $g : Y' \to Y$, we put $X' = X \times_Y Y'$, $f' = f_{(Y)}$ and $\mathscr{F}' = \mathscr{F} \otimes_Y \mathscr{O}_{Y'}$, so that the following diagram commutes:

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Y \end{array}$$

Our purpose is to understanding the higher direct images $R^p f'_*(\mathscr{F}')$ (assume giving the information of $R^p f_*(\mathscr{F})$), which (as we will see) can be reduced to considering the behaviour of $R^p f_*(\mathscr{F} \otimes_{\mathscr{O}_Y} \mathscr{G})$ for quasi-coherent $\mathscr{O}_Y$-modules $\mathscr{G}$, or that of the functor $R^p f_*(\mathscr{F} \otimes_{\mathscr{O}_Y} (-))$. If $\mathscr{F}$ is flat over Y, the functor $\mathscr{F} \otimes_{\mathscr{O}_Y} (-)$ is exact so the composition $R^\bullet f_*(\mathscr{F} \otimes_{\mathscr{O}_Y} (-))$ is a cohomological functor. But in general this is not true, and we want to replace $R^\bullet f_*(\mathscr{F} \otimes_{\mathscr{O}_Y} (-))$ by some good bahaved cohomological functor. We shall see in ([?], 6.10) that this new cohomological functor $\mathscr{G} \mapsto \mathscr{T}^\bullet(\mathscr{G})$ is defined locally (over Y) by $H^\bullet(\mathscr{L}^\bullet \otimes_Y \mathscr{G})$, where $\mathscr{L}^\bullet$ is a complex of locally free $\mathscr{O}_Y$-modules (uniquely determined up to homotopy). It is then interesting to forget about $\mathscr{T}^\bullet$ and consider the properties of functors of the form $H^\bullet(\mathscr{L}^\bullet \otimes_Y (-))$ (with appropriate finiteness conditions on $\mathscr{L}^\bullet$ of $H^i(\mathscr{L}^\bullet)$). The most important properties of these functors concern the exactness of the component $\mathscr{T}^i$ of $\mathscr{T}^\bullet$, and we will give various criterions for establishing such properties. As an application, we will obtain conditions allowing us to confirm that the functor $R^p f_*(\mathscr{F} \otimes_{\mathscr{O}_Y} (-))$ is exact (what we will express by saying that $\mathscr{F}$ is *cohomologically flat* on Y in dimension p). Another important property for the components $\mathscr{T}^i$ of $\mathscr{T}^\bullet$ is the semi-continuous property of the function $y \mapsto \dim_{\kappa(y)}(\mathscr{T}^i(\kappa(y)))$. If $\mathscr{T}^i$ is exact, this property is replaced by a continuity (hence locally constant) property, the converse being true according to Grauert when Y is reduced.

12.5.1 Functor on $\mathbf{Mod}(A)$

Let A be a ring (not necessarily commutative) and $\mathbf{Mod}(A)$ be the category of left A-modules. Consider an additive covariant functor $T : \mathbf{Mod}(A) \to \mathbf{Ab}$, and let M be a (A, A)-bimodule. Then the abelian group $T(M)$ is naturally endowed with a right A-module structure. In fact, for any $a \in A$, let $h_{a,M}$ be the endomorphism $x \mapsto xa$ induced on M. Then by hypothesis, $T(h_{a,M})$ is an endomorphism of $T(M)$. Moreover, since T is additive, for $a, b \in A$ we have

$$T(h_{ab}) = T(h_b \circ h_a) = T(h_b) \circ T(h_a), \quad T(h_{a+b}) = T(h_a) + T(h_b)$$

so the map $(a, y) \mapsto T(h_a)(y)$ is a right A-module structure on $T(M)$. In particular, $T(A)$ is a right A-module.

If A is a commutative ring, then for any A-module M, $T(M)$ is also an A-module. If $u : M \to N$ is a homomorphism of A-modules, we have, for $a \in A$, $u \circ h_a = h_{a,N} \circ u$, so $T(u) \circ T(h_{a,M}) = T(h_{a,N}) \circ T(u)$, which means $T(u) : T(M) \to T(N)$ is an A-homomorphism. We then say that T can be considered as a covariant endofunctor on $\mathbf{Mod}(A)$. More precisely, we define an equivalence from the category of additive covariant functors $\mathbf{Mod}(A) \to \mathbf{Ab}$ to the category of A-linear covariant functors $\mathbf{Mod}(A) \to \mathbf{Mod}(A)$, given by $T(h_{a,M}) = h_{a,T(M)}$ for

$a \in A$. As the forgetful functor $\mathbf{Mod}(A) \to \mathbf{Ab}$ is exact and faithful, the exactness of these functors are therefore equivalent.

Assume that A is commutative and let B be an A-algebra (not necessarily commutative) and $\rho : A \to B$ be the ring homomorphism. Then we have an additive covariant functor $\rho^* : N \mapsto \rho^*(N)$ from the category of B-modules to that of A-modules. By composition with T, we then deduce a functor $T_{(B)} : \mathbf{Mod}(B) \to \mathbf{Ab}$, which is evidently additive and covariant, and called the **extension of scalar** of T by B. It is clear that the extension of scalars is functorial and additive on T. Moreover, if T commutes with inductive limits or direct sums (resp. is left exact, resp. is right exact), so is the functor $T_{(B)}$, since the functor ρ^* is exact and commutes with inductive limits and direct sums.

Suppose that A is commutative and T commutes with inductive limits. Then for any multiplicative subset S of A and any A-module M, we have a canonical isomorphism of A-modules

$$T(S^{-1}M) \xrightarrow{\sim} S^{-1}T(M). \tag{12.5.1}$$

In fact, suppose first that S is of the form f^n for $f \in A$. Then we see that $M_f = \varinjlim M_n$, where (M_n, φ_{nm}) is the inductive system of A-module $M_n = M$ with $\varphi_{nm} = h_{f^{n-m}}$, so the isomorphism (12.5.1) follows from the hypothesis on T. In the general case, we have $S^{-1}M = \varinjlim_{f \in S} M_f$, so we can similarly conclude the isomorphism. Moreover, the functoriality of (12.5.1) shows that it is an isomorphism of $S^{-1}A$-modules, and we can then write

$$T_{(S^{-1}A)}(S^{-1}M) = S^{-1}T(M) = T(S^{-1}M). \tag{12.5.2}$$

If $S = A - \mathfrak{p}$ is the complement of a prime ideal $\mathfrak{p}$ of A, we then write $T_{\mathfrak{p}}$ instead of $T_{(A_{\mathfrak{p}})}$.

Proposition 12.5.1. *Under the above hypothesis, if $T_{\mathfrak{m}}$ is left exact (resp. right exact) for any maximal ideal $\mathfrak{m}$ of A, then T is left exact (resp. right exact).*

Proof. This is a direct concequence of that fact that two submodules N, P of an A-module M are equal if and only if $N_{\mathfrak{m}} = P_{\mathfrak{m}}$ for any maximal ideal $\mathfrak{m}$ of A (Corollary 1.3.21). ☐

12.5.2 Characterization of tensor product functor

Let A be a ring (not necessarily commutative) and $T : \mathbf{Mod}(A) \to \mathbf{Ab}$ be an additive covariant functor. For any left A-module M, we note that $\mathrm{Hom}_{\mathbb{Z}}(T(A), T(M))$ is canonically endowed with a left A-module structure such that $(a \cdot u)(y) = u(ya)$ for $y \in T(A)$, $a \in A$, $v \in \mathrm{Hom}_{\mathbb{Z}}(T(A), T(M))$. We define a homomorphism $\tilde{t}_M$ from M into $\mathrm{Hom}_{\mathbb{Z}}(T(A), T(M))$ to be the composition

$$M \xrightarrow{\sim} \mathrm{Hom}_A(A, M) \xrightarrow{T} \mathrm{Hom}_{\mathbb{Z}}(T(A), T(M))$$

where the first arrow is the canonical isomorphism $x \mapsto \theta_x$, given by $\theta_x(\xi) = \xi x$ for $\xi \in A$ and $x \in M$. Note that we have $\theta_{ax} = \theta_x \circ h_a$, so $T(\theta_{ax}) = T(\theta_x \circ h_a) = T(\theta_x) \circ T(h_a)$ and for $y \in T(A)$,

$$T(\theta_{ax})(y) = T(\theta_x)(T(h_a)(y)) = T(\theta_x)(ya).$$

We therefore conclude that $\tilde{t}_M$ induces a homomorphism

$$t_M : T(A) \otimes_A M \to T(M) \tag{12.5.3}$$

such that $t_M(a \otimes x) = T(\theta_x)(a)$ for $a \in T(A)$, $x \in M$. It is immediate to verify that t_M is functorial on M, which means for any homomorphism $u : M \to N$ of left A-modules, the diagram

$$\begin{array}{ccc} T(A)\otimes_A M & \xrightarrow{t_M} & T(M) \\ {\scriptstyle 1\otimes u}\downarrow & & \downarrow{\scriptstyle T(u)} \\ T(A)\otimes_A M & \xrightarrow{t_N} & T(N) \end{array} \tag{12.5.4}$$

is commutative.

More generally, if A is commutative, then for any A-module N we can define a canonical homomorphism

$$t_{N,M} : T(N)\otimes_A M \to T(N\otimes_A M). \tag{12.5.5}$$

For this, it suffices to replace θ_x by the A-module homomorphism $N \to N\otimes_A M$, which sends $y \in N$ to $y\otimes x$. It is clear that this functor is functorial on M and N. In particular, if B is an A-algebra (not necessairly commutative), we have a functorial homomorphism

$$T(M)_{(B)} = T(M)\otimes_A B \to T_{(B)}(M_{(B)}) \tag{12.5.6}$$

which is a homomorphism of B-modules. Moreover, the following diagram is commutative

$$\begin{array}{ccc} T(A)\otimes_A M & \xrightarrow{\quad t_M \quad} & T(M) \\ \downarrow & & \downarrow \\ T_{(B)}(B)\otimes_B M_{(B)} & \xrightarrow{t_{M_{(B)}}} & T_{(B)}(M_{(B)}) \end{array}$$

where the right vertical arrow is the composition

$$T(M) \to T(M)\otimes_A B \to T(M\otimes_A B) = T_{(B)}(M_{(B)})$$

and the left vertical arrow is the homomorphism

$$T(A)\otimes_A M \to T_{(B)}(B)\otimes_B (B\otimes_A M) = T_{(B)}(B)\otimes_A M,$$

where $T(A) \to T_{(B)}(B) = T(B)$ is the homomorphism induced by $A \to B$.

Lemma 12.5.2. *If $T : \mathbf{Mod}(A) \to \mathbf{Ab}$ is a covariant additive functor which commutes with direct sums, the canonical homomorphism t_L is an isomorphism for any free A-module L.*

Proof. In fact, we can write $L = \bigoplus_i L_i$, where L_i are isomorphic to A for each $i \in I$. The definition of t_L shows that $t_L = \bigoplus_i t_{L_i}$, since

$$T : \mathrm{Hom}_A(A, L) \to \mathrm{Hom}_{\mathbb{Z}}(T(A), T(L))$$

is the direct sum of the $\mathbb{Z}$-linear maps $T_i : \mathrm{Hom}_A(A, L_i) \to \mathrm{Hom}_{\mathbb{Z}}(T(A), T(L_i))$ in view of the hypothesis on T. We are then reduced to the case where $L = A$, and t_A is none other than the canonical isomorphism $T(A)\otimes_A A \cong T(A)$ for the A-module $T(A)$. □

Lemma 12.5.3. *Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories, F, G be covariant additive functors from $\mathcal{A}$ to $\mathcal{B}$, and $\gamma : F \Rightarrow G$ be a morphism such that, for any object $A \in \mathcal{A}$, $\gamma_A : F(A) \to G(A)$ is an epimorphism. Then, if F is right exact and G is semi-exact, G is right exact.*

Proof. Since G is semi-exact, it suffices to prove that for any epimorphism $v : A \to B$ in $\mathcal{A}$, $G(v) : G(A) \to G(B)$ is an epimorphism. Now, we have a commutative diagram

$$\begin{array}{ccc} F(A) & \xrightarrow{F(v)} & F(B) \\ {\scriptstyle \gamma_A}\downarrow & & \downarrow{\scriptstyle \gamma_B} \\ G(A) & \xrightarrow{G(v)} & G(B) \end{array}$$

in which $F(v)$, γ_A, γ_B are epimorphisms. It then follows from $G(v)$ is an epimorphism. □

Proposition 12.5.4. *Let $T : \mathbf{Mod}(A) \to \mathbf{Ab}$ be a covariant additive functor which commutes with direct sums. Then the following conditions are equivalent:*

(i) *T is right exact;*

(ii) *the canonical homomorphism t_M is an isomorphism for any A-module M;*

(ii') *T is semi-exact and the homomorphism t_M is surjective for any A-module M;*

(iii) *T is isomorphic to the functor $N \otimes_A (-)$, where N is an A-module.*

Proof. It is clear that (ii) implies (iii) and (iii) implies (i), since we can then take $N = T(A)$. We now prove that (i)⇒(ii); for this, put $F(M) = T(A) \otimes_A M$ for any A-module M. There exists an exact sequence $P \to L \to M \to 0$, where P and L are free A-modules. As T and F are right exact, we then have a commutative diagram

$$\begin{tikzcd} F(P) \arrow[r] \arrow[d, "t_P"] & F(L) \arrow[r] \arrow[d, "t_L"] & F(M) \arrow[r] \arrow[d, "t_M"] & 0 \\ T(P) \arrow[r] & T(L) \arrow[r] & T(M) \arrow[r] & 0 \end{tikzcd}$$

with exact rows. As t_P and t_L are isomorphism by Lemma 12.5.2, so is t_M by five lemma. Finally, it is clear that (ii) implies (ii'), and to see that (ii') implies (ii), it suffices to apply Lemma 12.5.3. □

Remark 12.5.5. For any right A-module N, let $T_N(M) = N \otimes_A M$, where M is a left A-module. Then T_N is a right exact covariant additive functor from $\mathbf{Mod}(A)$ to $\mathbf{Ab}$, and commutes with direct sums. If we canonically identify $T_N(A)$ with N, then the corresponding homomorphism (12.5.5) is the identity. We therefore conclude that the right A-module N in Proposition 12.5.4 is uniquely determined up to isomorphisms, and is isomorphic to $T(A)$. In other words, the morphisms $T \mapsto T(A)$ and $N \mapsto T_N$ define a equivalence from the category of covariant additive functors which are right exact and commute with direct sums to the category of right A-modules.

Proposition 12.5.6. *Let A be a left Artinian ring with Jacobson radical $\mathfrak{m}$ such that $A/\mathfrak{m}$ is a field k. Let $T : \mathbf{Mod}(A) \to \mathbf{Ab}$ be a covariant additive functor which commutes with direct sums. Then the conditions of Proposition 12.5.4 are equivalent to the following:*

(v) *T is semi-exact and the homomorphism $T(\varepsilon) : T(A) \to T(k)$ induced by the canonical homomorphism $\varepsilon : A \to k$ is surjective.*

12.5.3 Exactness of cohomological functors on Mod(A)

Proposition 12.5.7. *Let A be a ring (not necessarily commutative) and $T^\bullet$ be a covariant cohomological functor from $\mathbf{Mod}(A)$ to $\mathbf{Ab}$ which commutes with direct sums. Let p be an integer such that T^p and T^{p-1} are defined. Then the following conditions are equivalent:*

(i) *T^p is right exact;*

(ii) *T^{p+1} is left exact;*

(iii) *for any left A-module M, the canonical homomorphism*

$$T^p(A) \otimes_A M \to T^p(M) \tag{12.5.7}$$

is an isomorphism.

(iv) *for any A-module M, the homomorphism (12.5.7) is an epimorphism;*

(v) *T^p is isomorphic to a functor $N \otimes_A (-)$, where N is a (uniqely determined) right A-module and isomorphic to $T^p(A)$.*

If the conditions of Proposition 12.5.6 are satisfied, then these conditions are equivalent to:

(vi) *the canonical homomorphism $T^p(\varepsilon) : T^p(A) \to T^p(k)$ is an epimorphism.*

Proof. By the definition of cohomological functors, T^i are semi-exact for any i such that T^i is defined. Moreover, for any exact sequence

$$0 \longrightarrow A \xrightarrow{u} B \xrightarrow{v} C \longrightarrow 0$$

we have $\ker(T^{p+1}(u)) = \operatorname{coker}(T^p(v))$, so it is clear that (i) and (ii) are equivalent. The rest of the proposition follows from Proposition 12.5.4 and Proposition 12.5.6. □

Corollary 12.5.8. *Let A be a commutative ring. With the notations of Proposition 12.5.7, suppose that T^p is right exact. If $f \in A$ does not belong to the annihilator of any nonzero element of an A-module M, then f doe not belong to the annihilator of any nonzero element of the A-module $T^{p+1}(M)$. In particular, if A is an integral domain, then the A-module $T^{p+1}(A)$ is torsion free.*

Proof. Let h_f be the homothety with ratio f on M. Then the hypothesis signifies that h_f is injective, and by Proposition 12.5.7, the homomorphism $T^{p+1}(h_f)$ is also injective, whence the corollary. □

Proposition 12.5.9. *Let A be a ring and $T^\bullet : \mathbf{Mod}(A) \to \mathbf{Ab}$ be a cohomological functor which commutes with direct sums. Let p be an integer such that T^{p-1}, T^p and T^{p+1} are defined. Then the following conditions are satisfied:*

(i) *T^p is exact;*

(ii) *T^{p-1} is right exact and T^{p+1} is left exact;*

(iii) *for any left A-module M, the canonical homomorphism*

$$T^i(A) \otimes_A M \to T^i(M) \tag{12.5.8}$$

is an isomorphism for $i = p, p-1$;

(iii′) *for any left A-module M, the canonical homomorphism (12.5.8) is an epimorphism for $i = p, p-1$;*

(iv) *for any left A-module M, the canonical homomorphism (12.5.8) is an isomorphism for $i = p$ and $T^p(A)$ is a flat right A-module;*

(iv′) *for any left A-module M, the canonical homomorphism (12.5.8) is an epimorphism for $i = p$ and $T^p(A)$ is a flat right A-module.*

Proof. The equivalence of (i) and (ii) follows from the equivalence of conditions (i) and (ii) of Proposition 12.5.7, and the equivalence of (ii), (iii) and (iii′) follows from that of conditions (i), (iii) and (iv) of Proposition 12.5.7. Finally, to sya that $T^p(A)$ is flat signifies that the functor $T^p(A) \otimes_A M$ is left exact, so the equivalence of (i), (iv′) and (iv) also follows from that of conditions (i), (iii) and (iv) of Proposition 12.5.7. □

Corollary 12.5.10. *Suppose that A is commutative, T^p is exact, and that $T^p(A)$ is an A-module of finite presentation. Then the function $x \mapsto \dim_{\kappa(x)}(T^p(\kappa(x)))$ is locally free over $X = \operatorname{Spec}(A)$, hence constant if $\operatorname{Spec}(A)$ is connected.*

Proof. As $T^p(A)$ is a flat A-module by Proposition 12.5.9, it is projective and finitely generated (Corollary 1.5.7), so $\widetilde{T^p(A)}$ is a locally free $\mathscr{O}_X$-module. We have $T^p(\kappa(x)) = T^p(A) \otimes_A \kappa(x)$, so the rank function is locally constant by Theorem 1.5.5. □

Proposition 12.5.11. *Let A be a left Artinian ring with Jacobson radical $\mathfrak{m}$ such that $A/\mathfrak{m}$ is a field k. Let $T^\bullet : \mathbf{Mod}(A) \to \mathbf{Ab}$ be a cohomological functor which commutes with direct sums. Then the conditions of Proposition 12.5.9 are equivalent to the following:*

(v) *the canonical homomorphism $T^i(\varepsilon) : T^i(A) \to T^i(k)$ is an epimorphism for $i = p, p+1$;*

(vi) *$T^p(\varepsilon)$ is an epimorphism and $T^p(A)$ is a flat right A-module.*

Suppose that A is commutative. Then the above conditions are equivalent to the following:

(vii) *for any A-module M of finite length, we have $\ell(T^p(M)) = \ell(T^p(k)) \cdot \ell(M)$;*

(viii) *$\ell(T^p(A)) = \ell(T^p(k)) \cdot \ell(A)$.*

Proof. The equivalence of (v), (vi) and the conditions of Proposition 12.5.9 can be deduced from Proposition 12.5.7. □

Proposition 12.5.12. *Let A be a ring and $T^\bullet : \mathbf{Mod}(A) \to \mathbf{Ab}$ be a cohomological functor which commutes with direct sums. Suppose that there exists an integer n_0 such that T^p is exact for $p \geq n_0$. Then, for an integer $n < n_0$, the following conditions are equivalent:*

(i) *T^p is exact for $p \geq n$;*

(ii) *$T^p(A)$ is a flat right A-module for $p \geq n$;*

(iii) *for any left A-module M, the canonical homomorphism $T^p(A) \otimes_A M \to T^p(M)$ is surjective for $p \geq n-1$.*

Proof. The equivalence of (i) and (ii) follows from Proposition 12.5.9 by induction, and (iii) follows from Proposition 12.5.9(iii'). □

If A is a commutative ring, B is an A-algebra (not necessarily) commutative, and $T^\bullet : \mathbf{Mod}(A) \to \mathbf{Ab}$ be a cohomological functor, then since the functor ρ^* is exact and faithful ($\rho : A \to B$ being the ring homomorphism), we see that the extension of scalar $T^\bullet_{(B)}$ is also a cohomological functor.

Corollary 12.5.13. *Suppose that $T^\bullet$ satisfies the conditions of Proposition 12.5.12 and commutes with inductive limits. Moreover, assume that A is an integral domain and the A-modules $T^p(A)$ are of finite presentation. Then for any integer $n < n_0$, there exists a nonzero element $f \in A$ such that the functor $T^p_{(A_f)} : \mathbf{Mod}(A_f) \to \mathbf{Ab}$ is exact for $p \geq n$. In particular, if there are finitely many indices p such that $T^p \neq 0$, then there exists a nonzero element $f \in A$ such that the functors $T^p_{(A_f)}$ are all exact.*

Proof. By hypothesis, T^p is exact for $p \geq n_0$, so $T^p(A)$ is flat for these values of p. In view of Proposition 12.5.12 and (12.5.1), it suffices to choose f such that $T^p_{(A_f)}(A_f) = T^p(A_f) = (T^p(A))_f$ is a free A_f-module for $n \leq p < n_0$. If x is the generic point of $\mathrm{Spec}(A)$, $(T^p(A))_x$ is a vector space of finite dimension over the fraction field of A. As each $T^p(A)$ is of finite presentation, there exists an element $f \in A$ satisfying the desired property (Corollary 1.5.2). □

Corollary 12.5.14. *Suppose that $T^\bullet$ satisfies the conditions of Proposition 12.5.12 and commutes with inductive limits. Moreover, assume that A is Noetherian and the A-modules $T^p(A)$ are finitely generated. Then for any integer n, there exists an open dense subset U of $\mathrm{Spec}(A)$ such that, for any $p \geq n$, the function $x \mapsto \dim_{\kappa(x)}(T^p(\kappa(x)))$ is constant on U.*

Proof. Let $\mathfrak{p}$ be a minimal prime ideal of A. By hypothesis, the ring $B = A/\mathfrak{p}$ is integral and $\mathrm{Spec}(B)$ is identified with an irreducible component of $\mathrm{Spec}(A)$. We now prove by induction on the integer $p \geq n$ that, for each p, there exists $f_p \in B - \{0\}$ such that if we put $B' = B_{f_p}$, then $T^i_{(B')}$ is exact and the $T_i(B')$ are finitely generated B'-modules for $i \geq p$. By our hypothesis, this is true for $p \geq n_0$, since in this case T^p is exact and $T^p(B) \cong T^p(A)/T^p(\mathfrak{p})$, hence is a finitely generated A-module (and a fortiori B-module), so we can choose $f_p = 1$ (so that $B' = B = A/\mathfrak{p}$). We preceed by induction on p, so let f_p be such an element. Then f_p is the canonical image of an element $g_p \in A$, and if we put $A' = A_{g_p}$, we have $B' = A'/\mathfrak{p}'$ where $\mathfrak{p}'$ is the minimal prime ideal $\mathfrak{p}_{g_p}$ of A'. As $T^i(A_{g_p}) = (T_i(A))_{g_p}$, the $T^i(A')$ are finitely generated A'-modules, so $T^\bullet_{(A')}$ satisfies the same condition as $T^\bullet$, with n_0 replaced by p. We can then reduce the proof to the case where $A' = A$ and T^p is exact. The exact sequence $0 \to \mathfrak{p} \to A \to A/\mathfrak{p} \to 0$ then gives an exact sequence

$$T^{p-1}(A) \longrightarrow T^{p-1}(A/\mathfrak{p}) \xrightarrow{\delta} T^p(\mathfrak{p}) \longrightarrow T^p(A)$$

and as T^p is exact, the last arrow is injective, so $T^{p-1}(A/\mathfrak{p})$ is a quotient of $T^{p-1}(A)$ and therefore finitely generated. We note that in the proof of Corollary 12.5.13, we only use the fact that $T^p(A)$ are finitely generated for $p \geq n$, so we can apply it to the integral ring B and the functor $T^\bullet_{(B)}$, with $n = p - 1$, which produces the desired element f_N. If the corollary is proved for the ring B_{f_N}, then there exists an open dense subset W of $\mathrm{Spec}(B_{f_N})$ such that the function $\dim_{\kappa(x)}(T^p(\kappa(x)))$ is constant, since $A_x = (B_{f_N})_x$ for any $x \in W$. By applying this reasoning to each irreducible component of $\mathrm{Spec}(A)$, the corollary is then proved. To summarize, the corollary is now reduced to the case where A is an integral domain, so by Corollary 12.5.13 there exists an element $f \in A - \{0\}$ such that the A_f-modules $T^p(A_f)$ are free of finite rank for $p \geq n$. The corollary then follows from Corollary 12.5.10. □

Proposition 12.5.15. *Let A be a commutative local ring, k be its residue field, $T^\bullet : \mathbf{Mod}(A) \to \mathbf{Ab}$ be a cohomological functor which commutes with direct sums. Suppose that there exists an integer n_0 such that T^p is exact for $i \geq n_0$, and that the $T^p(A)$ are finitely presented A-modules. Then the conditions of Proposition 12.5.12 imply the following conditions, and they are equivalent if A is reduced:*

(iv) *for any $x \in \mathrm{Spec}(A)$, we have*

$$\dim_{\kappa(x)}(T^p(\kappa(x))) = \mathrm{rank}_k(T^p(k)) \quad \text{for} \quad p \geq n;$$

(iv′) *for any generic point x_j of an irreducible component of $\mathrm{Spec}(A)$, we have*

$$\mathrm{rank}_{\kappa(x_j)}(T^p(\kappa(x_j))) = \mathrm{rank}_k(T^p(k)) \quad \text{for} \quad p \geq n.$$

Proof. As $T^p(A)$ is a finitely presented A-module, condition (ii) of Proposition 12.5.12 is equivalent to that $T^p(A)$ is a free A-module for $p \geq n$ (Corollary 1.5.7). Condition (iii) implies that $T^p(\kappa(x)) = T^p(A) \otimes_A \kappa(x)$ for $p \geq n$, so the conditions of Proposition 12.5.12 implies (iv), and it is clear that (iv) implies (iv′). Conversely, we need to show that (iv′) implies (i) if A is reduced. We preceed by descending induction on the integer $p \geq n$, since we know that T^p is exact for $p \geq n_0$. Suppose that T^i is exact for $i \geq p > n$, we show that $T^{p-1}(A)$ is a free A-module. In view of the induction hypothesis, $T^{p-1}(A) \otimes_A M$ is isomorphic to $T^{p-1}(M)$ for any A-module M, by condition (iii) of Proposition 12.5.12 and Proposition 12.5.9. Applying this property to $M = \kappa(x_j)$ and $M = k$, we conclude that for each j, we have

$$\mathrm{rank}_{\kappa(x_j)}(T^{p-1}(A) \otimes_A \kappa(x_j)) = \mathrm{rank}_k(T^{p-1}(k)).$$

But this implies that $T^{p-1}(A)$ is free (Proposition 1.3.14), whence the proposition. □

12.5.4 Exactness of the fonctor $H^\bullet(P^\bullet \otimes_A M)$

Let A be a ring (notn ecessarily commutative) and $P^\bullet$ be a complex of flat right A-modules. As the functor $P^i \otimes_A (-)$ is then exact for each i, the δ-functor

$$T^\bullet(M) = H^\bullet(P^\bullet \otimes_A M) \tag{12.5.9}$$

is a cohomological functor from $\mathbf{Mod}(A)$ to $\mathbf{Ab}$, which is A-linear if A is commutative and commutes with inductive limits.

If A is commutative and B is an A-algebra, then the cohomological functor $T^\bullet_{(B)}$ can be defined by

$$T^\bullet_{(B)} = H^\bullet(P^\bullet \otimes_A \rho^*(N))$$

where $\rho : A \to B$ is the ring homomorphism. As we can write $P^\bullet \otimes_A \rho^*(N) = P^\bullet \otimes_A \rho^*(B \otimes_B N) = (P^\bullet \otimes_A B) \otimes_B N$, we see that

$$T^\bullet_{(B)}(N) = H^\bullet(P^\bullet_{(B)} \otimes_B N) \tag{12.5.10}$$

where $P^\bullet_{(B)} = P^\bullet \otimes_A B$ is a complex of flat B-modules (**??**).

Proposition 12.5.16. *Let $T^\bullet$ be the cohomological functor defined by (12.5.9). Then for an integer p, the following conditions are equivalent:*

(i) *T^p is left exact (or equivalently, T^{p-1} is right exact);*

(ii) *$W^p(P^\bullet) = \operatorname{coker}(P^{p-1} \to P^p)$ is a flat right A-module;*

(iii) *there exists a complex $Q^\bullet$ of flat right A-modules such that $d^{p-1} : Q^{p-1} \to Q^p$ is zero and an isomorphism $H^\bullet(P^\bullet \otimes_A M) \xrightarrow{\sim} H^\bullet(Q^\bullet \otimes_A M)$ of cohomological functors.*

Proof. By definition, we have a canonical exact sequence

$$0 \longrightarrow T^p(M) \longrightarrow W^p(P^\bullet \otimes_A M) \longrightarrow P^{p+1} \otimes_A M$$

where $W^p(P^\bullet \otimes_A M) = \operatorname{coker}(P^{p-1} \otimes_A M \to P^p \otimes_A M) = W^p(P^\bullet) \otimes_A M$ in view of the flatness of $P^\bullet$. For any homomorphism $f : M \to N$, we have a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & T^p(M) \arrow[r] \arrow[d, "u"] & W^p(P^\bullet) \otimes_A M \arrow[r] \arrow[d, "v"] & P^{p+1} \otimes_A M \arrow[d, "w"] \\ 0 \arrow[r] & T^p(N) \arrow[r] & W^p(P^\bullet) \otimes_A N \arrow[r] & P^{p+1} \otimes_A N \end{tikzcd} \tag{12.5.11}$$

with exact rows. If f is a monomorphism, so is w since P^{p+1} is flat; if T^p is left exact, then u is also a monomorphism. We then conclude that v is a monomorphism, which implies that $W^p(P^\bullet)$ is flat. Conversely, if this is the case, then v is a monomorphism for any monomorphism $f : M \to N$, so the diagram (12.5.11) shows that u is a monomorphism, and therefore T^p is left exact. This proves (i) implies (ii), and it is immediate that (iii) implies (i), because if $P^\bullet$ has zero differentials and $0 \to M' \to M \to M'' \to 0$ is an exact sequence of A-modules, the connecting homomorphism of the sequence

$$H^{p-1}(P^\bullet \otimes_A M'') \xrightarrow{\delta} H^p(P^\bullet \otimes_A M') \longrightarrow H^p(P^\bullet \otimes_A M)$$

is zero by definition, so T^p is left exact. Conversely, we prove that (ii) implies (iii). If $Z^p(P^\bullet) = \ker(P^p \to P^{p+1})$, we have an exact sequence

$$0 \longrightarrow Z^{p-1}(P^\bullet) \longrightarrow P^{p-1} \longrightarrow P^p \longrightarrow W^p(P^\bullet) \longrightarrow 0$$

in which P^{p-1}, P^p and $W^p(P^\bullet)$ are flat, so $Z^{p-1}(P^\bullet)$ is flat. We can then define $Q^\bullet$ to be the following complex

$$\cdots \longrightarrow P^{p-2} \longrightarrow Z^{p-1}(P^\bullet) \xrightarrow{0} W^p(P^\bullet) \longrightarrow P^{p+1} \longrightarrow P^{p+2} \longrightarrow \cdots$$

As the P^i are flat, we have

$$W^i(P^\bullet \otimes_A M) = W^i(P^\bullet) \otimes_A M, \quad Z_i(P^\bullet \otimes_A M) = Z_i(P^\bullet) \otimes_A M, \quad B^i(P^\bullet \otimes_A M) = B^i(P^\bullet) \otimes_A M$$

from which we conclude that for any M the functorial homomorphisms $H^i(P^\bullet \otimes_A M) \xrightarrow{\sim} H^i(Q^\bullet \otimes_A M)$ for each i, which proves the assertion. □

Remark 12.5.17. We note that the conditions of Proposition 12.5.16 imply that each $B^p(P^\bullet)$ is flat, because we have an exact sequence

$$0 \longrightarrow B^p(P^\bullet) \longrightarrow P^p \longrightarrow W^p(P^\bullet) \longrightarrow 0$$

in which P^p and $W^p(P^\bullet)$ are flat.

Corollary 12.5.18. *Suppose that A is a regular Noetherian ring of dimension 1. Then, for the T^p to be left exact, it is necessary and sufficient that $T^p(A)$ is a flat A-module. For T^p to be exact, it is necessary and sufficient that $T^p(A)$ and $T^{p+1}(A)$ are flat A-modules.*

Proof. Recall that for a module M over a Dedekind ting, flatness is equivalent to torsion free, so under our hypothesis, any submodule of a flat A-module is flat. The second assertion follows from the first, since T^p is exact if and only if T^p and T^{p+1} are left exact. To prove the frist assertion, we consider the following exact sequence

$$0 \longrightarrow H^p(P^\bullet) \longrightarrow W^p(P^\bullet) \longrightarrow B^{p+1}(P^\bullet) \longrightarrow 0$$

where $B^{p+1}(P^\bullet)$ is a submodule of the flat module P^{p+1}, hence flat, so $H^p(P^\bullet)$ is flat if and only if $W^p(P^\bullet)$ is flat. □

The most important application of Proposition 12.5.16 is the following result:

Proposition 12.5.19. *Let A be a (commutative) Noetherian ring and $P^\bullet$ be a complex of flat A-modules. Suppose that the P^i are finitely generated A-modules, or that the $H^i(P^\bullet)$ are finitely generated and there exists an integer n_0 such that $H^i(P^\bullet) = 0$ for $i > n_0$. Let $T^\bullet$ be the cohomological functor defined by (12.5.9), then the set U of $y \in \mathrm{Spec}(A)$ such that $(T^p)_y$ is left exact (resp. right exact) is open in $\mathrm{Spec}(A)$.*

Proof. In any case, by ([?] 0_{III}, 11.9.3), we can replace $P^\bullet$ by a complex $Q^\bullet$ consists of finitely generated A-modules such that the functors $H^\bullet(P^\bullet \otimes_A (-))$ and $H^\bullet(Q^\bullet \otimes_A (-))$ are isomorphic; in particular, the $W^i(P^\bullet)$ are finitely generated. Moreover, in view of Proposition 12.5.7, we only need to prove the assertion concerning left exactness of T^p. Now let $x \in U$; as the functor $M \mapsto M_x$ is exact, we have $(W^p(P^\bullet))_x = W^p(P^\bullet_x)$, and in view of (12.5.10) and Proposition 12.5.16, the hypothesis implies that $(W^p(P^\bullet))_x$ is a flat A_x-module, hence free (Theorem 1.5.5). We then conclude that there exists $f \in A$ such that $(W^p(P^\bullet))_f$ is a free A_f-module (Corollary 1.5.2), and a fortiori $(W^p(P^\bullet))_y$ is free over A_y for any $y \in D(f)$. This then proves our assertion, since $D(f) \subseteq U$. □

Corollary 12.5.20. *Under the hypothesis of Proposition 12.5.19, assume that A is integral. Then the set U of $\mathrm{Spec}(A)$ such that $(T^p)_x$ is exact is nonempty in $\mathrm{Spec}(A)$.*

Proof. It suffices to note that $(T^p)_x$ is exact for the generic point x of $\operatorname{Spec}(A)$, since A_x is then the fraction field. □

Proposition 12.5.21. *Under the hypothesis of Proposition 12.5.19, the conditions of Proposition 12.5.16 are equivalent to the following:*

(iv) *there exsist an A-module Q and an isomorphism*

$$T^p(M) \xrightarrow{\sim} \operatorname{Hom}_A(Q, M). \tag{12.5.12}$$

Moreover, the A-module Q is uniquely determined up to isomorphisms, and is finitely generated.

Proof. The uniqueness of Q follows easily from Yoneda's lemma, and it is clear that $\operatorname{Hom}_A(Q, -)$ is left exact. Conversely, assume that T^p is left exact, we prove the existence of Q. By ([?] 0_{III}, 11.9.3), we can assume that each P^i is free of finite rank, hence projective. The dual $\check{P}^i$ of P^i is then also a finitely generated projective module, P^i is isomorphic to the dual of $\check{P}^i$, and the canonical homomorphism $P^i \otimes_A M \to \operatorname{Hom}_A(\check{P}^i, M)$ is bijective. We see on the other hand (Proposition 12.5.16) that we can suppose that $d^{p-1} : P^{p-1} \to P^p$ is zero, so we have an exact sequence

$$0 \longrightarrow T^p(M) \xrightarrow{u} P^p \otimes_A M \longrightarrow P^{p+1} \otimes_A M$$

where $v = d^p \otimes 1$. Put $Q' = \ker d^p$, so that we have an exact sequence

$$0 \longrightarrow Q' \xrightarrow{w} P^p \xrightarrow{d^p} P^{p+1}$$

Then by transposition, the sequence

$$\check{P}^{p+1} \xrightarrow{(d^p)^t} \check{P}^p \xrightarrow{w^t} \check{Q}' \longrightarrow 0$$

is exact. We then claim that the module $Q = \check{Q}' = \operatorname{coker}(d^p)^t$ satisfies our requirement. In fact, we have the exact sequence

$$0 \longrightarrow \operatorname{Hom}_A(Q, M) \longrightarrow \operatorname{Hom}_A(\check{P}^p, M) \xrightarrow{v'} \operatorname{Hom}_A(\check{P}^{p+1}, M)$$

where $v' = \operatorname{Hom}((d^p)^t, 1)$. If we canonically identify $P^i \otimes_A M$ with $\operatorname{Hom}(\check{P}^i, M)$, then v' is identified with $v = d^p \otimes 1$, and we therefore have a functorial isomorphism $T^p(M) \xrightarrow{\sim} \operatorname{Hom}_A(Q, M)$. Morever, Q, being a quotient of $\check{P}^p$, is finitely generated. □

Proposition 12.5.22. *Under the hypothesis of Proposition 12.5.19, for any finitely generated A-module M:*

(a) *the $T^i(M)$ are finitely generated A-modules;*

(b) *for any ideal $\mathfrak{I}$ of A, the canonical homomorphism*

$$\widehat{T^i(M)} \to \varprojlim_n T^i(M \otimes_A (A/\mathfrak{I}^{n+1})) \tag{12.5.13}$$

is bijective.

Proof. As in Proposition 12.5.19, we can assume that each P^i is finitely generated. Then every submodule of $P^i \otimes_A M$ is finitely generated, whence assertion (a). To prove (b), it suffices to show that, if $u : E \to F$ is a homomorphism of finitely generated A-modules and M is a finitely generated A-module, then, if we set $K(M) = \ker u \otimes 1_M$ and $C(M) = \operatorname{coker} u \otimes 1_M$, we have canonical isomorphisms

$$\widehat{K(M)} \to \varprojlim K(M_n), \quad \widehat{C(M)} = \varprojlim C(M_n).$$

To this end, we note that since $E \otimes_A M$ and $F \otimes_A M$ are finitely generated A-module, $\widehat{K(M)}$ and $\widehat{C(M)}$ are respectively the kernel and cokernel of the homomorphism $\widehat{u \otimes 1_M} : \widehat{E \otimes_A M} \to \widehat{F \otimes_A M}$. By the left exactness of $\varprojlim$, we then have $\widehat{K(M)} = \varprojlim K(M_n)$, and $\widehat{K(M)} = \varprojlim K(M_n)$; on the other hand, since $C(M_n) = C(M) \otimes_A (A/\mathfrak{I}^{n+1})$, we see from defintion that $\widehat{C(M)} = \varprojlim C(M_n)$. □

12.5.5 The case of Noetherian local rings

We now consider the case where A is a Noetherian local ring. Let $\mathfrak{I}$ be the maximal ideal of A. Let $T : \mathbf{Mod}(A) \to \mathbf{Ab}$ be a covariant additive functor; then the canonical homomorphisms (12.5.3)

$$T(M) \otimes_A (A/\mathfrak{m}^{n+1}) \to T(M \otimes_A (A/\mathfrak{m}^{n+1}))$$

form a projective system of A-homomorphisms, hence give a functorial homomorphism of $\widehat{A}$-modules

$$\widehat{T(M)} \to \varprojlim T(M_n) \tag{12.5.14}$$

where $M_n = M \otimes_A (A/\mathfrak{m}^{n+1})$ and $A_n = A/\mathfrak{m}^{n+1}$.

Proposition 12.5.23. *Let A be a Noetherian local ring with maximal ideal $\mathfrak{m}$, residue field k, T : $\mathbf{Mod}(A) \to \mathbf{Ab}$ be a covariant additive functor, which is semi-exact and commutes with inductive limits. Suppose that for every finitely generated A-module M, $T(M)$ is a finitely generated A-module and the canonical homomorhism (12.5.14) is an isomorphism. Under these conditions, the following properties are equivalent:*

(i) *T is right exact;*

(ii) *for each n, the functor $N \mapsto T(N)$ is right exact on the category of finitely generated A_n-modules (which means T is exact on the category of A-modules of finite length);*

(iii) *the canonical homomorphism $T(\varepsilon) : T(A) \to T(k)$ is surjective;*

(iv) *for sufficiently large n, the canonical homomorphism $T(A_n) \to T(k)$ is surjective.*

Proof. It is clear that (i) implies (ii); to see that (ii) implies (i), let $u : M \to N$ be an epimorphism of A-modules. As T commutes with inductive limits and the functor $\varinjlim$ is exact on the category of A-modules (indexed by filtered sets), we can also suppose that M and N are finitely generated, so by hypothesis $T(M)$ and $T(N)$ are also finitely generated. As A is a Noethertian local ring (hence a Zariski ring with the $\mathfrak{m}$-adic topology), it suffices to prove that $\widehat{T(u)} : \widehat{T(M)} \to \widehat{T(N)}$ is surjective (Proposition 2.4.28). By hypothesis, $\widehat{T(M)}$ and $\widehat{T(N)}$ are isomorphic to $\varprojlim T(M_n)$ and $\varprojlim T(N_n)$, respectively, and $\widehat{T(u)}$ can be identified with the projective limit of the homomorphisms $T(u \otimes 1_{A_n}) : T(M_n) \to T(N_n)$, which are all surjective by condition (ii). We then conclude that $\widehat{T(u)}$ is surjective, so (ii) implies (i). It is clear that (i) implies (iii), and since $T(\varepsilon)$ can be written as $T(A) \to T(A_n) \to T(k)$, (iii) implies (iv). Finally, by Proposition 12.5.6 we know that (ii) is equivalent to (iv), since T is semi-exact on $\mathbf{Mod}(A_n)$. □

Corollary 12.5.24. *Under the hypotheses of Proposition 12.5.23, if $T(k) = 0$, then $T = 0$.*

Proof. As k is the only simple A-module, we deduce from (**??** 7.3.5.4) that $T(E) = 0$ for any A-module E of finite length. If M is a finitely generated A-module, then $\widehat{T(M)}$ is isomorphic to $\varprojlim T(M_n)$, and as M_n are of finite length, we have $\widehat{T(M)} = 0$, so $T(M) = 0$ since it is finitely generated (Proposition 2.4.28). Finally, for any A-module M, $T(M)$ is the inductive limit of $T(N_\alpha)$, where N_α are the finitely generated submodules of M, whence the assertion. □

Proposition 12.5.25. *Let A be a Noetherian local ring with maximal ideal $\mathfrak{m}$, residue field k, $T^\bullet : \mathbf{Mod}(A) \to \mathbf{Ab}$ be a cohomological functor which is semi-exact and commutes with inductive limits. Suppose that for any integer i and any finitely generated A-module M, $T^i(M)$ is finitely generated and the canonical homomorphism $\widehat{T^i(M)} \to \varprojlim T^i(M_n)$ is bijective. For an integer p, the following conditions are equivalent:*

(i) *T^p is exact;*

(ii) *T^p is right exact and $T^p(A)$ is a free A-module.*

(iii) *the canonical homomorphism $T^i(A) \to T^i(k)$ is surjective for $i = p, p-1$;*

(iv) *for each n, the canonical homomorphism $T^i(A_n) \to T^i(k)$ is surjective for $i = p, p-1$;*

(v) *for each n, the functor $N \mapsto T^p(N)$ is exact on the category of finitely generated A_n-modules.*

Proof. It follows from Proposition 12.5.7 that (i) is equivalent to that T^p and T^{p-1} are right exact; by the same reasoning, (v) is equivalent to say that T^p and T^{p-1} are right exact on the category of finitely generated A_n-modules. We then deduce form Proposition 12.5.23 that (i) is equivalent to (v); the equivalence of (i), (iii) and (iv) also follows from Proposition 12.5.23. Finally, we see that any finitely generated flat A-module is projective, the equivalence of (i) and (ii) follows from Proposition 12.5.9. □

Corollary 12.5.26. *Suppose the hypotheses of Proposition 12.5.25.*

(a) *If $T^p(k) = 0$, then $T^p = 0$, T^{p-1} is right exact and T^{p+1} is left exact.*

(b) *If $T^{p-1}(k) = T^{p+1}(k) = 0$, then T^p is exact, the canonical homomorphism*

$$T^p(A) \otimes_A M \to T^p(M)$$

is bijective and $T^p(A)$ is a free A-module.

Proof. Assertion (a) follows directly from Corollary 12.5.24 since T^p is semi-exact, and the first two assertions of (b) also follows from this. Finally, the last assertion follows from Proposition 12.5.23. □

Corollary 12.5.27. *Retain the the hypotheses of Proposition 12.5.25 and suppose that A is a DVR.*

(a) *For T^p to be right exact, it is necessary and sufficient that $T^{p+1}(A)$ is a free A-module.*

(b) *For T^p to be exact, it is necessary and sufficient that $T^p(A)$ and $T^{p+1}(A)$ are free A-modules.*

Proof. It is clear that (a) implies (b), in view of Proposition 12.5.7. To prove (a), let π be a uniformizer of A; for a finitely generated A-module M to be free (or equivalently, flat), it is necessary and sufficient that the homothety $h_\pi : x \mapsto \pi x$ of M is injective, because this means M is torsion free. Consider the exact sequence

$$0 \longrightarrow A \xrightarrow{h_f} A \longrightarrow k \longrightarrow 0$$

which induces an exact sequence

$$T^p(A) \longrightarrow T^p(k) \longrightarrow T^{p+1}(A) \xrightarrow{h_f} T^{p+1}(A)$$

We then see that $T^{p+1}(A)$ is free if and only if $T^p(A) \to T^p(k)$ is surjective, which proves our assertion (Proposition 12.5.23). □

12.5.6 Descent of exactness and the semi-continuity theorem

Proposition 12.5.28. *Under the hypothesis of Proposition 12.5.16, let B be a commutative A-algebra. If T^p is right exact (resp. left exact), then so is $T^p_{(B)}$. The converse is true if B is a fiathfully flat A-module.*

Proof. The first assertion is trivial since the forgetful functor is exact and faithful. Conversely, suppose first that B is a flat A-module. We then have, for any A-module M, $H^\bullet(P^\bullet \otimes_A (M \otimes_A B)) = (H^\bullet(P^\bullet \otimes_A M)) \otimes_A B$, wich means, for each p,

$$T^p(M) \otimes_A B = T^p_{(B)}(M_{(B)}). \tag{12.5.15}$$

Suppose that $T^p_{(B)}$ is right exact (resp. right exact), then as $M \to M_{(B)}$ is an exact functor, the first member of (12.5.15) is a right exact (resp. left exact) functor on M; if B is assumed to be faithfully flat, then we deduce that T^p has the same exactness property. □

Proposition 12.5.29. *Under the hypothesis of Proposition 12.5.16, suppose that A is a reduced Noetherian ring and the P^i are finitely generated A-modules. For T^p to be right exact (resp. left exact), it is necessary and sufficient that, for any A-algebra B which is a DVR, $T^p_{(B)}$ is right exact (resp. left exact).*

Proof. In view of Proposition 12.5.7, we only need to consider the left exactness, hence prove the sufficiency of the condition of the corollary. In view of Proposition 12.5.16, it suffices to prove that $W^p(P^\bullet)$ is a flat A-module; as P^p is finitely generated, $W^p(P^\bullet)$ is also finitely generated, so the criterion of ([?] 0_{III}, 10.2.8) shows that it suffices to show that $W^p(P^\bullet) \otimes_A B$ is a flat B-module for any A-algebra B which is a DVR. Now, as $P^\bullet$ is a complex of flat A-modules, we have

$$W^p(P^\bullet) \otimes_A B = W^p(P^\bullet \otimes_A B);$$

where $P^\bullet \otimes_A B$ is a complex of flat B-modules and for any B-module N, we have $H^\bullet(P^\bullet \otimes_A N) = H^\bullet((P^\bullet \otimes_A B) \otimes_B N)$, so $T^p_{(B)}(N) = H^p((P^\bullet \otimes_A B) \otimes_B N)$. Applying Proposition 12.5.16 to $T^p_{(B)}$, we see that the hypothesis that $T^p_{(B)}$ is left exact is equivalent to that $W^p(P^\bullet \otimes_A B)$ is a flat B-module, whence our assertion. □

The criterion of Proposition 12.5.29 reduces the exactness of T^p to the case of DVRs. In this case, we have the following useful result, which can be considered as a "universal coefficient theorem".

Proposition 12.5.30. *Under the hypothesis of Proposition 12.5.16, suppose that A is a one dimensional regular Noetherian ring (in other words, A is Noetherian and for any $x \in \mathrm{Spec}(A)$, A_x is a field or a DVR). Then for any integer p and A-module M, we have a canonical exact sequence*

$$0 \longrightarrow T^p(A) \otimes_A M \xrightarrow{t_M} T^p(M) \longrightarrow \mathrm{Tor}_1^A(T^{p+1}(A), M) \longrightarrow 0 \tag{12.5.16}$$

Proof. We simplify our notation by writing H^p, Z^p, B^p for $H^p(P^\bullet)$, $Z^p(P^\bullet)$, $B^p(P^\bullet)$. Then we have the following exact sequences

$$0 \longrightarrow H^p \longrightarrow W^p \longrightarrow B^{p+1} \longrightarrow 0$$
$$0 \longrightarrow B^{p+1} \longrightarrow Z^{p+1} \longrightarrow H^{p+1} \longrightarrow 0$$
$$0 \longrightarrow Z^p \longrightarrow P^p \longrightarrow B^{p+1} \longrightarrow 0$$

As P^p and P^{p+1} are flat, so are their submodules (since for any $x \in \mathrm{Spec}(A)$, a A_x-module is flat if and only if it is torsion free). By tensoring with M, we then obtain exact sequences

$$0 \longrightarrow H^p \otimes_A M \longrightarrow W^p \otimes_A M \xrightarrow{u} B^{p+1} \otimes_A M \longrightarrow 0 \tag{12.5.17}$$
$$0 \longrightarrow \mathrm{Tor}_1^A(H^{p+1}, M) \longrightarrow B^{p+1} \otimes_A M \xrightarrow{v} Z^{p+1} \otimes_A M \longrightarrow H^{p+1} \otimes_A M \longrightarrow 0 \tag{12.5.18}$$
$$0 \longrightarrow Z^p \otimes_A M \xrightarrow{w} P^p \otimes_A M \longrightarrow B^{p+1} \otimes_A M \longrightarrow 0 \tag{12.5.19}$$

By definition, $T^p(M) = \ker(d^p \otimes 1_M)/\operatorname{im}(d^{p-1} \otimes 1_M)$, which is the kernel of the homomorphism

$$(P^p \otimes_A M)/\operatorname{im}(d^{p-1} \otimes 1_M) \to P^{p+1} \otimes_A M$$

obtained from $d^p \otimes 1_M$ by passing to quotient. By definition we have $W^p = P^p/B^p$, so this homomorphism can also be written as $W^p \otimes_A M \to P^{p+1} \otimes_A M$, which is exactly the composition

$$W^p \otimes_A M \xrightarrow{u} B^{p+1} \otimes_A M \xrightarrow{v} Z^{p+1} \otimes_A M \xrightarrow{w} P^{p+1} \otimes_A M$$

By (12.5.19), the homomorphism w is injective, so we get an exact sequence

$$0 \longrightarrow \ker u \longrightarrow T^p(M) \longrightarrow \ker v \longrightarrow 0$$

This is exactly (12.5.16), since we have $H^p = T^p(A)$ by (12.5.17) and (12.5.18). □

Remark 12.5.31. Since $H^\bullet(P^\bullet \otimes_A M)$ is the cohomology of the double complex $P^\bullet \otimes_A M$ (where M is considered as a complex with only zero-th term), it is the limit of a regular homological spectral sequence with term E^2 given by

$$E^2_{p,q} = \mathrm{Tor}_p^A(H_q(P^\bullet), M) = \mathrm{Tor}_p^A(T^q(A), M).$$

The hypothesis of Proposition 12.5.30 implies that for any A-module E, F, we have $\mathrm{Tor}_p^A(E, F) = 0$ for $p \geq 2$ (Corollary 9.4.12), so we have an exact sequence

$$0 \longrightarrow E^2_{0,q} \longrightarrow H^q(P^\bullet \otimes_A M) \longrightarrow E^2_{1,q-1}$$

which is none other than (12.5.16).

Corollary 12.5.32. *Under the hypothesis of Proposition 12.5.16, suppose that A is a DVR with fraction field K, residue field k, and that the $T^i(A)$ are finitely generated A-modules. Then*

$$\dim_k(T^p(k)) \geq \dim_k(T^p(A) \otimes_A k) \geq \mathrm{rank}_A(T^p(A)) = \dim_K(T^p(K)). \tag{12.5.20}$$

Moreover, for the equality hold, it is necessary and sufficient that T^p is exact, or that $T^p(A)$ and $T^{p+1}(A)$ are free A-modules.

Proof. By taking $M = k$ in the exact sequence (12.5.16), we see that

$$\dim_k(T^p(k)) = \dim_k(T^p(A) \otimes_A k) + \dim_k(\mathrm{Tor}_1^A(T^{p+1}(A), k)).$$

On the other hand, as $T^p(A)$ is a finitely generated module over the integral local ring A, we have (Proposition 1.3.14)

$$\dim_k(T^p(A) \otimes_A k) \geq \mathrm{rank}_A(T^p(A)) = \mathrm{rank}_K(T^p(A) \otimes_A K)$$

and the equality holds if and only if $T^p(A)$ is a free A-module. On the other hand, since K is a flat A-module, we have $T^p(A) \otimes_A K = H^p(P^\bullet) \otimes_A K = H^p(P^\bullet \otimes_A K) = T^p(K)$, whence the inequalities in (12.5.20). Moreover, by our remarks, the equality holds if and only if $T^p(A)$ is free and $\mathrm{Tor}_1^A(T^{p+1}(A), k) = 0$, where ther later conditions means $T^{p+1}(A)$ is a free A-module. Finally, the last assertion follows from Corollary 12.5.27. □

Retain the hypothesis of Proposition 12.5.16, we now consider the function

$$d_p(x) = d_p^T(x) = \dim_{\kappa(x)}(T^p(\kappa(x)))$$

on the space $\mathrm{Spec}(A)$.

Lemma 12.5.33. *Let $\rho : A \to A'$ be a ring homomorphism and $f : \mathrm{Spec}(A') \to \mathrm{Spec}(A)$ be the corresponding morphism. If we set $T'^\bullet = T^\bullet_{(A')}$, then*

$$d_p^{T'} = d_p^T \circ f.$$

Proof. For any $x' \in \mathrm{Spec}(A')$, we have (setting $x = f(x')$)

$$H^\bullet(P^\bullet \otimes_A \kappa(x')) = H^\bullet((P^\bullet \otimes_A \kappa(x)) \otimes_{\kappa(x)} \kappa(x')) = H^\bullet((P^\bullet \otimes_A \kappa(x))) \otimes_{\kappa(x)} \kappa(x')$$

since $\kappa(x')$ is flat over $\kappa(x)$, whence the assertion. □

Lemma 12.5.34. *If A is a Noetherian ring and $P^\bullet$ is a complex of finitely generated flat A-modules. Then the function $d_p^T(x)$ is constructible.*

Proof. It suffices to prove that for any closed irreducible subset Y of $\mathrm{Spec}(A)$, there exists a nonempty open subset U of Y such that d_p is constant on U ([?] 0_{III}, 9.2.2). As $Y = \mathrm{Spec}(A/\mathfrak{a})$ for some radical ideal $\mathfrak{a}$, we can, in view of Lemma 12.5.34, reduce to the case where $Y = X$ and A is an integral Noetherian domain. But the conclusion then follows from Corollary 12.5.14. □

Theorem 12.5.35. *Let A be a Noetherian ring, $P^\bullet$ be a complex of finitely generated flat A-modules, and $T^\bullet(M) = H^\bullet(P^\bullet \otimes_A M)$ the cohomological functor defined by $P^\bullet$. For each $x \in \mathrm{Spec}(A)$, let $d_p(x) = \dim_{\kappa(x)}(T^p(\kappa(x)))$.*

(a) *The function d_p is constructible and upper semi-continuous on* $\mathrm{Spec}(A)$.

(b) *If T^p is exact, then d_p is continuous (hence locally constant) on* $\mathrm{Spec}(A)$. *The converse is true if A is reduced.*

Proof. The fisrt part of assertion (a) follows from Lemma 12.5.34. To prove the second one, it suffices ([?] 0_{III}, 9.3.4) to show that if $y \rightsquigarrow x$ is a generalization of $x \in \mathrm{Spec}(A)$, then $d_p(y) \leq d_p(x)$. Now, there exists a discrete valuation ring B and a morphism $f : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ such that, if a denote the closed point of $\mathrm{Spec}(B)$ and b is the genetric point, we have $f(a) = x$ and $f(b) = y$ (Proposition 11.7.1). In view of Lemma 12.5.34, we are then reduced to prove the inequality $d_p(a) \geq d_p(b)$ in $\mathrm{Spec}(B)$, which is none other than the inequality (12.5.20).

The first assertion of (b) follows similarly from Corollary 12.5.32 by passing to DVRs. For the converse, we use the criterion of Proposition 12.5.29. In view of Lemma 12.5.34, we can reduce to the case where A is a DVR. But as $\mathrm{Spec}(A)$ consists of two points, the hypothesis that d_p is continuous implies that T^p is exact, in view of Corollary 12.5.32. □

Corollary 12.5.36. *Let A be a Noetherian ring, $(\mathfrak{p}_i)_{1\leq i\leq r}$ be its minimal prime ideals, and k_i be the residue field of $\mathfrak{p}_i$.*

(a) *For any $x \in \mathrm{Spec}(A)$, there exists an index i such that $d_p(x) \geq \dim_{k_i}(T^p(k_i))$. In particular, if A is integral with fraction field K, we have $d_p(x) \geq \dim_K(T^p(K))$.*

(b) *Suppose that A is local and reduced, and let k be its residue field. Then, for T^p to be exact, it is necessary and sufficient that for each $1 \leq i \leq r$, we have*

$$\dim_k(T^p(k)) = \dim_{k_i}(T^p(k)). \tag{12.5.21}$$

Proof. Assertion (a) is immediate since any neighborhood of x contains one of the $\mathfrak{p}_i$, and it suffices to apply the definition of upper semi-continuity. On the other hand, if A is local, the only neighborhood in $\mathrm{Spec}(A)$ of the maximal ideal $\mathfrak{m}$ is $\mathrm{Spec}(A)$, so we have $d_p(x) \leq \dim_k(T^p(k))$ for any $x \in \mathrm{Spec}(A)$. Now, the relation (12.5.21) then implies that $d_p(x)$ is constant on $\mathrm{Spec}(A)$, and therefore T^p is exact in view of Theorem 12.5.35(b). The converse is also evident in view of this, since the specturm of a local ring is connected. □

12.5.7 Application to proper morphisms

12.5.7.1 The semicontinuity and exchange property Let $f: X \to Y$ be a quasi-compact and separated morphis of schemes, and let $\mathscr{P}^\bullet$ be a complex of quasi-coherent $\mathscr{O}_X$-modules. Suppose that each $\mathscr{P}^i$ is a Y-flat $\mathscr{O}_X$-module, so that we can consider the δ-functor defined by

$$\mathscr{T}^p(\mathscr{P}^\bullet, \mathscr{M}) = \mathscr{T}^p(\mathscr{P}^\bullet, \mathscr{M}) = R^p f_*(\mathscr{P}^\bullet \otimes_{\mathscr{O}_Y} \mathscr{M}) \tag{12.5.22}$$

where $\mathscr{M}$ is a quasi-coherent $\mathscr{O}_Y$-module. It follows from our hypothesis that $\mathscr{T}^\bullet$ is a cohomological functor on $\mathbf{Qcoh}(Y)$.

Now let $g: Y' \to Y$ be a base change morphism, and put $X' = X \times_Y Y'$, $f' = f_{(Y')}: X' \to Y'$, which is a quasi-compact and separated morphism. Let $\mathscr{P}'^\bullet = \mathscr{P}^\bullet \otimes_{\mathscr{O}_Y} \mathscr{O}_{Y'}$, which is a complex of Y'-flat $\mathscr{O}_{X'}$-modules. We can then define

$$\mathscr{T}^\bullet_{Y'} = Rf'_*(\mathscr{P}^\bullet \otimes_{\mathscr{O}_Y} \mathscr{M}')$$

as a cohomological functor on $\mathbf{Qcoh}(Y')$. If Y' is an affine scheme with $\widetilde{\text{ring } A'}$, then we also write $\mathscr{T}^\bullet_{(A')}$ for $\mathscr{T}^\bullet_{Y'}$. For any A'-module M', we then have $\mathscr{T}^\bullet_{A'}(\widetilde{M}') = T^\bullet_{A'}(M')$, where we set $T^\bullet_{A'}(M') = \Gamma(Y', \mathscr{T}^\bullet_{A'}(\widetilde{M}'))$. Then $T^\bullet_{A'}$ is a cohomological functor on the category of A'-modules. We observe that if $Y = \mathrm{Spec}(A)$ is also affine, the functor $T^\bullet_{A'}$ is then the extension of scalars of the functor $T^\bullet_A$: in fact, let $g': X' \to X$ be the bese change morphism of g; if $\mathfrak{U}$ is an affine open covering of X, $\mathfrak{U}' = g'^{-1}(\mathfrak{U})$ is then an affine open covering of X'. In view of ([?], 6.2.2), it boils down to show that

$$\mathscr{C}^\bullet(\mathfrak{U}, \mathscr{P}^\bullet \otimes_{\mathscr{O}_Y} g_*(\mathscr{M}')) = \mathscr{C}^\bullet(\mathfrak{U}', \mathscr{P}^\bullet \otimes_{\mathscr{O}_Y} \mathscr{M}),$$

and finally, that for any affine open U of X, if $U' = g^{-1}(U)$, then we have $\Gamma(U, \mathscr{P}^\bullet \otimes_{\mathscr{O}_Y} g_*(\mathscr{M}')) = \Gamma(U', \mathscr{P}^\bullet \otimes_{\mathscr{O}_Y} \mathscr{M})$, which is trivial. In particular, if U is an open subset of Y, we then have

$$\mathscr{T}^\bullet_U(\mathscr{M}|_U) = (\mathscr{T}^\bullet(\mathscr{M}))|_U.$$

For any quasi-coherent $\mathscr{O}_Y$-module $\mathscr{M}$, we have a canonical homomorphism

$$\mathscr{T}^p(\mathscr{O}_Y) \otimes_{\mathscr{O}_Y} \mathscr{M} \to \mathscr{T}^p(\mathscr{M}). \tag{12.5.23}$$

In fact, if Y is affine, this is the canonical homomorphism (12.5.3); this definition extends to the general case, since if U, V are two affine opens of Y such that $V \subseteq U$, the diagram

$$\begin{tikzcd}
(\mathscr{T}^p(\mathscr{O}_Y)\otimes_{\mathscr{O}_Y}\mathscr{M})|_U = \mathscr{T}^p_U(\mathscr{O}_Y|_U)\otimes_{\mathscr{O}_Y|_U}(\mathscr{M}|_U) \arrow[r] \arrow[d] & \mathscr{T}^p(\mathscr{M}|_U) = (\mathscr{T}^p(\mathscr{M}))|_U \arrow[d] \\
(\mathscr{T}^p(\mathscr{O}_Y)\otimes_{\mathscr{O}_Y}\mathscr{M})|_V = \mathscr{T}^p_V(\mathscr{O}_Y|_V)\otimes_{\mathscr{O}_Y|_V}(\mathscr{M}|_V) \arrow[r] & \mathscr{T}^p(\mathscr{M}|_V) = (\mathscr{T}^p(\mathscr{M}))|_V
\end{tikzcd}$$

is commutative by (12.5.4). Similarly, for any morphism $g : Y' \to Y$, we have a canonical homomorphism

$$\mathscr{T}^p(\mathscr{O}_Y)\otimes_{\mathscr{O}_Y} \to \mathscr{T}^p_{Y'}(\mathscr{O}_{Y'})$$

which is none other than the homomorphism (12.5.6).

If f is a proper morphism, $Y = \mathrm{Spec}(A)$ is a Noetherian affine scheme and $\mathscr{P}^\bullet$ is a *bounded above* complex of Y-flat coherent $\mathscr{O}_Y$-modules, we then have ([?], 6.10.5) an isomorphism $\mathscr{T}^p \xrightarrow{\sim} H^\bullet(\mathscr{L}^\bullet\otimes_{\mathscr{O}_Y}\mathscr{M})$, where $\mathscr{L}^\bullet = \widetilde{L}^\bullet$ and $L^\bullet$ is a bounded above complex of free A-modules of finite rank. The functor $\mathscr{T}^\bullet$ is then of the type we encountered in Section 12.5.4, so the corresponding results can be applied.

Theorem 12.5.37. *Let Y be a locally Noetherian scheme, $f : X \to Y$ be a proper morphism, and $\mathscr{P}^\bullet$ be a bounded above complex of Y-flat coherent $\mathscr{O}_Y$-modules. Then the functor $\mathscr{T}^\bullet$ defined by (12.5.22) satisfies the following conditions.*

(a) *(**Semi-continuity**) The function $y \mapsto d_p(y) = \dim_{\kappa(y)}(T^p_{(\kappa(y))}(\kappa(y)))$ is upper semi-continuous on Y.*

(b) *(**Exchange property**) For any integer p, the following conditions are equivalent:*

  (i) $\mathscr{T}^p$ is right exact;

  (i') $\mathscr{T}^p$ is isomorphic to the functor $\mathscr{M} \mapsto \mathscr{N}\otimes_{\mathscr{O}_Y}\mathscr{M}$, where $\mathscr{N}$ is isomorphic to $\mathscr{T}^p(\mathscr{O}_Y) = R^pf_*(\mathscr{P}^\bullet)$;

  (i'') the canonical homomorphism (12.5.23) is an isomorphism;

  (ii) $\mathscr{T}^{p+1}$ is left exact;

  (ii') $\mathscr{T}^{p+1}$ is isomorphic to the functor $\mathscr{M} \mapsto \mathcal{H}om_{\mathscr{O}_Y}(\mathscr{Q}, \mathscr{M})$, where $\mathscr{M}$ is a coherent $\mathscr{O}_Y$-module uniquely determined up to isomorphism;

  (iii) for any affine open $U = \mathrm{Spec}(A)$ of Y, the functor T^p_A is right exact;

  (iv) for any morphism $g : Y \to Y'$, the canonical homomorphism

$$\mathscr{T}^p(\mathscr{O}_Y)\otimes_{\mathscr{O}_Y} \to \mathscr{T}^p_{Y'}(\mathscr{O}_{Y'})$$

  is an isomorphism.

Proof. The semi-continuous property is local over Y, hence follows from Theorem 12.5.35. It is clear that (i'') implies (i') and (i') implies (i), and the equivalence of (i), (i''), (ii) and (ii') is proved in Proposition 12.5.16 and Proposition 12.5.7 if Y is affine. To pass to general case, we first prove that (i)⇔(iii), which shows that property (i) is local over Y; our demonstration will also apply to prove that properties (i'') and (ii) are local over Y. Now it is clear that (iii) implies (i), so we only need to consider the converse. It evidently suffices to show that for any affine open U of Y and any exact sequence $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ of quasi-coherent $(\mathscr{O}_Y|_U)$-modules, there exists an exact sequence $\mathscr{O}_Y$-modules whose restriction to U is the given sequence. Now, this follows from the hypothesis that Y is locally Noetherian and (Proposition 10.6.58): we can extend $\mathscr{F}$ (resp. $\mathscr{F}'$) to a quasi-coherent $\mathscr{O}_X$-module $\mathscr{G}$ (resp. sub-$\mathscr{O}_X$-module $\mathscr{G}'$ of $\mathscr{G}$), and it suffices to put $\mathscr{G}'' = \mathscr{G}/\mathscr{G}'$.

To prove the equivalence of (ii) and (ii') in the general case, we note that if Y is affine, the $\mathscr{O}_Y$-module $\mathscr{Q}$ is determined uniquely up to isomorphism; if U is an affine open of the affine scheme Y, then there is a functorial isomorphism

$$\mathscr{T}_U^{p+1}(\mathscr{M}|_U) \xrightarrow{\sim} \mathcal{H}om_{\mathscr{O}_Y|_U}(\mathscr{Q}|_U, \mathscr{M}|_U).$$

In the general case, for any affine open U of Y, there exists a coherent $(\mathscr{O}_Y|_U)$-module $\mathscr{Q}_U$ and a functorial isomorphism $\mathscr{T}_U^{p+1}(\mathscr{M}|_U) \xrightarrow{\sim} \mathcal{H}om_{\mathscr{O}_Y|_U}(\mathscr{Q}_U, \mathscr{M}|_U)$; the preceding remark allows us to glue the $\mathscr{Q}_U$ to produce a coherent $\mathscr{O}_Y$-module which satisfies (ii').

Finally, we prove the equivalence of (i) and (iv); it is clear that (iv) is local over Y, and we have seen that so is (i); moreover, (iv) is also local over Y'. If $Y = \operatorname{Spec}(A)$ and $Y' = \operatorname{Spec}(A')$, then $T^\bullet_{A'}$ is the extension of scalars of $T^\bullet_A$, and it is clear that (i') implies (iv). Conversely, suppose that $Y = \operatorname{Spec}(A)$ and let A' be the A-algebra $A \oplus M$, where M is a given A-module and the multiplication on A' is defined by $(a_1, m_1)(a_2, m_2) = (a_1, a_2, a_1m_2 + a_2m_1)$. Then we have

$$T^p_{A'}(A') = T^p(A \oplus M) = T^p(A) \oplus T^p(M)$$

and the hypothesis (iv) implies that the canonical homomorphism $T^p(A) \otimes_A M \to T^p(M)$ is also bijective, so (iv) implies (i'') and this completes the proof. □

Theorem 12.5.38. *Let Y be a locally Noetherian scheme, $f : X \to Y$ be a proper morphism, and $\mathscr{F}$ be a Y-flat coherent $\mathscr{O}_Y$-module. Then there exists a coherent $\mathscr{O}_Y$-module $\mathscr{Q}$ (uniquely determined up to isomorphism) such that there exists an isomorphism of functors on quasi-coherent $\mathscr{O}_Y$-modules $\mathscr{M}$:*

$$f_*(\mathscr{F} \otimes_{\mathscr{O}_Y} \mathscr{M}) \xrightarrow{\sim} \mathcal{H}om_{\mathscr{O}_Y}(\mathscr{Q}, \mathscr{M}).$$

Proof. In fact, the functor $\mathscr{T}^0 = f_*$ is left exact, so the assertion follows from Theorem 12.5.37. □

Corollary 12.5.39. *Let Y be a locally Noetherian scheme, $f : X \to Y$ be a proper morphism, $\mathscr{F}, \mathscr{F}'$ be Y-flat coherent $\mathscr{O}_Y$-modules, and $u : \mathscr{F} \to \mathscr{F}'$ be a homomorphism. Consider the following functors on quasi-coherent $\mathscr{O}_Y$-modules $\mathscr{M}$:*

$$\mathscr{T}(\mathscr{M}) = \ker(f_*(\mathscr{F} \otimes_{\mathscr{O}_Y} \mathscr{M}) \to f_*(\mathscr{F}' \otimes_{\mathscr{O}_Y} \mathscr{M})),$$
$$T(\mathscr{M}) = \Gamma(Y, \mathscr{T}(\mathscr{M})) = \ker(\Gamma(X, \mathscr{F} \otimes_{\mathscr{O}_Y} \mathscr{M}) \to \Gamma(X, \mathscr{F}' \otimes_{\mathscr{O}_Y} \mathscr{M})).$$

Then there exists a coherent $\mathscr{O}_Y$-module $\mathscr{R}$ (uniquely determined up to isomorphism) and isomorphisms of functors

$$\mathscr{T}(\mathscr{M}) \xrightarrow{\sim} \mathcal{H}om_{\mathscr{O}_Y}(\mathscr{R}, \mathscr{M}), \quad T(\mathscr{M}) \xrightarrow{\sim} \operatorname{Hom}_{\mathscr{O}_Y}(\mathscr{R}, \mathscr{M}). \tag{12.5.24}$$

Proof. It suffices to consider the functor $T(\mathscr{M})$, so by Theorem 12.5.38 there exists two coherent $\mathscr{O}_Y$-modules $\mathscr{Q}$, $\mathscr{Q}'$ defining functorial isomorphisms

$$\Gamma(X, \mathscr{F} \otimes_{\mathscr{O}_Y} \mathscr{M}) \xrightarrow{\sim} \operatorname{Hom}_{\mathscr{O}_Y}(\mathscr{Q}, \mathscr{M}), \quad \Gamma(X, \mathscr{F}' \otimes_{\mathscr{O}_Y} \mathscr{M}) \xrightarrow{\sim} \operatorname{Hom}_{\mathscr{O}_Y}(\mathscr{Q}', \mathscr{M}).$$

Now, $u : \mathscr{F} \to \mathscr{F}'$ defines a morphism of functors

$$\Gamma(X, \mathscr{F} \otimes_{\mathscr{O}_Y} \mathscr{M}) \to \Gamma(X, \mathscr{F}' \otimes_{\mathscr{O}_Y} \mathscr{M})$$

which corresponds to a unique homomorphism $v : \mathscr{Q}' \to \mathscr{Q}$ such that the diagram

$$\begin{tikzcd}
\Gamma(X, \mathscr{F} \otimes_{\mathscr{O}_Y} \mathscr{M}) \arrow[r] \arrow[d, "\sim"] & \Gamma(X, \mathscr{F}' \otimes_{\mathscr{O}_Y} \mathscr{M}) \arrow[d, "\sim"] \\
\operatorname{Hom}_{\mathscr{O}_Y}(\mathscr{Q}, \mathscr{M}) \arrow[r] & \operatorname{Hom}_{\mathscr{O}_Y}(\mathscr{Q}', \mathscr{M})
\end{tikzcd}$$

is commutative. As the covariant functor $\mathscr{N} \mapsto \mathrm{Hom}_{\mathscr{O}_Y}(\mathscr{N},\mathscr{M})$ is left exact, it then suffices to put $\mathscr{R} = \operatorname{coker} v$. □

Corollary 12.5.40. *Let Y be a locally Noetherian scheme, $f : X \to Y$ be a proper morphism, $\mathscr{F},\mathscr{G}$ be coherent $\mathscr{O}_X$-modules satisfying the following conditions:*

(a) *$\mathscr{F}$ is Y-flat;*

(b) *$\mathscr{G}$ is isomorphic to the cokernel of a homomorphism of locally free $\mathscr{O}_X$-modules $\mathscr{E}_1 \to \mathscr{E}_0$.*

Consider the following functors on quasi-coherent $\mathscr{O}_Y$-modules $\mathscr{M}$:

$$\mathscr{T}(\mathscr{M}) = f_*(\mathcal{H}om_{\mathscr{O}_Y}(\mathscr{G},\mathscr{F}\otimes_{\mathscr{O}_Y}\mathscr{M})),$$
$$T(\mathscr{M}) = \Gamma(Y,\mathscr{T}(\mathscr{M})) = \mathrm{Hom}_{\mathscr{O}_X}(\mathscr{G},\mathscr{F}\otimes_{\mathscr{O}_Y}\mathscr{M}).$$

Then there exsits a coherent $\mathscr{O}_Y$-module $\mathscr{N}$ (uniquely deteremined up to isomorphism) and isomorphisms of functors
$$\mathscr{T}(\mathscr{M}) \xrightarrow{\sim} \mathcal{H}om_{\mathscr{O}_Y}(\mathscr{N},\mathscr{M}), \quad T(\mathscr{M}) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{O}_Y}(\mathscr{N},\mathscr{M}).$$

Proof. We have functorial isomorphisms

$$\mathcal{H}om_{\mathscr{O}_X}(\mathscr{E}_i,\mathscr{F}\otimes_{\mathscr{O}_Y}\mathscr{M}) \xrightarrow{\sim} \check{\mathscr{E}}_i\otimes_{\mathscr{O}_X}(\mathscr{F}\otimes_{\mathscr{O}_Y}\mathscr{M}) \xrightarrow{\sim} (\check{\mathscr{E}}_i\otimes_{\mathscr{O}_X}\mathscr{F})\otimes_{\mathscr{O}_Y}\mathscr{M} \xrightarrow{\sim} \mathcal{H}om_{\mathscr{O}_X}(\mathscr{E}_i,\mathscr{F})\otimes_{\mathscr{O}_Y}\mathscr{M}$$

for $i = 0,1$. Put $\mathscr{F}_i = \mathcal{H}om_{\mathscr{O}_X}(\mathscr{E}_i,\mathscr{F})$, which are Y-flat coherent $\mathscr{O}_X$-modules, and let $u : \mathcal{H}om(v,1_{\mathscr{F}}) : \mathscr{F}_0 \to \mathscr{F}_1$. In view of the left exactness of the functor $\mathcal{H}om_{\mathscr{O}_X}(-,\mathscr{F}\otimes_{\mathscr{O}_Y}\mathscr{M})$, we have functorial isomorphisms

$$\begin{aligned}\mathcal{H}om_{\mathscr{O}_X}(\mathscr{G},\mathscr{F}\otimes_{\mathscr{O}_Y}\mathscr{M}) &\xrightarrow{\sim} \ker(\mathcal{H}om_{\mathscr{O}_X}(\mathscr{E}_0,\mathscr{F}\otimes_{\mathscr{O}_Y}\mathscr{M}) \to \mathcal{H}om_{\mathscr{O}_X}(\mathscr{E}_1,\mathscr{F}\otimes_{\mathscr{O}_Y}\mathscr{M}))\\ &\xrightarrow{\sim} \ker(\mathscr{F}\otimes_{\mathscr{O}_Y}\mathscr{M} \to \mathscr{F}_1\otimes_{\mathscr{O}_Y}\mathscr{M}).\end{aligned}$$

Since f_* is left exact, we then deduce functorial isomorphisms

$$f_*(\mathcal{H}om_{\mathscr{O}_X}(\mathscr{G},\mathscr{F}\otimes_{\mathscr{O}_Y}\mathscr{M})) \xrightarrow{\sim} \ker(f_*(\mathscr{F}_0\otimes_{\mathscr{O}_Y}\mathscr{M}) \to f_*(\mathscr{F}_1\otimes_{\mathscr{O}_Y}\mathscr{M}))$$

and it suffices to apply Corollary 12.5.39. □

Remark 12.5.41. In Theorem 12.5.38, Corollary 12.5.39 and Corollary 12.5.40, the formation of the $\mathscr{O}_Y$-modules $\mathscr{Q}$, $\mathscr{R}$, $\mathscr{N}$ commutes with base changes. For example, in Theorem 12.5.38, let $g : Y' \to Y$ be a base change morphism. Then we have, for any quasi-coherent $\mathscr{O}_{Y'}$-module $\mathscr{M}'$, the isomorphisms
$$f'_*(\mathscr{F}\otimes_{\mathscr{O}_Y}\mathscr{M}) \xrightarrow{\sim} \mathcal{H}om_{\mathscr{O}_{Y'}}(g^*(\mathscr{Q}),\mathscr{M}')$$
because in view of the adjunction property, we have

$$\mathrm{Hom}_{\mathscr{O}_Y}(\mathscr{Q},g_*(\mathscr{M}')) = \mathrm{Hom}_{\mathscr{O}Y'}(g^*(\mathscr{Q}),\mathscr{M}').$$

Similarly, if in Corollary 12.5.39 we replace $Y,f,\mathscr{M},\mathscr{F},\mathscr{F}'$ by $Y',f',\mathscr{M}',\mathscr{F}\otimes_{\mathscr{O}_X}\mathscr{O}_{X'},\mathscr{F}'\otimes_{\mathscr{O}_X}\mathscr{O}_{X'}$, then it is necessary to replace $\mathscr{R}$ by $g^*(\mathscr{R})$.

Remark 12.5.42. The condition (b) of Corollary 12.5.40 on $\mathscr{G}$ is satisfied for any $\mathscr{G}$ if there exists a Y-ample $\mathscr{O}_X$-module. In fact, it suffices to note that there exists a locally free $\mathscr{O}_X$-module $\mathscr{E}_0$ such that $\mathscr{G}$ is isomorphic to a quotient of $\mathscr{E}_0$ (Proposition 11.4.29). As $\mathscr{E}_0$ and $\mathscr{G}$ are coherent, the kernel $\mathscr{G}_1$ of $\mathscr{E}_0 \to \mathscr{G}$ is also coherent, and by applying the same reasoning to $\mathscr{G}_1$, we obtain an exact sequence $\mathscr{E}_1 \to \mathscr{E}_0 \to \mathscr{G} \to 0$, where $\mathscr{E}_0$ and $\mathscr{E}_1$ are locally free of finite type.

Proposition 12.5.43. *Under the hypothesis of Theorem 12.5.37, let y be a point of Y and p be an integer. Then the following conditions are equivalent:*

(i) *the functor $T^p_{\mathscr{O}_{Y,y}}$ is right exact;*

(ii) *the canonical homomorphism $T^p_{\mathscr{O}_{Y,y}}(\mathscr{O}_{Y,y}) \to T^p_{\mathscr{O}_{Y,y}}(\kappa(y))$ is surjective;*

(iii) *for any integer n, the canonical homomorphism $T^p_{\mathscr{O}_{Y,y}}(\mathscr{O}_{Y,y}/\mathfrak{m}_y^{n+1}) \to T^p_{\mathscr{O}_{Y,y}}(\kappa(y))$ is surjective.*

Moreover, the set of $y \in Y$ verifying these conditions is exactly the open subset U of Y such that $\mathscr{T}^p_U$ is right exact.

Proof. The equivalence of (i), (ii), (iii) follows from Proposition 12.5.22 and Proposition 12.5.23. The fact that the set U where $T^p_{\mathscr{O}_{Y,y}}$ is right exact is open is a concequence of Proposition 12.5.19, and conversely if $\mathscr{T}^p_V$ is right exact, so is $T^p_{\mathscr{O}_{Y,y}}$ for any $y \in V$, by the condition (iii) of Theorem 12.5.37 and Proposition 12.5.28. □

Corollary 12.5.44. *If $\mathscr{T}^p$ is right exact (resp. left exact), then, for any morphism $g : Y' \to Y$, $\mathscr{T}^p_{Y'}$ is right exact (resp. left exact). The converse is ture if the morphism g is faithfully flat.*

Proof. The first assertion follows immediately from Proposition 12.5.28 and the fact that the question is local on Y and Y', in view of Theorem 12.5.37(ii) and (iii). To prove the second assertion, it suffices to show that for any $y \in Y$, $T^p_{\mathscr{O}_{Y,y}}$ is right exact (resp. left exact). But by our hypothesis, there exists $y' \in Y'$ such that $g(y) = y'$ and $\mathscr{O}_{Y',y'}$ is fiathfully flat over $\mathscr{O}_{Y,y}$, so the assertion follows from Proposition 12.5.28. □

12.5.7.2 Cohomological flatness Let X, Y be schemes, $f : X \to Y$ be a quasi-compact and separated morphism, $\mathscr{P}^\bullet$ be a complex of Y-flat quasi-coherent $\mathscr{O}_X$-modules, $\mathscr{T}^\bullet$ be the cohomological functor defined by $\mathscr{P}^\bullet$, and y be a point of Y. We say that $\mathscr{P}^\bullet$ is **cohomologically flat over Y at the point y, in dimension p**, if there exists an open neighborhood U of y in Y such that $\mathscr{T}^p_U$ is exact. We say $\mathscr{P}^\bullet$ is **cohomologically flat in dimension p over Y** if it is cohomologically flat over Y at every point $y \in Y$, in dimension p.

If $\mathscr{P}^\bullet$ is cohomologically flat over Y (resp. over Y at point y) for any dimension p, we simply say that $\mathscr{P}^\bullet$ is **cohomologically flat over Y** (resp. over Y at y). By definition, the notion of cohomologically flatness is local over Y. If Y is locally Noetherian, then for $\mathscr{P}^\bullet$ to be cohomologically flat over Y in dimension p, it is necessary and sufficient that the functor $\mathscr{T}^p$ is exact (for this, mimic the proof Theorem 12.5.37).

Proposition 12.5.45. *Let Y be a locally Noetherian scheme, $f : X \to Y$ be a proper morphism, $\mathscr{P}^\bullet$ be a bounded above complex of Y-flat coherent $\mathscr{O}_X$-modules, $\mathscr{T}^\bullet$ be the cohomological functor defined by $\mathscr{P}^\bullet$. For any $y \in Y$, the following conditions are equivalent:*

(i) *$\mathscr{P}^\bullet$ is cohomologically flat over Y at y in dimension p;*

(ii) *the functor $\mathscr{T}^p$ is exact;*

(iii) *there exists an integer n_0 such that for $n \geq n_0$, we have*

$$\ell_{\mathscr{O}_{Y,y}}(T^p_{\mathscr{O}_{Y,y}}(\mathscr{O}_{Y,y}/\mathfrak{m}_y^{n+1})) = \ell_{\mathscr{O}_{Y,y}}(T^p_{\mathscr{O}_{Y,y}}(\kappa(y))) \cdot \ell_{\mathscr{O}_{Y,y}}(\mathscr{O}_{Y,y}/\mathfrak{m}_y^{n+1}). \tag{12.5.25}$$

(iv) *there exists an open neighborhood U of y such that $R^p f_*(\mathscr{P}^\bullet)|_U$ is isomorphic to $(\mathscr{O}_Y|_U)^n$ and that, for any quasi-coherent $(\mathscr{O}_Y|_U)$-module $\mathscr{M}$, the canonical homomorphism*

$$((R^p f_*(\mathscr{P}^\bullet))|_U) \otimes_{\mathscr{O}_Y|_U} \mathscr{M} \to R^p f_*((\mathscr{P}^\bullet|_U) \otimes_{\mathscr{O}_Y|_U} \mathscr{M}) \tag{12.5.26}$$

is bijective.

If these conditions are verified, then we have the following:

(v) *There exists an open neighborhood U of y such that the function d_p of Theorem 12.5.35 is constant in U.*

Moreover, if Y is reduced at the point y, then (v) is equivalent to the above conditions.

Proof. In fact, condition (ii) is equivalent to say that $T^p_{\mathscr{O}_{Y,y}}$ and $T^{p-1}_{\mathscr{O}_{Y,y}}$ are right exact, so the equivalence of (i) and (ii) are clear. As $\mathscr{O}_{Y,y}/\mathfrak{m}_y^{n+1}$ is Artinian, and that $T^p_{\mathscr{O}_{Y,y}}(\mathscr{O}_{Y,y}/\mathfrak{m}_y^{n+1})$ and $T^p_{\mathscr{O}_{Y,y}}(\kappa(y))$ are finitely generated $(\mathscr{O}_{Y,y}/\mathfrak{m}_y^{n+1})$-modules, hence of finite length, so the equivalence of (ii) and (iii) follows from Proposition 12.5.43 and Proposition 12.5.11. The fact that (i) implies (v), and the equivalence if Y is reduced at y, all results from Theorem 12.5.35. Finally, (i) implies that $(R^pf_*(\mathscr{P}^\bullet))_y$ is a flat $\mathscr{O}_{Y,y}$-module (Proposition 12.5.9(iv′)), hence free since it is finitely generated over $\mathscr{O}_{Y,y}$. Since $R^pf_*(\mathscr{P}^\bullet)$ is coherent, this implies $R^pf_*(\mathscr{P}^\bullet)$ is free over an open neighborhood of y (**??**). Conversely, it is clear that (iv) implies (i) by the definition of $\mathscr{T}^p_U$. □

Proposition 12.5.46. *Under the hypothesis of Proposition 12.5.45, the following conditions are equivalent:*

(i) *$\mathscr{P}^\bullet$ is cohomologically flat over Y in dimension $i \geq p$;*

(ii) *for $i \geq p-1$, the functors $\mathscr{T}^i$ are right exact;*

(iii) *for $i \geq p$, the $\mathscr{O}_X$-modules $R^if_*(\mathscr{P}^\bullet)$ is locally free.*

Proof. The equivalence of (i) and (ii) is verified in Proposition 12.5.9 and (i) implies (iii) in view of Proposition 12.5.45(iv). Conversely, assume that (iii) is satisfied; since the question is local over Y, we may assume that $Y = \operatorname{Spec}(A)$ is affine. In this case, since $\mathscr{P}^\bullet$ is bounded above, we then have $\mathscr{T}^\bullet \xrightarrow{\sim} H^\bullet(\mathscr{L}^\bullet \otimes_{\mathscr{O}_Y} \mathscr{M})$, where $\mathscr{L}^\bullet = \widetilde{L}^\bullet$ and $L^\bullet$ is a bounded above complex of free A-modules of finite rank. Then the functor T^i is exact for $i \gg 0$, and by our hypothesis, $T^i(A)$ is a free A-module for $i \geq p$. We conclude from **??** that $\mathscr{T}^i = T^i$ is exact for $i \geq p$. □

We usually apply the criterions of cohomologically flatness to the case where the complex $\mathscr{P}^\bullet$ has a single nonzero term $\mathscr{F}$ at degree 0. In this case, we have $\mathscr{T}^p(\mathscr{M}) = T^pf_*(\mathscr{F} \otimes_{\mathscr{O}_Y} \mathscr{M})$.

Proposition 12.5.47. *Let Y be a locally Noetherian scheme, $f : X \to Y$ be a flat and proper morphism, y be a point of Y. Suppose that $\Gamma(X_y, \mathscr{O}_{X_y}) = R$ is a separated $\kappa(y)$-algebra. Then $\mathscr{O}_X$ is cohomologically flat over Y at point y in dimension 0.*

Proof. In view of Proposition 12.5.45, we may assume that Y is the spectrum of $A = \mathscr{O}_{Y,y}$. Since f is exact, we can choose $\mathscr{F} = \mathscr{O}_X$ to define the functor $T^\bullet$, and it is clear that T^0 is left exact. It then remains to prove that T^0 is right exact, to which we can reduce to the case where $A = \mathscr{O}_{Y,y}$ is an Artinian ring (Proposition 12.5.43(iii)). Let k' be a finite extension of $\kappa(y)$ so that $R \otimes_{\kappa(y)} k'$ is a direct product of finitely many fields isomorphic to k'. Since there exists a local homomorphism from A to a local ring A', which is a free A-algebra and such that the residue field of A' is isomorphic to k' ([**?**] 0_{III}, 10.3.2), we can, in view of Proposition 12.5.28, assume further that R is isomorphic to a direct product of m fields isomorphic to $\kappa(y)$. In this case, it is easy to see that the fiber X_y has exactly m connected components X'_i, and $\Gamma(X'_i, \mathscr{O}_{X'_i}) = \kappa(y)$ for each i. Since A is now a local Artinian ring, its spectrum is reduced to a singleton, so X and X_y have the same underlying space; in particular, X also has m connected components X_i, and $X'_i = X_i \times_Y \kappa(y)$. We can then reduce to the case where $R = \kappa(y)$, and in view of Proposition 12.5.43(ii), it suffices to prove that the canonical homomorphism $\Gamma(X, \mathscr{O}_X) \to \Gamma(X_y, \mathscr{O}_{X_y})$ is surjective. But this is trivial, since the composition

$$\Gamma(Y, \mathscr{O}_Y) = A \to \Gamma(X, \mathscr{O}_X) \to \Gamma(X_y, \mathscr{O}_{X_y}) = \kappa(y)$$

is surjective. □

Corollary 12.5.48. *Under the hypothesis of Proposition 12.5.47, there exists an open neighborhood U of y such that:*

(a) $f_*(\mathscr{O}_X)|_U$ *is isomorphic to* $(\mathscr{O}_Y|_U)^m$;

(b) *for any $z \in U$, the canonical homomorphism*

$$(f_*(\mathscr{O}_X))_z \otimes_{\mathscr{O}_{Y,z}} \kappa(z) \to \Gamma(X_z, \mathscr{O}_{X_z})$$

is bijective.

(c) *there exists a coherent $\mathscr{O}_U$-module $\mathscr{Q}$ and a functorial isomorphism*

$$R^1 f_*(f^*(\mathscr{M})) \xrightarrow{\sim} \mathcal{H}om_{\mathscr{O}_U}(\mathscr{Q}, \mathscr{M})$$

Proof. Assertion (a) follows from Proposition 12.5.47 and Proposition 12.5.45, and (b), (c) follows from the fact that $\mathscr{T}_U^0$ is exact and Theorem 12.5.37. $\square$

Corollary 12.5.49. *Under the Proposition 12.5.47, assume that $\Gamma(X_y, \mathscr{O}_{Y,y}) = \kappa(y)$. Then there exists an open neighborhood U of y such that the canonical homomorphism $\mathscr{O}_Y|_U$.*

Proof. In fact, it follows from Corollary 12.5.48 that the integer m in (a) is necessarily equal to 1. $\square$

Remark 12.5.50. Under the conditions of Proposition 12.5.47, consider the Stein factorization

$$X \xrightarrow{f'} Y' \xrightarrow{g} Y$$

where $Y' = \mathrm{Spec}(f_*(\mathscr{O}_X))$. The finite morphism g is then such that $g_*(\mathscr{O}_{Y'}) = f_*(\mathscr{O}_X)$ is locally free in an neighborhood of y, and its fiber at y is the specturm of a separated algebra over $\kappa(y)$ (Proposition 11.1.29). We shall see later that, there exists an open neighborhood U of y such that for any $z \in U$, the fiber $g^{-1}(z)$ is the spectrum of a separatble $\kappa(z)$-algebra (this is called an étale covering of U). Therefore, the hypothesis of Proposition 12.5.47 made on y is in fact valid in an open neighborhood of y.

12.5.7.3 Invariance of Euler characteristic and Hilbert function

Let A be a ring, M be a finitely generated projective A-module (which means $\widetilde{M}$ is locally free on $X = \mathrm{Spec}(A)$). For any $\mathfrak{p} \in \mathrm{Spec}(A)$, the rank of M at $\mathfrak{p}$ is defined to be the rank of the $A_{\mathfrak{p}}$-module $M_{\mathfrak{p}}$ (or the rank of the locally free $\mathscr{O}_X$-module $\widetilde{M}$ at $\mathfrak{p}$). We then have

$$\mathrm{rank}_{\mathfrak{p}}(M) = \mathrm{rank}_{\mathfrak{p}}(M_{\mathfrak{p}}) = \dim_{\kappa(\mathfrak{p})}(M \otimes_A \kappa(\mathfrak{p})). \tag{12.5.27}$$

Proposition 12.5.51. *Let $P^\bullet$ be a bounded complex of finitely generated projective A-modules, and for any module M, let $T^\bullet(M) = H^\bullet(P^\bullet \otimes_A M)$. Then for any $\mathfrak{p} \in \mathrm{Spec}(A)$, we have*

$$\sum_i (-1)^i \dim_{\kappa(\mathfrak{p})}(T^i(\kappa(\mathfrak{p}))) = \sum_i (-1)^i \mathrm{rank}_{\mathfrak{p}}(P^i). \tag{12.5.28}$$

Proof. In fact, by definition we have $T^i(\kappa(\mathfrak{p})) = H^i(P^\bullet \otimes_A \kappa(\mathfrak{p}))$, and in view of 12.5.27, the formula (12.5.28) is none other than the invariance of Euler characteristic of a bounded complex of finite dimensional vector spaces when passing to homology. $\square$

Corollary 12.5.52. *The function*

$$\mathfrak{p} \mapsto \sum_i (-1)^i \dim_{\kappa(\mathfrak{p})}(T^i(\kappa(\mathfrak{p})))$$

is locally constant on $\mathrm{Spec}(A)$.

Proof. This follows from (12.5.28), since the rank of a finitely projective module P^i is locally constant (Theorem 1.5.5). □

Theorem 12.5.53. *Let Y be a locally Noetherian scheme, $f : X \to Y$ be a proper morphism, $\mathscr{P}^\bullet$ be a bounded complex of Y-flat coherent $\mathscr{O}_X$-modules. Let $T^\bullet$ be the functor defined by (12.5.22), then the function*

$$y \mapsto \sum_i (-1)^i \dim_{\kappa(y)}(T^i(\kappa(y))) \tag{12.5.29}$$

is locally constant on Y.

Proof. We may assume that $Y = \mathrm{Spec}(A)$ is affine with ring A Noetherian. Then there exists a complex $L^\bullet$ of finitely generated A-modules such that $\mathscr{T}^p(\mathscr{M}) \xrightarrow{\sim} H^p(\mathscr{L}^\bullet \otimes_{\mathscr{O}_Y} \mathscr{M})$, where $\mathscr{L}^\bullet = \widetilde{L}^\bullet$. As the complex $\mathscr{P}^\bullet$ is bounded (take a finite affine covering $\mathfrak{U}$ of X), the double complex $\mathscr{C}^\bullet(\mathfrak{U}, \mathscr{P}^\bullet \otimes_{\mathscr{O}_Y} \mathscr{M})$ is bounded. More precisely, there exists a finite subset E (independent of $\mathscr{M}$), such that for any $(i,j) \notin E$, we have $\mathscr{C}^i(\mathfrak{U}, \mathscr{P}^j \otimes_{\mathscr{O}_Y} \mathscr{M}) = 0$. From this, we see that there exists an integer i_0 such that $\mathscr{T}^i(\mathscr{M}) = 0$ for any quasi-coherent $\mathscr{O}_Y$-module $\mathscr{M}$ and $i \geq i_0$. In particular, for such values of i, $\mathscr{T}^i$ is trivially an exact functor, so by Proposition 12.5.16, $W^i(L^\bullet)$ is a finitely generated flat (hence projective) A-module for such i. Consider the complex $(Q^\bullet)$, where $Q^i = L^i$ for $i < i_0$, $Q^{i_0} = W^{i_0}(L^\bullet)$, and $Q^i = 0$ for $i > i_0$, and set $\mathscr{Q}^\bullet = \widetilde{Q}^\bullet$. It is clear that $H^i(\mathscr{Q}^\bullet \otimes_{\mathscr{O}_Y} \mathscr{M}) = H^i(\mathscr{L}^\bullet \otimes_{\mathscr{O}_Y} \mathscr{M})$ for $i < i_0 - 1$ and also for $i \geq i_0$ (the two members being both zero). On the other hand, as $\mathrm{im}(W^{i_0} \otimes_A M) = \mathrm{im}(L^{i_0} \otimes_A M)$ by defintion, we also have $H^i(\mathscr{Q}^\bullet \otimes_{\mathscr{O}_Y} \mathscr{M}) = H^i(\mathscr{L}^\bullet \otimes_{\mathscr{O}_Y} \mathscr{M})$ for $i = i_0 - 1$. We then see that we may assume that $\mathscr{L}^\bullet$ is a bounded complex, with the weaker assumption that each $\mathscr{L}^i$ is a locally free $\mathscr{O}_Y$-module. The theorem now follows from module bounded complex of projective Euler characteristic locally constant. □

Under the condition of Theorem 12.5.53, the function (12.5.29) is constant if Y is connected. If Y is connected and nonempty, we denote its value by $\chi(f, \mathscr{P}^\bullet)$ or $\chi(Y, \mathscr{P}^\bullet)$, or simply $\chi(\mathscr{P}^\bullet)$ if there is no confusion, and call it the **Euler characteristic of $\mathscr{P}^\bullet$ relative to f**. In the general case, we denote by $\chi(f, \mathscr{P}^\bullet; y)$ or $\chi(Y, \mathscr{P}^\bullet; y)$, or simple $\chi(\mathscr{P}^\bullet; y)$ the integer in (12.5.29).

Under the hypothesis of Theorem 12.5.53, let

$$0 \longrightarrow \mathscr{P}'^\bullet \xrightarrow{u} \mathscr{P}^\bullet \xrightarrow{v} \mathscr{P}''^\bullet \longrightarrow 0$$

be an exact sequence of bounded complexes of Y-flat coherent $\mathscr{O}_X$-modules, where the homomorphisms u, v are of even degrees $2d, 2d'$. Then as $\mathscr{T}^\bullet$ is a cohomological functor, we have an exact sequence

$$\cdots \to \mathscr{T}^i(\mathscr{P}'^\bullet, \kappa(y)) \to \mathscr{T}^{i+2d}(\mathscr{P}'^\bullet, \kappa(y)) \to \mathscr{T}^{i+2d+2d'}(\mathscr{P}'^\bullet, \kappa(y)) \to \mathscr{T}^{i+1}(\mathscr{P}'^\bullet, \kappa(y)) \to \cdots$$

with only finitely many nonzero terms. Since the Euler characteristic of this complex is zero, we conclude that

$$\chi(\mathscr{P}^\bullet; y) = \chi(\mathscr{P}'^\bullet; y) + \chi(\mathscr{P}''^\bullet; y) \tag{12.5.30}$$

for any $y \in Y$. By an induction process, it is not hard to see that (by (12.5.30))

$$\chi(\mathscr{P}^\bullet; y) = \sum_i (-1)^i \chi(\mathscr{P}^i; y) \tag{12.5.31}$$

where for any coherent $\mathscr{O}_X$-module $\mathscr{F}$, flat over Y, we denote by $\chi(\mathscr{F}; y)$ (or $\chi(f, \mathscr{F}; y)$) the function $\chi(\mathscr{L}^\bullet; y)$ corresponding to the complex $\mathscr{L}^\bullet$ with only zero-th term $\mathscr{F}$. Therefore, the study of Euler characteristics is reduced to that of a single coherent $\mathscr{O}_X$-module.

Proposition 12.5.54. *Under the hypothesis of Theorem 12.5.53, let Y' be a locally Noetherian scheme, $g : Y' \to Y$ be a morphism, $X' = X \times_Y Y'$, $f' = f_{(Y')} : X' \to Y'$, $\mathscr{P}'^{\bullet}$ be the bounded complex $\mathscr{P}^{\bullet} \otimes_{\mathscr{O}_Y} \mathscr{O}_{Y'}$ of coherent Y'-flat $\mathscr{O}_{X'}$-modules. Then for $y' \in Y'$, we have*

$$\chi(\mathscr{P}'^{\bullet}; y') = \chi(\mathscr{P}^{\bullet}; g(y')). \tag{12.5.32}$$

Proof. The $\mathscr{O}_{X'}$-modules $\mathscr{P}'^i$ is the inverse image off $\mathscr{P}^i$ under the projection $X' \to X$, hence Y'-flat. On the other hand, we see that f' is proper, so the first member of (12.5.32) is well-defined. The formula (12.5.32) follows from ([?], 6.10.4.2) and Lemma 12.5.34, by reducing to the case where Y' and Y are affine. □

Proposition 12.5.55. *Under the hypothesis of Theorem 12.5.53, suppose that there exists an integer i_0 such that $T^i(\kappa(y)) = 0$ for $i \neq i_0$ and any $y \in Y$. Then $\mathscr{T}^{i_0}(\mathscr{O}_Y) = R^{i_0} f_*(\mathscr{P}^{\bullet})$ is a locally free $\mathscr{O}_Y$-module, whose rank n is equal to $(-1)^{i_0}\chi(f, \mathscr{P}^{\bullet}; y)$.*

Proof. Note that the hypothesis of Proposition 12.5.19 is verified, so we can apply Proposition 12.5.22, and the hypothesis implies that $T^i_{\mathscr{O}_{Y,y}}$ is zero for $i \neq i_0$ in view of Corollary 12.5.24. By Proposition 12.5.9, $\mathscr{T}^{i_0}$ is then exact, so by ([?], 7.8.4), $R^{i_0} f_*(\mathscr{P}^{\bullet})$ is locally free with rank at $y \in Y$ equal to

$$\dim_{\kappa(y)}(T^{i_0}(\kappa(y))) = (-1)^{i_0}\chi(f, \mathscr{P}^{\bullet}; y)$$

by definition, since $T^i(\kappa(y)) = 0$ for $i \neq i_0$. □

Corollary 12.5.56. *Let Y be a locally Noetherian scheme, $f : X \to Y$ be a proper morphism, $\mathscr{F}$ be a Y-flat coherent $\mathscr{O}_X$-module. Suppose that $R^i f_*(\mathscr{F}) = 0$ for any $i > 0$. Then $f_*(\mathscr{F})$ is a locally free $\mathscr{O}_Y$-module, with rank equal to $\chi(f, \mathscr{F}; y)$.*

Proof. It suffices to prove that $H^i(X_y, \mathscr{F} \otimes_{\mathscr{O}_Y} \kappa(y)) = 0$ for any $i > 0$ and $y \in Y$. For this, we may assume that $Y = \mathrm{Spec}(A)$ is affine. With the notations of Theorem 12.5.53, and $\mathscr{P}^{\bullet}$ being reduced to a single term $\mathscr{F}$, we have in fact $\mathscr{T}^p(\mathscr{O}_Y) = 0$ for $p > 0$ by hypothesis. We then conclude from Proposition 12.5.9 that $\mathscr{T}^p$ is exact for $p > 0$, and the assertion then follows from the equivalence of Theorem 12.5.37(i) and (iv). □

Proposition 12.5.57. *Under the hypothesis of Theorem 12.5.53, let $\mathscr{L}$ be a very ample $\mathscr{O}_X$-module relative to Y, and put $\mathscr{P}^{\bullet}(n) = \mathscr{P}^{\bullet} \otimes_{\mathscr{O}_X} \mathscr{L}^{\otimes n}$ for each $n \in \mathbb{Z}$. Then for any $y \in Y$, the function*

$$n \mapsto \chi(f, \mathscr{P}^{\bullet}(n); y)$$

is a polynomial with coefficients in $\mathbb{Q}$, and is locally constant on Y.

Proof. It is clear that $\mathscr{P}^{\bullet}(n)$ is a complex of Y-flat $\mathscr{O}_X$-modules. In view of (12.5.31), we can assume that $\mathscr{P}^{\bullet}$ has a single nonzero term $\mathscr{F}$ at degree 0. Moreover, as the question is local on Y, we may reduce to the case where Y is affine and f is projective. Put $X_y = f^{-1}(y)$ and let $\mathscr{L}_y = \mathscr{L} \otimes_{\mathscr{O}_Y} \kappa(y)$, which is a very ample $\mathscr{O}_{X_y}$-module (Proposition 11.4.25). For the functor $\mathscr{T}^{\bullet}$ defined by $\mathscr{P}^{\bullet}(n)$, we then have

$$T^i(\kappa(y)) = H^i(X_y, \mathscr{F}_y \otimes \mathscr{L}_y^{\otimes n})$$

where $\mathscr{F}_y = \mathscr{F} \otimes_{\mathscr{O}_Y} \kappa(y)$. In this case, $\chi(f, \mathscr{F}(n); y)$ is none other than the Euler characteristic $\chi_{\kappa(y)}(\mathscr{F}_y(n))$ defined in ([?], 2.5.1). The fact that $\chi(f, \mathscr{P}^{\bullet}(n); y)$ is a rational polynomial then follows from ([?], 2.5.3). Moreover, for each n, this number is locally constant by Corollary 12.5.52, whence our assertion. □

We denote by $P(f,\mathscr{P}^\bullet;y)$ or $P(\mathscr{P}^\bullet;y)$ the polynomial of Proposition 12.5.57, with rational coefficients, and call it the **Hilbert polynomial** at y relative to $\mathscr{P}^\bullet$, f and $\mathscr{L}$ (or simply the Hilbert polynomial of $\mathscr{P}^\bullet$ at y, or of f, if there is no confusion). From the properties of $\chi(f,\mathscr{P}^\bullet;y)$, we have

$$P(\mathscr{P}^\bullet;y)=P(\mathscr{P}'^\bullet;y)+P(\mathscr{P}''^\bullet;y) \tag{12.5.33}$$

for an exact sequence $0\to\mathscr{P}^\bullet\to\mathscr{P}'^\bullet\to\mathscr{P}''^\bullet\to 0$, and in particular

$$P(\mathscr{P}^\bullet;y)=\sum_i(-1)^iP(\mathscr{P}^\bullet;y). \tag{12.5.34}$$

Similarly, with the hypotheses and notations of Proposition 12.5.54, we have

$$P(\mathscr{P}'^\bullet;y')=P(\mathscr{P}^\bullet;g(y')). \tag{12.5.35}$$

The formula (12.5.34) reduces the study of Hilber functions to that of a single Y-flat $\mathscr{O}_X$-module. In this case, this polynomial has an interpretation which does not depend on cohomology.

Corollary 12.5.58. *Let Y be a Noetherian scheme, $f:X\to Y$ be a projective morphism, $\mathscr{L}$ be a very ample $\mathscr{O}_X$-module relative to Y, $\mathscr{F}$ be a Y-flat coherent $\mathscr{O}_X$-module. Then there exists an integer n_0 such that for $n\geq n_0$, $f_*(\mathscr{F}(n))$ is a locally free $\mathscr{O}_Y$-module whose rank at $t\in Y$ is equal to $P(f,\mathscr{F};y)(n)$.*

Proof. As the morphism f is projective, by Theorem 12.2.13 there exists n_0 such that for $n\geq n_0$, we have $R^if_*(\mathscr{F}(n))=0$ for $i>0$. The conclusion then follows from Corollary 12.5.56. □

Proposition 12.5.59. *Let Y be a Noetherian scheme, $f:X\to Y$ be a projective morphism, $\mathscr{L}$ be a ample $\mathscr{O}_X$-module relative to f, and put $\mathscr{F}(n)=\mathscr{F}\otimes_{\mathscr{O}_X}\mathscr{L}^{\otimes n}$ for any $\mathscr{O}_X$-module $\mathscr{F}$ and $n\in\mathbb{Z}$. For a coherent $\mathscr{O}_X$-module $\mathscr{F}$ to be Y-flat, it is necessary and sufficient that there exists an integer n_0 such that, for $n\geq n_0$, $f_*(\mathscr{F}(n))$ is a locally free $\mathscr{O}_Y$-module.*

Proof. The necessarity of this condition is proved in Corollary 12.5.58. To prove the converse, we can assume that Y is affine with ring A. In view of the hypothesis and Theorem 12.2.13, the A-modules $\Gamma(X,\mathscr{F}(n))$ are finitely generated and projective (Theorem 1.5.5). Let S be the graded ring $\Gamma_{n\geq 0}\Gamma(X,\mathscr{L}^{\otimes n})$; then X is canonically idetified with $\mathrm{Proj}(S)$ (Theorem 11.4.27(ii) and Corollary 11.5.25). Let $M=\bigoplus_{n\geq n_0}\Gamma(X,\mathscr{F}(n))$; by replacing $\mathscr{L}$ with a power $\mathscr{L}^{\otimes d}$, we may assume that S is generated by finitely many elements of degree 1 (Proposition 11.2.18), and it then follows from Theorem 11.2.39 and Proposition 11.2.36 that $\mathscr{F}$ is identified with $\widetilde{M}$. For a homogeneous element $g\in S$ of positive degere, we then have $\Gamma(X_g,\mathscr{F})=M_{(g)}$. Now M, a sum of projective modules, is a flat A-module, hence so is M_g, and therefore also $M_{(g)}$, which is a direct factor of M_g. We then conclude that $\mathscr{F}$ is Y-flat at any point of X_g, and as the X_g cover X, the assertion is proved. □

Chapter 13

Local study of schemes and morphisms of schemes

13.1 Unramified morphisms, smooth morphisms and étale morphisms

In this section, we introduce the notions of unramified morphisms, smooth morphisms and étale morphisms between schemes. These three classes of morphisms are analogues of the following types of maps of manifolds in differential geometry.

- **Submersions** are maps inducing surjections of tangent spaces everywhere. They are useful in the notion of a fibration. (Perhaps a more relevant notion from differential geometry, allowing singularities, is "locally on the source a smooth fibration".)
- **Immersions** are maps inducing injections of tangent spaces. They can be thought as a generalized notion for submanifolds.
- **Local isomorphisms** are maps inducing isomorphisms of tangent spaces. They are be viewed as covering spaces of manifolds.

13.1.1 Formally unramification and formally smoothness

Let $f : X \to Y$ be a morphism of schemes. We say that f is **formally smooth** (resp. **formally unramified**) if for any affine scheme Y' over Y and any closed subscheme Y'_0 of Y' defined by a square zero ideal $\mathscr{I}$ of $\mathscr{O}_{Y'}$, the canonical map

$$\mathrm{Hom}_Y(Y', X) \to \mathrm{Hom}_Y(Y'_0, X) \tag{13.1.1}$$

is surjective (resp. injective). In this case, we say that X is formally smooth (resp. formally unramified) over Y. If X is both formally smooth and formally unramified over Y, then it is said to be **étale** over Y.

Example 13.1.1. Suppose that $Y = \mathrm{Spec}(A)$ and $X = \mathrm{Spec}(B)$ are affine schemes, so that the morphism f corresponds to a ring homomorphism $\varphi : A \to B$. Then X is formally smooth (resp. formally unramified) over Y if and only if B is a formally smooth (resp. formally unramified) A-algebra.

Example 13.1.2. In view of the proof of Proposition 9.7.17, if a morphism $f : X \to Y$ is formally smooth (resp. formally unramified), then for any affine scheme Z over Y and any closed subscheme Z_0 of Z defined by a nilpotent ideal $\mathscr{I}$ of $\mathscr{O}_Z$, the canonical map

$$\mathrm{Hom}_Y(Z, X) \to \mathrm{Hom}_Y(Z_0, X)$$

is surjective (resp. injective), so we can also take this as the definition of formally unramification (resp. formally smoothness). We also note that if f is formally smooth (resp. formally étale), then for an *arbitrary* scheme Z over Y and any closed subscheme Z_0 of Z defined by a locally nilpotent ideal $\mathscr{I}$ of $\mathscr{O}_Z$, the canonical map

$$\mathrm{Hom}_Y(Z,X) \to \mathrm{Hom}_Y(Z_0,X)$$

is surjective (resp. bijective). To see this, let (U_α) be an open affine covering of Z such that the ideal $\mathscr{I}|_{U_\alpha}$ is nilpotent, and for each α, let U_α^0 be the inverse image of U_α in Z_0, which is the closed subscheme of U_α defined by $\mathscr{I}|_{U_\alpha}$. Let $f_0 : Z_0 \to X$ be a Y-morphism, then by the hypotheses and Proposition 9.7.17, for any α, there is a (resp. unique) Y-morphism $f_\alpha : U_\alpha \to X$ whose restriction to U_α^0 equals to $f_0|_{U_\alpha^0}$. Since f_α and f_β then coincide on any affine open of $U_\alpha \cap U_\beta$, we conclude that there exists a (resp. unique) morpism $f : Z \to X$ whose restriction to Z_0 is equal to f_0.

Proposition 13.1.3 (Properties of Formally Unramified and Formally Smooth Morphisms).

(i) *A monomorphism is formally unramified, and an open immersion is formally étale.*

(ii) *The composition of two formally smooth (resp. formally unramified) morphisms is formally smooth (resp. formally unramified).*

(iii) *If $f : X \to Y$ is a formally smooth (resp. formally unramified) S-morphism, so is $f_{(S')} : X_{(S')} \to Y_{(S')}$ for any base change $S' \to S$.*

(iv) *If $f : X \to X'$ and $g : Y \to Y'$ are two formally smooth (resp. formally unramified) S-morphisms, $f \times_S g$ is formally smooth (resp. formally unramified).*

(v) *Let $f : X \to Y$ and $g : Y \to Z$ be two morphisms. If $g \circ f$ is formally unramified, so is f.*

(vi) *If $f : X \to Y$ is a formally unramified morphism, so is $f_{\mathrm{red}} : X_{\mathrm{red}} \to Y_{\mathrm{red}}$.*

Proof. In view of Proposition 10.5.14 and Proposition 10.5.22, it suffices to prove (i), (ii) and (iii). The two assertions of (i) are trivial by definition. To prove (ii), consider two morphisms $f : X \to Y$, $g : Y \to Z$, an affine scheme Z' over Z, a closed subscheme Z_0' of Z' defined by a square zero ideal. Suppose that f and g are formally smooth, and consider a Z-morphism $u_0 : Z_0' \to X$:

$$\begin{tikzcd}
Z_0' \arrow[r, "u_0"] \arrow[dd, "j"'] & X \arrow[d, "f"] \\
 & Y \arrow[d, "g"] \\
Z' \arrow[r] \arrow[uur, dashed, "u"] \arrow[ur, dashed, "v"] & Z
\end{tikzcd}$$

The hypothesis over g implies that there exists a Z-morphism $v : Z' \to Y$ such that $f \circ u_0 = v \circ j$ (where $j : Z_0' \to Z'$ is the canonical injection), and the hypothesis over f implies that there exists a morphism $u : Z \to X$ such that $f \circ u = v$ and $u \circ j = u_0$, so $(g \circ f) \circ u$ is equal to the structural morphism $Z' \to Z$ and $u \circ j = u_0$, which proves that $g \circ f$ is formally smooth. By a smilar reasoning, we see that $f \circ g$ is formally unramified if both morphisms are formally unramified.

Finally, to prove (iii), put $X' = X_{(S')}$, $Y' = Y_{(S')}$, $f' = f_{(S')}$. Consider an affine scheme Y'' over Y' and a closed subscheme Y_0'' of Y'' defined by a square zero ideal. Then $\mathrm{Hom}_{Y'}(Y'',X')$ is canonically identified with $\mathrm{Hom}_Y(Y'',X)$, and $\mathrm{Hom}_{Y'}(Y_0'',X')$ with $\mathrm{Hom}_Y(Y'',X)$, so our conclusion follows from the definition. $\square$

Proposition 13.1.4. *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms and suppose that g is formally unramified. Then, if $g \circ f$ is formally smooth (resp. formally étale), so is f.*

Proof. Let Y' be an affine scheme, $h : Y' \to Y$ be a morphism, Y'_0 be a closed subscheme of Y' defined by a square zero ideal, and $u_0 : Y'_0 \to X$ be a Y-morphism.

$$\begin{tikzcd} Y'_0 \arrow[r, "u_0"] \arrow[d, "j"'] & X \arrow[d, "f"] & \\ Y' \arrow[r, "h"] \arrow[ur, dashed, "u"] & Y \arrow[r, "g"] & Z \end{tikzcd}$$

If $g \circ f$ is formally smooth, there exists a Z-morphism $u : Y' \to X$ such that $u \circ j = u_0$ (where $j : Y'_0 \to Y'$ is the canonical injection) and $(g \circ f) \circ u = g \circ h$. But this then implies that $f \circ u$ and h are Z-morphisms from Y' to Y such that $(f \circ u) \circ j = h \circ j$, so since g is formally unramified, we conclude that $f \circ u = h$, so u is a Y-morphism. In view of Proposition 13.1.3(v), the proposition is therefore proved. □

Corollary 13.1.5. *Suppose that g is formally étale. Then for the composition $g \circ f$ to be formally smooth (resp. formally unramified), it is necessary and sufficient that f is formally smooth (resp. formally unramified).*

Proof. This follows from Proposition 13.1.4 and Proposition 13.1.3(ii) and (v). □

Proposition 13.1.6. *Let $f : X \to Y$ be a morphism of schemes.*

(i) *Let (U_α) be an open covering of X and, for each α, let $i_\alpha : U_\alpha \to X$ be the canonical injection. For f to be formally smooth (resp. formally unramified), it is necessary and sufficient that each morphism $f \circ i_\alpha$ is formally smooth (resp. formally unramified).*

(ii) *Let (V_λ) be an open covering of Y. For f to be formally smooth (resp. formally unramified), it is necessary and sufficient that each restriction $f|_{f^{-1}(V_\lambda)} : f^{-1}(V_\lambda) \to V_\lambda$ is formally smooth (resp. formally unramified).*

Proof. We first note that (ii) is a concequence of (i): in fact, if $j_\lambda : V_\lambda \to Y$ and $i_\lambda : f^{-1}(V_\lambda) \to X$ are the canonical injections, the restriction $f_\lambda : f^{-1}(V_\lambda) \to V_\lambda$ satisfies $j_\lambda \circ f_\lambda = f \circ i_\lambda$; if f is formally smooth (resp. formally unramified), then so is $f \circ i_\lambda$ (Proposition 13.1.3); but as j_λ is formally étale, this implies that f_λ is formally smooth (resp. formally unramified) by Corollary 13.1.5. Conversely, if each restriction f_λ is formally smooth (resp. formally unramified), then so is $j_\lambda \circ f_\lambda$ (Proposition 13.1.3), and hence is f in view of (i). Now since each i_α is formally étale, it then boils down to prove that if the $f \circ i_\alpha$ is formally smooth (resp. formally unramified), then so is f.

Let Y' be an affine scheme, Y'_0 be a closed subscheme of Y' defined by a square zero ideal, and $g : Y' \to Y$ be a morphism. Let $u_0 : Y'_0 \to X$ be a morphism, and denote by W_α (resp. W^0_α) the open subscheme of Y' (resp. Y'_0) induced on $u_0^{-1}(U_\alpha)$ (note that Y' and Y'_0 have the same underlying space). First suppose that $f \circ i_\alpha$ is formally unramified. We show that if u_1 and u_2 are two Y-morphisms from Y' to X whose restriction on Y'_0 coincide, then $u_1 = u_2$. To see this, note that by Proposition 13.1.3 (iv), the hypothesis that $f \circ i_\alpha$ is formally unramified implies that $u_1|_{W_\alpha} = u_2|_{W_\alpha}$ for each α, so the assertion is true in this case.

Now suppose that $f \circ i_\alpha$ is formally smooth for each α, we prove that there exists a Y-morphism $u : Y' \to X$ whose restriction to Y'_0 is u_0. Now since Y' is affine, we can apply ([?], 16.5.17), whose conclusion precisely proves the existence of u. □

13.1.2 Differential properties and characterizations

Proposition 13.1.7. *For a morphism $f : X \to Y$ to be formally unramified, it is necessary and sufficient that $\Omega^1_{X/Y} = 0$.*

Proof. By Proposition 13.1.6, the question is local on source and target, so we can assume that $X = \mathrm{Spec}(B)$ and $Y = \mathrm{Spec}(A)$ are affine, hence reduce to proving that the A-algebra B is formally unramified if and only if $\Omega^1_{B/A} = 0$. For this, recall that $\mathrm{Hom}_B(\Omega^1_{B/A}, M)$ is isomorphic to $\mathrm{Der}_A(B, M)$ for any B-module M, so if $\Omega^1_{B/A} = 0$, Proposition 9.7.1 implies that B is formally unramified over A. Conversely, assume that B is formally unramified over A and consider the multiplication map $\mu : B \otimes_A B \to B$ of the A-algebra B. Let $\mathfrak{J}$ be the kernel of μ and set $C = (B \otimes_A B)/\mathfrak{J}^2$, $N = \mathfrak{J}/\mathfrak{J}^2$. Then N is a square zero ideal of C, and since B is formally unramified over A, we get $\mathrm{Der}_A(B, N) = 0$ by Proposition 9.7.1. But N is by definition the differential module $\Omega^1_{B/A}$, so we conclude that $\mathrm{Hom}_B(\Omega^1_{B/A}, \Omega^1_{B/A}) = 0$, whence $\Omega^1_{B/A} = 0$. □

Corollary 13.1.8. *Let $f : X \to Y$ and $g : Y \to Z$ be two morphism. For f to be formally unramified, it is necessary and sufficient that the canonical homomorphism $f^*(\Omega^1_{Y/Z}) \to \Omega^1_{X/Z}$ is surjective.*

Proof. This is an immediate concequence of Proposition 13.1.7 and the exact sequence

$$f^*(\Omega^1_{Y/Z}) \longrightarrow \Omega^1_{X/Z} \longrightarrow \Omega^1_{X/Y} \longrightarrow 0$$

□

Proposition 13.1.9. *Let $f : X \to Y$ be a formally smooth morphism.*

(i) *The $\mathscr{O}_X$-module $\Omega^1_{X/Y}$ is locally projective. If f is locally of finite type, then $\Omega^1_{X/Y}$ is locally free of finite rank.*

(ii) *For any morphism $g : Y \to Z$, the sequence of $\mathscr{O}_X$-modules*

$$0 \longrightarrow f^*(\Omega^1_{Y/Z}) \longrightarrow \Omega^1_{X/Z} \longrightarrow \Omega^1_{X/Y} \longrightarrow 0 \tag{13.1.2}$$

is exact. Moreover, for any $x \in X$, there exists an open neighborhood U of x such that the restriction of these homomorphisms to U form a splict exact sequence.

Proof. If f is locally of finite type, then the diagonal map $\Delta_{X/Y} : X \to X \times_Y X$ is of finite presentation (Corollary 10.6.25), so $\Omega^1_{X/Y}$ is an $\mathscr{O}_X$-module of finite type in view of Corollary 10.6.27. To see that $\Omega^1_{X/Y}$ is locally projective, it suffices to assume that $Y = \mathrm{Spec}(A)$ and $X = \mathrm{Spec}(B)$ are affine, and the conclusion then follows from Corollary 9.7.22. Now to prove (ii), we can also assume that X, Y, Z are affine schemes, and in this case the conclusion then follows from the interpretation of the modules appearing in the sequence (13.1.2) and ([?] 0_{IV}, 20.5.7). □

Corollary 13.1.10. *If $f : X \to Y$ is a formally étale morphism, then for any morphism $g : Y \to Z$, the canonical homomorphism $f^*(\Omega^1_{Y/Z}) \to \Omega^1_{X/Z}$ is bijective.*

Proof. This follows from the exact sequence (13.1.2) and the fact that we have $\Omega^1_{X/Y} = 0$ (Proposition 13.1.7). □

Proposition 13.1.11. *Let $f : X \to Y$ be a morphism, X' be a closed subscheme of X such that the composition morphism $X' \to X \to Y$ is formally smooth. Then the sequence of $\mathscr{O}_{X'}$-modules*

$$0 \longrightarrow \mathscr{N}_{X'/X} \longrightarrow \Omega^1_{X/Y} \otimes_{\mathscr{O}_X} \mathscr{O}_{X'} \longrightarrow \Omega^1_{X'/Y} \longrightarrow 0 \tag{13.1.3}$$

is exact. Moreover, for any $x \in X$, there exists an open neighborhood U of x such that the restriction of these homomorphisms to U form a splict exact sequence.

Proof. In view of Proposition 13.1.6, we can assume that $X = \operatorname{Spec}(B)$, $X' = \operatorname{Spec}(B/\mathfrak{I})$ and $Y = \operatorname{Spec}(A)$ are affine, where $\mathfrak{I}$ is an idela of B. Then the conormal sheaf $\mathscr{N}_{X'/X}$ corresponds to the B-module $\mathfrak{I}/\mathfrak{I}^2$, and the conclusion follows from Example 9.7.5. □

Proposition 13.1.12. *Let $f : X \to Y$ be a morphism locally of finite type. Then the following conditions are equivalent:*

(i) *f is a monomorphism;*

(ii) *f is radical and formally unramified;*

(iii) *for any $y \in Y$, the fiber $f^{-1}(y)$ is either empty or $\kappa(y)$-isomorphic to $\operatorname{Spec}(\kappa(y))$.*

Proof. The fact that (i) implies (iii) follows from Proposition 10.3.39. It is clear that (iii) implies that f is radical, and we prove that $\Omega^1_{X/Y} = 0$ in this case. Note that the $\mathscr{O}_X$-module $\Omega^1_{X/Y}$ is quasi-coherent of finite type ([?], 16.3.9), so it follows from Corollary 10.3.23 that, for $(\Omega^1_{X/Y})_x = 0$, it is necessary and sufficient that if we put $Y_1 = \operatorname{Spec}(\kappa(y))$, $X_1 = f^{-1}(y) = X \times_Y Y_1$, then $(\Omega^1_{X_1/Y_1})_x = 0$. But as the morphism $f_1 : X_1 \to Y_1$ induced from f is formally unramified in view of the hypotheses of (iii) (Proposition 13.1.3), the conclusion follows from Proposition 13.1.7.

Finally, we show that (ii) implies (i). For this, consider the diagonal morphism $g = \Delta_f : X \to X \times_Y X$; since f is radical, g is surjective (Proposition 10.3.31). On the other hand, $\Omega^1_{X/Y}$ is defined by the conormal sheaf of the immersion g, and the hypothesis that f is unramified implies $\Omega^1_{X/Y} = 0$. Moerover, g is locally of finite presentation (Corollary 10.6.25), so it is an open immersion by ([?], 16.1.10). Being surjective, this immersion is then an isomorphism, so f is a monomorphism by Proposition 10.5.5. □

13.1.3 Unramified morphisms and smooth morphisms

We say that a morphism $f : X \to Y$ is **smooth** (resp. **unramified**) if it is locally of finite presentation and formally smooth (resp. formally unramified), and **étale** if it is both smooth and unramified.

Example 13.1.13. Let A be a ring and B be an A-algebra. Then B is a smooth (resp. unramified) algebra if the corresponding morphism $\operatorname{Spec}(B) \to \operatorname{Spec}(A)$ is smooth (resp. unramified).

Proposition 13.1.14 (Properties of Unramified and Smooth Morphisms).

(i) *An open immersion is étale. For an immersion to be unramified, it is necessary and sufficient that it is locally of finite presentation.*

(ii) *The composition of two smooth (resp. unramified) morphisms is formally smooth (resp. formally unramified).*

(iii) *If $f : X \to Y$ is a smooth (resp. unramified) S-morphism, so is $f_{(S')} : X_{(S')} \to Y_{(S')}$ for any base change $S' \to S$.*

(iv) *If $f : X \to X'$ and $g : Y \to Y'$ are two smooth (resp. unramified) S-morphisms, $f \times_S g$ is smooth (resp. unramified).*

(v) *Let $f : X \to Y$ and $g : Y \to Z$ be two morphisms. If g is locally of finite type and $g \circ f$ is unramified, so is f.*

Proof. This follows from Proposition 10.6.38 and Proposition 13.1.3. □

Proposition 13.1.15. *Let $f : X \to Y$, $g : Y \to Z$ be morphisms, and suppose that g is unramified. Then, if $g \circ f$ is smooth (resp. étale), so is f.*

Proof. In fact, as g and $g \circ f$ are locally of finite presentation, so is f (Proposition 10.6.24(v)), and the conclusion then follows from Proposition 13.1.4. □

Corollary 13.1.16. *Suppose that g is étale, then for f to be smooth (resp. étale), it is necessary and sufficient that $g \circ f$ is smooth (resp. étale).*

Proof. This follows from Proposition 13.1.15 and Proposition 13.1.14(ii). □

Proposition 13.1.17. *Let $g : Y \to S$ and $h : X \to S$ be morphisms locally of finite presentation. For an S-morphism $f : X \to Y$ to be unramified, it is necessary and sufficient that the canonical homomorphism $f^*(\Omega^1_{Y/S}) \to \Omega^1_{X/S}$ is surjective.*

Proof. As f is then locally of finite presentation (Proposition 13.1.14(ii)), the proposition follows from Corollary 13.1.8. □

In view of Proposition 13.1.6 and the local nature of morphisms locally of finite presentation, we say a morphism $f : X \to Y$ is smooth (resp. unramified) at a point $x \in X$ if there exists an open neighborhood U of x in X such that the restriction $f|_U$ is a smooth (resp. unramified) morphism from U into Y. Then a morphism $f : X \to Y$ is smooth (resp. unramified) if and only if it is smooth (resp. unramified) at every point of X. Moreover, it is clear from this definition that the points of X where f is smooth (resp. unramified) is open in X.

Proposition 13.1.18. *For any scheme Y and any locally free $\mathscr{O}_X$-module $\mathscr{E}$ of finite rank, the vector bundle $\boldsymbol{V}(\mathscr{E})$ is a smooth Y-scheme.*

Proof. In fact, by Proposition 13.1.6 we can assume that $Y = \operatorname{Spec}(A)$ is affine and $\boldsymbol{V}(\mathscr{E}) = \operatorname{Spec}(A[T_1, \dots, T_n])$. As $A[T_1, \dots, T_n]$ is formally smooth over A and finitely presented, this proves the proposition. □

Corollary 13.1.19. *Under the hypothesis of Proposition 13.1.18, the projective bundle $\mathbb{P}(\mathscr{E})$ is a smooth Y-scheme.*

Proof. We can assume that $Y = \operatorname{Spec}(A)$ and $\mathbb{P}(\mathscr{E}) = \mathbb{P}^n_Y$. Then there is a finite open covering of $\mathbb{P}^n_A$ by the $D_+(T_i)$. But the ring of $D_+(T_i)$, being $A[T_1, \dots, T_n]_{(T_i)}$, is isomorphic to the polynomial ring $A[T_0, \dots, \widehat{T}_i, \dots, T_n]$, and hence smooth, so the corollary follows from Proposition 13.1.6. □

13.1.4 Characterization of unramification and smoothness

13.1.4.1 Characterization of unramified morphisms

Theorem 13.1.20. *Let $f : X \to Y$ be a morphism locally of finite presentation and x be a point of X. Then the following conditions are equivalent:*

(i) *f is unramified at x;*

(ii) *the diagonal morphism $\Delta_f : X \to X \times_Y X$ is a local isomorphism at x.*

(ii′) *if $Z = X \times_Y X$ and $z = \Delta_f(x)$, the morphism $(\Delta_f)^\#_x : \mathscr{O}_{Z,z} \to \mathscr{O}_{X,x}$ is bijective;*

(ii″) *for any morphism $g : Y' \to Y$ and any point $y' \in Y'$ lying over $y = f(x)$, any Y'-section s' of $X' = X \times_Y Y'$ with $x' = s'(y')$ lying over x is a local isomorphism at y';*

(iii) *$(\Omega^1_{X/Y})_x = 0$;*

(iv) *the $\kappa(y)$-scheme X_y is unramified over $\kappa(y)$ at x;*

(iv′) *the point x is isolated in X_y and the local ring $\mathscr{O}_{X_y,x}$ is a field and separable over $\kappa(y)$;*

(iv″) *$\mathscr{O}_{X_y,x} = \mathscr{O}_{X,x}/\mathfrak{m}_y\mathscr{O}_{X,x}$ is a field and a finite separable extension of $\kappa(y)$;*

(v) *$\mathscr{O}_{X,x}$ is a formally unramified $\mathscr{O}_{Y,y}$-algebra.*

Proof. As f is locally of finite type, the $\mathscr{O}_X$-module $\Omega^1_{X/Y}$ is of finite type ([**?**], 16.3.9), so $(\Omega^1_{X/Y})_x = 0$ if and only if there exists an open neighborhood U of x such that $\Omega^1_{X/Y}|_U = 0$. In view of Proposition 13.1.7, this proves the equivalence of (i) and (iii). On the other hand, if we put $A = \mathscr{O}_{Y,y}$, $B = \mathscr{O}_{X,x}$, then $(\Omega^1_{X/Y})_x = \Omega^1_{B/A}$ ([**?**], 16.4.15), so the equivalence of (iii) and (v) also follows from the affine case of Proposition 13.1.7.

By the very definition of unramified morphism at x, we see that (i) is equivalent to (iv). Also, as properties (iv) and (iv′) only involve the morphism $X_y \to \mathrm{Spec}(\kappa(y))$, this also implies the equivalence of (iv) and (iv′). On the other hand, (iv′) and (iv″) are equivalent, because it amounts to the same thing to say that $\mathscr{O}_{X_y,x}$ is a finite $\kappa(y)$-algebra or that x is an isolated point of X_y, since X_y is a $\kappa(y)$-scheme locally of type (Proposition 10.6.44).

We now prove the equivalence of (ii) and (ii′). We can limit ourselves to the case where $Y = \mathrm{Spec}(R)$, $X = \mathrm{Spec}(S)$ are affine and f is finitely presented. Then we have $Z = \mathrm{Spec}(S \otimes_R S)$ and Δ_f corresponds to the multiplication map $S \otimes_R S \to S$, whose kernel $\mathfrak{I}$ is a finitely generated ideal ([**?**] 0_{IV}, 20.4.4). If we put $\mathscr{I} = \widetilde{\mathfrak{I}}$, the $\mathscr{O}_Z$-module $\Delta_f^*(\mathscr{O}_X) = \mathscr{O}_Z/\mathscr{I}$ is then of finite presentation, and the hypothesis that the homomorphism $(\Delta_f)^\#_x : \mathscr{O}_{Z,z} \to \mathscr{O}_{X,x}$ is bijective implies that, by replacing X by an open neighborhood of x, the homomorphism $(\Delta_f)^\# : \mathscr{O}_Z \to \Delta_f^*(\mathscr{O}_X)$ is itself bijective (**??**). This then proves that (ii′) implies (ii), and the converse is evident.

On the other hand, the equivalence of (ii) and (ii″) follows from Corollary 10.5.4 even without the finiteness hypothesis on f: in fact, giving a Y'-section $s' : Y' \to X'$ is equivalent to giving a Y-morphism $h = g' \circ s' : Y' \to X$ (where $g' : Y' \to X$ is the canonical projection) such that $s' = \Gamma_h$, and the diagram

$$\begin{tikzcd} Y' \arrow[r, "s'"] \arrow[d, "h"'] & X' = Y' \times_Y X \arrow[d, "h \times_X 1_X"] \\ X \arrow[r, "\Delta_f"] & X \times_Y X \end{tikzcd}$$

is then cartesian. Therefore if Δ_f is a local isomorphism at x, then s' is a local isomorphism at y' (since $x = h(y')$), and this proves (ii)⇒(ii″). The converse is obtained by applying (ii″) to the case where $Y' = X$, $y' = x$, $g = f$ and $s' = \Delta_f$.

To finish the proof of the theorem, it then suffices to prove the following implications

$$\text{(iv″)} \Rightarrow \text{(iii)} \Rightarrow \text{(ii)} \Rightarrow \text{(iv″)}.$$

First, as $\Omega^1_{X/Y}$ is an $\mathscr{O}_X$-module of finite type, it follows from Nakayama lemma that condition (iii) is equivalent to $(\Omega^1_{X/Y})_x/\mathfrak{m}_y(\Omega^1_{X/Y})_x = 0$, which means $(\Omega^1_{X_y/\mathrm{Spec}(\kappa(y))})_x = 0$ ([**?**], 16.4.5). We are therefore reduced to the case where Y is the specturm of a field k and X is an algebraic k-scheme. The hypothesis that $\mathscr{O}_{X,x}$ is a finite field extension k' of k implies that x is closed in X (Corollary 10.6.45), and hence isolated in X. But then the hypothsis that k' is separable over k implies that $\Omega^1_{k'/k} = 0$ ([**?**] 0_{IV}, 20.6.20), which proves (iii). Moreover, in this case $\Omega^1_{X/Y}|_U = 0$ for an open neighborhood of x in X, so assertion (ii) follows from the definition of $\Omega^1_{X/Y}$.

Finally, to show that (ii)⇒(iv″), we can, by replacing X by an open neighborhood of x, suppose that Δ_f is an open immersion. If we denote by $f_y : X_y \to \mathrm{Spec}(\kappa(y))$ the morphism induced by f on fiber, then Δ_{f_y} is also an open immersion (Corollary 10.4.14), and as condition (iv″) only concerns the $\kappa(y)$-scheme X_y, we can therefore assume that $Y = \mathrm{Spec}(k)$ and $X = \mathrm{Spec}(A)$, where k is a field and A is a finite type k-algebra. Condition (iv″) is then established if we can show that A is a finite and separable k-algebra. If K is an algebraic closure of k, this amounts to saying that $A \otimes_k K$ is a finite and separable K-algebra ([**?**], 4.6.1), so we can further assume that k is algebraically closed. We first prove that A is a finite k-algebra. For this, it suffices to show that every closed point x of X is isolated, since then the set of such points is open in X and discrete, hence finite (X is Noetherian), and the assertion then follows from Proposition 10.6.44. Now for such a point x, we have $\kappa(x) = k$ since k is algebraically closed (Proposition 10.6.42), so by Corollary 10.6.43 there is a Y-section of X such that $s(Y) = \{x\}$, and in view of ([?], 17.4.1.1), $\{x\}$ is the inverse image of the diagonal $\Delta_X(X)$ under a morphism $X \to X \times_Y X$, hence is open in X in view of the hypothesis of (ii). The k-algebra A is therefore finite, hence is isomorphic to a direct product of finite local k-algebras. Since our question is local, we can then assume that A is a finite local k-algebra, so that $X = \operatorname{Spec}(A)$ is reduced to a singleton. The residue field of A, being a finite extension of k, is necessarily equal to k, so in view of Proposition 10.3.27, the product $X \times_k X$ is reduced to a singleton, and Δ_f is then an isomorphism. Since Δ_f corresponds to the multiplication map $\mu : A \otimes_k A \to A$, we conclude that μ is an isomorphism, which means $\Omega^1_{A/k} = 0$ and A is separable over k. □

Remark 13.1.21. If we onlt assume that f is locally of finite type, then $\Omega^1_{X/Y}$ is an $\mathscr{O}_X$-module of finite type ([?], 16.3.9), and Δ_f is a morphism locally of finite presentation (Corollary 10.6.25). The proof of Theorem 13.1.20 is still valid, provided that condition (i) is replaced by the following: the restriction of f to an open neighborhood of x is formally unramified. We also see that in this case the restriction of f to an open neighborhood of x is a locally quasi-finite morphism.

In fact, many authors require unramified morphisms to be (locally of finite type), rather than (locally) of finite presentation. The benifits of this definition is that any closed immersion is then unramified, which is reasonable under our intuition about unramification. The requirement of being locally of finite presentation, however, is necessary when we consider étale morphisms, so we add it to the definition of unramified morphism for consistency (following Grothendieck's definition). However, one should note that most of the proofs involving unramified morphisms work through finite type cases, and the restriction of being locally of finite presentation in fact unnecessary in most of the cases when we talk about unramified morphisms.

Corollary 13.1.22. *Let $f : X \to Y$ be a morphism locally of finite presentation. Then the following conditions are equivalent:*

(i) *f is unramified;*

(ii) *the diagonal morphism $\Delta_f : X \to X \times_Y X$ is an open immersion;*

(ii') *for any morphism $Y' \to Y$, any Y'-section of $X' \times_Y Y'$ is an open immersion;*

(iii) $\Omega^1_{X/Y} = 0$;

(iv) *for any $y \in Y$, the $\kappa(y)$-scheme X_y is unramified over $\kappa(y)$;*

(iv') *for any $y \in Y$, the $\kappa(y)$-scheme X_y is isomorphic to $\coprod_{\lambda \in I} \operatorname{Spec}(K_\lambda)$, where for each λ, K_λ is a finite separable extension of $\kappa(y)$;*

(v) *for any $x \in X$, the local ring $\mathscr{O}_{X,x}$ is a formally unramified $\mathscr{O}_{Y,f(x)}$-algebra.*

Proof. This follows from the observation that a morphism $f : X \to Y$ is unramified if and only if it is unramified at every point of X. □

Corollary 13.1.23. *If $f : X \to Y$ is unramified, then it is locally quasi-finite.*

Proof. By Theorem 13.1.20, for any point $x \in X$, the $\kappa(y)$-algebra $\mathscr{O}_{X_y,x}$ is finite, so x is isolated in X_y and the conclusion follows from ([?], 13.1.4). □

Proposition 13.1.24. *Let Y be a locally Noetherian scheme, $f : X \to Y$ be a morphism locally of finite type, x be a point of X. If $A = \mathscr{O}_{Y,y}$ and $\mathscr{O}_{X,x}$, then the conditions of Theorem 13.1.20 are equivalent to the following:*

(vi) *$\widehat{B}\otimes_{\widehat{A}}\kappa_A$ is a finite and separable field extension of κ_A (which implies that $\widehat{B}$ is a finite $\widehat{A}$-algebra);*

(vi') *$\widehat{B}$ is a formally unramified $\widehat{A}$-algebra.*

Moreover, if $\kappa(x)=\kappa(y)$, or if k is separably closed, these conditions are also equivalent to:

(vi") *the homomorphism $\widehat{A}\to\widehat{B}$ is surjective.*

Proof. Note that by the same reasoning of ([?] 0_{IV}, 19.3.6), the A-alebra B is formally unramified if and only if $\widehat{B}$ is formally unramified over $\widehat{A}$ for the adic topology. On the other hand, since f is locally of finite type, $\Omega^1_{B/A}$ is a finitely generated B-module ([?], 16.3.9), $\Omega^1_{B/A}=0$ if and only if $\widehat{\Omega}_{B/A}=0$ (Proposition 2.4.28), so B is a formally unramified A-algebra if and only if it is formally unramified over A for the adic topology ([?] 0_{IV}, 20.7.4), and this proves the equivalence of (v) and (vi').

Since $\kappa_A=A/\mathfrak{m}_A=\widehat{A}/\mathfrak{m}_A\widehat{A}$, we have $\widehat{B}\otimes_{\widehat{A}}\kappa_A=\widehat{B}/\mathfrak{m}_A\widehat{B}=\widehat{B}\otimes_B(B/\mathfrak{m}_AB)$, and $\widehat{B}/\mathfrak{m}_A\widehat{B}$ is therefore the completion of $B/\mathfrak{m}_AB=B\otimes_A\kappa_A$ for the $\mathfrak{m}_B$-adic topology (Theorem 2.4.19). Since $\widehat{B}$ is a fiathfully flat B-module, we see that $B/\mathfrak{m}_AB$ is a field if and only if $\widehat{B}/\mathfrak{m}_A\widehat{B}$ is a field. Moreover, since completion does not change the residue field, in this case $B/\mathfrak{m}_AB$ is separable over κ_A if and only if $\widehat{B}/\mathfrak{m}_A\widehat{B}$ is separable over κ_A; this proves the equivalence of (iv") and (vi).

Finally, if $\kappa(x)=\kappa(y)$ or κ_A is separably closed, condition (vi) implies that the homomorphism $\widehat{A}/\mathfrak{m}_A\widehat{A}\to\widehat{B}/\mathfrak{m}_A\widehat{B}$ is bijective and that $\widehat{B}$ is a quasi-finite $\widehat{A}$-algebra ([?] 0_{I}, 7.4.4), hence finite (since $\widehat{A}$ is complete, $\widehat{B}$ is separated for the $\mathfrak{m}_A$-adic topology, and $\mathfrak{m}_A\widehat{B}$ is a defining ideal of $\widehat{B}$ ([?] 0_{I}, 7.4.1)). The homomorphism $\widehat{A}\to\widehat{B}$ is then surjective by Nakayama lemma, so (vi) implies (vi"); the converse of this is evident. ☐

Given an S-scheme Y and two S-morphisms $f:X\to Y$, $g:X\to Y$, we define the **coincidence scheme** of f and g to be the inverse image of the diagonal $\Delta_{Y/S}$ under the S-morphism $(f,g)_S$. Moreover, by Proposition 10.5.11, this subscheme is canonically identified with the kernel $\ker(f,g)$.

Proposition 13.1.25. *Let $h:Y\to S$ be an unramified morphism and $f:X\to Y$, $g:X\to Y$ be two S-morphisms. Then the coincidence scheme of f and g is an open subscheme of X.*

Proof. In fact, since $\Delta_h:Y\to Y\times_S Y$ is an open immersion by Corollary 13.1.22, the inverse image of $\Delta_{Y/S}$ under $(f,g)_S$ is an open subscheme of X. ☐

Corollary 13.1.26. *Under the hypotheses of Proposition 13.1.25, let x be a point of X such that the following diagram commutes*

$$\operatorname{Spec}(\kappa(x))\longrightarrow X\overset{f}{\underset{g}{\rightrightarrows}}Y$$

Then there exists an open neighborhood U of x such that $f|_U=g|_U$. If Y is also separated over S, then there also exists an open neighborhood Z of x such that $f|_Z=g|_Z$. In particular, if X is also connected, then $f=g$.

Proof. By Corollary 10.5.12, x belongs to the subscheme $\ker(f,g)$, and the assertion then follows from Proposition 13.1.25. ☐

Corollary 13.1.27. *Under the hypotheses of Proposition 13.1.25, suppose that the structural morphism $\varphi:X\to S$ is closed. Let s be a point of X and suppose that the compositions the following diagram commutes*

$$X_s\longrightarrow X\overset{f}{\underset{g}{\rightrightarrows}}Y$$

Then there exists an open neighborhood V of s in S such that $f|_{\varphi^{-1}(V)}=g|_{\varphi^{-1}(V)}$. If Y is also separated over S and φ is open, then we can choose V to be clopen. In particular, if S is also connected, then $f=g$.

Proof. It follows from Corollary 13.1.26 that the subscheme $C = \ker(f, g)$ is open in X and contains X_s. As φ is closed, there exists an open neighborhood V of s such that $\varphi^{-1}(V) \subseteq C$. If Y is also separated over S, then C is also closed, so $\varphi(X - C)$ is clopen in S, and its complement V in S is then a clopen neighborhood of s such that $\varphi^{-1}(V) \subseteq C$. □

Proposition 13.1.28. *Let Y be a connected scheme, $f : X \to Y$ be a unramified and separated morphism. Then any Y-section g of X is an isomorphism from Y onto an connected component of X, and the map $g \mapsto g(Y)$ is a bijection from $\Gamma(X/Y)$ to the set of connected components Z of X (necessarily open in X) such that the restriction of f to Z is an isomorphism from Z onto Y. In particular, if g_1 and g_2 and two Y-sections of X such that $g_1(y) = g_2(y)$ for a point $y \in Y$, then $g_1 = g_2$.*

Proof. It follows from Corollary 13.1.22 that any Y-section s of X is an open immersion, and as X is a separated Y-scheme, s is also a closed immersion (Corollary 10.5.19). Then s is an isomorphism from Y onto a clopen subscheme of X, and as $s(Y)$ is connected, this is necessarily a connected component of X. The rest of the proposition is then immediate. □

Example 13.1.29. Let k be a field and $X = \coprod_{n\in\mathbb{N}} \mathrm{Spec}(k)$ be an infinite direct sum of $\mathrm{Spec}(k)$. Then the morphism $f : X \to \mathrm{Spec}(k)$ is étale. This can be considered as a trivial covering space formed by infinitely many copies of the base space.

Example 13.1.30. Let k be a field with $\mathrm{char}(k) \neq 2$ and consider the normalization $f : \mathbb{A}^1_k \to C$ morphism, where $C = \mathrm{Spec}(k[T^2, T^3])$ and $\mathbb{A}^k_1 = \mathrm{Spec}(k[T])$. The morphism f corresponds to the ring homomorphism

$$\varphi : k[T^2, T^3] \to k[T], \quad (X, Y) \mapsto (T^2, T^3).$$

If we set $A = k[T^2, T^3]$ and $B = k[T]$, then it is easy to see that $B = A[X]/(X^2 - T^2)$, so the B-module $\Omega^1_{B/A}$ is generated by the symbol dX subject to the relation

$$0 = d(X^2 - T^2) = 2XdX - dT^2 = 2XdX.$$

and it is therefore isomorphic, as a B-module, to

$$A[X]/(2X, X^2 - T^2) = k[T^2, T^3, X]/(2X, X^2 - T^2) = k[T]/(2T) = k[T]/(T).$$

From this, we see that the support of $\Omega^1_{B/A}$ is equal to $\{(T)\}$, so by Corollary 13.1.22 the unramified locus of f is $\mathbb{A}^1_k - \{0\}$. This ramification can also be detected from the local homomorphism $\varphi_{\mathfrak{p}} : A_{\mathfrak{p}} \to B_{\mathfrak{P}}$, where $\mathfrak{p} = (T^2)$ and $\mathfrak{P} = (T)$. The induced homomorphism on residue fields is an isomorphism on $k(T)$, but the image of the maximal ideal of $\mathfrak{p}A_{\mathfrak{p}}$ to $\mathfrak{P}B_{\mathfrak{P}}$ is equal to $\mathfrak{P}^2 B_{\mathfrak{P}}$, so it is not unramified.

Geometrically, the ramification of f at origin is resulted from the fact that the tangent vectors (two different directions) of $\mathbb{A}^1_k$ are both bended to the same tangent vector (the same direction) of the origin of C. This justifies our intuition of unramification by unicity of tangent vector liftings.

Example 13.1.31. Let k be a field and consider the morphism $f : \mathbb{A}^k_1 \to \mathbb{A}^k_1$ corresponding to the ring homomorphism

$$\varphi : k[X] \to k[Y], \quad X \mapsto Y^2.$$

This can be considered as the projection of the parabola onto $\mathbb{A}^1_k$. Let $A = k[X]$ and $B = k[Y]$, then we have $B = A[T]/(T^2 - X)$, so the B-module $\Omega^1_{B/A}$ is isomorphic to $k[Y]/(2T) = k[Y]/(Y)$. Therefore the ramification locus of f is again $\mathbb{A}^1_k - \{0\}$. This is not surprising, since geometrically the tangent map of f at the origin is zero, so it does not satisfy the unicity of tangent vector liftings.

We also remark that the morphism f is flat. In fact, for any $\lambda \in k$, the fiber of f over the closed point $(X-\lambda)$ is $\mathrm{Spec}(k[Y]/(Y^2-\lambda))$, which is the disjiont union of 2 points if $\lambda \neq 0$, and is the tangen vector $\mathrm{Spec}(k[\varepsilon]/(\varepsilon^2))$ if $\lambda = 0$. In all cases, we see that $\dim_k(k[Y]/(Y^2-\lambda)) = 2$, so f is flat. Another way to see this is to use the miracle flatness: since f is a morphism between regular schemes of equal dimension, it must be flat.

13.1.4.2 Characterization of smooth morphisms

Theorem 13.1.32. *Let $f : X \to Y$ be a morphism locally of finite presentation, x be a point of X and $y = f(x)$. Then the following conditions are equivalent:*

(i) *f is smooth at x;*

(ii) *f is flat at x and the $\kappa(y)$-scheme X_y is smooth over $\kappa(y)$ at x;*

(ii′) *f is flat and geometrically regular at x;*

(iii) *$\mathscr{O}_{X,x}$ is a formally flat $\mathscr{O}_{Y,y}$-algebra.*

Proof. By the hypothesis on f, we can assume that $Y = \mathrm{Spec}(A)$, $X = \mathrm{Spec}(C)$, where $C = B/\mathfrak{I}$, $B = A[T_1, \dots, T_n]$ being a polynomial algebra and $\mathfrak{I}$ a finitely generated ideal of B. The equivalence of (i) and (iii) then follows from Theorem 9.7.28. On the other hand, apply this result to the morphism $f_y : X_y \to \mathrm{Spec}(\kappa(y))$ (which is locally of finite type), we see that the equivalence of (i), (ii) and (ii′) follows from Theorem 9.7.36. □

Corollary 13.1.33. *Let $f : X \to Y$ be a morphism locally of finite presentation. For f to be smooth, it is necessary and sufficient that f is flat and for any $y \in Y$, X_y is a geometrically regular $\kappa(y)$-scheme.*

Proof. This follows from the definition of flat morphisms and Theorem 13.1.32. □

Proposition 13.1.34. *Let Y be a locally Noetherian scheme, $f : X \to Y$ be a morphism locally of finite type, x be a point of X and $y = f(x)$. If $A = \mathscr{O}_{Y,y}$ and $\mathscr{O}_{X,x}$, then the conditions of Theorem 13.1.32 are equivalent to the following:*

(iv) *B is a formally smooth A-algebra;*

(iv′) *$\widehat{B}$ is a formally unramified $\widehat{A}$-algebra.*

Moreover, if $\kappa(x) = \kappa(y)$, these conditions are also equivalent to:

(iv″) *$\widehat{B}$ is isomorphic to a power series $\widehat{A}$-algebra $\widehat{A}[\![T_1, \dots, T_n]\!]$.*

Proof. The equivalence of condition (iii) of Theorem 13.1.32 and (iv) follows from Jacobian criterion (Theorem 9.7.28), and that of (iv) and (iv′) follows from ([**?**] 0_{IV} 19.3.6). On the other hand, (iv″) implies (iv′) without the hypothesis on residue fields (Example 9.7.12). Finally, if $\mathfrak{m}$ is the maximal ideal of $\widehat{A}$, condition (iv′) implies that $\widehat{B}/\mathfrak{m}\widehat{B}$ is a formally smooth complete Noetherian local $\kappa(y)$-algebra. The hypothesis $\kappa(x) = \kappa(y)$ then shows that $\widehat{B}/\mathfrak{m}\widehat{B}$ is $\kappa(y)$-isomorphic to a formal series algebra $\kappa(y)[\![T_1, \dots, T_n]\!]$ (Proposition 9.7.25). On the other hand, as $\widehat{A}[\![T_1, \dots, T_n]\!]$ is a flat $\widehat{A}$-module and a complete Noetherian local $\widehat{A}$-algebra, we conclude from ([**?**] 0_{IV}, 19.6.4) that it is isomorphic to $\widehat{B}$, so (iv′) imples (iv″). □

Proposition 13.1.35. *Let Y be a locally Noetherian scheme, $f : X \to Y$ be a morphism locally of finite type, x be a point of X and $y = f(x)$. Suppose that Y is reduced at y. Then for f to be smooth at x, it is necessary and sufficient that f is universally open in an open neighborhood of x in X_y and that the $\kappa(y)$-scheme X_y is a geometrically regular at x.*

Proof. In view of, it amounts to show that if X_y is geometrically regular at x, then f is flat at x if and only if it is universally open in an open neighborhood of x in X_y. Now if f is flat at x, so is it in an open neighborhood of x in X ([**?**], 11.1.1) and hence universally open in this neighborhood ([**?**], 2.4.6). Conversely, the hypotheses that X_y is geometrically regular at x and f is universally open in an open neighborhood of x in X_y together imply that f is flat at x, since $\mathscr{O}_{Y,y}$ is reduced ([**?**], 15.2.2). □

Corollary 13.1.36. *Let Y be a locally Noetherian scheme, $f : X \to Y$ be a morphism locally of finite type, x be a point of X and $y = f(x)$. Suppose that Y is reduced and geometrically unibrach at y. Then for f to be smooth at x, it is necessary and sufficient that f is equidimensional at x and that the $\kappa(y)$-scheme X_y is a geometrically regular at x.*

Proof. Since the set of points where f is equidimensional is open ([**?**], 13.3.2), this follows from Proposition 13.1.35 and Chevalley's criterion ([**?**], 14.4.4). □

Proposition 13.1.37. *Let $f : X \to Y$ be a morphism locally of finite presentation which is smooth at a point $x \in X$, and $y = f(x)$. Then, for the local ring $\mathscr{O}_{X,x}$ to be reduced (resp. integrally closed, resp. geometrically unibrach), it is necessary and sufficient that $\mathscr{O}_{Y,y}$ is.*

Proposition 13.1.38. *Let Y be a locally Noetherian scheme, $f : X \to Y$ be a morphism locally of finite type which is smooth at a point $x \in X$. Put $y = f(x)$, then*

- $\dim(\mathscr{O}_{X,x}) = \dim(\mathscr{O}_{Y,y}) + \dim(\mathscr{O}_{X,x} \otimes_{\mathscr{O}_{Y,y}} \kappa(y))$;
- $coprof(\mathscr{O}_{X,x}) = coprof(\mathscr{O}_{Y,y})$;
- *for the local ring $\mathscr{O}_{X,x}$ to possesses property (S_n) or (R_n), it is necessary and sufficient that $\mathscr{O}_{Y,y}$ to possesses property (S_n) or (R_n). In particular, if $\mathscr{O}_{X,x}$ is regular (resp. normal), so is $\mathscr{O}_{Y,y}$.*

13.1.4.3 Characterization of étale morphisms

Theorem 13.1.39. *Let $f : X \to Y$ be a morphism locally of finite presentation, x be a point of X and $y = f(x)$. then the following conditions are equivalent:*

(i) *f is étale at x.*

(i') *f is smooth and unramified at x;*

(ii) *f is smooth and quasi-finite at x;*

(iii) *f is flat and unramified at x;*

(iii') *f is flat at x and the ring $\mathscr{O}_{X,x}/\mathfrak{m}_y\mathscr{O}_{X,x}$ is a finite and separable field extension of $\kappa(y)$;*

(iv) *$\mathscr{O}_{X,x}$ is a formally étale $\mathscr{O}_{Y,y}$-algebra.*

Proof. The equivalence of (i) and (i') follows from definition, and that of (i) and (iv) follows from Theorem 13.1.20 and Theorem 13.1.32. The equivalence of (iii) and (iii') also follows from Theorem 13.1.20. The fact that (i')⇒(iii) follows from Theorem 13.1.32. Conversely, if (iii') is satisfied, then f is geometrically regular at x by Example 9.6.12, and hence smooth at x by Theorem 13.1.32. Also, the implication (i)⇒(ii) follows from Corollary 13.1.23.

It then remains to prove that (ii)⇒(iii), and as we have seen that f is flat at x (13.1.32), it suffices to show that X_y is unramified over $\kappa(y)$. In other words, we are reduced to the case where $Y = \mathrm{Spec}(k)$. As the question is local over X, we can also assume that $X = \mathrm{Spec}(A)$, where A is a finite local k-algebra ([**?**] 0_{I}, 7.4.1). In view of the hypothesis (ii), A is a formally smooth k-algebra, and hence geometrically regular (Proposition 9.7.25). Since it is Artinian, we then conclude $\mathfrak{m}_A = 0$, so A a finite and separable extension of k. □

Corollary 13.1.40. *Let f be a morphism locally of finite presentation. Then the following conditions are equivalent:*

(i) *f is étale.*

(i') *f is smooth and unramified;*

(ii) *f is smooth and locally quasi-finite;*

(iii) *f is flat and unramified;*

(iii') *f is flat and for any $y \in Y$, the fiber X_y is a sum of specturms of finite and separable field extensions of $\kappa(y)$;*

(iii'') *f is flat and for any $y \in Y$ and any separably closed extension k' of $\kappa(y)$, the geometric fiber $X_y \otimes_{\kappa(y)} k'$ is a sum of specturms of fields isomorphic to k'.*

Proposition 13.1.41. *Let Y be a locally Noetherian scheme, $f : X \to Y$ be a morphism locally of finite presentation, x be a point of X and $y = f(x)$. If $A = \mathscr{O}_{Y,y}$ and $\mathscr{O}_{X,x}$, then the conditions of Theorem 13.1.39 are equivalent to the following:*

(iv) *$\widehat{B}$ is a formally étale $\widehat{A}$-algebra.*

(iv') *$\widehat{B}$ is a free $\widehat{A}$-module and $\widehat{B} \otimes_{\widehat{A}} \kappa_A$ is a finite and separable field extension of κ_A (which implies that $\widehat{B}$ is a finite $\widehat{A}$-algebra).*

Moreover, if $\kappa(x) = \kappa(y)$ or $\kappa(y)$ is separably closed, these conditions are also equivalent to:

(iv'') *the canonical homomorphism $\widehat{A} \to \widehat{B}$ is bijective.*

Proposition 13.1.42. *Let Y be a locally Noetherian scheme, $f : X \to Y$ be a morphism locally of finite type, x be a point of X and $y = f(x)$. If f is étale at x, then $\dim(\mathscr{O}_{X,x}) = \dim(\mathscr{O}_{Y,y})$.*

Proof. This follows from Proposition 13.1.38 since x is isolated in the fiber X_y. □

13.2 Galois categories

13.2.1 The axioms of Galois theory

13.3 The étale fundamental group

13.3.1 Finite group quotients of schemes

Let G be a (fixed) finite group and S be a scheme. By a ***G*-scheme** over S, we mean an S-scheme X with a right action of G on X. By definition, this means we have a homomorphism $\rho : G \to \mathrm{Aut}_S(X)$ from G into the set of S-automorphisms of G. For any S-scheme Z, G has an induced left action on the set $\mathrm{Hom}_S(X, Z)$, so we can consider the set $\mathrm{Hom}_S(X, Z)^G$ of G-invariant S-morphisms. Since this set depends functorially by Z, we then obtain a functor $\mathrm{Hom}(X, -)^G$, for which we can ask the representability. By Yoneda lemma, this is equivalent to the existence a S-scheme Y and a G-invariant S-morphism $p : X \to Y$ such that, for any S-scheme Z, the corresponding map

$$\mathrm{Hom}_S(Y, Z) \to \mathrm{Hom}_S(X, Z)^G, \quad g \mapsto g \circ p$$

is bijective. In this case, we say that (Y, p) (or the S-morphism $p : X \to Y$) is a **quotient scheme** of X by G, and denote it by X/G. It is clear that the pair (Y, p) is determined up to isomorphism.

If the scheme X is affine, then the quotient scheme of X always exists and has a simple interpretation. In fact, in this case G has a left action on the ring A of X, and the invariant subring A^G then provides such a quotient.

Proposition 13.3.1. *Let R be a ring, A be an R-algebra with an R-linear action by G. Let $X = \operatorname{Spec}(A)$, $Y = \operatorname{Spec}(A^G)$, and $p : X \to Y$ be the canonical morphism.*

(a) *The morphism p is integral and surjective.*

(b) *The fibers of p are orbits of G, and p is a quotient map.*

(c) *Let $x \in X$, $y = p(x)$, and G_x be the stabilizer of x. Then $\kappa(x)$ is a normal extension of $\kappa(y)$ and the canonical map $G_x \to \operatorname{Gal}(\kappa(x)/\kappa(y))$ is surjective.*

(d) *(Y, p) is a quotient scheme of X by G.*

Proof. The assertions (a), (b) and (c) follows from Proposition 4.2.1, Proposition 4.2.6 and the fact that an integral morphism is closed (Proposition 11.6.7). Finally, assertion (d) follows from Theorem 10.1.17 and the fact that for any ring B, we have a canonical bijection

$$\operatorname{Hom}_{\mathbf{Ring}}(B, A)^G \xrightarrow{\sim} \operatorname{Hom}_{\mathbf{Ring}}(B, A^G).$$ □

Corollary 13.3.2. *Under the hypotheses of Proposition 13.3.1, the canonical homomorphism $\mathscr{O}_Y \to p_*(\mathscr{O}_X)^G$ is an isomorphism.*

Proof. By Proposition 10.1.12, the sheaf $p_*(\mathscr{O}_X)$ corresponds to the A^G-module A, so the corollary follows from the isomorphism $(S^{-1}A)^G = S^{-1}A^G$ of Proposition 4.2.3. □

Proposition 13.3.3. *Let S be a scheme, X be a G-scheme over S and $p : X \to Y$ be a G-invariant affine S-morphism such that $\mathscr{O}_Y \cong p_*(\mathscr{O}_X)^G$. Then the conditions of Proposition 13.3.1 are satisfied.*

Proof. Since conditions (a), (b) and (c) are local on X and Y, we can assume that $Y = \operatorname{Spec}(B)$ and $X = \operatorname{Spec}(A)$ are both affine (since the morphism p is affine). Then the hypothesis implies that $B = A^G$, so it suffices to apply Proposition 13.3.1. As for assertion (d), it suffices to note that $p : X \to Y$ is a quotient map. □

Corollary 13.3.4. *Under the hypotheses of Proposition 13.3.3, for any open subset U of Y, the restriction $p^{-1}(U) \to U$ is a quotient U by G.*

Proof. This follows from the fact that the restriction $p|_{p^{-1}(U)} : p^{-1}(U) \to U$ also satisfies the hypotheses of Proposition 13.3.3. □

Corollary 13.3.5. *Retain the hypotheses of Proposition 13.3.3.*

(a) *For X to be affine (resp. separated) over S, it is necessary and sufficient that Y is affine (resp. separated) over S.*

(b) *If X is of finite type over S, then it is finite over Y. If S is also locally Noetherian, then Y is of finite type over S.*

Proof. As X is affine (and a fortiori separated) over Y, we see that Y is affine (resp. separated) over S if and only if X is (Proposition 10.5.26 and Proposition 11.1.33). Now if X is of finite type over Z, so is it over Y, and hence finite over Y (since $p : X \to Y$ is integral). If S is also locally Noetherian, then since Y is already quasi-compact over S (Proposition 10.6.3), it suffices to show that Y is locally of finite type over S, so we may assume that $X = \operatorname{Spec}(A)$, $Y = \operatorname{Spec}(B)$ and $S = \operatorname{Spec}(R)$ are affine. Then the ring A is of finite type over R and the conclusion follows from Theorem 4.2.2. □

Let X be a G-scheme over a scheme S. If there exists a quotient scheme (Y, p) of X by G, we then say that X is **admissible** (or an **admissible G-scheme** over S). By Proposition 13.3.1, we see that any affine G-scheme over S is admissible. In the general case, we have the following characterization for admissible G-schemes.

Proposition 13.3.6. *Let X be a G-scheme over a scheme S. Then the following conditions are equivelent:*

(i) *X is admissible;*

(ii) *X is a union of affine open G-invariant subsets;*

(iii) *any orbit of G in X is contained in an affine open subset.*

Proof. It is clear that (ii)⇒(iii), and conversely, if an orbit T of G is contained in an affine open subset U, then the intersection $U' = \bigcap_{g \in G} g \cdot U$ is a G-invariant open subset containing T and contained in the affine open U. As in U, any finite subset has a fundamental system of open affine neighborhoods, there exists an open affine neighborhood V of T contained in U'. The transforms of V by G are therefore affine and contained in U', which is separate, so their intersection U'' is an affine open subset which is invariant under G and contains T. Since X is the union of orbits of G, we conclude that (iii)⇒(ii).

We note that condition (ii) is necessary for (i), since if $p : X \to Y$ is a quotient scheme of X and (V_α) is an affine open covering of Y, then $U_\alpha = p^{-1}(V_\alpha)$ is G-invariant and affine in X, and they cover X. Conversely, if (X_α) is a covering of X by G-invariant affine opens, then by Proposition 13.3.1, we can form the quotient $Y_\alpha = X_\alpha / G$; in each Y_i, the image of $X_i \cap X_j$ is an open subset Y_{ij}, which is identified with X_{ij}/G in view of Corollary 13.3.4. In particular, we deduce an isomorphism $Y_{ij} \cong Y_{ji}$, so we can glue Y_i to construct Y. □

Corollary 13.3.7. *If X is an admissible G-scheme, then it is admissible for any subgroup H of G.*

Corollary 13.3.8. *Let S be a scheme and suppose that X is a G-scheme over S which is affine over S. Then X is admissible; in fact, if X is defined by the quasi-coherent $\mathscr{O}_S$-algebra $\mathscr{A}$, then its quotient Y is defined by the quasi-coherent algebra $\mathscr{A}^G$.*

Proof. We may assume that $X = \mathrm{Spec}(A)$ and $S = \mathrm{Spec}(R)$ are affine, where A is an R-algebra. Then by Proposition 13.3.1 the quotient is defined by the R-algebra A^G, so the conclusion follows. □

Proposition 13.3.9. *Let X be an admissible G-scheme over S and $p : X \to Y$ be its quotient scheme. Consider a base change $S' \to S$ and put $X' = X \times_S S'$, $Y' = Y \times_S S'$, so that X' is a G-scheme over S' and the morphism $p' : X' \to Y'$ is G-invariant. If S' is flat over S, then (Y', p') is the quotient of X' by G. In other words, we have an isomorphism*

$$(X/G) \times_S S' \cong (X \times_S S')/G$$

Proof. We can evidently assume that S, X and Y are affine. It then suffices to prove that, if A is an R-algebra acted by G and $B = A^G$ is the invariant subring, then for any flat R-algebra R', the invairant subring of $A' = A \times_R R'$ is identified with $B' = B \otimes_R R'$. To see this, note that the subring B is characterized by the exact sequence

$$0 \longrightarrow B \longrightarrow A \xrightarrow{\varphi_A} \prod_{g \in G} A \longrightarrow 0$$

where the homomorphism φ_A is defined by $x \mapsto (gx - x)_{g \in G}$. Since the induced homomorphism $\varphi_A \otimes 1_{R'} : A' \to A'$ is identified with $\varphi_{A'}$ and R' is flat over R, the conclusion follows. □

Remark 13.3.10. We note that the flatness assumption is essential for Proposition 13.3.9. For example, if Y' is a closed subscheme of Y and X' is its inverse image under p, then Y' is not necessarily isomorphic to the quotient X'/G. However, as we shall see, this is true if X is étale over Y.

13.3.2 Decomposition groups and inertia groups

Let G be a finite group and X be a G-scheme. For $x \in X$, the stabilizer subgroup of x is called the **decomposition group** of x, and denoted by $G^Z(x)$. This group has a canonical action on the residue field $\kappa(x)$, and the kernel of the canonical homomorphism $G^Z(x) \to \mathrm{Aut}(\kappa(x))$ is called the **inertia group** of x, and denoted by $G^T(x)$.

Let Ω be a separably closed field and $\xi : \mathrm{Spec}(\Omega) \to X$ be a geometric point of X which image x. Then G acts on the set of Ω-points $X(\Omega)$, and the inertia group of x is identified with the stabilizer of ξ:

$$G^T(x) = \{g \in G : g\xi = \xi\}.$$

Using this interpretation of the inertia subgroup, we can prove the following fact:

Proposition 13.3.11. *For any base change morphism $S' \to S$, let $X' = X \times_S S'$, x' be a point of X', and x be its image in X. Then we have $G^T(x') = G^T(x)$.*

Proof. It suffices to choose Ω large enough so that it is an extension of $\kappa(s')$ (where s' is the image of x' in S'). □

Proposition 13.3.12. *Let X be an admissible G-scheme, $p : X \to Y$ be its quotient, and suppose that Y is locally Noetherian and p is finite. Let H be a subgroup of G and consider $X' = X/H$. Let $x \in X$, x' be its image in X', and $y = p(x)$.*

(a) *If $H \supseteq G^T(x)$, then the homomorphism $p^{\#}_{x'} : \mathscr{O}_{Y,y} \to \mathscr{O}_{X',x'}$ is étale.*

(b) *If $H \supseteq G^Z(x)$, then the homomorphism $p^{\#}_{x'} : \mathscr{O}_{Y,y} \to \mathscr{O}_{X',x'}$ induces an isomorphism on completions.*

Proof. Since Y is locally Noetherian, the local scheme $Y' = \mathrm{Spec}(\widehat{\mathscr{O}_{Y,y}})$ is flat over Y, so by considering the base change $Y' \to Y$ and use Proposition 13.3.9, we may assume that Y is the spectrum of a complete Noetherian local ring and X is the specturm of a finite A-algebra B. If $H \supseteq G^Z(x)$, then by Proposition 4.2.8(b), $\kappa(x')$ is identified with $\kappa(y)$ and $\mathfrak{m}_y$ generates the maximal ideal of $\mathfrak{m}_{x'}$, so the homomorphism $p^{\#}_{x'} : \mathscr{O}_{Y,y} \to \mathscr{O}_{X',x'}$ induces an isomorphism on completions. Assertion (a) then follows from (b) if we make a flat base change of Y so that $G^Z(x) = G^T(x)$ (Proposition 13.3.11). □

Corollary 13.3.13. *Under the conditions of Proposition 13.3.12, suppose that $G^T(x)$ is trivial. Then X is étale over Y at x.*

Corollary 13.3.14. *Suppose that X is connected and the action of G is faithful on X. For X to be étale over Y, it is necessary and sufficient that the inertia groups of X are trivial. In this case, G is identified with the group of Y-automorphisms of X.*

Proof. In view of Corollary 13.3.13, it suffices to suppose that X is étale over Y. Let $x \in X$ and $g \in G^T(x)$, then it follows from Corollary 13.1.26 that g acts trivially on X, and hence equals to the identity since G acts faithfully on X. □

Proposition 13.3.15. *Let S be a locally Noetherian scheme, X be a separated and étale scheme of finite type over S, and G be a finite group of S-automorphisms of X. Then the G-scheme X is admissible and the quotient scheme X/G is étale over S.*

Proof. Since X is separated and étale over S, it is quasi-projective over S ([?], 8.11.2), so the existence of X/G follows from Proposition 13.3.6(iii). To see that X/G is étale over S, we may assume that G acts transitively on the set of connected components of X, and by considering the stabilizer of a connected component, that X is connected. Finally, we can assume that G acts faithfully on X. But then the inertia groups of X are trivial, so it follows from Corollary 13.3.13 that $p : X \to X/G$ is étale. To see that X/G is then étale over S, we consider a point $x \in X$ and let $y = p(x)$, $s = \varphi(x)$ (where $\varphi : X \to S$ is the structural morphism). By taking a flat base change, we may assume that $\kappa(s)$ is separably closed, so by Proposition 13.1.41, the induced homomorphisms $\widehat{\mathscr{O}_{Y,y}} \to \widehat{\mathscr{O}_{X,x}}$ and $\widehat{\mathscr{O}_{S,s}} \to \widehat{\mathscr{O}_{X,x}}$ are bijective, so $\widehat{\mathscr{O}_{S,s}} \to \widehat{\mathscr{O}_{Y,y}}$ is bijective and X/S is étale over S by Proposition 13.1.41. □

Corollary 13.3.16. *If X is finite étale over S, then X/G is finite étale over S.*

Proof. By Corollary 13.3.8, if X is defined by the quasi-coherent $\mathscr{O}_S$-algebra $\mathscr{A}$, then X/G is defined by $\mathscr{A}^G$. By assumption $\mathscr{A}$ is a finite $\mathscr{O}_S$-algebra, so its subalgebra $\mathscr{A}^G$ is also finite over S, since S is locally Noetherian. □

13.3.3 The Galois category $\mathbf{FEt}_S$

Let S be a locally Noetherian and connected scheme. We denote by $\mathbf{FEt}_S$ the category of finite étale coverings of S, with morphisms given by S-morphisms of finite étale coverings of S. Any object X of $\mathbf{FEt}_X$ is often call a finite étale cover of S (even if X is empty). It is clear that there is a canonical functor $p : \mathbf{FEt}_S \to \mathbf{Sch}_S$.

Example 13.3.17. Let k be a separably closed field and $S = \mathrm{Spec}(k)$. In this case $\mathbf{FEt}_S$ is equivalent to the category of finite sets: in fact, a scheme étale over k is the disjoint union of spectra of fields finite separable over k, whose underlying space is finite. Conversely, a finite set with n points is uniquely endowed with the structure of a scheme étale over k, that is, the specturm $\mathrm{Spec}(k^n)$.

Consider now a geometric point $\bar{s} : \mathrm{Spec}(\Omega) \to S$ of S, with values in an algebraically closed field Ω. For an étale covering X of S, we define $F_{\bar{s}}(X)$ to be the underlying set of the fiber $X_{\bar{s}} = X \times_S \bar{s}$. Since $X_{\bar{s}}$ is finite and étale over $\bar{s}$, it is a disjoint union of copies of $\bar{s}$ (Example 13.3.17), so $F_{\bar{s}}(X)$ can also be considered as the set of geometric points $\mathrm{Spec}(\Omega) \to X$ lying over $\bar{s}$, in other words,

$$F_{\bar{s}}(X) = \mathrm{Hom}_S(\bar{s}, X).$$

The assignment $X \mapsto F(X)$ is clearly functorial on X, hence defines a functor $F_{\bar{s}} : \mathbf{FEt}_S \to \mathbf{FSet}$.

Proposition 13.3.18. *The pair $(\mathbf{FEt}_S, F_{\bar{s}})$ form a Galois category.*

Proof. After identifying $\mathbf{FEt}_{\bar{x}}$ with the category of finite sets (Example 13.3.17), we see that our functor $F_{\bar{s}}$ is nothing but the base change functor under the morphism $\bar{s} \to S$. Condition (G1) follows from Proposition 13.1.14, (G2) follows from (Corollary 13.3.16), (G3) from ([?], 5.3.5), and (G4) is trivial by definition. On the other hand, (G5) follows from ([?], 5.3.5) and the begining of Section 13.3.2, and (G6) is proved in ([?], 5.3.7). □

We can therefore apply the results proved in Section 13.2, which permits us in particular to define a pro-object P of $\mathbf{FEt}_S$ representing $F_{\bar{s}}$, which is called the **universal covering** of S at s, and a topological group $\pi = \mathrm{Aut}(F_{\bar{s}}) = \mathrm{Aut}^0(P)$, called the **étale fundamental group** of S at s, denoted by $\pi_1(S, \bar{s})$. The functor F then defines an equivalence from $\mathbf{FEt}_S$ to the category of finite π-sets, where $\pi = \pi_1(S, \bar{s})$. This equivalence allows us to interpret the operations of projective limits and finite inductive limits on coverings (products, fiber products, sums, passing to quotients, etc.) in terms of the analogues operations on π-$\mathbf{FSet}$, i.e. in terms of the obvious operations over finite π-sets. We also note that, since the topological connected components of an étale covering are also étale coverings, an object X in $\mathbf{FEt}_S$ is connected in $\mathbf{FEt}_S$ if and only if it is topological connected; in view of ([?], 5.5.3), this signifies that π acts transitively on $F_{\bar{s}}(X)$. Note that for an object X in $\mathbf{FEt}_S$ to be faithfully flat and quasi-compact over S (it is already flat and quasi-compact over S), it is necessary and sufficient that the structural morphism $X \to S$ is surjective, i.e., is an epimorphism in $\mathbf{FEt}_S$, or that $X \neq \varnothing$.

If $\bar{s}'$ is another geometric point of S (corresponding to an algebraically closed field Ω'), then it defines a fiber functor $F' = F_{\bar{s}'}$ over $\mathbf{FEt}_S$, which is necessarily isomorphic to $F = F_{\bar{s}}$. Therefore, the fundamental group $\pi_1(S, s)$ is independent of $\bar{s}$ up to isomorphisms. If we denote by $\pi_1(S; s, s')$ the set of these isomorphisms (which can also be regareded as the set of isomorphisms $F_{\bar{s}} \to F_{\bar{s}'}$), then we obtain a groupoid whose objects are geometric points of S, and the fundamental group being the automorphism of these objects. The set $\pi_1(S; \bar{s}, \bar{s}')$ can also be considered as the set of path classes from $\bar{s}$ to $\bar{s}'$, which has an evident composition law. Finally, we can define a pro-group Π_1^S of $\mathbf{FEt}_S$, called the **fundamental pro-group of S** or **local system of fundamental groups over S**, by the condition that we have a functorial isomorphism at the geometric point $\bar{s}$ of S:

$$F_{\bar{s}}(\Pi_1^S) = \pi_1(S, \bar{s}).$$

(cf. [?], remarique 5.5.10). In particular, if s is an ordinary point of S, the fiber of Π_1^S at s is a pro-group over $\kappa(s)$, which is the projective limit of finite étale groups over $\kappa(s)$. This pro-group is the called the **fundamental group of S at the point s**, and denoted by $\pi_1(S, s)$. By definition, the points with values in an algebraically closed extension Ω of $\kappa(s)$ are exactly the elements of $\pi_1(S, \bar{s})$, where $\bar{s}$ is the geometric point of S defined by this extension. In particular, (by taking S to be the spectrum of a field) any field k is canonically associated with a pro-group over k, denoted by $\pi_1(k)$, which is the projective limit of finite étale groups over k, and whose points in an algebraically closed extension Ω of k is identified with the elements of the Galois group of $\bar{k}/k$, where $\bar{k}$ is the Galois closure of k in Ω.

Now let $f : S' \to S$ be a morphism of locally Noetherian connected schemes, $\bar{s}'$ be a geometric point of S', and $\bar{s} = f(\bar{s}')$ be its image in S. Then the inverse image functor induces a functor of categories:

$$f^* : \mathbf{FEt}_S \to \mathbf{FEt}_{S'}$$

and we have a functorial isomorphism $F_{\bar{s}} \xrightarrow{\sim} F_{\bar{s}'} \circ f^*$, so that f^* is an exact functor and we have a commutative diagram

$$\begin{tikzcd}
\mathbf{FEt}_S \arrow[r, "\text{base change}"] \arrow[d, "F_{\bar{s}}"] & \mathbf{FEt}_{S'} \arrow[d, "F_{\bar{s}'}"] \\
\pi(S, \bar{s})\text{-}\mathbf{FSet} \arrow[r, "f^*"] & \pi(S', \bar{s}')\text{-}\mathbf{FSet}
\end{tikzcd}$$

We have in particular a canonical homomorphism

$$f_* = \pi_1(f, \bar{s}') : \pi_1(S', \bar{s}') \to \pi_1(S, \bar{s})$$

which allows us to consider the inverse image functor as an operation of restriction of the group action. The properties of the function f^* are therefore expressed in a simple way by the properties of the homomorphism of the associated groups. If in particular S' is an étale covering of S, then the homomorphism f_* is an isomorphism from $\pi_1(S', \bar{s}')$ onto an open subgroup of $\pi_1(S, \bar{s})$ defining the étale covering S' of S (i.e. the stabilizer of $\bar{s}' \in F_{\bar{s}}(S')$ in $\pi_1(S, \bar{s})$).

Proposition 13.3.19. *Let S be the spectrum of a field k, and Ω be an algebraically closed extension of k, defining a geometric point $\bar{s}$ of S with values in Ω. Let k^{sep} be the separable closure of k in Ω. Then there exists a canonical isomorphism $\pi_1(S, \bar{s}) \xrightarrow{\sim} \mathrm{Gal}(k^{\mathrm{sep}}/k)$.*

Proof. Let $\bar{k}$ be the algebraic closure of k in Ω, which corresponds to a geometric point $\bar{s}'$ of S with values in k'. The natural homomorphism $F_{\bar{s}'} \to F_{\bar{s}}$ is evidently an isomorphism, because any scheme X étale over S we have

$$\mathrm{Hom}_S(\bar{s}, Y) = \mathrm{Hom}_S(\bar{s}', Y)$$

In fact, X is then isomorphic to the spectrum of finite separable fields extensions of k in Ω, and any such extension necessairly lies in k^{sep}, hence in $\bar{k}$. On the other hand, the group $G = \mathrm{Gal}(\bar{k}/k) = \mathrm{Gal}(k^{\mathrm{sep}}/k)$ clearly acts on $F_{\bar{s}'}$, so we have a homomorphism

$$G \to \mathrm{Aut}(F_{\bar{s}'}) \xrightarrow{\sim} \mathrm{Aut}(F_{\bar{s}}) = \pi_1(S, \bar{s}).$$

We then obtain a continuous homomorphism $G \to \pi_1(S, \bar{s})$, and it remains to prove that this is an isomorphism. To this end, note that this homomorphism is injective because any element in its kernel is an autumorphism of $\bar{k}/k$ which induces the identity on any finite separated extension of k, hence trivial. It is surjective because if X is a connected étale covering of S, hence defined by a finite separable extension L/k, then G is transitive on the set of k-homorphisms of L into $\bar{k}$. □

Remark 13.3.20. We can in fact write out the explicit isomorphism of Remark 13.3.20. Observe that $\mathrm{Gal}(k^{\mathrm{sep}}/k) = \mathrm{Gal}(\bar{k}/k)$ and $F_{\bar{s}}(X) = \mathrm{Hom}_S(\mathrm{Spec}(\bar{k}), Y)$, so we can consider the map

$$\mathrm{Gal}(\bar{k}/k) \times F_{\bar{s}}(X) \to F_{\bar{s}}(X), \quad (\sigma, \bar{x}) \mapsto \bar{x} \circ {}^a\sigma$$

where ${}^a\sigma$ is the induced automorphism on $\mathrm{Spec}(\bar{k})$. This defines an action of $\mathrm{Gal}(\bar{k}/k)$ on $F_{\bar{s}}$ since

$$\sigma\tau \cdot \bar{x} = \bar{x} \circ {}^a(\sigma\tau) = \bar{x} \circ {}^a\tau \circ {}^a\sigma = \sigma \cdot (\tau \cdot \bar{x}).$$

The action is clearly functorial on X, so we obtain the canonical homomorpism $\mathrm{Gal}(\bar{k}/k) \to \pi_1(S, \bar{s})$.

Proposition 13.3.21. *Let S be a connected, locally Noetherian and integral normal scheme, K be the function field of S, Ω be an algebraically closed extension of K corresponding to a geometric point $\bar{s}'$ of $S' = \mathrm{Spec}(K)$ and a geometric point $\bar{s}$ of S. Then the canonical homomorphism $\pi_1(S', \bar{s}') \to \pi_1(S, \bar{s})$ is surjective. If we identify $\pi_1(S', \bar{s}')$ with the absolute Galois group $\mathrm{Gal}(K^{\mathrm{sep}}/K)$ of K in Ω, then the kernel of this homomorphism corresponds to the Galois group of the subextension M/K, where M is the composite of finite extensions L of k in Ω, which are unramified over S.*

Proof. The first assertion signifies that the inverse image under S' of a connected étale covering X of S is connected, i.e. that X is integral; this follows from ([?] 1.10.1). The kernel of this homomorphism can be interpreted as the automorphisms of K^{sep}/K which induces identity on the set $F_{\bar{s}}(X)$, where we can assume that the étale covering X of S is connected. But this then signifies, in view of the fact that $X_{\bar{s}}$ is étale over $\bar{s}$, that these automotphisms induce identity on the finite subextensions of K^{sep}/K which are unramified over S, whence our assertion. □

Chapter 14

Group schemes

14.1 Algebraic structures

14.1.1 Algebraic structures on the category of presheaves

Given a kind of algebraic structure in the category of sets, we propose to extend it to the category $\mathcal{C}$. Let us first consider an example: the case of groups.

14.1.1.1 Group objects in $\widehat{\mathcal{C}}$ Let $G \in \widehat{\mathcal{C}}$, a **group structure on G** is defined to be the assignment of a group structure on the set $G(S)$ for any $S \in \mathrm{Ob}(\mathcal{C})$, so that for any morphism $f : S' \to S$ in $\mathcal{C}$, the map $G(f) : G(S) \to G(S')$ is a homomorphism of groups. If G and H are groups in $\widehat{\mathcal{C}}$, a **group homomorphism** from G to H is defined to be a morphism $\theta \in \mathrm{Hom}(G, H)$ such that for any object $S \in \mathrm{Ob}(\mathcal{C})$, the map $\theta(S) : G(S) \to H(S)$ is a homomorphism of groups. We denote by $\mathrm{Hom}_{\mathbf{Grp}}(G, H)$ the set of group homomorphisms from G to H, and by $\mathbf{Grp}_{\widehat{\mathcal{C}}}$ the category of groups in $\widehat{\mathcal{C}}$.

Example 14.1.1. Let $E \in \widehat{\mathcal{C}}$, then the object $\mathcal{A}ut(E)$ is endowed with a group structure. The final object e also possesses a unique group structure and is a final object in $\mathbf{Grp}_{\widehat{\mathcal{C}}}$.

Let G be a group in $\widehat{\mathcal{C}}$. For any $S \in \mathrm{Ob}(\mathcal{C})$, let $e_G(S)$ be the unit element in $G(S)$. The family $e_G(S)$ then defines an element $e_G \in \Gamma(G) = \mathrm{Hom}(e, G)$, which is a morphism of groups $e \to G$ and called the **unit section** of G. We also note that giving a group structure over G amounts to given a composition law over G, which is a morphism

$$\pi_G : G \times G \to G$$

such that for any $S \in \mathrm{Ob}(\mathcal{C})$, $\pi_G(S)$ is a group structure on $G(S)$. With the same manner, $f : G \to H$ is a group homomorphism is and only if the following diagram is commutative:

$$\begin{tikzcd} G \times G \arrow[r, "\pi_G"] \arrow[d, "{(f,f)}"'] & G \arrow[d, "f"] \\ H \times H \arrow[r, "\pi_H"] & H \end{tikzcd}$$

A sub-object H of G such that for any $S \in \mathrm{Ob}(\mathcal{C})$, $H(S)$ is a subgroup of $G(S)$ possessing evidently a group structure induced by that of G: that is, such that the monomorphism $H \to G$ is a morphism of groups. The group H endowed with this structure is called a **subgroup** of G.

If G and H are two groups in $\widehat{\mathcal{C}}$, the product $G \times H$ is endowed with a group structure such that for any $S \in \mathrm{Ob}(\mathcal{C})$, $G(S) \times H(S)$ is endowed with the product group structure. The group $G \times H$ endowed with this structure is called the product group of G and H (and this is also the product in the category $\mathbf{Grp}_{\widehat{\mathcal{C}}}$).

If G is a group in $\widehat{\mathcal{C}}$ then for any $S \in \mathrm{Ob}(\mathcal{C})$, G_S is also a group in $\widehat{\mathcal{C}_{/S}}$. If G and H are groups in $\widehat{\mathcal{C}}$, then we can define an object $\mathcal{H}om_{\mathbf{Grp}}(G,H)$ of $\widehat{\mathcal{C}}$ by

$$\mathcal{H}om_{\mathbf{Grp}}(G,H)(S) = \mathrm{Hom}_{\mathbf{Grp}}(G_S, H_S).$$

One should note that $\mathcal{H}om_{\mathbf{Grp}}(G,H)$ is in general not a group, nor a fortiori the object $\mathcal{H}om$ in the category $\mathbf{Grp}_{\widehat{\mathcal{C}}}$. We define similarly objects $\mathcal{I}so_{\mathbf{Grp}}(G,H)$, $\mathcal{E}nd_{\mathbf{Grp}}(G)$ and $\mathcal{A}ut_{\mathbf{Grp}}(G)$.

Definition 14.1.2. Let $G \in \mathrm{Ob}(\mathcal{C})$. A **group structure over G** is defined to be a group structure over $h_G \in \widehat{\mathcal{C}}$. If G and H are groups in $\mathcal{C}$, a group homomorphism from G to H is defined to be an element $f \in \mathrm{Hom}(G,H) \cong \mathrm{Hom}(h_G, h_H)$ which is a group homomorphism from h_G to h_H. We denote by $\mathbf{Grp}_{\mathcal{C}}$ the category of groups in $\mathcal{C}$. Note that there is a Cartesian square in **Cat**:

$$\begin{tikzcd} \mathbf{Grp}_{\mathcal{C}} \arrow[r] \arrow[d] & \mathbf{Grp}_{\widehat{\mathcal{C}}} \arrow[d] \\ \mathcal{C} \arrow[r, "h"] & \widehat{\mathcal{C}} \end{tikzcd}$$

The preceding definitions and constructions carries over to groups in $\mathcal{C}$, provided that the corresponding functors (products, $\mathcal{H}om$ objects, etc.) are representable in $\mathcal{C}$. They also applies to categories of the form $\mathcal{C}_{/S}$, and in this case, we denote by $\mathcal{H}om_{S\text{-}\mathbf{Grp}}$ for $\mathcal{H}om_{\mathbf{Grp}}$, etc.

More generally, if $\mathcal{T}$ is a kind of structure over n base sets defined by finite projective limits (for example, by the commutativity of some diagrams constructed from Cartesian products: monoid, group, action by group, module over a ring, Lie algebra over a ring, etc.), we can define the notion of $\mathcal{T}$ structure over n objects $F_1, \dots, F_n$ over $\widehat{\mathcal{C}}$: such a structure is the assignment of a $\mathcal{T}$ structure over the sets $F_1(S), \dots, F_n(S)$ for each $S \in \mathrm{Ob}(\mathcal{C})$, so that for any morphism $S' \to S$ in $\mathcal{C}$, the family of maps $(F_i(S) \to F_i(S'))$ is a poly-homomorphism for the $\mathcal{T}$ structure. We define in a similar way the morphisms of the $\mathcal{T}$ structure, whence a category of $\mathcal{T}$ objects in $\widehat{\mathcal{C}}$. The fully faithful functor h permits us to define the category of $\mathcal{T}$ objects in $\mathcal{C}$ as a fiber product in **Cat**.

Suppose now that in $\mathcal{C}$ the pullbacks exist, and let $\mathcal{T}$ be an algebraic structure defined by the data of certain morphisms between Cartesian products satisfying some axioms consisting of the commutativity of certain diagrams constructed by the previous arrows. A $\mathcal{T}$ structure on a family of objects of $\mathcal{C}$ will therefore be defined by certain morphisms between Cartesian products satisfying certain commutation conditions. It follows that if $\mathcal{C}$ and $\mathcal{C}'$ are two categories with products and $\lambda : \mathcal{C} \to \mathcal{C}'$ is a functor commuting with products, then for any family of objects (F_i) of $\mathcal{C}$ equipped with a $\mathcal{T}$ structure, the family $(f(F_i))$ of objects of $\mathcal{C}'$ will thereby be endowed with a $\mathcal{T}$ structure. For example, any group in $\mathcal{C}$ will be transformed into a group in $\mathcal{C}'$, any pair of a ring in $\mathcal{C}$ and a module over this ring will be transformed into an analogous pair in $\mathcal{C}'$, etc.

In particular, let $\mathcal{C}$ be a category, then the constant functor $E \mapsto E_S$ commutes with finite projective limits, and hence transforms groups into S-groups (i.e. groups in $\mathcal{C}_{/S}$), rings to S-rings, etc.

Remark 14.1.3. It is worth noting that the previous construction, applied to the category $\widehat{\mathcal{C}}$, restores the notions that have already been defined there. In others words, it amounts to the same thing to give oneself a $\mathcal{T}$ structure over an object of $\widehat{\mathcal{C}}$ when we consider this object as a functor on $\mathcal{C}$, or to give ourselves a $\mathcal{T}$ structure on the representable functor over $\mathcal{C}$ defined by this object. For example, let $G \in \widehat{\mathcal{C}}$; if the functor $F \mapsto \mathrm{Hom}_{\widehat{\mathcal{C}}}(F,G)$ is endowed with a group structure, then so is its restriction to $\mathcal{C}$. Conversely, if G is a group in $\widehat{\mathcal{C}}$, then the multiplication morphism $\pi_G : G \times G \to G$ induces for each $F \in \widehat{\mathcal{C}}$ a group structure over $\mathrm{Hom}_{\widehat{\mathcal{C}}}(F,G)$, which is functorial on F.

14.1.1.2 Group action in $\widehat{\mathcal{C}}$ Let $E \in \widehat{\mathcal{C}}$ and $G \in \mathbf{Grp}_{\widehat{\mathcal{C}}}$. A ***G*-object structure** over E is defined to be an assignment over $E(S)$, for each $S \in \mathrm{Ob}(\mathcal{C})$, a $G(S)$-set structure on $G(S)$, so that for any morphism $S' \to S$ in $\mathcal{C}$, the map $E(S) \to E(S')$ is compatible with the group homomorphism $G(S) \to G(S')$. As usual, this is equivalent to giving a morphism

$$\mu : G \times E \to E$$

which for each S endows $E(S)$ with a $G(S)$-set structure. On the other hand, since $\mathrm{Hom}(G \times E, E) \cong \mathrm{Hom}(G, \mathcal{E}nd(E))$, the morphism μ defines also a morphism $G \to \mathcal{E}nd(E)$ and it is immediate to see that this is a group homomorphism which sends G into $\mathcal{A}ut(E)$. Therefore, giving a G-object structure over E is equivalent to giving a group homomorphism

$$\rho : G \to \mathcal{A}ut(E).$$

In particular, any element $g \in G(S)$ defines an automorphism $\rho(g)$ of the functor E_S, that is, an automorphism of $E \times h_S$ which commutes with the projection $E \times h_S \to h_S$, and in particular an automorphism of $E(S')$ for any morphism $S' \to S$.

Definition 14.1.4. Let G be a group in $\widehat{\mathcal{C}}$ and E be a G-object. We denote by E^G the sub-object of E defined by

$$E^G(S) = \{x \in E(S) : x_{S'} \text{ is invariant under } G(S') \text{ for any morphism } S' \to S\}.$$

Here $x_{S'}$ is the image of x under $E(S) \to E(S')$. It is clear that E^G (called the **invariant sub-object** of E) is the largest sub-object of E on which G acts trivially. If F is a sub-object of E, we denote by $N_G(F)$ and $Z_G(F)$ the subgroups of G defined by

$$\begin{aligned} N_G(F)(S) &= \{g \in G(S) : \rho(g)F_S = F_S\} \\ &= \{g \in G(S) : \rho(S)F(S') = F(S') \text{ for any morphism } S' \to S\}, \\ Z_G(F)(S) &= \{g \in G(S) : \rho(g)|_{F_S} = \mathrm{id}\} \\ &= \{g \in G(S) : \rho(g)|_{F(S')} = \mathrm{id} \text{ for any morphism } S' \to S\}. \end{aligned}$$

In particular, let $x \in \Gamma(E)$, i.e. a collection of elements $x_S \in E(S)$, $S \in \mathrm{Ob}(\mathcal{C})$, such that for any morphism $f : S' \to S$, we have $E(f)(x_s) = x_{S'}$ (if $\mathcal{C}$ admits a final object S_0, then we have $\Gamma(E) = E(S_0)$). Then x can be considered as a sub-functor of E, also denoted by x, and we have $N_G(x) = Z_G(x)$. This common functor is also denoted by $\mathrm{Stab}_G(x)$ and called the **stabilizer** of x. For any $S \in \mathrm{Ob}(\mathcal{C})$, we then have

$$\mathrm{Stab}_G(x)(S) = \{g \in G(S) : \rho(g)x_S = x_S\}.$$

Suppose that fiber products exist in $\mathcal{C}$. If $G = h_G$ (resp. $E = h_E$), where G is a group in $\mathcal{C}$ (resp. $E \in \mathrm{Ob}(\mathcal{C})$), and if $\mathcal{C}$ possesses a final object S_0, so that x is a morphism $S_0 \to E$, then the stabilizer $\mathrm{Stab}_G(x)$ is represented by the fiber product $G \times_E S_0$, where $G \to E$ is the composition of $\mathrm{id}_G \times x : G = G \times S_0 \to G \times E$ and $\mu : G \times E \to E$.

Remark 14.1.5. The formation of E^G, $N_G(F)$ and $Z_G(F)$ commute with base changes, so for any $S \in \mathrm{Ob}(\mathcal{C})$, weh ave

$$(E^G)_S = (E_S)^{G_S}, \quad N_G(F)_S \cong N_{G_S}(F_S), \quad Z_G(F)_S \cong Z_{G_S}(F_S).$$

If G is a group in $\mathcal{C}$ and E is an object of $\widehat{\mathcal{C}}$ (resp. an object of $\mathcal{C}$), a G-object structure over E is defined to be an h_G-object structure over E (resp. h_E). With this definition, the above notations carries to $\mathcal{C}$, if the corresponding functors are representable. For example, if $N_{h_G}(h_F)$ is representable, then it is represented by a unique sub-object of G, which is then a subgroup of G and denoted by $N_G(F)$.

We say that the group G in $\widehat{\mathcal{C}}$ acts on a group H in $\widehat{\mathcal{C}}$ if H is endowed with a G-object structure such that, for any $g \in G(S)$, the automorphism of $H(S)$ defined by g is a group automorphism. This is the same to say that for any $g \in G(S)$, the automorphism $\rho(g)$ of H_S is an automorphism of groups in $\widehat{\mathcal{C}_{/S}}$, or that the morphism $G \to \mathcal{A}ut(H)$ sends G into $\mathcal{A}ut_{\mathbf{Grp}}(H)$.

In the above situation, there exists over $H \times G$ a unique group structure such that, for any $S \in \mathrm{Ob}(\mathcal{C})$, $(H \times G)(S)$ is the semi-direct product of the groups $H(S)$ and $G(S)$ relative to the given action of $G(S)$ on $H(S)$. This group is denoted by $H \rtimes G$ and called the semi-direct product of H by G. By definition, we then have

$$(H \rtimes G)(S) = H(S) \rtimes G(S).$$

Let G be a group in $\widehat{\mathcal{C}}$. For any morphism $S' \to S$ of $\mathcal{C}$ and any $g \in G(S)$, let $\mathrm{Inn}(g)$ be the automorphism of $G(S')$ defined by $\mathrm{Inn}(g)h = ghg^{-1}$. This definition extends to a morphism of groups in $\widehat{\mathcal{C}}$:

$$\mathrm{Inn} : G \to \mathcal{A}ut_{\mathbf{Grp}}(G) \subseteq \mathcal{A}ut(G).$$

The above definitions then apply to H and we have subgroups $N_G(E)$ and $Z_G(E)$ for any subobject E of G.

Definition 14.1.6. We define the **center** of G and denote by $Z(G)$ the subgroup $Z_G(G)$ of G. We say that G is **abelian** if $Z_G(G) = G$ or, equivalently, if $G(S)$ is abelian for any $S \in \mathrm{Ob}(\mathcal{C})$. A subgroup H of G is called **invariant** in G if $N_G(H) = G$, or equivalently, if $H(S)$ is invariant in $G(S)$ for any S. Moreover, we say that H is **cental** in G if $Z_G(H) = G$, or equivalently, if $H(S)$ is cental in $G(S)$ for any S.

Definition 14.1.7. Let $f : G \to G'$ be a group homomorphism. The kernel of f is the subgroup of G defined by

$$(\ker f)(S) = \{x \in G(S) : f(S)x = 1\} = \ker f(S)$$

for any $S \in \mathrm{Ob}(\mathcal{C})$. This is an invariant subgroup of G. Note that if G and G' belongs to $\mathcal{C}$, $\mathcal{C}$ possesses a final object S_0 and fiber products exist in $\mathcal{C}$, then $\ker(f)$ is represented by $S_0 \times_{G'} G$.

Definition 14.1.8. Let $E \in \widehat{\mathcal{C}}$ and G be a group acting on E. We say that the action of G on E is faithful if the kernel of the morphism $G \to \mathcal{A}ut(E)$ is trivial, that is, if for any $S \in \mathrm{Ob}(\mathcal{C})$ and $g \in G(S)$, the condition $g_{S'} \cdot x = x$ for any morphism $S' \to S$ and $x \in E(S')$ implies $g = 1$.

Many definitions and propositions of elementary group theory are easily transported to the setting of groups in $\widehat{\mathcal{C}}$. Let us simply point out the following which will be useful to us:

Proposition 14.1.9. *Let $f : W \to G$ be a group homomorphism and put $H(S) = \ker f(S)$ for $S \in \mathrm{Ob}(\mathcal{C})$. Let $u : G \to W$ be a group homomorphism which is a section of f. Then W is identified with a semi-direct product of H by G for the action of G over H defined by $(g, h) \mapsto \mathrm{Inn}(u(g))h$ for $g \in G(S)$, $h \in H(S)$ and $S \in \mathrm{Ob}(\mathcal{C})$.*

All the definitions and propositions are transported as usual to $\mathcal{C}$. We define in particular the semi-product of two groups H and G in $\mathcal{C}$, with G acting on H, when the Cartesian product $H \times G$ exists in $\mathcal{C}$. We have the following analogue of Proposition 14.1.9:

Proposition 14.1.10. *Let $f : W \to G$ and $i : H \to W$ be group homomorphisms in $\mathcal{C}$ such that for any $S \in \mathrm{Ob}(\mathcal{C})$, $(H(S), i(S))$ is a kernel of $f(S) : W(S) \to G(S)$. Let $u : G \to W$ be a homomorphism of groups in $\mathcal{C}$ which is a section of f. Then W is identified with the semi-direct product of H by G for the action of G over H such that if $S \in \mathrm{Ob}(\mathcal{C})$, $g \in G(S)$ and $h \in H(S)$, we have $\mathrm{Inn}(u(g))i(h) = i(ghg^{-1})$.*

To end this paragraph, we breifly introduce the concept of modules over a ring in $\widehat{\mathcal{C}}$. So let A and M be objects of $\widehat{\mathcal{C}}$, we say that F is a **module over the ring A**, of simply an A-module, if for each $S \in \mathrm{Ob}(\mathcal{C})$ the et $A(S)$ is endowed with a ring structure and $M(S)$ with a module structure over this ring, so that for any morphism $S' \to S$, the map $A(S) \to A(S')$ is a ring homomorphism and $M(S) \to M(S')$ is a bi-homomorphism. If the ring A is fixed, we define as usual morphisms of A-modules M, M', whence the abelian group $\mathrm{Hom}_A(M, M')$, and the category of A-modules, which we denote by $\mathbf{Mod}(A)$.

Proposition 14.1.11. *The category $\mathbf{Mod}(A)$ is endowed with an abelian category structure defined "argument by argument". Moreover, $\mathbf{Mod}(A)$ is an (AB5) category, that is, arbitrary direct sums exist in $\mathbf{Mod}(A)$ and if M is an A-module, N is a submodule, and $(M_i)_{i\in I}$ is a filtrant family of increasing submodules of M, then*

$$\bigcup_{i\in I}(M_i \cap N) = \Big(\bigcup_{i\in I} M_i\Big) \cap N.$$

Proof. In fact, let $f : M \to M'$ be a morphism of A-modules. We define the A-modules $\ker f$ (resp. $\operatorname{im} f$ and $\operatorname{coker} f$) so that for any $S \in \mathrm{Ob}(\mathcal{C})$, $(\ker f)(S) = \ker f(S)$ (resp. $\cdots$). Then $\ker f$ (resp. $\operatorname{coker} f$) is a kernel (resp. cokernel) of f, and we have an isomorphism of A-modules $M/\ker f \cong \operatorname{im} f$. This proves that $\mathbf{Mod}(A)$ is an abelian category.

Arbitrary direct sums exist in $\mathbf{Mod}(A)$ and are defined "argument by argument". Finally, if M is an A-module, N is a submodule, and $(M_i)_{i\in I}$ is a filtrant family of increasing submodules of M, then the inclusion

$$\bigcup_{i\in I}(M_i \cap N) \subseteq \Big(\bigcup_{i\in I} M_i\Big) \cap N$$

is an equality: in fact, if $S \in \mathrm{Ob}(\mathcal{C})$ and $x \in N(S) \cap \bigcup_i M_i(S)$, then there exists $i \in I$ such that $x \in N(S) \cap M_i(S)$. □

Proposition 14.1.12. *If the category $\mathcal{C}$ is $\mathcal{U}$-small, then A is a generator for the category $\mathbf{Mod}(A)$. Concequently, $\mathbf{Mod}(A)$ is a Grothendieck category, hence possesses enough injectives.*

Proof. Let M be an A-module. For any $S \in \mathrm{Ob}(\mathcal{C})$, let $M_0(S)$ be a system of generators of the $A(S)$-module $M(S)$. Since, by hypothesis, $\mathcal{C}$ is small, we can consider the set $I = \coprod_{S\in\mathrm{Ob}(\mathcal{C})} M_0(S)$. We then have an epimorphism $A^{\oplus I} \to M$. This proves that A is a generator for $\mathbf{Mod}(A)$ (cf. [?] 1.9.1). As $\mathbf{Mod}(A)$ satisfies (AB5), it then follows from (cf. [?] 1.10.2) that $\mathbf{Mod}(A)$ has enough injectives. □

Remark 14.1.13. If we consider $\mathbb{Z}$ as a constant functor on $\mathcal{C}$, then the category of $\mathbb{Z}$-modules is isomorphic to the category of abelian groups.

Definition 14.1.14. If M is an A-module, then for any $S \in \mathrm{Ob}(\mathcal{C})$, M_S is an A_S-module, so we can define an abelian group $\mathcal{H}om_A(M, N)$ by

$$\mathcal{H}om_A(M, N)(S) = \mathrm{Hom}_{A_S}(M_S, N_S).$$

We define similarly objects $\mathcal{I}so_A(M, N)$, $\mathcal{E}nd_A(M)$ and $\mathcal{A}ut_A(M)$, which are groups in $\widehat{\mathcal{C}}$ endowed with the structure of composition.

Definition 14.1.15. Let A be a ring in $\widehat{\mathcal{C}}$, M be an A-module and G be a group in $\widehat{\mathcal{C}}$. We denote by $A[G]$ the group algebra in $\widehat{\mathcal{C}}$ of G over A, so that for any $S \in \mathrm{Ob}(\mathcal{C})$, we have

$$(A[G])(S) = A(S)[G(S)].$$

An ***A*[*G*]-module structure** on M is defined to be a G-object structure such that for any $S \in \mathrm{Ob}(\mathcal{C})$ and $g \in G(S)$, the automorphim of $F(S)$ defined by g is an automorphism of $A(S)$-module. Equivalently, this means the group homomorphism

$$\rho : G \to \mathcal{A}ut(M)$$

sends G to the subgroup $\mathcal{A}ut_A(M)$ of $\mathcal{A}ut(M)$. Therefore, given an $A[G]$-module structure on M, we have a group homomorphism

$$\rho : G \to \mathcal{A}ut_A(M).$$

We define similarly the abelian group $\mathrm{Hom}_{A[G]}(M, N)$ for $A[G]$-modules M, N, whence an additive category $\mathbf{Mod}(A[G])$.

The constructions above are immediately specialized in the case where G (or A, or both) are representable by objects of $\mathcal{C}$ which are thereby endowed with corresponding algebraic structures.

14.1.2 Algebraic structures on the category of schemes

We now apply the constructions of the previous paragraph to the category of schemes **Sch**, and more generally to categories $\mathbf{Sch}_{/S}$. We will simplify the notations in the following way: a group in **Sch** will also be called a **group scheme**, and a group scheme in $\mathbf{Sch}_{/S}$ will be called a **group scheme over *S***, or an ***S*-group**, or A-group when S is the spectrum of a ring A.

14.1.2.1 Constant schemes The category of schemes admits direct sums and fiber products, while direct sums commute with base changes. We can then define the constant objects: for any set E, we have a scheme $E_{\mathbb{Z}}$ and for any scheme S, an S-scheme $E_S = (E_{\mathbb{Z}})_S$. Recall that for any S-scheme T, $\mathrm{Hom}_S(T, E_S)$ is the set of locally constant maps from the space T to E.

The functor $E \mapsto E_S$ commutes with finite projective limits. In particular, if G is a group, then G_S is a group scheme over S; if A is a ring, then A_S is a ring scheme over S, etc.

14.1.2.2 Affine *S*-groups Let T be an affine S-scheme, or an S-scheme that is affine over S. Then the $\mathscr{O}_S$-algebra $f_*(\mathscr{O}_T)$ (also denoted by $\mathscr{A}(T)$), where $f : T \to S$ is the structural morphism, is then quasi-coherent. Conversely, any quasi-coherent $\mathscr{O}_S$-algebra $\mathscr{A}$ corresponds to an affine S-scheme $\mathrm{Spec}(\mathscr{A})$, and the constructions $T \mapsto \mathscr{A}(T)$, $\mathscr{A} \mapsto \mathrm{Spec}(\mathscr{A})$ are quasi-inverses of each other. It follows that giving an algebraic structure on an affine S-scheme T is equivalent to giving the corresponding structure on $\mathscr{A}(T)$ in the opposite category to that of quasi-coherent $\mathscr{O}_S$-algebras. In particular, if G is an affine S-group over S, $\mathscr{A}(G)$ is endowed with an augmented $\mathscr{O}_S$-bialgebra structure, that is, we have the following homomorphisms of $\mathscr{O}_S$-algebras

$$\Delta : \mathscr{A}(G) \to \mathscr{A}(G) \otimes_{\mathscr{O}_S} \mathscr{A}(G), \quad \varepsilon : \mathscr{A}(G) \to \mathscr{O}_S, \quad \tau : \mathscr{A}(G) \to \mathscr{A}(G)$$

corresponding to the morphisms of S-schemes

$$\pi : G \times G \to G, \quad e_G : S \to G, \quad i : G \to G.$$

The maps Δ, ε and τ satisfy the following conditions (which express that G is an S-monoid):

(HA1) Δ is coassociative: the following diagram is commutative

$$\begin{tikzcd}
\mathscr{A}(G) \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & \mathscr{A}(G) \otimes_{\mathscr{O}_S} \mathscr{A}(G) \arrow[d, "\mathrm{id} \otimes \Delta"] \\
\mathscr{A}(G) \otimes_{\mathscr{O}_S} \mathscr{A}(G) \arrow[r, "\Delta \otimes \mathrm{id}"] & \mathscr{A}(G) \otimes_{\mathscr{O}_S} \mathscr{A}(G) \otimes_{\mathscr{O}_S} \mathscr{A}(G)
\end{tikzcd}$$

(HA2) Δ is compatible with ε: the following compositions are identities:

$$\mathscr{A}(G) \xrightarrow{\Delta} \mathscr{A}(G) \otimes_{\mathscr{O}_S} \mathscr{A}(G) \xrightarrow{\mathrm{id}\otimes\varepsilon} \mathscr{A}(G) \otimes_{\mathscr{O}_S} \mathscr{O}_S \xrightarrow{\sim} \mathscr{A}(G)$$

$$\mathscr{A}(G) \xrightarrow{\Delta} \mathscr{A}(G) \otimes_{\mathscr{O}_S} \mathscr{A}(G) \xrightarrow{\varepsilon\otimes\mathrm{id}} \mathscr{O}_S \otimes_{\mathscr{O}_S} \mathscr{A}(G) \xrightarrow{\sim} \mathscr{A}(G)$$

Also, in this case $(\mathscr{A}(G), \Delta, \varepsilon, \tau)$ is a Hopf algebra. Let us take advantage of the circumstance to notice once again that it follows from the definition of an S-group structure that in order to give such a structure on a S-scheme G affine over S, it is not necessary to verify anything on $\mathscr{A}(G)$, but simply endow each $G(S')$ for S' above S with a group structure functorial in S'. This remark applies mutatis mutandis to morphisms.

14.1.2.3 The groups $\mathbb{G}_a$ and $\mathbb{G}_m$ We consider the **additive group functor** $\mathbb{G}_a : \mathbf{Sch}^{\mathrm{op}} \to \mathbf{Set}$ defined by the formula

$$\mathbb{G}_a(S) = \Gamma(S, \mathscr{O}_S),$$

endowed with the group structure defined by the additive group structure of the ring $\Gamma(S, \mathscr{O}_S)$. This is represented by the affine scheme, which we denote also by $\mathbb{G}_a$, and which is then a group scheme

$$\mathbb{G}_a = \mathrm{Spec}(\mathbb{Z}[T]).$$

In fact, we have bijections

$$\mathrm{Hom}(S, \mathbb{G}_a) = \mathrm{Hom}_{\mathbf{Alg}}(\mathbb{Z}[T], \Gamma(S, \mathscr{O}_S)) \cong \Gamma(S, \mathscr{O}_S).$$

For any scheme S, we then have an affine S-group over S, which we denote by $\mathbb{G}_{a,S}$, and it corresponds to the $\mathscr{O}_S$-bigebra $\mathscr{O}_S[T]$ with the comultiplication and counit given by

$$\Delta(T) = T \otimes 1 + 1 \otimes T, \quad \varepsilon(T) = 0.$$

Let $\mathbb{G}_m : \mathbf{Sch}^{\mathrm{op}} \to \mathbf{Set}$ be the **multiplication group functor** defined by

$$\mathbb{G}_m(S) = \Gamma(S, \mathscr{O}_S)^\times,$$

where $\Gamma(S, \mathscr{O}_S)^\times$ denotes the multiplication group of invertible elements in the ring $\Gamma(S, \mathscr{O}_S)$, endowed with the canonical group structure. This is represented by an affine group, which is still denoted by $\mathbb{G}_m$:

$$\mathbb{G}_m = \mathrm{Spec}(\mathbb{Z}[T, T^{-1}]) = \mathrm{Spec}(\mathbb{Z}[\mathbb{Z}])$$

where $\mathbb{Z}[\mathbb{Z}]$ is the group algebra of the additive group $\mathbb{Z}$ over the ring $\mathbb{Z}$. In fact,

$$\mathrm{Hom}(S, \mathrm{Spec}(\mathbb{Z}[T, T^{-1}])) = \mathrm{Hom}_{\mathbf{Alg}}(\mathbb{Z}[T, T^{-1}], \Gamma(S, \mathscr{O}_S)) \cong \Gamma(S, \mathscr{O}_S)^\times.$$

For any scheme S, we then have an affine S-group $\mathbb{G}_{m,S}$ over S, which corresponds to the $\mathscr{O}_S$-bigebra $\mathscr{O}_S[\mathbb{Z}]$, with the comultiplication and counit given by

$$\Delta(x) = x \otimes x, \quad \varepsilon(x) = 1 \ \text{ for } \ x \in \mathbb{Z}.$$

We also note that the set $\Gamma(S, \mathscr{O}_S)$ is a ring for each scheme S, so we can endow the functor $\mathbb{G}_a$ with a natural ring structure, which we denote by $\mathbb{O}$. The ring $\mathbb{O}$ is represented by the scheme $\mathrm{Spec}(\mathbb{Z}[T])$, which is also denoted by $\mathbb{O}$, which is then a ring scheme in $\widehat{\mathbf{Sch}}$. For any scheme S, $\mathbb{O}_S = S \times_{\mathrm{Spec}(\mathbb{Z})} \mathrm{Spec}(\mathbb{Z}[T]) = \mathrm{Spec}(\mathscr{O}_S[T])$ is then an affine ring scheme over S. Note that this ring is also denoted by $S[T]$.

For any object F in $\widehat{\mathbf{Sch}}$, the set $\mathbb{O}(F) := \mathrm{Hom}(F, \mathbb{O})$ is then endowed with a ring structure and is functorial on F. In particular, if X is a scheme and we are given morphisms $x : X \to F$ and $f : F \to \mathbb{O}$ (that is, $x \in F(X)$ and $f \in \mathbb{O}(F)$), then $f(x) := f \circ x$ is an element in $\mathbb{O}(X) = \Gamma(X, \mathscr{O}_X)$.

Definition 14.1.16. Let $\pi : M \to X$ be a morphism in $\widehat{\mathbf{Sch}}$, and $\mathbb{O}_X = \mathbb{O} \times X$. We say that M is an $\mathbb{O}_X$**-module** if for each X-scheme X', we are given an $\mathbb{O}(X')$-module structure on $\mathrm{Hom}_X(X', M)$, which is functorial on X'. Equivalently, this amounts to giving oneself an X-abelian group structure $\mu : M \times_X M \to M$ on M and an "external law"

$$\mathbb{O} \times M = \mathbb{O}_X \times_X M \to M, \quad (f, m) \mapsto f \cdot m$$

which is an X-morphism and for any $x \in X(S)$, endows $M(x) = \{m \in M(S) : \pi(m) = x\}$ an $\mathbb{O}(S)$-module structure. In this case, for any $Y \in \widehat{\mathbf{Sch}}_{/X}$ (not necessarily representable), the set $\mathrm{Hom}_X(Y, M) = \Gamma(M_Y/Y)$ is an $\mathbb{O}(Y)$-module, which is functorial on Y.

Example 14.1.17. Let k be a field of characteristic zero and A be a k-algebra. Then the set $\mathrm{Hom}_{\mathbf{Grp}}(\mathbb{G}_a, \mathbb{G}_m)(\mathrm{Spec}(A))$ consists of nilpotent elements of A; more precisely, all group homomorphisms from $\mathbb{G}_{a,\mathrm{Spec}(A)}$ to $\mathbb{G}_{m,\mathrm{Spec}(A)}$ are of the form $x \mapsto e^{ax}$ with $a \in A$ nilpotent. To see that, note that the underlying schemes of $\mathbb{G}_{a,\mathrm{Spec}(A)}$ and $\mathbb{G}_{m,\mathrm{Spec}(A)}$ are $\mathrm{Spec}(A[X])$ and $\mathrm{Spec}(A[Y, Y^{-1}])$, so any group homomorphism is of the form $Y \mapsto \sum_i f_i X^i$ for some $f_i \in A$. The condition that this is a group homomorphism is that

$$\sum_i f_i (X_1 + X_2)^i = \Big(\sum_i f_i X_1^i\Big)\Big(\sum_j f_j X_2^j\Big).$$

Expanding this, we conclude that $f_{i+j}/(i+j)! = f_i/i! f_j/j!$, so every such homomorphism is of the form $f_i = a^i/i!$, and a must be nilpotent since the sum is finite.

Now, we conclude that the functor $\mathcal{H}om_{\mathbf{Grp}}(\mathbb{G}_a, \mathbb{G}_m)$ is not representable. For any positive integer n, let $A_n = k[t]/(t^n)$. Then the morphism $x \mapsto e^{tx}$ is in $\mathrm{Hom}_{\mathbf{Grp}}(\mathbb{G}_a, \mathbb{G}_m)(\mathrm{Spec}(A_n))$ for each n. However, if A is the inverse limit $k[\![t]\!]$, then there is no corresponding morphism in $\mathrm{Hom}_{\mathbf{Grp}}(\mathbb{G}_a, \mathbb{G}_m)(\mathrm{Spec}(A))$, so $\mathcal{H}om_{\mathbf{Grp}}(\mathbb{G}_a, \mathbb{G}_m)$ is not representable.

14.1.2.4 Diagonalizable groups The construction of $\mathbb{G}_m$ can be generalized in the following manner. Let M be an abelian group and $M_{\mathbb{Z}}$ be the constant group scheme associated with M. We then consider the functor $D(M) : \mathbf{Sch}^{\mathrm{op}} \to \mathbf{Set}$ defined by

$$D(M)(S) = \mathrm{Hom}_{\mathbf{Grp}}(M_{\mathbb{Z}}(S), \mathbb{G}_m(S)) \cong \mathrm{Hom}_{S\text{-}\mathbf{Grp}}(M_S, \mathbb{G}_{m,S}) \cong \mathcal{H}om_{\mathbf{Grp}}(M_{\mathbb{Z}}, \mathbb{G}_m)(S).$$

This is an abelian group in $\widehat{\mathbf{Sch}}$ and is represented by the group scheme $\mathrm{Spec}(\mathbb{Z}[M])$, which is still denoted by $D(M)$. In fact, for any scheme S, we have

$$\mathrm{Hom}(S, \mathrm{Spec}(\mathbb{Z}[M])) = \mathrm{Hom}_{\mathbf{Alg}}(\mathbb{Z}[M], \Gamma(S, \mathscr{O}_S)) \cong \mathrm{Hom}_{\mathbf{Grp}}(M, \Gamma(S, \mathscr{O}_S)^\times).$$

For any scheme S, we then obtain an affine group scheme over S:

$$D_S(M) = D(M)_S = \mathcal{H}om_{\mathbf{Grp}}(M_{\mathbb{Z}}, \mathbb{G}_m)_S = \mathcal{H}om_{\mathbf{Grp}}(M_S, \mathbb{G}_{m,S}).$$

This is associated with the $\mathscr{O}_S$-bigebra $\mathscr{O}_S[M]$, whose comultiplication and counit are defined by

$$\Delta(x) = x \otimes x, \quad \varepsilon(x) = 1 \ \text{ for } \ x \in M.$$

If $f : M \to N$ is a homomorphism of abelian groups, we then have obtain a morphism of S-groups

$$D_S(f) : D_S(N) \to D_S(M),$$

whence a functor $D_S : M \mapsto D_S(M)$ from the category of abelian groups to the category of affine groups over S, which can also be described as the composition of the functor $M \mapsto M_S$ with the functor $M_S \mapsto \mathcal{H}om_{\mathbf{Grp}}(M_S, \mathbb{G}_{m,S})$. This functor clearly commutes with base changes. An S-group isomorphic to a group of them form $D_S(M)$ is called **diagonalizable**. We note that the elements of M can be interpreted as some characters of $D_S(M)$, that is, certain elements of $\mathrm{Hom}_{\mathbf{Grp}}(D_S(M), \mathbb{G}_{m,S})$.

Example 14.1.18. It is clear that we have $D(\mathbb{Z}) = \mathbb{G}_m$ and $D(\mathbb{Z}^n) = (\mathbb{G}_m)^n$. We now consider the group scheme

$$\boldsymbol{\mu}_n = D(\mathbb{Z}/n\mathbb{Z})$$

which is called the **group of n-th roots of unity**. In fact, we have

$$\boldsymbol{\mu}_n(S) = \mathrm{Hom}_{\mathbf{Grp}}(\mathbb{Z}/n\mathbb{Z}, \Gamma(S, \mathscr{O}_S)^\times) = \{f \in \Gamma(S, \mathscr{O}_S) : f^n = 1\}.$$

The S-group $\boldsymbol{\mu}_{n,S}$ corresponds to the $\mathscr{O}_S$-algebra $\mathscr{O}_S[T]/(T^n - 1)$. Suppose in particular that S is the spectrum of a field k of characteristic p. Then by putting $T - 1 = s$, we have

$$k[T]/(T^p - 1) = k[s]/(s^p),$$

which shows that the underlying space of $\boldsymbol{\mu}_{p,S}$ is reduced to a single point, and the local ring of this point is the Artinian k-algebra $k[s]/(s^p)$. By the same ideas, we see that the S-schemes $\mathbb{G}_{a,S}$, $\mathbb{G}_{m,S}$, $\mathbb{O}_S$ are smooth on S, that $D_S(M)$ is flat on S and that it is formally smooth (resp. smooth) on S if and only if the residual characteristic of S does not divide the torsion of M (resp. and if moreover M is finite type).

Example 14.1.19. The above procedure applies to "classical groups" (linear groups GL_n, symplectic groups Sp_n, orthogonal groups O_n, etc.). We define for example GL_n as representing the functor such that

$$\mathrm{GL}_n(S) = \mathrm{GL}(n, \Gamma(S, \mathscr{O}_S)) = \mathrm{Aut}_{\mathscr{O}_S}(\mathscr{O}_S^n).$$

We can construct it for example as the open set of $\mathrm{Spec}(\mathbb{Z}[T_{ij}])$ ($1 \leq i, j \leq n$) defined by the function $f = \det(T_{ij})$, which is $\mathrm{Spec}(\mathbb{Z}[T_{ij}, f^{-1}])$.

14.1.2.5 Module functors in the category of schemes We now associate with any $\mathscr{O}_S$-module over the schema S, an $\mathbb{O}_S$-module (where $\mathbb{O}_S$ denotes the ring functor introduced in 14.1.2.3). This can be done in two different ways, as we shall now define.

Definition 14.1.20. Let S be a scheme. For any $\mathscr{O}_S$-module $\mathscr{F}$, we denote by $\Gamma_{\mathscr{F}}$ and $\check{\Gamma}_{\mathscr{F}}$ the contravariant functors over $\mathbf{Sch}_{/S}$ defined by

$$\Gamma_{\mathscr{F}}(S') = \Gamma(S', \mathscr{F} \otimes_{\mathscr{O}_S} \mathscr{O}_{S'}), \quad \check{\Gamma}_{\mathscr{F}}(S') = \mathrm{Hom}_{\mathscr{O}_{S'}}(\mathscr{F} \otimes_{\mathscr{O}_S} \mathscr{O}_{S'}, \mathscr{O}_{S'}).$$

Then $\Gamma_{\mathscr{F}}$ and $\check{\Gamma}_{\mathscr{F}}$ are endowed with natural structures of $\mathbb{O}_S$-modules (we note that $\mathbb{O}_S(S') = \Gamma(S', \mathscr{O}_{S'}) = \Gamma_{\mathscr{O}_S}(S')$), so that we obtain functors Γ and $\check{\Gamma}$ from the category of $\mathscr{O}_S$-modules to that of $\mathbb{O}_S$-modules, Γ being convariant and $\check{\Gamma}$ being contracovariant.

We often restrict ourselves to the category of quasi-coherent $\mathscr{O}_S$-modules, so that Γ and $\check{\Gamma}$ are considered as functors from $\mathbf{Qcoh}(\mathscr{O}_S)$ to the category of $\mathbb{O}_S$-modules:

$$\Gamma : \mathbf{Qcoh}(\mathscr{O}_S) \to \mathbf{Mod}(\mathbb{O}_S), \quad \check{\Gamma} : \mathbf{Qcoh}(\mathscr{O}_S)^{\mathrm{op}} \to \mathbf{Mod}(\mathbb{O}_S).$$

The reader should however note that most of the propositions in this paragraph do not rely on the quasi-coherence hypothesis.

Proposition 14.1.21. *Let S be a scheme.*

(a) *The functors Γ and $\check{\Gamma}$ commute with base changes: if $S' \to S$ is a morphism and $\mathscr{F}$ is a quasi-coherent $\mathscr{O}_S$-module, then $\Gamma_{\mathscr{F} \otimes \mathscr{O}_{S'}} \cong (\Gamma_{\mathscr{F}})_{S'}$ and $\check{\Gamma}_{\mathscr{F} \otimes \mathscr{O}_{S'}} \cong (\check{\Gamma}_{\mathscr{F}})_{S'}$.*

(b) *The functors Γ and $\check{\Gamma}$ are fully faithful: the canonical maps*

$$\mathrm{Hom}_{\mathscr{O}_S}(\mathscr{F}, \mathscr{F}') \to \mathrm{Hom}_{\mathbb{O}_S}(\Gamma_{\mathscr{F}}, \Gamma_{\mathscr{F}'}),$$
$$\mathrm{Hom}_{\mathscr{O}_S}(\mathscr{F}, \mathscr{F}') \to \mathrm{Hom}_{\mathbb{O}_S}(\check{\Gamma}_{\mathscr{F}'}, \check{\Gamma}_{\mathscr{F}})$$

are bijective.

(c) *The functors* Γ *and* $\check{\Gamma}$ *are additive: we have* $\Gamma_{\mathscr{F}\oplus\mathscr{F}'} \cong \Gamma_{\mathscr{F}} \times_S \Gamma_{\mathscr{F}'}$ *and* $\check{\Gamma}_{\mathscr{F}\oplus\mathscr{F}'} \cong \check{\Gamma}_{\mathscr{F}} \times_S \check{\Gamma}_{\mathscr{F}'}$.

Proof. Assertions (a) and (c) are clear from the definitions. As for (b), we note that by taking S' to be the open subsets of S, we can construct a homomorphism $u : \mathscr{F} \to \mathscr{F}'$ from an $\mathbb{O}_S$-homomorphism $f : \Gamma_{\mathscr{F}} \to \Gamma_{\mathscr{F}'}$, and it is immediate to verify that this gives an inverse of the canonical map $\mathrm{Hom}_{\mathscr{O}_S}(\mathscr{F}, \mathscr{F}') \to \mathrm{Hom}_{\mathbb{O}_S}(\Gamma_{\mathscr{F}}, \Gamma_{\mathscr{F}'})$. A similar argument shows that the canonical map $\mathrm{Hom}_{\mathscr{O}_S}(\mathscr{F}, \mathscr{F}') \to \mathrm{Hom}_{\mathbb{O}_S}(\check{\Gamma}_{\mathscr{F}'}, \check{\Gamma}_{\mathscr{F}})$ is also bijective. □

We recall that if F, F' are $\mathbb{O}_S$-modules, then $\mathcal{H}om_{\mathbb{O}_S}(F, F')$ denote that S-functor which associates any morphism $S' \to S$ with $\mathrm{Hom}_{\mathbb{O}_{S'}}(F_{S'}, F'_{S'})$.

Proposition 14.1.22. *We have the following canonical morphisms in* $\mathbf{Mod}(\mathbb{O}_S)$*:*

$$\begin{tikzcd} \mathcal{H}om_{\mathbb{O}_S}(\Gamma_{\mathscr{F}}, \Gamma_{\mathscr{F}'}) \arrow[rr, "\sim"] & & \mathcal{H}om_{\mathbb{O}_S}(\check{\Gamma}_{\mathscr{F}'}, \check{\Gamma}_{\mathscr{F}}) \\ & \Gamma_{\mathcal{H}om_{\mathscr{O}_S}(\mathscr{F}, \mathscr{F}')} \arrow[ul] \arrow[ur] & \end{tikzcd}$$

Proof. For each S-scheme S', we have a canonical homomorphism

$$\Gamma_{\mathcal{H}om_{\mathscr{O}_S}(\mathscr{F}, \mathscr{F}')}(S') = \Gamma(\mathcal{H}om_{\mathscr{O}_S}(\mathscr{F}, \mathscr{F}') \otimes \mathscr{O}_{S'}) \to \mathrm{Hom}_{\mathscr{O}_{S'}}(\mathscr{F} \otimes \mathscr{O}_{S'}, \mathscr{F}' \otimes \mathscr{O}_{S'}).$$

The proposition then follows from Proposition 14.1.21 (a) and (b). □

Remark 14.1.23. Let $\mathscr{F}$ be a quasi-coherent $\mathscr{O}_S$-module. Recall that the S-functor $\check{\Gamma}_{\mathscr{F}}$ is represented by an affine S-scheme which is denoted by $\mathbb{V}(\mathscr{F})$ and called the vector bundle defined by $\mathscr{F}$:

$$\mathbb{V}(\mathscr{F}) = \mathrm{Spec}(\boldsymbol{S}(\mathscr{F})),$$

where $\boldsymbol{S}(\mathscr{F})$ denotes the symmetric algebra over $\mathscr{F}$. On the other hand, the article ([**?**]) shows that if S is Noetherian and $\mathscr{F}$ is a coherent $\mathscr{O}_S$-module, then $\Gamma_{\mathscr{F}}$ is representable if and only if $\mathscr{F}$ is locally free, and in this case we have an isomorphism $\Gamma_{\mathscr{F}} \cong \check{\Gamma}_{\mathscr{F}}$.

Proposition 14.1.24. *Let* $\mathscr{F}$ *and* $\mathscr{F}'$ *be quasi-coherent* $\mathscr{O}_S$*-modules and* $\mathscr{A}$ *be a quasi-coherent* $\mathscr{O}_S$*-algebra. Then we have a functorial isomorphism*

$$\mathrm{Hom}_S(\mathrm{Spec}(\mathscr{A}), \mathcal{H}om_{\mathbb{O}_S}(\Gamma_{\mathscr{F}'}, \Gamma_{\mathscr{F}})) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{O}_S}(\mathscr{F}', \mathscr{F} \otimes_{\mathscr{O}_S} \mathscr{A}).$$

Proof. If we put $X = \mathrm{Spec}(\mathscr{A})$, then the LHS is canonically isomorphic to $\mathcal{H}om_{\mathbb{O}_S}(\Gamma_{\mathscr{F}'}, \Gamma_{\mathscr{F}})(X)$, which by Proposition 14.1.21 is given by

$$\begin{aligned} \mathcal{H}om_{\mathbb{O}_S}(\Gamma_{\mathscr{F}'}, \Gamma_{\mathscr{F}})(X) &\cong \mathrm{Hom}_{\mathbb{O}_X}(\Gamma_{\mathscr{F}'\otimes\mathscr{O}_X}, \Gamma_{\mathscr{F}\mathscr{O}_X}) \cong \mathrm{Hom}_{\mathscr{O}_X}(\mathscr{F}' \otimes \mathscr{O}_X, \mathscr{F} \otimes \mathscr{O}_X) \\ &\cong \mathrm{Hom}_{\mathscr{O}_S}(\mathscr{F}', \varphi_*(\varphi^*(\mathscr{F}))) \end{aligned}$$

where $\varphi : X \to S$ is the structural morphism. On the other hand, by Corollary 11.1.24 we have $\varphi_*(\varphi^*(\mathscr{F})) \cong \mathscr{F} \otimes \mathscr{A}$, so the assertion follows. □

Corollary 14.1.25. *We have a canonical isomorphism* $\Gamma_{\mathscr{F}\otimes\mathscr{A}} \cong \mathcal{H}om_S(\mathrm{Spec}(\mathscr{A}), \Gamma_{\mathscr{F}})$.

Proof. Let $f : S' \to S$ be an S-scheme and $X' = X \times_S S'$, we then have a Cartesian diagram

$$\begin{tikzcd} X' \arrow[r, "\varphi'"] \arrow[d, "f'"'] & S' \arrow[d, "f"] \\ X \arrow[r, "\varphi"] & S \end{tikzcd}$$

By Proposition 11.1.29 and Corollary 11.1.30, X' is affine over S' and $\varphi'_*(\mathscr{O}_{X'}) = f^*(\mathscr{A})$, so

$$\mathcal{H}om_S(\operatorname{Spec}(\mathscr{A}), \Gamma_{\mathscr{F}})(S') = \operatorname{Hom}_{S'}(\operatorname{Spec}(f^*(\mathscr{A})), \Gamma_{f^*(\mathscr{F})})$$

and by Proposition 14.1.24 applied to $f^*(\mathscr{F})$, $\mathscr{F}' = \mathscr{O}_{S'}$ and $f^*(\mathscr{A})$, this is equal to

$$\Gamma(S', f^*(\mathscr{F}) \otimes f^*(\mathscr{A})) = \Gamma(S', f^*(\mathscr{F} \otimes \mathscr{A})) = \Gamma_{\mathscr{F}\otimes\mathscr{A}}(S'). \qquad \square$$

Proposition 14.1.26. *If $\mathscr{F}$ and $\mathscr{F}'$ are locally free of finite type, then the morphisms in Proposition 14.1.22 are isomorphisms.*

Proof. In fact, for any morphism $S' \to S$, we then have

$$\Gamma_{\mathcal{H}om_{\mathscr{O}_S}(\mathscr{F},\mathscr{F}')}(S') = \Gamma(S', \mathcal{H}om_{\mathscr{O}_S}(\mathscr{F}, \mathscr{F}') \otimes \mathscr{O}_{S'}) = \operatorname{Hom}_{\mathscr{O}_S}(\mathscr{F}, \mathscr{F}').$$

But this is also isomorphic to $\mathcal{H}om_{\mathbb{O}_S}(\Gamma_{\mathscr{F}}, \Gamma_{\mathscr{F}'})(S')$ and to $\mathcal{H}om_{\mathbb{O}_S}(\Gamma_{\mathscr{F}}, \Gamma_{\mathscr{F}'})(S')$, in view of Proposition 14.1.21 (b). $\square$

Corollary 14.1.27. *Let $\mathscr{F}$ be a locally free $\mathscr{O}_S$-module of finite type and put $\check{\mathscr{F}} = \mathcal{H}om_{\mathscr{O}_S}(\mathscr{F}, \mathscr{O}_S)$. Then we have canonical isomorphisms*

$$\Gamma_{\check{\mathscr{F}}} \cong \mathcal{H}om_{\mathbb{O}_S}(\Gamma_{\mathscr{F}}, \mathbb{O}_S) \cong \check{\Gamma}_{\mathscr{F}}, \quad \check{\Gamma}_{\check{\mathscr{F}}} \cong \mathcal{H}om_{\mathbb{O}_S}(\check{\Gamma}_{\mathscr{F}}, \mathbb{O}_S) \cong \Gamma_{\mathscr{F}},$$

Proof. This follows from Proposition 14.1.26 by taking $\mathscr{F}' = \mathscr{O}_S$ and note that $\Gamma_{\mathscr{O}_S} = \mathbb{O}_S$. $\square$

Proposition 14.1.28. *If $u : \mathscr{F} \to \mathscr{F}'$ is a morphism of locally free $\mathscr{O}_S$-modules of finite rank, then for $\Gamma_u : \Gamma_{\mathscr{F}} \to \Gamma_{\mathscr{F}'}$ to be a monomorphism, it is necessary and sufficient that f identifies $\mathscr{F}$ locally as a direct factor of $\mathscr{F}'$.*

Proof. One direction follows essentially from **??**. Conversely, if $\mathscr{F}$ is a direct factor of $\mathscr{F}'$, then for any $f : S' \to S$, $f^*(\mathscr{F})$ is a submodule of $f^*(\mathscr{F}')$, so $\Gamma_{\mathscr{F}}(S') = \Gamma(S', f^*(\mathscr{F}))$ is a submodule of $\Gamma_{\mathscr{F}'}(S') = \Gamma(S', f^*(\mathscr{F}'))$. $\square$

14.1.2.6 The category of $\mathscr{O}_S[G]$-modules Let G be an S-group and $\mathscr{F}$ be an $\mathscr{O}_S$-module. Then an **$\mathscr{O}_S[G]$-module structure** on $\mathscr{F}$ is defined to be an $\mathbb{O}_S[h_G]$-module structure on $\Gamma_{\mathscr{F}}$. A morphism of $\mathscr{O}_S[G]$-modules is by definition a morphism of the associated $\mathbb{O}_S[h_G]$-modules. We thus obtain a category $\mathbf{Mod}(\mathscr{O}_S[G])$ of $\mathscr{O}_S[G]$-modules and the full subcategory $\mathbf{Qcoh}(\mathscr{O}_S[G])$ formed by quasi-coherent $\mathscr{O}_S$-modules. By definition, giving an $\mathscr{O}_S[G]$-module structure on $\mathscr{F}$ is equivalent to giving a morphism of groups

$$\rho : h_G \to \mathcal{A}ut_{\mathbb{O}_S}(\Gamma_{\mathscr{F}}).$$

Remark 14.1.29. Since by Proposition 14.1.21 we have an anti-isomorphism

$$\mathcal{A}ut_{\mathbb{O}_S}(\Gamma_{\mathscr{F}}) \cong \mathcal{A}ut_{\mathbb{O}_S}(\check{\Gamma}_{\mathscr{F}}),$$

we see that an $\mathbb{O}_S[h_G]$-module structure on $\Gamma_{\mathscr{F}}$ is equivalent to an $\mathbb{O}_S[h_G]$-module structure on $\check{\Gamma}_{\mathscr{F}}$, and these two structures are connected by the operation $\rho(g) \mapsto \rho^*(g^{-1})$, where ρ^* denotes the image of $\rho : h_G \to \mathcal{A}ut_{\mathbb{O}_S}(\Gamma_{\mathscr{F}})$ under the above isomorphism.

Remark 14.1.30. The categories we have just constructed can also be defined by the following Cartesian squares:

$$\begin{tikzcd}
\mathbf{Qcoh}(\mathscr{O}_S[G]) \arrow[r, hook] \arrow[d] & \mathbf{Mod}(\mathscr{O}_S[G]) \arrow[r] \arrow[d] & \mathbf{Mod}(\mathbb{O}_S[h_G]) \arrow[d, "\text{forget}"] \\
\mathbf{Qcoh}(\mathscr{O}_S) \arrow[r, hook] & \mathbf{Mod}(\mathscr{O}_S) \arrow[r, "\Gamma"] & \mathbf{Mod}(\mathbb{O}_S)
\end{tikzcd}$$

The categories $\mathbf{Mod}(\mathscr{O}_S)$ and $\mathbf{Mod}(\mathbb{O}_S)$ are abelian, but one should be careful that in general the functor Γ is not exact, neither left nor right.

Remark 14.1.31. Let $\mathscr{F}$ be an $\mathscr{O}_S[G]$-module. The **subsheaf of invariants** $\mathscr{F}^G$ is defined as follows: for any open subset U of S,

$$\mathscr{F}^G(U) = \Gamma^G_{\mathscr{F}}(U) = \{x \in \mathscr{F}(U) : g \cdot x_{S'} = x_{S'} \text{ for any morphism } f : S' \to U \text{ and } g \in G(S')\}$$

where $x_{S'}$ denotes the image of x in $\Gamma(S', f^*(\mathscr{F})) = \Gamma(U, f_*(f^*(\mathscr{F})))$.

Be careful that the natural morphism $\Gamma_{\mathscr{F}^G} \to \Gamma^G_{\mathscr{F}}$ is not an isomorphism in general. For example, if $S = \mathrm{Spec}(\mathbb{Z})$ and G is the constant group $\mathbb{Z}/2\mathbb{Z} = \{1, \tau\}$ acting on $\mathscr{F} = \mathscr{O}_S$ via $\tau \cdot 1 = -1$, then we have $\mathscr{F}^G = 0$ since the ring $\Gamma(U, \mathscr{F})$ has characteristic zero for any standard open U of S. However, it is clear that $\Gamma^G_{\mathscr{F}}(\mathrm{Spec}(R)) = R$ for any $\mathbb{F}_2$-algebra R.

From now on, we restrict ourselves to the case where the group scheme G is affine over S. Then, in view of Proposition 14.1.24, giving a morphism of S-functors

$$\rho : h_G \to \mathcal{A}ut_{\mathbb{O}_S}(\Gamma_{\mathscr{F}})$$

is equivalent to giving a morphism of $\mathscr{O}_S$-modules

$$\mu : \mathscr{F} \to \mathscr{F} \otimes_{\mathscr{O}_S} \mathscr{A}(G).$$

The condition that ρ is a group homomorphism is then translated into the folllowing conditions on μ:

(CM1) the following diagram is commutative:

$$\begin{tikzcd}
\mathscr{F} \arrow[r, "\mu"] \arrow[d, "\mu"'] & \mathscr{F} \otimes_{\mathscr{O}_S} \mathscr{A}(G) \arrow[d, "\mathrm{id} \otimes \Delta"] \\
\mathscr{F} \otimes_{\mathscr{O}_S} \mathscr{A}(G) \arrow[r, "\mu \otimes \mathrm{id}"] & \mathscr{F} \otimes_{\mathscr{O}_S} \mathscr{A}(G) \otimes_{\mathscr{O}_S} \mathscr{A}(G)
\end{tikzcd}$$

(CM2) the following composition is the identity:

$$\mathscr{F} \xrightarrow{\mu} \mathscr{F} \otimes_{\mathscr{O}_S} \mathscr{A}(G) \xrightarrow{\mathrm{id} \otimes \varepsilon} \mathscr{F} \otimes \mathscr{O}_S \xrightarrow{\sim} \mathscr{F}$$

These two axioms then endow a *comodule structure* on $\mathscr{F}$ over the bigebra $\mathscr{A}(G)$.

Put $\mathscr{A} = \mathscr{A}(G)$. If $\mathscr{F}$ and $\mathscr{F}'$ are $\mathscr{A}$-comodules, a morphism $f : \mathscr{F} \to \mathscr{F}'$ of comodules is then defined to be a morphism of $\mathscr{O}_S$-modules such that the following diagram is commutative:

$$\begin{tikzcd}
\mathscr{F} \arrow[r, "f"] \arrow[d, "\mu_{\mathscr{F}}"'] & \mathscr{F}' \arrow[d, "\mu_{\mathscr{F}'}"] \\
\mathscr{F} \otimes \mathscr{A} \arrow[r, "f \otimes \mathrm{id}"] & \mathscr{F}' \otimes \mathscr{A}
\end{tikzcd}$$

We thus obtain a category $\mathbf{CoMod}(\mathscr{A})$ of comodules over $\mathscr{A}$, and we denote by $\mathbf{CoQcoh}(\mathscr{A})$ the full subcategory formed by quasi-coherent $\mathscr{O}_S$-modules. From the above remarks, it is also clear that we have the following:

Proposition 14.1.32. *Let G be an affine S-group. Then we have equivalences of categories:*

$$\mathbf{Mod}(\mathscr{O}_S[G]) \cong \mathbf{CoMod}(\mathscr{A}(G)), \quad \mathbf{Qcoh}(\mathscr{O}_S[G]) \cong \mathbf{CoQcoh}(\mathscr{A}(G)).$$

If moreover $S = \mathrm{Spec}(A)$ is affine and we put $A[G] = \Gamma(S, \mathscr{A}(G))$, then we have an equivalence of categories

$$\mathbf{CoQcoh}(\mathscr{A}(G)) \cong \mathbf{CoMod}(A[G]).$$

Proposition 14.1.33. *Suppose that G is affine and flat over S. Then the category* $\mathbf{Mod}(\mathscr{O}_S[G])$ *(resp.* $\mathbf{Qcoh}(\mathscr{O}_S[G])$*), being equivalent to the category of $\mathscr{A}(G)$-comodules (resp. quasi-coherent over $\mathscr{O}_S$), is abelian.*

Proof. Suppose that $\mathscr{A} = \mathscr{A}(G)$ is a flat $\mathscr{O}_S$-module. Let $\mathscr{E}$ be an $\mathscr{A}$-comodule and $\mathscr{F}$ be a sub-$\mathscr{O}_S$-module of $\mathscr{E}$. As $\mathscr{A}$ is flat over $\mathscr{O}_S$, we can identify $\mathscr{F} \otimes \mathscr{A}$ (resp. $\mathscr{F} \otimes \mathscr{A} \otimes \mathscr{A}$) as a sub-$\mathscr{O}_S$-module of $\mathscr{E}$ (resp. $\mathscr{E} \otimes \mathscr{A} \otimes \mathscr{A}$). Assume that $\mu_{\mathscr{E}}$ sends $\mathscr{F}$ into $\mathscr{F} \otimes \mathscr{A}$, then the restriction $\mu_{\mathscr{F}} : \mathscr{F} \to \mathscr{F} \otimes \mathscr{A}$ induces a comodule structure on $\mathscr{F}$, and we say that $\mathscr{F}$ is a sub-comodule of $\mathscr{E}$. By passing to quotient, $\mu_{\mathscr{E}}$ then defies a morphism of $\mathscr{O}_S$-modules $\mathscr{E}/\mathscr{F} \to \mathscr{E}/\mathscr{F} \otimes \mathscr{A}$, which endows $\mathscr{E}/\mathscr{F}$ with an $\mathscr{A}$-comodule structure.

Now if $f : \mathscr{E} \to \mathscr{E}'$ is a morphism of $\mathscr{A}$-comodules, then $\ker f$ (resp. $\operatorname{im} f$) is a sub-$\mathscr{A}$-comodule of $\mathscr{E}$ (resp. $\mathscr{E}'$), and f induces an isomorphism $\mathscr{E}/\ker f \xrightarrow{\sim} \operatorname{im} f$ of $\mathscr{A}$-comodules. Moreover, if $\mathscr{E}$ and $\mathscr{E}'$ are quasi-coherent $\mathscr{O}_S$-modules, then so are $\ker f$ and $\operatorname{im} f$. Therefore, we conclude that $\mathbf{CoMod}(\mathscr{A})$ and $\mathbf{CoQcoh}(\mathscr{A})$ are abelian categories. □

We now suppose further that G is a diagonalizable group, which means $\mathscr{A}(G)$ is the algebra of an abelian group M over the ring $\mathscr{O}_S$. If $\mathscr{F}$ is an $\mathscr{O}_S$-module, we then have

$$\mathscr{F} \otimes \mathscr{A}(G) = \coprod_{m \in M} \mathscr{F} \otimes m\mathscr{O}_S,$$

so giving a morphism $\mu : \mathscr{F} \to \mathscr{F} \otimes \mathscr{A}(G)$ is equivalent to giving a family of endomorphisms $(\mu_m)_{m \in M}$ of $\mathscr{F}$ such that for any section x of $\mathscr{F}$ over an open subset S, $(\mu_m(x))$ is a section of the direct sum $\coprod_{m \in M} \mathscr{F}$ (this means that over any sufficiently small open subset, there are only a finite number of restrictions of the $\mu_m(x)$ which are non-zero). For a morphism μ defined by

$$\mu(x) = \sum_{m \in M} \mu_m(x) \otimes m$$

to satisfy (CM1) and (CM2), it is necessary and sufficient that we have

$$\mu_m \circ \mu_n = \delta_{mn}\mu_m, \quad \sum_{m \in M} \mu_m = \mathrm{id}_{\mathscr{F}}$$

which signify that the μ_m are orthogonal projections adding up to the identity. We have therefore proved the following result:

Proposition 14.1.34. *If $G = D_S(M)$ is a diagonalizable group over S, then the category of $\mathscr{O}_S[G]$-modules (resp. quasi-coherent $\mathscr{O}_S[G]$-modules) is equivalent to the category of graded $\mathscr{O}_S$-modules (resp. quasi-coherent $\mathscr{O}_S[G]$-modules) of type M.*

Corollary 14.1.35. *The functor $\mathscr{A} \mapsto \operatorname{Spec}(\mathscr{A})$ induces an equivalence from the category of graded quasi-coherent $\mathscr{O}_S$-algebras of type M to the opposite category of that of affine S-schemes acted by the group $G = D_S(M)$.*

Proof. If X is an affine scheme over S acted by the affine S-group $D_S(M)$, then $\mathscr{A}(S)$ is a quasi-coherent $\mathscr{O}_S$-algebra which is acted by G, whence a graded $\mathscr{O}_S$-algebra of type M. The converse of this is immediate. □

Proposition 14.1.36. *Let G be a diagonalizable group over S. If*

$$0 \longrightarrow \mathscr{F}_1 \longrightarrow \mathscr{F}_2 \longrightarrow \mathscr{F}_3 \longrightarrow 0$$

is an exact sequence of quasi-coherent $\mathscr{O}_S[G]$-modules which split as a sequence of $\mathscr{O}_S$-modules, then it splits as a sequence of $\mathscr{O}_S[G]$-modules..

Proof. If $G = D_S(M)$, then each $\mathscr{F}_i$ is graded by the $(\mathscr{F}_i)_m$ and for each $m \in M$ the sequence

$$0 \longrightarrow (\mathscr{F}_1)_m \longrightarrow (\mathscr{F}_2)_m \longrightarrow (\mathscr{F}_3)_m \longrightarrow 0$$

of $\mathscr{O}_S$-modules is splitting. The proposition then follows from Proposition 14.1.34, since the corresponding result for graded modules is true. □

14.1.3 Cohomology of groups

14.1.3.1 The standard complex Let $\mathcal{C}$ be a category, G be a group in $\widehat{\mathcal{C}}$, A be a ring and M be a $A[G]$-module. For $n \geq 0$, we put

$$C^n(G,M) = \mathrm{Hom}(G^n, M), \quad \mathcal{C}^n(G,M) = \mathcal{H}om(G^n, M),$$

where G^0 is the final object e of $\widehat{\mathcal{C}}$. Then $\mathcal{C}^n(G,M)$ (resp. $C^n(G,M)$) is endowed evidently with a structure of $\mathbb{O}$-module (resp. $\Gamma(\mathbb{O})$-module), and we have

$$C^n(G,M) \cong \Gamma(\mathcal{C}^n(G,M)), \quad \mathcal{C}^n(G,M)(S) = C^n(G_S, M_S).$$

Giving an element of $C^n(G,M)$ is then equivalent to giving for each $S \in \mathrm{Ob}(\mathcal{C})$ an n-cochain of $G(S)$ in $M(S)$, which is functorial on S. The boundary operator

$$d : C^n(G(S), M(S)) \to C^{n+1}(G(S), M(S)),$$

which is defined by the formula

$$\begin{aligned}(df)(g_1,\dots,g_{n+1}) = g_1 \cdot f(g_2,\dots,g_{n+1}) &+ \sum_{i=1}^{n}(-1)^i f(g_1,\dots,g_ig_{i+1},\dots,g_{n+1}) \\ &+ (-1)^{n+1} f(g_1,\dots,g_n)\end{aligned}$$

is then functorial on S and hence defines a homomorphism

$$d : C^n(G,M) \to C^{n+1}(G,M)$$

such that $d \circ d = 0$. We then obtain a complex of abelian groups, which we denote by $C^\bullet(G,M)$. We define similarly a complex of A-modules $\mathcal{C}^n(G,M)$, and we have

$$C^\bullet(G,M) = \Gamma(\mathcal{C}^n(G,M)).$$

We denote by $H^n(G,M)$ (resp. $\mathcal{H}^n(G,M)$) the cohomology group of the complex $C^\bullet(G,M)$ (resp. $\mathcal{C}^\bullet(G,M)$). In particular, we have

$$\mathcal{H}^0(G,M) = M^G, \quad H^0(G,M) = \Gamma(M^G).$$

Remark 14.1.37. The set-theoretic definition of d is given to verify that $d \circ d = 0$. We can also define d in terms of the multiplication $m : G \times G \to G$ and the action $\mu : G \times M \to M$ as follows: for any $f \in C^n(G,M)$,

$$df = \mu \circ (\mathrm{id}_G \times f) + \sum_{i=1}^{n}(-1)^i f \circ (\mathrm{id}_{G^{i-1}} \times m \times \mathrm{id}_{G^{n-i}}) + (-1)^{n+1} f \circ \mathrm{pr}_{[1,n]},$$

where $\mathrm{pr}_{[1,n]}$ is the projection of $G^{n+1} = G^n \times G$ to G^n. Similarly, for any $S \in \mathrm{Ob}(\mathcal{C})$ and $f \in \mathrm{Ob}(\mathcal{C})^n(G,M)(S) = C^n(G_S, M_S)$, we have

$$df = \mu_S \circ (\mathrm{id}_G \times f) + \sum_{i=1}^{n}(-1)^i f \circ (\mathrm{id}_{G_S^{i-1}} \times m_S \times \mathrm{id}_{G_S^{n-i}}) + (-1)^{n+1} f \circ \mathrm{pr}_{[1,n]},$$

where m_S and μ_S are defined by base change.

We recall that $\mathbf{Mod}(A[G])$ is endowed with an abelian category structure, defined "argument by argument" (Proposition 14.1.11); therefore a sequence of $A[G]$-modules

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

is exact if and only the sequence of abelian groups

$$0 \longrightarrow M'(S) \longrightarrow M(S) \longrightarrow M''(S) \longrightarrow 0$$

is exact for any $S \in \mathrm{Ob}(\mathcal{C})$. If $\mathcal{C}$ is $\mathcal{U}$-small, then by Proposition 14.1.12, $\mathbf{Mod}(A[G])$ possesses enough injectives, so that the derived functors of the left exact functors $\mathcal{H}^0$ and H^0 can be defined. We now show that the functors $\mathcal{H}^n$ and H^n are isomorphic to the derived functors of $\mathcal{H}^0$ and H^0, respectively.

Definition 14.1.38. For any A-module P, we denote by $\mathrm{CoInd}(P)$ the object $\mathcal{H}om(G,P)$ of $\widehat{\mathcal{C}}$ endowed with the structure of an $A[G]$-module defined as follows: for any $S \in \mathrm{Ob}(\mathcal{C})$, we have $\mathcal{H}om(G,P)(S) = \mathrm{Hom}_S(G_S, P_S)$, and we act $g \in G(S)$ and $a \in A[S]$ on $\phi \in \mathrm{Hom}_S(G_S,P_S)$ by the formule

$$(g\cdot\phi)(h) = \phi(hg), \quad (a\cdot\phi)(h) = a\phi(h),$$

for any $h \in G(S')$ and $S' \to S$. Moreover, for any $\phi \in \mathrm{Hom}_S(G_S,P_S)$, we set

$$\varepsilon(\phi) = \phi(1) \in P(S)$$

where 1 denotes the unit element of $G(S)$. Then it is clear that the construction of $\mathrm{CoInd}(P)$ is functorial on P, and we have thus defined a functor $\mathrm{CoInd} : \mathbf{Mod}(A) \to \mathbf{Mod}(A[G])$ and a natural transform $\iota \circ \mathrm{CoInd} \to \mathrm{id}$, where ι denotes the forgetful functor.

Remark 14.1.39. Let G_1 and G_2 be two copies of G. Then the morphism

$$G_1 \times \mathrm{CoInd}(P) \to \mathrm{CoInd}(P), \quad (g_1,\phi) \mapsto (g_2 \mapsto \phi(g_2 g_1))$$

corresponds via the isomorphisms

$$\begin{aligned}\mathrm{Hom}(G_1 \times \mathrm{CoInd}(P), \mathrm{CoInd}(P)) &\cong \mathrm{Hom}(\mathrm{CoInd}(P), \mathcal{H}om(G_1, \mathcal{H}om(G_2,P)))\\ &\cong \mathrm{Hom}(\mathrm{CoInd}(P), \mathcal{H}om(G_2 \times G_1, P))\end{aligned}$$

to the morphism $\phi \mapsto ((g_2,g_1) \mapsto \phi(g_2g_1))$, i.e. to the morphism

$$\mathcal{H}om(G,P) \to \mathcal{H}om(G_2 \times G_1, P)$$

induced by the multiplication $\mu_G : G \times G \to G, (g_2, g_1) \mapsto g_2 g_1$.

Lemma 14.1.40. *The functor* CoInd *is right adjoint to the forgetful functor* $\iota : \mathbf{Mod}(A[G]) \to \mathbf{Mod}(A)$. *More precisely,* $\varepsilon : \iota \circ \mathrm{CoInd} \to \mathrm{id}$ *induces for any* $M \in \mathbf{Mod}(A[G])$ *and* $P \in \mathbf{Mod}(A)$ *a bijection*

$$\mathrm{Hom}_{A[G]}(M, \mathrm{CoInd}(P)) \xrightarrow{\sim} \mathrm{Hom}_A(M,P).$$

Therefore, if I *is an injective object of* $\mathbf{Mod}(A)$, *then* $\mathrm{CoInd}(I)$ *is an injective object of* $\mathbf{Mod}(A[G])$.

Proof. To any A-morphism $f : M \to P$, we associate an element $\phi_f \in \mathrm{Hom}_A(M, \mathrm{CoInd}(P))$ defined as follows: for $S \in \mathrm{Ob}(\mathcal{C})$ and $m \in M(S)$, $\phi_f(m)$ is the element of $\mathrm{Hom}_S(G_S,P_S)$ such that for any $g \in G(S')$, $S' \to S$,

$$\phi_f(m)(g) = f(gm) \in P(S').$$

Then for any $h \in G(S)$, we have $\phi_f(hm) = h \cdot f(m)$, i.e. $\phi_f \in \mathrm{Hom}_{A[G]}(M, \mathrm{CoInd}(P))$. Now if $\phi \in \mathrm{Hom}_{A[G]}(M, \mathrm{CoInd}(P))$ and we denote, for $m \in M(S)$, $f(m) = \phi(m)(1)$, then

$$\phi_f(m)(g) = f(gm) = \phi(gm)(1) = (g\cdot\phi(m)) = \phi(m)(g),$$

so $\phi_f = \phi$. Conversely, it is clear that $\phi_f(m)(1) = f(m)$, whence the first claim. The second claim then follows since the forgetful functor ι is exact. $\square$

Definition 14.1.41. Let M be an $A[G]$-module; the identity map on M (considered as an A-module) corresponds by adjunction to a morphism of $A[G]$-modules

$$\eta_M : M \to \mathrm{CoInd}(M)$$

such that for $S \in \mathrm{Ob}(\mathcal{C})$ and $m \in M(S)$, $\eta_M(m)$ is the morpism $G_S \to M_S$ such that for any $S' \to S$ and $g \in G(S')$, $\eta_M(m)(g) = g \cdot m_{S'} \in M(S')$. Note that η_M is a monomorphism: in fact, $\varepsilon_M : \mathrm{CoInd}(M) \to M$ is a morphism of A-modules such that $\varepsilon_M \circ \eta_M = \mathrm{id}_M$. Therefore, M is a direct factor of the A-module $\mathrm{CoInd}(M)$.

Lemma 14.1.42. *For any $P \in \mathbf{Mod}(A)$, we have*

$$H^n(G, \mathcal{H}om(G, P)) = 0, \quad \mathcal{H}^n(G, \mathcal{H}om(G, P)) = 0 \quad \textit{for} \quad n > 0.$$

Therefore, the functors $H^n(G, -)$ and $\mathcal{H}^n(G, -)$ are effacable for $n > 0$.

Proof. It suffices to prove that $\mathcal{C}^\bullet(G, \mathcal{H}om(G, P))$ and $C^\bullet(G, \mathcal{H}om(G, P))$ are null-homotopic at positive degrees. To this end, we only need to consider the second one, since the corresponding result can be derived via base changes. Now, we define for $n \geq 0$ a morphism

$$\sigma : C^{n+1}(G, \mathcal{H}om(G, P)) \to C^n(G, \mathcal{H}om(G, P)).$$

Let $f \in C^{n+1}(G, \mathcal{H}om(G, P))$; for any $S \in \mathrm{Ob}(\mathcal{C})$ and $g_1, \dots, g_n \in G(S)$, $\sigma(f)(g_1, \dots, g_n)$ is the element of $\mathrm{Hom}_S(G_S, P_S)$ such that for any $S' \to S$ and $x \in G(S')$,

$$\sigma(f)(g_1, \dots, g_n)(x) = f(x, g_1, \dots, g_n)(1) \in P(S'),$$

where 1 denotes the unit element of $G(S')$. Then σ is a null homotopy at positive degrees. In fact, for any $g_1, \dots, g_{n+1} \in G(S)$ and $x \in G(S')$, we have, on the one hand,

$$\begin{aligned} d\sigma(f)(g_1, \dots, g_{n+1})(x) = f(xg_1, g_2, \dots, g_{n+1})(1) + \sum_{i=1}^n (-1)^i f(x, g_1, \dots, g_i g_{i+1}, \dots, g_{n+1})(1) \\ + (-1)^{n+1} f(x, g_1, \dots, g_n)(1), \end{aligned}$$

and on the other hand,

$$\begin{aligned} \sigma(df)(g_1, \dots, g_{n+1})(x) &= (xf(g_1, \dots, g_{n+1}))(1) - f(xg_1, g_2, \dots, g_{n+1})(1) \\ &\quad + \sum_{i=1}^n (-1)^{i+1} f(x, g_1, \dots, g_i g_{i+1}, g_{n+1}) + (-1)^{n+2} f(x, g_1, \dots, g_n)(1), \end{aligned}$$

whence

$$(d\sigma(f) + \sigma(df))(g_1, \dots, g_{n+1})(x) = (xf(g_1, \dots, g_{n+1}))(1) = f(g_1, \dots, g_{n+1})(x),$$

i.e. $d\sigma + \sigma d$ is the identity map on $C^{n+1}(G, \mathcal{H}om(G, P))$, for any $n \geq 0$. □

Proposition 14.1.43. *Suppose that $\mathcal{C}$ is $\mathcal{U}$-small, finite products exist in $\mathcal{C}$, and that G is representable. Then the functors $H^n(G, -)$ (resp. $\mathcal{H}^n(G, -)$) are the derived functors of $H^0(G, -)$ (resp. $\mathcal{H}^n(G, -)$) over the category of $A[G]$-modules.*

Proof. In view of ([**?**] 2.2.1 and 2.3), it suffices to show that the $H^n(G)$ (resp. $\mathcal{H}^n(G, -)$) form a cohomological functors, since they are effacable for $n > 0$ in view of Lemma 14.1.42. Let

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

be an exact sequence of $A[G]$-modules, and let $S \in \mathrm{Ob}(\mathcal{C})$. By hypothesis, G is represented by an object $G \in \mathrm{Ob}(\mathcal{C})$, and finite products exist in $\mathcal{C}$. In particular, $\mathcal{C}$ possesses a final object e. For each $n \geq 0$, the product $G^n \times h_S$ is then represented by $G^n \times S$ (where $G^0 = e$), and the sequence

$$0 \longrightarrow M'(G^n \times S) \longrightarrow M(G^n \times S) \longrightarrow M''(G^n \times S) \longrightarrow 0$$

is exact. Therefore, the sequence of A-modules

$$0 \longrightarrow \mathcal{C}^n(h_G, M') \longrightarrow \mathcal{C}^n(h_G, M) \longrightarrow \mathcal{C}^n(h_G, M'') \longrightarrow 0$$

is exact, which means $\mathcal{C}^\bullet(G, -)$, considered as a functor from $\mathbf{Mod}(A[G])$ to the category of complexes of $\mathbf{Mod}(A)$, is exact. It then follows from the induced long exact sequence that $\mathcal{H}^n(G, -)$ form a cohomological functor. As the functor Γ is exact, the same holds for the functors $H^n(G, -)$. □

14.1.3.2 Cohomology of $\mathscr{O}_S[G]$-modules Let S be a scheme, G be an S-group and $\mathscr{F}$ be a quasi-coherent $\mathscr{O}_S[G]$-module. We define the cohomology groups of G with values in $\mathscr{F}$ by

$$H^n(G, \mathscr{F}) = H^n(h_G, \Gamma_{\mathscr{F}}).$$

Suppose that G is affine over S, then by Corollary 14.1.25, this cohomology can be calculated in the following way: $H^n(G, \mathscr{F})$ is the n-th cohomology group of the complex $C^\bullet(G, \mathscr{F})$ whose n-th term is

$$C^n(G, \mathscr{F}) = \Gamma(S, \mathscr{F} \otimes \underbrace{\mathscr{A}(G) \otimes \cdots \otimes \mathscr{A}(G)}_{n\text{-fold}}).$$

If f (resp. a_i) is a section of $\mathscr{F}$ (resp. $\mathscr{A}(G)$) over an open subset of S, we then have

$$\begin{aligned} d(f \otimes a_1 \otimes \cdots \otimes a_n) = \mu_{\mathscr{F}}(f) \otimes a_1 \otimes \cdots \otimes a_n + \sum_{i=1}^{n} (-1)^i f \otimes a_1 \cdots \otimes \Delta a_i \otimes \cdots \otimes a_n \\ + (-1)^{n+1} f \otimes a_1 \otimes \cdots \otimes a_n \otimes 1 \end{aligned}$$

where $\Delta : \mathscr{A}(G) \to \mathscr{A}(G) \otimes \mathscr{A}(G)$ and $\mu_{\mathscr{F}} : \mathscr{F} \to \mathscr{F} \otimes \mathscr{A}(G)$ are induced from the cogebrea structure of $\mathscr{A}(G)$ and the comodule structure on $\mathscr{F}$. Note in passing that the cohomology of G with values in $\mathscr{F}$ therefore depends only on the comodule structure of $\mathscr{F}$ and the monoid structure of G. In particular, we obtain a functor

$$H^0(G, \mathscr{F}) = \Gamma(S, \mathscr{F}^G)$$

where $\mathscr{F}^G$ is the invariant sheaf of $\mathscr{F}$ defined in Remark 14.1.31.

Theorem 14.1.44. *Let S be an affine scheme and G be an affine and flat group over S. Then the functors $H^n(G, -)$ are the derived functors of $H^0(G, -)$ over the category of quasi-coherent $\mathscr{O}_S[G]$-modules.*

If G is affine and flat over S, then by Proposition 14.1.33, the category $\mathbf{Qcoh}(\mathscr{O}_S[G])$ is equivalent to the category $\mathbf{CoQcoh}(\mathscr{A}(G))$ of quasi-coherent $\mathscr{A}(G)$-comodules over $\mathscr{O}_S$ and is abelian. On the other hand, $\mathscr{A}(G)$ being a flat $\mathscr{O}_S$-module, the functor $\mathscr{F} \mapsto \mathscr{F} \otimes_{\mathscr{O}_S} \mathscr{A}(G)^{\otimes n}$ is exact; as S is also affine, we conclude that $C^\bullet(G, -)$ is an exact functor over $\mathbf{Qcoh}(\mathscr{O}_S[G])$.

We denote by Δ (resp. η) the coultiplication (resp. counit) of $\mathscr{A}(G)$. For any quasi-coherent $\mathscr{O}_S$-module $\mathscr{P}$, we denote by $\mathrm{Ind}(\mathscr{P}) = \mathscr{P} \otimes_{\mathscr{O}_S} \mathscr{A}(G)$ endowed with the $\mathscr{A}(G)$-comodule structure defined by

$$\mathrm{id}_{\mathscr{P}} \otimes \Delta : \mathscr{P} \otimes_{\mathscr{O}_S} \mathscr{A}(G) \to \mathscr{P} \otimes_{\mathscr{O}_S} \mathscr{A}(G) \otimes_{\mathscr{O}_S} \mathscr{A}(G);$$

this defines a functor $\mathrm{Ind} : \mathbf{Qcoh}(\mathscr{O}_S) \to \mathbf{Qcoh}(\mathscr{O}_S[G])$. It follows from Corollary 14.1.25 that we have an isomorphism of $\mathbb{O}_S[G]$-modules

$$\Gamma_{\mathrm{Ind}(\mathscr{P})} \cong \mathrm{CoInd}(\Gamma_{\mathscr{P}}) = \mathcal{H}om(G, \Gamma_{\mathscr{P}}). \tag{14.1.1}$$

Via this identification, the morphism $\varepsilon : \mathrm{CoInd}(\Gamma_{\mathscr{P}}) \to \Gamma_{\mathscr{P}}$ then corresponds to the morphism $\mathrm{id}_{\mathscr{P}} \otimes \eta : \mathrm{Ind}(\mathscr{P}) \to \mathscr{P}$ of $\mathscr{O}_S$-modules, where we use Proposition 14.1.21. From Lemma 14.1.40, we then conclude the following corolalry:

Corollary 14.1.45. *Let S be a scheme and G be an affine group over S. Then the functor* Ind *is right adjoint to the forgetful functor $\iota : \mathbf{Qcoh}(\mathscr{O}_S[G]) \to \mathbf{Qcoh}(\mathscr{O}_S)$. More precisely, the map $\mathrm{id}_{\mathscr{P}} \otimes \eta$: $\mathrm{Ind}(\mathscr{P}) \to \mathscr{P}$ induces for any object $\mathscr{M}$ of $\mathbf{Qcoh}(\mathscr{O}_S[G])$ a bijection*

$$\mathrm{Hom}_{\mathscr{O}_S[G]}(\mathscr{M}, \mathrm{Ind}(\mathscr{P})) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{O}_S}(\mathscr{M}, \mathscr{P}).$$

Therefore, if $\mathscr{I}$ is an injective object in $\mathbf{Qcoh}(\mathscr{O}_S)$, then $\mathrm{Ind}(\mathscr{I})$ is an injective object in $\mathbf{Qcoh}(\mathscr{O}_S)$.

Let $\mathscr{F}$ be an $\mathscr{O}_S[G]$-module and $\mu_{\mathscr{F}} : \mathscr{F} \to \mathrm{Ind}(\mathscr{F})$ be the map defining the $\mathscr{A}(G)$-comodule structure. It follows from the axioms (CM1) and (CM2) that $\mu_{\mathscr{F}}$ is a morphism of $\mathscr{O}_S[G]$-modules, and that $(\mathrm{id}_{\mathscr{F}} \otimes \eta) \circ \mu_{\mathscr{F}} = \mathrm{id}_{\mathscr{F}}$, so that $\mathscr{F}$ is a direct factor of $\mathrm{Ind}(\mathscr{F})$ considered as $\mathscr{O}_S$-modules. In particular, $\mu_{\mathscr{F}}$ is a monomorphism. As we have, by (14.1.1) and Lemma 14.1.42,

$$H^n(G, \Gamma_{\mathrm{Ind}(\mathscr{F})}) \cong H^n(G, \mathcal{H}om_S(G, \Gamma_{\mathscr{F}})) = 0 \quad \text{for} \quad n > 0$$

we conclude that $H^n(G, -)$ is effacable for $n > 0$.

Finally, as S is affine, $\mathbf{Qcoh}(\mathscr{O}_S)$ possesses enough injectives. Let $\mathscr{F} \rightarrowtail \mathscr{I}$ be a monomorphism of $\mathscr{O}_S$-modules where $\mathscr{I}$ is injective object of $\mathbf{Qcoh}(\mathscr{O}_S)$; then, $\mathscr{A}(G)$ being flat over $\mathscr{O}_S$, $\mathrm{Ind}(\mathscr{F})$ is a sub-$\mathscr{O}_S[G]$-module of $\mathrm{Ind}(\mathscr{I})$, so we conclude that

Corollary 14.1.46. *Under the hypothesis of Theorem 14.1.44, the abelian category $\mathbf{Qcoh}(\mathscr{O}_S[G])$ possesses enough injectives.*

In view of ([?] 2.2.1 and 2.3), we then conclude that proof of Theorem 14.1.44.

Remark 14.1.47. We can also prove Corollary 14.1.45by the following calculation. To any morphism of $\mathscr{O}_S[G]$-modules $\phi : \mathscr{M} \to \mathscr{P} \otimes_{\mathscr{O}_S} \mathscr{A}(G)$, we associate the $\mathscr{O}_S$-morphism $(\mathrm{id}_{\mathscr{P}} \otimes \eta) \circ \phi$: $\mathscr{M} \to \mathscr{P}$. Conversely, to any $\mathscr{O}_S$-morphism $f : \mathscr{M} \to \mathscr{P}$ we associate the $\mathscr{O}_S[G]$-morphism $(f \otimes \mathrm{id}_{\mathscr{A}(G)}) \circ \mu_{\mathscr{M}} : \mathscr{M} \to \mathrm{Ind}(\mathscr{P})$. On the one hand, from axiom (CM2) we see that

$$(\mathrm{id}_{\mathscr{P}} \otimes \eta) \circ (f \circ \mathrm{id}_{\mathscr{A}(G)}) \circ \mu_{\mathscr{M}} = (f \circ \mathrm{id}_{\mathscr{O}_S}) \circ (\mathrm{id}_{\mathscr{P}} \otimes \eta) \circ \mu_{\mathscr{M}} = f.$$

On the other hand, for any ϕ the following diagram is commutative:

$$\begin{tikzcd}
\mathscr{M} \arrow[r, "\phi"] \arrow[d, "\mu_{\mathscr{M}}"'] & \mathscr{P} \otimes_{\mathscr{O}_S} \mathscr{A}(G) \arrow[d, "\mathrm{id}_{\mathscr{P}} \otimes \Delta"] \\
\mathscr{M} \otimes_{\mathscr{O}_S} \mathscr{A}(G) \arrow[r, "\phi \otimes \mathrm{id}_{\mathscr{A}(G)}"] & \mathscr{P} \otimes_{\mathscr{O}_S} \mathscr{A}(G) \otimes_{\mathscr{O}_S} \mathscr{A}(G)
\end{tikzcd}$$

so it follows that

$$\begin{aligned}
(((\mathrm{id}_{\mathscr{P}} \otimes \eta) \circ \phi) \otimes \mathrm{id}_{\mathscr{A}(G)}) \circ \mu_{\mathscr{M}} &= (\mathrm{id}_{\mathscr{P}} \otimes \eta \otimes \mathrm{id}_{\mathscr{A}(G)}) \circ (\phi \otimes \mathrm{id}_{\mathscr{A}(G)}) \circ \mu_{\mathscr{M}} \\
&= (\mathrm{id}_{\mathscr{P}} \otimes \eta \otimes \mathrm{id}_{\mathscr{A}(G)}) \circ (\mathrm{id}_{\mathscr{P}} \otimes \Delta) \circ \phi = \phi.
\end{aligned}$$

This proves the first claim of Corollary 14.1.45, and the second one then follows.

Let $\mathscr{F}$ be an $\mathscr{O}_S[G]$-module. We have seen that the axiom (CM2) shows that considered as $\mathscr{O}_S$-modules, $\mathscr{F}$ is a direct factor of $\mathrm{CoInd}(\mathscr{F})$. This implies the following proposition:

Proposition 14.1.48. *Let S be an affine scheme and G be an affine and flat group scheme over S. Suppose that for any exact sequence*

$$0 \longrightarrow \mathscr{F}_1 \longrightarrow \mathscr{F}_2 \longrightarrow \mathscr{F}_3 \longrightarrow 0$$

of quasi-coherent $\mathscr{O}_S[G]$-modules, which splits as a sequence of $\mathscr{O}_S$-modules, also split as $\mathscr{O}_S[G]$-modules. Then the functors $H^n(G,-)$ are zero for $n > 0$.

Proof. In fact, by the hypothesis, the sequence of $\mathscr{O}_S[G]$-modules

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathrm{CoInd}(\mathscr{F}) \longrightarrow \mathrm{CoInd}(\mathscr{F})/\mathscr{F} \longrightarrow 0$$

is splitting, so $\mathscr{F}$ is a direct factor of $\mathrm{CoInd}(\mathscr{F})$ as an $\mathscr{O}_S[G]$-module. Since $\mathrm{CoInd}(\mathscr{F})$ has trivial higher cohomology, so does $\mathscr{F}$. $\square$

Theorem 14.1.49. *Let S be an affine scheme and G be a diagonalizable S-group. Then for any quasi-coherent $\mathscr{O}_S[G]$-module $\mathscr{F}$, we have $H^n(G,\mathscr{F}) = 0$ for $n > 0$.*

Proof. This follows from Proposition 14.1.48 and Proposition 14.1.36. $\square$

14.1.4 G-equivariant objects and modules

Let $\mathcal{C}$ be a category with a final object e and such that fiber products exist in $\mathcal{C}$. Let G be a group in $\widehat{\mathcal{C}}$, $\pi : M \to X$ be a morphism in $\widehat{\mathcal{C}}$, and $\lambda = \lambda_X : G \times X \to X$ be an action of G on X. In this paragraph, we denote by $Y \times_f M$ the fiber product of $\pi : M \to X$ and an X-functor $f : Y \to X$.

For any $U \in \mathrm{Ob}(\mathcal{C})$ and $x \in X(U)$, the **fiber** of M at x is defined by $M_x = U \times_x M$, i.e. for any $\phi : U' \to U$, we have

$$M_x(U') = \{m \in M(U') : \pi(m) = x_{U'} = \phi^*(x)\}.$$

Finally, if $g \in G(U)$, we denote by $g(x)$ the element $\lambda(g,x)$ in $X(U)$.

Definition 14.1.50. We say that M is a ***G*-equivariant object over *X***, or a ***G*-equivariant *X*-object**, if we are given an action $\Lambda : G \times M \to M$ of G on M compatible with λ, i.e. such that the following diagram is commutative:

$$\begin{tikzcd} G \times M \arrow[r, "\Lambda"] \arrow[d, "\mathrm{id}_G \times \pi"] & M \arrow[d, "\pi"] \\ G \times X \arrow[r, "\lambda"] & X \end{tikzcd}$$

This is equivalent to saying that we are given, for any morphism $(g,x) : U \to G \times X$, morphisms

$$\Lambda_x^U(g) : M_x(U) \to M_{g(x)}(U), \quad m \mapsto g \cdot m$$

satisfying $1 \cdot m = m$ and $g \cdot (h \cdot m) = (gh) \cdot m$ and functorial on the $(G \times X)$-object U. Alternatively, this means we are given morphisms of U-objects

$$\Lambda_x(g) : M_x \to M_{g(x)}$$

such that $\Lambda_x(1) = \mathrm{id}$ and $\Lambda_{h(x)}(g) \circ \Lambda_x(h) = \Lambda_x(gh)$.

Now let A be a ring in $\widehat{\mathcal{C}}$ and $A_X = A \times X$. Under the condition described above, we say that M is a ***G*-equivariant A_X-module** if it is an A_X-module and the action Λ is compatible with the A_X-module structure on M, that is, if for any morphism $(g,x) : U \to G \times X$, the map $\Lambda_x(g) : M_x \to M_{g(x)}$ is a morphism of A_U-modules.

Remark 14.1.51. In the above definition for G-equivariant objects, the conditions $\Lambda_x(1) = \mathrm{id}$ and $\Lambda_{h(x)}(g) \circ \Lambda_x(h) = \Lambda_x(gh)$ implies that $\Lambda_x(g)$ is an isomorphism, with inverse $\Lambda_{g(x)}(g^{-1})$. Conversely, if we suppose that each $\Lambda_x(g)$ is an isomorphism, the condition $\Lambda_{h(x)}(g) \circ \Lambda_x(h) = \Lambda_x(gh)$, applied to $h = 1$, then implies that $\Lambda_x(1) = \mathrm{id}$.

Remark 14.1.52. If M is an A_X-module, then in view of the universal property of fiber products, giving a morphism $\Lambda : G \times M \to M$ which is compatible with λ is equivalent to giving a homomorphism of $A_{G\times X}$-modules

$$\theta : G \times M = (G \times X) \times_{\mathrm{pr}_X} M \to (G \times X) \times_\lambda M, \quad (g, x, m) \mapsto (g, g(x), g \cdot m),$$

and the morphisms $\Lambda_x(g) : M_x \to M_{g(x)}, m \mapsto g \cdot m$ are isomorphisms of A_U-modules if and only if θ is an isomorphism. As we have supposed that each $\Lambda_x(h)$ is an isomorphism, the equality $\Lambda_x(1) = \mathrm{id}$ follows from the equality $\Lambda_{h(x)}(g) \circ \Lambda_x(h) = \Lambda_x(gh)$. Therefore, Λ is an action of G over M if and only the following diagram of $(G \times G \times X)$-isomorphisms is commutative (where we denote by m the multiplication of G and $f^*(\theta)$ is the isomorphism induced from θ under a base change $f : G \times G \times X \to G \times X$)

$$\begin{tikzcd}
(G \times G \times X) \times_{\mathrm{pr}_X \circ \mathrm{pr}_{23}} M \arrow[r, "\mathrm{pr}_{23}^*(\theta)", "\sim"'] \arrow[d, equal] & (G \times G \times X) \times_{\lambda \circ \mathrm{pr}_{23}} M \arrow[d, equal] \\
(G \times G \times X) \times_{\mathrm{pr}_X \circ (m \times \mathrm{id}_X)} M \arrow[d, "(m\times \mathrm{id}_X)^*(\theta)"', "\sim"] & (G \times G \times X) \times_{\mathrm{pr}_X \circ (\mathrm{id}_G \times \lambda)} M \arrow[d, "\sim"', "(\mathrm{id}_G \times \lambda)^*(\theta)"] \\
(G \times G \times X) \times_{\lambda \times (m \times \mathrm{id}_X)} M \arrow[r, equal] & (G \times G \times X) \times_{\lambda \circ (\mathrm{id}_G \times \lambda)} M
\end{tikzcd}$$

Remark 14.1.53. The above definitions extend to the case where G is only a monoid. In this case, giving an action $\Lambda : G \times M \to M$ that is compatible with λ and such that each $\Lambda_x(g) : M_x \to M_{g(x)}$ is a morphism of A_U-modules is equivalent to giving a morphism

$$\theta : G \times M = (G \times X) \times_{\mathrm{pr}_X} M \to (G \times X) \times_\lambda M, \quad (g, x, m) \mapsto (g, g(x), g \cdot m),$$

such as the diagram in Remark 14.1.52 (without the signs $\sim$ under the arrows) is commutative, and such that $\mathrm{pr}_M \circ \theta \circ (\varepsilon_G \times \mathrm{id}_M) = \mathrm{id}_M$, where ε_G denotes the unit section of G and pr_M the projection on M (this is added since in this case the equality $\Lambda_x(1) = \mathrm{id}$ can not be derived).

Let Y be another object of $\widehat{\mathcal{C}}$ which is endowed with an action $\lambda_Y : G \times Y \to Y$ by G and N be a G-equivariant A_X-module. A morphism $f : Y \to X$ in $\widehat{\mathcal{C}}$ (resp. a homomorphism of A_X-modules $\phi : M \to X$) is called G-equivariant if it commutes with the action of G, i.e. if we have $f(g \cdot y) = g \cdot f(y)$ (resp. $\phi(g \cdot m) = g \cdot \phi(m)$), which is equivalent to $f \circ \lambda_Y = \lambda_X \circ \mathrm{id}_G \times f$ (resp. $\phi \circ \Lambda_M = \Lambda_N \circ (\mathrm{id}_G \times \phi)$). We then obtain the following lemma:

Lemma 14.1.54. *Let $f : Y \to X$ be a G-equivariant morphism and M be a G-equivariant A-module. Then the inverse image $f^*(M) = Y \times_f M$ is a G-equivariant A_Y-module.*

Proof. □

14.2 Tangent spaces and Lie algebras

In this section, we construct the tangent spaces and Lie algebras in scheme theory. It will be useful not to restrict oneself to the diagrams themselves, but to also be intersted to certain functors on the category of schemes which are not necessarily representable. The exposition we give here easily generalize beyond the theory of schemes. For example, it is valid for the theory of complex analytic spaces, with suitable modifications.

14.2.1 The tangent bundle and tangent space

14.2.1.1 The functor $\mathcal{H}om_{Z/S}(X,Y)$ Let $\mathcal{C}$ be a category and S be an object of $\mathcal{C}$. We consider objects X, Y, Z in $\widehat{\mathcal{C}}$ with X, Y lying over Z and Z lying over S:

$$\begin{tikzcd} X \arrow[dr, "p_X"'] & & Y \arrow[dl, "p_Y"] \\ & Z \arrow[d] & \\ & S & \end{tikzcd}$$

Definition 14.2.1. We define an object $\mathcal{H}om_{Z/S}(X,Y)$ in $\widehat{\mathcal{C}_{/S}}$ by the formula

$$\mathcal{H}om_{Z/S}(X,Y)(S') = \mathrm{Hom}_{Z_{S'}}(X_{S'}, Y_{S'}) = \mathrm{Hom}_Z(X \times_S S', Y),$$

where S' is an object of $\mathcal{C}_{/S}$. We see that $\mathcal{H}om_{Z/S}(X,Y)$ is none other than the sub-object of $\mathcal{H}om_S(X,Y)$ formed by morphisms compatible with p_X and p_Y, that is, it is the kernel of the morphisms

$$\mathcal{H}om_S(X,Y) \rightrightarrows \mathcal{H}om_S(X,Z)$$

where the first map is defined by composing with p_Y and the seond one is the constant map of p_X.

On the other hand, we see as in (**??**) that, for any object T of $\widehat{\mathcal{C}}$ over S, we have a natural bijection

$$\mathrm{Hom}_S(T, \mathcal{H}om_{Z/S}(X,Y)) \cong \mathrm{Hom}_Z(X \times_S T, Y).$$

Moreover, by (**??**), if E, F are objects of $\widehat{\mathcal{C}}$ lying over Z, then

$$\mathrm{Hom}_Z(E, \mathcal{H}om_Z(F,Y)) \cong \mathrm{Hom}_Z(E \times_Z F, Y) \cong \mathrm{Hom}_Z(F, \mathcal{H}om_Z(E,Y)).$$

Apply this to $E = X$ and $F = Z \times_S T$, we then obtain the following bijections for any object T of $\widehat{\mathcal{C}_{/S}}$:

$$\mathrm{Hom}_S(T, \mathcal{H}om_{Z/S}(X,Y)) \cong \mathrm{Hom}_Z(X \times_S T, Y) \cong \begin{cases} \mathrm{Hom}_Z(Z \times_S T, \mathcal{H}om_Z(X,Y)), \\ \mathrm{Hom}_Z(X, \mathcal{H}om_Z(Z \times_S T, Y)). \end{cases} \tag{14.2.1}$$

Since these bijections are functorial over T, we then obtain isomorphisms of S-functors

$$\begin{tikzcd} \mathcal{H}om_S(T, \mathcal{H}om_{Z/S}(X,Y)) \arrow[rr, "\sim"] \arrow[dr, "\sim"] & & \mathcal{H}om_{Z/S}(X, \mathcal{H}om_Z(Z \times_S T, Y)) \\ & \mathcal{H}om_{Z/S}(X \times_S T, Y) \arrow[ur, "\sim"] & \end{tikzcd} \tag{14.2.2}$$

We also note that, by definition, for $Z = S$ we have $\mathcal{H}om_{S/S}(X,Y) = \mathcal{H}om_S(X,Y)$. On the other hand, if $X = Z$, we put

$$\mathrm{Res}_{Z/S}Y = \mathcal{H}om_{Z/S}(Z,Y),$$

by definition, we then have

$$\mathrm{Res}_{Z/S}(Y)(S') = \mathrm{Hom}_Z(Z \times_S S', Y) = \Gamma(Y_{S'}/Z_{S'}).$$

The functor $\mathrm{Res}_{Z/S} : \widehat{\mathcal{C}_{/Z}} \to \widehat{\mathcal{C}_{/S}}$ is a right adjoint of the base change functor from S to Z. In fact, for any S-functor U, by (14.2.1) we have

$$\mathrm{Hom}_S(U, \mathrm{Res}_{Z/S}Y) = \mathrm{Hom}_S(U, \mathcal{H}om_{Z/S}(Z,Y)) \cong \mathrm{Hom}_Z(U \times_S Z, Y).$$

(If $\mathcal{C} = \mathbf{Sch}$ and Z is an S-scheme, the functor $\mathrm{Res}_{Z/S}$ is called the **Weil restriction**.) We also ntoe that since for any $S' \in \mathrm{Ob}(\mathcal{C}_{/S})$ we have

$$\mathcal{H}om_{Z/S}(X,Y)(S') = \mathrm{Hom}_Z(X_{S'},Y) \cong \mathrm{Hom}_X(X_{S'},Y\times_Z X) = \mathcal{H}om_{X/S}(X,Y\times_Z X),$$

so we obtain an isomorphism

$$\mathcal{H}om_{Z/S}(X,Y) \cong \mathcal{H}om_{X/S}(X,Y\times_Z X) = \mathrm{Res}_{X/S}(Y\times_Z X),$$

which for $Z = S$ gives an isomorphism

$$\mathcal{H}om_S(X,Y) \cong \mathrm{Res}_{X/S} Y_X.$$

Remark 14.2.2. The functore $Y \mapsto \mathcal{H}om_{Z/S}(X,Y)$ commutes with products in the sense that we have a functorial isomorphism

$$\mathcal{H}om_{Z/S}(X,Y\times_Z Y') \cong \mathcal{H}om_{Z/S}(X,Y)\times_S \mathcal{H}om_{Z/S}(X,Y') \tag{14.2.3}$$

It follows that if Y is a Z-group (resp. Z-ring, etc.), then $\mathcal{H}om_{Z/S}(X,Y)$ is an S-group (resp. S-ring, etc.).

Moreover, let $\pi : M \to Y$ be an Y-functor in $\mathbb{O}_Y$-modules. Put $H = \mathcal{H}om_{Z/S}(X,Y)$, then $\mathcal{H}om_{Z/S}(X,M)$ is endowed with a natrual structure of $\mathbb{O}_H$-module. More precisely, for any $H' \to H$, $\mathrm{Hom}_H(H',\mathcal{H}om_{Z/S}(X,M))$ is endowed with a natural structure of $\mathbb{O}(H'\times_S X)$-module.

Remark 14.2.3. Moreover, let $\pi : M \to Y$ be a Y-functor in $\mathbb{O}_Y$-modules. Put $H = \mathcal{H}om_{Z/S}(X,Y)$, then $\mathcal{H}om_{Z/S}(X,M)$ is endowed with a natural $\mathbb{O}_H$-module structure; more precisely, for any $H' \to H$, $\mathrm{Hom}_H(H',\mathcal{H}om_{Z/S}(X,M))$ is endowed with a natural $\mathbb{O}(H'\times_S X)$-structure.

In fact, denote by $m : M\times_Y M \to M$ and $\lambda : \mathbb{O}_Y \times_Y M \to M$ the defining morphisms of abelian group structure and module structure of M. Let H' be an S-scheme over H, that is, we are given a Z-morphism $f : X\times_S H' \to Y$, which makes $X\times_S H'$ a Y-object. Then $\mathrm{Hom}_H(H',\mathcal{H}om_{Z/S}(X,M))$ is the set of Z-morphisms $\phi : X\times_S H' \to M$ such that $\pi\circ\phi = f$, that is, the Y-morphisms $X\times_S H' \to M$.

Let ϕ,ψ be two such morphisms, we define $\phi+\psi$ as the composition of Y-morphisms

$$X\times_S H' \xrightarrow{\phi\times\psi} M\times_Y M \xrightarrow{m} M$$

and this endows $\mathcal{H}om_{Z/S}(X,M)$ an abelian group structure over $H = \mathcal{H}om_{Z/S}(X,Y)$.

Similarly, if a is an element of $\mathbb{O}(X\times_S H')$, i.e. an S-morphism $a : X\times_S H' \to \mathbb{O}_S$, we define $a\phi$ as the composition $\lambda\circ(a\times\phi)$, where $a\times\phi$ denotes the Y-morphism from $X\times_S H$ to $\mathbb{O}_Y\times_Y M \cong \mathbb{O}_S\times_S M$ with components a and ϕ. We verify that this endows $\mathrm{Hom}_H(H',\mathcal{H}om_{Z/S}(X,M))$ with an $\mathbb{O}(X\times_S H')$-module structure, which is functorial on H'.

14.2.1.2 The scheme $I_S(\mathscr{M})$

Definition 14.2.4. Let S be a scheme and $\mathscr{M}$ be a quasi-coherent $\mathscr{O}_S$-module. We denote by $\mathscr{D}_{\mathscr{O}_S}(\mathscr{M})$ the quasi-coherent algebra $\mathscr{O}_S\oplus\mathscr{M}$ (where $\mathscr{M}$ is considered as a square zero ideal). We denote by $I_S(\mathscr{M})$ the S-scheme $\mathrm{Spec}(\mathscr{D}_{\mathscr{O}_S}(\mathscr{M}))$. In particular, we have $\mathscr{D}_{\mathscr{O}_S} = \mathscr{D}_{\mathscr{O}_S}(\mathscr{O}_S)$, $I_S = I_S(\mathscr{O}_S)$, which are called the **algebra of dual numbers over S** and the **dual number scheme over S**.

We then obtain a contravariant functor $\mathscr{M} \mapsto I_S(\mathscr{M})$ from the category of quasi-coherent $\mathscr{O}_S$-modules to the category of S-schemes. In particular, the morphisms $0\to\mathscr{M}$ and $\mathscr{M}\to 0$ define respectively the structural morphism $\rho : I_S(\mathscr{M}) \to I_S(0) = S$ and a section $\varepsilon_{\mathscr{M}} : S \to I_S(\mathscr{M})$, which is called the **zero section** of $I_S(\mathscr{M})$.

As $\mathscr{M} \mapsto I_S(\mathscr{M})$ is a contravariant functor, for any endomorphism $a \in \mathrm{End}_{\mathscr{O}_S}(\mathscr{M})$, we have an S-endomorphism a^* of $I_S(\mathscr{M})$, and

$$1^* = \mathrm{id}, \quad (ab)^* = b^* \circ a^*, \quad 0^* = \varepsilon_{\mathscr{M}} \circ \rho, \quad a^* \circ \varepsilon_{\mathscr{M}} = \varepsilon_{\mathscr{M}}.$$

Therefore, the S-scheme $I_S(\mathscr{M})$ is endowed with a right action of the multiplicative monoid $\mathrm{End}_{\mathscr{O}_S}(\mathscr{M})$, which commutes with S-morphisms $I_S(\mathscr{M}) \to I_S(\mathscr{M}')$ induced by morphisms $\mathscr{M} \to \mathscr{M}'$. In particular, the operations a^* preserves the zero section of $I_S(\mathscr{M})$.

For any endomorphism $a \in \mathrm{End}_{\mathscr{O}_S}(\mathscr{M})$, $f : S' \to S$ and $m \in I_S(\mathscr{M})(S')$, we write $m \cdot a = a^*(m)$. Then we have

$$m \cdot 1 = m, \quad (m \cdot a) \cdot b = m \cdot (ab), \quad m \cdot 0 = \varepsilon_{\mathscr{M}}(\rho(m))$$

and, if $m = \varepsilon_{\mathscr{M}}(f)$, then $m \cdot a = m$.

Remark 14.2.5. The formation of $I_S(\mathscr{M})$ commutes with base changes: we have a canonical isomorphism

$$I_S(\mathscr{M})_{S'} \cong I_{S'}(\mathscr{M} \otimes_{\mathscr{O}_S} \mathscr{O}_{S'}).$$

For simplicity, we shall write $I_{S'}(\mathscr{M})$ for $I_S(\mathscr{M})_{S'}$. More generally, if X is an S-functor (not necessarily representable), then we define $I_X(\mathscr{M}) := I_S(\mathscr{M}) \times_S X$.

Remark 14.2.6. By consider the homotheties on $\mathscr{M}$, we see that the multiplicative monoid $\mathbb{O}(S')$ acts on the S'-scheme $I_{S'}(\mathscr{M})$, which is functorial on $\mathscr{M}$, i.e. the S-scheme $I_S(\mathscr{M})$ is endowed with a structure of an $\mathbb{O}_S$-object, which is functorial on $\mathscr{M}$. We then have a morphism of S-schemes

$$\lambda : I_S(\mathscr{M}) \times_S \mathbb{O}_S \to I_S(\mathscr{M}),$$

which satisfies the evident conditions. For any S-functor X, we then obtain by base change a morphism of X-functors

$$\lambda_X : I_X(\mathscr{M}) \times_S \mathbb{O}_S \to I_X(\mathscr{M})$$

which makes the S-functor $I_X(\mathscr{M})$ an object acted by the monoid $\mathbb{O}(X)$: any element a of $\mathbb{O}_X = \mathrm{Hom}_S(X, \mathbb{O}_S)$ defines an X-endomorphism a^* of $I_X(\mathscr{M})$. More precisely, if $x \in X(S')$ and $m \in I_S(\mathscr{M})(S') = I_{S'}(\mathscr{M})(S')$, then $a(x) = a \circ x$ belongs to $\mathbb{O}(S')$ and we have

$$(m, x) \cdot a = (m \cdot a(x), x).$$

This operation is functorial on $\mathscr{M}$ and preserves the zero section $\varepsilon_{\mathscr{M}} : X \to I_X(\mathscr{M})$, i.e. $a^* \circ \varepsilon_{\mathscr{M}} = \varepsilon_{\mathscr{M}}$ for any $a \in \mathbb{O}(X)$.

Even further, this operation is functorial on X in the following sense: if $\pi : Y \to X$ is a morphism of S-functors and $u : \mathbb{O}(X) \to \mathbb{O}(Y)$ is the corresponding ring homomorphism (i.e. $u(a) = a \circ \pi$ for $a \in \mathbb{O}(X)$), then the following diagram is commutative

$$\begin{tikzcd} I_Y(\mathscr{M}) \arrow[r, "u(a)^*"] \arrow[d, "\pi"'] & I_Y(\mathscr{M}) \arrow[d, "\pi"] \\ I_X(\mathscr{M}) \arrow[r, "a^*"] & I_X(\mathscr{M}) \end{tikzcd}$$

Let $\mathscr{M}$ and $\mathscr{N}$ be quasi-coherent $\mathscr{O}_S$-modules. The commutative diagram

$$\begin{tikzcd} & \mathscr{M} \oplus \mathscr{N} \arrow[dl] \arrow[dr] & \\ \mathscr{M} \arrow[dr] & & \mathscr{N} \arrow[dl] \\ & 0 & \end{tikzcd}$$

then defines a commutative diagram of S-schemes

$$\begin{tikzcd} & I_S(\mathscr{M}\oplus\mathscr{N}) & \\ I_S(\mathscr{M}) \arrow[ur] & & I_S(\mathscr{N}) \arrow[ul] \\ & S \arrow[ul, "\varepsilon_{\mathscr{M}}"] \arrow[uu, "\varepsilon_{\mathscr{M}\oplus\mathscr{N}}"] \arrow[ur, "\varepsilon_{\mathscr{N}}"'] & \end{tikzcd} \tag{14.2.4}$$

Proposition 14.2.7. *For any S-scheme X, the diagram of functors over S obtained by applying the functor $\mathcal{H}om_S(-, X)$ to (14.2.4) is Cartesian:*

$$\begin{tikzcd} \mathcal{H}om_S(I_S(\mathscr{M}\oplus\mathscr{N}), X) \arrow[r] \arrow[d] & \mathcal{H}om_S(I_S(\mathscr{N}), X) \arrow[d] \\ \mathcal{H}om_S(I_S(\mathscr{M}), X) \arrow[r] & \mathcal{H}om_S(S, X) = X \end{tikzcd}$$

Proof. It suffices to verify that for any $S' \to S$, the diagram obtained by applying the functors on S' is Cartesian. As the formation of $I_S(\mathscr{P})$ commutes with base change, it then suffices to prove this for $S' = S$, hence to verify that the following diagram is Cartesian:

$$\begin{tikzcd} X(I_S(\mathscr{M}\oplus\mathscr{N})) \arrow[r] \arrow[d] \arrow[dr, "X(\varepsilon_{\mathscr{M}\oplus\mathscr{N}})"] & X(I_S(\mathscr{N})) \arrow[d, "X(\varepsilon_{\mathscr{N}})"] \\ X(I_S(\mathscr{M})) \arrow[r, "X(\varepsilon_{\mathscr{M}})"] & X(S) \end{tikzcd}$$

Now if $x \in X(S)$, it follows from ([?] III, 5.1) that the fiber $X(\varepsilon_{\mathscr{M}})^{-1}(x)$ is isomorphic to $\mathrm{Hom}_{\mathscr{O}_S}(x^*(\Omega^1_{X/S}), \mathscr{M})$. Since this latter functor clearly commutes with finite direct sums of $\mathscr{O}_S$-modules, our assertion follows. □

Corollary 14.2.8. *Let X be an S-scheme and $\mathscr{M}$ be a free $\mathscr{O}_X$-module of finite type. Then the S-functor $\mathcal{H}om_S(I_S(\mathscr{M}), X)$ is isomorphic to a finite product of copies of $\mathcal{H}om_S(I_S, X)$.*

Remark 14.2.9. It follows from the proof of Proposition 14.2.7 that $\mathcal{H}om_S(I_S, X)$ is isomorphic to the X-functor $\check{\Gamma}_{\Omega^1_{X/S}}$, and hence represented by the vector bundle $\mathbb{V}(\Omega^1_{X/S})$.

14.2.1.3 The tangent bundle and condition (E)

Definition 14.2.10. Let S be a scheme and $\mathscr{M}$ be a free $\mathscr{O}_S$-module of finite rank. Let X be a functor over S. The **tangent bundle of X over S relative to the $\mathscr{O}_S$-module $\mathscr{M}$** is defined to be the S-functor

$$T_{X/S}(\mathscr{M}) = \mathcal{H}om_S(I_S(\mathscr{M}), X).$$

In particular, the **tangent bundle of X over S** is the functor

$$T_{X/S} = T_{X/S}(\mathscr{O}_S) = \mathcal{H}om_S(I_S, X).$$

The construction $\mathscr{M} \mapsto T_{X/S}(\mathscr{M})$ is then a covariant functor from the category of free $\mathscr{O}_S$-modules of finite type to the category of S-functors. In particular, the morphisms $\mathscr{M} \to 0$ and $0 \to \mathscr{M}$ define respectively an S-morphism $\pi_{\mathscr{M}} : T_{X/S}(\mathscr{M}) \to T_{X/S}(0) \cong X$ and a section $\tau : X \to T_{X/S}(\mathscr{M})$, called the **zero section**. Moreover, it follows from the preceding remarks that $\mathbb{O}(S)$ is a monoid acting on the X-functor $T_{X/S}(\mathscr{M})$, which is functorial on $\mathscr{M}$.

Remark 14.2.11. We note that the projection $\pi_{\mathscr{M}} : T_{X/S}(\mathscr{M}) \to X$ is induced by the zero section $\varepsilon_{\mathscr{M}} : S \to I_S(\mathscr{M})$, while the zero section $\tau : X \to T_{X/S}(\mathscr{M})$ is induced by the structural morphism $\rho : I_S(\mathscr{M}) \to S$. For any point $t \in T_{X/S}(\mathscr{M})(S')$ (resp. $x \in X(S')$), which corresponds to an S-morphism $f : I_{S'}(\mathscr{M}) \to X$ (resp. $g : S' \to X$), we have

$$\pi(t) = f \circ (\mathrm{id}_{S'} \times \varepsilon_{\mathscr{M}}), \quad (\text{resp. } \tau(x) = g \circ (\mathrm{id}_{S'} \times \rho)).$$

It follows from the above definition that $\mathscr{M} \mapsto T_{X/S}(\mathscr{M})$ is a covariant functor from the category of free $\mathscr{O}_X$-modules of finite rank to that of functors over X. In particular, $\mathbb{O}(S)$ is a monoid operating on the X-functor $T_{X/S}(\mathscr{M})$, which *respects the functoriality of* $\mathscr{M}$.

Remark 14.2.12. In particular, the above arguments motivates the following construction. For any S-morphism $X' \to X$, we put

$$\Sigma(X', \mathscr{M}) = \mathrm{Hom}_X(X', T_{X/S}(\mathscr{M})).$$

We have an action of the multiplicative monoid $\mathrm{End}_{\mathscr{O}_S}(\mathscr{M})$ over $\Sigma(X', \mathscr{M})$, denoted by $(\lambda, x) \mapsto \lambda * x$, such that

$$\lambda * (\mu * x) = (\lambda\mu) * x, \quad 1 * x = x, \quad 0 * x = \tau_0 * \phi \tag{14.2.5}$$

where τ_0 is the zero section $X \to T_{X/S}(\mathscr{M})$. We have similarly an action of $\mathrm{End}_{\mathscr{O}_S}(\mathscr{M} \oplus \mathscr{M})$ over $\Sigma(X', \mathscr{M} \oplus \mathscr{M})$.

Moreover, let $m : \mathscr{M} \oplus \mathscr{M} \to \mathscr{M}$ (resp. $\delta : \mathscr{M} \to \mathscr{M} \oplus \mathscr{M}$) the addition (resp. diagonal map) of $\mathscr{M}$, and put $m_{X'} : \Sigma(X', \mathscr{M} \oplus \mathscr{M}) \to \Sigma(X', \mathscr{M})$ and $\delta_{X'} : \Sigma(X', \mathscr{M}) \to \Sigma(X', \mathscr{M} \oplus \mathscr{M})$ be the induced morphisms. For $\lambda, \mu \in \mathbb{O}(S)$, let h_λ (resp. $h_{\lambda,\mu}$) be the multiplication by λ on $\mathscr{M}$ (resp. by (λ, μ) on $\mathscr{M} \oplus \mathscr{M}$). Since $m \circ h_{\lambda,\lambda} = h_\lambda \circ m$ and $m \circ h_{\lambda,\mu} = h_{\lambda+\mu}$, we have, for $z \in \Sigma(X', \mathscr{M} \oplus \mathscr{M})$ and $x \in \Sigma(X', \mathscr{M})$:

$$\lambda * m(z) = m((\lambda, \lambda) * z), \quad m((\lambda, \mu) * \delta(x)) = (\lambda + \mu) * x. \tag{14.2.6}$$

Definition 14.2.13. Let $x \in X(S) = \mathrm{Hom}_S(S, X) = \Gamma(X/S)$. We then define the tangent space of X over S at the point x relative to $\mathscr{M}$ to be the S-functor obtained from $T_{X/S}(\mathscr{M})$ by base change via the morphism $x : S \to X$:

$$\begin{tikzcd} T_{X/S,x}(\mathscr{M}) \arrow[r] \arrow[d] & T_{X/S}(\mathscr{M}) \arrow[d, "\pi"] \\ S \arrow[r, "x"] & X \end{tikzcd}$$

In particular, $T_{X/S,x}(\mathscr{O}_X)$ is denoted by $T_{X/S,x}$, which is called the **tangent space of X over S at the point x**.

Remark 14.2.14. It follows from Remark 14.2.11 that, for any $t : S' \to S$, $T_{X/S,x}(\mathscr{M})(S')$ is the set of S-morphisms $f : I_{S'}(\mathscr{M}) \to X$ such that $f \circ (\mathrm{id}_{S'} \times \varepsilon_{\mathscr{M}}) = x \circ t$, where $\varepsilon_{\mathscr{M}} : S \to I_S(\mathscr{M})$ is the zero section.

Proposition 14.2.15. *If X is representable, then $T_{X/S}(\mathscr{M})$ and $T_{X/S,x}(\mathscr{M})$ are representable. In particular, $T_{X/S}$ and $T_{X/S,x}$ are represented by the vector bundles $\mathbb{V}(\Omega^1_{X/S})$ and $\mathbb{V}(x^*(\Omega^1_{X/S}))$.*

Proof. It suffices to prove for $T_{X/S}(\mathscr{M})$, since the analogous result follows from base change. By Corollary 14.2.8, it suffices to consider $T_{X/S}$, which follows from Remark 14.2.9. □

Remark 14.2.16. By Proposition 14.2.15, we can give a simple description of the vector bundle representing $T_{X/S,x}$: if $x : S \to X$ is an S-morphism, then the image of x is locally closed in S by Corollary 10.5.8, hence defined by a quasi-coherent ideal $\mathscr{I}$ of an open subscheme of X. The quotient $\mathscr{I}/\mathscr{I}^2$ can then be considered as a quasi-coherent module over S, whose vector bundle $\mathbb{V}(\mathscr{I}/\mathscr{I}^2)$ is the desired representing scheme.

For example, let X be an algebraic scheme over a field X and x be a rational point of X over k. Let $\mathfrak{m}_x$ be the maximal ideal of the local ring $\mathscr{O}_{X,x}$, then we have $T_{X/k,x} = \mathbb{V}(\mathfrak{m}_x/\mathfrak{m}_x^2)$.

We now retun to the general situation. We first note that $T_{X/S,x}$ is a covariant functor from the category of free $\mathscr{O}_S$-modules of finite rank to that of functors over S. In particular, $\mathbb{O}_S$ is a set of perators of the functor $T_{X/S,x}(\mathscr{M})$, which respects the functoriality on $\mathscr{M}$.

Proposition 14.2.17. *The formulation of $T_{X/S}(\mathscr{M})$ and $T_{X/S,x}(\mathscr{M})$ commutes with base changes: for any S-scheme S', we have functorial isomorphisms*

$$T_{X_{S'}/S'}(\mathscr{M} \otimes \mathscr{O}_S) \xrightarrow{\sim} T_{X/S}(\mathscr{M})_{S'},$$
$$T_{X_{S'}/S',x'}(\mathscr{M} \otimes \mathscr{O}_S) \xrightarrow{\sim} T_{X/S,x}(\mathscr{M})_{S'}$$

where $x' = x_{S'}$.

Proof. This follows from the fact that $\mathcal{H}om$ commutes with base changes. □

Corollary 14.2.18. *The X-functor $T_{X/S}(\mathscr{M})$ (resp. the S-functor $T_{X/S,x}(\mathscr{M})$) is naturally endowed with an $\mathbb{O}_X$-object (resp. $\mathbb{O}_S$-object) structure, which is functorial on $\mathscr{M}$, and the isomorphism of Proposition 14.2.17 are isomorphism of $\mathbb{O}_{X_{S'}}$-objects (resp. $\mathbb{O}_{S'}$-objects).*

Proof. We first prove the case for $T_{X/S,x}(\mathscr{M})$. For any S' over S, $\mathbb{O}(S')$ acts on $\mathscr{M} \otimes \mathscr{O}_{S'}$, and hence on $T_{X_{S'}/S',x'}(\mathscr{M} \otimes \mathscr{O}_{S'}) = T_{X/S,x}(\mathscr{M})_{S'}$. It is easy to verify that this operation is functorail on S', so $T_{X/S,x}(\mathscr{M})$ is endowed with an $\mathbb{O}_S$-object structure.

For $T_{X/S}(\mathscr{M})$ this is more complicated. For each X' over X, put $T_{X/S}(\mathscr{M})_{X'} = T_{X/S}(\mathscr{M}) \times_X X'$; we need to endow $T_{X/S}(\mathscr{M})_{X'}(X') = \mathrm{Hom}_X(X', T_{X/S}(\mathscr{M}))$ with a structure of $\mathbb{O}(X')$-set which is functorial in X'. For this we construct the following diagram, where $X_{X'} = X \times_S X'$ and f' is the section of $X_{X'}$ over X' defined by $f : X' \to X$:

$$\begin{tikzcd}
 & T_{X_{X'}/X'}(\mathscr{M}) \arrow[dl] \arrow[dd] & \\
T_{X/S}(\mathscr{M}) \arrow[dd] & & T_{X/S}(\mathscr{M})_{X'} \arrow[ul] \arrow[ll] \arrow[dd] \\
 & X_{X'} \arrow[dl] & \\
X \arrow[dr] & & X' \arrow[ul, "f'"'] \arrow[ll, "f"'] \arrow[dl] \\
 & S &
\end{tikzcd}$$

This diagram, together with Remark 14.2.14, shows that $T_{X/S}(\mathscr{M})_{X'}(X')$ is identified with

$$T_{X_{X'}/X',f'}(\mathscr{M})(X') = \{X'\text{-morphisms } \psi : I_{X'}(\mathscr{M}) \to X_{X'} \text{ such that } \psi \circ \varepsilon_{\mathscr{M}} = f'\}, \qquad (14.2.7)$$

over which any $a \in \mathbb{O}(X')$ operates via the action over $I_{X'}(\mathscr{M})$, i.e. with the notations of 14.2.1.2, we have $a\psi = \psi \circ a^*$, so for any $X'' \to X'$ and $x \in I_{X'}(\mathscr{M})(X'')$, $(a\psi)(x) = \psi(x \cdot a)$. We then verify that this construction is functorial on X'. □

Remark 14.2.19. The operation of $\mathbb{O}_X$ over $T_{X/S}(\mathscr{M})$ can be simply defined as follows. For any $f : X' \to X$, by (14.2.7) we have[1]

$$\mathrm{Hom}_X(X', T_{X/S}(\mathscr{M})) = T_{X/S}(\mathscr{M})_{X'}(X') = \{\phi \in \mathrm{Hom}_S(I_{X'}(\mathscr{M}), X) \mid \phi \circ \varepsilon_{\mathscr{M}} = f\},$$

[1]If X' is representable, this equality can also be deduced from Remark 14.2.11 and the equivalence $\widehat{\mathbf{Sch}}_{/X} \xrightarrow{\sim} \widehat{\mathbf{Sch}_{/X}}$. In fact, the equivalence $\alpha : \widehat{\mathbf{Sch}}_{/X} \to \widehat{\mathbf{Sch}_{/X}}$ commutes with Yoneda embedding, so we have

$$\mathrm{Hom}_X(X', T_{X/S}(\mathscr{M})) \cong \mathrm{Hom}_X(X', \alpha(T_{X/S}(\mathscr{M}))) = \alpha(T_{X/S}(\mathscr{M}))(X') = \{\phi \in \mathrm{Hom}_S(I_{X'}(\mathscr{M}), X) : \pi_{\mathscr{M}}(\phi) = f\}.$$

and Remark 14.2.11 shows that $\pi_{\mathscr{M}}(\phi) = \phi \circ \varepsilon_{\mathscr{M}}$.

and we have seen in Remark 14.2.6 that $I_{X'}(\mathscr{M})$, considered as an S-functor, is endowed with an operation by the monoid $\mathbb{O}(X')$ which conserve the zero section $\varepsilon_{\mathscr{M}} : X' \to I_{X'}(\mathscr{M})$. Therefore, if we denote by a^* the endomorphism of $I_{X'}(\mathscr{M})$ defined by $a \in \mathbb{O}(X')$, then we have $a^*\phi = \phi \circ a$, which means for any $S' \to S$ and $(m, x') \in \mathrm{Hom}_S(S', I_S(\mathscr{M}) \times_S X')$,

$$(a\phi)(m, x') = \phi(m \cdot a(x'), x')$$

(note that $a^* \circ \varepsilon_{\mathscr{M}} = \varepsilon_{\mathscr{M}}$, whence $(a\phi) \circ \varepsilon_{\mathscr{M}} = f$). Similarly, the operation of $\mathbb{O}_S$ over $T_{X/S,x}(\mathscr{M})$ can be described as follows. For any $t : S' \to S$, $T_{X/S,x}(\mathscr{M})(S')$ is the set of S-morphisms $\phi : I_{S'}(\mathscr{M}) \to X$ such that $\phi \circ \varepsilon_{\mathscr{M}} = u \circ t$; for such a ϕ and $a \in \mathbb{O}(S')$, we have $a\phi = \phi \circ a^*$.

Let S be a scheme and X be an S-functor. We say that ***X* satisfies conditon (E) relative to *S*** if, for any $S' \to S$ and any free $\mathscr{O}_{S'}$-module $\mathscr{M}$ and $\mathscr{N}$ of finite rank, the diagram of sets

$$\begin{tikzcd} & X(I_{S'}(\mathscr{M} \oplus \mathscr{N})) \arrow[dl] \arrow[dr] & \\ X(I_{S'}(\mathscr{M})) \arrow[dr] & & X(I_{S'}(\mathscr{N})) \arrow[dl] \\ & X(S') & \end{tikzcd}$$

obtained by applying X to the diagram (14.2.4), is Cartesian. Equivalently, this means the functor $\mathscr{M} \mapsto T_{X/S}(\mathscr{M})$ transforms direct sums of free $\mathscr{O}_S$-modules of finite rank to products of X-functors. If this is the case, the same holds for the functor $\mathscr{M} \mapsto T_{X/S,x}(\mathscr{M}) = S \times_X T_{X/S}(\mathscr{M})$, for any $x \in \Gamma(X/S)$. By Proposition 14.2.7, we see that any representable functor satisfies condition (E).

We often say that "X/S satisfies condition (E)" to abbreviate that X satisfies condition (E) relative to S. In this case, the functor $\mathscr{M} \mapsto T_{X/S}(\mathscr{M})$ commutes with products, hence transforms groups to groups. In particular, $T_{X/S}(\mathscr{M})$ is an abelian X-group, and for the same reason $T_{X/S,x}(\mathscr{M})$ is an abelian S-group.

Proposition 14.2.20. *If X/S satisfies condition (E), the abelian group structure over $T_{X/S}(\mathscr{M})$ (resp. $T_{X/S,x}(\mathscr{M})$) and the operation of $\mathbb{O}_X$ (resp. $\mathbb{O}_S$) endow $T_{X/S}(\mathscr{M})$ (resp. $T_{X/S,x}(\mathscr{M})$) with the structure of an $\mathbb{O}_X$-module (resp. $\mathbb{O}_S$-module).*

Proof. The operation of $\mathbb{O}_X$ (resp. $\mathbb{O}_S$) is functorial on $\mathscr{M}$, so it respects the abelian group structure induced by the functoriality of $\mathscr{M}$. In fact, retain the notations of Remark 14.2.12. The structure of (abelian) X-group of $T_{X/S}(\mathscr{M})$ is deduced by the composition

$$T_{X/S}(\mathscr{M}) \times_X T_{X/S}(\mathscr{M}) \cong T_{X/S}(\mathscr{M} \oplus \mathscr{M}) \xrightarrow{m} T_{X/S}(\mathscr{M}),$$

and on the other hand the morphism

$$T_{X/S}(\mathscr{M}) \xrightarrow{\delta} T_{X/S}(\mathscr{M} \oplus \mathscr{M}) \cong T_{X/S}(\mathscr{M}) \times_X T_{X/S}(\mathscr{M})$$

is the diagonal morphism. We then deduce from the equality (14.2.6) and Remark 14.2.12 that

$$\lambda(x + y) = \lambda x + \lambda y, \quad (\lambda + \mu)x = \lambda x + \mu x,$$

for any $f : X' \to X$, $x, y \in \mathrm{Hom}_X(X', T_{X/S}(\mathscr{M}))$ and $\lambda, \mu \in \mathbb{O}(X')$. □

Remark 14.2.21. If X is representable, then it satisfies (E) and $T_{X/S}$ and $T_{X/S,x}$ are represented by vector bundles. The previous laws are the same as those which are deduced from the vector bundle structures.

Proposition 14.2.22. *If X/S satisfies condition (E), then $X_{S'}/S'$ satisfies condition (E) and the isomorphisms of Proposition 14.2.20 respects the $\mathbb{O}_{X_{S'}}$-module (resp. $\mathbb{O}_{S'}$-module) structure.*

Proof. The formulation of $I_S(\mathscr{M})$ commutes with base change, so the first assertion is immediate. The second one follows from the proof of Proposition 14.2.20. □

Proposition 14.2.23. *The functors $T_{X/S}(\mathscr{M})$ and $T_{X/S,x}(\mathscr{M})$ are functorial on X, which means if $f : X \to X'$ is an S-morphism, we have commutative diagrams*

$$\begin{tikzcd} T_{X/S}(\mathscr{M}) \arrow[r, "T(f)"] \arrow[d] & T_{X'/S}(\mathscr{M}) \arrow[d] \\ X \arrow[r] & X' \end{tikzcd} \qquad \begin{tikzcd} T_{X/S,x}(\mathscr{M}) \arrow[rr, "T_x(f)"] \arrow[dr] & & T_{X'/S,f\circ x}(\mathscr{M}) \arrow[dl] \\ & S & \end{tikzcd}$$

Moreover, if f is a monomorphism, so are $T(f)$ and $T_x(f)$.

Proof. The existence of $T(f)$ and $T_x(f)$, as well as the last assertion, follow immediately from definition. The commutativity of the diagrams then follows from the functoriality of these morphisms with respect to $\mathscr{M}$ and of the fact that $X = T_{X/S}(0)$. □

Remark 14.2.24. In the situation of Proposition 14.2.23, suppose that X and X' are representable and r is the rank of the free $\mathscr{O}_S$-module $\mathscr{M}$. Then by Corollary 14.2.8, $T_{X/S}(\mathscr{M})$ is isomorphic to the product over X of r copies of $\mathbb{V}(\Omega^1_{X/S})$, and similarly for $T_{X'/S}(\mathscr{M})$. Therefore, the square in Proposition 14.2.23 are Cartesian if f is an open immersion, of more generally if $f^*(\Omega^1_{X'/S}) = \Omega^1_{X/S}$ (for example if f is étale). In this case, we have an isomorphism of S-functors

$$T_{X/S,x}(\mathscr{M}) \xrightarrow{\sim} T_{X'/S,f\circ x}(\mathscr{M}).$$

More generally, the Cartesian square of Proposition 14.2.23 defines a morphism of X-functors

$$\begin{tikzcd} T_{X/S}(\mathscr{M}) \arrow[rr] \arrow[dr] & & T_{X'/S}(\mathscr{M}) \times_{X'} X \arrow[dl] \\ & X & \end{tikzcd}$$

Proposition 14.2.25. *Let $f : X \to X'$ be an S-morphism. If X and X' satisfy condition (E) relative to S, then*

$$T_{X/S}(\mathscr{M}) \xrightarrow{T(f)} T_{X'/S}(\mathscr{M})_X \qquad (\text{resp.}\quad T_{X/S,x}(\mathscr{M}) \xrightarrow{T_x(f)} T_{X'/S,f\circ x}(\mathscr{M}))$$

is a morphism of $\mathbb{O}_X$-modules (resp. $\mathbb{O}_S$-modules).

Proof. This follows from Proposition 14.2.23 by the functoriality on $\mathscr{M}$. □

Proposition 14.2.26. *Let X and Y be functors over S. We have isomorphisms functorial on $\mathscr{M}$:*

$$T_{X/S}(\mathscr{M}) \times_S T_{Y/S}(\mathscr{M}) \xrightarrow{\sim} T_{(X\times_S Y)/S}(\mathscr{M}), \tag{14.2.8}$$
$$T_{X/S,x}(\mathscr{M}) \times_S T_{Y/S,y}(\mathscr{M}) \xrightarrow{\sim} T_{(X\times_S Y)/S,(x,y)}(\mathscr{M}), \tag{14.2.9}$$

Proof. The first isomorphism follows from (14.2.3), and the second one is deduced by base change via $(x,y) : S \to X \times_S Y$. □

Corollary 14.2.27. *If X/S is endowed with an algebraic structure defined by finite Cartesian products, then $T_{X/S}(\mathscr{M})$ is endowed with the same structure and the projection $T_{X/S}(\mathscr{M}) \to X$ is a morphism of that structure.*

Proposition 14.2.28. *If X/S and Y/S satisfy condition (E), then $(X \times_S Y)/S$ satisfies condition (E) and (14.2.8) (resp. (14.2.9)) is an isomorphism of $\mathbb{O}_{X\times_S Y}$-modules (resp. $\mathbb{O}_S$-modules).*

Proof. Suppose that X/S and Y/S satisfy condition (E). Then by (14.2.8), so does $(X \times_S Y)/S$. Let $(x, y) : Z \to X \times_S Y$ be an S-morphism. To see that (14.2.8) is a morphism of $\mathbb{O}_{X\times_S Y}$-modules, in view of Remark 14.2.19, it suffices to show that the map

$$\begin{aligned}\{\phi \in \mathrm{Hom}_S(I_Z(\mathscr{M}), X) : \phi \circ \varepsilon_{\mathscr{M}} = x\} \times \{\psi \in \mathrm{Hom}_S(I_Z(\mathscr{M}), Y) : \psi \circ \varepsilon_{\mathscr{M}} = y\} \\ \to \{\theta \in \mathrm{Hom}_S(I_Z(\mathscr{M}), X \times_S Y) : \theta \circ \varepsilon_{\mathscr{M}} = (X, y)\}\end{aligned}$$

which to (ϕ, ψ) associated $\phi \times \psi$, is a morphism of $\mathbb{O}(Z)$-modules. But this is immediate, since for $a \in \mathbb{O}(Z)$ we have $a \cdot (\phi, \psi) = (\phi \circ a^*, \psi \circ a^*)$, and

$$(\phi \circ a^*) \times (\psi \circ a^*) = (\phi \times \psi) \circ a^* = a \cdot (\phi \times \psi).$$

Similarly, by using Remark 14.2.14, we can show that (14.2.9) is a morphism of O_S-modules. □

If X is an S-group and $e : S \to X$ is the unit section, we define

$$\mathfrak{Lie}(X/S, \mathscr{M}) = T_{X/S,e}(\mathscr{M}),$$

that is, $\mathfrak{Lie}(X/S, \mathscr{M})$ is defined by the Cartesian square

$$\begin{array}{ccc} \mathfrak{Lie}(X/S,\mathscr{M}) & \xrightarrow{i} & T_{X/S}(\mathscr{M}) \\ \downarrow & & \downarrow{\scriptstyle \pi} \\ S & \xrightarrow{e} & X \end{array}$$

By Corollary 14.2.27, the projection $\pi : T_{X/S}(\mathscr{M}) \to X$ is a morphism of S-groups, and it then follows that $\mathfrak{Lie}(X/S, \mathscr{M})$ is endowed with an S-group structure, and is isomorphic via i to the kernel of π.

If, moreover, X/S satisfies condition (E), we shall see in Proposition 14.2.29 that the S-group structure of $\mathfrak{Lie}(X/S, \mathscr{M})$, induced by that of X, coincides with the abelian group structure induced by functoriality of $\mathscr{M}$. To this end we introduce the following terminology: an **H-set** is a set X endowed with a composition law with a two-sided unit, denoted by e_X or simply e. If $f : X \to Y$ is a morphism of H-sets, its kernel $\ker f$ is defined to be $f^{-1}(e_Y)$, which is a sub-H-set of X.

An H-object in a category $\mathcal{C}$ is defined by the usual manner: this is an object X of $\mathcal{C}$, endowed with a morphism $X \times X \to X$ such that there exists a section of X (over the final object) possessing the property of being a two-sided unit. Any $\mathcal{C}$-monoid, and in particular any $\mathcal{C}$-group is therefore an H-object. In particular, an H-object of the category of functors over a scheme S is called an ***S*-H-functor**. If X is an S-H-functor (for example, an S-group), and $e : S \to X$ is the unit section of X, we define

$$\mathfrak{Lie}(X/S, \mathscr{M}) = T_{X/S,e}(\mathscr{M}), \quad \mathfrak{Lie}(X/S) = \mathfrak{Lie}(X/S, \mathscr{O}_S).$$

By Corollary 14.2.27, we see that $T_{X/S}(\mathscr{M})$ and $\mathfrak{Lie}(X/S, \mathscr{M})$ are also S-H-functors, and we have morphisms of S-H-functors

$$\mathfrak{Lie}(X/S, \mathscr{M}) \xrightarrow{i} T_{X/S}(\mathscr{M}) \underset{\tau}{\overset{\pi}{\rightleftarrows}} X \tag{14.2.10}$$

where i is an isomorphism from $\mathfrak{Lie}(X/S, \mathscr{M})$ to $\ker \pi$ and τ is a section of π.

Proposition 14.2.29. *Let X be an S-H-object satisfying condition (E) relative to S. Then the S-H-object structure of $\mathfrak{Lie}(X/S,\mathscr{M})$ induced by that of X coincides with the S-group structure induced by functoriality on $\mathscr{M}$.*

Since X satisfies condition (E), we see that $\mathfrak{Lie}(X/S,\mathscr{M})$ is an H-object in the category of $\mathbb{O}_S$-modules. The proposition then follows from the following lemma:

Lemma 14.2.30. *Let $\mathcal{C}$ be a category. Let G be an H-object in the category of $\mathcal{C}$-H-objects (i.e. G is a $\mathcal{C}$-H-object endowed with a morphism of $\mathcal{C}$-H-objects $h: G\times G\to G$). Then h coincides with the composition law of G and is commutative.*

Proof. By taking the values of the functors on a variable argument, we are reduced to the case where $\mathcal{C}$ is the category of sets. We then have a set G and two maps $f,h: G\times G\to G$ such that

$$h(f(x,y),f(z,t)) = f(h(x,z),h(y,t)), \tag{14.2.11}$$

and we have two elements e,u of G such that $f(e,x)=f(X,e)=x$ and $h(u,x)=h(x,u)=x$. This is the famous Eckmann-Hilton argument[2], which we now provide a proof. We first note that by (14.2.11),

$$h(f(u,y),f(x,u)) = f(x,y) = h(f(x,u),f(u,y)). \tag{14.2.12}$$

In particular, for $y=e$ (resp. $x=e$), we obtain, respectively,

$$\begin{aligned} x = f(x,e) = h(f(u,e),f(x,u)) = h(u,f(x,u)) = f(x,u),\\ y = f(e,y) = h(f(e,u),f(u,y)) = h(u,f(u,y)) = f(u,y), \end{aligned}$$

whence the equality $h(y,x)=f(x,y)=h(x,y)$ in view of (14.2.12). This proves the lemma, whence Proposition 14.2.29. ☐

Remark 14.2.31. The assertion of Proposition 14.2.29 can also be interpreted as follows: if we endow $\mathfrak{Lie}(X/S,\mathscr{M})$ with the abelian group structure induced by functoriality on $\mathscr{M}$, then the morphism $i:\mathfrak{Lie}(X/S,\mathscr{M})\to T_{X/S}(\mathscr{M})$ is a morphism of S-H-objects.

Corollary 14.2.32. *If X is an S-H-functor satisfying condition (E) relative to S, any element of $X(I_S(\mathscr{M}))$, which projects to the unit element of $X(S)$, is invertible.*

Proof. This follows from the sequence (14.2.10) and Proposition 14.2.29, since $\mathfrak{Lie}(X/S,\mathscr{M})$ is a group hence any element has an inverse. ☐

Corollary 14.2.33. *If X is an S-monoid satisfying condition (E) relative to S, an element of $X(I_S(\mathscr{M}))$ is invertible if and only if its image in $X(S)$ is invertible.*

Proof. One direction is immediate, so assume that $x\in X(I_S(\mathscr{M}))$ is an element whose projection s to $X(S)$ is invertible in $X(S)$. Let s^{-1} be the inverse of s in $X(S)$, then $y=x\tau(s^{-1})=x\tau(s)^{-1}$ is projective to the unit element of $X(S)$, and hence is invertible in $X(I_S(\mathscr{M}))$. If y^{-1} is this inverse, we then have

$$x\cdot\tau(s)^{-1}y^{-1} = (x\tau(s)^{-1})\cdot(x\tau(s)^{-1})^{-1} = e,$$

so x is right invertible. Similarly, by considering $y'=\tau(s^{-1})x=\tau(s)^{-1}x$, we see that x is also left invertible, so it is invertible in $X(I_S(\mathscr{M}))$. ☐

Corollary 14.2.34. *If X is an S-group satisfying condition (E) relative to S, the two S-group laws on $\mathfrak{Lie}(X/S,\mathscr{M})$ coincide.*

[2]This argument is used to prove, for example, that higher homotopy groups are abelian.

Corollary 14.2.35. *Let G be an S-group satisfying condition (E) relative to S. For $n \in \mathbb{Z}$, let $n_G : G \to G$ be the morphism of S-functors defined by $g \mapsto g^n$. Then the induced morphism $\mathfrak{Lie}(n_G) : \mathfrak{Lie}(G/S) \to \mathfrak{Lie}(G/S)$ is the multiplication by n, i.e. the map which to any $x \in \mathfrak{Lie}(G/S)(S')$ associates nx.*

Proof. We first note that n_G is in general not a morphism of groups, but it perverses the unit section $e : S \to G$, hence the induced morphism $\mathfrak{Lie}(n_G) = T_e(n_G)$ sends $\mathfrak{Lie}(G/S)$ into itself. If we denote by $i : \mathfrak{Lie}(G/S) \to T_{G/S}$ the inclusion, then $\mathfrak{Lie}(n_G)$ is defined by the equality $i(\mathfrak{Lie}(n_G)(x)) = i(x)^n$, for any $S' \to S$ and $x \in \mathfrak{Lie}(G/S)(S')$. Now by Remark 14.2.31 we have $i(x)^n = i(nx)$, whence $\mathfrak{Lie}(n_G)(x) = nx$. □

Before deducing other consequences from Proposition 14.2.29, let us prove another result of functoriality:

Proposition 14.2.36. *In the situation of 14.2.1.1, we have a functorial isomorphism on $\mathscr{M}$:*

$$T_{\mathcal{H}om_{Z/S}(X,Y)/S}(\mathscr{M}) \xrightarrow{\sim} \mathcal{H}om_{Z/S}(X, T_{Y/Z}(\mathscr{M})).$$

Proof. In fact, by definition we have

$$T_{\mathcal{H}om_{Z/S}(X,Y)/S}(\mathscr{M}) = \mathcal{H}om_S(I_S(\mathscr{M}), \mathcal{H}om_{Z/S}(X,Y)) \cong \mathcal{H}om_{Z/S}(X, \mathcal{H}om_Z(Z \times_S I_S(\mathscr{M}), Y)),$$

where we have used the isomorphism (14.2.1) with $T = I_S(\mathscr{M})$. In view of the isomorphism $Z \times_S I_S(\mathscr{M}) \cong I_Z(\mathscr{M})$, we then obtain

$$T_{\mathcal{H}om_{Z/S}(X,Y)/S}(\mathscr{M}) \cong \mathcal{H}om_{Z/S}(X, \mathcal{H}om_Z(I_Z(\mathscr{M}), Y)) = \mathcal{H}om_{Z/S}(X, T_{Y/Z}(\mathscr{M})). \qquad \square$$

Corollary 14.2.37. *If Y/Z satisfies condition (E), then $\mathcal{H}om_{Z/S}(X,Y)/S$ satisfies condition (E) and the isomorphism of Proposition 14.2.36 respects the $\mathbb{O}$-module structure over $\mathcal{H}om_{Z/S}(X,Y)$.*

Proof. Let $\mathscr{M}, \mathscr{N}$ be two free $\mathscr{O}_S$-modules of finite rank. If Y/Z satisfies condition (E), then

$$T_{Y/Z}(\mathscr{M} \oplus \mathscr{N}) \cong T_{Y/Z}(\mathscr{M}) \times_Y T_{Y/Z}(\mathscr{N}).$$

The right side is a sub-functor of $T_{Y/Z}(\mathscr{M}) \times_S T_{Y/Z}(\mathscr{N})$ and via the isomorphism (14.2.3), we obtain an isomorphism

$$\mathcal{H}om_{Z/S}(X, T_{Y/Z}(\mathscr{M} \oplus \mathscr{N})) \cong \mathcal{H}om_{Z/S}(X, T_{Y/Z}(\mathscr{M})) \times_{\mathcal{H}om_{Z/S}(X,Y)} \mathcal{H}om_{Z/S}(X, T_{Y/Z}(\mathscr{N})).$$

Combined with Proposition 14.2.36, this implies

$$T_{\mathcal{H}om_{Z/S}(X,Y)/S}(\mathscr{M} \oplus \mathscr{N}) \cong T_{\mathcal{H}om_{Z/S}(X,Y)/S}(\mathscr{M}) \times_{\mathcal{H}om_{Z/S}(X,Y)} T_{\mathcal{H}om_{Z/S}(X,Y)/S}(\mathscr{N}),$$

so $\mathcal{H}om_{Z/S}(X,Y)$ satisfies condition (E).

For the second assertion, let $H = \mathcal{H}om_{Z/S}(X,Y)$ and consider an S-morphism $\Delta : H' \to \mathcal{H}om_{Z/S}(X,Y)$, that is, an Z-morphism $\delta : H' \times_S X \to Y$, which makes $H' \times_S X$ a Y-object. We then have a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_H(H', \mathcal{H}om_{Z/S}(X, T_{Y/Z}(\mathscr{M}))) \arrow[r, hook] \arrow[d, equal] & \mathrm{Hom}_S(H', \mathcal{H}om_{Z/S}(X, T_{Y/Z}(\mathscr{M}))) \arrow[d, equal] \\
\mathrm{Hom}_Y(H' \times_S X, T_{Y/Z}(\mathscr{M})) \arrow[r, hook] \arrow[d, equal] & \mathrm{Hom}_Z(H' \times_S X, T_{Y/Z}(\mathscr{M})) \arrow[d, equal] \\
\{\psi \in \mathrm{Hom}_Z(I_{H' \times_S X}(\mathscr{M}), Y) : \psi \circ \varepsilon_{\mathscr{M}} = \delta\} \arrow[r, hook] & \mathrm{Hom}_Z(I_{H' \times_S X}(\mathscr{M}), Y).
\end{tikzcd}$$

By Remark 14.2.3, the action of $\alpha \in \mathbb{O}(H' \times_S X)$ over $\Psi \in \mathrm{Hom}_Y(H' \times_S X, T_{Y/Z}(\mathscr{M}))$ is given as follows: for any $U \to S$ and $(h, x) \in \mathrm{Hom}_S(U, H' \times_S X)$ (U is then an Y-object via $\delta \circ (h, x)$), we have

$$(\alpha\Psi)(h, x) = \alpha(h, x)\Psi(h, x),$$

where $\alpha(h, x) \in \mathbb{O}(U)$ acts on $\Psi(h, x) \in T_{Y/Z}(\mathscr{M})(U)$ via the $\mathbb{O}_Y$-module structure of $T_{Y/Z}(\mathscr{M})$. By Remark 14.2.19, the latter is given, via the identification

$$\mathrm{Hom}_Y(H' \times_S X, T_{Y/Z}(\mathscr{M})) = \{\psi \in \mathrm{Hom}_Z(I_{H' \times_S X}(\mathscr{M}), Y) : \psi \circ \varepsilon_{\mathscr{M}} = \delta\},$$

by the following: for any $(m, h, x) \in \mathrm{Hom}_S(U, I_S(\mathscr{M}) \times_S H' \times_S X)$,

$$(\alpha\psi)(m, h, x) = \psi(m \cdot \alpha(h, x), h, x). \tag{14.2.13}$$

On the other hand, consider the tangent space $T_{H/S}(\mathscr{M}) = \mathcal{H}om_S(I_S(\mathscr{M}), H)$; we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_H(H', T_{H/S}(\mathscr{M})) \arrow[r, hook] \arrow[d, equal] & \mathrm{Hom}_S(H', T_{H/S}(\mathscr{M})) \arrow[d, equal] \\
\{\Phi \in \mathrm{Hom}_S(I_{H'}(\mathscr{M}), H) : \Phi \circ \varepsilon_{\mathscr{M}} = \Delta\} \arrow[r, hook] \arrow[d, equal, "(*)"] & \mathrm{Hom}_S(I_{H'}(\mathscr{M}), H) \arrow[d, equal] \\
\{\phi \in \mathrm{Hom}_Z(I_{H' \times_S X}(\mathscr{M}), Y) : \phi \circ \varepsilon_{\mathscr{M}} = \delta\} \arrow[r, hook] & \mathrm{Hom}_Z(I_{H' \times_S X}(\mathscr{M}), Y)
\end{tikzcd}$$

where the bijection $(*)$ is given as follows: for any $U \to S$ and $(m, h, x) \in \mathrm{Hom}(U, I_S(\mathscr{M}) \times_S H' \times_S X)$ (so that U is over Z via $U \xrightarrow{x} X \to Z$), we have $\Phi(m, h) \in \mathrm{Hom}_Z(X \times_S U, Y)$ and

$$\phi(m, h, x) = \Phi(m, h) \circ (x \times \mathrm{id}_U) \in \mathrm{Hom}_Z(U, Y). \tag{14.2.14}$$

By Remark 14.2.19 (where we replace X by $\mathcal{H}om_{Z/S}(X, Y)$ and X' by H'), the action of $a \in \mathbb{O}(H')$ over $\Phi \in \mathrm{Hom}_S(I_{H'}(\mathscr{M}), H)$ is given by

$$(a\Phi)(m, h) = \Phi(m \cdot a(h), h)$$

where $U \to S$ and $(m, h) \in \mathrm{Hom}_S(U, I_S(\mathscr{M}) \times_S H')$. Therefore, if ϕ (resp. $a\phi$) is the element of $\mathrm{Hom}_Z(I_{H' \times_S X}(\mathscr{M}), Y)$ associated with Φ (resp $a\Phi$), we have, by (14.2.14),

$$(a\Phi)(m, h, x) = \Phi(m \cdot a(h), h) \circ (x \times \mathrm{id}_U) = \phi(m \cdot a(h), h, x). \tag{14.2.15}$$

Together with (14.2.13), this shows that the isomorphism $T_{H/S}(\mathscr{M}) \xrightarrow{\sim} \mathcal{H}om_{Z/S}(X, T_{Y/Z}(\mathscr{M}))$ of Proposition 14.2.36 is an isomorphism of $\mathbb{O}(H)$-modules. Moreover, for any $H' \to H$, the $\mathbb{O}(H')$-module structure of $\mathrm{Hom}_H(H', T_{H/S}(\mathscr{M}))$ extends, in a functorial way on H', to an $\mathbb{O}(H' \times_S X)$-module structure. $\square$

In particular, for $Z = S$, we obtain the following corollary:

Corollary 14.2.38. *We have a functorial isomorphism on $\mathscr{M}$:*

$$T_{\mathcal{H}om_S(X,Y)/S}(\mathscr{M}) \xrightarrow{\sim} \mathcal{H}om_S(X, T_{Y/S}(\mathscr{M})).$$

Moreover, if Y/S satisfies condition (E), then $\mathcal{H}om_S(X, Y)/S$ satisfies condition (E) and the preceding isomorphism respects the $\mathbb{O}$-module structure over $\mathcal{H}om_S(X, Y)$.

Let $u : X \to Y$ be an S-morphism, which can be identified with a constant morphism $\boldsymbol{u} : S \to \mathcal{H}om_S(X, Y)$ such that $\boldsymbol{u}(f) = u_{S'}$ for any $f : S' \to S$. The fiber product of $\boldsymbol{u}$ and $\mathcal{H}om_S(X, T_{Y/S}(\mathscr{M})) \to \mathcal{H}om_S(X, Y)$ is then identified with $\mathcal{H}om_{Y/S}(X, T_{Y/S}(\mathscr{M}))$, where X is over Y via u. Therefore, we deduce from the definition of $T_{\mathcal{H}om_S(X,Y)/S,\boldsymbol{u}}(\mathscr{M})$ and Corollary 14.2.38 the following:

Corollary 14.2.39. *Let $u : X \to Y$ be an S-morphism. We have a functorial isomorphism on $\mathscr{M}$ (where X is over Y via u):*

$$T_{\mathcal{H}om_S(X,Y)/S,\boldsymbol{u}}(\mathscr{M}) \xrightarrow{\sim} \mathcal{H}om_{Y/S}(X, T_{Y/S}(\mathscr{M})).$$

This is an isomorphism of $\mathbb{O}_S$-modules if Y/S satisfies condition (E).

In particular, for $Y = X$, $\mathcal{E}nd_S(X)$ is an S-functor in monoids, hence a fortiori an S-H-functor. Since $\mathfrak{Lie}(\mathcal{E}nd_S(X)/S, \mathscr{M})$ is by definition $T_{\mathcal{E}nd_S(X)/S,e}(\mathscr{M})$, where e is the unit section, we obtain (recall that $\mathcal{H}om_{X/S}(X, T_{X/S}(\mathscr{M})) \cong \mathrm{Res}_{X/S}T_{X/S}(\mathscr{M})$):

Corollary 14.2.40. *We have a functorial isomorphism on $\mathscr{M}$:*

$$\mathfrak{Lie}(\mathcal{E}nd_S(X)/S, \mathscr{M}) \xrightarrow{\sim} \mathrm{Res}_{X/S}T_{X/S}(\mathscr{M}).$$

This is an isomorphism of $\mathbb{O}_S$-modules if X/S satisfies condition (E).

Remark 14.2.41. Suppose that X/S satisfies condition (E). Then the functor $\mathrm{Res}_{X/S}T_{X/S}(\mathscr{M}) = \mathcal{H}om_{X/S}(X, T_{X/S}(\mathscr{M}))$ is endowed with a $\mathrm{Res}_{X/S}\mathbb{O}_X$-module structure, i.e. for any $S' \to S$,

$$\mathcal{H}om_{X/S}(X, T_{X/S}(\mathscr{M}))(S') = \{\psi \in \mathrm{Hom}_X(I_{S'}(\mathscr{M}) \times_S X, X) : \psi \circ (\varepsilon_{\mathscr{M}} \times \mathrm{id}_X) = \mathrm{pr}_X\}$$

is endowed with a $\mathbb{O}(X \times_S S')$-module structure, which is functorial on S'. This follows either from Proposition 14.2.20 and the properties of the functor $\mathrm{Res}_{X/S}$, or from the proof of Corollary 14.2.37.

We now give a geometric interpretation of the tangent bundle. Let U be an S-functor; by (**??**), we have isomorphism functorial on $\mathscr{M}$:

$$\begin{aligned} T_{X/S}(\mathscr{M})(U) &= \mathrm{Hom}_S(U, \mathcal{H}om_S(I_S(\mathscr{M}), X)) \cong \mathrm{Hom}_S(I_S(\mathscr{M}), \mathcal{H}om_S(U, X)) \\ &= \mathrm{Hom}_{I_S(\mathscr{M})}(U_{I_S(\mathscr{M})}, X_{I_S(\mathscr{M})}). \end{aligned}$$

In particular, the morphism $\mathscr{M} \to 0$ induces a commutative diagram

$$\begin{tikzcd} \mathrm{Hom}_S(U, T_{X/S}(\mathscr{M})) \arrow[r, "\sim"] \arrow[d, "\circ \pi_{\mathscr{M}}"] & \mathrm{Hom}_{I_S(\mathscr{M})}(U_{I_S(\mathscr{M})}, X_{I_S(\mathscr{M})}) \arrow[d] \\ \mathrm{Hom}_S(U, X) \arrow[r, equal] & \mathrm{Hom}_S(U, X) \end{tikzcd}$$

where the second vertical arrow is given by base change $\varepsilon_{\mathscr{M}} : S \to I_S(\mathscr{M})$. We therefore obtain the following proposition:

Proposition 14.2.42. *Let $h_0 : U \to X$ be an S-morphism. Then $\mathrm{Hom}_X(U, T_{X/S}(\mathscr{M}))$ is identified with the set of $I_S(\mathscr{M})$-morphisms $h : U_{I_S(\mathscr{M})} \to X_{I_S(\mathscr{M})}$ that extend h_0 (we view U (resp. X) as a sub-object of $U \times_S I_S(\mathscr{M})$ (resp. $X \times_S I_S(\mathscr{M})$) via $\mathrm{id}_U \times_S \varepsilon_{\mathscr{M}}$ (resp. $\mathrm{id}_X \times_S \varepsilon_{\mathscr{M}}$)).*

In particular, for $U = X$ and $h_0 = \mathrm{id}_X$, we obtain:

Corollary 14.2.43. *The set $\Gamma(T_{X/S}(\mathscr{M})/X)$ is identified with the set of $I_S(\mathscr{M})$-endomorphisms ϕ of $X_{I_S(\mathscr{M})}$ which induce identity on X, i.e. such that the following diagram is commutative:*

$$\begin{tikzcd} I_X(\mathscr{M}) \arrow[rr, "\phi"] & & I_X(\mathscr{M}) \\ & X \arrow[ul, "\varepsilon_{\mathscr{M}}"] \arrow[ur, "\varepsilon_{\mathscr{M}}"'] & \end{tikzcd}$$

On the other hand, by Corollary 14.2.39, $\Gamma(T_{X/S}(\mathscr{M})/X) \cong \mathfrak{Lie}(\mathcal{E}nd_S(X)/S, \mathscr{M})(S)$. If X/S satisfies condition (E), then $\mathcal{E}nd_S(X)/S$ satisfies condition (E) and $\mathfrak{Lie}(\mathcal{E}nd_S(X)/S, \mathscr{M})$ is then an $\mathbb{O}_S$-module (and in fact a $\mathrm{Res}_{X/S}\mathbb{O}_X$-module). Applying Proposition 14.2.29, we then deduce that

Proposition 14.2.44. *If X/S satisfies condition (E), the abelian group $\Gamma(T_{X/S}(\mathscr{M})/X)$ is identified with the set of $I_S(\mathscr{M})$-endomorphisms of $X_{I_S}(\mathscr{M})$ which induce identity on X. In particular, any $I_S(\mathscr{M})$-endomorphism of $X_{I_S}(\mathscr{M})$ which induces the identity on X is an automorphism.*

Corollary 14.2.45. *Let $u : X \to Y$ be an S-isomorphism with Y/S satisfying condition (E). Any $I_S(\mathscr{M})$-morphism of $X_{I_S(\mathscr{M})}$ to $Y_{I_S(\mathscr{M})}$ which extends u is an isomorphism.*

Proof. By Proposition 14.2.42 the considered set is identified with $\mathrm{Hom}_Y(X, T_{Y/S}(\mathscr{M}))$, which is isomorphic to $\Gamma(T_{Y/S}(\mathscr{M})/Y)$ by our hypothesis. □

Corollary 14.2.46. *If Y/S satisfies condition (E), the monomorphism $\mathcal{I}so_S(X,Y) \to \mathcal{H}om_S(X,Y)$ induces, for any $u \in \mathrm{Iso}_S(X,Y)$, an isomorphism*

$$T_{\mathcal{I}so_S(X,Y)/S,u}(\mathscr{M}) \xrightarrow{\sim} T_{\mathcal{H}om_S(X,Y)/S,u}(\mathscr{M}).$$

Proof. It suffices to see that $T_{\mathcal{I}so_S(X,Y)/S,u}(\mathscr{M}) \xrightarrow{\sim} T_{\mathcal{H}om_S(X,Y)/S,u}(\mathscr{M})$ is a bijection, for any $S' \to S$. By base change (cf. Proposition 14.2.26), it suffices to consider $S' = S$. In this case, we note that $T_{\mathcal{H}om_S(X,Y)/S,u}(\mathscr{M})(S)$ (resp. $T_{\mathcal{I}so_S(X,Y)/S,u}(\mathscr{M})(S)$) is the set of $I_S(\mathscr{M})$-morphisms (resp. automorphims) $X_{I_S(\mathscr{M})} \to Y_{I_S(\mathscr{M})}$ which extends u, and we can apply Corollary 14.2.45. □

Corollary 14.2.47. *If X/S satisfies (E), the monomorphism $\mathcal{A}ut_S(X) \to \mathcal{E}nd_S(X)$ induces, for any $u \in \mathcal{A}ut_S(X)$, an isomorphism $T_{\mathcal{A}ut_S(X)/S,u}(\mathscr{M}) \xrightarrow{\sim} T_{\mathcal{E}nd_S(X)/S,u}(\mathscr{M})$. In particular, we have*

$$\mathfrak{Lie}(\mathcal{A}ut_S(X)/S, \mathscr{M}) \xrightarrow{\sim} \mathfrak{Lie}(\mathcal{E}nd_S(X)/S, \mathscr{M}) \xrightarrow{\sim} \mathrm{Res}_{X/S}T_{X/S}(\mathscr{M})$$

so that $\mathfrak{Lie}(\mathcal{A}ut_S(X)/S, \mathscr{M})$ is endowed with a $\mathrm{Res}_{X/S}\mathbb{O}_X$-module structure.

Example 14.2.48. There exist functors possessing infinitesimal endomorphisms which are not automorphisms, and hence a fortiori do not satisfy condition (E). For any pointed set (E, x_0), let $M(E)$ be the free commutative monoid generated by E and $M_P(E, x_0)$ be the commutative monoid obtained by quotient $M(E)$ by the equivalence relation generated by $m \sim x_0 + m$. Then $(E, x_0) \to M_P(E, x_0)$ is the left adjoint of the forgetful functor from the category of commutative monoid to that of pointed sets. We say that $M_P(E, x_0)$ is the **free commutative monoid over the pointed set** (E, x_0).

Let X be the functor which associates any scheme S to the free commutative monoid over the set $\mathbb{O}(S)$, pointed by the zero element. A morphism $f : S \to I_{\mathbb{Z}} = \mathrm{Spec}(\mathbb{Z}[t])$ corresponds to a square zero element u_f of $\mathbb{O}(S)$, hence defines an endomorphism of $X(S)$ by $x \mapsto x + u_f$ (taken in $M_P(\mathbb{O}(S), 0)$). We thus obtain an endomorphism ϕ of $X_{I_{\mathbb{Z}}} = X \times_{\mathbb{Z}} I_{\mathbb{Z}}$, defined as follows. For any $f \in I_{\mathbb{Z}}(S)$ and $x \in X(S)$,

$$\phi(x, f) = (x + u_f, f).$$

If $f_0 : S \to I_{\mathbb{Z}}$ is the composition of the structural morphism $S \to \mathrm{Spec}(\mathbb{Z})$ and the zero section of $I_{\mathbb{Z}}$, the corresponding element $u_{f_0} = 0$, and hence $\phi(x, f_0) = (x, f_0)$ (since $x + 0 = x$ in $M_P(\mathbb{O}(S), 0)$). Since the map $X(S) \to X_{I_{\mathbb{Z}}}(S)$ is given by $x \mapsto (x, f_0)$, this shows that ϕ induces the identity on X, hence is an infinitesimal endomorphism of X which is evidently not an automorphism.

Suppose that X is representable. In this case, we have seen in Proposition 14.2.15 that the X-functor $T_{X/S}$ is represented by $\mathbb{V}(\Omega^1_{X/S})$, whence the bijections

$$\Gamma(T_{X/S}/X) \cong \mathrm{Hom}_X(\Omega^1_{X/S}, \mathscr{O}_S) \cong \mathrm{Der}_{\mathscr{O}_S}(\mathscr{O}_X). \tag{14.2.16}$$

This can also be deduced as follows. According to Proposition 14.2.44, $\Gamma(T_{X/S}/X)$ is identified with the set of **infinitesimal endomorphisms** of X (i.e. I_S-endomorphisms of X_{I_S} inducing the identity on X). Now X and X_{I_S} have the same underlying topological space, with structural sheaves being $\mathscr{O}_X$ and $\mathscr{D}_{\mathscr{O}_X} = \mathscr{O}_X \oplus \mathscr{M}$, where $\mathscr{M} = \mathscr{O}_X$ is considered as a square zero ideal. Let $\pi : \mathscr{D}_{\mathscr{O}_X} \to \mathscr{O}_X$ be the morphism of $\mathscr{O}_X$-algebras which is zero on $\mathscr{M}$, we then deduce that giving an infinitesimal endomorphism of X is equivalent to giving a morphism of $\mathscr{O}_S$-algebras $\phi : \mathscr{O}_X \to \mathscr{D}_{\mathscr{O}_X}$ such that $\pi \circ \phi = \mathrm{id}_{\mathscr{O}_X}$, which then amouts to giving an $\mathscr{O}_S$-derivation of the sheaf of rings $\mathscr{O}_X$.

Moreover, we see that if $D, D' \in \mathrm{Der}_{\mathscr{O}_S}(\mathscr{O}_X)$ and if we denote by ϕ_D the infinitesimal endomorphism correponding to D, then

$$\phi_{D+D'} = \phi_D \circ \phi_{D'}.$$

This shows that the identification

$$\{\text{infinitesimal endomorphisms of } X\} \cong \mathrm{Der}_{\mathscr{O}_S}(\mathscr{O}_X)$$

is an isomorphism of abelian groups. In view of Proposition 14.2.44 (and Remark 14.2.41), we have then isomorphism of abelian groups (as well as $\mathbb{O}(X)$-modules)

$$\Gamma(T_{X/S}/X) \xrightarrow{\sim} \mathrm{Der}_{\mathscr{O}_S}(\mathscr{O}_X)$$

which ressume the classical interpretation of tangent vectors in view of derivations of the structural sheaf. Recall also that $\Gamma(T_{X/S}/X)$ is equal to $H^0(X, \mathfrak{g}_{X/S})$, where $\mathfrak{g}_{X/S}$ is the dual of $\Omega^1_{X/S}$.

14.2.2 Tangent space of a group

Let G be a functor in groups over S. By Corollary 14.2.27, $T_{G/S}(\mathscr{M})$ and $\mathfrak{Lie}(G/S, \mathscr{M})$ are endowed with group structures over S and we have group morphisms

$$\mathfrak{Lie}(G/S, \mathscr{M}) \xrightarrow{i} T_{G/S}(\mathscr{M}) \underset{\tau}{\overset{\pi}{\rightleftarrows}} G \tag{14.2.17}$$

By definition i is an isomorphism from $\mathfrak{Lie}(G/S)(\mathscr{M})$ onto the kernel of π, and τ is a section of π. It then follows from Proposition 14.1.10 that we can identify $T_{G/S}(\mathscr{M})$ with a semi-direct product of G by $\mathfrak{Lie}(G/S, \mathscr{M})$.

Definition 14.2.49. The corresponding operation of G on $\mathfrak{Lie}(G/S, \mathscr{M})$ is denoted by

$$\mathrm{Ad} : G \to \mathcal{A}ut_{\mathbf{Grp}}(\mathfrak{Lie}(G/S, \mathscr{M}))$$

and called the adjoint representation (relative to $\mathscr{M}$) of G. For any $S' \to S$, we then have by definition, for $x \in G(S')$ and $X \in \mathfrak{Lie}(G/S, \mathscr{M})(S')$, that

$$\mathrm{Ad}(x)X = i^{-1}(\tau(x)i(x)\tau(x)^{-1}).$$

Definition 14.2.50. If G and H are two functors in groups over S and if $f : G \to H$ is a group morphism, then we have an induced morphism of exact sequences which is compatible with sections:

$$\begin{tikzcd}
1 \arrow[r] & \mathfrak{Lie}(G/S, \mathscr{M}) \arrow[r] \arrow[d, "\mathfrak{Lie}(f)"] & T_{G/S}(\mathscr{M}) \arrow[r] \arrow[d, "T(f)"] & G \arrow[r] \arrow[d, "f"] & 1 \\
1 \arrow[r] & \mathfrak{Lie}(H/S, \mathscr{M}) \arrow[r] & T_{H/S}(\mathscr{M}) \arrow[r] & H \arrow[r] & 1
\end{tikzcd}$$

The morphism $\mathfrak{Lie}(f) = T_e(f)$ is the derived morphism of f. If G/S and H/S satisfy condition (E), then $\mathfrak{Lie}(f)$ respects the $\mathbb{O}_S$-module structure induced by functoriality on $\mathscr{M}$ (cf. Proposition 14.2.25).

Proposition 14.2.51. *Let $g \in G(S)$, then $\mathrm{Ad}(g) : \mathfrak{Lie}(G/S, \mathscr{M}) \to \mathfrak{Lie}(G/S, \mathscr{M})$ is the derived morphism of $\mathrm{Inn}(g) : G \to G$.*

Proof. In fact, $\mathrm{Ad}(g)X = i^{-1}(\mathrm{Inn}(g)i(X))$, which is none other than $T(\mathrm{Inn}(g))X$ by the definition of the derived morphism. □

Suppose that G/S satisfies condition (E). Then, by Proposition 14.2.29, the group structure of $\mathfrak{Lie}(G/S, \mathscr{M})$ defined from G coincides with that induced by the $\mathbb{O}_S$-module structure of $\mathscr{M}$. We then deduce from the preceding proposition and the functoriality of the operation of $\mathbb{O}_S$ (Proposition 14.2.25) that:

Corollary 14.2.52. *Suppose that G/S satisfies condition (E). Then Ad sends G into the subgroup $\mathcal{A}ut_{\mathbb{O}_S}(\mathfrak{Lie}(G/S, \mathscr{M}))$ of $\mathcal{A}ut_{\mathbf{Grp}}(\mathfrak{Lie}(G/S), \mathscr{M})$, that is, for any $g \in G(S')$, $\mathrm{Ad}(g)$ respects the $\mathbb{O}(S')$-module structure of $\mathfrak{Lie}(G_{S'}/S', \mathscr{M})$. In other words, Ad is a linear representation of G on the $\mathbb{O}_S$-module $\mathfrak{Lie}(G/S, \mathscr{M})$.*

Remark 14.2.53. Suppose that G/S satisfies condition (E). Then the derived morphism of the group law $m : G \times_S G \to G$ is none other than the addition law of $\mathfrak{Lie}(G/S, \mathscr{M})$ (m is not a morphism of groups, but $m(e, e) = e$, so the derived morphism $\mathfrak{Lie}(m)$ sends $T_{(G\times_S G)/S,(e,e)}(\mathscr{M}) = \mathfrak{Lie}(G/S, \mathscr{M}) \times_S \mathfrak{Lie}(G/S, \mathscr{M})$ into $\mathfrak{Lie}(G/S, \mathscr{M})$). For any $n \in \mathbb{Z}$, we show similarly that if $n_G : G \to G$ is the morphism of S-functors defined by $g \mapsto g^n$, then the derived morphism $\mathfrak{Lie}(n_G)$ is the multiplication by n on $\mathfrak{Lie}(G/S)$, cf. Corollary 14.2.35.

Now consider the S-functor $\mathcal{H}om_{G/S}(G, T_{G/S}(\mathscr{M}))$; for any $S' \to S$, we have $T_{G/S}(\mathscr{M})_{S'} \cong T_{G_{S'}/S'}(\mathscr{M})$ and hence

$$\mathcal{H}om_{G/S}(G, T_{G/S}(\mathscr{M}))(S') \cong \mathrm{Hom}_{G_{S'}}(G_{S'}, T_{G_{S'}/S'}(\mathscr{M})) = \Gamma(T_{G_{S'}/S'}(\mathscr{M})/G_{S'}).$$

Note that we have an isomorphism, functorial on S',

$$\mathrm{Hom}_{S'}(G_{S'}, \mathfrak{Lie}(G_{S'}/S', \mathscr{M})) \xrightarrow{\sim} \Gamma(T_{G_{S'}/S'}(\mathscr{M})/G_{S'}) \tag{14.2.18}$$

which to any $f : G_{S'} \to \mathfrak{Lie}(G_{S'}/S', \mathscr{M})$ associates the section $s_f : G_{S'} \to T_{G_{S'}/S'}(\mathscr{M})$ such that, for any $S'' \to S'$ and $g \in G(S'')$,

$$s_f(g) = i(f(g))\tau(g).$$

Let h be an automorphism of the functor $G_{S'}$ over S' (not necessarily respects the group structure). To any section s of $T_{G_{S'}/S'}(\mathscr{M})$, we can associate $h(s)$ defined by transport the structure: this for example the only section of $T_{G_{S'}/S'}(\mathscr{M})$ fitting into the commutative diagram

$$\begin{tikzcd} G_{S'} \arrow[r, "s"] \arrow[d, "h"'] & T_{G_{S'}/S'}(\mathscr{M}) \arrow[d, "T(h)"] \\ G_{S'} \arrow[r, "h(s)"] & T_{G_{S'}/S'}(\mathscr{M}) \end{tikzcd}$$

In particular, we can take h to be the right translation t_x by an element x of $G(S')$, that is, $h(g) = t_x(g) = g \cdot x$, for any $g \in G(S'')$, $S'' \to S'$. We have immediately

$$t_x(s_f) = s_{t_x(f)},$$

where $t_x(f) : G_{S'} \to \mathfrak{Lie}(G_{S'}/S', \mathscr{M})$ is defined by

$$t_x(f)(g) = f(g \cdot x^{-1})$$

for any $g \in G(S'')$, $S'' \to S'$. It follows that if we operate G on $\mathcal{H}om_{G/S}(G, T_{G/S}(\mathscr{M}))$ and $\mathcal{H}om_S(G, \mathfrak{Lie}(G/S, \mathscr{M}))$ by right translation in the following way: for any $S' \to S$, $x \in G(S')$, $\sigma \in \Gamma(T_{G_{S'}/S'}(\mathscr{M}/G_{S'}))$ and $f \in \mathrm{Hom}_{S'}(G_{S'}, \mathfrak{Lie}(G_{S'}/S', \mathscr{M}))$,

$$(\sigma \cdot x)(g) = \sigma(g \cdot x^{-1}) \cdot \tau(x), \quad (f \cdot x)(g) = f(g \cdot x^{-1}),$$

for any $g \in G(S'')$, $S'' \to S'$, then the isomorphism (14.2.18) respects the action of G.

In particular, by this isomorphism, the elements of $\mathcal{H}om_{G/S}(G, T_{G/S}(\mathscr{M}))^G(S')$ (called **right invariant sections** of $T_{G_{S'}/S'}(\mathscr{M})$) corresponds to constant morphisms of $G_{S'}$ into $\mathfrak{Lie}(G_{S'}/S', \mathscr{M})$ (i.e. which factors through the projection $G_{S'} \to S'$), or to elements of $\mathfrak{Lie}(G_{S'}/S', \mathscr{M})(S') = \mathfrak{Lie}(G/S, \mathscr{M})(S')$. We then have the following proposition:

Proposition 14.2.54. *The map $\mathfrak{Lie}(G/S, \mathscr{M})(S) \to \Gamma(T_{G/S}(\mathscr{M})/G)$ which associates an element $X \in \mathfrak{Lie}(G/S, \mathscr{M})(S)$ the section $x \mapsto X(\pi(x))$ is a bijection from $\mathfrak{Lie}(G/S, \mathscr{M})(S)$ onto the set of right invariant sections of $\Gamma(T_{G/S}(\mathscr{M})/G)$.*

Similarly, we can act G on $\mathcal{E}nd_{I_S(\mathscr{M})/S}(G_{I_S(\mathscr{M})})$ as follows: for any $S' \to S$, $x \in G(S')$ and $u \in \mathcal{E}nd_{I_S(\mathscr{M})/S}(G_{I_S(\mathscr{M})})(S') = \mathrm{End}_{I_{S'}}(G_{I_{S'}(\mathscr{M})})$,

$$(u \cdot x)(g) = u(g \cdot x^{-1}) \cdot x,$$

for any $g \in G(S'')$, $S'' \to I_{S'}(\mathscr{M})$. Then the morphism of Corollary 14.2.43

$$\mathcal{H}om_{G/S}(G, T_{G/S}(\mathscr{M})) \to \mathcal{E}nd_{I_S(\mathscr{M})/S}(G_{I_S(\mathscr{M})})$$

respects the operation of G and induces for any $S' \to S$ a bijection from $\Gamma(T_{G_{S'}/S'}(\mathscr{M})/G_{S'})$ and the set of $I_{S'}(\mathscr{M})$-endomorphisms u of $G_{I_{S'}(\mathscr{M})}$ inducing the identity on G and are invariant under right translations, i.e. satisfies $u_{S''} \cdot x = u_{S''}$ for any $S'' \to S'$ and $x \in G(S'')$. By Proposition 14.2.44, we then conclude the following theorem:

Proposition 14.2.55. *There exists a bijection (functorail on G) from the set $\mathfrak{Lie}(G/S, \mathscr{M})(S)$ to the set of $I_S(\mathscr{M})$-endomorphisms of $G_{I_S(\mathscr{M})}$ inducing the identity on G and commutes with right translations of G, and this is a group isomorphism if G/S satisfies condition (E).*

By considering the case $\mathscr{M} = \mathscr{O}_S$, we thus obtain the classical definitions of the Lie algebra of a group.

Before going further, let us establish some new corollaries of Proposition 14.2.36. Let X, Y be over Z and Z be over S, as in 14.2.1.1. As we have seen in Proposition 14.2.36, the isomorphisms (14.2.2):

$$\begin{tikzcd}
\mathcal{H}om_S(I_S(\mathscr{M}), \mathcal{H}om_{Z/S}(X, Y)) \arrow[rr, "\cong"] \arrow[dr, "\cong"'] & & \mathcal{H}om_{Z/S}(X, \mathcal{H}om_Z(I_Z(\mathscr{M}), Y)) \\
& \mathcal{H}om_{Z/S}(X \times_S I_S(\mathscr{M}), Y) \arrow[ur, "\cong"'] &
\end{tikzcd} \tag{14.2.19}$$

induces the isomorphism θ below

$$\begin{tikzcd}
T_{\mathcal{H}om_{Z/S}(X,Y)}(\mathscr{M}) \arrow[rr, "\cong", "\theta"'] \arrow[dr, "\cong"'] & & \mathcal{H}om_{Z/S}(X, T_{Y/Z}(\mathscr{M})) \\
& \mathcal{H}om_{Z/S}(X \times_S I_S(\mathscr{M}), Y) \arrow[ur, "\cong"'] &
\end{tikzcd} \tag{14.2.20}$$

By Remark 14.2.2, if Y is a Z-group, so is $\mathcal{H}om_Z(V, Y)$ for any $V \to Z$ (in particular for $V = I_Z(\mathscr{M})$); explicitly, if $Z'' \to Z' \to Z$ and $\phi, \psi \in \mathrm{Hom}_Z(V_{Z'}, Y)$, then $\phi \cdot \psi$ is defined by

$$(\phi \cdot \psi)(v) = \phi(v)\psi(v)$$

for any $v \in V_{Z'}(Z'')$.

Definition 14.2.56. Suppose that X and Y are Z-groups. Let $\mathcal{H}om_{(Z/S)\text{-}\mathbf{Grp}}(X,Y)$ be the sub-functor of $\mathcal{H}om_{Z/S}(X,Y)$ defined as follows: for any $S' \to S$,

$$\mathcal{H}om_{(Z/S)\text{-}\mathbf{Grp}}(X,Y)(S') = \mathrm{Hom}_{Z_{S'}\text{-}\mathbf{Grp}}(X_{S'}, Y_{S'}). \tag{14.2.21}$$

This definition applies equally if we replace Y by the Z-group $T_{Y/Z}(\mathscr{M})$.

We then easily see that $T_{\mathcal{H}om_{(Z/S)\text{-}\mathbf{Grp}}(X,Y)/S}(\mathscr{M})(S')$ corresponds, under the isomorphisms of (14.2.20), to $Z_{S'}$-morphisms $\phi : X_{S'} \times_{S'} I_{S'}(\mathscr{M}) \to Y_{S'}$ which is multiplicative on X, that is, which satisfies $\phi(x_1x_2, m) = \phi(x_1, m)\phi(x_2, m)$, and these correspond to $Z_{S'}$-group morphisms $X_{S'} \to T_{Y/Z}(\mathscr{M})_{S'}$. We then obtain the following:

Proposition 14.2.57. *Let X, Y be Z-groups and Z be over S. We have an isomorphism of S-functors, functorial on $\mathscr{M}$:*

$$T_{\mathcal{H}om_{(Z/S)\text{-}\mathbf{Grp}}(X,Y)}(\mathscr{M}) \xrightarrow{\sim} \mathcal{H}om_{(Z/S)\text{-}\mathbf{Grp}}(X, T_{Y/Z}(\mathscr{M})).$$

In particular, for $Z = S$, we obtain the following corollary. Before stating it, we note that if Y is an abelian S-group, then so is $T_{Y/S}(\mathscr{M})$, and hence $H = \mathrm{Hom}_{S\text{-}\mathbf{Grp}}(X,Y)$ and $\mathrm{Hom}_{S\text{-}\mathbf{Grp}}(X, T_{Y/S}(M))$, and finally is $T_{H/S}(\mathscr{M})$.

Corollary 14.2.58. *Let X, Y be S-groups. We have an isomorphism of S-functors, functorial on $\mathscr{M}$:*

$$T_{\mathcal{H}om_{S\text{-}\mathbf{Grp}}(X,Y)/S}(\mathscr{M}) \xrightarrow{\sim} \mathcal{H}om_{S\text{-}\mathbf{Grp}}(X, T_{Y/S}(\mathscr{M})).$$

If Y is commutative, then this is an isomorphism of abelian S-groups.

If Y is an $\mathbb{O}_S$-module, the functor $T_{Y/S}(\mathscr{M})$ (resp. $\mathfrak{Lie}(Y/S, \mathscr{M})$) is endowed with an $\mathbb{O}_S$-module structure deduced by that of Y, which we denote by $T'_{Y/S}(\mathscr{M})$ (resp. $\mathfrak{Lie}'(Y/S, \mathscr{M})$). Therefore, if X, Y are $\mathbb{O}_S$-modules, then $T'_{Y/S}(\mathscr{M}) = \mathcal{H}om_S(I_S(\mathscr{M}), Y)$ and $H = \mathcal{H}om_{\mathbb{O}_S}(X,Y)$, and hence $\mathcal{H}om_{\mathbb{O}_S}(X, T'_{Y/S}(\mathscr{M}))$ and $T'_{H/S}(\mathscr{M})$, are endowed with $\mathbb{O}_S$-module structures, and we have:

Corollary 14.2.59. *If X, Y are $\mathbb{O}_S$-modules, we have an isomorphism of $\mathbb{O}_S$-modules, functorial on $\mathscr{M}$:*

$$T'_{\mathcal{H}om_{\mathbb{O}_S}(X,Y)/S}(\mathscr{M}) \xrightarrow{\sim} \mathcal{H}om_{\mathbb{O}_S}(X, T_{Y/S}(\mathscr{M})).$$

Definition 14.2.60. Let X, L be S-groups and X acts on L by groups automorphisms. We define the sub-functor $\mathcal{Z}^1_S(X, L)$ of $\mathcal{H}om_S(X, L)$ as follows: for any $S' \to S$, $\mathcal{Z}^1_S(X, L)(S')$ is defined to be the set

$$\{\phi \in \mathrm{Hom}_{S'}(X_{S'}, L_{S'}) : \phi(x_1x_2) = \phi(x_1)(x_1 \cdot \phi(x_2)) \text{ for any } x_1, x_2 \in X(S''),\, S'' \to S'\}.$$

The functor $\mathcal{Z}^1_S(X, L)$ is called the **functor of cross homomorphisms** from X to L.

If L is an $\mathbb{O}_S[X]$-module, then $\mathcal{Z}^1_S(X, L)$ coincides with the kernel of the differential

$$d : \mathcal{H}om_S(X, L) \to \mathcal{H}om_S(X^2, L)$$

defined in 14.1.3.1. In particular, $\mathcal{Z}^1_S(X, L)$ is an $\mathbb{O}_S$-module in this case.

Let $u : X \to Y$ be a morphism of S-groups. We have seen in Corollary 14.2.39 that we have an isomorphism of S-functors, functorial on $\mathscr{M}$:

$$T_{\mathcal{H}om_S(X,Y)/S,u}(\mathscr{M}) \xrightarrow{\sim} \mathcal{H}om_{Y/S}(X, T_{Y/S}(\mathscr{M})). \tag{14.2.22}$$

On the other hand, as Y is an S-group, we have $T_{Y/S}(\mathscr{M}) = \mathfrak{Lie}(Y/S, \mathscr{M}) \rtimes Y$, whence an isomorphism

$$\begin{aligned}\mathcal{H}om_{Y/S}(X, T_{Y/S}(\mathscr{M})) &\xrightarrow{\sim} \mathcal{H}om_{Y/S}(X, \mathfrak{Lie}(Y/S, \mathscr{M}) \rtimes Y) \\ &\xrightarrow{\sim} \mathcal{H}om_{Y/S}(X, \mathfrak{Lie}(Y, S, \mathscr{M})_Y) \\ &\xrightarrow{\sim} \mathcal{H}om_S(X, \mathfrak{Lie}(Y, S, \mathscr{M})).\end{aligned} \tag{14.2.23}$$

For any $S' \to S$, denote by $u' : X' \to Y'$ the morphism induced by u from base change. Consider the S-functor defined as follows:

$$\begin{aligned}\mathcal{H}om_{(Y/S)\text{-}\mathbf{Grp}}(X, \mathfrak{Lie}(Y/S,\mathscr{M}) \rtimes Y)(S') &= \mathrm{Hom}_{Y'\text{-}\mathbf{Grp}}(X', (\mathfrak{Lie}(Y/S,\mathscr{M}) \rtimes Y)_{S'}) \\ &= \mathrm{Hom}_{Y'\text{-}\mathbf{Grp}}(X', \mathfrak{Lie}(Y'/S',\mathscr{M}) \rtimes Y').\end{aligned}$$

The isomorphism (14.2.22) then induces an isomorphism

$$T_{\mathcal{H}om_{S\text{-}\mathbf{Grp}}(X,Y)/S,u}(\mathscr{M}) \xrightarrow{\sim} \mathcal{H}om_{(Y/S)\text{-}\mathbf{Grp}}(X, \mathfrak{Lie}(Y/S,\mathscr{M}) \rtimes Y). \tag{14.2.24}$$

The isomorphism (14.2.23) can be made explicit as follows. If $\Phi \in \mathcal{H}om_{Y/S}(X, \mathfrak{Lie}(Y/S,\mathscr{M}) \rtimes Y)$, then for any $S'' \to S' \to S$ and $x \in X(S'')$, we can write

$$\Phi(S')(x) = \phi(S')(x) \cdot u'(x) \quad \text{where} \quad \phi(S')(x) \in \mathfrak{Lie}(Y'/S',\mathscr{M})(S''),$$

which determines an element ϕ of $\mathcal{H}om_S(X, \mathfrak{Lie}(Y/S,\mathscr{M}))$. On the other hand, the composition of the morphisms

$$X \xrightarrow{u} Y \xrightarrow{\mathrm{Ad}} \mathrm{Aut}_{S\text{-}\mathbf{Grp}}(\mathfrak{Lie}(Y/S,\mathscr{M}))$$

defines an operation of X on $L = \mathfrak{Lie}(Y/S,\mathscr{M})$ by group automorphisms, and we note that $\Phi(S')$ is a group morphism if and only if for any $x_1, x_2 \in X(S'')$, we have

$$\phi(S')(x_1x_2) = \phi(S')(x_1)(u(x_1)\phi(S')(x_2)u(x_1)^{-1}) = \phi(S')(x_1)(x_1 \cdot \phi(S')(x_2)),$$

that is, if and only if $\phi \in \mathcal{Z}^1_S(X, \mathfrak{Lie}(Y/S,\mathscr{M}))$. We therefore obtain the following result:

Proposition 14.2.61. *Let $u : X \to Y$ be a morphism of S-groups. We have an isomorphism of S-functors, functorial on $\mathscr{M}$:*

$$T_{\mathcal{H}om_{S\text{-}\mathbf{Grp}}(X,Y)/S,u}(\mathscr{M}) \xrightarrow{\sim} \mathcal{Z}^1_S(X, \mathfrak{Lie}(Y/S,\mathscr{M})).$$

Suppose moreover that Y/S satisfies condition (E). Then it folows from Corollary 14.2.58, by the same proof of Corollary 14.2.37, that $\mathcal{H}om_{S\text{-}\mathbf{Grp}}(X,Y)/S$ satisfies condition (E). We then have (this also follows from Proposition 14.2.61)

$$T_{\mathcal{H}om_{S\text{-}\mathbf{Grp}}(X,Y)/S,u}(\mathscr{M} \oplus \mathscr{N}) \cong T_{\mathcal{H}om_{S\text{-}\mathbf{Grp}}(X,Y)/S,u}(\mathscr{M}) \times_S T_{\mathcal{H}om_{S\text{-}\mathbf{Grp}}(X,Y)/S,u}(\mathscr{N}).$$

Therefore, $T_{\mathcal{H}om_{S\text{-}\mathbf{Grp}}(X,Y)/S,u}(\mathscr{M})$ is endowed, as $\mathcal{Z}^1_S(X, \mathfrak{Lie}(Y/S,\mathscr{M}))$, with an $\mathbb{O}_S$-module structure, induced by functoriality on $\mathscr{M}$. We then deduce that the isomorphism Proposition 14.2.61 is an isomorphism of $\mathbb{O}_S$-modules in this case:

Proposition 14.2.62. *Let $u : X \to Y$ be a morphism of S-groups and suppose that Y/S satisfies condition (E). We have an isomorphism of $\mathbb{O}_S$-modules, functorial on $\mathscr{M}$:*

$$T_{\mathcal{H}om_{S\text{-}\mathbf{Grp}}(X,Y)/S,u}(\mathscr{M}) \xrightarrow{\sim} \mathcal{Z}^1_S(X, \mathfrak{Lie}(Y/S,\mathscr{M})).$$

Moreover, if Y/S satisfies condition (E), we deduce from Corollary 14.2.45, as the proof of Corollary 14.2.46, that for any $u \in \mathrm{Isos}_{S\text{-}\mathbf{Grp}}(X,Y)$ we have an isomorphism functorial on $\mathscr{M}$

$$T_{\mathcal{I}so_{S\text{-}\mathbf{Grp}}(X,Y)/S,u}(\mathscr{M}) \xrightarrow{\sim} T_{\mathcal{H}om_{S\text{-}\mathbf{Grp}}(X,Y)/S,u}(\mathscr{M}).$$

We then deduce the following corollaries:

Corollary 14.2.63. *Let $u : X \to Y$ be a morphism of S-groups. If Y/S satisfies condition (E), we have an isomorphism of $\mathbb{O}_S$-modules, functorial on $\mathscr{M}$:*

$$T_{\mathcal{I}so_{S\text{-}\mathbf{Grp}}(X,Y)/S,u}(\mathscr{M}) \xrightarrow{\sim} \mathcal{Z}^1_S(X, \mathfrak{Lie}(Y/S,\mathscr{M})).$$

Corollary 14.2.64. *Let X be an S-group. If X/S satisfies condition (E), we have an isomorphism of $\mathbb{O}_S$-modules, functorial on $\mathscr{M}$:*

$$\mathfrak{Lie}(\mathcal{A}ut_{S\text{-}\mathbf{Grp}}(X)/S,\mathscr{M}) \xrightarrow{\sim} \mathcal{Z}^1_S(X,\mathfrak{Lie}(X/S,\mathscr{M})).$$

If Y is abelian, then the adjoint representation of Y on $L=\mathfrak{Lie}(Y/S,\mathscr{M})$ is trivial, so we have $\mathcal{Z}^1_S(X,L)=\mathcal{H}om_{S\text{-}\mathbf{Grp}}(X,L)$. We thus have:

Corollary 14.2.65. *Let Y be an abelian S-group. We have an isomorphism of S-functors, functorial on $\mathscr{M}$:*

$$T_{\mathcal{H}om_{S\text{-}\mathbf{Grp}}(X,Y)/S,u}(\mathscr{M}) \xrightarrow{\sim} \mathcal{H}om_{S\text{-}\mathbf{Grp}}(X,\mathfrak{Lie}(Y/S,\mathscr{M})).$$

If Y/S satisfies condition (E), this is an isomorphism of $\mathbb{O}_S$-modules.

Consider now the case where X,Y are $\mathbb{O}_S$-modules. Recall that we denote by $T'_{Y/S}(\mathscr{M})$ (resp. $\mathfrak{Lie}'(Y/S,\mathscr{M})$) the functor $T_{Y/S}(\mathscr{M})$ (resp. $\mathfrak{Lie}(Y/S,\mathscr{M})$) endowed with the $\mathbb{O}_S$-module structure induced by that of Y. If Y/S satisfies condition (E), we always endow $\mathfrak{Lie}(Y/S,\mathscr{M})$ the $\mathbb{O}_S$-module structure defined by functoriality on $\mathscr{M}$. In this case, the abelian group structures of $\mathfrak{Lie}(Y/S,\mathscr{M})$ and $\mathfrak{Lie}'(Y/S,\mathscr{M})$ coincide (cf. Proposition 14.2.29), but this is in general not true for the module structures. For any $S'\to S$ and $a\in\mathbb{O}(S')$, we denote by $a\cdot' m$ (resp. $a\cdot m$) the action of a on $m\in\mathfrak{Lie}'(Y/S,\mathscr{M})(S')$ (resp. $m\in\mathfrak{Lie}(Y/S,\mathscr{M})(S')$), and similarly for the actions of a on $T'_{Y/S}(\mathscr{M})$ and $T_{Y/S}(\mathscr{M})$.

We have $T'_{Y/S}(\mathscr{M})\cong\mathfrak{Lie}'(Y/S,\mathscr{M})\oplus Y$ as $\mathbb{O}_S$-modules; therefore, we obtain, as in Corollary 14.2.65, that:

Proposition 14.2.66. *Let $u:X\to Y$ be a morphism of $\mathbb{O}_S$-modules. We have an isomorphism of S-functors, functorial on $\mathscr{M}$:*

$$T_{\mathcal{H}om_{\mathbb{O}_S}(X,Y)/S,u}(\mathscr{M}) \xrightarrow{\sim} \mathcal{H}om_{\mathbb{O}_S}(X,\mathfrak{Lie}'(Y/S,\mathscr{M})). \tag{14.2.25}$$

If Y/S satisfies condition (E), then $\mathcal{H}om_{\mathbb{O}_S}(X,Y)/S$ satisfies condition (E) and (14.2.25) is an isomorphism of $\mathbb{O}_S$-modules if we endow both sides the $\mathbb{O}_S$-module structure induced by functoriality on $\mathscr{M}$.

Remark 14.2.67. Let $u:X\to Y$ be a morphism of $\mathbb{O}_S$-modules. Denote by τ_u the map which to any morphism $\phi:X\to\mathfrak{Lie}'(Y/S,\mathscr{M})$ of $\mathbb{O}_S$-modules associates the morphism

$$\phi\oplus u: X\to T'_{Y/S}(\mathscr{M})=\mathfrak{Lie}'(Y/S,\mathscr{M})\oplus Y.$$

Then the isomorphism of Proposition 14.2.66 fits into the following diagram, functorial on $\mathscr{M}$:

$$\begin{tikzcd}
T_{\mathcal{H}om_{\mathbb{O}_S}(X,Y)/S,u} \arrow[r, "\cong"] \arrow[d] & \mathcal{H}om_{\mathbb{O}_S}(X,\mathfrak{Lie}'(Y/S,\mathscr{M})) \arrow[d, hook, "\tau_u"] \\
T_{\mathcal{H}om_{\mathbb{O}_S}(X,Y)/S}(\mathscr{M}) \arrow[r, "\cong"] & \mathcal{H}om_{\mathbb{O}_S}(X,T'_{Y/S}(\mathscr{M}))
\end{tikzcd}$$

Moreover, if Y/S satisfies condition (E), we deduce from Corollary 14.2.45, as the proof of Corollary 14.2.46, that for any $u\in\mathrm{Iso}_{\mathbb{O}_S}(X,Y)$, we have

$$T_{\mathcal{I}so_{\mathbb{O}_S}(X,Y)/S}(\mathscr{M})\cong T_{\mathcal{H}om_{\mathbb{O}_S}(X,Y)/S}(\mathscr{M}). \tag{14.2.26}$$

Corollary 14.2.68. *Let X be an $\mathbb{O}_S$-module satisfying condition (E) relative to S. We have an isomorphism, functorial on $\mathscr{M}$:*

$$\mathfrak{Lie}(\mathcal{A}ut_{\mathbb{O}_S}(X)/S,\mathscr{M}) \xrightarrow{\sim} \mathcal{H}om_{\mathbb{O}_S}(X,\mathfrak{Lie}'(X/S,\mathscr{M}))$$

which respects the $\mathbb{O}_S$-module structure induced by functoriality on $\mathscr{M}$. In particular, $\mathcal{A}ut_{\mathbb{O}_S}(X)/S$ satisfies condition (E).

Proof. The first assertion follows from (14.2.25) and (14.2.26). For the second one, as X/S satisfies condition (E), we have an isomorphism of $\mathbb{O}_S$-modules $\mathfrak{Lie}'(X/S, \mathscr{M} \oplus \mathscr{N}) \cong \mathfrak{Lie}'(X/S, \mathscr{M}) \times_S \mathfrak{Lie}'(X/S, \mathscr{N})$, and hence

$$\mathfrak{Lie}(\mathcal{A}ut_{\mathbb{O}_S}(X)/S, \mathscr{M} \oplus \mathscr{N}) \cong \mathfrak{Lie}(\mathcal{A}ut_{\mathbb{O}_S}(X)/S, \mathscr{M}) \times_S \mathfrak{Lie}(\mathcal{A}ut_{\mathbb{O}_S}(X)/S, \mathscr{N}).$$

In view of the sequence (14.2.17), this proves that $\mathcal{A}ut_{\mathbb{O}_S}(X)/S$ satisfies condition (E). □

Before going further towards this direction, let us take a closer look at the relations between Y, $\mathfrak{Lie}(Y/S)$ and $\mathfrak{Lie}'(Y/S)$. We first notice that (cf. Remark 14.2.14)

$$\mathfrak{Lie}(\mathbb{O}_S/S, \mathscr{M}) = \mathfrak{Lie}'(\mathbb{O}_S/S, \mathscr{M}) = \Gamma_{\mathscr{M}} \tag{14.2.27}$$

and that we have a canonical isomorphism

$$d : \mathbb{O}_S \xrightarrow{\sim} \mathfrak{Lie}(\mathbb{O}_S/S). \tag{14.2.28}$$

Now let F be an $\mathbb{O}_S$-module. For any $S_2 \to S_1 \to S$, we have a bihomomorphism

$$F(S_1) \to F(S_2), \quad \mathbb{O}(S_1) \to \mathbb{O}(S_2), \tag{14.2.29}$$

whence a morphism of $\mathbb{O}(S_2)$-modules

$$F(S_1) \otimes_{\mathbb{O}(S_1)} \mathbb{O}(S_2) \to F(S_2).$$

In particular, for $S_1 = S'$ and $S_2 = I_{S'}(\mathscr{M})$, we deduce a morphism of $\mathbb{O}(S')$-modules, functorial on $\mathscr{M}$

$$F(S') \otimes_{\mathbb{O}(S')} T_{\mathbb{O}_S/S}(\mathscr{M})(S') \to T'_{F/S}(\mathscr{M})(S').$$

With S' varies, we obtain morphisms of $\mathbb{O}_S$-modules, functorial on $\mathscr{M}$:

$$F \otimes_{\mathbb{O}_S} T_{\mathbb{O}_S/S}(\mathscr{M}) \to T'_{F/S}(\mathscr{M}). \tag{14.2.30}$$

These morphisms are functorial on $\mathscr{M}$, hence compatible with the projections of tangent bundles onto their bases; they then define morphisms of $\mathbb{O}_S$-modules

$$F \otimes_{\mathbb{O}_S} \mathfrak{Lie}(\mathbb{O}_S/S, \mathscr{M}) \to \mathfrak{Lie}'(F/S, \mathscr{M}) \tag{14.2.31}$$

such that the following diagram is commutative:

$$\begin{tikzcd}
0 \arrow[r] & F \otimes_{\mathbb{O}_S} \mathfrak{Lie}(\mathbb{O}_S/S, \mathscr{M}) \arrow[r] \arrow[d] & F \otimes_{\mathbb{O}_S} T_{\mathbb{O}_S/S}(\mathscr{M}) \arrow[r] \arrow[d] & F \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathfrak{Lie}'(F/S, \mathscr{M}) \arrow[r] & T'_{F/S}(\mathscr{M}) \arrow[r] & F \arrow[r] & 0
\end{tikzcd}$$

We can consider the morphisms (14.2.31) as morphisms of abelian S-groups:

$$F \otimes_{\mathbb{O}_S} \mathfrak{Lie}(\mathbb{O}_S/S, \mathscr{M}) \to \mathfrak{Lie}(F/S, \mathscr{M}). \tag{14.2.32}$$

By tensoring F with the isomorphism $d : \mathbb{O}_S \xrightarrow{\sim} \mathfrak{Lie}(\mathbb{O}_S/S)$, we then deduce (for $\mathscr{M} = \mathscr{O}_S$) a morphism of abelian S-groups

$$d : F \xrightarrow{\sim} F \otimes_{\mathbb{O}_S} \mathfrak{Lie}(\mathbb{O}_S/S) \to \mathfrak{Lie}(F/S). \tag{14.2.33}$$

Remark 14.2.69. If F/S satisfies condition (E), the morphisms (14.2.32) and (14.2.33) are not necessarily morphisms of $\mathbb{O}_S$-modules, if we endow both sides the module structure induced by functoriality on $\mathscr{M}$. For example, let k be a field with characteristic $p > 0$, $S = \mathrm{Spec}(k)$, and F be the $\mathbb{O}_S$-module which to any S-scheme T associates $F(T) = \Gamma(T, \mathscr{O}_T)$, endowed with the $\mathbb{O}(T)$-module structure obtained by acting a scalar via its p-th power, that is, $r \cdot f = r^p f$ for $r \in \mathbb{O}(T)$ and $f \in F(T)$. As an S-group, F is isomorphic to $\mathbb{G}_{a,S}$, so F satisfies condition (E) and $\mathfrak{Lie}(F/S)$ is identified with $\mathfrak{Lie}(\mathbb{G}_{a,S}/S) \cong \mathbb{O}_S$. Then, the morphism $d : F \to \mathfrak{Lie}(F/S)$ is, for any $T \to S$, the identity map $F(T) \to \mathbb{O}(T)$; it respects the abelian group structure, but not the $\mathbb{O}_S$-module structure.

Remark 14.2.70. We can explicit the morphisms (14.2.30) and (14.2.31) as follows. The morphism $\Theta : F \otimes_{\mathbb{O}_S} T_{\mathbb{O}_S/S}(\mathscr{M}) \to T'_{F/S}(\mathscr{M}) = \mathcal{H}om_S(I_S(\mathscr{M}), F)$ is defined so that for any $S' \to S$, $\alpha \in \mathbb{O}(I_{S'}(\mathscr{M}))$, and $f : S' \to F$,

$$\Theta(f \otimes \alpha) = \alpha(\tau_0 \circ f) = \alpha \cdot (f \circ \rho)$$

where $\tau_0 : F \to T'_{F/S}(\mathscr{M})$ is the zero section and $\rho : I_{S'}(\mathscr{M}) \to S'$ is the structural morphism.

Definition 14.2.71. We say that F is a **good** $\mathbb{O}_S$**-module** if the morphisms $F \otimes_{\mathbb{O}_S} T_{\mathbb{O}_S/S}(\mathscr{M}) \to T_{F/S}(\mathscr{M})$ (or equivalently, the morphisms $F \otimes_{\mathbb{O}_S} \mathfrak{Lie}(\mathbb{O}_S/S, \mathscr{M}) \to \mathfrak{Lie}(F/S, \mathscr{M})$) are isomorphisms of abelian S-groups (so that F/S satisfies condition (E)) and if moreover they respect the $\mathbb{O}_S$-module structures induced by functoriality on $\mathscr{M}$.

Proposition 14.2.72. *Let F be an $\mathbb{O}_S$-module. Consider the following conditions:*

(i) *F is a good $\mathbb{O}_S$-module.*

(ii) *F/S satisfies condition (E) and $d : F \to \mathfrak{Lie}(F/S)$ is an isomorphism of $\mathbb{O}_S$-modules.*

(iii) *$\mathfrak{Lie}(F/S, \mathscr{M}) = \mathfrak{Lie}'(F/S, \mathscr{M})$.*

Then we have (i)⇔(ii)⇒(iii).

Proof. The implication (i)⇒(ii) follows from definition. To see that (ii)⇒(ii), it suffices to show that the morphisms of abelian S-groups

$$F \otimes_{\mathbb{O}_S} \mathfrak{Lie}(\mathbb{O}_S/S, \mathscr{M}) \xrightarrow{\sim} \mathfrak{Lie}(F/S, \mathscr{M})$$

are isomorphisms of $\mathbb{O}_S$-modules. As F/S satisfies condition (E), the two members transform finite direct sums of copies of $\mathscr{O}_S$ into finite products of abelian S-groups. We are then reduced to the case where $\mathscr{M} = \mathscr{O}_S$, which follows by the hypothesis.

Finally, (i)⇒(iii) follows from the definition and the fact that the isomorphisms

$$F \otimes_{\mathbb{O}_S} \mathfrak{Lie}(\mathbb{O}_S/S, \mathscr{M}) \xrightarrow{\sim} \mathfrak{Lie}'(F/S, \mathscr{M})$$

of (14.2.31) is an isomorphism of $\mathbb{O}_S$-modules. □

Example 14.2.73. For any quasi-coherent $\mathscr{O}_S$-module $\mathscr{E}$, the $\mathbb{O}_S$-module $\Gamma_{\mathscr{E}}$ and $\check{\Gamma}_{\mathscr{E}}$ are good. In fact, for any $f : S' \to S$, the morphisms

$$\Gamma_{\mathscr{E}}(S') \otimes_{\mathbb{O}(S')} \mathbb{O}(I_{S'}(\mathscr{M})) \to T_{\Gamma_{\mathscr{E}}/S}(\mathscr{M})(S')$$
$$\check{\Gamma}_{\mathscr{E}}(S') \otimes_{\mathbb{O}(S')} \mathbb{O}(I_{S'}(\mathscr{M})) \to T_{\check{\Gamma}_{\mathscr{E}}/S}(\mathscr{M})(S')$$

correspond, respectively, to morphisms

$$\Gamma(S', f^*(\mathscr{E})) \otimes_{\mathbb{O}(S')} \Gamma(S', \mathscr{D}_{\mathscr{O}_{S'}}(\mathscr{M})) \to \Gamma(S', f^*(\mathscr{E}) \otimes_{\mathscr{O}_{S'}} \mathscr{D}_{\mathscr{O}_{S'}}(\mathscr{M})),$$
$$\mathrm{Hom}_{\mathscr{O}_{S'}}(f^*(\mathscr{E}), \mathscr{O}_{S'}) \otimes_{\mathbb{O}(S')} \Gamma(S', \mathscr{D}_{\mathscr{O}_{S'}}(\mathscr{M})) \to \mathrm{Hom}_{\mathscr{O}_{S'}}(f^*(\mathscr{E}), \mathscr{D}_{\mathscr{O}_{S'}}(\mathscr{M}));$$

which are both isomorphisms since $\mathscr{D}_{\mathscr{O}_{S'}}(\mathscr{M})$ is isomorphic, as $\mathscr{O}_{S'}$-module, to a finite direct sum of copies of $\mathscr{O}_{S'}$ (recall that $\mathscr{M}$ is assumed to be free).

Proposition 14.2.74. *Let F be a good $\mathbb{O}_S$-module. Then $\mathcal{A}ut_{\mathbb{O}_S}(F)/S$ satisfies condition (E) and we have a isomorphism (functorial on $\mathscr{M}$)*

$$\mathfrak{Lie}(\mathcal{A}ut_{\mathbb{O}_S}(F)/S,\mathscr{M}) \xrightarrow{\sim} \mathcal{H}om_{\mathbb{O}_S}(F,\mathfrak{Lie}(F/S,\mathscr{M}))$$

which respects the $\mathbb{O}_S$ induced by the functoriality on $\mathscr{M}$. In particular, we have an isomorphism of $\mathbb{O}_S$-modules

$$\mathfrak{Lie}(\mathcal{A}ut_{\mathbb{O}_S}(F)/S) \xrightarrow{\sim} \mathcal{E}nd_{\mathbb{O}_S}(F).$$

Morover, $\mathcal{E}nd_{\mathbb{O}_S}(F)$ is a good $\mathbb{O}_S$-module.

Proof. In fact, by Proposition 14.2.72, F/S satisfies condition (E) and

$$\mathfrak{Lie}(F/S,\mathscr{M}) = \mathfrak{Lie}'(F/S,\mathscr{M}) \cong F\otimes_{\mathbb{O}_S}\mathfrak{Lie}(\mathbb{O}_S/S,\mathscr{M}). \tag{14.2.34}$$

The first assertion then follows from Corollary 14.2.68. Put $E=\mathcal{E}nd_{\mathbb{O}_S}(F)$; by (14.2.34) and ([?] remarque 4.3.5), we have the following commutative diagram of abelian groups

$$\begin{tikzcd}
\mathcal{E}nd_{\mathbb{O}_S}(F)\otimes_{\mathbb{O}_S}\mathfrak{Lie}(\mathbb{O}_S/S,\mathscr{M}) \arrow[r, "d_E"] \arrow[d, equal] & \mathfrak{Lie}(\mathcal{E}nd_{\mathbb{O}_S}(F)/S,\mathscr{M}) \\
\mathcal{H}om_{\mathbb{O}_S}(F,F\otimes_{\mathbb{O}_S}\mathfrak{Lie}(\mathbb{O}_S/S,\mathscr{M})) \arrow[r, "d_F"] & \mathcal{H}om_{\mathbb{O}_S}(F,\mathfrak{Lie}(\mathcal{E}nd_{\mathbb{O}_S}(F)/S,\mathscr{M})) \arrow[u, "\cong"', "(*)"]
\end{tikzcd}$$

where d_F and $(*)$ are isomorphisms of $\mathbb{O}_S$-modules; therefore, so is d_E, and this proves the proposition. ☐

Remark 14.2.75. Put $\mathscr{O}_{I_S}=\mathscr{O}_S\oplus t\mathscr{O}_S$ (with $t^2=0$) and let F be a good $\mathbb{O}_S$-module. Then, for any $S'\to S$, the morphism

$$F(S')\oplus tF(S') = F(S')\otimes_{\mathbb{O}(S')}\mathbb{O}(I_{S'}) \to F(I_{S'}) = F(S')\oplus\mathfrak{Lie}(F/S)(S')$$

(which is the identity on $F(S')$) induces an isomorphism of $\mathbb{O}(S')$-modules $tF(S')\cong\mathfrak{Lie}(F/S)(S')$. By varying S', we then obtain an isomorphism $\mathfrak{Lie}(F/S)\cong tF$. For any $S'\to S$, we have, by Proposition 14.2.74, a commutative diagram

$$\begin{tikzcd}
\mathrm{End}_{\mathbb{O}_{S'}}(F_{S'}) \arrow[r, "\cong"] & \mathrm{Hom}_{\mathbb{O}_{S'}}(F_{S'},tF_{S'}) \arrow[r, "\cong"] \arrow[d, hook] & \mathfrak{Lie}(\mathcal{A}ut_{\mathbb{O}_S}(F)/S)(S') \arrow[d, hook] \\
& \mathrm{Aut}_{\mathbb{O}(I_{S'})}(F_{I_{S'}}) \arrow[r, equal] & T_{\mathcal{A}ut_{\mathbb{O}_S}(F)/S}(S')
\end{tikzcd}$$

and we deduce from Remark 14.2.67 that any $X\in\mathrm{End}_{\mathbb{O}_{S'}}(F_{S'})$ corresponds to the element $\mathrm{id}+tX$ of $\mathrm{Aut}_{\mathbb{O}_{I_{S'}}}(F_{I_{S'}})$.

We say that the S-group G is **good** if G/S satisfies condition (E) and $\mathfrak{Lie}(G/S)$ is a good $\mathbb{O}_S$-module. Note that if F is a good $\mathbb{O}_S$-module, it is also a good S-groups: in fact, F/S satisfies condition (E) and $\mathfrak{Lie}(F/S)\cong F$ (cf. Proposition 14.2.72 (ii)) is a good $\mathbb{O}_S$-module.

Example 14.2.76. If G is representable, then it is good. In fact, G/S satisfies condition (E) and $\mathfrak{Lie}(G/S)$ is of the form $\mathbb{V}(\mathscr{E})$ by Proposition 14.2.15, hence good by Example 14.2.73.

Lemma 14.2.77. *Let G be an S-group such that G/S satisfies condition (E), and $F=\mathfrak{Lie}(G/S)$. Then F/S satisfies condition (E) and the abelian group morphism $d:F\to\mathfrak{Lie}(F/S)$ respects the $\mathbb{O}_S$-module structure. Therefore, G is good if and only $d:F\to\mathfrak{Lie}(F/S)$ is bijective.*

Proof. ☐

Theorem 14.2.78. *If F is a good $\mathbb{O}_S$-module, the S-group $\mathcal{A}ut_{\mathbb{O}_S}(F)$ is good.*

Proof. In fact, by Proposition 14.2.74, $\mathcal{A}ut_{\mathbb{O}_S}(F)/S$ satisfies condition (E) and $\mathfrak{Lie}(\mathcal{A}ut_{\mathbb{O}_S}(F)/S) \cong \mathcal{E}nd_{\mathbb{O}_S}(F)$ is a good $\mathbb{O}_S$-module. □

Example 14.2.79. Let F be the $\mathbb{O}_S$-module defined in Remark 14.2.69. Then, the canonical morphism $d : F \to \mathfrak{Lie}(F/S)$ is, for any $T \to S$, the identity morphism $F(T) \to \mathbb{O}(T)$. It respects the abelian group structure, but not the module structure, so F is not good.

Let G be an S-group and F be a good $\mathbb{O}_S$-module. Suppose that we are given a linear representation of G on F, that is, an S-group morphism

$$\rho : G \to \mathcal{A}ut_{\mathbb{O}_S}(F).$$

If G/S satisfies condition (E), we deduce from Proposition 14.2.74 and Proposition 14.2.25 a morphism of $\mathbb{O}_S$-modules

$$d\rho : \mathfrak{Lie}(G/S) \to \mathfrak{Lie}(\mathcal{A}ut_{\mathbb{O}_S}(F)/S) \cong \mathcal{E}nd_{\mathbb{O}_S}(F).$$

Moreover, put $\mathscr{O}_{I_S} = \mathscr{O}_S \oplus t\mathscr{O}_S$ (with $t^2 = 0$), we deduce from Remark 14.2.75 that, if $S' \to S$ and $X \in \mathfrak{Lie}(G/S)(S') \subseteq G(I_{S'})$, then we have the following equality in $\mathrm{Aut}_{\mathbb{O}_{I_{S'}}}(F_{I_{S'}})$:

$$\rho(X) = \mathrm{id} + td\rho(X), \tag{14.2.35}$$

i.e. for any $S'' \to I_{S'}$ and $f \in F(S')$, we have $\rho(X)(f) = f + td\rho(X)(f)$ in $F(S'')$.

Let G be a good S-group. Then $\mathfrak{Lie}(G/S)$ is a good $\mathbb{O}_S$-module, and we have a morphism of S-groups

$$\mathrm{Ad} : G \to \mathcal{A}ut_{\mathbb{O}_S}(\mathfrak{Lie}(G/S)).$$

We then deduce a morphism of $\mathbb{O}_S$-modules

$$\mathrm{ad} : \mathfrak{Lie}(G/S) \to \mathcal{E}nd_{\mathbb{O}_S}(\mathfrak{Lie}(G/S)),$$

or equivalently, an $\mathbb{O}_S$-bilinear morphism

$$\mathfrak{Lie}(G/S) \times_S \mathfrak{Lie}(G/S) \to \mathfrak{Lie}(G/S), \quad (x, y) \mapsto [x, y] := \mathrm{ad}(x) \cdot y$$

where x, y denote elements of $\mathfrak{Lie}(G/S)(S') = \mathfrak{Lie}(G_{S'}/S')(S')$. If G is commutative, then the action Ad is trivial, and we have $[x, y] = 0$.

Remark 14.2.80. We can give an equivalent definition of the bracket: note first that it suffices to do this for $x, y \in \mathfrak{Lie}(G/S)(S)$. We then note that there is a canonical isomorphism $I_S \times_S I_S \cong I_{I_S}$; to avoid confusions, we denote by I and I' the two copies of I_S and put $\mathscr{O}_I = \mathscr{O}_S[t]$, $\mathscr{O}_{I'} = \mathscr{O}_S[t']$, where $t^2 = t'^2 = 0$. We then have a commutative diagram

$$\begin{array}{ccc} I \times I' & \longrightarrow & I' \\ \downarrow & & \downarrow \\ I & \longrightarrow & S \end{array}$$

(the two arrows from $I \times I'$ identifying it as the dual number scheme over I or over I'), which gives a commutative diagram of abelian groups (where $L = \mathfrak{Lie}(G/S)$) by (14.2.17):

$$\begin{tikzcd} & 1 \arrow[d] & 1 \arrow[d] & \\ & L(I) \arrow[r] \arrow[d] & L(S) \arrow[r] \arrow[d] & 1 \\ 1 \arrow[r] & L(I') \arrow[r] \arrow[d] & G(I \times I') \arrow[r] \arrow[d] & G(I') \arrow[r] \arrow[d] & 1 \\ 1 \arrow[r] & L(S) \arrow[r] \arrow[d] & G(I) \arrow[r] \arrow[d] & G(S) \arrow[r] \arrow[d] & 1 \\ & 1 & 1 & 1 \end{tikzcd} \tag{14.2.36}$$

The ninith vertex of this diagram is none other than $\mathfrak{Lie}(L/S)(S)$. If G is good, this is isomorphic to $L(S)$ and we then have the following diagram, where the rows and columns are exact sequences of groups and in view of the identification $L(I) = L(S) \oplus tL(S)$ (resp. $L(I') = L(S) \oplus t'L(S)$), the injection $L(S) \hookrightarrow L(I)$ (resp. $L(S) \hookrightarrow L(I')$) is given by $u \mapsto tu$ (resp. $u \mapsto t'u$):

$$\begin{tikzcd} L(S) \arrow[r, "t"] \arrow[d, "t'"] & L(I) \arrow[r] \arrow[d] & L(S) \arrow[d] \\ L(I') \arrow[r] \arrow[d] & G(I \times I') \arrow[r] \arrow[d] & G(I') \arrow[d] \\ L(S) \arrow[r] & G(I) \arrow[r] & G(S) \end{tikzcd} \tag{14.2.37}$$

Now in this diagram, if we take two elements x and y in $L(S)$ and choose arbitrarily element $\tilde{x} \in L(I)$ (resp. $\tilde{y} \in L(I')$) which maps to x (resp. to y), then the commutator $\tilde{x}\tilde{y}\tilde{x}^{-1}\tilde{y}^{-1}$ in $G(I \times I')$ does not depend on the choice of $\tilde{x}$ and $\tilde{y}$, and it is the image of an element $z \in L(S)$. In fact, if we identify x with its image under the canonical section $L(S) \to L(I)$ (and similarly for y), then $\tilde{x} = xu$ and $\tilde{y} = yv$, with $u, v \in L(S) = L(I) \cap L(I')$, and since $L(I)$, $L(I')$ are abelian, we have

$$\tilde{x}\tilde{y}\tilde{x}^{-1}\tilde{y}^{-1} = xuyvu^{-1}x^{-1}v^{-1}y^{-1} = xuyu^{-1}vx^{-1}v^{-1}y^{-1} = xyx^{-1}y^{-1}.$$

Moreover, this element is send to the unit element of $G(I)$ and of $G(I')$, hence comes from an element $z \in L(S)$. Finally, consider y (resp. x) as element of $L(I')$ (resp. $L(S) \subseteq G(I')$), by (14.2.35) we have

$$xyx^{-1} = \operatorname{Ad}(x)(y) = (\mathrm{id} + t'\operatorname{ad}(x))(y) = y + t'[x, y],$$

so the element $xyx^{-1}y^{-1}$ of $L(I')$ is the iamge of $z = [x, y] \in L(S)$.

From the above construction, we see that the bracket has the following properties:

(i) The bracket is functorial on G: more precisely, $G \mapsto \mathfrak{Lie}(G/S)$ is a functor from the category of good S-groups to the category of good $\mathcal{O}_S$-modules endowed with an $\mathcal{O}_S$-bilinear composition law.

(ii) We have $[x, y] + [y, x] = 0$. In fact, the diagram is symmetric, and by exchanging x and y we are considering the element $\tilde{y}\tilde{x}\tilde{y}^{-1}\tilde{x}^{-1}$, which is the inverse of $\tilde{x}\tilde{y}\tilde{x}^{-1}\tilde{y}^{-1}$.

Proposition 14.2.81. *Let F be a good $\mathbb{O}_S$-module. Via the identification $\mathfrak{Lie}(\mathcal{A}ut_{\mathbb{O}_S}(F)/S) = \mathcal{E}nd_{\mathbb{O}_S}(F)$, we have*

$$\mathrm{Ad}(g) \cdot Y = g \circ Y \circ g^{-1}, \quad [X, Y] = X \circ Y - Y \circ X,$$

for any $S' \to S$, $g \in \mathrm{Aut}_{\mathbb{O}_S}(F_{S'})$ and $X, Y \in \mathfrak{Lie}(\mathcal{A}ut_{\mathbb{O}_S}(F)/S)(S') = \mathrm{End}_{\mathbb{O}_S}(F_{S'})$.

Proof. By base change, we can assume that $S' = S$, which makes it possible to simplify the notations. Put $I = I_S$ and $\mathscr{O}_I = \mathscr{O}_S[t]$ (with $t^2 = 0$). Recall that the inclusion $i : \mathrm{End}_{\mathbb{O}_S}(F) \hookrightarrow \mathcal{A}ut_{\mathbb{O}_I}(F_I)$ sends Y to $\mathrm{id} + tY$, so by the definition of $\mathrm{Ad}(g)$, we have

$$\mathrm{id} + t\mathrm{Ad}(g)(Y) = g \circ (\mathrm{id} + tY) \circ g^{-1} = \mathrm{id} + t(g \circ Y \circ g^{-1}),$$

whence $\mathrm{Ad}(g)(Y) = g \circ Y \circ g^{-1}$.

Let I' be a second copy of I_S, and put $\mathscr{O}_{I'} = \mathscr{O}_S[t']$ (with $t'^2 = 0$). Apply the result of Remark 14.2.80 to $G = \mathcal{A}ut_{\mathbb{O}_S}(F)$ and $L = \mathfrak{Lie}(G/S) = \mathcal{A}ut_{\mathbb{O}_S}(F)$, where we identify X with its image under the canonical section $L(S) \hookrightarrow L(I)$; its image in $G(I \times I')$ is then $\mathrm{id} + t'X$, hence the inverse is $\mathrm{id} - t'X$. Similarly, Y is send to $\mathrm{id} + tY$, so the inverse is $\mathrm{id} - tY$. Then the commutator

$$(\mathrm{id} + t'X) \circ (\mathrm{id} + tY) \circ (\mathrm{id} - t'X) \circ (\mathrm{id} - tY) = \mathrm{id} + tt'(X \circ Y - Y \circ X)$$

is the image of $Z = X \circ Y - Y \circ X$ in $G(I \times I')$ (in fact, Z is send to $tZ \in L(I)$, hence to $\mathrm{id} + tt'Z \in G(I \times I')$). By Remark 14.2.80, we conclude that $[X, Y] = X \circ Y - Y \circ X$. □

Corollary 14.2.82. *Let G be a good S-group and $x, y, z \in \mathfrak{Lie}(G/S)(S')$. We have*

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0.$$

Proof. In fact, as G is good, $\mathfrak{Lie}(G/S)$ is a good $\mathbb{O}_S$-module and hence, by Theorem 14.2.78, $\mathcal{A}ut_{\mathbb{O}_S}(\mathfrak{Lie}(G/S))$ is a good S-group. Then, the morphism of S-groups

$$\mathrm{Ad} : G \to \mathcal{A}ut_{\mathbb{O}_S}(\mathfrak{Lie}(G/S))$$

gives by functoriality $\mathrm{ad}([x, y]) = [\mathrm{ad}(x), \mathrm{ad}(y)]$. Combined with Proposition 14.2.81, this shows that

$$\mathrm{ad}([x, y]) = [\mathrm{ad}(x), \mathrm{ad}(y)] = \mathrm{ad}(x) \circ \mathrm{ad}(y) - \mathrm{ad}(y) \circ \mathrm{ad}(x),$$

which implies the Jacobi identity by applied to an element z. □

Corollary 14.2.83. *Let G be a good S-group linearly acted on a good $\mathbb{O}_S$-module F (i.e. F is an $\mathbb{O}_S[G]$-module, G and F being good). Then the linear map $d\rho : \mathfrak{Lie}(G/S) \to \mathcal{E}nd_{\mathbb{O}_S}(F)$ is a representation, that is, we have*

$$d\rho([x, y]) = d\rho(x) \circ d\rho(y) - d\rho(y) \circ d\rho(x).$$

Proof. This follows from the functoriality of bracket and Proposition 14.2.81. □

To any good S-group (for example representable), we have associated a good $\mathbb{O}_S$-module $\mathfrak{Lie}(G/S)$ endowed functorially a bilinear map verifying

$$[x, y] + [y, x] = 0, \quad [x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0.$$

We therefore say that $\mathfrak{Lie}(G/S)$, endowed with this structure, is the **quasi-Lie algebra** of G over S. For any linear representation of G over a good $\mathbb{O}_S$-module F, we can associate a representation of the quasi-Lie algebra $\mathfrak{Lie}(G/S)$. In particular, the adjoint representation of G is associated to the adjoint representation of the quasi-Lie algebra.

Example 14.2.84. A group functor G over S is called very good if it is good and $\mathfrak{Lie}(G/S)$ is a Lie algebra over $\mathbb{O}_S$ (that is, if we have the identity $[x,x]=0$). The following S-groups are very good: $\mathcal{A}ut_{\mathbb{O}_S}(F)$ for any good $\mathbb{O}_S$-module F (cf. Proposition 14.2.81 and Corollary 14.2.82), any representable group (see below), any good S-group admitting a monomorphism into a very good S-group (cf. Proposition 14.2.23), for example any good subgroup of a very good representable group, or any good S-group admitting a faithful representation over a good $\mathbb{O}_S$-module, for example any good S-group such that Ad is faithful.

Now suppose that G is a group scheme. By Proposition 14.2.55, $\mathfrak{Lie}(G/S)(S)$ is identified with right invariant infinitesimal automorphisms of G, hence by (14.2.16) with derivations of $\mathscr{O}_G$ over $\mathscr{O}_S$ invariant under right translations. Moreover, this identification respects the module structure and is an *anti-isomorphism* of Lie algebras: put $\mathscr{O}_I=\mathscr{O}_S[t]$ and $\mathscr{O}_{I'}=\mathscr{O}_S[t']$ and let $x\in L(I)$ and $y\in L(I')$. The left translation λ_x (resp. λ_y) is an S-automorphism of $G_{I\times I'}$ which induces the identity on $G_{I'}$ (resp. G_I) and which corresponds to an $\mathscr{O}_S$-automorphism

$$u=\mathrm{id}+td_x,\qquad (\text{resp.}\quad v=\mathrm{id}+t'd_y)$$

of $\mathscr{O}_{G_{I\times I'}}=\mathscr{O}_G[t,t']/(t^2,t'^2)$, where d_x,d_y are $\mathscr{O}_S$-derivations of $\mathscr{O}_G$ invariant under right translations. As the correspondence of S-automorphisms of $G_{I\times I'}$ and $\mathscr{O}_S$-automorphisms of $\mathscr{O}_{G_{I\times I'}}$ is contravariant, $\lambda_x\lambda_y\lambda_x^{-1}\lambda_y^{-1}$ corresponds to $v^{-1}u^{-1}vu=\mathrm{id}+tt'(d_yd_x-d_xd_y)$. We then deduce from Remark 14.2.80 that the map $x\mapsto -d_x$ is an isomorphism of Lie algebras. The preceding argument is valid for $\mathfrak{Lie}(G/S)(S')=\mathfrak{Lie}(G_{S'}/S')(S')$ for any $S'\to S$, so we recover the following classical definition:

Proposition 14.2.85. *Via the isomorphism $x\mapsto -d_x$, $\mathfrak{Lie}(G/S)$ is identified with the functor which associates any $S'\to S$ to the Lie algebra of derivations of $G_{S'}$ over S' invariant under right translations.*

As we have seen in Example 14.2.76 that any representable group is good, we conclude the following corollary:

Corollary 14.2.86. *Any representable grop is very good.*

Let $e:S\to G$ be the unit section of G. Put $\omega^1_{G/S}=e^*(\Omega^1_{G/S})$ and recall that (cf. Proposition 14.2.15) $\mathfrak{Lie}(G/S)$ is represented by the vector bundle $\mathrm{Lie}(G/S)=\mathbb{V}(\omega^1_{G/S})$. We then have assocaited functorially to any S-group scheme G a vector bundle $\mathbb{V}(\omega^1_{G/S})$ over S, which represents the functor $\mathfrak{Lie}(G/S)$, hence is endowed with the structure of a Lie algebra S-scheme over $\mathbb{O}_S$. Moreover, this construction commutes with base change and finite products.

Remark 14.2.87. Let $\pi:G\to S$ be the structural morphism. The $\mathscr{O}_G$-module $\Omega^1_{G/S}$ is evidently $(G\times_S G)$-equivariant and hence, by ([?] I, 6.8.1), we have $\Omega^1_{G/S}\cong\pi^*(\omega^1_{G/S})$. It follows for example that $\Omega^1_{G/S}$ is locally free (resp. locally free of finite rank) if $\omega^1_{G/S}$ is, which is in particular the case if S is the spectrum of a field (resp. if S is the spectrum of a field and G is locally of finite type over S). Moreover, by ([?] I, 6.8.2), $\omega^1_{G/S}$ is endowed with a canonical $\mathbb{O}_S[G]$-module structure, which induces over $\mathbb{V}(\omega^1_{G/S})=\mathrm{Lie}(G/S)$ the adjoint representation.

On the other hand, e is an immersion, and is a closed immersion if G is separated over S (cf. Corollary 10.5.4). Hence $\omega^1_{G/S}$ is identified with $\mathscr{J}/\mathscr{J}^2$, where $\mathscr{J}$ is the quasi-coherent ideal defining $e(S)$ in an open subset U of G in which $e(G)$ is closed (if G is separated over S, we can put $U=G$, and if $G=\mathrm{Spec}(\mathscr{A}(G))$ is affine over S, $\mathscr{J}$ is none other than the augmented ideal of $\mathscr{A}(G)$, i.e. the kernel of $e^\sharp:\mathscr{A}(G)\to\mathscr{O}_S$).

Remark 14.2.88. We deduce from the isomorphism $\Omega^1_{G/S}\cong\pi^*(\omega^1_{G/S})$ that the $\mathscr{O}_S$-module $\omega^1_{G/S}$ is identified with the sheaf $\pi^G_*(\Omega^1_{G/S})$ of right invariant differentials of G over S, that is, the sheaf whose sections over an open subset U of S are the sections of $\Omega^1_{G/S}$ over $\pi^{-1}(U)$ which are invariant under right translations (cf. ([?] I, 6.8.3)).

We denote by $\mathscr{L}ie(G/S)$ the sheaf of sections of the vector bundle $\mathrm{Lie}(G/S) \to S$, which is the $\mathscr{O}_S$-module $(\omega^1_{G/S})^\vee = \mathcal{H}om_{\mathscr{O}_S}(\omega^1_{G/S}, \mathscr{O}_S)$ dual to $\omega^1_{G/S}$ (cf. Proposition 11.1.39). It is endowed with a Lie algebra structure over $\mathscr{O}_S$. As this construction does not commute with base change (in general), the Lie algebra structure on $\mathscr{L}ie(G/S)$ does not allow us to reconstruct the S-scheme structure on the $\mathbb{O}_S$-Lie algebra $\mathrm{Lie}(G/S)$. However, we have:

Proposition 14.2.89. *Suppose that $\omega^1_{G/S}$ is locally free of finite type. Then $\mathscr{L}ie(G/S)^\vee \cong (\omega_{G/S})^{\vee\vee} \cong \omega^1_{G/S}$ and hence*

$$\mathrm{Lie}(G/S) = \mathbb{V}(\omega^1_{G/S}) = \mathbb{V}(\mathscr{L}ie(G/S)^\vee) = \Gamma_{\mathscr{L}ie(G/S)}.$$

Proof. In fact, $\omega^1_{G/S}$ is reflexive if it is locally free of finite type, and the assertion follows from Corollary 14.1.27. □

Finally, let $G \to H$ be a monomorphism of group functors. Then $\mathfrak{Lie}(G/S) \to \mathfrak{Lie}(H/S)$ is also a monomorphism (cf. Proposition 14.2.23). As any monomorphism of vector bundles is a closed immersion[3], we obtain:

Corollary 14.2.90. *Let $G \to H$ be a monomorphism of S-groups.*

(i) $\mathrm{Lie}(G/S) \to \mathrm{Lie}(H/S)$ *is a closed immersion and hence $\omega^1_{H/S} \to \omega^1_{G/S}$ is an epimorphism.*

(ii) *If $\omega^1_{G/S}$ is locally free of finite type, then the corresponding morphism $\mathscr{L}ie(G/S) \to \mathscr{L}ie(H/S)$ is an isomorphism from $\mathscr{L}ie(G/S)$ to a submodule of $\mathscr{L}ie(H/S)$ which is locally a direct factor.*

Example 14.2.91. Let $S = \mathrm{Spec}(k)$ with k a field of characteristic p. Let $\boldsymbol{\alpha}_{p,S}$ be the S-functor which to any S-scheme T associates

$$\boldsymbol{\alpha}_{p,S}(T) = \{x \in \mathscr{O}(T) : x^p = 0\}.$$

Then $\boldsymbol{\alpha}_{p,S}$ is represented by $\mathrm{Spec}(\mathscr{O}_S[X]/(X^p))$, and hence is a very good S-group. It is also endowed with an $\mathbb{O}_S$-module structure, which is not very good, because the canonical morphism $\boldsymbol{\alpha}_{p,S} \to \mathfrak{Lie}(\boldsymbol{\alpha}_{p,S}/S) = \mathbb{G}_{a,S}$[4] is not bijective.

Example 14.2.92. Let Nil be the $\mathbb{Z}$-functor defined as follows: for any scheme S, $\mathrm{Nil}(S)$ is the nilideal of $\mathscr{O}_S$:

$$\mathrm{Nil}(S) = \{x \in \mathscr{O}(S) : \text{there exists } n \in \mathbb{N} \text{ such that } x^n = 0\}.$$

Let Nil^2, $\mathbb{O}_{\mathrm{red}}$ and F be the $\mathbb{Z}$-functors in groups which associate to any scheme S, respectively, the ideal $\mathrm{Nil}(S)^2$ and

$$\mathbb{O}_{\mathrm{red}}(S) = \mathscr{O}(S)/\mathrm{Nil}(S), \quad F(S) = \mathscr{O}(S)/\mathrm{Nil}(S)^2.$$

It is easily seen that $\mathfrak{Lie}(\mathbb{O}_{\mathrm{red}}/\mathbb{Z}) = 0$, hence the $\mathbb{O}_{\mathbb{Z}}$-module $\mathbb{O}_{\mathrm{red}}$ is not good (although it is a good $\mathbb{Z}$-group). If M, N are free $\mathbb{Z}$-modules of finite rank, we have

$$\mathrm{Nil}^2(I_S(M \oplus N)) = \mathrm{Nil}^2(S) \oplus \mathrm{Nil}^2(S) \otimes_{\mathbb{Z}} M \oplus \mathrm{Nil}(S) \otimes_{\mathbb{Z}} N$$

and hence

$$F(I_S(M \oplus N)) = F(S) \oplus \mathbb{O}_{\mathrm{red}}(S) \otimes_{\mathbb{Z}} M \oplus \mathbb{O}_{\mathrm{red}}(S) \otimes_{\mathbb{Z}} N.$$

We then deduce, on the one hand, that the $\mathbb{Z}$-functor F satisfies condition (E) and, on the other hand, that $\mathfrak{Lie}(F/\mathbb{Z}) = \mathbb{O}_{\mathrm{red}}$ (cf. (14.2.17)); as the latter is not a good $\mathbb{O}_{\mathbb{Z}}$-module, this shows that F is a $\mathbb{Z}$-group which satisfies condition (E) but is not good.

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[3] Let $f : \mathscr{M} \to \mathscr{N}$ be a morphism of $\mathscr{O}_S$-modules and $\mathscr{P} = \operatorname{coker} f$. If $\mathbb{V}(\mathscr{N}) \to \mathbb{V}(\mathscr{M})$ is a monomorphism, the surjective morphism $\boldsymbol{S}(\mathscr{N}) \to \boldsymbol{S}(\mathscr{P})$ factors through $\mathscr{O}_S$, hence $\mathscr{P} = 0$.

[4] This can be deduced from the exact sequence (14.2.17), or we can also note that $\omega^1_{G/k} = k[X]$.

14.2.3 Calculation of some Lie algebras

14.2.3.1 Lie algebras of diagonalizable groups Let $G = D_S(M)$ be a diagonalizable group over S (cf. 14.1.2.4). The formation of $\mathfrak{Lie}(G/S)$ commutes with base change, so it suffices to consider this construction for $G = D(M)$. We then have

$$G(I_S) = \mathrm{Hom}_{\mathbf{Grp}}(M, \Gamma(I_S, \mathscr{O}_{I_S})^\times) = \mathrm{Hom}_{\mathbf{Grp}}(M, \Gamma(S, \mathscr{D}_{\mathscr{O}_S})^\times).$$

Now the section $S \to I_S$ induces a split exact sequence

$$1 \longrightarrow \Gamma(S, \mathscr{O}_S) \longrightarrow \Gamma(S, \mathscr{D}_{\mathscr{O}_S})^\times \longrightarrow \Gamma(S, \mathscr{O}_S)^\times \longrightarrow 0$$

which implies that $\mathfrak{Lie}(G)(S)$ is identified with $\mathrm{Hom}_{\mathbf{Grp}}(M, \mathbb{O}_S)$, endowed with the evident $\mathbb{O}(S)$-module structure. We then obtain by base change the following:

Proposition 14.2.93. *We have isomorphisms*

$$\mathcal{H}om_{S\text{-}\mathbf{Grp}}(M_S, \mathbb{O}_S) \xrightarrow{\sim} \mathfrak{Lie}(D_S(M)/S), \quad \mathcal{H}om_{\mathbf{Grp}}(\widetilde{M}_S, \mathscr{O}_S) \xrightarrow{\sim} \mathscr{L}ie(D_S(M)/S),$$

where, in the second isomorhism, $\widetilde{M}_S$ is the sheaf of constant group over S defined by M, and $\mathcal{H}om_{\mathbf{Grp}}$ is the sheaf of homomorphisms of groups.

Corollary 14.2.94. *If M is free of finite rank, then*

$$\Gamma_{\mathscr{L}ie(D_S(M)/S)} \xrightarrow{\sim} \mathfrak{Lie}(D_S(M)/S), \quad M^\vee \otimes_{\mathbb{Z}} \mathscr{O}_S \xrightarrow{\sim} \mathscr{L}ie(D_S(M)/S).$$

In particular, $\mathbb{O}_S \cong \mathfrak{Lie}(\mathbb{G}_{m,S}/S)$ and $\mathscr{O}_S \cong \mathscr{L}ie(\mathbb{G}_{m,S}/S)$.

Proof. The second isomorphism follows from Proposition 14.2.93 the isomorphism

$$M^\vee \otimes_{\mathbb{Z}} \mathscr{O}_S = \mathrm{Hom}_{\mathbb{Z}}(\widetilde{M}_S, \mathscr{O}_S) = \mathrm{Hom}_{\mathbf{Grp}}(\widetilde{M}_S, \mathscr{O}_S),$$

which it implies that $\Gamma_{\mathscr{L}ie(D_S(M)/S)} = \mathcal{H}om_{S\text{-}\mathbf{Grp}}(M_S, \mathbb{O}_S)$, whence the first isomorphism. □

14.2.3.2 Normalizers and centralizers Recall that a sequence $0 \to F' \to F \to F'' \to 0$ of $\mathbb{O}_S$-modules is called **exact** if for any $S' \to S$ the sequence $0 \to F'(S') \to F(S') \to F''(S') \to 0$ of $\mathbb{O}(S')$-modules is exact. Similarly, a sequence $1 \to G' \to G \to G'' \to 1$ of S-groups is exact if for any $S' \to S$ the sequence of groups $1 \to G'(S') \to G(S') \to G''(S') \to 1$ is exact.

Lemma 14.2.95. *Let $1 \to G' \to G \to G'' \to 1$ be an exact sequence of S-groups.*

(i) *The sequences $1 \to T_{G'/S}(\mathscr{M}) \to T_{G/S}(\mathscr{M}) \to T_{G''/S}(\mathscr{M}) \to 1$ and $1 \to \mathfrak{Lie}(G'/S, \mathscr{M}) \to \mathfrak{Lie}(G/S, \mathscr{M}) \to \mathfrak{Lie}(G''/S, \mathscr{M}) \to 1$ are exact.*

(ii) *Let $1 \to H' \to H \to H'' \to 1$ be a second exact sequence of groups; it is exact if and only if the following sequence is exact:*

$$1 \longrightarrow G' \times_S H' \longrightarrow G \times_S H \longrightarrow G'' \times_S H'' \longrightarrow 1$$

(iii) *If two of the S-groups G', G, G'' satisfy condition (E), then the third one satisfies condition (E).*

(iv) *If $0 \to F' \to F \to F' \to 0$ is an exact sequence of $\mathbb{O}_S$-modules and two of the modules F', F, F'' are good, the third one is good.*

(v) *If two of the S-groups are good, the third one is good.*

Lemma 14.2.96. *Let G be an S-group, E, H be G-objects, F be an $\mathbb{O}_S[G]$-module.*

(a) *The canonical homomorphism* $E^G \times_S H^G \to (E \times_S H)^G$ *is an isomorphism.*

(b) *If* F *is a good* $\mathbb{O}_S$*-module, so is* F^G.

If E is an S-group and F is a sub-S-group of E, we denote by E/F the S-functor which to any $S' \to S$ associates the set $E(S')/F(S')$ of classes $\bar{x} = xF(S')$, $x \in E(S')$. If E is an abelian group over S, then E/F is endowed with an abelian group structure.

Now let G be an S-group and K be a sub-S-group of G; put $E = \mathfrak{Lie}(G/S, \mathscr{M})$ and $F = \mathfrak{Lie}(K/S, \mathscr{M})$. The adjoint action of K on E stablize F, hence induces an action of K over the S-functor E/F. For any $S' \to S$, we then have

$$(E/F)^K(S') = \{\bar{x} \in E(S')/F(S') : f^*(x^{-1})\mathrm{Ad}(k)(f^*(x)) \in F(S'') \text{ for } f : S'' \to S', k \in K(S'')\}$$

where $f^*(x)$ denotes the image of x in $E(S'')$.

Theorem 14.2.97. *Let* G *be an* S*-group,* K *be a sub-*S*-group of* G*,* $N = N_G(K)$ *and* $Z = Z_G(K)$.

(i) *If the group law of* $\mathfrak{Lie}(G/S, \mathscr{M})$ *is abelian, then*

$$\mathfrak{Lie}(N/S, \mathscr{M})/\mathfrak{Lie}(K/S, \mathscr{M}) = \big(\mathfrak{Lie}(G/S, \mathscr{M})/\mathfrak{Lie}(K/S, \mathscr{M})\big)^K.$$

(ii) *If the group law of* $\mathfrak{Lie}(G/S, \mathscr{M})$ *is abelian, then* $\mathfrak{Lie}(Z/S, \mathscr{M}) = \mathfrak{Lie}(G/S, \mathscr{M})^K$.

(iii) *If* G *satisfies condition (E) (resp. if* G *and* K *satisfy condition (E)), then* Z *satisfies condition (E) (resp.* N *satisfies condition (E)).*

(iv) *If* G *is good (resp. very good), then* Z *is good (resp. very good).*

(v) *If* G *and* K *are good, then* N *is good. If moreover* G *is very good, then* N *is very good.*

Corollary 14.2.98. *We have* $\mathfrak{Lie}(Z(G)/S) = \mathfrak{Lie}(G/S)^G$ *if the group law of* $\mathfrak{Lie}(G/S)$ *is abelian.*

Corollary 14.2.99. *If the group law of* $\mathfrak{Lie}(G/S)$ *is abelian and* K *is a normal subgroup of* G*, then*

$$\mathfrak{Lie}(G/S, \mathscr{M})/\mathfrak{Lie}(K/S, \mathscr{M}) = \big(\mathfrak{Lie}(G/S, \mathscr{M})/\mathfrak{Lie}(K/S, \mathscr{M})\big)^K.$$

Let G be a good S-group acting linearly on a good $\mathbb{O}_S$-module F via

$$\rho : G \to \mathcal{A}ut_{\mathbb{O}_S}(F).$$

We have defined a corresponding linear representation

$$d\rho : \mathfrak{Lie}(G/S) \to \mathcal{E}nd_{\mathbb{O}_S}(F).$$

The subgroups $N_G(E)$ and $Z_G(E)$ are defined for any subset E of F. Similarly, for any $S' \to S$, we define

$$N_{\mathfrak{Lie}(G/S)}(E)(S') = \{X \in \mathfrak{Lie}(G/S) : d\rho(X)E_{S'} \subseteq E_{S'}\},$$
$$Z_{\mathfrak{Lie}(G/S)}(E)(S') = \{X \in \mathfrak{Lie}(G/S) : d\rho(X)E_{S'} = 0\}.$$

called the **normalizer** and **centralizer**, respectively, of E in F.

Theorem 14.2.100. *Let* G *be a good* S*-group acting linearly on a good* $\mathbb{O}_S$*-module* F*, and* E *be a sub-*$\mathbb{O}_S$*-module of* F.

(a) *We have* $\mathfrak{Lie}(Z_G(E)/S) = Z_{\mathfrak{Lie}(G/S)}(E)$ *and* $Z_G(E)$ *is a good* S*-group; it is very good if* G *is.*

(b) *Suppose that E is a good $\mathbb{O}_S$-module. Then $\mathfrak{Lie}(N_G(E)/S) = N_{\mathfrak{Lie}(G/S)}(E)$ and $N_G(E)$ is a good S-group; it is very good if G is.*

Example 14.2.101. Let G be a good S-group. Then Theorem 14.2.100 can be applied to the adjoint representation of G. Let E be a good submodule of $\mathfrak{Lie}(G/S)$, for which we can associate the normalizer and centralizer. By Theorem 14.2.100, their Lie algebras are respectively the normalizer and centralizer of E in $\mathfrak{Lie}(G/S)$, given by the usual definition:

$$N_{\mathfrak{Lie}(G/S)}(E)(S') = \{X \in \mathfrak{Lie}(G/S) : [X, E_{S'}] \subseteq E_{S'}\},$$
$$Z_{\mathfrak{Lie}(G/S)}(E)(S') = \{X \in \mathfrak{Lie}(G/S) : d\rho[X, E_{S'}] = 0\}.$$

Example 14.2.102. Let K be a sub-S-group of G, then $\mathfrak{Lie}(K/S)$ is a sub-$\mathbb{O}_S$-module of $\mathfrak{Lie}(G/S)$. Suppose that $\mathfrak{Lie}(K/S)$ is a good $\mathbb{O}_S$-module; we evidently have

$$N_G(K) \subseteq N_G(\mathfrak{Lie}(K/S)), \quad Z_G(K) \subseteq Z_G(\mathfrak{Lie}(K/S))$$

whence, by Theorem 14.2.100, we obtain

$$\mathfrak{Lie}(N_G(K)/S) \subseteq N_{\mathfrak{Lie}(G/S)}(\mathfrak{Lie}(K/S)), \quad \mathfrak{Lie}(Z_G(K)/S) \subseteq Z_{\mathfrak{Lie}(G/S)}(\mathfrak{Lie}(K/S)),$$

but none of these four inclusions is a priori an identity. In particular, if K is a normal subgroup of G, then $\mathfrak{Lie}(K/S)$ is an ideal of $\mathfrak{Lie}(G/S)$.

Example 14.2.103. Let S be a scheme, F be the good $\mathbb{O}_S$-module $\mathbb{O}_S^2$ endowed with the natural action of the good S-group $G = \mathrm{GL}_{2,S}$, and E be the sub-$\mathbb{O}_S$-module of F formed by couples (x_1, x_2) such that x_2 is nilpotent. Put $N = N_G(E)$, then $\mathfrak{Lie}(N/S) = \mathfrak{Lie}(G/S)$ while, for any $S' \to S$, we have

$$N_{\mathfrak{Lie}(G/S)}(E)(S') = \left\{ \begin{pmatrix} a & b \\ x & c \end{pmatrix} : a, b, c, x \in \mathscr{O}(S'), x \text{ nilpotent} \right\}$$

hence $\mathfrak{Lie}(N_G(E)/S) \neq N_{\mathfrak{Lie}(G/S)}(E)$.

By considering the semi-direct product $G' = F \rtimes G$, we obtain a similar conter-example where E is a sub-$\mathbb{O}_S$-module of $\mathfrak{Lie}(G'/S)$. We also note that with the notations above, $E = \mathfrak{Lie}(K/S)$ where K is the subgroup $\mathbb{O}_S \oplus \mathrm{Nil}^2$ of F (that is, for any $S' \to S$, $K(S')$ is formed by couples (x_1, x_2) where $x_2 \in \mathrm{Nil}(S')^2$).

14.3 Equivalence relations and passing to quotient

14.3.1 Universally effective equivalence relations

14.3.1.1 Equivalence relations

Definition 14.3.1. Let $\mathcal{C}$ be a category. A **$\mathcal{C}$-equivalence relation** over $X \in \mathrm{Ob}(\mathcal{C})$ is defined to be a representable sunfunctor R of $X \times X$, such that for any $S \in \mathrm{Ob}(\mathcal{C})$, $R(S)$ is the graph of an equivalence relation over $X(S)$.

This definition is applicable in particular to the category $\widehat{\mathcal{C}}$. If we consider X as an object of $\widehat{\mathcal{C}}$, then a $\widehat{\mathcal{C}}$-equivalance relation over X is none other than a subfunctor R of $X \times X$ (not necessarily representable in $\mathcal{C}$) such that $R(S)$ is the graph of an equivalence relation on $X(S)$ for any $S \in \mathrm{Ob}(\mathcal{C})$. In fact, this conditon is evidently necessary. Conversely, if for any $S \in \mathrm{Ob}(\mathcal{C})$, $R(S)$ is the graph of an equivalence relation, then this equivalence relation extends to $R(F)$ for any $F \in \mathrm{Ob}(\widehat{\mathcal{C}})$ by declearing two morphisms $\phi, \psi : F \to R$ to be equivalent if, for any $S \in \mathrm{Ob}(\mathcal{C})$ and $x \in F(S)$, $\phi(x)$ is equivalent to $\psi(x)$ in $X(S)$.

If R is a $\mathcal{C}$-equivalence relation on X, we denote by $p_i : R \to X$ the morphism induced by the projection $\mathrm{pr}_i : X \times X \to X$. We then have a diagram

$$p_1, p_2 : R \rightrightarrows X.$$

A morphism $u : X \to Z$ is called **compatible with** R if $up_1 = up_2$. The cokernel in $\mathcal{C}$ of the couple (p_1, p_2) is also called the **quotient object** of X by R, and denoted by X/R. We then have an exact diagram

$$R \underset{p_2}{\overset{p_1}{\rightrightarrows}} X \xrightarrow{p} X/R$$

and X/R represents the covariant functor

$$\mathrm{Hom}_{\mathcal{C}}(X/R, Z) = \{\text{morphisms } X \to Z \text{ compatible with } R\}.$$

Since the quotient objects have been chosen in $\mathcal{C}$, the quotient X/R is unique (when it exists).

These definitions immediately generalize to $\widehat{\mathcal{C}}$-equivalence relations on X, but note that the Yoneda embedding functor $\mathcal{C} \to \widehat{\mathcal{C}}$ does not commutes with the formation of quotients, so the quotient X/R of X by R in $\mathcal{C}$ if not a priori a quotient of X by R in $\widehat{\mathcal{C}}$. Therefore, we will be careful not to identity $\mathcal{C}$ indiscriminetly with its image in $\widehat{\mathcal{C}}$ when dealing with questions involving passages to the quotient. In the following, by "equivalence relation", we simply mean $\widehat{\mathcal{C}}$-equivalence relations.

If X is an object of $\mathcal{C}$ over S, an **equivalence relation on X over S** is defined to be an equivalence relation R over X such that the structural morphism $X \to S$ is compatible with R. In this case, the canonical morphism $R \to X \times X$ then factors through the monomorphism

$$X \times_S X \to X \times X$$

and defines an equivalence relation over the object $X \to S$ of $\mathcal{C}_{/S}$. If the quotient X/R exists, it is endowed with a canonical morphism to S and the corresponding object of $\mathcal{C}_{/S}$ is a quotient of $X \in \mathrm{Ob}(\mathcal{C}_{/S})$ by the preceding equivalence relation. Conversely, if S is a squarable object of $\mathcal{C}$ and $Y \to S$ is a quotient of X by this equivalence relation (in $\mathcal{C}_{/S}$), then Y is a quotient by R in $\mathcal{C}$.

Definition 14.3.2. If X (resp. X') is an object of $\mathcal{C}$ endowed with an equivalence relation R (resp. R'), a morphism $u : X \to X'$ is called compatible with R and R' if the following equivalence relations are satisfied:

(i) For any $S \in \mathrm{Ob}(\mathcal{C})$, two points of $X(S)$ congruent modulo $R(S)$ are transformed by u to two points of $X'(S)$ congruent modulo $R'(S)$

(ii) There exists a morphism $R \to R'$ (necessarily unique) fitting into the diagram

$$\begin{tikzcd} R \arrow[r] \arrow[d] & R' \arrow[d] \\ X \times X \arrow[r, "u \times u"] & X' \times X' \end{tikzcd}$$

By the universal property of X/R, there then exists (if the quotients X/R and X'/R' exists) a unique morphism v fitting into the commutative diagram

$$\begin{tikzcd} X \arrow[r, "p"] \arrow[d, "u"] & X/R \arrow[d, "v"] \\ X' \arrow[r, "p'"] & X'/R' \end{tikzcd}$$

Definition 14.3.3. A sub-object Y of X is called **stable** under the equivalence relation R if the following equivalent conditions are satisfied:

(i) For any $S \in \mathrm{Ob}(\mathcal{C})$, the subset $Y(S)$ of $X(S)$ is stable under $R(S)$.

(ii) The inverse images of Y under p_1 and p_2 are identical.

A particular important case is the following: the quotient X/R exists and Y is the inverse image of a sub-object of X/R in X.

Definition 14.3.4. Let R be an equivalence relation over X and $X' \to X$ ve a morphism. The equivalence relation R' over X obtained by the Cartesian diagram

$$\begin{tikzcd} R' \arrow[r] \arrow[d] & R \arrow[d] \\ X' \times X' \arrow[r] & X \times X \end{tikzcd}$$

is called the inverse image of R in X'. In particular, if X' is a sub-object of X, the corresponding equivalence relation is called the induced relation on X', and denoted by $R_{X'}$.

The morphism $X' \to X$ is compatible with R' and R; we then have, if the quotients exist, a morphism $X'/R' \to X/R$. If X' is a sub-object of X, we shall see that in certain case we can prove that $X'/R' \to X/R$ is a monomorphism, hence identifies X'/R' with a sub-object of X/R. If this is the case, the inverse image of this sub-object in X will be a sub-object of X containing X' and stable under R, called the **saturation** of X' for the equivalence relation R.

Proposition 14.3.5. *If the sub-object Y of X is stable under R, we have two Cartesian squares for $i = 1,2$:*

$$\begin{tikzcd} R_Y \arrow[r] \arrow[d, "p_i"'] & R \arrow[d, "p_i"] \\ Y \arrow[r] & X \end{tikzcd}$$

Proof. This follows from the definition of R_Y and the stability of Y under R. □

14.3.1.2 Equivalence relation defiend by a free group action

Definition 14.3.6. Let X be an object of $\mathcal{C}$ and H be a $\mathcal{C}$-group acting on X. We say that H acts freely on X if the following conditions are satisfied:

(i) For any $S \in \mathrm{Ob}(\mathcal{C})$, the group $H(S)$ acts freely on $X(S)$.

(ii) The morphism of functors $H \times X \to X \times X$ defined by $(h,x) \mapsto (hx,x)$ is a monomorphism.

If H acts freely on X, the image of $H \times X$ by the morphism in (ii) is an equivalence relation on X, called the **equivalence relation defined by the action of H over X**. The quotient of X by this equivalence relation, if exists, is denoted by $H\backslash X$. It represents the following covariant functor: if Z is an object of $\mathcal{C}$, we have

$$\mathrm{Hom}(H\backslash X, Z) = \{\text{morphisms } X \to Z \text{ invariant under } H\}$$

where a morphism $f : X \to Z$ is invariant under H if for any $S \in \mathrm{Ob}(\mathcal{C})$, the corresponding morphism $X(S) \to Z(S)$ is invariant under the group $H(S)$.

Lemma 14.3.7. *Let H be a group acting freely on X and Y be a sub-object of X. The following conditions are equivalent:*

(i) *Y is stable under the equivalence relation defined by H.*

(ii) *For any $S \in \mathrm{Ob}(\mathcal{C})$, the subset $Y(S)$ of $X(S)$ is stable under $H(S)$.*

(iii) *There exists a morphism f (necessarily unique) fitting into the commutative diagram*

$$\begin{tikzcd} H \times Y \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ H \times X \arrow[r] & X \end{tikzcd}$$

Under these conditions, f defines a morphism of $\widehat{\mathcal{C}}$-groups $H \to \mathcal{A}ut(Y)$ and the equivalence relation over Y defined by H is the induced one from X.

Proof. The proof is immediate, by the definition of stable objects and the equivalence relation induced by H. The operation of H on Y is called the induced action. $\square$

Now consider the following situation: H and G are two $\mathcal{C}$-groups and we are given a group morphism $u : H \to G$. Then H acts on G by translations (we put $h \cdot g = u(h)g$) and it acts freely on G if and only if u is a monomorphism. In this case, the quotient of G by this action of H is denoted (if exists) by $H\backslash G$. Similarly, we can define a right action of H on G, and a quotient G/H. These quotients are functorial relative to the two groups. More precisely, we have the following lemma for right actions of H:

Lemma 14.3.8. *Let $u : H \to G$ and $u' : H' \to G'$ be two monomorphisms of $\mathcal{C}$-groups. Suppose that we are given a morphism of $\mathcal{C}$-groups $f : G \to G'$, then the following conditions are equivalent:*

(i) *f is compatible with the equivalence relations defined by H and H'.*

(ii) *For any $S \in \mathrm{Ob}(\mathcal{C})$, we have $f(u(H(S))) \subseteq u'(H(S))$.*

(iii) *There exists a morphism $g : H \to H'$ (necessarily unique and multiplicative) such that the following diagram is commutative:*

$$\begin{tikzcd} H \arrow[r, "g"] \arrow[d, "u"'] & H' \arrow[d, "u'"] \\ G \arrow[r, "f"] & G' \end{tikzcd}$$

Under these conditions, if the quotients G/H and G'/H' exist, there is a unique morphism $\bar{f}$ fitting into the commutative diagram

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[d, "p"'] & G' \arrow[d, "p'"] \\ G/H \arrow[r, "\bar{f}"] & G'/H' \end{tikzcd}$$

Proof. The first assertion can be verified element-wisely, and the second one then follows from (i). $\square$

We can then translate the notions introduced above for general equivalence relations to the present situation. Let us simply point out the following lemma, whose proof is immediate by reduction to the set case:

Lemma 14.3.9. *Let $u : H \to G$ be a monomorphism of $\mathcal{C}$-groups and G' be a sub-group of G. For a sub-object G' of G to be stable under the equivalence relation defined by H, it is necessary and sufficient that u factors through the canonical monomorphism $G' \to G$. In this case, the induced action of H on G' is none other than that deduced by the monomorphism $H \to G'$ factorizing u.*

14.3.1.3 Universally effective equivalence relations

Definition 14.3.10. Let $f : X \to Y$ be a morphism. The image of the canonical monomorphism

$$X \times_Y X \to X \times X$$

then defines a $\widehat{\mathcal{C}}$-equivalence relation on X, called the **equivalence relation defined by f over X** and denoted by $R(f)$.

Definition 14.3.11. Let R be an equivalence relation over X. We say that R is effective if

(i) R is representable (i.e. is a $\mathcal{C}$-equivalence relation);

(ii) the quotient Y/R exists in $\mathcal{C}$;

(iii) the diagram

$$R \underset{p_2}{\overset{p_1}{\rightrightarrows}} X \xrightarrow{p} Y$$

makes R the fiber product of X over Y, that is, R is the equivalence relation defined by p.

If R is an effective equivalence relation over X, then p is an effective epimorphism. If $f : X \to Y$ is an effective epimorphism, then $R(f)$ is an effective equivalence relation over X whose quotient is Y. There then exists a correspondence bewteen effective equivalence relations over X and effective quotients of X.

Definition 14.3.12. An equivalence relation R over X is called **universally effective** if the quotient $Y = X/R$ exists and if, for any $Y' \to Y$, the fiber product $X' = X \times_Y Y'$ and $R' = R \times_Y Y'$ exist and R' is a fiber product of X' over Y'. Equivalently, this amouts to saying that R is effective and $p : X \to X/R$ is a universally effective epimorphism.

Remark 14.3.13. Suppose that $\mathcal{C}$ is the category of S-schemes and denote by $\mathbb{G}_{a,S}$ the additive group over S. Let $R \subseteq X \times_S X$ be a universally effective equivalence relation and $p : X \to Y$ be the quotient. Then, for any open subset U of Y, $\mathscr{O}(U) = \mathrm{Hom}_S(U, \mathbb{G}_{a,S})$ is the set of elements $\phi \in \mathscr{O}(p^{-1}(U)) = \mathrm{Hom}_S(p^{-1}(U), \mathbb{G}_{a,S})$ such that $\phi \circ p_1 = \phi \circ p_2$. In particular, if R is given by a free right action over X of a group H, then $\mathscr{O}(U)$ is the set of $\phi \in \mathscr{O}(p^{-1}(U))$ such that $\phi(xh) = \phi(x)$ for any $S' \to S$ and $x \in X(S')$, $h \in H(S')$.

Proposition 14.3.14. *Let R be a universally effective equivalence relation over X, $f : X \to Z$ be a morphism compatible with R, with a factorization $g : X/R \to Z$. The following conditions are equivalent:*

(i) *g is a monomorphism;*

(ii) *R is the equivalence relation defined by f.*

Proof. In fact, (i) clearly implies (ii), and the converse follows from **??**. □

Definition 14.3.15. Let H be a $\mathcal{C}$-group acting freely on X. We say that H acts **effectively** on X, or the action of H on X is **effective** (resp. **universally effective**), if the equivalence relation defined by H is effective (resp. universally effective).

In practive, it is often difficult to characterize universally effective epimorphisms. We often have, however, a certain number of morphisms of this type, for example, faithfully flat and quasi-compact morphisms of schemes. This leads to the following definition: Let $\mathcal{M}$ be a family of morphisms of $\mathcal{C}$ satisfying the following properties:

(M1) $\mathcal{M}$ is *stable under base change*, i.e. for any morphism $u : T \to S$ in $\mathcal{M}$ is squarable and for any $S' \to S$, $u' : T \times_S S' \to S'$ belongs to $\mathcal{M}$.

(M2) The composition of two morphisms in $\mathcal{M}$ belongs to $\mathcal{M}$.

(M3) Any isomorphism belongs to $\mathcal{M}$.

(M4) Any morphism in $\mathcal{M}$ is an effective epimorphism.

Note that (M1) and (M2) imply:

(M1′) The Cartesian product of two morphisms in $\mathcal{M}$ is in $\mathcal{M}$: Let $u : X \to Y$ and $u' : X' \to Y'$ be two S-morphisms belonging to $\mathcal{M}$. If $Y \times_S Y'$ exsits, then $X \times_S X'$ exists and $u \times_S u'$ belongs to $\mathcal{M}$.

This follows from the diagram

$$\begin{tikzcd}
 & Y' & X' \arrow[l, "u'"'] \\
X \arrow[d, "u"] & X \times_S Y' \arrow[l] \arrow[u] \arrow[d] & X \times_S X' \arrow[l] \arrow[u] \arrow[dl, "u \times_S u'"] \\
Y & Y \times_S Y' \arrow[l] &
\end{tikzcd}$$

Similarly, (M1) and (M4) imply:

(M4′) Any morphism of $\mathcal{M}$ is a universally effective epimorphism.

The family $\mathcal{M}_0$ of universally effective morphisms verifies the conditions (M1)–(M4). In fact, (M1), (M3) and (M4) follows by definition, (M2) follows from ([**?**] IV, 1.8). In the following, we suppose that $\mathcal{M}$ is a family of morphisms in $\mathcal{C}$ verifying the above conditions. In particular, our result is applicable to the family $\mathcal{M}_0$.

Definition 14.3.16. We say that an equivalence relation R over X is **of type** $\mathcal{M}$ if it is representable and if $p_1 \in \mathcal{M}$[5]. We say that R is $\mathcal{M}$**-effective** if it is effective and if the canonical morphism $X \to X/R$ belongs to $\mathcal{M}$. Finally, we say the quotient Y of X is $\mathcal{M}$**-effective** if the canonical morphism $X \to Y$ belongs to $\mathcal{M}$.

Proposition 14.3.17. *Let $\mathcal{M}$ be a family of morphisms in $\mathcal{C}$ as above.*

(a) *An $\mathcal{M}$-effective equivalence relation is of type $\mathcal{M}$ and universally effective.*

(b) *An $\mathcal{M}$-effective quotient is universally effective.*

(c) *The map $R \mapsto X/R$ and $p \mapsto R(p)$ is a bijective correspondence from the set of effective equivalence relations over X to the set of $\mathcal{M}$-effective quotients of X.*

(d) *$\mathcal{M}_0$-effectivity is equivalent to universally effectivity.*

[5]This by (M2) and (M3) implies $p_2 \in \mathcal{M}$, since p_1 and p_2 are exchanged by an isomorphism of $X \times X$.

Proof. Let R be $\mathcal{M}$-effective, so that we have a Cartesian square

$$\begin{tikzcd} R \arrow[r, "p_2"] \arrow[d, "p_1"'] & X \arrow[d, "p"] \\ X \arrow[r, "p"] & X/R \end{tikzcd}$$

and $p \in \mathcal{M}$. By (M1), p_1 and p_2 belong to $\mathcal{M}$, so R is of type $\mathcal{M}$.

Put $Y = X/R$ and let $Y' \to Y$ be a morphism. By (M1), the fiber products $X' = X \times_Y Y'$ and $R' = R \times_Y Y'$ exist and the morphisms $X' \to Y'$ and $p_i' : R' \to X'$ belong to $\mathcal{M}$. Finally, as $R = X \times_Y X$, we obtain, by associativity of fiber products:

$$R' = X \times_Y X \times_Y Y' = X' \times_{Y'} X'$$

so R' is $\mathcal{M}$-effective and in particular R is universally effective. This proves (a) and also (d). The assertions of (b) and (c) then follows from this and the definition. □

Example 14.3.18. Let H be an S-group whose structural morphism belongs to $\mathcal{M}$. If H acts freely on the S-object X, then it defines an equivalence relation of type $\mathcal{M}$. In fact, by (M1) the fiber product $H \times_S X$ exists and $p_2 : H \times_S X \to X$ belongs to $\mathcal{M}$. We say that the operation of H is **$\mathcal{M}$-effective** if the equivalence relation over X defined by H is $\mathcal{M}$-effective.

Proposition 14.3.19 (**$\mathcal{M}$-effectivity and Base Change**). *Let R be an $\mathcal{M}$-effective equivalence relation on X over S and put $Y = X/R$. Let $S' \to S$ be a base change morphism such that $Y' = Y \times_S S'$ exists. Then $X' = X \times_S S'$ exists, $R' = R \times_S S'$ exists and is an $\mathcal{M}$-effective equivalence relation on X' over S' and $X'/R' \cong (X/R)'$.*

Proof. In fact, the canonical morphisms $X \to Y$ and $R \to Y$ belong to $\mathcal{M}$, hence by (M1′), X' and R' are representable. By associativity of fiber products, R' is the equivalence relation defined by the canonical morphism $X' \to Y'$ which belongs to $\mathcal{M}$, whence the conclusion. □

Proposition 14.3.20 (**$\mathcal{M}$-effectivity and Cartesian Product**). *Let R (resp. R') be an $\mathcal{M}$-effective equivalence relation on X (resp. X') over S. If $(X/R) \times_S (X'/R')$ exists, then $X \times_S X'$ exists, $R \times_S R'$ is an $\mathcal{M}$-effective equivalence relation on $X \times_S X'$ over S and*

$$(X \times_S X')/(R \times_S R') \cong (X/R) \times_S (X'/R').$$

Proof. Put $Y = X/R$ and $Y' = X'/R'$. By (M1′), the fiber product $X \times_S X'$ exists and the canonical morphism $q : X \times_S X' \to Y \times_S Y'$ belongs to $\mathcal{M}$. Now the formula

$$(X \times X') \times_{Y \times Y'} (X \times X') \cong (X \times_Y X) \times (X' \times_{Y'} X')$$

(where the product without subscript is taken over S) shows that $R \times_S R'$ is the equivalence relation defined by q on $X \times_S X'$, whence the proposition. □

Suppose that $\mathcal{C}$ possesses a final object e and let $f : G \to G'$ be a morphism of $\mathcal{C}$-groups such that $f \in \mathcal{M}$. Then by (M1), the kernel $\ker f$ is representable by $e \times_{G'} G$, and the morphism $\ker f \to e$ belongs to $\mathcal{M}$. On the other hand, the equivalence relation defined by f is the same as that defined by the action of $\ker f$ (right, say) over G, that is, the image of the morphism $G \times \ker f \to G \times G$, defined by $(g, h) \mapsto (g, gh)$.

Corollary 14.3.21. *Suppose that $\mathcal{C}$ possesses a final object e and let $f : G \to G'$ be a morphism of $\mathcal{C}$-groups such that $f \in \mathcal{M}$. Then the action of* $\ker f$ *on G is $\mathcal{M}$-effective and G' is the the quotient $G/\ker f$.*

Proof. Since f is a universally effective epimorphism by (M4′), G' is identified with the quotient of G by the equivalence relation defined by f, that is, by the action of $\ker f$. Since $\ker \to e$ belongs to $\mathcal{M}$, this equivalence relation is therefore representable by (M1), and we conclude the corollary. □

14.3.1.4 Construction of quotients by descent

Definition 14.3.22. We say that a descent data over X' relative to $S' \to S$ is **effective** if X' endowed with the descent data is isomorphic to the inverse image over S' of an object X over S.

If $S' \to S$ is a descent morphism, then the X in the above definition is unique up to unique isomorphism. The morphism $S' \to S$ is an effective descent morphism if it is a descent morphism and any descent data relative to $S' \to S$ is effective.

Now consider an equivalence relation R over an object X over S. Let X' (resp. X'', resp. X''') be the inverse image of X over S', $S'' = S' \times_S S'$ and $S''' = S' \times_S S' \times_S S'$ and let R', R'', R''' be the induced equivalence relations of R by inverse image. Suppose that the equivalence relation R' on X' is $\mathcal{M}$-effective, and consider the quotient $Y' = X'/R'$ which is an object over S'. Its inverse images under the two projections from S'' are isomorphic to X''/R'' by Proposition 14.3.19, so the S'-object Y' is endowed with a canonical glueing data. Using the same uniqueness for X'''/R''', we see that this is a descent data (note that we have implicitly assumed have all these fiber products exist, for example if $S' \to S$ is squarable).

Proposition 14.3.23. *Let R be an equivalence relation on an object X over S, and $S' \to S$ be a universally effective epimorphism. Suppose that any S-morphism whose inverse image over S' belongs to $\mathcal{M}$ is itself in $\mathcal{M}$. Then the following conditions are equivalent:*

(i) *R is $\mathcal{M}$-effective on X;*

(ii) *R' is $\mathcal{M}$-effective and the canonical descent date over X'/R' is effective.*

Moreover, if this is the case, the descent object of X'/R' is canonically isomorphic to X/R.

Proof. The fact that (i) implies (ii) follows directly from the definition of $\mathcal{M}$-effectivity and Proposition 14.3.17 (a). If the converse is true, then the last assertion follows from the fact that a universally effective epimorphism is a descent morphism, so the descent object is unique (up to isomorphism).

We now prove that (ii)⇒(i). Let $Y' = X'/R'$ and Y be the descent object. As the canonical morphism $p' : X' \to X'/R' = Y'$ is compatible with the descent data (its inverse image over S'' coincides with the canonical morphism $X'' \to X''/R''$ by Proposition 14.3.19), it comes from an S-morphism $p : X \to Y$. As p' belongs to $\mathcal{M}$, it follows from the hypothesis made on the morphism $S' \to S$ that p also belongs to $\mathcal{M}$. As p' is compatible with the equivalence relation R', p is compatible with R, since a universally effective epimorphism is a descent morphism. We then have a morphism

$$R \to X \times_Y X.$$

To see that R is $\mathcal{M}$-effective and that Y is isomorphic to X/R, it suffices to prove that this morphism is an isomorphism. Now since R' is effective, this becomes an isomorphism after base change to S', and it is therefore an isomorphism for the same reason. □

We note that the hypothesis of Proposition 14.3.23 is verified if we take $\mathcal{M} = \mathcal{M}_0$ to be the family of universally effective epimorphisms and if $\mathcal{C}$ possesses fiber products (cf. [**?**], IV, Corollaire 1.10). We then deduce the following corollary:

Corollary 14.3.24. *Suppose that $\mathcal{C}$ possesses fiber products (over S). Let R be an equivalence relation on X over S and $S' \to S$ be a universally effective epimorphism. Then the following conditions are equivalent:*

(i) *R is universally effective on X;*

(ii) *R' is universally effective and the canonical descent date over X'/R' is effective.*

Moreover, if this is the case, the descent object of X'/R' is canonically isomorphic to X/R.

14.3.2 Equivalence relations in the category of sheaves

14.3.2.1 Equivalence relations in $\widetilde{\mathcal{C}}$ Let $\mathcal{C}$ be a site and $\widetilde{\mathcal{C}}$ be the category of sheaves over $\mathcal{C}$. Let $i : \mathcal{C} \to \widehat{\mathcal{C}}$ be the inclusion functor.

Proposition 14.3.25. *Any equivalence relation in $\widetilde{\mathcal{C}}$ is universally effective: let R be a $\widetilde{\mathcal{C}}$-equivalence relation on the sheaf X, then the sheaf associated with the separated presheaf*

$$i(X)/i(R) : S \mapsto X(S)/R(S)$$

is a universally effective quotient sheaf of X by R.

Proof. Let $X/R = (i(X)/i(R))^{\#}$ be the quotient sheaf of X by R, which exists by ([**?**], IV, 4.4.1(ii)). It is necessary to show that $X \to X/R$ is a universally effective epimorphism, and that the morphism $f : R \to X \times_{X/R} X$ is an isomorphism. The first assertion follows from the proof of ([**?**], IV, 4.4.3). As for f, it comes from the sheafification of the morphism $i(R) \to i(X) \times_{i(X/R)} i(X)$, or, as $i(X)/i(R)$ is separated ([**?**], IV, 4.4.5(ii)) so that $i(X)/i(R) \to i(X/R)$ is a monomorphism, from the canonical morphism $i(R) \to i(X) \times_{i(X)/i(R)} i(X)$.

We are therefore reduced to the same assertion for the category of presheaves. But $i(X)/i(R)$ is the presheaf $S \mapsto X(S)/R(S)$ and we are then reduced to the same assertion for the category of sets, which is immediate. $\square$

Proposition 14.3.26. *Under the conditions of Proposition 14.3.25, let Y be a subsheaf of X. Denote by R_Y the equivalence relation induced on Y by R, then the canonical morphism $Y/R_Y \to X/R$ is a monomorphism: it identifies Y/R_Y with the subsheaf of X/R, which is the image sheaf of the composition morphism $Y \to X \to X/R$.*

Proof. The morphism of presheaves

$$i(Y)/i(R_Y) = i(Y)/i(R)_{i(Y)} \to i(X)/i(R)$$

is a monomorphism. As the functor $\#$ is left exact, it preserves monomorphisms and hence $Y/R_Y \to X/R$ is a monomorphism. The last assertion then follows from the commutative diagram

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & X \arrow[d] \\ Y/R_Y \arrow[r] & X/R \end{tikzcd}$$

and the fact that $Y \to Y/R_Y$ is covering. $\square$

In view of 14.3.26, we can identify Y/R_Y with a subsheaf of X/R.

Proposition 14.3.27. *Let R be a $\widetilde{\mathcal{C}}$-equivalence relation on a sheaf X. For any subsheaf Y of X stable under R, denote by $Y' = Y/R_Y$ the quotient considered as a subsheaf of $X' = X/R$. Then $Y = Y' \times_{X'} X$ and the maps $Y \mapsto Y/R_Y$ and $Y' \mapsto Y' \times_{X'} X$ give a bijective correspoondence between the set of subsheaves Y of X stable under R and the set of subsheaves Y' of X'.*

Proof. If Y' is a subsheaf of X', then $Y' \times_{X'} X$ is a subsheaf of X stable under R, and we have $(Y' \times_{X'} X)/R = Y'$. If Y' is obtained by passing to quotient of a subsheaf Y of X, then Y is a subobject of $Y' \times_{X'} X$. It then suffices to show that if we have two subobjects Y_1 and Y_2 of X, stable under R and $Y_1 \subseteq Y_2$, and if the quotients Y_1/R_{Y_1} and Y_2/R_{Y_2} are identical, then $Y_1 = Y_2$.

For this, we are evidently reduced to the same assertion in the case $Y_2 = X$. Denote then by P (resp. Q) the presheaf $i(X)/i(R)$ (resp. $i(Y)/i(R_Y)$), the diagram

$$\begin{tikzcd} Y \arrow[r] \arrow[d, hook] & Q \arrow[d, hook] \\ X \arrow[r] & P \end{tikzcd}$$

is Cartesian. As we have a commutative diagram

$$\begin{tikzcd} Q \arrow[r, hook] \arrow[d, hook] & Q^{\#} \arrow[d, equal] \\ P \arrow[r, hook] & P^{\#} \end{tikzcd}$$

and $Q \mapsto Q^{\#}$ is covering, the monomorphism $Q \hookrightarrow P$ is covering, so Q is a refinement of P. By base change, Y is then a refinement of X. As X and Y are both sheaves, we conclude that $Y = X$. ☐

In particular, if Y is a subsheaf of X and $Y' = Y/R_Y$, then the preceding correspondence ddefines a subsheaf $\overline{Y}$ of X, stable under R, containing Y and minimal with these properties; this subsheaf is called the saturation of Y for the equivalence relation R.

14.3.2.2 Description of the quotient of a sheaf by an equivalence relation Now assume that the topology of $\mathcal{C}$ is subcanonical. In this case, we know that any covering sieve is universally effective epimorphic, and the canonical functor $i : \mathcal{C} \to \widehat{\mathcal{C}}$ factors through $\widetilde{\mathcal{C}}$.

Proposition 14.3.28. *Let R be a $\widetilde{\mathcal{C}}$-equivalence relation on a sheaf X. Let $F \in \mathrm{Ob}(\widehat{\mathcal{C}})$ be the presheaf defined as follows: for any $S \in \mathrm{Ob}(\mathcal{C})$*[6]*,*

$$F(S) = \{\textit{sub-}S\textit{-sheaves } Z \textit{ of } X \times S \textit{ stable under } R \times S \textit{ whose quotient by } R_Z \textit{ is } S\}.$$

Then for any sheaf Y, $\mathrm{Hom}(Y, F)$ is identified with the set

$$\{\textit{sub-}Y\textit{-sheaves of } X \times Y \textit{ stable under } R \times Y \textit{ whose quotient is } Y\}.$$

In particular, the subsheaf R of $X \times X$ corresponds to a morphism $p : X \to F$ and the diagram

$$R \underset{p_2}{\overset{p_1}{\rightrightarrows}} X \xrightarrow{p} F$$

is exact, hence identifies F with the quotient sheaf X/R.

Proof. Let $Q = X/R$. For any sheaf Y and any morphism $f : Y \to Q$ corresponding to a section $s : Y \to Q \times Y$, consider the diagram

$$\begin{tikzcd} & Z \arrow[r] \arrow[d, hook] & Y \arrow[d, hook, "s"] \\ R \times Y \arrow[r, shift left] \arrow[r, shift right] & X \times Y \arrow[r] & Q \times Y \end{tikzcd} \tag{14.3.1}$$

where the square is Cartesian. It is immediate from Proposition 14.3.27 that Z is a sub-Y-sheaf of $X \times Y$ stable under $R \times Y$ whose quotient is Y; conversely, any Z with these properties

[6] $R \times S$ is the equivalence relation on $X \times S$ defined by $R \times S \subseteq X \times X \times S \times S$ (induced by the diagonal) and R_Z is the equivalence relation induced over Z.

provides a unique section of $Q \times Y$ over Y. Taking Y to be representable or arbitrary, we obtain an isomorphism $Q \cong F$ and the desired form of $\mathrm{Hom}(Y, F)$. Finally, consider the canonical morphism $X \to Q$, we immdediately see that it corresponds to the sub-X-sheaf R of $X \times X$, which proves our assertion. □

Corollary 14.3.29. *Let G be a subfunctor of F such that* $\mathrm{Hom}(X, G) \subseteq \mathrm{Hom}(X, F)$ *contains R. Then the canonical morphism $p : X \to F$ factors through G. As p is covering, it follows that G is a refinement of F. In particular, any subsheaf G of F verifying the preceding condition is equal to F.*

Proof. By the identification of Proposition 14.3.28, the hypothesis implies that $p : X \to F$ belongs to the image of $\mathrm{Hom}(X, G)$, whence it factors through G. □

We now consider the case where X and R are representable. Let's first introduce some terminology. In addition to the conditions (M1)–(M4) introduced in 14.3.1.3, we will use other conditions on a family $\mathcal{M}$ of morphisms of $\mathcal{C}$ (for completeness, we recall conditions (M1)–(M3)):

(M1) $\mathcal{M}$ is stable under base change.

(M2) The composition of two elements of $\mathcal{M}$ belongs to $\mathcal{M}$.

(M3) Any isomorphism belongs to $\mathcal{M}$.

(M4$_{\mathcal{T}}$) Any element of $\mathcal{M}$ is covering.

(M5$_{\mathcal{T}}$) Let $f : X \to Y$ be a morphism in $\mathcal{C}$. If there exists a covering sieve $R \hookrightarrow Y$ such that for any $Y' \to R$, $X \times_Y Y' \to Y'$ belongs to $\mathcal{M}$, then f belongs to $\mathcal{M}$.

Recall that (M1) and (M2) implies

(M1') The Cartesian product of two morphisms in $\mathcal{M}$ belongs to $\mathcal{M}$.

and (M1) and (M4$_{\mathcal{T}}$) implies (by [**?**], IV, 4.3.9):

(M4') Any morphism in $\mathcal{M}$ is a universally effective epimorphism.

The preceding conditions are verified for the family of covering morphisms, denoted by $\mathcal{M}_{\mathcal{T}}$, if $\mathcal{C}$ possesses fiber products. The results we are going to establish for a family $\mathcal{M}$ satisfying these conditions will apply in particular to the family $\mathcal{M}_{\mathcal{T}}$. In particular, we can take for $\mathcal{T}$ the canonical topology and for $\mathcal{M}$ the family of universally effective epimorphisms.

Proposition 14.3.30. *Let $\mathcal{M}$ be a family of morphisms verifying conditions (M1)–(M5$_{\mathcal{T}}$). Let R be a $\widetilde{\mathcal{C}}$-equivalence relation on $X \in \mathrm{Ob}(\mathcal{C})$ of type $\mathcal{M}$, $\widetilde{X}$ be the sheaf associated with X, $\widetilde{R}$ the $\widetilde{\mathcal{C}}$-equivalence relation on $\widetilde{X}$ defined by R, and $\widetilde{X}/\widetilde{R}$ the quotient sheaf. For R to be $\mathcal{M}$-effective, it is necessary and sufficient that $\widetilde{X}/\widetilde{R}$ is representable, and in this case it is represented by the quotient X/R.*

Proof. Suppose that R is $\mathcal{M}$-effective and let $Y = X/R$. The canonical morphism $p : X \to Y$ belongs to $\mathcal{M}$, hence is covering by (M4$_{\mathcal{T}}$). The corresponding morphism

$$\tilde{p} : \widetilde{X} \to \widetilde{Y}$$

is then a universally effective epimorphism in $\widetilde{\mathcal{C}}$, hence identifies $\widetilde{Y}$ with the quotient of $\widetilde{X}$ by the equivalence relation R' defined by $\tilde{p}$. As the functor $\mathcal{C} \to \widetilde{\mathcal{C}}$ commutes with fiber products, R' is none other than $\widetilde{R}$, because R is the equivalence relation defined by R (since it is effective). We then conclude that $\widetilde{X}/\widetilde{R}$ is represented by Y.

Conversely, suppose that $\widetilde{X}/\widetilde{R}$ is represented by an object Y of $\mathcal{C}$. Let $p : X \to Y$ be the morphism induced by the canonical morphism $\widetilde{X} \to \widetilde{X}/\widetilde{R}$, which is a covering morphism by

([**?**], IV, 4.4.3). It is clear as before that R is the equivalence relation defined by p, so it remains to show that $p \in \mathcal{M}$. But the Cartesian square

$$\begin{tikzcd} R \arrow[r, "\cong"] & X \times_Y X \arrow[r, "p_1"] \arrow[d, "p_2"] & X \arrow[d, "p"] \\ & X \arrow[r, "p"] & Y \end{tikzcd}$$

shows that the base change of p by the covering morphism p belongs to $\mathcal{M}$ (since $p_2 \in \mathcal{M}$ by our hypothesis). We then conclude from (M1) and (M5$_{\mathcal{T}}$) that $p \in \mathcal{M}$. □

Corollary 14.3.31. *Let $\mathcal{M}$ be a family of morphisms verifying conditions (M1)–(M5$_{\mathcal{T}}$) and $f : G \to G'$ be a morphism of $\mathcal{C}$-groups belonging to $\mathcal{M}$. Suppose that* $\ker f$ *is representable (for example, if $\mathcal{C}$ has a final object e), then the equivalence relation on G defined by $H =$* $\ker f$ *is $\mathcal{M}$-effective and G' represents the quotient sheaf $\widetilde{G}/\widetilde{H}$ for the topology $\mathcal{T}$.*

Proof. This follows from Corollary 14.3.21 and Proposition 14.3.30. □

We are now in a position to state the main theroem of this paragraph. Before this, let recall the following result:

Proposition 14.3.32. *Let $\{S_i \to S\}$ be a covering family and Z be a sheaf over S. Suppose that for each i, the S_i-functor $Z \times_S S_i$ is represented by an object T_i. Then the family T_i is endowed with a canonical descent data relative to $\{S_i \to S\}$. For Z to be representable, it is necessary and sufficient that this descent data is effective, and in this case the descent object represents Z.*

Proof. By ([**?**], IV, 4.4.3), $\{S_i \to S\}$ is universally effective epimorphic in $\widetilde{\mathcal{C}}$, hence is a descent family in $\widetilde{\mathcal{C}}$. If Z is represented by an object T, the $T \times_S S_i$ (considered as sheaves) is isomorphic to $Z \times_S S_i$, hence the descent data over T_i is effective and the descent object (necessarily unique) is isomorphic to Z. Conversely, suppose that the canonical descent data over T_i is effective and let T be a descent object. As the family $\{S_i \to S\}$ is a descent family, there exsits an S-morphism $T \to Z$ whose base change to S_i is the canonical morphism $T_i \to Z \times_S S_i$. This morphism is therefore locally an isomorphism, and it follows from ([**?**], IV, 4.4.8) that it is an isomorphism. □

Theorem 14.3.33. *Let $\mathcal{M}$ be a family of morphisms verifying conditions (M1)–(M5$_{\mathcal{T}}$) and R be a $\mathcal{C}$-equivalence relation of type $\mathcal{M}$ on an object X of $\mathcal{C}$. Consider the functor $F \in \mathrm{Ob}(\widehat{\mathcal{C}})$ defined as follows:*

$$F(S) = \{\textit{sub-}S\textit{-sheaf } Z \textit{ of } X \times S \textit{ stable under } R \times S \textit{ whose quotient by } R_Z \textit{ is } S\}.$$

Let F_0 be the sub-functor of F such that $F_0(S)$ is formed by representable $Z \in F(S)$, that is,

$$F_0(S) = \left\{\begin{matrix}\textit{sub-}\mathcal{C}_{/S}\textit{-objects } Z \textit{ of } X \times S \textit{ stable under } R \times S \textit{ such that } R_Z \textit{ is} \\ \mathcal{M}\textit{-effective and the quotient of } Z \textit{ by } R_Z \textit{ is } S\end{matrix}\right\}.$$

(a) *The morphism $p : X \to F$ defined by the sub-object R of $X \times X$ identifies F with the quotient sheaf of X by R.*

(b) *The following conditions are equivalent:*

 (i) F is representable.

 (ii) F_0 is representable.

 (iii) R is $\mathcal{M}$-effective.

and under these conditions, we have $F = F_0 = X/R$.

(c) *Let $\mathcal{N}$ be a family of morphisms which is stable under base change and such that for any covering family $\{S_i \to S\}$ and any family $\{T_i \to S_i\}$ of morphisms in $\mathcal{N}$, any descent data on the T_i relative to $\{S_i \to S\}$ is effective. Suppose that X is squarable and the morphism $R \to X \times X$ belongs to $\mathcal{N}$, then $F_0 = F$.*

Proof. The proof of (i) follows from Proposition 14.3.28. As for (ii), we have seen the equivalence of (i) and (iii) as well as the equality $F = X/R$ (cf. Proposition 14.3.30). It remains to prove that (ii) or (iii) implies $F_0 = F$, but for this we first note that F_0 is indeed a sub-functor of F. In fact, for any $S \in \mathrm{Ob}(\mathcal{C})$ and $Z \in F_0(S)$, the morphism $Z \to S$ belongs to $\mathcal{M}$ and hence is squarable, so $Z \times_S S'$ belongs to $F_0(S')$ for any $S' \to S$. As $R \in F(X)$ belongs to $F_0(X)$, Corollary 14.3.29 shows that (ii) implies $F_0 = F$.

Now suppose that (iii) is satisfied and let Q be an object of $\mathcal{C}$ representing X/R. Then the morphism $X \to Q$ belongs to $\mathcal{M}$ and, for any $S \in \mathrm{Ob}(\mathcal{C})$ and any $Z \in F(S)$, the diagram (14.3.1) of Proposition 14.3.28 shows that $Z = S \times_{(Q\times S)} X \times S$ is representable, and $Z \to S$ belongs to $\mathcal{M}$, hence $Z \in F_0(S)$.

Finally, to prove (c), let $f : S \to F$ be a morphism corresponding to $Z \in F(S)$. We must show that f factors through F_0, which means Z is representable. For this, we first note that if f factors through X, then it is the image of an element $x_0 \in X(S)$, and the corresponding sheaf Z is defined by the Cartesian squares (since the morphism $p : X \to F$ corresponds to the subsheaf R of $X \times X$)

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & R_S \arrow[r] \arrow[d] & R \arrow[d] \\ X_S \arrow[r, "\mathrm{id}_{X_S}\times\tau_{x_0}"] & X_S \times_S X_S \arrow[r] & X \times X \end{tikzcd}$$

where τ_{x_0} is the morphism $X_S \to X_S$ defined by $(x,s) \mapsto (x_0(s), s)$[7]. Moreover, as $R \to X \times X$ belongs to $\mathcal{N}$, so is $Z \to X_S$.

For the general case, as $X \to F$ is a covering morphism, there exists a covering family $\{S_i \to S\}$ and for each i a morphism $S_i \to X$ fitting into the diagram

$$\begin{tikzcd} X \arrow[r] & F \\ S_i \arrow[r] \arrow[u] & S \arrow[u, "f"'] \end{tikzcd}$$

By the preceding arguments, the morphism $f_i : S_i \to F$ defined by this diagram belongs to $F_0(S_i)$ and corresponds to the subsheaf $Z \times_S S_i$ of X_{S_i}, and the morphism $Z \times_S S_i \to X_{S_i}$ belongs to $\mathcal{N}$. As the family $\{X_{S_i} \to X_S\}$ is covering, the descent data on $Z \times_S S_i$ provides a descent object over S by our hypothesis, which must represents Z in view of Proposition 14.3.32. □

Corollary 14.3.34. *Let R be an $\mathcal{M}$-effective equivalence relation on X. For any sheaf F, the map*

$$\mathrm{Hom}(X/R, F) \to \mathrm{Hom}(X, F)$$

identifies $\mathrm{Hom}(X/R, F)$ *with the subset formed by morphisms $X \to F$ compatible with R.*

Proof. By Theorem 14.3.33, X/R represents the quotient sheaf $\widetilde{X}/\widetilde{R}$, and the defining property of $\widetilde{X}/\widetilde{R}$ gives the assertion. □

[7] Unwinding the definitions, we see that for any $S' \to S$, $Z(S)$ consists of elements $(x,s) \in X(S') \times S'$ such that $(x_0(s), x) \in R(S')$, so its quotient by R_Z is S.

Remark 14.3.35. In the hypothesis of Theorem 14.3.33 (iii), if we further suppose that the descent object T is such that the morphism $T \to S$ belongs to $\mathcal{N}$, then the inclusion morphism $Z \to X_S$ also belongs to $\mathcal{N}$, as it is obtained by the descent data on the morphisms $Z \times_S S_i \to X_{S_i}$, which are in $\mathcal{N}$.

Remark 14.3.36. We have proved the implications (iii)⇒(ii)⇒(i) and (iii)⇒$[F_0 = F = X/R]$ in Theorem 14.3.33 without resorting the "sufficient" part of Theorem 14.3.33, which is the only place we use condition (M5$_\mathcal{T}$). Therefore, they remain valid if $\mathcal{M}$ only satisfies conditions (M1)–(M4$_\mathcal{T}$). An example of such a family of that of squarable covering morphisms.

Corollary 14.3.37. *Under the conditions of Theorem 14.3.33 (ii), X/R is also the quotient sheaf of X by R for any intermediate topology between $\mathcal{T}$ and the canonical topology.*

Proof. If $\mathcal{T}'$ is an intermediate topology between $\mathcal{T}$ and the canonical topology, then $\mathcal{M}$ satisfies (M1)–(M4$_{\mathcal{T}'}$), so F_0 is identified with the quotient sheaf of X by R for $\mathcal{T}'$ (Remark 14.3.36), which is X/R. □

Corollary 14.3.38. *Let R be a universally effective equivalence relation on X. Then the object X/R of $\mathcal{C}$ represents the quotient sheaf of X by R for the canonical topology. Moreover, $(X/R)(S)$ is the set of sub-$\mathcal{C}_{/S}$-objects Z of X_S stable under $R \times S$ such that R_Z is universally effective and the quotient of Z by R_Z is S.*

Corollary 14.3.39. *Let $\mathcal{M}$ be the family of squarable covering morphisms. If R is an $\mathcal{M}$-effective equivalence relation on X, then X/R of $\mathcal{C}$ represents the quotient sheaf of X by R and it also represents the functor F_0 of Theorem 14.3.33.*

While in questions involving exclusively projective limits (fiber products, algebraic structures, etc.) we can identify $\mathcal{C}$ indiscriminately with a full subcategory of $\widetilde{\mathcal{C}}$ or of $\widehat{\mathcal{C}}$, it is not the same in those which combine projective and inductive limits. In questions involving both projective limits and inductive limits (in particular passages to the quotient), we should consider the given category as embedded in the category of sheaves. Thus if R is a $\mathcal{C}$-equivalence relation on the object X of $\mathcal{C}$, X/R will denote the quotient sheaf of X by R (designated previously by $(i(X)/i(R))^\#$), so in the case where this sheaf is representable, the object representing it. The previous results show that in the most important cases, a quotient in $\mathcal{C}$ will also be a quotient in the category of sheaves.

We now give an example of the usage of effectivity criteria. As before, let $\mathcal{T}$ be a subcanonical topology on $\mathcal{C}$ and choose a family $\mathcal{M}$ of morphisms satisfying conditions (M1)–(M5$_\mathcal{T}$). We consider a family $\mathcal{N}$ of morphisms in $\mathcal{C}$ with the following properties:

(N1) $\mathcal{N}$ is stable under base change.

(N$_\mathcal{T}$) The morphisms of $\mathcal{N}$ have descent property for the given topology. That is, for any $S \in \mathrm{Ob}(\mathcal{C})$, any covering family $\{S_i \to S\}$ and any family $\{T_i \to S_i\}$ of morphisms in $\mathcal{N}$, any dascent data on T_i relative to $\{S_i \to S\}$ is effective, and if T is the descent object, the morphism $T \to S$ belongs to $\mathcal{N}$.

As any element of $\mathcal{M}$ is covering, (N$_\mathcal{T}$) implies the following property:

(N$_\mathcal{M}$) If $Y' \to X'$ belongs to $\mathcal{N}$ and $X' \to X$ belongs to $\mathcal{M}$, any descent data over Y' relative to $X' \to X$ is effective. If Y is the descent object, then $Y \to X$ belongs to $\mathcal{N}$.

A particular important example is the following: $\mathcal{C}$ is the category of schemes, $\mathcal{T}$ is the fpqc topology, $\mathcal{M}$ is the family of faithfully flat and quasi-compact morphisms, $\mathcal{N}$ is the family of closed immersions, or that of quasi-compact immersions.

By Theorem 14.3.33, we then have the following result (cf. Remark 14.3.35):

Proposition 14.3.40. *Let X be a squarable object in $\mathcal{C}$ and R be an equivalence relation on X of type $\mathcal{M}$ such that $R \to X \times X$ belongs to $\mathcal{N}$. Then the quotient sheaf X/R is defined by*

$$(X/R)(S) = \left\{ \begin{matrix} \text{sub-}\mathcal{C}_{/S}\text{-objects } Z \text{ of } X \times S \text{ stable under } R \times S \text{ such that } Z \to X_S \\ \text{belongs to } \mathcal{N}, \text{ that } R_Z \text{ is } \mathcal{M}\text{-effective, and that the quotient of } Z \\ \text{by } R_Z \text{ is } S \end{matrix} \right\}.$$

Moreover, we have the following correspondence of stable subobjects of X and $\mathcal{M}$-effective equivalence relations.

Proposition 14.3.41. *Let $X \in \mathrm{Ob}(\mathcal{C})$ and R be an $\mathcal{M}$-effective equivalence relation on X.*

(a) *For any sub-object Y of X, stable under R and such that $Y \to X$ belongs to $\mathcal{N}$, the equivalence relation induced on Y by R is $\mathcal{M}$-effective and the quotient $Y/R_Y = Y'$ is a sub-object of $X' = X/R$ such that $Y' \to X'$.*

(b) *The map $Y \mapsto Y'$ is a bijection from the set of sub-objects Y of X stable under R such that $Y \to X$ belongs to $\mathcal{N}$ to the set of sub-objects Y' of X' such that $Y' \to X'$ belongs to $\mathcal{N}$. The inverse map is given by $Y' \to Y' \times_{X'} X$.*

Proof. As R is $\mathcal{M}$-effective, the morphism $X \to X'$ belongs to $\mathcal{M}$. Let Y' be a sub-object of X' such that the canonical morphism $Y' \to X'$ belongs to $\mathcal{N}$. Then, the sub-object $Y = Y' \times_{X'} X$ of X is stable under R, and the morphism $Y \to X$ (resp. $Y \to Y'$) belongs to $\mathcal{N}$ (resp. $\mathcal{M}$) since $\mathcal{N}$ and $\mathcal{M}$ are stable under base change. By Proposition 14.3.27, the quotient sheaf R/R_Y is represented by Y' and hence, by Proposition 14.3.30, R_Z is $\mathcal{M}$-effective.

Conversely, any sub-object Y of X, stable under R and such that the morphism $Y \to X$ belongs to $\mathcal{N}$, is obtained in this way. In fact, if Y is stable under R, its two inverse images in $R = X \times_{X'} X$ are identical and Y is endowed with a descent data relative to $X \to X'$; our assertion then follows from $(\mathrm{N}_{\mathcal{M}})$. □

Corollary 14.3.42. *Let $X \in \mathrm{Ob}(\mathcal{C})$ and R be an $\mathcal{M}$-effective equivalence relation on X. Suppose that $R \to X \times X$ belongs to $\mathcal{N}$, then for any Y as in Proposition 14.3.41, $R_Y \to Y \times Y$ belongs to $\mathcal{N}$ and hence, by Proposition 14.3.40, we have*

$$(Y/R_Y)(S) = \left\{ \begin{matrix} \text{sub-}\mathcal{C}_{/S}\text{-objects } Z \text{ of } Y \times S \text{ stable under } R_Y \times S \text{ such that } Z \to Y_S \\ \text{belongs to } \mathcal{N}, \text{ that } R_Z \text{ is } \mathcal{M}\text{-effective, and that the quotient of } Z \\ \text{by } R_Z \text{ is } S \end{matrix} \right\}.$$

14.3.3 Passage to quotient and algebraic structures

14.3.3.1 Principal homogeneous bundles We recall that an object X in $\widehat{\mathcal{C}}$ with a (right) group action by a group functor H is called **formally principal homogeneous** under H if the canonical morphism

$$X \times H \to X \times X, \quad (x, h) \mapsto (x, xh)$$

is an isomorphism. Equivalently, this means for any $S \in \mathrm{Ob}(\mathcal{C})$, $X(S)$ is formally principal homogeneous under $H(S)$, which is therefore empty or principal homogeneous under $H(S)$. In particular, if we act H on itself by (right) translations, then H is formally principal homogeneous under itself. The H-object X is called trivial if it is isomorphic to H acted by right translations.

Proposition 14.3.43. *If X be formally principal homogeneous under H, we have an isomorphism*

$$\Gamma(X) \xrightarrow{\sim} \mathrm{Iso}_H(H, X)$$

of principal homogeneous sets under $\Gamma(H)$.

Proof. To any section x of X, we can associate the morphism $H \to X$ defined set-wise by $h \mapsto xh$. The assertion is then immediate. □

Corollary 14.3.44. *We have an isomorphism of H-objects*

$$X \xrightarrow{\sim} \mathcal{I}so_H(H, X).$$

Moreover, for X to be trivial, it is necessary and sufficient that X is formally principal homogeneous and possesses a global section.

Definition 14.3.45. Let $\mathcal{C}$ be a site. An S-object X with an action by H is called a **principal homogeneous bundle under** H if it is **locally trivial**, that is, if the following equivalent conditions are satisfied:

(i) The set of morphisms $T \to S$ such that (the functor) $X \times_S T$ is trivial under $H \times_S T$ is a refinement of S.

(ii) There exists a covering family $\{S_i \to S\}$ such that for each i, the S_i-functor $X \times_S S_i$ is trivial under $H \times_S S_i$.

Proposition 14.3.46. *Let $\mathcal{C}$ be a site and $\mathcal{M}$ be a family of morphisms in $\mathcal{C}$ satisfying conditions (M1)–(M5$_\mathcal{T}$) of 14.3.2.2. Let H be an S-group such that the structural morphism $H \to S$ belongs to $\mathcal{M}$ and P be an S-object acted by H. The following conditions are equivalent:*

(i) *P is a principal homogeneous bundle under H.*

(ii) *P is formally principal homogeneous under H and the structural morphism $P \to S$ belongs to $\mathcal{M}$.*

(iii) *There exists a morphism $S' \to S$ in $\mathcal{M}$ such that the base change of P to S' is trivial, that is, $P \times_S S'$ is trivial under $H \times_S S'$.*

(iv) *H acts freely and $\mathcal{M}$-effectively on P and the quotient P/H is isomorphic to S.*

Proof. We first note that (ii) and (iv) are equivalent, in view of the fact that, in either case, $P \to S$ belongs to $\mathcal{M}$, hence is squarable, which ensures the representability of $H \times_S P$ and $P \times_S P$. It is clear that (ii) implies (iii), because we can take $S' = P$, and the hypothesis that P is formally principal homogeneous implies that $P \times_S P$ is trivial under $H \times_S P$, since it has a section (the diagonal section $P \to P \times_S P$). On the other hand, (iii) implies (i), since $\{S' \to S\}$ is a covering family by condition (M4$_\mathcal{T}$). It then remains to show that (i)⇒(ii). In this case, the morphism of sheaves $P \times_S H \to P \times_S P$ is locally an isomorphism, hence an isomorphism ([**?**] IV, 4.5.8); P is then formally principal homogeneous. The structural morphism $P \to S$ is locally isomorphic to the structural morphism $H \to S$, which belongs to $\mathcal{M}$. It is then an element of $\mathcal{M}$ by (M1) and (M5$_\mathcal{T}$). □

We note that if H acts freely on an S-object X and $p : X \to Y = X/H$ is the quotient morphism, then we have an induced morphism

$$(H \times_S Y) \times_Y X = H \times_S X \to X.$$

Therefore, $H \times_S Y$ has an induced action on X over Y, and the quotient $X/H \times_S Y$ is Y. The equivalence of (i) and (iv) in Proposition 14.3.46 can therefore be generalized to the following proposition:

Proposition 14.3.47. *Under the same hypothesis of Proposition 14.3.46, assume that the topology $\mathcal{T}$ is subcanonical. Let H be an S-group and X be an S-object over which H acts (on right). Suppose that the structural morphism $H \to S$ belongs to $\mathcal{M}$, then the following conditions are equivalent:*

(i) *H acts freely and $\mathcal{M}$-effectively on X.*

(ii) *There exists an S-morphism $p : X \to Y$ compatible with the equivalence relation on X defined by H and such that the induced action of $H \times_S Y$ on X over Y makes X a principal homogeneous bundle under H_Y over Y.*

Under these conditions, p identifies Y with the quotient X/H.

Corollary 14.3.48. *Let $\mathcal{C}$ be a category possessing a final object, arbitrary fiber products, and endowed with a subcanonical topology $\mathcal{T}$. Let $f : G \to H$ be a morphism of $\mathcal{C}$-groups and $K = \ker f$, and suppose that f belongs to a family $\mathcal{M}$ satisfying conditions (M1)–(M5$_{\mathcal{T}}$). Then H represents the quotient sheaf G/K, and f is a K_H-torsor*[8].

Proof. In fact, as f is covering, it is a universally effective epimorphism, so H is the quotient of G by the equivalence relation $R(f) = G \times_H G$, which is also the equivalence relation defined by K. On the other hand, the morphism $G \times K \to G \times_H G$, $(g, k) \mapsto (g, gk)$ is an isomorphism of $K_G = G \times_H K_H$-objects. Since the morphism $f : G \to H$ is covering, f is a K_H-torsor by Proposition 14.3.46 (ii). □

We can now specify Theorem 14.3.33 in the case of passage to quotient by a group action:

Proposition 14.3.49. *Under the hypothesis of Proposition 14.3.46, assume that the topology $\mathcal{T}$ is subcanonical and denote by F_0 the functor over S defined as follows: for any $S' \to S$, $F_0(S')$ is the set of representable sub-S'-functors Z of $X \times_S S'$, stable under $H \times_S S'$ and is a principal homogeneous bundle under the induced S'-group action.*

(a) *The following conditions are equivalent:*

   (i) The action of H on X is $\mathcal{M}$-effective and free.

   (ii) F_0 is representable

   Under these conditions, we have $F_0 = X/H$.

(b) *Let $\mathcal{N}$ be a family of morphisms which is stable under base change and such that for any covering family $\{S_i \to S\}$ and any family $\{T_i \to S_i\}$ of morphisms in $\mathcal{N}$, any descent data on the T_i relative to $\{S_i \to S\}$ is effective. Suppose that X is squarable and the morphism $X \times_S H \to X \times_S X$ belongs to $\mathcal{N}$, then the morphism $p : X \to F_0$ corresponding to the sub-object $X \times_S H$ of $X \times_S X$ identifies F_0 with the quotient sheaf X/H.*

14.3.3.2 Group structure and passage to quotient

We are now intersted in the algebraic structure induced on the quotient G/H of a group by a subgroup. We first consider category of sheaves over $\mathcal{C}$ for an arbitrary topology. By taking the canonical topology and apply Remark 14.3.36, we then obtain results for the passage to universally effective quotients in $\mathcal{C}$.

Proposition 14.3.50. *Let $u : H \to G$ be a monomorphisms of sheaves of groups. Then there exists a unique G-object structure on the quotient sheaf G/H such that the canonical morphism*

$$p : G \to G/H$$

is a morphism of G-objects. This structure is functorial relative to (G, H): if we have a commutative diagram

$$\begin{tikzcd} H \arrow[r] \arrow[d] & G \arrow[d] \\ H' \arrow[r] & G' \end{tikzcd}$$

Then the induced morphism $G/H \to G'/H'$ is compatible with $G \to G'$.

[8] We also say that G is a K-torsor over H.

Proof. The sheaf G/H is the sheaf associated with the presheaf

$$i(G)/i(H) : S \mapsto G(S)/H(S);$$

as # is left exact, it transforms objects acted by groups into objects acted by groups. Since the presheaf $i(G)/i(H)$ is endowed with an action by $i(G)$, $G/H = (i(G)/i(H))^{\#}$ is endowed with an action by $(i(G))^{\#} = G$. This structure clearly has the required properties. □

Corollary 14.3.51. *Let $u : H \to G$ be a monomorphism of $\mathcal{C}$-groups. Suppose that the action of H on G is universally effective, then there exists a uniqur G-object structure on the quotient object G/H in $\mathcal{C}$ such that $p : G \to G/H$ is a morphism of G-objects. This structure is functorial relative to (G, H).*

Proposition 14.3.52. *Let $u : H \to G$ be a monomorphism of sheaves of groups which identifies H with a normal subsheaf of G. Then there exists a unique group structure on the quotient sheaf G/H such that the caonical morphism $p : G \to G/H$ is a group morphism. This structure is functorial relative to the couple (G, H) (H being normal).*

Proof. The proof is the same as Proposition 14.3.50. □

Corollary 14.3.53. *Let $u : H \to G$ be a monomorphism of $\mathcal{C}$-groups identifying H with a normal subgroup of G. Suppose that the action of H on G is universally effective, then there exists a group structure on the quotient object G/H in $\mathcal{C}$ such that $p : G \to G/H$ is a morphism of groups. This structure is functorial relative to (G, H) (H being normal and acts universally effectively).*

We can characterize the group structure of G/H in the following way:

Proposition 14.3.54. *Under the conditions of Proposition 14.3.52, let K be a $\mathcal{C}$-group and $f : G \to K$ be a morphism. The following conditions are equivalent:*

(i) *f is a morphism of groups compatible with the equivalence relation defined by H.*

(ii) *f is a morphism of groups inducing the trivial morphism $H \to K$.*

(iii) *f factors into a morphism of groups $G/H \to K$.*

Proof. The equivalence of (i) and (ii) is proved set-wisely. We evidently have (iii)⇒(ii). The equivalence of (iii) and (ii) then follows from the formula

$$\mathrm{Hom}(G/H, K) \cong \mathrm{Hom}(i(G)/i(H), K)$$

and the definition of the group structure of G/H. □

Remark 14.3.55. In the preceding situation, if the kernel of f is exactly H, then the morphism $G/H \to K$ which factors f is a monomorphism. This follows from Proposition 14.3.14.

In the case of sheaves of groups, we can precise Proposition 14.3.27 as following:

Proposition 14.3.56. *Let G be a sheaf of groups, H be a normal subsheaf of groups. For any subsheaf of groups K of G containing H, let K' be the quotient group K/H considered as a a subgroup of $G' = G/H$. Then we have $K = K' \times_{G'} G$, and the maps $K \mapsto K/H$, $K' \mapsto K' \times_{G'} G$ define a bijection between the set of subsheaves of groups of G containing H and the set of subsheaves of groups of G'. In this correspondence, normal subgroups of G corresponds to that of G'.*

Proof. The first assertion follows equally from Proposition 14.3.27 and Lemma 14.3.9. It remains to show that K is normal in G if and only if K' is normal in G'. If K is normal in G, then the presheaf $i(K)/i(H)$ is normal in $i(G)/i(H)$, and the same is true for the associated sheaves. Conversely, if K' is normal in G', then the fiber product $K' \times_{G'} G$ is normal in G, which is equal to K. □

Now if L is a subsheaf of groups of G, then there exists a smallest normal subsheaf of groups $\overline{L}$ of G containing L, called the saturation of L. In fact, we have $\overline{L} = L \cdot H$.

Proposition 14.3.57. *Under the preceding conditions, $L \cdot H$ is a subsheaf of groups of G containing H and the image of L in G/H is identified with*

$$(L \cdot H)/H \cong L/(H \cap L).$$

Proof. Denote by L' the image sheaf of L in G/H. This is a subsheaf of groups of G/H corresponding to $L \cdot H$ by Proposition 14.3.56. As the morphism $L \to L'$ is covering, hence a universally effective epimorphism of sheaves, it follows from Proposition 14.3.25 that L' is identified with the quotient of L by the kernel of $L \to L'$, which is evidently $H \cap L$. $\square$

Finally, we consider the following case: we have a sheaf of groups G, a subsheaf of groups K of G and a subsheaf of groups H of K, which is normal in K. Let us first define a (right) action of the sheaf in groups $H \backslash K$ ($= K/H$) on G/H. The group K operates by right translations on G. As H is normal in K, this operation is compatible with the equivalence relation defined by the action of H and thus defines an operation of K on G/H, that is, a morphism of the opposite group K^{op} to $\mathcal{A}ut(G/H)$. Since the latter is a sheaf (cf. [?], IV, 4.5.13) and that this morphism is trivial on H, it factors through K/H and defines the desired operation. Since the right and left operations of G on itself commute, the operations of G and K/H on G/H commute.

Proposition 14.3.58. *Under the preceding conditions, K/H acts freely on G/H (on the right) and we have a canonical isomorphism of sheaves operated by G:*

$$(G/H)/(K/H) \cong G/K.$$

If K is normal in G, then K/H is normal in G/H and this isomorphism is a group isomorphism.

Proof. We have an isomorphism of presheaves

$$i(G)/i(K) \xrightarrow{\sim} (i(G)/i(H))/(i(K)/i(H))$$

which respects the action of $i(G)$. The result then follows by applying # on both sides. $\square$

Corollary 14.3.59. *Let G be a $\mathcal{C}$-group, K be a sub-$\mathcal{C}$-group of G, H be a normal sub-$\mathcal{C}$-group of K. Let $\mathcal{M}$ be a family of morphisms in $\mathcal{C}$ verifying the conditions (M1)–(M5$_{\mathcal{T}}$). Suppose that the right action of H on G (resp. K) is $\mathcal{M}$-effective, then K/H acts freely on G/H, and this action commutes with that of G. The following conditions are equivalent:*

(i) *The action of K on G is $\mathcal{M}$-effective.*

(ii) *The action of K/H on G/H is $\mathcal{M}$-effective.*

Under these conditions, we have an isomorphism of G-objects in $\mathcal{C}$:

$$(G/H)/(K/H) \cong G/K.$$

Proof. Since H acts $\mathcal{M}$-effectively on G and K, by Proposition 14.3.58 we have a diagram

$$\begin{tikzcd} H \arrow[r, hook] & K \arrow[r, hook] \arrow[d] & G \arrow[d] \\ & K/H \arrow[r, hook] & G/H \arrow[d] \\ & & G/K \end{tikzcd}$$

where the square is Cartesian. Since $\mathcal{M}$ is stable under composition, if K/H acts on G/H $\mathcal{M}$-effectively, then we conclude that $G \to G/K \in \mathcal{M}$, so K acts on G $\mathcal{M}$-effectively. Conversely, if $G \to G/K$ belongs to $\mathcal{M}$, then consider the Cartesian diagram

$$\begin{tikzcd} K \times G \arrow[r] \arrow[d] & G \arrow[d] \\ (K/H) \times (G/H) \arrow[r] & (G/H) \end{tikzcd}$$

By hypothesis, the morphism $K \times G \to G \times G$ belongs to $\mathcal{M}$ (Proposition 14.3.17), and $G \to G/H$ belongs to $\mathcal{M}$. We then conclude from (M1) and (M5$_{\mathcal{T}}$) that $(K/H) \times (G/H) \to G/H$ belongs to $\mathcal{M}$, so the equivalence relation on G/H defined by K/H is of type $\mathcal{M}$. Since the quotient of G/H by this equivalence relation is represented by G/K, we conclude from Theorem 14.3.33 that the action of K/H on G/H is $\mathcal{M}$-effective. □

Now let $\mathcal{N}$ be a family of morphisms in $\mathcal{C}$ verifying conditions (N1) and (N$_{\mathcal{M}}$) of **??** 14.3.2.2. By Proposition 14.3.56 and Proposition 14.3.41, we obtain:

Proposition 14.3.60. *Let G be a $\mathcal{C}$-group and H be a normal sub-$\mathcal{C}$-group of G whose action on G is $\mathcal{M}$-effective.*

(a) *For any sub-$\mathcal{C}$-group K of G containing H and such that the morphism $K \to G$ belongs to $\mathcal{N}$, H acts $\mathcal{M}$-effectively on K and the quotient $K/H \to K'$ is a sub-$\mathcal{C}$-group of $G' = G/H$ such that the morphism $K' \to G'$ belongs to $\mathcal{N}$.*

(b) *The map $K \mapsto K' = K/H$ is a bijection from the set of sub-$\mathcal{C}$-groups K of G containing H and such that the morphism $K \to G$ belongs to $\mathcal{N}$, H acts $\mathcal{M}$-effectively on K to the set of sub-$\mathcal{C}$-groups K' of G' such that the morphism $K' \to G'$ belongs to $\mathcal{N}$. Under this correspondence, the normal subgroups of G correspond to that of G'.*

Corollary 14.3.61. *If $H \to G$ belongs to $\mathcal{N}$, then $\mathcal{C}$ possesses a final object e and the unit section $e \to G/H$ belongs to $\mathcal{N}$.*

Proof. This follows form Proposition 14.3.60 by taking $K = H$. □

14.3.4 Applications to the category of schemes

Let **Sch** be the category of schemes, to which we can assocate the Zariski topology, that is, the topology generated by the family of morphisms $\{S_i \to S\}$, where each $S_i \to S$ is an open immersion and the union of images of S_i is equal to S. A sheaf over the Zariski topology is also called of local nature: this is a contravariant functor $F : \mathbf{Sch}^{\mathrm{op}} \to \mathbf{Set}$ such that for any scheme S and any covering $\{S_i \to S\}$, we have an exact diagram

$$F(S) \longrightarrow \prod_i F(S_i) \rightrightarrows \prod_{i,j} F(S_i \cap S_j)$$

In particular, a functor of local natura transforms direct sums to products. As any representable functor is a sheaf, this topology is coarser than the canonical topology.

To introduce (and handle) more topologies on **Sch**, we need a general criterion to identify the covering families of the topology generated by certain family of morphisms. This is contained in the following proposition.

Proposition 14.3.62. *Let $\mathcal{C}$ be a category and $\mathcal{C}'$ be a full subcategory. Let P be a set of families of morphisms of $\mathcal{C}$ with the same codomains, which is stable under composition and base change, and P' be a set of families of morphisms of $\mathcal{C}'$ containing the families of identity morphisms. We endow $\mathcal{C}$ with the topology generated by P and P' and suppose that the following conditions are satisfied:*

(a) *If $\{S_i \to S\} \in P'$ (hence $S_i, S \in \mathrm{Ob}(\mathcal{C}')$) and $T \to S$ is a morphism in $\mathcal{C}'$, then the fiber products $S_i \times_S T$ (in $\mathcal{C}$) exist and the family $\{S_i \times_S T \to T\}$ belongs to P'.*

(b) *For any $S \in \mathrm{Ob}(\mathcal{C})$, there exists $\{S_i \to S\} \in P$ with $S_i \in \mathrm{Ob}(\mathcal{C}')$ for each i.*

(c) *In the following situation*

$$S_{ijk} \xrightarrow{(P')} S_{ij} \xrightarrow{(P)} S_i \xrightarrow{(P')} S$$

where $S, S_i, S_{ij}, S_{ijk} \in \mathrm{Ob}(\mathcal{C}')$, $\{S_i \to S\} \in P'$, $\{S_{ij} \to S_i\} \in P$ for each i, $\{S_{ijk} \to S_{ij}\} \in P'$ for any i, j, there exists a family $\{T_n \to S\} \in P'$ and for each n a multi-index ijk and a commutative diagram

$$\begin{tikzcd} T_n \arrow[rr] \arrow[dr] & & S_{ijk} \\ & S \arrow[ur] & \end{tikzcd}$$

Then for a sieve R of $S \in \mathrm{Ob}(\mathcal{C})$ to be covering, it is necessary and sufficient that there exists a composite family

$$\begin{tikzcd} S_{ij} \arrow[r, dashed] \arrow[d, "(P')"'] & R \arrow[d] \\ S_i \arrow[r, "(P)"] & S \end{tikzcd} \tag{14.3.2}$$

where $S_i, S_{ij} \in \mathrm{Ob}(\mathcal{C}')$, $\{S_i \to S\} \in P$, $\{S_{ij} \to S_i\} \in P'$ for each i, and the morphisms $S_{ij} \to S$ factors through R.

Proof. Since the families in P and P' are covering, any family which is the composite of such families is again covering, so a sieve of the form indicated in the proposition is covering for $\mathcal{C}$, since it contains a covering sieve. Conversely, it suffices to prove that sieves of the form (14.3.2) form a topology, i,e, it suffices to verify the axioms (T1)–(T3).

To verify (T3), let $S \in \mathrm{Ob}(\mathcal{C})$. There exists by (b) a family $\{S_i \to S\} \in P$ with $S_i \in \mathrm{Ob}(\mathcal{C}')$. The families $\{\mathrm{id}_{S_i} : S_i \to S_i\}$ belong to P' by hypothesis, so the sieve S of S is of the following form:

$$\begin{tikzcd} S_i \arrow[r] \arrow[d, "(P')"'] & S \arrow[d, "\mathrm{id}_S"] \\ S_i \arrow[r, "(P)"] & S \end{tikzcd}$$

Now let R be a sieve of S with desired form (14.3.2) and R' be a sieve such that for any $T \to R$ in $\mathcal{C}$, the sieve $R' \times_T S$ is of the desired from. Then as the morphism $S_{ij} \to S$ factors through R, the sieve $R'_{ij} = R' \times_S S_{ij}$ of S_{ij} is of the desicred form by hypothesis:

$$\begin{tikzcd} R'_{ij} \arrow[r, hook] \arrow[dd] & S_{ij} \arrow[d, "(P')"'] \arrow[dr] & \\ & S_i \arrow[d, "(P)"'] & R \arrow[dl] \\ R' \arrow[r, hook] & S & \end{tikzcd}$$

so for each ij, we have a diagram of the form

$$\begin{tikzcd}
S_{ijkl} \arrow[d, "(P')"'] \arrow[dr] & \\
S_{ijk} \arrow[d, "(P)"'] & R'_{ij} \arrow[dl] \\
S_{ij} &
\end{tikzcd}$$

We have thus proved that there exists a composite family

$$S_{ijkl} \xrightarrow{(P')} S_{ijk} \xrightarrow{(P)} S_{ij} \xrightarrow{(P')} S_i \xrightarrow{(P)} S$$

belonging to $P \circ P' \circ P \circ P'$, which factors through R' and with all objects (except S) belong to $\mathcal{C}'$. Applying condition (c) to the family $\{S_{ijkl} \to S_i\}$, we then obtain for each i a family $\{T_{in} \to S_i\} \in P'$, such that $T_{in} \to S$ factors through one of the S_{ijkl}, hence through R':

$$\begin{tikzcd}
T_{in} \arrow[r] \arrow[d, "(P')"'] & S_{ijkl} \arrow[dd] \\
S_i \arrow[d, "(P)"'] & \\
S & R' \arrow[l, hook']
\end{tikzcd}$$

The sieve R' of S is therefore of the desired form (14.3.2), which verifies axiom (T2).

Fianlly, as for axiom (T1), let R be a sieve of S of the desired form and $T \to S$ be a morphism in $\mathcal{C}$. Let $T_i = S_i \times_S T$; the family $\{T_i \to T\}$ then belongs to P, and applying condition (b), we obtain for each i a family $\{U_{ik} \to T_i\} \in P$, with $U_{ik} \in \mathrm{Ob}(\mathcal{C}')$. By the hypotesis on P, we have $\{U_{ik} \to T\} \in P$, so by condition (a), $U_{ik} \times_{S_i} S_{ij} = U_{ikj}$ is an object of $\mathcal{C}'$ and for each ik, $\{U_{ikj} \to U_{ik}\} \in P'$.

$$\begin{tikzcd}
U_{ikj} \arrow[rr] \arrow[d, "(P')"'] & & S_{ij} \arrow[d, "(P')"] \arrow[ddr, bend left] & \\
U_{ik} \arrow[r, "(P)"] \arrow[dr, "(P)"'] & T_i \arrow[r] \arrow[d, "(P)"] & S_i \arrow[d, "(P)"] & \\
& T \arrow[r] & S & R \arrow[l, hook']
\end{tikzcd}$$

We therefore conclude that the family $\{U_{ikj} \to T\}$ factors through the sieve $T \times_S R$ of T, which is hence of the desired form. This proves axiom (T1) and completes the proof. □

Corollary 14.3.63. *If $S \in \mathrm{Ob}(\mathcal{C}')$ and R is a sieve of S, then R is covering if and only if there exists a family $\{T_i \to S\} \in P'$ which factors through R.*

Proof. In fact, any such sieve is covering. Conversely, it suffices to apply (c) to the family $\{S_i \to S\}$ and the identity morphisms of S_i to deduce that any covering sieve is of the indicated form. □

Corollary 14.3.64. *For a presheaf $F \in \mathrm{PSh}(\mathcal{C})$ to be separated (resp. a sheaf), it is necessary and sufficient that the morphisms*

$$F(S) \longrightarrow \textstyle\prod_i F(S_i)$$

is injective (resp. that the diagram

$$F(S) \longrightarrow \prod_i F(S_i) \rightrightarrows \prod_{i,j} F(S_i \times_S S_j)$$

is exact) for $\{S_i \to S\} \in P$ and $\{S_i \to S\} \in P'$, respectively.

Proof. In fact, these conditions are necessary, because the families above are covering. Conversely, if R is the sieve of S of a family of morphisms $\{S_{ij} \overset{(P')}{\to} S_i \overset{(P)}{\to} S\}$, a diagram chasing shows that the above conditions imply that $\mathrm{Hom}(S, F) \to \mathrm{Hom}(R, F)$ is injective (resp. bijective). But any covering sieve R' of S contains a sieve generated by such a family and we have a commutative diagram

$$\begin{tikzcd} \mathrm{Hom}(S,F) \arrow[rr, "f"] \arrow[dr, "g"'] & & \mathrm{Hom}(R,F) \arrow[dl, "h"] \\ & \mathrm{Hom}(R',F) & \end{tikzcd}$$

If g is injective, then so is f, so in this case F is separated. In this case, since the morphism $R' \to R$ is covering, we see that h is also injective (cf. **??**). Therefore, if g is bijective, so is f, hence F is a sheaf. □

Remark 14.3.65. The condition (c) of Proposition 14.3.62 is satisfies if P' is stable under composition and if any family $\{S_i \to S\}$ of morphisms in P with $S_i, S \in \mathrm{Ob}(\mathcal{C}')$ has a subfamily belonging to P'.

We now let $\mathcal{C} = \mathbf{Sch}$ be the category of schemes, and $\mathcal{C}'$ be the full subcategory formed by affine schemes. We shall consider the following sets P':

P_1': finite and surjective families, formed by flat morphisms;

P_2': finite and surjective families, formed by flat morphisms of finite presentation;

P_3': finite and surjective families, formed by étale morphisms;

P_4': finite and surjective families, formed by finite étale morphisms;

For each of these sets P_i' (except P_4'), the conditions of Proposition 14.3.62 are satisfied (as for (c), note that an affine scheme is quasi-compact, so any family of morphisms of $\mathcal{C}'$, belonging to P, contains a finite subfamily which is equally in P, hence in P_i' for $i = 1, 2, 3$). The corresponding topology $\mathcal{T}_i$ generated by P and P_i' is denoted and called by the following manner:

$\mathcal{T}_1$ is the faifufully flat and quasi-compact topology (fpqc);

$\mathcal{T}_2$ is the faifufully flat and finite presented topology (fppf);

$\mathcal{T}_3$ is the étale topology (ét);

$\mathcal{T}_4$ is the finite étale topology (étf).

As $P_1' \supseteq P_2' \supseteq P_3' \supseteq P_4'$, we have

$$\text{fpqc} \geq \text{fppf} \geq \text{ét} \geq \text{étf} \geq \text{Zar}.$$

Proposition 14.3.66. *Let $\mathcal{T}_i$ $(i = 1, 2, 3, 4)$ be the topologies on* **Sch** *defined above.*

(a) *For a sieve R of S to be covering for $\mathcal{T}_i$ $(1 \leq i \leq 3)$, it is necessary and sufficient that there exists a covering (S_α) of S by affine opens and for each α a family $\{S_{\alpha\beta} \to S_\alpha\} \in P_i'$, with $S_{\alpha\beta}$ affine, such that the the family $\{S_{\alpha\beta} \to S\}$ factors through R.*

(b) *For a presheaf F over* **Sch** *to be a sheaf for the fpqc topology (resp. fppf, étale, finite étale), it is necessary and sufficient that*

   (i) F is a sheaf over the Zariski topology, i.e. a functor of local nature.

   (ii) For any faithfully flat morphism (resp. faithfully flat morphism of finite presentation, resp. surjective étale, resp. finite surjective étale) $T \to S$ of affine schemes, we have an exact diagram

$$F(S) \longrightarrow F(T) \rightrightarrows F(T \times_S T)$$

(c) *The topologies $\mathcal{T}_i$ $(1 \leq i \leq 4)$ are subcanonical.*

(d) *Any surjective family formed by open and flat morphisms (resp. flat and locally of finite presentation, resp. étale, resp. finite and étale) is covering for the fpqc topology (resp. fppf, resp. étale, resp. finite étale).*

(e) *Any finite and surjective family, formed by flat and quasi-compact morphisms, is covering for the fpqc topology.*

Proof. Assertion (a) follows from Proposition 14.3.62, and (b) follows from Corollary 14.3.64, since a sheaf for the Zariski topology transforms direct sums into products. Any representable functor is a sheaf for Zariski topology, and satisfies condition (ii) by ([**?**], VIII, 5.3), so $\mathcal{T}_1$ is subcanonical, which proves (c).

Let $\{S_i \to S\}$ be a family of morphisms as in (d). By considering a covering of S by affine opens, we are reduced to the case where S is affine. We first deal with the case where each $S_i \to S$ is flat and open (resp. étale). Let S_{ij} be a covering og S_i be affine opens. As the morphisms considered are open, the images T_{ij} of S_{ij} in S form an open covering of S. As S is affine, hence quasi-compact, there exists a finite subcover of T_{ij}, with i, j belongs to a finite set F. Then $S' = \coprod_F S_{ij}$ is affine, and the morphism $S' \to S$ belongs to P_1' (resp. P_3'), hence is covering. As this factors through the given family $\{S_i \to S\}$, the latter is also covering.

In the case of the finite étale topology, each S_i is finite over S, hence is affine; in the preceding argument, we can then take for $\{S_{ij}\}$ the covering $\{S_i\}$ of S_i, and we obtain a morphism $S' \to S$ belonging to P_4'.

Now consider the case where $f_i : S_i \to S$ are flat and locally of finite presentation. For any $s \in S$, there exists (by the proof of ([**?**], IV_4, 17.16.2)) an affine subscheme $X(s)$ of one of the S_i such that $s \in f_i(X(s))$ and that the morphism $g_i : X(s) \to S$, restriction of f_i, is flat and quasi-finite. Then $g_i(X(s))$ is an open neighborhood $U(s)$ of s (([**?**], IV_2, 2.4.6)), and as S is affine, it is covered by a finite number of such opens $U(s_j)$, $j = 1, \dots, n$. Therefore, $X' = \coprod_j X(s_j)$ is affine, and the morphism $X' \to S$ is surjective, flat, of finite presentation and quasi-finite, hence belongs to P_2', which completes the proof of (d).

Finally, let $\{S_i \to S\}$ be a finite and surjective family of flat and quasi-compact morphisms. Let T_j be a covering of S be affine opens. Then $S_{ij} = T_j \times_S S_i$ is quasi-compact and hence has a finite affine covering T_{ijk}. Each morphism $T_{ijk} \to T_j$ is flat, and the family $\{T_{ijk} \to T_j\}$ is finite and surjective, hence covering for $\mathcal{T}_1$. The family $\{T_{ijk} \to S\}$ is hence also, by composition, covering, and it factors through the given family:

$$\begin{tikzcd} T_{ijk} \arrow[r] \arrow[dr] & S_{ij} \arrow[r] \arrow[d] & S_i \arrow[d] \\ & T_j \arrow[r] & S \end{tikzcd}$$

so the given family $\{S_i \to S\}$ is also covering for $\mathcal{T}_1$. □

Proposition 14.3.67. *Let $\mathcal{M}_i$ be the family of the following morphisms:*

$\mathcal{M}_1$: *faithfully flat and quasi-compact morphisms.*

$\mathcal{M}_2$: *faithfully flat and locally of finite presentaion morphisms.*

$\mathcal{M}_3$: *surjective étale morphisms.*

$\mathcal{M}_4$: *finite surjective étale morphisms.*

Then the family $\mathcal{M}_i$ verifies conditions (M1)–(M$_{\mathcal{T}_i}$) of 14.3.2.2.

Proof. For (M1)–(M3), these are classical results. By Proposition 14.3.66 (d) and (e), we see that $\mathcal{M}_i$ satisfies (M4$_{\mathcal{T}_i}$), so it remains to verify (M5$_{\mathcal{T}_i}$). For this, it suffices to show that each $\mathcal{M}_i$ satisfies condition (M5$_{\mathcal{T}_1}$), since it implies the others. This follows from ([**?**], VIII, n4 and n5). $\square$

Corollary 14.3.68. *If X is a scheme and R is an equivalence relation of type $\mathcal{M}_i$, then R is $\mathcal{M}_i$-effective if and only if the quotient sheaf X by R for $\mathcal{T}_i$ is representable and in this case it is represented by X/R.*

Proof. In fact, this is a concequence of Proposition 14.3.30. $\square$

We also consider families $\mathcal{N}$ of morphisms verifying conditions (N1) and (N$_{\mathcal{T}_i}$). But note, as above, that condition (N$_{\mathcal{T}_1}$) implies the others.

Proposition 14.3.69. *The following families satisfy conditions (N1) and (N$_{\mathcal{T}_1}$) of 14.3.2.2, that is, have descent property for the fpqc topology:*

$\mathcal{N}$: *open immersions.*

$\mathcal{N}'$: *closed immersions.*

$\mathcal{N}''$: *quasi-compact immersions.*

Proof. In view of Proposition 14.3.66 (b), it suffices to consider the descent data relative to the Zariski topology and to a faithfully flat and quasi-compact morphism. The first assertion is clear; for the second one, the case for $\mathcal{N}$ and $\mathcal{N}'$ follows from ([**?**], VIII, 4.4) and ([**?**], VIII, 1.9). The case for $\mathcal{N}''$ can be deduce as in ([**?**], VIII, 5.5), by using the previous two results. $\square$

We can therefore apply the results of the previous subsections to the families of morphisms given above. Let us give one as an example (Corollary 14.3.37 and Proposition 14.3.40):

Corollary 14.3.70. *Let X be a scheme and R be an equivalence relation on X. Suppose that $R \to X$ is faithfully flat and quasi-compact and $R \to X \times X$ is a closed immersion (resp. open immersion, resp. quasi-compact immersion). Then the quotient sheaf X/R is the same for the fpqc topology and the canonical topology, and for any scheme S, we have*

$$(X/R)(S) = \left\{\begin{array}{l}\text{closed (resp. open, resp. quasi-compact) subschemes } Z \text{ of } X \times S \\ \text{stable under } R \times S \text{ such that } Z \to X_S \text{ belongs to } \mathcal{N}\text{, that } R_Z \text{ is} \\ \text{faithfully flat and quasi-compact, and that diagram } R_Z \rightrightarrows Z \to S \\ \text{is exact}\end{array}\right\}.$$

Remark 14.3.71. The corresponding principal homogeneous bundles for the étale topology (resp. finite étale topology, resp. Zariski topology) is called **quasi-isotrivial** (resp. **locally isotrivial**, resp. **locally trivial**).

14.3.4.1 Homogeneous spaces Let G be an S-group scheme, X an S-scheme acted (on right) by G, and

$$\Phi : G \times_S X \to X \times_S X$$

be the S-morphism defined setwise by $(g, x) \mapsto (gx, x)$. Recall that X is called formally principal homogeneous under G if the following equivalent conditions are satisfied[9]:

(i) For any $T \to S$, the set $X(T)$ is either empty of pincipal homogeneous under $G(T)$,

(ii) Φ is an isomorphism of S-functors,

(iii) Φ is an isomorphism of S-schemes.

The definition of **formally homogeneous spaces** (not necessarily principal) is obtained by demanding that Φ is an epimorphism in the category of sheaves for an appropriate topology $\mathcal{T}$. On the other hand, the condition that Φ is an epimorphism of S-functors is equivalent to that, for any $T \to S$, the set $X(T)$ is empty or homogeneous (not necessarily principal) under $G(T)$. But this condition is too restrictive, as shown in the following example.

Example 14.3.72. Let $S = \operatorname{Spec}(\mathbb{R})$, $G = \mathbb{G}_{m,\mathbb{R}}$ and $X = \mathbb{G}_{m,R}$ over which G acts by $t \cdot x = t^2 x$. Then the morphism Φ is étale, finite and surjective, hence an epimorphism in the category of sheaves for the finite étale topology (a fortiori, an epimorphism of S-schemes). However, the points 1 and -1 of $X(\mathbb{R})$ are not conjugate under $G(\mathbb{R})$, so that the morphism $G(\mathbb{R}) \times X(\mathbb{R}) \to X(\mathbb{R}) \times X(\mathbb{R})$ is not surjective[10].

We are then proposed to the following definition:

Definition 14.3.73. Let G be an S-group, X be an S-group scheme acted by G and $\mathcal{T}$ be a subcanonical topology over $\mathbf{Sch}_{/S}$. We say that X is a **formally homogeneous space under** G (relative to the topology $\mathcal{T}$) if the following equivalent conditions are satisfied:

(i) the morphism $\Phi : G \times_S X \to X \times_S X$ is an epimorphism in the category of sheaves for the topology $\mathcal{T}$.

(ii) for any $T \to S$ and $x, y \in X(T)$, there exists a covering morphism $T' \to T$ for the topology $\mathcal{T}$ and $g \in G(T')$ such that $y_{T'} = g \cdot x_{T'}$.

Remark 14.3.74. Condition (i) implies that Φ is a universally effective epimorphism in $\mathbf{Sch}_{/S}$ (cf. [?], IV, 4.4.3). This implies, in particular, that Φ is a surjective morphism of schemes.

Proposition 14.3.75. *Let G be an S-group, X be an S-scheme acted by G, and $\mathcal{T}$ be a subcanonical topology over $\mathbf{Sch}_{/S}$. The following conditions are equivalent:*

(i) *X verifies the following conditions:*

  (a) *the morphism $\Phi : G \times_S X \to X \times_S X$ is covering, i.e. X is a formally homogeneous G-space.*

  (b) *the morphism $X \to S$ is covering, i.e. locally for the topology $\mathcal{T}$, it possesses a section*[11].

[9] The equivalence of (i) and (ii) is clear, and (ii)⇔(iii) since $\mathcal{C}$ is a full subcategory of $\widehat{\mathcal{C}}$.

[10] Obviously, this difficulty comes from the fact that if $\mathcal{C}'$ is a full subcategory of $\widehat{\mathcal{C}}$ containing $\mathcal{C}$, for example, the category of sheaves on $\mathcal{C}$ for a subcanonical topology $\mathcal{T}$, and if $f : X \to Y$ is a morphism in $\mathcal{C}$, then the implications

$$f \text{ is an epimorphism in } \widehat{\mathcal{C}} \Rightarrow f \text{ is an epimorphism in } \mathcal{C}' \Rightarrow f \text{ is an epimorphism in } \mathcal{C}$$

is in general strict.

[11] Note that the morphism $X \to S$ is an epimorphism in $\widehat{\mathbf{Sch}_{/S}}$ if and only if for any $T \to S$, the morphism $X(T) \to S(T) = \{T \to S\}$ is surjective, that is, the morphism $T \to S$ factors through X. But since $T \to S \in S(T)$ is the image of the identity morphism $\mathrm{id}_S : S \to S$, this is true if and only if id_S factors through X, which means X admits a section.

(ii) *X is locally isomorphic (as a G-scheme) to the quotient sheaf (for $\mathcal{T}$) of G by a subgroup scheme H, that is, there exists a covering family $\{S_i \to S\}$ such that each $X \times_S S_i$ represents the quotient sheaf of $G \times_S S_i$ by a subgroup scheme H_i.*

*Under these conditions, we say that X is a **homogeneous G-space** (relative to the topology $\mathcal{T}$).*

Proof. Suppose that (ii) is satisfied. Put $G_i = G \times_S S_i$ and $X_i = X \times_S S_i$, then X_i possesses a section over S_i, namely the composition of the unit section $\varepsilon_i : S_i \to G_i$ and the projection $\pi_i : G_i \to X_i = G_i/H_i$. We then conclude that $X \to S$ is covering.

On the other hand, π_i is covering, so $\pi_i \times \pi_i$ is also covering, and we have a commutative diagram

$$\begin{tikzcd} G_i \times_{S_i} G_i \arrow[r, "\Psi_i"] \arrow[d, "\mathrm{id} \times \pi_i"] & G_i \times_{S_i} X_i \arrow[d, "\pi_i \times \pi_i"] \\ G_i \times_{S_i} X_i \arrow[r, "\Phi_i", "\cong"'] & G_i \times_{S_i} X_i \end{tikzcd}$$

where Φ_i is induced by Φ by base change $S_i \to S$ and Ψ_i is the isomorphism defined by $(g, g') \mapsto (gg', g)$. Then $(\pi_i \times \pi_i) \circ \Psi_i$ is covering, hence Φ_i is a covering. This shows that Φ is locally covering, hence is covering, whence (ii)⇒(i).

Conversely, suppose that (i) is satisfied, and moreover the structural morphism $X \to S$ possesses a section $\sigma \in X(S)$. By Corollary 10.5.8, σ is an immersion. Define $H = G \times_X S$ by the diagram below, where the two squares are Cartesian:

$$\begin{tikzcd} H \arrow[r] & G \arrow[r, "\mathrm{id}_G \boxtimes \sigma"] \arrow[d, "\pi"] & G \times_S X \arrow[d, "\Phi"] \\ S \arrow[r, "\sigma"] & X \arrow[r, "\mathrm{id}_X \boxtimes \sigma"] & X \times_S X \end{tikzcd}$$

where π, $\mathrm{id}_G \boxtimes \sigma$ and $\mathrm{id}_X \boxtimes \sigma$ denote the morphisms defined setwisely, for $T \to S$ and $g \in G(T)$, $x \in X(T)$, by

$$\pi(g) = g \cdot \sigma_T, \quad (\mathrm{id}_G \boxtimes \sigma)(g) = (g, \sigma_T), \quad (\mathrm{id}_X \boxtimes \sigma)(x) = (x, \sigma_T).$$

Then π is covering, and H is a subgroup scheme of G, represented by the stabilizer $\mathrm{Stab}_G(\sigma)$ of σ, that is, for any $T \to S$, we have

$$H(T) = \{g \in G(T) : g \cdot \sigma_T = \sigma_T\}.$$

Denote by G/H the presheaf $T \mapsto G(T)/H(T)$, and $(G/H)^\#$ the associated sheaf for the topology $\mathcal{T}$. From the above arguments, we obtain a commutative diagram of morphisms of presheaves acted by G:

$$\begin{tikzcd} G \arrow[r, "\pi"] \arrow[d] & X \\ G/H \arrow[ur, "\bar{\pi}"'] & \end{tikzcd}$$

where $\bar{\pi}$ is a monomorphism (cf. Proposition 14.3.14). As π is covering, $\bar{\pi}$ is also a covering, so $\bar{\pi}$ induces an isomorphism $(G/H)^\# \cong X$. Therefore, we have shown that: if X is a homogeneous G-space such that $X \to S$ admits a section σ, then X represents the quotient sheaf G/H, where $H = G \times_X S$ is the stabilizer of σ.

In the general case, by hypothesis there exists a covering family $\{S_i \to S\}$ such that each morphism $X_i = X \times_S S_i \to S_i$ possesses a section σ_i. Put $G_i = G \times_S S_i$, then the morphism $\Phi_i = G_i \times_{S_i} X_i \to X_i \times_{S_i} X_i$ deduced from Φ by base change $S_i \to S$ is covering, hence, by the preceding arguments, $X_i \cong G_i/H_i$ where $H_i = \mathrm{Stab}_{G_i}(\sigma_i)$. This proves the implication (i)⇒(ii). □

14.4 Construction of quotient schemes

14.4.1 $\mathcal{C}$-groupoids

Let $\mathcal{C}$ be a category which has finite products and coproducts. Recall that a diagram

$$X_1 \underset{d_0}{\overset{d_1}{\rightrightarrows}} X_0 \xrightarrow{p} Y$$

in $\mathcal{C}$ is called **exact** if $pd_0 = pd_1$ and if, for any $T \in \mathcal{C}$, $T(p)$ is a bijection from $T(Y)$ to the subset of $T(X_0)$ formed by morphisms $f : X_0 \to T$ such that $fd_0 = fd_1$. We also say that (Y, p) is the cokernel of (d_0, d_1), and write

$$(Y, p) = \operatorname{coker}(d_0, d_1).$$

Let $\mathcal{C}$ be the category **Rsp** of ringed spaces. In this case, there always exists a cokernel (Y, p), of which we can give the following description: the underlying topological space Y is obtained from X_0 by identifying the points $d_0(x)$ and $d_1(x)$, endowed with the quotient topology. The canonical morphisms $\pi : X_0 \to Y$ and d_0, d_1 then induce a double arrow of sheaves of rings over Y:

$$\pi_*(\mathscr{O}_0) \underset{\delta_0}{\overset{\delta_1}{\rightrightarrows}} \pi_*((d_0)_*(\mathscr{O}_1)) = \pi_*((d_1)_*(\mathscr{O}_1))$$

where $\mathscr{O}_i$ is the structural sheaf of X_i. We choose $\mathscr{O}_Y$ to be the sheaf of rings over Y whose sections s are such that $\delta_0(s) = \delta_1(s)$. The morphism $p : X_0 \to Y$ is defined in the evident way.

Let $d_0, d_1 : X_1 \rightrightarrows X_0$ be a diagram in **Rsp** and (Y, p) be the cokernel. We say that an open subset U of X_0 is saturated if $d_0^{-1}(U) = d_1^{-1}(U)$, which is equivalent to that $U = p^{-1}(p(U))$. In this case, as Y is endowed with the quotient topology, $p(U)$ is an open subset of Y.

Lemma 14.4.1. *Let U be a saturated open subset of X and $V = p(U)$. If we denote by $U_1 = d_0^{-1}(U) = d_1^{-1}(U)$ the open subset of X_1, and $\tilde{d}_0$, $\tilde{d}_1$ and $\tilde{p}$ the restriction of d_0, d_1 to U_1 and p to U, then $(V, \tilde{p})$ is the cokernel of the following diagram in* **Rsp***:*

$$U_1 \underset{\tilde{d}_0}{\overset{\tilde{d}_1}{\rightrightarrows}} U \xrightarrow{\tilde{p}} V$$

Proof. Since U is saturated, the morphisms d_0, d_1 and p restricts to give the desired diagram. The claim that $(V, \tilde{p})$ is the cokernel is an immediate verification. □

Remark 14.4.2. The result of Lemma 14.4.1 is not true in the category of schemes. For example, let $S = \operatorname{Spec}(\mathbb{C})$, $X_0 = \mathbb{A}^2_S = \operatorname{Spec}(\mathbb{C}[x_1, x_2])$, $d_1 : \mathbb{G}_{m,S} \times_S \mathbb{A}^2_S \to \mathbb{A}^2_S$ be the action of $\mathbb{G}_{m,S}$ on $\mathbb{A}^2_S$ by multiplication, and $d_0 : \mathbb{G}_{m,S} \times_S \mathbb{A}^2_S \to \mathbb{A}^2_S$ the projection to the second factor. Let $U = \mathbb{A}^2_S - \{\mathfrak{m}\}$, where $\mathfrak{m}$ is the point $(0, 0)$. Then the projective space $\mathbb{P}^1_S$ is the cokernel of $(\tilde{d}_0, \tilde{d}_1)$ in **Rsp** and $\mathbf{Sch}_{/S}$, and the cokernel Y of (d_0, d_1) in **Rsp** is the union of $\mathbb{P}^1_S$ and the point $y_0 = \{p(\mathfrak{m})\}$, with the unique open subset containing y_0 being Y and we have $\Gamma(Y, \mathscr{O}_Y) = \mathbb{C}$ (note that Y is not a scheme). If $f : \mathbb{A}^2_S \to T$ is a morphism of S-schemes such that $fd_0 = fd_1$ and $\bar{f} : Y \to T$ is the induced morphism of ringed spaces, then for any affine open subset $V = \operatorname{Spec}(A)$ of T containing the point $t_0 = f(\mathfrak{m})$, we have $f^{-1}(V) = Y$, so the ring homomorphism $A \to \mathbb{C}[x_1, x_2]$ factors through $\mathbb{C}$. This shows that $S = \operatorname{Spec}(\mathbb{C})$ is the cokernel of (d_0, d_1) in the category $\mathbf{Sch}_{/S}$.

Lemma 14.4.3. *Let $d_0, d_1 : X_1 \rightrightarrows X_0$ be a diagram in* **Sch** *and (Y, p) be the cokernel in* **Rsp***.*

(a) *If Y is a scheme and p is a morphism of schemes, then (Y, p) is a cokernel of (d_0, d_1) in* **Sch***.*

(b) *Suppose that any point of X_0 has a saturated open neighborhood U such that, if $(\tilde{d}_0, \tilde{d}_1)$ is the induced diagram to $d_0^{-1}(U) = d_1^{-1}(U)$ and (Q, q) is the cokernel of $(\tilde{d}_0, \tilde{d}_1)$ in* **Rsp***, then Q is a scheme and q is a morphism of schemes. Then (Y, p) is a cokernel of (d_0, d_1) in* **Sch***.*

Proof. In the situation of (a), since (Y, p) is a cokernel of (d_0, d_1) in **Rsp**, every morphism $f : X_0 \to T$ of schemes such that $fd_0 = fd_1$ factors into a morphism $\bar{f} : Y \to T$ of ringed spaces. Now we know that $f = \bar{f}p$, and f, p are both morphisms of locally ringed spaces with p being surjective; it follows that $\bar{f}$ is also a morphism of locally ringed spaces, hence (Y, p) is a cokernel on (d_0, d_1) in **Sch**. Now (b) follows from (a) and Lemma 14.4.1 by glueing. □

In this section, we consider the existence of $\mathrm{coker}(d_0, d_1)$ if the two morphisms arise from a groupoid. More precisely, denote by $X_2 = X_1 \times_{d_1,d_0} X_1$ the fiber product and d'_0, d'_2 the two projections of X_2 to X_1, so that we have a Cartesian square:

$$\begin{tikzcd} X_2 \arrow[r, "d'_0"] \arrow[d, "d'_2"'] & X_1 \arrow[d, "d_1"] \\ X_1 \arrow[r, "d_0"] & X_0 \end{tikzcd} \tag{14.4.1}$$

Moreover, suppose that we are given a third morphism $d'_1 : X_2 \to X_1$. We say that $(d_0, d_1 : X_1 \rightrightarrows X_0, d'_1)$ is a $\mathcal{C}$**-groupoid** if for any object T of $\mathcal{C}$, $X_1(T)$ is the set of morphisms of a groupoid $X_*(T)$ whose set of objects is $X_0(T)$, with source map $d_1(T)$, target map $d_0(T)$, and whose composition map is $d'_1(T)$ (we identify $(X_1 \times_{d_1,d_0} X_1)(T)$ with $X_1(T) \times_{d_1(T),d_0(T)} X_1(T)$)[12].

If φ is a morphism of the groupid $X_*(T)$, the map $f \mapsto \varphi f$ is a bijection from the set of morphisms f whose target coincides with the source of φ to the set of morphisms with the same target as φ. We then conclude that there is a Cartesian square

$$\begin{tikzcd} X_2 \arrow[r, "d'_1"] \arrow[d, "d'_0"'] & X_1 \arrow[d, "d_0"] \\ X_1 \arrow[r, "d_0"] & X_0 \end{tikzcd} \tag{14.4.2}$$

Moreover, this square is also cocartesian: if $\varphi : X_1 \to Y$ is a morphism in $\mathcal{C}$ such that $\varphi d'_1 = \varphi d'_0$, then for any $T \in \mathrm{Ob}(\mathcal{C})$, the value of the morphism $\varphi(T) : X_1(T) \to Y(T)$ on $f \in X_1(T)$ only depends on the target of f (if g is another morphism with the same target as f, then $f^{-1}g$ is in the image of d'_1 and we have $\varphi(f) = \varphi d'_1(g, g^{-1}f) = \varphi d'_0(g, g^{-1}f) = \varphi(g)$), so it factors through $d_0(T)$.

Similarly, the map $g \mapsto g \circ \varphi$ is a bijection from the set of morphisms g whose source coincides with the target of φ to the set of morphisms with the same source as φ. We then conclude that there is a Cartesian square

$$\begin{tikzcd} X_2 \arrow[r, "d'_1"] \arrow[d, "d'_2"'] & X_1 \arrow[d, "d_1"] \\ X_1 \arrow[r, "d_1"] & X_0 \end{tikzcd} \tag{14.4.3}$$

which is also cocartesian.

On the other hand, let $s : X_0 \to X_1$ be the unique morphis in $\mathcal{C}$ such that, for any $T \in \mathrm{Ob}(\mathcal{C})$, $s(T) : X_0(T) \to X_1(T)$ associates to any object of $X_*(T)$ the identity morphism of this object. The morphism s satisfies the following equalities:

$$d_1 s = \mathrm{id}_{X_0}, \tag{14.4.4}$$

[12]Therefore, in this case, $X_2(T)$ is the set of pairs of composable morphisms (f_2, f_1), that is, such that $d_0(f_1) = d_1(f_2)$, and d'_0, d'_1 and d'_2 send (f_2, f_1) to f_2, $f_2 \circ f_1$, f_1, respectively.

$$d_0 s = \mathrm{id}_{X_0}\,. \tag{14.4.5}$$

Finally, the associativity of the composition maps is expressed by the commutativity of the following diagram

$$\begin{tikzcd} X_1 \times_{d_1,d_0} X_1 \times_{d_1,d_0} X_1 \arrow[r, "d_1' \times \mathrm{id}_{X_1}"] \arrow[d, "\mathrm{id}_{X_1} \times d_1'"'] & X_1 \times_{d_1,d_0} X_1 \arrow[d, "d_1"] \\ X_1 \times_{d_1,d_0} X_1 \arrow[r, "d_1'"] & X_0 \end{tikzcd} \tag{14.4.6}$$

Conversely, the conditions (14.4.2), (14.4.3) and (14.4.6) and the existence of a morphism s satisfying (14.4.4) imply that $(d_0, d_1 : X_1 \rightrightarrows X_0, d_1')$ is a groupoid. In the rest of this section, we mainly use the squares (14.4.1), (14.4.2) and (14.4.3), which are summarized into the following diagram:

$$\begin{tikzcd} X_2 \arrow[r, shift left, "d_0'"] \arrow[r, shift right, "d_1'"'] \arrow[d, shift right, "d_2'"'] \arrow[d, shift left, "d_1'"] & X_1 \arrow[r, "d_0"] \arrow[d, "d_1"] & X_0 \\ X_1 \arrow[r, "d_0"] \arrow[d, "d_1"'] & X_0 & \\ X_0 & & \end{tikzcd} \tag{14.4.7}$$

where the square is Cartesian and the first row and first column are exact.

We only use associativity in an indirect way, for example to ensure the existence of a morphism s satisfying (14.4.4) and (14.4.5), or to ensure the existence of a morphism $\sigma : X_1 \to X_1$ such that

$$d_0\sigma = d_1, \quad d_1\sigma = d_0. \tag{14.4.8}$$

(We choose σ so that $\sigma(T) : X_1(T) \to X_1(T)$ sends a morphism of $X_*(T)$ to its inverse.)

By abusing of languages, a $\mathcal{C}$-groupoid is also defined to be a diagram

$$\begin{tikzcd} X_2 \arrow[r, shift left=2, "d_0'"] \arrow[r, "d_1'" description] \arrow[r, shift right=2, "d_2'"'] & X_1 \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] & X_0 \end{tikzcd}$$

such that (14.4.1), (14.4.2) and (14.4.3) are Cartesian, that (14.4.6) is commutative and that there exists s satisfying (14.4.4) and (14.4.5)[13].

Example 14.4.4. Let X be an object in $\mathcal{C}$ and G be a $\mathcal{C}$-group acting on X (on the left). We denote by $d_0 : G \times X \to X$ the morphism defining the action of G over X, by $d_1 : G \times X \to X$ the projection onto the second factor, by $\mu : G \times G \to G$ the multiplication of G, and finally by $\mathrm{pr}_{2,3}$ the projection of $G \times G \times X = G \times (G \times X)$ onto the second and third factors. Then

$$\begin{tikzcd} G \times G \times X \arrow[r, shift left=2, "\mathrm{pr}_{2,3}"] \arrow[r, "\mu \times \mathrm{id}_X" description] \arrow[r, shift right=2, "\mathrm{id}_G \times d_0"'] & G \times X \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] & X_0 \end{tikzcd}$$

is a groupoid in $\mathcal{C}$. For any $T \in \mathrm{Ob}(\mathcal{C})$, the groupoid $X_*(T)$ has object set $X(T)$ and morphisms (g, x), where $g \in G(T)$ and $x \in X(T)$. Moreover, $X_*(T)$ is a setoid if and only if for any $x \in G(T)$, the automorphism group $\mathrm{Aut}(x)$ is trivial, that is, if and only if $G(T)$ acts freely on $X(T)$.

[13]For a gropoid X_*, we often say that X_0 is the base of the groupoid and X_1 is the equvialence prerelation.

Example 14.4.5. Let $d_0, d_1 : X_1 \to X_0$ be an **equivalence couple**, that is, if $d_0 \boxtimes d_1 : X_1 \to X_0 \times X_0$ is the morphism with components d_0, d_1, then for any $T \in \mathrm{Ob}(\mathcal{C})$, $(d_0 \boxtimes d_1)(T)$ is a bijection from $X_1(T)$ to the graph of an equivalence relation on $X_0(T)$. The set $X_1(T)$ is then identified with the set of couples (x, y) formed by elements of $X_1(T)$ such that $x \sim y$; similarly, the set $X_2(T) = (X_1 \times_{d_1, d_0} X_1)(T)$ is identified with the set of triples (x, y, z) of elements of $X_0(T)$ such that $x \sim y$ and $y \sim z$. There is then a unique morphism $d_1' : X_2 \to X_1$ fitting into the squares (14.4.2) and (14.4.3): $d_1'(T)$ sends $(x, y, z) \in X_2(T)$ to $(x, z) \in X_1(T)$. For this choice of d_1', $(d_0, d_1 : X_1 \rightrightarrows X_0, d_1')$ is a $\mathcal{C}$-groupoid.

Conversely, consider a $\mathcal{C}$-groupoid X_* such that $d_0 \boxtimes d_1 : X_1 \to X_0 \times X_0$ is a monomorphism (in other words, for any $T \in \mathrm{Ob}(\mathcal{C})$ and $x, y \in X_0(T)$, there exists a unique morphism from x to y). Then (d_0, d_1) is an equivalence couple and X_* can be reconstructed from (d_0, d_1) as explained above[14].

Example 14.4.6. Let $p : X \to Y$ be a morphism in $\mathcal{C}$ and $\mathrm{pr}_1, \mathrm{pr}_2$ be two projections from $X \times_{p,p} X$ to X. Then $(\mathrm{pr}_1, \mathrm{pr}_2) : X \times_{p,p} X \rightrightarrows X$ is an equivalence couple. We say that p is an **effective epimorphism** if the diagram

$$X \times_{p,p} X \underset{\mathrm{pr}_2}{\overset{\mathrm{pr}_1}{\rightrightarrows}} X \xrightarrow{p} Y$$

is exact, that is, if $(Y, p) = \mathrm{coker}(\mathrm{pr}_1, \mathrm{pr}_2)$.

For example, let S be a Noetherian scheme and $\mathcal{C}$ be the category of schemes finite over S. We show that an epimorphism in $\mathcal{C}$ is not necassarily effective: we choose $S = \mathrm{Spec}(k[T^3, T^5])$, where k is a field, $Y = S$ and $X = \mathrm{Spec}(k[T])$. If i denotes the inclusion of $B = k[T^3, T^5]$ to $A = k[T]$ and $p = \mathrm{Spec}(i)$, then $X \times_{p,p} X$ is identified with $\mathrm{Spec}(A \otimes A)$ and $\mathrm{coker}(\mathrm{pr}_1, \mathrm{pr}_2)$ is equal to $\mathrm{Spec}(B')$, where B' is the subring of A formed by elements a such that $a \otimes_B 1 = 1 \otimes_B a$. Now

$$T^7 \otimes_B 1 = (T^2 T^5) \otimes_B 1 = T^2 \otimes_B T^5 = T^2 \otimes_B (T^3 T^2) = T^5 \otimes_B T^2 = 1 \otimes_B T^7$$

hence $T^7 \in B'$, $T^7 \notin B$ and $\mathrm{Spec}(B) \neq \mathrm{Spec}(B')$, whence a couterexample[15].

Consider a $\mathcal{C}$-groupoid

$$\begin{tikzcd} X_2 \arrow[r, shift left=2, "d_0'"] \arrow[r, "d_1'" description] \arrow[r, shift right=2, "d_2'"'] & X_1 \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] & X_0 \end{tikzcd}$$

and let $f_0 : Y_0 \to X_0$ be a morphism in $\mathcal{C}$. Then by base change to Y_0, we obtain a $\mathcal{C}$-groupoid

$$\begin{tikzcd} Y_2 \arrow[r, shift left=2, "e_0'"] \arrow[r, "e_1'" description] \arrow[r, shift right=2, "e_2'"'] & Y_1 \arrow[r, shift left, "e_0"] \arrow[r, shift right, "e_1"'] & Y_0 \end{tikzcd}$$

which is said to be **induced** by X_* and f_0. We also say that Y_* is the **inverse image** of X_* by the base change morphism $f_0 : Y_0 \to X_0$. More precisely, we choose for Y_1 the fiber product of the diagram

$$\begin{tikzcd} Y_1 \arrow[r, dashed, "f_1"] \arrow[d, dashed] & X_1 \arrow[d, "d_0 \boxtimes d_1"] \\ Y_0 \times Y_0 \arrow[r, "f_0 \times f_0"] & X_0 \times X_0 \end{tikzcd}$$

[14]In particular, if G is a $\mathcal{C}$-group acting on the left on an object X of $\mathcal{C}$ and X_* is the associated groupoid, then (d_0, d_1) is an equivalence couple if and only if G acts freely on X.

[15]The same arguments apply to $B = k[T^n, T^{n+r}]$ and the element $T^{n+2r} \otimes_B 1$, provided that $2r \nmid n$.

for e_0 and e_1 the composition of the canonical morphism $Y_1 \to Y_0 \times Y_0$ and the first and second projections of $Y_0 \times Y_0$. The morphism $Y_1 \to Y_0 \times Y_0$ is then $e_0 \boxtimes e_1$, and we have $f_0 \circ e_i = d_i \circ f_1$ for $i = 0, 1$, where we denote by f_1 the projection of Y_1 to X_1. We put $Y_2 = Y_1 \times_{e_0, e_1} Y_1$. We can say that the couple (e_0, e_1) is defined such that, for any $T \in \mathrm{Ob}(\mathcal{C})$, and any couple (y, x) of elements of $Y_0(T)$, there is a one-to-one correspondence $\psi \mapsto {}_y\psi_x$ between the set of morphisms ψ of $X_*(T)$ with source $f_0(x)$, target $f_0(y)$ and the arrows ${}_y\psi_x$ of $Y_*(T)$ with source x and target y. We therefore determine $e_1' : Y_2 \to Y_1$ by defining for all $T \in \mathrm{Ob}(\mathcal{C})$ the composition of the morphism of $Y_*(T)$ using the formula

$$ {}_z\psi_y \circ {}_y\varphi_x = {}_z(\psi \circ \varphi)_x. $$

It is then clear that this makes $Y_*(T)$ a groupoid.

From the $\mathcal{C}$-groupoid X_* and the base change $f_0 : Y_0 \to X_0$, we can reconstruct the couple $(e_0, e_1) : Y_1 \rightrightarrows Y_0$ in another way: consider $Y_0 \times_{X_0} X_1$ and $\mathrm{pr}_1, \mathrm{pr}_2$ such that the square

$$ \begin{array}{ccc} Y_0 \times_{X_0} X_1 & \xrightarrow{\mathrm{pr}_2} & X_1 \\ {\scriptstyle \mathrm{pr}_1}\downarrow & & \downarrow{\scriptstyle d_0} \\ Y_0 & \xrightarrow{f_0} & X_0 \end{array} \tag{14.4.9} $$

is Cartesian. We then verify that we have a Cartesian square

$$ \begin{array}{ccc} Y_1 & \xrightarrow{e_0 \boxtimes f_1} & Y_0 \times_{X_0} X_1 \\ {\scriptstyle e_1}\downarrow & & \downarrow{\scriptstyle d_1 \circ \mathrm{pr}_2} \\ Y_0 & \xrightarrow{f_0} & X_0 \end{array} \tag{14.4.10} $$

where f_1 denotes the canonical projection from $Y_1 = (Y_0 \times Y_0) \times_{X_0 \times X_0} X_1$ to X_1.

Example 14.4.7. Let's take $Y_0 = X_1$, $f_0 = d_0$. For any object T of $\mathcal{C}$, $Y_1(T)$ is then identified with the set of diagrams of the form

$$ \begin{array}{ccc} b & \xrightarrow{\varphi} & d \\ {\scriptstyle f}\uparrow & & \uparrow{\scriptstyle g} \\ a & & c \end{array} $$

of $X_*(T)$. The source of this diagram is the morphism f, the target is the morphism g, and the composition of two diagrams is clear (by taking the compositin of the horizontal morphisms).

Similarly, by choosing $Y_0' = X_1$ and $f_0' = d_1$, the set $Y_1'(T)$ is identified for any $T \in \mathrm{Ob}(\mathcal{C})$ with the set of diagrams of the form

$$ \begin{array}{ccc} b & & d \\ {\scriptstyle f}\uparrow & & \uparrow{\scriptstyle g} \\ a & \xrightarrow{\psi} & c \end{array} $$

of the groupoid $X_*(T)$. The source of this diagram is the morphism f, the target is the morphism g, and the composition of two diagrams is evident.

Now since $X_*(T)$ is a groupoid for any $T \in \mathrm{Ob}(\mathcal{C})$, the identity map on $Y_0(T)$ and the map

$$ \begin{array}{ccc} b & \xrightarrow{\varphi} & d \\ {\scriptstyle f}\uparrow & & \uparrow{\scriptstyle g} \\ a & & c \end{array} \quad \mapsto \quad \begin{array}{ccc} b & & d \\ {\scriptstyle f}\uparrow & & \uparrow{\scriptstyle g} \\ a & \xrightarrow{g^{-1}\varphi f} & c \end{array} $$

from $Y_1(T)$ to $Y_1'(T)$ define an isomorphism of groupoids $Y_*(T)$ and $Y_*'(T)$. Moreover, this isomorphism depends functorial on T, hence is an isomorphism of the $\mathcal{C}$-groupoids Y_* and Y_*'.

Proposition 14.4.8. *Let X_* be a $\mathcal{C}$-groupoid and suppose that $f_0 : Y_0 \to X_0$ is a universally effective epimorphism. Then* $\operatorname{coker}(d_0, d_1)$ *exists if and only if* $\operatorname{coker}(e_0, e_1)$ *exists. Moreover, in this case f_0 induces an isomorphism*

$$\operatorname{coker}(d_0, d_1) \xrightarrow{\sim} \operatorname{coker}(e_0, e_1).$$

Proof. We denote by $C(d_0, d_1)$ the covariant functor from $\mathcal{C}$ to the category of sets which associates to any $T \in \operatorname{Ob}(\mathcal{C})$ the kernel of the couple $T(d_0), T(d_1) : T(X_0) \rightrightarrows T(X_1)$ (in **Set**), and similarly for $C(e_0, e_1)$. For any $T \in \mathcal{C}$, we then have a commutative diagram

$$\begin{tikzcd}
C(d_0,d_1)(T) \arrow[r] \arrow[d, "T(f)"] & T(X_0) \arrow[r, shift left, "T(d_1)"] \arrow[r, shift right, "T(d_0)"'] \arrow[d, "T(f_0)"] & T(X_1) \arrow[d, "T(f_1)"] \\
C(e_0,e_1)(T) \arrow[r] & T(Y_0) \arrow[r, shift left, "T(e_1)"] \arrow[r, shift right, "T(e_0)"'] & T(Y_1)
\end{tikzcd}$$

where $T(f)$ is the injection induced by the injection $T(f_0)$. If we can show that $T(f)$ is a surjection for any T, then we obtain a functorial isomorphsim $f : C(d_0, d_1) \xrightarrow{\sim} C(e_0, e_1)$, which implies the proposition. For this, consider the diagram

$$\begin{tikzcd}
 & Y_1 \arrow[r, "f_1"] \arrow[d, shift right, "e_1"'] \arrow[d, shift left, "e_0"] & X_1 \arrow[d, shift right, "d_1"'] \arrow[d, shift left, "d_0"] \\
Y_0 \times_{X_0} Y_0 \arrow[ur, "\Delta"] \arrow[r, shift left, "\mathrm{pr}_2"] \arrow[r, shift right, "\mathrm{pr}_1"'] & Y_0 \arrow[r, "f_0"] \arrow[d, "g"] & X_0 \arrow[dl, dashed, "h"] \\
 & T &
\end{tikzcd}$$

where Δ is the section of $Y_1 \to Y_0 \times Y_0$ defined by the morphism $s \circ f_0 \circ \mathrm{pr}_1 : Y_0 \times Y_0 \to X_1$, the morphism $s : X_0 \to X_1$ satisfying the equalities (14.4.4) and (14.4.5). If a morphism $g : Y_0 \to T$ is such that $ge_0 = ge_1$, we then have $ge_0\Delta = ge_1\Delta$, hence $g\mathrm{pr}_1 = g\mathrm{pr}_2$. As f_0 is an effective epimorphism, g is then the composition of f_0 with a morphism $h : X_0 \to T$, that is, we have $g = T(f_0)(h)$. It remains to show that h belongs to $C(d_0, d_1)(T)$, which means $hd_0 = hd_1$; now we have

$$hd_0 f_1 = hf_0 e_0 = ge_0 = ge_1 = hf_0 e_1 = hd_1 f_1$$

whence the desired equality since f_1 is an epimorphism (because f_0 is a universally epimorphism). □

Consider now a scheme S and choose $\mathcal{C} = \mathbf{Sch}_{/S}$. Then a $\mathcal{C}$-groupoid

$$\begin{tikzcd}
X_2 \arrow[r, shift left=2, "d_0'"] \arrow[r, "d_1'" description] \arrow[r, shift right=2, "d_2'"'] & X_1 \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] & X_0
\end{tikzcd}$$

permits us to define an equivalence relation on the underlying set $|X_0|$: if $x, y \in |X_0|$, we write $x \sim y$ if there exists $z \in |X_1|$ such that $x = d_1(z)$ and $y = d_0(z)$. The reflecxivity and symmetricity of this equation is evident[16]. As for the transtivity, if $x \sim y$ and $y \sim z$, then there exists $u, v \in |X_1|$ such that $x = d_1(u)$, $y = d_0(u)$, $y = d_1(v)$, $z = d_0(v)$. It then follows that (v, u) belongs to the set $|X_1| \times_{d_1, d_0} |X_1|$. As the canonical map

$$|X_1 \times_{d_1,d_0} X_1| \to |X_1| \times_{d_1,d_0} |X_1|$$

[16]The reflecxivity follows from the existence of $s : X_0 \to X_1$ which is a section of d_0 and d_1; the symmetricity follows from the existence of an involution σ of X_1 which exchanges d_0 and d_1.

on underlying sets is surjective, (v, u) is the image of some $w \in |X_2|$. We then have $x = d_1 d_1'(w)$ and $z = d_0 d_1'(w)$, then $x \sim z$.

Now let $f_0 : Y_0 \to X_0$ be a base change morphism of schemes over S. If x, y are points of $|Y_0|$, we see that $x \sim y$ if and only if $f_0(x) \sim f_0(y)$. In fact, if $x \sim y$, then there exists $z \in |Y_1|$ such that $x = e_1(z)$ and $y = e_0(z)$. As $f_0 \circ e_i = d_i \circ f_1$ for $i = 0, 1$, we then have $f_0(x) = d_1 f_1(z)$ and $f_0(y) = d_0 f_1(z)$, whence $f_0(x) \sim f_0(y)$.

Conversely, if $f_0(x) \sim f_0(y)$ and $z \in |X_1|$ is such that $f_0(y) = d_1(z)$ and $f_0(x) = d_0(z)$, then by the square (14.4.9), there exists a point $t \in |Y_0 \times_{X_0} X_1|$ such that $\mathrm{pr}_1(t) = x$ and $\mathrm{pr}_2(t) = z$. Similarly, as $f_0(y) = d_1 \mathrm{pr}_2(t)$, there exists $s \in |Y_1|$ such that $y = e_1(s)$ and $(e_0 \boxtimes f_1)(s) = t$ (cf. the square (14.4.10)). We then have $e_0(s) = \mathrm{pr}_1(e_0 \boxtimes f_1)(s) = \mathrm{pr}_1(t) = x$, whence $x \sim y$.

14.4.2 Passage to quotient for a finite and locally free groupoid

Let S be a scheme and consider a $\mathbf{Sch}_{/S}$-grupoid

$$\begin{tikzcd} X_2 \arrow[r, shift left=2, "d_0'"] \arrow[r, "d_1'" description] \arrow[r, shift right=2, "d_2'"'] & X_1 \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] & X_0 \end{tikzcd}$$

In this subsection, we prove the existence of a quotient of X_* under the hypothesis that the strucutral morphism is finite and locally free. More precisely, we shall prove the following theorem:

Theorem 14.4.9. *Suppose that X_* satisfies the following conditions*[17]*:*

(i) *d_1 is locally free and finite.*

(ii) *For any $x \in X_0$, the set $d_0 d_1^{-1}(x)$ is contained in an affine open subset of X_0.*

Then we have the following:

(a) *There exists a cokernel (Y, p) of (d_0, d_1) in $\mathbf{Sch}_{/S}$. Moreover, such a pair (Y, p) is a cokernel of (d_0, d_1) in the category of ringed spaces.*

(b) *The morphism p is open and integral, and Y is affine if X_0 is affine.*

(c) *The morphism $X_1 \to X_0 \times_Y X_0$ with components d_0 and d_1 is surjective.*

(d) *If (d_0, d_1) is an equivalence couple, then the morphism $X_1 \to X_0 \times_Y X_0$ in (c) is an isomorphism and $p : X_0 \to Y$ is locally free and finite. Further, (Y, p) is a cokernel of (d_0, d_1) in the category of sheaves for the fppf topology and, for any base change $Y' \to Y$, Y' is the cokernel of the groupoid $X_* \times_Y Y'$ induced from X_* by base change.*

In particular, for any base change $S' \to S$, $Y' = Y \times_S S'$ is the cokernel of the S'-groupoid $X'_* = X_* \times_S S'$. Hence, in this case, the formation of quotient commutes with base change.

It follows from Theorem 14.4.9 (a) that the underlying topological space of Y is the quotient of that of X_0 by the equivalence relation defined by the groupoid X_*. The rest of this subsection is devoted to the proof of Theorem 14.4.9.

[17] As d_0 and d_1 are exchanged by the involution σ, these conditions are symemtric on d_0 and d_1. Moreover, for any $x \in X_0$ we have $d_0 d_1^{-1}(x) = d_1 d_0^{-1}(x)$.

14.4.2.1 Quotient by a finite and locally free groupoid (affine case) We first prove the theorem under the asssumption that X_0 is affine and d_1 is locally free of constant rank n (then we shall see how to reduce the general case to this particular one). In this case, X_0, X_1 and X_2 are all affine, so we can suppose that $X_i = \mathrm{Spec}(A_i)$, $d_j = \mathrm{Spec}(\delta_j)$, $d'_k = \mathrm{Spec}(\delta'_k)$, where δ_j, δ'_k are homomorphism of rings. From (14.4.7), we then obtain a diagram

$$\begin{tikzcd} A_2 & A_1 \arrow[l, shift left, "\delta'_1"'] \arrow[l, shift right, "\delta'_0"] & A_0 \arrow[l, "\delta_0"'] \\ A_1 \arrow[u, "\delta'_2"] & A_0 \arrow[l, shift left, "\delta_1"'] \arrow[l, shift right, "\delta_0"] \arrow[u, "\delta_1"] & \end{tikzcd} \tag{14.4.11}$$

where the two squares are cocartesian and the first row is exact. Denote by B the subring of A_0 formed by $a \in A_0$ such that $\delta_0(a) = \delta_1(a)$. If $a_0 \in A_0$, let

$$P_{\delta_1}(T, \delta_0(a)) = T^n - \sigma_1 T^{n-1} + \cdots + (-1)^n \sigma_n$$

be the characteristic polynomial of $\delta_0(a)$ if we consider A_1 as an A_0-algebra via the homomorphism δ_1. As the two squares of (14.4.11) are cocartesian, we have

$$\delta_0(P_{\delta_1}(T, \delta_0(a))) = P_{\delta'_2}(T, \delta'_0\delta_0(a)), \quad \delta_1(P_{\delta_1}(T, \delta_0(a))) = P_{\delta'_2}(T, \delta'_1\delta_0(a)).$$

As $\delta'_0\delta_0 = \delta'_1\delta_0$, we then conclude that

$$\delta_0(P_{\delta_1}(T, \delta_0(a))) = \delta_1(P_{\delta_1}(T, \delta_0(a)))$$

that is, $\delta_0(\sigma_i) = \delta_1(\sigma_i)$ for any i. On the other hand, Hamilton-Cayley theorem shows that we have

$$\delta_0(a)^n - \delta_1(\sigma_1)\delta_0(a)^{n-1} + \cdots + (-1)^n \delta_1(\sigma_n) = 0.$$

As $\delta_1(\sigma_i) = \delta_0(\sigma_i)$, we also have

$$\delta_0(a)^n - \delta_0(\sigma_1)\delta_0(a)^{n-1} + \cdots + (-1)^n \delta_0(\sigma_n) = 0,$$

whence

$$a^n - \sigma_1 a^{n-1} + \cdots + (-1)^n \sigma_n = 0$$

because δ_0 has a section $\tau : A_1 \to A_0$ such that $\tau\delta_0 = \mathrm{id}_{A_0}$, so it is injective. We then conclude that A_0 *is integral over* B.

Now consider two prime ideals $\mathfrak{p}, \mathfrak{q}$ of A_0; we show that the equality $\mathfrak{p} \cap B = \mathfrak{q} \cap B$ implies the existence of a prime ideal $\mathfrak{r}$ of A_1 such that $\mathfrak{p} = d_0(\mathfrak{r})$ and $\mathfrak{q} = d_1(\mathfrak{r})$. In fact, if the assertion was not true, $\mathfrak{p}$ would be distinct from $\delta_0^{-1}(\mathfrak{n})$ any prime ideal $\mathfrak{n}$ of A_1 such that $\delta_1^{-1}(\mathfrak{n}) = \mathfrak{q}$. For such an ideal $\mathfrak{n}$ we would have $\delta_0^{-1}(\mathfrak{n}) \cap B = \delta_1^{-1}(\mathfrak{n}) \cap B = \mathfrak{q} \cap B = \mathfrak{p} \cap B$, so by Corollary 4.1.66, $\mathfrak{p}$ is not contained in $\delta_0^{-1}(\mathfrak{n})$. But there are finitely many prime ideals $\mathfrak{n}$ of A_1 such that $\delta_1^{-1}(\mathfrak{n}) = \mathfrak{q}$ (Proposition 4.1.78), hence, by prime aviodence, there exists $a \in \mathfrak{p}$ which is not in any of the $\delta_0^{-1}(\mathfrak{n})$. Therefore, $\delta_0(a)$ is not conained in these ideals $\mathfrak{n}$, and hence, by the lemma bolow, the norm $N_{\delta_1}(\delta_0(a))$ does not belong to $B \cap \mathfrak{q}$ (the norm is calculated by considering A_1 as an A_0-algebra via the homomorphism δ_1, and we have $N_{\delta_1}(\delta_0(a)) = \sigma_n$ with the notations above). As $(-1)^{n-1}\sigma_n = a^n + \sum_{i=1}^{n-1}(-1)^i \sigma_i a^{n-i}$, this norm belongs to $B \cap \mathfrak{p} = B \cap \mathfrak{q}$, which is a contradiction.

Lemma 14.4.10. *Let $A \to A'$ be a ring homomorphism such that A' is a projective A-module of rank n. Let $\mathfrak{p}$ be a prime ideal of A and $\mathfrak{q}_1, \dots, \mathfrak{q}_r$ be the prime ideals of A' lying over $\mathfrak{p}$. Let $a \in A'$, then a belongs to $\mathfrak{q}_1 \cup \cdots \cup \mathfrak{q}_r$ if and only if its norm $N(a)$ belongs to $\mathfrak{p}$.*

Proof. By replacing A and A' by the localizations $A_{\mathfrak{p}}$ and $A'_{\mathfrak{p}}$, we may assume that $(A,\mathfrak{p})$ is local and A' is semi-local, with $\mathrm{Spec}(A') = \{\mathfrak{q}_1,\dots,\mathfrak{q}_r\}$. In this case, A' is a free A-module of rank n, and $N(a)$ is the determinant of the endomorphism $h_a : A' \to A'$ with ratio a. We then conclude that $N(a) \notin \mathfrak{p}$ if and only if $N(a)$ is invertible, if and only if h_a is invertible, and this is equivalent to that $a \notin \mathfrak{q}_1 \cup \cdots \cup \mathfrak{q}_r$. □

We now prove assertion (a) of Theorem 14.4.9 in this case. Let $Y = \mathrm{Spec}(B)$ and $p = \mathrm{Spec}(i)$, where $i : B \to A_0$ is the inclusion. By the preceding arguments, $p : X_0 \to Y$ is integral, hence surjective, and the underlying space of $\mathrm{Spec}(B)$ is obtained from that of X_0 by identifying points x, y such that thre exists $z \in X_1$ such that $d_1(z) = y$, $d_0(z) = x$. Moreover, as i is integral, p is closed (Corollary 4.1.74) so that Y is endowed with the quotient topology of that of X_0. In particular, p is open: if U is an open subset of X_0, as d_1 is surjective and locally free and finite (hence faithfully flat and finitely presented), hence open, the saturation $U' = d_1 d_0^{-1}(U')$ of U' under the equivalence relation defined by X_* is open, so $p(U') = p(U)$ is open, since Y is endowed with the quotient topology.

Finally, it follows from the choice of B and the fact that p, d_0, d_1 are affine that the canonical sequence of sheaf of rings

$$\mathscr{O}_Y \longrightarrow p_*(\mathscr{O}_{X_0}) \underset{p_*(\delta_0)}{\overset{p_*(\delta_1)}{\rightrightarrows}} p_*((d_0)_*(\mathscr{O}_{X_1})) = p_*((d_1)_*(\mathscr{O}_{X_1}))$$

is exact. It remains to prove that (Y, p) is also the cokernel of (d_0, d_1) in the category of schemes (or more generally in the category of locally ringed spaces). Let $f : X_0 \to Z$ be a morphism of schemes such that $fd_0 = fd_1$. By the above arguments, there exists a unique morphism of ringed spaces $\tilde{f} : Y \to Z$ such that $f = \tilde{f}p$. Since the composition f and p are both local morphisms, we conclude that $\tilde{f}$ is a local morphism, hence a morphism of schemes.

Now the assertion (b) of Theorem 14.4.9 follows immediately. On the other hand, since $p : |X_0| \to |Y|$ is a quotient map, the following map

$$d_0 \boxtimes d_1 : |X_1| \to |X_0| \times_{|Y|} |X_0|$$

is surjective. This map factors into

$$|X_1| \xrightarrow{d_0 \boxtimes d_1} |X_0 \times_Y X_0| \xrightarrow{q} |X_0| \times_{|Y|} |X_0|$$

where q is the canonical map. We therefore conclude that the image of $d_0 \boxtimes d_1$ then contains points $v \in X_0 \times_Y X_0$ such that $\{v\} = q^{-1}(q(v))$, which is staisfied if v is a rational point over Y (that is, if the residue field $\kappa(v)$ is identified with $\kappa(w)$, where w is the image of v in Y). If $v \in X_0 \times_Y X_0$ is not rational over Y, let w be the image of v in Y. By ([**?**] 0_{III}, 10.3.1) there exists a local ring C and a flat local homomorphism $\rho : \mathscr{O}_w \to C$ such that $C/\mathfrak{m}_w C$ is isomorphic to $\kappa(v)$ as $\kappa(w)$-algebras. If we put $Y' = \mathrm{Spec}(C)$ and $\pi : Y' \to Y$ is the morphism induced by ρ, it is clear that the canonical projection of $(X_0 \times_Y X_0) \times_Y Y'$ onto $X_0 \times_Y X_0$ sends v to a point v' of $(X_0 \times_Y X_0) \times_Y Y'$ which is rational over Y'. As

$$(X_0 \times_Y X_0) \times_Y Y' \cong (X_0 \times_Y Y') \times_{Y'} (X_0 \times_Y Y'),$$

and as the hypothesis of Theorem 14.4.9 and the preceding results, in particular that of (b), is valid under base change $\pi : Y' \to Y$, we conclude that v' is the image of an element $u' \in X_1 \times_Y Y'$ under the morphism deduced from $d_0 \boxtimes d_1$ by base change. If u is the image of u' in X_1, we then have $v = (d_0 \boxtimes d_1)(u)$.

Finally, we prove assertion (d) of Theorem 14.4.9. By hypothesis, $X_0 = \mathrm{Spec}(A_0)$, $X_1 = \mathrm{Spec}(A_1)$, and for $i = 0, 1$, the morphism $\delta_i : A_0 \to A_1$ is finite; hence the morphism $A_0 \otimes_B A_0 \to A_1$ is finite. Since (d_0, d_1) is assumed to be an equivalence couple, we may further assume that $d_0 \boxtimes d_1 : X_1 \to X_0 \times_Y X_0$ is a monomorphism; then, by ([?] IV$_4$, 18.12.6), $d_0 \boxtimes d_1$ is a closed immersion, so $A_0 \otimes_B A_0 \to A_1$ is surjective. We will show that this is an isomorphism (and also that $p : X_0 \to Y$ is finite and locally free). For this, it suffices to show that for any prime ideal $\mathfrak{p}$ of B, the homomorphism $(A_0)_\mathfrak{p} \otimes_{B_\mathfrak{p}} (A_0)_\mathfrak{p} \to (A_1)_\mathfrak{p}$ with components $(\delta_0)_\mathfrak{p}$ and $(\delta_1)_\mathfrak{p}$ is bijective. In other words, we may assume that B is local. It then follows from Proposition 4.1.78 that $(A_0)_\mathfrak{p}$ is semi-local. By applying ([?] 0_{III}, 10.3.1) to perform a faithfully flat base change, we may also assume that the residue field of B is infinite, so that we can use the following lemma:

Lemma 14.4.11. *Let B be a local ring with infinite residue field, A be a semi-local ring and $i : B \to A$ be a homomorphism wich sends the maximal ideal $\mathfrak{m}$ of B into the radical $\mathfrak{r}$ of A. Let M be a free A-module of rank n and N be a sub-B-module of M which generates M as an A-module. Then N contains a basis of M over A.*

Proof. We recall that, by Nakayama lemma, a sequence $m_1, \dots, m_n$ of elements of M is an A-basis of M if and only if the canonical images of $m_1, \dots, m_n$ in $M/\mathfrak{r}M$ form a basis of $M/\mathfrak{r}M$ over $A/\mathfrak{r}$. We can then replace M by $M/\mathfrak{r}M$, N by $N/(N \cap \mathfrak{r}M)$, A by $A/\mathfrak{r}$ and B by $B/\mathfrak{m}$, in which case the lemma is immediate. In fact, we then have $A = K_1 \times \cdots \times K_r$, and M can be identified with $K_1^n \times \cdots \times K_r^n$. We can choose elements $(x_{ij})_{1 \leq i \leq r, 1 \leq j \leq n}$ of N such that for each $1 \leq i \leq r$, the i-th component of $(x_{i,1}, \dots, x_{i,n})$ in K_i^n is linearly independent over K_i. Then, we can consider the polynomial

$$f(k_{1,1}, \dots, k_{n,r}) = \prod_{i=1}^{r} \det_{K_i}\Big(\sum_{j=1}^{n} k_{j,1} x_{j,1}, \dots, \sum_{j=1}^{n} k_{j,r} x_{j,r}\Big)$$

where $\det_{K_i}(z_1, \dots, z_n)$ denotes the determinant of the i-th components of $z_1, \dots, z_n$. As k is an infinite field and each polynomial $\det_{K_i}(\sum_{j=1}^{n} k_{j,1} x_{j,1}, \dots, \sum_{j=1}^{n} k_{j,r} x_{j,r})$ with coefficient in A is nonzero (take $k_{i,1} = \cdots = k_{i,n} = 1$ and others to be zero), we conclude that there exists a family $(y_\ell)_{1 \leq \ell \leq n}$ of k-linear combinations of the x_{ij} such that for any $1 \leq i \leq n$, the i-th component of $(y_\ell)_{1 \leq \ell \leq n}$ is linearly independent over K_i. We therefore conclude that $(y_\ell)_{1 \leq \ell \leq n}$ is a basis of M over A. □

We shall apply Lemma 14.4.11 for the ring homomorphism $B \to A_0$ and $M = A_1$ (as an A_0-module via the homomorphism δ_1), and $N = \delta_0(A_0)$. In fact, as $d_0 \boxtimes d_1 : X_1 \to X_0 \times_Y X_0$ is a closed immersion, the homomorphism $A_0 \otimes_B A_0 \to A_1$ with components δ_0 and δ_1 is surjective; this signifies that $\delta_0(A_0)$ generates the A_0-module A_1.

Let $a_1, \dots, a_n$ be elements of A_0 such that $\delta_0(a_1), \dots, \delta_0(a_n)$ form a basis of A_1 over A_0. If we can show that $a_1, \dots, a_n$ is a basis of A_0 over B, then the homomorphism $A_0 \otimes_B A_0 \to A_1$ sends the basis $(1 \otimes a_i)_{1 \leq i \leq n}$ to the basis $(\delta_0(a_i))_{1 \leq i \leq n}$, hence is bijective. Therefore, if $\varepsilon : \mathbb{Z}^n \to A_0$ is a homomorphism of abelian groups which sends the natural basis of $\mathbb{Z}^n$ to $a_1, \dots, a_n$, it suffices to prove that the map $B \otimes_\mathbb{Z} \mathbb{Z}^n \to A_0$ with components i and ε is bijective. Now the diagram (14.4.11) gives the following commutative diagram:

$$\begin{tikzcd}
A_2 & A_1 \arrow[l, shift left, "\delta_1'"] \arrow[l, shift right, "\delta_0'"'] & A_0 \arrow[l, "\delta_0"'] \\
A_1 \otimes_\mathbb{Z} \mathbb{Z}^n \arrow[u, "u_2"] & A_0 \otimes_\mathbb{Z} \mathbb{Z}^n \arrow[l, shift left, "\delta_1 \otimes 1"] \arrow[l, shift right, "\delta_0 \otimes 1"'] \arrow[u, "u_1", "\cong"'] & B \otimes_\mathbb{Z} \mathbb{Z}^n \arrow[l, "i \otimes 1"'] \arrow[u, "u_0"]
\end{tikzcd}$$

where u_0, u_1 and u_2 have components i and ε, δ and $\delta_0\varepsilon$, δ_2' and $\delta_0'\delta_0\varepsilon$, respectively. We see that u_1 is an isomorphism. As the two squares in (14.4.11) are cocartesian, u_2 is then an isomorphism. Since the horizontal row of the above diagram is exact, we conclude that u_0 is bijective. This shows that A_0 is a locally free B-module of rank n, whence $\delta_0 \otimes \delta_1 : A_0 \otimes_B A_0 \to A_1$ is an isomorphism. This proves Theorem 14.4.9 in the particular case where X_0 is affine and d_1 is locally free of rank n.

14.4.2.2 Quotient by a finite and locally free groupoid (general case) Let U^n be the largest open subset of X_0 over which d_1 is finite and locally free of rank n. We know that X_0 is the direct sum of these U^n. On the other hand, it follows from the Cartesian squares

$$\begin{tikzcd} X_2 \arrow[r, "d_0'"] \arrow[d, "d_2'"'] & X_1 \arrow[d, "d_1"] \\ X_1 \arrow[r, "d_0"] & X_0 \end{tikzcd} \qquad \begin{tikzcd} X_2 \arrow[r, "d_1'"] \arrow[d, "d_2'"'] & X_1 \arrow[d, "d_1"] \\ X_1 \arrow[r, "d_1"] & X_0 \end{tikzcd}$$

that the inverse images of U^n under d_0 and d_1 both coincide with the lagest open subset of X_1 over which d_2' is locally free of rank n: in fact, as d_0 (resp. d_1) is surjective, finite and flat, hence faithfully flat and affine, d_2' is of rank n at a point x of X_1 if and only if d_1 is of rank n on a neighborhood of $d_0(x)$ (resp. $d_1(x)$). We then have $d_0^{-1}(U^n) = d_1^{-1}(U^n)$ so that the groupoid X_* is the direct sum of the groupoids X_*^n induced from X_* on the open and closed subsets U^n. It then suffices to prove Theorem 14.4.9 for each X_*^n, so we can assume that d_1 is locally free of finite rank n.

We now prove the theorem in the general case. By the above arguments, we can assume that d_1 is locally free of rank n. Let (Y, p) be a cokernel of (d_0, d_1) in the category of ringed spaces. Then as before, for (a) it suffices to prove that Y is a scheme and $p : X_0 \to Y$ is a morphism of schemes. By Lemma 14.4.3, this question is local over Y: let $y \in Y$ and $x_0 \in X_0$ such that $p(x) = y$; if x has a saturated affine open neighborhood U, then $p(U)$ is an affine open of Y by Theorem 14.4.9 (b) in the affine case and $p|_U$ is a morphism of schemes. It then suffices to prove that any $x \in X_0$ has a saturated affine open neighborhood U. Here is how we proceed (the proof is taken from [**?**], VIII, cor. 7.6). By condition (ii) of Theorem 14.4.9, there exists an open affine subset V of X_0 containing $d_1(d_0^{-1}(X))$; if $F = X_0 - V$, $d_1(d_0^{-1}(F))$ is closed because d_1 is integral and $V' = X_0 - d_1(d_0^{-1}(F))$ is the largest saturated open subset containing V. As V' is an neighborhood of the finite set $d_1(d_0^{-1}(x))$ (d_0 is alsl finite, hence has finite fiber), there exists a section f of the structural sheaf of V which vanishes over $V - V'$ and such that $d_1(d_0^{-1}(x))$ is contained in the open subset $V_f \subseteq V$ formed by points where f is non-vanishing. We then see that the largest saturated open subset V_f' of V_f is affine: in fact, let $Z(f) = V' - V_f$. Then $d_0^{-1}(Z(f))$ is the set of points of $d_0^{-1}(V') = d_1^{-1}(V')$ where the image $d_0^*(f)$ of f under the map induced by d_0 vanishes. On the other hand, as d_1 induces a locally free morphism of rank n from $d_0^{-1}(V') = d_1^{-1}(V')$ to V', by Lemma 14.4.10, $d_1(d_0^{-1}(Z(f)))$ is the set of points where the norm $N(d_0^*(f))$ relative to the morphism d_1 vanishes. It follows that $V_f' = V' - d_1(d_0(Z(f)))$ is the set of points of V_f where $N(d_0^*(f))$ does not vanish, so V_f' is affine.

We therefore conclude assertion (a), and (b), (c), together with the first part of (d), are then clear from the affine case. It remains to see the other concequences in (d). By hypothesis, the groupoid X_* is given by an equivalence relation $R \to X_0 \times_S X_0$, and we have proved that R is effective and that $p : X_0 \to Y = X_0/R$ is surjective, finite and locally free, hence, in particular, faithfully flat and finitely presented. Therefore, if $\mathcal{M}$ is the family of faithfully flat morphisms locally of finite presentaion, then R is $\mathcal{M}$-effective. By Corollary 14.3.68, we conclude that (Y, p) represents the quotient sheaf of X_0 by R for the fppf topology, and the assertion concerning base change follows from Proposition 14.3.17.

Remark 14.4.12. With the hypothesis and notations of Theorem 14.4.9, suppose that S is locally Noetherian and $\pi_0 : X_0 \to S$ is quasi-projective. Let $\mathscr{A}$ be an ample $\mathscr{O}_{X_0}$-module relative to π_0. By Proposition 11.6.11, $p_*(\mathscr{A})$ is an invertible $p_*(\mathscr{O}_{X_0})$-module, so there exists a covering $(V_i)_{i \in I}$ of Y by affine opens, such that $\mathscr{A}$ is trivial over each saturated affine open subset $U_i = p^{-1}(V_i)$. For each $i \in I$, let $A_{i,0} = \mathscr{O}_{X_0}(U_i)$, $A_{i,1} = \mathscr{O}_{X_1}(d_0^{-1}(U_i)) = \mathscr{O}_{X_1}(d_1^{-1}(U_i))$, $\delta_{i,0}$ (resp. $\delta_{i,1}$) be the morphism $A_{i,0} \to A_{i,1}$ induced by d_0 (resp. d_1), and $B_i = \mathscr{O}_Y(V_i) = \{b \in A_{i,0} : \delta_{i,0}(b) = \delta_{i,1}(b)\}$.

Following §§11.6.5, consider the invertible $\mathscr{O}_{X_0}$-module $N_{d_1}(d_0^*(\mathscr{A}))$, the norm of $d_0^*(\mathscr{A})$ relative to the finite and locally free morphism $d_1 : X_1 \to X_0$. If $\mathscr{A}$ is given, relative to the open covering $(U_i)_{i\in I}$, by the transition functions $c_{ij} \in \mathscr{O}_{X_0}(U_i \cap U_j)^\times$, then $N_{d_1}(d_0^*(\mathscr{A}))$ is given by the transition functions $N_{\delta_1}(\delta_0(c_{ij})) \in \mathscr{O}_{X_0}(U_i \cap U_j)^\times$. As, by the proof of 14.4.2.1, these elements belong to $\mathscr{O}_Y(U_i \cap U_j)^\times$, they define an invertible $\mathscr{O}_Y$-module $\mathscr{L}$, such that $p^*(\mathscr{L}) = N_{d_1}(d_0^*(\mathscr{A}))$. We also note that, for any $n \in \mathbb{N}$, we have $p^*(\mathscr{L}^n) = N_{d_1}(d_0^*(\mathscr{A}^n))$.

We now prove that $\mathscr{L}$ is ample for the morphism $\pi : Y \to S$ (the proof that $\pi : Y \to S$ is of finite type follows from that of Lemma 14.4.13 (b)). For this, by replacing S with an affine open, we may assume that S is affine. Let $y \in Y$, $x \in X_0$ such that $p(x) = y$, V be an open subset of Y containing y, and $U = p^{-1}(V)$. As $\mathscr{A}$ is π_0-ample, there exists an integer $n \geq 1$ and a section $s \in \Gamma(X_0, \mathscr{A}^n)$ such that the open subset $(X_0)_s$ satisfies $x \in (X_0)_s \subseteq U$ (cf. Theorem 11.4.27). With the preceding notations, s is given by the sections $a_i \in A_{i,0} = \mathscr{O}_{X_0}(U_i)$ such that $a_i = c_{ij}a_j$ over $U_i \cap U_j$, and $(X_0)_s$ is the union of $U_i' = \{p \in \mathrm{Spec}(A_{i,0}) : a_i \notin \mathfrak{p}\}$. For each $i \in I$, put $N(a_i) = N_{\delta_1}(\delta_0(a_i)) \in B_i$. By Theorem 14.4.9 (a) and Lemma 14.4.10, we have

$$p(U_i') = pd_1d_0^{-1}(U_i') = pd_1(\{\mathfrak{q} \in \mathrm{Spec}(A_{i,1}) : \delta_{i,0}(a_i) \notin \mathfrak{q}\})$$

and $d_1(\{\mathfrak{q} \in \mathrm{Spec}(A_{i,1}) : \delta_{i,0}(a_i) \notin \mathfrak{q}\}) = \{\mathfrak{p} \in \mathrm{Spec}(A_{i,0}) : N_{\delta_1}(\delta_{i,0}(a_i)) \notin \mathfrak{p}\}$, whence

$$p(U_i') = \{\mathfrak{p} \in \mathrm{Spec}(B_i) : N(a_i) \notin \mathfrak{p}\}.$$

It then follows that $p((X_0)_s) = Y_{N(s)}$, so if we denote by $N(s)$ the section of $\mathscr{L}^n$ over Y defined by the sections $N(a_i) \in \mathscr{O}_Y(V_i)$, we then have

$$y \in p((X_0)_s) = Y_{N(s)} \subseteq p(U) = V. \tag{14.4.12}$$

This proves that $\mathscr{L}$ is ample for $\pi : Y \to S$ (Theorem 11.4.27), so $\pi : Y \to S$ is quasi-projective.

14.4.3 Passage to quotient if there exists a quasi-section

We now prove a techniqucal lemma which will be used in the proof of the forecoming two theorems. Let S be a scheme and

$$\begin{tikzcd} X_2 \arrow[r, shift left=2, "d_0'"] \arrow[r, "d_1'" description] \arrow[r, shift right=2, "d_2'"'] & X_1 \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] & X_0 \end{tikzcd}$$

be a $\mathbf{Sch}_{/S}$-groupoid. A **quasi-section** of X_* is defined to be a subscheme U of X_0 such that

(a) The restriction of d_1 to $d_0^{-1}(U)$ is a finite, locally free and surjective morphism from $d_0^{-1}(U)$ to X_0.

(b) Any subset E of U formed by equivalent points for the equivalence relation defined by X_* is contained in an affine open subset of U[18].

If U is a quasi-section of X_*, the $\mathbf{Sch}_{/S}$-groupoid

$$\begin{tikzcd} U_2 \arrow[r, shift left=2, "u_0'"] \arrow[r, "u_1'" description] \arrow[r, shift right=2, "u_2'"'] & U_1 \arrow[r, shift left, "u_0"] \arrow[r, shift right, "u_1"'] & U \end{tikzcd}$$

[18]If $x, y \in E$, then there exists $z \in X_1$ such that $d_1(z) = x$, $d_0(z) = y$, that is, $z \in (d_1|_{d_0^{-1}(U)})^{-1}(x)$, which is a finite set by (a). Hence E is contained in the finite subset $d_0d_1^{-1}(x) \cap U$

induced from X_* and the inclusion $U \to X_0$ verifies the hypotheses of Theorem 14.4.9. In fact, put $V = d_0^{-1}(U)$ and let u, v be morphisms induced by d_0 and d_1:

$$X_0 \xleftarrow{v} V \xrightarrow{u} U$$

By (14.4.10), we then have a Cartesian square

$$\begin{tikzcd} U_1 \arrow[r] \arrow[d, "u_1"'] & V \arrow[d, "v"] \\ U \arrow[r, hook] & X_0 \end{tikzcd}$$

hence u_1 is surjective and finite locally free by (a). With (b), condition (a) then assures that the groupoid U_* satisfies the hypotheses of Theorem 14.4.9. In particular, $\operatorname{coker}(u_0, u_1)$ exists in $\mathbf{Sch}_{/S}$. Moreover, as d_0 possesses a section (the morphism $s : X_0 \to X_1$), u is a universally effective epimorphism by ([?], IV, 1.12); this ensures, by Proposition 14.4.8, that $\operatorname{coker}(u_0, u_1)$ coincides with the cokernel $\operatorname{coker}(v_0, v_1)$ of the groupoid V_*:

$$\begin{tikzcd} V_2 \arrow[r, shift left=2, "v_0'"] \arrow[r, "v_1'" description] \arrow[r, shift right=2, "v_2'"'] & V_1 \arrow[r, shift left, "v_0"] \arrow[r, shift right, "v_1"'] & V \end{tikzcd}$$

induced by U_* and the base change $u : V \to U$, which is also the inverse image of X_* under the base change

$$V \hookrightarrow X_1 \xrightarrow{d_0} X_0.$$

By Example 14.4.7, V_* is isomorphic to the groupoid V'_*, the inverse image of X_* under the base change

$$V \hookrightarrow X_1 \xrightarrow{d_1} X_0$$

and hence V'_* admits a cokernel in $\mathbf{Sch}_{/S}$. Now, being flat, surjective and finite, $v : V \to X_0$ is faithfully flat and quasi-compact, hence a universally effective epimorphism by Proposition 14.3.67. Therefore by Example 14.4.7, the groupoid X_* also admits a cokernel $\operatorname{coker}(d_0, d_1)$ in $\mathbf{Sch}_{/S}$. We have therefore proved the first assertion of (a) in the following lemma:

Lemma 14.4.13. *Suppose that the $\mathbf{Sch}_{/S}$-groupoid X_* possesses a quasi-section. Then:*

(a) *There exists a cokernel (Y, p) of (d_0, d_1) in $\mathbf{Sch}_{/S}$. Moreover, such a couple (Y, p) is also a cokernel of (d_0, d_1) in the category of ringed spaces.*

(a') *p is surjective, and is open (resp. universally closed) if d_0 is.*

(b) *Suppose that S is locally Noetherian and X_0 is locally of finite type (resp. of finite type) over S. Then p and $Y \to S$ are locally of finite presentation (resp. of finite presentation).*

(c) *The morphism $X_1 \to X_0 \times_Y X_0$ with components d_0 and d_1 is surjective.*

(d) *If (d_0, d_1) is an equivalence couple, $X_1 \to X_0 \times_Y X_0$ is an isomorphism. Moreover, if $d_0 : X_1 \to X_0$ is flat, p is faithfully flat.*

Proof. Before proving the second assertion of (a), let us first consider (a'), (b) and (c). We have seen that (Y, p) is identified with $\operatorname{coker}(v_0, v_1)$ and $\operatorname{coker}(u_0, u_1)$. Let q and r be the canonical epimorphisms of U and Y onto Y:

$$\begin{tikzcd} X_0 \arrow[dr, "p"'] & V \arrow[l, "v"'] \arrow[r, "u"] \arrow[d, "r"] & U \arrow[dl, "q"] \\ & Y & \end{tikzcd} \tag{14.4.13}$$

By hypothesis, v is surjective and finite locally free, hence open. On the other hand, if $d_0 : X_1 \to X_0$ is open (resp. universally closed), then u, which is induced from d_0 by restriction, is also open (resp. universally closed). As, by Theorem 14.4.9, q is surjective, integral and open, we conclude that r is surjective, and open (resp. universally closed) if d_0 is. The same assertion then holds for p, since v is surjective. This proves (a').

Now suppose that S is locally Noetherian and X_0 is locally of finite type over S, so that X_0 is locally Noetherian. Let $S' = \operatorname{Spec}(R)$ be an open affine subset of S, $Y' = \operatorname{Spec}(B)$ an open affine subset of Y which projections into S', and $U' = \operatorname{Spec}(A)$ be the inverse image of Y' in U. As R is Noetherian, it suffices to show that B is a finite type R-algebra, and this follows from the fact that R is Noetherian and A is integral over B (cf. (4.2.1)). Finally, as $X_0 \to S$ is locally of finite type, so is p (Proposition 10.6.21), hence p is locally of finite presentation since Y is locally Neotherian.

It remains to see that second assertion of (b). Suppose that X_0 is of finite type over S. Then, as p is surjective, Y is also quasi-compact over S, hence of finite type over S. As S is locally Noetherian, $X_0 \to S$ and $Y \to S$ are then finitely presented, and hence $p : X_0 \to Y$ is also finitele presented (Proposition 10.6.38).

Finally, as the groupoid V_* with base V is isomorphic to the inverse image of U_* by the base change u and to the inverse image of X_* by the base change v, we have Cartesian squares

$$\begin{tikzcd}
X_1 \arrow[d, "d_0\boxtimes d_1"] & V_1 \arrow[l] \arrow[r] \arrow[d, "v_0\boxtimes v_1"] & U_1 \arrow[d, "u_0\boxtimes u_1"] \\
X_0\times_Y X_0 & V\times_Y V \arrow[l, "v\times v"'] \arrow[r, "u\times u"] & U\times_Y U
\end{tikzcd} \tag{14.4.14}$$

As $u_0 \boxtimes u_1$ is surjective, so is $v_0 \boxtimes v_1$. Since $v \times v$ is surjective, the composition $V_1 \to X_0 \times_Y X_0$ is also surjective, whence so is $d_0 \boxtimes d_1$.

We now turn to the proof of the second assertion of (a). To see that (Y, p) is a cokernel of (d_0, d_1) in **Rsp**, we first show that Y is obtained from X_0 by identifying points x, y such that there exists $z \in X_1$ with $d_0(z) = x$ and $d_1(z) = y$. In fact, p is surjective and we have $pd_0 = pd_1$; conversely, if $p(x) = p(y)$, there exists a point z' of $X_0 \times_Y X_0$ whose first projection is x and second projection is y. If z is a point of X_1 such that $(d_0 \boxtimes d_1)(z) = z'$, then $d_0(z) = x$ and $d_1(z) = y$, whence our assertion. Also, if W is a saturated open subset of X_0, $W \cap U$ is saturated open in U, so by Theorem 14.4.9, $q(W \cap U)$ is open in Y. As $q(W \cap U) = p(W)$, we see that Y is endowed with the quotient topology of that of X_0.

It remains to show that the canonical sequence

$$\mathscr{O}_Y \longrightarrow p_*(\mathscr{O}_{X_0}) \rightrightarrows p_*((d_0)_*(\mathscr{O}_{X_1})) = p_*((d_1)_*(\mathscr{O}_{X_1}))$$

is exact. Let Y' be an open subset of Y and put $U' = q^{-1}(Y')$, $X_0' = p^{-1}(Y')$, etc. Then, U' is a saturated open subset of U for the equivalence relation defined by the groupoid U_*, and it follows from **??** and **??** that Y' is its cokernel, in $\mathbf{Sch}_{/S}$ and in **Rsp**. Similarly, X_0' is saturated open in X_0 for the equivalence relation defined by X_* and we have the following Cartesian squares

$$\begin{tikzcd}
X_0' \arrow[d] & V' = d_0^{-1}(U') \arrow[l, "\tilde{d}_1"'] \arrow[r, "\tilde{d}_0"] \arrow[d] & U' \arrow[d] \\
X_0 & V = d_0^{-1}(U) \arrow[l, "d_1"'] \arrow[r, "d_0"] & U'
\end{tikzcd}$$

Hence $\tilde{d}_1$ is surjective and finite locally free. On the other hand, let $x \in U'$. As U is a quasi-section, the set $E = d_0 d_1^{-1}(x) \cap U$ is finite and contained in an affine open W of U. The intersection $E' = E \cap U'$ is then a finite subset contained in a quasi-affine open subset $W \cap U'$.

Therefore, there exists an affine open W' of $W \cap V$ containing E', so U' is a quasi-section of the groupoid X'_* induced by X_* over X'_0. The first assertion of (a), applied to X'_* and U', shows that Y' is the cokernel of X'_* in $\mathbf{Sch}_{/S}$. In particular, for any S-scheme T, we have an exact sequence

$$T(Y') \xrightarrow{T(p|_{X'_0})} T(X'_0) \underset{T(d_0|_{X'_1})}{\overset{T(d_1|_{X'_1})}{\rightrightarrows}} T(X'_1)$$

Now if $T = \mathbb{G}_{a,S}$, this sequence is identified with

$$\Gamma(Y', \mathscr{O}_Y) \longrightarrow \Gamma(p^{-1}(Y'), \mathscr{O}_{X_0}) \underset{\delta_0}{\overset{\delta_1}{\rightrightarrows}} \Gamma(d_0^{-1}p^{-1}(Y'), \mathscr{O}_{X_1}) = \Gamma(d_1^{-1}p^{-1}(Y'), \mathscr{O}_{X_1})$$

which is then exact for any open subset Y' of Y. This conclude the proof of (a).

Finally, if (d_0, d_1) is an equivalence couple, so is (u_0, u_1) and the morphism $u_0 \boxtimes u_1 : U_1 \to U \times_Y U$ is an isomorphism (Theorem 14.4.9). As $v \times v$ is faithfully flat and quasi-compact, we conclude from (14.4.14) that $d_0 \boxtimes d_1$ is an isomorphism ([**?**], VIII, 5.4). Moreover, if d_0 is flat, u is also flat. Now q is flat by Theorem 14.4.9, so r is also flat from the diagram (14.4.13). As v is faithfully flat, p is then flat, and hence faithfully flat since it is surjective. □

14.4.4 Passage to quotient for a proper and flat groupoid

14.4.5 Quotients by a group scheme

We now consider the action of a group scheme G over S on an S-scheme X, and use the preceding results to construct the quotient $G \backslash X$. We first recall the following result:

Theorem 14.4.14. *Let S be a scheme and $f : X \to Y$ be an S-morphism. Suppose that one of the following conditions is satisfied:*

(α) *The morphism f is locally of finite presentation.*

(β) *The scheme S is locally Noetherian and X is locally of finite type over S.*

Then the following conditions are equivalent:

(i) *There exists an S-scheme X' and a factorization of f:*

$$f : X \xrightarrow{f'} X' \xrightarrow{i} Y$$

where f' is a faithfully flat S-morphism locally of finite presentation and i is a monomorphism.

(ii) *The first projection $\mathrm{pr}_1 : X \times_Y X \to X$ is a flat morphism.*

Moreover, under these conditions, (X', f') is a quotient of X by the equivalence relation induced by f (for the fppf topology), so that the factorization $f = i \circ f'$ is unique up to isomorphisms.

Proof. The implication (i)⇒(ii) is trivial. In fact, the first projection $\mathrm{pr}'_1 : X \times_{X'} X \to X$ factors through $X \times_Y X$:

$$\mathrm{pr}'_1 : X \times_{X'} X \xrightarrow{u} X \times_Y X \xrightarrow{\mathrm{pr}_1} X.$$

The morphism u is an isomorphism, since i is a monomorphism, and pr'_1 is flat since f' is flat, hence pr_1 is flat.

To prove that (ii)⇒(i), we first note that the assertions of Theorem 14.4.14 are local on Y (hence are local on S); they are also local on X, as it easily follows from the fact that a flat morphism locally of finite presentation is open ([**?**], IV$_3$, 11.3.1).

The case where Y is locally Noetherian and X is of finite type over Y is treated in ([**?**], cor.2 du th.2). We now see how to reduce ourselves to this case. Under the hypothesis (α), we can therefore assume that X, Y are affine and f is finitely presented. By replacing S with Y, we may also assume that X and Y are finitely presented over S. We then reduce to the case where S is Noetherian thanks to ([**?**], IV$_3$, 11.2.6).

Under the hypothesis (β), we can suppose that X, Y, S are affine, S is Noetherian and X is of finite type over S. Consider Y as a filtered projective limit of affine schemes Y_i which are of finite type over S. The schemes $X \times_{Y_i} X$ then form a descreasing filtered system of closed subschemes of $X \times_S X$, whose projective limit is $X \times_Y X$. As $X \times_S X$ is Noetherian, we have $X \times_{Y_i} X = X \times_Y X$ for i large enough, so that the composition

$$f_i : X \xrightarrow{f} Y \to Y_i$$

satisfies the hypothesis of (ii) if so does for f. As the equivalence relation defined by f over X coincides with that by f_i, it is clear that it suffices to prove (ii)⇒(i) for f_i, which means we can reduce to the case where Y is of finite type over S. □

Now let S be a scheme, G be a group scheme over S which is locally of finite presentation over S, and X be an S-scheme acted by G. If $X \to S$ possesses a section ξ, we recall that the stabilizer $\mathrm{Stab}_G(\xi)$ is represented by a group subscheme of G (in fact, by the group subscheme $G \times_X S$, cf. 14.1.1.2).

Theorem 14.4.15. *Let S be a scheme and G be a S-group scheme locally of finite presentation over S, which acts on an S-scheme X. Suppose that $X \to S$ possesses a section ξ such that the stabilizer H of ξ in G is flat over S. If one of the following conditions is satisfied:*

(a) *X is locally of finite type over S;*

(b) *S is locally Noetherian,*

then the fppf quotient sheaf G/H is representable by an S-scheme which is locally of finite presentation over S, and the S-morphism induced by ξ:

$$f : G = G \times_S S \to G \times_S X \to X, \quad g \mapsto g \cdot \xi$$

factors into

$$\begin{tikzcd} G \arrow[rr, "f"] \arrow[dr, "p"'] & & X \\ & G/H \arrow[ur, hook, "j"'] & \end{tikzcd}$$

where p is the canonical projection, which is a faithfully flat morphism locally of finite presentation, and j is a monomorphism.

Proof. The morphism f makes G an X-scheme. By the definition of the stabilizer of ξ, the morphism

$$G \times_S H \to G \times_X G, \quad (g, h) \mapsto (g, gh)$$

is an isomorphism. As H is flat over S, $G \times_S H$ is flat over G, hence the first projection $\mathrm{pr}_1 : G \times_S G \to G$ is flat. Therefore, if X is locally of finite type over S, then f is locally of finite presentation (Proposition 10.6.24), and otherwise S is supposed to be Noetherian. It then suffices to apply Theorem 14.4.14 to the morphism f. Also, it follows from ([**?**], IV$_4$, 17.7.5) that G/H is locally of finite presentation over S. □

Corollary 14.4.16. *Let S be a scheme and $u : G \to H$ be a morphism of S-group schemes. Suppose that G is locally of finite presentation over S and that, either H is locally of finite type over S, or S is locally Noetherian. Then, if $K = \ker u$ is flat over S, the quotient group G/K is representable by an S-group scheme which is of finite presentation over S, and u factors into*

$$\begin{tikzcd} G \arrow[rr, "u"] \arrow[dr, "p"'] & & H \\ & G/K \arrow[ur, hook, "j"'] & \end{tikzcd}$$

where p is the canonical projection which is faithfully flat and locally of finite presentation, and j is a monomorphism.

Proof. We can apply Theorem 14.4.15 to $X = H$ and the unit section of H. □