Algebra

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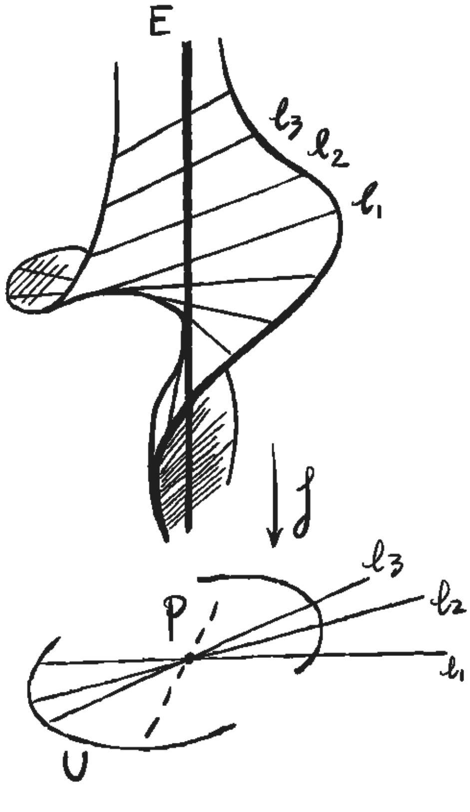

Contents

Chapter 1

D-modules and perverse sheaves

1.1 Elementary properties of D-modules

In this section we introduce several standard operations for D-modules over smooth algebraic varieties over $\mathbb{C}$ and present some fundamental results concerning them such as Kashiwara's equivalence theorem. Our main reference is [**?**].

1.1.1 Differential operators

Let X be a smooth (non-singular) algebraic variety over the complex number field $\mathbb{C}$ and $\mathscr{O}_X$ be the sheaf of rings of regular functions (structure sheaf) on it. We denote by Θ_X the **sheaf of vector fields** (**tangent sheaf**) on X:

$$\begin{aligned}\Theta_X &= \mathcal{D}er_{\mathbb{C}_X}(\mathscr{O}_X) = \mathcal{H}om_{\mathscr{O}_X}(\Omega^1_X, \mathscr{O}_X)\\ &= \{P \in \mathcal{E}nd_{\mathbb{C}_X}(\mathscr{O}_X) : P(fg) = P(f)g + fP(g) \text{ for } f, g \in \mathscr{O}_X\}.\end{aligned}$$

Hereafter, if there is no risk of confusion, we use the notation $f \in \mathscr{O}_X$ for a local section f of $\mathscr{O}_X$. Since X is smooth, the sheaf Θ_X is locally free of rank $n = \dim(X)$ over $\mathscr{O}_X$. We will identify $\mathscr{O}_X$ with a subsheaf of $\mathcal{E}nd_{\mathbb{C}_X}(\mathscr{O}_X)$ by identifying $f \in \mathscr{O}_X$ with $() \in \mathcal{E}nd_{\mathbb{C}_X}(\mathscr{O}_X)$. We define the sheaf $\mathscr{D}_X$ of **differential operators** on X as the subalgebra of $\mathcal{E}nd_{\mathbb{C}_X}(\mathscr{O}_X)$ generated by $\mathscr{O}_X$ and Θ_X. For any point of X we can take its affine open neighborhood U and a local coordinate system $\{x_i, \partial_i\}$ on it satisfying

$$x_i \in \mathscr{O}_X(U), \quad \Theta_U = \bigoplus_{i=1}^{n} \mathscr{O}_U \partial_i, \quad [\partial_i, \partial_j] = 0, \quad [\partial_i, x_j] = \delta_{ij}.$$

Hence we have

$$\mathscr{D}_U = \mathscr{D}_X|_U = \bigoplus_{\alpha \in \mathbb{N}^n} \mathscr{O}_U \partial^\alpha.$$

The ring $\mathscr{D}_X$ is generated by $\mathscr{O}_X$ and Θ_X, and their fundamental relations are the following:

- $\mathscr{O}_X \to \mathscr{D}_X$ is a ring homomorphism.
- $\Theta_X \to \mathscr{D}_X$ is left $\mathscr{O}_X$-linear.
- $\Theta_X \to \mathscr{D}_X$ is a Lie algebra homomorphism.
- $[v, f] = v(f)$ for $v \in \Theta_X$ and $f \in \mathscr{O}_X$.

where we denote by $v(f)$ the element of $\mathscr{O}_X$ obtained by differentiating f with respect to v. To be more precise, we formulate the above as follows:

Proposition 1.1.1. *Let $\mathscr{A}$ be a sheaf of rings on X, and $\iota : \mathscr{O}_X \to \mathscr{A}$, $\varphi : \Theta_X \to \mathscr{A}$ be sheaf morphisms such that*

(a) *$\iota : \mathscr{O}_X \to \mathscr{A}$ is a ring homomorphism;*

(b) *$\varphi : \Theta_X \to \mathscr{A}$ is left $\mathscr{O}_X$-linear;*

(c) *$\varphi : \Theta_X \to \mathscr{A}$ is a Lie algebra homomorphism;*

(d) *$[\varphi(v), \iota(f)] = \iota(v(f))$ for $v \in \Theta_X$, $f \in \mathscr{O}_X$.*

Then there exists a unique ring homomorpism $\Phi : \mathscr{D}_X \to \mathscr{A}$ such that the following diagram is commutative:

$$\begin{tikzcd}
\mathscr{O}_X \arrow[d] \arrow[dr, "\iota"] & \\
\mathscr{D}_X \arrow[r, "\Phi"] & \mathscr{A} \\
\Theta_X \arrow[u] \arrow[ur, "\varphi"'] &
\end{tikzcd}$$

Proof. Since the question is local, we can choose local coordinate system, so by the assumptions on ι and φ, we have

$$\Phi(\sum c_\alpha \partial^\alpha) = \sum_\alpha \iota(c_\alpha)\varphi(\partial_1^{\alpha_1}) \cdots \varphi(\partial_n^{\alpha_n}).$$

Conversely, if we define Φ as above, then it is easy to see that Φ is a ring homomorphism. $\square$

The ring $\mathscr{D}_X$ is non-commutative, so the study of its structure is more complicated than algebraic geometry (the study of commutative rings). However, we can derive objects in the theory of commutative algrbra from $\mathscr{D}_X$ as follows.

Let U be an affine open subset of X with local coordinates $\{x_i, \partial_i\}$. For a differential operator $P = \sum a_\alpha \partial^\alpha$, the **total symbol** of P is defined to be

$$\sigma(P)(x, \xi) := \sum a_\alpha(x)\xi^\alpha.$$

This is a function in $(x, \xi) = (x_1, \ldots, x_n, \xi_1, \ldots, \xi_n)$, which depends on the choice of a coordinate system. Let $Q = \sum b_\alpha \partial^\alpha$ be another differential operator and $\sigma(Q)(x, \xi) = \sum b_\alpha(x)\xi^\alpha$ be its total symbol, then the total symbol of PQ is given by the formula

$$\sigma(PQ)(x, \xi) = \sum_\alpha \frac{1}{\alpha!} \partial_\xi^\alpha \sigma(P)(x, \xi) \cdot \partial_x^\alpha \sigma(Q)(x, \xi). \tag{1.1.1}$$

Since $\sigma(P)(x, \xi)$ is a polynomial in ξ, we have $\partial_\xi^\alpha \sigma(P)(x, \xi) = 0$ except for finitely many α, and thus the right hand side of (1.1.1) is a finite sum over α, and it is easily deduced from Leibniz's rule

$$\partial^\alpha(fg) = \sum_\beta \binom{\alpha}{\beta} (\partial^\beta f)(\partial^{\alpha-\beta} g). \tag{1.1.2}$$

We now define an order filtration $F_p(\mathscr{D}_X)$ of $\mathscr{D}_X$ by

$$F_p(\mathscr{D}_X) = \{P \in \mathscr{D}_X : P = \sum_{|\alpha| \le p} a_\alpha \partial^\alpha\}.$$

Although this definition uses a coordinate system, the following proposition shows that it does not depend on the choice of this coordinate system.

Proposition 1.1.2. *We have $F_p(\mathscr{D}_X) = 0$ for $p < 0$, and for $p \geq 0$,*

$$F_p(\mathscr{D}_X) = \{P \in \mathscr{D}_X : [P, \mathscr{O}_X] \subseteq F_{p-1}(\mathscr{D}_X)\}.$$

Moreover, $F_0(\mathscr{D}_X) = \mathscr{O}_X$ and $F_1\mathscr{D}_X = \mathscr{O}_X \oplus \Theta_X$.

Proof. If locally we write $P = \sum_\alpha a_\alpha \partial^\alpha$ and $f \in \mathscr{O}_X$, then the operator $[P, f]$ is given by

$$\begin{aligned}[P, f](g) = P(fg) - fP(g) &= \sum_\alpha \left[a_\alpha \sum_\beta \binom{\alpha}{\beta} (\partial^\beta f)(\partial^{\alpha-\beta} g) - f a_\alpha (\partial^\alpha g)\right] \\ &= \sum_\alpha a_\alpha \sum_{0 \prec \beta \preceq \alpha} \binom{\alpha}{\beta} (\partial^\beta f)(\partial^{\alpha-\beta} g),\end{aligned}$$

which shows that $[P, \mathscr{O}_X]$ belongs to $F_{p-1}\mathscr{D}_X$ if and only if $P \in F_p(\mathscr{D}_X)$. □

Proposition 1.1.3. *The collection $\{F_p(\mathscr{D}_X)\}$ is an exhaustive increasing filtration of $\mathscr{D}_X$ and each $F_p(\mathscr{D}_X)$ is a locally free module over $\mathscr{O}_X$. Moreover, we have*

$$(F_p(\mathscr{D}_X))(F_q(\mathscr{D}_X)) = F_{p+q}(\mathscr{D}_X), \quad [F_p(\mathscr{D}_X), F_q(\mathscr{D}_X)] \subseteq F_{p+q-1}\mathscr{D}_X.$$

Proof. The first assertion is clear, and the second one can be checked locally. In fact, let $P \in F_p(\mathscr{D}_X)$ and $Q \in F_q(\mathscr{D}_X)$ be two differential operators and write $\sigma_j(W) = \sum_{|\alpha|=j} c_\alpha(x)\xi^\alpha$ for a differential operator W. We then have

$$\begin{aligned}\sigma(PQ) =& \sigma_p(P)\sigma_q(Q) + \left(\sigma_{p-1}(P)\sigma_q(Q) + \sigma_p(P)\sigma_{q-1}(Q) + \sum_i \frac{\partial\sigma_p(P)}{\partial\xi_i}\frac{\partial\sigma_q(Q)}{\partial x_i}\right) \\ &+ \text{terms of degree less than } p+q-1 \text{ in } \xi,\end{aligned} \tag{1.1.3}$$

whence the assertions. □

Let us now consider the graded ring

$$\operatorname{gr}(\mathscr{D}_X) = \bigoplus_{p=0}^{\infty} \operatorname{gr}_p(\mathscr{D}_X).$$

By Proposition 1.1.3, this is a sheaf of commutative algebras of finite type over $\mathscr{O}_X$. Since $\operatorname{gr}_0(\mathscr{D}_X) = \mathscr{O}_X$ and $\operatorname{gr}_1(\mathscr{D}_X) = \Theta_X$, we obtain an $\mathscr{O}_X$-algebra homomorphism

$$\boldsymbol{S}_{\mathscr{O}_X}(\Theta_X) \to \operatorname{gr}(\mathscr{D}_X) \tag{1.1.4}$$

Take an affine chart U with a coordinate system $\{x_i, \partial_i\}$ and set $\xi_i := \partial_i \bmod F_0\mathscr{D}_U = \mathscr{O}_U$, we then have

$$\operatorname{gr}(\mathscr{D}_U) = \mathscr{O}_U[\xi_1, \dots, \xi_n],$$

so the homomorphism (1.1.4), which is given by $\xi^\alpha \mapsto \partial^\alpha$, is an isomorphism. We denote by σ_p the homomorphism defined by

$$\sigma_p : F_p(\mathscr{D}_X) \to \operatorname{gr}_p(\mathscr{D}_X) \subseteq \operatorname{gr}(\mathscr{D}_X) \cong \boldsymbol{S}_{\mathscr{O}_X}(\Theta_X).$$

By using local coordinates, we have

$$\sigma_p(P) = \sum_{|\alpha|=p} a_\alpha(x)\xi^\alpha \in \boldsymbol{S}^p_{\mathscr{O}_X}(\Theta_X)$$

for $P = \sum_\alpha a_\alpha \partial^\alpha$. The corresponding section $\sigma_p(P)$ is called the **principal symbol** of P.

Proposition 1.1.4. *Assume that $A = \mathscr{D}_X(U)$ for some affine open subset U of X or $A = \mathscr{D}_{X,x}$ for some $x \in X$. Then A is a left (and right) Noetherian ring.*

Proof. We have seen that A admits a Hausdorff and complete filtration so that the associated graded ring is (commutative and) Noetherian. The assertion thus follows from **??**. $\square$

We have thus succeeded to derive an element of the commutative algebra $\boldsymbol{S}_{\mathscr{O}_X}(\Theta_X)$, namely the principal symbol, from a differential operator. Although the principal symbol is only a part of a differential operator (indeed, it is the part of the highest degree of the total symbol), it carries a great deal of information on $\mathscr{D}_X$, as seen in the following.

Let $\pi : T^*X \to X$ denote the cotangent bundle of X. Then we may regard $\xi_1, \dots, \xi_n$ as the coordinate system of the cotangent space $\bigoplus_{i=1}^n \mathbb{C}dx_i$, and hence $\boldsymbol{S}_{\mathscr{O}_X}(\Theta_X)$ is canonically identified with the sheaf $\pi_*(\mathscr{O}_{T^*X})$. A section of $\boldsymbol{S}_{\mathscr{O}_X}(\Theta_X)$ can therefore be regarded as a function on T^*X. If $P \in D_p\mathscr{D}_X$ and $Q \in D_q\mathscr{D}_X$, then $[P, Q] \in D_{p+q-1}\mathscr{D}_X$, so the commutator induces a multiplication map on $\mathrm{gr}(\mathscr{D}_X)$. This can be explicitly calculated by equation (1.1.3) in local coordinates:

$$\sigma_{p+q-1}([P,Q]) = \sum_i \Big(\frac{\partial\sigma_p(P)}{\partial\xi_i}\frac{\partial\sigma_q(Q)}{\partial x_i} - \frac{\partial\sigma_q(Q)}{\partial\xi_i}\frac{\partial\sigma_p(P)}{\partial x_i}\Big). \tag{1.1.5}$$

Now for two function f, g in x and ξ, the **Possion bracket** of f and g is defined by

$$\{f, g\} = \sum_i \Big(\frac{\partial f}{\partial\xi_i}\frac{\partial g}{\partial x_i} - \frac{\partial g}{\partial\xi_i}\frac{\partial f}{\partial x_i}\Big). \tag{1.1.6}$$

With the notation, we can then write (1.1.5) into the following form:

$$\sigma_{p+q-1}([P,Q]) = \{\sigma_p(P), \sigma_q(Q)\}. \tag{1.1.7}$$

Considering f and g as functions on T^*X, we see that $\{\cdot,\cdot\}$ is independent of the choice of a local coordinate system as below. We have a canonical 1-form ω_X on T^*X. For every point $p \in T^*X$, a 1-form ω_p at a point $\pi(p)$ of X is determined by the definition of T^*X. The canonical 1-form ω_X is then defined by $\omega_X(p) = \pi^*\omega_p$. In local coordinates, we have

$$\omega_X = \sum_i \xi_i dx_i.$$

At each point p, the 2-form $\theta_X = d\omega_X$ gives an anti-symmetric bilinear form on $T_p(T^*X)$, which is nondegenerate and induces an isomorphism $H : T_p^*(T^*X) \cong T_p(T^*)X$ by

$$\theta_X(v, H(\eta)) = \langle \eta, v\rangle, \quad \eta \in T_p^*(T^*X), v \in T_p(T^*X).$$

Explicitly in local coordinates, the isomorphism H is given by

$$H : T_p^*(T^*X) \cong T_p(T^*)X, \quad d\xi_i \mapsto \partial/\partial_i, dx_i \mapsto -\partial/\partial\xi_i. \tag{1.1.8}$$

In particular, $H_f = H(df)$ is a vector field on T^*X for any function f on T^*X; this is called the **Hamiltonian** of f.

Definition 1.1.1. For functions f, g on T^*X, the **Poisson bracket** of f and g is defined to be $\{f, g\} = H_f(g)$.

By (1.1.8), the Possion bracket $\{\cdot,\cdot\}$ is expressed as (1.1.6) in local coordinates so it is determined by (T^*X, θ_X). A pair (M, θ) of a manifold M and a closed 2-form θ on M is called a **symplectic manifold** if θ is a nondegenerate anti-symmetric bilinear form on T_pM for every p. For such a manifold, we can define a Possion bracket in the same way as above, and this is a notion depending on the symplectic structure of M. By tracing back the above arguments, we can determine a 2-form from the Poisson bracket.

In the formula (1.1.7), the commutator of $\mathscr{D}_X$ expresses the noncommutatitity of $\mathscr{D}_X$. Hence, symbolically speaking, the noncommutatitity of $\mathscr{D}_X$ determines a symplectic structure of T^*X.

1.1.2 D-modules

Let X be a smooth algebraic variety. We say that a sheaf $\mathscr{M}$ on X is a left $\mathscr{D}_X$-module if $\Gamma(U, \mathscr{M})$ is endowed with a left $\Gamma(U, \mathscr{D}_X)$-module structure for each open subset U of X and these actions are compatible with restriction morphisms. Note that $\mathscr{O}_X$ is a left $\mathscr{D}_X$-module via the canonical action of $\mathscr{D}_X$. We have the following very easy (but useful) interpretation of the notion of left $\mathscr{D}_X$-modules.

Lemma 1.1.5. *Let $\mathscr{M}$ be an $\mathscr{O}_X$-module. Giving a left $\mathscr{D}_X$-module structure on $\mathscr{M}$ extending the $\mathscr{O}_X$-module structure is equivalent to giving a $\mathbb{C}$-linear morphism*

$$\nabla : \Theta_X \to \mathcal{E}nd_{\mathbb{C}}(\mathscr{M}), \quad v \mapsto \nabla_v$$

satisfying the following conditions:

(a) $\nabla_{fv}(s) = f\nabla_v(s)$ *for* $f \in \mathscr{O}_X$, $v \in \Theta_X$, $s \in M$;

(b) $\nabla_v(fs) = v(f)s + f\nabla_v(s)$ *for* $f \in \mathscr{O}_X$, $v \in \Theta_X$, $s \in M$;

(c) $\nabla_{[v,w]}(s) = [\nabla_v, \nabla_w](s)$ *for* $v, w \in \Theta_X$, $s \in M$.

In terms of ∇ the left $\mathscr{D}_X$-module structure on $\mathscr{M}$ is given by $v \cdot s = \nabla_v(s)$ for $v \in \Theta_X$, $s \in \mathscr{M}$.

Proof. The proof is immediate, because $\mathscr{D}_X$ is generated by $\mathscr{O}_X$, Θ_X and satisfies the relation $[v, f] = v(f)$. □

For a locally free left $\mathscr{O}_X$-module $\mathscr{M}$ of finite rank, a $\mathbb{C}$-linear morphism $\nabla : \Theta_X \to \mathcal{E}nd_{\mathbb{C}}(\mathscr{M})$ satisfying the conditions (a), (b) is usually called a **connection** (of the corresponding vector bundle). If it also satisfies the condition (c), it is called an **integrable** (or **flat**) **connection**. Hence we may regard a (left) $\mathscr{D}_X$-module as an integrable connection of an $\mathscr{O}_X$-module which is not necessarily locally free of finite rank. We say that a $\mathscr{D}_X$-module $\mathscr{M}$ is an **integrable connection** if it is locally free of finite rank over $\mathscr{O}_X$, and we denote by $\mathrm{Conn}(X)$ the category of integrable connections on X. Integrable connections are the most elementary left D-modules. Nevertheless, they are especially important because they generate (in a categorical sense) the category of holonomic systems, as we see later.

Example 1.1.1 (Ordinary Differential Equations). Consider an ordinary differential operator

$$P = a_n(x)\partial^n + \cdots + a_0(x), \quad \partial = d/dx, a_i \in \mathscr{O}_{\mathbb{C}}$$

on $\mathbb{C}$ and the corrresponding $\mathscr{D}_{\mathbb{C}}$-module $\mathscr{M} = \mathscr{D}_{\mathbb{C}}/\mathscr{D}_{\mathbb{C}}P = \mathscr{D}_{\mathbb{C}}u$, where $u \equiv 1 \bmod \mathscr{D}_{\mathbb{C}}P$, and hence $Pu = 0$. Then on $U = \{x \in \mathbb{C} : a_n(x) \neq 0\}$ we have $\mathscr{M}|_U \cong \bigoplus_{i=0}^{n-1} \mathscr{O}_U u_i$ (where $u_i = \partial^i u$), so $\mathscr{M}$ is an integrable connection of rank n on U.

1.1.2.1 Differential homomorphisms

Let X be a smooth algebraic variety and $\mathscr{M}, \mathscr{N}$ be $\mathscr{O}_X$-modules. A $\mathbb{C}$-linear sheaf homomorphism $\varphi : \mathscr{M} \to \mathscr{N}$ is called a **differential homomorphism** if for every $s \in \mathscr{M}$ there exists finitely many $P_i \in \mathscr{D}_X$ and $v_i \in \mathscr{N}$ such that

$$\varphi(fs) = \sum_i P_i(f)v_i$$

for any $f \in \mathscr{O}_X$. In other words, φ is a differential homomorphism if it can be generated by differential operators. Let $\mathcal{D}iff(\mathscr{M}, \mathscr{N})$ be the the sheaf of differential homomorphisms from $\mathscr{M}$ to $\mathscr{N}$. From our definition, it is clear that $\mathscr{D}_X = \mathcal{D}iff(\mathscr{O}_X, \mathscr{O}_X)$.

Lemma 1.1.6. *Let $\mathscr{M}, \mathscr{N}, \mathscr{H}$ be $\mathscr{O}_X$-modules and $\varphi : \mathscr{M} \to \mathscr{N}$, $\psi : \mathscr{N} \to \mathscr{H}$ be differential homomorphisms. Then $\psi \circ \varphi : \mathscr{M} \to \mathscr{N}$ is also a differential homomorphism.*

Proof. By definition, for every $s \in \mathscr{M}$, there exists $P_i, Q_j \in \mathscr{D}_X$ and $v_i \in \mathscr{N}$, $w_{ij} \in \mathscr{H}$ such that

$$\varphi(fs) = \sum_i P_i(f)v_i, \quad \psi(gv_i) = \sum_j Q_j(g)w_{ij}.$$

It then follows that

$$(\psi \circ \varphi)(fs) = \psi\Big(\sum_i P_i(f)v_i\Big) = \sum_i jQ_i(P_i(f))w_{ij}$$

so $\psi \circ \varphi$ is also a differential homomorphism. □

Let us consider the right $\mathscr{D}_X$-module $\mathscr{N} \otimes_{\mathscr{O}_X} \mathscr{D}_X$ for an $\mathscr{O}_X$-module $\mathscr{N}$. The right $\mathscr{D}_X$-module structure gives $\mathscr{N} \otimes_{\mathscr{O}_X} \mathscr{D}_X$ an $\mathscr{O}_X$-module structure. By tensoring $\mathscr{N}$ with the left $\mathscr{O}_X$-linear homomorphism[1]

$$\mathscr{D}_X \to \mathscr{O}_X, \quad P \mapsto P(1) \in \mathscr{O}_X,$$

we obtain a $\mathbb{C}$-linear homomorphism (which is not $\mathscr{O}_X$-linear)

$$P_{\mathscr{N}} : \mathscr{N} \otimes_{\mathscr{O}_X} \mathscr{D}_X \to \mathscr{N}$$

which induces a map

$$\mathcal{H}om_{\mathscr{O}_X}(\mathscr{M}, \mathscr{N} \otimes_{\mathscr{O}_X} \mathscr{D}_X) \to \mathcal{H}om_{\mathbb{C}}(\mathscr{M}, \mathscr{N}), \quad \varphi \mapsto P_{\mathscr{N}} \circ \varphi.$$

If $\varphi(s) = \sum_i v_i \otimes P_i$ for $s \in \mathscr{M}$, then we have

$$P_{\mathscr{N}} \circ \varphi(fs) = \sum_i P_i(f)v_i.$$

Therefore, we see that $P_{\mathscr{N}} \circ \varphi$ is a differential homomorphism.

Proposition 1.1.7. *The homomorphism $\mathcal{H}om_{\mathscr{O}_X}(\mathscr{M}, \mathscr{N} \otimes_{\mathscr{O}_X} \mathscr{D}_X) \to \mathcal{D}iff(\mathscr{M}, \mathscr{N})$ is an isomorphism.*

Proof. We first prove the proposition for $\mathscr{M} = \mathscr{O}_X$. By taking a coordinate system $\{x_i, \partial_i\}$, we have $\mathscr{N} \otimes_{\mathscr{O}_X} \mathscr{D}_X \cong \bigoplus_\alpha \mathscr{N} \otimes \partial^\alpha$, and

$$\mathcal{H}om_{\mathscr{O}_X}(\mathscr{O}_X, \mathscr{N} \otimes_{\mathscr{O}_X} \mathscr{D}_X) \cong \bigoplus_\alpha \mathscr{N} \otimes \partial^\alpha.$$

Let $\mathscr{K}$ be the kernel of the homomorphism in question, and put

$$F_p\mathscr{K} = \{\sum v_\alpha \otimes \partial^\alpha \in \mathscr{K} : v_\alpha = 0 \text{ for } |\alpha| > p\}.$$

Then we have $\mathscr{K} = \bigcup_p F_p\mathscr{K}$, so it suffices to prove inductively that $F_p\mathscr{K} = 0$ for each $p \geq 0$. Let $\sum v_\alpha \otimes \partial^\alpha \in F_p\mathscr{K}$; then we have $\sum (\partial^\alpha f)v_\alpha = 0$ for any $f \in \mathscr{O}_X$. First we see that $F_0\mathscr{K} = 0$ for $f = 1$, so assume that $p > 0$ and $F_{p-1}\mathscr{K} = 0$. For each fixed integer i, by replacing f with $x_i f$, we have

$$0 = \sum \partial^\alpha (x_i f)v_\alpha - \sum x_i(\partial^\alpha f)v_\alpha = \sum_{\alpha_i > 0} (\partial^{\alpha - \delta_i} f)v_\alpha,$$

where δ_i is the i-th unit vector $(0, \ldots, 1, \ldots, 0)$. The induction hypothesis leads to $v_\alpha = 0$ for $\alpha_i > 0$. Fianlly, we also see that $v_0 = 0$, since $v_0 \otimes \partial^0 \in F_0\mathscr{K} = 0$.

On the other hand, if $\varphi \in \mathcal{D}iff(\mathscr{O}_X, \mathscr{N})$ and $\varphi(f) = \sum_i P_i(f)v_i$, then $\varphi = P_{\mathscr{N}} \circ \tilde{\varphi}$ with $\tilde{\varphi}(1) = \sum v_i \otimes P_i$.

[1] This is in fact the projection from $\mathscr{D}_X$ to $\mathscr{O}_X$, which is the identity on $\mathscr{O}_X$ and zero on Θ_X.

We have thus proved the proposition in the case when $\mathscr{M} = \mathscr{O}_X$. For the general case, let $\varphi \in \mathcal{D}\mathit{iff}(\mathscr{M}, \mathscr{N})$; since the map $\mathscr{O}_X \ni f \mapsto \varphi(fs)$ belongs to $\mathcal{D}\mathit{iff}(\mathscr{O}_X, \mathscr{N})$ for every $s \in \mathscr{M}$, there exists a unique homomorphism $\tilde{\varphi}(s) \in \mathscr{N} \otimes_{\mathscr{O}_X} \mathscr{D}_X$ satisfying

$$P_{\mathscr{N}}(\tilde{\varphi}(s)f) = \varphi(fs), \quad f \in \mathscr{O}_X.$$

It is immediate that $\tilde{\varphi} \in \mathcal{H}\mathit{om}(\mathscr{M}, \mathscr{N} \otimes_{\mathscr{O}_X} \mathscr{D}_X)$ is $\mathscr{O}_X$-linear. If $\psi \in \mathcal{H}\mathit{om}(\mathscr{M}, \mathscr{N} \otimes_{\mathscr{O}_X} \mathscr{D}_X)$ satisfies $P_{\mathscr{N}} \circ \psi = 0$, then for every $s \in \mathscr{M}$ we have $P_{\mathscr{N}}\psi(fs) = 0$ for $f \in \mathscr{O}_X$, and therefore $\psi(s) = 0$. □

Corollary 1.1.8. *There is a canonical isomorphism*

$$\mathcal{H}\mathit{om}_{\mathscr{D}_X^{\mathrm{op}}}(\mathscr{M} \otimes_{\mathscr{O}_X} \mathscr{D}_X, \mathscr{N} \otimes_{\mathscr{O}_X} \mathscr{D}_X) \xrightarrow{\sim} \mathcal{D}\mathit{iff}(\mathscr{M}, \mathscr{N}).$$

Proof. It suffices to note that

$$\mathcal{H}\mathit{om}_{\mathscr{D}_X^{\mathrm{op}}}(\mathscr{M} \otimes_{\mathscr{O}_X} \mathscr{D}_X, \mathscr{N} \otimes_{\mathscr{O}_X} \mathscr{D}_X) \cong \mathcal{H}\mathit{om}_{\mathscr{O}_X}(\mathscr{M}, \mathscr{N} \otimes_{\mathscr{O}_X} \mathscr{D}_X).$$
□

Remark 1.1.1. The inverse map of Proposition 1.1.7 is given as follows. Let $\varphi \in \mathcal{D}\mathit{iff}(\mathscr{M}, \mathscr{N})$ be a differential homomorphism. Then for each $s \in \mathscr{M}$, there exists $P_i \in \mathscr{D}_X$ and $v_i \in \mathscr{N}$ such that

$$\varphi(fs) = \sum_i P_i(f)v_i \quad \text{for} \quad f \in \mathscr{O}_X. \tag{1.1.9}$$

The inverse image $\tilde{\varphi} \in \mathcal{H}\mathit{om}_{\mathscr{O}_X}(\mathscr{M}, \mathscr{N} \otimes_{\mathscr{O}_X} \mathscr{D}_X)$ is then defined to be

$$\tilde{\varphi}(s) = \sum_i v_i \otimes P_i \in \mathscr{N} \otimes \mathscr{D}_X.$$

In fact, it is easy to verify that $P_{\mathscr{N}} \circ \tilde{\varphi} = \varphi$ from equation (1.1.9).

Proposition 1.1.9. *Let $\mathscr{M}$, $\mathscr{N}$ and $\mathscr{H}$ be $\mathscr{O}_X$-modules. Then the diagram*

$$\begin{tikzcd}
\mathcal{H}\mathit{om}_{\mathscr{D}_X^{\mathrm{op}}}(\mathscr{M} \otimes_{\mathscr{O}_X} \mathscr{D}_X, \mathscr{N} \otimes_{\mathscr{O}_X} \mathscr{D}_X) \otimes_{\mathbb{C}} \mathcal{H}\mathit{om}_{\mathscr{D}_X^{\mathrm{op}}}(\mathscr{M} \otimes_{\mathscr{O}_X} \mathscr{D}_X, \mathscr{N} \otimes_{\mathscr{O}_X} \mathscr{D}_X) \arrow[r] \arrow[d] & \mathcal{D}\mathit{iff}(\mathscr{M}, \mathscr{N}) \otimes_{\mathbb{C}} \mathcal{D}\mathit{iff}(\mathscr{N}, \mathscr{H}) \arrow[d] \\
\mathcal{H}\mathit{om}_{\mathscr{D}_X^{\mathrm{op}}}(\mathscr{M} \otimes_{\mathscr{O}_X} \mathscr{D}_X, \mathscr{H} \otimes_{\mathscr{O}_X} \mathscr{D}_X) \arrow[r] & \mathcal{D}\mathit{iff}(\mathscr{M}, \mathscr{H})
\end{tikzcd}$$

is commutative, where the vertical arrows are the homomorphisms obtained by composition.

Proof. This follows from Proposition 1.1.7, since the isomorphsim in it is compatible with compositions. □

Let us apply the above argument to the de Rham complex

$$\cdots \longrightarrow 0 \longrightarrow \Omega_X^0 \longrightarrow \Omega_X^1 \longrightarrow \cdots \longrightarrow \Omega_X^n \longrightarrow 0 \longrightarrow \cdots$$

Take a coordinate system $\{x_i, \partial_i\}$, we then have

$$d(f\omega) = df \wedge \omega + fd\omega = \sum_i \frac{\partial f}{\partial x_i} dx_i \wedge \omega + fd\omega,$$

so the exterior differential d is a differential homomorphism, and Corollary 1.1.8 gives a complex of right $\mathscr{D}_X$-modules

$$\cdots \longrightarrow 0 \longrightarrow \Omega_X^0 \otimes_{\mathscr{O}_X} \mathscr{D}_X \longrightarrow \cdots \longrightarrow \Omega_X^n \otimes_{\mathscr{O}_X} \mathscr{D}_X \longrightarrow 0 \longrightarrow \cdots \tag{1.1.10}$$

By definition, the differential in this complex is given by

$$d(\omega \otimes P) = \sum_{i=1}^{n} dx_i \wedge \omega \otimes \partial_i P + d\omega \otimes P \tag{1.1.11}$$

where $\omega \in \Omega_X^{\bullet}$, $P \in \mathscr{D}_X$. In particular, choosing $\omega = 1 \in \Omega_X^0$, we obtain

$$dP = \sum_{i=1}^{n} dx_i \otimes \partial_i P, \tag{1.1.12}$$

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^p \omega \wedge d\eta, \tag{1.1.13}$$

where $\omega \in \Omega_X^p$ and $\eta \in \Omega_X^q \otimes_{\mathscr{O}_X} \mathscr{D}_X$.

Conversely, these two formule characterize the differential d of $\Omega_X^{\bullet} \otimes_{\mathscr{O}_X} \mathscr{D}_X$. For every left $\mathscr{D}_X$-module $\mathscr{M}$, we then obtain a complex

$$\cdots \longrightarrow 0 \longrightarrow \Omega_X^0 \otimes_{\mathscr{O}_X} \mathscr{M} \longrightarrow \cdots \longrightarrow \Omega_X^n \otimes_{\mathscr{O}_X} \mathscr{M} \longrightarrow 0 \longrightarrow \cdots$$

by applying the functor $- \otimes_{\mathscr{D}_X} \mathscr{M}$ to (1.1.10). In other words, we have

$$ds = \sum_{i=1}^{n} dx_i \otimes \partial_i s, \tag{1.1.14}$$

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^p \omega \wedge d\eta, \tag{1.1.15}$$

where $s \in \mathscr{M}$, $\omega \in \Omega_X^p$, $\eta \in \Omega_X^q \otimes \mathscr{M}$. The complex $(\Omega_X^{\bullet} \otimes_{\mathscr{O}_X} \mathscr{M}, d)$ is called the **de Rham complex** of $\mathscr{M}$ and denoted by $DR_X(\mathscr{M})$.

By applying the functor

$$\mathcal{H}om_{\mathscr{D}_X^{\mathrm{op}}}(-, \mathscr{D}_X) : \mathbf{Mod}(\mathscr{D}_X^{\mathrm{op}}) \to \mathbf{Mod}(\mathscr{D}_X)^{\mathrm{op}}$$

to (1.1.10), we obtain a complex of left $\mathscr{D}_X$-modules

$$\cdots \longrightarrow 0 \longrightarrow \mathscr{D}_X \otimes_{\mathscr{O}_X} \textstyle\bigwedge^n \Theta_X \longrightarrow \cdots \longrightarrow \mathscr{D}_X \otimes_{\mathscr{O}_X} \bigwedge^0 \Theta_X \longrightarrow 0 \longrightarrow \cdots \tag{1.1.16}$$

since we have

$$\mathcal{H}om_{\mathscr{D}_X^{\mathrm{op}}}(\Omega_X^p \otimes_{\mathscr{O}_X} \mathscr{D}_X, \mathscr{D}_X) \cong \mathcal{H}om_{\mathscr{O}_X}(\Omega_X^p, \mathscr{D}_X) \cong \mathscr{D}_X \otimes_{\mathscr{O}_X} \textstyle\bigwedge^p \Theta_X.$$

Explicitly, the differential d is given by

$$\begin{aligned} d(P \otimes v_1 \wedge \cdots \wedge v_p) &= \sum_i (-1)^{i-1} P v_i \otimes v_1 \wedge \cdots \wedge \hat{v}_i \wedge \cdots \wedge v_p \\ &\quad + \sum_{i<j} (-1)^{i+j} P \otimes [v_i, v_j] \wedge v_1 \wedge \cdots \wedge \hat{v}_i \wedge \cdots \wedge \hat{v}_j \wedge \cdots \wedge v_p. \end{aligned}$$

Proposition 1.1.10. *We have the following locally free resolutions of the left $\mathscr{D}_X$-module $\mathscr{O}_X$ and the right $\mathscr{D}_X$-module Ω_X.*

$$0 \longrightarrow \Omega_X^0 \otimes_{\mathscr{O}_X} \mathscr{D}_X \longrightarrow \cdots \longrightarrow \Omega_X^n \otimes_{\mathscr{O}_X} \mathscr{D}_X \longrightarrow \Omega_X \longrightarrow 0 \tag{1.1.17}$$

$$0 \longrightarrow \mathscr{D}_X \otimes_{\mathscr{O}_X} \textstyle\bigwedge^n \Theta_X \longrightarrow \cdots \longrightarrow \mathscr{D}_X \otimes_{\mathscr{O}_X} \bigwedge^0 \Theta_X \longrightarrow \mathscr{O}_X \longrightarrow 0 \tag{1.1.18}$$

Proof. The assertion for Ω_X follows from the one for $\mathscr{O}_X$ using the side-changing operation. Later we will show that the complex (1.1.10) is a resolution of $\mathscr{O}_X$ in a general setting (cf. Proposition 1.2.2). □

1.1.2.2 Correspondence between left and right D-modules Take a local coordinate system $\{x_i, \partial_i\}$ on an affine open subset U of X. For $P = \sum_\alpha a_\alpha \partial^\alpha \in \mathscr{D}_U$, consider its formal adjoint

$$P^* := \sum_\alpha (-1)^{|\alpha|} \partial^\alpha a_\alpha(x) \in \mathscr{D}_U.$$

Then we have $(PQ)^* = Q^* P^*$, and an anti-automorphism $P \mapsto P^*$ of $\mathscr{D}_U$. Therefore, for a left $\mathscr{D}_U$-module $\mathscr{M}$ we can define a right action of $\mathscr{D}_U$ on $\mathscr{M}$ by $sP := P^* s$ for $s \in \mathscr{M}$, and obtain a right $\mathscr{D}_U$-module $\mathscr{M}^*$. However, this notion depends on the choice of a local coordinate. In order to globalize this correspondence to arbitrary smooth algebraic variety X we need to use the canonical sheaf $\Omega_X := \Omega_X^n$, where $n = \dim(X)$, since the formal adjoint of a differential operator naturally acts on Ω_X.

Recall that there are two natural actions of Θ on the sheaf $\Omega_X^\bullet = \bigoplus_i \Omega_X^i$. For $v \in \Theta_X$, its **inner derivation** $i_v \in \mathcal{E}nd_{\mathbb{C}}(\Omega_X^\bullet)$ is characterized by the following properties:

(I1) $i_{fv} = f i_v = i_v f$ for $f \in \mathscr{O}_X$, $v \in \Theta_X$;

(I2) $i_v(\omega \wedge \eta) = (i_v \omega) \wedge \eta + (-1)^{\deg(\omega)} \omega \wedge i_v \eta$ for $\omega, \eta \in \Omega_X^\bullet$;

(I3) $i_v(\mathscr{O}_X) = 0$;

(I4) $i_v(\omega) = \langle v, \omega \rangle \in \mathscr{O}_X$ for $\omega \in \Omega_X^1$.

In contrast, the Lie derivative $\mathfrak{L}_v \in \mathcal{E}nd_{\mathbb{C}}(\Omega_X^\bullet)$ is characterized by the following properties:

(L1) $\mathfrak{L}_v(\omega \wedge \eta) = \mathfrak{L}_v \omega \wedge \eta + \omega \wedge \mathfrak{L}_v \eta$ for $\omega, \eta \in \Omega_X^\bullet$;

(L2) $\mathfrak{L}_v f = v(f)$ for $f \in \mathscr{O}_X$,

(L3) $d\mathfrak{L}_v = \mathfrak{L}_v d$.

Hence $\mathfrak{L}_V$ is an operator of degree 0 on $\Omega_X^\bullet$, and locally it is given by

$$\mathfrak{L}_v(\omega)(v_1, \ldots, v_n) = v(\omega(v_1, \ldots, v_n)) - \sum_{i=1}^n \omega(v_1, \ldots, [v, v_i], \ldots, v_n).$$

The Lie derivatives also satisfy $[\mathfrak{L}_v, \mathfrak{L}_w] = \mathfrak{L}_{[v,w]}$ for $v, w \in \Theta_X$, and these two derivatives are related by

$$\mathfrak{L}_v = d i_v + i_v d.$$

Lemma 1.1.11. *For every $f \in \mathscr{O}_X$ and $v \in \Theta_X$, we have*

$$\mathfrak{L}_{fv}(\omega) = \mathfrak{L}_v(f\omega) = f\mathfrak{L}_v(\omega) + v(f)\omega.$$

where $\omega \in \Omega_X := \Omega_X^n$.

Proof. For $\omega \in \Omega_X^\bullet$, we have

$$\mathfrak{L}_{fv}(\omega) = d(f i_v \omega) + f i_v(d\omega) = df \wedge i_v \omega + f d i_v \omega + f i_v d\omega = df \wedge i_v \omega + f \mathfrak{L}_v(\omega).$$

Since $i_v(df \wedge \omega) = i_v(df) \wedge \omega - df \wedge i_v \omega = v(f)\omega - df \wedge i_v \omega$, we then conclude that

$$\mathfrak{L}_{fv}(\omega) = f\mathfrak{L}_v(\omega) + v(f)\omega - i_v(df \wedge \omega) = \mathfrak{L}_v(f\omega) - i_v(df \wedge \omega),$$

and this gives the desired result since $df \wedge \omega = 0$ for $\omega \in \Omega_X$. $\square$

Lemma 1.1.11 shows that the map $\varphi : \Theta_X \to \mathcal{E}nd_{\mathbb{C}}(\Omega_X)^{\mathrm{op}}$, $v \mapsto -\mathfrak{L}_v$ is left $\mathscr{O}_X$-linear, so by Proposition 1.1.1 we obtain an extended homomorphism $\mathscr{D}_X \to \mathcal{E}nd_{\mathbb{C}}(\Omega_X)^{\mathrm{op}}$, which gives a right $\mathscr{D}_X$-module structure on Ω_X. By definition, the equality

$$\omega v = -\mathfrak{L}_v(\omega)$$

holds for $v \in \Theta_X$ and $\omega \in \Omega_X$.

Remark 1.1.2. The action of Ω_X is related to integration by parts. Namely, $f \in \mathscr{O}_X$, $\omega \in \Omega_X$ and $P \in \mathscr{D}_X$ formally satisfy

$$\int_X (\omega P) f = \int_X \omega P(f).$$

That is, there exists a differential form η of degree $(n-1)$ such that

$$(\omega P) f - \omega P(f) = d\eta.$$

For example, if $P = v \in \Theta_X$, then it is easy to see that

$$(\omega v) f - \omega v(f) = -\mathfrak{L}_v(\omega) f - \omega v(f) = -d(f i_v \omega).$$

Remark 1.1.3. In terms of an affine open U and a local coordinate system $\{x_i, \partial_i\}$, we have

$$(f dx_1 \wedge \cdots \wedge dx_n) P = P^* f \, dx_1 \wedge \cdots \wedge dx_n, \tag{1.1.19}$$

where $f \in \mathscr{O}_X$ and $P \in \mathscr{D}_U$. To see this, it suffices to note that if $v = \sum_i v_i \partial_i$, then we have

$$\begin{aligned}(f dx_1 \wedge \cdots \wedge dx_n) v &= (f dx_1 \wedge \cdots \wedge dx_n)(\sum_i v_i \partial_i) = -\sum_i v_i \mathfrak{L}_{\partial_i}(f dx_1 \wedge \cdots dx_n) \\ &= -\sum_i v_i f \mathfrak{L}_{\partial_i}(x_1 \wedge \cdots \wedge x_n) - \sum_i v_i \frac{\partial f}{\partial x_i} x_1 \wedge \cdots \wedge x_n = v^* f \, dx_1 \wedge \cdots \wedge dx_n.\end{aligned}$$

For an invertible $\mathscr{O}_X$-module $\mathscr{L}$, we denote by $\mathscr{L}^{\otimes -1}$ the dual $\mathcal{H}om_{\mathscr{O}_X}(\mathscr{L}, \mathscr{O}_X)$. For $t \in \mathscr{L}^{\otimes -1}$ and $s \in \mathscr{L}$, we denote by $\langle t, s \rangle \in \mathscr{O}_X$ the image of $t \otimes s$ under the isomorphism $\mathscr{L}^{\otimes -1} \otimes_{\mathscr{O}_X} \mathscr{L} \cong \mathscr{O}_X$. Let $\mathscr{A}$ be an $\mathscr{O}_X$-algebra, that is, a sheaf of rings with a ring homomorphism $\mathscr{O}_X \to \mathscr{A}$. Then there exists a natural ring structure on $\mathscr{L} \otimes_{\mathscr{O}_X} \mathscr{A} \otimes_{\mathscr{O}_X} \mathscr{L}^{\otimes -1}$, given by

$$(s_1 \otimes a_1 \otimes t_1) \cdot (s_2 \otimes a_2 \otimes t_2) = s_1 \otimes a_1 \langle t_1, s_2 \rangle a_2 \otimes t_2$$

where $s_i \in \mathscr{L}$, $t_i \in \mathscr{L}^{\otimes -1}$, $a_i \in \mathscr{A}$. If $\mathscr{M}$ is an $\mathscr{A}$-module, then $\mathscr{L} \otimes_{\mathscr{O}_X} \mathscr{M}$ is a left $\mathscr{L} \otimes_{\mathscr{O}_X} \mathscr{A} \otimes_{\mathscr{O}_X} \mathscr{L}^{\otimes -1}$-module, whose action is given by

$$(s \otimes a \otimes t) \cdot (s' \otimes u) = s \otimes a \langle t, s' \rangle u.$$

In view of the isomorphism $\mathscr{L}^{\otimes -1} \otimes_{\mathscr{O}_X} \mathscr{L} \cong \mathscr{O}_X$, it is clear that we have the following proposition:

Proposition 1.1.12. *Let $\mathscr{L}$ be an invertible $\mathscr{O}_X$-module, and $\mathscr{A}$ an $\mathscr{O}_X$-algebra. Then the category* $\mathbf{Mod}(\mathscr{A})$ *of left $\mathscr{A}$-modules and the category* $\mathbf{Mod}(\mathscr{L} \otimes_{\mathscr{O}_X} \mathscr{A} \otimes_{\mathscr{O}_X} \mathscr{L}^{\otimes -1})$ *of left $\mathscr{L} \otimes_{\mathscr{O}_X} \mathscr{A} \otimes_{\mathscr{O}_X} \mathscr{L}^{\otimes -1}$-modules are equivalent to each other by*

$$\mathbf{Mod}(\mathscr{A}) \to \mathbf{Mod}(\mathscr{L} \otimes_{\mathscr{O}_X} \mathscr{A} \otimes_{\mathscr{O}_X} \mathscr{L}^{\otimes -1}), \quad \mathscr{M} \mapsto \mathscr{L} \otimes_{\mathscr{O}_X} \mathscr{M}.$$

We now apply this result to the invertible sheaf Ω_X over X. For thism, we note the following result.

Proposition 1.1.13. *We have a canonical isomorphism*

$$\mathscr{D}_X^{\mathrm{op}} \cong \Omega_X \otimes_{\mathscr{O}_X} \mathscr{D}_X \otimes_{\mathscr{O}_X} \Omega_X^{\otimes -1}.$$

Proof. The right $\mathscr{D}_X$-module structure on Ω_X gives a homomorphism $\mathscr{D}_X^{\mathrm{op}} \to \mathcal{E}nd_{\mathbb{C}}(\Omega_X)$, and its image is in $\mathcal{D}iff(\Omega_X, \Omega_X) = \Omega_X \otimes_{\mathscr{O}_X} \mathscr{D}_X \otimes_{\mathscr{O}_X} \Omega_X^{\otimes -1}$ in view of (1.1.19). We therefore obtain a homomorphism

$$\varphi : \mathscr{D}_X^{\mathrm{op}} \to \Omega_X \otimes_{\mathscr{O}_X} \mathscr{D}_X \otimes_{\mathscr{O}_X} \Omega_X^{\otimes -1}.$$

Note that $\Omega_X^{\otimes -1} \otimes_{\mathscr{O}_X} \mathscr{D}_X^{\mathrm{op}} \otimes_{\mathscr{O}_X} \Omega_X = (\Omega_X \otimes_{\mathscr{O}_X} \mathscr{D}_X \otimes_{\mathscr{O}_X} \Omega_X^{\otimes -1})^{\mathrm{op}}$, so we also obtain a homomorphism

$$\psi := (\Omega_X^{\otimes -1} \otimes \varphi \otimes \Omega_X)^{\mathrm{op}} : \Omega_X \otimes_{\mathscr{O}_X} \mathscr{D}_X \otimes_{\mathscr{O}_X} \to \mathscr{D}_X^{\mathrm{op}}.$$

Using Remark 1.1.3, it is easy to check that ψ and φ are inverses of each other. $\square$

Remark 1.1.4. In an affine open U and a local coordinate system $\{x_i, \partial_i\}$, from the proof of Proposition 1.1.13 we see that the isomorphism of Proposition 1.1.13 is explicitly given by

$$\mathscr{D}_X^{\mathrm{op}} \to \Omega_X \otimes_{\mathscr{O}_X} \mathscr{D}_X \otimes_{\mathscr{O}_X} \Omega_X^{\otimes -1}, \quad P \mapsto dx \otimes P^* \otimes dx^{\otimes -1}$$

where $dx = dx_1 \wedge \cdots dx_n$ and P^* is the formal adjoint of P.

We can identify $\mathbf{Mod}(\mathscr{D}_X^{\mathrm{op}})$ with the category of right $\mathscr{D}_X$-modules. Using tensor products and Hom sheaf, we have the following construction between left and right $\mathscr{D}_X$-modules.

Proposition 1.1.14. *Let $\mathscr{M}, \mathscr{N}$ be left $\mathscr{D}_X$-modules and $\mathscr{M}', \mathscr{N}'$ be right $\mathscr{D}_X$-modules. Then we have the following induced $\mathscr{D}_X$-module structures:*

(i) *$\mathscr{M} \otimes_{\mathscr{O}_X} \mathscr{N}$ has a left $\mathscr{D}_X$-module defined by $v(s \otimes t) = vs \otimes t + s \otimes vt$.*

(ii) *$\mathscr{M}' \otimes_{\mathscr{O}_X} \mathscr{N}$ has a right $\mathscr{D}_X$-module defined by $(s' \otimes t)v = s'v \otimes t - s' \otimes vt$.*

(iii) *$\mathcal{H}om_{\mathscr{O}_X}(\mathscr{M}, \mathscr{N})$ has a left $\mathscr{D}_X$-module defined by $(v\varphi)(s) = v(\varphi(s)) - \varphi(v(s))$.*

(iv) *$\mathcal{H}om_{\mathscr{O}_X}(\mathscr{M}', \mathscr{N}')$ has a left $\mathscr{D}_X$-module defined by $(v\varphi)(s) = -v(\varphi(s)) - \varphi(v(s))$.*

(v) *$\mathcal{H}om_{\mathscr{O}_X}(\mathscr{M}, \mathscr{N}')$ has a right $\mathscr{D}_X$-module defined by $(v\varphi)(s) = v(\varphi(s)) + \varphi(v(s))$.*

The verifications of Proposition 1.1.14 are straightforward, for which we just need to check the conditions of Proposition 1.1.1. A good way to memorize these results is by using the corespondence "left" $\leftrightarrow$ 0, "right" $\leftrightarrow$ 1, and $\mathcal{H}om(\bullet, \bigstar) = -\bullet + \bigstar$.

Proposition 1.1.15. *Given a left $\mathscr{D}_X$-module $\mathscr{M}$ and an $\mathscr{O}_X$-module $\mathscr{N}$, there exists a canonical isomorphism of left $\mathscr{D}_X$-modules*

$$\mathscr{D}_X \otimes_{\mathscr{O}_X} (\mathscr{N} \otimes_{\mathscr{O}_X} \mathscr{M}) \xrightarrow{\sim} (\mathscr{D}_X \otimes_{\mathscr{O}_X} \mathscr{N}) \otimes_{\mathscr{O}_X} \mathscr{M},$$

where the left $\mathscr{D}_X$-module structure on $\mathscr{D}_X \otimes_{\mathscr{O}_X} -$ is induced by left multiplication on $\mathscr{D}_X$.

Proof. An $\mathscr{O}_X$-module homomorphism

$$\mathscr{N} \otimes_{\mathscr{O}_X} \mathscr{M} \to (\mathscr{D}_X \otimes_{\mathscr{O}_X} \mathscr{N}) \otimes_{\mathscr{O}_X} \mathscr{M}, \quad s \otimes u \mapsto (1 \otimes s) \otimes u$$

can be extended to a $\mathscr{D}_X$-module homomorphism

$$\varphi : \mathscr{D}_X \otimes_{\mathscr{O}_X} (\mathscr{N} \otimes_{\mathscr{O}_X} \mathscr{M}) \to (\mathscr{D}_X \otimes_{\mathscr{O}_X} \mathscr{N}) \otimes \mathscr{M}.$$

It is easy to see that the image of $F_p(\mathscr{D}_X \otimes_{\mathscr{O}_X} (\mathscr{N} \otimes_{\mathscr{O}_X} \mathscr{M})) := F_p(\mathscr{D}_X) \otimes_{\mathscr{O}_X} (\mathscr{N} \otimes_{\mathscr{O}_X} \mathscr{M})$ under φ is contained in $F_p((\mathscr{D}_X \otimes_{\mathscr{O}_X} \mathscr{N}) \otimes \mathscr{M}) := (F_p(\mathscr{D}_X) \otimes_{\mathscr{O}_X} \mathscr{N}) \otimes_{\mathscr{O}_X} \mathscr{M}$, so φ induces a homomorphism from

$$\mathrm{gr}_p(\varphi) : \mathrm{gr}_p(\mathscr{D}_X \otimes_{\mathscr{O}_X} (\mathscr{N} \otimes_{\mathscr{O}_X} \mathscr{M})) \to \mathrm{gr}_p((\mathscr{D}_X \otimes_{\mathscr{O}_X} \mathscr{N}) \otimes_{\mathscr{O}_X} \mathscr{M}).$$

It is easy to check that this is the identity, using the fact that $\mathrm{gr}_p(\mathscr{D}_X) \cong \boldsymbol{S}^p_{\mathscr{O}_X}(\Theta_X)$, so by induction on p, it is easy to see that φ is an isomorphism. $\square$

Remark 1.1.5. Take a local coordinate system $\{x_i, \partial_i\}$. Then the homomorphism φ in Proposition 1.1.15 is given by

$$\varphi(\partial^\alpha \otimes (s \otimes u)) = \partial^\alpha((1 \otimes s) \otimes u) = \sum_\beta \binom{\alpha}{\beta} (\partial^\beta \otimes s) \otimes \partial^{\alpha-\beta} u.$$

Therefore, the inverse map ψ of φ is given by

$$\psi((\partial^\alpha \otimes s) \otimes u) = \sum (-1)^{|\beta|} \binom{\alpha}{\beta} \partial^{\alpha-\beta} \otimes (s \otimes \partial^\beta u).$$

Remark 1.1.6. By setting $\mathscr{N} = \mathscr{O}_X$ in Proposition 1.1.15, we see that the two left $\mathscr{D}_X$-module structures on $\mathscr{D}_X \otimes_{\mathscr{O}_X} \mathscr{M}$ are isomorphic. This is a nontrivial fact since the isomorphism of these two modules is not the identity map.

Proposition 1.1.16. *Let $\mathscr{N}$ be a right $\mathscr{D}_X$-module and $\mathscr{M}_1$, $\mathscr{M}_2$ be left $\mathscr{D}_X$-modules. Then*

$$\mathscr{N} \otimes_{\mathscr{D}_X} (\mathscr{M}_1 \otimes_{\mathscr{O}_X} \mathscr{M}_2) \cong (\mathscr{N} \otimes_{\mathscr{O}_X} \mathscr{M}_1) \otimes_{\mathscr{D}_X} \mathscr{M}_2.$$

Proof. Each side is a quotient of $\mathscr{N} \otimes_{\mathscr{O}_X} \mathscr{M}_1 \otimes_{\mathscr{O}_X} \mathscr{M}_2$. The left hand side is the one divided by the submodule generated by

$$sv \otimes (t_1 \otimes t_2) - s \otimes v(t_1 \otimes t_2) = sv \otimes t_1 \otimes t_2 - s \otimes vt_1 \otimes t_2 - s \otimes t_1 \otimes vt_2,$$

and the right hand side is the one divided by the submodule generated by

$$(s \otimes t_1)v \otimes t_2 - (s \otimes t_1) \otimes vt_2 = sv \otimes t_1 \otimes t_2 - s \otimes vt_1 \otimes t_2 - s \otimes t_1 \otimes vt_2,$$

so they are isomorphic. □

Proposition 1.1.17. *Let $\mathscr{M}_1, \mathscr{M}_2$ and $\mathscr{M}_3$ be left $\mathscr{D}_X$-modules. Then we have a canonical homomorphism of left $\mathscr{D}_X$-modules*

$$\mathscr{M}_1 \otimes_{\mathscr{O}_X} \mathcal{H}om_{\mathscr{O}_X}(\mathscr{M}_1, \mathscr{M}_2) \to \mathscr{M}_2,$$

and an isomorphism

$$\mathcal{H}om_{\mathscr{D}_X}(\mathscr{M}_1 \otimes \mathscr{M}_2, \mathscr{M}_3) \cong \mathcal{H}om_{\mathscr{D}_X}(\mathscr{M}_1, \mathcal{H}om_{\mathscr{O}_X}(\mathscr{M}_2, \mathscr{M}_3)). \tag{1.1.20}$$

Proof. It is clear that we have a canonical homomorphism $\Phi : \mathscr{M}_1 \otimes_{\mathscr{O}_X} \mathcal{H}om_{\mathscr{O}_X}(\mathscr{M}_1, \mathscr{M}_2) \to \mathscr{M}_2$ which is $\mathscr{O}_X$-linear. To see that it is $\mathscr{D}_X$-linear, we note that

$$\Phi(v(s_1 \otimes \varphi)) = \Phi(vs_1 \otimes \varphi + s_1 \otimes v\varphi) = \varphi(vs_1) + (v\varphi)(s_1) = v(\varphi(s_1)) = v(\Phi(s_1 \otimes \varphi)).$$

where $v \in \Theta_X$, $s_i \in \mathscr{M}_i$, $\varphi \in \mathcal{H}om_{\mathscr{O}_X}(\mathscr{M}_1, \mathscr{M}_2)$. This proves the first claim, and the second one can be proved similarly, by a detailed computation. □

Proposition 1.1.18. *The correspondence*

$$\Omega_X \otimes_{\mathscr{O}_X} \bullet : \mathbf{Mod}(\mathscr{D}_X) \to \mathbf{Mod}(\mathscr{D}_X^{\mathrm{op}}).$$

gives an equivalence of categories, whose quasi-inverse is given by

$$\Omega_X^{\otimes -1} \otimes_{\mathscr{O}_X} \bullet = \mathcal{H}om_{\mathscr{O}_X}(\Omega_X, \bullet) : \mathbf{Mod}(\mathscr{D}_X)^{\mathrm{op}} \to \mathbf{Mod}(\mathscr{D}_X).$$

*These operations are called **side-changing operations** of $\mathscr{D}_X$-modules.*

Proof. By Proposition 1.1.17, we have a canonical homomorphism $\Omega_X^{\otimes -1} \otimes_{\mathscr{O}_X} \Omega_X \otimes_{\mathscr{O}_X} \mathscr{M} \to \mathscr{M}$, which is an isomorphism since Ω_X is locally free. The other direction follows from (1.1.20) since $\mathcal{H}om_{\mathscr{O}_X}(\mathscr{O}_X, \mathscr{M}) \cong \mathscr{M}$. □

1.1.2.3 Quasi-coherent and coherent D-modules On algebraic varieties, the category of quasi-coherent sheaves (over $\mathscr{O}_X$) is sufficiently wide and suitable for various algebraic operations. Since our sheaf $\mathscr{D}_X$ is locally free over $\mathscr{O}_X$, it is quasi-coherent over $\mathscr{O}_X$. We mainly deal with $\mathscr{D}_X$-modules which are quasi-coherent over $\mathscr{O}_X$. For a smooth algebraic variety X, the category of $\mathscr{D}_X$-modules that are quasi-coherent over $\mathscr{O}_X$ is denoted by $\mathbf{Qcoh}(\mathscr{O}_X)$. This is clearly an abelian category.

It is well known that for affine algebraic varieties X, the global section functor $\Gamma(X, -)$ is exact and $\Gamma(X, \mathscr{M}) = 0$ if and only if $\mathscr{M} = 0$ for $\mathscr{M} \in \mathbf{Qcoh}(\mathscr{M})$. Replacing $\mathscr{O}_X$ by $\mathscr{D}_X$, we come to the following notion.

Definition 1.1.2. A smooth algebraic variety X is called ***D*-affine** if the following conditions are satisfied:

(A1) The global section functor $\Gamma(X, -) : \mathbf{Qcoh}(\mathscr{D}_X) \to \mathbf{Mod}(\Gamma(X, \mathscr{D}_X))$ is exact.

(A2) If $\Gamma(X, \mathscr{M}) = 0$ for $\mathscr{M} \in \mathbf{Qcoh}(\mathscr{D}_X) = 0$, then $\mathscr{M} = 0$.

It is clear that any smooth affine algebraic variety is D-affine. As in the case of quasi-coherent $\mathscr{O}_X$-modules on affine varieties we have the following.

Proposition 1.1.19. *Assume that X is a smooth algebraic variety that is D-affine.*

(a) *Any $\mathscr{M} \in \mathbf{Qcoh}(\mathscr{D}_X)$ is generated over $\mathscr{D}_X$ by its global sections.*

(b) *The functor $\Gamma(X, -) : \mathbf{Qcoh}(\mathscr{D}_X) \to \mathbf{Mod}(\Gamma(X, \mathscr{D}_X))$ is an equivalence of categories.*

Remark 1.1.7. The D-affinity holds also for certain non-affine varieties. For example, we will see that projective spaces are D-affine. We will also show that flag manifolds for semi-simple algebraic groups are D-affine. This fact was one of the key points in the settlement of the Kazhdan-Lusztig conjecture.

Remark 1.1.8. If X is affine, we can replace D_X with $\mathscr{D}_X^{\mathrm{op}}$ in the above argument. In other words, smooth affine varieties are also D^{op}-affine. Note that D-affine varieties are not necessarily D^{op}-affine in general. For example, $\mathbb{P}^1$ is not D^{op}-affine by $\Gamma(\mathbb{P}^1, \Omega_{\mathbb{P}^1}) = 0$, but it is D-affine as we shall see.

Recall that an $\mathscr{A}$-module $\mathscr{M}$ over a ringed space $(X, \mathscr{A})$ is called *coherent* if it is of finite type over $\mathscr{A}$ and any finite generating relation of $\mathscr{M}$ is finitely presented. We now consider coherent $\mathscr{D}_X$-modules over X.

Proposition 1.1.20. *Let X be a smooth algebraic variety. Then $\mathscr{D}_X$ is a coherent sheaf of rings. Moreover, a $\mathscr{D}_X$-module is coherent if and only if it is quasi-coherent over $\mathscr{O}_X$ and of finite type over $\mathscr{D}_X$.*

Proof. The first statement follows from the second one, so we only need to prove the second one. If $\mathscr{M}$ is a coherent $\mathscr{D}_X$-module, then $\mathscr{M}$ is of finite type over $\mathscr{D}_X$. Moreover, $\mathscr{M}$ is quasi-coherent over $\mathscr{O}_X$ since it is locally finitely presented as a $\mathscr{D}_X$-module and $\mathscr{D}_X$ is quasi-coherent over $\mathscr{O}_X$. Conversely, assume that $\mathscr{M}$ is of finite type over $\mathscr{D}_X$ and quasi-coherent over $\mathscr{O}_X$. To see that $\mathscr{M}$ is coherent over $\mathscr{D}_X$, it suffices to show that for any affine open subset U of X, the kernel of any homomorphism $\varphi : \mathscr{D}_U^m \to \mathscr{M}|_U$ of $\mathscr{D}_U$ is finitely presented over $\mathscr{D}_U$. Since $\mathscr{D}_U(U)$ is a left Noetherian ring, the kernel of $\mathscr{D}_U(U)^m \to \mathscr{M}(U)$ is a finitely generated $\mathscr{D}_U(U)$-module, and this proves the assertion in view of Proposition 1.1.19. $\square$

Theorem 1.1.21. *A $\mathscr{D}_X$-module is coherent over $\mathscr{O}_X$ if and only if it is an integrable connection.*

Proposition 1.1.22. *Let X be a smooth algebraic variety that is D-affine. Then the global section functor $\Gamma(X, -)$ induces the equivalence*

$$\mathbf{Coh}(\mathscr{D}_X) \xrightarrow{\sim} \mathbf{Mod}_f(\Gamma(X, \mathscr{D}_X))$$

where for a ring A, we denote by $\mathbf{Mod}_f(A)$ the category of finitely generated A-modules.

Proposition 1.1.23. *Any* $\mathbf{Qcoh}(\mathscr{D}_X)$ *is embedded into an injective object* $\mathscr{I}$ *of* $\mathbf{Qcoh}(\mathscr{D}_X)$ *which is flasque.*

Corollary 1.1.24. *If* X *is D-affine, then for any* $\mathscr{M} \in \mathbf{Qcoh}(\mathscr{D}_X)$*, we have* $H^i(X, \mathscr{M}) = 0$ *for* $i > 0$.

Proposition 1.1.25. *Let* X *be a smooth quasi-projective algebraic variety.*

(a) *Any* $\mathscr{M} \in \mathbf{Qcoh}(\mathscr{D}_X)$ *is a quotient of a locally free* $\mathscr{D}_X$*-module.*

(b) *Any* $\mathscr{M} \in \mathbf{Coh}(\mathscr{D}_X)$ *is a quotient of a locally free* $\mathscr{D}_X$*-module of finite rank.*

Corollary 1.1.26. *Let* X *be a smooth quasi-projective algebraic variety. Then any* $\mathscr{M} \in \mathbf{Qcoh}(\mathscr{D}_X)$ *admits a resolution by locally free* $\mathscr{D}_X$*-modules and a finite resolution by locally projective* $\mathscr{D}_X$*-modules. If* $\mathscr{M}$ *is a coherent* $\mathscr{D}_X$*-module, then we can assume that each term of these resolutions has finite rank.*

Remark 1.1.9. In view of Proposition 1.1.25 and Corollary 1.1.26, we may assume that our variety X is quasi-projective throughout this section. One shall see that this assumption is harmless for most of our applications.

1.1.3 Characteristic varieties

Recall that we have obtained from $\mathscr{D}_X$, first a commutative object $\boldsymbol{S}_{\mathscr{O}_X}(\Theta_X)$ by taking graded ring, next the cotangent bundle T^*X as its geometric object, and finally the symplectic structure on T^*X as a reflection of the noncommutativity of $\mathscr{D}_X$.

Now we consider $\mathscr{D}_X$-modules and derive a commutative object from each of them. Let $\mathscr{M}$ be a $\mathscr{D}_X$-module which is quasi-coherent over $\mathscr{O}_X$. We consider an *increasing* filtration of $\mathscr{M}$ by quasi-coherent $\mathscr{O}_X$-submdules $F_i(\mathscr{M})$ satisfying the following conditions:

(F1) $\mathscr{M} = \bigcup_i F_i(\mathscr{M})$.

(F2) $F_i(\mathscr{D}_X)F_j(\mathscr{M}) \subseteq F_{i+j}(\mathscr{M})$.

(F3) $F_i(\mathscr{M}) = 0$ for $i \ll 0$.

In this case, we say that M is a **filtered** $\mathscr{D}_X$**-module**. The associated graded module

$$\mathrm{gr}(\mathscr{M}) = \bigoplus_{i\in\mathbb{Z}} F_i(\mathscr{M})/F_{i-1}(\mathscr{M})$$

is then a graded module over $\mathrm{gr}(\mathscr{D}_X) = \pi_*(\mathscr{O}_{T^*X})$, which is clearly quasi-coherent over $\mathscr{O}_X$.

Proposition 1.1.27. *Let* M *be a filtered* $\mathscr{D}_X$*-module. Then the following conditions are equivalent:*

(i) $\mathrm{gr}(\mathscr{M})$ *is coherent over* $\pi_*(\mathscr{O}_{T^*X})$.

(ii) *There exist locally a surjective* $\mathscr{D}_X$*-linear homorphism* $\Phi : \mathscr{D}_X^{\oplus r} \to \mathscr{M}$ *and integers* n_j *(*$j = 1, \dots, r$*) such that for each* $i \in \mathbb{Z}$,

$$\Phi\Big(\bigoplus_{j=1}^{r} F_{i-n_j}(\mathscr{D}_X)\Big) = F_i(\mathscr{M}).$$

(iii) $F_i(\mathscr{M})$ *is coherent over* $\mathscr{O}_X$ *for each* i*, and there exists an integer* i_0 *such that locally we have*

$$F_j(\mathscr{D}_X)F_i(\mathscr{M}) = F_{i+j}(\mathscr{M}) \quad \textit{for} \quad i \geq i_0, j \geq 0.$$

Proof. Since the question is local, we can reduce to affine case, from which the equivalence of (i) and (ii) is clear. Now it is easily checked that (ii) holds if and only if $F_i(\mathscr{M})$ is coherent over $\mathscr{O}_X$ and one can find i_0 as in (iii) locally on X. □

A filtration $\{F_i(\mathscr{M})\}$ is called **good** if it satisfies the conditions of Proposition 1.1.27. By Proposition 1.1.27, a good filtration induces a coherent module $\mathrm{gr}(\mathscr{M})$ over $\pi_*(\mathscr{O}_{T^*X})$.

Theorem 1.1.28. *Let X be a smooth algebraic variety.*

(a) *Any coherent $\mathscr{D}_X$-module admits a (locally defined) good filtration. Conversely, a quasi-coherent $\mathscr{D}_X$-module endowed with a good filtration is coherent.*

(b) *Let F, F' be two filtrations of a $\mathscr{D}_X$-module $\mathscr{M}$ and assume that F is good. Then there exists an integer i_0 such that locally we have*

$$F_i(\mathscr{M}) \subseteq F'_{i+i_0}(\mathscr{M}) \quad \text{for} \quad i \in \mathbb{Z}.$$

If, moreover, F' is also a good filtration, there exists i_0 such that locally

$$F'_{i-i_0}(\mathscr{M}) \subseteq F_i(\mathscr{M}) \subseteq F'_{i+i_0}(\mathscr{M}) \quad \text{for} \quad i \in \mathbb{Z}.$$

Proof. If $\mathscr{M}$ is a coherent $\mathscr{D}_X$-module, then $\mathscr{M}$ is locally generated by a finite number of sections $u_1, \ldots, u_N$, and we can define $F_i(\mathscr{M})$ by

$$F_i(\mathscr{M}) = \sum_{\nu=1}^{N} F_i(\mathscr{D}_X) u_\nu.$$

It is easy to see that this is a good filtration of $\mathscr{M}$. Conversely, if $\mathscr{M}$ admits a good filtration, then it is locally generated by finitely many sections, hence coherent. The second assertion is local, and hence follows from the corresponding result for graded modules. □

Let $\mathscr{M}$ be a coherent $\mathscr{D}_X$-module with a good filtration. Let $\pi : T^*X \to X$ be the cotangent bundle of X. Since we have $\mathrm{gr}(\mathscr{D}_X) \cong \pi_*(\mathscr{O}_{T^*X})$, the graded module $\mathrm{gr}(\mathscr{M})$ is a coherent module over $\pi_*(\mathscr{O}_{T^*X})$ by Proposition 1.1.27. The support of the coherent $\mathscr{O}_{T^*X}$-module

$$\widetilde{\mathrm{gr}(\mathscr{M})} := \mathscr{O}_{T^*X} \otimes_{\pi^{-1}(\mathscr{O}_{T^*X})} \pi^*(\mathrm{gr}(\mathscr{M}))$$

is called the **characteristic variety** of $\mathscr{M}$ and denoted by $\mathrm{Ch}(M)$ (it is sometimes called the **singular support** of M). Since $\mathrm{gr}(\mathscr{M})$ is a graded module over the graded ring $\mathrm{gr}(\mathscr{D}_X)$, we see that $\mathrm{Ch}(M)$ is a closed conic (i.e., stable by the scalar multiplication of complex numbers on the fibers) algebraic subset in T^*X.

Theorem 1.1.29. *The characteristic variety of a coherent $\mathscr{D}_X$-module $\mathscr{M}$ does not depend on the choice of a good filtration.*

Proof. We say two good filtrations F and G are "adjacent" if they satisfy the condition

$$F_i(\mathscr{M}) \subseteq G_i(\mathscr{M}) \subseteq F_{i+1}(\mathscr{M}) \quad \text{for} \quad i \in \mathbb{Z}.$$

We first show the assertion in this case. Consider the natural homomorphism

$$\varphi_i : F_i(\mathscr{M})/F_{i-1}(\mathscr{M}) \to G_i(\mathscr{M})/G_{i-1}(\mathscr{M}).$$

Then we have $\ker \varphi_i \cong G_{i-1}(\mathscr{M})/F_{i-1}(\mathscr{M}) \cong \operatorname{coker} \varphi_{i-1}$, so the homorphism $\varphi : \mathrm{gr}^F(\mathscr{M}) \to \mathrm{gr}^G(\mathscr{M})$ entails an isomorphism $\ker \varphi \cong \operatorname{coker} \varphi$. Consider the exact sequence

$$0 \longrightarrow \ker \varphi \longrightarrow \mathrm{gr}^F(\mathscr{M}) \longrightarrow \mathrm{gr}^G(\mathscr{M}) \longrightarrow \mathrm{gr}^G(\mathscr{M}) \longrightarrow \operatorname{coker} \varphi \longrightarrow 0$$

of coherent $\mathrm{gr}^F(\mathscr{D}_X)$-modules. From this we obtain

$$\mathrm{supp}(\mathrm{gr}^F(\mathscr{M})) = \mathrm{supp}(\ker \varphi) \cup \mathrm{supp}(\mathrm{im}\, \varphi),$$

$$\operatorname{supp}(\operatorname{gr}^G(\mathscr{M})) = \operatorname{supp}(\operatorname{im}\varphi) \cup \operatorname{supp}(\operatorname{coker}\varphi).$$

Hence $\ker\varphi \cong \operatorname{coker}\varphi$ implies $\operatorname{supp}(\operatorname{gr}^F(\mathscr{M})) = \operatorname{supp}(\operatorname{gr}^G(\mathscr{M}))$, so the assertion is proved for adjacent good filtrations.

Let us consider the general case. Namely, assume that F and G are arbitrary good filtrations of M. For $k \in \mathbb{Z}$ set

$$F_i^{(k)}(\mathscr{M}) = F_i(\mathscr{M}) + G_{i+k}(\mathscr{M}), \quad i \in \mathbb{Z}.$$

By Theorem 1.1.28, $F^{(k)}$ is a good filtration of $\mathscr{M}$, and $F^{(k)}$, $F^{(k+1)}$ are adjacent for each $k \in \mathbb{Z}$. Since $F^{(k)} = F$ for $k \ll 0$ and $F^{(k)} = G[k]$ for $k \gg 0$, we conclude the assertion from the adjacent case. □

Let U be an affine open subset of X. Then T^*U is an affine open subset of T^*X, and $\operatorname{Ch}(\mathscr{M}) \cap T^*U$ coincides with the support of the coherent $\mathscr{O}_{T^*U}$-module associated to the finitely generated $\operatorname{gr}(\Gamma(U, \mathscr{D}_U))$-module $\operatorname{gr}(\Gamma(U, \mathscr{M}))$. We then have

$$\operatorname{Ch}(\mathscr{M}) \cap T^*U = \{p \in T^*U : f(p) = 0 \text{ for } f \in \mathfrak{J}_{\Gamma(U,\mathscr{M})}\},$$

where $\mathfrak{J}_{\Gamma(U,\mathscr{M})}$ is the characteristic ideal of $\Gamma(U, \mathscr{M})$, defined by

$$\mathfrak{J}_{\Gamma(U,\mathscr{M})} = \sqrt{\operatorname{Ann}(\operatorname{gr}(\Gamma(U, \mathscr{M})))} = \bigcap_{\mathfrak{p} \in \operatorname{SS}_0(\Gamma(U,\mathscr{M}))} \mathfrak{p}.$$

The decomposition of $\operatorname{Ch}(\mathscr{M}) \cap T^*U$ into irreducible components is given by

$$\operatorname{Ch}(\mathscr{M}) \cap T^*U = \bigcup_{\mathfrak{p} \in \operatorname{SS}_0(\Gamma(U,\mathscr{M}))} \{p \in T^*U : f(p) = 0 \text{ for } f \in \mathfrak{p}\}.$$

Proposition 1.1.30. *Let $\mathscr{M}$ be a coherent $\mathscr{O}_X$-module and $\pi : T^*X \to X$ be the cotangent bundle.*

(a) $\operatorname{supp}(\mathscr{M}) = \pi(\operatorname{Ch}(\mathscr{M}))$.

(b) $\operatorname{Ch}(\mathscr{M})$ *is a closed conic and algebraic subset.*

(c) *If $0 \to \mathscr{M}' \to \mathscr{M} \to \mathscr{M}'' \to 0$ is an exact sequence of coherent $\mathscr{D}_X$-modules, then*

$$\operatorname{Ch}(\mathscr{M}) = \operatorname{Ch}(\mathscr{M}') \cup \operatorname{Ch}(\mathscr{M}'').$$

Proof. Since $\mathscr{O}_{T^*X}$ is faithfully flat over $\pi^{-1}(\operatorname{gr}(\mathscr{D}_X))$, we see that $\pi(\operatorname{Ch}(\mathscr{M})) = \operatorname{supp}(\operatorname{gr}(\mathscr{M}))$, so the first assertion follows from the easily seen fact $\operatorname{supp}(\mathscr{M}) = \operatorname{supp}(\operatorname{gr}(\mathscr{M}))$. Now we have remarked (b), and to prove (c), we may assume that $\mathscr{M}$ has a good filtration F. If we endow $\mathscr{M}'$ and $\mathscr{M}''$ the induced filtration, then they are good in view of Theorem 1.1.28, and the sequence

$$0 \longrightarrow \operatorname{gr}(\mathscr{M}') \longrightarrow \operatorname{gr}(\mathscr{M}) \longrightarrow \operatorname{gr}(\mathscr{M}'') \longrightarrow 0$$

The assertion then follows from the corresponding property of supp. □

Let X be a smooth algebraic variety and assume that we are given a coherent $\mathscr{O}_X$-module $\mathscr{G}$. Then we can define an algebraic cycle $\operatorname{Cyc}(\mathscr{G})$ associated to $\mathscr{G}$ as follows. For each irreducible component V of $\operatorname{supp}(\mathscr{G})$, with generic point η, the local ring $\mathscr{O}_{X,\eta}$ is an Artinian ring, and we define the **multiplicity of $\mathscr{G}$ along V** to be

$$\operatorname{mult}_X(\mathscr{G}) := \ell_{\mathscr{O}_{X,\eta}}(\mathscr{G}_\eta).$$

For irreducible subvariety V with $V \not\subseteq \operatorname{supp}(\mathscr{G})$, we set $\operatorname{mult}_V(\mathscr{G}) = 0$; we then define the formal sum

$$\operatorname{Cyc}(\mathscr{G}) = \sum \operatorname{mult}_V(\mathscr{G}) \cdot V$$

which is called the associated cycle of $\mathscr{G}$.

Let $\mathscr{M}$ be a coherent $\mathscr{D}_X$-module. By choosing a good filtration of $\mathscr{M}$, we can consider the coherent $\mathscr{O}_{T^*X}$-module $\widetilde{\mathrm{gr}(\mathscr{M})}$. From the proof of Theorem 1.1.29, it is easy to see that the cycle $\mathrm{Cyc}(\mathrm{gr}(\mathscr{M}))$ is independent of the choice of the filtration of $\mathscr{M}$.

Definition 1.1.3. For a coherent $\mathscr{D}_X$-module $\mathscr{M}$ we define the **characteristic cycle** of $\mathscr{M}$ by

$$\mathrm{Cyc}(\mathscr{M}) := \mathrm{Cyc}(\widetilde{\mathrm{gr}(\mathscr{M})}) = \sum_V \mathrm{mult}_V(\widetilde{\mathrm{gr}(\mathscr{M})}) \cdot V.$$

For $d \in \mathbb{N}$, we denote its degree d part by

$$\mathrm{Cyc}_d(\mathscr{M}) := \sum_{\dim(V)=d} \mathrm{mult}_V(\widetilde{\mathrm{gr}(\mathscr{M})}) \cdot V.$$

Proposition 1.1.31. *If $0 \to \mathscr{M}' \to \mathscr{M} \to \mathscr{M}'' \to 0$ is an exact sequence of coherent $\mathscr{D}_X$-modules, then for any irreducible component V of $\mathrm{Ch}(\mathscr{M})$, we have*

$$\mathrm{mult}_V(\widetilde{\mathrm{gr}(\mathscr{M})}) = \mathrm{mult}_V(\widetilde{\mathrm{gr}(\mathscr{M}')}) + \mathrm{mult}_V(\widetilde{\mathrm{gr}(\mathscr{M}'')}).$$

In particular, for $d = \dim(\mathrm{Ch}(\mathscr{M}))$, we have

$$\mathrm{Cyc}_d(\mathscr{M}) = \mathrm{Cyc}_d(\mathscr{M}') + \mathrm{Cyc}_d(\mathscr{M}'').$$

Proof. The first assertion follows from the above remarks, and the second one follows from this. □

Proposition 1.1.32. *Let $\mathscr{M}$ be a coherent $\mathscr{D}_X$-module, and $f : \mathscr{M} \to \mathscr{M}$ be a monomorphism of $\mathscr{D}_X$-modules. Then $\mathrm{Ch}(\mathscr{M}/f(\mathscr{M}))$ is a nowhere dense subset of $\mathrm{Ch}(\mathscr{M})$.*

Proof. We choose a good filtration on $\mathscr{N} := \mathscr{M}/f(\mathscr{M})$. By the exact sequence $0 \to \mathscr{M} \to \mathscr{M} \to \mathscr{N} \to 0$, we have

$$\mathrm{mult}_V(\mathscr{N}) = \mathrm{mult}_V(\mathscr{M}) - \mathrm{mult}_V(\mathscr{M}) = 0$$

for any irreducible component V of $\mathrm{Ch}(\mathscr{M})$, so the multiplicity of $\widetilde{\mathrm{gr}(\mathscr{N})}$ on V is 0. It follows that the support of $\widetilde{\mathrm{gr}(\mathscr{N})}$ cannot contain V, so $\mathrm{Ch}(\mathscr{M}/f(\mathscr{M}))$ is nowhere dense. □

Example 1.1.2. Let $\mathscr{M}$ be an integrable connection of rank $r > 0$ on X. We can then define a good filtration on $\mathscr{M}$ by setting

$$F_i(\mathscr{M}) = \begin{cases} 0 & i < 0, \\ \mathscr{M} & i \geq 0, \end{cases}$$

and we have $\mathrm{gr}(\mathscr{M}) \cong \mathscr{M} \cong \mathscr{O}_X^r$ locally. Moreover, since Θ_X annihilates $\mathrm{gr}(\mathscr{M})$ by degree consideration, we get $\mathrm{Ch}(\mathscr{M}) = T_X^*X = s(X) \cong X$ (the zero section of T^*X), and $\mathrm{Cyc}(\mathscr{M}) = r \cdot T_X^*X$.

Proposition 1.1.33. *For a non-zero coherent $\mathscr{D}_X$-module $\mathscr{M}$ the following three conditions are equivalent:*

(i) *$\mathscr{M}$ is an integrable connection.*

(ii) *$\mathscr{M}$ is coherent over $\mathscr{O}_X$.*

(iii) $\mathrm{Ch}(\mathscr{M}) = T_X^*X \cong X$ *(the zero section of T^*X).*

Proof. We have seen that (i) is equivalent to (ii) ([**?**] Theorem 1.4.10), so it remains to prove that (iii)$\Rightarrow$(ii). Since the question is local, we may assume that X is affine with local coordinate system $\{x_i, \partial_i\}$, so that we have $T^*X = X \times \mathbb{C}^n$. Assume that $\mathrm{Ch}(\mathscr{M}) = T^*_X X$, then for a good filtration of $\mathscr{M}$ we have

$$\sqrt{\mathrm{Ann}_{\mathscr{O}_X[\xi_1,\dots,\xi_n]}(\mathrm{gr}(\mathscr{M}))} = \sum_{i=1}^{n} \mathscr{O}_X[\xi]\xi_i.$$

Here we denote by ξ_i the principal symbol of ∂_i, and we identify $\pi_*(\mathscr{O}_{T^*X})$ with $\mathscr{O}_X[\xi_1, \dots, \xi_n]$. Note that if we set $\mathfrak{I} = \sum_{i=1}^{n} \mathscr{O}_X[\xi]\xi_i$, then since $\mathscr{O}_X[\xi_1, \dots, \xi_n]$ is Noetherian, there exists an integer $m > 0$ such that we have

$$\mathfrak{I}^m \subseteq \mathrm{Ann}_{\mathscr{O}_X[\xi_1,\dots,\xi_n]}(\mathrm{gr}(\mathscr{M})) \subseteq \mathfrak{I}$$

for $m \gg 0$. Since the monomials ξ^α of degree m generate $\mathfrak{I}^m$, we conclude that

$$\partial^\alpha F_j(\mathscr{M}) \subseteq F_{j+m-1}(\mathscr{M}), \quad |\alpha| = m, \quad j \in \mathbb{Z}.$$

On the other hand, since the filtration is good, we have $F_i(\mathscr{D}_X)F_j(\mathscr{M}) = F_{i+j}(\mathscr{M})$ for $j \gg 0$, so it follows that

$$F_{m+j}(\mathscr{M}) = F_m(\mathscr{D}_X)F_j(\mathscr{M}) = \sum_{|\alpha| \le m} \mathscr{O}_X \partial^\alpha F_j(\mathscr{M}) \subseteq F_{j+m-1}(\mathscr{M}).$$

This implies $F_{j+1}(\mathscr{M}) = F_j(\mathscr{M}) = \mathscr{M}$ for $j \gg 0$, which means $\mathscr{M}$ is coherent over $\mathscr{O}_X$ (since each $F_i(\mathscr{M})$ is coherent over $\mathscr{O}_X$). □

Example 1.1.3. For a coherent $\mathscr{D}_X$-module $\mathscr{M} = D_X u \cong \mathscr{D}_X/\mathscr{I}$, where $\mathscr{I} = \mathrm{Ann}(u)$, consider the good filtration $F_i(\mathscr{M}) = F_i(\mathscr{D}_X)u$ on $\mathscr{M}$. If we define a filtration on $\mathscr{I}$ by $F_i(\mathscr{I}) = F_i(\mathscr{D}_X) \cap \mathscr{I}$, then we have $\mathrm{gr}(\mathscr{M}) \cong \mathrm{gr}(\mathscr{D}_X)/\mathrm{gr}(\mathscr{I})$. In this case, the graded ideal $\mathrm{gr}(\mathscr{I})$ is generated by the principal symbols $\sigma(P)$ of $P \in \mathfrak{I}$. Therefore, for an arbitrary set $\{\sigma(P_i)\}$ of generators of $\mathrm{gr}(\mathscr{I})$, we have $\mathfrak{I} = \sum_i \mathscr{D}_X P_i$ and

$$\mathrm{Ch}(\mathscr{M}) = \{(x, \xi) \in T^*X : \sigma(P_i)(x, \xi) = 0 \text{ for each } i\}.$$

However, for a set $\{Q_i\}$ of generators of $\mathfrak{I}$, the similar equality does not always hold. In general, we have only the inclusion

$$\mathrm{Ch}(\mathscr{M}) \subseteq \{(x, \xi) \in T^*X : \sigma(Q_i)(x, \xi) = 0 \text{ for each } i\}.$$

This failure is due to the fact that in general we do note have

$$\mathrm{gr}(\mathscr{I}) = \sum \mathrm{gr}(\mathscr{D}_X)\sigma_i(P_i).$$

One of the most fundamental results in the theory of D-modules is the following result about the characteristic varieties of coherent D-modules.

Theorem 1.1.34. *The characteristic variety of any coherent $\mathscr{D}_X$-module is involutive (or coisotropic) with respect to the symplectic structure of the cotangent bundle T^*X.*

Let us admit this theorem for the time being and proceed with our arguments. Since the dimension of any involutive closed analytic subset is greater than or equal to $\dim(X)$, we obtain the following proposition.

Proposition 1.1.35. *For every coherent $\mathscr{D}_X$-module $\mathscr{M}$, the dimension of $\mathrm{Ch}(\mathscr{M})$ at every point is greater than or equal to $\dim(X)$.*

A coherent $\mathscr{D}_X$-module $\mathscr{M}$ is called a **holonomic** $\mathscr{D}_X$**-module** (or a holonomic system, or a **maximally overdetermined system**) if it satisfies $\dim(\mathrm{Ch}(\mathscr{M})) = \dim(X)$. By Theorem 1.1.34, characteristic varieties of holonomic $\mathscr{D}_X$-modules are $\mathbb{C}^\times$-invariant Lagrangian subset of T^*X. Holonomic $\mathscr{D}_X$-modules are the coherent $\mathscr{D}_X$-modules whose characteristic variety has minimal possible dimension. Assume that the dimension of the characteristic variety $\mathrm{Ch}(M)$ is "small", then this means that the ideal defining the corresponding system of differential equations is "large", and hence the space of the solutions should be "small". In fact, we will see later that the holonomicity is related to the finite dimensionality of the solution space.

1.1.4 Codimension filtration

Theorem 1.1.36. *Let $\mathscr{M}$ be a coherent $\mathscr{D}_X$-module.*

(a) $\mathcal{E}xt^i_{\mathscr{D}_X}(\mathscr{M}, \mathscr{D}_X) = 0$ *for* $i < \mathrm{codim}(\mathrm{Ch}(\mathscr{M}))$.

(b) $\mathrm{codim}(\mathrm{Ch}(\mathcal{E}xt^i_{\mathscr{D}_X}(\mathscr{M}, \mathscr{D}_X))) \geq i$.

(c) $\mathrm{Ch}(\mathcal{E}xt^i_{\mathscr{D}_X}(\mathscr{M}, \mathscr{D}_X)) \subseteq \mathrm{Ch}(\mathscr{M})$.

Proof. □

1.1.5 Global dimension of $\mathscr{D}_X$

Recall that we have defined a functor

$$\mathcal{H}om_{\mathscr{O}_X} : \mathbf{Mod}(\mathscr{D}_X)^{\mathrm{op}} \times \mathbf{Mod}(\mathscr{D}_X) \to \mathbf{Mod}(\mathscr{D}_X),$$

whose derived functor is denoted by

$$R\mathcal{H}om_{\mathscr{O}_X} : D^-(\mathscr{D}_X)^{\mathrm{op}} \times D^+(\mathscr{D}_X) \to D^+(\mathscr{D}_X).$$

Since injective $\mathscr{D}_X$-modules are injective $\mathscr{O}_X$-modules, the diagram

$$\begin{tikzcd} D^-(\mathscr{D}_X)^{\mathrm{op}} \times D^+(\mathscr{D}_X) \arrow[r, "R\mathcal{H}om_{\mathscr{O}_X}"] \arrow[d] & D^+(\mathscr{D}_X) \arrow[d] \\ D^-(\mathscr{O}_X)^{\mathrm{op}} \times D^+(\mathscr{O}_X) \arrow[r, "R\mathcal{H}om_{\mathscr{O}_X}"] & D^+(\mathscr{O}_X) \end{tikzcd}$$

is commutative.

Lemma 1.1.37. *There is a canonical isomorphism*

$$R\mathcal{H}om_{\mathscr{D}_X}(\mathscr{O}_X, R\mathcal{H}om_{\mathscr{O}_X}(\mathscr{M}, \mathscr{N})) \cong R\mathcal{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{N}).$$

Let $\mathscr{M}, \mathscr{N} \in \mathbf{Mod}(\mathscr{D}_X)$ and $d_X = \dim(X)$. Since $H^i R\mathcal{H}om_{\mathscr{O}_X}(\mathscr{M}, \mathscr{N}) = 0$ for $i > d_X + 1$ by Golovin's theorem ([**?**]), $R\mathcal{H}om_{\mathscr{O}_X}(\mathscr{M}, \mathscr{N})$ can be represented by a complex $\mathscr{K}^\bullet$ with $\mathscr{K}^i = 0$ for $i > d_X + 1$ and $i < 0$. We therefore obtain an isomorphism

$$R\mathcal{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{N}) \cong R\mathcal{H}om_{\mathscr{D}_X}(\mathscr{O}_X, \mathscr{K}^\bullet) \cong \mathcal{H}om_{\mathscr{D}_X}(\mathscr{D}_X \otimes_{\mathscr{O}_X} \textstyle\bigwedge^\bullet \Theta_X, \mathscr{K}^\bullet).$$

This leads us to the following theorem about the global dimension of $\mathscr{D}_X$.

Theorem 1.1.38. *The global dimension of $\mathscr{D}_X$ is bounded by* $2d_X + 1$.

Proof. Since the complex $\mathscr{D}_X \otimes_{\mathscr{O}_X} \bigwedge^\bullet \Theta_X$ is concentrated at degrees $[0, d_X]$ and $\mathscr{K}^\bullet$ at degrees $[0, 1 + d_X]$, we conclude that $H^i R\mathcal{H}om_{\mathscr{D}_X}(\mathscr{O}_X, \mathscr{K}^\bullet) = 0$ for $i > 2d_X + 1$, which proves our assertion. □

Remark 1.1.10. Let $X = \mathbb{C}^n$ with $n > 1$ and I be an infinite set. Since $R\mathcal{H}om_{\mathscr{D}_X}(\mathscr{O}_X, \mathscr{D}_X^{\oplus I}) \cong \mathscr{O}_X^{\oplus I}[-n]$, we have

$$\mathrm{Ext}^{2n+1}_{\mathscr{D}_X}(\mathscr{O}_{X,0}, \mathscr{D}_X^{\oplus I}) \cong H^{n+1}_{\{0\}}(X, \mathscr{O}_X^{\oplus I}).$$

We prove that this does not vanish by induction on n. Suppose that $n > 1$, and let $p : X \to Y = \mathbb{C}^{n-1}$ be the projection map. Then we have an exact sequence

$$0 \longrightarrow p^{-1}(\mathscr{O}_Y^{\oplus I}) \longrightarrow \mathscr{O}_X^{\oplus I} \xrightarrow{\partial_n} \mathscr{O}_X^{\oplus I} \longrightarrow 0$$

and accordingly an exact sequence

$$H^{n+1}_{\{0\}}(X, \mathscr{O}_X^{\oplus I}) \longrightarrow H^{n+2}_{\{0\}}(X, p^{-1}(\mathscr{O}_Y^{\oplus I})) \longrightarrow H^{n+2}_{\{0\}}(X, \mathscr{O}_X^{\oplus I}) = 0$$

Note that $H^i_{\{0\}}(X, p^{-1}(\mathscr{F})) = H^{i-2}_{\{0\}}(Y, \mathscr{F})$ for any sheaf $\mathscr{F}$ over Y, so we conclude that

$$H^{n+2}_{\{0\}}(X, p^{-1}(\mathscr{O}_Y^{\oplus I})) = H^n_{\{0\}}(Y, \mathscr{O}_Y^{\oplus I}) \neq 0$$

and therefore $H^{n+1}_{\{0\}}(X, \mathscr{O}_X^{\oplus I}) \neq 0$.

We shall next consider the global dimension of $\mathscr{D}_{X,x}$ for any point $x \in X$. For this, it suffices to consider the following lemma.

Lemma 1.1.39. *Suppose that the projective dimension of a finitely generated module M over a Noetherian ring A is finite, and that $\mathrm{Ext}^i_A(M, A) = 0$ for $i > r$. Then the projective dimension of M is less than or equal to r.*

Proof. □

Theorem 1.1.40. *Let X be an n-dimensional smooth algebra variety. Then for any affine open subset U and $x \in X$, the global dimension of $\Gamma(U, \mathscr{D}_X)$ and $\mathscr{D}_{X,x}$ are equal n.*

Proof. By Lemma 1.1.39, it suffices to show that $\mathcal{E}xt^i_{\mathscr{D}_X}(\mathscr{M}, \mathscr{D}_X) = 0$ for $i > n$ and any coherent $\mathscr{D}_X$-module. By Theorem 1.1.36 (b), we have

$$\mathrm{codim}(\mathrm{Ch}(\mathcal{E}xt^i_{\mathscr{D}_X}(\mathscr{M}, \mathscr{D}_X))) \geq i > n$$

so $\mathcal{E}xt^i_{\mathscr{D}_X}(\mathscr{M}, \mathscr{D}_X) = 0$ by Proposition 1.1.35. We therefore conclude that $\mathrm{gl.dim}(\mathscr{D}_{X,x}) \leq n$ and $\mathrm{gl.dim}(\Gamma(U, \mathscr{D}_X))$, and the equality follows from

$$\mathrm{Ext}^n_{\Gamma(U,\mathscr{D}_X)}(\Gamma(U, \mathscr{O}_X), \Gamma(U, \mathscr{D}_X)) = \Gamma(U, \Omega_X) \neq 0$$

(cf. Proposition 1.1.44). □

1.1.6 Duality functor

Let $D(\mathscr{D}_X)$ be the derived category of $\mathbf{Mod}(\mathscr{D}_X)$, and $D^*(\mathscr{D}_X)$ ($* \in \{+, -, b\}$) denote the full subcategory of $D(\mathscr{D}_X)$ consisting of complexes bounded above, below, and bounded, respectively. Let $D^b_{\mathrm{coh}}(\mathscr{D}_X)$ denote the full subcategory of $\mathscr{M} \in D^b(\mathscr{D}_X)$ such that $H^i(\mathscr{M})$ are coherent $\mathscr{D}_X$-modules. Then since $\mathbf{Coh}(\mathscr{D}_X)$ is a Serre subcategory of $\mathbf{Mod}(\mathscr{D}_X)$, we see that $D^b_{\mathrm{coh}}(\mathscr{D}_X)$ is a triangulated subcategory of $D^b(\mathscr{D}_X)$.

We now try to find heuristically a candidate for the "dual" of a left $\mathscr{D}$-module. Let $\mathscr{M}$ be a left $\mathscr{D}_X$-module. Since $\mathscr{D}_X$ is a $(\mathscr{D}_X, \mathscr{D}_X)$-bimodule, we see $\mathcal{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{D}_X)$ is a right $\mathscr{D}_X$-module by right multiplication of $\mathscr{D}_X$ on $\mathscr{D}_X$. By the side-changing functor $\otimes_{\mathscr{O}_X} \Omega_X^{\otimes -1}$, we then obtain a left $\mathscr{D}_X$-module $\mathcal{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{D}) \otimes_{\mathscr{O}_X} \Omega_X^{\otimes -1}$. Since the functor $\mathcal{H}om_{\mathscr{D}_X}(-, \mathscr{D}_X)$ is not exact, a more natural choice is the complex $R\mathcal{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{D}_X) \otimes_{\mathscr{O}_X} \Omega_X^{\otimes -1}$ of left $\mathscr{D}_X$-modules.

Example 1.1.4. Let $X = \mathbb{C}$ (or an open subset of $\mathbb{C}$) and $\mathscr{M} = \mathscr{D}_X/\mathscr{D}_X P$ ($P \neq 0$). By applying the functor $\mathcal{H}om_{\mathscr{D}_X}(-, \mathscr{D}_X)$ to the exact sequence

$$0 \longrightarrow \mathscr{D}_X \xrightarrow{\cdot P} \mathscr{D}_X \longrightarrow \mathscr{M} \longrightarrow 0$$

of left $\mathscr{D}_X$-modules, we get an exact sequence

$$0 \longrightarrow 0 \longrightarrow \mathcal{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{D}_X) \longrightarrow \mathscr{D}_X \xrightarrow{P\cdot} \mathscr{D}_X$$

Hence in this case, we have

$$\mathcal{E}xt^0_{\mathscr{D}_X}(\mathscr{M}, \mathscr{D}_X) = \mathcal{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{D}_X) = \ker(P : \mathscr{D}_X \to \mathscr{D}_X) = 0,$$

and the only non-vanishing cohomology group is the first one

$$\mathcal{E}xt^1_{\mathscr{D}_X}(\mathscr{M}, \mathscr{D}_X) = \mathscr{D}_X/P\mathscr{D}_X.$$

The left DX-module obtained by the side changing $\otimes\Omega_X^{\otimes -1}$ is isomorphic to

$$\mathcal{E}xt^1_{\mathscr{D}_X}(\mathscr{M}, \mathscr{D}_X) \otimes_{\mathscr{O}_X} \Omega_X^{\otimes -1} \cong \mathscr{D}_X/\mathscr{D}_X P^*,$$

where P^* is the formal adjoint of P. From this calculation, we see that $\mathcal{E}xt^1$ is more suited than $\mathcal{E}xt^0$ to be called a "dual" of $\mathscr{M}$. More generally, if $d_X = \dim(X)$ and $\mathscr{M}$ is a holonomic $\mathscr{D}_X$-module, then we can (and will) prove that only the term $\mathcal{E}xt^n_{\mathscr{D}_X}(\mathscr{M}, \mathscr{D}_X)$ survives and the resulting left $\mathscr{D}_X$-module $\mathcal{E}xt^n_{\mathscr{D}_X}(\mathscr{M}, \mathscr{D}_X) \otimes_{\mathscr{O}_X} \Omega_X^{\otimes -1}$ is also holonomic. Hence the correct definition of the dual $D_X(\mathscr{M})$ of a holonomic $\mathscr{D}_X$-module $\mathscr{M}$ is given by $D_X(\mathscr{M}) = \mathcal{E}xt^n_{\mathscr{D}_X}(\mathscr{M}, \mathscr{D}_X) \otimes_{\mathscr{O}_X} \Omega_X^{\otimes -1}$. For a non-holonomic $\mathscr{D}_X$-module, one may have other non-vanishing cohomology groups, and hence the duality functor should be defined as follows for the derived categories.

We now define a contravariant functor $D_X : D^-(\mathscr{D}_X) \to D^+(\mathscr{D}_X)$ by

$$D_X(\mathscr{M}) = R\mathcal{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{D}_X) \otimes_{\mathscr{O}_X} \Omega_X^{\otimes -1}[d_X] = R\mathcal{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{D}_X \otimes_{\mathscr{O}_X} \Omega_X^{\otimes -1})[d_X].$$

The shift $[d_X]$ is added so that D_X sends $\mathscr{O}_X$ to itself. Since the cohomological dimension of $\mathscr{D}_X$ is finite, D_X preserves $D^b(\mathscr{D}_X)$.

Example 1.1.5. We have

$$H^i(D_X(\mathscr{D}_X)) = \begin{cases} \mathscr{D}_X \otimes_{\mathscr{O}_X} \Omega_X^{\otimes -1} & i = -d_X, \\ 0 & i \neq -d_X. \end{cases}$$

Lemma 1.1.41. *Let $\mathscr{M}$ be a coherent $\mathscr{D}_X$-module. Then for any affine open subset U of X, we have*

$$\Gamma(U, \mathcal{E}xt^i_{\mathscr{D}_X}(\mathscr{M}, \mathscr{D}_X)) = \mathrm{Ext}^i_{\Gamma(U,\mathscr{D}_X)}(\Gamma(U, \mathscr{M}), \Gamma(U, \mathscr{D}_X)).$$

Proof. □

Proposition 1.1.42. *The functor D_X sends $D^b_{\mathrm{coh}}(\mathscr{D}_X)$ to $D^b_{\mathrm{coh}}(\mathscr{D}_X)^{\mathrm{op}}$ and $D_X^2 \cong \mathrm{id}$ on $D^b_{\mathrm{coh}}(\mathscr{D}_X)$. In particular, D_X is fully faithful on $D^b_{\mathrm{coh}}(\mathscr{D}_X)$.*

Proof. We see from Lemma 1.1.41 that $H^i(D_X(\mathscr{M}))$ is coherent for each i, whence the first claim. Now we construct a canonical morphism $\mathscr{M} \to D_X^2(\mathscr{M})$ for $\mathscr{M} \in D^b(\mathscr{D}_X)$. First note that

$$D_X^2(\mathscr{M}) \cong R\mathcal{H}om_{\mathscr{D}_X^{\mathrm{op}}}(R\mathcal{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{D}_X), \mathscr{D}_X),$$

where $R\mathcal{H}om_{D_X}(\mathscr{M}, \mathscr{D}_X)$ and $\mathscr{D}_X$ are regareded as objects of $D^b(\mathscr{D}_X^{\mathrm{op}})$ byy the right multiplication of $\mathscr{D}_X$, and the left $\mathscr{D}_X$-action on the right hand side is induced from the left multiplication of $\mathscr{D}_X$. Set $\mathscr{K}^\bullet = R\mathcal{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{D}_X) \in D^b_{\mathrm{coh}}(\mathscr{D}_X^{\mathrm{op}})$. Then we have an isomorphism (note that $\otimes_{\mathbb{C}}$ is exact, hence has no derived functors)

$$R\mathcal{H}om_{\mathscr{D}_X \otimes_{\mathbb{C}} \mathscr{D}_X^{\mathrm{op}}}(\mathscr{M} \otimes_{\mathbb{C}} \mathscr{K}^\bullet, \mathscr{D}_X) \cong R\mathcal{H}om_{\mathscr{D}_X}(\mathscr{M}, R\mathcal{H}om_{\mathscr{D}_X^{\mathrm{op}}}(\mathscr{K}^\bullet, \mathscr{D}_X)).$$

Applying $H^0(R\Gamma(X, -))$, we then obtain

$$\mathrm{Hom}_{\mathscr{D}_X \otimes_{\mathbb{C}} \mathscr{D}_X^{\mathrm{op}}}(\mathscr{M} \otimes_{\mathbb{C}} \mathscr{K}^\bullet, \mathscr{D}_X) \cong \mathrm{Hom}_{\mathscr{D}_X}(\mathscr{M}, R\mathcal{H}om_{\mathscr{D}_X^{\mathrm{op}}}(\mathscr{K}^\bullet, \mathscr{D}_X)).$$

Hence the canonical homomorphism $\mathscr{M} \otimes_{\mathbb{C}} \mathscr{K}^\bullet \to \mathscr{D}_X$ gives rise to a canonical morphism

$$\mathscr{M} \to R\mathcal{H}om_{\mathscr{D}_X^{\mathrm{op}}}(\mathscr{K}^\bullet, \mathscr{D}_X).$$

To show that this morphism is an isomorphism for $\mathscr{M} \in D^b_{\mathrm{coh}}(\mathscr{D}_X)$, we may assume that X is affine. Then we can replace $\mathscr{M}$ with $\mathscr{D}_X$ by a five lemma argument, and the claim is then clear. □

Proposition 1.1.43. *Let $\mathscr{M}$ be a coherent $\mathscr{D}_X$-module.*

(a) $H^i(D_X(\mathscr{M})) = 0$ *unless* $\mathrm{codim}(\mathrm{Ch}(\mathscr{M})) - d_X \le i \le 0$.

(b) $\mathrm{codim}(\mathrm{Ch}(H^i(D_X(\mathscr{M})))) \ge d_X + i$.

(c) *$\mathscr{M}$ is holonomic if and only if $H^i(D_X(\mathscr{M})) = 0$ for $i \ne 0$.*

(d) *If $\mathscr{M}$ is holonomic, then $D_X(\mathscr{M}) \cong H^0(D_X(\mathscr{M}))$ is also holonomic.*

Proof. Since $D_X(\mathscr{M}) \cong R\mathcal{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{D}_X)[d_x]$ as $\mathscr{O}_X$-modules, the first two assertions follow from Theorem 1.1.36. The statement (d) and the only if part of (c) follow follow from (a), (b) and Proposition 1.1.35, so it suffices to prove that if part of (c). Assume that $H^i(D_X(\mathscr{M})) = 0$ for $i \ne 0$, that is, $D_X(\mathscr{M}) \cong H^0(D_X(\mathscr{M}))$. If we write $\mathscr{M}^* = H^0(D_X(\mathscr{M}))$, then $D_X(\mathscr{M}^*) = D_X^2(\mathscr{M}) \cong \mathscr{M}$ and $H^0(D_X(\mathscr{M}^*)) \cong \mathscr{M}$. On the other hand, by (b) we have $\mathrm{codim}(\mathrm{Ch}(H^0(D_X(\mathscr{M}^*)))) \ge d_X$, and hence $D_X(\mathscr{M}^*) \cong \mathscr{M}$ is a holonomic $\mathscr{D}_X$-module. □

Proposition 1.1.44. *Let $\mathscr{M}$ be an integrable connection. Then*

$$D_X(\mathscr{M}) \cong \mathcal{H}om_{\mathscr{O}_X}(\mathscr{M}, \mathscr{O}_X).$$

Proof. We consider the locally free resolution

$$0 \longrightarrow \mathscr{D}_X \otimes_{\mathscr{O}_X} \textstyle\bigwedge^{d_X}\Theta_X \longrightarrow \cdots \longrightarrow \mathscr{D}_X \otimes_{\mathscr{O}_X} \Theta_X \longrightarrow \mathscr{D}_X \longrightarrow \mathscr{O}_X \longrightarrow 0 \tag{1.1.21}$$

of $\mathscr{O}_X$. Since $\mathscr{M}$ is locally free over $\mathscr{O}_X$, $\mathscr{D}_X \otimes_{\mathscr{O}_X} \bigwedge^\bullet\Theta_X \otimes_{\mathscr{O}_X} \mathscr{M}$ is a locally free resolution of $\mathscr{M}$. Using this resolution, we can calculate $D_X(\mathscr{M})$ by the complex

$$\begin{aligned}
&\mathcal{H}om_{\mathscr{D}_X}(\mathscr{D}_X \otimes_{\mathscr{O}_X} (\textstyle\bigwedge^\bullet\Theta_X \otimes_{\mathscr{O}_X} \mathscr{M}), \mathscr{D}_X \otimes_{\mathscr{O}_X} \Omega_X^{\otimes -1})[d_X] \\
&\quad\cong \mathcal{H}om_{\mathscr{O}_X}(\textstyle\bigwedge^\bullet\Theta_X \otimes_{\mathscr{O}_X} \mathscr{M}, \mathscr{D}_X \otimes_{\mathscr{O}_X} \Omega_X^{\otimes -1})[d_X] \\
&\quad\cong \mathcal{H}om_{\mathscr{O}_X}(\Omega_X \otimes_{\mathscr{O}_X} \textstyle\bigwedge^\bullet\Theta_X \otimes_{\mathscr{O}_X} \mathscr{M}, \Omega_X \otimes_{\mathscr{O}_X} \mathscr{D}_X \otimes_{\mathscr{D}_X} \Omega_X^{\otimes -1})[d_X].
\end{aligned}$$

Since $\Omega_X \otimes_{\mathscr{O}_X} \bigwedge^\bullet\Theta_X \cong \Omega_X^\bullet$, this is isomorphic to

$$\begin{aligned}
&\mathcal{H}om_{\mathscr{O}_X}(\Omega_X^\bullet \otimes_{\mathscr{O}_X} \mathscr{M}, \mathscr{O}_X) \otimes_{\mathscr{O}_X} (\Omega_X \otimes_{\mathscr{O}_X} \mathscr{D}_X \otimes_{\mathscr{D}_X} \Omega_X^{\otimes -1})[d_X] \\
&\quad\cong (\textstyle\bigwedge^\bullet\Theta_X \otimes_{\mathscr{O}_X} \mathcal{H}om_{\mathscr{O}_X}(\mathscr{M}, \mathscr{O}_X)) \otimes_{\mathscr{O}_X} (\Omega_X \otimes_{\mathscr{O}_X} \mathscr{D}_X \otimes_{\mathscr{D}_X} \Omega_X^{\otimes -1})[d_X] \\
&\quad\cong \mathscr{D}_X \otimes (\textstyle\bigwedge^\bullet\Theta_X \otimes_{\mathscr{O}_X} \mathcal{H}om_{\mathscr{O}_X}(\mathscr{M}, \mathscr{O}_X))[d_X].
\end{aligned}$$

This complex is quasi-isomorphic to $\mathcal{H}om_{\mathscr{O}_X}(\mathscr{M}, \mathscr{O}_X)$ in view of (1.1.21). □

For a complex $\mathscr{M} \in D^b(\mathscr{D}_X)$, we define the support of $\mathscr{M}$ by

$$\operatorname{supp}(\mathscr{M}) := \bigcup_i \operatorname{supp}(H^i(\mathscr{M})).$$

and for $\mathscr{M} \in D^b_{\mathrm{coh}}(\mathscr{D}_X)$, we set

$$\mathrm{Ch}(\mathscr{M}) := \bigcup_i \mathrm{Ch}(H^i(\mathscr{M})).$$

Proposition 1.1.45. *For a coherent $\mathscr{M} \in D^b_{\mathrm{coh}}(\mathscr{D}_X)$, we have*

$$\mathrm{Ch}(D_X(\mathscr{M})) = \mathrm{Ch}(\mathscr{M}).$$

Proof. The proof of the inclusion $\mathrm{Ch}(D_x(\mathscr{M})) \subseteq \mathrm{Ch}(\mathscr{M})$ is reduced to the case $\mathscr{M} \in \mathbf{Mod}(\mathscr{D}_X)$, which is nothing but Theorem 1.1.36. Then, by applying this to $D_X(\mathscr{M})$, we obtain the inverse inclusion, since $D_X^2(\mathscr{M}) \cong \mathscr{M}$. □

In the rest of this paragraph, we give a description of $R\mathcal{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{N})$ for $M \in D^b_{\mathrm{coh}}(\mathscr{D}_X)$, $N \in D^b(\mathscr{D}_X)$, in terms of the duality functor.

Lemma 1.1.46. *For $M \in D^b_{\mathrm{coh}}(\mathscr{D}_X)$ and $N \in D^b(\mathscr{D}_X)$, we have*

$$R\mathcal{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{N}) \cong R\mathcal{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{D}_X) \otimes^L_{\mathscr{D}_X} \mathscr{N}.$$

Proof. Note that there is a canonical morphism

$$R\mathcal{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{N}) \to R\mathcal{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{D}_X) \otimes^L_{\mathscr{D}_X} \mathscr{N}.$$

Hence we may assume that $\mathscr{M} = \mathscr{D}_X$, in which case the assertion is obvious since both sides are isomorphic to $\mathscr{N}$. □

Proposition 1.1.47. *For $M \in D^b_{\mathrm{coh}}(\mathscr{D}_X)$ and $N \in D^b(\mathscr{D}_X)$, we have*

$$\begin{aligned} R\mathcal{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{N}) &\cong (\Omega_X \otimes^L_{\mathscr{O}_X} D_X(\mathscr{M})) \otimes^L_{\mathscr{D}_X} \mathscr{N}[-d_X] &\quad (1.1.22)\\ &\cong \Omega_X \otimes^L_{\mathscr{D}_X} (D_X(\mathscr{M}) \otimes^L_{\mathscr{O}_X} \mathscr{N})[-d_X] &\quad (1.1.23)\\ &\cong R\mathcal{H}om_{\mathscr{D}_X}(\mathscr{O}_X, D_X(\mathscr{M}) \otimes^L_{\mathscr{O}_X} \mathscr{N}) &\quad (1.1.24)\end{aligned}$$

in $D^b(\mathbb{C}_X)$. In particular, we have

$$R\mathcal{H}om_{\mathscr{D}_X}(\mathscr{O}_X, \mathscr{N}) \cong \Omega_X \otimes^L_{\mathscr{O}_X} \mathscr{N}[-d_X]. \quad (1.1.25)$$

Proof. We first prove (1.1.25). By Lemma 1.1.46, we may assume that $\mathscr{N} = \mathscr{D}_X$. In this case, by Proposition 1.1.10 we have

$$\begin{aligned} R\mathcal{H}om_{\mathscr{D}_X}(\mathscr{O}_X, \mathscr{D}_X) &\cong \mathcal{H}om_{\mathscr{D}_X}(\mathscr{D}_X \otimes_{\mathscr{O}_X} \textstyle\bigwedge^\bullet \Theta_X, \mathscr{D}_X) \cong \mathcal{H}om_{\mathscr{O}_X}(\textstyle\bigwedge^\bullet \Theta_X, \mathscr{D}_X)\\ &\cong \Omega_X^\bullet \otimes_{\mathscr{O}_X} \mathscr{D}_X \cong \Omega_X[-d_X].\end{aligned}$$

Now in the general case, by Lemma 1.1.46 we have

$$R\mathcal{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{N}) \cong R\mathcal{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{D}_X) \otimes^L_{\mathscr{D}_X} \mathscr{N} \cong (\Omega_X \otimes^L_{\mathscr{O}_X} D_X(\mathscr{M})) \otimes^L_{\mathscr{D}_X} \mathscr{N}[-d_X].$$

The second and the third isomorphisms follow from the derived version of Proposition 1.1.16 and (1.1.25), respectively. □

1.2 Functorial properties of D-modules

1.2.1 Inverse images of D-modules

1.2.1.1 The transfer module $\mathscr{D}_{X\to Y}$ Let $f : X \to Y$ be a morphism of smooth algebraic varieties. We want to construct a $\mathscr{D}_X$-module by lifting a $\mathscr{D}_Y$-module. Let $\mathscr{M}$ be a (left) $\mathscr{D}_Y$-module and consider its inverse image

$$f^*(\mathscr{M}) = \mathscr{O}_X \otimes_{f^{-1}(\mathscr{O}_Y)} f^{-1}(\mathscr{M}).$$

We can endow $f^*(\mathscr{M})$ with a (left) $\mathscr{D}_X$-module structure as follows. First, note that we have a canonical $\mathscr{O}_X$-linear homomorphism

$$\Theta_X \to f^*(\Theta_Y) = \mathscr{O}_X \otimes_{f^{-1}(\mathscr{O}_Y)} f^{-1}(\Theta_Y), \quad v \mapsto \tilde{v} \tag{1.2.1}$$

obtained by taking the $\mathscr{O}_X$-dual of the homomorphism $\mathscr{O}_X \otimes_{f^{-1}(\mathscr{O}_Y)} f^{-1}(\Omega^1_Y) \to \Omega^1_X$. Then we can define a left $\mathscr{D}_X$-module structure on $f^*(\mathscr{M})$ by

$$v(a \otimes s) = v(a) \otimes s + a\tilde{v}(s), \quad v \in \Theta_X, a \in \mathscr{O}_X, s \in \mathscr{M}. \tag{1.2.2}$$

Here, if we write $\tilde{v} = \sum_i a_i \otimes w_i$ with $a_i \in \mathscr{O}_X$ and $w_i \in \Theta_Y$, then

$$\tilde{v}(s) = \sum_i a_i \otimes w_i(s). \tag{1.2.3}$$

This definition is independent of the choice of $\sum_i a_i \otimes w_i$. Indeed, since we can define a morphism

$$(\mathscr{O}_X \otimes_{f^{-1}(\mathscr{O}_Y)} f^{-1}(\Theta_Y)) \otimes_{\mathbb{C}} f^{-1}(\mathscr{M}) \to \mathscr{O}_X \otimes_{f^{-1}(\mathscr{O}_Y)} f^{-1}(\mathscr{M}), \quad (f \otimes v) \otimes s \mapsto f \otimes vs$$

we obtain from (1.2.1) a homomorphism

$$\Theta_X \otimes_{\mathbb{C}} (\mathscr{O}_X \otimes_{\mathbb{C}} f^{-1}(\mathscr{M})) \to \mathscr{O}_X \otimes_{f^{-1}(\mathscr{O}_Y)} f^{-1}(\mathscr{M})$$

which can be checked to be given by

$$v \otimes a \otimes s \mapsto \sum_i aa_i \otimes w_i(s).$$

Moreover, it is easily seen that $v(af^*(b) \otimes s) = v(a \otimes bs)$ for $b \in \mathscr{O}_Y$, so this provides us a homomorphism $\Theta_X \otimes_{\mathbb{C}} f^*(\mathscr{M}) \to f^*(\mathscr{M})$. Since this action satisfies the conditions of Proposition 1.1.1, it can be extended to an action of $\mathscr{D}_X$. If we are given a local coordinate system $\{y_i, \partial_i\}$ of Y, then the action of $v \in \Theta_X$ can be written more explicitly as

$$v(a \otimes s) = v(a) \otimes s + a \sum_{i=1}^{n} v(y_i \circ f) \otimes \partial_i s. \tag{1.2.4}$$

Regarding $\mathscr{D}_Y$ as a left $\mathscr{D}_Y$-module by the left multiplication, we obtain a left $\mathscr{D}_X$-module $f^*(\mathscr{D}_Y) = \mathscr{O}_X \otimes_{f^{-1}(\mathscr{O}_Y)} f^{-1}(\mathscr{D}_Y)$. Then the right multiplication of $\mathscr{D}_Y$ on $\mathscr{D}_Y$ induces a right $f^{-1}(\mathscr{D}_Y)$-module structure on $f^*(\mathscr{D}_Y)$:

$$(a \otimes P)Q = a \otimes PQ, \quad a \in \mathscr{O}_X, P, Q \in \mathscr{D}_Y,$$

and it is immediate that these two actions commute, so $f^*(\mathscr{D}_Y)$ turns out to be a $(\mathscr{D}_X, f^{-1}(\mathscr{D}_Y))$-bimodule, which is denote by $\mathscr{D}_{X\to Y}$. Let $1_{X\to Y}$ denote the canonical element $1 \otimes 1$ of $\mathscr{D}_{X\to Y}$. Then for $b \in \mathscr{O}_Y$, we have

$$v1_{X\to Y} = \sum_i a_i 1_{X\to Y} w_i, \quad 1_{X\to Y} b = f^*(b) 1_{X\to Y}. \tag{1.2.5}$$

Therefore, if $\mathscr{M}$ is a left $\mathscr{D}_Y$-module, we have an isomorphism

$$f^*(\mathscr{M}) \cong \mathscr{D}_{X\to Y} \otimes_{f^{-1}(\mathscr{D}_Y)} f^{-1}(\mathscr{M})$$

of left $\mathscr{D}_X$-modules. We have thus defined a functor

$$f^* : \mathbf{Mod}(\mathscr{D}_Y) \to \mathbf{Mod}(\mathscr{D}_X), \quad \mathscr{M} \mapsto \mathscr{D}_{X\to Y} \otimes_{f^{-1}(\mathscr{D}_Y)} f^{-1}(\mathscr{M}).$$

Example 1.2.1. Let $X = \mathbb{C}$ and $Y = \mathbb{C}$ with coordinates $\{x, \partial_x\}$ and $\{y, \partial_y\}$, respectively. Let f be the morphism given by

$$f : X \to Y, \quad x \mapsto x^2.$$

Then

$$\mathscr{D}_Y = \bigoplus \mathscr{O}_Y \partial_y^n, \quad \mathscr{D}f^*(\mathscr{D}_Y) = \mathscr{D}_{X\to Y} = \bigoplus_{n\geq 0} \mathscr{O}_X \otimes_{\mathbb{C}} \mathbb{C}\partial_y^n$$

and we have

$$\partial_x(a \otimes \partial_y^n) = \frac{\partial a}{\partial x} \otimes \partial_y^n + 2ax \otimes \partial_y^{n+1}.$$

While $\mathscr{D}_{X\to Y}$ is isomorphic to $\mathscr{D}_X$ in $X \setminus \{0\}$, we see that it is not coherent in a neighborhood of $x = 0$. In fact, we have

$$\mathscr{D}_{X\to Y} = \bigoplus_{n\geq 0} \mathscr{O}_X (x^{-1}\partial_x)^n \subseteq \mathscr{D}_X[x^{-1}].$$

Example 1.2.2. Let $i : X \to Y$ be a closed immersion of smooth algebraic varieties. At each point $x \in X$, we can choose a local coordinate $\{y_i, \partial_{y_i}\}$ on an affine open subset of Y such that $y_{r+1} = \cdots = y_n = 0$ gives a local defining equation of X. We set $x_i = y_i \circ i$ for $i = 1, \ldots, r$, which gives a local coordinate $\{x_i, \partial_{x_i}\}$ of an affine open subset of X. The canonical morphism $\Theta_X \to \mathscr{O}_X \otimes_{i^{-1}(\mathscr{O}_Y)} i^{-1}(\Theta_Y)$ is then given by $\partial_{x_i} \mapsto \partial_{y_i}$ for $i = 1, \ldots, r$. Now consider

$$\mathscr{D} = \bigoplus_{\alpha_1,\ldots,\alpha_r} \mathscr{O}_Y \partial_{y_1}^{\alpha_1} \cdots \partial_{y_r}^{\alpha_r} \subseteq \mathscr{D}_Y.$$

Since $[\partial_{y_i}, \partial_{y_j}] = 0$, $\mathscr{D}$ is a subring of $\mathscr{D}_Y$, and we have $\mathscr{D}_Y \cong \mathscr{D} \otimes_{\mathbb{C}} \mathbb{C}[\partial_{y_{r+1}}, \ldots, \partial_{y_n}]$ as a left $\mathscr{D}$-module. We therefore conclude that

$$\mathscr{D}_{X\to Y} \cong \mathscr{O}_X \otimes_{i^{-1}(\mathscr{O}_Y)} i^{-1}(\mathscr{D}) \otimes_{\mathbb{C}} \mathbb{C}[\partial_{y_{r+1}}, \ldots, \partial_{y_n}].$$

On the other hand, it is easily seen that $\mathscr{D}_X$ is isomorphic to the submodule $\mathscr{O}_X \otimes_{i^{-1}(\mathscr{O}_Y)} i^{-1}(\mathscr{D})$ of $\mathscr{D}_{X\to Y}$, so we conclude that

$$\mathscr{D}_{X\to Y} \cong \mathscr{D}_X \otimes_{\mathbb{C}} \mathbb{C}[\partial_{y_{r+1}}, \ldots, \partial_{y_n}]. \tag{1.2.6}$$

as a left $\mathscr{D}_X$-module. In particular, $\mathscr{D}_{X\to Y}$ is a locally free $\mathscr{D}_X$-module of infinite rank (unless $r = n$).

Proposition 1.2.1. *Let $f : X \to Y$ be a smooth morphism. Then $\mathscr{D}_{X\to Y}$ is generated by $1_{X\to Y}$ as a left $\mathscr{D}_X$-module.*

Proof. We can locally write $f(x, y) = y$ in appropriate coordinate systems $y = (y_1, \ldots, y_m)$ of Y and $(x, y) = (x_1, \ldots, x_n, y_1, \ldots, y_m)$ of X. It then follows from (1.2.4) that

$$\partial_{x_i}(a \otimes P) = 0, \quad \partial_{y_i}(a \otimes P) = a \otimes \partial_{y_i} P \quad \text{for } a \in \mathscr{O}_X, P \in \mathscr{D}_Y.$$

In particular, we see that $1_{X\to Y}\partial_{y_i} = \partial_{y_i} 1_{X\to Y}$, so $\mathscr{D}_{X\to Y}$ is generated by $1_{X\to Y}$ as a left $\mathscr{D}_X$-module. $\square$

Let $f : X \to Y$ be a smooth morphism of smooth algebraic varieties and $\Omega^\bullet_{X/Y}$ denote the sheaf of differential forms relative to f, that is,

$$\Omega^1_{X/Y} = \operatorname{coker}(f^*\Omega^1_Y \to \Omega^1_X), \tag{1.2.7}$$

$$\Omega^p_{X/Y} = \textstyle\bigwedge^p \Omega^1_{X/Y} = \Omega^p_X / \big(\operatorname{im}(f^*(\Omega^1_Y) \to \Omega^1_X) \wedge \Omega^{p-1}_X\big). \tag{1.2.8}$$

We then have the relative de Rham complex

$$\Omega^\bullet_{X/Y} : \cdots \longrightarrow 0 \longrightarrow \Omega^0_{X/Y} \xrightarrow{d^0_{X/Y}} \Omega^1_{X/Y} \longrightarrow \cdots \longrightarrow \Omega^n_{X/Y} \longrightarrow 0 \longrightarrow \cdots$$

(n is the relative dimension of f.) Since the differential $d_{X/Y}$ are differential homomorphisms of degree 1, we obtain a complex of right $\mathscr{D}_X$-modules

$$\Omega^\bullet_{X/Y} \otimes_{\mathscr{O}_X} \mathscr{D}_X : \cdots \longrightarrow 0 \longrightarrow \Omega^0_{X/Y} \otimes_{\mathscr{O}_X} \mathscr{D}_X \longrightarrow \cdots \longrightarrow \Omega^n_{X/Y} \otimes_{\mathscr{O}_X} \mathscr{D}_X \longrightarrow 0 \longrightarrow \cdots \tag{1.2.9}$$

by Corollary 1.1.8. Applying $\mathcal{H}om_{\mathscr{D}_X}(-, \mathscr{D}_X)$ to this complex, we get

$$\mathscr{D}_X \otimes_{\mathscr{O}_X} \textstyle\bigwedge^\bullet \Theta_{X/Y} : \cdots \longrightarrow \mathscr{D}_X \otimes_{\mathscr{O}_X} \bigwedge^n \Theta_{X/Y} \longrightarrow \cdots \longrightarrow \mathscr{D}_X \otimes_{\mathscr{O}_X} \bigwedge^1 \Theta_{X/Y} \longrightarrow \cdots \tag{1.2.10}$$

where

$$\Theta_{X/Y} = \mathcal{H}om_{\mathscr{O}_X}(\Omega^1_{X/Y}, \mathscr{O}_X) = \ker(\Theta_X \to f^*(\Theta_Y)).$$

Proposition 1.2.2. *Let $f : X \to Y$ be a smooth morphism of relative dimension n between smooth algebraic varieties. Then the following sequence is exact*

$$0 \longrightarrow \mathscr{D}_X \otimes_{\mathscr{O}_X} \textstyle\bigwedge^n \Theta_{X/Y} \longrightarrow \cdots \longrightarrow \mathscr{D}_X \otimes_{\mathscr{O}_X} \bigwedge^0 \Theta_{X/Y} \longrightarrow \mathscr{D}_{X\to Y} \longrightarrow 0 \tag{1.2.11}$$

Proof. We define a good filtration of $\mathscr{D}_X \otimes_{\mathscr{O}_X} \bigwedge^k \Theta_{X/Y}$ by

$$F_i(\mathscr{D}_X \otimes_{\mathscr{O}_X} \textstyle\bigwedge^k \Theta_{X/Y}) = F_{i-k}(\mathscr{D}_X) \otimes_{\mathscr{O}_X} \bigwedge^k \Theta_{X/Y},$$

and equip $\mathscr{D}_{X\to Y}$ with the quotient filtration from $\mathscr{D}_X$. Then (1.2.11) is a complex of filtered modules. Taking associated graded modules, we obtain a complex of $\operatorname{gr}(\mathscr{D}_X)$-modules

$$0 \longrightarrow \operatorname{gr}(\mathscr{D}_X \otimes_{\mathscr{O}_X} \textstyle\bigwedge^n \Theta_{X/Y}) \longrightarrow \cdots \longrightarrow \operatorname{gr}(\mathscr{D}_X \otimes_{\mathscr{O}_X} \bigwedge^0 \Theta_{X/Y}) \longrightarrow \operatorname{gr}(\mathscr{D}_{X\to Y}) \longrightarrow 0 \tag{1.2.12}$$

If we write locally $\Theta_{X/Y} = \bigoplus_{i=1}^n \mathscr{O}_X v_i$, then (1.2.12) is the Koszul complex with respect to $(v_1, \ldots, v_n)$, which is exact since the codimension of the zero set of $\{v_1, \ldots, v_n\}$ in T^*X equals to n. We then conclude that (1.2.11) is exact. □

1.2.1.2 The derived inverse images of D-modules The language of derived categories is most suitable for the systematic study of inverse images of D-modules. Let $f : X \to Y$ be a morphism of smooth algebraic varieties, then the functor

$$f^* = \mathscr{O}_X \otimes_{f^{-1}(\mathscr{O}_Y)} f^{-1}(-) : \mathbf{Mod}(\mathscr{D}_Y) \to \mathbf{Mod}(\mathscr{D}_X)$$

is a right exact functor, hence extends, via left derivation, to a triangulated functor

$$Lf^* : D^-(\mathscr{D}_Y) \to D^-(\mathscr{D}_X).$$

Since the flat dimension of $\mathscr{O}_Y$ is finite, this functor sends $D^b(\mathscr{D}_Y)$ to $D^b(\mathscr{D}_X)$, and hence induces a functor

$$Lf^* : D^b(\mathscr{D}_Y) \to D^b(\mathscr{D}_X).$$

For a bounded complex $\mathscr{N}$ of $\mathscr{D}_Y$-modules, by taking a bounded flat resolution $\mathscr{L} \to \mathscr{N}$, we can express $Lf^*(\mathscr{N})$ as $f^*(\mathscr{L})$.

Proposition 1.2.3. *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of smooth algebraic varieties, then*

$$Lf^* \circ Lg^* \cong L(g \circ f)^*.$$

Proof. We first note that

$$\begin{aligned} Lf^*(\mathscr{D}_{Y\to Z}) &= \mathscr{O}_X \otimes^L_{f^{-1}(\mathscr{O}_Y)} f^{-1}(\mathscr{D}_{Y\to Z}) \\ &= \mathscr{O}_X \otimes^L_{f^{-1}(\mathscr{O}_Y)} f^{-1}(\mathscr{O}_Y \otimes_{g^{-1}(\mathscr{O}_Z)} g^{-1}(\mathscr{D}_Z)) \\ &= \mathscr{O}_X \otimes^L_{f^{-1}(\mathscr{O}_Y)} f^{-1}(\mathscr{O}_Y) \otimes^L_{(g\circ f)^{-1}(\mathscr{O}_Z)} (g \circ f)^{-1}(\mathscr{D}_Z) \\ &= \mathscr{O}_X \otimes^L_{(g\circ f)^{-1}(\mathscr{O}_Z)} (g \circ f)^{-1}(\mathscr{D}_Z) = \mathscr{D}_{X\to Z} \end{aligned}$$

where we have used the fact that $\mathscr{D}_Z$ is a locally free $\mathscr{O}_Z$-module. We thus obtain isomorphisms

$$\mathscr{D}_{X\to Z} \cong \mathscr{D}_{X\to Y} \otimes^L_{f^{-1}(\mathscr{D}_Y)} f^{-1}(\mathscr{D}_{Y\to Z}) \cong \mathscr{D}_{X\to Y} \otimes^L_{f^{-1}(\mathscr{D}_Y)} f^{-1}(\mathscr{D}_{Y\to Z})$$

and therefore

$$\begin{aligned} L(g \circ f)^*(\mathscr{M}) &= \mathscr{D}_{X\to Z} \otimes^L_{(g\circ f)^{-1}(\mathscr{D}_Y)} (g \circ f)^{-1}(\mathscr{M}) \\ &\cong (\mathscr{D}_{X\to Y} \otimes_{f^{-1}(\mathscr{D}_Y)} f^{-1}(\mathscr{D}_{Y\to Z})) \otimes^L_{f^{-1}(g^{-1}(\mathscr{D}_Y))} f^{-1}(g^{-1}(\mathscr{M})) \\ &\cong \mathscr{D}_{X\to Y} \otimes^L_{f^{-1}(\mathscr{D}_Y)} f^{-1}(\mathscr{D}_{Y\to Z} \otimes^L_{g^{-1}(\mathscr{D}_Y)} g^{-1}(\mathscr{M})) \cong Lf^*(Lg^*(\mathscr{M})), \end{aligned}$$

whence our claim. □

Proposition 1.2.4. *Let $f : X \to Y$ be a morphism of smooth algebraic varieties, then Lf^* sends $D^b_{\mathrm{qcoh}}(\mathscr{D}_Y)$ to $D^b_{\mathrm{qcoh}}(\mathscr{D}_X)$.*

Proof. Let $\mathscr{M} \in D^b_{\mathrm{qcoh}}(\mathscr{D}_Y)$, then as a complex of $\mathscr{O}_X$-modules, we have

$$Lf^*(\mathscr{M}) = \mathscr{O}_X \otimes^L_{f^{-1}(\mathscr{O}_Y)} f^{-1}(\mathscr{M}).$$

The assertion then follows from the corresponding result for the derived tensor product on the category $\mathbf{Qcoh}(\mathscr{O}_X)$. □

Remark 1.2.1. We note that $Lf^*(\mathscr{D}_Y) = \mathscr{D}_{X\to Y} \otimes^L_{f^{-1}(\mathscr{D}_Y)} f^{-1}(\mathscr{D}_Y) = \mathscr{D}_{X\to Y}$. If f is a closed immersion with $\dim(X) < \dim(Y)$, then the $\mathscr{D}_X$-module $\mathscr{D}_{X\to Y}$ is locally of infinite rank (cf. Example 1.2.2). Therefore, the functor Lf^* does not necessarily send $D^b_{\mathrm{coh}}(\mathscr{D}_Y)$ to $D^b_{\mathrm{coh}}(\mathscr{D}_X)$.

Proposition 1.2.5. *Let $f : X \to Y$ be a smooth morphism between smooth algebraic varieties.*

(a) *If $\mathscr{M}$ is a left $\mathscr{D}_Y$-module, then $L^i f^*(\mathscr{M}) = 0$ for $i \neq 0$ (hence we can write f^* instead of Lf^*).*

(b) *If $\mathscr{M}$ is a coherent $\mathscr{D}_Y$-module, then $f^*(\mathscr{M})$ is a coherent $\mathscr{D}_X$-module.*

Proof. The first assertion follows from the flatness of $\mathscr{O}_X$ over $f^{-1}(\mathscr{O}_Y)$, since f is flat. To see that $f^*(\mathscr{M})$ is a coherent $\mathscr{D}_X$-module if $\mathscr{M}$ is coherent, we use Proposition 1.1.20, so let $\mathscr{D}_X^{\oplus m} \to \mathscr{M}$ be a surjective homomorphism. Then by applying f^*, we obtain a surjective homomorphism $\mathscr{D}_{X\to Y}^{\oplus m} \to f^*(\mathscr{M})$, and it suffices to note that there is a surjective homomorphism $\mathscr{D} \to \mathscr{D}_{X\to Y}$ given by $P \mapsto P1_{X\to Y}$ (cf. Proposition 1.2.1). □

Proposition 1.2.6. *Let $i : X \to Y$ be a closed immersion and set $d = \mathrm{codim}_Y(X)$. Then for any $\mathscr{M} \in \mathbf{Mod}(\mathscr{D}_Y)$, we have $i^*(\mathscr{M}) \in D^{[-d,0]}(\mathscr{D}_X)$.*

Proof. Let $n = \dim(Y)$, then we have a locally free resolution

$$0 \longrightarrow \mathscr{K}_{n-d} \longrightarrow \cdots \longrightarrow \mathscr{K}_1 \longrightarrow 0 \longrightarrow \mathscr{K}_0 \longrightarrow \mathscr{O}_X \longrightarrow 0 \tag{1.2.13}$$

of the $i^{-1}(\mathscr{O}_Y)$-module $\mathscr{O}_X$ (Koszul resolution, see, eg. [Matm Theorem 43]). From this resolution, we then obtain a locally free resolution of the right $i^{-1}(\mathscr{D}_Y)$-module $\mathscr{D}_{X\to Y}$:

$$0 \longrightarrow \mathscr{K}_{n-d} \otimes_{i^{-1}(\mathscr{O}_Y)} \mathscr{D}_Y \longrightarrow \cdots \longrightarrow \mathscr{K}_0 \otimes_{i^{-1}(\mathscr{O}_Y)} \mathscr{D}_Y \longrightarrow \mathscr{D}_{X\to Y} \longrightarrow 0$$

Since $Li^*(\mathscr{M}) = \mathscr{D}_{X\to Y} \otimes^L_{i^{-1}(\mathscr{D}_Y)} i^{-1}(\mathscr{M})$, it is then represented by the complex

$$0 \longrightarrow \mathscr{K}_{n-d} \otimes_{i^{-1}(\mathscr{O}_Y)} i^{-1}(\mathscr{M}) \longrightarrow \cdots \longrightarrow \mathscr{K}_0 \otimes_{i^{-1}(\mathscr{O}_Y)} i^{-1}(\mathscr{M}) \longrightarrow 0$$

This proves our assertion, since this complex is concentrated at degrees $[-d, 0]$. □

1.2.2 External tensor product

Let X and Y be two smooth algebraic varieties and $\mathrm{pr}_1 : X \times Y \to X$, $\mathrm{pr}_2 : X \times Y \to Y$ be the canonical projections. For an $\mathscr{O}_X$-module $\mathscr{F}$ and an $\mathscr{O}_Y$-module $\mathscr{G}$, we can define an $\mathscr{O}_{X\times Y}$-module $\mathscr{F} \boxtimes \mathscr{G}$ by

$$\begin{aligned}\mathscr{F} \boxtimes \mathscr{G} &:= (\mathscr{O}_{X\times Y} \otimes_{\mathrm{pr}_1^{-1}(\mathscr{O}_X)} \mathscr{F}) \otimes_{\mathrm{pr}_2^{-1}(\mathscr{O}_Y)} \mathrm{pr}_2^{-1}(\mathscr{G}) \\ &= \mathscr{O}_{X\times Y} \otimes_{\mathrm{pr}_1^{-1}(\mathscr{O}_X) \otimes_{\mathbb{C}} \mathrm{pr}_2^{-1}(\mathscr{O}_Y)} (\mathrm{pr}_1^{-1}(\mathscr{F}) \otimes_{\mathbb{C}} \mathrm{pr}_2^{-1}(\mathscr{G})).\end{aligned}$$

It is well known that this functor is exact with respect to $\mathscr{F}$ and $\mathscr{G}$, hence extends to a functor

$$(-) \boxtimes (-) : D^b(\mathscr{O}_X) \times D^b(\mathscr{O}_Y) \to D^b(\mathscr{O}_{X\times Y}).$$

Furthermore, we have $\mathrm{supp}(\mathscr{F} \boxtimes \mathscr{G}) = \mathrm{supp}(\mathscr{F}) \times \mathrm{supp}(\mathscr{G})$.

We note that the external tensor product of $\mathscr{D}_X$ and $\mathscr{D}_Y$ is given by

$$\mathscr{D}_X \boxtimes \mathscr{D}_Y = \mathscr{O}_{X\times Y} \otimes_{\mathrm{pr}_1^{-1}(\mathscr{O}_X) \otimes_{\mathbb{C}} \mathrm{pr}_2^{-1}(\mathscr{O}_Y)} (\mathrm{pr}_1^{-1}(\mathscr{D}_X) \otimes_{\mathbb{C}} \mathrm{pr}_2^{-1}(\mathscr{D}_Y)) \cong \mathscr{D}_{X\times Y},$$

so for a $\mathscr{D}_X$-module $\mathscr{M}$ and a $\mathscr{D}_Y$-module $\mathscr{N}$, we have

$$\begin{aligned}\mathscr{M} \boxtimes \mathscr{N} &= \mathscr{O}_{X\times Y} \otimes_{\mathrm{pr}_1^{-1}(\mathscr{O}_X) \otimes_{\mathbb{C}} \mathrm{pr}_2^{-1}(\mathscr{O}_Y)} (\mathrm{pr}_1^{-1}(\mathscr{M}) \otimes_{\mathbb{C}} \mathrm{pr}_2^{-1}(\mathscr{N})) \\ &\cong \mathscr{D}_{X\times Y} \otimes_{\mathrm{pr}_1^{-1}(\mathscr{D}_X) \otimes_{\mathbb{C}} \mathrm{pr}_2^{-1}(\mathscr{D}_Y)} (\mathrm{pr}_1^{-1}(\mathscr{M}) \otimes_{\mathbb{C}} \mathrm{pr}_2^{-1}(\mathscr{N})).\end{aligned}$$

This means the $\mathscr{O}_X$-module $\mathscr{M} \boxtimes \mathscr{N}$ is canoncially endowed with a left $\mathscr{D}_{X\times Y}$-module structure, so we obtain a bifunctor

$$(-) \boxtimes (-) : D^b(\mathscr{D}_X) \times D^b(\mathscr{D}_Y) \to D^b(\mathscr{D}_{X\times Y})$$

for derived categories such that the following diagram is commutative:

$$\begin{tikzcd} D^b(\mathscr{D}_X) \otimes D^b(\mathscr{D}_Y) \arrow[r, "(-)\boxtimes(-)"] \arrow[d] & D^b(\mathscr{D}_{X\times Y}) \arrow[d] \\ D^b(\mathscr{O}_X) \otimes D^b(\mathscr{O}_Y) \arrow[r, "(-)\boxtimes(-)"] & D^b(\mathscr{O}_{X\times Y}) \end{tikzcd}$$

It is easily seen that the functor $(-) \boxtimes (-)$ sends $D^b_{\mathrm{qcoh}}(\mathscr{D}_X) \times D^b_{\mathrm{qcoh}}(\mathscr{D}_Y)$ (resp $D^b_{\mathrm{coh}}(\mathscr{D}_X) \times D^b_{\mathrm{coh}}(\mathscr{D}_Y)$) to $D^b_{\mathrm{qcoh}}(\mathscr{D}_{X\times Y})$ (resp. $D^b_{\mathrm{coh}}(\mathscr{D}_{X\times Y})$), and we have

$$\mathrm{pr}_1^*(\mathscr{M}) \cong \mathscr{M} \boxtimes \mathscr{O}_Y, \quad \mathrm{pr}_2^*(\mathscr{N}) \cong \mathscr{O}_X \boxtimes \mathscr{N}.$$

Let X be a smooth algebraic variety and $\Delta_X : X \to X \times X$ be the diagonal morphism. For $\mathscr{M}, \mathscr{N} \in \mathbf{Mod}(\mathscr{D}_X)$, we easily seen that $\mathscr{M} \otimes_{\mathscr{O}_X} \mathscr{N}$ is isomorphic to $\Delta_X^*(\mathscr{M} \boxtimes \mathscr{N})$ as a $\mathscr{D}_X$-module. Moreover, the external tensor products of flat modules are flat, so for $\mathscr{M}, \mathscr{N} \in D^b(\mathscr{D}_X)$ we have a canonical isomorphism

$$\mathscr{M} \otimes^L_{\mathscr{O}_X} \mathscr{N} \cong L\Delta_X^*(\mathscr{M} \boxtimes \mathscr{N}).$$

Proposition 1.2.7. *Let $f : X \to Y$ and $f' : X' \to Y'$ be morphisms of smooth algebraic varieties. Then for $\mathscr{M}, \mathscr{N} \in D^b(\mathscr{D}_Y)$ and $\mathscr{M}' \in D^b(\mathscr{D}_{Y'})$, we have*

$$\begin{aligned} L(f \times f')^*(\mathscr{M} \boxtimes \mathscr{M}') &\cong Lf^*(\mathscr{M}) \boxtimes Lf'^*(\mathscr{M}'), \\ Lf^*(\mathscr{M} \otimes_{\mathscr{O}_Y} \mathscr{N}) &\cong Lf^*(\mathscr{M}) \otimes^L_{\mathscr{O}_X} Lf^*(\mathscr{N}). \end{aligned}$$

Proof. The first statement follows from $(f \times f')^*(\mathscr{M} \boxtimes \mathscr{N}') \cong f^*(\mathscr{M}) \boxtimes f'^*(\mathscr{M}')$ for $\mathscr{M} \in \mathbf{Mod}(\mathscr{D}_Y)$, $\mathscr{M}' \in \mathbf{Mod}(\mathscr{D}_Y)$. The second one then follows as follows:

$$\begin{aligned} Lf^*(\mathscr{M} \otimes_{\mathscr{O}_Y} \mathscr{N}) &\cong Lf^* L\Delta_Y^*(\mathscr{M} \boxtimes \mathscr{N}) \cong L\Delta_X^* L(f \times f)^*(\mathscr{M} \boxtimes \mathscr{N}) \\ &\cong L\Delta_X^*(Lf^*(\mathscr{M}) \boxtimes Lf^*(\mathscr{N})) \cong Lf^*(\mathscr{M}) \otimes^L_{\mathscr{O}_X} Lf^*(\mathscr{N}). \end{aligned}$$

where we have used the equality $(f, f) \circ \Delta_X = \Delta_Y \circ f$. □

Proposition 1.2.8. *Let $\mathscr{M}, \mathscr{N} \in D^b(\mathscr{D}_X)$ and $\mathscr{L} \in D^b(\mathscr{D}_X^{\mathrm{op}})$. Then we have canonical isomorphisms of $\mathbb{C}_X$-modules*

$$(\mathscr{L} \otimes^L_{\mathscr{O}_X} \mathscr{N}) \otimes^L_{\mathscr{D}_X} \mathscr{M} \cong \mathscr{L} \otimes^L_{\mathscr{D}_X} (\mathscr{M} \otimes^L_{\mathscr{O}_X} \mathscr{N})$$

Proof. By taking flat resolutions, we may assume that $\mathscr{M}, \mathscr{N} \in \mathbf{Mod}(\mathscr{D}_X)$ and $\mathscr{L} \in \mathbf{Mod}(\mathscr{D}_X^{\mathrm{op}})$. The assertion then follows from Proposition 1.1.16. □

Proposition 1.2.9. *For a coherent $\mathscr{D}_X$-module $\mathscr{M}$ and a coherent $\mathscr{D}_Y$-module $\mathscr{N}$, we have*

$$\mathrm{Ch}(\mathscr{M} \boxtimes \mathscr{N}) = \mathrm{Ch}(\mathscr{M}) \times \mathrm{Ch}(\mathscr{N}).$$

Proof. Let $F(\mathscr{M})$ and $F(\mathscr{N})$ be good filtrations of $\mathscr{M}$ and $\mathscr{N}$, respectively. We define a good filtration of $\mathscr{M} \boxtimes \mathscr{N}$ by

$$F_k(\mathscr{M} \boxtimes \mathscr{N}) = \sum_{i+j=k} F_i(\mathscr{M}) \boxtimes F_j(\mathscr{N}).$$

Then we have $\mathrm{gr}(\mathscr{M} \boxtimes \mathscr{N}) = \mathrm{gr}(\mathscr{M}) \boxtimes \mathrm{gr}(\mathscr{N})$, hence

$$\widetilde{\mathrm{gr}(\mathscr{M} \boxtimes \mathscr{N})} = \widetilde{\mathrm{gr}(\mathscr{M})} \boxtimes \widetilde{\mathrm{gr}(\mathscr{N})}.$$

This proves the assertion, as we have remarked. □

1.2.3 Direct image of D-modules

1.2.3.1 The transfer module $\mathscr{D}_{Y \leftarrow X}$ Suppose that $v(y) = \int u(x, y)dx$ makes sense for a function $u(x, y)$, let us consider how to derive differential equations for $v(y)$ from those for $u(x, y)$. Supposing, in addtion, that Stokes's theorem $\int \frac{\partial u(x,y)}{\partial x_i} dx = 0$ holds, we obtain

$$\int \frac{\partial}{\partial x_i} S(x, y, \partial_x, \partial_y) u(x, y) = 0$$

for all differential operators $S(x, y, \partial_x, \partial_y)$, so for

$$Q(y, \partial_y) = \sum_i \partial_{x_i} S_i(x, y, \partial_x, \partial_y) + P(x, y, \partial_x, \partial_y),$$

we have

$$Q(y,\partial_y)v(y) = \int P(x,y,\partial_x,\partial_y)u(x,y)dx.$$

Furthermore, if $P(x,y,\partial_x,\partial_y)u = 0$, then $Q(y,\partial_y)v(y) = 0$.

We now describe the above consideration in the languate of D-modules. Let X be a smooth algebraic variety with a coordinate system (x,y) and Y be a submanifold with a coordinate system y. Then the above consideration means that we can associate a $\mathscr{D}_X$-module $\mathscr{M}$ with a $\mathscr{D}_Y$-module

$$\mathscr{M}/(\sum_i \partial_{x_i}\mathscr{M}) = (\mathscr{D}_X/\sum_i \partial_{x_i}\mathscr{D}_X) \otimes_{\mathscr{D}_X} \mathscr{M}.$$

This can be generalized to an arbitrary morphism, which is the subject of this paragraph.

Let $f : X \to Y$ be a morphisms of smooth algebraic varieties. The right $\mathscr{D}_Y$-module structure of $\mathscr{D}_Y$ gives via side-changing a left $\mathscr{D}_Y$-module structure on $\mathscr{D}_Y \otimes_{\mathscr{O}_Y} \Omega_Y^{\otimes -1}$, whose inverse image by f is the left $\mathscr{D}_X$-module

$$f^{-1}(\mathscr{D}_Y \otimes_{\mathscr{O}_Y} \Omega_Y^{\otimes -1}) = f^{-1}(\mathscr{D}_Y \otimes_{\mathscr{O}_Y} \Omega_Y^{\otimes -1}) \otimes_{f^{-1}(\mathscr{O}_Y)} \mathscr{O}_X.$$

By side-changing again, we then obtain a right $\mathscr{D}_X$-module

$$\begin{aligned} f^{-1}(\mathscr{D}_Y \otimes_{\mathscr{O}_Y} \Omega_Y^{\otimes -1}) \otimes_{f^{-1}(\mathscr{O}_Y)} \Omega_X &= f^{-1}(\mathscr{D}_Y) \otimes_{f^{-1}(\mathscr{O}_Y)} f^{-1}(\Omega_Y^{\otimes -1}) \otimes_{f^{-1}(\mathscr{O}_Y)} \Omega_X \\ &\cong f^{-1}(\mathscr{D}_Y) \otimes_{f^{-1}(\mathscr{O}_Y)} \Omega_{X/Y} \end{aligned}$$

where we write $\Omega_{X/Y} = \Omega_X \otimes_{f^{-1}(\mathscr{O}_Y)} f^{-1}(\Omega_Y^{\otimes -1})$. Since the right and left action of $\mathscr{D}_Y$ commute, we see that $f^{-1}(\mathscr{D}_Y) \otimes_{f^{-1}(\mathscr{O}_Y)} \Omega_{X/Y}$ is an $(f^{-1}(\mathscr{D}_Y), \mathscr{D}_X)$-bimodule, which is denoted by $\mathscr{D}_{Y\leftarrow X}$.

Recall that we have defined a $(\mathscr{D}_X, f^{-1}(\mathscr{D}_Y))$-bimodule $\mathscr{D}_{X\to Y} = \mathscr{O}_X \otimes_{f^{-1}(\mathscr{O}_Y)} f^{-1}(\mathscr{D}_Y)$. Since the category of left $\mathscr{D}$-modules and that of right $\mathscr{D}_X$-modules are equivalent, we can switch $\mathscr{D}_{X\to Y}$ to a $(f^{-1}(\mathscr{D}_Y), \mathscr{D}_X)$-bimodule, which is explicitly given by

$$\begin{aligned} \Omega_X \otimes_{\mathscr{O}_X} \mathscr{D}_{X\to Y} \otimes_{f^{-1}(\mathscr{O}_Y)} f^{-1}(\Omega_Y^{\otimes -1}) &= \Omega_X \otimes_{f^{-1}(\mathscr{O}_Y)} f^{-1}(\mathscr{D}_Y) \otimes_{f^{-1}(\mathscr{O}_Y)} f^{-1}(\Omega_Y^{\otimes -1}) \\ &\cong f^{-1}(\mathscr{D}_Y) \otimes_{f^{-1}(\mathscr{O}_Y)} \Omega_{X/Y} \cong \mathscr{D}_{X\leftarrow Y}. \end{aligned} \tag{1.2.14}$$

In other words, the bimodule $\mathscr{D}_{Y\leftarrow X}$ is obtained by side-changing from $\mathscr{D}_{X\to Y}$.

Let us explicitly describe the action of $\mathscr{D}_X$ on $\mathscr{D}_{Y\leftarrow X}$. The action of $\mathscr{O}_X \subseteq \mathscr{D}_X$ is obtained from the $\mathscr{O}_X$-module structure of $\Omega_{X/Y}$, and the action of $\Theta_X \subseteq \mathscr{D}_X$ is given as follows. Let $\tilde{v} = \sum a_i \otimes w_i$ be the image of $v \in \Theta_X$ under the canonical homomorphism $\Theta_X \to f^*(\Theta_Y) = \mathscr{O}_X \otimes_{f^{-1}(\mathscr{O}_Y)} f^{-1}(\Theta_X)$. Then for $\omega \in \Omega_X$, a generator θ of Ω_Y, and $P \in \mathscr{D}_Y$, we have

$$(P \otimes (\omega/\theta))v = \sum_i Pw_i \otimes a_i\omega/\theta - P \otimes \mathfrak{Lie}_v(\omega)/\theta + \sum_i P \otimes a_i(\mathfrak{Lie}_{w_i}(\theta)/\theta)\omega/\theta. \tag{1.2.15}$$

Taking a coordinate system $\{x_i, \partial_{x_i}\}$ of X and a coordinate system $\{y_j, \partial_{y_j}\}$ of Y, we then have

$$(P \otimes a(x)\, dx/dy)\partial_{x_i} = \sum_j P\partial_{y_j} \otimes \frac{\partial f_j(x)}{\partial x_i} a(x)\, dx/dy - P \otimes \frac{\partial a(x)}{\partial x_i} dx/dy \tag{1.2.16}$$

where $dx = dx_1 \wedge \cdots \wedge dx_n$ and $dy = dy_1 \wedge \cdots dy_m$.

Example 1.2.3. Let $i : X \to Y$ be a closed immersion of smooth algebraic varieties. As in the situation of Example 1.2.2, we take a local coordinate $y = (y_1, \ldots, y_n)$ and let $x_i = y_i \circ i$ for $i = 1, \ldots, r$. Then we have

$$\mathscr{D}_{Y\leftarrow X} = i^{-1}(\mathscr{D}_Y) \otimes_{i^{-1}(\mathscr{O}_Y)} i^{-1}(\Omega_Y^{\otimes -1}) \otimes_{i^{-1}(\mathscr{O}_Y)} \Omega_X.$$

We can locally identify $i^{-1}(\Omega_Y^{\otimes -1}) \otimes_{i^{-1}(\mathscr{O}_Y)} \Omega_X$ with $\mathscr{O}_X$ via the section dx/dy, so if we set

$$\mathscr{D} = \bigoplus_{\alpha_1,\dots,\alpha_n} \mathscr{O}_Y \partial_{y_1}^{\alpha_1} \cdots \partial_{y_n}^{\alpha_n} \subseteq \mathscr{D}_Y,$$

then $\mathscr{D}_Y \cong \mathbb{C}[\partial_{y_{r+1}}, \dots, \partial_{y_n}] \otimes_{\mathbb{C}} \mathscr{D}$ as a right $\mathscr{D}$-module and we have

$$\mathscr{D}_{Y\leftarrow X} \cong \mathbb{C}[\partial_{y_{n+1}}, \dots, \partial_{y_n}] \otimes_{\mathbb{C}} i^{-1}(\mathscr{D}) \otimes_{i^{-1}(\mathscr{O}_Y)} \mathscr{O}_X. \tag{1.2.17}$$

From (1.2.16), we see that the right $\mathscr{D}_X$-action of the right-hand side of (1.2.17) is induced by the right $\mathscr{D}_X$-action on $i^{-1}(\mathscr{D}) \otimes_{i^{-1}(\mathscr{O}_Y)} \mathscr{O}_Y$ given by

$$(P \otimes 1)\partial_{x_i} = (P\partial_{y_i}) \otimes 1, \quad (P \otimes 1)a = P \otimes a,$$

where $P \in \mathscr{D}$ and $a \in \mathscr{O}_X$. We therefore conclude that $i^{-1}(\mathscr{D}) \otimes_{i^{-1}(\mathscr{O}_Y)} \mathscr{O}_X \cong \mathscr{D}_X$, so we obtain a local isomorphism

$$\mathscr{D}_{Y\leftarrow X} \cong \mathbb{C}[\partial_{y_{r+1}}, \dots, \partial_{y_n}] \otimes_{\mathbb{C}} \mathscr{D}_X.$$

In particular, $\mathscr{D}_{X\to Y}$ and $\mathscr{D}_{Y\leftarrow X}$ are isomorphic as $\mathscr{O}_X$-modules.

Proposition 1.2.10. *Let $f : X \to Y$ be a smooth morphism of relative dimension n. Then*

(a) *The right $\mathscr{D}_X$-module $\mathscr{D}_{Y\leftarrow X}$ is coherent.*

(b) *We have an exact sequence*

$$0 \longrightarrow \Omega^0_{X/Y} \otimes_{\mathscr{O}_X} \mathscr{D}_X \longrightarrow \cdots \longrightarrow \Omega^d_{X/Y} \otimes_{\mathscr{O}_X} \mathscr{D}_X \longrightarrow \mathscr{D}_{Y\leftarrow X} \longrightarrow 0$$

(c) *We have canonical isomorphisms*

$$R\mathcal{H}om_{\mathscr{D}_X}(\mathscr{D}_{X\to Y}, \mathscr{D}_X) \cong \mathscr{D}_{Y\leftarrow X}[-n] \in D^b(f^{-1}(\mathscr{D}_Y) \otimes \mathscr{D}_X^{\mathrm{op}}),$$
$$R\mathcal{H}om_{\mathscr{D}_X}(\mathscr{D}_{Y\leftarrow X}, \mathscr{D}_X) \cong \mathscr{D}_{X\to Y}[-n] \in D^b(\mathscr{D}_X \otimes f^{-1}(\mathscr{D}_Y)).$$

Proof. The first assertion follows from Proposition 1.2.5 and the isomorphism (1.2.14), and the second assertion is proved just like Proposition 1.2.2, by replacing $\Theta_{X/Y}$ with $\Omega_{X/Y}$. Now since $\Omega_{X/Y}$ is locally free, we see from (b) that $R\mathcal{H}om_{\mathscr{D}_X}(\mathscr{D}_{X\to Y}, \mathscr{D}_X)$ is quasi-isomorphic to the complex

$$0 \longrightarrow \mathscr{D}_X \otimes_{\mathscr{O}_X} \Theta^n_{X/Y} \longrightarrow \cdots \longrightarrow \mathscr{D}_X \otimes_{\mathscr{O}_X} \Theta^n_{X/Y} \longrightarrow 0$$

which is quasi-isomorphic to $\mathscr{D}_{X\to Y}[-n]$ by Proposition 1.2.2. The second statement of (c) can be proved similarly. □

1.2.3.2 The case for a closed immersion $i : X \to Y$

Proposition 1.2.11. *Let $i : X \to Y$ be a closed immersion and $d = \mathrm{codim}_Y(X)$. Then*

(a) *$\mathscr{D}_{Y\leftarrow X}$ and $\mathscr{D}_{X\to Y}$ are coherent over $i^{-1}(\mathscr{D}_Y)$.*

(b) *For any $\mathscr{M} \in D^+(\mathscr{D}_Y)$, we have a canonical isomorphism*

$$R\mathcal{H}om_{i^{-1}(\mathscr{D}_Y)}(\mathscr{D}_{Y\leftarrow X}, i^{-1}(\mathscr{M})) \cong Li^*(\mathscr{M})[-d] \in D^b(\mathscr{D}_X).$$

Proof. The first assertion is clear from the general property of i^{-1}. To see (b), it suffices to show the isomorphism

$$R\mathcal{H}om_{i^{-1}(\mathscr{D}_Y)}(\mathscr{D}_{Y\leftarrow X}, i^{-1}(\mathscr{D}_Y)) \cong \mathscr{D}_{X\to Y}[-d]. \tag{1.2.18}$$

Indeed, from (1.2.18) we conclude that

$$\begin{aligned} Li^*(\mathscr{M}) &= \mathscr{D}_{X\to Y} \otimes^L_{i^{-1}(\mathscr{D}_Y)} i^{-1}(\mathscr{M}) \cong R\mathcal{H}om_{i^{-1}(\mathscr{D}_Y)}(\mathscr{D}_{Y\leftarrow X}) \otimes^L_{i^{-1}(\mathscr{D}_Y)} i^{-1}(\mathscr{M})[d] \\ &\cong R\mathcal{H}om_{i^{-1}(\mathscr{D}_Y)}(\mathscr{D}_{Y\leftarrow X}, i^{-1}(\mathscr{M}))[d]. \end{aligned}$$

We also note that (1.2.18) is equivalent to

$$R\mathcal{H}om_{i^{-1}(\mathscr{D}_Y^{\mathrm{op}})}(\mathscr{D}_{X\to Y}, i^{-1}(\mathscr{D}_Y)) \cong \mathscr{D}_{Y\leftarrow X}[-d] \tag{1.2.19}$$

by side changing operation. To this end, we have

$$\begin{aligned} R\mathcal{H}om_{i^{-1}(\mathscr{D}_Y^{\mathrm{op}})}(\mathscr{D}_{X\to Y}, i^{-1}(\mathscr{D}_Y)) &\cong R\mathcal{H}om_{i^{-1}(\mathscr{D}_Y^{\mathrm{op}})}(\mathscr{O}_X \otimes_{i^{-1}(\mathscr{O}_Y)} i^{-1}(\mathscr{D}_Y), i^{-1}(\mathscr{D}_Y)) \\ &\cong R\mathcal{H}om_{i^{-1}(\mathscr{O}_Y)}(\mathscr{O}_X, i^{-1}(\mathscr{D}_Y)) \\ &\cong i^{-1}(\mathscr{D}_Y) \otimes_{i^{-1}(\mathscr{O}_Y)} R\mathcal{H}om_{i^{-1}(\mathscr{O}_Y)}(\mathscr{O}_X, i^{-1}(\mathscr{O}_Y)). \end{aligned}$$

By using the Koszul resolution (1.2.13), we see that $R\mathcal{H}om_{i^{-1}(\mathscr{O}_Y)}(\mathscr{O}_X, i^{-1}(\mathscr{O}_Y))$ is represented by the complex $\check{\mathscr{K}}_0 \to \cdots \to \check{\mathscr{K}}_d$, where $\check{\mathscr{K}}_j = \mathcal{H}om_{i^{-1}(\mathscr{O}_Y)}(\mathscr{K}_j, i^{-1}(\mathscr{O}_Y))$. note that $\mathscr{K}_d$ is a locally free $i^{-1}(\mathscr{O}_Y)$-module of rank one and we have a canonical prefect pairing $\mathscr{K}_j \otimes_{i^{-1}(\mathscr{O}_Y)} \mathscr{K}_{d-j} \to \mathscr{K}_d$ for each j. We then conclude that $\check{\mathscr{K}}_j \cong \mathscr{K}_{d-j} \otimes_{i^{-1}(\mathscr{O}_Y)} \check{\mathscr{K}}_d$, so

$$\begin{aligned} R\mathcal{H}om_{i^{-1}(\mathscr{O}_Y)}(\mathscr{O}_X, i^{-1}(\mathscr{O}_Y)) &\cong [\mathscr{K}_d \to \cdots \to \mathscr{K}_0] \otimes_{i^{-1}(\mathscr{O}_Y)} \check{\mathscr{K}}_d \\ &= \mathscr{O}_X \otimes_{i^{-1}(\mathscr{O}_Y)} \check{\mathscr{K}}_d[-d] \cong i^{-1}(\Omega_Y^{\otimes -1}) \otimes_{i^{-1}(\mathscr{O}_Y)} \Omega_X[-d]. \end{aligned}$$

Therefore, we have

$$R\mathcal{H}om_{i^{-1}(\mathscr{O}_Y)}(\mathscr{O}_X, i^{-1}(\mathscr{O}_Y)) \cong i^{-1}(\mathscr{D}_Y) \otimes_{i^{-1}(\mathscr{O}_Y)} i^{-1}(\Omega_Y^{\otimes -1}) \otimes_{i^{-1}(\mathscr{O}_Y)} \Omega_X[-d] \cong \mathscr{D}_{Y\leftarrow X}[-d]$$

in view of the definition of $\mathscr{D}_{Y\leftarrow X}$. □

Inspired by Proposition 1.2.11, for a closed immersion $i : X \to Y$ of smooth algebraic varieties we define a left exact functor

$$i^! : \mathbf{Mod}(\mathscr{D}_Y) \to \mathbf{Mod}(\mathscr{D}_X), \quad \mathscr{M} \mapsto \mathcal{H}om_{i^{-1}(\mathscr{D}_Y)}(\mathscr{D}_{Y\leftarrow X}, i^{-1}(\mathscr{M})).$$

It turns out that this is the "right" definition of the extrodinary inverse image functor for D-modules. On the other hand, we also note that following result.

Proposition 1.2.12. *Let $i : X \to Y$ be a closed immersion with $d = \mathrm{codim}_Y(X)$. Then for $\mathscr{M} \in D^+(\mathscr{D}_Y)$ we have*

$$Ri^!(\mathscr{M}) \cong R\mathcal{H}om_{i^{-1}(\mathscr{D}_Y)}(\mathscr{D}_{Y\leftarrow X}, i^{-1}(\mathscr{M})) \cong Li^*(\mathscr{M})[d].$$

Proof. The second isomorphism is proved in Proposition 1.2.11. To see the first one, we first show that

$$i^!(\mathscr{M}) \cong \mathcal{H}om_{i^{-1}(\mathscr{D}_Y)}(\mathscr{D}_{Y\leftarrow X}, i^{-1}(\Gamma_X(\mathscr{M}))), \quad \mathscr{M} \in \mathbf{Mod}(\mathscr{D}_Y) \tag{1.2.20}$$

where $\Gamma_X(\mathscr{M})$ denotes the subsheaf of $\mathscr{M}$ consisting of sections supported in X. To this end, it suffices to show that $\psi(s) \in i^{-1}(\Gamma_X(\mathscr{M}))$ for any $\psi \in \mathcal{H}om_{i^{-1}(\mathscr{D}_Y)}(\mathscr{D}_{Y\leftarrow X}, i^{-1}(\mathscr{M}))$ and $s \in \mathscr{D}_{Y\leftarrow X}$. Since this question is local, we may take a local coordinate as in Example 1.2.2. Then we have

$$\mathscr{D}_{Y\leftarrow X} \cong \mathbb{C}[\partial_{y_{r+1}}, \ldots, \partial_{y_n}] \otimes_{\mathbb{C}} \mathscr{D}_X.$$

Since the $i^{-1}(\mathscr{D}_Y)$-module $\mathbb{C}[\partial_{y_{r+1}}, \dots, \partial_{y_n}] \otimes_{\mathbb{C}} \mathscr{D}_X$ is generated by $1 \otimes 1$, we may assume that $s = 1 \otimes 1$. Let $\mathscr{J} \subseteq \mathscr{O}_Y$ be the defining ideal of X. By $i^{-1}(\mathscr{J})s = 0$, we have $i^{-1}(\mathscr{J})\psi(s) = 0$, so $\psi(s) \in i^{-1}(\Gamma_X(\mathscr{M}))$ and we get (1.2.20).

We next show that

$$Ri^!(\mathscr{M}) = R\mathcal{H}om_{i^{-1}(\mathscr{D}_Y)}(\mathscr{D}_{Y\leftarrow X}, i^{-1}R\Gamma_X(\mathscr{M})), \quad \mathscr{M} \in D^+(\mathscr{D}_Y). \tag{1.2.21}$$

For this, it is sufficient to show that if $\mathscr{I}$ is an injective $\mathscr{D}_Y$-module, then $i^{-1}(\Gamma_X(\mathscr{I}))$ is an injective $i^{-1}(\mathscr{D}_Y)$-module. But this follows from

$$\begin{aligned}
\mathrm{Hom}_{i^{-1}(\mathscr{D}_Y)}(\mathscr{K}, i^{-1}(\Gamma_X(\mathscr{I}))) &\cong \mathrm{Hom}_{i^{-1}(\mathscr{D}_Y)}(i^{-1}i_*(\mathscr{K}), i^{-1}\Gamma_X(\mathscr{I})) \\
&\cong \mathrm{Hom}_{\mathscr{D}_Y}(i_*(\mathscr{K}), i_*i^{-1}\Gamma_X(\mathscr{I})) \\
&\cong \mathrm{Hom}_{\mathscr{D}_Y}(i_*(\mathscr{K}), \Gamma_X(\mathscr{I})) \cong \mathrm{Hom}_{\mathscr{D}_Y}(i_*(\mathscr{K}), \mathscr{I})
\end{aligned}$$

for any $i^{-1}(\mathscr{D}_Y)$-module $\mathscr{K}$.

It remains to show that the canonical morphism

$$R\mathcal{H}om_{i^{-1}(\mathscr{D}_Y)}(\mathscr{D}_{Y\leftarrow X}, i^{-1}R\Gamma_X(\mathscr{M})) \to R\mathcal{H}om_{i^{-1}(\mathscr{D}_Y)}(i^{-1}(\mathscr{D}_{Y\leftarrow X}), i^{-1}(\mathscr{M}))$$

is an isomorphism. Let $j : Y \setminus X \to Y$ be the complementary open immersion. By the distinguished triangle

$$R\Gamma_X(\mathscr{M}) \longrightarrow \mathscr{M} \longrightarrow Rj_*j^{-1}(\mathscr{M}) \xrightarrow{+1}$$

it then suffices to show that $R\mathcal{H}om_{i^{-1}(\mathscr{D}_Y)}(\mathscr{D}_{Y\leftarrow X}, i^{-1}Rj_*j^{-1}(\mathscr{M})) = 0$, which is equivalent the the assertion

$$R\mathcal{H}om_{i^{-1}(\mathscr{D}_Y)}(\mathscr{D}_{X\to Y}, i^{-1}Rj_*j^{-1}(\mathscr{M})) = 0 \quad \text{for} \quad \mathscr{M} \in D^+(\mathscr{D}_Y^{\mathrm{op}})$$

by side-changing. On the other hand, note that

$$\begin{aligned}
R\mathcal{H}om_{i^{-1}(\mathscr{D}_Y^{\mathrm{op}})}(\mathscr{D}_{X\to Y}, i^{-1}Rj_*j^{-1}(\mathscr{M})) &= R\mathcal{H}om_{i^{-1}(\mathscr{D}_Y^{\mathrm{op}})}(\mathscr{O}_X \otimes_{i^{-1}(\mathscr{O}_Y)} i^{-1}(\mathscr{D}_Y), i^{-1}Rj_*j^{-1}(\mathscr{M})) \\
&\cong R\mathcal{H}om_{i^{-1}(\mathscr{O}_Y)}(\mathscr{O}_X, i^{-1}R_j * j^{-1}(\mathscr{M})) \\
&\cong \mathscr{O}_X \otimes^L_{i^{-1}(\mathscr{O}_Y)} i^{-1}Rj_*j^{-1}(\mathscr{M}),
\end{aligned}$$

so the claim follows from Lemma 1.2.13. $\square$

Lemma 1.2.13. *Let $i : X \to Y$ be a closed immersion of algebraic varieties. Set $U = Y \setminus X$ and denote by $j : U \to X$ the complementary open immersion. Then for any $\mathscr{K} \in D^b(\mathscr{O}_U)$, we have $\mathscr{O}_X \otimes^L_{i^{-1}(\mathscr{O}_Y)} i^{-1}Rj_*(\mathscr{K}) = 0$.*

Proof. For any $\mathscr{K} \in D^b(\mathscr{O}_U)$, we have

$$i_*(\mathscr{O}_X \otimes^L_{i^{-1}(\mathscr{O}_Y)} i^{-1}Rj_*(\mathscr{K})) = i_*(\mathscr{O}_X) \otimes^L_{\mathscr{O}_Y} Rj_*(\mathscr{K}) = Rj_*(j^{-1}i_*(\mathscr{O}_X) \otimes^L_{\mathscr{O}_U} \mathscr{K}) = 0,$$

where we have used the projection formula and the fact that $j^{-1}i_* = 0$. Since i_* is fully faithful, this completes the proof. $\square$

Remark 1.2.2. In view of Proposition 1.2.12, it is therefore quite convenient to use the shifted functor $Lf^*[d_{Y/X}]$ which in the case of closed immersions coincides with $Ri^!$, we thus define

$$f^! = Lf^*[d_{Y/X}] : D^b(\mathscr{D}_X) \to D^b(\mathscr{D}_Y)$$

for an arbitrary morphism $f : X \to Y$. As we have remarked, this definition turns out to give the "right" extrodinary inverse image functor for D-modules.

1.2.3.3 The derived direct image of D-modules Let $f : X \to Y$ be a morphism of smooth algebraic varieties and $\mathscr{M}$ be a $\mathscr{D}_X$-module. If we define its integral (or direct image) to be the $\mathscr{D}_Y$-module $f_*(\mathscr{D}_{Y\leftarrow X} \otimes_{\mathscr{D}_X} \mathscr{M})$, it would equal to

$$f_*(\operatorname{coker}(\Omega_{X/Y}^{d_{X/Y}-1}) \otimes_{\mathscr{O}_X} \mathscr{M} \to \Omega_{X/Y}^{d_{X/Y}-1} \otimes_{\mathscr{O}_X} \mathscr{M})$$

for f smooth. However, this operation is better behaved if we consider the setting of derived categories. By taking a flat resolution of $\mathscr{M}$ and an injective resolution of $\mathscr{N}$, we therefore define a functor

$$f_+ : D^b(\mathscr{D}_X) \to D^b(\mathscr{D}_Y), \quad \mathscr{M} \mapsto Rf_*(\mathscr{D}_{Y\leftarrow X} \otimes^L_{\mathscr{D}_{\mathscr{X}}} \mathscr{M})$$

called the **integral of $\mathscr{M}$ along fibers of f**. Note that we also have a functor $f_+ : D^b(\mathscr{D}_X^{\mathrm{op}}) \to D^b(\mathscr{D}_Y^{\mathrm{op}})$ given by

$$f_+(\mathscr{M}) = Rf_*(\mathscr{M} \otimes^L_{\mathscr{D}_X} \mathscr{D}_{X\to Y}), \quad \mathscr{M} \in D^b(\mathscr{D}_X^{\mathrm{op}}).$$

In view of (1.2.14), it is not hard to see that we have a commutative diagram

$$\begin{tikzcd} D^b(\mathscr{D}_X) \arrow[r, "f_+"] \arrow[d, "\Omega_X \otimes_{\mathscr{O}_X}(-)"'] & D^b(\mathscr{D}_Y) \arrow[d, "\Omega_Y \otimes_{\mathscr{O}_Y}(-)"] \\ D^b(\mathscr{D}_X^{\mathrm{op}}) \arrow[r] & D^b(\mathscr{D}_Y^{\mathrm{op}}) \end{tikzcd}$$

Proposition 1.2.14. *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of smooth algebraic varieties. Then we have $(g \circ f)_+ = g_+ \circ f_+$.*

Proof. Similar to the proof of Proposition 1.2.3, we have isomorphisms

$$\mathscr{D}_{Z\leftarrow X} \cong f^{-1}(\mathscr{D}_{Z\leftarrow Y}) \otimes_{f^{-1}(\mathscr{D}_Y)} \mathscr{D}_{Y\leftarrow X} \cong f^{-1}(\mathscr{D}_{Z\leftarrow Y}) \otimes^L_{f^{-1}(\mathscr{D}_Y)} \mathscr{D}_{Y\leftarrow X}$$

of complexes of $((g \circ f)^{-1}(\mathscr{D}_Z), \mathscr{D}_X)$-modules. For $\mathscr{M} \in D^b(\mathscr{D}_X)$, by definition we have

$$g_+ f_+(\mathscr{M}) = Rg_*(\mathscr{D}_{Z\leftarrow X} \otimes^L_{\mathscr{D}_Y} Rf_*(\mathscr{D}_{Y\leftarrow X} \otimes^L_{\mathscr{D}_X} \mathscr{M})).$$

We now claim that the canonical morphism

$$\mathscr{D}_{Z\leftarrow X} \otimes^L_{\mathscr{D}_Y} Rf_*(\mathscr{D}_{Y\leftarrow X} \otimes^L_{\mathscr{D}_X} \mathscr{M}) \to Rf_*(f^{-1}(\mathscr{D}_{Z\leftarrow X}) \otimes^L_{f^{-1}(\mathscr{D}_Y)} (\mathscr{D}_{Y\leftarrow X} \otimes^L_{\mathscr{D}_X} \mathscr{M}))$$

is an isomorphism. In fact, we have a canonical isomorphism

$$\mathscr{F} \otimes^L_{\mathscr{D}_Y} Rf_*(\mathscr{G}) \cong Rf_*(f^{-1}(\mathscr{F}) \otimes^L_{f^{-1}(\mathscr{D}_Y)} \mathscr{G})$$

for $\mathscr{F} \in D^-_{\mathrm{qcoh}}(\mathscr{D}_Y^{\mathrm{op}})$, $\mathscr{G} \in D^b(f^{-1}(\mathscr{D}_Y))$. To see this, we may assume that Y is affine, so that we can then replace $\mathscr{F}$ by a complex of free right $\mathscr{D}_Y$-modules belonging to $D^-_{\mathrm{qcoh}}(\mathscr{D}_Y^{\mathrm{op}})$. In this case, it suffices to prove our claim for $\mathscr{F} = \mathscr{D}_Y^{\oplus I}$. Now we have

$$\mathscr{F} \otimes^L_{\mathscr{D}_Y} Rf_*(\mathscr{G}) \cong Rf_*(\mathscr{G})^{\oplus I}, \quad Rf_*(f^{-1}(\mathscr{F}) \otimes^L_{f^{-1}(\mathscr{D}_Y)} \mathscr{G}) \cong Rf_*(\mathscr{G}^{\oplus I})$$

so the claim follows from the fact that Rf_* commutes with direct sums. With these, we finally conclude that

$$\begin{aligned} g_+ f_+(\mathscr{M}) &\cong Rg_* Rf_*(f^{-1}(\mathscr{D}_{Z\leftarrow Y}) \otimes^L_{f^{-1}(\mathscr{D}_Y)} (\mathscr{D}_{Y\leftarrow X} \otimes^L_{\mathscr{D}_X} \mathscr{M})) \\ &\cong R(g \circ f)_*((f^{-1}(\mathscr{D}_{Z\leftarrow Y}) \otimes^L_{f^{-1}(\mathscr{D}_Y)} \mathscr{D}_{Y\leftarrow X}) \otimes^L_{\mathscr{D}_X} \mathscr{M}) \\ &\cong R(g \circ f)_*(\mathscr{D}_{Z\leftarrow X} \otimes^L_{\mathscr{D}_X} \mathscr{M}) \cong (g \circ f)_*(\mathscr{M}) \end{aligned}$$

and this completes the proof. $\square$

Example 1.2.4. Let $j : U \to X$ be an open immersion of algebraic varieties. Then we have $\mathscr{D}_{X\leftarrow U} = j^{-1}(\mathscr{D}_X) = \mathscr{D}_U$, so $j_+ = Rj_*$.

Example 1.2.5. Let $i : X \to Y$ be a closed immersion of smooth algebraic varieties. Take a local coordinate $\{y_i, \partial_{y_i}\}$ as in Example 1.2.3, for $\mathscr{M} \in \mathbf{Mod}(\mathscr{D}_X)$, we then have $H^p(i_+(\mathscr{M})) = 0$ for $p \neq 0$, and

$$H^0(i_+(\mathscr{M})) \cong \mathbb{C}[\partial_{y_{r+1}}, \dots, \partial_{y_n}] \otimes_{\mathbb{C}} i_*(\mathscr{M}) \in \mathbf{Mod}(\mathscr{D}_Y).$$

Proposition 1.2.15. *Let $i : X \to Y$ be a closed immersion of smooth algebraic varieties.*

(a) *The restriction $i_+ : \mathbf{Mod}(\mathscr{D}_X) \to \mathbf{Mod}(\mathscr{D}_Y)$ is an exact functor.*

(b) *The functor i_+ sends $\mathbf{Qcoh}(\mathscr{D}_X)$ to $\mathbf{Qcoh}(\mathscr{D}_Y)$.*

Proof. This follows from the explicit local description given in Example 1.2.5. □

Proposition 1.2.16. *Let $i : X \to Y$ be a closed immersion of smooth algebraic varieties.*

(a) *There exists a functorial isomorphism*

$$R\mathcal{H}om_{\mathscr{D}_Y}(i_+(\mathscr{M}), \mathscr{N}) \cong i_* R\mathcal{H}om_{\mathscr{D}_X}(\mathscr{M}, i^!(\mathscr{N})),$$

where $\mathscr{M} \in D^-(\mathscr{D}_X)$, $\mathscr{N} \in D^+(\mathscr{D}_Y)$.

(b) *The functor $i^! : D^b(\mathscr{D}_Y) \to D^b(\mathscr{D}_X)$ is right adjoint to $i_+ : D^b(\mathscr{D}_X) \to D^b(\mathscr{D}_Y)$.*

Proof. Since $H^0(R\mathrm{Hom}_{\mathscr{D}_Y}(\mathscr{K}, \mathscr{L})) \cong \mathrm{Hom}_{D^b(\mathscr{D}_Y)}(\mathscr{K}, \mathscr{L})$, the second statement follows from the first one by taking $H^0(R\Gamma(Y, -))$. We note that for $\mathscr{M} \in \mathbf{Mod}(\mathscr{D}_X)$ and $\mathscr{N} \in \mathbf{Mod}(\mathscr{D}_Y)$, there is a canonical isomorphism

$$\mathcal{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathcal{H}om_{i^{-1}(\mathscr{D}_Y)}(\mathscr{D}_{Y\leftarrow X}, i^{-1}(\mathscr{N}))) \cong \mathcal{H}om_{i^{-1}(\mathscr{D}_Y)}(\mathscr{D}_{Y\leftarrow X} \otimes_{\mathscr{D}_X} \mathscr{M}, i^{-1}(\mathscr{N}))$$

from which we obtain

$$R\mathcal{H}om_{\mathscr{D}_X}(\mathscr{M}, R\mathcal{H}om_{i^{-1}(\mathscr{D}_Y)}(\mathscr{D}_{Y\leftarrow X}, i^{-1}(\mathscr{N}))) \cong R\mathcal{H}om_{i^{-1}(\mathscr{D}_Y)}(\mathscr{D}_{Y\leftarrow X} \otimes^L_{\mathscr{D}_X} \mathscr{M}, i^{-1}(\mathscr{N}))$$

for $\mathscr{M} \in D^-(\mathscr{D}_X)$, $\mathscr{N} \in D^+(\mathscr{D}_Y)$. Therefore, we have

$$\begin{aligned}
R\mathcal{H}om_{\mathscr{D}_Y}(i_+(\mathscr{M}), \mathscr{N}) &\cong R\mathcal{H}om_{\mathscr{D}_Y}(i_*(\mathscr{D}_{Y\leftarrow X} \otimes^L_{\mathscr{D}_X} \mathscr{M}), \mathscr{N}) \\
&\cong R\mathcal{H}om_{\mathscr{D}_Y}(i_*(\mathscr{D}_{Y\leftarrow X} \otimes^L_{\mathscr{D}_X} \mathscr{M}), R\Gamma_X(\mathscr{N})) \\
&\cong R\mathcal{H}om_{\mathscr{D}_Y}(i_*(\mathscr{D}_{Y\leftarrow X} \otimes^L_{\mathscr{D}_X} \mathscr{M}), i_* i^{-1} R\Gamma_X(\mathscr{N})) \\
&\cong i_* R\mathcal{H}om_{\mathscr{D}_Y}(i^{-1} i_*(\mathscr{D}_{Y\leftarrow X} \otimes^L_{\mathscr{D}_X} \mathscr{M}), i^{-1} R\Gamma_X(\mathscr{N})) \\
&\cong i_* R\mathcal{H}om_{\mathscr{D}_Y}(\mathscr{D}_{Y\leftarrow X} \otimes^L_{\mathscr{D}_X} \mathscr{M}, i^{-1} R\Gamma_X(\mathscr{N})) \\
&\cong i_* R\mathcal{H}om_{\mathscr{D}_Y}(\mathscr{M}), R\mathcal{H}om_{i^{-1}(\mathscr{D}_Y)}(\mathscr{D}_{Y\leftarrow X}, i^{-1} R\Gamma_X(\mathscr{N})) \\
&\cong i_* R\mathcal{H}om_{\mathscr{D}_X}(\mathscr{M}, i^!(\mathscr{N})),
\end{aligned}$$

where we have used Proposition 1.2.12 in the last isomorphism and its proof. □

We now consider a special case: let Z be a smooth algebraic variety and assume that $f : X = Y \times Z \to Y$ is the canonical projection. To compute the derived tensor product $\mathscr{D}_{Y\leftarrow X} \otimes^L_{\mathscr{D}_X} \mathscr{M}$, we use the resolution of the right $\mathscr{D}_X$-module $\mathscr{D}_{Y\leftarrow X}$ given in Proposition 1.2.10. Let $n = \dim(Z)$; for $\mathscr{M} \in \mathbf{Qcoh}(\mathscr{D}_X)$, we define its **relative de Rham complex** $DR_{X/Y}(\mathscr{M})$ by

$$DR_{X/Y}(\mathscr{M})^i = \Omega^{n+i}_{X/Y} \otimes_{\mathscr{O}_X} \mathscr{M}.$$

In view of Remark 1.1.1, the differential of $DR_{X/Y}(\mathscr{M})$ is given by

$$d(\omega s) = d\omega \otimes s + \sum_{i=1}^{n} (dz_i \wedge \omega) \otimes \partial_i s,$$

where $\{z_i, \partial_i\}$ is a local coordinate of Z. Note that each term of $DR_{X/Y}(\mathscr{M})$ is an $f^{-1}(\mathscr{D}_Y)$-module given by

$$P(\omega \otimes s) = \omega \otimes ((P \otimes 1)s),$$

where we denote by $P \mapsto P \otimes 1$ the canonical homomorphism $f^{-1}(\mathscr{D}_Y) \to \mathscr{D}_X$. By Proposition 1.2.10, we have

$$\mathscr{D}_{Y \leftarrow X} \otimes^{L}_{\mathscr{D}_X} \mathscr{M} \cong DR_{X/Y}(\mathscr{M})$$

in the derived category of $f^{-1}(\mathscr{D}_Y)$-modules.

Proposition 1.2.17. *Let Y and Z be smooth algebraic varieties and $f : X = Y \times Z \to Y$ be the projection. Let $n = \dim(Z)$.*

(a) *For $\mathscr{M} \in \mathbf{Mod}(\mathscr{D}_X)$, we have $f_+(\mathscr{M}) \cong Rf_*(DR_{X/Y}(\mathscr{M})) \in D^{[-n,n]}(\mathscr{D}_Y)$.*

(b) *The functor f_+ sends $D^b_{\mathrm{qcoh}}(\mathscr{D}_X)$ to $D^b_{\mathrm{qcoh}}(\mathscr{D}_Y)$.*

Proof. The first assertion in (a) follows from the above consideration and the definition of f_+, and the second one follows from the fact that f_* has cohomological dimension n. In order to prove (b), it is sufficient to show that $R^i f_*(DR_{X/Y}(\mathscr{M}))$ is a quasi-coherent $\mathscr{O}_Y$-module for each i, which is true since $DR_{X/Y}(\mathscr{M})$ is a complex of quasi-coherent $\mathscr{O}_X$-modules. □

Corollary 1.2.18. *Let $f : X \to Y$ be a morphism of smooth algebraic varieties. Then f_+ sends $D^b_{\mathrm{qcoh}}(\mathscr{D}_X)$ to $D^b_{\mathrm{qcoh}}(\mathscr{D}_Y)$.*

Proof. Note that any morphism $f : X \to Y$ is a composition of a closed immersion $i : X \to X \times Y$ (the graph morphism) and a projection $X \times Y \to Y$. Therefore the corollary follows from Proposition 1.2.14, Proposition 1.2.15 and Proposition 1.2.17. □

To conclude our discussion, we prove a Künneth formula for the direct image functor. This can also be considered as a generalization of Fubini's theorem on integrations. Before this, we establish the following lemma, which is a special case of Künneth formula:

Lemma 1.2.19. *Let $f : X \to Y$ be a morphism of algebraic varieties and S be an algebraic variety. For $\mathscr{M} \in D^b_{\mathrm{qcoh}}(\mathscr{O}_X)$ and $\mathscr{N} \in D^b_{\mathrm{qcoh}}(\mathscr{O}_S)$, the canonical homomorphism*

$$Rf_*(\mathscr{M}) \boxtimes \mathscr{N} \to R(f \times \mathrm{id}_S)_*(\mathscr{M} \boxtimes \mathscr{N}).$$

Proof. Since the question is local, we may assume that S is affine. Then there exists an isomorphism $\mathscr{F} \cong \mathscr{N}$ such that $\mathscr{F}^i$ is a projective $\mathscr{O}_S$-module for each i and $\mathscr{F}^i = 0$ for $|i| \gg 0$. Hence we may assume from the beginning that $\mathscr{N} = \mathscr{O}_S$. Consider the cartesian square

$$\begin{array}{ccc} X \times S & \xrightarrow{p} & X \\ {\scriptstyle f \times \mathrm{id}_S}\downarrow & & \downarrow{\scriptstyle f} \\ Y \times S & \xrightarrow{q} & Y \end{array}$$

where p, q are projections. By the base change theorem, we then have

$$Rf_*(\mathscr{M}) \boxtimes \mathscr{O}_S \cong q^* Rf_*(\mathscr{M}) \cong R(f \times \mathrm{id}_S)_* p^*(\mathscr{M}) \cong R(f \times \mathrm{id}_S)_*(\mathscr{M} \boxtimes \mathscr{O}_T).$$ □

Proposition 1.2.20. *Let $f : X \to Y$ and $f' : X' \to Y'$ be morphisms of smooth algebraic varieties. Then for $\mathscr{M} \in D^b_{\mathrm{qcoh}}(\mathscr{D}_X)$ and $\mathscr{M}' \in D^b_{\mathrm{qcoh}}(\mathscr{D}_{X'})$, we have a canonical isomorphism*

$$f_+(\mathscr{M}) \boxtimes f'_+(\mathscr{M}') \cong (f \times f')_+(\mathscr{M} \boxtimes \mathscr{M}').$$

Proof. By decomposing $f \times f'$ into the composite of $X \times X' \to Y \times X' \to Y \times Y'$, it is sufficient to show that for a morphism $f : X \to Y$ of smooth algebraic varieties and a smooth algebraic variety S the canonical morphism

$$f_+(\mathscr{M}) \boxtimes \mathscr{N} \to (f \times \mathrm{id}_S)_+(\mathscr{M} \boxtimes \mathscr{N})$$

is an isomorphism, where $\mathscr{M} \in D^b_{\mathrm{qcoh}}(\mathscr{D}_X)$ and $\mathscr{N} \in D^b_{\mathrm{qcoh}}(\mathscr{S})$. By decomposing f into the composition of $\Gamma_f : X \to X \times Y$ and the projection $X \times Y \to Y$, we may assume that f is either a closed immersion or a projection. Moreover, we may assume that $\mathscr{M} \in \mathbf{Qcoh}(\mathscr{D}_X)$ and $\mathscr{N} \in \mathbf{Qcoh}(\mathscr{D}_Y)$.

Assume that $i : X \to Y$ is a closed immersion. Since the question is local, we may take a local coordinate $\{y_i, \partial_{y_i}\}$ of Y so that $y_{r+1}, \cdots, y_n$ give the defining equations of X. Then by Example 1.2.3, we have

$$\begin{aligned} i_+(\mathscr{M}) \boxtimes \mathscr{N} &\cong (\mathbb{C}[\partial_{y_{r+1}}, \ldots, \partial_{y_n}] \otimes_{\mathbb{C}} i_*(\mathscr{M})) \boxtimes \mathscr{N} \\ &\cong \mathbb{C}[\partial_{y_{r+1}}, \ldots, \partial_{y_n}] \otimes_{\mathbb{C}} (i \times 1)_*(\mathscr{M} \boxtimes \mathscr{N}) \\ &\cong (i \times \mathrm{id}_S)(\mathscr{M} \boxtimes \mathscr{N}). \end{aligned}$$

On the other hand, if $f : X \to Y$ is a projection, then we have seen that

$$\begin{aligned} f_+(\mathscr{M}) \boxtimes \mathscr{N} &\cong Rf_*(DR_{X/Y}(\mathscr{M})) \boxtimes \mathscr{N}, \\ (f \times \mathrm{id}_S)_*(\mathscr{M} \boxtimes \mathscr{N}) &\cong R(f \times \mathrm{id}_S)_*(DR_{X\times S/Y\times S}(\mathscr{M} \boxtimes \mathscr{N})). \end{aligned}$$

Now, since $DR_{X/Y}(\mathscr{M})$ is a complex of quasi-coherent $\mathscr{O}_X$-modules, by Lemma 1.2.19 we have

$$\begin{aligned} Rf_*(DR_{X/Y}(\mathscr{M})) \boxtimes \mathscr{N} &\cong R(f \times \mathrm{id}_S)_*(DR_{X/Y}(\mathscr{M})) \boxtimes \mathscr{N} \\ &\cong R(f \times \mathrm{id}_S)_*(DR_{X\times S/Y\times S}(\mathscr{M} \boxtimes \mathscr{N})) \end{aligned}$$

and therefore

$$Rf_*(DR_{X/Y}(\mathscr{M})) \boxtimes \mathscr{N} \cong R(f \times \mathrm{id}_S)_*(DR_{X\times S/Y\times S}(\mathscr{M} \boxtimes \mathscr{N})).$$

This completes the proof. □

1.2.4 Kashiwara's equivalence

We have seen in Proposition 1.2.15 that for a closed immersion $i : X \to Y$, the direct image functor $i_+ : \mathbf{Qcoh}(\mathscr{D}_X) \to \mathbf{Qcoh}(\mathscr{D}_Y)$ is an exact functor. In this case, the image of a $\mathscr{D}_X$-module under i_+ is a $\mathscr{D}_Y$-module supported in X. Let us denote by $\mathbf{Qcoh}^X(\mathscr{D}_Y)$ (resp. $\mathbf{Coh}^X(\mathscr{D}_Y)$) the full subcategory of $\mathbf{Qcoh}(\mathscr{D}_Y)$ (resp. $\mathbf{Coh}(\mathscr{D}_Y)$) consisting of $\mathscr{D}_Y$-modules that are supported in X. We then have the following characterization of the essential image of i_+, which plays a fundamental role in various studies of D-modules.

Theorem 1.2.21 (Kashiwara's equivalence). *Let $i : X \to Y$ be a closed immersion of smooth algebraic varieties. Then the functor i_+ induces equivalences*

$$\mathbf{Qcoh}(\mathscr{D}_X) \xrightarrow{\sim} \mathbf{Qcoh}^X(\mathscr{D}_Y), \quad \mathbf{Coh}(\mathscr{D}_X) \xrightarrow{\sim} \mathbf{Coh}^X(\mathscr{D}_Y)$$

of abelian categories, whose quasi-inverse is given by $i^!$. Moreover, for any $\mathscr{N} \in \mathbf{Qcoh}^X(\mathscr{D}_Y)$, we have $H^p(i^!(\mathscr{N})) = 0$ for $p \neq 0$.

Similar to the case of abelian categories, we denote by $D^{b,X}_{\mathrm{qcoh}}(\mathscr{D}_Y)$ (resp. $D^{b,X}_{\mathrm{coh}}(\mathscr{D}_Y)$) the full subcategory of $D^b_{\mathrm{qcoh}}(\mathscr{D}_Y)$ (resp. $D^b_{\mathrm{coh}}(\mathscr{D}_Y)$) consisting of complexes $\mathscr{N}$ whose coholomogy class $H^*(\mathscr{N})$ are supported in X.

Corollary 1.2.22. *For $\mathcal{N} = \mathbf{Coh}$ or $\mathbf{Qcoh}$, the functor*

$$i_+ : D^b_{\mathcal{N}}(\mathscr{D}_X) \to D^{b,X}_{\mathcal{N}}(\mathscr{D}_Y)$$

gives an equivalence of triangulated categories, whose quasi-inverse is given by $i^!$.

Example 1.2.6. We consider the $\mathscr{D}_Y$-module

$$\mathscr{B}_{X|Y} = i_+(\mathscr{O}_X) \in \mathbf{Qcoh}^X(\mathscr{D}_Y).$$

If $\{y_i, \partial_i\}$ is a local coordinate system of Y such that $y_{r+1}, \ldots, y_n$ give the defining equations of X, then by Example 1.2.5 we have

$$\mathscr{B}_{X|Y} = \mathbb{C}[\partial_{y_{r+1}}, \ldots, \partial_{y_n}] \otimes_{\mathbb{C}} i_*(\mathscr{O}_X) = \mathscr{D}_Y / \Big(\sum_{i=1}^{r} \mathscr{D}_Y \partial_i + \sum_{j=r+1}^{n} \mathscr{D}_Y y_j \Big).$$

In particular, for $X = \{x\}$ where $x \in Y$, we get

$$\mathscr{B}_{\{x\}|Y} = \mathscr{D}_Y / \mathscr{D}_Y \mathfrak{m}_x = \mathscr{D}_Y \delta_x \cong \mathbb{C}[\partial_1, \ldots, \partial_n]\delta_x,$$

where $\mathfrak{m}_x = (y_1, \ldots, y_n)$ is the maximal ideal at x and $\delta_x \equiv 1 \bmod \mathfrak{m}_x \in \mathscr{B}_{\{x\}|Y}$. Here we use the notation δ_x since the corresponding system $y_j u = 0$ $(1 \leq j \leq n)$ of differential equations is the one satisfied by the Dirac delta function supported at $\{x\}$. By Kashiwara's equivalence, we have the correspondence

$$\mathbf{Qcoh}^{\{x\}}(\mathscr{D}_Y) \cong \{\text{the category of } \mathbb{C}\text{-vector spaces}\},$$

so objects of $\mathbf{Qcoh}^{\{x\}}(\mathscr{D}_Y)$ are direct sums of $\mathscr{B}_{\{x\}|Y}$.

We now give an application of Kashiwara's equivalence theorem.

Theorem 1.2.23. *A product of a projective space and a smooth affine variety is D-affine.*

Theorem 1.2.24. *Let $f : X \to Y$ be a proper morphism. Then for any $\mathscr{M} \in D^b_{\mathrm{coh}}(\mathscr{D}_X)$, the direct image $f_+(\mathscr{M})$ belongs to $D^b_{\mathrm{coh}}(\mathscr{D}_Y)$.*

1.2.5 Base change theorem for direct images

Let X be a topological space, Z a closed subset, and $U = X \setminus Z$ the complementary open subset of X. We denote by $i : Z \to X$ and $j : U \to X$ the immersions. Then by **??**, for an injective sheaf $\mathscr{F}$ on X we get an exact sequence

$$0 \longrightarrow \Gamma_Z(\mathscr{F}) \longrightarrow \mathscr{F} \longrightarrow j_* j^{-1}(\mathscr{F}) \longrightarrow 0$$

where $\Gamma_Z(\mathscr{F})$ is the sheaf of sections of $\mathscr{F}$ supported in Z. For any $\mathscr{F} \in D^b(\mathbb{C}_X)$, we then obtain a distinguished triangle

$$R\Gamma_Z(\mathscr{F}) \longrightarrow \mathscr{F} \longrightarrow Rj_* j^{-1}(\mathscr{F}) \xrightarrow{+1}$$

Considering this distinguished triangle in the case where X is a smooth algebraic variety and $\mathscr{F} \in D^b(\mathscr{D}_X)$, we obtain the following result.

Proposition 1.2.25. *Let X be a smooth algebraic variety and Z be a closed smooth subvariety of X. Set $U = X \setminus Z$ and denote by $i : Z \to X$ and $j : U \to X$ the immersions.*

(a) *For $\mathscr{M} \in D^b_{\mathrm{qcoh}}(\mathscr{D}_X)$, we have a canonical distinguished triangle*

$$R\Gamma_Z(\mathscr{M}) \longrightarrow \mathscr{M} \longrightarrow j_+j^!(\mathscr{M}) \xrightarrow{+1}$$

(b) *We have $i^!j_+(\mathscr{N}) = 0$ for $\mathscr{N} \in D^b_{\mathrm{qcoh}}(\mathscr{D}_U)$ and $R\Gamma_Z(\mathscr{M}) = i_+i^!(\mathscr{M})$ for $\mathscr{M} \in D^b_{\mathrm{qcoh}}(\mathscr{D}_X)$.*

Proof. For an open immersion $j : U \to X$, we have $j^! = j^{-1}$ and $j_+ = Rj_*$, whence assertion (a). The isomorphism $i^!j_+ = 0$ follows from Lemma 1.2.13. To see that $R\Gamma_Z = i_+i^!$ holds on $D^b_{\mathrm{qcoh}}(\mathscr{D}_X)$, we note that since $j_+j^!(\mathscr{M}) \in D^b_{\mathrm{qcoh}}(\mathscr{D}_X)$, we have $R\Gamma_Z(\mathscr{M}) \in D^{b,Z}_{\mathrm{qcoh}}(\mathscr{D}_X)$, and therefore $R\Gamma_Z(\mathscr{M}) \cong i_+i^!R\Gamma_Z(\mathscr{M})$ by Corollary 1.2.22. It then suffices to show that $i^!R\Gamma_Z(\mathscr{M}) \cong i^!(\mathscr{M})$. For this, we can apply $i^!$ to the distinguished triangle in (a), and note that $i^!j_+ = 0$ by the first assertion of (b). □

Theorem 1.2.26 (Base Change Theorem). *Let $f : X \to Y$ and $g : Y' \to Y$ be morphisms of smooth algebraic varieties and consider the fiber product*

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "g'"'] & Y' \arrow[d, "g"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

Then there is a canonical isomorphism of functors

$$g^!f_+ \cong f'_+g'^! : D^b_{\mathrm{qcoh}}(\mathscr{D}_X) \to D^b_{\mathrm{qcoh}}(\mathscr{D}_{Y'}).$$

Corollary 1.2.27 (Projection Formula). *Let $f : X \to Y$ be a morphism of smooth varieties. Then for $\mathscr{M} \in D^b_{\mathrm{qcoh}}(\mathscr{D}_X)$ and $\mathscr{N} \in D^b_{\mathrm{qcoh}}(\mathscr{D}_Y)$, we have a canonical isomorphism*

$$f_+(\mathscr{M} \otimes^L_{\mathscr{O}_X} Lf^*(\mathscr{N})) \cong f_+(\mathscr{M}) \otimes^L_{\mathscr{O}_Y} \mathscr{N}.$$

Proof. Applying Theorem 1.2.26 to the Cartesian square

$$\begin{tikzcd} X \arrow[r, "\Gamma_f"] \arrow[d, "f"'] & X \times Y \arrow[d, "f \times \mathrm{id}_Y"] \\ Y \arrow[r, "\Delta_Y"] & Y \times Y \end{tikzcd}$$

we obtain

$$f_+(\mathscr{M} \otimes^L_{\mathscr{O}_X} Lf^*(\mathscr{N})) \cong f_+(L\Gamma_f^*(\mathscr{M} \boxtimes \mathscr{N})) \cong L\Delta_Y^*((f \times \mathrm{id}_Y)_+(\mathscr{M} \boxtimes \mathscr{N})) \cong f_+(\mathscr{M}) \otimes^L_{\mathscr{O}_Y} \mathscr{N}. \quad \square$$

1.2.6 Inverse images in the non-characteristic case

We have shown in Proposition 1.2.5 that the inverse image of a coherent D-module with respect to a smooth morphism is again coherent. However, this is not true in general for non-smooth morphisms, as we saw in Remark 1.2.1. We now give a sufficient conidtion on the coherence of the inverse image of a coherent D-module, which is more general than smoothness.

For a morphism $f : X \to Y$ of smooth algebraic varieties, we have a canonical morphism

$$\rho_f : X \times_Y T^*Y \to T^*X$$

which is the morphism gathering all homomorphisms $df_x^* : T^*_{f(x)}Y \to T^*_x X$ dual to $df_x : T_x X \to T_{f(x)}Y$. Let $\varpi_f : X \times_Y T^*Y \to T^*Y$ denote the canonical projection, then we have

$$\varphi_f^*(\omega_Y) = \rho_f^*(\omega_X)$$

where ω_X and ω_Y denote the canonical 1-forms on T^*X and T^*Y, respectively. Consider the diagram

$$\begin{tikzcd} X \arrow[d, "f"'] & X \times_Y T^*Y \arrow[l] \arrow[r, "\rho_f"] \arrow[d, "\varpi_f"'] & T^*X \\ Y & T^*Y \arrow[l] & \end{tikzcd}$$

Let $T^*_X X$ be the zero section of X and

$$T^*_X Y := \rho_f^{-1}(T^*_X X) \subseteq X \times_Y T^*Y.$$

If we choose a local cooredinate for X and Y, then $T^*_X Y$ can be written locally as

$$T^*_X Y = \{(x, y, \xi) \in X \times_Y T^*Y : y = f(x) \text{ and } \xi \in \ker(df_x^*)\}.$$

In particular, if f is a closed immersion, then $T^*_X Y$ is the conormal bundle of X in Y.

Lemma 1.2.28.

Let $f : X \to Y$ be a morphism of smooth algebraic varieties and $\mathscr{N}$ be a coherent $\mathscr{D}_Y$-module. We say that f is **non-characteristic with respect to $\mathscr{M}$** if

$$\varpi_f^{-1}(\mathrm{Ch}(\mathscr{N})) \cap T^*_X Y \subseteq X \times_Y T^*_Y Y.$$

This definition is motivated by the theory of linear partial differential equations, as we see below.

Example 1.2.7.

Example 1.2.8. A smooth morphism $f : X \to Y$ is non-characteristic with respect to any coherent $\mathscr{D}_Y$-module. In fact, in this case the associated homomorphism $df_x^* : T^*_{f(x)}Y \to T^*_x X$ is injective for any $x \in X$, so $T^*_X Y$ is contaiend in $X \times_Y T^*_Y Y$.

Theorem 1.2.29. *Let $f : X \to Y$ be a morphism of smooth algebraic varieties and $\mathscr{N}$ be a coherent $\mathscr{D}_Y$-module such that f is non-characteristic with respect to $\mathscr{N}$.*

(a) $H^i(Lf^*(\mathscr{N})) = 0$ *for* $i \neq 0$.

(b) $H^0(Lf^*(\mathscr{N}))$ *is a coherent* $\mathscr{D}_X$*-module.*

(c) $\mathrm{Ch}(H^0(Lf^*(\mathscr{M}))) = \rho_f \varpi_f^{-1}(\mathrm{Ch}(\mathscr{M}))$.

Remark 1.2.3. Note that by the assumption of being non-characteristic, the morphism

$$\rho_f : \varpi_f^{-1}(\mathrm{Ch}(\mathscr{N})) \to T^*X$$

is finite (i.e. it is closed and all fibers are finite). Therefore, $\rho_f \varpi_f^{-1}(\mathrm{Ch}(\mathscr{N}))$ is a closed algebraic subset of T^*X.

1.2.7 Relations with the duality functors

In this paragraph, we consider the relations between the duality functor and various functors on the derived category of D-modules. In particular, we shall prove that f_+ and $f^!$ commute with the duality functor D, and establish an adjunction formula for f_+.

Theorem 1.2.30. *Let $f : X \to Y$ be a morphism of smooth algebraic varieties, and $\mathscr{M}$ be a coherent $\mathscr{D}_Y$-module.*

(a) *Assume that $Lf^*(\mathscr{M}) \in D^b_{\mathrm{coh}}(\mathscr{D}_X)$, then there exists a canonical isomorphism*

$$D_X(Lf^*(\mathscr{M})) \to Lf^*(D_Y(\mathscr{M})).$$

(b) *If f is non-characteristic with respect to $\mathscr{M}$ (hence $Lf^*(\mathscr{M}) = f^*(\mathscr{M})$ is coherent by Theorem 1.2.29), then the morphism in (a) is an isomorphism.*

Proposition 1.2.31. *Let $f : X \to Y$ be a proper morphism of smooth algebraic varieties. Then we have a trace homomorphism*

$$\mathrm{tr}_f : f_+(\mathscr{O}_X)[d_X] \to \mathscr{O}_Y[d_Y].$$

Theorem 1.2.32. *Let $f : X \to Y$ be a proper morphism of smooth algebraic varieties. Then we have a canonical isomorphism of functors*

$$f_+ D_X \xrightarrow{\sim} D_Y f_+ : D^b_{\mathrm{coh}}(\mathscr{D}_X) \to D^b_{\mathrm{coh}}(\mathscr{D}_Y).$$

Corollary 1.2.33 (Adjunction formula). *Let $f : X \to Y$ be a proper morphism of smooth algebraic varieties. Then we have an isomorphism*

$$R\mathcal{H}om_{\mathscr{D}_Y}(f_+(\mathscr{M}), \mathscr{N}) \xrightarrow{\sim} Rf_* R\mathcal{H}om_{\mathscr{D}_X}(\mathscr{M}, f^!(\mathscr{N}))$$

for $\mathscr{M} \in D^b_{\mathrm{coh}}(\mathscr{D}_X)$ and $\mathscr{N} \in D^b(\mathscr{D}_Y)$.

Theorem 1.2.34. *Let $f : X \to Y$ be a smooth morphism of smooth algebraic varieties. Then we have an isomorphism*

$$R\mathcal{H}om_{\mathscr{D}_Y}(\mathscr{N}, f_+(\mathscr{M})) \xrightarrow{\sim} Rf_* R\mathcal{H}om_{\mathscr{D}_X}(f^!(\mathscr{N}), \mathscr{M})$$

for $\mathscr{M} \in D^b(\mathscr{D}_X)$ and $\mathscr{N} \in D^b_{\mathrm{coh}}(\mathscr{D}_Y)$.

1.3 Holonomic D-modules

In this section we study functorial behaviors of holonomic systems and show that any simple object in the abelian category of holonomic $\mathscr{D}_X$-modules is a minimal extension of an integrable connection on a locally closed smooth subvariety Y of X.

1.3.1 The category of holonomic D-modules

Recall that the dimension of the characteristic variety $\mathrm{Ch}(\mathscr{M})$ of a (nontrivial) coherent $\mathscr{D}_X$-module $\mathscr{M}$ satisfies the inequality $\dim(\mathrm{Ch}(\mathscr{M})) \geq \dim(X)$ and that a coherent $\mathscr{D}_X$-module $\mathscr{M}$ is called **holonomic** if $\dim(\mathrm{Ch}(\mathscr{M})) = \dim(X)$ or $\mathscr{M} = 0$. We denote by $\mathbf{Mod}_h(\mathscr{D}_X)$ the full subcategory of $\mathbf{Coh}(\mathscr{D}_X)$ consisting of holonomic $\mathscr{D}_X$-modules.

Proposition 1.3.1. *Let X be a smooth algebraic variety.*

(a) *For an exact sequence $0 \to \mathscr{M}' \to \mathscr{M} \to \mathscr{M}'' \to 0$ in $\mathbf{Coh}(\mathscr{D}_X)$, $\mathscr{M}$ is holonomic if and only if $\mathscr{M}', \mathscr{M}''$ are holonomic.*

(b) *Any holonomic $\mathscr{D}_X$-module has finite length. In other words, the category* $\mathbf{Mod}_h(\mathscr{D}_X)$ *is Artinian.*

Proof. The first statement follows from $\mathrm{Ch}(\mathscr{M}) = \mathrm{Ch}(\mathscr{M}') \cup \mathrm{Ch}(\mathscr{M}'')$, and the second one is proved using the characteristic cycle as follows. For a holonomic $\mathscr{D}_X$-module $\mathscr{M}$ consider its characteristic cycle

$$\mathrm{Cyc}(\mathscr{M}) = \sum_V \mathrm{mult}_V(\mathscr{M}) \cdot V.$$

Note that $\dim(V) = d_X$ for any $V \in \mathrm{Ch}(\mathscr{M})$. We define the total multiplicity of $\mathscr{M}$ to be

$$\mathrm{mult}(\mathscr{M}) = \sum_V m_V(\mathscr{M}).$$

Then by Proposition 1.1.31, the total multiplicity is addigive, and $\mathrm{mult}(\mathscr{M}) = 0$ if and only if $\mathrm{Ch}(\mathscr{M}) = \varnothing$, if and only if $\mathscr{M} = 0$. The second assertion then follows by induction on $\mathrm{mult}(\mathscr{M})$. □

We denote by $D^b_h(\mathscr{D}_X)$ the full subcategory of $D^b_{\mathrm{coh}}(\mathscr{D}_X)$ consisting of objects $\mathscr{M} \in D^b_{\mathrm{coh}}(\mathscr{D}_X)$ whose cohomology groups are holonomic. By Proposition 1.3.1, this is a full triangulated subcategory of $D^b_{\mathrm{coh}}(\mathscr{D}_X)$.

The following result is the first important step in the study of holonomic D-modules. It suggests that we can say that a holonomic $\mathscr{D}$-module is generically an integrable connection.

Proposition 1.3.2. *Let $\mathscr{M}$ be a holonomic $\mathscr{D}_X$-module. Then there exists an open dense subset $U \subseteq X$ such that $\mathscr{M}|_U$ is coherent over $\mathscr{O}_U$. In other words, $\mathscr{M}|_U$ is an integral connection on U.*

Proof. Let $T^*_X X$ be the zero section of X in T^*X and set $S = \mathrm{Ch}(\mathscr{M}) \setminus T^*_X X$. If $S = \varnothing$, then $\mathrm{Ch}(\mathscr{M})$ is contained in $T^*_X X$ and $\mathscr{M}$ is an integral connection by Theorem 1.1.21, so we may assume that $S \neq \varnothing$. Since S is conic, the dimension of each fiber of $\pi|_S : S \to \pi(S)$ is strictly positive and hence $\dim(\pi(S)) < \dim(S) \leq \dim(X)$. There then exists a nonempty open subset $U \subseteq X$ such that $U \subset X \setminus \pi(S)$, and we have $\mathrm{Ch}(\mathscr{M}|_U) \setminus T^*_U U = \varnothing$ in view of the local nature of the characteristic variety. We then conclude that $\mathscr{M}|_U$ is coherent over $\mathscr{O}_U$ by Theorem 1.1.21, whence the proposition. □

Recall that if $U \subseteq X$ is an open subset and $\mathscr{N}$ is a coherent submodule of $\mathscr{M}|_U$, then there exists a coherent submodule $\widetilde{\mathscr{N}}$ of $\mathscr{M}$ whose restriction on U is equal to $\mathscr{N}$ (cf. **??**). We now prove that this result is also true for holonomic modules, that is, we can assume that both $\mathscr{N}$ and $\widetilde{\mathscr{N}}$ are holonomic.

Proposition 1.3.3. *Let $\mathscr{M} \in \mathbf{Qcoh}(\mathscr{D}_X)$. For an open subset $U \subseteq X$, suppose that we are given a holonomic submodule $\mathscr{N}$ of $\mathscr{M}|_U$. Then there exists a holonomic submodule $\widetilde{\mathscr{N}}$ of $\mathscr{M}$ such that $\widetilde{\mathscr{N}}|_U = \mathscr{N}$.*

Proof. By **??** and **??**, we may assume that $\mathscr{M}$ is coherent and $\mathscr{M}|_U = \mathscr{N}$. Set $\mathscr{L} = H^0(D_X(\mathscr{M}))$. By Proposition 1.1.43 (b), we have $\mathrm{codim}(\mathrm{Ch}(\mathscr{L})) \geq d_X$, so $\mathscr{L}$ is a holonomic $\mathscr{D}_X$. Moreover, its dual module $\widetilde{\mathscr{N}} = D_X(\mathscr{L})$ is also holonomic by Proposition 1.1.43 (d). Since we have $\mathscr{L} = H^0(D_X(\mathscr{M})) \cong \tau^{\geq 0}(D_X(\mathscr{M}))$ by Proposition 1.1.43 (a), there is a distinguished triangle

$$\mathscr{K} \longrightarrow D_X(\mathscr{M}) \longrightarrow \mathscr{L} \xrightarrow{+1}$$

where $\mathscr{K} = \tau^{-\leq -1}(D_X(\mathscr{M}))$. By applying D_X, we obtain a distinguished triangle

$$\widetilde{\mathscr{N}} \longrightarrow \mathscr{M} \longrightarrow D_X(\mathscr{K}) \xrightarrow{+1}$$

Since the duality functor commutes with restrictions to open subsets, we conclude that

$$\widetilde{\mathscr{N}}|_U = D_U(\mathscr{L}|_U) = D_U^2(\mathscr{M}|_U) = \mathscr{M}|_U = \mathscr{N}.$$

It remains to show that the canonical morphism $\widetilde{\mathscr{N}} \to \mathscr{M}$ is injective, for which we will show that

$$H^i(D_X(\tau^{\geq -k}\mathscr{K})) = 0 \quad \text{for } \ i > 0, k > 0. \tag{1.3.1}$$

(Note that $\tau^{\geq -k}\mathscr{K} \cong \mathscr{K}$ for $k \gg 0$, in view of Proposition 1.1.43.) To this end, let us first prove that

$$H^i(D_X(H^{-k}(\mathscr{K})[k])) = H^{i-k}(D_X(H^{-k}(\mathscr{K}))) = 0 \quad \text{for } \ i < 0, k > 0. \tag{1.3.2}$$

For $k > 0$ we have $H^{-k}(\mathscr{K}) \cong H^{-k}(D_X(\mathscr{M}))$ and $\operatorname{codim}(\operatorname{Ch}(H^{-k}(\mathscr{K}))) \geq d_X - k$ by Proposition 1.1.43 (b), so (1.3.2) is a concequence of Proposition 1.1.43 (a). We now prove (1.3.1) by induction on k. If $k = 1$, then we have $\tau^{\geq -k}(\mathscr{K}) = H^{-k}(\mathscr{K})[k]$, and the assertion follows from (1.3.2). In the general case $k \geq 2$, by applying D_X to the distinguished triangle

$$H^{-k}(\mathscr{K})[k] \longrightarrow \tau^{\geq -k}\mathscr{K} \longrightarrow \tau^{\geq -(k-1)}\mathscr{K} \xrightarrow{+1}$$

we obtain a distinguished triangle

$$D_X(\tau^{\geq -(k-1)}\mathscr{K}) \longrightarrow D_X(\tau^{\geq -k}\mathscr{K}) \longrightarrow D_X(H^{-k}(\mathscr{K})[k]) \xrightarrow{+1}$$

so the assertion follows from (1.3.2) and the induction hypothesis. □

1.3.2 Functors for holonomic D-modules

We now consider the bahavaior of holonomic D-modules under various functors. We first note the following result, which is a concequence of Proposition 1.1.43.

Proposition 1.3.4. *The duality functor D_X induces isomorphisms*

$$\mathbf{Mod}_h(\mathscr{D}_X) \xrightarrow{\sim} \mathbf{Mod}_h(\mathscr{D}_X^{\mathrm{op}}), \quad D_h^b(\mathscr{D}_X) \xrightarrow{\sim} D_h^b(\mathscr{D}_X^{\mathrm{op}}).$$

Let X and Y be smooth algebraic varieties. Since $\operatorname{Ch}(\mathscr{M} \boxtimes \mathscr{N}) = \operatorname{Ch}(\mathscr{M}) \times \operatorname{Ch}(\mathscr{N})$ for $\mathscr{M} \in \mathbf{Coh}(\mathscr{D}_X)$ and $\mathscr{N} \in \mathbf{Coh}(\mathscr{D}_Y)$, the following is also immediate:

Proposition 1.3.5. *The external tensor product $\boxtimes$ induces functors*

$$(-) \boxtimes (-) : \mathbf{Mod}_h(\mathscr{D}_X) \times \mathbf{Mod}_h(\mathscr{D}_Y) \to \mathbf{Mod}_h(\mathscr{D}_{X\times Y}),$$
$$(-) \boxtimes (-) : D_h^b(\mathscr{D}_X) \times D_h^b(\mathscr{D}_Y) \to D_h^b(\mathscr{D}_{X\times Y}).$$

Now recall that for a morphism $f : X \to Y$ of smooth algebraic varieties, we have defined functors

$$f_+ : D_{\mathrm{qcoh}}^b(\mathscr{D}_X) \to D_{\mathrm{qcoh}}^b(\mathscr{D}_Y), \quad f^! : D_{\mathrm{qcoh}}^b(\mathscr{D}_Y) \to D_{\mathrm{qcoh}}^b(\mathscr{D}_X).$$

Moreover, if f is proper (resp. smooth), the functor f_+ (resp. $f^!$) preserves the coherency (cf. Theorem 1.2.24 and Proposition 1.2.5) and we obtain functors

$$f_+ : D_{\mathrm{coh}}^b(\mathscr{D}_X) \to D_{\mathrm{coh}}^b(\mathscr{D}_Y), \quad (\text{resp. } f^! : D_c^b(\mathscr{D}_Y) \to D_c^b(\mathscr{D}_X)).$$

However, neither f_+ nor $f^!$ preserves the coherency for general morphisms f. A surprising fact, which we will show in this paragraph, is that the holonomicity is nevertheless preserved by these functors for any morphism $f : X \to Y$. In other words, the coherence for f_+ and $f^!$ is guaranteed if we restrict ourselves to holonomic D-modules.

Theorem 1.3.6. *Let $f : X \to Y$ be a morphism of smooth algebraic varieties. Then f_+ sends $D_h^b(\mathscr{D}_X)$ to $D_h^b(\mathscr{D}_Y)$ and $f^!$ sends $D_h^b(\mathscr{D}_Y)$ to $D_h^b(\mathscr{D}_X)$.*

Corollary 1.3.7. *The internal tensor product $\otimes^L_{\mathscr{O}_X}$ induces a functor*

$$(-) \otimes^L_{\mathscr{O}_X} (-) : D_h^b(\mathscr{D}_X) \times D_h^b(\mathscr{D}_X) \to D_h^b(\mathscr{D}_X).$$

Proof. This follows from Proposition 1.3.5 and Theorem 1.3.6, since $(-) \otimes^L_{\mathscr{O}_X} (-) = L\Delta_X^*((-) \boxtimes (-))$, where Δ_X is the diagonal morphism. □

The proof of Theorem 1.3.6 is rather involved, so we need some reduction. We first note that taking direct image under a closed immersion preserves holonomic D-modules.

Lemma 1.3.8. *Let $i : X \to Y$ be a closed immersion. Then for $\mathscr{M} \in D^b_{\mathrm{coh}}(\mathscr{D}_X)$, we have $\mathscr{M} \in D_h^b(\mathscr{D}_X)$ if and only if $i_+(\mathscr{M}) \in D_h^b(\mathscr{D}_Y)$.*

Proof. Since i_+ is exact (cf. Proposition 1.2.15), we may assume that $\mathscr{M} \in \mathbf{Coh}(\mathscr{D}_X)$. Let

$$T^*Y \xleftarrow{\varpi} X \times_Y T^*Y \xrightarrow{\rho} T^*X$$

be the canonical morphisms. Then we have $\mathrm{Ch}(i_+(\mathscr{M})) = \varpi\rho^{-1}(\mathrm{Ch}(\mathscr{M}))$ by ([?], Lemma 2.3.5). Since ϖ is a closed immersion and ρ is a smooth surjective morphisms with d-dimension fibers (d is the codimension of X in Y), we conclude that

$$\dim(\mathrm{Ch}(i_+(\mathscr{M}))) = \dim(\mathrm{Ch}(\mathscr{M})) + d$$

from which the claim follows. □

1.3.3 Adjunction formulas

Let $f : X \to Y$ be a morphism of smooth algebraic varieties. Recall that we have defined functors f_+ and $f^!$ on the derived category of D-modules. We now define new functors using the duality functor D:

$$\begin{aligned} f_! &:= D_Y f_+ D_X : D_h^b(\mathscr{D}_X) \to D_h^b(\mathscr{D}_Y), \\ f^+ &:= D_X f^! D_Y : D_h^b(\mathscr{D}_Y) \to D_h^b(\mathscr{D}_X). \end{aligned}$$

We now justify our notations by showing that $(f_!, f^!)$ and (f^+, f_+) are adjoint pairs.

Theorem 1.3.9. *For $\mathscr{M} \in D_h^b(\mathscr{D}_X)$ and $\mathscr{N} \in D_h^b(\mathscr{D}_Y)$, we have canonical isomorphisms*

$$\begin{aligned} R\mathcal{H}om_{\mathscr{D}_Y}(f_!(\mathscr{M}), \mathscr{N}) &\xrightarrow{\sim} Rf_* R\mathcal{H}om_{\mathscr{D}_X}(\mathscr{M}, f^!(\mathscr{N})), \\ Rf_* R\mathcal{H}om_{\mathscr{D}_X}(f^+(\mathscr{N}), \mathscr{M}) &\xrightarrow{\sim} R\mathcal{H}om_{\mathscr{D}_Y}(\mathscr{N}, f_+(\mathscr{M})). \end{aligned}$$

Proof. We first note that by (1.2.14) and the projection formula for $\mathscr{O}_X$-modules, for $\mathscr{E} \in D^b(\mathscr{D}_X)$ we have

$$\begin{aligned} Rf_*((\Omega_X \otimes^L_{\mathscr{O}_X} \mathscr{E}) \otimes^L_{\mathscr{D}_X} \mathscr{D}_{X\to Y}) &\cong Rf_*(\mathscr{E} \otimes^L_{\mathscr{D}_X} (\Omega_X \otimes^L_{\mathscr{O}_X} \mathscr{D}_{X\to Y})) \\ &\cong Rf_*(\mathscr{E} \otimes^L_{\mathscr{D}_X} \mathscr{D}_{Y\leftarrow X} \otimes^L_{f^{-1}(\mathscr{O}_Y)} f^{-1}(\Omega_Y)) \\ &\cong Rf_*(\mathscr{E} \otimes^L_{\mathscr{D}_X} \mathscr{D}_{Y\leftarrow X}) \otimes^L_{\mathscr{O}_Y} \Omega_Y, \end{aligned}$$

so by (1.1.22), we have

$$Rf_* R\mathcal{H}om_{\mathscr{D}_X}(\mathscr{M}, f^!(\mathscr{N})) \cong Rf_*((\Omega_X \otimes^L_{\mathscr{O}_X} D_X(\mathscr{M})) \otimes^L_{\mathscr{D}_X} f^!(\mathscr{N}))[-d_X]$$

$$
\begin{aligned}
&\cong Rf_*((\Omega_X \otimes^L_{\mathscr{O}_X} D_X(\mathscr{M})) \otimes^L_{\mathscr{D}_X} Lf^*(\mathscr{N}))[-d_Y] \\
&\cong Rf_*((\Omega_X \otimes^L_{\mathscr{O}_X} D_X(\mathscr{M})) \otimes^L_{\mathscr{D}_X} \mathscr{D}_{X\to Y} \otimes^L_{f^{-1}(\mathscr{D}_Y)} f^{-1}(\mathscr{N}))[-d_Y] \\
&\cong Rf_*((\Omega_X \otimes^L_{\mathscr{O}_X} D_X(\mathscr{M})) \otimes^L_{\mathscr{D}_X} \mathscr{D}_{X\to Y}) \otimes^L_{\mathscr{D}_Y} \mathscr{N}[-d_Y] \\
&\cong (Rf_*(D_X(\mathscr{M}) \otimes^L_{\mathscr{D}_X} \mathscr{D}_{Y\leftarrow X}) \otimes^L_{\mathscr{O}_X} \Omega_Y) \otimes^L_{\mathscr{D}_Y} \mathscr{N}[-d_Y] \\
&\cong R\mathcal{H}om_{\mathscr{D}_Y}(f_+(D_X(\mathscr{M})), \mathscr{N}).
\end{aligned}
$$

This proves the first assertion, and the second one follows by taking duality. □

Corollary 1.3.10. *For $\mathscr{M} \in D^b_h(\mathscr{D}_X)$ and $\mathscr{N} \in D^b_h(\mathscr{D}_Y)$, we have canonical isomorphisms*

$$
\begin{aligned}
\mathrm{Hom}_{\mathscr{D}_Y}(f_!(\mathscr{M}), \mathscr{N}) &\xrightarrow{\sim} \mathrm{Hom}_{\mathscr{D}_X}(\mathscr{M}, f^!(\mathscr{N})), \\
\mathrm{Hom}_{\mathscr{D}_X}(f^+(\mathscr{N}), \mathscr{M}) &\xrightarrow{\sim} \mathrm{Hom}_{\mathscr{D}_Y}(\mathscr{N}, f_+(\mathscr{M})).
\end{aligned}
$$

Proof. This follows from Theorem 1.3.9 by applying $H^0R\Gamma(Y, -)$. □

Theorem 1.3.11. *There is a morphism of functors*

$$
f_! \to f_+ : D^b_h(\mathscr{D}_X) \to D^b_h(\mathscr{D}_Y).
$$

Moreover, if f is proper, then this morphism is an isomorphism.

Proof. By Hironaka's desingularization theorem ([**?**]), there exists a smooth completion $\widetilde{X}$ of X. Since X is quasi-projective, a desingularization $\widetilde{X}$ of the Zariski closure $\bar{X}$ of X in the projective space is such a completion (even if X is not quasi-projective, there exists a smooth completion by a theorem due to Nagata). Therefore, the map $f : X \to Y$ factorizes as

$$
X \xhookrightarrow{g} X \times Y \xhookrightarrow{j} \widetilde{X} \times Y \xrightarrow{p} Y
$$

where g is the graph morphism of f and p is the canonical projection. In this situation, g and p are proper and j is an open immersion, so we can reduce our problem to the cases of proper morphisms and open immersions. If f is proper, we have an isomorphism $f_! = D_Y f_+ D_X \cong f_+$ by Theorem 1.2.32, so let us consider the case where $f = j : X \to Y$ is an open immersion. Let $\mathscr{M} \in D^b_h(\mathscr{D}_X)$, then by Corollary 1.3.10 and Example 1.2.4, we have

$$
\begin{aligned}
\mathrm{Hom}_{D^b_h(\mathscr{D}_Y)}(j_!(\mathscr{M}), j_+(\mathscr{M})) \cong \mathrm{Hom}_{D^b_h(\mathscr{D}_X)}(\mathscr{M}, j^! j_+(\mathscr{M})) &\cong \mathrm{Hom}_{D^b_h(\mathscr{D}_X)}(\mathscr{M}, j^{-1}Rj_*(\mathscr{M})) \\
&\cong \mathrm{Hom}_{D^b_h(\mathscr{D}_X)}(\mathscr{M}, \mathscr{M}),
\end{aligned}
$$

from which we obtain the desired morphism $j_!(\mathscr{M}) \to j_+(\mathscr{M})$ as the image of the identity morphism on $\mathscr{M}$. □

1.3.4 Finiteness property

In this paragraph we prove the coherence of solution spaces for holonomic D-modules. For this, we need the following lemma.

Lemma 1.3.12. *Let $\mathscr{M}$ be a coherent $\mathscr{D}_X$-module. Then there exists an open dense subset $U \subseteq X$ such that $\mathscr{M}|_U$ is projective over $\mathscr{O}_U$.*

Proof. Take a good filtration of $\mathscr{M}$, then $\mathrm{gr}(\mathscr{M})$ is coherent over $\pi_*(\mathscr{O}_{T^*X})$, so there exists an open dense subset $U \subseteq X$ such that $\mathrm{gr}(\mathscr{M})|_U$ is free over $\pi_*(\mathscr{O}_{T^*U})$. By shrinking U is necessary, we may assume that $\mathrm{gr}(\mathscr{M})|_U$ is free over $\mathscr{O}_U$. This implies that each $F_i(\mathscr{M})/F_{i-1}(\mathscr{M})|_U$ (and hence each $F_i(\mathscr{M})|_U$) is projective over $\mathscr{O}_U$, so $\mathscr{M}|_U$ is projective over $\mathscr{O}_U$. □

Theorem 1.3.13. *The following conditions on $\mathscr{M} \in D^b_{\mathrm{coh}}(\mathscr{D}_X)$ are equivalent:*

(i) $\mathscr{M} \in D^b_h(\mathscr{D}_X)$.

(ii) *There exists a decreasing sequence*

$$X = X_0 \supset X_1 \supset \cdots \supset X_m \supset \varnothing$$

of closed subsets of X such that $X_k \setminus X_{k+1}$ is smooth and the cohomology sheaves $H^p(i^!_k(\mathscr{M}))$ are integrable connections, where $i_k : X_k \setminus X_{k+1} \to X$ is the canonical inclusion.

(iii) *For any $x \in X$, the cohomology groups $H^p(i^!_x(\mathscr{M}))$ are finite dimensional over $\mathbb{C}$, where $i_x : \{x\} \to X$ is the canonical inclusion.*

1.3.5 Minimal extensions

A non-zero coherent D-module $\mathscr{M}$ is called simple if it contains no coherent D-submodules other than $\mathscr{M}$ or 0. Proposition 1.3.1 implies that for any holonomic D-module $\mathscr{M}$ there exists a finite sequence

$$\mathscr{M} = \mathscr{M}_0 \supset \mathscr{M}_1 \supset \cdots \supset \mathscr{M}_n \supset 0$$

of holonomic D-submodules such that $\mathscr{M}_i/\mathscr{M}_{i+1}$ is simple for each i (Jordan-Hölder series of $\mathscr{M}$). In this paragrpah, we will give a classification of simple holonomic D-modules. More precisely, we will construct simple holonomic D-modules from integrable connections on locally closed smooth subvarieties using functors introduced in earlier sections, and show that any simple holonomic D-module is of this type. This construction corresponds via the Riemann-Hilbert correspondence to the minimal extension (Deligne-Goresky-MacPherson extension) in the category of perverse sheaves.

Let Y be a (locally closed) smooth subvariety of a smooth algebraic variety X. Assume that the inclusion map $i : Y \to X$ is affine. Then $\mathscr{D}_{X\leftarrow Y}$ is locally free over $\mathscr{D}_Y$ and $Ri_* = i_*$ (higher cohomology groups vanish). Therefore, for a holonomic $\mathscr{D}_Y$-module $\mathscr{M}$ we have $H^j(i_+(\mathscr{M})) = H^j(i_!(\mathscr{M})) = 0$ for $j \neq 0$. Namely, we may regard $i_+(\mathscr{M})$ and $i_!(\mathscr{M})$ as $\mathscr{D}_X$-modules. These $\mathscr{D}_X$-modules are holonomic by Theorem 1.3.6 and Proposition 1.3.4. By Theorem 1.3.11 we have a morphism $i_!(\mathscr{M}) \to i_+(\mathscr{M})$ in $\mathbf{Mod}_h(\mathscr{D}_X)$, whose image $L(Y, \mathscr{M})$ is called the minimal extension of $\mathscr{M}$. By Proposition 1.3.1, $L(Y, \mathscr{M})$ is a holonomic $\mathscr{D}_X$-module

Theorem 1.3.14. *Let X be a smooth algebraic variety.*

(a) *Let Y be a locally closed smooth connected subvariety of X such that $i : Y \to X$ is affine, and $\mathscr{M}$ be a simple holonomic $\mathscr{D}_Y$-module. Then the minimal extension $L(Y, \mathscr{M})$ is also simple, and it is characterized as the unique simple submodule (resp. unique simple quotient module) of $i_+(\mathscr{M})$ (resp. of $i_!(\mathscr{M})$)*

(b) *Any simple holonomic $\mathscr{D}_X$-module is isomorphic to the minimal extension $L(Y, \mathscr{M})$ for some pair $(Y, \mathscr{M})$, where Y is as in (a) and $\mathscr{M}$ is a simple integrable connection on Y.*

(c) *Let $(Y, \mathscr{M})$ and $(Y', \mathscr{M}')$ be pairs as in (a). Then $L(Y, \mathscr{M}) \cong L(Y', \mathscr{M}')$ if and only if $\bar{Y} = \bar{Y}'$ and $\mathscr{M}|_U \cong \mathscr{M}'|_U$ for an open dense subset U of $Y \cap Y'$.*

Proposition 1.3.15. *Let Y be a locally closed smooth subvariety of X such that $i : Y \to X$ is affine, and let $\mathscr{M}$ be an integrable connection on Y. Then we have*

$$D_X(L(Y, \mathscr{M})) \cong L(Y, D_Y(\mathscr{M})).$$

Proof. By the exactness of the duality functor we obtain

$$\begin{aligned} D_X(L(Y,\mathscr{M})) &\cong \operatorname{im}(D_X(i_+(\mathscr{M})) \to D_X(i_!(\mathscr{M}))) \cong \operatorname{im}(i_!(D_Y(\mathscr{M})) \to i_+(D_Y(\mathscr{M}))) \\ &= L(Y, D_Y(\mathscr{M})), \end{aligned}$$

which proves our claim. □

1.4 Analytic D-modules and the de Rham functor

1.4.1 Analytic D-modules

In the algebraic case holonomicity is preserved under the inverse and direct images; however, in our analytic situation this is true for inverse images but not for general direct images.

Theorem 1.4.1. *Let $f : X \to Y$ be a morphism of complex manifolds, and $\mathscr{M}$ be a holonomic $\mathscr{D}_Y$-module. Then we have $Lf^*(\mathscr{M}) \in D^b_h(\mathscr{D}_X)$.*

Theorem 1.4.2. *Let $f : X \to Y$ be a proper morphism of complex manifolds. Assume that a holonomic $\mathscr{D}_X$-module $\mathscr{M}$ admits a good filtration locally on Y. Then we have $f_+(\mathscr{M}) \in D^b_h(\mathscr{D}_Y)$.*

Theorem 1.4.1 is proved using the theory of b-functions (see Kashiwara [Kas7]), and Theorem 1.4.2 can be proved using $\mathrm{Ch}(f_+(\mathscr{M})) = \varpi_f \rho_f^{-1}(\mathrm{Ch}(\mathscr{M}))$ and some results from symplectic geometry. We note that in both theorems if we only consider the situation where f comes from a morphism of smooth algebraic varieties and $\mathscr{M}$ is associated to an algebraic holonomic D-module, then they are consequences of the corresponding facts on algebraic D-modules.

Example 1.4.1. Let us give an example so that the holonomicity is not preserved by the direct image with respect to a non-proper morphism of complex manifolds even if it comes from a morphism of smooth algebraic varieties. Set $X = \mathbb{C} \setminus \{0\}$, $Y = \mathbb{C}$ and let x be the canonical coordinate of $Y = \mathbb{C}$. Let $j : X \to Y$ be the embedding, which is regarded as a morphism of algebraic varieties. If we regard it as a morphism of complex manifolds, we denote it by $j^{\mathrm{an}} : X^{\mathrm{an}} \to Y^{\mathrm{an}}$. Then we have

$$H^0(j_+(\mathscr{O}_X)) = j_*(\mathscr{O}_X), \quad H^0(j^{\mathrm{an}}_+(\mathscr{O}_{X^{\mathrm{an}}})) = j^{\mathrm{an}}_*(\mathscr{O}_{X^{\mathrm{an}}}).$$

Note that $j_*(\mathscr{O}_X) = \mathscr{O}_Y[x^{-1}]$ is holonomic, while $j^{\mathrm{an}}_*(\mathscr{O}_{X^{\mathrm{an}}})$ contains non-meromorphisc functions like $e^{x^{-1}}$ and is much larger than $\mathscr{O}_{Y^{\mathrm{an}}}[x^{-1}]$. The $\mathscr{D}_{Y^{\mathrm{an}}}$-module $\mathscr{O}_{Y^{\mathrm{an}}}[x^{-1}]$ is holonomic, but $j^{\mathrm{an}}_*(\mathscr{O}_{X^{\mathrm{an}}})$ is not even a coherent $\mathscr{D}_{Y^{\mathrm{an}}}$-module.

1.4.2 Solution complexes and de Rham functors

Let X be a compact manifold. For $\mathscr{M} \in D^b(\mathscr{D}_X)$, we set

$$DR_X(\mathscr{M}) := \Omega_X \otimes^L_{\mathscr{D}_X} \mathscr{M}, \quad \mathrm{Sol}_X(\mathscr{M}) := R\mathcal{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_X).$$

We call $DR_X(\mathscr{M}) \in D^b(\mathbb{C}_X)$ (resp. $\mathrm{Sol}_X(\mathscr{M}) \in D^(\mathbb{C}_X)$) the de Rham complex (resp. the solution complex) of $\mathscr{M} \in D^b(\mathscr{D}_X)$. Then $DR_X(-)$ and $\mathrm{Sol}_X(-)$ define functors

$$DR_X : D^b(\mathscr{D}_X) \to D^b(\mathbb{C}_X), \quad \mathrm{Sol}_X : D^b(\mathscr{D}_X) \to D^b(\mathbb{C}_X)^{\mathrm{op}}.$$

A motivation for introducing the solution complexes $\mathrm{Sol}_X(\mathscr{M}) = R\mathcal{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_X)$ came from the theory of linear partial differential equations. In fact, for a coherent $\mathscr{D}_X$-module $\mathscr{M}$ the sheaf $\mathrm{Hom}_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_X)$ (on X) is the sheaf of holomorphic solutions to the system of linear PDEs corresponding to $\mathscr{M}$. By (an analogue in the analytic situation of) Proposition 1.1.47, we have the following.

Proposition 1.4.3. *For $\mathscr{M} \in D^b_{\mathrm{coh}}(\mathscr{D}_X)$ we have*

$$DR_X(\mathscr{M}) \cong R\mathcal{H}om_{\mathscr{D}_X}(\mathscr{O}_X, \mathscr{M})[d_X] \cong \mathrm{Sol}_X(D_X(\mathscr{M}))[d_X].$$

Hence properties of Sol_X can be deduced from those of DR_X. The functor DR_X has the advantage that it can be computed using a resolution of the right D_X-module Ω_X. In fact, similar to Proposition 1.1.10 we have a locally free resolution

$$0 \longrightarrow \Omega_X^0 \otimes_{\mathscr{O}_X} \mathscr{D}_X \longrightarrow \cdots \longrightarrow \Omega_X^{d_X} \otimes_{\mathscr{O}_X} \mathscr{D}_X \longrightarrow \Omega_X \longrightarrow 0$$

of the right $\mathscr{D}_X$-module Ω_X. It follows that for $\mathscr{M} \in \mathbf{Mod}(\mathscr{D}_X)$ the object $DR_X(\mathscr{M})[-d_X]$ of the derived category is represented by the complex

$$\Omega_X^\bullet \otimes_{\mathscr{O}_X} \mathscr{M} = [\Omega_X^0 \otimes_{\mathscr{O}_X} \mathscr{M} \to \cdots \to \Omega_X^{d_X} \otimes_{\mathscr{O}_X} \mathscr{M}]$$

where the differential $d^p : \Omega_X^p \otimes_{\mathscr{O}_X} \mathscr{M} \to \Omega_X^{p+1} \otimes_{\mathscr{O}_X} \mathscr{M}$ is given by

$$d^p(\omega \otimes s) = d\omega \otimes s + \sum_i dx_i \wedge \omega \otimes \partial_i s, \quad \omega \in \Omega_X^p, s \in \mathscr{M}$$

where $\{x_i, \partial_i\}$ is a local coordinate system of X.

Let us consider the case where $\mathscr{M}$ is an integrable connection of rank r (a coherent $\mathscr{D}_X$-module which is locally free of rank r over $\mathscr{O}_X$). In this case the 0-th cohomology sheaf $\mathscr{L} := H^0(\Omega_X^\bullet \otimes_{\mathscr{O}_X} \mathscr{M}) \cong \mathcal{H}om_{\mathscr{D}_X}(\mathscr{O}_X, \mathscr{M})$ of $\Omega_X^\bullet \otimes_{\mathscr{O}_X} \mathscr{M}$ coincides with the kernel of the sheaf homomorphism

$$d^0 = \nabla : \mathscr{M} \cong \Omega_X^0 \otimes_{\mathscr{O}_X} \mathscr{M} \to \Omega_X^1 \otimes_{\mathscr{O}_X} \mathscr{M}$$

which is the sheaf

$$\mathscr{M}^\nabla = \{s \in \mathscr{M} : \nabla s = 0\} = \{s \in \mathscr{M} : \Theta_X s = 0\}$$

of horizontal sections of the integrable connection $\mathscr{M}$. It is a locally free $\mathbb{C}_X$-module of rank r by the classical Frobenius theorem.

A locally free $\mathbb{C}_X$-module of finite rank is often called a **local system** on X. We denote by $\mathrm{Loc}(X)$ the category of local systems on X. Using the local system $\mathscr{L} = \mathscr{M}^\nabla$, we have a $\mathscr{D}_X$-linear isomorphism $\mathscr{O}_X \otimes_{\mathbb{C}_X} \mathscr{L} \cong \mathscr{M}$. Conversely, for a local system $\mathscr{L}$ we can define an integrable connection $\mathscr{M}$ by $\mathscr{M} = \mathscr{O}_X \otimes_{\mathbb{C}_X} \mathscr{L}$ and

$$\nabla = d \otimes \mathrm{id}_{\mathscr{L}} : \mathscr{O}_X \otimes_{\mathbb{C}_X} \mathscr{L} \cong \Omega_X^0 \otimes_{\mathscr{O}_X} \mathscr{M} \to \Omega_X^1 \otimes_{\mathbb{C}_X} \mathscr{L} \cong \Omega_X^1 \otimes_{\mathscr{O}_X} \mathscr{M}$$

such that $\mathscr{M}^\nabla = \mathscr{L}$. As a result, the category of integrable connections on X is equivalent to that of local systems on X.

Under the identification $\mathscr{O}_X \otimes_{\mathbb{C}_X} \mathscr{L} \cong \mathscr{M}$, the differentials in the complex $\Omega_X^\bullet \otimes_{\mathscr{O}_X} \mathscr{M}$ are written explicitly by

$$d \otimes \mathrm{id}_{\mathscr{L}} : \Omega_X^p \otimes_{\mathbb{C}_X} \mathscr{L} \to \Omega_X^{p+1} \otimes_{\mathbb{C}_X} \mathscr{L}.$$

Therefore, the higher cohomology groups $H^i(\Omega_X^\bullet \otimes_{\mathscr{O}_X} \mathscr{M})$ of the complex $\Omega_X^\bullet \otimes_{\mathscr{O}_X} \mathscr{M}$ vanish by the holomorphic Poincaré lemma, and we get finally a quasi-isomorphism $\Omega_X^\bullet \otimes_{\mathscr{O}_X} \mathscr{M} \cong \mathscr{L} = \mathscr{M}^\nabla$ for an integrable connection $\mathscr{M}$. We have therefore proved the following theorem.

Theorem 1.4.4. *Let $\mathscr{M}$ be an integrable connection of rank r on a complex manifold X. Then $DR_X(\mathscr{M})$ is concentrated at degree $-d_X$ and $H^{-d_X}(DR_X(\mathscr{M}))$ is a local system on X. Moreover, we have an equivalence*

$$H^{-d_X}(DR_X(-)) : \mathrm{Conn}(X) \xrightarrow{\sim} \mathrm{Loc}(X)$$

of categories.

We also note the following relation of DR_X and the higher direct image functor.

Theorem 1.4.5. *Let $f : X \to Y$ be a morphism of complex manifolds. For $\mathscr{M} \in D^b(\mathscr{D}_X)$ we have an isomorphism in $D^b(\mathbb{C}_Y)$:*

$$Rf_*(DR_X(\mathscr{M})) \cong DR_Y(f_+(\mathscr{M})).$$

Proof. We note that $Rf_*(DR_X(\mathscr{M})) = Rf_*(\Omega_X \otimes^L_{\mathscr{D}_X} \mathscr{M})$ and

$$DR_Y(f_+(\mathscr{M})) = \Omega_Y \otimes^L_{\mathscr{D}_Y} Rf_*(\mathscr{D}_{Y\leftarrow X} \otimes^L_{\mathscr{D}_X} \mathscr{M}) \cong Rf_*(f^{-1}(\Omega_Y) \otimes^L_{f^{-1}(\mathscr{D}_Y)} \mathscr{D}_{Y\leftarrow X} \otimes^L_{\mathscr{D}_X} \mathscr{M}).$$

It is therefore sufficient to show that $\Omega_X \cong f^{-1}(\Omega_Y) \otimes^L_{f^{-1}(\mathscr{D}_Y)} \mathscr{D}_{Y\leftarrow X}$, which follows from the definition of $\mathscr{D}_{Y\leftarrow X}$. □

1.4.3 Constructible sheaves

For a morphism $f : X \to Y$ of analytic spaces we have functors

$$Rf_* : D^b(\mathbb{C}_X) \to D^b(\mathbb{C}_Y), \qquad f^{-1} : D^b(\mathbb{C}_Y) \to D^b(\mathbb{C}_X),$$
$$Rf_! : D^b(\mathbb{C}_X) \to D^b(\mathbb{C}_Y), \qquad f^! : D^b(\mathbb{C}_Y) \to D^b(\mathbb{C}_X),$$

where $f^!$ is right adjoint to $Rf_!$. For $\mathscr{K} \in D^b(\mathbb{C}_X)$ and $\mathscr{L} \in D^b(\mathbb{C}_Y)$, we define

$$\mathscr{K} \boxtimes_{\mathbb{C}} \mathscr{L} = \mathrm{pr}_1^{-1}(\mathscr{K}) \otimes_{\mathbb{C}_{X\times Y}} \mathrm{pr}_2^{-1}(\mathscr{L}),$$

where $\mathrm{pr}_1 : X \times Y \to X$ and $\mathrm{pr}_2 : X \times Y \to Y$ are projections.

For an analytic space X, the dualizing complex of X is defined by

$$\omega_X = p_X^!(\mathbb{C}) \in D^b(\mathbb{C}_X)$$

where $p_X : X \to \mathrm{pt}$ is the unique morphism. If X is a compact manifold, ω_X is isomorphic to $\mathbb{C}_X[2d_X]$. The **Verdier dual** $D_X(\mathscr{F})$ of $\mathscr{F} \in D^b(\mathbb{C}_X)$ is defined by

$$D_X(\mathscr{F}) := R\mathcal{H}om_{\mathbb{C}_X}(\mathscr{F}, \omega_X) \in D^b(\mathbb{C}_X)$$

and we thus obtain a functor

$$D_X : D^b(\mathbb{C}_X) \to D^b(\mathbb{C}_X)^{\mathrm{op}}.$$

We recall that a locally finite partition $X = \coprod_{\alpha\in A} X_\alpha$ of an analytic space X by locally closed analytic subsets X_α is called a **stratification** of X if, for any $\alpha \in A$, X_α is smooth (hence a complex manifold) and $\bar{X}_\alpha = \coprod_{\beta\in B} X_\beta$ for a subset B of A. Each complex manifold X_α for $\alpha \in A$ is called a **stratum** of the stratification $X = \coprod_{\alpha\in A} X_\alpha$.

A $\mathbb{C}_X$-module $\mathscr{F}$ is called a **constructible sheaf** on X if there exists a stratification $X = \coprod_{\alpha\in A} X_\alpha$ of X such that the restriction $F|_{X_\alpha}$ is a local system on X_α for each $\alpha \in A$. For an analytic space X, we denote by $D^b_c(X)$ the full subcategory of $D^b(\mathbb{C}_X)$ consisting of bounded complexes of $\mathbb{C}_X$-modules whose cohomology groups are constructible.

Example 1.4.2. On the complex plane $X = \mathbb{C}$ let us consider the ordinary differential equation

$$x(\frac{d}{dx} - \lambda)u = 0$$

where $\lambda \in \mathbb{C}$ is a constant. Denote by $\mathscr{O}_X$ the sheaf of holomorphic functions on X and define a subsheaf $\mathscr{F} \subseteq \mathscr{O}_X$ of holomorphic solutions to this ordinary equation by

$$\mathscr{F} = \{u \in \mathscr{O}_X : (x\frac{d}{dx} - \lambda)u = 0\}.$$

Then the sheaf $\mathscr{F}$ is constructible with respect to the stratification $X = (\mathbb{C} - \{0\}) \amalg \{0\}$ of X. Indeed, the restriction $\mathscr{F}|_{\mathbb{C}-\{0\}} \cong \mathbb{C}x^\lambda$ is a locally free sheaf of rank one over $\mathbb{C}_{\mathbb{C}-\{0\}}$ and the stalk of $\mathscr{F}$ at 0 is given by

$$\mathscr{F}_0 = \begin{cases} \mathbb{C} & \lambda = 0, 1, 2, \ldots, \\ 0 & \text{otherwise.} \end{cases}$$

For an algebraic variety X, we denote the underlying analytic space by X^{an}. For a morphism $f : X \to Y$ of algebraic varieties we denote the corresponding morphism for analytic spaces by $f^{\mathrm{an}} : X^{\mathrm{an}} \to Y^{\mathrm{an}}$. A locally finite partition $X = \coprod_{\alpha \in A} X_\alpha$ of an algebraic variety X by locally closed subvarieties X_α is called a stratification of X if each X_α is smooth and $\bar{X}_\alpha = \coprod_{\beta \in B} X_\beta$ for a subset B of A. It is clear that a stratification $X = \coprod_{\alpha \in A} X_\alpha$ of an algebraic variety X induces a stratification $X^{\mathrm{an}} = \coprod_{\alpha \in A} X_\alpha^{\mathrm{an}}$ of the corresponding analytic space X^{an}.

Definition 1.4.1. Let X be an algebraic variety. A $\mathbb{C}_{X^{\mathrm{an}}}$-module $\mathscr{F}$ is called an **algebraically constructible sheaf** if there exists a stratification $X = \coprod_{\alpha \in A} X_\alpha$ of X such that $\mathscr{F}|_{X_\alpha^{\mathrm{an}}}$ is a locally constant sheaf on X_α^{an} for each $\alpha \in A$.

For an algebraic variety X, we denote by $D_c^b(X)$ the full subcategory of $D^b(\mathbb{C}_{X^{\mathrm{an}}})$ consisting of bounded complexes of $\mathbb{C}_{X^{\mathrm{an}}}$-modules whose cohomology groups are algebraically constructible (note that $D_c^b(X)$ is not a subcategory of $D^b(\mathbb{C}_X)$ but of $D^b(\mathbb{C}_{X^{\mathrm{an}}})$). We write $\omega_{X^{\mathrm{an}}}$ and $D_{X^{\mathrm{an}}}$ simply as ω_X and D_X. For a morphism $f : X \to Y$ of algebraic varieties, we write $(f^{\mathrm{an}})^{-1}$, $(f^{\mathrm{an}})^!$, Rf_*^{an}, $Rf_!^{\mathrm{an}}$ as f^{-1}, $f^!$, Rf_*, $Rf_!$, respectively.

Theorem 1.4.6. *Let X be an algebraic variety or an analytic space.*

(a) *We have $\omega_X \in D_c^b(X)$ and the functor D_X preserves the category $D_c^b(X)$ and $D_X \circ D_X \cong \mathrm{id}$ on $D_c^b(X)$.*

(b) *Let $f : X \to Y$ be a morphism of algebraic varieties or analytic spaces. Then f^{-1} and $f^!$ induces functors $D_c^b(Y) \to D_c^b(X)$ and we have $f^! = D_X \circ f^{-1} \circ D_Y$ on $D_c^b(Y)$.*

(c) *Let $f : X \to Y$ be a morphism of algebraic varieties or analytic spaces, where we assume that f is proper in the case where f is a morphism of analytic spaces. Then Rf_* and $Rf_!$ induce functors $D_c^b(X) \to D_c^b(Y)$, and we have $Rf_! = D_Y \circ Rf_* \circ D_X$ on $D_c^b(X)$.*

(d) *The functor $(-) \otimes_{\mathbb{C}} (-)$ induces*

$$(-) \otimes_{\mathbb{C}} (-) : D_c^b(X) \times D_c^b(X) \to D_c^b(X).$$

We also recall that an object $\mathscr{F} \in D_c^b(X)$ is called a perverse sheaf if we have

$$\dim(\mathrm{supp}(H^i(\mathscr{F}))) \leq -i, \quad \dim(\mathrm{supp}(H^i(D_X(\mathscr{F})))) \leq -i$$

for any $i \in \mathbb{Z}$. We denote by $\mathrm{Perv}(\mathbb{C}_X)$ the full subcategory of $D_c^b(X)$ consisting of perverse sheaves.

1.4.4 Kashiwara's constructibility theorem

We now prove some basic properties of holomorphic solutions to holonomic D-modules. If $\mathscr{M}$ is a holonomic $\mathscr{D}_X$-module on a complex manifold X, its holomorphic solution complex $\mathrm{Sol}_X(\mathscr{M}) = R\mathcal{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_X)$ possesses very rigid structures. Namely, all the cohomology groups of $\mathrm{Sol}_X(\mathscr{M})$ are constructible sheaves on X, so we have $\mathrm{Sol}_X(\mathscr{M}) \in D_c^b(X)$. This is the famous constructibility theorem due to Kashiwara. In particular, we obtain

$$\dim(H^i(\mathrm{Sol}_X(\mathscr{M}))_x) < +\infty$$

for every $i \in \mathbb{Z}$ and $x \in X$. Moreover, in his Ph.D. thesis [Kas3], Kashiwara essentially proved that $\mathrm{Sol}_X(\mathscr{M})[d_X]$ satisfies the conditions of perverse sheaves, although the theory of perverse sheaves did not exist at that time. Let us give a typical example. Let Y be a complex submanifold of X with codimension $d = d_X - d_Y$. Then for the holonomic $\mathscr{D}_X$-module $\mathscr{M} = \mathscr{B}_{Y|X}$, the complex

$$\mathrm{Sol}_X(\mathscr{M})[d_X] \cong (\mathbb{C}_Y[-d])[d_X] = \mathbb{C}_Y[d_Y]$$

is a perverse sheaf on X. Before giving the proof of Kashiwara's results, let us recall the following fact. It was shown by Kashiwara that for any holonomic $\mathscr{D}_X$-module there exists a Whitney stratification $X = \coprod_{\alpha \in A} X_\alpha$ of X such that $\mathrm{Ch}(\mathscr{M}) \subseteq \coprod_{\alpha\in A} T^*_{X_\alpha} X$. This follows from the geometric fact that $\mathrm{Ch}(\mathscr{M})$ is a $\mathbb{C}^\times$-invariant Lagrangian analytic subset of T^*X. Let us fix such a stratification for a holonomic system $\mathscr{M}$.

Proposition 1.4.7. *Let $\mathscr{F} = R\mathcal{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_X) \in D^b(\mathbb{C}_X)$. Then $H^i(\mathscr{F})|_{X_\alpha}$ is a locally constant sheaf on X_α for any $i \in \mathbb{Z}$ and $\alpha \in A$.*

Proposition 1.4.8. *Let $\mathscr{M}$ be a holonomic $\mathscr{D}_X$-module. Then for each $i \in \mathbb{Z}$ and $x \in X$, the stalk $H^i(R\mathcal{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_X)_x)$ at x is a finite-dimensional vector space over $\mathbb{C}$.*

By Proposition 1.4.3, Proposition 1.4.7 and Proposition 1.4.8, we obtain Kashiwara's constructibility theorem:

Theorem 1.4.9. *Let $\mathscr{M}$ be a holonomic $\mathscr{D}_X$-module over a complex manifold X. Then $\mathrm{Sol}_X(\mathscr{M}) = R\mathcal{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_X)$ and $DR_X(\mathscr{M}) = \Omega_X \otimes^L_{\mathscr{D}_X} \mathscr{M}$ are objects of $D^b_c(X)$.*

Corollary 1.4.10. *Let $\mathscr{M}$ be a holonomic $\mathscr{D}_X$-module and $D_X(\mathscr{M})$ its dual. Then we have isomorphisms*

$$D_X(DR_X(\mathscr{M})) \cong DR_X(D_X(\mathscr{M})), \quad D_X(\mathrm{Sol}_X(\mathscr{M})[d_X]) \cong \mathrm{Sol}_X(D_X(\mathscr{M}))[d_X].$$

Theorem 1.4.11. *Let X be a complex manifold and $\mathscr{M}$ a holonomic $\mathscr{D}_X$-module. Then $\mathrm{Sol}_X(\mathscr{M})[d_X]$ and $DR_X(\mathscr{M})$ are perverse sheaves on X.*

Proof. By $DR_X(\mathscr{M}) \cong \mathrm{Sol}_X(D_X(\mathscr{M}))[d_X]$, it is sufficient to prove that $\mathscr{F} = \mathrm{Sol}_X(\mathscr{M})[d_X]$ is a perverse sheaf for any holonomic $\mathscr{D}_X$-module $\mathscr{M}$. Moreover, we have $D_X(\mathrm{Sol}_X(\mathscr{M})[d_X]) \cong \mathrm{Sol}_X(D_X(\mathscr{M}))[d_X]$ by Corollary 1.4.10, so it suffices to prove that $\dim(\mathrm{supp}(H^i(\mathscr{F}))) \leq -i$ for $i \in \mathbb{Z}$. We take a Whitney stratification $X = \coprod_{\alpha\in A} X_\alpha$ of X such that $\mathrm{Ch}(\mathscr{M}) \subseteq \coprod_{\alpha \in A} T^*_{X_\alpha} X$ and let $i_\alpha : X_\alpha \hookrightarrow X$ for $\alpha \in A$. Then by Proposition 1.4.7, the complex $i_\alpha^{-1}(\mathscr{F})$ has locally constant cohomology groups for $\alpha \in A$. For $i \in \mathbb{Z}$, set $Z = \mathrm{supp}(H^i(\mathscr{F}))$, then Z is a union of connected components of strata X_α's and we need to prove that $\dim(Z) = d_Z \leq -i$. Choose a smooth point z of Z contained in a stratum X_α such that $\dim(X_\alpha) = \dim(Z)$ and take a germ of complex submanifold Y of X at z which intersects with Z transversally at $z \in Z$ ($\dim(Y) = d_Y = d_X - d_Z$). We can choose the pair (z, Y) so that Y is non-characteristic for $\mathscr{M}$, because we have the estimate $\mathrm{Ch}(\mathscr{M}) \subseteq \coprod_{\alpha\in A} T^*_{X_\alpha} X$. Therefore, by the Cauchy-Kowalevski-Kashiwara theorem, we obtain

$$\mathscr{F}|_Y = R\mathcal{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_X)|_Y[d_X] \cong R\mathcal{H}om_{\mathscr{D}_Y}(\mathscr{M}_Y, \mathscr{O}_Y)[d_X].$$

Our assumption $H^i(\mathscr{F})_z \neq 0$ implies that $\mathcal{E}xt^{i+d_X}_{\mathscr{D}_Y}(\mathscr{M}_Y, \mathscr{O}_Y)_z \neq 0$. On the other hand, by Theorem 1.1.40 and

$$R\mathcal{H}om_{\mathscr{D}_Y}(\mathscr{M}_Y, \mathscr{O}_Y) \cong R\mathcal{H}om_{\mathscr{D}_Y}(\mathscr{M}_Y, \mathscr{D}_Y) \otimes^L_{\mathscr{D}_Y} \mathscr{O}_Y,$$

we have $\mathcal{E}xt^i_{\mathscr{D}_Y}(\mathscr{M}_Y, \mathscr{O}_Y) = 0$ for $i > d_Y$. We then conclude that $i + d_X \leq d_Y$, which means $d_Z = d_X - d_Y \leq -i$ and this completes the proof. □

1.5 Meromorphic connections

1.5.1 Meromorphic connections in the one-dimensional case

1.5.1.1 Systems of ODEs and meromorphic connections We start from the classical theory of ordinary differential equations (ODEs for short). We always consider the problem in an open neighborhood of $x = 0 \in \mathbb{C}$. Here the complex plane $\mathbb{C}$ is considered as a complex manifold and we use only the classical topology. Set $\mathscr{O} = \mathscr{O}_{\mathbb{C},0}$ and denote by K its quotient field. Then K is the field of meromorphic functions with possible poles at $x = 0$. Note that $\mathscr{O}$ and K are identified with the ring of convergent power series $\mathbb{C}\{\{x\}\}$ at $x = 0$ and its quotient field $\mathbb{C}\{\{x\}\}[x^{-1}]$, respectively.

For a matrix $A(x) = (a_{ij}(x)) \in \mathcal{M}_n(K)$, let us consider the system of ODEs

$$\frac{d}{dx}u(x) = A(x)u(x) \tag{1.5.1}$$

where $u(x) = (u_1(x), \dots, u_n(x))^t$ is a column vector of unknown functions. Setting $v(x) = T^{-1}u(x)$ for an invertible matrix $T = T(x) \in \mathrm{GL}_n(K)$, the equation (1.5.1) can be rewritten as

$$\frac{d}{dx}v(x) = \Big(T^{-1}AT - T^{-1}\frac{d}{dx}T\Big)v(x).$$

Therefore, we say that two systems

$$\frac{d}{dx}u(x) = A(x)u(x), \quad A(x) \in \mathcal{M}_n(K)$$

and

$$\frac{d}{dx}v(x) = B(x)v(x), \quad B(x) \in \mathcal{M}_n(K)$$

are equivalent if there exists $T \in \mathrm{GL}_n(K)$ such that $B = T^{-1}AT - T^{-1}\frac{d}{dx}T$.

As solutions to (1.5.1) we consider holomorphic (possibly multivalued) solutions on a punctured disk $B_\varepsilon^* = \{x \in \mathbb{C} : 0 < |x| < \varepsilon\}$, where ε is a sufficiently small positive number. Namely, let $\widetilde{K}$ denote the ring consisting of possibly multivalued holomorphic functions defined on a punctured disk B_ε^* for a sufficiently small $\varepsilon > 0$. Then we say that $u(x) = (u_1(x), \dots, u_n(x))^t$ is a solution to (1.5.1) if it belongs to $\widetilde{K}^n$ and satisfies (1.5.1).

Let us now reformulate these classical notions by the modern language of meromorphic connections. Let M be a finite-dimensional vector space over K endowed with a $\mathbb{C}$-linear map $\nabla : M \to M$. Then M (or more precisely the pair (M, ∇)) is called a **meromorphic connection** (at $x = 0$) if it satisfies the condition

$$\nabla(fu) = \frac{df}{dx}u + f\nabla u, \quad f \in K, u \in M. \tag{1.5.2}$$

If (M, ∇) and (N, ∇) are meromorphic connections, a K-linear map $\varphi : M \to N$ is called a morphism of meromorphic connections if it satisfies $\varphi \circ \nabla = \nabla \circ \varphi$. In this case we write $\varphi : (M, \nabla) \to (N, \nabla)$.

Remark 1.5.1. The condition (1.5.2) can be replaced with the weaker one

$$\nabla(fu) = \frac{df}{dx}u + f\nabla u, \quad f \in \mathscr{O}, u \in M. \tag{1.5.3}$$

Indeed, if the condition (1.5.3) holds, then for $f \in \mathscr{O} - \{0\}$, $g \in \mathscr{O}$, $u \in M$, we have

$$\nabla(gu) = \nabla(ff^{-1}gu) = f'f^{-1}gu + f\nabla(f^{-1}gu)$$

and hence

$$\begin{aligned}\nabla(f^{-1}gu) &= -f^{-1}f'gu + f^{-1}\nabla(gu) = (-f^{-1}f'g + f^{-1}g')u + f^{-1}g\nabla u \\ &= (f^{-1}g)'u + f^{-1}g\nabla u.\end{aligned}$$

Meromorphic connections naturally form an abelian category. Note that for a meromorphic connection (M, ∇) the vector space M is a left $\mathscr{D}_{\mathbb{C},0}$-module by the action $\frac{d}{dx}u = \nabla u$. Note also that ∇ is uniquely extended to an element of $\mathrm{End}_{\mathbb{C}}(\widetilde{K} \otimes_K M)$ satisfying

$$\nabla(fu) = \frac{df}{dx}u + f\nabla u, \quad f \in \widetilde{K}, u \in M,$$

and $\widetilde{K} \otimes_K M$ is also a left $\mathscr{D}_{\mathbb{C},0}$-module. We say that $u \in \widetilde{K} \otimes_K M$ is a **horizontal section** of (M, ∇) if $\nabla u = 0$.

Let (M, ∇) be a meromorphic connection and choose a K-basis $\{e_i\}_{1 \le i \le n}$ of M. Then the matrix $A(x) = (a_{ij}(x)) \in \mathcal{M}_n(K)$ defined by

$$\nabla e_j = -\sum_{i=1}^{n} a_{ij}(x)e_i \tag{1.5.4}$$

is called the connection matrix of (M, ∇) with respect to the basis $\{e_i\}_{1 \le i \le n}$. In terms of this basis the action of ∇ is described by

$$\nabla\Big(\sum_{i=1}^{n} u_i e_i\Big) = \sum_{i=1}^{n}\Big(\frac{du_i}{dx} - \sum_{j=1}^{n} a_{ij}u_j\Big)e_i.$$

Hence the condition $\nabla u = 0$ for $u = \sum_{i=1}^{n} u_i e_i \in \widetilde{K} \otimes_K M$ is equivalent to the equation

$$\frac{d}{dx}u(x) = A(x)u(x) \tag{1.5.5}$$

for $u(x) = (u_1(x), \dots, u_n(x)) \in \widetilde{K}^n$. We have seen that to each meromorphic connection (M, ∇) endowed with a K-basis $\{e_i\}_{1 \le i \le n}$ of M we can associate a system (1.5.5) of ODEs and that the horizontal sections of (M, ∇) correspond to solutions of (1.5.5). Conversely, to any $A = (a_{ij}(x)) \in \mathcal{M}_n(K)$ we can associate a meromorphic connection (M_A, ∇_A) given by

$$M_A = \bigoplus_{i=1}^{n} Ke_i, \quad \nabla e_j = -\sum_{i=1}^{n} a_{ij}(x)e_i.$$

Under this correspondence we easily see the following.

Lemma 1.5.1. *Two systems of ODEs*

$$\frac{d}{dx}u(x) = A(x)u(x), \quad A(x) \in \mathcal{M}_n(K)$$

and

$$\frac{d}{dx}v(x) = B(x)v(x), \quad B(x) \in \mathcal{M}_n(K)$$

and equivalent if and only if the associated meromorphic connections (M_A, ∇_A) and (M_B, ∇_B) are isomorphic.

Let (M, ∇) and (N, ∇) be meromorphic connections. Then $M \otimes_K N$ and $\mathrm{Hom}_K(M, N)$ are endowed with structures of meromorphic connections by

$$\nabla(u \otimes v) = \nabla u \otimes v + u \otimes \nabla v, \quad (\nabla\phi)u = \nabla(\phi(u)) - \phi(\nabla u)$$

where $\phi \in \mathrm{Hom}_K(M, N)$, $u \in M$, $v \in N$. Note that the one-dimensional K-module K is naturally endowed with a structure of a meromorphic connection by $\nabla f = \frac{df}{dx}$. In particular, for a meromorphic connection (M, ∇), the dual space $M^* = \mathrm{Hom}_K(M, K)$ is endowed with a structure of meromorphic connection by

$$\langle \nabla\phi, u\rangle = \frac{d}{dx}\langle \phi, u\rangle - \langle \phi, \nabla u\rangle, \quad \phi \in M^*, u \in M.$$

If $A = (a_{ij}(x)) \in \mathcal{M}_n(K)$ is the connection matrix of M with respect to a K-basis $\{e_i\}_{1 \le i \le n}$ of M, then the connection matrix A^* of M^* with respect to the dual basis $\{e_i^*\}_{1 \le i \le n}$ is given by $A^* = -A^t$.

1.5.1.2 Meromorphic connections with regular singularities For an open interval $(a,b) \subseteq \mathbb{R}$ and $\varepsilon > 0$ we set

$$S^{\varepsilon}_{(a,b)} = \{x : 0 < |x| < \varepsilon, \arg(x) \in (a,b)\}.$$

This is a subset of (the universal covering of) $\mathbb{C} \setminus \{0\}$, called an **open angular sector**. A function $f \in \widetilde{K}$ is said to have **moderate growth** (or to be in the **Nilsson class**) at $x = 0$ if it satisfies the following condition: For any open interval $(a,b) \subseteq \mathbb{R}$ and $\varepsilon > 0$ such that f is defined on $S^{\varepsilon}_{(a,b)}$, there exist constants $C > 0$ and $N \gg 0$ such that $|f(x)| \leq C|x|^{-N}$ for $x \in S^{\varepsilon}_{(a,b)}$. We denote by $\widetilde{K}^{mod}$ the set of $f \in \widetilde{K}$ which have moderate growth at $x = 0$. Note that in the case where f is single-valued f has moderate growth if and only if it is meromorphic.

Let us consider a system of ODEs:

$$\frac{d}{dx}u(x) = A(x)u(x) \tag{1.5.6}$$

for $A(x) = (a_{ij}(x)) \in \mathcal{M}_n(K)$. It is well known in the theory of linear ODEs that the set of solutions $u \in \widetilde{K}^n$ to (1.5.6) forms a vector space of dimension n over $\mathbb{C}$. Let us take n linearly independent solutions $u_1(x), \dots, u_n(x)$ to this equation. Then the matrix $S(x) = (u_1(x), \dots, u_n(x))$ is called a **fundamental solution matrix** of (1.5.6). Since the analytic continuation of $S(x)$ along a circle around $x = 0 \in \mathbb{C}$ is again a solution matrix of (1.5.6), there exists an invertible matrix $G \in \mathrm{GL}_n(\mathbb{C})$ such that

$$\lim_{t \to 2\pi} S(e^{it}x) = S(x)G.$$

The matrix G is called the **monodromy matrix** of the equation (1.5.6). Let us take a matrix $\Gamma \in \mathcal{M}_n(\mathbb{C})$ such that $\exp(2\pi i \Gamma) = G$ and set $T(x) = S(x)\exp(-\Gamma \log(x))$. Then we can easily check that the entries of $T(x)$ are singlevalued functions. Thus we obtained a decomposition $S(x) = T(x)\exp(\Gamma \log(x))$ of $S(x)$, in which the last part $\exp(\Gamma \log(x))$ has the same monodromy as that of $S(x)$. The following well-known fact is fundamental.

Theorem 1.5.2. *The following three conditions on the system (1.5.6) are equivalent:*

(i) *The system (1.5.6) is equivalent to the system*

$$\frac{d}{dx}v(x) = \frac{\Gamma(x)}{x}v(x)$$

for some $\Gamma(x) \in \mathcal{M}_n(\mathcal{O})$.

(ii) *The system (1.5.6) is equivalent to the system*

$$\frac{d}{dx}v(x) = \frac{\Gamma}{x}v(x)$$

for some $\Gamma \in \mathcal{M}_n(\mathbb{C})$.

(iii) *All solutions to (1.5.6) in* $\widetilde{K}^n$ *belong to* $(\widetilde{K}^{mod})^n$.

Proof. □

On a neighborhood of $x = 0$ in $\mathbb{C}$ consider an ordinary differential equation

$$P(x,\partial)u = 0, \quad P(x,\partial) = \sum_{i=0}^{n} a_i(x)\partial^i, \quad \partial = \frac{d}{dx} \tag{1.5.7}$$

where $a_i(x)$ is holomorphic on a neighborhood of $x = 0$ and $a_n(x)$ is not identically zero (i.e., the order of $P(x,\partial)$ is n). We can rewrite P in the form

$$P(x,\partial) = \sum_{i=0}^{n} b_i(x)\theta^i, \quad b_i(x) \in K$$

with $b_n(x) = 0$ and $\theta = x\partial$. Recall the following classical result.

Theorem 1.5.3 (Fuchs, 1866)**.** *For P as above the following conditions are equivalent:*

(i) *All solutions of (1.5.7) belong to $\widetilde{K}^{mod}$.*

(ii) *We have $\mathrm{ord}_0(a_i/a_n) \geq -(n-i)$ for $0 \leq i \leq n$.*

(iii) *b_i/b_n are holomorphic for $0 \leq i \leq n$.*

We say that a meromorphic connection (M, ∇) at $x = 0$ is regular if there exists a finitely generated $\mathcal{O}$-submodule $L \subseteq M$ which is stable by the action of $\theta = x\nabla$ and generates M over K. We call such an $\mathcal{O}$-submodule L an **$\mathcal{O}$-lattice** of (M, ∇).

Lemma 1.5.4. *Let (M, ∇) be a regular meromorphic connection. Then any $\mathcal{O}$-lattice L of (M, ∇) is a free $\mathcal{O}$-module of rank $\dim_K(M)$.*

Proof. Since L is a torsion free finitely generated module over the principal ideal domain $\mathcal{O}$, it is free of finite-rank. Hence it is sufficient to show that the canonical homomorphism $K \otimes_{\mathcal{O}} L \to M$ is an isomorphism. The surjectivity is clear; to show the injectivity take a free basis $\{e_i\}_{1\leq i\leq n}$ of L. It is sufficient to show that $\{e_i\}_{1\leq i\leq n}$ is linearly independent over K. Assume $\sum_{i=1}^n f_i e_i = 0$ for $f_i \in K$. For $N \gg 0$ we have $a_i := x^N f_i \in \mathcal{O}$ for any $i = 1, \dots, n$. Then from $\sum_{i=1}^n a_i e_i = 0$ we obtain $a_i = 0$, and hence $f_i = 0$. □

By this lemma and the invariance of $\mathcal{O}$-lattices under $\theta = x\nabla$, we easily see that a meromorphic connection (M, ∇) is regular if and only if there exists a K-basis $\{e_i\}_{1\leq i\leq n}$ of M such that the associated system of ODEs is of the form

$$\frac{d}{dx}u(x) = \frac{\Gamma(x)}{x}u(x), \quad \Gamma(x) \in \mathcal{M}_n(\mathcal{O}).$$

Proposition 1.5.5. *For a meromorphic connection (M, ∇) at $x = 0$, the following three conditions are equivalent:*

(i) *(M, ∇) is regular.*

(ii) *All the horizontal sections of (M, ∇) belong to $\widetilde{K}^{mod} \otimes_K M$.*

(iii) *For any $u \in M$ there exists a finitely generated $\mathcal{O}$-submodule L of M such that $u \in L$ and $\theta L \subseteq L$, i.e., M is a union of θ-stable finitely generated $\mathcal{O}$-submodules.*

(iv) *For any $u \in M$ there exists a polynomial $F(t) \in \mathcal{O}[t]$ such that $F(\theta)u = 0$.*

Proof. We have remarked the equivalence of (i) and (ii) from Theorem 1.5.2. Also, if (M, ∇) is regular and L is an $\mathcal{O}$-lattice, then $M = \bigcup_{n\geq 0} x^{-n}L$ and each $x^{-n}L$ is a θ-stable finitely generated $\mathcal{O}$-submodule of M, whence (iii). Conversely, if (iii) is satisfied, then by taking a K-basis $\{e_1, \dots, e_n\}$ of M and choose a family of θ-stable finitely generated $\mathcal{O}$-submodule L_i of M such that $e_i \in L_i$, we see that the sum $L = \sum_{i=1}^n L_i \subseteq M$ is an $\mathcal{O}$-lattice of (M, ∇).

Finally, we note that if (iii) is true, then for each $u \in M$ we can take a θ-stable finitely generated $\mathcal{O}$-submodule $L \subseteq M$ such that $u \in L$, and set $L_i = \mathcal{O}u + \mathcal{O}\theta u + \cdots + \mathcal{O}\theta^{i-1}u$. Then $(L_i)_{i\geq 1}$ is an increasing sequence of $\mathcal{O}$-submodules of L. Since L is Noetherian over $\mathcal{O}$, there exists $m > 0$ such that $\bigcup_{i\geq 1} L_i = L_m$. The condition $L_m = L_{m+1}$ then implies that $\theta^m u = -\sum_{i=0}^{m-1} a_{m-i}\theta^i u$ for some $a_i \in \mathcal{O}$, whence the assertion of (iv). Conversely, if (iv) is satisfied and $F(t) = t^m + a_1 t^{m-1} + \cdots + a_m$, then $L = \mathcal{O}u + \mathcal{O}\theta u + \cdots + \mathcal{O}\theta^{m-1}u$ is a θ-stable finitely generated $\mathcal{O}$-submodule such that $u \in L$. □

Proposition 1.5.6. *Let $0 \to (M', \nabla) \to (M, \nabla) \to (M'', \nabla) \to 0$ be an exact sequence of meromorphic connections. Then (M, ∇) is regular if and only if (M', ∇) and (M'', ∇) are regular.*

Proof. By the condition (iv) of Proposition 1.5.5, (M', ∇) and (M'', ∇) are regular if (M, ∇) is regular. Let us prove the converse. For $u \in M$ there exist $m \geq 0$ and $F(t) \in \mathscr{O}[t]$ such that

$$F(\theta)u \in M'$$

by the regularity of (M'', ∇). Also by the regularity of (M', ∇), there exists $G(t) \in \mathscr{O}[t]$ such that

$$G(\theta)F(\theta)u = 0.$$

It then follows from Proposition 1.5.5 that (M, ∇) is regular. □

Proposition 1.5.7. *Assume that M and N are regular meromorphic connections. Then $\mathrm{Hom}_K(M, N)$ and $M \otimes_K N$ are also regular meromorphic connections.*

Proof. This can be easily checked by examining the connection matrices. □

1.5.1.3 Regularity of D-modules on algebraic curves

We also have the notion of meromorphic connections in the algebraic category. In the algebraic situation the ring $\mathscr{O} = \mathbb{C}\{\{x\}\}$ is replaced by the stalk $\mathscr{O}_{C,p}$, where C is a smooth algebraic curve and p is a point of C. We denote by $K_{C,p}$ the quotient field of $\mathscr{O}_{C,p}$. Note that $\mathscr{O}_{C,p}$ is a discrete valuation ring and hence a principal ideal domain.

Let M be a finite dimensional $K_{C,p}$-module and

$$\nabla : M \to \Omega^1_{C,p} \otimes_{\mathscr{O}_{C,p}} M$$

be a $\mathbb{C}$-linear map. The pair (M, ∇) is called an **algebraic meromorphic connection** at $p \in C$ if

$$\nabla(fu) = df \otimes u + f\nabla u, \quad f \in K_{C,p}, u \in M.$$

By a morphism $\varphi : (M, \nabla) \to (N, \nabla)$ of algebraic meromorphic connections at $p \in C$ we mean a $K_{C,p}$-linear map $\varphi : M \to N$ satisfying $\nabla \circ \varphi = (\mathrm{id} \otimes \varphi) \circ \nabla$.

Algebraic meromorphic connections at $p \in C$ naturally form an abelian category. Choose a local parameter $x \in \mathscr{O}_{C,p}$ at p and set $\partial = \frac{d}{dx}$. Then we have $K_{C,p} = \mathscr{O}_{C,p}[x^{-1}]$. Identifying $\Omega^1_{C,p}$ with $\mathscr{O}_{C,p}$ by $\mathscr{O}_{C,p} \ni f \mapsto f dx \in \Omega^1_{C,p}$, an algebraic meromorphic connection at $p \in C$ is a finite-dimensional $K_{C,p}$-module endowed with a $\mathbb{C}$-linear map $\nabla : M \to M$ satisfying

$$\nabla(fu) = \frac{df}{dx}u + f\nabla u, \quad f \in K_{C,p}, u \in M.$$

An algebraic meromorphic connection (M, ∇) at $p \in C$ is called **regular** if there exists a finitely generated $\mathscr{O}_{C,p}$-submodule L of M such that $M = K_{C,p}L$ and $x\nabla(L) \subseteq \Omega^1_{C,p} \otimes_{\mathscr{O}_{C,p}} L$ for some (and hence any) local parameter x at p. We call such an $\mathscr{O}_{C,p}$-submodule L an $\mathscr{O}_{C,p}$**-lattice** of (M, ∇). Algebraic meromorphic connections share some basic properties with analytic ones. For example, Lemma 1.5.4, Proposition 1.5.5 and Proposition 1.5.6 remain valid also in the algebraic category.

Lemma 1.5.8. *Let (M, ∇) be an algebraic meromorphic connection at $p \in C$. Choose a local parameter x at p, and denote by $(M^{\mathrm{an}}, \nabla)$ the corresponding (analytic) meromorphic connection at $x = 0$, i.e., $M^{\mathrm{an}} = \mathbb{C}\{\{x\}\}[x^{-1}] \otimes_{K_{C,p}} M$. Then (M, ∇) is regular if and only if $(M^{\mathrm{an}}, \nabla)$ is as well.*

Proof. We may identify $\Omega^1_{C,p}$ with the local parameter x. If (M, ∇) is regular, then we can take an $\mathscr{O}_{C,p}$-lattice L of (M, ∇), and $\mathbb{C}\{\{x\}\}[x^{-1}] \otimes_{K_{C,p}} L$ is an $\mathbb{C}\{\{x\}\}$ lattice of $(M^{\mathrm{an}}, \nabla)$, so $(M^{\mathrm{an}}, \nabla)$ is regular.

Conversely, assume that $(M^{\mathrm{an}}, \nabla)$ is regular. We take a finitely generated $\mathscr{O}_{C,p}$-submodule L_0 of M which generates M over $K_{C,p}$. By Proposition 1.5.5, L_0 and hence $L = \mathscr{O}_{C,p}[\theta]L_0$ must be contained in a θ-stable finitely generated $\mathbb{C}\{\{x\}\}$-submodule, so $L^{\mathrm{an}} = \mathbb{C}\{\{x\}\} \otimes_{\mathscr{O}} L$ is also finitely generated over $\mathbb{C}\{\{x\}\}$. Since $\mathbb{C}\{\{x\}\}$ is faithfully flat over $\mathscr{O}_{C,p}$, this implies the finiteness of L over $\mathscr{O}_{C,p}$, so (M, ∇) is regular. □

Let us globalize the above definition of regularity. Let $\mathscr{M}$ be an integrable connection on an algebraic curve C. Take a smooth completion $\bar{C}$ of C and denote by $j : C \hookrightarrow \bar{C}$ the open immersion. Note that C is unique up to isomorphisms because it is a curve. Let us consider the $\mathscr{D}_{\bar{C}}$-module $j_*(\mathscr{M})$ (note that $R^i j_*(\mathscr{M}) = 0$ for $i \neq 0$ since j is affine open immersion). Since $\mathscr{M}$ is locally free over $\mathscr{O}_C$, it is free on a non-trivial (Zariski) open subset $U = C \setminus V$ of C, where V consists of finitely many points, hence $j_*(\mathscr{M})|_{\bar{C}\setminus V}$ is also free over $\bar{C} \setminus V$. In particular, $j_*(\mathscr{M})$ is locally free over $j_*(\mathscr{O}_C)$ (of finite rank). Let $p \in \bar{C} \setminus C$, then since $j_*(\mathscr{M})_p$ is a $\mathscr{D}_{\bar{C},p}$-module, it is naturally endowed with a structure of an algebraic meromorphic connection at $p \in C$ by $\nabla(m) = dx \otimes \partial m$, where x is a local parameter at p and $\partial = dx$. We call this $\mathscr{D}_{\bar{C}}$-module $j_*(\mathscr{M})$ the **algebraic meromorphic extension** of M.

Definition 1.5.1. Let $\mathscr{M}$ be an integrable connection on a smooth algebraic curve C. For a boundary point $p \in \bar{C} \setminus C$, we say that $\mathscr{M}$ has **regular singularity at p** (or p is a **regular singular point** of $\mathscr{M}$) if the algebraic meromorphic connection $(j_*(\mathscr{M})_p, \nabla)$ is regular. Moreover, an integrable connection $\mathscr{M}$ on C is called **regular** if it has regular singularity at any boundary point $p \in \bar{C} \setminus C$.

Proposition 1.5.9. *Let $\mathscr{M}$ be an integrable connection on C. Then for any open subset U of C the restriction $\mathscr{M}|_U$ has regular singularity at any point of $C \setminus U$.*

Proof. This is easily checked from the definition of regularity. □

Proposition 1.5.10. *Let $0 \to \mathscr{M}' \to \mathscr{M} \to \mathscr{M}'' \to 0$ be an exact sequence of integrable connections on C. Then $\mathscr{M}$ is regular if and only if $\mathscr{M}'$ and $\mathscr{M}''$ are regular.*

Proof. This follows easily from Proposition 1.5.5 and Lemma 1.5.8. □

Proposition 1.5.11. *Let $\mathscr{M}$ and $\mathscr{N}$ be regular integrable connections on C. Then the integrable connections $\mathscr{M} \otimes_{\mathscr{O}_C} \mathscr{N}$ and $\mathcal{H}om_{\mathscr{O}_C}(\mathscr{M}, \mathscr{N})$ are also regular.*

Proof. For $p \in \bar{C} \setminus C$, we have

$$j_*(\mathscr{M} \otimes_{\mathscr{O}_C} \mathscr{N})_p \cong j_*(\mathscr{M})_p \otimes_{K_{C,p}} j_*(\mathscr{N})_p, \quad j_*(\mathcal{H}om_{\mathscr{O}_C}(\mathscr{M}, \mathscr{N}))_p \cong \mathrm{Hom}_{K_{C,p}}(j_*(\mathscr{M})_p, j_*(\mathscr{N})_p).$$

Hence the assertion follows from Lemma 1.5.8 and Proposition 1.5.7. □

Proposition 1.5.12. *Let V be a subset of $\bar{C} \setminus C$ and set $C' = C \cup V$. We denote by $j : C \to C'$ the embedding. For each point $p \in V$, we fix a local parameter x_p and set $\theta_p = x_p \frac{d}{dx_p}$. Then the following three conditions for an integrable connection $\mathscr{M}$ on C are equivalent:*

(i) *$\mathscr{M}$ has regular singularity at any $p \in V$.*

(ii) *$j_*(\mathscr{M})$ is a union of coherent $\mathscr{O}_{C'}$-submodules which are stable under the action of θ_p for any $p \in V$.*

(iii) *There exists a coherent $\mathscr{D}_{C'}$-module $\mathscr{M}'$ such that $\mathscr{M}'|_C \cong \mathscr{M}$ and $\mathscr{M}'$ is a union of coherent $\mathscr{O}_{C'}$-submodules which are stable under the action of θ_p for any $p \in V$.*

Proof. It is clear that (ii)⇒(iii), and for $\mathscr{M}'$ as in (iii) we have $j_*(\mathscr{M})_p = K_{C',p} \otimes_{\mathscr{O}_{C',p}} \mathscr{M}'_p$ for $p \in V$. Therefore, Proposition 1.5.5 implies that $\mathscr{M}$ has regular singularity at p. Now assume that $\mathscr{M}$ has regular singularity at any $p \in V$. For $p \in V$ we take an $\mathscr{O}_{C',p}$-lattice L_p of $(j_*(\mathscr{M})_p, \nabla)$. Then $x_p^{-i} L_p$ is also an $\mathscr{O}_{C',p}$-lattice of $(j_*(\mathscr{M})_p, \nabla)$ for each integer $i \geq 1$, and we have $j_*(\mathscr{M})_p = \bigcup_i x_p^{-i} L_p$. Note that there exists an open subset U_p of $C \cup \{p\}$ containing p such that $x_p^{-i} L_p$ is extended to a coherent $\mathscr{O}_{U_p}$-submodule $\mathscr{L}_p^i$ of $j_*(\mathscr{M})|_{U_p}$ satisfying $\mathscr{L}_p^i|_{U_p \cap C} = \mathscr{M}|_{U_p \cap C}$. Then $\mathscr{L}_p^i$ for $p \in V$ can be glued together and we obtain a coherent $\mathscr{O}_{C'}$-submodule $\mathscr{L}^i$ of $j_*(\mathscr{M})$ which is satable under the action of θ_p for $p \in V$. We then have $j_*(\mathscr{M}) = \bigcup_i \mathscr{L}^i$. □

Lemma 1.5.13. *Let C be an algebraic curve. Then a coherent $\mathscr{D}_C$-module $\mathscr{M}$ is holonomic if and only if it is generically an integrable connection.*

Proof. The only if part follows from Proposition 1.3.2. Assume that $\mathscr{M}$ is generically an integrable connection, i.e., there exists an open dense subset U of C such that $\mathscr{M}|_U$ is an integrable connection. Note that $V = C \setminus U$ consists of finitely many points. We see from our assumption that the characteristic variety $\mathrm{Ch}(\mathscr{M})$ of $\mathscr{M}$ is contained in $T^*_C C \cup \bigcup_{p \in V} T^*_p C$. By $\dim(T^*_C C) = \dim(T^*_p C) = 1$, we have $\dim(\mathrm{Ch}(\mathscr{M})) \leq 1$, and hence $\mathscr{M}$ is holonomic. $\square$

A holonomic $\mathscr{D}_C$-module $\mathscr{M}$ on an algebraic curve C is said to be regular if there exists an open dense subset U of C such that $\mathscr{M}|_U$ is a regular integrable connection on U. An object $\mathscr{M}$ of $D^b_h(\mathscr{D}_C)$ is said to be regular if all of the cohomology sheaves $\mathscr{M}$ are regular. By definition, a holonomic $\mathscr{D}_C$-module supported on a finite set is regular.

Example 1.5.1. Consider an algebraic differential equation $P(x, \partial)u = 0$ on $\mathbb{A}^1 = \mathbb{C}$. Then the holonomic $\mathscr{D}_C$-module $\mathscr{M} = \mathscr{D}_{\mathbb{C}} u = \mathscr{D}_{\mathbb{C}} / \mathscr{D}_{\mathbb{C}} P(x, \partial)$ is regular if and only if the equation $P(x, \partial)u = 0$ has a regular singular point (in the classical sense) at any point in $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$ (i.e. $P(x, \partial)u = 0$ is a Fuchsian ODE).

The following proposition plays a crucial role in defining the regularity of holonomic D-modules over higher-dimensional varieties.

Proposition 1.5.14. *Let $f : C \to C'$ be a dominant morphism of algebraic curves. Then*

(a) *$\mathscr{M} \in \mathbf{Mod}_h(\mathscr{D}_{C'})$ is regular if and only if $f^!(\mathscr{M})$ is regular.*

(b) *$\mathscr{N} \in \mathbf{Mod}_h(\mathscr{D}_C)$ is regular if and only if $f_+(\mathscr{M})$ is regular.*

Proof. We may assume that $C = \bar{C}$ and $C' = \bar{C}'$, and take an open subset C'_0 of C' such that $f_0 : C_0 \to C'_0$ is étale and $\mathscr{M}|_{C'_0}$ and $\mathscr{N}_{C_0}$ are integrable connection (where $C_0 = f^{-1}(C'_0)$). For $p \in C \setminus C_0$ (resp $p' \in C' \setminus C'_0$). we take a local parameter x_0 (resp. $y_{p'}$) and set $\theta_p = x_p \partial_{x_p}$ (resp. $\theta_{p'} = y_{p'} \partial_{y_{p'}}$). If $p' = f(p)$, then we may assume that $y_{p'} = x_p^{m_p}$ for some integer m_p, so that $\theta_p = m_p \theta_{p'}$. We denote by $j : C_0 \to C$ and $j : C'_0 \to C'$ the canonical immersions.

To prove (a), we may assume that $\mathscr{M} = j'_*(\mathscr{M}|_{C'_0})$. Note that $Lf^*(\mathscr{M})|_{C_0} = f_0^*(\mathscr{M}|_{C'_0})$ is an integrable connection, so $\mathscr{M}$ (resp. $f^!(\mathscr{M})$) is regular if and only if $\mathscr{M}_{p'}$ (resp. $j_* f_0^*(\mathscr{M}|_{C'_0})_p$) is a regular algebraic meromorphic connection for any $p' \in C' \setminus C'_0$ (resp. $p \in C \setminus C_0$). Suppose that $p \in C \setminus C_0$ and $p' = f(p)$, then we have $\mathscr{O}_{C',p'} \subseteq \mathscr{O}_{C,p}$ by the hypothesis on f, and

$$\begin{aligned} j_* f_0^*(\mathscr{M}|_{C'_0})_p &\cong K_{C,p} \otimes_{\mathscr{O}_{C,p}} f^*(\mathscr{M})_p \cong K_{C,p} \otimes_{\mathscr{O}_{C',p'}} \mathscr{M}_{p'} \\ &\cong (K_{C,p} \otimes_{\mathscr{O}_{C',p'}} K_{C',p'}) \otimes_{K_{C',p'}} \mathscr{M}_{p'} \\ &\cong K_{C,p} \otimes_{K_{C',p'}} \mathscr{M}_{p'}. \end{aligned}$$

Therefore, if $\mathscr{M}_{p'}$ has an $\mathscr{O}_{C',p'}$-lattice L, then $\mathscr{O}_{C,p} \otimes_{\mathscr{O}_{C',p'}} L$ is an $\mathscr{O}_{C,p}$-lattice of $K_{C,p} \otimes_{K_{C',p'}} \mathscr{M}_{p'}$. COnversely, assume that $K_{C,p} \otimes_{K_{C',p'}} \mathscr{M}_{p'}$ is a regular algebraic meromorphic connection at p. Then by Proposition 1.5.5 we have $K_{C,p} \otimes_{K_{C',p'}} \mathscr{M}_{p'} = \bigcup_i L_i$, where L_i is a θ_p-stable finitely generated $\mathscr{O}_{C,p}$-module. Since $\mathscr{O}_{C,p}$ is finitely generated over $\mathscr{O}_{C',p'}$, $L'_i := L_i \cap (1 \otimes M_{p'})$ is finitely generated over $\mathscr{O}_{C',p'}$. Moreover, by the relation $\theta_p = m_p \theta_{p'}$, we see that L'_i is $\theta_{p'}$-stable. It then follows from Proposition 1.5.5 that $\mathscr{M}_{p'} = 1 \otimes \mathscr{M}_{p'}$ is also regular, and this proves assertion (a).

Now consider assertion (b). We have $f_+(\mathscr{N})|_{C'_0} = (f_0)_*(\mathscr{N}|_{C_0})$ and $(f_0)_*(\mathscr{N}|_{C_0})$ is an integrable connection. Moreover, $\mathscr{N}$ (resp. $f_+(\mathscr{N})$) is regular if and only if $j_*(\mathscr{N}|_{C_0})$ (resp. $j'_*(f_0)_*(\mathscr{N}|_{C_0})$) is a union of coherent $\mathscr{O}_C$ (resp. $\mathscr{O}_{C'}$)-modules which are stable under θ_p (resp. $\theta_{p'}$) for any $p \in C \setminus C_0$ (resp. $p' \in C' \setminus C'_0$). Note that $j'_*(f_0)_*(\mathscr{N}|_{C_0}) \cong f_* j_*(\mathscr{N}|_{C_0})$, so if $j_*(\mathscr{N}|_{C_0})$

is a union of coherent $\mathscr{O}_C$-modules $\mathscr{L}_i$ which are stable udner θ_p for $p \in C \setminus C_0$, then $f_*j_*(\mathscr{N}|_{C_0})$ is a union of $\mathscr{O}_{C'}$-modules $f_*(\mathscr{L}_i)$ which are stable under $\theta_{p'}$ for $p' \in C' \setminus C'_0$. Conversely, if $f_+(\mathscr{N})$ is regular, then $Lf^*f_+(\mathscr{N})$ is also regular by (a). The restriction of the canonical morphism $\mathscr{N} \to Lf^*f_+(\mathscr{N})$ to C_0 is given by $\mathscr{N}|_{C_0} \to f^*f_*(\mathscr{N}_0)$, whence a monomorphism. This implies the regularity of $\mathscr{N}$ and completes the proof. □

Let us give comments on the difference of the notion of regularity in algebraic and analytic situations. Let C be a one-dimensional complex manifold and V be a finite subset of C. We denote by $j : U := C \setminus V \to C$ the canonical embedding. Let $\mathscr{M}$ be an integrable connection on U. We say that a coherent $\mathscr{D}_C$-module $\widetilde{\mathscr{M}}$ is a meromorphic extension of $\mathscr{M}$ if $\widetilde{\mathscr{M}}|_U \cong \mathscr{M}$ and $\widetilde{\mathscr{M}}$ is isomorphic as an $\mathscr{O}_C$-module to a locally free $\mathscr{O}_C(V)$-module, where $\mathscr{O}_C(V)$ denotes the sheaf of meromorphic functions on C with possible poles on V. The following example shows that in the analytic situation a meromorphic extension of an integrable connection is not uniquely determined and one cannot define the notion of the regularity of an integrable connection at a boundary point unless its meromorphic extension is specified. Nevertheless, as we see later the uniqueness of a regular meromorphic extension in the analytic situation holds true as a part of the Riemann-Hilbert correspondence.

Example 1.5.2. We regard $C = \mathbb{C}$ as an algebraic curve, and let $j : U := C \setminus \{0\} \to C$ be the embedding. Let us consider two (algebraic) integrable connections $\mathscr{M} = \mathscr{D}_U/\mathscr{D}_U\partial$ and $\mathscr{N} = \mathscr{D}_U/\mathscr{D}_U(x^2\partial - 1)$ on U. We have an isomorphism $\mathscr{M}^{\mathrm{an}} \cong \mathscr{N}^{\mathrm{an}}$ given by

$$\mathscr{M}^{\mathrm{an}} \to \mathscr{N}^{\mathrm{an}}, \quad [P \mod \mathscr{D}_{U^{\mathrm{an}}}\partial] \mapsto [P\exp(1/x) \mod \mathscr{D}_{U^{\mathrm{an}}}(x^2\partial - 1)].$$

We consider meromorphic extensions $j_*(\mathscr{M})^{\mathrm{an}}$ and $j_*(\mathscr{N})^{\mathrm{an}}$ of $\mathscr{M}^{\mathrm{an}}$ and $\mathscr{N}^{\mathrm{an}}$, respectively. Let us show that they are not isomorphic. Note that $\mathscr{M}$ is regular since it is isomorphic to $\mathscr{O}_U$ as a $\mathscr{D}_U$-module, so $j_*(\mathscr{M})_0^{\mathrm{an}}$ is a regular meromorphic connection. It is then sufficient to show that $j_*(\mathscr{N})_0^{\mathrm{an}}$ is not regular as a meromorphic connection, which can be easily shown by checking that its horizontal sections do not have moderate growth.

1.5.2 Regular meromorphic connections on complex manifolds

1.5.2.1 Meromorphic connections in higher dimensions Let X be a complex manifold and D be a divisor (complex hypersurface) of X. We denote by $\mathscr{O}_X(D)$ the sheaf of meromorphic functions associated to D. For a local defining equation $g \in \mathscr{O}_X$ of D, we have $\mathscr{O}_X(D) = \mathscr{O}_X[g^{-1}]$, so $\mathscr{O}_X(D)$ is a coherent sheaf of rings. A **meromorphic connection $\mathscr{M}$ along the divisor D** is defined to be a coherent $\mathscr{O}_X(D)$-module $\mathscr{M}$ endowed with a $\mathbb{C}$-linear morphism $\nabla : \mathscr{M} \to \Omega^1_X \otimes_{\mathscr{O}_X} \mathscr{M}$ satisfying the conditions

$$\nabla(fs) = df \otimes s + f\nabla s, \qquad f \in \mathscr{O}_X(D), s \in \mathscr{M}, \tag{1.5.8}$$

$$[\nabla_v, \nabla_w] = \nabla_{[v,w]}, \qquad v, w \in \Theta_X. \tag{1.5.9}$$

A morphism $\varphi : (\mathscr{M}, \nabla) \to (\mathscr{N}, \nabla)$ of meromorphic connections along D is defined to be an $\mathscr{O}_X(D)$-linear morphism $\varphi : M \to N$ satisfying $\nabla \circ \varphi = (\mathrm{id} \otimes \varphi) \circ \nabla$. Finally, for a meromorphic connection $(\mathscr{M}, \nabla)$ along D, we set $\mathscr{M}^{\nabla} = \{s \in \mathscr{M} : \nabla s = 0\}$, whose elements are called **horizontal sections** of $(\mathscr{M}, \nabla)$.

We denote by $\mathrm{Conn}(X, D)$ the category of meromorphic connections along D. Note that $\mathrm{Conn}(X, D)$ is an abelian category and an object of $\mathrm{Conn}(X, D)$ is a $\mathscr{D}_X$-module which is isomorphic as an $\mathscr{O}_X$-module to a coherent $\mathscr{O}_X(D)$-module, and a morphism $(\mathscr{M}, \nabla) \to (\mathscr{N}, \nabla)$ is just a morphism of the corresponding $\mathscr{D}_X$-modules. Therefore, $\mathrm{Conn}(X, D)$ is naturally regarded as a subcategory of $\mathbf{Mod}(\mathscr{D}_X)$. For $(\mathscr{M}, \nabla) \in \mathrm{Conn}(X, D)$, the restriction $\mathscr{M}|_{X \setminus D}$ of $\mathscr{M}$ to $X \setminus D$ belongs to $\mathrm{Conn}(X)$; i.e., $\mathscr{M}|_{X \setminus D}$ is locally free over $\mathscr{O}_{X \setminus D}$ (cf. Theorem 1.1.21). In that sense, a meromorphic connection is an extension of a vector bundle with integrable connection on $X \setminus D$ to an object on X with singularities along D.

Remark 1.5.2. Assume that $\dim(X) = 1$ and D has simple multiplicities. Let $p \in D$ and take a local coordinate x such that $x(p) = 0$. Then the stalk $\mathscr{O}_X(D)_p$ at $p \in D$ is identified with the quotient field $\mathbb{C}\{\{x\}\}[x^{-1}]$ of $\mathscr{O}_{X,p} \cong \mathbb{C}\{\{x\}\}$. Since $\mathscr{O}_X(D)_p$ is a field, any coherent $\mathscr{O}_X(D)$-module is free on a open neighborhood of p. Hence the stalk $(\mathscr{M}_p, \nabla_p)$ of $(\mathscr{M}, \nabla) \in \mathrm{Conn}(X, D)$ at $p \in D$ turns out to be a meromorphic connection in the sense of 1.5.1.1 by identifying Ω^1_X with $\mathscr{O}_X$ via $dx \in \Omega^1_X$ (note that condition (1.5.9) is automatically satisfied in the one-dimensional situation).

For $(\mathscr{M}, \nabla), (N, \nabla) \in \mathrm{Conn}(X, D)$, the $\mathscr{O}_X(D)$-modules $\mathscr{M} \otimes_{\mathscr{O}_X(D)} \mathscr{N}$ and $\mathcal{H}om_{\mathscr{O}_X(D)}(\mathscr{M}, \mathscr{N})$ are endowed with structures of meromorphic connections along D by

$$\nabla(s \otimes t) = \sum_i \omega_i \otimes (s_i \otimes t) + \sum_j \alpha_j \otimes (s \otimes t_j),$$
$$(\nabla\varphi)(s) = (\mathrm{id} \otimes \varphi)(\nabla s) - \nabla(\varphi(s)),$$

where $\nabla s = \sum_i \omega_i \otimes s_i$ and $\nabla t = \sum_j \alpha_j \otimes t_j$. In particular, for $(\mathscr{M}, \nabla) \in \mathrm{Conn}(X, D)$, its dual $\mathscr{M}^* = \mathcal{H}om_{\mathscr{O}_X(D)}(\mathscr{M}, \mathscr{O}_X(D))$ is naturally endowed with a structure of a meromorphic connection along D. The following simple observation will be effectively used in proving the classical Riemann-Hilbert correspondence.

Lemma 1.5.15. *For $(\mathscr{M}, \nabla), (N, \nabla) \in \mathrm{Conn}(X, D)$, we have*

$$\Gamma(X, \mathcal{H}om_{\mathscr{O}_X(D)}(\mathscr{M}, \mathscr{N})^{\nabla}) = \mathrm{Hom}_{\mathrm{Conn}(X,D)}((\mathscr{M}, \nabla), (\mathscr{N}, \nabla)).$$

Proof. This follows from the definition of the connection ∇ for $\mathcal{H}om_{\mathscr{O}_X(D)}(\mathscr{M}, \mathscr{N})$. □

Proposition 1.5.16. *Let $\varphi : \mathscr{M} \to \mathscr{N}$ be a morphism of meromorphic connections along D. If $\varphi|_{X \setminus D}$ is an isomorphism, then φ is an isomorphism.*

Proof. We note that the kernel and cokernel of φ are coherent $\mathscr{O}_X(D)$-modules supported by D, so it suffices to prove that any coherent $\mathscr{O}_X(D)$-module $\mathscr{M}$ whose support is contained in D is trivial. To this end, take a local defining equation g of D. For a section $s \in \mathscr{M}$ whose support is contained in D, consider the $\mathscr{O}_X$-coherent submodule $\mathscr{O}_X s \subseteq \mathscr{M}$. Since the support of $\mathscr{O}_X s$ is contained in D, we have $g^N s = 0$ for $N \gg 0$ by Hilbert's Nullstellensatz, so $s = g^{-N} g^N s = 0$. □

Corollary 1.5.17. *Any meromorphic connection $\mathscr{M}$ along D is reflexive in the sense that the canonical morphism $\mathscr{M} \to \mathscr{M}^{**}$ is an isomorphism.*

Proof. This follows from Proposition 1.5.16 and the fact that $\mathscr{M}|_{X \setminus D}$ is locally free. □

Let $f : Y \to X$ be a morphism of complex manifolds such that $f^{-1}(D)$ is a divisor on Y. Then we have

$$\mathscr{O}_Y(f^{-1}(D)) \cong \mathscr{O}_Y \otimes_{f^{-1}(\mathscr{O}_X)} f^{-1}(\mathscr{O}_X(D)) \cong \mathscr{O}_Y \otimes^L_{f^{-1}(\mathscr{O}_X)} f^{-1}(\mathscr{O}_X(D)).$$

Indeed, since $\mathscr{O}_X(D)$ is flat over $\mathscr{O}_X$, we have

$$H^i(\mathscr{O}_Y \otimes^L_{f^{-1}(\mathscr{O}_X)} f^{-1}(\mathscr{O}_X(D))) = 0, \quad i > 0.$$

Moreover, for a local defining equation g of D, we have $\mathscr{O}_X(D) = \mathscr{O}_X[g^{-1}]$ and $\mathscr{O}_Y(f^{-1}(D)) = \mathscr{O}_Y[(g \circ f)^{-1}]$, whence

$$\mathscr{O}_Y(f^{-1}(D)) \cong \mathscr{O}_Y \otimes_{f^{-1}(\mathscr{O}_X)} f^{-1}(\mathscr{O}_X(D)).$$

Proposition 1.5.18. *Let $f : Y \to X$ be a morphism of complex manifolds such that $f^{-1}(D)$ is a divisor on Y. Then for any $\mathscr{M} \in \mathrm{Conn}(X, D)$ we have $H^i(Lf^*(\mathscr{M})) = 0$ for $i > 0$ and $H^0(Lf^*(\mathscr{M})) \in \mathrm{Conn}(Y, f^{-1}(D))$. In particular, the inverse image functor of the category of $\mathscr{O}$-modules induces an exact functor*

$$f^* : \mathrm{Conn}(X, D) \to \mathrm{Conn}(Y, f^{-1}(D)).$$

Proof. In fact, we have

$$\begin{aligned} Lf^*(\mathscr{M}) &\cong \mathscr{O}_Y \otimes^L_{f^{-1}(\mathscr{O}_X)} f^{-1}(\mathscr{M}) \cong \mathscr{O}_Y(f^{-1}(D)) \otimes^L_{f^{-1}(\mathscr{O}_X(D))} f^{-1}(\mathscr{M}) \\ &\cong \mathscr{O}_Y(f^{-1}(D)) \otimes_{f^{-1}(\mathscr{O}_X(D))} f^{-1}(\mathscr{M}), \end{aligned}$$

which proves the claim. □

We consider the unit disk $\mathbb{D} = \{x \in \mathbb{C} : |x| < 1\}$. For a morphism $i : \mathbb{D} \to X$ such that $i^{-1}(D) = \{0\}$, the stalk $i^*(\mathscr{M})_0$ at $0 \in \mathbb{D}$ is a meromorphic connection of one-variable in the sense of 1.5.1.1. A meromorphic connection $\mathscr{M}$ on X along D is called **regular** if $i^*(\mathscr{M})_0$ is regular in the sense of 1.5.1.2 for any morphism $i : \mathbb{D} \to X$ such that $i^{-1}(D) = \{0\}$. We denote by $\mathrm{Conn}^{\mathrm{reg}}(X, D)$ the category of regular meromorphic connections along D.

Proposition 1.5.19. *Let D be a divisor of the complex manifold X.*

(a) *Let $0 \to \mathscr{M}' \to \mathscr{M} \to \mathscr{M}'' \to 0$ be a short exact sequence of meromorphic connections along D. Then $\mathscr{M}$ is regular if and only if $\mathscr{M}'$ and $\mathscr{M}''$ are regular.*

(b) *Let $\mathscr{M}$ and $\mathscr{N}$ be regular meromorphic connections along D. Then the meromorphic connections $\mathscr{M} \otimes_{\mathscr{O}_X(D)} \mathscr{N}$ and $\mathcal{H}om_{\mathscr{O}_X(D)}(\mathscr{M}, \mathscr{N})$ are also regular.*

Proof. By definition we can reduce the problem to the case when X is the unit disk $\mathbb{D} \subseteq \mathbb{C}$. Then (a) follows from Proposition 1.5.10 and we can prove (b) by using Proposition 1.5.7. □

Ameromorphic connection on X along D is called **effective** if it is generated as an $\mathscr{O}_X(D)$-module by a coherent $\mathscr{O}_X$-submodule. We will see later that any regular meromorphic connection is effective.

Proposition 1.5.20. *Let $f : X' \to X$ be a proper surjective morphism of complex manifolds such that $D' = f^{-1}(D)$ is a divisor on X' and $f|_{X' \setminus D'} : X' \setminus D' \to X \setminus D$ is an isomorphism. Assume that $\mathscr{N}$ is an effective meromorphic connection on X' along D'.*

(a) *We have $H^i(f_+(\mathscr{N})) = 0$ for $i > 0$ and $H^0(f_+(\mathscr{N}))$ is an effective meromorphic connection on X along D.*

(b) *If $\mathscr{N}$ is regular, then so is $H^0(f_+(\mathscr{N}))$.*

Proof. We denote by $\mathscr{D}_{X'}(D')$ the subalgebra of $\mathcal{E}nd_{\mathbb{C}}(\mathscr{O}_{X'}(D'))$ generated by $\mathscr{D}_{X'}$ and $\mathscr{O}_{X'}(D')$. Then we have

$$\mathscr{D}_{X'}(D') \cong \mathscr{D}_{X'} \otimes_{\mathscr{O}_{X'}} \mathscr{O}_{X'}(D') \cong \mathscr{O}_{X'}(D') \otimes_{\mathscr{O}_{X'}} \mathscr{D}_{X'}.$$

We first show that

$$\mathscr{D}_{X \leftarrow X'} \otimes^L_{\mathscr{D}_{X'}} \mathscr{D}_{X'}(D') \cong \mathscr{D}_{X'}(\mathscr{D}'). \tag{1.5.10}$$

Note that the canonical morphism $f^{-1}(\Omega_X) \to \Omega_{X'}$ induces an isomorphism $\mathscr{O}_{X'}(D') \otimes_{f^{-1}(\mathscr{O}_X)} f^{-1}(\Omega_X) \to \mathscr{O}_{X'}(D') \otimes_{\mathscr{O}_{X'}} \Omega_{X'}$ by Proposition 1.5.16 and the hypothesis on f. We therefore have

$$\begin{aligned} \mathscr{D}_{X \leftarrow X'} \otimes^L_{\mathscr{D}_{X'}} \mathscr{D}_{X'}(D') &\cong \mathscr{D}_{X \leftarrow X'} \otimes^L_{\mathscr{O}_{X'}} \mathscr{O}_{X'}(D') \cong \mathscr{D}_{X \leftarrow X'} \otimes_{\mathscr{O}_{X'}} \mathscr{O}_{X'}(D') \\ &\cong f^{-1}(\mathscr{D}_X) \otimes_{f^{-1}(\mathscr{O}_X)} \mathscr{O}_{X'}(D'). \end{aligned}$$

Let us show that the canonical morphism $\mathscr{D}_{X'}(D') \to f^{-1}(\mathscr{D}_X) \otimes_{f^{-1}(\mathscr{O}_X)} \mathscr{O}_{X'}(D')$ induced by the canonical section $1 \otimes 1$ of the right $\mathscr{D}_{X'}(D')$-module $f^{-1}(\mathscr{D}_X) \otimes_{f^{-1}(\mathscr{O}_X)} \mathscr{O}_{X'}(D')$ is an isomorphism. For this, it suffices to show that $F_p(\mathscr{D}_{X'})(D') \to f^{-1}(F_p(\mathscr{D}_X)) \otimes_{f^{-1}(\mathscr{O}_X)} \mathscr{O}_{X'}(D')$ is an isomorphism for each $p \in \mathbb{Z}$, which follows from Proposition 1.5.16. This completes the proof of (1.5.10).

By hypothesis, there exists a coherent $\mathscr{O}_{X'}$-submodule $\mathscr{L}$ of $\mathscr{N}$ such that $\mathscr{N} \cong \mathscr{O}_{X'}(D') \otimes_{\mathscr{O}_{X'}} \mathscr{L}$. Then by (1.5.10) and $f^{-1}(\mathscr{O}_X(D)) \otimes_{f^{-1}(\mathscr{O}_X)} \mathscr{O}_{X'} \cong \mathscr{O}_{X'}(D')$, we have

$$\begin{aligned} f_+(\mathscr{N}) &= Rf_*(\mathscr{D}_{X \leftarrow X'} \otimes^L_{\mathscr{D}_{X'}} \mathscr{N}) \cong Rf_*(\mathscr{D}_{X \leftarrow X'} \otimes^L_{\mathscr{D}_{X'}} \mathscr{D}_{X'}(D') \otimes^L_{\mathscr{D}_{X'}(D')} \mathscr{N}) \\ &\cong Rf_*(\mathscr{N}) \cong Rf_*(\mathscr{O}_{X'}(D') \otimes_{\mathscr{O}_{X'}} \mathscr{L}) \cong Rf_*(f^{-1}(\mathscr{O}_X(D)) \otimes_{f^{-1}(\mathscr{O}_X)} \mathscr{L}) \\ &\cong \mathscr{O}_X(D) \otimes_{\mathscr{O}_X} Rf_*(\mathscr{L}) \end{aligned}$$

so $H^i(f_+(\mathscr{N})) \cong \mathscr{O}_X(D) \otimes_{\mathscr{O}_X} R^i f_*(\mathscr{L})$ and $H^0(f_+(\mathscr{N}))$ is effective. Since $R^i f_*(\mathscr{L})$ is coherent over $\mathscr{O}_X$ for any i by Grauert direct image theorem, we see that $H^i(f_+(\mathscr{N}))$ is coherent over $\mathscr{O}_X(D)$. Moreover, we have $H^i(f_+(\mathscr{N})) = 0$ for $i > 0$ by $R^i f_*(\mathscr{L})|_{X \setminus D} = 0$ and Proposition 1.5.16, whence statement (a).

Now assume that $\mathscr{N}$ is regular. Let $i : \mathbb{D} \to X$ be a morphism from the unit disk $\mathbb{D}$ satisfying $i^{-1}(D) = \{0\}$. Since f is proper, there exists a lift $j : \mathbb{D} \to X'$ satisfying $f \circ j = i$. Then we have

$$i^* H^0(f_+(\mathscr{N})) \cong i^* f_+(\mathscr{N}) \cong j^* f^*(f_+(\mathscr{N})).$$

Since the canonical morphism $\mathscr{N} \to f^*(f_+(\mathscr{N}))$ is an isomorphism on $X' \setminus D'$, it is an isomorphism on X' by Proposition 1.5.16. We thus obtain that $i^* H^0(f_+(\mathscr{N})) \cong j^*(\mathscr{N})$, so $H^0(f_+(\mathscr{N}))$ is regular. □

1.5.2.2 Meromorphic connections with logarithmic poles We now consider the case where D is a normal crossing divisor on a complex manifold, i.e. we assume that D is locally defined by a function of the form $x_1 \cdots x_r$, where $(x_1, \ldots, x_n)$ is a local coordinate for X. Let $p \in D$ and fix such a coordinate at $p \in D$. We denote by D_k the (local) irreducible component of D defined by x_k for each k.

The meromorphic connections $\mathscr{M}$ on X along D which we will consider in this paragraph are also of very special type. First, we assume that there exists a holomorphic vector bundle (locally free $\mathscr{O}_X$-module of finite rank) $\mathscr{L}$ on X such that $\mathscr{M} = \mathscr{O}_X(D) \otimes_{\mathscr{O}_X} \mathscr{L}$ as an $\mathscr{O}_X(D)$-module. Hence taking a local defining equation $x_1 \cdots x_r = 0$ of D and choosing a basis $e_1, \ldots, e_m$ of $\mathscr{L}$ around a point $P \in D$, the associated $\mathbb{C}$-linear morphism $\nabla : \mathscr{M} \to \Omega^1_X \otimes_{\mathscr{O}_X} \mathscr{M}$ can be expressed as

$$\nabla e_i = \sum_{k,j} a^k_{ij} \frac{dx_k}{x_k} \otimes e_j \tag{1.5.11}$$

where $a^k_{ij} \in \mathscr{O}_X(D)$. If the functions a^k_{ij} are holomorphic, we say the meromorphic connection $\mathscr{M}$ along the normal crossing divisor D has a **logarithmic pole** with respect to the lattice $\mathscr{L}$ at p. If this is the case at any $p \in M$, we say that $\mathscr{M}$ has a **logarithmic pole** along D with respect to $\mathscr{L}$. Note that this definition does not depend on the choice of the coordinates $\{x_k\}$ and the basis $\{e_i\}$ of $\mathscr{L}$.

Let $\mathscr{M}$ be a meromorphic connection along D which has a logarithmic pole with respect to the lattice $\mathscr{L}$. Take a basis $\{e_1, \ldots, e_m\}$ of $\mathscr{L}$ and set $A_k = (a^k_{ij})$ for $1 \le k \le n$ as in (1.5.11). Since A_k belongs to $\mathcal{M}_m(\mathscr{O}_X)$ by hypothesis, we can consider its restriction $A_k|_{D_k} \in \mathcal{M}_m(\mathscr{O}_{D_k})$. Then $A_k|_{D_k}$ defines a canonical section $\mathrm{Res}^{\mathscr{L}}_{D_k}(\nabla)$ of the vector bundle $\mathcal{E}nd_{\mathscr{O}_{D_k}}(\mathscr{L}|_{D_k})$ on D_k. Indeed, we can check easily that $A_k|_{D_k} \in \mathcal{M}_m(\mathscr{O}_{D_k}) \cong \mathcal{E}nd_{\mathscr{O}_{D_k}}(\mathscr{L}|_{D_k})$ does not depend on the choice of a local coordinate and a basis of $\mathscr{L}$. The section $\mathrm{Res}^{\mathscr{L}}_{D_k}(\nabla)$ is called the **residue** of $(\mathscr{M}, \nabla)$ along D_k.

Proposition 1.5.21. *Let $\mathscr{M}$ be a meromorphic connection along D which has a logarithmic pole with respect to the lattice $\mathscr{L}$.*

(a) *On $D_k \cap D_l$ we have $[\mathrm{Res}^{\mathscr{L}}_{D_k}(\nabla), \mathrm{Res}^{\mathscr{L}}_{D_l}(\nabla)] = 0$.*

(b) *The eigenvalues of $\mathrm{Res}^{\mathscr{L}}_{D_k}(\nabla)(x) \in \mathrm{End}_{\mathbb{C}}(\mathscr{L}_x/\mathfrak{m}_x\mathscr{L}_x)$ are locally constant along D_l.*

Proof. With the notations of (1.5.11), we have $\nabla_{\partial_k} = \frac{A^k}{x_k}$, so the integrability condition $[\nabla_{\partial_k}, \nabla_{\partial_l}]$ expands out to

$$\frac{\partial}{\partial x_k}\Big(\frac{A^l}{x_l}\Big) - \frac{\partial}{\partial x_l}\Big(\frac{A^k}{x_k}\Big) = [\frac{A^k}{x_k}, \frac{A^l}{x_l}].$$

After rearranging the terms, this becomes

$$x_k\frac{\partial A^l}{\partial x_k} - x_l\frac{\partial A^k}{\partial x_l} = [A^k, A^l],$$

so the restriction of the two matrices A^k and A^l to the set $x_k = x_l = 0$ commute with each other.

For the proof of (b), let $\overline{\mathscr{L}}$, $\bar{A}^k$ be the restriction of $\mathscr{L}$, A^k to D_l, respectiely. Then the formula

$$\bar{\nabla}\bar{e}_i = \sum_{j,k\neq l} \bar{a}^k_{ij}\frac{dx_k}{x_k} \otimes \bar{e}_j, \quad \bar{A}^k = (\bar{a}^k_{ij})$$

defines an integrable connection $\bar{\nabla}$ on $\mathscr{L}|_{D_l}$, and one checks that $\bar{A}^l = \mathrm{Res}^{\mathscr{L}}_{D_l}(\nabla)$ is a horizontal section of $\mathcal{E}nd_{\mathscr{O}_{D_l}}(\overline{\mathscr{L}})$ with respect to the induced connection. From this we can easily check that the eigenvalues of the matrix $\bar{A}^l$ are locally free along D_l. □

Proposition 1.5.22. *Let $\mathscr{M}$ be a meromorphic connection which has a logarithmic pole along D with respect to a lattice $\mathscr{L}$. Then $\mathscr{M}$ is regular.*

Proof. For any morphism $i : \mathbb{D} \to X$ from the unit disk $\mathbb{D}$ such that $i^{-1}(D) = \{0\}$, we easily see that the meromorphic connection $i^*(\mathscr{M})$ on $\mathbb{D}$ has a logarithmic pole along $\{0\}$ with respect to the lattice $i^*(\mathscr{L})$. Then the stalk $i^*(\mathscr{M})_0$ is regular by Theorem 1.5.2. □

The following construction enables us to extend analytic integrable connections on $X \setminus D$ to regular meromorphic connections along the divisor $D \subseteq X$.

Theorem 1.5.23. *Let D be a normal crossing divisor on X. We fix a section $\tau : \mathbb{C}/\mathbb{Z} \to \mathbb{C}$ of the projection $\mathbb{C} \to \mathbb{C}/\mathbb{Z}$. Then for any integrable connection $\mathscr{M}$ on $X \setminus D$, there exists a unique locally free $\mathscr{O}_X$-module $\mathscr{L}_\tau$ satisfying the following two conditions:*

(a) *We have $\mathscr{L}_\tau|_{X\setminus D} = \mathscr{M}$.*

(b) *The connection $\nabla : \mathscr{M} \to \Omega^1_{X\setminus D} \otimes_{\mathscr{O}_{X\setminus D}} \mathscr{M}$ can be uniquely extended to a $\mathbb{C}$-linear morphism*

$$\nabla : \mathscr{M}_\tau \to \Omega^1_X \otimes_{\mathscr{O}_X} \mathscr{M}_\tau,$$

where $\mathscr{M}_\tau = \mathscr{O}_X(D) \otimes_{\mathscr{O}_X} \mathscr{L}_\tau$, so that $(\mathscr{M}_\tau, \nabla)$ is a meromorphic connection which has a logarithmic pole along D with respect to $\mathscr{L}_\tau$.

(c) *At any irreducible component D_0 of D, the eigenvalues of the residue $\mathrm{Res}^{\mathscr{L}_\tau}_{D_0}(\nabla)$ of $(\mathscr{M}_\tau, \nabla)$ along D_0 are contained in $\tau(\mathbb{C}/\mathbb{Z}) \subseteq \mathbb{C}$.*

Moreover, with the above choice of $\mathscr{L}_\tau$, the restriction map

$$\Gamma(X, \mathscr{M}_\tau^\nabla) \to \Gamma(X \setminus D, \mathscr{M}^\nabla)$$

is an isomorphism.

Proof. Let us start with the local existence, since that is easier. Since we are working locally, we can assume that $X = \mathbb{D}^n$, where $\mathbb{D}$ is the unit disk in $\mathbb{C}$. The divisor D is given by the equation $x_1 \cdots x_r = 0$, so $X \setminus D = (\mathbb{D}^*)^r \times \mathbb{D}^{n-r}$. Let $\mathscr{L}$ and $\mathscr{L}'$ be locally free $\mathscr{O}_X$-modules of rank m satisfying the above conditions, and

$$\nabla : \mathscr{O}_X(D) \otimes_{\mathscr{O}_X} \mathscr{L} \to \Omega_X^1 \otimes_{\mathscr{O}_X} \mathscr{O}_X(D) \otimes_{\mathscr{O}_X} \mathscr{L}, \quad \nabla : \mathscr{O}_X(D) \otimes_{\mathscr{O}_X} \mathscr{L}' \to \Omega_X^1 \otimes_{\mathscr{O}_X} \mathscr{O}_X(D) \otimes_{\mathscr{O}_X} \mathscr{L}'$$

be the correponding connections. Take a basis $\{e_i\}$ of $\mathscr{L}$ and a basis $\{e_i'\}$ of $\mathscr{L}'$. Then we can write

$$\nabla e_i = \sum_{k,j} a_{ij}^k \frac{dx_k}{x_k} \otimes e_j, \quad \nabla' e_i' = \sum_{k,j} a'^k_{ij} \frac{dx_k}{x_k} \otimes e_j'.$$

We set $\omega = \sum_k A^k \frac{dx_k}{x_k}$ and $\omega' = \sum_k A'^k \frac{dx_k}{x_k}$. By assumption $\mathscr{L}|_{X\setminus D}$ and $\mathscr{L}'|_{X\setminus D}$ are isomorphic. After a short calculation, the isomorphism between the two bundles with connection translates into the existence of an invertible matrix $S \in \mathrm{GL}_m(\mathscr{O}_{X\setminus D})$ such that

$$dS = S\omega - \omega' S. \tag{1.5.12}$$

The entries of S are holomorphic functions on $X \setminus D$, possibly with essential singularities along D. To prove the uniqueness statement, it is enough to show that $S \in \mathrm{GL}_m(\mathscr{O}_X)$, meaning that the entries of S can be extended to holomorphic functions on X. By Hartog's theorem, holomorphic functions extend over subsets of codimension ≥ 2, so we only need to to prove that the entries of S extend over the generic point of each irreducible component of D. To keep the notation simple, we will check this at points of $D_1 \setminus \bigcup_{k\neq 1} D_k$, meaning at points where $x_1 = 0$ but $x_2 \cdots x_r \neq 0$. Now in view of the relation (1.5.12), we have

$$x_1 \frac{\partial S}{\partial x_1} = SA^1 - A'^1 S, \tag{1.5.13}$$

and after taking the matrix norm of both sides, we obtain that

$$|x_1| \cdot \|\frac{\partial S}{\partial x_1}\| \leq C\|S\|,$$

where $C > 0$ is a constant that depends on the size of the (holomorphic) entries of the two matrices A^1 and A'^1. We can now apply Grönwall's inequality to deduce that the entries of S have moderate growth near x_1, hence are meromorphic functions on the set where $x_2 \cdots x_r \neq 0$.

To show that S is actually holomorphic on $D_1 \setminus \bigcup_{k\neq 1} D_k$, we consider the Laurent expansion

$$S = \sum_{j=p}^{\infty} S_j x^j$$

where $S_p \neq 0$ is the leading term. After substituting this into (1.5.13), we get

$$\sum_{j=p}^{\infty} jS_j x_1^j = \sum_{j=p}^{\infty} (S_j A^1 - A'^1 S_j) x_1^j,$$

in which the coefficients at x^p equals to

$$pS_p = S_p \cdot A^1|_{x_1=0} - A'^1|_{x_1=0} \cdot S_p = S_p \cdot \mathrm{Res}_{D_1}^{\mathscr{L}}(\nabla) - \mathrm{Res}_{D_1}^{\mathscr{L}'}(\nabla') \cdot S_p.$$

Since $\mathrm{Res}^{\mathscr{L}}_{D_1}(\nabla)$ and $\mathrm{Res}^{\mathscr{L}'}_{D_1}(\nabla')$ have their eigenvalues contained in $\tau(\mathbb{C}/\mathbb{Z})$, this relation forces $p = 0$. Indeed, suppose that v is a nontrivial eigenvector for $\mathrm{Res}^{\mathscr{L}}_{D_1}(\nabla)$, with eigenvalue λ. Then

$$p(S_p v) = \lambda(S_p v) - \mathrm{Res}^{\mathscr{L}'}_{D_1}(\nabla')(S_p v),$$

so $S_p v$ is an eigenvector for $\mathrm{Res}^{\mathscr{L}'}_{D_1}(\nabla')$, with eigenvalue $\lambda - p$. As the difference of these two eigenvalues is an integer, this can only happen for $p = 0$. The conclusion is that S extends holomorphically to all of X, proving the desired uniqueness.

Next we prove the existence of the extension (as a vector bundle) $\mathscr{L}_\tau$ of $\mathscr{M}$. If there exists locally such an extension $\mathscr{L}_\tau$, we can glue these local extensions to get the global one by the uniqueness of L proved above. Hence we may assume that $X \setminus D = (\mathbb{D}^*)^r \times \mathbb{D}^{n-r} \subseteq X = \mathbb{D}^n$. By Theorem 1.4.4 the integrable connection $\mathscr{M}$ on $X \setminus D$ is uniquely determined by the monodromy representation

$$\rho : \pi_1(X \setminus D) \to \mathrm{GL}_m(\mathbb{C})$$

defined by the local system corresponding to $\mathscr{M}$. Note that $\pi_1(X \setminus D) = \mathbb{Z}^r$, so this corresponds to r commuting matrices $C_1, \dots, C_r$ of $\mathrm{GL}_m(\mathbb{C})$. It is a simple exercise to show that there are uniquely determined matrices $\Gamma^i \in \mathcal{M}_m(\mathbb{C})$ such that

(a) $\exp(2\pi \mathrm{i} \Gamma^i) = C_i$,

(b) all the eigenvalues of Γ^i belong to $\tau(\mathbb{C}/\mathbb{Z})$,

(c) the Γ^i's are mutuallt commuting matrices.

We can now define $\mathscr{L}_\tau = \mathscr{O}_X^{\otimes m}$, and put a meromorphic connection on $\mathscr{M}_\tau = \mathscr{O}_X(D) \otimes_{\mathscr{O}_X} \mathscr{L}_\tau$ by the formula

$$\nabla e_i = \sum_{j,k} \Gamma^k_{i,j} \frac{dx_k}{x_k} \otimes e_j.$$

From the construction, it is clear that this has the three properties in the statement of the theorem. Moreover, a horizontal section of $\mathscr{M}$ is the same thing as a monodromy invariant vector $v \in \mathbb{C}^m$, meaning one with $C^1 v = \cdots = C^r v = 0$. This is equivalent to $\Gamma^1 v = \cdots = \Gamma^r v = 0$, so v also represents a horizontal section of $\mathscr{M}_\tau$. □

1.5.2.3 Deligne's Riemann-Hilbert correspondence In Theorem 1.5.23 we proved that an integrable connection $\mathscr{M}$ defined on the complement $X \setminus D$ of a normal crossing divisor D on X can be extended to a meromorphic connection $\mathscr{M}_\tau = \mathscr{O}_X(D) \otimes_{\mathscr{O}_X} \mathscr{L}_\tau$ on X, regular along D. We now generalize this result to arbitrary divisors on X.

Let D be a (not necessarily normal crossing) divisor on X and $(\mathscr{M}, \nabla)$ be a meromorphic connection on X which is meromorphic along D. We consider the following condition (R) on $(\mathscr{N}, \nabla)$, which is a priori weaker than the regularity along D (in fact, we will see that these two conditions are equivalent).

(R) There exists an open subset U of the regular part D_{reg} of D which intersects with each connected component of D_{reg} and satisfies the following condition: There exist an open neighborhood $\widetilde{U}$ of U in X and an isomorphism $\varphi : \mathbb{D} \times U \xrightarrow{\sim} \widetilde{U}$ such that $\varphi|_{\{0\}\times U} = \mathrm{id}_U$ and for each $x \in U$ the pull-back $(\varphi_x^*(\mathscr{M}), \varphi_x^*(\nabla))$ with respcet to $\varphi_x = \varphi|_{\mathbb{D}\times\{x\}} : \mathbb{D} \times \{x\} \to X$ is regular along $\{0\} \times \{x\}$.

It is clear that if $(\mathscr{M}, \nabla)$ is regular along D, then it satisfies the condition (R).

Lemma 1.5.24. *Assume that a meromorphic connection $(\mathscr{M}, \nabla)$ satisfies the condition (R). Then the restriction map*

$$\Gamma(X, \mathscr{M}^\nabla) \to \Gamma(X \setminus D, \mathscr{M}^\nabla)$$

Proof. The injectivity follows from Proposition 1.5.16. We need to show that any $s \in \Gamma(X \setminus D, \mathscr{M}^\nabla)$ can be extended to a section of $\mathscr{M}^\nabla$ on X. By Corollary 1.5.17, it is sufficient to show that for any $p \in D$ and any $u \in \mathscr{N}_p^* = \mathcal{H}om_{\mathscr{O}_X(D)}(\mathscr{N}, \mathscr{O}_X(D))_p$, the function $g = \langle u, s \rangle$ is meromorphic at p. By Hartgos' theorem, it suffices to consider the case $p \in D_{\mathrm{reg}}$.

We first consider the case where $p \in U \subseteq D_{\mathrm{reg}}$, where U is as in condition (R). We take a local coordinate $(x_1, \dots, x_n)$ at p such that D is defined by $x_1 = 0$, and we may assume that $U = \{0\} \times \mathbb{D}^{n-1}$. Let

$$g(x) = \sum_{k \in \mathbb{Z}} g_k(x_2, \dots, x_n) x_1^k$$

by the Laurent expansion of g with respect to x_1. Condition (R) then implies that for each $x' = (x_2, \dots, x_n) \in \mathbb{D}^{n-1}$ the restriction $g|_{\mathbb{D}^* \times \{x'\}}$ is meromorphic at the point $\{0\} \times \{x'\}$, which means for each $x' = (x_2, \dots, x_n) \in \mathbb{D}^{n-1}$, we have $g_k(x') = 0$ for $k \ll 0$. For $k \in \mathbb{Z}$, we set

$$U_k = \{x' = (x_2, \dots, x_n) \in \mathbb{D}^{n-1} : g_i(x') = 0 \text{ for } i \leq k\}.$$

Then we have $\mathbb{D}^{n-1} = \bigcup_{k \in \mathbb{Z}} U_k$ and each U_k is a closed analytic subset of $\mathbb{D}^{n-1}$. It then follows Baire's category theorem that U_k has nonempty interior for some k, and hence g is meromorphic at p.

Let us consider the general case $p \in D_{\mathrm{reg}}$. We denote by K the subset of D_{reg} consisting of $p \in D_{\mathrm{reg}}$ such that $g = \langle u, s \rangle$ is meromorphic at p for any $u \in \mathscr{N}_p^*$. Note that K is an open subset of D_{reg} containing U; in particular, it intersects with any connected component of D_{reg}, so it suffices to show that K is also a closed subset of D_{reg}. Let $q \in \bar{K}$; we take a local coordinate $(x_1, \dots, x_n)$ at q such that D is defined by $x_1 = 0$. For $u \in \mathscr{N}_p^*$, we consider the expansion

$$g(x) = \sum_{k \in \mathbb{Z}} g_k(x_2, \dots, x_n) x_1^k$$

of $g = \langle u, s \rangle$. Since g is meromorphic on K, there exists some integer $r \in \mathbb{Z}$ such that $g_k = 0$ on K for $k \leq r$. It follows from continuity that $g_k = 0$ for any $k \leq r$ on an open neighborhood of q, so $q \in K$. $\square$

Lemma 1.5.25. *Let $(\mathscr{M}, \nabla)$ and $(\mathscr{N}, \nabla)$ be meromorphic connections along D satisfying the condition (R). Then the restriction to $X \setminus D$ induces an isomorphism*

$$\mathrm{Hom}_{\mathrm{Conn}(X,D)}((\mathscr{M}, \nabla), (\mathscr{N}, \nabla)) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Conn}(X \setminus D)}((\mathscr{M}|_{X \setminus D}, \nabla), (\mathscr{N}|_{X \setminus D}, \nabla))$$

Proof. By Lemma 1.5.15 we have

$$\Gamma(X, \mathcal{H}om_{\mathscr{O}_X(D)}(\mathscr{M}, \mathscr{N})^\nabla) = \mathrm{Hom}_{\mathrm{Conn}(X,D)}((\mathscr{M}, \nabla), (\mathscr{N}, \nabla)).$$

Therefore, it suffices to apply Lemma 1.5.24 to the meromorphic connection $\mathcal{H}om_{\mathscr{O}_X(D)}(\mathscr{M}, \mathscr{N})$, which satisfies the condition (R) by Proposition 1.5.19 (b). $\square$

Theorem 1.5.26 (Deligne). *Let X be a complex manifold and D be a (not necessarily normal crossing) divisor on X. Then the restriction functor $\mathscr{N} \mapsto \mathscr{N}|_{X \setminus D}$ induces an equivalence*

$$\mathrm{Conn}^{\mathrm{reg}}(X, D) \xrightarrow{\sim} \mathrm{Conn}(X \setminus D)$$

of categories.

Proof. Since a regular meromorphic connection along D satisfies the condition (R), the restriction functor is fully faithful by Lemma 1.5.25. Let us prove the essential surjectivity. We take an integrable connection $\mathscr{M}$ on $X \setminus D$ and consider the problem of extending $\mathscr{M}$ to a regular meromorphic connection on the whole X. By Hironaka's theorem there exists a proper surjective morphism $f : X' \to X$ of complex manifolds such that $D' = f^{-1}(D)$ is a normal crossing divisor on X and the restriction $g : X' \setminus D' \to X \setminus D$ of f is an isomorphism. By Theorem 1.5.23 we can extend the integrable connection $g^*(\mathscr{M})$ to a meromorphic connection $\mathscr{N}$ on X' along D' which has a logarithmic pole with respect to a lattice $\mathscr{L}$. Then $H^0(f_+(\mathscr{N}))$ satisfies the desired property by Proposition 1.5.20. □

When D is normal crossing, we proved in Theorem 1.5.23 the uniqueness of the regular meromorphic extension of an integrable connection on $X \setminus D$ under an additional condition about the lattice $\mathscr{L}_\tau$. Theorem 1.5.26 above asserts that this condition was not really necessary.

By Theorem 1.4.4, we have the following topological interpretation of Theorem 1.5.26.

Corollary 1.5.27. *Let X be a complex manifold and D be a divisor on X. Then we have an equivalence*

$$\mathrm{Conn}^{\mathrm{reg}}(X, D) \xrightarrow{\sim} \mathrm{Loc}(X \setminus D)$$

of categories.

We call this result **Deligne's Riemann-Hilbert correspondence**. This classical Riemann-Hilbert correspondence became the prototype of the Riemann-Hilbert correspondence for analytic regular holonomic D-modules.

Corollary 1.5.28. *Let D be a (not necessarily normal crossing) divisor on a complex manifold X and let $\mathscr{M}$ be a meromorphic connection along D.*

(a) *If $\mathscr{M}$ satisfies the condition (R), then $\mathscr{N}$ is regular along D.*

(b) *If $\mathscr{M}$ is regular along D, then $\mathscr{M}$ is an effective meromorphic connection along D.*

Proof. Assume that $\mathscr{M}$ satisfies condition (R). By Theorem 1.5.26, there exists a regular meromorphic connection $\mathscr{N}$ along D such that $\mathscr{N}|_{X\setminus D} \cong \mathscr{M}|_{X\setminus D}$. Since both $\mathscr{M}$ and $\mathscr{N}$ satisfy the condition (R), this isomorphism can be extended to a morphism $\mathscr{M} \xrightarrow{\sim} \mathscr{N}$ of meromorphic connections on X by Lemma 1.5.25. This is in fact an isomorphism by Proposition 1.5.16, and hence $\mathscr{M}$ is regular along D.

Now if $\mathscr{M}$ is regular along D, then in the proof of Theorem 1.5.26 we explicitly constructed a regular meromorphic extension of $\mathscr{M}|_{X\setminus D}$, which is isomorphic to $\mathscr{M}$ and has the required property. □

Assume that D is a normal crossing divisor on X. We define a subsheaf of the sheaf Θ_X of holomorphic vector fields on X by

$$\Theta_X[D] = \{v \in \Theta_X : v\mathscr{I} \subseteq \mathscr{I}\},$$

where $\mathscr{I}$ is the defining ideal of D. If $\{x_i, \partial_i\}$ is a local coordinate system of X in which D is defined by $x_1 \cdots x_r = 0$, then $\Theta_X[D]$ is generated by $x_1\partial_1, \ldots, x_r\partial_r, \partial_{r+1}, \ldots, \partial_n$ over $\mathscr{O}_X$.

Corollary 1.5.29. *Let D be a normal crossing divisor on a complex manifold X. Then the following conditions on a meromorphic connection $\mathscr{M}$ along D are equivalent:*

(i) *$\mathscr{M}$ is regular along D.*

(ii) *$\mathscr{M}$ is a union of $\Theta_X[D]$-stable coherent $\mathscr{O}_X$-submodules.*

Proof. Assume that $\mathscr{M}$ is regular along D. By Theorem 1.5.23, for a section $\tau : \mathbb{C}/\mathbb{Z} \to \mathbb{C}$ of $\mathbb{C} \to \mathbb{C}/\mathbb{Z}$ we have a locally free $\mathscr{O}_X$-module $\mathscr{L}_\tau$ such that $\mathscr{N}|_{X\setminus D} \cong \mathscr{L}_\tau|_{X\setminus D}$ and $\mathscr{O}_X(D) \otimes_{\mathscr{O}_X} \mathscr{L}_\tau$ is a regular meromorphic connection along D. Then by Theorem 1.5.23 we have $\mathscr{M} \cong \mathscr{O}_X(D) \otimes_{\mathscr{O}_X} \mathscr{L}_\tau$. Take a local coordinate system $\{x_i\}$ of X such that D is defined by $g(x) = x_1 \cdots x_r = 0$, we have

$$\mathscr{O}_X(D) \otimes_{\mathscr{O}_X} \mathscr{L}_\tau = \bigcup_{k \geq 0} g^{-k}\mathscr{O}_X \otimes_{\mathscr{O}_X} \mathscr{L}_\tau.$$

Since for each k the definition of the $\mathscr{O}_X$-coherent subshaef $g^{-k}\mathscr{O}_X \otimes_{\mathscr{O}_X} \mathscr{L}_\tau$ of $\mathscr{O}_X(D) \otimes_{\mathscr{O}_X} \mathscr{L}_\tau$ does not depend on the local coordinate $\{x_i\}$ and the defining equation g, it is globally defined on X. It is clear that each $g^{-k}\mathscr{O}_X \otimes_{\mathscr{O}_X} \mathscr{L}_\tau$ is $\Theta_X[D]$-stable, since $\mathscr{L}_\tau$ is $\Theta_X[D]$-stable.

Conversely, assume that $\mathscr{M}$ is a union of $\Theta_X[D]$-stable coherent $\mathscr{O}_X$-submodules. By Corollary 1.5.28 (a) it suffices to check that $\mathscr{M}$ satisfies the condition (R). If we restrict $\mathscr{M}$ to a unit disk $\mathbb{D}$ in the condition (R), then the restricted meromorphic connection satisfies condition (ii) on the unit disk $\mathbb{D}$. By Proposition 1.5.5, this means that the restriction is a regular meromorphic connection at $0 \in \mathbb{D}$. So $\mathscr{N}$ satisfies the condition (R), and hence regular. □

Theorem 1.5.30 (Deligne). *Let D be a divisor on a complex manifold X and let $j : Y = X \setminus D \to X$ be the embedding. Let $\mathscr{M}$ be a regular meromorphic connection along D. Then the natural morphisms*

$$DR_X(\mathscr{M}) \to Rj_*j^{-1}DR_X(\mathscr{M}), \quad R\Gamma(X, DR_X(\mathscr{M})) \to R\Gamma(Y, DR_Y(\mathscr{M}|_Y))$$

are isomorphisms.

Proof. It suffices to prove that $DR_X(\mathscr{M}) \to Rj_*j^{-1}DR_X(\mathscr{M})$ is an isomorphism, since the second assertion follows by taking $R\Gamma$. We first consider the case where D is normal crossing. The problem being local on X, we may assume that $X = \mathbb{D}^n$ and $Y = (\mathbb{D}^*)^r \times \mathbb{D}^{n-r}$, where $\mathbb{D}$ is the unit disk in $\mathbb{C}$ and $\mathbb{D}^* = \mathbb{D} \setminus \{0\}$. Recall that $\mathrm{Conn}^{\mathrm{reg}}(X, D)$ is equivalent to $\mathrm{Loc}(Y)$, so by $\pi_1(Y) = \pi_1(((\mathbb{C}^\times)^r \times \mathbb{C}^{n-r})^{\mathrm{an}})$, we can assume that $X = (\mathbb{C}^n)^{\mathrm{an}}$ and $Y = ((\mathbb{C}^\times)^r \times \mathbb{C}^{n-r})^{\mathrm{an}}$, where $\mathbb{C}^n, \mathbb{C}^\times$ and $\mathbb{C}^{n-r}$ are regareded as algebraic varieties. We can assume also that $\mathscr{M}$ is a simple object of $\mathrm{Conn}^{\mathrm{reg}}(X, D)$. Indeed, let

$$0 \longrightarrow \mathscr{M}_1 \longrightarrow \mathscr{M}_2 \longrightarrow \mathscr{M}_3 \longrightarrow 0$$

be an exact sequence in this category, and denote by $\Phi_i : DR_X(\mathscr{M}_i) \to Rj_*j^{-1}(\mathscr{M}_i)$ the natural morphisms, where $i = 1, 2, 3$. Then Φ_2 is an isomorphism if Φ_1 and Φ_3 are as well. By $\pi_1(Y) \cong \mathbb{Z}^r$, we see that there exists $\lambda_1, \ldots, \lambda_r \in \mathbb{C}$ such that

$$\mathscr{M} \cong (\mathscr{M}_{\lambda_1} \boxtimes \cdots \boxtimes \mathscr{M}_{\lambda_r} \boxtimes \mathscr{O}_{\mathbb{C}^{n-r}})^{\mathrm{an}} \tag{1.5.14}$$

where for $\lambda \in \mathbb{C}$ we denote by $\mathscr{M}_\lambda$ the (algebraic) $\mathscr{D}_{\mathbb{C}}$-module given by $\mathscr{M}_\lambda = \mathscr{D}_{\mathbb{C}}/\mathscr{D}_{\mathbb{C}}(x\partial - \lambda)$. By ([?] Proposition 4.7.8), we then obtain

$$DR_X(\mathscr{M}) \cong DR_{\mathbb{C}}(\mathscr{M}_{\lambda_1}) \boxtimes_{\mathbb{C}} \cdots \boxtimes_{\mathbb{C}} DR_{\mathbb{C}}(\mathscr{M}_{\lambda_r}) \boxtimes_{\mathbb{C}} \mathbb{C}_{(\mathbb{C}^{n-r})^{\mathrm{an}}}.$$

Therefore, it suffices to show that $DR_{\mathbb{C}}(\mathscr{M}_\lambda) \cong Rj_*j^{-1}(\mathscr{M}_\lambda)$ where $j : \mathbb{C}^\times \to \mathbb{C}$ is the canonical embedding. Since the canonical morphism $DR_{\mathbb{C}}(\mathscr{M}_\lambda) \cong Rj_*j^{-1}(\mathscr{M}_\lambda)$ is an isomorphism outside of the origin, we only need to show the isomorphism $DR_{\mathbb{C}}(\mathscr{M}_\lambda)_0 \cong Rj_*j^{-1}(\mathscr{M}_\lambda)_0$ for the stalks at 0. Set $\nabla = \partial - \frac{\lambda}{x}$, then $DR_{\mathbb{C}}(\mathscr{M}_\lambda)_0$ and $Rj_*j^{-1}(\mathscr{M}_\lambda)_0$ are represented, respectively, by the complexes

$$[\mathscr{K} \xrightarrow{\nabla} \mathscr{K}], \quad [\widetilde{\mathscr{K}} \xrightarrow{\nabla} \widetilde{\mathscr{K}}],$$

where $\mathscr{K} = \mathscr{O}_{\mathbb{C}^{\mathrm{an}}}[x^{-1}]$ and $\widetilde{K} = j_*(\mathscr{O}_{(\mathbb{C}^\times)^{\mathrm{an}}})$. From this we easily see by considering the Laurent series expansions of functions in $\mathscr{K}$ and $\widetilde{\mathscr{K}}$ that $DR_{\mathbb{C}}(\mathscr{M}_\lambda)_0 \to Rj_*j^{-1}(\mathscr{M}_\lambda)_0$ is an isomorphism.

Now we consider the general case where D is an arbitrary divisor on X. By Hironaka's theorem there exists a proper surjective morphism $f : X' \to X$ of complex manifolds such that $D' = f^{-1}(D)$ is a normal crossing divisor on X' and the restriction $X' \setminus D' \to X \setminus D$ of f is an isomorphism. We denote by $j' : X' \setminus D' \to X'$ the embedding. By Theorem 1.5.26 and Proposition 1.5.20, there exists a regular meromorphic connection $\mathscr{M}'$ on X' along D' such that $\mathscr{N} \cong f_+(\mathscr{M}')$. Then we have

$$DR_X(\mathscr{M}) \cong DR_X(f_+(\mathscr{M}')) \cong Rf_*DR_{X'}(\mathscr{M}') \cong Rf_*Rj'_*j'^{-1}DR_{X'}(\mathscr{M}') \cong Rh_*j^{-1}DR_X(\mathscr{M})$$

by Theorem 1.4.5, which completes the proof. □

1.5.3 Regular integrable connections on algebraic varieties

In this subsection X denotes a smooth algebraic variety. The corresponding complex manifold is denoted by X^{an}. Assume that we are given an open embedding $j : X \to V$ of X into a smooth variety V such that $D := V \setminus X$ is a divisor on V. We set $\mathscr{O}_V(D) = j_*(\mathscr{O}_X)$. As in the analytic situation, $\mathscr{O}_V(D)$ is a coherent sheaf of rings. We say that a $\mathscr{D}_V$-module is an **algebraic meromorphic connection along D** if it is isomorphic as an $\mathscr{O}_V$-module to a coherent $\mathscr{O}_V(D)$-module. We denote by $\mathrm{Conn}(V, D)$ the category of algebraic meromorphic connections along D, which is an abelian category. Unlike the analytic situation, an extension of an integrable connection on X to an algebraic meromorphic connection on V is unique as follows.

Lemma 1.5.31. *The functor $j^{-1} : \mathrm{Conn}(V, D) \to \mathrm{Conn}(X)$ induces an equivalence of categories, whose quasi-inverse is given by j_*.*

Proof. This follows easily from the fact that the category of coherent $\mathscr{O}_V(D)$-modules is naturally equivalent to that of coherent $\mathscr{O}_X$-modules. □

It follows that $\mathrm{Conn}(V, D)$ is a subcategory of $\mathbf{Mod}_h(\mathscr{D}_V)$ by Theorem 1.3.6. An integrable connection $\mathscr{M}$ on X is called **regular** if for any morphism $i_C : C \to X$ from a smooth algebraic curve C the induced integrable connection $i_C^*(\mathscr{M})$ on C is regular in the sense of 1.5.1.3. We denote by $\mathrm{Conn}^{\mathrm{reg}}(X)$ the full subcategory $\mathrm{Conn}(X)$ consisting of regular integrable connections.

Proposition 1.5.32.

(a) *Let $0 \to \mathscr{M}' \to \mathscr{M} \to \mathscr{M}'' \to 0$ be an exact sequence of integrable connections on X. Then $\mathscr{M}$ is regular if and only if $\mathscr{M}'$ and $\mathscr{M}''$ are regular.*

(b) *Let $\mathscr{M}$ and $\mathscr{N}$ be regular integrable connections on X. Then the integrable connections $\mathscr{M} \otimes_{\mathscr{O}_X} \mathscr{N}$ and $\mathcal{H}om_{\mathscr{O}_X}(\mathscr{M}, \mathscr{N})$ are regular.*

Proof. This follows from Proposition 1.5.6 and Proposition 1.5.7. □

We give below some criteria for the regularity of integrable connections. Let us take a smooth completion $j : X \hookrightarrow \bar{X}$ of X such that $D = \bar{X} \setminus X$ is a divisor on X. Such a completion always exists thanks to Hironaka"s theorem (in fact, we can take a completion X so that $D = \bar{X} \setminus X$ is a normal crossing divisor). We call such a completion $\bar{X}$ of X a **divisor completion**. For a divisor completion $j : X \hookrightarrow \bar{X}$ of X, we can consider the analytic meromorphic connection

$$j_*(\mathscr{M})^{\mathrm{an}} = \mathscr{O}_{\bar{X}^{\mathrm{an}}} \otimes_{\mathscr{O}_{\bar{X}}} j_*(\mathscr{M}) \in \mathrm{Conn}(\bar{X}^{\mathrm{an}}, D^{\mathrm{an}})$$

on $\bar{X}^{\mathrm{an}}$ along D^{an}.

Proposition 1.5.33. *The following three conditions on an integrable connection $\mathscr{M}$ on X are equivalent:*

(i) *$\mathscr{M}$ is a regular integrable connection.*

(ii) *For some divisor completion $j : X \hookrightarrow \bar{X}$ of X the analytic meromorphic connection $j_*(\mathscr{M})^{\mathrm{an}}$ is regular.*

(iii) *For any divisor completion $j : X \hookrightarrow \bar{X}$ of X the analytic meromorphic connection $j_*(\mathscr{M})^{\mathrm{an}}$ is regular.*

Proof. We first prove the part (ii)⇒(i). Assume that for a divisor completion $j : X \hookrightarrow \bar{X}$ the meromorphic connection $j_*(\mathscr{M})^{\mathrm{an}}$ is regular. We need to show that for any morphism $i_C : C \to X$ from an algebraic curve C the induced integrable connection $i_C^*(\mathscr{M})$ is regular. We may assume that the image of C is not a single point. We take a smooth completion $j_C : C \hookrightarrow \bar{C}$ of C and a morphism $i_{\bar{C}} : \bar{C} \to \bar{X}$ so that the following diagram

$$\begin{tikzcd} C \arrow[r, "j_C"] \arrow[d, "i_C"'] & \bar{C} \arrow[d, "i_{\bar{C}}"] \\ X \arrow[r, "j"] & \bar{X} \end{tikzcd}$$

is commutative. We may also assume that this diagram is cartesian by replacing C with $i_{\bar{C}}^{-1}(X)$ (see Proposition 1.5.9). In this situation we have a natural isomorphism

$$((j_C)_* i_C^*(\mathscr{M}))^{\mathrm{an}} \cong i_{\bar{C}}^* j_*(\mathscr{M})^{\mathrm{an}} \cong (i_{\bar{C}}^{\mathrm{an}})^*(j_*(\mathscr{M})^{\mathrm{an}}).$$

Since $j_*(\mathscr{M})^{\mathrm{an}}$ is regular, $(i_{\bar{C}}^{\mathrm{an}})^*(j_*(\mathscr{M})^{\mathrm{an}})$ (and hence $((j_C)_* i_C^*(\mathscr{M}))^{\mathrm{an}}$) is regular by the definition of (analytic) regular meromorphic connections. It follows that $i_C^*(\mathscr{M})$ is regular by Lemma 1.5.8.

It remains to show that (i)$\Rightarrow$(iii). By Corollary 1.5.28 (a), it is sufficient to verify the condition (R) for $j_*(\mathscr{M})^{\mathrm{an}}$. We can take φ in the condition (R) so that $\varphi_x = \varphi|_{\mathbb{D}\times\{x\}}$ comes from an algebraic morphism, for which the condition (R) can be easily checked by the argument used in the proof of (ii)$\Rightarrow$(i). □

Let $j : X \hookrightarrow \bar{X}$ be a divisor completion of X such that $D = \bar{X} - X$ is normal crossing. In this situation we give another criterion of the regularity of an integrable connection $\mathscr{M}$ on X. Let $\mathscr{D}$ be the defining ideal of D and consider the sheaf

$$\Theta_{\bar{X}}[D] = \{v \in \Theta_X : v\mathscr{I} \subseteq \mathscr{I}\}$$

as in the analytic case. We denote by $\mathscr{D}_{\bar{X}}[D]$ the subalgebra of $\mathscr{D}_{\bar{X}}$ generated by $\Theta_{\bar{X}}$ and $\mathscr{O}_{\bar{X}}$. In terms of a local coordinate $\{x_i, \partial_i\}$ of $\bar{X}$ for which D is defined by $x_1 \cdots x_r = 0$, $\Theta_{\bar{X}}[D]$ is generated by $x_1\partial_1, \ldots, x_r\partial_r, \partial_{r+1}, \ldots, \partial_n$ over $\mathscr{O}_{\bar{X}}$.

Theorem 1.5.34 (**Deligne**). *Under the above notation the following three conditions on an integrable connection $\mathscr{M}$ on X are equivalent to each other:*

(i) *$\mathscr{M}$ is regular.*

(ii) *The $\mathscr{D}_{\bar{X}}$-module $j_*(\mathscr{M})$ is a union of $\mathscr{O}_{\bar{X}}$-coherent $\mathscr{D}_{\bar{X}}[D]$-submodules.*

(iii) *For any irreducible component D_0 of D, there exists an open dense subset $D' \subseteq D_0$ satisfying the condition: For each point $p \in D'$, there exists an algebraic curve $\bar{C} \subseteq \bar{X}$ which intersects with D' transversally at p and such that the integrable connection $i_C^*(\mathscr{M})$ on $C = \bar{C} - \{p\}$ (where $i_C : C \hookrightarrow X$) has a regular singularity at $p \in \bar{C}$.*

Proof. It is clear that (i)$\Rightarrow$(iii). Conversely, under the condition (iii), the corresponding analytic meromorphic connection $\mathscr{N} = j_*(\mathscr{M})^{\mathrm{an}}$ satisfies the condition (R), so $j_*(\mathscr{M})^{\mathrm{an}}$ is regular by Corollary 1.5.28 (a).

Now assume that $\mathscr{M}$ is regular. Then $\mathscr{N} = j_*(\mathscr{M})^{\mathrm{an}}$ is an analytic meromorphic connection along D^{an}. By Corollary 1.5.29, $\mathscr{N}$ is a union of $\Theta_{\bar{X}^{\mathrm{an}}}[D^{\mathrm{an}}]$-stable coherent $\mathscr{O}_{\bar{X}^{\mathrm{an}}}$-submodules $\widetilde{\mathscr{L}}_i$. Since $\bar{X}$ is projective, for each i there exists a coherent $\mathscr{O}_{\bar{X}}$-submodule $\mathscr{L}_i$ of $j_*(\mathscr{M})$ such that $\mathscr{L}_i^{\mathrm{an}} = \widetilde{\mathscr{L}}_i$. Denote by $\mathscr{N}_i$ the image of $\Theta_{\bar{X}}[D] \otimes_{\mathbb{C}} \mathscr{L}_i \to j_*(\mathscr{M})$, then by $\mathscr{N}_i^{\mathrm{an}} \subseteq \mathscr{L}_i^{\mathrm{an}}$, we obtain $\mathscr{N}_i \subseteq \mathscr{L}_i$ by GAGA. Namely, each $\mathscr{L}_i$ is $\Theta_{\bar{X}}[D]$-stable. Since we have $\bigcup_i \mathscr{L}_i = j_*(\mathscr{M})$, the condition of (ii) holds.

Conversely, if (ii) holds, then the $\mathscr{D}_{\bar{X}^{\mathrm{an}}}$-module $j_*(\mathscr{M})^{\mathrm{an}}$ is a union of $\Theta_{\bar{X}^{\mathrm{an}}}[D^{\mathrm{an}}]$-stable coherent $\mathscr{O}_{\bar{X}^{\mathrm{an}}}$-submodules, and hence is a regular meromorphic connection along D^{an} by Corollary 1.5.29. Therefore, $\mathscr{M}$ is regular by Proposition 1.5.33. □

The following version of the Riemann-Hilbert correspondence in the algebraic situation will play fundamental roles in establishing more general correspondence for algebraic regular holonomic $\mathscr{D}_X$-modules.

Theorem 1.5.35 (Deligne). *Let X be a smooth algebraic variety. Then the functor $\mathscr{M} \mapsto \mathscr{M}^{\mathrm{an}}$ induces an equivalence of categories*

$$\mathrm{Conn}^{\mathrm{reg}}(X) \xrightarrow{\sim} \mathrm{Conn}(X^{\mathrm{an}})$$

Corollary 1.5.36. *For a smooth algebraic variety X we have an equivalence of categories*

$$\mathrm{Conn}^{\mathrm{reg}}(X) \xrightarrow{\sim} \mathrm{Loc}(X^{\mathrm{an}}).$$

Proof. This follows from Theorem 1.5.35 and Theorem 1.4.4. □

The rest of this paragraph is devoted to the proof of Theorem 1.5.35. We fix a divisor completion $j : \bar{X} \hookrightarrow X$ of X and set $D = \bar{X} \setminus X$. We denote by $\mathrm{Conn}^{\mathrm{reg}}(\bar{X}, D)$ the full subcategory of $\mathrm{Conn}(\bar{X}, D)$ consisting of $\mathscr{M} \in \mathrm{Conn}(\bar{X}, D)$ such that $\mathscr{M}|_X$ is regular. Then we have the following commutative diagram

$$\begin{tikzcd} \mathrm{Conn}^{\mathrm{reg}}(\bar{X}, D) \arrow[r] \arrow[d] & \mathrm{Conn}^{\mathrm{reg}}(\bar{X}^{\mathrm{an}}, D^{\mathrm{an}}) \arrow[d] \\ \mathrm{Conn}^{\mathrm{reg}}(X) \arrow[r] & \mathrm{Conn}(X^{\mathrm{an}}) \end{tikzcd}$$

where vertical arrows are given by restrictions and horizontal arrows are given by $\mathscr{M} \mapsto \mathscr{M}^{\mathrm{an}}$. Since the vertical arrows are equivalences by Lemma 1.5.31 and Theorem 1.5.26, our assertion is equivalent to the equivalence of

$$\mathrm{Conn}^{\mathrm{reg}}(\bar{X}, D) \to \mathrm{Conn}^{\mathrm{reg}}(\bar{X}^{\mathrm{an}}, D^{\mathrm{an}}). \tag{1.5.15}$$

We denote by $\mathbf{Coh}(\mathscr{O}_{\bar{X}}(D))$ (resp. $\mathbf{Coh}^e(\mathscr{O}_{\bar{X}^{\mathrm{an}}}(D^{\mathrm{an}}))$) the category of coherent $\mathscr{O}_{\bar{X}}(D)$-modules (resp. the category of coherent $\mathscr{O}_{\bar{X}^{\mathrm{an}}}(D^{\mathrm{an}})$-modules generated by a coherent $\mathscr{O}_{\bar{X}^{\mathrm{an}}}$-submodule.) Note that any coherent $\mathscr{O}_{\bar{X}}(D)$-module is generated by its coherent $\mathscr{O}_X$-submodule.

By Corollary 1.5.28, any regular meromorphic connection on $\bar{X}^{\mathrm{an}}$ along D^{an} is effective. We denote by $\mathrm{Conn}^e(\bar{X}^{\mathrm{an}}, D^{\mathrm{an}})$ the full subcategory of effective meromorphic connections on $\bar{X}^{\mathrm{an}}$ along D^{an}. By Proposition 1.5.33 the equivalence of (1.5.15) can be deduced from the equivalence of

$$\mathrm{Conn}(\bar{X}, D) \to \mathrm{Conn}^e(\bar{X}^{\mathrm{an}}, D^{\mathrm{an}}). \tag{1.5.16}$$

To see the equivalence (1.5.16), we first prove the following lemma.

Lemma 1.5.37. *The functor*

$$\mathbf{Coh}(\mathscr{O}_{\bar{X}}(D)) \to \mathbf{Coh}^e(\mathscr{O}_{\bar{X}^{\mathrm{an}}}(D^{\mathrm{an}}))$$

given by $\mathscr{M} \mapsto \mathscr{M}^{\mathrm{an}}$ is an equivalence of categories.

Proof. □

Note that $\mathrm{Conn}(\bar{X}, D)$ consists of pairs $(\mathscr{M}, \nabla)$ of $\mathscr{M} \in \mathbf{Coh}(\mathscr{O}_{\bar{X}}(D))$ and a flat connection $\nabla \in \mathrm{Hom}_{\mathbb{C}}(\mathscr{M}, \Omega^1_{\bar{X}} \otimes_{\mathscr{O}_{\bar{X}}} \mathscr{M})$. In view of Lemma 1.5.37, $\mathrm{Conn}^e(\bar{X}^{\mathrm{an}}, D^{\mathrm{an}})$ is equivalent to the category consisting of pairs $(\mathscr{M}, \widetilde{\nabla})$ of $\mathscr{M} \in \mathbf{Coh}(\mathscr{O}_{\bar{X}}(D))$ and a flat connection $\widetilde{\nabla} \in \mathrm{Hom}_{\mathbb{C}}(\mathscr{M}^{\mathrm{an}}, \Omega^1_{\bar{X}^{\mathrm{an}}} \otimes_{\mathscr{O}_{\bar{X}^{\mathrm{an}}}} \mathscr{M}^{\mathrm{an}})$. It then suffices to show that for $\mathscr{M} \in \mathbf{Mod}(\mathscr{O}_{\bar{X}}(D))$ the set Λ of flat connections $\nabla \in \mathrm{Hom}_{\mathbb{C}}(\mathscr{M}, \Omega^1_{\bar{X}} \otimes_{\mathscr{O}_{\bar{X}}} \mathscr{M})$ are in bijective correspondence to the set $\tilde{\Lambda}$ flat connections $\widetilde{\nabla} \in \mathrm{Hom}_{\mathbb{C}}(\mathscr{M}^{\mathrm{an}}, \Omega^1_{\bar{X}^{\mathrm{an}}} \otimes_{\mathscr{O}_{\bar{X}^{\mathrm{an}}}} \mathscr{M}^{\mathrm{an}})$. Since these two sets are defined by $\mathbb{C}$-linear morphisms (not by $\mathscr{O}$-linear morphisms), we cannot directly use GAGA. We will show the correspondence by rewriting the conditions in terms of $\mathscr{O}$-linear morphisms.

We first show that the set Λ_1 of connections $\nabla \in \mathrm{Hom}_{\mathbb{C}}(\mathscr{M}, \Omega^1_{\bar{X}} \otimes_{\mathscr{O}_{\bar{X}}} \mathscr{M})$ are in bijective correspondence to the set $\tilde{\Lambda}_1$ of connections $\widetilde{\nabla} \in \mathrm{Hom}_{\mathbb{C}}(\mathscr{M}^{\mathrm{an}}, \Omega^1_{\bar{X}^{\mathrm{an}}} \otimes_{\mathscr{O}_{\bar{X}^{\mathrm{an}}}} \mathscr{M}^{\mathrm{an}})$. For this, we need the notaion of differential operators. Let Y be a complex manifold or a smooth algebraic variety. For $\mathscr{O}_Y$-modules $\mathscr{K}$ and $\mathscr{L}$, we define subsheaves $F_p\mathscr{D}(\mathscr{K}, \mathscr{L})$ (for $p \in \mathbb{Z}$) of $\mathcal{H}om_{\mathbb{C}}(\mathscr{K}, \mathscr{L})$ recursively by $F_p\mathscr{D}(\mathscr{K}, \mathscr{L}) = 0$ for $p < 0$ and

$$F_p\mathscr{D}(\mathscr{K}, \mathscr{L}) = \{P \in \mathcal{H}om_{\mathbb{C}}(\mathscr{K}, \mathscr{L}) : [P, f] \in F_{p-1}\mathscr{D}(\mathscr{K}, \mathscr{L}) \text{ for } f \in \mathscr{O}_Y\}$$

for $p \geq 0$. The sections of $F_p\mathscr{D}(\mathscr{K}, \mathscr{L})$ are called differential operators of order p.

We can also give a different description of $F_p\mathscr{D}(\mathscr{K}, \mathscr{L})$. Let $\Delta : Y \to Y \times Y$ be the diagonal embedding and let $\mathrm{pr}_i : Y \times Y \to Y$ $(i = 1, 2)$ be the projections. We denote by $\mathfrak{I} \subseteq \mathscr{O}_{Y\times Y}$ the defining ideal of $\Delta(Y)$. By taking Δ^{-1} of the canonical morphism $\mathrm{pr}_i^{-1}(\mathscr{O}_Y) \to \mathscr{O}_{Y\times Y}$, we obtain two ring homomorphisms

$$\alpha_i : \mathscr{O}_Y = \Delta^{-1}\mathrm{pr}_i^{-1}(\mathscr{O}_Y) \to \Delta^{-1}(\mathscr{O}_{Y\times Y}).$$

In particular, we have tow $\mathscr{O}_Y$-module structures on $\Delta^{-1}(\mathscr{O}_{Y\times Y})$. Since $\mathfrak{I}$ is an ideal of $\mathscr{O}_{Y\times Y}$, we also have two $\mathscr{O}_Y$-module structure on each $\Delta^{-1}(\mathfrak{I}^k)$ and $\Delta^{-1}(\mathfrak{I}^k/\mathfrak{I}^l)$ for $k < l$. Note that the two $\mathscr{O}_Y$-module structure on $\Delta^{-1}(\mathfrak{I}^k/\mathfrak{I}^{k+1})$ coincide, and that $\Delta^{-1}(\mathfrak{I}/\mathfrak{I}^2)$ is identified with Ω^1_Y.

Now consider the sheaf $\mathcal{H}om_{\mathscr{O}_Y}(\Delta^{-1}(\mathscr{O}_{Y\times Y}) \otimes_{\mathscr{O}_Y} \mathscr{K}, \mathscr{L})$, where the tensor $\Delta^{-1}(\mathscr{O}_{Y\times Y}) \otimes_{\mathscr{O}_Y} \mathscr{K}$ is taken with respect to the $\mathscr{O}_Y$-module structure induced by α_2, and it is regarded as an $\mathscr{O}_Y$-module via the $\mathscr{O}_Y$-module structure on $\Delta^{-1}(\mathscr{O}_{Y\times Y})$ induced by α_1. Define

$$\beta : \mathcal{H}om_{\mathscr{O}_Y}(\Delta^{-1}(\mathscr{O}_{Y\times Y}) \otimes_{\mathscr{O}_Y} \mathscr{K}, \mathscr{L}) \to \mathcal{H}om_{\mathbb{C}}(\mathscr{K}, \mathscr{L}), \quad \psi \mapsto (s \mapsto \psi(1 \otimes s)).$$

Lemma 1.5.38. *The morphism β induces an isomorphism*

$$\mathcal{H}om_{\mathscr{O}_Y}(\Delta^{-1}(\mathscr{O}_{Y\times Y}/\mathfrak{I}^{p+1}) \otimes_{\mathscr{O}_Y} \mathscr{K}, \mathscr{L}) \cong F_p\mathscr{D}(\mathscr{K}, \mathscr{L}).$$

We now return to the proof of (1.5.16). Note that we have

$$\Lambda_1 \subseteq F_1\mathscr{D}(\mathscr{M}, \Omega^1_{\bar{X}} \otimes_{\mathscr{O}_{\bar{X}}} \mathscr{M}), \quad \tilde{\Lambda}_1 \subseteq F_1\mathscr{D}(\mathscr{M}^{\mathrm{an}}, \Omega^1_{\bar{X}^{\mathrm{an}}} \otimes_{\mathscr{O}_{\bar{X}^{\mathrm{an}}}} \mathscr{M}^{\mathrm{an}}).$$

By examining the connection condition, we see by Lemma 1.5.38 that Λ_1 is in bijective correspondence with the set

$$\{\varphi \in \mathrm{Hom}_{\mathscr{O}_{\bar{X}}}(\Delta^{-1}(\mathscr{O}_{\bar{X}\times\bar{X}}/\mathfrak{I}^2) \otimes_{\mathscr{O}_{\bar{X}}} \mathscr{M}, \Omega^1_{\bar{X}} \otimes_{\mathscr{O}_{\bar{X}}} \mathscr{M}) : \varphi|_{\Omega^1_{\bar{X}} \otimes_{\mathscr{O}_{\bar{X}}} \mathscr{M}} = \mathrm{id}\},$$

where we identify $\Delta^{-1}(\mathfrak{I}/\mathfrak{I}^2)$ with $\Omega^1_{\bar{X}}$. The same argument holds true in the analytic category and we have a similar description of $\tilde{\Lambda}_1$. Now we can apply GAGA to conclude that $\tilde{\Lambda}_1$ is in bijective correspondence with Λ_1.

We finaly give a reformulation of the flatness condition for connections. For a connection $\nabla \in \mathrm{Hom}_{\mathbb{C}}(\mathscr{M}, \Omega^1_{\bar{X}} \otimes_{\mathscr{O}_{\bar{X}}} \mathscr{M})$, we define $\nabla^1 \in \mathrm{Hom}_{\mathbb{C}}(\Omega^1_{\bar{X}} \otimes_{\mathscr{O}_{\bar{X}}} \mathscr{M}, \Omega^2_{\bar{X}} \otimes_{\mathscr{O}_{\bar{X}}} \mathscr{M})$ by the formula

$$\nabla^1(\omega \otimes s) = d\omega \otimes s - \omega \wedge \nabla s.$$

Then we have $\nabla^1 \circ \nabla \in \mathrm{Hom}_{\mathscr{O}_{\bar{X}}}(\mathscr{M}, \Omega^2_{\bar{X}} \otimes_{\mathscr{O}_{\bar{X}}} \mathscr{M})$, and ∇ is a flat connection if and only if $\nabla^1 \circ \nabla = 0$ (in fact, ∇ can be seen as an extension of ∇, and the curvature operator is given by $\nabla^1 \circ \nabla$). This gives a reformulation of the flatness condition in terms of an $\mathscr{O}$-linear morphism. In the analytic category we also have a similar reformulation, and we can then apply GAGA to obtain the desired bijection $\Lambda \cong \tilde{\Lambda}$. The proof of Theorem 1.5.35 is now complete.

1.5.4 Regular holonomic D-modules