Algebra

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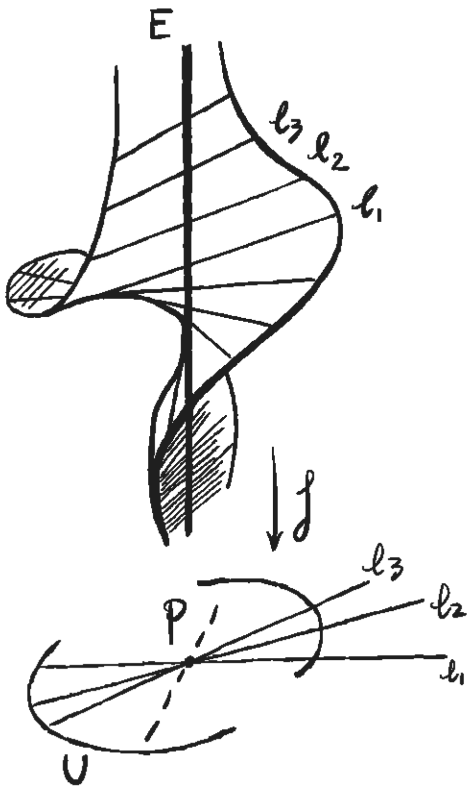


Contents

Chapter 1

Lie algebras

1.1 Definition of Lie algebras

In this section, $\mathbb{K}$ will denote a commutative ring with unit not reduced to zero.

1.1.1 Algebras and Lie algebras

Let A be a $\mathbb{K}$-module with a bilinear map $(x, y) \mapsto xy$ of $A \times A$ into A. All the axioms for algebras are satisfied except associativity of multiplication. By an abuse of language, A is called a (not necessarily associative) **algebra** over $\mathbb{K}$, or sometimes, when no confusion can arise, an algebra over $\mathbb{K}$. In this paragraph we shall use the latter notation. If the $\mathbb{K}$-module A is given the multiplication $(x, y) \mapsto yx$, an algebra is obtained called the **opposite** of the above algebra.

A sub-$\mathbb{K}$-module I of A which is stable under multiplication is given the structure of an algebra over $\mathbb{K}$ in an obvious way. Then I is called a subalgebra of A. Also, I is called a left (resp. right) **ideal** of A if the conditions $x \in I, y \in A$ imply $yx \in A$ (resp. $xy \in I$). If I is both a left ideal and a right ideal of A, I is called a two-sided ideal of A. In this case the multiplication on A enables us to define, on passing to the quotient, a bilinear multiplication on the quotient space A/I such that A/I has an algebra structure. The algebra A/I is called the quotient algebra of A by I.

Let A_1 and A_2 be two algebras over $\mathbb{K}$ and ϕ a map of A_1 into A_2. Then ϕ is called a homomorphism if ϕ is $\mathbb{K}$-linear and $\phi(xy) = \phi(x)\phi(y)$ for $x, y \in A_1$. The kernel I of ϕ is a two-sided ideal of A_1 and the image of ϕ is a subalgebra of A_2. On passing to the quotient, ϕ defines an isomorphism of the algebra A_1/I onto the algebra $\phi(A_1)$.

Let A be an algebra over $\mathbb{K}$. A map D of A into A is called a **derivation** of A if it is $\mathbb{K}$-linear and $D(xy) = (Dx)y + x(Dy)$ for all $x, y \in A$. The kernel of a derivation of A is a subalgebra of A. If D_1 and D_2 are derivations of A, then $D_1D_2 - D_2D_1$ is a derivation of A.

Let A_1 and A_2 be two algebras over $\mathbb{K}$. On the product $\mathbb{K}$-module $A = A_1 \times A_2$ we define a multiplication by writing

$$(x_1, y_1)(x_2, y_2) = (x_1x_2, y_1y_2)$$

for all $x_1, x_2 \in A_1$ and $y_1, y_2 \in A_2$. The algebra thus defined is called the **product algebra** of A_1 and A_2. The map $x_1 \mapsto (x_1, 0)$ (resp. $y_1 \mapsto (0, y_1)$) is an isomorphism of A_1 (resp. A_2) onto a two-sided ideal of A. Under these isomorphisms A_1 and A_2 are identified with two-sided ideals of A. The $\mathbb{K}$-module A is then the direct sum of A_1 and A_2 Conversely, let A be an algebra over $\mathbb{K}$ and A_1, A_2 two two-sided ideals of A such that A is the direct sum of A_1 and A_2. Then $A_1A_2 \subseteq A_1 \cap A_2 = \{0\}$; then, if x_1, x_2 belong to A_1 and y_1, y_2 to A_2, then $(x_1 + y_1)(x_2 + y_2) = x_1x_2 + y_1y_2$, so that A is identified with the product algebra $A_1 \times A_2$. Every left (resp. right, twosided) ideal of A_1 is a left (resp. right, two-sided) ideal of A. We leave to the reader the task of formulating the analogous results in the case of an arbitrary finite family of algebras.

Let A be an algebra over $\mathbb{K}$ and suppose that the $\mathbb{K}$-module A admits a basis $(a_i)_{i \in I}$. There exists a unique system $(\gamma_{ijk})_{(i,j,k) \in I \times I \times I}$ of elements of $\mathbb{K}$ such that $a_ia_j = \sum_k \gamma_{ijk}a_k$ for all a_i, a_j in I. The γ_{ijk} are called the **constants of structure** of A with respect to the basis $(a_i)_{i \in I}$.

Definition 1.1.1. An algebra $\mathfrak{g}$ over $\mathbb{K}$ is called a **Lie algebra** over $\mathbb{K}$ if its multiplication (denoted by $(x, y) \mapsto [x, y]$) satisfies the identities:

- (**Anti-symmetric**) $[x,y] = -[y,x]$ for any $x \in \mathfrak{g}$.
- (**Jacobi identity**) $[[x,y],z] + [[y,z],x] + [[z,x],y] = 0$ for all $x,y,z \in \mathfrak{g}$.

The product $[x,y]$ is called the bracket of x and y.

The bracket $[x,y]$ is an alternating bilinear function of x and y. We have the identity $[x,y] = -[y,x]$. Every subalgebra and every quotient algebra of a Lie algebra is a Lie algebra. Every product of Lie algebras is a Lie algebra. If $\mathfrak{g}$ is a Lie algebra, the opposite algebra $\mathfrak{g}^{op}$ is a Lie algebra and the map $x \mapsto -x$ an isomorphism of $\mathfrak{g}$ onto $\mathfrak{g}^{op}$.

Example 1.1.2. Let A be an associative algebra over $\mathbb{K}$. The bracket $[x,y] = xy - yx$ is a bilinear function of x and y. It is easily verified that the law of composition $(x,y) \mapsto [x,y]$ on the $\mathbb{K}$-module A makes A into a Lie algebra over $\mathbb{K}$, which is denoted by $\mathfrak{Lie}(A)$.

Example 1.1.3. Choose A to be the associative algebra of endomorphisms of a $\mathbb{K}$-module M. We obtain the Lie algebra of endomorphisms of M, denoted by $\mathfrak{gl}(M)$. (If $M = \mathbb{K}^n$, the Lie algebra $\mathfrak{gl}(M)$ is denoted by $\mathfrak{gl}(n,\mathbb{K})$.) Every Lie subalgebra of $\mathfrak{gl}(M)$ is a Lie algebra over $\mathbb{K}$. In particular

(a) If M is given a (not necessarily associative) algebra structure, the derivations of M form a Lie algebra over $\mathbb{K}$.

(b) If M admits a finite basis, the endomorphisms of M of zero trace form a Lie algebra over $\mathbb{K}$ denoted by $\mathfrak{sl}(M)$ (or $\mathfrak{sl}(n,\mathbb{K})$ if $M = \mathbb{K}^n$).

(c) The set $\mathcal{M}_n(\mathbb{K})$ of square matrices of order n can be considered as a Lie algebra over $\mathbb{K}$ canonically isomorphic to $\mathfrak{gl}(n,\mathbb{K})$. Let (E_{ij}) be the canonical basis of $\mathcal{M}_n(\mathbb{K})$. It follows easily that:

$$\begin{cases} [E_{ij}, E_{kl}] = 0 & \text{if } j \neq k \text{ and } i \neq l \\ [E_{ij}, E_{jl}] = E_{il} & \text{if } i \neq l \\ [E_{ij}, E_{ki}] = -E_{kj} & \text{if } j \neq k \\ [E_{ij}, E_{ji}] = E_{ii} - E_{jj} \end{cases} \tag{1.1.1}$$

The Lie subalgebra of $\mathcal{M}_n(\mathbb{K})$ consisting of the triangular matrices (resp. triangular matrices of zero trace, resp. triangular matrices of zero diagonal) is denoted by $\mathfrak{t}(n,\mathbb{K})$ (resp. $\mathfrak{st}(n,\mathbb{K})$, resp. $\mathfrak{n}(n,\mathbb{K})$).

Example 1.1.4. Let M be a $\mathbb{K}$-module. A tuple $\mathcal{F} = (M_0, \ldots, M_n)$ of sub-$\mathbb{K}$-modules with

$$\{0\} = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M$$

is called a **flag** in M. Then

$$\mathfrak{g}(\mathcal{F}) = \{x \in \mathfrak{gl}(M) : x(M_i) \subseteq M_i \text{ for all } i\}$$

is a Lie subalgebra of $\mathfrak{gl}(M)$. To visualize this Lie algebra, we shall describe linear maps by suitable block matrices. If M is a $\mathbb{K}$-module which is a direct sum $M = M_1 \oplus \cdots \oplus M_n$, then we write an endomorphism $A \in \mathrm{End}(M)$ as an $n \times n$ block matrix

$$A = (A_{ij}) = \begin{pmatrix} A_{11} & \cdots & A_{1n} \\ A_{21} & \cdots & A_{2n} \\ \vdots & \ddots & \vdots \\ A_{n1} & \cdots & A_{nn} \end{pmatrix}$$

where $A_{ij} \in \mathrm{Hom}(M_j, M_i)$ is uniquely determined by the requirement that the image of $v = (v_1, \ldots, v_n) \in M$ is

$$Av = \Big(\sum_{j=1}^{n} A_{ij} v_j\Big)_{i=1,\ldots,n}.$$

Two elements x, y of a Lie algebra $\mathfrak{g}$ are said to commute if $[x,y] = 0$. The Lie algebra $\mathfrak{g}$ is said to be **abelian** if any two of its elements commute. Let A be an associative algebra and $\mathfrak{g}$ the Lie algebra defined by it. Two elements x, y commute in $\mathfrak{g}$ if and only if $xy = yx$ in A. Every $\mathbb{K}$-space can obviously be given a unique commutative Lie algebra structure over $\mathbb{K}$. If $\mathfrak{g}$ is a Lie algebra, every one-dimensional sub-$\mathbb{K}$-module of $\mathfrak{g}$ is an abelian Lie subalgebra of $\mathfrak{g}$.

Example 1.1.5 (**Lie algebras of dimension one**)**.** A one dimensional Lie algebra $\mathfrak{g}$ over $\mathbb{K}$ must have $[x,x] = 0$ if $\{x\}$ is a basis. Thus $\mathfrak{g}$ must be abelian and is unique up to isomorphism.

Example 1.1.6 (**Lie algebras of dimension two**)**.** Let $\mathfrak{g}$ be a 2-dimensional Lie algebra and $\{v,w\}$ be a basis of $\mathfrak{g}$. The expansion of $[v,w]$ in terms of u and v determines the Lie algebra up to isomorphism. If $[v,w] = 0$, then $\mathfrak{g}$ is abelian. Otherwise let $[v,w] = \alpha v + \beta w$. We shall produce a basis $\{x,y\}$ with $[x,y] = y$. We set $x = av + bw$ and $y = cv + dw$, then

$$[x,y] = [av + bw, cv + dw] = (ad - bc)[v,w] = (ad - bc)(\alpha v + \beta w).$$

We can choose a, b such that $ad - bc = 1$. Then to obtain $[x,y] = y$, we can set $\alpha = c$ and $\beta = d$. Then we conclude that the only possible 2-dimensional Lie algebras $\mathfrak{g}$ over the field $\mathbb{K}$, up to isomorphism, are

- $\mathfrak{g}$ abelian.
- $\mathfrak{g}$ with a basis $\{x,y\}$ such that $[x,y] = y$.

When $\mathbb{K} = \mathbb{R}$, the second example arises as follows

$$G = \{\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a, b \in \mathbb{R}\}, \quad \mathfrak{Lie}(G) = \{\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} : a, b \in \mathbb{R}\}$$

The group G is isomorphic to the group of affine transformations $t \mapsto at + b$ of the line.

Example 1.1.7 (**Lie algebras of dimension three**)**.** We give some examples in dimension 3 with $\mathbb{K} = \mathbb{R}$.

(i) The **Heisenberg Lie algebra** which is defined to be the set of all matrices

$$\begin{pmatrix} 0 & a & c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix}$$

is an example of what will be called a *nilpotent Lie algebra*. This Lie algebra is used even when the field is not $\mathbb{R}$.

(ii) The Lie algebra of all matrices

$$\begin{pmatrix} t & 0 & x \\ 0 & t & y \\ 0 & 0 & 0 \end{pmatrix}$$

is an example of what will be called a *split solvable Lie algebra*. It is isomorphic with the Lie algebra of the group of translations and dilations of the plane.

(iii) The Lie algebra of all matrices

$$\begin{pmatrix} 0 & \theta & x \\ -\theta & 0 & y \\ 0 & 0 & 0 \end{pmatrix}$$

is an example of a *solvable Lie algebra* that is not split solvable. It is isomorphic with the Lie algebra of the group of translations and rotations of the plane.

(iv) The vector product Lie algebra has a basis $\mathbf{i}, \mathbf{j}, \mathbf{k}$ with bracket relations

$$[\mathbf{i}, \mathbf{j}] = \mathbf{k}, \quad [\mathbf{j}, \mathbf{k}] = \mathbf{i}, \quad [\mathbf{k}, \mathbf{i}] = \mathbf{j}$$

It is an example of what will be called a *simple Lie algebra*, and it is isomorphic to the Lie algebra $\mathfrak{so}(3)$ of the group of rotations in $\mathbb{R}^3$, via the isomorphism

$$\mathbf{i} \mapsto \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \quad \mathbf{j} \mapsto \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad \mathbf{k} \mapsto \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{1.1.2}$$

Example 1.1.8. Let $\mathfrak{h}_3(\mathbb{R})$ be the 3-dimensional vector space with the basis p, q, z equipped with the Lie bracket determined by

$$[p,q] = z, \quad [p,z] = [q,z] = 0.$$

Then $\mathfrak{h}_3(\mathbb{R})$ is exactly the three-dimensional Heisenberg algebra introduced in Example 1.1.7. The linear endomorphism of $\mathfrak{h}_3(\mathbb{R})$ defined by

$$Dz = 0, \quad Dp = q, \quad Dq = -p$$

is then a derivation of $\mathfrak{h}_3(\mathbb{R})$, so that we obtain a Lie algebra $\mathfrak{g} := \mathfrak{h}_3(\mathbb{R}) \rtimes_D \mathbb{R}$, called the **oscillator algebra**. Writing $h := (0,1)$ for the additional basis element in $\mathfrak{g}$, the nonzero brackets of basis elements are

$$[p,q] = z, \quad [h,p] = q, \quad [h,q] = -p.$$

The algebras have the following interperations. On the algebra $A := C^\infty(\mathbb{R}, \mathbb{C})$, consider the operators

$$Pf(x) = i\tilde{F}(x), \quad Qf(x) = xf(x), \quad Zf(x) = if(x).$$

Then the Lie subalgebra of $\mathfrak{gl}(A)$ generated by P, Q, and Z is isomorphic to the Heisenberg algebra $\mathfrak{h}_3(\mathbb{R})$, i.e.,

$$[P,Q] = Z, \quad [P,Z] = [Q,Z] = 0.$$

Adding also the operator

$$Hf(x) = \frac{i}{2}(P^2 + Q^2)f(x) = \frac{i}{2}\Big(-\frac{d^2 f}{dx^2}(x) + x^2 f(x)\Big)$$

we obtain a four-dimensional subalgebra, whose bracket is given by

$$[P,Q] = Z, \quad [H,P] = Q, \quad [H,Q] = -P$$

so this is isomorphic to the oscillator algebra.

Definition 1.1.9. Let $\mathfrak{g}$ be a Lie algebra and x an element of $\mathfrak{g}$. The linear map $y \mapsto [x,y]$ of $\mathfrak{g}$ into $\mathfrak{g}$ is called the **adjoint representation** of x and is denoted by $\mathrm{ad}_{\mathfrak{g}}(x)$ or $\mathrm{ad}(x)$.

Proposition 1.1.10. *Let $\mathfrak{g}$ be a Lie algebra. For all $x \in \mathfrak{g}$, $\mathrm{ad}(x)$ is a derivation. The map $x \mapsto \mathrm{ad}(x)$ is a homomorphism of the Lie algebra $\mathfrak{g}$ into the Lie algebra $\mathfrak{d}$ of derivations of $\mathfrak{g}$. If $D \in \mathfrak{d}$ and $x \in \mathfrak{g}$, then $[D, \mathrm{ad}(x)] = \mathrm{ad}(Dx)$. The map $\mathrm{ad}(x)$ is also called the* ***inner derivation*** *defined by x.*

Proof. The Jacobi identity can be written as

$$\mathrm{ad}(x)([y,z]) = [\mathrm{ad}(x)(y), z] + [y, \mathrm{ad}(x)(z)]$$

or

$$\mathrm{ad}([x,y])(z) = \mathrm{ad}(x)(\mathrm{ad}(y)(z)) - \mathrm{ad}(y)(\mathrm{ad}(x)(z))$$

whence the first two assertions. On the other hand, if $D \in \mathfrak{d}$, $x \in \mathfrak{g}$, $y \in \mathfrak{g}$, then

$$[D, \mathrm{ad}(x)]y = D([x,y]) - [x, Dy] = [Dx, y] = \mathrm{ad}(Dx)(y),$$

whence the last assertion. $\square$

Proposition 1.1.11. *Let $\mathfrak{g}$ be a Lie algebra and $\phi \in \mathrm{Aut}(\mathfrak{g})$. Then for $x \in \mathfrak{g}$, we have $\mathrm{ad}(\phi(x)) = \phi \circ \mathrm{ad}(x) \circ \phi^{-1}$.*

Proof. By the definition of a homomorphism, we have

$$\phi([x,y]) = [\phi(x), \phi(y)]$$

for $x, y \in \mathfrak{g}$. This means $\phi(\mathrm{ad}(x)(y)) = \mathrm{ad}(\phi(x))(\phi(y))$, whence $\phi \circ \mathrm{ad}(x) = \mathrm{ad}(\phi(x)) \circ \phi$, so the claim follows. $\square$

1.1.2 Ideals of Lie algebras

It follows from the anti-symmetric property of Lie brackets that in a Lie algebra $\mathfrak{g}$ there is no distinction between left ideals and right ideals, every ideal being two-sided. We therefore speak simply of ideals.

Let $\mathfrak{a}$ and $\mathfrak{b}$ be two sub-$\mathbb{K}$-modules of $\mathfrak{g}$. By an abuse of notation, the sub-$\mathbb{K}$-module of $\mathfrak{g}$ generated by the elements of the form $[x, y]$ with $x \in \mathfrak{a}$, $y \in \mathfrak{b}$ is denoted by $[\mathfrak{a}, \mathfrak{b}]$. If $z \in \mathfrak{g}$, then $[z, \mathfrak{a}]$, or $[\mathfrak{a}, z]$, denotes the sub-$\mathbb{K}$-module $[\mathbb{K}z, \mathfrak{a}] = \mathrm{ad}(z)(\mathfrak{a})$.

Proposition 1.1.12. *Let $\mathfrak{g}$ be a Lie algebra. If $\mathfrak{a}$ and $\mathfrak{b}$ are ideals of $\mathfrak{g}$, $\mathfrak{a} + \mathfrak{b}$ and $\mathfrak{a} \cap \mathfrak{b}$ are ideals of $\mathfrak{g}$.*

Proof. Be definition, an ideal of $\mathfrak{g}$ is a sub-$\mathbb{K}$-module of $\mathfrak{g}$ which is stable under the inner derivations of $\mathfrak{g}$, so the claim follows. □

Definition 1.1.13. A sub-$\mathbb{K}$-module of $\mathfrak{g}$ which is stable under every derivation of $\mathfrak{g}$ is called a **characteristic ideal** of $\mathfrak{g}$.

Proposition 1.1.14. *Let $\mathfrak{g}$ be a Lie algebra, $\mathfrak{a}$ an ideal (resp. a characteristic ideal) of $\mathfrak{g}$ and $\mathfrak{b}$ a characteristic ideal of $\mathfrak{a}$. Then $\mathfrak{b}$ is an ideal (resp. a characteristic ideal) of $\mathfrak{g}$.*

Proof. Every inner derivation (resp. every derivation) of $\mathfrak{g}$ leaves $\mathfrak{a}$ stable and induces on $\mathfrak{a}$ a derivation and hence leaves $\mathfrak{b}$ stable. □

Example 1.1.15. Let $\mathrm{Der}(\mathfrak{g})$ be the Lie algebra of derivations of a Lie algebra $\mathfrak{g}$. Then by Proposition 1.1.10, we see $\mathrm{ad}(\mathfrak{g})$ is a characteristic ideal of $\mathrm{Der}(\mathfrak{g})$.

Proposition 1.1.16. *If $\mathfrak{a}$ and $\mathfrak{b}$ are ideals (resp. characteristic ideals) of $\mathfrak{g}$, then $[\mathfrak{a}, \mathfrak{b}]$ is an ideal (resp. a characteristic ideal) of $\mathfrak{g}$.*

Proof. Let D be an inner derivation (resp. a derivation) of $\mathfrak{g}$. If $x \in \mathfrak{a}$ and $y \in \mathfrak{b}$, then

$$D([x, y]) = [Dx, y] + [x, Dy] \in [\mathfrak{a}, \mathfrak{b}].$$

Hence the proposition. □

Example 1.1.17. Let $\mathfrak{g}$ be a Lie algebra. The sub-$\mathbb{K}$-module $[\mathfrak{g}, \mathfrak{g}]$ is an ideal, called the **commutator algebra** of $\mathfrak{g}$. Since $\mathfrak{g}$ is stable under each derivation of $\mathfrak{g}$, by Proposition 1.1.16, $[\mathfrak{g}, \mathfrak{g}]$ is a characteristic ideal of $\mathfrak{g}$.

If $\mathfrak{a}$ is a sub-$\mathbb{K}$-module of $\mathfrak{g}$, the set of $x \in \mathfrak{g}$ such that $[x, \mathfrak{a}] \subseteq \mathfrak{a}$ is a subalgebra $\mathfrak{n}$ of $\mathfrak{g}$ called the normalizer of $\mathfrak{a}$ in $\mathfrak{g}$. If further $\mathfrak{a}$ is a subalgebra of $\mathfrak{g}$, then $\mathfrak{a} \subseteq \mathfrak{n}$ and $\mathfrak{a}$ is an ideal of $\mathfrak{n}$.

1.1.3 Extensions of Lie algebras

Definition 1.1.18. Let $\mathfrak{a}$ and $\mathfrak{b}$ be two Lie algebras over $\mathbb{K}$. An **extension** of $\mathfrak{b}$ by $\mathfrak{a}$ is an exact sequence

$$0 \longrightarrow \mathfrak{a} \xrightarrow{\lambda} \mathfrak{g} \xrightarrow{\mu} \mathfrak{b} \longrightarrow 0$$

where $\mathfrak{g}$ is a Lie algebra over $\mathbb{K}$.

The kernel $\mathfrak{n}$ of μ is called the **kernel** of the extension. The homomorphism λ is an isomorphism of $\mathfrak{a}$ onto $\mathfrak{n}$ and the homomorphism λ defines an isomorphism of $\mathfrak{g}/\mathfrak{n}$ onto $\mathfrak{b}$ when passing to the quotient. By an abuse of language, $\mathfrak{g}$ is also called an extension of $\mathfrak{b}$ by $\mathfrak{a}$.

Two extensions

$$0 \to \mathfrak{a} \xrightarrow{\lambda} \mathfrak{g} \xrightarrow{\mu} \mathfrak{b} \to 0, \quad 0 \to \mathfrak{a} \xrightarrow{\tilde{\lambda}} \tilde{\mathfrak{g}} \xrightarrow{\tilde{\mu}} \mathfrak{b} \to 0$$

are said to be **equivalent** if there exists a homomorphism ϕ of $\mathfrak{g}$ into $\tilde{\mathfrak{g}}$ such that the following diagram commutes:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathfrak{a} & \xrightarrow{\lambda} & \mathfrak{g} & \xrightarrow{\mu} & \mathfrak{b} & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle\phi} & & \| & & \\
0 & \longrightarrow & \mathfrak{a} & \xrightarrow{\tilde{\lambda}} & \tilde{\mathfrak{g}} & \xrightarrow{\tilde{\mu}} & \mathfrak{b} & \longrightarrow & 0
\end{array}$$

By the five lemma, such an homomorphism must by bijective. It follows from this that the relation just defined between two extensions of $\mathfrak{b}$ by $\mathfrak{a}$ is an equivalence relation.

Proposition 1.1.19. *Let*

$$0 \longrightarrow \mathfrak{a} \xrightarrow{\lambda} \mathfrak{g} \xrightarrow{\mu} b \longrightarrow 0$$

be an extension of $\mathfrak{b}$ *by* $\mathfrak{a}$ *and* $\mathfrak{n}$ *its kernel.*

(a) If there exists a subalgebra $\mathfrak{m}$ *of* $\mathfrak{g}$ *supplementary to* $\mathfrak{n}$ *in* $\mathfrak{g}$*, the restriction of* μ *to* $\mathfrak{m}$ *is an isomorphism of* $\mathfrak{m}$ *onto* $\mathfrak{b}$*. If* ν *denotes the inverse isomorphism of this restriction, then* ν *is a homomorphism of* $\mathfrak{b}$ *into* $\mathfrak{g}$ *and* $\mu \circ \nu$ *is the identity automorphism of* $\mathfrak{b}$*.*

(b) Conversely, if there exists a homomorphism ν *of* $\mathfrak{b}$ *into* $\mathfrak{g}$ *such that* $\mu \circ \nu$ *is the identity automorphism of* $\mathfrak{b}$*, then* $\nu(\mathfrak{b})$ *is a supplementary subalgebra of* $\mathfrak{n}$ *in* $\mathfrak{g}$*.*

Proof. The assertions of (a) are immediate. On the other hand, let ν be a homomorphism of $\mathfrak{b}$ into $\mathfrak{g}$ such that $\mu \circ \nu$ is the identity automorphism of $\mathfrak{b}$. Then $\nu^{-1}(\mathfrak{b})$ is a subalgebra of $\mathfrak{g}$ and $\mathfrak{g}$ is the sum of $\nu(\mathfrak{b})$ and $\mu^{-1}(0) = \mathfrak{n}$. □

Definition 1.1.20. Let

$$0 \longrightarrow \mathfrak{a} \xrightarrow{\lambda} \mathfrak{g} \xrightarrow{\mu} \mathfrak{b} \longrightarrow 0$$

be an extension of $\mathfrak{b}$ by $\mathfrak{a}$ and $\mathfrak{n}$ its kernel. This extension is called **inessential** (resp. **trivial**) if there exists a subalgebra (resp. an ideal) of $\mathfrak{g}$ supplementary to $\mathfrak{n}$ in $\mathfrak{g}$. This extension is called **central** if $\mathfrak{n}$ is contained in the centre of $\mathfrak{g}$.

If the extension is trivial, let $\mathfrak{m}$ be an ideal of $\mathfrak{g}$ supplementary to $\mathfrak{n}$ in $\mathfrak{g}$. Then $\mathfrak{g}$ is canonically identified with the Lie algebra $\mathfrak{m} \times \mathfrak{n}$ and hence with the Lie algebra $\mathfrak{a} \times \mathfrak{b}$. Conversely, let $\mathfrak{a}$ and $\mathfrak{b}$ be two Lie algebras; then $\mathfrak{a} \times \mathfrak{b}$ is a trivial extension of $\mathfrak{a}$ by $\mathfrak{b}$. An inessential central extension is trivial, for let $\mathfrak{g}$ be a Lie algebra, $\mathfrak{n}$ an ideal of $\mathfrak{g}$ contained in the centre of $\mathfrak{g}$ and $\mathfrak{m}$ a subalgebra of $\mathfrak{g}$ supplementary to $\mathfrak{n}$ in $\mathfrak{g}$. Then

$$[\mathfrak{m}, \mathfrak{g}] = [\mathfrak{m}, \mathfrak{m}] + [\mathfrak{m}, \mathfrak{n}] = [\mathfrak{m}, \mathfrak{m}] \subseteq \mathfrak{m}$$

and hence $\mathfrak{m}$ is an ideal of $\mathfrak{g}$.

1.1.4 Semi-direct product of Lie algebras

Let $\mathfrak{a}$ and $\mathfrak{b}$ be two Lie algebras over $\mathbb{K}$. It is not easy to construct all the extensions of $\mathfrak{b}$ by $\mathfrak{a}$, but we shall describe quite simply all the inessential extensions of $\mathfrak{b}$ by $\mathfrak{a}$.

Let $\mathfrak{g}$ be an inessential extension of $\mathfrak{b}$ by $\mathfrak{a}$. We identify $\mathfrak{a}$ with an ideal of $\mathfrak{g}$, $\mathfrak{b}$ with a subalgebra of $\mathfrak{g}$ supplementary to $\mathfrak{a}$ and the $\mathbb{K}$-module $\mathfrak{g}$ with the $\mathbb{K}$-module $\mathfrak{a} \oplus \mathfrak{b}$. For all $b \in \mathfrak{b}$, let ϕ_b be the restriction to $\mathfrak{a}$ of $\mathrm{ad}_{\mathfrak{g}}(b)$; this is a derivation of $\mathfrak{a}$ and the map $b \mapsto \phi_b$ is a homomorphism of $\mathfrak{b}$ into the Lie algebra of derivations of $\mathfrak{a}$. On the other hand, for $a_1, a_2 \in \mathfrak{a}$ and $b_1, b_2 \in \mathfrak{b}$, we have:

$$\begin{aligned}[(a_1, b_1), (a_2, b_2)] &= [a_1 + b_1, a_2 + b_2] \\ &= [a_1, a_2] + [a_1, b_2] + [b_1, a_2] + [b_1, b_2] \\ &= [a_1, a_2] + \phi_{b_1}(a_2) - \phi_{b_2}(a_1) + [b_1, b_2].\end{aligned}$$

Conversely, let $\mathfrak{a}$ and $\mathfrak{b}$ be Lie algebras over $\mathbb{K}$ and $\phi : \mathfrak{b} \to \mathrm{Der}(\mathfrak{a})$ a homomorphism of $\mathfrak{b}$ into the Lie algebra of derivations of $\mathfrak{a}$. On the direct sum $\mathfrak{g}$ of the $\mathbb{K}$-modules $\mathfrak{a}$ and $\mathfrak{b}$ we define the bracket of two elements by writing:

$$[(a_1, b_1), (a_2, b_2)] = [a_1, a_2] + \phi_{b_1}(a_2) - \phi_{b_2}(a_1) + [b_1, b_2].$$

for all $a_1, a_2 \in \mathfrak{a}$ and $b_1, b_2 \in \mathfrak{b}$. It is immediate that this bracket is an alternating bilinear function on $\mathfrak{g}$. A simple but tedious verification shows that this defines a Lie algebra structure on $\mathfrak{g}$, for with we will denoted by $\mathfrak{a} \rtimes_\phi \mathfrak{b}$. The map $(a, b) \mapsto b$ of $\mathfrak{g}$ onto $\mathfrak{b}$ is a homomorphism μ whose kernel $\mathfrak{n}$ is the ideal of elements of $\mathfrak{g}$ of the form $(a, 0)$. The map $a \mapsto (a, 0)$ is an isomorphism λ of $\mathfrak{a}$ onto $\mathfrak{n}$. Hence

$$0 \longrightarrow \mathfrak{a} \xrightarrow{\lambda} \mathfrak{g} \xrightarrow{\mu} b \longrightarrow 0$$

is an extension of $\mathfrak{b}$ by $\mathfrak{a}$ of kernel $\mathfrak{n}$, which is said to be canonically defined by $\mathfrak{a}$, $\mathfrak{b}$, and the homomorphism ϕ. The map $\mathfrak{b} \mapsto (0, \mathfrak{b})$ is an isomorphism ν of $\mathfrak{b}$ onto a subalgebra of $\mathfrak{g}$ supplementary to $\mathfrak{n}$ in $\mathfrak{g}$, hence the extension is inessential.

If $\mathfrak{a}$ is identified with $\mathfrak{n}$ under λ and $\mathfrak{b}$ with $\nu(\mathfrak{b})$ under ν, then, for $a \in \mathfrak{a}$ and $b \in \mathfrak{b}$,

$$\mathrm{ad}(b)(a) = [(0,b),(a,0)] = (\phi_b(a),0) = \phi_b(a).$$

When $\phi = 0$, $\mathfrak{g}$ is then the product Lie algebra of $\mathfrak{b}$ and $\mathfrak{a}$. In the general case, $\mathfrak{g}$ is called the **semi-direct product** of $\mathfrak{b}$ by $\mathfrak{a}$ (corresponding to the homomorphism ϕ). We have therefore established the following proposition:

Proposition 1.1.21. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be two Lie algebras over $\mathbb{K}$ and*

$$0 \longrightarrow \mathfrak{a} \xrightarrow{\lambda} \mathfrak{g} \xrightarrow{\mu} b \longrightarrow 0$$

an inessential extension of $\mathfrak{b}$ by $\mathfrak{a}$, and ϕ the corresponding homomorphism of $\mathfrak{b}$ into the Lie algebra of derivations of $\mathfrak{a}$. Then this extension is equivalent to the semi-direct product of $\mathfrak{b}$ by $\mathfrak{a}$ corresponding to the homomorphism ϕ.

Example 1.1.22. Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{K}$ and D a derivation of $\mathfrak{g}$. Let $\mathfrak{h}$ be the abelian Lie algebra $\mathbb{K}$. The map $\lambda \mapsto \lambda D$ is then a homomorphism of $\mathfrak{h}$ into the Lie algebra of derivations of $\mathfrak{g}$. We form the corresponding semidirect product $\mathfrak{k}$ of $\mathfrak{h}$ by $\mathfrak{g}$. Let x_0 be the element $(0,1)$ of $\mathfrak{k}$. Then for all $x \in \mathfrak{g}$, we have $Dx = [x_0, x]$.

Example 1.1.23. Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{K}$, M a $\mathbb{K}$-module and ρ a homomorphism of $\mathfrak{g}$ into $\mathfrak{gl}(M)$. If M is considered as an abelian Lie algebra, the Lie algebra of derivations of M is $\mathfrak{gl}(M)$. We can therefore form the semi-direct product $\mathfrak{k}$ of $\mathfrak{g}$ by M corresponding to ρ.

In particular, let $\mathfrak{g} = \mathfrak{gl}(M)$ and ρ be the identity map of $\mathfrak{gl}(M)$. The semi-direct product of $\mathfrak{g}$ by M is then denoted by $\mathfrak{aff}(M)$ (or $\mathfrak{aff}(n,\mathbb{K})$ if $M = \mathbb{K}^n$). An element of $\mathfrak{aff}(M)$ is an ordered pair (v,x), where $v \in M$, $x \in \mathfrak{gl}(M)$; and the bracket is defined by

$$[(v,x),(w,y)] = (x(w) - y(v)), [x,y]).$$

When M is a finite-dimensional vector space over $\mathbb{R}$, $\mathfrak{aff}(M)$ is canonically identified with the Lie algebra of the affine group of M.

Let $\mathfrak{t}$ be a Lie algebra over $\mathbb{K}$. A linear map θ of $\mathfrak{t}$ into $\mathfrak{aff}(M)$ can be written $x \mapsto ((\zeta(x),\eta(x))$, where ζ is a linear map of $\mathfrak{t}$ into M and η a linear map of $\mathfrak{t}$ into $\mathfrak{gl}(M)$. We examine the conditions that ζ and η must satisfy for θ to be a homomorphism. For $x,y \in \mathfrak{t}$, we must have $\theta([x,y]) = [\theta(x),\theta(y)]$, which is to say

$$\begin{aligned}(\zeta([x,y]),\eta([x,y])) &= [(\zeta(x),\eta(x)),(\zeta(y),\eta(y))] \\ &= (\eta(x)\cdot\zeta(y) - \eta(y)\cdot\zeta(x),[\eta(x),\eta(y)]).\end{aligned}$$

Hence for θ to be a homomorphism of $\mathfrak{t}$ into $\mathfrak{aff}(M)$, it is necessary and sufficient that η be a homomorphism of $\mathfrak{t}$ into $\mathfrak{gl}(M)$ and that ζ satisfy the relation

$$\zeta([x,y]) = \eta(x)\cdot\zeta(y) - \eta(y)\cdot\zeta(x). \tag{1.1.3}$$

Let W be the $\mathbb{K}$-module $M \oplus \mathbb{K}$. We take $\mathfrak{t}$ to be the subalgebra of $\mathfrak{gl}(W)$ consisting of the $x \in \mathfrak{gl}(W)$ such that $x(W) \subseteq M$. For all $x \in \mathfrak{t}$, let $\eta(x)$ be the restriction of x to M and $\zeta(x) = x(0,1) \in M$. For $x,y \in \mathfrak{t}$,

$$\zeta([x,y]) = [x,y](0,1) = \eta(x)\zeta(y) - \eta(y)\zeta(x).$$

Hence the map $x \mapsto (\zeta(x),\eta(x))$ is a homomorphism θ of $\mathfrak{t}$ into $\mathfrak{aff}(M)$. Clearly θ is bijective. Let $\phi = \theta^{-1}$. If $(v,x) \in \mathfrak{aff}(M)$, $\phi(v,x)$ is the element $\tilde{x}$ of $\mathfrak{t}$ defined by

$$\tilde{x}(w,\lambda) = (x(w) + \lambda v, 0).$$

The algebra $\mathfrak{aff}(M)$ is often identified with the subalgebra $\mathfrak{t}$ of $\mathfrak{gl}(W)$ under the isomorphism ϕ.

When M is a finite-dimensional vector space over $\mathbb{R}$, the homomorphism ϕ of $\mathfrak{aff}(M)$ into $\mathfrak{gl}(M)$ corresponds to a canonical homomorphism Φ of the affine group A of M into the group $\mathrm{GL}(W)$. If $a \in A$, $\Phi(a)$ is the unique element g of $\mathrm{GL}(W)$ such that $g(v,1) = (a(v),1)$ for all $v \in M$. This homomorphism is injective and $\Phi(A)$ is the set of automorphisms of W which leave invariant all the linear varieties of W parallel to M.

Example 1.1.24. Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{d}$ be the Lie algebra of derivations of $\mathfrak{g}$. Then the identity map of $\mathfrak{d}$ defines a semi-direct product $\mathfrak{D}$ of $\mathfrak{d}$ by $\mathfrak{g}$ called the **holomorph** of $\mathfrak{g}$. Then $\mathfrak{g}$ is identified with an ideal and $\mathfrak{d}$ with a subalgebra of $\mathfrak{D}$.

1.1.5 Extension of scalars

Let $\mathbb{K}_0$ be a commutative ring and ρ a homomorphism of $\mathbb{K}_0$ into $\mathbb{K}$. Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{K}$. Let $\rho^*(\mathfrak{g})$ be the algebra obtained by considering $\mathfrak{g}$ as an algebra over $\mathbb{K}_0$ by means of ρ. Then $\rho^*(\mathfrak{g})$ is a Lie algebra. The subalgebras (resp. ideals) of $\mathfrak{g}$ are subalgebras (resp. ideals) of $\rho^*(\mathfrak{g})$. If $\mathfrak{a}$ and $\mathfrak{b}$ are sub-$\mathbb{K}$-modules of $\mathfrak{g}$, the bracket $[\mathfrak{a}, \mathfrak{b}]$ is the same in $\mathfrak{g}$ and in $\rho^*(\mathfrak{g})$, for $[\mathfrak{a}, \mathfrak{b}]$ is the set of elements of the form $\sum_i [x_i, y_i]$ where $x_i \in \mathfrak{a}$, $y_i \in \mathfrak{b}$.

Let $\mathbb{K}'$ be a commutative ring and σ a homomorphism of $\mathbb{K}$ into $\mathbb{K}'$. Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{K}$. Let $\sigma_*(\mathfrak{g}) = \mathfrak{g} \otimes_{\mathbb{K}} \mathbb{K}'$ be the algebra over $\mathbb{K}'$ derived from $\mathfrak{g}$ by extending scalars. Then $\sigma_*(\mathfrak{g})$ is a Lie algebra over $\mathbb{K}'$. If $\mathfrak{a}$ is a subalgebra (resp. an ideal) of $\mathfrak{g}$, the canonical image of $\sigma_*(\mathfrak{a})$ in $\sigma_*(\mathfrak{g})$ is a subalgebra (resp. an ideal) of $\sigma_*(\mathfrak{g})$. If $\mathfrak{a}$ and $\mathfrak{b}$ are submodules of $\mathfrak{g}$, the canonical image in $\sigma_*(\mathfrak{g})$ of $\sigma_*([\mathfrak{a}, \mathfrak{b}])$ is equal to the bracket of the canonical images of $\sigma_*(\mathfrak{a})$ and $\sigma_*(\mathfrak{b})$.

If $\mathbb{K}$ is a field, $\mathbb{K}'$ is an extension field of $\mathbb{K}$, and σ the canonical injection of $\mathbb{K}$ into $\mathbb{K}'$, then with the usual identifications we have

$$[\mathfrak{a}, \mathfrak{b}] \otimes_{\mathbb{K}} \mathbb{K}' = [\mathfrak{a} \otimes_{\mathbb{K}} \mathbb{K}', \mathfrak{b} \otimes_{\mathbb{K}} \mathbb{K}'].$$

If M is a finite-dimensional vector space over the field $\mathbb{K}$, $M \otimes_{\mathbb{K}} \mathbb{K}'$ is a finite-dimensional vector space over $\mathbb{K}'$ and the associative algebra $\mathrm{End}(M \otimes_{\mathbb{K}} \mathbb{K}')$ is canonically identified with the associative algebra $\mathrm{End}(M) \otimes_{\mathbb{K}} \mathbb{K}'$. Hence the Lie algebra $\mathfrak{gl}(M \otimes_{\mathbb{K}} \mathbb{K}')$ is canonically identified with the Lie algebra $\mathfrak{gl}(M) \otimes_{\mathbb{K}} \mathbb{K}'$.

Definition 1.1.25. Let $\mathfrak{g}$ be a complex Lie algebra. A real Lie algebra $\mathfrak{h}$ with $\mathfrak{h}_{(\mathbb{C})} \cong \mathfrak{g}$ is called a **real form** of $\mathfrak{g}$.

We have seen that to every real Lie algebra we can assign a complexification in a unique way. However, nonisomorphic real algebras can have isomorphic complexifications, resp., complex Lie algebras can have nonisomorphic real forms.

Example 1.1.26. The complexifications of $\mathfrak{so}(3)$ and $\mathfrak{sl}_2(\mathbb{R})$ are both isomorphic to $\mathfrak{sl}_2(\mathbb{C})$. To see this, we consider the bases

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad u = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad t = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

of $\mathfrak{sl}_2(\mathbb{R})$, and

$$x = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad z = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

of $\mathfrak{so}(3)$. Then

$$[h, u] = 2t, \quad [u, t] = 2h, \quad [t, h] = -2u$$

and

$$[x, y] = z, \quad [y, z] = x, \quad [z, x] = y.$$

Let $\mathfrak{h} = \mathbb{R}ih + \mathbb{R}u + \mathbb{R}it$. Then $\mathfrak{h}$ is a Lie algebra with $\mathfrak{h}_{(\mathbb{C})} = \mathfrak{sl}_2(\mathbb{C})$ and is isomorphic to $\mathfrak{so}(3)$ via

$$\mathfrak{h} \to \mathfrak{so}(3), \quad ih \mapsto 2x, u \mapsto 2y, it \mapsto 2z.$$

We note that $\mathfrak{h}$ coincides with $\mathfrak{su}_2(\mathbb{C})$. Since, obviously, $\mathfrak{sl}_2(\mathbb{R})_{(\mathbb{C})} = \mathfrak{sl}_2(\mathbb{C})$, it only remains to show that $\mathfrak{sl}_2(\mathbb{R})$ and $\mathfrak{so}(3)$ are not isomorphic. For this, it suffices to check that $\mathbb{R}h + \mathbb{R}(u + t)$ is a two-dimensional subalgebra of $\mathfrak{sl}_2(\mathbb{R})$, while $\mathfrak{so}(3)$ has no two-dimensional subalgebra. Namely, the latter is isomorphic to $\mathbb{R}^3$ with the vector product, and since the vector product of two vectors is orthogonal to these vectors, a plane cannot be a subalgebra.

Example 1.1.27 (A complex Lie algebra with no real form). On the abelian Lie algebra $V := \mathbb{C}^2$, we consider the linear operator D defined by $De_1 = 2e_1$ and $De_2 = ie_2$ with respect to the canonical basis. Then we form the three-dimensional complex Lie algebra $g := V \rtimes_D \mathbb{C}$ and note that $V = [\mathfrak{g}, \mathfrak{g}]$ is a 2-dimensional ideal of $\mathfrak{g}$.

Suppose that $\mathfrak{g}$ has a real form. Let $\sigma \in \mathrm{Aut}(\mathfrak{g})$ be the corresponding complex conjugation, which is an involutive automorphism of $\mathfrak{g}$. Then

$$\sigma(V) = \sigma([\mathfrak{g}, \mathfrak{g}]) = [\sigma(\mathfrak{g}), \sigma(\mathfrak{g})] = [\mathfrak{g}, \mathfrak{g}] = V,$$

so that σ induces an antilinear involution θ_V on V. Let $\sigma(0,1) = (v_0, \lambda)$ and note that $\sigma(V) = V$ implies that $\lambda \neq 0$. Applying σ again, we see that

$$(0,1) = \sigma^2(0,1) = \theta_V(v_0) + \bar{\lambda}\sigma(0,1) = (\theta_V(v_0) + \bar{\lambda}v_0, |\lambda|^2).$$

We conclude that $|\lambda| = 1$. Further, since σ is a homomorphism, we have

$$\sigma \circ \mathrm{ad}(0,1) = \sigma([(0,1),\cdot]) = [\sigma(0,1),\sigma] = \mathrm{ad}(\sigma(0,1)) \circ \sigma = \mathrm{ad}(v_0,\lambda) \circ \sigma$$

which implies by restricting to V that

$$\theta_V \circ D \circ \theta_V = \lambda D.$$

If $v \in V$ is a D-eigenvector with $D_v = \alpha v$, then

$$D(\theta_V v) = \theta_V(\lambda D v) = \overline{\lambda\alpha}\theta_V(v).$$

This means that $\theta_V(V_\alpha(D)) = V_{\overline{\lambda\alpha}}(D)$. In particular, θ_V permutes the D-eigenspaces. Now $|\lambda| = 1$, so θ_V preserves both eigenspaces. For $\alpha = 2$, this leads to $\lambda = 1$. For $\alpha = i$, this implies $\lambda = -i$. This is a contradiction.

This example is minimal because each complex Lie algebra of dimension 2 has a real form.

1.2 Enveloping algebra of a Lie algebra

1.2.1 The enveloping algebra

Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{K}$. For any associative algebra with unit element A over $\mathbb{K}$, a homomorphism of $\mathfrak{g}$ into A is a $\mathbb{K}$-linear map σ of $\mathfrak{g}$ into A such that

$$\sigma([x,y]) = [\sigma(x), \sigma(y)].$$

(in other words a homomorphism of $\mathfrak{g}$ into the Lie algebra associated with A). If B is another associative algebra with unit element over $\mathbb{K}$ and τ a homomorphism of A into B, then $\tau \circ \sigma$ is an a-map of $\mathfrak{g}$ into B. We shall look for an associative algebra with unit element and a homomorphism of $\mathfrak{g}$ into this algebra which are universal.

Definition 1.2.1. Let $\mathfrak{g}$ be a Lie algebra. A pair $(U(\mathfrak{g}), \iota)$, consisting of a unital associative algebra $U(\mathfrak{g})$ and a homomorphism $\iota : \mathfrak{g} \to U(\mathfrak{g})$ of Lie algebras, is called a **(universal) enveloping algebra** of $\mathfrak{g}$ if it has the following universal property. For each homomorphism $\sigma : \mathfrak{g} \to \mathfrak{Lie}(A)$ of $\mathfrak{g}$ into the Lie algebra A, where A is a unital associative algebra, there exists a unique homomorphism $\tilde{\sigma} : U(\mathfrak{g}) \to A$ of unital associative algebras with $\tilde{\sigma} \circ \iota = \sigma$.

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\sigma"] \arrow[d, "\iota"'] & A \\ U(\mathfrak{g}) \arrow[ur, "\tilde{\sigma}"'] & \end{tikzcd}$$

The universal property determines a universal enveloping algebra uniquely in the following sense:

Proposition 1.2.2 (Uniqueness of the Enveloping Algebra). *If $(U(\mathfrak{g}), \iota)$ and $(\tilde{U}(\mathfrak{g}), \tilde{\iota})$ are two enveloping algebras of the Lie algebra $\mathfrak{g}$, then there exists an isomorphism $\phi : U(\mathfrak{g}) \to \tilde{U}(\mathfrak{g})$ of unital associative algebras satisfying $\phi \circ \iota = \tilde{\iota}$.*

Proof. Since $\tilde{\iota} : \mathfrak{g} \to \tilde{U}(\mathfrak{g})$ is a homomorphism of Lie algebras, the universal property of the pair $(U(\mathfrak{g}), \iota)$ implies the existence of a unique algebra homomorphism $\phi : U(\mathfrak{g}) \to \tilde{U}(\mathfrak{g})$ with $\phi \circ \iota = \tilde{\iota}$. Similarly, the universal property of $(\tilde{U}(\mathfrak{g}), \tilde{\iota})$ implies the existence of an algebra homomorphism $\psi : \tilde{U}(\mathfrak{g}) \to U(\mathfrak{g})$ with $\psi \circ \tilde{\iota} = \iota$. Then $\psi \circ \phi : U(\mathfrak{g}) \to U(\mathfrak{g})$ is an algebra homomorphism with $(\psi \circ \phi) \circ \iota = \iota$, so that the uniqueness part of the universal property of $(U(\mathfrak{g}), \iota)$ yields $\psi \circ \phi = \mathrm{id}_{U(\mathfrak{g})}$. We likewise get $\phi \circ \psi = \mathrm{id}_{\tilde{U}(\mathfrak{g})}$, showing that ϕ is an isomorphism of unital algebras. □

Proposition 1.2.3 (Existence of the Enveloping Algebra). *Each Lie algebra $\mathfrak{g}$ has an enveloping algebra $(U(\mathfrak{g}), \iota)$.*

Proof. Let $T(\mathfrak{g})$ be the tensor algebra of $\mathfrak{g}$ and consider the ideal J of $T(\mathfrak{g})$ generated by the tensors $x \otimes y - y \otimes x - [x,y]$, where $x, y \in \mathfrak{g}$. Then $U(\mathfrak{g}) = T(\mathfrak{g})/J$ is a unital associative algebra and the quotient map $\iota : \mathfrak{g} \to U(\mathfrak{g})$ is a linear map satisfying

$$\iota([x,y]) = [x,y] + J = x \otimes y - y \otimes x + J = [\iota(x), \iota(y)].$$

so that ι is a homomorphism of Lie algebras $\mathfrak{g} \to \mathfrak{Lie}(U(\mathfrak{g}))$.

To verify the universal property for $(U(\mathfrak{g}), \iota)$, let $\phi : \mathfrak{g} \to \mathfrak{Lie}(A)$ be a homomorphism of Lie algebras, where A is a unital associative algebra. In view of the universal property of $T(\mathfrak{g})$, there exists an algebra homomorphism $\tilde{\phi} : T(\mathfrak{g}) \to A$ with $\tilde{\phi}(x) = \phi(x)$ for all $x \in \mathfrak{g}$. Then $J \subseteq \ker \tilde{\phi}$ and so $\tilde{\phi}$ factors through an algebra homomorphism $\tilde{\phi} : U(\mathfrak{g}) \to A$ with $\tilde{\phi} \circ \iota = \phi$. To see that $\tilde{\phi}$ is unique, it suffices to note that $\iota(\mathfrak{g})$ and 1 generate $U(\mathfrak{g})$ as an associative algebra because $\mathfrak{g}$ and 1 generate $T(\mathfrak{g})$ as an associative algebra. □

Let $T_+(\mathfrak{g})$ be the two-sided ideal of $T(\mathfrak{g})$ consisting of the tensors whose component of order 0 is zero, and $T_0(\mathfrak{g}) = \mathbb{K}$ be the set of elements of $T(\mathfrak{g})$ of order 0. Let $U_+(\mathfrak{g})$ and $U_0(\mathfrak{g})$ be the canonical images of $T_+(\mathfrak{g})$ and $T_0(\mathfrak{g})$ in $U(\mathfrak{g})$. As $J \subseteq T_+(\mathfrak{g})$, the decomposition $T(\mathfrak{g}) = T_0(\mathfrak{g}) \oplus T_+(\mathfrak{g})$ implies a decomposition into a direct sum $U(\mathfrak{g}) = U_0(\mathfrak{g}) \oplus U_+(\mathfrak{g})$. The algebra $U(\mathfrak{g})$ therefore has a unit element distinct from 0 and $U_0(\mathfrak{g}) = \mathbb{K} \cdot 1$. For all $x \in U(\mathfrak{g})$, the component of x in $U_0(\mathfrak{g})$ is called the **constant term** of x. The elements with constant term zero form a two-sided ideal of $U(\mathfrak{g})$, namely the two-sided ideal $U_+(\mathfrak{g})$ generated by the canonical image of $\mathfrak{g}$ in $U(\mathfrak{g})$.

1.2.2 Enveloping algebra of the opposite Lie algebra

Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{K}$, $\mathfrak{g}^{op}$ the opposite Lie algebra and ι and ι^{op} the canonical maps of $\mathfrak{g}$ and $\mathfrak{g}^{op}$ into their enveloping algebras $U(\mathfrak{g})$ and $U(\mathfrak{g}^{op})$. Then ι is a homomorphism of $\mathfrak{g}^{op}$ into the associative algebra $U(\mathfrak{g})^{op}$ opposite to the associative algebra $U(\mathfrak{g})$. Hence there exists one and only one homomorphism ϕ of $U(\mathfrak{g}^{op})$ into $U(\mathfrak{g})^{op}$ such that $\iota = \phi \circ \iota^{op}$.

Proposition 1.2.4. *The homomorphism ϕ is an isomorphism of $U(\mathfrak{g}^{op})$ onto $U(\mathfrak{g})^{op}$.*

Proof. There exists a homomorphism ψ of $U(\mathfrak{g})$ into $U(\mathfrak{g}^{op})$ and such that $\iota^{op} = \psi \circ \iota$. Also ψ can be considered as a homomorphism of $U(\mathfrak{g})^{op}$ into $U(\mathfrak{g}^{op})$. Then

$$\iota^{op} = \psi \circ \phi \circ \iota^{op} \qquad \iota = \phi \circ \psi \circ \iota$$

so $\psi \circ \phi$ and $\phi \circ \psi$ are the identity maps of $U(\mathfrak{g}^{op})$ and $U(\mathfrak{g})$. Hence the proposition. □

By Proposition 1.2.4, $U(\mathfrak{g}^{op})$ is identified with $U(\mathfrak{g})^{op}$ under the isomorphism ϕ. Then ι^{op} is identified with ι. With this identification, the isomorphism $\theta : x \mapsto -x$ of $\mathfrak{g}$ onto $\mathfrak{g}^{op}$ defines an isomorphism $\tilde{\theta}$ of $U(\mathfrak{g})$ onto $U(\mathfrak{g}^{op}) = U(\mathfrak{g})^{op}$. This isomorphism can be considered as an antiautomorphism of $U(\mathfrak{g})$. It is called the **principal antiautomorphism** of $U(\mathfrak{g})$. If $x_1, \ldots, x_n$ are in $\mathfrak{g}$, then:

$$\tilde{\theta}(\iota(x_1) \cdots \iota(x_n)) = \tilde{\theta}(\iota(x_n)) \cdots \tilde{\theta}(\iota(x_n)) = (-\iota(x_n)) \cdots (-\iota(x_1)) = (-1)^n \iota(x_n) \cdots \iota(x_1).$$

1.2.3 Enveloping algebra of product of algebras

Let $\mathfrak{g}_1, \ldots, \mathfrak{g}_n$ be Lie algebras over $\mathbb{K}$, $U(\mathfrak{g}_i)$ the enveloping algebra of $\mathfrak{g}_i$ and ι_i the canonical map of $\mathfrak{g}_i$ into $U(\mathfrak{g}_i)$. Let $\mathfrak{g} = \mathfrak{g}_1 \times \cdots \times \mathfrak{g}_n$ be the product, $U(\mathfrak{g})$ be its enveloping algebra and ι the canonical map of $\mathfrak{g}$ into $U(\mathfrak{g})$. The canonical injections of $\mathfrak{g}_i$ into $\mathfrak{g}$ define canonical homomorphisms of $U(\mathfrak{g}_i)$ into $U(\mathfrak{g})$ whose images commute and hence a homomorphism $\phi : U(\mathfrak{g}_1) \otimes_{\mathbb{K}} \cdots \otimes_{\mathbb{K}} U(\mathfrak{g}_n) \to U(\mathfrak{g})$.

Proposition 1.2.5. *The homomorphism ϕ is an algebra isomorphism.*

Proof. For simplicity we may assume that $n = 2$. The map

$$\sigma : (x_1, x_2) \mapsto \iota_1(x_1) \otimes 1 + 1 \otimes \iota_2(x_2)$$

is a homomorphism of $\mathfrak{g}$ into $U(\mathfrak{g}_1) \otimes U(\mathfrak{g}_2)$ and hence there exists a unique homomorphism $\tilde{\sigma} : U(\mathfrak{g}) \to U(\mathfrak{g}_1) \otimes_{\mathbb{K}} U(\mathfrak{g}_2)$ such that $\tilde{\sigma} \circ \iota = \sigma$.

$$\begin{tikzcd}
\mathfrak{g} \arrow[r, "\sigma"] \arrow[d, "\iota"'] & U(\mathfrak{g}_1) \otimes_{\mathbb{K}} U(\mathfrak{g}_2) \arrow[dl, bend left, "\phi"] \\
U(\mathfrak{g}) \arrow[ur, bend left, "\tilde{\sigma}"] &
\end{tikzcd}$$

Now by definition we have $\iota = \phi \circ \sigma$, whence

$$\phi \circ \tilde{\sigma} \circ \iota = \phi \circ \sigma = \iota, \quad \tilde{\sigma} \circ \phi \circ \sigma = \tilde{\sigma} \circ \iota = \sigma,$$

so $\phi \circ \tilde{\sigma}$ and $\tilde{\sigma} \circ \phi$ are the identity maps of $U(\mathfrak{g})$ and $U(\mathfrak{g}_1) \otimes_{\mathbb{K}} U(\mathfrak{g}_2)$ respectively. Hence the proposition. □

By Proposition 1.2.5, if $\mathfrak{g}_1, \ldots, \mathfrak{g}_n$ are Lie algebras over $\mathbb{K}$ with enveloping algebras $U(\mathfrak{g}_1), \ldots, U(\mathfrak{g}_n)$, the enveloping algebra U of $\mathfrak{g} = \mathfrak{g}_1 \times \cdots \times \mathfrak{g}_n$ is canonically identified with $U(\mathfrak{g}_1) \otimes_{\mathbb{K}} \cdots \otimes_{\mathbb{K}} U(\mathfrak{g}_n)$ and the canonical map of $\mathfrak{g}_1 \times \cdots \times \mathfrak{g}_n$ into $U(\mathfrak{g})$ is identified with the map:

$$(x_1, \ldots, x_n) \mapsto \iota_1(x_1) \otimes 1 \otimes \cdots \otimes 1 + \cdots + 1 \otimes \cdots \otimes 1 \otimes \iota_2(x_2)$$

where ι_i is the canonical map of $\mathfrak{g}_i$ into $U(\mathfrak{g}_i)$.

1.2.4 Extension of scalars

Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{K}$, $T(\mathfrak{g})$ its tensor algebra, J the two-sided ideal of $T(\mathfrak{g})$ generated by the $x \otimes y - y \otimes x - [x, y]$ with $x, y \in \mathfrak{g}$ and $U(\mathfrak{g}) = T(\mathfrak{g})/J$. Let $\mathbb{K}'$ be a commutative ring with unit element and ρ a homomorphism of $\mathbb{K}$ into $\mathbb{K}'$. Then the tensor algebra of $\rho_*(\mathfrak{g})$ is canonically identified with $\rho_*(T(\mathfrak{g}))$. Let $\tilde{J}$ be the two-sided ideal of $\rho_*(T(\mathfrak{g}))$ generated by the $x \otimes y - y \otimes x - [x, y]$ with $x, y \in \rho_*(\mathfrak{g})$. Clearly the canonical image of $\rho_*(J)$ in $\rho_*(T(\mathfrak{g}))$ is contained in $\tilde{J}$. To see that it is equal to $\tilde{J}$, it suffices to show that, if x and y denote two elements of $\rho_*(\mathfrak{g})$, then $x \otimes y - y \otimes x - [x, y]$ belongs to this image. Now

$$x = \sum_i x_i \otimes \lambda_i, \quad y = \sum_j y_j \otimes \mu_j, \quad x_i, y_j \in \mathfrak{g}, \lambda_i, \mu_j \in \mathbb{K}'.$$

whence

$$x \otimes y - y \otimes x - [x, y] = \sum_{i,j} (x_i \otimes y_j - y_j \otimes x_j - [x_i, y_j]) \otimes \lambda_i \mu_j$$

which proves our assertion. Then it can be seen that $\rho_*(U(\mathfrak{g})) = \rho_*(T(\mathfrak{g})/J)$ is canonically identified with $\rho_*(T(\mathfrak{g}))/\tilde{J}$, so the enveloping algebra of $\rho_*(\mathfrak{g})$ is canonically identified with $\rho_*(U(\mathfrak{g}))$ and the canonical map of $\rho_*(\mathfrak{g})$ into its enveloping algebra is identified with $\iota \otimes 1$ (where ι denotes the canonical map of $\mathfrak{g}$ into $U(\mathfrak{g})$).

1.2.5 Filtration on the enveloping algebra

Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{K}$ and $T(\mathfrak{g})$ the tensor algebra of the $\mathbb{K}$-module $\mathfrak{g}$. Let $T^n(\mathfrak{g})$ be the sub-$\mathbb{K}$-module of $T(\mathfrak{g})$ consisting of homogeneous tensors of order n and $T_n(\mathfrak{g}) = \bigoplus_{i \le n} T^i(\mathfrak{g})$ (for $n < 0$ we set $T_n(\mathfrak{g}) = \{0\}$). Let $U_n(\mathfrak{g})$ be the canonical image of $T_n(\mathfrak{g})$ in the enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$. Then the family $(U_n(\mathfrak{g}))$ is a filtration on $U(\mathfrak{g})$ induced by that on $T(\mathfrak{g})$. With this filtration, we can form the graded algebra $\mathrm{gr}(U(\mathfrak{g}))$ by defining $\mathrm{gr}_n(U(\mathfrak{g})) = U_n(\mathfrak{g})/U_{n-1}(\mathfrak{g})$. Since $T(\mathfrak{g})$ is a graded algebra isomorphic to $\mathrm{gr}(T(\mathfrak{g}))$, we get a homomorphism $\phi : T(\mathfrak{g}) \to \mathrm{gr}(U(\mathfrak{g}))$, induced by the compositions

$$T_n(\mathfrak{g}) \to U_n(\mathfrak{g}) \to \mathrm{gr}_n(U(\mathfrak{g})) = U_n(\mathfrak{g})/U_{n-1}(\mathfrak{g}).$$

Note that with the corresponding topology, the algebra $U(\mathfrak{g})$ is discrete and hence Hausdorff and complete.

Proposition 1.2.6. *The map ϕ of $T(\mathfrak{g})$ onto $\mathrm{gr}(U(\mathfrak{g}))$ is an algebra homomorphism map 1 to 1 and is zero on the two-sided ideal generated by the tensors $x \otimes y - y \otimes x$ with $x, y \in \mathfrak{g}$.*

Proof. If $s \in T^p(\mathfrak{g})$ and $t \in T^q(\mathfrak{g})$, then $\phi(s)\phi(t) = \phi(st)$ by definition of the multiplication on $\mathrm{gr}(U(\mathfrak{g}))$. Hence ϕ is an algebra homomorphism and clearly $\phi(1) = 1$. If x, y are in $\mathfrak{g}$, then $x \otimes y - y \otimes x \in T^2(\mathfrak{g})$ and the canonical image of this element in $U_2(\mathfrak{g})$ is equal to that of $[x, y]$ and therefore belongs to $U_1(\mathfrak{g})$. Hence $\phi(x \otimes y - y \otimes x) = 0$, which proves the proposition. □

Let $\boldsymbol{S}(\mathfrak{g})$ be the symmetric algebra of the $\mathbb{K}$-module $\mathfrak{g}$ and τ the canonical homomorphism of $T(\mathfrak{g})$ onto $\boldsymbol{S}(\mathfrak{g})$. Then Proposition 1.2.6 shows that there exists a canonical homomorphism $\varpi : \boldsymbol{S}(\mathfrak{g}) \to$

$\mathrm{gr}(U(\mathfrak{g}))$ which is surjective and such that $\phi = \varpi \circ \tau$. Then we have the following commutative diagram:

$$\begin{tikzcd} & U_n(\mathfrak{g}) \arrow[dr, "\pi_n"] & \\ T^n(\mathfrak{g}) \arrow[ur, "\psi_n"] \arrow[rr, "\phi_n"] \arrow[dr, "\tau_n"'] & & \mathrm{gr}_n(U(\mathfrak{g})) \\ & \boldsymbol{S}^n(\mathfrak{g}) \arrow[ur, "\varpi_n"'] & \end{tikzcd} \tag{1.2.1}$$

Proposition 1.2.7. *If $\mathbb{K}$ is Noetherian and $\mathfrak{g}$ is a finitely generated module, then the ring $U(\mathfrak{g})$ is right and left Noetherian.*

Proof. The algebra $\boldsymbol{S}(\mathfrak{g})$ is a finitely generated algebra over $\mathbb{K}$ and hence a Noetherian ring. Hence $\mathrm{gr}(U(\mathfrak{g}))$, which is isomorphic to a quotient ring of $\boldsymbol{S}(\mathfrak{g})$, is Noetherian. Hence $U(\mathfrak{g})$ is right and left Noetherian (Corollary **??**). □

Corollary 1.2.8. *Suppose that $\mathbb{K}$ is a field and that $\mathfrak{g}$ is finite-dimensional over $\mathbb{K}$. Let $I_1, \ldots, I_r$ be right (resp. left) ideals of finite codimension in $U(\mathfrak{g})$. Then the product ideal $I_1 \cdots I_r$ is of finite codimension.*

Proof. By induction on r it suffices to consider the case of, for example, two right ideals. The right $U(\mathfrak{g})$-module I_1 is generated by a finite number of elements $u_1, \ldots, u_p$ by Proposition 1.2.7. Let $v_1, \ldots, v_q$ be elements of $U(\mathfrak{g})$ whose classes modulo I_2 generate the $\mathbb{K}$-module $U(\mathfrak{g})/I_2$. Then the canonical images in I_1/I_1I_2 of the u_iv_i generate the $\mathbb{K}$-module I_1/I_1I_2, which is therefore finite-dimensional. Hence

$$\dim_{\mathbb{K}}(U/I_1I_2) = \dim_{\mathbb{K}}(U/I_1) + \dim_{\mathbb{K}}(I_1/I_1I_2) < +\infty$$

which proves the claim. □

1.2.6 The Poincaré-Birkhoff-Witt theorem

Let $\mathfrak{g}$ be a Lie $\mathbb{K}$-algebra which is a free $\mathbb{K}$-module and $(x_\lambda)_{\lambda\in\Lambda}$ be a basis of the $\mathbb{K}$-module $\mathfrak{g}$. We give Λ a total ordering. Let P be the polynomial algebra $\mathbb{K}[\{z_\lambda\}_{\lambda\in\Lambda}]$ in indeterminates z_i in one-to-one correspondence with the x_i. For every sequence $I = (\lambda_1, \ldots, \lambda_n)$ of elements of I, let z_I denote the monomial $z_{\lambda_1} \cdots z_{\lambda_n}$ and x_I the tensor $x_{\lambda_1} \otimes \cdots \otimes x_{\lambda_n}$.

The z_λ's, for λ increasing, form a basis of the $\mathbb{K}$-module P (we make the convention that $\varnothing$ is an increasing sequence and that $z_\varnothing = 1$) . Let P_n be the sub-$\mathbb{K}$-module of polynomials of degree $\leq n$. We shall first prove several lemmas. (To abbreviate, we write $\lambda \leq I$ if $\lambda \leq \mu$ for every index μ of the sequence I.)

Lemma 1.2.9. *For every integer $n \geq 0$, there exists a unique homomorphism σ_n of the $\mathbb{K}$-module $\mathfrak{g} \otimes_{\mathbb{K}} P_n$ into the $\mathbb{K}$-module P satisfying the following conditions:*

(A_n) *$\sigma_n(x_\lambda \otimes z_I) = z_\lambda z_I$ for $\lambda \leq I$ and $z_I \in P_n$.*

(B_n) *$\sigma_n(x_\lambda \otimes z_I) - z_\lambda z_I \in P_k$ for $z_I \in P_k$ with $k \leq n$.*

(C_n) *$\sigma_n(x_\lambda \otimes \sigma_n(x_\mu \otimes z_J)) = \sigma_n(x_\mu \otimes \sigma_n(x_\lambda \otimes z_J)) + \sigma_n([x_\lambda, x_\mu] \otimes z_J)$ for $z_J \in P_{n-1}$.*

Moreover, the restriction of σ_n to $\mathfrak{g} \otimes_{\mathbb{K}} P_{n-1}$ coincides with σ_{n-1}.

Proof. The last assertion follows from the others since the restriction of σ_n to $\mathfrak{g} \otimes_{\mathbb{K}} P_{n-1}$ satisfies conditions (A_{n-1}), (B_{n-1}) and (C_{n-1}). We shall prove the existence and uniqueness of σ_n by induction on n. For $n = 0$, condition (A_0) gives $\sigma_0(x_i \otimes 1) = z_i$, and conditions ($B_0$) and ($C_0$) are then obviously satisfied. Suppose now that the existence and uniqueness of σ_{n-1} are proved. We show that σ_{n-1} admits a unique extension σ_n to $\mathfrak{g} \otimes_{\mathbb{K}} P_n$ satisfying conditions (A_n), (B_n) and (C_n).

To extend σ_{n-1}, we must define $\sigma_n(x_\lambda \otimes z_I)$ for an increasing sequence I of n elements. If $\lambda \leq I$, the value is given by condition (A_n). Otherwise, I can be written uniquely in the form (μ, J), where $\mu < \lambda$ and $\mu \leq J$. Now we observe that, if the map σ_n is already defined, then by condition (C_n) we must have

$$\sigma_n(x_\lambda \otimes z_I) = \sigma_n(x_\lambda \otimes \sigma_n(x_\mu \otimes z_J)) = \sigma_n(x_\mu \otimes \sigma_n(x_\lambda \otimes z_J)) + \sigma_n([x_\lambda, x_\mu] \otimes z_J).$$

Also note that, since $\mu \leq (\lambda, J)$, we have $\sigma_n(x_\mu \otimes (z_\lambda z_J)) = z_\mu z_\lambda z_J = z_\lambda z_I$. So the above equation can be written as

$$\sigma_n(x_\lambda \otimes z_I) = z_\lambda z_I + \sigma_n(x_\mu \otimes (\sigma_n(x_\lambda \otimes z_J) - z_\lambda z_J)) + \sigma_n([x_\lambda, x_\mu] \otimes z_J).$$

This is our definition of $\sigma_n(x_\lambda \otimes z_I)$; namely, we set

$$\sigma_n(x_\lambda \otimes z_I) := \begin{cases} z_\lambda z_I & \text{if } \lambda \leq I \\ z_\lambda z_I + \sigma_{n-1}(x_\mu \otimes (\sigma_{n-1}(x_\lambda \otimes z_J) - z_\lambda z_J)) + \sigma_{n-1}([x_\lambda, x_\mu] \otimes z_J) & \text{otherwise} \end{cases}$$

By (B_{n-1}) we have $\sigma_{n-1}(x_\lambda \otimes z_J) - z_\lambda z_J \in P_{n-1}$, so conditions (A_n) and (B_n) are satisfied. It remains to verify condition (C_n).

Let J be an increasing sequence such that $z_J \in P_{n-1}$. In the above case $\mu < \lambda$ and $\mu \leq J$, since $\mu \leq (\lambda, J)$, we have $\sigma_n(x_\mu \otimes (z_\lambda z_J)) = z_\mu z_\lambda z_J$, so

$$\begin{aligned} \sigma_n(x_\mu \otimes \sigma_n(x_\lambda \otimes z_J)) &= \sigma_{n-1}(x_\mu \otimes (z_\lambda z_J)) + \sigma_{n-1}(x_\mu \otimes (\sigma_n(x_\lambda \otimes z_J) - z_\lambda z_J)) \\ &= z_\mu z_\lambda z_J + \sigma_{n-1}(x_\mu \otimes (\sigma_{n-1}(x_\lambda \otimes z_J) - z_\lambda z_J)). \end{aligned} \tag{1.2.2}$$

So from our definition on $\sigma_n(x_\lambda \otimes z_I)$, we see condition (C_n) is satisfied in this case. As condition (C_n) is trivially satisfied for $\lambda = \mu$, we see (C_n) holds if $\lambda \leq J$ or $\mu \leq J$. If none of these inequalities holds, $J = (\nu, Q)$, where $\nu < Q$, $\nu < \lambda$, and $\nu < \mu$. Writing henceforth to abbreviate $\sigma_n(x \otimes z) = xz$, it follows from (A_n) and the previous argument that

$$x_\mu(z_J) = x_\mu x_\nu(z_Q) = x_\nu x_\mu(z_Q) + [x_\mu, x_\nu](z_Q).$$

Now $x_\mu(z_Q)$ is of the form $z_\mu z_Q + w$, where $w \in P_{n-2}$. The condition (C_n) can be applied to $x_\lambda x_\nu(z_\mu z_Q)$ since $\nu < (\mu, Q)$, and to $x_\lambda x_\nu(w)$ by the induction hypothesis, and hence to $x_\lambda x_\nu x_\mu(z_Q)$. Hence

$$\begin{aligned} x_\lambda x_\mu(z_J) &= x_\lambda x_\nu x_\mu(z_Q) + x_\lambda[x_\mu, x_\nu](z_Q) \\ &= x_\nu x_\lambda x_\mu(z_Q) + [x_\lambda, x_\nu](x_\mu(z_Q)) + [x_\mu, x_\nu](x_\lambda(z_Q)) + [x_\lambda, [x_\mu, x_\nu]](z_Q). \end{aligned}$$

Since λ, μ are both larger than ν, exchanging λ and μ and subtracting, we get

$$\begin{aligned} &x_\lambda x_\mu(z_J) - x_\mu x_\lambda(z_J) = x_\nu[x_\lambda, x_\mu](z_Q) + [x_\lambda, [x_\nu, x_\mu]](z_Q) - [x_\mu, [x_\lambda, x_\nu]](z_Q) \\ &= [x_\lambda, x_\mu](x_\nu(z_Q)) + [x_\nu, [x_\lambda, x_\mu]](z_Q) + [x_\lambda, [x_\nu, x_\mu]](z_Q) - [x_\mu, [x_\lambda, x_\nu]](z_Q) \end{aligned}$$

and hence by the Jacobi identity

$$x_\lambda x_\mu(z_J) - x_\mu x_\lambda(z_J) = [x_\lambda, x_\mu](x_\nu(z_Q)) = [x_\lambda, x_\mu](z_J)$$

which completes the proof of the lemma. □

Lemma 1.2.10. *There exists a homomorphism $\rho : \mathfrak{g} \to \mathrm{End}(P)$ such that*

(a) $\rho(x_\lambda)z_I = z_\lambda z_I$ *for* $\lambda \leq I$*;*

(b) $\rho(x_\lambda)z_I \equiv z_\lambda z_I$ *mod* P_n *if I has length n.*

Proof. By Lemma 1.2.9 there exists a homomorphism σ of the $\mathbb{K}$-module $\mathfrak{g} \otimes_{\mathbb{K}} P$ into P satisfying, for all n, conditions (A_n), (B_n), (C_n) (where σ_n is replaced by σ). This homomorphism defines a homomorphism ρ of the $\mathbb{K}$-module $\mathfrak{g}$ into the $\mathbb{K}$-module $\mathrm{End}(P)$ and ρ is a homomorphism because of condition (C_n). Finally, ρ satisfies properties (a) and (b) of the lemma because of conditions (A_n) and (B_n). □

Lemma 1.2.11. *Let $\tilde{s}$ be a tensor in $T_n(\mathfrak{g})$ that is mapped to zero in $U_n(\mathfrak{g})$. Then the homogeneous component t_n of $\tilde{s}$ of order n is in the kernel of the canonical homomorphism $T(\mathfrak{g}) \to \boldsymbol{S}(\mathfrak{g})$.*

Proof. We write t_n in the form $t_n = \sum_{i=1}^r x_{I_i}$, where the I_i are sequences of n elements of Λ. The map ρ extends to a homomorphism of the algebra $T(\mathfrak{g})$ into the algebra $\mathrm{End}(P)$ (which we shall also denote by ρ). By Lemma 1.2.10, $\rho(t)$ is a polynomial whose terms of highest degree are $\sum_{i=1}^r z_{I_i}$. Since $\tilde{s}$ is mapped to zero in $U_n(\mathfrak{g})$, $\rho(t) = 0$ and hence $\sum_{i=1}^r z_{I_i} = 0$ in P. Now P is canonically identified with $\boldsymbol{S}(\mathfrak{g})$, since $\mathfrak{g}$ has basis (x_λ). Hence the canonical image of t_n in $\boldsymbol{S}(\mathfrak{g})$ is zero. □

Theorem 1.2.12 (Poincaré-Birkhoff-Witt Theorem). *Let $\mathfrak{g}$ be a Lie $\mathbb{K}$-algebra, $U(\mathfrak{g})$ its enveloping algebra, $\mathrm{gr}(U(\mathfrak{g}))$ the graded algebra associated with the filtered algebra $U(\mathfrak{g})$ and $\boldsymbol{S}(\mathfrak{g})$ the symmetric algebra of the $\mathbb{K}$-module $\mathfrak{g}$. If $\mathfrak{g}$ is a free $\mathbb{K}$-module, the canonical homomorphism $\boldsymbol{S}(\mathfrak{g}) \to \mathrm{gr}(U(\mathfrak{g}))$ is an isomorphism.*

Proof. It is necessary to prove that the canonical homomorphism of $\boldsymbol{S}(\mathfrak{g})$ onto $\mathrm{gr}(U(\mathfrak{g}))$ is injective. In other words, in the notations of (1.2.1), it is necessary to show that the condition $\psi(t) \in U_{n-1}(\mathfrak{g})$ implies $\tau(t) = 0$. Now $\psi(t) \in U_{n-1}(\mathfrak{g})$ means that there exists a tensor $s \in T_{n-1}(\mathfrak{g})$ such that $t - s$ is mapped to zero in $U_n(\mathfrak{g})$. The tensor $t - s$ admits $\tilde{s}$ as homogeneous component of order n and hence $\tilde{s}$ is zero in $\boldsymbol{S}(\mathfrak{g})$ by Lemma 1.2.11. □

Corollary 1.2.13. *Suppose that $\mathfrak{g}$ is a free $\mathbb{K}$-module. Let W be a sub-$\mathbb{K}$-module of $T^n(\mathfrak{g})$. If in the notation of diagram (1.2.1), the restriction of τ_n to W is an isomorphism of W onto $\boldsymbol{S}^n(\mathfrak{g})$, then the restriction of ψ_n to W is an isomorphism of W onto a supplement of $U_{n-1}(\mathfrak{g})$ in $U_n(\mathfrak{g})$.*

Proof. The restriction to W of $\varpi_n \circ \tau_n$ is a bijection of W onto $\mathrm{gr}_n(U(\mathfrak{g}))$; so is the restriction $\theta_n \circ \psi_n$ to W. Hence the corollary. □

Corollary 1.2.14. *If $\mathfrak{g}$ is a free $\mathbb{K}$-module, then the canonical map of $\mathfrak{g}$ into its enveloping algebra is injective.*

Proof. This follows from Corollary 1.2.13 bytaking $W = T^1(\mathfrak{g})$. □

When $\mathfrak{g}$ is a free $\mathbb{K}$-module (in particular when $\mathbb{K}$ is a field), $\mathfrak{g}$ is identified with a submodule of $U(\mathfrak{g})$ under the canonical map of $\mathfrak{g}$ into $U(\mathfrak{g})$. This convention is adopted from the following corollary onwards.

Corollary 1.2.15. *If $\mathfrak{g}$ admits a totally ordered basis $(x_\lambda)_{\lambda \in \Lambda}$, then the elements $x_{\lambda_1} \cdots x_{\lambda_n}$ of the enveloping algebra $U(\mathfrak{g})$, where $(\lambda_1, \dots, \lambda_n)$ is an arbitrary increasing finite sequence of elements of Λ, form a basis of the $\mathbb{K}$-module $U(\mathfrak{g})$.*

Proof. Let Λ_n be the set of increasing sequences of n elements of Λ. For $I = (\lambda_1, \dots, \lambda_n) \in \Lambda_n$, let $y_I = x_{\lambda_1} \otimes \cdots \otimes x_{\lambda_n}$. Let W be the submodule of $T^n(\mathfrak{g})$ with basis $(y_I)_{I \in \Lambda_n}$. Then Corollary 1.2.13 shows that the restriction of ψ_n to W is an isomorphism of W onto a supplement of $U_{n-1}(\mathfrak{g})$ in $U_n(\mathfrak{g})$. But $\psi_n(y_I) = x_{\lambda_1} \cdots x_{\lambda_n}$, whence the corollary. □

Corollary 1.2.16. *Let $\tilde{S}^n(\mathfrak{g}) \subseteq T^n(\mathfrak{g})$ be the set of homogeneous symmetric tensors of order n. Suppose that $\mathbb{K}$ is a field of characteristic 0. Then the composite map of the canonical maps*

$$\boldsymbol{S}^n(\mathfrak{g}) \to \tilde{S}^n(\mathfrak{g}) \to U_n(\mathfrak{g})$$

is an isomorphism of the vector space $\boldsymbol{S}^n(\mathfrak{g})$ onto a supplement of $U_{n-1}(\mathfrak{g})$ in $U_n(\mathfrak{g})$.

Proof. This follows from Corollary 1.2.13 by taking $W = \tilde{S}^n(\mathfrak{g})$. □

Suppose henceforth that $\mathbb{K}$ is a field of characteristic 0. Let η_n be the map of $\boldsymbol{S}^n(\mathfrak{g})$ into $U_n(\mathfrak{g})$ just defined. Let $U^n(\mathfrak{g}) = \eta_n(\boldsymbol{S}^n(\mathfrak{g}))$. The vector space $U(\mathfrak{g})$ is then the direct sum of the $U^n(\mathfrak{g})$. The η_n define an isomorphism η of the vector space $\boldsymbol{S}(\mathfrak{g}) = \bigoplus_n \boldsymbol{S}^n(\mathfrak{g})$ onto the vector space $U(\mathfrak{g}) = \bigoplus_n U^n(\mathfrak{g})$, called the **canonical isomorphism** of $\boldsymbol{S}(\mathfrak{g})$ onto $U(\mathfrak{g})$; this is not an algebra isomorphism. We have the commutative diagram

$$\begin{tikzcd}
 & U^n(\mathfrak{g}) \arrow[dr, "\pi_n"] & \\
\tilde{S}^n(\mathfrak{g}) \arrow[ur, "\psi_n"] \arrow[dr, "\tau_n"'] & & \mathrm{gr}_n(U(\mathfrak{g})) \\
 & \boldsymbol{S}^n(\mathfrak{g}) \arrow[uu, "\eta_n"] \arrow[ur, "\varpi_n"'] &
\end{tikzcd} \tag{1.2.3}$$

where each arrow represents a vector space isomorphism. If $x_1, \dots, x_n$ are in $\mathfrak{g}$, then η_n maps the product $x_1 x_2 \cdots x_n$, calculated in $\boldsymbol{S}(\mathfrak{g})$, to the element

$$\frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} x_{\sigma(1)} \cdots x_{\sigma(n)}$$

calculated in $U(\mathfrak{g})$.

Corollary 1.2.17. *If $\mathbb{K}$ is an integral domain and $\mathfrak{g}$ is a free $\mathbb{K}$-module, the algebra $U(\mathfrak{g})$ has no divisors of zero.*

Proof. The algebra $\mathrm{gr}_n(U(\mathfrak{g}))$ is isomorphic to a polynomial algebra over $\mathbb{K}$ and is therefore an integral domain. Hence the corollary by Corollary **??**. □

1.2.7 Enveloping algebra of subalgebras and quotients

Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{K}$, $\mathfrak{h}$ a subalgebra of $\mathfrak{g}$ and $\iota_{\mathfrak{g}}$, $\iota_{\mathfrak{h}}$ the canonical maps of $\mathfrak{g}$, $\mathfrak{g}$ into their enveloping algebras $U(\mathfrak{g})$, $U(\mathfrak{h})$. Then the canonical injection i of $\mathfrak{h}$ into $\mathfrak{g}$ defines a homomorphism $\tilde{i}$, called canonical, of $U(\mathfrak{h})$ into $U(\mathfrak{g})$ such that $\iota_{\mathfrak{g}} \circ i = \tilde{i} \circ \iota_{\mathfrak{h}}$. The algebra $\tilde{i}(U(\mathfrak{h}))$ is generated by 1 and $\iota_{\mathfrak{g}}(\mathfrak{h})$.

Proposition 1.2.18. *Let $\mathfrak{h}$ be a subalgebra of the Lie algebra $\mathfrak{g}$ and $U(\mathfrak{h})$ its enveloping algebra. Suppose that the $\mathbb{K}$-modules $\mathfrak{h}$ and $\mathfrak{g}/\mathfrak{h}$ are free (for example if $\mathbb{K}$ is a field) . Let $(x_\alpha)_{\alpha\in\Lambda}$ be a basis of $\mathfrak{h}$ and $(y_\beta)_{\beta\in\Delta}$ a family of elements of $\mathfrak{g}$ whose canonical images in $\mathfrak{g}/\mathfrak{h}$ form a basis of $\mathfrak{g}/\mathfrak{h}$.*

(a) The canonical homomorphism of $U(\mathfrak{h})$ into $U(\mathfrak{g})$ is injective.

(b) If Δ is totally ordered, the elements $y_{\beta_1}\cdots y_{\beta_n}$, with β_i increasing, form a basis of $U(\mathfrak{g})$ considered as a left or right module over $U(\mathfrak{h})$.

Proof. We give $\Lambda\cup\Delta$ a total ordering such that every element of Λ is less than every element of Δ. The elements $x_{\alpha_1}\cdots x_{\alpha_p}$ in $U(\mathfrak{h})$, with α_i increasing, form a basis of $U(\mathfrak{h})$. The elements $x_{\alpha_1}\cdots x_{\alpha_p} y_{\beta_1}\cdots y_{\beta_q}$ calculated in $U(\mathfrak{g})$ similarly form a basis of $U(\mathfrak{g})$. Hence the canonical homomorphism of $U(\mathfrak{h})$ into $U(\mathfrak{h})$ maps the elements of a basis of $U(\mathfrak{h})$ to linearly independent elements of $U(\mathfrak{g})$ and is therefore injective. It is moreover seen that the $y_{\beta_1}\cdots \beta_{\beta_q}$ form a basis of $U(\mathfrak{g})$ considered as a left $U(\mathfrak{h})$-module. Ordering $\Lambda\cup\Delta$ so that every element of Δ is less than every element of Λ, it is similarly seen that the $y_{\beta_1}\cdots y_{\beta_q}$ form a basis of $U(\mathfrak{g})$ considered as a right $U(\mathfrak{h})$-module. □

Under the conditions of Proposition 1.2.18, $U(\mathfrak{h})$ is identified with the subalgebra of $U(\mathfrak{h})$ generated by $\mathfrak{h}$ by means of the canonical homomorphism of $U(\mathfrak{h})$ into $U(\mathfrak{g})$.

If $\mathfrak{h}$ is an ideal of $\mathfrak{g}$, the left ideal of $U(\mathfrak{g})$ generated by $\iota_{\mathfrak{g}}(\mathfrak{h})$ coincides with the right ideal generated by $\iota_{\mathfrak{g}}(\mathfrak{h})$, in other words it is a two-sided ideal. This follows since, for $x\in\mathfrak{h}$ and $y\in\mathfrak{g}$,

$$\iota_{\mathfrak{g}}(x)\iota_{\mathfrak{g}}(y) = \iota_{\mathfrak{g}}(y)\iota_{\mathfrak{g}}(x) + \iota_{\mathfrak{g}}([x,y])$$

and $[x,y]\in\mathfrak{h}$.

Proposition 1.2.19. *Let $\mathfrak{h}$ be an ideal of $\mathfrak{g}$, π the canonical homomorphism of $\mathfrak{g}$ onto $\mathfrak{g}/\mathfrak{h}$. Then we have an exact sequence*

$$U(\mathfrak{h}) \xrightarrow{\tilde{i}} U(\mathfrak{g}) \xrightarrow{\tilde{\pi}} U(\mathfrak{g}/\mathfrak{h}) \longrightarrow 0$$

of unitial algebras.

Proof. Let $\iota_{\mathfrak{g}/\mathfrak{h}}$ be the canonical map of $\mathfrak{g}/\mathfrak{h}$ into $U(\mathfrak{g}/\mathfrak{h})$. The commutative diagram

$$\begin{tikzcd}
\mathfrak{h} \arrow[r, "i"] \arrow[d, "\iota_{\mathfrak{h}}"] & \mathfrak{g} \arrow[r, "\pi"] \arrow[d, "\iota_{\mathfrak{g}}"] & \mathfrak{g}/\mathfrak{h} \arrow[d, "\iota_{\mathfrak{g}/\mathfrak{h}}"] \\
U(\mathfrak{h}) \arrow[r, "\tilde{i}"] & U(\mathfrak{g}) \arrow[r, "\tilde{\pi}"] & U(\mathfrak{g}/\mathfrak{h})
\end{tikzcd} \tag{1.2.4}$$

proves that $\tilde{\pi}$ is zero on $\iota_{\mathfrak{g}}(\mathfrak{h})$ and hence on the ideal I it generates. Let $\psi : U(\mathfrak{g}) \to U(\mathfrak{g})/I$ be the canonical homomorphism. There exists a homomorphism $\phi : U(\mathfrak{g})/I \to U(\mathfrak{g}/\mathfrak{h})$ such that $\tilde{\pi} = \phi\circ\psi$.

$$\begin{tikzcd}
\mathfrak{g} \arrow[rr, "\pi"] \arrow[d, "\iota_{\mathfrak{g}}"] & & \mathfrak{g}/\mathfrak{h} \arrow[d, "\iota_{\mathfrak{g}/\mathfrak{h}}"] \arrow[dl, "\theta"'] \\
U(\mathfrak{g}) \arrow[r, "\psi"] \arrow[rr, bend right, "\tilde{\pi}"'] & U(\mathfrak{g})/I \arrow[r, shift left, "\phi"] & U(\mathfrak{g}/\mathfrak{h}) \arrow[l, shift left, "\tilde{\theta}"]
\end{tikzcd}$$

The map $\psi\circ\iota_{\mathfrak{g}}$ of $\mathfrak{g}$ into $U(\mathfrak{g})/I$ is a homomorphism and is zero on $\mathfrak{h}$ and hence defines a homomorphism $\theta : \mathfrak{g}/\mathfrak{h} \to U(\mathfrak{g})/I$ such that $\theta\circ\pi = \psi\circ\iota_{\mathfrak{g}}$. Now by the commutativity of (1.2.19), we see $\phi\circ\psi\circ\iota_{\mathfrak{g}} = \tilde{\pi}\circ\iota_{\mathfrak{g}} = \iota_{\mathfrak{g}/\mathfrak{h}}\circ\pi$, whence $\phi\circ\theta = \iota_{\mathfrak{g}/\mathfrak{h}}$. By the universal property of $U(\mathfrak{g}/\mathfrak{h})$, there exist a unique homomorphism $\tilde{\theta} : U(\mathfrak{g}/\mathfrak{h}) \to U(\mathfrak{g})/I$ such that $\theta = \tilde{\theta}\circ\iota_{\mathfrak{g}/\mathfrak{h}}$. Then

$$\tilde{\theta}\circ\phi\circ\theta = \tilde{\theta}\circ\iota_{\mathfrak{g}/\mathfrak{h}} = \theta, \quad \phi\circ\tilde{\theta}\circ\iota_{\mathfrak{g}/\mathfrak{h}} = \phi\circ\theta = \iota_{\mathfrak{g}/\mathfrak{h}}$$

whence $\tilde{\theta}\circ\phi$ and $\phi\circ\tilde{\theta}$ are the identity maps of $U(\mathfrak{g})/I$ and $U(\mathfrak{g}/\mathfrak{h})$ respectively. This completes the proof. □

1.2.8 Extension by derivations

Lemma 1.2.20. *Let V be a $\mathbb{K}$-module and $T(V)$ the tensor algebra of V. Let u be an endomorphism of V. There exists one and only one derivation of $T(V)$ which extends u. This derivation commutes with the symmetry operators on $T(V)$.*

Proof. Let u_n be the endomorphism of $T^n(V)$ defined by

$$x_1 \otimes \cdots \otimes x_n \mapsto \sum_{i=1}^{n} x_1 \otimes \cdots \otimes u(x_i) \otimes \cdots \otimes x_n.$$

It is easy to check that u_n is a derivation, and the map $\bigoplus_n u_n$ then satisfies the requirements. □

Proposition 1.2.21. *Let $\mathfrak{g}$ be a Lie algebra, $U(\mathfrak{g})$ its enveloping algebra, ι the canonical map of $\mathfrak{g}$ into $U(\mathfrak{g})$ and D a derivation of $\mathfrak{g}$.*

(a) There exists one and only one derivation D_U of $U(\mathfrak{g})$ such that $\iota \circ D = D_U \circ \iota$ (that is, such that D_U extends D, when $\mathfrak{g}$ can be identified with a submodule of $U(\mathfrak{g})$ under ι).

(b) D_U leaves stable $U_n(\mathfrak{g})$ and $U^n(\mathfrak{g})$ (the latter can be identified with the images in $U(\mathfrak{g})$ of the homogeneous symmetric tensors of order n over $\mathfrak{g}$).

(c) D_U commutes with the principal antiautomorphism of $U(\mathfrak{g})$.

(d) If D is the inner derivation of $\mathfrak{g}$ defined by an element x of $\mathfrak{g}$, D_U is the inner derivation of $U(\mathfrak{g})$ defined by $\iota(x)$.

Proof. Let D_T be the derivation of the tensor algebra $T(\mathfrak{g})$ of $\mathfrak{g}$ which extends D. The two-sided ideal J of $T(\mathfrak{g})$ generated by the $x \otimes y - y \otimes x - [x,y]$ with $x, y \in \mathfrak{g}$ is stable under D_T, for

$$D_T(x \otimes y - y \otimes x - [x,y]) = Dx \otimes y - y \otimes Dx - [Dx,y] + x \otimes Dy - Dy \otimes x - [x, Dy].$$

On passing to the quotient, D_T defines a derivation D_U of $U(\mathfrak{g})$ such that $\iota \circ D = D_U \circ \iota$. The uniqueness of D_U is immediate since 1 and $\iota(\mathfrak{g})$ generate the algebra $U(\mathfrak{g})$. Assertion (b) is obvious. Let ω be the principal antiautomorphism of $U(\mathfrak{g})$. We now prove (c). If $x_1, \ldots, x_n$ are in $\mathfrak{g}$, then

$$\begin{aligned} D_U\omega(\iota(x_1)\cdots\iota(x_n)) &= D_U((-1)^n \iota(x_n)\cdots\iota(x_1)) \\ &= (-1)^n \sum_{i=1}^{n} \sigma(x_n)\cdots D_U(\iota(x_i))\cdots\iota(x_1) \\ &= \omega D_U(\iota(x_1)\cdots\iota(x_n)). \end{aligned}$$

Finally, let $x \in \mathfrak{g}$. Let $\mathrm{ad}_U(\iota(x))$ be the inner derivation $y \mapsto \iota(x)y - y\iota(x)$ of $U(\mathfrak{g})$. Then, for $y \in \mathfrak{g}$,

$$(\mathrm{ad}_U(\iota(x)) \circ \iota)(y) = \iota(x)\iota(y) - \iota(y)\iota(x) = \iota([x,y]) = (\iota \circ \mathrm{ad}_{\mathfrak{g}}(x))(y)$$

whence $\mathrm{ad}_U(\iota(x)) \circ \iota = \iota \circ \mathrm{ad}_{\mathfrak{g}}(x)$. This completes the proof. □

Applying Proposition 1.2.21 to the case of an abelian Lie algebra, it is seen that every endomorphism u of a $\mathbb{K}$-module can be extended uniquely to a derivation of the symmetric algebra of this module; this derivation is derived on passing to the quotient from the derivation of the tensor algebra which extends u.

We again take a Lie algebra $\mathfrak{g}$ over $\mathbb{K}$ and let D be a derivation of $\mathfrak{g}$. Let D_T, D_S be the derivations of $T(\mathfrak{g})$, $\boldsymbol{S}(\mathfrak{g})$ which extend D and let D_U be the unique derivation of $U(\mathfrak{g})$ such that $\iota \circ D = D_U \circ \iota$. Since D_U leaves the $U_n(\mathfrak{g})$ stable, D_U defines on taking quotients a derivation D_G of $\mathrm{gr}(U(\mathfrak{g}))$. Also, the commutative diagram (1.2.1) proves that D_G can also be derived from D_S by the homomorphism ϖ_n. If further $\mathbb{K}$ is a field of characteristic 0, the isomorphisms of diagram (1.2.3) map one into another the restrictions of D_T, D_S, D_U, D_G to $\tilde{S}^n(\mathfrak{g})$, $\boldsymbol{S}^n(\mathfrak{g})$, $U^n(\mathfrak{g})$, $\mathrm{gr}_n(U(\mathfrak{g}))$. Hence the canonical isomorphism of $\boldsymbol{S}(\mathfrak{g})$ onto $U(\mathfrak{g})$ maps D_S to D_U. This proves, in particular, that *the canonical isomorphism $\eta : \boldsymbol{S}(\mathfrak{g}) \to U(\mathfrak{g})$ is in fact an isomorphism of $\mathfrak{g}$-modules*.

1.3 Representations of Lie algebras

1.3.1 Representations

Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{K}$ and M a $\mathbb{K}$-module. A homomorphism of $\mathfrak{g}$ into the Lie Algebra $\mathfrak{gl}(M)$ is called a **representation** of $\mathfrak{g}$ on the module M. An injective representation is called faithful. If $\mathbb{K}$ is a field, the dimension (finite or infinite) of M over $\mathbb{K}$ is called the **dimension** of the representation. The representation $x \mapsto \mathrm{ad}(x)$ of $\mathfrak{g}$ on the $\mathbb{K}$-module $\mathfrak{g}$ is called the **adjoint representation** of $\mathfrak{g}$. If M is a $\mathfrak{g}$-module and $x \in \mathfrak{g}$, we denote by x_M the homothety of M defined by x.

Example 1.3.1. Let G be a real Lie group, $\mathfrak{g}$ its Lie algebra and θ a smooth representation of G on a finite-dimensional real vector space M. Then the corresponding homomorphism of $\mathfrak{g}$ into $\mathfrak{gl}(M)$ is a representation of $\mathfrak{g}$ on M.

Let $U(\mathfrak{g})$ be the enveloping algebra of $\mathfrak{g}$. Then we have a one-to-one correspondence between the set of representations of $\mathfrak{g}$ on M and the set of representations of $U(\mathfrak{g})$ on M. On the other hand we know that there is an equivalence between the notion of representation of the associative algebra $U(\mathfrak{g})$ and that of left $U(\mathfrak{g})$-module.

Definition 1.3.2. Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{K}$ and $U(\mathfrak{g})$ its enveloping algebra. A unitary left module over $U(\mathfrak{g})$ is called a **left $\mathfrak{g}$-module**, or simply a **$\mathfrak{g}$-module**.

A unitary right module over $U(\mathfrak{g})$ is called a right $\mathfrak{g}$-module. Such a module is identified with a left $U(\mathfrak{g})^{op}$-module, that is, with a left $\mathfrak{g}^{op}$-module. Let ω be the principal antiautomorphism of $U(\mathfrak{g})$. If M is a right $\mathfrak{g}$-module, a left $\mathfrak{g}$-module structure is defined on M by writing $a \cdot v = v \cdot \omega(a)$ for $v \in M$ and $a \in U(\mathfrak{g})$.

The notions and results of the theory of modules can be translated into the language of representations:

- Two representations ρ and η of $\mathfrak{g}$ on M and W are called **similar** or **isomorphic** if the $\mathfrak{g}$-modules M and W are isomorphic. For this it is necessary and sufficient that there exist an isomorphism u of the $\mathbb{K}$-module M onto the $\mathbb{K}$-module W such that $u\rho u^{-1} = \eta$.

- For all $i \in I$, let ρ_i be a representation of $\mathfrak{g}$ on M_i. Let M be the $\mathfrak{g}$-module the direct sum of the $\mathfrak{g}$-modules M_i. There is a corresponding representation ρ of $\mathfrak{g}$ on M, called the direct sum of the ρ_i and denoted by $\sum_i \rho_i$. If $v = (v_i)_{i \in I}$ is an element of M and $x \in \mathfrak{g}$, then $\rho(x) \cdot v = (\rho_i(x) \cdot v_i)_{i \in I}$.

- A representation ρ of $\mathfrak{g}$ on M is called **simple** or **irreducible** if the associated $\mathfrak{g}$-module is simple. It amounts to the same to say that there exists no sub-$\mathbb{K}$-module of M (other than $\{0\}$ and M) stable under all the $\rho(x)$, $x \in \mathfrak{g}$. A class of simple g-modules defines a class of simple representations of $\mathfrak{g}$.

- A representation ρ of $\mathfrak{g}$ on M is called semi-simple or completely reducible if the associated $\mathfrak{g}$-module is semi-simple. It amounts to the same to say that ρ is similar to a direct sum of simple representations or that every sub-$\mathbb{K}$-module of M stable under $\rho(\mathfrak{g})$ has a supplement stable under $\rho(\mathfrak{g})$.

- Let δ be a class of simple representations of $\mathfrak{g}$ corresponding to a class C of simple $\mathfrak{g}$-modules. Let ρ be a representation of $\mathfrak{g}$ on M. The isotypical component M_C of species C of the $\mathfrak{g}$-module M is also called the isotypical component of M of species δ. This component is the sum of the sub-$\mathbb{K}$-modules of M stable under the $\rho(x)$ and on which the $\rho(x)$ induce a representation of class δ. If M_δ is of length n, then δ is said to have multiplicity n. The sum of the different M_δ is direct, and it is equal to M if and only if ρ is semi-simple.

Let M be a $\mathfrak{g}$-module. The quotient $\mathfrak{g}$-modules of the sub-$\mathfrak{g}$-modules of M are also the sub-$\mathfrak{g}$-modules of the quotient modules of M: they are obtained by considering two sub-$\mathfrak{g}$-modules U, M of M such that $U \supseteq V$ and forming the $\mathfrak{g}$-module U/V. If all the simple modules of the above type are isomorphic to a given simple $\mathfrak{g}$-module S, M is called a **pure $\mathfrak{g}$-module of species S**. If ρ and η are the representations of $\mathfrak{g}$ corresponding to M and S, we also say that ρ is pure of species η.

Let N be a sub-$\mathfrak{g}$-module of M. For M to be pure of species S, it is necessary and sufficient that M and M/N be pure of species S. Henceforth let M be a g-module and suppose that the set of sub-$\mathfrak{g}$-modules of M which are pure of species S admits a maximal element N. Then every submodule of M which is pure of species S is contained in N, for if P is any such module, then $P + N$ is also pure of species S.

Suppose that the $\mathfrak{g}$-module M admits a Jordan-Hölder series $(M_i)_{0\leq i\leq n}$. For M to be pure of species N, it is necessary and sufficient that the successive quotients M_i/M_{i-1} be isomorphic to N.

Proposition 1.3.3. *Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{K}$ and $\mathfrak{a}$ an ideal of $\mathfrak{g}$. Let M be a $\mathfrak{g}$-module and S a simple $\mathfrak{a}$-module. Consider M as an $\mathfrak{a}$-module and suppose that the set of sub-$\mathfrak{a}$-modules of M which are pure of species S admits a maximal element N. Then N is a sub-$\mathfrak{g}$-module of M.*

Proof. Let π be the canonical map of M onto M/N considered as $\mathfrak{a}$-modules. For $y\in\mathfrak{g}$, denote by f_y the map $v\mapsto\pi(y\cdot v)$ of N into M/N. It suffices to show that $f_y(N)=\{0\}$. Let $x\in\mathfrak{a}$ and $v\in N$, then

$$x_M\cdot f_y(v)=x_M\cdot\pi(y_M\cdot v)=\pi(x_My_M\cdot v)=\pi(y_Mx_M\cdot v)+\pi([x,y]_M\cdot v).$$

Since $\mathfrak{a}$ is an ideal, we have $[x,y]\in\mathfrak{a}$, whence

$$x_M\cdot f_y(v)=\pi(y_Mx_M\cdot v)+\pi([x,y]_M\cdot v)=f_y(x_M\cdot v).$$

This shows f_y is a homomorphism of $\mathfrak{a}$-modules, so it follows that $f_y(N)$ is a sub-$\mathfrak{a}$-module of M/N isomorphic to a quotient of N and hence pure of species S; hence $f_y(N)=\{0\}$. □

Corollary 1.3.4. *Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{K}$ and $\mathfrak{a}$ an ideal of $\mathfrak{g}$. Let M be a simple $\mathfrak{g}$-module, and of finite length as a $\mathbb{K}$-module. Then there exists a simple $\mathfrak{a}$-module S such that M is a pure a-module of species S.*

Proof. Since the $\mathfrak{a}$-module M is of finite length, there exists a minimal element S in the set of sub-$\mathfrak{a}$-modules of M: it is a simple sub-$\mathfrak{a}$-module of M. The largest sub-$\mathfrak{a}$-module of M which is pure of species S is therefore nonzero and is a sub-$\mathfrak{g}$-module of M (Proposition 1.3.3) and is therefore identical with M. □

1.3.2 Tensor product of $\mathfrak{g}$-modules

We have defined the direct sum of a family of representations of $\mathfrak{g}$. We shall now define other operations on representations. Let $\mathfrak{g}_1,\mathfrak{g}_2$ be two Lie algebras over $\mathbb{K}$ and M_i a $\mathfrak{g}_i$-module. Let $U(\mathfrak{g}_i)$ be the enveloping algebra of $\mathfrak{g}_i$ and ι_i the canonical map of $\mathfrak{g}_i$ into $U(\mathfrak{g}_i)$. Then M_i is a left $U(\mathfrak{g}_i)$-module and hence $M_1\otimes_{\mathbb{K}}M_2$ has a canonical left $U(\mathfrak{g}_1)\otimes_{\mathbb{K}}U(\mathfrak{g}_2)$-module structure. Now $U(\mathfrak{g}_1)\otimes_{\mathbb{K}}U(\mathfrak{g}_2)$ is the enveloping algebra of $\mathfrak{g}_1\times\mathfrak{g}_2$ and the map $(x_1,x_2)\mapsto\iota_1(x_1)\otimes 1+1\otimes\iota_2(x_2)$ is the canonical map of $\mathfrak{g}_1\times\mathfrak{g}_2$ into this enveloping algebra. Hence there exists a $(\mathfrak{g}_1\times\mathfrak{g}_2)$-module structure on $M=M_1\otimes_{\mathbb{K}}M_2$ such that:

$$\begin{aligned}(x_1,x_2)\cdot(v_1\otimes v_2)&=(\iota_1(x_1)\otimes 1+1\otimes\iota_2(x_2))\cdot(v_1\otimes v_2)\\&=(x_1)\cdot v_1\otimes v_2+v_1\otimes(x_2)\cdot v_2.\end{aligned}$$

This structure defines a representation of $\mathfrak{g}_1\otimes\mathfrak{g}_2$ on M. If now $\mathfrak{g}_1=\mathfrak{g}_2=\mathfrak{g}$, the homomorphism $x\mapsto(x,x)$ of $\mathfrak{g}$ into $\mathfrak{g}\times\mathfrak{g}$, composed with the above representation, defines a representation of $\mathfrak{g}$ on M and hence a $\mathfrak{g}$-module structure on M such that:

$$x\cdot(v_1\otimes v_2)=(x_{M_1}\cdot v_1)\otimes v_2+v_1\otimes(x_{M_2}\cdot v_2).$$

By an analogous argument we see that:

Proposition 1.3.5. *Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{K}$ and $M_1,\dots,M_n$ be $\mathfrak{g}$-module. On the tensor product $M_1\otimes_{\mathbb{K}}\cdots\otimes_{\mathbb{K}}M_n$, there exists one and only one $\mathfrak{g}$-module structure such that*

$$x\cdot(v_1\otimes v_n)=\sum_{i=1}^{n}v_1\otimes\cdots\otimes x\cdot v_i\otimes\cdots\otimes v_n \tag{1.3.1}$$

for all $x\in\mathfrak{g}$, $x_i\in M_i$.

The corresponding representation is called the tensor product of the given representations of $\mathfrak{g}$ on the M_i. In particular, if M is a $\mathfrak{g}$-module, we have a $\mathfrak{g}$-module structure on each $M_n=\bigotimes^n M$ and hence on the tensor algebra $T(M)$ of M.

Note that formula (1.3.1) shows that, for all $x\in\mathfrak{g}$, x_T is the unique derivation of the algebra $T(M)$ which extends x_M. We know that x_T defines on passing to the quotient a derivation of the symmetric algebra $\boldsymbol{S}(M)$ of M. Hence $\boldsymbol{S}(M)$ can be considered as a quotient $\mathfrak{g}$-module of $T(M)$ and the x_S are derivations of $\boldsymbol{S}(M)$.

Still more particularly, consider $\mathfrak{g}$ as a $\mathfrak{g}$-module by means of the adjoint representation of $\mathfrak{g}$. Let $U(\mathfrak{g})$ be the enveloping algebra of $\mathfrak{g}$. By Proposition 1.2.21, x_M defines on passing to the quotients a derivation of U which is just the inner derivation defined by $\iota(x)$ (ι denoting the canonical map of $\mathfrak{g}$ into $U(\mathfrak{g})$). Then $U(\mathfrak{g})$ can be considered as a quotient $\mathfrak{g}$-module of $T(\mathfrak{g})$. If $\mathbb{K}$ is a field of characteristic 0, the canonical isomorphism of $\boldsymbol{S}(\mathfrak{g})$ onto $U(\mathfrak{g})$ is a $\mathfrak{g}$-module isomorphism.

1.3.3 Representation on homomorphism modules

Again let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be two Lie algebras over $\mathbb{K}$ and M_i a $\mathfrak{g}_i$-module. Let $U(\mathfrak{g}_i)$ be the enveloping algebra of $\mathfrak{g}_i$ and ι_i the canonical map of $\mathfrak{g}_i$ into $U(\mathfrak{g}_i)$. Then M_i is a left $U(\mathfrak{g}_i)$-module and hence $\mathrm{Hom}_{\mathbb{K}}(M_1, M_2)$ has a canonical left $U(\mathfrak{g}_1)^{op} \otimes_{\mathbb{K}} U(\mathfrak{g}_2)$-module structure. Now $U(\mathfrak{g}_1)^{op} \otimes_{\mathbb{K}} U(\mathfrak{g}_2)$ is the enveloping algebra of $\mathfrak{g}_1^{op} \times \mathfrak{g}_2$ and the map

$$(x_1, x_2) \mapsto \iota_1(x_1) \otimes 1 + 1 \otimes \iota_2(x_2)$$

is the canonical map of $\mathfrak{g}_1^{op} \times \mathfrak{g}_2$ into this enveloping algebra. Hence there exists a $\mathfrak{g}_1^{op} \times \mathfrak{g}_2$-module structure on $M = \mathrm{Hom}_{\mathbb{K}}(M_1, M_2)$ such that

$$\begin{aligned}((x_1, x_2) \cdot u)(v) &= ((\iota_1(x_1) \otimes 1 + 1 \otimes \iota_2(x_2)) \cdot u)(v) \\ &= u(x_1 \cdot v) + x_2 \cdot u(v).\end{aligned}$$

for all $u \in \mathrm{Hom}_{\mathbb{K}}(M_1, M_2)$ and $v \in M_1$. This structure defines a representation of $\mathfrak{g}_1^{op} \times \mathfrak{g}_2$ on M. If now $\mathfrak{g}_1 = \mathfrak{g}_2 = \mathfrak{g}$ the homomorphism $x \mapsto (-x, x)$ of $\mathfrak{g}$ into $\mathfrak{g}^{op} \times \mathfrak{g}$ composed with the above representation, defines a representation of $\mathfrak{g}$ on M and hence a $\mathfrak{g}$-module structure on M such that

$$(x \cdot u)(v) = x \cdot u(v) - u(x \cdot v).$$

Combining these results with Proposition 1.3.5, we see that:

Proposition 1.3.6. *Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{K}$ and $M_1, \dots, M_{n+1}$ be $\mathfrak{g}$-modules. Let N be the $\mathbb{K}$-module $\mathrm{Hom}_{\mathbb{K}}(M_1, \dots, M_n; M_{n+1})$ of multilinear maps of $\prod_{i=1}^n M_i$ into M_{n+1}. Then there exists one and only one $\mathfrak{g}$-module structure on N such that*

$$(x \cdot u)(v_1, \dots, v_n) = x \cdot u(v_1, \dots, v_n) - \sum_{i=1}^{n} u(v_1, \dots, x \cdot v_i, \dots, v_n) \tag{1.3.2}$$

for all $x \in \mathfrak{g}$, $u \in N$, and $v_i \in M_i$.

In particular, let $\mathfrak{g}$ be a Lie algebra over $\mathbb{K}$ and M a $\mathfrak{g}$-module and consider $\mathbb{K}$ as a trivial $\mathfrak{g}$-module. Then Proposition 1.3.6 defines a $\mathfrak{g}$-module structure on $\mathrm{Hom}_{\mathbb{K}}(M, \mathbb{K}) = M^*$. The corresponding representation is called the **dual representation** of the representation M. We have

$$(x \cdot f)(v) = -f(x \cdot v).$$

for all $x \in \mathfrak{g}$, $f \in M^*$, $v \in M$. In other words, $x_{M^*} = -x_M^t$. When $\mathbb{K}$ is a field and M is finite-dimensional, the $\mathfrak{g}$-module M is simple (resp. semi-simple) if and only if the $\mathfrak{g}$-module M^* is simple (resp. semi-simple).

Proposition 1.3.7. *Let M_1, M_2 be two $\mathfrak{g}$-modules. Then the canonical $\mathbb{K}$-linear maps*

$$M_1^* \otimes_{\mathbb{K}} M_2 \xrightarrow{\phi} \mathrm{Hom}_{\mathbb{K}}(M_1, M_2), \quad \mathrm{Hom}_{\mathbb{K}}(M_1, M_2^*) \xrightarrow{\psi} (M_1 \otimes_{\mathbb{K}} M_2)^*$$

(where the second is bijective) are $\mathfrak{g}$-module homomorphisms.

Proof. For $x \in \mathfrak{g}$, $f \in M_1^*$, $v_1 \in M_1$, $v_2 \in M_2$, we have

$$\begin{aligned}\phi(x \cdot (f \otimes v_2))(v_1) &= (\phi((x \cdot f) \otimes v_2 + f \otimes (x \cdot v_2)))(v_1) \\ &= \langle x \cdot f, v_1 \rangle v_2 + \langle f, v_1 \rangle (x \cdot v_2), \\ (x \cdot \phi(f \otimes v_2))(v_1) &= x \cdot \phi(f \otimes v_2)(v_1) - \phi(f \otimes v_2)(x \cdot v_1)\end{aligned}$$

$$= \langle f, v_1\rangle(x\cdot v_2) - \langle f, x\cdot v_1\rangle v_2$$
$$= \langle f, v_1\rangle(x\cdot v_2) + \langle x\cdot f, v_1\rangle v_2.$$

Similarly, for $x \in \mathfrak{g}$, $u \in \mathrm{Hom}_{\mathbb{K}}(M_1, M_2^*)$, and $v_1 \in M_1, v_2 \in M_2$, we have

$$\begin{aligned}(\psi(x\cdot u))(v_1\otimes v_2) &= \langle (x\cdot u)(v_1), v_2\rangle = \langle x\cdot u(v_1), v_2\rangle - \langle u(x\cdot v_1), v_2\rangle\\ &= -\langle u(v_1), x\cdot v_2\rangle - \langle u(x\cdot v_1), v_2\rangle\\ (x\cdot\psi(u))(v_1\otimes v_2) &= -\psi(u)(x\cdot(v_1\otimes v_2)) = -\psi(u)((x\cdot v_1)\otimes v_2 + v_1\otimes(x\cdot v_2))\\ &= -\langle u(x\cdot v_1), v_2\rangle - \langle u(v_1), x\cdot v_2\rangle.\end{aligned}$$

These completes the proof. □

The $\mathfrak{g}$-modules $\mathrm{Hom}_{\mathbb{K}}(M_1, M_2^*)$ and $(M_1\otimes M_2)^*$ are identified under the isomorphism ψ. If M_1 and M_2 have finite bases, ϕ is an isomorphism, which allows us to identify the $\mathfrak{g}$-modules $M_1^*\otimes_{\mathbb{K}} M_2$ and $\mathrm{Hom}_{\mathbb{K}}(M_1, M_2)$; in that case, we can therefore identify the $\mathfrak{g}$-modules $M_1^*\otimes_{\mathbb{K}} M_2^*$, $\mathrm{Hom}_{\mathbb{K}}(M_1, M_2^*)$ and $(M_1\otimes_{\mathbb{K}} M_2)^*$.

Example 1.3.8. Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{K}$ and M a $\mathfrak{g}$-module. The $\mathfrak{g}$-module structure on M and the trivial $\mathfrak{g}$-module structure on $\mathbb{K}$ define a $\mathfrak{g}$-module structure on the $\mathbb{K}$-module $N = \mathrm{Hom}_{\mathbb{K}}(M, M; \mathbb{K})$ of bilinear forms on M. Then

$$(x\cdot\beta)(v, w) = -\beta(x\cdot v, w) - \beta(v, x\cdot w).$$

for all $x \in \mathfrak{g}$, $v, w \in M$, and $\beta \in N$. If β is a given element of N, the set of $x \in \mathfrak{g}$ such that $x\cdot\beta = 0$ is a subalgebra of $\mathfrak{g}$.

Let M be a $\mathbb{K}$-module and β a bilinear form on M. By the above, the set of $x \in \mathfrak{gl}(M)$ such that for all $v, w \in M$,

$$\beta(x\cdot v, w) + \beta(v, x\cdot w) = 0,$$

is a Lie subalgebra of $\mathfrak{gl}(M)$. Suppose that $\mathbb{K}$ is a field, M is finite-dimensional and β is non-degenerate. Then every $x \in \mathfrak{gl}(M)$ admits a left adjoint x^* (relative to β) which is everywhere defined and the subalgebra in question is the set of $x \in \mathfrak{gl}(M)$ such that $x^* = -x$. By this process we can construct two important examples of Lie algebras:

(a) Take $M = \mathbb{K}^n$ and β be the standard inner product on M given by

$$\beta((\zeta_1, \dots, \zeta_n), (\eta_1, \dots, \eta_n)) = \sum_{i=1}^n \zeta_i\eta_i.$$

We canonically identify $\mathfrak{gl}(\mathbb{K}^n)$ with $\mathcal{M}_n(\mathbb{K})$. Then the Lie algebra obtained is the Lie algebra of skew-symmetric matrices, denoted by $\mathfrak{so}(n, \mathbb{K})$. (When $\mathbb{K} = \mathbb{R}$, this algebra is the Lie algebra of the orthogonal group $\mathrm{O}(n, \mathbb{R})$).

(b) Take $M = \mathbb{K}^{2n}$ and let β be the bilinera form

$$\beta((\zeta_1, \dots, \zeta_{2n}), (\eta_1, \dots, \eta_{2n})) = \sum_{i=1}^n \zeta_i\eta_i - \sum_{i=1}^n \zeta_{n+i}\eta_{n+i}.$$

The matrix of β with respect to the canonical basis of $\mathbb{K}^{2n}$ is $\left(\begin{smallmatrix} 0 & I_n\\ -I_n & 0\end{smallmatrix}\right)$. Let $U = \left(\begin{smallmatrix} A & B\\ C & D\end{smallmatrix}\right)$ be the matrix with respect to the canonical basis of $\mathbb{K}^{2n}$ of an element u of $\mathfrak{gl}(M)$ (where A, B, C, D lying in $\mathcal{M}_n(\mathbb{K})$). Then u^* has with respect to the same basis the matrix

$$\begin{pmatrix} 0 & I_n\\ -I_n & 0\end{pmatrix}\begin{pmatrix} A & B\\ C & D\end{pmatrix}\begin{pmatrix} 0 & -I_n\\ I_n & 0\end{pmatrix} = \begin{pmatrix} D^T & -B^T\\ -C^T & A^T\end{pmatrix}$$

The condition $u^* = -u$ is therefore equivalent to the conditions

$$D = -A^T, \quad B = B^t, \quad C = C^t.$$

The Lie algebra defined above is usually denoted by $\mathfrak{sp}(2n, \mathbb{K})$. Note that when $\mathbb{K} = \mathbb{R}$, the Lie algebra obtained is the Lie algebra of the symplectic group $\mathrm{Sp}(2n, \mathbb{R})$.

Example 1.3.9. The $\mathfrak{g}$-module structure on M defines on the $\mathbb{K}$-module $\mathrm{End}_{\mathbb{K}}(M, M)$ of endomorphisms of M a $\mathfrak{g}$-module structure. For all $x \in \mathfrak{g}$ and $u \in \mathrm{End}_{\mathbb{K}}(M)$,

$$x\cdot u = [x_M, u] = \mathrm{ad}(x_M)(u).$$

where $\mathrm{ad}(x_M)$ denotes the image of x_M under the adjoint representation of $\mathfrak{gl}(M)$.

1.3.4 Invariant elements and bilinear forms

Definition 1.3.10. Let $\mathfrak{g}$ be a Lie algebra and M a $\mathfrak{g}$-module. An element $v \in M$ is called **invariant** (with respect to the $\mathfrak{g}$-module structure on M or with respect to the corresponding representation of $\mathfrak{g}$) if $x \cdot v = 0$ for all $x \in \mathfrak{g}$.

Example 1.3.11. Let M, N be two $\mathfrak{g}$-modules and $P = \mathrm{Hom}_{\mathbb{K}}(M, N)$. For an element f of P to be invariant, it is necessary and sufficient that f be a homomorphism of the $\mathfrak{g}$-module M into the $\mathfrak{g}$-module N. In particular, if $M = N$, f is invariant if and only if f is permutable with the x_M for all $x \in \mathfrak{g}$.

Example 1.3.12. Let M be a $\mathbb{K}$-module with a finite basis. If M has a $\mathfrak{g}$-module structure, $\mathrm{Hom}_{\mathbb{K}}(M, M)$ and $M^* \otimes_{\mathbb{K}} M$ have $\mathfrak{g}$-module structures and the canonical map of $M^* \otimes_{\mathbb{K}} M$ into $\mathrm{Hom}_{\mathbb{K}}(M, M)$ is a $\mathfrak{g}$-module isomorphism. As $1 \in \mathrm{Hom}_{\mathbb{K}}(M, M)$ is obviously invariant, the corresponding element u of $M^* \otimes_{\mathbb{K}} M$ is an invariant. If $(e_1, \ldots, e_n)$ is a basis of M and $(e_1^*, \ldots, e_n^*)$ is the dual basis, we have $u = \sum_{i=1}^n e_i^* \otimes e_i$.

Example 1.3.13. Let M be a $\mathfrak{g}$-module. Let β be a bilinear form on M and f the corresponding element of $\mathrm{Hom}_{\mathbb{K}}(M, M^*)$. For β to be invariant, it is necessary and sufficient that f be a $\mathfrak{g}$-module homomorphism. Suppose that $\mathbb{K}$ is a field and that M is finite dimensional. A non-degenerate invariant bilinear form β on M defines an isomorphism of the $\mathfrak{g}$-module M onto the $\mathfrak{g}$-module M^* and hence an isomorphism of the $\mathfrak{g}$ -module $M \otimes_{\mathbb{K}} M$ onto the $\mathfrak{g}$-module $M^* \otimes_{\mathbb{K}} M$. Thus, by Example 1.3.12, giving β defines canonically an invariant element Ω in the $\mathfrak{g}$-module $M \otimes_{\mathbb{K}} M$, which can be constructed as follows: let $(e_1, \ldots, e_n)$ be a basis of M and $(e^1, \ldots, e^n)$ the basis of M such that $\beta(e_i, e^j) = \delta_{ij}$. Then $\Omega = \sum_{i=1}^n e^i \otimes e_i$.

Proposition 1.3.14. *Let $\mathfrak{g}$ be a Lie $\mathbb{K}$-algebra, $\mathfrak{a}$ an ideal of $\mathfrak{g}$, ρ a representation of $\mathfrak{g}$ on M. Then the set N of elements of M invariant under $\mathfrak{a}$ is a sub-$\mathfrak{g}$-module.*

Proof. Let $v \in N$ and $y \in \mathfrak{g}$. For all $x \in \mathfrak{g}$, $[x, y] \in \mathfrak{a}$ and hence

$$\rho(x)\rho(y)v = \rho([x, y])v + \rho(y)\rho(x)v = 0,$$

hence $\rho(y)v \in N$. □

Proposition 1.3.15. *Let M be a semi-simple $\mathfrak{g}$-module. Then the sub-$\mathfrak{g}$-module $M^{\mathfrak{g}}$ of invariant elements of M admits one and only one supplement $\mathfrak{g}$-module, namely the sub-$\mathfrak{g}$-module $M_{\mathfrak{eff}} = \mathfrak{g} \cdot M$.*

Proof. Let N be a submodule of M which is stable under $\mathfrak{g}$ and a supplement of $M^{\mathfrak{g}}$ in M. Then for any $v \in M$, we have $v = v_0 + v_1$ with $v_0 \in M^{\mathfrak{g}}$ and $v_1 \in N$, so

$$x \cdot v = x \cdot v_1 \in N.$$

Since x is arbitrary, this implies $M_{\mathfrak{eff}} \subseteq N$. Now let N_1 be a sub-$\mathfrak{g}$-module of N supplementary to $M_{\mathfrak{eff}}$ in N. For all $v \in N_1$, $x \cdot v \in M_{\mathfrak{eff}} \cap N_1 = \{0\}$ for all $x \in \mathfrak{g}$, hence $v \in M^{\mathfrak{g}}$ and $v = 0$. This proves that $M_{\mathfrak{eff}} = N$. □

Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{K}$. The adjoint representation of $\mathfrak{g}$ on $\mathfrak{g}$ and the zero representation of $\mathfrak{g}$ on $\mathbb{K}$ define a $\mathfrak{g}$-module structure on the $\mathbb{K}$-module $P = \mathrm{Hom}_{\mathbb{K}}(\mathfrak{g}, \mathfrak{g}; \mathbb{K})$ of bilinear forms on $\mathfrak{g}$. Briefly we say that a bilinear form β on $\mathfrak{g}$ is **invariant** if it is invariant under the representation of P. By definition, the necessary and sufficient condition for this to be so is that

$$\beta([x, y], z) = \beta(x, [y, z]).$$

for all x, y, z in $\mathfrak{g}$. Now let $\mathfrak{d}$ be the Lie algebra of derivations of $\mathfrak{g}$. The identity representation of $\mathfrak{d}$ and the zero representation of $\mathfrak{d}$ on $\mathbb{K}$ define a representation of $\mathfrak{b}$ on P. Briefly we say that a bilinear form on $\mathfrak{g}$ is **completely invariant** if it is invariant under the representation of P. A completely invariant bilinear form is invariant. For a bilinear form β on $\mathfrak{g}$ to be completely invariant, it is necessary and sufficient that

$$\beta(Dx, y) + \beta(x, Dy) = 0.$$

Proposition 1.3.16. *Let $\mathfrak{g}$ be a Lie algebra, β an invariant symmetric bilinear form on $\mathfrak{g}$ and $\mathfrak{a}$ an ideal of $\mathfrak{g}$.*

(a) The orthogonal $\mathfrak{a}^{\perp}$ of $\mathfrak{a}$ with respect to β is an ideal of $\mathfrak{g}$.

(b) If $\mathfrak{a}$ is characteristic and β is completely invariant, then $\mathfrak{a}^{\perp}$ is characteristic.

(c) *If β is non-degenerate, then $\mathfrak{a} \cap \mathfrak{a}^{\perp}$ is abelian.*

Proof. Let D be a derivation of $\mathfrak{g}$. Suppose that $\mathfrak{a}$ is stable under D and that $\beta(Dx, y) + \beta(x, Dy) = 0$ for x, y in $\mathfrak{g}$. Then $z \in \mathfrak{a}^{\perp}$ implies $Dz \in \mathfrak{a}^{\perp}$, since, for all $y \in \mathfrak{a}$, $Dy \in \mathfrak{a}$ and hence $\beta(Dz, y) = -\beta(z, Dy) = 0$. Thus $\mathfrak{a}^{\perp}$ is stable under D. This establishes (a) and (b).

Now let $\mathfrak{b}$ be an ideal of $\mathfrak{g}$ and suppose that the restriction of β to $\mathfrak{b}$ is zero. For x, y in $\mathfrak{b}$ and $z \in \mathfrak{g}$, since $[y, z] \in \mathfrak{b}$, we have

$$\beta([x, y], z) = \beta(x, [y, z]) = 0,$$

Thus $[\mathfrak{b}, \mathfrak{b}]$ is orthogonal to $\mathfrak{g}$. If β is non-degenerate, $\mathfrak{b}$ is therefore abelian. This result applied to $\mathfrak{a} \cap \mathfrak{a}^{\perp}$ proves (c). $\square$

Definition 1.3.17. Let $\mathfrak{g}$ be a Lie $\mathbb{K}$-algebra and M a $\mathfrak{g}$-module. Suppose that M, considered as a $\mathbb{K}$-module, admits a finite basis. The bilinear form **associated** with the $\mathfrak{g}$-module M (or with the corresponding representation) is the symmetric bilinear form $(x, y) \mapsto \operatorname{tr}(x_M y_M)$ on $\mathfrak{g}$. If the representation in question is the adjoint representation, the associated bilinear form is called the **Killing form** of $\mathfrak{g}$.

Example 1.3.18 (Killing forms for some Lie algebras).

(a) Let (h, u, t) be the basis for $\mathfrak{sl}_2(\mathbb{K})$ given in Example 1.1.26. The bracket ralation is given by

$$[h, u] = 2t, \quad [u, t] = 2h, \quad [t, h] = -2u.$$

Therefore the matrix of $\operatorname{ad}(h), \operatorname{ad}(u), \operatorname{ad}(t)$ are

$$\operatorname{ad}(h) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix} \quad \operatorname{ad}(u) = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ -2 & 0 & 0 \end{pmatrix} \quad \operatorname{ad}(t) = \begin{pmatrix} 0 & -2 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and the Killing form has the matrix

$$\kappa = \begin{pmatrix} 8 & 0 & 0 \\ 0 & -8 & 0 \\ 0 & 0 & 8 \end{pmatrix}$$

(b) Let (x, y, z) be the basis for $\mathfrak{so}(3)$ given in Example 1.1.26. Then the Killing form has the matrix

$$\kappa = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

(c) Let (h, p, q, z) be the basis for the oscillator algebra given in Example 1.1.8, The nontrivial bracket ralation is given by

$$[p, q] = z, \quad [h, p] = q, \quad [h, q] = -p.$$

Therefore the matrix of $\operatorname{ad}(h)$, $\operatorname{ad}(p)$, $\operatorname{ad}(q)$ are

$$\operatorname{ad}(h) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \operatorname{ad}(p) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \operatorname{ad}(q) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

while $\operatorname{ad}(z) = 0$. Killing form has the matrix

$$\kappa = \begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Proposition 1.3.19. *Let $\mathfrak{g}$ be a Lie algebra and M a $\mathfrak{g}$-module. Supose that M, considered as a $\mathbb{K}$-module, admits a finite basis. Then the bilinear form associated with M is invariant.*

Proof. For x, y, z in $\mathfrak{g}$, we have:

$$\operatorname{tr}([x,y]z) = \operatorname{tr}(xyz) - \operatorname{tr}(yxz) = \operatorname{tr}(xyz) - \operatorname{tr}(xzy) = \operatorname{tr}(x[y,z]),$$

whence the claim. □

Proposition 1.3.20. *Suppose that $\mathbb{K}$ is a field and that the Lie algebra $\mathfrak{g}$ is finite-dimensional over $\mathbb{K}$. Let $\mathfrak{a}$ be an ideal of $\mathfrak{g}$, κ the Killing form of $\mathfrak{g}$. Then $\kappa|_{\mathfrak{a}\times\mathfrak{a}}$ is the Killing form on $\mathfrak{a}$.*

Proof. Let u be an endomorphism of the vector space $\mathfrak{g}$ which leaves $\mathfrak{a}$ stable. Let $u|_{\mathfrak{a}}$ be the restriction of u to $\mathfrak{a}$ and $\bar{u}$ the endomorphism of the vector space $\mathfrak{g}/\mathfrak{a}$ derived from u when passing to the quotient. Then $\operatorname{tr}(u) = \operatorname{tr}(u|_{\mathfrak{a}}) + \operatorname{tr}(\bar{u})$. Let $x \in \mathfrak{a}, y \in \mathfrak{a}$ and apply the above formula to the case where $u = \operatorname{ad}_{\mathfrak{g}}(x)\operatorname{ad}_{\mathfrak{g}}(y)$, we see $u|_{\mathfrak{a}} = \operatorname{ad}_{\mathfrak{a}}(x)\operatorname{ad}_{\mathfrak{a}}(y)$ and $\bar{u} = 0$, hence $\operatorname{tr}(\operatorname{ad}_{\mathfrak{a}}(x)\operatorname{ad}_{\mathfrak{a}}(y)) = \beta(x,y)$. □

Proposition 1.3.21. *Suppose that $\mathbb{K}$ is a field and that the Lie algebra $\mathfrak{g}$ is inite-dimensional over $\mathbb{K}$. The Killing form κ of $\mathfrak{g}$ is completely invariant.*

Proof. Let D be a derivation of $\mathfrak{g}$. There exists a Lie algebra $\tilde{\mathfrak{g}}$ containing $\mathfrak{g}$ as an ideal of codimension 1 and an element x_0 of $\tilde{\mathfrak{g}}$ such that $Dx = [x_0, x]$ for all $x \in \mathfrak{g}$ (Example 1.1.22). Let $\tilde{\kappa}$ be the Killing form of $\tilde{\mathfrak{g}}$. For x, y in $\mathfrak{g}$, we have $\tilde{\kappa}([x, x_0], y) = \tilde{\kappa}(x, [x_0, y])$, that is, $\tilde{\kappa}(Dx, y) + \tilde{\kappa}(x, Dy) = 0$. Now the restriction of $\tilde{\kappa}$ to $\mathfrak{g}$ is κ by Proposition 1.3.21. Hence the proposition. □

Proposition 1.3.22. *Let $\mathfrak{g}$ be a Lie algebra and κ the Killing form on $\mathfrak{g}$. Then for every $\phi \in \operatorname{Aut}(\mathfrak{g})$, we have*

$$\kappa(\phi(x), y) = \kappa(x, \phi^{-1}(y))$$

for $x, y \in \mathfrak{g}$.

Proof. By Proposition 1.1.11, we have

$$\kappa(\phi(x), y) = \operatorname{tr}(\operatorname{ad}(\phi(x))\operatorname{ad}(y)) = \operatorname{tr}(\phi\operatorname{ad}(x)\phi^{-1}\operatorname{ad}(y))$$
$$\kappa(x, \phi^{-1}(y)) = \operatorname{tr}(\operatorname{ad}(x)\operatorname{ad}(\phi^{-1}(y))) = \operatorname{tr}(\operatorname{ad}(x)\phi^{-1}\operatorname{ad}(y)\phi).$$

Since $\phi\operatorname{ad}(x)\phi^{-1}\operatorname{ad}(y)$ and $\operatorname{ad}(x)\phi^{-1}\operatorname{ad}(y)\phi$ are conjugate, they have the same trace, so the claim follows. □

1.3.5 Casimir element

Proposition 1.3.23. *Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{K}$, $U(\mathfrak{g})$ its enveloping algebra, $\mathfrak{a}$ a finite-dimensional ideal of $\mathfrak{g}$ and β an invariant bilinear form on $\mathfrak{g}$, whose restriction to $\mathfrak{a}$ is non-degenerate. Let $(e_1, \dots, e_n)$ and $(e^1, \dots, e^n)$ be two bases of $\mathfrak{a}$ such that $\beta(e_i, e^j) = \delta_{ij}$. Then the element*

$$\Omega(\beta, \mathfrak{a}) = \sum_{i=1}^{n} e_i e^i$$

of $U(\mathfrak{g})$ belongs to the centre of $U(\mathfrak{g})$ and is independent of the choice of basis (e_i).

Proof. For $x \in \mathfrak{g}$ let $x_{\mathfrak{a}}$ be the restriction to $\mathfrak{a}$ of $\operatorname{ad}_{\mathfrak{g}}(x)$. Then $x \mapsto x_{\mathfrak{a}}$ is a representation of $\mathfrak{g}$ on the vector space $\mathfrak{a}$ and the restriction $\kappa|_{\mathfrak{a}}$ of κ to $\mathfrak{a}$ is invariant under this representation. By Example 1.3.13, the tensor $\sum_{i=1}^{n} e_i \otimes f_i$ is independent of the choice of basis (e_i) and is an invariant element of the tensor algebra of $\mathfrak{a}$. It is also an element of the tensor algebra $T(\mathfrak{g})$ of $\mathfrak{g}$, which is invariant for the representation derived from the adjoint representation of $\mathfrak{g}$. Its canonical image in $U(\mathfrak{g})$, that is Ω, is therefore independent of the choice of basis (e_i) and is an invariant for the representation of $\mathfrak{g}$ on $U(\mathfrak{g})$. This element is therefore permutable with every element of $\mathfrak{g}$ and therefore belongs to the centre of $U(\mathfrak{g})$. □

When β is the bilinear form associated with a $\mathfrak{g}$-module M, the element c of Proposition 1.3.23 is called the Casimir element associated with M (or with the corresponding representation). This element exists if the restriction of β to $\mathfrak{a}$ is nondegenerate.

Proposition 1.3.24. *Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{K}$, $\mathfrak{a}$ an ideal of $\mathfrak{g}$ of finite dimension n and M a $\mathfrak{g}$-module of finite dimension over $\mathbb{K}$. Let Ω be the Casimir element (assumed to exist) associated with M and $\mathfrak{a}$, then $\operatorname{tr}(\Omega_M) = n$. In particular, if M is simple and n is not divisible by the characteristic of $\mathbb{K}$, then Ω_M is an automorphism of M.*

Proof. In the notation of Proposition 1.3.23,

$$\operatorname{tr}(\Omega_M) = \sum_{i=1}^{n} \operatorname{tr}(e_i e^i) = \sum_{i=1}^{n} \beta(e_i, e^i) = n.$$

Hence, if n is not divisible by the characteristic of $\mathbb{K}$, then $\Omega_M \neq 0$. On the other hand, as Ω belongs to the centre of $U(\mathfrak{g})$, Ω_M is permutable with all the x_M, $x \in \mathfrak{g}$. If further M is simple, Ω_M is therefore invertible in $\operatorname{End}_{\mathfrak{g}}(M)$ by Schur's lemma. □

1.3.6 Extension of scalars

Let $\mathbb{K}'$ be a commutative ring with unit element and σ a homomorphism of $\mathbb{K}$ into $\mathbb{K}'$. Let $\mathfrak{g}$ be a Lie $\mathbb{K}$-algebra, $U(\mathfrak{g})$ its enveloping algebra and M a left $\mathfrak{g}$-module, that is a left $U(\mathfrak{g})$-module. Then $\sigma_*(M)$ has a canonical left $\sigma_*(U(\mathfrak{g}))$-module structure and hence a left $\sigma_*(\mathfrak{g})$-module structure. Let ρ and $\sigma_*\rho$ be the representations of $\mathfrak{g}$ and $\sigma_*(\mathfrak{g})$ corresponding to M and $\sigma_*(M)$. Then $\sigma_*\rho$ is said to be derived from σ by extending scalars. If $x \in \mathfrak{g}$, $\sigma_*\rho(x)$ is just the endomorphism $\rho(x) \otimes 1$ of $\rho_*(M) = M \otimes_{\mathbb{K}} \mathbb{K}'$.

Suppose that $\mathbb{K}$ is a field, that $\mathbb{K}'$ is an extension of $\mathbb{K}$ and that σ is the canonical injection of $\mathbb{K}$ into $\mathbb{K}'$. Let V and W be vector subspaces of M. Let $\mathfrak{a}$ be the vector subspace of $\mathfrak{g}$ consisting of the $x \in \mathfrak{g}$ such that $\rho(x)(V) \subseteq W$. Let $\tilde{\mathfrak{a}}$ be the vector subspace of $\sigma_*(\mathfrak{g})$ consisting of the $x \in \sigma_*(\mathfrak{g})$ such that $\sigma_*\rho(x)(\sigma_*(V)) \subseteq \sigma_*(W)$. Then $\tilde{\mathfrak{a}} = \sigma_*(\mathfrak{a})$, for clearly $\sigma_*(\mathfrak{a}) \subseteq \tilde{\mathfrak{a}}$, and each $x \in \tilde{\mathfrak{a}}$ can be written as $x = \sum_{i=1}^{n} \lambda_i x_i$, where $x_i \in \mathfrak{g}$ and $\lambda_i \in \mathbb{K}'$ are linearly independent over $\mathbb{K}$. For all $v \in V$, $\rho(x)v \in \sigma_*(W)$, that is, $\sum_{i=1}^{n} \lambda_i \rho(x_i)v \in \sigma_*(W)$, whence $\rho(x_i)v \in W$ for each i.

In particular, the centre of $\sigma_*(\mathfrak{g})$ is derived from the centre of $\mathfrak{g}$ by extending $\mathbb{K}$ to $\mathbb{K}'$. Similarly, let $\mathfrak{h}$ be a subalgebra of $\mathfrak{g}$ and $\mathfrak{n}$ the normalizer of $\mathfrak{h}$ in $\mathfrak{g}$. Then the normalizer of $\sigma_*(\mathfrak{h})$ in $\sigma_*(\mathfrak{g})$ is $\sigma_*(\mathfrak{n})$.

Let $\mathfrak{b}$ be a vector subspace of $\mathfrak{g}$ and W a vector subspace of M. Let V be the vector subspace of M consisting of the $v \in M$ such that $\rho(\mathfrak{b}) \cdot v \subseteq W$. Let $\tilde{V}$ be the vector subspace of $\sigma_*(M)$ consisting of the $v \in \sigma_*(M)$ such that $(\sigma_*\rho)(\sigma_*(\mathfrak{b})) \cdot v \subseteq \sigma_*(W)$. As above it is seen that $\tilde{V} = \sigma_*(V)$. In particular, the vector subspace of invariants of $\sigma_*(M)$ is derived from the vector subspace of invariants of M by extending the base field from $\mathbb{K}$ to $\mathbb{K}'$.

Let ρ be a representation of $\mathfrak{g}$ on a $\mathbb{K}$-module M with a finite basis $(x_1, \dots, x_n)$. Then the bilinear form on $\iota_*(M)$ associated with $\iota_*\rho$ is derived from the bilinear form associated with ρ by extending the base ring to $\mathbb{K}'$ (for, if $u \in \operatorname{End}_{\mathbb{K}}(M)$, then u has the same matrix with respect to $(x_1, \dots, x_n)$ as $u \otimes 1$ with respect to $(x_1 \otimes 1, \dots, x_n \otimes 1)$ and hence u and $u \otimes 1$ have the same trace) . In particular, if the $\mathbb{K}$-module $\mathfrak{g}$ has a finite basis, the Killing form of $\sigma_*(\mathfrak{g})$ is derived from that of $\mathfrak{g}$ by extending the base ring to $\mathbb{K}'$.

Let $\mathfrak{g}$ be a Lie $\mathbb{K}$-algebra and M and N be $\mathfrak{g}$-modules. If M and N are isomorphic $\mathfrak{g}$-modules, $\sigma_*(M)$ and $\sigma_*(N)$ are isomorphic $\iota_*(\mathfrak{g})$-modules. The converse also holds if $\mathbb{K}$ and $\mathbb{K}'$ are fields.

Proposition 1.3.25. *Let $\mathbb{K}$ be a field, $\mathbb{K}'$ an extension of $\mathbb{K}$, $\mathfrak{g}$ a Lie $\mathbb{K}$-algebra and M, N two $\mathfrak{g}$-modules of finite dimension over $\mathbb{K}$. If $\sigma_*(M)$ and $\sigma_*(M)$ are isomorphic $\sigma_*(\mathfrak{g})$-modules, then M and N are isomorphic $\mathfrak{g}$-modules.*

Proof. Suppose first that $\mathbb{K}'$ is an extension of $\mathbb{K}$ of finite degree n. Let $U(\mathfrak{g})$ be the enveloping algebra of $\mathfrak{g}$, so that the enveloping algebra of $\sigma_*(\mathfrak{g})$ is $\sigma_*(U(\mathfrak{g}))$. As $\sigma_*(M)$ and $\sigma_*(N)$ are isomorphic as $\sigma_*(\mathfrak{g})$-modules, they are a fort-iori isomorphic as $U(\mathfrak{g})$-modules; but as $U(\mathfrak{g})$-modules they are respectively isomorphic to M^n and N^n. Now M and N are $U(\mathfrak{g})$-modules of finite length; M (resp. N) is therefore the direct sum of a family $(P_i^{r_i})_{1 \le i \le p}$ (resp. $(Q_j^{s_j})_{1 \le j \le q}$) of submodules such that the P_i (resp. Q_j) are indecomposable and two P_i (resp. Q_j) of different indices are not isomorphic. Then M^n (resp. N^n) is isomorphic to the direct sum of the $P_i^{nr_i}$ (resp. $Q_j^{ns_j}$); it follows that $p = q$ and that after permuting the Q_j if necessary $nr_i = ns_i$ and P_i is isomorphic to Q_i for all i. Hence M is isomorphic to N.

In the general case, let P be the $\mathfrak{g}$-module $\operatorname{Hom}_{\mathbb{K}}(M, N)$ and Q the subspace of invariants of P, that is, the set of homomorphisms of the $\mathfrak{g}$-module M into the $\mathfrak{g}$-module N. In the $\iota_*(\mathfrak{g})$-module $\operatorname{Hom}_{\mathbb{K}'}(\sigma_*(M), \sigma_*(N))$ the subspace of invariants is $\sigma_*(Q)$. The hypothesis that $\sigma_*(M)$ and $\sigma_*(N)$ are isomorphic implies that M and N have the same dimension over $\mathbb{K}$ and that there exists in $\sigma_*(Q)$ an element g which is an isomorphism of $\sigma_*(M)$ onto $\sigma_*(N)$. Let $(f_1, \dots, f_d)$ be a basis of Q over $\mathbb{K}$ and choose bases of M and N over $\mathbb{K}$. If $\lambda_k \in \mathbb{K}$, the matrix of $f = \sum_{k=1}^{d} \lambda_k f_k$ with respect to these bases has determinant which is a polynomial $D(\lambda_1, \dots, \lambda_d)$ with coefficients in $\mathbb{K}$. When $f = g$, this determinant is non-zero and hence the coefficients of D are not all non-zero. Therefore, if $\overline{\mathbb{K}}$ is the algebraic closure of

$\mathbb{K}$, there exists (since $\overline{\mathbb{K}}$ is infinite) elements $\mu_k \in \overline{\mathbb{K}}$ such that $D(\mu_1, \ldots, \mu_d) \neq 0$. If $\mathbb{K}_2$ is the algebraic extension of $\mathbb{K}$ generated by the μ_k, it follows that $\sum_{k=1}^d \mu_k f_k$ is an isomorphism of $\sigma_*(M)$ onto $\sigma_*(M)$; but $\mathbb{K}_2$ is of finite degree over $\mathbb{K}$ and hence M and N are isomorphic by the first part of the argument. □

1.4 Nilpotent Lie algebras

In the following, we shall encounter several important classes of Lie algebras that play a central role in the structure theory of finite-dimensional Lie algebras. The first of these classes, nilpotent Lie algebras, are those for which iterated brackets $[x_1, [x_2, [x_3, [x_4, \ldots]]]]$ of sufficiently large order vanish. The most important result on nilpotent Lie algebras is Engel's Theorem which translates nilpotency of a Lie algebra into a pointwise condition. Typical examples of nilpotent Lie algebras are Lie algebras of strictly upper triangular (block) matrices.

Henceforth $\mathbb{K}$ denotes a field. In the rest of this section the Lie algebras are assumed to be finite-dimensional over $\mathbb{K}$.

1.4.1 Central series and nilpotent Lie algebras

1.4.1.1 Lower central series and upper central series Let $\mathfrak{g}$ be a Lie algebra. The **lower central series** of $\mathfrak{g}$ is the decreasing sequence of characteristic ideals of $\mathfrak{g}$ defined inductively as follows

$$C^1(\mathfrak{g}) = \mathfrak{g}, \quad C^{p+1}(\mathfrak{g}) = [\mathfrak{g}, C^p(\mathfrak{g})].$$

Proposition 1.4.1. *Let $\mathfrak{g}$ and $\mathfrak{h}$ be two Lie algebras over $\mathbb{K}$ and f a surjective homomorphism of $\mathfrak{g}$ onto $\mathfrak{h}$. Then $f(C^p(\mathfrak{g})) = C^p(\mathfrak{h})$.*

Proof. If $\mathfrak{a}$ and $\mathfrak{b}$ are submodules of $\mathfrak{g}$, it follows immediately that $(f[\mathfrak{a}, \mathfrak{b}]) = [f(\mathfrak{a}), f(\mathfrak{b})]$. The proposition is then immediate by induction on p. □

Let $\mathfrak{s}$ a subset of $\mathfrak{g}$. The **centralizer** of $\mathfrak{s}$ in $\mathfrak{g}$ is the set of elements of $\mathfrak{g}$ which commute with those of $\mathfrak{s}$. This is the intersection of the kernels of the $\mathrm{ad}(y)$, where y runs through $\mathfrak{s}$; it is therefore a subalgebra of $\mathfrak{g}$.

Proposition 1.4.2. *Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{a}$ an ideal (resp. a characteristic ideal) of $\mathfrak{g}$. The centralizer $\mathfrak{z}(\mathfrak{a})$ of $\mathfrak{a}$ in $\mathfrak{g}$ is an ideal (resp. a characteristic ideal) of $\mathfrak{g}$.*

Proof. Let D be an inner derivation (resp. a derivation) of $\mathfrak{g}$. If $x \in \mathfrak{z}(\mathfrak{a})$ and $y \in \mathfrak{a}$, then

$$[Dx, y] = D([x, y]) - [x, Dy] = 0$$

hence $Dx \in \mathfrak{z}(\mathfrak{a})$ and the proposition follows. □

The centralizer of $\mathfrak{g}$ in g is called the **centre** of $\mathfrak{g}$, that is the characteristic ideal of $x \in \mathfrak{g}$ such that $[x, y] = 0$ for all $y \in \mathfrak{g}$. The **upper central series** of $\mathfrak{g}$ is the increasing sequence of characteristic ideals of $\mathfrak{g}$ defined inductively as follows: $C_0(\mathfrak{g}) = \{0\}$ and $C_{p+1}(\mathfrak{g})$ is the inverse image of the centre of $\mathfrak{g}/C_p(\mathfrak{g})$ in $\mathfrak{g}$. In particular, $C_1(\mathfrak{g})$ is the centre of $\mathfrak{g}$.

Definition 1.4.3. A Lie algebra $\mathfrak{g}$ is called **nilpotent** if there exists a decreasing finite sequence of ideals $(\mathfrak{g}_i)_{0 \le i \le p}$ of $\mathfrak{g}$ with $\mathfrak{g}_0 = \mathfrak{g}$, $\mathfrak{g}_p = \{0\}$, such that $[\mathfrak{g}, \mathfrak{g}_i] \subseteq \mathfrak{g}_{i+1}$.

Example 1.4.4 (Examples of nilpotent Lie algebras).

(a) The Heisenberg algebra $\mathfrak{h}_3(\mathbb{R})$ is nilpotent

(b) Every abelian Lie algebra is nilpotent.

(c) If $\mathcal{F} = (V_0, \ldots, V_n)$ is a flag in the vector space V, then $\mathfrak{g}_n(\mathcal{F})$ is a nilpotent Lie algebra.

Proposition 1.4.5. *Let $\mathfrak{g}$ be a Lie algebra. The following conditions are equivalent:*

(i) $\mathfrak{g}$ is nilpotent;

(ii) $C^k(\mathfrak{g}) = \{0\}$ for sufficiently large k;

(iii) *$C_k(\mathfrak{g}) = \mathfrak{g}$ for sufficiently large k;*

(iv) *there exists an integer $k > 0$ such that $\mathrm{ad}(x_1) \circ \cdots \circ \mathrm{ad}(x_k) = 0$ for all elements $x_1, \dots, x_k$ in $\mathfrak{g}$;*

(v) *there exists a decreasing sequence of ideals $(\mathfrak{g}_i)_{0 \le i \le n}$ of $\mathfrak{g}$ with $\mathfrak{g}_0 = \mathfrak{g}$, $\mathfrak{g}_n = \{0\}$, such that $[\mathfrak{g}, \mathfrak{g}_i] \subseteq \mathfrak{g}_{i+1}$ and $\dim(\mathfrak{g}/\mathfrak{g}_{i+1}) = 1$ for $0 \le i < n$.*

Proof. First we note that, if $C^k(\mathfrak{g}) = \{0\}$ (resp. $C_k(\mathfrak{g}) = \mathfrak{g}$), then the sequence $C^0(\mathfrak{g}), \dots, C^p(\mathfrak{g})$ (resp. $C_k(\mathfrak{g}), \dots, C_0(\mathfrak{g})$) has the above properties and hence $\mathfrak{g}$ is nilpotent. Conversely, suppose that there exists a sequence $(\mathfrak{g}_i)_{0 \le i \le n}$ with the properties prescribed. It is seen by induction on n that $\mathfrak{g}_i \supseteq C^{i+1}(\mathfrak{g})$ and $\mathfrak{g}_{i-1} \subseteq C_i(\mathfrak{g})$. Hence $C^{p+1}(\mathfrak{g}) = \{0\}$ and $C_p(\mathfrak{g}) = \mathfrak{g}$. We have thus proved that conditions (i), (ii) and (iii) are equivalent. On the other hand, $C^i(\mathfrak{g})$ is the set of linear combinations of elements of the form

$$[x_1[x_2, [\cdots, [x_{i-2}, [x_{i-1}, x_i]] \cdots]]]$$

where $x_1, \dots, x_i$ run through $\mathfrak{g}$. Hence conditions (ii) and (iv) are equivalent. Finally, it is clear that (v) implies (i); conversely, if there exists a sequence $(\mathfrak{g}_i)_{0 \le i \le p}$ of ideals with the properties in the definition of nilpotency, then there exists a decreasing sequence $(\mathfrak{h}_i)_{0 \le i \le n}$ of vector subspaces of $\mathfrak{g}$ of dimensions $n, n-1, \dots, 0$ and a sequence of indices $i_0 < i_1 < \cdots < i_p$ with $\mathfrak{g}_0 = \mathfrak{h}_{i_0}$, $\mathfrak{g}_1 = \mathfrak{h}_{i_1}, \dots, \mathfrak{g}_p = \mathfrak{h}_{i_p}$; then as $[\mathfrak{g}, \mathfrak{g}_i] \subseteq \mathfrak{g}_{i+1}$, the $\mathfrak{h}_i$ are ideals and $[\mathfrak{g}, \mathfrak{h}_i] \subseteq \mathfrak{h}_{i+1}$ for all i. Hence conditions (i) and (v) are equivalent. □

Corollary 1.4.6. *The centre of a non-zero nilpotent Lie algebra is non-zero.*

Proof. If k is the largest integer such that $C^k(\mathfrak{g}) \neq \{0\}$, then we have $[\mathfrak{g}, C^k(\mathfrak{g})] = \{0\}$, whence $\mathfrak{z}(\mathfrak{g}) \supset C^k(\mathfrak{g})$ and is nonzero. □

Corollary 1.4.7. *The Killing form of a nilpotent Lie algebra is zero. Moreover, if $\mathfrak{g}$ is a Lie algebra and $\mathfrak{a}$ is a nilpotent ideal of $\mathfrak{g}$, then $\mathfrak{a}$ is orthogonal to $\mathfrak{g}$ under the Killing form.*

Proof. For all x and y in a nilpotent Lie algebra $\mathrm{ad}(x)\mathrm{ad}(y)$ is nilpotent and hence of zero trace, so the Killing form is trivial. Now let $\mathfrak{a}$ be a nilpotent ideal of a Lie algebra $\mathfrak{g}$; then for $x \in \mathfrak{g}$ and $y \in \mathfrak{a}$, we can show by induction that

$$(\mathrm{ad}(x)\mathrm{ad}(y))^n(\mathfrak{g}) \subseteq C^n(\mathfrak{a}).$$

Since $\mathfrak{a}$ is nilpotent, this shows $\mathrm{ad}(x)\mathrm{ad}(y)$ is nilpotent, so $\mathfrak{a} \perp \mathfrak{g}$ under the Killing form. □

Proposition 1.4.8. *Subalgebras, quotient algebras and central extensions of a nilpotent Lie algebra are nilpotent. A finite sum of nilpotent Lie algebras is a nilpotent Lie algebra.*

Proof. Let $\mathfrak{g}$ be a Lie algebra, $\mathfrak{h}$ a subalgebra of $\mathfrak{g}$, $\mathfrak{a}$ an ideal of $\mathfrak{g}$, $\mathfrak{k} = \mathfrak{g}/\mathfrak{a}$ and π the canonical map of $\mathfrak{g}$ onto $\mathfrak{k}$. If $\mathfrak{g}$ is nilpotent, then $C^k(\mathfrak{g}) = \{0\}$ for some integer k, hence $C^k(\mathfrak{h}) \subseteq C^k(\mathfrak{g}) = \{0\}$ and $C^k(\mathfrak{k}) = \pi(C^k(\mathfrak{g})) = \{0\}$ and hence $\mathfrak{h}$ and $\mathfrak{k}$ are nilpotent. If $\mathfrak{k}$ is nilpotent and $\mathfrak{a}$ is contained in the centre of $\mathfrak{g}$, then $C^k(\mathfrak{k}) = \{0\}$ for some integer k, hence $C^k(\mathfrak{g}) \subseteq \mathfrak{a}$ and therefore $C^{k+1}(\mathfrak{g}) = \{0\}$, so that $\mathfrak{g}$ is nilpotent. Finally, the assertion concerning products follows for example from the equivalence of (i) and (iv) of Proposition 1.4.5. □

Note that the result of Proposition 1.4.8 can not be strengthed: in fact, there exsit non-nilpotent Lie algebras with a nilpotent ideal and nilpotent quotient. Here is an example.

Example 1.4.9. Let $\mathfrak{g}$ be a 2-dimensional nonabelian Lie algebra with basis x, y such that $[x, y] = y$. Then $\mathfrak{a} = \mathbb{K}y$ is an nilpotent ideal in $\mathfrak{g}$, and $\mathfrak{g}/\mathfrak{a}$ is 1-dimensional and hence also nilpotent. However, $\mathfrak{g}$ is not nilpotent, since $C^n(\mathfrak{g}) = \mathfrak{a}$ for $n \ge 2$.

Proposition 1.4.10. *Let $\mathfrak{g}$ be a nilpotent Lie algebra and $\mathfrak{h}$ a subalgebra of $\mathfrak{g}$ distinct from $\mathfrak{g}$. Then the normalizer of $\mathfrak{h}$ in $\mathfrak{g}$ is distinct from $\mathfrak{h}$.*

Proof. Let k be the greatest integer such that $C^k(\mathfrak{g}) + \mathfrak{h} \neq \mathfrak{h}$. Then

$$[C^k(\mathfrak{g}) + \mathfrak{h}, \mathfrak{h}] \subseteq C^{k+1}(\mathfrak{g}) + \mathfrak{h} \subseteq \mathfrak{h},$$

so the normalizer of $\mathfrak{h}$ in $\mathfrak{g}$ contains $C^k(\mathfrak{g})$, which implies it is distinct from $\mathfrak{h}$. □

1.4.2 Engel's theorem

Lemma 1.4.11. *Let V be a vector space over $\mathbb{K}$. If x is a nilpotent endomorphism of V, the map $y \mapsto [x,y]$ of* $\mathrm{End}_{\mathbb{K}}(V)$ *into* $\mathrm{End}_{\mathbb{K}}(V)$ *is nilpotent.*

Proof. If $\mathrm{ad}(x)$ denotes this map, then $\mathrm{ad}(x)^n(y)$ is a sum of terms of the form $\pm x^i y x^j$ with $i+j=n$. If $x^k=0$, then $\mathrm{ad}(x)^{2k-1}(y)=0$ for all y. □

Theorem 1.4.12 (Engel). *Let V be a vector space over $\mathbb{K}$ and $\mathfrak{g}$ a finite-dimensional subalgebra of $\mathfrak{gl}(V)$ whose elements are nilpotent endomorphisms of V. If V is nonzero, then there exists nonzero v in V such that $x \cdot v = 0$ for all $x \in \mathfrak{g}$.*

Proof. We proceed by induction on the dimension n of $\mathfrak{g}$. The theorem is obvious if $n=0$, so suppose that it is true for algebras of dimension smaller than n. Let $\mathfrak{h}$ be a Lie subalgebra of $\mathfrak{g}$ of dimension $m<n$. If $x \in \mathfrak{h}$, then $\mathrm{ad}_{\mathfrak{g}}(x)$ maps $\mathfrak{h}$ into itself and defines on passing to the quotient an endomorphism $\mathrm{ad}_{\mathfrak{g}/\mathfrak{h}}(x)$ of the space $\mathfrak{g}/\mathfrak{h}$. By Lemma 1.4.11, $\mathrm{ad}_{\mathfrak{g}}(x)$ is nilpotent and hence $\mathrm{ad}_{\mathfrak{g}/\mathfrak{h}}(x)$ is nilpotent. By the induction hypothesis there exists a non-zero element of $\mathfrak{g}/\mathfrak{h}$ which is annihilated by all the $\mathrm{ad}_{\mathfrak{g}/\mathfrak{h}}(x)$, $x \in \mathfrak{h}$. It follows that $\mathfrak{h}$ is an ideal in a certain $(m+1)$-dimensional subalgebra of $\mathfrak{g}$.

We conclude (by iteration starting with $\mathfrak{h}=\{0\}$) that $\mathfrak{g}$ has an ideal $\mathfrak{h}$ of dimension $n-1$. Let $y \in \mathfrak{g}-\mathfrak{h}$. We again use the induction hypothesis: the $v \in V$ such that $x \cdot v = 0$ for all $x \in \mathfrak{g}$ form a non-zero vector subspace U of V. This subspace is stable under y by Proposition 1.3.14. Since y is a nilpotent endomorphism of V, there exists a non-zero element of U which is annihilated by y and hence by every element of $\mathfrak{g}$. □

Corollary 1.4.13. *For a Lie algebra $\mathfrak{g}$ to be nilpotent, it is necessary and sufficient that, for all $x \in \mathfrak{g}$, $\mathrm{ad}(x)$ be nilpotent.*

Proof. The condition is necessary by Proposition 1.4.5. Suppose that its sufficiency has been proved for Lie algebras of dimension smaller than n. Let $\mathfrak{g}$ be an n-dimensional Lie algebra such that, for all $x \in \mathfrak{g}$, $\mathrm{ad}(x)$ is nilpotent. Then by Theorem 1.4.12 applied to the set of $\mathrm{ad}(x)$, $x \in \mathfrak{g}$, the centre $\mathfrak{z}$ of $\mathfrak{g}$ is non-zero. Then $\mathfrak{g}$ is a central extension of the Lie algebra $\mathfrak{g}/\mathfrak{z}$, which is nilpotent by our induction hypothesis. The proof is completed by applying Proposition 1.4.8. □

Corollary 1.4.14. *Let $\mathfrak{g}$ be a nilpotent Lie algebra. Then the centre of $\mathfrak{g}$ is nonzero.*

Proof. By applying $\mathfrak{g}$ on itself by the adjoint representation and consider its image in $\mathfrak{gl}(\mathfrak{g})$, we may assume that $\mathfrak{g}$ is a subalgebra of $\mathfrak{gl}(V)$ consists of nilpotent endomorphisms of V (Corollary 1.4.13). Then we can apply Theorem 1.4.12. □

Corollary 1.4.15. *Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{a}$ an ideal of $\mathfrak{g}$. Suppose that $\mathfrak{g}/\mathfrak{a}$ is nilpotent and that, for all $x \in \mathfrak{g}$, the restriction of $\mathrm{ad}(x)$ to $\mathfrak{a}$ is nilpotent. Then $\mathfrak{g}$ is nilpotent.*

Proof. Let $x \in \mathfrak{g}$. As $\mathfrak{g}/\mathfrak{a}$ is nilpotent, there exists an integer n such that $\mathrm{ad}(x)^n(\mathfrak{g}) \subseteq \mathfrak{a}$. By hypothesis there exists an integer m such that $\mathrm{ad}(x)^m(\mathfrak{a}) = \{0\}$. Hence $\mathrm{ad}(x)^{m+n}(\mathfrak{g}) = \{0\}$, which shows $\mathfrak{g}$ is nilpotent by Corollary 1.4.13. □

Example 1.4.16. We consider the vector space $V = \mathbb{K}^{\mathbb{N}}$ with the basis $\{e_i : i \in \mathbb{N}\}$. In terms of the rank-one-operators $E_{ij} \in \mathrm{End}(V)$ defined by $E_{ij}e_k = \delta_{jk}e_i$, we consider the Lie algebra

$$\mathfrak{g} = \mathrm{span}\{E_{ij} : i > j\}.$$

This is an inifinite version of strictly lower triangular matrices. Then by a computation just as the ordinary case, we see that

$$C^n(\mathfrak{g}) = \mathrm{span}\{E_{ij} : i - j > n - 1\}$$

and this implies that $\mathfrak{g}$ is not nilpotent. But we have $C^\infty(\mathfrak{g}) = \bigcap_n C^n(\mathfrak{g}) = \{0\}$, i.e., $\mathfrak{g}$ is *residually nilpotent*. As a vector space, $\mathfrak{g}$ consists of endomorphisms of finite rank, so every element in $\mathfrak{g}$ is nilpotent. Thus Engel's theorem fails in infinite case.

Proposition 1.4.17. *If $\mathfrak{a}, \mathfrak{b}$ are nilpotent ideals of $\mathfrak{g}$, then $\mathfrak{a}+\mathfrak{b}$, $\mathfrak{a} \cap \mathfrak{b}$, and $[\mathfrak{a}, \mathfrak{b}]$ are all nilpotent.*

Proof. Since $\mathfrak{a} \cap \mathfrak{b}$ and $[\mathfrak{a}, \mathfrak{b}]$ are subalgebras of $\mathfrak{a}$, they are nilpotent. As for $\mathfrak{a} + \mathfrak{b}$, we claim that

$$C^{2n}(\mathfrak{a} + \mathfrak{b}) \subseteq C^n(\mathfrak{a}) + C^n(\mathfrak{b}).$$

which implies the assertion because $C^n(\mathfrak{a}) = C^n(\mathfrak{b}) = \{0\}$ holds if n is sufficiently large. To prove this claim, note that the space $C^{2n}(\mathfrak{a} + \mathfrak{b})$ is spanned by elements of the form

$$y = [x_1, [x_2, \dots [x_{2n-1}, x_{2n}] \dots]]$$

with $x_i \in \mathfrak{a} \cup \mathfrak{b}$. If at least n of the x_i are contained in $\mathfrak{a}$, then $y \in C^n(\mathfrak{a})$. If this is not the case, then n of the x_i are contained in $\mathfrak{b}$, which leads to $y \in C^n(\mathfrak{b})$. □

Definition 1.4.18. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra, then by Proposition 1.4.17, there is a largest nilpotent ideal of $\mathfrak{g}$. The maximal nilpotent ideal is called the **nilradical of** $\mathfrak{g}$, and is denoted by $\mathfrak{nil}(\mathfrak{g})$.

Now let $\mathfrak{g}$ be a Lie $\mathbb{K}$-algebra, $\mathbb{K}'$ an extension of $\mathbb{K}$ and σ the canonical embedding of $\mathbb{K}$ to $\mathbb{K}'$. As $C^p(\iota_*(\mathfrak{g})) = \sigma_*(C^p(\mathfrak{g}))$, $\mathfrak{g}$ is nilpotent if and only if $\sigma_*(\mathfrak{g})$ is nilpotent.

Let M be a $\mathfrak{g}$-module of finite dimension over $\mathbb{K}$, $\mathfrak{n}$ the largest nilpotency ideal for M. Let $(M_i)_{0 \le i \le n}$ be a Jordan-Hölder series of the $\mathfrak{g}$-module M. Then $x_M(M_i) \subseteq M_{i+1}$ for all i and all $x \in \mathfrak{n}$, hence

$$x_{\sigma_*(M)}(\sigma_*(M_i)) \subseteq \sigma_*(M_{i+1})$$

for all i and all $x \in \sigma_*(\mathfrak{g})$; hence $x_{\sigma_*(M)}$ is nilpotent for $x \in \sigma_*(\mathfrak{g})$ so that $\iota_*(\mathfrak{n})$ is contained in the largest nilpotency ideal $\tilde{\mathfrak{n}}$ for $\sigma_*(M)$. We shall now see that, if $\mathbb{K}'$ is separable over $\mathbb{K}$, then $\tilde{\mathfrak{n}} = \sigma_*(\mathfrak{n})$.

Proposition 1.4.19. *Let $\mathbb{K}'$ be a separable extension of $\mathbb{K}$. Then the largest nilpotency ideal of $\sigma_*(\mathfrak{g})$ for $\sigma_*(M)$ is $\sigma_*(\mathfrak{n})$.*

Proof. Let A be the associative $\mathbb{K}$-algebra generated by 1 and the x_M ($x \in \mathfrak{g}$), $\tilde{A}$ the associative $\mathbb{K}$-algebra generated by 1 and the $x_{\sigma_*(M)}$ ($x \in \sigma_*(\mathfrak{g})$) and R and $\tilde{r}$ the Jacobson radicals of A and $\tilde{A}$. The algebra $\tilde{A}$ is canonically identified with $\sigma_*(A)$. Then let $y \in \tilde{\mathfrak{n}}$ and write $y = \sum_{i=1}^n \lambda_i y_i$, where the y_i are in $\mathfrak{g}$ and the $\lambda_i \in \mathbb{K}'$ are linearly independent over $\mathbb{K}$. Then $y_{\sigma_*(M)} = \sum_{i=1}^n \lambda_i (y_i)_{\sigma_*(M)}$ and $y_{\sigma_*(M)} \in \tilde{r} = \sigma_*(R)$. Hence $(y_i)_M \in R$ and therefore $y_i \in \mathfrak{n}$ for all i. It follows that $y \in \sigma_*(\mathfrak{n})$, whence $\tilde{\mathfrak{n}} = \sigma_*(\mathfrak{n})$. □

In particular, if $\mathbb{K}'$ is separable over $\mathbb{K}$, the nilradical of $\sigma_*(\mathfrak{g})$ is derived from that of $\mathfrak{g}$ by extending the base field from $\mathbb{K}$ to $\mathbb{K}'$.

1.4.3 The largest nilpotency ideal of a representation

In Corollary 1.4.13 we have seen that nilpotency of a Lie algebra is intensively connected with the nilpotency of the endomorphisms it induces (via the adjoint representation). In this subsection we generalize this concept and consider a Lie algebra $\mathfrak{g}$ with an arbitrary representation M of $\mathfrak{g}$. We will see the subset $\mathfrak{a}$ of $\mathfrak{g}$ with act nilpotently on M is a nilpotent ideal, which will be called the largest nilpotency ideal associated with M.

Lemma 1.4.20. *Let $\mathfrak{g}$ be a Lie algebra, $\mathfrak{a}$ an ideal of $\mathfrak{g}$ and M a simple $\mathfrak{g}$ -module. If for all $x \in \mathfrak{a}$, x_M is nilpotent, then $x_M = 0$ for all $x \in \mathfrak{a}$.*

Proof. Let N be the subspace of M consisting of the $v \in M$ such that $x_M \cdot v = 0$ for all $x \in \mathfrak{a}$. By Theorem 1.4.12, $N \neq \{0\}$. On the other hand, for all $y \in \mathfrak{g}$, N is stable under $\mathfrak{g}$ by Proposition 1.3.14. Hence $N = M$, which proves the lemma. □

Proposition 1.4.21. *Let $\mathfrak{g}$ be a Lie algebra, $\mathfrak{a}$ an ideal of $\mathfrak{g}$, M a $\mathfrak{g}$-module of finite dimension over $\mathbb{K}$ and $(M_i)_{0 \le i \le n}$ a Jordan-Hölder series of the $\mathfrak{g}$-module M. Then the following conditions are equivalent:*

(i) *for all $x \in \mathfrak{a}$, x_M is nilpotent;*

(ii) *for all $x \in \mathfrak{a}$, x_M is in the Jacobson radical of the associative algebra A generated by 1 and the y_M where $y \in \mathfrak{g}$;*

(iii) *for all $x \in \mathfrak{a}$, $x_M(M_i) \subseteq M_{i+1}$ for each $0 \le i < n$.*

If these conditions are fulfilled, then $\mathfrak{a}$ is orthogonal to $\mathfrak{g}$ with respect to the bilinear form associated with the $\mathfrak{g}$-module M.

Proof. As A is finite-dimensional over $\mathbb{K}$, the Jacobson radical of A is a nilpotent ideal (**??**) and hence every element of this radical is nilpotent. This proves (ii)⇒(i). Now assume (i). Then since each $Q_i = M_i/M_{i+1}$ is a simple $\mathfrak{g}$-module and the endomorphism x_{Q_i} on it induced by x is nilpotent, we see $x_{Q_i} = 0$, whence (i) implies (iii).

For (iii)⇒(i), suppose condition (iii) holds; let $x \in \mathfrak{a}$ and $z \in A$. Then $z(M_i) \subseteq M_i$ and hence $(zx_M)^n(M) = \{0\}$; thus Ax_M is a left nilideal of A and hence is contained in the Jacobson radical of A (**??**).

Finally, suppose conditions (i), (ii) and (iii) hold. Let $x \in \mathfrak{a}$ and $y \in \mathfrak{g}$. We have just seen that $y_M x_M$ is nilpotent and hence $\mathrm{tr}(y_M x_M) = 0$, which proves the last assertion of the proposition. □

Proposition 1.4.22. *Let $\mathfrak{g}$ be a Lie algebra, M a $\mathfrak{g}$-module of finite dimension over $\mathbb{K}$ and A the associative algebra generated by 1 and the x_M.*

(a) The ideals $\mathfrak{a}$ of $\mathfrak{g}$ such that x_M is nilpotent for all $x \in \mathfrak{a}$ has a largest element $\mathfrak{n}$.

(b) The ideal $\mathfrak{n}$ is the set of $x \in \mathfrak{g}$ such that x_M belongs to the Jacobson radical of A.

(c) Let $(M_i)_{0\le i\le n}$ be a Jordan-Hölder series of the $\mathfrak{g}$-module M; then $\mathfrak{n}$ is also the set of $x \in \mathfrak{g}$ such that $x_{M_i/Mi+1} = 0$ for all i.

(d) $\mathfrak{n}$ is orthogonal to $\mathfrak{g}$ with respect to the bilinear form associated with M.

*The ideal $\mathfrak{n}$ is called the **largest nilpotency ideal** for the $\mathfrak{g}$-module M or the largest nilpotency ideal of the corresponding representation.*

Proof. The set of $x \in \mathfrak{g}$ such that x_M belongs to the Jacobson radical of A is obviously an ideal of $\mathfrak{g}$. The proposition then follows immediately from Proposition 1.4.21. □

Clearly $\mathfrak{n}$ contains the kernel of the representation on M. It equals it when M is semi-simple (Proposition 1.4.22(c)), but not in general. It should be noted that an element of x of $\mathfrak{g}$ such that x_M is nilpotent does not necessarily belong to $\mathfrak{n}$. We also note that a particular case of Proposition 1.4.21 immediately gives the following result:

Corollary 1.4.23. *Let V be a finite-dimensional vector space and $\mathfrak{g}$ be a nilpotent subalgebra of $\mathfrak{gl}(V)$ such that all elements of $\mathfrak{g}$ are nilpotent. Then there exists a complete flag $\mathcal{F}$ in V with $\mathfrak{g} \subseteq \mathfrak{g}_n(\mathcal{F})$. In particular, $\mathfrak{g}$ is nilpotent and there is a basis for V with respect to which the elements of $\mathfrak{g}$ correspond to strictly upper triangular matrices.*

Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{a}$ an ideal of $\mathfrak{g}$. For $\mathfrak{a}$ to be nilpotent, it is necessary and sufficient that, for all $x \in \mathfrak{a}$, $\mathrm{ad}_{\mathfrak{g}}(x)$ be nilpotent; the condition is obviously sufficient and is necessary, for, if $\mathfrak{a}$ is nilpotent and $x \in \mathfrak{a}$, $\mathrm{ad}_{\mathfrak{a}}(x)$ is nilpotent and $\mathrm{ad}_{\mathfrak{g}}(x)$ maps $\mathfrak{g}$ into $\mathfrak{a}$, hence $\mathrm{ad}_{\mathfrak{g}}(x)$ is nilpotent. Then Proposition 1.4.22 applied to the adjoint representation of $\mathfrak{g}$ gives the following result:

Proposition 1.4.24. *Let $\mathfrak{g}$ be a Lie algebra and A the associative subalgebra of $\mathfrak{gl}(\mathfrak{g})$ generated by 1 and the $\mathrm{ad}_{\mathfrak{g}}(x)$.*

(a) The set $\mathfrak{n}$ of $y \in \mathfrak{g}$ such that $\mathrm{ad}_{\mathfrak{g}}(y)$ belongs to the Jacobson radical of A is the nilradical of $\mathfrak{g}$.

(b) $\mathfrak{n}$ is orthogonal to $\mathfrak{g}$ under the Killing form.

1.5 Solvable Lie algebras

In this section, we turn to the class of solvable Lie algebras. They are defined in a similar fashion as nilpotent ones, and indeed every nilpotent Lie algebra is solvable. The central results on solvable Lie algebras are Lie's Theorem on representations of solvable Lie algebras and Cartan's Solvability Criterion in terms of vanishing of $\mathrm{tr}(\mathrm{ad}[x,y]\mathrm{ad}z)$ for $x, y, z \in \mathfrak{g}$. As we shall see later on, similar techniques apply to semi-simple Lie algebras.

In this section $\mathbb{K}$ henceforth denotes a field of characteristic 0 and all Lie algebras are assumed to be finite-dimensional over $\mathbb{K}$.

1.5.1 Solvable Lie algebras

The derived series of $\mathfrak{g}$ is the decreasing sequence of characteristic ideals of $\mathfrak{g}$ defined inductively as follows

$$D^0(\mathfrak{g}) = \mathfrak{g}, \quad D^{p+1}(\mathfrak{g}) = [D^p(\mathfrak{g}), D^p(\mathfrak{g})].$$

A Lie algebra $\mathfrak{g}$ is called **solvable** if its p-th derived algebra $D^p(\mathfrak{g})$ is zero for sufficiently large p. Clearly a nilpotent Lie algebra is solvable.

Proposition 1.5.1. *Subalgebras and quotient algebras of a solvable Lie algebra are solvable. Every extension of a solvable algebra by a solvable algebra is solvable. Every finite sum of solvable algebras is solvable.*

Proof. Let $\mathfrak{g}$ be a Lie algebra, $\mathfrak{h}$ be a subalgebra, $\mathfrak{a}$ be an ideal of $\mathfrak{g}$, $\mathfrak{k} = g/\mathfrak{a}$ and $\pi : \mathfrak{g} \to \mathfrak{k}$ be the canonical map. If $\mathfrak{g}$ is solvable then $D^p(\mathfrak{g})$ is trivial for some integer $p > 0$, hence $D^p(\mathfrak{h}) \subseteq D^p(\mathfrak{g}) = \{0\}$ and $D^p(\mathfrak{k}) = \pi(D^p(\mathfrak{g})) = \{0\}$, and $\mathfrak{h}$, $\mathfrak{k}$ are solvable. If $\mathfrak{a}$ and $\mathfrak{k}$ are solvable then there exist integers s, t such that $D^s(\mathfrak{a}) = D^t(\mathfrak{k}) = \{0\}$; then $D^t(\mathfrak{g}) \subseteq \mathfrak{a}$, hence $D^{s+t}(\mathfrak{g}) = D^s(D^t(\mathfrak{g})) \subseteq D^s(\mathfrak{a}) = \{0\}$ and $\mathfrak{g}$ is solvable. The last assertion follows from the second by induction on the number of factors. □

Proposition 1.5.2. *Let $\mathfrak{g}$ be a Lie algebra. The following conditions are equivalent:*

(i) *$\mathfrak{g}$ is solvable;*

(ii) *there exists a decreasing sequence $(\mathfrak{g}_i)_{0\leq i\leq n}$ of ideals of $\mathfrak{g}$ such that the algebras $\mathfrak{g}_i/\mathfrak{g}_{i+1}$ are abelian;*

(iii) *there exists a decreasing sequence $(\mathfrak{g}_i)_{0\leq i\leq n}$ of subalgebras of $\mathfrak{g}$ such that $\mathfrak{g}_{i+1}$ is an ideal of $\mathfrak{g}$ and $\mathfrak{g}_i/\mathfrak{g}_{i+1}$ is abelian;*

(iv) *there exists a decreasing sequence $(\mathfrak{g}_i)_{0\leq i\leq n}$ of subalgebras of $\mathfrak{g}$ such that $\mathfrak{g}_{i+1}$ is an ideal of $\mathfrak{g}_i$ of codimension 1.*

Proof. Clearly (i) implies (ii) by considering the sequence of derived ideals of $\mathfrak{g}$, and (ii) implies (iii). Suppose that condition (iii) holds; every vector subspace of $\mathfrak{g}_i$ containing $\mathfrak{g}_{i+1}$ is an ideal of $\mathfrak{g}_i$, whence immediately (iv). Finally, (iv) implies (i) since an extension of a solvable algebra by a solvable algebra is solvable. □

Example 1.5.3 (Examples of solvable Lie algebras).

(a) The oscillator algebra is solvable, but not nilpotent.

(b) Every nilpotent Lie algebra is solvable because $D^n(\mathfrak{g}) \subseteq C^n(\mathfrak{g})$ follows easily by induction.

(c) Consider $\mathbb{R}$ and $\mathbb{C}$ as abelian real Lie algebras and write $I \in \mathrm{End}_{\mathbb{R}}(\mathbb{C})$ for the multiplication with i. Then the Lie algebra $\mathbb{C} \rtimes_I \mathbb{R}$ is solvable, but not nilpotent. It is isomorphic to $\mathfrak{g}/\mathfrak{z}(\mathfrak{g})$, where $\mathfrak{g}$ is the oscillator algebra.

(d) Let $\mathfrak{g}$ be a 2-dimensional nonabelian Lie algebra with basis x, y and bracket $[x, y] = y$. Then $D^1(\mathfrak{g}) = \mathbb{K}y$ and $D^2(\mathfrak{g}) = \{0\}$, so that $\mathfrak{g}$ is solvable. On the other hand, $C^n(\mathfrak{g}) = \langle y\rangle$ for each $n \geq 2$, so that $\mathfrak{g}$ is not nilpotent.

(e) It is not hard to verify that $D(\mathfrak{t}(n,\mathbb{K})) = \mathfrak{n}(n,\mathbb{K})$. As $\mathfrak{n}(n,\mathbb{K})$ is nilpotent and hence solvable, $\mathfrak{t}(n,\mathbb{K})$ is solvable. Therefore $\mathfrak{st}(n,\mathbb{K})$ is solvable.

Example 1.5.4. If $\mathcal{F} = (V_0, \ldots, V_n)$ is a complete flag in the n-dimensional vector space V, then $\mathfrak{g}(\mathcal{F})$ is a solvable Lie algebra. In fact,

$$\mathfrak{g}(\mathcal{F}) \cong \mathfrak{g}_n(\mathcal{F}) \rtimes \mathfrak{gl}(1,\mathbb{K})^n \cong \mathfrak{g}_n(\mathcal{F}) \rtimes \mathbb{K}^n$$

Since $\mathbb{K}^n \cong \mathfrak{g}(\mathcal{F})/\mathfrak{g}_n(\mathcal{F})$ is abelian and $\mathfrak{g}_n(\mathcal{F})$ nilpotent, the solvability of $\mathfrak{g}(\mathcal{F})$ follows. Below we shall see that Lie's Theorem provides a converse for solvable subalgebras of $\mathfrak{gl}(V)$.

Let $\mathfrak{a}$, $\mathfrak{b}$ be two solvable ideals of a Lie algebra $\mathfrak{g}$. The algebra $(\mathfrak{a} + \mathfrak{b})/\mathfrak{b}$ is isomorphic to $\mathfrak{a}/(\mathfrak{a} \cap \mathfrak{b})$ and hence is solvable and $\mathfrak{a} + \mathfrak{b}$, which is an extension of $(\mathfrak{a} + \mathfrak{b})/\mathfrak{b}$ by $\mathfrak{b}$, is also solvable. It follows that a maximal solvable ideal of $\mathfrak{g}$ contains every solvable ideal of $\mathfrak{g}$ and hence $\mathfrak{g}$ has a largest solvable ideal, which is called that **radical** of $\mathfrak{g}$.

Proposition 1.5.5. *The radical $\mathfrak{r}$ of a Lie algebra $\mathfrak{g}$ is the smallest ideal of $\mathfrak{g}$ such that $\mathfrak{g}/\mathfrak{r}$ has radical $\{0\}$.*

Proof. Let $\mathfrak{a}$ be an ideal of $\mathfrak{g}$ and π the canonical map of $\mathfrak{g}$ onto $\mathfrak{g}/\mathfrak{a}$. If the radical of $\mathfrak{g}/\mathfrak{a}$ is zero, then $\pi(\mathfrak{r})$, which is a solvable ideal of $\mathfrak{g}/\mathfrak{a}$, is zero; hence $\mathfrak{a} \subseteq \mathfrak{r}$.

On the other hand, the inverse image $\pi^{-1}(\tilde{\mathfrak{r}})$ of the radical $\tilde{\mathfrak{r}}$ of $\mathfrak{g}/\mathfrak{r}$ is an ideal of $\mathfrak{g}$ which is solvable by Proposition 1.5.1 and hence is equal to $\mathfrak{r}$; therefore $\tilde{\mathfrak{r}} = \{0\}$. □

Proposition 1.5.6. *Let $\mathfrak{g}_1, \dots, \mathfrak{g}_n$ be Lie algebras. The radical $\mathfrak{r}$ of the product of the $\mathfrak{r}_i$ is the product of the radicals $\mathfrak{r}_i$ of the $\mathfrak{g}_i$.*

Proof. The product $\tilde{\mathfrak{r}}$ of the $\mathfrak{r}_i$ is a solvable ideal and hence $\tilde{\mathfrak{r}} \subseteq \mathfrak{r}$. The canonical image of $\mathfrak{r}$ in $\mathfrak{g}_i$ is a solvable ideal of $\mathfrak{g}_i$ and hence is contained in $\mathfrak{r}_i$; hence $\mathfrak{r} \subseteq \tilde{\mathfrak{r}}$. □

1.5.2 Nilpotent radical of a Lie algebra

Let $\mathfrak{g}$ be a Lie algebra and M a $\mathfrak{g}$-module. Recall that the largest nilpotency ideal of $\mathfrak{g}$ for M is defined to be the largest ideal $\mathfrak{a}$ such that x_M is nilpotent for all $x \in \mathfrak{a}$.

Proposition 1.5.7. *Let $\mathfrak{g}$ be a Lie algebra. Then the following ideals coincide:*

(i) *the intersection of the largest nilpotency ideals of the finite-dimensional representations of $\mathfrak{g}$;*

(ii) *the intersection of the kernels of the finite-dimensional semi-simple representations of $\mathfrak{g}$;*

(iii) *the intersection of the kernels of the finite-dimensional simple representations of $\mathfrak{g}$;*

Proof. Let $\mathfrak{s}$ be the intersection of the kernels of the finite-dimensional simple representations of $\mathfrak{g}$, then $\mathfrak{s}$ is the intersection in (i) by Proposition 1.4.22 (c), hence the claim. □

The common ideal $\mathfrak{s}$ in Proposition 1.5.7 is called the **nilpotent radical** of $\mathfrak{g}$. Since the nilradical of $\mathfrak{g}$ is the largest nilpotency ideal for the adjoint representation, we see $\mathfrak{s}$ is contained in the nilradical of $\mathfrak{g}$, hence a nilpotent ideal of $\mathfrak{g}$.

Lemma 1.5.8. *Let V be a finite-dimensional vector space over $\mathbb{K}$, $\mathfrak{g}$ a subalgebra of $\mathfrak{gl}(V)$ such that V is a simple $\mathfrak{g}$-module and $\mathfrak{a}$ an abelian ideal of $\mathfrak{g}$. Then $\mathfrak{a} \cap [\mathfrak{g}, \mathfrak{g}] = [\mathfrak{g}, \mathfrak{a}] = \{0\}$.*

Proof. Let A be the subalgebra of $\mathrm{End}_{\mathbb{K}}(V)$ generated by 1 and $\mathfrak{a}$. Let $\mathfrak{b}$ be an ideal of $\mathfrak{g}$ contained in $\mathfrak{a}$ such that $\mathrm{tr}(bs) = 0$ for all $b \in \mathfrak{b}, s \in A$. Then $\mathrm{tr}(b^n) = 0$ for every integer $n > 0$ and hence b is nilpotent. As the elements of $\mathfrak{b}$ are all nilpotent, $\mathfrak{b}$ is nilpotent and hence $\mathfrak{b} = \{0\}$ by Lemma 1.4.20.

Now if $x \in \mathfrak{g}$, $a \in \mathfrak{a}$, $s \in A$, then since $\mathfrak{a}$ is abelian,

$$\mathrm{tr}([x,a]s) = \mathrm{tr}(xas - axs) = \mathrm{tr}(x[a,s]) = 0$$

Therefore by taking $\mathfrak{b} = [\mathfrak{g}, \mathfrak{a}]$ we get $[\mathfrak{g}, \mathfrak{a}] = \{0\}$. The elements of $\mathfrak{g}$ then commute with those of A, so if x, y belong to $\mathfrak{g}$ and $s \in A$,

$$\mathrm{tr}([x,y]s) = \mathrm{tr}(xys - yxs) = \mathrm{tr}(x[y,s]) = 0.$$

Again by taking $\mathfrak{b} = [\mathfrak{g}, \mathfrak{g}] \cap \mathfrak{a}$, it follows that $[\mathfrak{g}, \mathfrak{g}] \cap \mathfrak{a} = \{0\}$. □

Theorem 1.5.9. *Let $\mathfrak{g}$ be a Lie algebra, $\mathfrak{r}$ its radical and $\mathfrak{s}$ its nilpotent radical. Then $\mathfrak{s} = [\mathfrak{g}, \mathfrak{g}] \cap \mathfrak{r}$.*

Proof. Every linear form λ on $\mathfrak{g}$ which is zero on $[\mathfrak{g}, \mathfrak{g}]$ is a simple representation (with space $\mathbb{K}$) of $\mathfrak{g}$, whence $\lambda(\mathfrak{s}) = \{0\}$. It follows that $\mathfrak{s} \subseteq [\mathfrak{g}, \mathfrak{g}]$. On the other hand, $\mathfrak{s}$ is contained in the radical $\mathfrak{r}$ of $\mathfrak{g}$ since it is nilpotent, so we get $\mathfrak{s} \subseteq [\mathfrak{g}, \mathfrak{g}] \cap \mathfrak{r}$.

Let ρ is a finite-dimensional simple representation of $\mathfrak{g}$. Let k be the least non-negative integer such that $\rho(D^{k+1}(\mathfrak{r})) = \{0\}$, and write

$$\tilde{\mathfrak{g}} = \rho(\mathfrak{g}), \quad \tilde{\mathfrak{a}} = \rho(D^k(\mathfrak{r})).$$

As $D^k(\mathfrak{r})$ is an ideal of $\mathfrak{g}$ and $D^{k+1}(\mathfrak{g}) = \{0\}$, we see $\tilde{\mathfrak{a}}$ is an abelian ideal of $\tilde{\mathfrak{g}}$. If V is the space of ρ, then $\tilde{\mathfrak{g}} \subseteq \mathfrak{gl}(V)$ and V is a simple $\tilde{\mathfrak{g}}$-module. By Lemma 1.5.8, we have

$$\rho([\mathfrak{g}, \mathfrak{g}] \cap D^k(\mathfrak{r})) \subseteq [\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}] \cap \tilde{\mathfrak{a}} = \{0\}. \tag{1.5.1}$$

Since $\rho(D^k(\mathfrak{r})) \neq \{0\}$, we must have $D^k(\mathfrak{r}) \nsubseteq [\mathfrak{g}, \mathfrak{g}]$ in view of (1.5.1), whence $k = 0$. That is, $\rho([\mathfrak{g}, \mathfrak{g}] \cap \mathfrak{r}) = \{0\}$. By the definition of $\mathfrak{s}$, this implies $[\mathfrak{g}, \mathfrak{g}] \cap \mathfrak{r} \subseteq \mathfrak{s}$, which completes the proof. □

Corollary 1.5.10. *Let $\mathfrak{g}$ be a solvable Lie algebra. Then the nilpotent radical of $\mathfrak{g}$ is $[\mathfrak{g},\mathfrak{g}]$. If ρ is a finite-dimensional simple representation of $\mathfrak{g}$, then $\rho(\mathfrak{g})$ is abelian and the associative algebra L generated by 1 and $\rho(\mathfrak{g})$ is a field of finite degree over $\mathbb{K}$.*

Proof. Here we have $\mathfrak{r} = \mathfrak{g}$, so $\mathfrak{s} = [\mathfrak{g},\mathfrak{g}]$ and $\rho([\mathfrak{g},\mathfrak{g}]) = \{0\}$ by the definition of $\mathfrak{s}$. This shows that $\tilde{\mathfrak{g}} = \rho(\mathfrak{g})$ is abelian. Every nonzero element of L is invertible by Schur's Lemma, so L is a field. □

Corollary 1.5.11 (Lie's theorem). *Let $\mathfrak{g}$ be a solvable Lie algebra and assume that $\mathbb{K}$ is algebraically closed. Let M be a $\mathfrak{g}$-module of finite dimension over $\mathbb{K}$ and $(M_i)_{0\le i\le n}$ be a Jordan-Hölder series of M. Then M_i/M_{i+1} is of dimension 1 over $\mathbb{K}$ for $0 \le i < n$ and, for all $x \in \mathfrak{g}$,*

$$\rho_i(x) = \lambda_i(x)\cdot 1,$$

where λ_i is a linear form on $\mathfrak{g}$ which is zero on $[\mathfrak{g},\mathfrak{g}]$ and ρ_i is the representation of $\mathfrak{g}$ on M_i/M_{i+1}. In particular, every simple $\mathfrak{g}$-module of finite dimension over $\mathbb{K}$ is of dimension 1.

Proof. The associative algebra L_i generated by 1 and $\rho_i(\mathfrak{g})$ is a finite field extension of $\mathbb{K}$ and therefore equal to $\mathbb{K}$, so $\rho_i(\mathfrak{g})$ acts on M_i/M_{i+1} by scalars. Moreover, M_i/M_{i+1} is a simple L_i-module, whence $\dim(M_i/M_{i+1}) = 1$. The rest of the corollary is obvious. □

Corollary 1.5.12. *Suppose that $\mathbb{K}$ is algebraically closed. If $\mathfrak{g}$ is an n-dimensional solvable Lie algebra, every ideal of $\mathfrak{g}$ is a term of a decreasing sequence of ideals of dimensions $n, n-1, \ldots, 0$.*

Proof. Every ideal is part of a Jordan-Hölder series of $\mathfrak{g}$, considered as the space of the adjoint representation. Then it suffices to apply Corollary 1.5.11. □

Corollary 1.5.13. *Suppose that $\mathbb{K} = \mathbb{R}$. Let $\mathfrak{g}$ be a solvable Lie algebra. Every simple representation of $\mathfrak{g}$ is of dimension ≤ 2. Every ideal of $\mathfrak{g}$ is a term of a decreasing sequence $(\mathfrak{g}_i)_{0\le i\le n}$ of ideals such that $\dim(\mathfrak{g}_i/\mathfrak{g}_{i+1}) \le 2$.*

Proof. This is proved in a similar way to that for Corollaries 1.5.11, using the fact that every algebraic extension of $\mathbb{R}$ is of degree ≤ 2. □

Corollary 1.5.14. *For a Lie algebra $\mathfrak{g}$ to be solvable, it is necessary and sufficient that $[\mathfrak{g},\mathfrak{g}]$ be nilpotent.*

Proof. If $[\mathfrak{g},\mathfrak{g}]$ is nilpotent, then $\mathfrak{g}$ is solvable because $\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$ is abelian and solvability is an extension property. If, conversely, $\mathfrak{g}$ is solvable, then Theorem 1.5.9 implies that the adjoint representation of $[\mathfrak{g},\mathfrak{g}]$ on $\mathfrak{g}$, and hence on $[\mathfrak{g},\mathfrak{g}]$, is nilpotent. Then we derive that $[\mathfrak{g},\mathfrak{g}]$ is nilpotent. □

Corollary 1.5.15. *Let ρ be a finite-dimensional representation of a Lie algebra $\mathfrak{g}$ and $\mathfrak{r}$ be the radical of $\mathfrak{g}$. Then the largest nilpotency ideal $\mathfrak{a}$ of $\mathfrak{r}$ for ρ is contained in the largest nilpotency ideal $\mathfrak{n}$ of $\mathfrak{g}$ for ρ.*

Proof. It is clear that $\mathfrak{n}\cap\mathfrak{r} \subseteq \mathfrak{a}$, so by Theorem 1.5.9, $[\mathfrak{g},\mathfrak{a}] \subseteq [\mathfrak{g},\mathfrak{g}]\cap\mathfrak{r} \subseteq \mathfrak{n}\cap\mathfrak{r} \subseteq \mathfrak{a}$ and hence $\mathfrak{a}$ is an ideal of $\mathfrak{g}$. This proves $\mathfrak{a} \subseteq \mathfrak{n}$. □

Corollary 1.5.16. *Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{r}$ its radical. The following sets are identical:*

(i) *the nilradical of $\mathfrak{g}$;*

(ii) *the nilradical of $\mathfrak{r}$;*

(iii) *the set of $x \in \mathfrak{r}$ such that $\mathrm{ad}_{\mathfrak{g}}(x)$ is nilpotent;*

(iv) *the set of $x \in \mathfrak{r}$ such that $\mathrm{ad}_{\mathfrak{r}}(x)$ is nilpotent.*

Proof. Let these sets be denoted by $\mathfrak{a},\mathfrak{b},\mathfrak{c},\mathfrak{d}$. The inclusions $\mathfrak{a}\subseteq\mathfrak{b}\subseteq\mathfrak{d}\subseteq\mathfrak{c}$ are clear. We have $\mathfrak{c}\subseteq\mathfrak{a}$ by Corollary 1.5.15 applied to the adjoint representation of $\mathfrak{g}$, so the claim follows. □

1.5.3 Cartan's criterion for solvability

Lemma 1.5.17. *Let M be a finite-dimensional vector space, U and V two vector subspaces of $\mathfrak{gl}(M)$ such that $V \subseteq U$ and $\mathfrak{t}$ the set of $t \in \mathfrak{gl}(M)$ such that $[t, U] \subseteq V$. If $z \in \mathfrak{t}$ is such that $\mathrm{tr}(zt) = 0$ for all $t \in \mathfrak{t}$, then z is nilpotent.*

Proof. It suffices to prove this when $\mathbb{K}$ is algebraically closed, which we shall assume henceforth. Let z_s and z_n be the semi-simple and nilpotent components of z and let (e_i) be a basis of M such that $z_s(e_i) = \lambda_i e_i$ where $\lambda_i \in \mathbb{K}$. Let Q be the vector space over $\mathbb{Q}$ generated by the λ_i. We need to prove that $Q = \{0\}$. For this, let f be a $\mathbb{Q}$-linear form on Q and let $\tilde{s}$ be the endomorphism of M such that $t(e_i) = f(\lambda_i)e_i$. If (E_{ij}) is the canonical basis of $\mathfrak{gl}(M)$ defined by $E_{ij}e_k = \delta_{jk}e_i$, then

$$\mathrm{ad}(z_s)E_{ij} = (\lambda_i - \lambda_j)E_{ij}, \quad \mathrm{ad}(t)E_{ij} = f(\lambda_i - \lambda_j)E_{ij}.$$

Now there exists a polynomial P with no constant term and with coefficients in $\mathbb{K}$ such that $P(\lambda_i - \lambda_j) = f(\lambda_i) - f(\lambda_j)$ for all i and j. Then

$$P(\mathrm{ad}(z_s))E_{ij} = f(\lambda_i - \lambda_j)$$

$\mathrm{ad}(t) = P(\mathrm{ad}(z_s))$. On the other hand, $\mathrm{ad}(z_s)$ is a polynomial with no constant term in $\mathrm{ad}(z)$. Now $\mathrm{ad}(z)(U) \subseteq V$, whence also $\mathrm{ad}(t)(U) \subseteq V$ and $t \in \mathfrak{t}$. By the hypothesis $0 = \mathrm{tr}(zt) = \sum_i \lambda_i f(\lambda_i)$, so $0 = f(\mathrm{tr}(zt)) = \sum_i f(\lambda_i)^2$. Since the $f(\lambda_i)$ are rational numbers, $f = 0$, which completes the proof. □

Theorem 1.5.18 (Cartan's Criterion). *Let $\mathfrak{g}$ be a Lie algebra, M a finite-dimensional vector space, ρ a representation of $\mathfrak{g}$ on M and β the bilinear form on $\mathfrak{g}$ associated with ρ. Then $\rho(\mathfrak{g})$ is solvable if and only if $[\mathfrak{g}, \mathfrak{g}]$ is orthogonal to $\mathfrak{g}$ with respect to β.*

Proof. It can obviously be reduced to the case where $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{gl}(M)$ and ρ is the identity map. If $\mathfrak{g}$ is solvable, $[\mathfrak{g}, \mathfrak{g}]$ is contained in the largest nilpotency ideal of the identity representation of $\mathfrak{g}$ (Theorem 1.5.9) and hence is orthogonal to $\mathfrak{g}$ with respect to β (Proposition 1.4.21).

Suppose that $[\mathfrak{g}, \mathfrak{g}]$ is orthogonal to $\mathfrak{g}$ with respect to β. We prove that $\mathfrak{g}$ is solvable. Let $\mathfrak{t}$ be the set of $t \in \mathfrak{gl}(M)$ such that $[t, \mathfrak{g}] \subseteq [\mathfrak{g}, \mathfrak{g}]$. If $t \in \mathfrak{t}$ and x, y belong to $\mathfrak{g}$, then $[t, x] \in [\mathfrak{g}, \mathfrak{g}]$ and hence

$$\mathrm{tr}(t[x, y]) = \beta(t, [x, y]) = 0$$

whence by linearity $\mathrm{tr}(tu) = 0$ for all $u \in [\mathfrak{g}, \mathfrak{g}]$. Also, clearly $[\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{t}$. Hence (Lemma 1.5.17) every element of $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent. It follows that $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent and hence that $\mathfrak{g}$ is solvable. □

1.5.4 Properties of the radical ideal

Proposition 1.5.19. *Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{r}$ its radical.*

(a) *If ρ is a finite-dimensional representation of $\mathfrak{g}$ and β is the associated bilinear form, then $\mathfrak{r}$ and $[\mathfrak{g}, \mathfrak{g}]$ are orthogonal with respect to β.*

(b) *$\mathfrak{r}$ is the orthogonal of $[\mathfrak{g}, \mathfrak{g}]$ with respect to the Killing form.*

Proof. Let x, y be in $\mathfrak{g}$, $z \in \mathfrak{r}$. Then $[y, z] \in [\mathfrak{g}, \mathfrak{g}] \cap \mathfrak{r}$ and hence

$$\beta([x, y], z) = \beta(x, [y, z]) = 0$$

by Theorem 1.5.9. Hence (a). Let $\tilde{\mathfrak{r}}$ be the orthogonal of $[\mathfrak{g}, \mathfrak{g}]$ with respect to the Killing form. It is an ideal of $\mathfrak{g}$ which contains $\mathfrak{r}$ by the above. On the other hand, the image $\mathfrak{s}$ of $\tilde{\mathfrak{r}}$ under the adjoint representation of $\mathfrak{g}$ is solvable by Theorem 1.5.18 and hence $\tilde{\mathfrak{r}}$ is solvable being a central extension of $\mathfrak{s}$. Hence $\tilde{\mathfrak{r}} = \mathfrak{r}$. □

Corollary 1.5.20. *Let $\mathfrak{g}$ be a Lie algebra. Then $\mathfrak{g}$ is solvable if and only if $[\mathfrak{g}, \mathfrak{g}]$ is orthogonal to $\mathfrak{g}$ under the Killing form.*

Proof. This is a concequence of Proposition 1.5.19. □

Corollary 1.5.21. *The radical $\mathfrak{r}$ of a Lie algebra $\mathfrak{g}$ is a characteristic ideal.*

Proof. Recall that $[\mathfrak{g}, \mathfrak{g}]$ is a characteristic ideal and the Killing form is completely invariant (Proposition 1.3.21). Hence the orthogonal of $[\mathfrak{g}, \mathfrak{g}]$ with respect to the Killing form is a characteristic ideal by Proposition 1.3.16. □

Corollary 1.5.22. *Let $\mathfrak{g}$ be a Lie algebra, $\mathfrak{r}$ its radical and $\mathfrak{a}$ an ideal of $\mathfrak{g}$. Then the radical of $\mathfrak{a}$ is equal to $\mathfrak{r} \cap \mathfrak{a}$.*

Proof. The set $\mathfrak{r} \cap \mathfrak{a}$ is a solvable ideal of $\mathfrak{a}$ and hence is contained in the radical $\tilde{\mathfrak{r}}$ of $\mathfrak{a}$. Conversely, $\tilde{\mathfrak{r}}$ is an ideal of $\mathfrak{g}$ (Corollary 1.5.21) and hence $\tilde{\mathfrak{r}} = \mathfrak{r}$. □

Corollary 1.5.23. *Let $\mathfrak{g}$ be a Lie algebra, $\mathfrak{r}$ its radical and $\mathfrak{n}$ its nilradical. Then for any derivation D of $\mathfrak{g}$, we have $D(\mathfrak{r}) \subseteq \mathfrak{n}$. In particular, $\mathfrak{n}$ is a characteristic ideal.*

Proof. Consider the Lie algebra $\tilde{\mathfrak{g}} = \mathfrak{g} \rtimes_D \mathbb{K}$ extended by D, where $D = \mathrm{ad}(x_0)$ for some $x_0 \in \tilde{\mathfrak{g}}$. We identify $\mathfrak{g}$ with an ideal of $\tilde{\mathfrak{g}}$. Let $\tilde{\mathfrak{r}}$ be the radical of $\tilde{\mathfrak{g}}$ and $\tilde{\mathfrak{n}}$ the nilradical of $\tilde{\mathfrak{g}}$. Then $\mathfrak{r}$ is a solvable ideal of $\tilde{\mathfrak{g}}$, hence contained in $\tilde{\mathfrak{r}}$. By Theorem 1.5.9, we thus obtain

$$D(\mathfrak{r}) = [x_0, \mathfrak{r}] \subseteq [\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}] \cap \tilde{\mathfrak{r}} = \tilde{\mathfrak{s}}$$

where $\tilde{\mathfrak{s}}$ is the nilpotent radical of $\tilde{\mathfrak{g}}$. For all $x \in \tilde{\mathfrak{s}}$, $\mathrm{ad}_{\tilde{\mathfrak{g}}}(x)$ is nilpotent, hence $\mathrm{ad}_{\mathfrak{g}}(x)$ is nilpotent and $\tilde{\mathfrak{s}} \cap \mathfrak{g}$ is therefore a nilpotent ideal. This shows $D(\mathfrak{r}) \subseteq \mathfrak{n}$, and the second claim follows since $\mathfrak{n} \subseteq \mathfrak{r}$. □

Corollary 1.5.24. *Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{a}$ an ideal of $\mathfrak{g}$. If the nilradical of $\mathfrak{g}$ is $\mathfrak{n}$, then the nilradical of $\mathfrak{a}$ is $\mathfrak{a} \cap \mathfrak{n}$.*

Proof. Clearly, $\mathfrak{a} \cap \mathfrak{n}$ is a nilpotent ideal of $\mathfrak{a}$, hence contained in the nilradical of $\mathfrak{a}$, denoted by $\tilde{\mathfrak{n}}$. Conversely, $\tilde{\mathfrak{n}}$ is an ideal of $\mathfrak{g}$ because it is invariant under all the derivations $\mathrm{ad}_{\mathfrak{a}}(x)$, $x \in \mathfrak{a}$, by Corollary 1.5.23. This implies that $\tilde{\mathfrak{n}} \subseteq \mathfrak{a} \cap \mathfrak{n}$. □

Remark 1.5.25. To summarize some of the above results, note that, if $\mathfrak{r}$, $\mathfrak{n}$, $\mathfrak{s}$, $\mathfrak{g}^{\perp}$ denote respectively the radical of $\mathfrak{g}$, the largest nilpotent ideal of $\mathfrak{g}$, the nilpotent radical of $\mathfrak{g}$ and the orthogonal of $\mathfrak{g}$ with respect to the Killing form, then

$$\mathfrak{r} \supset \mathfrak{g}^{\perp} \supset \mathfrak{n} \supset \mathfrak{s}.$$

The inclusion $\mathfrak{r} \supset \mathfrak{g}^{\perp}$ follows from Proposition 1.5.19. The inclusion $\mathfrak{g}^{\perp} \supset \mathfrak{n}$ and $\mathfrak{n} \supset \mathfrak{s}$ follow from Proposition 1.4.24.

Let $\mathfrak{g}$ be a Lie $\mathbb{K}$-algebra and $\sigma : \mathbb{K} \to \mathbb{K}'$ an extension of fields. Clearly $\sigma_*(\mathfrak{g})$ is solvable if and only if $\mathfrak{g}$ is solvable, since $D^p(\sigma_*(\mathfrak{g})) = \sigma_*(D^p(\mathfrak{g}))$. Let $\mathfrak{r}$ be the radical of $\mathfrak{g}$. Then $\sigma_*(\mathfrak{r})$ is the radical of $\sigma_*(\mathfrak{g})$. For let β be the Killing form of $\mathfrak{g}$. As $\mathfrak{r}$ is the orthogonal of $[\mathfrak{g}, \mathfrak{g}]$ with respect to β, $\sigma_*(\mathfrak{g})$ is the orthogonal of $[\sigma_*(\mathfrak{g}), \sigma_*(\mathfrak{g})]$ with respect to the form derived from β by extension from $\mathbb{K}$ to $\mathbb{K}'$, that is, the Killing form of $\sigma_*(\mathfrak{g})$. Our assertion then follows from a further application of Proposition 1.5.19.

1.6 Semi-simple Lie algebras

In this section, we encounter a third class of Lie algebras. Semisimple Lie algebras are a counterpart to the solvable and nilpotent Lie algebras because their ideal structure is quite simple. They can be decomposed as a product of simple ideals. On the other hand, they have a rich geometric structure which even makes a complete classification of finite-dimensional semi-simple Lie algebras possible. One can even show that every finite-dimensional Lie algebra is a semidirect product of its radical and a semi-simple subalgebra.

Recall that $\mathbb{K}$ denotes a field of characteristic 0 and that all Lie algebras are assumed to be finite-dimensional over $\mathbb{K}$.

1.6.1 Semi-simple Lie algebras

Definition 1.6.1. A Lie algebra $\mathfrak{g}$ is called **semi-simple** if the only abelian ideal of $\mathfrak{g}$ is $\{0\}$. The Lie algebra $\mathfrak{g}$ is called **simple** if it is not abelian and it contains no ideals other than $\mathfrak{g}$ and $\{0\}$.

A semi-simple algebra has zero centre and hence its adjoint representation is faithful. The algebra $\{0\}$ is semi-simple, and a Lie algebra of dimension 1 or 2 is not semi-simple. If $\mathfrak{g}_1, \ldots, \mathfrak{g}_n$ are semi-simple, then $\mathfrak{g} = \mathfrak{g}_1 \times \cdots \mathfrak{g}_n$ is semi-simple: for if $\mathfrak{a}$ is an abelian ideal of $\mathfrak{g}$, then projections of $\mathfrak{a}$ onto $\mathfrak{g}_1, \ldots, \mathfrak{g}_n$ reduce to $\{0\}$.

Example 1.6.2. Any 3-dimensional Lie algebra $\mathfrak{g}$ is either solvable or simple. In fact, we have already seen that any Lie algebra of dimension 1 or 2 is solvable. If $\mathfrak{g}$ is not simple, then it has a nontrivial ideal $\mathfrak{a}$. This $\mathfrak{a}$ is solvable, and so is $\mathfrak{g}/\mathfrak{a}$. Hence $\mathfrak{g}$ is solvable by Proposition 1.5.1.

To decide whether such a $\mathfrak{g}$ is solvable or simple, we have only to compute $[\mathfrak{g},\mathfrak{g}]$. If $[\mathfrak{g},\mathfrak{g}] = \mathfrak{g}$, then $\mathfrak{g}$ is simple (because the commutator series cannot end in 0), while if $[\mathfrak{g},\mathfrak{g}] \neq \mathfrak{g}$, then $\mathfrak{g}$ is solvable (because $[\mathfrak{g},\mathfrak{g}]$ has dimension at most 2 and is therefore solvable). It follows from this analysis that $\mathfrak{sl}_2(\mathbb{K})$ and $\mathfrak{so}(3)$ are both simple.

Proposition 1.6.3. *Let $\mathfrak{g}$ be a Lie algebra. The following conditions are equivalent:*

(i) *$\mathfrak{g}$ is semi-simple.*

(ii) *The radical $\mathfrak{r}$ of $\mathfrak{g}$ is zero.*

(iii) *The Killing form β of $\mathfrak{g}$ is non-degenerate.*

Moreover, a semi-simple Lie algebra is equal to its derived ideal.

Proof. If $\mathfrak{g}$ is semi-simple then $\mathfrak{r} = \{0\}$, for otherwise the last non-zero derived algebra of $\mathfrak{r}$ is an abelian ideal of $\mathfrak{g}$. Conversely, since every abelian ideal of $\mathfrak{g}$ is contained in $\mathfrak{g}$, if $\mathfrak{r} = \{0\}$ then $\mathfrak{g}$ is semi-simple. The last claim follows from Proposition 1.5.19, since $[\mathfrak{g},\mathfrak{g}] = \mathfrak{r}^{\perp}$. □

Corollary 1.6.4. *Let $\mathfrak{g}$ be a semi-simple Lie algebra and (M, ρ) a finite-dimensional representation of $\mathfrak{g}$. Then $\rho(\mathfrak{g}) \subseteq \mathfrak{sl}(M)$.*

Proof. The linear form $x \mapsto \mathrm{tr}(\rho(x))$ is zero on $[\mathfrak{g},\mathfrak{g}] = \mathfrak{g}$. □

Proposition 1.6.5. *Let $\mathfrak{g}$ be a semi-simple Lie algebra and ρ a finite-dimensional faithful representation of $\mathfrak{g}$. Then the bilinear form on $\mathfrak{g}$ associated with ρ is non-degenerate.*

Proof. We may identify $\mathfrak{g}$ with $\rho(\mathfrak{g})$. Then since $[\mathfrak{g},\mathfrak{g}] = \mathfrak{g}$, the orthogonal of $\mathfrak{g}$ with respect to this form is a solvable ideal by Theorem 1.5.18 and hence is zero. □

Corollary 1.6.6. *Let $\mathfrak{g}$ be a Lie algebra, κ its Killing form and $\mathfrak{a}$ a semi-simple subalgebra of $\mathfrak{g}$. The orthogonal $\mathfrak{a}^{\perp}$ of $\mathfrak{a}$ with respect to κ is a supplementary subspace of $\mathfrak{a}$ in $\mathfrak{g}$ and $[\mathfrak{a},\mathfrak{a}^{\perp}] \subseteq \mathfrak{a}^{\perp}$. If $\mathfrak{a}$ is an ideal of $\mathfrak{g}$, so is $\mathfrak{a}^{\perp}$, which is then the centralizer of $\mathfrak{a}$ in $\mathfrak{g}$.*

Proof. Since $\mathfrak{a}$ is semi-simple, the adjoint representation of $\mathfrak{a}$ on $\mathfrak{g}$ is faithful and $\kappa|_{\mathfrak{a}\times\mathfrak{a}}$ is non-degenerate. Hence $\mathfrak{a}^{\perp}$ is a supplementary subapce to $\mathfrak{a}$ in $\mathfrak{g}$. On the other hand, if x, y are in $\mathfrak{a}$ and $z \in \mathfrak{a}^{\perp}$, then

$$\kappa(x, [y,z]) = \kappa([x,y],z) = 0$$

which proves that $[\mathfrak{a},\mathfrak{a}^{\perp}] \subseteq \mathfrak{a}^{\perp}$. If $\mathfrak{a}$ is an ideal of $\mathfrak{g}$, we know that $\mathfrak{a}^{\perp}$ is an ideal of $\mathfrak{g}$ by Proposition 1.3.16 and therefore $\mathfrak{g}$ is identified with $\mathfrak{a} \times \mathfrak{a}^{\perp}$. As the centre of $\mathfrak{a}$ is zero, the centralizer of $\mathfrak{a}$ in $\mathfrak{g}$ is $\mathfrak{a}^{\perp}$. □

Corollary 1.6.7. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be Lie algebras and $\mathfrak{g}$ be an extension of $\mathfrak{b}$ by $\mathfrak{a}$. If $\mathfrak{a}$ is semi-simple, then this extension is trivial.*

Proof. Since $\mathfrak{a}$ is an ideal of $\mathfrak{g}$, by Corollary 1.6.6 $\mathfrak{a}^{\perp}$ is an ideal of $\mathfrak{g}$ and $\mathfrak{g} \cong \mathfrak{a} \times \mathfrak{a}^{\perp}$. since $\mathfrak{a}^{\perp}$ is isomorphic to $\mathfrak{b}$, this extension is trivial and $\mathfrak{g}$ is therefore semi-simple. □

Corollary 1.6.8. *If $\mathfrak{g}$ is semi-simple, every derivation of $\mathfrak{g}$ is inner.*

Proof. Since $\mathfrak{z}(\mathfrak{g}) = \{0\}$, the space $\mathrm{ad}(\mathfrak{g})$ is isomorphic to $\mathfrak{g}$ and hence semi-simple and is an ideal of the Lie algebra $\mathfrak{d}$ of derivations of $\mathfrak{g}$. Corollary 1.6.6 then implies $\mathfrak{d} = \mathrm{ad}(\mathfrak{g}) \times \mathrm{ad}(\mathfrak{g})^{\perp}$ and $\mathrm{ad}(\mathfrak{g})^{\perp}$ is the centralizer of $\mathrm{ad}(\mathfrak{g})$ in $\mathfrak{d}$. But if $D \in \mathfrak{d}$ commutes with the elements of $\mathrm{ad}(\mathfrak{g})$, then, for all $x \in \mathfrak{g}$, $\mathrm{ad}(D(x)) = [D, \mathrm{ad}(x)] = 0$, whence $D(x) = 0$; hence $D = 0$. This shows $\mathrm{ad}(\mathfrak{g})^{\perp} = \{0\}$, whence the claim follows. □

1.6.2 Semi-simplicity of representations

Proposition 1.6.9. *Let $\mathfrak{g}$ be a semi-simple Lie algebra. Then the adjoint representation of $\mathfrak{g}$ is semi-simple. Every ideal and every quotient algebra of $\mathfrak{g}$ is semi-simple.*

Proof. Let $\mathfrak{a}$ be an ideal of $\mathfrak{g}$. Then orthogonal $\mathfrak{a}^\perp$ of $\mathfrak{a}$ in $\mathfrak{g}$ with respect to the Killing form is an ideal of $\mathfrak{g}$ and $\mathfrak{a}^\perp \cap \mathfrak{a}$ is an abelian ideal (Proposition 1.3.16) and hence zero. Hence $\mathfrak{g} = \mathfrak{a} \times \mathfrak{a}^\perp$ as $\mathfrak{g}$-modules. Moreover, as the Killing form of $\mathfrak{g}$ is non-degenerate, so are its restrictions to $\mathfrak{a}$ and $\mathfrak{a}^\perp$ and hence $\mathfrak{a}$ and $\mathfrak{a}^\perp$ are semi-simple. □

Lemma 1.6.10. *Let $\mathfrak{g}$ be a Lie algebra. Then the following two conditions are equivalent:*

(i) *All finite-dimensional representations of $\mathfrak{g}$ are semi-simple.*

(ii) *Given a linear representation ρ of $\mathfrak{g}$ on a finite-dimensional vector space V and a vector subspace W of codimension 1 such that $\rho(x)(V) \subseteq W$ for all $x \in \mathfrak{g}$, there exists a supplementary line of W which is stable under $\rho(\mathfrak{g})$ (and hence annihilated by $\rho(\mathfrak{g})$).*

Proof. Since a subspace W of V as in (ii) is stable under $\mathfrak{g}$, we see (i) implies (ii). Suppose that (ii) holds. Let (M, ρ) be a finite-dimensional representation of $\mathfrak{g}$ and N a $\mathfrak{g}$-invariant subspace. Let μ be the representation of $\mathfrak{g}$ on $\mathrm{End}_{\mathbb{K}}(M)$ canonically derived from ρ, given by $\mu(x) = \mathrm{ad}_{\mathrm{End}_{\mathbb{K}}(M)}(\rho(x))$. Let V (resp. W) be the subspace of $\mathrm{End}_{\mathbb{K}}(M)$ consisting of the linear maps of M into N whose restriction to N is a homothety (resp. zero); then W is of codimension 1 in V and for all $x \in \mathfrak{g}$ we have $\mu(x)(V) \subseteq W$. By condition (ii), there exists $u \in V$ which is annihilated by $\mu(x)$ for all $x \in \mathfrak{g}$ and whose restriction to N is a non-zero homothety. By multiplying u by a suitable scalar, it can be assumed that u is a projection of M onto N. Now to say that $\mu(x) \cdot u = 0$ means that u commutes with $\rho(x)$, so it is a homomorphism of $\mathfrak{g}$-modules. Hence the kernel of u is a supplement sub-$\mathfrak{g}$-module of N in M, which shows M is semi-simple. □

Lemma 1.6.11. *Let $\mathfrak{g}$ be a semi-simple Lie algebra, ρ a representation of $\mathfrak{g}$ on a finite-dimensional vector space V and W a subspace of V of codimension 1 such that $\rho(x)(V) \subseteq W$ for all $x \in \mathfrak{g}$. Then there exists a supplementary line of W which is stable under $\rho(\mathfrak{g})$.*

Proof. For all $x \in \mathfrak{g}$ let $\tilde{\rho}(x)$ be the restriction of $\rho(x)$ to W. Suppose first that $\tilde{\rho}$ is simple. If $\tilde{\rho} = 0$, then $\rho(x)\rho(y) = 0$ for all x, y in $\mathfrak{g}$, hence $\rho(\mathfrak{g}) = \rho([\mathfrak{g}, \mathfrak{g}]) = \{0\}$ and our assertion is obvious. If $\tilde{\rho} \neq 0$, let $\mathfrak{a}$ be the kernel of $\tilde{\rho}$ and let $\mathfrak{b}$ be a supplementary ideal of $\mathfrak{a}$ in $\mathfrak{g}$; then $\mathfrak{b} \neq \{0\}$ and the restriction of $\tilde{\rho}$ to $\mathfrak{b}$ is faithful. The restriction to $\mathfrak{b}$ of the bilinear form associated with $\tilde{\rho}$ is non-degenerate (Proposition 1.6.5) and hence the Casimir element Ω associated with $\mathfrak{b}$ and $\tilde{\rho}$ can be formed. By Proposition 1.3.24, $\tilde{\rho}(\Omega)$ is an automorphism of W. On the other hand, $\rho(\Omega)(V) \subseteq W$, hence the kernel Z of $\rho(\Omega)$ is a supplementary line of W. Since by Proposition 1.3.23, Ω belongs to the centre of the enveloping algebra of $\mathfrak{g}$, $\rho(\Omega)$ is permutable with $\rho(x)$ for all $x \in \mathfrak{g}$ and hence Z is stable under $\rho(\mathfrak{g})$.

In the general case we argue by induction on the dimension of V. Let U be a minimal non-zero stable subspace of W. Let $\bar{\rho}$ be the quotient representation on $\overline{V} = V/U$. Then, for all $x \in \mathfrak{g}$, $\bar{\rho}(x)(\overline{V}) \subseteq \overline{W}$, where $\overline{W} = W/U$ is of codimension 1 in $\overline{V}$. By the induction hypothesis there exists a line $\overline{Z}$ which is supplementary to $\overline{W}$ and stable under $\bar{\rho}(\mathfrak{g})$. Let Z be the inverse image of $\overline{Z}$ in V, which is stable under $\rho(\mathfrak{g})$. Then Z contains U as subspace of codimension 1 and $\rho(x)(Z) \subseteq W \cap Z = U$, so by inductional hypothesis there exists a supplementary line of T in Z which is stable under $\rho(\mathfrak{g})$; this line is supplementary to W in V, which completes the proof. □

Theorem 1.6.12 (Weyl). *A Lie algebra is semi-simple if and only if every finite-dimensional representation is completely reducible.*

Proof. One direction is already proved by Lemma 1.6.10 and Lemma 1.6.11.

Conversely, assume that every finite-dimensional representation of a Lie algebra $\mathfrak{g}$ is completely reducible. Since the adjojnt representation is semi-simple, every ideal of $\mathfrak{g}$ admits a supplementary ideal and hence can be considered as a quotient of $\mathfrak{g}$. If $\mathfrak{g}$ is not semi-simple then $\mathfrak{g}$ admits a non-zero abelian quotient and therefore a quotient of dimension 1. Now the Lie algebra $\mathbb{K}$ of dimension 1 admits non-semi-simple representations, for example

$$\lambda \mapsto \begin{pmatrix} 0 & 0 \\ \lambda & 0 \end{pmatrix}$$

This induces a non-semi-simple representation of $\mathfrak{g}$, which is a contradiction. □

A Lie algebra $\mathfrak{g}$ is called **simple** if the only ideals of $\mathfrak{g}$ are $\{0\}$ and $\mathfrak{g}$ and if further $\mathfrak{g}$ is nonabelian. Clearly a simple Lie algebra is semi-simple. The algebra $\{0\}$ is not simple.

Proposition 1.6.13. *For a Lie algebra $\mathfrak{g}$ to be semi-simple, it is necessary and sufficient that it be a product of simple algebras.*

Proof. The condition is sufficient. Conversely, suppose that $\mathfrak{g}$ is semi-simple. Since the adjoint representation of $\mathfrak{g}$ is semi-simple, $\mathfrak{g}$ is the product of minimal non-zero ideals $\mathfrak{g}_1, \ldots, \mathfrak{g}_k$. Then by Corollary 1.6.7 $\mathfrak{g}$ is identified with the product algebra of the $\mathfrak{g}_i$. Every ideal of $\mathfrak{g}_i$ is then an ideal of $\mathfrak{g}$ and hence zero or equal to $\mathfrak{g}_i$. On the other hand $\mathfrak{g}_i$ is non-abelian. Hence the $\mathfrak{g}_i$ are simple Lie algebras. □

Corollary 1.6.14. *A semi-simple Lie algebra is the product of its simple ideals $\mathfrak{g}_i$. Every ideal of $\mathfrak{g}$ is the sum of certain of the $\mathfrak{g}_i$.*

Proof. We have $\mathfrak{g} = \mathfrak{g}_1 \times \cdots \times \mathfrak{g}_n$, where the $\mathfrak{g}_i$ are simple. As the centre of $\mathfrak{g}_i$ is zero, the centralizer of $\mathfrak{g}_i$ in $\mathfrak{g}$ is the product of the $\mathfrak{g}_j$ for $j \neq i$. Then let $\mathfrak{a}$ be an ideal of $\mathfrak{g}$. If it does not contain $\mathfrak{g}_i$, then $\mathfrak{a} \cap \mathfrak{g}_i = \{0\}$, hence $[\mathfrak{a}, \mathfrak{g}_i] = \{0\}$ and $\mathfrak{a}$ is contained in the sum of the $\mathfrak{g}_i$ for $j \neq i$. It follows that $\mathfrak{a}$ is the product of certain of the $\mathfrak{g}_i$. Hence the simple ideals of $\mathfrak{g}$ are precisely the $\mathfrak{g}_i$. □

The simple ideals of a semi-simple Lie algebra $\mathfrak{g}$ are called the **simple components** of $\mathfrak{g}$.

Corollary 1.6.15. *Let $\mathfrak{g}_1$, $\mathfrak{g}_2$ be two Lie algebras, $\mathfrak{r}_1$ and $\mathfrak{r}_2$ their radicals and $f : \mathfrak{g}_1 \to \mathfrak{g}_2$ a surjective homomorphism. Then $\mathfrak{r}_2 = f(\mathfrak{r}_1)$.*

Proof. As $f(\mathfrak{r}_1)$ is solvable, $f(\mathfrak{r}_1) \subseteq \mathfrak{r}_2$. On the other hand, $\mathfrak{g}_1/\mathfrak{r}_1$ is semi-simple, hence $\mathfrak{g}_2/f(\mathfrak{r}_2)$, which is isomorphic to a quotient of $\mathfrak{g}_1/\mathfrak{r}_1$, is semi-simple and hence $f(\mathfrak{r}_1) \supseteq \mathfrak{r}_2$. □

Example 1.6.16. Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{K}$ and ρ a representation of $\mathfrak{g}$ on a vector space M. On the other hand let f be a $\mathbb{K}$-linear map of $\mathfrak{g}$ into M such that:

$$f([x,y]) = \rho(x)f(y) - \rho(y)f(x).$$

for all $x, y \in \mathfrak{g}$. By Example 1.1.23, being given ρ and f is equivalent to being given a homomorphism $x \mapsto (f(x), \rho(x))$ of $\mathfrak{g}$ into $\mathfrak{aff}(M)$. On the other hand we have seen that the element $(f(x), \rho(x))$ of $\mathfrak{aff}(M)$ is canonically identified with the element $\tau(x)$ of $\mathfrak{gl}(N)$ (where $N = M \oplus \mathbb{K}$) which induces $\rho(x)$ on M and maps the element $(0,1)$ of N to $f(x)$. And τ is then a representation of $\mathfrak{g}$ on N such that $\tau(x)(N) \subseteq M$ for all $x \in \mathfrak{g}$.

Then, if $\mathfrak{g}$ is semi-simple, there exists by Lemma 1.6.11 a line Z which is supplementary to M in N and annihilated by $\tau(\mathfrak{g})$. In other words, there exists an element $v_0 \in M$ such that $(-v_0, 1) \in N$ is annihilated by $\tau(x)$, that is such that

$$f(x) = \rho(x) \cdot v_0$$

for all $x \in \mathfrak{g}$.

1.6.3 Jordan decomposition

Proposition 1.6.17. *Let M be a finite-dimensional vector space over $\mathbb{K}$ and $\mathfrak{g}$ a semi-simple subalgebra of $\mathfrak{gl}(M)$. Then $\mathfrak{g}$ contains the semi-simple and nilpotent components of its elements.*

Proof. If $\sigma : \mathbb{K} \to \mathbb{K}'$ is an extension of $\mathbb{K}$, the Killing form of $\sigma_*(\mathfrak{g})$ is the extension to $\sigma_*(\mathfrak{g})$ of that of $\mathfrak{g}$ and hence is non-degenerate; therefore $\sigma_*(\mathfrak{g})$ is semi-simple. It therefore suffices to prove the proposition when the base field is algebraically closed, which we shall henceforth assume to be the case.

For every subspace N of M, let $\mathfrak{g}_N$ be the subalgebra of $\mathfrak{gl}(M)$ consisting of the elements which leave N stable and whose restriction to N has trace zero. As $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$, $\mathfrak{g} \subseteq \mathfrak{g}_N$ if N is stable under $\mathfrak{g}$. Then let $\mathfrak{g}^*$ be the intersection of the normalizer of $\mathfrak{g}$ in $\mathfrak{gl}(M)$ and the algebras $\mathfrak{g}_N$ where N runs through the set of subspaces of M which are stable under $\mathfrak{g}$. As the semi-simple (resp. nilpotent) component x_s (resp. x_n) of $x \in \mathfrak{gl}(M)$ is a polynomial in x with no constant term and $\mathrm{ad}(x_s)$ (resp. $\mathrm{ad}(x_n)$) is the semi-simple (resp. nilpotent) part of $\mathrm{ad}(x)$, clearly $x \in \mathfrak{g}^*$ implies $x_s \in \mathfrak{g}^*$ and $x_n \in \mathfrak{g}^*$; it therefore suffices to show that $\mathfrak{g}^* = \mathfrak{g}$.

Since $\mathfrak{g}$ is a semi-simple ideal of $\mathfrak{g}^*$, $\mathfrak{g}^* = \mathfrak{a} \times \mathfrak{g}$ by Corollary 1.6.6. Let $a \in \mathfrak{a}$ and let N be a subspace which is minimal among the non-zero subspaces of M which are stable under $\mathfrak{g}$. The restriction of a to N is a scalar multiple of the identity by Schur's lemma, has trace zero by construction and hence is zero since $\mathbb{K}$ is of characteristic 0. As M is the direct sum of subspaces such as N, it follows that $a = 0$ and hence $\mathfrak{g}^* = \mathfrak{g}$. □

Corollary 1.6.18. *Let M be a finite-dimensional vector space over $\mathbb{K}$ and $\mathfrak{g}$ a semi-simple subalgebra of $\mathfrak{gl}(M)$. Then an element x of $\mathfrak{g}$ is a semi-simple (resp. nilpotent) endomorphism of M if and only if $\mathrm{ad}_{\mathfrak{g}}(x)$ is a semi-simple (resp. nilpotent) endomorphism of $\mathfrak{g}$.*

Proof. Let x_s (resp. x_n) be the semi-simple (resp. nilpotent) component of $x \in \mathfrak{g}$. Then $x_s \in \mathfrak{g}$ and $x_n \in \mathfrak{g}$ (Proposition 1.6.17). Then $\mathrm{ad}_{\mathfrak{g}}(x_s)$ (resp. $\mathrm{ad}_{\mathfrak{g}}(x_n)$) is the semi-simple (resp. nilpotent) component of $\mathrm{ad}_{\mathfrak{g}}(x)$. If x is semi-simple (resp. nilpotent) so then is $\mathrm{ad}_{\mathfrak{g}}(x)$. If now $\mathrm{ad}_{\mathfrak{g}}(x)$ is semi-simple (resp. nilpotent), it is equal to $\mathrm{ad}_{\mathfrak{g}}(x_s)$ (resp. $\mathrm{ad}_{\mathfrak{g}}(x_n)$) and hence $x = x_s$ (resp. $x = x_n$) since the adjoint representation of $\mathfrak{g}$ is faithful ($\mathfrak{g}$ is semi-simple). □

Definition 1.6.19. Let $\mathfrak{g}$ be a semi-simple Lie algebra. An element x of $\mathfrak{g}$ is called **semi-simple** (resp. **nilpotent**) if for any finite dimension representation M of $\mathfrak{g}$ over $\mathbb{K}$, x_M is a semi-simple (resp. nilpotent) endomorphism of M.

Proposition 1.6.20. *Let $\mathfrak{g}$, $\tilde{\mathfrak{g}}$ be semi-simple Lie algebras and φ a homomorphism of $\mathfrak{g}$ into $\tilde{\mathfrak{g}}$. If $x \in \mathfrak{g}$ is semi-simple (resp. nilpotent), so is $\varphi(x)$. If φ is surjective, every semi-simple (resp. nilpotent) element of $\tilde{\mathfrak{g}}$ is the image under f of a semi-simple (resp. nilpotent) element of $\mathfrak{g}$.*

Proof. If ρ is a representation of $\tilde{\mathfrak{g}}$, $\rho \circ \varphi$ of is a representation of $\mathfrak{g}$, whence the first assertion. If φ is surjective, there exists a homomorphism $\mathfrak{g}$ of $\tilde{\mathfrak{g}}$ into $\mathfrak{g}$ such that $\varphi \circ \rho$ is the identity homomorphism of $\tilde{\mathfrak{g}}$ (Corollary 1.6.7) and the second assertion then follows from the first assertion. □

Theorem 1.6.21. *Let $\mathfrak{g}$ be a semi-simple Lie algebra.*

(a) Let $x \in \mathfrak{g}$. If there exists a faithful representation ρ of $\mathfrak{g}$ such that $\rho(x)$ is a semi-simple (resp. nilpotent) endomorphism, then x is semi-simple (resp. nilpotent) .

(b) Every element of $\mathfrak{g}$ can be written uniquely as the sum of a semi-simple element and a nilpotent element which commute with each other.

Proof. Suppose that the hypothesis of (a) holds, so that we can identify $\mathfrak{g}$ with a subalgebra of $\mathfrak{gl}(V)$, where V is the space of ρ. Then by hypothesis $\rho(x)$ is a semi-simple (resp. nilpotent) endomorphism, whence $\mathrm{ad}_{\mathfrak{g}}(x)$ is semi-simple (resp. nilpotent) by Corollary 1.6.18. Now let (M, τ) be a representation of $\mathfrak{g}$, $\mathfrak{b}$ the supplementary ideal of the kernel $\mathfrak{a}$ of τ, and π the projection of $\mathfrak{g}$ onto $\mathfrak{b}$. Since $\mathrm{ad}_{\mathfrak{g}}(x)$ is semi-simple (resp. nilpotent) and $\mathfrak{g} = \mathfrak{a} \times \mathfrak{b}$, we see $\mathrm{ad}_{\mathfrak{b}}(x) = \mathrm{ad}_{\mathfrak{b}}(\pi(x))$ is semi-simple (resp. nilpotent), and therefore $\tau(x)$ is a semi-simple (resp. nilpotent) endomorphism of M (we have $\tau(\mathfrak{g}) \cong \mathfrak{b}$). This shows the first assertion in view of Corollary 1.6.18. The second result then follows from Proposition 1.6.17 applied to a faithful representation. □

1.6.4 Reductive Lie algebras

We give a brief discussion of reductive Lie algebras. This class of Lie algebras is only slightly larger than the class of semi-simple Lie algebras, but they occur naturally. It contains the abelian Lie algebras. Reductive Lie algebras often occur as stabilizer subalgebras inside semi-simple Lie algebras, thus they appear frequently in proofs by induction on the dimension.

Definition 1.6.22. A Lie algebra is called **reductive** if its adjoint representation is semi-simple.

We now that a semisimple Lie algebra has semi-simple representations. Therefore, any semi-simple Lie algebra is reductive. On the other hand, most results for semi-simple Lie algebras can be translated without diffculty to reductive Lie algebras.

Proposition 1.6.23. *Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{r}$ its radical. The following conditions are equivalent:*

(i) *$\mathfrak{g}$ is reductive;*

(ii) *$[\mathfrak{g}, \mathfrak{g}]$ is semi-simple;*

(iii) *$\mathfrak{g}$ is the product of a semi-simple algebra and an abelian algebra;*

(iv) *$\mathfrak{g}$ has a finite-dimensional representation whose associated bilinear form is non-degenerate;*

(v) *$\mathfrak{g}$ has a faithful semi-simple finite-dimensional representation;*

(vi) *The nilpotent radical of* $\mathfrak{g}$ *is zero.*

(vii) $\mathfrak{r}$ *is the centre of* $\mathfrak{g}$.

Proof. If the adjoint representation of $\mathfrak{g}$ is semi-simple, then $\mathfrak{g}$ is a direct sum of minimal non-zero ideals $\mathfrak{a}_i$ and hence is isomorphic to the product of the $\mathfrak{a}_i$; the $\mathfrak{a}_i$ have no ideals other than $\{0\}$, so $\mathfrak{a}_i$ and hence is simple or abelian of dimension 1. Therefore, $[\mathfrak{g},\mathfrak{g}]$ is equal to the product of those $\mathfrak{a}_i$ which are simple and hence is semi-simple. Conversely, if $[\mathfrak{g},\mathfrak{g}]$ is semi-simple, $\mathfrak{g}$ is isomorphic to the product of $[\mathfrak{g},\mathfrak{g}]$ by a Lie algebra $\mathfrak{h}$ (Corollary 1.6.7); $\mathfrak{h}$ is isomorphic to $\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$ and hence is abelian. We have therefore proves (i)$\Leftrightarrow$(ii).

Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be two Lie algebras, ρ_i a finite-dimensional representation of $\mathfrak{g}_i$ and β_i the bilinear form on $\mathfrak{g}_i$ associated with ρ_i; ρ_1 and ρ_2 can be considered as representations of $\mathfrak{g}=\mathfrak{g}_1\times\mathfrak{g}_2$; let ρ be their direct sum. Clearly the bilinear form on $\mathfrak{g}$ associated with ρ is the direct sum of β_1 and β_2 and hence is non-degenerate if β_1 and β_2 are non-degenerate. Then to prove the implication (iii)$\Rightarrow$(iv) it suffices to consider the following cases: (a) $\mathfrak{g}$ is semi-simple; then the adjoint representation admits as associated form the Killing form, which is non-degenerate; (b) $\mathfrak{g}=\mathbb{K}$; then the identity representation of $\mathfrak{g}$ on $\mathbb{K}$ has an associated bilinear form which is non-degenerate.

Let (M,ρ) be a finite-dimensional representation of $\mathfrak{g}$ and β the associated bilinear form. By Proposition 1.4.22, the largest nilpotency ideal $\mathfrak{n}$ of $\mathfrak{g}$ for ρ is orthogonal to $\mathfrak{g}$ with respect to $\mathfrak{g}$. Let $(M_i)_{0\le i\le n}$ be a Jordan-Hölder series for the $\mathfrak{g}$-module M and let ρ_i be the representation on M_i/M_{i+1}. Then by Proposition 1.4.22(c), $\mathfrak{n}$ is the set of $x\in\mathfrak{g}$ such that $\rho_i(x)=0$ for all i. Let $\tau=\sum_i\rho_i$, then τ has kernel $\mathfrak{n}$ and is a semi-simple representation of $\mathfrak{g}$. If β is non-degenerate, then $\mathfrak{n}=\{0\}$ and hence τ is faithful. This proves (iv)$\Rightarrow$(v). Also, it is clear that (v)$\Rightarrow$(vi).

If the nilpotent radical of $\mathfrak{g}$ is zero, then $[\mathfrak{g},\mathfrak{g}]\cap\mathfrak{r}=\{0\}$ by Theorem 1.5.9; as $[\mathfrak{g},\mathfrak{r}]\subseteq[\mathfrak{g},\mathfrak{g}]\cap\mathfrak{r}$, $\mathfrak{r}$ is the centre of $\mathfrak{g}$. Finally, if $\mathfrak{r}$ is the centre of $\mathfrak{g}$, then the adjoint representation of $\mathfrak{g}$ is identified with a representation of $\mathfrak{g}/\mathfrak{r}$, which is a semi-simple Lie algebra; this representation is therefore semi-simple. □

Remark 1.6.24. If a Lie algebra $\mathfrak{g}$ can be decomposed as a product $\mathfrak{a}\times\mathfrak{b}$ of an abelian Lie algebra $\mathfrak{a}$ and a semi-simple Lie algebra $\mathfrak{b}$, this decomposition is unique. More precisely, the centre of $\mathfrak{g}$ is equal to the product of the centres of $\mathfrak{a}$ and $\mathfrak{b}$ and is hence equal to $\mathfrak{a}$. And $[\mathfrak{g},\mathfrak{g}]=[\mathfrak{a},\mathfrak{a}]\times[\mathfrak{b},\mathfrak{b}]=\mathfrak{b}$.

Corollary 1.6.25. *Let* $\mathfrak{g}$ *be a Lie algebra.*

(a) Every finite product of reductive algebras is a reductive algebra.

(b) If $\mathfrak{g}$ *is a reductive Lie algebra of centre* $\mathfrak{z}$*, every ideal of* $\mathfrak{g}$ *is a direct factor, the product of its intersections with* $\mathfrak{z}$ *and* $[\mathfrak{g},\mathfrak{g}]$*, and is a reductive Lie algebra.*

(c) Every quotient of a reductive Lie algebra is a reductive Lie algebra.

Proof. Assertion (a) follows for example from condition (c) of Proposition 1.6.23. Suppose that $\mathfrak{g}$ is reductive. Let $\mathfrak{a}$ be an ideal of $\mathfrak{g}$. Since the adjoint representation of $\mathfrak{g}$ is semi-simple, $\mathfrak{a}$ has a supplementary ideal $\mathfrak{b}$ and $\mathfrak{g}$ is identified with $\mathfrak{a}\times\mathfrak{b}$. For all $x\in\mathfrak{g}$, let $\rho(x)$ be the restriction of $\mathrm{ad}_{\mathfrak{g}}(x)$ to $\mathfrak{a}$. Then ρ is a semi-simple representation of $\mathfrak{g}$ which is zero on $\mathfrak{b}$ and defines on passing to the quotient the adjoint representation on $\mathfrak{a}$. Hence $\mathfrak{a}$ is reductive. Similarly, $\mathfrak{g}/\mathfrak{a}$ and $\mathfrak{b}$, which are isomorphic, are reductive. Finally, let $\mathfrak{z}(\mathfrak{a})$, $\mathfrak{z}(\mathfrak{b})$ be the centres of $\mathfrak{a}$ and $\mathfrak{b}$; then $\mathfrak{a}=\mathfrak{z}(\mathfrak{a})\times[\mathfrak{a},\mathfrak{a}]$, $\mathfrak{b}=\mathfrak{z}(\mathfrak{b})\times[\mathfrak{b},\mathfrak{b}]$, $\mathfrak{z}(\mathfrak{g})=\mathfrak{z}(\mathfrak{a})\times\mathfrak{z}(\mathfrak{b})$, and $[\mathfrak{g},\mathfrak{g}]=[\mathfrak{a},\mathfrak{a}]\times[\mathfrak{b},\mathfrak{b}]$; hence $\mathfrak{a}=(\mathfrak{a}\cap\mathfrak{z}(\mathfrak{a}))+(\mathfrak{a}\cap[\mathfrak{g},\mathfrak{g}])$. □

Proposition 1.6.26. *Let* $\mathfrak{g}$ *be a Lie algebra,* $\mathfrak{r}$ *its radical and* $\mathfrak{s}$ *its nilpotent radical. Then:*

(a) $\mathfrak{s}=[\mathfrak{g},\mathfrak{r}]=[\mathfrak{g},\mathfrak{g}]\cap\mathfrak{r}$.

(b) $\mathfrak{s}$ *is the intersection of the orthogonals of* $\mathfrak{g}$ *with respect to the bilinear forms associated with the finite-dimensional representations of* $\mathfrak{g}$.

Proof. Clearly $[\mathfrak{g},\mathfrak{r}]\subseteq[\mathfrak{g},\mathfrak{g}]\cap\mathfrak{r}$, and $[\mathfrak{g},\mathfrak{g}]\cap\mathfrak{r}=\mathfrak{s}$ by Theorem 1.5.9. Let $\tilde{\mathfrak{g}}=\mathfrak{g}/[\mathfrak{g},\mathfrak{r}]$ and π be the canonical homomorphism of $\mathfrak{g}$ onto $\tilde{\mathfrak{g}}$; then $\pi(\mathfrak{r})$ is the radical $\tilde{\mathfrak{r}}$ of $\tilde{\mathfrak{g}}$ (Corollary 1.6.15), hence $[\tilde{\mathfrak{g}},\tilde{\mathfrak{r}}]=\{0\}$ and $\tilde{\mathfrak{r}}$ is the centre of $\tilde{\mathfrak{g}}$; therefore (Proposition 1.6.23) $\tilde{\mathfrak{g}}$ has a finite-dimensional faithful semi-simple representation, whence $\mathfrak{s}\subseteq[\mathfrak{g},\mathfrak{r}]$, and this proves (a).

Now let $\mathfrak{t}$ be the intersection of the orthogonals of $\mathfrak{g}$ with respect to the bilinear forms associated with the finite-dimensional representations of $\mathfrak{g}$. Then $\mathfrak{s}\subseteq\mathfrak{t}$ by Proposition 1.4.22. On the other hand,

$\mathfrak{g}/\mathfrak{s}$ has a finite-dimensional faithful semi-simple representation and hence (Corollary 1.6.25) a finite-dimensional representation ρ such that the associated bilinear form is non-degenerate. Considered as a representation of $\mathfrak{g}$, ρ has an associated bilinear form β on $\mathfrak{g}$ and the orthogonal of $\mathfrak{g}$ with respect to $\mathfrak{g}$ is $\mathfrak{s}$, whence $\mathfrak{t} \subseteq \mathfrak{s}$. ☐

Corollary 1.6.27. *Let $\mathfrak{g}_1$, $\mathfrak{g}_2$ be Lie algebras, $\mathfrak{s}_1$ (resp. $\mathfrak{s}_2$) the nilpotent radical of $\mathfrak{g}_1$ (resp. $\mathfrak{g}_2$) and f a surjective homomorphism of $\mathfrak{g}_1$ to $\mathfrak{g}_2$. Then $\mathfrak{s}_2 = f(\mathfrak{s}_1)$ and $\mathfrak{g}_2$ is reductive if and only if the kernel of f contains $\mathfrak{s}$.*

Proof. If $\mathfrak{r}_1, \mathfrak{r}_2$ are the radicals of $\mathfrak{g}_1, \mathfrak{g}_2$, then $\mathfrak{s}_2 = [\mathfrak{g}_1, \mathfrak{r}_1] = [f(\mathfrak{g}_1), f(\mathfrak{r}_1)] = f([\mathfrak{g}_1, \mathfrak{r}_1]) = f(\mathfrak{s}_1)$. Assertion (b) is an immediate consequence of (a). ☐

As an application, we establish the following useful criterion for the semi-simplicity of representations of a Lie algebra.

Theorem 1.6.28. *Let $\mathfrak{g}$ be a Lie algebra, $\mathfrak{r}$ its radical, ρ a finite-dimensional representation of $\mathfrak{g}$, $\tilde{\mathfrak{g}} = \rho(\mathfrak{g})$ and $\tilde{\mathfrak{r}} = \rho(\mathfrak{r})$. Then the following conditions are equivalent:*

(i) ρ is semi-simple;

(ii) $\tilde{\mathfrak{g}}$ is reductive and its centre consists of semi-simple endomorphisms;

(iii) $\tilde{\mathfrak{r}}$ consists of semi-simple endomorphisms;

(iv) the restriction of ρ to $\mathfrak{r}$ is semi-simple.

Proof. If ρ is semi-simple, then $\tilde{\rho}$ is reductive (Proposition 1.6.23(v)); the associative algebra generated by 1 and $\tilde{\mathfrak{g}}$ is semi-simple by **??**, hence its centre is semi-simple (**??**) and the elements of this centre are semi-simple endomorphisms (**??**).

If $\tilde{\mathfrak{g}}$ is reductive, its centre is equal to its radical, that is $\tilde{\mathfrak{r}}$, whence the implication (ii)⇒(iii). Suppose that $\tilde{\mathfrak{r}}$ consists of semi-simple endomorphisms. As $[\tilde{\mathfrak{g}}, \tilde{\mathfrak{r}}]$ consists of nilpotent endomorphisms, $[\tilde{\mathfrak{g}}, \tilde{\mathfrak{r}}] = \{0\}$. Then the implication (iii)⇒(iv) follows from **??**.

Finally, assume the restriction of ρ on $\mathfrak{r}$ is semi-simple. Let $\mathfrak{s}$ be the nilpotent radical of $\mathfrak{g}$ and $\tilde{\rho}$ the restriction of ρ to $\mathfrak{r}$. The elements of $\rho(\mathfrak{s})$ are nilpotent and hence $\mathfrak{s}$ is contained in the largest nilpotency ideal of $\tilde{\rho}$. As $\tilde{\rho}$ is semi-simple, $\tilde{\rho}(\mathfrak{s}) = \{0\}$ and $\tilde{\mathfrak{g}}$ is reductive (Corollary 1.6.27), so that $\tilde{\mathfrak{g}} = [\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}] \times \tilde{\mathfrak{r}}$ with $[\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}]$ semi-simple. Let A (resp. B) be the associative algebra generated by 1 and $[\tilde{\mathfrak{g}}, \tilde{\mathfrak{g}}]$ (resp. $\tilde{\mathfrak{r}}$). Then A is semi-simple by **??**, hence $A \otimes_{\mathbb{K}} B$ is semi-simple (Corollary **??**) and hence the associative algebra generated by 1 and $\tilde{\mathfrak{g}}$, which is a quotient of $A \otimes_{\mathbb{K}} B$, is semi-simple, which proves that ρ is semi-simple. ☐

Corollary 1.6.29. *Let $\mathfrak{g}$ be a Lie algebra and ρ and τ two finite-dimensional semi-simple representations of $\mathfrak{g}$. Then the tensor product of ρ and τ is semi-simple.*

Proof. Let $\mathfrak{r}$ be the radical of $\mathfrak{g}$. For $x \in \mathfrak{r}$, $\rho(x)$ and $\tau(x)$ are semi-simple (Theorem 1.6.28), hence $\rho(x) \otimes 1 + 1 \otimes \tau(x)$ is semi-simple and hence the tensor product of ρ and τ is semi-simple (Theorem 1.6.28). ☐

Corollary 1.6.30. *Let $\mathfrak{g}$ be a Lie algebra and ρ and τ two finite-dimensional semi-simple representations of $\mathfrak{g}$ on spaces M and N. Then the representation of $\mathfrak{g}$ on $\mathrm{End}_{\mathbb{K}}(M, N)$ canonically derived from ρ and τ is semi-simple.*

Proof. The $\mathfrak{g}$-module $\mathrm{End}_{\mathbb{K}}(M, N)$ is canonically identified with the $\mathfrak{g}$-module $M^* \otimes_{\mathbb{K}} N$ (Proposition 1.3.7), so the claim follows from Corollary 1.6.29. ☐

Corollary 1.6.31. *Let $\mathfrak{g}$ be a Lie algebra, ρ a semi-simple representation of $\mathfrak{g}$ on a finite-dimensional vector space V, $T(V)$ and $\boldsymbol{S}(V)$ the tensor and symmetric algebras of V and ρ_T, ρ_S the representations of $\mathfrak{g}$ on $T(V)$ and $\boldsymbol{S}(V)$ canonically derived from ρ. Then ρ_T and ρ_S are semi-simple and, more precisely, direct sums of finite-dimensional simple representations.*

Proof. Let $T^n(V)$ be the subspace of T consisting of the homogeneous tensors of order n. This subspace is stable under ρ_T and the representation defined by ρ_T on $T^n(V)$ is semi-simple by Corollary 1.6.29. Hence the corollary for ρ_T and therefore for ρ_S, which is a quotient representation of ρ_T. ☐

Corollary 1.6.32. *Let $\mathfrak{g}$ be a Lie algebra, $\mathfrak{a}$ an ideal of $\mathfrak{g}$ and ρ a semi-simple representation of $\mathfrak{g}$.*

(a) The restriction $\tilde{\rho}$ of ρ to $\mathfrak{a}$ is semi-simple.

(b) If ρ is simple, $\tilde{\rho}$ is a sum of simple representations isomorphic to one another.

Proof. Passing to the quotient by the kernel of ρ, ρ can be assumed to be faithful. Then $\mathfrak{g}$ is reductive. Let $\mathfrak{g} = \mathfrak{g}_1 \times \mathfrak{g}_2$, where $\mathfrak{g}_1$ is the centre of $\mathfrak{g}$ and $\mathfrak{g}_2$ is semi-simple. Then $\mathfrak{a} = \mathfrak{a}_1 \times \mathfrak{a}_2$, where $\mathfrak{a}_i \subseteq \mathfrak{g}_i$ and $\mathfrak{a}_1$ is the centre of $\mathfrak{a}$. The elements of $\rho(\mathfrak{g}_1)$, and in particular those of $\rho(\mathfrak{a}_1)$, are semi-simple by Theorem 1.6.28 and hence $\tilde{\rho}$ is semi-simple. Hence (a). Assertion (b) follows from (a), using Corollary 1.3.4. □

Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{h}$ a Lie subalgebra of $\mathfrak{g}$. Then $\mathfrak{h}$ is called **reductive in** $\mathfrak{g}$ if the representation $x \mapsto \mathrm{ad}_{\mathfrak{g}}(x)$ of $\mathfrak{h}$ is semi-simple. This representation admits as subrepresentation the adjoint representation of $\mathfrak{h}$. Hence, if $\mathfrak{h}$ is reductive in $\mathfrak{g}$, $\mathfrak{h}$ is reductive. On the other hand, to say that a Lie algebra is reductive in itself is equivalent to saying that it is reductive.

Proposition 1.6.33. *Let $\mathfrak{g}$ be a Lie algebra, $\mathfrak{h}$ a subalgebra reductive in $\mathfrak{g}$, ρ a representation of $\mathfrak{g}$ on a vector space V and W the sum of the finite-dimensional subspaces of V which are simple $\mathfrak{h}$-modules. Then W is stable under $\rho(\mathfrak{g})$.*

Proof. Let W_0 be a finite-dimensional simple sub-$\mathfrak{h}$-module of V. We need to prove that $\rho(x)(W_0) \subseteq W$ for all $x \in \mathfrak{g}$. Let M denote the vector space $\mathfrak{g}$ considered as an $\mathfrak{h}$-module by means of the representation $x \mapsto \mathrm{ad}_{\mathfrak{g}}(x)$ of $\mathfrak{h}$ on $\mathfrak{g}$. Then $M \otimes_{\mathbb{K}} W_0$ is a semi-simple $\mathfrak{h}$-module (Corollary 1.6.29). Let θ be the $\mathbb{K}$-linear map of $M \otimes_{\mathbb{K}} W_0$ into V defined by $\theta(x \otimes w) = \rho(x)w$. This is an $\mathfrak{g}$-module homomorphism, for if $y \in \mathfrak{g}$ then:

$$\theta([y,x] \otimes w + x \otimes \rho(y)w) = \rho([y,x])w + \rho(x)\rho(y)w = \rho(y)\rho(x)w = \rho(y)\theta(x \otimes w).$$

Hence $\theta(M \otimes_{\mathbb{K}} W_0)$ is a finite-dimensional semi-simple $\mathfrak{h}$ -module. Hence $\theta(M \otimes_{\mathbb{K}} W_0) \subseteq W$, that is, $\rho(x)(W_0) \subseteq W$ for all $x \in \mathfrak{g}$. □

Corollary 1.6.34. *Let $\mathfrak{g}$ be a Lie algebra, $\mathfrak{h}$ a subalgebra reductive in $\mathfrak{g}$ and ρ a finite-dimensional semi-simple representation of $\mathfrak{g}$. Then the restriction of ρ to $\mathfrak{h}$ is semi-simple.*

Proof. It suffices to consider the case where ρ is simple. We adopt the notation V, W of Proposition 1.6.33. Let W_1 be a nonzero minimal sub-$\mathfrak{h}$-module of V. Then $W_1 \subseteq W$, hence $W \neq \{0\}$ and $W = V$. □

Corollary 1.6.35. *Let $\mathfrak{g}$ be a Lie algebra, $\mathfrak{h}$ a subalgebra reductive in $\mathfrak{g}$ and $\mathfrak{k}$ a subalgebra of $\mathfrak{h}$ reductive in $\mathfrak{h}$. Then $\mathfrak{k}$ is reductive in $\mathfrak{g}$.*

Proof. The representation $x \mapsto \mathrm{ad}_{\mathfrak{g}}(x)$ of $\mathfrak{h}$ on $\mathfrak{g}$ is semi-simple and hence its restriction to $\mathfrak{k}$ is semi-simple by Corollary 1.6.34. □

To conclude this paragraph, we now present a technique for recognizing certain Lie algebras of matrices as semi-simple. It begins with the following proposition.

Proposition 1.6.36. *Let V be a finite-dimensional vector space over $\mathbb{K}$ and β a nondegenerate symmetric (resp. alternating) bilinear form on V. Let $\mathfrak{g}$ be the Lie algebra consisting of the $x \in \mathfrak{gl}(V)$ such that*

$$\beta(xv, w) = -\beta(v, xw)$$

for all $v, w \in V$. Then $\mathfrak{g}$ is reductive; $\mathfrak{g}$ is even semi-simple except in the case where β is symmetric and $\dim(V) = 2$.

Proof. For all $x \in \mathfrak{gl}(V)$ let x^* denote its adjoint relative to β; then $\mathrm{tr}(x) = \mathrm{tr}(x^*)$. The condition $\beta(xv, w) = -\beta(v, xw)$ means $x = -x^*$. In particular, if $y \in \mathfrak{gl}(V)$ then $y - y^* \in \mathfrak{g}$. Then let x be an element of $\mathfrak{g}$ orthogonal to $\mathfrak{g}$ with respect to the bilinear form κ associated with the identity representation of $\mathfrak{g}$. For all $y \in \mathfrak{gl}(V)$, $\mathrm{tr}(x(y - y^*)) = 0$, hence

$$\mathrm{tr}(xy) = \mathrm{tr}(xy^*) = \mathrm{tr}((xy^*)^*) = \mathrm{tr}(yx^*) = -\mathrm{tr}(yx) = -\mathrm{tr}(xy)$$

and hence $\mathrm{tr}(xy) = 0$. It follows that $x = 0$, so that κ is non-degenerate. Hence $\mathfrak{g}$ is reductive by Proposition 1.6.23. It remains to show that the centre of $\mathfrak{g}$ is zero (except when β is symmetric and $n = 2$). By extending the base field, we can assume that $\mathbb{K}$ is algebraically closed.

When β is symmetric, it can be identified with the bilinear form on $\mathbb{K}^n$ with matrix I_n with respect to the canonical basis. Under these conditions $\mathfrak{g}$ is identified with the Lie algebra $\mathfrak{so}(n, \mathbb{K})$ of skew-symmetric matrices. Let $X = (x_{ij}) \in \mathfrak{g}$ be in the center of $\mathfrak{g}$ and E_{ij} be a standard basis for $\mathcal{M}_n(\mathbb{K})$. We use the fact that X commutes with the matrix $Y_{kl} = E_{kl} - E_{lk}$, which implies

$$x_{ik} = x_{il} = x_{kj} = x_{kl} = 0 \quad \text{for } i \neq k, l \text{ and } j \neq k, l.$$

If $n > 2$, there exist, for all distinct indices i and l, distinct indices k and j such that $k \neq i, j \neq l, k \neq j$; hence $x_{il} = 0$. This proves that an element of the centre of $\mathfrak{g}$ is zero. Note that when $n = 2$, the Lie algebra $\mathfrak{so}(2, \mathbb{K})$ is abelian.

When β is alternating, β can be identified with the bilinear form on $\mathbb{K}^{2n}$ with matrix $\left(\begin{smallmatrix} 0 & I_n \\ -I_n & 0 \end{smallmatrix}\right)$ with respect to the canonical basis. Under these conditions $\mathfrak{g}$ is identified with the Lie algebra $\mathfrak{sp}(2n, \mathbb{K})$ of matrices of the form $X = \left(\begin{smallmatrix} A & B \\ C & -A^t \end{smallmatrix}\right)$ where B and C are symmetric. Assume that X is an element in the center of $\mathfrak{g}$. We use first the fact that X commutes with the matrix $\left(\begin{smallmatrix} Y & 0 \\ 0 & -Y^t \end{smallmatrix}\right)$, where $Y \in \mathcal{M}_n(\mathbb{K})$. Then

$$AX = XA, \quad CX = -X^t C, \quad XB = -BX^t. \tag{1.6.1}$$

As these equalities must hold for all Y, it follows that A is a scalar matrix λI_n. We now use the fact that X commutes with the matrix $\left(\begin{smallmatrix} 0 & Y \\ 0 & 0 \end{smallmatrix}\right)$ where Y is a symmetric matrix of $\mathcal{M}_n(\mathbb{K})$. Then $\lambda Y = YC = CY = 0$, which proves first that $\lambda = 0$.

For all $Y \in \mathcal{M}_n(\mathbb{K})$, $Y + Y^t$ is symmetric, hence we have $YC = -Y^t C$. Using the equation $CX = -X^t C$ in (1.6.1), we see that C commutes with every element of $\mathcal{M}_n(\mathbb{K})$ and hence is a scalar matrix, necessarily zero since $YC = 0$. It is similarly shown that $B = 0$. □

Example 1.6.37. As another example, let $\mathfrak{g}$ be a real Lie algebra of real or complex or quaternion matrices closed under conjugate transpose. We show that $\mathfrak{g}$ is reductive. For this, let $(-)^*$ denote the conjugate transpose and define a bilinear form on $\mathfrak{g}$ by $(X, Y) = \operatorname{Re}\operatorname{tr}(XY^*)$. This bilinear form is a real inner product since

$$(X, X) = \operatorname{Re}\operatorname{tr}(XX^*) = \operatorname{Re}\sum_{i=1}^{n}(AA^*)_{ii} = \sum_{i=1}^{n}\sum_{j=1}^{n} X_{ij}\bar{X}_{ji} = \sum_{i=1}^{n}\sum_{j=1}^{n} |X_{ij}|^2 \geq 0$$

with equality iff $X = 0$. Now let $\mathfrak{a}$ be an ideal in $\mathfrak{g}$, and let $\mathfrak{a}^\perp$ be the orthogonal complement of $\mathfrak{a}$ in $\mathfrak{g}$. Then $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a}^\perp$ as vector spaces. To see that $\mathfrak{a}^\perp$ is an ideal in $\mathfrak{g}$, let X be in $\mathfrak{a}^\perp$, Y be in $\mathfrak{g}$, and Z be in $\mathfrak{a}$. Then (recall that $\operatorname{Re}(x\bar{y}) = \operatorname{Re}(y\bar{x})$)

$$\begin{aligned}
([X, Y], Z) &= \operatorname{Re}\operatorname{tr}(XYZ^* - YXZ^*) \\
&= -\operatorname{Re}\operatorname{tr}(XZ^*Y) + \operatorname{Re}\operatorname{tr}(XYZ^*) \\
&= -\operatorname{Re}\operatorname{tr}(XZ^*Y - XYZ^*) \\
&= -\operatorname{Re}\operatorname{tr}(X(Y^*Z)^* - X(ZY^*)^*) \\
&= -(X, [Y, Z]).
\end{aligned}$$

Since Y^* is in $\mathfrak{g}$, $[Y^*, Z]$ is in $\mathfrak{a}$. Thus the right side is 0 for all Z, and $[X, Y]$ is in $\mathfrak{a}^\perp$. Hence $\mathfrak{a}^\perp$ is an ideal, and $\mathfrak{g}$ is reductive.

1.6.5 The Levi-Malcev theorem

In the preceding expositions, we dealt in particular with solvable and semi-simple Lie algebras separately. Now we shall address the question how a finite-dimensional Lie algebra $\mathfrak{g}$ decomposes into its maximal solvable ideal $\mathfrak{r}$ and the semi-simple quotient $\mathfrak{g}/\mathfrak{r}$. The theorems of Levi and Malcev are fundamental for the structure theory of finite-dimensional Lie algebras. Levi's Theorem asserts the existence of a semi-simple subalgebra $\mathfrak{s}$ of $\mathfrak{g}$ complementing the radical $\mathfrak{r}$, also called a **Levi complement**. As a consequence, $\mathfrak{g} \cong \mathfrak{r} \rtimes \mathfrak{s}$ is a semidirect sum. Malcev's Theorem asserts that all Levi complements are conjugate under the group of inner automorphisms of $\mathfrak{g}$, which is a uniqueness result.

Let E be a vector space over the field $\mathbb{K}$ and u a nilpotent endomorphoism of E. The series $\sum_n u^n/n!$ has only a finite number of non-zero terms and we can therefore write

$$e^u = \exp(u) = \sum_n \frac{u^n}{n!}.$$

This definition agrees with the usual one if $\mathbb{K} = \mathbb{R}$ and if E is complete and normed. If v is another nilpotent endomorphism of E which commutes with u, then

$$e^u e^v = \sum_{i,j} \frac{u^i}{i!}\frac{v^j}{j!} = \sum_n \frac{(u+v)^n}{n!} = e^{u+v}$$

In particular, $e^u e^{-u} = e^{-u} e^u = 1$ and hence e^u is always an automorphism of V.

If further E is a (not necessarily associative) algebra and u is a (nilpotent) derivation of E, then e^u is an automorphism of the algebra V. For if $x, y \in E$ then

$$u^n(xy) = \sum_{i+j=n} \binom{n}{i} u^i(x) u^j(y)$$

for every integer $n \geq 0$ (Leibniz's formula). It follows that

$$e^u(xy) = \sum_n \frac{u^n(xy)}{n!} = \sum_n \sum_{i+j=n} \frac{u^i(x)}{i!} \frac{u^j(y)}{j!} = e^u(x) e^u(y)$$

whence our assertion.

Now let $\mathfrak{g}$ be a Lie algebra. If x belongs to the nilpotent radical of $\mathfrak{g}$, the derivation $\mathrm{ad}_{\mathfrak{g}}(x)$ of $\mathfrak{g}$ is nilpotent. Such an automorphism is called a **special automorphism** of $\mathfrak{g}$. Clearly a special automorphism leaves every ideal of $\mathfrak{g}$ stable.

Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{r}$ its radical. A **Levi subalgebra** of $\mathfrak{g}$ is any subalgebra of $\mathfrak{g}$ supplementary to $\mathfrak{r}$. A Levi subalgebra is isomorphic to $\mathfrak{g}/\mathfrak{r}$ and hence is semi-simple. As a semi-simple subalgebra has only 0 in common with $\mathfrak{r}$, every semi-simple subalgebra $\mathfrak{s}$ such that $\mathfrak{g} = \mathfrak{r} + \mathfrak{s}$ is a Levi subalgebra; consequently the image of a Levi subalgebra under a surjective homomorphism is a Levi subalgebra.

Theorem 1.6.38 (Levi-Malcev). *Every Lie algebra $\mathfrak{g}$ possesses has a Levi subalgebra $\mathfrak{s}$, and every Levi subalgebra of $\mathfrak{g}$ is the image of $\mathfrak{s}$ under a special automorphism.*

Proof. Let $\mathfrak{r}$ denote the radical of $\mathfrak{g}$. We first treat two special cases. If $[\mathfrak{g}, \mathfrak{r}] = \{0\}$, then by Proposition 1.6.23, $\mathfrak{g}$ is then the product of its centre $\mathfrak{r}$ by $[\mathfrak{g}, \mathfrak{g}]$, which is semi-simple. Hence $[\mathfrak{g}, \mathfrak{g}]$ is a Levi subalgebra. Moreover, if $\tilde{\mathfrak{s}}$ is a semi-simple subalgebra, then $\tilde{\mathfrak{s}} = [\tilde{\mathfrak{s}}, \tilde{\mathfrak{s}}]$, hence $\mathfrak{s} \subseteq [\mathfrak{g}, \mathfrak{g}]$ and $[\mathfrak{g}, \mathfrak{g}]$ is the unique Levi subalgebra of $\mathfrak{g}$.

Then assume that $[\mathfrak{g}, \mathfrak{r}] \neq \{0\}$ and the only ideals of $\mathfrak{g}$ contained in $\mathfrak{r}$ are $\{0\}$ and $\mathfrak{r}$. Then $[\mathfrak{g}, \mathfrak{r}] = \mathfrak{r}$, $[\mathfrak{r}, \mathfrak{r}] = \{0\}$ and the centre of $\mathfrak{g}$ is zero. Let M (resp. N) be the subspace of $\mathrm{End}_{\mathbb{K}}(\mathfrak{g})$ consisting of the linear maps of $\mathfrak{g}$ into $\mathfrak{r}$ whose restriction to $\mathfrak{r}$ is a homothety (resp. zero); N is therefore of codimension 1 in M. For $u \in M$, let $\lambda(u)$ denote the ratio of the homothety of $\mathfrak{r}$ defined by u. Let ρ be the representation of $\mathfrak{g}$ on $\mathrm{End}_{\mathbb{K}}(\mathfrak{g})$ canonically derived from the adjoint representation: that is, $\rho(x) \cdot u = [\mathrm{ad}_{\mathfrak{g}}(x), u]$ for $x \in \mathfrak{g}$ and $u \in \mathrm{End}_{\mathbb{K}}(\mathfrak{g})$. Clearly $\rho(x)(M) \subseteq N$ for all $x \in \mathfrak{g}$. Moreover, if $x \in \mathfrak{r}, y \in \mathfrak{g}$ and $u \in M$, then

$$(\rho(x) \cdot u)(y) = [x, u(y)] - u([x, y]) = -\lambda(u)[x, y] \tag{1.6.2}$$

since $u(y) \in \mathfrak{r}$ and $[\mathfrak{r}, \mathfrak{r}] = \{0\}$. This equation can also be written as $\rho(x) \cdot u = -\mathrm{ad}_{\mathfrak{g}}(\lambda(u)x)$.

As the centre of $\mathfrak{g}$ is zero, the map $x \mapsto \mathrm{ad}_{\mathfrak{g}}(x)$ defines a bijection ϕ of $\mathfrak{r}$ onto a subspace P of $\mathrm{End}_{\mathbb{K}}(\mathfrak{g})$. This subspace is stable under $\rho(\mathfrak{g})$ and contained in N since $\mathfrak{r}$ is an abelian ideal and (1.6.2) shows that $\rho(x)(M) \subseteq P$ for $x \in \mathfrak{r}$. The representation of $\mathfrak{g}$ on M/P derived from ρ is therefore zero on $\mathfrak{r}$ and defines a representation $\bar{\rho}$ of the semi-simple algebra $\mathfrak{g}/\mathfrak{r}$. For all $y \in \mathfrak{g}/\mathfrak{r}$, the space $\bar{\rho}(y)(M/P)$ is contained in N/P, which is of codimension 1 in M/P. Consequently by Lemma 1.6.11 there exists $u_0 \in M$ such that $\lambda(u_0) = -1$ and $\rho(x)u_0 \in P$ for all $x \in \mathfrak{g}$. The map $x \mapsto \phi^{-1}(\rho(x) \cdot u_0)$ is a linear map of $\mathfrak{g}$ into $\mathfrak{r}$. By (1.6.2) its restriction to $\mathfrak{r}$ is the identity map of $\mathfrak{r}$. Hence its kernel is a subspace $\mathfrak{s}$ of $\mathfrak{g}$ supplementary to $\mathfrak{r}$ in $\mathfrak{g}$. As $\mathfrak{s}$ is the set of $x \in \mathfrak{g}$ such that $\rho(x) \cdot u_0 = 0$, $\mathfrak{s}$ is a subalgebra of $\mathfrak{g}$ and therefore a Levi subalgebra of $\mathfrak{g}$.

Let $\tilde{\mathfrak{s}}$ be another Levi subalgebra. For all $x \in \tilde{\mathfrak{s}}$, let $h(x)$ be the unique element of $\mathfrak{r}$ such that $x + h(x) \in \mathfrak{s}$. Since $\mathfrak{s}$ is a subalgebra and $\mathfrak{r}$ is abelian, for x, y in $\tilde{\mathfrak{s}}$,

$$[x + h(x), y + h(y)] = [x, y] + [x, h(y)] + [h(x), y] \in \mathfrak{s}$$

hence

$$h([x, y]) = [x, h(y)] + [h(x), y] = \mathrm{ad}(x)(h(y)) - \mathrm{ad}(y)(h(x)).$$

By Example 1.6.16, there exists $a \in \mathfrak{r}$ such that $h(x) = -[x, a]$ for all $x \in \mathfrak{g}$. Then:

$$x + h(x) = x + [a, x] = (1 + \mathrm{ad}(a)) \cdot x. \tag{1.6.3}$$

As $\mathfrak{r}$ is commutative, we have $1 + \mathrm{ad}(a) = e^{\mathrm{ad}(a)}$. As $\mathfrak{r} = [\mathfrak{g}, \mathfrak{r}]$, $e^{\mathrm{ad}(a)}$ is a special automorphism of $\mathfrak{g}$. By (1.6.3) this special automorphism maps $\tilde{\mathfrak{s}}$ to $\mathfrak{s}$.

In the general case, we argue by induction on the dimension n of the radical. There is nothing to prove if $n = 0$ and hence it can be assumed that the theorem holds for Lie algebras whose radical is of dimension $< \dim(\mathfrak{g})$. Also, it suffices to consider the case where $[\mathfrak{g}, \mathfrak{r}] \neq \{0\}$. As $[\mathfrak{g}, \mathfrak{r}]$ is nilpotent (Proposition 1.6.26), its centre $\mathfrak{z}$ is nonzero. Let $\mathfrak{m}$ be a minimal non-zero ideal of $\mathfrak{g}$ contained in $\mathfrak{z}$ (whence contained in $\mathfrak{m}$). If $\mathfrak{m} = \mathfrak{r}$, we return to the case prescribed above. Therefore assume that $\mathfrak{m} \neq \mathfrak{r}$ and let π be the canonical map of $\mathfrak{g}$ onto $\bar{\mathfrak{g}} = \mathfrak{g}/\mathfrak{m}$. The radical of $\bar{\mathfrak{g}}$ is $\bar{\mathfrak{r}} = \mathfrak{r}/\mathfrak{m}$. By the induction hypothesis, $\bar{\mathfrak{g}}$ has a Levi subalgebra $\bar{\mathfrak{h}}$. Then $\mathfrak{h} = \pi^{-1}(\bar{\mathfrak{h}})$ is a subalgebra of $\mathfrak{g}$ containing $\mathfrak{m}$ such that $\mathfrak{h}/\mathfrak{m} = \bar{\mathfrak{h}}$ is semi-simple and hence having $\mathfrak{m}$ as radical. By the induction hypothesis $\mathfrak{h} = \mathfrak{m} + \mathfrak{s}$ where $\mathfrak{s}$ is a semi-simple subalgebra. Then the equality $\bar{\mathfrak{g}} = \bar{\mathfrak{r}} + \bar{\mathfrak{h}}$ implies

$$\mathfrak{g} = \mathfrak{r} + \mathfrak{h} = \mathfrak{r} + \mathfrak{m} + \mathfrak{s} = \mathfrak{r} + \mathfrak{s}$$

hence $\mathfrak{s}$ is a Levi subalgebra of $\mathfrak{g}$.

Let $\tilde{\mathfrak{s}}$ be another Levi subalgebra of $\mathfrak{g}$. Then $\pi(\mathfrak{s})$ and $\pi(\tilde{\mathfrak{s}})$ are two Levi subalgebras of $\bar{\mathfrak{g}}$ and there exists by the induction hypothesis $\bar{a} \in [\bar{\mathfrak{g}}, \bar{\mathfrak{r}}]$ such that $e^{\mathrm{ad}(\bar{a})}(\pi(\tilde{\mathfrak{s}})) = \pi(\mathfrak{s})$. If $a \in [\mathfrak{g}, \mathfrak{r}]$ is such that $\pi(a) = \bar{a}$, it follows that:

$$\mathfrak{s}_1 := e^{\mathrm{ad}(a)}(\tilde{\mathfrak{s}}) \subseteq \mathfrak{m} + \mathfrak{s} = \mathfrak{h}$$

Then $\mathfrak{s}_1$ and $\mathfrak{s}$ are two Levi subalgebras of $\mathfrak{h}$ and by the induction hypothesis there exists $b \in \mathfrak{m}$ such that $e^{\mathrm{ad}(b)}(\mathfrak{s}_1) = \mathfrak{s}$. Hence $\mathfrak{s} = e^{\mathrm{ad}(b)}e^{\mathrm{ad}(a)}(\tilde{\mathfrak{s}})$. Finally, as $\mathfrak{m}$ is in the centre of $[\mathfrak{g}, \mathfrak{r}]$, $e^{\mathrm{ad}(b)}e^{\mathrm{ad}(a)} = e^{\mathrm{ad}(b+a)}$ and $b + a \in [\mathfrak{g}, \mathfrak{r}]$, which completes the proof. □

Corollary 1.6.39. *Let $\mathfrak{s}$ be a Levi subalgebra of $\mathfrak{g}$ and $\mathfrak{h}$ a semi-simple subalgebra of $\mathfrak{g}$.*

(a) There exists a special automorphism of $\mathfrak{g}$ map $\mathfrak{h}$ onto a subalgebra of $\mathfrak{s}$.

(b) $\mathfrak{h}$ is contained in a Levi subalgebra of $\mathfrak{g}$.

Proof. Let $\mathfrak{r}$ be the radical of $\mathfrak{g}$ and $\mathfrak{k} = \mathfrak{h} + \mathfrak{r}$, which is a subalgebra of $\mathfrak{g}$. Then $\mathfrak{k}/\mathfrak{r}$ is semi-simple and $\mathfrak{r}$ is solvable, hence $\mathfrak{r}$ is the radical of $\mathfrak{k}$ and $\mathfrak{h}$ is a Levi subalgebra of $\mathfrak{k}$. On the other hand, $\mathfrak{k} \cap \mathfrak{s} = \tilde{\mathfrak{h}}$ is a supplementary subalgebra to $\mathfrak{r}$ in $\mathfrak{k}$ and hence also a Levi subalgebra of $\mathfrak{k}$. Then there exists $a \in [\mathfrak{a}, \mathfrak{r}]$ such that $e^{\mathrm{ad}_{\mathfrak{a}}(a)}$ maps $\mathfrak{h}$ onto $\tilde{\mathfrak{h}}$. Now $a \in [\mathfrak{g}, \mathfrak{r}]$; $e^{\mathrm{ad}_{\mathfrak{a}}(a)}$ maps $\mathfrak{h}$ onto a subalgebra of $\mathfrak{s}$ and therefore $e^{-\mathrm{ad}_{\mathfrak{a}}(a)}(\mathfrak{s})$ is a Levi subalgebra of $\mathfrak{g}$ containing $\mathfrak{h}$. □

Corollary 1.6.40. *For a subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ to be a Levi subalgebra of $\mathfrak{g}$, it is necessary and sufficient that $\mathfrak{h}$ be a maximal semi-simple subalgebra of $\mathfrak{g}$.*

Corollary 1.6.41. *Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{a}$ an ideal of $\mathfrak{g}$ such that $\mathfrak{g}/\mathfrak{a}$ is semi-simple. Then $\mathfrak{g}$ contains a subalgebra supplementary to $\mathfrak{a}$ in $\mathfrak{g}$. In other words, every extension of a semi-simple Lie algebra is inessential.*

Proof. Let $\mathfrak{s}$ be a Levi subalgebra of $\mathfrak{g}$. Its canonical image in $\mathfrak{g}/\mathfrak{a}$ is a Levi subalgebra and therefore equal to $\mathfrak{g}/\mathfrak{a}$, hence $\mathfrak{g} = \mathfrak{s} + \mathfrak{a}$. Then an ideal of $\mathfrak{s}$ supplementary in $\mathfrak{s}$ to the ideal $\mathfrak{a} \cap \mathfrak{s}$ is a subalgebra of $\mathfrak{g}$ supplementary to $\mathfrak{a}$ in $\mathfrak{g}$. □

Corollary 1.6.42. *Let $\mathfrak{g}$ be a Lie algebra, $\mathfrak{r}$ its radical, $\mathfrak{s}$ a Levi subalgebra of $\mathfrak{g}$ and $\mathfrak{a}$ an ideal of $\mathfrak{g}$. Then $\mathfrak{a}$ is the direct sum of $\mathfrak{a} \cap \mathfrak{r}$ which is its radical and $\mathfrak{a} \cap \mathfrak{s}$ which is a Levi subalgebra of $\mathfrak{a}$.*

Proof. We know that $\mathfrak{a} \cap \mathfrak{r}$ is the radical of $\mathfrak{a}$ (Corollary 1.5.22). Let $\mathfrak{h}$ be a Levi subalgebra of $\mathfrak{a}$ and $\mathfrak{s}_1$ a Levi subalgebra of $\mathfrak{g}$ containing $\mathfrak{h}$ (Corollary 1.6.39). The algebra $\mathfrak{a} \cap \mathfrak{s}_1$ is an ideal of $\mathfrak{s}_1$, is therefore semi-simple, and contains $\mathfrak{h}$ and is therefore equal to $\mathfrak{h}$. Hence $\mathfrak{a}$ is the direct sum of $\mathfrak{a} \cap \mathfrak{s}_1$ and $\mathfrak{a} \cap \mathfrak{r}$. There exists a special automorphism map $\mathfrak{s}_1$ onto $\mathfrak{s}$; this automorphism leaves $\mathfrak{r}$ and $\mathfrak{a}$ invariant; hence $\mathfrak{a}$ is the direct sum of $\mathfrak{a} \cap \mathfrak{r}$ and $\mathfrak{a} \cap \mathfrak{s}$ and $\mathfrak{a} \cap \mathfrak{s}$ is a Levi subalgebra of $\mathfrak{a}$. □

Corollary 1.6.43. *Let $\mathfrak{g}$ be a Lie algebra, $\mathfrak{r}$ the radical of $\mathfrak{g}$ and $\mathfrak{s}$ a Levi subalgebra of $\mathfrak{g}$. Then*

$$[\mathfrak{g}, \mathfrak{g}] = [\mathfrak{g}, \mathfrak{r}] \rtimes \mathfrak{s}$$

In particular, $\mathfrak{s}$ is a Levi-subalgebra of $[\mathfrak{g}, \mathfrak{g}]$.

Proof. Since $\mathfrak{s}$ is semi-simple, we have $[\mathfrak{s}, \mathfrak{s}] = \mathfrak{s}$, which implies $\mathfrak{s} \subseteq [\mathfrak{g}, \mathfrak{g}]$. Therefore Corollary 1.6.42 shows $[\mathfrak{g}, \mathfrak{g}] = ([\mathfrak{g}, \mathfrak{g}] \cap \mathfrak{r}) + \mathfrak{s}$. But we know that $[\mathfrak{g}, \mathfrak{g}] \cap \mathfrak{r} = [\mathfrak{g}, \mathfrak{r}]$, so the claim follows. □

1.6.6 The invariant theorem

Let $\mathfrak{g}$ be a Lie algebra and ρ a representation of $\mathfrak{g}$ on a vector space M. For every class δ of simple representations of $\mathfrak{g}$ let M_δ be the isotypical component of M of species δ. The subspace M_0 of invariant elements of M is just M_{δ_0}, where δ_0 denotes the class of the zero representation of $\mathfrak{g}$ on a space of dimension 1. In this paragraph, we deduce some of the properties about this subspace.

Lemma 1.6.44. *Let (M,ρ), (N,σ), and (P,τ) be representations of $\mathfrak{g}$. Suppose that we are given a $\mathbb{K}$-bilinear map $(\cdot\,\cdot)$ of $M\times N$ into P such that*

$$(\rho(x)m,n)+(m,\sigma(x)n)=\tau(x)(m,n)$$

for all $m\in M$, $n\in N$, $x\in\mathfrak{g}$.

(a) If $m\in M_0$, then the map $n\mapsto(m,n)$ is a $\mathfrak{g}$-module homomorphism.

(b) If $n\in N_\delta$, then $(m,n)\in P_\delta$.

(c) If M is a (not necessarily associative) algebra and the $\rho(x)$ are derivations of M, M_0 is a subalgebra of M and each M_δ is a right and left M_0-module.

Proof. For $m\in M_0$, $n\in N$ and $x\in\mathfrak{g}$,

$$\tau(x)(m,n)=(m,\sigma(x)n),$$

whence (a). Assertion (b) follows from (a). If $M=N=P$ and $\rho=\sigma=\tau$, assertion (b) gives assertion (c) as a special case. ☐

Lemma 1.6.45. *Suppose further that (N,σ) and (P,τ) are semi-simple. For all $n\in N$ (resp. $p\in P$), let $n^\sharp$ (resp. $p^\sharp$) be its component in N_0 (resp. P_0). Let $m\in M_0$, then for all $n\in N$, $(m,n)^\sharp)=(m,n^\sharp)$.*

Proof. By linearity it suffices to consider the case where $n\in N_\delta$. If $\delta\neq\delta_0$, $n^\sharp=0$ and $(m,n)\in P_\delta$ by Lemma 1.6.44, hence $(m,n)^\sharp)=0=(m,n^\sharp)$. If $\delta=\delta_0$, $n^\sharp=n$ and $(m,n)\in P_0$ (Lemma 1.6.44), hence $(m,n)^\sharp=(m,n)=(m,n^\sharp)$. ☐

Theorem 1.6.46. *Let $\mathfrak{g}$ be a Lie algebra, V a semi-simple $\mathfrak{g}$-module of finite dimension over $\mathbb{K}$, $S=\boldsymbol{S}(V)$ the symmetric algebra of V and x_S the derivation of S which extends s_V (so that $x\mapsto x_S$ is a representation of $\mathfrak{g}$ on S).*

(a) The algebra S_0 of invariants of S is generated by a finite number of elements.

(b) For every class δ of simple representations of $\mathfrak{g}$ of finite dimension over $\mathbb{K}$, let S_δ be the isotypical component of S of species δ. Then S_δ is a finitely generated S_0- module.

1.6.7 Extension of scalars

Let $\sigma:\mathbb{K}\to\mathbb{K}'$ be an extension of $\mathbb{K}$. For a Lie algebra $\mathfrak{g}$ over $\mathbb{K}$ to be semi-simple, it is necessary and sufficient that $\sigma_*(\mathfrak{g})$ be semi-simple; for the Killing form of $\sigma_*(\mathfrak{g})$ is derived from the Killing form of $\mathfrak{g}$ by extending the base field from $\mathbb{K}$ to $\mathbb{K}'$; hence is non-degenerate if and only if that of $\mathfrak{g}$ is non-degenerate.

If $\sigma_*(\mathfrak{g})$ is simple, $\mathfrak{g}$ is semi-simple by the above and cannot be a product of two non-zero ideals, hence $\mathfrak{g}$ is simple. On the other hand if $\mathfrak{g}$ is simple $\sigma_*(\mathfrak{g})$ (which is semi-simple) may be not simple.

Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{r}$ its radical. Then $\sigma_*(\mathfrak{r})$ is the radical of $\sigma_*(\mathfrak{g})$. Therefore, if $\mathfrak{s}$ denotes the nilpotent radical of $\mathfrak{g}$, the nilpotent radical of $\sigma_*(\mathfrak{g})$ is $[\sigma_*(\mathfrak{g}),\sigma_*(\mathfrak{r})]=\sigma_*(\mathfrak{s})$. It follows that $\mathfrak{g}$ is reductive if and only if $\sigma_*(\mathfrak{g})$ is reductive.

Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{h}$ a subalgebra. Recall that a representation of $\mathfrak{g}$ is semi-simple if and only if the representation of $\sigma_*(\mathfrak{g})$ derived by extending the base field to $\mathbb{K}'$ is semi-simple. Hence $\mathfrak{h}$ is reductive in $\mathfrak{g}$ if and only if $\sigma_*(\mathfrak{h})$ is reductive in $\sigma_*(\mathfrak{g})$.

Now let $\mathbb{K}_0$ be a subfield of $\mathbb{K}$ such that $[\mathbb{K}:\mathbb{K}_0]$ is finite. Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{g}_0$ the (finite-dimensional) Lie algebra derived from $\mathfrak{g}$ by restricting the base field from $\mathbb{K}$ to $\mathbb{K}_0$. Every abelian ideal of $\mathfrak{g}$ is an abelian ideal of $\mathfrak{g}_0$; conversely, if $\mathfrak{a}_0$ is an abelian ideal of $\mathfrak{g}_0$, the smallest vector subspace over $\mathbb{K}$ of $\mathfrak{g}$ containing $\mathfrak{a}_0$ is an abelian ideal of $\mathfrak{g}$; hence $\mathfrak{g}$ is semi-simple if and only if $\mathfrak{g}_0$ is semi-simple. If

$\mathfrak{g}_0$ is simple, clearly $\mathfrak{g}$ is simple. Conversely, if the Lie algebra $\mathfrak{g}$ is simple, then $\mathfrak{g}_0$ is also simple. To see this, let $\mathfrak{a}_0$ be a simple component of $\mathfrak{g}_0$. For all $\lambda \in \mathbb{K}^\times$, $\lambda\mathfrak{a}_0$ is an ideal of $\mathfrak{g}_0$ and

$$[\mathfrak{a}_0, \lambda\mathfrak{a}_0] = \lambda[\mathfrak{a}_0, \mathfrak{a}_0] = \lambda\mathfrak{a}_0 \neq \{0\}$$

hence $\lambda\mathfrak{a}_0 \supseteq \mathfrak{a}_0$ and therefore $\lambda\mathfrak{a}_0 = \mathfrak{a}_0$ since $\dim_{\mathbb{K}_0}(\lambda\mathfrak{a}_0) = \dim_{\mathbb{K}_0}(\mathfrak{a}_0)$. Now the vector subspace of $\mathfrak{g}$ generated by $\mathfrak{a}_0$ is a non-zero ideal of $\mathfrak{g}$ and hence is the whole of $\mathfrak{g}$. Hence $\mathfrak{g}_0 = \mathfrak{a}_0$, which proves our assertion.

1.7 Ado's theorem

Recall that $\mathbb{K}$ denotes a field of characteristic 0 and that all Lie algebras are assumed to be finite-dimensional over $\mathbb{K}$.

1.7.1 Coefficients of a representation

Let U be an associative algebra with unit element over $\mathbb{K}$, U^* the dual of the vector space U and ρ a representation of U on a vector space E. For $e \in E$ and $e^* \in E^*$, let $\theta(e^*, e) \in U^*$ be the corresponding coefficient of ρ, defined by $\theta(e^*, e)(x) = \langle e^*, \rho(x)e\rangle$ for $x \in U$. If U^* is endowed with the **coregular representation**, defined by

$$(y \cdot f)(x) = f(xy)$$

where $x, y \in U$ and $f \in U^*$, then for fixed e^*, the map $e \mapsto \theta(e^*, e)$ is a homomorphism of the U-module E into the U-module U^*. In fact, for $x, y \in U$,

$$\theta(e^*, \rho(y)e)(x) = \langle e^*, \rho(x)\rho(y)e\rangle = \langle e^*, \rho(xy)e\rangle = \theta(e^*, e)(xy) = (y \cdot \theta(e^*, e))(x).$$

Therefore the vector subspace $C(\rho)$ of U^* generated by the coefficients of ρ is a sub-U-module of U^*. If $(e_i^*)_{i\in I}$ is a family of elements generating E^* over $\mathbb{K}$, the map $e \mapsto (\theta(e_i^*, e))$ is an injective U-homomorphism of E into $C(\rho)^I$, for $\theta(e_i^*, e) = 0$ for all i implies

$$\langle e_i^*, e\rangle = \theta(e_i^*, e)(1) = 0$$

for all i and hence $e = 0$.

In particular, if U is the enveloping algebra of a Lie algebra $\mathfrak{g}$ and ρ is a representation of $\mathfrak{g}$ (identified with a representation of U) on an n-dimensional vector space E, the $\mathfrak{g}$-module E is isomorphic to a sub-$\mathfrak{g}$-module of $(C(\rho))^n$.

1.7.2 The extension theorem

Proposition 1.7.1. *Let $\mathfrak{g} = \mathfrak{a} + \mathfrak{h}$ be a Lie algebra which is the direct sum of an ideal $\mathfrak{a}$ and a subalgebra $\mathfrak{h}$, $U(\mathfrak{g})$ the enveloping algebra of $\mathfrak{g}$ and $U(\mathfrak{a}) \subseteq U(\mathfrak{g})$ the enveloping algebra of $\mathfrak{a}$. Then there exists one and only one g-module structure on $U(\mathfrak{a})$ such that:*

(a) for $x \in \mathfrak{a}$ and $u \in U(\mathfrak{a})$, $x_{U(\mathfrak{a})} \cdot u = -ux$;

(b) for $x \in \mathfrak{h}$ and $u \in U(\mathfrak{a})$, $x_{U(\mathfrak{a})} \cdot u = xu - ux$ (the latter element is certainly in $U(\mathfrak{a})$ since the inner derivation of $U(\mathfrak{g})$ defined by x leaves $\mathfrak{a}$ and hence $U(\mathfrak{a})$ stable).

Proof. For conditions (a) and (b) define uniquely a linear map $x \mapsto x_{U(\mathfrak{a})}$ of $\mathfrak{g}$ into $\mathrm{End}_{\mathbb{K}}(U(\mathfrak{a}))$. It therefore suffices to verify that $[x, y]_{U(\mathfrak{a})} = [x_{U(\mathfrak{a})}, y_{U(\mathfrak{a})}]$. If $x, y \in \mathfrak{a}$ or $x, y \in \mathfrak{h}$, this is immediate from the definition. Otherwise, assume that $x \in \mathfrak{h}$ and $y \in \mathfrak{a}$. Then

$$\begin{aligned}[x_{U(\mathfrak{a})}, y_{U(\mathfrak{a})}] \cdot u &= x_{U(\mathfrak{a})}(-uy) - y_{U(\mathfrak{a})}(xu - ux) = x(-uy) - (-uy)x - (-(xu - ux)y) \\ &= uyx - uxy = u(yx - xy) = [x, y]_{U(\mathfrak{a})} \cdot u.\end{aligned}$$

This proves the claim. □

We shall also consider the dual representation $x \mapsto -x^t_{U(\mathfrak{a})}$ of $\mathfrak{g}$ on $U(\mathfrak{a})^*$. For $x \in \mathfrak{a}$, the endomorphism $-x^t_{U(\mathfrak{a})}$ is the transpose of right multiplication by x in $U(\mathfrak{a})$; the corresponding representation of $U(\mathfrak{a})$ is therefore the coregular representation on $U(\mathfrak{a})^*$.

Definition 1.7.2. Let $\mathfrak{g}$ be a Lie algebra, $\mathfrak{h}$ a subalgebra of $\mathfrak{g}$ and τ a representation of $\mathfrak{h}$ on W. A representation ρ of $\mathfrak{g}$ on V is called an **extension** of τ to $\mathfrak{g}$ if there exists an injective homomorphism of the $\mathfrak{h}$-module W into the $\mathfrak{h}$-module V. We also say that the $\mathfrak{g}$-module V is an extension of the $\mathfrak{h}$-module W.

Lemma 1.7.3. *Let $\mathfrak{g} = \mathfrak{a} + \mathfrak{h}$ be a Lie algebra which is the sum of an ideal $\mathfrak{a}$ and a subalgebra $\mathfrak{h}$. Let ρ be a finite-dimensional representation of $\mathfrak{g}$. Suppose that $\rho(x)$ is nilpotent for all $x \in \mathfrak{a}$ and all $x \in \mathfrak{h}$. Then $\rho(x)$ is nilpotent for all $x \in \mathfrak{g}$.*

Proof. Passing to the quotient by the kernel of ρ, ρ may be assumed to be faithful. Then $\mathfrak{a}$ and $\mathfrak{h}$ are nilpotent and hence $\mathfrak{g}$, which is an extension of a quotient of $\mathfrak{h}$ by $\mathfrak{a}$, is solvable. Then $\mathfrak{h}$ and $\mathfrak{a}$ are contained in the largest nilpotency ideal of ρ by Corollary 1.5.15, which then equals to $\mathfrak{g}$ since $\mathfrak{g} = \mathfrak{a} + \mathfrak{h}$. □

Theorem 1.7.4 (Zassenhaus). *Let $\mathfrak{g} = \mathfrak{a} + \mathfrak{h}$ be a Lie algebra which is the direct sum of an ideal $\mathfrak{a}$ and a subalgebra $\mathfrak{h}$ and τ a finite-dimensional representation of $\mathfrak{a}$ whose largest nilpotency ideal contains $[\mathfrak{a}, \mathfrak{h}]$.*

(a) There exists a finite-dimensional extension ρ of τ to $\mathfrak{g}$ whose largest nilpotency ideal contains that of τ.

(b) If for all $x \in \mathfrak{h}$ the restriction to $\mathfrak{a}$ of $\mathrm{ad}(x)$ is nilpotent, ρ can be chosen such that moreover the largest nilpotency ideal of ρ contains $\mathfrak{h}$.

Proof. Let $U(\mathfrak{a})$ be the enveloping algebra of $\mathfrak{a}$. Suppose that $U(\mathfrak{a})$ and $U(\mathfrak{a})^*$ have the $\mathfrak{g}$-module structures defined at the beginning of this paragraph. Let $I \subseteq U(\mathfrak{a})$ be the kernel of τ (identified with a representation of $U(\mathfrak{a})$). It is a two-sided ideal of $U(\mathfrak{a})$ of finite codimension. By definition the subspace $C(\tau)$ of $U(\mathfrak{a})^*$ is orthogonal from I. Let T be the sub-$\mathfrak{g}$-module of $U(\mathfrak{a})^*$ generated by $C(\tau)$.

We now show that T is finite-dimensional over $\mathbb{K}$. Let W be the space on which τ operates and $W = W_0 \supset W_1 \supset \cdots \supset W_r = \{0\}$ a Jordan-Hölder series of the $\mathfrak{a}$-module W. Let τ_i be the representation of $\mathfrak{a}$ on W_i/W_{i+1} derived from τ. Let $J \subseteq U(\mathfrak{a})$ be the intersection of the kernels of the τ_i (identified with representations of $U(\mathfrak{a})$). Then

$$J^r \subseteq I \subseteq J$$

and $J \cap \mathfrak{a}$ is the largest nilpotency ideal of τ. By Corollary 1.2.8, J^r is of finite codimension in $U(\mathfrak{a})$. If $x \in \mathfrak{h}$, the derivation $u \mapsto xu - ux$ of $U(\mathfrak{a})$ maps $\mathfrak{a}$ into $[\mathfrak{h}, \mathfrak{a}] \subseteq J$, hence $U(\mathfrak{a})$ into J and hence J^r into J^r. On the other hand, clearly J^r is a sub-$\mathfrak{a}$-module of $U(\mathfrak{a})$. Hence J^r is a sub-$\mathfrak{g}$-module of $U(\mathfrak{a})$. The orthogonal of J^r in $U(\mathfrak{a})^*$ is then a finite-dimensional sub-$\mathfrak{g}$-module which contains $C(\tau)$ and therefore T. This shows that T is finite-dimensional over $\mathbb{K}$.

We have seen that the $\mathfrak{a}$-module W is isomorphic to a sub-$\mathfrak{a}$-module of a product $(C(\tau))^n$. Hence the $\mathfrak{g}$-module T^n provides a finite-dimensional extension ρ of τ to $\mathfrak{g}$. Moreover, $\rho(x)$ is nilpotent for $x \in J \cap \mathfrak{a}$, since x^r is contained in J^r and therefore kills T. Since $J \cap \mathfrak{a}$ is an ideal of $\mathfrak{g}$ (for it contains $[\mathfrak{h}, \mathfrak{a}]$ by hypothesis), we see that $J \cap \mathfrak{a}$ is contained in the largest nilpotency ideal of ρ. Thus (a) is proved.

Suppose finally that for all $x \in \mathfrak{h}$ the restriction to $\mathfrak{a}$ of $\mathrm{ad}_{\mathfrak{g}}(x)$ is nilpotent. As the elements of $\mathfrak{h}$ operate on $U(\mathfrak{a})$ by derivations, there exists an integer n such that $(x_{U(\mathfrak{a})})^n = 0$ for all $x \in \mathfrak{h}$; the endomorphisms derived from $x_{U(\mathfrak{a})}$ on $U(\mathfrak{a})/J^r$ and on T (which are finite-dimensional spaces) are therefore nilpotent. Thus $\rho(x)$ is nilpotent for all $x \in \mathfrak{h}$. We have seen earlier that this is also true for $x \in J \cap \mathfrak{a}$. As $J \cap \mathfrak{a}$ is an ideal of $\mathfrak{a}$ containing $[\mathfrak{h}, \mathfrak{a}]$, the sum $\mathfrak{h} + (J \cap \mathfrak{a})$ is also an ideal of $\mathfrak{g}$. Assertion (b) of then results from Lemma 1.7.3. □

1.7.3 Ado's theorem

Proposition 1.7.5. *Let $\mathfrak{g}$ be a Lie algebra, $\mathfrak{n}$ its nilradical, $\mathfrak{a}$ a nilpotent ideal of $\mathfrak{g}$ and τ a finite-dimensional representation of $\mathfrak{a}$ such that every element of $\tau(\mathfrak{a})$ is nilpotent. Then τ admits a finite-dimensional extension ρ to $\mathfrak{g}$ such that every element of $\rho(\mathfrak{n})$ is nilpotent.*

Proof. Let $\mathfrak{a} = \mathfrak{n}_0 \subseteq \mathfrak{n}_1 \subseteq \cdots \subseteq \mathfrak{n}_p = \mathfrak{n}$ be a sequence of subalgebras of $\mathfrak{n}$ such that $\mathfrak{n}_{i-1}$ is an ideal of $\mathfrak{n}_i$ of codimension 1 for $1 \leq i \leq p$ (Proposition 1.4.5). The algebra $\mathfrak{n}_i$ is therefore the direct sum of $\mathfrak{n}_{i-1}$ and a 1-dimensional subalgebra. As $\mathrm{ad}_{\mathfrak{n}}(x)$ is nilpotent for all $x \in \mathfrak{n}$, it is possible by Theorem 1.7.4(b) to find one by one finite-dimensional extensions $\tau_1, \tau_2, \ldots, \tau_p$ of τ to $\mathfrak{n}_1, \ldots, \mathfrak{n}_p = \mathfrak{n}$ of τ such that every element of $\tau_p(\mathfrak{n})$ is nilpotent.

Let $\mathfrak{r}$ be the radical of $\mathfrak{g}$ and let $\mathfrak{n} = \mathfrak{r}_0 \subseteq \mathfrak{r}_1 \subseteq \cdots \subseteq \mathfrak{r}_q = \mathfrak{r}$ be a sequence of subalgebras of $\mathfrak{r}$ such that $\mathfrak{r}_{i-1}$ is an ideal of $\mathfrak{r}_i$ of codimension 1 for $1 \leq i \leq q$ (Proposition 1.5.2). The algebra $\mathfrak{r}_i$ is thus the direct sum of $\mathfrak{r}_{i-1}$ and a 1-dimensional subalgebra. As $[\mathfrak{r}, \mathfrak{r}]$ is nilpotent and hence contained in $\mathfrak{n}$, it is possible by Theorem 1.7.4(a) to find one by one finite-dimensional extensions $\tau_{p+1}, \tau_{p+2}, \ldots, \tau_{p+q}$ of τ_p to $\mathfrak{r}_1, \mathfrak{r}_2, \ldots, \mathfrak{r}_q = \mathfrak{r}$ such that every element of $\tau_{p+q}(\mathfrak{n})$ is nilpotent.

Finally $\mathfrak{g}$ is the direct of $\mathfrak{r}$ and a Levi subalgebra $\mathfrak{s}$. As $[\mathfrak{s}, \mathfrak{r}] \subseteq [\mathfrak{g}, \mathfrak{r}] \subseteq \mathfrak{n}$ (Theorem 1.5.9), it is possible by Theorem 1.7.4(a) to find a finite-dimensional extension ρ of τ_{p+q} to $\mathfrak{g}$ such that every element of $\rho(\mathfrak{n})$ is nilpotent. □

Theorem 1.7.6 (**Ado**). *Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{n}$ its nilradical. Then there exists a finite-dimensional faithful representation ρ of $\mathfrak{g}$ such that every element of $\rho(\mathfrak{n})$ is nilpotent.*

Proof. The 1-dimensional Lie algebra $\mathbb{K}$ admits finite-dimensional faithful representations η such that every element of $\eta(\mathbb{K})$ is nilpotent, for example the representation

$$\lambda \mapsto \begin{pmatrix} 0 & 0 \\ \lambda & 0 \end{pmatrix}.$$

It is easily deduced that the centre $\mathfrak{z}$ of $\mathfrak{g}$, which is a product of 1-dimensional algebras, admits a finite-dimensional faithful representation τ such that every element of $\tau(\mathfrak{z})$ is nilpotent. Let ρ_1 be a finite-dimensional extension of τ to $\mathfrak{g}$ such that every element of $\rho_1(\mathfrak{n})$ is nilpotent (Proposition 1.7.5); if $\mathfrak{a}$ denotes the kernel of ρ_1, then $\mathfrak{a} \cap \mathfrak{z} = \{0\}$. On the other hand, let ρ_2 be the adjoint representation of $\mathfrak{g}$, whose kernel is $\mathfrak{z}$; every element of $\rho_2(\mathfrak{n})$ is nilpotent. The direct sum ρ of ρ_1 and ρ_2 is finite-dimensional; every element of $\rho(\mathfrak{n})$ is nilpotent; and the kernel of ρ, contained in $\mathfrak{a}$ and in $\mathfrak{z}$, is zero, so that ρ is faithful. □

Chapter 2

Free Lie algebras

In this section, the letter $\mathbb{K}$ denotes a non-zero commutative ring. The unit element of $\mathbb{K}$ is denoted by 1. Unless otherwise mentioned, all cogebras, algebras and bigebras, all modules and all tensors products are over $\mathbb{K}$. However, when necessary, $\mathbb{K}$ will be assumed to be a field of characteristic 0.

2.1 Enveloping bigebra of a Lie algebra

Throughout this section, $\mathfrak{g}$ will denote a Lie algebra over $\mathbb{K}$, $U(\mathfrak{g})$ or simply U its enveloping algebra, ι the canonical map of $\mathfrak{g}$ into $U(\mathfrak{g})$ and (U_n) the canonical filtration of U.

2.1.1 Primitive elements

We consider a cogebra E with coproduct $c : E \to E \otimes E$ and counit ε. Recall that ε is a linear form on the $\mathbb{K}$-module E such that (with the canonical identification of $E \otimes \mathbb{K}$ and $\mathbb{K} \otimes E$ with E):

$$1_E = (\varepsilon \otimes 1_E) \circ c = (1_E \otimes \varepsilon) \circ c.$$

Let E^+ denote the kernel of ε and let e be an element of E such that

$$c(e) = e \otimes e, \quad \varepsilon(e) = 1.$$

The $\mathbb{K}$-module E is then the direct sum of E^+ and the submodule $\mathbb{K}e$ which is free with basis e (**??**); let $\pi_e : E \to E^+$ and $\eta_e : E \to \mathbb{K}e$ denote the projectors associated with this decomposition:

$$\pi_e(x) = x - \varepsilon(x)e, \quad \eta_e(x) = \varepsilon(x)e.$$

With these settings, an element x of E is called ***e*-primitive** if

$$c(x) = x \otimes e + e \otimes x. \tag{2.1.1}$$

The e-primitive elements of E form a submodule of E, which is denoted by $P_e(E)$.

Proposition 2.1.1. *Every e-primitive element of E belongs to E^+.*

Proof. We note that (2.1.1) implies

$$x = \varepsilon(x)e + \varepsilon(e)x = \varepsilon(x)e + x$$

whence $\varepsilon(x) = 0$. □

Remark 2.1.2. We note that, conversely, if $c(x) = y \otimes e + e \otimes z$ with $y, z \in E^+$, then

$$x = \varepsilon(y)e + \varepsilon(e)z = z, \quad x = y\varepsilon(e) + e\varepsilon(z) = y$$

so x is e-primitive. In other words, x is e-primitive if and only if $c(x) \in e \otimes E^+ \oplus E^+ \otimes e$.

Proposition 2.1.3. *For $x \in E^+$, write*

$$c_e^+(x) = c(x) - x \otimes e - e \otimes x$$

so that the e-primitive elements are the kenrel of c_e^+. Then

$$(\pi_e \otimes \pi_e) \circ c = c_e^+ \circ \pi_e. \tag{2.1.2}$$

Proof. Let x be in e. Then

$$\begin{aligned}(\pi_e \circ \pi_e)(c(x)) &= ((1-\eta_e) \otimes (1-\eta_e))(c(x)) \\ &= c(x) - (1 \otimes \eta_e)(c(x)) - (\eta_e \otimes 1)(c(x)) + (\eta_e \otimes \eta_e)(c(x)).\end{aligned}$$

As ε is the counit of E, we have

$$(1 \otimes \eta_e)(c(x)) = x \otimes e, \quad (\eta_e \otimes 1)(c(x)) = e \otimes x$$

whence

$$(\eta_e \otimes \eta_e)(c(x)) = (\eta_e \otimes 1)(1 \otimes \eta_e)(c(x)) = \varepsilon(x)e \otimes e$$

from this we conclude

$$(\pi_e \otimes \pi_e)(c(x)) = c(x) - x \otimes e - e \otimes x + \varepsilon(x)e \otimes e.$$

On the other hand,

$$\begin{aligned}c_e^+(\pi_e(x)) = c_e^+(x - \varepsilon(x)e) &= c(x - \varepsilon(x)e) - (x - \varepsilon(x)e) \otimes e - e \otimes (x - \varepsilon(x)e) \\ &= c(x) - x \otimes e - e \otimes x + \varepsilon(x)e \otimes e\end{aligned}$$

whence the claim. □

As E^+ is a direct factor submodule of E, $E^+ \otimes E^+$ can be identified with a direct factor submodule of $E \otimes E$. With this identification, $\pi_e \otimes \pi_e$ is a projector of $E \otimes E$ onto $E^+ \otimes E^+$. By formula (2.1.2), c_e^+ maps E^+ into $E^+ \otimes E^+$ and π_e is a morphism of the cogebra (E, c) into the cogebra (E^+, c_e^+).

Proposition 2.1.4. *If the cogebra (E, c) is coassociative (resp. cocommutative), so is the cogebra (E^+, c_e^+).*

Proof. Let $\pi : E \to F$ be a surjective cogebra morphism. Then for any associative $\mathbb{K}$-algebra B, the map $f \mapsto f \circ \pi$ is an injective algebra homomorphism of $\mathrm{Hom}_{\mathbb{K}}(F, B)$ into $\mathrm{Hom}_{\mathbb{K}}(E, B)$. By **??** and **??**, we see if E is coassociative (resp. cocommutative), then so is F. Now it amounts to apply this result to the morphism $\pi_e : E \to E^+$. □

Now let E be a bigebra, c its coproduct, ε its counit and 1 its unit element. As $\varepsilon(1) = 1$ and $c(1) = 1 \otimes 1$, the preceding results can be applied with $e = 1$. The 1-primitive elements of E are simply called primitive, that is the elements x of E such that

$$c(x) = x \otimes 1 + 1 \otimes x.$$

We write simply π, η, $P(E)$, c^+ instead of π_1, η_1, $P_1(E)$, c_1^+.

Proposition 2.1.5. *The set $P(E)$ of primitive elements of E is a Lie subalgebra of E.*

Proof. If x, y are in $P(E)$, then

$$\begin{aligned}c(xy) = c(x)c(y) &= (x \otimes 1 + 1 \otimes x)(y \otimes 1 + 1 \otimes y) \\ &= xy \otimes 1 + 1 \otimes xy + x \otimes y + y \otimes x\end{aligned}$$

whence

$$c([x, y]) = [x, y] \otimes 1 + 1 \otimes [x, y]$$ □

Proposition 2.1.6. *Let $\phi : E \to F$ be a bigebra morphism. If x is a primitive element of E, then $\phi(x)$ is a primitive element of F and the restriction of ϕ to $P(E)$ is a Lie algebra homomorphism $P(\phi) : P(E) \to P(F)$.*

Proof. Let c_E (resp. c_F) be the coproduct of E (resp. F). Since ϕ is a cogebra morphism, $c_F \circ \phi = (\phi \otimes \phi) \circ c_E$, whence

$$\begin{aligned} c_F(\phi(x)) &= (\phi \otimes \phi)(c_E(x)) = (\phi \otimes \phi)(x \otimes 1 + 1 \otimes x) \\ &= \phi(x) \otimes 1 + 1 \otimes \phi(x) \end{aligned}$$

so ϕ maps $P(E)$ to $P(F)$ and $\phi([x,y]) = [\phi(x), \phi(y)]$ since ϕ is an algebra homomorphism. □

Remark 2.1.7. Let p be a prime number such that $p \cdot 1 = 0$ in $\mathbb{K}$. The binomial formula and the congruences $\binom{p}{i} \equiv 0 \pmod{p}$ for $1 \leq i \leq p-1$ imply that $P(E)$ is stable under the map $x \mapsto x^p$.

Remark 2.1.8. By definition, the diagram

$$0 \longrightarrow P(E) \longrightarrow E^+ \xrightarrow{c^+} E^+ \otimes E^+$$

is an exact sequence. If $\mathbb{K}'$ is a commutative ring and $\rho : \mathbb{K} \to \mathbb{K}'$ a ring homomorphism, $E_{(\mathbb{K}')} = E \otimes_{\mathbb{K}} \mathbb{K}'$ is a K'-bigebra and the inclusion $P(E) \to E$ defines a homomorphism of Lie $\mathbb{K}'$-algebras

$$\alpha : P(E) \otimes_{\mathbb{K}} \mathbb{K}' \to P(E \otimes_{\mathbb{K}} \mathbb{K}').$$

If $\mathbb{K}'$ is flat over $\mathbb{K}$, it follows that the diagram

$$0 \longrightarrow P(E) \otimes_{\mathbb{K}} \mathbb{K}' \longrightarrow E^+ \otimes_{\mathbb{K}} \mathbb{K}' \xrightarrow{c^+ \otimes 1_{\mathbb{K}'}} (E^+ \otimes_{\mathbb{K}} \mathbb{K}') \otimes_{\mathbb{K}'} (E^+ \otimes_{\mathbb{K}} \mathbb{K}')$$

is an exact sequence, which implies that α is an isomorphism.

Recall that we a graduation on a bigebra is that compatible with the bialgebra structure. Similarly, a **filtration** on the bigebra E is that compatible with the bigebra structure on E: that is, an increasing sequence $(E_n)_{n \geq 0}$ of submodules of E such that

- $E_0 = \mathbb{K} \cdot 1$ and $E = \bigcup_{n \geq 0} E_n$;
- $E_m \cdot E_n \subseteq E_{m+n}$ for $m, n \geq 0$;
- $c(E_n) \subseteq \sum_{i+j=n} \operatorname{im}(E_i \otimes E_j)$ for $n \geq 0$;[1]

Note that if E is a graded bigebra and $(E^n)_{n \geq 0}$ is its graduation, the sequence (E_n) where $E_n = \sum_{i=0}^{n} E^i$ is then a filtration compatible with the bigebra structure on E.

Proposition 2.1.9. *Let E be a filtered bigebra and $(E_n)_{n \geq 0}$ its filtration. For every integer n, let $E_n^+ = E_n \cap E^+$. Then $E_0^+ = \{0\}$ and*

$$c^+(E_n^+) \subseteq \sum_{i=1}^{n-1} \operatorname{im}(E_i^+ \otimes E_{n-i}^+). \tag{2.1.3}$$

Proof. As $E_0 = \mathbb{K} \cdot 1$ we have $E_0^+ = 0$. If $x \in E_n$, then $\pi(x) = x - \varepsilon(x) \cdot 1$, whence $\pi(x) \in E_n^+$ and $\pi(E_n) \subseteq E_n^+$. It follows that $\pi \otimes \pi$ maps $\operatorname{im}(E_i \otimes E_j)$ into $\operatorname{im}(E_i^+ \otimes E_j^+)$ for $i, j \geq 0$. As $c^+ = (\pi \otimes \pi) \circ c$ in E^+ (Proposition 2.1.3),

$$c^+(E_n^+) \subseteq \sum_{i=0}^{n} \operatorname{im}(E_i^+ \otimes E_{n-i}^+) = \sum_{i=1}^{n-1} \operatorname{im}(E_i^+ \otimes E_{n-i}^+)$$

which proves the claim. □

Corollary 2.1.10. *The elements of E_1^+ are primitive.*

Proof. If $x \in E_1^+$, then $c^+(x) = 0$ by (2.1.3), so x is primitive. □

Example 2.1.11. Let M be a $\mathbb{K}$-module and $E = T(M)$ (resp. $S(M)$ or $\bigwedge(M)$) be its tensor algebra (resp. symmetric or exterior). Then the elements of degree 1 are primitive elements of E, which justifies Corollary 2.1.10. The converse is also true, for example, if $\mathbb{K}$ is a field of characteristic zero.

[1]If A and B are two submodules of E, we denote by $\operatorname{im}(A \otimes B)$ the image of the canonical map $A \otimes B \to E \otimes E$.

2.1.2 Enveloping bigebra of a Lie algebra

Recall that $\mathfrak{g}$ denotes a Lie algebra and U its enveloping algebra, with its canonical filtration $(U_n)_{n\geq 0}$. We now define a coproduct c on U such that U, with the filtration (U_n), becomes a filtered bigebra. The bigebra (U, c) will be called the **enveloping bigebra** of $\mathfrak{g}$.

Proposition 2.1.12. *There exists on the algebra U one and only one coproduct c which makes U into a bigebra such that the elements of $\iota(\mathfrak{g})$ are primitive. The bigebra (U, c) is cocommutative; its counit is the linear form ε: such that the constant term of every element x of U is $\varepsilon(x)\cdot 1$. The canonical filtration $(U_n)_{n\geq 0}$ of U is compatible with this bigebra structure.*

Proof. Let $x \in \mathfrak{g}$; we write $c_0(x) = \iota(x)\otimes 1 + 1\otimes\iota(x) \in U\otimes U$. If x, y are in $\mathfrak{g}$, then

$$c_0(x)c_0(y) = \iota(x)\iota(y)\otimes 1 + 1\otimes\iota(x)\iota(y) + \iota(x)\otimes\iota(y) + \iota(y)\otimes\iota(x)$$

whence $[c_0(x), c_0(y)] = c_0([x, y])$. By the universal property of U, there exists a unique unital algebra homomorphism

$$c : U \to U\otimes U$$

such that $c(\iota(x)) = \iota(x)\otimes 1 + 1\otimes\iota(x)$ for $x\in\mathfrak{g}$. This proves the uniqueness assertion.

We need to show that c is coassociative and cocommutative. First, the linear maps c_1 and c_2 of U into $U\otimes U\otimes U$ defined by

$$c_1 = (c\otimes 1_U)\circ c, \quad c_2 = (1_U\otimes c)\circ c$$

are unital algebra homomorphisms which coincide on $\iota(\mathfrak{g})$ since, for $a\in\iota(\mathfrak{g})$,

$$c_1(a) = a\otimes 1 + 1\otimes a\otimes 1 + 1\otimes 1\otimes a = c_2(a)$$

so c is indeed coassociative. Now let σ be the automorphism of $U\otimes U$ such that $\sigma(a\otimes b) = b\otimes a$ for $a, b\in U$. The map $\sigma\circ c$ and c of U into $U\otimes U$ are unital algebra homomorphisms which coincide on $\iota(\mathfrak{g})$, so they coincide on U. This proves the cocommutativity of c.

Let ε be such that the constant term of every element x of U is $\varepsilon(x)\cdot 1$. Then the maps $(1_U\otimes\varepsilon)\circ c$ and $(\varepsilon\otimes 1_U)\circ c$ of U into U are unital algebra homomorphisms which coincide with 1_U on $\iota(\mathfrak{g})$. Thus they are all identity on U, and therefore ε is a counit for c.

Finally, we know that $U_0 = \mathbb{K}\cdot 1$, $U = \bigcup_{n\geq 0} U_n$ and $U_n\cdot U_m \subseteq U_{m+n}$. Let $a_1, \ldots, a_n$ be in $\iota(\mathfrak{g})$. Then

$$\begin{aligned} c(a_1\cdots a_n) &= \prod_{i=1}^{n} c(a_i) = \prod_{i=1}^{n}(a_i\otimes 1 + 1\otimes a_i) \\ &= \sum_{i=0}^{n}\sum_{\sigma\in\mathrm{Sh}(i,n-i)} (a_{\sigma(1)}\cdots a_{\sigma(i)})\otimes(a_{\sigma(i+1)}\cdots a_{\sigma(n)}) \end{aligned} \tag{2.1.4}$$

As U_n is the $\mathbb{K}$-module generated by the products of at most n elements of $\iota(\mathfrak{g})$, formula (2.1.12) implies that the filtration (U_n) is compatible with the bigebra structure of (U, c). This completes the proof. □

Proposition 2.1.13. *Let E be a bigebra with coproduct denoted by c_E and let ϕ be a Lie algebra homomorphism of $\mathfrak{g}$ into $P(E)$. Then the unital algebra homomorphism $\tilde{\phi} : U \to E$ such that $\tilde{\phi}\circ\iota = \phi$ is a bigebra morphism.*

Proof. We show that $(\phi\otimes\phi)\circ c = c_E\circ\phi$. These are two unital algebra homomorphisms of U into $E\otimes E$ and, for $a\in\iota(\mathfrak{g})$,

$$(\phi\otimes\phi)(c(a)) = \phi(a)\otimes 1 + 1\otimes\phi(a) = c_E(\phi(a))$$

since $\phi(a)\in P(E)$. Similarly if ε_E is the counit of E, $\varepsilon_E\circ\phi$ is a unital algebra homomorphism $U\to\mathbb{K}$ which is zero on $\iota(\mathfrak{g})$ (Proposition 2.1.1) and therefore coincides with ε. □

It follows from Propositions 2.1.13 that the map $\phi \mapsto \phi\circ\iota$ defines a one-to-one correspondence between bigebra homomorphisms $U(\mathfrak{g})\to E$ and Lie algebra homomorphisms $\mathfrak{g}\to P(E)$.

Corollary 2.1.14. *Let $\mathfrak{g}_i$ be a Lie algebra, $U(\mathfrak{g}_i)$ its enveloping bigebra and $\iota_i : \mathfrak{g}_i \to U(\mathfrak{g}_i)$ the canonical map ($i = 1, 2$). For every Lie algebra homomorphism $\varphi : \mathfrak{g}_1\to\mathfrak{g}_2$, the unital algebra homomorphism $U(\varphi)$: $U(\mathfrak{g}_1)\to U(\mathfrak{g}_2)$ such that $U(\varphi)\circ\iota_1 = \iota_2\circ\varphi$ is a bigebra morphism.*

2.1.3 Structure of the cogebra $U(\mathfrak{g})$ in characteristic zero

We now explore the structure of the bigebra $U(\mathfrak{g})$, for which we will from now on assume that $\mathbb{K}$ is *a field of characteristic* 0. Let $\boldsymbol{S}(\mathfrak{g})$ be the symmetric algebra of the vector space $\mathfrak{g}$, $c_{\boldsymbol{S}}$ its coproduct and η the canonical isomorphism of $\boldsymbol{S}(\mathfrak{g})$ onto the vector space U. Recall that if $x_1, \dots, x_n$ are in $\mathfrak{g}$, then

$$\eta(x_1 \cdots x_n) = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \iota(x_{\sigma(1)}) \cdots \iota(x_{\sigma(n)}). \tag{2.1.5}$$

In particular, for $x \in \mathfrak{g}$ and $n \geq 0$, we have

$$\eta(x^n) = \iota(x)^n.$$

Note that by **??**, η is the unique linear map of $\boldsymbol{S}(\mathfrak{g})$ into U satisfying condition (2.1.5).

Proposition 2.1.15. *For every integer $n \in \mathbb{N}$, let U^n be the vector subspace of U generated by the $\iota(x)^n$ for $x \in \mathfrak{g}$.*

(a) The sequence $(U^n)_{n \geq 0}$ is a graduation of the vector space U compatible with its cogebra structure.

(b) The canonical map $\eta : \boldsymbol{S}(\mathfrak{g}) \to U$ is an isomorphism of graded cogebras.

The graduation $(U^n)_{n\geq 0}$ of U is called the ***canonical graduation****.*

Proof. Let $x \in \mathfrak{g}$ and $n \in \mathbb{N}$. Then since $c_{\boldsymbol{S}}$ is an algebra homomorphism,

$$c_{\boldsymbol{S}}(x^n) = c_{\boldsymbol{S}}(x)^n = (x \otimes 1 + 1 \otimes x)^n = \sum_{i=0}^{n} \binom{n}{i} x^i \otimes x^{n-i}.$$

Similarly, by (2.1.5),

$$\begin{aligned} c(\eta(x^n)) &= c(\iota(x)^n) = c(\iota(x))^n = (\iota(x) \otimes 1 + 1 \otimes \iota(x))^n \\ &= \sum_{i=0}^{n} \binom{n}{i} \sigma(x)^i \otimes \iota(x)^{n-i} = \sum_{i=0}^{n} \binom{n}{i} \eta(x^i) \otimes \eta(x^{n-i}), \end{aligned} \tag{2.1.6}$$

whence

$$(\eta \otimes \eta)(c_{\boldsymbol{S}}(x^n)) = c(\eta(x^n)).$$

As the x^n, for $x \in \mathfrak{g}$ and $n \in \mathbb{N}$, generate the vector space $\boldsymbol{S}(\mathfrak{g})$, $(\eta \otimes \eta) \circ c_{\boldsymbol{S}} = c \circ \eta$ and η is a cogebra isomorphism. On the other hand, formula (2.1.5) shows that $\eta(\boldsymbol{S}^n(\mathfrak{g})) = U^n$, which completes the proof of (a) and (b) taking account of the fact that the graduation of $\boldsymbol{S}(\mathfrak{g})$ is compatible with its cogebra structure. □

Corollary 2.1.16. *The canonical map ι defines an isomorphism of $\mathfrak{g}$ onto the Lie algebra $P(U)$ of primitive elements of U.*

Proof. As c^+ is a graded homomorphism of degree 0,

$$P(U) = \sum_{n \geq 1} (P(U) \cap U^n).$$

It suffices to prove that if $n > 1$ and $a \in U^n$ is primitive, then $a = 0$. Now a can be written as $\sum_i \lambda_i a_i^n$, where $\lambda_i \in \mathbb{K}$, $a_i \in \iota(\mathfrak{g})$. By (2.1.6), the term of bi-degree $(1, n-1)$ in $c^+(a)$ is $n \sum_i \lambda_i a_i \otimes a_i^{n-1}$. Hence $\sum_i \lambda_i a_i \otimes a_i^{n-1} = 0$. If $m : U \otimes U \to U$ is the linear map defined by multiplication on U, then

$$a = \sum_i \lambda_i a_i^n = m\Big(\sum_i \lambda_i a_i \otimes a_i^{n-1}\Big) = 0$$

which proves the claim. □

Example 2.1.17. Let V be a vector space. The primitive elements of the bigebra $\boldsymbol{S}(V)$ are the elements of degree 1, as we have already seen. This also follows from Corollary 2.1.16 applied to the commutative Lie algebra V.

Remark 2.1.18. The map η is the unique morphism of graded cogebras of $\boldsymbol{S}(\mathfrak{g})$ into U such that $\eta(1) = 1$ and $\eta(x) = \iota(x)$ for $x \in \mathfrak{g}$. For if $\tilde{\eta}$ is a morphism satisfying these conditions, we can prove by induction on n that $\tilde{\eta}(x^n) = \eta(x^n)$ for $x \in \mathfrak{g}$ and $n > 1$. To this end, assume the hypothesis for degree smaller than n; as $c^+_{\boldsymbol{S}}(x^n) = \sum_{i=1}^{n-1}\binom{n}{i}x^i \otimes x^{n-i}$, by the induction hypothesis we have

$$(\eta \otimes \eta)(c^+_{\boldsymbol{S}}(x^n)) = (\tilde{\eta} \otimes \tilde{\eta})(c^+_{\boldsymbol{S}}(x^n)).$$

It then follows that $c^+(\eta(x^n)) = c^+(\tilde{\eta}(x^n))$, so $\eta(x^n) - \tilde{\eta}(x^n)$ is a primitive element of degree n and hence is zero (Corollary to Corollary 2.1.16).

Let $(e_i)_{i\in I}$ be a basis of the vector $\mathbb{K}$-space $\mathfrak{g}$, where the indexing set I is totally ordered. For all $\alpha \in \mathbb{N}^{\oplus I}$, we write

$$e_\alpha = \prod_{i\in I} \frac{\iota(e_i)^{\alpha_i}}{\alpha_i!}.$$

The e_α, for $|\alpha| \le n$, form a basis of the vector $\mathbb{K}$-space U_n (Corollary 1.2.15). Then

$$e_0 = 1, \quad e_{\varepsilon_i} = \iota(e_i) \quad \text{for} \quad i \in I.$$

As the graded algebra associated with the filtered algebra U is commutative, for α, β in $\mathbb{N}^{\oplus I}$,

$$e_\alpha \cdot e_\beta \equiv (\alpha, \beta) e_{\alpha+\beta} \mod U_{|\alpha|+|\beta|-1}$$

where

$$(\alpha, \beta) = \prod_{i\in I} \frac{(\alpha_i + \beta_i)}{\alpha_i!\beta_i!}.$$

On the other hand, we have immediately

$$\varepsilon(e_0) = 1, \quad \varepsilon(e_\alpha) = 0 \quad \text{for} \quad |\alpha| \ge 1.$$

Finally, formula (2.1.6) implies that, for $\alpha \in \mathbb{N}^{\oplus I}$,

$$c(e_\alpha) = \sum_{\beta+\gamma=\alpha} e_\beta \otimes e_\gamma. \tag{2.1.7}$$

This formula allows us to determine the algebra $U^* = \operatorname{Hom}(U, \mathbb{K})$ dual to the cogebra U. For let $\mathbb{K}[\![X_i]\!]_{i\in I}$ be the algebra of formal power series in indeterminates $(X_i)_{i\in I}$; if $\lambda \in U^*$, let f_λ denote the formal power series

$$f_\lambda = \sum_\alpha \langle f, e_\alpha\rangle X^\alpha \quad \text{where } X^\alpha = \prod_{i\in I} X_i^{\alpha_i}$$

and the summation index α runs through $\mathbb{N}^{\oplus I}$.

Proposition 2.1.19. *The map $\lambda \mapsto f_\lambda$ is an isomorphism of the algebra U^* onto the algebra of formal power series $\mathbb{K}[\![X_i]\!]_{i\in I}$.*

Proof. Because (e_α) is a basis of U, the map $\lambda \mapsto f_\lambda$ is $\mathbb{K}$-linear and bijective. On the other hand, for λ, μ in U^*, by (2.1.7),

$$\begin{aligned}
f_{\lambda\mu} &= \sum_\alpha \langle \lambda\mu, e_\alpha\rangle X^\alpha = \sum_\alpha \langle \lambda \otimes \mu, c(e_\alpha)\rangle x^\alpha \\
&= \sum_\alpha \langle \lambda \otimes \mu, \sum_{\beta+\gamma=\alpha} e_\beta \otimes e_\gamma\rangle X^\alpha \\
&= \sum_{\beta,\gamma} \langle \lambda, e_\beta\rangle \langle \mu, e_\gamma\rangle X^{\beta+\gamma} = f_\lambda f_\mu,
\end{aligned}$$

which shows that $\lambda \mapsto f_\lambda$ is an algebra isomorphism and completes the proof. □

Recall that, if E is a bigebra, the subset $P(E)$ is a Lie algebra, which also has a universal enveloping bigebra $U(P(E))$. Therefore, the canonical injection $P(E) \to E$ can be extended to a bigebra morphism $\phi_E : U(P(E)) \to E$ (Proposition 2.1.13). Using the structural results for the enveloping bigebra $U(P(E))$, we are able to prove the following theorem:

Theorem 2.1.20. *Let E be a cocommutative bigebra.*

(a) The bigebra morphism $\phi_E : U(P(E)) \to E$ is injective.

(b) If there exists on E a filtration compatible with its bigebra structure, the morphism ϕ_E is an isomorphism.

Proof. Let c_E (resp. ε_E) be the coproduct (resp. counit) of E. we write $\mathfrak{g} = P(E)$; let $(e_i)_{i\in I}$ be a basis of the vector $\mathbb{K}$-space $\mathfrak{g}$, where the indexing set I is totally ordered, and let $(e_\alpha)_{\alpha\in\mathbb{N}^{\oplus I}}$ be the basis for $U(\mathfrak{g})$. We write $X_\alpha = \phi_E(e_\alpha)$ for $\alpha \in \mathbb{N}^{\oplus I}$. Since ϕ_E is a cogebra morphism, we have:

$$\varepsilon_E(X_0) = 1, \quad \varepsilon_E(X_\alpha) = 0 \quad \text{for} \quad |\alpha| \geq 1, \tag{2.1.8}$$

$$c_E(X_\alpha) = \sum_{\beta+\gamma=\alpha} X_\beta \otimes X_\gamma \quad \text{for} \quad \alpha \in \mathbb{N}^{\oplus I} \tag{2.1.9}$$

We show that ϕ_E is injective. In fact, we will show that, if $\phi : \boldsymbol{S}(\mathfrak{g}) \to E$ is a cogebra morphism that is injective on $\boldsymbol{S}^0(\mathfrak{g}) + \boldsymbol{S}^1(\mathfrak{g})$, then ϕ is injective. To prove this, let $n \geq 0$; we write $\boldsymbol{S}_n(\mathfrak{g}) = \sum_{i\leq n} \boldsymbol{S}^i(\mathfrak{g})$ and $c_{\boldsymbol{S}}$ for the coproduct of $\boldsymbol{S}(\mathfrak{g})$ and show by induction on n that $\phi|_{\boldsymbol{S}_n(\mathfrak{g})}$ is injective. Since the assertion is trivial for $n = 0$ and $n = 1$, we assume that $n \geq 2$ and let $u \in \boldsymbol{S}_n(\mathfrak{g})$ be such that $\phi(u) = 0$. Then

$$\begin{aligned} 0 = c_E(\phi(u)) &= (\phi \otimes \phi)(c_{\boldsymbol{S}}(u)) \\ &= \phi(u) \otimes 1 + 1 \otimes \phi(u) + (\phi \otimes \phi)(c_{\boldsymbol{S}}^+(u)) \\ &= (\phi \otimes \phi)(c_{\boldsymbol{S}}^+(u)). \end{aligned}$$

As $c_{\boldsymbol{S}}^+(u) \in \boldsymbol{S}_{n-1}(\mathfrak{g}) \otimes \boldsymbol{S}_{n-1}(\mathfrak{g})$, the induction hypothesis shows that u is a primitive element of $\boldsymbol{S}(V)$, hence is of degree 1 and hence is zero, since $\phi|_{\boldsymbol{S}^1(V)}$ is injective. Now since $U(\mathfrak{g})$ is isomorphic to $\boldsymbol{S}(\mathfrak{g})$ as cogebras and $\phi_E|_{\boldsymbol{S}^1(\mathfrak{g})}$ is identified with the canonical inclusion from $P(E)$ to E, we see the first assertion follows. In particular, the family (X_α) is free.

We show that ϕ_E is surjective if E has a filtration compatible with its bigebra structure. Let $(E_n)_{n\geq 0}$ be such a filtration and write $E_n^+ = E_n \cap \ker(\varepsilon_E)$. We show by induction on n that E_n^+ is contained in the image of ϕ_E. As $E = \mathbb{K} \cdot 1 + \bigcup_{n\geq 0} E_n^+$, this will imply the surjectivity of ϕ_E. The assertion is trivial for $n = 0$ and follows from Corollary 2.1.10 if $n = 1$; suppose henceforth that $n \geq 2$ and let $x \in E_n^+$. By Proposition 2.1.9,

$$c_E^+(x) \in \sum_{i=1}^{n-1} E_i^+ \otimes E_{n-i}^+$$

and by the induction hypothesis there exist scalars $\lambda_{\alpha,\beta}$, where α, β are in $\mathbb{N}^{\oplus I}$, which are zero except for a finite number, such that

$$c_E^+(x) = \sum_{\alpha,\beta\neq 0} \lambda_{\alpha,\beta} X_\alpha \otimes X_\beta.$$

Hence by (2.1.9),

$$\begin{aligned} (c_E^+ \otimes 1_E)(c_E^+(x)) &= \sum_{\alpha,\beta,\gamma\neq 0} \lambda_{\alpha+\beta,\gamma} X_\alpha \otimes X_\beta \otimes X_\gamma, \\ (1_E \otimes c_E^+)(c_E^+(x)) &= \sum_{\alpha,\beta,\gamma\neq 0} \lambda_{\alpha,\beta+\gamma} X_\alpha \otimes X_\beta \otimes X_\gamma. \end{aligned}$$

By Proposition 2.1.4 the coproduct c_E^+ is coassociative, so the linear independence of the X_α implies that

$$\lambda_{\alpha+\beta,\gamma} = \lambda_{\alpha,\beta+\gamma} \quad \text{for} \quad \alpha, \beta, \gamma \in \mathbb{N}^{\oplus I} \setminus \{0\}. \tag{2.1.10}$$

On the other hand, since the coproduct c_E is cocommutative, the same argument as above implies

$$\lambda_{\alpha,\beta} = \lambda_{\beta,\alpha} \quad \text{for} \quad \alpha, \beta \in \mathbb{N}^{\oplus I} \setminus \{0\}. \tag{2.1.11}$$

Suppose that there exists a family of scalars (μ_α) with $|\alpha| \geq 2$ such that

$$\mu_{\alpha+\beta} = \lambda_{\alpha,\beta} \quad \text{for} \quad \alpha, \beta \in \mathbb{N}^{\oplus I} \setminus \{0\}. \tag{2.1.12}$$

Then

$$c_E^+(x) = \sum_{\alpha,\beta\neq 0} \mu_{\alpha+\beta} X_\alpha \otimes X_\beta = \sum_{|\gamma|\geq 2} \mu_\gamma c_E^+(X_\gamma)$$

by (2.1.9), hence $y - x - \sum_{|\gamma|\geq 2} \mu_\gamma X_\gamma$ is primitive and hence belongs to $P(E) \subseteq \operatorname{im} \phi_E$. Then

$$x = y + \sum_{|\gamma|\geq 2} \mu_\gamma \phi_E(e_\gamma) \in \operatorname{im}(\phi_E).$$

The proof will therefore be complete when we have proved the existence of the family (μ_α). For this, it suffices to prove that

$$\alpha + \beta = \gamma + \delta$$

implies $\lambda_{\alpha,\beta} = \lambda_{\gamma,\delta}$ for $\alpha, \beta, \gamma, \delta$ non-zero. By Riesz's Decomposition Lemma there exist π, ρ, σ, τ in $\mathbb{N}^{\oplus I}$ such that

$$\alpha = \pi + \sigma, \quad \beta = \rho + \tau, \quad \gamma = \pi + \rho, \quad \delta = \sigma + \tau.$$

Supose $\pi \neq 0$; as $\sigma + \beta = \rho + \delta$, relation (2.1.10) implies

$$\lambda_{\alpha,\beta} = \lambda_{\pi+\sigma,\beta} = \lambda_{\pi,\sigma+\beta} = \lambda_{\pi,\rho+\delta} = \lambda_{\pi+\rho,\delta} = \lambda_{\gamma,\delta}.$$

If on the other hand $\pi = 0$, then $\beta = \gamma + \tau$ and $\delta = \alpha + \tau$, whence

$$\lambda_{\alpha,\beta} = \lambda_{\alpha,\gamma+\tau} = \lambda_{\alpha+\tau,\gamma} = \lambda_{\delta,\gamma}$$

by (2.1.10), but also $\lambda_{\delta,\gamma} = \lambda_{\gamma,\delta}$ by (2.1.11), whence $\lambda_{\alpha,\beta} = \lambda_{\gamma,\delta}$. □

2.2 Free Lie algebra over a set X

2.2.1 The free Lie algebra $L(X)$

Let X be a set. Recall the construction of the free magma $M(X)$ constructed on X. By induction on the integer $n \geq 1$, we define the sets X_n by writing $X_1 = X$ and taking

$$X_n = \coprod_{p=1}^{n-1} X_p \times X_{n-p}$$

if X is finite, so is each X_n. The disjoint union of the family $(X_n)_{n\geq 1}$ is denoted by $M(X)$; each of the sets X_n (and in particular X) is identified with a subset of $M(X)$. Let x and y be in $M(X)$; let p and q denote the integers such that $x \in X_p$ and $y \in X_q$ and let $n = p + q$; the image of the ordered pair (x, y) under the canonical injection of $X_p \times X_{n-p}$ into Xn is denoted by $\cdot y$ and called the product of x and y. Every map of X into a magma M can be extended in a unique way to a magma homomorphism of $M(X)$ into M.

Let x be in $M(X)$; the unique integer n such that $x \in X_n$ is called the length of x and denoted by $\ell(x)$. Then $\ell(x \cdot y) = \ell(x) + \ell(y)$ for $x, y \in M(X)$. The set X is the subset of $M(X)$ consisting of the elements of length 1. Every element x of length ≥ 2 can be written uniquely in the form $x = y \cdot z$.

Now suppose we are given a commutative ring $\mathbb{K}$ with unity and a nonempty set X. Let $M(X)$ be the free magma on X. We introduce a free $\mathbb{K}$-module whose basis is $M(X)$, denoted by $\operatorname{Lib}(X)$. We will define multiplication in $\operatorname{Lib}(X)$ to turn it into a $\mathbb{K}$-algebra. So let $f, g \in \operatorname{Lib}(X)$; then we can write f and g as

$$f = \sum_{u\in M(X)} \lambda_u u, \quad g = \sum_{v\in M(X)} \mu_v v.$$

Then we can define a multiplication fg by the following formula

$$fg = \sum_{\mu,\nu} \lambda_\mu \mu$$

It is clear that $\operatorname{Lib}(X)$ is a $\mathbb{K}$-algebra (it is denoted by $\operatorname{Lib}_{\mathbb{K}}(X)$ when it is necessary to indicate the ring $\mathbb{K}$). The set $M(X)$ is a basis of the $\mathbb{K}$-module $\operatorname{Lib}(X)$ and X will therefore be identified with a subset of $\operatorname{Lib}(X)$. If A is an algebra, every map of X into A can be extended uniquely to a homomorphism of $\operatorname{Lib}(X)$ into A.

It is obvious that any homomorphism from $\mathrm{Lib}(X)$ into a Lie algebra sends to zero the elements of either two forms, $a \cdot a$ and $a \cdot (b \cdot c) + b \cdot (c \cdot a) + c \cdot (a \cdot b)$, where $a, b, c \in \mathrm{Lib}(X)$. The **free Lie algebra** over the set X is then defined to be the quotient algebra

$$L(X) = \mathrm{Lib}(X)/\mathfrak{a}$$

where $\mathfrak{a}$ is the two-sided ideal of $\mathrm{Lib}(X)$ generated by the elements of one of the forms

$$Q(a) = a \cdot a \quad \text{for} \quad a \in \mathrm{Lib}(X),$$

$$J(a,b,c) = a \cdot (b \cdot c) + b \cdot (c \cdot a) + c \cdot (a \cdot b)$$

for $a, b, c \in \mathrm{Lib}(X)$. Clearly $L(X)$ is a Lie $\mathbb{K}$-lagebra. The product of two elements u, v of $L(X)$ will be denoted by $[u, v]$, like a usual Lie algebra. When it is necessary to indicate the ring $\mathbb{K}$, we write $L_{\mathbb{K}}(X)$ for $L(X)$. The following proposition justifies the name *free* Lie algebra given to $L(X)$.

Proposition 2.2.1. *Let ι be the canonical map of X to $L(X)$. For every map f of X into a Lie algebra $\mathfrak{g}$, there exists one and only one homomorphism $\varphi : L(X) \to \mathfrak{g}$ such that $f = \varphi \circ \iota$.*

$$\begin{tikzcd} L(X) \arrow[r, "\varphi"] & \mathfrak{g} \\ X \arrow[u, "\iota"] \arrow[ur, "f"'] & \end{tikzcd}$$

Proof. let h be the homomorphism of $\mathrm{Lib}(X)$ into $\mathfrak{g}$ extending f. For all $a \in \mathrm{Lib}(X)$, $h(Q(a)) = h(a \cdot a) = [h(a), h(a)] = 0$. Similarly, the Jacobi identity of $\mathfrak{g}$ implies that $h(J(a,b,c)) = 0$ for a, b, c in $\mathrm{Lib}(X)$. It follows that $h(\mathfrak{a}) = 0$, whence there is a homomorphism φ of $L(X)$ into $\mathfrak{g}$. By restricting to X, we obtain $f = \varphi \circ \iota$. □

Corollary 2.2.2. *The family $(\iota(x))_{x \in X}$ is free over $\mathbb{K}$ in $L(X)$.*

Proof. Let $x_1, \ldots, x_n$ be distinct elements in X and $\lambda_1, \ldots, \lambda_n$ be elements of $\mathbb{K}$ such that

$$\sum_{i=1}^{n} \lambda_i \iota(x_i) = 0. \tag{2.2.1}$$

Let $\mathfrak{g}$ be the commutative Lie algebra with $\mathbb{K}$ as underlying module. For $i = 1, \ldots, n$, there exists a homomorphism φ_i of $L(X)$ into $\mathfrak{g}$ such that $\varphi_i(\iota(x_i)) = 1$ and $\varphi_i(\iota(x)) = 0$ for $x \neq x_i$ (Proposition 2.2.1); applying φ_i to relation (2.2.1), we obtain $\lambda_i = 0$. □

Corollary 2.2.3. *Let $\mathfrak{a}$ be a Lie algebra. Every extension of $L(X)$ by $\mathfrak{a}$ is inessential.*

Proof. Let $\mathfrak{a} \xrightarrow{\lambda} \mathfrak{g} \xrightarrow{\mu} L(X)$ be an extension. As μ is surjective, there exists a map f of X into $\mathfrak{g}$ such that $\iota = \mu \circ f$. Let φ be the homomorphism of $L(X)$ into $\mathfrak{g}$ such that $f = \varphi \circ \iota$. Then $(\mu \circ \varphi) \circ \iota = \mu \circ f = \iota$ and Proposition 2.2.1 shows that $\mu \circ \varphi$ is the identity of $L(X)$. The given extension is therefore inessential. □

As the ring $\mathbb{K}$ is non-zero, Corollary 2.2.2 shows that ι is injective. Hence the set X can be identified by means of ι with its image in $L(X)$; with this convention, X generates $L(X)$ and every map of X into a Lie algebra $\mathfrak{g}$ can be extended to a Lie algebra homomorphism of $L(X)$ into $\mathfrak{g}$.

Proposition 2.2.4. *Let X be a set, let M be an $L(X)$-module and let d be a map of X into M. There exists a unique linear map D of $L(X)$ into M extending d and satisfying the relation*

$$D([a,b]) = a \cdot D(b) - b \cdot D(a) \quad \textit{for} \quad a, b \in L(X). \tag{2.2.2}$$

Proof. We define a Lie algebra $\mathfrak{g}$ with underlying module $M \times L(X)$ by means of the bracket

$$[(m,a),(n,b)] = (a \cdot n - b \cdot m, [a,b]), \tag{2.2.3}$$

for a, b in $L(X)$ and m, n in M. Let φ be the homomorphism of $L(X)$ into $\mathfrak{g}$ such that $\varphi(x) = (d(x), x)$ for all x in X; let $\varphi(a) = (D(a), u(a))$ for all a in $L(X)$. By formula (2.2.3), u is a homomorphism of $L(X)$ into itself; as $u(x) = x$ for x in X, $u(a) = a$ for all a in $L(X)$, whence

$$\varphi(a) = (D(a), a). \tag{2.2.4}$$

By (2.2.3) and (2.2.4), relation (2.2.2) then follows from $\varphi([a,b]) = [\varphi(a), \varphi(b)]$. The uniqueness of D follows from the universal property of $L(X)$. □

Corollary 2.2.5. *Every map of X into $L(X)$ can be extended uniquely into a derivation of $L(X)$.*

Proof. When M is equal to $L(X)$ with the adjoint representation, relation (2.2.2) means that D is a derivation. □

Proposition 2.2.6. *Let X and Y be two sets. Every map $f : X \to Y$ can be extended uniquely to a Lie algebra homomorphism $L(f) : L(X) \to L(Y)$. For every map $g : Y \to Z$, we have : $(g \circ f) = L(g) \circ L(f)$.*

Proof. The existence and uniqueness of $L(f)$ follow from Proposition 2.2.1. The homomorphisms $L(g \circ f)$ and $L(g) \circ L(f)$ have the same restriction to X and hence are equal (Proposition 2.2.1). □

Corollary 2.2.7. *If f is injective (resp. surjective, bijective), so is $L(f)$.*

Proof. Since the assertion is trivial for $X = \varnothing$, we assume $X \neq \varnothing$. If f is injective there exists a map g of Y into X such that $g \circ f$ is the identity map of X; by Proposition 2.2.6, $L(g) \circ L(f)$ is the identity of automorphism of $L(X)$ and hence $L(f)$ is injective. When f is surjective, there exists a map g of Y into X such that $f \circ g$ is the identity map of Y; then $L(f) \circ L(g)$ is the identity map of $L(Y)$, which proves that $L(f)$ is surjective. □

Let X be a set and S a subset of X. The above corollary shows that the canonical injection of S into X can be extended to an isomorphism of $L(S)$ onto the Lie subalgebra $\tilde{L}(S)$ of $L(X)$ generated by S; we shall identify $L(S)$ and $\tilde{L}(S)$ by means of this isomorphism.

Let $(S_\alpha)_{\alpha \in I}$ be a right directed family of subsets of X with union S. The relation $S_\alpha \subseteq S_\beta$ implies $L(S_\alpha) \subseteq L(S_\beta)$ and hence the family of Lie subalgebras $L(S_\alpha)$ of $L(X)$ is right directed. Therefore $\mathfrak{g} = \bigcup_{\alpha \in I} L(S_\alpha)$ is a Lie subalgebra of $L(X)$; then $S \subseteq \mathfrak{g}$, whence $L(S) \subseteq \mathfrak{g}$, and, as $L(S_\alpha) \subseteq L(S)$ for all $\alpha \in I$, $\mathfrak{g} \subseteq L(S)$. Hence

$$L\Big(\bigcup_{\alpha \in I} S_\alpha\Big) = \bigcup_{\alpha \in I} L(S_\alpha) \tag{2.2.5}$$

for every right directed family $(S_\alpha)_{\alpha \in I}$ of subsets of X.

Applying the above to the family of finite subsets of X, we see that every element of $L(X)$ is of the form $P(x_1, \dots, x_n)$ where P is a Lie polynomial in n indeterminates and $x_1, \dots, x_n$ are elements of X.

Proposition 2.2.8. *Let $\mathbb{K}'$ be a non-zero commutative ring and $\rho : \mathbb{K} \to \mathbb{K}'$ a ring homomorphism. For every set X there exists one and only one Lie $\mathbb{K}'$-algebra isomorphism*

$$\nu : L_{\mathbb{K}}(X) \otimes \mathbb{K}' \to L_{\mathbb{K}'}(X)$$

such that $\nu(x \otimes 1) = x$ for $x \in X$.

Proof. Applying Proposition 2.2.1 to $\mathfrak{g} = L_{\mathbb{K}'}(X)$ considered as a Lie $\mathbb{K}$-algebra and the map $x \mapsto x$ of X into $\mathfrak{g}$, we obtain a $\mathbb{K}$-homomorphism $L_{\mathbb{K}}(X) \to L_{\mathbb{K}'}(X)$, whence there is a $\mathbb{K}'$-homomorphism $\nu : L_{\mathbb{K}}(X) \otimes \mathbb{K}' \to L_{\mathbb{K}'}(X)$. The fact that ν is unique and is an isomorphism follows from the fact that the ordered pair $(L_{\mathbb{K}}(X) \otimes \mathbb{K}', x \mapsto x \otimes 1)$ is a solution of the same universal map problem as the ordered pair $(L_{\mathbb{K}'}(X), x \mapsto x)$. □

Let $\mathfrak{g}$ be a Lie algebra and $\mathscr{A} = (a_i)_{i \in I}$ a family of elements of $\mathfrak{g}$. Let $\varphi_{\mathscr{A}}$ be the homomorphism of $L(I)$ into $\mathfrak{g}$ map each $i \in I$ to a_i. The image of $\varphi_{\mathscr{A}}$ is the subalgebra of $\mathfrak{g}$ generated by $\boldsymbol{a}$; the elements of the kernel of $\varphi_{\mathscr{A}}$ are called the **relators** of the family $\mathfrak{a}$. The family $\boldsymbol{a}$ is called **generating** (resp. **free**, **basic**) if $\varphi_{\mathscr{A}}$ is surjective (resp. injective, bijective). A **presentation** of $\mathfrak{g}$ is then defined to be an ordered pair $(\mathscr{A}, \mathscr{R})$ consisting of a generating family $\mathscr{A} = (a_i)_{i \in I}$ and a family $\mathscr{R} = (r_j)_{J \in J}$ of relators of a generating the ideal of $L(I)$ the kernel of $\varphi_{\mathscr{A}}$. We also say that $\mathfrak{g}$ is **presented** by the family $\mathscr{A}$ related by the relators $\mathscr{R}$.

Let I be a set and $\mathscr{R} = (r_j)_{j \in J}$ be a family of elements of the free Lie algebra $L(I)$; let $\mathfrak{a}_{\mathscr{R}}$ be the ideal of $L(I)$ generated by $\mathscr{R}$. The quotient algebra $L(I, \mathscr{R}) = L(I)/\mathfrak{a}_{\mathscr{R}}$ is called the *Lie algebra defined by I and the family of relators $\mathscr{R}$*; we also say that $L(I, \mathscr{R})$ is defined by the presentation $(I, \mathscr{R})$, or also by $(I, (r_j = 0)_{j \in J})$. When the family $\mathscr{R}$ is empty, $L(I, \mathscr{R}) = L(I)$.

Let I and $\mathscr{R}$ be as above; let ξ_i denote the image of i in $L(I, \mathscr{R})$. The generating family $\mathscr{X} = (\xi_i)_{i \in I}$ and the family of relators $\mathscr{R}$ constitute a presentation of $L(I, \mathscr{R})$. Conversely, if $\mathfrak{g}$ is a Lie algebra and $(\mathscr{A}, \mathscr{R})$, where $\mathscr{A} = (a_i)_{i \in I}$ is a presentation of $\mathfrak{g}$, there exists a unique isomorphism $\varphi : L(I, \mathscr{R}) \to \mathfrak{g}$ such that $\varphi(\xi_i) = a_i$ for all $i \in I$.

Let I be a set. Let T_i denote the canonical image of the element i of I in $L(I)$ (which is also sometimes denoted by $L((T_i)_{i\in I})$); the elements of $L(I)$ are called **Lie polynomials** in the indeterminates $(T_i)_{i\in I}$. Let $\mathfrak{g}$ be a Lie algebra. If $\mathscr{T} = (t_i)_{i\in I}$ is a family of elements of $\mathfrak{g}$, let $\varphi_{\mathscr{T}}$ denote the homomorphism of $L(I)$ into $\mathfrak{g}$ such that $\varphi_{\mathscr{T}}(T_i) = t_i$ for $i \in I$. The image under $\varphi_{\mathscr{T}}$ of the element P of $L(I)$ is denoted by $P(\{t_i\}_{i\in I})$. In particular, $P(\{T_i\}_{i\in I}) = P$; the above element $P(\{t_i\}_{i\in I})$ is sometimes called the element of $\mathfrak{g}$ obtained by **substituting** the t_i for the T_i in the Lie polynomial $P((T_i)_{i\in I})$.

Let $\sigma : \mathfrak{g} \to \tilde{\mathfrak{g}}$ be a Lie algebra homomorphism. For every family $\mathscr{T} = (t_i)_{i\in I}$ of elements of $\mathfrak{g}$ and all $P \in L(I)$,

$$\sigma(P(\{t_i\}_{i\in I})) = P(\{\sigma(t_i)\}_{i\in I}) \tag{2.2.6}$$

for $\sigma \circ \varphi_{\mathscr{T}}$ maps T_i to $\sigma(t_i)$ for $i \in I$.

Let $(Q_j)_{j\in J}$ be a family of elements of $L(I)$ and let $P \in L(J)$. By substituting the Q_j for the T_j in P, we obtain a Lie polynomial $R = P(\{Q_j\}_{j\in J}) \in L(I)$. Then

$$R(\{t_i\}_{i\in I}) = P(\{Q_j(\{t_i\}_{i\in I})\}_{j\in J}) \tag{2.2.7}$$

for every family $\mathscr{T} = (t_i)_{i\in I}$ of elements of a Lie algebra $\mathfrak{g}$, as is seen by operating by the homomorphism $\varphi_{\mathscr{T}}$ on the equation $R = P((Q_j)_{j\in J})$ and using (2.2.6).

Let $\mathfrak{g}$ be a Lie algebra, I a finite set and $P \in L(I)$. Suppose that $\mathfrak{g}$ is a free $\mathbb{K}$-module. The map

$$\tilde{P} : \mathfrak{g}^I \to \mathfrak{g}$$

defined by $\tilde{P}(\{t_i\}_{i\in I}) = P(\{t_i\}_{i\in I})$ is then polynomial[2]. For the set F of mapps of $\mathfrak{g}^I$ to $\mathfrak{g}$ is a Lie algebra with bracket defined by

$$[\phi, \psi](t) = [\phi(t), \psi(t)];$$

the set $\tilde{F}$ of polynomial maps of $\mathfrak{g}^I$ into $\mathfrak{g}$ is a Lie subalgebra of F by the bilinearity of the bracket. Our assertion then follows from the fact that the map $P \mapsto \tilde{P}$ is a Lie algebra homomorphism and $\tilde{T}_i = \pi_i \in \tilde{F}$ for all $i \in I$.

2.2.2 Graduation of the free Lie algebra

Let Δ be a commutative monoid, written additively. Let ϕ_0 denote a map of X into Δ and ϕ the homomorphism of the free magma $M(X)$ into Δ which extends ϕ_0. For all $\delta \in \Delta$, let $\mathrm{Lib}^\delta(X)$ be the submodule of $\mathrm{Lib}(X)$ with basis the subset $\phi^{-1}(\delta)$ of $M(X)$. The family $(\mathrm{Lib}^\delta(X))_{\delta\in\Delta}$ is a graduation of the algebra $\mathrm{Lib}(X)$, that is

$$\mathrm{Lib}(X) = \bigoplus_{\delta\in\Delta} \mathrm{Lib}^\delta(X)$$

$$\mathrm{Lib}^\delta(X) \cdot \mathrm{Lib}^\gamma(X) \subseteq \mathrm{Lib}^{\delta+\gamma}(X) \quad \text{for} \quad \delta, \gamma \in \Delta.$$

We calim that the graduation of $\mathrm{Lib}(X)$ descents to $L(X)$, so that we get a graduation on $L(X)$. For this, it suffices to show the following lemma.

Lemma 2.2.9. *The ideal $\mathfrak{a}$ defining $L(X)$ is graded.*

Proof. For $a, b \in \mathrm{Lib}(X)$, let $B(a,b) = a \cdot b + b \cdot a$. The formulae

$$B(a,b) = Q(a+b) - Q(a) - Q(b)$$

$$Q(\lambda_1 w_1 + \cdots + \lambda_n w_n) = \sum_i \lambda_i^2 Q(w_i) + \sum_{i<j} \lambda_i\lambda_j B(w_i, w_j)$$

for $w_1, \ldots, w_n \in M(X)$ and $\lambda_1, \ldots, \lambda_n$ in $\mathbb{K}$, show that the family $(Q(a))$ and $(Q(w), B(w, \tilde{w}))$ generate the same submodule of $\mathrm{Lib}(X)$. As J is trilinear the ideal $\mathfrak{a}$ is generated by the homogeneous elements $Q(w)$, $B(v,w)$ and $J(u,v,w)$, where u, v, w are in $M(X)$, and hence is graded (**??**). □

[2] Recall the definition of polynomial maps of a free module M into a module N: if $p \geq 0$ is an integer, a map $f : M \to N$ is called **homogeneous polynomial** of degree p if there exists a multilinear map u of M^p into N such that

$$f(x) = u(x, \cdots, x) \quad \text{for} \quad x \in M.$$

A map of M into N is called **polynomial** if it is a finite sum of homogeneous polynomial map of suitable degrees.

Let the Lie algebra $L(X) = \mathrm{Lib}(X)/\mathfrak{a}$ be given the quotient graduation. The homogeneous component of $L(X)$ of degree δ is denoted by $L^\delta(X)$; it is the submodule of $L(X)$ generated by the image of $w \in M(X)$ such that $\phi(w) = \delta$. We shall make special use of the following two cases:

(a) Total graduation: we take $\Delta = \mathbb{N}$ and $\phi_0(x) = 1$ for all $x \in X$, whence $\phi(w) = \ell(w)$ for $w \in M(X)$. The $\mathbb{K}$-module $L^n(X)$ is generated by the images of the elements of length n in $M(X)$, which we shall call alternants of degree n. We shall see later that the module $L^n(X)$ is free and admits a basis consisting of alternants of degree n. Then $L(X) = \bigoplus_{n\geq 1} L^n(X)$ and $L^1(X)$ admits X as a basis. By the construction of $M(X)$,

$$L^n(X) = \sum_{p=1}^{n-1} [L^p(X), L^{n-p}(X)] \tag{2.2.8}$$

and in particular

$$[L^m(X), L^n(X)] \subseteq L^{m+n}(X). \tag{2.2.9}$$

(b) Multigraduation: we take Δ to be the free commutative monoid $\mathbb{N}^{\oplus X}$ constructed on X. The map ϕ_0 of X into Δ is defined by $\phi_0(x)_y = \delta_{xy}$, where δ_{xy} is the Kronecker symbol. For $w \in M(X)$ and $x \in X$, the integer $(\phi(w))_x$ is "the number of occurrences of the letter x in w." For $\alpha \in \mathbb{N}^{\oplus X}$, we write $|\alpha| = \sum_{x \in X} \alpha_x$, whence $|\phi(w)| = \ell(w)$ for all $w \in M(X)$. It follows that

$$L^n(X) = \bigoplus_{|\alpha|=n} L^\alpha(X); \tag{2.2.10}$$

$$[L^\alpha(X), L^\beta(X)] \subseteq L^{\alpha+\beta}(X) \quad \text{for} \quad \alpha, \beta \in \mathbb{N}^{\oplus X}. \tag{2.2.11}$$

In the following, when we talk about canonical graduations, we always mean the above two graduations on $L(X)$.

Proposition 2.2.10. *Let S be a subset of X. If $\mathbb{N}^{\oplus S}$ is identified with its canonical image in $\mathbb{N}^{\oplus X}$, then $L(S) = \sum_{\alpha \in \mathbb{N}^{\oplus}} L^\alpha(X)$. Further, for all $\alpha \in \mathbb{N}^{\oplus S}$, the homogeneous component of degree α under the multigraduation on $L(S)$ is equal to $L^\alpha(X)$.*

Proof. Let $\alpha \in \mathbb{N}^{\oplus S}$. The module $L^\alpha(S)$ is generated by the image in $L(X)$ of the elements w in $M(S)$ such that $\phi(w) = \alpha$, that is, the set of w of $M(X)$ such that $\phi(w) = \alpha$. Hence $L^\alpha(S) = L^\alpha(X)$. The proposition follows from this and the relation $L(S) = \sum_{\alpha \in \mathbb{N}^{\oplus S}} L^\alpha(S)$. □

Corollary 2.2.11. *For every family $(S_i)_{i\in I}$ of subsets of X,*

$$L\Big(\bigcap_{i\in I} S_i\Big) = \bigcap_{i\in I} L(S_i).$$

Proof. This follows from Proposition 2.2.10 and the obvious formula $\mathbb{N}^{\oplus S} = \bigcap_{i\in I} \mathbb{N}^{\oplus S_i}$, where we have written $S = \bigcap_{i\in I} S_i$. □

It turns out that the total graduation on the free Lie algebra $L(X)$ can be identified with the lower central series: recall that the latter is defined by the following inductive formula:

$$C^1(L(X)) = L(X), \quad C^{n+1}(L(X)) = [L(X), L^n(X)].$$

Proposition 2.2.12. *Let $\mathfrak{g}$ be a Lie algebra and P a submodule of $\mathfrak{g}$. Define the submodule P_n of $\mathfrak{g}$ by $P_1 = P$ and $P_{n+1} = [P, P_n]$ for $n \geq 1$. Then*

$$[P_m, P_n] \subseteq P_{n+m}, \tag{2.2.12}$$

$$P_n = \sum_{k=1}^{n-1} [P_k, P_{n-k}] \quad \textit{for} \quad n \geq 2. \tag{2.2.13}$$

Proof. We prove (2.2.12) by induction on m. The case $m = 1$ is obvious. By the Jacobi identity,

$$[[P, P_m], P_n] \subseteq [P_m, [P, P_n]] + [P, [P_m, P_n]],$$

which is

$$[P_{m+1}, P_n] \subseteq [P_m, P_{n+1}] + [P, [P_m, P_n]].$$

The induction hypothesis implies $[P_m, P_{n+1}] \subseteq P_{m+n+1}$ and $[P_m, P_n] \subseteq P_{m+n}$, whence

$$[P_{m+1}, P_n] \subseteq P_{m+n+1} + [P, P_{m+n}] = P_{m+n+1}.$$

With (2.2.12), $P_n \supseteq \sum_{k=1}^{n-1}[P_k, P_{n-k}] \supseteq [P_1, P_n] = P_{n+1}$, whence (2.2.13). □

Corollary 2.2.13. *Let $\mathfrak{g}$ be a Lie algebra and $(C^n(\mathfrak{g}))_{n\geq 1}$ the lower central series of $\mathfrak{g}$. Then*

$$[C^m(\mathfrak{g}), C^n(\mathfrak{g})] \subseteq C^{m+n}(\mathfrak{g}) \quad \textit{for} \quad m, n \geq 1.$$

Proof. When we take $P = \mathfrak{g}$, it is clear that the sequence (P_n) is the lower central series $(C^n(\mathfrak{g}))$ of $\mathfrak{g}$. Hence we get the implication. □

Proposition 2.2.14. *Let X be a set and $n \geq 0$ an integer.*

(a) $L^{n+1}(X) = [L^1(X), L^n(X)]$.

(b) The module $L^n(X)$ is generated by the elements $[x_1, [x_2, \ldots, [x_{n-1}, x_n] \ldots]]$, where $x_1, \ldots, x_n$ runs through the set of sequences of n elements of X.

(c) The lower central series of $L(X)$ is given by $C^n(L(X)) = \sum_{p\geq n} L^p(X)$.

Proof. We apply Proposition 2.2.12 with $\mathfrak{g} = L(X)$ and $P = L^1(X)$. By induction on n, we deduce from (2.2.8) and (2.2.12) the equality $P_n = L^n(X)$. The desired relation in (a) is equivalent to the definition $[P, P_n] = P_{n+1}$. Assertion (b) then follows from (a) by induction on n.

Let $\mathfrak{g} = L(X)$ and $\mathfrak{g}_n = \sum_{p\geq n} L^p(X)$. Then $\mathfrak{g} = \mathfrak{g}_0$ and formula (2.2.9) implies $[\mathfrak{g}_n, \mathfrak{g}_m] \subseteq \mathfrak{g}_{n+m}$ and in particular $[\mathfrak{g}, \mathfrak{g}_n] \subseteq \mathfrak{g}_{n+1}$. By induction on n, $C^n(\mathfrak{g}) \subseteq \mathfrak{g}_n$. On the other hand, from (a) we deduce that $L^n(X) \subseteq \mathfrak{g}_n$ by induction on n. As $C^n(\mathfrak{g})$ is an ideal of $\mathfrak{g}$, the relation $L^p(X) \subseteq C^n(\mathfrak{g})$ implies that

$$L^{p+1}(X) = [L^1(X), L^p(X)] \subseteq C^n(\mathfrak{g})$$

by (a). Hence $L^p(X) \subseteq C^n(\mathfrak{g})$ for $p \geq n+1$, whence $\mathfrak{g}_n \subseteq C^n(\mathfrak{g})$. □

Corollary 2.2.15. *Let $\mathfrak{g}$ be a Lie algebra and $(x_i)_{i\in I}$ a generating family of g. The n-th term $C^n(\mathfrak{g})$ of the lower central series of $\mathfrak{g}$ is generated by the iterated brackets $[x_{i_1}, [x_{i_2}, \ldots, [x_{i_{p-1}}, x_{i_p}] \ldots]]$ for $p \geq n$, and $i_1, \ldots, i_p$ in I.*

Proof. Let φ be the homomorphism of $L(I)$ into $\mathfrak{g}$ such that $\varphi(i) = x_i$ for all $i \in I$. As $(x_i)_{i\in I}$ generates $\mathfrak{g}$, $\mathfrak{g} = \varphi(L(I))$, whence $C^n(\mathfrak{g}) = \varphi(C^n(L(I)))$ by Proposition 1.4.1. The corollary then follows from assertions (b) and (c) of Proposition 2.2.14. □

2.2.3 Elimination theorem

Proposition 2.2.16. *Let S_1 and S_2 be two disjoint sets and $d : S_1 \times S_2 \to L(S_2)$ a map. Let $\mathfrak{g}$ be the quotient Lie algebra of $L(S_1 \cup S_2)$ by the ideal generated by the elements $[s_1, s_2] - d(s_1, s_2)$ with $s_1 \in S_1, s_2 \in S_2$; let $\psi : L(S_1 \cup S_2) \to \mathfrak{g}$ be the canonical quotient map.*

(a) The restriction ι_i of ψ to S_i can be extended to an isomorphism of $L(S_i)$ onto a subalgebra $\mathfrak{a}_i$ of $\mathfrak{g}$;

(b) $\mathfrak{g} = \mathfrak{a}_1 + \mathfrak{a}_2$, $\mathfrak{a}_1 \cap \mathfrak{a}_2 = \{0\}$ *and $\mathfrak{a}_2$ is an ideal of $\mathfrak{g}$.*

Proof. For $i = 1, 2$, let ψ_i denote the homomorphism of $L(S_i)$ into $\mathfrak{g}$ which extends ι_i and $\mathfrak{a}_i$ its image. Clearly $\iota_i(S_i)$ generates $\mathfrak{a}_i$. Let $s_1 \in S_1$; we write $D = \mathrm{ad}(\iota_1(s_1))$. The derivation D of $\mathfrak{g}$ maps $\iota_2(S_2)$ into $\mathfrak{a}_2$ by the relation

$$[\iota_1(s_1), \iota_2(s_2)] = \psi_2(d(s_1, s_2)) \quad \text{for} \quad s_2 \in S_2;$$

as the subalgebra $\mathfrak{a}_2$ of $\mathfrak{g}$ is generated by $\iota_2(S_2)$, therefore $D(\mathfrak{a}_2) \subseteq \mathfrak{a}_2$. The normalizer of $\mathfrak{a}_2$ in $\mathfrak{g}$ then contains $\iota_1(S_1)$ by the above and hence also $\mathfrak{a}_1$. Hence

$$[\mathfrak{a}_1, \mathfrak{a}_2] \subseteq \mathfrak{a}_2. \tag{2.2.14}$$

Therefore $\mathfrak{a}_1 + \mathfrak{a}_2$ is a Lie subalgebra of $\mathfrak{g}$ and, as it contains the generating set $\iota_1(S_1) \cup \iota_2(S_2)$,

$$\mathfrak{a}_1 + \mathfrak{a}_2 = \mathfrak{g}. \tag{2.2.15}$$

For all $s_1 \in S_1$ there exists a derivation D_{s_1} of $L(S_2)$ such that

$$D_{s_1}(s_2) = d(s_1, s_2)$$

for all $s_2 \in S_2$ (Corollary 2.2.5). The map $s_1 \mapsto D_{s_1}$ can be extended to a homomorphism D of $L(S_1)$ into the Lie algebra of derivations of $L(S_2)$. Let $\mathfrak{h}$ be the semi-direct product of $L(S_1)$ by $L(S_2)$ corresponding to D. As a module $\mathfrak{h}$ is equal to $L(S_1) \times L(S_2)$ and in particular

$$[(s_1, 0), (0, s_2)] = (0, d(s_1, s_2)) \tag{2.2.16}$$

for $s_1 \in S_1$ and $s_2 \in S_2$. From (2.2.16) we deduce the existence of a homomorphism f of $\mathfrak{g}$ into $\mathfrak{h}$ such that $f(\iota_1(s_1)) = (s_1, 0)$ and $f(\iota_2(s_2)) = (0, s_2)$ for $s_1 \in S_1$ and $s_2 \in S_2$. We deduce immediately the relation

$$f(\psi_1(a_1) + \psi_2(a_2)) = (a_1, a_2) \tag{2.2.17}$$

for $a_1 \in L(S_1)$ and $a_2 \in L(S_2)$. Formula (2.2.17) shows ψ_1 and ψ_2 are injective and $\mathfrak{a}_1 \cap \mathfrak{a}_2 = \{0\}$. Formulae (2.2.14) and (2.2.15) then imply the proposition. □

Proposition 2.2.17 (Elimination Theorem). *Let X be a set, S a subset of X and T the set of sequences $(s_1, \ldots, s_n, x)$ with $n \geq 0$, $s_1, \ldots, s_n$ in S and x in $X \setminus S$.*

(a) The module $L(X)$ is the direct sum of the subalgebra $L(S)$ of $L(X)$ and the ideal $\mathfrak{a}$ of $L(X)$ generated by $X \setminus S$.

(b) There exists a Lie algebra isomorphism ϕ of $L(T)$ onto $\mathfrak{a}$ which maps $(s_1, \ldots, s_n, x)$ to $(\operatorname{ad}(s_1) \circ \cdots \circ \operatorname{ad}(s_n))(x)$.

Proof. Let $\mathfrak{g}$ be the Lie algebra constructed as in Proposition 2.2.16 given

$$S_1 = S, \quad S_2 = T, \quad d(s, t) = (s, s_1, \ldots, s_n, x) \in T \subseteq L(T)$$

for $t = (s_1, \ldots, s_n, x)$ in T and $s \in S_1$. We identify $L(S)$ and $L(T)$ with their canonical images in $\mathfrak{g}$ (Proposition 2.2.16(a)). Let ψ be the map $(s_1, \ldots, s_n, x) \mapsto (\operatorname{ad}(s_1) \circ \cdots \circ \operatorname{ad}(s_n))(x)$ of T into $L(X)$. Obviously $\psi(d(s,t)) = [s, \psi(t)]$ for $s \in S$ and $t \in T$ and hence there exists a homomorphism $\alpha : \mathfrak{g} \to L(X)$ whose restriction to S is the identity and whose restriction to T is ψ. Now $X \setminus S \subseteq T$, whence there is a homomorphism $\beta : L(X) \to \mathfrak{g}$ whose restriction to $X = S \cup (X \setminus S)$ is the identity.

We show that α is an isomorphism and β the inverse isomorphism. As $\psi(x) = x$ for x in $X \setminus S$, we see that $\alpha \circ \beta$ coincides with the identity on X, whence $\alpha \circ \beta = \operatorname{id}_{L(X)}$. On the other hand, $[s, t] = d(s, t)$ in $\mathfrak{g}$ for $s \in S$, $t \in T$ by construction; it follows that $t = (s_1, \ldots, s_n, x)$ is equal in $\mathfrak{g}$ to $(\operatorname{ad}(s_1) \circ \cdots \circ \operatorname{ad}(s_n))(x)$ whence $t = \beta(\alpha(t))$. As $\beta(\alpha(s)) = s$ for $s \in S$ and $S \cup T$ generates $\mathfrak{g}$, $\beta \circ \alpha = \operatorname{id}_{\mathfrak{g}}$.

As α is an isomorphism of $\mathfrak{g}$ onto $L(X)$, Proposition 2.2.16 shows that the restriction of α to $L(T)$ is an isomorphism ϕ of $L(T)$ onto an ideal $\mathfrak{b}$ of $L(X)$ such that the module $L(X)$ is the direct sum of $L(S)$ and $\mathfrak{b}$. Obviously

$$\phi(s_1, \ldots, s_n, x) = (\operatorname{ad}(s_1) \circ \cdots \circ \operatorname{ad}(s_n))(x)$$

for $(s_1, \ldots, s_n, x)$ in T. Hence $\phi(T) \subseteq \mathfrak{a}$, whereas $\mathfrak{b} \subseteq \mathfrak{a}$ since $\phi(T)$ generates the subalgebra $\mathfrak{b}$ of $L(X)$. But $\mathfrak{b}$ is an ideal and $X \setminus S \subseteq \phi(T) \subseteq \mathfrak{b}$, whence $\mathfrak{a} = \mathfrak{b}$. □

Applying the elimination theorem for $S = \{y\}$, we obtain the following corollary.

Corollary 2.2.18. *Let $y \in X$. The free Lie algebra $L(X)$ is the direct sum of the free submodule $\mathbb{K}y$ and the Lie subalgebra admitting as basic family the family of $\operatorname{ad}(y)^n(z)$ where $n \geq 0$ and $z \in X \setminus \{y\}$.*

2.2.4 Bases for the free Lie algebra

Let X be a set, $M(X)$ the free magma constructed on X and $M^n(X)$, where $n \in \mathbb{N}^+$, the set of elements of $M(X)$ of length n. If $w \in M(X)$ and $\ell(w) \geq 2$, let $\alpha(w)$ and $\beta(w)$ denote the elements of $M(X)$ determined by the relation $w = \alpha(w) \cdot \beta(w)$; then $\ell(\alpha(w)) < \ell(w)$, $\ell(\beta(w)) < \ell(w)$. Finally, for u, v in $M(X)$, let $u^n v$ denote the element defined by induction on the integer $n \geq 0$ by $u^0 v = v$ and $u^{n+1}v = u(u^n v)$.

We now extract a special family for the free Lie algebra $L(X)$, which turns out to be a basis for the $\mathbb{K}$-module $L(X)$ (called the *Hall basis of* $L(X)$). For this, we shall first define the *Hall set* of a free magma: Let X be a set, a **Hall set** relative to X is any totally ordered subset H of $M(X)$ satisfying the following conditions:

(H1) If $u \in H$, $v \in H$ and $\ell(u) < \ell(v)$, then $u < v$.

(H2) $X \subseteq H$ and $H \cap M^2(X)$ consists of the products xy with x, y in X and $x < y$.

(H3) An element w of $M(X)$ of length ≥ 3 belongs to H if and only if it is of the form $a(bc)$ with a, b, c in H, bc in H, $b \leq a < bc$ and $b < c$.

Proposition 2.2.19. *Let H be a Hall set relative to X and let x, y be in X.*

(a) $H \cap M(\{x\}) = \{x\}$.

(b) Suppose that $x < y$ and let d_y be the homomorphism of $M(X)$ into $\mathbb{N}$ such that $d_y(y) = 1$ and $d_y(z) = 0$ for $z \in X$, $z \neq y$. The set of elements $w \in H \cap M(\{x, y\})$ such that $d_y(w) = 1$ consists of the elements $x^n y$ with $n \geq 0$ an integer.

Proof. For every subset S of X, we identify the free magma $M(S)$ with its canonical image in $M(X)$. By condition (H2) , $x \in H$ and $H \cap M^2(\{x\}) = 0$. If $w \in H \cap M(\{x\})$, where $n = \ell(w) \geq 3$, the elements $\alpha(w)$ and $\beta(w)$ also belong to $H \cap M(\{x\})$ by condition (H3). It immediately follows by induction on n that $H \cap M^n(\{x\}) = 0$ for $n \geq 2$, whence (a).

We now prove (b). By condition (H2), $y \in H$ and $xy \in H$. We show by induction on n that $x^n y \in H$ for $n \geq 2$ an integer. Now $x^n y = x(x(x^{n-2}y))$ and the induction hypothesis implies that $x^{n-2}y \in H$. Now $\ell(x) < \ell(x^{n-2}y)$ for $n > 2$ and $x < y$, whence $x < x^{n-2}y$ in any case; condition (H3) shows that $x^n y \in H$. On the other hand, certainly $d_y(x^n y) = 1$. Conversely, let $w \in H \cap M(\{x, y\})$, with $d_y(w) = 1$. If $\ell(w) = 1$, then $w = y$; if $\ell(w) = 2$, then $w = xy$ by condition (H2). If $\ell(w) \geq 3$, then $w = a(bc)$ with a, b, c in $H \cap M(\{x, y\})$. The case $d_y(bc) = 0$ is impossible, since this would imply $bc \in M(\{x\})$, which is impossible by (a). Hence $d_y(bc) = 1$ and $d_y(a) = 0$, whence $a = x$ by (a). It follows immediately by induction on $n = \ell(w)$ that $w = x^{n-1}y$, which completes the proof of (b). □

Corollary 2.2.20. *If $|X| \geq 2$, then $H \cap M^n(X) \neq \varnothing$ for every integer $n \geq 1$.*

Proposition 2.2.21. *Let X be a finite set with at least two elements. Let H be a Hall set relative to X. Then there exist a strictly increasing bijection $p \mapsto w_p$ of $\mathbb{N}$ onto H and a sequence $(P_p)_{p \in \mathbb{N}}$ of subsets of H with the following properties:*

(a) $P_0 = X$.

(b) For every integer $p \geq 0$, $w_p \in P_p$.

(c) For every integer $n \geq 1$, there exists an integer $p(n)$ such that every element of P_p is of length $> n$ for all $p \geq p(n)$.

(d) For every integer $p \geq 0$, the set P_{p+1} consists of the elements of the form $w_p^i w$, where $i \geq 0$, $w \in P_p$ and $w \neq w_p$.

Proof. As X is finite, each of the sets $M^n(X)$ is finite. Let $H_n = H \cap M^n(X)$ for all $n \geq 1$. Then Corollary 2.2.20 shows that the finite set H_n is non-empty. Let u_n be the cardinal of H_n; let $v_0 = 0$ and $v_n = u_1 + \cdots + u_n$ for $n \geq 1$. As H_n is a totally ordered finite set, there exists a strictly increasing bijection $p \mapsto w_p$ of the interval $[v_{n-1}, v_n - 1]$ of $\mathbb{N}$ onto H_n. It is immediate that $p \mapsto w_p$ is a strictly increasing bijection of $\mathbb{N}$ onto H.

Let $P_0 = X$ and for every integer $p \geq 1$ define

$$P_p = \{w \in H : w \neq w_p, \text{ and, either } w \in X \text{ or } \alpha(w) < w_p\}.$$

(Note that if w is of length ≥ 2 the relation $w \in H$ implies $\alpha(w) \in H$ by condition (H3).) Then $w_p \in P_p$: this is clear if $w_p \in X$ and follows from the inequality $\ell(\alpha(w_p)) < \ell(w_p)$ and condition (H1) when $w_p \notin X$. Hence conditions (a) and (b) are satisfied.

Let $n \geq 1$ be an integer and let $p \geq v_n$. For all $w \in P_p$, $\ell(w) \geq \ell(w_p) > n$ by the very definition of the map $p \mapsto w_p$. This establishes (c). We now show that every element of the form $u = w_p^i w$ with $i \geq 0$, $w \in P_p$ and $w \neq w_p$ belongs to P_{p+1}. If $i \neq 0$, then $\ell(u) > \ell(w_p)$, whence $u > w_p$ and $u \geq w_{p+1}$; then $u \notin X$ and $\alpha(u) = w_p < w_{p+1}$, whence $u \in P_{p+1}$. If $i = 0$, then $u \in P_p$ and $u \neq w_p$; then $u > w_p$, whence $u \geq w_{p+1}$; if u does not belong to X, then $\alpha(w) < w_p$ (by the definition of P_p), whence $\alpha(w) < w_{p+1}$; then again $u \in P_{p+1}$.

Conversely, let $u \in P_{p+1}$. We distinguish two cases:

(α) There exists no element v of $M(X)$ such that $u = w_p v$. By definition of P_{p+1}, $u > w_p$. Further, if $u \notin X$, then $\alpha(u) \neq w_p$ by the given hypothesis and $\alpha(u) < w_{p+1}$ since $u \in P_{p+1}$; hence $\alpha(u) < w_p$. This shows $u \in P_p$ and $u \neq w_p$.

(β) There exists v in $M(X)$ such that $u = w_p v$. By condition (H3), of necessity, either $w_p \in X$, $v \in X$ and $w_p < v$, or $v \notin X$ and $\alpha(v) \leq w_p < v$. In either case, $v \in P_{p+1}$.

Then there exist an integer $i \geq 0$ and an element w of $M(X)$ such that $u = w_p^i w$, and either $w \in X$ or $w \notin X$ and $\alpha(w) \neq w_p$. If $i = 0$, we have case (α) above, whence $w \in P_p$ and $w \neq w_p$. If $i > 0$, the proof of (β) above establishes, by induction on i, the relations $w \in P_{p+1}$ and $w \neq w_p$. Suppose $w \notin X$; from $w \in P_{p+1}$ it follows that $\alpha(w) \leq w_p$ and as $\alpha(w) \neq w_p$, we conclude that $w \in P_p$. This completes the proof of (d). $\square$

We now arrive the most important theorem in this paragraph, which shows that any Hall set H relative to X gives arise to a basis for the free Lie algebra $L(X)$. More precisely, we have the following result:

Theorem 2.2.22. *Let H be a Hall set relative to X and ε the canonical map of $M(X)$ into the free Lie algebra $L(X)$. Then the image of H is a basis of the module $L(X)$.*

Proof. For every element w of H we write $\varpi = \varepsilon(w)$. We first consider the case where X is finite. If X is empty, so are $M(X)$ and therefore H and $L(X)$ is zero. If X has a single element x, $H \cap M_n(X)$ is empty for $n \geq 2$ (Proposition 2.2.19(a)). Therefore, $H = \{x\}$; we know also that the module $L(X)$ is free and has basis $\{x\}$. The theorem is therefore true when X has at most one element.

Suppose henceforth that X has at least two elements; choose sequences (w_p) and (P_p) with the properties stated in Proposition 2.2.21. For every integer $p \geq 0$, let L_p denote the submodule of $L(X)$ generated by the elements w_i with $0 \leq i < p$ and $\mathfrak{g}_p$ the Lie subalgebra of $L(X)$ generated by the family $\varepsilon(P_p)$.

We claim that, for every integer $p \geq 0$, the module L_p admits the family $(\varpi_i)_{0 \leq i < p}$ as basis, the Lie algebra $\mathfrak{g}_p$ admits $\varepsilon(P_p)$ as basic family and the module $L(X)$ is the direct sum of L_p and $\mathfrak{g}_p$: $L_0 = \{0\}$ and $\mathfrak{g}_0 = L(X)$, so the claim is true for $p = 0$. We argue by induction on p. Suppose then that the claim is true for some integer $p \geq 0$. Let $u_{i,w} = \operatorname{ad}(\varpi_p)^i(\varpi) = \varepsilon(w_p^i w)$ for $i \geq 0$, $w \in P_p$, $w \neq w_p$. By Corollary 2.2.18, the free Lie algebra $\mathfrak{g}_p$ is the direct sum of the module T_p of basis $\{\varpi_p\}$ and a Lie subalgebra $\mathfrak{h}_p$ admitting

$$\mathscr{F} = (u_{i,w})_{i \geq 0, w \in P_p, w \neq w_p}$$

as basic family. By Proposition 2.2.21(b), the family $(\bar{u})_{u \in P_{p+1}}$ is equal to $\mathscr{F}$ and hence is a basic family for $\mathfrak{h}_p = \mathfrak{g}_{p+1}$. Hence $L(X) = L_p \oplus T_p \oplus \mathfrak{g}_{p+1}$ and, as $L_{p+1} = L_p + T_p$, we see $L(X) = L_{p+1} \oplus \mathfrak{g}_{p+1}$ and $(\varpi_i)_{0 \leq i \leq p}$ is a basis of the module L_{p+1}.

Let n be a positive integer. By Proposition 2.2.21(c) there exists an integer $p(n)$ such that P_p has only elements of length $> n$ for $p \geq p(n)$. For $p \geq p(n)$, the Lie subalgebra $\mathfrak{g}_p$ of $L(X)$ is generated by elements of degree $> n$ and hence $L^n(X) \cap \mathfrak{g}_p = \{0\}$. On the other hand, the elements ϖ_i of $L(X)$ are homogeneous and the family $(w_i)_{0 \leq i < p}$ is a basis of a supplementary module of $\mathfrak{g}_p$. It follows immediately that the family of elements ϖ_i of degree n is a basis of the module $L^n(X)$ and that the sequence $(\varpi_i)_{i \geq 0}$ is a basis of the module $L(X)$.

Now we consider the general case. If S is a subset of X, recall that $M(S)$ is identified with the submagna of $M(X)$ generated by S and $L(S)$ is identified with the Lie subalgebra of $L(X)$ generated by S; we have seen that if $w \in M(S)$ is of length ≥ 2 then $\alpha(w) \in M(S)$ and $\alpha(w) \in M(S)$. It follows immediately that $H \cap M(S)$ is a Hall set relative to S. For every finite subset Φ of H there exists a finite subset S of X such that $\Phi \subseteq M(S)$. The finite case then shows that the elements ϖ with $w \in \Phi$ are linearly independent in $L(S)$ and hence in $L(X)$. Therefore the family $(\bar{w})_{w\in H}$ is free. For every element a of $L(X)$ there exists a finite subset S of X such that $a \in L(S)$. By the finite case, the subset $\varepsilon(H \cap M(S))$ of $\varepsilon(H)$ generates the module $L(S)$ and hence a is a linear combination of elements of $\varepsilon(H)$. Hence $\varepsilon(H)$ generates the module $L(X)$, which completes the proof. □

Corollary 2.2.23. *The module $L(X)$ is free and so is each of the submodules $L^\alpha(X)$ where $\alpha \in \mathbb{N}^{\oplus X}$ and $L^n(X)$ where $n \in \mathbb{N}$. The modules $L^\alpha(X)$ are of finite rank and so are the modules $L^n(X)$ if X is finite.*

Proof. There exists a Hall set H relative to X. For all $w \in H$, the element $\varepsilon(w)$ of $L(X)$ belongs to one of the modules $L^\alpha(X)$ (with $\alpha \in \mathbb{N}^{\oplus X}$) and the module $L(X)$ is the direct sum of the submodules $L^\alpha(X)$. Further, for all $\alpha \in \mathbb{N}^{\oplus X}$, the set of elements of $M(X)$ whose canonical image in $\mathbb{N}^{\oplus X}$ is equal to α is finite; this shows that each of the modules $L^\alpha(X)$ is free and of finite rank and that $L(X)$ is free. Now $L^n(X) = \sum_{|\alpha|=n} L^\alpha(X)$ and hence $L^n(X)$ is free; when X is finite, the set of $\alpha \in \mathbb{N}^{\oplus X}$ such that $|\alpha| = n$ is finite and hence $L^n(X)$ is then of finite rank. □

Example 2.2.24. Suppose that X consists of two distinct elements x and y and let $L^{(-,1)}$ be the submodule of $L(X)$ the sum of the $L^\alpha(X)$ where $\alpha \in \mathbb{N}^{\oplus X}$ and $\alpha(y) = 1$. It follows immediately from Theorem 2.2.22 and Proposition 2.2.19 that the elements of $\operatorname{ad}(x)^n(y)$ where $n \geq 0$ is an integer form a basis of the submodule $L^{(-,1)}$. It follows that the restriction to $L^{(-,1)}$ of the map $\operatorname{ad}(x)$ is injective.

2.2.5 Subalgebras of the free Lie algebra

A well-known theorem in the group theory due to NielsenSchreier says that every subgroup of a free group is itself free. This result has no full counterpart in the theory of Lie algebras over arbitrary commutative rings. For instance, let $L = L(\{x,y\})$ be a free *Lie ring*, which is a Lie algebra over integers. Further, let $\mathfrak{g}$ denote its Lie subalgebra generated by $2x$, y, $[x,y]$. Suppose that $\mathfrak{g}$ is a free Lie ring, then $\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$ must be a free abelian group. But $\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$ contains an element $[x,y] + [\mathfrak{g},\mathfrak{g}]$ of additive order 2, a contradiction. Hence, $\mathfrak{g}$ is not a free Lie ring.

However, if $\mathfrak{g}$ is generated by a subset of $\mathbb{K}X$ which can be included into a basis of this free module, then $\mathfrak{g}$ is a free Lie algebra. This is an easy consequence of an auxiliary lemma, which follows.

Lemma 2.2.25. *Let $L(X)$ be a free Lie algebra over a commutative ring $\mathbb{K}$, Y being a basis of the free $\mathbb{K}$-module $\mathbb{K}X$. Then Y is the free generating set for $L(X)$.*

Proof. Let φ denote an automorphism of the free module $\mathbb{K}X = \mathbb{K}Y$ sending X into Y, and ψ its inverse. The restriction $\varphi|_X$ and $\psi|_X$ extend to homomorphisms $\tilde{\varphi}, \tilde{\psi} : L(X) \to L(Y)$. Thus we have

$$(\tilde{\varphi} \circ \tilde{\psi})|_{\mathbb{K}X} = \tilde{\varphi}|_{\mathbb{K}X} \circ \tilde{\psi}|_{\mathbb{K}X} = \varphi|_{\mathbb{K}X} \circ \psi|_{\mathbb{K}Y} = (\varphi \circ \psi)|_{\mathbb{K}X} = 1_{\mathbb{K}X}.$$

Simiarly, $(\tilde{\psi} \circ \tilde{\varphi})|_{\mathbb{K}X}$. Since $\mathbb{K}X$ generates $L(X)$, it follows that $\tilde{\varphi}$ is an isomorphism with inverse $\tilde{\psi}$ transforming X to Y. Since X is a free generating set, the same is true for Y. □

In this paragraph we are going to prove a proposition which will give the freeness of some special subalgebras of free Lie algebras. This will enable us to prove the main theorem concerning the freeness of homogeneous subalgebras in the case where the base ring satisfies some rather mild restrictions (all projective modules over this ring must be free). Then, as corollaries, we will derive well-known theorems of Shirshov and Witt in which the base ring is respectively a field or the ring of integers.

We will require some more notation. Let X be a nonempty set and $M(X)$ the free magma over X. A subset S is called a **left ideal** of $M(X)$ if for all $u \in M(X)$, $v \in S$, we have $uv \in S$. Suppose that H is a Hall family in $M(X)$ satisfying the following condition: for any $u \in H \setminus S$, $v \in H \cap S$, we have $u \leq v$. An element $w \in S$ is called ***S*-reducible** if it can be represented in the form $w = uv$, where $u, v \in S$; all the other elements $w \in S$ are called ***S*-irreducible**.

Proposition 2.2.26. *Let S be a left ideal in $M(X)$, Y the set of S-irreducible elements in $H \cap S$ (H as above). Then the $\mathbb{K}$-submodule M of the free Lie algebra $L(X)$ generated over $\mathbb{K}$ by $H \cap S$ is a free Lie algebra with Y as a free generating set.*

Proof. Let $\tilde{Y}$ denote a set which is in a bijective correspondence with Y. We consider $M(\tilde{Y})$ and let $\phi : M(\tilde{Y}) \to M(Y)$ be the the isomorphism of magmas. We will be constructing a basic family $\tilde{H}$ in $M(\tilde{Y})$ as follows. We set $\tilde{H}_1 = \tilde{Y}$ and using ϕ^{-1}, we transfer $\tilde{Y}$ the ordering of Y (which is a subset of the well-ordered set H). Supose that we managed to construct $\tilde{H}_1, \ldots, \tilde{H}_{n-1}$ so that $\phi(\tilde{H}_i) \subseteq H \cap S$. We also suppose that $\tilde{H}_1 \cup \cdots \cup \tilde{H}_{n-1}$ is a linearly ordered set in such a way that ϕ is a homomorphism of ordered sets on $\tilde{H}_1 \cup \cdots \cup \tilde{H}_{n-1}$. We will construct $\tilde{H}_n$ from these data by

$$\tilde{H}_n = \{uv : u \in \tilde{H}_t, v \in \tilde{H}_{n-t}, u < v, \text{and if } \ell(v) > 1 \text{ then } u \geq \alpha(v)\}.$$

Let us show first that $\phi(\tilde{H}_n) \subseteq H \cap S$. For this, we first note that $\phi(u), \phi(v) \in H \cap S$ and $\phi(u) < \phi(v)$ since $u < v$. We now distinguish two cases:

(a) If $\ell(v) > 1$, then $v = \alpha(v)\beta(v)$ and $\alpha(v), \beta(v) \in \phi^{-1}(S)$. Since $u \geq \alpha(v)$ we have $\phi(u) \geq \phi(\alpha(v))$. Hence $w \in H \cap S$ by condition (H3)

(b) If $\ell(v) = 1$ then $\ell(\phi(v)) = 1$, so $\phi(uv) = \phi(u)\phi(v) \in H$ be condition (H2). Moreover, since S is a left ideal we have $\phi(uv) \in S$, so $w \in H \cap S$.

Setting $\tilde{H} = \bigcup_n \tilde{H}_n$, we obtain a Hall set $\tilde{H}$ in $M(\tilde{Y})$. Hence $\tilde{H}$ becomes a basis of the free $\mathbb{K}$-module $L(\tilde{Y})$. The image of $L(\tilde{Y})$ under a homomorphism $\tilde{\phi}$ induced by ϕ is a subalgebra $\mathfrak{g}$ of $L(X)$ which coincides with $\mathbb{K}(H \cap S)$ since S is an ideal in $M(X)$. The elements of $\tilde{H}$, which is a basis of the $\mathbb{K}$-module $L(\tilde{Y})$, map injectively into the elements of the $\mathbb{K}$-basis H of $L(X)$. Thus $\tilde{\phi} : L(\tilde{Y}) \to \mathfrak{g}$ is an isomorphism of the free Lie $L(\tilde{Y})$ and the subalgebra $\mathbb{K}(H \cap S)$ map $\tilde{Y}$ into Y. Hence Y is a set of free generators for $\mathbb{K}(H \cap S)$. ☐

Corollary 2.2.27. *Let $L(X)$ be a free Lie algebra over an arbitrary commutative ring $\mathbb{K}$. Then the derived algebra $[L(X), L(X)]$ is a free Lie algebra provided $|X| > 1$, and its free generating set consists of all right-normed products of the form*

$$x_1 x_2 \cdots x_{s-1} x_s, \quad x_i \in X, x_1 \geq x_2 \geq \cdots \geq x_{s-1} < x_s, s \geq 2. \tag{2.2.18}$$

Proof. In this case we set S to be the set of elements of length ≥ 2 in $M(X)$ and choose for H any of Hall bases. Then the irreducible elements in $H \cap S$ will be precisely of the form as in (2.2.18) and $\mathbb{K}(H \cap S) = [L(X), L(X)]$. ☐

Theorem 2.2.28. *Let $\mathbb{K}$ be a commutative ring over which all projective modules are free. Suppose that $L = L(X)$ is a free Lie algebra with some $\mathbb{N}$-grading in which X is homogeneous and $\mathfrak{g}$ is a homogeneous subalgebra which is a direct summand of the $\mathbb{K}$-module $L(X)$. Then $\mathfrak{g}$ is a free Lie algebra with a homogeneous set of free generators.*

The hypotheses of Theorem 2.2.28 are satisfied by any principal ideal domain (e.g., by any field), by the polynomial rings over fields, etc. In some cases it is possible to abandon restrictions imposed on the subalgebra or weaken them somewhat.

Theorem 2.2.29 (A.I.Shirshov). *Let $\mathfrak{g}$ be a nonzero subalgebra of a free Lie algebra $L(X)$ over a field $\mathbb{K}$. Then $\mathfrak{g}$ is a free Lie algebra.*

Proof. We consider the natural grading of $L(X)$ (with respect to degrees in X). ☐

Corollary 2.2.30. *Let $L(X)$ be a free Lie algebra over a field $\mathbb{K}$ and $\mathfrak{a}$ a proper nonzero ideal in $L(X)$. Then $\mathfrak{a}$ is a free Lie algebra of infinite rank.*

Proof. Since $\mathfrak{a} \neq L(X)$ there exists $x \in L(X) \setminus \mathfrak{a}$. We set $\mathfrak{g} = \mathfrak{a} \oplus \mathbb{K}x$. Since $[\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{a}$, a free generating set Y for $\mathfrak{g}$ can be chosen using Lemma 2.2.25 so that $x \in Y$ and $Y \setminus \{x\} \subseteq \mathfrak{a}$. Using the Elimination theorem, we find that a free generating set for $\mathfrak{a}$ can be chosen to include all products of the form

$$\operatorname{ad}(x)^n(y), \quad n = 0, 1, \cdots, y \in Y \setminus \{x\}.$$

By the invariance of the rank of a free Lie algebra, $\mathfrak{a}$ cannot be generated by a finite set. ☐

Theorem 2.2.31 (E.Witt). *Let $L(X)$ be a free Lie ring and $\mathfrak{g}$ a homogeneous subring. If the abelian group $L(X)/\mathfrak{g}$ is free then $\mathfrak{g}$ is a free Lie ring.*

Proof. We apply natural grading of $L(X)$. By our hypotheses, $\mathfrak{g}$ is the direct summand of the abelian group $L(X)$. Each projective module over $\mathbb{Z}$ being free follows using the theorem about subgroups of abelian groups. Thus Theorem 2.2.28 applies. ☐

2.3 Enveloping algebra of the free Lie algebra

In this section, $A(X) = A_{\mathbb{K}}(X)$ denote the *free associative algebra* of the set X over the ring $\mathbb{K}$. The set X is identified with its canonical image in $A(X)$; recall that the $\mathbb{K}$-module $A(X)$ admits as basis the free monoid $\Delta(X)$ derived from X; $A^+(X)$ denotes the submodule of $A(X)$ generated by the non-empty words.

2.3.1 The enveloping algebra of $L(X)$

Theorem 2.3.1. *Let $\alpha : L(X) \to A(X)$ be the unique Lie algebra homomorphism extending the canonical injection of X into $A(X)$. Let $\iota : L(X) \to U(L(X))$ be the canonical map of $L(X)$ into its enveloping algebra and $\tilde{\alpha} : U(L(X)) \to A(X)$ the unique unital algebra homomorphism such that $\tilde{\alpha} \circ \iota = \alpha$. Then*

(a) α is injective and $\alpha(L(X))$ is a direct factor submodule of $A(X)$.

(b) $\tilde{\alpha}$ is bijective.

Proof. Let B be a unital $\mathbb{K}$-algebra and $\phi : X \to B$ a map; by Proposition 2.2.1, there exists a Lie algebra homomorphism $\psi : L(X) \to B$ such that $\psi|_X = \phi$; by the universal property of enveloping algebra, there exists a unital algebra homomorphism $\theta : U(L(X)) \to B$ such that $\theta \circ \iota = \psi$ and hence such that $(\theta \circ \iota)|_X = \phi$. As $\iota(X)$ generates the unital algebra $U(L(X))$, the homomorphism θ is the unique unital algebra homomorphism satisfying the latter condition. This shows that the ordered pair $(U(L(X))), \iota|_X)$ is a solution of the same universal map problem as $A(X)$; taking ϕ to be the canonical injection of X into $A(X)$, we deduce that β is an isomorphism, which shows (b).

Finally, as $L(X)$ is a free $\mathbb{K}$-module (Corollary 2.2.23), ι is injective and $\iota(L(X))$ is a direct factor submodule of $U(L(X))$ (Corollary 1.2.15). By (b), this proves (a). □

Corollary 2.3.2. *There exists on the algebra $A(X)$ a unique coproduct making $A(X)$ into a bigebra such that the elements of X are primitive. Further, β is an isomorphism of the bigebra $U(L(X))$ onto $A(X)$ with this bigebra structure.*

Proof. This follows from assertion (b) of the theorem and the fact that X generates the unital algebra $A(X)$ (c.f. Proposition 2.1.12). □

Henceforth $A(X)$ is given this bigebra structure and $L(X)$ is identified with its image under α, that is, with the Lie subalgebra of $A(X)$ generated by X. For x, y in $L(X)$, we write $[x, y]$ for the Lie bracket computed in $L(X)$; then under the map α, $[x, y]$ is mapped to $xy - yx$. Hence we may identify the element $[x, y]$ with $xy - yx$.

Corollary 2.3.3. *If $\mathbb{K}$ is a field of characteristic 0, $L(X)$ is the Lie algebra ef primitive elements of $A(X)$.*

Proof. This follows from Corollary 2.3.2 and Corollary 2.1.16. □

Remark 2.3.4. Let $\mathbb{K}'$ be a commutative ring containing $\mathbb{K}$. If $A(X)$, $L(X)$ and $L_{\mathbb{K}'}(X)$ are identified with subsets of $A_{\mathbb{K}'}(X)$, we deduce from assertion (a) of Theorem 2.3.1 the relation

$$L(X) = L_{\mathbb{K}'}(X) \cap A(X). \tag{2.3.1}$$

Remark 2.3.5. Corollary 2.3.3 remains valid if it only assumes that the additive group of the ring $\mathbb{K}$ is torsion-free. For suppose first that $\mathbb{K} = \mathbb{Z}$; every primitive element of $A(X)$ is a primitive element of $A_{\mathbb{Q}}(X)$ and hence is in $L_{\mathbb{Q}}(X) \cap A(X) = L(X)$ (Corollary 2.3.3 and (2.3.1)). In the general case, $\mathbb{K}$ is flat over $\mathbb{Z}$ and we apply Remark 2.1.8 and Proposition 2.2.8.

Remark 2.3.6. Let Δ be a commutative monoid, ϕ_0 a map of X into Δ and $\phi : \Delta(X) \to \Delta$ the homomorphism of the associated monoid; if $A(X)$ is given the graduation $(A^\delta(X))_{\delta\in\Delta}$ and $L(X)$ the graduation $(L^\delta(X))_{\delta\in\Delta}$, we have immediately, for $\delta \in \Delta$, $L^\delta(X) \subseteq L(X) \cap A^\delta(X)$. As L is the sum of the $L^\delta(X)$ for $\delta \in \Delta$, and the sum of the $L(X) \cap A^\delta(X)$ for $\delta \in \Delta$ is direct, this implies

$$L^\delta(X) = L(X) \cap A^\delta(X). \tag{2.3.2}$$

2.3.2 Projection of $A^+(X)$ onto $L(X)$

Given an element u of $A(X)$, it is typically hard to tell that whether u belongs to $L(X)$. It is the point that such an element u is usually represented as products in the algebra $A(X)$, rather than brackets. In this paragraph, we give a conventient way to solve this problem: namely, we will give a projection of $A(X)$ onto $L(X)$.

Proposition 2.3.7. *Let π be the linear map of $A^+(X)$ into $L(X)$ defined by*

$$\pi(x_1 \cdots x_n) = (\mathrm{ad}(x_1) \circ \cdots \circ \mathrm{ad}(x_{n-1}))(x_n)$$

for $x_1, \ldots, x_n$ in X. Then

(a) the restriction π_0 of π to $L(X)$ is a derivation of $L(X)$;

(b) for every integer $n \geq 1$ and $u \in L^n(X)$, $\pi(u) = n \cdot u$.

Proof. Let E be the endomorphism algebra of the module $L(X)$ and θ the homomorphism of $A(X)$ into E such that $\theta(x) = \mathrm{ad}(x)$ for all $x \in X$. The restriction of θ to $L(X)$ is a Lie algebra homomorphism of $L(X)$ into E, which coincides on X with the adjoint representation of $L(X)$, whence

$$\theta(u) \cdot v = [u, v] \quad \text{for} \quad u, v \in L(X). \tag{2.3.3}$$

Let a be in $A(X)$ and b in $A^+(X)$; then

$$\pi(a \cdot b) = \theta(a) \cdot \pi(b). \tag{2.3.4}$$

(It suffices to consider the case $a = x_1 \cdots x_p$ and $b = x_{p+1} \cdots x_{p+q}$ with $x_1, \ldots, x_{p+q}$ in X; but then (2.3.4) follows immediately from (2.3.3) since $\theta(x) = \mathrm{ad}(x)$ for $x \in x$.) Let u and v be in $L(X)$; by (2.3.3) and (2.3.4),

$$\begin{aligned}\pi_0([u,v]) &= \pi(uv - vu) = \theta(u) \cdot \pi(v) - \theta(v) \cdot \pi(u) \\ &= [u, \pi(v)] - [v, \pi(u)] = [u, \pi_0(v)] + [\pi_0(u), v]\end{aligned}$$

hence π_0 is a derivation of $L(X)$. Let π_1 be the endomorphism of the module $L(X)$ which coincides on $L^n(X)$ with multiplication by the integer n. The formula $[L^n(X), L^m(X)] \subseteq L^{m+n}(X)$ shows that π_1 is a derivation. The derivation $\pi_1 - \pi_0$ of $L(X)$ is zero on X and, as X generates $L(X)$, $\pi_0 = \pi_1$, whence (b). □

Remark 2.3.8. We have also a more straightforward proof for assertion (b) of Proposition 2.3.7, using inductions: Note that $\pi(x) = x$ for $x \in X$; moreover, if π coincides with π_0 on $L^n(X)$, then for $u \in L^1(X)$ and $v \in L^n(X)$,

$$\pi([u,v]) = [u, \pi(v)] + [\pi(u), v] = n[u,v] + [u,v] = (n+1)[u,v]$$

which shows that $\pi = \pi_0$ on $L^{n+1}(X)$ since $L^{n+1}(X) = [L^1(X), L^n(X)]$.

Corollary 2.3.9. *Suppose that $\mathbb{K}$ is a $\mathbb{Q}$-algebra. Let P be the linear map of $A^+(X)$ into itself such that*

$$P(x_1 \cdots x_n) = \frac{1}{n}(\mathrm{ad}(x_1) \circ \cdots \circ \mathrm{ad}(x_{n-1}))(x_n)$$

for $x_1, \ldots, x_n$ in X. Then P is a projection of $A^+(X)$ onto $L(X)$.

Proof. The image of P is contained in $L(X)$. Further, for all $n \geq 1$ and all u in $L^n(X)$, $P(u) = u$ by Proposition 2.3.7. As $L(X) = \bigoplus_{n \geq 1} L^n(X)$, we see that the restriction of P to $L(X)$ is the identity. □

Remark 2.3.10. Suppose that $\mathbb{K}$ is a field of characteristic zero and let Q be the projection of $A(X) = U(L(X))$ onto $L(X)$ associated with the canonical graduation of $U(L(X))$. For $\alpha \in \mathbb{N}^{\oplus X}$, $Q(A^\alpha(X)) \subseteq L^\alpha(X)$: It suffices to verify that the image and the kernel of Q are graded submodules of $A(X)$ with the graduation of type $\mathbb{N}^{\oplus X}$. This is obvious for the image, which is equal to $L(X)$. On the other hand, let $n \geq 1$ be an integer. The vector subspace of $A(X)$ generated by the y^n, where $y \in L(X)$, is equal to the vector subspace of $A(X)$ generated by the $\sum_{\sigma \in \mathfrak{S}_n} y_{\sigma(1)} \cdots y_{\sigma(n)}$, where $y_1, \ldots, y_n$ are homogeneous elements of $L(X)$; then this subspace is a graded submodule of $A(X)$.

Note that, if $|X| \geq 2$, the projectors P and Q do not coincide on $A^+(X)$. For let x, y be in X with $x \neq y$ and write

$$z = x[x,y] + [x,y]x = x^2y - yx^2.$$

Then $Q(z) = 0$ and $P(z) = \frac{1}{3}[x, [x,y]] \neq 0$.

2.3.3 Dimension of the homogeneous components

Let X be a set, α an element of $N^{\oplus X}$ and $d > 0$ an integer. We write $d \mid \alpha$ if there exists $\beta \in \mathbb{N}^{\oplus X}$ such that $\alpha = d\beta$. The element β which is unique, is then denoted by α/d.

Lemma 2.3.11. *Let $n > 0$ be an integer, $T_1, \dots, T_n$ indeterminates and $u_1, \dots, u_n$ elements of $\mathbb{Z}$. Let $(c(\alpha))_{\alpha \in \mathbb{N}^n \setminus \{0\}}$ be a family of elements of $\mathbb{Z}$ such that*

$$1 - \sum_{i=1}^{n} u_i T_i = \prod_{\alpha \neq 0} (1 - T^\alpha)^{c(\alpha)} \tag{2.3.5}$$

Then for $\alpha \in \mathbb{N}^n \setminus \{0\}$,

$$c(\alpha) = \frac{1}{|\alpha|} \sum_{d|\alpha} \mu(d) \frac{(|\alpha|/d)!}{(\alpha/d)!} u^{\alpha/d} \tag{2.3.6}$$

where μ is the Möbius function.

Proof. Formula (2.3.5) is equivalent, on taking logarithms on both sides

$$\log\left(1 - \sum_{i=1}^{n} u_i T_i\right) = \sum_{\alpha \neq 0} c(\alpha) \log(1 - T^\alpha).$$

Now

$$-\log\left(1 - \sum_{i=1}^{n} u_i T_i\right) = \sum_{j \neq 1} \frac{1}{j} \left(\sum_{i=1}^{n} u_i T_i\right)^j = \sum_{j \neq 1} \frac{1}{j} \sum_{|\beta| = j} \frac{|\beta|!}{\beta!} u^\beta T^\beta = \sum_{|\beta| > 0} \frac{1}{|\beta|} \frac{|\beta|!}{\beta!} u^\beta T^\beta.$$

On the other hand,

$$-\sum_{\alpha \neq 0} c(\alpha) \log(1 - T^\alpha) = \sum_{|\alpha| > 0, k \geq 1} \frac{1}{k} c(\alpha) T^{k\alpha} = \sum_{|\beta| > 0, k|\beta} \frac{1}{k} c(\beta/k) T^\beta.$$

Hence (2.3.5) is equivalent to

$$\sum_{k|\beta} \left|\frac{\beta}{k}\right| c\left(\frac{\beta}{k}\right) = \frac{|\beta|!}{\beta!} u^\beta \quad \text{for} \quad \beta \in \mathbb{N}^n \setminus \{0\}. \tag{2.3.7}$$

Let Λ be the set of $(\lambda_1, \dots, \lambda_n) \in \mathbb{N}^n \setminus \{0\}$ such that the g.c.d. of $\lambda_1, \dots, \lambda_n$ is equal to 1. Every element of $\mathbb{N}^n \setminus \{0\}$ can be written uniquely in the form $m\lambda$, where $m \neq 1$ is an integer and $\lambda \in \Lambda$. Condition (2.3.7) is then equivalent to

$$\sum_{k|m} \left|\frac{m\lambda}{k}\right| c\left(\frac{m\lambda}{k}\right) = \frac{(m|\lambda|)!}{(m\lambda)!} u^{m\lambda} \quad \text{for} \quad \lambda \in \Lambda, m \geq 1. \tag{2.3.8}$$

By the Mobius inversion formula, condition (2.3.8) is equivalent to

$$|m\lambda| c(m\lambda) = \sum_{d|m} \mu(d) \frac{\left|\frac{m\lambda}{d}\right|!}{\left(\frac{m\lambda}{d}\right)!} u^{\frac{m\lambda}{d}}$$

for all $\lambda \in \Lambda$ and $m \geq 1$. This proves the lemma. □

Theorem 2.3.12. *Let X be a finite set of n elements.*

(a) For every $r \geq 1$, the $\mathbb{K}$-module $L^r(X)$ is free of rank

$$c(r) = \frac{1}{r} \sum_{d|r} \mu(d) n^{r/d}, \tag{2.3.9}$$

where μ is the Möbius function.

(b) For all $\alpha \in \mathbb{N}^{\oplus X} \setminus \{0\}$, the $\mathbb{K}$-module $L^\alpha(X)$ is free of rank

$$c(\alpha) = \frac{1}{|\alpha|} \sum_{d|\alpha} \mu(d) \frac{(|\alpha|/d)!}{(\alpha/d)!}. \tag{2.3.10}$$

Proof. We already know that the modules $L^r(X)$, where $r \in \mathbb{N}$, and $L^\alpha(X)$, where $\alpha \in \mathbb{N}^{\oplus X}$, are free. Consider the multigraduation $(A^\alpha(X))_{\alpha \in \mathbb{N}^{\oplus X}}$ of $A(X)$ defined by the canonical homomorphism ϕ of $\Delta(X)$ into $\mathbb{N}^{\oplus X}$; then $A^\alpha(X) \cap L(X) = L^\alpha(X)$. For $\alpha \in \mathbb{N}^{\oplus X}$, the $\mathbb{K}$-module $A^\alpha(X)$ admits as basis the set of words in which each letter x of X appears α_x times. Let $d(\alpha)$ be the number of these words, that is, the rank of $A^\alpha(X)$. We shall calculate in two different ways the formal power series $P((T_x)_{x\in X}) \in \mathbb{Z}[\![(T_x)_{x\in X}]\!]$ defined by

$$P(T) = \sum_{\alpha \in \mathbb{N}^{\oplus X}} d(\alpha) T^\alpha. \tag{2.3.11}$$

First, we have

$$P(T) = \sum_{m \in \Delta(X)} T^{\phi(m)} = \sum_{r=0}^{\infty} \sum_{x_1,\ldots,x_r} T_{x_1} \cdots T_{x_r} = \sum_{r=0}^{\infty} \Big(\sum_{x \in X} T_x \Big)^r$$

whence

$$P(T) = \Big(1 - \sum_{x\in X} T_x\Big)^{-1}. \tag{2.3.12}$$

On the other hand, for all $\alpha \in \mathbb{N}^{\oplus X} \setminus \{0\}$, let $(e_{\alpha,j})_{1\le j\le c(\alpha)}$ be a basis of $L^\alpha(X)$ and give the set I of ordered pairs (α, j) such that $\alpha \in \mathbb{N}^{\oplus X} \setminus \{0\}$ and $1 \le j \le c(\alpha)$ a total ordering. By Theorem 2.3.1 and the Poincaré-Birkhoff-Witt Theorem. the elements

$$y_{\boldsymbol{m}} = \prod_{(\alpha,j)\in I} (e_{\alpha,j})^{\boldsymbol{m}(\alpha,j)}$$

where the index $\boldsymbol{m}$ runs through $\mathbb{N}^{\oplus I}$, form a basis for $A(X)$. Each y_m is multidegree $u(\boldsymbol{m}) := \sum_{(\alpha,j)\in I} \boldsymbol{m}(\alpha,j)\alpha$. It follows that

$$P(T) = \sum_{\boldsymbol{m}\in\mathbb{N}^{\oplus I}} T^{u(\boldsymbol{m})} = \sum_{\boldsymbol{m}\in\mathbb{N}^{\oplus I}} \prod_{(\alpha,j)\in I} T^{\boldsymbol{m}(\alpha,j)\alpha} = \prod_{(\alpha,j)\in I} \sum_{r=0}^{\infty} T^{r\alpha} = \prod_{(\alpha,j)\in I} (1 - T^\alpha)^{-1},$$

so finally

$$P(T) = \prod_{\alpha\in\mathbb{N}^{\oplus X}\setminus\{0\}} (1 - T^\alpha)^{-c(\alpha)}. \tag{2.3.13}$$

Comparing (2.3.12) and (2.3.13), we obtain

$$1 - \sum_{x\in X} T_x = \prod_{\alpha\in\mathbb{N}^{\oplus X}\setminus\{0\}} (1 - T^\alpha)^{c(\alpha)}. \tag{2.3.14}$$

Lemma 2.3.11 then gives (b).

If we now substitute the same indeterminate U for the T_x for $x \in X$ in formula (2.3.14), we obtain

$$1 - nU = \prod_{\alpha\in\mathbb{N}^{\oplus X}\setminus\{0\}} (1 - U^{|\alpha|})^{c(\alpha)} = \prod_{r>0} (1 - U^r)^{c(r)}$$

Applying Lemma 2.3.11 again, we deduce (a). $\square$

2.4 Central filtrations

2.4.1 Real filtrations and order functions

Let G be a group. A **real filtration** on G is a family $(G_\alpha)_{\alpha\in\mathbb{R}}$ of subgroups of G such that

$$G_\alpha = \bigcap_{\beta<\alpha} G_\beta \quad \text{for} \quad \alpha \in \mathbb{R}. \tag{2.4.1}$$

Note that this implies $G_\alpha \subseteq G_\beta$ for $\beta < \alpha$ and hence the family (G_α) is decreasing. The filtration (G_α) is called separated if $\bigcap_\alpha G_\alpha$ reduces to the identity element and is called exhaustive if $G = \bigcup_\alpha G_\alpha$.

Example 2.4.1. Let $(G_n)_{n\in\mathbb{Z}}$ be a decreasing sequence of subgroups of G. It is a decreasing filtration in the usual sense. For every integer n and all α in the interval $(n-1, n]$ of $\mathbb{R}$, we write $H_\alpha = G_n$, in particular $H_n = G_n$. It is immediate that we thus obtain a real filtration $(H_\alpha)_{\alpha\in\mathbb{R}}$ on G; such a filtration will be called an **integral filtration**. Hence decreasing filtrations can be identified with integral filtrations.

Let A be an algebra; a real filtration (A_α) on the additive group of A is called **compatible with the algebra structure** if $A_\alpha \cdot A_\beta \subseteq A_{\alpha+\beta}$ for α, β in $\mathbb{R}$ and $\mathbb{K} \cdot A_\alpha \subseteq A_\alpha$ for $\alpha \in \mathbb{R}$. If the filtration is exhaustive, (A_α) is a fundamental system of neighbourhoods of 0 under the topology on A which is compatible with the algebra structure. Let B be a unital algebra; a real filtration (B_α) on the additive group of B is called **compatible with the unital algebra structure** if it is compatible with the algebra structure and $1 \in B_0$.

Let G be a group with identity element e. Let (G_α) be a real filtration on G. For all x in G let I_x denote the set of real numbers α such that $x \in G_\alpha$. If $\alpha \in I_x$ and $\beta < \alpha$, then $\beta \in I_x$ and hence I_x is an interval. Using relation (2.4.1), we see that I_x contains its least upper bound when this is finite. Therefore I_x is of the form $(-\infty, v(x)] \cap \mathbb{R}$ with $v(x) \in \overline{\mathbb{R}}$; we have $v(x) = \sup\{\alpha : x \in G_\alpha\}$.

The map v of G into $\overline{\mathbb{R}}$ is called the **order function** associated with the real filtration (G_α) and $v(x)$ is called the **order** of x. This map has the following properties:

(a) For $x \in G$ and $\alpha \in \mathbb{R}$, the relations $x \in G_\alpha$ and $v(x) \geq \alpha$ are equivalent.

(b) For x, y in G,

$$v(e) = +\infty, \quad v(x^{-1}) = v(x), \quad v(xy) \geq \inf\{v(x), v(y)\}. \tag{2.4.2}$$

and we have the equality $v(xy) = \inf\{v(x), v(y)\}$ if $v(x) > v(y)$.

(c) For all $\alpha \in \mathbb{R}$, let G_α^+ denote the set of $x \in G$ such that $v(x) > \alpha$. Then $G_\alpha^+ = \bigcup_{\beta>\alpha} G_\beta$ and in particular G_α^+ is a subgroup of G.

Conversely, let v be a map of G into $\overline{\mathbb{R}}$ satisfying relations (2.4.2). For all $\alpha \in \mathbb{R}$, let G_α be the set of $x \in G$ such that $v(x) \geq \alpha$. Then $(G_\alpha)_{\alpha\in\mathbb{R}}$ is a real filtration of G and v is the order function associated with this filtration. For the filtration (G_α) to be integral, it is necessary and sufficient that v map G into $\mathbb{Z} \cup \{\pm\infty\}$. For it to be exhaustive (resp. separated), it is necessary and sufficient that $v^{-1}(-\infty) = 0$ (resp. $v(+\infty) = \{e\}$).

Let A be a $\mathbb{K}$-algebra (resp. unital $\mathbb{K}$-algebra). By the above, the relation

$$x \in A_\alpha \Leftrightarrow v(x) \geq \alpha \quad \text{for} \quad x \in A, \alpha \in \mathbb{R}$$

defines a bijection of the set of exhaustive real filtrations $(A_\alpha)_{\alpha\in\mathbb{R}}$ compatible with the algebra (resp. unital algebra) structure on A onto the set of maps $v : A \to \overline{\mathbb{R}}$ not taking the value $-\infty$ and satisfying axioms (2.4.3) to (2.4.6) (resp. (2.4.4) to (2.4.7)) below: ($x, y \in A$ and $\lambda \in \mathbb{K}$)

$$v(x+y) \geq \inf\{v(x), v(y)\} \tag{2.4.3}$$
$$v(-x) = v(x) \tag{2.4.4}$$
$$v(\lambda x) \geq v(x) \tag{2.4.5}$$
$$v(xy) \geq v(x) + v(y) \tag{2.4.6}$$
$$v(1) \geq 0 \tag{2.4.7}$$

Let G be a commutative group with a real filtration $(G_\alpha)_{\alpha\in\mathbb{R}}$. As before we write

$$G_\alpha^+ = \bigcup_{\beta>\alpha} G_\alpha.$$

Clearly G_α^+ is a subgroup of G_α. We write $\mathrm{gr}_\alpha(G) = G_\alpha = G_\alpha^+$ and

$$\mathrm{gr}(G) = \bigoplus_{\alpha\in\mathbb{R}} \mathrm{gr}_\alpha(G).$$

The graded group associated with the filtered group G is the group $\mathrm{gr}(G)$ with its natural graduation of type $\mathbb{R}$.

Example 2.4.2. When the filtration (G_α) is integral, $\mathrm{gr}_\alpha(G) = \{0\}$ for non-integral α and $\mathrm{gr}_n(G) = G_n/G_{n-1}$ for every integer n. The definition of the associated graded group therefore coincides essentially with the usual one.

Let A be an algebra (resp. unital algebra) and $(A_\alpha)_{\alpha\in\mathbb{R}}$ a real filtration compatible with the algebra (resp. unital algebra) structure. Then

$$A_\alpha \cdot A_\beta \subseteq A_{\alpha+\beta}, \quad A_\alpha^+ \cdot A_\beta + A_\alpha \cdot A_\beta^+ \subseteq A_{\alpha+\beta}^+,$$

and the bilinear map of $A_\alpha \times A_\beta$ into $A_{\alpha+\beta}$ the restriction of multiplication on A defines on taking quotients a bilinear map

$$\mathrm{gr}_\alpha(A) \times \mathrm{gr}_\beta(A) \to \mathrm{gr}_{\alpha+\beta}(A).$$

We derive a bilinear map of $\mathrm{gr}(A) \times \mathrm{gr}(A)$ into $\mathrm{gr}(A)$ which makes it a graded algebra (resp. unital graded algebra) of type $\mathbb{R}$. If A is an associative (resp. commutative, resp. Lie) algebra, so is $\mathrm{gr}(A)$.

2.4.2 Central filtrations on a group

Let G be a group. A real filtration $(G_\alpha)_{\alpha\in\mathbb{R}}$ on G is called **central** if $G = \bigcup_{\alpha>0} G_\alpha$ and the commutator $[x,y] = x^{-1}y^{-1}xy$ of an element x of G_α and an element y of G_β belongs to $G_{\alpha+\beta}$. In terms of the order function v, the above definition translates into the relations

$$v(x) > 0, \quad v([x,y]) \geq v(x) + v(y) \quad \text{for} \quad x, y \in G. \tag{2.4.8}$$

We deduce that $v([x,y]) > v(x)$ if $v(x) \neq +\infty$; if we write $x^y = y^{-1}xy$, then $x^y = x \cdot [x,y]$, whence

$$v(x^y) = v(x).$$

This relation expresses the fact that each of the subgroups G_α of G is normal. The G_α form a fundamental system of neighbourhoods of e for a topology compatible with the group structure on G said to be defined by the filtration (G_α).

In the rest of our discussion, G denotes a group with a *central filtration* (G_α). For all $\alpha \in \mathbb{R}$, we define the subgroup G_α^+ of G by

$$G_\alpha^+ = \bigcup_{\beta>\alpha} G_\alpha.$$

In particular $G_\alpha^+ = G_\alpha = G$ for $\alpha \leq 0$. Recall that if A and B are two subgroups of G, $[A,B]$ denotes the subgroup of G generated by the commutators $[a,b]$ with $a \in A$ and $b \in B$. With this notation we have the formulae

$$[G_\alpha, G_\beta] \subseteq G_{\alpha+\beta} \tag{2.4.9}$$

$$[G_\alpha^+, G_\beta] \subseteq G_{\alpha+\beta}^+ \tag{2.4.10}$$

$$[G, G_\alpha] \subseteq G_\alpha^+. \tag{2.4.11}$$

By (2.4.11), G_α^+ is a normal subgroup of G_α for all $\alpha \in \mathbb{R}$ and the quotient group $\mathrm{gr}_\alpha(G) = G_\alpha/G_\alpha^+$ is commutative. We write $\mathrm{gr}(G) = \mathrm{gr}_\alpha(G)$ and give this group the graduation of type $\mathbb{R}$ in which $\mathrm{gr}_\alpha(G)$ consists of elements of degree α. Then $\mathrm{gr}_\alpha(G) = \{0\}$ for $\alpha \leq 0$.

Proposition 2.4.3. *Let G be a group with a central filtration (G_α).*

(a) Let α, β be in $\mathbb{R}$. There exists a biadditive map

$$\phi_{\alpha\beta} : \mathrm{gr}_\alpha(G) \times \mathrm{gr}_\beta(G) \to \mathrm{gr}_{\alpha+\beta}(G)$$

which maps $(xG_\alpha^+, yG_\beta^+)$ onto $[x,y]G_{\alpha+\beta}^+$.

(b) Let ϕ be the biadditive map of $\mathrm{gr}(G) \times \mathrm{gr}(G)$ into $\mathrm{gr}(G)$ whose restriction to $\mathrm{gr}_\alpha(G) \times \mathrm{gr}_\beta(G)$ is $\phi_{\alpha\beta}$ for every ordered pair (α, β). Then the map ϕ gives $\mathrm{gr}(G)$ a Lie $\mathbb{Z}$-algebra structure, called the ***graded Lie algebra associated with the filtered group G****.*

Proof. Recall that identity

$$[xy, z] = [x,z]^y[y,z] \tag{2.4.12}$$

for x, y, z in G. For $x \in G_\alpha$ and $y \in G_\beta$ the class modulo $G_{\alpha+\beta}^+$ of the element $[x,y]$ of $G_{\alpha+\beta}$ will be denoted by $f(x,y)$. For $a \in G_{\alpha+\beta}$ and $z \in G$, $a^{-1} \cdot a^z = [a,z] \in G_{\alpha+\beta}^+$; in particular $f(x,y)$ is equal to the class modulo $G_{\alpha+\beta}^+$ of $[x,y]^z$. Formula (2.4.12) therefore implies

$$f(xy, z) = f(x,z)f(y,z). \tag{2.4.13}$$

Now $[y,x] = [x,y]^{-1}$, whence

$$f(y,x) = f(x,y)^{-1}. \tag{2.4.14}$$

Formula (2.4.13) and (2.4.14) we deduce

$$f(x, yz) = f(x,y)f(x,z). \tag{2.4.15}$$

We have to prove that the map $f : G_\alpha \times G_\beta \to \mathrm{gr}_{\alpha+\beta}(G)$ defines on taking quotients a map $\phi_{\alpha\beta}$: $\mathrm{gr}_\alpha(G) \times \mathrm{gr}_\beta(G) \to \mathrm{gr}_{\alpha+\beta}(G)$. By (2.4.13) and (2.4.15) it suffices to prove that $f(x,y) = 0$ if $x \in G_\alpha^+$ or $y \in G_\beta^+$, which follows from (2.4.10).

As $[x,x] = e$, it follows from (2.4.14) that ϕ is an alternating $\mathbb{Z}$-bilinear map. Hence it remains to prove that, for $u \in \mathrm{gr}_\alpha(G)$, $v \in \mathrm{gr}_\beta(G)$ and $w \in \mathrm{gr}_\gamma(G)$,

$$\phi(u, \phi(v,w)) + \phi(v, \phi(w,u)) + \phi(w, \phi(u,v)) = 0. \tag{2.4.16}$$

Let $x \in G_\alpha$, $y \in G_\beta$, and $z \in G_\gamma$ be elements representing respectively u, v and w. We know that x^y and x are two elements of G_α^+ which are congruent modulo G; and hence x^y is a representative of u in G_α; as $[y,z]$ is a representative of $\phi(v,w)$ in $G_{\beta+\gamma}$, we see that $[x^y, [y,z]]$ is a representative of $\phi(u, \phi(v,w))$ in $G_{\alpha+\beta+\gamma}$. By cyclic permutation, we see that $[y^z, [z,x]]$ and $[z^x, [x,y]]$ represent respectively $\phi(v, \phi(w,u))$ and $\phi(w, \phi(u,v))$ in $G_{\alpha+\beta+\gamma}$. Relation (2.4.16) is then a consequence of the following identity

$$[x^y, [y,z]] \cdot [y^z, [z,x]] \cdot [z^x, [x,y]] = e.$$

□

Now we give an example of central filtrations. Let A be a unital associative algebra with a unital algebra filtration (A_α) such that $A_0 = A$; then A_α is a two-sided ideal of A for all $\alpha \in \mathbb{R}$. Let A^* denote the multiplicative group of invertible elements of A. For all $\alpha > 0$, let Γ_α denote the set of $x \in A^*$ such that $x - 1 \in A_\alpha$; we write $\Gamma = \bigcup_{\alpha>0} \Gamma_\alpha$ and $\Gamma_\beta = \Gamma$ for $\beta \leq 0$.

Proposition 2.4.4. *The set Γ is a subgroup of A^* and (Γ_α) is a central filtration on Γ.*

Proof. We have $\Gamma = \bigcup_{\alpha>0} \Gamma_\alpha$ by construction and the realtion $\Gamma_\alpha = \bigcap_{\beta<\alpha} \Gamma_\beta$ follows from $A_\alpha = \bigcap_{\beta<\alpha} A_\beta$. We show that Γ_α is a subgroup of A^*. Now $1 \in \Gamma_\alpha$; let x, y be in Γ_α, whence $x - 1 \in A_\alpha$, $y - 1 \in A_\alpha$. As A_α is a two-sided ideal of A, the formulae

$$xy - 1 = (x-1)(y-1) + (x-1) + (y-1) \tag{2.4.17}$$
$$x^{-1} - 1 = -x^{-1}(x-1) \tag{2.4.18}$$

imply that $xy - 1 \in A_\alpha$ and $x^{-1} - 1 \in A_\alpha$, whence $xy \in \Gamma_\alpha$ and $x^{-1} \in \Gamma_\alpha$. As $\Gamma = \bigcup_{\alpha>0} \Gamma_\alpha$, this is a subgroup of A^*.

Finally, let $\alpha > 0$, $\beta > 0$, $x \in \Gamma_\alpha$ and $y \in \Gamma_\beta$. Let $\xi = x - 1$ and $\eta = y - 1$. Then

$$[x, y] - 1 = x^{-1}y^{-1}(\xi\eta - \eta\xi) \tag{2.4.19}$$

by hypothesis, $\xi \in A_\alpha$ and $\eta \in A_\beta$, whence $\xi\eta - \eta\xi \in A_{\alpha+\beta}$. As $A_{\alpha+\beta}$ is a two-sided ideal of A, $[x, y] - 1 \in A_{\alpha+\beta}$, whence $[x, y] \in \Gamma_{\alpha+\beta}$. □

Remark 2.4.5. Let $\alpha, \beta \geq 0$. By formulae (2.4.17), (2.4.18) and (2.4.19),

$$x^{-1} - 1 \equiv -(x - 1) \mod A_{2\alpha} \tag{2.4.20}$$
$$xy - 1 \equiv (x - 1) + (y - 1) \mod A_{\alpha+\beta} \tag{2.4.21}$$
$$[x, y] - 1 \equiv [(x - 1), (y - 1)] \mod A_{\alpha+\beta+\inf\{\alpha,\beta\}}. \tag{2.4.22}$$

We prove for example (2.4.22). If $x - 1 = \xi$ and $y - 1 = \eta$, (2.4.19) gives:

$$[x, y] - 1 - [\xi, \eta] = \big((x^{-1} - 1) + (y^{-1} - 1) + (x^{-1} - 1)(y^{-1} - 1)\big)[\xi, \eta]$$

Now $[\xi, \eta] \in A_{\alpha+\beta}$, $x^{-1} - 1 \in A_\alpha$, $y^{-1} - 1 \in A_\beta$, whence we obtain (2.4.22).

Let G be a group and $\rho : G \to \Gamma$ a homomorphism. For all real α we write $G_\alpha = \rho^{-1}(\Gamma_\alpha)$. As (Γ_α) is a central filtration on Γ, it is immediate that (G_α) is a central filtration on G.

Proposition 2.4.6. *Let G be endowed with the central filtration induced by ρ.*

(a) For all $\alpha \in \mathbb{R}$, there exist a unique group homomorphism $\phi_\alpha : \mathrm{gr}_\alpha(G) \to \mathrm{gr}_\alpha(A)$ which maps the class modulo G^+_α of an element $a \in G_\alpha$ to the class modulo A^+_α of $\rho(a) - 1$.

(b) Let ϕ be the group homomorphism of $\mathrm{gr}(G)$ into $\mathrm{gr}(A)$ whose restriction to $\mathrm{gr}_\alpha(G)$ is ϕ_α. Then ϕ is an injective homomorphism of $\mathbb{Z}$-Lie algebras.

Proof. Let $\alpha > 0$. By hypothesis, for all a in G_α, $\rho(a) - 1 \in A_\alpha$; let $p_\alpha(a)$ denote the class of $\rho(a) - 1$ modulo A^+_α. As $A_{2\alpha} \subseteq A^+_\alpha$, relation (2.4.21) implies $p_\alpha(ab) = p_\alpha(a) + p_\alpha(b)$. Then $a \in G^+_\alpha$ if and only if $\rho(a) - 1 \in A^+_\alpha$; therefore G^+_α is the kernel of the homomorphism p_α of G_α into $\mathrm{gr}_\alpha(A)$. On passing to the quotient, p_α then defines an injective homomorphism ϕ_α of $\mathrm{gr}_\alpha(G)$ into $\mathrm{gr}_\alpha(A)$. For $\alpha \leq 0$, $\mathrm{gr}_\alpha(G) = \{0\}$ and the only choice is $\phi_\alpha = 0$.

As ϕ_α is injective for all α, ϕ is injective. We show that ϕ is a Lie algebra homomorphism. As $\mathrm{gr}_\alpha(G) = \{0\}$ for $\alpha \leq 0$, it suffices to establish the formula

$$p_{\alpha+\beta}([a, b]) = [p_\alpha(a), p_\beta(b)]$$

for $\alpha > 0$, $\beta > 0$, $a \in G_\alpha$ and $b \in G_\beta$. which follows from (2.4.22). □

2.4.3 Integral central filtrations

Recall that a filtration (G_α) on the group G is called integral if $G_\alpha = G_n$ for every integer n and all $\alpha \in (n - 1, n]$. To be given an integral central filtration on a group G is equivalent to being given a sequence $(G_n)_{n\geq 1}$ of subgroups of G satisfying the conditions

$$G_1 = G$$
$$G_n \supseteq G_{n+1} \quad \text{for} \quad n \geq 1$$
$$[G_m, G_n] \subseteq G_{m+n} \quad \text{for} \quad m, n \geq 1$$

For every integer $n \geq 1$, G_n is a normal subgroup of G and the quotient $\mathrm{gr}_n(G) = G_n/G_{n+1}$ is commutative. On taking quotients, the map $(x, y) \mapsto [x, y] = x^{-1}y^{-1}xy$ of $G_m \times G_n$ into G_{m+n} allows us to define on $\mathrm{gr}(G) = \bigoplus_{n\geq 1} \mathrm{gr}_n(G)$ a graded Lie algebra structure of type $\mathbb{N}$ over the ring $\mathbb{Z}$.

Example 2.4.7. For a group G, recall that the lower central series of G is defined by

$$C^1(G) = G, \quad C^{n+1}(G) = [G, C^n(G)] \quad \text{for} \quad n \geq 1.$$

The corresponding filtration is called the **lower central filtration** of G.

Example 2.4.8. Let $\mathbb{K}$ be a ring, $n > 0$ an integer and A the set of upper triangular matrices with n rows and n columns and elements in $\mathbb{K}$. For $p \geq 0$, let

$$A_p = \{(x_{ij}) \in A : x_{ij} = 0 \text{ for } i - j > p\}.$$

Then $A_0 = A$ and $A_p A_q \subseteq A_{p+q}$. Let $\Gamma_p = 1 + A_p$. Then Γ_1 is a subgroup of $\mathrm{GL}(n, \mathbb{K})$ called the **strict upper triangular group** of order n over $\mathbb{K}$. By Proposition 2.4.4, (Γ_p) is an integral filtration on Γ_1. As $\Gamma_n = \{1\}$, we see that the group Γ_1 is nilpotent.

Proposition 2.4.9. *The lower central series of G is an integral central filtration on G. Moreover, if $(G_n)_{n\geq 1}$ is an integral central filtration on G, then $C^n(G) \subseteq G_n$ for all $n \geq 1$.*

Proof. We prove the second claim by induction on n; $C^1(G) = G = G_1$; for $n > 1$,

$$C^n(G) = [G, C^{n-1}(G)] \subseteq [G, G_{n-1}] \subseteq G_n.$$

This completes the proof. □

Proposition 2.4.10. *Let G be a group and $\mathrm{gr}(G)$ the graded Lie $\mathbb{Z}$-algebra associated with the lower central filtration on G. Then $\mathrm{gr}(G)$ is generated by $\mathrm{gr}_1(G) = G/[G, G]$.*

Proof. Let L be the Lie subalgebra of $\mathrm{gr}(G)$ generated by $\mathrm{gr}_1(G)$; we show that $L \supseteq \mathrm{gr}_n(G)$ by induction on n, the assertion being trivial for $n = 1$. Suppose that $n > 1$ and $L \supseteq \mathrm{gr}_{n-1}(G)$. As $C^n(G) = [G, C^{n-1}(G)]$, the construction of the Lie algebra law on $\mathrm{gr}(G)$ shows immediately that

$$\mathrm{gr}_n(G) = [\mathrm{gr}_1(G), \mathrm{gr}_{n-1}(G)] \subseteq L.$$

Thus the claim follows. □

The above proof shows that the lower central series of the Lie algebra $\mathrm{gr}(G)$ is given by

$$C^n(\mathrm{gr}(G)) = \sum_{m\geq n} \mathrm{gr}_m(G).$$

2.4.4 Magnus algebras

In this section and the following, X denotes a set, $F(X)$ the free group constructed on X and $A(X)$ the free associative algebra constructed on X with its total graduation $(A_n(X))_{n\geq 0}$. The set X is identified with its images in $F(X)$ and $A(X)$.

Let $\widehat{A}(X)$ be the product module $\prod_{n\geq 0} A^n(X)$. We define on $\widehat{A}(X)$ a multiplication by the rule

$$(a \cdot b)_n = \sum_{i=0}^{n} a_i \cdot b_{n-i}$$

where $a = (a_n)$ and $b = (b_n)$ are in $\widehat{A}(X)$. We know that $\widehat{A}(X)$ is an associated algebra and that $A(X)$ is identified with the subalgebra of $\widehat{A}(X)$ consisting of the sequence all of whose terms are zero except for a finite number.

The algebra $\widehat{A}(X)$ is given the product topology of the discrete topologies on the factors $A^n(X)$; this topology makes $A(X)$ into a complete Hausdorff topological algebra, when the ring $\mathbb{K}$ has the discrete topology, and $A(X)$ is dense in $A(X)$. Let $a = (a_n) \in \widehat{A}(X)$; the family $(a_n)_{n\geq 0}$ is summable and $a = \sum_{n\geq 0} a_n$.

For every integer $m \geq 0$, let $\widehat{A}_m(X)$ denote the ideal consisting of the series $a = \sum_{n\geq m} a_n$ such that $a_n \in A^n(X)$ for all $n \geq m$. This sequence of ideals is a fundamental system of neighbourhoods of 0 in $A(X)$ and an integral filtration on $A(X)$. The order function associated with the above filtration is denoted by ω; then $\omega(0) = +\infty$ and $\omega(a) = m$ if $a = \sum_{n\geq m} a_n$ with $a_n \in A_n(X)$ for all $n \geq m$ and $a_m \neq 0$.

The algebra $\widehat{A}(X)$ is called the Magnus algebra of the set X with coefficients in $\mathbb{K}$. If there is any ambiguity over $\mathbb{K}$ we write $\widehat{A}_{\mathbb{K}}(X)$.

Proposition 2.4.11. *Let B be a unital associative algebra with a real filtration $(B_\alpha)_{\alpha\in\mathbb{R}}$ such that B is Hausdorff and complete. Let $f : X \to B$ be a mapp such that there exists $\lambda > 0$ for which $f(X) \subseteq B_\lambda$. Then f can be extended in a unique way to a continuous unital homomorphism $\hat{f}$ of $\widehat{A}(X)$ into B.*

Proof. Let $\tilde{f}$ be the unique unital algebra homomorphism of $A(X)$ into B extending f. We show that $\tilde{f}$ is continuous: $\tilde{f}(A^n(X)) \subseteq B_{n\lambda}$ whence $\tilde{f}(\widehat{A}_n(X) \cap A(X)) \subseteq B_{n\lambda}$. Therefore $\tilde{f}$ can be extended in a unique wat by continuity to a homomorphism $\hat{f} : \widehat{A}(X) \to B$. $\square$

We preserve the hypotheses and notation of Proposition 2.4.11 and let $u \in \widehat{A}(X)$. The element $\hat{f}(u)$ is denoted by $u(\{f(x)\}_{x\in X})$ and called the **result of substituting the $\boldsymbol{f(x)}$ for the $\boldsymbol{x}$ in $\boldsymbol{u}$**. In particular, $u(\{x\}_{x\in X}) = u$. Now let $\boldsymbol{u} = (u_y)_{y\in Y}$ be a family of elements of $\widehat{A}_1(X)$ and let $u \in A(Y)$. The above allows us to define the element $v(\{u_y\}_{y\in Y}) \in \widehat{A}(X)$. It is denoted by $v \circ \boldsymbol{u}$. As $u_y(\{f(x)\}) \in B_\lambda$, the elements $u_y(\{f(x)\})$ can be substituted for the y in v. The maps $v \mapsto (v \circ \boldsymbol{u})(\{f(x)\})$ and $v \mapsto v(\{u_y(\{f(x)\})\})$ are then two continuous homomorphisms of unital algebras of $\widehat{A}(X)$ into B taking the same value $u_y(\{f(x)\})$ at the element $y \in Y$. Therefore (Proposition 2.4.11)

$$(v \circ \boldsymbol{u})(\{f(x)\}) = v(\{u_y(\{f(x)\})\}) \tag{2.4.23}$$

for all $v \in \widehat{A}(Y)$.

2.4.5 Magnus group

For all $a = (a_n)_{n\geq 0}$ in $\widehat{A}(X)$, the element a_0 of $\mathbb{K}$ will be called the **constant term** of a and denoted by $\varepsilon(a)$. Formula (2.4.23) shows that ε is an algebra homomorphism of $\widehat{A}(X)$ into $\mathbb{K}$.

Lemma 2.4.12. *For an element a of $\widehat{A}(X)$ to be invertible, it is necessary and sufficient that its constant term be invertible in $\mathbb{K}$.*

Proof. If a is invertible in $\widehat{A}(X)$, $\varepsilon(a)$ is invertible in $\mathbb{K}$. Conversely, if $\varepsilon(a)$ is invertible in $\mathbb{K}$, there exists $u \in \widehat{A}_1(X)$ such that $a = \varepsilon(a)(1-u)$; we write $b = \varepsilon(a)^{-1}\sum_{n\geq 0} u^n$. Then $ab = ba = 1$ and a is invertible. $\square$

The set of elements of $\widehat{A}(X)$ of constant term 1 is therefore a subgroup of the multiplicative monoid $\widehat{A}(X)$, called the **Magnus group** constructed on X (relative to $\mathbb{K}$). In this paragraph it will be denoted by $\Gamma(X)$ or simply Γ. For every integer $n \geq 1$, we denote by Γ_n the set of $a \in \Gamma$ such that $\omega(a-1) \geq n$. By Proposition 2.4.4, the sequence $(\Gamma_n)_{n\geq 0}$ is an integral central filtration on Γ.

We now explore the relation between the Magnus group $\Gamma(X)$ and the free group $F(X)$. For this we first prove some lemmas.

Lemma 2.4.13. *Let n be an nonzero rational integer. In the ring $\mathbb{K}[\![t]\!]$ of formal power series we write $(1+t)^n = \sum_{j\geq 0} c_{j,n}t^j$. Then there exists an integer $j \geq 1$ such that $c_{j,n} \neq 0$.*

Proof. If $n > 0$, then $c_{n,n} = 1$ by the binomial formula. Suppose that $n < 0$ and let $m = -n$. If $c_{j,n} = 0$ for all $j \geq 1$, then $(1+t)^n = 1$, whence, taking the inverse, $(1+t)^m = 1$, which contradicts the formula $c_{m,m} = 1$. $\square$

Lemma 2.4.14. *Let $x_1, \ldots, x_s$ be elements of X such that $x_i \neq x_{i+1}$ for $1 \leq i \leq s-1$. Let $n_1, \ldots, n_s$ be nonzero rational integers. Then the element $\prod_{i=1}^s (1+x_i)^{n_i}$ of $\widehat{A}$ is unequal to 1.*

Proof. Let $\mathfrak{m}$ be a maximal ideal of $\mathbb{K}$ and κ the field $\mathbb{K}/\mathfrak{m}$. Let $\pi : \widehat{A}_{\mathbb{K}}(X) \to \widehat{A}_\kappa(X)$ be the unique continuous homomorphism of unital $\mathbb{K}$-algebras such that $\pi(x) = x$ for $x \in X$. It suffices to prove that $\pi(\prod_i (1+x_i)^{n_i}) \neq 1$ and the problem is reduced to the case where $\mathbb{K}$ is a field. In the notation of Lemma 2.4.13:

$$\prod_{i=1}^s (1+x_i)^{n_i} = \sum_{b_i \geq 0} c_{b_1,n_1} \cdots c_{b_s,n_s} x_1^{b_1} \cdots x_s^{b_s}.$$

By Lemma 2.4.13, there exist integers $a_i > 0$ such that $c_{a_i,n_i} \neq 0$ for $1 \leq i \leq s$. Now no monomial $x_1^{b_1} \cdots x_s^{b_s}$ such that $b_i \geq 0$ and $(b_1, \ldots, b_s) \neq (a_1, \ldots, a_s)$ can be equal to $x_1^{a_1} \cdots x_s^{a_s}$. It follows that the coefficient of $x_1^{a_1} \cdots x_s^{a_s}$ in $\prod_{i=1}^s (1+x_i)^{n_i}$ is $c_{a_1,n_1} \cdots c_{a_s,n_s} \neq 0$, which implies the result. $\square$

Lemma 2.4.15. *Let r be a map of X into $\widehat{A}(X)$ such that $\omega(r(x)) \geq 2$ for all $x \in X$. Let σ be the continuous endomorphism of $\widehat{A}(X)$ such that $\sigma(x) = x + r(x)$ for $x \in X$. Then σ is an automorphism and $\sigma(\widehat{A}_m(X)) = \widehat{A}_m(X)$ for all $m \in \mathbb{N}$.*

Proof. We have $\sigma(x) \equiv x \bmod \widehat{A}_2(X)$ for $x \in X$, whence, for $n \geq 1$ and $x_1, \ldots, x_n$ in X,

$$\sigma(x_1) \cdots \sigma(x_n) \equiv x_1 \cdots x_n \mod \widehat{A}_{n+1}(X);$$

it follows by linearity that $\sigma(a) \equiv a$ modulo $\widehat{A}_{n+1}(X)$ for all $a \in A^n(X)$ and in particular $\sigma(A^n(X)) \subseteq \widehat{A}_n(X)$. It follows that $\sigma(A^m(X)) \subseteq \widehat{A}_n(X)$ for $m \geq n$, whence $\sigma(\widehat{A}_n(X)) \subseteq \widehat{A}_n(X)$. In other words, σ is compatible with the filtration $(\widehat{A}_m(X))$ on $A(X)$ and its restriction to the associated graded ring is the identity. Hence σ is bijective. □

Theorem 2.4.16. *Let $r : X \to \widehat{A}(X)$ be a map such that $\omega(r(x)) \geq 2$ for all $x \in X$. Then the unique homomorphism $g : F(X) \to \Gamma(X)$ such that $g(x) = 1 + x + f(x)$ for all $x \in X$ is injective.*

Proof. Let $w \neq 1$ be an element on $F(X)$. By **??**, there exist $x_1, \ldots, x_s$ in X and nonzero integers $n_1, \ldots, n_s$ such that $x_i \neq x_{i+1}$ for $1 \leq i \leq s-1$ and

$$w = x_1^{n_1} \cdots x_s^{n_s}.$$

In the notation of Lemma 2.4.15,

$$g(w) = \prod_i (1 + \sigma(x_i))^{n_i} = \sigma\Big(\prod_i (1 + x_i)^{n_i}\Big)$$

hence $g(w) \neq 1$ by Lemma 2.4.14 and 2.4.15. □

Theorem 2.4.17. *Suppose that in the ring $\mathbb{K}$ the relation $n \cdot 1 = 0$ implies $n = 0$ for every integer n. Let $r : X \to \widehat{A}(X)$ such that $\omega(r(x)) \geq 2$ for $x \in X$ and let η be the homomorphism of $F(X)$ into the Magnus group $\Gamma(X)$ such that $\eta(x) = 1 + x + r(x)$ for $x \in X$.*

(a) For all $n \geq 0$, $C^n(F(X))$ is the inverse image under η the subgroup $1 + \widehat{A}_n(X)$ of $\Gamma(X)$.

(b) Let $c(x)$ be the canonical image of x in $F(X)/[F(X), F(X)]$ for $x \in X$. Let $\mathfrak{g}$ be the graded Lie $\mathbb{Z}$-algebra associated with the filtration $(C^n(F(X)))_{n \geq 1}$ of $F(X)$. Then the unique homomorphism of the free Lie $\mathbb{Z}$-algebra $L_{\mathbb{Z}}(X)$ into $\mathfrak{g}$ which extends c is an isomorphism.

Proof. We write $F(X) = F$, $\Gamma(X) = \Gamma$, $\widehat{A}(X) = \widehat{A}$, $\widehat{A}_{\mathbb{Z}}(X) = \widehat{A}_{\mathbb{Z}}$, $C^n(F(X)) = C^n$, $\Gamma_n = 1 + \widehat{A}_n(X)$ and let $\alpha : L_{\mathbb{Z}}(X) \to \mathfrak{g}$ be the homomorphism introduced in the statement of (b).

Let γ denote the homomorphism of F into Γ defined by $\gamma(x) = 1 + x$ for $x \in X$. By Lemma 2.4.15 there exists an automorphism σ of the algebra A compatible with the filtration on A and such that $\sigma(1 + x) = \eta(x)$ for all $x \in X$; then $\sigma(\Gamma_n) = \Gamma_n$ for all n. As the homomorphisms η and $\sigma \circ \gamma$ of F into Γ coincide on X, $\eta = \sigma \circ \gamma$ and hence $\eta^{-1}(\Gamma_n) = \gamma^{-1}(\Gamma_n)$. Under the hypotheses of $\mathbb{K}$, $\mathbb{Z}$ can be identified with a subring of $\mathbb{K}$; the Magnus algebra $A_{\mathbb{Z}}$ is therefore identified with a subring of A and the filtration on $A_{\mathbb{Z}}$ is induced by that on A. As γ maps F into $A_{\mathbb{Z}}$, we see that it suffices to prove the claims under the supplementary hypotheses $\mathbb{K} = \mathbb{Z}$, $r = 0$ and hence $\eta = \gamma$, hypotheses which we shall henceforth make.

As X generates the group $F = C^1$, the set $c(X)$ generates the $\mathbb{Z}$-module $\mathfrak{g}^1 = C^1/C^2$. But $\mathfrak{g}^1$ generates the Lie $\mathbb{Z}$-algebra $\mathfrak{g}$ (Proposition 2.4.10) and hence $c(X)$ generates $\mathfrak{g}$, which proves that α is surjective. We now identify $\mathrm{gr}(\widehat{A})$ with $A(X)$ under the canonical isomorphisms $A^n(X) \cong \widehat{A}_n/\widehat{A}_{n+1}$. For every integer $n \geq 1$, we write $F^n = \gamma^{-1}(\Gamma_n)$; then $(F^n)_{n \geq 1}$ is an integral central filtration on F. Let $\tilde{\mathfrak{g}}$ denote the associated graded Lie $\mathbb{Z}$-algebra. Let f be the Lie algebra homomorphism of $\tilde{\mathfrak{g}}$ into $A(X)$ associated with γ (Proposition 2.4.6). Now $C^n \subseteq F^n$ for every integer $n \geq 1$ (Proposition 2.4.9) and hence there is a canonical homomorphism ε of $\mathfrak{g} = \bigoplus_n C^n/C^{n+1}$ into $\tilde{\mathfrak{g}} = \bigoplus_n F^n/F^{n+1}$.

$$L_{\mathbb{Z}}(X) \xrightarrow{\alpha} \mathfrak{g} \xrightarrow{\varepsilon} \tilde{\mathfrak{g}} \xrightarrow{f} A(X)$$

We write $\beta = f \circ \varepsilon$; we give β explicitly as follows: if u is the class modulo C^{n+1} of an element w of C^n, then $\gamma(w) - 1$ is of order $\geq n$ in A and $\beta(u)$ is the homogeneous component of $\gamma(w) - 1$ of degree n. In particular,

$$\beta(c(x)) = x \quad \text{for} \quad x \in X. \tag{2.4.24}$$

The Lie algebra homomorphism $\beta \circ \alpha : L_{\mathbb{Z}}(X) \to A(X)$ restricted to X is the identity by (2.4.24) and hence is the canonical injection. Therefore α is injective and hence bijective; this proves (b). As $\beta \circ \alpha = f \circ \varepsilon \circ \alpha$ is injective and α is bijective, ε is injective. For all $n \geq 1$,

$$\varepsilon_n : C^n / C^{n+1} \to F^n / F^{n+1}.$$

Now $C^1 = F = F^1$; moreover, if $C^n = F^n$, then $C^n \cap F^{n+1} = F^{n+1}$ whence $C^{n+1} = F^{n+1}$. This proves (a) by induction on $n \geq 1$. □

Corollary 2.4.18. *Under the hypothesis of Theorem 2.4.17, we have* $\bigcap_n C^n(F(X)) = \{e\}$.

Proof. Apply Theorem 2.4.17 with $\mathbb{K} = \mathbb{Z}$ and $r = 0$, we get

$$\bigcap_n C^n(F(X)) = \bigcap_n \eta^{-1}(1 + \widehat{A}_n(X)) = \eta^{-1}\Big(\bigcap_n (A + \widehat{A}_n(X))\Big) = \eta^{-1}(1) = \{e\}.$$

This proves the claim. □

Let H be a Hall set relative to X. Let M be the magma defined by the law of composition $(x, y) \mapsto [x, y] = x^{-1}y^{-1}xy$ on $F(X)$ and let ϕ be the homomorphism of $M(X)$ into M whose restriction to X is the identity. The elements of $\phi(H)$ are called the **basic commutators** of $F(X)$ associated with the Hall set H. For every integer $n \geq 1$, let H_n be the subset of H consisting of the elements of length n; we know that the canonical map of H_n into $L_{\mathbb{Z}}(X)$ is a basis of the Abelian group $L^n_{\mathbb{Z}}(X)$. Moreover, $\phi(H_n) \subseteq C^n$; for all $u \in H_n$, let $\phi_n(u)$ denote the class mod C^{n+1} of $\phi(u) \in C^n$. Theorem 2.4.17 then shows that ϕ_n is a bijection of H_n onto a basis of the Abelian group C^n / C^{n+1}. It follows immediately that, for all $w \in F(X)$ and all $i \geq 1$, there exists a unique element α_i of $\mathbb{Z}^{\oplus H_i}$ such that, for $n \geq 1$,

$$w = \prod_{i=1}^{n} \prod_{u \in H_i} \phi(u)^{\alpha_i(u)} \mod C^{n+1},$$

where the product is calculated according to the total ordering given on H.

Example 2.4.19. Suppose that X is a set with two elements x, y and let $H_1 = \{x, y\}$, $H_2 = \{xy\}$. Every element w of $F(X)$ can therefore be written

$$w = x^a y^b [x, y]^c \mod C^3.$$

2.5 The Hausdorff series

In this section we assume that $\mathbb{K}$ is a field of characteristic 0. We shall consider the exponential and logarithm maps on Lie algebras and derive the famous Hausdorff formula for Lie algebras.

2.5.1 Exponential and logarithm in filtered algebras

Let A be a unital associative algebra which is Hausdorff and complete under a real filtration (A_α). We write $\mathfrak{m} = A_0^+ = \bigcup_{\alpha > 0} A_\alpha$. For $x \in \mathfrak{m}$, the family $(x^n / n!)_{n \in \mathbb{N}}$ is then summable. We write

$$e^x = \exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Then $\exp(x) \in 1 + \mathfrak{m}$ and the map $\exp : \mathfrak{m} \to 1 + \mathfrak{m}$ is called the **exponential map** of A.

For all $y \in 1 + \mathfrak{m}$, the family $((-1)^{n-1}(y-1)^n / n)_{n \geq 1}$ is summable. We write

$$\log(y) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(y-1)^n}{n}.$$

Then $\log(y) \in \mathfrak{m}$ and the map $\log : 1 + \mathfrak{m} \to \mathfrak{m}$ is called the **logarithmic map** of A.

Proposition 2.5.1. *The exponential map is a homeomorphism of* $\mathfrak{m}$ *to* $1 + \mathfrak{m}$ *and the logarithmic map is the inverse homeomorphism.*

Proof. For $x \in A_\alpha$, the term $x^n/n!$ is in $A_{n\alpha}$. It follows that the series defining the exponential converges uniformly on each of the sets A_α for $\alpha > 0$; as A_α is open in $\mathfrak{m}$ and $\mathfrak{m} = \bigcup_{\alpha>0} A_\alpha$, the exponential map is continuous. It can be similarly shown that the logarithmic map is continuous.

Let e and l be the formal power series with

$$e(X) = \sum_{n=1}^{\infty} \frac{X^n}{n!}, \quad l(X) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{X^n}{n}.$$

We know that $e(l(X)) = l(e(X)) = X$ on $\widehat{A}(\{X\}) = \mathbb{K}[\![X]\!]$. By substitution, we deduce that

$$e(l(x)) = l(e(x)) = x$$

for $x \in \mathfrak{m}$; as

$$\exp(x) = e(x) + 1, \quad \log(1+x) = l(x)$$

it follows immediately that $\log(\exp(x)) = x$ and $\exp(\log(1+x)) = 1+x$ for x in $\mathfrak{m}$, whence the proposition. □

Example 2.5.2. If $x, y \in \mathfrak{m}$ and x and y commute, then

$$\exp(x+y) = \exp(x)\exp(y)$$

since the family $(x^i y^j / i! j!)$ is summable.

Remark 2.5.3. Let B be a complete Hausdorff filtered unital associative algebra and $\mathfrak{n} = \bigcup_{\alpha>0} B_\alpha$. Let f be a continuous unital homomorphism of A into B such that $f(\mathfrak{m}) \subseteq \mathfrak{n}$. Then $f(\exp(x)) = \exp(f(x))$ for $x \in \mathfrak{m}$ and $f(\log(y)) = \log(f(y))$ for $y \in 1 + \mathfrak{m}$; we show for example the first of these formulae:

$$f(\exp(x)) = \sum_{n=0}^{\infty} \frac{f(x^n)}{n!} = \sum_{n=0}^{\infty} \frac{f(x)^n}{n!} = \exp(f(x)).$$

Remark 2.5.4. As the series e and l are without constant term and A_α is a closed ideal of A, $\exp(A_\alpha) \subseteq 1 + A_\alpha$ and $\log(1+A_\alpha) \subseteq A_\alpha$ whence $\exp(A_\alpha) = 1 + A_\alpha$ and $\log(1 + A_\alpha) = A_\alpha$ for $\alpha > 0$.

Remark 2.5.5. Let E be a unital associative algebra. If a is a nilpotent element of E, the family $(a^n/n!)$ has finite support and we write $\exp(a) = \sum_{n=0}^{\infty} a^n/n!$. An element b is called **unipotent** if $b-1$ is nilpotent; then we write

$$\log(b) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(b-1)^n}{n}.$$

We deduce from the relations $e(l(X)) = l(e(X)) = X$ that the map $a \mapsto \exp(a)$ is a bijection of the set of nilpotent elements of E onto the set of unipotent elements of E and that $b \mapsto \log(b)$ is the inverse map.

Let X be a set. We identify the free Lie algebra $L(X)$ with its canonical image in $A(X)$ and denote by $\widehat{L}(X)$ the closure of $L(X)$ in $\widehat{A}(X)$, that is the set of elements of $\widehat{A}(X)$ of the form $a = \sum_{n\geq 1} a_n$ such that $a_n \in L^n(X)$ for all $n \geq 1$; this is a filtered Lie subalgebra of $\widehat{A}(X)$.

Theorem 2.5.6. *The restriction of the exponential map of $\widehat{A}$ to $\widehat{L}(X)$ is a bijection of $\widehat{L}(X)$ onto a closed subgroup of the Magnus group $\Gamma(X)$.*

Proof. We write $A(X) = A$, $A^n(X) = A^n$, $\widehat{A}(X) = \widehat{A}$, $L^n(X) = L^n$, $\widehat{L}(X) = \widehat{L}$, $\Gamma(X) = \Gamma$. Let B be the algebra $A \otimes A$ with the graduation of type $\mathbb{N}$ defined by $B_n = \sum_{i+j=n} A^i \otimes A^j$. Let $\widehat{B} = \prod_n B^n$ be the associated complete filtered algebra. The coproduct $c : A \to A \otimes A$ defined in Corollary 2.3.2 is graded of degree 0 and hence extends by continuity to a homomorphism $\hat{c} : \widehat{A} \to \widehat{B}$ given by

$$\hat{c}\Big(\sum_n a_n\Big) = \sum_n c(a_n) \quad \text{for} \quad a_n \in A^n.$$

We also define continuous homomorphisms δ_1 and δ_2 of $\widehat{A}$ into $\widehat{B}$ by

$$\delta_1\Big(\sum_{n=0}^{\infty} a_n\Big) = \sum_{n=0}^{\infty} (a_n \otimes 1), \quad \delta_2\Big(\sum_{n=0}^{\infty} a_n\Big) = \sum_{n=0}^{\infty} (1 \otimes a_n)$$

for $a_n \in A^n$. By Corollary 2.3.3, L^n is the set of $a_n \in A^n$ such that $c(a_n) = a_n \otimes 1 + 1 \otimes a_n$. It follows that $\widehat{L}$ is the set of $a \in \widehat{A}$ such that

$$\hat{c}(a) = \delta_1(a) + \delta_2(a). \tag{2.5.1}$$

Let Δ be the set of $b \in \widehat{A}$ of constant term equal to 1 and satisfying the relation

$$\hat{c}(b) = \delta_1(b) \cdot \delta_2(b), \tag{2.5.2}$$

in other words, the set of $b = \sum_n b_n$ such that $b_n \in A^n$ for all $n \geq 0$, $b_0 = 1$ and $c(b_n) = \sum_{i+j=n} b_i \otimes b_j$. The latter characterization shows that Δ closed subset of Γ; as $\hat{c}$, δ_1 and δ_2 are ring homomorphisms and every element of $\delta_1(\widehat{A})$ commutes with every element of $\delta_2(\widehat{A})$, the restrictions to Γ of the maps c and $\delta_1\delta_2$ are group homomorphisms and Δ is a subgroup of Γ.

By Proposition 2.5.1, the exponential map of $\widehat{A}$ is a bijection of the set $\widehat{A}^+$ of elements of $\widehat{A}$ with no constant term onto Γ. Let $a \in \widehat{A}^+$ and $b = \exp(a)$. As c is a continuous ring homomorphism,

$$\hat{c}(b) = \hat{c}\Big(\sum_{n=0}^{\infty} \frac{a^n}{n!}\Big) = \sum_{n=0}^{\infty} \frac{\hat{c}(a)^n}{n!} = \exp(\hat{c}(a)).$$

The relations $\delta_1(b) = \exp(\delta_1(a))$ and $\delta_2(b) = \exp(\delta_2(a))$ are proved similarly and, as $\delta_1(a)$ commutes with $\delta_2(a)$,

$$\delta_1(b)\delta_2(b) = \exp(\delta_1(a) + \delta_2(a)).$$

Therefore a satisfies (2.5.1) if and only if b satisfies (2.5.2), which proves the theorem. $\square$

By the proof, the set $\widehat{L}(X)$ is mapped bijectively to the closed subgroup Δ. Hence the group law of Δ can be transported by the exponential map to $\widehat{L}(X)$. In other words, $\widehat{L}(X)$ is a complete topological group with the law of composition $(a,b) \mapsto ab$ given by

$$a \star b = \log(\exp(a)\exp(b)).$$

The topological group thus obtained is called the **Hausdorff group** (derived from X relative to $\mathbb{K}$).

Let η be the homomorphism of the free group $F(X)$ into Γ such that $\eta(x) = \exp(x)$ for $x \in X$. As $\exp(x) - 1 - x$ is of order ≥ 2, η is injective by Theorem 2.4.17. Therefore the map $\log \circ \eta$ is an injective homomorphism of $F(X)$ into the Hausdodf group which extends the canonical jnjection $X \to \widehat{L}(X)$.

For every integer $n \geq 1$ we denote by $\widehat{L}_n$ the set of elements of order $\geq n$ in $\widehat{L}$ and by Γ_n the set of $u \in \Gamma$ such that $u - 1$ is of order $\geq n$. Then $\widehat{L}_n = \exp^{-1}(\Gamma_n)$; as $(\Gamma_n)_{n\geq 1}$ is an integral central filtration on Γ, $(\widehat{L}_n)_{n\geq 1}$ is an integral central filtration on the group $\widehat{L}$.

Lemma 2.5.7. *Let $\mathfrak{g}$ be a filtered Lie algebra, $(\mathfrak{g}_\alpha)_{\alpha\in\mathbb{R}}$ its filtration and let $\alpha \in \mathbb{R}$. Let P be a homogeneous Lie polynomial of degree n in the indeterminates $(T_i)_{i\in I}$. Then $P(\{a_i\}) \in \mathfrak{g}_{n\alpha}$ every family $(a_i)_{i\in I}$ of elements of $\mathfrak{g}_\alpha$.*

Proof. Every Lie polynomial of degree $n \geq 2$ is a finite sum of terms of the form $[Q, R]$ where Q and R are of degree $< n$ and the sum their degrees is equal to n (Proposition 2.2.14). The lemma follows by induction on n. $\square$

A **Lie formal power series** (with coefficients in $\mathbb{K}$) in the indeterminates $(T_i)_{i\in I}$ is any element of the Lie algebra $\widehat{L}(\{T_i\}_{i\in I}) = \widehat{L}(I)$. Such an element P can be written uniquely as the sum of a summable family $(P_\nu)_{\nu\in\mathbb{N}}$ where $P_\nu \in L^\nu(I)$.

Proposition 2.5.8. *Suppose that I is finite. Let $\mathfrak{g}$ be a complete Hausdorff filtered Lie algebra such that $\mathfrak{g} = \bigcup_{\alpha>0}\mathfrak{g}_\alpha$; let $\mathscr{T} = (t_i)_{i\in I}$ be a family of elements of $\mathfrak{g}$. Then the homomorphism $\phi_{\mathscr{T}} : L(I) \to \mathfrak{g}$ such that $\phi_{\mathscr{T}}(T_i) = t_i$ can be extended by continuity to a unique continuous homomorphism $\hat{\phi}_{\mathscr{T}}$ of $\widehat{L}(I)$ into $\mathfrak{g}$.*

Proof. There exists $\alpha > 0$ such that $t_i \in \mathfrak{g}_\alpha$ for all $i \in I$; hence $\phi_{\mathscr{T}}(L^\nu(I)) \subseteq \mathfrak{g}_{|\nu|\alpha}$ for all ν (Lemma 2.5.7) , which implies the continuity of $\phi_{\mathscr{T}}$. $\square$

If $u \in \widehat{L}(I)$, we write $u(\{t_i\})$ for $\hat{\phi}_{\mathscr{T}}(u)$. In particular, taking $\mathfrak{g} = \widehat{L}(I)$, $u = u(\{T_i\})$; in the general case, $u(\{t_i\})$ is called the result of **substituting** the t_i for the T_i in the Lie formal power series $u(\{T_i\})$. If $u = \sum_{\nu\in\mathbb{N}^{\oplus I}} u_\nu$, where $u_\nu \in L^\nu(X)$, the family $(u_\nu(\{t_i\}))_{\nu\in\mathbb{N}^{\oplus I}}$ is summable and

$$u(\{t_i\}) = \sum_{\nu\in\mathbb{N}^{\oplus I}} u_\nu(\{t_i\}).$$

Let σ be a continuous homomorphism of $\mathfrak{g}$ into a complete Hausdorff filtered Lie algebra $\tilde{\mathfrak{g}}$ such that $\tilde{\mathfrak{g}} = \bigcup_{\alpha>0} \tilde{\mathfrak{g}}_\alpha$. For every finite family $\mathscr{T} = (t_i)_{i\in I}$ of elements of $\mathfrak{g}$ and all $u \in \widehat{L}(I)$,

$$\sigma(u(\{t_i\})) = u(\{\sigma(t_i)\}), \tag{2.5.3}$$

for the homomorphism $\sigma \circ \hat{\phi}_{\mathscr{T}}$ is continuous and maps T_i to $\sigma(t_i)$ for $i \in I$.

Let $\mathscr{U} = (u_j)_{j\in J}$ be a *finite* family of elements of $\widehat{L}(I)$ and let $v \in \widehat{L}(J)$; by substituting the u_j for the T_j in v, we obtain an element $w = v(\{u_j\}_{j\in J})$ denoted by $v \circ \mathscr{U}$. Then

$$w(\{t_i\}_{i\in I}) = v(\{u_j(\{t_i\}_{i\in I})\}_{j\in J}) \tag{2.5.4}$$

for every finite family $\mathscr{T} = (t_i)_{i\in I}$ of elements of $\mathfrak{g}$, as is seen by operating with the continuous homomorphism $\hat{\phi}_{\mathscr{T}}$ on the equation $w = v(\{u_j\}_{j\in J})$.

2.5.2 The Hausdorff series

Let $\{U, V\}$ be a set of two elements. In this paragraph, we consider the element $H = U \star V = \log(\exp(U)\exp(V))$ of the Lie algebra $\widehat{L}_{\mathbb{Q}}(\{U, V\})$, which is called the **Hausdoiff series** in the indeterminates U and V. We denote by H_n (resp. $H_{r,s}$) the homogeneous component of H of total degree n (resp. multidegree (r, s)). Then

$$H = \sum_{n\geq 0} H_n = \sum_{r,s\geq 0} H_{r,s}, \quad H_n = \sum_{\substack{r+s=n\\ r,s\geq 0}} H_{r,s}. \tag{2.5.5}$$

Theorem 2.5.9. *If r, s are two positive integers such that $r + s \geq 1$, then $H_{r,s} = H^{(1)}_{r,s} + H^{(2)}_{r,s}$, where*

$$H^{(1)}_{r,s} = \frac{1}{r+s}\sum_{n\geq 1}\frac{(-1)^{n-1}}{n}\sum_{\substack{r_1+\cdots+r_{n-1}=r\\ s_1+\cdots+s_{n-1}=s-1\\ r_1+s_1\geq 1,\ldots,r_{n-1}+s_{n-1}\geq 1}}\Big(\Big(\prod_{i=1}^{n-1}\frac{\mathrm{ad}(U)^{r_i}}{r_i!}\frac{\mathrm{ad}(V)^{s_i}}{s_i!}\Big)\frac{\mathrm{ad}(U)^{r_n}}{r_n!}\Big)(V) \tag{2.5.6}$$

$$H^{(2)}_{r,s} = \frac{1}{r+s}\sum_{n\geq 1}\frac{(-1)^{n-1}}{n}\sum_{\substack{r_1+\cdots+r_{n-1}=r-1\\ s_1+\cdots+s_{n-1}=s\\ r_1+s_1\geq 1,\ldots,r_{n-1}+s_{n-1}\geq 1}}\Big(\prod_{i=1}^{n-1}\frac{\mathrm{ad}(U)^{r_i}}{r_i!}\frac{\mathrm{ad}(V)^{s_i}}{s_i!}\Big)(U) \tag{2.5.7}$$

Proof. In the algebra $\widehat{A}_{\mathbb{Q}}(\{U, V\})$ we have $\exp(U)\exp(V) = 1 + W$, where $W = \sum_{r+s\geq 1}\frac{U^r}{r!}\frac{V^s}{s!}$, whence $H = \sum_{n\geq 1}(-1)^{n-1}W^n/n$; that is,

$$H_{r,s} = \sum_{n\geq 1}\frac{(-1)^{n-1}}{n}\sum_{\substack{r_1+\cdots+r_n=r\\ s_1+\cdots+s_n=s\\ r_1+s_1\geq 1,\ldots,r_n+s_n\geq 1}}\prod_{i=1}^{n}\frac{U^{r_i}}{r_i!}\frac{V^{s_i}}{s_i!}. \tag{2.5.8}$$

The linear map P_n defined by (Corollary 2.3.9)

$$P_n(x_1, \ldots, x_n) = \frac{1}{n}(\mathrm{ad}(x_1)\circ\cdots\circ\mathrm{ad}(x_{n-1}))(x_n) \quad \text{for} \quad x_1, \ldots, x_n \in \{U, V\}$$

is a projection of $A^n_{\mathbb{Q}}(\{U, V\})$ onto $L^n_{\mathbb{Q}}(\{U, V\})$; as $H_{r,s}$ belongs to $L^{r+s}_{\mathbb{Q}}(\{U, V\})$, we then have $H_{r,s} = P_{r+s}(H_{r,s})$. Now

$$P_{r+s}\Big(\prod_{i=1}^{n}\frac{U^{r_i}}{r_i!}\frac{V^{s_i}}{s_i!}\Big) = \frac{1}{r+s}\Big(\Big(\prod_{i=1}^{n-1}\frac{\mathrm{ad}(U)^{r_i}}{r_i!}\frac{\mathrm{ad}(V)^{s_i}}{s_i!}\Big)\frac{\mathrm{ad}(U)^{r_n}}{r_n!}\frac{\mathrm{ad}(V)^{s_n-1}}{s_n!}\Big)(V) \tag{2.5.9}$$

when $s_n \geq 1$ and

$$P_{r+s}\Big(\prod_{i=1}^{n}\frac{U^{r_i}}{r_i!}\frac{V^{s_i}}{s_i!}\Big) = \frac{1}{r+s}\Big(\Big(\prod_{i=1}^{n-1}\frac{\mathrm{ad}(U)^{r_i}}{r_i!}\frac{\mathrm{ad}(V)^{s_i}}{s_i!}\Big)\frac{\mathrm{ad}(U)^{r_n-1}}{r_n!}\Big)(U) \tag{2.5.10}$$

when $r_n \geq 1$ and $s_n = 0$. Moreover, obviously $\mathrm{ad}(t)^{p-1}(t) = 0$ if $p \geq 2$ and $\mathrm{ad}(t)^0(t) = t$. It follows that the two sides of (2.5.9) are zero when $s_n \geq 2$ and those of (2.5.10) are zero when $r_n \geq 2$. The theorem then follows since $H^{(1)}_{r,s}$ is the sum of the terms of type (2.5.9) and $H^{(2)}_{r,s}$ is the sum of the terms of type (2.5.10). ☐

Example 2.5.10. By a few computation, we get that

$$H(U,V) \equiv U + V + \frac{1}{2}[U,V] + \frac{1}{12}[U,[U,V]] + \frac{1}{2}[V,[V,U]] + \frac{1}{24}[V,[U,[V,U]]]$$

modulo $\sum_{n\geq 5} L^n(\{U,V\})$.

Remark 2.5.11. We note that $H_{0,n} = H_{n,0} = 0$ for every integer $n \neq 1$, whence

$$H(U,0) = H(0,U) = U.$$

On the other hand, as $[U,-U] = 0$, we have $H(U,-U) = 0$.

Remark 2.5.12. We have defined a projector Q of $A(X)$ onto $L(X)$ such that $Q(a^n) = 0$ for $a \in L(X)$ and $n \geq 2$ and $Q(1) = 0$. From this we see $H = Q(\exp(H)) = Q(\exp(U)\exp(V))$, whence immediately

$$H_{r,s} = Q\Big(\frac{U^r}{r!}\frac{V^s}{s!}\Big)$$

since Q preserves the graduation on $L(X)$.

As $\mathbb{K}$ is a field containing $\mathbb{Q}$, the Hausdorff series can be considered as a Lie formal power series with coefficients in $\mathbb{K}$. Therefore, if $\mathfrak{g}$ is a complete Hausdorff filtered Lie algebra with $\mathfrak{g} = \bigcup_\alpha \mathfrak{g}_\alpha$ then, for a,b in $\mathfrak{g}$, a and b can be substituted for U and V in H.

In particular, let A be a complete Hausdorff filtered unital associative algebra. We write $\mathfrak{m} = \bigcup_{\alpha>0} A_\alpha$ and $\mathfrak{m}_\alpha = A_\alpha \cap \mathfrak{m}$ for $\alpha \in \mathbb{R}$. With the bracket $[a,b] = ab - ba$, $\mathfrak{m}$ is a complete Hausdorff filtered Lie algebra, to which the above can be applied. With this notation, we have the following result which completes Proposition 2.5.1.

Proposition 2.5.13. *If $a,b \in \mathfrak{m}$, then* $\exp(H(a,b)) = \exp(a)\exp(b)$.

Proof. Let a,b be in $\mathfrak{m}$; there exists $\alpha > 0$ such that $a \in A_\alpha$ and $b \in A_\alpha$. Then there exists a continuous homomorphism θ from the Magnus algebra $\widehat{A}(\{U,V\})$ into A map U to a and V to b (Proposition 2.4.11). The restriction of θ to $\widehat{L}(\{U,V\})$ is a continuous homomorphism of Lie algebras of $L(\{U,V\})$ into $\mathfrak{m}$ which maps U (resp. V) to a (resp. b). By formula (2.5.3), therefore $\theta(H) = H(a,b)$. It then suffices to apply the continuous homomorphism θ to the two sides of the relation

$$\exp(H(U,V)) = \exp(U)\exp(V)$$

to obtain the claim. □

Remark 2.5.14. If a and b commute, then $H_{r,s}(a,b) = 0$ for $r+s \geq 2$, for every homogeneous Lie polynomial of degree ≥ 2 is zero at (a,b). Then $H(a,b) = a+b$ and Proposition 2.5.13 recovers the formula

$$\exp(a+b) = \exp(a)\exp(b).$$

Proposition 2.5.15. *Let $\mathfrak{g}$ be a complete Hausdorff filtered Lie algebra such that $\mathfrak{g} = \bigcup_{\alpha>0}\mathfrak{g}_\alpha$. The map $(a,b) \mapsto H(a,b)$ is a group law on $\mathfrak{g}$ compatible with the topology on $\mathfrak{g}$ under which 0 is the identity element and $-a$ is the inverse of a for all $a \in \mathfrak{g}$.*

Proof. The map $(a,b) \mapsto H(a,b)$ of $\mathfrak{g}\times\mathfrak{g}$ into $\mathfrak{g}$ is continuous (Proposition 2.5.8); as the map $a \mapsto -a$ is obviously continuous, it suffices to prove the relations

$$H(H(a,b),c) = H(a,H(b,c)) \tag{2.5.11}$$
$$H(a,-a) = 0 \tag{2.5.12}$$
$$H(a,0) = H(0,a) = 0 \tag{2.5.13}$$

for a,b,c in $\mathfrak{g}$. By formula (2.5.4), it suffices to prove these formulae when a,b,c are three indeterminates and $\mathfrak{g} = \widehat{L}(\{a,b,c\})$. Now the restriction of the exponential map to $\widehat{L}(\{a,b,c\})$ is an injection into the Magnus algebra $\widehat{A}(\{a,b,c\})$ and by Proposition 2.5.13:

$$\exp(H(H(a,b),c)) = \exp(H(a,b))\exp(c) = \exp(a)\exp(b)\exp(c)$$
$$\exp(H(a,H(b,c))) = \exp(a)\exp(H(b,c)) = \exp(a)\exp(b)\exp(c)$$

$$\begin{aligned}
\exp(H(a,-a)) &= \exp(a)\exp(-a) = \exp(a-a) = \exp(0)\\
\exp(H(a,0)) &= \exp(a)\exp(0) = \exp(a)\\
\exp(H(0,a)) &= \exp(0)\exp(a) = \exp(a).
\end{aligned}$$

This establishes relations (2.5.11) to (2.5.13) □

Remark 2.5.16. Take $\mathfrak{g}$ to be the Lie algebra $\widehat{L}(X)$. The group law introduced in the above proposition coincides with the law defined for $\widehat{L}(X)$. In other words,

$$a \star b = H(a,b) \quad \text{for} \quad a,b \in \widehat{L}(X).$$

Thus the Hausdorff group law is given by the Hausdorff series.

Remark 2.5.17. Let $\mathfrak{g}$ be a Lie algebra with the integral filtration $(C^n(\mathfrak{g}))$ defined by the lower central series. Suppose that there exists $N \geq 1$ such that $C^N(\mathfrak{g}) = \{0\}$. With the topology derived from the filtration $(C^n(\mathfrak{g}))_{n\geq 1}$, the Lie algebra $\mathfrak{g}$ is Hausdorff, complete and even discrete. Then $P(a_1,\dots,a_r) = 0$ for $a_1,\dots,a_r$ in $\mathfrak{g}$ and for every homogeneous Lie polynomial P of degree $\geq N$; in particular, $H_{r,s}(a,b) = 0$ for $r+s \geq N$ and the series $H(a,b) = \sum_{r,s} H_{r,s}(a,b)$ has only a finite number of non-zero terms. The group law $(a,b) \mapsto H(a,b)$ on $\mathfrak{g}$ is then a polynomial map.

Proposition 2.5.18. *Let $K_{r,s}$ be the component of $H(U+V,-U)$ of multidegree (r,s). Then*

$$H_{n,1}(U,V) = \frac{1}{(n+1)!}\mathrm{ad}(U)^n(V) \quad \textit{for} \quad n \geq 0.$$

Proof. We write $K(U,V) = H(U+V,-U)$, $K_1(U,V) = \sum_n K_{n,1}(U,V)$. We denote by L (resp. R) left (resp. right) multiplication by U on $\widehat{A}(\{U,V\})$. We can write

$$e^U V e^{-U} = \sum_{p,q} \frac{U^p}{p!} V \frac{(-U)^q}{q!} = \sum_{n=0}^{\infty} \frac{1}{n!}\Big(\sum_{p+q=n} \frac{n!}{p!q!} L^p(-R)^q(V)\Big) = \sum_{n=0}^{\infty} \frac{1}{n!}(L-R)^n(V),$$

and therefore

$$e^U V e^{-U} = \sum_{n=0}^{\infty} \frac{1}{n!}\mathrm{ad}(U)^n(V). \tag{2.5.14}$$

We now calculate modulo the ideal $\sum_{m=0}^{\infty}\sum_{n\geq 2} \widehat{A}^{m,n}(\{U,V\})$ of $\widehat{A}(\{U,V\})$. For $n \geq 1$,

$$(U+V)^n \equiv U^n + \sum_{i=1}^{n-1} U^i V U^{n-1-i}$$

whence

$$\mathrm{ad}(U)(U+V)^n \equiv (L-R)\sum_{i=1}^{n-1} L^i R^{n-i}(V) \equiv (L^n - R^n)(V) \equiv U^n V - V U^n.$$

Summing over n, we conclude that

$$\mathrm{ad}(U)e^{U+V} \equiv e^U V - V e^U. \tag{2.5.15}$$

On the other hand, $K_1(U,V) \equiv K(U,V)$ and $e^{K_1(U,V)} \equiv 1 + K_1(U,V)$ and hence

$$K_1 \equiv e^K - 1 \equiv e^{U+V}e^{-U} - 1$$

by Proposition 2.5.13. We deduce that

$$\begin{aligned}
\mathrm{ad}(U)(K_1) &= Ue^{U+V}e^{-U} - E^{U+V}e^{-U}U \equiv (Ue^{U+V} - e^{U+V}U)e^{-U}\\
&\equiv e^U V - V e^U \equiv e^U V e^{-U} - V\\
&\equiv \sum_{n=1}^{\infty} \frac{1}{n!}\mathrm{ad}(U)^n(V) \equiv \mathrm{ad}(U)\Big(\sum_{n=0}^{\infty} \frac{\mathrm{ad}(U)^n}{(n+1)!}\Big)(V).
\end{aligned}$$

It then suffices to apply Example 2.2.24. □

2.5.3 Convergence of the Hausdorff series

2.5.3.1 Real or complex case In this paragraph we assume that $\mathbb{K}$ is one of the fields $\mathbb{R}$ or $\mathbb{C}$ with its usual absolute value. Recall that a **normable algebra** over $\mathbb{K}$ is a (not necessarily associative) algebra over $\mathbb{K}$ with a topology $\mathcal{T}$ with the following properties:

(a) $\mathcal{T}$ can be defined by a norm;

(b) the map $(x, y) \mapsto xy$ of $A \times A$ into A is continuous.

We denote by $\mathfrak{g}$ a complete normable Lie algebra over $\mathbb{K}$. We choose a norm on $\mathfrak{g}$ and a number $M > 0$ such that

$$\|[x, y]\| \leq M\|x\|\|y\|$$

for $x, y \in \mathfrak{g}$.

Let I be a finite set and let $P(\mathfrak{g}^I, \mathfrak{g})$ (resp. $\widehat{P}(\mathfrak{g}^I, \mathfrak{g})$) be the vector space of continuous-polynomials (resp. formal power series with continuous components) on $\mathfrak{g}^I$ with values in $\mathfrak{g}$. Recall $P(\mathfrak{g}^I, \mathfrak{g})$ has a graduation of type $\mathbb{N}^I$ and that $\widehat{P}(\mathfrak{g}^I, \mathfrak{g})$ is identified with the completion of the vector space $P(\mathfrak{g}^I, \mathfrak{g})$ with the topology defined by the filtration associated with the graduation of $P(\mathfrak{g}^I, \mathfrak{g})$. Moreover, $P(\mathfrak{g}^I, \mathfrak{g})$ is a graded Lie algebra with the bracket defined by $[f, g](x) = [f(x), g(x)]$ for f, g in $P(\mathfrak{g}^I, \mathfrak{g})$, $x \in \mathfrak{g}^I$; this Lie algebra structure can be extended by continuity to $\widehat{P}(\mathfrak{g}^I, \mathfrak{g})$ and makes it into a complete Hausdorff filtered Lie algebra.

By Proposition 2.5.8, there exists one and only one continuous Lie algebra homomorphism $\phi_I : u \mapsto \tilde{u}$ of $\widehat{L}(I)$ into $\widehat{P}(\mathfrak{g}^I, \mathfrak{g})$ map the indeterminate of index i to the projection π_i for all $i \in I$, since $\pi_i \in P(\mathfrak{g}^I, \mathfrak{g})$. It follows that $\tilde{u} \in P(\mathfrak{g}^I, \mathfrak{g})$ for $u \in L(I)$; more precisely, when $u \in L(I)$, $\tilde{u}$ is just the polynomial map $(t_i) \mapsto u(\{t_i\})$. On the other hand, clearly ϕ_I is compatible with the multigraduations of $L(I)$ and $P(\mathfrak{g}^I, \mathfrak{g})$. If $u = \sum_{\nu \in \mathbb{N}^I} u_\nu$ where $u_\nu \in L^\nu(I)$ for $\nu \in \mathbb{N}^I$, then

$$\tilde{u} = \sum_{\nu \in \mathbb{N}^I} \tilde{u}_\nu, \quad \tilde{u}_\nu \in P_\nu(\mathfrak{g}^I, \mathfrak{g}).$$

Let $\mathscr{U} = (u_j)_{j \in J}$ be a *finite* family of elements of $\widehat{L}(I)$, let $v \in \widehat{L}(J)$ and let $w = v \circ \mathscr{U}$. We write $\tilde{\mathscr{U}} = (\tilde{u}_j)_{j \in J}$. Then again we have

$$\tilde{v} \circ \tilde{\mathscr{U}} = \widetilde{v \circ \mathscr{U}}. \tag{2.5.16}$$

Let $H = \sum_{r,s} H_{r,s} \in \widehat{L}(U, V)$ be the Hausdorff series. We shall show that the corresponding formal power series

$$\tilde{H} = \sum_{r,s} \tilde{H}_{r,s} \in \widehat{P}(\mathfrak{g} \times \mathfrak{g}, \mathfrak{g})$$

is convergent. For this, we introduce the following formal power series $\eta \in \mathbb{Q}[\![U, V]\!]$:

$$\begin{aligned} \eta(U, V) &= -\log(2 - \exp(U + V)) = \sum_{n=1}^{\infty} \frac{1}{n}(\exp(U + V) - 1) \\ &= \sum_{n=1}^{\infty} \frac{1}{n} \sum_{\substack{r_1, \dots, r_n \\ s_1, \dots, s_n \\ r_i + s_i \geq 1}} \frac{U^{r_1}}{r_1!} \frac{V^{s_1}}{s_1!} \cdots \frac{U^{r_n}}{r_n!} \frac{V^{s_n}}{s_n!} \end{aligned} \tag{2.5.17}$$

To simplify our notation, we set

$$\eta(U, V) = \sum_{r,s} \eta_{r,s} U^r V^s \tag{2.5.18}$$

where

$$\eta_{r,s} = \sum_{n=1}^{\infty} \sum_{\substack{r_1, \dots, r_n \\ s_1, \dots, s_n \\ r_i + s_i \geq 1}} \frac{1}{r_1! \cdots r_n! s_1! \cdots s_n!}. \tag{2.5.19}$$

Now let u and v be two positive real numbers such that $u + v < \log 2$; then $0 \leq \exp(u + v) - 1 < 1$; the series derived from (2.5.17) by substituting u for U and v for V are then convergent and the above calculations imply that

$$\sum_{r,s} \eta_{r,s} u^r v^s = -\log(2 - \exp(u + v)) < +\infty. \tag{2.5.20}$$

Lemma 2.5.19. *Let r, s be non-negative integers and $\|\tilde{H}_{r,s}\|$ be the norm of the continuous polynomial $\tilde{H}_{r,s}$. Then*

$$\|\tilde{H}_{r,s}\| \leq M^{r+s-1}\eta_{r,s}.$$

Proof. Let r_i, s_i be in $\mathbb{N}$ for $1 \leq i \leq n$, with $s_n = 1$; we write $r = \sum_i r_i$, $s = \sum_i s_i$ and consider the following element of $L(\{U, V\})$:

$$Z = \Big(\Big(\sum_{i=1}^{n-1} \mathrm{ad}(U)^{r_i}\mathrm{ad}(V)^{s_i}\Big)\mathrm{ad}(U)^{r_n}\Big)(V).$$

Then $\tilde{Z} = f \circ \gamma$, where f is the following $(r+s)$-linear map of $\mathfrak{g}^{r+s}$ into $\mathfrak{g}$:

$$(x_1, \dots, x_r, y_1, \dots, y_s) \mapsto (\mathrm{ad}(x_1) \circ \cdots \circ \mathrm{ad}(x_{r_1}) \circ \mathrm{ad}(y_1) \circ \cdots \circ \mathrm{ad}(y_{s_1}) \circ \mathrm{ad}(x_{r_1}+1) \circ \cdots \circ \mathrm{ad}(x_r))(y_s)$$

and where γ is the following map of $\mathfrak{g}^2$ to $\mathfrak{g}^{r+s}$:

$$(x, y) \mapsto (\underbrace{x, \overset{r}{\dots}, x}, \underbrace{y, \overset{r}{\dots}, y});$$

whence $\|\tilde{Z}\| \leq \|f\| \leq M^{r+s-1}$. Applying these inequalities to the various terms on the right hand side of formulae in Theorem 2.5.9, we obtain:

$$\|\widetilde{H^{(1)}_{r,s}}\| \leq \frac{M^{r+s-1}}{r+s} \sum_{n=1}^{\infty} \frac{1}{n} \sum_{\substack{r_1+\cdots+r_n=r \\ s_1+\cdots+s_{n-1}=s-1 \\ r_1+s_1\geq 1,\dots,r_{n-1}+s_{n-1}\geq 1}} \frac{1}{r_1!\cdots r_n! s_1!\cdots s_{n-1}!}. \tag{2.5.21}$$

A similar argument gives

$$\|\widetilde{H^{(2)}_{r,s}}\| \leq \frac{M^{r+s-1}}{r+s} \sum_{n=1}^{\infty} \frac{1}{n} \sum_{\substack{r_1+\cdots+r_{n-1}=r-1 \\ s_1+\cdots+s_n=s \\ r_1+s_1\geq 1,\dots,r_{n-1}+s_{n-1}\geq 1}} \frac{1}{r_1!\cdots r_{n-1}! s_1!\cdots s_n!}; \tag{2.5.22}$$

whence, by (2.5.19)

$$\|\tilde{H}_{r,s}\| \leq \eta_{r+s}\frac{M^{r+s-1}}{r+s} \leq \eta_{r,s}M^{r+s-1}$$

which proves the lemma. □

Proposition 2.5.20. *The formal power series $\tilde{H}$ is a convergent series; its domain of absolute convergence contains the open set*

$$\Omega = \{(x, y) \in \mathfrak{g} \times \mathfrak{g} : \|x\| + \|y\| < \frac{1}{M}\log 2\}.$$

Proof. Let u, v be two positive real numbers such that $u + v < \frac{1}{M}\log 2$; then by Lemma 2.5.19,

$$M\sum_{r,s} \|\tilde{H}_{r,s}\| u^r v^s \leq \sum_{r,s} \eta_{r,s} M^{r+s} u^r v^s = -\log(2 - \exp(u+v)) < +\infty \tag{2.5.23}$$

where we use formula (2.5.18). □

Let $h : \Omega \to \mathfrak{g}$ denote the analytic function defined by $\tilde{H}$, that is, by the formula

$$h(x, y) = \sum_{r,s} \tilde{H}_{r,s}(x, y) = \sum_{r,s} H_{r,s}(x, y) \quad \text{for} \quad (x, y) \in \Omega. \tag{2.5.24}$$

This function is called the Hausdorff function of $\mathfrak{g}$ relative to M (or simply the Hausdorff function of $\mathfrak{g}$ if no confusion can arise). Note that $H_{r,s}(U, -U) = 0$ if $r + s \geq 2$ and hence

$$h(x, -x) = 0 \quad \text{for} \quad \|x\| < \frac{1}{2M}\log 2. \tag{2.5.25}$$

Similarly,

$$h(0, x) = h(x, 0) = x \quad \text{for} \quad \|x\| < \frac{1}{M}\log 2. \tag{2.5.26}$$

Proposition 2.5.21. *Let Ξ be the subset of $\mathfrak{g} \times \mathfrak{g} \times \mathfrak{g}$ defined by*

$$\Xi = \{(x,y,z) \in \mathfrak{g} \times \mathfrak{g} \times \mathfrak{g} : \|x\| + \|y\| + \|z\| < \frac{1}{M} \log \frac{3}{2}\}.$$

If $(x,y,z) \in \Xi$, then

$$(x,y) \in \Omega, \quad (h(x,y),z) \in \Omega, \quad (y,z) \in \Omega, \quad (x,h(y,z)) \in \Omega, \tag{2.5.27}$$

and

$$h(h(x,y),z) = h(x,h(y,z)). \tag{2.5.28}$$

Proof. Let $(x,y,z) \in \Xi$; clearly $(x,y) \in \Omega$ and $(y,z) \in \Omega$. Moreover:

$$\|h(x,y)\| \leq \sum_{r,s} \|\tilde{H}_{r,s}\| \|x\|^r \|y\|^s,$$

and hence by (2.5.24),

$$\|h(x,y)\| \leq -\frac{1}{M} \log(2 - \exp(M(\|x\| + \|y\|))).$$

Now $M(\|x\| + \|y\|) < \log \frac{3}{2} - M\|z\|$; we write $u = \exp(M\|z\|)$; then $1 \leq u \leq \frac{3}{2}$ and

$$\begin{aligned} M(\|h(x,y)\| + \|z\|) &< -\log(2 - \exp(\log \frac{3}{2} - M\|z\|)) + M\|z\| \\ &= -\log\left(2 - \frac{3}{2u}\right) + \log u = \log \frac{2u^2}{4u-3} \\ &= \log\left(2 + \frac{2(u-1)(u-3)}{4u-3}\right) \leq \log 2. \end{aligned}$$

We see similarly that $(x,h(y,z)) \in \Omega$. We now prove (2.5.28). Recall that by Proposition 2.5.15, in the Lie algebra $\widehat{L}(\{U,V,W\})$ we have

$$H(H(U,V),W) = H(U,H(V,W)).$$

By formula (2.5.16), we therefore have in $\widehat{P}(\mathfrak{g} \times \mathfrak{g} \times \mathfrak{g}, \mathfrak{g})$ the relation

$$\tilde{H} \circ (\tilde{H} \times 1_{\mathfrak{g}}) = \tilde{H} \circ (1_{\mathfrak{g}} \times \tilde{H}).$$

By **??**, there exists a number $\varepsilon > 0$ such that formula (2.5.28) is true when $\|x\|$, $\|y\|$, and $\|z\|$ are $\leq \varepsilon$. But the functions $(x,y,z) \mapsto h(h(x,y),z)$ and $(x,y,z) \mapsto h(x,h(y,z))$ are analytic functions on Ξ with values in $\mathfrak{g}$. As Ξ is connected and they coincide in a neighbourhood of 0, they are equal. □

Let α be a real number such that $0 < \alpha \leq \frac{1}{3M} \log \frac{3}{2}$; set

$$G = \{x \in \mathfrak{g} : \|x\| < \alpha\}, \quad \Theta = \{(x,y) \in G \times G : h(x,y) \in G\}$$

and $m : \Theta \to G$ be the restriction of h to Θ. Then

(a) Θ is open in $G \times G$ and m is analytic.

(b) $x \in G$ implies $(0,x) \in \Theta$, $(x,0) \in \Theta$ and $m(0,x) = m(x,0) = x$.

(c) $x \in G$ implies $-x \in G$, $(x,-x) \in \Theta$, $(-x,x) \in \Theta$ and

$$m(x,-x) = m(-x,x) = 0.$$

(d) Let x,y,z be elements of G such that $(x,y) \in \Theta$, $(m(x,y),z) \in \Theta$, $(y,z) \in \Theta$ and $(x,m(y,z)) \in \Theta$. Then $m(m(x,y),z) = m(x,m(y,z))$.

In other words, if we write $-x = i(x)$, the quadruple $(G,0,i,m)$ is a *Lie group germ* over $\mathbb{K}$.

Example 2.5.22. Let A be a complete normed unital associative algebra. Then $\|xy\| \leq \|x\|\|y\|$ for x, y in A. Let I be a finite set and let $\widehat{P}(A^I, A)$ be the vector space of formal power series with continuous components on A^I with values in A with the algebra structure obtained by writing

$$f \cdot g = m \circ (f, g) \quad \text{for} \quad f, g \in \widehat{P}(A^I, A),$$

where $m : A \times A \to A$ denotes multiplication on A. Similarly, we define a continuous homomorphism of unital algebras $u \mapsto \tilde{u}$ of $\widehat{A}(I)$ into $\widehat{P}(A^I, A)$ map the indeterminate of index i to π_i; this homomorphism extends the Lie algebra homomorphism of $\widehat{L}(I)$ into $\widehat{P}(A^I; A)$. If $u = \sum_\nu u_\nu$ with $u_\nu \in A^\nu(I)$ for $\nu \in \mathbb{N}^I$, then $\tilde{u} = \sum_\nu \tilde{u}_\nu$, where $\tilde{u}_\nu$ is the polynomial map $(t_i)_{i \in I} \mapsto u_\nu(\{t_i\})$.

Let $\mathscr{U} = (u_j)_{j \in J}$ be a finite family of elements of $\widehat{A}(J)$, let $v \in \widehat{A}(J)$ and write $w = v \circ \mathscr{U}$. Then

$$\widetilde{(v \circ \mathscr{U})} = \tilde{v} \circ \tilde{\mathscr{U}}. \tag{2.5.29}$$

In particular we take $I = \{U\}$, identify A and $A^{\{U\}}$ and consider the images e and l of the series $e(U) = \sum_{n=1}^{\infty} U^n / n!$ and $l(U) = \sum_{n=1}^{\infty} (-1)^{n-1} U^n / n$ in $\widehat{P}(A, A)$. Then $\|\tilde{U}^n\| \leq 1$ for $\|x_1 \cdots x_n\| \leq \|x_1\| \cdots \|x_n\|$ for $x_1, \ldots, x_n$ in A. Therefore the radius of absolute convergence of e (resp. l) is infinite (resp. ≥ 1). We shall denote by e_A (resp. l_A) the analytic map of A into A (resp. of B into A, where B is the open unit ball of A) defined by the convergent series $\tilde{e}$ (resp. $\tilde{l}$) and we shall write $\exp_A(x) = 1 + e_A(x)$ (for $x \in A$) and $\log_A(x) = l_A(x-1)$ (for $x \in A$ and $\|x - 1\| < 1$). Then

$$\exp_A(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad \text{for} \quad x \in A,$$

$$\log_A(x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(x-1)^n}{n}, \quad \text{for} \quad x \in A, \|x - 1\| < 1.$$

As $(e \circ l)(U) = (l \circ e)(U) = U$, we conclude from (2.5.29) that $\tilde{e} \circ \tilde{l} = \tilde{l} \circ \tilde{e} = 1_A$, therefore

$$\exp_A(\log_A(x)) = x, \quad \text{for} \quad x \in A, \|x - 1\| < 1, \log_A(\exp_A(x)) = x, \quad \text{for} \quad x \in A, \|x\| < \log 2,$$

(because $\|x\| < \log 2$ implies $\|\exp_A(x) - 1\| \leq \exp(\|x\|) - 1 < 1$.)

Finally, we consider A as a complete normed Lie algebra. In this case, we have

$$\|[x, y]\| = \|xy - yx\| \leq 2\|x\|\|y\|,$$

so Proposition 2.5.20 implies that the domain of absolute convergence of the formal power series $\tilde{H}$ contains the open set

$$\Omega = \{(x, y) \in A \times A : \|x\| + \|y\| < \frac{1}{2} \log 2\}.$$

Hence $\tilde{H}$ defines an analytic function $h : \Omega \to A$. Then $h(x, y) = \sum_{r,s} H_{r,s}(x, y)$. Moreover, for $\|x\| + \|y\| < \frac{1}{2} \log 2$ we have

$$\exp_A(x) \exp_A(y) = \exp_A(h(x, y)). \tag{2.5.30}$$

In fact, it follows from (2.5.29) and the relation $e^U e^V = e^{H(U,V)}$ that

$$m \circ (1 + \tilde{e}, 1 + \tilde{e}) = (1 + \tilde{e}) \circ \tilde{H}$$

in $\widehat{P}(A \times A, A)$. We therefore deduce that (2.5.30) is true for (x, y) sufficiently close to $(0, 0)$, whence the claim follows by analytic continuation.

2.5.3.2 Ultrametric case We now consider the case where $\mathbb{K}$ is a non-discrete complete valued field of characteristic zero, with an ultrametric absolute value. We denote by p the characteristic of the residue field of $\mathbb{K}$.

We assume that $p \neq 0$ and write $a = |p|$ (where $|\cdot|$ is the absolute value on $\mathbb{K}$); we know that $0 < a < 1$ and that there exists one and only one valuation v on $\mathbb{K}$ with values in $\mathbb{R}$ whose restriction to $\mathbb{Q}$ is the p-adic valuation v_p and which is such that $|x| = a^{v(x)}$ for all $x \in \mathbb{K}$. Also we write

$$\theta = \frac{1}{p-1}.$$

Lemma 2.5.23. *Let n be a non-negative integer and $n = n_0 + n_1 p + \cdots + n_k p^k$ be the p-adic expansion of n. Let $S(n) = n_0 + n_1 + \cdots + n_k$. Then*

$$v_p(n!) = \frac{n - S(n)}{p-1}.$$

Proof. Recall that $v_p(n!) = \sum_{i=1}^{n} v_p(i)$ and the number of integers i between 1 and n for which $v_p(i) \geq j$ is equal to the integral part $[n/p^j]$ of n/p^j. Then

$$v_p(n!) = \sum_{j=0}^{\infty} j([n/p^j] - [n/p^{j+1}]) = \sum_{j=1}^{\infty} [n/p^j].$$

As $[n/p^j] = \sum_{i=j}^{\infty} n_i p^{i-j}$, we then conclude that

$$v_p(n!) = \sum_{j=1}^{\infty}\sum_{i=j}^{\infty} n_i p^{i-j} = \sum_{i=1}^{\infty}\sum_{j=1}^{i} n_i p^{i-j} = \sum_{i=1}^{\infty} n_i \frac{1-p^i}{1-p} = \frac{n-S(n)}{p-1}.$$

This proves the lemma. □

Lemma 2.5.24. *For every integer $n \geq 1$, we have*

$$v_p(n) \leq v_p(n!) \leq (n-1)\theta \quad \textit{and} \quad v_p(n) \leq \frac{\log n}{\log p}.$$

Proof. We have $v_p(n!) = (n - S(n))\theta \leq (n-1)\theta$ by Lemma 2.5.23, which proves the first inequality. On the other hand, $n \geq p^{v_p(n)}$ by the definition of v_p, whence $v(n) \leq (\log n)/(\log p)$. □

Let $I = \{U, V\}$ be a set of two elements and let $H = \sum_{r,s} H_{r,s} \in \widehat{L}_{\mathbb{Q}}(I)$ be the Hausdorff series. Let $\mathbb{Z}_{(p)}$ be the local ring of $\mathbb{Z}$ relative to the prime ideal (p) and $(e_b)_{b\in B}$ a basis of $L_{\mathbb{Z}_{(p)}}(I)$ over $\mathbb{Z}$. It is also a basis of $L_{\mathbb{Q}}(I)$ over $\mathbb{Q}$.

Proposition 2.5.25. *Let r, s be two non-negative integers. If $H_{r,s} = \sum_b \lambda_b e_b$, where $\lambda_b \in \mathbb{Q}$, is the decomposition of H with respect to the basis $(e_b)_{b\in B}$, then for all $b \in B$,*

$$v_p(\lambda_b) \geq -(r+s-1)\theta. \tag{2.5.31}$$

Proof. The ring $A_{\mathbb{Z}_{(p)}}(I)$ is identified with the sub-$\mathbb{Z}_{(p)}$-module of $A_{\mathbb{Q}}(I)$ generated by the words $w \in \Delta(I)$ (the free monoid over I). As $L_{\mathbb{Z}_{(p)}}(I)$ is a direct factor of $A_{\mathbb{Z}_{(p)}}(I)$ (c.f. Remark 2.3.4),

$$L_{\mathbb{Z}_{(p)}}(I) = A_{\mathbb{Z}_{(p)}}(I) \cap L_{\mathbb{Q}}(I). \tag{2.5.32}$$

Let f be the integer such that $f \leq (r+s-1)\theta < f+1$. Relation (2.5.31) is then equivalent to $v_p(\lambda_b) \geq -f$ for all $b \in B$, that is, $H_{r,s} \in p^{-f} L_{\mathbb{Z}_{(p)}}$. But this is equivalent also by (2.5.32) to $H_{r,s} \in p^{-f} A_{\mathbb{Z}_{(p)}}(I)$.

By formula (2.5.8), it suffices to show that, for every integer $n \geq 1$ and all integers $r_1, \ldots, r_n$ and $s_1, \ldots, s_n$ such that $r_1 + \cdots + r_n = r$, $s_1 + \cdots + s_n = s$, and $r_i + s_i \geq 1$ for $1 \leq i \leq n$, we have

$$v_p(n \cdot r_1! \cdots r_n! s_1! \cdots s_n!) \leq f. \tag{2.5.33}$$

But by Lemma 2.5.24, $v_p(r_i! s_i!) \leq (r_i + s_i - 1)\theta$ and $v_p(n) \leq v_p(n!) \leq (n-1)\theta$; the left hand side of (2.5.33) is therefore bounded above by

$$\theta(n - 1 + \sum_{i=1}^{n}(r_i + s_i - 1)) = \theta(r+s-1);$$

as it is an integer, it is bounded by f, which completes the proof. □

Now let $\mathfrak{g}$ be a **normed Lie algebra over** $\mathbb{K}$, by which we mean a Lie algebra with a norm $\|\cdot\|$ such that

$$\|x+y\| \leq \sup\{\|x\|, \|y\|\}, \quad \|[x,y]\| \leq \|x\|\|y\|,$$

for $x, y \in \mathfrak{g}$. For every finite set I we define a continuous Lie algebra homomorphism $u \mapsto \tilde{u}$ of $\widehat{L}(I)$ into $\widehat{P}(\mathfrak{g}^I, \mathfrak{g})$. We see that if $u = \sum_\nu u_\nu$ with $u_\nu \in L^\nu(I)$ for $\nu \in \mathbb{N}^I$, then $u = \sum_\nu \tilde{u}_\nu$, where $\tilde{u}_\nu$ is the polynomial map $(t_i)_{i\in I} \mapsto u_\nu(\{t_i\})$. The composition formula (2.5.16) remains valid.

Let $H = \sum_{r,s} H_{r,s} \in \widehat{L}(\{U, V\})$ be the Hausdorff series. We shall show that the corresponding formal power series with continuous components

$$\tilde{H} = \sum_{r,s} \tilde{H}_{r,s} \in \widehat{P}(\mathfrak{g} \times \mathfrak{g}, \mathfrak{g})$$

is convergent.

Lemma 2.5.26. *Let r, s be non-negative integers such that $r + s \neq 0$ and let $\|\tilde{H}_{r,s}\|$ be the norm of the continuous polynomial $\tilde{H}_{r,s}$. Then*

$$\|\tilde{H}_{r,s}\| \leq a^{-(r+s-1)\theta}.$$

Proof. Let B be a Hall set relative to I and let $H_{r,s} = \sum_{b\in B} \lambda_b e_b$ be the decomposition of $H_{r,s}$ with respect to the corresponding basis of $L(\{U, V\})$. Then

$$|\lambda_b| \leq a^{-(r+s-1)\theta}. \tag{2.5.34}$$

Moreover, for $b \in B$ we have

$$\|\tilde{e}_b\| \leq 1. \tag{2.5.35}$$

We show more generally by induction on n that, for every alternant b of degree n in the two indeterminates U and V, $\|\tilde{b}\| \leq 1$. If $n = 1$, b is one of the projections of $\mathfrak{g} \times \mathfrak{g}$ onto $\mathfrak{g}$ and hence is of norm ≤ 1; if $n > 1$, there exist two alternants b_1 and b_2 of degrees $< n$ such that $b = [b_1, b_2]$. As the map $\gamma : (x, y) \mapsto [x, y]$ of $\mathfrak{g} \times \mathfrak{g}$ into $\mathfrak{g}$ is bilinear of norm ≤ 1, we have

$$\|\tilde{b}\| = \|\gamma \circ (\tilde{b}_1, \tilde{b}_2)\| \leq \|\tilde{b}_1\| \|\tilde{b}_2\| \leq 1.$$

Relations (2.5.34) and (2.5.35) imply the lemma. □

Proposition 2.5.27. *The formal power series $\tilde{H}$ is a convergent series. If G is the ball $\{x \in \mathfrak{g} : \|x\| < a^\theta\}$, the domain of absolute convergence of $\tilde{H}$ contains $G \times G$.*

Proof. If u and v are two positive real numbers such that $u < a^\theta$ and $v < a^\theta$, then by Lemma 2.5.26,

$$\|\tilde{H}_{r,s}\| u^r v^s \leq a^{-(r+s-1)\theta} u^r v^s = a^\theta (u a^{-\theta})^r (v a^{-\theta})^s$$

and therefore $\|\tilde{H}_{r,s}\| u^r v^s$ tends to 0 when $r + s$ tends to infinity. □

We denote by $h : G \times G \to \mathfrak{g}$ the analytic function defined by H, that is by the formula

$$h(x, y) = \sum_{r,s} \tilde{H}_{r,s}(x, y) = \sum_{r,s} H_{r,s}(x, y) \quad \text{for} \quad (x, y) \in G \times G.$$

This function is called the **Hausdorff function** of $\mathfrak{g}$. Let $(x, y) \in G \times G$, then

$$\|\tilde{H}_{r,s}(x, y)\| \leq \sup\{\|x\|, \|y\|\} \tag{2.5.36}$$

$$\|h(x, y)\| \leq \sup\{\|x\|, \|y\|\} \tag{2.5.37}$$

To see this, we note that (2.5.37) follows immediately from (2.5.36) and (2.5.37) is trivial for $r = s = 0$; if $r \geq 1$, then

$$\|\tilde{H}_{r,s}(x, y)\| \leq \tilde{H}_{r,s}\| \|x\|^r \|y\|^s \leq \|x\| \left(\frac{\|x\|}{a^\theta}\right)^{r-1} \left(\frac{\|y\|}{a^\theta}\right)^s \leq \|x\|;$$

we argue similarly if $s \geq 1$. In particular, $h(x, y) \in G$ for $(x, y) \in G \times G$.

Proposition 2.5.28. *Let G be the ball $\{x \in \mathfrak{g} : \|x\| < a^\theta\}$. The analytic map*

$$h : G \times G \to G$$

makes G into a group in which 0 is the identity element and $-x$ is the inverse of x for all $x \in G$. Moreover, if r is a real number such that $0 < r < a^\theta$, the ball

$$G_r = \{x \in \mathfrak{g} : \|x\| < r\}$$

is an open subgroup of G.

Proof. As $H(U, -U) = 0$ and $H(0, U) = H(U, 0) = U$, $h(x, -x) = 0$ and

$$h(0, x) = h(x, 0) = x$$

for $x \in G$. It therefore remains to prove the associativity formula

$$h(h(x, y), z) = h(x, h(y, z)) \quad \text{for} \quad x, y, z \in G. \tag{2.5.38}$$

Now recall that

$$H(H(U, V), W) = H(U, H(V, W))$$

in $\widehat{L}(\{U, V, W\})$ (Proposition 2.5.15), so we have

$$\tilde{H} \circ (\tilde{H} \times 1_{\mathfrak{g}}) = \tilde{H} \circ (1_{\mathfrak{g}} \times \tilde{H}) \tag{2.5.39}$$

in $\widehat{P}(\mathfrak{g} \times \mathfrak{g} \times \mathfrak{g}, \mathfrak{g})$ and (2.5.39) implies (2.5.38). □

Example 2.5.29. For an example, we let A be a unital associative algebra with a norm $\|\cdot\|$ satisfying the conditions:

$$\|x + y\| \leq \sup\{\|x\|, \|y\|\}$$
$$\|xy\| \leq \|x\| \|y\|$$
$$\|1\| = 1$$

for x, y in A, and complete with this norm. Again we have the substituting $u \mapsto \tilde{u}$ as in Example 2.5.22 and the composition formula is valid.

We take $I = \{U\}$ and consider the image $\tilde{e}$ and $\tilde{l}$ of the series $e(U) = \sum_{n=1}^{\infty} U^n/n!$ and $l(U) = \sum_{n=1}^{\infty} (-1)^{n-1} U^n/n$ in $\widehat{P}(A, A)$. Then

$$\left\| \widetilde{\left(\frac{U^n}{n!}\right)} \right\| \leq a^{-(n-1)\theta}, \quad \left\| \widetilde{\left(\frac{U^n}{n}\right)} \right\| \leq a^{-\frac{\log n}{\log p}}$$

by Lemma 2.5.24. Hence the radius of absolute convergence of the series $\tilde{e}$ (resp. $\tilde{l}$) is $\geq a^\theta$ (resp. ≥ 1). For $r > 0$, let $G_r = \{x \in A : \|x\| < r\}$; we write $G = G_{a^\theta}$. The series $\tilde{e}$ (resp. $\tilde{l}$) defines an analytic map e_A (resp. l_A) of G (resp. G_1) into A. We write:

$$\exp_A(x) = 1 + e_A(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \text{for} \quad x \in G$$

$$\log_A(x) = l_A(x - 1) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(x-1)^n}{n} \quad \text{for} \quad x - 1 \in G_1.$$

For $x \in G_r$ and $n \geq 1$,

$$\left\| \frac{x^n}{n} \right\| \leq \left\| \frac{x^n}{n!} \right\| < r^n a^{-(n-1)\theta} = r \left(\frac{r}{a^\theta}\right)^{n-1} < r$$

whence $e_A(G_r) \subseteq G_r$ and $l_A(G_r) \subseteq G_r$ for $r < a^\theta$. Since $e(l(X)) = X$ and $l(e(X)) = X$, we deduce that $e_A(l_A(x)) = l_A(e_A(x)) = x$ for $x \in G_r$. Then

$$\exp_A(\log_A(x)) = x \quad \text{for} \quad x \in 1 + G_r,$$

$$\log_A(\exp_A(x)) = x \quad \text{for} \quad x \in G_r.$$

That is, *the map* $\exp_A$ *defines an analytic isomorphism of* G_r *onto* $1 + G_r$ *and the inverse isomorphism is the restriction of* $\log_A$ *to* $1 + G_r$.

If A is given the bracket $[x, y] = xy - yx$, A becomes a complete normed Lie algebra, for

$$\|xy - yx\| \leq \sup\{\|xy\|, \|yx\|\} \leq \|x\| \|y\|.$$

Proposition 2.5.27 then implies that the domain of absolute convergence of $\tilde{H}$ contains $G \times G$ and $\tilde{H}$ therefore defines an analytic function $h : G \times G \to A$, which is

$$h(x, y) = \sum_{r,s} H_{r,s}(x, y).$$

Again, we have $\exp_A(x) \exp_A(y) = \exp_A(h(x, y))$ for x, y in G as in Example 2.5.22.

Chapter 3

Coxeter systems and tits systems

3.1 Coxeter systems

In this section, W denotes a group written multiplicatively, with identity element 1, and S denotes a set of generators of W such that $S = S^{-1}$ and $1 \notin S$. Every element of W is the product of a finite sequence of elements of S.

3.1.1 Length and reduced decompositions

Let $w \in W$. The **length** of w (with respect to S), denoted by $\ell_S(w)$ or simply by $\ell(w)$, is the smallest positive integer p such that w is the product of a sequence of p elements of S. A reduced decomposition of w (with respect to S) is any sequence $\boldsymbol{s} = (s_1, \dots, s_p)$ of elements of S such that $w = s_1 \cdots s_p$ and $p = \ell(w)$. In particular, we see 1 is the unique element of length 0 and S consists of the elements of length 1.

Proposition 3.1.1. *Let w and $\tilde{w}$ be in W. Then*

$$\ell(w\tilde{w}) \leq \ell(w) + \ell(\tilde{w}), \quad \ell(w^{-1}) = \ell(w), \quad |\ell(w) - \ell(\tilde{w})| \leq \ell(w\tilde{w}^{-1}). \tag{3.1.1}$$

Proof. Let $(s_1, \dots, s_p)$ and $(\tilde{s}_1, \dots, \tilde{s}_q)$ be reduced decompositions of w and $\tilde{w}$ respectively. Thus $\ell(w) = p$ and $\ell(\tilde{w}) = q$. Since, we see $\ell(w\tilde{w}) \leq p + q$. Since $S = S^{-1}$ and $w^{-1} = s_p^{-1} \cdots s_1^{-1}$ we have $\ell(w^{-1}) \leq p = \ell(w)$. Replacing w by w^{-1} gives the opposite inequality, so $\ell(w) = \ell(w^{-1})$. Replacing w by $w\tilde{w}^{-1}$ gives the relations

$$\ell(w) - \ell(\tilde{w}) \leq \ell(w\tilde{w}^{-1}), \quad \ell(w\tilde{w}^{-1}) = \ell(\tilde{w}w^{-1}). \tag{3.1.2}$$

Exchanging w and $\tilde{w}$ in (3.1.1) gives $\ell(\tilde{w}) - \ell(w) \leq \ell(\tilde{w}w^{-1})$, proving the last claim. $\square$

Corollary 3.1.2. *Let $\boldsymbol{s} = (s_1, \dots, s_p)$ and $\tilde{\boldsymbol{s}} = (\tilde{s}_1, \dots, \tilde{s}_q)$ be two sequences of elements of S such that $w = s_1 \cdots s_p$ and $\tilde{w} = \tilde{s}_1 \cdots \tilde{w}_q$. If the sequence $(s_1, \dots, s_p, \tilde{s}_1, \dots, \tilde{s}_q)$ is a reduced decomposition of $w\tilde{w}$, then $\boldsymbol{s}$ is a reduced decomposition of w and $\tilde{\boldsymbol{s}}$ is one of $\tilde{w}$.*

Proof. By hypothesis, $\ell(w) \leq p$, $\ell(\tilde{w}) \leq q$ and $\ell(w\tilde{w}) = p + q$. By Proposition 3.1.1, we must have $\ell(w) = p$ and $\ell(\tilde{w}) = q$, hence the corollary. $\square$

Corollary 3.1.3. *The map $d(w, \tilde{w}) = \ell(w\tilde{w}^{-1})$ defines a metric on W, which is invariant under right translations.*

Let n be a positive integer (possibly infinite) larger than 2 and consider the dihedral group D_n, which is defined by the following presentations

$$D_n = \langle \tau, \sigma \mid \tau^2 = \sigma^n = 1, (\tau\sigma)^2 = 1 \rangle = \langle \tau, \tilde{\tau} \mid \tau^2 = \tilde{\tau}^2 = 1, (\tau\tilde{\tau})^n = 1 \rangle.$$

(where the second presentation is obtained from the first one by setting $\tilde{\tau} = \tau\sigma$.)

Proposition 3.1.4. *Assume that S consists of two distinct elements s and $\tilde{s}$ of order 2.*

(a) *The subgroup C of W generated by $r = s\tilde{s}$ is normal, and W is the semi-direct product of the subgroup $N = \{1, s\}$ and C. Moreover, $[W : C] = 2$.*

(b) *Let n be the order (finite or infinite) of r. Then W is of order $2n$ and there is a unique isomorphism $\varphi : D_n \to W$ such that $\varphi(\tau) = s$ and $\varphi(\tilde{\tau}) = \tilde{s}$.*

Proof. We have $srs^{-1} = ss\tilde{s}s = \tilde{s}s = r^{-1}$, and hence $sr^n s^{-1} = r^{-n}$ for every integer n. Since W is generated by $\{s, \tilde{s}\}$ and hence by $\{s, r\}$, the subgroup C is normal. It follows that NC is a subgroup of W, and since NC contains s and $\tilde{s} = sr$, we have $W = NC = C \cup sC$. To prove (a), it is therefore enough to show that $W \neq C$. If $W = C$, the group W would be abelian, so $r^2 = s^2\tilde{s}^2 = 1$. But then the only elements of $W = C$ would be 1 and r, contradicting the hypothesis that W contains at least three elements, namely 1, s and $\tilde{s}$.

Since $s \neq \tilde{s}$, we have $r \neq 1$ and so $n \geq 2$. Since C is of order n and $[W : C] = 2$, the order of W is $2n$. The group W is the semi-direct product of N and C. In view of the formulas

$$sr^n s^{-1} = r^{-n}, \quad \tau\sigma^n\tau^{-1} = \sigma^{-n}$$

There exists an isomorphism $\varphi : D_n \to W$ such that $\varphi(\tau) = s$ and $\varphi(\sigma) = r$, and hence $\varphi(\tilde{\tau}) = \tilde{s}$. The uniqueness of φ follows from the fact that D_n is generated by $\{\tau, \tilde{\tau}\}$. □

In view of Proposition 3.1.4, a group generated by distinct elements of order two is also called a **dihedral group**. For such a group W of order $2n$ generated by s and $\tilde{s}$, denote by $\boldsymbol{s}_p$ (resp. $\tilde{\boldsymbol{s}}_p$) the sequence of length p whose odd (resp. even) numbered terms are equal to s and whose even (resp. odd) numbered terms are equal to $\tilde{s}$. If w_p (resp. $\tilde{w}_p$) be the product of the elements in $\boldsymbol{s}_p$ (resp. $\tilde{\boldsymbol{s}}_p$). We have

$$w_{2k} = (s\tilde{s})^k, \quad w_{2k+1} = (s\tilde{s})^k s, \quad \tilde{w}_{2k} = (\tilde{s}s)^k = (s\tilde{s})^{-k}, \quad \tilde{w}_{2k+1} = (s\tilde{s})^{-k-1}s. \tag{3.1.3}$$

If $\boldsymbol{s} = (s_1, \dots, s_p)$ is a reduced decomposition (with respect to $\{s, \tilde{s}\}$) of an element w of W, then clearly $s_i \neq s_{i+1}$ for $1 \leq i < p$. Hence, $\boldsymbol{s} = \boldsymbol{s}_p$ or $\boldsymbol{s} = \tilde{\boldsymbol{s}}_p$.

If $n = +\infty$, the elements $(s\tilde{s})^k$ and $(s\tilde{s})^k s$ for $n \in \mathbb{Z}$ are distinct. Hence, the elements w_p and $\tilde{w}_p$ are distinct, and if $\boldsymbol{s}$ is a reduced decomposition of w_p (resp. $\tilde{w}_p$) we necessarily have $\boldsymbol{s} = \boldsymbol{s}_p$ (resp. $\boldsymbol{s} = \tilde{\boldsymbol{s}}_p$). It follows from this that $\ell(w_p) = \ell(\tilde{w}_p) = p$ and that the set of reduced decompositions of the elements of W consists of the $\boldsymbol{s}_p$ and the $\tilde{\boldsymbol{s}}_p$. Moreover, every element of W has a unique reduced decomposition.

Suppose now that n is finite. If $p \geq 2n$, we have $w_p = w_{p-2n}$ and $\tilde{w}_p = \tilde{w}_{p-2n}$; if $n \leq p \leq 2n$, we have $w_p = \tilde{w}_{2n-p}$ and $\tilde{w}_p = w_{2n-p}$. Hence, neither $\boldsymbol{s}_p$ nor $\tilde{\boldsymbol{s}}_p$ are reduced decompositions if $p > n$. It follows that each elements of W is one of the $2n$ elements $w_0 = \tilde{w}_0$, w_p, and $\tilde{w}_p$ for $1 \leq p \leq n-1$, and $w_n = \tilde{w}_n$. These $2n$ elements are thus distinct and it follows as above that $\ell(w_p) = \ell(\tilde{w}_p) = p$ for $p \leq n$ and that the set of reduced decompositions of elements of W consists of the $\boldsymbol{s}_p$ and the $\tilde{\boldsymbol{s}}_p$ for $0 \leq p \leq n$. Every element of W except w_n has a unique reduced decomposition; w_n has two.

3.1.2 Coxeter systems and Coxeter groups

From now on we assume that the elements of S are of order 2. The system (W, S) is said to be a **Coxeter system** if it satisfies the following condition:

(C) For $s, \tilde{s}$ in S, let $m(s, \tilde{s})$ be the order of $s\tilde{s}$ and let I be the set of pairs $(s, \tilde{s})$ such that $m(s, \tilde{s})$ is finite. Then the generating set S and the relations $(s\tilde{s})^{m(s,\tilde{s})} = 1$ for $(s, \tilde{s})$ in I form a presentation of the group W.

When (W, S) is a Coxeter system, one also says, by abuse of language, that W is a **Coxeter system**.

Example 3.1.5. Let $S = \{s, \tilde{s}\}$ and consider the group W given by the following presentation

$$W = \langle s, \tilde{s} \mid s^2 = \tilde{s}^2 = 1, (s\tilde{s})^n = 1 \rangle.$$

Consider on the other hand the dihedral group D_n, which is given by the same presentation with $\{s, \tilde{s}\}$ replaced by $\{\tau, \tilde{\tau}\}$. Therefore (W, S) is a Coxeter system and there is an isomorphism $\psi : W \to D_n$ such that $\psi(s) = \tau$, $\psi(\tilde{s}) = \tilde{\tau}$.

Example 3.1.6. Suppose that (W, S) is a Coxeter system. There exists a homomorphism $\epsilon : W \to \{\pm 1\}$ characterized by $\epsilon(s) = -1$ for all $s \in S$. We call $\epsilon(w)$ the **signature** of w; it is equal to $(-1)^{\ell}(w)$. The formula $\epsilon(w\tilde{w}) = \epsilon(w)\epsilon(\tilde{w})$ thus translates into $\ell(w\tilde{w}) = \ell(w) + \ell(\tilde{w})$ mod 2.

Let I be a set. A **Coxeter matrix** of type I is a symmetric square matrix $M = (m_{ij})_{i,j\in I}$ whose entries are integers or $+\infty$ satisfying the relations

$$m_{ii} = 1, \quad m_{ij} \geq 2 \quad \text{for } i, j \in I, i \neq j.$$

A **Coxeter graph** of type I is (by abuse of language) a pair consisting of a graph Γ having I as its set of uertices and a map f from the set of edges of this graph to the set consisting of $+\infty$ and the set of integers ≥ 3.

A Coxeter graph (Γ, f) is associated to any Coxeter matrix M of type I as follows: the graph Γ has set of vertices I and set of edges the set pairs $\{i, j\}$ of elements of I such that $m_{ij} \geq 3$, and the map f associates to the edge $\{i, j\}$ the corresponding element m_{ij} of M. It is clear that this gives rise to a bijection between the set of Coxeter matrices of type I and the set of Coxeter graphs of type I.

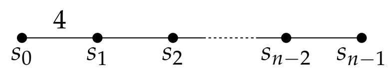
If (W, S) is a Coxeter system, the matrix $M = (m(s, \tilde{s}))_{s,\tilde{s}\in S}$, where $m(s, \tilde{s})$ is the order of $s\tilde{s}$, is a Coxeter matrix of type S which is called the **Coxeter matrix of (W, S)**: indeed, $m(s, s) = 1$ since $s^2 = 1$ for all $s \in S$, and $m(s, \tilde{s}) = m(\tilde{s}, s) \geq 2$ if $s \neq \tilde{s}$ since $s\tilde{s} = (\tilde{s}s)^{-1}$ is then $\neq 1$. The Coxeter graph (Γ, f) associated to M is called the Coxeter graph of (W, S). We remark that two vertices s and $\tilde{s}$ of S are joined if and only if s and $\tilde{s}$ do not commute.

Example 3.1.7. The Coxeter matrix of the dihedral group D_n is $\left(\begin{smallmatrix} 1 & n \\ n & 1 \end{smallmatrix}\right)$ and its Coxeter graph is represented by $\bullet\overset{n}{\text{———}}\bullet$ when $n \geq 3$, and by $\bullet \quad \bullet$ when $n = 2$.

Example 3.1.8. The path

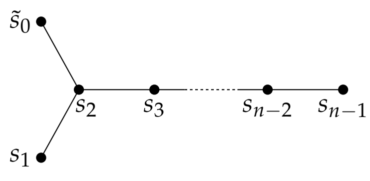
is the Coxeter graph of the symmetric group $\mathfrak{S}_n$ with respect to the generating system of adjacent transpositions $s_i = (i, i+1)$, $1 \leq i \leq n-1$ (We will see this later). An understanding of this particular example is very valuable, both because of the importance of the symmetric group as such and its role as the most accessible nontrivial example of a Coxeter group. We will frequently return to $\mathfrak{S}_n$ in order to concretely illustrate various general concepts and constructions.

Example 3.1.9. The graph



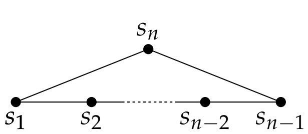
is the Coxeter graph of the group $\mathfrak{S}_n^B$ of all signed permutations of the set $[n] = \{1, 2, \ldots, n\}$. It can be thought of in terms of the following combinatorial model. Suppose that we have a deck of n cards, such that the j-th card has "$+j$" written on one side and "$-j$" on the other. The elements of $\mathfrak{S}_n^B$ can then be identified with the possible rearrangements of stacks of cards; that is, a group element is a permutation of $[n] = \{1, 2, \ldots, n\}$ (the order of the cards in the stack) together with the sign information $[n] \mapsto \{+, -\}$ (telling which side of each card is up). The Coxeter generators s_i $(1 \leq i \leq n-1)$ interchange the card in position i with that in position $i+1$ in the stack (preserving orientation), and s_0 flips the first card (the top card).

The group $\mathfrak{S}_n^B$ has a subgroup, denoted $\mathfrak{S}_n^D$, with Coxeter graph



where $\tilde{s}_0 = s_0 s_1 s_0$. In terms of the card model this group consists of the stacks with an even number of turned-over cards (i.e., with minus side up). The generators s_i $(1 \leq i \leq n-1)$ are adjacent interchanges as before, and $\tilde{s}_0$ flips cards first and the second cards together (as a package).

Example 3.1.10. The circuit



is the Coxeter graph of the group $\widetilde{\mathfrak{S}}_n$ of **affine permutations** of the integers. This is the group of all permutations σ of the set $\mathbb{Z}$ such that

$$\sigma(j+n) = \sigma(j) + n \quad \text{for } j \in \mathbb{Z}$$

and

$$\sum_{i=1}^{n} \sigma(i) = \binom{n+1}{2}$$

with composition as group operation. The Coxeter generators are the periodic adjacent transpositions $\tilde{s}_i = \prod_{j\in\mathbb{Z}}(i+jn, i+1+jn)$ for $i = 1, \dots, n$.

Example 3.1.11. The one-way infinite path

is the Coxeter graph of the group of permutations with finite support of the positive integers (i.e., permutations that leave all but a finite subset fixed). The generators are the adjacent transpositions $s_i = (i, i+1)$ with $i \geq 1$.

A Coxeter system (W, S) is said to be **irreducible** if the underlying graph of its Coxeter graph is connected and non-empty. Equivalently, S is non-empty and there exists no partition of S into two distinct subsets S_1 and S_2 of S such that every element of S_1 commutes with every element of S_2. More generally, let $(\Gamma_i)_{i\in I}$ be the family of connected components of Γ and let S_i be the set of vertices of Γ_i. Let $W_i = W_{S_i}$ be the subgroup of W generated by S_i. Then the (W_i, S_i) are irreducible Coxeter systems (Proposition 3.1.34) called the irreducible components of (W, S). Moreover, the group W is the restricted direct product[1]. Indeed, this follows from the following more general proposition:

Proposition 3.1.12. *Let $(S_i)_{i\in I}$ be a partition of S such that every element of S_i commutes with every element of S_j if $i \neq j$. For all $i \in I$, let W_i be the subgroup generated by S_i. Then W is the restricted direct product of the family $(W_i)_{i\in I}$.*

Proof. It is clear that for all $i \in I$ the subgroup $\tilde{W}_i$ generated by the union of the W_j for $j \neq i$ is also generated by $\tilde{S}_i = \bigcup_{j\neq i} S_i$. Thus $W_i \cap \widetilde{W}_i = W_\varnothing = 1$. Since W is generated by the union of the W_i, this proves the proposition. □

3.1.3 Reduced decompositions for a Coxeter system

Suppose that (W, S) is a Coxeter system. Let T be the set of conjugates in W of elements of S. The elements of T (i.e., the elements conjugate to some Coxeter generator) are called **reflections**. The definition shows that $S \subseteq T$ and that $t^2 = 1$ for $t \in T$. The elements of S are sometimes called **simple reflections**.

For any finite sequence $\boldsymbol{s} = (s_1, \dots, s_p)$ of elements of S, let $w_i = s_1 \cdots s_i$ and denote by $\widehat{T}(\boldsymbol{s})$ the sequence $(t_1, \dots, t_p)$ of elements of T defined by

$$t_j = (s_1 \cdots s_{j-1}) s_j (s_1 \cdots s_{j-1})^{-1} = w_j s_j w_j^{-1} \quad \text{where } w_j = s_1 \cdots s_j \text{ for } 1 \leq j \leq p.$$

Note that we have the following formulas for the sequence $\widehat{T}(\boldsymbol{s})$:

$$t_j w = s_1 \cdots \hat{s}_j \cdots s_p, \quad s_1 \cdots s_p = t_p t_{p-1} \cdots t_1. \tag{3.1.4}$$

For any element $t \in T$, denote by $n(\boldsymbol{s}, t)$ the number of integers j such that $t_j = t$.

Proposition 3.1.13. *Let (W, S) be a Coxeter system and T be the set of conjugates in W of elements of S.*

(a) Let $w \in W$ and $t \in T$. Then the number $(-1)^{n(\boldsymbol{s},t)}$ has the same value $\eta(w, t)$ for all sequences $\boldsymbol{s} = (s_1, \dots, s_p)$ of elements of S such that $w = s_1 \dots s_p$.

[1] A group G is the restricted direct product of a family $(G_i)_{i\in I}$ of subgroups if, for any finite subset J of I, the subgroup G_J of G generated by the G_i for $i \in J$ is the direct product of the G_i for $i \in J$ and if G is the union of the G_J. Equivalently, every element of G_i commutes with every element of G_j for $i \neq j$ and every element of G can be written uniquely as a product $\prod_{i\in I} g_i$ with $g_i \in G_i$ and $g_i = 1$ for all but finitely many indices i.

(b) For $w \in W$, let π_w be the map from $T \times \{\pm 1\}$ to itself defined by

$$\pi_w(t, \varepsilon) = (wtw^{-1}, \varepsilon \cdot \eta(w^{-1}, t)). \tag{3.1.5}$$

Then the map $w \mapsto \pi_w$ is a homomorphism from W to $\mathfrak{S}_{T \times \{\pm 1\}}$.

Proof. For $s \in S$, define a map π_s from $T \times \{\pm 1\}$ to itself by

$$\pi_s(t, \varepsilon) = (sts^{-1}, \varepsilon \cdot (-1)^{\delta_{st}}) \tag{3.1.6}$$

where $(t, \varepsilon) \in T \times \{\pm 1\}$. It is immediate that $\pi_s^2 = 1$, which shows that π_s is a permutation of $T \times \{\pm 1\}$. Let $\boldsymbol{s} = (s_1, \dots, s_p)$ be a sequence of elements of S. Put $w = s_p \cdots s_1$ and $\pi_{\boldsymbol{s}} = \pi_{s_p} \cdots \pi_{s_1}$. We shall show, by induction on p, that

$$\pi_{\boldsymbol{s}}(t, \varepsilon) = (wtw^{-1}, \varepsilon \cdot (-1)^{n(\boldsymbol{s}, t)}). \tag{3.1.7}$$

This is clear for $p = 0$. If $p > 1$, put $\tilde{\boldsymbol{s}} = (s_1, \dots, s_{p-1})$ and $\tilde{w} = s_{p-1} \cdots s_1$. Using the induction hypothesis, we obtain

$$\pi_{\boldsymbol{s}}(t, \varepsilon) = \pi_{s_q}(\tilde{w} t \tilde{w}^{-1}, \varepsilon \cdot (-1)^{n(\tilde{\boldsymbol{s}}, t)}) = (wtw^{-1}, \varepsilon \cdot (-1)^{n(\tilde{\boldsymbol{s}}, t) + \delta_{s_p, \tilde{w} t \tilde{w}^{-1}}}).$$

But $\widehat{T}(\boldsymbol{s}) = (\widehat{T}(\tilde{\boldsymbol{s}}), \tilde{w}^{-1} s_p \tilde{w})$ and $n(\boldsymbol{s}, t) = n(\tilde{\boldsymbol{s}}, t) + \delta_{s_p, \tilde{w} t \tilde{w}^{-1}}$, proving formula (3.1.7).

Now let $s, \tilde{s} \in S$ be such that $r = s\tilde{s}$ has finite order n. Let $\boldsymbol{s} = (s_1, \dots, s_{2n})$ be the sequence of elements of S defined by $s_j = s$ for j odd and $s_j = \tilde{s}$ for j even. Then $s_{2n} \cdots s_1 = r^{-n} = 1$ and by definition we have

$$t_j = r^{j-1} s \quad \text{for} \quad 1 \leq j \leq 2n.$$

Since r is of order n, the elements $t_1, \dots, t_n$ are distinct and $t_{j+n} = t$, for $1 \leq j \leq n$. For all $t \in T$, the integer $n(\boldsymbol{s}, t)$ is thus equal to 0 or 2 and (3.1.7) shows that $\pi_{\boldsymbol{s}} = 1$. In other words, $(\pi_s \pi_{\tilde{s}})^n = 1$. Thus, by the definition of Coxeter systems, there exists a homomorphism $w \mapsto \pi_w$ from W to $\mathfrak{S}_{T \times \{\pm 1\}}$ such that π_s is given by the right-hand side of (3.1.7). Then by the homomorphism property, we have $\pi_w = \pi_{\boldsymbol{s}}$ for every sequence $\boldsymbol{s} = (s_1, \dots, s_p)$ such that $w = s_p \cdots s_1$ and from (3.1.7) we see $(-1)^{n(\boldsymbol{s}, t)}$ shares a common value $\eta(w, t)$ for such a sequence $\boldsymbol{s}$. This proves the proposition. ☐

We now come to the so-called "exchange property," which is a fundamental combinatorial property of a Coxeter group. In its basic version, the condition $t \in T$ in the statement below is weakened to $t \in S$, hence the adjective "strong" for the version given here.

Proposition 3.1.14 (Strong Exchange Property). *Let (W, S) be a Coxeter system, $w \in W$ and $t \in T$. Let $\eta(w, t)$ be the number defined in Proposition 3.1.13. Then the following conditions are equivalent.*

(i) $\ell(tw) < \ell(w)$;

(ii) $\eta(w, t) = -1$.

If these conditions are satisfied, then for any sequence $\boldsymbol{s} = (s_1, \dots, s_p)$ of elements of S with $w = s_1 \cdots s_p$ there exists an integer j such that

$$t_j = s_1 \cdots s_{j-1} s_j (s_1 \cdots s_{j-1})^{-1} = t.$$

Proof. First, assume that $\eta(w, t) = -1$, and choose a reduced decomposition $\boldsymbol{s} = (s_1, \dots, s_p)$ of w. Then by Proposition 3.1.13 we have $\eta(w, t) = (-1)^{n(\boldsymbol{s}, t)}$, whence $n(\boldsymbol{s}, t)$ is odd. In particular, since $n(\boldsymbol{s}, t) \neq 0$, we have $t = t_j$ for some j, so

$$\ell(tw) = \ell(s_1 \cdots \hat{s}_j \cdots s_p) \leq p - 1 < p = \ell(w).$$

Conversely, if $\eta(w, t) = 1$, then using (3.1.5), we get

$$\begin{aligned} \pi_{(tw)^{-1}}(t, \varepsilon) &= \pi_{w^{-1}} \pi_t(t, \varepsilon) = \pi_{w^{-1}}(t, -\varepsilon) \\ &= (w^{-1} tw, -\varepsilon \cdot \eta(w, t)) = (w^{-1} tw, -\varepsilon). \end{aligned}$$

This means that $\eta(tw, t) = -1$, which by the implication (ii)⇒(i) already proved shows that $\ell(ttw) < \ell(tw)$, that is, $\ell(w) < \ell(tw)$.

Fianlly, assume the given conditions, and choose a sequence $\boldsymbol{s} = (s_1, \dots, s_p)$ of elements of S with $w = s_1 \cdots s_p$. Then, since $\eta(w, t) = (-1)^{n(\boldsymbol{s}, t)}$, we deduce that $n(\boldsymbol{s}, t)$ is odd, and hence that $t = t_j$ for some j. Therefore, $tw = s_1 \cdots \hat{s}_j \cdots s_p$. ☐

Corollary 3.1.15. *Let (W, S) be a Coxeter system, $w \in W$ and $t \in T$. If $\boldsymbol{s} = (s_1, \ldots, s_p)$ is a reduced sequence for w, then the follwing are equivalent:*

(i) $\ell(tw) < \ell(w)$;

(ii) $tw = s_1 \cdots \hat{s}_j \cdots s_p$ *for some j;*

(iii) $t = (s_1 \cdots s_{j-1})s_j(s_1 \cdots s_{j-1})^{-1}$ *for some j.*

Proof. The equivalence (ii)$\Leftrightarrow$(iii) is easy to see (and does not require the hypothesis that $\boldsymbol{s}$ is reduced). Proposition 3.1.14 shows that (i)$\Rightarrow$(ii), and the converse is obvious. $\square$

We now consider the following sets:

$$T_L(w) = \{t \in T : \ell(tw) < \ell(w)\}, \quad T_R(w) = \{t \in T : \ell(wt) < \ell(w)\}. \tag{3.1.8}$$

In this notation "L" and "R" are mnemonic for "left" and "right." The set $T_L(w)$ is called the set of **left associated reflections** to w, and similarly for $T_R(w)$. Corollary 3.1.15 gives some useful characterizations of the set $T_L(w)$, and we have a similar description for $T_R(w)$ by applying these to w^{-1}, since clearly $T_R(w) = T_L(w^{-1})$.

Proposition 3.1.16. *Let (W, S) be a Coxeter system, $w \in W$, $\boldsymbol{s} = (s_1, \ldots, s_p)$ be a sequence such that $w = s_1 \cdots s_p$, and $\widehat{T}(\boldsymbol{s}) = (t_1, \ldots, t_p)$. Then the sequence $\boldsymbol{s}$ is a reduced decomposition of w if and only if the t_j are distinct, and in that case $T_L(w) = \{t_1, \ldots, t_p\}$ and $|T_L(w)| = \ell(w)$.*

Proof. If $t \in T_L(w)$, then $n(\boldsymbol{s}, t) = -1$ so t equals to at least one of the t_j's, hence $T_w \subseteq \{t_1, \ldots, t_p\}$, whence $|T_w| \leq \ell(w)$ since $\boldsymbol{s}$ is arbitrary. If the t_j are distinct, then $n(\boldsymbol{s}, t)$ is equal to 1 or 0 according to whether t does or does not belong to $\{t_1, \ldots, t_p\}$. It follows that $T_w = \{t_1, \ldots, t_p\}$ and that $p = |T_w| \leq \ell(w)$, which implies that $\boldsymbol{s}$ is reduced.

Suppose finally that $t_i = t_j$ with $i < j$. This gives $s_i = us_ju^{-1}$ with $u = s_{i+1} \cdots s_{j-1}$, hence

$$w = w_{i-1}s_ius_js_{j+1} \cdots s_p = w_{i-1}us_js_js_{j+1} \cdots s_p = s_1 \cdots \hat{s}_i \cdots \hat{s}_j \cdots s_p.$$

This shows that $\boldsymbol{s}$ is not a reduced decomposition of w. $\square$

Corollary 3.1.17. *Let (W, S) be a Coxeter system and $w \in W$. Then the set $T_L(w)$ consists of elements in T of the form $w_2sw_2^{-1}$ corresponding to the triples $(w_1, w_2, s) \in W \times W \times S$ such that $w = w_2sw_1$ and $\ell(w_1) + \ell(w_2) + 1 = \ell(w)$.*

Proof. By Proposition 3.1.16, if $\boldsymbol{s} = (s_1, \ldots, s_p)$ is a reduced decomposition of w and $\widehat{T}(\boldsymbol{s}) = (t_1, \ldots, t_p)$, then $T_L(w) = \{t_1, \ldots, t_p\}$. Now each element t_j can be written as

$$t_j = w_2s_jw_2^{-1} \quad \text{where } w_2 = s_1 \cdots s_{j-1}$$

and therefore

$$w = w_2s_jw_1 \quad \text{where } w_1 = s_{j+1} \cdots s_p.$$

Moreover, since $\boldsymbol{s}$ is reduced, we have $\ell(w_2) = j - 1$ and $\ell(w_1) = p - j$, whence $\ell(w) = \ell(w_1) + \ell(w_2) + 1$.

Conversely, if $t = w_2sw_s^{-1}$ where (w_1, w_2, s) is a triple such that $w = w_2sw_1$ and $\ell(w_1) + \ell(w_2) + 1 = \ell(w)$, then

$$\ell(tw) = \ell(w_2sw_2^{-1}w) = \ell(w_2w_1) \leq \ell(w_2) + \ell(w_1) = \ell(w) - 1 < \ell(w)$$

whence $t \in T_L(w)$. This completes the proof. $\square$

We will quite often need to refer to the associated simple reflections, for which we introduce the following special notation and terminology:

$$D_L(w) = T_L(w) \cap S = \{s \in S : \ell(sw) < \ell(w)\}, \quad D_R(w) = \{s \in S : \ell(ws) < \ell(w)\}.$$

The set $D_R(w)$ is called the **right descent set**, and similarly for $D_L(w)$. Their elements are called **right** (resp. **left**) **descents**. The reason for this terminology will become clear when we specialize to the symmetric groups. Still, we have $D_R(w) = D_L(w^{-1})$.

Corollary 3.1.18. *Let (W, S) be a Coxeter system. For all $s \in S$ and $w \in W$, the following hold:*

(a) $s \in D_L(w)$ if and only if some reduced expression for w begins with the letter s.

(b) $s \in D_R(w)$ if and only if some reduced expression for w ends with the letter s.

Proof. One direction is clear, and the other direction follows easily from Corollary 3.1.15. $\square$

Proposition 3.1.19 (Deletion Property). *Let (W, S) be a Coxeter system, $w \in W$, and $\boldsymbol{s} = (s_1, \dots, s_p)$ be a sequence such that $w = s_1 \cdots s_p$. If $\boldsymbol{s}$ is not reduced for w, then there exists indices $i < j$ such that $w = s_1 \cdots \hat{s}_i \cdots \hat{s}_j \cdots s_p$.*

Proof. This is contained in the proof of Proposition 3.1.16. $\square$

Corollary 3.1.20. *Let (W, S) be a Coxeter system. Then any expression $w = s_1 \cdots s_p$ contains a reduced expression for w as a subword, obtainable by deleting an even number of letters.*

Proposition 3.1.21. *Assume that (W, S) is a Coxeter system. Then, two elements s and $\tilde{s}$ of S are conjugate in W if and only if the following condition is satisfied: there exists a finite sequence $(s_1, \dots, s_p)$ of elements of S such that $s_1 = s, s_p = \tilde{s}$ and $s_i s_{i+1}$ is of finite odd order for $1 \leq i < p$.*

Proof. Let s and $\tilde{s}$ in S be such that $r = s\tilde{s}$ is of finite order $2n + 1$. Then we have $srs = r^{-1}$, hence

$$r^n s r^{-n} = r^n r^n s = r^{-1} s = \tilde{s} s s = \tilde{s}$$

and $\tilde{s}$ is conjugate to s.

For a fixed s in S, let C_s be the set of $\tilde{s} \in S$ satisfying the prescribed condition. Then by the above argument, the elements s_i and s_{i+1} are conjugate for $1 \leq i < p$ by the above, hence every element $\tilde{s}$ of C_s is conjugate to s. Now define a map $f : S \to \{\pm 1\}$ which equals to 1 on C_s and to -1 on $S \setminus C_s$. Let s_1 and s_2 in S be such that $s_1 s_2$ is of finite order n. Then $f(s_1)f(s_2) = 1$ if s_1 and s_2 are both in C_s or both in $S \setminus C_s$. In the other case, $f(s_1)f(s_2) = -1$, but n is even. Thus $(f(s_1)f(s_2))^n = 1$ in all cases. Since (W, S) is a Coxeter system, there exists a extension of f on W, which we still denote by f. Let $\tilde{s} \in S$ be a conjugate of s. Since s belongs to the kernel of f, so does $\tilde{s}$, hence $f(\tilde{s}) = 1$ and finally $\tilde{s} \in C_s$. $\square$

3.1.4 Characterization of Coxeter systems

The statement that the Exchange Property is fundamental in the combinatorial theory of Coxeter groups can be made very precise: It characterizes such groups! This is often a convenient way to prove that a given group is a Coxeter group.

Throughout this paragraph we assume that W is an arbitrary group and that $S \subseteq W$ is a generating subset such that $s^2 = 1$ for all $s \in S$. For such a pair, we define the "Exchange Property" and "Deletion Property" as follows.

(E) Let $w \in W$ and $s \in S$ be such that $\ell(sw) \leq \ell(w)$. Then for any reduced decomposition $(s_1, \dots, s_p)$ of w, there exists an integer j such that $s = s_1 \cdots s_{j-1} s_j (s_1 \cdots s_{j-1})^{-1}$.

(D) Let $w \in W$ and $s \in S$. If $(s_1, \dots, s_p)$ is not a reduced decomposition of w, then there exists integers $i < j$ such that $w = s_1 \cdots \hat{s}_i \cdots \hat{s}_j \cdots s_p$.

By Proposition 3.1.14, we see (W, S) satisfies this condition if it is a Coxeter system.

Proposition 3.1.22. *Let (W, S) be a system satisfying (E). Let $s \in S$, $w \in W$ and $\boldsymbol{s} = (s_1, \dots, s_p)$ be a reduced decomposition of w. Then one of the following must hold:*

(i) *$\ell(sw) = \ell(w) + 1$ and $(s, s_1, \dots, s_p)$ is a reduced decomposition of sw.*

(ii) *$\ell(sw) = \ell(w) - 1$ and there exists an integer j such that $(s_1, \dots, \hat{s}_j, \dots, s_p)$ is a reduced decomposition of sw and $(s, s_1, \dots, \hat{s}_j, \dots, s_p)$ is a reduced decomposition of w.*

Proof. Put $\tilde{w} = sw$, then we have $\ell(\tilde{w}) = \ell(w) \pm 1$. We distinguish two cases. If $\ell(\tilde{w}) > \ell(w)$ then $\ell(\tilde{w}) = p + 1$ and $\tilde{w} = ss_1 \cdots s_p$, so $(s, s_1, \dots, s_p)$ is a reduced decomposition of $\tilde{w}$. If $\ell(\tilde{w}) \leq \ell(w)$, then by (E), there exists an integer j such that $s = t_j$. Then $\tilde{w} = s_1 \cdots \hat{s}_j \cdots s_p$ and hence $w = ss_1 \cdots \hat{s}_j \cdots s_p$. It then follows that $\ell(\tilde{w}) = p - 1$ and that $(s_1, \dots, \hat{s}_j, \dots, s_p)$ is a reduced decomposition of $\tilde{w}$. $\square$

Lemma 3.1.23. *Let (W, S) be a system satisfying (E). Let $w \in W$ have length $p \geq 1$, let D be the set of reduced decompositions of w and let $f : D \to E$ be a map from D to a set E. Assume that $f(\boldsymbol{s}) = f(\tilde{\boldsymbol{s}})$ if the elements $\boldsymbol{s} = (s_1, \ldots, s_p)$, $\tilde{\boldsymbol{s}} = (\tilde{s}_1, \ldots, \tilde{s}_p)$ of D satisfy one of the following hypotheses:*

(a) $s_1 = \tilde{s}_1$ *or* $s_p = \tilde{s}_p$.

(b) There exist s and $\tilde{s}$ in S such that $s_i = \tilde{s}_j = s$ and $s_j = \tilde{s}_i = \tilde{s}$ for every i odd and j even.

Then f is constant.

Proof. Let $\boldsymbol{s}, \tilde{\boldsymbol{s}} \in D$ and put $\boldsymbol{t} = (\tilde{s}_1, s_1, \ldots, s_{p-1})$. We are going to show that if $f(\boldsymbol{s}) \neq f(\tilde{\boldsymbol{s}})$ then $\boldsymbol{t} \in D$ and $f(\boldsymbol{t}) \neq f(\boldsymbol{s})$. Indeed, $w = \tilde{s}_1 \cdots \tilde{s}_p$; and hence $\tilde{s}_1 w = \tilde{s}_2 \cdots \tilde{s}_p$ is of length $< p$. By Proposition 3.1.22, there exists an integer j such that the sequence $\boldsymbol{u} = (\tilde{s}_1, s_1, \ldots, \hat{s}_j, \ldots, s_p)$ belongs to D. We have $f(\boldsymbol{u}) = f(\tilde{\boldsymbol{s}})$ by condition (a); if $j \neq p$ we would have $f(\boldsymbol{s}) = f(\boldsymbol{u})$ for the same reason, and hence $f(\boldsymbol{s}) = f(\tilde{\boldsymbol{s}})$ contrary to our hypothesis. Thus $j = p$ and hence $\boldsymbol{t} = \boldsymbol{u} \in D$ and $f(\boldsymbol{t}) = f(\tilde{\boldsymbol{s}}) \neq f(\boldsymbol{s})$.

For any integer j with $0 \leq j \leq p+1$, define a sequence $\boldsymbol{s}_j$ of p elements of S as follows:

$$\begin{aligned}
\boldsymbol{s}_0 &= (\tilde{s}_1, \ldots, \tilde{s}_p) \\
\boldsymbol{s}_{p-k+1} &= (s_1, \tilde{s}_1, \ldots, s_1, \tilde{s}_1, s_1, \ldots, s_k) \quad \text{for } p-k \text{ even and } 0 \leq k \leq p \\
\boldsymbol{s}_{p-k+1} &= (\tilde{s}_1, s_1, \ldots, s_1, \tilde{s}_1, s_1, \ldots, s_k) \quad \text{for } p-k \text{ odd and } 0 \leq k \leq p.
\end{aligned}$$

Denote by (H_j) the assertion "$\boldsymbol{s}_j, \boldsymbol{s}_{j+1} \in D$ and $f(\boldsymbol{s}_j) \neq f(\boldsymbol{s}_{j+1})$." By the preceding argument, (H_j) implies (H_{j+1}) for $0 \leq j < p$, and (H_p) is not satisfied by condition (b). Hence, (H_0) is not satisfied. Since $\boldsymbol{s}_0 = \tilde{\boldsymbol{s}}$ and $\boldsymbol{s}_1 = \boldsymbol{s}$, it follows that $f(\boldsymbol{s}) = f(\tilde{\boldsymbol{s}})$. □

Proposition 3.1.24. *Let (W, S) be a system satisfying (E). Let M be a monoid (with unit element 1) and $f : S \to M$ a map. For $s, \tilde{s} \in S$, let $m(s, \tilde{s})$ be the order of $s\tilde{s}$ and put*

$$\alpha(s, \tilde{s}) = \begin{cases} (f(s)f(\tilde{s}))^k & \text{if } m(s, \tilde{s}) = 2k, k \text{ finite} \\ (f(s)f(\tilde{s}))^k f(s) & \text{if } m(s, \tilde{s}) = 2k+1, k \text{ finite} \\ 1 & \text{if } m(s, \tilde{s}) = \infty. \end{cases}$$

If $\alpha(s, \tilde{s}) = \alpha(\tilde{s}, s)$ whenever $s \neq \tilde{s}$ are in S, then there exists a map $f : W \to M$ such that

$$g(w) = f(s_1) \cdots f(s_p)$$

for all $w \in W$ and any reduced decomposition $(s_1, \ldots, \tilde{s}_p)$ of w.

Proof. For any $w \in W$, let D_w be the set of reduced decompositions of w and $f_w : D_w \to M$ the map defined by

$$f_w(s_1, \ldots, s_p) = f(s_1) \cdots f(s_p).$$

We are going to prove by induction on the length of w that f_w is constant, which will establish the proposition. The cases $\ell(w) = 0, 1$ being trivial, we assume that $p \geq 2$ and that our assertion is proved for the elements to with $\ell(w) < p$. Let w be of length p and $\boldsymbol{s}, \tilde{\boldsymbol{s}} \in D_w$; by Lemma 3.1.23 it suffices to prove that $f_w(\boldsymbol{s}) = f_w(\tilde{\boldsymbol{s}})$ in cases (a) and (b) of that Lemma.

In case (a), the formula

$$f_w(s_1, \ldots, s_p) = f(s_1) f_{w_2}(s_2, \ldots, s_p) = f_{w_1}(s_1, \ldots, s_{p-1}) f(s_p)$$

for $w_1 = s_1 \ldots s_{p-1}$ and $w_2 = s_2 \ldots s_p$ and the induction hypothesis show that $f_w(\boldsymbol{s}) = f_w(\boldsymbol{s})$ if $s_1 = \tilde{s}_1$ or $s_p = \tilde{s}_p$.

Suppose that there exist two elements s and $\tilde{s}$ of S such that $s_i = \tilde{s}_j = s$ and $s_j = \tilde{s}_i = \tilde{s}$ for every i odd and j even. It suffices to treat the case $s \neq \tilde{s}$, since otherwise $\boldsymbol{s} = \tilde{\boldsymbol{s}}$. In this case, the sequences $\boldsymbol{s}$ and $\tilde{\boldsymbol{s}}$ are then two distinct reduced decompositions of w in the dihedral group generated by s and $\tilde{s}$, whence the order n of $s\tilde{s}$ is finite. In the notation of (3.1.3), we have $\boldsymbol{s} = \boldsymbol{s}_n$ and $\tilde{\boldsymbol{s}} = \tilde{\boldsymbol{s}}_n$. Consequently, $f_w(\boldsymbol{s}) = \alpha(\boldsymbol{s}, \tilde{\boldsymbol{s}})$ and $f_w(\tilde{\boldsymbol{s}}) = \alpha(\tilde{\boldsymbol{s}}, \boldsymbol{s})$, hence $f_w(\boldsymbol{s}) = f_w(\tilde{\boldsymbol{s}})$. □

Theorem 3.1.25. *For a system (W, S), the following conditions are equivalent.*

(i) (W, S) is a Coxeter system;

(ii) (W, S) *has the Exchange Property;*

(iii) (W, S) *has the Deletion Property.*

Proof. Denote by (E) the exchange Property, and by (D) the Deletion Property. Proposition 3.1.14 shows that any Coxeter system satisfies (E), and Proposition 3.1.19 shows that any coxeter system satisfies (D). Conversely, suppose that (E) is satisfied. Let G be a group and $f : S \to G$ a map such that $(f(s)f(\tilde{s}))^n = 1$ whenever s and $\tilde{s}$ belong to S and $s\tilde{s}$ is of finite order n. Then by f satisfies the condition in Proposition 3.1.24, so there exists a map $g : W \to G$ such that

$$g(w) = f(w_1) \cdots f(w_p)$$

whenever $w = s_1 \cdots s_p$ is of length p. To prove that (W, S) is a Coxeter system, it suffices to prove that g is a homomorphism, which is a consequence of the formula

$$g(sw) = f(s)g(w) \quad \text{for} \quad s \in S, w \in W.$$

since S generates W. This follows by applying Proposition 3.1.22.

Fianlly, we show that (iii)⇒(i). Suppose that $\ell(sw) < \ell(w) = p$. Then, by the Deletion Property, two letters can be deleted from $ss_1 \cdots s_p$, giving a new expression for sw. If s is not one of these letters, then $ss_1 \cdots s_p = ss_1 \cdots \hat{s}_i \cdots \hat{s}_j \cdots s_p$ would give

$$\ell(w) = \ell(s_1 \cdots \hat{s}_i \cdots \hat{s}_j \cdots s_p) < p,$$

a contradiction. Hence, s must be one of the deleted letters and we obtain $sw = ss_1 \cdots s_p = s_1 \cdots \hat{s}_j \cdots s_p$. This completes the proof. □

Remark 3.1.26. The Exchange Property is stated above in its "left" version, since we are acting with s on the left of w. There is, of course, a "right" version (replace sw by ws), which is equivalent as a consequence of Theorem 3.1.25.

3.1.5 Coxeter system defined by partitions

Proposition 3.1.27. *Suppose that (W, S) is a Coxeter system. For any $s \in S$, let P_s be the set of elements $w \in W$ such that $\ell(sw) > \ell(w)$. We have the following properties:*

(A) $\bigcap_{s \in S} P_s = \{1\}$.

(B) *For any s in S, the sets P_s and sP_s form a partition of W.*

(C) *Let $s, \tilde{s}$ be in S and let w be in W. If $w \in P_s$ and $w\tilde{s} \notin P_s$ then $s = w\tilde{s}w^{-1}$.*

Proof. For (A), let $w \neq 1$ be in W and let $(s_1, \ldots, s_p)$ be a reduced decomposition of w. Then $p \geq 1$ and $(s_2, \ldots, s_p)$ is a reduced decomposition of $s_1 w$, so $\ell(w) = p$ and $\ell(s_1 w) = p - 1$. Hence $w \notin P_{s_1}$.

To prove (B), let $w \in W$ and $s \in S$. If $\ell(sw) = l(w) + 1$ then $w \in P_s$. On the other hand, if $\ell(sw) = \ell(w) - 1$, put $\tilde{w} = sw$ so that $w = s\tilde{w}$; then $\ell(\tilde{w}) < \ell(s\tilde{w})$ hence $\tilde{w} \in P_s$ and so $w \in sP_s$.

Let $s, \tilde{s} \in S$ and $w \in W$ be as in (C); let p be the length of w. From $w \in P_s$ it follows that $\ell(sw) = p + 1$; and from $w\tilde{s} \notin P_s$ it follows that $\ell(sw\tilde{s}) = \ell(w\tilde{s}) - 1 \leq p$. Since $\ell(sw\tilde{s}) = \ell(sw) \pm 1$, we have finally that $\ell(w\tilde{s}) = p + 1$ and $\ell(sw\tilde{s}) = p$. Let $(s_1, \ldots, s_p)$ be a reduced decomposition of w and $s_{p+1} = \tilde{s}$; then $(s_1, \ldots, s_{p+1})$ is a reduced decomposition of the element $w\tilde{s}$ of length $p + 1$. Since $\ell(sw\tilde{s}) = p$, by the exchange condition there exists an integer j with $1 \leq j \leq p + 1$ such that

$$s = (s_1 \cdots s_{j-1}) s_j (s_1 \cdots s_{j-1})^{-1}.$$

If $1 \leq j \leq p$, we would have $sw = s_1 \cdots \hat{s}_j \cdots s_p$, contradicting the formula $\ell(sw) = p + 1$. Thus $j = p + 1$ and we get $s = w\tilde{s}w^{-1}$. □

Conversely, we have the following result:

Proposition 3.1.28. *Let $(P_s)_{s \in S}$ be a family of subsets of W satisfying (C) and the following conditions:*

(A′) $1 \in P_s$ *for all* $s \in S$.

(B′) *The sets P_s and sP_s are disjoint for all $s \in S$.*

Then (W, S) is a Coxeter system and P_s consists of the elements $w \in W$ such that $\ell(sw) > \ell(w)$.

Proof. Let $s \in S$ and $w \in W$. If $w \notin P_s$, let $(s_1, \dots, s_p)$ be a reduced decomposition of w and $w_j = s_1 \cdots s_j$ for $1 \leq i \leq p$; also put $w_0 = 1$. Since $w_0 \in P_s$ by (A') and since $w_p = w$ is not in P_s, there exists an integer j with $1 \leq j < p$ such that $w_{j-1} \in P_s$ and $w_j = w_{j-1}s_j$ does not belong to P_s. By (C), we therefore have

$$ss_1 \cdot ss_{j-1} = s_1 \cdots s_{j-1}s_j.$$

Which implies that $sw = s_1 \cdots \hat{s}_j \cdots s_p$ and $\ell(sw) < \ell(w)$. Note that we have also showed that the exchange condition holds for (W, S), so it is a Coxeter system.

On the other hand, assume that $w \in P_s$. Put $\tilde{s} = sw$, so that $\tilde{w} \notin P_s$ by (B'). By the preceding argument, we then have $\ell(s\tilde{w}) < \ell(\tilde{w})$, that is $\ell(w) < \ell(sw)$. It then follows that P_s consists of those $w \in W$ such that $\ell(sw) > \ell(w)$. □

3.1.6 Subgroups of a Coxeter group

Let (W, S) is a Coxeter system. For any subset X of S, we denote by W_X the subgroup of W generated by X.

Proposition 3.1.29. *Let (W, S) be a a Coxeter system and w be in W. There exists a subset S_w of S such that $\{s_1, \dots, s_p\} = S_w$ for any reduced decomposition $(s_1, \dots, s_p)$ of w.*

Proof. Denote by M the monoid consisting of the subsets of S with the composition law $(A, B) \mapsto A \cup B$; the identity element of M is $\varnothing$. Put $f(s) = \{s\}$ for $s \in S$. We are going to apply Proposition 3.1.24 to M and f. We have $\alpha(s, \tilde{s}) = \{s, \tilde{s}\}$ for $s, \tilde{s}$ in S if $m(s, \tilde{s})$ is finite, hence there exists a map $g : w \mapsto S_w$ from W to M such that $g(w) = f(s_1) \cup \cdots \cup (s_p)$. Therefore $S_w = \{s_1, \dots, s_p\}$ for any $w \in W$ and any reduced decomposition $(s_1, \dots, s_p)$ of w. □

Example 3.1.30. Let n be an integer and consider the Coxeter group D_n generated by $\{s, \tilde{s}\}$. Recall that if n is finite then the element w_n in (3.1.3) have two reduced decomposition, and we have $S_{w_n} = \{s, \tilde{s}\}$, verifying Proposition 3.1.29.

Corollary 3.1.31. *Let (W, S) is a Coxeter system. For any subset X of S, the subgroup W_X of W consists of the elements w of W such that $S_w \subseteq X$.*

Proof. If $w = (s_1, \dots, s_p)$ with $s_1, \dots, s_p$ in S, then $w^{-1} = s_p \cdots s_1$, hence $S_w = S_{w^{-1}}$. Proposition 3.1.22 shows that $S_{sw} \subseteq \{s\} \cup S_w$ for $s \in S$ and $w \in W$, which implies the formula $S_{w\tilde{w}} \subseteq S_w \cup S_{\tilde{w}}$ by induction on the length of w. Therefore, the set U of $w \in W$ such that $S_w \subseteq X$ is a subgroup of W; we have $X \subseteq U \subseteq W_X$, hence $U = W_X$. □

Corollary 3.1.32. *Let (W, S) is a Coxeter system. For any subset X of S, we have $W_X \cap S = X$.*

Proof. This follows from Corollary 3.1.31 and the formula $S_s = \{s\}$ for s in S. □

Corollary 3.1.33. *Let (W, S) is a Coxeter system. For any subset X of S and any w in W_X, the length of w with respect to the generating set X of W_X is equal to $\ell_S(w)$.*

Proof. Let $(s_1, \dots, s_p)$ be a reduced decomposition of w considered as an element of W. We have $w = s_1 \dots s_p$ and $s_i \in X$ for $1 \leq i \leq p$. Moreover, w cannot be a product of $q < p$ elements of $X \subseteq S$ by definition of $p = \ell_S(w)$, so $\ell_X(w) = p = \ell_S(w)$. □

Proposition 3.1.34. *Let (W, S) is a Coxeter system.*

(a) For any subset X of S, the pair (W_X, X) is a Coxeter system.

(b) Let $(X_\alpha)_{\alpha \in A}$ be a family of subsets of S. If $X = \bigcap_\alpha X_\alpha$, then $W_X = \bigcap_\alpha W_{X_\alpha}$.

(c) Let X and Y be two subsets of S. Then $W_X \subseteq W_Y$ if and only if $X \subseteq Y$.

Proof. Every element of X is of order 2 and X generates W_X. Let $s \in X$ and $w \in W_X$ with $\ell_X(sw) \leq \ell_X(w) = p$. By Corollary 3.1.33, we have $\ell_S(sw) \leq \ell_S(w) = p$. Let $s_1, \dots, s_p$ be elements of X such that $w = s_1 \cdots s_p$. Since (W, S) satisfies the exchange condition, there exists an integer j such that $s = (s_1 \cdots s_{j-1})s_j(s_1 \cdots s_{j-1})^{-1}$. Thus, (W_X, X) satisfies the exchange condition and is therefore a Coxeter system. This proves (a), and assertions (b) and (c) follow immediately from Corollary 3.1.31. □

3.2 Tits systems

3.2.1 Tits systems and Bruhat decompositions

Let G be a group and B a subgroup of G. The group $B \times B$ acts on G by $(b, \tilde{b}) \cdot g = bg\tilde{b}^{-1}$ for $b, \tilde{b} \in B$ and $g \in G$. The orbits of $B \times B$ on G are the sets BgB for $g \in G$, and are called the double cosets of G with respect to B. They form a partition of G; the corresponding quotient is denoted by $B \backslash G / B$. If C and $\tilde{C}$ are double cosets, $C\tilde{C}$ is a union of double cosets.

Definition 3.2.1. A **Tits system** is a quadruple (G, B, N, S), where G is a group, B and N are two subgroups of G and S is a subset of $N/(B \cap N)$, satisfying the following axioms:

(T1) The set $B \cup N$ generates G and $B \cap N$ is a normal subgroup of N.

(T2) The set S generates the group $W = N/(B \cap N)$ and consists of elements of order 2.

(T3) $sBw \subseteq BwB \cup BswB$ for $s \in S$ and $w \in W$[2].

(T4) For all $s \in S$, $sBs \nsubseteq B$.

Remark 3.2.2. Let (G, B, N, S) be a Tits system, and let Z be a normal subgroup of G contained in B. Let $\tilde{G} = G/Z$, $\tilde{B} = B/Z$, $\tilde{N} = N/(Z \cap N)$, and let $\tilde{S}$ be the image of S in $\tilde{N}/(\tilde{B} \cap \tilde{N})$. Then one sees immediately that $(\tilde{G}, \tilde{B}, \tilde{N}, \tilde{S})$ is a Tits system.

Throughout this paragraph, let (G, B, N, S) be a Tits system. We set $T = B \cap N$ and $W = N/T$. A double coset means a double coset of G with respect to B. For any $w \in W$, we set $C(w) = BwB$; this is a double coset.

We first deduce several elementary consequences of the axioms (T1) to (T4). We denote by $w, \tilde{w}, \ldots$ elements of W and by $s, \tilde{s}, \ldots$ elements of S. The following relations are clear:

$$C(1) = B, \quad C(w\tilde{w}) \subseteq C(w)C(\tilde{w}), \quad C(w^{-1}) = C(w)^{-1} \tag{3.2.1}$$

Axiom (T3) can also be written in the form

$$C(s) \cdot C(w) \subseteq C(w) \cup C(sw). \tag{3.2.2}$$

Moreover, since $C(sw) \subseteq C(s) \cdot C(w)$ by (3.2.1) and since $C(s) \cdot C(w)$ is a union of double cosets, it either contains $C(w)$ or is disjoint with $C(w)$. Therefore, there are only two possibilities:

$$C(s) \cdot C(w) = \begin{cases} C(sw) & \text{if } C(w) \nsubseteq C(s) \cdot C(w) \\ C(w) \cup C(sw) & \text{if } C(w) \subseteq C(s) \cdot C(w). \end{cases} \tag{3.2.3}$$

By (T4), $B \neq C(s) \cdot C(s)$; putting $w = s$ in (3.2.3) and using the relation $s^2 = 1$, we obtain

$$C(s) \cdot C(s) = B \cup C(s). \tag{3.2.4}$$

This formula shows that $B \cup C(s)$ is a subgroup of G. Multiplying both sides of (3.2.4) on the right by $C(w)$, and using formula (3.2.3) and the relation $B \cdot C(w) = C(w)$, we obtain

$$C(s) \cdot C(s) \cdot C(w) = C(w) \cup C(sw). \tag{3.2.5}$$

Taking the inverses of the sets entering into formulas (3.2.2), (3.2.3) and (3.2.5) and then replacing w by w^{-1}, we obtain the formulas

$$C(w) \cdot C(s) \subseteq C(w) \cup C(ws) \tag{3.2.6}$$

$$C(w) \cdot C(s) = \begin{cases} C(ws) & \text{if } C(w) \nsubseteq C(w) \cdot C(s) \\ C(w) \cup C(ws) & \text{if } C(w) \subseteq C(w) \cdot C(s) \end{cases} \tag{3.2.7}$$

$$C(w) \cdot C(s) \cdot C(s) = C(w) \cup C(ws). \tag{3.2.8}$$

[2] Every element of W is a coset modulo $B \cap N$, and is thus a subset of G; hence products such as BwB make sense. More generally, for any subset A of W, we denote by BAB the subset $\bigcup_{w \in A} BwB$.

Proposition 3.2.3. *Let $s_1, \dots, s_p \in S$ and let $w \in W$. We have*

$$C(s_1 \cdots s_p) \cdot C(w) \subseteq \bigcup_{(i_1,\dots,i_k)} C(s_{i_1} \cdots s_{i_p} w)$$

where $(i_1, \dots, i_k)$ denotes the set of strictly increasing sequences of integers in the interval $[1, p]$.

Proof. We argue by induction on p, the case $p = 0$ being trivial. If $p \geq 1$, we have $C(s_1 \cdots s_p) \cdot C(w) \subseteq C(s_1) \cdot C(s_2 \cdots s_p) \cdot C(w)$. By the induction hypothesis, $C(s_2 \cdots s_p) \cdot C(w)$ is contained in the union of the $C(s_{j_1} \cdots s_{j_k} w)$, where $2 \leq j_1 < \cdots < j_k \leq p$. By (T3), the set $C(s_1) \cdot C(s_{j_1} \cdots j_k w)$ is contained in the union of the sets $C(s_1 s_{j_1} \cdots s_{j_p} w)$ and $C(s_{j_1} \cdots s_{j_k} w)$. This proves the claim. □

Example 3.2.4. Let $\mathbb{K}$ be a field, n a positive integer, and (e_i) the canonical basis of $\mathbb{K}^n$. Let $G = \mathrm{GL}(n, \mathbb{K})$, B be the upper triangular subgroup of G, and let N be the subgroup of G consisting of the matrices having exactly one non-zero element in each row and column. An element of N permutes the lines $\mathbb{K}e_i$; this gives rise to a surjective homomorphism $N \to \mathfrak{S}_n$, whose kernel is the subgroup $T = B \cap N$ of diagonal matrices, and allows us to identify $W = N/T$ with $\mathfrak{S}_n$. We denote by s_j the element of W corresponding to the transposition of j and $j+1$; let S be the set of s_j.

Now we show that the quadruple (G, B, N, S) is a Tits system. We only need to verify (T3), i.e.

$$s_j B w \subseteq BwB \cup B s_j w B \quad \text{for } 1 \leq j \leq n-1, w \in W.$$

or equivalently,

$$s_j B \subseteq B\widetilde{B} \cup B s_j \widetilde{B} \quad \text{with } \widetilde{B} = wBw^{-1}.$$

Let G_j be the subgroup of G consisting of the elements that fix the e_i for $i \neq j, j+1$ and stabilize the plane spanned by e_j and e_{j+1}; this group is isomorphic to $\mathrm{GL}(2, \mathbb{K})$. One checks that $G_j B = B G_j$. Since $s_j \in G_j$, we have $s_j B \subseteq B G_j$, and it suffices to prove that

$$G_j \subseteq (B \cap G_j)(\widetilde{B} \cap G_j) \cup (B \cap G_j) s_j (\widetilde{B} \cap G_j).$$

Identify G_j with $\mathrm{GL}(2, \mathbb{K})$; the group $B \cap G_j$ is then identified with the upper triangular subgroup B_2 of $\mathrm{GL}(2, \mathbb{K})$, while the group $\widetilde{B} \cap G_j$ is identified with B_2 when $w(j) < w(j+1)$ and with the lower triangular subgroup B_2^- otherwise. In the first case, the formula to be proved can be written

$$\mathrm{GL}(2, \mathbb{K}) = B_2 \cup B_2 s B_2 \quad s = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

this follows for example from the fact that B_2 is the stabilizer of a point for the action of $\mathrm{GL}(2, \mathbb{K})$ on the projective line $\mathbb{P}^1(\mathbb{K})$, and acts transitively on the complement of this point. In the second case, the formula to be proved can be written

$$\mathrm{GL}(2, \mathbb{K}) = B_2 B_2^- \cup B_2 s B_2^-$$

since $B_2^- = s B_2 s$, this follows from the preceding formula by multiplying on the right by s.

Theorem 3.2.5 (Bruhat Decomposition). *Let (G, B, N, S) be a Tits system. Then we have*

$$G = BWB = \bigcup_{w \in W} BwB.$$

Moreover the map $w \mapsto C(w)$ is a bijection from W to the set $B \backslash G / B$ of double cosets of G with respect to B.

Proof. It is clear that BWB is stable under $x \mapsto x^{-1}$, and Proposition 3.2.3 shows that it is stable under the product. Since it contains B and N, it is equal to G. It remains to prove that $C(w) \neq C(\tilde{w})$ if $w \neq \tilde{w}$. For this, we shall prove by induction on the integer p the following assertion:

(A_p) If w and $\tilde{w}$ are distinct elements of W such that $\ell(w) \geq \ell(\tilde{w}) = p$, then $C(w) \neq C(\tilde{w})$.

This assertion is clear for $p = 0$, since then $\tilde{w} = 1$ and $w \neq 1$, hence $C(\tilde{w}) = B$ and $C(w) \neq B$. Assume that $p \geq 1$ and that $w, \tilde{w}$ satisfy the hypotheses of (A_p). There exists $s \in S$ such that $s\tilde{w}$ is of length $p - 1$. We have

$$\ell(w) > \ell(s\tilde{w})$$

hence $w \neq s\tilde{w}$. Moreover, $sw \neq s\tilde{w}$; by formula (3.1.1), we have

$$\ell(sw) \geq \ell(w) - 1 \geq \ell(s\tilde{w}) = p - 1$$

By the induction hypothesis, $C(s\tilde{w})$ is distinct from $C(w)$ and from $C(sw)$; from formula (2) it follows that

$$C(s\tilde{w}) \cap C(s) \cdot C(w) = \varnothing.$$

Since $C(s\tilde{w}) \subseteq C(s) \cdot C(\tilde{w})$, we have finally that $C(w) \neq C(\tilde{w})$. □

3.2.2 Relation with Coxeter systems

Theorem 3.2.6. *Let (G, B, N, S) be a Tits system. Then the pair (W, S) is a Coxeter system. Moreover, for $s \in S$ and $w \in W$, the relations $C(sw) = C(s) \cdot C(w)$ and $\ell(sw) > \ell(w)$ are equivalent.*

Proof. For any $s \in S$, let P_s be the set of elements $w \in W$ such that $C(s) \cdot C(w) = C(sw)$. We are going to verify that the P_s satisfy conditions (A'), (B') and (C) of Proposition 3.1.28. The two assertions of the theorem will then follow.

Condition (A') is clear, and we verify (B'). If P_s and sP_s had an element w in common, we would have $w \in P_s$ and $sw \in P_s$, and hence

$$C(s) \cdot C(s) = C(sw), \quad C(s) \cdot C(sw) = C(w).$$

It would follow that $C(s) \cdot C(s) \cdot C(w) = C(w)$ and, by formula (3.2.5), this would imply that $C(w) = C(sw)$, which would contradict Theorem 3.2.5.

We verify (C). Let $s, \tilde{s} \in S$ and $w \in W$ be such that $w \in P_s$ and $w\tilde{s} \notin P_s$. We show that $C(sw) = C(w\tilde{s})$, which then implies $sw = w\tilde{s}$ by Theorem 3.2.5 and verifies (C). To this end, we first note that, by the definition of P_s and (3.2.3), we have

$$C(sw) = C(s) \cdot C(w) = C(s)\tilde{w}\tilde{s}B \tag{3.2.9}$$
$$C(\tilde{w}) \subseteq C(s) \cdot C(\tilde{w}) \tag{3.2.10}$$

where we use the relation $w = \tilde{w}\tilde{s}$. On the other hand, formula (3.2.6) implies

$$C(\tilde{w}) \cdot C(\tilde{s}) \subseteq C(\tilde{w}) \cup C(\tilde{w}\tilde{s}) = C(\tilde{w}) \cup C(w). \tag{3.2.11}$$

Since $C(\tilde{w})$ is a union of left cosets of B and since $C(s) \cdot C(\tilde{w}) = C(s)\tilde{w}B$, formula (3.2.10) shows that $C(s)\tilde{w}$ meets $C(\tilde{w})$ and a fortiori that $C(s)\tilde{w}\tilde{s}B$ meets $C(\tilde{w})\tilde{s}B = C(\tilde{w}) \cdot C(\tilde{s})$. It follows from formulas (3.2.9) and (3.2.11) that the double coset $C(sw)$ is equal to one of the double cosets $C(w\tilde{s})$ and $C(w)$; since $sw \neq w$, we conclude that $C(sw) = C(w\tilde{s})$, which finishes the proof. □

Corollary 3.2.7. *Let (G, B, N, S) be a Tits system and (W, S) be the corresponding Coxeter system. Let $w_1, \ldots, w_p$ be elements of W and $w = w_1 \cdots w_p$. If*

$$\ell(w) = \ell(w_1) + \cdots + \ell(w_p)$$

then $C(w) = C(w_1) \cdots C(w_p)$.

Proof. On taking reduced decompositions of the w_i, one is reduced to the case of a reduced decomposition $w = s_1 \cdots s_p$ with $s_i \in S$. If $u = s_2 \cdots s_p$, then $w = s_1 u$ and $\ell(s_1 u) > \ell(u)$, so $C(w) = C(s) \cdot C(u)$ by Theorem 3.2.6. The required formula follows from this by induction on p. □

Corollary 3.2.8. *Let (G, B, N, S) be a Tits system and (W, S) be the corresponding Coxeter system. Let $w \in W$ and let $T_L(w)$ be the set of left associated reflections to w. If $t \in T_L(w)$, then $C(t) \subseteq C(w) \cdot C(w^{-1})$.*

Proof. If $t \in T_L(w)$, there exist by Corollary 3.1.17 elements $w_1, w_2 \in W$ and $s \in S$ such that

$$w = w_1 s w_2, \quad \ell(w) = \ell(w_1) + \ell(w_2) + 1, \quad t = w_1 s w_1^{-1}.$$

By Corollary 3.2.7, we then have

$$C(w) \cdot C(w)^{-1} = C(w_1) \cdot C(s) \cdot C(w_2) \cdot C(w_2^{-1}) \cdot C(s) \cdot C(w_1^{-1}) \supseteq C(w_1) \cdot C(s) \cdot C(s) \cdot C(w_1^{-1}).$$

By (3.2.8), we have $C(s) \subseteq C(s) \cdot C(s)$. Hence

$$C(w) \cdot C(w^{-1}) \supseteq C(w_1) \cdot C(s) \cdot C(w_1^{-1}) \supset C(t).$$

This proves the claim. □

Corollary 3.2.9. *Let (G, B, N, S) be a Tits system and (W, S) be the corresponding Coxeter system. Let $w \in W$ and let H_w be the subgroup of G generated by $C(w) \cdot C(w^{-1})$. Then:*

(a) For any reduced decomposition $(s_1, \ldots, s_p)$ of w, we have $C(s_j) \subseteq H_w$ for each j.

(b) The group H_w contains $C(w)$ and is generated by $C(w)$.

Proof. We prove (a) by induction on j. Assume that $C(s_k)$ is contained in H_w for $k < j$. Let $t = t_j = (s_1 \cdots s_{j-1})s_j(s_1 \cdot s_{j-1})^{-1}$, which belongs to the subset $T_L(w)$ of W by Proposition 3.1.16. By Corollary 3.2.8 we then have $C(t) \subseteq H_w$, whence $C(s_j) \subseteq H_w$ by the induction hypothesis. Since $C(w) = C(s_1) \cdot C(s_p)$ by Corollary 3.2.7, we have $C(w) \subseteq H_w$, so (b) follows since $C(w^{-1}) = C(w)^{-1}$. □

Example 3.2.10. From the Tits system for $\mathrm{GL}(n, \mathbb{K})$, we see the symmetric group $\mathfrak{S}_n$, with the set of transpositions of consecutive elements, is a Coxeter group.

3.2.3 Parabolic subgroups in a Tits system

For any subset X of S, we denote by W_X the subgroup of W generated by X and by G_X the union BW_XB of the double cosets $C(w)$, $w \in W_X$. We have $G_\varnothing = B$ and $G_S = G$.

Theorem 3.2.11. *Let (G, B, N, S) be a Tits system and (W, S) be the corresponding Coxeter system.*

(a) For any subset X of S, the set G_X is a subgroup of G, generated by $\bigcup_{s \in X} C(s)$.

(b) The map $X \mapsto G_X$ is a bijection from $\mathscr{P}(S)$ to the set of subgroups of G containing B.

(c) Let $(X_i)_{i \in I}$ be a family of subsets of X. If $X = \bigcap_{i \in I} X_i$, then $G_X = \bigcap_{i \in I} G_{X_i}$.

(d) Let X and Y be two subsets of S. Then $G_X \subseteq G_Y$ if and only if $X \subseteq Y$.

Proof. It is clear that $G_X = G_X^{-1}$; Proposition 3.2.3 shows that $G_X \cdot G_X \subseteq G_X$, and hence (a) follows, taking into account Corollary 3.2.7.

The injectivity of $X \mapsto G_X$ follows from that of $X \mapsto W_X$ and Theorem 3.2.5. Conversely, let H be a subgroup of G containing B. Let U be the set of $w \in W$ such that $C(w) \subseteq H$. We have $H = BUB$ since H is a union of double cosets. Let $X = U \cap S$; we show that $H = G_X$. Clearly, $G_X \subseteq H$. On the other hand, let $u \in U$ and let $(s_1, \ldots, s_p)$ be a reduced decomposition of u. Corollary 3.2.9 implies that $C(s_j) \subseteq H$, and hence that $s_j \in X$ for each j. Thus, $u \in W_X$, and since H is the union of the $C(u)$ for $u \in U$, we have $H \subseteq G_X$, which proves (b). Assertions (c) and (d) follow from analogous properties of W_X. □

Corollary 3.2.12. *Let (G, B, N, S) be a Tits system and (W, S) be the corresponding Coxeter system. Then the set S consists of the elements $w \in W$ such that $w \neq 1$ and $B \cup C(w)$ is a subgroup of G.*

Proof. The elements $w \in W$ such that $B \cup C(w)$ is a subgroup of G are those for which there exists $X \subseteq S$ with $W_X = \{1, w\}$. Moreover, if $w \neq 1$, we necessarily have $|X| = 1$, i.e. $w \in S$. □

Remark 3.2.13. The above corollary shows that S is determined by (G, B, N); for this reason, we sometimes allow ourselves to say that (G, B, N) is a Tits system, or that (B, N) is a Tits system in G.

Proposition 3.2.14. *Let (G, B, N, S) be a Tits system and (W, S) be the corresponding Coxeter system. Let X be a subset of S and N_X a subgroup of N whose image in W is equal to W_X. Then (G_X, B, N_X, X) is a Tits system.*

Proof. We have $G_X = BW_XB = BN_XB$, which shows that G_X is generated by $B \cup N_X$. The verification of the axioms (T1) to (T4) is now immediate. □

Proposition 3.2.15. *Let (G, B, N, S) be a Tits system and (W, S) be the corresponding Coxeter system. Let $X, Y \subseteq S$ and $w \in W$. We have $G_X w G_Y = BW_X w W_Y B$.*

Proof. Let $s_1, \ldots, s_p \in X$ and $t_1, \ldots, t_q \in Y$. Proposition 3.2.3 shows that

$$C(s_1 \cdots s_p) \cdot C(w) \cdot C(t_1 \cdots t_q) \subseteq BW_X w W_Y B.$$

and hence that $G_X w G_Y \subseteq BW_X w W_Y B$. The opposite inclusion is obvious. □

Remark 3.2.16. Denote by $G_X\backslash G/G_Y$ the set of subsets of G of the form G_XgG_Y, $g \in G$; and define $W_X\backslash W/W_Y$ analogously. The preceding proposition shows that the canonical bijection $w \mapsto C(w)$ from W to $B\backslash G/B$ defines a bijection $W_X\backslash W/W_Y \mapsto G_X\backslash G/G_Y$.

Proposition 3.2.17. *Let (G, B, N, S) be a Tits system and (W, S) be the corresponding Coxeter system. Let $X \subseteq S$ and $g \in G$. Then the relation $gBg^{-1} \subseteq G_X$ implies that $g \in G_X$.*

Proof. Let $w \in W$ be such that $g \in C(w)$. Since B is a subgroup of G_X, the hypothesis $gBg^{-1} \subseteq G_X$ implies that $C(w) \cdot C(w^{-1}) \subseteq G_X$, and hence that $C(w) \subseteq G_X$ by Corollary 3.2.9, so g belongs to G_X. $\square$

Definition 3.2.18. Let (G, B, N, S) be a Tits system. A subgroup of G is said to be **parabolic** if it contains a conjugate of B.

Proposition 3.2.19. *Let (G, B, N, S) be a Tits system and (W, S) be the corresponding Coxeter system. Let P be a subgroup of G.*

(a) P is parabolic if and only if there exists a subset X of S such that P is conjugate to G_X.

(b) Let $X, \tilde{X} \subseteq S$ and $g, \tilde{g} \in G$ be such that $P = gG_Xg^{-1} = \tilde{g}G_{\tilde{X}}\tilde{g}^{-1}$. Then, $X = \tilde{X}$ and $g\tilde{g}^{-1} \in P$.

Proof. Assertion (a) follows from Theorem 3.2.11(b). Under the hypotheses of (b), we have

$$g^{-1}\tilde{g}B\tilde{g}^{-1}g \subseteq g^{-1}\tilde{g}G_{\tilde{X}}\tilde{g}^{-1}g = G_X$$

and Proposition 3.2.17 shows that $g^{-1}\tilde{g} \in G_X$. Hence, $G_{\tilde{X}} = G_X$ and $\tilde{X} = X$ by Theorem 3.2.11(b). Finally,

$$\tilde{g}g^{-1} = g \cdot g^{-1}\tilde{g}g^{-1} \in gG_Xg^{-1},$$

which finishes the proof of (b). $\square$

In view of Proposition 3.2.19, if the parabolic subgroup P is conjugate to G_X, where $X \subseteq S$, then P is said to be of **type** X.

Theorem 3.2.20. *Let (G, B, N, S) be a Tits system and (W, S) be the corresponding Coxeter system.*

(a) Let P_1 and P_2 be two parabolic subgroups of G whose intersection is parabolic and let $g \in G$ be such that $gP_1g^{-1} \subseteq P_2$. Then $g \in P_2$ and $P_1 \subseteq P_2$.

(b) Two distinct parabolic subgroups whose intersection is parabolic are not conjugate.

(c) Let Q_1 and Q_2 be two parabolic subgroups of G contained in a subgroup Q of G. Then any $g \in G$ such that $gQ_1g^{-1} = Q_2$ belongs to Q.

(d) Every parabolic subgroup is its own normaliser.

Proof. Assertion (a) follows from Proposition 3.2.17 and 3.2.19, and implies (b). Under the hypotheses of (c), we have $gQ_1g^{-1} \subseteq Q$, which implies that $g \in Q$ by (a). Finally, (d) follows from (c) by taking $Q_1 = Q_2 = Q$. $\square$

Proposition 3.2.21. *Let (G, B, N, S) be a Tits system and $T = B \cap N$. Let P_1 and P_2 be two parabolic subgroups of G. Then $P_1 \cap P_2$ contains a conjugate of T.*

Proof. By first transforming P_1 and P_2 by an inner automorphism of G, we may assume that $B \subseteq P_1$. Let $g \in G$ be such that $gBg^{-1} \subseteq P_2$. By Theorem 3.2.5, there exist $n \in N$ and $b, \tilde{b} \in B$ such that $g = bn\tilde{b}$. Since T is normal in N,

$$P_2 \supseteq gBg^{-1} = bnBn^{-1}b^{-1} \supseteq bnTn^{-1}b^{-1} = bTb^{-1}, \quad P_1 \supseteq B \supseteq bTb^{-1},$$

which proves the proposition. $\square$

Chapter 4

Groups generated by reflections

4.1 Hyperplanes, chambers and facets

In this section, E denotes a real affine space of finite dimension d and T the space of translations of E. If x and y are two points of E, $[x, y]$ (resp. (x, y), resp. $(x, y]$) will denote the closed segment (resp. open segment, resp. segment open at x and closed at y) with extremities x, y. The space T is provided with its unique separated topological vector space topology; it is isomorphic to $\mathbb{R}^d$. The space E is provided with the unique topology such that, for all $e \in E$, the map $t \mapsto t + e$ from T to E is a homeomorphism.

Let H be a hyperplane of E. Recall that $E \setminus H$ has two connected components, called the **open half-spaces** bounded by H. Their closures are called the **closed half-spaces** bounded by H. Let $x, y \in E$. Then x and y are said to be **strictly on the same side** of H if they are contained in the same open half-space bounded by H, or equivalently, if the closed segment with extremities x and y does not meet H; x and y are said to be **on opposite sides** of H if x belongs to one of the open half-spaces bounded by H and y to the other. If $x \in E$ and $t \in T$, then x and t are said to be **strictly on the same side** of H if this is so for x and $h + t$ for all $h \in H$.

Let A be a non-empty connected subset of E. For any hyperplane H of E that does not meet A, $D_H(A)$ denotes the unique open half-space bounded by H that contains A. If $\mathcal{H}$ is a set of hyperplanes of E, none of which meet A, put

$$D_{\mathcal{H}}(A) = \bigcap_{H \in \mathcal{H}} D_H(A).$$

If A consists of a single point a, we write $D_H(a)$ and $D_{\mathcal{H}}(a)$ instead of $D_H(\{a\})$ and $D_{\mathcal{H}}(\{a\})$.

4.1.1 Facets and chambers

Let $\mathcal{H}$ be a locally finite subset of hyperplanes of E. Then since each hyperplane is closed, the union of $\mathcal{H}$ is closed. In other words, the set of points of E that do not belong to any hyperplane H of the set $\mathcal{H}$ is open since $\mathcal{H}$ is locally finite. More precisely, we have the following result:

Proposition 4.1.1. *Let a be a point of E. There exists a connected open neighbourhood of a that does not meet any hyperplane H that belongs to $\mathcal{H}$ and does not pass through a. Moreover, there exist only finitely many hyperplanes that belong to $\mathcal{H}$ and pass through a.*

Proof. The set $\mathcal{R}$ of hyperplanes H such that $H \in \mathcal{H}$ and $a \notin H$ is locally finite since it is contained in $\mathcal{H}$. Hence, the set U of points of E that do not belong to any of the hyperplanes of the set $\mathcal{R}$ is open. Since $a \in U$, there is a connected open neighbourhood of a contained in U. The remainder of the proposition is clear. □

Given two points x and y of E, denote by xRy the relation "for any hyperplane $H \in \mathcal{H}$, either $x \in H$ and $y \in H$ or x and y are strictly on the same side of H." Clearly R is an equivalence relation on E. A **facet** of E relative to $\mathcal{H}$ is defined to be an equivalence class of the equivalence relation R. Since $\mathcal{H}$ is locally finite, it is clear that the set of facets is locally finite.

Let F be a facet and a a point of F. A hyperplane $H \in \mathcal{H}$ contains F if and only if $a \in H$; the set $\mathcal{F}$ of these hyperplanes is thus finite; their intersection is an affine subspace L of E, which we shall call the **affine support** of F (if this family is empty, we set $L = E$). The dimension of L will be called the **dimension** of F.

If $\mathcal{R}$ is the set of hyperplanes $H \in \mathcal{H}$ not containing F, then

$$F = L \cap \bigcap_{H \in \mathcal{R}} D_H(a). \tag{4.1.1}$$

In fact, the closure of F is given by

$$\overline{F} = L \cap \bigcap_{H \in \mathcal{R}} \overline{D_H(a)}. \tag{4.1.2}$$

It is clear that the right-hand side contains the left-hand side. Conversely, let x be in the right-hand side. The open segment with extremities a and x is contained in L and in each of the $D_H(a)$ for $H \in \mathcal{H}$, and hence in F. It follows that x is in the closure of F, hence the formula.

Proposition 4.1.2. *Let F be a facet and L its afffine support.*

(a) The set F is a convex open subset of the affine subspace L of E.

(b) The closure of F is the union of F and facets of dimension strictly smaller than that of F.

(c) In the topological space L, the set F is the interior of its closure.

Proof. Since every open half-space and every hyperplane are convex subsets of E, formula (4.1.1) shows that F is the intersection of a family of convex subsets, and hence is convex. On the other hand, let a be in F, and let U be a convex open neighbourhood of a in E that does not meet any of the hyperplanes in the set $\mathcal{R}$ of $H \in \mathcal{H}$ such that $a \notin H$. For any $H \in \mathcal{R}$, we thus have $U \subseteq D_H(a)$, hence $L \cap U \subseteq F$, so that F is open in the topological space L.

Let b be a point of $\overline{F} \setminus F$, belonging to a facet $\tilde{F}$, and let $\mathcal{R}_1$ be the set of hyperplanes $H \in \mathcal{R}$ passing through b. Put $\mathcal{R}_2 = \mathcal{R} \setminus \mathcal{R}_1$. For any H in $\mathcal{R}_2$, we have $b \notin H$ and $b \in \overline{D_H(a)}$, hence $b \in D_H(a)$ and $D_H(b) = D_H(a)$. By the definition of a facet, we thus have

$$\tilde{F} = L \cap \bigcap_{H \in \mathcal{R}_1} H \cap \bigcap_{H \in \mathcal{R}_2} D_H(a). \tag{4.1.3}$$

whereas (4.1.2) implies that

$$\overline{F} = L \cap \bigcap_{H \in \mathcal{R}_1} \overline{D_H(a)} \cap \bigcap_{H \in \mathcal{R}_2} \overline{D_H(a)}. \tag{4.1.4}$$

hence $\tilde{F} \subseteq F$. We cannot have $\mathcal{R}_1 = \varnothing$, for this would imply that $F = \tilde{F}$ by (4.1.1) and (4.1.3), contrary to the hypothesis that $b \notin F$ and $b \in \tilde{F}$. The support of $\tilde{F}$ is the set $\tilde{L} = L \cap \bigcap_{H \in \mathcal{R}_1} H$; we have $a \in L$, but $a \notin H$ for H in $\mathcal{R}_1$, so $\tilde{L} \neq L$ and finally $\dim(\tilde{L}) < \dim(L)$. This proves (b).

Let H be in $\mathcal{R}_1$ and let D be the open half-space bounded by H and distinct from $D_H(a)$; we have $b \in H \cap L$, and it is immediate that $D \cap L$ is a half-space of L bounded by the hyperplane $H \cap L$ of L. Consequently, every neighbourhood of b in L meets $D \cap L$, and since $D \cap L$ is disjoint from $\overline{F}$ by (4.1.2), we see that the point b of $\overline{F} \setminus F$ cannot be in the interior of F in the topological space L. Since F is open in L, we have (c). □

Corollary 4.1.3. *Let F_1 and F_2 be two facets. If $\overline{F}_1 = \overline{F}_2$, the facets F_1 and F_2 are equal.*

Proof. This follows from Proposition 4.1.2(c). □

Proposition 4.1.4. *Let F be a facet, and let L be an affine subspace of E that is an intersection of hyperplanes belonging to $\mathcal{H}$. Denote by $\mathcal{R}$ the set of hyperplanes $H \in \mathcal{H}$ that do not contain L. Then the following conditions are equivalent:*

(i) There exists a facet $\tilde{F}$ with support L that meets $\overline{F}$.

(ii) There exists a facet $\tilde{F}$ with support L contained in $\overline{F}$.

(iii) There exists a point x in $L \cap \overline{F}$ that does not belong to any of the hyperplanes of $\mathcal{R}$.

If these conditions are satisfied, $L \cap D_{\mathcal{R}}(F)$ is the unique facet with support L contained in $\overline{F}$.

Proof. Since $\overline{F}$ is a union of facets (Proposition 4.1.2(b)), every facet that meets $\overline{F}$ meets a facet contained in $\overline{F}$, and so is equal to it. This shows (i)⇒(ii). Moreover, in this case every point x of $\tilde{F}$ satisfies (c) since every hyperplane of $\mathcal{H}$ containing x contains $\tilde{F}$, and hence L, so (iii) is fullfilled.

Let x be a point satisfying (iii) and let $\tilde{F}$ be the facet containing x; it is clear that $\tilde{F}$ meets $\overline{F}$. Let $H \in \mathcal{H}$, then $x \notin H$ if $H \in \mathcal{R}$ and clearly $x \in H$ if $H \notin \mathcal{R}$. Consequently the support of $\tilde{F}$ is the intersection of the hyperplanes of $\mathcal{H} \setminus \mathcal{R}$, and is equal to L.

Finally, let $\tilde{F}$ be a facet with support L contained in F, and let x be a point of $\tilde{F}$. Since no hyperplane of $\mathcal{R} \subseteq \mathcal{H}$ passes through x, Proposition 4.1.1 shows that there exists a convex open neighbourhood U of x that does not meet any hyperplane of $\mathcal{R}$. Since x is in the closure of F, $U \cap F \neq \varnothing$. Now $\mathcal{R}$ is the set of hyperplanes $H \in \mathcal{H}$ that do not contain $\tilde{F}$, and for all H in $\mathcal{R}$ we have $D_H(x) = D_H(U) = D_H(U \cap F) = D_H(F)$ and formula (4.1.1) implies that $\tilde{F} = L \cap D_{\mathcal{R}}(F)$. □

Definition 4.1.5. A **chamber** of E relative to $\mathcal{H}$ (or simply a chamber if there is no ambiguity regarding $\mathcal{H}$) is a facet of E relative to $\mathcal{H}$ that is not contained in any hyperplane belonging to $\mathcal{H}$.

Let U be the open set in E consisting of the points that do not belong to any hyperplane of $\mathcal{H}$. Since a hyperplane of $\mathcal{H}$ must contain any facet that it meets, the chambers of E are exactly the facets contained in U. Every chamber is a convex (hence connected) open subset of E by Proposition 4.1.2(a). Since the chambers form a partition of U, they are exactly the connected components of U. Every convex subset A of U is connected, and thus contained in a chamber, which is unique if A is non-empty. It is clear that the chambers are the facets with support E, and Proposition 4.1.2(c) shows that every chamber is the interior of its closure. Finally, let C be a chamber and A a non-empty subset of C; formulas (4.1.1) and (4.1.2) imply that

$$C = \bigcap_{H \in \mathcal{H}} D_H(A) = D_{\mathcal{H}}(A), \quad \bar{C} = \bigcap_{H \in \mathcal{H}} \overline{D_H(A)}. \tag{4.1.5}$$

since $D_H(A) = D_H(a)$ for all $a \in A$.

Proposition 4.1.6. *Let C be a non-empty subset of E. Assume that there exists a subset $\mathcal{R}$ of $\mathcal{H}$ with the following properties:*

(a) For any $H \in \mathcal{R}$, there exists an open half-space D_H bounded by H such that $C = \bigcap_{H \in \mathcal{R}} D_H$.

(b) The set C does not meet any hyperplane belonging to $\mathcal{H} \setminus \mathcal{R}$.

Under these conditions, C is a chamber defined by $\mathcal{H}$ in E, and $D_H = D_H(C)$ for all $H \in \mathcal{H}$.

Proof. Properties (a) and b) show that C is a convex subset of U; hence there is a chamber $\tilde{C}$ with $C \subseteq \tilde{C}$. Since $C \subseteq D_H$, we have $D_H = D_H(C)$ for all H in $\mathcal{R}$, hence $C = D_{\mathcal{R}}(C) \supseteq D_{\mathcal{H}}(C)$. We have $D_{\mathcal{H}}(C) = \tilde{C}$ by (4.1.5), hence $C \supseteq \tilde{C}$ and therefore $C = \tilde{C}$. □

Proposition 4.1.7. *Every point of E is in the closure of at least one chamber.*

Proof. If E reduces to a single point, this is clear. Otherwise, let $a \in E$ and let $H_1, \dots, H_m$ be the hyperplanes of $\mathcal{H}$ containing a. Since $\mathcal{H}$ is locally finite, there exists a neighbourhood V of a that does not meet any hyperplane of $\mathcal{H}$ other than $H_1, \dots, H_m$. Let D be a straight line passing through a and not contained in any of the H_i; if $x \in D$, $x \neq a$, and x is sufficiently close to a, then the open segment (a, x) is contained in V and does not meet any of the H_i. Then $(a, x) \subseteq U$; since (a, x) is connected, it is contained in a chamber C, hence $a \in \bar{C}$. □

Proposition 4.1.8. *Let L be an affine subspace of E and Ω a non-empty open subset of L.*

(a) There exists a point a in Ω that does not belong to any of the hyperplanes of $\mathcal{H}$ that do not contain L.

(b) If L is a hyperplane and $L \notin \mathcal{H}$, there exists a chamber that meets Ω.

(c) If L is a hyperplane and $L \in \mathcal{H}$, there exists a point a in Ω that does not belong to any hyperplane $H \neq L$ of $\mathcal{H}$.

Proof. Denote by $\mathcal{R}$ the set of hyperplanes H with $H \in \mathcal{H}$ and $L \nsubseteq H$, and by $\mathcal{L}$ the set of hyperplanes of the affine space L of the form $L \cap H$ with $H \in \mathcal{R}$. It is clear that $\mathcal{L}$ is a locally finite set of hyperplanes in L, and Proposition 4.1.7 shows that Ω meets a chamber Γ defined by $\mathcal{L}$ in L. If a is a point of $\Gamma \cap \Omega$, then $a \notin H$ for all $H \in \mathcal{R}$, hence (a).

Assume now that L is a hyperplane; any hyperplane containing L is then equal to it, so we may distinguish two cases. If $L \notin \mathcal{H}$, then $\mathcal{R} = \mathcal{H}$, and we have $a \notin H$ for all $H \in \mathcal{H}$; thus a belongs to a chamber defined by $\mathcal{H}$ in E, hence (b). If $L \in \mathcal{H}$, then $\mathcal{R} = \mathcal{H} \setminus \{L\}$, so (c) follows. □

4.1.2 Walls and faces

Definition 4.1.9. Let C be a chamber of E. A **face** of C is defined to be a facet contained in the closure of C whose support is a hyperplane. A **wall** of C is a hyperplane that is the support of a face of C.

Every wall of C belongs to $\mathcal{H}$. Proposition 4.1.4 shows that a hyperplane $L \in \mathcal{H}$ is a wall of C if and only if $C \neq D_{\mathcal{H}\setminus\{L\}}(C)$. Moreover, every wall of C is the support of a single face of C.

Proposition 4.1.10. *Every hyperplane H belonging to $\mathcal{H}$ is the wall of at least one chamber.*

Proof. Every hyperplane H belonging to $\mathcal{H}$ is the wall of at least one chamber. □

Proof. By Proposition 4.1.8(c), there exists a point a of H that does not belong to any hypgplane $L \neq H$ of $\mathcal{H}$; by Proposition 4.1.7, there exists a chamber C such that $a \in \bar{C}$. Proposition 4.1.4 then shows that H is a wall of C. □

Proposition 4.1.11. *Let C be a chamber and $\mathcal{W}$ the set of walls of C. Then $C = D_{\mathcal{W}}(C)$ and every subset $\mathcal{L}$ of $\mathcal{H}$ such that $C = D_{\mathcal{W}}(C)$ contains $\mathcal{W}$. A subset F of $\bar{C}C$ is a facet if and only if it is a facet of E relative to the family $\mathcal{W}$.*

Proof. Let $\mathcal{L}$ be a subset of $\mathcal{H}$ such that $C = D_{\mathcal{L}}(C)$. Consider a hyperplane $L \in \mathcal{H} \setminus \mathcal{L}$ and let $\mathcal{R} = \mathcal{H} \setminus \{L\}$. Then $\mathcal{L} \subseteq \mathcal{R}$, and we get $C = D_{\mathcal{R}}(C)$ because

$$C \subseteq D_{\mathcal{R}}(C) \subseteq D_{\mathcal{L}}(C).$$

In particular, L does not meet $D_{\mathcal{R}}(C)$ since it does not meet C. By Proposition 4.1.4, this implies L is not a wall. Consequently, every wall of C belongs to $\mathcal{L}$.

Now let H be a hyperplane belonging to $\mathcal{L}$ that is not a wall of C. Since $C = D_{\mathcal{L}}(C)$, by applying Proposition 4.1.4 to $\mathcal{L}$ and H, we see the convex set $D_{\mathcal{L}\setminus\{H\}}(C)$ does not meet H, so $D_{\mathcal{L}\setminus\{H\}}(C) \subseteq D_H(C)$ and $C = D_{\mathcal{L}\setminus\{H\}}(C)$. If $\mathcal{F}$ is a finite subset of $\mathcal{L}$ that does not contain any wall of C, we conclude by induction on the cardinal of $\mathcal{F}$ that $C = D_{\mathcal{L}\setminus\mathcal{F}}(C)$.

Let a be a point of C and b be a point of $D_{\mathcal{W}}(a)$. Since the closed segment $[a, b]$ is compact, the set $\mathcal{S}$ of hyperplanes $H \in \mathcal{H}$ that meet $[a, b]$ is finite. Since a and b are strictly on the same side of every wall of C, no wall of C belongs to $\mathcal{S}$, so by the preceding paragraph we have $C = D_{\mathcal{L}\setminus\mathcal{S}}(C)$, whence $b \in D_{\mathcal{L}\setminus\mathcal{S}}(C) = C$. We have therefore proved that $D_{\mathcal{W}}(a) \subseteq C$, whence $C = D_{\mathcal{W}}(a)$ and the first part of the proposition is established.

To prove the last assertion of the proposition, it clearly suffices to show that a subset F of $\bar{C}$ that is a facet of E relative to $\mathcal{W}$ is a facet of E relative to $\mathcal{H}$, or that every hyperplane $H \in \mathcal{H}$ that meets F contains F. So let H be a hyperplane that meets F but does not contain it. Since F is open in its affine support, it is not completely on one side of H. It follows that $\bar{C}$ is not completely on one side of H and hence that the hyperplane H does not belong to $\mathcal{H}$, which completes the proof. □

Remark 4.1.12. It follows from formula (4.1.2) and Proposition 4.1.11 that the closure of a chamber C is the intersection of the closed half-spaces that are bounded by a wall of C and contain C.

Let F be a facet whose support is a hyperplane L; then F is a face of two chambers. In fact, let $\mathcal{R} = \mathcal{H} \setminus \{L\}$ and denote by D^+ and D^- the two open half-spaces bounded by L. The set $D_{\mathcal{R}}(F)$ is open and contains $F \subseteq L$, and since every point of F is in the closure of D^+ and D^-, the sets $C^+ = D_{\mathcal{R}}(F) \cap D^+$ and $C^- = D_{\mathcal{R}}(F) \cap D^-$ are nonempty, so by Proposition 4.1.6 they are chambers. Moreover, the hyperplane L meets $D_{\mathcal{R}}(F) = D_{\mathcal{R}}(C^+)$; Proposition 4.1.4 shows that L is a wall of C^+ and that F, which meets $L \cap D_{\mathcal{R}}(F)$, is a face of C^+; similarly, F is a face of C^-. Finally, let C be a chamber of which F is a face, and suppose for example that $D^+ = D_L(C)$. By Proposition 4.1.4, the set $D_{\mathcal{R}}(C)$ meets F, and hence is equal to $D_{\mathcal{R}}(F)$, and we have

$$C = D_{\mathcal{H}}(C) = D_L(C) \cap D_{\mathcal{R}}(C) = D^+ \cap D_{\mathcal{R}}(F) = C^+.$$

This shows the uniqueness of the chambers C^+ and C^-.

4.1.3 Intersection hyperplanes

Recall that two affine subspaces P_1 and P_2 of E are said to be **parallel** if they have the same direction, or equivalently there exists a vector t in T such that $P_2 = t + P_1$. It is clear that the relation "P_1 and P_2 are parallel" is an equivalence relation on the set of affine subspaces of E.

Lemma 4.1.13. *Any two non-parallel hyperplanes have nonempty intersection.*

Proof. Let H_1 and H_2 be two non-parallel hyperplanes, $x_1 \in H_1$ and $x_2 \in H_2$; there exist two hyperplanes M_1 and M_2 of the vector space T of translations such that $H_1 = M_1 + x_1$ and $H_2 = M_2 + x_2$. Since H_1 and H_2 are not parallel, we have $M_1 \neq M_2$ and hence $T = M_1 + M_2$; hence there exist $u_1 \in M_1$ and $u_2 \in M_2$ such that $x_1 - x_2 = u_1 - u_2$, and the point $u_1 + x_1 = u_2 + x_2$ belongs to $H_1 \cap H_2$. $\square$

Proposition 4.1.14. *Let H_1 and H_2 be two distinct hyperplanes of E, and f_1, f_2 two affine functions on E such that H_1 (resp. H_2) consists of the points $x \in E$ such that $f_1(x) = 0$ (resp. $f_2(x) = 0$). Finally, let L be a hyperplane of E. Assume that one of the following hypotheses is satisfied:*

(a) The hyperplanes H_1, H_2 and L are parallel.

(b) The hyperplanes H_1 and H_2 are not parallel, and $H_1 \cap H_2 \subseteq L$.

Then there exist real numbers λ_1, λ_2, not both zero, such that L consists of those points $x \in E$ at which the affine function $g = \lambda_1 f_1 + \lambda_2 f_2$ vanishes.

Proof. The claim being trivial when $L = H_1$, we assume that there exists a point a in L with $a \notin H_1$. Put $\lambda_1 = f_2(a)$, $\lambda_2 = -f_1(a)$ and

$$g = \lambda_1 f_1 + \lambda_2 f_2.$$

then $\lambda_2 \neq 0$ since $a \notin H_1$; moreover, since $H_1 \neq H_2$, there exists $b \in H_1$ such that $b \notin H_2$, so that $f_1(b) = 0$, $f_2(b) \neq 0$, and thus $g(b) = -f_1(a) f_2(b)$ is nonzero. The set $\tilde{L}$ of points where the affine function g vanishes is then a hyperplane of E; we have $g(a) = 0$, so $a \in \tilde{L}$.

Assume that H_1 and H_2 are parallel. Then since g and f_1 both vanish at every point of $\tilde{L} \cap H_1$, so does f_2 (since $\lambda_2 \neq 0$); thus every point of $\tilde{L} \cap H_1$ belongs to H_2; but since H_1 and H_2 are parallel and distinct, they are disjoint, so $\tilde{L} \cap H_1 = \varnothing$, and Lemma 4.1.13 shows that $\tilde{L}$ is parallel to H_1. Since $a \in L$ and $a \in \tilde{L}$, we thus have $L = \tilde{L}$.

Assume that H_1 and H_2 are not parallel. Then by Lemma 4.1.13, there is a point $c \in H_1 \cap H_2$; we give E the vector space structure obtained by taking c as the origin. Then $H_1 \cap H_2$ is a vector subspace of E of codimension 2, and since $a \notin H_1$, the vector subspace M of E generated by $H_1 \cap H_2$ and a is a hyperplane. Since $H_1 \cap H_2 \subseteq L \cap \tilde{L}$ and $a \in L \cap \tilde{L}$, we have $M \subseteq L \cap \tilde{L}$, hence $M = L = \tilde{L}$. $\square$

Proposition 4.1.15. *Let C be a chamber, let H_1 and H_2 be two walls of C, and let L be a hyperplane meeting $D_{H_1}(C) \cap D_{H_2}(C)$. Assume that H_1 is distinct from H_2 and that one of the following conditions is satisfied:*

(a) The hyperplanes H_1, H_2 and L are parallel.

(b) The hyperplanes H_1 and H_2 are not parallel, and $H_1 \cap H_2 \subseteq L$.

Then L meets C.

Proof. Let b_1 (resp. b_2) be a point of the face of C with support H_1 (resp. H_2); it is immediate that every point of the segment (b_1, b_2) belongs to C. Introduce an affine function f_1 that vanishes at every point of H_1 and is such that $f_1(x) > 0$ for x in $D_{H_1}(C)$; similarly, introduce an affine function f_2 having an analogous property with respect to H_2. By applying Proposition 4.1.14, we can find numbers λ_1 and λ_2 and an affine function g having the properties stated in the proposition. We have $(\lambda_1, \lambda_2) \neq (0, 0)$, and for every point x of $L \cap D_{H_1}(C) \cap D_{H_2}(C)$, we have $f_1(x) > 0$, $f_2(x) > 0$ and $\lambda_1 f_1(x) + \lambda_2 f_2(x) = 0$, so $\lambda_1 \lambda_2 < 0$. On the other hand, we have $g(b_1) = \lambda_2 f_2(b_1)$ and $g(b_2) = \lambda_1 f_1(b_2)$, and since $f_1(b_2) > 0$, $f_2(b_1) > 0$, we have $g(b_1) g(b_2) < 0$. The points b_1 and b_2 are thus strictly on opposite sides of the hyperplane L, hence there exists a point c of L which belongs to (b_1, b_2), hence that belongs to C. $\square$

4.2 Reflections over a vector space

In this section, $\mathbb{K}$ will denote a field with characteristic not equal to 2. We denote by V a vector space over $\mathbb{K}$.

4.2.1 Pseudo-reflections and reflections

Definition 4.2.1. An endomorphism s of the vector space V is said to be a **pseudo-reflection** if $1 - s$ is of rank 1.

Let s be a pseudo-reflection in V, and let D be the image of $1 - s$. By definition, D is of dimension 1; thus, given $a \neq 0$ in D, there exists a non-zero linear form a^* on V such that $x - s(x) = \langle a^*, x \rangle a$ for all $x \in V$.

Conversely, given $a \neq 0$ in V and a linear form $a^* \neq 0$ on V, the formula

$$s_{a,a^*}(x) = x - \langle a^*, x \rangle a \quad \text{for} \quad x \in V \tag{4.2.1}$$

defines a pseudo-reflection s_{a,a^*}. The image of $1 - s_{a,a^*}$ is generated by a and the kernel of $1 - s_{a,s^*}$ is the hyperplane of V consisting of those x such that $\langle a^*, x \rangle = 0$. If V^* is the dual of V, it is immediate that the transpose $s_{a^*,a}$ of s_{a,a^*} is the pseudo-reflection of V^* given by the formula

$$s_{a^*,a}(x^*) = x^* - \langle x, a^* \rangle a^* \quad \text{for} \quad x^* \in V^*. \tag{4.2.2}$$

If a is a non-zero vector, a **pseudo-reflection with vector a** is any pseudo-reflection s such that a belongs to the image of $1 - s$. The **hyperplane of a pseudo-reflection** s is the kernel of $1 - s$, the set of vectors x such that $s(x) = x$.

Proposition 4.2.2. *Let G be a group and ρ an irreducible representation of G on a vector space V; assume that there exists an element g of G such that $\rho(g)$ is a pseudo-reflection.*

(a) Every endomorphism of V commuting with $\rho(G)$ is a homothety, and ρ is absolutely irreducible.

(b) Assume that V is finite dimensional. Let β be a non-zero bilinear form on V invariant under $\rho(G)$. Then β is non-degenerate, either symmetric or skew-symmetric, and every bilinear form on V invariant under $\rho(G)$ is proportional to β.

Proof. Let u be an endomorphism of V commuting with $\rho(G)$. Let g be an element of G such that $\rho(g)$ is a pseudo-reflection and let D be the image of $1 - \rho(g)$. Since D is of dimension 1 and $u(D) \subseteq D$, there exists $\lambda \in \mathbb{K}$ such that $u - \lambda I$ vanishes on D; the kernel N of $u - \lambda I$ is then a vector subspace of V invariant under $\rho(G)$ and is non-zero as it contains D; since ρ is irreducible, $N = V$ and $u = \lambda I$. The second part of (a) follows from the first.

Let N be the subspace of V consisting of those $x \in V$ such that $\beta(x, y) = 0$ for all y in V. Since β is invariant under $\rho(G)$, the subspace N is stable under $\rho(G)$ and distinct from V since $\beta \neq 0$. Since ρ is irreducible, $N = 0$ and β is therefore non-degenerate.

If V is finite dimensional, every bilinear form on V is given by the formula

$$\tilde{\beta}(x, y) = \beta(u(x), y)$$

for some endomorphism u of V. If $\tilde{\beta}$ is invariant under $\rho(G)$, then u commutes with $\rho(G)$. Indeed, let x, y be in V and let g be in G; since β and $\tilde{\beta}$ are invariant under $\rho(G)$, we have

$$\begin{aligned}\beta(u(\rho(g)(x)), y) &= \tilde{\beta}(\rho(g)(x), y) = \tilde{\beta}(x, \rho(g^{-1})(y)) \\ &= \beta(u(x), \rho(g^{-1})(y)) = \beta(\rho(g)(u(x)), y).\end{aligned}$$

hence $u(\rho(g)(x)) = \rho(g)(u(x))$ since β is non-degenerate. By (a), there exists $\lambda \in \mathbb{K}$ with $u = \lambda I$, so $\tilde{\beta} = \lambda\beta$.

In particular, we can apply this to the bilinear form $\tilde{\beta}(x, y) = \beta(y, x)$; then $\beta(y, x) = \lambda\beta(x, y) = \lambda^2\beta(y, x)$ for all x, y in V, and since β is nonzero we have $\lambda^2 = 1$, hence $\lambda = 1$ or $\lambda = -1$. Thus, β is either symmetric or skew-symmetric. $\square$

Recall that the field $\mathbb{K}$ is assumed to be of characteristic different from 2. A **reflection** in V is a pseudo-reflection s such that $s^2 = 1$. If s is a reflection, we denote by $V^+(s)$ the kernel of $s - 1$ and by $V^-(s)$ that of $s + 1$.

Proposition 4.2.3. *Let s be an endomorphism of V.*

(a) If s is a reflection, V is the direct sum of the hyperplane $V^+(s)$ and the line $V^-(s)$.

(b) *Conversely, assume that V is the direct sum of a hyperplane H and a line D such that $s(x) = x$ and $s(y) = -y$ for $x \in H$ and $y \in D$. Then s is a reflection and $H = V^+(s)$, $D = V^-(s)$. Finally, D is the image of $1 - s$.*

Proof. If s is a reflection, $V^+(s)$ is a hyperplane. If x belongs to $V^+(s) \cap V^-(s)$, then $x = s(x) = -s(x)$, so $x = 0$ since K is of characteristic $\neq 2$. Finally, for $x \in V$, the vector $y = s(x) + x$ (resp. $z = s(x) - x$) belongs to $V^+(s)$ (resp. $V^-(s)$) since $s^2 = 1$, and $2x = y - z$. Thus V is the direct sum of $V^+(s)$ and $V^-(s)$, and $V^-(s)$ is necessarily of dimension 1 since $V^+(s)$ is a hyperplane.

Conversely, under the hypotheses of (b), every element of V can be written uniquely in the form $v = x + y$ with $x \in H$ and $y \in D$ and we have $s(v) = x - y$; assertion (b) follows immediately from this. □

Corollary 4.2.4. *If V is finite dimensional, every reflection is of determinant -1.*

Proof. Let s be a reflection in V. Proposition 4.2.3 shows that there exists a basis $(e_1, \ldots, e_n)$ of V such that $s(e_1) = e_1, \ldots, s(e_{n-1}) = e_{n-1}$ and $s(e_n) = -e_n$, hence $\det(s) = -1$. □

Proposition 4.2.5. *Let s be a reflection in V.*

(a) *A subspace U of V is stable under s if and only if $V^-(s) \subseteq U$ or $U \subseteq V^+(s)$.*

(b) *An endomorphism u of V commutes with s if and only if $V^+(s)$ and $V^-(s)$ are stable under u.*

Proof. If $U \subseteq V^+(s)$ then $s(x) = x$ for all $x \in U$, so $s(U) \subseteq U$. Assume that $V^-(s) \subseteq U$; then, for any $x \in U$, $s(x) - x \in V^-(s) \subseteq U$, hence $s(x) \in U$; thus $s(U) \subseteq U$. Conversely, assume that $s(U) \subseteq U$; if $U \nsubseteq V^+(s)$, there exists $x \in W$ with $s(x) \neq x$; the non-zero vector $a = s(x) - x$ belongs to the line $V^-(s)$, and hence generates this space; since $a \in U$, we have $V^-(s) \subseteq U$.

Now assume first that u commutes with s. If x is a vector such that $s(x) = \varepsilon x$ (where $\varepsilon = \pm 1$), then $s(u(x)) = \varepsilon u(x)$, so $V^+(s)$ and $V^-(s)$ are stable under u. Conversely, assume that $V^+(s)$ and $V^-(s)$ are stable under u. It is clear that $us - su$ vanishes on $V^+(s)$ and on $V^-(s)$, and since V is the direct sum of $V^+(s)$ and $V^-(s)$, we have $us - su = 0$. □

Corollary 4.2.6. *Two distinct reflections s_1 and s_2 commute if and only if $V^-(s_1) \subseteq V^+(s_2)$ and $V^-(s_2) \subseteq V^+(s_1)$.*

Proof. If $V^-(s_1) \subseteq V^+(s_2)$ and $V^-(s_2) \subseteq V^+(s_1)$, Proposition 4.2.5(a) shows that $V^+(s_2)$ and $V^-(s_2)$ are stable under s_1, hence $s_1 s_2 = s_2 s_1$ by Proposition 4.2.5(b).

Conversely, if $s_1 s_2 = s_2 s_1$, the subspace $V^-(s_1)$ is stable under s_2 by Proposition 4.2.5(b); by Proposition 4.2.5(a), there are two possible cases: if $V^-(s_2) \subseteq V^-(s_1)$, then since they are both of dimension 1, these spaces are therefore equal, hence $V^-(s_1) \nsubseteq V^+(s_2)$ since $V^+(s_2)$ is stable under s_1, we have $V^+(s_2) \subseteq V^+(s_1)$ and so these two hyperplanes are equal. But then $s_1 = s_2$, contrary to our assumptions.

If $V^-(s_1) \subseteq V^+(s_2)$, then the image of $1 - s_1$ is contained in the kernel of $1 - s_2$, so $(1 - s_2)(1 - s_1) = 0$. Since s_1 and s_2 commute, we have $(1 - s_1)(1 - s_2) = 0$, in other words $V^-(s_2) \subseteq V^+(s_1)$. □

Example 4.2.7. Let a be a nonzero vector in V and a^* be a nonzero linear form on V. It follows from formula (4.2.1) that

$$s_{a,a^*}^2(x) = x + (\langle a^*, a\rangle - 2)\langle a^*, x\rangle a$$

and hence that s_{a,a^*} is a reflection if and only if $\langle a^*, a\rangle = 2$. In this case, we have $s_{a,a^*}(a) = -a$.

4.2.2 Orthogonal reflections in a Euclidean affine space

Assume that V is finite dimensional and let β be a non-degenerate bilinear form on V. Then β is invariant under a reflection s in V if and only if the subspaces $V^+(s)$ and $V^-(s)$ of V are orthogonal with respect to β; they are then non-isotropic. Moreover, for any non-isotropic hyperplane H in V, there is a unique reflection s that preserves β and induces the identity on H. This is the **symmetry** with respect to H. If a is a nonzero vector orthogonal to H, we have $\beta(a, a) \neq 0$ and the reflection s is given by the formula

$$s(x) = x - 2\frac{\beta(x, a)}{\beta(a, a)}a \quad \text{for} \quad x \in V. \tag{4.2.3}$$

The reflection s is also called the **orthogonal reflection with respect to H**.

Proposition 4.2.8. *Assume that V is finite dimensional. Let β be a nondegenerate symmetric bilinear form on V, U a subspace of V and $U^\perp$ the orthogonal complement of U with respect to β. Finally, let s be the orthogonal reflection with respect to a non-isotropic hyperplane H of V. Then the following conditions are equivalent:*

(i) *U is stable under s.*

(ii) *$U^\perp$ is stable under s.*

(iii) *H contains U or $U^\perp$.*

Proof. We have $V^+(s) = H$, and by what we have said, $V^-(s)$ is the orthogonal complement $H^\perp$ of H with respect to β. By Proposition 4.2.5, U is stable under s if and only if $U \subseteq H$ or $H^\perp \subseteq U$; but the relation $H^\perp \subseteq U$ is equivalent to $U^\perp \subseteq H$. This proves the equivalence of (i) and (iii); that of (ii) and (iii) follows by interchanging the roles of U and $U^\perp$, since $(U^\perp)^\perp = U$. □

Now let E be an affine space of which V is the space of translations. Giving the form β on V provides E with the structure of a Euclidean space. Let H be a non-isotropic hyperplane of E. The symmetry with respect to H is also called the orthogonal reflection with respect to H; we often denote it by s_H. We have $s_H^2 = 1$ and s_H is the unique displacement of E, distinct from the identity and leaving fixed the elements of H. The automorphism of V associated to s_H is the orthogonal reflection with respect to the direction of H (which is a non-isotropic hyperplane of V).

Every x in E can be written uniquely in the form $x = h + v$, with $h \in H$ and $u \in V$ orthogonal to H; we have

$$s_H(h+v) = h - v.$$

Proposition 4.2.9. *Let H_1 and H_2 be two parallel, non-isotropic hyperplanes of E. There exists a unique vector $v \in V$ orthogonal to H and such that $H_2 = H_1 + v$. The displacement $s_{H_1}s_{H_2}$ is the translation by the vector $2v$.*

Proof. The existence and uniqueness of v are immediate. The automorphism of V associated to $s_{H_2}s_{H_1}$ is the identity; thus $s_{H_2}s_{H_1}$ is a translation. On the other hand, let $a \in H_2$; then $a - v \in H_1$ and

$$s_{H_2}s_{H_1}(a-v) = s_{H_2}(a-v) = a + v = (a-v) + 2v$$

showing that $s_{H_2}s_{H_1}$ is the translation by the vector $2v$. □

Proposition 4.2.10. *Let H_1 and H_2 be two distinct, parallel, non-isotropic hyperplanes. If $\mathbb{K}$ is of characteristic zero (resp. $p > 0$, with $p \neq 2$), the group of displacements of E generated by s_{H_1} and s_{H_2} is an infinite dihedral group (resp. a dihedral group of order $2p$).*

Proof. Indeed, by Proposition 3.1.4, it suffices to show that $s_{H_2}s_{H_1}$ is of infinite order (resp. order $2p$), which is clear. □

Example 4.2.11. We retain the notation of Proposition 4.2.9 and assume in addition that $\mathbb{K} = \mathbb{R}$. Put $s_1 = s_{H_1}$ and $s_2 = s_{H_2}$. Let H_n be the hyperplane $H + nv$ and let C_n be the set of points of E of the form $a + \xi v$ with $a \in H$ and $n < \xi < n+1$. The C_n are connected open sets forming a partition of $E - a$. They are therefore the chambers defined by the system $\mathcal{H} = (H_n)$ in E. The translation $(s_2s_1)^n$ transforms the chamber $C = C_0$ into the chamber C_{2n} and since $s_1(C_0) = C_{-1}$, we have $(s_2s_1)^n s_1(C) = C_{2n-1}$. It follows that the dihedral group W generated by s_1 and s_2 permutes the chambers C_n simply-transitively.

Moreover, as we shall now show, if the chambers C and $w(C)$ are on opposite sides of H_1 (for $w \in W$), we have $\ell(s_1w) = \ell(w) - 1$ (the lengths being taken with respect to $S = \{s_1, s_2\}$). Indeed, we then have $w(C) = C_n$ for some $n < 0$, and we have two cases:

(a) If $n = -2k$, then $w = (s_1s_2)^k$ and

$$s_1w = s_1 \cdot \underbrace{s_1s_2 \cdots s_1s_2}_{k\text{-times}} = \underbrace{s_2s_1 \cdots s_2s_1}_{k-1\text{-times}} \cdot s_2 = (s_2s_1)^{k-1}s_2,$$

so $\ell(w) = 2k$ and $\ell(s_1w) = 2k-1$.

(b) If $n = -2k-1$, then $w = (s_1s_2)^k s_1$ and

$$s_1w = s_1 \cdot \underbrace{s_1s_2 \cdots s_1s_2}_{k\text{-times}} \cdot s_1 = \underbrace{s_2s_1 \cdots s_2s_1}_{k\text{-times}} = (s_2s_1)^k,$$

so $\ell(w) = 2k+1$ and $\ell(sw) = 2k$.

4.2.3 Complements on plane rotations

In this paragraph V denotes a real vector space of dimension 2, provided with a scalar product (that is, a non-degenerate, positive, symmetric bilinear form) and an orientation. The measures of angles will be taken with respect to the base 2π. The principal measure of an angle between half-lines (resp. lines) is thus a real number θ such that $0 \le \theta < 2\pi$ (resp. $0 \le \theta < \pi$). By abuse of language, for any real number θ, we shall use θ to denote an angle whose measure is θ and denote by ρ_θ the rotation with angle θ.

Proposition 4.2.12. *Let s be the orthogonal reflection with respect to a line D of V. If Δ_1 and Δ_2 are two half-lines starting at the origin (resp. two lines passing through the origin) of V, we have*

$$(\widehat{s(\Delta_1), s(\Delta_2)}) \equiv -(\widehat{\Delta_1, \Delta_2}) \mod 2\pi \quad (\textit{resp.} \mod \pi).$$

Proof. Let u be a rotation transforming Δ_1 into Δ_2. Since su is an orthogonal transformation of V of determinant -1, it is a reflection and thus $(su)^2 = 1$. Consequently, $u^{-1} = sus^{-1}$ transforms $s(\Delta_1)$ into $s(\Delta_2)$, hence the proposition. □

Corollary 4.2.13. *Let D_1 and D_2 be two lines of V and let θ be a measure of the angle (D_1, D_2). Then $s_{D_2}s_{D_1} = \rho_{2\theta}$.*

Proof. We know that $s_{D_2}s_{D_1}$ is a rotation since it is of determinant 1. Let Δ_1 and Δ_2 be two half-lines starting at the origin carried by D_1 and D_2. We have

$$\begin{aligned}(\widehat{\Delta, s_{D_2}s_{D_1}(\Delta)}) &\equiv (\widehat{\Delta_1, s_{D_2}(\Delta_1)}) \equiv (\widehat{\Delta_1, \Delta_2}) + (\widehat{\Delta_2, s_{D_2}(\Delta_1)}) \\ &\equiv (\widehat{\Delta_1, \Delta_2}) + (\widehat{s_{D_2}(\Delta_2), s_{D_2}(\Delta_1)}) \\ &\equiv (\widehat{\Delta_1, \Delta_2}) - (\widehat{\Delta_2, \Delta_1}) \equiv 2(\widehat{\Delta_1, \Delta_2}) \mod 2\pi.\end{aligned}$$

hence the corollary. □

Now let Δ_1 and Δ_2 be two half-lines of V such that $\Delta_2 \neq \pm\Delta_1$, and let s_1 and s_2 be the orthogonal reflections with respect to the lines D_1 and D_2 containing Δ_1 and Δ_2. Let θ be the principal measure of the angle (D_1, D_2). If $\theta \in \pi\mathbb{Q}$, denote by m the smallest positive integer such that $m\theta \in \pi\mathbb{Z}$. If $\theta \notin \pi\mathbb{Q}$, put $m = \infty$. Let W be the group generated by s_1 and s_2.

Proposition 4.2.14. *The group W is dihedral of order $2m$. It consists of the rotations $\rho_{2n\theta}$ and the products $\rho_{2n\theta}s_1$ for $n \in \mathbb{Z}$. The transforms of D_1 and D_2 by the elements of W are the transforms of D_1 by the rotations $\rho_{n\theta}$ for $n \in \mathbb{Z}$.*

Proof. The Corollary to Proposition 4.2.12 shows that s_2s_1 is of order m, which gives the first assertion. The elements of W are thus of the form $(s_2s_1)^n = \rho_{2n\theta}$ or $(s_2s_1)^n s_1 = \rho_{2n\theta}s_1$. The last assertion follows from this, since $D_2 = \rho_\theta(D_1)$. □

Corollary 4.2.15. *Let C be the open angular sector formed by the half-lines Δ_1 and Δ_2, with angular θ. Then no transform of D_1 or D_2 by an element of W meets C if and only if m is finite and $\theta = \pi/m$.*

Proof. If $m = \infty$, the image of the set $n\theta$ ($n \in \mathbb{Z}$) is dense in $\mathbb{R}/2\pi\mathbb{Z}$; the union of the transforms of D_1 by the elements of W is thus dense in V and meets C. If m is finite and if $\theta = k\pi/m$ with $1 < k < m$, the integers k and m being refittively prime, there exists an integer j such that $kj = 1 \mod m$. Then $(D_1, \rho_{j\theta}(D_1)) \equiv \pi/m \mod \pi$, and $\rho_{j\theta}(D_1)$ meets C. This shows that the condition is necessary. The converse is immediate. □

Example 4.2.16. Assume that m is finite and that $\theta = \pi/m$. If $n \in \mathbb{Z}$, let C_n be the open angular sector formed by $\rho_{n\theta}(\Delta_1)$ and $\rho_{n\theta}(\Delta_2)$. The C_n for $-m < n < m$ are connected open subsets forming a partition of $E \setminus \bigcup_n D_n$ (where $D_n = \rho_{n\theta}(D_1)$). These are therefore the chambers determined in E by the system of m lines D_n with $1 \le n \le m$. We have $C_{2k} = \rho_{2k\theta}(C)$. Moreover, $C_{2k-1} = \rho_{2k\theta}s_1(C)$ if and only if $n \in 2m\mathbb{Z}$. Consequently, the group W permutes the chambers C_n simply-transitively.

We show finally that, if $w \in W$ is such that the chambers C and $w(C)$ are on opposite sides of the line D, then $\ell(sw) = \ell(w) - 1$ (the lengths being taken with respect to $S = \{s, \tilde{s}\}$). Indeed, the assumption implies that $w(C) = C_n$ with $-m \le n < 0$. If $n = -2k$, we have $w = (s\tilde{s})^k$ and $sw = \tilde{s}(s\tilde{s})^{k-1}$, so $\ell(w) = 2k$ and $\ell(sw) = 2k - 1$ (Example 4.2.11). If $n = -2k + 1$, we have $w = (s\tilde{s})^{k-1}s$ and $sw = (\tilde{s}s)^{k-1}$, hence $\ell(w) = 2k - 1$ and $\ell(sw) = 2k - 2$.

4.3 Reflection group over affine spaces

In this paragraph, we denote by E a real affine space of finite dimension l, and by T the space of translations of E. We assume that T is provided with a scalar product (that is, a non-degenerate, positive, symmetric bilinear form), denoted by $(\cdot,\cdot)$. For $t \in T$, put $\|t\| = \sqrt{(t,t)}$. The function $d(x,y) = \|x-y\|$ is a distance on E, which defines the topology of E.

4.3.1 The associated Coxeter system

We denote by $\mathcal{H}$ a collection of hyperplanes of E and by W the group of displacements of the Euclidean space E generated by the orthogonal reflections s_H with respect to the hyperplanes $H \in \mathcal{H}$. We assume that the following **discreteness conditions** are satisfied:

(D1) For any $w \in W$ and any $H \in \mathcal{H}$, the hyperplane $w(H)$ belongs to $\mathcal{H}$;

(D2) The group W, provided with the discrete topology, acts properly on E.

Since E is locally compact, it follows that condition (D2) is equivalent to the following condition:

(D2′) For any two compact subsets K and L of E, the set of $w \in W$ such that $w(K)$ meets L is finite.

Lemma 4.3.1. *The set of hyperplanes $\mathcal{H}$ is locally finite.*

Proof. Indeed, let K be a compact subset of E. If a hyperplane $H \in \mathcal{H}$ meets K, the set $s_H(K)$ also meets K, since every point of $K \cap H$ is fixed by s_H. The set of $H \in \mathcal{H}$ meeting K is thus finite by (D2′). □

We can thus apply to E and $\mathcal{H}$ the definitions and results of the previous parts. We shall call the chambers, facets, walls, etc. defined in E by $\mathcal{H}$ simply the chambers, facets, walls, etc. relative to W. Any displacement $w \in W$ permutes the chambers, facets, walls, etc.

Proposition 4.3.2. *Let C be a chamber.*

(a) For any $x \in E$, there exists an element $w \in W$ such that $w(x) \in \bar{C}$.

(b) For any chamber $\tilde{C}$, there is an element $w \in W$ such that $w(\tilde{C}) = C$.

(c) The group W is generated by the set of orthogonal reflections with respect to the walls of C.

Proof. Let $\mathcal{W}$ be the set of walls of C and let W_C be the subgroup of W generated by the reflections with respect to the walls of C.

Let $x \in E$ and let O_x be the orbit of x under the group W_C. It suffices to prove that O_x meets $\bar{C}$. Let a be a point of C; there is a closed ball B with centre a meeting O_x; since B is compact, property (D2′) shows that $B \cap O_x$ is finite. Hence, there exists a point y of O_x such that

$$d(a,y) \leq d(a,z) \quad \text{for} \quad z \in O_x. \tag{4.3.1}$$

We shall prove that $y \in \bar{C}$. For this, it suflices to show that if H is a wall of C then $y \in \overline{D_H(C)}$ (Proposition 4.1.11). Since $s_H \in W_C$, we have $s_H(y) \in O_x$ and so

$$d(a,y)^2 \leq d(a,s_H(y))^2 \tag{4.3.2}$$

by (4.3.1). There exist $b \in H$ and two vectors t and u such that $a = b+t$ and $y = b+u$, the vector u being orthogonal to H; then $s_H(y) = b-u$, and (4.3.2) is equivalent to $(t-u,t-u) \leq (t+u,t+u)$, or to $(t,u) \geq 0$. This inequality implies that $y \in \overline{D_H(C)}$.

Let $\tilde{C}$ be a chamber and $b \in \tilde{C}$. By what we have proved, there exist $w \in W_C$ such that $w^{-1}(b) \in \bar{C}$; hence, the chamber $\tilde{C}$ meets $\overline{w(C)}$; since $\overline{w(C)}$ is the union of $w(C)$ and facets with empty interior, we have $\tilde{C} = w(C)$.

Finally, we have to prove that $W = W_C$ and for this it suffices to prove that $s_{\tilde{H}} \in W_C$ for all $\tilde{H} \in \mathcal{H}$. Now $\tilde{H}$ is a wall of at least one chamber $\tilde{C}$; we have seen that there exists $w \in W_C$ such that $\tilde{C} = w(C)$; consequently, there exists a wall H of C such that $\tilde{H} = w(H)$; hence $s_{\tilde{H}} = w s_H w^{-1} \in W_C$. □

Theorem 4.3.3. *Let C be a chamber and let S be the set of reflections with respect to the walls of C.*

(a) The pair (W,S) is a Coxeter system.

(b) *Let $w \in W$ and let H be a wall of C. Then the relation $\ell(s_H w) > \ell(w)$ implies that the chambers C and $w(C)$ are on the same side of H.*

(c) *For any chamber $\tilde{C}$, there exists a unique element $w \in W$ such that $w(C) = \tilde{C}$.*

(d) *The set of hyperplanes H such that $s_H \in W$ is equal to $\mathcal{H}$.*

Proof. Every element of S is of order 2 and Proposition 4.3.2 shows that S generates W. For any wall H of C, denote by P_H the set of elements $w \in W$ such that the chambers C and $w(C)$ (which do not meet H) are on the same side of H. We shall verify conditions (A'), (B') and (C) of Proposition 3.1.28.

Since $1 \in P_H$, condition (A') is trivial. For (B'), we show that P_H and $s_H \cdot P_H$ are disjoint: Indeed, $w(C)$ and $s_H w(C)$ are on opposite sides of H, so if $w(C)$ is on the same side of H as C, it is not on the same side as $s_H w(C)$.

To prove condition (C), let $w \in P_H$ and let $\tilde{H}$ be a wall of C such that $ws_{\tilde{H}} \notin P_H$. By assumption, $w(C)$ is on the same side of H as C and $ws_{\tilde{H}}(C)$ is on the other side. Thus, $ws_{\tilde{H}}(C)$ and $w(C)$ are on opposite sides of H; hence, the chambers $s_{\tilde{H}}(C)$ and C are on opposite sides of the hyperplane $w^{-1}(H)$. Let a be a point of the face of C with support $\tilde{H}$. The point $a = s_{\tilde{H}}(a)$ is in the closure of the two chambers C and $s_{\tilde{H}}(C)$ that are contained respectively in the two open half-spaces bounded by $w^{-1}(H)$; thus, $a \in w^{-1}(H)$, so $\tilde{H} = w^{-1}(H)$. From this we deduce that $s_{\tilde{H}} = w^{-1}s_H w$, hence (C) is verified.

Assertions (a) and (b) follow from this and Proposition 3.1.28. Moreover, we have (by condition (A) of Proposition 3.1.27)

$$\bigcap_{H \in \mathcal{W}} P_H = \{1\}. \tag{4.3.3}$$

Proposition 4.3.2 shows that W acts transitively on the set of chambers. Moreover, if $w \in W$ is such that $w(C) = C$, then $w \in P_H$ for every wall H of C, hence $w = 1$ by (4.3.3). This proves (c).

Finally, let H be a hyperplane such that $s_H \in W$. If H did not belong to $\mathcal{H}$, there would be at least one chamber $\tilde{C}$ meeting H (Proposition 4.1.8). Every point of $H \cap \tilde{C}$ is invariant under s_H, and thus belongs to the chambers $\tilde{C}$ and $s_H(\tilde{C})$; thus, $\tilde{C} = s_H(\tilde{C})$, which contradicts (c) since $s_H \neq 1$. □

Corollary 4.3.4. *Let Σ be a set of reflections generating W. Then every reflection belonging to W is conjugate to an element of Σ.*

Proof. Let $\tilde{\mathcal{H}}$ be the set of hyperplanes of the form $w(H)$ with $w \in W$ and $H \in \mathcal{H}$ such that $s_H \in \Sigma$. Since W is generated by the family $(s_H)_{H \in \tilde{\mathcal{H}}}$ and since $\tilde{\mathcal{H}}$ is stable under W, we can apply all the results of this paragraph to $\tilde{\mathcal{H}}$ instead of $\mathcal{H}$. But Theorem 4.3.3(d) shows that every reflection s_H in W is of the form $s_{\tilde{H}} = ws_H w^{-1}$ with $\tilde{H} = w(H) \in \tilde{\mathcal{H}}$, hence the corollary. □

4.3.2 Fundamental domain and stabilisers

Recall that a subset D of E is called a fundamental domain for the group W if every orbit of W in E meets D in exactly one point. This is equivalent to the following conditions:

(a) For every $x \in E$, there exists to $w \in W$ such that $w(x) \in D$.

(b) If $x, y \in D$ and $w \in W$ are such that $y = w(x)$, then $x = y$ (even though we may have $w \neq 1$).

Theorem 4.3.5. *For any chamber C, the closure $\overline{C}$ of C is a fundamental domain for the action of W on E.*

Proof. By Proposition 4.3.2, it suffices to prove the following assertion:

Let C be a chamber, let x and y be two points of C and let $w \in W$ be such that $w(x) = y$. Then $x = y$ and w belongs to the subgroup $W_{\mathcal{W}}$, where $\mathcal{W}$ is the set of walls of C containing x. (4.3.4)

We argue by induction on the length p of w (relative to the set S of reflections with respect to the walls of C), the case $p = 0$ being obvious. If $p \geq 1$, there exist a wall H of C and an element $\tilde{w} \in W$ such that $w = s_H\tilde{w}$ and $\ell(s_H\tilde{w}) = p - 1$. Since $\ell(s_H w) < \ell(w)$, the chambers C and $w(C)$ are on opposite sides of H by Theorem 4.3.3. Thus, $\overline{C} \cap w(\overline{C}) \subseteq H$, so $y \in H$. Thus $y = \tilde{w}(x)$ and the induction hypothesis implies that $x = y$ and $\tilde{w} \in W_{\mathcal{W}}$. Since $y \in H$, it follows that $H \in \mathcal{W}$, and hence that $w = s_H\tilde{w} \in W_{\mathcal{W}}$, completing the proof. □

Corollary 4.3.6. *If C is a chamber and F a facet, there exists a unique facet $\tilde{F}$ contained in C that is transformed into F by an element of W.*

Proposition 4.3.7. *Let F be a facet and C a chamber such that $F \subseteq \overline{C}$. Let $w \in W$. The following conditions are equivalent:*

(i) *$w(F)$ meets F;*

(ii) *$w(F) = F$;*

(iii) *$w(\overline{F}) = \overline{F}$;*

(iv) *w fixes at least one point of F;*

(v) *w fixes every point of F;*

(vi) *w fixes every point of $\overline{F}$;*

(vii) *w belongs to the subgroup of W generated by the reflections with respect to the walls of C containing F.*

Proof. We know two distinct facets are disjoint and have distinct closures (Corollary 4.1.3). The equivalence of (i), (ii) and (iii) follows. On the other hand, it is clear that

$$\text{(vii)} \Rightarrow \text{(vi)} \Rightarrow \text{(v)} \Rightarrow \text{(iv)} \Rightarrow \text{(i)}$$

and assertion (4.3.4) shows that (i)⇒(vii). □

For every subset X of E, denote by $W(X)$ the subgroup of W consisting of the elements that fix every point of X.

Proposition 4.3.8. *Let X be a non-empty subset of E, let $\mathcal{H}_X$ be the set of hyperplanes $H \in \mathcal{H}$ containing X, let L_X be the intersection of the $H \in \mathcal{H}_X$, and let F be a facet of E open in L_X. Then $W(X) = W(L_X) = W(F)$, and $W(X)$ is generated by the reflections with respect to the hyperplanes in $\mathcal{H}_X$.*

Proof. let $\tilde{X}$ be the affine subspace of E generated by X. Clearly, $W(X) = W(\tilde{X})$. By Proposition 4.1.8, there exists a point $x \in \tilde{X}$ that does not belong to any hyperplane $H \in \mathcal{H} \setminus \mathcal{H}_X$. Let F_x be the facet containing x; it is open in $\tilde{X}$ and Proposition 4.3.7 shows that

$$W(F_x) \subseteq W(L_X) \subseteq W(X) = W(\tilde{X}) \subseteq W(\{x\}) = W(F_x)$$

hence $W(X) = W(L_X) = W(F_x)$. Replacing X by F, we also have $W(X) = W(F)$, hence the proposition. □

Corollary 4.3.9. *For any non-empty subset X of E, there exists a point $a \in E$ such that $W(X) = W(\{a\})$. Moreover, the group $W(X)$ is a Coxeter group.*

Proof. By the proof of Proposition 4.3.8, we see $W(X) = W(a)$ for some $a \in X$. Moreover, if $\mathcal{R}$ is the set of hyperplanes $H \in \mathcal{H}$ such that $a \in H$, then $W(X)$ is generated by reflections s_H with $H \in \mathcal{R}$, whence $W(X)$ is a Coxeter group by Theorem 4.3.3. □

Corollary 4.3.10. *Let C be a chamber of E and S the set of reflections With respect to the walls of C. Let $w \in W$ and let $(s_1, \dots, s_p)$ be a reduced decomposition of w with respect to S. If $x \in \overline{C}$ is fixed by w, then $s_j(x) = x$ for all j.*

Proof. Since W acts freely on chambers, we see x is contained in a facet F of C. Therefore by Proposition 4.3.7 w is contained in the group $W(F)$, which is generated by the set $S(F)$ of reflections with respect to the hyperplanes in $\mathcal{H}_F$ (Proposition 4.3.8). By Corollary 3.1.31, we see $w \in W(F)$ if and only if $S_w \subseteq S(F)$, whence the claim. □

4.3.3 Coexter matrix and Coxeter graph of reflection groups

Let C be a chamber, $S = S(C)$ the set of orthogonal reflections with respect to the walls of C and $M = (m(s,\tilde{s}))$ the Coxeter matrix of the Coxeter system (W,S): recall that $m(s,\tilde{s})$ is the order (finite or infinite) of the element $s\tilde{s}$ of W (for $s,\tilde{s} \in S$). If $\tilde{C}$ is another chamber, the unique element $w \in W$ such that $w(C) = \tilde{C}$ defines a bijection

$$s \mapsto f(s) = wsw^{-1}$$

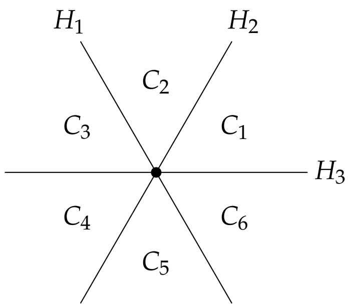
from S to $\tilde{S} = S(\tilde{C})$, and we have $m(f(s), f(\tilde{s})) = m(s,\tilde{s})$. It follows that, if W acts on the set $X(W)$ of pairs (C,s), where C is a chamber and $s \in S(C)$, by

$$w \cdot (C,s) = (w(C), wsw^{-1}),$$

each orbit i of W in X meets each of the sets $\{C\} \times S(C)$ in exactly one point, which we denote by $(C, s_i(C))$. Thus, if I is the set of orbits and $i,j \in I$, the number $m_{ij} = m(s_i(C), s_j(C))$ is independent of the choice of chamber C. The matrix $M(W) = (m_{ij})_{i,j\in I}$ is a Coxeter matrix called the **Coxeter matrix of *W***. The Coxeter graph associated to $M(W)$ is called the **Coxeter graph of *W***.

Let C be a chamber. For any $i \in I$, denote by $H_i(C)$ the wall of C such that $s_i(C)$ is the reflection with respect to $H_i(C)$ and by $e_i(C)$ the unit vector orthogonal to $H_i(C)$ on the same side of $H_i(C)$ as C. The map $i \mapsto H_i(C)$ is called the **canonically indexed family** of walls of C.

Example 4.3.11. Consider the space plane $\mathbb{R}^2$ and the following hyperplanes:



and let W be its Weyl group. If we denote by $\{s,\tilde{s}\}$ the orbit of W in $X(W)$ and set $s(C_1) = s_{H_2}$ and $\tilde{s}(C_1) = s_{H_3}$, then

$$\begin{array}{llllll} s(C_1) = s_{H_2} & s(C_2) = s_{H_2} & s(C_3) = s_{H_3} & s(C_4) = s_{H_3} & s(C_5) = s_{H_1} & s(C_6) = s_{H_1} \\ \tilde{s}(C_1) = s_{H_3} & \tilde{s}(C_2) = s_{H_1} & \tilde{s}(C_3) = s_{H_1} & \tilde{s}(C_4) = s_{H_2} & \tilde{s}(C_5) = s_{H_2} & \tilde{s}(C_6) = s_{H_3} \end{array}$$

With this, we see the Coxeter matrix of W is given by

$$M(W) = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$$

We then recognize that this matrix is the Coxeter matrix of $\mathfrak{S}_2$, as in Example 3.1.8.

Proposition 4.3.12. *Let C be a chamber and let $i,j \in I$ with $i \neq j$. Put $s_i = s_i(C)$, $H_i = H_i(C)$, $e_i = e_i(C)$ and define s_j, H_j, and e_j similarly.*

(a) If H_i and H_j are parallel, then $m_{ij} = \infty$ and $e_i = -e_j$.

(b) If H_i and H_j are not parallel, then m_{ij} is finite and

$$(e_i, e_j) = -\cos(\pi/m_{ij}). \tag{4.3.5}$$

In particular, we have $(e_i, e_j) \leq 0$.

Proof. If H_i and H_j are parallel, then $s_i s_j$ is a translation, so $m_{ij} = \infty$. Moreover, $e_i = \pm e_j$. Now, there exists a point a (resp. b) in the closure of C that belongs to H_i (resp. H_j) but not to H_j (resp. H_i). Then $(a - b, e_i) > 0$ and $(b - a, e_j) > 0$, which excludes the case $e_i = e_j$ and proves (a).

Assume now that H_i and H_j are not parallel. Choose an origin $a \in H_i \cap H_j$ and identify T with E by the bijection $t \mapsto a + t$. Let V be the plane orthogonal to $H_i \cap H_j$ and passing through a. Put

$\Gamma = V \cap D_{H_i}(C) \cap D_{H_j}(C)$ (where $D_H(C)$ denotes the open half-space bounded by H and containing C) and let D_i (resp. D_j) be the half-line in V contained in $H_i \cap V$ (resp. $H_j \cap V$) and in the closure of F. For a suitable orientation of V, the set F is the union of the open half-lines Δ in V such that

$$0 < \widehat{(D, \Delta)} < \widehat{(D_1, D_2)}.$$

Let $\tilde{W}$ be the subgroup of W generated by s_i and s_j. For any $w \in \tilde{W}$, the hyperplanes $w(H_i)$ and $w(H_j)$ belong to $\mathcal{H}$, contain $H_i \cap H_j$ and do not meet C. It follows that they do not meet Γ (Proposition 4.1.15). Then Corollary 4.2.15 thus implies (b). □

4.3.4 Systems of obtuse vectors

Let q be a positive quadratic form on a real vector space V and let β be the associated symmetric bilinear form. A subset A of V is called **obtuse** with respect to β if $\beta(a, b) \leq 0$ for all $a, b \in A$.

Lemma 4.3.13. *Let $\{a_1, \ldots, a_n\}$ be a obtuse subset of V with respect to β.*

(a) If $c_1, \ldots, c_n$ are real numbers such that $q(\sum_i c_i a_i) = 0$, then $q(\sum_i |c_i| a_i) = 0$.

(b) If q is non-degenerate and if there exists a linear form f on V such that $f(a_i) > 0$ for all i, the vectors $a_1, \ldots, a_n$ are linearly independent.

Proof. The relation $\beta(a_i, a_j) \leq 0$ for $i \neq j$ immediately implies that

$$q(\sum_i |c_i| a_i) = \sum_{i,j} |c_i c_j| \beta(a_i, a_j) \leq \sum_{i,j} c_i c_j \beta(a_i, a_j) = q(\sum_i c_i a_i),$$

whence (a). If q is non-degenerate, the relation $\sum_i c_i a_i = 0$ thus implies that $\sum_i |c_i| a_i = 0$; it follows that, for any linear form f on V, we have $\sum_i |c_i| f(a_i) = 0$, and hence $c_i = 0$ for all i if we also have $f(a_i) > 0$ for all i. This proves (b). □

Lemma 4.3.14. *Let $Q = (q_{ij})$ be a real, symmetric, square matrix of order n such that:*

(i) $q_{ij} \leq 0$ for $i \neq j$;

(ii) there does not exist a partition of $\{1, 2, \ldots, n\}$ into two non-empty subsets I and J such that $q_{ij} = 0$ for $i \in I, j \in J$;

(iii) the quadratic form $q(\boldsymbol{x}) = \boldsymbol{x}^T Q \boldsymbol{x}$ induced by Q on $\mathbb{R}^n$ is positive.

Then

(a) The kernel N of q is of dimension 0 or 1. If $\dim(N) = 1$, N is generated by a vector all of whose coordinates are positive.

(b) The smallest eigenvalue of Q is of multiplicity 1 and a corresponding eigenvector has all its coordinates positive or all its coordinates negative.

Proof. Since q is a positive quadratic form, the kernel N of q is the set of isotropic vectors for q. Let $e_1, \ldots, e_n$ be the canonical basis of $\mathbb{R}^n$. If $\sum_i c_i e_i \in N$, Lemma 4.3.13 shows that we also have $\sum_i |c_i| e_i \in N$, and hence $\sum_i q_{ij} |c_i| = 0$ for all j. Let I be the set of i such that $c_i \neq 0$. If $j \notin I$, then $q_{ij} |c_i| < 0$ for $i \in I$ (since $i \neq j$) and $q_{ij} |c_i| = 0$ for $i \notin I$, so $q_{ij} = 0$ for $i \in I$ and all j. In particular, assumption (ii) implies that either $I = \varnothing$ or $I = \{1, \ldots, n\}$. Consequently, every non-zero vector in N has nonzero coordinates. If $\dim(N) \geq 2$, the intersection of N with the hyperplane with equation $x_i = 0$ would be of dimension ≥ 1, contrary to what we have just shown. The preceding argument also shows that, if $\dim(N) = 1$, then N contains a vector all of whose coordinates are positive. This proves (a).

On the other hand, we know that the eigenvalues of Q are real and positive since q is positive. Let λ be the smallest of them. The matrix $\tilde{Q} = Q - \lambda I_n$ is then the matrix of a degenerate positive form $\tilde{q}$ and the off-diagonal elements of $\tilde{Q}$ are the same as those of Q. Consequently, $\tilde{Q}$ satisfies conditions (i), (ii) and (iii) of the statement of the lemma. Since the kernel $\tilde{N}$ of $\tilde{q}$ is the eigenspace of Q corresponding to the eigenvalue λ, assertion (b) follows from (a). □

Now we return to our real affine vector space E and its space of translations T. Recall that $\dim(E) = l$ and we have a scalar product on T, the results above can be applied to T.

Proposition 4.3.15. *Let $\{e_1, \dots, e_n\}$ be a obtuse subset of T such that*

(i) *$\{e_1, \dots, e_n\}$ generates T;*

(ii) *there does not exist a partition of $\{1, \dots, n\}$ into two non-empty subsets I and J such that $(e_i, e_j) = 0$ for $i \in I$ and $j \in J$.*

Then there are two possibilities:

(a) $(e_1, \dots, e_n)$ is a basis of T;

(b) $n = l + 1$, and there exists a family $(c_i)_{1 \leq i \leq n}$ of positive real numbers such that $\sum_i c_i e_i = 0$, and any family $(\tilde{c}_i)_{1 \leq i \leq n}$ of real numbers such that $\sum_i \tilde{c}_i e_i = 0$ is proportional to $(c_i)_{1 \leq i \leq n}$.

Proof. Put $q_{ij} = (e_i, e_j)$. The matrix $Q = (q_{ij})$ then satisfies the hypotheses of Lemma 4.3.14. The kernel N of the quadratic form q on $\mathbb{R}^n$ with matrix Q is the set of $(c_1, \dots, c_n)$ such that $\sum_i c_i e_i = 0$. If $N = \{0\}$, the e_i are linearly dependent and we are in case (a). If $\dim(N) = 1$, Lemma 4.3.14 shows that we are in case (b). □

Proposition 4.3.16. *Let $(e_1, \dots, e_n)$ be a obtuse basis of T.*

(a) If $x = \sum_i c_i e_i$ is such that $(x, e_i) \geq 0$ for all i, then $c_i \geq 0$ for all i.

(b) If x and y are two elements of T such that $(x, e_i) \geq 0$ and $(y, e_i) \geq 0$ for all i, then $(x, y) \geq 0$. If $(x, e_i) > 0$ and $(y, e_i) > 0$ for all i, then $(x, y) > 0$.

Proof. Under the hypotheses of (a), assume that $c_i < 0$ for some i. Let f be the linear form on T defined by $f(e_i) = 1$ and $f(e_j) = -c_i / \sum_{k=1}^{n} |c_k|)^{-1}$ for $j \neq i$. The vectors $-x, e_1, \dots, e_n$ then satisfy the hypotheses of Lemma 4.3.13(b) (taking for q the metric form on T). We conclude that they are linearly independent, which is absurd. Hence (a). Moreover, if $x = \sum_i c_i e_i$ and $y \in T$, then $(x, y) = \sum_i c_i (e_i, y)$, so (b) follows immediately from (a). □

4.3.5 Finiteness theorems

Lemma 4.3.17. *Let A be a set of unit vectors in T. If there exists a real number $\lambda < 1$ such that $(a, b) \leq \lambda$ for $a, b \in A$ and $a \neq b$, then the set A is finite.*

Proof. For $a, b \in A$ such that $a \neq b$, we have

$$\|a - b\|^2 = 2 - 2(a, b) \geq 2 - 2\lambda.$$

Now the unit sphere S of T being compact, there exists a finite covering of S by sets of diameter $< \sqrt{(2 - 2\lambda)}$ each of these sets contains at most one point of A, hence the lemma. □

Denote by $U(w)$ the automorphism of T associated to the affine map $w \in W$ from E to itself. We have

$$w(x + t) = w(x) + U(w) \cdot t \quad \text{for } t \in T \text{ and } x \in E$$

This defines a homomorphism U from the group W to the orthogonal group of T; the kernel of U is the set of translations belonging to W.

Theorem 4.3.18. *Let U be the homomorphism defined above.*

(a) The set of walls of a chamber is finite.

(b) The set of directions of hyperplanes belonging to $\mathcal{H}$ is finite.

(c) The group $U(W)$ is finite.

Proof. Assertion (a) follows immediately from Proposition 4.3.12(b) and Lemma 4.3.17. We prove (b). Let C be a chamber and $\mathcal{W}$ the set of its walls. The facets of $\overline{C}$ (relative to $\mathcal{H}$) are the same as those relative to $\mathcal{W}$ (Proposition 4.1.11). Since $\mathcal{W}$ is finite, they are finite in number. Since a facet meets only finitely many hyperplanes belonging to $\mathcal{H}$, the set of hyperplanes belonging to $\mathcal{H}$ and meeting $\overline{C}$ is finite, hence so is the set $A(C)$ of unit vectors in T orthogonal to some hyperplane belonging to $\mathcal{H}$ and meeting $\overline{C}$. Consequently, there exists a real number $\lambda < 1$ such that $(a, b) \leq \lambda$ for $a, b \in A(C)$ and $a \neq b$.

Let A be the set of unit vectors in T orthogonal to a hyperplane belonging to $\mathcal{H}$. Let $a, b \in A$ with $a \neq b$. If a and b are parallel, then $a = -a$ and $(a, b) = -1$. Otherwise, let $H \in \mathcal{H}$ (resp. $L \in \mathcal{H}$) be such that a (resp. b) is orthogonal to H (resp. L). We have $H \cap L \neq \varnothing$, and if $x \in H \cap L$ there exists an element $w \in W$ such that $x \in w(\overline{C})$. The vectors $U(w) \cdot a$ and $U(w) \cdot b$ then belong to $A(C)$, we have

$$(a, b) = (U(w) \cdot a, U(w) \cdot b) \leq \lambda$$

and the set A is finite by Lemma 4.3.17. Hence (b).

Now let $w \in W$ be such that $U(w) \cdot a = a$ for all $a \in A$. Then $U(w) \cdot t = t$ for all t belonging to the subspace of T generated by A. On the other hand, if $t \in T$ is orthogonal to A, we have $U(s_H) \cdot t = t$ for all $H \in \mathcal{H}$, hence $U(w) \cdot t = t$ and therefore $U(w) = 1$. Since $U(w)(A) = A$ for all $w \in W$, we deduce that $U(W)$ is isomorphic to a group of permutations of the finite set A, hence (c). □

Proposition 4.3.19. *Let C be a chamber and $\mathcal{R}$ a set of walls of C. Let $W_{\mathcal{R}}$ be the subgroup of W generated by the orthogonal reflections with respect to the elements of $\mathcal{R}$. For $H \in \mathcal{R}$, denote by e_H the unit vector orthogonal to H on the same side of H as C. Then the following conditions are equivalent:*

(i) *The group $W_{\mathcal{R}}$ is finite.*

(ii) *There exists a point of E invariant under every element of $W_{\mathcal{R}}$.*

(iii) *The hyperplanes belonging to $\mathcal{R}$ have non-empty intersection.*

(iv) *The family of vectors $(e_H)_{H \in \mathcal{R}}$ is free in T.*

Proof. If $W_{\mathcal{R}}$ is finite, then the element $\sum_{w \in W_{\mathcal{R}}} w(x)$ is invariant under $W_{\mathcal{R}}$, so $W_{\mathcal{R}}$ has a fixed point. Converselt, by property (D2) at the beginning of this paragraph, the stabiliser in W of every point of E is finite, so (ii) implies (i).

Since the group $W_{\mathcal{R}}$ is generated by the set of reflections with respect to the hyperplanes belonging to $\mathcal{R}$, the fixed points of $W_{\mathcal{R}}$ are the points of E belonging to every hyperplane $H \in \mathcal{R}$, hence the equivalence of (ii) and (iii).

Assume that there exists a point a of E such that $a \in H$ for all $H \in \mathcal{R}$ and let $t \in T$ be such that $a + t \in C$. Since $(e_{H_1}, e_{H_2}) \leq 0$ for $H_1, H_2 \in \mathcal{R}$ such that $H_1 \neq H_3$ (Proposition 4.3.12), and since $(t, e_H) > 0$ for all $H \in \mathcal{R}$, Lemma 4.3.13(b) implies that the e_H for $H \in \mathcal{R}$ are linearly independent. Consequently, (iii) implies (iv).

Suppose finally that the family $(e_H)_{H \in \mathcal{R}}$ is free. Let a be a point of E. For any hyperplane $H \in \mathcal{R}$, there exists a real number c_H such that H consists of the points $a + t$ of E with $(t, e_H) = c_H$. Since the family (e_H) is free, there exists $t \in T$ such that $(t, e_H) = c_H$ for all $H \in \mathcal{R}$, and the point $a + t$ of E belongs to all the hyperplanes $H \in \mathcal{R}$. Thus, (iv) implies (iii). □

Remark 4.3.20. Since W is generated by reflections with respect to the walls of the chamber C, the preceding proposition gives a criterion for W to be finite. We shall return to this question later.

Example 4.3.21. Let F be a finite-dimensional real affine space and G a group of automorphisms of F. For all $g \in G$, denote by $U(g)$ the automorphism of the space of translations V of F associated to g. Assume that the image $U(G)$ is a finite subgroup of $\mathrm{GL}(V)$; then V has a scalar product invariant under $U(G)$. If, in addition, G acts properly on F when it is provided with the discrete topology, and if it is generated by reflections, we can apply to G the results of this paragraph.

4.3.6 Representation of the Weyl group on the underlying space

Let I be the set of vertices of the Coxeter graph of W and let J be a subset of I such that no vertex in J is linked to any vertex in $I \setminus J$. Ley C be a chamber, s the canonical bijection from I to the set of reflections with respect to the walls of C, and let $W_{J,C}$ be the subgroup generated by the image $s(J)$. It follows from Proposition 3.1.12 that W is the direct product of the two subgroups $W_{J,C}$ and $W_{I \setminus J,C}$. Let $\tilde{C}$ be another chamber and $\tilde{s}$ the corresponding injection of I into W. We have seen that if $w \in W$ transforms C into $\tilde{C}$, then $\tilde{s}(i) = ws(i)w^{-1}$ for $i \in I$. Since $W_{J,C}$ is normal in W, it follows that $\tilde{s}(i) \in W_{J,C}$ for all $i \in J$. We deduce that the subgroup $W_{J,C}$ does not depend on C. We denote it simply by W_J from now on.

Remark 4.3.22. The definition of $W_{J,C}$ clearly extends to an arbitrary subset J of I. But if there exist a vertex in J and a vertex in $I \setminus J$ that are linked, then $W_{J,C}$ is not normal and depends on the choice of C.

Let T_J be the subspace of T consisting of the vectors invariant under every element of $U(W_J)$ and let $T_J^\perp$ be the subspace orthogonal to T_J. Since W_J is a normal subgroup of W, it is clear that T_J is invariant under $U(W)$, and hence so is $T_J^\perp$.

Proposition 4.3.23. *Let $J_1, \ldots, J_r$ be the sets of vertices of the connected components of the Coxeter graph of W. For $1 \leq p \leq r$, put*

$$W_p = W_{J_p}, \quad T_p = T_{J_p}, \quad T_p^\perp = T_{J_p}^\perp, \quad \text{and} \quad T_0 = \bigcap_{p=1}^{r} T_p.$$

(a) The group W is the direct product of the groups W_p.

(b) The space T is the orthogonal direct sum of the subspaces $T_0, T_1^\perp, \ldots, T_r^\perp$, which are all stable under $U(W)$.

(c) For all q, the subspace T_q of T consists of the vectors invariant under $U(W_q)$; it is the direct sum of the $T_p^\perp$ for $p \neq q$.

(d) Let C be a chamber. The subspace $T_p^\perp$ is generated by the vectors $e_i(C)$ for $i \in J_p$.

(e) The representations of W in the subspaces $T_p^\perp$ are absolutely irreducible, non-trivial, and pairwise inequivalent.

Proof. Assertion (a) follows from Proposition 3.1.12. On the other hand, we have already seen that the subspaces $T_p^\perp$, are invariant under $U(W)$, and so is T_0. Let C be a chamber; since W_p is generated by the reflections $s_i(C)$ for $i \in J_p$, it is clear that T_p is the subspace orthogonal to the $e_i(C)$ (which we simply denote by e_i henceforth) for $i \in J_p$, hence (d). Moreover, if $i \in J_p$, $j \in J_q$ with $p \neq q$, then $m_{ij} = 2$ since $\{i, j\}$ is not an edge of the Coxeter graph of W, so $(e_i, e_j) = 0$. Assertion (b) is now immediate. And assertion (c) follows, since T_p is the orthogonal complement of $T_p^\perp$.

Finally, let V be a subspace of $T_p^\perp$ invariant under $U(W_p)$. For all $i \in J_p$, either $e_i \in V$ is orthogonal to V (Proposition 4.2.5). Let A (resp. B) be the set of $i \in J_p$ such that $e_i \in V$ (resp. e_i is orthogonal to V). Clearly $(e_i, e_j) = 0$ for $i \in A$ and $j \in B$, and since J_p is the vertices of a connected component of the Coxeter graph, it follows that either $A = \varnothing$ and $V = \{0\}$, or $A = J_p$ and $V = T_p^\perp$. Consequently, the representation of W_p on $T_p^\perp$ is irreducible, hence absolutely irreducible (Proposition 4.2.2). It is non-trivial by the very definition of $T_p^\perp$. Finally, the last assertion in (e) follows immediately from (c). □

If the subspace T_0 of vectors in T invariant under $U(W)$ reduces to $\{0\}$, then W is said to be **essential**; if the representation U of W on T is irreducible, then W is said to be **irreducible**.

Corollary 4.3.24. *Assume that $W \neq \{1\}$. Then W is irreducible if and only if it is essential and its Coxeter graph is connected.*

We retain the notation of Proposition 4.3.23. For $1 \leq p \leq r$, let E_p be the set of orbits of the group T_p in E, and let π_p be the canonical map from E to E_p. Moreover, let E_0 be the set of orbits of $T_0^\perp$. The action of T on E passes to the quotient; in particular, $T_p^\perp$ acts on E_p and it is immediate (for example by taking an origin in E) that E_p is an affine space admitting $T_p^\perp$ as its space of translations. Put $\tilde{E} = E_0 \times E_1 \times \cdots \times E_r$; this is an affine space having $T = T_0 \oplus T_1^\perp \oplus \cdots \oplus T_r^\perp$ as its space of translations. Let $\pi : E \to \tilde{E}$ be the product map of the π_p; since π commutes with the action of T, this is a bijection and even an isomorphism of affine spaces. In what follows, we identify E and $\tilde{E}$ by means of π; the map π_p is then identified with the canonical projection of $\tilde{E}$ onto E_p.

Since W leaves T_p stable, the action of W on E passes to the quotient and defines an action of W on E_p, and hence by restriction an action of W_p on E_p. On the other hand, let C be a chamber, let $i \in I$ and let p be the integer such that $i \in J_p$. For any $x \in E$, we have

$$s_i(C)(x) = x - \lambda e_i(C) \quad \text{with} \quad \lambda \in \mathbb{R}.$$

Since $e_i \in T_q$ for $q \neq p$, it follows that $\pi_q(w(x)) = \pi_q(x)$ for $x \in E$, $w \in W_p$ and $q \neq p$. Consequently, if $w = w_1 \cdot w_r$ with $w_p \in W_p$ for $1 \leq p \leq r$, then

$$w(x_0, \ldots, x_r) = (x_0, w_1(x_1), \ldots, x_r(x_r)) \tag{4.3.6}$$

for all $x_p \in E_p$ and $0 \leq p \leq r$. In other words, the action of W on E is exactly the product of the actions of the W_p on E (we put $W_0 = \{1\}$). It follows that W_p acts faithfully on E_p and that W_p can be identified with a group of displacements of the Euclidean space E_p (the space $T_p^\perp$ of translations of E_p being provided, of course, with the scalar product induced by that on T).

Proposition 4.3.25. *Assume the notations in Proposition 4.3.23.*

(a) The group W_p is a group of displacements of the Euclidean affine space E_p; it is generated by reflections; provided with the discrete topology, it acts properly on E_p; it is irreducible.

(b) Let $\mathcal{H}_p$ be the set of hyperplanes H of E_p such that $s_H \in W_p$. The set $\mathcal{H}$ consists of the hyperplanes of the form

$$H = E_0 \times E_1 \times \cdots \times E_{p-1} \times H_p \times E_{p+1} \times \cdots E_r$$

with $1 \leq p \leq r$ and $H_p \in \mathcal{H}_p$.

(c) Every chamber C is of the form $E_0 \times C_1 \times \cdots \times C_p$, where for each p the set C_p is a chamber defined in E_p by the set of hyperplanes $\mathcal{H}_p$; moreover, the walls of C_p are the hyperplanes $\pi_p(H_i(C))$ for $i \in J_p$.

Proof. Let C be a chamber. Put $H_i = H_i(C)$, $e_i = e_i(C)$ and $s_i = s_i(C)$ for $i \in I$. Let i be in J_p. Since $e_i \in T_p^\perp$ and T is the direct sum of the mutually orthogonal subspaces $T_0, T_1^\perp, \dots, T_r^\perp$, the hyperplane of T orthogonal to e_i is of the form $L_i + T_p$, where L_i is the hyperplane of $T_p^\perp$ orthogonal to e_i. The affine hyperplane H_i of E is of the form $L_i + T_p + x$, with $x \in E$, and we have

$$H_i = E_0 \times E_1 \times \cdots \times E_{p-1} \times \tilde{H}_i \times E_{p+1} \times \cdots \times E_r \tag{4.3.7}$$

with $\tilde{H}_i = L_i + \pi_p(x) = \pi_p(H_i)$. It is now immediate that s_i acts in E_p by the reflection associated to the hyperplane $\tilde{H}_i$ of E_p. Thus, the group W_p is a group of displacements generated by reflections in E_p; the verification of the properness criterion (D2′) is immediate. Finally, Proposition 4.3.23(e) shows that W_p is irreducible. This proves (a).

By Corollary 4.3.4, the set $\mathcal{H}_p$ consists of the hyperplanes of the form $w_p(\tilde{H}_i)$ for $i \in J_p$ and $w \in W_p$. Further, if $w = w_1 \cdots w_r$ with $w_p \in W_p$ for all p, formulas (4.3.6) and (4.3.7) imply that

$$w(H_i) = E_0 \times E_1 \times \cdots \times E_{p-1} \times w_p(\tilde{H}_i) \times E_{p+1} \times \cdots \times E_r \tag{4.3.8}$$

from which (b) follows immediately.

Let $i \in J_p$. By formula (4.3.8), the open half-space D_i bounded by H_i and containing C is of the form

$$D_i = E_0 \times E_1 \times \cdots \times E_{p-1} \times \tilde{D}_i \times E_{p+1} \times \cdots \times E_r$$

Where $\tilde{D}_i$ is an open half-space bounded by $\tilde{H}_i$ in E_p. Put $C_p = \bigcap_{i \in J_p} \tilde{D}_i$; since $C = \bigcap_{i \in I} D_i$, it follows immediately that

$$C = E_0 \times C_1 \times \cdots \times C_r$$

consequently, none of the sets C_p is empty, and since C does not meet any hyperplane belonging to $\mathcal{H}$, the set C_p does not meet any hyperplane belonging to $\mathcal{H}_p$. Proposition 4.1.6 now shows that C_p is one of the chambers defined by $\mathcal{H}_p$ in E_p. By using Proposition 4.1.4, it is easy to see that the walls of C_p are the hyperplanes $\tilde{H}_i = \pi_p(H_i)$ for $i \in J_p$. □

4.3.7 Structure of chambers

Let C be a chamber, let $\mathcal{W}$ be the set of walls of C, and for $H \in \mathcal{W}$ let e_H be the unit vector orthogonal to H on the same side of H as C.

Proposition 4.3.26. *Assume that the group W is essential and finite. Then:*

(a) There exists a unique point a of E invariant under W.

(b) The family $(e_H)_{H \in \mathcal{W}}$ is a basis of T.

(c) The chamber C is the open simplicial cone with vertex a defined by the dual basis of $(e_H)_{H \in \mathcal{W}}$ in T.

Proof. By Proposition 4.3.19, there exists a point $a \in E$ invariant under W. Let $t \in T$ be such that $t + a$ is invariant under W. For all $w \in W$,

$$U(w) \cdot t + a = w(t + a) = t + a$$

so $U(w) \cdot t = t$. Since W is essential, this implies that $t = 0$, showing the uniqueness of a.

Since W is essential, $T_0 = \{0\}$ in the notation of Proposition 4.3.23, and Proposition 4.3.23 shows that the family $(e_H)_{H \in \mathcal{W}}$ generates the vector space T. The existence of a point of E invariant under W shows that the family $(e_H)_{H \in \mathcal{W}}$ is free (Proposition 4.3.19).

Let a be the unique point of E invariant under W. Since $(e_H)_{H \in \mathcal{W}}$ is a basis of T, and since the scalar product is a non-degenerate bilinear form on T, there exists a unique basis $(f_H)_{H \in \mathcal{W}}$ of T such that $(e_H, f_L) = \delta_{H,L}$ for H, L in $\mathcal{W}$. Every point x of E can be written uniquely in the form $x = t + a$ with $t = \sum_H \xi_H f_H$. Then x belongs to C if and only if, for every hyperplane $H \in \mathcal{W}$, x is on the same side of H as e_H, or in other words $(t, e_H) = \xi_H$ is strictly positive. Hence (c). □

Proposition 4.3.27. *Assume that the group W is essential, irreducible and infinite. Then:*

(a) No point of E is invariant under W.

(b) We have $|\mathcal{W}| = \dim(T) + 1$, and there exist real numbers $c_H > 0$ such that $\sum_H c_H e_H = 0$. Moreover, any family $(\tilde{c}_H)_{H \in \mathcal{W}}$ of real numbers such that $\sum_H \tilde{c}_H e_H = 0$ is proportional to $(c_H)_{H \in \mathcal{W}}$.

(c) The chamber C is an open simplex.

Proof. Assertion (a) follows from Proposition 4.3.19. On the other hand, since W is essential, the vectors $(e_H)_{H \in \mathcal{W}}$ generate T. We have $(e_H, e_{\tilde{H}}) \leq 0$ for $H, \tilde{H} \in \mathcal{H}$ and $H \neq \tilde{H}$ (Proposition 4.3.12) and, since W is irreducible, there does not exist a partition of $\mathcal{W}$ into two disjoint subsets $\mathcal{W}_1$ and $\mathcal{W}_2$ such that $H_1 \in \mathcal{W}_1$ and $H_2 \in \mathcal{W}_2$ imply that $e_{H_1}, e_{H_2}) = 0$. We can thus apply Proposition 4.3.15 and case (a) of that lemma is excluded; indeed, the e_H are not linearly independent, since W has no fixed point. Assertion (b) follows.

We now prove (c). Number the walls of C as $H_0, H_1, \ldots, H_l$ and put $t_i = e_{H_i}$. By (b), the vectors $t_1, \ldots, t_l$ form a basis of T, so the hyperplanes $H_1, \ldots, H_l$ have a point a_0 in common, and there exists a basis $(t_1^*, \ldots, t_l^*)$ of T such that $(t_i, t_j^*) = \delta_{ij}$. Moreover, again by (b) there exist positive real numbers $c_1, \ldots, c_l$ such that

$$t_0 = -(c_1 t_1 + \cdots + c_l t_l).$$

Since the vector t_0 is orthogonal to the hyperplane H_0, there exists a real number c such that H_0 is the set of points $x = t + a_0$ of E with $(t, t_0) = -c$.

Every point of E can be written uniquely in the form $x = t + a_0$ with $t = \xi_1 t_1^* + \cdots + \xi_l t_l^*$ and $\xi_1, \ldots, \xi_l$ real. The point x belongs to C if and only if it is on the same side of H_i as t_i for $0 \leq i \leq l$; this translates into the inequalities $(t, t_i) > 0$ for $1 \leq i \leq l$ and $(t, t_0) > -c$, or equivalently by $\xi_i > 0$ for $1 \leq i \leq l$ and $c_1\xi_1 + \cdots + c_l\xi_l < c$. Since C is non-empty, we therefore have $c > 0$. Put $a_i = a_0 + \frac{c}{c_i} t_i^*$ for $1 \leq i \leq l$. Then the chamber C consists of the l points of E of the form $a_0 + \sum_{i=1}^{l} \lambda_i (a - a_0)$ with $\lambda_i > 0$ and $\lambda_1 + \cdots + \lambda_l < 1$, so C is the open simplex with vertices $a_0, \ldots, a_l$. □

Remark 4.3.28. Identify E with $E_0 \times E_1 \times \cdots E_r$ and W with $W_1 \times \cdots \times W_r$. By Proposition 4.3.25, the chamber C is then identified with $E_0 \times C_1 \times \cdots \times C_r$ where C_p is a chamber in E_p with respect to the set of hyperplanes $\mathcal{H}_p$. By Proposition 4.3.26 and Proposition 4.3.27, each of the chambers C_p is an open simplicial cone or an open simplex.

Remark 4.3.29. Assume that W is irreducible and essential. If H_1 and H_2 are two walls of C, then m_{H_1,H_2} if and only if $e_{H_1} = -e_{H_2}$ (Proposition 4.3.12). By Proposition 4.3.26 and Proposition 4.3.27, this can happen only if H_1 and H_2 are the only walls of C and E is of dimension 1. Thus, the only case in which one of the m_{H_1,H_2} is infinite is that in which E is of dimension 1 and the group W is generated by the reflections associated to two distinct points.

In the general case, the entries of the Coxeter matrix associated to W are finite unless at least one of $E_1, \ldots, E_r$ is of the preceding type.

4.3.8 Special points

Let W_T be the set of translations belonging to W and let Λ be the set of $t \in T$ such that the translation $x \mapsto x + t$ belongs to W_T. It is immediate that Λ is stable under $U(W)$ and that W_T is a normal subgroup of W. Since W acts properly on E, the same holds for W_T, and it follows easily that Λ is a discrete subgroup of T. For any point x of E, denote by W_x the stabiliser of x in W.

Proposition 4.3.30. *Let $x \in E$. The following conditions are equivalent:*

(i) *We have $W = W_x \cdot W_T$.*

(ii) *The restriction of the homomorphism U to W_x is an isomorphism from W_x to $U(W)$.*

(iii) *For every hyperplane $H \in \mathcal{H}$, there exists a hyperplane $\tilde{H} \in \mathcal{H}$ parallel to H and such that $x \in \mathcal{H}$.*

Proof. It is clear that (i)⇔(ii), since W_T is the kernel of U and $W_T \cap W_x = \{1\}$. Assume (i) and let $H \in \mathcal{H}$; then $s_H \in W_x \cdot W_T$ so there exists a vector $t \in \Lambda$ such that $x = s_H(x) + t$. The vector t is therefore orthogonal to H, and if $\tilde{H} = H + \frac{1}{2}t$ then $s_{\tilde{H}}(x) = s_H(x) + t$ for all $x \in E$. Since $t \in \Lambda$ and $s_H \in W$, we have $s_{\tilde{H}} \in W$, and so $\tilde{H} \in \mathcal{H}$; we also have $x = s_{\tilde{H}}(x)$, and so $x \in \tilde{H}$. Thus, (i) implies (iii).

Assume (iii). Let $H \in \mathcal{H}$, take $\tilde{H}$ as in (iii). Then $s_{\tilde{H}}(x) = x$, so $s_{\tilde{H}} \in W_x$. Since H is parallel to $\tilde{H}$, the element $w = s_{\tilde{H}} s_H$ of W is a translation, so $w \in W_T$; then $s_H = s_{\tilde{H}} w \in W_x \cdot W_T$. Since W is generated by the family $(s_H)_{H \in \mathcal{H}}$, it follows that $W = W_x \cdot W_T$ and hence (iii) implies (i). □

A point x of E is called **special** for W if it satisfies the equivalent conditions in Proposition 4.3.30. It is clear that the set of special points of E is stable under W.

Proposition 4.3.31. *There exists a special point for W.*

Proof. By Proposition 4.3.23, we need only consider the case when W is essential. The group $U(W)$ of automorphisms of T is finite (Theorem 4.3.18) and $U(s_H)$ is an orthogonal reflection for every hyperplane H; moreover, $U(W)$ is generated by the family $(U(s_H))_{H \in \mathcal{H}}$. By Proposition 4.3.26, there exists a basis $(e_i)_{i \in I}$ of T such that the group $U(W)$ is generated by the set of reflections $(s_i)_{i \in I}$ defined by

$$s_i(t) = t - 2(t, e_i)e_i$$

The Corollary 4.3.4 shows that every reflection $s \in U(W)$ is of the form $s = U(s_H)$ with $H \in \mathcal{H}$. We can thus find in $\mathcal{H}$ a family of hyperplanes $(H_i)_{i \in I}$ such that $s_i = U(s_{H_i})$ for all i. Since the vectors e_i are linearly independent, there exists $x \in E$ such that $x \in H_i$ for all $i \in I$ (Proposition 4.3.19). We have $s_{H_i} \in W_x$, so $U(W) = U(W_x)$, which means that $W = W_x \cdot W_T$ since W_T is the kernel of U. Thus, x is a special point. □

Remark 4.3.32. When W is finite and essential, there is only one special point for W, namely the unique point invariant under W. Thus, the consideration of special points is interesting mainly when W is infinite.

Proposition 4.3.33. *Assume that W is essential. Let x be a special point for W. Then the chambers relative to W_x are the open simplicial cones with vertex x. For every chamber C_x relative to W_x, there exists a unique chamber C relative to W contained in C_x and such that $x \in \overline{C}$. The union of the $\tilde{w}(\overline{C})$ for $\tilde{w} \in W_x$ is a closed neighbourhood of x in E. Every wall of C_x is a wall of C. If W is infinite and irreducible, the walls of C are the walls of C_x together with an affine hyperplane not parallel to the walls of C_x.*

Proof. Let $\mathcal{H}_x$ be the set of $H \in \mathcal{H}$ containing x. The group W_x is generated by the H for $H \in \mathcal{H}_x$ Proposition 4.3.8). The chambers relative to W_x are the open simplicial cones with vertex x (Proposition 4.3.26). Let C_x be such a chamber and let U be a non-empty open ball with centre x not meeting any element of $\mathcal{H} \setminus \mathcal{H}_x$. Since $x \in \overline{C}_x$, there exists a point y in $U \cap C_x$. Then $y \notin H$ for all $H \in \mathcal{H}$, so y belongs to a chamber C relative to $\mathcal{H}$. Since $\mathcal{H}_x \subseteq \mathcal{H}$, we have $C \subseteq C_x$. The set $U \cap C_x$ does not meet any $H \in \mathcal{H}$ and is convex, so $U \cap C_x \subseteq C$; thus $x \in C$. Conversely, let C_1 be a chamber relative to W contained in C_x and such that $x \in \overline{C}_1$; then C_1 meets U and $U \cap C_1 \subseteq U \cap C_x = U \cap C$; the chambers C and C_1, having a point in common, must coincide. For any $\tilde{w} \in W_x$, we have $\tilde{w}(U) = U$, so

$$U \cap \tilde{w}(C) = \tilde{w}(U \cap C) = \tilde{w}(U \cap C_x) = U \cap \tilde{w}(C_x).$$

Since the union of the $\tilde{w}(C_x)$ for $\tilde{w} \in W_x$ is dense in E, the union of the $U \cap \tilde{w}(C) = U \cap \tilde{w}(C_x)$ is dense in U, and the union of the $\tilde{w}(\overline{C})$ for $\tilde{w} \in W_x$ thus contains U. Finally, if H is a wall of C_x, there exist a point $z \in U \cap H$ and an open neighbourhood $V \subseteq U$ of z such that $V \cap C_x$ is the intersection of V and the open half-space bounded by H containing C_x; since $V \cap C_x = V \cap U \cap C_x = V \cap U \cap C = V \cap C$, it follows that H is awall of C. If W is infinite and irreducible, C is an open simplex by Proposition 4.3.27 and so has one more wall than the open simplicial cone C_x. □

Corollary 4.3.34. *Assume that W is essential.*

(a) If $x \in E$ is special, there exists a chamber C such that x is an extremal point of C.

(b) If C is a chamber, there exists an emtremal point of C that is special.

Proof. The first assertion follows from Proposition 4.3.33. The second follows from the first and the fact that W acts transitively on the set of chambers. □

Remark 4.3.35. Assume that W is essential, irreducible and infinite and retain the notation of Proposition 4.3.33. Since U is an isomorphism from W_x to $U(W)$, it follows that the Coxeter graph of the group of displacements $U(W)$ (which is generated by the reflections $U(s_H)$ for $H \in \mathcal{H}$ can be obtained from the Coxeter graph of W by omitting the vertex i corresponding to the unique wall of C that is not a wall of C_x.

Proposition 4.3.36. *Assume that W is essential. Let x be a special point, Ω_x be the set of its transforms under the group of translations W_T, and let C be a chamber. Then C meets Ω_x in a unique point, which is an extremal point of C.*

Proof. There exists a chamber C_0 such that x is an extremal point of C_0 (Corollary 4.3.34). Every chamber is of the form $C = t\tilde{w}(C_0)$ with $\tilde{w} \in W_x$ and $t \in W_T$ since $W = W_x \cdot W_T$. Thus $t\tilde{w}(x) = t(x) \in \Omega_x$ is an extremal point of C. On the other hand, C cannot contain two distinct points of Ω_x since C is a fundamental domain for W. □

Remark 4.3.37. The set Ω_x is contained in the set of special points, but in general is distinct from it.

4.4 Geometric representation of a Coxeter matrix

In this section, all vector spaces will be assumed to be real.

4.4.1 Associated form and reflections of a Coxeter matrix

Let S be a set and let $M = (m(s,\tilde{s}))$ be a Coxeter matrix of type S. Let E be the real vector space spanned by S, $(e_s)_{s\in S}$ be the canonical basis of E, and let β_M be the bilinear form on E such that

$$\beta_M(e_s, e_{\tilde{s}}) = -\cos\frac{\pi}{m(s,\tilde{s})}.$$

The form β_M is symmetric. It is called the associated form of the matrix M. We have $\beta_M(s,s) = 1$ and $\beta_M(s,\tilde{s}) \leq 0$ for $s, \tilde{s} \in S$ and $s \neq \tilde{s}$. Let $s \in S$ and let f_s be the linear form $x \mapsto 2\beta_M(e_s, x)$. We denote by σ_s the pseudo-reflection defined by the pair (e_s, f_s); since $(e_s, f_s) = 2$, it is a reflection. We have

$$\sigma_s(x) = x - 2\beta_M(e_s, x)e_s$$

and in particular

$$\sigma_s(e_{\tilde{s}}) = e_{\tilde{s}} - 2\cos\frac{\pi}{m(s,\tilde{s})}e_s.$$

Since e_s is not isotropic for β_M, the space E is the direct sum of the line $\mathbb{R}e_s$ and the hyperplane H_s orthogonal to e_s. Since σ_s is equal to -1 on $\mathbb{R}e_s$ and to 1 on H_s, it follows that σ_s preserves the form β_M. When S is finite and β_M is non-degenerate (a case to which we shall examine), it follows that σ_s is an orthogonal reflection.

Now let $s, \tilde{s} \in S$ with $s \neq \tilde{s}$, and denote by $E_{s,\tilde{s}}$ the plane $\mathbb{R}e_s \oplus \mathbb{R}e_{\tilde{s}}$.

Proposition 4.4.1. *The restriction of β_M to $E_{s,\tilde{s}}$ is positive, and it is nondegenerate if and only if $m(s,\tilde{s})$ is finite.*

Proof. Let $z = xe_s + ye_{\tilde{s}}$ with $x, y \in \mathbb{R}$ be an element of $E_{s,\tilde{s}}$. We have

$$\beta_M(z,z) = x^2 - 2xy\cos\frac{\pi}{m(s,\tilde{s})} + y^2 = (x - \cos\frac{\pi}{m(s,\tilde{s})}y)^2 + \sin^2\frac{\pi}{m(s,\tilde{s})}y^2,$$

which shows that β_M is positive on $E_{s,\tilde{s}}$, and that it is non-degenerate there if and only if $\sin(\pi/m(s,\tilde{s})) \neq 0$. The proposition follows. □

The reflections σ_s and $\sigma_{\tilde{s}}$ leave $E_{s,\tilde{s}}$ stable. We are going to determine the order of $\sigma_s\sigma_{\tilde{s}}$.

Proposition 4.4.2. *The subgroup of* $\mathrm{GL}(E)$ *generated by* σ_s *and* $\sigma_{\tilde{s}}$ *is a dihedral group of order* $2m(s,\tilde{s})$.

Proof. First assume that $m(s,\tilde{s}) = \infty$. Let $u = e_s + e_{\tilde{s}}$. We have $\beta_M(u, e_s) = \beta_M(u, e_{\tilde{s}}) = 0$, so u is invariant under σ_s and $\sigma_{\tilde{s}}$. Moreover,

$$\sigma_s(\sigma_{\tilde{s}})(e_s) = \sigma_s(e_s + 2e_{\tilde{s}}) = 3e_s + 2e_{\tilde{s}} = 2u + e_s$$

hence $(\sigma_s\sigma_{\tilde{s}})^n(e_s) = 2nu + e_s$ for all $n \in \mathbb{Z}$. It follows that the restriction of $\sigma_s\sigma_{\tilde{s}}$ to $E_{s,\tilde{s}}$ is of infinite order.

Now consider the case $m(s,\tilde{s})$ is finite. The form β_M provides $E_{s,\tilde{s}}$ with the structure of a Euclidean plane. Since the scalar product of e_s and $e_{\tilde{s}}$ is equal to $-\cos(\pi/m(s,\tilde{s})) = \cos(\pi - \pi/m(s,\tilde{s}))$, we can orient $E_{s,\tilde{s}}$ so that the angle between the half-lines $\mathbb{R}e_s$ and $\mathbb{R}e_{\tilde{s}}$ is equal to $\pi - \pi/m(s,\tilde{s})$. If D and $\tilde{D}$ denote the lines orthogonal to e_s and $e_{\tilde{s}}$,

$$\widehat{(\tilde{D}, D)} = \pi - \widehat{(D, \tilde{D})} = \pi/m(s,\tilde{s}).$$

Now, the restrictions $\bar{\sigma}_s$ and $\bar{\sigma}_{\tilde{s}}$ of σ_s and $\sigma_{\tilde{s}}$ to $E_{s,\tilde{s}}$ are the orthogonal symmetries with respect to D and $\tilde{D}$. By the Corollary 4.2.14, it follows that $\bar{\sigma}_s\bar{\sigma}_{\tilde{s}}$ is the rotation with angle $2\pi/m(s,\tilde{s})$. In particular, its order is $m(s,\tilde{s})$.

Finally we return to the space E. Since σ_s and $\sigma_{\tilde{s}}$ are of order 2, and are distinct, it is enough to show that their product $\sigma_s\sigma_{\tilde{s}}$ is of order $m(s,\tilde{s})$. When $m(s,\tilde{s})$ is infinite, this follows the argument above. When $m(s,\tilde{s})$ is finite, it follows from Proposition 4.4.1 that E is the direct sum of $E_{s,\tilde{s}}$ and its orthogonal complement $E_{s,\tilde{s}}^{\perp}$. Since σ_s and $\sigma_{\tilde{s}}$ act as the identity on $E_{s,\tilde{s}}^{\perp}$, and since the restriction of $\sigma_s\sigma_{\tilde{s}}$ to $E_{s,\tilde{s}}$ is of order $m(s,\tilde{s})$, the order of $\sigma_s\sigma_{\tilde{s}}$ is indeed equal to $m(s,\tilde{s})$. □

Mow let $W = W(M)$ be the group defined by the family of generators $(g_s)_{s\in S}$ and the relations

$$(g_s g_{\tilde{s}})^{m(s,\tilde{s})} = 1 \quad \text{for} \quad s, \tilde{s} \in S, m(s,\tilde{s}) \neq +\infty.$$

Proposition 4.4.3. *There exists a unique homomorphism* $\sigma : W \to \mathrm{GL}(E)$ *such that* $\sigma(g_s) = \sigma_s$ *for all* $s \in S$. *The elements of* $\sigma(W)$ *preserve the bilinear form* β_M.

Proof. To prove the existence and uniqueness of σ, it is enough to show that $(\sigma_s\sigma_{\tilde{s}})^{m(s,\tilde{s})} = 1$ if $m(s,\tilde{s}) \neq +\infty$. Now, if $s = \tilde{s}$, this follows from the fact that σ_s is of order 2; if $s \neq \tilde{s}$, it follows from what we proved in Proposition 4.4.2. Finally, since the reflections σ_s preserve β_M, so do the elements of $\sigma(W)$. □

Proposition 4.4.4. *Let* $\sigma : W \to \mathrm{GL}(E)$ *be the homomorphism in Proposition 4.4.3.*

(a) The map $s \mapsto g_s$ *from* S *to* W *is injective.*

(b) Each of the g_s *is of order* 2.

(c) If $s, \tilde{s} \in S$, *then* $g_s g_{\tilde{s}}$ *is of order* $m(s,\tilde{s})$.

In particular, S *can be identified with a subset of* W.

Proof. Assertion (a) follows from the fact that the composite map $s \mapsto g_s \mapsto \sigma_s$ from S to $\mathrm{GL}(E)$ is injective. For (b) (resp. (c)), we remark that the order of g_s (resp. the order of $g_s g_{\tilde{s}}$) is at most 2 (resp. at most $m(s,\tilde{s})$). Since we have seen in Proposition 4.4.2 that the order of σ_s (resp. of $\sigma_s\sigma_{\tilde{s}}$) is 2 (resp. $m(s,\tilde{s})$), we must have equality. □

Corollary 4.4.5. *The pair* (W, S) *is a Coxeter system with matrix* M. *In particular, every Coxeter matrix corresponds to a Coxeter group.*

Proof. This is simply the content of properties (b) and (c) in Proposition 4.4.4, together with the definition of W. □

4.4.2 Contragradient representation

Let E^* be the dual of E. Since W acts on E via σ, it also acts, by transport of structure, on E^*. The corresponding representation

$$\sigma^* : W \to \mathrm{GL}(E^*)$$

is called the **contragredient representation** of σ. We have

$$\sigma^*(w) = (\sigma(w)^{-1})^t \quad \text{for} \quad w \in W.$$

If $x^* \in E^*$ and $w \in W$, we denote by $w(x^*)$ the action of w on x^*.

For $s \in S$, we denote by A_s the set of $x^* \in E^*$ such that $x^*(e_s) > 0$. Let C be the intersection of the A_s for $s \in S$. When S is finite, C is an open simplicial cone in E^*. Also, if $s, \tilde{s} \in S$, let $W_{s,\tilde{s}}$ be the subgroup of W generated by s and $\tilde{s}$.

Lemma 4.4.6. *Let $s, \tilde{s} \in S$ with $s \neq \tilde{s}$, and let $w \in W_{s,\tilde{s}}$. Then $w(A_s \cap A_{\tilde{s}})$ is contained in either A_s or in $s(A_s)$, and in the second case $\ell(sw) = \ell(w) - 1$.*

Proof. Let $E^*_{s,\tilde{s}}$ be the dual of the plane $E_{s,\tilde{s}} = \mathbb{R}e_s \oplus \mathbb{R}e_{\tilde{s}}$. The transpose of the injection $E_{\tilde{s},\tilde{s}} \to E$ is a surjection $\pi : E^* \to E^*_{s,\tilde{s}}$ that commutes with the action of the group $W_{s,\tilde{s}}$. It is clear that A_s, $A_{\tilde{s}}$, and $A_s \cap A_{\tilde{s}}$ are the inverse images under π of corresponding subsets of $E^*_{s,\tilde{s}}$ (considered as the space of the contragredient representation of the Coxeter group $W_{s,\tilde{s}}$). Moreover, since the length of an element of $W_{s,\tilde{s}}$ is the same with respect to $\{s, \tilde{s}\}$ and with respect to S (Corollary 3.1.33), we are reduced finally to the case where $S = \{s, \tilde{s}\}$. If $m = m(s, \tilde{s})$, the group W is then a dihedral group of order $2m$.

We first deal with the case $m = +\infty$. In this case $\beta_M(e_s, e_{\tilde{s}}) = -1$, so

$$\begin{aligned} s(e_s) &= -e_s & \tilde{s}(e_s) &= e_s + 2e_{\tilde{s}}, \\ s(e_{\tilde{s}}) &= e_{\tilde{s}} + 2e_s & \tilde{s}(e_{\tilde{s}}) &= -e_{\tilde{s}}. \end{aligned}$$

Let $(\varepsilon, \tilde{\varepsilon})$ be the dual basis of $(e_s, e_{\tilde{s}})$. Then

$$\begin{aligned} s(\varepsilon) &= -\varepsilon + 2\tilde{\varepsilon} & \tilde{s}(\varepsilon) &= \varepsilon, \\ s(\tilde{\varepsilon}) &= \tilde{\varepsilon} & \tilde{s}(\tilde{\varepsilon}) &= -\tilde{\varepsilon} + 2\varepsilon. \end{aligned}$$

Let D be the affine line of E^* containing ε and $\tilde{\varepsilon}$. The formulas above show that D is stable under s and $\tilde{s}$ and that the restriction of s (resp. $\tilde{s}$) to D is the reflection with respect to the point $\tilde{\varepsilon}$ (resp. ε). Let $\theta : \mathbb{R} \to D$ be the affine bijection

$$t \mapsto \theta(t) = t\varepsilon + (1 - t)\tilde{\varepsilon}.$$

Let I_n be the image under θ of the open interval $(n, n+1)$, and let C_n be the union of the λI_n for $\lambda > 0$ (that is, the simplicial cone induced by I_n from the origin). Then $C_0 = C$. Moreover, by Example 4.2.11, on the affine space D, the I_n are permuted simply-transitively by W; hence so are the C_n. If $w \in W$, $w(C)$ is equal to one of the C_n hence is contained in A_s if $n \geq 0$ and in $s(A_s)$ if $n < 0$. In the second case, I_0 and I_n are on opposite sides of the point $\tilde{\varepsilon}$; hence $\ell(sw) = \ell(w) - 1$.

Now assume that m is finite. The form β_M is then non-degenerate so we can identify E^* with E. We have seen that E can be oriented so that the angle between the half-lines $\mathbb{R}_+e_s$ and $\mathbb{R}_+e_{\tilde{s}}$ is equal to $\pi - \pi/m$. Let D (resp. $\tilde{D}$) be the half-line obtained from $\mathbb{R}_+e_s$ (resp. $\mathbb{R}_+e_{\tilde{s}}$) by a rotation of $\pi/2$ (resp. $-\pi/2$), cf. Fig. **??**. The chamber C is the set of $x \in C$ whose scalar product with e_s and $e_{\tilde{s}}$ is positive. This is the open angular sector with origin $\tilde{D}$ and extremity D. By Example 4.2.16, every element w of W transforms C into an angular sector that is on the same side of D as C (i.e. is contained in A_s) or on the opposite side (i.e. contained in $s(A_s)$), and in the latter case $\ell(sw) = \ell(w) - 1$, which completes the proof of the lemma. □

Theorem 4.4.7 (Tits). *If $w \in W$ and $C \cap w(C) \neq \varnothing$, then $w = 1$. Therefore, the group W acts simply-transitively on the set of $w(C)$ for $w \in W$.*

Proof. We are going to prove the following assertions, where n denotes an non-negative integer:

(P_n) Let $w \in W$ with $\ell(w) = n$ and $s \in S$. Then either $w(C) \subseteq A_s$, or $w(C) \subseteq s(A_s)$ and $\ell(sw) = \ell(w) - 1$.

(Q_n) Let $w \in W$ with $\ell(w) = n$ and $s, \tilde{s} \in S$, $s \neq \tilde{s}$. Then there exists $u \in W_{s,\tilde{s}}$ such that

$$w(C) \subseteq u(A_s \cap A_{\tilde{s}}) \quad \text{and} \quad \ell(w) = \ell(u) + \ell(u^{-1}w).$$

These assertions are trivial for $n = 0$. We prove them by induction on n according to the scheme

$$((P_n) \wedge (Q_n)) \Rightarrow (P_{n+1}) \quad \text{and} \quad ((P_{n+1}) \wedge (Q_n)) \Rightarrow Q_{n+1}.$$

Let us first prove the implication $((P_n) \wedge (Q_n)) \Rightarrow (P_{n+1})$. Let $w \in W$ with $\ell(w) = n+1$ and $s \in S$. We can write w in the form $w = \tilde{s}\tilde{w}$ with $\tilde{s} \in S$ and $\ell(\tilde{w}) = n$. If $\tilde{s} = s$, (P_n) applied to $\tilde{s}$ shows that $\tilde{w}(C) \subseteq A_s$, hence $w(C) \subseteq s(A_s)$ and $\ell(sw) = \ell(\tilde{w}) = \ell(w) - 1$. If $\tilde{s} \neq s$, (Q_n) applied to $\tilde{w}$ shows that there exists $u \in W_{s,\tilde{s}}$ such that

$$\tilde{w}(C) \subseteq u(A_s \cap A_{\tilde{s}}) \quad \text{and} \quad \ell(\tilde{w}) = \ell(u) + \ell(u^{-1}\tilde{w}),$$

whence $w(C) = \tilde{s}\tilde{w}(C) \subseteq \tilde{s}u(A_s \cap A_{\tilde{s}})$. We now apply Lemma 4.4.6 to the element $v = \tilde{s}u$. There are two possibilities: either

$$\tilde{s}u(A_s \cap A_{\tilde{s}}) \subseteq A_s \quad \text{and a fortoiri} \quad w(C) \subseteq A_s,$$

or

$$\tilde{s}u(A_s \cap A_{\tilde{s}}) \subseteq s(A_s) \quad \text{and a fortoiri} \quad w(C) \subseteq s(A_s).$$

Moreover, in the second case $\ell(s\tilde{s}u) = \ell(\tilde{s}u) - 1$, hence

$$\begin{aligned}\ell(sw) = \ell(s\tilde{s}\tilde{w}) = \ell(s\tilde{s}uu^{-1}\tilde{w}) &\leq \ell(s\tilde{s}u) + \ell(u^{-1}\tilde{w}) = \ell(\tilde{s}u) - 1 + \ell(u^{-1}\tilde{w}) \\ &= \ell(\tilde{s}u) - 1 - \ell(u) + \ell(\tilde{w}) \leq \ell(\tilde{w}) = \ell(w) - 1,\end{aligned}$$

which implies $\ell(sw) = \ell(w) - 1$.

Now we turn to the proof of $((P_{n+1}) \wedge (Q_n)) \Rightarrow (Q_{n+1})$. Let $w \in W$ with $\ell(w) = n+1$ and $s, \tilde{s} \in S$, $s \neq \tilde{s}$. If $w(C)$ is contained in $A_s \cap A_{\tilde{s}}$, condition (Q_{n+1}) is satisfied with $u = 1$. For if not, suppose for example that $w(C)$ is not contained in A_s. By (P_{n+1}), $w(C) \subseteq s(A_s)$ and $\ell(sw) = n$. By (Q_n) applied to sw, there exists $v \in W_{s,\tilde{s}}$ such that

$$sw(C) \subseteq v(A_s \cap A_{\tilde{s}}) \quad \text{and} \quad \ell(sw) = \ell(v) + \ell(v^{-1}sw).$$

Then $w(C) \subseteq sv(A_s \cap A_{\tilde{s}})$ and

$$\ell(w) = 1 + \ell(sw) = 1 + \ell(v) + \ell(v^{-1}sw) \geq \ell(sv) + \ell((sv)^{-1}w) \geq \ell(w),$$

so the inequalities above must be equalities. It follows that (Q_{n+1}) is satisfied with $u = sv$.

Finally we are left to prove the theorem using (P_n) and (Q_n). Let $w \in W$ with $w \neq 1$. We can write w in the form $s\tilde{w}$ with $s \in S$ and $\ell(\tilde{w}) = \ell(w) - 1$. By (P_n) applied to $\tilde{w}$ and $n = \ell(\tilde{w})$, we have $\tilde{w}(C) \subseteq A_s$, since the case $\tilde{w}(C) \subseteq s(A_s)$ is excluded because $\ell(s\tilde{w}) = \ell(w) = \ell(\tilde{w}) + 1$. Therefore $w(C) = w\tilde{w}(C) \subseteq s(A_s)$, and since A_s and $s(A_s)$ are disjoint, we have $C \cap w(C) = \varnothing$. This completes the proof. □

Corollary 4.4.8. *The representations σ and σ^* are injective.*

Proof. Indeed, if $\sigma^*(w) = 1$, then $w(C) = C$, so $w = 1$ by Corollary 4.4.8. The injectivity of σ follows from that of σ^*. □

Corollary 4.4.9. *If S is finite, $\sigma(W)$ is a discrete subgroup of $\mathrm{GL}(E)$ provided with its canonical Lie group structure. Similarly, $\sigma^*(W)$ is a discrete subgroup of $\mathrm{GL}(E^*)$.*

Proof. Let $x^* \in C$. Since S is finite, the set U of $g \in \mathrm{GL}(E^*)$ such that $g(x^*) \in C$ is a neighbourhood of the identity element in $\mathrm{GL}(E^*)$ (it is an intersection of finitely many open sets). By Theorem 4.4.7, $\sigma^*(W) \cap U = \{1\}$, thus $\sigma^*(W)$ is a discrete subgroup of $\mathrm{GL}(E^*)$. By transport of structure, it follows that $\sigma(W)$ is discrete in $\mathrm{GL}(E)$. □

For $s \in S$, we define

$$H_s = \{x^* \in E^* : \langle x^*, e_s \rangle = 0\}, \quad \bar{A}_s = \{x^* \in E^* : \langle x^*, e_s \rangle \geq 0\}, \quad \bar{C} = \bigcap_{s \in S} \bar{A}_s.$$

For the weak topology $\sigma(E^*, E)$ defined by the duality between E^* and E, the $\bar{A}$ are closed half-spaces and $\bar{C}$ is a closed convex cone. Moreover, $\bar{C}$ is the weak* closure of C; indeed, if $x^* \in \bar{C}$ and $y^* \in C$, then $x^* + ty^* \in C$ for every real number $t > 0$ and $x^* = \lim_{t \to 0^+}(x^* + ty^*)$.

For $X \subseteq S$, put

$$C_X = \Big(\bigcap_{s\in X} H_s\Big) \cap \Big(\bigcap_{s\in S\setminus X} A_s\Big).$$

We have $C_X \subseteq C$, $C_\varnothing = C$ and $C_S = \{0\}$. The sets C_X, for $X \in \mathscr{P}(S)$, form a partition of $\bar{C}$. On the other hand, recall that W_X denotes the subgroup of W generated by X. Clearly $w(x^*) = x^*$ for $w \in W_X$ and $x^* \in C_X$.

Proposition 4.4.10. *Let $X, \tilde{X} \subseteq X$ and $w, \tilde{w} \in W$. If $w(C_X) \cap \tilde{w}(C_{\tilde{X}}) \neq \varnothing$, then $X = \tilde{X}$, $w = \tilde{w}$, and $w(C_X) = \tilde{w}(C_{\tilde{X}})$.*

Proof. We are reduced immediately to the case $\tilde{w} = 1$. The proof is by induction on the length n of w. If $n = 0$, the assertion is clear. If $\ell(w) > 0$, there exists $s \in S$ such that $\ell(sw) = \ell(w) - 1$ and then $w(C) \subseteq s(A_s)$, hence $w(\bar{C}) \subseteq s(\bar{A}_s)$. Since $\bar{C} \subseteq \bar{A}_s$, it follows that

$$\bar{C} \cap w(\bar{C}) \subseteq H_s.$$

Then $s(x^*) = x^*$ for all $x^* \in \bar{C} \cap w(\bar{C})$, and a fortiori for all $x^* \in C_{\tilde{X}} \cap w(C_X)$. Consequently, the relation $C_{\tilde{X}} \cap w(C_X) \neq \varnothing$ implies on the one hand that $C_{\tilde{X}} \cap H_s \neq \varnothing$, and hence that $s \in \tilde{X}$, and on the other hand that $C_{\tilde{X}} \cap sw(C_X) \neq \varnothing$. The induction hypothesis then implies that $X = \tilde{X}$ and $swW_X = W_{\tilde{X}} = W_X$, so $sw \in W_X$ and $w \in W_X$ since $s \in W_X$. It follows that $w \in W_X$ and that $w(C_X) = C_X = C_{\tilde{X}}$. ☐

Corollary 4.4.11. *Let X be a subset of S and x^* an element of C_X. The stabiliser of x^* in W is W_X.*

Now let U be the union of the $w(\bar{C})$ for $w \in W$, and let $\mathcal{F}$ be the set of subsets of U of the form $w(C_X)$, with $X \subseteq S$ and $w \in W$. By the above, $\mathcal{F}$ is a partition of U.

Proposition 4.4.12. *Let U and $\mathcal{F}$ be defined as above.*

(a) The cone U is convex.

(b) Every closed segment contained in U meets only finitely many elements of $\mathcal{F}$.

(c) The cone $\bar{C}$ is a fundamental domain for the action of W on U.

Proof. To prove (c), it is enough to show that, if $x^*, y^* \in C$ and $w \in W$ are such that $w(x^*) = y^*$, then $x^* = y^*$. Now there exist two subsets X and Y of S such that $x^* \in C_X$ and $y^* \in C_Y$; we have $w(C_X) \cap C_Y \neq \varnothing$, and Proposition 4.4.10 shows that $X = Y$ and $w \in W_X$, which implies that $x^* = y^*$.

Now let $x^*, y^* \in U$; we shall show that the segment $[x^*, y^*]$ is covered by *finitely many* elements of $\mathcal{F}$, which will establish both (a) and (b). By transforming x^* and y^* by the same element of W, we can assume that $x^* \in \bar{C}$. Let $w \in W$ be such that $y^* \in w(\bar{C})$. We argue by induction on the length of w. For $s \in S$, the relation $w(\bar{C}) \nsubseteq \bar{A}_s$ is equivalent to $w(C) \nsubseteq A_s$ and hence to $\ell(sw) < \ell(w)$. Proposition 3.1.29 now implies that there exist only finitely many $s \in S$ such that $w(\bar{C}) \nsubseteq \bar{A}_s$. The set T of $s \in S$ such that $\langle y^*, e_s\rangle < 0$ is thus finite.

On the other hand, the intersection $\bar{C} \cap [x^*, y^*]$ is a closed segment $[x^*, z^*]$. If $z^* = y^*$, that is, if $y^* \in \bar{C}$, then there exist subsets X and Y of S such that $x^* \in C_X$ and $y^* \in C_Y$. The open segment (x^*, y^*) is then contained in $W_{X\cap Y}$ (follows from the definition of $W_{X\cap Y}$), so $[x^*, y^*] \subseteq C_X \cup C_Y \cup C_{X\cap Y}$. If $z^* \neq y^*$, choose $s \in T$ such that $z^* \in H_s$. Then $w(C) \nsubseteq A_s$ and $\ell(sw) < \ell(w)$. Since $s(y^*) \in sw(\bar{C})$ and $\ell(sw) < \ell(w)$, the induction hypothesis thus implies that the segment $[z^*, y^*] = s([z^*, s(y^*)])$ is covered by a finite number of elements of $\mathcal{F}$, and hence so is $[x^*, y^*] = [x^*, z^*] \cup [z^*, y^*]$ since $[x^*, z^*] \subseteq \bar{C}$. ☐

4.4.3 Irreducibility and finiteness criterion

We retain the preceding notations, and assume that S is finite from now on, so that the space E has finite dimension.

Proposition 4.4.13. *Assume that (W, S) is irreducible. Let $E^\perp$ be the subspace of E orthogonal to E with respect to β_M. The group W acts trivially on $E^\perp$, and every proper subspace of E stable under W is contained in $E^\perp$.*

Proof. If $x \in E^\perp$, then $\sigma_s(x) = x$ for all $s \in S$. Since W is generated by S, it follows that W acts trivially on $E^\perp$.

Let H be a subspace of E stable under W. Let $s, \tilde{s} \in S$ be two elements that are linked in the graph Γ of (W, S) (recall that this means that $m(s, \tilde{s}) \geq 3$). Suppose that $e_s \in H$. Then $\sigma_{\tilde{s}}(e_s) = e_s - 2\beta_M(e_s, e_{\tilde{s}})e_{\tilde{s}} \in H$ and since $\beta_M(e_s, e_{\tilde{s}})$ is non-zero, we have $e_{\tilde{s}} \in H$. Since Γ is connected, it follows that, if H contains one of the e_s it contains them all and coincides with E. Except in this case, it follows from Proposition 4.2.5 that, for all $s \in S$, H is contained in the hyperplane H_s orthogonal to e_s. Since the intersection of the H_s is equal to $E^\perp$, this proves the proposition. □

Corollary 4.4.14. *Assume that (W, S) is irreducible. Then:*

(a) If β_M is non-degenerate, the W-module E is absolutely simple.

(b) If β_M is degenerate, the W-module E is not semi-simple.

Proof. In case (a), Proposition 4.4.13 shows that E is simple, hence also absolutely simple (Proposition 4.2.2). In case (b), $E^\perp \neq \{0\}$ and $E \neq E^\perp$ (since $\beta_M \neq 0$), and Proposition 4.4.13 shows that $E^\perp$ has no complement stable under W, so the W-module E is not semi-simple. □

Theorem 4.4.15. *The following properties are equivalent:*

(i) *W is finite.*

(ii) *The form β_M is positive and non-degenerate.*

Proof. First assume that W is finite. Let $S = \bigcup_i S_i$ be the decomposition of S into connected components and let $W = \prod_i W_i$ be the corresponding decomposition of W. The space E can be identified with the direct sum of the spaces $E_i = \mathbb{R}^{S_i}$, and β_M can be identified with the direct sum of the corresponding forms β_{M_i}. We are thus reduced to the case when (W, S) is irreducible. Since W is assumed to be finite, the group algebra $\mathbb{R}[W]$ is semi-simple, whence E is a semi-simple W-module. By Corollary 4.4.14, it follows that E is absolutely simple. Let $\tilde{\beta}$ be a positive non-degenerate form on E, and let $\tilde{\beta}_W$ be the sum of its transforms under W. Since $\tilde{\beta}_W$ is invariant under W, it is proportional to β_M by Proposition 4.2.2. Since $\beta_M(e_s, e_s) = 1$ for all $s \in S$, the coefficient of proportionality is positive, and since $\tilde{\beta}_W$ is positive so is β_M, which proves (ii).

Converselt, if β_M is positive non-degenerate, the orthogonal group $\mathrm{O}(\beta_M)$ is compact. Since $\sigma(W)$ is a discrete subgroup of $\mathrm{O}(\beta_M)$ (Corollary 4.4.9), it follows that $\sigma(W)$ is finite, hence so is W. □

Corollary 4.4.16. *If (W, S) is irreducible and finite, E is an absolutely simple W-module.*

The finiteness criterion provided by Theorem 4.4.15 permits the classification of all finite Coxeter groups. We restrict ourselves here to the following preliminary result:

Proposition 4.4.17. *If W is finite, the graph of (W, S) is a forest (a disjoint union of trees).*

Proof. Otherwise, this graph would contain a loop $(s_1, s_2, \ldots, s_n, s_1)$ with $n \geq 3$. Putting $m_i = m(s_i, s_{i+1})$ for $i = 1, \ldots, n-1$ and $m_n = m(s_n, s_1)$, this means that $m_i \geq 3$ for all i. Let $x = e_{s_1} + \cdots + e_{s_n}$. Then $x \neq 0$ and $\beta_M(x, x) = n + 2\sum_{i<j} \beta_M(e_{s_i}, e_{s_j})$. Now for $i < j$,

$$\beta_M(e_{s_i}, e_{s_{i+1}}) = -\cos\frac{\pi}{m_i} \leq -\cos\frac{\pi}{3} \leq -\frac{1}{2}.$$

and similarly for $\beta_M(e_{s_n}, e_{s_1})$. Since the other terms in the sum are non-positive, we obtain

$$\beta_M(x, x) \leq n - n = 0,$$

contrary to the fact that β_M is positive non-degenerate. □

Let (W, S) be a finite Coxeter group and denote by (x, y) the form $\beta_M(x, y)$. By Theorem 4.4.15, this is a scalar product on E. For all $s \in S$, let H_s be the hyperplane associated to the orthogonal reflection σ_s, and let $\mathcal{H}$ be the family of hyperplanes $w(H_s)$, for $s \in S$, $w \in W$. Let C be the set of $x \in E$ such that $(x, e_s) > 0$ for all $s \in S$. Finally, identify W (by means of σ) with a subgroup of the orthogonal group $\mathrm{O}(E)$ of the space E.

Proposition 4.4.18. *With the preceding notations, W is the subgroup of $\mathrm{O}(E)$ generated by the reflections with respect to the hyperplanes of $\mathcal{H}$. It is an essential group and C is a chamber of E relative to $\mathcal{H}$.*

Proof. The first assertion is trivial. On the other hand, if $x \in E$ is invariant under W, it is orthogonal to all the e_s, and hence is zero; this shows that W is essential. Finally, the isomorphism $E \to E^*$ defined by β_M transforms C to the set C of the previous part; the property (P_n) proved there shows that, for all $w \in W$ and all $s \in S$, $w(C)$ does not meet H_s. It follows that C is contained in the complement U of the union of the hyperplanes of $\mathcal{H}$, and since C is connected, open and closed in U, it is a chamber of E relative to $\mathcal{H}$. □

We can apply to W and C all the properties proved for reflection groups. In particular, C is a fundamental domain for the action of W on E. Conversely, let E be a finite dimensional real vector space, provided with a scalar product $(\cdot, \cdot)$ and let W be an essential finite group of displacements of E leaving 0 fixed; assume that W is generated by reflections. Let C be a chamber of E with respect to W, and let S be the set of orthogonal reflections relative to the walls of C. Then (W, S) is a finite Coxeter system. Moreover, if $s \in S$, denote by H_s the wall of C corresponding to s, and denote by e_s the unit vector orthogonal to H_s and on the same side of H_s as C. If $M = (m(s, \tilde{s}))$ denotes the Coxeter matrix of (W, S), Proposition 4.3.12 and Proposition 4.3.26 show that

$$(e_s, e_{\tilde{s}}) = -\cos(\pi/m(s, \tilde{s}))$$

and that the e_s form a basis of E. The natural representation of W on E can thus be identified with the representation σ.

4.4.4 Coxeter matrices with positive degenerate bilinear form

To conclude this section, we consider the case where the bilinear form β_M is positive and degenerate. Still, we assume that S is finite.

Lemma 4.4.19. *The orthogonal complement $E^\perp$ of E for β_M is of dimension 1. It is generated by an element $e = \sum_{s \in S} \lambda_s e_s$ with $\lambda_s > 0$ for all s.*

Proof. This follows from Lemma 4.3.14, applied to the matrix with entries $\beta_M(e_s, e_{\tilde{s}})$. □

Let $e_M = \sum_s \lambda_s e_s$ be the unique vector satisfying the conditions of Lemma 4.4.19 and such that $\sum_s \lambda_s = 1$, and let A be the affine hyperplane of E^* consisting of the $y^* \in E^*$ such that $\langle y^*, e_M \rangle = 1$. If T denotes the orthogonal complement of e_M in E^*, A has a natural structure of an affine space with space of translations T. Moreover, the form β_M defines by passage to the quotient a non-degenerate scalar product on $E/E^\perp$, hence also on its dual T. This gives a Euclidean structure on the affine space A.

Let G be the subgroup of $\mathrm{GL}(E)$ consisting of the automorphisms leaving e_M and β_M invariant. If $g \in G$, the contragredient map $g^* = (g^{-1})^t$ leaves A and T stable, and defines by restriction to A a displacement $i(g)$ of A. It is immediate that this gives an isomorphism from G to the group of displacements of A. Moreover, the stabiliser G_a of a point a of A can be identified with the orthogonal group of the Hilbert space T and is thus compact. On the other hand, G is a locally compact group countable at infinity and A is a Baire space, so it follows that the map $g \mapsto g(a)$ defines a homeomorphism from G/G_a to A, whence G acts properly on A. Since W is a subgroup of G, it can be identified with a group of displacements of A. We are going to show that this group satisfies the discreteness conditions. More precisely:

Proposition 4.4.20. *The group W provided with the discrete topology acts properly on A and is generated by orthogonal reflections. It is infinite, irreducible and essential. The intersection $C \cap A$ is a chamber of A for W. The walls of $C \cap A$ are the hyperplane of A formed by the intersection of A with the hyperplane of E^* orthogonal to e_s (which we denote by L_s), with $s \in S$. If ε_s is the unit vector of T orthogonal to L_s on the same side of L_s as $C \cap A$, then $(\varepsilon_s, \varepsilon_{\tilde{s}}) = -\cos(\pi/m(s, \tilde{s}))$ and the Coxeter matrix of W is identified with M.*

Proof. By Corollary 4.4.9, W is discrete in $\mathrm{GL}(E)$, and hence in G, and acts properly on A. Let $s \in S$. Since $|S| \geq 2$, the hyperplane of E^* orthogonal to e_s is not orthogonal to e_M and its intersection L_s with A is indeed a hyperplane. The displacement corresponding to s is thus a displacement of order 2 leaving fixed all the points of L_s: it is necessarily the orthogonal reflection associated to L_s. It follows that W is generated by orthogonal reflections. Theorem 4.4.15 now shows that it is infinite and Proposition 4.4.13 that it is essential and irreducible.

Since C is an open simplicial cone, whose walls are the hyperplanes orthogonal to e_s with $s \in S$, the intersection $C \cap A$ is a convex, hence connected, open and closed subset of the complement of the union of the L_s in A. Moreover, $C \cap A$ is non-empty, for if $x^* \in C$ we have $\langle x^*, e_M \rangle = \sum_s \lambda_s \langle x^*, e_s \rangle > 0$

and therefore $\langle x^*, e_M\rangle^{-1}x^* \in A$. It follows that $C \cap A$ is a chamber of A relative to the system of the L_s. Moreover, $w(C \cap A) \cap L_s = \varnothing$ for all $w \in W$ (cf. property (P_n) of Theorem 4.4.7) and it follows that $C \cap A$ is a chamber of A relative to the system consisting of the transforms of the L_s by the elements of W; by Corollary 4.3.4, it follows that $C \cap A$ is a chamber of A relative to W.

Let a_s^* be the vertex of the simplex $C \cap A$ not contained in L_s. We have $\langle a_s^*, e_{\tilde{s}}\rangle = 0$ for $s, \tilde{s} \in S$ and $s \neq \tilde{s}$, and since $a_s^* \in A$, we have $\langle a_s^*, e_M\rangle = 1$, whence

$$\langle a_s^*, e_s\rangle = \lambda_s^{-1}\langle a_s^*, e_M\rangle = \lambda_s^{-1}.$$

Let ε_s be the vector in T defined by the relations:

$$(\varepsilon_s, a_s^* - a_{\tilde{s}}^*) = \lambda_s^{-1} \quad \text{for} \quad s, \tilde{s} \in S, s \neq \tilde{s}.$$

The vector ε_s is orthogonal to L_s and is on the same side of the hyperplane L_s as $C \cap A$. Moreover,

$$(\varepsilon_s, a_s^* - a_{\tilde{s}}^*) = \langle e_s, a_s^* - a_{\tilde{s}}^*\rangle \quad \text{for} \quad s, \tilde{s} \in S.$$

which shows that ε_s is the image of the class of e_s under the isomorphism from $E/E^{\perp}$ to T given by the quadratic form β_M. It follows that $(\varepsilon_s, \varepsilon_{\tilde{s}}) = \beta_M(e_s, e_{\tilde{s}})$. Consequently ε_s is a unit vector and the last assertion of Proposition 4.4.20 is proved. □

The Euclidean affine space A provided with the group W is called **the space associated to the Coaceter matrix M** and we denote it by A_M (and by T_M the space of translations of A_M). Proposition 4.4.20 admits a converse:

Proposition 4.4.21. *Let W be a group of displacements of a Euclidean affine space A with translation space T satisfying the discreteness conditions. Assume that W is infinite, essential and irreducible. Then the form β_M attached to the Coxeter matrix M of W is positive degenerate and there exists a unique isomorphism from the affine space A_M associated to M to A, commuting with the action of W. This isomorphism transforms the scalar product of A_M into a multiple of the scalar product of A.*

Proof. Let C_0 be a chamber of A and let S be the set of orthogonal reflections with respect to the walls of C_0. If η_s denotes the unit vector orthogonal to the hyperplane N_s associated to the reflection s and on the same side of N_s as C_0, the form β_M is such that $\beta_M(e_s, e_{\tilde{s}}) = (\eta_s, \eta_{\tilde{s}})$ for $s, \tilde{s} \in S$. It is thus positive. Since the η_s are linearly dependent (Proposition 4.3.27), it is degenerate.

We can thus apply the preceding constructions to M. With the same notation as above, $(e_s, e_{\tilde{s}}) = (\eta_s, \eta_{\tilde{s}})$ and there exists a unique isomorphism $\varphi : T_M \to T$ such that $\varphi(\varepsilon_s) = \eta_s$. Let a and b be two distinct vertices of C_0 and s_0 the reflection in S such that $a \notin N_{s_0}$. Put $\lambda = (\eta_s, a - b)$ and let ψ be the affine bijection from A_M to A defined by

$$\psi(a_{s_0}^* + x) = a + \lambda\varphi(x) \quad \text{for} \quad x \in T_M.$$

It is then immediate that $\psi(L_s) = N_s$ for all $s \in S$ and that ψ transforms the scalar product of A_M into a multiple of that on A. It follows at once that ψ commutes with the action of W. Finally, the uniqueness of ψ is evident, for a_s for example is the unique point of A_M invariant under the reflections $\tilde{s} \in S$ with $\tilde{s} \neq s$. □

4.5 Invariants in the symmetric algebra

4.5.1 Poincaré series of graded algebras

Let $\Bbbk$ be a field. Let M be a graded $\Bbbk$-module of type $\mathbb{Z}$, and M_n the set of homogeneous elements of M of degree n. Assume that each M_n is finite-dimensional. Recall that the filtration on M is called bounded below if there exist an integer n_0 such that $M_n = 0$ for $n \leq n_0$.

Definition 4.5.1. If the filtration on M is bounded below, the formal series $\sum_n \dim_{\Bbbk}(M_n)T^n$, which is an element of $\mathbb{Q}((T))$, is called the **Poincaré series** of M and denoted by $P_M(T)$.

Let N be another graded $\Bbbk$-module of type $\mathbb{Z}$, and $(N_n)_{n\in\mathbb{Z}}$ its grading. Assume that N_n. Then

$$P_{M\oplus N}(T) = P_M(T) + P_N(T), \tag{4.5.1}$$

and, if $M \otimes_{\Bbbk} N$ is provided with the total grading, then

$$P_{M\otimes_{\Bbbk} N}(T) = P_M(T)P_N(T). \tag{4.5.2}$$

Proposition 4.5.2. *Let $S = \bigoplus_{n\in\mathbb{N}} S_n$ be a commutative graded $\Bbbk$-algebra with a system of generators $(x_1, \ldots, x_l)$ consisting of homogeneous and algebraically independent elements. Let d_i be the degree of x_i and assume that $d_i > 0$ for all i. Then the S_n are free and of finite rank over $\Bbbk$, and*

$$P_n(S) = \prod_{i=1}^{l}(1 - T^{d_i})^{-1}. \tag{4.5.3}$$

Proof. Indeed, S can be identified with the tensor product $\Bbbk[x_1] \otimes \cdots \otimes \Bbbk[x_l]$ provided with the total grading. The Poincaré series of $\Bbbk[x_i]$ is $\sum_{n\in\mathbb{N}} T^{nd_i} = (1 - T^{d_i})^{-1}$, and it suffices to apply (4.5.2). □

Under the assumptions of Proposition 4.5.2, we shall say that S is a **graded polynomial algebra** over $\Bbbk$.

Corollary 4.5.3. *The degrees d_i are determined up to order by S.*

Proof. Indeed, the inverse of $P_S(T)$ is the polynomial $N(T) = \prod_{i=1}^{l}(1 - T^{d_i})$, which is thus uniquely determined. If q is an integer ≥ 1 and if $\zeta \in \mathbb{C}$ is a primitive q-th root of unity, the multiplicity of the root ζ of $N(T)$ is equal to the number of d_i that are multiples of q. This number is zero for q sufficiently large. The number of d_i equal to q is thus determined uniquely by descending induction. □

The integers d_i are called the **characteristic degrees** of S. The number of them is equal to the transcendence degree of S over $\Bbbk$. It is the multiplicity of the root 1 of the polynomial $N(T)$.

Let $S = \bigoplus_{n\in\mathbb{N}} S_n$ be a commutative graded $\Bbbk$-algebra, and $R = \bigoplus_{n\in\mathbb{N}} R_n$ a graded subalgebra of S. Assume that each R_n is finite-dimensional and that the R-module S admits a finite basis consisting of homogeneous elements $z_1, \ldots, z_s$ of degrees $f_1, \ldots, f_s$. Then, if M denotes the graded $\Bbbk$-module $\sum_i \Bbbk z_i$, the graded $\Bbbk$-module S is isomorphic to $R \otimes_{\Bbbk} M$, so each S_n is free and of finite type and

$$P_S(T) = P_M(T)P_R(T) = \Big(\sum_{i=1}^{s} T^{f_i}\Big) P_R(T). \tag{4.5.4}$$

Proposition 4.5.4. *Retain the preceding notation and assume that S and R are graded polynomial algebras.*

(a) The graded $\Bbbk$-algebras R and S have the same transcendence degree r over $\Bbbk$.

(b) Let $p_1, \ldots, p_l$ (resp. $q_1, \ldots, q_l$) be the characteristic degrees of S (resp. R). Then

$$\prod_{i=1}^{l}(1 - T^{q_i}) = \Big(\sum_{i=1}^{s} T^{f_i}\Big) \prod_{i=1}^{l}(1 - T^{p_i}). \tag{4.5.5}$$

In particular, we have $sp_1 \cdots p_l = q_1 \cdots q_l$.

Proof. Formula (4.5.4) shows first of all that the multiplicity of the root 1 is the same in the polynomials $P_S(T)^{-1}$ and $P_R(T)^{-1}$ and, taking (4.5.3) into account, proves both (a) and the first part of (b). It follows from (4.5.5) that

$$\prod_{i=1}^{l}(1 + T + T^2 + \cdots + T^{q_i - 1}) = \Big(\sum_{i=1}^{s} T^{f_i}\Big) \prod_{i=1}^{l}(1 + T + T^2 + \cdots + T^{p_i - 1}).$$

Putting $T = 1$ in this relation gives the last claim. □

Remark 4.5.5. Let $S = \Bbbk[X_1, \ldots, X_l]$ be a graded polynomial algebra over $\Bbbk$, d_i the degree of X_i, and $F(X_1, \ldots, X_l)$ a homogeneous element of degree d of S. Then

$$\sum_{i=1}^{l} d_i X_i \frac{\partial F}{\partial X_i} = dF.$$

Indeed, it is immediate that the $\Bbbk$-linear map D from S to S that transforms every homogeneous element z of degree p into pz is a derivation of S. Thus

$$dF(X_1, \ldots, X_l) = D(F(X_1, \ldots, X_l)) = \sum_{i=1}^{l} D(X_i)\frac{\partial F}{\partial X_i} = \sum_{i=1}^{l} d_i X_i \frac{\partial F}{\partial X_i}. \tag{4.5.6}$$

4.5.2 Invariants of finite linear groups

Let $\Bbbk$ be a field, V a $\Bbbk$-vector space, and G a group acting on V. We know that every automorphism of V extends uniquely to an automorphism of the symmetric algebra $S = \mathcal{S}(V)$, and thus G acts on this algebra. If $x \in S$ and $g \in G$, we denote by $g \cdot x$ the transform of x by g. Let R be the subalgebra S^G of S formed by the elements invariant under G.

Assume that G is finite and V has finite dimension. Then S is an R-module of finite type, and R is an integral $\Bbbk$-algebra of finite type (**??**). Let L be its field of fractions of S. The field of fractions K of R is then the set of elements of L invariant under G (Corollary **??**), so L is a Galois extension of R (**??**). Every element of L can be written z/t with $z \in S$ and $t \in R$ (**??**). Therefore the rank of the R-module S is $[L : K]$. Assume that G acts faithfully on V. The Galois group of L over K can then be identified with G, so $[L : K] = |G|$; thus

$$\operatorname{rank}_R(S) = [L : K] = |G|. \tag{4.5.7}$$

In this paragraph, we consider the case where G is generated by pseudo-reflections. In particular, this is true for finite Coxeter groups. We shall see that in this case the algebra R has a bunch of good properties.

Theorem 4.5.6. *Let $\Bbbk$ be a field, V a finite dimensional vector space over $\Bbbk$, $S = \mathcal{S}(V)$ the symmetric algebra of V, G a finite group of automorphisms of V, and R the graded subalgebra of S consisting of the elements invariant under G. Assume that G is generated by pseudo-reflections and that $q = |G|$ is coprime to the characteristic of $\Bbbk$. Then the R-module S has a basis consisting of q homogeneous elements.*

Proof. Since every submodule of $S/(R_+S)$ is free over $R_0 = \Bbbk$, it is enough to show (in view of **??**) that the canonical homomorphism from $R_+ \otimes_R S$ to S is injective. For any R-module M, let $T(M)$ be the kernel of the map $R_+ \otimes_R M \to M$ (in other words, $T(M) = \operatorname{Tor}_1^R(R/R_+, M)$). It is clear that $T(-)$ is a functor on the category R-modules, thus, if G acts R-linearly on M, then G acts on $T(M)$.

The group G acts R-linearly on S, and hence also on $T(S)$. Moreover, $T(S)$ has a natural structure of graded S-module. We show first that, if $g \in G$, then

$$g(x) \equiv x \mod S_1 T(S). \tag{4.5.8}$$

For this it is enough to do this when g is a pseudo-reflection. Then there exists a non-zero vector v in V such that $g(x) - x \in \Bbbk v$ for all $x \in V$. Since V generates S, it follows that g acts trivially on S/Sv. Thus, for any $y \in S$, there exists an element $h(y)$ in S such that

$$g(y) - y = h(y)v$$

Since S is integral and v is non-zero, this element is determined uniquely by y; it is immediate that h is an endomorphism of degree -1 of the R-module S. Thus, $g - 1 = m_v \circ h$, where m_v denotes the homothety with ratio v in S. Hence,

$$T(g) - 1 = T(g-1) = T(m_v) \circ T(h)$$

the image of which is contained in $T(S)v$, proving our assertion.

Next we show that any element of $T(S)$ invariant under G is zero. Indeed, let φ be the endomorphism of the R-module S defined by

$$\varphi(y) = q^{-1} \sum_{g \in G} g(y).$$

for all $y \in S$. Then $\varphi(S) = R$ and it factors through the canonical injection $i : R \to S$. Since $R_+ \otimes_R R \to R$ is injective, we see $T(R) = 0$. This implies $T(\varphi) = 0$, so

$$0 = T(\varphi) = q^{-1} \sum_{g \in G} T(g).$$

But $q^{-1} \sum_{g \in G} T(g)$ leaves fixed the elements of $T(S)$ invariant under G. These elements are therefore zero.

Assume that $T(S) \neq 0$. There exists in $T(S)$ a homogeneous element $x \neq 0$ of minimum degree. Since each g is a homomorphism of degree zero, (4.5.8) implies $g(x) = x$, so x is invariant under G, which must by zero since $T(R) = 0$. This is a contradiction, so $T(S) = 0$. $\square$

Example 4.5.7. Let g be a pseudo-reflection of V whose order $n \geq 2$ is finite and coprime to the characteristic of $\Bbbk$. By Maschke's theorem, V can be decomposed as $D \oplus H$, where H is the hyperplane consisting of the elements of V invariant under g and D is a line on which g acts by multiplication by a primitive n-th root of unity. When $\Bbbk = \mathbb{R}$, this is possible only when $n = 2$, and g is then a reflection. In this case, the groups to which Theorem 4.5.6 applies are the finite Coxeter groups. (For $\Bbbk = \mathbb{C}$, on the other hand, Theorem 4.5.6 applies to certain groups that are not Coxeter groups.)

Theorem 4.5.8. *Retain the assumptions and notation of Theorem 4.5.6.*

(a) There exists a graded vector subspace of S forming a complement to R_+S in S and stable under G.

(b) Let U be such a complement. Then the canonical homomorphism from $U \otimes_{\Bbbk} R$ to S is an isomorphism of G-modules, and the representation of G in U (resp. S) is isomorphic to the regular representation of G on $\Bbbk$ (resp. R).

Proof. Indeed, for any integer $n \geq 0$, the $\Bbbk$-vector spaces S_n and $(R_+S) \cap S_n$ are stable under G, and it follows from Maschke's theorem that there exists a G-stable complement U_n of $(R_+S) \cap S_n$ in S_n. Then $\sum_n U_n$ is a G-stable complement of R_+S in S, hence (a).

Let U be a graded vector subspace of S forming a complement of R_+S in S and stable under G. By **??**, every basis of the $\Bbbk$-vector space U is also a basis of the R-module S, and consequently is a basis of the field of fractions L of S over the field of fractions K of R. Thus, the K-vector space L can be identified with $U \otimes_{\Bbbk} K$. Since U is stable under G, this identification is compatible with the action of G. The group algebra $K[G]$ of G over K can be identified with the algebra $\Bbbk[G] \otimes_{\Bbbk} K$. The Galois extension L of K admits a normal basis (that is, a basis of the form $\{g(\beta) : g \in G\}$ with $\beta \in L$), which can be interpreted as saying that L, considered as an $K[G]$-module, is isomorphic to $K[G]$. Since U is finite dimensional over K, it follows that the $\Bbbk[G]$-module U is isomorphic to $\Bbbk[G]$. Our assertions follow immediately from this. □

We retain the assumptions and notation of Theorem 4.5.6. In the following we will establish some ring theoretic properties for the invariant ring.

Theorem 4.5.9. *Let $(\alpha_1, \ldots, \alpha_l)$ be a minimal generating set of the ideal R_+ of R consisting of homogeneous elements. Let k_i be the degree of α_i and assume that the k_i are coprime to the characteristic exponent of $\Bbbk$. Then $l = \dim(V)$, the α_i generate the $\Bbbk$-algebra R, and are algebraically independent over $\Bbbk$. In particular, R is a graded $\Bbbk$-algebra of polynomials of transcendence degree l over $\Bbbk$.*

Theorem 4.5.9 follows from Proposition 4.5.4(a), Theorem 4.5.6 and the following lemma:

Lemma 4.5.10. *Let $\Bbbk$ be a field, S a graded $\Bbbk$-algebra of polynomials, and R a graded subalgebra of S of finite type such that the R-module S admits a basis $(z_\lambda)_{\lambda \in \Delta}$ consisting of homogeneous elements. Let $(\alpha_1, \ldots, \alpha_l)$ be a minimal generating set of the ideal R_+ of R consisting of homogeneous elements and assume that, for all i, the degree k_i of α_i is coprime to the characteristic exponent p of $\Bbbk$. Then the α_i generate the $\Bbbk$-algebra R and are algebraically independent ouer $\Bbbk$. In particular, R is a graded $\Bbbk$-algebra of polynomials.*

Proof. By **??**, the assumption made about the α_i is equivalent to saying that they are homogeneous and that their images in the $\Bbbk$-vector space $R_+/(R_+)^2$ form a basis of this space. This condition is invariant under extension of the base field; we can thus reduce to the case where $\Bbbk$ is perfect.

The family $(\alpha_1, \ldots, \alpha_l)$ generates the algebra R by **??**. We argue by contradiction and assume that this family is not algebraically independent over $\Bbbk$. We show first of all that there exist families $(\beta_i)_{1 \leq i \leq l}$, $(y_j)_{1 \leq j \leq s}$, and $(d_{ik})_{1 \leq i \leq l, 1 \leq k \leq s}$ of homogeneous elements of S with the following properties:

(i) $\beta_i \in R$ for all i and β_i are not all zero;

(ii) $\deg(y_j) > 0$ for all j;

(iii) $\alpha_i = \sum_{i=1}^{s} d_{ik} y_j$;

(iv) $\sum_{i=1}^{l} \beta_i d_{ik} = 0$ for all k.

Let $X_1, \ldots, X_s$ be indeterminates and give $\Bbbk[X_1, \ldots, X_s]$ with the graded algebra structure for which X_i has degree k_i. Then there exist non-zero homogeneous elements $H(X_1, \ldots, X_l)$ of $\Bbbk[X_1, \ldots, X_s]$ such that $H(\alpha_1, \ldots, \alpha_l) = 0$; choosbe H to be of minimum degree. If $\partial H / \partial X_i \neq 0$, then $(\partial H / \partial X_i)(\alpha_1, \ldots, \alpha_l)$

is a non-zero homogeneous element of R; if $p \neq 1$, H is not of the form H_1^p with $H_1 \in \Bbbk[X_1, \ldots, X_s]$. Take

$$\beta_i = k_i \frac{\partial H}{\partial X_i}(\alpha_1, \ldots, \alpha_l).$$

Since $\Bbbk$ is perfect, the polynomials $\partial H/\partial X_i \in \Bbbk[X_1, \ldots, X_s]$ are not all zero. In view of the assumption made about the k_i, neither are the β_i.

On the other hand, S can be identified with a graded algebra of polynomials $\Bbbk[x_1, \ldots, x_s]$ for appropriate indeterminates $x_1, \ldots, x_l$, with suitable degrees $m_i > 0$. Let D_i be the partial derivative with respect to x_i on S. Take $d_{ik} = k_i^{-1} D_k(\alpha_i)$, then the condition (iv) holds because we have

$$0 = D_k(H(\alpha_1, \ldots, \alpha_l)) = \sum_{i=1}^{l} \frac{\partial H}{\partial X_i}(\alpha_1, \ldots, \alpha_l) D_k(\alpha_i).$$

On the other hand, if we put $y_i = m_i x_i$, the condition (iii) follows from the equality (4.5.6).

Let $\mathfrak{b}$ be the ideal of R generated by the β_i; there exists a subset J of $I = \{1, \ldots, l\}$ such that $(\beta_i)_{i \in J}$ is a minimal system of generators of $\mathfrak{b}$. Then $J \neq \varnothing$ since $\mathfrak{b} \neq 0$. We shall deduce from conditions (iii) and (iv) that, if $i \in J$, then α_i is an R-linear combination of the α_j for $j \neq i$, which will contradict the minimality of $(\alpha_1, \ldots, \alpha_l)$ and will complete the proof.

To this end, we first note that, by the definition of J, there exist homogeneous elements $\gamma_{ij} \in R$ ($j \in J$, $i \in I \setminus J$) such that

$$\beta_i = \sum_{j \in J} \gamma_{ij} \beta_j \tag{4.5.9}$$

for $i \in I \setminus J$. Then condition (iv) gives for each k that

$$\sum_{j \in J} \beta_j (d_{jk} + \sum_{i \in I \setminus J} \gamma_{ij} d_{ik}) = 0. \tag{4.5.10}$$

Put $u_{jk} = d_{jk} + \sum_{i \in I \setminus J} \gamma_{ij} d_{ik}$, we thus obtain

$$\sum_{j \in J} \beta_j u_{jk} = 0. \tag{4.5.11}$$

Write $u_{jk} = \sum_{\lambda \in \Delta} \delta_{jk\lambda} z_\lambda$ where $\delta_{jk\lambda} \in R$. Relation (4.5.11) implies that $\sum_{j \in J} \beta_j \delta_{jk\lambda} = 0$ for all k and λ. If one of the $\delta_{jk\lambda}$ had a nonzero homogeneous component of degree 0, the preceding equality would imply that one of the β_j ($j \in J$) is an R-linear combination of the others, contradicting the minimality of $(\beta_j)_{j \in J}$. Thus $\delta_{jk\lambda} \in R_+$ and consequently $u_{jk} \in R_+ S$ for all j and k. Thus, there exist $u_{jkt} \in S$ such that $u_{jk} = \sum_{t=1}^{l} u_{jkt} \alpha_t$, in other words,

$$d_{jk} + \sum_{i \in I \setminus J} \gamma_{ij} d_{ik} = \sum_{t=1}^{l} u_{jkt} \alpha_t. \tag{4.5.12}$$

Multiply both sides of (4.5.12) by y_k and add for $i \in J$ fixed and $1 \leq k \leq l$; in view of condition (iii), we find that

$$\alpha_i + \sum_{i \in I \setminus J} \gamma_{ij} \alpha_i = \sum_{t=1}^{l} \sum_{k=1}^{s} u_{jkt} y_k \alpha_t.$$

Take the homogeneous components of degree k_i of both sides. Since $\deg(y_k) > 0$, α_i is then an S-linear combination of the α_j for $j \neq i$. Since S is free over R and $\alpha_1, \ldots, \alpha_l \in R$, it follows that α_i is an R-linear combination of the α_j with $j \neq i$. □

Example 4.5.11. Let G be the symmetric group $\mathfrak{S}_l$, which is a finite Coxeter group. By choosing a basis for V, we can identify $S = \mathcal{S}(V)$ as the polynomial algebra $\Bbbk[X_1, \ldots, X_l]$. The subalgebra R is then identified with the algebra of symmetric polynomials in $\Bbbk[X_1, \ldots, X_l]$. The fundamental theorem of symmetric functions states that $\Bbbk[X_1, \ldots, X_l]$ is generated by the elementary symmetric polynomials $e_1, \ldots, e_n$, with $\deg(e_i) = i$. Also, these polynomials are algebraically independent over $\Bbbk$. This justifies the claims in Theorem 4.5.9.

Corollary 4.5.12. *With the assumptions and notations of Theorem 4.5.9, the product of the characteristic degrees of R is $|G|$.*

Proof. Indeed, $\operatorname{rank}_R(S) = |G|$ by (4.5.7). The characteristic degrees of S are equal to 1. Then the corollary follows from Proposition 4.5.4(b). □

Lemma 4.5.13. *Let $\Bbbk$ be a field, V a finite dimensional vector space over $\Bbbk$, $S = \bigoplus_n S_n$ the symmetric algebra of V, ρ an endomorphism of V, and $\rho^{(n)}$ the canonical extension of ρ to S_n. Then, with T denoting an indeterminate, we have in $\Bbbk[\![T]\!]$*

$$\sum_{n=0}^{\infty} \operatorname{tr}(\rho^{(n)})T^n = \det(1 - \rho T)^{-1}.$$

Proof. Extending the base field if necessary, we can assume that $\Bbbk$ is algebraically closed. Let $(e_1, \dots, e_l)$ be a basis of V with respect to which the matrix of ρ is lower triangular, and let $\lambda_1, \dots, \lambda_l$ be the diagonal elements of this matrix. With respect to the basis $\{e_1^{i_1} \cdots e_l^{i_l} : i_1 + \cdots + i_l = n\}$ of S_n, ordered lexicographically, the matrix of $\rho^{(n)}$ is lower triangular and its diagonal elements are the products $\lambda_1^{i_1}, \dots, \lambda_l^{i_l}$. Thus

$$\operatorname{tr}(\rho^{(n)}) = \sum_{i_1+\cdots+i_l=n} \lambda_1^{i_1} \cdots \lambda_l^{i_l}$$

and consequently

$$\begin{aligned}\sum_{n=0}^{\infty} \operatorname{tr}(\rho^{(n)})T^n &= \Big(\sum_{n=0}^{\infty} \lambda_1^n T^n\Big) \cdots \Big(\sum_{n=0}^{\infty} \lambda_l^n T^n\Big) \\ &= \prod_{i=1}^{l} (1 - \lambda_i T)^{-1} = \det(1 - \rho T)^{-1}.\end{aligned}$$ □

Proposition 4.5.14. *Let $\Bbbk$ be a field, V a finite dimensional vector space over $\Bbbk$, $S = \bigoplus_n S_n$ the symmetric algebra of V. Let G be a finite group acting on V, q the order of G, and R the graded subalgebra of S consisting of the elements invariant under G. Assume that $\Bbbk$ is of characteristic 0. Then the Poincaré series of R is*

$$P_R(T) = q^{-1} \sum_{g \in G} \det(1 - Tg)^{-1}.$$

Proof. Indeed, the endomorphism $\phi = q^{-1} \sum_{g \in G} g^{(n)}$ is a projection of S_n onto R_n, so $\operatorname{tr}(f) = \dim_{\Bbbk}(R_n)$. Thus, the Poincaré series of R is

$$q^{-1} \sum_{g \in G} \sum_{n=0}^{\infty} \operatorname{tr}(g^{(n)})T^n$$

and it suffices to apply Lemma 4.5.13. □

Lemma 4.5.15. *Let $\Bbbk$ be a field with characteristic p and V a vector space. Let G be a group acting on V. Then the set of pseudo-reflections of G forms a cyclic subgroup of G.*

Proof. Let g be a pseudo-reflection on V contained in G. Then since g has finite order, it is diagonalizable: there exist a basis $(x_1, \dots, x_l)$ of V such that $g(x_i) = \lambda_i x_i$, where λ_i are l-th roots of unity. Since g is a pseudo-reflection, the rank of $g - 1$ is 1, which means $\lambda_i = 1$ for all but at most one, say $\lambda_1 = \zeta$. Since the group $\mu_l(\Bbbk)$ of l-th roots of unity is cyclic, we conclude that the group of pseudo-reflection on V is cyclic. Since any subgroup of a cyclic group is again cyclic, we see the claim follows. □

Proposition 4.5.16. *With the assumptions and notation of Theorem 4.5.9, let H be the set of pseudo-reflections of G. Assume that $\Bbbk$ is of characteristic 0. Then $|H| = \sum_{i=1}^{l}(k_i - 1)$.*

Proof. By **??**, we can assume that $\Bbbk$ is algebraically closed. For any $g \in G$, let $\lambda_1(g), \dots, \lambda_l(g)$ be its eigenvalues. Since every $g \in G$ is diagonalizable (each g is idempotent), $g = 1$ if and only if all of the $\lambda_i(g)$ are equal to 1, and $g \in H$ if and only if the number of $\lambda_i(g)$ equal to 1 is $r - 1$ (we then denote by $\lambda(g)$ the eigenvalue distinct from 1). Proposition 4.5.2 and Proposition 4.5.14 prove that

$$q \prod_{i=1}^{l} (1 - T^{k_i})^{-1} = \sum_{g \in G} \det(1 - gT)^{-1}. \tag{4.5.13}$$

in $\Bbbk[\![T]\!]$, and hence in $\Bbbk(T)$. Consequently, we have in $\Bbbk(T)$

$$q\frac{1}{\prod_{i=1}^{l}(1-T^{k_i})} = \frac{1}{(1-T)^l} + \sum_{g\in H}\frac{1}{(1-\lambda(g)T)(1-T)^{l-1}} + \sum_{\substack{g\notin H\\ g\neq 1}}\frac{1}{\det(1-gT)},$$

whence

$$q\frac{(1-T)^{l-1}}{\prod_{i=1}^{l}(1-T^{k_i})} = \frac{1}{(1-T)^l} + \sum_{g\in H}\frac{1}{1-\lambda(g)T} + \sum_{\substack{g\notin H\\ g\neq 1}}\frac{1}{\det(1-gT)}. \tag{4.5.14}$$

Now (4.5.14) can be rewritten into

$$\frac{q-\prod_{i=1}^{l}(1+T+\cdots+T^{k_i-1})}{(1-T)\prod_{i=1}^{l}(1+T+\cdots+T^{k_i-1})} = \sum_{g\in H}\frac{1}{1-\lambda(g)T} + \sum_{\substack{g\notin H\\ g\neq 1}}\frac{(1-T)^{l-1}}{\det(1-gT)}. \tag{4.5.15}$$

It follows that $q-\prod_{i=1}^{l}(1+T+\cdots+T^{k_i-1})$ vanishes for $T=1$, so $q=k_1\cdots k_l$, which we knew already by Corollary 4.5.12. This granted, let $Q(T)$ be the polynomial in the left-hand of (4.5.15). Differentiating the equality $(1-T)Q(T) = q-\prod_{i=1}^{l}(1+T+\cdots+T^{k_i-1})$ and putting $T=1$ we see that $-Q(1)$ is the value for the following polynomial at $T=1$:

$$-\frac{d}{dt}\Big(\prod_{i=1}^{l}(1+T+\cdots+T^{k_i-1})\Big) = -\sum_{i=1}^{l}(1+2T+\cdots+(k_i-1)T^{k_i-2})\prod_{j\neq i}(1+T+\cdots+T^{k_j-1}),$$

whence

$$Q(1) = \sum_{i=1}^{l}\frac{(k_i-1)k_i}{2}\prod_{j\neq i}k_i = \Big(\prod_{i=1}^{l}k_i\Big)\Big(\sum_{i=1}^{l}\frac{k_i-1}{2}\Big).$$

Returning to (4.5.15), we have on the other hand

$$Q(1) = \Big(\prod_{i=1}^{l}k_i\Big)\Big(\sum_{g\in H}\frac{1}{1-\lambda(g)}\Big).$$

Thus we finally get

$$\sum_{i=1}^{l}\frac{k_i-1}{2} = \sum_{g\in G}\frac{1}{1-\lambda(g)}. \tag{4.5.16}$$

Now the subgroup C of pseudo-reflections of G is cyclic by Lemma 4.5.15. Let N be the order of C; the values of $\lambda(g)$ for $g\in C$ are thus $1,\theta,\theta^2,\ldots,\theta^{N-1}$ with θ a primitive N-th root of unity. We have $\frac{1}{1-\theta^i}+\frac{1}{1-\theta^{N-i}}=1$, so

$$\sum_{\substack{g\in C\\ g\neq 1}}\frac{1}{1-\lambda(g)} = \frac{1}{2}(N-1) = \frac{1}{2}|H|.$$

The equality (4.5.16) thus proves the proposition. □

Remark 4.5.17. When $\Bbbk=\mathbb{R}$, G is a Coxeter group and H is the set of reflections belonging to G, we know that the elements of H are in one-to-one correspondence with the walls of V.

Example 4.5.18. Again consider the case $G=\mathfrak{S}_l$. We have already remarked that $k_i=i$, so $\sum_{i=1}^{l}(k_i-1)=l(l-1)/2$, which is exactly the number of transpositions in $\mathfrak{S}_l$.

Proposition 4.5.19. *With the assumptions and notation of Theorem 4.5.9, assume that $\Bbbk$ is of characteristic $\neq 2$. Then $-1\in G$ if and only if the characteristic degrees $k_1,\ldots,k_l$ of R are all even.*

Proof. Let ϕ be the automorphism of the algebra S that extends the automorphism -1 of V. Then $\phi(z)=(-1)^{\deg(z)}z$ for all homogeneous z in S. Thus, if $-1\in G$, every homogeneous element of odd degree of R is zero, and the k_i are all even. Conversely, if the k_i are all even, every element of R is invariant under ϕ, and Galois theory shows that $-1\in G$. □

We retain the assumptions and notation of Theorem 4.5.9, and assume that $\Bbbk$ is of characteristic 0. An element z of S is said to be **anti-invariant** under G if

$$g(z) = \det(g)^{-1}z$$

for all $g \in G$. Let H be the set of pseudo-reflections belonging to G and distinct from 1. For all $g \in H$, there exist $e_g \in V$ and $f_g \in V^*$ such that

$$g(x) = x + f_g(x)e_g \quad \text{for} \quad x \in V.$$

Proposition 4.5.20. *Let $e = \prod_{g \in H} e_g$ be in S.*

(a) The elements of S which are anti-invariant under G are the elements of Re.

(b) Identify S with the polynomial algebra $\Bbbk[x_1, \dots, x_l]$ by choosing a basis $(x_1, \dots, x_l)$ of V, and let $(\alpha_1, \dots, \alpha_l)$ be algebraically independent homogeneous elements of S generating the algebra R. Then the Jacobian $J = \det(\partial P_i / \partial x_j)$ is of the form λe, where $\lambda \in \Bbbk^\times$.

(c) $\sum_{i=1}^{l}(\deg(P_i) - 1) = |H|$.

Proof. With the notation in (b), we have

$$dP_1 \wedge \cdots dP_l = J dx_1 \wedge \cdots \wedge dx_l.$$

So for all $g \in G$,

$$\begin{aligned} g(J)(\det(g))dx_1 \wedge \cdots \wedge dx_l &= g(J)d(gx_1) \wedge \cdots \wedge d(gx_l) = g(dP_1 \wedge \cdots \wedge dP_l) \\ &= dP_1 \wedge \cdots \wedge dP_l = J\, dx_1 \wedge \cdots \wedge dx_l. \end{aligned}$$

hence J is anti-invariant under G. On the other hand, the field of fractions L of S is a Galois extension of the field of fractions K of R; if D is a derivation of K with values in an extension field Ω of L, then D extends to a derivation of L with values in Ω. Since the P_i are algebraically independent, it follows that $dP_1 \wedge \cdots \wedge dP_l \neq 0$, hence $J \neq 0$.

Let z be an element of S anti-invariant under G. We show that z is divisible by e in S. Let v be a non-zero vector in V. The elements of G that leave $\Bbbk v$ stable leave stable a complementary hyperplane L (by semi-simplicity); an element of G leaving $\Bbbk v$ stable is 1 or a pseudo-reflection with vector v if and only if it induces 1 on L; the pseudo-reflections with vector v belonging to G thus constitute, together with 1, a cyclic subgroup C of G; let N be its order. There exists a basis $(x_1, \dots, x_l)$ of V such that $v = x_1$, $x_2, \dots, x_l \in L$, and z can be identified with a polynomial $P(x_1, \dots, x_l)$ with coefficients in $\Bbbk$. From $g(z) = \det(g)^{-1}z$ for $g \in C$, we see that x_1 only appears in $P(x_1, \dots, x_l)$ with exponents congruent to -1 modulo N. Thus, $P(x_1, \dots, x_l)$ is divisible by $x_1^{N-1} = v^{N-1}$. Now e is, up to a scalar factor, the product of the v^{N-1} for those $v \in V$ such that $N > 1$, and these elements of S are mutually coprime. Since S is factorial, z is divisible by e.

By the previous arguments, J is divisible by e in S. Now

$$\deg(J) = \sum_{i=1}^{l}(k_i - 1) = |H|$$

so $\deg(J) = \deg(e)$, hence $J = \lambda D$ with $\lambda \in \Bbbk$. Since $J \neq 0$, $\lambda \notin \Bbbk^\times$. This proves (b).

Now e is anti-invariant under G. Next, if $y \in R$, it is clear that ye is anti-invariant under G. Finally, if $z \in S$ is anti-invariant under G, we have seen that there exists $y \in S$ such that $z = ye$. Since S is integral, by applying G on this equality we see $y \in R$. This proves (a). Assertion (c) is obtained by writing down the fact that the homogeneous polynomials e and J are of the same degree. □

Proposition 4.5.21. *Let $\Bbbk$ be a field, V a finite dimensional vector space over $\Bbbk$, G a finite group of automorphisms of V whose order q is invertible in $\Bbbk$, S the symmetric algebra of V, and R the subalgebra of S consisting of the elements invariant under G. Then a prime ideal $\mathfrak{P}$ of height 1 of S is ramified over $\mathfrak{p} = \mathfrak{P} \cap R$ if and only if there exist a non-zero element a of V and a non-zero element a^* of V^* such that $\mathfrak{P} = Sa$ and the pseudo-reflection s_{a,a^*} belongs to G. Moreover, in this case we have*

(b) the decomposition group $G^Z(\mathfrak{P})$ is the subgroup of elements of G leaving $\Bbbk a$ stable, and the inertia group $G^T(\mathfrak{P})$ is the cyclic subgroup H_a of G consisting of the pseudo-reflections of G with vector a;

(c) the residue field $\kappa(\mathfrak{P})$ of S at $\mathfrak{P}$ is separable over the residue field $\kappa(\mathfrak{p})$ of R at $\mathfrak{p}$, and the ramification index $e(\mathfrak{P}/p)$ equal to the coefficient of $\mathfrak{P}$, augmented by 1, in the divisor $\operatorname{div}(D_{S/R})$ of the different, is equal to $|H_a|$.

Proof. To say that $\mathfrak{P}$ is ramified over R means that its inertia group $G^T(\mathfrak{P})$ does not reduce to the identity, in other words that there exists $g \neq 1$ in G such that $g(z) \equiv z \bmod \mathfrak{P}$ for all $z \in S$. Since S is a factorial ring, $\mathfrak{P}$ is a principal ideal Sa, and a must divide all the elements $g(z) - z$ ($z \in S$); now, for $z \in V$, these elements are homogeneous of degree 1 and are all non-zero (since $g \neq 1$); thus, a must be homogeneous of degree 1, in other words $a \in V$. Then there exists a linear form a^* on V such that $g = s_{a,a^*}$. Conversely, if g is a pseudo-reflection s_{a,a^*} different from 1, then $g(z) \equiv z \bmod Sa$ for all $z \in S$, so g belongs to the inertia group of the prime ideal $\mathfrak{P} = Sa$. This proves the first assertion of the and the characterizations of $G^Z(\mathfrak{P})$ and $G^T(\mathfrak{P})$.

Since q is coprime to the characteristic p of $\Bbbk$ (which is also that of $\kappa(\mathfrak{P})$), the extension $\kappa(\mathfrak{P})/\kappa(\mathfrak{p})$ is separable (Corollary **??**). This implies $e(\mathfrak{P}/p) = |H_a| = |G^T(\mathfrak{P})|$. Since $e(\mathfrak{P}/\mathfrak{p})$ is coprime to p, the coefficient of $\mathfrak{P}$ in $\operatorname{div}(D_{S/R})$ is $e(\mathfrak{P}/p) - 1$. ☐

Theorem 4.5.22. *Let $\Bbbk$ be a field, V a finite dimensional vector space over $\Bbbk$, S the symmetric algebra of V, G a finite group of automorphisms of V, and R the subalgebra of S consisting of the elements invariant under G. Assume that $q = |G|$ is invertible in $\Bbbk$. Then the following conditions are equivalent:*

(i) *G is generated by pseudo-reflections;*

(ii) *S is a free graded R-module;*

(iii) *R is a graded polynomial algebra over $\Bbbk$.*

Proof. The equivalence of (ii) and (iii) follows from **??**. The implication (i)⇒(ii) follows from Theorem 4.5.6. We show that (iii)⇒(i). Let G' be the subgroup of G generated by the pseudo-reflections belonging to G, and let R' be the subalgebra of S consisting of the elements invariant under G'. We have $R \subseteq R' \subseteq S$. By Proposition 4.5.21, $\operatorname{div}(D_{S/R}) = \operatorname{div}(D_{S/R'})$, so $\operatorname{div}(D_{R'/R}) = 0$. Assume then that R is a graded polynomial algebra. Since this is also the case for R' (since G' is generated by pseudo-reflections), **??** shows that the R-module R' admits a homogeneous basis $(Q_1, \dots, Q_m)$; let $q_i = \deg(Q_i)$. Put

$$d = \det(\operatorname{tr}_{R'/R}(Q_i Q_j)).$$

The fact that $\operatorname{div}(D_{R'/R})$ is zero shows that $\operatorname{div}(d) = 0$, which means that d belongs to $\Bbbk^*$. On the other hand $\operatorname{tr}_{R'/R}(Q_i Q_j)$ is a homogeneous element of degree $q_i + q_j$, and d is homogeneous of degree $2\sum_i q_i$. Thus $q_i = 0$ for all i, which means that $R' = R$, and so $G' = G$ by Galois theory. This proves that G is generated by pseudo-reflections. ☐

Remark 4.5.23. Retain the assumptions and notation of Theorem 4.5.22. Let H be the set of pseudo-reflections belonging to G. Assume that H generates G. For all $g \in G$, put $g(x) = x + f_g(x)e_g$ with $e_g \in V$, $f_g \in V^*$. Put $e = \prod_{g \in H} e_g \in S$. The the difierent of S over R is the principal ideal Se.

4.6 Coxeter transformations for reflection groups

In this section, let V be a real vector space of finite dimension l and W a finite subgroup of $\mathrm{GL}(V)$ generated by reflections and essential. Provide V with a scalar product (x, y) invariant under W. Denote by $\mathcal{H}$ the set of hyperplanes H of V such that the corresponding orthogonal reflection s_H belongs to W. Then we can apply the results in Section 4.3. For example, the group W acts transitively on the set of chambers determined by $\mathcal{H}$, and the number of walls of a chamber C is also d (Proposition 4.3.26.)

4.6.1 Ordered chamber and Coxeter transformations

An **ordered chamber** relative to W is a pair consisting of a chamber C determined by $\mathcal{H}$ and a bijection $i \mapsto H_i$ from $\{1, 2, \dots, l\}$ onto the set of walls of C. The **Coxeter transformation** determined by an ordered chamber $(C, (H_i)_{1 \le i \le l})$ is the element $w_C = s_{H_1} s_{H_2} \dots s_{H_l}$ of W. This definition of w_C clearly depends on the chamber C and the ordering (H_i). However, as we will see, the conjugate class of w_C are all equal, for any ordered chamber $(C, (H_i)_{1 \le i \le l})$. For this, we need the following lemmas.

Lemma 4.6.1. *Let X be a forest having only a finite number of vertices.*

(a) *If X has at least one vertex, it has a terminal vertex.*

(b) *If X has at least two vertices, there is a partition (X_1, X_2) of its set of vertices into two non-empty subsets such that two distinct vertices that both belong to X_1 or both belong to X_2 are never joined.*

Proof. Let $(x_0, \ldots, x_n)$ be an injective path of maximal length in X. The vertex x_0 cannot be joined to any vertex y distinct from $x_1, \ldots, x_n$, since otherwise there would exist an injective path in X of length $n+1$, namely $(y, x_0, \ldots, x_n)$. The vertex x_0 is not joined to any vertex x_i with $2 \leq i \leq n$, since otherwise $(x_0, x_1, \ldots, x_i)$ would be a circuit in the forest X. Thus, x_0 is terminal.

We shall prove (b) by induction of the number m of vertices of X, the case $m = 2$ being trivial. Suppose then that $m \geq 3$ and that assertion (b) is proved for graphs with $m-1$ vertices. Let a be a terminal vertex of X (cf. (a)). We apply the induction hypothesis to the full subgraph of X whose vertices are $\tilde{X} = X \setminus \{a\}$. Thus, there exist two non-empty disjoint subsets $\tilde{X}_1$ and $\tilde{X}_2$ of $\tilde{X}$ such that $\tilde{X} = \tilde{X}_1 \cup \tilde{X}_2$ and two distinct vertices in $\tilde{X}_1$ (resp. $\tilde{X}_2$) are never joined. Since a is joined to at most one vertex of $\tilde{X}$, we can suppose for example that it is not joined to any vertex in $\tilde{X}_1$. The partition $(\tilde{X}_1 \cup \{a\}, \tilde{X}_2)$ then has the required property. □

Lemma 4.6.2. *Let X be a finite forest, and $\rho : X \to G, x \mapsto g_x$ a map from X to a group G such that g_x and g_y commute whenever x and y are not linked in X. Let $\mathcal{T}$ be the set of total orderings on X. For all $\xi \in \mathcal{T}$, let p_ξ be the product in G of the sequence $(g_x)_{x \in X}$ defined by ρ. Then the elements p_ξ are conjugate in G.*

Proof. We proceed by induction on $n = |X|$. The case $n = 1$ is immediate, so assume that $n \geq 2$. For a terminal vertex a (exists by Lemma 4.6.1), let $b \in X \setminus \{a\}$ be a vertex linked to a if one exists; if a is not linked to any vertex in $X \setminus \{a\}$, let b in $X \setminus \{a\}$ be arbitrary. In all cases, since a is not linked to any other vertices, we see g_a commutes with g_x for $x \neq b$. Let $\eta \in \mathcal{T}$ be such that a is the largest element of X and b the largest element of $X \setminus \{a\}$, we let $\xi \in \mathcal{T}$ and prove that p_ξ, p_η are conjugate.

Case 1. Assume first that, for the order ξ on X, a is the largest element of X and b the largest element of $X \setminus \{a\}$. Let $\tilde{X}$ be the full subgraph $X \setminus \{a\}$ of X, which is also a forest. Define a map $\tilde{\rho} : \tilde{X} \to G$ from $\tilde{X}$ to G by

$$\tilde{\rho}(x) = \tilde{g}_x = \begin{cases} g_x & x \neq b, \\ g_b g_a & x = b. \end{cases}$$

Let $\tilde{\xi}, \tilde{\eta}$ be the restrictions of ξ, η to $\tilde{X}$. Then the induction hypothesis applies to the map $\tilde{\rho}$, so $p_{\tilde{\eta}}$ and $p_{\tilde{\xi}}$ are conjugate. But it is clear that $p_{\tilde{\xi}} = p_\xi$ and $p_{\tilde{\eta}} = p_\eta$, proving the lemma in this case.

Case 2. Now we weaken the assumption and suppose that a is the largest element of X for ξ. Let X_1 (resp. X_2) be the set of elements of $X \setminus \{a\}$ strictly larger (resp. smaller) than b; let ξ_i be the restriction of ξ to X_i. Then since g_a commutes with g_x for $x \in X_2$, we have

$$p_\xi = p_{\xi_1} g_b p_{\xi_2} g_a = p_{\xi_1} g_b g_a p_{\xi_2},$$

and this element is conjugate to $p_{\xi_2} p_{\xi_1} g_b g_a$. We are thus reduced to *Case 2*.

Case 3. In the general case, let X_3 (resp. X_4) be the set of elements of X strictly larger (resp. smaller) than a; let ξ_i be the restriction of ξ to X_i. Then $p_\xi = p_{\xi_3} g_a p_{\xi_4}$, and this element is conjugate to $p_{\xi_4} p_{\xi_3} g_a$. We are thus reduced to *Case 3*. □

Proposition 4.6.3. *All Coxeter transformations are conjugate in W.*

Proof. Since W permutes the chambers determined by $\mathcal{H}$ transitively, we are reduced to proving the following: let $(C, (H_i)_{1 \leq i \leq d})$ be an ordered chamber and $\pi \in \mathfrak{S}_d$, then w_C and $w_{\pi(C)}$ are conjugate in W. Taking into account Proposition 4.4.17, this will follow immediately from Lemma 4.6.2: Note that two elements s_H and $s_{\tilde{H}}$ commute if they are not linked in the Coxeter graph, by definition (in this case $m(s_H, s_{\tilde{H}}) = 2$). □

It follows from Proposition 4.6.3 that all the Coxeter transformations are of the same order $h = h(W)$. This number is called the **Coxeter number** of W.

Remark 4.6.4. Let $W_1, \ldots, W_p$ be essential finite groups acting in the spaces $V_1, \ldots, V_p$ and generated by reflections. Let C_j be a chamber relative to W_j. Let W be the group $W_1 \times \cdots \times W_p$ acting in the space $V_1 \times \cdots \times V_p$. Then $C_1 \times \cdots \times C_p$ is a chamber relative to W. The Coxeter transformations of W defined by C are the products $w_{C_1} \cdots w_{C_p}$, where w_{C_j} is a Coxeter transformation of W_j defined by C_j.

4.6.2 Eigenvalues of Coxeter transformations

Since all Coxeter transformations are conjugate, they all have the same characteristic polynomial $P(T)$. Let h be the Coxeter number of W. Then since w_C has order h, its minimal polynomial must divides the polynomial $T^h - 1$, and has the same roots as $P(T)$. Therefore we can write

$$P(T) = \prod_{i=1}^{l}(T - e^{\frac{2\pi i m_j}{h}}).$$

where $0 \le m_1 \le \cdots \le m_l \le h$ are integers. The integers $m_1, \ldots, m_l$ are called the **exponents** of W.

Let C be a chamber determined by $\mathcal{H}$, $H_1, \ldots, H_l$ its walls, and put $s_i = s_{H_i}$. Denote by e_i the unit vector orthogonal to H_i and on the same side of H_i as C. By Lemma 4.6.1, we can assume that the H_i are numbered so that $e_1, \ldots, e_r$ are pairwise orthogonal and $e_{r+1}, \ldots, e_l$ are pairwise orthogonal. Then $w_1 = s_1 \cdots s_r$ and $w_2 = s_{r+1} \cdots s_l$ are the orthogonal symmetries with respect to the subspaces

$$W_1 = H_1 \cap \cdots \cap H_r, \quad W_2 = H_{r+1} \cap \cdots \cap H_l.$$

And $w_C = w_1 w_2$ is a Coxeter transformation. Since $(e_1, \ldots, e_l)$ is a basis of V by Proposition 4.3.26, V is the direct sum of W_1 and W_2.

We deduce first that 1 is not an eigenvalue of w_C. For if $v \in V$ is such that $w_C(v) = v$, then $w_1(v) = w_2(v)$, so $v - w_1(v) = v - w_2(v)$ is orthogonal to W_1 and W_2, and hence is zero. Thus $v = w_1(v) = w_2(v) \in W_1 \cap W_2 = \{0\}$. Consequently,

$$0 < m_1 \le m_2 \le \cdots \le m_l < h. \tag{4.6.1}$$

The characteristic polynomial of w_C has real coefiicients. Thus, for all j, the power of $T - e^{2\pi i m_j/h}$ in $P(T)$ is equal to that of $T - e^{2\pi i(h-m_j)/h}$. Hence

$$m_j + m_{l+1-j} = h \quad 1 \le j \le l. \tag{4.6.2}$$

Adding the equalities (4.6.2) we obtain

$$m_1 + m_2 + \cdots + m_l = \frac{1}{2} lh. \tag{4.6.3}$$

Lemma 4.6.5. *Assume that W is irreducible and $l \ge 2$. With the preceding notation, there exist two linearly independent vectors z_1, z_2 such that*

(a) the plane P generated by z_1, z_2 is stable under w_1 and w_2;

(b) $w_1|_P$ and $w_2|_P$ are orthogonal reflections with respect to $\mathbb{R}z_1$ and $\mathbb{R}z_2$;

(c) $z_1, z_2 \in \bar{C}$, and $P \cap C$ is the set of linear combinations of z_1, z_2 with positive coefiicients.

Proof. Let $(e^1, \ldots, e^l)$ be the basis of V such that $(e^i, e_j) = \delta_{ij}$. Then C is the open simplicial cone determined by the e^i (Proposition 4.3.26). For any $x \in V$, we have

$$x = \sum_{j=1}^{l}(x, e_j)e^j.$$

So it is clear that W_1 is generated by $e^{r+1}, \ldots, e^l$ and W_2 by $e^1, \ldots, e^r$. Let q be the endomorphism of V such that $q(e_i) = e^i$. Its matrix with respect to $(e^1, \ldots, e^l)$ is $Q = ((e_i, e_j))$. We have $(e_i, e_j) < 0$ for $i \ne j$ (Proposition 4.3.12). Since W is irreducible, there does not exist any partition $\{1, 2, \ldots, l\} = I \cup J$ such that $(e_i, e_j) = 0$ for $i \in I$ and $j \in J$. Thus by Lemma 4.3.14 Q has an eigenvector $z = a_1 e^1 + \cdots + a_l e^l$, all of whose coordinates are positive; let λ be the corresponding eigenvalue. Put

$$z_1 = a_{r+1}e^{r+1} + \cdots + a_l e^l \in W_1 \cap \bar{C}, \quad z_2 = a_1 e^1 + \cdots + a_r e^r \in W_2 \cap \bar{C}.$$

and let P be the plane generated by z_1 and z_2. Then $P \cap C$ is the set of linear combinations of z_1 and z_2 with positive coefficients. The relation $q(z) = \lambda z$ gives $\sum_{j=1}^{l} a_j e_j = \sum_{j=1}^{l} \lambda a_j e^j$; taking inner product with e_k (where $k \le r$) gives

$$a_k + \sum_{j=r+1}^{l} a_j (e_j, e_k) = \lambda a_k$$

and therefore

$$\begin{aligned}(\lambda-1)z_2 &= \sum_{k=1}^{r}\Big(\sum_{j=r+1}^{l} a_j(e_j,e_k)\Big)e^k = \sum_{j=r+1}^{l} a_j\Big(\sum_{k=1}^{r}(e_j,e_k)e^k\Big)\\ &= \sum_{j=r+1}^{l} a_j\Big(-\sum_{k=r+1}^{l}(e_j,e_k)e^k+\sum_{k=1}^{l}(e_j,e_k)e^k\Big)\\ &= \sum_{j=r+1}^{l} a_j\Big(-e^j+\sum_{k=1}^{l}(e_j,e_k)e^k\Big)\\ &= -\sum_{j=r+1}^{l} a_je^j+\sum_{j=r+1}^{l} a_je_j\\ &= -z_1+\sum_{j=r+1} a_je_j.\end{aligned} \tag{4.6.4}$$

By th definition of w_2, it follows that w_2 fixes the line generated by $(\lambda-1)z_2+z_1$. It is clear that w_2 fix z_2, so it leaves stable that plane generated by z_2 and $(\lambda-1)z_2+z_1$, that is, P. Similarly, w_1 leaves P stable. Since $z_1 \in P\cap W_1$ and $z_2 \in P\cap W_2$, $w_1|_P$ fixes z_1 and $w_2|_P$ fixes z_2. Thus $w_1|_P$ and $w_2|_P$ are the reflections with respect to $\mathbb{R}z_1$ and $\mathbb{R}z_2$. □

Theorem 4.6.6. *Assume that W is irreducible. Then:*

(a) $m_1=1$ *and* $m_l=h-1$.

(b) $|\mathcal{H}|=\frac{1}{2}lh$.

Proof. We retain the precegrlg notation. The restriction of $w=w_C=w_1w_2$ to P is the rotation with angle $2(\mathbb{R}_+z_2,\mathbb{R}_+z_1)$. Since ρ has order h, the h elements $1,w,\dots,w^{h-1}$ of W are pairwise distinct; the elements $w_1,w_1w,\dots,w_1w^{h-1}$ are thus pairwise distinct, and are distinct from the preceding elements since $w^i|_P$ is a rotation and $w_1w^j|_P$ is a reflection. The set

$$\{1,w,\dots,w^{h-1},w_1,w_1w,\dots,w_1w^{h-1}\}$$

is the subgroup $\tilde{W}$ of W generated by w_1 and w_2, and induces on P the group D generated by the orthogonal reflections with respect to $\mathbb{R}z_1,\mathbb{R}z_2$. The transform of C by an element of $\tilde{W}$ is either disjoint from $-C$ or equal to $-C$. Thus, the transform of $P\cap C$ by an element of D is either disjoint from $-(P\cap C)$ or equal to $-(P\cap C)$. Hence, for a suitable orientation of P, there exists an integer $m>0$ such that $(\mathbb{R}_+z_2,\mathbb{R}_+z_1)=\pi/m$ (Proposition 4.2.14). Moreover, the sets $\tilde{w}(C)$, for $\tilde{w}\in\tilde{W}$, are pairwise disjoint; the sets $\sigma(P\cap C)$, for $\sigma\in D$, are thus pairwise disjoint; so D is of order $2h$. Hence $m=h$. By definition, $w|_P$ is a rotation with angle $2\pi/m$, and so has eigenvalues $e^{2\pi i/h}$ and $e^{2\pi i(h-1)/h}$. This proves that $m_1=1$ and $m_l=h-1$.

The transforms of $\mathbb{R}z_1$ and $\mathbb{R}z_2$ by $\tilde{W}$ are h lines $L_1,\dots,L_h$ of P, and the points of $P\setminus\bigcup_{i=1}^{h}L_i$ are transforms by the elements of $\tilde{W}$ of points of $P\cap C$. Thus, a hyperplane of $\mathcal{H}$ necessarily cuts P along one of the lines L_i, and consequently is a transform by an operation of $\tilde{W}$ of a hyperplane of $\mathcal{H}$ containing $\mathbb{R}z_1$ or $\mathbb{R}z_2$.

Now, any $H\in\mathcal{H}$ that contains $\mathbb{R}z_1$ is one of the hyperplanes $H_1,\dots,H_r$. Indeed, assume otherwise and let e_H be the unit vector orthogonal to H and on the same side of H as C. Then since H_i are the walls of C, we have $(e_H,e_i)\geq 0$ for all i. Thus by Proposition 4.3.16. Then $e_H=\lambda_1e_1+\cdots+\lambda_le_l$ with the $\lambda_i\geq 0$. Now $0=(e_H,z_1)=\lambda_{r+1}a_1+\cdots+\lambda_la_l$, so $\lambda_{r+1}=\cdots=\lambda_l=0$ and $e_H=\lambda_1e_1+\cdots+\lambda_re_r$. Suppose that two of the λ_j were nonzero, for example λ_1 and λ_2; since $e_1,\dots,e_r$ are pairwise orthogonal, we would have

$$s_1(e_H)=-\lambda_1e_1+\lambda_2e_2+\cdots+\lambda_re_r$$

and the coordinates of $s_1(e_H)$ would not be of the same sign, which is absurd (the reflection s_1 preserves chambers of $\mathcal{H}$). Thus, e_H is proportional to one of the vectors $e_1,\dots,e_r$, which proves our assertion. Similarly, any $H\in\mathcal{H}$ that contains $\mathbb{R}z_2$ is one of the hyperplanes $H_{r+1},\dots,H_l$.

The number of elements of $\mathcal{H}$ containing $\mathbb{R}z_1$ or $\mathbb{R}z_2$ is therefore l. If h is even, $|\mathcal{H}|$ thus equal to hl. If h is odd, $|\mathcal{H}|$ is equal to $\frac{h-1}{2}l+r$, and also to $\frac{h-1}{2}l+(l-r)$; hence $r=l-r$, and $|\mathcal{H}|=\frac{h-1}{2}l+\frac{l}{2}=\frac{h}{2}l$. □

Remark 4.6.7. Retain the notation of the preceding proof. Let $w_C \otimes 1$ be the $\mathbb{C}$-linear extension of w_C to $V \otimes_{\mathbb{R}} \mathbb{C}$, and $w_C \otimes 1|_P$ the restriction of $w_C \otimes 1$ to $P \otimes_{\mathbb{R}} \mathbb{C}$. From our study of $w_C|_P$, $w_C \otimes 1|_P$ has an eigenvector v corresponding to the eigenvalue $e^{2\pi i/h}$, and this eigenvector does not belong to any of the sets $L \otimes_{\mathbb{R}} \mathbb{C}$, where L denotes a line of P (since L is not stable under w_C). Now, for any $H \in \mathcal{H}$, we have seen that $H \cap P$ is a line; thus, $v \notin H \otimes_{\mathbb{R}} \mathbb{C}$.

Corollary 4.6.8. *Let $\mathcal{R}$ be the set of unit vectors of V orthogonal to an element of $\mathcal{H}$. If W is irreducible, then*

$$(x,x) = \frac{1}{h} \sum_{e \in \mathcal{R}} (x,e)^2 \tag{4.6.5}$$

for all $x \in V$.

Proof. Put $f(x) = \sum_{e \in \mathcal{H}} (x,e)^2$. It is clear that f is a positive quadratic form invariant under W, and non-degenerate since the e_i form a basis of V. Since W is irreducible, there exists a constant λ such that $f(x) = \lambda(x,x)$ (Proposition 4.2.2). If $(x_i)_{1 \le i \le l}$ is an orthonormal basis of V for the scalar product (x,y), then

$$\begin{aligned} \lambda l = \sum_{i=1}^{l} \lambda(x_i,x_i) &= \sum_{i=1}^{l} f(x_i) = \sum_{i=1}^{l} \sum_{e \in \mathcal{H}} (x_i,e)^2 \\ &= \sum_{e \in \mathcal{H}} \sum_{i=1}^{l} (x_i,e)^2 = \sum_{e \in \mathcal{H}} 1 = |\mathcal{R}| = 2|\mathcal{H}| = hl. \end{aligned}$$

Thus $\lambda = h$, which proves (4.6.5). □

Proposition 4.6.9. *If W is irreducible and h is even, the unique element of W that transforms C to $-C$ is $w_C^{h/2}$.*

Proof. We use the notation in the proof of Theorem 4.6.6. Since $w_C|_P$ is a rotation through an angle $2\pi/h$, $w_C^{h/2}$ transforms z_1 to $-z_1$ and z_2 to $-z_2$, and hence $z = z_1 + z_2$ to $-z$. Now $z \in C$, so the chamber $w_C^{h/2}(C)$ is necessarily $-C$. □

Proposition 4.6.10. *Assume that W is irreducible. Let $\alpha_1, \dots, \alpha_l$ be homogeneous elements of the symmetric algebra $S = \mathcal{S}(V)$, algebraically independent over $\mathbb{R}$ and generating the algebra R of elements of S invariant under W. If p_j is the degree of α_j, then the exponents of W are $p_1 - 1, \dots, p_l - 1$.*

Proof. Put $V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$, $S_{\mathbb{C}} = \mathcal{S}(V_{\mathbb{C}}) = S \otimes_{\mathbb{R}} \mathbb{C}$ and extend the scalar product on V to a hermitian form on $V_{\mathbb{C}}$. If w_C is a Coxeter transformation of W, there exists an orthonormal basis $(x_i)_{1 \le i \le l}$ of $V_{\mathbb{C}}$ consisting of eigenvectors of $w_C \otimes 1$ (i.e., $w_C \otimes 1$ is diagonalizable); moreover, we can assume that, for $1 \le j \le l$, x_j corresponds to the eigenvalue $e^{2\pi i m_j/h}$ of $w_C \otimes 1$. It is clear that $S_{\mathbb{C}}$ can be identified with the algebra $\mathbb{C}[x_1, \dots, x_l]$, and that we can write $\alpha_j \otimes 1 = f_j(x_1, \dots, x_d)$ where f_j is a homogeneous polynomial of degree p_j in $\mathbb{C}[x_1, \dots, x_l]$. Put $D_j = \partial/\partial x_j$ and $J = \det(D_k f_j)$. Recall (Proposition 4.5.20) that $J(x_1, \dots, x_l)$ is proportional to the product in $S_{\mathbb{C}}$ of $|\mathcal{H}|$ vectors y_k of V, each of which is orthogonal to a hyperplane of $\mathcal{H}$. Since we can assume that $x_1 \notin H \otimes \mathbb{C}$ for all $H \in \mathcal{H}$ (Remark 4.6.7), the x_1 component of each of the vectors y_k is nonzero, so $J(1,0,\dots,0) \neq 0$. The rule for expanding determinants now proves the existence of a permutation $\sigma \in \mathfrak{S}_l$ such that $(D_{\sigma(j)} f_j)(1,0,\dots,0) \neq 0$ for all j. Since $D_{\sigma(j)} f_j$ is homogeneous of degree $p_j - 1$, the coefficient of $x_1^{p_j-1} x_{\sigma(j)}$ in $f_j(x_1, \dots, x_l)$ is non-zero. Now $f_j(x_1, \dots, x_l)$ is invariant under $w_C \otimes 1$, and

$$(w_C \otimes 1)(x_1^{p_j-1} x_{\sigma(j)}) = \left(e^{\frac{2\pi i (p_j - 1 + m_{\sigma(j)})}{h}}\right) x_1^{p_j-1} x_{\sigma(j)}.$$

This proves that $p_j - 1 + m_{\sigma(j)} \equiv 0 \bmod h$. Now $h - m_{\sigma(j)}$ is an exponent by (4.6.2). Permuting the α_j if necessary, we can assume that $p_j - 1 \equiv m_j \bmod h$ for all j. Since $p_j - 1 \ge 0$ and $m_j < h$, we have $p_j - 1 = m_j + \mu_j h$ with μ_j a non-negative integer. By Proposition 4.5.16, we see that

$$|\mathcal{H}| = \sum_{j=1}^{l} (p_j - 1) = \sum_{j=1}^{l} m_j + h \sum_{j=1}^{l} \mu_j.$$

Taking into account formula (4.6.3) and Theorem 4.6.6(b), we obtain $h \sum_{j=1}^{l} \mu_j = 0$, so $\mu_j = 0$ for all j, and finally $p_j - 1 = m_j$ for all j. □

Corollary 4.6.11. *If $(m_j)_{1\leq i\leq l}$ is the increasing sequence of exponents of W, the order of W is equal to $\prod_{j=1}^{l}(m_j+1)$.*

Proof. This follows from the relations $p_j - 1 = m_i$ and Corollary 4.5.12. $\square$

Corollary 4.6.12. *If w_C is a Coxeter transformation of W, then $e^{2\pi i/h}$ and $e^{-2\pi i/h}$ are eigenvalues of w_C of multiplicity* 1*.*

Proof. If $e^{2\pi i/h}$ has multiplicity bigger than 1, then there would exist two non-proportional homogeneous invariants of degree 2 in S, and hence two non-proportional quadratic forms on V^* invariant under W, contrary to Proposition 4.2.2. Since the multiplicity of $e^{2\pi i/h}$ and $e^{-2\pi i/h}$ are the same, this implies $e^{-2\pi i/h}$ also has multiplicity 1. $\square$

Corollary 4.6.13. *The homothety with ratio -1 of V belongs to W if and only if all the exponents of W are odd. In that case, h is even and $w_C^{h/2} = -1$ for any Cometer transformation of W.*

Proof. The first assertion follows from Proposition 4.5.19. Assume that the exponents of W are odd. Then h is even by formula (4.6.2), and

$$(e^{\frac{2\pi i m_j}{h}})^{h/2} = e^{\pi i m_j} = -1.$$

thus $w_C^{h/2} = -1$ since w_C is a semi-simple automorphism of V. $\square$

Chapter 5

Root systems and Weyl groups

5.1 Abstract root systems

In this section, $\mathbb{K}$ denotes a field of characteristic zero and V denotes a vector space over $\mathbb{K}$ of finite dimension.

5.1.1 Root systems in a vector space

Lemma 5.1.1. *Let V be a vector space over $\mathbb{K}$, Φ a finite subset of V generating V. For any $\alpha \in \Phi$ such that $\alpha \neq 0$, there exists at most one reflection s of V such that $s(\alpha) = -\alpha$ and $s(\Phi) = \Phi$.*

Proof. Let G be the group of automorphisms of V leaving Φ stable. Since Φ generates V, G is isomorphic to a subgroup of the symmetric group of Φ, and hence is finite. Let $s, \tilde{s}$ be reflections of V such that $s(\alpha) = \tilde{s}(\alpha) = -\alpha$ and $s, \tilde{s} \in G$. Then $w = s\tilde{s}$ belongs to G, and hence is of finite order m. On the other hand, since s and $\tilde{s}$ are reflections for α, we have $w(x) \equiv x \bmod \mathbb{K}\alpha$, so there exists a linear form f on V such that

$$w(x) = x + f(x)\alpha \quad \text{for} \quad x \in V$$

and $f(\alpha) = 0$. By induction on n, it follows that

$$w^n(x) = x + nf(x)\alpha \quad \text{for} \quad x \in V.$$

Taking n equal to m, we see that $mf(m) = 0$ for all $x \in V$, so $f = 0$. This shows $w = 1$ and therefore $s = \tilde{s}$. $\square$

Let V be a vector space over $\mathbb{K}$ and Φ a subset of V. Then Φ is said to be a **root system** in V if the following conditions are satisfied:

(R1) Φ is finite, does not contain 0, and generates V.

(R2) For all $\alpha \in \Phi$, there exists an element $\check{\alpha}$ of the dual V^* of V such that $\langle \check{\alpha}, \alpha \rangle = 2$ and that the reflection $s_{\alpha,\check{\alpha}}$ leaves Φ stable.

(R3) For all $\alpha \in \Phi$, $\check{\alpha}(\Phi) \subseteq \mathbb{Z}$.

By Lemma 5.1.1, the reflection $s_{\alpha,\check{\alpha}}$ (and hence also the linear form $\check{\alpha}$) is determined uniquely by α, so (R3) makes sense. We put $s_{\alpha,\check{\alpha}} = s_\alpha$. Then $s_\alpha(x) = x - \langle \check{\alpha}, x \rangle \alpha$ for all $x \in V$. The elements of Φ are called the **roots** (of the system considered). The *dimension of* V is called the **rank** of the system.

The automorphisms of V that leave Φ stable are called the **automorphisms of** Φ. They form a finite group denoted by $\mathrm{Aut}(\Phi)$. The subgroup of $\mathrm{Aut}(\Phi)$ generated by the s_α, $\alpha \in \Phi$, is called the **Weyl group of** Φ and is denoted by $W(\Phi)$, or simply by W.

Example 5.1.2. Let Φ be a root system in V. Let $\mathbb{K}'$ be an extension of $\mathbb{K}$. Identify V canonically with a subset of $V \otimes_{\mathbb{K}} \mathbb{K}'$ and V^* with a subset of $V^* \otimes_{\mathbb{K}} \mathbb{K}' = (V \otimes_{\mathbb{K}} \mathbb{K}')^*$. Then Φ is a root system in $V \otimes_{\mathbb{K}} \mathbb{K}'$, and the $\check{\alpha}$ are the same as before.

Example 5.1.3. Let Φ be a root system in V. Let $(\cdot,\cdot)$ be a symmetric non-degenerate bilinear form on V which is invariant under $W(\Phi)$. Identify V with V^* by means of this form. If $\alpha \in \Phi$, then α is non-isotropic and by formula (4.2.3), $\check{\alpha}$ is given by

$$\check{\alpha} = \frac{2\alpha}{(\alpha,\alpha)}.$$

We note that the $\check{\alpha}$ generate V^* in this case (by axiom (R1)), so the group $W(\Phi)$, considered as a subgroup of $\mathrm{GL}(V)$, is essential. Moreover, Corollary 4.3.4 shows that the only reflections belonging to $W(\Phi)$ are the s_α.

Proposition 5.1.4. *Let $V_{\mathbb{Q}}$ (resp. $V_{\mathbb{Q}}^*$) be the $\mathbb{Q}$-vector subspace of V (resp. V^*) generated by the α (resp. the $\check{\alpha}$). Then $V_{\mathbb{Q}}$ (resp. $V_{\mathbb{Q}}^*$) is a $\mathbb{Q}$-structure on V (resp. V^*) and the set Φ is a root system in $V_{\mathbb{Q}}$.*

The restriction to $V_{\mathbb{Q}} \times V_{\mathbb{Q}}^*$ of the canonical bilinear form on $V \times V^*$ gives an identification of each of the spaces $V_{\mathbb{Q}}$, $V_{\mathbb{Q}}^*$ with the dual of the other.

Proof. If $\mathbb{K} = \mathbb{R}$, there exists a scalar product on V invariant under $W(\Phi)$ since the later is finite; Example 5.1.3 now shows that the $\check{\alpha}$ generate V^*. By Example 5.1.2, the $\check{\alpha}$ again generate V^* if $\mathbb{K} = \mathbb{Q}$ (by extending to $\mathbb{R}$). We now go to the general case. Put $E = V_{\mathbb{Q}}$. By (R3), each $\check{\alpha}$ maps E to $\mathbb{Q}$, and so defines an element $\check{\alpha}_{\mathbb{Q}}$ of E^*. It is immediate that Φ is a root system in E, and that the element corresponding to α in E^* is $\check{\alpha}_{\mathbb{Q}}$. By what we said above, the $\check{\alpha}_{\mathbb{Q}}$ generate the vector space E^*. Consider the canonical homomorphism $i : E \otimes_{\mathbb{Q}} \mathbb{K} \to V$, and its transpose $i^t : V^* \to E^* \otimes_{\mathbb{Q}} \mathbb{K}$. Since Φ generates V, i^t is injective; but the image of i^t contains the $\check{\alpha}_{\mathbb{Q}}$, so i^t is surjective. From this we conclude finally that i and i^t are isomorphisms. We can therefore identify V with $E \otimes_{\mathbb{Q}} \mathbb{K}$, V^* with $E^* \otimes_{\mathbb{Q}} \mathbb{K}$, $\check{\alpha}$ with $\check{\alpha}_{\mathbb{Q}} \otimes 1$, and $V_{\mathbb{Q}}^*$ with E^*. Thus, $V_{\mathbb{Q}}$ (resp. $V_{\mathbb{Q}}^*$) is a $\mathbb{Q}$-structure on V (resp. V^*). The restriction to $V_{\mathbb{Q}} \times V_{\mathbb{Q}}^*$ of the canonical bilinear form on $V \times V^*$ can be identified with the canonical bilinear form on $E \times E^*$, hence the proposition. ☐

Proposition 5.1.4 reduces the study of root systems to the case $\mathbb{K} = \mathbb{Q}$. Example 5.1.2 reduces it further to the study of root systems in the real vector space $V_{\mathbb{R}} = V_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{R}$. The Weyl groups associated to these different systems are canonically identified.

Proposition 5.1.5. *Let Φ be a root system in V. Then the set $\check{\Phi} = \{\check{\alpha} : \alpha \in \Phi\}$ forms a root system in V^*, and $\check{\check{\alpha}} = \alpha$ for all $\alpha \in \Phi$.*

Proof. The set $\check{\Phi}$ is finite and does not contain 0. We first show that it generates V^*. Since by Proposition 5.1.4, $V_{\mathbb{Q}}$ (resp. $V_{\mathbb{Q}}^*$) is a $\mathbb{Q}$-structure on V (resp. V^*), it suffices to reduces the case for $\mathbb{K} = \mathbb{Q}$, which is shown in the proof of Proposition 5.1.4. Thus $\check{\Phi}$ satisfies axiom (R1).

Since $s_{\alpha,\check{\alpha}}$ is an automorphism of the vector space V equipped with the subset Φ, $(s_{\alpha,\check{\alpha}})^{-t}$ leaves the set $\check{\Phi}$ stable; but $(s_{\alpha,\check{\alpha}})^{-t} = s_{\check{\alpha},\alpha}$, which proves that $\check{\Phi}$ satisfies (R2) and that $\check{\check{\alpha}} = \alpha$. Finally, $\langle \check{\alpha}, \beta \rangle \in \mathbb{Z}$ for all $\check{\alpha} \in \check{\Phi}$ and $\beta \in \Phi$, so $\check{\Phi}$ satisfies (R3). ☐

The set $\check{\Phi}$ is called the coroot system of Φ. The map $\alpha \mapsto \check{\alpha}$ is a bijection from Φ to $\check{\Phi}$, called the canonical bijection from Φ to $\check{\Phi}$. Note that, if α, β are elements of Φ such that $\alpha + \beta \in \Phi$, then $(\alpha+\beta)\check{} \neq \check{\alpha} + \check{\beta}$ in general. That is, the map $\alpha \mapsto \check{\alpha}$ is not linear.

Since $s_\alpha(\alpha) = -\alpha$, axiom (R2) shows that $-\Phi = \Phi$. Evidently $(-\alpha)\check{} = -\check{\alpha}$ and $-1 \in \mathrm{Aut}(\Phi)$ (note that it is *not* always true that $-1 \in W(\Phi)$) The equality $(s_{\alpha,\check{\alpha}})^{-t} = s_{\check{\alpha},\alpha}$ shows that the map $u \mapsto u^{-t}$ is an isomorphism from the group $W(\Phi)$ to the group $W(\check{\Phi})$. We identify these two groups by means of this isomorphism; in other words, we consider $W(\Phi)$ as acting both in V and in V^*. Similarly for $\mathrm{Aut}(\Phi)$.

Proposition 5.1.6. *Let Φ be a root system in V. For $x, y \in V$, put*

$$(x,y) = \sum_{\alpha \in \Phi} \langle \check{\alpha}, x \rangle \langle \check{\alpha}, y \rangle.$$

Then $(\cdot,\cdot)$ is a non-degenerate symmetric bilinear form on V that is invariant under $\mathrm{Aut}(\Phi)$. Moreover, for $x, y \in V_{\mathbb{Q}}$ we have $(x,y) \in \mathbb{Q}$, and the canonical extension of $(\cdot,\cdot)$ to $V_{\mathbb{R}}$ is also non-degenerate and positive.

Proof. It is clear that $(\cdot,\cdot)$ is a symmetric bilinear form on V. If $\tau \in \mathrm{Aut}(\Phi)$,

$$(\tau(x),\tau(y)) = \sum_{\alpha \in \Phi} \langle \check{\alpha}, \tau(x) \rangle \langle \check{\alpha}, \tau(y) \rangle = \sum_{\alpha \in \Phi} \langle \tau^t(\check{\alpha}), x \rangle \langle \tau^t(\check{\alpha}), y \rangle = (x,y)$$

since $(\tau^t)(\check{\Phi}) = \check{\Phi}$. If $x, y \in V_{\mathbb{Q}}$ then $(x,y) \in \mathbb{Q}$ by (R3). If $z \in V_{\mathbb{R}}$, then $(z,z) = \sum_{\alpha\in\Phi}\langle\check{\alpha}, z\rangle \geq 0$, and $(z,z) > 0$ if $z \neq 0$ by Proposition 5.1.5, so the canonical extension of $(\cdot\,,\cdot)$ to $V_{\mathbb{R}}$ is positive and non-degenerate. The restriction of $(\cdot\,,\cdot)$ to $V_{\mathbb{Q}}$ is thus non-degenerate, and hence the form on V is non-degenerate. □

Proposition 5.1.7. *Let X be a nonempty subset of Φ, let V_X be the vector subspace of V generated by X, and V_X^* the vector subspace of V^* generated by the $\check{\alpha}$, where $\alpha \in X$. Then*

(a) V is the direct sum of V_X and the orthogonal complement of V_X, V^* is the direct sum of V_X^* and the orthogonal complement of V_X, and V_X^* is identified with the dual of V_X^*;

(b) $\Phi \cap V_X$ is a root system in V_X, and the canonical bijection from $\Phi \cap V_X$ to its coroot system is identified with the restriction of the map $\alpha \mapsto \check{\alpha}$ to $\Phi \cap V_X$.

Proof. By Proposition 5.1.4, we can assume that $\mathbb{K} = \mathbb{R}$. Identify V with V^* by means of the symmetric bilinear form of Proposition 5.1.6. We have $\check{\alpha} = 2\alpha/(\alpha,\alpha)$ for all $\alpha \in \Phi$. Every vector subspace of V is non-isotropic, and the proposition is now clear. □

Corollary 5.1.8. *Let U be a vector subspace of V such that $\Phi \cap U \neq \varnothing$, and V_U be the vector subspace generated by $\Phi \cap U$. Then $\Phi \cap U$ is a root system in V_U.*

Proof. This follows from Proposition 5.1.7 applied to $X = \Phi \cap U$, which is nonempty by hypothesis. □

Let Φ be a root system in V. For $\alpha, \beta \in \Phi$, put

$$n(\alpha,\beta) = \langle\alpha, \check{\beta}\rangle. \tag{5.1.1}$$

Then by (R2) and (R3), we have

$$\begin{cases} n(\alpha,\alpha) = 2, \\ n(\alpha,\beta) \in \mathbb{Z}, \\ n(-\alpha,\beta) = n(\alpha,-\beta) = -n(\alpha,\beta). \end{cases} \tag{5.1.2}$$

By the definition of $n(\alpha,\beta)$, we can write

$$s_\alpha(\beta) = \alpha - n(\beta,\alpha)\alpha. \tag{5.1.3}$$

And by Proposition 5.1.5, we have

$$n(\alpha,\beta) = n(\check{\beta},\check{\alpha}). \tag{5.1.4}$$

Let $(\cdot,\cdot)$ be a symmetric bilinear form on V, non-degenerate and invariant under $W(\Phi)$. By Example 5.1.3,

$$n(\alpha,\beta) = \frac{2(\alpha,\beta)}{(\beta,\beta)}. \tag{5.1.5}$$

It then follows that $n(\alpha,\beta) = 0$ if and only if $(\beta,\alpha) = 0$, if and only if s_α commutes with s_β. Also, if $n(\alpha,\beta) \neq 0$, then

$$\frac{n(\alpha,\beta)}{n(\beta,\alpha)} = \frac{(\alpha,\alpha)}{(\beta,\beta)}. \tag{5.1.6}$$

Proposition 5.1.9. *Let Φ be a root system in V. If $\tau \in \mathrm{Aut}(\Phi)$, then*

(a) $\tau s_\alpha \tau^{-1} = s_{\tau(\alpha)}$ for all $\alpha \in \Phi$.

(b) $n(\beta,\alpha) = n(\tau(\beta),\tau(\alpha))$ for all $\alpha,\beta \in \Phi$.

Proof. We have $\tau(\Phi) = \Phi$. Hence we also have $\tau s_\alpha \tau^{-1}(\Phi) \subseteq \Phi$. Further, $\tau s_\alpha \tau^{-1}$ keeps the hyperplane $\tau(\alpha^\perp)$ pointwise fixed, and it maps $\tau(\alpha)$ to $-\tau(\alpha)$. Hence by linearity we get $\tau s_\alpha \tau^{-1} = s_{\tau(\alpha)}$. Now (b) follows from the equations

$$\tau s_\alpha \tau^{-1}(\tau(\beta)) = \tau(\beta - n(\beta,\alpha)\alpha) = \tau(\beta) - n(\beta,\alpha)\tau(\alpha),$$

and

$$s_{\tau(\alpha)}(\tau(\beta)) = \tau(\beta) - n(\tau(\beta),\tau(\alpha))\tau(\alpha).$$

This completes the proof. □

Remark 5.1.10. Note that in the proof of Proposition 5.1.9(b), we do not use the hypothesis that $\beta \in \Phi$. In fact, if $\tau \in \mathrm{Aut}(\Phi)$, then for any $x \in V$, $\alpha \in \Phi$, we have

$$n(x, \alpha) = n(\tau(x), \tau(\alpha)).$$

This observation will be useful when we define weights of the root system Φ.

5.1.2 Irreducibility of root systems

Let $V = \bigoplus_{i=1}^{r} V_i$ be a vector space over $\mathbb{K}$ that is the direct sum of a family of vector spaces. Identify V^* with the direct sum of the V^*. For all i, let Φ_i be a root system in V_i. Then $\Phi = \bigcup_i \Phi_i$ is a root system in V whose coroot system is $\check{\Phi} = \bigcup_i \check{\Phi}_i$; the canonical bijection from Φ to $\check{\Phi}$ extends, for all i, the canonical bijection from Φ_i to $\check{\Phi}_i$. The set Φ is called the **direct sum** of the root systems Φ_i. Let $\alpha \in \Phi_i$. If $j \neq i$, the kernel of $\check{\alpha}$ contains V_j, so s_α induces the identity on V_j. On the other hand, $\mathbb{K}\alpha \in V_i$, so s_α leaves V_i stable. These remarks show that $W(\Phi)$ can be identified with $\prod_{i=1}^{r} W(\Phi_i)$.

A root system Φ is said to be **irreducible** if $\Phi \neq \varnothing$ and if Φ is not the direct sum of two non-empty root systems.

Proposition 5.1.11. *Let V be a vector space over $\mathbb{K}$ that is the direct sum of vector spaces $V_1, \dots, V_r$. Let Φ be a root system in V and put $\Phi_i = \Phi \cap V_i$. Then the following three conditions are equivalent:*

(i) *the V_i are stable under $W(\Phi)$;*

(ii) *$\Phi \subseteq V_1 \cup \cdots \cup V_r$;*

(iii) *for all i, Φ_i is a root system in V_i, and Φ is the direct sum of the Φ_i.*

Proof. Clearly (iii)⇒(i). Assume that the V_i are stable under $W(\Phi)$. Let $\alpha \in \Phi$ and let H be the kernel of $\check{\alpha}$. By Proposition 4.2.3, each V_i is the sum of a subspace of H and a subspace of $\mathbb{K}\alpha$. Since $V = \bigoplus_i V_i$, one of the V_i contains $\mathbb{K}\alpha$, so $\Phi \subseteq V_1 \cup \cdots \cup V_r$. This shows (i)⇒(ii).

If condition (ii) is satisfied, then by axiom (R1), Φ_i is nonempty for each i and generates V_i, so it is a root system in V_i by Proposition 5.1.7. Since $\Phi = \bigcup_i \Phi_i$, it is the direct sum of the Φ_i. □

Corollary 5.1.12. *Let Φ be a root system in V. Then the following conditions are equivalent:*

(i) *Φ is irreducible;*

(ii) *the $W(\Phi)$-module V is simple;*

(iii) *the $W(\Phi)$-module V is absolutely simple;*

Proof. This follows from Proposition 5.1.11 and Proposition 4.2.2. □

Proposition 5.1.13. *Every root system Φ in V is the direct sum of a family $(\Phi_i)_{i \in I}$ of irreducible root systems that is unique up to a bijection of the index set.*

Proof. The existence of the Φ_i is proved by induction on $|\Phi|$: if Φ is nonempty and not irreducible, Φ is the direct sum of two root systems Φ', Φ'' such that $|\Phi'| < |\Phi|$, $|\Phi''| < |\Phi|$, and the induction hypothesis applies to Φ' and Φ''. To prove uniqueness, it suffices to prove that, if Φ is the direct sum of Φ' and Φ'', every Φ_i is necessarily contained either in Φ' or in Φ''. Let V', V'' and V_i', V_i'' be the vector subspaces of V generated by Φ', Φ'' and $\Phi' \cap \Phi_i, \Phi'' \cap \Phi_i$. Since the sum $V' + V''$ is direct, the sum $V_i' + V_i''$ is direct. Since $\Phi_i \subseteq \Phi' \cup \Phi''$, Φ_i is the direct sum of the root systems $\Phi'' \cap \Phi_i$ and $\Phi'' \cap \Phi_i$, hence either $\Phi' \cap \Phi_i = \varnothing$ or $\Phi'' \cap \Phi_i = \varnothing$, which proves the assertion. □

The Φ_i in Proposition 5.1.13 are called the irreducible components of Φ. Note that for any non-zero scalars λ_i, the union of the $\lambda_i \Phi_i$ is a root system in V, whose coroot system is the union of the $\lambda_i \check{\Phi}_i$, and whose Weyl group is $W(\Phi)$.

Proposition 5.1.14. *Let Φ be a root system in V, (Φ_i) the family of its irreducible components, V_i the vector subspace of V generated by Φ_i, β the invariant symmetric bilinear form on V induced by Φ, and $\tilde{\beta}$ a symmetric bilinear form on V invariant under $W(\Phi)$. Then the V_i are pairwise orthogonal with respect to $\tilde{\beta}$, and, for all i, the restrictions of β and $\tilde{\beta}$ to V_i are proportional.*

Proof. If $v_i \in V_i$, $v_j \in V_j$, $i \neq j$ and if $w \in W(\Phi_j)$, then

$$\tilde{\beta}(v_i, w(v_j)) = \tilde{\beta}(v_i, v_j).$$

which shows that $w(v_j) - v_j$ is orthogonal to v_i with respect to $\tilde{\beta}$. Since V_j is irreducible for $W(\Phi_j)$, it is generated by the $w(v_j) - v_j$, and it is therefore orthogonal to V_j. The fact that the restrictions of β and $\tilde{\beta}$ to each of the V_i are proportional follows from Proposition 4.2.2. $\square$

Let Φ be a root system in V. We have seen (Proposition 5.1.6) that the symmetric bilinear form

$$(x, y) \mapsto \beta_\Phi(x, y) = \sum_{\alpha \in \Phi} \langle \check{\alpha}, x \rangle \langle \check{\alpha}, y \rangle$$

on V is non-degenerate and invariant under $\mathrm{Aut}(\Phi)$. Interchanging the roles of Φ and $\check{\Phi}$, it follows that the symmetric bilinear form

$$(x^*, y^*) \mapsto \beta_{\check{\Phi}}(x^*, y^*) = \sum_{\alpha \in \Phi} \langle \alpha, x^* \rangle \langle \alpha, y^* \rangle$$

on V^* is non-degenerate and invariant under $\mathrm{Aut}(\Phi)$. Now, the bilinear form $\beta_{\check{\Phi}}$ on V^* determines an isomorphism $\sigma : V \to V^*$, and we have an inverse form on V induced by $\beta_{\check{\Phi}}$, which is defined by (for $x, y \in V$)

$$(x, y)_\Phi = \beta_{\check{\Phi}}(\sigma(x), \sigma(y)) = \sum_{\alpha \in \Phi} \langle \alpha, \sigma(x) \rangle \langle \beta, \sigma(y) \rangle. \tag{5.1.7}$$

Similarly, the bilinear form β_Φ on V determines an isomorphism $\tau : V^* \to V$, and we get an inverse form on V^* induced by β_Φ, given by (for $x^*, y^* \in V^*$)

$$(x^*, y^*)_{\check{\Phi}} = \beta_\Phi(\tau(x^*), \tau(y^*)) = \sum_{\alpha \in \Phi} \langle \check{\alpha}, \tau(x^*) \rangle \langle \check{\beta}, \tau(y^*) \rangle. \tag{5.1.8}$$

The forms $(\cdot, \cdot)_\Phi$ and $(\cdot, \cdot)_{\check{\Phi}}$ will be called the **canonical bilinear form** on V and V^*, respectively. It is clear that they are non-degenerate and invariant under $\mathrm{Aut}(\Phi)$.

By definition of σ, we have $\langle \alpha, \sigma(x) \rangle = \beta_{\check{\Phi}}(\sigma(\alpha), \sigma(x)) = (\alpha, x)_\Phi$, so

$$(x, y)_\Phi = \beta_{\check{\Phi}}(\sigma(x), \sigma(y)) = \sum_{\alpha \in \Phi} (\alpha, x)_\Phi (\alpha, y)_\Phi. \tag{5.1.9}$$

In view of Proposition 5.1.14, $(\cdot, \cdot)_\Phi$ is the only non-zero symmetric bilinear form invariant under $W(\Phi)$ which satisfies the identity (5.1.9). For $\beta \in \Phi$, (5.1.9) gives

$$(\beta, \beta)_\Phi = \sum_{\alpha \in \Phi} (\alpha, \beta)_\Phi^2 = \frac{1}{4} (\beta, \beta)_\Phi^2 \sum_{\alpha \in \Phi} n(\alpha, \beta)^2$$

where we note that the number $n(\alpha, \beta)$ is independent of the choice of the bilinear form. Therefore,

$$4(\beta, \beta)_\Phi^{-1} = \sum_{\alpha \in \Phi} n(\alpha, \beta)^2.$$

Moreover, by Example 5.1.3 we have, for $x, y \in V$, that

$$\beta_\Phi(x, y) = \sum_{\alpha \in \Phi} \left(\frac{2\alpha}{(\alpha, \alpha)_\Phi}, x \right)_\Phi \left(\frac{2\alpha}{(\alpha, \alpha)_\Phi}, y \right)_\Phi = 4 \sum_{\alpha \in \Phi} (\alpha, x)_\Phi (\alpha, y)_\Phi (\alpha, \alpha)_\Phi^{-2}.$$

and similarly for $x^*, y^* \in V^*$,

$$\beta_{\check{\Phi}}(x^*, y^*) = \sum_{\alpha \in \Phi} \left(\frac{2\check{\alpha}}{(\check{\alpha}, \check{\alpha})_{\check{\Phi}}}, x^* \right)_{\check{\Phi}} \left(\frac{2\check{\alpha}}{(\check{\alpha}, \check{\alpha})_{\check{\Phi}}}, y^* \right)_{\check{\Phi}} = 4 \sum_{\alpha \in \Phi} (\check{\alpha}, x^*)_{\check{\Phi}} (\check{\alpha}, y)_{\check{\Phi}} (\check{\alpha}, \check{\alpha})_{\check{\Phi}}^{-2}.$$

It follows that, if Φ is irreducible, there exists a constant $\gamma(\Phi) > 0$ and a constant $\gamma(\check{\Phi}) > 0$ such that

$$\sum_{\alpha \in \Phi} (\alpha, x)_\Phi (\alpha, y)_\Phi (\alpha, \alpha)_\Phi^{-2} = \gamma(\Phi)(x, y)_\Phi. \tag{5.1.10}$$

By the definition of $\gamma(\Phi)$, we have $\beta_\Phi(x, y) = 4\gamma(\Phi)(x, y)_\Phi$. Similarly, there is a constant $\gamma(\check{\Phi})$ such that $\beta_{\check{\Phi}}(x^*, y^*) = 4\gamma(\check{\Phi})(x^*, y^*)_{\check{\Phi}}$. We claim that $\gamma(\Phi) = \gamma(\check{\Phi})$. For this, we need the following lemma.

Lemma 5.1.15. *Let $(\cdot,\cdot)_1$ and $(\cdot,\cdot)_2$ be two inner products on V such that*

$$(x,y)_1 = \lambda(x,y)_2 \quad \text{for} \quad x,y \in V.$$

Taking the inverse inner product on V^*, then

$$(x^*,y^*)_1 = \lambda^{-1}(x^*,y^*)_2 \quad \text{for} \quad x^*,y^* \in V.$$

Proof. Let σ_1, σ_2 be the isomorphisms of V to V^* given by $(\cdot,\cdot)_1$ and $(\cdot,\cdot)_2$. Then for $x,y \in V$,

$$\langle \sigma_1(x), y\rangle = (x,y)_1 = \lambda(x,y)_2 = \lambda\langle \sigma_2(x), y\rangle$$

whence $\sigma_1 = \lambda\sigma_2$. Now for $x^*, y^* \in V^*$, we have

$$\begin{aligned}(x^*,y^*)_1 &= (\sigma_1^{-1}(x^*), \sigma_1^{-1}(y^*))_1 = (\lambda^{-1}\sigma_2^{-1}(x^*), \lambda^{-1}\sigma_2^{-1}(y^*))_1 \\ &= \lambda^{-1}(\sigma_2(x^*), \sigma_2(y^*)) = \lambda^{-1}(x^*,y^*)_2.\end{aligned}$$

This completes the proof. □

Now we have $\beta_\Phi = 4\gamma(\Phi)(\cdot,\cdot)_\Phi$. Taking inverse bilinear form on V^* and note that the inverse bilinear form of $(\cdot,\cdot)_\Phi$ is $\beta_{\check{\Phi}}$, we conclude by Lemma 5.1.15 that

$$(\cdot,\cdot)_{\check{\Phi}} = (4\gamma(\Phi))^{-1}\beta_{\check{\Phi}}.$$

Compare this with the definition of $\gamma(\check{\Phi})$, we conclude that $\gamma(\Phi) = \gamma(\check{\Phi})$.

In view of this result, for $\beta \in \Phi$, we observe

$$(\check{\beta},\check{\beta})_{\check{\Phi}} = (4\gamma(\Phi))^{-1}\sum_{\alpha\in\Phi}\langle\check{\beta},\alpha\rangle = \gamma(\Phi)^{-1}\sum_{\alpha\in\Phi}\frac{(\alpha,\beta)_\Phi^2}{(\beta,\beta)_\Phi^2},$$

so by (5.1.9),

$$(\check{\beta},\check{\beta})_{\check{\Phi}}(\beta,\beta)_\Phi = \gamma(\Phi)^{-1}.$$

5.1.3 Relation between two roots

Choose a scalar product on $V_\mathbb{R}$ invariant under $W(\Phi)$. It is then possible to speak of the **length** of a root and the **angle** between two roots. Proposition 5.1.14 shows that this angle is independent of the choice of scalar product, as is the ratio of the lengths of two roots, provided they belong to the same irreducible component of Φ.

From now on, we assume that $\mathbb{K} = \mathbb{R}$, so that we can consider the length and angles of roots in Φ. Throughout the following, Φ denotes a root system in a vector space V; and V is equipped with a scalar product $(\cdot,\cdot)$ invariant under $W(\Phi)$.

Let $\alpha, \beta \in \Phi$. By formula (5.1.3),

$$n(\alpha,\beta)n(\beta,\alpha) = 4\cos^2\widehat{(\alpha,\beta)} \le 4.$$

Thus, the integer $n(\alpha,\beta)n(\beta,\alpha)$ must take one of the values $0,1,2,3,4$. In view of Corollary 4.2.13, we see that the only possibilities are the following, up to interchanging α and β: In particular, we get the

$n(\alpha,\beta)$	0	1	-1	1	-1	1	-1	1	-1	2	-2
$n(\beta,\alpha)$	0	1	-1	2	-2	3	-3	4	-4	2	-2
θ	$\frac{\pi}{2}$	$\frac{\pi}{3}$	$\frac{2\pi}{3}$	$\frac{\pi}{4}$	$\frac{3\pi}{4}$	$\frac{\pi}{6}$	$\frac{5\pi}{6}$	0	π	0	π
$\frac{\lVert\beta\rVert}{\lVert\alpha\rVert}$	arb.	1	1	$\sqrt{2}$	$\sqrt{2}$	$\sqrt{3}$	$\sqrt{3}$	2	2	1	1

Table 5.1: Possible angles in a root system.

following proposition:

Theorem 5.1.16. *Let Φ be a root system in V.*

(a) If two roots are proportional, the factor of proportionality can only be $\pm 1, \pm\frac{1}{2}, \pm 2$.

(b) If α and β are two non-proportional roots, and if $\|\alpha\| \leq \|\beta\|$, then $n(\alpha, \beta)$ takes one of the values $0, \pm 1$.

Proposition 5.1.17. *Let α, β be two roots.*

(a) If $n(\alpha, \beta) > 0$, $\alpha - \beta$ is a root unless $\alpha = \beta$.

(b) If $n(\alpha, \beta) < 0$, $\alpha + \beta$ is a root unless $\alpha = -\beta$.

Proof. If $n(\alpha, \beta) > 0$, the possibilities, by the table above, are the following:

(i) $n(\alpha, \beta) = 1$, and $\alpha - \beta = s_\beta(\alpha) \in \Phi$;

(ii) $n(\beta, \alpha) = 1$, and $\beta - \alpha = s_\alpha(\beta) \in \Phi$;

(iii) $\beta = \alpha$.

This proves (a), and (b) follows by changing β to $-\beta$. □

Corollary 5.1.18. *Let α and β be two roots.*

(a) If $(\alpha, \beta) > 0$, $\alpha - \beta$ is a root unless $\alpha = \beta$;

(b) If $(\alpha, \beta) < 0$, $\alpha + \beta$ is a root unless $\alpha = -\beta$;

(c) If $\alpha - \beta \notin \Phi \cup \{0\}$ and $\alpha + \beta \notin \Phi \cup \{0\}$, then $(\alpha, \beta) = 0$.

Proof. Assertions (a) and (b) follow from Theorem 5.1.16 and formula (5.1.5). Assertion (c) follows from (a) and (b). □

It is possible that $\alpha + \beta \in \Phi$ but $(\alpha, \beta) = 0$. When $\alpha - \beta \notin \Phi \cup \{0\}$ and $\alpha + \beta \notin \Phi \cup \{0\}$, α and β are said to be **strongly orthogonal**.

Proposition 5.1.19. *Let α and β be two non-proportional roots.*

(a) The set I of integers j such that $\beta + j\alpha \in \Phi$ is a root is an interval $[-p, q]$ in $\mathbb{Z}$ containing 0.

(b) Let S be the set of $\beta + j\alpha$ for $j \in I$. Then,

$$s_\alpha(S) = S \quad \textit{and} \quad s_\alpha(\beta + q\alpha) = \beta - p\alpha.$$

(c) $n(\beta, \alpha) = p - q$.

Proof. Clearly $0 \in I$. If for some $-p < k < q$ we have $\beta + k\alpha \notin \Phi$, then we can find $i < j \in [-p, q] \cap \mathbb{Z}$ such that

$$\beta + i\alpha \in \Phi, \quad \beta + (i+1)\alpha \notin \Phi, \quad \beta + (j-1)\alpha \notin \Phi, \quad \beta + j\alpha \in \Phi.$$

But then Corollary 5.1.18 implies both $(\beta + i\alpha, \alpha) \geq 0$, $(\beta + j\alpha, \alpha) \leq 0$. Since $i < j$ and $(\alpha, \alpha) > 0$, this is absurd.

We have

$$s_\alpha(\beta + j\alpha) = \beta - n(\beta, \alpha)\alpha - j\alpha = \beta + j'\alpha$$

with $j' = -j - n(\beta, \alpha)$. Thus $s_\alpha(S) \subseteq S$ and consequently $s_\alpha(S) = S$. Now $j \mapsto -j - n(\beta, \alpha)$ is a decreasing bijection from I to I. It follows that, $j' = -p$ when $j = p$, so that $-p = -q - n(\beta, \alpha)$. This proves (b) and (c). □

The set S is called the **α-chain** of roots defined by β; $\beta - p\alpha$ is its origin, $\beta + q\alpha$ is its end, and $p + q$ is its length.

Corollary 5.1.20. *Let S be an α-chain of roots, and γ the origin of S. Then the length of S is $-n(\gamma, \alpha)$; it is equal to $0, 1, 2$ or 3.*

Proof. The first assertion follows from Proposition 5.1.19(c), applied to $\beta = \gamma$, and using the fact that $p = 0$. On the other hand, since γ is not proportional to α, Table 5.1 shows that $|n(\gamma, \alpha)| \leq 3$, hence the corollary. □

Proposition 5.1.21. *Let α, β be two non-proportional roots such that $\beta+\alpha$ is a root. Let p, q be the integers in Proposition 5.1.19. Then*

$$\frac{(\beta+\alpha, \beta+\alpha)}{(\beta, \beta)} = \frac{p+1}{q}.$$

Proof. Let S be the a-chain defined by α, γ its origin; its length ℓ is ≥ 1 since $\beta+\alpha$ is a root. By Table 5.1, the following cases are possible:

ℓ	1	2	2	3	3	3
β	γ	γ	$\gamma+\alpha$	γ	$\gamma+\alpha$	$\gamma+2\alpha$
p	0	0	1	0	1	2
q	1	2	1	3	2	1
$\frac{\|\beta+\alpha\|^2}{\|\beta\|^2}$	1	$\frac{1}{2}$	2	$\frac{1}{3}$	1	3

In each case, the formula to be proved is satisfied. □

Proposition 5.1.22. *Assume that Φ is irreducible. Let α and β be two roots such that $\|\alpha\| = \|\beta\|$. Then there eaists $w \in W(\Phi)$ such that $w(\alpha) = \beta$.*

Proof. The transforms of α by $W(\Phi)$ generate V by Corollary 5.1.12. Hence there exists $w \in W(\Phi)$ such that $(w(\alpha), \beta) \neq 0$, and we can then assume that $(\alpha, \beta) \neq 0$. By formula (5.1.6), $n(\alpha, \beta) = n(\beta, \alpha)$. Replacing β if necessary by $s_\beta(\beta) = -\beta$, we can assume that $n(\alpha, \beta) > 0$. Then, by Table 5.1, either $\alpha = \beta$ (in which case the proposition is clear), or $n(\alpha, \beta) = n(\beta, \alpha) = 1$; in that case

$$s_\alpha s_\beta s_\alpha(\beta) = s_\alpha s_\beta(\beta - \alpha) = s_\alpha(-\beta - \alpha + \beta) = \alpha.$$

This completes the proof. □

If a root $\alpha \in \Phi$ is such that $\pm\frac{1}{2}\alpha \notin \Phi$, then α is called an **indivisible** root. A root system Φ is called **reduced** if every root is indivisible.

Proposition 5.1.23. *Assume that Φ is irreducible and reduced.*

(a) The ratio $(\beta, \beta)/(\alpha, \alpha)$ for $\alpha, \beta \in \Phi$ must take one of the values $1, 2, \frac{1}{2}, 3, \frac{1}{3}$.

(b) The set of the (α, α) for $\alpha \in \Phi$ has at most two elements.

Proof. Since Φ is irreducible, the transforms of a root by $W(\Phi)$ generate V. Hence, for any roots α, β, there exists a root β' such that $(\alpha, \beta') \neq 0$ and $(\beta', \beta') = (\beta, \beta)$. By formula (5.1.5) and Table 5.1 takes one of the values $1, 2, \frac{1}{2}, 3, \frac{1}{3}$ (recall that the system is assumed to be reduced). By multiplying $(\cdot, \cdot)$ by a suitable scalar, we can assume that $(\alpha, \alpha) = 1$ for certain roots and that the other possible values of (β, β) for $\beta \in \Phi$ are 2 and 3. The values 2 and 3 cannot both be attained, since in that case there would exist $\beta, \gamma \in \Phi$ such that $(\beta, \beta)/(\gamma, \gamma) = 3/2$, contrary to what we have seen above. □

Proposition 5.1.24. *Assume that Φ is irreducible, non-reduced and of rank ≥ 2.*

(a) The set Φ_0 of indivisible roots is a root system in V. This system is irreducible and reduced, and $W(\Phi_0) = W(\Phi)$.

(b) Let A be the set of roots α for which (α, α) takes the smallest value λ. Then any two non-proportional elements of A are orthogonal.

(c) Let B be the set of $\beta \in \Phi$ such that $(\beta, \beta) \in 2\lambda$. Then $B \neq \varnothing$, $\Phi_0 = A \cup B$, and $\Phi = A \cup B \cup 2A$.

Proof. If $\alpha \in \Phi \setminus \Phi_0$, then $\frac{1}{2}\alpha \in \Phi$, but $\frac{1}{2}(\frac{1}{2}\alpha) \notin \Phi$ (Theorem 5.1.16), so $\frac{1}{2}\alpha \in \Phi_0$. This proves that Φ_0 satisfies (R1). It is clear that, for all $\alpha \in \Phi$, $s_\alpha(\Phi_0) = \Phi_0$, so Φ_0 satisfies (R2) and (R3). Since $\alpha \in \Phi \setminus \Phi_0$ implies that $\frac{1}{2}\alpha \in \Phi_0$, and since $s_\alpha = s_{\alpha/2}$, we have $W(\Phi) = W(\Phi_0)$. Thus Φ_0 is irreducible (Corollary 5.1.12), and it is evidently reduced.

Since Φ is not reduced, there exists $\alpha \in \Phi_0$ such that $2\alpha \in \Phi$. Since Φ_0 is irreducible and $\dim(V) \geq 2$, α cannot be proportional or orthogonal to every root. Let $\beta \in \Phi_0$ be such that $n(\beta, \alpha) \geq 0$ and β is not proportional to α. Changing β to $-\beta$ if necessary, we can assume that $n(\beta,\alpha) > 0$. Now $\frac{1}{2}n(\beta,\alpha) = n(\beta, 2\alpha) \in \mathbb{Z}$, so $n(\beta,\alpha) \in 2\mathbb{Z}$. From Table 5.1, $n(\beta,\alpha) = 2$ and $(\beta,\beta) = 2(\alpha,\alpha)$. Now since Φ_0 is reduced, Proposition 5.1.23 then shows that, for all $\gamma \in \Phi_0$, either $(\gamma,\gamma) = (\alpha,\alpha)$ or $(\gamma,\gamma) = 2(\alpha,\alpha)$. Also, for any $\gamma \in \Phi \setminus \Phi_0$, the vector $\frac{1}{2}\gamma$ is an element of Φ_0 such that $(\frac{1}{2}\gamma, \frac{1}{2}\gamma) = (\alpha,\alpha)$ or $(\frac{1}{2}\gamma, \frac{1}{2}\gamma) = 2(\alpha,\alpha)$. In the later case, we have $(\gamma,\gamma) = 8(\alpha,\alpha)$, which is excluded by Table 5.1. It then follows that $\lambda = (\alpha,\alpha)$, $B \neq \varnothing$, $\Phi_0 = A \cup B$, and $\Phi \subseteq A \cup B \cup 2A$. On the other hand, if $\gamma \in A$, there exists $w \in W(\Phi)$ such that $\gamma = w(\alpha)$ (Proposition 5.1.22), so $2\gamma = w(2\alpha) \in \Phi$; thus $2A \subseteq \Phi$ and $\Phi = A \cup B \cup 2A$.

Finally, let $\gamma, \tilde{\gamma}$ be two non-proportional elements of A. Then

$$n(2\gamma, \tilde{\gamma}) = 2n(\gamma, \tilde{\gamma}) = 4n(\gamma, 2\tilde{\gamma}) \in 4\mathbb{Z}$$

and therefore $|n(\gamma, \tilde{\gamma})| \leq 1$. Since γ and $\tilde{\gamma}$ have the same length, by Table 5.1 we conclude that $n(\gamma, \tilde{\gamma}) = 0$ and therefore $(\gamma, \tilde{\gamma}) = 0$. □

Proposition 5.1.25. *Assume that Φ is irreducible and reduced, and that (α,α) takes the values λ and 2λ for $\alpha \in \Phi$. Let A be the set of roots α such that $(\alpha,\alpha) = \lambda$. Assume that any two non-proportional elements of A are orthogonal. Then $\Phi_1 = \Phi \cup 2A$ is an irreducible non-reduced root system and Φ is the set of indivisible roots of Φ_1.*

Proof. It is clear that Φ_1 satisfies (R1) and (R3). We show that, if $\alpha, \beta \in \Phi_1$, then $n(\beta,\alpha) = \langle \check{\alpha}, \beta \in \mathbb{Z}$. This is clear if $\alpha \in \Phi$. Since $(2\alpha)^\vee = \frac{1}{2}\check{\alpha}$ for $\alpha \in A$, it is also immediate if $\alpha, \beta \in 2A$. Finally, assume that $\beta \in \Phi$ and that $\alpha = 2\gamma$ with $\gamma \in A$.

(1) If $\gamma = \pm\beta$, then $\langle \check{\alpha}, \beta \rangle = \pm\frac{1}{2}\langle \check{\gamma}, \gamma \rangle = \pm 1$.

(2) If γ is not proportional to β and if $\beta \in A$, the assumption on A implies that $\langle \check{\gamma}, \beta \rangle = 0$, so $\langle \check{\alpha}, \beta \rangle = 0$.

(3) If $\beta \in \Phi \setminus A$, then $(\beta,\beta) = 2\lambda = 2(\gamma,\gamma)$, so $\langle \check{\gamma}, \beta \rangle$ is equal to $0, \pm 2$ by Table 5.1. Thus $\langle \check{\alpha}, \beta \rangle = \frac{1}{2}\langle \check{\gamma}, \beta \rangle \in \mathbb{Z}$.

Thus Φ_1 is a root system in V, and the other assertions are clear. □

5.1.4 Chambers and bases

Recall that we have assumed $\mathbb{K} = \mathbb{R}$. Let Φ be a root system in V. For all $\alpha \in \Phi$, let H_α be the hyperplane of V consisting of the points invariant under s_α. The chambers in V determined by the set of the H_α are called the **chambers** of Φ. The bijection $V \mapsto V^*$ defined by the scalar product $(\cdot,\cdot)$ takes α to $2\check{\alpha}/(\check{\alpha},\check{\alpha})$ for $\alpha \in \Phi$, hence H_α to $H_{\check{\alpha}}$, and hence the chambers of Φ to those of $\check{\Phi}$. If C is a chamber of Φ, the corresponding chamber of $\check{\Phi}$ is denoted by $\check{C}$. By Proposition 5.1.14, $\check{C}$ depends only on C and not on the choice of $(\cdot,\cdot)$.

Theorem 5.1.26. *Let Φ be a root system in V and C a chamber of Φ.*

(a) The group $W(\Phi)$ acts simply-transitively on the set of chambers.

(b) C is an open simplicial cone and $\overline{C}$ is a fundamental domain for $W(\Phi)$.

(c) Let $H_1, \ldots, H_l$ be the walls of C. Then for all i, there exists a unique indivisible root α_i such that $H_i = H_{\alpha_i}$, and such that α_i is on the same side of H_i as C.

(d) The set $\Delta(C) = \{\alpha_1, \ldots, \alpha_l\}$ is a basis of V.

(e) C is the set of $x \in V$ such that $\langle \check{\alpha}_i, x \rangle > 0$ for all i (or, equivalently, the set of $x \in V$ such that $(x, \alpha_i) > 0$ for all i).

(f) Let S be the set of the s_{α_i}. Then the pair $(W(\Phi), S)$ is a Coxeter system.

Proof. Assertions (a) and (f) follow from Theorem 4.3.3. The second part of assertion (b) follows from Theorem 4.3.5. The root α_i is orthogonal to H_i, and $\check{\alpha}_i$ is identified with $2\alpha_i/(\alpha_i,\alpha_i)$. Since $W(\Phi)$ is essential, the remaining parts follow from Proposition 4.3.26. □

Remark 5.1.27. Let $m(\alpha, \beta)$ be the order of $s_\alpha s_\beta$ for $\alpha, \beta \in \Delta(C)$. The matrix $(m(\alpha, \beta))$ is identified with the Coxeter matrix of (W, S). If $\alpha \neq \beta$, Proposition 4.3.12 shows that the angle $\widehat{(\alpha, \beta)}$ is equal to $\pi - \pi/m(\alpha, \beta)$. In particular, this angle is either obtuse or equal to π, and $(\alpha, \beta) \leq 0$. By using Table 5.1, it follows that $m(\alpha, \beta)$ is equal to $2, 3, 4$ or 6.

Definition 5.1.28. A subset Δ of Φ is called a **basis** of Φ if there exists a chamber C of Φ such that $\Delta = \Delta(C)$. If C is a chamber, $\Delta(C)$ is called the basis of Φ defined by C.

By Theorem 5.1.26, the map $C \mapsto \Delta(C)$ is a bijection from the set of chambers to the set of bases. Consequently, $W(\Phi)$ acts simply-transitively on the set of bases.

Proposition 5.1.29. *Let C be a chamber of Φ and Δ be the corresponding basis. If $\alpha \in \Delta$, set*

$$\phi(\alpha) = \begin{cases} \check{\alpha} & \text{if } 2\alpha \notin \Phi, \\ \frac{1}{2}\check{\alpha} & \text{if } 2\alpha \in \Phi \end{cases}$$

Then $\phi(\Delta)$ is the basis of $\check{\Phi}$ defined by $\check{C}$.

Proof. Since α and $\phi(\alpha)$ determine the same hyperplanes H_α, the walls of C are $H_{\phi\alpha}$ for $\alpha \in \Delta$. It remains to show that $\phi(\alpha)$ is indivisible. This follows from the observation that $(k\alpha)\check{} = k^{-1}\check{\alpha}$. □

Let Δ be a basis of Φ. The **Cartan matrix** of Φ (relative to Δ) is the matrix $(n(\alpha, \beta))_{\alpha, \beta \in \Delta}$. For $\alpha, \beta \in \Delta$, we have $n(\alpha, \alpha) = 2$ and

$$n(\alpha, \beta) = 2\frac{(\alpha, \beta)}{(\beta, \beta)} = -2\frac{\|\alpha\|}{\|\beta\|} \cos \frac{\pi}{m(\alpha, \beta)}$$

where $m(\alpha, \beta)$ denotes as above the order of $s_\alpha s_\beta$. By Table 5.1, we then conclude that, if $\alpha \neq \beta$ then $n(\alpha, \beta) = -1, -2$ or -3.

Remark 5.1.30. The Cartan matrix $(n(\alpha, \beta))$ should not be confused with the Coxeter matrix $(m(\alpha, \beta))$. Note in particular that the Cartan matrix is not necessarily symmetric.

If Δ and Δ' are two bases of Φ, there exists a unique element $w \in W$ such that $w(\Delta) = \Delta'$. We have

$$n(w(\alpha), w(\beta)) = n(\alpha, \beta), \quad m(w(\alpha), w(\beta)) = m(\alpha, \beta)$$

for $\alpha, \beta \in \Delta$. Consequently, the Cartan and Coxeter matrices associated to Δ can be obtained from those associated to Δ' by composition with the bijection $\alpha \mapsto w(\alpha)$ from Δ to Δ'.

The Cartan and Coxeter matrices can actually be defined canonically in the following way. Let X be the set of pairs (Δ, α), where Δ is a basis of Φ and $\alpha \in \Delta$. The group W acts in an obvious way on X and each orbit of W on X meets each of the sets $\{\Delta\} \times \Delta$ in exactly one point. If I is the set of these orbits, each basis Δ admits a **canonical indexing** $(\alpha_i)_{i \in I}$. Moreover, there exists a unique matrix $N = (n_{ij})$ (resp. $M = (m_{ij})$), of type $I \times I$, such that for any basis Δ, the Cartan (resp. Coxeter) matrix associated to Δ can be obtained from N (resp. M) by composing with the canonical indexing of Δ; it is called the **canonical Cartan matrix** (resp. **Coxeter matrix**) of Φ.

Proposition 5.1.31. *Let Δ be a basis of Φ and α an indivisible root. Then there exist $\beta \in \Delta$ and $w \in W(\Phi)$ such that $\alpha = w(\beta)$.*

Proof. Let C be the chamber such that $\Delta = \Delta(C)$. The hyperplane H_α is a wall of a chamber C' of Φ, and there exists an element of $W(\Phi)$ that transforms C' to C. We are therefore reduced to the case where H_α is a wall of C. Then α is proportional to an element β of Φ. Since α and β are indivisible, $\alpha = \pm\beta$. If $\alpha = -\beta$, then $\alpha = s_\beta(\beta)$, hence the proposition. □

Corollary 5.1.32. *Let Φ_1 and Φ_2 be two reduced root systems in vector spaces V_1 and V_2, and let Δ_1 and Δ_2 be bases of Φ_1 and Φ_2. Let $f : \Delta_1 \to \Delta_2$ be a bijection that transforms the Cartan matrix of Φ_1 to that of Φ_2. Then there exists an isomorphism $u : V_1 \to V_2$ that transforms Φ_1 to Φ_2 and α to $f(\alpha)$ for all $\alpha \in \Delta_1$.*

Proof. Recall that Δ_i is a basis for V_i. Let u be the isomorphism from V_1 to V_2 that takes α to $f(\alpha)$ for all $\alpha \in \Delta_1$. Then u transforms s_α to $s_{f(\alpha)}$, hence $W(\Phi_1)$ to $W(\Phi_2)$ (Theorem 5.1.26), and hence Φ_1 to Φ_2 (Proposition 5.1.31). □

Proposition 5.1.33. *Let Δ be a basis of Φ, and G the subgroup of $\mathrm{Aut}(\Phi)$ consisting of the elements leaving Δ stable. Then $W(\Phi)$ is a normal subgroup of $\mathrm{Aut}(\Phi)$ and $\mathrm{Aut}(\Phi)$ is the semi-direct product of G and $W(\Phi)$.*

Proof. If $\alpha \in \Phi$ and $\tau \in \mathrm{Aut}(\Phi)$, then $\tau s_\alpha \tau^{-1} = s_{\tau(\alpha)}$ by Proposition 5.1.9. Since $W(\Phi)$ is generated by the s_α, we see that $W(\Phi)$ is a normal subgroup of $\mathrm{Aut}(\Phi)$. By transport of structure, τ transforms a basis Δ of Φ to a basis Δ' of Φ. Since $W(\Phi)$ acts simply-transitively on the set of bases, there exist a unique $w \in W(\Phi)$ such that $w(\Delta) = \Delta'$. The element $w^{-1}\tau$ then leaves Δ stable, hence is in G; and we see $\tau = w(w^{-1}\tau)$. This completes the proof. □

Example 5.1.34. Let $\Phi_1, \ldots, \Phi_r$ be root systems in vector spaces $V_1, \ldots, V_r$. Let Φ be the direct sum of the Φ_i in $V = \bigoplus_{i=1}^r V_i$, C_i a chamber of Φ, and $\Delta_i = \Delta(C_i)$. It is immediate that $C = \prod_i C_i$ is a chamber of Φ and that $\Delta(C) = \bigcup_i \Delta_i$. It follows from Theorem 5.1.26 that all the chambers and bases of Φ are obtained in this way.

5.1.5 Positive roots

Let C be a chamber of Φ, and let $\Delta = \{\alpha_1, \ldots, \alpha_l\}$ be the corresponding basis of Φ. The **order relation on V (resp. V^*) defined by C** is the order relation campatible with the vector space structure of V (resp. V^*) for which the positive elements are the linear combinations of the α_i (resp. the $\check{\alpha}_i$) with real positive coefficients. An element that is positive for one of these relations is said to be **positive for C**, or **positive for the basis Δ**. These order relations are also defined by $\check{C}$, as one sees by identifying V with V^* by using a scalar product invariant under $W(\Phi)$. In view of Theorem 5.1.26, an element of V^* is positive if and only if it takes positive values on C, and an element x of V is positive if and only if it takes positive values on $\check{C}$, or, equivalently, if $(x, y) \geq 0$ for all $y \in C$.

The elements of $\overline{C}$ are positive for C by Proposition 4.3.16 and Theorem 5.1.26. But the set of positive elements for C is in general distinct from $\overline{C}$ (cf. the root systems A_2, B_2, G_2).

Theorem 5.1.35. *Every root is a linear combination with integer coefiicients of the same sign of elements of $\Delta(C)$. In particular, every root is either positive or negative for C.*

Proof. If $\alpha \in \Phi$, the kernel H_α of $\check{\alpha}$ does not meet C, so $\check{\alpha}$ is either positive or negative on the whole of C, hence the second assertion. It remains to show that α is contained in the $\mathbb{Z}$-subgroup P of V generated by $\Delta(C)$; we can assume that α is indivisible. Now the group P is clearly stable under the s_α for $\alpha \in \Delta(C)$, hence also under $W(\Phi)$ by Theorem 5.1.26. Since α is of the form $w(\beta)$ with $w \in W(\Phi)$ and $\beta \in \Delta(C)$ (cf. Proposition 5.1.31), we have $\alpha \in P$. □

Denote by $\Phi^+(C)$ the set of roots that are positive for C. Then by Theorem 5.1.35 we have a partition $\Phi = \Phi^+(C) \cup \Phi^-(C)$ of Φ, where $\Phi^-(C) = -\Phi^+(C)$.

Corollary 5.1.36. *Let γ be a linear combination of roots with integer coefficients, and α an indivisible root. If γ is proportional to α, then $\gamma \in \mathbb{Z}\alpha$.*

Proof. By Proposition 5.1.31, C can be chosen so that $\alpha \in \Delta(C)$. By Theorem 5.1.35,

$$\gamma = \sum_{\beta \in \Delta(C)} n_\beta \beta \quad \text{where } n_\beta \in \mathbb{Z}.$$

Thus, if γ is proportional to α, then $\gamma = n_\alpha \alpha$, which proves the corollary. □

Now let S be the set of reflections s_α for $\alpha \in \Delta(C)$ and let T be the union of the conjugates of S under W (which is the set of *all reflections* in W). For $\alpha \in \Delta(C)$ and $w \in W$, the element $t = w s_\alpha w^{-1}$ of T is the orthogonal reflection $s_{w(\alpha)}$ associated to the root $\beta = w(\alpha)$; conversely, for any indivisible root β, there exists an element $w \in W$ such that $\alpha = w^{-1}(\beta) \in \Delta(C)$ (Proposition 5.1.31) and $s_\beta = w s_\alpha w^{-1} \in T$. It follows that a bijection ψ from the set of indivisible roots to $T \times \{\pm 1\}$ is obtained by associating to an indivisible root β the pair (s_β, ε), where $\varepsilon = 1$ if β is positive and $\varepsilon = -1$ if β is negative.

On the other hand, (W, S) is a Coxeter system and the results of paragraph 3.1.3 can be applied. We have seen that, if w is an element of W of length (with respect to S) equal to p, there exists a subset $T_L(w)$ of T, with p elements, such that, if $w = s_1 \cdots s_p$ with $s_i \in S$ and if

$$t_j = (s_1 \cdots s_{j-1}) s_j (s_1 \cdots s_{j-1})^{-1} = w_j s_j w_j^{-1}$$

for $1 \leq i \leq p$, then $T_L(w) = \{t_1, \dots, t_p\}$. Recall that we have also defined in Proposition 3.1.13 a number $\eta(w,t)$ (for $w \in W$ and $t \in T$) equal to $+1$ if $t \notin T_L(w)$, and to -1 if $t \in T_L(w)$ (Proposition 3.1.14 and (3.1.8)). Finally, recall that, if we define a map π_w from the set $T \times \{\pm 1\}$ to itself by the formula

$$\pi_w(t,\varepsilon) = (wtw^{-1}, \varepsilon \cdot \eta(w^{-1}, t))$$

then the map $w \mapsto \pi_w$ is a homomorphism from W to the group of permutations of the set $T \times \{\pm 1\}$.

Proposition 5.1.37. *Assume that Φ is reduced and let $w \in W$, $\alpha \in \Phi$. Let $\psi : \Phi \to T \times \{\pm 1\}$ be the bijection given above.*

(a) We have $\psi(w(\alpha)) = \pi_w(\psi(\alpha))$.

(b) Assume that α is positive. Then the root $w(\alpha)$ is negative if and only if $\eta(w^{-1}, s_\alpha) = -1$, in other words, if and only if $s_\alpha \in T_L(w^{-1})$.

(c) We have $\eta(w, s_\alpha) = -1$ if and only if the chambers C and $w(C)$ are on opposite sides of the hyperplane H_α. In other words, the set $T_L(w)$ consists of the reflections with respect to the walls separating C and $w(C)$.

Proof. Let $\beta \in \Delta(C)$ and put $s = s_\beta$. Clearly $T_L(s) = \{s\}$ and consequently

$$\pi_s(t,\varepsilon) = \begin{cases} (sts^{-1}, \varepsilon) & \text{if } t \neq s, \\ (s, -\varepsilon) & \text{if } t = s. \end{cases} \tag{5.1.11}$$

On the other hand, let $\rho = \sum_{\gamma \in \Delta(C)} n_\gamma(\rho)\gamma$ be a positive root. Put

$$s(\rho) = \sum_{\gamma \in \Delta(C)} n_\gamma(s(\rho))\gamma.$$

If $\rho \neq \beta$, there exists an element $\gamma \in \Delta(C)$ with $\gamma \neq \beta$ such that $n_\gamma(\rho) > 0$, and we have $n_\gamma(s(\rho)) = n_\gamma(\rho) > 0$ by formula (5.1.3). Hence $s(\rho)$ is positive. We deduce immediately that

$$\psi(s(\varepsilon \cdot \rho)) = \begin{cases} (ss_\rho s^{-1}, \varepsilon) & \text{if } \rho \neq \beta, \\ (s, -\varepsilon) & \text{if } \rho = \beta. \end{cases} \tag{5.1.12}$$

Comparison of (5.1.11) and (5.1.12) now shows that $\pi_s(\psi(\gamma)) = \psi(s(\gamma))$ for all roots γ and all $s \in S$. Since S generates W, assertion (a) follows.

On the other hand, saying that $w(\alpha)$ is negative is equivalent to saying that

$$\psi(w(\alpha)) = (ws_\alpha w^{-1}, -1).$$

or, by assertion (a), that $\pi_w(\psi(\alpha)) = (ws_\alpha w^{-1}, -1)$. If in addition α is positive, then $\psi(\alpha) = (s_\alpha, 1)$ and $\pi_w(\psi(\alpha)) = (ws_\alpha w^{-1}, \eta(w^{-1}, s_\alpha))$, hence (b).

Finally, by assertion (b), $\eta(w, s_\alpha) = -1$ if and only if one of the roots α and $w^{-1}(\alpha)$ is positive and the other negative. This is equivalent to saying that $(\alpha, x)(w^{-1}(\alpha), x) < 0$ for all $x \in C$, hence the first assertion in (c). The second assertion in (c) follows immediately. $\square$

Corollary 5.1.38. *Let $\beta \in \Delta(C)$. Then the reflection s_β permutes the positive roots not proportional to β.*

Proof. We reduce immediately to the case in which Φ is reduced. In that case, our assertion follows from Proposition 5.1.37(b) and the fact that $T_L(s_\beta) = \{s_\beta\}$. $\square$

Corollary 5.1.39. *Assume that Φ is reduced. Let $w \in W$, let p be the length of w with respect to S, and let $w = s_1 \cdots s_p$ be a reduced decomposition of w. Let $\alpha_1, \dots, \alpha_p$ be the elements of $\Delta(C)$ corresponding to $s_1, \dots, s_p$. For $1 \leq i \leq p$, put*

$$\theta_i = s_p s_{p-1} \cdots s_{i+1}(\alpha_i).$$

Then the roots θ_i are positive, pairwise distinct, and such that $w(\theta_i)$ are negative. Moreover, every positive root α such that $w(\alpha)$ is negative is equal to one of the θ_i. Concequently, the length of w is equal to the number of positive roots that are sent to negative roots by w.

Proof. Let X be the set of positive roots that are sent to negative roots by w. By Proposition 5.1.37(b),

$$|X| = |T_L(w^{-1})| = \ell(w^{-1}) = \ell(w) = p.$$

On the other hand, if $\alpha \in X$ it is clear that there exists $1 \leq i \leq p$ such that

$$s_{i+1} \cdots s_p(\alpha) \succ 0 \quad \text{and} \quad s_i s_{i+1} \cdots s_p(\alpha) \prec 0.$$

By Corollary 5.1.38, this implies that $s_i s_{i+1} \cdots s_p(\alpha) = \alpha_i$ and hence that $\alpha = \theta_i$. The set X is thus contained in the set of θ_i. Since $|X| = p$, this is possible only if X is equal to the set of θ_i, and these are pairwise distinct. Hence the corollary. □

Corollary 5.1.40. *Assume that Φ is reduced. Then there exists a unique longest element w_0 in W such that*

(a) $\ell(w)$ is equal to the number of positive roots;

(b) w_0 transforms the chamber C to $-C$ and the basis Δ to $-\Delta$;

(c) we have $w_0^2 = 1$ and $\ell(ww_0) = \ell(w_0) - \ell(w)$ for all $w \in W$.

Proof. It is clear that $-C$ is a chamber. Hence there exists an element w_0 of W that transforms C to $-C$. Then $w_0(\alpha) < 0$ for all positive roots α and assertion (a) is a consequence of Corollary 5.1.39. To see $w_0(\Delta) = -\Delta$, we recall that (Theorem 5.1.26) $-\Delta$ is the vectors orthogonal to a wall H of $-C$ and on the same side of H as $-C$. Since C and $-C$ have the same wall and are on the opposite side of each wall of C, we conclude that $\Delta(-C) = -\Delta(C)$, whence $w_0(\Delta) = -\Delta$. We have $w_0^2(C) = C$, so $w_0^2 = 1$.

Finally, if $w \in W$, the length $\ell(w)$ (resp. $\ell(ww_0)$) is equal, by Proposition 5.1.37(c), to the number of walls separating C and $w(C)$ (resp. $ww_0(C) = -w(C)$). Since $w(C)$ and $-w(C)$ are on opposite sides of every wall, the sum $\ell(w) + \ell(ww_0)$ is equal to the total number of walls, that is to $\ell(w_0)$. □

Proposition 5.1.41. *Let $x \in V$. Then the following three properties are equivalent:*

(i) *$x \in \bar{C}$;*

(ii) *$x \geq s_\alpha(x)$ for all $\alpha \in \Delta(C)$ (with respect to the order relation defined by C);*

(iii) *$x \geq w(x)$ for all $w \in W$.*

Proof. Since $s_\alpha(x) = x - \langle\check{\alpha}, x\rangle\alpha$ and since $\bar{C}$ is the set of elements $x \in V$ such that $\langle\check{\alpha}, x\rangle \geq 0$ for all $\alpha \in \Delta(C)$, the equivalence of (i) and (ii) is obvious. On the other hand, it is clear that (iii)⇒(ii). We show that (i)⇒(iii). Let $x \in \bar{C}$, and let $w \in W$. We argue by induction on the length $\ell(w)$ of w. The case $\ell(w) = 0$ is trivial. If $\ell(w) \geq 1$, it can be written in the form $w = \tilde{w}s_\alpha$, with $\alpha \in \Delta(C)$ and $\ell(\tilde{w}) = \ell(w) - 1$. Then

$$x - w(x) = x - \tilde{w}(x) + \tilde{w}(x - s_\alpha(x)).$$

The induction hypothesis shows that $x - \tilde{w}(x)$ is positive. On the other hand,

$$w\tilde{w}(x - s_\alpha(x)) = w(s_\alpha(x) - x) = -\langle\check{\alpha}, x\rangle w(\alpha).$$

Now $s_\alpha \in T_L(w^{-1})$, and Proposition 5.1.37(b) shows that $w(\alpha) < 0$. Hence the result. □

Corollary 5.1.42. *An element $x \in C$ if and only if $x > w(x)$ for all in $w \in W$ such that $w \neq 1$.*

Proof. This follows because W acts simply transitively on chambers, and if $x \in \bar{C} \setminus C$ then x is contained in a hyperplane, whence $s_\alpha(x) = x$ for some $\alpha \in \Phi$. □

Proposition 5.1.43. *Let $(\beta_i)_{1\leq i\leq n}$ be a sequnce of positive roots for the chamber C such that $\beta_1 + \cdots + \beta_n$ is a root. Then there exists a permutation $\pi \in \mathfrak{S}_n$ such that, for all $1 \leq i \leq n$, $\beta_{\pi(1)} + \cdots + \beta_{\pi(i)}$ is a root.*

Proof. We argue by induction on n, the proposition being clear for $n \leq 2$. Put $\beta = \beta_1 + \cdots + \beta_n$. Then $\sum_{i=1}^n (\beta, \beta_i) = (\beta, \beta) > 0$, so there exists an index k such that $(\beta, \beta_k) > 0$. If $\beta = \beta_k$, then $n = 1$ since β_i are positive for all i. Otherwise $\beta - \beta_k$ is a root by Corollary 5.1.18; it then suffices to apply the induction hypothesis to $\beta - \beta_k = \sum_{i\neq k} \beta_i$. □

Corollary 5.1.44. *Let $\alpha \in \Phi^+(C)$. Then $\alpha \notin \Delta(C)$ if and only if α is the sum of two positive roots. In otherwords, a positive root is in $\Delta(C)$ is and only if it is indecomposable.*

Proof. If α is the sum of two positive roots, then Theorem 5.1.35 shows that $\alpha \notin \Delta(C)$. If $\alpha \notin \Delta(C)$, Theorem 5.1.35 shows that $\alpha = \sum_{i=1}^{n} \beta_i$ with $\beta_i \in \Delta(C)$ and $n \geq 2$. Permuting the β_i if necessary, we can assume that $\sum_{i=1}^{n-1} \beta_i$ is a root (Proposition 5.1.43), and hence that α is the sum of the positive roots $\sum_{i=1}^{n-1} \beta_i$ and β_n. □

Corollary 5.1.45. *Let ϕ be a map from Φ to a abelian group G having the following properties:*

(a) $\phi(-\alpha) = -\phi(\alpha)$ for all $\alpha \in \Phi$;

(b) if $\alpha, \beta \in \Phi$ are such that $\alpha + \beta \in \Phi$, then $\phi(\alpha + \beta) = \phi(\alpha) + \phi(\beta)$.

Let Q be the subgroup of V generated by Φ. Then ϕ extends to a homomorphism from Q to G.

Proof. Let Δ be a basis of Φ. Let ψ be the unique homomorphism from Q to G that coincides with ϕ on Δ. It suffices to show that $\phi(\alpha) = \psi(\alpha)$ when α is a positive root relative to Δ. We have $\alpha = \beta_1 + \cdots + \beta_n$ with $\beta_i \in \Delta$ for all i, and $\beta_1 + \cdots + \beta_i$ for all i (Proposition 5.1.43). We show that $\phi(\alpha) = \psi(\alpha)$ by induction on n. This is clear if $n = 1$. The induction hypothesis gives

$$\psi(\beta_1 + \cdots + \beta_{n-1}) = \phi(\beta_1 + \cdots + \beta_{n-1})$$

and we have $\phi(\beta_n) = \psi(\beta_n)$, hence $\phi(\alpha) = \psi(\alpha)$, which proves the corollary. □

For any root $\alpha = \sum_{\beta \in \Delta(C)} n_\beta \beta$, denote by $Y(\alpha)$ the set of $\beta \in \Delta(C)$ such that $n_\beta \neq 0$. Moreover, observe that $\Delta(C)$ can be identified with the set of vertices of the graph of the Coxeter system $(W(\Phi), S)$.

Corollary 5.1.46. *Let Φ be a root system in V and $Y(\alpha)$ the set defined above.*

(a) Let $\alpha \in \Phi$. Then $Y(\alpha)$ is a connected subset of $\Delta(C)$.

(b) Let Y be a nonempty connected subset of $\Delta(C)$. Then $\sum_{\beta \in Y} \beta$ belongs to Φ.

Proof. To prove (a), we can assume that α is positive. We argue by induction on $|Y(\alpha)|$, the assertion being trivial if $|Y(\alpha)| = 1$. By Proposition 5.1.43, there exists $\beta \in \Delta(C)$ such that $\alpha - \beta \in \Phi$. Let p be the largest positive integer such that $\gamma := \alpha - p\beta \in \Phi$. Since $\gamma - \beta \notin \Phi$ and $\gamma + \beta \in \Phi$, $(\gamma, \beta) \neq 0$ by Proposition 5.1.19, thus β is linked to at least one element of $Y(\gamma)$. But $Y(\alpha) = Y(\gamma) \cup \{\beta\}$, and $Y(\gamma)$ is connected by the induction hypothesis. Thus $Y(\alpha)$ is connected, which proves (a).

Now let Y be a non-empty connected subset of $\Delta(C)$; we show by induction on $|Y|$ that $\sum_{\beta \in Y} \beta$ is a root. The case in which $|Y| = 1$ is trivial. Assume that $|Y|$. Since X is a forest (Proposition 4.4.17, Y is a tree and has a terminal vertex β (Lemma 4.6.2). The set $Y \setminus \{\beta\}$ is then connected, and one of its element is linked to β. By the induction hypothesis, $\alpha = \sum_{\gamma \in Y \setminus \{\beta\}} \gamma \in \Phi$, and since $(\alpha, \beta) < 0$ (Proposition 4.3.12), it follows that $\alpha + \beta \in \Phi$. □

5.1.6 Closed and parabolic subsets

Let P be a subset of Φ. Then P is said to be **closed** if the conditions $\alpha, \beta \in P$ and $\alpha + \beta \in \Phi$ imply $\alpha + \beta \in P$; and is **parabolic** if P is closed and if $P \cup (-P) = \Phi$. Closed subsets of root systems play an important role in the study of subalgebras of finite dimensional semi-simple Lie algebras and in the theory of reductive algebraic groups. In this paragraph, we establish some of its basic properties.

Let C be a chamber of Φ and $\Phi^+(C)$ the positive roots relative to C. Then $\Phi^+(C)$ is clearly parabolic, and every closed subset containing $\Phi^+(C)$ is also parabolic. Conversely, let P be a parabolic subset of Φ. Since $P \cup (-P) = \Phi$, one may wonder that if there exist a chamber C such that $P \supseteq \Phi^+(C)$. This is in fact true, and characterizes parabolic subsets from closed subsets of Φ, as we will now show.

Proposition 5.1.47. *Let C be a chamber of Φ and P a closed subset of Φ containing $\Phi^+(C)$. Let $\Sigma = \Delta(C) \cap P$ and Q be the set of roots that are linear combinations of elements of Σ with non-positive integer coefficient. Then $P = \Phi^+(C) \cup Q$.*

Proof. It is enough to show that $P \cap \Phi^-(C) = Q$ since $\Phi = \Phi^+(C) \cup \Phi^-(C)$. Let $-\alpha \in Q$. Then α is the sum of n elements of Σ. We show, by induction on n, that $-\alpha \in P$. This is clear if $n = 1$. If $n > 1$, then by Proposition 5.1.43 we can write $\alpha = \beta + \gamma$ with $\gamma \in \Sigma$ and β the sum of $n-1$ elements of Σ. By the induction hypothesis, $-\beta \in P$ and $-\gamma \in P$; since P is closed, $-\alpha \in P$. Thus, $Q \subseteq P \cap \Phi^-(C)$.

Conversely, let $-\alpha \in P \cap \Phi^-(C)$. Then α is the sum of p elements of $\Delta(C)$. We show, by induction on p, that $-\alpha \in Q$. This is clear if $p = 1$. If $p > 1$, then by Proposition 5.1.43, we can write $\alpha = \beta + \gamma$

with $\gamma \in \Delta(C)$ and β a root that is the sum of $p-1$ elements of $\Delta(C)$. Since $-\gamma = \beta + (-\alpha)$ and since P is closed, $-\gamma \in P$, hence $\gamma \in \Sigma$. Moreover, $-\beta = \gamma + (-\alpha)$ so $-\beta \in P$ since P is closed. By the induction hypothesis, $-\beta \in Q$, so $-\alpha = -\beta - \gamma \in Q$. Thus, $P \cap \Phi^-(C) = Q$. □

Proposition 5.1.48. *Let P be a subset of Φ. The following conditions are equivalent:*

(i) *P is parabolic;*

(ii) *P is closed and there exists a chamber C of Φ such that $P \supseteq \Phi^+(C)$;*

(iii) *there exist a chamber C of Φ and a subset Σ of $\Delta(C)$ such that P is the union of $\Phi^+(C)$ and the set Q of roots that are linear combinations of elements of Σ with non-positive integer coefiicients.*

Proof. By Proposition 5.1.47 we have (ii)⇒(iii). We adopt the assumptions and notation of (iii). It is clear that $P \cup (-P) = \Phi$. We show that, if $\alpha, \beta \in P$ are such that $\alpha + \beta \in \Phi$, then $\alpha + \beta \in P$. This is obvious if the root $\alpha + \beta$ is positive. Assume that $\alpha + \beta$ is negative. Then $\alpha + \beta = \sum_{\gamma \in \Delta(C)} n_\gamma \gamma$, with $n_\gamma \leq 0$. But the coeflicient of every element γ of $\Delta(C) \setminus \Sigma$ in α or β is positive, hence $n_\gamma = 0$ if $\gamma \in \Delta(C) \setminus \Sigma$, so $\alpha + \beta \in Q$. This proves (iii)⇒(i).

Finally, assume that P is parabolic. Let C be a chamber such that $|P \cap \Phi^+(C)|$ is as large as possible. Let $\alpha \in \Delta(C)$ and assume that $\alpha \notin P$, so that $-\alpha \in P$. For all $\beta \in P \cap \Phi^+(C)$, β is not proportional to α (for the hypothesis $\beta = 2\alpha$ would imply that $\alpha = 2\alpha + (-\alpha) \in P$ since P is closed). Thus $s_\alpha(\beta) \in \Phi^+(C)$ by Corollary 5.1.38. If we put $C' = s_\alpha(C)$, then $\beta = s_\alpha(s_\alpha(\beta)) \in s_\alpha(\Phi^+(C)) = \Phi^+(C')$, so $P \cap \Phi^+(C) \subseteq P \cap \Phi^+(C')$. Moreover, $-\alpha = s_\alpha(\alpha) \in s_\alpha(\Phi^+(C)) = \Phi^+(C')$, hence $-\alpha \in P \cap \Phi^+(C')$, and $|P \cap \Phi^+(C')| > |P \cap \Phi^+(C)|$. This contradiction shows $\alpha \in P$, whence $\Delta(C) \subseteq P$. Since P is closed, we then have $\Phi^+(C) \subseteq P$, which shows (i)⇒(ii). □

Corollary 5.1.49. *Let P be a subset of Φ. Then the following conditions are equivalent:*

(i) *there exists a chamber C such that $P = \phi^+(C)$;*

(ii) *P is closed and $P \cup (-P) = \Phi$ is a partition of Φ;*

Moreover, the chamber C such that $P = \Phi^+(C)$ is unique, if it exists.

Proof. Clearly (i) implies (ii), and the converse follows from Proposition 5.1.48. If $P = \phi^+(C)$, then $\check{C}$ is the set of $x^* \in V^*$ such that $\langle x^*, x\rangle > 0$ for $x \in P$, whence the uniqueness of C. □

Corollary 5.1.50. *Assume that V is equipped with the structure of an ordered vector space such that, for this structure, every root of Φ is either positive or negative. Let P be the set of positive roots for this structure. Then there exists a unique chamber C of Φ such that $P = \Phi^+(C)$.*

Proof. Indeed, P satisfies condition (ii) of Corollary 5.1.49. □

This corollary applies in particular when the order being considered is total, the condition on Φ then being automatically satisfied. Recall that such an order can be obtained, for example, by choosing a basis of V and taking the *lexicographic* order on V.

Corollary 5.1.51. *A subset Δ of Φ is a basis of Φ if and only if the following conditions are satisfied:*

(a) *the elements of Δ are linearly independent;*

(b) *every root of Φ is a linear combination of elements of Φ in which the coefficients are either all positive or all negative;*

(c) *every root of Δ is indivisible.*

Proof. We already know that the conditions are necessary. Assume that the above conditions are satisfied. Let P be the set of roots that are linear combinations of elements of Δ with positive coeflicients. Since P satisfies condition (ii) of Corollary 5.1.49, there exists a chamber C such that $P = \Phi^+(C)$; let $\Delta' = \Delta(C)$, and let X and X' be the convex cones generated by Δ and Δ'. Then

$$\Delta \subseteq P \subseteq X, \quad \Delta' \subseteq P \subseteq X',$$

which shows that X and X' are both generated by P, and hence coincide. But the half-lines generated by the elements of Δ (resp. by Δ') are the extreme generators of X (resp. X'); since such a half-line contains only one indivisible root, $\Delta = \Delta'$. □

Corollary 5.1.52. *Let Δ be a basis of Φ, Σ a subset of Δ, V_Σ the vector subspace of V generated by Σ, and $\Phi_\Sigma = \Phi \cap V_\Sigma$. Then Σ is a basis of the root system Φ_Σ. We call Φ_Σ the root system **generated** by Σ.*

Proof. This follows immediately from Corollary 5.1.51 and the Corollary 5.1.8. □

Corollary 5.1.53. *Let Δ be a basis for Φ, $\Sigma_1, \dots, \Sigma_r$ pairwise orthogonal subsets of Δ, and $\Sigma = \Sigma_1 \cup \cdots \cup \Sigma_r$. Then every root α that is a linear combination of elements of Σ is actually a linear combination of elements of one of the Σ_i. In particular, if Φ is irreducible, there is no partition of Δ into pairwise orthogonal subsets.*

Proof. Let $E_1, \dots, E_r, E$ be the vector subspaces ofV generated by $\Sigma_1, \dots, \Sigma_r$, Σ, respectively. By Corollary 5.1.52, we can assume that $E = V$. Then, by Theorem 5.1.26 and the hypothesis, the E_i are stable under $W(\Phi)$, so Φ is the union of the $\Phi \cap E_i$ by Proposition 5.1.11. □

Corollary 5.1.54. *Let P be a parabolic subset of Φ, C a chamber of Φ, Σ a subset of $\Delta(C)$ such that P is the union of $\Phi^+(C)$ and the set Q of roots that are linear combinations of elements of Σ with non-positive integer coefiicients. Let V_Σ be the vector subspace of V generated by Σ. Then $P \cap (-P) = Q \cup (-Q) = \Phi \cap V_\Sigma$ is a root system in V_Σ with basis Σ.*

Proof. We have $P \cap (-P) = (\Phi^+(C) \cup Q) \cup (\Phi^-(C) \cup (-Q)) = Q \cup (-Q)$, and Theorem 5.1.26 proves that $Q \cup (-Q) = \Phi \cap V_\Sigma$. Finally, Σ is a basis of the root system $\Phi \cap V_\Sigma$ by Corollary 5.1.52. □

Proposition 5.1.55. *Let C (resp. C') be a chamber of Φ, Σ (resp. Σ') a subset of $\Delta(C)$ (resp. $\Delta(C')$), Q (resp. Q') the set of linear combinations of elements of Σ (resp. Σ') with negative integer coefiicients, and $P = Q \cup \Phi^+(C)$ (resp. $P' = Q' \cup \Phi^+(C')$). If there exists an element of $W(\Phi)$ transforming P to P', then there exists an element of $W(\Phi)$ transforming C to C' and Σ to Σ.*

Proof. We reduce immediately to the case $P = P'$. Let $\tilde{V}$ be the vector subspace of V generated by $P \cap (-P)$. Then Σ and Σ' are bases of the root system $\tilde{\Phi} = P \cap (-P)$ in $\tilde{V}$ (Corollary 5.1.54). Hence there exists $w_1 \in W(\tilde{\Phi})$ such that $w_1(\Sigma) = \Sigma'$. It is clear that w_1 is induced by an element w of $W(\Phi)$ that is a product of the reflections s_σ with $\sigma \in \Sigma$. Let $\gamma = \sum_{\beta \in \Delta(C)} c_\beta \beta$ be an element of $P \setminus \tilde{\Phi}$. Then $c_\beta > 0$ for at least one $\beta \in \Delta(C) \setminus \Sigma$. Moreover, if $\sigma \in \Sigma$, then $s_\sigma(\gamma) - \gamma \in \mathbb{K}\sigma \subseteq \tilde{V}$, so $s_\sigma(\gamma)$ has at least one coordinate > 0 with respect to $\Delta(C)$ (formula (5.1.3)), hence $s_\sigma(\gamma) \in \Phi^+(C) \subseteq P$ and finally $s_\sigma(\gamma) \in P \setminus \tilde{\Phi}$. It follows that $P \setminus \tilde{\Phi}$ is stable under the reflections s_σ, with $\sigma \in \Sigma$, and hence under w, so $w(P) = P$. We are thus reduced to proving the proposition when $P = P'$ and $\Sigma = \Sigma'$. In this case, $Q = Q'$, so $\Phi^+(C) = P \setminus Q = P \setminus Q' = \Phi^+(C')$, and hence $C = C'$ by Corollary 5.1.49. □

Corollary 5.1.56. *Let P, P' be two parabolic subsets of Φ transformed into each other by an element of the Weyl group. If there exists a chamber C of Φ such that $\Phi^+(C) \subseteq P$ and $\Phi^+(C) \subseteq P'$, then $P = P'$.*

Proof. This follows from Proposition 5.1.47 and Proposition 5.1.55 since the only element of $W(\Phi)$ transforming C to C is 1. □

Corollary 5.1.57. *Let $\tilde{\Phi}$ be the intersection of Φ with a vector subspace of V, so that $\tilde{\Phi}$ is a root system in the vector subspace $\tilde{V}$ that it generates. Let $\tilde{\Delta}$ be a basis of $\tilde{\Phi}$, then there exists a basis of Φ containing $\tilde{\Delta}$, and $\tilde{\Phi}$ is the set of elements of Φ that are linear combinations of elements of $\tilde{\Delta}$.*

Proof. Let $(e_1, \dots, e_l)$ be a basis of V such that $\Delta = (e_{r+1}, \dots, e_l)$. The lexicographic order on V corresponding to this basis defines a chamber C of Φ, by Corollary 5.1.50. It is clear that every element of $\tilde{\Delta}$ is minimal in $\Phi^+(C)$. Thus $\tilde{\Delta} \subseteq \Delta$, and the second claim is clear. □

Proposition 5.1.58. *Let P be a closed subset of Φ such that $P \cap (-P) = \varnothing$. Then there exists a chamber C of Φ such that $P \subseteq \Phi^+(C)$.*

Proof. We show that no sum $\alpha_1 + \cdots + \alpha_p$ of elements of P is zero. We proceed by induction on p. The assertion being clear for $p = 1$, assume that $p \geq 2$. If $\alpha_1 + \cdots + \alpha_p = 0$, then

$$-\alpha_1 = \alpha_2 + \cdots + \alpha_p$$

so $(\alpha_1, \alpha_2 + \cdots + \alpha_p) < 0$, hence there exists $2 \leq j \leq p$ such that $(\alpha_1, \alpha_j) < 0$. By Corollary 5.1.18, $\alpha_1 + \alpha_j \in P$, and the relation $(\alpha_1 + \alpha_j) + \sum_{i \neq 1, j} \alpha_i = 0$ contradicts the induction hypothesis.

Next, we show that there exists a non-zero element γ in V such that $(\gamma, \alpha) \geq 0$ for all $\alpha \in P$. If not, Corollary 5.1.18 would show that an infinite sequence $\{\alpha_i\}$ of elements of P could be found such that

$$\beta_n = \alpha_1 + \cdots + \alpha_n \in P.$$

for all n. Since P is finite, there would exist two distinct integers i, j such that $\beta_i = \beta_j$, which would contradict the previous result that no sum of elements of P is zero.

To prove the proposition, it is enough (Corollary 5.1.50) to show that there exists a basis $(\alpha_i)_{1\leq i\leq l}$ of V such that, for the lexicographic order defined by this basis, every element of P is positive. We proceed by induction on $l = \dim(V)$, and assume that the proposition is established for all dimensions $< l$. Let $\gamma \in V$ be such that $\gamma \neq 0$ and $(\gamma, \alpha) \geq 0$ for all $\alpha \in P$. Let H be the hyperplane orthogonal to γ, and U the subspace of H generated by $\Phi \cap H$. Then $\Phi \cap H$ is a root system in U and $P \cap H$ is closed in $\Phi \cap H$. By the induction hypothesis, there exists a basis $(\beta_1, \ldots, \beta_r)$ of U such that the elements of $P \cap H$ are positive for the lexicographic order defined by this basis. Then any basis of V whose first $r+1$ elements are $\gamma, \beta_1, \ldots, \beta_r$ has the required property. □

Proposition 5.1.59. *Let P be a subset of Φ and V_P (resp. $\Gamma(P)$) the vector subspace (resp. the subgroup) of V generated by P. The following conditions are equivalent:*

(i) *P is closed and symmetric;*

(ii) *P is closed, and P is a root system in V_P;*

(iii) *$\Gamma(P) \cap \Phi = P$.*

Assume that these conditions are satisfied. For any $\alpha \in P$, let $\check{\alpha}|_P$ be the restriction of α to V_P. Then the map $\alpha \mapsto \check{\alpha}|_P$ is the canonical bijection from the root system P to $\check{P}$.

Proof. Clearly (iii)⇒(i). Assume that P is closed and symmetric. First, P satisfies (R1) in V_P. We show that, if $\alpha, \beta \in P$, then $s_\alpha(\beta) \in P$. This is clear if α and β are proportional. Otherwise, $s_\alpha(\beta) = \beta - n(\beta,\alpha)\alpha$ and $\beta - p\alpha$ for all rational integers p between 0 and $n(\beta, \alpha)$ (Proposition 5.1.19), so

$$\beta - n(\beta, \alpha)\alpha \in P$$

since P is closed and symmetric. Thus, $s_{\alpha,\check{\alpha}_P}(\beta) \in P$ and P satisfies (R2). It is clear that P satisfies (R3). Thus, P satisfies (ii), and we have proved the last assertion of the proposition at the same time.

To see (ii)⇒(iii), we show that, if condition (ii) is satisfied, then $\Gamma \cap \Phi = P$. It is clear that $P \subseteq \Gamma \cap \Phi$. Let $\beta \in \Gamma \cap \Phi$. Since $\beta \in P$ and $P = -P$, $\beta = \alpha_1 + \cdots + \alpha_n$ with $\alpha_i \in P$. We shall prove that $\beta \in P$. This is clear if $n = 1$. We argue by induction on n. We have

$$0 < (\beta, \beta) = \sum_{i=1}^{n} (\beta, \alpha_i)$$

so $(\beta, \alpha_i) > 0$ for some index i. If $\beta = \alpha_i$, then $\beta \in P$. Otherwise, $\beta - \alpha_i \in \Phi$ by Corollary 5.1.18, so $\beta - \alpha_i \in P$ by the induction hypothesis, hence $\beta \in P$ since P is closed. □

Remark 5.1.60. The conditions of Proposition 5.1.59 can be realised with $V_P = V$ and yet $P \neq \Phi$. For example, this is the case when Φ is a system of type G_2 and P a system of type A_2 (c.f. Figure 5.4).

Let V be a finite dimensional real vector space, Φ a root system in V^*, and $\mathcal{P}$ the set of parabolic subsets of Φ. Let $\mathcal{H}$ be the set of kernels of the roots in Φ and $\mathcal{F}$ the set of facets of V relative to $\mathcal{H}$. For each $P \in \mathcal{P}$, we associate with P a facet defined by

$$\mathscr{F}(P) = \{v \in V : \alpha(v) \geq 0 \text{ for all } \alpha \in P\}.$$

Simialrly, for $F \in \mathcal{F}$, we associate with F a parabolic subset define

$$\mathscr{P}(F) = \{\alpha \in \Phi : \alpha(v) \geq 0 \text{ for all } v \in F\}.$$

It is clear from the definition that for $P \in \mathcal{P}$ and $F \in \mathcal{F}$ we have the following relations

$$\mathscr{F}(\mathscr{P}(\mathscr{F}(P))) = \mathscr{F}(P), \quad \mathscr{P}(\mathscr{F}(\mathscr{P}(F))) = \mathscr{P}(F).$$

Proposition 5.1.61. *Let $\mathscr{F}$ and $\mathscr{P}$ be the operators defined above. Then*

(a) *for $P \in \mathcal{P}$, $\mathscr{P}(\mathscr{F}(P)) = P$.*

(b) *for $F \in \mathcal{F}$, $F(P(F))$ is the closure of F.*

Proof. Let $P \in \mathcal{P}$. There exists a chamber C of Φ and a subset Σ of the basis $\Delta(C)$ such that $P = \Phi^+(C) \cup Q$ where Q is the set of linear combinations of elements of Σ with non-positive integer coefficients. Put

$$\Delta(C) = \{\alpha_1, \dots, \alpha_l\}, \quad \Sigma = \{\alpha_1, \dots, \alpha_m\}.$$

If $v \in V$, we have the following equivalences:

$$\begin{aligned} v \in F &\Leftrightarrow \alpha_1(v) \geq 0, \dots, \alpha_l(v) \geq 0, \alpha_1(v) \leq 0, \dots, \alpha_m(v) \leq 0 \\ &\Leftrightarrow \alpha_1(v) = 0, \dots, \alpha_m(v) = 0, \alpha_{m+1}(v) \geq 0, \dots, \alpha_l(v) \geq 0. \end{aligned}$$

Therefore $\mathscr{F}(P)$ is characterized by

$$\mathscr{F}(P) = \{v \in V : \alpha_1(v) = \dots = \alpha_m(v) = 0, \alpha_{m+1}(v) \geq 0, \dots, \alpha_l(v) \geq 0\}. \tag{5.1.13}$$

If $\beta = \sum_{i=1}^l b_i \alpha_i \in \Phi$, then by (5.1.13) we conclude that

$$\begin{aligned} \beta \in \mathscr{P}(\mathscr{F}(P)) &\Leftrightarrow b_{m+1} \geq 0, \dots, b_l \geq 0 \\ &\Leftrightarrow \beta \in \Phi^+(C) \text{ or } (-\beta \in \Phi^+(C) \text{ and } b_{m+1} = \dots = b_l = 0) \\ &\Leftrightarrow \beta \in \Phi^+(C) \cup Q = P. \end{aligned}$$

Thus we conclude that $P = \mathscr{P}(\mathscr{F}(P))$.

Let $F \in \mathcal{F}$. It is clear that $\mathscr{P}(F) \in \mathcal{P}$. On the other hand, F is contained in the closure of a chamber relative to $\mathcal{H}$, and so is a facet relative to the set of walls of this chamber (Proposition 4.1.11). Consequently, $\overline{F}$ is of the form

$$\{v \in V : \alpha(v) \geq 0 \text{ for all } \alpha \in T\},$$

where T is a subset of Φ which we can clearly take to be $\mathscr{P}(F)$. Thus, $\overline{F} = \mathscr{F}(\mathscr{P}(F))$. □

If $P \in \mathcal{P}$, the facet F such that $P = \mathscr{P}(F)$ is said to be associated to P; we denote it by F_P. By Proposition 5.1.61, we see $F \mapsto \mathscr{P}(F)$ is a bijection from $\mathcal{F}$ to $\mathcal{P}$.

As an application of the results in this paragraph, we show the existence of a maximal root in an irreducible root system. More precisely, we have the following proposition.

Proposition 5.1.62. *Assume that Φ is irreducible. Let C be a chamber of Φ, and let $\Delta(C) = \{\alpha_1, \dots, \alpha_l\}$ be the corresponding basis. There exists a root $\tilde{\alpha} = \sum_{i=1}^l n_i \alpha_i$ such that*

(a) for every root $\sum_{i=1}^l p_i \alpha_i$, we have $n_i \geq p_i$ for all i. In other words, Φ has a largest element for the ordering defined by C;

(b) $\tilde{\alpha} \in \overline{C}$, and $(\tilde{\alpha}, \tilde{\alpha}) \geq (\alpha, \alpha)$ for every root α;

(b) for every positive root α not proportional to $\tilde{\alpha}$, we have $n(\alpha, \tilde{\alpha}) = 0$ or 1.

The root $\tilde{\alpha}$ is said to be the highest root of Φ (with respect to C).

Proof. Let $\alpha = \sum_{i=1}^l n_i \alpha_i$ and $\beta = \sum_{i=1}^l p_i \alpha_i$ be two maximal roots for the ordering defined by C. We shall prove that $\alpha = \beta$, which will establish (a). Note that if $(\alpha, \alpha_i) < 0$ for some index i, it follows that either $\alpha + \alpha_i \in \Phi$ or $\alpha = -\alpha_i$, and both possibilities are absurd by the maximality of α. Thus $(\alpha, \alpha_i) \geq 0$ for all i.

If $\alpha < 0$, then $\alpha < -\alpha$, which is absurd. Thus $n_i \geq 0$ for all i. Let I be the set of $\{1, \dots, l\}$ such that $n_i > 0$, and J the complement of I. Then $I \neq \varnothing$. If J were nonempty, there would exist an $i \in I$ and $j \in J$ such that $(\alpha_i, \alpha_j) < 0$ (Corollary 5.1.55); we would then have

$$(\alpha, \alpha_j) = \sum_{i=1}^n n_i (\alpha_i, \alpha_j) < 0$$

since $(\alpha_i, \alpha_j) < 0$ whenever i and j are distinct, which is a contradiction. Thus $J = \varnothing$ and $n_i > 0$ for all i. Similarly, we have $(\beta, \alpha_i) \geq 0$ for all i, and we can not have $(\beta, \alpha_i) = 0$ for all i since $\beta \neq 0$, so

$$(\beta, \alpha) = \sum_{i=1}^l n_i (\beta, \alpha_i) > 0.$$

If $\gamma = \beta - \alpha \in \Phi$, either $\alpha > \beta$ or $\beta > \alpha$ since γ is either positive or negative, which contradicts the maximality of α and β. Thus $\alpha = \beta$ (Corollary 5.1.18). This proves the existence of $\tilde{\alpha}$.

Now since $(\alpha, \alpha_i) \geq 0$ for all i, we have $\alpha \in \bar{C}$. We shall prove that $(\alpha, \alpha) \leq (\tilde{\alpha}, \tilde{\alpha})$ for every $\alpha \in \Phi$. Since $\bar{C}$ is a fundamental domain for $W(\Phi)$, we can assume that $\alpha \in \bar{C}$. We have $\tilde{\alpha} - \alpha \succ 0$, so $(\tilde{\alpha} - \alpha, x) \geq 0$ for all $x \in \bar{C}$. In particular $(\tilde{\alpha} - \alpha, \tilde{\alpha}) \succ 0$ and $(\tilde{\alpha} - \alpha, \alpha) \succ 0$, hence

$$(\tilde{\alpha}, \tilde{\alpha}) \geq (\alpha, \tilde{\alpha}) \geq (\alpha, \alpha).$$

Finally, to prove (c), we note that, if α is any root of Φ, then $(\tilde{\alpha}, \tilde{\alpha}) \geq (\alpha, \tilde{\alpha})$ still holds since $\tilde{\alpha} - \alpha \succ 0$ and $\tilde{\alpha} \in \bar{C}$. Thus $n(\alpha, \tilde{\alpha}) = 2(\alpha, \tilde{\alpha})/(\tilde{\alpha}, \tilde{\alpha}) \leq 1$ if α is not proportional to $\tilde{\alpha}$. If $\alpha \succ 0$, then $(\tilde{\alpha}, \alpha) \geq 0$ since $\tilde{\alpha} \in \bar{C}$, so $n(\alpha, \tilde{\alpha}) \geq 0$ and therefore $n(\alpha, \tilde{\alpha})$ must be either 0 or 1. □

Example 5.1.63. We consider several irreducible root systems of rank 2 and draw their highest roots. In Figure 5.1 to Figure 5.4, the highest roots are computed by the given basis.

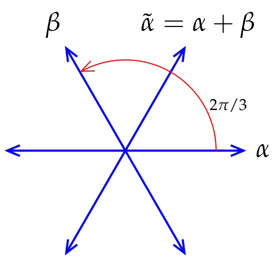


Figure 5.1: The root system A_2.

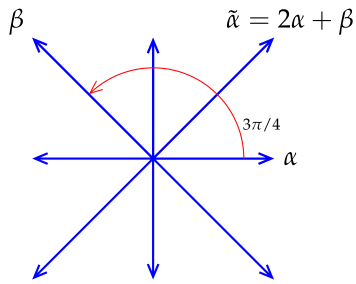


Figure 5.2: The root system B_2.

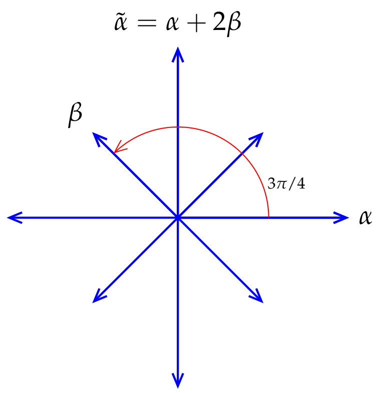


Figure 5.3: The root system C_2.

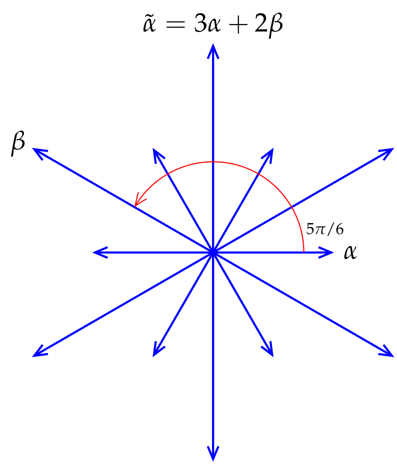


Figure 5.4: The root system G_2.

5.1.7 Weights and dominant weights

Let Φ be a root system in V with $l = \dim(V)$. Denote by $\Lambda_r(\Phi)$ the lattice of V generated by Φ (which is called the **root lattice**); the elements of $\Lambda_r(\Phi)$ are called the **radical weights** of Φ. By Theorem 5.1.35, $\Lambda_r(\Phi)$ is a discrete subgroup of V of rank l, and every basis of Φ is a basis of $\Lambda_r(\Phi)$. Similarly, the group $\Lambda_r(\check{\Phi})$ is a discrete subgroup of V^* of rank l.

Proposition 5.1.64. *The set of $x \in V$ such that $\langle z, y^* \rangle \in \mathbb{Z}$ for all $y^* \in \Lambda_r(\check{\Phi})$ (or, equivalently, for all $y^* \in \check{\Phi}$) is a discrete subgroup G of V containing $\Lambda_r(\Phi)$. If $\check{\Delta}$ is a basis of $\check{\Phi}$, the basis of V dual to $\check{\Delta}$ is a basis of G.*

Proof. Let $x \in V$. The following three properties are equivalent:

(i) $\langle x, y^* \rangle \in \mathbb{Z}$ for all $y^* \in \check{\Phi}$;

(ii) $\langle x, y^* \rangle \in \mathbb{Z}$ for all $y^* \in \check{\Delta}$;

(iii) the coordinates of x with respect to the dual basis of $\check{\Delta}$ are in $\mathbb{Z}$.

We then conclude that the basis dual to $\check{\Delta}$ is a basis of G. On the other hand, (R3) proves that $\Phi \subseteq G$, so $\Lambda_r(\Phi) \subseteq G$. □

The group G of Proposition 5.1.64 is denoted by $\Lambda(\Phi)$, and its elements are called the **weights** of Φ. We can also consider the group $\Lambda(\check{\Phi})$ of weights of $\check{\Phi}$. Since

$$\operatorname{rank}(\Lambda(\Phi)) = \operatorname{rank}(\Lambda_r(\Phi)) \quad \text{and} \quad \operatorname{rank}(\Lambda(\check{\Phi})) = \operatorname{rank}(\Lambda_r(\check{\Phi})),$$

we see the quotient groups $\Lambda(\Phi)/\Lambda_r(\Phi)$ and $\Lambda(\check{\Phi})/\Lambda_r(\check{\Phi})$ are finite groups in duality over $\mathbb{Q}/\mathbb{Z}$, and hence are isomorphic (this common group is called the **fundamental group** of Φ). The common order of these two groups is called the **connection index** of Φ (or of $\check{\Phi}$). It is clear that if Φ is a direct sum of root systems Φ_i, then the group $\Lambda_r(\Phi)$ (resp. $\Lambda(\Phi)$) is identified canonically with the direct sum of the $\Lambda_r(\Phi_i)$ (resp. $\Lambda(\Phi_i)$).

Proposition 5.1.65. *Let $\tilde{\Phi}$ be a subset of Φ, $\tilde{\Lambda}_r$ the subgroup of $\Lambda_r(\Phi)$ generated by $\tilde{\Phi}$, and $\tilde{W}$ the subgroup of $W(\Phi)$ generated by the s_α, with $\alpha \in \tilde{\Phi}$. If $x \in \Lambda(\Phi)$ and $w \in \tilde{W}$, then $x - w(x) \in \tilde{\Lambda}_r$.*

Proof. If $w = s_\alpha$ for $\alpha \in \tilde{\Phi}$, then

$$x - w(x) = \langle x, \check{\alpha}\rangle\alpha \in \mathbb{Z}\alpha \subseteq \tilde{\Lambda}_r.$$

If $w = s_{\alpha_1} \cdots s_{\alpha_p}$ with $\alpha_i \in \tilde{\Phi}$, it is still true that $x - w(x) \in \tilde{\Lambda}_r$, as we can see by induction. □

Corollary 5.1.66. *Let Φ be a root system. Then the quotient group* $\operatorname{Aut}(\Phi)/W(\Phi)$ *(cf. Proposition 5.1.33) acts canonically on $\Lambda(\Phi)/\Lambda_r(\Phi)$.*

Proof. Let $\tau \in \operatorname{Aut}(\Phi)$. Recall that by the remark follows Proposition 5.1.9, for any weight $x \in V$ and $\alpha \in \Phi$, we have

$$\langle \tau(x), \check{\alpha}\rangle = n(\tau(x), \alpha) = n(x, \tau^{-1}(\alpha)).$$

Therefore the group $\operatorname{Aut}(\Phi)$ leaves $\Lambda(\Phi)$ and $\Lambda_r(\Phi)$ invariant, and hence acts on the quotient group $\Lambda(\Phi)/\Lambda_r(\Phi)$. By Proposition 5.1.65 (take $\tilde{\Phi} = \Phi$), the group $W(\Phi)$ acts trivially on $\Lambda(\Phi)/\Lambda_r(\Phi)$. Passing to the quotient, we see that the quotient group $\operatorname{Aut}(\Phi)/W(\Phi)$ (cf. Proposition 5.1.33) acts canonically on $\Lambda(\Phi)/\Lambda_r(\Phi)$. □

From now on, we assume that Φ is *reduced*. Let C be a chamber of Φ, and let Δ be the corresponding basis of Δ. Since Φ is reduced, $\check{\Delta} = \{\check{\alpha}\}_{\alpha\in\Delta}$ is a basis of $\check{\Phi}$. The dual basis $(\varpi_\alpha)_{\alpha\in\Delta}$ of $\check{\Delta}$ is thus a basis of the group of weights; its elements are called the **fundamental weights** (relative to Δ, or to C); if the elements of Δ are denoted by $(\alpha_1, \dots, \alpha_l)$, the corresponding fundamental weights are denoted by $(\varpi_1, \dots, \varpi_l)$. For any element $x \in V$, $x \in C$ if and only if $\langle x, \check{\alpha}\rangle > 0$ for all $\alpha \in \Delta$. It follows that C is the set of linear combinations of the ϖ_i, with positive coefficients.

The entries $n(\alpha, \beta) = \langle \alpha, \check{\beta}\rangle$ of the Cartan matrix are, for fixed α, the coordinates of a with respect to the basis $(\varpi_\beta)_{\beta\in\Delta}$:

$$\alpha = \sum_{\beta\in\Delta} n(\alpha, \beta)\varpi_\beta.$$

The Cartan matrix is thus the transpose of the matrix of the canonical injection $\Lambda_r(\Phi) \to \Lambda(\Phi)$ with respect to the bases Δ and $(\varpi_\beta)_{\beta\in\Delta}$ of the $\mathbb{Z}$-moduhas $\Lambda_r(\Phi)$ and $\Lambda(\Phi)$. Therefore, to express the fundamental weights in means of the basis Δ, one only need to invert the Cartan matrix for Δ. In particular, we conclude that the connection index is just the determinant of the Cartan matrix.

Note that we have

$$\langle \varpi_\alpha, \check{\beta}\rangle = (\varpi_\alpha, \frac{2\beta}{(\beta, \beta)}) = \delta_{\alpha\beta}$$

for $\alpha, \beta \in \Delta$ ($\delta_{\alpha\beta}$ denoting the Kronecker symbol) since $(\varpi_\alpha)_{\alpha\in\Delta}$ is the dual basis for $\check{\Delta}$, hence

$$s_\beta(\varpi_\alpha) = \varpi_\alpha - \delta_{\alpha\beta}\beta \quad \text{and} \quad (\varpi_\alpha, \beta) = \frac{1}{2}(\beta, \beta)\delta_{\alpha,\beta}. \tag{5.1.14}$$

In other words, ϖ_α is orthogongl to β for $\beta \neq \alpha$, and its orthogonal projection onto $\mathbb{R}\alpha$ is $\frac{1}{2}\alpha$. Since $\varpi_\alpha \in \overline{C}$, $(\varpi_\alpha, \varpi_\beta) \geq 0$ for $\alpha, \beta \in \Delta$, i.e., the angle $\widehat{(\varpi_\alpha, \varpi_\beta)}$ is acute or a right angle.

An element $\lambda \in V$ is said to be **dominant** if it belongs to $\overline{C}$, in other words if its coordinates with respect to $(\varpi_\beta)_{\beta\in\Delta}$ are non-negative integers, or equivalently if $2(\lambda, \alpha)/(\alpha, \alpha)$ is positive for all $\alpha \in \Delta$; it is **strictly dominant** if $\lambda \in C$. Since $\overline{C}$ is a fundamental domain for $W(\Phi)$, there exists, for any element $\lambda \in V$, a unique dominant element $\tilde{\lambda}$ such that $\tilde{\lambda}$ is a transform of λ by $W(\Phi)$. We denote by $\Lambda^+(\Phi)$ the set of weights for Φ that are positive with respect to the order on V defined by Δ, and $\Lambda^{++}(\Phi)$ the set of dominant weights for Φ.

Proposition 5.1.67. *Let Δ be a basis of Φ, Σ a subset of Δ, V_Σ the vector subspace of V generated by Σ, $\Phi_\Sigma = \Phi \cap V_\Sigma$ (which is a root system in V_Σ), and π the orthogonal projection of V onto V_Σ. Then $\Lambda_r(\Phi_\Sigma) = \Lambda_r(\Phi) \cap V_\Sigma$, $\Lambda(\Phi_\Sigma) = \Lambda(\Phi) \cap V_\Sigma$, and the set of dominant weights of Φ_Σ is the image under π of the set of dominant weights of Φ.*

Proof. Indeed, $\Lambda_r(\Phi)$ is the subgroup of V with basis Δ, $\Lambda_r(\Phi_\Sigma)$ is the subgroup of V_Σ with basis Σ, from which $\Lambda_r(\Phi_\Sigma) = \Lambda_r(\Phi) \cap V_\Sigma$ is immediate. If $\varpi \in \Lambda(\Phi)$ and $\alpha \in \Phi_\Sigma$, then

$$\langle \pi(\varpi), \check{\alpha} \rangle = \frac{2(\pi(\varpi), \alpha)}{(\alpha, \alpha)} = \frac{2(\varpi, \pi(\alpha))}{(\alpha, \alpha)} = \frac{2(\varpi, \alpha)}{(\alpha, \alpha)} = \langle \varpi, \check{\alpha} \rangle \in \mathbb{Z},$$

so $\pi(\varpi) \in \Lambda(\Phi_\Sigma)$ and hence $\pi(\Lambda(\Phi)) \subseteq \Lambda(\Phi_\Sigma)$. If $\tilde{\varpi} \in \Lambda(\Phi_\Sigma)$, $\tilde{\varpi}$ extends to a linear form ϖ on V^* vanishing on $\check{\Delta} \setminus \check{\Sigma}$; then we have $\langle \varpi, \check{\alpha} \rangle \in \mathbb{Z}$ for all $\alpha \in \Delta$, so $\alpha \in \Lambda(\Phi)$, and $\tilde{\varpi} = \pi(\varpi)$; hence $\Lambda(\Phi_\Sigma) = \pi(\Lambda(\Phi))$. The assertion about dominant weights is proved in the same way. □

Proposition 5.1.68. *Let $\delta = \frac{1}{2}\sum_{\beta \in \Phi^+(C)} \beta$ be the half sum of the positive roots.*

(a) $\delta = \sum_{\beta \in \Delta} \varpi_\beta$, which is an element in C.

(b) $s_\alpha(\delta) = \delta - \alpha$ for all $\alpha \in \Delta$.

(c) $(\delta, \alpha) = \frac{1}{2}(\alpha, \alpha)$ for all $\alpha \in \Delta$.

*The vector δ is called the **Weyl vector** for Φ.*

Proof. Since Φ is reduced, $s_\alpha(\Phi^+ \setminus \{\alpha\}) = \Phi^+ \setminus \{\alpha\}$ by Corollary 5.1.38 and $s_\alpha(\alpha) = -\alpha$ for all $\alpha \in \Delta$, so $s_\alpha(\delta) = \delta - \alpha$. Since $s_\alpha(\delta) = \delta - \langle \delta, \check{\alpha} \rangle \alpha$, this implies

$$\langle \delta, \check{\alpha} \rangle = 1 = \langle \sum_{\beta \in \Delta} \varpi_\beta, \check{\alpha} \rangle.$$

Hence $\delta = \sum_\beta \varpi_\beta$, and consequently $\delta \in C$. Finally, (c) is equivalent to $\langle \delta, \check{\alpha} \rangle = 1$. □

Corollary 5.1.69. *Let ρ be half the sum of positive roots of $\check{\Phi}$ (for $\check{\Delta}$). For all $\alpha \in V$, the sum of the coordinates of α with respect to the basis Δ is $\langle \alpha, \rho \rangle$. If $\alpha \in \Phi$, this sum is equal to $\frac{1}{2}\sum_{\beta \in \Phi^+(C)} n(\alpha, \beta)$.*

Proof. Interchanging the roles of Φ and $\check{\Phi}$ above, we have $\langle \alpha, \rho \rangle = 1$ for all $\alpha \in \Delta$, hence the corollary. □

From the description (5.1.14), it is easy to determine the fundamental weights for rank 2 root systems, and the Weyl vector δ can be then computed using Proposition 5.1.68(a). We now give the precise result for these root systems.

Example 5.1.70 (The weight lattice of A_2). Consider the root system A_2, which is given in Figure 5.1. We know that $\widehat{(\alpha, \beta)} = 2\pi/3$, so the fundamental weights ϖ_α and ϖ_β satisfy $\widehat{(\varpi_\alpha, \alpha)} = \widehat{(\varpi_\beta, \beta)} = \pi/6$. By some geometric computations, we find that

$$\varpi_\alpha = \frac{1}{3}(2\alpha + \beta), \quad \varpi_\beta = \frac{1}{3}(\alpha + 2\beta).$$

The weight lattice of A_2 is presented in Figure 5.5.

Example 5.1.71 (The weight lattice of B_2). Now consider the root system B_2, which is given in Figure 5.2. The basis $\{\alpha, \beta\}$ of B_2 satisfies $\widehat{(\alpha, \beta)} = 3\pi/4$ and $\|\beta\| = \sqrt{2}\|\alpha\|$, so the fundamental weights ϖ_α and ϖ_β satisfy $\widehat{(\varpi_\alpha, \alpha)} = \widehat{(\varpi_\beta, \beta)} = \pi/4$. It is then easy to compute that

$$\varpi_\alpha = \frac{1}{2}(2\alpha + \beta), \quad \varpi_\beta = \alpha + \beta.$$

The weight lattice of B_2 is presented in Figure 5.6.

Note that the root system C_2 has a similar pattern as B_2, and is given in Figure 5.3. In fact, B_2 is isomorphic to B_2, so they have similar weight lattices.

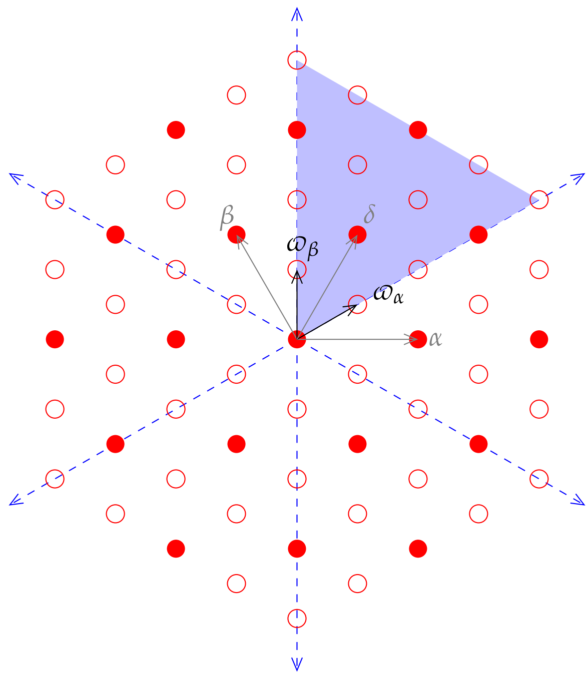


Figure 5.5: The weight lattice of A_2.

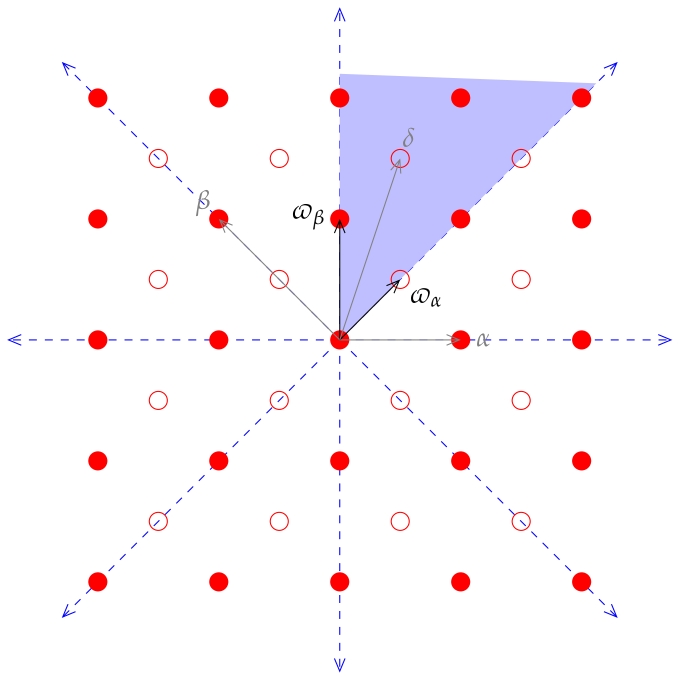


Figure 5.6: The weight lattice of B_2.

Example 5.1.72. Finally, consider the root system G_2, which is given in Figure 5.4. Let $\{\alpha, \beta\}$ be the basis for G_2, then $\widehat{(\alpha, \beta)} = 5\pi/6$ and $\|\beta\| = \sqrt{3}\|\alpha$. From these, we see the fundamental weights ϖ_α and ϖ_β satisfy $\widehat{(\varpi_\alpha, \alpha)} = \widehat{(\varpi_\beta, \beta)} = \pi/3$. Now some geometric computations show that

$$\varpi_\alpha = 2\alpha + \beta, \quad \varpi_\beta = 3\alpha + 2\beta \tag{5.1.15}$$

The weight lattice of G_2 is then available, and is presented in Figure 5.7.

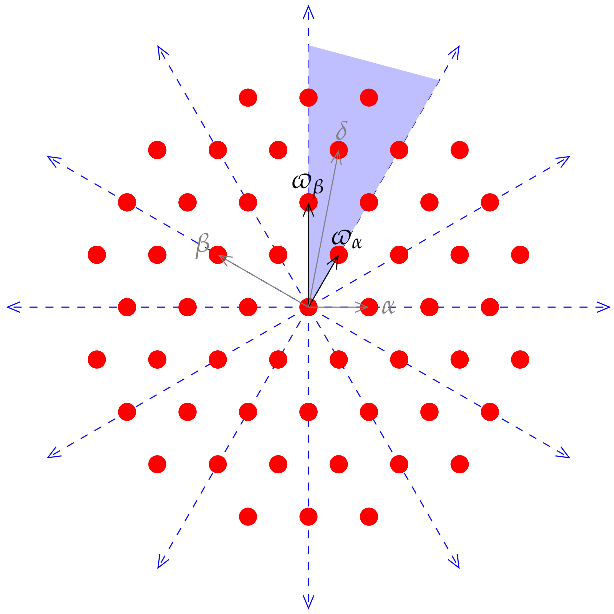


Figure 5.7: The weight lattice of G_2.

Recall that we have introduced a partial ordering on the space V compactible with the basis of Φ: Let Δ be a basis for Φ and $\lambda, \mu \in E$. Then we say λ is **higher** than μ, written $\lambda \preceq \mu$, if $\mu - \lambda$ is a nonnegative $\mathbb{R}$-linear combination of elements in Δ. We equivalently say that μ is **lower** than λ. We now develop various useful properties of this relation, which will be used later to formulate the theorem of the highest weight for representations of a semi-simple Lie algebra. From now on, we assume a basis Δ for Φ has been chosen, and that the notions of higher, lower, and dominant are defined relative to Δ.

Proposition 5.1.73. *Let $\lambda \in V$ be a dominant element. Then*

(a) $\lambda \succeq 0$;

(b) $w(\lambda) \preceq \lambda$ for all $w \in W$;

(c) if λ is a strict dominant weight, then $\lambda \succeq \delta$, where δ is the Weyl vector for Φ.

Proof. Assertion (b) follows from Proposition 5.1.41, by the definition of dominant. For any basis $\{\alpha_1, \dots, \alpha_l\}$ of V, we can form a dual basis $\{\alpha_1^*, \dots, \alpha_l^*\}$ of V by the relation $(\alpha_i^*, \alpha_j) = \delta_{ij}$. By Proposition 4.3.16, since $\{\alpha_1, \dots, \alpha_j\}$ is an obtuse basis, we see $\{\alpha_1^*, \dots, \alpha_j^*\}$ is a acute basis for V, meaning that $(\alpha_i^*, \alpha_j^*) \geq 0$ for all i, j. If λ is dominant, the coefficients in the expansion $\lambda = \sum_j c_j \alpha_j$ are given by

$$c_j = (\alpha_j^*, \lambda) = \sum_{i=1}^{r} (\alpha_j^*, \alpha_i^*)(\alpha_i, \lambda).$$

Since λ is dominant, we have $(\alpha_i, \lambda) \geq 0$. Furthermore, the α_i^*'s form an obtuse basis for E and thus $(\alpha_i^*, \alpha_j^*) \geq 0$ for each i, j. This proves $c_j \geq 0$, hence $\lambda \succeq 0$.

Suppose that λ is a strictly dominant weight. Then $\lambda - \delta$ will still be dominant in light of Proposition 5.1.68(a). Thus, by assertion (a) we have $\lambda - \delta \succeq 0$, which is equivalent to $\lambda \succeq \delta$. ☐

Recall that the *convex hull* of a subset $X \subseteq V$ is the smallest convex subset of E containing X. For $\lambda \in V$, we let $W \cdot \lambda$ denote the Weyl-group orbit of $\lambda \in V$ and we let $\operatorname{conv}(W \cdot \lambda)$ denote the convex hull of $W \cdot \lambda$.

Proposition 5.1.74. *Let $\lambda \in V$ be a dominant weight. Then the convex hull of the orbit of λ under W is given by*

$$\operatorname{conv}(W \cdot \lambda) = \{\mu \in V : w(\mu) \preceq \lambda \text{ for all } w \in W\}.$$

By the characterization of the convex hull, we see $\operatorname{conv}(W \cdot \lambda)$ is invariant under the Weyl group. Thus if μ belongs to $\operatorname{conv}(W \cdot \lambda)$, then every point in $\operatorname{conv}(W \cdot \mu)$ also belongs to $\operatorname{conv}(W \cdot \lambda)$. Then the proposition may be restated as follows: if λ and μ are dominant, then

$$\mu \preceq \lambda \Leftrightarrow \operatorname{conv}(W \cdot \mu) \subseteq \operatorname{conv}(W \cdot \lambda).$$

We establish two lemmas that will lead to a proof of Proposition 5.1.74.

Lemma 5.1.75. *Suppose K is a compact, convex subset of V and μ is an element of V that it is not in K. Then there is an element γ of V such that we have $(\gamma, \mu) > (\gamma, \eta)$ for all $\eta \in K$.*

Proof. Since K is compact, we can choose an element η_0 of K that minimizes the distance to μ. Set $\gamma = \mu - \eta_0$, so that

$$(\gamma, \mu - \eta_0) = (\mu - \eta_0, \mu - \eta_0) \geq 0.$$

and thus $(\gamma, \mu) \geq (\gamma, \eta_0)$. Now, for any $\eta \in K$, the vector $\eta_0 + t(\eta - \eta_0)$ belongs to K for $0 \leq t \leq 1$, and we compute that

$$\|\eta_0 + t(\eta - \eta_0) - \mu\|^2 = (\mu - \eta_0, \mu - \eta_0) - 2t(\mu - \eta_0, \eta - \eta_0) + t^2(\eta - \eta_0, \eta - \eta_0).$$

The only way this quantity can be greater than or equal to $(\mu - \eta_0, \mu - \eta_0)$ for small positive t is if

$$(\mu - \eta_0, \eta - \eta_0) = (\gamma, \eta - \eta_0) \leq 0.$$

Thus $(\gamma, \eta) \leq (\gamma, \eta_0) \leq (\gamma, \mu)$, which completes the proof. □

Lemma 5.1.76. *If λ and μ are dominant and $\mu \notin \operatorname{conv}(W \cdot \lambda)$, then there exists a dominant element $\gamma \in V$ such that*

$$(\gamma, \mu) > (\gamma, w(\lambda))$$

for all $w \in W$.

Proof. By Lemma 5.1.75, we can find some $\gamma \in V$, not necessarily dominant, such that $(\gamma, \mu) \geq (\gamma, \eta)$ for all $\eta \in \operatorname{conv}(W \cdot \lambda)$. In particular, we have $(\gamma, \mu) > (\gamma, w(\lambda))$ for all $w \in W$. Choose some $w_0 \in W$ so that $\tilde{\gamma} := w_0(\gamma)$ is dominant. We will show that replacing γ by $\tilde{\gamma}$ makes (γ, μ) bigger while permuting the values of $(\gamma, w(\lambda))$.

By Proposition 5.1.41, $\gamma \preceq \tilde{\gamma}$, meaning that $\tilde{\gamma}$ equals γ plus a non-negative linear combination of positive simple roots. But since μ is dominant, it has non-negative inner product with each positive simple root, and we see that $(\tilde{\gamma}, \mu) > (\gamma, \mu)$. Thus

$$(\tilde{\gamma}, \mu) > (\gamma, \mu) > (\gamma, w(\mu))$$

for all $w \in W$. On the other hand,

$$(\tilde{\gamma}, w(\lambda)) = (w_0(\gamma), w(\lambda)) = (\gamma, (w_0^{-1}w)(\lambda)).$$

Thus, as w ranges over W, the values of $(\gamma, w(\lambda))$ and $(\tilde{\gamma}, w(\lambda))$ range through the same set of real numbers. Thus, $(\tilde{\gamma}, \mu) > (\tilde{\gamma}, w(\lambda))$ for all w as claimed. □

Proof of Proposition 5.1.74. Since each $\mu \in V$ is W-conjugate to a dominant element which is maximal in the orbit of μ, to prove the proposition, we may assume that μ is dominant. Assume first that $\mu \in \operatorname{conv}(W \cdot \lambda)$. By Proposition 5.1.41, every element of the form $w(\lambda)$, for $w \in W$, is lower than λ. But it is easy to see a convex combination of elements lower than λ is still lower than λ, thus $\mu \preceq \lambda$.

Conversely, assume that $\mu \notin \operatorname{conv}(W \cdot \lambda)$. Let γ be a dominant element as in Lemma 5.1.76. Then since $(\gamma, \lambda) > (\gamma, \mu)$, we see $(\gamma, \lambda - \mu) < 0$. Since γ is dominant, this rules out the case $\mu \preceq \lambda$ (recall that elements in $\bar{C}$ have acute angle by Proposition 4.3.16). □

5.1.8 Coxeter transformation

Let C be a chamber of Φ, let $\{\alpha_1,\ldots,\alpha_l\}$ be the corresponding basis of Φ, and let $w_C = s_{\alpha_1}\cdots s_{\alpha_l}$. The element w_C of W is called the Coxeter transformation of W defined by C and the bijection $i \mapsto \alpha_i$. Its order h is called the Coweter number of W (or of Φ).

Proposition 5.1.77. *Assume that Φ is irreducible. Let m be an integer between 1 and $h-1$ and prime to h. Then $e^{2\pi i m/h}$ is an eigenvalue of w_C of multiplicity 1.*

For the proof of this result, we need the following lemma:

Lemma 5.1.78. *For all $w \in W$, the characteristic polynomial of w has integer coefiicients.*

Proof. We know that $\{\alpha_1,\ldots,\alpha_l\}$ is a basis of the subgroup $\Lambda_r(\Phi)$ of V generated by Φ. Since w leaves $\Lambda_r(\Phi)$ stable, its matrix with respect to $\{\alpha_1,\ldots,\alpha_l\}$ has integer entries; hence its characteristic polynomial has integer coeflicients. □

Proof of Proposition 5.1.77. Let $P(T)$ be the characteristic polynomial of w_C. The above lemma shows that the coeflicients of $P(T)$ are integers. By Corollary 4.6.12, the primitive h-th root of unity $\zeta = e^{2\pi i/h}$ is a simple root of $P(T)$. Every conjugate of ζ over $\mathbb{Q}$ is therefore also a simple root of $P(T)$. But we know that the primitive h-th roots of unity are conjugate over $\mathbb{Q}$. They are thus all simple roots of $P(T)$, which proves the proposition. □

Proposition 5.1.79. *Assume that Φ is irreducible and reduced, and let $\tilde{\alpha} = \sum_{i=1}^{l} n_i\alpha_i$ be the highest root of Φ. Then $n_1+\cdots+n_l = h-1$.*

Proof. Let Φ^+ be the set of positive roots relative to C. Then by Corollary 5.1.69,

$$\sum_{i=1}^{l} = \frac{1}{2}\sum_{\alpha\in\Phi^+} n(\tilde{\alpha},\alpha) = 1+\frac{1}{2}\sum_{\alpha\in\Phi^+,\alpha\neq\tilde{\alpha}} n(\tilde{\alpha},\alpha) = 1+\sum_{\alpha\in\Phi^+,\alpha\neq\tilde{\alpha}}\frac{(\alpha,\tilde{\alpha})}{(\alpha,\alpha)}.$$

By Proposition 5.1.62, for any $\alpha\in\Phi^+$ and $\alpha\neq\tilde{\alpha}$, $n(\alpha,\tilde{\alpha}) = 0$ or 1, so $n(\tilde{\alpha},\alpha)^2 = n(\tilde{\alpha},\alpha)$, that is,

$$\frac{4(\alpha,\tilde{\alpha})}{(\tilde{\alpha},\tilde{\alpha})} = \frac{2(\alpha,\tilde{\alpha})}{(\tilde{\alpha},\tilde{\alpha})}.$$

Hence

$$\sum_{i=1}^{l} +1 = 2+2\sum_{\alpha\in\Phi^+,\alpha\neq\tilde{\alpha}}\frac{(\alpha,\tilde{\alpha})^2}{(\alpha,\alpha)(\tilde{\alpha},\tilde{\alpha})} = 2\sum_{\alpha\in\Phi^+}\frac{(\alpha,\tilde{\alpha})^2}{(\alpha,\alpha)(\tilde{\alpha},\tilde{\alpha})} = (\tilde{\alpha},\tilde{\alpha})^{-1}\sum_{\alpha\in\Phi}(\frac{\alpha}{\|\alpha\|},\tilde{\alpha})^2.$$

By Corollary 4.6.8, we have

$$\sum_{\alpha\in\Phi}(\frac{\alpha}{\|\alpha\|},\tilde{\alpha})^2 = h(\tilde{\alpha},\tilde{\alpha}),$$

so $n_1+\cdots+n_l+1 = h$. □

Proposition 5.1.80. *Assume that Φ is irreducible, and that all roots have the same length (such a group is called* ***simply laced****). Let $\alpha\in\Phi$, then the number of elements of Φ not orthogonal to α is $4h-6$.*

Proof. Note that the assumption implies Φ is reduced. Let $\tilde{\Phi}$ be the set of roots not orthogonal to and not proportional to α. By Corollary 4.6.8,

$$(\alpha,\alpha)^2+(\alpha,-\alpha)^2+\sum_{\beta\in\tilde{\Phi}}(\alpha,\beta)^2 = h(\alpha,\alpha)^2.$$

That is,

$$\sum_{\beta\in\tilde{\Phi}}(\alpha,\beta)^2 = (h-2)(\alpha,\alpha)^2.$$

If $\beta\in\tilde{\Phi}$, then $(\alpha,\alpha) = \pm\frac{1}{2}(\alpha,\alpha)$ by Table 5.1. Hence $\frac{1}{4}|\tilde{\Phi}| = h-2$ and so $|\tilde{\Phi} = 4h-8$. The number of roots not orthogonal to α is then $|\tilde{\Phi}+2 = 4h-6$ (plus α and $-\alpha$). □

Proposition 5.1.81. *Assume that Φ is irreducible and reduced. Put $s_{\alpha_i} = s_i$ and let $w_C = s_1\cdots s_l$.*

(a) *Let $\theta_i = s_l s_{l-1} \cdots s_{i+1}(\alpha_i)$ for each i, then θ_i is positive and $w_C(\theta_i)$ is negative. Moreover, if α is a positive root such that $w_C(\alpha)$ is negative, then α is equal to one of the θ_i.*

(b) *The family $(\theta_i)_{i \leq i \leq l}$ is a basis for $\Lambda_r(\Phi)$.*

(c) *Let Ω_i be the orbit of θ_i under G. Then the sets Ω_i are pairwise disjoint, they are all the orbits of w_C on Φ, and each has h elements.*

Proof. Observe first that $(s_1, \ldots, s_l)$ is a reduced decomposition of w_C with respect to the set S of the s_i. Indeed, otherwise there would exist a subset $X = S \setminus \{s_j\}$ of $l-1$ elements of S such that $w_C \in W_X$, which implies $s_j = (s_1 \cdots s_{j-1}) w (s_{j+1} \cdots s_l) \in W_X$, contradicting to Corollary 3.1.32. Applying Corollary 5.1.39 to w_C gives assertion (a).

Let Λ_i be the subgroup of $\Lambda_r(\Phi)$ generated by the α_j, $j > i$. It is immediate that Λ_i is stable under the s_j, $j > i$, and that $s_j(\alpha_i) = \alpha_i \bmod \Lambda_i$ for $j > i$. Hence:

$$\theta_i = s_l \cdots s_{i+1}(\alpha_i) \equiv \alpha_i \mod Q_i.$$

In other words, there exist integers c_{ij} such that

$$\theta_i = \alpha_i + \sum_{j>i} c_{ij} \alpha_j.$$

Assertion (b) then follows immediately.

Finally, let α be a root. The element $\sum_{k=0}^{h-1} w_C^k(\alpha)$ is invariant under w_C, and hence zero (recall that 1 is not an eigenvalue of w_C). The $c^k(\alpha)$ cannot therefore all have the same sign, and there exists a k such that $w_C^k(\alpha) > 0$ and $w_C^{k+1}(\alpha) < 0$. By (a), $c^k(\alpha)$ is one of the θ_i. Thus every orbit of w_C on Φ is one of the Ω_i. Each orbit of w_C in Φ has at most h elements, and there are at most l distinct orbits, by the above remarks. Now by Theorem 4.6.6(b), the cardinal of Φ is equal to hl, which immediately implies (c). □

5.1.9 Exponential invariants

In this paragraph, R denotes a commutative ring, with a unit element, and not reduced to 0. Let Λ be a free $\mathbb{Z}$-module of finite rank l. We denote by $R[\Lambda]$ the group algebra of the additive group of Λ over R. For any $\lambda \in \Lambda$, denote by e^λ the corresponding element of $R[\Lambda]$. Then $(e^\lambda)_{\lambda \in \Lambda}$ is a basis of the R-module $R[\Lambda]$, and, for any $\lambda, \mu \in \Lambda$, we have

$$e^\lambda e^\mu = e^{\lambda+\mu}, \quad (e^\lambda)^{-1} = e^{-\lambda}, \quad e^0 = 1.$$

Lemma 5.1.82. *Assume that R is a factorial domain.*

(a) *The ring $R[\Lambda]$ is also factorial.*

(b) *If λ, μ are non-proportional elements of Λ, then the elements $1 - e^\lambda$ and $1 - e^\mu$ of $R[\Lambda]$ are relatively prime.*

Proof. Let $(\lambda_1, \ldots, \lambda_l)$ be a basis of Λ, and $X_1, \ldots, X_l$ be indeterminates. The R-linear map from $R[X_1, \ldots, X_l, X_1^{-1}, \ldots, X_l^{-}$ to $R[\Lambda]$ that takes $X_1^{n_1} \cdots X_l^{n_l}$ to $e^{n_1\lambda_1 + \cdots + n_l\lambda_l}$ is an isomorphism of rings. Now $R[X_1, \ldots, X_l]$ is a factorial ring, and $R[X_1, \ldots, X_l, X_1^{-1}, \ldots, X_l^{-1}]$ is a localization of $R[X_1, \ldots, X_l]$, hence is also factorial.

Let Γ_1 (resp. Γ_2) be the set of elements of Λ of which some multiple belongs to $\mathbb{Z}\lambda + \mathbb{Z}\mu$) (resp. $\mathbb{Z}\lambda$). Then the groups Λ/Γ_1 and Λ/Γ_2 are torsion-free, so there exists a complement of Γ_2 in Γ_1 and a complement of Γ_1 in Λ. Consequently, there exists a basis $(\gamma_1, \ldots, \gamma_l)$ of the $\mathbb{Z}$-module Λ and rational integers j, m, n such that $\lambda = j\gamma_1$, $\mu = m\gamma_1 + n\gamma_2$, with $j > 0$ and $n > 0$. Putting $X_i = e^{\gamma_i}$, we then have $1 - e^\lambda = 1 - X_1^j$ and $1 - e^\mu = 1 - X_1^m X_2^n$. Let K be an algebraic closure of the field of fractions of R, so that $R[\Lambda]$ can be identified with a subring of the ring $B = K[X_1, \ldots, X_l, X_1^{-1}, \ldots, X_l^{-1}]$. For any j-th root of unity ζ, $1 - \zeta X_1$ is irreducible in $K[X_1, \ldots, X_l]$; moreover, the ideal generated by $1 - \zeta X_1$ contains no monomial in the X_i. We conclude that the ideal $(1 - \zeta X_1)B$ of B is a prime ideal of height 1, hence that $1 - \zeta X_1$ is irreducible in B. The irreducible factors of $1 - X_1^j$ in B are thus of the form $1 - \zeta X_1$. Now none of these factors divide $1 - X_1^m X_2^n$ in B: for the homomorphism f from B to B such that $f(X_1) = \zeta^{-1}$, $f(X_i) = X_i$ for $i \geq 2$ satisfies

$$f(1 - \zeta X_1) = 0, \quad f(1 - X_1^m X_2^n) = 1 - z^{-m} X_2^n \neq 0.$$

Thus, $1 - X_1^j$ and $1 - X_1^m X_2^n$ are relatively prime in B. Since $R[\Lambda]$ is a subring of B, it follows that they are relatively prime in $R[\Lambda]$. □

Now let Φ be a reduced root system in a real vector space V. In the remaining, we take for Λ the group of weights of Φ. The group $W = W(\Phi)$ acts on Λ, hence also on the algebra $R[\Lambda]$; we have $w(e^{\lambda}) = e^{w(\lambda)}$ for $w \in W$ and $x \in \Lambda$. Let C be a chamber of Φ and let $\Delta = (\alpha_i)_{1\leq i\leq l}$ be the corresponding basis of Φ. We provide V (and hence also Λ) with the order structure defined by C.

Let $x = \sum_{\lambda\in\Lambda} x_\lambda e^\lambda$ be an element of $R[\Lambda]$. The set of $\lambda \in \Lambda$ such that $x_\lambda \neq 0$ is called the **support** of x and denoted by $\mathrm{supp}(x)$. The set X of maximal elements of $\mathrm{supp}(x)$ is called the **maximal support** of x. A term $x_\lambda e^\lambda$ with λ a maximal element in $\mathrm{supp}(x)$ is then called a **maximal term** of x.

Lemma 5.1.83. *Let $x \in R[\Lambda]$ and let $(x_\lambda e^\lambda)_{\lambda\in X}$ be the family of maximal terms of x. Let $\mu \in \Lambda$ and let $y \in R[\Lambda]$ be such that e^μ is the unique maximal term of y. Then, the family of maximal terms of xy is $(x_\lambda e^{\lambda+\mu})_{\lambda\in X}$.*

Proof. Put $x = \sum_\lambda x_\lambda e^\lambda$, $y = \sum_\gamma y_\gamma e^\gamma$, and $xy = \sum_\nu z_\nu e^\nu$. Then $\gamma \leq \mu$ for all $\gamma \in \Lambda$ such that $y_\gamma \neq 0$, and $z_\nu = \sum_{\lambda+\gamma=\nu} x_\lambda y_\gamma$. If $\nu = \lambda + \mu = \lambda' + \gamma$ with $\lambda \in X$ and $x_{\lambda'} y_\gamma \neq 0$, then $\gamma \preceq \mu$, hence $\lambda' \geq \lambda$ and concequently $\lambda' = \lambda$. Thus $z_{\lambda+\mu} = x_\lambda y_\mu = x_\lambda \neq 0$. This shows that $X + \mu$ is contained in the support of xy.

On the other hand, if $\nu = \lambda' + \gamma$ with $x_{\lambda'} y_\gamma \neq 0$, there exists $\lambda \in X$ such that $\lambda' \leq \lambda$ and we have $\lambda' + \gamma \leq \lambda + \gamma$. The maximal support of xy is therefore contained in $X + \mu$. Since no two elements of $X + \mu$ are comparable, it follows that $X + \mu$ is exactly the maximal support of any and we have seen above that $z_{\lambda+\mu} = x_\lambda$ for $\lambda \in X$, which completes the proof of the lemma. □

Corollary 5.1.84. *Let $y \in R[\Lambda]$ admits a unique maximal term. Then y is not a divisor of zero.*

Proof. Since $x \neq 0$ means that the maximal support of x is non-empty, Lemma 5.1.83 shows that $x \neq 0$ implies $xy \neq 0$ whenever y admits a unique maximal term of the form e^μ. □

For $w \in W$, let $\varepsilon(w) = (-1)^{\ell(w)}$ be the determinant of w. An element $x \in R[\Lambda]$ is said to be anti-invariant under W if

$$w(x) = \varepsilon(w)x$$

for all $w \in W$, where the length $\ell(w)$ is taken relative to the family of reflections s_{α_i}.. The anti-invariant elements of $R[\Lambda]$ form an R-submodule of $R[\Lambda]$. For any $x \in R[\Lambda]$, put

$$J(x) = \sum_{w\in W} \varepsilon(w)w(x) \tag{5.1.16}$$

For $x \in R[\Lambda]$ and $w \in W$, we have

$$w(J(x)) = \sum_{\tilde{w}\in W} \varepsilon(\tilde{w})w\tilde{w}(x) = \varepsilon(w)J(x).$$

so $J(x)$ is anti-invariant. On the other hand, for any anti-invariant element x of $R[\Lambda]$, we have $J(x) = |W| \cdot x$. It then follows that, if $|W|$ is invertible in R, the map $|W|^{-1}J$ is a projection from $R[\Lambda]$ onto the submodule of anti-invariant elements.

Let $\varpi_1, \dots, \varpi_l$ be the fundamental weights corresponding to the chamber C and

$$\delta = \varpi_1 + \cdots + \varpi_l$$

be half the sum of the positive roots. By definition, a strict dominant weight λ is of the form $\delta + \mu$ with μ a dominant weight. If λ is strict dominant, then $w(\lambda) < \lambda$ for all $w \neq 1$ and e^λ is thus the unique maximal term of $J(e^\lambda)$.

Proposition 5.1.85. *If 2 is not a zero divisor in R, then the elements $J(e^\lambda)$ for $\lambda \in \Lambda \cap C$ (i.e., the set of strict dominant weights) form a basis of the module of anti-invariant elements of $R[\Lambda]$.*

Proof. The weights $w(\lambda)$ for $w \in W$ and λ a strict dominant weight are pairwise distinct. It follows that the $J(e^\lambda)$ are linearly independent (by comparing the maximal term). On the other hand, let $x = \sum_\lambda x_\lambda e^\lambda$ be an anti-invariant element of $R[\Lambda]$. If λ_0 belongs to a wall, it is invariant under a reflection $s \in W$ and

$$x = \sum_\lambda x_\lambda e^\lambda = -s(x) = -\sum_\lambda x_\lambda e^{s(\lambda)}$$

It follows that $2x_{\lambda_0} = 0$, so $x_{\lambda_0} = 0$ by hypothesis. Since every element in Λ that does not belong to any wall can be written uniquely in the form $w(\lambda)$ with $w \in W$ and λ a strict dominant (i.e., $\lambda \in \Lambda \cap C$), we thus have

$$x = \sum_{\lambda\in\Lambda\cap C} \sum_{w\in W} x_{w(\lambda)} e^{w(\lambda)}. \tag{5.1.17}$$

Since $w(x) = \varepsilon(w)x$, we have $x_{w(\lambda)} = \varepsilon(w)x_\lambda$ and we deduce from (5.1.17) that

$$x = \sum_{\lambda \in \Lambda \cap C} x_\lambda J(e^\lambda)$$

which completes the proof. □

Consider now the element q of the algebra $R[\frac{1}{2}\Lambda]$ defined by

$$q = \prod_{\alpha \in \Phi^+} (e^{\alpha/2} - e^{-\alpha/2}) = e^\delta \prod_{\alpha \in \Phi^+} (1 - e^{-\alpha}) = e^{-\delta} \prod_{\alpha \in \Phi^+} (e^\alpha - 1). \tag{5.1.18}$$

Since $\delta \in \Lambda$, we see q is in fact an element of $R[\Lambda]$. It is called the **Weyl denominator** of Φ (because it appears as an denominator in the Weyl character formula).

Proposition 5.1.86. *Let q be the Weyl denominator defined by (5.1.18).*

(a) The element q is an anti-invariant element of $R[\Lambda]$; its unique maximal term is e^δ and $q = J(e^\delta)$.

(b) For any $\lambda \in \Lambda$, the element $J(e^\lambda)$ is divisible uniquely by q and the quotient $J(e^\lambda)/q$ is an element of $R[\Lambda]$ invariant under W. Moreover, if λ is strict dominant, then the unique maximal term of $J(e^\lambda)/q$ is $e^{\lambda-\delta}$.

(c) If 2 is not a zero divisor in R, multiplication by q is a bijection from the set of elements of $R[\Lambda]$ invariant under W to the set of anti-invariant elements of $R[\Lambda]$.

Proof. We know that, for each i, the reflection $s_i = s_{\alpha_i}$ leaves stable the set of positive roots other than α_i and that $s_i(\alpha_i) = -\alpha_i$. Hence,

$$s_i(q) = (e^{-\alpha_i/2} - e^{\alpha_i}/2) \prod_{\alpha \in \Phi^+, \alpha \neq \alpha_i} (e^{\alpha/2} - e^{-\alpha/2}) = -q = (-1)^{\ell(s_i)} q.$$

Since the s_i generate W, this proves the first assertion in (a). The second assertion in (a) follows immediately from (5.1.18) and Lemma 5.1.83, noting that 1 is the unique maximal term of $1 - e^{-\alpha}$ for $\alpha \in \Phi^+$.

Now assume that $R = \mathbb{Z}$. By Proposition 5.1.85,

$$q = \prod_{\lambda \in \Lambda \cap C} c_\lambda J(e^\lambda) \quad c_\lambda \in \mathbb{Z}. \tag{5.1.19}$$

On the other hand, it is clear from (5.1.18) that

$$q = e^\delta + \sum_{\mu \prec \delta} d_\lambda e^\mu. \tag{5.1.20}$$

If $\lambda \in \Lambda \cap C$ with $\lambda \neq \delta$, then $\lambda \succ \delta$ and the coefficient of e^λ in q is zero by (5.1.20). Thus $c_\lambda = 0$. Moreover, comparison of the coefficients of e^δ in (5.1.19) and (5.1.20) shows that $c_\delta = 1$ and hence that $q = J(e^\delta)$.

We continue to assume that $R = \mathbb{Z}$. Let $\lambda \in \Lambda$, $\alpha \in \Phi$ and M be a system of representatives of the right cosets of W with respect to the subgroup $\{1, s_\alpha\}$. Then,

$$J(e^\lambda) = \sum_{w \in M} \varepsilon(w) e^{w(\lambda)} + \sum_{w \in M} (-1)^{\ell(s_\alpha w)} e^{s_\alpha w(\lambda)}.$$

Now $s_\alpha w(\lambda) = w(\lambda) - \langle \check{\alpha}, w(\lambda) \rangle \alpha = w(\lambda) + n_w \alpha$, with $n_w \in \mathbb{Z}$. Thus

$$J(e^\lambda) = \sum_{w \in M} \varepsilon(w) e^{w(\lambda)} (1 - e^{n_w \alpha}) = \sum_{w \in M, n_w \neq 0} \varepsilon(w) e^{w(\lambda)} (1 - e^{n_w \alpha}).$$

If $n_w > 0$, it is clear that $1 - e^{n_w \alpha}$ is divisible by $1 - e^\alpha$ and this is also true when $n_w < 0$ since $1 - e^{n_w \alpha} = -e^{n_w \alpha}(1 - e^{-n_w \alpha})$. Hence, $J(e^\lambda)$ is divisible by $1 - e^\alpha$ in $\mathbb{Z}[\Lambda]$.

By Lemma 5.1.82, $\mathbb{Z}[\Lambda]$ is factorial and the elements $1 - e^\alpha$ for $\alpha \in \Phi$ and $\alpha > 0$ are mutually prime. It follows that $J(e^\lambda)$ is divisible in $\mathbb{Z}[\Lambda]$ by the product $\prod_{\alpha \in \Phi^+}(1 - e^\alpha)$, and hence also by $q = e^{-\delta} \prod_{\alpha \in \Phi^+}(1 - e^\alpha)$. If λ is strict dominant, then the unique maximal term in $J(e^\lambda)$ is e^λ. By Lemma 5.1.83, we conclude that the unique maximal term of $J(e^\lambda)/q$ is $e^{\lambda - \delta}$.

Returning to the general case, by extension of scalars from $\mathbb{Z}$ to R, we deduce from the above that $q = J(e^\delta)$ and that every element $J(e^\lambda)$ is divisible by q. Since e^δ is the unique maximal term of q, Corollary 5.1.84 show that there exists a unique element $y \in R[\Lambda]$ such that $J(e^\delta) = qy$ and it follows immediately that y is invariant under W. This proves (a) and (b). Finally, if 2 is not a zero divisor in R, Corollary 5.1.84 and Proposition 5.1.85 imply (c). □

Let $R[\Lambda]^W$ be the subalgebra of $R[\Lambda]$ consisting of the elements invariant under W. For $\lambda \in \Lambda$, let $W \cdot \lambda$ be the orbit of λ under W. Let

$$S(e^\lambda) = \sum_{w \in W} e^{w(\lambda)}$$

be the sum of the W-transforms of e^λ, which is a W-invariant element. If λ is dominant, then $w(\lambda) \preceq \lambda$ for all $w \in W$, so any $\mu \in W \cdot \lambda$ distinct from λ is lower than λ, and e^λ is then the unique maximal term in $S(e^\lambda)$.

Let $x = \sum_\lambda x_\lambda e^\lambda$. Then $x_{w(\lambda)} = x_\lambda$ for all $\lambda \in \Lambda$ and all $w \in W$. On the other hand, every orbit of W in Λ meets $\Lambda^{++} = \Lambda \cap \bar{C}$ in exactly one point. Hence,

$$x = \sum_{\lambda \in \Lambda^{++}} x_\lambda S(e^\lambda).$$

From this, we deduce that the set $S(e^\lambda)$ for $\lambda \in \Phi^+$ form a basis of the R-module $R[\Lambda]^W$. In fact, we have the following general result:

Proposition 5.1.87. *For any $\lambda \in \Lambda^{++}$, let x_λ be an element of $A[\Lambda]^W$ with unique maximal term e^λ. Then the family $(x_\lambda)_{\lambda \in \Lambda^{++}}$ is a basis of the R-module $R[\Lambda]^W$.*

To prove this, we need the following lemma:

Lemma 5.1.88. *Let I be an ordered set satisfying the following condition:*

(Min) Every non-empty subset of I contains a minimal element.

Let M be a free R-module, $(e_i)_{i \in I}$ a basis of M, and $(x_i)_{i \in I}$ a family of elements of M such that

$$x_i = e_i + \sum_{j<i} a_{ij} e_j$$

for all $i \in I$ (with $a_{ij} \in R$, the support of the family (a_{ij}) being finite for all i). Then $(x_i)_{i \in I}$ is a basis of M.

Proof. For any subset J of I, let M_J be the submodule of M with basis $(e_i)_{i \in J}$. Let $\mathfrak{S}$ be the set of subsets J of I with the following properties:

(a) If $j \leq i$ and $i \in J$, then $j \in J$;

(b) $(x_i)_{i \in J}$ is a basis of M_J.

It is immediate that $\mathfrak{S}$, ordered by inclusion, is inductive and non-empty. It therefore has a maximal element J. If $J \neq I$, let i_0 be a minimal element of $I \setminus J$ and put $\tilde{J} = J \cup \{i_0\}$. Every element $i \in I$ such that $i < i_0$ then belongs to J, so $\tilde{J}$ satisfies (a). On the other hand, $\tilde{J}$ also satisfies (b): indeed,

$$e_{i_0} = x_{i_0} - \sum_{j<i_0} a_{i_0 j} e_j$$

from which (b) follows. Hence $\tilde{J} \in \mathfrak{S}$, a contradiction. Thus, $J = I$ and the lemma is proved. □

Proof of Proposition 5.1.87. We now prove Proposition 5.1.87. We apply Lemma 5.1.88 with $I = \Lambda^{++}$. Let $\mu \in I$, and let I_μ be the set of $\lambda \in I$ such that $\lambda \preceq \mu$. If $\lambda \in I_\mu$, the relations

$$\mu - \lambda \succeq 0, \quad \lambda, \mu \in \bar{C}$$

imply that

$$(\mu - \lambda, \lambda) \geq 0, \quad (\mu - \lambda, \mu) \geq 0$$

and hence that

$$(\lambda, \lambda) \leq (\lambda, \mu) \leq (\mu, \mu).$$

The set I_μ is thus bounded. Since I is discrete, it follows that I_μ is finite, and it is clear that I satisfies the condition (Min). On the other hand, for all $\lambda \in I$,

$$x_\lambda = e^\lambda + \sum_{\mu \prec \lambda} c_{\lambda\mu} e^\mu$$

and since x_λ is invariant,

$$x_\lambda = S(e^\lambda) + \sum_{\mu \prec \lambda, \mu \in I} c_{\lambda\mu} S(e^\mu).$$

The proposition now follows from Lemma 5.1.88. □

Theorem 5.1.89. *Let $\varpi_1, \ldots, \varpi_l$ be the fundamental weights corresponding to the chamber C, and, for each i, let x_i be an element of $R[\Lambda]^W$ with e^{ϖ_i} as its unique maximal term. Let*

$$\phi : R[X_1, \ldots, X_l] \to R[\Lambda]^W$$

be the homomorphism from the polynomial algebra $R[X_1, \ldots, X_l]$ to $R[\Lambda]^W$ that takes X_i to x_i. Then the map ϕ is an isomorphism.

Proof. Lemma 5.1.83 implies that the image under ϕ of the monomial $X_1^{n_1} \cdots X_l^{n_l}$ is an element with unique maximal term $e^{n_1\varpi_1+\cdots+n_l\varpi_l}$. Since every element of Λ^{++} can be written uniquely in the form $n_1\varpi_1 + \cdots + n_l\varpi_l$, Proposition 5.1.87 shows that the images under ϕ of the monomials $X_1^{n_1} \cdots X_l^{n_l}$ are a basis of $R[\Lambda]^W$, hence the theorem. □

As an example, we can take $x_i = S(e^{\varpi_i})$ in Theorem 5.1.89; note that $x_i = J(e^{\delta+\varpi_i})/q$ is another choice, in view of Proposition 5.1.86.

5.2 Affine Weyl groups

5.2.1 The affine Weyl group

From now on, we denote by Φ a reduced root system in a real vector space V. We denote by W the Weyl group of Φ, identify it with a group of automorphisms of the dual V^* of V, and we provide V^* with a scalar product invariant under W. In other words, for $\alpha \in \Phi$, we denote by s_α the reflection actting on both V and V^*, given by

$$s_\alpha(x) = x - \langle x, \alpha \rangle \check{\alpha}, \quad s_\alpha(y) = y - \langle y, \check{\alpha} \rangle \alpha$$

for $x \in V^*$ and $y \in V$. Let E be the affine space underlying V^*; for $v \in V^*$, we denote by t_v the translation of E by the vector v. Finally, we denote by Λ (resp. Λ_r) the group of translations t_v whose vector v belongs to the group of weights $\Lambda(\check{\Phi})$ (resp. to the group of radical weights $\Lambda_r(\check{\Phi})$) of the inverse root system $\check{\Phi}$ of Φ.

For $\alpha \in \Phi$ and $k \in \mathbb{Z}$, let $H_{\alpha,k}$ be the hyperplane of E defined by

$$H_{\alpha,k} = \{x \in E : \langle \alpha, x \rangle = k\} \tag{5.2.1}$$

and let $s_{\alpha,k}$ be the orthogonal reflection with respect to $H_{\alpha,k}$. Then

$$s_{\alpha,k}(x) = x - (\langle \alpha, x \rangle - k)\check{\alpha} = s_{\alpha,0} + k\check{\alpha}$$

for all $x \in E$. In other words,

$$s_{\alpha,k} = t_{k\check{\alpha}} \circ s_\alpha. \tag{5.2.2}$$

where s_α is the orthogonal reflection with respect to the hyperplane $H_\alpha = H_{\alpha,0}$, i.e. the reflection associated to the root α (recall that we identify $W(\Phi)$ with $W(\check{\Phi})$ via $u \mapsto u^{-t}$, under which s_α is mapped to $s_{\check{\alpha}}$). Formula (5.2.2) shows that $s_{\alpha,k}$ does not depend on the choice of scalar product.

The group of affine transformations of E generated by the reflections $s_{\alpha,k}$, for $\alpha \in \Phi$ and $k \in \mathbb{Z}$, is called the **affine Weyl group** of the root system Φ and denoted by $W_a(\Phi)$ (or simply by W_a).

Proposition 5.2.1. *The group W_a is the semi-direct product of W by Λ_r.*

Proof. Since W is generated by the reflections s_α, it is contained in W_a. On the other hand, $t_{\check{\alpha}} = s_{\alpha,1} \circ s_\alpha$ if $\alpha \in \Phi$, which shows that $\Lambda_r \subseteq W_a$. Since W leaves $\Lambda_r(\check{\Phi})$ stable, the group G of affine transformations generated by W and Q is the semi-direct product of W by Λ_r. Now $G \subseteq W_a$ from above and $s_{\alpha,k} \in G$ for all $\alpha \in \Phi$ and $k \in \mathbb{Z}$ by (5.2.2). It follows that $W_a = G$. □

Proposition 5.2.2. *The group W_a, with the discrete topology, acts properly on E and permutes the hyperplanes $H_{\alpha,k}$, for $\alpha \in \Phi$ and $k \in \mathbb{Z}$. In other words, W_a satisfies the discreteness condition for the collection $\mathcal{H} = \{H_{\alpha,k}\}$.*

Proof. Since $\Lambda_r(\check{\Phi})$ is a discrete subgroup of V^*, the group Λ_r acts properly on E. Hence, so does $W_a = W \cdot \Lambda_r$, since W is finite. Moreover, for $\alpha \in \Phi$ and $k \in \mathbb{Z}$:

$$s_\beta(H_{\alpha,k}) = H_{s_\beta(\alpha),k}, \quad t_{\check{\beta}}(H_{\alpha,k}) = H_{\alpha,k+n(\alpha,\beta)}$$

where $n(\alpha, \beta)$ is an integer, hence the second assertion. □

We can thus apply the results in Subsection 4.3 to W_a acting on E. To avoid any confusion with the chambers of the Weyl group W in V^*, we shall call the chambers determined by the system of hyperplanes $H_{\alpha,k}$ (for $\alpha \in \Phi$ and $k \in \mathbb{Z}$) in E **alcoves**. The group W_a thus acts simply transitively on the set of alcoves and the closure of an alcove is a fundamental domain for W_a acting on E. It is clear that the Weyl group W is identified with the canonical image $U(W_a)$ of W_a in the orthogonal group of V^*. It follows that W_a is essential and that W_a is irreducible if and only if the root system Φ is. If Φ is irreducible, every alcove is an open simplex (Proposition 4.3.27). In the general case, the canonical product decomposition of the affine space E corresponds to the decomposition of R into irreducible components. In particular, the alcoves are products of open simplexes. Note also that Corollary 4.3.4 shows that the $s_{\alpha,k}$ are the only reflections in W_a.

By the discription $W_a = W \cdot \Lambda_r$, we see the set of $v \in T$ such that the translation $x \mapsto x + v$ belongs to W is exactly $\check{\Phi}$, and the translations belonging to W_a are exactly those in Λ_r. Reclal that a point $x \in E$ is called **special** if $W_a = W_x \cdot \Lambda_r$, where W_x is the subset of W_a fixing x (Proposition 4.3.30). In our context, it turns out that the set of special points are exactly weights of Φ.

Proposition 5.2.3. *The special points of W_a are exactly the weights of $\check{\Phi}$.*

Proof. Let $x_0 \in E$, $\alpha \in \Phi$, and $k \in \mathbb{Z}$. Let $H_{\alpha,k}$ be the hyperplane in E given by (5.2.1). Then hyperplane H parallel to $H_{\alpha,k}$ and passing through x has the following equation

$$H = \{x \in E : \langle x, \alpha\rangle = \langle x_0, \alpha\rangle\}.$$

To be equal to some $H_{\beta,j}$, it is necessary on the one hand that α and β are proportional, and so, since Φ is reduced, that $\alpha = \pm\beta$, and on the other hand that $\langle \alpha, x_0\rangle$ is an integer. In view of Proposition 4.3.30(iii), it follows immediately that w_0 is a special point of Wu if and only if $\langle \alpha, x_0\rangle \in \mathbb{Z}$ for all $\alpha \in \Phi$, in other words, if and only if $x_0 \in \Lambda(\check{\Phi})$. □

Corollary 5.2.4. *Let Φ be a root system in V.*

(a) If $x \in \Lambda(\check{\Phi})$, then there exists an alcove A such that x is an extremal point of A.

(b) If A is an alcove, then $\bar{A} \cap \Lambda(\check{\Phi})$ reduces to a single point and this is an extremal point of $\bar{A}$.

Proof. This is a concequence of Proposition 5.2.3, in view of Corollary 4.3.34 and Proposition 4.3.36. □

Proposition 5.2.5. *Let $\check{C}$ be a chamber of $\check{\Phi}$.*

(a) There exists a unique alcove A contained in $\check{C}$ such that $0 \in \bar{A}$.

(b) The union of the $w(\bar{A})$ for $w \in W$ is a closed neighbourhood of 0 in E.

(c) Every wall of $\check{C}$ is a wall of A.

Proof. This follows from Proposition 4.3.33. □

Assume now that Δ is irreducible. Let $(\alpha_i)_{i\in I}$ be a basis of Φ, and let $(\check{\varpi}_i)_{i\in I}$ be the dual basis. The $\check{\varpi}_i$ are the fundamental weights of $\check{\Phi}$ for the chamber $\check{C}$ of $\check{\Phi}$ corresponding to the basis $(\alpha_i)_{i\in I}$. Let

$$\tilde{\alpha} = \sum_i n_i\alpha_i$$

be the highest root of Φ, and let J be the set of $i \in I$ such that $n_i = 1$.

Proposition 5.2.6. *Let A be the alcove contained in $\check{C}$ and containing 0 in its closure. Then*

(a) A is the set of $x \in E$ such that $\langle \alpha_i, x\rangle > 0$ for all $i \in I$ and $\langle \tilde{\alpha}, x\rangle < 1$, which is also the set of $x \in E$ such that $0 < \langle x, \alpha\rangle < 1$ for all $\alpha \in \Phi^+$;

(b) the set $\bar{A} \cap \Lambda(\check{\Phi})$ consists of 0 and the $\check{\varpi}_i$ for $i \in J$.

Proof. First, it is clear that the alcove A can be characterized by

$$A = \{x \in V^* : 0 < \langle \alpha, x\rangle < 1 \text{ for all } \alpha \in \Phi^+\}.$$

In fact, the right-hand side is clearly included in A. On the other hand, it is convex (hence connected), but any element outside it is separated from it by one of the hyperplanes H_α or $H_{\alpha,1}$, so it equals to A.

Now we prove the first assertion of (a). Let D be the set of $x \in E$ such that $\langle \tilde{\alpha}, x\rangle < 1$ and set $\tilde{A} = \check{C} \cap D$. Since $0 \in \bar{A}$, we have $A \subseteq D$ and hence $A \subseteq \tilde{A}$. We are going to show that, for all $\alpha \in \Phi$ and all $k \in \mathbb{Z}$, the sets A and $\tilde{A}$ are on the same side of the hyperplane $H_{\alpha,k}$. This will prove that $\tilde{A} \subseteq A$ and so will establish assertion (a). If $k = 0$, the whole of the chamber $\check{C}$ is on one side of $H_{\alpha,0}$, which establishes our assertion in this case. If $k \neq 0$, we may, by replacing α by $-\alpha$, assume that $k > 0$. Then $\langle \alpha, x\rangle < k$ on A, since $0 \in \bar{A}$. On the other hand, $\tilde{\alpha} - \alpha$ is positive on (Proposition 5.1.62). Thus, for $y \in \tilde{A}$, we have $\langle \alpha, y\rangle \leq \langle \tilde{\alpha}, y\rangle < 1 \leq k$. Consequently, A and $\tilde{A}$ are on the same side of $H_{\alpha,k}$.

Now let $\check{\omega} \in \Lambda(\check{\Phi})$. Then $\check{\omega} = \sum_i p_i\check{\omega}_i$ with $p_i \in \mathbb{Z}$, and $\check{\omega} \in \overline{C}\check{}$ if and only if the integers p_i are positive. If $\check{\omega} \in \overline{C}\check{}$, then $\check{\omega} \in \bar{A}$ if and only if $\langle \tilde{\alpha}, \check{\omega}\rangle = \sum_i p_in_i \leq 1$, hence (b). □

Corollary 5.2.7. *The alcove A is an open simplex with vertices 0 and the $\check{\omega}_i/n_i$.*

Proof. This follows from Proposition 5.2.5 and that $(\check{\omega}_i/n_i)$ is the dual basis for $(n_i\alpha_i)$. □

Recall that the chosen scalar product on V is invariant not only under W but under the whole of the group $\mathrm{Aut}(\Phi)$. Again we identify $\mathrm{Aut}(\Phi)$ and $\mathrm{Aut}(\check{\Phi})$. Let N be the normaliser of W_a in the group of displacements of the affine Euclidean space E. If g is a displacement of E, and s is the orthogonal reflection with respect to a hyperplane H, the displacement gsg^{-1} is the orthogonal reflection with respect to the hyperplane $g(H)$. It follows that N is the set of displacements of E that permute the hyperplanes $H_{\alpha,k}$, (for $\alpha \in \Phi$ and $k \in \mathbb{Z}$).

Now, the group of automorphisms of E is the semi-direct product of the orthogonal group U of V^* and the group T of translations. If $u \in U$ and $v \in V^*$, the hyperplane $H_{\alpha,k}$ is transformed by $g = u \circ t_v$ into the hyperplane with equation

$$\begin{aligned} g(H_{\alpha,k}) &= \{u(x+v) : \langle \alpha, x\rangle = k\} = \{y \in E : \langle \alpha, u^{-1}(y) - v\rangle = k\} \\ &= \{y \in E : \langle u^{-t}(\alpha), x\rangle = k + \langle \alpha, v\rangle\}. \end{aligned}$$

Consequently, $g \in N$ if and only if, on the one hand u^t permutes the roots, in other words belongs to $\mathrm{Aut}(\Phi)$, and on the other hand $\langle \alpha, v\rangle \in \mathbb{Z}$ for all $\alpha \in \Phi$, that is, $u \in \Lambda(\check{\Phi})$. In other words, the group N is the semi-direct product of $\mathrm{Aut}(\Phi)$ by Λ. Since $\Lambda_r \subseteq \Lambda$ and $W \subseteq \mathrm{Aut}(\Phi)$, the quotient group N/W_a is the semi-direct product of $\mathrm{Aut}(\Phi)/W$ by $\Lambda(\check{\Phi})/\Lambda_r(\check{\Phi})$; it is immediately checked that the corresponding action of $\mathrm{Aut}(\Phi)/W$ on $\Lambda(\check{\Phi})/\Lambda_r(\check{\Phi})$ is the canonical action. We denote by $\widehat{W}_a$ the subgroup of N formed by the semi-direct product of W by Λ. This is a normal subgroup of N, and $N/\widehat{W}_a$ is canonically isomorphic to $\mathrm{Aut}(\Phi)/W$; moreover, the canonical map from $\Lambda(\check{\Phi})$ to $\widehat{W}_a/W_a$ gives by passing to the quotient an isomorphism from $\Lambda(\check{\Phi})/\Lambda_a(\check{\Phi})$ to $\widehat{W}_a/W_a$.

Let A be the alcove of E contained in $\check{C}$ and containing 0, and let N_A be the subgroup consisting of the elements $g \in N$ such that $g(A) = A$. Since W_a is simply-transitive on the alcoves, the group N is the semi-direct product of N_A by W_a. The corresponding isomorphism from N/W_a to N_A gives rise in particular to a canonical isomorphism from $\Lambda(\check{\Phi})/\Lambda_r(\check{\Phi})$ to the group $\Gamma_A = N_A \cap \widehat{W}_a$.

Assume that Φ is irreducible, and retain the notation of Proposition 5.2.6. Put $\Delta_0 = \Delta$, and let Δ_i be the subset $\Delta \setminus \{\alpha_j\}$; also, let Φ_i be the root system generated by Δ_i. For $i = 0$ (resp. $i \in I$), let w_i be the unique element of $W(\Phi_i)$ (identified with a subgroup of W) which transforms the positive roots of Φ_i relative to the basis $(\alpha_j)_{j\neq i}$ into negative roots (Corollary 5.1.40). Note that $w_i(\Delta_i) = -\Delta_i$ by Corollary 5.1.40.

Proposition 5.2.8. *For each $i \in I$, define $\gamma_i = t_{\check{\omega}_i}w_iw_0$. Then the element γ_i belongs to Γ_A for $i \in J$, and the map $i \mapsto \gamma_i$ is a bijection from J to $\Gamma_A \setminus \{1\}$. In particular, $|\Gamma_A| = |J| + 1$.*

Proof. We remark first of all that the root $w_i(\tilde{\alpha})$ is of the form

$$n_i\alpha_i + \sum_{j\neq i} b_{ij}\alpha_i$$

and hence is positive. We show that if $i \in J$ then $\gamma_i \in \Gamma_A$. Let $a \in A$; then $b = w_0(a) \in -A$ in view of the second characterization of Proposition 5.2.6(a) (since w_0 maps positive roots to negative roots). It is enough to show that $w_i(b) + \check{\omega}_i \in A$. For $1 \leq j \leq l$ and $j \neq i$:

$$\langle w_i(b) + \check{\omega}_i, \alpha_j\rangle = \langle \check{\omega}_i + w_i(b), \alpha_j\rangle = \langle b, w_i(\alpha_j)\rangle > 0. \tag{5.2.3}$$

since $b \in -A$ and $w_i(\alpha_j)$ is negative. On the other hand,

$$\langle w_i(b) + \check{\omega}_i, \alpha_i\rangle = 1 + \langle b, w_i(\alpha_i)\rangle > 0 \tag{5.2.4}$$

since $b \in -A$ and $w_i(\tilde{\alpha}) \in \Phi^+$ imply $\langle b, w_i(\tilde{\alpha}) > -1$. Finally,

$$\langle w_i(b) + \check{\varpi}_i, \tilde{\alpha}\rangle = n_i + \langle b, w_i(\tilde{\alpha})\rangle = 1 + \langle b, w_i(\tilde{\alpha})\rangle < 1 \tag{5.2.5}$$

since $b \in -A$ and $w_i(\tilde{\alpha})$ is positive. The relation (5.2.3), (5.2.4), and (5.2.5) then imply that $w_i(b) + \check{\varpi}_i \in A \in \mathcal{A}$, and hence that $\gamma_i \in \Gamma_A$.

It is clear that the map $i \mapsto \gamma_i$ is injective, since $\gamma_i(0) = \check{\varpi}_i$. Finally, let $\gamma \in \Gamma_A$ with $\gamma \neq 1$, and put $\gamma = tw$ with $t \in \Lambda$ and $w \in W$. Then $t \neq 1$ since $\Gamma_A \cap W = \{1\}$. On the other hand, $t(0) = \gamma(0) \in \overline{A} \cap \Lambda(\check{\Phi})$ and Proposition 5.2.6 implies that there exists $i \in J$ such that $t(0) = \check{\varpi}_i$. Then $\gamma_i^{-1}\gamma(0) = 0$, hence $\gamma = \gamma_i$ (recall that $\Gamma_A \cap W = \{1\}$). This completes the proof. □

Corollary 5.2.9. *The $(\check{\varpi}_i)_{i \in I}$ form a system of representatives in $\Lambda(\check{\Phi})$ of the non-zero elements of $\Lambda(\check{\Phi})/\Lambda_r(\check{\Phi})$.*

Proof. Indeed, if we identify Γ_A with $\Lambda(\check{\Phi})/\Lambda_r(\check{\Phi})$, the element γ_i is identified with the class of $\check{\varpi}_i$ mod $\Lambda_r(\check{\Phi})$, via the map $\gamma \mapsto \gamma(0)$. □

We shall now consider the automorphism $g \mapsto \gamma g \gamma^{-1}$ defined by $\gamma \in \Gamma_A$. Since any element $\gamma \in \Gamma_A$ fixes A, this automorphism induces a permutation of the set $\{s_0, s_1, \dots, s_l\}$ (where we set $s_0 = s_{\tilde{\alpha},1}$ and $s_i = s_{\alpha_i}$). Thus we get a homomorphism from Γ_A onto the permutation group $\mathfrak{S}_{l+1}$. This homomorphism is injective. In fact, if a non-trivial element $\gamma = t_{\check{\varpi}_i} w_i w_0$ with $i \in I$ induces the identity, we get $\gamma s_j \gamma^{-1} = s_j$ for all j. In particular we get

$$s_j t_{\check{\varpi}_i} w_i w_0 s_j = t_{\check{\varpi}_i} w_i w_0 \quad j = 1, \dots, l.$$

This implies $s_j(\check{\varpi}_i) = \check{\varpi}_i$ and therefore $\langle \alpha_j, \check{\varphi}_i\rangle = 0$ for all j, which is a contradiction.

Proposition 5.2.10. *Let $U : \widehat{W}_a \to W$ be the natrual homomorphism.*

(a) *If $\gamma_i = t_{\check{\varpi}_i} w_i w_0 \in \Gamma_A$ for $i \in J$, then $\gamma_i s_0 \gamma_i^{-1} = s_i$.*

(b) *The map U is injective on Γ_A and the set $\{\alpha_1, \dots, \alpha_l, -\tilde{\alpha}\}$ is stable under the subgroup $W_A = U(\Gamma_A)$ of W.*

Proof. Let us show first that $\gamma_i s_0 \gamma_i^{-1}(0) = 0$, which is equivalent to $\gamma_i^{-1}(0) \in H_{\tilde{\alpha},1}$. Now $\gamma_i^{-1}(0) = w_0 w_i(-\check{\varpi}_i)$, and since $s_j(\check{\varpi}_i) = \check{\varpi}_i$ for $j \neq i$, we have $w_i(\check{\varpi}_i) = \check{\varpi}_i$, hence $\gamma_i^{-1}(0) = -w_0(\check{\varpi}_i)$. It then remains to show $\langle \tilde{\alpha}, -w_0(\check{\varpi}_i)\rangle = 1$. Now $w_0(C) = -C$ implies $w_0(\Delta) = -\Delta$, whence $w_0(\tilde{\alpha}) = -\tilde{\alpha}$ and we have

$$\langle \tilde{\alpha}, -w_0(\check{\varpi}_i)\rangle = \langle w_0(\tilde{\alpha}), -\check{\varpi}_i\rangle = \langle \tilde{\alpha}, \check{\varpi}_i\rangle = n_i = 1.$$

Hence we get $\gamma_i s_0 \gamma_i^{-1} \in W$, which implies $\gamma_i s_0 \gamma_i^{-1} \in \{s_1, \dots, s_l\}$. Now the natrual homomorphism $U : \widehat{W}_a \to W$ is injective on Γ_A since $\Gamma_A \cap \Lambda = \{1\}$ by Proposition 5.2.8. Hence to determine the element $\gamma_i s_0 \gamma_i^{-1} \in W$, it is enough to determine the image of $\gamma_i s_0 \gamma_i^{-1}$ under this homomorphism $U : \widehat{W}_a \to W$. Now this image is clearly given by

$$w_i w_0 s_{\tilde{\alpha}} w_0 w_i = w_i s_{w_0(\tilde{\alpha})} w_i = w_i s_{-\tilde{\alpha}} w_i = s_{-w_i(\tilde{\alpha})} = s_{w_i(\tilde{\alpha})}.$$

Thus the image is equal to s_β where $\beta = w_i(\tilde{\alpha})$. On other hand, $\beta \in \pm\Delta$ since $\gamma_i s_0 \gamma_i^{-1} \in \{s_1, \dots, s_l\}$. As was remarked in the proof of Proposition 5.2.8, $\beta = w_i(\tilde{\alpha})$ is of the form $\alpha_i + \sum_{j \neq i} m_j \alpha_j$. Thus β must coincide with α_i and we finally conclude that $\gamma_i s_0 \gamma_i^{-1} = s_i$.

Now let $\gamma_i = t_{\check{\varpi}_i} w_i w_0$ be a nontrivial element in Γ_A. We have seen above that

$$U(\gamma_i)(-\tilde{\alpha}) = w_i w_0(-\tilde{\alpha}) = w_i(\tilde{\alpha}) = \alpha_i.$$

Put $\gamma_i^{-1} = \gamma_j = t_{\check{\varpi}_j} w_j w_0$, then $w_j w_0 = (w_i w_0)^{-1} = w_0 w_i$, hence

$$U(\gamma_i)(\alpha_j) = w_i w_0(\alpha_j) = w_0 w_j(\alpha_j) = -\tilde{\alpha}$$

since $w_j w_0(-\tilde{\alpha}) = \alpha_j$. Also, for $\alpha_k \in \Delta \setminus \{\alpha_j\}$, we get

$$U(\gamma_i)(\alpha_k) = w_0 w_j(\alpha_k) \in w_0(-\Delta_j) \subseteq \Delta.$$

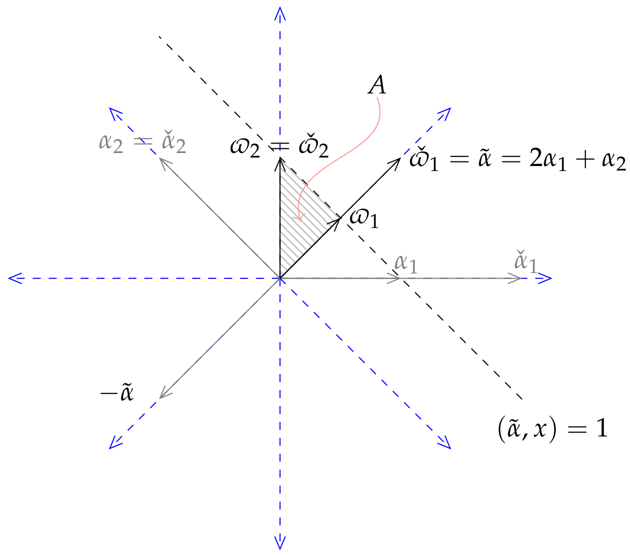
Thus $U(\gamma_i)$ keeps the set $\{\alpha_1, \dots, \alpha_l, -\tilde{\alpha}\}$ stable. □

Corollary 5.2.11. *For each $i \in J$, we have $w_i(\tilde{\alpha}) = \alpha_i$, and the element γ_i maps the hyperplane of equation $\langle \tilde{\alpha}, x \rangle < 1$ to the hyperplane H_{α_i}.*

Example 5.2.12. Consider the root system B_2 endowed with the inner product such that $\|\alpha_1\| = 1$ and $\|\alpha_2\| = \sqrt{2}$. Using this inner product, we can identify V^* with V, hence draw the alcove in the space V. First we compute the coroots and the dual basis:

$$\check{\alpha}_1 = \frac{2\alpha_1}{(\alpha_1, \alpha_1)^2} = 2\alpha_1, \quad \check{\alpha}_2 = \frac{2\alpha_2}{(\alpha_2, \alpha_2)} = \alpha_2, \quad \check{\varpi}_1 = 2\varpi_1, \quad \check{\varpi}_2 = \varpi_2.$$

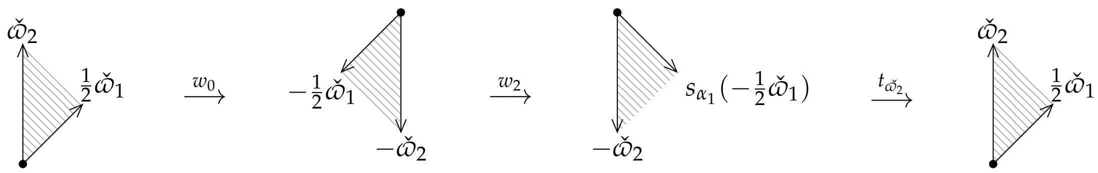
The alcove can be then deduced:



Recall that $\tilde{\alpha} = 2\alpha_1 + \alpha_2$ is the highest root for B_2, so by definition, $J = \{2\}$ and the elements w_0 and w_2 are given by

$$w_0 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad w_2 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

It then follows that the alcove A is mapped by γ_2 in the following way:



Note that $w_2 w_0$ is just the reflection along x-axis. This justifies the results of Proposition 5.2.8 and Proposition 5.2.10. Also, we note that the wall corresponding to $(\alpha, x) < 1$ is mapped to the wall orthogonal to α_2.

Proposition 5.2.13. *Assume that Φ is irreducible. Let $\Delta = \{\alpha_1, \ldots, \alpha_l\}$ be a basis of Φ, f the connection index of Φ and $\tilde{\alpha} = \sum_{i=1}^{l} n_i \alpha_i$ the highest root of Φ (for the order defined by Δ). Then the order of W is equal to*

$$|W| = (\ell!) n_1 \cdots n_l f.$$

Proof. Let $(\check{\varpi}_1, \ldots, \check{\varpi}_l)$ be the basis of $\Lambda(\check{\Phi})$ dual to Δ. By Corollary 5.2.7, the open simplex A with vertices $0, n_1^{-1} w_1, \ldots, n_l^{-1} \check{\varpi}_l$ is an alcove of E. Choose a Haar measure μ on the additive group V^*. Let

$$U = \{a_1 \check{\varpi}_1 + \cdots + a_l \check{\varpi}_l : 0 < a_i < 1 \text{ for all } i\}.$$

Then we have

$$\mu(U)/\mu(A) = (l!) n_1 \cdots n_l. \tag{5.2.6}$$

On the other hand, let
$$\tilde{U} = \{a_1\check{\alpha}_1 + \cdots + a_l\check{\alpha}_l : 0 < a_i < 1 \text{ for all } i\}.$$
Then since $(\check{\alpha}_1, \ldots, \check{\alpha}_l)$ is a basis for the $\mathbb{Z}$-module $\Lambda_r(\check{\Phi})$, we see $\tilde{w}\tilde{U}$ for $\tilde{w} \in \Lambda_r$ are pairwise disjoint. Now let (w_λ) is a family of representatives of right cosets of Λ_r in Λ and U_1 be the union of the $w_\lambda U$. Then the sets $\tilde{w}U_1$, for $\tilde{w} \in \Lambda_r$, are pairwise disjoint and
$$\bigcup_{\tilde{w}\in\Lambda_r} \tilde{w}U_1 = \bigcup_{w\in\Lambda} wU =: M.$$
Let $\tilde{M} = \bigcup_{\tilde{w}\in\Lambda_r} \tilde{w}\tilde{U}$, then the union of $\tilde{U}$ (resp. U_1) and a suitable subset of $V^* \setminus \tilde{M}$ (resp. $V^* \setminus M$) is a fundamental domain, evidently μ-measurable, for Λ_r. Since M and $\tilde{M}$ have negligible complements under μ, this shows that
$$\mu(\tilde{U})/\mu(U) = (\Lambda : \Lambda_r) = f. \tag{5.2.7}$$
Fianlly, by a similar argument, we can show that
$$\mu(\tilde{U})/\mu(A) = (W_a : \Lambda_r) = |W|. \tag{5.2.8}$$
The claim then follows by comparing formulas (5.2.6), (5.2.7), and (5.2.8). □

5.2.2 Characterization of Weyl groups

We consider the question of determining what kind of finite Coxeter group can be realized as the Weyl group of a root system in some vector space. Since we have already seen that any Coxeter group can be realized as finite group of $\mathrm{GL}(V)$ generated by reflections, we may restrict our concern to this case. First, we show that any such group, under a suitable condition, is isomorphic to the affine Weyl group of some root system.

Proposition 5.2.14. *Let X be an Euclidean space of finite dimension l. Let $\mathcal{H}$ be a collection of affine hyperplanes of V and G the group generated by the orthogonal reflections s_H with respect to the hyperplanes $H \in \mathcal{H}$. Assume that G satisfies the discreteness condition (i.e. that $g(H) \in \mathcal{H}$ for all $H \in \mathcal{H}$ and $g \in G$, and that G acts properly on V). Assume also that 0 is a special point for G and that the group T of translations belonging to G is of rank l. Then there exists a unique reduced root system Φ in X^* such that the canonical isomorphism from X to V^* transforms G to the affine Weyl group W_a of Φ.*

Proof. We remark first of all that the assumption on T implies that G is essential: otherwise, the affine space X would decompose into a product $X_0 \times X_1$, with $\dim(X_1) < l$, the group G being identified with a group of displacements acting properly on X_1 (Proposition 4.3.25), and T would not be of rank l.

Define
$$\mathcal{H}_0 = \{H \in \mathcal{H} : 0 \in H\}, \quad \mathcal{H}_H = \{L \in \mathcal{H} : L \text{ is parallel to } H\}.$$
Since 0 is a special point, $\mathcal{H}$ is the union of the $\mathcal{H}_H$ for $H \in \mathcal{H}_0$ by Proposition 4.3.30(iii). Now fix a hyperplane $H \in \mathcal{H}$. Since T is of rank l, there exists a $v \in X$ such that the translation by the vector v belongs to T and $v \notin H$. The hyperplanes $H + kv$ for $k \in \mathbb{Z}$ are pairwise distinct and belong to $\mathcal{H}_H$. Now let e be a unit vector of X orthogonal to H. Then $H + (v,e)e \in \mathcal{H}_H$ and since $\mathcal{H}$ is locally finite, there is a smallest real number $\lambda > 0$ such that $H + \lambda e \in \mathcal{H}$. Let $e_H = \lambda e$, we are going to show that $\mathcal{H}_H$ is the set of hyperplanes $H + ke_H$ for $k \in \mathbb{Z}$. Indeed,
$$\tilde{H} = H + e_H \in \mathcal{H}_H$$
and the element $s_{\tilde{H}} \circ s_H$ of G is the translation by the vector $2e_H$ (Proposition 4.2.9). Consequently, $H + 2ne_H = (s_{\tilde{H}}s_H)^n(H)$ and $H + (2n+1))e_H = (s_{\tilde{H}}s_H)n(L)$ belongs to $\mathcal{H}_H$. On the other hand, if $L \in \mathcal{H}_H$, there exists $\xi \in \mathbb{R}$ such that $L = H + \xi e_H$ and there exists an integer n such that either $2n < \xi \leq 2n+1$ or $2n-1 < \xi \leq 2n$. In the first case, $(s_H s_{\tilde{H}})^n(L) = H + (\xi - 2n)e_H$ with $0 < (\xi - 2n) < 1$, and the definition of λ implies that $\xi = 2n+1$; in the second case,
$$s_H(s_H s_{\tilde{H}})^n(L) = H + (2n - \xi)e_H$$
with $0 \leq 2n - \xi < 1$, and the definition of λ implies $\xi = 2n$.

It follows that if α_H is the linear form on X such that

$$\tilde{H} = \{x \in X : \langle \alpha_H, x\rangle = 1\}$$

then the set $\mathcal{H}_H$ is the set of hyperplanes $H_{\alpha_H,k} = \{x \in X : \langle \alpha_H, x\rangle = k\}$ for $k \in \mathbb{Z}$, and α_H and $-\alpha_H$ are the only linear forms with this property.

Consequently, the proposition will be proved if we show that the set Φ of elements of V of the form $\pm\alpha_H$ is a reduced root system in V.

It is clear that Φ is finite (since $\mathcal{H}_0$ is finite) and does not contain 0. Moreover, Φ generates V. Indeed, if $x \in X$ is orthogonal to Φ, then $x \in H$ for all $H \in \mathcal{H}_0$ and the translation by the vector x commutes with every element of G. Since G is essential, this implies that $x = 0$. We now prove (R2). For $v \in V$ and $\xi \in \mathbb{R}$, put $H_{v,\xi} = \{x \in X : \langle v, x\rangle = \xi\}$ as above; if $\alpha \in \Phi$, put $H_\alpha = H_{\alpha,0}$, and let s_α be the invert of the transpose of s_{H_α}. There exists a unique element $\check{\alpha} \in X$ orthogonal to H_α and such that $\langle \check{\alpha}, \alpha\rangle = 2$. Then $s_{H_\alpha} = s_{\check{\alpha},\alpha}$ and $s_\alpha = s_{\alpha,\check{\alpha}}$. For $\beta \in \Phi$, we have

$$H_{s_\alpha(\eta),1} = s_{H_\alpha}(H_{\beta,1}) \in \mathcal{H}$$

so there exist $\gamma \in \Phi$ and $n \in \mathbb{N}_+$ such that $H_{s_\alpha(\beta),1} = H_{\gamma,n}$. Then

$$s_{H_\alpha}(H_{\gamma,1}) = H_{\beta,1/n}.$$

and so $1/n \in \mathbb{Z}$. Thus $n = 1$ and $s_\alpha(\beta) \in \Phi$. This proves (R2).

Finally, we show that Φ satisfies (R3). Let $\alpha \in \Phi$ and set $\tilde{H}_\alpha = H_{\alpha,1}$. By the definition of $\check{\alpha}$, we have

$$\tilde{H}_\alpha = H_\alpha + \frac{1}{2}\check{\alpha}.$$

Therefore the translation $t_{\check{\alpha}}$ by the vector $\check{\alpha}$ is the product $s_{\tilde{H}_\alpha} s_{H_\alpha}$ (Proposition 4.2.9), it belongs to T and $\check{\alpha} = t_{\check{\alpha}}(0)$ is a special point for G. Consequently, for all $\beta \in \Phi$, there exists a hyperplane $H_{\beta,k}$ passing through $\check{\alpha}$, with k an integer, which shows that $\langle \beta, \check{\alpha}\rangle \in \mathbb{Z}$, and proves (R3). It is clear that Φ is reduced, for if $H_1, H_2 \in \mathcal{H}_0$ and $H_1 \neq H_2$, then the linear forms α_{H_1} and α_{H_2} are not proportional. □

Remark 5.2.15. The assumption that T is of rank l is satisfied in particular when G is irreducible and infinite. Indeed, the vector space generated by vectors corresponding to the translations in T is invariant under the canonical image of G in the linear group of X (note that $t_{g(v)} = g t_v g^{-1}$ for $v \in X$ and $g \in G$). It is different from $\{0\}$ if G is infinite and is thus equal to the whole of X if G is infinite and irreducible.

We must note that, however, a finite group generated by reflections is not always the Weyl group of a root system. More precisely, we have the following characterizations for this.

Proposition 5.2.16. *Let V be a real vector space of finite dimension l, and let G be a finite subgroup of* $\mathrm{GL}(V)$*, generated by reflections and essential. Equip V with a scalar product invariant under G, then the following conditions are equivalent:*

(i) *There exists a discrete subgroup of V of rank l that is stable under G.*

(ii) *There exists a $\mathbb{Q}$-structure on V invariant under G.*

(iii) *There exists a root system in V whose Weyl group is G.*

(iv) *There exists a discrete group $\tilde{G}$ of displacements of V, acting properly on V, and generated by reflections, such that $\tilde{G}$ is the semi-direct product of G and a group of translations of rank l.*

A group G satisfying the equivalent conditions above is called a ***crystallographic group****.*

Proof. Clearly (iii)⇒(ii). Let $V_{\mathbb{Q}} \subseteq V$ be a $\mathbb{Q}$-structure on V invariant under G. Let A be a finite subset of $V_{\mathbb{Q}}$ generating the $\mathbb{Q}$-vector space $V_{\mathbb{Q}}$. Replacing A by $\bigcup_{s\in G} s(A)$, we can assume that A is stable under G (recall that G is finite). Let B be the subgroup of V generated by A. Then B is stable under G, of finite type and torsion-free, so has a basis over $\mathbb{Z}$ which is both a basis of $V_{\mathbb{Q}}$ over $\mathbb{Q}$ and a basis of V over $\mathbb{R}$. This proves (ii)⇒(i).

Let $\tilde{G}$ be a group satisfying condition (iv). The group of translations of $\tilde{G}$ is of rank l, and 0 is a special point for $\tilde{G}$ by Proposition 4.3.30. Proposition 5.2.14 then shows that there exists a reduced root system Φ_0 in V^* such that $\tilde{G}$ is identified with $W_a(\Phi)$; the group G is then the Weyl group of the inverse root system of Φ. Therefore (iv)⇒(iii).

Finally, we show that (i)⇒(iv). Assume that G leaves stable a discrete subgroup M of V, of rank l. For any reflection $s \in G$, $s(x) - x \in M$ for all $x \in M$, so the line D_s orthogonal to H_s meets M; let $\alpha_s, -\alpha_s$ be the generators of the cyclic group $C_s \cap M$. The set A of the $\pm\alpha_s$ is stable under G, hence generates a subgroup $\tilde{M}$ of M stable under G. If $x \in V$ is orthogonal to the α_s, then it belongs to H_s for all reflections $s \in G$, whence $s(x) = x$. Because G is essential and generated by reflections, this implies $x = 0$, so $\tilde{M}$ generates V. Since $\tilde{M}$ is discrete, this implies $\tilde{M}$ is of rank l. Let $\tilde{G}$ be the group of affine transformations of V that is the semi-direct product of G and the group of translations whose vectors belong to $\tilde{M}$. We shall show that $\tilde{G}$ is generated by the reflections in it, which will complete the proof. First, since G is generated by reflections, we see the group generated by reflection in $\tilde{G}$ contains the group G. On the other hand, for any reflection s of G, let t_s be the translation with vector α_s. The transformation $s \circ t_s$ is a reflection, and $s \circ t_s \in \tilde{G}$; thus t_s is a product of two reflections of $\tilde{G}$; this being true for every reflection s of G, the translations whose vector belongs to $\tilde{M}$ are all generated by reflections, whence our claim. □

Remark 5.2.17. Let G be a finite group generated by reflections and essential. We will see that G is crystallographic if and only if every element of its Coxeter matrix is one of the integers $1, 2, 3, 4, 6$.

5.3 Classification of root systems

In this section, we are going to determine, up to isomorphism, all root systems, and consequently all crystallographic groups. More generally, we shall start by determining all finite groups generated by reflections in a finite-dimensional real vector space: this is equivalent to determining all finite Coxeter groups, or to determining all Coxeter matrices of finite order such that the associated bilinear form is positive and nondegenerate.

5.3.1 Finite Coxeter groups

Let $M = (m_{ij})_{i,j \in I}$ be a Coxeter matrix of finite order l and put

$$q_{ij} = -\cos(\pi/m_{ij}).$$

Recall that $q_{ii} = 1$ and that $q_{ij} = q_{ji}$ is zero or $\leq -1/2$ for $i \neq j$. Put $E = \mathbb{R}^I$ and let $(e_i)_{i \in I}$ be the canonical basis of E. Denote by $(\cdot, \cdot)$ the bilinear form on E associated to M and q the quadratic form $x \mapsto (x, x)$ on E. For $x = \sum_{i \in I} \xi_i e_i$,

$$\|x\|^2 = q(x) = \sum_{i,j \in I} q_{ij} \xi_i \xi_j.$$

Denote by (Γ, f) the Coxeter graph of M. If a is an edge of Γ, $f(a)$ is called the **order** of a.

In the remainder of this paragraph, the Coxeter group $W(M)$ defined by M is assumed to be finite, so that q is positive and non-degenerate and Γ is a forest (Proposition 4.4.17). We also assume that Γ is connected (in other words that the Coxeter group $W(M)$ is irreducible), so that Γ is a tree.

From the condition that q is positive and non-degenerate, we shall obtain conditions on the m_{ij} that will enable us to list all the possibilities for the corresponding Coxeter graphs; it will only remain to show that these possibilities are actually realised, in other words that the corresponding groups $W(M)$ are finite. We will arrange our discussion into a list of lemmas.

Lemma 5.3.1. *For all $i \in I$, we have $\sum_{j \neq i} q_{ij}^2 < 1$.*

Proof. Fix $i \in I$. Let J be the set $j \in I$ such that $q_{ij} \neq 0$, in other words, that $\{i, j\}$ is linked in Γ. If $j_1, j_2 \in J$ and $j_1 \neq j_2$, then $\{j_1, j_2\}$ is not an edge: otherwise i, j_1, j_2 would form a circuit, contradicting to the fact that Γ is a tree; so $(e_{j_1}, e_{j_2}) = 0$. Let $F = \sum_{j \in J} \mathbb{R} e_j$, then $(e_j)_{j \in J}$ is an orthonormal basis for F. The distance $d = d(e_i, F)$ is given by

$$0 < d^2 = 1 - \sum_{j \in J} (e_i, e_j)^2 = 1 - \sum_{j \in J} q_{ij}^2 = 1 - \sum_{j \neq i} q_{ij}^2,$$

hence the lemma. □

Lemma 5.3.2. *Any vertex of Γ belongs to at most 3 edges.*

Proof. Recall that two vertices i, j is linked if and only if $m_{ij} \geq 3$, which means $q_{ij} \geq 1/4$. Therefore, if i is linked to h other vertices, the relations $q_{ij}^2 \geq 1/4$ for these other vertices implies that $h/4 < 1$ by Lemma 5.3.1, so $h \leq 3$. □

Lemma 5.3.3. *If i belongs to 3 edges, these edges are of order 3.*

Proof. If not, we wuold have, in view of $\cos \pi/4 = \sqrt{2}/2$, that

$$\sum_{j \neq i} q_{ij} \geq \frac{1}{4} + \frac{1}{4} + \left(\frac{\sqrt{2}}{2}\right)^2 = 1$$

which is impossible by Lemma 5.3.1. □

Lemma 5.3.4. *If there exists an edge of order ≥ 6, then $l = 2$.*

Proof. Indeed, let $\{i, j\}$ be such an edge. If $l > 2$, one of the edges i, j (say i) would be linked to a third vertex j', since Γ is connected. In View of the relation $\cos \pi/6 = \sqrt{3}/2$, we would have

$$\sum_{k \neq i} q_{ik} \geq \frac{1}{4} + \left(\frac{\sqrt{3}}{2}\right)^2 = 1$$

which is impossible by Lemma 5.3.1. □

Lemma 5.3.5. *A vertex cannots belong to two distinct edges of order ≥ 4.*

Proof. If i is such a vertex, then since $\cos \pi/4 = \sqrt{2}/2$, we have

$$\sum_{j \neq i} q_{ij} \geq \left(\frac{\sqrt{2}}{2}\right)^2 + \left(\frac{\sqrt{2}}{2}\right)^2 = 1,$$

which is impossible by Lemma 5.3.1. □

Let $\{i, j\}$ be an edge of Γ. We are going to define a new Coxeter graph, which will be obtained from the graph of M by identifying i and j. The set $\tilde{I}$ of its vertices is the quotient of I obtained by identifying i and j. Then $s = \{i, j\}$ is an element of $\tilde{I}$, and we identify the elements of I distinct from i and j with their canonical images in $\tilde{I}$. Let k, k' be two distinct elements of $\tilde{I}$. Then $\{k, k\}$ is an dege of the new graph in the following cases:

(1) k and k' are distinct from x and $\{k, k'\}$ is an edge of Γ; in this case, the order of this edge is defined to be $m_{k,k'}$;

(2) $k = s$, and one of the sets $\{i, k'\}$, $\{j, k'\}$ is an edge of Γ; the order of $\{k, k'\}$ is defined to be $m_{i,k'}$ if $\{i, k'\}$ is an edge of Γ, and $m_{j,k'}$ is an edge of Γ (these two possibilities are mutually exclusive because Γ is a tree).

Let $\tilde{M} = (\tilde{m}_{ij})_{i,j \in \tilde{I}}$ be the new Coxeter thus defined, and put $\tilde{q}_{ij} = -\cos(\pi/\tilde{m}_{ij})$. Then, for $k \neq \{i, j\}$, $\tilde{q}_{sk} = q_{ik} + q_{jk}$. Thus, if $(\xi_i) \in \mathbb{R}^{\tilde{I}}$,

$$\sum_{k,k' \in \tilde{I}} \tilde{q}_{k,k'} \xi_k \xi_{k'} = \sum_{k,k' \in I \setminus \{i,j\}} q_{k,k'} \xi_k \xi_{k'} + 2 \sum_{k \in I \setminus \{i,j\}} (q_{ik} + q_{jk}) \xi_k \xi_s + \xi_s^2. \tag{5.3.1}$$

Setting, $\xi_i = \xi_j = \xi_s$, we can rewrite (5.3.1) as

$$\sum_{k,k' \in \tilde{I}} \tilde{q}_{k,k'} \xi_k \xi_{k'} = \sum_{k,k' \in I} q_{k,k'} \xi_k \xi_{k'} + \xi_s^2 - \xi_i^2 - \xi_j^2 - 2q_{ij} \xi_i \xi_j = \sum_{k,k' \in I} q_{k,k'} \xi_k \xi_{k'} - (1 + 2q_{ij}) \xi_s^2. \tag{5.3.2}$$

Lemma 5.3.6. *If $\{i, j\}$ is or order 3, then $W(\tilde{M})$ is a finite Coxeter group.*

Proof. In fact, $q_{ij} = -1/2$, so by (5.3.2),

$$\sum_{k,k' \in \tilde{I}} \tilde{q}_{k,k'} \xi_k \xi_{k'} = \sum_{k,k' \in I} q_{k,k'} \xi_k \xi_{k'}$$

and $(\xi_k)_{k \in \tilde{I}} \mapsto \sum_{k,k' \in \tilde{I}} \tilde{q}_{k,k'} \xi_k \xi_{k'}$ is a positive non-degenerate quadratic form. It then suffices to apply Theorem 4.4.15. □

Lemma 5.3.7. *We have one of the following:*

(a) Γ *has a unique ramification point, and all the edges of* Γ *are of order* 3.

(b) Γ *is a chain and has at most one dege of order* ≥ 4.

Proof. We prove by induction on l. Assume that Γ has a ramification point i. Then i belongs to 3 edges of order 3, say $\{i,k_1\}$, $\{i,k_2\}$, $\{i,k_3\}$ (Lemma 5.3.2 and 5.3.3). If $l=4$ then the lemma is proved. If not, then k_1, say, belongs to an edge distinct from those just mentioned since Γ is connected. Identify i and k_1 in the Coxeter graph of M. This gives a new graph to which the induction hypothesis can be applied, in view of Lemma 5.3.6. Now the image s of i is a ramification point of the new graph $\tilde{\Gamma}$. Thus $\tilde{\Gamma}$ has no other ramification point and all its edges are of order 3. Thus all the edges of Γ are of order 3, and i and k_1 are its only possible ramification points. But if k_1 were a ramification point of Γ, s would belong to at least 4 edges in $\tilde{\Gamma}$, contrary to Lemma 5.3.2.

Assume that Γ has no ramification point, so that Γ is a chain. Let $\{i,j\}$ be an edge of order ≥ 4. If $\ell=2$, the lemma is trivial. If not, then j, say, belongs to an edge $\{j,k\}$ with $k\neq j$ (since X is connected). This edge is of order 3 by Lemma 5.3.5. Identify j and k in the Coxeter graph of M. By Lemma 5.3.6, the induction hypothesis can be applied. Let s be the image of $\{j,k\}$ in the new graph $\tilde{\Gamma}$. In $\tilde{\Gamma}$, $\{i,s\}$ is an edge of order ≥ 4, so $\tilde{\Gamma}$ has no other edge of order ≥ 4, and hence $\{i,j\}$ is the only edge of order ≥ 4 in Γ. $\square$

Lemma 5.3.8. *Let* $i_1,\dots,i_p$ *be vertices of* Γ *such that* $\{i_1,i_2\},\dots,\{i_{p-1},i_p\}$ *are edges of order* 3. *Then*

$$q(\sum_{r=1}^{p} re_{i_r}) = \frac{p(p+1)}{2}.$$

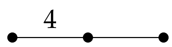
Proof. We have $(e_{i_r},e_{i_r})=1$, $e_{i_r},e_{i_{r+1}})=-1/2$, and $(e_{i_r},e_{i_s})=0$ if $|r-s|>1$. Thus

$$q(\sum_{r=1}^{p} re_{i_r}) = \sum_{r=1}^{p} r^2 - 2\sum_{r=1}^{p-1}\frac{1}{2r(r+1)} = p^2-\sum_{r=1}^{p-1} r = \frac{p(p+1)}{2}.$$

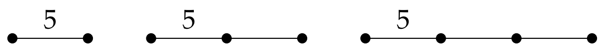
This proves the lemma. $\square$

Lemma 5.3.9. *Assume that* Γ *is a chain with vertices* $1,2,\dots,l$ *and edges* $\{1,2\},\{2,3\},\dots,\{l-1,l\}$.

(a) If one of the edges $\{2,3\},\{3,4\},\dots,\{l-2,l-1\}$ *is of order* ≥ 4, *then this edge is of order* 4 *and the graph is the following:*



(b) If the edge $\{1,2\}$ *is of order* 5, *the graph is one of the following:*



Proof. We can assume that $l>2$, since the lemma is trivial if $l=2$. First assume that $\{i,i+1\}$ is of order ≥ 4 with $1\leq i\leq l-1$. Put

$$x=e_1+2e_2+\cdots+ie_i,\quad y=e_l+2e_{l-1}+\cdots+(l-i)e_{i+1}$$

and $j=l-i$. By Lemma 5.3.8, we have

$$\|x\|^2=\frac{1}{2}i(i+1),\quad \|y\|^2=\frac{1}{2}j(j+1).$$

On the other hand, $(x,y)=ij(e_i,e_{i+1})=-ij\cos(\pi/m)$ with $m=4$ or 5 (Lemma 5.3.4). Now

$$(x,y)^2\leq\|x\|^2\|y\|^2$$

which gives

$$\frac{1}{4}ij(i+1)(j+1)>i^2j^2\cos^2(\pi/m)$$

so

$$(i+1)(j+1) > 4ij\cos^2(\pi/m) \geq 2ij.$$

This gives, first of all, that $ij - i - j - 1 < 0$, or equivalently $(i-1)(j-1) < 2$. If $1 < i < l-1$, then $1 < j < l-1$, so $i = j = 2$ and

$$16\cos^2(\pi/m) < 9$$

since $\cos(\pi/5) = (1+\sqrt{5})/4$, this implies $m = 4$ and proves (a). If $i = 1$ and $m = 5$, then $2j + 2 > 4j\cos^2(\pi/5)$, or $j < \sqrt{5} + 1 < 4$. This shows $l = j + 1 \leq 4$, and proves (b). □

Lemma 5.3.10. *If Γ has a ramification point i, the full subgraph $\Gamma \setminus \{i\}$ is the union of three chains, and if $r-1, s-1, t-1$ are the lengths of these chains, the triple $\{r, s, t\}$ is equal, up to a permutation, to one of the triples $\{1,2,2\}$, $\{1,2,3\}$, $\{1,2,4\}$, $\{1,1,m\}$ (for some $m \geq 1$).*

Proof. The vertex i belongs to 3 edges by Lemma 5.3.2, and there is no other ramification point by Lemma 5.3.7, so the full subgraph $\Gamma \setminus \{i\}$ is the sum of 3 chains Γ_1, Γ_2, and Γ_3, each of which has a terminal vertex linked to i in Γ. Let $\{i_1, i_2\}, \ldots, \{i_{r-1}, i_r\}$ be the edges Of Γ_2, $\{j_1, j_2\}, \ldots, \{j_{s-1}, j_s\}$ those of Γ_2, and $\{k_1, k_2\}, \ldots, \{k_{t-1}, k_t\}$ those of Γ_3, with i_1, j_1, k_1 linked to i in Γ. We can assume that $r \geq s \geq t \geq 1$. Put

$$x = e_{i_r} + 2e_{i_{r-1}} + \cdots + re_{i_1}, \quad y = e_{j_s} + 2e_{j_{s-1}} + \cdots + se_{j_1}, \quad z = e_{k_t} + 2e_{k_{t-1}} + \cdots + te_{k_1}.$$

Since all the edges of Γ are of order 3 (Lemma 5.3.7), Lemma 5.3.8 gives

$$\|x\|^2 = \frac{1}{2}r(r+1), \quad \|y\|^2 = \frac{1}{2}s(s+1), \quad \|z\|^2 = \frac{1}{2}t(t+1).$$

On the other hand, e_i is orthogonal to vectors other than $e_{i_1}, e_{j_1}, e_{k_1}$, so

$$(e_i, x) = -\frac{1}{2}p, \quad (e_i, y) = -\frac{1}{2}s, \quad (e_i, z) = -\frac{1}{2}t.$$

The unit vectors $\|x\|^{-1}x$, $\|y\|^{-1}y$, and $\|z\|^{-1}z$ are mutually orthogonal, and e_i does not belong to the subspace F they generate; the suqre of the distance from e_i to F is

$$\begin{aligned} 1 - (e_i, \frac{x}{\|x\|})^2 - (e_i, \frac{y}{\|y\|})^2 - (e_i, \frac{z}{\|z\|})^2 &= 1 - \frac{r}{2(r+1)} - \frac{s}{2(s+1)} - \frac{t}{2(t+1)} \\ &= \frac{1}{2(r+1)} + \frac{1}{2(s+1)} + \frac{1}{2(t+1)} - \frac{1}{2}. \end{aligned}$$

Since this quantity is positive, we have

$$(r+1)^{-1} + (s+1)^{-1} + (t+1)^{-1} > 1. \tag{5.3.3}$$

Hence $3(t+1)^{-1} > 1$ and finally $r = 1$. Then (5.3.3) gives

$$(r+1)^{-1} + (s+1)^{-1} > \frac{1}{2} \tag{5.3.4}$$

hence $2(s+1)^{-1} > 1/2$, so $s \leq 2$. Finally, if $s = 2$, (5.3.4) gives $(r+1)^{-1} > 1/6$, hence $r \leq 4$. □

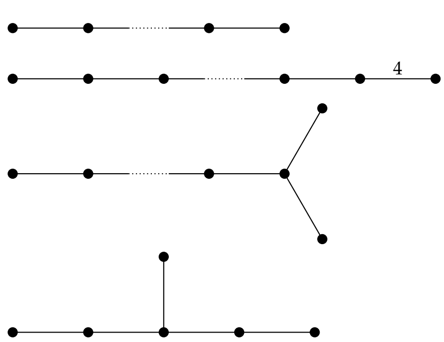
Theorem 5.3.11. *The graph of any irreducible finite Coxeter system (W, S) is isomorphic to one of the following:*

A_l $(l \geq 1)$

B_l $(l \geq 1)$

D_l $(l \geq 4)$

E_6



E_7

E_8

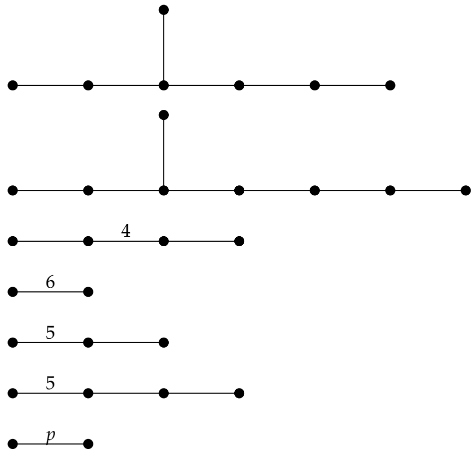
F_4

G_2

H_3

H_4

$I_2(p)$



Moreover, no two of these graphs are isomorphic.

Proof. Indeed, let $W = (m_{ij})$ be the Coxeter matrix of (W, S), and let $I = |S|$. If one of the m_{ij} is ≥ 6, then $l = 2$ by Lemma 5.3.4 and the Coxeter graph of (W, S) is of type G_2 or $I_2(p)$ with $p \geq 7$. Assume now that all the $m_{ij} \leq 5$.

If the m_{ij} are not all equal to 3, then the graph Γ of (W, S) is a chain and exactly one of the m_{ij} is equal to 4 or 5 by Lemma 5.3.7. If one of the m_{ij} is equal to 5, Lemma 5.3.9 shows that we have one of the types H_3, H_4 or $I_2(5)$. If one of the m_{ij} is equal to 4, Lemma 5.3.9 shows that we have one of the types B_l or F_4.

Assume now that all the m_{ij} are equal to 3. If X is a chain, the Coxeter graph is of type A_l. If not, Lemma 5.3.10 shows that it is of type E_6, E_7, E_8 or D_l. The fact that no two of the Coxeter graphs listed are isomorphic is clear. □

Theorem 5.3.12. *The Coxeter groups defined by the Coxeter graphs of Theorem 5.3.11 are finite.*

Proof. This is clear for $I_2(p)$, the corresponding group being the dihedral group of order $2p$. For H_4 the matrix (q_{ij}) is

$$Q = \begin{pmatrix} 1 & -\frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 1 & -\cos\frac{\pi}{5} \\ 0 & 0 & -\cos\frac{\pi}{5} & 1 \end{pmatrix}$$

By computing the principal minors for this matrix, we see Q is positive and non-degenerate, so the corresponding Coxeter group is finite. The same holds for that corresponding to H_3, since it is isomorphic to a subgroup of the preceding.

For the types $A_l, B_l, \ldots, G_2$, we shall construct root systems having the corresponding groups as Weyl groups. We shall see that these groups are not only finite, but crystallographic. □

5.3.2 Dynkin graphs

By abuse of language, we shall call a **normed graph** a pair (Γ, f) having the following properties:

(a) Γ is a graph (called the **underlying graph** of (Γ, f)).

(b) If E denotes the set of pairs (i, j) such that $\{i, j\}$ is an edge of Γ, f is a map from E to $\mathbb{R}$ such that $f(i, j)f(j, i) = 1$ for all $(i, j) \in E$.

There is an obvious notion of isomorphism of normed graphs.

Let Φ be a reduced root system in a real vector space V. We are going to associate to it a normed graph (Γ, f), called the Dynkln graph of Φ. The vertices of Γ will be the elements of the set I of orbits of $W(\Phi)$ in the union of the sets $\{\Delta\} \times \Delta$ (where Δ is a basis of Φ). If $N = (n_{ij})_{i,j \in I}$ (resp. $M = (m_{ij})_{i,j \in I}$ is the canonical Cartan matrix (resp. the Coxeter matrix) of Φ, two vertices i and j of Γ are linked if and only if $n_{ij} \neq 0$ and we then put

$$f(i, j) = \frac{n_{ij}}{n_{ji}}.$$

Since $n_{ij} = 0$ implies $n_{ji} = 0$, this defines a normed graph (Γ, f).

Let $(\cdot, \cdot)$ be a scalar product on V, invariant under $W(\Phi)$, and let $\Delta = (\alpha_i)_{i \in I}$ be a basis of Φ, indexed canonically. Then that vertices i and j of the graph Γ are linked if and only if $(\alpha, \alpha_j) \neq 0$, and

$$f(i,j) = \frac{n(\alpha_i, \alpha_j)}{n(\alpha_j, \alpha_i)} = \frac{(\alpha_i, \alpha_i)}{(\alpha_j, \alpha_j)}.$$

In view of Table 5.1, the only possibilities are the following, up to interchanging i and j:

(1) i and j are not linked, and $n_{ij} = n_{ji} = 0$, $m_{ij} = 2$;

(2) $f(i,j) = f(j,i) = 1$ and $n_{ij} = n_{ji} = -1$, $m_{ij} = 3$;

(3) $f(i,j) = 2$, $f(j,i) = 1/2$, and $n_{ij} = n_{ji} = -2$, $m_{ij} = 4$;

(4) $f(i,j) = 3$, $f(j,i) = 1/3$, and $n_{ij} = n_{ji} = -3$, $m_{ij} = 6$;

We see from this that the Dynkin graph Φ determines the Cartan matrix and the Coxeter matrix of Φ, and hence determines Φ up to isomorphism. More precisely, Corollary 5.1.32 implies the following result:

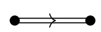
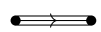
Proposition 5.3.13. *Let Φ_1 and Φ_2 be two reduced root systems in vector spaces V_1 and V_2. Let $\Delta_1 = (\alpha_i)_{i \in I_1}$ and $\Delta_2 = (\alpha_i)_{i \in I_2}$ be bases of Δ_1 and Δ_2, indexed canonically. Let θ be an isomorphism from the Dynkin graph of Φ_1 to the Dynkin graph of Φ_2. Then, there exists a unique isomorphism from V_1 to V_2 transforming Φ_1 into Φ_2 and α_i into $\alpha_{\theta(i)}$ for all $i \in I_1$.*

It is clear that an automorphism of Φ defines an automorphism of the Dynkin graph of Φ, and hence a homomorphism Π from the group $\mathrm{Aut}(\Phi)$ to the group of automorphisms of the Dynkin graph of Φ.

Corollary 5.3.14. *The homomorphism Π defines by passage to the quotient an isomorphism from the group $\mathrm{Aut}(\Phi)/W(\Phi)$ to the group of automorphisms of the Dynkin graph of Φ.*

Proof. Recall that an vertex i represents an orbit of W in the union of the sets $\{\Delta\} \times \Delta$, where Δ is a basis. Therefore $\Pi(w) = 1$ for all $w \in W(\Phi)$. On the other hand, Proposition 5.3.13 shows that there exists an isomorphism Ξ from the group of isomorphisms of the Dynkin graph of Φ to the subgroup G of elements of $\mathrm{Aut}(\Phi)$ leaving fixed a given basis Δ of Φ, such that $\Pi \circ \Xi = 1$. Since $\mathrm{Aut}(\Phi)$ is the semi-direct product of G and $W(\Phi)$, the corollary follows. $\square$

In practice, the Dynkin graph (Γ, f) is represented by a diagram composed of nodes and bonds in the following way. The nodes correspond to the vertices of Γ; two nodes corresponding to two distinct vertices i and j are joined by 0, 1, 2 or 3 bonds in cases (1), (2), (3) and (4) above (up to interchanging i and j). Moreover, in cases (3) and (4), that is when $f(i,j) > 1$, or when the roots α_i and α_j are not orthogonal and not of the same length, an inequality sign $>$ is placed on the double or triple bond joining the nodes corresponding to i and j oriented towards the node corresponding to j (that is, the shortest root):

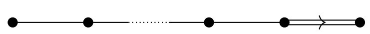
 if $f(i,j) = 2$  if $f(i,j) = 3$.

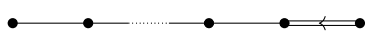
It is clear that the Dynkin graph (Γ, f) can be recovered from this diagram.

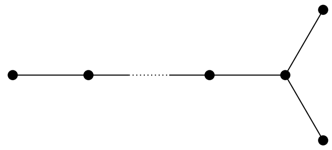
We remark that the diagram associated to the Coxeter graph of $W(\Phi)$ can be obtained from that associated to the Dynkin graph of Φ by keeping the nodes and the single bonds and replacing the double (resp. triple) bonds by a bond surmounted by the number 4 (resp. 6). Conversely, given the Coxeter graph of $W(\Phi)$, the diagram associated to the Dynkin graph of Φ can be recovered using the inverse of this procedure, except for the inequality signs on the double or triple bonds. Theorem 5.3.11 thus gives immediately the list of possible Dynkin graphs. More precisely:

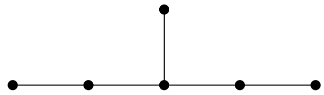
Theorem 5.3.15. *If Φ is an irreducible reduced root system, its Dynkin graph is isomorphic to one of the graphs represented by the following diagrams:*

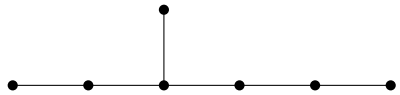
A_l $(l \geq 1)$

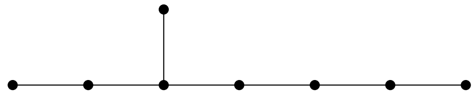
B_l  $(l \geq 2)$

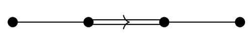
C_l  $(l \geq 3)$

D_l  $(l \geq 4)$

E_6 

E_7 

E_8 

F_4 

G_2 

Moreover, no two of these graphs are isomorphic and there are irreducible reduced root systems having each of them as their Dynkin graph (up to isomorphism).

Proof. The first assertion follows immediately from Theorem 5.3.11, in view of the preceding remarks, the fact that the Coxeter groups of the graphs H_3, H_4 and $I_2(p)$ (for $p = 5$ and $p \geq 7$) are not crystallographic, and the fact that the two possible inequalities for the double (resp. triple) bond of the Dynkin graph associated to the Coxeter graph F_4 (resp. G_2) give isomorphic Dynkin graphs. The second assertion is clear and the third will follow from the explicit construction of an irreducible reduced root system for each type. $\square$

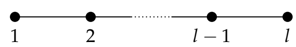
Remark 5.3.16. The graph A_1 reduces to a single node; we denote it also by B_1 or C_1. The graph B_2 is also denoted by C_2, since they are isomorphic. The graph A_3 is also denoted by D_3. Finally, D_2 denotes the graph consisting of two unconnected nodes. (These conventions are derived from the properties of the corresponding root systems)

Remark 5.3.17. If (Γ, f) is the Dynkin graph of a reduced root system Φ, the Dynkin graph of the inverse system can be identified with (Γ, f^{-1}). In other words, the diagram associated to the Dynkin graph of $\check{\Phi}$ can be obtained from that associated to the Dynkin graph of Φ by reversing the inequality signs. If Φ is irreducible, we see that Φ is isomorphic to $\check{\Phi}$ unless Φ is of type B_l or C_l, in which case $\check{\Phi}$ is of type C_l or B_l.

Example 5.3.18 (Automorphism groups for connected Dynkin graphs). As an application of Theorem 5.3.15, we compute the automorphim of all connected Dynkin diagrams, so that by Corollary 5.3.14 we can determine the automorphim of reduced irreducible root systems. Let Γ be one of the graphs in Theorem 5.3.15. First we note that, if Γ is not simply-laced (which means it has $>$ in it), then the automorphim group of Γ is trivial. In fact, in this case any automorphim τ of Γ must preserve the edge $\{i, j\}$ such that $n(i, j) \neq 1$ and therefore fixes i and j. With this, since we see in any case the vertex i or j is linked to another unique vertex k, this vertex is also fixed by τ. Continuing this argument, we see that τ eventually fixes all vertices of Γ, so $\tau = 1$. Comparing with Theorem 5.3.15, we get

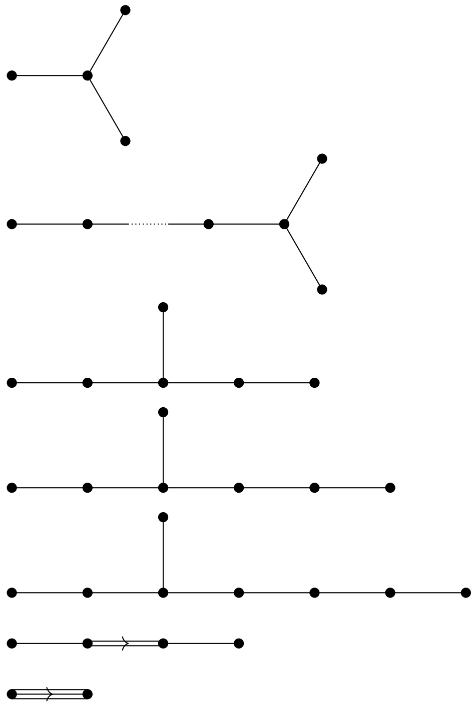
$$\mathrm{Aut}(B_l) = \mathrm{Aut}(C_l) = \mathrm{Aut}(F_4) = \mathrm{Aut}(G_2) = \{1\}.$$

As an example, we compute the automorphims of the Dynkin graph A_l, which is given by



Let $\tau \in \mathrm{Aut}(A_l)$. Then τ sends terminal vertices to terminal vertices, so $\tau(1) = 1$ or $\tau(1) = l$. From this and the fact that τ send edges to edges, we conclude that $\tau(i) = i$ or $\tau(i) = l + 1 - i$ for all i. If we denote the second automorphim by -1, then $\mathrm{Aut}(A_l) = \{1, -1\} \cong \mathbb{Z}/2\mathbb{Z}$.

Following this partten, one easily computes the automorphim groups for the rest simply-laced Dynkin graphs. We list our result in the following table:



A_l	$\mathrm{Aut}(A_l) = \mathbb{Z}/2\mathbb{Z}$
B_l	$\mathrm{Aut}(B_l) = \{1\}$
C_l	$\mathrm{Aut}(C_l) = \{1\}$
D_4	$\mathrm{Aut}(D_4) = \mathfrak{S}_3$
D_l	$\mathrm{Aut}(D_l) = \mathbb{Z}/2\mathbb{Z} \quad (l > 4)$
E_6	$\mathrm{Aut}(E_6) = \mathbb{Z}/2\mathbb{Z}$
E_7	$\mathrm{Aut}(E_7) = \{1\}$
E_8	$\mathrm{Aut}(E_8) = \{1\}$
F_4	$\mathrm{Aut}(F_4) = \{1\}$
G_2	$\mathrm{Aut}(G_2) = \{1\}$

Let Φ be an irreducible reduced root system and (Γ, f) be its Dynkin graph. We are going to define another normed graph $(\tilde{\Gamma}, \tilde{f})$ that we shall call the **completed Dynkin graph** of Φ. The set $\tilde{I}$ of vertices of Γ consists of the set I of vertices of Γ and a vertex denoted by 0, not belonging to I. To define $\tilde{f}$, choose a basis $\Delta = (\alpha_i)_{i\in I}$ of Φ and a scalar product $(\cdot,\cdot)$ invariant under $W(\Phi)$. Let $\alpha_0 = -\tilde{\alpha}$ be the negative of the highest root $\tilde{\alpha}$ for the order defined by Δ. Two distinct vertices $i, j \in I$ are linked if and only if $(\alpha_i, \alpha_j) \neq 0$ and we then put

$$\tilde{f} = \frac{(\alpha_i, \alpha_i)}{(\alpha_j, \alpha_j)}.$$

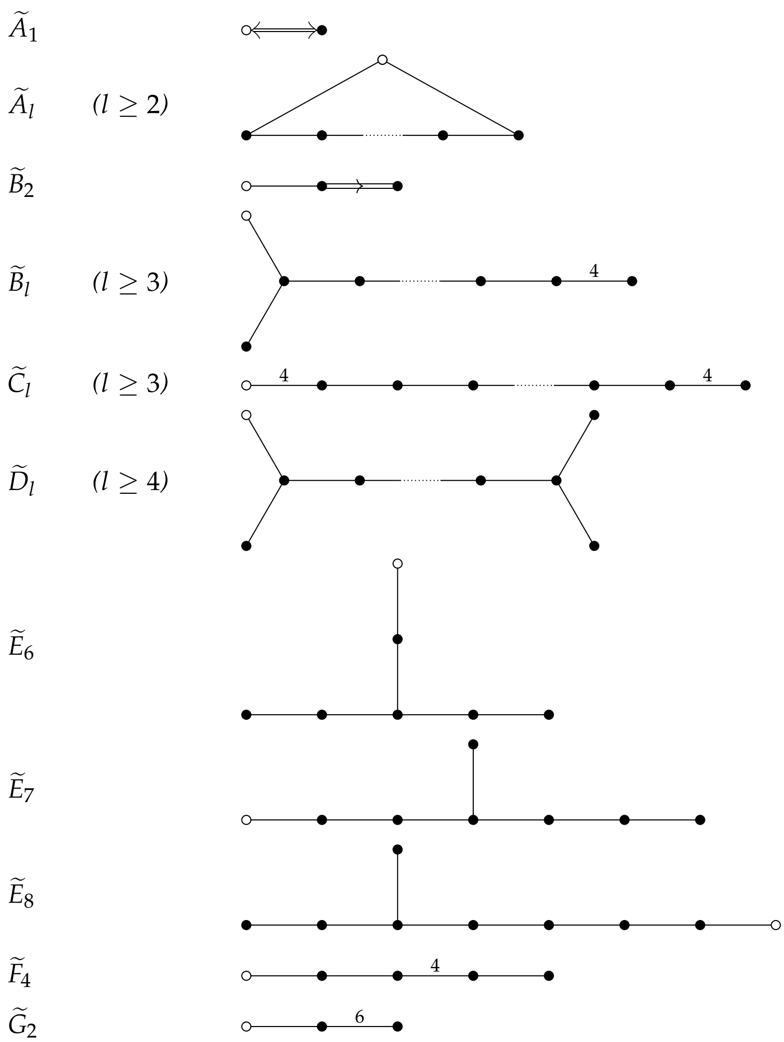
It is immediate that the graph $\tilde{\Gamma}$ and the map $\tilde{f}$ thus defined do not depend on the choice of Δ or the scalar product.

If the rank l of Φ is equal to 1, then $I = \{i\}$ and $\alpha_0 = -\alpha_1$; hence $\tilde{f}(0, i) = 1$. If $l \geq 2$, α_0 is not proportional to any of the α_i and (α_0, α_i) is negative (Proposition 5.1.62). For any pair (i, j) of distinct elements of I, the only possibilities are those mentioned above (putting, for example, $n_{0i} = n(\alpha_0, \alpha_i)$ and $m_{0,i}$ be order of $s_{\alpha_0} s_{\alpha_i}$ for all $i \in I$).

In the case $l \geq 2$, the completed Dynkin graph is represented by a diagram with the same conventions as before, sometimes indicating by dotted lines the bonds joining the vertex 0 to the other vertices. We remark that the inequality sign $>$ on such a bond, if it exists, is always directed towards the vertex distinct from 0, since α_0 is a longest root (Proposition 5.1.62). The graph (Γ, f) is the subgraph obtained from $(\tilde{\Gamma}, \tilde{f})$ by deleting the vertex 0. The action of $\mathrm{Aut}(\Phi)$ on (Γ, f) extends to an action on $(\tilde{\Gamma}, \tilde{f})$, leaving 0 fixed, and $W(\Phi)$ acts trivially on $(\tilde{\Gamma}, \tilde{f})$. Let $W_a(\Phi)$ be the affine Weyl group of Φ. Then Proposition 5.2.6 together with Theorem 4.3.3 shows that the Coxeter graph Σ of the affine Weyl group $W_a(\Phi)$ can be obtained from (Γ, f), and the index set $\tilde{I}$ can be viewed as the set of orbits of $W_a(\Phi)$ in the union of sets $\{\Delta \cup \{\tilde{\alpha}\}\} \times (\Delta \cup \{\tilde{\alpha}\})$.

On the other hand, let N be the normaliser of $W_a(\Phi)$. To any $g \in N$ corresponds an automorphism $\Pi(g)$ of Σ and $\Pi(g) = 1$ if $g \in W_a(\Phi)$. Conversely, given an automorphism θ of Σ there is, by Proposition 4.4.21, a unique element $g = \Xi(\theta)$ preserving a given alcove A and such that $\Pi(g) = \theta$. Since N is the semi-direct product of the subgroup N_A of elements preserving A and $W_a(\Phi)$, we deduce that Π defines by passage to the quotient an isomorphism from $N/W_a(\Phi) = N_A$ to $\mathrm{Aut}(\Sigma)$. Recall that the group $N/W_a(\Phi)$ can be identified with the semi-direct product of $\mathrm{Aut}(\Phi)/W(\Phi)$ and $\Lambda(\check{\Phi})/\Lambda_r(\check{\Phi})$, therefore the group $\operatorname{Aut}(\Sigma)$ is also isomorphic to this semi-direct product. We also recall that the group $\Lambda(\check{\Phi})/\Lambda_r(\check{\Phi})$ is isomorphic to the group $\Gamma_A = N_A \cap \widehat{W}_a(\Phi)$, and that the elements γ_i in Proposition 5.2.8 correspond to nontrivial representatives in $\Lambda(\check{\Phi})/\Lambda_r(\check{\Phi})$. The element of $\operatorname{Aut}(\Sigma)$ corresponding to the element γ_i of Γ_A transforms the vertex 0 to the vertex i of Σ, in view of Proposition 5.2.10.

Theorem 5.3.19. *Let (W, S) be an irreducible Coxster system with S finite. Then the associated quadratic form is positive and degenerate if and only if the Coxeter graph of (W, S) is isomorphic to one of the following:*



Moreover, no two of these Coxeter graphs are isomorphic.

Proof. By Proposition 4.4.21, the Coxeter systems whose quadratic form is positive and degenerate are those which correspond to the affine Weyl groups of irreducible reduced root systems. The theorem therefore follows from the determination of the completed Dynkin graphs. $\square$

Chapter 6

Semi-simple Lie algebras and its representations

In this section, $\mathbb{K}$ denotes a field. By a vector space, we mean a vector space over $\mathbb{K}$, and similarly for "Lie algebra", etc. All Lie algebras are assumed to be finite dimensional.

6.1 Cartan subalgebras and regular elements

6.1.1 Root decomposition of representations

Let V be a vector space and S a set. A **representation** of S on V is defined to be a map $\rho : S \to \mathrm{End}(V)$. Let $\mathcal{P}(S)$ (or $\mathcal{P}$ if the set S is understood) denote the set of all maps from S to $\mathbb{K}$. For any $\lambda \in \mathcal{P}$, we denote by $V_\lambda(S)$ (resp. $V^\lambda(S)$) the subspaces of V defined by

$$V_\lambda(S) = \{v \in V : \rho(s) \cdot v = \lambda(s)v \text{ for all } s \in S\}$$

and

$$V^\lambda(S) = \{v \in V : (\rho(s) - \lambda(s))^n v = 0 \text{ for } n \text{ sufficiently large}\}.$$

We say that $V_\lambda(S)$ is the **eigenspace** of V relative to λ (and to ρ), that $V^\lambda(S)$ is the **primary subspace** of V relative to λ (and to ρ). Also, the space $V_0(S)$ is called that **nilspace** of V (relative to ρ). We say λ is a **weight** of S in V if $V^\lambda(S) \neq 0$, and the set of all weights of V is denoted by $\mathcal{P}(V)$.

In particular, if S reduces to a single element s, then the set $\mathcal{P}$ can be identified with $\mathbb{K}$, and we may use the notations $V_\lambda(s)$ and $V^\lambda(s)$ instead of $V_\lambda(\{s\})$ and $V^\lambda(\{s\})$. We speak of eigenspaces, primary subspaces and the nilspace of $\rho(s)$; an element v of $V_\lambda(s)$ is called an **eigenvector** of $\rho(s)$, and $v \neq 0$, $\lambda(s)$ is called the corresponding **eigenvalue** of $\rho(s)$. For any map $\lambda : S \to \mathbb{K}$, the following relations are immediate:

$$V_\lambda(S) = \bigcap_{s \in S} V_\lambda(s), \quad V^\lambda(S) = \bigcap_{s \in S} V^\lambda(S).$$

Let $\mathbb{K}'$ be an extension of $\mathbb{K}$. The canonical map from $\mathrm{End}(V)$ to $\mathrm{End}(V \otimes_{\mathbb{K}} \mathbb{K}')$ gives, by composition with ρ, a map $\tilde{\rho} : S \to \mathrm{End}(V \otimes_{\mathbb{K}} \mathbb{K}')$. Similarly, every map $\lambda : S \to \mathbb{K}$ can be viewed as a map from S to $\mathbb{K}'$. With these notations, we have the following proposition:

Proposition 6.1.1. *For any map $\lambda \in \mathcal{P}$, we have*

$$(V \otimes_{\mathbb{K}} \mathbb{K}')_\lambda(S) = V_\lambda(S) \otimes_{\mathbb{K}} \mathbb{K}', \quad (V \otimes_{\mathbb{K}} \mathbb{K}')_\lambda(S) = V_\lambda(S) \otimes_{\mathbb{K}} \mathbb{K}'$$

Proof. Let (α_i) be a basis of the $\mathbb{K}$-vector space $\mathbb{K}'$. If $v \in V \otimes_{\mathbb{K}} \mathbb{K}'$, then v can be expressed uniquely in the form $\sum_i v_i \otimes \alpha_i$ where (v_i) is a finitely-supported family of elements of V. For all $s \in S$ and $n \in \mathbb{N}$,

$$(\tilde{\rho}(s) - \lambda(s))^n v = \sum (\rho(s) - \lambda(s))^n v_i \otimes \alpha_i$$

from which the claim follows. □

For two representations (V,ρ) and (W,η) of S, we define an **S-homomorphism** to be a linear map $\varphi : V \to W$ such that for each $s \in S$ and $v \in V$ we have

$$\varphi(\rho(s)v) = \eta(s)\varphi(v).$$

We say the S-homomorphism φ is injective, surjective, or bijective if the map φ is injective, surjective, bijective respectively.

Proposition 6.1.2. *Let (U,θ), (V,ρ), and (W,η) be representations of S.*

(a) Let $\varphi : V \to W$ be an S-homomorphism. Then φ maps $V_\lambda(S)$ (resp. $V^\lambda(S)$) into $W_\lambda(S)$ (resp. $W^\lambda(S)$) for any $\lambda \in \mathcal{P}$.

(b) Let $B : V \times W \to U$ be a bilinear map such that

$$\theta(s)B(v,w) = B(\rho(s)v,w) + B(v,\eta(s)w)$$

for $s \in S$ and $v \in V$, $w \in W$. Then for any $\lambda,\mu \in \mathcal{P}$, B maps $V_\lambda(S) \times W_\mu(S)$ (resp. $V^\lambda(S) \times W^\mu(S)$) into $U_{\lambda+\mu}(S)$ (resp. $U^{\lambda+\mu}(S)$).

(c) Let $B : V \times W \to U$ be a bilinear map such that

$$\theta(s)B(v,w) = B(\rho(s)v,\eta(s)w)$$

for $s \in S$ and $v \in V$, $w \in W$. Then for any $\lambda,\mu \in \mathcal{P}$, B maps $V_\lambda(S) \times W_\mu(S)$ (resp. $V^\lambda(S) \times W^\mu(S)$) into $U_{\lambda\mu}(S)$ (resp. $U^{\lambda\mu}(S)$).

Proof. In case (a), for any $s \in S$ and $v \in V$ we have

$$(\eta(s) - \lambda(s))^n\varphi(v) = \varphi((\rho(s) - \lambda(s))^n v)$$

hence the conclusion. In case (b), for any $s \in S$ and $v \in V$,

$$(\theta(s) - \lambda(s) - \mu(s))B(v,w) = B((\rho(s) - \lambda(s))v,w) + B(v,(\eta(s) - \mu(s))w)$$

hence by induction on n we get

$$(\theta(s) - \lambda(s) - \mu(s))B(v,w) = \sum_{i+j=n}\binom{n}{i}B((\rho(s) - \lambda(s))^i v,w) + B(v,(\eta(s) - \mu(s))^j w).$$

The assertions in (b) follow immediately. In case (c), however, we have

$$(\theta(s) - \lambda(s)\mu(s))B(v,w) = B((\rho(s) - \lambda(s))v,\eta(s)w) + B(\lambda(s)v,(\eta(s) - \mu(s))w)$$

for $s \in S$, $v \in V$, $w \in W$. Again by induction on n, we find

$$(\theta(s) - \lambda(s)\mu(s))^n B(v,w) = \sum_{i+j=n}\binom{n}{i}B(\lambda(s)^j(\rho(s) - \lambda(s))^i v,\eta(s)^i(\eta(s) - \mu(s))^j w).$$

The assertions in (c) follow immediately. □

Proposition 6.1.3. *The sums $\sum_{\lambda\in\mathcal{P}} V_\lambda(S)$ and $\sum_{\lambda\in\mathcal{P}} V^\lambda(S)$ are direct.*

Proof. Since $V_\lambda(S) \subseteq V^\lambda(S)$, we only need to prove $\sum_{\lambda\in\mathcal{P}} V^\lambda(S)$ is direct. For this, we will show that any linera combination of weight vectors can not be a weight vector again unless it is zero.

If S is empty, the claim is trivial. If S is reduced to a single element s, let $\lambda_0,\lambda_1,\dots,\lambda_n$ be distinct elements of $\mathbb{K}$. For each $i = 0,\dots,n$ let $v_i \in V^{\lambda_i}(s)$ and assume that $v_0 = a_1v_1 + \cdots + a_nv_n$. It suffices to prove that $v_0 = 0$. For each i, there exists an integer $r_i > 0$ such that $(\rho(s) - \lambda_i)^{r_i}(v_i) = 0$. Consider the polynomials

$$P(T) = \prod_{i=1}^{n}(T - \lambda_i)^{r_i}, \quad Q(T) = (T - \lambda_0)^{r_0}.$$

We then have

$$Q(\rho(s))v_0 = 0, \quad P(\rho(s))v_0 = \sum_{i=1}^{n} a_i P(\rho(s))v_i = 0.$$

Since P and Q are relatively prime, the Bezout identity proves that $v_0 = 0$.

Now we assume that S is finite and prove the claim by induction on the cardinality of S. Let $s \in S$ and $\widetilde{S} = S \setminus \{s\}$. Let $(v_\lambda)_{\lambda\in\mathcal{P}}$ be a finitely-supported family of elements of V and (a_λ) a finitely-supported family of elements of $\mathbb{K}$ such that

$$v_{\lambda_0} = \sum_{\lambda\in\mathcal{P}} a_\lambda v_\lambda \in V^{\lambda_0}(S), \quad v_\lambda \in V^\lambda(S), \tag{6.1.1}$$

where $\lambda_0 \in \mathcal{P}$. Let $\tilde{\mathcal{P}}$ be the set of $\lambda \in \mathcal{P}$ such that $\lambda|_{\widetilde{S}} = \lambda_0|_{\widetilde{S}}$. Then for any $\lambda \in \widetilde{S}$ we have $V^\lambda(\widetilde{S}) = V^{\lambda_0}(\widetilde{S})$, so by restricting (6.1.1) on $\widetilde{S}$ we get

$$\sum_{\lambda\in\mathcal{P}\setminus\tilde{\mathcal{P}}} a_\lambda v_\lambda \in V^{\lambda_0}(\widetilde{S}), \quad v_\lambda \in V^\lambda(\widetilde{S}).$$

By the induction hypothesis applied to $\widetilde{S}$, we have $\sum_{\lambda\in\mathcal{P}\setminus\tilde{\mathcal{P}}} a_\lambda v_\lambda = 0$, so equation (6.1.1) reduces to

$$v_{\lambda_0} = \sum_{\lambda\in\tilde{\mathcal{P}}} a_\lambda v_\lambda, \quad v_\lambda \in V^\lambda(s). \tag{6.1.2}$$

If λ, μ are distinct elements of $\tilde{\mathcal{P}}$, then $\lambda(s) \neq \mu(s)$. Since the sum $\sum_{\lambda\in s} V^\alpha(s)$ is direct, from (6.1.2) we see $v_{\lambda_0} = 0$, so the claim is proved in this case.

In the general case, let $(v_\lambda)_{\lambda\in\mathcal{P}}$ and $(a_\lambda)_{\lambda\in\mathcal{P}}$ be finitely-supported families of elements of V and $\mathbb{K}$, respectively, such that

$$v_{\lambda_0} = \sum_{\lambda\in\mathcal{P}} a_\lambda v_\lambda \in V^{\lambda_0}(S), \quad v_\lambda \in V^\lambda(S).$$

Let $\tilde{\mathcal{P}}$ be the finite set of $\lambda \in \mathcal{P}$ such that $v_\lambda \neq 0$, and let $\widetilde{S}$ be a finite subset of S such that the conditions $\lambda, \mu \in \tilde{\mathcal{P}}$ and $\lambda|_{\widetilde{S}} = \mu|_{\widetilde{S}}$ imply that $\lambda = \mu$. We have $v_\lambda \in V^\lambda(\widetilde{S})$, so applying the preceding arguments we see $v_\lambda = 0$ for $\lambda \in \tilde{\mathcal{P}}$, which completes the proof. □

Since the sum $\sum_{\lambda\in\mathcal{P}} V^\lambda(S)$ is direct, we may seek that if there is any chance that $V = \bigoplus_{\lambda\in\mathcal{P}} V^\lambda(S)$, so that we have a nice discription of the action of S on V. First,ecall that, if $x \in \mathrm{End}(V)$, we denote by $\mathrm{ad}(x)$ the map $y \mapsto xy - yx = [x, y]$ from $\mathrm{End}(V)$ to itself.

Lemma 6.1.4. *Let V be a vector space and $x, y \in \mathrm{End}(V)$.*

(a) Assume that V is finite dimensional. Then x is triangularizable if and only if we have

$$V = \bigoplus_{\lambda\in\mathbb{K}} V^\alpha(x).$$

(b) If there exists an integer n such that $\mathrm{ad}(x)^n(y) = 0$, each $V^\alpha(x)$ is stable under y.

(c) Assume that V is finite dimensional. If $V = \bigoplus_{\alpha\in\mathbb{K}} V^\alpha(x)$ and each $V^\alpha(x)$ is stable under y, then there exists an integer n such that $\mathrm{ad}(x)^n(y) = 0$.

Proof. Let $E = \mathrm{End}(V)$ and B be the bilinear map defined by

$$B : E \times V \to V, \quad (x, v) \mapsto x(v).$$

By the definition of $\mathrm{ad}(x)$, for $x, y \in E$ and $v \in V$ we have

$$x(B(y, v)) = xy(v) = yx(v) + [x, y](v) = B(y, x(v)) + B([x, y], v).$$

Let x operate on E via $\mathrm{ad}(x)$, then by Proposition 6.1.2 we have $B(E^0(x), V^\alpha(x)) \subseteq V^\alpha(x)$ for all $\alpha \in \mathbb{K}$. If $\mathrm{ad}(x)^n(y) = 0$, then $y \in E^0(x)$, so that $y(V^\alpha(x)) \subseteq V^\alpha(x)$, which proves (b).

To prove (c), we first note that since each $V^\alpha(x)$ is stable under y, we can replace V by $V^\alpha(x)$ and x (resp. y) by its restriction to $V^\alpha(x)$. Replacing x by $x - \alpha$, we can assume that x is nilpotent. Then, writting $\mathrm{ad}(x)$ as $L_x - R_x$, where L_x is the left multiplication by x and R_x the right multiplication by x, we have

$$\mathrm{ad}(x)^n = (L_x - R_x)^n = \sum_{i+j=n} (-1)^i \binom{n}{i} L_x^i R_x^j. \tag{6.1.3}$$

Since x is nilpotent we see L_x and R_x are both nilpotent, so the claim in (c) follows from (6.1.3). □

Corollary 6.1.5. *If V is a finite dimensional vector space and there exists an integer n such that $\mathrm{ad}(x)^n(y) = 0$, then $\mathrm{ad}(x)^{2\dim(V)-1}(y) = 0$.*

Proof. This follows from (6.1.3) adn the fact that if x is nilpotent, then $x^{\dim(V)} = 0$. □

In the sequel, we shall say that a map $\rho : S \to \mathrm{End}(V)$ is **almost commutative** if for every pair (s,t) of elements of S, there exists an integer n such that

$$\mathrm{ad}(\rho(s))^n(\rho(t)) = 0.$$

An S-representation (V,ρ) is called **almost commutative** if the map ρ is almost commutative.

Theorem 6.1.6. *Let (V,ρ) be a finite dimensional S-representation. Then the following conditions are equivalent:*

(i) *The map ρ is almost commutative and for each $s \in S$ the endomorphism $\rho(s)$ is triangularizable.*

(ii) *For all $\lambda \in \mathcal{P}$, $V^\lambda(S)$ is stable under $\rho(S)$, and $V = \bigoplus_{\lambda\in\mathcal{P}} V^\lambda(S)$.*

Proof. If $V = \bigoplus_{\lambda\in\mathcal{P}} V^\lambda(S)$, then for each $s \in S$ we have $V = \bigoplus_{\lambda\in\mathcal{P}} V^\lambda(s)$, so it follows from Lemma 6.1.4 that (ii) implies (i). Conversely, assume that condition (i) is satisfied. Then Lemma 6.1.4 imply that each $V^\lambda(S)$ is stable under $\rho(S)$. It remains to prove that $V = \bigoplus_{\lambda\in\mathcal{P}} V^\lambda(S)$. We argue by induction on $\dim(V)$. We distinguish two cases.

If for all $s \in S$, $\rho(s)$ has a single eigenvalue $\lambda(s)$, then $V = V^\lambda(S)$, so the claim is true. On the other hand, if there exists $s \in S$ such that $\rho(s)$ has at least two distinct eigenvalues, then V is the direct sum of the $V^\alpha(s)$ for $\alpha \in \mathbb{K}$, and $\dim(V^\alpha(s)) < \dim(V)$ for all α. Since each $V^\alpha(s)$ is stable under $\rho(S)$, it suffices to apply the induction hypothesis to complete the proof. □

Corollary 6.1.7. *Let (V,ρ) be a finite dimensional almost commutative representation of S. Let $\mathbb{K}'$ be an extension of $\mathbb{K}$. Assume that, for all $s \in S$, the endomorphism $\rho(s)\otimes 1$ of $V \otimes_{\mathbb{K}} \mathbb{K}'$ is triangularizable. Then*

$$V \otimes_{\mathbb{K}} \mathbb{K}' = \bigoplus_{\lambda\in\mathcal{P}} (V \otimes_{\mathbb{K}} \mathbb{K}')^\lambda(S).$$

Proof. Let $\tilde{\rho} : S \to \mathrm{End}(V \otimes_{\mathbb{K}} \mathbb{K}')$ be the map induced by ρ. If $s_1, s_2 \in S$, then there exists an integer n such that $\mathrm{ad}(\rho(s_1))^n(\rho(s_2)) = 0$, hence $\mathrm{ad}(\tilde{\rho}(s_1))^n(\tilde{\rho}(s_2)) = 0$. It now suffices to apply Theorem 6.1.6. □

Proposition 6.1.8. *Let (V,ρ) be a finite dimensional almost commutative representation of S. Let $V^+(S)$ be the vector subspace defined by $V^+(S) = \sum_{s\in S}(\bigcap_{i\geq 1}\rho(s)^i V)$.*

(a) *The space $V^+(S)$ is the unique subspace of V that is stable under $\rho(S)$ and such that $V = V^0(S) \oplus V^+(S)$. Moreover, we have*
$$\sum_{s\in S}\rho(s)V^+(S) = V^+(S).$$

(b) *Every vector subspace W of V, stable under $\rho(S)$ and such that $W^0(S) = \{0\}$, is contained in $V^+(S)$.*

(c) *For any extension $\mathbb{K}'$ of $\mathbb{K}$, we have $(V \otimes_{\mathbb{K}} \mathbb{K}')^+(S) = V^+(S) \otimes_{\mathbb{K}} \mathbb{K}'$.*

Proof. The last assertion is immediate. Thus, taking Proposition 6.1.1 into account, in proving the others we can assume that $\mathbb{K}$ is algebraically closed. By Theorem 6.1.6, $V = \bigoplus_{\lambda\in\mathcal{P}} V^\lambda(S)$, and the $V^\lambda(S)$ are stable under $\rho(S)$. If $s \in S$, the characteristic polynomial of $\rho(s)|_{V^\lambda(S)}$ is $(T-\lambda(s))^{\dim(V^\lambda(S))}$. It follows that $\bigcap_{i\geq 1}\rho(s)^i V^\lambda(s)$ is zero if $\lambda(s) = 0$ and is equal to $V^\lambda(S)$ if $\lambda(s) \neq 0$ (in this case $\rho(s)$ is an isomorphism when restricted to $V^\lambda(S)$). Hence for each $s \in S$ we get

$$\bigcap_{i\geq 1}\rho(s)^i V = \bigoplus_{\lambda\in\mathcal{P},\lambda(s)\neq 0} V^\lambda(S).$$

Summing over $s \in S$, we see $V^+(S) = \bigoplus_{\lambda \neq 0} V^\lambda(S)$, so the claims in (a) follow.

If W is a vector subspace of V stable under $\rho(S)$, then $W = \bigoplus_{\lambda \in \mathcal{P}} W^\lambda(S)$ and $W^\lambda(S) = W \cap V^\lambda(S)$. If $W^0(S) = 0$, we see that $W \subseteq V^+(S)$, which proves (b).

Finally, let $\widetilde{V}$ be a vector subspace of V stable under $\rho(S)$ and such that $\widetilde{V} \cap V^0(S) = \{0\}$. Then $\widetilde{V}^0(S) = 0$, so $\widetilde{V} \subseteq V^+(S)$ by (b). If, in addition, $V = V^0(S) + \widetilde{V}$, we see that $\widetilde{V} = V^+(S)$. □

We call $(V^0(S), V^+(S))$ the Fitting decomposition of V, or of the map $\rho : S \to \mathrm{End}(V)$. If S reduces to a single element s, we write $V^+(s)$ or $V^+(\rho(s))$ instead of $V^+(\{s\})$. Note that $\rho(s)|_{V^0(s)}$ is nilpotent and $\rho(s)|_{V^+(s)}$ is bijective.

Corollary 6.1.9. *Let V and W be finite dimensional representations of S that are almost commutative. Let $\varphi : V \to W$ be a surjective S-homomorphism. Then $\varphi(V^\lambda(S)) = W^\lambda(S)$ for all $\lambda \in \mathcal{P}$.*

Proof. In view of Proposition 6.1.1, we may assume that $\mathbb{K}$ is algebraically closed. By Theorem 6.1.6 and Proposition 6.1.2 we have $V = \bigoplus_{\lambda \in \mathcal{P}} V^\lambda(S)$, $W = \bigoplus_{\lambda \in \mathcal{P}} W^\lambda(S)$, and

$$W = \varphi(V) = \sum_{\lambda \in \mathcal{P}} \varphi(V^\lambda(S)) \subseteq \bigoplus_{\lambda \in \mathcal{P}} W^\lambda(S) = W,$$

whence the corollary. □

Proposition 6.1.10. *Assume that $\mathbb{K}$ is perfect. Let V be a finite dimensional vector space, x an element of* $\mathrm{End}(V)$, *x_s and x_n the semi-simple and nilpotent components of x.*

(a) *For all $\lambda \in \mathbb{K}$, $V^\lambda(x) = V^\lambda(x_n) = V_\lambda(x_s)$.*

(b) *If V has an algebra structure and if x is a derivation of V, then x_s and x_n are derivations of V.*

(c) *If V has an algebra structure and if x is an algebra automorphism of V, then x_s and $1 + x_s^{-1}x_n$ are automorphisms of V.*

Proof. In view of Proposition 6.1.1, we can assume that $\mathbb{K}$ is algebraically closed, so that $V = \bigoplus_{\lambda \in \mathcal{P}} V^\lambda(x)$. The semi-simple component of $x|_{V^\lambda(x)}$ is the homothety with ratio λ in $V^\lambda(x)$, which proves (a).

Assume from now on that V has an algebra structure. Let $v \in V^\lambda(x)$ and $w \in V^\mu(x)$. If x is a derivation of V, then $vw \in V^{\lambda+\mu}(x)$ (take the bilinear map in Proposition 6.1.2 to be the product map on V), so

$$x_s(vw) = (\lambda + \mu)(vw) = (\lambda v)w + x(\mu w) = (x_s v) \cdot w + v \cdot (x_s w).$$

This proves that x_s is a derivation of V, so that $x_n = x - x_s$ is also a derivation of V.

If x is an automorphism of V, then $\ker(x_s) = V^0(x) = 0$, so x_s is bijective. On the other hand, $vw \in V^{\lambda\mu}(x)$ by Proposition 6.1.2, so

$$x_s(vw) = (\lambda\mu)(vw) = (\lambda v)(\mu w) = x_s(v) \cdot x_s(w).$$

This proves that x_s is an algebra automorphism of V, but then so is $x_s^{-1}x = 1 + x_s^{-1}x_n$. □

Now let S be a finite dimensional vector space. In this case a representation of S is required to be *linear*, i.e., a linear map from S to $\mathrm{End}(V)$.

Proposition 6.1.11. *Let (V, ρ) be a finite dimensional representation of S that is almost commutative, and λ a weight of V.*

(a) *If $\mathbb{K}$ has characteristic zero, then the map λ is linear.*

(b) *If $\mathbb{K}$ has nonzero characteristic p, then there exists a power q of p dividing* $\dim(V^\lambda(S))$, *and a homogeneous polynomial function P of degree q such that $\lambda(s)^q = P(s)$ for all $s \in S$.*

Proof. Since $V^\lambda(S)$ is stable under $\rho(S)$ by Lemma 6.1.4, we can assume that $V = V^\lambda(S)$. Let $n = \dim(V)$, then for $s \in S$,

$$\det(X - \rho(s)) = (X - \lambda(s))^n.$$

On the other hand, the expansion of the determinant shows that

$$\det(X - \rho(s)) = X^n + a_1(s)X^{n-1} + \cdots + a_{n-1}(s)X^1 + a_n(s).$$

where $a_i : S \to \mathbb{K}$ is a homogeneous polynomial function of degree i. If $\mathbb{K}$ has characteristic zero, then we get $-n\lambda(s) = a_1(s)$, so λ is a linear function. On the other hand, if $\mathrm{char}(\mathbb{K}) = p > 0$, then write $n = qm$ where q is a power of the characteristic exponent of k and $(q, m) = 1$. Since $(X - \lambda(s))^n = (X^q - \lambda(s)^q)^m$, we see $-m\lambda(s)^q = a_q(s)$, which implies the result. $\square$

Proposition 6.1.12. *Assume that $\mathbb{K}$ is infinite and let (V, ρ) be a finite dimensional representation of S that is almost commutative. Let L be an extension of $\mathbb{K}$. Let $\widetilde{V} = V \otimes_{\mathbb{K}} \mathbb{K}'$, $\widetilde{S} = S \otimes_{\mathbb{K}} \mathbb{K}'L$, and $\tilde{\rho} : \widetilde{S} \to \mathrm{End}(\widetilde{V})$ be the map obtained from ρ by extension of scalars. Then*

$$V^0(S) \otimes_{\mathbb{K}} \mathbb{K}' = \widetilde{V}^0(S) = \widetilde{V}^0(\widetilde{S}).$$

Proof. The first equality follows from Proposition 6.1.1. To prove the second, we can assume that $V = V^0(S)$ and so $\widetilde{V} = \widetilde{V}^0(S)$. Let $(s_1, \ldots, s_m)$ be a basis of S and $(e_1, \ldots, e_n)$ a basis of V. Then there exist polynomials $P_{ij}(X_1, \ldots, X_m)$ such that

$$\tilde{\rho}(a_1 s_1 + \cdots + a_m s_m)^n e_j = \sum_{i=1}^{n} P_{ij}(a_1, \ldots, a_m) e_i.$$

for $1 \le j \le n$ and $a_1, \ldots, a_m \in L$. By hypothesis, $\tilde{\rho}(s)^n = 0$ for all $s \in S$, in other words $P_{ij}(a_1, \ldots, a_m) = 0$ for $1 \le i, j \le n$ and $a_1, \ldots, a_m \in L$. Since $\mathbb{K}$ is infinite, $P_{ij} = 0$. Consequently, every element of $\tilde{\rho}(\widetilde{S})$ is nilpotent and $\widetilde{V} = \widetilde{V}^0(\widetilde{S})$. $\square$

Proposition 6.1.13. *Assume that $\mathbb{K}$ is infinite and let (V, ρ) be a finite dimensional representation of S that is almost commutative. Let S' be the set of $s \in S$ such that $V^0(s) = V^0(S)$. If $s \in S$, let $P(s)$ be the determinant of the endomorphism of $V/V^0(S)$ defined by $\rho(s)$.*

(a) The function $s \mapsto P(s)$ is a polynomial on S. We have $S' = \{s \in S : P(s) \neq 0\}$, and this is an open subset of S in the Zariski topology.

(b) The set S' is non-empty, and $V^+(s) = V^+(S)$ for all $s \in S'$.

Proof. The fact that $s \mapsto P(s)$ is polynomial follows from the linearity of ρ. If $s \in S$, $V^0(s) \supseteq V^0(S)$, with equality if and only if $\rho(s)$ defines an automorphism of $V/V^0(S)$, hence (a).

Now let $\mathbb{K}'$ be an algebraic closure of $\mathbb{K}$, and introduce $\widetilde{V}$, $\widetilde{S}$, $\tilde{\rho}$ as in Proposition 6.1.12. We remark that $\tilde{\rho}$ is almost commutative by continuation of the polynomial identity

$$\mathrm{ad}(\rho(s_1)^{2\dim(V)-1}(\rho(s_2)) = 0$$

valid for $s_1, s_2 \in S$. Applying Theorem 6.1.6, we deduce a decomposition

$$\widetilde{V} = \widetilde{V}^0(\widetilde{S}) \oplus \bigoplus_{i=1}^{r} \widetilde{V}^{\lambda_i}(\widetilde{S}).$$

with $\lambda_i \neq 0$ for $1 \le i \le r$. By Proposition 6.1.11, for each i there exists a polynomial function P_i non-zero on $\widetilde{S}$ and an integer q_i such that $\lambda_i^{q_i} = P_i$. Since $\mathbb{K}$ is infinite, there exists $s \in S$ such that $P_i(s) \neq 0$ for all i. Then $\lambda_i(s) \neq 0$ for all i, so $\widetilde{V}^0(\widetilde{S}) = \widetilde{V}^0(s)$ and consequently $V^0(S) = V^0(s)$, which shows that $\widetilde{S} \neq \varnothing$. If $s \in \widetilde{S}$, the fact that $V^+(S)$ is stable under $\rho(s)$ and is a complement of $V^0(s)$ in V implies that $V^+(S) = V^+(s)$. $\square$

Let $\mathfrak{h}$ be a Lie algebra and V an $\mathfrak{h}$-module. Let $\mathcal{P}$ be the set of all maps from $\mathfrak{h}$ to $\mathbb{K}$. For each $\lambda \in \mathcal{P}$ we have defined the subspaces $V_\lambda(\mathfrak{h})$ and $V^\lambda(\mathfrak{h})$. In particular, if $\mathfrak{g}$ is a Lie algebra containing $\mathfrak{h}$ as a subalgebra, and if $x \in \mathfrak{g}$, we shall often employ the notations $\mathfrak{g}_\lambda(\mathfrak{h})$ and $\mathfrak{g}^\lambda(\mathfrak{h})$. It will then be understood that $\mathfrak{h}$ operates on $\mathfrak{g}$ by the adjoint representation $\mathrm{ad}_{\mathfrak{g}}$. Recall that all Lie algebras in this section is assumed to be finite dimensional.

Proposition 6.1.14. *Let $\mathfrak{h}$ be a Lie algebra, and V, W and M be $\mathfrak{g}$-modules.*

(a) The sum $\sum_{\lambda \in \mathcal{P}} V^\lambda(\mathfrak{h})$ is direct.

(b) If $\varphi : V \to W$ is a homomorphism of $\mathfrak{h}$-modules, then $\varphi(V^\lambda(\mathfrak{h})) \subseteq W^\lambda(\mathfrak{h})$ for all $\lambda \in \mathcal{P}$.

(c) *If $B : V \times W \to M$ is a bilinear $\mathfrak{h}$-invariant map, then $B(V^\lambda(\mathfrak{h}) \times W^\mu(\mathfrak{h})) \subseteq M^{\lambda+\mu}(\mathfrak{h})$.*

Proof. This follows from Proposition 6.1.2 and Proposition 6.1.3. □

Let (V, ρ) be a representation of $\mathfrak{h}$. Then for $x, y \in \mathfrak{h}$, by induction on n we have

$$\mathrm{ad}(\rho(x))^n(\rho(y)) \in C^{n+1}(\rho(\mathfrak{h})).$$

By Proposition 1.4.5, if $\mathfrak{h}$ is a nilpotent algebra, then any representation ρ is almost commutative, so the results in the preceding parts are available.

Proposition 6.1.15. *Let $\mathfrak{h}$ be a nilpotent Lie algebra and (V, ρ) a finite dimensional $\mathfrak{h}$-module.*

(a) *Each $V^\lambda(\mathfrak{h})$ is an $\mathfrak{h}$-submodule of V. If $\rho(x)$ is triangularizable for all $x \in \mathfrak{h}$, then $V = \bigoplus_{\lambda \in \mathcal{P}} V^\lambda(\mathfrak{h})$.*

(b) *If $\mathbb{K}$ is infinite, then there exists $x \in \mathfrak{h}$ such that $V^0(x) = V^0(\mathfrak{h})$.*

(c) *If $\mathbb{K}$ is of characteristic 0, and if $\lambda \in \mathcal{P}(V)$, then λ is a linear form on $\mathfrak{h}$ vanishing on $[\mathfrak{h}, \mathfrak{h}]$ and $V_\lambda(\mathfrak{h}) \neq \{0\}$.*

(d) *If $\varphi : V \to W$ is a surjective homomorphism of finite dimensional $\mathfrak{h}$-modules, then $\varphi(V^\lambda(\mathfrak{h})) = W^\lambda(\mathfrak{h})$ for all $\lambda \in \mathcal{P}$.*

(e) *If W is a finite dimensional $\mathfrak{h}$-module, and B a bilinear form on $V \times W$ invariant under $\mathfrak{h}$, then $V^\lambda(\mathfrak{h})$ and $W^\mu(\mathfrak{h})$ are orthogonal relative to B if $\lambda + \mu \neq 0$. Moreover, if B is non-degenerate then so is its restriction to $V^\lambda(\mathfrak{h}) \times W^{-\lambda}(\mathfrak{h})$ for all $\lambda \in \mathcal{P}$.*

Proof. We have remarked that if $\mathfrak{h}$ is nilpotent, then ρ is almost commutative, so assertions (a), (b), and (d) follow. We now prove (c). We can assume that $V = V^\lambda(\mathfrak{h})$. Then, for all $x \in \mathfrak{h}$, $\lambda(x) = \dim(V)^{-1}\mathrm{tr}(\rho(x))$, which proves that λ is linear and that λ vanishes on $[\mathfrak{h}, \mathfrak{h}]$. Consider the map $\eta : \mathfrak{h} \to \mathrm{End}(V)$ defined by

$$\eta(x) = \rho(x) - \lambda(x).$$

This is a representation of $\mathfrak{h}$ on V, and by part (a) $\eta(x)$ is nilpotent for all $x \in \mathfrak{h}$. By Engel's theorem, there exists $v \neq 0$ in V such that $\eta(x)v = 0$ for all $x \in \mathfrak{h}$, so $v \in V_\lambda(\mathfrak{h})$.

The first assertion of (e) follows from Proposition 6.1.2. To prove the second, we can assume that $\mathbb{K}$ is algebraically closed in view of Proposition 6.1.1. It then follows from the first and the fact that $V = \bigoplus_{\lambda \in \mathcal{P}} V^\lambda(\mathfrak{h})$ and $W = \bigoplus_{\lambda \in \mathcal{P}} W^\lambda(\mathfrak{h})$, so the second assertion follows. □

Example 6.1.16. Let $\mathbb{K}$ be a field with characteristic 2. Let $\mathfrak{h} = \mathfrak{sl}(2, \mathbb{K})$ and V be the $\mathfrak{h}$-module $\mathbb{K}^2$ given by left multiplication. Since $\mathfrak{h}$ is a nilpotent Lie algebra we see the representation is almost commutative. If $x = \left(\begin{smallmatrix} a & b \\ c & a \end{smallmatrix}\right)$ is an arbitrary element of $\mathfrak{h}$, denote by $\lambda(x)$ the unique $\lambda \in \mathbb{K}$ such that $\lambda^2 = a^2 + bc$. A calculation shows immediately that

$$(x - \lambda(x))^2 = x^2 - \lambda(x)^2 = 0$$

whence $V = V^\lambda(\mathfrak{h})$. Note that the map λ is not linear.

Corollary 6.1.17. *Let $\mathfrak{h}$ be a nilpotent Lie algebra, and V a finite dimensional $\mathfrak{h}$-module such that $V^0(\mathfrak{h}) = 0$. Let $f : \mathfrak{h} \to V$ be a linear map such that*

$$f([x, y]) = x \cdot f(y) - y \cdot f(x)$$

for $x, y \in \mathfrak{h}$. Then there exists $v_0 \in V$ such that $f(x) = x \cdot v_0$ for all $x \in \mathfrak{h}$.

Proof. Let $W = V \times \mathbb{K}$ and operate $\mathfrak{h}$ on W by the formula

$$x \cdot (v, \alpha) = (x \cdot v - \alpha f(x), 0).$$

The identity satisfied by f implies that W is an $\mathfrak{h}$-module. The map $(v, \alpha) \mapsto \alpha$ from W to $\mathbb{K}$ is a surjective homomorphism from W to the trivial $\mathfrak{h}$-module $\mathbb{K}$. By Proposition 6.1.15, it follows that $W^0(\mathfrak{h})$ contains an element of the form $(v_0, 1)$ with $v_0 \in V$. In view of the hypothesis on V, we have

$$(V \times \{0\}) \cap W^0(\mathfrak{h}) = \{0\}$$

whence $W^0(\mathfrak{h}) \subseteq \{0\} \times \mathbb{K}$. Since $(v_0, 1) \in W^0(\mathfrak{h})$, we see $W^0(\mathfrak{h})$ is of dimension 1 and hence is annihilated by $\mathfrak{h}$. Therefore, $x \cdot v_0 - f(x) = 0$ for all $x \in \mathfrak{h}$, which proves the corollary. □

Proposition 6.1.18. *Let $\mathfrak{g}$ be a Lie algebra, $\mathfrak{h}$ a nilpotent subalgebra of $\mathfrak{g}$.*

(a) If V is a $\mathfrak{g}$-module, then $\mathfrak{g}^\lambda(\mathfrak{h})V^\mu(\mathfrak{h}) \subseteq V^{\lambda+\mu}(\mathfrak{h})$ for $\lambda, \mu \in \mathcal{P}$. In particular, each $V^\lambda(\mathfrak{h})$ is a $\mathfrak{g}^0(\mathfrak{h})$-module.

(b) For $\lambda, \mu \in \mathcal{P}$, $[\mathfrak{g}^\lambda(\mathfrak{h}), \mathfrak{g}^\mu(\mathfrak{h})] \subseteq \mathfrak{g}^{\lambda+\mu}(\mathfrak{h})$. In particular, $\mathfrak{g}^0(\mathfrak{h})$ is a Lie subalgebra of $\mathfrak{g}$ containing $\mathfrak{h}$, and the $\mathfrak{g}^\lambda(\mathfrak{h})$ are stable under $\mathfrak{g}^0(\mathfrak{h})$. Moreover, $\mathfrak{g}^0(\mathfrak{h})$ is its own normalizer in $\mathfrak{g}$.

(c) If B is a bilinear form on $\mathfrak{g}$ invariant under $\mathfrak{h}$, then $\mathfrak{g}^\lambda(\mathfrak{h})$ and $\mathfrak{g}^\mu(\mathfrak{h})$ are orthogonal relative to B for $\lambda + \mu \neq 0$. Assume that B is non-degenerate, then for all $\lambda \in \mathcal{P}$ the restriction of B to $\mathfrak{g}^\lambda(\mathfrak{h}) \times \mathfrak{g}^{-\lambda}(\mathfrak{h})$ is non-degenerate. In particular, the restriction of B to $\mathfrak{g}^0(\mathfrak{h}) \times \mathfrak{g}^0(\mathfrak{h})$ is non-degenerate.

(d) Assume that $\mathbb{K}$ is of characteristic zero. Then, if $x \in \mathfrak{g}^\lambda(\mathfrak{h})$ with $\lambda \neq 0$, $\mathrm{ad}(x)$ is nilpotent.

Proof. The map $(x, y) \mapsto [x, y]$ from $\mathfrak{g} \times \mathfrak{g}$ to $\mathfrak{g}$ is $\mathfrak{g}$-invariant by the Jacobi identity, hence $\mathfrak{h}$-invariant. The first part of (b) thus follows from Proposition 6.1.2, and (a) is proved similarly. If x belongs to the normalizer of $\mathfrak{g}^0(\mathfrak{h})$ in $\mathfrak{g}$, then $\mathrm{ad}(y)(x) = -[x, y] \in \mathfrak{g}^0(\mathfrak{h})$ for all $y \in \mathfrak{h}$, so $\mathrm{ad}(y)^n(x) = 0$ for n sufficiently large. This proves that $x \in \mathfrak{g}^0(\mathfrak{h})$.

To prove (d), we can assume that $\mathbb{K}$ is algebraically closed. Let $x \in \mathfrak{g}^\lambda(\mathfrak{h})$, with $\lambda \neq 0$. For all $\mu \in \mathcal{P}$ and any integer $n > 0$, $\mathrm{ad}(x)^n \cdot \mathfrak{g}^\mu(\mathfrak{h}) \subseteq \mathfrak{g}^{\mu+n\lambda}(\mathfrak{h})$. Let $\mathcal{P}_1$ be the finite set of $\mu \in \mathcal{P}$ such that $\mathfrak{g}^\mu(\mathfrak{h}) \neq 0$. If $\mathbb{K}$ is of characteristic zero and $\lambda \neq 0$, then $(\mathcal{P}_1 + n\lambda) \cap \mathcal{P}_1 = \varnothing$ for n sufficiently large, so $\mathrm{ad}(x)^n = 0$. □

Recall that in a semi-simple Lie algebra $\mathfrak{g}$, every element $x \in \mathfrak{g}$ can be written uniquely as the sum of a semi-simple element x_s and a nilpotent element x_n that commute with each other. The element x_s (resp. x_n) is called the semi-simple (resp. nilpotent) component of x.

Lemma 6.1.19. *Assume that $\mathbb{K}$ is of characteristic zero. Let $\mathfrak{g}$ be a semi-simple Lie algebra over $\mathbb{K}$, κ the Killing form of $\mathfrak{g}$, $\mathfrak{h}$ a subalgebra of $\mathfrak{g}$. Assume that the following conditions are satisfied:*

(i) the restriction of κ to $\mathfrak{h}$ is non-degenerate;

(ii) if $x \in \mathfrak{h}$, the semi-simple and nilpotent components of x in $\mathfrak{g}$ belong to $\mathfrak{h}$.

Then $\mathfrak{h}$ is reductive in $\mathfrak{g}$.

Proof. By Proposition 1.6.23, $\mathfrak{h}$ itself is reductive. Let $\mathfrak{z}$ be the centre of $\mathfrak{h}$. If $x \in \mathfrak{z}$ is nilpotent, then $x = 0$; indeed, for all $y \in \mathfrak{h}$, $\mathrm{ad}(x)$ and $\mathrm{ad}(y)$ commute, their composition $\mathrm{ad}(x) \circ \mathrm{ad}(y)$ is nilpotent, and $\kappa(x, y) = 0$, so $x = 0$.

Now let x be an arbitrary element of $\mathfrak{z}$ and x_s, x_n be its semi-simple and nilpotent components. We have $x_s, x_n \in \mathfrak{h}$ by hypothesis. Since $\mathrm{ad}(x_n)$ is of the form $P(\mathrm{ad}(x))$, where P is a polynomial with no constant term, we see $(\mathrm{ad}(x_n)) \cdot \mathfrak{h} = 0$ and so $x_n \in \mathfrak{z}$. Then $x_n = 0$ by the above arguments, so $\mathrm{ad}(x)$ is semi-simple. Consequently, the restriction to $\mathfrak{h}$ of the adjoint representation of $\mathfrak{g}$ is semi-simple. □

Proposition 6.1.20. *Assume that $\mathbb{K}$ is of characteristic zero. Let $\mathfrak{g}$ be a semi-simple Lie algebra and $\mathfrak{h}$ a nilpotent subalgebra of $\mathfrak{g}$. Then the algebra $\mathfrak{g}^0(\mathfrak{h})$ satisfies the conditions in Lemma 6.1.19, so it is reductive in $\mathfrak{g}$.*

Proof. Let $x, y \in \mathfrak{g}$, x_s and y_s be their semi-simple components, x_n and y_n their nilpotent components. By Proposition 6.1.10, we see

$$y \in \mathfrak{g}^0(x) \Leftrightarrow \mathrm{ad}(x_s)(y) = 0 \Leftrightarrow \mathrm{ad}(y)(x_s) = 0 \Leftrightarrow \mathrm{ad}(y_s)(x_s) = 0 \Leftrightarrow y_s \in \mathfrak{g}^0(x).$$

so the condition (ii) in Lemma 6.1.19 are proved. The Killing form of $\mathfrak{g}$ is nondegenerate, so its restriction to $\mathfrak{g}^0(\mathfrak{h})$ is non-degenerate by Lemma 6.1.19. The fact that $\mathfrak{g}^0(\mathfrak{h})$ is reductive in $\mathfrak{g}$ thus follows from Lemma 6.1.19. □

Proposition 6.1.21. *Assume that $\mathbb{K}$ is of characteristic zero. Let $\mathfrak{g}$ be a semi-simple Lie algebra, $\mathfrak{h}$ a subalgebra of $\mathfrak{g}$ reductive in $\mathfrak{g}$, and $\mathfrak{z}$ the centralizer of $\mathfrak{h}$ in $\mathfrak{g}$. Then the subalgebra $\mathfrak{z}$ satisfies the conditions of Lemma 6.1.19, so it is reductive in $\mathfrak{g}$.*

Proof. By Proposition 1.3.15 applied to the $\mathfrak{h}$-module $\mathfrak{g}$, we have $\mathfrak{g} = \mathfrak{z} \oplus [\mathfrak{h}, \mathfrak{g}]$. Let κ be the Killing form of $\mathfrak{g}$, and let $x \in \mathfrak{g}$, $y \in \mathfrak{h}$, $z \in \mathfrak{z}$. Then since $[y, z] = 0$,

$$\kappa([x, y], z) = \kappa(x, [y, z]) = 0$$

which shows that $\mathfrak{z}$ is orthogonal to $[\mathfrak{h}, \mathfrak{g}]$ relative to κ. Since κ is nondegenerate, and since $\mathfrak{g} = \mathfrak{z} \oplus [\mathfrak{h}, \mathfrak{g}]$, this implies that the restriction of κ to $\mathfrak{z}$ is non-degenerate. Thus condition (i) of Lemma 6.1.19 is thus satisfied.

Now let $x \in \mathfrak{h}$ and let x_s and x_n be its semi-simple and nilpotent components. Then the semi-simple component of $\mathrm{ad}(x)$ is $\mathrm{ad}(x_s)$. Since $\mathrm{ad}(x)$ is zero on $\mathfrak{z}$, so is $\mathrm{ad}(x_s)$, by Proposition 6.1.10. Thus $x_s \in \mathfrak{z}$, so $x_n \in \mathfrak{z}$ and condition (ii) of Lemma 6.1.19 is satisfied. □

Proposition 6.1.22. *Let $\mathfrak{g}$ be a Lie algebra, G a group, and ρ a homomorphism from G to* $\mathrm{Aut}(\mathfrak{g})$.

(a) *For $\lambda, \mu \in \mathbb{K}$, $[\mathfrak{g}^{\lambda}(G), \mathfrak{g}^{\mu}(G)] \subseteq \mathfrak{g}^{\lambda\mu}(G)$. In particular, $\mathfrak{g}^1(G)$ is a subalgebra of $\mathfrak{g}$.*

(b) *If κ is a symmetric bilinear form on $\mathfrak{g}$ invariant under $\rho(G)$, then $\mathfrak{g}^{\lambda}(G)$ and $\mathfrak{g}^{\mu}(G)$ are orthogonal relative to B for $\lambda\mu \neq 1$. Assume that B is nondegenerate. Then, if $\lambda \neq 0$, the restriction of B to $\mathfrak{g}^{\lambda}(G) \times \mathfrak{g}^{1/\lambda}(G)$ is nondegenerate.*

Proof. Assertion (a) and the first half of (b) follow from Proposition 6.1.2 applied to the bilinear map $(x, y) \mapsto [x, y]$. To prove the second half of (b), we can assume that $\mathbb{K}$ is algebraically closed. Then $\mathfrak{g} = \bigoplus_{\alpha \in \mathbb{K}} \mathfrak{g}^{\alpha}(G)$. In view of the above, $\mathfrak{g}^{\lambda}(G)$ is orthogonal to $\mathfrak{g}^{\mu}(G)$ if $\lambda\mu \neq 1$. Since B is non-degenerate, it follows that its restriction to $\mathfrak{g}^{\lambda}(G) \times \mathfrak{g}^{1/\lambda}(G)$ is also. □

Proposition 6.1.23. *Assume that $\mathbb{K}$ is of characteristic zero. Let $\mathfrak{g}$ be a semi-simple Lie algebra, G a group and ρ a homomorphism from G to* $\mathrm{Aut}(\mathfrak{g})$. *Let $\mathfrak{h}$ be the subalgebra of $\mathfrak{g}$ consisting of the elements invariant under $\rho(G)$. Assume that the representation ρ is semi-simple, then $\mathfrak{h}$ satisfies the conditions of Lemma 6.1.19, so it is reductive in $\mathfrak{g}$.*

Proof. Let $\mathfrak{g}^+$ be the vector subspace of $\mathfrak{g}$ generated by the $\rho(g)x - x$, $g \in G$, $x \in \mathfrak{g}$. The vector space $\mathfrak{k} = \mathfrak{h} + \mathfrak{g}^+$ is stable under $\rho(G)$. Let $\mathfrak{n}$ be a complement of $\mathfrak{k}$ in $\mathfrak{g}$ stable under $\rho(G)$, so that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{n}$. If $x \in \mathfrak{n}$ and $g \in G$, then $\rho(a)x - x \in \mathfrak{n} \cap \mathfrak{g}^+ = \{0\}$, so $x \in \mathfrak{h}$ and then $x = 0$ since $\mathfrak{h} \cap \mathfrak{n} = \{0\}$. Thus, $\mathfrak{g} = \mathfrak{k} = \mathfrak{h} + \mathfrak{g}^+$. Let κ be the Killing form of $\mathfrak{g}$ and let $x \in \mathfrak{g}$, $y \in \mathfrak{h}$, $g \in G$. Then by Proposition 1.3.22,

$$\kappa(y, \rho(g)x - x) = \kappa(y, \rho(g)x) - \kappa(y, x) = \kappa(\rho(g^{-1})y, x) - \kappa(y, x) = 0.$$

Thus $\mathfrak{h}$ and $\mathfrak{g}^+$ are orthogonal relative to κ. It follows that the restriction of κ to $\mathfrak{h}$ is non-degenerate, hence condition (i) of Lemma 6.1.19.

To see condition (ii), we may assume that $\mathbb{K}$ is algebraically closed. Let $x \in \mathfrak{g}^1(G)$ and $x = x_s + x_n$ be the Jordan decomposition of x in $\mathfrak{g}$. First we note that, for each $g \in G$, since $\rho(g)(x) = x$, the map $\mathrm{ad}(x)$ commutes with $\rho(g)$ (Proposition 1.1.11), whence each $\mathfrak{g}^{\lambda}(x)$ is stable under $\rho(g)$. For $y \in \mathfrak{g}^{\lambda}(x) = \mathfrak{g}_{\lambda}(x_s)$ and $z \in \mathfrak{g}$, we then have (Proposition 1.3.22)

$$\begin{aligned}\kappa(\rho(g^{-1})x_s, [y, z]) &= \kappa(x_s, \rho(g)[y, z]) = \kappa(x_s, [\rho(g)y, \rho(g)z]) \\ &= \kappa([x_s, \rho(g)y], \rho(g)z) = \lambda(x)\kappa(\rho(g)y, \rho(g)z) \\ &= \lambda(x)\kappa(y, z) = \kappa([x_s, y], z) = \kappa(x_s, [y, z]).\end{aligned}$$

Since the Killing form κ is nondegenerate on $\mathfrak{g}$ and $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$ ($\mathfrak{g}$ is semi-simple), this implies $\rho(g)x_s = x_s$, whence $\rho(g)x_n = x_n$ and condition (ii) is satisfied. □

6.1.2 Cartan subalgebras and regular elements

Let $\mathfrak{h}$ be a subalgebra of a Lie algebra $\mathfrak{g}$. We say $\mathfrak{h}$ is **self-normalizing** if $\mathfrak{h}$ equals its normalizer in $\mathfrak{g}$. A **Cartan subalgebra** of $\mathfrak{g}$ is defined to be a nilpotent self-normalizing subalgebra.

Example 6.1.24 (Example of Cartan subalgebras).

(a) If $\mathfrak{g}$ is nilpotent, the only Cartan subalgebra of $\mathfrak{g}$ is $\mathfrak{g}$ itself. In fact, if $\mathfrak{h}$ is a proper subalgebra of $\mathfrak{g}$ and k is the greatest integer such that $C^k(\mathfrak{g}) + \mathfrak{h} \neq \mathfrak{h}$, then

$$[C^k(\mathfrak{g}) + \mathfrak{h}, \mathfrak{h}] \subseteq C^{k+1}(\mathfrak{g}) + \mathfrak{h} \subseteq \mathfrak{h}$$

whence the normalizer of $\mathfrak{h}$ in $\mathfrak{g}$ contains $C^k(\mathfrak{g}) + \mathfrak{h}$.

(b) Let $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{K})$, and let $\mathfrak{h}$ be the set of diagonal matrices belonging to $\mathfrak{g}$. We show that $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$. First, $\mathfrak{h}$ is commutative, hence nilpotent. Let (E_{ij}) be the canonical basis of $\mathfrak{gl}(n, \mathbb{K})$, and let $x = \sum \mu_{ij} E_{ij}$ be an element of the normalizer of $\mathfrak{h}$ in $\mathfrak{g}$. If $i \neq j$, then the coefficient of E_{ij} in $[E_{ii}, x]$ is μ_{ij}. Since $E_{ii} \in \mathfrak{h}$, $[E_{ii}, x] \in \mathfrak{h}$, and the coefficient in question is zero. Thus $\mu_{ij} = 0$ for $i \neq j$, so $x \in \mathfrak{h}$, which shows that $\mathfrak{h}$ is indeed a Cartan subalgebra of $\mathfrak{g}$.

(c) Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ and let $\mathfrak{g}_1$ be a subalgebra of $\mathfrak{g}$ containing $\mathfrak{h}$. Then $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}_1$.

(d) Let $\mathfrak{g}$ be a Lie algebra, $\mathfrak{h}$ a nilpotent subalgebra of $\mathfrak{g}$. We already know that $\mathfrak{g}^0(\mathfrak{h})$ is self-normalizing by Proposition 6.1.18, so if $\mathfrak{g}^0(\mathfrak{h})$ is nilpotent, then it is a Cartan subalgebra of $\mathfrak{g}$.

Proposition 6.1.25. *Let $\mathfrak{g}$ be a Lie algebra and let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. Then $\mathfrak{h}$ is a maximal nilpotent subalgebra of $\mathfrak{g}$.*

Proof. Let $\mathfrak{k}$ be a nilpotent subalgebra of $\mathfrak{g}$ containing $\mathfrak{h}$. Then $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{k}$, so $\mathfrak{h} = \mathfrak{k}$ by Example 6.1.24(a). □

Example 6.1.26. There exist maximal nilpotent subalgebras that are not Cartan subalgebras. For example, let $\mathbb{K}$ be a field with $\mathrm{char}(\mathbb{K}) \neq 2$. Consider the Lie algebras $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{K})$ and $\mathfrak{h} = \mathfrak{n} \oplus \mathbb{K} I_n$, where $\mathfrak{n}$ is the subalgebra of strictly upper triangular matrices. Then $\mathfrak{h}$ is a maximal nilpotent subalgebra but not a Cartan subalgebra.

First, it is easily computed that, if $\mathfrak{b}$ is the Lie subalgebra of upper triangular matrices, then $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{b}$ and thus $\mathfrak{h}$ is not Cartan. It is clear that $\mathfrak{h}$ is nilpotent, so let $\tilde{\mathfrak{h}}$ be a nilpotent subalgebra of $\mathfrak{g}$ containing $\mathfrak{h}$. Since $\mathfrak{h} \subsetneq \tilde{\mathfrak{h}}$, we have $\mathfrak{h} \subsetneq \mathfrak{n}_{\tilde{\mathfrak{h}}}(\mathfrak{h}) = \mathfrak{b} \cap \tilde{\mathfrak{h}}$, so there must exist an element $x \in (\mathfrak{b} \cap \tilde{\mathfrak{h}}) \setminus \mathfrak{h}$. But all elements of $\mathfrak{b} \setminus \mathfrak{h}$ have at least two distinct eigenvalues, and thus are not ad-nilpotent. Thus we find a contradiction to Engel's theorem and this shows $\mathfrak{h}$ is a maximal nilpotent subalgebra.

Proposition 6.1.27. *Let $(\mathfrak{g}_i)_{i \in I}$ be a finite family of Lie algebras and $\mathfrak{g} = \prod_{i \in I} \mathfrak{g}_i$. Then the Cartan subalgebras of $\mathfrak{g}$ are the subalgebras of the form $\prod_{i \in I} \mathfrak{h}_i$, where each $\mathfrak{h}_i$ is a Cartan subalgebra of $\mathfrak{g}_i$.*

Proof. If $\mathfrak{h}_i$ is a subalgebra of $\mathfrak{g}_i$ with normalizer $\mathfrak{n}_i$, then $\mathfrak{h}_i$ is a subalgebra of $\mathfrak{g}$ with normalizer $\prod_i \mathfrak{n}_i$. If the $\mathfrak{h}_i$ are nilpotent, then $\prod_i \mathfrak{h}_i$ is nilpotent. Thus, if $\mathfrak{h}_i$ is a Cartan subalgebra of $\mathfrak{g}_i$ for all i, $\prod_i \mathfrak{h}_i$ is a Cartan subalgebra of $\mathfrak{g}$. Conversely, let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. The projection $\mathfrak{h}_i$ of $\mathfrak{h}$ onto $\mathfrak{g}_i$ is a nilpotent subalgebra of $\mathfrak{g}_i$, and $\prod_i \mathfrak{h}_i$ is a nilpotent subalgebra of $\mathfrak{g}$ containing $\mathfrak{h}$, hence $\mathfrak{h} = \prod_i \mathfrak{h}_i$ by Proposition 6.1.25. Thus, for all i, $\mathfrak{h}_i$ is its own normalizer in $\mathfrak{g}_i$, and so is a Cartan subalgebra of $\mathfrak{g}_i$. □

Example 6.1.28. If $\mathbb{K}$ is of characteristic zero, then $\mathfrak{gl}(n, \mathbb{K})$ is the product of the ideals $\mathfrak{sl}(n, \mathbb{K})$ and $\mathbb{K}$. It follows from Example 6.1.24 and Proposition 6.1.27 that the set of diagonal matrices of trace 0 in $\mathfrak{sl}(n, \mathbb{K})$ is a Cartan subalgebra of $\mathfrak{sl}(n, \mathbb{K})$.

Proposition 6.1.29. *Let $\mathfrak{g}$ be a Lie algebra, $\mathfrak{h}$ a subalgebra of $\mathfrak{g}$, and L an extension of $\mathbb{K}$. Then $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ if and only if $\mathfrak{h} \otimes_{\mathbb{K}} \mathbb{K}'$ is a Cartan subalgebra of $\mathfrak{g} \otimes_{\mathbb{K}} \mathbb{K}'$.*

Proof. Indeed, $\mathfrak{h}$ is nilpotent if and only if $\mathfrak{h} \otimes_{\mathbb{K}} \mathbb{K}'$ is nilpotent. On the other hand, if $\mathfrak{n}$ is the normalizer of $\mathfrak{h}$ in $\mathfrak{g}$, the normalizer of $\mathfrak{h} \otimes_{\mathbb{K}} \mathbb{K}'$ in $\mathfrak{g} \otimes_{\mathbb{K}} \mathbb{K}'$ is $\mathfrak{n} \otimes_{\mathbb{K}} \mathbb{K}'$. □

Proposition 6.1.30. *Let $\mathfrak{g}$ be a Lie algebra, $\mathfrak{h}$ a subset of $\mathfrak{g}$. Let $\mathfrak{h}$ operate on $\mathfrak{g}$ by the adjoint representation. Then $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ if and only if $\mathfrak{h} = \mathfrak{g}^0(\mathfrak{h})$.*

Proof. First let $\mathfrak{h}$ be a Cartan subalgebra and assume that $\mathfrak{g}^0(\mathfrak{h}) \neq \mathfrak{h}$. Consider the representation of $\mathfrak{h}$ on $\mathfrak{g}^0(\mathfrak{h})/\mathfrak{h}$ obtained from the adjoint representation by passage to the quotient. By applying Engel's theorem, we see that there exists $x \in \mathfrak{g}^0(\mathfrak{h})$ such that $x \notin \mathfrak{h}$ and $[\mathfrak{h}, x] \subseteq \mathfrak{h}$. Then x belongs to the normalizer of $\mathfrak{h}$ in $\mathfrak{g}$, so $\mathfrak{h}$ is not a Cartan subalgebra of $\mathfrak{g}$, contradiction.

Assume now that $\mathfrak{h} = \mathfrak{g}^0(\mathfrak{h})$. By Proposition 6.1.18, $\mathfrak{h}$ is then a subalgebra of $\mathfrak{g}$. If $x \in \mathfrak{h}$, $\mathrm{ad}(x)$ is nilpotent on $\mathfrak{h}$ since $\mathfrak{h} \subseteq \mathfrak{g}^0(\mathfrak{h})$; hence the algebra $\mathfrak{h}$ is nilpotent. But then $\mathfrak{h}$ is a Cartan subalgebra by Proposition 6.1.18. □

Corollary 6.1.31. *Let $\mathfrak{g}$ be a Lie algebra, $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$. If $\mathbb{K}$ is infinite, there exists $x \in \mathfrak{h}$ such that $\mathfrak{h} = \mathfrak{g}^0(x)$.*

Proof. In fact, we have $\mathfrak{h} = \mathfrak{g}^0(\mathfrak{h})$ and we can apply Proposition 6.1.13. □

Proposition 6.1.32. *Let $\varphi : \mathfrak{g} \to \tilde{\mathfrak{g}}$ be a surjective homomorphism of Lie algebras. If $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$, $\varphi(\mathfrak{h})$ is a Cartan subalgebra of $\tilde{\mathfrak{g}}$.*

Proof. Indeed, $\varphi(\mathfrak{h})$ is a nilpotent subalgebra of $\tilde{\mathfrak{g}}$. On the other hand, consider the representation $x \mapsto \mathrm{ad}(\varphi(x))$ of $\mathfrak{h}$ on $\tilde{\mathfrak{g}}$. By Proposition 6.1.15(d), $\varphi(\mathfrak{g}^0(\mathfrak{h})) = \tilde{\mathfrak{g}}^0(\mathfrak{h})$. Now $\mathfrak{g}^0(\mathfrak{h}) = \mathfrak{h}$, and on the other hand it is clear that $\tilde{\mathfrak{g}}^0(\mathfrak{h}) = \tilde{\mathfrak{g}}^0(\varphi(\mathfrak{h}))$. Hence, $\varphi(\mathfrak{h}) = \tilde{\mathfrak{g}}^0(\varphi(\mathfrak{h}))$ and it suffices to apply Proposition 6.1.30. □

Corollary 6.1.33. *Let $\mathfrak{h}$ be a Cartan subalgebra of a Lie algebra $\mathfrak{g}$, and let $C^n(\mathfrak{g})$ be a term of the descending central series of $\mathfrak{g}$. Then $\mathfrak{g} = \mathfrak{h} + C^n(\mathfrak{g})$.*

Proof. Indeed, Corollary 6.1.31 shows that the image of $\mathfrak{h}$ in $\mathfrak{g}/C^n(\mathfrak{g})$ is a Cartan subalgebra of $\mathfrak{g}/C^n(\mathfrak{g})$, hence is equal to $\mathfrak{g}/C^n(\mathfrak{g})$ since $\mathfrak{g}/C^n(\mathfrak{g})$ is nilpotent. □

Corollary 6.1.34. *Let $\mathfrak{g}$ be a Lie algebra, $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$, and $\mathfrak{k}$ a subalgebra of $\mathfrak{g}$ containing $\mathfrak{h}$.*

(a) $\mathfrak{k}$ is self-normalizing. In particular, if x an element of $\mathfrak{k}$ and x_s and x_n are its semi-simple and nilpotent components. Then $x_s \in \mathfrak{k}$ and $x_n \in \mathfrak{k}$.

(b) Assume that $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$; let G be a Lie group with Lie algebra $\mathfrak{g}$, T the integral subgroup of G with Lie algebra $\mathfrak{k}$. Then T is a Lie subgroup of G, and it is the identity component of the normalizer of T in G.

Proof. Let $\mathfrak{n}$ be the normalizer of $\mathfrak{k}$ in $\mathfrak{g}$. Since $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{n}$, $\{0\}$ is a Cartan subalgebra of $\mathfrak{n}/\mathfrak{k}$ by Corollary 6.1.32, hence is equal to its normalizer in $\mathfrak{n}/\mathfrak{k}$; in other words, $\mathfrak{n} = \mathfrak{k}$. Finally, if $x \in \mathfrak{k}$ then we have $\mathrm{ad}(x)\mathfrak{k} \subseteq \mathfrak{k}$, so $\mathrm{ad}(x_s)\mathfrak{k} \subseteq \mathfrak{k}$ and $\mathrm{ad}(x_n)\mathfrak{k} \subseteq \mathfrak{k}$. Since $\mathfrak{k}$ is its own normalizer in $\mathfrak{g}$, $x_s \in \mathfrak{a}$ and $x_n \in \mathfrak{a}$. Assertion (b) follows from (a) and ([?] III, no.9, cor. of prop.11). □

Corollary 6.1.35. *Let $\mathfrak{g}$ be a Lie algebra, let $\mathbb{K}_0$ be a subfield of $\mathbb{K}$ such that $[\mathbb{K} : \mathbb{K}_0] < +\infty$, and let $\mathfrak{g}_0$ be the Lie algebra obtained from $\mathfrak{g}$ by restriction of scalars to $\mathbb{K}_0$. Let $\mathfrak{h}$ be a subset of $\mathfrak{g}$. Then $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ if and only if $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}_0$.*

Proof. This follows from Corollary 6.1.30, since the condition $\mathfrak{h} = \mathfrak{g}^0(\mathfrak{h})$ does not involve the base field. □

Proposition 6.1.36. *Let $\mathfrak{g}$ be a Lie algebra, $\mathfrak{z}$ its centre, $\mathfrak{h}$ a vector subspace of $\mathfrak{g}$. Then $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ if and only if $\mathfrak{h}$ contains $\mathfrak{z}$ and $\mathfrak{h}/\mathfrak{z}$ is a Cartan subalgebra of $\mathfrak{g}/\mathfrak{z}$.*

Proof. Assume that $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$. Since $[\mathfrak{z}, \mathfrak{g}] \subseteq \mathfrak{h}$, we have $\mathfrak{z} \subseteq \mathfrak{h}$. On the other hand, $\mathfrak{h}/\mathfrak{z}$ is a Cartan subalgebra of $\mathfrak{g}/\mathfrak{z}$ by Proposition 6.1.32.

Assume that $\mathfrak{h} \supseteq \mathfrak{z}$ and that $\mathfrak{h}/\mathfrak{z}$ is a Cartan subalgebra of $\mathfrak{g}/\mathfrak{z}$. Let π be the canonical morphism from $\mathfrak{g}$ to $\mathfrak{g}/\mathfrak{z}$. The algebra $\mathfrak{h}$, which is a central extension of $\mathfrak{h}/\mathfrak{z}$, is then nilpotent. Let $\mathfrak{n}$ be the normalizer of $\mathfrak{h}$ in $\mathfrak{g}$. If $x \in \mathfrak{n}$, $[\pi(x), \mathfrak{h}/\mathfrak{z}] \subseteq \mathfrak{h}/\mathfrak{z}$, hence $\pi(x) \in \mathfrak{h}/\mathfrak{z}$, and so $x \in \mathfrak{h}$. This proves that $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$. □

Corollary 6.1.37. *Let $C_\infty(\mathfrak{g})$ be the union of the ascending central series of the Lie algebra $\mathfrak{g}$. The Cartan subalgebras of $\mathfrak{g}$ are the inverse images of the Cartan subalgebras of $\mathfrak{g}/C_\infty(\mathfrak{g})$.*

Proof. Indeed, the centre of $\mathfrak{g}/C_i(\mathfrak{g})$ is $C_{i+1}(\mathfrak{g})/C_i(\mathfrak{g})$, and the corollary follows immediately from Proposition 6.1.36 by induction. □

From now on we suppose the field $\mathbb{K}$ has characteristic zero. Let $\mathfrak{g}$ be a Lie algebra of dimension n. If $x \in \mathfrak{g}$, we write the characteristic polynomial of $\mathrm{ad}(x)$ in the form

$$\det(T - \mathrm{ad}(x)) = \sum_{i=0}^{n} a_i(x) T^i$$

where $a_i(x)$ is a homogeneous polynomial of degree $n - i$ from $\mathfrak{g}$ to $\mathbb{K}$.

Definition 6.1.38. The **rank** of $\mathfrak{g}$, denoted by $\mathrm{rank}(\mathfrak{g})$, is the smallest integer i such that $a_i \neq 0$. An element x of $\mathfrak{g}$ is called **regular** if $a_r(x) \neq 0$, where $r = \mathrm{rank}(\mathfrak{g})$.

For all $x \in \mathfrak{g}$, $\mathrm{rank}(\mathfrak{g}) \leq \dim(\mathfrak{g}^0(x))$, and equality holds if and only if x is regular. The set of regular elements is dense and open in $\mathfrak{g}$ for the Zariski topology.

Example 6.1.39 (Examples of regular elements).

(a) If $\mathfrak{g}$ is nilpotent then $\operatorname{rank}(\mathfrak{g}) = \dim(\mathfrak{g})$ and all elements of $\mathfrak{g}$ are regular. The converse is also true since $\operatorname{rank}(\mathfrak{g}) = \dim(\mathfrak{g})$ implies $\operatorname{ad}(x)^n = 0$ for all $x \in \mathfrak{g}$, where $n = \dim(\mathfrak{g})$.

(b) Let $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{K})$. If $x = \left(\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right)$, an easy calculation gives

$$\det(T - \operatorname{ad}(x)) = T^3 - 4(bc + a^2)T.$$

Then $\operatorname{rank}(\mathfrak{g}) = 1$ and the regular elements are those x such that $bc + a^2 \neq 0$.

(c) Let V be a vector space of finite dimension n, and $\mathfrak{g} = \mathfrak{gl}(V)$. Let $x \in \mathfrak{g}$, and let $\lambda_1, \dots, \lambda_n$ be the roots of the characteristic polynomial of x in an algebraic closure of $\mathbb{K}$ (each root being written a number of times equal to its multiplicity). The canonical isomorphism from $V^* \otimes V$ to $\mathfrak{g}$ is compatible with the $\mathfrak{g}$-module structures of these two spaces, in other words it takes $1 \otimes x - x^t \otimes 1$ to $\operatorname{ad}(x)$. In view of Proposition 6.1.10(a), it follows that the roots of the characteristic polynomial of $\operatorname{ad}(x)$ are the $\lambda_i - \lambda_j$ for $1 \leq i, j \leq n$ (each root being written a number of times equal to its multiplicity). Thus, the rank of $\mathfrak{g}$ is n, and x is regular if and only if each λ_i is a simple root of the characteristic polynomial of x.

Proposition 6.1.40. *Let $\mathfrak{g}$ be a Lie algebra, $\mathbb{K}'$ an extension of $\mathbb{K}$, and $\tilde{\mathfrak{g}} = \mathfrak{g} \otimes_{\mathbb{K}} \mathbb{K}'$.*

(a) An element x of $\mathfrak{g}$ is regular in $\mathfrak{g}$ if and only if $x \otimes 1$ is regular in $\tilde{\mathfrak{g}}$.

(b) $\operatorname{rank}(\mathfrak{g}) = \operatorname{rank}(\tilde{\mathfrak{g}})$.

Proof. This follows from the observation that, if $\det(T - \operatorname{ad}(x')) = \sum_{i=0}^{n} a_i'(x')T^i$ for $x' \in \tilde{\mathfrak{g}}$, then $a_i'|_{\mathfrak{g}} = a_i$ for all i. □

Proposition 6.1.41. *Let $(\mathfrak{g}_i)_{i \in I}$ be a finite family of Lie algebras, and let $\mathfrak{g} = \prod_{i \in I} \mathfrak{g}_i$.*

(a) An element $(x_i)_{i \in I}$ of $\mathfrak{g}$ is regular in $\mathfrak{g}$ if and only if, for all $i \in I$, x_i is regular in $\mathfrak{g}_i$.

(b) $\operatorname{rank}(\mathfrak{g}) = \sum_{i \in I} \operatorname{rank}(\mathfrak{g}_i)$.

Proof. Indeed, for any $x = (x_i)_{i \in I} \in \mathfrak{g}$, the characteristic polynomial of $\operatorname{ad}_{\mathfrak{g}}(x)$ is the product of the characteristic polynomials of the $\operatorname{ad}_{\mathfrak{g}_i}(x_i)$. □

Proposition 6.1.42. *Let $\varphi : \mathfrak{g} \to \tilde{\mathfrak{g}}$ be a surjective homomorphism of Lie algebras.*

(a) If x is a regular element of $\mathfrak{g}$, $\varphi(x)$ is regular in $\tilde{\mathfrak{g}}$. The converse is true if $\ker \varphi$ *is contained in the centre of $\mathfrak{g}$.*

(b) $\operatorname{rank}(\mathfrak{g}) \geq \operatorname{rank}(\tilde{\mathfrak{g}})$.

Proof. Let $x \in \mathfrak{g}$, then the characteristic polynomials of $\operatorname{ad}(x)$, $\operatorname{ad}(\varphi(x))$ and $\operatorname{ad}(x)|_{\ker \varphi}$ satisfy

$$\chi_{\operatorname{ad}(x)|_{\ker \varphi}} \cdot \chi_{\operatorname{ad}(\varphi(x))} = \chi_{\operatorname{ad}(x)}.$$

This proves (b) and the first assertion of (a). If $\ker \varphi$ is contained in the centre of $\mathfrak{g}$ then we have $\chi_{\operatorname{ad}(x)|_{\ker \varphi}} = T^{\dim(\ker \varphi)}$, hence the second assertion of (a). □

Corollary 6.1.43. *Let $C_n(\mathfrak{g})$ be a term of the ascending central series of $\mathfrak{g}$. The regular elements of $\mathfrak{g}$ are those whose image in $\mathfrak{g}/C_n(\mathfrak{g})$ is regular.*

Proposition 6.1.44. *Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{h}$ a subalgebra of $\mathfrak{g}$. Every element of $\mathfrak{h}$ regular in $\mathfrak{g}$ is regular in $\mathfrak{h}$.*

Proof. For $x \in \mathfrak{h}$, the restriction of $\operatorname{ad}_{\mathfrak{g}}(x)$ to $\mathfrak{h}$ is $\operatorname{ad}_{\mathfrak{h}}(x)$, and so defines an endomorphism $u(x)$ of the vector space $\mathfrak{g}/\mathfrak{h}$ by passage to the quotient. Let $d_0(x)$ (resp. $d_1(x)$) be the dimension of the nilspace of $\operatorname{ad}_{\mathfrak{h}}(x)$ (resp. of $u(x)$), and let n_0 (resp. n_1) be the minimum of $d_0(x)$ (resp. $d_1(x)$) when x belongs to $\mathfrak{h}$. There exist non-zero polynomial maps p_0, p_1 from $\mathfrak{h}$ to $\mathbb{K}$ such that

$$d_0(x) = n_0 \Leftrightarrow p_0(x) \neq 0, \quad d_1(x) = n_1 \Leftrightarrow p_1(x) \neq 0.$$

Since $\mathbb{K}$ is infinite, the set S of $x \in \mathfrak{h}$ such that $d_0(x) = n_0$ and $d_1(x) = n_1$ is then non-empty. Every element of S is regular in $\mathfrak{h}$. On the other hand, S is the set of elements of $\mathfrak{h}$ such that the nilspace of $\operatorname{ad}_{\mathfrak{g}}(x)$ has minimum dimension, and thus contains every element of $\mathfrak{h}$ regular in $\mathfrak{g}$. □

Theorem 6.1.45. *Let $\mathfrak{g}$ be a Lie algebra.*

(a) If x is a regular element of $\mathfrak{g}$, $\mathfrak{g}^0(x)$ is a Cartan subalgebra of $\mathfrak{g}$.

(b) If $\mathfrak{h}$ is a maximal nilpotent subalgebra of $\mathfrak{g}$, and if $x \in \mathfrak{h}$ is regular in $\mathfrak{g}$, then $\mathfrak{h} = \mathfrak{g}^0(x)$.

(c) If $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$, then $\dim(\mathfrak{h}) \geq \operatorname{rank}(\mathfrak{g})$.

(d) The Cartan subalgebras of $\mathfrak{g}$ of dimension $\operatorname{rank}(\mathfrak{g})$ are the $\mathfrak{g}^0(x)$ where x is a regular element.

Proof. Let x be a regular element of $\mathfrak{g}$ and let $\mathfrak{h} = \mathfrak{g}^0(x)$. Clearly $\mathfrak{h}^0(x) = \mathfrak{h}$. Since x is regular in $\mathfrak{h}$ by Proposition 6.1.44, we have $\operatorname{rank}(\mathfrak{h}) = \dim(\mathfrak{h})$, so $\mathfrak{h}$ is nilpotent. On the other hand, $\mathfrak{h} = \mathfrak{g}^0(x) \supseteq \mathfrak{g}^0(\mathfrak{h}) \supseteq \mathfrak{h}$, so $\mathfrak{h} = \mathfrak{g}^0(\mathfrak{h})$ is a Cartan subalgebra of $\mathfrak{g}$. This proves (a).

If $\mathfrak{h}$ is a maximal nilpotent subalgebra of $\mathfrak{g}$, and if $x \in \mathfrak{h}$ is regular in $\mathfrak{g}$, then $\mathfrak{h} \subseteq \mathfrak{g}^0(x)$ and $\mathfrak{g}^0(x)$ is nilpotent by (a), so $\mathfrak{h} = \mathfrak{g}^0(x)$, which proves (b).

If $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$, there exists $x \in \mathfrak{h}$ such that $\mathfrak{h} = \mathfrak{g}^0(x)$ (Corollary 6.1.31), so $\dim(\mathfrak{h}) \geq \operatorname{rank}(\mathfrak{g})$, which proves (c). If in addition $\dim(\mathfrak{h}) = \operatorname{rank}(\mathfrak{g})$, then x is regular. Finally, if y is regular in $\mathfrak{g}$, $\mathfrak{g}^0(y)$ is a Cartan subalgebra by (a), and is obviously of dimension $\operatorname{rank}(\mathfrak{g})$. This proves (d). □

Corollary 6.1.46. *Every Lie algebra $\mathfrak{g}$ has Cartan subalgebras, and the rank of $\mathfrak{g}$ is the minimum dimension of a Cartan subalgebra.*

Corollary 6.1.47. *Let $\varphi : \mathfrak{g} \to \tilde{\mathfrak{g}}$ be a surjective homomorphism of Lie algebras. If $\tilde{\mathfrak{h}}$ is a Cartan subalgebra of $\tilde{\mathfrak{g}}$, there exists a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ such that $\tilde{\mathfrak{h}} = \varphi(\mathfrak{h})$.*

Proof. By Corollary 6.1.46, $\varphi^{-1}(\tilde{\mathfrak{h}})$ has a Cartan subalgebra $\mathfrak{h}$. Since $\varphi(\mathfrak{h})$ is a Cartan subalgebra of $\tilde{\mathfrak{h}}$ by Corollary 6.1.32, we see $\varphi(\mathfrak{h}) = \tilde{\mathfrak{h}}$. Then

$$\varphi(\mathfrak{n}_{\mathfrak{g}}(\mathfrak{h})) \subseteq \mathfrak{n}_{\tilde{\mathfrak{g}}}(\varphi(\mathfrak{h})) = \mathfrak{n}_{\tilde{\mathfrak{g}}}(\tilde{\mathfrak{h}}) = \tilde{\mathfrak{h}}$$

whence $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{h}) \subseteq \varphi^{-1}(\tilde{\mathfrak{h}})$. But $\mathfrak{h}$ is its own normalizer in $\varphi^{-1}(\tilde{\mathfrak{h}})$, so $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{h}$ and $\mathfrak{h}$ is a Cartan subalgebra og $\mathfrak{g}$. □

Corollary 6.1.48. *Every Lie algebra $\mathfrak{g}$ is the sum of its Cartan subalgebras.*

Proof. The sum $\mathfrak{s}$ of the Cartan subalgebras of $\mathfrak{g}$ contains the set of regular elements of $\mathfrak{g}$. Since this set is dense in $\mathfrak{g}$ for the Zariski topology, we have $\mathfrak{s} = \mathfrak{g}$ (any subspace in the Zariski topology is closed). □

Proposition 6.1.49. *Let $\mathfrak{g}$ be a Lie algebra, $\mathfrak{a}$ an abelian subalgebra of $\mathfrak{g}$. Assume that $\operatorname{ad}_{\mathfrak{g}}(x)$ is semi-simple for all $x \in \mathfrak{a}$, then the Cartan subalgebras of the centralizer $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{a})$ are exactly the Cartan subalgebras of $\mathfrak{g}$ containing $\mathfrak{a}$. In particular, such subalgebra exists.*

Proof. Let $\mathfrak{c} := \mathfrak{z}_{\mathfrak{g}}(\mathfrak{a})$ and fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{c}$. Since $\mathfrak{a}$ is contained in the centre of $\mathfrak{c}$, we have $\mathfrak{a} \subseteq \mathfrak{z}(\mathfrak{c}) \subseteq \mathfrak{h}$ by Proposition 6.1.36. Let $\mathfrak{n}$ be the normalizer of $\mathfrak{h}$ in $\mathfrak{g}$. Then we have

$$[\mathfrak{a}, \mathfrak{n}] \subseteq [\mathfrak{h}, \mathfrak{n}] \subseteq \mathfrak{h}.$$

Since for each $x \in \mathfrak{a}$, the endomorphism $\operatorname{ad}_{\mathfrak{g}}(x)$ is semi-simple and commutes with each other, $\mathfrak{n}$ is a semi-simple $\mathfrak{a}$-module, so that there exists an $\mathfrak{a}$-invariant subspace $\mathfrak{l} \subseteq \mathfrak{n}$ with $\mathfrak{n} = \mathfrak{h} \oplus \mathfrak{l}$. Then

$$[\mathfrak{a}, \mathfrak{l}] \subseteq [\mathfrak{a}, \mathfrak{n}] \cap \mathfrak{l} \subseteq \mathfrak{h} \cap \mathfrak{l} = \{0\}$$

implies $\mathfrak{l} \subseteq \mathfrak{c}$. Thus, $\mathfrak{n}$ is the normalizer of $\mathfrak{h}$ in $\mathfrak{c}$, and hence $\mathfrak{n} = \mathfrak{h}$, so $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ containing a.

Conversely, let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ containing $\mathfrak{a}$. Then $\mathfrak{h} = \mathfrak{g}^0(\mathfrak{h}) \subseteq \mathfrak{g}^0(\mathfrak{a})$, and by hypothesis $\mathfrak{g}^0(\mathfrak{a}) = \mathfrak{g}_0(\mathfrak{a}) = \mathfrak{c}$. Hence $\mathfrak{a} \subseteq \mathfrak{h} \subseteq \mathfrak{c}$ and $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{c}$ (for it is equal to its own normalizer in $\mathfrak{g}$, and so a fortiori in $\mathfrak{c}$). □

Proposition 6.1.50. *Let $\mathfrak{n}$ be a nilpotent subalgebra of a Lie algebra $\mathfrak{g}$. Then there exists a Cartan subalgebra of $\mathfrak{g}$ contained in $\mathfrak{g}^0(\mathfrak{n})$.*

Proof. Put $\mathfrak{a} = \mathfrak{g}^0(\mathfrak{n})$. Then $\mathfrak{n} \subseteq \mathfrak{a}$ since $\mathfrak{n}$ is nilpotent. If $x \in \mathfrak{a}$, let $P(x)$ be the determinant of the endomorphism of $\mathfrak{g}/\mathfrak{a}$ defined by $\mathrm{ad}(x)$. Denote by $\tilde{\mathfrak{a}}$ the set of $x \in \mathfrak{a}$ such that $P(x) \neq 0$, which is an open subset of $\mathfrak{a}$ in the Zariski topology, so that the relations $x \in \tilde{\mathfrak{a}}$ and $\mathfrak{g}^0(x) \subseteq \mathfrak{a}$ are equivalent. By Proposition 6.1.13 applied to the action of $\mathfrak{n}$ on $\mathfrak{g}$, there exists $y \in \mathfrak{n}$ such that $\mathfrak{g}^0(y) = \mathfrak{a}$, and $y \in \tilde{\mathfrak{a}}$ so $\tilde{\mathfrak{a}}$ is non-empty. Since $\tilde{\mathfrak{a}}$ is open, its intersection with the set of regular elements of $\mathfrak{a}$ is non-empty. Let x be an element of this intersection. Then $\mathfrak{g}^0(x) \subseteq \mathfrak{a}$ and $\mathfrak{g}^0(x)$ is a Cartan subalgebra of $\mathfrak{a}$, hence is nilpotent. On the other hand, Proposition 6.1.18 shows that $\mathfrak{g}^0(x)$ is its own normalizer in $\mathfrak{g}$. It is therefore a Cartan subalgebra of $\mathfrak{g}$, which completes the proof. □

Now we let $\mathfrak{g}$ be a semi-simple Lie algebra and consider the Cartan subalgebras of $\mathfrak{g}$. In this case, since each $\mathrm{ad}(x)$ is semi-simple, we have in fact a eigenspace decomposition for the adjoint representation of on $\mathfrak{g}$. This will be our starting point for the theory of roots and weights.

Theorem 6.1.51. *Let $\mathfrak{g}$ be a semi-simple Lie algebra, $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$. Then $\mathfrak{h}$ is abelian, and all of its elements are semi-simple in $\mathfrak{g}$.*

Proof. Since $\mathfrak{h} = \mathfrak{g}^0(\mathfrak{h})$, $\mathfrak{h}$ is reductive by Proposition 6.1.20, hence abelian since it is nilpotent. On the other hand, the restriction of the adjoint representation of $\mathfrak{g}$ to $\mathfrak{h}$ is semi-simple, so the elements of $\mathfrak{h}$ are semi-simple in $\mathfrak{g}$. □

Corollary 6.1.52. *Let $\mathfrak{g}$ be a semi-simple Lie algebra, $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$. If $x \in \mathfrak{h}$ and $y \in \mathfrak{g}^\lambda(\mathfrak{h})$, we have $[x, y] = \lambda(x)y$.*

Proof. Indeed, $\mathfrak{g}^\lambda(x) = \mathfrak{g}_\lambda(x)$ since $\mathrm{ad}(x)$ is semi-simple. □

Corollary 6.1.53. *Let $\mathfrak{g}$ be a semi-simple Lie algebra. Then every regular element of $\mathfrak{g}$ is semi-simple.*

Proof. Any such an element belongs to a Cartan subalgebra. □

Corollary 6.1.54. *Let $\mathfrak{h}$ be a Cartan subalgebra of a reductive Lie algebra $\mathfrak{g}$. Then*

(a) $\mathfrak{h}$ is abelian;

(b) if ρ is a finite dimensional semi-simple representation of $\mathfrak{g}$, then the elements of $\rho(\mathfrak{h})$ are semi-simple.

Proof. Let $\mathfrak{c}$ be the centre of $\mathfrak{g}$, and $\mathfrak{s}$ its derived algebra. Then $\mathfrak{g} = \mathfrak{c} \times \mathfrak{s}$, so $\mathfrak{h} = \mathfrak{c} \times \mathfrak{t}$ where $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{s}$ (Proposition 6.1.36). In view of Theorem 6.1.51, $\mathfrak{t}$ is abelian, hence so is $\mathfrak{h}$. Moreover, $\rho(\mathfrak{t})$ consists of semi-simple elements and so does $\rho(\mathfrak{c})$, so assertion (b) follows. □

6.1.3 Conjugacy of Cartan subalgebras

Through out this paragraph, the field $\mathbb{K}$ is assumed to have characteristic zero. Let $\mathfrak{g}$ be a Lie algebra. Denote its group of automorphisms by $\mathrm{Aut}(\mathfrak{g})$. We recall that for each nilpotent inner derivation $\mathrm{ad}(x)$, the map $e^{\mathrm{ad}(x)}$ is an automorphism of $\mathfrak{g}$. The elements of the group

$$\mathrm{Aut}_e(\mathfrak{g}) = \langle e^{\mathrm{ad}(x)} : x \in \mathfrak{g}, \mathrm{ad}(x) \text{ is nilpotent} \rangle$$

generated by these automorphisms are called **elementray automorphisms**. If $\phi \in \mathrm{Aut}(\mathfrak{g})$, then we have $\phi e^{\mathrm{ad}(x)} \phi^{-1} = e^{\mathrm{ad}(\phi(x))}$. It follows that $\mathrm{Aut}_e(\mathfrak{g})$ is a normal subgroup of $\mathrm{Aut}(\mathfrak{g})$. If $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, $\mathrm{Aut}_e(\mathfrak{g})$ is contained in the group $\mathrm{Inn}(\mathfrak{g})$ of inner automorphisms of $\mathfrak{g}$.

Lemma 6.1.55. *Let V be a finite dimensional vector space, $\mathfrak{n}$ a Lie subalgebra of $\mathfrak{g} = \mathfrak{gl}(V)$ consisting of nilpotent elements.*

(a) The map $x \mapsto \exp(x)$ is a bijection from $\mathfrak{n}$ to a subgroup N of $\mathrm{GL}(V)$ consisting of unipotent elements. We have $\mathfrak{n} = \log(\exp(\mathfrak{n}))$. The map $f \mapsto f \circ \log$ is an isomorphism from the algebra of polynomial functions on $\mathfrak{n}$ to the algebra of restrictions to N of polynomial functions on $\mathrm{End}(V)$.

(b) Let W be a finite dimensional vector space, $\mathfrak{m}$ a Lie subalgebra of $\mathfrak{gl}(W)$ consisting of nilpotent elements, $\pi : \mathfrak{n} \to \mathfrak{m}$ a homomorphism. Let Π be the map $\exp(x) \mapsto \exp(\pi(x))$ from $\exp(\mathfrak{n})$ to $\exp(\mathfrak{m})$. Then Π is a group homomorphism.

(c) If $x \in \mathfrak{n}$ and $y \in \mathfrak{g}$, then

$$e^{\mathrm{ad}(x)}(y) = e^x y e^{-x}.$$

Proof. By Engel's theorem, we can identify V with $\mathbb{K}^n$ in such a way that $\mathfrak{n}$ is a subalgebra of $\mathfrak{n}$ (the Lie subalgebra of $\mathfrak{gl}(n,\mathbb{K})$ consisting of the upper triangular matrices with zeros on the diagonal). For $s \geq 0$, let $\mathfrak{n}_s$ be the set of $(x_{ij}) \in \mathfrak{gl}(n,\mathbb{K})$ such that $x_{ij} = 0$ for $j - i < s$. Then

$$[\mathfrak{n}_s, \mathfrak{n}_t] \subseteq \mathfrak{n}_{s+t}$$

and the Hausdorff series defines a polynomial map $(x,y) \mapsto H(x,y)$ from $\mathfrak{n} \times \mathfrak{n} \to \mathfrak{n}$. This map makes $\mathfrak{n}$ into a group. By **??**, the maps $x \mapsto \exp(x)$ from $\mathfrak{n}$ to $1 + \mathfrak{n}$ and $y \mapsto \log(y)$ from $1+\mathfrak{n}$ to $\mathfrak{n}$ are inverse bijections and are polynomial. By **??**, these maps are isomorphisms of groups if $\mathfrak{n}$ is given the group law $(x,y) \mapsto H(x,y)$ and if $1+\mathfrak{n}$ is considered as a subgroup of $\mathrm{GL}(n,\mathbb{K})$. Assertions (a) and (b) of the lemma now follow. Let $x \in \mathfrak{n}$ and denote by L_x, R_x the left multiplication and right multiplication, which commute and are nilpotent. We have $\mathrm{ad}(x) = L_x - R_x$, so, for all $y \in \mathfrak{g}$,

$$e^{\mathrm{ad}(x)}(y) = e^{L_x - R_x}(y) = e^{L_x}e^{-R_x}(y) = e^x y e^{-x}.$$

This completes the proof. □

With the notation in Lemma 6.1.55, Π is called the **linear representation** of $\exp(\mathfrak{n})$ compatible with the given representation π of $\mathfrak{n}$ on W.

Proposition 6.1.56. *Let $\mathfrak{g}$ be a Lie algebra, $\mathfrak{n}$ a subalgebra of $\mathfrak{g}$ such that $\mathrm{ad}(x)$ is nilpotent for all $x \in \mathfrak{n}$. Then $e^{\mathrm{ad}(\mathfrak{n})}$ is a subgroup of $\mathrm{Aut}_e(\mathfrak{g})$.*

Proof. This follows immediately from Lemma 6.1.55(a). □

Let $\mathfrak{g}$ be a Lie algebra, $\mathfrak{h}$ a nilpotent subalgebra of $\mathfrak{g}$ and Φ the set of non-zero weights of $\mathfrak{h}$ in $\mathfrak{g}$, in other words the set of linear forms $\lambda \neq 0$ on $\mathfrak{h}$ such that $\mathfrak{g}^\lambda(\mathfrak{h}) \neq 0$. Assume that

$$\mathfrak{g} = \mathfrak{g}^0(\mathfrak{h}) \oplus \bigoplus_{\lambda \in \Phi} \mathfrak{g}^\lambda(\mathfrak{h})$$

which is the case if $\mathbb{K}$ is algebraically closed. For $\lambda \in \Phi$ and $x \in \mathfrak{g}^\lambda(\mathfrak{h})$, $\mathrm{ad}(x)$ is nilpotent by Proposition 6.1.18(d). Denote by $E(\mathfrak{h})$ the subgroup of $\mathrm{Aut}_e(\mathfrak{g})$ generated by the $e^{\mathrm{ad}(x)}$ where x is of the form above. If $\phi \in \mathrm{Aut}(\mathfrak{g})$, it is immediate that $\phi E(\mathfrak{h})\phi^{-1} = E(\phi(\mathfrak{h}))$.

Lemma 6.1.57. *Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{h}$ a nilpotent subalgebra of $\mathfrak{g}$.*

(a) Let $\mathfrak{h}_r$ be the set of $x \in \mathfrak{h}$ such that $\mathfrak{g}^0(x) = \mathfrak{g}^0(\mathfrak{h})$. Then $\mathfrak{h}_r$ is the set of $x \in \mathfrak{h}$ such that $\lambda(x) \neq 0$ for all $\lambda \in \Phi$, and $\mathfrak{h}_r$ is open and dense in $\mathfrak{h}$ in the Zariski topology.

(b) Put $\Phi = \{\lambda_1, \dots, \lambda_s\}$ where the λ_i are mutually distinct, and define

$$F : \mathfrak{g}^0(\mathfrak{h}) \times \mathfrak{g}^{\lambda_1}(\mathfrak{h}) \times \cdots \times \mathfrak{g}^{\lambda_s}(\mathfrak{h}) \to \mathfrak{g}, \quad (h, x_1, \dots, x_s) \mapsto e^{\mathrm{ad}(x_1)} \cdots e^{\mathrm{ad}(x_s)} h.$$

Then F is a dominant polynomial map.

Proof. Assertion (a) is clear, so we prove (b). Let $n = \dim(\mathfrak{g})$. If $\lambda \in \Phi$ and $x \in \mathfrak{g}^\lambda(\mathfrak{h})$, we have $\mathrm{ad}(x)^n = 0$. It follows that $(y,x) \mapsto e^{\mathrm{ad}(x)}y$ is a polynomial map from $\mathfrak{g} \times \mathfrak{g}^\lambda(\mathfrak{h})$ to $\mathfrak{g}$, and it follows by induction that F is polynomial. Let $h_0 \in \mathfrak{h}_r$ and let dF be the tangent linear map of F at $(h_0, 0, \dots, 0)$. It suffices to prove that dF is surjective. For $h \in \mathfrak{g}^0(\mathfrak{h})$, we have

$$F(h_0 + h, 0, \dots, 0) = h_0 + h,$$

so $dF(h, 0, \dots, 0) = h$ and $\mathfrak{g}^0(\mathfrak{h})$ is in the image of dF. On the other hand, for $x \in \mathfrak{g}^{\lambda_i}(\mathfrak{h})$,

$$F(h_0, 0, \dots, x, \dots, 0) = e^{\mathrm{ad}(x)}h_0 = h + \mathrm{ad}(x)(h_0) + \cdots$$

so $dF(h_0, 0, \dots, x, \dots, 0) = \mathrm{ad}(x)(h_0)$. Since $\mathrm{ad}(h_0)$ induces an automorphism of $\mathfrak{g}^{\lambda_1}(\mathfrak{h})$, we see $\mathfrak{g}^{\lambda_i}(\mathfrak{h})$ is also contained in the image of dF, so dF is surjective. This implies F is dominant. □

Proposition 6.1.58. *Assume that $\mathbb{K}$ is algebraically closed. Let $\mathfrak{g}$ be a Lie algebra, $\mathfrak{h}$ and $\tilde{\mathfrak{h}}$ Cartan subalgebras of $\mathfrak{g}$. Then there exist $\phi \in E(\mathfrak{h})$ and $\tilde{\phi} \in E(\tilde{\mathfrak{h}})$ such that $\phi(\mathfrak{h}) = \tilde{\phi}(\tilde{\mathfrak{h}})$.*

Proof. We retain the notation of Lemma 6.1.57. From the fact that $\mathfrak{h}$ and $\tilde{\mathfrak{h}}$ are Cartan subalgebras, it follows that $\mathfrak{g}^0(\mathfrak{h}) = \mathfrak{h}$ and $\mathfrak{g}^0(\tilde{\mathfrak{h}}) = \tilde{\mathfrak{h}}$. By Lemma 6.1.57, $E(\mathfrak{h}) \cdot \mathfrak{h}_r$ and $E(\tilde{\mathfrak{h}}) \cdot \tilde{\mathfrak{h}}_r$ contain open dense subsets of $\mathfrak{g}$ in the Zariski topology, so their intersection is nonempty. In other words, there exist $\phi \in E(\mathfrak{h})$, $h \in \mathfrak{h}_r$ and $\tilde{\phi} \in E(\tilde{\mathfrak{h}})$, $h' \in \mathfrak{h}'_r$ such that $\phi(h) = \tilde{\phi}(h')$. Then we have

$$\phi(\mathfrak{h}) = \phi(\mathfrak{g}^0(h)) = \mathfrak{g}^0(\phi(h)) = \mathfrak{g}^0(\tilde{\phi}(h')) = \tilde{\phi}(\mathfrak{g}^0(h')) = \tilde{\phi}(\tilde{\mathfrak{h}})$$

whence the proposition. □

Corollary 6.1.59. *Assume that $\mathbb{K}$ is algebraically closed. Let $\mathfrak{g}$ be a Lie algebra, $\mathfrak{h}$ and $\tilde{\mathfrak{h}}$ Cartan subalgebras of $\mathfrak{g}$. Then $E(\mathfrak{h}) = E(\tilde{\mathfrak{h}})$.*

Proof. Let ϕ and $\tilde{\phi}$ be as in Proposition 6.1.58. Then

$$E(\mathfrak{h}) = \phi E(\mathfrak{h})\phi^{-1} = E(\phi(\mathfrak{h})) = E(\tilde{\phi}(\tilde{\mathfrak{h}})) = \tilde{\phi}E(\tilde{\mathfrak{h}})\tilde{\phi}^{-1} = E(\tilde{\mathfrak{h}})$$

whence the corollary. □

Because of this result, if $\mathbb{K}$ is algebraically closed we shall denote simply by E the group $E(\mathfrak{h})$, where $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$.

Remark 6.1.60. In general, $\mathrm{Aut}_e(\mathfrak{g}) \neq E$ (for example, if $\mathfrak{g}$ is nilpotent, E reduces to the identity element, even though non-trivial elementary automorphisms exist provided $\mathfrak{g}$ is non-abelian). However, it can be shown that $\mathrm{Aut}_e(\mathfrak{g}) = E$ for g semi-simple.

Theorem 6.1.61. *Assume that $\mathbb{K}$ is algebraically closed. Let $\mathfrak{g}$ be a Lie algebra. The group E is normal in $\mathrm{Aut}(\mathfrak{g})$ and operates transitively on the set of Cartan subalgebras of $\mathfrak{g}$.*

Proof. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, and $\phi \in \mathrm{Aut}(\mathfrak{g})$. Then by Corollary 6.1.59 we have $\phi E(\mathfrak{h})\phi^{-1} = E(\phi(\mathfrak{h})) = E(\mathfrak{h})$, so $E(\mathfrak{h}) = E$ is normal in $\mathrm{Aut}(\mathfrak{g})$. If $\tilde{\mathfrak{h}}$ is another Cartan subalgebra of $\mathfrak{g}$, then, in the notation of Proposition 6.1.58, $\tilde{\phi}^{-1}\phi(\mathfrak{h}) = \tilde{\mathfrak{h}}$, and we have $\tilde{\phi}^{-1}\phi \in E$. □

Theorem 6.1.62. *Let $\mathfrak{g}$ be a Lie algebra.*

(a) *The Cartan subalgebras of $\mathfrak{g}$ are all of the same dimension, namely $\mathrm{rank}(\mathfrak{g})$, and the same nilpotency class.*

(b) *An element $x \in \mathfrak{g}$ is regular if and only if $\mathfrak{g}^0(x)$ is a Cartan subalgebra of $\mathfrak{g}$. Every Cartan subalgebra is obtained in this way.*

Proof. To prove (a), we can assume that $\mathbb{K}$ is algebraically closed (Proposition 6.1.29 and Proposition 6.1.40), in which case it follows from the Conjugacy Theorem. Assertion (b) follows from (a) and Theorem 6.1.45(a) and (d). □

Proposition 6.1.63. *Let $\mathfrak{g}$ be a Lie algebra and $\tilde{\mathfrak{g}}$ a subalgebra of $\mathfrak{g}$. Then the following conditions are equivalent:*

(i) *$\tilde{\mathfrak{g}}$ contains a regular element of $\mathfrak{g}$, and $\mathrm{rank}(\mathfrak{g}) = \mathrm{rank}(\tilde{\mathfrak{g}})$.*

(ii) *$\tilde{\mathfrak{g}}$ contains a Cartan subalgebra of $\mathfrak{g}$.*

(iii) *Every Cartan subalgebra of $\tilde{\mathfrak{g}}$ is a Cartan subalgebra of $\mathfrak{g}$.*

Proof. Assume that $\mathrm{rank}(\mathfrak{g}) = \mathrm{rank}(\tilde{\mathfrak{g}})$ and that there exists $x \in \tilde{\mathfrak{g}}$ regular in $\mathfrak{g}$. Put $h = \mathfrak{g}^0(x)$ and $\tilde{\mathfrak{h}} = \tilde{\mathfrak{g}}^0(x) = \mathfrak{h} \cap \tilde{\mathfrak{g}}$. Then

$$\mathrm{rank}(\tilde{\mathfrak{g}}) \leq \dim(\tilde{\mathfrak{h}}) \leq \dim(\mathfrak{h}) = \mathrm{rank}(\mathfrak{g}) = \mathrm{rank}(\tilde{\mathfrak{g}})$$

whence $\mathfrak{h} = \tilde{\mathfrak{h}} \subseteq \tilde{\mathfrak{g}}$ and (b) follows.

Assume that $\tilde{\mathfrak{g}}$ contains a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, and let $\mathfrak{h}_1$ be a Cartan subalgebra of $\tilde{\mathfrak{g}}$. To prove that $\mathfrak{h}_1$ is a Cartan subalgebra of $\mathfrak{g}$, we can assume that $\mathbb{K}$ is algebraically closed. Let $E(\mathfrak{h})$ and $\tilde{E}(\mathfrak{h})$ be the groups of automorphisms of $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$ associated to $\mathfrak{h}$. By Theorem 6.1.62, there exists $\phi \in \tilde{E}(\mathfrak{h})$ such that $\phi(\mathfrak{h}) = \mathfrak{h}_1$. Now every element of $\tilde{E}(\mathfrak{h})$ is induced by an element of $E(\mathfrak{h})$; indeed, it suffices to verify this for $e^{\mathrm{ad}(x)}$, with $x \in \tilde{\mathfrak{g}}^\lambda(\mathfrak{h})$, $\lambda \neq 0$, in which case it follows from the inclusion $\tilde{\mathfrak{g}}^\lambda(\mathfrak{h}) \subseteq \mathfrak{g}^\lambda(\mathfrak{h})$. Thus $\mathfrak{h}_1$ is a Cartan subalgebra of $\mathfrak{g}$.

Assume that condition (iii) is satisfied. Let $\mathfrak{h}$ be a Cartan subalgebra of $\tilde{\mathfrak{g}}$. Since this is a Cartan subalgebra of $\mathfrak{g}$, it contains a regular element of $\mathfrak{g}$, and on the other hand $\mathrm{rank}(\mathfrak{g}) = \dim(\mathfrak{h}) = \mathrm{rank}(\tilde{\mathfrak{g}})$. □

Corollary 6.1.64. *Let $\mathfrak{n}$ be a nilpotent subalgebra of $\mathfrak{g}$. Then the subalgebra $\mathfrak{g}^0(\mathfrak{n})$ has the properties in Proposition 6.1.63.*

Proof. Indeed, Proposition 6.1.50 shows $\mathfrak{g}^0(\mathfrak{n})$ satisfies the property (ii). □

Proposition 6.1.65. *Let $\mathfrak{g}$ be a Lie algebra, r the rank of $\mathfrak{g}$, n the nilpotency class of the Cartan subalgebras of $\mathfrak{g}$, and $x \in \mathfrak{g}$. There exists an r-dimensional subalgebra of $\mathfrak{g}$ whose nilpotency class is smaller than n and which contains x.*

Proof. Let T be an indeterminate. Let $\mathbb{K}' = \mathbb{K}(T)$ and $\tilde{\mathfrak{g}} = \mathfrak{g} \otimes_{\mathbb{K}} \mathbb{K}'$. If $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$, then $\tilde{\mathfrak{h}} = \mathfrak{h} \otimes_{\mathbb{K}} \mathbb{K}'$ is a Cartan subalgebra of $\tilde{\mathfrak{g}}$, hence the rank of $\tilde{\mathfrak{g}}$ is r and the nilpotency class of the Cartan subalgebras of $\tilde{\mathfrak{g}}$ is n.

Choose a regular element y of $\mathfrak{g}$. Then we have $a_r(y) \neq 0$, where a_i's are the polynomials in the characteristic polynomial of $\mathrm{ad}(y)$. Denote also by a_r the polynomial function on $\tilde{\mathfrak{g}}$ that extends a_r. Then the element $a_r(x+Ty)$ of $\mathbb{K}[T]$ has dominant coefficient $a_r(y)$. In particular, $x+Ty$ is regular in $\tilde{\mathfrak{g}}$. Let $\tilde{\mathfrak{h}}$ be the nilspace of $\mathrm{ad}(x+Ty)$ in $\tilde{\mathfrak{g}}$. Then $\dim(\tilde{\mathfrak{h}}) = r$ and the nilpotency class of $\tilde{\mathfrak{h}}$ is n. Put $\mathfrak{k} = \tilde{\mathfrak{h}} \cap (\mathfrak{g} \otimes_{\mathbb{K}} \mathbb{K}[T])$; then $\mathfrak{k} \otimes_{\mathbb{K}[T]} \mathbb{K}(T) = \tilde{\mathfrak{h}}$.

Let φ be the homomorphism from $\mathbb{K}[T]$ to $\mathbb{K}$ such that $\varphi(T) = 0$, and let ψ be the homomorphism $1 \otimes \varphi$ from $\mathfrak{g} \otimes_{\mathbb{K}} \mathbb{K}[T]$ to $\mathfrak{g}$. Then $\psi(\mathfrak{k})$ is a subalgebra of $\mathfrak{g}$ whose nilpotency class is smaller than n and which contains $\psi(x+Ty) = x$. In the free $\mathbb{K}[T]$-module $\mathfrak{g} \otimes_{\mathbb{K}} \mathbb{K}[T]$, $\mathbb{K}$ is a submodule of rank r, and $(\mathfrak{g} \otimes_{\mathbb{K}} \mathbb{K}[T])/\mathbb{K}$ is torsion free, so the submodule $\mathbb{K}$ is a direct summand of $\mathfrak{g} \otimes_{\mathbb{K}} \mathbb{K}[T]$. Hence $\dim_{\mathbb{K}} \psi(k) = 1$, which completes the proof. □

Let $\mathfrak{g}$ be a solvable Lie algebra. Denote by $C^\infty(\mathfrak{g})$ the intersection of the terms of the descending central series of $\mathfrak{g}$. This is a characteristic ideal of $\mathfrak{g}$, and is the smallest ideal $\mathfrak{a}$ of $\mathfrak{g}$ such that $\mathfrak{g}/\mathfrak{a}$ is nilpotent. Since $C^\infty(\mathfrak{g}) \subseteq [\mathfrak{g}, \mathfrak{g}]$, it is a nilpotent ideal of $\mathfrak{g}$ (Corollary 1.5.14). By Proposition 6.1.56, the set of $e^{\mathrm{ad}(x)}$ for $x \in C^\infty(\mathfrak{g})$ is a subgroup of $\mathrm{Aut}(\mathfrak{g})$.

Theorem 6.1.66. *Let $\mathfrak{g}$ be a solvable Lie algebra, and let $\mathfrak{h}$, $\tilde{\mathfrak{h}}$ be Cartan subalgebras of $\mathfrak{g}$. There exists $x \in C^\infty(\mathfrak{g})$ such that $e^{\mathrm{ad}(x)}\mathfrak{h} = \tilde{\mathfrak{h}}$.*

Proof. We argue by induction on $\dim(\mathfrak{g})$, the case where $\mathfrak{g} = 0$ being trivial. Let $\mathfrak{n}$ be a minimal non-zero abelian ideal of $\mathfrak{g}$. Let $\pi : \mathfrak{g} \to \mathfrak{g}/\mathfrak{n}$ be the canonical morphism. Then $\pi(C^\infty(\mathfrak{g})) = C^\infty(\mathfrak{g}/\mathfrak{n})$ (Proposition). Since $\pi(\mathfrak{h})$ and $\pi(\tilde{\mathfrak{h}})$ are Cartan subalgebras of $\mathfrak{g}/\mathfrak{n}$ (Proposition 6.1.32), there exists, by the induction hypothesis, an $x \in C^\infty(\mathfrak{g})$ such that $e^{\mathrm{ad}(\pi(x))}\pi(\mathfrak{h}) = \pi(\tilde{\mathfrak{h}})$. Replacing $\mathfrak{h}$ by $e^{\mathrm{ad}(x)}\mathfrak{h}$, we can assume that $\pi(\mathfrak{h}) = \pi(\tilde{\mathfrak{h}})$, in other words that

$$\mathfrak{h} + \mathfrak{n} = \tilde{\mathfrak{h}} + \mathfrak{n}$$

Then $\mathfrak{h}$ and $\tilde{\mathfrak{h}}$ are Cartan subalgebras of $\mathfrak{h} + \mathfrak{n}$. If $\mathfrak{h} + \mathfrak{n} \neq \mathfrak{g}$, the assertion to be proved follows from the induction hypothesis. Assume from now on that $\mathfrak{h} + \mathfrak{n} = \tilde{\mathfrak{h}} + \mathfrak{n} = \mathfrak{g}$. By the minimality of $\mathfrak{n}$, $[\mathfrak{g}, \mathfrak{n}] = \{0\}$ or $[\mathfrak{g}, \mathfrak{n}] = \mathfrak{n}$.

If $[\mathfrak{g}, \mathfrak{n}] = \{0\}$, then $\mathfrak{n}$ is contained in the center of $\mathfrak{g}$ and therefore $\mathfrak{n} \subseteq \mathfrak{h}$ and $\mathfrak{n} \subseteq \tilde{\mathfrak{h}}$ (Proposition 6.1.36). In this case we have $\mathfrak{h} = \mathfrak{h} + \mathfrak{n} = \tilde{\mathfrak{h}} + \mathfrak{n} = \tilde{\mathfrak{h}}$.

It remains to consider the case where $[\mathfrak{g}, \mathfrak{n}] = \mathfrak{n}$, so $\mathfrak{n} \subseteq C^\infty(\mathfrak{g})$. The ideal $\mathfrak{n}$ is a simple $\mathfrak{g}$-module; since $\mathfrak{g} = \mathfrak{h} + \mathfrak{n}$ and $[\mathfrak{n}, \mathfrak{n}] = \{0\}$, it follows that $\mathfrak{n}$ is a simple $\mathfrak{h}$-module. If $\mathfrak{h} \cap \mathfrak{n} \neq \{0\}$, then by the simplicity $\mathfrak{n} \subseteq \mathfrak{h}$, so $\mathfrak{g} = \mathfrak{h}$ and $\tilde{\mathfrak{h}} = \mathfrak{h}$. Assume now that $\mathfrak{h} \cap \mathfrak{n} = \{0\}$. Then $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{n}$ and hence $\mathfrak{g} = \tilde{\mathfrak{h}} \oplus \mathfrak{n}$, since $\mathfrak{h}$ and $\tilde{\mathfrak{h}}$ have the same dimension. For all $x \in \mathfrak{h}$, let $f(x)$ be the unique element of $\mathfrak{n}$ such that $x - f(x) \in \tilde{\mathfrak{h}}$. If $x, y \in \mathfrak{h}$,

$$[x, y] - [x, f(y)] - [f(x), y] = [x - f(x), y - f(y)] \in \tilde{\mathfrak{h}},$$

so $f([x, y]) = [x, f(y)] + [f(x), y]$. By Corollary 6.1.17, there exists $z \in \mathfrak{n}$ such that $f(x) = [x, z]$ for all $x \in \mathfrak{h}$. We have $\mathrm{ad}(z)^2(\mathfrak{g}) \subseteq \mathrm{ad}(z)(\mathfrak{n}) = 0$, so, for all $x \in \mathfrak{h}$,

$$e^{\mathrm{ad}(z)}(x) = x + [z, x] = x - f(x).$$

Thus $e^{\mathrm{ad}(z)}(\mathfrak{h}) = \tilde{\mathfrak{h}}$. Since $z \in C^\infty(\mathfrak{g})$, this completes the proof. □

Lemma 6.1.67. *Let $\mathfrak{g}$ be a Lie algebra, $\mathfrak{r}$ its radical, π the canonical homomorphism from $\mathfrak{g}$ to $\mathfrak{g}/\mathfrak{r}$, ϕ an elementary automorphism of $\mathfrak{g}/\mathfrak{r}$. Then there exists an elementary automorphism ψ of $\mathfrak{g}$ such that $\pi \circ \psi = \phi \circ \pi$.*

Proof. We can assume that ϕ is of the form $e^{\mathrm{ad}(\bar{x})}$, where $\bar{x} \in \mathfrak{g}/\mathfrak{r}$ and $\mathrm{ad}(\bar{x})$ is nilpotent. Let $\mathfrak{s}$ be a Levi subalgebra of $\mathfrak{g}$ and let x be the element of $\mathfrak{s}$ such that $\pi(x) = \bar{x}$. Since $\mathrm{ad}_{\mathfrak{s}}(x)$ is nilpotent, $\mathrm{ad}_{\mathfrak{g}}(x)$ is nilpotent, and $\phi = e^{\mathrm{ad}_{\mathfrak{g}}(x)}$ is an elementary automorphism of $\mathfrak{g}$ such that $\pi \circ \psi = \phi \circ \pi$. □

Proposition 6.1.68. *Let $\mathfrak{g}$ be a Lie algebra, $\mathfrak{r}$ its radical, $\mathfrak{h}$ and $\tilde{\mathfrak{h}}$ Cartan subalgebras of $\mathfrak{g}$, and π the canonical homomorphism from $\mathfrak{g}$ to $\mathfrak{g}/\mathfrak{r}$. Then the following conditions are equivalent:*

(i) *$\mathfrak{h}$ and $\tilde{\mathfrak{h}}$ are conjugate by an elementary automorphism of $\mathfrak{g}$.*

(ii) *$\pi(\mathfrak{h})$ and $\pi(\tilde{\mathfrak{h}})$ are conjugate by an elementary automorphism of $\mathfrak{g}/\mathfrak{r}$.*

Proof. It is clear that (i) implies (ii). Conversely, we assume that condition (ii) is satisfied and prove (i). By Lemma 6.1.67, we are reduced to the case where $\pi(\mathfrak{h}) = \pi(\tilde{\mathfrak{h}})$. Put $\mathfrak{k} = \mathfrak{h} + \mathfrak{r} = \tilde{\mathfrak{h}} + \mathfrak{r}$, which is a solvable subalgebra of $\mathfrak{g}$. Then $\mathfrak{h}$ and $\tilde{\mathfrak{h}}$ are Cartan subalgebras of $\mathfrak{k}$, so there exists $x \in C^{\infty}(\mathfrak{k})$ such that $e^{\mathrm{ad}_{\mathfrak{k}}(x)}\mathfrak{h} = \tilde{\mathfrak{h}}$ (Theorem 6.1.66). Since $\mathfrak{k}/\mathfrak{r}$ is nilpotent, $C^{\infty}(\mathfrak{k}) \subseteq \mathfrak{r}$. On the other hand, $C^{\infty}(\mathfrak{k}) \subseteq [\mathfrak{k},\mathfrak{k}] \subseteq [\mathfrak{g},\mathfrak{g}]$, so $x \in \mathfrak{r} \cap [\mathfrak{g},\mathfrak{g}]$. By Theorem , $\mathrm{ad}_{\mathfrak{g}}(x)$ is nilpotent, so $e^{\mathrm{ad}_{\mathfrak{g}}(x)}$ is an elementary automorphism of $\mathfrak{g}$ transforming $\mathfrak{h}$ to $\tilde{\mathfrak{h}}$. □

6.2 Split semi-simple Lie algebras

In this section, $\mathbb{K}$ denotes a field of characteristic zero. Unless otherwise stated, by a vector space, we mean a vector space over $\mathbb{K}$; similarly for Lie algebra, etc.

6.2.1 The Lie algebra $\mathfrak{sl}(2,\mathbb{K})$ and its representations

Recall that we denote by $\mathfrak{sl}(2,\mathbb{K})$ the Lie algebra consisting of the square matrices of order 2, trace zero, and with entries in $\mathbb{K}$. This Lie algebra is semi-simple of dimension 3. The canonical basis of $\mathfrak{sl}(2,k)$ is the basis (e,f,h), where

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and the commutator relations are

$$[h,e] = 2e, \quad [h,f] = -2f, \quad [e,f] = h.$$

Since the identity representation of $\mathfrak{sl}(2,\mathbb{K})$ is injective, h is a semi-simple element of $\mathfrak{sl}(2,\mathbb{K})$ and e,f are nilpotent elements of $\mathfrak{sl}(2,\mathbb{K})$. Also, $\mathbb{K}h$ is a Cartan subalgebra of $\mathfrak{sl}(2,\mathbb{K})$. The map $x \mapsto -x^t$ is an involutive automorphism of the Lie algebra $\mathfrak{sl}(2,\mathbb{K})$, called the canonical involution of $\mathfrak{sl}(2,\mathbb{K})$; it transforms (e,h,f) into $(-f,-h,-e)$.

Lemma 6.2.1. *Let (e,h,f) be a triple of elements of an associative algebra A satisfying the commutator relations*

$$[h,e] = 2e, \quad [h,f] = -2f, \quad [e,f] = h.$$

Then the following assertions hold:

(a) $[h,e^n] = 2ne^n$ and $[h,f] = -2nf^n$ for any positive integer n.

(b) For any positive integer n,

$$[f,e^n] = -ne^{n-1}(h+(n-1)) = -n(h-(n-1))e^{n-1}. \tag{6.2.1}$$

and

$$[e,f^n] = nf^{n-1}(h-(n-1)) = n(h+(n-1))f^{n-1}. \tag{6.2.2}$$

Proof. Since $\mathrm{ad}(h)$ is a derivation of A and $[h,e] = 2e$ commutes with e, we obtain inductively $[h,e^n] = ne^{n-1}[h,e] = 2ne^n$. The second part of (a) is obtained similarly.

Note that by induction we can prove that $[f,e^n] = \sum_{i=0}^{n-1} e^i[f,e]e^{n-i-1}$, and by assertion (a) we have

$$[f,e^n] = \sum_{i=0}^{n-1} e^i(-h)e^{n-i-1} = -\sum_{i=0}^{n-1} e^i([h,e^{n-i-1}] + e^{n-i-1}h)$$

$$= -n(n-1)e^{n-1} - ne^{n-1}h.$$

This proves the first part of (6.2.1), and the second part can be done using (a). The other part of (b) is reduced to the first by using the canonical involution of $\mathfrak{sl}(2,\mathbb{K})$. □

Let (V,ρ) be an $\mathfrak{sl}(2,\mathbb{K})$-module. Let $\lambda \in \mathbb{K}$. If $h \cdot v = \lambda v$ we say, by abuse of language, that v is an element of V of weight λ, or that λ is the **weight** of v. If V is finite dimensional, then $\rho(h)$ is semi-simple, so the set of elements of weight λ is the primary subspace of V relative to $\rho(h)$ and λ.

Lemma 6.2.2. *Let (V,ρ) be an $\mathfrak{sl}(2,\mathbb{K})$-module. If $v \in V$ is an element of weight λ, then $e \cdot v$ is an element of weight $\lambda+2$ and $f \cdot v$ is an element of weight $\lambda - 2$.*

Proof. Indeed, we have

$$h \cdot e \cdot v = e \cdot h \cdot v + [h,e] \cdot v = e \cdot \lambda v + 2e \cdot v = (\lambda+2)(e \cdot v),$$

and similarly $h \cdot f \cdot v = (\lambda-2)(f \cdot v)$. □

Definition 6.2.3. Let (V,ρ) be an $\mathfrak{sl}(2,\mathbb{K})$-module. An element v of V is said to be **primitive** if it is a non-zero eigenvector of $\rho(h)$ and belongs to the kernel of $\rho(e)$.

In other words, a non-zero element v of V is primitive if and only if $\mathbb{K}v$ is stable under the operation of $\mathbb{K}h + \mathbb{K}e$. This follows for example from Lemma 6.2.2.

Example 6.2.4. The element e is primitive of weight 2 for the adjoint representation of $\mathfrak{sl}(2,\mathbb{K})$. The element $(1,0)$ of $\mathbb{K}^2$ is primitive of weight 1 for the identity representation of $\mathfrak{sl}(2,\mathbb{K})$ on $\mathbb{K}^2$.

Lemma 6.2.5. *Let (V,ρ) be a non-zero finite dimensional $\mathfrak{sl}(2,\mathbb{K})$-module. Then V has primitive elements.*

Proof. Since e is a nilpotent element of $\mathfrak{sl}(2,\mathbb{K})$, $\rho(e)$ is nilpotent, say with exponent n. By (6.2.1), we then have

$$n(\rho(h) - n + 1)\rho(e)^{n-1} = [\rho(f), \rho(e)^n] = 0$$

and hence the nonzero elements of $\rho(e)^{n-1}(V)$ are primitive. □

Proposition 6.2.6. *Let (V,ρ) be an $\mathfrak{sl}(2,\mathbb{K})$-module, and $v \in V$ a primitive element of weight λ. For each $n \geq 0$, define*

$$v_n = \frac{1}{n!}\rho(f)^n \cdot v.$$

Then we have (set $v_{-1} = 0$)

$$h \cdot v_n = (\lambda - 2n)v_n, \quad f \cdot v_n = (n+1)v_{n+1}, \quad e \cdot v_n = (\lambda - n + 1)v_{n-1}.$$

Proof. The first formula follows from Lemma 6.2.2, and the second from the definition of the v_n. We prove the third by induction on n. It is satisfied for $n = 0$ since $v_{-1} = 0$. If $n > 0$, then

$$\begin{aligned}
n\rho(e)v_n &= \rho(e)\rho(f)v_{n-1} = [\rho(e),\rho(f)]v_{n-1} + \rho(f)\rho(e)v_{n-1}\\
&= \rho(h)v_{n-1} + \rho(f)(\lambda - n + 2)v_{n-2}\\
&= (\lambda - 2n + 2)v_{n-1} + (\lambda - n + 2)(n-1)v_{n-1}\\
&= n(\lambda - n + 1)v_{n-1}.
\end{aligned}$$

This proves the proposition. □

The integers $n \geq 0$ such that $v_n \neq 0$ constitute an interval in $\mathbb{N}$, and the corresponding elements v_n form a basis over $\mathbb{K}$ of the submodule generated by v (indeed, they are linearly independent because they are non-zero elements of distinct weights). This basis will be said to be **associated** to the primitive element v.

Proposition 6.2.7. *Let (V,ρ) be an $\mathfrak{sl}(2,\mathbb{K})$-module, and $v \in V$ a primitive element of weight λ. Assume that the submodule W of V generated by the primitive element v is finite dimensional.*

(a) The dimension of W is $\lambda+1$, and in particular λ is an integer.

(b) $\{v_0, v_1, \ldots, v_\lambda\}$ is a basis for W and $v_n = 0$ for $n > \lambda$.

(c) *The eigenvalues of $\rho(h)$ on W are $\lambda, \lambda-2, \lambda-4, \dots, -\lambda$, all with multiplicity* 1.

(d) *Every primitive element of W is proportional to v.*

(e) *The commutant of the module W is reduced to the scalars. In particular, W is absolutely simple.*

Proof. Let m be the largest integer such that $v_m \neq 0$. Then $0 = e \cdot v_{m+1} = (\lambda - m)v_m$, so $\lambda = m$. Since $(v_0, v_1, \dots, v_m)$ is a basis of W, this proves (a) and (b). Assertion (c) follows from the equality $h \cdot v_n = (\lambda - 2n)v_n$, and (d) is clear by Proposition 6.2.6.

Let ϕ be an element of the commutant of the module W. Then $\rho(h)\phi(v) = \phi\rho(h)(v) = \lambda\phi(v)$, so there exists $\mu \in \mathbb{K}$ such that $\phi(v) = \mu v$. Then

$$\phi\rho(f)^n(v) = \rho(f)^n\phi(v) = \mu\rho(f)^n(v)$$

for all $n \geq 0$, so $\phi = \mu \cdot 1$, proving (e). □

Corollary 6.2.8. *Let (V, ρ) be a finite dimensional $\mathfrak{sl}(2, \mathbb{K})$-module.*

(a) *The endomorphism $\rho(h)$ is diagonalizable and its eigenvalues are integers.*

(b) *For any $p \in \mathbb{Z}$, let V_p be the eigenspace of h_V corresponding to the eigenvalue p. Let t be a non-negative integer. Then the map $f^t|_{V_p} : V_p \to V_{p-2t}$ is injective for $t \leq p$, bijective for $t = p$, and surjective for $t \geq p$. The map $e^t|_{V_{-p}} : V_{-p} \to V_{-p+2t}$ is injective for $t \leq p$, bijective for $t = p$, and surjective for $t \geq p$.*

(c) *The length of V is equal to $\dim(\ker \rho(e))$ and to $\dim(\ker \rho(f))$.*

(d) *Let V_e (resp. V_o) be the sum of the V^μ for μ even (resp. odd). Then V_e (resp. V_o) is the sum of the simple submodules of V of odd (resp. even) dimension, and $V = V_e \oplus V_o$. The length of V_e is $\dim(V^0)$, and that of V_o is $\dim(V^1)$.*

(e) $\ker \rho(e) \cap \operatorname{im} \rho(e) \subseteq \bigoplus_{\mu>0} V^\mu$ *and* $\ker \rho(f) \cap \operatorname{im} \rho(f) \subseteq \bigoplus_{\mu<0} V^\mu$.

Proof. If V is simple, then V is generated by a primitive element, and it suffices to apply Proposition 6.2.6 and Proposition 6.2.7. The general case follows since every finite dimensional $\mathfrak{sl}(2, \mathbb{K})$-module is semi-simple. □

Let (u, v) be the canonical basis of $\mathbb{K}^2$. For the identity representation of $\mathfrak{sl}(2, \mathbb{K})$, we have

$$\begin{aligned} e(u) &= 0, \quad h(u) = 1, \quad f(u) = v \\ e(v) &= u, \quad h(v) = -v, \quad f(v) = 0. \end{aligned}$$

Consider the symmetric algebra $\mathcal{S}(\mathbb{K}^2)$ of $\mathbb{K}^2$. The elements of $\mathfrak{sl}(2, \mathbb{K})$ extend uniquely to derivations of $\mathcal{S}(\mathbb{K}^2)$, giving $\mathcal{S}(\mathbb{K}^2)$ the structure of an $\mathfrak{sl}(2, \mathbb{K})$-module. Let $V(\lambda)$ be the set of homogeneous elements of $\mathcal{S}(\mathbb{K}^2)$ of degree λ. If μ, λ are integers such that $0 \leq \mu \leq \lambda$, put

$$\omega_\mu^\lambda = \binom{\lambda}{\mu} u^{\lambda-\mu} v^\mu \in V(\lambda).$$

Proposition 6.2.9. *For any integer $\lambda \geq 0$, $V(\lambda)$ is a simple $\mathfrak{sl}(2, \mathbb{K})$-module. In this module, $\omega_0^\lambda = u^\lambda$ is primitive of weight λ.*

Proof. We have $e(u^\lambda) = 0$ and $h(u^\lambda) = \lambda u^\lambda$, so u^λ is primitive of weight λ. The submodule of $V(\lambda)$ generated by u^λ is of dimension $\lambda + 1$ by Proposition 6.2.7, and so is equal to $V(\lambda)$. By Proposition 6.2.7, $V(\lambda)$ is absolutely simple. □

Theorem 6.2.10 (Classification of Simple $\mathfrak{sl}(2, \mathbb{K})$-Modules). *For each $n \in \mathbb{N}$, there exists a simple $\mathfrak{sl}(2, \mathbb{K})$-module of dimension n which is unique up to isomorphism. Moreover, every finite dimensional simple $\mathfrak{sl}(2, \mathbb{K})$-module is isomorphic to some $V(\lambda)$ with $\lambda \in \mathbb{N}$.*

Proof. This follows from Lemma 6.2.5 and Propositions 6.2.6, 6.2.7 and 6.2.9. □

Remark 6.2.11. Let β be the bilinear form on $V(\lambda)$ such that

$$\beta(\omega_\mu^\lambda, \omega_\nu^\lambda) = \begin{cases} 0 & \text{if } \mu + \nu \neq \lambda, \\ (-1)^\mu \binom{\lambda}{\mu} & \text{if } \mu + \nu = \lambda. \end{cases}$$

It is now easy to check that β is invariant, and that β is symmetric for m even, and alternating for m odd.

Example 6.2.12. The adjoint representation of $\mathfrak{sl}(2,\mathbb{K})$ defines on $\mathfrak{sl}(2,\mathbb{K})$ the structure of a simple $\mathfrak{sl}(2,\mathbb{K})$-module. This module is isomorphic to $V^{(2)}$ by an isomorphism that takes u^2 to e, $2uv$ to $-h$, and v^2 to $-f$.

Example 6.2.13. Let $V = \mathbb{K}[X,Y]$ be the vector space of polynomials in two variables. For each $\lambda > 0$ we let V^λ be the vector subspace of all homogeneous polynomials of degree λ. This has a basis given by the monomials $\binom{\lambda}{\mu} X^\mu Y^{\lambda-\mu}$ for $0 \le \mu \le \lambda$. We turn this vector subspace into a module for $\mathfrak{sl}(2,\mathbb{K})$ by defining a Lie algebra homomorphism $\varphi : \mathfrak{sl}(2,\mathbb{K}) \to \mathfrak{gl}(V^\lambda)$ in the following way

$$\varphi(e) = X\frac{\partial}{\partial Y}, \quad \varphi(f) = Y\frac{\partial}{\partial X}, \quad \varphi(h) = X\frac{\partial}{\partial X} - Y\frac{\partial}{\partial Y}.$$

Note that this construction in fact coincides with that of $V(\lambda)$, by the definition of the induced representation of $\mathfrak{sl}(2,\mathbb{K})$ on $\mathcal{S}(\mathbb{K}^2)$.

Proposition 6.2.14. *Let V be a finite dimensional $\mathfrak{sl}(2,K)$-module, λ a postive integer, P_λ the set of primitive elements of weight λ. Then the map*

$$\epsilon : \mathrm{Hom}_{\mathfrak{sl}(2,\mathbb{K})}(V(\lambda), V) \to V, \quad \phi \mapsto \phi(u^\lambda)$$

is linear, injective, and its image is $P_\lambda \cup \{0\}$.

Proof. This map is clearly linear, and it is injective because u^λ generates the $\mathfrak{sl}(2,\mathbb{K})$-module $V(\lambda)$. If $\phi \in \mathrm{Hom}(V(\lambda), V)$,

$$e \cdot (\phi(u^\lambda)) = \phi(e(u^\lambda)) = 0, \quad h \cdot (\phi(u^\lambda)) = \phi(h(u^\lambda)) = \lambda\phi(u^\lambda)$$

so $\phi(u^\lambda) \in P_\lambda \cup \{0\}$. Let $v \in P_\lambda$ and W the submodule of V generated by v. By Proposition 6.2.6, there exists an isomorphism from the module $V(\lambda)$ to the module W that takes u^λ to v. Thus the image of ϵ is $P_\lambda \cup \{0\}$. □

Corollary 6.2.15. *The multiplicity of $V(\lambda)$ in V is* $\dim(P_\lambda \cup \{0\})$.

Proof. This follows because $\dim(\mathrm{Hom}_{\mathfrak{sl}(2,\mathbb{K})}(V(\lambda), V))$ is the multiplicity of $V(\lambda)$ in V. □

Proposition 6.2.16 (String Property for $\mathfrak{sl}(2,\mathbb{K})$-modules). *Let V be a finite-dimensional representation of $\mathfrak{sl}(2,\mathbb{K})$. If $\alpha, \alpha + 2k \in \mathcal{P}_V$ for some $k \in \mathbb{N}$, then $\alpha + 2j \in \mathcal{P}_V$ for $j = 0, \dots, 2k$.*

Proof. We may assume that $\beta := \alpha + 2k$ satisfies $|\beta| \ge |\alpha|$. Then we pick some simple submodule $V_i \cong V^{(\lambda_i)}$ of V such that β is an eigenvalue of $\rho(h)|_{V_i}$. Then $\lambda_i - \beta \in 2\mathbb{N}$ and all integers in $[-|\beta|, |\beta|] \cap (\beta + 2\mathbb{Z})$ are eigenvalues of $\rho(h)|_{V_i}$. This contains in particular the set of all integers of the form $\alpha + 2j$, $j = 0, 1, \dots, k$, between α and β. □

To conclude this paragraph, we prove an important property for $\mathfrak{sl}(2,\mathbb{K})$-modules. We first consider the element $\theta \in \mathrm{Aut}(\mathfrak{sl}(2,\mathbb{K}))$ defined by

$$\theta = \exp(\mathrm{ad}(e)) \exp(-\mathrm{ad}(f)) \exp(\mathrm{ad}(e)) = \mathrm{Ad}(\vartheta) = \mathrm{Inn}(\theta)$$

where ϑ is the element

$$\vartheta = \exp(e)\exp(-f)\exp(e) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathrm{SL}(2,\mathbb{K}).$$

A calculation shows that θ maps (e,h,f) to $(-f,-h,-e)$, so it is the canonical involution on $\mathfrak{sl}(2,\mathbb{K})$. Further, we recall that, since $\mathrm{ad}(e)$ and $\mathrm{ad}(f)$ are nilpotent automorphims of $\mathfrak{g}$, by Theorem 1.6.21 the elements e, f are nilpotent elements in $\mathfrak{sl}(2,\mathbb{K})$ (the adjoint representation is faithful since $\mathfrak{sl}(2,\mathbb{K})$ is semi-simple). Therefore, by Corollary 1.6.18, if (V,ρ) is a finite dimensional representation of $\mathfrak{sl}(2,\mathbb{K})$ then $\rho(e)$ and $\rho(f)$ are nilpotent automorphims in $\mathrm{GL}(V)$, so we can consider

$$\theta_V := \exp(\rho(e)) \exp(-\rho(f)) \exp(\rho(e)) \in \mathrm{GL}(V).$$

Proposition 6.2.17. *Let (V, ρ) be a finite-dimensional representation of $\mathfrak{sl}_2(\mathbb{K})$ and define θ_V as above.*

(a) For each $z \in \mathfrak{sl}(2, \mathbb{K})$, we have

$$\theta_V \rho(z) \theta_V^{-1} = \rho(\vartheta z \vartheta^{-1}).$$

(b) For each integer μ we have $\theta_V(V^\mu) = V^{-\mu}$. In particular, $\dim(V^\mu) = \dim(V^{-\mu})$.

Proof. For $z \in \mathfrak{sl}(2, \mathbb{K})$, we obtain from $\exp \circ \operatorname{ad} = \operatorname{Ad} \circ \exp$ that

$$\begin{aligned} \operatorname{Ad}(\theta_V) &= \exp(\operatorname{ad}(\rho(e))) \exp(\operatorname{ad}(\rho(-f))) \exp(\operatorname{ad}(\rho(e))) \\ &= \rho \circ \exp(\operatorname{ad}(e)) \circ \exp(-\operatorname{ad}(f)) \circ \exp(\operatorname{ad}(e)) \\ &= \rho \circ \theta = \rho \circ \operatorname{Ad}(\vartheta) = \rho \circ \operatorname{Inn}(\theta). \end{aligned}$$

This implies the second claim because for $v \in V_\mu$, we have

$$\rho(h)(\theta_V(v)) = \theta_V(\theta_V^{-1} \rho(h) \theta_V)(v) = \theta_V \rho(\vartheta h \vartheta^{-1})(v) = \theta_V \rho(-h)(v) = -\mu \theta_V(v).$$

Simialrly, for $w \in V_{-\mu}$ we have

$$\rho(h)(\theta_V^{-1}(w)) = \theta_V^{-1}(\theta_V \rho(h) \theta_V^{-1})(w) = \theta_V^{-1} \rho(\vartheta h \vartheta^{-1})(w) = \theta_V^{-1} \rho(-h)(w) = \mu \theta_V^{-1}(w).$$

This completes the proof. □

6.2.2 Root system of a split semi-simple Lie algebra

Let $\mathfrak{g}$ be a semi-simple Lie algebra. A Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is called **splitting** if, for all $x \in \mathfrak{h}$, $\operatorname{ad}_{\mathfrak{g}}(x)$ is triangularizable. A semi-simple Lie algebra is called **splittable** if it has a splitting Cartan subalgebra. A **split semi-simple Lie algebra** is a pair $(\mathfrak{g}, \mathfrak{h})$ where $\mathfrak{g}$ is a semi-simple Lie algebra and $\mathfrak{h}$ is a splitting Cartan subalgebra of $\mathfrak{g}$.

Example 6.2.18. Let $\mathfrak{g}$ be a semi-simple Lie algebra, $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$. For all $x \in \mathfrak{h}$, $\operatorname{ad}_{\mathfrak{g}}(x)$ is semi-simple. Thus, to say that $\mathfrak{h}$ is splitting means that $\operatorname{ad}_{\mathfrak{g}}(x)$ is diagonalizable for all $x \in \mathfrak{h}$. In particular, if $\mathbb{K}$ is algebraically closed, every semi-simple Lie algebra $\mathfrak{g}$ is splittable, and every Cartan subalgebra of $\mathfrak{g}$ is splitting. When $\mathbb{K}$ is not algebraically closed, there exist non-splittable semi-simple Lie algebras. Moreover, if $\mathfrak{g}$ is splittable, there may exist Cartan subalgebras of $\mathfrak{g}$ that are not splitting.

Remark 6.2.19. Let $\mathfrak{g}$ be a reductive Lie algebra. Then $\mathfrak{g} = \mathfrak{z} \times \mathfrak{s}$ where $\mathfrak{z}$ is the centre of $\mathfrak{g}$ and $\mathfrak{s} = [\mathfrak{g}, \mathfrak{g}]$ is semi-simple. The Cartan subalgebras of $\mathfrak{g}$ are the subalgebras of the form $\mathfrak{h} = \mathfrak{c} \times \mathfrak{t}$ where $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{s}$ (Proposition 6.1.36). Then $\mathfrak{h}$ is called splitting if $\mathfrak{t}$ is splitting relative to $\mathfrak{s}$. This leads in an obvious way to the definition of splittable or split reductive algebras.

From now on, we write $(\mathfrak{g}, \mathfrak{h})$ for a split semi-simple Lie algebra. Since $\mathfrak{g}$ is semi-simple, we have $\mathfrak{g}^\lambda(\mathfrak{h}) = \mathfrak{g}_\lambda(\mathfrak{h})$ for all $\lambda \in \mathfrak{h}^*$, which we simply denote by $\mathfrak{g}^\lambda$. Then the weights of $\mathfrak{h}$ on $\mathfrak{g}$ are the linear forms λ on $\mathfrak{h}$ such that $\mathfrak{g}^\lambda \neq 0$. A **root** of $(\mathfrak{g}, \mathfrak{h})$ is defined to be a non-zero weight of $\mathfrak{h}$ on $\mathfrak{g}$. Denote by $\Phi(\mathfrak{g}, \mathfrak{h})$, or simply by Φ, the set of roots of $(\mathfrak{g}, \mathfrak{h})$. We have

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\lambda \in \Phi} \mathfrak{g}^\lambda.$$

Proposition 6.2.20. *Let α, β be roots of $(\mathfrak{g}, \mathfrak{h})$ and let $\langle \cdot, \cdot \rangle$ be a non-degenerate invariant symmetric bilinear form on $\mathfrak{g}$ (for example the Killing form of $\mathfrak{g}$).*

(a) If $\alpha + \beta \neq 0$, $\mathfrak{g}^\alpha$ and $\mathfrak{g}^\beta$ are orthogonal. The restriction of $\langle \cdot, \cdot \rangle$ to $\mathfrak{g}^\alpha \times \mathfrak{g}^{-\alpha}$ is non-degenerate. The restriction of $\langle \cdot, \cdot \rangle$ to $\mathfrak{h}$ is non-degenerate.

(b) Let $x \in \mathfrak{g}^\alpha$, $y \in \mathfrak{g}^{-\alpha}$ and $h \in \mathfrak{h}$. Then $[x, y] \in \mathfrak{h}$ and $\langle h, [x, y] \rangle = \alpha(h) \langle x, y \rangle$.

Proof. Assertion (a) is a special case of Proposition 6.1.15. If $x \in \mathfrak{g}^\alpha$, $y \in \mathfrak{g}^{-\alpha}$ and $h \in \mathfrak{h}$, we have $[x, y] \in \mathfrak{g}^{\alpha - \alpha} = \mathfrak{h}$, and

$$\langle h, [x, y] \rangle = \langle [h, x], y \rangle = \langle \alpha(h) x, y \rangle = \alpha(h) \langle x, y \rangle.$$

This completes the proof. □

Theorem 6.2.21. *Let α be a root of $(\mathfrak{g}, \mathfrak{h})$ and $\langle \cdot, \cdot \rangle$ be a non-degenerate invariant symmetric bilinear form on $\mathfrak{g}$*

(a) *The vector space $\mathfrak{g}^\alpha$ is of dimension 1 and $\mathbb{Z}\alpha \cap \Phi = \{\pm\alpha\}$.*

(b) *The vector subspace $\mathfrak{h}_\alpha = [\mathfrak{g}^\alpha, \mathfrak{g}^{-\alpha}]$ of $\mathfrak{h}$ is of dimension 1. It contains a unique element h_α such that $\alpha(h_\alpha) = 2$.*

(c) *The vector subspace $\mathfrak{s}_\alpha = \mathfrak{h}_\alpha + \mathfrak{g}^\alpha + \mathfrak{g}^{-\alpha}$ is a Lie subalgebra of $\mathfrak{g}$.*

(d) *If e_α is a non-zero element of $\mathfrak{g}^\alpha$, there exists a unique $f_\alpha \in \mathfrak{g}^{-\alpha}$ such that $[e_\alpha, f_\alpha] = h_\alpha$. Let π be the linear map from $\mathfrak{sl}(2, \mathbb{K})$ to $\mathfrak{g}$ that takes (e, h, f) to $(e_\alpha, h_\alpha, f_\alpha)$. Then π is an isomorphism from the Lie algebra $\mathfrak{sl}(2, \mathbb{K})$ to the Lie algebra $\mathfrak{s}_\alpha$.*

Proof. Let t_α be the unique element of $\mathfrak{h}$ such that $\alpha(h) = \langle t_\alpha, h \rangle$ for all $h \in \mathfrak{h}$. By Proposition 6.2.20, $[x, y] = \langle x, y \rangle t_\alpha$ for all $x \in \mathfrak{g}^\alpha, y \in \mathfrak{g}^{-\alpha}$. On the other hand $\langle \mathfrak{g}^\alpha, \mathfrak{g}^{-\alpha} \rangle \neq 0$. Hence $\mathfrak{h}_\alpha = [\mathfrak{g}^\alpha, \mathfrak{g}^{-\alpha}] = \mathbb{K}t_\alpha$.

Choose $x \in \mathfrak{g}^\alpha$, $y \in \mathfrak{g}^{-\alpha}$ such that $\langle x, y \rangle = 1$, so $[x, y] = t_\alpha$. Recall that $[t_\alpha, x] = \alpha(t_\alpha)x$ and $[t_\alpha, y] = -\alpha(t_\alpha)y$. If $\alpha(t_\alpha) = 0$, it follows that $\mathbb{K}x + \mathbb{K}y + \mathbb{K}t_\alpha$ is a nilpotent subalgebra $\mathfrak{t}$ of $\mathfrak{g}$. Since $t_\alpha \in [\mathfrak{t}, \mathfrak{t}]$, $\mathrm{ad}_\mathfrak{g}(t_\alpha)$ is nilpotent, which is absurd since $\mathrm{ad}_\mathfrak{g}(t_\alpha)$ is non-zero semi-simple. So $\alpha(t_\alpha) \neq 0$. Hence there exists a unique $h_\alpha \in \mathfrak{h}_\alpha$ such that $\alpha(h_\alpha) = 2$, which proves (b).

Choose a non-zero element e_α of $\mathfrak{g}^\alpha$. There exists $f_\alpha \in \mathfrak{g}^{-\alpha}$ such that $[e_\alpha, f_\alpha] = h_\alpha$ (since $[e_\alpha, \mathfrak{g}^{-\alpha}] = \mathfrak{h}_\alpha$ by (b)). Then

$$[h_\alpha, e_\alpha] = \alpha(h_\alpha)e_\alpha = 2e_\alpha, \quad [h_\alpha, f_\alpha] = \alpha(h_\alpha)f_\alpha = -2f_\alpha$$

hence $\mathbb{K}e_\alpha + \mathbb{K}f_\alpha + \mathbb{K}h_\alpha$ is a subalgebra of $\mathfrak{g}$ and the linear map π from $\mathfrak{sl}(2, \mathbb{K})$ to $\mathbb{K}e_\alpha + \mathbb{K}f_\alpha + \mathbb{K}h_\alpha$ is an isomorphism of Lie algebras.

Now we consider the subspace

$$V = \mathbb{K}f_\alpha + \mathfrak{h} + \bigoplus_{n=1}^{\infty} \mathfrak{g}^{n\alpha}$$

of $\mathfrak{g}$. Since V is invariant under $\mathrm{ad}(e_\alpha)$ and $\mathrm{ad}(f_\alpha)$, it is invariant under $\mathfrak{s}$. According to Proposition 6.2.17, we then have

$$\dim V_n(\mathrm{ad}(h_\alpha)) = \dim V_{-n}(\mathrm{ad}(h_\alpha))$$

for all $n \in \mathbb{Z}$. Since $\alpha(h_\alpha) = 2$ and $[h_\alpha, \mathfrak{h}] = \{0\}$, this leads to

$$\dim(\mathfrak{g}^\alpha) = \dim V_2(\mathrm{ad}(h_\alpha)) = \dim V_{-2}(\mathrm{ad}(h_\alpha)) = 1$$

and $\dim(\mathfrak{g}^{n\alpha}) = 0$ for $n > 1$. This proves (a).

If e_α is a non-zero element of $\mathfrak{g}^\alpha$, the element f_α constructed is the unique element of $\mathfrak{g}^{-\alpha}$ such that $[e_\alpha, f_\alpha] = h_\alpha$ since $\dim(\mathfrak{g}^\alpha) = 1$. The last assertion of (d) therefore follows. □

The notations $\mathfrak{h}_\alpha$, h_α, $\mathfrak{s}_\alpha$ will be retained in what follows, and the triple $(e_\alpha, h_\alpha, f_\alpha)$ is often refered as a $\mathfrak{sl}(2, \mathbb{K})$-triple. (To define h_α, we take $\langle \cdot, \cdot \rangle$ to be the Killing form.) If e_α is a non-zero element of $\mathfrak{g}^\alpha$, the isomorphism π of Theorem 6.2.21 and the representation $x \mapsto \mathrm{ad}_\mathfrak{g}(\pi(x))$ of $\mathfrak{sl}(2, \mathbb{K})$ on $\mathfrak{g}$ will be said to be **associated** to e_α. We also note that, since $\alpha(h_\alpha) = 2$, the coroot of α is exactly h_α, and therefore $\beta(h_\alpha) = n(\beta, \alpha) = \langle \beta, \check{\alpha} \rangle$.

Corollary 6.2.22. *Let κ be the Killing form of $\mathfrak{g}$. For all $x, y \in \mathfrak{h}$,*

$$\kappa(x, y) = \sum_{\gamma \in \Phi} \gamma(x)\gamma(y).$$

Proof. Indeed, $\mathrm{ad}(x)\mathrm{ad}(y)$ leaves each $\mathfrak{g}^\gamma$ stable, and its restriction to $\mathfrak{g}^\gamma$ is the homothety with ratio $\gamma(x)\gamma(y)$. □

Proposition 6.2.23 (The Root String Lemma). *Let $\alpha, \beta \in \Phi$ with $\beta \neq \pm\alpha$.*

(a) *The set $\{k \in \mathbb{Z} : \beta + k\alpha \in \Phi\}$ is an interval in $\mathbb{Z}$. If it is of the form $[-p, q] \cap \mathbb{Z}$ with $p, q \in \mathbb{Z}$, then $p - q = \beta(h_\alpha)$. In particular, $\beta(h_\alpha) \in \mathbb{Z}$.*

(b) *If $\beta(h_\alpha) < 0$ then $\beta + \alpha \in \Phi$; and if $\beta(h_\alpha) > 0$ then $\beta - \alpha \in \Phi$.*

(c) *We have $[\mathfrak{g}^\alpha, \mathfrak{g}^\beta] = \mathfrak{g}^{\alpha+\beta}$, and $[\mathfrak{g}^\alpha, \mathfrak{g}^\beta] = \{0\}$ implies $\beta(h_\alpha) \geq 0$.*

Proof. We consider the subspace $V = \sum_{k\in\mathbb{Z}} \mathfrak{g}^{\beta+k\alpha}$. Note that $\beta \neq \pm\alpha$ implies that 0 is not contained in $\beta + \mathbb{Z}\alpha$ (since $\mathbb{Z}\alpha \cap \Phi = \{\pm\alpha\}$). From $[\mathfrak{g}^\delta, \mathfrak{g}^\gamma] \subseteq \mathfrak{g}^{\delta+\gamma}$, we see that V is a $\mathfrak{s}_\alpha$-submodule of $\mathfrak{g}$. The eigenvalues of h_α on V are given by

$$\mathcal{P}_V = \{(\beta + k\alpha)(h_\alpha) : \beta + k\alpha \in \Phi\} = \beta(h_\alpha) + 2\{k : \beta + k\alpha \in \Phi\}.$$

Hence the string property of $\mathfrak{sl}_2(\mathbb{K})$-modules implies the string property of the root system.

Next we note that $\beta \in \Phi$ leads to $0 \in [-p, q]$, so $p, q \geq 0$. Recall that $\mathcal{P}_V = -\mathcal{P}_V$, therefore

$$\beta(h_\alpha) - 2p = -(\beta(h_\alpha) + 2q)$$

which implies $\beta(h_\alpha) = p - q$. Now $\beta(h_\alpha) < 0$ means $0 \leq p < q$, so $1 \in [-p, q]$ and $\alpha + \beta \in \Phi$. On the other hand, if $\beta(h_\alpha) > 0$ then $-p < q < p$. Since p can not be zero, it follows that $-1 \in [-p, q]$ and $\alpha - \beta \in \Phi$.

Since $[\mathfrak{g}^\delta, \mathfrak{g}^\gamma] \subseteq \mathfrak{g}^{\delta+\gamma}$ and all multiplicities of the eigenvalues of $\mathrm{ad}(h_\alpha)$ on V are 1, apply Proposition 6.2.6 to a nonzero element of $\mathfrak{g}^{\beta+q\alpha}$ implies that $V \cong V^{(\beta(h_\alpha)+2q)}$. This immediately shows that

$$[\mathfrak{g}^\alpha, \mathfrak{g}^{\beta+k\alpha}] = \mathfrak{g}^{\beta+(k+1)\alpha} \quad \text{for} \quad k = -p, -p+1, \ldots, q-1.$$

Now assume that $[\mathfrak{g}^\alpha, \mathfrak{g}^\beta] = \{0\}$. Then $W = \sum_{k\leq 0} \mathfrak{g}^{\beta+k\alpha}$ is invariant under $\mathfrak{s}_\alpha$ and so a submodule of V. Then $\beta(h_\alpha)$ is the maximal eigenvalue of $\mathrm{ad}(h_\alpha)$ on W, so $\beta(h_\alpha) \geq 0$ by Proposition 6.2.16. $\square$

Corollary 6.2.24. *Each root $\alpha \in \Phi$ satisfies $\mathbb{K}\alpha \cap \Phi = \{\pm\alpha\}$.*

Proof. Suppose that α and $c\alpha$ lie in Φ for some $c \in \mathbb{K}$. By Proposition 6.2.23, $2c = c\alpha(h_\alpha) \in \mathbb{Z}$, so that $c \in 2\mathbb{Z}$. Simialrly we have $c^{-1} \in \frac{1}{2}\mathbb{Z}$, so that $c \in \{\pm 2, \pm 1, \pm\frac{1}{2}\}$. From the $\mathfrak{sl}_2$-Theorem, we know already that $\mathbb{Z}\alpha \cap \Phi = \{\pm\alpha\}$, which rules out the cases $c = \pm 2$. The cases $c = \pm\frac{1}{2}$ are likewise ruled out by applying the same argument to $c\alpha$ instead. $\square$

Let $\alpha \in \Phi$ and $(e_\alpha, h_\alpha, f_\alpha)$ be a $\mathfrak{sl}(2, \mathbb{K})$-triple. Since $\mathrm{ad}(e_\alpha)$ is nilpotent, $\exp(\mathrm{ad}(e_\alpha))$ is an elementary automorphism of $\mathfrak{g}$. Similarly, $\exp(\mathrm{ad}(f_\alpha))$ is an elementary automorphism of $\mathfrak{g}$. As in the case of $\mathfrak{sl}(2, \mathbb{K})$, consider the following element in $\mathrm{Aut}(\mathfrak{g})$ defined by

$$\theta_\alpha = e^{\mathrm{ad}(e_\alpha)} e^{-\mathrm{ad}(f_\alpha)} e^{\mathrm{ad}(e_\alpha)} \in \mathrm{Aut}(\mathfrak{g}).$$

Proposition 6.2.25. *Let $\alpha \in \Phi$ and θ_α be defined as above. Then*

(a) For all $h \in \mathfrak{h}$, $\theta_\alpha(h) = h - \alpha(h)h_\alpha$.

(b) For all $\beta \in \Phi$, $\theta_\alpha(\mathfrak{g}^\beta) = \mathfrak{g}^{\beta-\beta(h_\alpha)\alpha}$.

(c) If $\beta \in \Phi$ then $\beta - \beta(h_\alpha)\alpha \in \Phi$.

Proof. Let $h \in \mathfrak{h}$. If $\alpha(h) = 0$, then $[e_\alpha, h] = [f_\alpha, h] = 0$, so $\theta_\alpha(h) = h$. On the other hand, we have $\theta_\alpha(h_\alpha) = -h_\alpha$. This proves assertion (a) and it follows that $\theta_\alpha^2|_\mathfrak{h} = 1$. If $x \in \mathfrak{g}^\beta$ and $h \in \mathfrak{h}$,

$$\begin{aligned}
[h, \theta_\alpha(x)] &= [\theta_\alpha^2(h), \theta_\alpha(x)] = \theta_\alpha([\theta_\alpha(h), x]) \\
&= \theta_\alpha([h, x]) - \alpha(h)\theta_\alpha([h_\alpha, x]) \\
&= [\beta(h) - \alpha(h)\beta(h_\alpha)]\theta_\alpha(x)
\end{aligned}$$

so $\theta_\alpha(x) \in \mathfrak{g}^{\beta-\beta(h_\alpha)\alpha}$. This proves (b), and assertion (c) follows from (b). $\square$

Theorem 6.2.26. *Let $(\mathfrak{g}, \mathfrak{h})$ be a split semi-simple Lie algebra and Φ the set of its roots.*

(a) The set $\Phi = \Phi(\mathfrak{g}, \mathfrak{h})$ is a reduced root system in $\mathfrak{h}^*$.

(b) For each $\alpha \in \Phi$, the map $s_\alpha : \lambda \mapsto \lambda - \lambda(h_\alpha)\alpha$ from $\mathfrak{h}^*$ to $\mathfrak{h}^*$ is the unique reflection of $\mathfrak{h}^*$ such that $s_\alpha(\alpha) = -\alpha$ and $s_\alpha(\Phi) = \Phi$. Moreover, s_α is the transpose of $\theta_\alpha|_\mathfrak{h}$.

Proof. First, Φ generates $\mathfrak{h}^*$, for if $h \in \mathfrak{h}$ is such that $\alpha(h) = 0$ for all $\alpha \in \Phi$, then $\mathrm{ad}(h) = 0$ and hence $\mathfrak{h} = 0$ since the centre of $\mathfrak{g}$ is zero. By definition, $0 \notin \Phi$. Let $\alpha \in \Phi$. Since $\alpha(h_\alpha) = 2$, we see s_α is a reflection such that $s(\alpha) = -\alpha$. Then $s_\alpha(\Phi) = \Phi$ by Proposition 6.2.25, and $\beta(h_\alpha) \in \mathbb{Z}$ for all $\beta \in \Phi$. This shows that Φ is a root system in $\mathfrak{h}^*$, and Φ is reduced by Corollary 6.2.24. For all $h \in \mathfrak{h}$ and $\lambda \in \mathfrak{h}^*$,

$$\langle s_\alpha(\lambda), h\rangle = \langle \lambda - \lambda(h_\alpha)\alpha, h\rangle = \lambda(h) - \lambda(h_\alpha)\alpha(h) = \langle \lambda, h - \alpha(h)h_\alpha\rangle = \langle \lambda, \theta_\alpha(h)\rangle$$

so s_α is the transpose of $\theta_\alpha|_\mathfrak{h}$. This completes the proof. $\square$

Identify $\mathfrak{h}$ canonically with $\mathfrak{h}^{**}$. We then have, by Theorem 6.2.26, $h_\alpha = \alpha^\vee$ for all $\alpha \in \Phi$. The h_α thus form the root system $\Phi^\vee$ in $\mathfrak{h}$ inverse to Φ. We shall call $\Phi(\mathfrak{g}, \mathfrak{h})$ the **root system** of $(\mathfrak{g}, \mathfrak{h})$. The Weyl group, group of weights, Coxeter number of $\Phi(\mathfrak{g}, \mathfrak{h})$ are called the Weyl group, group of weights, Coxeter number of $(\mathfrak{g}, \mathfrak{h})$. As before, we consider the Weyl group as operating not only on $\mathfrak{h}^*$, but also on $\mathfrak{h}$ by transport of structure, so that $s_\alpha = \theta_\alpha|_{\mathfrak{h}}$. Since the θ_α are elementary automorphisms of $\mathfrak{g}$, we have:

Corollary 6.2.27. *Every element of the Weyl group of $(\mathfrak{g}, \mathfrak{h})$, operating on $\mathfrak{h}$, is the restriction to $\mathfrak{h}$ of an elementary automorphism of $\mathfrak{g}$.*

Corollary 6.2.28. *Let $(\mathfrak{g}, \mathfrak{h})$ be a split semi-simple Lie algebra, β an invariant symmetric bilinear form on $\mathfrak{g}$, and W the Weyl group of $(\mathfrak{g}, \mathfrak{h})$. Then the restriction $\tilde{\beta}$ of β to $\mathfrak{h}$ is invariant under W.*

Proof. Let $\alpha \in \Phi$, π the associated representation of $\mathfrak{sl}(2, \mathbb{K})$ on $\mathfrak{g}$. Then β is invariant under π. In particular, $\tilde{\beta}$ is invariant under $\theta_\alpha|_{\mathfrak{h}}$, and hence under W. □

Remark 6.2.29. If $\mathfrak{h}_{\mathbb{Q}}$ (resp. $\mathfrak{h}^*_{\mathbb{Q}}$) denotes the $\mathbb{Q}$-vector subspace of $\mathfrak{h}$ (resp. $\mathfrak{h}^*$) generated by the h_α (resp. the α), where $\alpha \in \Phi$, then $\mathfrak{h}$ (resp. $\mathfrak{h}^*$) can be identified canonically with $\mathfrak{h}_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{K}$ (resp. with $\mathfrak{h}^* \otimes_{\mathbb{Q}} \mathbb{K}$) and $\mathfrak{h}^*_{\mathbb{Q}}$ can be identified with the dual of $\mathfrak{h}_{\mathbb{Q}}$. We call $\mathfrak{h}_{\mathbb{Q}}$ and $\mathfrak{h}^*_{\mathbb{Q}}$ the canonical $\mathbb{Q}$-structures on $\mathfrak{h}$ and $\mathfrak{h}^*$. When we mention $\mathbb{Q}$-rationality for a vector subspace of $\mathfrak{h}$, for a linear form on $\mathfrak{h}$, etc., we shall mean these structures, unless we indicate otherwise. When we mention Weyl chambers, or facets, of $\Phi(\mathfrak{g}, \mathfrak{h})$, we shall work in $\mathfrak{h}_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{R}$ or $\mathfrak{h}^* \otimes_{\mathbb{Q}} \mathbb{R}$, that we shall denote by $\mathfrak{h}_{\mathbb{R}}$ and $\mathfrak{h}^*_{\mathbb{R}}$.

Remark 6.2.30. The root system $\check{\Phi}$ in $\mathfrak{h}$ defines a non-degenerate symmetric bilinear form β on $\mathfrak{h}$, namely the form $(x, y) \mapsto \sum_{\alpha \in \Phi} \alpha(x)\alpha(y)$. By Corollary 6.2.22, this form is just the restriction of the Killing form to $\mathfrak{h}$. The extension of $\beta|_{\mathfrak{h}_{\mathbb{Q}} \times \mathfrak{h}_{\mathbb{Q}}}$ to $\mathfrak{h}_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{R}$ is positive nondegenerate. On the other hand, we see that the inverse form on $\mathfrak{h}^*$ of the restriction to $\mathfrak{h}$ of the Killing form on $\mathfrak{g}$ is the canonical bilinear $\langle \cdot, \cdot \rangle_\Phi$ form of Φ.

Proposition 6.2.31. *Let $(\mathfrak{g}, \mathfrak{h})$ be a split semi-simple Lie algebra, β an invariant symmetric bilinear form on $\mathfrak{g}$, and W the Weyl group of $(\mathfrak{g}, \mathfrak{h})$. Then the restriction $\tilde{\beta}$ of β to $\mathfrak{h}$ is invariant under W. Moreover, if β is non-degenerate, so is $\tilde{\beta}$.*

Proof. Let $\alpha \in \Phi$, let x_α be a non-zero element of $\mathfrak{g}^\alpha$, π the associated representation of $\mathfrak{sl}(2, \mathbb{K})$ on $\mathfrak{g}$, and Π the representation of $\mathrm{SL}(2, \mathbb{K})$ on $\mathfrak{g}$ compatible with Π. Then β is invariant under π, and hence under Π. In particular, $\tilde{\beta}$ is invariant under $\theta_\alpha(t)|_{\mathfrak{h}}$, and hence under W. The last assertion follows from Proposition 6.2.20(a). □

Proposition 6.2.32. *Let $(\mathfrak{g}, \mathfrak{h})$ be a split semi-simple Lie algebra, $\langle \cdot, \cdot \rangle$ a nondegenerate invariant symmetric bilinear form on $\mathfrak{g}$. For all $\alpha \in \Phi$, let x_α be a non-zero element of $\mathfrak{g}_\alpha$. Let $(h_i)_{i \in I}$ be a basis of $\mathfrak{h}$, and $(h'_i)_{i \in I}$ the basis of $\mathfrak{h}$ such that $\langle h_i, h'_j \rangle = \delta_{ij}$. The Casimir element associated to $\langle \cdot, \cdot \rangle$ in the enveloping algebra of $\mathfrak{g}$ is then*

$$\Omega = \sum_{\alpha \in \Phi} \frac{1}{\langle x_\alpha, x_{-\alpha} \rangle} x_\alpha x_{-\alpha} + \sum_{i \in I} h_i h'_i.$$

Proof. Indeed, by Proposition 6.2.20(a), $\langle h_i, x_\alpha \rangle = \langle h'_i, x_\alpha \rangle = 0$ for all $i \in I$, $\alpha \in \Phi$, and

$$\langle \frac{1}{\langle x_\alpha, x_{-\alpha} \rangle} x_\alpha, x_\beta \rangle = \delta_{\alpha\beta}$$

for all $\alpha, \beta \in \Phi$. Thus the claim follows. □

Let $(\mathfrak{g}_1, \mathfrak{h}_1)$, $(\mathfrak{g}_2, \mathfrak{h}_2)$ be split semi-simple Lie algebras, φ an isomorphism from $\mathfrak{g}_1$ to $\mathfrak{g}_2$ such that $\varphi(\mathfrak{h}_1) = \mathfrak{h}_2$. By transport of structure, the transpose of the map $\varphi|_{\mathfrak{h}_1}$ takes $\Phi(\mathfrak{g}_2, \mathfrak{h}_2)$ to $\Phi(\mathfrak{g}_1, \mathfrak{h}_1)$.

Proposition 6.2.33. *Let $\mathfrak{g}$ be a semi-simple Lie algebra, $\mathfrak{h}_1$ and $\mathfrak{h}_2$ splitting Cartan subalgebras of $\mathfrak{g}$. There exists an isomorphism from $\mathfrak{h}_1^*$ to $\mathfrak{h}_2^*$ that takes $\Phi(\mathfrak{g}, \mathfrak{h}_1)$ to $\Phi(\mathfrak{g}, \mathfrak{h}_2)$.*

Proof. Let $\mathbb{K}'$ be an algebraic closure of $\mathbb{K}$, and set $\tilde{\mathfrak{g}} = \mathfrak{g} \otimes_{\mathbb{K}} \mathbb{K}'$, $\tilde{h}_i = \mathfrak{h}_i \otimes_{\mathbb{K}} \mathbb{K}'$. Then $\Phi(\tilde{\mathfrak{g}}, \tilde{\mathfrak{h}}_i)$ is the image of $\Phi(\mathfrak{g}, \mathfrak{h}_i)$ under the map $\lambda \mapsto \lambda \otimes 1$ from $\mathfrak{h}_i^*$ to $\mathfrak{h}_i^* \otimes_{\mathbb{K}} \mathbb{K}' = \tilde{\mathfrak{h}}_i^*$. By Theorem 6.1.61, there exists an automorphism of $\tilde{\mathfrak{g}}$ taking $\tilde{\mathfrak{h}}_1$ to $\tilde{\mathfrak{h}}_2$, hence an isomorphism φ from $\tilde{\mathfrak{h}}_1^*$ to $\tilde{\mathfrak{h}}_2^*$ that takes $\Phi(\tilde{\mathfrak{g}}, \tilde{\mathfrak{h}}_1)$ to $\Phi(\tilde{\mathfrak{g}}, \tilde{\mathfrak{h}}_2)$. Then $\varphi|_{\mathfrak{h}_1^*}$ takes $\Phi(\mathfrak{g}, \mathfrak{h}_1)$ to $\Phi(\mathfrak{g}, \mathfrak{h}_2)$, and hence $\mathfrak{h}_1^*$ to $\mathfrak{h}_2^*$. □

Let $(\mathfrak{g}, \mathfrak{h})$ be a split semi-simple Lie algebra and Φ be its roots. We have seen that each $\mathfrak{g}^\alpha$ is one-dimensional, and for each $\alpha \in \Phi$ there exist a $\mathfrak{sl}(2, \mathbb{K})$-triple. Moreover, for $\alpha, \beta \in \Phi$ we have

$$[\mathfrak{g}^\alpha, \mathfrak{g}^\beta] = \mathfrak{g}^{\alpha+\beta}.$$

Now we consider the problem of choosing a nice basis for $\mathfrak{g}$ with respect to this root space decomposition. That is, we want to extracta a family $(x_\alpha)_{\alpha\in\Phi}$ of elements of $\mathfrak{g}$ with $x_\alpha \in \mathfrak{g}^\alpha$ for each $\alpha \in \Phi$ with nice properties. For this, we need some definitions. A family $(x_\alpha)_{\alpha\in\Phi}$ of elements of $\mathfrak{g}$ with $x_\alpha \in \mathfrak{g}^\alpha$ is called **distinguished** if for each $\alpha \in \Phi$ we have $[x_\alpha, x_{-\alpha}] = h_\alpha$. The existence of such families follows from Theorem 6.2.21. In fact, let Δ be a basis for $(\mathfrak{g}, \mathfrak{h})$. Then every family $(x_\alpha)_{\alpha\in\Phi^+}$ of nonzero elements in $\mathfrak{g}$, can be extended into a distinguished family $(x_\alpha)_{\alpha\in\Phi}$.

Proposition 6.2.34. *Let $\langle\cdot,\cdot\rangle$ a non-degenerate invariant symmetric bilinear form on $\mathfrak{g}$. Let $(x_\alpha)_{\alpha\in\Phi}$ be a distinguished family. Then for all $\alpha \in \Phi$, we have*

$$\langle x_\alpha, x_{-\alpha}\rangle = \frac{1}{2}\langle h_\alpha, h_\alpha\rangle.$$

Proof. Indeed, since $\langle\cdot,\cdot\rangle$ is invariant, we have

$$2\langle x_\alpha, x_{-\alpha}\rangle = \langle \alpha(h_\alpha)x_\alpha, x_{-\alpha}\rangle = \langle [h_\alpha, x_\alpha], x_{-\alpha}\rangle = \langle h_\alpha, [x_\alpha, x_{-\alpha}]\rangle = \langle h_\alpha, h_\alpha\rangle,$$

which proves the claim. □

Let $(x_\alpha)_{\alpha\in\Phi}$ be a distinguished family for $(\mathfrak{g}, \mathfrak{h})$. We define the **transition coefficients** $N_{\alpha,\beta}$ of (x_α) by the following formula:

$$[x_\alpha, x_\beta] = N_{\alpha,\beta}x_{\alpha+\beta}$$

where $N_{\alpha,\beta}$ is defined to be zero if $\alpha + \beta \notin \Phi$. Since $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}^\alpha$, every $x \in \mathfrak{g}$ can be written uniquely as

$$x = h + \sum_{\alpha\in\Phi} m_\alpha x_\alpha$$

and the bracket of two such elements can be then calculated using the transition coefficients. First, let us note the the transition coefficients satisfy some basic equations.

Proposition 6.2.35. *Let $(x_\alpha)_{\alpha\in\Phi}$ be a distinguished family. Let $\alpha, \beta \in \Phi$ be such that $\alpha + \beta \in \Phi$ and let $[-p, q]$ be the interval of k such that $\beta + k\alpha \in \Phi$. Then,*

$$N_{\alpha,\beta}N_{-\alpha,\alpha+\beta} = q(p+1), \tag{6.2.3}$$
$$N_{-\alpha,\alpha+\beta}\langle h_\beta, h_\beta\rangle = -N_{-\alpha,-\beta}\langle h_{\alpha+\beta}, h_{\alpha+\beta}\rangle, \tag{6.2.4}$$
$$N_{\alpha,\beta}N_{-\alpha,-\beta} = (p+1)^2. \tag{6.2.5}$$

Proof. Let π be the representation of $\mathfrak{sl}(2, \mathbb{K})$ on $\mathfrak{g}$ defined by X_α. The element $v = x_{\beta+q\alpha}$ is then primitive of weight $p + q$ by Proposition 6.2.23. For $n \geq 0$, put

$$v_n = \frac{1}{n!}\rho(f)^n v.$$

By Proposition 6.2.6 we then have

$$\mathrm{ad}(x_\alpha)v_q = (p+1)v_{q-1}, \quad \mathrm{ad}(x_{-\alpha})\mathrm{ad}(x_\alpha)v_q = q(p+1)v_q.$$

This proves (6.2.3) since v_q is a non-zero element of $\mathfrak{g}^\beta$.

Since the form $\langle\cdot,\cdot\rangle$ is invariant, we have

$$N_{-\alpha,\alpha+\beta}\langle x_\beta, x_{-\beta}\rangle = \langle [x_{-\alpha}, x_{\alpha+\beta}], x_{-\beta}\rangle = -\langle x_{\alpha+\beta}, [x_{-\alpha}, x_{-\beta}]\rangle = -N_{-\alpha,-\beta}\langle x_{\alpha+\beta}, x_{-\alpha-\beta}\rangle \tag{6.2.6}$$

which, in view of Proposition 6.2.34, proves (6.2.4).

The restriction of $\langle\cdot,\cdot,\rangle$ to $\mathfrak{h}$ is non-degenerate and invariant under the Weyl group (Corollary 6.2.28). Identify $\mathfrak{h}$ and $\mathfrak{h}^*$ by means of this restriction. If $\gamma \in \Phi$, h_γ is identified with $2\gamma/\langle\gamma,\gamma\rangle$. Hence, for all $\gamma, \delta \in \Phi$,

$$\frac{\langle\gamma,\gamma\rangle}{\langle\delta,\delta\rangle} = \frac{\langle h_\alpha, h_\alpha\rangle}{\langle h_\gamma, h_\gamma\rangle}.$$

Now, by **????**, we have
$$\frac{\langle\alpha+\beta,\alpha+\beta\rangle}{\langle\beta,\beta\rangle}=\frac{p+1}{q}$$
so by (6.2.6) we see
$$N_{\alpha,\beta}N_{-\alpha,-\beta}=-N_{\alpha,\beta}N_{-\alpha,\alpha+\beta}\frac{\langle h_\beta,h_\beta\rangle}{\langle h_{\alpha+\beta},h_{\alpha+\beta}\rangle}=-N_{\alpha,\beta}N_{-\alpha,\alpha+\beta}\frac{p+1}{q}=(p+1)^2.$$
This completes the proof. □

Let $(x_\alpha)_{\alpha\in\Phi}$ be a distinguished family. We have an associated linear form $\varphi:\mathfrak{g}\to\mathfrak{g}$ which is equal to -1 on $\mathfrak{h}$ and which takes x_α to $-x_{-\alpha}$ for all $\alpha\in\Phi$. The family (x_α) is called a **Chevalley system** for $(\mathfrak{g},\mathfrak{h})$ if the associated form φ is an automorphism of $\mathfrak{g}$. As we will see, Chevalley systems possess integral transition coefficients.

Proposition 6.2.36. *Let $(x_\alpha)_{\alpha\in\Phi}$ be a Chevalley system for $(\mathfrak{g},\mathfrak{h})$. Then $N_{\alpha,\beta}=-N_{-\alpha,-\beta}$ and $N_{\alpha,\beta}=\pm(p+1)$ for any $\alpha,\beta\in\Phi$ such that $\alpha+\beta\in\Phi$.*

Proof. Let φ be the automorphism of $\mathfrak{g}$ associated with the Chevalley system (x_α). Then
$$\begin{aligned}N_{-\alpha,-\beta}x_{-\alpha,-\beta}&=[x_{-\alpha},x_{-\beta}]=[\varphi(x_\alpha),\varphi(x_\beta)]=\varphi([x_\alpha,x_\beta])\\&=\varphi(N_{\alpha,\beta}x_{\alpha+\beta})=-N_{\alpha,\beta}x_{-\alpha-\beta}.\end{aligned}$$
so $N_{-\alpha,-\beta}=-N_{\alpha,\beta}$. Now $N_{\alpha,\beta}=\pm(q+1)$ by (6.2.35). □

Proposition 6.2.37. *Let $(x_\alpha)_{\alpha\in\Phi}$ be a Chevalley system for $(\mathfrak{g},\mathfrak{h})$. Let V be a $\mathbb{Z}$-submodule of $\mathfrak{h}$ containing the h_α and contained in the group of weights of $\check{\Phi}$. Let $\mathfrak{g}_{\mathbb{Z}}$ be the $\mathbb{Z}$-submodule of $\mathfrak{g}$ generated by V and the x_α. Then $\mathfrak{g}_{\mathbb{Z}}$ is a $\mathbb{Z}$-Lie subalgebra of $\mathfrak{g}$, and the canonical map from $\mathfrak{g}_{\mathbb{Z}}\otimes_{\mathbb{Z}}\mathbb{K}$ to $\mathfrak{g}$ is an isomorphism.*

Proof. If $\alpha,\beta\in\Phi$ are such that $\alpha+\beta\in\Phi$, then $N_{\alpha,\beta}\in\mathbb{Z}$. On the other hand, if $\alpha\in\Phi$ and $h\in V$, then $\alpha(h)\in\mathbb{Z}$. This proves that $\mathfrak{g}_{\mathbb{Z}}$ is a $\mathbb{Z}$-Lie subalgebra of $\mathfrak{g}$. On the other hand, V is a free abelian group of rank $\dim(\mathfrak{h})$, so $\mathfrak{g}_{\mathbb{Z}}$ is a free abelian group of rank $\dim(\mathfrak{g})$; this implies the last assertion. □

The existence of Chevalley systems for $(\mathfrak{g},\mathfrak{h})$ is not easy to prove. We will return to this topic after we introduce the generating relation of a split semi-simple Lie algebra.

6.2.3 Subalgebras invariant under a Cartan subalgebra

In this paragraph, we consider cartain subalgebras of a split semi-simple Lie algebra $\mathfrak{g}$. Again, we assume that $\mathbb{K}$ has characteristic zero and $(\mathfrak{g},\mathfrak{h})$ is a split semi-simple Lie algebra, with root system Φ.

Let V be a vector subspace of $\mathfrak{g}$. Then since $\mathfrak{h}$ acts via the adjoint representation on $\mathfrak{g}$, it also acts on V. Although V may not be stable under $\mathrm{ad}(\mathfrak{h})$, it contains a largest invariant subspace. Such a subspace may be characterized using the root decomposation of $\mathfrak{g}$, which is the content of the following proposition.

Proposition 6.2.38. *Let V be a vector subspace of $\mathfrak{g}$ and $\Phi(V)$ the set of $\alpha\in\Phi$ such that $\mathfrak{g}^\alpha\subseteq V$. Then $(V\cap\mathfrak{h})+\bigoplus_{\alpha\in\Phi(V)}\mathfrak{g}^\alpha$ is the largest vector subspace of V stable under $\mathrm{ad}(\mathfrak{h})$.*

Proof. A vector subspace W of V is stable under $\mathfrak{h}$ if and only if
$$W=(W\cap\mathfrak{h})+\bigoplus_{\alpha\in\Phi}(W\cap\mathfrak{g}^\alpha).$$
The largest vector subspace of V stable under $\mathrm{ad}(\mathfrak{h})$ is thus $(V\cap\mathfrak{h})+\bigoplus_{\alpha\in\Phi}(V\cap\mathfrak{g}^\alpha)$. But since $\dim(\mathfrak{g}^\alpha)=1$, we have $V\cap\mathfrak{g}^\alpha=\{0\}$ if $\alpha\notin\Phi(V)$ and $V\cap\mathfrak{g}^\alpha=\mathfrak{g}^\alpha$ when $\alpha\in\Phi(V)$, whence the claim. □

Now we want to consider subalgebras of $\mathfrak{g}$ invariant under $\mathrm{ad}(\mathfrak{h})$. It turns our that such subalgebras are classified by certain subsets of the root system Φ. For this, we recall that a subset P of Φ is said to be **closed** if the conditions $\alpha,\beta\in P$, $\alpha+\beta\in\Phi$ imply $\alpha+\beta\in P$, in other words if $(P+P)\cap\Phi\subseteq P$. For any subset P of Φ, we put
$$\mathfrak{h}_P=\bigoplus_{\alpha\in P}\mathfrak{h}_\alpha,\quad \mathfrak{g}^P=\bigoplus_{\alpha\in P}\mathfrak{g}^\alpha.$$
If $P,Q\subseteq\Phi$, we clearly have
$$[\mathfrak{h},\mathfrak{g}^P]\subseteq\mathfrak{g}^P,\quad [\mathfrak{g}^P,\mathfrak{g}^Q]=\mathfrak{g}^{(P+Q)\cap\Phi}+\mathfrak{h}_{P\cap(-Q)}.$$

Lemma 6.2.39. *Let $\mathfrak{t}$ be a vector subspace of $\mathfrak{h}$ and P a subset of Φ. Then $\mathfrak{t}+\mathfrak{g}^P$ is a subalgebra of $\mathfrak{g}$ if and only if P is a closed subset of Φ and $\mathfrak{t} \supseteq \mathfrak{h}_{P\cap(-P)}$.*

Proof. Indeed, we have

$$[\mathfrak{t}+\mathfrak{g}^P, \mathfrak{t}+\mathfrak{g}^P] = [\mathfrak{t}, \mathfrak{g}^P] + [\mathfrak{g}^P, \mathfrak{g}^P] = \mathfrak{h}_{P\cap(-P)} + [\mathfrak{t}, \mathfrak{g}^P] + \mathfrak{g}^{(P+P)\cap\Phi}.$$

Hence $\mathfrak{t}+\mathfrak{g}^P$ is a subalgebra of $\mathfrak{g}$ if and only if $\mathfrak{t} \supseteq \mathfrak{h}_{P\cap(-P)}$ and $\mathfrak{g}^{(P+P)\cap\Phi} \subseteq \mathfrak{g}^P$, which proves the lemma. □

Proposition 6.2.40. *Let $(\mathfrak{g}, \mathfrak{h})$ be a split semi-simple Lie algebra and Φ its root system.*

(a) The subalgebras of $\mathfrak{g}$ stable under $\mathrm{ad}(\mathfrak{h})$ are the vector subspaces of the form $\mathfrak{t}+\mathfrak{g}^P$, where P is a closed subset of Φ and $\mathfrak{t}$ is a vector subspace of $\mathfrak{h}$ containing $\mathfrak{h}_{P\cap(-P)}$.

(b) Let $\mathfrak{t}_1$, $\mathfrak{t}_2$ be vector subspaces of $\mathfrak{h}$ and P, Q closed subsets of Φ, with $\mathfrak{t}_2 \subseteq \mathfrak{t}_1$, $Q \subseteq P$ and $\mathfrak{t}_1 \supseteq \mathfrak{h}_{P\cap(-P)}$. Then $\mathfrak{t}_2+\mathfrak{g}^Q$ is an ideal of $\mathfrak{t}_1+\mathfrak{g}^P$ if and only if

$$(P+Q)\cap\Phi \subseteq Q \quad \textit{and} \quad \mathfrak{h}_{P\cap(-Q)} \subseteq \mathfrak{t}_2 \subseteq \bigcap_{\alpha\in P\setminus Q} \ker\alpha.$$

Proof. Assertion (a) follows immediately from Proposition 6.2.38 and Lemma 6.2.39. Let $\mathfrak{t}_1, \mathfrak{t}_2$ and P, Q be as in (b). Then

$$[\mathfrak{t}_1+\mathfrak{g}^P, \mathfrak{t}_2+\mathfrak{g}^Q] = \mathfrak{h}_{P\cap(-Q)} + [\mathfrak{t}_1, \mathfrak{g}^Q] + [\mathfrak{t}_2, \mathfrak{g}^P] + \mathfrak{g}^{(P+Q)\cap\Phi}.$$

Hence, $\mathfrak{t}_2+\mathfrak{g}^Q$ is an ideal of $\mathfrak{t}_1+\mathfrak{g}^P$ if and only if

$$\mathfrak{h}_{P\cap(-Q)} \subseteq \mathfrak{t}_2, \quad [\mathfrak{t}_2, \mathfrak{g}^P] \subseteq \mathfrak{g}^Q, \quad \mathfrak{g}^{(P+Q)\cap\Phi} \subseteq \mathfrak{g}^Q.$$

This implies (b). □

By Proposition 6.2.40, the subalgebras of $\mathfrak{g}$ containing $\mathfrak{h}$ are the sets $\mathfrak{h}+\mathfrak{g}^P$ where P is a closed subset of Φ. We also note that, by Proposition 6.1.63, every Cartan subalgebra of $\mathfrak{h}+\mathfrak{g}^P$ is a Cartan subalgebra of $\mathfrak{g}$.

Proposition 6.2.41. *Let $\mathfrak{a} = \mathfrak{t}+\mathfrak{g}^P$ be a subalgebra of $\mathfrak{g}$ stable under $\mathrm{ad}(\mathfrak{h})$, where P is a closed subset of Φ.*

(a) Let $\mathfrak{k}$ be the set of $x \in \mathfrak{t}$ such that $\alpha(x) = 0$ for all $\alpha \in P\cap(-P)$. Then the radical of $\mathfrak{a}$ is $\mathfrak{k}+\mathfrak{g}^Q$, where Q is the set of $\alpha \in P$ such that $-\alpha \notin P$; moreover, $\mathfrak{g}^Q$ is a nilpotent ideal of $\mathfrak{a}$.

(b) $\mathfrak{a}$ is semi-simple if and only if $P = -P$ and $\mathfrak{t} = \mathfrak{h}_P$.

(c) $\mathfrak{a}$ is reductive in $\mathfrak{g}$ if and only if $P = -P$.

(d) $\mathfrak{a}$ is solvable if and only if $P\cap(-P) = \varnothing$; in this case $[\mathfrak{a}, \mathfrak{a}] = \mathfrak{g}^R$, where

$$R = ((P+P)\cap\Phi) \cup \{\alpha \in P : \alpha(\mathfrak{t}) \neq 0\}.$$

(e) $\mathfrak{a}$ consists of nilpotent elements if and only if $\mathfrak{t} = 0$; in this case $P\cap(-P) = \varnothing$, and $\mathfrak{a}$ is nilpotent.

Proof. We first prove (e). If $\mathfrak{a}$ consists of nilpotent elements, $\mathfrak{a}$ is clearly nilpotent, and $\mathfrak{t} = 0$ since the elements of $\mathfrak{h}$ are semi-simple. Assume that $\mathfrak{t} = 0$; then by Proposition 6.2.40, $P\cap(-P) = \varnothing$. By Proposition 5.1.58, there exists a base Δ of Φ such that $P \subseteq \Phi^+(\Delta)$. Hence, there exists an integer $n > 0$ with the following properties: if $\alpha_1, \dots, \alpha_n \in P$ and $\beta \in \Phi\cup\{0\}$, then

$$\alpha_1 + \cdots + \alpha_n + \beta \notin \Phi\cup\{0\}.$$

Then the adjoint representation is nilpotent on $\mathfrak{g}^P$, hence $\mathfrak{g}^P$ is nilpotent, hence (e).

Next we consider (d). If $P\cap(-P) = \varnothing$, $\mathfrak{g}^P$ is a subalgebra of $\mathfrak{g}$ by Proposition 6.2.40, and is nilpotent by (e). Now

$$[\mathfrak{a}, \mathfrak{a}] = [\mathfrak{t}, \mathfrak{g}^P] + [\mathfrak{g}^P, \mathfrak{g}^P] = [\mathfrak{t}, \mathfrak{g}^P] + \mathfrak{g}^{(P+P)\cap\Phi} \subseteq \mathfrak{g}^P.$$

so $\mathfrak{a}$ is solvable by (e) and $[\mathfrak{a}, \mathfrak{a}]$ is given by the formula in the proposition. If $P\cap(-P) \neq \varnothing$, let $\alpha \in P$ be such that $-\alpha \in P$. Then $\mathfrak{h}_\alpha + \mathfrak{g}^\alpha + \mathfrak{g}^{-\alpha}$ is a simple subalgebra of $\mathfrak{a}$ so $\mathfrak{a}$ is not solvable.

We prove (a). Let Q be the subset of P described in (a). Since P is closed, we have $(P+Q)\cap\Phi\subseteq P$. If $\alpha\in P$, $\beta\in Q$ and $\alpha+\beta\in\Phi$, we cannot have $\alpha+\beta\in -P$ since this would imply (P is closed) that

$$-\beta=-(\alpha+\beta)+\alpha\in P$$

whereas $\beta\in Q$; therefore $(P+Q)\cap\Phi\subseteq Q$. This shows that $\mathfrak{g}^Q$ is an ideal of $\mathfrak{a}$ (Proposition 6.2.40(b), since $P\cap(-Q)=\varnothing$ by definition), and is nilpotent by (e). We also note that $P\cap(-P)=P\setminus Q$, so (by the definition of $\mathfrak{k}$)

$$\{0\}=\mathfrak{h}_{P\cap(-Q)}\subseteq\mathfrak{k}\subseteq\bigcap_{\alpha\in P\setminus Q}\ker\alpha.$$

By Proposition 6.2.40(b), this implies $\mathfrak{k}+\mathfrak{g}^Q$ is an ideal of $\mathfrak{a}$. Since $Q\cap(-Q)=\varnothing$, this ideal is solvable by (c), and is therefore contained in the radical $\mathfrak{r}$ of $\mathfrak{a}$. Since $\mathfrak{r}$ is stable under every derivation of $\mathfrak{a}$ (Corollary 1.5.21), it is stable under $\mathrm{ad}(\mathfrak{h})$. Hence there exists a subset R of P such that $\mathfrak{r}=(\mathfrak{r}\cap\mathfrak{h})+\mathfrak{g}^R$. Suppose that $\alpha\in R$ and that $-\alpha\in P$. Then since $\mathfrak{r}$ is an ideal of $\mathfrak{a}$, we have

$$\mathfrak{g}^{-\alpha}=[\mathfrak{h}_\alpha,\mathfrak{g}^{-\alpha}]=[[\mathfrak{g}^\alpha,\mathfrak{g}^{-\alpha}],\mathfrak{g}^{-\alpha}]\subseteq[[\mathfrak{r},\mathfrak{a}],\mathfrak{a}]\subseteq\mathfrak{r},$$

therefore $-\alpha\in R$. By (c), this contradicts the fact that $\mathfrak{r}$ is solvable, consequently $R\subseteq Q$. Finally, if $x\in\mathfrak{r}\cap\mathfrak{h}$ and $\alpha\in P\cap(-P)$, then $[x,\mathfrak{g}^\alpha]\subseteq\mathfrak{g}^\alpha\cap\mathfrak{r}=0$, so $\alpha(x)=0$; this shows that $x\in\mathfrak{k}$. Hence $\mathfrak{r}\subseteq\mathfrak{k}+\mathfrak{g}^Q$ and the proof of (a) is complete.

By (a), the adjoint representation of $\mathfrak{a}$ on $\mathfrak{g}$ is semi-simple if and only if $\mathrm{ad}_{\mathfrak{g}}(x)$ is semi-simple for all $x\in\mathfrak{k}+\mathfrak{g}^Q$ (Theorem 1.6.28). This is the case if and only if $Q=\varnothing$, in other words $P=-P$. If moreover $\mathfrak{a}$ is semi-simple, then $\mathfrak{a}=[\mathfrak{a},\mathfrak{a}]\subseteq\mathfrak{h}_P+\mathfrak{g}^P$ and consequently $\mathfrak{t}=\mathfrak{h}_P$ (since $\mathfrak{t}\supset\mathfrak{h}_{P\cap(-P)}=\mathfrak{h}_P$ in this case). Conversely, if $P=-P$ and $\mathfrak{t}=\mathfrak{h}_P$, then $\mathfrak{a}$ is reductive, and $\mathfrak{a}=\bigoplus_{\alpha\in P}\mathfrak{s}_\alpha$, so $\mathfrak{a}=[\mathfrak{a},\mathfrak{a}]$ and $\mathfrak{a}$ is semi-simple. This proves the assertions of (b) and (c), and completes the proof. □

Corollary 6.2.42. *Let $\mathfrak{b}=\mathfrak{h}+\mathfrak{g}^P$ be a subalgebra of $\mathfrak{g}$ containing $\mathfrak{h}$, where P is a closed subset of Φ.*

(a) $\mathfrak{b}$ is solvable if and only if $P\cap(-P)=\varnothing$; in this case $[\mathfrak{b},\mathfrak{b}]=\mathfrak{g}^P$.

(b) $\mathfrak{b}$ is reductive if and only if $P=-P$.

Proposition 6.2.43. *Let $\mathfrak{g}^P=\mathfrak{h}_P+\mathfrak{g}^P$ be a semi-simple subalgebra of $\mathfrak{g}$ stable under $\mathrm{ad}(\mathfrak{h})$, where P is a closed subset of Φ. Then*

(a) $\mathfrak{h}_P$ is a splitting Cartan subalgebra of $\mathfrak{g}^P$;

(b) the root system of $(\mathfrak{g}^P,\mathfrak{h}_P)$ is the set of restrictions to $\mathfrak{h}_P$ of elements of P.

Proof. Let $\mathfrak{n}=\mathfrak{n}_{\mathfrak{g}^P}(\mathfrak{h}_P)$ be the normalizer of $\mathfrak{h}_P$ in $\mathfrak{g}^P$. Then since $\mathfrak{h}_P$ is stable under $\mathrm{ad}(\mathfrak{h})$, $\mathfrak{n}$ is also stable under $\mathrm{ad}(\mathfrak{h})$:

$$[[\mathfrak{h},\mathfrak{n}],\mathfrak{h}_P]=[\mathfrak{h},[\mathfrak{n},\mathfrak{h}_P]]+[\mathfrak{n},[\mathfrak{h}_P,\mathfrak{h}]]\subseteq[\mathfrak{h},\mathfrak{h}_P]+[\mathfrak{n},\mathfrak{h}_P]\subseteq\mathfrak{h}_P.$$

Hence $\mathfrak{n}$ is of the form $\mathfrak{h}_P+\mathfrak{g}^Q$ where $Q\subseteq P$ (Proposition 6.2.38). If $\alpha\in Q$, then

$$\mathfrak{g}^\alpha=[\mathfrak{h}_\alpha,\mathfrak{g}^\alpha]\subseteq[\mathfrak{h}_P,\mathfrak{g}^\alpha]\subseteq\mathfrak{h}_P.$$

which is absurd. Thus $Q=\varnothing$ and we get $\mathfrak{n}=\mathfrak{h}_P$. Therefore $\mathfrak{h}_P$ is a Cartan subalgebra of $\mathfrak{g}^P$. If $x\in\mathfrak{h}_P$, then $\mathrm{ad}_{\mathfrak{g}}(x)$, and a fortiori $\mathrm{ad}_{\mathfrak{g}^P}(x)$, are triangularizable; thus (a) is proved, and (b) is clear. □

Proposition 6.2.44. *Let $\mathfrak{b}=\mathfrak{h}+\mathfrak{g}^P$ be a subalgebra of $\mathfrak{g}$ containing $\mathfrak{h}$. Then the following conditions are equivalent:*

(i) $\mathfrak{b}$ is a maximal solvable subalgebra of $\mathfrak{g}$;

(ii) there exists a base Δ of Φ such that $P=\Phi^+(\Delta)$;

(iii) $P\cap(-P)=\varnothing$ and $P\cup(-P)=\Phi$.

Proof. If $\mathfrak{b}$ is solvable, $P\cap(-P)=\varnothing$. Then there exists a base Δ of R such that $P\subseteq\Phi^+(\Delta)$. Then $\mathfrak{h}+\mathfrak{g}^{\Phi^+(\Delta)}$ is a solvable subalgebra of $\mathfrak{g}$ containing $\mathfrak{b}$, hence equal to $\mathfrak{b}$ if $\mathfrak{b}$ is maximal. In particular, $P\cap(-P)=\varnothing$ and $P\cup(-P)=\Phi$.

Assume that $P\cap(-P)=\varnothing$ and that $P\cup(-P)=\Phi$. Then $\mathfrak{b}$ is solvable. Moreover, if $\tilde{\mathfrak{b}}$ be a solvable subalgebra of $\mathfrak{g}$ containing $\mathfrak{b}$, there exists a subset Q of Φ such that $\tilde{\mathfrak{b}}=\mathfrak{h}+\mathfrak{g}^Q$. Then $Q\cap(-Q)=\varnothing$ and $Q\supseteq P$, so $Q=P$ and $\tilde{\mathfrak{b}}=\mathfrak{b}$. □

A subalgebra of $\mathfrak{g}$ containing $\mathfrak{h}$ satisfying the equivalent conditions in Proposition 6.2.44 is called a **Borel subalgebra of** $(\mathfrak{g}, \mathfrak{h})$. A subalgebra $\mathfrak{b}$ of a splittable algebra $\mathfrak{g}$ is called a **Borel subalgebra of** $\mathfrak{g}$ if there exists a splitting Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ such that $\mathfrak{b}$ is a Borel subalgebra of $(\mathfrak{g}, \mathfrak{h})$.

With the notations of Proposition 6.2.44, we also say that $\mathfrak{b}$ is the Borel subalgebra of $\mathfrak{g}$ defined by $\mathfrak{h}$ and Δ. The map which associates $\Phi^+(\Delta)$ to a base Δ of Φ is injective. Consequently, $\Delta \mapsto \mathfrak{b}_\Delta = \mathfrak{h} + \mathfrak{g}^{\Phi^+(\Delta)}$ is a bijection from the set of bases of Φ to the set of Borel subalgebras of $(\mathfrak{g}, \mathfrak{h})$. Thus, the number of Borel subalgebras of $(\mathfrak{g}, \mathfrak{h})$ is equal to the number of basis of Φ, which is the order of the Weyl group of Φ.

Proposition 6.2.45. *Let $\mathfrak{b}$ be a subalgebra of $\mathfrak{g}$, $\mathbb{K}'$ an extension of $\mathbb{K}$. Then $\mathfrak{b} \otimes_{\mathbb{K}} \mathbb{K}'$ is a Borel subalgebra of $(\mathfrak{g} \otimes_{\mathbb{K}} \mathbb{K}', \mathfrak{h} \otimes_{\mathbb{K}} \mathbb{K}')$ if and only if $\mathfrak{b}$ is a Borel subalgebra of $(\mathfrak{g}, \mathfrak{h})$.*

Proof. This is clear from condition (iii) of Proposition 6.2.44, in view of the invariance of the root system under base changes. □

Proposition 6.2.46. *Let $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}$ be the Borel subalgebra of $(\mathfrak{g}, \mathfrak{h})$ defined by a base Δ of Φ. Let $r = \dim(\mathfrak{h})$.*

(a) If $h \in \mathfrak{h}$ and $x \in \mathfrak{n}$, then the characteristic polynomial of $\mathrm{ad}_{\mathfrak{g}}(h + x)$ is

$$\chi_{\mathrm{ad}_{\mathfrak{g}}(h+x)}(T) = T^r \prod_{\alpha \in \Phi} (T - \alpha(h)).$$

(b) The nilradical of $\mathfrak{b}$ is equal to $\mathfrak{n}$ and to $[\mathfrak{b}, \mathfrak{b}]$. This is also the set of elements of $\mathfrak{b}$ nilpotent in $\mathfrak{g}$.

(c) For all $\alpha \in \Delta$, let x_α be a non-zero element of $\mathfrak{g}^\alpha$. Then $(x_\alpha)_{\alpha \in \Delta}$ generates the Lie algebra $\mathfrak{n}$, and we have $[\mathfrak{n}, \mathfrak{n}] = \sum_{\alpha \in \Phi^+ \setminus \Delta} \mathfrak{g}^\alpha$.

Proof. There exists a total order on $\mathfrak{h}^*_{\mathbb{Q}}$ compatible with its vector space structure and such that the elements of $\Phi^+(\Delta)$ are > 0 (Proposition). Let h, x be as in (a) and $y \in \mathfrak{g}^\alpha$. Then $[h + x, y] = \alpha(h)y + z$ where $z \in \bigoplus_{\beta > \alpha} \mathfrak{g}^\beta$. Then, with respect to a suitable basis of $\mathfrak{g}$, the matrix of $\mathrm{ad}_{\mathfrak{g}}(h + x)$ has the following properties:

(i) it is upper triangular;

(ii) the diagonal entries of the matrix are the number 0 (r times) and the $\alpha(h)$ for $\alpha \in \Phi$.

This proves (a), and it also shows that the characteristic polynomial of $\mathrm{ad}_{\mathfrak{b}}(h + x)$ is

$$\chi_{\mathrm{ad}_{\mathfrak{b}}(h+x)}(T) = T^r \prod_{\alpha \in \Phi^+(\Delta)} (T - \alpha(h)).$$

It follows from the preceding that the set of elements of $\mathfrak{b}$ nilpotent in $\mathfrak{g}$, as well as the largest nilpotent ideal of $\mathfrak{b}$, are equal to $\mathfrak{n}$. We have $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$ by Proposition 6.2.41(c). Finally, assertion (c) follows from Proposition 6.2.23 and Proposition 5.1.43. □

Corollary 6.2.47. *Let $\mathfrak{b}$ be a Borel subalgebra of $\mathfrak{g}$.*

(a) Every Cartan subalgebra of $\mathfrak{b}$ is a splitting Cartan subalgebra of $\mathfrak{g}$.

(b) If $\mathfrak{h}_1, \mathfrak{h}_2$ are Cartan subalgebras of $\mathfrak{b}$, there exists $x \in [\mathfrak{b}, \mathfrak{b}]$ such that $e^{\mathrm{ad}(x)}\mathfrak{h}_1 = \mathfrak{h}_2$.

Proof. Assertion (a) follows from Proposition 6.2.46(a) and Proposition 6.1.63. Assertion (b) follows from Proposition 6.2.46(b) and Theorem 6.1.66. □

Proposition 6.2.48. *Let $\mathfrak{b}_1, \mathfrak{b}_2$ be Borel subalgebras of $\mathfrak{g}$. Then there exists a splitting Cartan subalgebra of $\mathfrak{g}$ contained in $\mathfrak{b}_1 \cap \mathfrak{b}_2$.*

Proof. We set

$$\mathfrak{n}_1 = [\mathfrak{b}_1, \mathfrak{b}_1], \quad \mathfrak{n}_2 = [\mathfrak{b}_2, \mathfrak{b}_2], \quad \mathfrak{p} = \mathfrak{b}_1 \cap \mathfrak{b}_2, \quad \mathfrak{a} = \mathfrak{b}_1 + \mathfrak{b}_2.$$

Denote by $\mathfrak{t}^\perp$ of the orthogonal complement of a subset $\mathfrak{t}$ of $\mathfrak{g}$ with respect to the Killing form of $\mathfrak{g}$. Then by Proposition 6.2.20 we have $\mathfrak{n}_1 = \mathfrak{b}_1^\perp$ and $\mathfrak{n}_2 = \mathfrak{b}_2^\perp$. Put

$$r = \dim(\mathfrak{h}), \quad s = \dim(\mathfrak{n}_1) = \dim(\mathfrak{n}_2), \quad t = \dim(\mathfrak{p}).$$

Then we have

$$\dim(\mathfrak{b}_1) = \dim(\mathfrak{b}_2) = r + s, \quad \dim(\mathfrak{a}) = 2(r+s) - t, \quad \dim(\mathfrak{g}) = r + 2s,$$

and therefore

$$\dim(\mathfrak{a} \cap \mathfrak{p}) \geq \dim(\mathfrak{a}) + \dim(\mathfrak{p}) - \dim(\mathfrak{g}) = 2(r+s) - t + t - (r+2s) = r. \tag{6.2.7}$$

The elements of $\mathfrak{p} \cap \mathfrak{n}_1$ are nilpotent in $\mathfrak{g}$ and belong to $\mathfrak{b}_2$, hence to $\mathfrak{n}_2$. Consequently

$$\mathfrak{p} \cap \mathfrak{n}_1 \subseteq \mathfrak{n}_1 \cap \mathfrak{n}_2 = \mathfrak{a}^{\perp},$$

so $\mathfrak{a} \cap \mathfrak{p} \cap \mathfrak{n}_1 = 0$. In view of (6.2.7), we see that $\mathfrak{a} \cap \mathfrak{p}$ is a complement of $\mathfrak{n}_1$ in $\mathfrak{b}_1$.

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{b}_1$ and z be an element of $\mathfrak{h}$ regular in $\mathfrak{g}$. Since $z \in \mathfrak{h} \subseteq \mathfrak{b}_1$ and $\mathfrak{a} \cap \mathfrak{p} + \mathfrak{n}_1 = \mathfrak{b}_1$, there exists $y \in \mathfrak{n}_1$ such that $y + z \in \mathfrak{a} \cap \mathfrak{p}$. By Proposition 6.2.46(i), $\mathrm{ad}_{\mathfrak{g}}(y+z)$ has the same characteristic polynomial as $\mathrm{ad}_{\mathfrak{g}}(z)$, so $x = y + z$ is regular in $\mathfrak{g}$ and a fortiori in $\mathfrak{b}_1$ and $\mathfrak{b}_2$ (Proposition 6.1.44). Since $\mathfrak{g}$, $\mathfrak{b}_1$, $\mathfrak{b}_2$ have the same rank, $\mathfrak{b}_1^0(x) = \mathfrak{g}^0(x) = \mathfrak{b}_2^0(x)$ is simultaneously a Cartan subalgebra of $\mathfrak{b}_1$, of $\mathfrak{g}$ and of $\mathfrak{b}_2$. Finally, this Cartan subalgebra of $\mathfrak{g}$ is splitting by Corollary 6.2.47. □

Corollary 6.2.49. *The group $\mathrm{Aut}_e(\mathfrak{g})$ operates transitively on the set of pairs $(\mathfrak{h}, \mathfrak{b})$ where $\mathfrak{h}$ is a splitting Cartan subalgebra of $\mathfrak{g}$ and $\mathfrak{b}$ is a Borel subalgebra of $(\mathfrak{g}, \mathfrak{h})$.*

Proof. Let $(\mathfrak{h}_1, \mathfrak{b}_1)$ and $(\mathfrak{h}_2, \mathfrak{b}_2)$ be two such pairs. There exists a splitting Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ contained in $\mathfrak{b}_1 \cap \mathfrak{b}_2$ (Proposition 6.2.48). By Corollary 6.2.47, we are reduced to the case in which $\mathfrak{h} = \mathfrak{h}_1 = \mathfrak{h}_2$. Let Φ be the root system of $(\mathfrak{g}, \mathfrak{h})$. There exists bases Δ_1, Δ_2 of Φ such that $\mathfrak{b}_i$ is associated to Δ_i, and there exists $w \in W(\Phi)$ which transforms Δ_1 into Δ_2. Finally, there exists $\phi \in \mathrm{Aut}_e(\mathfrak{g})$ such that $\phi|_{\mathfrak{h}} = w$ by Corollary 6.2.27. Then $\phi(\mathfrak{h}) = \mathfrak{h}$ and $\phi(\mathfrak{b}_1) = \mathfrak{b}_2$. □

Proposition 6.2.50. *Let $\mathfrak{p} = \mathfrak{h} + \mathfrak{g}^P$ be a subalgebra of $\mathfrak{g}$ containing $\mathfrak{h}$. Then the following conditions are equivalent:*

(i) *$\mathfrak{p}$ contains a Borel subalgebra of $(\mathfrak{g}, \mathfrak{h})$;*

(ii) *there exists a base Δ of Φ such that $P \supseteq \Phi^+(\Delta)$;*

(iii) *P is parabolic, in other words, $P \cup (-P) = \Phi$.*

Proof. Conditions (i) and (ii) are equivalent by Proposition 6.2.44. Conditions (ii) and (iii) are equivalent by Proposition . □

A subalgebra of $\mathfrak{g}$ containing $\mathfrak{h}$ and satisfying the conditions of Proposition 6.2.50 is called a **parabolic subalgebra of** $(\mathfrak{g}, \mathfrak{h})$. A parabolic subalgebra of $\mathfrak{g}$ is a parabolic subalgebra of $(\mathfrak{g}, \mathfrak{h})$ where $\mathfrak{h}$ is a splitting Cartan subalgebra of $\mathfrak{g}$.

Let Δ be a basis of Φ, and $\mathfrak{b}$ the corresponding Borel subalgebra. If $\Sigma \subseteq \Delta$, denote by Q_Σ the set of roots that are linear combinations of elements of Σ with coefficients ≤ 0. Put $P(\Sigma) = \Phi^+(\Delta) \cup Q_\Sigma$ and $\mathfrak{p}_\Sigma = \mathfrak{h} \oplus \mathfrak{g}^{P(\Sigma)}$. By Proposition 5.1.48 and Proposition 6.2.50, $\mathfrak{p}_\Sigma$ is a parabolic subalgebra containing $\mathfrak{b}$ and every parabolic subalgebra of $\mathfrak{g}$ containing $\mathfrak{b}$ is obtained in this way.

Proposition 6.2.51. *Let $\mathfrak{p} = \mathfrak{h} + \mathfrak{g}^P$ be a parabolic subalgebra of $(\mathfrak{g}, \mathfrak{h})$, Q the set of $\alpha \in P$ such that $-\alpha \notin P$, and $\mathfrak{s} = \mathfrak{h} + \mathfrak{g}^{P \cap (-P)}$. Then*

(a) *$\mathfrak{p} = \mathfrak{s} \oplus \mathfrak{g}^Q$;*

(b) *$\mathfrak{s}$ is reductive in $\mathfrak{g}$;*

(c) *$\mathfrak{g}^Q$ is the nilradical of $\mathfrak{p}$ and the nilpotent radical of $\mathfrak{p}$;*

(d) *The centre of $\mathfrak{p}$ is trivial.*

Proof. By Proposition 6.2.41, $\mathfrak{s}$ is reductive in $\mathfrak{g}$ and $\mathfrak{g}^Q$ is a nilpotent ideal of $\mathfrak{p}$. Since $P \cup (-P) = \Phi$, by Proposition 6.2.41(a) the radical of $\mathfrak{p}$ is $\mathfrak{h} + \mathfrak{g}^Q$. If $\mathfrak{n}$ is the nilradical of $\mathfrak{p}$, then $\mathfrak{g}^Q \subseteq \mathfrak{n} \subseteq \mathfrak{h} + \mathfrak{g}^Q$ (since $\mathfrak{n}$ is contained in the radical of $\mathfrak{p}$). If $x \in \mathfrak{n} \cap \mathfrak{h}$, $\mathrm{ad}_{\mathfrak{p}}(x)$ is nilpotent, so $\alpha(x) = 0$ for all $\alpha \in P$, and hence $x = 0$. This proves $\mathfrak{n} = \mathfrak{g}^Q$. By Theorem 1.5.9, the nilpotent radical of $\mathfrak{p}$ equals to $[\mathfrak{p}, \mathfrak{p}] \cap (\mathfrak{h} + \mathfrak{g}^Q)$, and is contained in the nilradical $\mathfrak{g}^Q$. Since $[\mathfrak{h}, \mathfrak{g}^Q] = \mathfrak{g}^Q$, we see $\mathfrak{g}^Q \subseteq [\mathfrak{p}, \mathfrak{p}]$, therefore $\mathfrak{g}^Q$ is contained in the nilpotent radical of $\mathfrak{p}$, and hence equals to it.

Let

$$z = h + \sum_{\alpha \in P} x_\alpha, \quad h \in \mathfrak{h}, x_\alpha \in \mathfrak{g}^\alpha$$

be an element of the centre of $\mathfrak{p}$. For all $y \in \mathfrak{h}$, we have $0 = [y, z] = \sum \alpha(y) x_\alpha$, so $x_\alpha = 0$ for all $\alpha \in P$. It follows that $[h, \mathfrak{g}^\beta] = 0$ for all $\beta \in P$, so $h = 0$. □

Finally, recall that closed subsets of Φ are in one-to-one correspondence to facets, so by Proposition 5.1.61 we have the following result:

Proposition 6.2.52. *Let $\mathcal{H}$ be the set of hyperplanes of $\mathfrak{h}_{\mathbb{R}}$ consisting of the kernels of the roots in Φ. Let $\mathcal{F}$ be the set of facets of $\mathfrak{h}_{\mathbb{R}}$ relative to $\mathcal{H}$. Let $\mathcal{S}$ be the set of parabolic subalgebras of $(\mathfrak{g}, \mathfrak{h})$. For every $\mathfrak{p} = \mathfrak{h} + \mathfrak{g}^P \in \mathcal{S}$, let $\mathscr{F}(\mathfrak{p})$ be the facet associated to P. Then $\mathfrak{p} \mapsto \mathscr{F}(\mathfrak{p})$ is a bijection from $\mathcal{S}$ to $\mathcal{F}$. Moreover, $\mathfrak{p}_1 \supseteq \mathfrak{p}_2$ if and only if $\mathscr{F}(\mathfrak{p}_1) \subseteq \overline{\mathscr{F}(\mathfrak{p}_2)}$.*

6.2.4 Lie algebra defined by a reduced root system

In this paragraph we denote by Φ a reduced root system in a vector space V and by Δ a basis of Φ. We denote by $(n(\alpha, \beta))_{\alpha, \beta \in \Delta}$ the Cartan matrix relative to Δ. Recall that $n(\alpha, \beta) = \langle \alpha, \beta \rangle$. We are going to show that Φ is the root system of a split semi-simple Lie algebra which is unique up to isomorphism. In fact, we will construct a Lie algebra using the Cartan matrix and show that every Lie algebra with Root system Φ is isomorphic to this constructed Lie algebra. In particular, this shows a split semi-simple Lie algebra is uniquely determined by its root system. Since we have already classified root systems, we thus obtain a classfication for split semi-simple Lie algebras.

To begin with, we first establish the following **Serre relations** for a split semi-simple Lie algebra. This relation turns out to characterizes split semi-simple Lie algebras.

Proposition 6.2.53. *Let $(\mathfrak{g}, \mathfrak{h})$ be a split semi-simple Lie algebra, Φ its root system, Δ a basis of Φ, and $(n(\alpha, \beta))_{\alpha, \beta \in \Delta}$ the corresponding Cartan matrix. Let $(e_\alpha, h_\alpha, f_\alpha)$ be the $\mathfrak{sl}(2, \mathbb{K})$-triple for $(\mathfrak{g}, \mathfrak{h})$. Then any family $(x_\alpha)_{\alpha \in \Delta \cup (-\Delta)}$ with $x_\alpha \in \mathfrak{g}^\alpha$ satisfies the following relations:*

$$[h_\alpha, h_\beta] = 0 \tag{6.2.8}$$
$$[h_\alpha, x_\beta] = n(\beta, \alpha) x_\beta \tag{6.2.9}$$
$$[h_\alpha, x_{-\beta}] = -n(\beta, \alpha) x_{-\beta} \tag{6.2.10}$$
$$[x_\alpha, x_{-\beta}] = 0 \text{ if } \alpha \neq \beta \tag{6.2.11}$$
$$\mathrm{ad}(x_\alpha)^{1-n(\beta,\alpha)}(x_\beta) = 0 \text{ if } \alpha \neq \beta \tag{6.2.12}$$
$$\mathrm{ad}(x_{-\alpha})^{1-n(\beta,\alpha)}(x_{-\beta}) = 0 \text{ if } \alpha \neq \beta. \tag{6.2.13}$$
$$\tag{6.2.14}$$

The family $(h_\alpha)_{\alpha \in \Delta}$ is a basis of $\mathfrak{h}$. If $x_\alpha \neq 0$ for all $\alpha \in \Delta \cup (-\Delta)$, then the Lie algebra $\mathfrak{g}$ is generated by the family $(x_\alpha)_{\alpha \in \Delta \cup (-\Delta)}$.

Proof. Formulas (6.2.8), (6.2.9) and (6.2.10) are clear. If $\alpha \neq \beta$, $\beta - \alpha$ is not a root since every element of Φ is a linear combination of elements of Δ with integer coefficients all of the same sign. This proves (6.2.11). In view of Proposition 5.1.19, this also proves that the α-chain defined by β is

$$\{\beta, \beta + \alpha, \dots, \beta - n(\beta, \alpha)\alpha\};$$

hence $\beta + (1 - n(\beta, \alpha))\alpha \notin \Phi$, which proves (6.2.12). The equality (6.2.13) is established in a similar way. The family $(h_\alpha)_{\alpha \in \Delta}$ is the coroot system of Φ, which a basis of $\check{\Phi}$, and hence of $\mathfrak{h}$. If $x_\alpha \neq 0$ for all $\alpha \in \Delta \cup (-\Delta)$, then $[x_\alpha, x_{-\alpha}] = \lambda_\alpha h_\alpha$ with $\lambda_\alpha \neq 0$, so the last assertion follows from Proposition 6.2.46. □

Let $(\mathfrak{g}, \mathfrak{h})$ be a split semi-simple Lie algebra, Φ its root system. A **framing** of $(\mathfrak{g}, \mathfrak{h})$ is a pair $(\Delta, (x_\alpha)_{\alpha\in\Delta})$, where Δ is a basis of Φ, and where, for all $\alpha \in \Delta$, x_α is a non-zero element of $\mathfrak{g}^\alpha$. A **framed semi-simple Lie algebra** is a sequence $(\mathfrak{g}, \mathfrak{h}, \Delta, (x_\alpha)_{\alpha\in\Delta})$ where $(\mathfrak{g}, \mathfrak{h})$ is a split semi-simple Lie algebra, and where $(\Delta, (x_\alpha)_{\alpha\in\Delta})$ is a framing of $(\mathfrak{g}, \mathfrak{h})$. A framing of $\mathfrak{g}$ is a framing of $(\mathfrak{g}, \mathfrak{h})$, where $\mathfrak{h}$ is a splitting Cartan subalgebra of $\mathfrak{g}$.

Let $(\mathfrak{g}_1, \mathfrak{h}_1, \Delta_1, (x^1_\alpha)_{\alpha\in\Delta_1})$ and $(\mathfrak{g}_2, \mathfrak{h}_2, \Delta_2, (x^2_\alpha)_{\alpha\in\Delta_2})$ be framed semi-simple Lie algebras. An framed isomorphism from $\mathfrak{g}_1$ to $\mathfrak{g}_2$ is an isomorphism φ from $\mathfrak{g}_1$ to $\mathfrak{g}_2$ that takes $\mathfrak{h}_1$ to $\mathfrak{h}_2$, Δ_1 to Δ_2, and x^1_α to $x^2_{\varphi(\alpha)}$ for all $\alpha \in \Delta_1$ (where $\varphi(\alpha)$ acts by the contragredient map of $\varphi|_{\mathfrak{h}_1}$). In this case, φ is said to transform the framing $(\Delta_1, (x^1_\alpha)_{\alpha\in\Delta_1})$ to the framing $(\Delta_2, (x^2_\alpha)_{\alpha\in\Delta_2})$.

If $(\Delta, (x_\alpha)_{\alpha\in\Delta})$ is a framing of $(\mathfrak{g}, \mathfrak{h})$, there exists, for all $\alpha \in \Delta$, a unique element $x_{-\alpha}$ of $\mathfrak{g}^{-\alpha}$ such that $[x_\alpha, x_{-\alpha}] = h_\alpha$ (Theorem 6.2.21). The family $(x_\alpha)_{\alpha\in\Delta\cup(-\Delta)}$ is called the **generating family defined by the framing**. This is also the generating family defined by the framing $(-\Delta, (x_\alpha)_{\alpha\in-\Delta})$. For all $\alpha \in \Delta\cup(-\Delta)$, let $t_\alpha \in \mathbb{K}^\times$, and assume that $t_\alpha t_{-\alpha} = 1$ for all $\alpha \in \Delta$. Then $(t_\alpha x_\alpha)_{\alpha\in\Delta\cup(-\Delta)}$ is the generating family defined by the framing $(\Delta, (t_\alpha x_\alpha)_{\alpha\in\Delta})$.

Now we begin the construction of our algebra. Let $(n(\alpha, \beta))_{\alpha,\beta\in\Delta}$ be the Cartan matrix of Φ (in fact, we will see that the construction applies to any square matrix $(n(\alpha, \beta))_{\alpha,\beta\in\Delta}$ over $\mathbb{K}$ with non-zero determinant and such that $n(\alpha, \alpha) = 2$ for all $\alpha \in \Delta$). Let E be the free associative algebra of the set Δ over $\mathbb{K}$. Recall that E is $\mathbb{N}$-graded (Example **??**). We are going to associate to each $\alpha \in \Delta$ endomorphisms $X^0_{-\alpha}, H^0_\alpha, X^0_\alpha$ of the vector space E, of degrees $1, 0, -1$ respectively. For any word $(\alpha_1, \dots, \alpha_n)$ in elements of Δ, put

$$X^0_{-\alpha}(\alpha_1, \dots, \alpha_n) = (\alpha, \alpha_1, \dots, \alpha_n), \tag{6.2.15}$$

$$H^0_\alpha(\alpha_1, \dots, \alpha_n) = -\sum_{i=1}^n n(\alpha_i, \alpha)(\alpha_1, \dots, \alpha_n). \tag{6.2.16}$$

On the other hand, $X^0_\alpha(\alpha_1, \dots, \alpha_n)$ is defined by induction on n using the following formula

$$X^0_\alpha(\alpha_1, \dots, \alpha_n) = (X^0_{-\alpha_1} X^0_\alpha + \delta_{\alpha,\alpha_1} H^0_\alpha)(\alpha_2, \dots, \alpha_n) \tag{6.2.17}$$

where δ_{α,α_1} is the Kronecker symbol and it is understood that $X^0_\alpha(\alpha_1, \dots, \alpha_n)$ is zero if $(\alpha_1, \dots, \alpha_n)$ is the empty word.

Lemma 6.2.54. *For $\alpha, \beta \in \Delta$, we have*

$$[X^0_\alpha, X^0_{-\alpha}] = H^0_\alpha \tag{6.2.18}$$

$$[H^0_\alpha, H^0_\beta] = 0 \tag{6.2.19}$$

$$[H^0_\alpha, H^0_\beta] = n(\beta, \alpha) X^0_\beta \tag{6.2.20}$$

$$[H^0_\alpha, H^0_{-\beta}] = -n(\beta, \alpha) X^0_{-\beta} \tag{6.2.21}$$

$$[X^0_\alpha, X^0_{-\beta}] = 0 \text{ if } \alpha \neq \beta. \tag{6.2.22}$$

Proof. Indeed, relation (6.2.17) can be written as

$$(X^0_\alpha X^0_{-\alpha_1})(\alpha_2, \dots, \alpha_n) = (X^0_{-\alpha_1} X^0_\alpha)(\alpha_2, \dots, \alpha_n) + \delta_{\alpha,\alpha_1} H^0_\alpha(\alpha_2, \dots, \alpha_n)$$

which proves (6.2.18) and (6.2.22). Relation (6.2.16) is clear. Next,

$$\begin{aligned}
[H^0_\alpha, X^0_{-\beta}](\alpha_1, \dots, \alpha_n) &= H^0_\alpha X^0_{-\beta}(\alpha_1, \dots, \alpha_n) - X^0_{-\beta} H^0_\alpha(\alpha_1, \dots, \alpha_n) \\
&= H^0_\alpha(\beta, \alpha_1, \dots, \alpha_n) + \sum_{i=1}^n n(\alpha_i, \alpha)(\beta, \alpha_1, \dots, \alpha_n) \\
&= -n(\beta, \alpha)(\beta, \alpha_1, \dots, \alpha_n) \\
&= -n(\beta, \alpha) X^0_{-\beta}(\alpha_1, \dots, \alpha_n).
\end{aligned}$$

This proves (6.2.21). Finally, note that by (6.2.18), (6.2.19) and (6.2.22) implies that

$$[H^0_\alpha, [X^0_\beta, X^0_{-\gamma}]] = 0,$$

whence

$$\begin{aligned}0 &= [H_\alpha^0, [X_\beta^0, X_{-\gamma}^0]] = [[H_\alpha^0, X_\beta^0], X_{-\gamma}^0] + [X_\beta^0, [H_\alpha^0, X_{-\gamma}^0]] \\ &= [[H_\alpha^0, X_\beta^0] - n(\gamma,\alpha)X_\beta^0, X_{-\gamma}^0] \\ &= [[H_\alpha^0, X_\beta^0] - n(\gamma,\alpha)X_\beta^0, X_{-\gamma}^0]\end{aligned} \tag{6.2.23}$$

where in the last step we use (6.2.22). Now, considering the empty word immediately gives

$$([H_\alpha, X_\beta^0] - n(\beta,\alpha)X_\beta^0)(\varnothing) = 0$$

so (6.2.23) implies that

$$([H_\alpha^0, X_\beta^0] - n(\beta,\alpha)X_\beta^0)X_{-\gamma_1}^0 \cdots X_{-\gamma_n}^0(\varnothing) = 0$$

for all $\gamma_1, \ldots, \gamma_n \in \Delta$. This proves (6.2.20). □

Lemma 6.2.55. *The endomorphisms $X_\alpha^0, H_\beta^0, X_{-\gamma}^0$ where $\alpha, \beta, \gamma \in \Delta$, are linearly independent.*

Proof. Since $X_{-\alpha}^0(\varnothing) = \alpha$, it is clear that the $X_{-\alpha}^0$ are linearly independent. Assume that $\sum_\alpha c_\alpha H_\alpha^0 = 0$; then, for all $\beta \in \Delta$,

$$0 = \Big[\sum_\alpha c_\alpha H_\alpha^0, X_{-\beta}^0\Big] = -\sum_\alpha c_\alpha n(\beta,\alpha) X_{-\beta}^0;$$

since $\det(n(\beta,\alpha)) \neq 0$, it follows that $c_\alpha = 0$ for all α. Assume that $\sum_\alpha c_\alpha X_\alpha^0 = 0$. In view of formulas (6.2.15), (6.2.16), (6.2.17),

$$X_\alpha^0(\beta) = 0. \quad X_\alpha^0(\beta,\beta) = 2\delta_{\alpha\beta}\beta$$

for all $\beta \in \Delta$. It follows that $c_\alpha = 0$ for all α. Since $X_\alpha^0, H_\alpha^0, X_{-\alpha}^0$ are of degree $-1, 0, 1$, respectively, the lemma follows from what has gone before. □

Let $X = B \times \{-1, 0, 1\}$. Put $X_\alpha = (\alpha, -1)$, $H_\alpha = (\alpha, 0)$, and $X_{-\alpha} = (\alpha, 1)$. Let $\hat{\mathfrak{g}}$ be the Lie algebra defined by the generating family X and the following set $\mathcal{R}$ of relators:

$$\begin{aligned}&[H_\alpha, H_\beta] \\ &[H_\alpha, X_\beta] - n(\beta,\alpha)X_\beta \\ &[H_\alpha, X_{-\beta}] + n(\beta,\alpha)X_{-\beta} \\ &[X_\alpha, X_{-\alpha}] + H_\alpha \\ &[X_\alpha, X_{-\beta}] \text{ if } \alpha \neq \beta.\end{aligned}$$

By Lemma 6.2.54, there exists a unique linear representation ρ of $\hat{\mathfrak{g}}$ on E such that

$$\rho(X_\alpha) = X_\alpha^0, \quad \rho(H_\alpha) = H_\alpha^0, \quad \rho(X_{-\alpha}) = X_{-\alpha}^0.$$

In view of Lemma 6.2.55, the canonical images in $\hat{\mathfrak{g}}$ of the elements X_α, H_β, $X_{-\gamma}$, where $\alpha, \beta, \gamma \in \Delta$ are linearly independent. In the following, we identify X_α, H_α, $X_{-\alpha}$ with their canonical images in $\hat{\mathfrak{g}}$.

Lemma 6.2.56. *There exists a unique involutive automorphism θ of $\hat{\mathfrak{g}}$ such that*

$$\theta(X_\alpha) = -X_{-\alpha}, \quad \theta(X_{-\alpha}) = -X_\alpha, \quad \theta(H_\alpha) = -H_\alpha$$

*for all $\alpha \in \Delta$. This automorphism is called the **canonical involutive automorphism** of $\hat{\mathfrak{g}}$.*

Proof. Indeed, there exists an involutive automorphism of the free Lie algebra $L(X)$ satisfying these conditions. It leaves $\mathcal{R} \cup (-\mathcal{R})$ stable, and hence defines by passage to the quotient an involutive automorphism of a satisfying the conditions of the lemma. The uniqueness follows from the fact that $\hat{\mathfrak{g}}$ is generated by the elements X_α, H_α, $X_{-\alpha}$ for $\alpha \in \Delta$. □

Let Λ_r be the root lattice of Φ; this is a free $\mathbb{Z}$-module with basis Δ. There exists a graduation of type Λ_r on the free Lie algebra $L(X)$ such that X_α, H_α, $X_{-\alpha}$ are of degrees $\alpha, 0, -\alpha$, respectively. Now the elements of Φ are homogeneous. Hence there exists a unique graduation of type Λ_r on $\hat{\mathfrak{g}}$ compatible with the Lie algebra structure of $\hat{\mathfrak{g}}$ and such that X_α, H_α, $X_{-\alpha}$ are of degrees $\alpha, 0, -\alpha$, respectively. For any $\mu \in \Lambda_r$, denote by $\hat{\mathfrak{g}}_\mu$ the set of elements of $\hat{\mathfrak{g}}$ homogeneous of degree μ.

Lemma 6.2.57. *Let $z \in \hat{\mathfrak{g}}$. Then $z \in \hat{\mathfrak{g}}_\mu$ if and only if $[H_\alpha, z] = \langle \mu, \check{\alpha} \rangle z$ for all $\alpha \in \Delta$.*

Proof. For $\mu \in \Lambda_r$, let $\hat{\mathfrak{g}}_{(\mu)}$ be the set of $x \in \hat{\mathfrak{g}}$ such that $[H_\alpha, x] = \langle \mu, \check{\alpha} \rangle x$ for all $\alpha \in \Delta$. The sum of the $\hat{\mathfrak{g}}_{(\mu)}$ is direct. To prove the lemma, it therefore suffices to show that $\hat{\mathfrak{g}}_\mu \subseteq \hat{\mathfrak{g}}_{(\mu)}$. Let $\alpha \in \Delta$. The endomorphism u of the vector space $\hat{\mathfrak{g}}$ such that $u|_{\hat{\mathfrak{g}}_\mu} = \langle \mu, \check{\alpha} \rangle \cdot 1$ is a derivation of $\hat{\mathfrak{g}}$ such that $ux = \mathrm{ad}(H_\alpha)(x)$ for $x = X_\beta$, $x = H_\beta$, $x = X_{-\beta}$; hence $u = \mathrm{ad}(H_\alpha)$, which proves our assertion. ☐

Denote by Λ_r^+ (resp. Λ_r^-) the set of linear combinations of elements of Δ with positive (resp. negative) integer coefficients, not all zero. Put $\hat{\mathfrak{g}}^+ = \sum_{\mu \in \Lambda_r^+} \hat{\mathfrak{g}}_\mu$ and $\hat{\mathfrak{g}}_- = \sum_{\mu \in \Lambda_r^-} \hat{\mathfrak{g}}_\mu$. Since $\Lambda_r^+ + \Lambda_r^+ \subseteq \Lambda_r^+$ and $\Lambda_r^- + \Lambda_r^- \subseteq \Lambda_r^-$, $\hat{\mathfrak{g}}^+$ and $\hat{\mathfrak{g}}^-$ are Lie subalgebras of $\hat{\mathfrak{g}}$.

Proposition 6.2.58. *Let $\hat{\mathfrak{g}}$ be the Lie algebra defined above.*

(a) The Lie algebra $\hat{\mathfrak{g}}^+$ is generated by the family $(X_\alpha)_{\alpha \in \Delta}$.

(b) The Lie algebra $\hat{\mathfrak{g}}^-$ is generated by the family $(X_{-\alpha})_{\alpha \in \Delta}$.

(c) The family $(H_\alpha)_{\alpha \in \Delta}$ is a basis of the vector space $\hat{\mathfrak{g}}_0$.

(d) The vector space $\hat{\mathfrak{g}}$ is the direct sum of $\hat{\mathfrak{g}}^+$, $\hat{\mathfrak{g}}^-$, and $\hat{\mathfrak{g}}_0$.

Proof. Let $\hat{\mathfrak{n}}^+$ (resp. $\hat{\mathfrak{n}}^-$) be the Lie subalgebra of $\hat{\mathfrak{g}}$ generated by $(X_\alpha)_{\alpha \in \Delta}$ (resp. $(X_{-\alpha})_{\alpha \in \Delta}$), and $\hat{\mathfrak{h}}$ the vector subspace of $\hat{\mathfrak{g}}$ generated by $(H_\alpha)_{\alpha \in \Delta}$. Since the X_α are homogeneous elements of $\hat{\mathfrak{g}}^+$, $\hat{\mathfrak{n}}^+$ is a graded subalgebra of $\hat{\mathfrak{g}}^+$; hence, $[\hat{\mathfrak{h}}, \hat{\mathfrak{n}}^+] \subseteq \hat{\mathfrak{n}}^+$, so $\hat{\mathfrak{h}} + \hat{\mathfrak{n}}^+$ is a subalgebra of $\hat{\mathfrak{g}}$; since

$$[X_{-\alpha}, X_\beta] = \delta_{\alpha\beta} H_\alpha$$

we see $[X_{-\alpha}, \hat{\mathfrak{n}}^+] \subseteq \hat{\mathfrak{h}} + \hat{\mathfrak{n}}^+$ for all $\alpha \in \Delta$. Similarly, $\hat{\mathfrak{n}}^-$ is a graded subalgebra of $\hat{\mathfrak{g}}^-$, one has $[\hat{\mathfrak{h}}, \hat{\mathfrak{n}}^-] \subseteq \hat{\mathfrak{n}}^-$, $\hat{\mathfrak{h}} + \hat{\mathfrak{n}}^-$ is a subalgebra of $\hat{\mathfrak{g}}$, and $[X_\alpha, \hat{\mathfrak{n}}^-] \subseteq \hat{\mathfrak{h}} + \hat{\mathfrak{n}}^-$ for all $\alpha \in \Delta$. Put $\hat{\mathfrak{a}} = \hat{\mathfrak{n}}^- + \hat{\mathfrak{h}} + \hat{\mathfrak{n}}^+$. The preceding shows that $\hat{\mathfrak{a}}$ is stable under $\mathrm{ad}(X_\alpha)$, $\mathrm{ad}(H_\alpha)$ and $\mathrm{ad}(X_{-\alpha})$ for all $\alpha \in \Delta$, and hence is an ideal of $\hat{\mathfrak{g}}$. Since $\hat{\mathfrak{a}}$ contains X_α, H_α, $X_{-\alpha}$ for all $\alpha \in \Delta$, $\hat{\mathfrak{a}} = \hat{\mathfrak{g}}$. It follows from this that the inclusions $\hat{\mathfrak{n}}^+ \subseteq \hat{\mathfrak{g}}^+$, $\hat{\mathfrak{h}} \subseteq \hat{\mathfrak{g}}_0$, $\hat{\mathfrak{n}}^- \subseteq \hat{\mathfrak{g}}^-$ are equalities, which proves the proposition. ☐

Proposition 6.2.59. *The Lie algebra $\mathfrak{g}^+$ (resp. $\mathfrak{g}^-$) is a free Lie algebra with basic family $(X_\alpha)_{\alpha \in \Delta}$ (resp. $(X_{-\alpha})_{\alpha \in \Delta}$).*

Proof. Let L be the Lie subalgebra of E generated by Δ. By **??**, L can be identified with the free Lie algebra generated by Δ. The left regular representation of E on itself is clearly injective, and defines by restriction to L an injective representation $\tilde{\rho}$ of the Lie algebra L on E. Let φ be the unique homomorphism from L to $\mathfrak{g}^-$ which takes α to $X_{-\alpha}$ for all $\alpha \in \Delta$.

$$\begin{tikzcd} L \arrow[r, "\tilde{\rho}"] \arrow[d, "\varphi"'] & \mathrm{End}(E) \\ \hat{\mathfrak{g}}^- \arrow[ur, "\rho"'] & \end{tikzcd}$$

Then, for all $\alpha \in \Delta$, $\rho(\varphi(\alpha)) = X^0_{-\alpha}$ is the endomorphism of left multiplication by α on E, so $\rho \circ \varphi = \tilde{\rho}$, which proves that φ is injective. Thus, $(X_{-\alpha})_{\alpha \in \Delta}$ is a basic family for $\hat{\mathfrak{g}}^-$. Since $\theta(X_{-\alpha}) = -X_\alpha$ for all α, $(X_\alpha)_{\alpha \in \Delta}$ is a basic family for $\hat{\mathfrak{g}}^+$. ☐

Recall that if $\alpha, \beta \in \Delta$ and if $\alpha \neq \beta$, then $n(\beta, \alpha) \leq 0$; moreover, if $n(\beta, \alpha) = 0$, then $n(\alpha, \beta) = 0$. For any pair (α, β) of distinct elements of Δ, put

$$X_{\alpha\beta} = \mathrm{ad}(X_\alpha)^{1-n(\beta,\alpha)}(X_\beta), \quad Y_{\alpha\beta} = \mathrm{ad}(X_{-\alpha})^{1-n(\beta,\alpha)}(X_{-\beta}).$$

Then $X_{\alpha\beta} \in \hat{\mathfrak{g}}^+$ and $Y_{\alpha\beta} \in \hat{\mathfrak{g}}^-$.

Lemma 6.2.60. *Let $\alpha, \beta \in \Delta$ with $\alpha \neq \beta$. Then*

$$[\hat{\mathfrak{g}}^+, Y_{\alpha\beta}] = 0, \quad [\hat{\mathfrak{g}}^-, X_{\alpha,\beta}] = 0.$$

Proof. The second formula follows from the first by using the automorphism θ. To prove the first, it suffices to show that $[X_\gamma, Y_{\alpha,\beta}] = 0$ for all $\gamma \in \Delta$. We first note that, if $\gamma \neq \alpha$ and $\gamma \neq \beta$, then X_γ commutes with $X_{-\alpha}$ and $X_{-\beta}$, and hence with $Y_{\alpha\beta}$.

If $\gamma = \beta$, then $\gamma \neq \alpha$ and X_γ commutes with $X_{-\alpha}$, so

$$\begin{aligned}[X_\gamma, Y_{\alpha\beta}] &= \mathrm{ad}(X_{-\alpha})^{1-n(\beta,\alpha)}([X_\gamma, X_{-\beta}]) = -\mathrm{ad}(X_{-\alpha})^{1-n(\beta,\alpha)}(H_\beta) \\ &= -n(\alpha,\beta)\mathrm{ad}(X_{-\alpha})^{-n(\beta,\alpha)}(X_{-\alpha}).\end{aligned}$$

If $n(\beta,\alpha) < 0$, this expression is zero since $\mathrm{ad}(X_{-\alpha})(X_{-\alpha}) = 0$. If $n(\beta,\alpha) = 0$, then $n(\alpha,\beta) = 0$. In both cases, $[X_\gamma, Y_{\alpha\beta}] = 0$.

If $\gamma = \alpha$, then $\gamma \neq \beta$. In the algebra of endomorphisms of $\hat{\mathfrak{g}}$,

$$\begin{aligned}[\mathrm{ad}(H_\alpha), \mathrm{ad}(X_\alpha)] &= \mathrm{ad}([H_\alpha, X_\alpha]) = 2\,\mathrm{ad}(X_\alpha), \\ [\mathrm{ad}(H_\alpha), \mathrm{ad}(X_{-\alpha})] &= \mathrm{ad}([H_\alpha, X_{-\alpha}]) = -2\,\mathrm{ad}(X_{-\alpha}), \\ [\mathrm{ad}(X_\alpha), \mathrm{ad}(X_{-\alpha})] &= \mathrm{ad}([X_\alpha, X_{-\alpha}]) = \mathrm{ad}(H_\alpha).\end{aligned}$$

Therefore, by Lemma 6.2.1, we get

$$[\mathrm{ad}(X_\alpha), \mathrm{ad}(X_{-\alpha})^{1-n(\beta,\alpha)}] = (1-n(\beta,\alpha))\mathrm{ad}(X_{-\alpha})^{-n(\beta,\alpha)}(\mathrm{ad}(H_\alpha) + n(\beta,\alpha)).$$

Concequently,

$$\begin{aligned}[X_\gamma, Y_{\alpha\beta}] &= [\mathrm{ad}(X_\alpha), \mathrm{ad}(X_{-\alpha})^{1-n(\beta,\alpha)}](X_{-\beta}) + \mathrm{ad}(X_{-\alpha})^{1-n(\beta,\alpha)}\mathrm{ad}(X_\alpha)(X_{-\beta}) \\ &= (1-n(\beta,\alpha))\mathrm{ad}(X_{-\alpha})^{-n(\beta,\alpha)}(\mathrm{ad}(H_\alpha) + n(\beta,\alpha))(X_{-\beta}) + \mathrm{ad}(X_{-\alpha}).\end{aligned}$$

Now $[H_\alpha, X_{-\beta}] + n(\beta,\alpha)X_{-\beta} = 0$ and $[X_\alpha, X_{-\beta}] = 0$, so $[X_\alpha, Y_{\alpha\beta}] = 0$. □

Lemma 6.2.61. *The ideal $\hat{\mathfrak{a}}^+$ of $\hat{\mathfrak{g}}^+$ generated by the $X_{\alpha\beta}$ ($\alpha, \beta \in \Delta$, $\alpha \neq \beta$) is an ideal of $\mathfrak{g}$. The ideal $\hat{\mathfrak{a}}^-$ of $\mathfrak{g}^-$ generated by the $Y_{\alpha\beta}$ ($\alpha, \beta \in \Delta$, $\alpha \neq \beta$) is an ideal of $\mathfrak{g}$ and is equal to $\theta(\hat{\mathfrak{a}}^+)$.*

Proof. Let $\hat{\mathfrak{a}} = \sum_{\alpha,\beta\in\Delta,\alpha\neq\beta} \mathbb{K}X_{\alpha\beta}$. Since each $X_{\alpha\beta}$ is homogeneous in $\hat{\mathfrak{g}}$, we have $[\hat{\mathfrak{g}}_0, \hat{\mathfrak{a}}] \subseteq \hat{\mathfrak{a}}$ by Lemma 6.2.57 and Proposition 6.2.58. Let U (resp. V) be the enveloping algebra of $\hat{\mathfrak{g}}$ (resp. $\hat{\mathfrak{g}}^+$), and σ the representation of U on $\hat{\mathfrak{g}}$ defined by the adjoint representation of $\hat{\mathfrak{g}}$. The ideal of $\hat{\mathfrak{g}}$ generated by $\hat{\mathfrak{a}}$ is $\sigma(U)\hat{\mathfrak{a}}$. Now $\hat{\mathfrak{g}} = \hat{\mathfrak{g}}^- + \hat{\mathfrak{g}}_0 + \hat{\mathfrak{g}}^+$ (Proposition 6.2.58), $\sigma(\hat{\mathfrak{g}}^-)\hat{\mathfrak{a}} = 0$, and $\sigma(\hat{\mathfrak{g}}_0) \subseteq \hat{\mathfrak{a}}$ by Lemma 6.2.60. By the Poincaré-Birkhoff-Witt theorem, $\sigma(U)\hat{\mathfrak{a}} = \sigma(V)\hat{\mathfrak{a}}$, which proves the first assertion of the lemma. It follows that the ideal of $\theta(\hat{\mathfrak{g}}^+) = \hat{\mathfrak{g}}^-$ generated by the $\theta(\mathbb{K}X_{\alpha\beta}) = \mathbb{K}Y_{\alpha\beta}$ ($\alpha, \beta \in \Delta$, $\alpha \neq \beta$) is the ideal $\theta(\hat{\mathfrak{a}})$ of $\hat{\mathfrak{g}}$. □

The ideal $\hat{\mathfrak{a}}^+ + \hat{\mathfrak{a}}^-$ of $\hat{\mathfrak{g}}$ is graded since it is generated by homogeneous elements. Consequently, the Lie algebra $\hat{\mathfrak{g}}/(\hat{\mathfrak{a}}^+ + \hat{\mathfrak{a}}^-)$ is a Λ_r-graded Lie algebra; in the remainder of this paragraph, it is denoted by $\mathfrak{g}_\Delta$, or simply by $\mathfrak{g}$. By Proposition 6.2.58, if $\mathfrak{g}^\mu \neq 0$ then $\mu \in \Lambda_r^+$, or $\mu \in \Lambda_r^-$, or $\mu = 0$. Denote by x_α (resp. h_α, $x_{-\alpha}$) the canonical image of X_α (resp. H_α, $X_{-\alpha}$) in $\mathfrak{g}$. In view of the definition of $\hat{\mathfrak{g}}$, $\hat{\mathfrak{a}}^+$ and $\hat{\mathfrak{a}}^-$, it follows that $\mathfrak{g}$ is the Lie algebra defined by the generating family $(x_\alpha, h_\alpha, x_{-\alpha})_{\alpha\in\Delta}$ and the relations

$$[h_\alpha, h_\beta] = 0 \tag{6.2.24}$$

$$[h_\alpha, X_\beta] - n(\beta,\alpha)x_\beta = 0 \tag{6.2.25}$$

$$[h_\alpha, x_{-\beta}] + n(\beta,\alpha)x_{-\beta} = 0 \tag{6.2.26}$$

$$[x_\alpha, x_{-\alpha}] - h_\alpha = 0 \tag{6.2.27}$$

$$[x_\alpha, x_{-\beta}] = 0 \text{ if } \alpha \neq \beta \tag{6.2.28}$$

$$\mathrm{ad}(x_\alpha)^{1-n(\beta,\alpha)}(x_\beta) = 0 \text{ if } \alpha \neq \beta \tag{6.2.29}$$

$$\mathrm{ad}(x_{-\alpha})^{1-n(\beta,\alpha)}(x_{-\beta}) = 0 \text{ if } \alpha \neq \beta. \tag{6.2.30}$$

Let $z \in \mathfrak{g}$ and $\mu \in \Lambda_r$. Then $z \in \mathfrak{g}^\mu$ if and only if $[h_\alpha, z] = \langle \mu, \check{\alpha} \rangle z$ for all $\alpha \in \Delta$. This follows from Lemma 6.2.57. Since $\hat{\mathfrak{g}}_0 \cap (\hat{\mathfrak{a}}^+ + \hat{\mathfrak{a}}^-)$, the canonical map from $\hat{\mathfrak{g}}^0$ to $\mathfrak{g}^0$ is an isomorphism. Consequently, $(h_\alpha)_{\alpha\in\Delta}$ is a basis of the vector space $\mathfrak{g}^0$. The commutative subalgebra $\mathfrak{g}^0$ of $\mathfrak{g}$ will be denoted by $\mathfrak{h}_\Delta$ or simply by $\mathfrak{h}$. We note that there exists a unique isomorphism $\mu \mapsto \mu_\Delta$ from V to $\mathfrak{h}^*$ such that $\langle \mu_\Delta, h_\alpha \rangle = \langle \mu, \check{\alpha} \rangle$ for all $\mu \in V$ and all $\alpha \in \Delta$.

The involutive automorphism θ of $\hat{\mathfrak{g}}$ defines by passage to the quotient an involutive automorphism of $\mathfrak{g}$ that will also be denoted by θ. We have $\theta(x_\alpha) = -x_{-\alpha}$ for $\alpha \in \Delta \cup (-\Delta)$, and $\theta(h_\alpha) = -h_\alpha$.

Lemma 6.2.62. *Let $\alpha \in \Delta \cup (-\Delta)$. Then $\mathrm{ad}(x_\alpha)$ is locally nilpotent.*

Proof. Assume that $\alpha \in \Delta$. Let $\mathfrak{g}'$ be the set of $z \in \mathfrak{g}$ such that $\mathrm{ad}(x_\alpha)^p(z) = 0$ for sufficiently large p. Since $\mathrm{ad}(x_\alpha)$ is a derivation of $\mathfrak{g}$, $\mathfrak{g}'$ is a subalgebra of $\mathfrak{g}$. By (6.2.13), $x_\beta \in \mathfrak{g}'$ for all $\beta \in \Delta$. By (6.2.9), (6.2.11), (6.2.12), $h_\beta \in \mathfrak{g}'$ and $x_{-\beta} \in \mathfrak{g}'$ for all $\beta \in \Delta$. Hence $\mathfrak{g}' = \mathfrak{g}$ and $\mathrm{ad}(x_\alpha)$ is locally nilpotent. Since $\mathrm{ad}(x_{-\alpha}) = \theta \circ \mathrm{ad}(x_\alpha) \circ \theta^{-1}$ we see that $\mathrm{ad}(x_{-\alpha})$ is locally nilpotent. □

Lemma 6.2.63. *Let $\mu, \nu \in \Lambda_r$ and $w \in W(\Phi)$ be such that $w(\mu) = \nu$. Then there exists an automorphim of $\mathfrak{g}$ that takes $\mathfrak{g}^\mu$ to $\mathfrak{g}^\nu$.*

Proof. For all $\alpha \in \Delta$, let s_α be the reflection in V defined by α. Since $W(\Phi)$ is generated by the s_α, it suffices to prove the lemma when $w = s_\alpha$. In view of Lemma 6.2.62, we can define an automorphim of $\mathfrak{g}$ by

$$\theta_\alpha = e^{\mathrm{ad}(x_\alpha)} e^{-\mathrm{ad}(x_{-\alpha})} e^{\mathrm{ad}(x_\alpha)}. \tag{6.2.31}$$

We have

$$\begin{aligned}
\theta_\alpha(h_\beta) &= e^{\mathrm{ad}(x_\alpha)} e^{-\mathrm{ad}(x_{-\alpha})} e^{\mathrm{ad}(x_\alpha)}(h_\beta) = e^{\mathrm{ad}(x_\alpha)} e^{-\mathrm{ad}(x_{-\alpha})}(h_\beta - n(\alpha,\beta)x_\alpha) \\
&= e^{\mathrm{ad}(x_\alpha)}\Big(h_\beta - n(\alpha,\beta)x_\alpha - n(\alpha,\beta)x_{-\alpha} - n(\alpha,\beta)h_\alpha + \frac{n(\alpha,\beta)}{2}2x_{-\alpha}\Big) \\
&= e^{\mathrm{ad}(x_\alpha)}(h_\beta - n(\alpha,\beta)h_\alpha - n(\alpha,\beta)x_\alpha) \\
&= h_\beta - n(\alpha,\beta)h_\alpha - n(\alpha,\beta)x_\alpha - n(\alpha,\beta)x_\alpha + 2n(\alpha,\beta)x_\alpha \\
&= h_\beta - n(\alpha,\beta)h_\alpha.
\end{aligned}$$

Therefore θ_α is involutive and if $z \in \mathfrak{g}^\mu$,

$$\begin{aligned}
[h_\beta, \theta_\alpha^{-1}(z)] &= \theta_\alpha^{-1}[\theta_\alpha(h_\beta), z] = [h_\beta - n(\alpha,\beta)h_\alpha, z] = \theta_\alpha^{-1}(\langle \mu, \check{\beta}\rangle z - n(\alpha,\beta)\langle \mu, \check{\alpha}\rangle z) \\
&= \theta_\alpha^{-1}(\langle \mu, \check{\beta}\rangle z - \langle \alpha, \check{\beta}\rangle\langle \mu, \check{\alpha}\rangle z) = \theta_\alpha^{-1}(\langle \mu - \langle \mu, \check{\alpha}\rangle\alpha, \check{\beta}\rangle z) = \langle s_\alpha\mu, \check{\beta}\rangle \theta_\alpha^{-1}(z),
\end{aligned}$$

so $\theta_\alpha^{-1}(z) \in \mathfrak{g}^{s_\alpha(\mu)}$. This shows that $\theta_\alpha^{-1}(\mathfrak{g}^\mu) \subseteq \mathfrak{g}^{s_\alpha(\mu)}$. Since θ_α is an involution and since this inclusion holds for all $\mu \in \Lambda_r$, we see that $\theta_\alpha(\mathfrak{g}^\mu) = \mathfrak{g}^{s_\alpha(\mu)}$, which proves the lemma. □

Lemma 6.2.64. *Let $\mu \in \Lambda_r$ and assume that μ is not a multiple of a root. Then there exists $w \in W(\Phi)$ such that certain of the coordinates of $w(\mu)$ with respect to the basis Δ are positive and certain of them are negative.*

Proof. Let $V_\mathbb{R}$ be the vector space $\Lambda_r \otimes_\mathbb{Z} \mathbb{R}$, in which Φ is a root system. By the assumption, there exists $f \in V_\mathbb{R}^*$ such that $\langle f, \alpha\rangle \neq 0$ for all $\alpha \in \Phi$, and $\langle f, \mu\rangle = 0$. There exists a chamber $\check{C}$ of $\check{\Phi}$ such that $f \in \check{C}$. By Proposition 5.1.41, there exists $w \in W(\Phi)$ such that $w(f)$ belongs to the chamber associated to Δ, in other words such that $\langle w(f), \alpha\rangle > 0$ for all $\alpha \in \Delta$. Write $w(\mu) = \sum_{\alpha\in\Delta} t_\alpha \alpha$. Then

$$0 = \langle f, \mu\rangle = \langle w(f), w(\mu)\rangle = \sum_{\alpha\in\Delta} t_\alpha \langle w(f), \alpha\rangle,$$

which proves that certain t_α are positive and others are negative. □

Lemma 6.2.65. *Let $\mu \in \Lambda_r$. If $\mu \notin \Phi \cup \{0\}$, then $\mathfrak{g}^\mu = 0$. If $\mu \in \Phi$, then $\dim(\mathfrak{g}^\mu) = 1$.*

Proof. If μ is not a multiple of an element of Φ, there exists $w \in W$ such that $w(\mu) \notin \Lambda_r^+ \cup \Lambda_r^-$ (Lemma 6.2.64), so $\hat{\mathfrak{g}}_{w(\mu)} = 0$, $\mathfrak{g}^{w(\mu)} = 0$, and hence $\mathfrak{g}^\mu = 0$ by Lemma 6.2.63.

Let $\alpha \in \Delta$ and let m be an integer. Since $\hat{\mathfrak{g}}^+$ is a free Lie algebra with basic family $(x_\alpha)_{\alpha\in\Delta}$, we have $\dim(\hat{\mathfrak{g}}_\alpha) = 1$ and $\mathfrak{g}^{\hat{m}} = 0$ for $m > 1$. Hence $\dim(\mathfrak{g}^\alpha) \leq 1$ and $\mathfrak{g}^{m\alpha} = 0$ for $m > 1$. We cannot have $\mathfrak{g}^\alpha = 0$, as this would imply that $x_\alpha \in \hat{\mathfrak{a}}^+ + \hat{\mathfrak{a}}^-$, and hence that $\hat{\mathfrak{a}}^+ + \hat{\mathfrak{a}}^-$ contains $H_\alpha = [X_\alpha, X_{-\alpha}]$, whereas $\hat{\mathfrak{g}}_0 \cap (\hat{\mathfrak{a}}^+ + \hat{\mathfrak{a}}^-) \neq 0$. Consequently, $\dim(\mathfrak{g}^\alpha) = 1$.

If $\mu \in \Phi$, there exists $w \in W(\Phi)$ such that $w(\mu) \in \Delta$, so $\dim(\mathfrak{g}^\mu) = \dim(\mathfrak{g}^{w(\mu)}) = 1$. Moreover, if $n > 1$ is an integer then $\mathfrak{g}^{nw(\mu)=0}$ and so $\mathfrak{g}^{n\mu} = 0$. □

Fianlly, we can address the following existence theorem for the semi-simple Lie algebra with a reduced root system Φ.

Theorem 6.2.66. *Let Φ be a reduced root system, Δ a basis of Φ. Let $\mathfrak{g}$ be the Lie algebra defined by the generating family $(x_\alpha, h_\alpha, x_{-\alpha})_{\alpha\in\Delta}$ and the Serre relations. Let $\mathfrak{h} = \sum_{\alpha\in\Delta} \mathbb{K}h_\alpha$. Then $(\mathfrak{g}, \mathfrak{h})$ is a split semi-simple Lie algebra. The isomorphism $\mu \mapsto \mu_\Delta$ from V to $\mathfrak{h}^*$ maps Φ to the root system of $(\mathfrak{g}, \mathfrak{h})$. For all $\mu \in \Delta$, $\mathfrak{g}^\mu$ is the eigenspace relative to the root μ.*

Proof. Since $\dim(\mathfrak{g}^0) = |\Delta|$, it follows from Lemma 6.2.65 that $\mathfrak{g}$ is of finite dimension equal to $|\Delta| + |\Phi|$. We show that $\mathfrak{g}$ is semi-simple. Let $\mathfrak{k}$ be an abelian ideal of $\mathfrak{g}$. Since $\mathfrak{k}$ is stable under $\mathrm{ad}(\mathfrak{h})$, $\mathfrak{k} = (\mathfrak{k}\cap\mathfrak{h}) + \sum_{\mu\in\Phi}(\mathfrak{k}\cap\mathfrak{g}^\mu)$. It is clear that, for all $\alpha \in \Delta$, $\mathfrak{g}^\alpha + \mathfrak{g}^{-\alpha} + \mathbb{K}h_\alpha$ is isomorphic to $\mathfrak{sl}(2,\mathbb{K})$. In view of Lemma 6.2.65, for all $\mu \in \Phi$, $\mathfrak{g}^\mu$ is contained in a subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}(2,\mathbb{K})$; consequently, $\mathfrak{k}\cap\mathfrak{g}^\mu = 0$ for all $\mu \in \Phi$, so $\mathfrak{k} \subseteq \mathfrak{h}$; hence $[\mathfrak{k}, \mathfrak{g}^\mu] \subseteq \mathfrak{k}\cap\mathfrak{g}^\mu = 0$, so $\mu_\Delta(\mathfrak{k}) = 0$ for all $\mu \in \Phi$. It follows that $\mathfrak{k} = 0$, which proves that $\mathfrak{g}$ is semi-simple.

Let $\mu \in \Phi$. There exists $\alpha \in \Delta$ such that $\langle \mu, \check{\alpha}\rangle \neq 0$, and $\mathrm{ad}(h_\alpha)|_{\mathfrak{g}^\mu}$ is then a non-zero homothety. Consequently, $\mathfrak{h}$ is equal to its own normalizer in $\mathfrak{g}$, and hence is a Cartan subalgebra of $\mathfrak{g}$. For all $h \in \mathfrak{h}$, $\mathrm{ad}(h)$ is diagonalizable, so $(\mathfrak{g}, \mathfrak{h})$ is a split semi-simple Lie algebra. For all $\mu \in \Phi$, it is clear that μ_Δ is a root of $(\mathfrak{g}, \mathfrak{h})$ and that $\mathfrak{g}^\mu$ is the corresponding eigenspace. The number of roots of $(\mathfrak{g}, \mathfrak{h})$ is $\dim(\mathfrak{g}) - \dim(\mathfrak{h}) = |\Phi|$. Hence, the map $\mu \mapsto \mu_\Delta$ from V to $\mathfrak{h}^*$ maps Φ to the root system of $(\mathfrak{g}, \mathfrak{h})$. □

With the existence theorem, we now establish the uniqueness of split semi-simple Lie algebras. More precisely, we show that any split semi-simple Lie algebra with root system Φ is isomorphic to our previous construction.

Proposition 6.2.67. *Let $(\mathfrak{g}, \mathfrak{h}, \Delta, (x_\alpha)_{\alpha\in\Delta})$ be a framed semi-simple Lie algebra. Let $(n(\alpha,\beta))_{\alpha,\beta\in\Delta}$ and $(x_\alpha)_{\alpha\in\Delta\cup(-\Delta)}$ be the corresponding Cartan matrix and generating family.*

(a) *The family $(x_\alpha, h_\alpha, x_{-\alpha})_{\alpha\in\Delta}$ and the Serre relations constitute a presentation of $\mathfrak{g}$.*

(b) *The family $(x_\alpha)_{\alpha\in\Delta}$ and the relations (6.2.29) constitute a presentation of the subalgebra of $\mathfrak{g}$ generated by $(x_\alpha)_{\alpha\in\Delta}$.*

Proof. Let Φ be the root system of $(\mathfrak{g}, \mathfrak{h})$. Applying to Φ and Δ the previous constructions, we obtain objects that we shall denote by $\hat{\mathfrak{g}}', \mathfrak{g}', x'_\alpha, h'_\alpha$ instead of $\hat{\mathfrak{g}}, \mathfrak{g}, x_\alpha, h_\alpha$. There exists a homomorphism φ from the Lie algebra $\mathfrak{g}'$ to the Lie algebra $\mathfrak{g}$ such that $\varphi(x'_\alpha) = x_\alpha$, $\varphi(h'_\alpha) = h_\alpha$, and $\varphi(x'_{-\alpha}) = x_{-\alpha}$ for all $\alpha \in \Delta$ (Proposition 6.2.53). Since $\dim(\mathfrak{g}') = |\Phi| + |\Delta| = \dim(\mathfrak{g})$, φ is bijective. This proves (a). The second claim follows from Proposition 6.2.59. □

Corollary 6.2.68. *Every framed semi-simple Lie algebra is obtained from a framed semi-simple $\mathbb{Q}$-Lie algebra by extension of scalars from $\mathbb{Q}$ to $\mathbb{K}$.*

Proof. This follows from Proposition 6.2.67 and the fact that our construction for $\mathfrak{g}_\Delta$ works for $\mathbb{Q}$. □

Theorem 6.2.69. *Let $(\mathfrak{g}_1, \mathfrak{h}_1, \Delta_1, (x^1_\alpha)_{\alpha\in\Delta_1})$ and $(\mathfrak{g}_2, \mathfrak{h}_2, \Delta_2, (x^2_\alpha)_{\alpha\in\Delta_2})$ be framed semi-simple Lie algebras, let Φ_1 and Φ_2 be the root systems of $(\mathfrak{g}_1, \mathfrak{h}_1)$ and $(\mathfrak{g}_2, \mathfrak{h}_2)$, let $(n_1(\alpha,\beta))$ (resp. $(n_2(\alpha,\beta))$) be the Cartan matrix of Φ_1 (resp. Φ_2) relative to Δ_1 (resp. Δ_2), and let Γ_1 (resp. Γ_2) be the Dynkin graph of Φ_1 (resp. Φ_2) relative to Δ_1 (resp. Δ_2).*

(a) *If φ is an isomorphism from $\mathfrak{h}_1^*$ to $\mathfrak{h}_2^*$ such that $\varphi(\Phi_1) = \Phi_2$ and $\varphi(\Delta_1) = \Delta_2$, there exists a unique isomorphism ψ from $(\mathfrak{g}_1, \mathfrak{h}_1, \Delta_1, (x^1_\alpha)_{\alpha\in\Delta_1})$ to $(\mathfrak{g}_2, \mathfrak{h}_2, \Delta_2, (x^2_\alpha)_{\alpha\in\Delta_2})$ such that $\psi|_{\mathfrak{h}_1} = (\varphi^{-1})^t$.*

(b) *If f is a bijection from Δ_1 to Δ_2 such that $n_2(f(\alpha), f(\beta)) = n_1(\alpha,\beta)$ for all $\alpha,\beta \in \Delta_1$, there exists an isomorphism from $(\mathfrak{g}_1, \mathfrak{h}_1, \Delta_1, (x^1_\alpha)_{\alpha\in\Delta_1})$ to $(\mathfrak{g}_2, \mathfrak{h}_2, \Delta_2, (x^2_\alpha)_{\alpha\in\Delta_2})$.*

(c) *If Γ_1 is isomorphic to Γ_2, then there exists an isomorphism from $(\mathfrak{g}_1, \mathfrak{h}_1, \Delta_1, (x^1_\alpha)_{\alpha\in\Delta_1})$ to $(\mathfrak{g}_2, \mathfrak{h}_2, \Delta_2, (x^2_\alpha)_{\alpha\in\Delta_2})$.*

Proof. This follows immediately from Proposition 6.2.67. □

Proposition 6.2.70. *Let $(\mathfrak{g}, \mathfrak{h}, \Delta, (x_\alpha)_{\alpha\in\Delta})$ be a framed semi-simple Lie algebra, and $(x_\alpha)_{\alpha\in\Delta\cup(-\Delta)}$ the corresponding generating family. There exists a unique involutive automorphism θ of $\mathfrak{g}$ such that $\theta(x_\alpha) = x_{-\alpha}$ for all $\alpha \in \Delta\cup(-\Delta)$ and $\theta(h) = -h$ for all $h \in \mathfrak{h}$.*

Proof. The uniqueness is clear since $(x_\alpha)_{\alpha\in\Delta\cup(-\Delta)}$ generates the Lie algebra $\mathfrak{g}$. In view of Proposition 6.2.67, the existence of θ follows from our construction. □

Corollary 6.2.71. *Let $(\mathfrak{g}, \mathfrak{h})$ be a split semi-simple Lie algebra. Then $(\mathfrak{g}, \mathfrak{h})$ possesses a Chevalley system.*

Proof. Let Φ be the root system of $(\mathfrak{g}, \mathfrak{h})$. For all $\alpha \in \Phi$, let x_α be a non-zero element of $\mathfrak{g}^\alpha$. Assume that the x_α are chosen so that $[x_\alpha, x_{-\alpha}] = h_\alpha$ for all $\alpha \in \Phi$. Let Δ be a basis of Φ and θ the automorphism of $\mathfrak{g}$ such that $\theta(x_\alpha) = -x_{-\alpha}$ for all $\alpha \in \Delta \cup (-\Delta)$. We have $\theta|_{\mathfrak{h}} = -\mathrm{id}_{\mathfrak{h}}$. Hence, for all $\alpha \in \Phi$ there exists $t_\alpha \in \mathbb{K}^\times$ such that $\theta(x_\alpha) = t_\alpha x_{-\alpha}$. We have

$$t_\alpha t_{-\alpha} h_\alpha = [t_{-\alpha} x_\alpha, t_\alpha x_{-\alpha}] = [\theta(x_{-\alpha}), \theta(x_\alpha)] = \theta([x_{-\alpha}, x_\alpha]) = \theta(-h_\alpha) = h_\alpha,$$

so $t_\alpha t_{-\alpha} = 1$ for all $\alpha \in \Phi$. Let $N_{\alpha\beta}$ be the transition constants for the basis $(x_\alpha)_{\alpha \in \Delta \cup (-\Delta)}$. If $\alpha, \beta, \alpha + \beta \in \Phi$, then

$$\begin{aligned} N_{-\alpha,-\beta} t_\alpha t_\beta x_{-\alpha-\beta} &= t_\alpha t_\beta [x_{-\alpha}, x_{-\beta}] = [\theta(x_\alpha), \theta(x_\beta)] = \theta([x_\alpha, x_\beta]) \\ &= N_{\alpha\beta} \theta(x_{\alpha+\beta}) = N_{\alpha\beta} t_{\alpha+\beta} x_{-\alpha-\beta}. \end{aligned}$$

so, in view of Proposition 6.2.35,

$$(p+1)^2 t_\alpha t_\beta = N_{\alpha\beta}^2 t_{\alpha+\beta} \tag{6.2.32}$$

where q is an integer. It follows that if t_α and t_β are squares in $\mathbb{K}^\times$ (up to a sign), so is $t_{\alpha+\beta}$ (up to a sign). Since $t_\alpha = 1$ for all $\alpha \in \Delta$, Theorem 5.1.35 proves that t_α is a square for all $\alpha \in \Phi$. Choose, for all $\alpha \in \Phi$, a $c_\alpha \in \mathbb{K}$ such that $\pm c_\alpha^2 = t_\alpha$. This choice can be made so that $c_\alpha c_{-\alpha} = 1$ for all $\alpha \in \Phi$. Put $x'_\alpha = c_\alpha^{-1} x_\alpha$. Then, for all $\alpha \in \Phi$,

$$x'_\alpha \in \mathfrak{g}^\alpha, \quad [x'_\alpha, x'_{-\alpha}] = [x_\alpha, x_\alpha] = h_\alpha;$$

and

$$\theta(x'_\alpha) = \theta(c_\alpha x_\alpha) = c_\alpha^{-1} t_\alpha x_{-\alpha} = \pm c_\alpha x_{-\alpha} = \pm c_\alpha c_{-\alpha} x'_{-\alpha} = \pm x'_{-\alpha}.$$

Therefore it follows from (6.2.71) that $(x'_\alpha)_{\alpha \in \Phi}$ is a Chevalley system for $(\mathfrak{g}, \mathfrak{h})$. □

6.2.5 Automorphism of split semi-simple Lie algebras

Recall that $\mathrm{Aut}(\mathfrak{g})$ denotes the group of automorphisms of $\mathfrak{g}$. If $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$, we denote by $\mathrm{Aut}(\mathfrak{g}, \mathfrak{h})$ the group of automorphisms of $\mathfrak{g}$ that leave $\mathfrak{h}$ stable. Assume that $\mathfrak{h}$ is splitting, and let Φ be the root system of $(\mathfrak{g}, \mathfrak{h})$. If $\varphi \in \mathrm{Aut}(\mathfrak{g}, \mathfrak{h})$, the contragredient map of $\varphi|_{\mathfrak{h}}$ is an element of $\mathrm{Aut}(\Phi)$ (the group of automorphisms of Φ) which we shall denote by $\varepsilon(\varphi)$ in this paragraph. Thus

$$\varepsilon : \mathrm{Aut}(\mathfrak{g}, \mathfrak{h}) \to \mathrm{Aut}(\Phi)$$

is a homomorphism of groups. For any root system Φ and any basis Δ of Φ, we denote by $\mathrm{Aut}(\Phi, \Delta)$ the group of automorphisms of Φ that leave Δ stable. Recall that $\mathrm{Aut}(\Phi)$ is the semi-direct product of $\mathrm{Aut}(\Phi, \Delta)$ and $W(\Phi)$, and that $\mathrm{Aut}(\Phi)/W(\Phi)$ is canonically isomorphic to the group of automorphisms of the Dynkin graph of Φ.

Proposition 6.2.72. *Let $(\mathfrak{g}, \mathfrak{h}, \Delta, (x_\alpha)_{\alpha \in \Delta})$ be a framed semi-simple Lie algebra, and Φ the root system of $(\mathfrak{g}, \mathfrak{h})$. Let G be the subgroup of $\mathrm{Aut}(\mathfrak{g}, \mathfrak{h})$ of automorphisms of $(\mathfrak{g}, \mathfrak{h}, \Delta, (x_\alpha)_{\alpha \in \Delta})$. Then the restriction of ε to G is an isomorphism from G to $\mathrm{Aut}(\Phi, \Delta)$.*

Proof. If $\varphi \in G$, then $\varphi(x_\alpha) = x_{\varepsilon(\varphi)(\alpha)}$ for all $\alpha \in \Delta$, so it is clear that $\varepsilon(\varphi) \in \mathrm{Aut}(\Phi, \Delta)$. On the other hand, the map $\varepsilon|_G : G \to \mathrm{Aut}(\Phi, \Delta)$ is bijective by Theorem 6.2.69. □

Let Γ be an abelian group, and $A = \bigoplus_{\gamma \in \Gamma} A_\gamma$ an Γ-graded algebra. For any homomorphism f from the group Γ to the multiplicative group $\mathbb{K}^\times$, let $\varsigma(f)$ be the $\mathbb{K}$-linear map from A to A whose restriction to each A_γ is the homothety with ratio $f(\gamma)$; it is clear that $\varsigma(f)$ is an automorphism of the graded algebra A, and that ς is a homomorphism from the group $\mathrm{Hom}(\Gamma, \mathbb{K}^\times)$ to the group of automorphisms of the graded algebra A.

Let $\mathfrak{h}$ be a splitting Cartan subalgebra of $\mathfrak{g}$, and Φ the root system of $(\mathfrak{g}, \mathfrak{h})$. Recall that $\Lambda(\Phi)$ (resp. $\Lambda_r(\Phi)$) denotes the group of weights (resp. radical weights) of Φ. Put

$$T_\Lambda = \mathrm{Hom}(\Lambda(\Phi), \mathbb{K}^\times), \quad T_{\Lambda_r} = \mathrm{Hom}(\Lambda_r(\Phi), \mathbb{K}^\times)$$

We can consider $\mathfrak{g} = \mathfrak{h} + \sum_{\alpha \in \Phi} \mathfrak{g}^\alpha$ as a $\Lambda_r(\Phi)$-graded algebra. The preceding remarks define a canonical homomorphism from T_{Λ_r} to $\mathrm{Aut}(\mathfrak{g}, \mathfrak{h})$, which will be denoted by f in this paragraph. In the other hand, the canonical injection from $\Lambda_r(\Phi)$ to $\Lambda(\Phi)$ defines a homomorphism from T_Λ to T_{Λ_r}, which will be denoted by ι:

$$T_\Lambda \xrightarrow{\iota} T_{\Lambda_r} \xrightarrow{\varsigma} \mathrm{Aut}(\mathfrak{g},\mathfrak{h}).$$

If $\varphi \in \mathrm{Aut}(\mathfrak{g},\mathfrak{h})$, let φ^* be the restriction of $(\varphi|_{\mathfrak{h}})^{-t}$ to $\Lambda_r(\Phi)$. Then for all $f \in T_{\Lambda_r}$,

$$\varsigma(f \circ \varphi^*) = \varphi^{-1} \circ \varsigma(f) \circ \varphi. \tag{6.2.33}$$

Indeed, let $\gamma \in \Lambda_r(\Phi)$ and $x \in \mathfrak{g}^\gamma$; then $\varphi(x) \in \mathfrak{g}^{\varphi^*(\gamma)}$ and

$$\varsigma(f \circ \varphi^*)x = (f \circ \varphi^*)(\gamma)x = \varphi^{-1}(f(\varphi^*(\gamma))\varphi(x)) = (\varphi^{-1} \circ \varsigma(f) \circ \varphi)(x).$$

Proposition 6.2.73. *The following sequence of homomorphisms is exact:*

$$1 \longrightarrow T_{\Lambda_r} \xrightarrow{\varsigma} \mathrm{Aut}(\mathfrak{g},\mathfrak{h}) \xrightarrow{\varepsilon} \mathrm{Aut}(\Phi) \longrightarrow 1$$

Proof. Let $f \in T_{\Lambda_r}$. The restriction of $\varsigma(f)$ to $\mathfrak{h} = \mathfrak{g}^0$ is the identity, so $\mathrm{im}\,\varsigma \subseteq \ker \varepsilon$. If $f \in \ker \varsigma$, then $f(\alpha) = 1$ for all $\alpha \in \Phi$, so f is the identity of T_{Λ_r}.

Let $\varphi \in \ker \varepsilon$; then $\varphi|_{\mathfrak{h}} = \mathrm{id}_{\mathfrak{h}}$. For all $\alpha \in \Phi$, we have $\varphi(\mathfrak{g}^\alpha) = \mathfrak{g}^\alpha$, and there exist a $t_\alpha \in \mathbb{K}^\times$ such that $\varphi(x) = t_\alpha x$ for all $x \in \mathfrak{g}^\alpha$. Writing down the condition that $\varphi \in \mathrm{Aut}(\mathfrak{g})$, we see

$$t_\alpha t_{-\alpha} = 1, \quad t_\alpha t_\beta = t_{\alpha+\beta}$$

whence there exists $f \in T_{\Lambda_r}$ such that $f(\alpha) = t_\alpha$ for all $\alpha \in \Phi$. Then $\varphi = \varsigma(f)$, hence $\ker \varepsilon \subseteq \mathrm{im}\,\varsigma$. Finally, the image of $\mathrm{Aut}(\mathfrak{g},\mathfrak{h})$ under ε contains $W(\Phi)$ by Corollary 6.2.27, and contains $\mathrm{Aut}(\Phi,\Delta)$ by Proposition 6.2.72. Hence this image is equal to $\mathrm{Aut}(\Phi)$. □

Corollary 6.2.74. *Let $(\Delta,(x_\alpha)_{\alpha\in\Delta})$ be a framing of $(\mathfrak{g},\mathfrak{h})$. Let G be the subsubgroup of $\mathrm{Aut}(\mathfrak{g},\mathfrak{h})$ of automorphims of the framed algebra. Then $\mathrm{Aut}(\mathfrak{g},\mathfrak{h})$ is the semidirect product of G and $\varepsilon^{-1}(W(\Phi))$.*

Proof. By Proposition 6.2.72, G is isomorphic to $\mathrm{Aut}(\Phi,\Delta)$ via ε. Recall that $\mathrm{Aut}(\Phi)$ is the semidirect product of $\mathrm{Aut}(\Phi,\Delta)$ and $W(\Phi)$, so the claim follows from the fact that ε is surjective. □

Remark 6.2.75. Note that an automorphim $\varphi \in \mathrm{Aut}(\mathfrak{g},\mathfrak{h})$ leaving the framing invariant may not be an automorphim of the framed algebra, since this holds if and only if x_α is mapped to $x_{\varepsilon(\varphi)(\alpha)}$ (the hypothesis only guarantees $\varphi(x_\alpha) = t_\alpha x_{\varepsilon(\varphi)(\alpha)}$. We also note that the quotient of these two groups is isomorphic to T_{Λ_r}.

Corollary 6.2.76. *The group $\varepsilon^{-1}(W(\Phi))$ operates simply-transitively on the set of framings of $(\mathfrak{g},\mathfrak{h})$.*

Proof. This follows immediately from Proposition 6.2.73. □

Let $\alpha \in \Phi$, $x_\alpha \in \mathfrak{g}^\alpha$, $x_{-\alpha} \in \mathfrak{g}^{-\alpha}$ be such that $[x_\alpha, x_{-\alpha}] = h_\alpha$. We have seen (Theorem 6.2.26) that the restriction of the elementary automorphism

$$\theta_\alpha(t) = e^{t\mathrm{ad}(e_\alpha)}e^{-t\mathrm{ad}(f_\alpha)}e^{t\mathrm{ad}(e_\alpha)}.$$

to $\mathfrak{h}$ is the transpose of s_α; so $\varepsilon(\theta_\alpha(t)) = s_\alpha$ and consequently $\theta_\alpha(t)^2 \in \ker \varepsilon$.

Lemma 6.2.77. *Let $\alpha \in \Phi$ and $t \in \mathbb{K}^\times$. Let f be the homomorphism $\lambda \mapsto t^\lambda(h_\alpha)$ from $\Lambda_r(\Phi)$ to $\mathbb{K}^\times$. Then $\varsigma(f) = \theta_\alpha(t)\theta_\alpha(-1)$.*

Proof. Let π be the representation of $\mathfrak{sl}(2,\mathbb{K})$ on $\mathfrak{g}$ associated to x_α and Π be the representation of $\mathrm{SL}(2,\mathbb{K})$ compatible with π. Introduce the notations $\theta(t), h(t)$. Since $\pi(h) = \mathrm{ad}(h_\alpha)$, the elements of $\mathfrak{g}^\alpha$ are of weight $\lambda(h_\alpha)$ for π. By (**??**), we have

$$\theta_\alpha(t)\theta_\alpha(-1) = \Pi(\theta(t)\theta(-1)) = \Pi(h(t)).$$

Hence the restriction of $\theta_\alpha(t)\theta_\alpha(-1)$ to $\mathfrak{g}^\alpha$ is the homothety of ratio $t^{\lambda(h_\alpha)}$ (**??**), and the lemma follows. □

Proposition 6.2.78. *The image of the composite homomorphism*

$$T_\Lambda \xrightarrow{\iota} T_{\Lambda_r} \xrightarrow{\varsigma} \mathrm{Aut}(\mathfrak{g}, \mathfrak{h})$$

is contained in $\mathrm{Aut}_e(\mathfrak{g})$.

Proof. Let Δ be a basis of Φ. Then $(h_\alpha)_{\alpha\in\Delta}$ is a basis of $\check{\Phi}$, and the dual basis of $(h_\alpha)_{\alpha\in\Delta}$ in $\mathfrak{h}^*$ is a basis of the group $\Lambda(\Phi)$, which we denote by $(\varpi_\alpha)_{\alpha\in\Delta}$. Then for $f \in T_\Lambda$,

$$f(\lambda) = f(\sum_{\alpha\in\Delta} n_\alpha \varpi_\alpha) = \prod_{\alpha\in\Delta} f(\varpi_\alpha)^{n_\alpha} = \prod_{\alpha\in\Delta} t_\alpha^{n_\alpha}$$

where $t_\alpha = f(\varpi_\alpha)$. Therefore the group T_Λ is generated by the homomorphisms $\lambda \mapsto t^{\lambda(h_\alpha)}$ ($t \in \mathbb{K}^\times$, $\alpha \in \Delta$). If f is the restriction of such a homomorphism to $\Lambda_r(\Phi)$, Lemma 6.2.77 proves that $\varsigma(f) \in \mathrm{Aut}_e(\mathfrak{g})$, hence the proposition. □

Let $\overline{\mathbb{K}}$ be an algebraic closure of $\mathbb{K}$. The map which associates to any automorphism φ of $\mathfrak{g}$ the automorphism $\varphi \otimes 1$ of $\mathfrak{g} \otimes_{\mathbb{K}} \overline{\mathbb{K}}$ is an injective homomorphism from $\mathrm{Aut}(\mathfrak{g})$ to $\mathrm{Aut}(\mathfrak{g} \otimes_{\mathbb{K}} \overline{\mathbb{K}})$. We denote by $\mathrm{Aut}_0(\mathfrak{g})$ the normal subgroup of $\mathrm{Aut}(\mathfrak{g})$ which is the inverse image of $\mathrm{Aut}_e(\mathfrak{g} \otimes_{\mathbb{K}} \overline{\mathbb{K}})$ under this homomorphism; this is the set of automorphisms of $\mathfrak{g}$ that become elementary on extending the base field from $\mathbb{K}$ to $\overline{\mathbb{K}}$. It is clear that $\mathrm{Aut}_0(\mathfrak{g})$ is independent of the choice of $\overline{\mathbb{K}}$, and that $\mathrm{Aut}_e(\mathfrak{g}) \subseteq \mathrm{Aut}_0(\mathfrak{g})$. The groups $\mathrm{Aut}_0(\mathfrak{g})$ and $\mathrm{Aut}_e(\mathfrak{g})$ can be distinct. If $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$, put

$$\mathrm{Aut}_e(\mathfrak{g}, \mathfrak{h}) = \mathrm{Aut}_e(\mathfrak{g}) \cap \mathrm{Aut}(\mathfrak{g}, \mathfrak{h}), \quad \mathrm{Aut}_0(\mathfrak{g}, \mathfrak{h}) = \mathrm{Aut}_0(\mathfrak{g}) \cap \mathrm{Aut}(\mathfrak{g}, \mathfrak{h}).$$

Lemma 6.2.79. *Let $\mathfrak{h}$ be a splitting Cartan subalgebra of $\mathfrak{g}$, and $\varphi \in \mathrm{Aut}_0(\mathfrak{g}, \mathfrak{h})$. Assume that the restriction of φ to $\sum_{\alpha\in\Phi} \mathfrak{g}^\alpha$ does not have 1 as an eigenvalue. Then $\varepsilon(\varphi) = 1$.*

Proof. By extension of $\mathbb{K}$, we are reduced to the case where $\varphi \in \mathrm{Aut}_e(\mathfrak{g}, \mathfrak{h})$. The dimension of the nilspace of $\varphi - 1$ is at least $\dim(\mathfrak{h})$ (**??**). Hence $(\varphi - 1)|_{\mathfrak{h}}$ is nilpotent. Since $\varphi|_{\mathfrak{h}} \in \mathrm{Aut}(\check{\Phi})$, $\varphi|_{\mathfrak{h}}$ is of finite order, and hence semi-simple. Consequently, $(\varphi - 1)|_{\mathfrak{h}} = 0$, which proves that $\varepsilon(\varphi) = 1$. □

Lemma 6.2.80. *Let $m = [\Lambda(\Phi) : \Lambda_r(\Phi)]$. If f is the m-th power of an element of T_Λ, then $f \in \iota(T_{\Lambda_r})$. In particular, if $\mathbb{K}$ is algebraically closed, then $\iota(T_\Lambda) = T_{\Lambda_r}$.*

Proof. There exist a basis $(\lambda_1, \dots, \lambda_l)$ of $\Lambda(\Phi)$ and integers $n_1, \dots, n_l$ bigger than 1 such that $(n_1\lambda_1, \dots, n_l\lambda_l)$ is a basis of $\Lambda_r(\Phi)$. We have $m = n_1 \cdots n_l$. Let $g \in T_{\Lambda_r}$ and put $t_i = g(n_i\lambda_i)$; for $1 \le i \le l$, set $m_i = \prod_{j\neq i} n_j$. Let χ be the element of T_Λ such that $\chi(\lambda_i) = t_i^{m_i}$. Then

$$\chi(n_i\lambda_i) = t_i^{m_i n_i} = t_i^m = (g^m)(n_i\lambda_i)$$

so $\chi|_{\Lambda_r} = g^m$, which shows $f = \chi|_{\Lambda_r} \in T_\Lambda$. If $\mathbb{K}$ is algebraically closed, every element of $\mathbb{K}^\times$ is the m-th power of an element of $\mathbb{K}^\times$, so every element of T_{Λ_r} is the m-th power of an element of T_Λ; hence $T_{\Lambda_r} = \iota(T_\Lambda)$. □

Proposition 6.2.81. *We have $\varsigma(T_{\Lambda_r}) \subseteq \mathrm{Aut}_0(\mathfrak{g}, \mathfrak{h})$ and $\varepsilon^{-1}(W(\Phi)) = \mathrm{Aut}_0(\mathfrak{g}, \mathfrak{h})$.*

Proof. Let $f \in T_{\Lambda_r}$ and let $\overline{\mathbb{K}}$ be an algebraic closure of k. By Lemma 6.2.80, f extends to an element of T_Λ. By Proposition 6.2.78,

$$\varsigma(f) \otimes 1 \in \mathrm{Aut}_e(\mathfrak{g} \otimes_{\mathbb{K}} \overline{\mathbb{K}}, \mathfrak{h} \otimes_{\mathbb{K}} \overline{\mathbb{K}}).$$

Hence $\varsigma(f) \in \mathrm{Aut}_0(\mathfrak{g}, \mathfrak{h})$ and $\ker\varepsilon \subseteq \mathrm{Aut}_0(\mathfrak{g}, \mathfrak{h})$. The image of $\mathrm{Aut}_e(\mathfrak{g}, \mathfrak{h})$ under ε contains $W(\Phi)$ (Corollary 6.2.27). In view of the exact sequence in Proposition 6.2.73, we see that $\varepsilon^{-1}(W(\Phi)) \subseteq \mathrm{Aut}_0(\mathfrak{g}, \mathfrak{h})$.

It remains to prove that $\mathrm{Aut}_0(\mathfrak{g}, \mathfrak{h}) \subseteq \varepsilon^{-1}(W(\Phi))$. In view of the previous result, it suffices to prove that $\varepsilon(\mathrm{Aut}_0(\mathfrak{g}, \mathfrak{h})) \cap \mathrm{Aut}(\Phi, \Delta)$, where Δ denotes a basis of Φ, reduces to $\{1\}$.

Let $\varphi \in \mathrm{Aut}_0(\mathfrak{g}, \mathfrak{h})$ be such that $\varepsilon(\varphi) \in \mathrm{Aut}(\Phi, \Delta)$. The subgroup of $\mathrm{Aut}(\Phi)$ generated by $\varepsilon(\varphi)$ has a finite number of orbits on Φ. Let U be such an orbit, of cardinal r, and $\mathfrak{g}^U = \sum_{\beta\in U} \mathfrak{g}^\beta$. Let $\beta_1 \in U$, and put $\beta_i = \varepsilon(\varphi)^{i-1}\beta_1$ for $i = 1, \dots, r$, so that $U = \{\beta_1, \dots, \beta_r\}$. Let x_{β_1} be a non-zero element of $\mathfrak{g}^{\beta_1}$, and put $x_{\beta_i} = \varphi^{i-1}x_{\beta_i}$ for $i = 1, \dots, r$. There exists $c_U \in \mathbb{K}^\times$ such that $\varphi^r x_{\beta_1} = c_U x_{\beta_1}$, hence $\varphi^r x_{\beta_i} = c_U x_{\beta_i}$, for all i, and consequently $\varphi^r|_{\mathfrak{g}^U} = c_U \cdot 1$. Let $f \in T_{\Lambda_r}$, and $\tilde{\varphi} = \varphi \circ \varsigma(f)$, which is an element of $\mathrm{Aut}_0(\mathfrak{g}, \mathfrak{h})$ by what we have proved. We have $\tilde{\varphi}|_{\mathfrak{g}^U} = \tilde{c}_U \cdot 1$, where

$$\tilde{c}_U = c_U \prod_{i=1}^{r} f(\beta_i) = c_U f(\sum_{i=1}^{r} \beta_i).$$

Put $\Delta = \{\alpha_1, \ldots, \alpha_l\}$ and $\sum_{i=1}^r \beta_i = \sum_{j=1}^l m_j^U \alpha_j$. Since $\varepsilon(\varphi) \in \mathrm{Aut}(\Phi, \Delta)$, the m_j^U are integers of the same sign and not all zero. We have

$$\tilde{c}_U = c_U \prod_{j=1}^{l} f(\alpha_j)^{m_j^U}.$$

Now f can be chosen so that $\tilde{c}_U \neq 1$ for every orbit U; indeed, this reduces to choosing elements $f(\alpha_1) = t_1, \ldots, f(\alpha_l) = t_l$ of $\mathbb{K}^\times$ which are not annihilated by a finite number of polynomials in $t_1, \ldots, t_l$, not identically zero. For such a choice of f, $\varepsilon(\tilde{\varphi}) = 1$ by Lemma 6.2.79, so

$$\varepsilon(\varphi) = \varepsilon(\tilde{\varphi})\varepsilon(\varsigma(f))^{-1} = 1.$$

This proves our claim and completes the proof. □

Corollary 6.2.82. *Let Δ be a basis of Φ. The group $\mathrm{Aut}(\mathfrak{g}, \mathfrak{h})$ is isomorphic to the semi-direct product of the groups $\mathrm{Aut}(\Phi, \Delta)$ and $\mathrm{Aut}_0(\mathfrak{g}, \mathfrak{h})$.*

Proof. This follows from Proposition 6.2.72, Corollary 6.2.74, and Lemma 6.2.80. □

Remark 6.2.83. Let $\varepsilon_0, \varepsilon_e$ be the restrictions of ε to $\mathrm{Aut}_0(\mathfrak{g}, \mathfrak{h})$, $\mathrm{Aut}_e(\mathfrak{g}, \mathfrak{h})$. Let $\tilde{\varsigma}$ be the homomorphism from T_Λ to $\mathrm{Aut}_e(\mathfrak{g}, \mathfrak{h})$ induced by ς via the canonical injection from $\Lambda_r(\Phi)$ to $\Lambda(\Phi)$. In the preceding we have established the following commutative diagram:

$$\begin{tikzcd}
1 \arrow[r] & T_{\Lambda_r} \arrow[r, "\varsigma"] & \mathrm{Aut}(\mathfrak{g}, \mathfrak{h}) \arrow[r, "\varepsilon"] & \mathrm{Aut}(\Phi) \arrow[r] & 1 \\
1 \arrow[r] & T_{\Lambda_r} \arrow[u] \arrow[r, "\varsigma"] & \mathrm{Aut}_0(\mathfrak{g}, \mathfrak{h}) \arrow[u] \arrow[r, "\varepsilon_0"] & W(\Phi) \arrow[u] \arrow[r] & 1 \\
 & T_{\Lambda} \arrow[u, "\iota"] \arrow[r, "\tilde{\varsigma}"] & \mathrm{Aut}_e(\mathfrak{g}, \mathfrak{h}) \arrow[u] \arrow[r, "\varepsilon_e"] & W(\Phi) \arrow[u] \arrow[r] & 1
\end{tikzcd}$$

in which the vertical arrows other than ι denote the canonical injections. We have seen (Proposition 6.2.73 and Proposition 6.2.81) that the first two rows are exact. In the third row, the homomorphism ε_e is surjective (Corollary 6.2.27); it can be shown that its kernel is $\tilde{\varsigma}(T_\Lambda)$.

Proposition 6.2.84. *For a splittable semi-simple Lie algera $\mathfrak{g}$, the group $\mathrm{Aut}_0(\mathfrak{g})$ operates simply-transitively on the set of framings of $\mathfrak{g}$.*

Proof. Let $e_1 = (\mathfrak{g}, \mathfrak{h}_1, \Delta_1, (x_\alpha^1)_{\alpha \in \Delta_1})$ and $e_2 = (\mathfrak{g}, \mathfrak{h}_2, \Delta_2, (x_\alpha^2)_{\alpha \in \Delta_2})$ be two framings of $\mathfrak{g}$. Let $\overline{\mathbb{K}}$ be an algebraic closure of $\mathbb{K}$. There exists an element of $\mathrm{Aut}_e(\mathfrak{g} \otimes_{\mathbb{K}} \overline{\mathbb{K}})$ that transforms $\mathfrak{h}_1 \otimes_{\mathbb{K}} \overline{\mathbb{K}}$ into $\mathfrak{h}_2 \otimes_{\mathbb{K}} \overline{\mathbb{K}}$ (Theorem 6.1.61). Hence, by Proposition 6.2.81 and Corollary 6.2.76, there exists an element φ of $\mathrm{Aut}_e(\mathfrak{g} \otimes_{\mathbb{K}} \overline{\mathbb{K}})$ that transforms the framing $e_1 \otimes_{\mathbb{K}} \overline{\mathbb{K}}$ of $\mathfrak{g} \otimes_{\mathbb{K}} \overline{\mathbb{K}}$ into the framing $e_2 \otimes_{\mathbb{K}} \overline{\mathbb{K}}$. Since $\mathfrak{h}_1$ and the x_α^1 (resp. $\mathfrak{h}_2$ and the x_α^2) generate $\mathfrak{g}$, we have $\varphi(\mathfrak{g}) = \mathfrak{g}$, so φ is of the form $\psi \otimes 1$ where $\psi \in \mathrm{Aut}_0(\mathfrak{g})$, and ψ transforms e_1 into e_2. The simplicity of the action follows from Corollary 6.2.76. □

Corollary 6.2.85. *Let $(\mathfrak{g}, \mathfrak{h}, \Delta, (x_\alpha)_{\alpha \in \Delta})$ be a framing of $\mathfrak{g}$, and G the group of automorphisms of the framing (isomorphic to $\mathrm{Aut}(\Phi, \Delta)$). Then $\mathrm{Aut}(\mathfrak{g})$ is the semi-direct product of G and $\mathrm{Aut}_0(\mathfrak{g})$.*

Proof. Indeed, every element of $\mathrm{Aut}(\mathfrak{g})$ transforms $(\mathfrak{g}, \mathfrak{h}, \Delta, (x_\alpha)_{\alpha \in \Delta})$ into a framing of $\mathfrak{g}$. By Proposition 6.2.84, every coset of $\mathrm{Aut}(\mathfrak{g})$ modulo $\mathrm{Aut}_0(\mathfrak{g})$ meets G in exactly one point. □

It follows from Corollary 6.2.85 that the group $\mathrm{Aut}(\mathfrak{g})/\mathrm{Aut}_0(\mathfrak{g})$ can be identified with $\mathrm{Aut}(\Phi, \Delta)$, and is isomorphic to the group of automorphisms of the Dynkin graph of Φ.

Corollary 6.2.86. *We have* $\mathrm{Aut}(\mathfrak{g}) = \mathrm{Aut}_0(\mathfrak{g})$ *when* $\mathfrak{g}$ *is a splittable simple Lie algebra of type* A_1, B_n $(n \geq 2)$, C_n $(n \geq 2)$, E_7, E_8, F_4, G_2. *The quotient* $\mathrm{Aut}(\mathfrak{g})/\mathrm{Aut}_0(\mathfrak{g})$ *is of order* 2 *when* $\mathfrak{g}$ *is of type* A_n $(n \geq 2)$, D_n $(n \geq 5)$, E_6; *it is isomorphic to* $\mathfrak{S}_3$ *when* $\mathfrak{g}$ *is of type* D_4.

Remark 6.2.87. Let $e = (\mathfrak{g}, \mathfrak{h}_1, \Delta_1, (x_\alpha^1)_{\alpha\in\Delta})$, $e_2 = (\mathfrak{g}, \mathfrak{h}_2, \Delta_2, (x_\alpha^2)_{\alpha\in\Delta_2})$, $\tilde{e}_2 = (\mathfrak{g}, \mathfrak{h}_2, \Delta_2, (y_\alpha^2)_{\alpha\in\Delta_2})$ be framings of $\mathfrak{g}$, and φ (resp. $\tilde{\varphi}$) an element of $\mathrm{Aut}_0(\mathfrak{g})$ that transforms e_1 to e_2 (resp. $\tilde{e}_2$). Then $\varphi|_{\mathfrak{h}_1} = \tilde{s}|_{\mathfrak{h}_1}$. Indeed, $\varphi\tilde{\varphi}^{-1} \in \mathrm{Aut}_0(\mathfrak{g}, \mathfrak{h}_1)$ and $\varphi\tilde{\varphi}^{-1}(\Delta_1) = \Delta_1$, so $\varepsilon(\varphi\tilde{\varphi}^{-1}) = 1$ (**??????**).

Remark 6.2.88. Let $\mathfrak{g}$ be a splittable semi-simple Lie algebra. Let X be the set of pairs $(\mathfrak{h}, \Delta)$ where $\mathfrak{h}$ is a splitting Cartan subalgebra of $\mathfrak{g}$ and Δ a basis of the root system of $(\mathfrak{g}, \mathfrak{h})$. If $x_1 = (\mathfrak{h}_1, \Delta_1)$ and $x_2 = (\mathfrak{h}_2, \Delta_2)$ are two elements of X, there exists $\varphi \in \mathrm{Aut}_0(\mathfrak{g})$ that transforms x_1 into x_2 (Proposition 6.2.84), and the restriction φ_{x_1,x_2} of φ to $\mathfrak{h}_1$ does not depend on the choice of φ (Remark 6.2.87). In particular, $\varphi_{x_3,x_2} \circ \varphi_{x_2,x_1} = \varphi_{x_3,x_1}$ if $x_1, x_2, x_3 \in X$, and $\varphi_{x,x} = 1$. The set of families $(h_x)_{x\in X}$ satisfying the conditions

(a) $h_x \in \mathfrak{h}$ if $x = (\mathfrak{h}, \Delta)$;

(b) $\varphi_{x_2,x_1}(h_{x_1}) = h_{x_2}$ if $x_1, x_2 \in X$

is in a natural way a vector space $\mathfrak{h}(\mathfrak{g})$ which we sometimes call the **canonical Cartan subalgebra** of $\mathfrak{g}$. For $x_1 = (\mathfrak{h}_1, \Delta_1)$ and $x_2 = (\mathfrak{h}_2, \Delta_2)$, φ_{x_2,x_1} takes Δ_1 to Δ_2, and hence the root system of $(\mathfrak{g}, \mathfrak{h}_1)$ to that of $(\mathfrak{g}, \mathfrak{h}_2)$; it follows that the dual $\mathfrak{h}(\mathfrak{g})^*$ of $\mathfrak{h}(\mathfrak{g})$ is naturally equipped with a root system $\Phi(\mathfrak{g})$ and with a basis $\Delta(\mathfrak{g})$ of $\Phi(\mathfrak{g})$. We sometimes say that $\Phi(\mathfrak{g})$ is the canonical root system of $\mathfrak{g}$ and that $\Delta(\mathfrak{g})$ is its canonical basis. The group $\mathrm{Aut}(\mathfrak{g})$ operates on $\mathfrak{h}(\mathfrak{g})$ leaving $\Phi(\mathfrak{g})$ and $\Delta(\mathfrak{g})$ stable; the elements of $\mathrm{Aut}(\mathfrak{g})$ that operate trivially on $\mathfrak{h}(\mathfrak{g})$ are those of $\mathrm{Aut}_0(\mathfrak{g})$ (the group $\mathrm{Aut}_0(\mathfrak{g})$ acts trivially on each family $(h_x)_{x\in X}$).

Proposition 6.2.89. *Let* $\mathfrak{h}$ *be a splitting Cartan subalgebra of* $\mathfrak{g}$. *Then* $\mathrm{Aut}_0(\mathfrak{g}) = \mathrm{Aut}_e(g) \cdot \ker\varepsilon = \mathrm{Aut}_e(\mathfrak{g}) \cdot \varsigma(T_{\Lambda_r})$.

Proof. By Corollary 6.2.49, $\mathrm{Aut}_0(\mathfrak{g}) = \mathrm{Aut}_e(\mathfrak{g}) \cdot \mathrm{Aut}_0(\mathfrak{g}, \mathfrak{h})$. On the other hand, $\varepsilon(\mathrm{Aut}_e(\mathfrak{g}, \mathfrak{h})) \supseteq W(\Phi)$, so $\mathrm{Aut}_0(\mathfrak{g}, \mathfrak{h}) = \mathrm{Aut}_e(\mathfrak{g}, \mathfrak{h}) \cdot \ker\varepsilon$. □

6.3 Modules over a split semi-simple Lie algebra

In this section, $(\mathfrak{g}, \mathfrak{h})$ denotes a split semi-simple Lie algebra, Φ its root system, W its Weyl group, Δ a basis of Φ, Φ^+ (resp. Φ^-) the set of positive (resp. negative) roots relative to Δ. Put

$$\mathfrak{n}^+ = \sum_{\alpha\in\Phi^+} \mathfrak{g}^\alpha, \quad \mathfrak{n}^- = \sum_{\alpha\in\Phi^-} \mathfrak{g}^\alpha, \quad \mathfrak{b}^+ = \mathfrak{h} + \mathfrak{n}^+, \quad \mathfrak{b}^- = \mathfrak{h} + \mathfrak{n}^-.$$

We have $\mathfrak{n}^+ = [\mathfrak{b}^+, \mathfrak{b}^+]$ and $\mathfrak{n}^- = [\mathfrak{b}^-, \mathfrak{b}^-]$. For all $\alpha \in \Phi$, we choose an element $x_\alpha \in \mathfrak{g}^\alpha$ such that $[x_\alpha, x_{-\alpha}] = h_\alpha$. It turns out that none of the definitions in this part will depend on this choice.

6.3.1 Weights and primitive elements

Let V be a $\mathfrak{g}$-module. For all $\lambda \in \mathfrak{h}^*$, denote by V^λ the primary subspace, relative to λ, of V considered as an $\mathfrak{h}$-module. The elements of V^λ are called the elements of weight λ of the $\mathfrak{g}$-module V. The sum of the V^λ is direct by Proposition 6.1.3. For all $\alpha \in \mathfrak{h}^*$ and $\lambda \in \mathfrak{h}^*$, $\mathfrak{g}^\alpha V^\lambda \subseteq V^{\alpha+\lambda}$ (Proposition 6.1.18(a)). The dimension of V^λ is called the **multiplicity** of λ in V; if it is ≥ 1, i.e. if $V^\lambda \neq 0$, λ is said to be a weight of V. If V is finite dimensional, the homotheties of V defined by the elements of $\mathfrak{h}$ are semi-simple, so $V^\lambda = V_\lambda$ is the set of $x \in V$ such that $h \cdot x = \lambda(h)x$ for all $h \in \mathfrak{h}$.

Lemma 6.3.1. *Let* V *be a* $\mathfrak{g}$*-module and* $v \in V$. *Then the following conditions are equivalent:*

(i) $\mathfrak{b}^+ \cdot v \subseteq \mathbb{K}v$;

(ii) $\mathfrak{h} \cdot v \subseteq \mathbb{K}v$ *and* $\mathfrak{n}^+ \cdot v = 0$;

(iii) $\mathfrak{h} \cdot v \subseteq \mathbb{K}v$ *and* $\mathfrak{g}^\alpha v = 0$ *for all* $\alpha \in \Delta$.

Proof. Assume that $\mathfrak{b}^+ \cdot v \subseteq \mathbb{K}v$. Then there exists $\lambda \in \mathfrak{h}^*$ such that $v \in V^\lambda$. Let $\alpha \in \Phi^+$. Then $\mathfrak{g}^\alpha \cdot v \subseteq V^\lambda \cap V^{\lambda+\alpha} = 0$. Hence $\mathfrak{n}^+ \cdot v = 0$. It is clear that (ii) implies (iii), and (iii)⇒(i) follows from the fact that $(x_\alpha)_{\alpha\in\Delta}$ generates $\mathfrak{n}^+$. □

Let V be a $\mathfrak{g}$-module and $v \in V$. Then v is said to be a **primitive element** of V if $v \neq 0$ and v satisfies the equivalent conditions of Lemma 6.3.1. A primitive element belongs to one of the V^λ. For all $\lambda \in \mathfrak{h}^*$, let V^λ_π denote the set of $v \in V^\lambda$ such that $\mathfrak{b}^+ v \subseteq \mathbb{K}v$. Thus, the primitive elements of weight λ are the non-zero elements of V^λ_π.

Proposition 6.3.2. *Let V be a $\mathfrak{g}$-module, v a primitive element of V and ω the weight of v. Assume that V is generated by v as a $\mathfrak{g}$-module.*

(a) If $U(\mathfrak{n}^-)$ denote the enveloping algebra of $\mathfrak{n}^-$, then $V = U(\mathfrak{n}^-) \cdot v$.

(b) For all $\lambda \in \mathfrak{h}^*$, V^λ is the set of $x \in V$ such that $h \cdot x = \lambda(h)x$ for all $h \in \mathfrak{h}$. We have $V = \bigoplus_{\lambda \in \mathfrak{h}^*} V^\lambda$ and each V^λ is finite dimensional. The space V^ω is of dimension 1, and every weight of V is of the form $\omega - \sum_{\alpha \in \Delta} n_\alpha \alpha$, where the n_α are non-negative integers.

(c) V is an indecomposable $\mathfrak{g}$-module, and its commutant reduces to the scalars.

*(d) Let $U(\mathfrak{g})$ be the enveloping algebra of $\mathfrak{g}$, and $Z(\mathfrak{g})$ the centre of $U(\mathfrak{g})$. Then there exists a unique homomorphism $\chi : Z \to \mathbb{K}$ such that, for all $z \in Z$, z_V is the homothety with ratio $\chi(z)$. This homomorphism is called the **cantral character** of the $\mathfrak{g}$-module V.*

Proof. Let $U(\mathfrak{b}^+)$ be the enveloping algebra of $\mathfrak{b}^+$. We have $U(\mathfrak{g}) = U(\mathfrak{n}^-) \cdot U(\mathfrak{b}^+)$. Hence

$$V = U(\mathfrak{g}) \cdot v = U(\mathfrak{n}^-) \cdot U(\mathfrak{b}^+) \cdot v = U(\mathfrak{n}^-) \cdot v.$$

Denote by $\alpha_1, \ldots, \alpha_n$ the distinct elements of Φ^+. Then

$$\{x_{-\alpha_1}^{p_1} \cdots x_{-\alpha_n}^{p_n} : (p_1, \ldots, p_n) \in \mathbb{N}^n\}$$

is a basis of $U(\mathfrak{n}^-)$, so

$$V = \sum_{(p_1,\ldots,p_n) \in \mathbb{N}^n} \mathbb{K} x_{-\alpha_1}^{p_1} \cdots x_{-\alpha_n}^{p_n} \cdot v. \tag{6.3.1}$$

For $\lambda \in \mathfrak{h}^*$, we put

$$T_\lambda = \sum_{\substack{(p_1,\ldots,p_n) \in \mathbb{N}^n \\ \omega - \sum_{i=1}^n p_i \alpha_n = \lambda}} \mathbb{K} x_{-\alpha_1}^{p_1} \cdots x_{-\alpha_n}^{p_n} \cdot v.$$

By Proposition 6.1.18(a), if $h \in \mathfrak{h}$, $h_V|_{T_\lambda}$ is the homothety with ratio $\lambda(h)$, so $T_\lambda \subseteq V^\lambda$. On the other hand, (6.3.1) implies that

$$V = \sum_{\lambda \in \omega - \mathbb{N}\alpha_1 - \cdots - \mathbb{N}\alpha_n} T_\lambda.$$

The sum of the V^λ is direct. From these observations it follows that $V^\lambda = T_\lambda$, that V is the direct sum of the V^λ, and that V^λ is the set of $x \in V$ such that $h \cdot x = \lambda(h)x$ for all $h \in \mathfrak{h}$. On the other hand, $\dim(V^\lambda)$ is at most the cardinal of the set of $(p_1, \ldots, p_n) \in \mathbb{N}^n$ such that $p_1\alpha_1 + \cdots + p_n\alpha_n = \omega - \lambda$. This proves that $V^\lambda = 0$ if $\omega - \lambda \notin \sum_{\alpha \in \Delta} \mathbb{N}\alpha$, that $\dim(V^\omega) = 1$, and that the V^λ are all finite dimensional.

Let s be an element of the commutant of V. For all $h \in \mathfrak{h}$, since h_V commutes with s, we see

$$h_V s(v) = s h_V(v) = \omega(h) s(v),$$

so $s(v) \in V^\omega$; hence there exists $t \in \mathbb{K}$ such that $s(v) = tv$. Now, for all $(p_1, \ldots, p_n) \in \mathbb{N}^n$,

$$s(x_{-\alpha_1}^{p_1} \cdots x_{-\alpha_n}^{p_n} \cdot v) = x_{-\alpha_1}^{p_1} \cdots x_{-\alpha_n}^{p_n} \cdot s(v) = t x_{-\alpha_1}^{p_1} \cdots x_{-\alpha_n}^{p_n} \cdot v.$$

so that $s = t \cdot 1$. Hence, the commutant of V reduces to the scalars. This implies (d) and the fact that V is indecomposable. □

Proposition 6.3.3. *Let V be a $\mathfrak{g}$-module generated by a primitive element v of weight ω, and W a semi-simple $\mathfrak{g}$-module. Let $E = \mathrm{Hom}_{\mathfrak{g}}(V, W)$ be the set of homomorphisms. Then $\varphi \mapsto \varphi(v)$ is an isomorphism from E to W^ω_π.*

Proof. It is clear that $\varphi(v) \in W_\pi^\omega$ for all $\varphi \in E$. If $\varphi \in E$ and $\varphi(v) = 0$, then $\varphi = 0$ since v generates the $\mathfrak{g}$-module V. We show that, if e is a non-zero element of W_π^ω, there exists $\varphi \in E$ such that $\varphi(v) = e$. Let W' be the submodule of W generated by e. By Proposition 6.3.2, W' is indecomposable, hence simple since W is semi-simple. The element (v, e) is primitive in the $\mathfrak{g}$-module $V \oplus W$. Let N be the submodule of $V \oplus W$ generated by (e, v). Then $N \cap W \subseteq \pi_2(N) = W'$, so $N \cap W = 0$ or W'; if $N \cap W = W'$, N contains the linearly independent elements (v, e) and $(0, e)$ which are primitive of weight ω; this is absurd (Proposition 6.3.2), so $N \cap W = 0$. Thus $\pi_1|_N$ is an injective map ψ from N to V; this map is surjective since its image contains v. Thus $\varphi = \pi_2 \circ \psi^{-1}$ is a homomorphism from the $\mathfrak{g}$-module V to the $\mathfrak{g}$-module W such that $\varphi(v) = e$. □

Next we consider primitive elements in a simple module. Recall that fixing Δ defines an order relation on $\mathfrak{h}_\mathbb{Q}^*$. The elements of $\mathfrak{h}_\mathbb{Q}^*$ that are positive are the linear combinations of elements of Δ with positive rational coefficients. More generally, we shall consider the following order relation between elements $\lambda, \mu \in \mathfrak{h}^*$: $\lambda \succeq \mu$ if and only if $\lambda - \mu$ is a linear combination of elements of Δ with positive rational coefficients.

Lemma 6.3.4. *Let V be a simple $\mathfrak{g}$-module, ω a weight of V. Then the following conditions are equivalent:*

(i) *every weight of V is of the form $\omega - \mu$ where μ is a positive radical weight;*

(ii) *ω is the highest weight of V;*

(iii) *for all $\alpha \in \Delta$, $\omega + \alpha$ is not a weight of V;*

(iv) *there exists a primitive element of weight ω.*

Proof. It is clear that (i)⇒(ii)⇒(iii). Assume that condition (iii) is satisfied. For all $h \in \mathfrak{h}$, $\ker(h_V - \omega(h))$ is non-zero, contained in V^ω, and stable under $\mathfrak{h}_V$. By induction on $\dim(\mathfrak{h})$, we see that there exists a non-zero v in V^ω such that $\mathfrak{h} \cdot v \subseteq \mathbb{K}v$. Condition (iii) implies that $\mathfrak{n}^+ \cdot v = 0$, so v is primitive.

Finally, let v be a primitive element of weight ω. Since V is simple, V is generated by v as a $\mathfrak{g}$-module, so assertion (i) follows from Proposition 6.3.2. □

Thus, for any simple $\mathfrak{g}$-module, the existence of a primitive element is equivalent to that of a highest weight, or to that of a maximal weight. Note that there exist simple $\mathfrak{sl}(2, \mathbb{C})$-modules V that have no weights for any Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{sl}(2, \mathbb{C})$. These modules are of infinite dimension over $\mathbb{C}$.

Proposition 6.3.5. *Let V be a simple $\mathfrak{g}$-module with a highest weight ω.*

(a) *The primitive elements of V are the non-zero elements of V^ω.*

(b) *V is semi-simple as an $\mathfrak{h}$-module.*

(c) *We have $V = \bigoplus_{\lambda \in \mathfrak{h}^*} V^\lambda$. For all $\lambda \in \mathfrak{h}^*$, V^λ is finite dimensional. We have $\dim(V^\omega) = 1$.*

(d) *The $\mathfrak{g}$-module V is absolutely simple.*

Proof. Assertions (a), (b) and (c) follow from Proposition 6.3.2 and Lemma 6.3.4. Assertion (d) follows from Proposition 6.3.2(c). □

Corollary 6.3.6. *Let V be a simple $\mathfrak{g}$-module with a highest weight ω. If V is finite dimensional, the canonical homomorphism $U(\mathfrak{g}) \to \mathrm{End}(V)$ is surjective.*

Proof. This follows from Proposition 6.3.5(d). □

Proposition 6.3.7. *Let V be a simple $\mathfrak{g}$-module with a highest weight ω, W a semi-simple $\mathfrak{g}$-module, and W' the isotypical component of type V in W. Then W' is the submodule of W generated by W_π^ω. Its length is equal to the dimension of W_π^ω.*

Proof. Let $\tilde{W}$ be the submodule of W generated by W_π^ω. It is clear that every submodule of W isomorphic to V is contained in $\tilde{W}$. Hence $W' \subseteq \tilde{W}$. On the other hand, every submodule of W generated by an element in W_π^ω is isomorphic to V, so we conclude that $\tilde{W} \subseteq W'$, hence they are equal. Let $E = \mathrm{Hom}_\mathfrak{g}(V, W)$; the length of W' is $\dim_\mathbb{K}(E)$, that is $\dim_\mathbb{K}(W_\pi^\omega)$ (Proposition 6.3.3). □

Let $\lambda \in \mathfrak{h}^*$. Since $\mathfrak{b}^+ = \mathfrak{h} + \mathfrak{n}^+$ and since $\mathfrak{n}^+ = [\mathfrak{b}^+, \mathfrak{b}^+]$, the map $h + n \mapsto \lambda(h)$ (where $h \in \mathfrak{h}$, $n \in \mathfrak{n}^+$) from $\mathfrak{b}^+$ to $\mathbb{K}$ is a 1-dimensional representation of $\mathfrak{b}^+$. Denote by $L(\lambda)$ the $\mathbb{K}$-vector space $\mathbb{K}$ equipped with the $\mathfrak{b}^+$-module structure defined by this representation. Let $U(\mathfrak{g})$, $U(\mathfrak{b}^+)$ be the enveloping algebras of $\mathfrak{g}$, $\mathfrak{b}^+$, so that $U(\mathfrak{b}^+)$ is a subalgebra of $U(\mathfrak{g})$; recall that $U(\mathfrak{g})$ is a free right $U(\mathfrak{b}^+)$-module (Proposition 1.2.18). Put

$$Z(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b}^+)} L(\lambda). \tag{6.3.2}$$

Then $Z(\lambda)$ is a $\mathfrak{g}$-module, called the Verma module of weight λ. Denote by e the element $1 \otimes 1$ of $Z(\lambda)$.

Proposition 6.3.8. *Let $Z(\lambda)$ be the module defined above.*

(a) The element e of $Z(\lambda)$ is primitive of weight λ and generates the $\mathfrak{g}$-module $Z(\lambda)$.

(b) Let $Z^+(\lambda) = \sum_{\mu \neq \lambda} Z(\lambda)^\mu$. Then every submodule of $Z(\lambda)$ distinct from $Z(\lambda)$ is contained in $Z^+(\lambda)$.

(c) There exists a largest proper submodule $F(\lambda)$ of $Z(\lambda)$. The quotient module $V(\lambda) = Z(\lambda)/F(\lambda)$ is simple and has highest weight λ.

Proof. It is clear that e generates the $\mathfrak{g}$-module $Z(\lambda)$. If $x \in \mathfrak{b}^+$,

$$x \cdot e = (x \cdot 1) \otimes 1 = (1 \cdot x) \otimes 1 = 1 \otimes (x \cdot 1) = \lambda(x)(1 \otimes 1) = \lambda(x)e.$$

So e is primitive and (a) follows. The $\mathfrak{h}$-module $Z(\lambda)$ is semi-simple (Proposition 6.3.2). If M is a $\mathfrak{g}$-submodule of $Z(\lambda)$, then $M = \sum_{\mu \in \mathfrak{h}^*} (M \cap Z(\lambda)^\mu)$. The hypothesis $M \cap Z(\lambda)^\lambda \neq 0$ implies that $M = Z(\lambda)$, since $\dim(Z(\lambda)^\lambda) = 1$ and e generates the $\mathfrak{g}$-module $Z(\lambda)$. If $M \neq Z(\lambda)$, then $G = \sum_{\mu \neq \lambda} G \cap Z(\lambda)^\mu \subseteq Z^+(\lambda)$.

Let $F(\lambda)$ be the sum of the $\mathfrak{g}$-submodules of $Z(\lambda)$ distinct from $Z(\lambda)$. By (b), $F(\lambda) \subseteq Z^+(\lambda)$. Hence $F(\lambda)$ is the largest submodule of $Z(\lambda)$ distinct from $Z(\lambda)$. It is clear that $Z(\lambda)/F(\lambda)$ is simple and that the canonical image of e in $Z(\lambda)/F(\lambda)$ is primitive of weight λ. □

Theorem 6.3.9. *Let $\lambda \in \mathfrak{h}^*$. The $\mathfrak{g}$-module $V(\lambda)$ is simple and has highest weight λ. Every simple $\mathfrak{g}$-module of highest weight λ is isomorphic to $V(\lambda)$.*

Proof. The first assertion follows from Proposition 6.3.8(c). The second follows from Proposition 6.3.7. □

Proposition 6.3.10. *Let V be a $\mathfrak{g}$-module, λ an element of $\mathfrak{h}^*$ and v a primitive element of V of weight λ.*

(a) There exists a unique homomorphism of $\mathfrak{g}$-modules $\psi : Z(\lambda) \to V$ such that $\psi(e) = v$.

(b) Assume that v generates V. Then ψ is surjective. Moreover, ψ is bijective if and only if, for every non-zero element u of $U(\mathfrak{n}^-)$, u_V is injective.

(c) The map $u \mapsto u \otimes 1$ from $U(\mathfrak{n}^-)$ to $Z(\lambda)$ is bijective.

Proof. Let K be the kernel of the representation of $U(\mathfrak{b}^+)$ on $L(\lambda)$; it is of codimension 1 in $U(\mathfrak{b}^+)$. Let $J = U(\mathfrak{g})K$ be the left ideal of $U(\mathfrak{g})$ generated by K; then $L(\lambda)$ can be identified with $U(\mathfrak{b}^+)/K$ as a left $U(\mathfrak{b}^+)$-module, and $Z(\lambda)$ can be identified with $U(\mathfrak{g})/J$ as a left $U(\mathfrak{g})$-module. We have $K \cdot v = 0$, so $J \cdot v = 0$, and the unique homomorphism $U(\mathfrak{g})/J \to V$ that maps e to v is well defined. This proves (a).

Now assume that v generates V. It is clear that ψ is surjective. Since $\mathfrak{g}/\mathfrak{b}^+ = \mathfrak{n}^-$, by Proposition 1.2.18, a basis of $U(\mathfrak{n}^-)$ over $\mathbb{K}$ is also a basis of $U(\mathfrak{g})$ as a right $U(\mathfrak{b}^+)$-module. Hence the map $\varphi : u \mapsto u \otimes 1$ from $U(\mathfrak{n}^-)$ to $U(\mathfrak{g}) \otimes_{U(\mathfrak{b}^+)} L(\lambda)$ is bijective. Let $u \in U(\mathfrak{n}^-)$. Then $\varphi^{-1} \circ u_{Z(\lambda)} \circ \varphi$ is left multiplication by u on $U(\mathfrak{n}^-)$. In view of Corollary 1.2.17, $u_{Z(\lambda)}$ is injective if $u \neq 0$. Consequently, if ψ is bijective, then u_V is injective for non-zero u in $U(\mathfrak{n}^-)$.

Assume that ψ is not injective. There exists $u \in U(\mathfrak{n}^-)$ such that $u \neq 0$ and $\psi(\varphi(u)) = 0$. Then

$$u_V \cdot v = u_V \cdot \psi(1 \otimes 1) = \psi(u \otimes 1) = \psi(\varphi(u)) = 1.$$

This shows u_V is not injective and completes the proof. □

Corollary 6.3.11. *Let $\lambda \in \mathfrak{h}^*$ and $\alpha \in \Delta$ be such that $\lambda(h_\alpha) + 1 \in \mathbb{N}$. Then $Z(-\alpha + s_\alpha(\lambda))$ is isomorphic to a $\mathfrak{g}$-submodule of $Z(\lambda)$.*

Proof. Put $m = \lambda(h_\alpha)$. Let $x = x_{-\alpha}^{m+1} \cdot e \in Z(\lambda)$, and let V be the submodule of $Z(\lambda)$ generated by x; then $x \neq 0$ by Proposition 6.3.10(c). On the other hand, $x \in Z(\lambda)^{\lambda-(m+1)\alpha}$. For $\beta \in \Delta$ and $\beta \neq \alpha$, $[\mathfrak{g}^{-\alpha}, \mathfrak{g}^{\beta}] = 0$ and $\mathfrak{g}^{\beta} \cdot e = 0$, so $\mathfrak{g}^{\beta} \cdot x = 0$. Finally, since $[x_\alpha, x_{-\alpha}] = h_\alpha$, we have (Lemma 6.2.1)

$$[x_\alpha, x_{-\alpha}^{m+1}] = (m+1)x_{-\alpha}^{m}(h_\alpha - m)$$

and therefore

$$\begin{aligned} x_\alpha \cdot x = x_\alpha x_{-\alpha}^{m+1} \cdot e = [x_\alpha, x_{-\alpha}^{m+1}] \cdot e + x_{-\alpha}^{m+1} x_\alpha \cdot e = [x_\alpha, x_{-\alpha}^{m+1}] \cdot e \\ = (m+1)x_{-\alpha}^{m}(h_\alpha - m) \cdot e = (m+1)x_{-\alpha}^{m}(\lambda(h_\alpha)e - me) = 0. \end{aligned}$$

Thus x is a primitive element of weight $\lambda - (m+1)\alpha$. In view of Proposition 6.3.10, since $u_{Z(\lambda)}$ is injective for $u \in U(\mathfrak{n}^-)$, the $\mathfrak{g}$-module V is isomorphic to $Z(-\alpha + \lambda - m\alpha) = Z(-\alpha + s_\alpha(\lambda))$. □

Remark 6.3.12. We note that $-\alpha + s_\alpha(\lambda) = \lambda$ if and only if $\lambda(h_\alpha) = -1$, so only the cases $\lambda(h_\alpha) \in \mathbb{N}$ in Corollary 6.3.11 are interesting.

Corollary 6.3.13. *Let $\delta = \frac{1}{2}\sum_{\alpha \in \Phi^+} \alpha$ be the Weyl vector, and $\lambda, \mu \in \mathfrak{h}^*$. Assume that $\lambda + \delta$ is a dominant weight in $\mathfrak{h}^*$, and that there exists $w \in W$ with $\mu + \delta = w(\lambda + \delta)$. Then $Z(\mu)$ is isomorphic to a submodule of $Z(\lambda)$.*

Proof. The assertion is clear when $w = 1$. Assume that it is established whenever w is of length $< p$. If w is of length p, there exists $\alpha \in \Delta$ such that $w = s_\alpha \tilde{w}^{-1}$ with $\ell(\tilde{w}) = p - 1$. We have $\tilde{w}(\alpha) \in \Phi^+$ by Corollary 5.1.39, and hence $\tilde{w}^{-1}(\lambda + \delta)(h_\alpha) = (\lambda + \delta)(h_{\tilde{w}(\alpha)})$ is a non-negative integer. Put

$$\tilde{\mu} = \tilde{w}^{-1}(\lambda + \delta) - \delta.$$

By the induction hypothesis, $Z(\tilde{\mu})$ is isomorphic to a submodule of $Z(\lambda)$. On the other hand, by Proposition 5.1.68,

$$-\alpha + s_\alpha(\tilde{\mu}) = -\alpha + s_\alpha \tilde{w}^{-1}(\lambda + \delta) - s_\alpha(\delta) = w(\lambda + \delta) - \delta = \mu.$$

Moreover, $\delta(h_\alpha) = 1$, so $\tilde{\mu}(h_\alpha) + 1 \in \mathbb{N}$. Corollary 6.3.11 now implies that $Z(\mu)$ is isomorphic to a submodule of $Z(\tilde{\mu})$, and hence also to a submodule of $Z(\lambda)$. □

Example 6.3.14. Let $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{K})$ and (e, h, f) be the root decomposition for $\mathfrak{g}$. Then $\mathfrak{h}$ is the vector space generated by h. We identify $\mathbb{K}$ with $\mathfrak{h}^*$ via the map $\lambda \mapsto (h \mapsto \lambda)$. Then the root α is identified with $2 \in \mathbb{K}$, and the weight lattice is identified with $\mathbb{Z} \subseteq \mathbb{K}$. The Verma modules of $\mathfrak{g}$ can be constructed explicitly as follows.

Recall that by the Poincaré-Birkhoff-Witt theorem, as a vector space $\mathfrak{sl}(2, \mathbb{K})$ has a basis

$$\{f^i h^j e^k : i, j, k \in \mathbb{N}\}.$$

Let $\lambda \in \mathbb{K}$ and v be a nonzero element in $L(\lambda)$. Then the tensor product over $U(\mathfrak{b}^+)$ means we identify $ab \otimes v$ and $a \otimes bv$ for $a \in \mathfrak{sl}(2, \mathbb{K})$ and $b \in \mathfrak{b}^+$, so

$$ah \otimes v = a \otimes hv = \lambda(a \otimes v), \quad ae \otimes v = a \otimes ev = 0.$$

Therefore, as a vector space, $Z(\lambda)$ is an infinite dimensional $\mathbb{K}$-vector space with a basis

$$\{f^n v : n \in \mathbb{N}\}.$$

Set $v_n = f^n v$, then the action of $\mathfrak{sl}(2, \mathbb{K})$ on $Z(\lambda)$ is given by (c.f. Proposition 6.2.6)

$$h \cdot v_j = (\lambda - 2j)v_j, \quad e \cdot v_j = j(\lambda - (j-1))v_{j-1}, \quad f \cdot v_j = v_{j+1}. \tag{6.3.3}$$

When λ is a non-negative integer, the space $F(\lambda)$ spanned by $v_{\lambda+1}, v_{\lambda+2}, \ldots$ is invariant under the action of $\mathfrak{sl}(2, \mathbb{K})$. After all, this space is clearly invariant under the action of f and h, and it is invariant under e because we have

$$e \cdot v_{\lambda+1} = (\lambda + 1)(\lambda - \lambda)v_\lambda = 0.$$

Moreover, we claim that $F(\lambda)$ is the largest proper sub-$\mathfrak{sl}(2, \mathbb{K})$-module of $Z(\lambda)$: In fact, if a sub-$\mathfrak{sl}(2, \mathbb{K})$-module M contains a linear combination $v = \sum_{i=0}^{\lambda} c_i v_i$, then $e^N v$ is a multiple of v_0, where N is the largest number such that $c_N \neq 0$; therefore $M = Z(\lambda)$. The quotient space $V(\lambda) = Z(\lambda)/F(\lambda)$ is then the unique simple $\mathfrak{sl}(2,\mathbb{K})$-module with highest weight λ, with dimension $\lambda+1$. We also note that, the submodule $F(\lambda)$ is isomorphic to $Z(-\lambda-2)$ since the element $v_{\lambda+1}$ is primitive with weight $-\lambda-2$. Since the Weyl group of $\mathfrak{sl}(2,\mathbb{K})$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ and the nontrivial element acts by $x\mapsto -x$ on $\mathbb{K}$, this verifies Corollary 6.3.11 and Corollary 6.3.13.

On the other hand, if λ is not a non-negative integer, then $e\cdot v_j = j(\lambda-(j-1))v_{j-1}$ is nonzero for all $j>0$, so from any linear combination $\sum_i c_i v_i$ we can get v_0 by repeatly applying e. In other words, the proper submodule $F(\lambda)$ equals to 0, and $Z(\lambda)$ itself is the simple $\mathfrak{sl}(2,\mathbb{K})$-module with highest weight λ.

6.3.2 Commutant of $\mathfrak{h}$ in the enveloping algebra

Let $(\mathfrak{g},\mathfrak{h})$ be a splitting semi-simple Lie algbra. Let $U(\mathfrak{g})$ be the enveloping algebra of $\mathfrak{g}$, $U(\mathfrak{h})\subseteq U(\mathfrak{g})$ the enveloping algebra of $\mathfrak{h}$. Since $\mathfrak{h}$ is abelian, the algebra $U(\mathfrak{h})$ can be identified with the symmetric algebra $\boldsymbol{S}(\mathfrak{h})$ of $\mathfrak{h}$, and also with the algebra of polynomial functions on $\mathfrak{h}^*$. Denote by $\alpha_1,\dots,\alpha_n$ the pairwise distinct positive roots. Let $(h_1,\dots,h_l)$ be a basis of $\mathfrak{h}$. By the Poincaré-Birkhoff-Witt theorem, the elements

$$u(\boldsymbol{q},\boldsymbol{m},\boldsymbol{p}) := x_{-\alpha_1}^{q_1}\cdots x_{-\alpha_n}^{q_n}h_1^{m_1}\cdots h_l^{m_l}x_{\alpha_1}^{p_1}\cdots x_{\alpha_n}^{p_n} \tag{6.3.4}$$

(where q_i,m_i,p_i are non-negative integers) form a basis of the vector space $U(\mathfrak{g})$. For all $h\in\mathfrak{h}$, we have

$$[h,u(\boldsymbol{q},\boldsymbol{m},\boldsymbol{p})] = \sum_{i=1}^n (p_i-q_i)\alpha_i(h)\cdot u(\boldsymbol{q},\boldsymbol{m},\boldsymbol{p}). \tag{6.3.5}$$

The vector space U is a $\mathfrak{g}$-module (hence also an $\mathfrak{h}$-module) under the adjoint representation. If $\lambda\in\mathfrak{h}^*$, the subspaces $U(\mathfrak{g})^\lambda$ and $U(\mathfrak{g})_\lambda$ are defined; formula (6.3.5) shows that $U(\mathfrak{g})^\lambda = U(\mathfrak{g})_\lambda$ and that $U(\mathfrak{g}) = \bigoplus_{\lambda\in\Lambda_r}U(\mathfrak{g})^\lambda$ (where Λ_r is the group of radical weights of Φ). In particular, $U(\mathfrak{g})^0$ is the commutant of $\mathfrak{h}$, or of $U(\mathfrak{h})$, in $U(\mathfrak{g})$.

Lemma 6.3.15. *Let $\mathfrak{J} = (\mathfrak{n}^-U(\mathfrak{g}))\cap U(\mathfrak{g})^0$.*

(a) We have $\mathfrak{J} = (U(\mathfrak{g})\mathfrak{n}^+)\cap U(\mathfrak{g})^0$, and $\mathfrak{J}$ is a two-sided ideal of $U(\mathfrak{g})^0$.

(b) We have $U(\mathfrak{g})^0 = U(\mathfrak{h})\oplus\mathfrak{J}$.

Proof. It is clear that $\mathfrak{n}^-U(\mathfrak{g})$ (resp. $U(\mathfrak{g})\mathfrak{n}^+$) is the set of linear combinations of the elements $u(\boldsymbol{q},\boldsymbol{m},\boldsymbol{p})$ such that $\sum q_i>0$ (resp. $\sum p_i>0$). On the other hand, $u(\boldsymbol{q},\boldsymbol{m},\boldsymbol{p})\in U(\mathfrak{g})^0$ if and only if $\sum_{i=1}^n p_i\alpha_i = \sum_{i=1}^n q_i\alpha_i$, which implies that $(\mathfrak{n}^-U(\mathfrak{g}))\cap U(\mathfrak{g})^0 = (U(\mathfrak{g})\mathfrak{n}^+)\cap U(\mathfrak{g})^0$. Finally, $(\mathfrak{n}^-U(\mathfrak{g}))\cap U(\mathfrak{g})^0$ (resp. $(U(\mathfrak{g})\mathfrak{n}^+)\cap U(\mathfrak{g})^0$) is a right (resp. left) ideal of $U(\mathfrak{g})^0$, hence (a). Further, an element $u(\boldsymbol{q},\boldsymbol{m},\boldsymbol{p})$ that is in $U(\mathfrak{g})^0$ belongs to $U(\mathfrak{h})$ (resp. to $\mathfrak{J}$) if and only if $\sum p_i+\sum q_i = 0$ (resp. $\sum p_i+\sum q_i>0$) hence (b). ☐

In view of Lemma 6.3.15, the projection of $U(\mathfrak{g})^0$ onto $U(\mathfrak{h})$ with kernel $\mathfrak{J}$ is a homomorphism of algebras. It is called the **Harish-Chandra homomorphism** from $U(\mathfrak{g})^0$ to $U(\mathfrak{h})$ (relative to Δ). Recall that $U(\mathfrak{h})$ can be identified with the algebra of polynomial functions on $\mathfrak{h}^*$.

Proposition 6.3.16. *Let $\lambda\in\mathfrak{h}^*$, V a $\mathfrak{g}$-module generated by a primitive element of weight λ, χ the central character of V, and φ the Harish-Chandra homomorphism from $U(\mathfrak{g})^0$ to $U(\mathfrak{h})$. Then $\chi(z) = \langle\varphi(z),\lambda\rangle$ for all $z\in Z(\mathfrak{g})$, the centre of $U(\mathfrak{g})$.*

Proof. Let v be a primitive element of V of weight λ, and z an element of the centre of $U(\mathfrak{g})$. Since $\mathfrak{J} = (U(\mathfrak{g})\mathfrak{n}^+)\cap U(\mathfrak{g})^0$, there exist $u_1,\dots,u_p\in U(\mathfrak{g})$ and $n_1,\dots,n_p\in\mathfrak{n}^+$ such that $z = \varphi(z)+\sum_{i=1}^p u_in_i$. Then

$$\chi(z)v = z\cdot v = \varphi(z)\cdot v+\sum_{i=1}^p u_in_i\cdot v = \varphi(z)\cdot v = \lambda(\varphi(z))v = \langle\varphi(z),\lambda\rangle v. \qquad ☐$$

We now use the Harish-Chandra homomorphism to establish an important property for the centre character of the Casimir element. More precisely, we will compute the eigenvalue of the Casimir element.

Proposition 6.3.17. *Let $(\mathfrak{g},\mathfrak{h})$ be a split semi-simple Lie algebra, $\langle\cdot,\cdot\rangle$ a nondegenerate invariant symmetric bilinear form on $\mathfrak{g}$, and Ω the Casimir element associated to $\langle\cdot,\cdot\rangle$. Let $\lambda\in\mathfrak{h}^*$, V a $\mathfrak{g}$-module generated by a primitive element of weight λ, and χ the central character of V. Then $\chi(\Omega) = \langle\lambda,\lambda+2\delta\rangle$, where δ is the Weyl vector.*

Proof. We retain the notations in Proposition 6.2.32. Then

$$\Omega = \sum_{\alpha\in\Phi^-} \frac{1}{\langle x_\alpha, x_{-\alpha}\rangle} x_\alpha x_{-\alpha} + \sum_{\alpha\in\Phi^+} \frac{1}{\langle x_\alpha, x_{-\alpha}\rangle} x_{-\alpha} x_\alpha + \sum_{\alpha\in\Phi^+} \frac{1}{\langle x_\alpha, x_{-\alpha}\rangle}[x_\alpha, x_{-\alpha}] + \sum_{i\in I} h_i h_i'.$$

Therefore

$$\varphi(\Omega) = \sum_{\alpha\in\Phi^+} \frac{1}{\langle x_\alpha, x_{-\alpha}\rangle}[x_\alpha, x_{-\alpha}] + \sum_{i\in I} h_i h_i'.$$

By Proposition 6.3.16,

$$\chi(\Omega) = \sum_{\alpha\in\Phi^+} \frac{1}{\langle x_\alpha, x_{-\alpha}\rangle}\lambda([x_\alpha, x_{-\alpha}]) + \sum_{i\in I} \lambda(h_i)\lambda(h_i').$$

Let h_λ be the element of $\mathfrak{h}$ such that $\langle h_\lambda, h\rangle = \lambda(h)$ for all $h \in \mathfrak{h}$. By Proposition 6.2.20,

$$\frac{1}{\langle x_\alpha, x_{-\alpha}\rangle}\lambda([x_\alpha, x_{-\alpha}]) = \langle h_\lambda, \frac{1}{\langle x_\alpha, x_{-\alpha}\rangle}[x_\alpha, x_{-\alpha}]\rangle = \alpha(h_\lambda) = \langle h_\lambda, h_\alpha\rangle = \langle\lambda, \alpha\rangle.$$

Hence

$$\chi(\Omega) = \sum_{\alpha\in\Phi^+}\langle\lambda, \alpha\rangle + \sum_{i\in I}\lambda(h_i)\lambda(h_i') = \langle\lambda, \alpha\rangle + \langle\lambda, \lambda\rangle = \langle\lambda, \lambda + 2\delta\rangle.$$ □

Example 6.3.18. Let $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{K})$ and consider the bilinear form $(x, y) \mapsto \operatorname{tr}(xy)$ on $\mathfrak{g}$. This form is nondegenerate and invariant as one can check. The Casimir element for $\mathfrak{sl}(2, \mathbb{K})$ with respect to this bilinear form is given by

$$\Omega = \frac{1}{2}h^2 + ef + fe.$$

Recall that we have discribed the Verma modules for $\mathfrak{sl}(2, \mathbb{K})$ in Example 6.3.14. By (6.3.3), for any $\lambda \in \mathbb{K}$ and $v_j \in Z(\lambda)$, we have

$$\begin{aligned}
\Omega \cdot v_j &= (\frac{1}{2}h^2 + ef + fe) \cdot v_j = \frac{1}{2}(\lambda - 2j)^2 v_j + e \cdot v_{j+1} + f \cdot j(\lambda - (j-1))v_{j-1} \\
&= \Big[\frac{1}{2}(\lambda - 2j)^2 + (j+1)(\lambda - j) + j(\lambda - (j-1))\Big] v_j \\
&= (\frac{1}{2}\lambda^2 + \lambda)v_j.
\end{aligned}$$

Thus Ω acts by $\frac{1}{2}\lambda^2 + \lambda$ on $Z(\lambda)$. Note that $\frac{1}{2}\lambda^2 + \lambda$ is invariant under the map $\lambda \mapsto \lambda - 2$, which corresponds the result that $Z(-\lambda - 2)$ is a submodule of $Z(\lambda)$ if λ is non-negative integral.

6.3.3 Finite dimensional $\mathfrak{g}$-modules

In this paragraph, we retain the previous general notations. We denote by Λ (resp. Λ_r) the group of weights of Φ (resp. radical weights of Φ). We denote by Λ^+ (resp. Λ_r^+) the set of elements of Λ (resp. Λ_r) that are positive for the order relation defined by Δ. We denote by Λ^{++} the set of dominant weights of Φ relative to Δ (recall that dominant weights are positive by Proposition 5.1.73). If $w \in W$, we denote by $\varepsilon(w)$ the determinant of w, which is equal to 1 or -1. We put $\delta = \frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha$ to be the Weyl vector.

We have seen in Example 6.3.14 that for any $\lambda \in \mathbb{K}$ there is an infinite dimensional $\mathfrak{sl}(2, \mathbb{K})$-modules with highest weight λ (namely $Z(\lambda)$), and the module $Z(\lambda)$ is simple if λ is not a non-negative integer. On the other hand, the weights of any finite dimensional simple $\mathfrak{sl}(2, \mathbb{K})$-module must be integers by Theorem 6.2.10. This result extends to the the general case: we will show that the weights of a finite dimensional module are weights in the root system, i.e., are elements of Λ. This can be viewed as a generalization for the observations above, and justifies our terminology for weights. Moreover, we shall construct automorphims of these modules and show that weights belonging to the same orbit under the Weyl group have the same multiplicity.

Proposition 6.3.19. *Let V be a finite dimensional $\mathfrak{g}$-module.*

(a) All weights of V belongs to Λ.

(b) $V = \bigoplus_{\mu\in\Lambda} V^\mu$.

(c) *For all $\mu \in \mathfrak{h}^*$, V^μ is the set of $x \in V$ such that $h \cdot x = \mu(h)x$ for all $h \in \mathfrak{h}$.*

Proof. For all $\alpha \in \Delta$, there exists a homomorphism from $\mathfrak{sl}(2, \mathbb{K})$ to $\mathfrak{g}$ that takes h to h_α. Thus, by Corollary 6.2.8, $(h_\alpha)_V$ is diagonalizable and its eigenvalues are integers. Hence, the set of $(h_\alpha)_V$, for $\alpha \in \Delta$, is diagonalizable. Consequently, for all $h \in \mathfrak{h}$, h_V is diagonalizable. By Theorem 6.1.6, $V = \bigoplus_{\mu \in \mathfrak{h}^*} V^\mu$. On the other hand, if $V^\mu \neq 0$, the preceding shows that $\mu(h_\alpha) \in \mathbb{Z}$ for all $\alpha \in \Delta$, so $\mu \in \Lambda$. This proves (a) and (b). We see in the same way that $\mathfrak{h}_V$ is diagonalizable, hence (c). □

Corollary 6.3.20. *Let ρ be a finite dimensional representation of $\mathfrak{g}$ and β the bilinear form associated to ρ.*

(a) *If $x, y \in \mathfrak{h}_\mathbb{Q}$, then $\beta(x, y) \in \mathbb{Q}$ and $\beta(x, x) \in \mathbb{Q}_+$.*

(b) *If ρ is injective, the restriction of β to $\mathfrak{h}$ is non-degenerate.*

Proof. Assertion (a) follows from Proposition 6.3.19 since the elements of Λ have rational values on $\mathfrak{h}_\mathbb{Q}$. If ρ is injective, β is non-degenerate (Proposition 1.6.5), so the restriction of β to $\mathfrak{h}$ is non-degenerate (Proposition 6.1.18(c)). □

Lemma 6.3.21. *Let V be a $\mathfrak{g}$-module and ρ the corresponding representation of $\mathfrak{g}$.*

(a) *If x is a nilpotent element of $\mathfrak{g}$, and if $\rho(x)$ is locally nilpotent,*

$$\rho(e^{\mathrm{ad}(x)}(y)) = e^{\rho(x)}\rho(y)e^{-\rho(x)}$$

for all $y \in \mathfrak{g}$.

(b) *If $\alpha \in \Phi$ and if the images under ρ of the elements of $\mathfrak{g}^\alpha$ and $\mathfrak{g}^{-\alpha}$ are locally nilpotent, the set of weights of V is stable under the reflection s_α.*

Proof. With the assumptions in (a), we have $\rho(\mathrm{ad}(a)^n(y)) = \mathrm{ad}(\rho(x)^n(\rho(y))$ for all $n \geq 0$, so $\rho(e^{\mathrm{ad}(x)}(y)) = e^{\mathrm{ad}(\rho(x))}\rho(b)$. On the other hand,

$$e^{\mathrm{ad}(\rho(x))}(\rho(y)) = e^{\rho(x)}\rho(y)e^{-\rho(x)}$$

is assertion (b) of Lemma 6.1.55.

We now adopt the assumptions in (b). Let $\theta_\alpha = e^{\mathrm{ad}(x_\alpha)}e^{-\mathrm{ad}(x_{-\alpha})}e^{\mathrm{ad}(x_\alpha)}$. By (a), there exists $s \in \mathrm{GL}(V)$ such that $\rho(\theta_\alpha(y)) = u\rho(b)u^{-1}$ for all $y \in \mathfrak{g}$. Now $\theta_\alpha|_\mathfrak{h}$ is the transpose of s_α (Proposition 6.2.25). Let λ be a weight of V. There exists a non-zero element x of V such that $\rho(h)x = \lambda(h)x$ for all $h \in \mathfrak{h}$. Then

$$\rho(h)u^{-1}x = u^{-1}\rho(\theta_\alpha(h))x = u^{-1}\rho(s_\alpha^t(h))x = u^{-1}\lambda(s_\alpha^t(h))x = s_\alpha(\lambda)(h)u^{-1}x$$

for all $h \in \mathfrak{h}$. Consequently, $s_\alpha(\lambda)$ is a weight of V. □

Proposition 6.3.22. *Let V be a finite dimensional $\mathfrak{g}$-module and $\varphi \in \mathrm{Aut}_0(\mathfrak{g})$.*

(a) *There exists $s \in \mathrm{GL}(V)$ such that $(\varphi(x))_V = sx_Vs^{-1}$ for all $x \in \mathfrak{g}$.*

(b) *If $\varphi \in \mathrm{Aut}_e(\mathfrak{g})$, s can be chosen to be an element of $\mathrm{SL}(V)$ leaving stable all the $\mathfrak{g}$-submodules of V.*

Proof. Assertion (b) follows from Lemma 6.3.21(a). Now let $\varphi \in \mathrm{Aut}_0(\mathfrak{g})$ and denote by ρ the representation of $\mathfrak{g}$ defined by V. By (b), the representations ρ and $\rho \circ \varphi$ become equivalent after extension of scalars. They are therefore equivalent (Proposition 1.3.25), hence the existence of s. □

Let s satisfy the condition in Proposition 6.3.22(a), and let $\tilde{\mathfrak{h}} = \varphi(\mathfrak{h})$; denote by φ^* the isomorphism $\lambda \mapsto \lambda \circ \varphi^{-1}$ from $\mathfrak{h}^*$ to $\tilde{\mathfrak{h}}^*$. It is clear that $s(V^\lambda) = V^{\varphi^*(\lambda)}$. In particular:

Corollary 6.3.23. *The isomorphism φ^* takes the weights of V with respect to $\mathfrak{h}$ to those of V with respect to $\tilde{\mathfrak{h}}$; corresponding weights have the same multiplicity.*

Corollary 6.3.24. *Let V be a finite dimensional $\mathfrak{g}$-module. Let $w \in W$. For all $\lambda \in \mathfrak{h}^*$, the vector subspaces V^λ and $V^{w(\lambda)}$ have the same dimension. The set of weights of V is stable under W.*

Proof. Indeed, w is of the form φ^* with $\varphi \in \mathrm{Aut}_e(\mathfrak{g}, \mathfrak{h})$ (Corollary 6.2.27). □

Finally we consider finite dimensional simple modules and show their weights are dominant. In fact, we shall prove that, such a simple module is isomorphic to $V(\lambda)$ with $\lambda \in \Lambda^{++}$.

Theorem 6.3.25. *A simple $\mathfrak{g}$-module is finite dimensional if and only if it has a highest weight belonging to Λ^{++}.*

Proof. We denote by V a simple $\mathfrak{g}$-module and by $\mathscr{X}$ its set of weights. Assume that V is finite dimensional. Then $\mathscr{X}$ is finite and non-empty (Proposition 6.3.19) and so has a maximal element ω. Let $\alpha \in \Delta$. Then $\omega + \alpha \notin \mathscr{X}$, which proves that ω is the highest weight of V (Lemma 6.3.4). On the other hand, there exists a homomorphism from $\mathfrak{sl}(2, \mathbb{K})$ to $\mathfrak{g}$ that takes h to h_α; by Proposition 6.2.7(a), $\omega(h_\alpha)$ is a non-negative integer, so $\omega \in \Lambda^{++}$.

Assume that V has a highest weight $\omega \in \Lambda^{++}$. Let $\alpha \in \Delta$ and let v be a primitive element of weight ω in V. Put $v_j = x_{-\alpha}^j v$ for $j \geq 0$, $m = \omega(h_\alpha) \in \mathbb{N}$, and $N = \sum_{j=0}^m \mathbb{K} v_j$. Note that by Proposition 6.2.6 we have $x_\alpha v_{m+1} = 0$. If $\beta \in \Delta$ and $\beta \neq \alpha$, then $[x_\beta, x_{-\alpha}] = 0$, so

$$x_\beta v_{m+1} = x_\beta x_{-\alpha}^{m+1} = x_{-\alpha}^{m+1} x_\beta v = 0.$$

If $v_{m+1} \neq 0$, we conclude that v_{m+1} is primitive, which is absurd (Proposition 6.3.5(a)); so $v_{m+1} = 0$. Thus the subspace N is invariant under the subalgebra $\mathfrak{s}_\alpha$. Now $\mathfrak{s}_\alpha$ is reductive in $\mathfrak{g}$, so the sum of the finite dimensional subspaces of V that are simple $\mathfrak{s}_\alpha$-modules is a $\mathfrak{g}$-submodule of V (Proposition 1.6.33); since this sum is non-zero, it is equal to V. It follows from this that $(x_\alpha)_V$ and $(x_{-\alpha})_V$ are locally nilpotent. In view of Lemma 6.3.21, $\mathscr{X}$ is stable under s_α, and this holds for all α. Hence $\mathscr{X}$ is stable under W. Now every orbit of W on Λ meets Λ^{++}. On the other hand, if $\lambda \in \Lambda^{++}$, then $\lambda = \omega - \sum_{\alpha \in \Delta} n_\alpha \alpha = \sum_{\alpha \in \Delta} m_\alpha \alpha$ with n_α, m_α non-negative integers for all $\alpha \in \Delta$. So $\mathscr{X} \cap \Lambda^{++}$ is finite and hence so is $\mathscr{X}$. Since each weight has finite multiplicity (Proposition 6.3.2(b)), V is finite dimensional. $\square$

Corollary 6.3.26. *If $\lambda \in \mathfrak{h}^*$ and $\lambda \notin \Lambda^{++}$, the $\mathfrak{g}$-module $V(\lambda)$ is infinite dimensional.*

Corollary 6.3.27. *The $\mathfrak{g}$-modules $V(\lambda)$ for $\lambda \in \Lambda^{++}$ constitute a set of representatives of the classes of finite dimensional simple $\mathfrak{g}$-modules.*

The $\mathfrak{g}$-modules $V(\lambda)$, where λ is a fundamental weight, are called the fundamental $\mathfrak{g}$-modules; the corresponding representations are called the fundamental representations of $\mathfrak{g}$; they are absolutely irreducible by Proposition 6.3.5. If V is a finite dimensional $\mathfrak{g}$-module and $\lambda \in \Lambda^{++}$, the isotypical component of V of type $V(\lambda)$ is called the isotypical component of highest weight λ of V.

Remark 6.3.28. Let $\lambda \in \Lambda^{++}$, ρ_λ the representation of $\mathfrak{g}$ on $V(\lambda)$, $\varphi \in \mathrm{Aut}(\mathfrak{g})$, and σ the canonical image of φ in $\mathrm{Aut}(\Phi, \Delta)$ (Corollary 6.2.85). Then $\rho_\lambda \circ \varphi$ is equivalent to $\rho_{\sigma(\lambda)}$; indeed, if $\varphi \in \mathrm{Aut}_0(\mathfrak{g})$, $\rho_\lambda \circ \varphi$ and $\rho_{\sigma(\lambda)}$ are equivalent to ρ_λ (Proposition 6.3.22); and, if φ leaves $\mathfrak{h}$ and Δ stable, $\rho_\lambda \circ \varphi$ is simple of highest weight $\sigma(\lambda)$. In particular, the fundamental representations are permuted by φ, and this permutation is the identity if and only if $\varphi \in \mathrm{Aut}_0(\mathfrak{g})$.

Proposition 6.3.29. *Let V be a finite dimensional $\mathfrak{g}$-module and $\mathscr{X}$ its set of weights. Let $\lambda \in \mathscr{X}$, $\alpha \in \Phi$, I the set of $k \in \mathbb{Z}$ such that $\lambda + k\alpha \in \mathscr{X}$, $-p$ (resp. q) the smallest (resp. largest) element of I. Let m_k be the multiplicity of $\lambda + k\alpha$.*

(a) $I = [-p, q] \cap \mathbb{Z}$ and $p - q = \lambda(h_\alpha)$.

(b) For any integer $t \in [0, p+q]$, $\lambda + (q-t)\alpha$ and $\lambda + (-p+t)\alpha$ are conjugate under s_α, and $m_{q-t} = m_{-p+t}$.

(c) If $t \in \mathbb{Z}$ and $t < (q-p)/2$, $(x_\alpha)_V$ maps $V^{\lambda+t\alpha}$ injectively into $V^{\lambda+(t+1)\alpha}$.

(d) The function $t \mapsto m_t$ is increasing on $[-p, (q-p)/2]$ and decreasing on $[(q-p)/2, q]$.

Proof. Let $\alpha \in \Delta$. Give V the $\mathfrak{sl}(2, \mathbb{K})$-module structure defined by the elements $x_\alpha, x_{-\alpha}, h_\alpha$ of $\mathfrak{g}$. Every non-zero element of $V^{\lambda+q\alpha}$ is then primitive. Consequently, $(\lambda + q\alpha)(h_\alpha) \geq 0$ and

$$(x_{-\alpha})^r V^{\lambda+q\alpha} \neq 0 \quad \text{for} \quad 0 \leq r \leq (\lambda + q\alpha)(h_\alpha) = \lambda(h_\alpha) + 2q$$

(Proposition 6.2.7). It follows that $V^{\lambda+t\alpha} \neq 0$ for $q - (\lambda(h_\alpha) + 2q) \leq t \leq q$, so

$$p + q \geq \lambda(h_\alpha) + 2q.$$

Applying this result to $-\alpha$ gives

$$p+q \geq \lambda(h_{-\alpha}) + 2q = -\lambda(h_\alpha) + 2q.$$

Hence $p - q = \lambda(h_\alpha)$ and $\lambda + t\alpha \in \mathscr{X}$ for $-p \leq t \leq q$, which proves (a).

We have $s_\alpha(\alpha) = -\alpha$, and $s_\alpha(\mu) = \mu + k\alpha$ for all $\mu \in \mathfrak{h}^*$. Since W leaves $\mathscr{X}$ stable (Corollary 6.3.24), s_α leaves $\{\lambda - p\alpha, \lambda - p\alpha + \alpha, \ldots, \lambda + q\alpha\}$ stable and takes $\lambda - p\alpha + t\alpha$ to $\lambda + q\alpha - t\alpha$ for all $t \in \mathbb{K}$. Using Corollary 6.3.24 again, we see that $m_{-p+t} = m_{q-t}$ for every integer $t \in [0, p+q]$. This proves (b).

By Corollary 6.2.8, $(x_\alpha)_V|_{V^{\lambda+t\alpha}}$ is injective for $t < (q-p)/2$. Now $(x_\alpha)_V$ maps $V^{\lambda+t\alpha}$ to $V^{\lambda+(t+1)\alpha}$. Hence $m_{t+1} \geq m_t$ for $t < (q-p)/2$. Changing α to $-\alpha$, we see that $m_{t+1} \leq m_t$ for $t > (q-p)/2$. This proves (c) and (d). □

Corollary 6.3.30. *Let V be a finite dimensional $\mathfrak{g}$-module and $\mathscr{X}$ its set of weights. If $\lambda \in \mathscr{X}$ and $\lambda(h_\alpha) \geq 1$, then $\lambda - \alpha \in \mathscr{X}$. If $\lambda + \alpha \in \mathscr{X}$ and $\lambda(h_\alpha) = 0$, then $\lambda \in \mathscr{X}$ and $\lambda - \alpha \in \mathscr{X}$.*

Proof. This follows immediately from Proposition 6.3.29(a). □

Corollary 6.3.31. *Let V be a finite dimensional $\mathfrak{g}$-module and $\mathscr{X}$ its set of weights. Let $\mu \in \Lambda^{++}$ and $\nu \in \Lambda_r^+$. If $\mu + \nu \in \mathscr{X}$, then $\mu \in \mathscr{X}$.*

Proof. Write $\nu = \sum_{\alpha\in\Delta} c_\alpha \alpha$, where $c_\alpha \in \mathbb{N}$ for all $\alpha \in \Delta$. The corollary is clear when $\sum_{\alpha\in\Delta} c_\alpha = 0$; assume that $\sum_{\alpha\in\Delta} c_\alpha > 0$ and argue by induction on $\sum_{\alpha\in\Delta} c_\alpha$. Let $(\cdot\,,\cdot)$ be a W-invariant non-degenerate positive symmetric bilinear form on $\mathfrak{h}_{\mathbb{R}}^*$. Then $(\nu, \nu) = (\nu, \sum_{\alpha\in\Delta} c_\alpha \alpha) > 0$, so there exists $\beta \in \Delta$ such that $c_\beta \geq 1$ and $(\nu, \beta) > 0$, hence $\nu(h_\beta) \geq 1$. Since $\mu \in \Lambda^{++}$, it follows that $(\mu + \nu)(h_\beta) \geq 1$. By Corollary 6.3.30, $\mu + (\nu - \beta) \in \mathscr{X}$, and it suffices to apply the induction hypothesis. □

Corollary 6.3.32. *Let V be a finite dimensional $\mathfrak{g}$-module and $v \in V$ be primitive of weight ω. Let Σ be the set of $\alpha \in \Delta$ such that $\omega(h_\alpha) = 0$. Then the stabilizer in $\mathfrak{g}$ of the line $\mathbb{K}v$ is the parabolic subalgebra $\mathfrak{p}_\Sigma$ associated to Σ.*

Proof. Let $\mathscr{X}$ be the set of weights of V. Replacing V by the $\mathfrak{g}$-submodule generated by v, if necessary, we can assume that V is simple (Proposition 6.3.2, since V is finite dimensional). In this case ω is the highest weight of V. Let $\mathfrak{s}$ be the stabilizer; we have $\mathfrak{b}^+ \cdot v \subseteq \mathbb{K}v$. Let $\alpha \in \Delta$ be such that $\omega(h_\alpha) = 0$. We have $\omega + \alpha \notin \mathscr{X}$, hence $\omega - \alpha \notin \mathscr{X}$ (Proposition 6.3.29) and consequently $\mathfrak{g}^{-\alpha} \cdot v = 0$. The preceding proves that $\mathfrak{p}_\Sigma \subseteq \mathfrak{s}$. If $\mathfrak{p}_\Sigma \neq \mathfrak{s}$, then $\mathfrak{s} = \mathfrak{p}_{\Sigma'}$, where Σ' is a subset of Δ strictly containing Σ. Let $\beta \in \Sigma' \setminus \Sigma$. Then $\mathfrak{g}^{-\beta}$ stabilizes $\mathbb{K}v$, and hence annihilates v. But $\omega(h_\alpha) > 0$ and $\omega + \alpha \notin \mathscr{X}$, so by Theorem 6.3.25(c) we obtain a contradiction. □

A subset $\mathscr{X}$ of Λ is called Φ-saturated if it satisfies the following condition: for all $\lambda \in \mathscr{X}$ and all $\alpha \in \Phi$, we have $\lambda - t\alpha \in \mathscr{X}$ for all integers t between 0 and $\lambda(h_\alpha)$. Since $s_\alpha() = \lambda - \lambda(h_\alpha)\alpha$, we see that an Φ-saturated subset of Λ is stable under W. Let $\mathscr{Y} \subseteq \Lambda$. An element λ of $\mathscr{Y}$ is called Φ-extremal in $\mathscr{Y}$ if, for all $\alpha \in \Phi$, either $\lambda + \alpha \notin \mathscr{Y}$ or $\lambda - \alpha \notin \mathscr{Y}$.

Proposition 6.3.33. *Let V be a finite dimensional $\mathfrak{g}$-module and $d \geq 1$ an integer. Then the set of weights of V of multiplicity $\geq d$ is Φ-saturated.*

Proof. This follows immediately from Corollary 6.3.24 and Theorem 6.3.25. □

The set $\mathscr{X}$ of weights for a finite dimensional $\mathfrak{g}$-module is Φ-saturated, as we will show now. Moreover, we shall see the set $\mathscr{X}$ can be characterized using this notation. But before that, let's establish the following analogoue for Corollary 6.3.31.

Lemma 6.3.34. *Let $\mathscr{X}$ be a Φ-saturated subset of Λ, with highest weight ω. If $\mu \in \Lambda^{++}$ and $\mu \preceq \omega$, then $\mu \in \mathscr{X}$.*

Proof. Since $\lambda \preceq \mu$ and $\lambda \in \mathscr{X}$, we can write $\tilde{\mu} = \mu + \sum_{\alpha\in\Delta} n_\alpha \alpha$ with $\tilde{\mu} \in \mathscr{X}$ and $n_\alpha \in \mathbb{N}$ for all $\alpha \in \Delta$. We shall show how to reduce one of the n_α by one while still remaining in $\mathscr{X}$, thus eve ntually arriving at the conclusion that $\mu \in \mathscr{X}$. Of course, our starting point is the fact that λ itself is such a $\tilde{\mu}$. Now suppose $\tilde{\mu} \neq \mu$, so some n_α is positive. Let $(\cdot\,,\cdot)$ be a W-invariant non-degenerate positive symmetric bilinear form on $\mathfrak{h}_{\mathbb{R}}^*$. From $(\sum_\alpha n_\alpha \alpha, \sum_\alpha n_\alpha \alpha) > 0$, we deduce that $(\sum_\alpha n_\alpha \alpha, \beta) > 0$ for some $\beta \in \Delta$, with $n_\beta > 0$. In particular, $\sum_\alpha n_\alpha \alpha(h_\beta) \geq 0$. Since μ is dominant, $\mu(h_\beta) \geq 0$. Therefore $\tilde{\mu}(h_\beta)$ is positive. By definition of saturated set, it is now possible to subtract β once from $\tilde{\mu}$ without leaving $\mathscr{X}$, thus reducing n_β by one. □

Proposition 6.3.35. *Let V be a finite dimensional simple $\mathfrak{g}$-module, ω its highest weight, $\mathscr{X}$ its set of weights. Choose a W-invariant non-degenerate positive symmetric bilinear form $(\cdot\,,\cdot)$ on $\mathfrak{h}_{\mathbb{R}}^*$.*

(a) $\mathscr{X}$ is the smallest Φ-saturated subset of Λ containing ω, which is characterized by

$$\mathscr{X} = W\cdot(\mathscr{X}\cap\Lambda^{++}) = \mathrm{conv}(W\cdot\omega) = \{\mu\in\Lambda : w(\mu)\preceq\omega \text{ for all } w\in W\}. \tag{6.3.6}$$

(b) The Φ-extremal elements of $\mathscr{X}$ are the W-transforms of ω.

(c) If $\mu\in\mathscr{X}$, we have $\|\mu\|\leq\|\omega\|$. If, in addition, $\mu\neq\omega$, we have $\|\mu+\delta\|<\|\omega+\delta\|$. If μ is not Φ-extremal in $\mathscr{X}$, then $\|\mu\|<\|\omega\|$.

(d) An element λ of Λ (resp. of Λ^{++}) belongs to $\mathscr{X}$ (resp. $\mathscr{X}\cap\Lambda^{++}$) if and only if $\omega-\lambda\in\Lambda_r^+$.

Proof. Let $\mathscr{Y}$ be the smallest Φ-saturated subset of Λ containing ω. We have $\mathscr{Y}\subseteq\mathscr{X}$ by Proposition 6.3.35. Assume that $\mathscr{X}\neq\mathscr{Y}$. Let λ be a maximal element of $\mathscr{X}\setminus\mathscr{Y}$. Since $\mathscr{X}$ is invariant under W, we may conjugate λ by W and therefore assume that λ is dominant. Since $\lambda\neq\omega$, there exists $\alpha\in\Delta$ such that $\lambda+\alpha\in\mathscr{X}$. Then by Lemma 6.3.34 we have $\lambda\in\mathscr{X}$, contradiction. This proves the first claim in (a). Also, Lemma 6.3.34 shows that $\mathscr{X}$ contains the W-conjugates of dominant weights μ such that $\mu\preceq\omega$, which proves $\mathscr{X}=W\cdot(\mathscr{X}\cap\Lambda^{++})$. The last statement $\mathscr{X}=\mathrm{conv}(W\cdot\omega)$ follows from Proposition 5.1.74.

It is clear that ω is an Φ-extremal element of $\mathscr{X}$; its W-transforms are therefore also Φ-extremal in $\mathscr{X}$. Let λ be an Φ-extremal element of $\mathscr{X}$; we shall prove that $\lambda\in W\cdot\omega$. By conjugation via W, we can assume that $\lambda\in\Lambda^{++}$. Let $\alpha\in\Delta$; introduce p and q as in Theorem 6.3.25. Since λ is Φ-extremal, either $p=0$ or $q=0$. Since $p-q=\lambda(h_\alpha)\geq 0$, we can not have $q>0$. Hence $q=0$ and $\lambda+\alpha\notin\mathscr{X}$. Since this holds for all α, we conclude that λ is the highest weight for $\mathscr{X}$, which is ω.

Let $\mu\in\mathscr{X}\cap\Lambda^{++}$. Then $\omega+\mu\in\Lambda^{++}$ and $\omega-\mu\in\Lambda_r^+$, so

$$0\leq(\omega-\mu,\omega+\mu)=(\omega,\omega)-(\mu,\mu);$$

hence, $(\mu,\mu)\leq(\omega|\omega)$, and this extends to all $\mu\in\mathscr{X}$ by using the Weyl group. If $\mu\neq\omega$,

$$\begin{aligned}(\mu+\delta,\mu+\delta)&=(\mu,\mu)+2(\mu,\delta)+(\delta,\delta)\leq(\omega,\omega)+2(\mu,\delta)+(\delta,\delta)\\&=(\omega+\delta,\omega+\delta)-2(\omega-\mu,\delta).\end{aligned}$$

Now $\omega-\mu\in\Lambda_r^+$ is nonzero and δ is strict dominant, so $(\omega-\mu,\delta)>0$. This shows $(\mu+\delta,\mu+\delta)<(\omega+\delta,\omega+\delta)$. If μ is not Φ-extremal in $\mathscr{X}$, there exists $\alpha\in\mathscr{X}$ such that $\mu\pm\alpha\in\mathscr{X}$; then

$$\|\mu\|<\sup(\|\mu+\alpha\|,\|\mu-\alpha\|)\leq\sup_{\lambda\in\mathscr{X}}\|\lambda\|=\|\omega\|$$

which proves (c).

If $\lambda\in\mathscr{X}$, then $\omega-\lambda\in\Lambda_r^+$ by Proposition 6.3.2. If $\lambda\in\Lambda$ and $\omega-\lambda\in\Lambda_r^+$, then $\lambda\in\mathscr{X}$ by Lemma 6.3.34. □

Corollary 6.3.36. *Let $\mathscr{X}$ be a finite Φ-saturated subset of Λ. There exists a finite dimensional $\mathfrak{g}$-module whose set of weights is $\mathscr{X}$.*

Proof. Since $\mathscr{X}$ is stable under W, $\mathscr{X}$ is the smallest Φ-saturated set containing $\mathscr{X}\cap\Lambda^{++}$. By Proposition 6.3.35(a), $\mathscr{X}$ is the set of weights of $\bigoplus_{\lambda\in\mathscr{X}\cap\Lambda^{++}}V(\lambda)$. □

Remark 6.3.37. Recall that there exists a unique element w_0 of W that transforms Δ into $-\Delta$; we have $w^2=1$ and $-w_0$ respects the order relation on Λ. With this in mind, let V be a finite dimensional simple $\mathfrak{g}$-module, ω its highest weight. Then $w_0(\omega)$ is the lowest weight of V, and its multiplicity is 1.

With Eq. (6.3.6), we want consider elements $\omega\in\Lambda^{++}$ such that $\mathscr{X}=W\cdot\omega$. These are the minimal elements in the set Λ^{++} of dominant weights, which will be called minuscule weights. We shall see they enjoy some nice properties.

Proposition 6.3.38. *Let $\lambda\in\Lambda$ and $\mathscr{X}$ the smallest Φ-saturated subset of Λ containing λ. Choose a norm on $\mathfrak{h}_{\mathbb{R}}$ as in Proposition 6.3.35. Then the following conditions are equivalent:*

(i) $\mathscr{X}=W\cdot\lambda$;

(ii) *all elements of $\mathscr{X}$ have the same norm;*

(iii) *for all $\alpha \in \Phi$, we have $\lambda(h_\alpha) \in \{0, 1, -1\}$.*

Every non-empty Φ-saturated subset of Λ contains an element λ satisfying the above conditions.

Proof. We first introduce the following condition:

(ii') for all $\alpha \in \Phi$ and for every integer t between 0 and $\lambda(h_\alpha)$, $\|\lambda - t\alpha\| \geq \|\lambda\|$.

It is clear that (i)⇒(ii)⇒(ii'). Now assume that condition (ii') is satisfied. Let $\alpha \in \Phi$. We have $\|\lambda\| = \|\lambda - \lambda(h_\alpha)\alpha\|$, so $\|\lambda - t\alpha\| < \|\lambda\|$ for every integer t strictly between 0 and $\lambda(h_\alpha)$; hence, there can be no such integers, so $|\lambda(h_\alpha)| \leq 1$. This shows (ii')⇒(iii).

Assume that condition (iii) is satisfied. Let $w \in W$ and $\alpha \in \Phi$. Then $w(\lambda)(h_\alpha) = \lambda(h_{w^{-1}\alpha}) \in \{0, 1, -1\}$; thus, if t is an integer between 0 and $w(\lambda)(h_\alpha)$, $w(\lambda) - t\alpha$ is equal to $w(\lambda)$ or $s_\alpha(w(\lambda))$. This proves that $W \cdot \lambda$ is Φ-saturated, so $\mathscr{X} = W \cdot \lambda$.

Let $\mathscr{Y}$ be a non-empty Φ-saturated subset of Λ. There exists in $\mathscr{Y}$ an element λ of minimum norm. It is clear that λ satisfies condition (ii'), hence the last assertion of the proposition. □

Proposition 6.3.39. *Let V be a finite dimensional simple $\mathfrak{g}$-module, $\mathscr{X}$ the set of weights of V, and λ the highest element of $\mathscr{X}$. Then the conditions of Proposition 6.3.38 are equivalent to:*

(iv) *for all $\alpha \in \Phi$ and all $x \in \mathfrak{g}^\alpha$, we have $(x_V)^2 = 0$.*

If these conditions are satisfied, all the weights of V have multiplicity 1.

Proof. If (i) in Proposition 6.3.38 is satisfied, then $\mathscr{X} = W \cdot \lambda$ and the weights all have the same multiplicity as λ, in other words, multiplicity 1. Moreover, if $w \in W$ and $\alpha \in \Phi$, $w(\lambda) + t\alpha$ cannot be a weight of V unless $|t| \leq 1$; thus, if $x \in \mathfrak{g}^\alpha$,

$$(x_V)^2(V^{w(\lambda)}) \subseteq V^{w(\lambda)+2\alpha} = 0$$

whence $(x_V)^2 = 0$, which proves (i)⇒(iv).

Conversely, assume that (iv) is satisfied. Let $\alpha \in \Phi$, and give V the $\mathfrak{sl}(2, \mathbb{K})$-module structure defined by the elements $x_\alpha, x_{-\alpha}, h_\alpha$ of $\mathfrak{g}$. Condition (iv), applied to $x = x_\alpha$, implies that the weights of the $\mathfrak{sl}(2, \mathbb{K})$-module V belong to $\{0, 1, -1\}$. In particular, $\lambda(h_\alpha) \in \{0, 1, -1\}$, so (iv)⇒(iii). □

Proposition 6.3.40. *Let $\mathfrak{g}$ be a simple Lie algebra. Denote by $\alpha_1, \ldots, \alpha_l$ the elements of Δ and let $\varpi_1, \ldots, \varpi_l$ be the corresponding fundamental weights. Let $h = \sum_{i=1}^l n_i h_{\alpha_i}$ be the highest root of $\check{\Phi}$, and J the set of $i \in \{1, \ldots, l\}$ such that $n_i = 1$. Let $\lambda \in \Lambda^{++} \setminus \{0\}$. Then the conditions of Proposition 6.3.38 are equivalent to each of the following conditions:*

(v) *$\lambda(h) = 1$;*

(vi) *there existence $i \in J$ such that $\lambda = \varpi_i$.*

The ϖ_i, for $i \in J$, form a system of representatives in $\Lambda(\Phi)$ of the nonzero elements of $\Lambda(\Phi)/\Lambda_r(\Phi)$.

Proof. Let $\lambda = \sum_{i=1}^l a_i \varpi_i$, where a_i are non-negative integers and not all zero. Then $\lambda(h) = \sum_{i=1}^l a_i n_i$ and $n_i \geq 1$ for all i, which gives the equivalence of (v) and (vi) immediately. On the other hand, $\lambda(H) = \sup_{\alpha \in \Phi^+} \lambda(h_\alpha)$, and $\lambda(h) > 0$ since λ is a non-zero element of Λ^{++}. Hence condition (v) is equivalent to the condition $\lambda(h_\alpha) \in \{0, 1\}$ for all $\alpha \in \Phi$, in other words to condition (iii) of Proposition 6.3.38. The last assertion of the proposition follows from Corollary 5.2.9. □

Assume that $\mathfrak{g}$ is simple. A **minuscule weight** of $(\mathfrak{g}, \mathfrak{h})$ is an element of $\Lambda^{++} \setminus \{0\}$ which satisfies the equivalent conditions of Proposition 6.3.38, 6.3.39, and 6.3.40. Let $\check{\Sigma}'$ be the Coxeter graph of the affine Weyl group $W_a(\check{\Phi})$. Recall that the vertices of $\check{\Sigma}'$ are the vertices of the Coxeter graph $\check{\Sigma}$ of $W(\check{\Phi})$, together with a supplementary vertex 0. The group $\mathrm{Aut}(\check{\Phi})$ operates on $\check{\Sigma}'$ leaving 0 fixed. The group $\mathrm{Aut}(\check{\Sigma}')$ is canonically isomorphic to the semi-direct product of $\mathrm{Aut}(\check{\Phi})/W(\check{\Phi})$ with a group Γ_A; clearly $(\mathrm{Aut}(\check{\Sigma}'))(0) = \Sigma_A(0)$ and $\Sigma_A(0)$ consists of 0 and the vertices of $\check{\Sigma}$ corresponding to h_{α_i} for $i \in J$. In summary, the minuscule weights are the fundamental weights corresponding to the vertices of $\check{\Sigma}$ which can be obtained from 0 by the operation of an element of $\mathrm{Aut}(\check{\Sigma}')$.

6.3.4 Tensor product and dual of $\mathfrak{g}$-modules

Let E, F be $\mathfrak{g}$-modules. Recall that $\mathfrak{g}$ has an action on the tensor product $E \otimes F$ of the vector spaces E, F by

$$x \cdot (v \otimes w) = x \cdot v \otimes w + v \otimes x \cdot w.$$

By Proposition 6.1.2(b), for $\lambda, \mu \in \mathfrak{h}^*$ we have $E^\lambda \otimes F^\mu \subseteq (E \otimes F)^{\lambda+\mu}$. If E and F are finite dimensional, then $E = \sum_{\lambda \in \Lambda} E^\lambda$ and $F = \sum_{\mu \in \Lambda} F^\mu$; consequently,

$$(E \otimes F)^\nu = \sum_{\substack{\lambda,\mu \in \Lambda \\ \lambda+\mu=\nu}} E^\lambda \otimes F^\mu.$$

In other words, equipped with its graduation of type Λ, $E \otimes F$ is the graded tensor product of the graded vector spaces E and F.

Proposition 6.3.41. *Let E, F be finite dimensional simple $\mathfrak{g}$-modules, with highest weights λ, μ, respectively.*

(a) The component of $E \otimes F$ of highest weight $\lambda + \mu$ is a simple submodule, generated by $(E \otimes F)^{\lambda+\mu} = E^\lambda \otimes F^\mu$.

(b) The highest weight of any simple submodule of $E \otimes F$ is lower than $\lambda + \mu$.

Proof. If $\alpha, \beta \in \Lambda$ and if $E^\alpha \otimes F^\beta \neq 0$, then $\alpha \preceq \lambda$ and $\beta \preceq \mu$. Consequently, $(E \otimes F)^{\lambda+\mu}$ is equal to $E^\lambda \otimes F^\mu$, and hence is of dimension 1, and $\lambda + \mu$ is the highest weight of $E \otimes F$. Every non-zero element of $E^\lambda \otimes F^\mu$ is primitive. By Proposition 6.3.7, the length of the isotypical component of $E \otimes F$ of highest weight $\lambda + \mu$ is 1. □

Retain the notations of Proposition 6.3.41. Let C be the isotypical component of $E \otimes F$ of highest weight $\lambda + \mu$. Then C depends only on E and F and not on the choice of $\mathfrak{h}$ and the basis Δ. In other words, let $\tilde{\mathfrak{h}}$ be a splitting Cartan subalgebra of $\mathfrak{g}$, $\tilde{\Phi}$ the root system of $(\mathfrak{g}, \tilde{\mathfrak{h}})$, and $\tilde{\Delta}$ a basis of $\tilde{\Phi}$; let $\tilde{\lambda}, \tilde{\mu}$ be the highest weights of E, F relative to $\tilde{\mathfrak{h}}$ and $\tilde{\Delta}$; let $\tilde{C}$ be the isotypical component of $E \otimes F$ of highest weight $\tilde{\lambda} + \tilde{\mu}$; then $\tilde{C} = C$. Indeed, to prove this we can assume, by extension of the base field, that $\mathbb{K}$ is algebraically closed. Then there exists $s \in \mathrm{Aut}_e(\mathfrak{g})$ that takes $\mathfrak{h}$ to $\tilde{\mathfrak{h}}$, Φ to $\tilde{\Phi}$, Δ to $\tilde{\Delta}$. Let $s \in \mathrm{SL}(E \otimes F)$ have the properties in Proposition 6.3.22. Then $s((E \otimes F)^{\lambda+\mu}) = (E \otimes F)^{\tilde{\lambda}+\tilde{\mu}}$ and $s(C) = C$. Hence

$$(E \otimes F)^{\tilde{\lambda}+\tilde{\mu}} \subseteq \tilde{C} \cap s(C) = \tilde{C} \cap C,$$

so $\tilde{C} = C$. Thus, to two classes of finite dimensional simple $\mathfrak{g}$-modules we can associate canonically a third; in other words, we have defined on the set $\mathscr{S}_\mathfrak{g}$ of classes of finite dimensional simple $\mathfrak{g}$-modules a composition law. Fith this structure, $\mathscr{S}_\mathfrak{g}$ is canonically isomorphic to the additive monoid Λ^{++}.

Corollary 6.3.42. *Let $(\varpi_\alpha)_{\alpha \in \Delta}$ be the fundamental weights relative to Δ. Let $\lambda = \sum_{\alpha \in \Delta} m_\alpha \varpi_\alpha$ be in Λ^{++}. For all $\alpha \in \Delta$, let E_α be a simple gmodule of highest weight ϖ_α. Then in the $\mathfrak{g}$-module $\bigotimes_{\alpha \in \Delta} E_\alpha^{\otimes m_\alpha}$, the isotypical component of highest weight λ is of length* 1.

Proof. This follows from Proposition 6.3.41 by induction on $\sum_{\alpha \in \Delta} m_\alpha$. □

Proposition 6.3.43. *Let $\lambda, \mu \in \Lambda^{++}$. Let E, F, M be simple $\mathfrak{g}$-modules with highest weights λ, μ, $\lambda + \mu$. Let $\mathscr{X}$ (resp. $\mathscr{Y}$, $\mathscr{Z}$) be the set of weights of E (resp. F, M). Then $\mathscr{Z} = \mathscr{X} + \mathscr{Y}$.*

Proof. Fe have $E = \bigoplus_{\nu \in \Lambda} E^\nu$ and $F = \bigoplus_{\sigma \in \Lambda} F^\sigma$, so $E \otimes F$ is the direct sum of the

$$(E \otimes F)^\tau = \sum_{\nu+\sigma=\tau} E^\nu \otimes F^\sigma.$$

By Proposition 6.3.41, M can be identified with a $\mathfrak{g}$-submodule of $E \otimes F$, so $\mathscr{Z} \subseteq \mathscr{X} + \mathscr{Y}$. Fe have $M^\tau = M \cap (E \otimes F)^\tau$, and it is enough to show that, for $\nu\mathscr{X}$ and $\sigma \in \mathscr{Y}$, we have $M \cap (E \otimes F)^{\nu+\sigma} \neq 0$.

Let $(e_1, \dots, e_n)$ (resp. $(f_1, \dots, f_m)$) be a basis of E (resp. F) consisting of elements each of which belong to some E^ν (resp. F^σ), and such that $e_1 \in E^\lambda$ (resp. $f_1 \in F^\mu$). The $e_i \otimes f_j$ form a basis of $V \otimes W$. Suppose that the result to be proved is false. Then there exists a pair (i, j) such that the coordinate of index (i, j) of every element M is zero. Let $U(\mathfrak{g})$ be the enveloping algebra of $\mathfrak{g}$, $U(\mathfrak{g})^*$ the dual of $U(\mathfrak{g})$, γ the coproduct of $U(\mathfrak{g})$. For all $u \in U(\mathfrak{g})$, let $x_i(u)$ (resp. $y_j(u)$) be the coordinate of $u(e_1)$ (resp. $u(f_1)$) of index i (resp. j); let $z_{ij}(u)$ be the coordinate of index (i, j) of $u(e_1 \otimes f_1)$. Then x_i, y_j, z_{ij} are elements of $U(\mathfrak{g})^*$. Now e_1 generates the $\mathfrak{g}$-module E, so $x_i \neq 0$, and similarly $y_j \neq 0$. By the definition of the $\mathfrak{g}$-module $E \otimes F$, if $\gamma(u) = \sum u_s \otimes v_s$, we have

$$z_{ij}(u) = \sum_s x_i(u_s) y_j(v_s) = \langle \gamma(u), x_i \otimes y_j \rangle.$$

In other words, z_{ij} is the product of x_i and y_j in the algebra $U(\mathfrak{g})^*$. But this algebra is an integral domain (**??**), so $z_{ij} \neq 0$. Since $u(e_1 \otimes f_1) \in M$ for all $u \in U$, this is a contradiction. □

We now turn to dual of $\mathfrak{g}$-modules. Let E, F be $\mathfrak{g}$-modules. Recall that $\mathrm{Hom}_{\mathbb{K}}(E, F)$ has a canonical $\mathfrak{g}$-module structure, defined by

$$(x \cdot f)(e) = x \cdot f(e) - f(x \cdot e).$$

Let f be an element of weight λ in $\mathrm{Hom}_{\mathbb{K}}(E, F)$. If $\mu \in \mathfrak{h}^*$, then $f(E^\mu) \subseteq F^{\lambda+\mu}$ (Proposition 6.1.2(b)). Thus, if E and F are finite dimensional, the elements of weight λ in $\mathrm{Hom}_{\mathbb{K}}(E, F)$ are the graded homomorphisms of degree λ.

Proposition 6.3.44. *Let E be a finite dimensional $\mathfrak{g}$-module, and consider the $\mathfrak{g}$-module $E^* = \mathrm{Hom}_{\mathbb{K}}(E, \mathbb{K})$.*

(a) An element $\lambda \in \Lambda$ is a weight of E^* if and only if $-\lambda$ is a weight of E, and the multiplicity of λ in E^* is equal to that of $-\lambda$ in E.

(b) If E is simple and has highest weight ω, E^* is simple and has highest weight $-w_0(\omega)$.

Proof. Consider $\mathbb{K}$ as a trivial $\mathfrak{g}$-module whose elements are of weight 0. By what was said above, the elements of E^* of weight λ are the homomorphisms from E to $\mathbb{K}$ which vanish on E^μ if $\mu \neq -\lambda$. This proves (a). If E is simple, E^* is simple, and the last assertion follows from Remark 6.3.37. □

Remark 6.3.45. Let E, E^* be as in Proposition 6.3.44, and $\sigma \in \mathrm{Aut}(\mathfrak{g}, \mathfrak{h})$ be such that $\varepsilon(\sigma) = -w_0$ in the notations of Proposition 6.2.73. Let ρ, ρ^* be the representations of $\mathfrak{g}$ associated to E, E^*. Then $\rho \circ \sigma$ is a simple representation of $\mathfrak{g}$ with highest weight $-w_0(\omega)$, so $\rho \circ \sigma$ is equivalent to ρ^*.

In particular, if $w_0 = -1$, then for any finite dimensional $\mathfrak{g}$-module E, E is isomorphic to E^*. Recall that, if $\mathfrak{g}$ is simple, $w_0 = -1$ in the following cases: $\mathfrak{g}$ of type A_1, B_l ($l \geq 2$), C_l ($l \geq 2$), D_l (l even ≥ 4), E_7, E_8, F_4, G_2.

Lemma 6.3.46. *Let $h_0 = \sum_{\alpha \in \Phi^+} h_\alpha$. Then $h_0 = \sum_{\alpha \in \Delta} a_\alpha h_\alpha$, where a_α are integers ≥ 1. Let $(b_\alpha)_{\alpha \in \Delta}$, $(c_\alpha)_{\alpha \in \Delta}$ be families of scalars such that $a_\alpha = b_\alpha c_\alpha$ for all $\alpha \in \Delta$. Put $x = \sum_{\alpha \in \Delta} b_\alpha x_\alpha$ and $y = \sum_{\alpha \in \Delta} c_\alpha x_{-\alpha}$. Then there existence a homomorphism $\varphi : \mathfrak{sl}(2, \mathbb{K}) \to \mathfrak{g}$ which maps (e, h, f) to (x, h_0, y).*

Proof. The fact that the a_α are integers ≥ 1 follows from the fact that $(h_\alpha)_{\alpha \in \Delta}$ is a basis of the root system $\check{\Phi}$. We have $\alpha(h_0) = 2$ for all $\alpha \in \Delta$ (Corollary 5.1.69), so

$$[h_0, x] = \sum_{\alpha \in \Delta} b_\alpha \alpha(h_0) x_\alpha = 2x,$$
$$[h_0, y] = \sum_{\alpha \in \Delta} c_\alpha (-\alpha(h_0)) x_{-\alpha} = -2y.$$

On the other hand,

$$[x, y] = \sum_{\alpha, \beta \in \Delta} b_\alpha c_\beta [x_\alpha, x_{-\beta}] = \sum_{\alpha \in \Delta} b_\alpha c_\alpha h_\alpha = h_0.$$

Hence the existence of the homomorphism φ. □

Proposition 6.3.47. *Let E be a finite dimensional simple $\mathfrak{g}$-module, ω its highest weight, and $\mathscr{B}$ the vector space of $\mathfrak{g}$-invariant bilinear forms on E. Let $m = \sum_{\alpha \in \Phi^+} \omega(h_\alpha)$, so that $m/2$ is the sum of the coordinates of ω with respect to Δ (Corollary 5.1.69). Let w_0 be the element of W such that $w_0(\Delta) = -\Delta$.*

(a) If $w_0(\omega) \neq -\omega$, then $\mathscr{B} = 0$.

(b) Assume that $w_0(\omega) = -\omega$. Then $\mathscr{B}$ is of dimension 1, and every nonzero element of $\mathscr{B}$ is non-degenerate. If m is even (resp. odd), every element of Δ is symmetric (resp. alternating).

Proof. Let $\beta \in \mathscr{B}$. The map φ from E to E^* defined, for $x, y \in E$, by $f(x)(y) = \beta(x,y)$ is a homomorphism of $\mathfrak{g}$-modules. If $\beta \neq 0$, then $\beta \neq 0$, so φ is an isomorphism by Schur's lemma, and hence β is non-degenerate. Consequently, the $\mathfrak{g}$-module E is isomorphic to the $\mathfrak{g}$-module E^*, so that $w_0(\omega) = -\omega$. We have thus proved (a).

Assume from now on that $w_0(\omega) = -\omega$. Then E is isomorphic to E^*. The vector space $\mathscr{B}$ is isomorphic to $\mathrm{Hom}_{\mathfrak{g}}(E, E^*)$, and hence to $\mathrm{Hom}_{\mathfrak{g}}(E, E)$ which is of dimension 1 (Proposition 6.3.2). Hence $\dim(\mathscr{B}) = 1$. Every non-zero element β of Δ is non-degenerate by (a). Put $\beta_1(x,y) = \beta(y,x)$ for $x, y \in E$. By the preceding, there exists $\lambda \in \mathbb{K}$ such that $\beta_1(x,y) = \lambda\beta(x,y)$ for all $x, y \in E$. Then $\beta(y,x) = \lambda\lambda(x,y) = \lambda^2(y,x)$, so $\lambda^2 = 1$ and $\lambda = \pm 1$. Thus, the elements in $\mathscr{B}$ are either symmetric or alternating.

Now by Proposition 6.3.47, there exists a homomorphism φ from $\mathfrak{sl}(2,\mathbb{K})$ onto a subalgebra of $\mathfrak{g}$ that takes h to $\sum_{\alpha\in\Phi^+} h_\alpha$. Consider E as an $\mathfrak{sl}(2,\mathbb{K})$-module via this homomorphism. Then the elements of E^λ are of weight $\lambda(\sum_{\alpha\in\Phi^+} h_\alpha) =$. If $\lambda \in \Lambda$ is such that $E^\lambda \neq 0$ and $\lambda \neq \pm\omega$, then $-\omega \preceq \lambda \preceq \omega$ (recall that we have assumed that $w_0(\omega) = -\omega$, so $-\omega$ is the lowest weight of E), so

$$-m = -\omega(\sum_{\alpha\in\Phi^+} h_\alpha) < \lambda(\sum_{\alpha\in\Phi^+} h_\alpha) < \omega(\sum_{\alpha\in\Phi^+} h_\alpha) = m.$$

Let S be the isotypical component of type $V(m)$ of the $\mathfrak{sl}(2,\mathbb{K})$-module E. By the preceding, S is of length 1 and contains E^ω, $E^{-\omega}$. Since E^ω and $E^{-\omega}$ are not orthogonal with respect to β, the restriction of β to S is non-zero. By Remark 6.2.11, m is even or odd according as this restriction is symmetric or alternating. This completes the proof. □

In view of Proposition 6.3.47, a finite dimensional irreducible representation ρ of $\mathfrak{g}$ is said to be **orthogonal** (resp. **symplectic**) if there exists on E a non-degenerate symmetric (resp. alternating) bilinear form invariant under ρ.

6.3.5 Representation ring and characters

Let $\mathfrak{g}$ be a finite dimensional Lie algebra. Let $\mathscr{F}_{\mathfrak{g}}$ (resp. $\mathscr{S}_{\mathfrak{g}}$) be the set of classes of finite dimensional (resp. finite dimensional simple) $\mathfrak{g}$-modules. Let $\mathscr{R}(\mathfrak{g})$ be the free abelian group $\mathbb{Z}^{(\mathscr{S}(\mathfrak{g}))}$. For any finite dimensional simple $\mathfrak{g}$-module E, denote its class by $[E]$. Let F be a finite dimensional $\mathfrak{g}$-module; let $(F_n, F_{n-1}, \dots, F_0)$ be a Jordan-Hölder series for F; the element $\sum_{i=1}^n [F_i/F_{i-1}]$ of $\mathscr{R}(\mathfrak{g})$ depends only on F and not on the choice of Jordan-Hölder series; we denote it by $[F]$. If

$$0 \longrightarrow F_1 \longrightarrow F_2 \longrightarrow F_3 \longrightarrow 0$$

is an exact sequence of finite dimensional $\mathfrak{g}$-modules, then $[F_2] = [F_1] + [F_3]$.

Let F be a finite dimensional semi-simple $\mathfrak{g}$-module; for all $E \in \mathscr{S}_{\mathfrak{g}}$, let n_E be the length of the isotypical component of F of type E; then $[F] = \sum_{E\in\mathscr{S}_{\mathfrak{g}}} n_E[E]$. If F_1, F_2 are finite dimensional semi-simple a-modules, and if $[F_1] = [F_2]$, then F_1 and F_2 are isomorphic.

Proposition 6.3.48. *Let G be an abelian group written additively, and $\Phi : \mathscr{F}_{\mathfrak{g}} \to G$ a map; by abuse of notation, we denote by $\varphi(F)$ the image under φ of the class of any finite dimensional $\mathfrak{g}$-module F. Assume that, for any exact sequence*

$$0 \longrightarrow F_1 \longrightarrow F_2 \longrightarrow F_3 \longrightarrow 0$$

of finite dimensional a-modules, we have $\varphi(F_2) = \varphi(F_1) + \varphi(F_3)$. Then, there exists a unique homomorphism $\theta : \mathscr{R}(\mathfrak{g}) \to G$ such that $\theta([F]) = \varphi(F)$ for every finite dimensional $\mathfrak{g}$-module F.

Proof. There exists a unique homomorphism θ from $\mathscr{R}(\mathfrak{g})$ to G such that $\theta([E]) = \varphi(E)$ for every finite dimensional simple $\mathfrak{g}$-module E. Let F be a finite dimensional $\mathfrak{g}$-module, and $(F_n, F_{n-1}, \dots, F_0)$ a Jordan-Hölder series of F; if $n > 0$, we have, by induction on n,

$$\theta([F]) = \sum_{i=1}^n \theta([F_i/F_{i-1}]) = \sum_{i=1}^n \varphi([F_i/F_{i-1}]) = \varphi(F).$$

If $n = 0$ then $[F] = 0$ so $\theta([F]) = 0$; on the other hand, by considering the exact sequence of zero modules we see that $\varphi(0) = 0$. □

Example 6.3.49. Take $G = \mathbb{Z}$ and $\varphi(F) = \dim(F)$. The corresponding homomorphism from $\mathscr{R}(\mathfrak{g})$ to $\mathbb{Z}$ is denoted by dim. Let $\mathbf{1}$ be the class of a trivial $\mathfrak{g}$-module of dimension 1, and let ψ be the homomorphism $n \mapsto n \cdot \mathbf{1}$ from $\mathbb{Z}$ to $\mathscr{R}(\mathfrak{g})$. It is immediate that $\dim \circ \psi = \mathrm{id}_{\mathbb{Z}}$, so that $\mathscr{R}(\mathfrak{g})$ is the direct sum of $\ker \dim$ and $\mathbb{Z}\mathbf{1}$.

Proposition 6.3.50. *There exists on the additive group $\mathscr{R}(\mathfrak{g})$ a unique multiplication distributive over addition such that $[E][F] = [E \otimes F]$ for all finite dimensional $\mathfrak{g}$-modules E, F. In this way $\mathscr{R}(\mathfrak{g})$ is given the structure of a commutative ring. The class of the trivial $\mathfrak{g}$-module of dimension 1 is the unit element of this ring.*

Proof. It is clear that there exists a commutative multiplication on $\mathscr{R}(\mathfrak{g}) = \mathbb{Z}^{(\mathscr{S}_{\mathfrak{g}})}$ that is distributive over addition and such that $[E][F] = [E \otimes F]$ for all $E, F \in \mathscr{S}_{\mathfrak{g}}$. Let E_1, E_2 be finite dimensional $\mathfrak{g}$-modules, l_1 and l_2 their lengths; we show that $[E_1][E_2] = [E_1 \otimes E_2]$ by induction on $l_1 + l_2$. This is clear if $l_1 + l_2 \leq 2$. On the other hand, let F_1 be a submodule of E_1 distinct from 0 and E_1. Then

$$[F_1][E_2] = [F_1 \otimes E_2], \quad [E_1/F_1][E_2] = [(E_1/F_1) \otimes E_2]$$

by the induction hypothesis. On the other hand, $(E_1 \otimes E_2)/(F_1 \otimes E_2)$ is isomorphic to $(E_1/F_1) \otimes E_2$. Hence

$$[E_1][E_2] = ([E_1/F_1] + [F_1])[E_2] = [(E_1/F_1) \otimes E_2] + [F_1 \otimes E_2] = [E_1 \otimes E_2]$$

which proves our assertion. It follows immediately that the multiplication defined above is associative, so $\mathscr{R}(\mathfrak{g})$ has the structure of a commutative ring. Finally, it is clear that the class of the trivial $\mathfrak{g}$-module of dimension 1 is the unit element of this ring. □

Proposition 6.3.51. *There exists a unique involutive automorphism $X \mapsto X^*$ of the ring $\mathscr{R}(\mathfrak{g})$ such that $[E]^* = [E^*]$ for every finite dimensional $\mathfrak{g}$-module E.*

Proof. The uniqueness is clear. By Proposition 6.3.48, there exists a homomorphism $X \mapsto X^*$ from the additive group $\mathscr{R}(\mathfrak{g})$ to itself such that $[E]^* = [E^*]$ for every finite dimensional $\mathfrak{g}$-module E. We have $(X^*)^* = X$, so this homomorphism is involutive. It is an automorphism of the ring $R(\mathfrak{g})$ since $(E \otimes F)^*$ is isomorphic to $E^* \otimes F^*$ for all finite dimensional $\mathfrak{g}$-modules E and F. □

Let $U(\mathfrak{g})$ be the enveloping algebra of $\mathfrak{g}$, $U(\mathfrak{g})^*$ the vector space dual of $U(\mathfrak{g})$. Recall that the coalgebra structure of $U(\mathfrak{g})$ defines on $U(\mathfrak{g})^*$ a commutative, associative algebra structure with unit element. For any finite dimensional $\mathfrak{g}$-module E, the map $u \mapsto \mathrm{tr}(u_E)$ from $U(\mathfrak{g})$ to $\mathbb{K}$ is an element τ_E of $U(\mathfrak{g})^*$. If

$$0 \longrightarrow E_1 \longrightarrow E_2 \longrightarrow E_3 \longrightarrow 0$$

is an exact sequence of finite dimensional $\mathfrak{g}$-modules, then $\tau_{E_2} = \tau_{E_1} + \tau_{E_3}$. Hence, by Proposition 6.3.48 there exists a unique homomorphism, which we denote by tr, from the additive group $\mathscr{R}(\mathfrak{g})$ to the group $U(\mathfrak{g})^*$ such that $\mathrm{tr}([E]) = \tau_E$ for every finite dimensional $\mathfrak{g}$-module E. If $\mathbb{K}$ denotes the trivial $\mathfrak{g}$-module of dimension 1, it is easy to check that $\mathrm{tr}([\mathbb{K}])$ is the unit element of $U(\mathfrak{g})^*$. Finally, let E and F be finite dimensional $\mathfrak{g}$-modules. Let $u \in U(\mathfrak{g})$ and let γ be the coproduct of $U(\mathfrak{g})$. By definition of the $U(\mathfrak{g})$-module $E \otimes F$, if $\gamma(u) = \sum_i u_i \otimes v_i$, then

$$u_{E \otimes F} = \sum_i (u_i)_E \otimes (v_i)_F.$$

Therefore

$$\tau_{E \otimes F}(u) = \sum_i \mathrm{tr}((u_i)_E)\mathrm{tr}((v_i)_F) = \sum_i \tau_E(u_i)\tau_F(v_i) = (\tau_E \otimes \tau_F)(\gamma(u)).$$

This means that $\tau_E \tau_F = \tau_{E \otimes F}$. Thus, $\mathrm{tr} : \mathscr{R}(\mathfrak{g}) \to U(\mathfrak{g})^*$ is a homomorphism of rings.

Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be Lie algebras, φ a homomorphism from $\mathfrak{g}_1$ to $\mathfrak{g}_2$. Every finite dimensional $\mathfrak{g}_2$-module E defines by means of φ an $\mathfrak{g}_1$-module, hence elements of $\mathscr{R}(\mathfrak{g}_2)$ and $\mathscr{R}(\mathfrak{g}_1)$ that we denote provisionally by $[E]_2$ and $[E]_1$. By Proposition 6.3.48, there exists a unique homomorphism, denoted by $\mathscr{R}(\varphi)$, from the group $\mathscr{R}(\mathfrak{g}_2)$ to the group $\mathscr{R}(\mathfrak{g}_1)$ such that $\mathscr{R}(\varphi)([E]_2) = [E]_1$ for every finite dimensional $\mathfrak{g}_2$-module E. Moreover, $\mathscr{R}(\varphi)$ is a homomorphism of rings. If $U(\varphi)$ is the homomorphism from $U(\mathfrak{g}_1)$ to $U(\mathfrak{g}_2)$ extending φ, the following diagram is commutative

$$\begin{tikzcd} \mathscr{R}(\mathfrak{g}_2) \arrow[r, "\mathscr{R}(\varphi)"] \arrow[d, "\mathrm{tr}"] & \mathscr{R}(\mathfrak{g}_1) \arrow[d, "\mathrm{tr}"] \\ U(\mathfrak{g}_2)^* \arrow[r, "U(\varphi)^t"] & U(\mathfrak{g}_1)^* \end{tikzcd}$$

In what follows we take for a the splittable semi-simple Lie algebra $\mathfrak{g}$. The ring $\mathscr{R}(\mathfrak{g})$ is called the **representation ring** of $\mathfrak{g}$. For all $\lambda \in \Lambda^{++}$, we denote by $[\lambda]$ the class of the simple $\mathfrak{g}$-module $V(\lambda)$ of highest weight λ.

Let Λ be a commutative monoid written additively, and $\mathbb{Z}[\Lambda] = \mathbb{Z}^{(\Lambda)}$ the algebra of the monoid Λ over $\mathbb{Z}$. Denote by $(e^{\lambda})_{\lambda\in\Lambda}$ the canonical basis of $\mathbb{Z}[\Lambda]$. For all $\lambda, \mu \in \Lambda$, we have $e^{\lambda+\mu} = e^{\lambda}e^{\mu}$. If 0 is the neutral element of Λ, then e^0 is the unit element of $\mathbb{Z}[\Lambda]$, which is denoted by 1. Let E be a Λ-graded vector space over a field $\mathbb{K}$, and let $(E^{\lambda})_{\lambda\in\Delta}$ be its graduation. If each E^{λ} is finite dimensional, the **character** of E, denoted by $\chi(E)$, is the element $(\dim(E^{\lambda}))_{\lambda\in\Lambda}$ of $\mathbb{Z}^{\Lambda}$. If E itself is finite dimensional, then this element has finite support, so (we regard $\mathbb{Z}[\Lambda]$ as a subset of $\mathbb{Z}^{\Lambda}$)

$$\chi(E) = \sum_{\lambda\in\Lambda} \dim(E^{\lambda}) \cdot e^{\lambda} \in \mathbb{Z}[\Lambda].$$

Let E_1, E_2, E_3 be Λ-graded vector spaces such that the $E_1^{\lambda}, E_2^{\lambda}, E_3^{\lambda}$ are finite dimensional over $\mathbb{K}$, and

$$0 \longrightarrow E_1 \longrightarrow E_2 \longrightarrow E_3 \longrightarrow 0$$

an exact sequence of graded homomorphisms of degree 0. It is immediate that

$$\chi(E_2) = \chi(E_1) + \chi(E_3). \tag{6.3.7}$$

In particular, if F_1, F_2 are Λ-graded vector spaces such that the F_1^{λ} and the F_2^{λ} are finite dimensional over $\mathbb{K}$, then

$$\chi(F_1 \oplus F_2) = \chi(F_1) + \chi(F_2). \tag{6.3.8}$$

If F_1 and F_2 are finite dimensional, we also have

$$\chi(F_1 \otimes F_2) = \chi(F_1)\chi(F_2). \tag{6.3.9}$$

Example 6.3.52. Assume that $\Lambda = \mathbb{N}$. Let T be an indeterminate. There exists a unique isomorphism from the algebra $\mathbb{Z}[\mathbb{N}]$ to the algebra $\mathbb{Z}[T]$ that takes e^n to T^n for all $n \in \mathbb{N}$. For any finite dimensional $\mathbb{N}$-graded vector space E, the image of $\chi(E)$ in $\mathbb{Z}[T]$ is the Poincaré polynomial of E.

Now let Λ be the weight lattice for $\mathfrak{g}$. Let E be a $\mathfrak{g}$-module such that $E = \sum_{\lambda\in\mathfrak{h}^*} E^{\lambda}$ and such that each E^{λ} is finite dimensional. We know that $(E^{\lambda})_{\lambda\in\mathfrak{h}^*}$ is a graduation of the vector space E. In what follows we shall reserve the notation $\chi(E)$ for the character of E considered as a $\mathfrak{h}^*$-graded vector space. Thus, the character $\chi(E)$ is an element of $\mathbb{Z}^{\mathfrak{h}^*}$. If E is finite dimensional, $\chi(E) \in \mathbb{Z}[\Lambda]$. By formula (6.3.7) and Proposition 6.3.48, there exists a unique homomorphism from the group $\mathscr{R}(\mathfrak{g})$ to $\mathbb{Z}[\Lambda]$ that takes E to $\chi(E)$, for any finite dimensional $\mathfrak{g}$-module E; this homomorphism will be denoted by χ. Relation (6.3.9) shows that χ is a homomorphism from the ring $\mathscr{R}(\mathfrak{g})$ to the ring $\mathbb{Z}[\Lambda]$.

Remark 6.3.53. Every element of Λ defines a simple $\mathfrak{h}$-module of dimension 1 (the module $L(\lambda)$, in the previous notation), hence a homomorphism from the group $Z[\Lambda]$ to the group $\mathscr{R}(\mathfrak{h})$, which is an injective homomorphism of rings. It is immediate that the composite

$$\mathscr{R}(\mathfrak{g}) \longrightarrow \mathbb{Z}[\Lambda] \longrightarrow \mathscr{R}(\mathfrak{h})$$

is the homomorphism defined by the canonical injection of $\mathfrak{h}$ into $\mathfrak{g}$.

The Weyl group W operates by automorphisms on the group Λ, and hence operates on $\mathbb{Z}^{\Lambda}$. For all $\lambda \in \Lambda$ and all $w \in W$, we have $we^{\lambda} = e^{w(\lambda)}$. Let $\mathbb{Z}[\Lambda]^W$ be the subring of $\mathbb{Z}[\Lambda]$ consisting of the elements invariant under W.

Lemma 6.3.54. *If $\lambda \in \Lambda^{++}$, then $\chi([\lambda]) \in \mathbb{Z}[\Lambda]^W$ and the unique maximal term of $\chi([\lambda])$ is e^{λ}.*

Proof. This follows from Corollary 6.3.24 and Proposition 6.3.2. □

Theorem 6.3.55. *Let $(\varpi_{\alpha})_{\alpha\in\Delta}$ be the family of fundamental weights relative to Δ. Let $(X_{\alpha})_{\alpha\in\Delta}$ be a family of indeterminates.*

(a) The following homomorphisms are isomorphisms of rings

$$\mathscr{R}(\mathfrak{g}) \xrightarrow{\chi} \mathbb{Z}[\Lambda]^W \xleftarrow{\phi} \mathbb{Z}[(X_\alpha)_{\alpha\in\Delta}]$$

where $\phi(X_\alpha) = \chi([\varpi_\alpha])$.

(b) Let E be a finite dimensional $\mathfrak{g}$-module. If $\chi(E) = \sum_\lambda m_\lambda \chi([\lambda])$, then the isotypical component of E of highest weight λ has length m_λ.

Proof. The family $([\lambda])_{\lambda\in\Lambda^{++}}$ is a basis of the $\mathbb{Z}$-module $\mathscr{R}(\mathfrak{g})$, and the family $(\chi[\lambda])_{\lambda\in\Lambda^{++}}$ is a basis of the $\mathbb{Z}$-module $\mathbb{Z}[\Lambda]^W$ (Lemma 6.3.54 and Proposition 5.1.87). This proves that χ is an isomorphism. The fact that φ is an isomorphism follows from Lemma 6.3.54 and Theorem 5.1.89. The claim in (b) follows from (a). □

Corollary 6.3.56. *Let E, F be finite dimensional $\mathfrak{g}$-modules. Then E is isomorphic to F if and only if $\chi(E) = \chi(F)$.*

Proof. This follows from Theorem 6.3.55 and the fact that E, F are semi-simple. □

6.3.6 Symmetric invariants

In this paragraph, we retain the notations in the previous part. We denote by $(\mathfrak{g},\mathfrak{h})$ a split semi-simple Lie algebra, by Φ its root system, by W its Weyl group, and by Λ its group of weights.

Let V be a finite dimensional vector space, $\boldsymbol{S}(V)$ its symmetric algebra. The coalgebra structure of $\boldsymbol{S}(V)$ defines on $\boldsymbol{S}(V)^*$ a commutative and associative algebra structure. The vector space $\boldsymbol{S}(V)^*$ can be identified canonically with $\prod_{n\geq 0} \boldsymbol{S}^n(V)^*$, and $\boldsymbol{S}^n(V)^*$ can be identified canonically with the space of symmetric n-linear forms on V. The canonical injection of $V^* = \boldsymbol{S}^1(V)^*$ into $\boldsymbol{S}(V)^*$ defines an injective homomorphism from the algebra $\boldsymbol{S}(V^*)$ to the algebra $\boldsymbol{S}(V)^*$, whose image is $\mathrm{gr}(\boldsymbol{S}(V)^*) = \sum_{n\geq 0} \boldsymbol{S}^n(V)^*$. We identify the algebras $\boldsymbol{S}(V^*)$ and $\mathrm{gr}(\boldsymbol{S}(V)^*)$ by means of this homomorphism; we also identify $\boldsymbol{S}(V^*)$ with the algebra of polynomial functions on V.

6.3.6.1 The exponential of a linear functional The elements $(u_n) \in \prod_{n\geq 0} \boldsymbol{S}^n(v)^*$ such that $u_0 = 0$ form an ideal $\mathfrak{m}$ of $\boldsymbol{S}(V)^*$; we give $\boldsymbol{S}(V)^*$ the $\mathfrak{m}$-adic topology, in which $\boldsymbol{S}(V)^*$ is complete and $\boldsymbol{S}(V^*)$ is dense in $\boldsymbol{S}(V)^*$. If $(e_i^*)_{1\leq i\leq l}$ is a basis of V^*, and if $T_1,\dots,T_l$ are indeterminates, the homomorphism from $\mathbb{K}[T_1,\dots,T_l]$ to $\boldsymbol{S}(V^*)$ that takes T_i to e_i^* is an isomorphism of algebras, and extends to a continuous isomorphism from the algebra $\mathbb{K}[\![T_1,\dots,T_l]\!]$ to the algebra $\boldsymbol{S}(V)^*$.

For all $\lambda \in V^*$, the family $\lambda^n/n!$ is summable in $\boldsymbol{S}(V)^*$; its sum is called the **exponential** of λ and is denoted by $\exp(\lambda)$. If $x_1,\dots,x_n \in V$, then we have

$$\langle \exp(\lambda), x_1\cdots x_n\rangle = \frac{1}{n!}\langle \lambda^n, x_1\cdots x_n\rangle = \frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n} \langle \lambda, x_{\sigma(1)}\rangle\cdots\langle\lambda, x_{\sigma(n)}\rangle = \langle\lambda, x_1\rangle\cdots\langle\lambda,x_n\rangle.$$

It follows immediately that $\exp(\lambda)$ is the unique homomorphism from the algebra $\boldsymbol{S}(V)$ to $\mathbb{K}$ that extends λ.

We have $\exp(\lambda+\mu) = \exp(\lambda)\exp(\mu)$ for all $\lambda,\mu\in V^*$. Thus, the map $\exp: V^* \to \boldsymbol{S}(V)^*$ is a homomorphism from the additive group V^* to the multiplicative group of invertible elements of $\boldsymbol{S}(V)^*$. The family $(\exp(\lambda))_{\lambda\in V^*}$ is a free family in the vector space $\boldsymbol{S}(V)^*$ (**??**), and in particular the map ψ is injective.

Proposition 6.3.57. *Let Π be a subgroup of V^* that generates the vector space V^*, and n an non-negative integer. Then $\pi_n(\exp(\Pi))$ generates the vector space $\boldsymbol{S}^n(V^*)$.*

Proof. By **??**, any product of p elements of V^* is a $\mathbb{K}$-linear combination of elements of the form x^p where $x\in\Pi$. But we have $x^n = n!\pi_n(\exp(x))$, whence the proposition. □

By transport of structure, every automorphism of V defines automorphisms of the algebras $\boldsymbol{S}(V)$ and $\boldsymbol{S}(V)^*$; this gives linear representations of $\mathrm{GL}(V)$ on $\boldsymbol{S}(V)$ and $\boldsymbol{S}(V)^*$.

Now consider the vector space $V = \mathfrak{h}$. The map $\lambda \mapsto \exp(\lambda)$ from Λ to $\boldsymbol{S}(\mathfrak{h})^*$ is a homomorphism from the additive group Λ to $\boldsymbol{S}(\mathfrak{h})^*$ equipped with its multiplicative structure. Consequently, there exists a unique homomorphism ψ from the algebra $\mathbb{K}[\Lambda]$ of the monoid Λ to the algebra $\boldsymbol{S}(\mathfrak{h})^*$ such that

$$\psi(e^\lambda) = \exp(\lambda) \quad \text{for} \quad \lambda \in \Lambda.$$

By transport of structure, $\psi(w(e^\lambda)) = w(\psi(e^\lambda))$ for all $\lambda \in \Lambda$ and all $w \in W$. Hence, if $\mathbb{K}[\Lambda]^W$ (resp. $\boldsymbol{S}(\mathfrak{h})^{*W}$) denotes the set of elements of $\mathbb{K}[\Lambda]$ (resp. $\boldsymbol{S}(\mathfrak{h})^*$) invariant under W, we have $\psi(\mathbb{K}[\Lambda]^W) \subseteq \boldsymbol{S}(\mathfrak{h})^{*W}$.

Proposition 6.3.58. *Let $\boldsymbol{S}^n(\mathfrak{h}^*)^W$ be the set of elements of $\boldsymbol{S}^n(\mathfrak{h}^*)$ invariant under W. Then*

$$\pi_n(\psi(\mathbb{K}[\Lambda]^W)) = \boldsymbol{S}^n(\mathfrak{h}^*)^W.$$

Proof. It is clear from the preceding that $\pi_n(\psi(\mathbb{K}[\Lambda]^W)) \subseteq \boldsymbol{S}^n(\mathfrak{h}^*)^W$. Every element of $\boldsymbol{S}^n(\mathfrak{h}^*)$ is a $\mathbb{K}$-linear combination of elements of the form

$$\pi_n(\exp(\lambda)) = (\pi_n \circ \psi)(e^\lambda)$$

where $\lambda \in \Lambda$ (Proposition 6.3.57). Hence every element of $\boldsymbol{S}^n(\mathfrak{h}^*)^W$ is a linear combination of elements of the form

$$\sum_{w\in W} w((\pi_n \circ \psi)(e^\lambda)) = (\pi_n \circ \psi)\Big(\sum_{w\in W} w(e^\lambda)\Big)$$

each of which belongs to $\pi_n(\psi(\mathbb{K}[\Lambda]^W))$. □

Proposition 6.3.59. *Let E be a finite dimensional $\mathfrak{g}$-module. Let $U(\mathfrak{h}) = \boldsymbol{S}(\mathfrak{h})$ be the enveloping algebra of $\mathfrak{h}$. If $u \in U(\mathfrak{h})$, then*

$$\mathrm{tr}(u_E) = \langle \psi(\chi(E)), u \rangle. \tag{6.3.10}$$

Proof. It suffices to treat the case in which $u = h_1 \cdots h_n$ with $h_1, \dots, h_n \in \mathfrak{h}$. For all $\lambda \in \Lambda$, let $d_\lambda = \dim(E^\lambda)$. Then $\chi(E) = \sum_\lambda d_\lambda e^\lambda$, so $\psi(\chi(E)) = \sum_\lambda d_\lambda \exp(\lambda)$, hence

$$\langle \psi(\chi(E)), u \rangle = \sum_\lambda d_\lambda \langle \exp(\lambda), h_1 \cdots h_n \rangle = \sum_\lambda d_\lambda \langle \lambda, h_1 \rangle \cdots \langle \lambda, h_n \rangle = \mathrm{tr}(u_E).$$

This proves the claim. □

Corollary 6.3.60. *Let $U(\mathfrak{g})$ be the enveloping algebra of $\mathfrak{g}$. Let $\zeta : U(\mathfrak{g})^* \to U(\mathfrak{h})^* = \boldsymbol{S}(\mathfrak{h})^*$ be the transpose of the canonical injection $U(\mathfrak{h}) \to U(\mathfrak{g})$. The following diagram commutes*

$$\begin{tikzcd} \mathscr{R}(\mathfrak{g}) \arrow[r, "\chi"] \arrow[d, "\mathrm{tr}"] & \mathbb{Z}[\Lambda] \arrow[d, "\psi"] \\ U(\mathfrak{g})^* \arrow[r, "\zeta"] & \boldsymbol{S}(\mathfrak{h})^* \end{tikzcd}$$

Proof. This is just a reformulation of Proposition 6.3.59. □

Corollary 6.3.61. *Let n be a non-negative integer. Then every element of $\boldsymbol{S}^n(\mathfrak{h}^*)^W$ is a linear combination of polynomial functions on $\mathfrak{h}$ of the form $x \mapsto \mathrm{tr}(\rho(x)^n)$, where ρ is a finite dimensional linear representation of $\mathfrak{g}$.*

Proof. By Proposition 6.3.58, $\boldsymbol{S}^n(\mathfrak{h}^*)^W = (\pi_n \circ \psi)(\mathbb{K}[\Lambda]^W)$. Now $\mathbb{Z}[\Lambda]^W = \chi(\mathscr{R}(\mathfrak{g}))$ (Theorem 6.3.55), so $\psi(\mathbb{K}[\Lambda]^W)$ is the $\mathbb{K}$-vector subspace of $\boldsymbol{S}(\mathfrak{h})^*$ generated by $\psi(\chi(\mathscr{R}(\mathfrak{g}))) = \zeta(\mathrm{tr}(\mathscr{R}(\mathfrak{g})))$. Consequently, $\boldsymbol{S}^n(\mathfrak{h}^*)^W$ is the vector subspace of $\boldsymbol{S}^n(\mathfrak{h}^*)$ generated by $(\pi_n \circ \zeta \circ \mathrm{tr})(\mathscr{R}(\mathfrak{g}))$. But, if ρ is a finite dimensional linear representation of $\mathfrak{g}$, for all $x \in \mathfrak{h}$ have

$$(\pi_n \circ \zeta \circ \mathrm{tr})(\rho)(x) = \langle (\zeta \circ \mathrm{tr})(\rho), \frac{x^n}{n!} \rangle = \frac{1}{n!}\mathrm{tr}(\rho(x)^n).$$ □

6.3.6.2 Invariant polynomial functions We now take a further step and consider invariant polynomial functions on $\mathfrak{g}$. Let $\mathfrak{g}$ be a finite dimensional Lie algebra. In accordance with the conventions before, we identify the algebra $\boldsymbol{S}(\mathfrak{g}^*)$, the algebra $\mathrm{gr}(\boldsymbol{S}(\mathfrak{g})^*)$, and the algebra of polynomial functions on $\mathfrak{g}$. For all $x \in \mathfrak{g}$, we have the adjoint representation $\mathrm{ad}(x)$ on $\mathfrak{g}$, which can be extended to a derivation on $\boldsymbol{S}(\mathfrak{g})$, still denoted by $\mathrm{ad}(x)$. Let $\mathrm{ad}^*(x)$ be the restriction of $-\mathrm{ad}(x)^t$ to $\mathfrak{g}^*$ (the coadjoint representation), and extend it to $\boldsymbol{S}(\mathfrak{g}^*)$ to obtain a derivation on $\boldsymbol{S}(\mathfrak{g}^*)$. Then ad and ad^* are representations of $\mathfrak{g}$. If $f \in \boldsymbol{S}^n(\mathfrak{g}^*)$, then $\mathrm{ad}^*(x)(f) \in \boldsymbol{S}^n(\mathfrak{g}^*)$ and, for $x_1, \dots, x_n \in \mathfrak{g}$,

$$\mathrm{ad}(x)(x_1 \cdots x_n) = \sum_{i=1}^{n} x_1 \cdots x_{i-1}[x, x_i]x_{i+1} \cdots x_n, \tag{6.3.11}$$

$$\mathrm{ad}^*(x)(f)(x_1, \dots, x_n) = -\sum_{i=1}^{n} f(x_1, \dots, x_{i-1}, [x, x_i], x_{i+1}, \dots, x_n). \tag{6.3.12}$$

We deduce easily from (6.3.11) and (6.3.12) that $\mathrm{ad}(x)$ and $\mathrm{ad}^*(x)$ are derivations of $\boldsymbol{S}(\mathfrak{g})$, $\boldsymbol{S}(\mathfrak{g}^*)$ respectively. An element of $\boldsymbol{S}(\mathfrak{g})$ (resp. $\boldsymbol{S}(\mathfrak{g}^*)$) that is invariant under the representation ad (resp. ad^*) of $\mathfrak{g}$ is called an **invariant element** of $\boldsymbol{S}(\mathfrak{g})$ (resp. $\boldsymbol{S}(\mathfrak{g}^*)$).

Remark 6.3.62. Let $U(\mathfrak{g})$ be the enveloping algebra of $\mathfrak{g}$ and recall that we have a canonical isomorphism $\eta : \boldsymbol{S}(\mathfrak{g}) \to U(\mathfrak{g})$ as $\mathfrak{g}$-modules (the comments after Proposition 1.2.21). In particular, the invariant elements in $\boldsymbol{S}(\mathfrak{g})$ corresponds to invariant elements of $U(\mathfrak{g})$ under $\mathfrak{g}$.

Lemma 6.3.63. *Let ρ be a finite dimensional linear representation of $\mathfrak{g}$, and n a non-negative integer. Then the function $x \mapsto \mathrm{tr}(\rho(x)^n)$ on $\mathfrak{g}$ is an invariant polynomial function.*

Proof. Put $g(x_1, \dots, x_n) = \mathrm{tr}(\rho(x_1) \cdots \rho(x_n))$ for $x_1, \dots, x_n \in \mathfrak{g}$. If $x \in \mathfrak{g}$, we have

$$\begin{aligned} -(\mathrm{ad}^*(x)g)(x_1, \dots, x_n) &= \sum_{i=1}^{n} \mathrm{tr}(\rho(x_1) \cdots \rho(x_{i-1})[\rho(x), \rho(x_i)]\rho(x_{i+1}) \cdots \rho(x_n)) \\ &= \mathrm{tr}(\rho(x)\rho(x_1) \cdots \rho(x_n)) - \mathrm{tr}(\rho(x_1) \cdots \rho(x_n)\rho(x)) = 0. \end{aligned}$$

so $\mathrm{ad}^*(x)g = 0$. Let h be the symmetric multilinear form defined by

$$h(x_1, \dots, x_n) = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} g(x_{\sigma(1)}, \dots, x_{\sigma(n)}).$$

For all $x \in \mathfrak{h}$, we have $\mathrm{ad}^*(x)h = 0$ and $\mathrm{tr}(\rho(x)^n) = h(x, \dots, x)$, hence the lemma. □

Lemma 6.3.64. *Let E be a finite dimensional $\mathfrak{g}$-module, and $e \in E$. Then x is an invariant element of the $\mathfrak{g}$-module E if and only if $\exp(x_E)e = e$ for every nilpotent element x of $\mathfrak{g}$.*

Proof. The condition is clearly necessary. Assume now that it is satisfied. Let e be a nilpotent element of $\mathfrak{g}$. There exists an integer N such that $(x_E)^N = 0$. For all $t \in \mathbb{K}$, we have

$$0 = \exp(tx_E)e - e = \sum_{n=0}^{N-1} \frac{(tx_E)^n e}{n!}$$

so $x \cdot e = 0$. But the Lie algebra $\mathfrak{g}$ is generated by its nilpotent elements (Proposition 6.2.53). Hence e is an invariant element of the $\mathfrak{g}$-module E. □

For any $\xi \in \mathrm{GL}(\mathfrak{g})$, let $\boldsymbol{S}(\xi)$ be the automorphism of $\boldsymbol{S}(\mathfrak{g})$ that extends ξ, and $\boldsymbol{S}^*(\xi)$ the restriction to $\boldsymbol{S}(\mathfrak{g}^*)$ of the contragredient automorphism of $\boldsymbol{S}(\xi)$. Then S and $\boldsymbol{S}^*$ are representations of $\mathrm{GL}(\mathfrak{g})$. If x is a nilpotent element of $\mathfrak{g}$, $\mathrm{ad}(x)$ is locally nilpotent on $\boldsymbol{S}(\mathfrak{g})$ and $\boldsymbol{S}(\exp(\mathrm{ad}(x))) = \exp(\mathrm{ad}(x))$, so

$$\boldsymbol{S}^*(\exp(\mathrm{ad}(x))) = \exp(\mathrm{ad}^*(x)). \tag{6.3.13}$$

Proposition 6.3.65. *Let f be a polynomial function on $\mathfrak{g}$. Then the following conditions are equivalent:*

(i) *$f \circ \varphi = f$ for all $\varphi \in \mathrm{Aut}_e(\mathfrak{g})$;*

(ii) *$f \circ \varphi = f$ for all $\varphi \in \mathrm{Aut}_0(\mathfrak{g})$;*

(iii) *f is invariant.*

Proof. The equivalence of (i) and (iii) follows from formula (6.3.13) and Lemma 6.3.64. It follows from this that (iii) implies (ii) by extension of the base field. The implication (ii)⇒(i) is clear. □

Theorem 6.3.66. *Let $I(\mathfrak{g}^*)$ be the algebra of invariant polynomial functions on $\mathfrak{g}$. Let $i : \boldsymbol{S}(\mathfrak{g}^*) \to \boldsymbol{S}(\mathfrak{h}^*)$ be the restriction homomorphism.*

(a) The map $i|_{I(\mathfrak{g}^*)}$ is an isomorphism from the algebra $I(\mathfrak{g}^*)$ to the algebra $\boldsymbol{S}(\mathfrak{h}^*)^W$.

(b) For any non-negative integer n, let $I^n(\mathfrak{g}^*)$ be the set of homogeneous elements of $I(\mathfrak{g}^*)$ of degree n. Then $I^n(\mathfrak{g}^*)$ is the set of linear combinations of functions on $\mathfrak{g}$ of the form $x \mapsto \operatorname{tr}(\rho(x)^n)$, where ρ is a finite dimensional linear representation of $\mathfrak{g}$.

(c) Let $l = \operatorname{rank}(\mathfrak{g})$. Then there exist l algebraically independent homogeneous elements of $I(\mathfrak{g}^*)$ that generate the algebra $I(\mathfrak{g}^*)$.

Proof. Let $f \in I(\mathfrak{g}^*)$ and $w \in W$. There exists $s \in \operatorname{Aut}_e(\mathfrak{g}, \mathfrak{h})$ such that $s|_{\mathfrak{h}} = w^*$. Since f is invariant under s (Proposition 6.3.65), $i(f)$ is invariant under w. Hence $i(I(\mathfrak{g}^*)) \subseteq \boldsymbol{S}(\mathfrak{h}^*)^W$. We prove that, if $f \in I(\mathfrak{g}^*)$ is such that $i(f) = 0$, then $f = 0$. Extending the base field if necessary, we can assume that $\mathbb{K}$ is algebraically closed. By Eq. (6.3.13), f vanishes on $\varphi(\mathfrak{h})$ for all $\varphi \in \operatorname{Aut}_e(\mathfrak{g})$. Hence f vanishes on every Cartan subalgebra of $\mathfrak{g}$, and in particular on the set of regular elements of $\mathfrak{g}$. But this set is dense in $\mathfrak{g}$ for the Zariski topology, so our claim follows.

Let n be a non-negative integer. Let I^n be the set of linear combinations of functions of the form $x \mapsto \operatorname{tr}(\rho(x)^n)$ on $\mathfrak{g}$, where ρ is a finite dimensional linear representation of $\mathfrak{g}$. By Lemma 6.3.63, $I^n \subseteq I^n(\mathfrak{g}^*)$. Thus

$$i(I^n) \subseteq i(I^n(\mathfrak{g}^*)) \subseteq \boldsymbol{S}^n(\mathfrak{h}^*)^W.$$

By Corollary 6.3.61, $\boldsymbol{S}^n(\mathfrak{h}^*)^W \subseteq i(I^n)$, hence $i(I^n(\mathfrak{g}^*)) = \boldsymbol{S}^n(\mathfrak{h}^*)^W$, which proves (a), and $i(I^n) = i(I^n(\mathfrak{g}^*))$, which shows (b). Assertion (c) follows from (a) and Theorem 4.5.9. □

Corollary 6.3.67. *Assume that $\mathfrak{g}$ is simple. Let $m_1, \dots, m_l$ be the exponents of the Weyl group of $\mathfrak{g}$. Then there exist elements $P_1, \dots, p_l$ of $I(\mathfrak{g}^*)$, homogeneous of degrees $m_1 + 1, \dots, m_l + 1$, which are algebraically independent and generate the algebra $I(\mathfrak{g}^*)$.*

Proof. This follows from Theorem 6.3.66(a) and Proposition 4.6.10. □

Corollary 6.3.68. *Let Δ be a basis of Φ, $\boldsymbol{S}(\mathfrak{h})$ the symmetric algebra of $\mathfrak{h}$, and J the ideal of $\boldsymbol{S}(\mathfrak{g})$ generated by $\mathfrak{n}^+ \cup \mathfrak{n}^-$.*

(a) $\boldsymbol{S}(\mathfrak{g}) = \boldsymbol{S}(\mathfrak{h}) \oplus J$.

(b) Let $j : \boldsymbol{S}(\mathfrak{g}) \to \boldsymbol{S}(\mathfrak{h})$ be the homomorphism defined by the preceding decomposition of $\boldsymbol{S}(\mathfrak{g})$. Let $I(\mathfrak{g})$ be the set of invariant elements of $\boldsymbol{S}(\mathfrak{g})$. Let $\boldsymbol{S}(\mathfrak{h})^W$ be the set of elements of $\boldsymbol{S}(\mathfrak{h})$ invariant under the operation of W. Then $j|_{I(\mathfrak{g})}$ is an isomorphism from $I(\mathfrak{g})$ to $\boldsymbol{S}(\mathfrak{h})^W$.

Proof. Assertion (a) is clear. The Killing form defines an isomorphism from the vector space $\mathfrak{g}^*$ to the vector space $\mathfrak{g}$, which extends to an isomorphism ξ from the $\mathfrak{g}$-module $\boldsymbol{S}(\mathfrak{g}^*)$ to the $\mathfrak{g}$-module $\boldsymbol{S}(\mathfrak{g})$. We have $\xi(I(\mathfrak{g}^*)) = I(\mathfrak{g})$. The orthogonal complement of $\mathfrak{h}$ with respect to the Killing form is $\mathfrak{n}^+ + \mathfrak{n}^-$. If we identify $\mathfrak{h}^*$ with the orthogonal complement of $\mathfrak{n}^+ + \mathfrak{n}^-$ in $\mathfrak{g}^*$, then $\xi(\mathfrak{h}^*) = \mathfrak{h}$, so $\xi(\boldsymbol{S}(\mathfrak{h}^*)) = \boldsymbol{S}(\mathfrak{h})$ and $\xi(\boldsymbol{S}(\mathfrak{h}^*)^W) = \boldsymbol{S}(\mathfrak{h})^W$. Finally, $\xi^{-1}(J)$ is the set of polynomial functions on $\mathfrak{g}$ that vanish on $\mathfrak{h}$. This proves that ξ transforms the homomorphism i of Theorem 6.3.66 into the homomorphism j. Thus assertion (b) follows from Theorem 6.3.66(i). □

Recall that W is a group generated by reflections, so by Theorem 4.5.9 the algebra $\boldsymbol{S}(\mathfrak{h})^W$ (resp. $\boldsymbol{S}(\mathfrak{h}^*)^W$) is a graded polynomial $\mathbb{K}$-algebra of transcendence degree $l = \operatorname{rank}(\mathfrak{g})$; therefore $I(\mathfrak{g})$ and $I(\mathfrak{g}^*)$ are both polynomial $\mathbb{K}$-algebras. On the other hand, recall that by the Poincaré-Birkhoff-Witt theorem, $\boldsymbol{S}(\mathfrak{g})$ is isomorphic to the graded algebra $\operatorname{gr}(U(\mathfrak{g}))$ associated with $U(\mathfrak{g})$, so $I(\mathfrak{g})$ can be identified with the graded algebra $\operatorname{gr}(Z(\mathfrak{g}))$ associated with $Z(\mathfrak{g})$, the centre of $U(\mathfrak{g})$. The preceding remarks show that $\operatorname{gr}(Z(\mathfrak{g}))$ is a graded polynomial algebra of degree l. We now show that $Z(\mathfrak{g})$ is a polynomial algebra and extend this to the case where $\mathfrak{g}$ is not splitting. The following lemma is the key for our arguments.

Lemma 6.3.69. *Let $\mathbb{K}$ be a field and $A = \bigoplus_{n\geq 0} A_n$ be a graded $\mathbb{K}$-algebra, $\mathbb{K}'$ an extension of $\mathbb{K}$, and $A' = A \otimes_{\mathbb{K}} \mathbb{K}'$. Assume that A' is a graded polynomial algebra over $\mathbb{K}'$. Then A is a graded polynomial algebra over $\mathbb{K}$.*

Proof. Since A' is a polynomial algebra, we have $A'_0 = \mathbb{K}'$, so $A_0 = \mathbb{K}$. Put $A_+ = \bigoplus_{n\geq 1} A_n$ and $V = A_+/A_+^2$. Then V is a graded vector space and there is a graded linear map $f : V \to A_+$ of degree zero such that the composite with the canonical projection $A_+ \to V$ is the identity on M. Give $\boldsymbol{S}(V)$ the graded structure induced by that of V. The homomorphism of $\mathbb{K}$-algebras $\tilde{f} : \boldsymbol{S}(V) \to A$ that extends f is a graded homomorphism of degree 0; an immediate induction on the degree shows that $\tilde{f}$ is surjective.

We claim that A is a graded polynomial algebra if and only if V is finite dimensional and $\tilde{f}$ is bijective. In fact, if V is finite dimensional, $\boldsymbol{S}(V)$ is clearly a graded polynomial algebra, and so is A if $\tilde{f}$ is bijective. Conversely, assume that A is generated by algebraically independent homogeneous elements $x_1, \dots, x_l$ of degrees $d_1, \dots, d_l$. Let $\bar{x}_i$ be the image of x_i in V. It is immediate that the $\bar{x}_i$ form a basis of V; since $\bar{x}_i$ is of degree d_i, it follows that $\boldsymbol{S}(V)$ and A are isomorphic; in particular, $\dim(\boldsymbol{S}^n(V)) = \dim(A_n)$ for all n. Since $\tilde{f}$ is surjective, it is necessarily bijective.

Finally, we show that $\tilde{f}$ is bijective in our case. In fact, the preceding result, appiled to the graded $\mathbb{K}'$-algebra A', shows that $\tilde{f} \otimes 1 : \boldsymbol{S}(V) \otimes \mathbb{K}' \to A \otimes \mathbb{K}'$ is bijective, and hence so is $\tilde{f}$. ☐

Proposition 6.3.70. *Let $\mathfrak{g}$ be a semi-simple Lie algebra, l its rank. Let $I(\mathfrak{g})$ (resp. $I(\mathfrak{g}^*)$) be the set of elements of $\boldsymbol{S}(\mathfrak{g})$ (resp. $\boldsymbol{S}(\mathfrak{g}^*)$) invariant under the representation induced by the adjoint representation of $\mathfrak{g}$. Let $Z(\mathfrak{g})$ be the centre of the enveloping algebra of $\mathfrak{g}$.*

(a) $I(\mathfrak{g})$ and $I(\mathfrak{g}^*)$ are graded polynomial algebras of transcendence degree l.

(b) $Z(\mathfrak{g})$ is isomorphic to the algebra of polynomials in l indeterminates over $\mathbb{K}$.

Proof. The canonical filtration of the enveloping algebra of $\mathfrak{g}$ induces a filtration of $Z(\mathfrak{g})$. so $\mathrm{gr}(Z(\mathfrak{g}))$ is isomorphic to $I(\mathfrak{g})$. In view of **??**, it follows that (i)⇒(ii). On the other hand, Theorem 6.3.66 and Corollary 6.3.68 show that (a) is true whenever $\mathfrak{g}$ is split. The general case reduces to that case in view of the previous lemma. ☐

Proposition 6.3.71. *We retain the notations of Proposition 6.3.70, and denote by $\mathfrak{p}$ the ideal of $\boldsymbol{S}(\mathfrak{g}^*)$ generated by the homogeneous elements of I of positive degree. Let $x \in \mathfrak{g}$. Then x is nilpotent if and only if $f(x) = 0$ for all $f \in \mathfrak{p}$.*

Proof. Extending the base field if necessary, we can assume that $\mathfrak{g}$ is splittable. Assume that x is nilpotent. For any finite dimensional linear representation ρ of $\mathfrak{g}$, and any positive integer n, we have $\mathrm{tr}(\rho(x)^n) = 0$, so $f(x) = 0$ for all homogeneous $f \in I(\mathfrak{g}^*)$ of positive degree, and hence $f(x) = 0$ for all $f \in \mathfrak{p}$. Conversely, if $f(x) = 0$ for all $f \in \mathfrak{p}$, then $\mathrm{tr}(\mathrm{ad}(x)^n) = 0$ for all $n \geq 1$, so $\mathrm{ad}(x)$ is nilpotent and thus x is nilpotent by Theorem 1.6.21. ☐

We have seen that a polynomial is invariant under $\mathrm{ad}(x)$ if and only if it is invariant under $\mathrm{Aut}_0(\mathfrak{g})$ (Proposition 6.3.65). Now we use the preceding results to prove a reverse of this. For this we need the following lemma.

Lemma 6.3.72. *Let V be a finite dimensional vector space, G a finite group of automorphisms of V, and v_1, v_2 elements of V such that $v_2 \notin G \cdot v_1$. Then there exists a G-invariant polynomial function f on V such that $f(v_1) \neq f(v_2)$.*

Proof. Indeed, for each $s \in G$ there exists a polynomial function f_s on V equal to 1 at v_1 and to 0 at sv_2. Then the function $f = 1 - \prod_{s\in G} f_s$ is equal to 0 at v_1 and to 1 on Gv_2. The polynomial function $g = \prod_{t\in G} t \cdot f$ is G-invariant, equal to 0 at v_1 and to 1 at v_2. ☐

Proposition 6.3.73. *Let $\mathfrak{g}$ be a semi-simple Lie algebra and $\varphi \in \mathrm{Aut}(\mathfrak{g})$. Then $\varphi \in \mathrm{Aut}_0(\mathfrak{g})$ if and only if for any invariant polynomial function f on $\mathfrak{g}$, we have $f \circ \varphi = f$.*

Proof. By extending scalars if necessary, we can assume that $\mathbb{K}$ is algebraically closed. One implication is proved in Proposition 6.3.65. We assume that $f \circ \varphi = f$ for all $f \in I(\mathfrak{g})$ and prove that $\varphi \in \mathrm{Aut}_0(\mathfrak{g})$. In view of Proposition 6.3.65 and Corollary 6.2.85, we can assume that $\varphi \in \mathrm{Aut}(\mathfrak{g}, \mathfrak{h})$ and that φ leaves stable a Weyl chamber C (i.e., $\varepsilon(\varphi) \in \mathrm{Aut}(\Phi, \Delta)$). Let $x \in C \cap \mathfrak{h}_{\mathbb{Q}}$. We have $\varphi(x) \in C$. If f is a W-invariant polynomial function on $\mathfrak{h}$, we have $f(x) = f(\varphi(x))$ (Theorem 6.3.66(a)). By Lemma 6.3.72, it follows that $\varphi(x) \in W \cdot x$. Since $\varphi(x) \in C$, we have $x = \varphi(x)$. Then $\varphi|_{\mathfrak{h}} = \mathrm{id}_{\mathfrak{h}}$ and therefore $\varphi \in \mathrm{Aut}_0(\mathfrak{g}, \mathfrak{h})$ (Corollary 6.2.82). ☐

Corollary 6.3.74. *The group* $\mathrm{Aut}_0(\mathfrak{g})$ *is open and closed in* $\mathrm{Aut}(\mathfrak{g})$ *in the Zariski topology.*

Proof. Proposition 6.3.73 shows that $\mathrm{Aut}_0(\mathfrak{g})$ is closed. Let $\overline{\mathbb{K}}$ be an algebraic closure of $\mathbb{K}$. The group $\mathrm{Aut}(\mathfrak{g}\otimes\overline{\mathbb{K}})/\mathrm{Aut}_0(\mathfrak{g}\otimes\overline{\mathbb{K}})$ is finite (Corollary 6.2.85); a fortiori, the group $\mathrm{Aut}(\mathfrak{g})/\mathrm{Aut}_0(\mathfrak{g})$ is finite. Since the cosets of $\mathrm{Aut}_0(\mathfrak{g})$ in $\mathrm{Aut}(\mathfrak{g})$ are closed, it follows that $\mathrm{Aut}_0(\mathfrak{g})$ is open in $\mathrm{Aut}(\mathfrak{g})$. □

6.3.6.3 Centre of the universal enveloping algebra To conclude this paragraph, we consider the centre $Z(\mathfrak{g})$ of the enveloping algebra $U(\mathfrak{g})$. We have seen that $Z(\mathfrak{g})$ and $\boldsymbol{S}(\mathfrak{h})^W$ are both isomorphic to a polynomial algebra over $\mathbb{K}$ of transcendence degree $l=\mathrm{rank}(\mathfrak{g})$, so they are isomorphic. In fact, with a bit efforts, we can construct a explicit isomorphism from $Z(\mathfrak{g})$ to $\boldsymbol{S}(\mathfrak{h})^W$, which is called the Harish-Chandra isomorphism. We also recall that we have a Harish-Chandra homomorphism, which is a homomorphism φ from the commutant $U(\mathfrak{g})^0$ of $U(\mathfrak{h})$ to $U(\mathfrak{h})=\boldsymbol{S}(\mathfrak{h})$.

Let $\delta=\frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha$ be the Weyl vector, and t_δ the automorphism of the algebra $\boldsymbol{S}(\mathfrak{h})$ that takes every $x\in\mathfrak{h}$ to $x-\delta(x)$, and hence the polynomial function f on $\mathfrak{h}^*$ to the function $\lambda\mapsto f(\lambda-\delta)$.

Theorem 6.3.75. *Let* $U(\mathfrak{g})$ *be the enveloping algebra of* $\mathfrak{g}$*,* $Z(\mathfrak{g})$ *its centre,* $U(\mathfrak{h})\subseteq U(\mathfrak{g})$ *the enveloping algebra of* $\mathfrak{h}$ *(identified with* $\boldsymbol{S}(\mathfrak{h})$*),* $U(\mathfrak{g})^0$ *the commutant of* $U(\mathfrak{h})$ *in* $U(\mathfrak{g})$*, and* φ *the Harish-Chandra homomorphism from* $U(\mathfrak{g})^0$ *to* $U(\mathfrak{h})$ *relative to* Δ*. Let* $\boldsymbol{S}(\mathfrak{h})^W$ *be the set of elements of* $\boldsymbol{S}(\mathfrak{h})$ *invariant under the action of* W*. Then* $(t_\delta\circ\varphi)|_{Z(\mathfrak{g})}$ *is an isomorphism from* $Z(\mathfrak{g})$ *to* $\boldsymbol{S}(\mathfrak{h})^W$*, independent of the choice of* Δ*.*

Proof. Let Λ^{++} be the set of dominant weights of Φ, $w\in W$, $\lambda\in\Lambda^{++}$, $\mu=w(\lambda)$. Then $Z(\mu-\delta)$ is isomorphic to a submodule of $Z(\lambda-\delta)$ by Corollary 6.3.13, and $\langle\varphi(u),\lambda-\delta\rangle=\langle\varphi(u),\mu-\delta\rangle$ for all $u\in Z(\mathfrak{g})$ (Proposition 6.3.16). Thus, the polynomial functions $(t_\delta\circ\varphi)(u)$ and $(\delta\circ\varphi)(u)\circ w$ on $\mathfrak{h}^*$ coincide on Λ^{++}. But Λ^{++} is dense in $\mathfrak{h}^*$ in the Zariski topology: this can be seen by identifying $\mathfrak{h}^*$ with $\mathbb{K}^\Delta$ by means of the basis consisting of the fundamental weights ϖ_α, and by applying **??**. Hence

$$(t_\delta\circ\varphi)(u)=(t_\delta\circ\varphi)(u)\circ w, \tag{6.3.14}$$

which shows that $(t_\delta\circ\varphi)(Z(\mathfrak{g}))\subseteq\boldsymbol{S}(\mathfrak{h})^W$.

Let $\zeta:I(\mathfrak{g})\to\boldsymbol{S}(\mathfrak{h})$ be the homomorphism defined in Corollary 6.3.68, which induces an isomorphism from $I(\mathfrak{g})$ to $\boldsymbol{S}(\mathfrak{h})^W$. Consider the canonical isomorphism from the $\mathfrak{g}$-module $U(\mathfrak{g})$ to the $\mathfrak{g}$-module $\boldsymbol{S}(\mathfrak{g})$, and let θ be its restriction to $Z(\mathfrak{g})$. Then $\theta(Z(\mathfrak{g}))=I(\mathfrak{g})$.

$$\begin{tikzcd}
Z(\mathfrak{g}) \arrow[r, "\theta"] \arrow[d, "\varphi"] & I(\mathfrak{g}) \arrow[d, "\zeta"] \\
\boldsymbol{S}(\mathfrak{h}) \arrow[r, "t_\delta"] & \boldsymbol{S}(\mathfrak{h})
\end{tikzcd}$$

Let z be an element of $Z(\mathfrak{g})$ with $\deg(z)\le d$ in $U(\mathfrak{g})$. Introduce the notations of (6.3.4), and put

$$z=\sum_{|\boldsymbol{q}|+|\boldsymbol{m}|+|\boldsymbol{p}|\le d}\lambda_{\boldsymbol{q},\boldsymbol{m},\boldsymbol{p}}u(\boldsymbol{q},\boldsymbol{m},\boldsymbol{p}).$$

Let $v(\boldsymbol{q},\boldsymbol{m},\boldsymbol{p})$ be the monomial

$$v(\boldsymbol{q},\boldsymbol{m},\boldsymbol{p})=x_{-\alpha_1}^{q_1}\cdots x_{-\alpha_n}^{q_n}h_1^{m_1}\cdots h_l^{m_l}x_{\alpha_1}^{m_1}\cdots x_{\alpha_n}^{p_n}$$

calculated in $\boldsymbol{S}(\mathfrak{g})$. Denoting by $\boldsymbol{S}_n(\mathfrak{g})$ the sum of the homogeneous components of $\boldsymbol{S}(\mathfrak{g})$ of degrees smaller than or equal to n, then by Corollary 1.2.16 we have

$$\theta(z)\equiv\sum_{|\boldsymbol{q}|+|\boldsymbol{m}|+|\boldsymbol{p}|=d}\lambda_{\boldsymbol{q},\boldsymbol{m},\boldsymbol{p}}v(\boldsymbol{q},\boldsymbol{m},\boldsymbol{p})\mod \boldsymbol{S}_{d-1}(\mathfrak{g})$$

so

$$(\zeta\circ\theta)(z)\equiv\sum_{|\boldsymbol{m}|=d}\lambda_{\boldsymbol{0},\boldsymbol{m},\boldsymbol{0}}v(\boldsymbol{0},\boldsymbol{m},\boldsymbol{0})\mod \boldsymbol{S}_{d-1}(\mathfrak{h})$$

and concequently

$$(\zeta\circ\theta)(z)\equiv\varphi(z)\mod \boldsymbol{S}_{d-1}(\mathfrak{h}). \tag{6.3.15}$$

The canonical filtrations on $U(\mathfrak{g})$ and $\boldsymbol{S}(\mathfrak{g})$ induce filtrations on $Z(\mathfrak{g})$, $I(\mathfrak{g})$ and $\boldsymbol{S}(\mathfrak{h})^W$, and θ, ζ are compatible with these filtrations, so that $\mathrm{gr}(\zeta \circ \theta)$ is an isomorphism from the vector space $\mathrm{gr}(Z(\mathfrak{g}))$ to the vector space $\mathrm{gr}(\boldsymbol{S}(\mathfrak{h})^W)$. By (6.3.15), $\mathrm{gr}(\varphi) = \mathrm{gr}(\zeta \circ \theta)$, and it is clear that $\mathrm{gr}(t_\delta)$ is an isomorphism. Hence $\mathrm{gr}(t_\delta \circ \varphi)$ is bijective, so $t_\delta \circ \varphi : Z(\mathfrak{g}) \to \boldsymbol{S}(\mathfrak{h})^W$ is bijective (since $Z(\mathfrak{g})$ and $\boldsymbol{S}(\mathfrak{h})^W$ are isomorphic to their associated graded algebras).

Now let E be a simple $\mathfrak{g}$-module of highest weight λ, and χ its central character. Let $w \in W$ and let $\tilde{\varphi}$ and $t_{\tilde{\delta}}$ be the homomorphisms analogous to φ and δ relative to the basis $w(\Delta)$. The highest weight of E relative to $w(\Delta)$ is $w(\lambda)$. By Proposition 6.3.16,

$$\langle \varphi(u), \lambda \rangle = \chi(u) = \langle \tilde{\varphi}(u), w(\lambda) \rangle$$

for $u \in Z(\mathfrak{g})$, and by (6.3.14)

$$\begin{aligned} \langle (t_\delta \circ \varphi)(u), w(\lambda + \delta) \rangle &= \langle (t_\delta \circ \varphi)(u), \lambda + \delta \rangle = \langle \varphi(u), \lambda \rangle = \langle \tilde{\varphi}(u), w(\lambda) \rangle \\ &= \langle (t_{\tilde{\delta}} \circ \tilde{\varphi})(u), w(\lambda + \delta) \rangle. \end{aligned}$$

Thus, the polynomial functions $(t_\delta \circ \varphi)(u)$ and $(t_{\tilde{\delta}} \circ \tilde{\varphi})(u)$ coincide on $w(\Lambda^{++}) + w(\delta)$, and hence are equal. □

Corollary 6.3.76. *For all $\lambda \in \mathfrak{h}^*$, let χ_λ be the homomorphism $z \mapsto \langle \varphi(z), \lambda \rangle$ from $Z(\mathfrak{g})$ to $\mathbb{K}$.*

(a) If $\mathbb{K}$ is algebraically closed, every homomorphism from $Z(\mathfrak{g})$ to $\mathbb{K}$ is of the form χ_λ for some $\lambda \in \mathfrak{h}^*$.

(b) Let $\lambda, \mu \in \mathfrak{h}^*$, then $\chi_\lambda = \chi_\mu$ if and only if $\mu + \delta \in W \cdot (\lambda + \delta)$.

Proof. If $\mathbb{K}$ is algebraically closed, every homomorphism from $\boldsymbol{S}(\mathfrak{h})^W$ to $\mathbb{K}$ extends to a homomorphism from $\boldsymbol{S}(\mathfrak{h})$ to $\mathbb{K}$ (**??** and Corollary **??**), and every homomorphism from $\boldsymbol{S}(\mathfrak{h})$ to $\mathbb{K}$ is of the form $f \mapsto f(\lambda)$ for some $\lambda \in \mathfrak{h}^*$. Hence, if χ is a homomorphism from $Z(\mathfrak{g})$ to $\mathbb{K}$, there exists a $\mu \in \mathfrak{h}^*$ such that, for all $z \in Z(\mathfrak{g})$,

$$\chi(z) = \langle (t_\delta \circ \varphi)(z), \lambda \rangle = \langle \varphi(z), \lambda - \delta \rangle$$

hence (a).

Let $\lambda, \mu \in \mathfrak{h}^*$ and assume that $\chi_\lambda = \chi_\mu$. Then, for all $z \in Z(\mathfrak{g})$,

$$\langle (t_\delta \circ \varphi)(z), \lambda + \delta \rangle = \chi_\lambda(z) = \chi_\mu(z) = \langle (t_\delta \circ \varphi)(z), \mu + \delta \rangle$$

in other words, the homomorphisms from $\boldsymbol{S}(\mathfrak{h})$ to $\mathbb{K}$ defined by $\lambda + \delta$ and $\mu + \delta$ coincide on $\boldsymbol{S}(\mathfrak{h})^W$; thus, assertion (b) follows from Corollary **??**. □

Corollary 6.3.77. *Let E, F be finite dimensional simple $\mathfrak{g}$-modules, and χ_E, χ_F their central characters. If $\chi_E = \chi_F$, then E and F are isomorphic.*

Proof. Let λ, μ be the highest weights of E, F, so that $\chi_E = \chi_\lambda$ and $\chi_F = \chi_\mu$. Then $\chi_\lambda = \chi_\mu$, so there exists $w \in W$ such that $\mu + \delta = w(\lambda + \delta)$. Since $\lambda + \delta$ and $\mu + \delta$ belong to the chamber defined by Δ, we have $w = 1$. Thus, $\lambda = \mu$, hence the corollary. □

Proposition 6.3.78. *For any class λ of finite dimensional simple $\mathfrak{g}$-modules, let $U(\mathfrak{g})_\lambda$ be the isotypical component of type λ of the $\mathfrak{g}$-module $U(\mathfrak{g})$ (for the adjoint representation of $\mathfrak{g}$ on $U(\mathfrak{g})$). Let λ_0 be the class of the trivial $\mathfrak{g}$-module of dimension 1. Let $[U(\mathfrak{g}), U(\mathfrak{g})]$ be the vector subspace of $U(\mathfrak{g})$ generated by the brackets of pairs of elements of $U(\mathfrak{g})$.*

(a) $U(\mathfrak{g})$ is the direct sum of the $U(\mathfrak{g})_\lambda$;

(b) $U(\mathfrak{g})_{\lambda_0} = Z(\mathfrak{g})$, and $\sum_{\lambda \neq \lambda_0} U(\mathfrak{g})_\lambda = [U(\mathfrak{g}), U(\mathfrak{g})]$.

(c) Let $u \mapsto u^\sharp$ be the projection of $U(\mathfrak{g})$ onto $Z(\mathfrak{g})$ defined by the decomposition $U(\mathfrak{g}) = Z(\mathfrak{g}) \oplus [U(\mathfrak{g}), U(\mathfrak{g})]$. Then for $u, v \in U(\mathfrak{g})$ and $z \in Z(\mathfrak{g})$, we have

$$(uv)^\sharp = (vu)^\sharp, \quad (uz)^\sharp = u^\sharp z.$$

(d) Let φ be the Harish-Chandra homomorphism. Let $\lambda \in \Lambda^{++}$, and let E be a finite dimensional simple $\mathfrak{g}$-module of highest weight λ. For all $u \in U(\mathfrak{g})$, we have

$$\mathrm{tr}(u_E) = \dim(E) \langle \varphi(u^\sharp), \lambda \rangle.$$

Proof. The $\mathfrak{g}$-module $U(\mathfrak{g})$ is a direct sum of finite dimensional submodules. This implies (a). It is clear that $U(\mathfrak{g})_{\lambda_0} = Z(\mathfrak{g})$. Let V be a vector subspace of $U(\mathfrak{g})$ defining a irreducible subrepresentation of class λ of the adjoint representation. Then either $[\mathfrak{g}, V] = V$ or $[\mathfrak{g}, V] = 0$. Thus, if $\lambda \neq \lambda_0$ then $[\mathfrak{g}, V] = V$, so $\sum_{\lambda\neq\lambda_0} U(\mathfrak{g})_\lambda \subseteq [U(\mathfrak{g}), U(\mathfrak{g})]$. On the other hand, if $u \in U(\mathfrak{g})$ and $x_1, \dots, x_n \in \mathfrak{g}$, then

$$[x_1 \cdots x_n, u] = \sum_{i=1}^{n} [x_i, x_{i+1} \cdots x_n u x_1 \cdots x_{i-1}] \in [\mathfrak{g}, U(\mathfrak{g})]$$

Hence $[U(\mathfrak{g}), U(\mathfrak{g})] \subseteq [\mathfrak{g}, \sum_\lambda U(\mathfrak{g})_\lambda] = [\mathfrak{g}, \sum_{\lambda\neq\lambda_0} U(\mathfrak{g})_\lambda] \subseteq \sum_{\lambda\neq\lambda_0} U(\mathfrak{g})_\lambda$. This proves (b). Under these conditions, (c) follows from Lemma 1.6.45.

Finally, let E, λ be as in (d). Then

$$\mathrm{tr}(u_E) = \mathrm{tr}((u^\sharp)_E) = \mathrm{tr}(\langle \varphi(u^\sharp), \lambda\rangle \cdot 1) = \dim(E) \angle \varphi(u^\sharp), \lambda\rangle.$$

This completes the proof. □

6.3.7 Weyl's character formula

Let $(e^\lambda)_{\lambda\in\mathfrak{h}^*}$ be the canonical basis of the ring $\mathbb{Z}[\mathfrak{h}^*]$. Give the space $\mathbb{Z}^{\mathfrak{h}^*}$ of all maps from $\mathfrak{h}^*$ to $\mathbb{Z}$ the product topology of the discrete topologies on the factors. If $f \in \mathbb{Z}^{\mathfrak{h}^*}$, the family $(f(\lambda)e^\lambda)_{\lambda\in\mathfrak{h}^*}$ is summable, and

$$f = \sum_{\lambda\in\mathfrak{h}^*} f(\lambda)e^\lambda.$$

Let $\mathbb{Z}\langle\Lambda\rangle$ be the set of $f \in \mathbb{Z}^{\mathfrak{h}^*}$ whose support is contained in a finite union of sets of the form $\nu - \Lambda^+$, where $\lambda \in \mathfrak{h}^*$. Then $\mathbb{Z}[\Lambda] \subseteq \mathbb{Z}\langle\Lambda\rangle \subseteq \mathbb{Z}^{\mathfrak{h}^*}$. Define on $\mathbb{Z}\langle\Lambda\rangle$ a ring structure extending that of $\mathbb{Z}[\Lambda]$ by putting, for $f, g \in \mathbb{Z}\langle\Lambda\rangle$ and $\lambda \in \mathfrak{h}^*$

$$(fg)(\lambda) = \sum_{\mu\in\mathfrak{h}^*} f(\mu)g(\lambda - \mu)$$

(the family $(f(\mu)g(\lambda-\mu))_{\mu\in\mathfrak{h}^*}$ has finite support, in view of the condition satisfied by the supports of f and g). If $f = \sum_\mu x_\mu e^\mu$ and $g = \sum_\nu y_\nu e^\nu$, then $fg = \sum_{\mu,\nu} x_\mu y_\nu e^{\mu+\nu}$.

Let $\lambda \in \mathfrak{h}^*$. We let $p(\lambda)$ denote the number of ways (possibly zero) that λ can be expressed as a non-negative integer combination of positive roots. The function p is known as the **Kostant partition function**. We have $p(\lambda) > 0$ if and only if $\lambda \in \Lambda_r^+$. Recall that the Weyl denominator (Proposition 5.1.86)

$$q = \prod_{\alpha\in\Phi^+} (e^{\alpha/2} - e^{-\alpha/2}) = e^\delta \prod_{\alpha\in\Phi^+} (1 - e^{-\alpha}) = e^{-\delta} \prod_{\alpha\in\Phi^+} (e^\alpha - 1) = \sum_{w\in W} e^{w(\delta)}$$

is an anti-invariant element of $\mathbb{Z}[\Lambda]$. We first deduce the inverse of q, which is an element of $Z\langle\Lambda\rangle$.

Lemma 6.3.79. *In the ring $Z\langle\Lambda\rangle$, we have*

$$q^{-1} = e^{-\delta} \sum_{\nu\in\Lambda^+} p(\nu)e^{-\nu} = e^{-\delta} \sum_{\nu\in\Lambda_r^+} p(\nu)e^{-\nu}.$$

Proof. We have $p(\nu) = 0$ if $\nu \notin \Lambda_r^+$, so the second equality holds. For the first, we note that

$$\sum_{\nu\in\Lambda_r^+} p(\nu)e^{-\nu} = \prod_{\alpha\in\Phi^+} (1 + e^{-\alpha} + e^{-2\alpha} + \cdots) = \prod_{\alpha\in\Phi^+} \frac{1}{1 - e^{-\alpha}}$$

whence

$$qe^{-\delta} \sum_{\nu\in\Lambda_r^+} p(\nu)e^{-\nu} = \prod_{\alpha\in\Phi^+} \frac{1}{1 - e^{-\alpha}} \prod_{\alpha\in\Phi^+} (1 - e^{-\alpha}) = 1$$

so the claim follows. □

Lemma 6.3.80. *Let $\lambda \in \mathfrak{h}^*$. Then the Verma module $Z(\lambda)$ admits a character that is an element of $Z\langle\Lambda\rangle$, and we have*

$$\chi(Z(\lambda)) = \frac{e^{\lambda+\delta}}{q}.$$

Proof. Let $\alpha_1, \ldots, \alpha_n$ be distinct elements of Φ^+. The elements $x_{-\alpha_1}^{p_1} \cdots x_{-\alpha_n}^{p_n} \otimes 1$ form a basis of $Z(\lambda)$ (Proposition 6.3.8). For $h \in \mathfrak{h}$, we have

$$\begin{aligned} h \cdot (x_{-\alpha_1}^{p_1} \cdots x_{-\alpha_n}^{p_n} \otimes 1) &= [h, x_{-\alpha_1}^{p_1} \cdots x_{-\alpha_n}^{p_n}] \otimes 1 + x_{-\alpha_1}^{p_1} \cdots x_{-\alpha_n}^{p_n} \otimes h \cdot 1 \\ &= (\lambda - \sum_{i=1}^{n} p_i \alpha_i)(h)(x_{-\alpha_1}^{p_1} \cdots x_{-\alpha_n}^{p_n} \otimes 1). \end{aligned}$$

Thus, the dimension of $Z(\lambda)^{\lambda-\mu}$ equals to $p(\mu)$. This proves that $\chi(Z(\lambda))$ is defined, is an element of $\mathbb{Z}\langle\Lambda\rangle$, and that

$$\chi(Z(\lambda)) = \sum_{\mu} p(\mu) e^{\lambda-\mu} = \frac{e^{\lambda+\delta}}{q}$$

in view of Lemma 6.3.79. □

Lemma 6.3.81. *Let M be a $\mathfrak{g}$-module which admits a character $\chi(M) \in \mathbb{Z}\langle\Lambda\rangle$. Let $U(\mathfrak{g})$ be the enveloping algebra of $\mathfrak{g}$, $Z(\mathfrak{g})$ the centre of $U(\mathfrak{g})$, $\lambda \in \mathfrak{h}^*$, and χ_λ the corresponding homomorphism from $Z(\mathfrak{g})$ to $\mathbb{K}$. Assume that, for all $z \in Z(\mathfrak{g})$, z_M is the homothety with ratio $\chi_\lambda(z)$. Let D_M be the set of $\mu \in W(\lambda+\delta) - \delta$ such that $\mu + \Lambda_r^+$ meets $\operatorname{supp}(\chi(M))$. Then $\chi(M)$ is a $\mathbb{Z}$-linear combination of the $\chi(Z(\mu))$ for $\mu \in D_M$.*

Proof. If $\operatorname{supp}(\chi(M))$ is empty, the lemma is clear. Assume that $\operatorname{supp}(\chi(M)) \neq \varnothing$. Let μ be a maximal element of this support, and put $\dim(M^\mu) = m$. There exists a $\mathfrak{g}$-homomorphism φ from $(Z(\mu))^m$ to M which maps $(Z(\mu)^\mu)^m$ bijectively onto M^μ (Proposition 6.3.10). Thus the central character of $Z(\mu)$ is χ_λ, so $\mu \in W \cdot (\lambda+\delta) - \delta$ by Corollary 6.3.76. This proves that $D_M \neq \varnothing$, and allows us to argue by induction on $|D_M|$. Let L and N be the kernel and cokernel of φ. Then we have an exact sequence of $\mathfrak{g}$-homomorphisms

$$0 \longrightarrow L \longrightarrow (Z(\mu))^m \longrightarrow M \longrightarrow N \longrightarrow 0$$

so

$$\chi(M) = -\chi(L) + m\chi(Z(\mu)) + \chi(N)$$

Note that the sets $\operatorname{supp}(\chi(L))$ and $\operatorname{supp}(\chi(N))$ are contained in a finite union of sets $\nu - \Lambda^+$. For $z \in Z$, z_L and z_N are homotheties with ratio $\chi_\lambda(z)$. Clearly, $D_N \subseteq D_M$. On the other hand, since μ is the highest weight of M, we have $(\mu + \Lambda_r^+) \cap \operatorname{supp}(\chi(M)) = \{\mu\}$, and $\mu \notin \operatorname{supp}(\chi(N))$, so $\mu \notin D_N$ and

$$|D_N| < |D_M|.$$

Now L is a submodule of $(Z(\mu))^m$; if $\nu \in D_L$, then $\mu - \nu \in \Lambda_r^+$ by Proposition 6.3.2, so $\nu \in D_M$ and $D_L \subseteq D_M$. Since $L \cap (Z(\mu)^\mu)^m = 0$, we have $\mu \notin D_L$, so

$$|D_L| < |D_M|.$$

It now suffices to apply the induction hypothesis. □

Theorem 6.3.82 (Weyl's Character Formula). *Let M be a finite dimensional simple $\mathfrak{g}$-module, and λ its highest weight. Then*

$$\chi(M) = \frac{\sum_{w\in W} \varepsilon(w) e^{w(\lambda+\delta)}}{\sum_{w\in W} \varepsilon(w) e^{w(\delta)}}. \tag{6.3.16}$$

Proof. With the notations of Lemma 6.3.81, the central character of M is χ_λ by Proposition 6.3.16. Hence, by Lemmas 6.3.80 and 6.3.81, $q\chi(M)$ is a $\mathbb{Z}$-linear combination of the $e^{\mu+\delta}$ such that

$$\mu + \delta \in W \cdot (\lambda+\delta).$$

On the other hand, by Lemma 6.3.54, $q\chi(M)$ is anti-invariant, and its unique maximal term is $e^{\lambda+\delta}$, whence the theorem. □

Example 6.3.83. Take $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{K})$, $\mathfrak{h} = \mathbb{K}h$. Let α be the root of $(\mathfrak{g},\mathfrak{h})$ such that $\alpha(h) = 2$. The $\mathfrak{g}$-module $V(\lambda)$ has highest weight $(\lambda/2)\alpha$. Hence

$$\chi(V(\lambda)) = \frac{e^{\frac{\lambda}{2}\alpha+\frac{1}{2}\alpha} - e^{-\frac{\lambda}{2}\alpha-\frac{1}{2}\alpha}}{e^{\frac{1}{2}\alpha} - e^{-\frac{1}{2}\alpha}} = e^{-\frac{\lambda}{2}\alpha}\frac{e^{(\lambda+1)\alpha} - 1}{e^{\alpha} - 1}$$

$$
\begin{aligned}
&= e^{-\frac{\lambda}{2}\alpha}(e^{\lambda\alpha} + e^{(\lambda-1)\alpha} + \cdots + 1) \\
&= e^{\frac{\lambda}{2}\alpha} + e^{-\frac{\lambda-2}{2}\alpha} + \cdots + e^{-\frac{\lambda}{2}\alpha}
\end{aligned}
$$

which can be easily derived from Proposition 6.2.16.

Example 6.3.84. Now consider the case $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{K})$. We consider the space

$$V = \mathbb{Z}^n / \langle \varepsilon_1 + \cdots + \varepsilon_n \rangle$$

and inner product on V is defined to be

$$\Big(\sum_{i=1}^n a_i \varepsilon_i, \sum_{i=1}^n b_i \varepsilon_i \Big) = \sum_{i=1}^n a_i b_i.$$

The root system of $\mathfrak{g}$ is given by

$$\Phi = \{\varepsilon_i - \varepsilon_j : 1 \le i, j \le n\}.$$

A basis for Φ can be chosen to be $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ where

$$\alpha_i = \varepsilon_i - \varepsilon_{i+1}$$

and the corresponding positive roots are given by $\Phi^+ = \{\varepsilon_i - \varepsilon_j : i < j\}$. The Weyl group is $W = \mathfrak{S}_n$, acting by $\pi(\sum_i a_i \varepsilon_i) = \sum_i a_i \varepsilon_{\sigma(i)}$. In particular, the fundamental weights $\{\varpi_1, \ldots, \varpi_n\}$ can be verified to be given by

$$\varpi_i = \varepsilon_1 + \cdots + \varepsilon_i.$$

Hence the Weyl vector δ can be then computed by

$$
\begin{aligned}
\delta &= \frac{1}{2} \sum_{i<j} (\varepsilon_i - \varepsilon_j) = \frac{1}{2} \sum_{i=1}^{n-1} \Big((n-i)\varepsilon_i - \sum_{j=i+1}^{n} \varepsilon_j \Big) = \frac{1}{2} \sum_{i=1}^{n} (n+1-2i)\varepsilon_i \\
&= \frac{1}{2} \Big((n-1) \sum_{i=1}^{n-1} \varepsilon_i + \sum_{i=1}^{n-1} (n+1-2i)\varepsilon_i \Big) = \sum_{i=1}^{n-1} (n-i)\varepsilon_i.
\end{aligned}
$$

Now consider a finite dimensional irreducible representation V of $\mathfrak{g}$ with highest weight λ. Since λ is dominant, we may write $\lambda = \sum_{i=1}^n \lambda_i \varepsilon_i$ with $\lambda_1 \ge \cdots \ge \lambda_n$ and $\lambda_n = 0$. Then the Weyl character formula takes the form

$$\chi(V) = \frac{\sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) e^{\sum_{i=1}^{n-1} (\lambda_i + n - i)\varepsilon_{\sigma(i)}}}{\sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) e^{\sum_{i=1}^{n-1} (n-i)\varepsilon_{\sigma(i)}}} = \frac{\sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^n z_{\sigma(i)}^{\lambda_i + n - i}}{\sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^n z_{\sigma(i)}^{n-i}} = \frac{\det(z_j^{\lambda_i + n - i})}{\det(z_j^{n-i})} \tag{6.3.17}$$

where we set $z_i := e^{\varepsilon_i}$. The right side of (6.3.17) is a famous formula for the Schur polynomial $s_\lambda(z_1, \ldots, z_n)$, which can be proved by Young tableaux.

We now establish some of the concequences of the Weyl character formula. First, we shall use it to compute the dimension of $V(\lambda)$, for a dominant weight λ.

Theorem 6.3.85 (Weyl Dimension Formula). *Let V be a finite dimensional simple $\mathfrak{g}$-module, λ its highest weight and $(\cdot, \cdot)$ a W-invariant non-degenerate positive symmetric bilinear form on $\mathfrak{h}^*$. Then the dimension of $V(\lambda)$ may be computed as*

$$\dim(V(\lambda)) = \frac{\prod_{\alpha \in \Phi^+} \langle \lambda + \delta, h_\alpha \rangle}{\prod_{\alpha \in \Phi^+} \langle \delta, h_\alpha \rangle} = \prod_{\alpha \in \Phi^+} \Big(1 + \frac{(\lambda, \alpha)}{(\delta, \alpha)} \Big). \tag{6.3.18}$$

Proof. Let T be an indeterminate. For all $\nu \in \Lambda$, we denote by f_ν the ring homomorphism from $\mathbb{Z}[\Lambda]$ to $\mathbb{R}[\![T]\!]$ that takes e^μ to $e^{(\nu,\mu)T}$ for all $\mu \in \Lambda$. Then by the definition of $\chi(V)$, $\dim(V)$ is the constant term of the series $f_\nu(\chi(V))$.

If $\mu \in \mathfrak{h}^*$, put $J(e^\mu) = \sum_{w \in W} \varepsilon(w) e^{w(\mu)}$. Then for all $\mu, \nu \in \Lambda$, we have

$$f_\nu(J(e^\nu)) = \sum_{w \in W} \varepsilon(w) e^{(\nu, w(\mu))T} = \sum_{w \in W} \varepsilon(w) e^{(w^{-1}(\nu), \mu)T} = f_\mu(J(e^\nu)).$$

In particular, in view of (5.1.18),

$$f_\delta(J(e^\mu)) = f_\mu(J(e^\delta)) = e^{(\mu,\delta)T} \prod_{\alpha\in\Phi^+} (1 - e^{-(\mu,\alpha)T}).$$

Setting $|\Phi^+| = N$, we conclude that

$$f_\delta(J(e^\mu)) \equiv T^N \prod_{\alpha\in\Phi} (\mu,\alpha) \mod T^{N+1}\mathbb{R}[\![T]\!].$$

The formula $J(e^{\lambda+\delta}) = \chi(V)J(e^\delta)$ thus implies

$$T^N \prod_{\alpha\in\Phi^+} (\lambda+\delta,\alpha) \equiv f_\delta(\chi(V))\cdot T^N \prod_{\alpha\in\Phi^+} (\delta,\alpha) \mod T^{N+1}\mathbb{R}[\![T]\!]$$

whence

$$\dim(V) = \frac{\prod_{\alpha\in\Phi^+}(\lambda+\delta,\alpha)}{\prod_{\alpha\in\Phi^+}(\delta,\alpha)} = \prod_{\alpha\in\Phi^+}\Big(1+\frac{(\lambda,\alpha)}{(\delta,\alpha)}\Big).$$

Now, if $\alpha\in\Phi^+$, α can be identified with an element of $\mathfrak{h}_\mathbb{R}$ proportional to h_α, so

$$\frac{(\lambda+\delta,\alpha)}{(\delta,\alpha)} = \frac{\langle\lambda+\delta,h_\alpha\rangle}{\langle\delta,h_\alpha\rangle}.$$

This completes the proof. □

Example 6.3.86. In the example of $\mathfrak{sl}(2,\mathbb{K})$, let λ be an integer, we find that

$$\dim(V(\lambda)) = \frac{(\frac{\lambda}{2}\alpha+\frac{1}{2}\alpha)(h_\alpha)}{(\frac{1}{2}\alpha)(h_\alpha)} = \lambda+1.$$

which is known in Theorem 6.2.10.

Example 6.3.87. Consider the application of Weyl dimension formula on $\mathfrak{sl}(n,\mathbb{K})$. For an irreducible representation $V(\lambda)$ of $\mathfrak{sl}(n,\mathbb{K})$ with highest weight $\lambda = \sum_{i=1}^n \lambda_i\varepsilon_i$, we have

$$\dim(V(\lambda)) = \frac{\prod_{i<j}[\lambda_i+(n-i)]-[\lambda_j+(n-j)]}{\prod_{i<j}[(n-i)-(n-j)]} = \frac{\prod_{i<j}\lambda_i-\lambda_j+(j-i)}{\prod_{i<j}(j-i)}. \tag{6.3.19}$$

Again, the right side of (6.3.19) is a famous formula for the dimension of an irreducible representation of $\mathfrak{sl}(n,\mathbb{K})$, in which case the representations are indexed by partitions of at most n parts.

Beside the dimension formula, we can also comupte the multiplicity of each weight μ in the module $V(\lambda)$. This is the following famous Kostant's Multiplicity Formula.

Theorem 6.3.88 (Kostant's Multiplicty Formula). *Let λ be dominant integral element and $V(\lambda)$ is the finite-dimensional irreducible representation with highest weight λ. Then if μ is a weight of $V(\lambda)$, the multiplicity of μ is given by*

$$m_\mu^\lambda = \sum_{w\in W}\varepsilon(w)p(w(\lambda+\delta)-(\mu+\delta)).$$

Proof. By the Weyl character formula and Lemma 6.3.79, we have

$$\chi(V(\lambda)) = \Big(\sum_{\eta\in\Lambda^+} p(\eta)e^{-(\eta+\delta)}\Big)\Big(\sum_{w\in W}\varepsilon(w)e^{w(\lambda+\delta)}\Big).$$

For a fixed weight μ, the coefficient of e^μ in the character $\chi(V(\lambda))$ is just the multiplicity of μ in $V(\lambda)$. This coefficient is the sum of the quantity $p(\eta)\varepsilon(w)$ over all pairs (η,w) for which

$$-\eta-\delta+w(\lambda+\delta) = \mu,$$

or equivalently

$$\eta = w(\lambda+\delta)-(\mu+\delta).$$

Substituting this into $p(\eta)$ and summing over W gives Kostant's formula. □

Corollary 6.3.89. *If μ is a weight of $V(\lambda)$ distinct from λ,*

$$m_\mu^\lambda = -\sum_{w\in W, w\neq 1} \varepsilon(w) m^\lambda_{\lambda+\delta-w(\delta)}.$$

Proof. Apply Kostant's Multiplicty Formula with $\lambda = 0$. If $\mu \in \Lambda \setminus \{0\}$, we find that

$$0 = m^\lambda_{-\mu} = \sum_{w\in W} \varepsilon(w) p(w(\delta) + \mu - \delta)$$

hence

$$p(\mu) = -\sum_{w\in W, w\neq 1} \varepsilon(w) p(w(\delta) + \mu - \delta).$$

Since $w(\mu+\delta) \neq \lambda+\delta$ for all $w \in W$, substitute this into Kostant's Multiplicty Formula gives

$$\begin{aligned} m_\mu^\lambda &= -\sum_{w_1\in W} \varepsilon(w_1) \sum_{w_2\in W, w_2\neq 1} \varepsilon(w_2) p(w_2(\delta) + w_1(\lambda+\delta) - (\mu+\delta) - \delta) \\ &= -\sum_{w_2\in W, w_2\neq 1} \varepsilon(w_2) \sum_{w_1\in W} \varepsilon(w_1) p(w_1(\lambda+\delta) - (\mu+\delta - w_2(\delta)+\delta)) \\ &= -\sum_{w_2\in W, w_2\neq 1} \varepsilon(w_2) m^\lambda_{w_2(\mu+\delta-w_2(\delta))}. \end{aligned}$$

This completes the proof. □

Finally, with the multiplicity formula, we consider the question of decomposing the tensor product $V(\lambda)\otimes V(\mu)$ into direct sum of simple $\mathfrak{g}$-modules. Recall that we have seen that the highest weight of $V(\lambda)\otimes V(\mu)$ is $\lambda+\mu$, with multiplicity 1.

Proposition 6.3.90. *Let $\lambda, \mu \in \Lambda^{++}$. Then in $\mathscr{R}(\mathfrak{g})$, the coefficient of $[\nu]$ in $[\lambda]\cdot[\mu]$ is given by*

$$m_\nu^{\lambda,\mu} = \sum_{w_1,w_2\in W} \varepsilon(w_1w_2) p(w_1(\lambda+\delta) + w_2(\mu+\delta) - (\nu+2\delta)). \tag{6.3.20}$$

Proof. Let E, F be finite dimensional simple $\mathfrak{g}$-modules of highest weights λ, μ. Let ℓ_ν be the length of the isotypical component of $E\otimes F$ of highest weight ν. It suffices to show that

$$\ell_\nu = \sum_{w_1,w_2\in W} \varepsilon(w_1w_2) p(w_1(\lambda+\delta) + w_2(\mu+\delta) - (\nu+2\delta)).$$

We have

$$\sum_{\nu\in\Lambda^{++}} \ell_\nu \chi([\nu]) = \chi(E\otimes F) = \chi_E\chi_F$$

so, after multiplying by q and using Weyl Character Formula,

$$\sum_{\nu\in\Lambda^{++}} \ell_\nu J(e^{\nu+\delta}) = \Big(\sum_{\sigma\in\Lambda} m_\sigma^\lambda e^\sigma\Big)\Big(\sum_{w\in W}\varepsilon(w)e^{w(\mu+\delta)}\Big) = \sum_{\tau\in\Lambda}\Big(\sum_{w\in W} m^\lambda_{\tau+\delta-w(\mu+\delta)}\Big)e^{\tau+\delta}. \tag{6.3.21}$$

Now, if $\nu \in \Lambda^{++}$, $\nu+\delta$ belongs to the chamber defined by Δ; thus, for all $w \in W$ distinct from 1, we have $w(\nu+\delta) \notin \Lambda^{++}$. Consequently, the coefficient of $e^{\nu+\delta}$ in $\sum_\nu \ell_\nu J(e^{\nu+\delta})$ is equal to ℓ_ν. In view of (6.3.21), we obtain

$$\ell_\nu = \sum_{w\in W} \varepsilon(w) m^\lambda_{\nu+\delta-w(\mu+\delta)}.$$

By Kostant's Multiplicity Formula, this shows

$$\ell_\nu = \sum_{w_1,w_2\in W} \varepsilon(w_1w_2) p(w_1(\lambda+\delta) + w_2(\mu+\delta) - (\nu+2\delta)).$$

This completes the proof. □

Example 6.3.91. We note that $w_1(\lambda+\delta) + w_2(\mu+\delta) \preceq \lambda+\mu+2\delta$ with the equality holds if and only if for $w_1 = w_2 = 1$. Hence by (6.3.21), the coefficient of $[\lambda+\mu]$ in $[\lambda]\cdot[\mu]$ is 1, and any weight of $[\lambda]\cdot[\mu]$ is lower than $\lambda+\mu$. This justifies Proposition 6.3.41

Example 6.3.92. We return to the case of $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{K})$. Let $\lambda = (p/2)\alpha$, $\mu = (q/2)\alpha$, and $\nu = (n/2)\alpha$ with $p \geq q$. We have $\delta = \alpha/2$, so

$$\begin{aligned} m_\nu^{\lambda+\mu} &= \sum_{w_1,w_2\in\mathfrak{S}_2} p\Big(w_1(\frac{p}{2}\alpha + \frac{1}{2}\alpha) + w_2(\frac{q}{2}\alpha + \frac{1}{2}\alpha) - (\frac{n}{2}\alpha + 2\frac{1}{2}\alpha)\Big) \\ &= p\Big(\frac{p+q-n}{2}\alpha\Big) - p\Big(\frac{p-q-n-2}{2}\alpha\Big) - p\Big(\frac{q-p-n-2}{2}\alpha\Big) + p\Big(\frac{-p-q-n-4}{2}\alpha\Big) \\ &= p\Big(\frac{p+q-n}{2}\alpha\Big) - p\Big(\frac{p-q-n-2}{2}\alpha\Big) \end{aligned}$$

This is zero if $p+q+n$ is not divisible by 2, or if $n \geq p+q$. If $p+q-n = 2r$ with $r \geq 0$ and integer, we have

$$m_\nu^{\lambda+\mu} = p(r\alpha) - p((r-q-1)\alpha)$$

hence $m_\nu^{\lambda+\mu} = 1$ if $r \leq q$ and $m_\nu^{\lambda+\mu} = 0$ if $r > q$. From this, we conclude that the $\mathfrak{g}$-module $V(p) \oplus V(q)$ can be decomposed as

$$V(p) \otimes V(q) = \bigoplus_{i=0}^{q} V(p+q-2i).$$

This result is usually refered as the **Clebsch-Gordan formula**.

6.4 Classical splittable simple Lie algebras

In this paragraph we describe explicitly, for each type of classical splittable simple Lie algebra: an algebra of this type, its dimension and its splitting Cartan subalgebras; its coroots; its Borel subalgebras and its parabolic subalgebras; its fundamental simple representations; those of its fundamental simple representations which are orthogonal or symplectic; the algebra of invariant polynomial functions.

6.4.1 Lie algebras of type A_l

6.4.2 Lie algebras of type B_l

Let V be a finite dimensional vector space, and β a non-degenerate symmetric bilinear form on V. The set of endomorphisms x of V such that

$$\beta(xv_1, v_2) + \beta(v_1, xv_2) = 0 \quad \text{for} \quad v_1, w_2 \in V$$

is a Lie subalgebra of $\mathfrak{sl}(V)$, and semisimple for $\dim(V) \neq 2$ (Proposition 1.6.36). We denote it by $\mathfrak{o}(\beta)$ and call it the **orthogonal Lie algebra** associated to ψ. Assume that V is of odd dimension $2l+1 \geq 3$ and that β is of maximum index l. Denote by Q the quadratic form such that β is associated to Q (that is, $Q(x) = \beta(x,x)$). Now V can be written as the direct sum of two maximal totally isotropic subspaces F^+ and F^- and the orthogonal complement of $F^+ + F^-$, which is non-isotropic and 1-dimensional. Up to multiplying β by a non-zero constant, we can assume that there exists $e_0 \in (F^+ + F^-)^\perp$ such that $\beta(e_0, e_0) = 1$. On the other hand, F^+ and F^- are in duality via β; let $(e_i)_{1\leq i\leq l}$ be a basis of F^+ and $(e_{-i})_{1\leq i\leq l}$ the dual basis of F^-. Then

$$(e_1, \ldots, e_l, e_0, e_{-l}, \ldots, e_{-1})$$

is a basis of V; we have

$$Q\Big(\sum x_i e_i\Big) = x_0^2 + 2\sum_{i=1}^{l} x_i x_{-i}.$$

and the matrix of β with respect to this basis is the square matrix of order $2l+1$

$$B = \begin{pmatrix} 0 & 0 & s \\ 0 & 1 & 0 \\ s & 0 & 0 \end{pmatrix}, \quad s = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

where s is the square matrix of order l all of whose entries are zero except those on the anti diagonal which are equal to 1. A basis of V with the preceding properties will be called a **Witt basis** of V. The algebra $\mathfrak{g} = \mathfrak{o}(\beta)$ can then be identified with the algebra $\mathfrak{o}_B(2l+1, \mathbb{K})$ of square matrices x of order $2l+1$ such that $Bx + xB = 0$.

Proposition 6.4.1. *The algebra $\mathfrak{o}_B(2l+1, \mathbb{K})$ is given by*

$$\mathfrak{o}_B(2l+1, \mathbb{K}) = \{x \in \mathfrak{gl}(2l+1, \mathbb{K}) : x + x' = 0\}$$

where x' is the transposition of x with respect to the anti-diagonal.

Proof. The algebra $\mathfrak{o}_B(2l+1, \mathbb{K})$ is the set $x \in \mathfrak{gl}(2l+1, \mathbb{K})$ such that $xB + Bx = 0$. Viewing B as a permutation matrix, then Bx permutes the rows by $i \mapsto n - i$, so $B^2 = I$ and

$$B(Bx)^T = x', \quad BBx = x.$$

Moreover, we observe that, since $B = B^T$,

$$x^T B = x^T B^T = (Bx)^T.$$

Hence we conclude that

$$x^T B + Bx = (Bx)^T + Bx = B(B(Bx)^T + BBx) = B(x' + x).$$

The matrix B is invertible, so claim follows. □

In particular, it follows from **????** that

$$\dim(\mathfrak{g}) = \frac{2l(2l+1)}{2} = l(2l+1).$$

Let $\mathfrak{h}$ be the set of diagonal elements of $\mathfrak{g}$. This is a commutative subalgebra of $\mathfrak{g}$, with basis the elements

$$h_i = e_{i,i} - e_{-i,-i}, \quad 1 \le i \le l.$$

Let (ε_i) be the basis of $\mathfrak{h}^*$ dual to (h_i). We note that, for an element $h \in \mathfrak{h}$ and $1 \le i < j \le l$, we have

$$[h, E_{ij}] = (h_i - h_j)E_{ij}, \quad [h, E_{i,-j}] = (h_i + h_j)E_{ij}, \quad [h, E_{-j,i}] = (-h_j - h_i)E_{-j,i}.$$

Thus, put

$$\begin{cases} x_{\varepsilon_i - \varepsilon_j} = E_{i,j} - E_{-j,-i} & (1 \le i < j \le l) \\ x_{\varepsilon_j - \varepsilon_i} = E_{j,i} - E_{-i,-j} & (1 \le i < j \le l) \\ x_{\varepsilon_i + \varepsilon_j} = E_{i,-j} - E_{j,-i} & (1 \le i < j \le l) \\ x_{-\varepsilon_i - \varepsilon_j} = E_{-j,i} - E_{-i,j} & (1 \le i < j \le l) \\ x_{\varepsilon_i} = E_{i,0} - E_{0,-i} & (1 \le i \le l) \\ x_{-\varepsilon_i} = E_{0,i} - E_{-i,0} & (1 \le i \le l) \end{cases}$$

It is easy to verify that these elements form a basis of a complement of $\mathfrak{h}$ in $\mathfrak{g}$ and that, for $h \in \mathfrak{h}$,

$$[h, x_\alpha] = \alpha(h)x_\alpha$$

for all $\alpha \in \Phi$, where Φ is the set

$$\Phi = \{\pm\varepsilon_i, \pm\varepsilon_i \pm \varepsilon_j : 1 \le i < j \le l\}.$$

It follows that $\mathfrak{h}$ is equal to its normalizer in $\mathfrak{g}$, and hence is a Cartan subalgebra of $\mathfrak{g}$, that $\mathfrak{h}$ is splitting, and that the roots of $(\mathfrak{g}, \mathfrak{h})$ are the elements of Φ. The root system Φ of $(\mathfrak{g}, \mathfrak{h})$ is of type B_l for $l \ge 2$, and of type A_1 (also said to be of type B_1) for $l = 1$. Consequently, $\mathfrak{g}$ is a splittable simple Lie algebra of type B_l.

Every splitting Cartan subalgebra of $\mathfrak{o}(\beta)$ is a transform of $\mathfrak{h}$ by an elementary automorphism of $\mathfrak{o}(\beta)$, and hence by an element of $\mathrm{O}(\beta)$, and consequently is the set $\mathfrak{h}_{\mathcal{B}}$ of elements of $\mathfrak{g}$ whose matrix with respect to a Witt basis $\mathcal{B}$ of V is diagonal. We verify immediately that the only vector subspaces invariant under $\mathfrak{h}_{\mathcal{B}}$ are those generated by a subset of $\mathcal{B}$. If $l = 1$, the algebras $\mathfrak{o}(\beta)$ and $\mathfrak{sl}(2, \mathbb{K})$ have the same root systems, and are thus isomorphic. From now on, we assume that $l \geq 2$.

The root system $\check{\Phi}$ is determined, and we find that

$$h_{\varepsilon_i} = 2h_i, \quad h_{\varepsilon_i - \varepsilon_j} = h_i - h_j, \quad h_{\varepsilon_i + \varepsilon_j} = h_i + h_j.$$

Put $\alpha_1 = \varepsilon_1 - \varepsilon_2, \ldots, \alpha_{l-1} = \varepsilon_{l-1} - \varepsilon_l$, $\alpha_l = \varepsilon_l$. Then $(\alpha_1, \ldots, \alpha_l)$ is a basis Δ of Φ; the positive roots relative to Δ are the ε_i and the $\varepsilon_i \pm \varepsilon_j$ $(i < j)$. The corresponding Borel subalgebra $\mathfrak{b}$ is the set of upper triangular matrices in $\mathfrak{g}$.

It is immediately verified that the only vector subspaces of V distinct from $\{0\}$ and V stable under $\mathfrak{b}$ are the elements of the maximal flag corresponding to the basis (e_i), that is, the totally isotropic subspaces $V_1, \ldots, V_l$, where V_i is generated by $e_1, \ldots, e_i$, together with their orthogonal complements $V_{-1}, \ldots, V_{-i}$: the orthogonal complement V_{-i} of V_i is generated by $e_1, \ldots, e_l, e_0, e_{-l}, \ldots, e_{-i-1}$ and is not totally isotropic. On the other hand, if an element of $\mathfrak{g}$ leaves stable a vector subspace, it leaves stable its orthogonal complement. Consequently, $\mathfrak{b}$ is the set of elements of $\mathfrak{g}$ leaving stable the elements of the flag $\{V_1, \ldots, V_l\}$.

A flag is said to be **isotropic** if each of its elements is totally isotropic. The flag $\{V_1, \ldots, V_l\}$ is a maximal isotropic flag. Since the group $\mathrm{O}(\beta)$ operates transitively both on the Borel subalgebras of $\mathfrak{g}$ and on the maximal isotropic flags, we see that, for any maximal isotropic flag $\mathcal{F}$ in V, the set $\mathfrak{b}_{\mathcal{F}}$ of elements of $\mathfrak{g}$ leaving stable the elements of $\mathcal{F}$ is a Borel subalgebra of g and that the map $\mathcal{F} \mapsto \mathfrak{b}_{\mathcal{F}}$ is a bijection from the set of maximal isotropic flags to the set of Borel subalgebras of $\mathfrak{g}$.

Let $\mathcal{F}$ be an isotropic flag and let $\mathfrak{p}_{\mathcal{F}}$ be the set of elements of $\mathfrak{g}$ leaving stable the elements of $\mathcal{F}$. If $\mathcal{F} \subseteq \{V_1, \ldots, V_l\}$, then $\mathfrak{p}_{\mathcal{F}}$ is a parabolic subalgebra of $\mathfrak{g}$ containing $\mathfrak{b}$, and it is easy to verify that the only nontrivial totally isotropic subspaces stable under $\mathfrak{p}_{\mathcal{F}}$ are the elements of $\mathcal{F}$. This gives 2^l parabolic subalgebras of $\mathfrak{g}$ containing $\mathfrak{b}$. We see as above that the map $\mathcal{F} \mapsto \mathfrak{p}_{\mathcal{F}}$ is a bijection from the set of isotropic flags in V to the set of parabolic subalgebras of $\mathfrak{g}$. Moreover, $\mathfrak{p}_{\mathcal{F}} \subseteq \mathfrak{p}_{\mathcal{G}}$ if and only if $\mathcal{F} \supseteq \mathcal{G}$.

The fundamental weights corresponding to $\alpha_1, \ldots, \alpha_l$ are

$$\begin{aligned} \varpi_i &= \varepsilon_1 + \cdots + \varepsilon_i, \quad 1 \leq i \leq l-1, \\ \varpi_l &= \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_l). \end{aligned}$$

Let σ be the identity representation of $\mathfrak{g}$ on V. Then the exterior power $\wedge^k \sigma$ operates on $\bigwedge^k V$. If $h \in \mathfrak{h}$,

$$\begin{aligned} \sigma(h) \cdot e_i &= \varepsilon_i(h) e_i \quad \text{for } \ 1 \leq i \leq l, \\ \sigma(h) \cdot e_0 &= 0, \\ \sigma(h) \cdot e_{-i} &= -\varepsilon_i(h) e_{-i} \quad \text{for } \ 1 \leq i \leq l. \end{aligned}$$

It follows that, for $1 \leq k \leq l$, $\varpi_k = \varepsilon_1 + \cdots + \varepsilon_k$ is the highest weight of $\wedge^k \sigma$, and the elements of weight ϖ_k being those proportional to $e_1 \wedge \cdots e_k$. We shall show that for $1 \leq k \leq l$, the representation $\wedge^k \sigma$ is a fundamental representation of $\mathfrak{g}$ of highest weight ϖ_k. For this, it is enough to show that $\wedge^k \sigma$ is irreducible for $0 \leq k \leq 2l+1$. But the bilinear form Ψ on $\bigwedge^k V \times \bigwedge^{2l+1-k} V$ defined by

$$x \wedge y = \Psi(x, y) e_1 \wedge \cdots e_l \wedge e_0 \wedge e_{-l} \wedge \cdots \wedge e_{-1}$$

is invariant under $\mathfrak{g}$ and puts $\bigwedge^k V$ and $\bigwedge^{2l+1-k} V$ in duality. Thus, the representation $\wedge^{2l+1-k} \sigma$ is the dual of $\wedge^i \sigma$ and it suffices to prove the irreducibility of $\wedge^k \sigma$ for $0 \leq k \leq l$, or that the smallest subspace T_k of $\bigwedge^i V$ containing $e_1 \wedge \cdots e_i$ and stable under $\mathfrak{g}$ is the whole of $\wedge^k V$. This is immediate for $k = 0$ and $k = 1$. For $k = 2$ (and hence $l \geq 2$), the representation $\wedge^2 \sigma$ and the adjoint representation of $\mathfrak{g}$ (which is irreducible) have the same dimension $l(2l+1)$ and the same highest weight $\varepsilon_1 + \varepsilon_2$ (recall that $\varepsilon_1 + \varepsilon_2$ is the highest root of B_l). We conclude that $\wedge^2 \sigma$ is equivalent to the adjoint representation, and hence is irreducible. This proves our assertion for $l = 1$ and $l = 2$.

We now argue by induction on l, and assume that $l \geq k \geq 3$. We remark first of all that if W is a non-isotropic subspace of V of odd dimension, with orthogonal complement $W^{\perp}$, the restriction β_W of β to W is non-degenerate and $\mathfrak{o}(\beta_W)$ can be identified with the subalgebra of $\mathfrak{g}$ consisting of the elements vanishing on $W^{\perp}$. If $\dim(W) < \dim(V)$, and if β_W is of maximal index, the induction hypothesis implies that if T_k contains a non-zero element of the form $v \wedge w$, with $v \in \bigwedge^{k-i} W^{\perp}$ and $w \in \bigwedge^i W$

$(0 \le i \le k)$, then T_k contains $v \wedge \bigwedge^k W$: indeed, we have $x \cdot (v \wedge w) = v \wedge x \cdot w$ for all $x \in \mathfrak{o}(\psi_W)$. We now show by induction on $p \in [0, r]$ that T_r contains the elements

$$x = e_{i_1} \wedge \cdots \wedge e_{i_{k-p}} \wedge e_{j_1} \wedge \cdots \wedge e_{j_p}$$

for $1 \le i_1 < \cdots < i_{k-p} \le l$ and $-l \le j_1 < \cdots < j_p \le 0$. For $p = 0$, this follows from the irreducibility of the operation of $\mathfrak{gl}(F)$ on $\bigwedge^k F$, since $\mathfrak{g}$ contains the elements leaving $F = V_l = \sum_{i=1}^{l} \mathbb{K}e_i$ fixed and inducing on it any endomorphism (c.f. **????**). If $p = 1$, let $q \in (1, l)$ be such that $q \neq -j_1$ and such that there exists $\lambda \in [1, k-p]$ with $q = i_\lambda$; if $p \ge 2$, let $q \in [1, l]$ be such that $-q \in \{j_1, \dots, j_p\}$. Permuting the e_i if necessary, we can assume that $q = 1$. Now take for W the orthogonal complement of $W^\perp = \mathbb{K}e_1 + \mathbb{K}e_{-1}$. If $p = 1$, we have $x \in e_1 \wedge \bigwedge^{k-1} W$; since T_k contains $e_1 \wedge \cdots \wedge e_k$, we see that T_k contains x. If $p \ge 2$, either $x \in e_{-1} \wedge \bigwedge^{k-1} W$ or $x \in e_1 \wedge e_{-1} \wedge \bigwedge^{k-2} W$; since T_k contains $e_{-1} \wedge e_2 \wedge \cdots \wedge e_{k-1}$ and $e_{-1} \wedge e_1 \wedge e_2 \wedge \cdots e_{k-2}$ by the induction hypothesis, we see that T_k contains x, which completes the proof.

We shall now determine the fundamental representation with highest weight ϖ_l. Recall that for the quadratic form Q on V, the Clifford algebra $\mathrm{Cl}(V, Q)$ associated with Q is defined by the following relations

$$v^2 + Q(v) \cdot 1 = 0 \quad \text{for} \quad v \in V.$$

Also, from this relation we deduce that, for $v, w \in V$, we have

$$vw + wv = -2\beta(v, w).$$

We note show that representations of the Clifford algebra $\mathrm{Cl}(V, Q)$ corresponds to representations of that of $\mathfrak{o}(\beta)$.

Lemma 6.4.2. *Let V be a finite dimensional vector space, Q a non-degenerate quadratic form on V, β the symmetric bilinear form associated to Q, $\mathrm{Cl}(V, Q)$ the Clifford algebra of V relative to Q, φ_0 the composite of the canonical maps*

$$\mathfrak{o}(\beta) \longrightarrow \mathfrak{gl}(V) \longrightarrow V \otimes V^* \longrightarrow V \otimes V \longrightarrow \mathrm{Cl}^+(V, Q)$$

(the 1st is the canonical injection, the 3rd is defined by the canonical isomorphism from V^* to V corresponding to β, the 4th is defined by the multiplication in $\mathrm{Cl}(V, Q)$). Put $\varphi = -\frac{1}{4}\varphi_0$.

- *(a) If (e_i), (e'_j) are bases of V such that $\beta(e_i, e'_j) = \delta_{ij}$, we have $\varphi_0(x) = \sum_i (xe_i)e'_i$ for all $x \in \mathfrak{o}(\beta)$.*
- *(b) If $x, y \in \mathfrak{o}(\beta)$, we have $\sum_i (xe_i)(ye'_i) = -\sum_i (xye_i)e'_i$.*
- *(c) If $x \in \mathfrak{o}(\beta)$ and $v \in V$, we have $[\varphi(x), v] = xv$.*
- *(d) If $x, y \in \mathfrak{o}(\beta)$, we have $[\varphi(x), \varphi(y)] = \varphi([x, y])$*
- *(e) $\varphi(\mathfrak{o}(\beta))$ generates the associative algebra $\mathrm{Cl}^+(V, Q)$.*
- *(f) Let N be a left $\mathrm{Cl}^+(V, Q)$-module and ρ the corresponding homomorphism from $\mathrm{Cl}^+(V, Q)$ to $\mathrm{End}_{\mathbb{K}}(N)$. Then $\rho \circ \varphi$ is a representation of $\mathfrak{o}(\beta)$ on N. If N is simple, $\rho \circ \varphi$ is irreducible.*

Proof. Assertion (a) is clear. If $x, y \in \mathfrak{o}(\beta)$, we have (putting $\beta(x, y) = (x, y)$):

$$\begin{aligned}
\sum_i (xe_i)(ye'_i) &= \sum_{ijk} (xe_i, e'_j)(be'_i, e_k)e_je'_k = \sum_{ijk} (e_i, xe'_j)(e'_i, ye_k)e_je'_k \\
&= \sum_{jk} (xe'_j, ye_k)e_je'_k = -\sum_{jk} (e'_j, xye_k)e_je'_k \\
&= -\sum_k (xye_k)e'_k
\end{aligned}$$

which proves (b). Next, for $v \in V$, we have by (a),

$$\begin{aligned}
[\varphi(x), v] &= -\frac{1}{4}\sum_i ((xe_i)e'_iv - v(xe_i)e'_i) = -\frac{1}{4}\sum_i ((xe_i)e'_iv + (xe_i)ve'_i - (xe_i)ve'_i - v(xe_i)e'_i) \\
&= \frac{1}{2}\sum_i ((xe_i)(e'_i, v) - (xe_i, v)e'_i) = \frac{1}{2}x\Big(\sum_i (e'_i, v)e_i\Big) + \frac{1}{2}\sum_i (e_i, xv)e'_i
\end{aligned}$$

$$= \frac{1}{2}xv + \frac{1}{2}xv = -xv$$

which proves (c). Then from (a), (b), (c), we conclude that

$$\begin{aligned}[\varphi(x), \varphi(y)] &= [\varphi(x), -\frac{1}{4}\sum_i (ye_i)e_i'] = -\frac{1}{4}\sum_i ([\varphi(x), ye_i]e_i' + (ye_i)[\varphi(x), e_i']) \\ &= -\frac{1}{4}\sum_i ((xye_i)e_i' + (ye_i)(xe_i')) = -\frac{1}{4}\sum_i ((xye_i)e_i' - (yxe_i)(e_i')) = \varphi([x,y])\end{aligned}$$

which proves (d). To prove (e), we can, by extending scalars, assume that $\mathbb{K}$ is algebraically closed. Choose then a basis (e_i) of V such that $\beta(e_i, e_j) = \delta_{ij}$, so $e_i' = e_i$. If $i \neq j$, then $E_{ij} - E_{ji} \in \mathfrak{o}(\beta)$ and

$$\varphi(E_{ij} - E_{ji}) = -\frac{1}{4}(e_ie_j - e_je_i) = -\frac{1}{2}e_ie_j$$

but the e_ie_j generate $\mathrm{Cl}^+(V, Q)$. Assertion (f) follows from (d) and (e). $\square$

Chapter 7

Compact real Lie groups

In this section, the expression "Lie group" means "finite dimensional Lie group over the field of real numbers", the expression "Lie algebra" means, unless stated otherwise, "finite dimensional Lie algebra over the field of real numbers", and the expression "real Lie algebra" (resp. "complex Lie algebra") means "finite dimensional Lie algebra over the field of real numbers" (resp. "finite dimensional Lie algebra over the field of complex numbers"). We denote by G_0 the identity component of a topological group G, by $Z(G)$ the centre of a group G, by $D(G)$ its derived group, and by $N_G(H)$ or $N(H)$ (resp. $Z_G(H)$ or $Z(H)$) the normalizer (resp. centralizer) of a subset H of a group G.

7.0.1 Connected commutative real Lie groups

We use the letter $\mathbb{K}$ to denote the field $\mathbb{R}$ or $\mathbb{C}$. Let V be a finite dimensional $\mathbb{K}$-vector space, $(\cdot\,,\cdot)$ be a *separating*[1] positive hermitian form on V, G be a group, $\mathfrak{g}$ be an $\mathbb{R}$-Lie algebra, $\rho : G \to \mathrm{GL}(V)$ be a group homomorphism, and $\varphi : \mathfrak{g} \to \mathfrak{gl}(V)$ be a homomorphism of $\mathbb{R}$-Lie algebras. The form $(\cdot\,,\cdot)$ is **invariant** under G (resp. $\mathfrak{g}$) if and only $\rho(\mathfrak{g})$ is unitary with respect to $(\cdot\,,\cdot)$ for all $\mathfrak{g} \in G$ (resp. $\varphi(x)$ is *anti-hermitian*[2] with respect to $(\cdot\,,\cdot)$ for all $x \in \mathfrak{g}$). Indeed, denote by x^* the adjoint of an endomorphism x of V with respect to $(\cdot\,,\cdot)$; for $\mathfrak{g}$ in G, x in $\mathfrak{g}$, v and w in V, we have

$$(\rho(g)v, \rho(g)w) = (\rho(g)^*\rho(g)v, w),$$
$$(\varphi(x)v, w) + (x, \varphi(x)w) = ((\varphi(x) + \varphi(x)^*)v, w).$$

If the form $(\cdot\,,\cdot)$ is invariant under G (resp. $\mathfrak{g}$), the orthogonal complement of a stable subspace of V is stable; in particular, the representation ρ (resp. φ) is then semi-simple. Moreover, for all $g \in G$ (resp. $x \in \mathfrak{g}$), the endomorphism $\rho(g)$ (resp. $\varphi(x)$) of V is then semi-simple, with eigenvalues of absolute value 1 (resp. with purely imaginary eigenvalues); indeed, $\rho(g)$ is unitary (resp. $\mathrm{i}\varphi(x)$ is hermitian).

Assume that $\mathbb{K} = \mathbb{R}$. If G is a connected Lie group, ρ a morphism of Lie groups, $\mathfrak{g}$ the Lie algebra of G and φ the homomorphism induced by ρ, then $(\cdot\,,\cdot)$ is invariant under G if and only if it is invariant under $\mathfrak{g}$ ([**?**] III, §6 no.5 cor.3). Moreover, there exists a separating positive hermitian form on V invariant under G if and only if the subgroup $\rho(G)$ of $\mathrm{GL}(V)$ is relatively compact ([**?**] VII §3, no.1 prop.1).

Let G be a connected commutative (real) Lie group. The exponential map $\exp_G : \mathfrak{Lie}(G) \to G$ is a morphism of Lie groups, surjective with discrete kernel ([**?**] III, §6 no.4 prop.11), so $\mathfrak{Lie}(G)$ is a universal covering of G. The following conditions are equivalent:

(i) G is simply-connected;

(ii) $\exp_G$ is an isomorphism;

(iii) G is isomorphic to $\mathbb{R}^n$ ($n = \dim(G)$).

In this case, transporting the vector space structure of $L(G)$ to G by the isomorphism $\exp_G$ gives a vector space structure on G, which is the only one compatible with the topological group structure of

[1] Recall that a hermitian form $(\cdot\,,\cdot)$ on V is said to be separating (or non-degenerate) if, for every non-zero element u of V, there exists $v \in V$ such that $(u, v) \neq 0$.

[2] An element $x \in \mathrm{End}(V)$ is said to be anti-hermitian with respect to $(\cdot\,,\cdot)$ if the adjoint x^* of x with respect to $(\cdot\,,\cdot)$ is equal to $-x$. When $\mathbb{K} = \mathbb{C}$ (resp. $\mathbb{K} = \mathbb{R}$) this also means that the endomorphism ix of V (resp. of $\mathbb{C} \otimes_{\mathbb{R}} V$) is hermitian.

G. Simply-connected commutative Lie groups are called **vector (Lie) groups**; unless stated otherwise, they are always given the $\mathbb{R}$-vector space structure defined above.

Denote by $\Gamma(G)$ the kernel of $\exp_G$. By ([?] VII, §1, no.1 Th. 1), the group G is compact if and only if $\Gamma(G)$ is a lattice in $\mathfrak{Lie}(G)$, i.e. (loc. cit.) if the rank of the free $\mathbb{Z}$-module $\Gamma(G)$ is equal to the dimension of G. Conversely, if V is a finite dimensional $\mathbb{R}$-vector space and Γ a lattice in V, the quotient topological group V/Γ is a compact connected commutative Lie group. The compact connected commutative Lie groups are called **real tori**, or **tori**.

In the general case, let E be the vector subspace of $\mathfrak{Lie}(G)$ generated by $\Gamma(G)$, and let V be a complementary subspace. Then G is the direct product of its Lie subgroups $\exp(E)$ and $\exp(V)$; the first is a torus, the second is vector. Finally, every compact subgroup of G is contained in $\exp(E)$ (since its projection onto $\exp(V)$ is necessarily reduced to the identity element); thus, the subgroup $\exp(E)$ is the unique maximal compact subgroup of G.

For example, take $G = \mathbb{C}^*$; identify $\mathfrak{Lie}(G)$ with $\mathbb{C}$ so that the exponential map of G is $z \mapsto e^z$. Then $\Gamma(G) = 2\pi i\mathbb{Z}$, $E = \mathrm{i}\mathbb{R}$, and so $\exp(E) = \mathbb{U}$, the upper half plane; if we take $V = \mathbb{R}$, then $\exp(V) = \mathbb{R}_+^\times$ and we recover the isomorphism $\mathbb{C}^\times \cong \mathbb{U} \times \mathbb{R}_+^\times$. Note finally that $\exp_G : \mathfrak{Lie}(G) \to G$ is a universal covering of G, hence $\Gamma(G)$ can be identified naturally with the fundamental group of G.

7.1 Compact Lie algebras and maximal tori

The Lie groups we are dealing with are *real* Lie groups, so its algebras are *real* Lie algebras. However, since $\mathbb{C}$ is algebraically closed, our previous theory for splitting semi-simple Lie algebras works nicely only for *complex* Lie algebras. Our strategy is that, for a real Lie group G and its Lie algebra $\mathfrak{g}$, we will consider the complexification $\mathfrak{g}_{(\mathbb{C})}$ of $\mathfrak{g}$, and apply the theory of roots on $\mathfrak{g}_{(\mathbb{C})}$. But first, we note that, the Lie algebra $\mathfrak{g}$ itself, being the Lie algebra of a compact Lie group, already has some extraordinary properties.

7.1.1 Compact Lie algebras

A **compact Lie algebra** is defined to be a Lie algebra that is isomorphic to the Lie algebra of a compact real Lie group. In this subsection we characterize these Lie algebras, and consider their basic properties.

Proposition 7.1.1. *Let $\mathfrak{g}$ be a (real) Lie algebra. The following conditions are equivalent:*

(i) *$\mathfrak{g}$ is isomorphic to the Lie algebra of a compact Lie group.*

(ii) *The group $\mathrm{Inn}(\mathfrak{g})$ is compact.*

(iii) *$\mathfrak{g}$ has an invariant bilinear form that is symmetric, positive and separating.*

(iv) *$\mathfrak{g}$ is reductive and for all $x \in \mathfrak{g}$, the endomorphism $\mathrm{ad}(x)$ is semi-simple, with purely imaginary eigenvalues.*

(v) *$\mathfrak{g}$ is reductive and its Killing form κ is negative.*

Proof. If $\mathfrak{g}$ is the Lie algebra of a compact Lie group G, the group $\mathrm{Inn}(\mathfrak{g})$ is separating and isomorphic to a quotient of the compact group G_0 ([?] III §6, no.4 cor.4), hence is compact. Moreover, in this case, there exists a symmetric bilinear form on $\mathfrak{g}$ that is positive, separating and invariant under $\mathrm{Inn}(\mathfrak{g})$, hence also invariant under the adjoint representation of $\mathfrak{g}$. This shows (i)⇒(ii)⇒(iii).

If (iii) is satisfied, the adjoint representation of $\mathfrak{g}$ is semisimple, hence $\mathfrak{g}$ is reductive; moreover, the endomorphisms $\mathrm{ad}(x)$, for $x \in \mathfrak{g}$, have the indicated properties in (iv). For all $x \in \mathfrak{g}$, $\kappa(x,x) = \mathrm{tr}(\mathrm{ad}(x)^2)$, so $\kappa(x,x)$ is the sum of the squares of the eigenvalues of $\mathrm{ad}(x)$, and hence is negative if these are purely imaginary. We then conclude that (iii)⇒(iv)⇒(v).

Finally, assume that $\mathfrak{g}$ is reductive, hence the product of a commutative subalgebra $\mathfrak{z}$ and a semi-simple subalgebra $\mathfrak{s}$. The Killing form of $\mathfrak{s}$ is the restriction of the form κ to $\mathfrak{s}$, hence is negative and separating if κ is negative. The subgroup $\mathrm{Inn}(\mathfrak{s})$ of $\mathrm{GL}(\mathfrak{s})$ is closed (it is the identity component of $\mathrm{Aut}(\mathfrak{s})$, cf. [?] III §10, no.2 cor.2) and leaves the separating positive form $-\beta$ invariant; thus, it is compact, and $\mathfrak{s}$ is isomorphic to the Lie algebra of the compact Lie group $\mathrm{Inn}(\mathfrak{s})$. Further, since $\mathfrak{z}$ is commutative, it is isomorphic to the Lie algebra of a torus T. Thus $\mathfrak{g}$ is isomorphic to the Lie algebra of the compact Lie group $\mathrm{Inn}(\mathfrak{s}) \times T$. □

Thus, the compact Lie algebras are the products of a commutative algebra (namely its centre) with a compact semi-simple algebra. In other words, a Lie algebra is compact if and only if it is reductive and its derived Lie algebra is compact. The Lie algebra of a compact Lie group is compact.

Proposition 7.1.2. *The product of a finite number of Lie algebras is a compact Lie algebra if and only if each factor is compact. A subalgebra of a compact Lie algebra is compact. Let $\mathfrak{h}$ be an ideal of a compact Lie algebra $\mathfrak{g}$. Then the algebra $\mathfrak{g}/\mathfrak{h}$ is compact and the extension $\mathfrak{h} \to \mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$ is trivial.*

Proof. The first two assertions follow from the characterization (iii) of Proposition 7.1.1. The last one follows from (a) and the fact that, in a reductive Lie algebra, every ideal is a direct factor. □

Proposition 7.1.3. *Let G be a Lie group such that the group of connected components of G is finite. Then the following conditions are equivalent:*

(i) *The Lie algebra $\mathfrak{Lie}(G)$ is compact.*

(ii) *The group $\mathrm{Ad}(G)$ is compact.*

(iii) *There exists a separating positive symmetric bilinear form on $\mathfrak{Lie}(G)$ invariant under the adjoint representation of G.*

(iv) *G has a Riemannian metric invariant under left and right translations.*

Proof. If $\mathfrak{Lie}(G)$ is compact, the group $\mathrm{Ad}(G_0) = \mathrm{Inn}(\mathfrak{Lie}(G))$ is compact; since it has finite index in $\mathrm{Ad}(G)$, this latter group is also compact. Further, if $\mathrm{Ad}(G)$ is compact, then clearly (iii) is ture. On the other hand, if (iii) holds, then since $\mathrm{Inn}(\mathfrak{Lie}(G)) \subseteq \mathrm{Ad}(G)$, (i) follows from Proposition 7.1.1. The equivalence of (iii) and (iv) is established in ([**?**] III, §3, no.13). □

One may wonder the reverse problem: what happens if a connected Lie group has a compact Lie algebra $\mathfrak{g}$? Is the group then compact? The following theorem answers this question, and give us a nice description for such Lie groups.

Theorem 7.1.4 (Weyl). *Let G be a connected Lie group whose Lie algebra is compact semi-simple. Then G is compact and its centre is finite.*

Proof. Since $\mathfrak{g}$ is semi-simple, the centre $Z = Z(G)$ of G is discrete (ite Lie algebra is $\mathfrak{z}(\mathfrak{g})$, which is zero). Moreover, the quotient group G/Z is isomorphic to $\mathrm{Ad}(G)$ ([**?**] III §6, no.4, cor.4), hence compact (Proposition 7.1.3). Finally, the group G/Z is equal to its derived group ([**?**] III §9, no.2, cor. of prop.4). The theorem now follows from ([**?**] VII §3, no.2, prop.5). □

Proposition 7.1.5. *Let G be a connected Lie group whose Lie algebra is compact. Then there exist a torus T, a simply-connected compact semi-simple Lie group S, a vector group E and a surjective morphism $\psi : E \times T \times S \to G$ with finite kernel. If G is compact, the group E is reduced to the identity.*

Proof. Let Z (resp. S) be a simply-connected Lie group whose Lie algebra is isomorphic to the centre (resp. the derived algebra) of $\mathfrak{Lie}(G)$. Then Z is a vector group, S is a compact group with finite centre (Theorem 7.1.4) and G can be identified with the quotient of $Z \times S$ by a discrete subgroup Γ, which is central ([**?**] VII §3, no.2, lemma 4). Since the image of the projection of Γ onto S is central, hence finite, $\Gamma \cap Z$ is of finite index in Γ. Let Z' be the vector subspace of Z generated by $\Gamma \cap Z$, and E a complementary subspace. Then the group $T = Z'/(\Gamma \cap Z)$ is compact and commutative, hence a torus, and G is isomorphic to the quotient of the product group $E \times T \times S$ by a finite group. If G is compact, so is $E \times T \times S$ ([**?**] III §4, no.1 cor.2 of prop.2), hence so is E, which implies that $E = \{e\}$. □

Corollary 7.1.6. *Let G be a connected compact Lie group. Then $Z(G)_0$ is a torus, $D(G)$ is a connected compact semi-simple Lie group and the morphism $(x, y) \mapsto xy$ from $Z(G)_0 \times D(G)$ to G is a finite covering.*

Proof. With the notation in Proposition 7.1.5, we have $E = \{e\}$ and the subgroups $\psi(T)$ and $\psi(S)$ of G are compact, hence closed. Thus it suffices to show that $\psi(T) = Z(G)_0$, $\psi(S) = D(G)$. Now, $\mathfrak{Lie}(G) = \mathfrak{Lie}(\psi(T)) \times \mathfrak{Lie}(\psi(S))$; since S is semi-simple and T is commutative, this implies that

$$\mathfrak{Lie}(\psi(T)) = \mathfrak{z}(\mathfrak{Lie}(G)) = \mathfrak{Lie}(Z(G)_0), \quad \mathfrak{Lie}(\psi(S)) = [\mathfrak{Lie}(G), \mathfrak{Lie}(G)] = \mathfrak{Lie}(D(G)),$$

hence the stated assertion. □

Corollary 7.1.7. *The centre and the fundamental group of a connected compact semi-simple Lie group are finite. Its universal covering is compact.*

Proof. With the notation in Proposition 7.1.5, the groups E and T are reduced to the identity, thus S is a universal covering of G, and the fundamental group of G is isomorphic to $\ker\psi$, hence finite. The centre Z of G is discrete since G is semi-simple, so Z is finite. □

Corollary 7.1.8. *The fundamental group of a connected compact Lie group G is a finitely generated $\mathbb{Z}$-module, of rank equal to the dimension of $Z(G)$.*

Proof. Indeed, with the notations in Corollary 7.1.6, the fundamental group of $Z(G)_0$ is isomorphic to $\mathbb{Z}^n$, with $n = \dim(Z(G)_0)$, and the fundamental group of $D(G)$ is finite by Corollary 7.1.7. □

Corollary 7.1.9. *Let G be a connected compact Lie group. The following conditions are equivalent:*

(i) *G is semi-simple;*

(ii) *$Z(G)$ is finite;*

(iii) *$\pi_1(G)$ is finite.*

Proof. This follows from Proposition 7.1.5 and Proposition 7.1.5. □

Corollary 7.1.10. *Let G be a connected compact Lie group. Then* $\mathrm{Inn}(G)$ *is the identity component of the Lie group* $\mathrm{Aut}(G)$.

Proof. Let $\varphi \in \mathrm{Aut}(G)_0$. Then φ induces an automorphism φ_1 of $Z(G)_0$ and an automorphism φ_2 of $D(G)$, and we have $\varphi_1 \in \mathrm{Aut}(Z(G)_0)_0$, $\varphi_2 \in \mathrm{Aut}(D(G))_0$. Since $\mathrm{Aut}(Z(G)_0)$ is discrete, we have $\varphi_1 = \mathrm{id}$; since $D(G)$ is semi-simple, by Corollary **??**, there exists an element g of $D(G)$ such that $\varphi_2(x) = \mathrm{Inn}(g)(x)$ for all $x \in D(G)$. For all $x \in Z(G)_0$, we have $\mathrm{Inn}(g)(x) = x = \varphi_1(x)$; since $G = Z(G)_0 \cdot D(G)$, it follows that $\mathrm{Ad}(g) = \varphi$. □

Proposition 7.1.11. *Let G be a Lie group whose Lie algebra is compact.*

(a) *Assume that G is connected. Then G has a largest compact subgroup K, which is connected. Moreover, there exists a closed central vector subgroup N of G such that G is the direct product $N \times K$.*

(b) *Assume that the group of connected components of G is finite. Then:*

 (i) *Every compact subgroup of G is contained in a maximal compact subgroup.*

 (ii) *If K_1 and K_2 are two maximal compact subgroups of G, there exists $g \in G$ such that $K_2 = gK_1g^{-1}$.*

 (iii) *Let K be a maximal compact subgroup of G. Then $K \cap G_0$ is equal to K_0, and is the largest compact subgroup of G_0.*

 (iv) *There exists a closed central vector subgroup H of G_0, normal in G, such that for any maximal compact subgroup K of G, G_0 is the direct product of K_0 by H, and G is the semi-direct product of K by H.*

Proof. Assume thet G is connected. We retain the notations of Proposition 7.1.5. The projection of $\ker\psi$ onto E is a finite subgroup of the vector group E, hence is reduced to the identity. It follows that $\ker\psi$ is contained in $T \times S$, hence that G is the direct product of the vector group $N = \psi(E)$ and the compact group $K = \psi(T \times S)$. Every compact subgroup of G has a projection onto N that is reduced to the identity element, hence is contained in K. This proves the assertions of (a).

Now assume that G/G_0 is finite. By (a), G_0 is the direct product of its largest compact subgroup L and a vector subgroup M; the subgroup L of G is clearly normal. Let $\mathfrak{n}$ be a vector subspace complement of $\mathfrak{Lie}(L)$ in $\mathfrak{Lie}(G)$, stable under the adjoint representation of G (Proposition 7.1.3); this is an ideal of $\mathfrak{Lie}(G)$ and we have $\mathfrak{Lie}(G) = \mathfrak{Lie}(L) \times \mathfrak{n}$. Let H be the integral subgroup of G with Lie algebra $\mathfrak{n}$; by ([**?**] III, §6, no.6, prop.14), this is normal in G. The projection of $\mathfrak{Lie}(G)$ onto $\mathfrak{Lie}(M)$ with kernel $\mathfrak{Lie}(L)$ induces an isomorphism from $\mathfrak{n}$ to $\mathfrak{Lie}(M)$, so it follows that the projection of G_0 to M induces an étale morphism from H to M. Since M is simply connected, this is an isomorphism, and H is a vector group. The morphism $(x, y) \mapsto xy$ from $M \times L$ to G_0 is an injective étale morphism (since $H \cap L$ is reduced to the identity), hence an isomorphism. It follows that H is a closed subgroup of G and the quotient G/H is compact, since G_0/H is compact and G/G_0 is finite ([**?**] III, §4, no.1 cor.2 of prop.2). By ([**?**] VII, §3, no.2 prop.3), every compact subgroup of G is contained in a maximal compact subgroup, which are conjugate, and for any maximal compact subgroup K of G, G is the semi-direct product of K by H.

Since G_0 contains H, it is the semi-direct product of H by $G_0 \cap K$. It follows that $G_0 \cap K$ is connected, hence equal to K_0, since $K/(G_0 \cap K)$ is isomorphic to G/G_0, hence finite. Finally, K_0 is clearly the largest compact subgroup of G_0 by (a). $\square$

Corollary 7.1.12. *If H satisfies the conditions of Proposition 7.1.11 (b)(iv), and if K_1 and K_2 are two maximal compact subgroups of G, there exists $h \in H$ such that $hK_1h^{-1} = K_2$.*

Proof. Indeed, by Proposition 7.1.11 (b)(ii) there exists an element $g \in G$ such that gK_1g^{-1}. On the other hand, by Proposition 7.1.11 (b)(iv), there exists $h \in H$ and $k \in K_1$ such that $g = hk$. The element h then has the required properties. $\square$

7.1.2 Maximal tori of compact Lie groups

Recall that for a Lie group G, the adjoint representation of $g \in G$ is the tangent map at e to $\mathrm{Inn}(g)$.

Lemma 7.1.13. *Let G be a Lie group, K a compact subgroup of G, and $(\cdot,\cdot)$ an invariant bilinear form on $\mathfrak{Lie}(G)$. Let $x, y \in \mathfrak{Lie}(G)$. There exists an element k of K such that $(u, [\mathrm{Ad}(k)(x), y]) = 0$ for all $u \in \mathfrak{Lie}(K)$.*

Proof. The function $v \mapsto (\mathrm{Ad}(v)(x), y)$ from K to $\mathbb{R}$ is continuous, and hence has a minimum at some point $k \in K$. Let $u \in \mathfrak{Lie}(K)$ and put

$$h(t) = (\mathrm{Ad}(\exp(tu)) \circ \mathrm{Ad}(k)(x), y), \quad t \in \mathbb{R}.$$

We have $h(t) \geq h(0)$ for all t; moreover,

$$\left.\frac{dh}{dt}\right|_{t=0} = ([u, \mathrm{Ad}(k)(x)], y) = (u, [\mathrm{Ad}(k)(x), y]),$$

hence the lemma. $\square$

Theorem 7.1.14. *Let $\mathfrak{g}$ be a compact Lie algebra. The Cartan subalgebras of $\mathfrak{g}$ are its maximal abelian subalgebras; in particular, $\mathfrak{g}$ is the union of its Cartan subalgebras. The group $\mathrm{Inn}(\mathfrak{g})$ operates transitively on the set of Cartan subalgebras of $\mathfrak{g}$.*

Proof. Since $\mathfrak{g}$ is reductive, its Cartan subalgebras are commutative (Corollary 6.1.54). Conversely, let $\mathfrak{t}$ be a commutative subalgebra of $\mathfrak{g}$. By Proposition 7.1.1, $\mathrm{ad}(x)$ is semi-simple for all $x \in \mathfrak{t}$; by Proposition 6.1.49, there exists a Cartan subalgebra of $\mathfrak{g}$ containing $\mathfrak{t}$. This proves the first assertion of the theorem.

Now let $\mathfrak{t}$ and $\tilde{\mathfrak{t}}$ be two Cartan subalgebras of $\mathfrak{g}$. We prove that there exists $f \in \mathrm{Inn}(\mathfrak{g})$ such that $f(\mathfrak{t}) = \tilde{\mathfrak{t}}$. By Proposition 7.1.1, we can assume that $\mathfrak{g}$ is of the form $\mathfrak{Lie}(G)$, where G is a connected compact Lie group, and can choose a separating invariant symmetric bilinear form $(\cdot,\cdot)$ on $\mathfrak{g}$. Let x (resp. $\tilde{x}$) be a regular element of $\mathfrak{g}$ such that $\mathfrak{t} = \mathfrak{g}^0(x)$ (resp. $\tilde{\mathfrak{t}} = \mathfrak{g}^0(\tilde{x})$). Applying Lemma 7.1.13 with $K = G$, we see that there exists $k \in G$ such that $[\mathrm{Ad}(k)(x), \tilde{y}]$ is orthogonal to $\mathfrak{g}$, and hence is zero; then $\mathrm{Ad}(k)(x) \in \mathfrak{g}^0(\tilde{x}) = \tilde{\mathfrak{t}}$, so $\mathfrak{g}^0(\mathrm{Ad}(k)(x)) = \tilde{\mathfrak{t}}$ since $\mathrm{Ad}(k)(x)$ is regular. We conclude that $\mathrm{Ad}(k)(\mathfrak{t}) = \tilde{\mathfrak{t}}$, hence the theorem. $\square$

Corollary 7.1.15. *Let $\mathfrak{g}$ be a compact Lie algebra. Let $\mathfrak{t}$ and $\tilde{\mathfrak{t}}$ be Cartan subalgebras of $\mathfrak{g}$, $\mathfrak{a}$ be a subset of $\mathfrak{t}$ and u an automorphism of $\mathfrak{g}$ that takes $\mathfrak{a}$ into $\tilde{\mathfrak{t}}$. There exists an element s of $\mathrm{Inn}(\mathfrak{g})$ such that $u \circ s$ takes $\mathfrak{t}$ to $\tilde{\mathfrak{t}}$, and coincides with u on $\mathfrak{a}$.*

Proof. Put $G = \mathrm{Inn}(\mathfrak{g})$, and consider the fixer $Z_G(\mathfrak{a})$ of $\mathfrak{a}$ in G; this is a Lie subgroup of G, whose Lie algebra $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{a})$ consists of the elements of $\mathfrak{g}$ that commute with every element of $\mathfrak{a}$ ([?] III §9, no.3, prop.7). Then $\mathfrak{t}$ and $u^{-1}(\tilde{\mathfrak{t}})$ are two Cartan subalgebras of the compact Lie algebra $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{a})$. By Theorem 7.1.14, there exists an element s of $Z_G(\mathfrak{a})$ such that $s(\mathfrak{t}) = u^{-1}(\tilde{\mathfrak{t}})$; any such element has the desired properties. $\square$

Let G be a Lie group. A **torus** of G is a closed subgroup that is a torus, in other words, any commutative connected compact subgroup. The maximal elements of the set of tori of G, ordered by inclusion, are called the **maximal tori** of G.

Theorem 7.1.16. *Let G be a connected compact Lie group.*

(a) *The Lie algebras of the maximal tori of G are the Cartan subalgebras of $\mathfrak{Lie}(G)$.*

(b) *Let T_1 and T_2 be two maximal tori of G. There exists $g \in G$ such that $T_2 = gT_1g^{-1}$.*

(c) *G is the union of its maximal tori.*

Proof. Let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{Lie}(G)$; the integral subgroup of G whose Lie algebra is $\mathfrak{t}$ is closed (Corollary 6.1.34) and commutative, and hence is a torus of G. If T is a maximal torus of G, its Lie algebra is commutative, hence is contained in a Cartan subalgebra of $\mathfrak{Lie}(G)$ (Theorem 7.1.14). It follows that the maximal tori of G are exactly the integral subgroups of G associated to the Cartan subalgebras of $\mathfrak{Lie}(G)$, whence (a). Assertion (b) follows from Theorem 7.1.14, since the canonical homomorphism $G \to \mathrm{Inn}(\mathfrak{Lie}(G))$ is surjective ([?] III, no.4 cor.4 of prop.10).

Denote by M the union of the maximal tori of G, and let T be a maximal torus of G. The continuous map $(g,t) \mapsto gtg^{-1}$ from $G \times T$ to G has image M, which is therefore closed in G; thus, to prove (c), it suffices to prove that M is open in G; since M is invariant under inner automorphisms, it suffices to show that, for all $a \in T$, M is a neighbourhood of a. We argue by induction on the dimension of G and distinguish two cases.

Suppose that a is not central in G. Let H be the identity component of the centralizer of a in G; this is a connected compact subgroup of G distinct from G, which contains T, and hence a. Since $\mathrm{Ad}(a)$ is semi-simple, the Lie algebra of H is the nilspace of $\mathrm{Ad}(a) - \mathrm{id}$; it now follows from ([?] VII §4, no.2, prop.4) that the union Y of the conjugates of H is a neighbourhood of a. By the induction hypothesis, $H \subseteq M$, and hence $Y \subseteq M$; thus, M is a neighbourhood of a.

Now assume that a is central in G. It suffices to prove that $a\exp(x)$ belongs to M for all x in $\mathfrak{Lie}(G)$. Now every element x of $\mathfrak{Lie}(G)$ belongs to a Cartan subalgebra of G (Theorem 7.1.14); the corresponding integral subgroup $\tilde{T}$ contains $\exp(x)$; since it is conjugate to T, it contains a and hence $a\exp(x)$, as required. ☐

Corollary 7.1.17. *Let G be a connected compact Lie group.*

(a) *The exponential map of G is surjective.*

(b) *For all $n \geq 1$, the map $g \mapsto g^n$ on G is surjective.*

Proof. Indeed, $\exp(\mathfrak{Lie}(G))$ contains all the maximal tori of G, hence (a). Assertion (b) follows from the formula $\exp(x)^n = \exp(nx)$ for x in $\mathfrak{Lie}(G)$. ☐

Corollary 7.1.18. *Let G be a connected compact Lie group. Then the intersection of the maximal tori of G is the centre of G.*

Proof. Let x be an element of the centre of G; by Theorem 7.1.16 (c), there exists a maximal torus T of G containing x; then x belongs to all the conjugates of T, hence to all the maximal tori of G. Conversely, if x belongs to all the maximal tori of G, it commutes with every element of G by Theorem 7.1.16 (c). ☐

Corollary 7.1.19. *Let G be a connected compact Lie group. Let $g \in G$ and $Z = Z_G(g)$ be its centralizer. Then g belongs to Z_0, and the group Z_0 is the union of the maximal tori of G containing g.*

Proof. There exists a maximal torus T of G containing g by Theorem 7.1.16 (c), and hence contained in Z_0. Moreover, the group Z_0 is a connected compact Lie group, and hence the union of its maximal tori Theorem 7.1.16 (c). If T is a maximal torus containing g, then since T is connected and commutative, it is contained in Z_0. This proves the claim. ☐

Corollary 7.1.20. *Let G be a connected compact Lie group, T a maximal torus of G, H a Lie group and $\varphi : G \to H$ a morphism. Then φ is injective if and only if its restriction to T is injective.*

Proof. Indeed, by Theorem 7.1.16 the normal subgroup $\ker\varphi$ of G reduces to the identity element if and only if its intersection with T reduces to the identity element. ☐

Corollary 7.1.21. *Let G be a connected compact Lie group.*

(a) *Let H be a torus of G. Then the centralizer of H is connected, and is the union of the maximal tori of G containing H.*

(b) *Let $\mathfrak{z}$ be a commutative subalgebra of $\mathfrak{Lie}(G)$. Then the fixer of $\mathfrak{z}$ in G is connected, and is the union of the maximal tori of G whose Lie algebras contain $\mathfrak{z}$.*

Proof. To prove (a), it suffices to prove that if an element g of G centralizes H, there exists a maximal torus of G containing H and g. Now, if $Z = Z_G(g)$ is the centralizer of g, we have $g \in Z_0$ by Corollary 7.1.19 and $H \subseteq Z_0$; if T is a maximal torus of the connected compact Lie group Z_0 containing H, we have $g \in T$ (Corollary 7.1.19), hence (a). Assertion (b) follows from (a) applied to the closure of the integral subgroup with Lie algebra $\mathfrak{z}$, in view of ([**?**] III §9, no.3, prop.9). □

Remark 7.1.22. It follows from Corollary 7.1.21 that a maximal torus of G is a maximal commutative subgroup. The converse is not true: for example, in the group $\mathrm{SO}(3,\mathbb{R})$, the maximal tori are of dimension 1, and thus cannot contain the subgroup of diagonal matrices, which is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$. Moreover, if $g \in \mathrm{SO}(3,\mathbb{R})$ is a non-scalar diagonal matrix, g is a regular element of $\mathrm{SO}(3,\mathbb{R})$ whose centralizer is not connected.

Corollary 7.1.23. *Let G be a connected compact Lie group. The maximal tori of G are their own centralizers, and are the fixers of their Lie algebras.*

Proof. Let T be a maximal torus of G and Z its centralizer; since $\mathfrak{Lie}(T)$ is a Cartan subalgebra of $\mathfrak{Lie}(G)$, we have $\mathfrak{Lie}(T) = \mathfrak{Lie}(Z)$, hence $T = Z$ since Z is connected. □

Corollary 7.1.24. *Let G be a connected compact Lie group. Let T and $\tilde{T}$ be two maximal tori of G, A be a subset of T and φ be an automorphism of G that takes A into $\tilde{T}$. There exists $g \in G$ such that $\varphi \circ \mathrm{Inn}(g)$ takes T to $\tilde{T}$ and coincides with φ on A.*

Proof. Let Z be the centralizer of A. Then T and $\varphi^{-1}(\tilde{T})$ are two maximal tori of Z_0; every element g of Z_0 such that $\mathrm{Inn}(g)(T) = \varphi^{-1}(\tilde{T})$ has the desired properties. □

Corollary 7.1.25. *Let G be a compact Lie group and T be a maximal torus of G. Then $G = N_G(T) \cdot G_0$, and the injection of $N_G(T)$ into G induces an isomorphism from $N_G(T)/N_{G_0}(T)$ to G/G_0.*

Proof. Let $h \in G$. Then $h^{-1}Th$ is a maximal torus of G_0, hence (Theorem 7.1.16) there exists $g \in G_0$ such that $hg \in N_G(T)$; thus h belongs to $N_G(T) \cdot G_0$, hence the first assertion. The second follows immediately. □

Proposition 7.1.26. *Let G and $\tilde{G}$ be two connected compact Lie groups.*

(a) *Let $\varphi : G \to \tilde{G}$ be a surjective morphism of Lie groups. The maximal tori of $\tilde{G}$ are the images under φ of the maximal tori of G. If the kernel of φ is central in G (for example discrete), the maximal tori of G are the inverse images under φ of the maximal tori of $\tilde{G}$.*

(b) *Let H be a connected closed subgroup of G. Every maximal torus of H is the intersection with H of a maximal torus of G.*

Proof. Let T be a maximal torus of G; then $\mathfrak{Lie}(T)$ is a Cartan subalgebra of $\mathfrak{Lie}(G)$ (Theorem 7.1.16 (a)), so $\mathfrak{Lie}(\varphi(T))$ is a Cartan subalgebra of $\mathfrak{Lie}(\tilde{G})$ by Corollary 6.1.32; it follows that $\varphi(T)$ is a maximal torus of $\tilde{G}$ (Theorem 7.1.16 (a)). If $\ker\varphi$ is central in G, it is contained in T by Corollary 7.1.18, so $T = \varphi^{-1}(\varphi(T))$.

Conversely, let $\tilde{T}$ be a maximal torus of $\tilde{G}$; we show that there exists a maximal torus T of G such that $\varphi(T) = \tilde{T}$. Let T_1 be a maximal torus of G; then $\varphi(T_1)$ is a maximal torus of $\tilde{G}$ and there exists $\tilde{g} \in \tilde{G}$ such that $\tilde{T} = \tilde{g}\varphi(T_1)\tilde{g}^{-1}$ (Theorem 7.1.16 (b)); if $g \in G$ is such that $\varphi(g) = \tilde{g}$, we have $\tilde{T} = \varphi(T)$ with $T = gT_1g^{-1}$.

Let H be a connected closed subgroup of G and S be a maximal torus of H; this is a torus of G so there exists a maximal torus T of G containing S. Then $T \cap H$ is a commutative subgroup of H containing S, hence is equal to S (Corollary 7.1.23). □

Remark 7.1.27. Proposition 7.1.26 generalizes immediately to connected groups with compact Lie algebras. In particular, if G is a connected Lie group whose Lie algebra is compact, the Cartan subgroups of G are exactly the inverse images of the maximal tori of the connected compact Lie group $\mathrm{Ad}(G)$ (under the canonical homomorphism from G to $\mathrm{Ad}(G)$).

We shall call the **rank** of a connected Lie group G the rank of its Lie algebra, and we shall denote it by $\mathrm{rank}(G)$. By Theorem 7.1.16, the rank of a connected compact Lie group is the common dimension of its maximal tori.

Let G be a connected compact Lie group and H a closed subgroup of G. If H is connected, then $\mathrm{rank}(H) \leq \mathrm{rank}(G)$ (since the maximal tori of H are tori in G). By Theorem 7.1.16 (c), to say that H is connected and of maximal rank (that is, of rank $\mathrm{rank}(G)$) means that H is a union of maximal tori of G.

Proposition 7.1.28. *Let G be a connected compact Lie group, and H a connected closed subgroup of maximal rank.*

(a) *The compact manifold G/H is simply connected.*

(b) *The homomorphism $\pi_1(H) \to \pi_1(G)$, induced by the canonical injection of H into G, is surjective.*

Proof. Since H is connected, we have an exact sequence

$$\pi_1(H) \longrightarrow \pi_1(G) \longrightarrow \pi_1(G/H, \bar{e}) \longrightarrow 0$$

where $\bar{e}$ is the image in G/H of the identity element of G. Since G/H is connected, this immediately implies the equivalence of assertions (a) and (b). Moreover, if $\varphi : \tilde{G} \to G$ is a surjective morphism of connected compact Lie groups whose kernel is central, proving the proposition (in the form (a)) for G is the same as proving it for $\tilde{G}$ (by Proposition 7.1.26). Thus, we can first of all replace G by $\mathrm{Ad}(G)$, then assume that G is semi-simple, and then by replacing G by a universal covering, assume that G is simply-connected. But then assertion (b) is trivial. □

Let G be a connected compact Lie group, T a maximal torus of G. For an application of Proposition 7.1.26, we consider the derived group $D(G)$ of G and its universal covering $\widetilde{D}(G)$; let $p : \widetilde{D}(G) \to G$ be the composite of the canonical morphisms $\widetilde{D}(G) \to D(G)$ and $D(G) \to G$. Then $\widetilde{D}(G)$ is a connected compact Lie group (Corollary 7.1.7); moreover, the inverse image $\widetilde{T}$ of T under p is a maximal torus of $\widetilde{D}(G)$.

Lemma 7.1.29. *Let H be a Lie group, $\varphi_T : T \to H$ and $\tilde{\varphi} : \widetilde{D}(G) \to H$ morphisms of Lie groups such that $\varphi_T(p(t)) = \tilde{\varphi}(t)$ for all $t \in \widetilde{T}$. Then there exists a unique morphism of Lie groups $\varphi : G \to H$ such that $\varphi \circ p = \tilde{\varphi}$ and such that the restriction of φ to T is φ_T.*

$$\begin{tikzcd}
\widetilde{D}(G) \arrow[r, "p"] \arrow[rr, bend left, "\tilde{\varphi}"] & G \arrow[r, dashed, "\varphi"] & H \\
\widetilde{T} \arrow[u, hook] \arrow[r, "p"] & T \arrow[u, hook] \arrow[ur, "\varphi_T"'] &
\end{tikzcd}$$

Proof. Put $Z = Z(G)_0$; by Corollary 7.1.6, the morphism of Lie groups $\psi : Z \times \widetilde{D}(G) \to G$ such that $g(z, x) = z^{-1}p(x)$ is a covering; its kernel consists of the pairs (z, x) such that $p(x) = z$, for which $x \in p^{-1}(Z) \subseteq \widetilde{T}$.

$$\begin{tikzcd}
Z \times \widetilde{D}(G) \arrow[r, "\psi"] \arrow[rr, bend left, "(z,x) \mapsto \varphi_T(z^{-1})\tilde{\varphi}(x)"] & G \arrow[r, "\varphi"] & H
\end{tikzcd}$$

Since the morphism $(z, x) \mapsto \varphi_T(z^{-1})\tilde{\varphi}(x)$ from $Z \times \widetilde{D}(G)$ to H maps $\ker \psi$ to $\{e\}$, there exists a unique morphism $\varphi : G \to H$ such that

$$\varphi(\psi(z, x)) = \varphi(z^{-1}p(x)) = \varphi_T(z^{-1})\tilde{\varphi}(x).$$

Since z and x are arbitrary, we conclude that $\varphi(z) = \varphi_T(z)$ for $z \in Z$ and $\varphi \circ p = \tilde{\varphi}$. But we also have $\varphi(t) = \varphi_T(t)$ for $t \in p(\widetilde{T})$; since $T = Z \cdot p(\widetilde{T})$, the restriction of φ to T is indeed φ_T. □

Proposition 7.1.30. *Let G be a connected compact Lie group and T a maximal torus of G. Let H be a Lie group and $f : \mathfrak{Lie}(G) \to \mathfrak{Lie}(H)$ a homomorphism of Lie algebras. Then there exists a morphism of Lie groups $\varphi : G \to H$ such that $\mathfrak{Lie}(\varphi) = f$ if and only if there exists a morphism of Lie groups $\varphi_T : T \to H$ such that $\mathfrak{Lie}(\varphi_T) = \varphi|_{\mathfrak{Lie}(T)}$. Moreover, in this case we have $\varphi_T = \varphi|_T$.*

Proof. If $\varphi : G \to H$ is a morphism of Lie groups such that $\mathfrak{Lie}(\varphi) = f$, the restriction φ_T of φ to T is the unique morphism from T to H such that $\mathfrak{Lie}(\varphi_T) = f|_{\mathfrak{Lie}(T)}$. Conversely, let $\varphi_T : T \to H$ be a morphism of Lie groups such that $\mathfrak{Lie}(\varphi_T) = f|_{\mathfrak{Lie}(T)}$. Let $\widetilde{D}(G)$ and p be as above; the map $\mathfrak{Lie}(p)$

induces an isomorphism from $\mathfrak{Lie}(\widetilde{D}(G))$ to the derived algebra $\mathfrak{d}$ of $\mathfrak{Lie}(G)$. There exists a morphism of Lie groups $\tilde{\varphi} : \widetilde{D}(G) \to H$ such that $\mathfrak{Lie}(\tilde{\varphi}) = (f|_{\mathfrak{d}}) \circ \mathfrak{Lie}(p)$ ([?] III §6, no.1 th.1). The morphisms $t \mapsto \tilde{\varphi}(t)$ and $t \mapsto \varphi_T(p(t))$ from $\widetilde{T}$ to H induce the same homomorphism of Lie algebras, and hence coincide. Applying Lemma 7.1.29, we deduce the existence of a morphism $\varphi : G \to H$ such that $\mathfrak{Lie}(\varphi)$ and f coincide on $\mathfrak{Lie}(T)$ and $\mathfrak{d}$. Since $\mathfrak{Lie}(G) = \mathfrak{d} + \mathfrak{Lie}(T)$, we have $\mathfrak{Lie}(\varphi) = f$. □

7.1.3 The Weyl group associated with a maximal torus

Proposition 7.1.31. *Let G be a compact Lie group, H a connected closed subgroup of G of maximal rank and N the normalizer of H in G. Then H is of finite index in N and is the identity component of N.*

Proof. The Lie algebra of H contains a Cartan subalgebra of $\mathfrak{Lie}(G)$. Thus, by Corollary 6.1.34, H is the identity component of N. Since N is compact, H is of finite index in N. □

Remark 7.1.32. Every integral subgroup H of G such that $\mathrm{rank}(H) = \mathrm{rank}(G)$ is closed: indeed, the preceding proof shows that H is the identity component of its normalizer, which is a closed subgroup of G.

Let G be a connected compact Lie group and T a maximal torus of G. Denote by $N_G(T)$ the normalizer of T in G. In view of Proposition 7.1.31, the quotient group $N_G(T)/T$ is finite. We denote it by $W_G(T)$, or by $W(T)$, and call it the **Weyl group** of the maximal torus T of G, or the Weyl group of G relative to T. Since T is commutative, the operation of $N_G(T)$ on T by inner automorphisms of G induces by passage to the quotient an operation, called the **canonical operation**, of the group $W_G(T)$ on the Lie group T. By Corollary 7.1.23, this operation is faithful: the associated homomorphism $W_G(T) \to \mathrm{Aut}(T)$ is injective.

If $\tilde{T}$ is another maximal torus of G and if $g \in G$ is such that $\mathrm{Inn}(g)$ maps T to $\tilde{T}$, then $\mathrm{Inn}(g)$ induces an isomorphism a_g from $W_G(T)$ to $W_G(\tilde{T})$ and

$$a_g(w)(gtg^{-1}) = gw(t)g^{-1}$$

for all $w \in W_G(T)$ and all $t \in T$.

Proposition 7.1.33. *Let G be a connected compact Lie group and T a maximal torus of G.*

(a) Every conjugacy class of G meets T.

(b) The intersections with T of the conjugacy classes of G are the orbits of the Weyl group.

Proof. The first assertion follows from Theorem 7.1.16. By definition of the Weyl group and the canonical operation, any two elements in the same orbit of $W_G(T)$ on T are conjugate in G; conversely, let a, b be two elements of T conjugate under G. There exists $h \in G$ such that $b = hah^{-1}$; applying Corollary 7.1.24 with $A = \{a\}$, $\varphi = \mathrm{Inn}(h)$, $\tilde{T} = T$, we see that there exists $g \in G$ such that $\mathrm{Inn}(hg)$ maps T to T and a to b. The class of hg in $W_G(T)$ then maps a to b, hence the proposition. □

Corollary 7.1.34. *Let G be a connected compact Lie group and T a maximal torus of G. The canonical injection of T into G defines by passage to the quotient a homeomorphism from $T/W_G(T)$ to the space $G/\mathrm{Inn}(G)$ of conjugacy classes of G.*

Proof. Indeed, this is a bijective continuous map between two compact spaces. □

Corollary 7.1.35. *Let G be a connected compact Lie group and T a maximal torus of G. Let C be a subset of G stable under inner automorphisms. Then C is open (resp. closed, resp. dense) in G if and only if $C \cap T$ is open (resp. closed, resp. dense) in T.*

Proof. This follows from Corollary 7.1.34 and the fact that the canonical maps $T \to T/W_G(T)$ and $G \to G/\mathrm{Inn}(G)$ are open. □

Denote the Lie algebra of G by $\mathfrak{g}$, and that of T by $\mathfrak{t}$. The operation of $W_G(T)$ on T induces a representation, called the **canonical representation**, of the group $W_G(T)$ on the $\mathbb{R}$-vector space t. We have the following counterpart of Proposition 7.1.33.

Proposition 7.1.36. *Let G be a connected compact Lie group and T a maximal torus of G. Let $\mathfrak{g} = \mathfrak{Lie}(G)$ and $\mathfrak{t} = \mathfrak{Lie}(T)$.*

(a) *Every orbit of G on $\mathfrak{g}$ (for the adjoint representation) meets $\mathfrak{t}$.*

(b) *The intersections with $\mathfrak{t}$ of the orbits of G are the orbits of $W_G(T)$ on $\mathfrak{t}$.*

Proof. Assertion (a) follows from Theorem 7.1.14. Let x, y be two elements of $\mathfrak{t}$ conjugate under $\mathrm{Ad}(G)$, and let $h \in G$ be such that $\mathrm{Ad}(h)(x) = y$. Applying Corollary 7.1.15 with $\mathfrak{a} = \{x\}$, $u = \mathrm{Ad}(h)$, $\tilde{\mathfrak{t}} = \mathfrak{t}$, we see that there exists $g \in G$ such that $\mathrm{Ad}(hg)$ maps $\mathfrak{t}$ to $\mathfrak{t}$ and x to y. Then $hg \in N_G(T)$ ([**?**] III §9, no.4 prop.11), and the class of hg in $W_G(T)$ maps x to y, hence the proposition. □

Corollary 7.1.37. *Let G be a connected compact Lie group and T a maximal torus of G. Then the canonical injection of $\mathfrak{t}$ into $\mathfrak{g}$ defines by passage to the quotient a homeomorphism from $\mathfrak{t}/W_G(T)$ to $\mathfrak{g}/\mathrm{Ad}(G)$.*

Proof. Denote the map by $\iota : \mathfrak{t} \to \mathfrak{g}$ and the induced map by $\bar{\iota}$. Then $\bar{\iota}$ is bijective and continuous by Proposition 7.1.36. We have a commutative diagram

$$\begin{tikzcd} \mathfrak{t} \arrow[r, "\iota"] \arrow[d] & \mathfrak{g} \arrow[d] \\ \mathfrak{t}/W_G(T) \arrow[r, "\bar{\iota}"] & \mathfrak{g}/\mathrm{Ad}(G) \end{tikzcd}$$

where the vertical are quotient maps. Since ι and $\pi_{\mathfrak{g}}$ are proper ([**?**] I §10, no. 1, prop.2 and [**?**] III §4, no. 1, prop.2 (c)) and since $\pi_{\mathfrak{t}}$ is surjective, it follows that $\bar{\iota}$ is proper ([**?**] I §10, no.1 prop.5), and hence is a homeomorphism. □

Proposition 7.1.38. *Let G be a connected compact Lie group and T a maximal torus of G. Let H be a closed subgroup of G containing T.*

(a) *Denote by $W_H(T)$ the subgroup $N_H(T)/T$ of $W_G(T)$. Then the group H/H_0 is isomorphic to the quotient group $W_H(T)/W_{H_0}(T)$.*

(b) *H is connected if and only if every element of $W_G(T)$ that has a representative in H belongs to $W_{H_0}(T)$.*

Proof. Assertion (a) follows from Corollary 7.1.25, and (b) is a particular case of (a). □

7.2 Compact real forms

If $\mathfrak{g}$ is a complex Lie algebra, we denote by $\mathfrak{g}_{[\mathbb{R}]}$ (or sometimes by $\mathfrak{g}$) the real Lie algebra obtained by restriction of scalars. If $\mathfrak{k}$ is a real Lie algebra, we denote by $\mathfrak{k}_{(\mathbb{C})}$ (or sometimes by $\mathfrak{k}_{\mathbb{C}}$) the complex Lie algebra $\mathbb{C} \otimes_{\mathbb{R}} \mathfrak{k}$ obtained by extension of scalars. The homomorphisms of real Lie algebras $\mathfrak{k} \to \mathfrak{g}_{[\mathbb{R}]}$ correspond bijectively to the homomorphisms of complex Lie algebras $\mathfrak{k}_{(\mathbb{C})} \to \mathfrak{g}$: if $f : \mathfrak{k} \to \mathfrak{g}_{[\mathbb{R}]}$ and $g : \mathfrak{k}_{(\mathbb{C})} \to \mathfrak{g}$ correspond, we have

$$f(x) = g(1 \otimes x), \quad g(\lambda \otimes x) = \lambda f(x)$$

for $x \in \mathfrak{k}$, $\lambda \in \mathbb{C}$.

Let $\mathfrak{g}$ be a complex Lie algebra. A **real form** of $\mathfrak{g}$ is a real subalgebra $\mathfrak{k}$ of $\mathfrak{g}$ that is an $\mathbb{R}$-structure on the $\mathbb{C}$-vector space $\mathfrak{g}$. This means that the homomorphism of complex Lie algebras $\mathfrak{k}_{(\mathbb{C})} \to \mathfrak{g}$ associated to the canonical injection $\mathfrak{k} \to \mathfrak{g}_{[\mathbb{R}]}$ is bijective. Thus, a real subalgebra $\mathfrak{k}$ of $\mathfrak{g}$ is a real form of $\mathfrak{g}$ if and only if the subspaces $\mathfrak{k}$ and $i\mathfrak{k}$ of the real vector space $\mathfrak{g}$ are complementary. The **conjugation** of a relative to the real form $\mathfrak{k}$ is the map $\sigma : \mathfrak{g} \to \mathfrak{g}$ such that

$$\sigma(x + \mathrm{i}y) = x - \mathrm{i}y, \quad x, y \in \mathfrak{k}.$$

Proposition 7.2.1. *Let $\mathfrak{g}$ be a complex Lie algebra.*

(a) *Let $\mathfrak{k}$ be a real form of $\mathfrak{g}$ and σ the conjugation of $\mathfrak{g}$ relative to $\mathfrak{k}$. Then:*

$$\sigma^2 = \mathrm{id}_{\mathfrak{g}}, \quad \sigma(\lambda x + \mu y) = \bar{\lambda}\sigma(x) + \bar{\mu}\sigma(y), \quad \sigma([x, y]) = [\sigma(x), \sigma(y)] \tag{7.2.1}$$

for $\lambda, \mu \in \mathbb{C}$, $x, y \in \mathfrak{g}$. An element x of $\mathfrak{g}$ belongs to $\mathfrak{k}$ if and only if $\sigma(x) = x$.

(b) *Let $\sigma : \mathfrak{g} \to \mathfrak{g}$ be a map satisfying (7.2.1). Then the set $\mathfrak{k}$ of fixed points of σ is a real form of $\mathfrak{g}$, and σ is the conjugation of $\mathfrak{g}$ relative to $\mathfrak{k}$.*

Proof. Recall that $\mathfrak{k}$ is a real Lie algera, so (a) follows from the definition of σ. For (b), it follows from (7.2.1) that the set $\mathfrak{k}$ is a Lie subalgebra of $\mathfrak{g}$. Moreover, for $x \in \mathfrak{g}$, we have

$$x = \frac{x + \sigma(x)}{2} + \mathrm{i}\frac{x - \sigma(x)}{2\mathrm{i}}$$

and $\frac{x+\sigma(x)}{2}$, $\frac{x-\sigma(x)}{2\mathrm{i}}$ are elements of $\mathfrak{k}$. □

Note that if κ denotes the Killing form of $\mathfrak{g}$, and if $\mathfrak{k}$ is a real form of $\mathfrak{g}$, the restriction of κ to $\mathfrak{k}$ is the Killing form of $\mathfrak{k}$; in particular, κ is real-valued on $\mathfrak{k} \times \mathfrak{k}$. Assume that $\mathfrak{g}$ is reductive; then the real Lie algebra $\mathfrak{k}$ is compact if and only if the restriction of κ to $\mathfrak{k}$ is negative. In that case we say that $\mathfrak{k}$ is a **compact real form** of $\mathfrak{g}$.

7.2.1 The real form associated with a Chevalley system

We now associate a real form for a Chevalley system. Consider a split semi-simple Lie algebra $(\mathfrak{g}, \mathfrak{h})$ over $\mathbb{C}$, with root system $\Phi(\mathfrak{g}, \mathfrak{h}) = \Phi$, and a Chevalley system $(x_\alpha)_{\alpha\in\Phi}$ of $(\mathfrak{g}, \mathfrak{h})$. Recall that the linear map $\theta : \mathfrak{g} \to \mathfrak{g}$ that coincides with $-\mathrm{id}_{\mathfrak{h}}$ on $\mathfrak{h}$ and maps x_α to $x_{-\alpha}$ for all $\alpha \in \Phi$ is an automorphism of $\mathfrak{g}$. Moreover, if $\alpha, \beta, \alpha + \beta$ are roots, then

$$[x_\alpha, x_\beta] = N_{\alpha,\beta} x_{\alpha+\beta} \tag{7.2.2}$$

with $N_{\alpha,\beta} \in \mathbb{R}^*$ and

$$N_{-\alpha,-\beta} = -N_{\alpha,\beta} \tag{7.2.3}$$

Denote by $\mathfrak{h}_0$ the real vector subspace of $\mathfrak{h}$ consisting of the $h \in \mathfrak{h}$ such that $\alpha(h) \in \Phi$ for all $\alpha \in \Phi$. Then $\mathfrak{h}_0$ is an $\mathbb{R}$-structure on the complex vector space $\mathfrak{h}$, we have $[x_\alpha, x_{-\alpha}] \in \mathfrak{h}_0$ for all $\alpha \in \Phi$ and the restriction of the Killing form κ of $\mathfrak{g}$ to $\mathfrak{h}_0$ is separating positive. Moreover, we have: (Proposition 6.2.34)

$$\kappa(h, x_\alpha) = 0, \quad \kappa(x_\alpha, x_\beta) = 0 \ \text{ if } \alpha + \beta \neq 0, \quad \kappa(x_\alpha, x_{-\alpha}) < 0. \tag{7.2.4}$$

Proposition 7.2.2. *Let $(x_\alpha)_{\alpha\in\Phi}$ be a Chevalley system for $(\mathfrak{g}, \mathfrak{h})$.*

(a) *The real vector subspace $\mathfrak{g}_0 = \mathfrak{h}_0 + \sum_{\alpha\in\Phi} \mathbb{R}x_\alpha$ of $\mathfrak{g}$ is a real form of $\mathfrak{g}$, of which $\mathfrak{h}_0$ is a Cartan subalgebra. The pair $(\mathfrak{g}_0, \mathfrak{h}_0)$ is a split semi-simple real Lie algebra, of which (x_α) is a Chevalley system.*

(b) *Let σ be the conjugation of $\mathfrak{g}$ relative to $\mathfrak{g}_0$. Then $\sigma \circ \theta = \theta \circ \sigma$. The set of fixed points of $\sigma \circ \theta$ is a compact real form $\mathfrak{g}_u$ of $\mathfrak{g}$, of which $\mathrm{i}\mathfrak{h}_0$ is a Cartan subalgebra.*

Proof. Assertion (a) follows immediately from the preceding. We prove (b). Since $\sigma \circ \theta$ and $\theta \circ \sigma$ are two semi-linear maps from $\mathfrak{g}$ to $\mathfrak{g}$ that coincide on $\mathfrak{g}_0$, they coincide. Now $\sigma \circ \theta$ satisfies conditions (7.2.1), hence is the conjugation of a relative to the real form $\mathfrak{g}_u$ consisting of the $x \in \mathfrak{g}$ such that $\sigma \circ \theta(x) = x$. For all $\alpha \in \Phi$ put

$$u_\alpha = x_\alpha - x_{-\alpha}, \quad v_\alpha = \mathrm{i}(x_\alpha + x_{-\alpha}) \tag{7.2.5}$$

Then the $\mathbb{R}$-vector space $\mathfrak{g}_u$ is generated by $\mathrm{i}\mathfrak{h}_0$, the u_α and the v_α. More precisely, if we choose a chamber C of Φ, then

$$\mathfrak{g}_u = \mathrm{i}\mathfrak{h}_0 \oplus \bigoplus_{\alpha\in\Phi^+(C)} (\mathbb{R}u_\alpha + \mathbb{R}v_\alpha). \tag{7.2.6}$$

It is clear that $\mathrm{i}\mathfrak{h}_0$ is a Cartan subalgebra of $\mathfrak{g}_u$, and it remains to prove that the restriction of Δ to $\mathfrak{g}_u$ is negative. Now $\mathrm{i}\mathfrak{h}_0$ and the different subspaces of the form $\mathbb{R}u_\alpha \oplus \mathbb{R}v_\alpha$ are orthogonal with respect to Δ, by (7.2.4); the restriction of Δ to $\mathrm{i}\mathfrak{h}_0$ is negative and

$$\kappa(u_\alpha, u_\alpha) = \kappa(v_\alpha, v_\alpha) = -2\kappa(x_\alpha, x_{-\alpha}) < 0, \quad \kappa(u_\alpha, v_\alpha) = 0, \tag{7.2.7}$$

whence the proposition. □

Remark 7.2.3. With the preceding notations, we have the following formulas:

$$\begin{aligned}
[h, u_\alpha] &= \mathrm{i}\alpha(h)v_\alpha \\
[h, v_\alpha] &= \mathrm{i}\alpha(h)u_\alpha \\
[u_\alpha, v_\alpha] &= 2\mathrm{i}h_\alpha \\
[u_\alpha, u_\beta] &= N_{\alpha,\beta}u_{\alpha+\beta} + N_{\alpha,-\beta}u_{\alpha-\beta} \quad \alpha \neq \pm\beta \\
[v_\alpha, v_\beta] &= -N_{\alpha,\beta}u_{\alpha+\beta} - N_{\alpha,-\beta}u_{\alpha-\beta} \quad \alpha \neq \pm\beta \\
[u_\alpha, v_\beta] &= N_{\alpha,\beta}v_{\alpha+\beta} + N_{\alpha,-\beta}v_{\alpha-\beta} \quad \alpha \neq \pm\beta
\end{aligned} \tag{7.2.8}$$

(in the last three formulas, it is understood, as usual, that $N_{\alpha,\beta} = 0$ if $\alpha+\beta$ is not a root). Note that $\sum \mathbb{R}u_\alpha$ is a real subalgebra of $\mathfrak{g}$, in view of (7.2.8), which equals to $\mathfrak{g}_0 \cap \mathrm{i}\mathfrak{g}_u$.

Let $\Lambda_r(\Phi)$ be the group of radical weights of Φ. Recall that to any homomorphism $f : \Lambda_r(\Phi) \to \mathbb{C}^\times$ is associated an elementary automorphism $\varsigma(f)$ of $\mathfrak{g}$ such that $\varsigma(f)(h) = h$ for all $h \in \mathfrak{h}$ and $\varsigma(f)x_\alpha = f(\alpha)x_\alpha$.

Proposition 7.2.4. *Let $\mathfrak{k}$ be a compact real form of $\mathfrak{g}$ such that $\mathfrak{k} \cap \mathfrak{h} = \mathrm{i}\mathfrak{h}_0$. Then there exists a homomorphism $f : \Lambda_r(\Phi) \to \mathbb{R}_+^\times$ such that $\mathfrak{k} = \varsigma(f)(\mathfrak{g}_u)$.*

Proof. Let τ be the conjugation of $\mathfrak{k}$ relative to $\mathfrak{g}$. By hypothesis $\tau(x) = x$ for $x \in \mathrm{i}\mathfrak{h}_0$, so $\tau(x) = -x$ for $x \in \mathfrak{h}_0$. Thus, for all $\alpha \in \Phi$ and all $h \in \mathfrak{h}_0$,

$$[h, \tau(x_\alpha)] = [-\tau(h), \tau(x_\alpha)] = -\tau([h, x_\alpha]) = -\tau(\alpha(h)x_\alpha);$$

it follows that $[h, \tau(x_\alpha)] = -\alpha(h)\tau(x_\alpha)$ for all $h \in \mathfrak{h}_0$, hence also for all $h \in \mathfrak{h}$. Hence there exists $c_\alpha \in \mathbb{C}^\times$ such that $\tau(x_\alpha) = c_\alpha x_{-\alpha}$. Since $[x_\alpha, x_{-\alpha}] \in \mathfrak{h}_0$, we have $[\tau(x_\alpha), \tau(x_{-\alpha})] = -[x_\alpha, x_{-\alpha}]$, so $c_\alpha c_{-\alpha} = 1$; similarly, formulas (7.2.2) and (7.2.3) give $c_{\alpha+\beta} = c_\alpha c_\beta$ if $\alpha, \beta, \alpha+\beta$ are roots. By Lemma 6.2.80, there exists a homomorphism $g : \Lambda_r(\Phi) \to \mathbb{C}^\times$ such that $g(\alpha) = c_\alpha$ for all $\alpha \in \Phi$.

We now show that each c_α is strictly positive. Indeed, $c_\alpha\kappa(x_\alpha, x_{-\alpha}) = \kappa(x_\alpha, \tau(x_\alpha))$, and since $\kappa(x_\alpha, x_{-\alpha})$ is negative, it suffices to show that $\kappa(z, \tau(z)) < 0$ for every non-zero element z of $\mathfrak{g}$; but every element of $\mathfrak{g}$ can be written as $x + \mathrm{i}y$, with x and y in $\mathfrak{k}$, and

$$\kappa(x + \mathrm{i}y, \tau(x + \mathrm{i}y)) = \kappa(x + \mathrm{i}y, x - \mathrm{i}y) = \kappa(x, x) + \kappa(y, y) < 0$$

hence the stated assertion, the restriction of κ to $\mathfrak{g}$ being negative and separating by hypothesis.

It follows that the homomorphism g takes values in $\mathbb{R}_+^\times$; hence there exists a homomorphism $f : \Lambda_r(\Phi) \to \mathbb{R}_+^\times$ such that $g = f^{-2}$. Then $\varsigma(f)^{-1}(\mathfrak{k})$ is a real form of $\mathfrak{g}$; the corresponding conjugation is $\tilde{\tau} = \varsigma(f)^{-1} \circ \tau \circ \varsigma(f)$. For all $\alpha \in \Phi$, we have

$$\tilde{\tau}(x_\alpha) = \varsigma(f)^{-1}(\tau(c_\alpha^{-1/2}x_\alpha)) = \varsigma(f)^{-1}(c_\alpha^{1/2}x_{-\alpha}) = x_{-\alpha}$$

and $\tilde{\tau} = \tau(h) = h$ for $h \in \mathrm{i}\mathfrak{h}_0$; it follows that $\tilde{\tau}$ is the conjugation with respect to $\mathfrak{g}_u$, and hence that $\varsigma(f)^{-1}(\mathfrak{k}) = \mathfrak{g}_u$. □

Theorem 7.2.5. *Let $\mathfrak{g}$ be a complex semi-simple Lie algebra.*

(a) $\mathfrak{g}$ has compact (resp. splittable) real forms.

(b) The group $\mathrm{Inn}(\mathfrak{g})$ operates transitively on the set of compact (resp. splittable) real forms of $\mathfrak{g}$.

Proof. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. Then $(\mathfrak{g}, \mathfrak{h})$ is split, and has a Chevalley system (x_α). Assertion (a) now follows from Proposition 7.2.2. Let $\mathfrak{k}$ be a compact real form of $\mathfrak{g}$; we show that there exists $s \in \mathrm{Inn}(\mathfrak{g})$ such that $s(\mathfrak{g}_u) = \mathfrak{g}$. Let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{k}$; then $\mathfrak{t}_{(\mathbb{C})}$ is a Cartan subalgebra of $\mathfrak{g}$; since $\mathrm{Inn}(\mathfrak{g})$ operates transitively on the set of Cartan subalgebras of $\mathfrak{g}$ (Theorem 6.1.61), we are reduced to the case in which $\mathfrak{t}_{(\mathbb{C})} = \mathfrak{h}$. Since $\mathfrak{k}$ is a compact form, the eigenvalues of the endomorphisms $\mathrm{ad}(h)$, for $h \in \mathfrak{t}$, are purely imaginary (Proposition 7.1.1), so the roots $\alpha \in \Phi$ map $\mathfrak{t}$ to $i\mathbb{R}$; this implies that $\mathfrak{t} = \mathrm{i}\mathfrak{h}_0$. Then, by Proposition 7.2.4, there exists $s \in \mathrm{Inn}(\mathfrak{g})$ such that $s(\mathfrak{g}_u) = \mathfrak{k}$, hence (b) in the case of compact forms.

Finally, let $\mathfrak{k}_1$ and $\mathfrak{k}_2$ be two splittable real forms of $\mathfrak{g}$. There exist framings $(\mathfrak{k}_1, \mathfrak{h}_1, \Delta_1, (x_\alpha^1))$ and $(\mathfrak{k}_2, \mathfrak{h}_2, \Delta_2, (x_\alpha^2))$. These extend in an obvious way to bases e_1 and e_2 of $\mathfrak{g}$. An automorphism of $\mathfrak{g}$ that maps e_1 to e_2 maps $\mathfrak{k}_1$ to $\mathfrak{k}_2$; thus, it suffices to apply Proposition 6.2.84 to obtain the existence of an element u of $\mathrm{Aut}_0(\mathfrak{g}) = \mathrm{Inn}(\mathfrak{g})$ such that $u(\mathfrak{k}_1) = \mathfrak{k}_2$. □

Corollary 7.2.6. *Let $\mathfrak{k}$ and $\tilde{\mathfrak{k}}$ be two compact real Lie algebras. Then $\mathfrak{k}$ and $\tilde{\mathfrak{k}}$ are isomorphic if and only if the complex Lie algebras $\mathfrak{k}_{(\mathbb{C})}$ and $\tilde{\mathfrak{k}}_{(\mathbb{C})}$ are isomorphic.*

Proof. The condition is clearly necessary. Conversely, assume that $\mathfrak{k}_{(\mathbb{C})}$ and $\tilde{\mathfrak{k}}_{(\mathbb{C})}$ are isomorphic. Let $\mathfrak{z}$ (resp. $\tilde{\mathfrak{z}}$) be the centre of $\mathfrak{k}$ (resp. $\tilde{\mathfrak{k}}$) and $\mathfrak{s}$ (resp. $\tilde{\mathfrak{s}}$) the derived algebra of $\mathfrak{k}$ (resp. $\tilde{\mathfrak{k}}$). Then $\mathfrak{z}_{(\mathbb{C})}$ and $\mathfrak{z}_{(\mathbb{C})}$ are the centres of $\mathfrak{k}_{(\mathbb{C})}$ and $\tilde{\mathfrak{k}}_{(\mathbb{C})}$, respectively, and hence are isomorphic; it follows that the commutative algebras $\mathfrak{z}$ and $\tilde{\mathfrak{z}}$ are isomorphic. Similarly, $\mathfrak{s}_{(\mathbb{C})}$ and $\tilde{\mathfrak{s}}_{(\mathbb{C})}$ are isomorphic, hence $\mathfrak{s}$ and $\tilde{\mathfrak{s}}$, which are compact real forms of two isomorphic complex semi-simple Lie algebras, are isomorphic by Theorem 7.2.5. □

Corollary 7.2.7. *Let $\mathfrak{g}$ be a complex Lie algebra. The following conditions are equivalent:*

(a) *$\mathfrak{g}$ is reductive;*

(b) *there exists a compact real Lie algebra $\mathfrak{k}$ such that $\mathfrak{g}$ is isomorphic to $\mathfrak{k}_{(\mathbb{C})}$;*

(c) *there exists a compact Lie group G such that $\mathfrak{g}$ is isomorphic to $\mathfrak{Lie}(G)_{(\mathbb{C})}$.*

Proof. By definition, conditions (ii) and (iii) are equivalent and imply (i). If $\mathfrak{g}$ is reductive, it is the direct product of $\mathfrak{g}$ commutative algebra, which clearly has a compact real form, and a semi-simple algebra which has one by Theorem 7.2.5, hence (i) implies (ii). □

Corollary 7.2.8. *Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be two complex semi-simple Lie algebras. Then the compact real forms of $\mathfrak{g}_1 \times \mathfrak{g}_2$ are the products $\mathfrak{t}_1 \times \mathfrak{t}_2$, where $\mathfrak{k}_i$ is a compact real form of $\mathfrak{g}_i$.*

Proof. Indeed, there exists a compact real form $\mathfrak{k}_1$ (resp. $\mathfrak{k}_2$) of $\mathfrak{g}_1$ (resp. $\mathfrak{g}_2$); then $\mathfrak{k}_1 \times \mathfrak{k}_2$ is a compact real form of $\mathfrak{g}_1 \times \mathfrak{g}_2$. The corollary now follows from Theorem 7.2.5(b), applied to $\mathfrak{g}_1$, $\mathfrak{g}_2$ and $\mathfrak{g}_1 \times \mathfrak{g}_2$. □

Note that it follows from Corollary 7.2.8 above that a compact real Lie algebra $\mathfrak{k}$ is simple if and only if the complex Lie algebra $\mathfrak{k}_{(\mathbb{C})}$ is simple. We say that $\mathfrak{k}$ is of type A_n, or B_n, ..., if $\mathfrak{k}_{(\mathbb{C})}$ is of type A_n, or B_n, ... By Corollary 7.2.6 above, two compact simple real Lie algebras are isomorphic if and only if they are of the same type.

Let G be an almost simple connected compact Lie group. We say that G is of type A_n, or B_n, ..., if its Lie algebra is of type A_n, or B_n, ... Two simply-connected almost simple compact Lie groups are isomorphic if and only if they are of the same type.

7.2.2 Examples of complex Lie algebras

7.2.2.1 Compact Lie algebras of type A_n Let V be a finite dimensional complex vector space and β a separating positive Hermitian form on V. The unitary group associated to β is the subgroup $\mathrm{U}(\beta)$ of $\mathrm{GL}(V)$ consisting of the automorphisms of the complex Hilbert space (V, β); this is a (real) Lie subgroup of the group $\mathrm{GL}(V)$, whose Lie algebra is the subalgebra $\mathfrak{u}(\beta)$ of the real Lie algebra $\mathfrak{gl}(V)$ consisting of the endomorphisms x of V such that $x^* = -x$, where x^* denotes the adjoint of x relative to β. Since the group $\mathrm{U}(\beta)$ is compact, $\mathfrak{u}(\beta)$ is a compact real Lie algebra. Similarly, the special unitary group $\mathrm{SU}(\beta) = \mathrm{U}(\beta) \cap \mathrm{SL}(V)$ is a compact Lie subgroup of $\mathrm{SL}(V)$, whose Lie algebra is $\mathfrak{su}(\beta) = \mathfrak{u}(\beta) \cap \mathfrak{sl}(V)$.

When $V = \mathbb{C}^n$ and β is the usual Hermitian form (for which the canonical basis of $\mathbb{C}^n$ is orthonormal), we write $\mathrm{U}(n, \mathbb{C})$, $\mathrm{SU}(n, \mathbb{C})$, $\mathfrak{u}(n, \mathbb{C})$, $\mathfrak{su}(n, \mathbb{C})$ instead of $\mathrm{U}(\beta)$, $\mathrm{SU}(\beta)$, $\mathfrak{u}(\beta)$, $\mathfrak{su}(\beta)$. The elements of $\mathrm{U}(n, \mathbb{C})$ (resp. $\mathfrak{u}(n, \mathbb{C})$) are the matrices $A \in \mathcal{M}_n(\mathbb{C})$ such that $A^*A = I_n$ (resp. $A^* + A = 0$), which are said to be unitary (resp. anti-Hermitian).

Proposition 7.2.9. *Let V be a complex vector space of finite dimension.*

(a) *The compact real forms of the complex Lie algebra $\mathfrak{sl}(V)$ are the algebras $\mathfrak{su}(\beta)$, where β belongs to the set of separating positive Hermitian forms on the complex vector space V.*

(b) *The algebras $\mathfrak{u}(\beta)$ are the compact real forms of $\mathfrak{gl}(V)$.*

Proof. Let β be a separating positive hermitian form on V. For all $x \in \mathfrak{gl}(V)$, put $\sigma(x) = -x^*$ (where x^* is the adjoint of x relative to Φ). Then σ satisfies conditions (7.2.1), so the set $\mathfrak{u}(\beta)$ (resp. $\mathfrak{su}(\beta)$) of fixed points of σ on $\mathfrak{gl}(V)$ (resp. $\mathfrak{sl}(V)$) is a compact real form of $\mathfrak{gl}(V)$ (resp. $\mathfrak{sl}(V)$). Since $\mathrm{GL}(V)$ operates transitively on the set of separating positive hermitian forms on V and on the set of compact real forms of $\mathfrak{sl}(V)$ (Theorem 7.2.5), the claim is proved. □

Corollary 7.2.10. *Every compact simple real Lie algebra of type A_n ($n \geq 1$) is isomorphic to $\mathfrak{su}(n+1, \mathbb{C})$.*

Proof. Indeed, every complex Lie algebra of type A_n is isomorphic to $\mathfrak{sl}(n+1, C)$. ☐

Remark 7.2.11. We have $\mathfrak{gl}(V) = \mathfrak{sl}(V) \times \mathbb{C} \cdot 1$, $\mathfrak{u}(\beta) = \mathfrak{su}(\beta) \times \mathbb{R} \cdot 1$; the compact real forms of $\mathfrak{gl}(V)$ are the $\mathfrak{su}(\beta) \times \mathbb{R} \cdot \alpha$, $\alpha \in \mathbb{C}^*$. If the complex Lie algebra $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$ is equipped with the usual splitting Cartan subalgebra and Chevalley system, then

$$\mathfrak{g}_u = \mathfrak{su}(n, \mathbb{C}), \quad \mathfrak{g}_0 = \mathfrak{sl}(n, \mathbb{R}), \quad \mathfrak{g}_0 \cap \mathfrak{g}_u = \mathfrak{o}(n, \mathbb{R}).$$

7.2.2.2 Compact Lie algebras of type B_n and D_n Let V be a finite dimensional real vector space and β a separating positive symmetric bilinear form on V. The orthogonal group associated to β is the subgroup $\mathrm{O}(\beta)$ of $\mathrm{GL}(V)$ consisting of the automorphisms of the Euclidean space (V, β); this is a Lie subgroup of $\mathrm{GL}(V)$, whose Lie algebra is the subalgebra $\mathfrak{o}(\beta)$ of $\mathfrak{gl}(V)$ consisting of the endomorphisms x of V such that $x^* = -x$, x^* denoting the adjoint of x relative to β. Since the group $\mathrm{O}(\beta)$ is compact, $\mathfrak{o}(\beta)$ is thus a compact real Lie algebra. Put $\mathrm{SO}(\beta) = \mathrm{O}(\beta) \cap \mathrm{SL}(V)$; this is a closed subgroup of finite index of $\mathrm{O}(\beta)$ (of index 2 if $\dim(V) \neq 0$), hence also with Lie algebra $\mathfrak{o}(\beta)$.

When $V = \mathbb{R}^n$ and β is the usual quadratic form (for which the canonical basis of $\mathbb{R}^n$ is orthonormal), we write $\mathrm{O}(n, \mathbb{R})$, $\mathrm{SO}(n, \mathbb{R})$, $\mathfrak{o}(n, \mathbb{R})$ instead of $\mathrm{O}(\beta)$, $\mathrm{SO}(\beta)$, $\mathfrak{o}(\beta)$. The elements of $\mathrm{O}(n, \mathbb{R})$ (resp. $\mathfrak{o}(n, \mathbb{R})$) are the matrices $A \in \mathcal{M}_n(\mathbb{R})$ such that $A^T A = I_n$ (resp. $A + A^T = 0$), which are said to be orthogonal (resp. anti-symmetric).

Let $V_{(\mathbb{C})}$ be the complex vector space associated to V and let $\beta_{(\mathbb{C})}$ be the bilinear form on $V_{(\mathbb{C})}$ associated to β. Identify $\mathfrak{gl}(V)_{(\mathbb{C})}$ with $\mathfrak{gl}(V_{(\mathbb{C})})$; then $\mathfrak{o}(\beta)_{(\mathbb{C})}$ is identified with $\mathfrak{o}(\beta_{(\mathbb{C})})$: this is clear since the map $x \mapsto x^* + x$ from $\mathfrak{gl}(V_{(\mathbb{C})})$ to itself is $\mathbb{C}$-linear. Since $\mathfrak{o}(\beta_{(\mathbb{C})})$ is of type B_n if $\dim(V) = 2n+1$, $n \geq 1$, and of type D_n if $\dim(V) = 2n$, $n \geq 3$, we deduce:

Proposition 7.2.12. *Every compact simple real Lie algebra of type B_n, $n \geq 1$ (resp. of type D_n, $n \geq 3$) is isomorphic to $\mathfrak{o}(2n+1, \mathbb{R})$ (resp. $\mathfrak{o}(2n, \mathbb{R})$).*

7.2.2.3 Compact groups of rank 1 Recall that the topological group $\mathrm{SU}(2, \mathbb{C})$ is isomorphic to the topological group S^3 of quaternions of norm 1, and the quotient of $\mathrm{SU}(2, \mathbb{C})$ by the subgroup $Z = \{I_2, -I_2\}$ is isomorphic to the topological group $\mathrm{SO}(3, \mathbb{R})$. Note that Z is the centre of $\mathrm{SU}(2, \mathbb{C})$: indeed, since $\mathbb{H} = \mathbb{R} \cdot S^3$, every element of the centre of the group S^3 is in the centre $\mathbb{R}$ of the algebra $\mathbb{H}$ and hence belongs to the group with two elements $S^3 \cap \mathbb{R} = \{I_2, -I_2\}$.

Proposition 7.2.13. *Every compact semi-simple real Lie algebra of rank 1 is isomorphic to $\mathfrak{su}(2, \mathbb{C})$ and to $\mathfrak{o}(3, \mathbb{R})$. Every connected semi-simple compact Lie group of rank 1 is isomorphic to $\mathrm{SU}(2, \mathbb{C})$ if it is simply-connected, and to $\mathrm{SO}(3, \mathbb{R})$ if not.*

Proof. The first assertion follows from Corollary 7.2.10 and Proposition 7.2.12. Now since $\mathrm{SU}(2, \mathbb{C})$ is homeomorphic to S^3, hence is simply-connected, every simply-connected compact semi-simple Lie group of rank 1 is isomorphic to $\mathrm{SU}(2, \mathbb{C})$; every connected compact semi-simple Lie group of rank 1 that is not simply-connected is isomorphic to a quotient of $\mathrm{SU}(2, \mathbb{C})$ by a subgroup of Z that does not reduce to the identity element, hence to $\mathrm{SO}(3, \mathbb{R})$. ☐

Recal that the canonical basis of $\mathfrak{sl}(2, \mathbb{C})$ is the basis (e, h, f), where

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

We thus obtain a basis (u, ih, v) of $\mathfrak{su}(2, \mathbb{C})$, also called canonical, by putting

$$u = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad ih = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad v = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

(note that by dividing by i, these matrices correspond to Pauli matrices occurs in quantum mechanics) We have

$$[ih, u] = 2v, \quad [ih, v] = -2u, \quad [u, v] = 2ih. \tag{7.2.9}$$

If κ denotes the Killing form of $\mathfrak{su}(2,\mathbb{C})$, an immediate calculation gives

$$\kappa(x_1u + x_2v + x_3ih, y_1u + y_2v + y_3ih) = -8\sum_{i=1}^{3} x_iy_i \tag{7.2.10}$$

so that, if we identify $\mathfrak{su}(2,\mathbb{C})$ with $\mathbb{R}^3$ by means of the canonical basis, the adjoint representation of $\mathrm{SU}(2,\mathbb{C})$ defines a homomorphism $\mathrm{SU}(2,\mathbb{C}) \to \mathrm{SO}(3,\mathbb{R})$.

Further, note that $\mathbb{R}ih$ is a Cartan subalgebra of $\mathfrak{su}(2,\mathbb{C})$, that the maximal torus T of $\mathrm{SU}(2,\mathbb{C})$ that corresponds to it consists of the diagonal matrices, and that the exponential map $\exp : \mathbb{R}ih \to T$ maps xh, for $x \in i\mathbb{R}$, to the matrix $\left(\begin{smallmatrix} e^x & 0 \\ 0 & e^{-x} \end{smallmatrix}\right)$, and thus has kernel $\mathbb{Z}\left(\begin{smallmatrix} 2\pi i & 0 \\ 0 & -2\pi i \end{smallmatrix}\right)$.

Recall that the element

$$\theta = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \mathrm{SU}(2,\mathbb{C})$$

induce an automorphim of the group $\mathfrak{sl}(2,\mathbb{K})$, and we have

$$\theta^2 = -I_2, \quad \mathrm{Inn}(\theta)t = t^{-1}, \quad t \in T. \tag{7.2.11}$$

Now for $t = \left(\begin{smallmatrix} z & 0 \\ 0 & \bar{z} \end{smallmatrix}\right)$, we have

$$\mathrm{Ad}(t)(e) = z^2e, \quad \mathrm{Ad}(t)(f) = z^{-2}f, \quad \mathrm{Ad}(t)(h) = h \tag{7.2.12}$$

$$\mathrm{Ad}(t)(u) = \mathrm{Re}(z^2)u + \mathrm{Im}(z^2)v, \quad \mathrm{Ad}(t)(v) = -\mathrm{Im}(z^2)u + \mathrm{Re}(z^2)v. \tag{7.2.13}$$

7.3 Root system associated to a compact Lie group

In this section we shall construct a root system from a compact Lie group and consider its relation with the root system of the Lie algebra associated. Here quite a bit notations in the Lie algebra setting generalize to compact Lie groups, and analogous statements are valid. However, there are still subtle points for the transition of Lie algebras to Lie groups, as we will see in the discussion.

7.3.1 The character group and nodal group

Let G be a compact Lie group with Lie algebra $\mathfrak{g}$. Denote by $X(G)$ the (commutative) group of continuous homomorphisms from G to the topological group $\mathbb{C}^\times$. By **??**, the elements of $X(G)$ are morphisms of Lie groups. We identify the Lie algebra of $\mathbb{C}^\times$ with $\mathbb{C}$ in such a way that the exponential map of $\mathbb{C}^\times$ coincides with the map $z \mapsto e^z$ from $\mathbb{C}$ to $\mathbb{C}^\times$. Then, to any element $\chi \in X(G)$ is associated an element $\mathfrak{Lie}(\chi) \in \mathrm{Hom}_{\mathbb{R}}(\mathfrak{g},\mathbb{C})$; we denote by χ_* the element of $\mathrm{Hom}_{\mathbb{C}}(\mathfrak{g}_{(\mathbb{C})},\mathbb{C})$ that corresponds to it (that is, whose restriction to $\mathfrak{g} \subseteq \mathfrak{g}_{(\mathbb{C})}$ is equal to $\mathfrak{Lie}(\chi)$). For all $x \in \mathfrak{g}$ and all $\chi \in X(G)$, we have

$$\chi(\exp_G(x)) = e^{\chi_*(x)},$$

by functoriality of the exponential map.

We shall often denote the group $X(G)$ additively; in that case, we denote the element $\chi(g)$ of $\mathbb{C}^\times$ by g^χ. With this notation, we have the formulas

$$g^{\chi+\tau} = g^\chi g^\tau \quad \text{for} \quad g \in G, \chi, \tau \in X(G)$$

and

$$(\exp_G(x))^\chi = e^{\chi_*(x)} \quad \text{for} \quad x \in \mathfrak{g}, \chi \in X(G).$$

Since G is compact, the elements of $X(G)$ take values in the subgroup S^1 of complex numbers of absolute value 1, so that $X(G)$ can be identified with the group of continuous (or analytic) homomorphisms from G to S^1. It follows that, for all $\chi \in X(G)$, the map $\mathfrak{Lie}(\chi)$ takes values in the subspace $\mathbb{R}i$ of $\mathbb{C}$, so χ_* maps $\mathfrak{g}$ to $\mathbb{R}i$.

If G is commutative, $X(G)$ is simply the (discrete) dual group of G. If G is commutative and finite, $X(G)$ can be identified with the dual finite group $\mathrm{Hom}_{\mathbb{Z}}(G,\mathbb{Q}/\mathbb{Z})$ (where we identify $\mathbb{Q}/\mathbb{Z}$ with a subgroup of $\mathbb{C}^\times$ by the homomorphism $x \mapsto \exp(2\pi ix)$).

For any morphism $\varphi : G \to K$ of compact Lie groups, we denote by $X(\varphi)$ the homomorphism $\chi \mapsto \chi \circ \varphi$ from $X(K)$ to $X(G)$. If N is a closed normal subgroup of the compact Lie group G, we have an exact sequence of $\mathbb{Z}$-modules

$$0 \longrightarrow X(G/N) \longrightarrow X(G) \longrightarrow X(N)$$

Proposition 7.3.1. *For any compact Lie group G, the $\mathbb{Z}$-module $X(G)$ is finitely generated. It is free if G is connected.*

Proof. Assume first that G is connected; every element of $X(G)$ vanishes on the derived group $D(G)$ of G, hence we have an isomorphism $X(G/D(G)) \cong X(G)$. But $G/D(G)$ is connected and commutative, hence is a torus, and $X(G/D(G))$ is a free $\mathbb{Z}$-module of finite rank. In the general case, it follows from the exactness of the sequence

$$0 \longrightarrow X(G/G_0) \longrightarrow X(G) \longrightarrow X(G_0)$$

where $X(G_0)$ is free of finite rank and $X(G/G_0)$ is finite (since G/G_0 is finite), that $X(G)$ is of finite type. □

Proposition 7.3.2. *Let G be a commutative compact Lie group, and $(\chi_i)_{i\in I}$ a family of elements of $X(G)$. Then the χ_i generate $X(G)$ if and only if the intersection of their kernels reduces to the identity element.*

Proof. By **??**, the orthogonal complement of the kernel of χ_i is the subgroup M_i of $X(G)$ generated by χ_i; by Corollary **??**, the orthogonal complement of $\ker \chi_i$ is the subgroup of $X(G)$ generated by the M_i, hence the proposition. □

We now consider a torus T in G, with Lie algebra $\mathfrak{t}$. The nodal group of T, denoted by $\Gamma(T)$, is the kernel of the exponential map of T. This is a discrete subgroup of $\mathfrak{t}$, whose rank is equal to the dimension of T, and the $\mathbb{R}$-linear map $\mathbb{R} \otimes_{\mathbb{Z}} \Gamma(T) \to \mathfrak{t}$ that extends the canonical injection of $\Gamma(T)$ into $\mathfrak{t}$ is bijective. It induces by passage to the quotient an isomorphism $\mathbb{R}/\mathbb{Z} \otimes_{\mathbb{Z}} \Gamma(T) \to T$. For example, the nodal group $\Gamma(S^1)$ of S^1 is the subgroup $2\pi i\mathbb{Z}$ of $\mathfrak{Lie}(S^1) = i\mathbb{R}$.

For any morphism of tori $\varphi : T \to H$, denote by $\Gamma(\varphi)$ the homomorphism $\Gamma(T) \to \Gamma(H)$ induced by $\mathfrak{Lie}(\varphi)$. We have a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \Gamma(T) \arrow[r] \arrow[d, "\Gamma(\varphi)"] & \mathfrak{Lie}(T) \arrow[r, "\exp_T"] \arrow[d, "\mathfrak{Lie}(\varphi)"] & T \arrow[r] \arrow[d, "\varphi"] & 0 \\ 0 \arrow[r] & \Gamma(H) \arrow[r] & \mathfrak{Lie}(H) \arrow[r] & H \arrow[r] & 0 \end{tikzcd} \tag{7.3.1}$$

Let $\chi \in X(T)$; applying the preceding to the morphism from T to S^1 defined by χ, we see that the $\mathbb{C}$-linear map $\chi_* : \mathfrak{t}_{(\mathbb{C})} \to \mathbb{C}$ maps $\Gamma(T)$ to $2\pi i\mathbb{Z}$. Thus, we can define a $\mathbb{Z}$-bilinear form on $X(T) \times \Gamma(T)$ by putting

$$\langle \chi, x\rangle = \frac{1}{2\pi i}\chi_*(x), \quad \chi \in X(T), x \in \Gamma(T).$$

Proposition 7.3.3. *The bilinear form $(\chi, x) \mapsto \langle \chi, x\rangle$ on $X(T) \times \Gamma(T)$ is invertible.*

Proof. Recall that, by definition, this means that the linear maps $X(T) \to \mathrm{Hom}_{\mathbb{Z}}(\Gamma(T), \mathbb{Z})$ and $\Gamma(T) \to \mathrm{Hom}_{\mathbb{Z}}(X(T), \mathbb{Z})$ associated to this bilinear form are bijective. It is immediate that if the conclusion of the proposition is true for two tori, it is also true for their product. Thus, since every torus of dimension n is isomorphic to $(S^1)^n$, we are reduced to the case in which $T = S^1$. In this particular case, the assertion is immediate. □

Let $\varphi : T \to H$ be a morphism of tori. Then under the bilinear form $\langle \cdot\, , \cdot\rangle$, each of the linear maps $X(\varphi) : X(H) \to X(T)$ and $\Gamma(\varphi) : \Gamma(T) \to \Gamma(H)$ is the transpose of the other: for all $\chi \in X(H)$ and all $x \in \Gamma(T)$,

$$\langle X(\varphi)(\chi), x\rangle = \langle \chi, \Gamma(\varphi)(x)\rangle. \tag{7.3.2}$$

In fact, both sides are equal to $\mathfrak{Lie}(\chi \circ \varphi)(x)$.

Proposition 7.3.4. *Let T and H be tori. Denote by* $\mathrm{Hom}(T, H)$ *the group of morphisms of Lie groups from T to H. Then the maps $\varphi \mapsto X(\varphi)$ and $\varphi \mapsto \Gamma(\varphi)$ are isomorphisms of groups from* $\mathrm{Hom}(T, H)$ *to* $\mathrm{Hom}_{\mathbb{Z}}(X(H), X(T))$ *and to* $\mathrm{Hom}_{\mathbb{Z}}(\Gamma(T), \Gamma(H))$*, respectively.*

Proof. If φ is a morphism of Lie groups from T to H, the homomorphism $X(\varphi)$ is simply the dual of φ. The map from $\mathrm{Hom}_{\mathbb{Z}}(X(H), X(T))$ to $\mathrm{Hom}(T, H)$ defined in (**??**) is the inverse of the map $\varphi \mapsto X(\varphi)$; the latter is thus bijective. If we identify $\Gamma(T)$ (resp. $\Gamma(H)$) with the dual $\mathbb{Z}$-module of $X(T)$ (resp. $X(H)$) (Proposition 7.3.3), $\Gamma(\varphi)$ coincides with the transpose of the homomorphism $X(\varphi)$, hence the proposition. □

Example 7.3.5. By Proposition 7.3.4, the map $\varphi \mapsto \Gamma(\varphi)(2\pi i)$ from $\mathrm{Hom}(S^1, T)$ to $\Gamma(T)$ is an isomorphism; if $\chi \in X(T) = \mathrm{Hom}(T, S^1)$ and $\varphi \in \mathrm{Hom}(S^1, T)$, then the composite $\chi \circ \varphi \in \mathrm{Hom}(S^1, S^1)$ is the endomorphism $z \mapsto z^n$, where $n = \langle \chi, \Gamma(\varphi)(2\pi i)\rangle$. We shall identify $\mathrm{Hom}(S^1, S^1) = X(S^1)$ with $\mathbb{Z}$ from now on, the element n of $\mathbb{Z}$ being associated to the endomorphism $z \mapsto z^n$; thus, with the notations above,

$$\chi \circ \varphi = \langle \chi, \Gamma(\varphi)(2\pi i)\rangle.$$

Remark 7.3.6. Let $\varphi : T \to H$ be a morphism of tori. The snake diagram associated to (7.3.1) gives an exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \ker\Gamma(\varphi) \arrow[r] & \ker\mathfrak{Lie}(\varphi) \arrow[r] & \ker\varphi \arrow[dll, "\delta"] \\ & \operatorname{coker}\Gamma(\varphi) \arrow[r] & \operatorname{coker}\mathfrak{Lie}(\varphi) \arrow[r] & \operatorname{coker}\varphi \arrow[r] & 0 \end{tikzcd} \tag{7.3.3}$$

In particular, assume that φ is surjective, with finite kernel Z, so that we have an exact sequence

$$0 \longrightarrow Z \xrightarrow{i} T \xrightarrow{\varphi} H \longrightarrow 0$$

where i is the canonical injection. Then, $\mathfrak{Lie}(\varphi)$ is bijective, and (7.3.3) gives an isomorphism $Z \cong \operatorname{coker}\Gamma(\varphi)$, hence an exact sequence

$$0 \longrightarrow \Gamma(T) \xrightarrow{\Gamma(\varphi)} \Gamma(H) \longrightarrow Z \longrightarrow 0 \tag{7.3.4}$$

Moreover, by **??**, the sequence

$$0 \longrightarrow X(H) \xrightarrow{\hat{\varphi}} X(T) \xrightarrow{\hat{i}} X(Z) \longrightarrow 0 \tag{7.3.5}$$

is exact.

Remark 7.3.7. To the exact sequence $0 \to \Gamma(T) \to \mathfrak{Lie}(T) \to T \to 0$ is associated an isomorphism from $\Gamma(T)$ to the fundamental group of T, called **canonical** in the sequel. For any morphism of tori $\varphi : T \to H$, $\Gamma(\varphi)$ can then be identified via the canonical isomorphisms $\Gamma(T) \to \pi_1(T)$ and $\Gamma(H) \to \pi_1(H)$ with the homomorphism $\varphi^* : \pi_1(T) \to \pi_1(H)$ induced by φ. Note that this gives another interpretation of the exact sequence (7.3.4).

Remark 7.3.8. The homomorphisms of $\mathbb{Z}$-modules $(-)_* : X(T) \to \mathrm{Hom}_{\mathbb{C}}(\mathfrak{t}_{(\mathbb{C})}, \mathbb{C})$ and $\iota : \Gamma(T) \to \mathfrak{t}_{(\mathbb{C})}$ (ι is induced by the canonical injection of $\Gamma(T)$ into $\mathfrak{t}$) extend to isomorphisms of $\mathbb{C}$-vector spaces

$$(-)_* : \mathbb{C} \otimes X(T) \cong \mathrm{Hom}_{\mathbb{C}}(\mathfrak{t}_{(\mathbb{C})}, \mathbb{C}), \quad \mathbb{C} \otimes \Gamma(T) \cong \mathfrak{t}_{(\mathbb{C})}.$$

which we shall call canonical in the sequel. Note that, if we extend the pairing between $X(T)$ and $\Gamma(T)$ by $\mathbb{C}$-linearity to a bilinear form $\langle \cdot, \cdot \rangle$ on $(\mathbb{C} \otimes X(T) \times (\mathbb{C} \otimes \Gamma(T))$, then

$$\langle \chi, x\rangle = 2\pi i \chi_*(x).$$

7.3.2 Weight of a representation

Now let G be a connected compact Lie group and T a maximal torus of G. We denote by $\mathfrak{g}$ (resp. $\mathfrak{t}$) the Lie algebra of G (resp. T), by $\mathfrak{g}_{\mathbb{C}}$ (resp. $\mathfrak{t}_{\mathbb{C}}$) the complexified Lie algebra of $\mathfrak{g}$ (resp. $\mathfrak{t}$), and by W the Weyl group of G relative to T. Let V be a finite dimensional vector space over $\mathbb{K}$ (recall that $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$), and $\rho : G \to \mathrm{GL}(V)$ a continuous (hence smooth) representation of the connected compact Lie group G on V. Define a complex vector space $\tilde{V}$ and a continuous representation $\tilde{\rho} : G \to \mathrm{GL}(\tilde{V})$ as follows: if $\mathbb{K}=\mathbb{C}$ we set $\tilde{V}=V$ and $\tilde{\rho}=\rho$; if $\mathbb{K}=\mathbb{R}$, set $\tilde{V}=V_{(\mathbb{C})}$ and $\tilde{\rho}$ to be the composite of ρ with the canonical homomorphism $\mathrm{GL}(V)\to\mathrm{GL}(\tilde{V})$. For all $\lambda\in X(G)$, denote by $\tilde{V}_\lambda(G)$ the vector subspace of $\tilde{V}$ consisting of the $v\in\tilde{V}$ such that $\tilde{\rho}(g)v=g^\lambda v$ for all $g\in G$. By Proposition 6.1.3, the sum of the $\tilde{V}_\lambda(G)$ (for λ belonging to $X(G)$) is direct. Moreover:

Lemma 7.3.9. *If G is commutative, then $\tilde{V}=\bigoplus_{\lambda\in X(G)}\tilde{V}_\lambda(G)$.*

Proof. Recall that ρ is semi-simple since G is compact, so it suffices to prove the lemma in the case in which ρ is simple. In that case, the commutant Z of $\rho(G)$ in $\mathrm{End}(\tilde{V})$ reduces to homotheties (**??**); thus, the image of the homomorphism $\tilde{\rho}$ is contained in the subgroup $\mathbb{C}^\times\cdot 1$ of $\mathrm{GL}(\tilde{V})$, and there exists $\lambda\in X(G)$ such that $\tilde{V}=\tilde{V}_\lambda(G)$. □

Let T be a mximal torus of G. Then the elements $\lambda\in X(T)$ such that $\tilde{V}_\lambda(T)\neq 0$ are called the **weights** of the representation ρ of G. Denote by $\mathcal{P}(\rho,T)$, or by $\mathcal{P}(\rho)$ if there is no possibility of confusion over the choice of T, the set of weights of ρ relative to T. By Lemma 7.3.9,

$$\tilde{V}=\bigoplus_{\lambda\in\mathcal{P}(\rho,T)}\tilde{V}_\lambda(T).$$

Let H be another maximal torus of G and g an element of G such that $\mathrm{Inn}(g)(T)=H$. For all $\lambda\in X(T)$,

$$\tilde{\rho}(g)(\tilde{V}_\lambda(T))=\tilde{V}_\mu(H),\quad \mu=X(\mathrm{Inn}(g^{-1}))(\lambda). \tag{7.3.6}$$

Concequently

$$X(\mathrm{Inn}(g^{-1}))(\mathcal{P}(\rho,H))=\mathcal{P}(\rho,T) \tag{7.3.7}$$

TheWeyl group $W=W_G(T)$ operates on the left on the $\mathbb{Z}$-module $X(T)$ by $w\mapsto X(w^{-1})$; thus, for $t\in T$, $\lambda\in X(T)$, $w\in W$, we have $t^{w(\lambda)}=(w^{-1}(t))^\lambda$.

Proposition 7.3.10. *The set $\mathcal{P}(\rho,T)$ is stable under the operation of the Weyl group W. Let $n\in N_G(T)$ and let w be its class in W; for $\lambda\in X(T)$, we have*

$$\rho(n)(\tilde{V}_\lambda(T))=\tilde{V}_{w(\lambda)}(T),\quad \dim(\tilde{V}_{w(\lambda)}(T))=\dim(\tilde{V}_\lambda(T)).$$

Proof. Formula (7.3.7), with $H=T$, $g=n$, implies that $\mathcal{P}(\rho,T)$ is stable under w; further, $\tilde{\rho}(n)$ induces an isomorphism from $\tilde{V}_\lambda(T)$ to $\tilde{V}_{w(\lambda)}(T)$ by formula (7.3.6), hence the proposition. □

Proposition 7.3.11. *The homomorphism $\rho : G\to\mathrm{GL}(V)$ is injective if and only if $\mathcal{P}(\rho,T)$ generates the $\mathbb{Z}$-module $X(T)$.*

Proof. The homomorphism ρ is injective if and only if its restriction to T is injective (Corollary 7.1.20). Further, since the canonical homomorphism $\mathrm{GL}(V)\to\mathrm{GL}(\tilde{V})$ is injective, we can replace ρ by $\tilde{\rho}$. It then follows that the kernel of the restriction of ρ to T is the intersection of the kernels of the elements of $\mathcal{P}(\rho,T)$. Thus, the conclusion follows from Proposition 7.3.2. □

The linear representation $\mathfrak{Lie}(\rho)$ of $\mathfrak{t}$ in $\mathfrak{gl}(\tilde{V})$ extends to a homomorphism of $\mathbb{C}$-Lie algebras

$$\tilde{\mathfrak{Lie}}(\rho):\mathfrak{t}_{\mathbb{C}}\to\mathfrak{gl}(\tilde{V}).$$

Moreover, recall that we have associated to every element λ of $X(T)$ a linear form λ_* on $\mathfrak{t}_{\mathbb{C}}$ such that

$$(\exp_T(x))^\lambda=e^{\lambda_*(x)},\quad x\in\mathfrak{t}.$$

Recall finally that, for any map $\mu:\mathfrak{t}_{\mathbb{C}}\to\mathbb{C}$, we denote by $\tilde{V}_\mu(\mathfrak{t}_{\mathbb{C}})$ the vector subspace of $\tilde{V}$ consisting of the v such that $(\tilde{\mathfrak{Lie}}(\rho)(x))(v)=\mu(x)v$ for all $x\in\mathfrak{t}_{\mathbb{C}}$.

Proposition 7.3.12. *With the notations above, we have:*

(a) for all $\lambda \in X(T)$ we have $\tilde{V}_\lambda(T) = \tilde{V}_{\lambda_*}(\mathfrak{t}_\mathbb{C})$;

(b) the map $(-)_* : X(T) \to \mathrm{Hom}_\mathbb{C}(\mathfrak{t}_\mathbb{C}, \mathbb{C})$ induces a bijection from $\mathcal{P}(\rho, T)$ to the set of weights of $\mathfrak{t}_\mathbb{C}$ on $\tilde{V}$.

Proof. Note first that, if W operates on $\mathfrak{t}_\mathbb{C}$ by associating to any element w of W the endomorphism $\mathfrak{Lie}(w)_{(\mathbb{C})}$ of $\mathfrak{t}_\mathbb{C}$, the map $(-)_*$ is compatible with the operation of W on $X(T)$ and $\mathrm{Hom}_\mathbb{C}(\mathfrak{t}_\mathbb{C}, \mathbb{C})$. Therefore, $\tilde{V}_\lambda(T) \subseteq \tilde{V}_{\lambda_*}(\mathfrak{t}_\mathbb{C})$. Since $\tilde{V}$ is the direct sum of the $\tilde{V}_\lambda(T)$ and the $\tilde{V}_{\lambda_*}(\mathfrak{t}_\mathbb{C})$, we conclude (a), and (b) follows from (a). □

Assume now that $\mathbb{K} = \mathbb{R}$. Denote by σ the conjugation of $\tilde{V}$ relative to V, defined by $\sigma(x + iy) = x - iy$ for x, y in V; for every complex vector subspace E of $\tilde{V}$, the smallest subspace of $\tilde{V}$ rational over $\mathbb{R}$ and containing E is $E + \sigma(E)$. In particular, for all $\lambda \in X(T)$, there exists a real vector subspace $V(\lambda)$ of V such that the subspace $V(\lambda)_{(\mathbb{C})}$ of $\tilde{V}$ is $\tilde{V}_\lambda(T) + \tilde{V}_{-\lambda}(T)$ (note that $\sigma(\tilde{V}_\lambda(T)) = \tilde{V}_{-\lambda}(T)$, since λ has value in S^1 and $\bar{z} = z^{-1}$ for $z \in S^1$). We have $V(\lambda) = V(-\lambda)$, and the $V(\lambda)$ are the isotypical components of the representation of T on V induced by ρ.

We now define the **roots** of G relative to T to be the non-zero weights of the adjoint representation of G. The set of roots of G relative to T is denoted by $\Phi(G, T)$, or simply by Φ if there is no risk of confusion. By Proposition 7.3.12, the map

$$(-)_* : X(T) \to \mathfrak{t}_\mathbb{C}^*$$

($\mathfrak{t}_\mathbb{C}^*$ denotes the dual of the complex vector space $\mathfrak{t}_\mathbb{C}$) maps $\Phi(G, T)$ bijectively onto the set $\Phi(\mathfrak{g}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$ of roots of the split reductive algebra $(\mathfrak{g}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$. If we put

$$\mathfrak{g}^\alpha = (\mathfrak{g}_\mathbb{C})_\alpha(T) = (\mathfrak{g}_\mathbb{C})_{\alpha_*}(\mathfrak{t}_\mathbb{C}) \tag{7.3.8}$$

for $\alpha \in \Phi$, then each $\mathfrak{g}^\alpha$ is of dimension 1 over $\mathbb{C}$ and

$$\mathfrak{g}_\mathbb{C} = \mathfrak{t}_\mathbb{C} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}^\alpha \tag{7.3.9}$$

For each $\alpha \in \Phi$, denote by $\mathfrak{g}(\alpha)$ the 2-dimensional subspace of $\mathfrak{g}$ such that $\mathfrak{g}(\alpha)_{(\mathbb{C})} = \mathfrak{g}^\alpha + \mathfrak{g}^{-\alpha}$; the non-zero isotypical components of $\mathfrak{g}$ for the adjoint representation of T are $\mathfrak{t}$ and the $\mathfrak{g}(\alpha)$. Further, let Q be the quadratic form associated to the Killing form of $\mathfrak{g}$; it is negative (Proposition 7.1.1) and its restriction $Q(\alpha)$ to $\mathfrak{g}(\alpha)$ is negative and separating. For each element $\mathfrak{t}$ of T, $\mathrm{Ad}(\mathfrak{t})$ leaves $Q(\alpha)$ stable, and hence gives a morphism of Lie groups

$$\iota_\alpha : T \to \mathrm{SO}(Q(\alpha)).$$

There exists a *unique isomorphism* $\rho_\alpha : S^1 \to \mathrm{SO}(\mathfrak{g}(\alpha))$ such that $\iota_\alpha = \rho_\alpha \circ \alpha$. Indeed, let x be a non-zero element of $\mathfrak{g}^\alpha$, and let y be the image of x under the conjugation of $\mathfrak{g}_\mathbb{C}$ relative to $\mathfrak{g}$; then $y \in \mathfrak{g}^{-\alpha}$, and we obtain a basis (u, v) of $\mathfrak{g}(\alpha)$ by putting

$$u = x + y, \quad y = i(x - y);$$

the matrix of the endomorphism of $\mathfrak{g}(\alpha)$ induced by $\mathrm{Ad}(\mathfrak{t})$, $t \in T$, with respect to the basis (u, v) is

$$\begin{pmatrix} \mathrm{Re}(t^\alpha) & -\mathrm{Im}(t^\alpha) \\ \mathrm{Im}(t^\alpha) & \mathrm{Re}(t^\alpha) \end{pmatrix}$$

hence the assertion.

Proposition 7.3.13. *Let $\Lambda_r(\Phi)$ be the subgroup of $X(T)$ generated by the roots of G.*

(a) The centre $Z(G)$ of G is a closed subgroup of T, equal to the intersection of the kernels of the roots. The canonical map $X(T/Z(G)) \to X(T)$ is injective with image $\Lambda_r(\Phi)$.

(b) The compact group $Z(G)$ is isomorphic to the dual of the discrete group $X(T)/\Lambda_r(\Phi)$.

(c) $Z(G)$ reduces to the identity element if and only if $\Lambda_r(\Phi)$ is equal to $X(T)$.

Proof. By Corollary 7.1.18 $Z(G)$ is contained in T. Since this is the kernel of the adjoint representation (G is connected), it is the intersection of the kernels of the roots, in other words the orthogonal complement of the subgroup $\Lambda_r(\Phi)$ of $X(T)$. Thus, the proposition follows from **??** and **??**. □

Proposition 7.3.14. *Every automorphism of the Lie group G that induces the identity on T is of the form* $\mathrm{Inn}(t)$, *with $t \in T$.*

Proof. Assume first of all that $Z(G)$ reduces to the identity element, in other words that $X(T) = \Lambda_r(\Phi)$ (Proposition 7.3.13). Let φ be an automorphism of G inducing the identity on T, and $f = \mathfrak{Lie}(\varphi)_{(\mathbb{C})}$; then f is an automorphism of $\mathfrak{g}_{\mathbb{C}}$ inducing the identity on $\mathfrak{t}_{\mathbb{C}}$. By Proposition 6.2.73, there exists a unique homomorphism $\tau : \Lambda_r(\Phi) \to \mathbb{C}^\times$ such that f induces on each $\mathfrak{g}^\alpha$ the homothety with ratio $\tau(\alpha)$. Since f leaves stable the real form $\mathfrak{g}$ of $\mathfrak{g}_{\mathbb{C}}$, it commutes with the conjugation σ of $\mathfrak{g}_{\mathbb{C}}$ with respect to $\mathfrak{g}$; but $\sigma(\mathfrak{g}^\alpha) = \mathfrak{g}^{-\alpha}$, so $\tau(-\alpha) = \overline{\tau(\alpha)}$ for all $\alpha \in \Phi$. This implies that $|\tau(\alpha)|^2 = \tau(\alpha)\tau(-\alpha) = 1$. It follows that τ takes values in S^1 (in other words, τ is a character on $X(T)$), and hence corresponds by duality to an element t of T such that $\tau(\alpha) = \alpha(t)$, or equivalently $\mathrm{Ad}(t)_{(\mathbb{C})} = f$, so $\mathrm{Inn}(t) = \varphi$.

In the general case, the preceding applies to the group $G/Z(G)$, whose centre reduces to the identity element, and to its maximal torus $T/Z(G)$. It follows that, if φ is an automorphism of G inducing the identity on T, there exists an element t of T such that φ and $\mathrm{Inn}(t)$ induce by passage to the quotient the same automorphism of $G/Z(G)$. But, since the canonical morphism $D(G) \to G/Z(G)$ is a finite covering (Corollary 7.1.6), φ and $\mathrm{Inn}(t)$ induce the same automorphism of $D(G)$, hence of $D(G) \times Z(G)$, and hence also of G (loc. cit.). □

Corollary 7.3.15. *Let φ be an automorphism of G and H the closed subgroup of G consisting of the fixed points of φ. Then, the automorphism φ is inner if and only if H_0 is of maximal rank.*

Proof. If φ is equal to $\mathrm{Inn}(g)$, with $g \in G$, the subgroup $H_0 = Z_G(g)_0$ is of maximal rank (Corollary 7.1.20). Conversely, if H contains a maximal torus T, the automorphism φ is of the form $\mathrm{Int}(t)$ with $t \in T$ (Proposition 7.3.14). □

7.3.3 Nodal vectors and coroots

Lemma 7.3.16. *Let H be a closed subgroup of T and $Z_G(H)$ its centralizer in G.*

(a) *$\Phi(Z_G(H)_0, T)$ is the set of $\alpha \in \Phi(G,T)$ such that $\alpha(H) = \{1\}$.*

(b) *The center of $Z_G(H)_0$ is the intersection of the kernels of $\alpha \in \Phi(Z_G(H)_0, T)$.*

(c) *If H is connected, then $Z_G(H)$ is connected.*

Proof. The Lie algebra $\mathfrak{Lie}(Z_G(H))_{(\mathbb{C})}$ consists of the invariants of H on $\mathfrak{g}_{\mathbb{C}}$ (**??**), and hence is the direct sum of $\mathfrak{t}_{\mathbb{C}}$ and the $\mathfrak{g}^\alpha$ for which $\alpha(H) = \{1\}$, hence (a). Assertion (b) follows from Proposition 7.3.13, and assertion (c) has already been proved (Corollary 7.1.21). □

Theorem 7.3.17. *Let $\alpha \in \Phi(G,T)$. Then the centralizer Z_α of the kernel of α is a connected closed subgroup of G, with centre* $\ker\alpha$. *Its derived group $D(Z_\alpha) = H_\alpha$ is a connected closed semi-simple subgroup of G of rank* 1. *We have $\Phi(Z_\alpha, T) = \{\alpha, -\alpha\}$ and* $\dim(Z_\alpha) = \dim(T) + 2$.

Proof. Let $\tilde{Z}_\alpha$ be the centralizer of $(\ker\alpha)_0$. By Lemma 7.3.16(c), this is a connected closed subgroup of G, and $\Phi(\tilde{Z}_\alpha, T)$ is the set of $\beta \in \Phi(G,T)$ such that $\beta((\ker\alpha)_0) = \{1\}$. Clearly, $\{\alpha, -\alpha\} \subseteq \Phi(\tilde{Z}_\alpha, T)$. Conversely, let $\beta \in \Phi(\alpha, T)$; since $(\ker\alpha)_0$ is of finite index in $\ker\alpha$, there exists an integer $r \neq 0$ such that $t^{r\beta} = 1$ for $t \in \ker\alpha$. From the exactness of the sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow X(T) \longrightarrow X(\ker\alpha) \longrightarrow 0$$

corresponding by duality to the exact sequence

$$0 \longrightarrow \ker\alpha \longrightarrow T \xrightarrow{\alpha} S^1 \longrightarrow 0$$

it follows that $r\beta$ is a multiple of α; by Theorem 6.2.21, this implies that $\beta \in \{\alpha, -\alpha\}$. Thus $\Phi(\tilde{Z}_\alpha, T) = \{\alpha, -\alpha\}$, and it follows from Lemma 7.3.16(b) that the centre of $\tilde{Z}_\alpha$ is $\ker\alpha$, so $\tilde{Z}_\alpha = Z_\alpha$. Finally, by Corollary 7.1.9, $D(Z_\alpha)$ is a connected closed semi-simple subgroup of G; it is of rank 1 because $[\mathfrak{Lie}(Z_\alpha)_{(\mathbb{C})}, \mathfrak{Lie}(Z_\alpha)_{(\mathbb{C})}] = \mathfrak{g}^\alpha + \mathfrak{g}^{-\alpha} + [\mathfrak{g}^\alpha, \mathfrak{g}^{-\alpha}]$. □

Corollary 7.3.18. *Let $\alpha \in \Phi(G,T)$. Then there exists a morphism of Lie groups $\phi : \mathrm{SU}(2,\mathbb{C}) \to G$ with the following properties:*

(a) *The image of ϕ lies in H_α, hence commutes with the kernel of α.*

(b) *For all $z \in S^1$, we have*

$$\phi\begin{pmatrix} z & 0 \\ 0 & \bar{z} \end{pmatrix} \in T, \quad \alpha \circ \phi\begin{pmatrix} z & 0 \\ 0 & \bar{z} \end{pmatrix} = z^2.$$

If ϕ_1 and ϕ_2 are two morphisms from $\mathrm{SU}(2,\mathbb{C})$ *to G with the preceding properties, there exists $z \in S^1$ such that $\phi_2 = \phi_1 \circ \mathrm{Inn}\left(\begin{smallmatrix} z & 0 \\ 0 & \bar{z} \end{smallmatrix}\right)$.*

Proof. By Theorem 7.3.17 and Proposition 7.2.13, there exists a morphism of Lie groups $\phi : \mathrm{SU}(2,\mathbb{C}) \to H_\alpha$ that is surjective with discrete kernel. Then $\phi^{-1}(T \cap H_\alpha)$ is a maximal torus of $\mathrm{SU}(2,\mathbb{C})$ (Proposition 7.1.26). Since the maximal tori of $\mathrm{SU}(2,\mathbb{C})$ are conjugate, we can assume, replacing ϕ by $\phi \circ \mathrm{Inn}(t)$ (with $t \in \mathrm{SU}(2,\mathbb{C})$) if necessary, that $\phi^{-1}(T \cap H_\alpha)$ is the group of diagonal matrices in $\mathrm{SU}(2,\mathbb{C})$. Then $\phi\left(\begin{smallmatrix} z & 0 \\ 0 & \bar{z} \end{smallmatrix}\right) \in T$ for all $z \in S^1$, and the map

$$\begin{pmatrix} z & 0 \\ 0 & \bar{z} \end{pmatrix} \mapsto \alpha \circ \phi\begin{pmatrix} z & 0 \\ 0 & \bar{z} \end{pmatrix}$$

is a root of $\mathrm{SU}(2,\mathbb{C})$, hence is equal to one of the two maps $\left(\begin{smallmatrix} z & 0 \\ 0 & \bar{z} \end{smallmatrix}\right) \mapsto z^2$ or $\left(\begin{smallmatrix} z & 0 \\ 0 & \bar{z} \end{smallmatrix}\right) \mapsto z^{-2}$ (formula (7.2.12)). In the first case, the homomorphism ϕ has the required properties; in the second case, the homomorphism $\phi \circ \mathrm{Inn}(\theta)$ has them (formula (7.2.11)).

If ϕ_1 and ϕ_2 are two morphisms from $\mathrm{SU}(2,\mathbb{C})$ to G satisfying the stated conditions, they both map $\mathrm{SU}(2,\mathbb{C})$ into H_α (condition (a)), hence are both universal coverings of H_α. Hence, there exists an automorphism ψ of $\mathrm{SU}(2,\mathbb{C})$ such that $\phi_2 = \phi_1 \circ \psi$, and we conclude by using Proposition 7.3.14. □

It follows from the preceding corollary (since $\mathrm{Inn}(t)$ acts trivially on T) that the homomorphism ϕ_T from S^1 to T, defined by

$$\phi_T(z) = \phi\begin{pmatrix} z & 0 \\ 0 & \bar{z} \end{pmatrix}$$

for $z \in S^1$, is independent of the choice of ϕ. Denote by $k_\alpha \in \Gamma(T)$ the image under $\Gamma(\phi_T)$ of the element $2\pi i$ of $\Gamma(S^1) = 2\pi i\mathbb{Z}$ (which is a generator for it); it is called the **nodal vector** associated to the root α. We have $\langle \alpha, k_\alpha \rangle = 2$, in other words $\alpha_*(k_\alpha) = 4\pi i$; since k_α belongs to the intersection of $\mathfrak{t}$ and the $\mathfrak{Lie}(H_\alpha)_{(\mathbb{C})}$, we have

$$k_\alpha = 2\pi i h_{\alpha_*}$$

where h_{α_*} is the inverse root associated to the root α_* of $(\mathfrak{g}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$. In other words, when $\Gamma(T) \otimes \mathbb{R}$ is identified with the dual of $X(T) \otimes \mathbb{R}$ via the pairing $\langle \cdot, \cdot \rangle$, k_α is identified with the coroot $\check{\alpha} \in (X(T) \otimes \mathbb{R})^*$.

Remark 7.3.19. For all $x \in \mathbb{R}$, we have

$$\phi\begin{pmatrix} e^{2\pi i x} & 0 \\ 0 & e^{-2\pi i x} \end{pmatrix} = \phi_T(e^{2\pi i x}) = \exp(x k_\alpha).$$

In particular,

$$\phi\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \phi_T(-1) = \exp(\frac{1}{2} k_\alpha).$$

We denote by $\check{\Phi}(G,T)$ the set of nodal vectors k_α for $\alpha \in \Phi(G,T)$. This is a subset of $\Gamma(T)$ whose image under the canonical injection of $\Gamma(T)$ into $\mathfrak{t}_\mathbb{C}$ is identified with the homothety with ratio $2\pi i$ of the inverse root system $\check{\Phi}(\mathfrak{g}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$ of Φ_*. It follows that $\check{\Phi}(G,T)$ generates the $\mathbb{R}$-vector space $\mathfrak{Lie}(T \cap D(G))$, and hence that its orthogonal complement in $X(T)$ is $X(T/(T \cap D(G)))$.

Denote by $\mathrm{Aut}(T)$ the group of automorphisms of the Lie group T; the Weyl group $W = W_G(T)$ can be identified with a subgroup of $\mathrm{Aut}(T)$. On the other hand, recall that the Weyl group $W(\mathfrak{g}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$ of the split reductive algebra $(\mathfrak{g}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$ operates on $\mathfrak{t}_\mathbb{C}$, and thus is canonically identified with a subgroup of $\mathrm{GL}(\mathfrak{t}_\mathbb{C})$.

Proposition 7.3.20. *The map $\varphi \mapsto \mathfrak{Lie}(\varphi)_{(\mathbb{C})}$ from $\mathrm{Aut}(T)$ to $\mathrm{GL}(\mathfrak{t}_\mathbb{C})$ induces an isomorphism from W to the Weyl group of the split reductive Lie algebra $(\mathfrak{g}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$. For all $\alpha \in \Phi$, $W_{Z_\alpha}(T)$ is of order 2, and the image under the preceding isomorphism of the non-identity element of $W_{Z_\alpha}(T)$ is the reflection $s_{h_{\alpha_*}}$.*

Proof. The map under consideration is injective. It remains to show that its image is equal to $W(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$. Let $g \in N_G(T)$; we have $\mathrm{Ad}(g) \in \mathrm{Aut}(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}}) \cap \mathrm{Inn}(\mathfrak{g}_{\mathbb{C}})$, so $\mathrm{Ad}(g) \in \mathrm{Aut}_0(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ (Corollary 6.2.85. By Proposition 6.2.81 the automorphism of $\mathfrak{t}_{\mathbb{C}}$ induced by $\mathrm{Ad}(g)$ belongs to $W(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$. Thus, the image of W in $\mathrm{GL}(\mathfrak{t}_{\mathbb{C}})$ is contained in $W(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$.

Let $\alpha \in \Phi(G,T)$, and let $\phi : \mathrm{SU}(2,\mathbb{C}) \to G$ be a morphism of Lie groups having the properties in Corollary 7.3.18. The image under ϕ of the element θ of $\mathrm{SU}(2,\mathbb{C})$ has the following properties (c.f. Corollary 7.3.18(a) and formula (7.2.11)):

$$\mathrm{Inn}(\phi(\theta))(t) = \begin{cases} t & \text{if } t \in \ker\alpha, \\ t^{-1} & \text{if } t \in T \cap H_\alpha. \end{cases}$$

It follows that $\mathrm{Ad}(\phi(\theta))$ induces the identity on $\ker\alpha_* \subseteq \mathfrak{t}_{\mathbb{C}}$ and induces the map $x \mapsto -x$ on $[\mathfrak{g}^\alpha, \mathfrak{g}^{-\alpha}]$, hence coincides with the reflection $s_{h_{\alpha*}}$. Thus, the image of W contains all the $s_{h_{\alpha*}}$, and hence is equal to $W(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$. In particular, $W_{Z_\alpha}(T)$ is of order 2, and hence consists of the identity and $\mathrm{Inn}(\phi(\theta))$. This completes the proof of the proposition. □

Corollary 7.3.21. *Assume that G is semi-simple. Then every element of G is the commutator of two elements of G.*

Proof. Let w_C be a Coxeter transformation of the Weyl group $W(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$, and let n be an element of $N_G(T)$ whose class in W is identified with w_C by the isomorphism defined in the proposition. Denote by φ_{w_C} the morphism $t \mapsto [n,t]$ from T to T; for $x \in \mathfrak{t}_{\mathbb{C}}$, we have

$$\mathfrak{Lie}(\varphi_{w_C})(x) = \mathrm{Ad}(n)(x) - x = w_C(x) - x.$$

Recall that the endomorphism w_C of $\mathfrak{t}_{\mathbb{C}}$ has no eigenvalue equal to 1. Consequently, $\mathfrak{Lie}(\varphi_{w_C})$ is surjective, and hence so is φ. It follows that every element of T is the commutator of two elements of G, which implies the corollary in view of Theorem 7.1.16. □

7.3.4 Application: the fundamental groups of G and H

Recall that since T is commutative, the map $\exp_T : \mathfrak{t} \to T$ is a universal covering of T and the group $\Gamma(T)$ can be viewed as the fundamental group $\pi_1(T)$. We now consider the fundamental group of the compact Lie group G; we denote by $\gamma(G,T)$ the homomorphism from $\Gamma(T)$ to $\pi_1(G)$ that is the composite of the canonical isomorphism from $\Gamma(T)$ to $\pi_1(T)$ and the homomorphism $\pi_1(\iota)$, where ι is the canonical injection $T \to G$.

Proposition 7.3.22. *The homomorphism $\gamma(G,T) : \Gamma(T) \to \pi_1(G)$ is surjective. Its kernel is the subgroup $N(G,T)$ of $\Gamma(T)$ generated by the family of nodal vectors $(k_\alpha)_{\alpha\in\Phi(G,T)}$.*

Proof. The homomorphism $\gamma(G,T)$ is surjective by Proposition 7.1.28. We denote by $P(G,T)$ the assertion: "the kernel of $\gamma(G,T)$ is generated by the k_α" which it remains to prove, and distinguish several cases.

First let G be simply-connected. Let $\rho : \mathfrak{g}_{\mathbb{C}} \to \mathfrak{gl}(V)$ be a linear representation of $\mathfrak{g}_{\mathbb{C}}$ on a finite dimensional complex vector space V. Restricting to $\mathfrak{g}$, we obtain a representation of $\mathfrak{g}$ on the real vector space $V(\mathbb{R})$; since G is simply-connected, there exists an analytic linear representation Π of G on $V(\mathbb{R})$ such that $\rho = \mathfrak{Lie}(\Pi)$. It follows from Proposition 7.3.12 that the image $X(T)_*$ of $X(T)$ in $\mathfrak{t}_{\mathbb{C}}^*$ contains all the weights of ρ on V. This being true for every representation ρ of $\mathfrak{g}_{\mathbb{C}}$, it follows from Theorem 6.3.25 that $X(T)_*$ contains the group of weights of Φ_*, which is by definition the set of $\lambda \in \mathfrak{t}_{\mathbb{C}}^*$ such that $\lambda(\alpha_*) \in \mathbb{Z}$ for all $\alpha \in \Phi$, in other words, $\lambda(k_\alpha) \in 2\pi i\mathbb{Z}$ for all $\alpha \in \Phi$. Thus, the group $X(T)$ contains all the elements λ of $X(T) \otimes \mathbb{Q}$ such that $\langle \lambda, k_\alpha \rangle \in \mathbb{Z}$ for all $\alpha \in \Phi$, which implies by duality that $\Gamma(T)$ is generated by the k_α, hence the assertion $P(G,T)$.

Next, suppose that G is the direct product of a simply-connected group $\tilde{G}$ and a torus H. Then T is the direct product of a maximal torus $\tilde{T}$ of $\tilde{G}$ with H, $\Gamma(T)$ can be identified with $\Gamma(\tilde{T}) \times \Gamma(H)$, $\pi_1(G)$ with $\pi_1(\tilde{G}) \times \pi_1(H)$, and $\gamma(G,T)$ with the homomorphisms with components $\gamma(\tilde{G},\tilde{T})$ and $\gamma(H,H)$. Since $\gamma(H,H)$ is bijective, the canonical map $\Gamma(\tilde{T}) \to \Gamma(T)$ maps $\ker\gamma(\tilde{G},\tilde{T})$ bijectively onto $\ker\gamma(G,T)$. Moreover, the k_α belong to the Lie algebra of the derived group of $\tilde{G}$, hence to the image of $\Gamma(\tilde{T})$, so it is immediate that $P(\tilde{G},\tilde{T})$ implies $P(G,T)$, hence assertion $P(G,T)$, in view of the first case.

For the general case, there exists a surjective morphism $p : \tilde{G} \to G$ with finite kernel, where $\tilde{G}$ is the direct product of a simply-connected group by a torus (Proposition 7.1.5). If $\tilde{T}$ is the inverse image of

T in $\tilde{G}$ (this is a maximal torus of $\tilde{G}$), and N the kernel of p, we have exact sequences $0 \to N \to \tilde{G} \to G \to 0$ and $0 \to N \to \tilde{T} \to T \to 0$, hence a commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & \Gamma(\tilde{T}) \arrow[r] \arrow[d, "\gamma(\tilde{G},\tilde{T})"] & \Gamma(T) \arrow[r] \arrow[d, "\gamma(G,T)"] & N \arrow[r] \arrow[d, "\mathrm{id}_N"] & 0 \\ 0 \arrow[r] & \pi_1(\tilde{G}) \arrow[r] & \pi_1(G) \arrow[r] & N \arrow[r] & 0 \end{tikzcd}$$

It follows immediately from the snake diagram that $P(\tilde{G}, \tilde{T})$ implies $P(G, T)$, hence the proposition. $\square$

Corollary 7.3.23. *G is simply-connected if and only if the family $(k_\alpha)_{\alpha\in\Phi(G,T)}$ generates $\Gamma(T)$.*

Corollary 7.3.24. *Let H be a connected closed subgroup of G containing T; there is an exact sequence*

$$0 \longrightarrow N(H,T) \longrightarrow N(G,T) \longrightarrow \pi_1(H) \longrightarrow \pi_1(G) \longrightarrow 0$$

Proof. This follows from the snake lemma, applied to the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & N(H,T) \arrow[r] \arrow[d] & \Gamma(T) \arrow[r] \arrow[d] & \pi_1(H) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & N(G,T) \arrow[r] & \Gamma(T) \arrow[r] & \pi_1(G) \arrow[r] & 0 \end{tikzcd}$$

(note that $N(H,T)$ is a subgroup of $N(G,T)$, and $\pi_1(H) \to \pi_1(G)$ is surjective by Proposition 7.3.22.) $\square$

Remark 7.3.25. It can be shown that $\pi_2(G)$ is zero. The exactness of the preceding sequence then gives an isomorphism from $\pi_2(G/H)$ to $N(G,T)/N(H,T)$.

7.3.5 Subgroups of maximal rank

Recall that a subset P of $\Phi = \Phi(G,T)$ is said to be closed if $(P+P)\cap\Phi \subseteq P$, and symmetric if $P = -P$. We know that closed subsets of the root system of $(\mathfrak{g}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$ corresponds to subalgebras of the corresponding Lie algebra. We now consider an analogous result for compact Lie groups, which concern its subgroups of maximal rank.

Proposition 7.3.26. *Let $\mathscr{H}$ be the set of connected closed subgroups of G containing T, ordered by inclusion. Then the map $H \mapsto \Phi(H,T)$ is an increasing bijection from H to the set of symmetric closed subsets of $\Phi(G,T)$, ordered by inclusion.*

Proof. If $H \in \mathscr{H}$, then $\mathfrak{Lie}(H)_{(\mathbb{C})}$ is the direct sum of $\mathfrak{t}_\mathbb{C}$ and the $\mathfrak{g}^\alpha$ for $\alpha \in \Phi(H,T)$; since this is a reductive subalgebra in $\mathfrak{g}_\mathbb{C}$, the subset $\Phi(H,T)$ of Φ satisfies the stated conditions (Proposition 6.2.41). Conversely, if P is a subset of Φ satisfying these conditions, then $\mathfrak{t}_\mathbb{C} \oplus \sum_{\alpha\in P}\mathfrak{g}^\alpha$ is a subalgebra of $\mathfrak{g}_\mathbb{C}$ (loc. cit.) which is rational over $\mathbb{R}$, and hence of the form $\mathfrak{h}_{(\mathbb{C})}$, where $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$. Let $H(P)$ be the integral subgroup of G defined by $\mathfrak{h}$; it is closed (Remark 7.1.32). We verify immediately that the maps $H \mapsto \Phi(H,T)$ and $P \mapsto H(P)$ are increasing and inverses of each other. $\square$

Corollary 7.3.27. *There are only finitely-many closed subgroups of G containing T.*

Proof. Let H be such a subgroup; then $H_0 \in \mathscr{H}$, and $\mathscr{H}$ is finite. Moreover, H is a subgroup of $N_G(H_0)$ containing H_0 (since H_0 is normal in H), and $N_G(H_0)/H_0$ is finite by Proposition 7.1.31. $\square$

Corollary 7.3.28. *Let H be a connected closed subgroup of G containing T, and let $W_G^H(T)$ be the stabilizer in $W_G(T)$ of the subset $\Phi(H,T)$ of Φ. Then the group $N_G(H)/H$ is isomorphic to the quotient group $W_G^H(T)/W_H(T)$.*

Proof. Indeed, it follows from Proposition 7.1.38, applied to $N_G(H)$ (recall that H is the identity component of $N_G(H)$), that $N_G(H)/H$ is isomorphic to $W_{N_G(H)}(T)/W_H(T)$, where $W_{N_G(H)}(T)$ is the set of elements of $W_G(T)$ whose representatives in $N_G(T)$ normalize H (i.e., belongs to $N_G(H)$). Let $n \in N_G(T)$ and w be its class in $W_G(T)$; let $\mathfrak{h}$ be the Lie algebra of H. By **??**, n normalizes H if and only if $\mathrm{Ad}(n)(\mathfrak{h}) = \mathfrak{h}$; in view of Proposition 7.3.10, this also means that the subset $\Phi(H,T)$ of Φ is stable under w, hence the corollary. $\square$

Proposition 7.3.29. *Let H be a connected closed subgroup of G of maximal rank, and C its centre. Then C contains the centre of G, and H is the identity component of the centralizer of C.*

Proof. Let T be a maximal torus of H. Since the centre of G is contained in T, it is contained in C. Put $L = Z_G(C)_0$; this is a connected closed subgroup of G containing H, hence is of maximal rank. Let $\tilde{C}$ be the centre of L; then $C \subseteq \tilde{C}$ by the definition of L. On the other hand, since the center of L is contained in T, it is contained in C. Thus the centre of L is C.

Denote by Φ_H and Φ_L the root systems of H and L, respectively, relative to T; then $\Phi_H \subseteq \Phi_L \subseteq \Phi(G,T)$. Since $Z(H) = Z(L)$, Proposition 7.3.13 implies the equality $\Lambda_r(\Phi_H) = \Lambda_r(\Phi_L)$; but $\Lambda_r(\Phi_H) \cap \Phi_L = \Phi_H$ (Proposition 5.1.59), so $\Phi_H = \Phi_L$ and $H = L$ (Proposition 7.3.26). $\square$

Example 7.3.30. Say that a subgroup C of G is **radical** if there exists a maximal torus T of G and a subset P of $\Phi(G,T)$ such that $C = \bigcap_{\alpha \in P} \ker \alpha$. It follows from Proposition 7.3.29 and Lemma 7.3.16 that the map $H \mapsto Z(H)$ induces a bijection from the set of connected closed subgroups of maximal rank to the set of radical subgroups of G. The inverse bijection is the map $C \mapsto Z_G(C)_0$.

Corollary 7.3.31. *The set of $g \in G$ such that $T \cap gTg^{-1} \neq Z(G)$ is the union of a finite number of closed smooth submanifolds of G distinct from G.*

Proof. Indeed, put $M_g = T \cap gTg^{-1}$; we have $T \subseteq Z_G(M_g)$ and $gTg^{-1} \subseteq Z_G(M_g)$. Hence, there exists $x \in Z_G(M_g)$ such that $xTx^{-1} = gTg^{-1}$ (Theorem 7.1.16), which implies that $g \in Z_G(M_g) \cdot N_G(T)$. Denote by $\mathscr{M}$ the finite set of closed subgroups of G containing T and distinct from G, and put $X = \bigcup_{H \in \mathscr{M}} H \cdot N_G(T)$; this is a finite union of closed submanifolds of G, distinct from G. If $M_g \neq Z(G)$, then $Z_G(M_g) \in \mathscr{M}$, and g belongs to X. Conversely, if $g \in H \cdot N_G(T)$, with $H \in \mathscr{M}$, then M_g contains $Z(H)$, so $M_g \neq Z(G)$ by Proposition 7.3.29. This proves the claim. $\square$

Proposition 7.3.32. *Let X be a subset of T, and let Φ_X be the set of roots $\alpha \in \Phi(G,T)$ such that $\alpha(X) = \{1\}$. Then the group $Z_G(X)/Z_G(X)_0$ is isomorphic to the quotient of the subgroup of $W_G(T)$ fixing X by the subgroup generated by the reflections s_α for $\alpha \in \Phi_X$.*

Proof. Put $H = Z_G(X)$; since $\mathfrak{Lie}(H)_{(\mathbb{C})}$ is the set of points of $\mathfrak{g}_{\mathbb{C}}$ fixed by $\mathrm{Ad}(X)$, it is the sum of $\mathfrak{t}_{\mathbb{C}}$ and the $\mathfrak{g}^\alpha$ for which $\alpha(X) = \{1\}$. Consequently, $\Phi(H_0, T) = \Phi_X$, so $W_{H_0}(T)$ is generated by the reflections s_α for $\alpha \in \Phi_X$. It now suffices to apply Proposition 7.1.38. $\square$

7.3.6 Root diagrams

A root diagram (or simply a diagram, if there is no risk of confusion) is a triple $D = (M, M_0, \Phi)$ where

(RD0) M is a free $\mathbb{Z}$-module of finite rank and the submodule M_0 is a direct factor of M;

(RD1) Φ is a finite subset of M, $\Phi \cup M_0$ generates the $\mathbb{Q}$-vector space $\mathbb{Q} \otimes M$;

(RD2) for all $\alpha \in \Phi$, there exists an element $\check{\alpha}$ of $M^* = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$ such that $\check{\alpha}(M_0) = 0$, $\check{\alpha}(\alpha) = 2$ and the endomorphism $x \mapsto x - \check{\alpha}(x)\alpha$ of M leaves Φ stable.

Since $\Phi \cup M_0$ generates $\mathbb{Q} \otimes M$, for all $\alpha \in \Phi$ the element $\check{\alpha}$ of M^* is uniquely determined by α; we denote by s_α the endomorphism $x \mapsto x - \check{\alpha}(x)\alpha$ of M. Moreover, the $\mathbb{Q}$-vector space $\mathbb{Q} \otimes M$ is the direct sum of $\mathbb{Q} \otimes M_0$ and the vector subspace $V(\Phi)$ generated by Φ, and Φ is a root system in $V(\Phi)$.

The elements of Φ are called the roots of the root diagram D, and the elements $\check{\alpha}$ of M^* the inverse roots. The group generated by the automorphisms s_α of M is called the Weyl group of D and is denoted by $W(D)$; the elements of $W(D)$ induce the identity on M_0, and induce on $V(\Phi)$ the transformations of the Weyl group of the root system Φ.

Example 7.3.33 (Example of root diagrams).

(a) For every free $\mathbb{Z}$-module of finite type M, the triple $(M, M, \varnothing)$ is a root diagram.

(b) If $D = (M, M_0, \Phi)$ is a root diagram, let M_0^* be the orthogonal complement of $V(\Phi)$ in M^*, and let $\check{\Phi}$ be the set of coroots of Φ. Then $\check{D} = (M^*, M_0^* \check{\Phi})$ is a root diagram, called the **inverse** of D. For all $\alpha \in \Phi$, the symmetry $s_{\check{\alpha}}$ of M^* is the contragredient automorphism of the symmetry s_α of M; the map $w \mapsto (w^{-1})^t$ is an isomorphism from $W(D)$ to $W(\check{D})$. Moreover, $V(\check{\Phi})$ can be naturally identified with the dual of the $\mathbb{Q}$-vector space $V(\Phi)$, $\check{\Phi}$ then being identified with the inverse root system of Φ.

(c) Let $(\mathfrak{g}, \mathfrak{h})$ be a split reductive $\mathbb{Q}$-Lie algebra, and $M \subseteq \mathfrak{h}$ a permissible lattice. Let M_0 be the subgroup of M orthogonal to the roots of $(\mathfrak{g}, \mathfrak{h})$ and $\check{\Phi}$ the set of the h_α, $\alpha \in \Phi(\mathfrak{g}, \mathfrak{h})$. Then $(M, M_0, \check{\Phi})$ is a root diagram, and $(M^*, M_0^*, \Phi(\mathfrak{g}, \mathfrak{h}))$ is the inverse diagram.

(d) Let V be a vector space over $\mathbb{Q}$ and Φ a root system in V; denote by $\Lambda(\Phi)$ the group of weights of Φ and by $\Lambda_r(\Phi)$ the group of radical weights of Φ. Then $(\Lambda_r(\Phi), 0, \Phi)$ and $(\Lambda(\Phi), 0, \Phi)$ are root diagrams. A root diagram (M, M_0, Ψ) is isomorphic to a diagram of the form $(\Lambda_r(\Phi), 0, \Phi)$ (resp. $(\Lambda(\Phi), 0, \Phi)$) if and only if M is generated by Ψ (resp. M^* is generated by $\check{\Psi}$).

For every subgroup X of $\Lambda(\Phi)$ containing $\Lambda_r(\Phi)$, $(X, 0, \Phi)$ is a root diagram and, up to isomorphism, every diagram (M, M_0, Ψ) such that $M_0 = 0$, in other words such that Ψ generates a subgroup of M of finite index, arises in this way.

The root diagram (M, M_0, Φ) is said to be **reduced** if the root system Φ is reduced (in other words if the relations $\alpha, \beta \in \Phi$, $\lambda \in \mathbb{Z}$, $\beta = \lambda\alpha$ imply that $\lambda = \pm 1$). The examples given above are all reduced.

Theorem 7.3.34. *Let G be a compact connected Lie group and T a maximal torus of G.*

(a) $(X(T), X(T/T \cap D(G)), \Phi(G, T))$ is a reduced root diagram; its Weyl group consists of the $X(w)$ for $w \in W$; the group $X(Z(G))$ is isomorphic to the quotient of $X(T)$ by the subgroup generated by $\Phi(G, T)$.

(b) $(\Gamma(T), \Gamma(Z(G)_0), \check{\Phi}(G, T))$ is a reduced root diagram; its Weyl group consists of the $\Gamma(w)$, for $w \in W$; the group $\pi_1(G)$ is isomorphic to the quotient of $\Gamma(T)$ by the subgroup generated by $\check{\Phi}(G, T)$.

(c) If each of the $\mathbb{Z}$-modules $X(T)$ and $\Gamma(T)$ is identified with the dual of the other, each of the preceding root diagrams is identified with the inverse of the other.

Proof. The first assertion follows from Proposition 7.3.13 and the observation that the orthogonal complement of $\check{\Phi}(G, T)$ in $X(T)$ is $X(T/(T \cap D(G)))$. The second part follows from Proposition 7.3.22. □

Denote by $D^*(G, T)$ the diagram $(X(T), X(T/(T \cap D(G)), \Phi(G, T)))$ and by $D_*(G, T)$ the diagram $(\Gamma(T), \Gamma(Z(G)_0), \check{\Phi}(G, T))$; these are called the **contravariant diagram** and the **covariant diagram** of G (relative to T), respectively.

Example 7.3.35 (Example of root diagrams for compact Lie groups).

(a) If G is semi-simple of rank 1, then $D^*(G, T)$ and $D_*(G, T)$ are necessarily isomorphic to one of the two diagrams

$$D_1 = (\mathbb{Z}, 0, \{1, -1\}), \quad D_2 = (\mathbb{Z}, 0, \{2, -2\}).$$

If G is isomorphic to $\mathrm{SU}(2, \mathbb{C})$, $D^*(G, T)$ is isomorphic to Δ_1 (since G is simply-connected) so $D_*(G, T)$ is isomorphic to Δ_2. If G is isomorphic to $\mathrm{SO}(3, \mathbb{R})$, $D^*(G, T)$ is isomorphic to Δ_1 (since $Z(G) = \{1\}$), so $D_*(G, T)$ is isomorphic to Δ_2.

(b) If G and $\tilde{G}$ are two connected compact Lie groups, with maximal tori T and $\tilde{T}$, respectively, and if $D^*(G, T) = (M, M_0, \phi)$ and $D^*(\tilde{G}, \tilde{T}) = (\tilde{M}, \tilde{M}_0, \tilde{\Phi})$, then $D^*(G \times \tilde{G}, T \times \tilde{T})$ can be identified with $(M \oplus \tilde{M}, M_0 \oplus \tilde{M}_0, \Phi \cup \tilde{\Phi})$. Similarly for the covariant diagrams.

(c) Let N be a closed subgroup of T, central in G, and let (M, M_0, Φ) be the contravariant diagram of G relative to T. Then the contravariant diagram of G/N relative to T/N can be identified with $(\tilde{M}, \tilde{M}_0, \Phi)$, where $\tilde{M}$ is the subgroup of M consisting of the λ such that $\lambda(N) = \{1\}$ and $\tilde{M}_0 = \tilde{M} \cap M_0$.

(d) Let N be a finite abelian group, and $\varphi : \pi_1(G) \to N$ a surjective homomorphism. Let $\tilde{G}$ be the covering of G associated to this homomorphism; this is a connected compact Lie group, of which N is a central subgroup, and G can be naturally identified with $\tilde{G}/N$. Let $\tilde{T}$ be the maximal torus of $\tilde{G}$ that is the inverse image of T. If $D_*(G, T) = (M, M_0, \Phi)$ is the covariant diagram of G relative to T, the covariant diagram of $\tilde{G}$ relative to $\tilde{T}$ can be identified with $(\tilde{M}, \tilde{M}_0, \Phi)$, where $\tilde{M}$ is the kernel of the composite homomorphism $\varphi \circ \gamma(G, T) : M \to N$ and $\tilde{M}_0 = M_0 \cap \tilde{M}$.

Example 7.3.36. Let $\mathfrak{z}$ be the centre of $\mathfrak{g}_{\mathbb{C}}$; then $\mathfrak{z} = \mathfrak{Lie}(Z(G))_{(\mathbb{C})}$. We have the following relations between the diagrams of G relative to T and the direct and inverse root systems of the split reductive algebra $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$:

(a) The canonical isomorphism from $\mathbb{C} \otimes \Gamma(T)$ to $\mathfrak{t}_{\mathbb{C}}$ induces a bijection from $\mathbb{C} \otimes \Gamma(Z(G)_0)$ to $\mathfrak{z}$ and a bijection from $1 \otimes \check{\Phi}(G,T)$ to $2\pi i \cdot \check{\Phi}(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$.

(b) The canonical isomorphism from $\mathbb{C} \otimes X(T)$ to the dual $\mathfrak{t}_{\mathbb{C}}^*$ of $\mathfrak{t}_{\mathbb{C}}$ induces a bijection from $\mathbb{C} \otimes X(T/(T \cap D(G)))$ to the orthogonal complement of $\mathfrak{t}_{\mathbb{C}} \cap \mathscr{D}(\mathfrak{g})_{\mathbb{C}}$, and a bijection from $1 \otimes \Phi(G,T)$ to $\Phi(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$.

Example 7.3.37. Assume that the group G is semi-simple. Then by Corollary 7.3.21 and the fact that $\mathfrak{z}(\mathfrak{g}) = 0$, we conclude that $X(T/T \cap D(G)) = \Gamma(Z(G)_0) = 0$, so the root diagrams of G relative to T are given by

$$D^*(G,T) = (X(T), 0, \Phi), \quad D_*(G,T) = (\Gamma(T), 0, \check{\Phi}).$$

where we denote by Φ (resp. $\check{\Phi}$) the root system $\Phi(G,T)$ (resp. $\check{\Phi}(G,T)$). Then we have inclusions

$$\Lambda_r(\Phi) \subseteq X(T) \subseteq \Lambda(\Phi), \quad \Lambda_r(\check{\Phi}) \subseteq \Gamma(T) \subseteq \Lambda(\check{\Phi}).$$

The finite abelian groups $\Lambda(\Phi)/\Lambda_r(\Phi)$ and $\Lambda(\check{\Phi})/\Lambda_r(\check{\Phi})$ are in duality; if $\widehat{M}$ denotes the dual group of the finite abelian group M, we deduce from the preceding canonical isomorphisms

$$\begin{aligned} \Gamma(T)/\Lambda_r(\check{\Phi}) &\to \pi_1(G) & \Lambda(\check{\Phi})/\Gamma(T) &\to Z(G), \\ \Lambda(\Phi)/X(T) &\to \widehat{\pi_1(G)} & X(T)/\Lambda_r(\Phi) &\to \widehat{Z(G)}. \end{aligned}$$

In particular, the product of the orders of $\pi_1(G)$ and $Z(G)$ is equal to the connection index f of $\Phi(G,T)$.

Now let $\tilde{G}$ be another connected compact Lie group, $\tilde{T}$ a maximal torus of $\tilde{G}$. Let $\varphi : G \to \tilde{G}$ be an isomorphism of Lie groups such that $\varphi(T) = \tilde{T}$; denote by φ_T the isomorphism from T to $\tilde{T}$ that it defines. Then $X(\varphi_T)$ is an isomorphism from $D^*(\tilde{G}, \tilde{T})$ to $D^*(G,T)$, denoted by $D^*(\varphi)$, and $\Gamma(\varphi_T)$ is an isomorphism from $D_*(G,T)$ to $D_*(\tilde{G}, \tilde{T})$, denoted by $D_*(\varphi)$. If $t \in T$, and if we put $\psi = \varphi \circ \mathrm{Inn}(t) = \mathrm{Inn}(\varphi(t)) \circ \varphi$, then $D^*(\psi) = D^*(\varphi)$ and $D_*(\psi) = D_*(\varphi)$.

Proposition 7.3.38. *Let f be an isomorphism from $D^*(\tilde{G}, \tilde{T})$ to $D^*(G,T)$ (resp. from $D_*(G,T)$ to $D_*(\tilde{G}, \tilde{T})$). Then there exists an isomorphism $\varphi : G \to \tilde{G}$ such that $\varphi(T) = \tilde{T}$ and $f = D^*(\varphi)$ (resp. $f = D_*(\varphi)$); if φ_1 and φ_2 are two such isomorphisms, there exists an element t of T such that $\varphi_2 = \varphi_1 \circ \mathrm{Inn}(t)$.*

Proof. The second assertion follows immediately from Proposition 7.3.14; we prove the first for the covariant diagrams, for example. Denote by $\tilde{\mathfrak{g}}$ (resp. $\tilde{\mathfrak{t}}$) the Lie algebra of $\tilde{G}$ (resp. $\tilde{T}$), and by $\tilde{\mathfrak{g}}_{\mathbb{C}}$ (resp. $\tilde{\mathfrak{t}}_{\mathbb{C}}$) its complexified Lie algebra. By Theorem 6.2.69(a), there exists an isomorphism $\psi : \mathfrak{g}_{\mathbb{C}} \to \tilde{\mathfrak{g}}_{\mathbb{C}}$ that maps $\mathfrak{t}_{\mathbb{C}}$ to $\tilde{\mathfrak{t}}_{\mathbb{C}}$ and induces on $\Gamma(T) \subseteq \mathfrak{t}_{\mathbb{C}}$ the given isomorphism $f : \Gamma(T) \to \Gamma(\tilde{T})$. Then $\mathfrak{g}$ and $\psi^{-1}(\tilde{\mathfrak{g}})$ are two compact forms of $\mathfrak{g}_{\mathbb{C}}$ that have the same intersection $\mathfrak{t}$ with $\mathfrak{t}_{\mathbb{C}}$; by Proposition 7.2.4, there exists an inner automorphism θ of $\mathfrak{g}_{\mathbb{C}}$ inducing the identity on $\mathfrak{t}_{\mathbb{C}}$ and such that $\theta(\mathfrak{g}) = \psi^{-1}(\tilde{\mathfrak{g}})$. By replacing ψ with $\psi \circ \theta$, we can assume that ψ maps $\mathfrak{g}$ to $\tilde{\mathfrak{g}}$. Further, by Proposition 7.3.4, there exists a unique morphism $\varphi_T : T \to \tilde{T}$ such that $\Gamma(\varphi_T) = f_T$. Then the restriction of ψ to $\mathfrak{t}$ is $\mathfrak{Lie}(\varphi_T)$, and by Proposition 7.1.30, there exists a unique morphism $\varphi : G \to \tilde{G}$ that induces φ_T on T and ψ on $\mathfrak{g}_{\mathbb{C}}$. Applying the preceding to f^{-1} and ψ^{-1} we obtain an inverse morphism to φ, which is therefore an isomorphism. Then $D_*(\varphi) = \Gamma(\varphi_T) = f$, hence the proposition. □

Note that, if T and $\tilde{T}$ are two maximal tori of G, the diagrams $D^*(G,T)$ and $D^*(G,\tilde{T})$ are isomorphic (if $g \in G$ is such that $gTg^{-1} = \tilde{T}$, then $\mathrm{Int}(g)$ is an isomorphism from G to G that maps T to $\tilde{T}$). Denote by $D^*(G)$ the isomorphism class of $D^*(G,T)$; this is a root diagram that depends only on G and is called the **contravariant diagram** of G. The covariant diagram $D_*(G)$ of G is defined similarly, and we obtain:

Corollary 7.3.39. *Two connected compact Lie groups G and $\tilde{G}$ are isomorphic if and only if the diagrams $D^*(G)$ and $D^*(\tilde{G})$ (resp. $D_*(G)$ and $D_*(\tilde{G})$) are equal.*

Proposition 7.3.40. *For every reduced root diagram D, there exists a connected compact Lie group G such that $D^*(G)$ (resp. $D_*(G)$) is isomorphic to D.*

Proof. By replacing D, if necessary, by its inverse diagram, we are reduced to constructing G such that $D^*(G)$ is isomorphic to D. Put $D = (M, M_0, \Phi)$; then $\mathbb{Q} \otimes M$ is the direct sum of $\mathbb{Q} \otimes M_0$ and the vector subspace $V(\Phi)$ generated by Φ. Moreover, since the inverse roots take integer values on M, the projection of M on $V(\Phi)$ parallel to $\mathbb{Q} \otimes M_0$ is contained in the group of weights $\Lambda(\Phi)$ of Φ, so that M is a subgroup of $M_0 \oplus \Lambda(\Phi)$ of finite index. Denote by $\tilde{D}$ the diagram $(M_0 \oplus \Lambda(\Phi), M_0, \Phi)$.

Let $\mathfrak{a}$ be a complex semi-simple Lie algebra whose canonical root system is isomorphic to $\Phi \subseteq \mathbb{C} \otimes V(\Phi)$, and let $\mathfrak{g}_1$ be a compact real form of $\mathfrak{a}$. Let G_1 be a simply-connected real Lie group whose Lie algebra is isomorphic to $\mathfrak{g}_1$; then G_1 is compact (Theorem 7.1.4). Let T_1 be a maximal torus of G_1. By Theorem 7.3.34, the diagram $D^*(G_1, T_1)$ is isomorphic to $(\Lambda(\Phi), 0, \Phi)$.

Let T_0 be a torus of dimension equal to the rank of M_0; then $D^*(T_0, T_0)$ is isomorphic to $(M_0, M_0, \varnothing)$, so $D^*(G_1 \times T_0, T_1 \times T_0)$ is isomorphic to $\tilde{D}$. Finally, let N be the finite subgroup of $T_1 \times T_0$ orthogonal to M (recall that M is of finite index in $M_0 \oplus \Lambda(\Phi) \cong X(T_1 \times T_0)$). Put $G = (G_1 \times T_0)/N$, $T = (T_1 \times T_0)/N$. Then G is a connected compact Lie group, T a maximal torus of G, and $D(G, T)$ is isomorphic to D (c.f. Example 7.3.35(b) and (c)). □

The classification of connected compact Lie groups up to isomorphism is thus reduced to that of reduced root diagrams. The connected compact semi-simple Lie groups correspond to the reduced root diagrams (M, M_0, Φ) such that $M_0 = 0$; giving such a diagram is equivalent to giving a reduced root system Φ in a vector space V over $\mathbb{Q}$ and a subgroup M of V such that $\Lambda_r(\Phi) \subseteq M \subseteq \Lambda(\Phi)$.

7.3.7 Automorphisms of a connected compact Lie group

Let G be a compact connected Lie group and T a maximal torus. Denote by $\mathrm{Aut}(G)$ the Lie group of automorphisms of G, and by $\mathrm{Aut}(G, T)$ the closed subgroup of $\mathrm{Aut}(G)$ consisting of the elements φ such that $\varphi(T) = T$. We have seen (Corollary 7.1.10) that the identity component of $\mathrm{Aut}(G)$ is the subgroup $\mathrm{Inn}(G)$ of inner automorphisms; denote by $\mathrm{Inn}_G(H)$ the image in $\mathrm{Inn}(G)$ of a subgroup H of G.

Let D be the covariant diagram of G relative to T; denote by $\mathrm{Aut}(D)$ the group of its automorphisms, and by $W(D)$ its Weyl group. The map $\varphi \mapsto D_*(\varphi)$ is a homomorphism from $\mathrm{Aut}(G, T)$ to $\mathrm{Aut}(D)$. Proposition 7.3.38 immediately gives:

Proposition 7.3.41. *The homomorphism* $\mathrm{Aut}(G, T) \to \mathrm{Aut}(D)$ *is surjective, with kernel* $\mathrm{Inn}_G(T)$.

Note that $\mathrm{Aut}(G, T) \cap \mathrm{Inn}(G) = \mathrm{Inn}_G(N_G(T))$ and that the image of $\mathrm{Inn}_G(N_G(T))$ in $\mathrm{Aut}(D)$ is $W(D)$ (Proposition 7.3.20). Thus, Proposition 7.3.41 gives an isomorphism

$$\mathrm{Aut}(G, T)/(\mathrm{Aut}(G, T) \cap \mathrm{Inn}(G)) \cong \mathrm{Aut}(D)/W(D).$$

Further, $\mathrm{Aut}(G) = \mathrm{Inn}(G) \cdot \mathrm{Aut}(G, T)$. Indeed, if φ belongs to $\mathrm{Aut}(G)$, $\varphi(T)$ is a maximal torus of T, hence is conjugate to T, and there exists an inner automorphism ν of G such that $\varphi(T) = \nu(T)$, in other words $\nu^{-1}\varphi \in \mathrm{Aut}(G, T)$. It follows that $\mathrm{Aut}(G)/\mathrm{Inn}(G)$ can be identified with $\mathrm{Aut}(G, T)/(\mathrm{Aut}(G, T) \cap \mathrm{Inn}(G))$, so in view of the preceding we have an exact sequence

$$1 \longrightarrow \mathrm{Inn}(G) \longrightarrow \mathrm{Aut}(G) \longrightarrow \mathrm{Aut}(D)/W(D) \longrightarrow 1$$

Consequently, we conclude the following result:

Proposition 7.3.42. *The group* $\mathrm{Aut}(G)/\mathrm{Inn}(G)$ *is isomorphic to* $\mathrm{Aut}(D)/W(D)$.

In particular, assume that G is semi-simple; the group $\mathrm{Aut}(D)$ can then be identified with the subgroup of $\mathrm{Aut}(\Phi(G, T))$ consisting of the elements f such that $f(X(T)) \subseteq X(T)$, and the subgroup $W(D)$ can be identified with $W(\Phi(G, T))$.

Corollary 7.3.43. *If G is simply-connected, or if G is semi-simple, the group* $\mathrm{Aut}(G)/\mathrm{Inn}(G)$ *is isomorphic to the group of automorphisms of the Dynkin graph of* $\Phi(G, T)$.

Proof. This follows from the preceding and Corollary 5.3.14. □

For all $\alpha \in \Phi(G, T)$, denote by $\mathfrak{g}(\alpha)$ the 2-dimensional vector subspace of $\mathfrak{g}$ such that $\mathfrak{g}(\alpha)_{(\mathbb{C})} = \mathfrak{g}^{\alpha} + \mathfrak{g}^{-\alpha}$; denote by κ the Killing form of $\mathfrak{g}$. A **framing** of (G, T) is a pair $(\Delta, (U_\alpha)_{\alpha \in \Phi})$, where Δ is a basis of $\Phi(G, T)$ and where, for all $\alpha \in \Delta$, u_α is an element of $\mathfrak{g}(\alpha)$ such that $\kappa(u_\alpha, u_\alpha) = -1$. A framing of G is a maximal torus T of G together with a framing of (G, T).

Lemma 7.3.44. *Let* Δ_0 *be a basis of* $\Phi(G, T)$*. The group* $\mathrm{Inn}_G(T)$ *operates simply-transitively on the set of framings of* (G, T) *of the form* $(\Delta_0, (u_\alpha)_{\alpha \in \Delta_0})$.

Proof. For all $\alpha \in \Delta_0$, denote by κ_α the restriction of the Killing form κ to $\mathfrak{g}(\alpha)$; the operation of T on $\mathfrak{g}(\alpha)$ defines a morphism $\iota_\alpha : T \to \mathrm{SO}(\kappa_\alpha)$. We have seen that $\mathrm{SO}(\kappa_\alpha)$ can be identified with S^1 in such a way that ι_α is identified with the root α. Since Δ_0 is a basis of Φ, it is a basis of the $\mathbb{Z}$-module $\Lambda_r(\Phi)$ generated by the roots, hence a basis of the submodule $X(T/Z(G))$ of $X(T)$. It follows that the product of the morphisms ι_α induces an isomorphism from $T/Z(G)$ to the product of the groups $\mathrm{SO}(\kappa_\alpha)$. But the latter group operates simply-transitively on the set of framings of (G, T) whose first component is Δ_0. $\square$

Proposition 7.3.45. *The group* $\mathrm{Inn}(G)$ *operates simply-transitively on the set of framings of* G.

Proof. Let $e = (T, \Delta, (u_\alpha))$ and $\tilde{e} = (\tilde{T}, \tilde{\Delta}, (\tilde{u}_\alpha))$ be two framings of G. There exist elements g in G such that $\mathrm{Inn}(g)(T) = \tilde{T}$, and these elements form a single coset modulo $N_G(T)$. Thus, we can assume that $T = \tilde{T}$, and we must prove that there exists a unique element of $\mathrm{Inn}_G(N_G(T))$ that transforms e to $\tilde{e}$. First, there exists a unique element w of $W(\Phi)$ such that $w(\Delta) = \tilde{\Delta}$. Since $W(\Phi)$ can be identified with $N_G(T)/T$, there exists $n \in N_G(T)$ such that $w = \mathrm{Inn}(n)$, and n is uniquely determined modulo T. Thus, we can assume that $\Delta = \tilde{\Delta}$, and we must prove that there exists a unique element of $\mathrm{Inn}_G(T)$ that transforms e to $\tilde{e}$, which is simply Lemma 7.3.44. $\square$

Corollary 7.3.46. *Let e be a framing of (G, T) and let E be the group of automorphisms of G that leave e stable. Then* $\mathrm{Aut}(G)$ *is the semi-direct product of E by* $\mathrm{Inn}(G)$*, and* $\mathrm{Aut}(G, T)$ *is the semi-direct product of E by* $\mathrm{Inn}(G) \cap \mathrm{Aut}(G, T) = \mathrm{Inn}_G(N_G(T))$.

Proof. Indeed, every element of $\mathrm{Aut}(G)$ transforms e into a framing of G. By Proposition 7.3.45, every coset of $\mathrm{Aut}(G)$ modulo $\mathrm{Inn}(G)$ meets E in a single point, hence the first assertion. The second is proved in the same way. $\square$

Remark 7.3.47. Let G and $\tilde{G}$ be two connected compact Lie groups, and let $e = (T, \Delta, (u_\alpha))$ and $\tilde{e} = (\tilde{T}, \tilde{\Delta}, (\tilde{u}_\alpha))$ be framings of G and $\tilde{G}$, respectively. Let X be the set of isomorphisms from G to $\tilde{G}$ that take e to $\tilde{e}$. The map $\varphi \mapsto D^*(\varphi)$ (resp. $D_*(\varphi)$) is a bijection from X to the set of isomorphisms from $D^*(\tilde{G}, \tilde{T})$ to $D^*(G, T)$ (resp. $D_*(G, T)$ to $D_*(\tilde{G}, \tilde{T})$) that map $\tilde{\Delta}$ to Δ (resp. Δ to $\tilde{\Delta}$). Indeed, this follows immediately from Proposition 7.3.38 and Lemma 7.3.44.

7.4 Conjugacy classes of maximal tori

Chapter 8

The language of schemes

8.1 Affine schemes

Let A be a ring. Recall that we have associate with A a topological space $\mathrm{Spec}(A)$, called the spectrum of A. In this section we shall make $\mathrm{Spec}(A)$ a locally ringed space and consider sheaf of modules over it; such spaces will be called **affine schemes**.

8.1.1 Sheaves associated with a module

Let A be a ring and M an A-module. For any element $f \in A$, let S_f be the multiplicative subset consisting of powers of f. Recall that the localization of M with respect to S_f is then denoted by M_f, and that of A by A_f. Let $\bar{S}_f$ be the saturation of S_f, which is defined to be the complement of the union of prime ideals of A that are disjoint from S_f, or equivalently not contains f. By **??**, the set $\bar{S}_f$ is also characterized by

$$\bar{S}_f = \{x \in A : \text{there exist } n, m \geq 0 \text{ such that } f^n x = f^m\}.$$

Also, by **??**, we have $\bar{S}_f A = A_f$ and $\bar{S}_f M = M_f$.

Lemma 8.1.1. *Let f, g be elements of A. Then the following conditions are equivalent:*

(i) *$g \in \bar{S}_f$, or equivalently $\bar{S}_g \subseteq \bar{S}_f$;*

(ii) *$f \in \sqrt{(g)}$, or equivalently $\sqrt{(f)} \subseteq \sqrt{(g)}$;*

(iii) *$D(f) \subseteq D(g)$, or equivalently $V(g) \subseteq V(f)$.*

Proof. We first note that $g \in \bar{S}_f$ is equivalent to $S_g \subseteq \bar{S}_f$, so the equivalence in (i). Also, the equivalence of (ii) and (iii) follows from **??**. Finally, if $g \in \bar{S}_f$, then there exist $n, m \geq 0$ such that $f^n g = f^m$, which is an element of (g), and thus $f \in \sqrt{(g)}$. Conversely, if $D(f) \subseteq D(g)$, then by the discriptions $S_f = \bigcup_{f \notin \mathfrak{p}} \mathfrak{p}$ and $S_g = \bigcup_{g \notin \mathfrak{p}} \mathfrak{p}$, we conclude that $\bar{S}_g \subseteq \bar{S}_f$, whence the lemma. □

If $D(g) \subseteq D(f)$ in $\mathrm{Spec}(A)$, then by Lemma 8.1.1, we have $\bar{S}_f \subseteq \bar{S}_g$, so there is a canonical homomorphism $\rho_{g,f} : M_f \to M_g$; moreover, if $D(f) \supseteq D(g) \supseteq D(h)$, we then have

$$\rho_{h,g} \circ \rho_{g,f} = \rho_{h,f}.$$

As f runs through $A - \mathfrak{p}$ (where $\mathfrak{p}$ is a point in $X = \mathrm{Spec}(A)$), the set S_f then constitute a filtered set indexed by $A - \mathfrak{p}$, since any two element f, g of $A - \mathfrak{p}$ contains S_{fg}; as the union of the S_f for $f \in A - \mathfrak{p}$ is $A - \mathfrak{p}$, we conclude from **??** that the $A_{\mathfrak{p}}$-module $M_{\mathfrak{p}}$ is canonically identified with the direct limit $\varinjlim M_f$, relative to the family $(\rho_{g,f})$ of homomorphisms. For each $f \in A - \mathfrak{p}$, we denote the canonical homomorphism from M_f to $M_{\mathfrak{p}}$ by

$$\rho^f_{\mathfrak{p}} : M_f \to M_{\mathfrak{p}}.$$

We now define the **structural sheaf** of the prime spectrum $X = \mathrm{Spec}(A)$, denoted by $\widetilde{A}$, to be the sheaf of rings associated with the presheaf $D(f) \mapsto A_f$ over the basis $\mathcal{B}$ of X, formed by $D(f)$ with

$f \in A$. Simialrly, for an A-module M,we define the **associated sheaf** $\widetilde{M}$ to be the sheaf associated presheaf $D(f) \mapsto M_f$ over the basis $\mathcal{B}$ of X. By the property of sheafification, it is clear that the stalk $\widetilde{A}_{\mathfrak{p}}$ (resp. $\widetilde{M}_{\mathfrak{p}}$) is identified with the ring $A_{\mathfrak{p}}$ (resp. with $A_{\mathfrak{p}}$-module $M_{\mathfrak{p}}$).

Theorem 8.1.2. *For each A-module M, the presheaf $D(f) \to M_f$ is a sheaf on the basis $\mathcal{B}$ of X, so for each $f \in A$ we have a canonical isomorphism*

$$M_f \to \Gamma(D(f), \widetilde{M}).$$

In particular, M is canonically identified with $\Gamma(X, \widetilde{M})$.

Proof. To show that the presheaf $D(f) \mapsto M_f$ is a sheaf on the basis $\mathcal{B}$ of X, we need to check the sheaf condition for the standard open coverings. If $D(f) = \bigcup_{i=1}^{n} D(g_i)$, then the sheaf condition for this covering is equivalent with the exactness of the sequence

$$0 \longrightarrow M_f \longrightarrow \bigoplus_i M_{g_i} \longrightarrow \bigoplus_{i,j} M_{g_i g_j} \; .$$

Note that $D(g_i) = D(fg_i)$, and hence we can rewrite this sequence as the sequence

$$0 \longrightarrow M_f \longrightarrow \bigoplus_i M_{fg_i} \longrightarrow \bigoplus_{i,j} M_{fg_i g_j} \; .$$

Since the $D(g_i)$'s cover $D(f)$ (which is identified with $\mathrm{Spec}(A_f)$), the elements $g_1, \dots, g_n$ generate the unit ideal in A_f, so we may apply **??** to the module M_f over A_f and the elements $g_1, \dots, g_n$ to conclude that the sequence is exact. □

Corollary 8.1.3. *Let M, N be A-modules. The canonical homomorphism*

$$\mathrm{Hom}_A(M, N) \to \mathrm{Hom}_{\widetilde{A}}(\widetilde{M}, \widetilde{N}), \quad \phi \mapsto \tilde{\phi}$$

is bijective. In particular, the relations $M = 0$ and $\widetilde{M} = 0$ are equivalent.

Proof. Consider the canonical homomorphism $\mathrm{Hom}_{\widetilde{A}}(\widetilde{M}, \widetilde{N}) \to \mathrm{Hom}_A(M, N)$, $\varphi \mapsto \Gamma(\varphi)$ (Theorem 8.1.2). It suffices to show that $\phi \mapsto \tilde{\phi}$ and $\varphi \mapsto \Gamma(\varphi)$ are inverses of each other. Now, it is evident that $\Gamma(\tilde{\phi}) = \phi$, by the definition of $\tilde{\phi}$. On the other hand, if we put $\phi = \Gamma(\varphi)$ for $\varphi \in \mathrm{Hom}_{\widetilde{A}}(\widetilde{M}, \widetilde{N})$, the map $\varphi_{D(f)} : \Gamma(D(f), \widetilde{M}) \to \Gamma(D(f), \widetilde{N})$ induced by φ is making the following diagram commutes:

$$\begin{tikzcd} M \arrow[r, "\phi"] \arrow[d, "\rho_{f,1}"'] & N \arrow[d, "\rho_{f,1}"] \\ M_f \arrow[r, "\varphi_{D(f)}"] & N_f \end{tikzcd}$$

We then have necessarily $\varphi_{D(f)} = \phi_f$ for each $f \in A$, which shows $\widetilde{\Gamma(\varphi)} = \varphi$. □

Proposition 8.1.4. *For each $f \in A$, the open set $D(f) \subseteq X$ is canonically identified with the spectrum $\mathrm{Spec}(A_f)$, and the sheaf $\widetilde{M}_f$ associated with the A_f-module M_f is canonically identified with the restriction $\widetilde{M}_{D(f)}$.*

Proof. The first assertion is proved in **??**. Now for $D(g) \subseteq D(f)$, then M_g is identified with the localization of M_f with respect to the canonical image of g in A_f, so the canonical identification of $\tilde{M}_f$ and $\widetilde{M}_{D(f)}$ follows by definition. □

Proposition 8.1.5. *The functor $M \mapsto \widetilde{M}$ is an exact functor from the category of A-modules to the category of $\widetilde{A}$-modules.*

Proof. Let M, N be two A-modules and $\phi : M \to N$ a homomorphism; for any $f \in A$, we have a corresponding homomorphism $\phi_f : M_f \to N_f$, and for $D(g) \subseteq D(f)$ the diagram

$$\begin{tikzcd} M_f \arrow[r, "\phi_f"] \arrow[d, "\rho_{g,f}"'] & N_f \arrow[d, "\rho_{g,f}"] \\ M_g \arrow[r, "\phi_g"] & N_g \end{tikzcd}$$

is commutative. These then give a homomorphism of $\widetilde{A}$-modules $\tilde{\phi}:\widetilde{M}\to\widetilde{N}$. Moreover, for each $\mathfrak{p}\in X$, $\tilde{\phi}_x$ is the direct limit of ϕ_f for $f\in A-\mathfrak{p}$, and concequently identified with the canonical homomorphism $\phi_{\mathfrak{p}}:M_{\mathfrak{p}}\to N_{\mathfrak{p}}$. If P is another A-module, $\psi:N\to P$ a homomorphism and $\eta=\psi\circ\phi$, then it is immediate that $\eta_{\mathfrak{p}}=\psi_{\mathfrak{p}}\circ\phi_{\mathfrak{p}}$, hence $\tilde{\eta}=\tilde{\psi}\circ\tilde{\phi}$. We thus get a covariant functor $\widetilde{(-)}$ from the category of A-modules to the category of $\widetilde{A}$-modules. This functor is exact since for each $\mathfrak{p}\in X$, $M\mapsto M_{\mathfrak{p}}$ is an exact functor; furthermore, we have $\operatorname{supp}(M)=\operatorname{supp}(\widetilde{M})$ by the definitions of these two members. □

Corollary 8.1.6. *Let M and N be two A-modules.*

(a) *If $\phi:M\to N$ is a homomorphism, then the sheaves associated with $\ker\phi$, $\operatorname{im}\phi$, and $\operatorname{coker}\phi$ are $\ker\tilde{\phi}$, $\operatorname{im}\tilde{\phi}$, and $\operatorname{coker}\tilde{\phi}$, respectively. In particular, $\tilde{\phi}$ is injective (resp. surjective, bijective) if and only if ϕ is injective (resp. surjective, bijective).*

(b) *If M is a filtered limit (resp. direct sum) of a family $(M_i)_{i\in I}$ of A-modules, then $\widetilde{M}$ is a filtered limit (resp. direct sum) of the family $(\widetilde{M}_i)$.*

Proof. For (a), it suffices to apply the exact functor $M\mapsto\widetilde{M}$ to the following exact sequences:

$$0\longrightarrow\ker\phi\longrightarrow M\longrightarrow\operatorname{im}\phi\longrightarrow 0$$

$$0\longrightarrow\operatorname{im}\phi\longrightarrow N\longrightarrow\operatorname{coker}\phi\longrightarrow 0$$

Now let (M_i,ρ_{ji}) be a filtered system of A-modules, with limit M, and let $\rho_i:M_i\to M$ be the canonical homomorphism. Since we have $\tilde{\rho}_{kj}\circ\tilde{\rho}_{ji}=\tilde{\rho}_{ji}$ and $\tilde{\rho}_i=\tilde{\rho}_j\circ\tilde{\rho}_{ji}$ for $i\leq j\leq k$, we see $(\widetilde{M},\tilde{\rho}_{ji})$ is a direct system of sheaves over X, and if we denote by $\eta_i:\widetilde{M}_i\to\varinjlim\widetilde{M}_i$ the canonical homomorphism, a unique homomorphism $\psi:\varinjlim\widetilde{M}_i\to\widetilde{M}$ such that $\psi\circ\eta_i=\tilde{\rho}_i$. For this ψ to be bijective, it suffices that for each $\mathfrak{p}\in X$, $\psi_{\mathfrak{p}}$ is a bijection from $(\varinjlim\widetilde{M}_i)_{\mathfrak{p}}$ to $\widetilde{M}_{\mathfrak{p}}$; but $\widetilde{M}_{\mathfrak{p}}=M_{\mathfrak{p}}$ and

$$(\varinjlim\widetilde{M}_i)_{\mathfrak{p}}=\varinjlim(\widetilde{M}_i)_{\mathfrak{p}}=\varinjlim(M_i)_{\mathfrak{p}}=M_{\mathfrak{p}}$$

Also, it follows by definition that $(\tilde{\rho}_i)_{\mathfrak{p}}$ and $(\eta_i)_{\mathfrak{p}}$ are equal to the canonical homomorphism from $(M_i)_{\mathfrak{p}}$ to $M_{\mathfrak{p}}$; since $(\tilde{\rho}_i)_{\mathfrak{p}}=\psi_{\mathfrak{p}}\circ(\eta_i)_{\mathfrak{p}}$, $\psi_{\mathfrak{p}}$ is therefore the identity.

Finally, if M is a direct sum of two modules N and P, it is immediate that $\widetilde{M}=\widetilde{M}\oplus\widetilde{P}$; by taking filtered limits, we then generalize this result for the direct sum of an arbitrary family. This completes the proof. □

Remark 8.1.7. By Proposition 8.1.5, we conclude that the sheaves which are isomorphic to the sheaves associated with A-modules form an abelian category. Note also that it follows from Corollary 8.1.6 that if M is a finitely generated A-module, that is, if there exists a surjective homomorphism $A^n\to M$, then there exists a homomorphism surjective $\widetilde{A}^n\to\widetilde{M}$, in other words, the $\widetilde{A}$-module $\widetilde{M}$ is generated by a finite family of sections over X, and vice versa.

Corollary 8.1.8. *Let N and P be submodules of M. The sheaves $\widetilde{N}$ and $\widetilde{P}$ can be identified with sub-$\widetilde{A}$-modules of $\widetilde{M}$, and we have*

$$\widetilde{N+P}=\widetilde{N}+\widetilde{P},\quad \widetilde{N\cap P}=\widetilde{N}\cap\widetilde{P}.$$

In particular, if $\widetilde{N}=\widetilde{P}$, then $N=P$.

Proof. If N is a submodule of an A-module M, the canonical injection $N\to M$ induced an injective homomorphism $\widetilde{N}\to\widetilde{M}$, hence identifies $\widetilde{N}$ with a sub-$\widetilde{A}$-module of $\widetilde{M}$. Now note that $N+P$ is the image of the canonical homomorphism $\alpha:N\oplus P\to M$, so by Corollary 8.1.6 we have

$$\widetilde{N+P}=\widetilde{\operatorname{im}\alpha}=\operatorname{im}\tilde{\alpha}=\widetilde{N}+\widetilde{P}$$

since $\tilde{\alpha}$ is equal to the canonical homomorphism $\widetilde{N}\oplus\widetilde{P}\to\widetilde{M}$. Similarly, since $N\cap P$ is the kenrel of the canonical homomorphism $M\to(M/N)\oplus(M/P)$, we also have $\widetilde{N\cap P}=\widetilde{N}\cap\widetilde{P}$. □

Corollary 8.1.9. *Over the category of sheaves isomorphic to sheaves associated with A-modules, the global section functor Γ is exact.*

Proof. In fact, let $\widetilde{M} \xrightarrow{\tilde{\phi}} \widetilde{N} \xrightarrow{\tilde{\psi}} P$ be an exact sequence corresponding to two homomorphisms $\phi : M \to N$ and $\psi : N \to P$. If $Q = \operatorname{im}\phi$ and $R = \ker\psi$, we have

$$\widetilde{Q} = \operatorname{im}\tilde{\phi} = \ker\tilde{\psi} = \widetilde{R}$$

by Corollary 8.1.6, so $Q = R$ and the sequence is exact. □

Corollary 8.1.10. *Let M and N be two A-modules.*

(a) *The sheaf associated with $M \otimes_A N$ is canonically identified with $\widetilde{M} \otimes_{\widetilde{A}} \widetilde{N}$.*

(b) *If moreover M is finitely presented, the sheaf associated with $\operatorname{Hom}_A(M, N)$ is canonically identified with $\mathcal{H}om_{\widetilde{A}}(\widetilde{M}, \widetilde{N})$.*

Proof. The sheaf $\mathscr{F} = \widetilde{M} \otimes_{\widetilde{A}} \widetilde{N}$ is the sheafification of the presheaf

$$U \mapsto \mathscr{F}(U) = \Gamma(U, \widetilde{M}) \otimes_{\Gamma(U,\widetilde{A})} \Gamma(U, \widetilde{N})$$

where U runs through the basis $\mathcal{B}$ of X formed by $D(f)$, $f \in A$. Now, $\mathscr{F}(D(f))$ is canonically identified with $M_f \otimes_{A_f} N_f$ by Theorem 8.1.2, which is isomorphic to $\Gamma(D(f), \widetilde{M \otimes_A N})$. Moreover, it is immediately verified that the canonical isomorphisms

$$\mathscr{F}(D(f)) \cong \Gamma(D(f), \widetilde{M \otimes_A N})$$

is compatible with the restriction maps, so they define a canonical isomorphism $\widetilde{M} \otimes_{\widetilde{A}} \widetilde{N} \cong \widetilde{M \otimes_A N}$.

Now assume that M is finitely presented. The sheaf $\mathscr{G} = \mathcal{H}om_{\widetilde{A}}(\widetilde{M}, \widetilde{N})$ is the sheafification of the presheaf

$$U \mapsto \mathscr{G}(U) = \operatorname{Hom}_{\widetilde{A}|_U}(\widetilde{M}|_U, \widetilde{N}|_U)$$

where U runs through the basis $\mathcal{B}$ of X. By Proposition 8.1.4 and Corollary 8.1.3, the module $\mathscr{G}(D(f))$ is then identified with $\operatorname{Hom}_{A_f}(M_f, N_f)$, which is isomorphic to $\Gamma(D(f), \widetilde{\operatorname{Hom}_A(M, N)})$ by **??**. It is clear that these isomorphisms are compatible with the restriction maps, so we conclude that $\mathcal{H}om_{\widetilde{A}}(\widetilde{M}, \widetilde{N}) \cong \widetilde{\operatorname{Hom}_A(M, N)}$. □

Now consider an A-algebra B (commutative); this can be interpreted by saying that B is an A-module and that we are given an element $e \in B$ and an A-homomorphism $\varphi : B \otimes_A B \to B$ so that the diagrams

$$\begin{tikzcd} B \otimes_A B \otimes_A B \arrow[r, "\varphi\otimes 1"] \arrow[d, "1\otimes\varphi"'] & B \otimes_A B \arrow[d, "\varphi"] \\ B \otimes_A B \arrow[r, "\varphi"] & B \end{tikzcd} \qquad \begin{tikzcd} B \otimes_A B \arrow[rr, "\sigma"] \arrow[dr, "\varphi"'] & & B \otimes_A B \arrow[dl, "\varphi"] \\ & B & \end{tikzcd}$$

(where σ is the canonical symmetry) commute, and that $\varphi(e \otimes x) = \varphi(x \otimes e) = x$. In view of Corollary 8.1.10, the homomorphism $\tilde{\varphi} : \widetilde{B} \otimes_{\widetilde{A}} \widetilde{B} \to \widetilde{B}$ of $\widetilde{A}$-modules satisfies similar conditions, hence defines a $\widetilde{A}$**-algebra** structure on $\widetilde{B}$. In the same way, the data of a B-module N amounts to giving an A-module N and an A-homomorphism $\psi : B \otimes_A N \to N$ such that the diagram

$$\begin{tikzcd} B \otimes_A B \otimes_A N \arrow[r, "\varphi\otimes 1"] \arrow[d, "1\otimes\psi"'] & B \otimes_A N \arrow[d, "\psi"] \\ B \otimes_A N \arrow[r, "\psi"] & N \end{tikzcd}$$

commutes and $\psi(e \otimes n) = n$; the homomorphism $\tilde{\psi} : \widetilde{B} \otimes_{\widetilde{A}} \widetilde{N} \to \widetilde{N}$ then satisfies similar conditions, and defines on $\widetilde{N}$ a $\widetilde{B}$-module structure.

If $\rho : B \to B'$ (resp. $\phi : N \to N'$) is a homomorphism of A-algebras (resp. a homomorphisms of B-modules), then $\tilde{\rho}$ (resp. $\tilde{\phi}$) is a homomorphism of $\widetilde{A}$-algebras (resp. a homomorphisms of $\widetilde{B}$-modules), $\ker\tilde{\rho}$ is an ideal of $\widetilde{B}$ (resp. $\ker\tilde{\phi}$, $\operatorname{coker}\tilde{\phi}$, and $\operatorname{im}\tilde{\phi}$ are $\widetilde{B}$-modules). Moreover, by **??**(b) if N is a B-module, then $\widetilde{N}$ is a finitely generated $\widetilde{B}$-module if and only if N is finitely generated over B.

If M and N are two B-modules, the $\widetilde{B}$-module $\widetilde{M} \otimes_{\widetilde{B}} \widetilde{N}$ is canonically identified with $\widetilde{M \otimes_B N}$; similarly, $\mathcal{H}om_{\widetilde{B}}(\widetilde{M}, \widetilde{N})$ is canonically identified with $\widetilde{\operatorname{Hom}_B(M, N)}$ if M is finitely presented. If $\mathfrak{b}$ is an ideal of B, then $\widetilde{\mathfrak{b}N} = \tilde{\mathfrak{b}}\widetilde{N}$.

Finally, if B is a graded A-algebra with (B_n) its graduation, the $\widetilde{A}$-algebra $\widetilde{B}$ is then the direct sum of the sub-$\widetilde{A}$-modules $\widetilde{B}_n$ (Corollary 8.1.6), so $(\widetilde{B}_n)$ is a graduation of $\widetilde{B}$. Similarly, if M is a graded B-module with graduation (M_n), then $\widetilde{M}$ is a graded $\widetilde{B}$-module with graduation $(\widetilde{M}_n)$.

8.1.2 Functorial properties of the associated sheaf

We now consider the functorial properties of the operation $M \mapsto \widetilde{M}$. Let A and B be rings and $\varphi : B \to A$ be a ring homomorphism. Then we have an associated map

$$^a\varphi : X = \operatorname{Spec}(A) \to Y = \operatorname{Spec}(B)$$

We will define a canonical homomorphism

$$\varphi^\# : \mathscr{O}_Y \to {}^a\varphi_*(\mathscr{O}_X)$$

of sheaf of rings. For any $g \in B$, we set $f = \varphi(g)$; we have $\varphi^{-1}(D(g)) = D(f)$ by **??**. Now the sections $\Gamma(D(g), \widetilde{B})$ and $\Gamma(D(f), \widetilde{A})$ are canonically identified with B_g and A_f, respectively, and we have an induced map $\varphi_g : B_g \to A_f$, which then gives a homomorphism of rings

$$\Gamma(D(g), \widetilde{B}) \to \Gamma(\varphi^{-1}(D(g)), \widetilde{A}) = \Gamma(D(g), {}^a\varphi_*(\widetilde{A})).$$

Moreover, these homomorphisms satisfy the following compatible conditions: for $D(g) \supseteq D(g')$ in Y, the diagram

$$\begin{tikzcd}
\Gamma(D(g), \widetilde{A}) \arrow[r] \arrow[d] & \Gamma(D(g), {}^a\varphi_*(\widetilde{A})) \arrow[d] \\
\Gamma(D(g'), \widetilde{A}) \arrow[r] & \Gamma(D(g'), {}^a\varphi_*(\widetilde{A}))
\end{tikzcd}$$

is commutative; we then get a morphism of $\mathscr{O}_Y$-algebras, since $D(g)$ form a basis for the topological space Y. The couple $({}^a\varphi, \varphi^\#)$ is called the **canonical morphism** of the locally ringed spaces induced by φ.

We also note that, if $y = {}^a\varphi(x)$, the homomorphism $\varphi^\#_x$ is no other that the homomorphism

$$\varphi_x : B_y \to A_x$$

induced by the homomorphism $\varphi : B \to A$. In fact, for $b/g \in B_y$, where $b, g \in B$ and $g \notin \mathfrak{p}_y$, $D(g)$ is then an open neighborhood of y in Y, and the homomorphism

$$\Gamma(D(g), \widetilde{B}) \to \Gamma(D(g), ({}^a\varphi)_*(\widetilde{A}))$$

induced by $\varphi^\#$ is just φ_g; by considering the section $\xi \in \Gamma(D(g), \widetilde{B})$ corresponding to b/g, we then obtain that $\varphi^\#_x(\xi) = \varphi(b)/\varphi(g)$ in A_x.

Example 8.1.11. Let S be a multiplicative subset of A and $\varphi : A \to S^{-1}A$ the canonical homomorphism. We have seen in **??** that ${}^a\varphi$ is a homeomorphism from $Y = \operatorname{Spec}(S^{-1}A)$ to the subspace $X = \operatorname{Spec}(A)$ formed by x such that $\mathfrak{p}_x \cap S = \varnothing$. Moreover, for any x in this subspace, hence of the form ${}^a\varphi(y)$ where $y \in Y$, the homomorphism $\varphi^\#_y : \mathscr{O}_{Y,y} \to \mathscr{O}_{X,x}$ is bijective; therefore, $\mathscr{O}_Y$ is identified with the sheaf induced over Y by $\mathscr{O}_X$.

Proposition 8.1.12. *Let $\varphi : B \to A$ be a ring homomorphism and $\Phi : X \to Y$ the associated morphism. For any A-module M, there exists a canonical functorial isomorphism of the $\mathscr{O}_Y$-module $\widetilde{\varphi^*(M)}$ to its direct image $\Phi_*(\widetilde{M})$.*

Proof. For $g \in B$, put $f = \varphi(g)$; the modules $\Gamma(D(g), \widetilde{\varphi^*(M)})$ and $\Gamma(D(f), \widetilde{M})$ are identified with $(\varphi^*(M))_g$ and M_f, respectively; moreover, the B_g-module $\varphi_g^*(M_f)$ is canonically isomorphic to $(\varphi^*(M))_g$. We then have a functorial isomorphism of $\Gamma(D(g), \widetilde{B})$-modules:

$$\Gamma(D(g), \widetilde{\varphi^*(M)}) \cong \varphi^*(\Gamma(D(\varphi(g)), \widetilde{M}))$$

and this isomorphism satisfies the compatible conditions with restrictions, hence define an isomorphism of sheaves. □

This proof also proves that for any A-algebra R, the canonical functorial isomorphism $\widetilde{\varphi^*(R)} \to \Phi_*(\widetilde{R})$ is an isomorphism of $\mathscr{O}_Y$-algebras. If M is an R-module, the canonical isomorphism $\widetilde{\varphi^*(M)} \cong \Phi_*(\widetilde{M})$ is an isomorphism of $\Phi_*(\widetilde{R})$-modules.

Corollary 8.1.13. *The direct image functor Φ_* is exact on the category of quasi-coherent sheaves.*

Proof. Recall that the functor φ^* is exact and $M \mapsto \widetilde{M}$ is an exact functor. □

Proposition 8.1.14. *Let $\varphi : B \to A$ be a ring homomorphism and $\Phi : X \to Y$ the associated morphism. Let N be a B-module and $\varphi_*(N)$ the A-module $N \otimes_B A$. Then there exist a canonical functorial isomorphism of the $\mathscr{O}_X$-module $\Phi^*(\widetilde{N})$ to $\widetilde{\varphi_*(N)}$.*

Proof. We first remark that $j : z \mapsto z \otimes 1$ is a B-homomorphism from N to $\varphi^*\varphi_*(N)$: this holds because for $g \in B$, we have

$$(gz) \otimes 1 = z \otimes \varphi(g) = \varphi(g)(z \otimes 1).$$

By Corollary 8.1.3, this corresponds to a homomorphism $\tilde{j} : \widetilde{N} \to \varphi^*\widetilde{(\varphi_*(N))}$ of $\mathscr{O}_Y$-modules, and via Proposition 8.1.12 we can think that $\tilde{j}$ maps $\widetilde{N}$ to $\Phi_*(\widetilde{\varphi_*(N)})$. From the adjointness of Φ^* and Φ_*, this canonically corresponds to a homomorphism

$$\theta : \Phi^*(\widetilde{N}) \to \widetilde{\varphi_*(N)}.$$

It then remains to show that θ is bijective, or equivalently that θ_x is bijective for every $x \in X$. For this, put $y = {}^a\varphi(x)$, choose $g \in B$ such that $y \in D(g)$, and let $f = \varphi(g)$. Then the ring $\Gamma(D(f), \widetilde{A})$ is identified with A_f, the module $\Gamma(D(f), \widetilde{\varphi_*(N)})$ is identified with $(\varphi_*(N))_f$, and $\Gamma(D(g), \widetilde{N})$ is identified with N_g. Let $s = n/g^p$ ($n \in N$) be a section of $\Gamma(D(g), \widetilde{N})$ and $t = a/f^q$ ($a \in A$) a section of $\Gamma(D(f), \widetilde{A})$. Then, since s is send to $(n \otimes 1)/f^p$ be $\tilde{j}$, by definition we have

$$\theta_x(s_x \otimes t_x) = t_x \cdot s_x.$$

Recall that we can identify $(\varphi_*(N))_f$ with $N_f \otimes_{B_g} \varphi^*(A_f)$, under which n/g^p is identified with $(n/g^p) \otimes 1$. So it is immediately seen that θ_x is none other than the canonical isomorphism

$$N_y \otimes_{B_y} \varphi_y^*(A_x) \cong (\varphi_*(N))_x = (N \otimes_B \varphi^*(A))_x.$$

Finally, let $v : N_1 \to N_2$ be a homomorphism of B-modules; since $\tilde{v}_y = v_y$ for any $y \in Y$, it follows immediately from the preceding argument that $\Phi^*(\tilde{v})$ is canonically identified to $\widetilde{v \otimes 1}$, which completes the proof. □

If S is an B-algebra, the canonical isomorphism of $\Phi^*(\widetilde{S})$ to $\widetilde{\varphi_*(S)}$ is an isomorphism of $\mathscr{O}_X$-algebras; if moreover N is a S-module, the canonical isomorphism of $\Phi^*(\widetilde{N})$ to $\widetilde{\varphi_*(N)}$ is an isomorphism of $\Phi^*(\widetilde{S})$-algebras.

Corollary 8.1.15. *The sections of $\Phi^*(\widetilde{N})$ which are cannical images of section of the B-module $\Gamma(\widetilde{N})$, generate the A-module $\Gamma(\Phi^*(\widetilde{N}))$.*

Proof. In fact, these images are identified with the elements $z \otimes 1$ of $\varphi_*(N)$, if we identify N and $\varphi_*(N)$ with $\Gamma(\widetilde{N})$ and $\Gamma(\widetilde{\varphi_*(N)})$. □

By the proof of Proposition 8.1.14, we see that the canonical map $\alpha : \widetilde{N} \to \Phi_*\Phi^*(\widetilde{N})$ is none other than the homomorphism $\tilde{j}$, where $j : N \to \varphi_*(N)$ is the canonical map $z \mapsto z \otimes 1$. Similarly, the canonical map $\beta : \Phi^*\Phi_*(\widetilde{M}) \to \widetilde{M}$ is none other than the homomorphism $\tilde{p}$, where $p : \varphi^*(M) \otimes_B \varphi^*(A) \to M$ is the canonical homomorphism that sends $m \otimes a$ to am.

Corollary 8.1.16. *Let N_1 and N_2 be B-modules and assume that N_1 is finitely presented. Then there is a canonical homomorphism*

$$\Phi^*(\mathcal{H}om_{\widetilde{B}}(\widetilde{N}_1, \widetilde{N}_2)) \to \mathcal{H}om_{\widetilde{A}}(\Phi^*(\widetilde{N}_1), \Phi^*(\widetilde{N}_2)).$$

This homomorphism is bijective if φ is a flat homomorphism.

Proof. By the above remarks and Corollary 8.1.10, this homomorphism is induced by the homomorphism

$$\mathrm{Hom}_B(N_1, N_2) \otimes_B A \to \mathrm{Hom}_A(N_1 \otimes_B A, N_2 \otimes_B A).$$

The last assertion follows from **??**. ☐

A locally ringed space $(X, \mathscr{O}_X)$ is called an **affine scheme** if it is isomorphic to the spectrum of a ring A. In this case, the ring $\Gamma(X, \mathscr{O}_X)$ is canonically identified with A. By absuing language, we often call $\mathrm{Spec}(A)$ an affine scheme, without mention the structural sheaf.

Let A and B be two rings and $(X, \mathscr{O}_X)$, $(Y, \mathscr{O}_Y)$ the corresponding affine schemes. Then any ring homomorphism $\varphi : B \to A$ corresponds to a morphism $({}^a\varphi, \varphi^\#) : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$. Note that the homomorphism φ is completely determined by $({}^a\varphi, \varphi^\#)$, since by definition we have $\varphi = \Gamma(\varphi^\#) : \Gamma(\widetilde{B}) \to \Gamma(\widetilde{A})$.

Theorem 8.1.17. *Let $(X, \mathscr{O}_X)$ and $(Y, \mathscr{O}_Y)$ be two affine schemes. Then any morphism $(\psi, \psi^\#) : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ of locally ringed spaces is of the form $({}^a\varphi, \varphi^\#)$, where $\varphi : \Gamma(Y, \mathscr{O}_Y) \to \Gamma(X, \mathscr{O}_X)$ is a ring homomorphism.*

Proof. Put $A = \Gamma(X, \mathscr{O}_X)$ and $B = \Gamma(Y, \mathscr{O}_Y)$. Let $(\psi, \psi^\#) : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ be a morphism of locally ringed spaces. By definition, $\psi^\#$ is a homomorphism from $\mathscr{O}_Y$ to $\psi_*\mathscr{O}_X$, and we then deduce a canonical homomorphism of rings

$$\varphi = \Gamma(\psi^\#) : B = \Gamma(Y, \mathscr{O}_Y) \to \Gamma(Y, \psi_*\mathscr{O}_X) = \Gamma(X, \mathscr{O}_X) = A.$$

Since $\psi^\#_x$ is a local homomorphism, by passing to quotients we deduce a monomorphism θ^x from the residue field $\kappa(\psi(x))$ into the residue field $\kappa(x)$ such that, for any section $f \in \Gamma(Y, \mathscr{O}_Y)$, we have $\theta^x(f(\psi(x))) = \varphi(f)(x)$ (we consider the elements of $\Gamma(Y, \mathscr{O}_Y)$ as functions on B). The relationship $f(\psi(x)) = 0$ is therefore equivalent to $\varphi(f)(x) = 0$, which means $\psi(x) = {}^a\varphi(x)$. Since this hold for any $x \in X$, we conclude that $\psi = {}^a\varphi$. We also know that the diagram

$$\begin{tikzcd}
B = \Gamma(Y, \mathscr{O}_Y) \arrow[r, "{}^a\varphi"] \arrow[d] & \Gamma(X, \mathscr{O}_X) = A \arrow[d] \\
B_{\psi(x)} \arrow[r, "\psi^\#_x"] & A_x
\end{tikzcd}$$

is commutative, so $\psi^\#_x$ is equal to the homomorphism $\varphi_x : B_{\psi(x)} \to A_x$ induced from φ. Since the morphism $\psi^\#$ is determined by $\psi^\#_x$, we obtain that $\psi^\# = \varphi^\#$. ☐

Corollary 8.1.18. *Let $(X, \mathscr{O}_X)$ and $(Y, \mathscr{O}_Y)$ be two affine schemes. Then there is a canonical bijection*

$$\mathrm{Mor}(X, Y) \to \mathrm{Hom}_{\mathbf{Ring}}(\Gamma(Y, \mathscr{O}_Y), \Gamma(X, \mathscr{O}_X))$$

which sends a morphism $(f, f^\#)$ to the global section of $f^\#$.

We can also say that the functors $(\mathrm{Spec}(A), \widetilde{A})$ in A and $\Gamma(X, \mathscr{O}_X)$ in $(X, \mathscr{O}_X)$ define an equivalence of the opposite category of commutative rings and of the category of the category of affine schemes.

Corollary 8.1.19. *If $\varphi : B \to A$ is surjective, then the corresponding morphism Φ is a monomorphism of locally ringed spaces.*

Proof. The map ${}^a\varphi$ is injective by **??**, and since φ is surjective, for any $x \in x$ the map $\varphi^\#_x : B_{{}^a\varphi(x)} \to A_x$, obtained by passing to localization, is surjective; these prove the assertion. ☐

8.1.3 Quasi-coherent sheaves over affine schemes

Recall that we have defined the abstract notion of a quasi-coherent sheaf. In this paragraph we show that any quasi-coherent sheaf on an affine scheme $\mathrm{Spec}(A)$ corresponds to the sheaf $\widetilde{M}$ associated with an A-module M.

Lemma 8.1.20. *Let $X = \mathrm{Spec}(A)$ and $V = \bigcup_{i=1}^{n} D(g_i)$ be a union of finitely may standard opens. Let $\mathscr{F}$ be an $\mathscr{O}_X$-module satisfying the conditions:*

(a) *For any $D(f) \subseteq D(g_i)$ (resp. $D(g_ig_j)$) and any section $s \in \Gamma(D(f), \mathscr{F})$, there exists $n \geq 0$ such that $f^n s$ can be extended to a section of $\mathscr{F}$ on $D(g_i)$ (resp. on $D(g_ig_j)$).*

(b) *For any $D(f) \subseteq D(g_i)$ (resp. $D(g_ig_j)$) and any section $t \in \Gamma(V, \mathscr{F})$ such that $t|_{D(f)} = 0$, there exists $n \geq 0$ such that $f^n t = 0$.*

Then we have the stronger conditions:

(α) *For any $f \in A$ and any section $s \in \Gamma(D(f) \cap V, \mathscr{F})$, there exists $n \geq 0$ such that $f^n s$ can be extended to a section of $\mathscr{F}$ on V.*

(β) *For any $f \in A$ and any section $t \in \Gamma(V, \mathscr{F})$ such that $t|_{D(f) \cap V} = 0$, there exists $n \geq 0$ such that $f^n t = 0$.*

Proof. First we prove condition (β). Since $D(f) \cap D(g_i) = D(fg_i)$, for each i we have an integer $n_i \geq 0$ such that $(fg_i)^{n_i} t$ restricts to zero on $D(g_i)$. Since g_i is invertible in $A_{g_i} = \Gamma(D(g_i), \mathscr{O}_X)$, this implies $f^{n_i} t = 0$ on $D(g_i)$. Take n such that $n \geq n_i$, then $f^n t = 0$ on each $D(g_i)$, whence $f^n t = 0$ and we get (β).

To show (α), we apply (a) on $\mathscr{F}|_{D(g_i)}$ to get an integer integers $n_i \geq 0$ and $s_i' \in \Gamma(D(g_i), \mathscr{F})$ such that

$$s_i'|_{D(fg_i)} = (fg_i)^{n_i} s|_{D(fg_i)}.$$

By inverting g_i, this produces sections $s_i \in \Gamma(D(g_i), \mathscr{F})$ such that

$$s_i|_{D(fg_i)} = f^{n_i} s|_{D(fg_i)}.$$

We may assume that all n_i take the same value n. Then each $s_i - s_j$ restricts to zero on $D(f) \cap D(g_i) \cap D(g_j) = D(fg_ig_j)$, so by applying (b) on $\mathscr{F}|_{D(g_ig_j)}$ we get an integer m_{ij} such that

$$(fg_ig_j)^{m_{ij}}(s_i - s_j)|_{D(g_ig_j)} = 0.$$

Then similarly, since g_ig_j is invertible in $A_{g_ig_j}$, this implies

$$f^{m_{ij}}(s_i - s_j)|_{D(g_ig_j)} = 0.$$

We can also assume that all m_{ij}'s take the same value m, so that by the sheaf condition there exists a section $u \in \Gamma(V, \mathscr{F})$ such that $u|_{D(g_i)} = f^m s_i$. Then $f^n u$ extends $f^{m+n} s$, as desired. ☐

Theorem 8.1.21. *Let $X = \mathrm{Spec}(A)$ be an affine scheme. Let V be a quasi-compact open subset and $\mathscr{F}$ be an $\mathscr{O}_X|_V$-module. Then the following are equivalent:*

(i) *There is a A-module M such that $\mathscr{F}$ is isomorphic to $\widetilde{M}|_V$.*

(ii) *There exists a finite open covering (V_i) of V by sets of the form $D(f_i)$ ($f_i \in A$) contained in V, such that, for each i, $\mathscr{F}|_{V_i}$ is isomorphic to a sheaf of the form $\widetilde{M}_i$, where M_i is an A_{f_i}-module.*

(iii) *The sheaf $\mathscr{F}$ is quasi-coherent.*

(iv) ***(Serre's lifting criterion)*** *The following conditions are satisfied:*

  (a) *For any $D(f) \subseteq V$ and any section $s \in \Gamma(D(f), \mathscr{F})$, there exists $n \geq 0$ such that $f^n s$ can be extended to a section of $\mathscr{F}$ on V.*

  (b) *For any $D(f) \subseteq V$ and any section $t \in \Gamma(V, \mathscr{F})$ such that $t|_{D(f)} = 0$, there exists $n \geq 0$ such that $f^n t = 0$.*

Proof. The implication (i)⇒(ii) is immediate from Proposition 8.1.4 since X can be covered by standard opens. Also, since any A-module is isomorphic to the kernel of a homomorphism $A^{\oplus I} \to A^{\oplus J}$, Corollary 8.1.6 shows that (ii)⇒(iii). Conversely, if $\mathscr{F}$ is quasi-coherent, any point $x \in V$ possesses a neighborhood of the form $D(f) \subseteq V$ such that $\mathscr{F}|_{D(f)}$ is isomorphic to the cokernel of a homomorphism $(\widetilde{A}_f)^{\oplus I} \to (\widetilde{A}_f)^{\oplus J}$, hence to the sheaf associated with the cokernel of the corresponding homomorphism $A_f^{\oplus I} \to A_f^{\oplus J}$ (Corollary 8.1.3 and Corollary 8.1.6); since V is quasi-compact, it then follows that (iii) implies (ii).

Now we prove that (ii)⇒(iv). First assume that $V = D(g)$ for some $g \in A$, and $\mathscr{F}$ is isomorphic to $\widetilde{N}$ for some A_g-module N. Since $D(g)$ can be identified with $\operatorname{Spec}(A_g)$, we can assume that $g = 1$ and $V = X$. In this case, the set $\Gamma(D(f), \mathscr{F})$ and N_f are canonically identified (Theorem 8.1.2), and it is clear that conditions (a) and (b) in (iv) are satisfied. To prove the general case, since V is quasi-compact we can choose a finite covering by standard opens $D(g_i)$ with $\mathscr{F}|_{D(g_i)}$ isomorphic to $\widetilde{M}_i$ for some A_{g_i}-module M_i. Then $\mathscr{F}$ satisfies the conditions (a) and (b) in Lemma 8.1.20, so by Lemma 8.1.20, $\mathscr{F}$ also satisfies conditions (α) and (β), which is what we want.

Finally, we show that (iv)⇒(i). First we prove that, if (a) and (b) hold for $\mathscr{F}$, then they hold for $\mathscr{F}|_{D(g)}$ with $D(g) \subseteq V$. This is evident for condition (a); as for (b), if $t \in \Gamma(D(g), \mathscr{F})$ restricts to zero on $D(f) \subseteq D(g)$, then by condition (a) there is an integer $m \geq 0$ such that $g^m t$ can be extended to V. By applying condition (b) on the extension of $g^m t$, we get another integer $n \geq 0$ such that $f^n g^m t = 0$. Since g is invertible in A_g, this gives $f^n t = 0$ as desired.

This being done, since V is quasi-compact, by Lemma 8.1.20 we know that conditions (α) and (β) holds for $\mathscr{F}$. Now consider the module $M = \Gamma(V, \mathscr{F})$; we shall define a morphism $\varphi : \widetilde{M} \to j_*\mathscr{F}$, where $j : V \hookrightarrow X$ is the inclusion. For this, it suffices to define

$$\varphi_f : M_f \to \Gamma(D(f), j_*\mathscr{F}) = \Gamma(D(f) \cap V, \mathscr{F})$$

for each $f \in A$. Since f is invertible in A_f and $\Gamma(D(f) \cap V, \mathscr{F})$ is a A_f-module, the restriction $M = \Gamma(V, \mathscr{F}) \to \Gamma(D(f) \cap V, \mathscr{F})$ factors into

$$M \longrightarrow M_f \xrightarrow{\varphi_f} \Gamma(D(f) \cap V, \mathscr{F})$$

which gives the desired maps φ_f. We now claim that the conditions (a) and (b) in (iv) imply that φ is an isomorphism. In fact, if $s \in \Gamma(D(f) \cap V, \mathscr{F})$, then by condition (a) there exist an integer $n \geq 0$ and $z \in \Gamma(V, \mathscr{F}) = M$ such that $z|_{D(f) \cap V} = f^n s$; then $\varphi_f(z/f^n) = s$, showing that φ is surjective. Similarly, if there is $z \in M$ such that $\varphi_f(z/1) = 0$ in $D(f) \cap V$, then by condition (b) there is $n \geq 0$ such that $f^n z = 0$, so that $z/1 = 0$ in M_f. This means φ_f is injective, so we get an isomorphism $\widetilde{M} \cong j_*\mathscr{F}$. By restriction, we then conclude that $\mathscr{F} \cong \widetilde{M}|_V$. □

Corollary 8.1.22. *Let $X = \operatorname{Spec}(A)$ be an affine scheme. Then the functors $M \mapsto \widetilde{M}$ and $\mathscr{F} \mapsto \Gamma(X, \mathscr{F})$ define equivalences of categories between the category of quasi-coherent $\mathscr{O}_X$-modules and the category of A-modules.*

Proof. The space X it self is quasi-compact, so we can apply Theorem 8.1.21. □

Corollary 8.1.23. *Let $X = \operatorname{Spec}(A)$ be an affine scheme. Then kernels and cokernels of maps of quasi-coherent $\mathscr{O}_X$-modules are quasi-coherent.*

Proof. This follows from the exactness of the functor $\widetilde{M}$ and Corollary 8.1.6. □

Corollary 8.1.24. *For $\widetilde{M}$ to be a $\mathscr{O}_X$-module of finite type (resp. of finite presentation), it is necessary and sufficient that M is a finitely generated A-module (resp. of finite presentation).*

Proof. In view of the exactness of the functor $M \mapsto \widetilde{M}$, it is immediate that if M is of finite type (resp. finite presentation), so is $\widetilde{M}$. Conversely, if $\widetilde{M}$ is of finite type (resp. finite presentation), since X is quasi-compact, there exists finitely many $f_i \in A$ such that $D(f_i)$ cover X and M_{f_i} is of finite type (resp. finite presentation) over A_{f_i}. It then follows from **??** that M is of finite type (resp. finite presentation). □

Corollary 8.1.25. *For an A-module M, the $\mathscr{O}_X$-module is locally free of finite rank if and only if M is a finitely generated projective A-module.*

Proof. Since X is quasi-compact, this follows from **??**. □

Corollary 8.1.26. *Let $X = \operatorname{Spec}(A)$ be an affine scheme. Then any quasi-coherent $\mathscr{O}_X$-algebra over X is isomorphic to an $\mathscr{O}_X$-algebra of the form $\widetilde{B}$, where B is an algebra over A. Moreover, any quasi-coherent $\widetilde{B}$-module is isomorphic to a $\widetilde{B}$-module of the form $\widetilde{N}$, where N is a B-module.*

Proof. In fact, a quasi-coherent $\mathscr{O}_X$-algebra is a quasi-coherent $\mathscr{O}_X$-module, hence of the form $\widetilde{B}$, where B is an A-module. The fact that B is an A-algebra follows from the structural morphism $\widetilde{B} \otimes_{\mathscr{O}_X} \widetilde{B} \to \widetilde{B}$ of $\mathscr{O}_X$-modules, which induces an A-algebra map $B \otimes_A B \to B$.

If $\mathscr{G}$ is a quasi-coherent $\widetilde{B}$-module, it suffices to show that $\mathscr{G}$ is also a quasi-coherent $\mathscr{O}_X$-module to then conclude in the same way. As the question is local, we can, by restricting ourselves to an open set of X of the form $D(f)$, over which $\mathscr{G}$ is the cokernel of a morphism $\widetilde{B}^{\oplus I} \to \widetilde{B}^{\oplus J}$ of $\widetilde{B}$-modules (and a fortiori $\mathscr{O}_X$-modules). The claim then follows from Corollary 8.1.3 and Corollary 8.1.6. □

Proposition 8.1.27. *Let $X = \operatorname{Spec}(A)$ be an affine scheme, Suppose that*

$$0 \longrightarrow \mathscr{F}_1 \longrightarrow \mathscr{F}_2 \longrightarrow \mathscr{F}_3 \longrightarrow 0$$

is a short exact sequence of $\mathscr{O}_X$-modules, and assume that $\mathscr{F}_1$ is quasi-coherent. Then the sequence

$$0 \longrightarrow \Gamma(X, \mathscr{F}_1) \longrightarrow \Gamma(X, \mathscr{F}_2) \longrightarrow \Gamma(X, \mathscr{F}_3) \longrightarrow 0$$

is exact.

Proof. We know already that Γ is a left-exact functor so we have only to show that the last map is surjective (which we denot by $\psi : \Gamma(X, \mathscr{F}_2) \to \Gamma(X, \mathscr{F}_3)$). Let $s \in \Gamma(X, \mathscr{F}_3)$ be a global section. Since the morphism $\mathscr{F}_2 \to \mathscr{F}_3$ is surjective, for any $x \in X$ there is an open neighborhood $D(f)$ of x such that

$$s|_{D(f)} = \psi(t)$$

where $t \in \mathscr{F}_2(D(f))$. We claim that for some $n > 0$, $f^n s = \psi(u)$ for some $u \in \Gamma(X, \mathscr{F}_2)$. Indeed, we can cover X with a finite number of open sets $D(g_i)$ such that for each i, $s|_{D(g_i)} = \psi(t_i)$ for a section $t_i \in \mathscr{F}_2(D(g_i))$. Then by the exactness of the original sequence, on $D(f) \cap D(g_i) = D(fg_i)$ we have

$$(t - t_i)|_{D(fg_i)} \in \mathscr{F}_1(D(fg_i))$$

where we identify $\mathscr{F}_1$ as the kernel of ψ. Since $\mathscr{F}_1$ is quasi-coherent, by Corollary 8.1.22, there is an integer $n \geq 0$ such that the $f^n(t - t_i)|_{D(fg_i)}$ can be extended to a section $u_i \in \mathscr{F}_1(D(g_i))$. Let

$$\tilde{t}_i = f^n t_i + u_i \in \mathscr{F}_2(D(g_i)).$$

Then $\tilde{t}_i|_{D(fg_i)} = f^n t_i|_{D(fg_i)} + f^n(t - t_i)|_{D(fg_i)} = f^n t|_{D(fg_i)}$ and we have

$$f^n s|_{D(g_i)} = f^n \psi(t_i) = \psi(\tilde{t}_i - u_i) = \psi(\tilde{t}_i). \tag{8.1.1}$$

Now on $D(g_i g_j)$ the two sections $\tilde{t}_i$ and $\tilde{t}_j$ of $\mathscr{F}_2$ are mapped to $f^n s|_{D(g_i g_j)}$ by ψ, so $\tilde{t}_i - \tilde{t}_j \in \mathscr{F}_1(D(g_i g_j))$. Furthermore, since $\tilde{t}_i$ and $\tilde{t}_j$ are both equal to $f^n t|_{D(fg_i g_j)}$ on $D(fg_i g_j)$, by Corollary 8.1.22 there exists $m \geq 0$ such that $f^m(\tilde{t}_i - \tilde{t}_j) = 0$ on $D(g_i g_j)$, which we may take to be independent of i and j. Then the sections $f^m \tilde{t}_i$ glue to give a global section of $\mathscr{F}_2$ over X, which lifts $f^{m+n} s$ by (8.1.1). This proves the claim.

Now cover X by a finite number of open sets $D(f_i)$ such that $s|_{D(f_i)}$ lifts to a section of $\mathscr{F}_2$ over $D(f_i)$ for each i. Then by the previous proof, we can find an integer $n \geq 0$ (one for all i) and global sections $t_i \in \Gamma(X, \mathscr{F}_2)$ such that $\psi(t_i) = f^n s$. Since the open sets $D(f_i)$ cover X, the ideal $(f_1^n, \dots, f_r^n)$ is the unit ideal of A, and we can write $1 = \sum a_i f_i^n$, with $a_i \in A$. Let $t = \sum a_i t_i$. Then t is a global section of $\mathscr{F}_2$ whose image under ψ is $\sum a_i f_i^n s = s$. □

Proposition 8.1.28. *Let $X = \operatorname{Spec}(A)$ be an affine scheme. Then the direct sum of an arbitrary collection of quasi-coherent sheaves on X is quasi-coherent. More generally, colimits of quasi-coherent sheaves are quasi-coherent.*

Proof. Let $\{\mathscr{F}_i\}_{i \in I}$ be a collection of quasi-coherent sheaves on X. By Theorem 8.1.21 we can write $\mathscr{F}_i = \widetilde{M}_i$ for A-modules M_i, so the assertion follows from Corollary 8.1.6. □

Proposition 8.1.29. *Let X be an affine scheme. Suppose that*

$$0 \longrightarrow \mathscr{F}_1 \longrightarrow \mathscr{F}_2 \longrightarrow \mathscr{F}_3 \longrightarrow 0$$

is a short exact sequence of sheaves $\mathscr{O}_X$-modules. If two out of three are quasi-coherent then so is the third.

Proof. The statement about kernels and cokernels follows from the fact that the functor $M \mapsto \widetilde{M}$ is exact and fully faithful from A-modules to quasi-coherent sheaves. Now let $\mathscr{F}_1$ and $\mathscr{F}_3$ be quasi-coherent. By Proposition 8.1.27, the corresponding sequence of global sections over X is exact, say $0 \to M_1 \to M_2 \to M_3 \to 0$. Applying the functor $\widetilde{M}$ we get an exact commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \widetilde{M}_1 \arrow[r] \arrow[d] & \widetilde{M}_2 \arrow[r] \arrow[d] & \widetilde{M}_3 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathscr{F}_1 \arrow[r] & \mathscr{F}_2 \arrow[r] & \mathscr{F}_3 \arrow[r] & 0
\end{tikzcd}$$

The two outside arrows are isomorphisms, since $\mathscr{F}_1$ and $\mathscr{F}_3$ are quasi-coherent. So by the five lemma, the middle one is also, showing that $\mathscr{F}_2$ is quasi-coherent. □

Theorem 8.1.30. *Let $X = \mathrm{Spec}(A)$ be an affine scheme where A is Noetherian. Let V be an open subset of X and $\mathscr{F}$ an $\mathscr{O}_X|_V$-module. Then the following conditions are equivalent:*

(i) *$\mathscr{F}$ is coherent.*

(ii) *$\mathscr{F}$ is of finite type and quasi-coherent.*

(iii) *There exists a finitely generated A-moduel M such that $\mathscr{F} \cong \widetilde{M}|_V$.*

Proof. It is clear that (i) implies (ii). To show (ii) implies (iii), we note that V is quasi-compact since X is Noetherian, so by Theorem 8.1.21, $\mathscr{F}$ is isomorphic to $\widetilde{M}|_V$, where M is an A-module. Now we have $M = \varinjlim M_\lambda$, where M_λ is the set of finitely generated sub-A-modules of M. Since the functor $\widetilde{(-)}$ is exact, this implies $\mathscr{F} = \widetilde{N}|_V = \varinjlim \widetilde{M_\lambda}|_V$. But $\mathscr{F}$ is of finite type and V is quasi-compact, so by **??** there exists an index λ such that $\mathscr{F} = \widetilde{M_\lambda}|_V$ (note that the canonical homomorphism $\widetilde{M}_\lambda \to \widetilde{M}$ is injective). This proves (iii).

It remains to show that $\widetilde{M}|_V$ is coherent if M is finitely generated. Since $\mathscr{F}$ is clearly of finite type, it suffices to show that for every open $U \subseteq X$ and $s_1, \dots, s_n \in \mathscr{F}(U)$, the associated map $\bigoplus_{i=1}^n \mathscr{O}_U \to \mathscr{F}|_U$ is of finite type. Since the question is local, we may assume $V = D(f)$ for $f \in A$. Then it suffices to show the kernel of a morphism $\bigoplus_{i=1}^n \widetilde{A}_f \to \widetilde{M}$ is of finite type. But this morphism corresponds to a homomorphism $A_f^n \to M$, whose kernel is finitely generated since A_f is Noetherian, so the claim follows. □

Corollary 8.1.31. *Let $X = \mathrm{Spec}(A)$ be an affine scheme where A is Noetherian. Then any quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$ is the inductive limit of coherent $\mathscr{O}_X$-modules.*

Proof. We have $\mathscr{F} = \widetilde{M}$ for an A-module M, and M is the inductive limit of its finitely generated submodules. □

Corollary 8.1.32. *Let $X = \mathrm{Spec}(A)$ be an affine scheme where A is Noetherian. Then the functor $M \mapsto \widetilde{M}$ gives an equivalence of categories between the category of finite generated A-modules and the category of coherent $\mathscr{O}_X$-modules.*

8.2 General schemes

8.2.1 Schemes and morphisms of schemes

Let $(X, \mathscr{O}_X)$ be a ringed space. An oepn subset V of X is said to be **affine** if the ringed space $(V, \mathscr{O}_X|_V)$ is an affine scheme (i.e. isomorphic to the spectrum of a ring). We say $(X, \mathscr{O}_X)$ is a **scheme** if every point of X admits an affine open neighborhood. If $(X, \mathscr{O}_X)$ is a scheme, then affine open subsets of X form a basis for X (because the standard opens form a basis for a spectrum $\mathrm{Spec}(A)$, and they are again affine), and in particular $(X, \mathscr{O}_X)$ is a locally ringed space. With this, for any open subset U of X, the ringed space $(U, \mathscr{O}_X|_U)$ is also a scheme, called the scheme **induced** on U by X, or the **restriction** of $(X, \mathscr{O}_X)$ on U.

Proposition 8.2.1. *The underlying space of a scheme is Kolmogoroff.*

Proof. In fact, if x and y are two points of a scheme X, then it is obvious that there exists an open neighborhood of one of these points not containing the other if x, y are not not in a same open affine; and if they are in the same open affine, this follows from the fact that the underlying spaces of affine schemes are Kolmogoroff (**??**). □

Proposition 8.2.2. *If $(X, \mathscr{O}_X)$ is a scheme, any irreducible closed subset of X admits a unique generic point, and the map $x \mapsto \overline{\{x\}}$ is a bijection of X to the family of irreducible closed subsets of X.*

Proof. Let Y is an irreducible closed subset of X and $y \in Y$. If U is an affine open neighborhood of y in X, then $U \cap Y$ is dense in Y and is irreducible (**??**), so it is the closure in U of a point $x \in U$, and therefore $Y \subseteq \overline{U}$ is the closure of x in X. The uniqueness of the generic point of X follows from Proposition 8.2.1 and ([**?**] 0_{I}, 2.1.3). □

If Y is an irreducible closed subset of X and y its generic point, the local ring $\mathscr{O}_{X,y}$ is then denoted by $\mathscr{O}_{X,Y}$ and called the **local ring of X along** Y, or the **local ring of** Y **in** X. We say a scheme $(X, \mathscr{O}_X)$ is **irreducible** (resp. **connected**) if the underlying space X is irreducible (resp. connected), and **integral** if it is irreducible and reduced. We say the scheme $(X, \mathscr{O}_X)$ is **locally integral** if each point $x \in X$ admits an open neighborhoods U such that the scheme induced over U by $(X, \mathscr{O}_X)$ is integral. If X is an irreducible scheme and x is its generic point, the local ring $\mathscr{O}_{X,x}$ is called **the ring of rational functions on X**.

Let $(X, \mathscr{O}_X)$ and $(Y, \mathscr{O}_Y)$ be schemes. A **morphism** (of schemes) from $(X, \mathscr{O}_X)$ to $(Y, \mathscr{O}_Y)$ is defined to be that of locally ringed space. That is, a pair $(f, f^\#)$ such that for each $x \in X$, the homomorphism $f^\#_x$ is local. In this case, by passing to quotients, $f^\#_x$ induces a monomorphism $f^x : \kappa(f(x)) \to \kappa(x)$, so $\kappa(x)$ can be considered as an extension of the field $\kappa(f(x))$.

The composition of two morphisms of schemes is defined in the same way with that of locally ringed spaces, and we then see that schemes form a category, denoted by **Sch**. Following the general notation, we denote by $\mathrm{Hom}_{\mathbf{Sch}}(X, Y)$ the set of morphisms from a scheme X to a scheme Y.

Example 8.2.3. Let U be an open subset of X. Then the canonical injection of $(U, \mathscr{O}_X|_U)$ to $(X, \mathscr{O}_X)$ is a morphism of schemes; it is moreover a monomorphism of ringed spaces (and a fortiori a monomorphism of schemes).

Proposition 8.2.4. *Let $(X, \mathscr{O}_X)$ be a scheme and $(Y, \mathscr{O}_Y)$ be an affine scheme. Then there exists a canonical bijection*
$$\mathrm{Hom}_{\mathbf{Sch}}(X, Y) \to \mathrm{Hom}_{\mathbf{Ring}}(\Gamma(X, \mathscr{O}_X), \Gamma(X, \mathscr{O}_X)).$$

Proof. Let $A = \Gamma(Y, \mathscr{O}_Y)$. Note first that, if $(X, \mathscr{O}_X)$ and $(Y, \mathscr{O}_Y)$ are any two ringed spaces, a morphism $(\psi, \psi^\#) : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ canonically defines a homomorphism of rings
$$\Gamma(Y, \mathscr{O}_Y) \to \Gamma(Y, \psi_*(\mathscr{O}_X)) = \Gamma(X, \mathscr{O}_X).$$
It then remains to see that any homomorphism $\rho : A \to \Gamma(X, \mathscr{O}_X)$ is of the form $\Gamma(\psi^\#)$ for a unique morphism $(\psi, \psi^\#)$. However, there is by hypothesis a covering (V_α) of X by affine open sets. By considering the composition
$$A \xrightarrow{\rho} \Gamma(X, \mathscr{O}_X) \longrightarrow \Gamma(V_\alpha, \mathscr{O}_X|_{V_\alpha})$$
we obtain a homomorphism $\rho_\alpha : A \to \Gamma(V_\alpha, \mathscr{O}_X|_{V_\alpha})$, which corresponds to a morphism $(\psi_\alpha, \psi^\#_\alpha)$ from the scheme $(V_\alpha, \mathscr{O}_X|_{V_\alpha})$ to $(Y, \mathscr{O}_Y)$ (Theorem 8.1.17). Moreover, for each pair (α, β) of indices, any point of $V_\alpha \cap V_\beta$ admits an affine open neighborhood W contained in $V_\alpha \cap V_\beta$; it is clear that by composing ρ_α and ρ_β with the restriction homomorphism to W, we obtain the same homomorphism $A \to \Gamma(W, \mathscr{O}_X|_W)$, so, by virtue of the relation $(\psi^\#_\alpha)_x = (\rho_\alpha)_x$ for any $x \in V_\alpha$ and any α, the restrictions of $(\psi_\alpha, \psi^\#_\alpha)$ and $(\psi_\beta, \psi^\#_\beta)$ to W coincide. By gluing we then get a unique morphism $(\psi, \psi^\#) : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ which extending $(\psi_\alpha, \psi^\#_\alpha)$ on each V_α. It is clear that $(\psi, \psi^\#)$ is a morphism of schemes, and we have $\Gamma(\psi^\#) = \rho$. □

Remark 8.2.5. Let $\rho : A \to \Gamma(X, \mathscr{O}_X)$ be a ring homomorphism, and let $(\psi, \psi^\#)$ be the corresponding morphism $(X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$. For each $f \in A$, we have

$$\psi^{-1}(D(f)) = \{x \in X : f \notin \mathfrak{m}_{\psi(x)}\} = \{x \in X : (\rho(x))_x \notin \mathfrak{m}_x\} = X_{\rho(f)}.$$

Note that this can be viewed as a generalization of **??**(a).

Proposition 8.2.6. *Under the hypothesis of Proposition 8.2.4, let $\rho : A \to \Gamma(X, \mathscr{O}_X)$ be a ring homomorphism and $f : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ the corresponding morphism of schemes. Let $\mathscr{F}$ (resp. $\mathscr{G}$) be a quasi-coherent $\mathscr{O}_X$-module (resp. $\mathscr{O}_Y$-module). Then there exist a canonical bijection*

$$\mathrm{Hom}_{\mathbf{Qcoh}(\mathbf{Y})}(\mathscr{G}, f_*(\mathscr{F})) \to \mathrm{Hom}_A(\Gamma(Y, \mathscr{G}), \rho^*(\Gamma(X, \mathscr{F}))).$$

Proof. Indeed, by reasoning as in Proposition 8.2.4, we are immediately reduced to the case where X is affine and the proposition then follows from Corollary 8.1.3 and Proposition 8.1.12. □

We say a morphism $f : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ is **open** (resp. **closed**) if for any open subset U of X (resp. any closed subset F of X), $f(U)$ is open in Y (resp. $f(F)$ is closed in Y). We say f is **dominant** if $f(X)$ is dense in Y, and **surjective** if f is surjective. It should be noted that these conditions only involve the continuous map f.

Proposition 8.2.7. *Let $f : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ and $g : (Y, \mathscr{O}_Y) \to (Z, \mathscr{O}_Z)$ be morphisms of schemes.*

(a) *If f and g are open (resp. closed, dominant, surjective), so is the composition $g \circ f$.*

(b) *If f is surjective and $g \circ f$ is closed, g is closed.*

(c) *If $g \circ f$ is surjective, g is surjective.*

Proof. The assertions (a) and (c) are evident. Put $h = g \circ f$. If F is closed in Y, then $f^{-1}(F)$ is closed in X, so $h(f^{-1}(F))$ is closed in Z. But since f is surjective, we have $f(f^{-1}(F)) = F$, so $h(f^{-1}(F)) = g(F)$, which shows g is closed. □

Proposition 8.2.8. *Let $f : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ be a morphism of schemes and (U_α) an open covering of Y. For f to be open (resp. closed, surjective, dominant), it is necessary and sufficient that for each U_α, the restrictions $f : f^{-1}(U_\alpha) \to U_\alpha$ is open (resp. closed, surjective, dominant).*

Proof. The proposition follows immediately from the definitions, taking into account the fact that a subset F of Y is closed (resp. open, dense) in Y if and only if each of the sets $F \cap U_\alpha$ is closed (resp. open, dense) in U_α. □

Let $(X, \mathscr{O}_X)$ and $(Y, \mathscr{O}_Y)$ be schemes; suppose that X and Y have a same finite number of irreducible components X_i (resp. Y_i) $(1 \le i \le n)$; let ξ_i (resp. η_i) be the generic point of X_i (resp. Y_i). We say that a morphism

$$f : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$$

is **birational** if, for any i, we have $f^{-1}(\eta_i) = \{\xi_i\}$ and $f^\#_{\xi_i} : \mathscr{O}_{Y,\eta_i} \to \mathscr{O}_{X,\xi_i}$ is an isomorphism. It is clear that any birational morphism is dominant, hence surjective if it is closed.

Remark 8.2.9. Throughout the remainder of this chapter and when there is no risk of confusions, we will omit in the notation of a scheme (resp. of a morphism) the structural sheaf (resp. the morphism of structural sheaf). If U is an open subset of the underlying space of a scheme X, when we speak of U as of a scheme, it will always be the scheme induced on U by X.

With the morphisms of schemes defined, we can talk about glueing schemes as in the case of ringed spaces. It follows immediately from the definition that any ringed space obtained by glueing schemes is again a scheme. In particular, since any scheme admits a basis of affine open subsets, we see any scheme is obtained by glueing affine schemes.

Example 8.2.10. Cosnider a field K, $A = K[s]$, $B = K[t]$ be two rings of polynomials over K with one indeterminate, and $X_1 = \mathrm{Spec}(A)$, $X_2 = \mathrm{Spec}(B)$. In X_1 (resp. X_2), let U_{12} (resp. U_{21}) be the affine open set $D(s)$ (resp. $D(t)$), whose ring A_s (resp. B_t) is formed by the rational fractions of the form $f(s)/s^m$ (resp. $g(t)/t^n$) with $f \in A$ (resp. $g \in B$). Let φ_{12} be the isomorphism of of schemes $U_{21} \to U_{12}$ corresponding to the isomorphism of A and B such that, $f(s)/s^m$ is mapped to the rational fraction $f(1/t)/(1/t^m)$ (i.e. we map s to $1/t$). We can then glue X_1 and X_2 along U_{12} and U_{21} by the isomorphism u_{12}, which evidently satisfies the glueing condition. We will see later the schem X thus obtained is a particular case of a general method of construction. We only show here that X is not an affine scheme, which will result from the fact that the ring $\Gamma(X, \mathscr{O}_X)$ is isomorphic to K, therefore has a spectrum reduced to a point. Indeed, a section of $\mathscr{O}_X$ above X has a restriction over X_1 (resp. X_2), identified with an open affine of X, which is a polynomial $f(s)$ (resp. $g(t)$), and it follows from the definition of u_{12} that we must have $g(t) = f(1/t)$, which is not possible only if $f = g \in K$.

8.2.2 Local schemes

Let X be a scheme and $A = \Gamma(X, \mathscr{O}_X)$. We say X is a local scheme if X is affine and the ring X is local. In this case, there then exists a unique closed point ξ in X, and for any point $x \in X$ we have $\xi \in \overline{\{x\}}$.

Following this notation, for a general scheme Y and $y \in Y$, the scheme $\mathrm{Spec}(\mathscr{O}_{Y,y})$ is called the **local scheme of Y at y**. Let V be an affine open subset of Y containing y, and B the ring of V. The local ring $\mathscr{O}_{Y,y}$ is then canonically identified with B_y, and the canonical homomorphism $B \to B_y$ then induces a morphism $\mathrm{Spec}(\mathscr{O}_{Y,y}) \to V$ of schemes. If we compose this with the canonical injection of V into Y, we then get a morphism $\mathrm{Spec}(\mathscr{O}_{Y,y}) \to Y$, which is independent of the choice of the affine open V containing y: in fact, if U is another affine neighborhood of y, there exists an affine open neighborhood W of y contained in $U \cap V$; we can then limit ourselves to the case $U \subseteq V$, and if A is the ring of U, we have a commutative diagram

$$\begin{tikzcd} A \arrow[rr] \arrow[dr] & & B \arrow[dl] \\ & \mathscr{O}_{Y,y} & \end{tikzcd}$$

The morphism $\mathrm{Spec}(\mathscr{O}_{Y,y}) \to Y$ thus defined is said to be **canonical**.

Proposition 8.2.11. *Let Y be a scheme, $y \in Y$, and $f : \mathrm{Spec}(\mathscr{O}_{Y,y}) \to Y$ be the canonical morphism.*

(a) *The map f is a homeomorphism of $\mathrm{Spec}(\mathscr{O}_{Y,y})$ onto the subspace S_y of points $z \in Y$ such that $y \in \overline{\{z\}}$ (i.e. the set of generalizations of y).*

(b) *For each $\mathfrak{p} \in \mathrm{Spec}(\mathscr{O}_{Y,y})$, the homomorphism $f^\#_{\mathfrak{p}} : \mathscr{O}_{Y,f(\mathfrak{p})} \to (\mathscr{O}_{Y,y})_{\mathfrak{p}}$ is an isomorphism.*

In particular, f is a monomorphism of locally ringed spaces.

Proof. Since the unique closed point η of $\mathrm{Spec}(\mathscr{O}_{Y,y})$ belongs to the closure of any other point in this space, and $f(\eta) = y$, the image of $\mathrm{Spec}(\mathscr{O}_{Y,y})$ by the continuous map f is contained in S_y. As S_y is contained in any affine open neighborhood of y, we can reduce to the case where Y is an affine scheme; but in this case the proposition follows immediately. □

Corollary 8.2.12. *There is a bijective correspondence between $\mathrm{Spec}(\mathscr{O}_{Y,y})$ and irreducible closed subsets of Y containing y.*

Proof. This follows from Proposition 8.2.11 and the fact that every irreducible closed set in Y has a unique generic point. □

Corollary 8.2.13. *For a point $y \in Y$ to be the generic point of an irreducible component of Y, it is necessary and suffcient that $\mathscr{O}_{Y,y}$ is zero-dimensional.*

Proof. This follows from the observation that y is the generic point of an irreducible component if and only if it is a maximal element under generalization, which is then equivalent by Corollary 8.2.12 to that $\mathrm{Spec}(\mathscr{O}_{Y,y})$ is a singleton. □

Proposition 8.2.14. *Let $(X, \mathscr{O}_X)$ be a local scheme with $A = \Gamma(X, \mathscr{O}_X)$, ξ its unique closed point, and $(Y, \mathscr{O}_Y)$ a scheme. Then any morphism $f : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ factors through $\mathrm{Spec}(\mathscr{O}_{Y,f(\xi)})$:*

$$f : X \to \mathrm{Spec}(\mathscr{O}_{Y,f(\xi)}) \to Y$$

where the second one is the canonical morphism, and the first one corresponds to a local homomorphism $\mathscr{O}_{Y,f(\xi)} \to A$.

Proof. In fact, for any $x \in X$, we have $\xi \in \overline{\{x\}}$, hence $f(\xi) \in \overline{\{f(x)\}}$. It then follows that $f(X)$ is contained in any affine open neighborhood of $f(\xi)$ (in fact any open neighborhood of $f(\xi)$). We can then reduce to the case that $(Y, \mathscr{O}_Y)$ is an affine scheme with ring $B = \Gamma(Y, \mathscr{O}_Y)$, and the morphism f corresponds to a ring homomorphism $\rho : B \to A$. We have $\rho^{-1}(\mathfrak{p}_\xi) = \mathfrak{p}_{f(\xi)}$, so the image under ρ of an element of $B - \mathfrak{p}_{f(\xi)}$ is invertible in the local ring A, and we get a canonical homomorphism $\rho_\xi : B_{f(\xi)} \to A$. □

Corollary 8.2.15. *There is a canonical bijection between* $\mathrm{Hom}_{\mathbf{Sch}}(X, Y)$ *to the set of local homomorphisms* $\mathscr{O}_{Y,y} \to A$*, where* $y \in Y$*.*

Proof. It suffices to note that any local homomorphism $\mathscr{O}_{Y,y} \to A$ corresponds to a unique morphism $f : X \to \mathrm{Spec}(\mathscr{O}_{Y,y})$ such that $f(\xi) = y$, and by composing this with the canonical morphism $\mathrm{Spec}(\mathscr{O}_{Y,y}) \to Y$, we get a morphism $X \to Y$. □

Example 8.2.16. The affine scheme whose ring is a field K have an underlying space reduced to one point. If A is a local ring of maximal ideal $\mathfrak{m}$, any local homomorphism $A \to K$ has a kernel equal to $\mathfrak{m}$, so factors into $A \to A/\mathfrak{m} \to K$, where the second arrow is a monomorphism. The morphisms $\mathrm{Spec}(K) \to \mathrm{Spec}(A)$ correspond therefore bijectively to the field extensions $A/\mathfrak{m} \to K$.

Let $(Y, \mathscr{O}_Y)$ be a scheme; for any $y \in Y$ and any ideal $\mathfrak{a}_y$ of $\mathscr{O}_{Y,y}$, the canonical homomorphism $\mathscr{O}_{Y,y}/\mathfrak{a}_y$ defines a morphism $\mathrm{Spec}(\mathscr{O}_{Y,y}/\mathfrak{a}_y) \to \mathrm{Spec}(\mathscr{O}_{Y,y})$; by composing this with the canonical morphism $\mathrm{Spec}(\mathscr{O}_{Y,y}) \to Y$, we obtain a morphism $\mathrm{Spec}(\mathscr{O}_{Y,y}/\mathfrak{a}_y) \to Y$, also called canonical. If $\mathfrak{a}_y = \mathfrak{m}_y$ is the maximal ideal of $\mathscr{O}_{Y,y}$, then $\mathscr{O}_{Y,y} = \kappa(y)$ and Corollary 8.2.15 then imply the following result:

Corollary 8.2.17. *Let* $(X, \mathscr{O}_X)$ *be a local scheme with* $K = \Gamma(X, \mathscr{O}_X)$ *a field,* ξ *its unique point, and* $(Y, \mathscr{O}_Y)$ *be a scheme. Then any morphism* $f : (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ *factors into*

$$f : X \to \mathrm{Spec}(\kappa(f(\xi))) \to Y$$

where the second arrow is the canonical morphism, and the first arrow corresponds to a field extension $\kappa(f(\xi)) \to K$*. This establishes a canonical bijection between* $\mathrm{Hom}_{\mathbf{Sch}}(X, Y)$ *to the set of field extensions* $\kappa(y) \to K$*, where* $y \in Y$*.*

Corollary 8.2.18. *For any* $y \in Y$*, the canonical morphism* $\mathrm{Spec}(\mathscr{O}_{Y,y}/\mathfrak{a}_y) \to Y$ *is a monomorphism of locally ringed spaces.*

Proof. This follows from Proposition 8.2.11 and Corollary 8.1.19. □

Remark 8.2.19. Let X be a local scheme, ξ its unique closed point. Since any affine open neighborhood of ξ is necessarily all of X, any invertible $\mathscr{O}_X$-module is necessarily isomorphic to $\mathscr{O}_X$ (in other words, is trivial). This property does not hold in general for any affine scheme $\mathrm{Spec}(A)$, but we will see that if A is a normal ring, this is true when A is factorial.

8.2.3 Shemes over a scheme

As in any category, for a scheme S we can define the category $\mathbf{Sch}/S$ of S-objects in the category of schemes, which will be a morphism $\varphi : X \to S$ where X is a scheme. In this case we also say that X is a **scheme over S**, or an **S-scheme**. We say that S is the **base scheme** of the S-scheme X and φ is called the structural morphism of the S-scheme X. When S is an affine scheme of the ring A, we also say that X is a **scheme over A** or an **A-scheme**.

It follows from Proposition 8.2.4 that giving a scheme over a ring A is equivalent to giving a scheme $(X, \mathscr{O}_X)$, where $\mathscr{O}_X$ is an A-algebra. In particular, any scheme can be considered as a scheme over $\mathbb{Z}$. In other words, the scheme $\mathrm{Spec}(\mathbb{Z})$ is a final object in the category of schemes (also a final object in the category of locally ringed spaces).

If $\varphi : X \to S$ is the structural morphism of an S-scheme X, we say a point $x \in X$ is **lying over** a point $s \in S$ if $\varphi(x) = s$. We say X **dominates** S if the morphism φ is dominant. Let X and Y be two S-schemes; a morphism $u : X \to Y$ is called a **morphism of schemes over S** (or **S-morphism**) if the diagram

$$\begin{tikzcd} X \arrow[rr, "u"] \arrow[dr] & & Y \arrow[dl] \\ & S & \end{tikzcd}$$

is commutative; in other words, if for any $s \in S$ and any $x \in X$ lying over s, the point $u(x)$ is lying over s. This definition immediately shows that the composite of two S-morphisms is an S-morphism, so the S-schemes thus form a category. We denote by $\mathrm{Hom}_S(X, Y)$ the set of S-morphisms from an S-scheme X to an S-scheme Y; the identity morphism of an S-scheme X is then denoted by 1_X or id_X. If S is an affine scheme S, we also say A-morphisms for S-morphisms.

If X is an S-scheme, $\varphi : X \to S$ the structural morphism, an ***S*-section** of X is defined to be an S-morphism of S to X, which is equivalently a morphism $\psi : S \to X$ of schemes such that $\varphi \circ \psi = \mathrm{id}_S$. We denote by $\Gamma(X/S)$ the set of S-sections of X.

Example 8.2.20. If X is an S-scheme and $v : X' \to X$ a morphism of schemes, the composition scheme

$$X' \xrightarrow{v} X \longrightarrow S$$

then defines X' as an S-scheme; in particular, any scheme induceed over an open subset U of X can be considered as an S-scheme by means of the canonical injection.

Example 8.2.21. Let $u : X \to Y$ be an S-morphism of S-schemes, the restriction of u on any open subset U of X is then an S-morphism $U \to Y$. Conversely, let (U_α) be a covering of X and for each α, let $u_\alpha : U_\alpha \to Y$ be an S-morphism; if for any pair (α, β) of indices, the restrictions of u_α and u_β on $U_\alpha \cap U_\beta$ coincide, then there exists a unique S-morphism $X \to Y$ whose restriction on U_α equals to u_α.

Let $S \to S'$ be a morphism of schemes; for any S'-scheme X, the composition morphism $X \to S' \to S$ then defines X as an S-scheme. Conversely, suppose that S' is the scheme induced over an open subset of S; let X be an S-scheme and suppose that the structural morphism $X \to S$ has image contained in S'; then we can consider X as an S'-scheme. In the latter case, if Y is an S-scheme whose structural morphism also maps the underlying space in S', any S-morphism from X in Y is also an S'-morphism.

8.2.4 Quasi-coherent sheaves on schemes

Proposition 8.2.22. *Let X be a scheme. For an $\mathscr{O}_X$-module $\mathscr{F}$ to be quasi-coherent, it is necessary and sufficient that, for any affine open subset V of X, the restriction $\mathscr{F}|_V$ is isomorphism to the sheaf associated with a $\Gamma(V, \mathscr{O}_X)$-module.*

Proof. We recall that being quasi-coherent is a local property, and affine opens form a basis for X. Also, by Theorem 8.1.21, a quasi-coherent sheaf on an affine open V is isomorphic to $\widetilde{M}$ for some $\Gamma(V, \mathscr{O}_X)$-module M. □

Corollary 8.2.23. *Let X be an arbitrary scheme.*

(i) *Let $0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{H} \to 0$ be an exact sequence of $\mathscr{O}_X$-modules; if two of them are quasi-coherent, then so is the third one.*

(ii) *The images, kernels and cokernels of homomorphisms of quasi-coherent $\mathscr{O}_X$-modules are quasi-coherent. The inductive limits and direct sums of quasi-coherent sheaves are quasi-coherent. If $\mathscr{G}$ and $\mathscr{H}$ are quasi-coherent $\mathscr{O}_X$-modules of a quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, then $\mathscr{G} + \mathscr{H}$ and $\mathscr{G} \cap \mathscr{H}$ are quasi-coherent.*

(iii) *Let $0 \to \mathscr{F}_1 \to \mathscr{F}_2 \to \mathscr{F}_3 \to \mathscr{F}_4 \to \mathscr{F}_5 \to 0$ be an exact sequence of $\mathscr{O}_X$-modules. If $\mathscr{F}_1, \mathscr{F}_2, \mathscr{F}_4, \mathscr{F}_5$ are quasi-coherent, so is $\mathscr{F}_3$.*

(iv) *If $\mathscr{F}$ and $\mathscr{G}$ are quasi-coherent $\mathscr{O}_X$-modules, $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G}$ is quasi-coherent. In particular, if $\mathscr{J}$ is a quasi-coherent ideal of $\mathscr{O}_X$, $\mathscr{J}\mathscr{F}$ is a quasi-coherent $\mathscr{O}_X$-module.*

(v) *Let $\mathscr{F}$ be a quasi-coherent $\mathscr{O}_X$-module with finite presentation. Then for any quasi-coherent $\mathscr{O}_X$-module $\mathscr{G}$, $\mathcal{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})$ is quasi-coherent.*

(vi) *If $\mathscr{F}$ is a quasi-coherent $\mathscr{O}_X$-module of finite type, the annihilator $\mathscr{J}$ of $\mathscr{F}$ is a quasi-coherent ideal of $\mathscr{O}_X$.*

Proof. By Proposition 8.2.22, assertions (i) to (v) follow from Proposition 8.1.29, Corollary 8.1.6, and Corollary 8.1.10. To prove (vi), we can assume that $X = \mathrm{Spec}(A)$ is affine, $\mathscr{F} = \widetilde{M}$, where M is a finitely generated A-module, with generators $t_1, \dots, t_r$. The ideal $\mathscr{J}$ is then the intersection of the annihilators of t_i. But the annihilator of t_i is by definition the kernel of the canonical morphism $\mathscr{O}_X \to \mathscr{F}$ corresponding to $s \mapsto st_i$ from A to M, hence quasi-coherent. It then follows that $\mathscr{J}$ is quasi-coherent, as an intersection of quasi-coherent $\mathscr{O}_X$-modules. □

Corollary 8.2.24. *Let X be a scheme, $0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{H} \to 0$ an exact sequence of quasi-coherent $\mathscr{O}_X$-modules. If $\mathscr{H}$ is finitely presented and $\mathscr{G}$ is of finite type, then $\mathscr{F}$ is of finite type.*

Proof. Since this question is local, we may assume that X is affine, and the corresponding result is then **??**. ◻

Proposition 8.2.25. *Let X be a scheme and $\mathscr{B}$ be a quasi-coherent $\mathscr{O}_X$-algebra. For a $\mathscr{B}$-module $\mathscr{F}$ to be quasi-coherent, it is necessary and sufficient that $\mathscr{F}$ is a quasi-coherent $\mathscr{O}_X$-module. In particular, if $\mathscr{F}$ and $\mathscr{G}$ are quasi-coherent $\mathscr{B}$-modules, $\mathscr{F} \otimes_{\mathscr{B}} \mathscr{G}$ is a quasi-coherent $\mathscr{B}$-module; the same holds for $\mathcal{H}om_{\mathscr{B}}(\mathscr{F}, \mathscr{G})$ if $\mathscr{F}$ is a finitely presented $\mathscr{B}$-module.*

Proof. Since the question is local, we can suppose that X is affine with ring A, and then $\mathscr{B} = \widetilde{B}$, where B is an A-algebra. If $\mathscr{F}$ is quasi-coherent over the space $(X, \mathscr{B})$, we can write $\mathscr{F}$ as the cokernel of $\mathscr{B}$-homomorphism $\mathscr{B}^{\oplus I} \to \mathscr{B}^{\oplus J}$; since this homomorphism is also an $\mathscr{O}_X$-homomorphism, and $\mathscr{B}^{\oplus I}$, $\mathscr{B}^{\oplus J}$ are quasi-coherent $\mathscr{O}_X$-modules, we conclude that $\mathscr{F}$ is also quasi-coherent.

Conversely, if $\mathscr{F}$ is a quasi-coherent $\mathscr{O}_X$-module, we have $\mathscr{F} = \widetilde{M}$ where M is a B-module (Corollary 8.1.26); M is isomorphic to the cokernel of a homomorphism $B^{\oplus I} \to B^{\oplus J}$, so $\mathscr{F}$ is a $\mathscr{B}$-module isomorphic to the cokernel of the corresponding homomorphism $\mathscr{B}^{\oplus I} \to \mathscr{B}^{\oplus J}$. This completes the proof. ◻

Let X be a scheme. A quasi-coherent $\mathscr{O}_X$-algebra $\mathscr{B}$ is **of finite type** (resp. **of finite presentation**) if for all $x \in X$, there exists an open affine neighborhood U of x such that $\Gamma(U, \mathscr{B}) = B$ is an algebra of type finite (resp. of finite presentation[1]) over $\Gamma(U, \mathscr{O}_X) = A$. If this is the case, we have $\mathscr{B}|_U = \widetilde{B}$, and for all $f \in A$, the $(\mathscr{O}_X|_{D(f)})$-algebra $\mathscr{B}|_{D(f)}$ induced on $D(f)$ is of finite type (resp. of finite presentation), because it is isomorphic to $\widetilde{B \otimes_A A_f}$. As the $D(f)$ form a basis of the topology of X, we deduce that for any open set V of X, $\mathscr{B}|_V$ is a $(\mathscr{O}_X|_V)$-algebra of finite type (resp. of finite presentation).

Proposition 8.2.26. *Let X be a scheme, $\mathscr{E}$ a locally free $\mathscr{O}_X$-module of rank r, Z a finite subset of X contained in an affine open V. Then there exists an open neighborhood $U \subseteq V$ of Z such that $\mathscr{E}|_U$ is isomorphic to $\mathscr{O}_U^r$.*

Proof. By replacing X by V, we can assume that $X = \operatorname{Spec}(A)$ is affine. For each $z_i \in Z$ there exists a closed point z_i' in the closure $\overline{\{z_i\}}$ (that is, a maximal ideal containing $\mathfrak{p}_{z_i}$); if Z' is the set of the z_i', any neighborhood of Z' is a neighborhood of Z, and we can then suppose that Z is closed in X. Now, the subset Z of X is defined by an ideal $\mathfrak{a}$ of A; consider the scheme $\operatorname{Spec}(A/\mathfrak{a})$, with Z its underlying space, and the injection $\iota : Z \to X$ corresponds to the canonical homomorphism $A \to A/\mathfrak{a}$. Then $\iota^*(\mathscr{E}) = \mathscr{E} \otimes_{\mathscr{O}_X} \mathscr{O}_Z$ is locally free with rank r over the discrete scheme Z, so is isomorphic to $\mathscr{O}_Z^r$. In other words, there exist sections $s_1, \dots, s_r$ of $\mathscr{E} \otimes_{\mathscr{O}_X} \mathscr{O}_Z$ over Z such that the homomorphism $\mathscr{O}_Z^r \to \mathscr{E} \otimes_{\mathscr{O}_X} \mathscr{O}_Z$ defined by these sections is bijective. On the other hand, we have $\mathscr{E} = \widetilde{M}$ where M is an A-module; then each s_i belongs to $M \otimes_A (A/\mathfrak{a})$, and is then the image of an element $t_i \in M = \Gamma(X, \mathscr{E})$. For each $z_j \in Z$, by **??**, there then exists a neighborhood V_i of z_i in X such that the restrictions of t_i to V_j define an isomorphism $\mathscr{O}_X^r|_{V_j} \to \mathscr{E}|_{V_j}$; the union U of the V_j's then satisfies the requirement. ◻

Proposition 8.2.27. *Let X a scheme and $\mathscr{F}$ be a quasi-coherent $\mathscr{O}_X$-module of finite type. Then there exists a closed subscheme Y of X with underlying space $\operatorname{supp}(\mathscr{F})$ and a quasi-coherent $\mathscr{O}_Y$-module $\mathscr{G}$ of finite type supported on Y such that, if $j : Y \to X$ is the canonical injection, $\mathscr{F}$ is isomorphic to $j_*(\mathscr{G})$.*

Proof. It suffices to note that the annihilator $\mathscr{J}$ of $\mathscr{F}$ is a quasi-coherent ideal of $\mathscr{O}_X$ (Corollary 8.2.23), so if Y is the closed subscheme of X defined by $\mathscr{J}$, as $\mathscr{J}\mathscr{F} = 0$, $\mathscr{F}$ is an $(\mathscr{O}_X/\mathscr{J})$-module, and we can take $\mathscr{G} = j^*(\mathscr{F})$. ◻

8.2.5 Noetherian schemes and locally Noetherian schemes

We say a scheme X is Noetherian (resp. locally Noetherian) if there is a finite covering (resp. a covering) of open affines V_α such that each ring $\Gamma(V_\alpha, \mathscr{O}_X)$ is Noetherian. The underlying space of a Noetherian (resp. locally Noetherian) scheme is then a Noetherian space (resp. locally Noetherian). Moreover, if X is locally Noetherian. the structural sheaf $\mathscr{O}_X$ is coherent, any quasi-coherent $\mathscr{O}_X$-module of finite type is coherent (Theorem 8.1.30), and any local ring $\mathscr{O}_{X,x}$ is Noetherian. Any quasi-coherent

[1] Recall that an algebra B is finitely presented over A if it is isomorphic to the quotient of a polynomial ring over A in finitely many variables by a finitely generated ideal.

sub-$\mathscr{O}_X$-module (resp. any quasi-coherent $\mathscr{O}_X$-quotient) of a coherent $\mathscr{O}_X$-module $\mathscr{F}$ is then coherent, because the question is local again, and we just apply Theorem 8.1.30, together with the fact that a sub-module (resp. quotient module) of a finitely generated module on a Noetherian ring is finitely generated. More particularly, any quasi-consistent ideal of $\mathscr{O}_X$ is consistent.

If a scheme X is a finite union (resp. a union) of open Noetherian (resp. locally Noetherian) subschemes W_λ, it is clear that X is then Noetherian (resp. locally Noetherian).

Proposition 8.2.28. *For a scheme X to be Noetherian, it is necessary and sufficient that it is locally Noetherian and its underlying space is quasi-compact.*

Proof. This follows from the definition, since a Noetherian space is quasi-compact. □

Proposition 8.2.29. *Let X be an affine scheme with ring A. Then the following conditions are equivalent:*

(i) *X is Noetherian;*

(ii) *X is locally Noetherian;*

(iii) *A is Noetherian.*

Proof. Since X is quasi-compact, it is clear that (i) and (ii) are equivalent. Also, (iii) implies (i) by definition. Now assume that X is Noetherian, then there is a finite covering (V_i) of X by affine opens where $A_i = \Gamma(V_i, \mathscr{O}_X)$ is Noetherian. Let $(\mathfrak{a}_n)$ be an increasing sequence of ideals of A; it corresponds to it canonically in a one-to-one way to an increasing sequence $(\tilde{\mathfrak{a}}_n)$ of ideals in $\widetilde{A} = \mathscr{O}_X$; to see that the sequence $(\mathfrak{a}_n)$ is stationary, it suffices to prove that the sequence $(\tilde{\mathfrak{a}}_n)$ is. However, the restriction $\tilde{\mathfrak{a}}_n|_{V_i}$ is a quasi-coherent ideal of $\mathscr{O}_X|_{V_i}$; $\tilde{\mathfrak{a}}_n|_{V_i}$ is then of the form $\tilde{\mathfrak{a}}_{n,i}$, where $\mathfrak{a}_{n,i}$ is an ideal of A_i. As A_i is Noetherian, the sequence $(\mathfrak{a}_{n,i})$ is stationary for all i, hence the proposition. □

Note that the above reasoning also proves that if X is a Noetherian scheme, any increasing sequence of coherent ideals of $\mathscr{O}_X$ is stationary.

Proposition 8.2.30. *Let X be a locally Noetherian scheme. Any quasi-coherent $\mathscr{O}_X$-algebra $\mathscr{B}$ of finite type is a coherent shaf of rings and an $\mathscr{O}_X$-algebra of finite presentation.*

Proof. We can assume that $X = \mathrm{Spec}(A)$ is affine, where A is a Noetherian ring, and $\mathscr{B} = \widetilde{B}$, where B is an A-algebra of finite type. It then follows that B is finitely presented over A, so $\mathscr{B}$ is of finite presentation. To show that $\mathscr{B}$ is coherent, we must prove that the kernel $\mathscr{N}$ of a $\mathscr{B}$-homomorphism $\mathscr{B}^m \to \mathscr{B}$ is a $\mathscr{B}$-module of finite type; but it is of the form $\widetilde{N}$, where N is the kernel of the corresponding homomorphism $B^m \to B$ of B-modules. Since B is also Noetherian, N is a finitely generated B-module. There then exists a surjective B-homomorphism $B^n \to N$, so a surjective homomorphism $\mathscr{B}^n \to \mathscr{N}$, which proves our assertion. □

Corollary 8.2.31. *Let X be a locally Noetherian scheme and $\mathscr{B}$ be a quasi-coherent $\mathscr{O}_X$-algebra of finite type. For a $\mathscr{B}$-module $\mathscr{F}$ to be coherent, it is necessary and sufficient that $\mathscr{F}$ is a quasi-coherent $\mathscr{O}_X$-module and a $\mathscr{B}$-module of finite type, and if $\mathscr{G}$ is a sub-$\mathscr{B}$-module or a quotient $\mathscr{B}$-module of $\mathscr{F}$, for $\mathscr{G}$ to be a coherent $\mathscr{B}$-module, it is necessary and sufficient that $\mathscr{G}$ is a quasi-coherent $\mathscr{O}_X$-module.*

Proof. Considering Proposition 8.2.25, the conditions on $\mathscr{F}$ is necessary. To prove the sufficiency, we can assume that $X = \mathrm{Spec}(A)$ is affine, where A is Noetherian, $\mathscr{B} = \widetilde{B}$, where B is an A-algebra of finite type, and $\mathscr{F} = \widetilde{M}$, where M is a B-module and there exists a surjective B-homomorphism $\mathscr{B}^m \to \mathscr{F}$. Then we get a corresponding homomorphism $B^m \to M$, so M is a finitely generated B-module; the kernel P of thie homomorphism is finitely generated since B is Noetherian, and $\mathscr{F}$ is therefore the cokernel of a morphism $\mathscr{B}^n \to \mathscr{B}^m$, so it is coherent (since $\mathscr{B}$ is a coherent sheaf of rings). The same reasoning shows that any quasi-coherent sub-$\mathscr{B}$-module (resp. quotient $\mathscr{B}$-module) of $\mathscr{F}$ is of finite type, whence the second part of the corollary. □

Proposition 8.2.32. *Let X be a locally Noetherian scheme and E be a subset of X. Any point $x \in E$ admits in E a maximal generalization y (i.e. y has no further generalization in Y). In particular, if $E \neq \varnothing$, there exists a maximal element $y \in E$ under generalization.*

Proof. The generalizations of x in X lie in the points of $\mathrm{Spec}(\mathscr{O}_{X,x})$ (Corollary 8.2.12), where $\mathscr{O}_{X,x}$ is a Noetherian local ring. We then know that the lengths of chains of prime ideals in this ring are bounded by $\dim(\mathscr{O}_{X,x})$, and to prove the proposition it suffices to consider a chain of prime ideals belonging to E and having the greatest possible length. □

Proposition 8.2.33. *Let X be a scheme. Then the following conditions are equivalent:*

(i) $X = \mathrm{Spec}(A)$ *is affine and A is Artinian;*

(ii) *X is Noetherian and has discrete underlying space;*

(iii) *X is Noetherian and every point in X is closed (in other words, X is T1).*

If these equivalent conditions hold, then X is finite and the ring A is a direct product of finitely many Artinian local rings.

Proof. We know that (i) implies the last asserion. It is clear that (i)⇒(ii)⇒(iii). To see that (iii) implies (i), let us first show that X is then finite; we can indeed reduce to case where X is affine, and we know that a Noetherian ring of which all the prime ideals are maximal is Artinian, hence our assertion. □

Note that a Noetherian scheme can have an underlying space finite without being artinian, as shown by the example of a spectrum discrete valuation ring.

8.3 Product of schemes

Let (X_α) be a family of schemes, and X be the topological space which is the **coproduct** of the underlying spaces of X. Then X is the union of its open subspaces U_α, and for each α we have a embedding $\iota_\alpha : X_\alpha \to X$ with image equal to U_α. If we endow each U_α the sheaf $(\iota_\alpha)_*(\mathscr{O}_{X_\alpha})$, it is clear that X becomes a scheme, which we will call the **coproduct** of the family (X_α), and denote by $\coprod_\alpha X_\alpha$. It is clear that the scheme X satisfies the universal property of coproducts of X_α's: for any scheme Y and morphisms $f_\alpha : X_\alpha \to Y$, there exists a unique morphism $f : X \to Y$ such that $f \circ \iota_\alpha = f_\alpha$. In other words, we have a functorial bijection

$$\mathrm{Hom}(\coprod_\alpha X_\alpha, Y) \to \prod_\alpha \mathrm{Hom}(X_\alpha, Y).$$

This fact can be also stated that $\coprod_\alpha X_\alpha$ represents the covariant functor $\prod_\alpha \mathrm{Hom}(X_\alpha, -)$ on the category of schemes. In particular, if X_α are S-schemes with structural morphisms ψ_α, then X is an S-scheme with structural morphism $\psi : X \to S$ such that $\psi \circ \iota_\alpha = \psi_\alpha$. We usually denote the coproduct of two schemes X and Y by $X \amalg Y$, and it is clear that if $X = \mathrm{Spec}(A)$ and $Y = \mathrm{Spec}(B)$, then $X \amalg Y$ is canonically identified with $\mathrm{Spec}(A \times B)$.

In this section, we shall consider product of schemes, which is far more complicated than coproducts. We will see that fiber products plays a central role of many construction and operations on schemes.

8.3.1 Product of schemes

Let X and Y be S-schemes. Recall that the object $X \times_S Y$ represents by definition the contravariant functor

$$T \mapsto F(T) = \mathrm{Hom}_S(T, X) \times \mathrm{Hom}_S(T, Y)$$

on the category of S-schemes. To prove the existence of $X \times_S Y$, we shall apply the methods used in **??**. We first verify condition (ii) in **??**, which means F is a sheaf over the category **Sch**: this is evident since the functors $T \mapsto \mathrm{Hom}_S(T, X)$ and $T \mapsto \mathrm{Hom}_S(T, Y)$ are sheaves, and a projective limit of sheaves over **Sch** is again a sheaf over **Sch**.

This already allows us to bring ourselves back to the case that the scheme S is affine. In fact, let (S_α) is a covering of S by affine open sets. In view of the above fact and of ([**?**] new, 0_{I}, 4.5.5), it suffices to show that each of the functors $F \times_{h_S} h_{S_\alpha}$ is representable. On the other hand, let $\varphi : X \to S$ and $\psi : Y \to S$ be the structural morphisms; it is immediate that when T is a locally ringed S_α-space (hence also an S-space), we have canonical identifies

$$\mathrm{Hom}_S(T, X) \xrightarrow{\sim} \mathrm{Hom}_{S_\alpha}(T, \varphi^{-1}(S_\alpha)), \quad \mathrm{Hom}_S(T, Y) \xrightarrow{\sim} \mathrm{Hom}_{S_\alpha}(T, \psi^{-1}(S_\alpha)).$$

Therefore, in view of ([**?**] new, 0_{I}, 4.5.5), we only need to show that F is representable when it is restricted to the subcategory of locally ringed S_α-spaces.

With these being done, assume that S is affine and consider a covering (X_λ) (resp. (Y_μ)) of X (resp. Y) by affine opens. We shall verify the conditions (i) and (iii) of **??** for the subfunctors $F_{\lambda\mu} : T \mapsto$

$\mathrm{Hom}_S(T, X_\lambda) \times \mathrm{Hom}_S(T, Y_\mu)$ of F. Let Z be a locally ringed S-space and (p, q) be an element of $F(Z)$, i.e. $p : Z \to X$ and $q : Z \to Y$ are S-morphisms. These determine by Yoneda Lemma a natural transform $h_Z \to F$ which associates a locally ringed S-space T the map

$$\mathrm{Hom}_S(T, Z) \to F(T), \quad g \mapsto (p \circ g, q \circ g)$$

and every natural transform $h_Z \to F$ is of this form. We now show that the functor

$$T \mapsto F_{\lambda\mu}(T) \times_{F(T)} h_Z(T) \tag{8.3.1}$$

is representable by a locally ringed S-space induced by Z on an open subset of Z. In fact, an element of the right side of (8.3.1) (which is a fiber product of sets) is a triple (u_λ, v_μ, g), where $g : T \to Z$, $u_\lambda : T \to X_\lambda$, and $v_\mu : T \to Y_\mu$ are S-morphisms such that the following diagram commutes:

$$\begin{tikzcd}
T \arrow[dr, "g"] \arrow[drr, bend left, "v_\mu"] \arrow[ddr, bend right, "u_\lambda"'] & & \\
& Z \arrow[r, "q"] \arrow[d, "p"] & Y \arrow[d] \\
& X \arrow[r] & S
\end{tikzcd}$$

Now this in particular implies that $g(T) \subseteq Z_{\lambda\mu} = p^{-1}(X_\lambda) \cap q^{-1}(Y_\mu)$, and conversely, any S-morphism $g : T \to Z$ verifying this condition corresponds to the unique triple $(p \circ g, q \circ g, g)$, since $p \circ g$ (resp $q \circ g$) can be viewed as a morphism from Z to X_λ (resp. Y_μ). In other word, we have a canonical bijection

$$F_{\lambda\mu}(T) \times_{F(T)} \mathrm{Hom}_S(T, Z) \xrightarrow{\sim} \mathrm{Hom}_S(T, Z_{\lambda\mu})$$

and the functor (8.3.1) is then represented by the couple $(Z_{\lambda\mu}, (p|_{Z_{\lambda\mu}}, q|_{Z_{\lambda,\mu}}), j_{\lambda\mu})$, where $j_{\lambda\mu} : Z_{\lambda\mu} \to Z$ is the cannical injection. Since the $Z_{\lambda\mu}$ form an open covering of Z, this proves both of the conditions (i) and (iii) of **??**.

It remains to show that the functors $F_{\lambda\mu}$ are representable, which means we need to construct $X \times_S Y$ when X, Y, and S are affine schemes. This is fairly easy, as we will now show.

Proposition 8.3.1. *Assume that $S = \mathrm{Spec}(A)$, $X = \mathrm{Spec}(B)$, and $Y = \mathrm{Spec}(C)$, where B and C are A-algebras. Then the scheme $Z = \mathrm{Spec}(B \otimes_A C)$, with p, q the S-morphisms corresponding to the canonical A-homomorphisms $B \to B \otimes_A C$ and $C \to B \otimes_A C$, representes the functor F in the category of locally rinfed S-spaces.*

Proof. In fact, in the category of rings, the tensor product $B \otimes_A C$ of two A-algebras B and C is a coproduct in the category of A-algebras, as can be easily verified. □

We then conclude that fiber products exist in the category of schemes. As always, the notation $X \times_S Y$ will be used to denote this product for two S-schemes X and Y. If $S = \mathrm{Spec}(A)$ is an affine scheme, we also write $X \times_A Y$. If $Y = \mathrm{Spec}(B)$ is an affine scheme, in view of Proposition 8.3.1, we use $X \otimes_S B$ to denote this product, and $X \otimes_A B$ if $S = \mathrm{Spec}(A)$ is also affine.

The general notations and results for fiber products in a category can be then used for the product of schemes. In particular, if $p_1 : X \times_X Y \to X$, $p_2 : X \times_S Y \to Y$ are the canonical projections, and $g : T \to X$, $h : T \to Y$ are two S-morphisms, we denote by $(g, h)_S$ the unique S-morphism fits into the following diagram:

$$\begin{tikzcd}
T \arrow[dr, "(g{,}h)_S"] \arrow[drr, bend left, "h"] \arrow[ddr, bend right, "g"'] & & \\
& X \times_S Y \arrow[r, "p_2"] \arrow[d, "p_1"] & Y \arrow[d] \\
& X \arrow[r] & S
\end{tikzcd}$$

If $S = \mathrm{Spec}(A)$, $X = \mathrm{Spec}(B)$, $Y = \mathrm{Spec}(C)$, and $T = \mathrm{Spec}(D)$ are all affine, and g, h correspond to homomorphisms $\rho : B \to D$, $\sigma : C \to D$ of A-algebras, then $(g, h)_S$ corresponds to the homomorphism $\tau : B \otimes_A C \to D$ such that

$$\tau(b \otimes c) = \rho(b)\sigma(c).$$

Again, if $S = \mathrm{Spec}(A)$ is affine, we also write $(g, h)_A$ instead of $(g, h)_S$.

Corollary 8.3.2. *Let $Z = X \times_S Y$ be the product of two S-schemes, $p : Z \to X$, $q : Z \to Y$ the canonical projections, φ (resp ψ) the structural morphisms of X (resp. Y). Let U, V be open subsets of X, Y respectively, and W be an open subset of S such that $p(U) \subseteq W$ and $p(V) \subseteq W$. Then the product $U \times_W V$ is canonically identified with the scheme induced by Z on the subset $p^{-1}(V) \cap q^{-1}(W)$ (considered as a U-scheme). Moreover, if $g : T \to X$, $h : T \to Y$ are S-morphisms such that $g(T) \subseteq V$, $h(T) \subseteq W$, the U-morphism $(g, h)_S$ is identified with $(g, h)_S$, considered as morphisms from T to $p^{-1}(V) \cap q^{-1}(W)$.*

Proof. We first note that, if U is an open set of S and $\varphi : X \to S$, $\psi : Y \to S$ are structural morphisms with images in U, then the fiber product $X \times_S Y$ is identified with $X \times_U Y$. Apply this to V and W, we conclude that $U \times_W V = U \times_W V$. So it suffices to prove that the subscheme $R = p^{-1}(U) \cap q^{-1}(V)$ with its restricted projections to U and V form a product of U and V. For this, we note that if T is an S-scheme, we can identify the S-morphisms $T \to R$ and the S-morphisms $T \to Z$ with image in R. If $g : T \to U$, $h : T \to V$ are two S-morphisms, we can consider them as S-morphisms of T in X and Y respectively, and by hypothesis there is therefore an S-morphism and there is a morphism $f : T \to Z$ such that $g = p \circ f$, $h = q \circ f$. Since $p(f(T)) \subseteq U$ and $q(f(T)) \subseteq V$, we have

$$f(T) \subseteq p^{-1}(U) \cap q^{-1}(V) = W$$

whence our claim. □

Corollary 8.3.3. *Let (X_λ) (resp. (Y_μ)) be a family of S-schemes and X (resp. Y) be their coproduct. Then $X \times_S Y$ is identified with the coproduct of the family $(X_\lambda \times_S Y_\mu)$.*

Proof. In fact, in the notations of Corollary 8.3.2, the underlying space of $X \times_S Y$ is the disjoint union of open sets $p^{-1}(X_\lambda) \cap q^{-1}(Y_\mu)$, and it suffices to apply Corollary 8.3.2. □

Remark 8.3.4. The product of two Noetherian S-schemes need not to be Noetherian, even if they are both spectrum of fields. For example, if k is a nonperfect field of characteristic $p > 0$, the tensor product $A = k^{p^{-\infty}} \otimes_k k^{p^{-\infty}}$ is not a Noetherian ring: in fact, for any integer $n > 0$, there exists $x_n \in k^{p^{-\infty}}$ such that $x_n^{p^n} \in k$ and $x_n^{p^{n-1}} \notin k$. If we consider the element $z_n = 1 \otimes x_n - x_n \otimes 1$ of A, we then have $z_n^{p^n} = 0$, and $z_n^{p^{n-1}} \neq 0$ since 1 and $x_n^{p^{n-1}}$ are linearly independent over k. We then conclude that the nilradical of A is not nilpotent, so A is not Noetherian.

Remark 8.3.5. We should note that the underlying topological space of the fiber product $X \times_S Y$ is not the fiber product of the underlying topological spaces. This can be seen from the tensor product of two fields, which can not be a field.

8.3.2 Base change of schemes

The functor $X \times_S Y$ is covariant in both of its variables, and this follows from the following commutative diagram:

$$\begin{tikzcd}
X \times Y \arrow[r, "f \times 1"] \arrow[d] & X' \times Y \arrow[r, "f' \times 1"] \arrow[d] & X'' \times Y \arrow[d] \\
X \arrow[r, "f"] & X' \arrow[r, "f'"] & X''
\end{tikzcd}$$

Proposition 8.3.6. *For any S-scheme X, the first (resp. second) projection $X \times_S S$ (resp. $S \times_S X$) is a functorial isomorphism of $X \times_S S$ (resp. $S \times_S X$) to X, with inverse isomorphism $(1_X, \varphi)_S$ (resp. $(\varphi, 1_X)_S$), where $\varphi : X \to S$ is the structural morphism. We can therefore write*

$$X \times_S S = S \times_S X = X.$$

Proof. It suffices to prove that the triple $(X, 1_X, \varphi)$ form a product of X and S, which is immediate. □

Corollary 8.3.7. *Let X and Y be S-schemes, $\varphi : X \to S$, $\psi : Y \to S$ the structural morphisms. If we identify canonically X with $X \times_S S$ and Y with $S \times_S Y$, the projections $X \times_S Y \to X$ and $X \times_S Y \to Y$ is identified respectively to $1_X \times \psi$ and $\varphi \times 1_Y$.*

We can define similarly the fiber product of S-schemes $X_1, \ldots, X_n$, whose existence can be proved by induction on n, which is isomorphic to $(X_1 \times_S \cdots \times_S X_{n-1}) \times_S X_n$. The uniqueness of the product entails, as in any category, its properties of commutativity and associativity. If, for example, p_1, p_2, p_3 denotes the projections of $X_1 \times_S X_2 \times_S X_3$, and if we identify this scheme with $(X_1 \times_S X_2) \times_S X_3$, the projection in $X_1 \times_S X_2$ is identified with $(p_1, p_2)_S$.

Let S, S' be two schemes, $\varphi : S \to S'$ an morphism, making S' an S-scheme. For any S-scheme X, consider the product $X \times_S S'$, and let p and π' the projections to X and S' respectively. Through the morphism π', this product is an S'-scheme, which we may denoted by $X_{(S')}$ or $X_{(\varphi)}$, and the obtained scheme is called the **base change** of X from S to S', or the inverse image of X via φ. We note that if π is the structural morphism of X and θ is the structural morphism of $X \times_S S'$, the following diagram is commutative:

$$\begin{tikzcd} X_{(S')} \arrow[r, "\pi'"] \arrow[d, "p"'] & S' \arrow[d, "\varphi"] \\ X \arrow[r, "\pi"] & S \end{tikzcd}$$

For any S-morphism $f : X \to Y$, we denote by $f_{(S')}$ the S'-morphism $f \times_S 1 : X_{(S')} \to Y_{(S')}$ and call it the inverse image of f by φ. The operation $X_{(S')}$ is clearly a covariant functor on X, from the category **Sch**$/S$ to **Sch**$/S'$.

Let S, S' be two affine schemes with rings A, A'; a morphism $S' \to S$ corresponds to a homomorphism $A \to A'$. If X is an S-scheme, we then denote by $X_{(A')}$ of $X \otimes_A A'$ by the S'-scheme $X_{(S')}$; if X is also affine with ring B, then $X_{(A')}$ is affine with ring $B_{(A')} = B \otimes_A A'$.

We point out that the scheme $X_{(S')}$ satisfies the following universal property: any S'-scheme T is an S-scheme via the morphism φ, and for any S-morphism $g : T \to X$ there exists a unique S'-morphism $f : T \to X_{(S')}$ such that $g = p \circ f$.

Proposition 8.3.8 (Transitivity). *Let $\varphi' : S'' \to S'$ and $\varphi : S' \to S$ be morphism of schemes. For any S-scheme X, there is a canonical functorial isomorphism of the S''-schemes $(X_{(\varphi)})_{(\varphi')}$ and $X_{(\varphi\circ\varphi')}$.*

Proof. In fact, let T be an S''-scheme, ψ its structural morphism, $g : T \to X$ an S-morphism (T is an S-scheme via the morphism $\varphi \circ \varphi' \circ \psi$). Since T is an S'-scheme with structural morphism $\varphi' \circ \psi$, we can write $g = p \circ g'$, where $g' : T \to X_{(\varphi)}$ is an S'-morphism. Then $g' = p' \circ g''$, where $g'' : T \to (X_{(\varphi)})_{\varphi'}$ is an S''-morphism:

$$\begin{tikzcd} (X_{(\varphi)})_{\varphi'} \arrow[r, "p'"] \arrow[d, "\pi''"] & X_{(\varphi)} \arrow[r, "p"] \arrow[d, "\pi'"] & X \arrow[d, "\pi"] \\ S'' \arrow[r, "\varphi'"] & S' \arrow[r, "\varphi"] & S \end{tikzcd}$$

The claim now follows from the definition of the universal property of $X_{(\varphi\circ\varphi')}$. □

The previous result can also be written as $(X_{(S')})_{(S'')} = X_{(S'')}$, if there is no risk of confusion. Moreover precisely, we have

$$(X \times_S S') \times_{S'} S'' = X \times_S S'';$$

the functorial of the isomorphism in Proposition 8.3.8 also shows the transitive of inverse image of morphisms:

$$(f_{(S')})_{(S'')} = f_{(S'')}$$

for any S-morphism $f : X \to Y$.

Corollary 8.3.9. *If X and Y are S-schemes, there exists a canonical functorial isomorphism of S'-schemes $X_{(S')} \times_{S'} Y_{(S')}$ and $(X \times_S Y)_{S'}$.*

Proof. In fact, we have, the following canonical isomorphisms:

$$(X \times_S S') \times_{S'} (Y \times_S S') = X \times_S (Y \times_S S') = (X \times_S Y) \times_S S'$$

where we use Proposition 8.3.8 and the associativity of fiber product. □

Again, the functorial isomorphism in Corollary 8.3.9 also gives the isomorphism

$$(u_{(S')}, v_{(S')})_{(S')} = ((u,v)_S)_{S'}$$

for any S-morphisms $u : T \to X$, $v : T \to Y$. In other words, the inverse image functor $X_{(S')}$ commutes on the formation of the products; note that it also commutes to the formation of coproducts.

Corollary 8.3.10. *Let Y be an S-scheme, $f : X \to Y$ a morphism making X a Y-scheme (and also an S-scheme). Then the scheme $X_{(S')}$ is canonically identified with the product $X \times_Y Y_{(S')}$, and the projection $X \times_Y Y_{(S')} \to Y_{(S')}$ is identified with $f_{(S')}$.*

Proof. Let $\psi : Y \to S$ be the structural morphism of Y; we have a commutative diagram

$$\begin{tikzcd}
X_{(S')} \arrow[r, "f_{(S')}"] \arrow[d] & Y_{(S')} \arrow[r, "\psi_{(S')}"] \arrow[d] & S' \arrow[d] \\
X \arrow[r, "f"] & Y \arrow[r, "\psi"] & S
\end{tikzcd}$$

Now $Y_{(S')}$ is identified with $S'_{(\psi)}$, and $X_{(S')}$ with $S'_{(\psi\circ f)}$, so by Proposition 8.3.8 and Proposition 8.3.6, we deduce the corollary. □

Example 8.3.11. Let A be a ring, X an A-scheme, and $\mathfrak{a}$ an ideal of A. Then $X_0 = X \otimes_A (A/\mathfrak{a})$ is an $(A/\mathfrak{a})$-scheme, called the scheme obtained from X by **reduction mod $\mathfrak{a}$**.

Proposition 8.3.12. *Let $f : X \to X'$, $g : Y \to Y'$ be two S-morphisms that are monomorphisms of schemes; then $f \times_S g$ is a monomorphism. In particular, for any extension $S' \to S$ of base scheme, the inverse image $f_{(S')} : X_{(S')} \to Y_{(S')}$ is a monomorphism.*

Proof. In fact, if p, q are the projections of $X \times_S Y$, and p', q' that of $X' \times_S Y'$:

$$\begin{tikzcd}
X \times_S Y \arrow[rr, "q"] \arrow[dd, "p"'] \arrow[dr, "f\times_S g"] & & Y \arrow[d, "g"] \\
& X' \times_S Y' \arrow[r, "q'"] \arrow[d, "p'"'] & Y' \arrow[d] \\
X \arrow[r, "f"] & X' \arrow[r] & S
\end{tikzcd}$$

then for any two morphisms $u, v : T \to X \times_S Y$, the relation $(f \times_S g) \circ u = (f \times_S g) \circ v$ implies

$$p' \circ (f \times_S g) \circ u = p' \circ (f \times_S g) \circ v$$

so $f \circ p \circ u = f \circ p \circ v$, and since f is a monomorphism, we conclude $p \circ u = p \circ v$. Similarly, since g is a monomorphism, we have $q \circ u = q \circ v$, whence $u = v$. □

For any S-morphism $f : S' \to X$, the morphism $f' = (f, 1_{S'})_S$ is then an S'-morphism from S' to $X' = X_{(S')}$ such that $p \circ f' = f$, $\pi' \circ f' = 1_{S'}$, which is called an ***S'*-section** of X':

$$\begin{tikzcd}
X' \arrow[r, "\pi'"] \arrow[d, "p"'] & S' \arrow[l, bend right, "f'"'] \arrow[d, "\varphi"] \arrow[dl, "f"] \\
X \arrow[r, "\pi"] & S
\end{tikzcd}$$

Conversely if f' is an S'-section, then $f = p \circ f'$ is an S-morphism $S' \to X$. We then deduce the following canonical correspondence

$$\mathrm{Hom}_S(S', X) \xrightarrow{\sim} \mathrm{Hom}_{S'}(S', X') \tag{8.3.2}$$

The morphism f' is called the **graph** of f, and denoted by Γ_f. A particularly important case is $S' = X$ and $f = 1_X$, where corresponding morphism $X \to X \times_S X$ is called the **diagonal morphism** of X, and denoted by Δ_X. Also, if $f : X \to Y$ is a morphism of schemes, we denote by Δ_f the diagonal map from X to $X \times_Y X$.

Example 8.3.13. Since any scheme X can be considered as a $\mathbb{Z}$-scheme, we can consider the X-sections of $X \otimes_{\mathbb{Z}} \mathbb{Z}[T]$ (where T is an indeterminate) corresponding to the $\mathbb{Z}$-morphisms $X \to \mathrm{Spec}(\mathbb{Z}[T])$.

$$\begin{tikzcd}
X \otimes_{\mathbb{Z}} \mathbb{Z}[T] \arrow[r] \arrow[d] & \mathrm{Spec}(\mathbb{Z}[T]) \arrow[d] \\
X \arrow[r] \arrow[u, dashed, bend left] \arrow[ur, dashed] & \mathrm{Spec}(\mathbb{Z})
\end{tikzcd}$$

We claim that such X-sections correspond to sections of the structural sheaf $\mathscr{O}_X$ of X. In fact, the morphisms $X \to \mathrm{Spec}(\mathbb{Z}[T])$ correspond to ring homomorphisms $\mathbb{Z}[T] \to \Gamma(X, \mathscr{O}_X)$, which in turn are entirely determined by the image of T, and can be an arbitrary element of $\Gamma(X, \mathscr{O}_X)$, whence our assertion.

8.3.3 Tensor product of quasi-coherent sheaves

Let S be a scheme, X, Y be two S-schemes, $Z = X \times_S Y$, and p, q be the projections of Z to X and Y, respectively. Let $\mathscr{F}$ (resp. $\mathscr{G}$) be an $\mathscr{O}_X$-module (resp. an $\mathscr{O}_Y$-module). Then the tensor product $p^*(\mathscr{F}) \otimes_{\mathscr{O}_Z} q^*(\mathscr{G})$ is called the **tensor product of $\mathscr{F}$ and $\mathscr{G}$ over $\mathscr{O}_S$** (or **over S**) and denoted by $\mathscr{F} \otimes_{\mathscr{O}_S} \mathscr{G}$ (or $\mathscr{F} \otimes_S \mathscr{G}$). More generaly, if $(X_i)_{1 \le i \le n}$ is a finite family of S-schemes and for each i, $\mathscr{F}_i$ is an $\mathscr{O}_X$-module, we can define the tensor product $\mathscr{F}_1 \otimes_S \cdots \otimes_S \mathscr{F}_n$ over the scheme $Z = X_1 \times_S \cdots \times_S X_n$. This is a quasi-coherent $\mathscr{O}_Z$-module if each $\mathscr{F}_i$ is quasi-coherent (Corollary 8.2.23), and is coherent if each $\mathscr{F}_i$ is coherent and Z is locally Noetherian in view of **??**.

We note that if $X = Y = S$, the above definition coincide with the usual one of tensor product of $\mathscr{O}_S$-modules. Moreover, as $q^*(\mathscr{O}_Y) = \mathscr{O}_{X \times_S Y}$, the product $\mathscr{F} \otimes_S \mathscr{O}_Y$ is canonically identified with $p^*(\mathscr{F})$, and similarly $\mathscr{O}_X \otimes_S \mathscr{G}$ is identified with $q^*(\mathscr{G})$. In particular, if $Y = S$ and $f : X \to Y$ is the structural morphism, then $\mathscr{O}_X \otimes_Y \mathscr{G} = f^*(\mathscr{G})$: the ordinary tensor product and the inverse image therefore appears as a special case of the general tensor product. We also note that if X and Y are fixed, the operation $\mathscr{F} \otimes_S \mathscr{G}$ is a covariant bifunctor and is right exact on $\mathscr{F}$ and $\mathscr{G}$.

Proposition 8.3.14. *Let S, X, Y be affine schemes with rings A, B, C, respectively, where B, C are A-algebras. Let M (resp. B) be a B-module (resp. C-module) and $\mathscr{F} = \widetilde{M}$ (resp. $\mathscr{G} = \widetilde{N}$) the associated quasi-coherent sheaf. Then $\mathscr{F} \otimes_S \mathscr{G}$ is canonically isomorphic to the sheaf associated with the $(B \otimes_A C)$-module $M \otimes_A N$.*

Proof. In fact, in view of Proposition 8.1.14 and Corollary 8.1.10, $\mathscr{F} \otimes_S \mathscr{G}$ is canonically isomorphic to the sheaf associated with the $(B \otimes_A C)$-module

$$(M \otimes_B (B \otimes_A C)) \otimes_{B \otimes_A C} ((B \otimes_A C) \otimes_C N)$$

and due to the canonical isomorphisms between tensor products, the latter is isomorphic to $M \otimes_B (B \otimes_A C) \otimes_C N = (M \otimes_B B) \otimes_A (C \otimes_C N) = M \otimes_A N$. $\square$

Proposition 8.3.15. *Let $f : T \to X$, $g : T \to Y$ be two S-morphisms, and $\mathscr{F}$ (resp. $\mathscr{G}$) be an $\mathscr{O}_X$-module (resp. $\mathscr{O}_Y$-module). Then we have $(f, g)_S^*(\mathscr{F} \otimes_S \mathscr{G}) = f^*(\mathscr{F}) \otimes_{\mathscr{O}_T} g^*(\mathscr{G})$.*

Proof. If p, q are the projections of $X \times_S Y$, the assertion follows from the relations $(f, g)_S^* \circ p^* = f^*$ and $(f, g)_S^* \circ q^* = g^*$, and the fact that the inverse image operation commutes with tensor products. $\square$

Corollary 8.3.16. *Let $f : X \to X'$, $g : Y \to Y'$ be two S-schemes and $\mathscr{F}'$ (resp. $\mathscr{G}'$) be an $\mathscr{O}_{X'}$-module (resp. $\mathscr{O}_{Y'}$-module). Then $(f \times_S g)^*(\mathscr{F}' \otimes_S \mathscr{G}') = f^*(\mathscr{F}') \otimes_S g^*(\mathscr{G}')$.*

Proof. This follows from Proposition 8.3.15 and the fact that $f \times_S g = (f \circ p, g \circ q)_S$, where p, q are the projections of $X \times_S Y$. $\square$

Corollary 8.3.17. *Let X, Y, Z be S-schemes and $\mathscr{F}$ (resp. $\mathscr{G}$, $\mathscr{H}$) be an $\mathscr{O}_X$-module (resp. $\mathscr{O}_Y$-module, $\mathscr{O}_Z$-module). Then the sheaf $\mathscr{F} \otimes_S \mathscr{G} \otimes_S \mathscr{H}$ is the inverse image of $(\mathscr{F} \otimes_S \mathscr{G} \otimes_S \mathscr{H})$ under the canonical isomorphism $X \times_S Y \times_S Z \to (X \times_S Y) \times_S Z$, and $\mathscr{G} \otimes_S \mathscr{F}$ is the inverse image of $\mathscr{F} \otimes_S \mathscr{G}$ under the canonical isomorphism $X \times_S \to Y \times_S X$.*

Proof. The first isomorphism is $(p_1, p_2)_S \times_S p_3$, where p_1, p_2, p_3 are the projections of $X \times_S Y \times_S Z$, and second one is similarly. □

Corollary 8.3.18. *If X is an S-scheme, any $\mathscr{O}_X$-module $\mathscr{F}$ is the inverse image of $\mathscr{F} \otimes_S \mathscr{O}_S$ under the canonical isomorphism from X to $X \times_S S$.*

Let X be an S-scheme, $\mathscr{F}$ be an $\mathscr{O}_X$-module, and $\varphi : S' \to S$ be an morphism. We denote by $\mathscr{F}_{(\varphi)}$ or $\mathscr{F}_{(S')}$ the sheaf $\mathscr{F} \otimes_S \mathscr{O}_{S'}$ over $X \times_S S' = X_{(\varphi)} = X_{(S')}$, so $\mathscr{F}_{(S')} = p^*(\mathscr{F})$, where p is the projection $X_{(S')} \to X$.

Proposition 8.3.19. *Let $\varphi' : S'' \to S'$ be a morphism. For any $\mathscr{O}_X$-module $\mathscr{F}$ over the S-scheme X, $(\mathscr{F}_{(\varphi)})_{(\varphi')}$ is the inverse image of $\mathscr{F}_{(\varphi\circ\varphi)}$ under the canonical isomorphism $(X_{(\varphi)})_{(\varphi')} \to X_{(\varphi\circ\varphi')}$.*

Proof. This follows from the definition and the associativity of base change, since $(\mathscr{F} \otimes_S \mathscr{O}_{S'}) \otimes_{S'} \mathscr{O}_{S''} = \mathscr{F} \otimes_S \mathscr{O}_{S''}$. □

Proposition 8.3.20. *Let Y be an S-scheme and $f : X \to Y$ be an S-morphism. For any $\mathscr{O}_Y$-module $\mathscr{G}$ and any morphism $S' \to S$, we have $(f_{(S')})^*(\mathscr{G}_{(S')}) = (f^*(\mathscr{G}))_{(S')}$.*

Proof. This follows from the diagram

$$\begin{tikzcd} X_{(S')} \arrow[r, "f_{(S')}"] \arrow[d] & Y_{(S')} \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

and the functoriality of inverse images. □

Corollary 8.3.21. *Let X, Y be S-schemes and $\mathscr{F}$ (resp. $\mathscr{G}$) be an $\mathscr{O}_X$-module (resp. $\mathscr{O}_Y$-module). For any morphism $S' \to S$, the inverse image of the sheaf $(\mathscr{F}_{(S')}) \otimes_{S'} (\mathscr{G}_{(S')})$ under the canonical isomorphism $(X \times_S Y)_{(S')} \cong (X_{(S')}) \times_{S'} (Y_{(S')})$ is equal to $(\mathscr{F} \otimes_S \mathscr{G})_{(S')}$.*

Proof. If p, q are the projections of $X \times_S Y$, the isomorphism is given by $(p_{(S')}, q_{(S')})'_S$, so the corollary follows from Proposition 8.3.15 and Proposition 8.3.20. □

Proposition 8.3.22. *Let X, Y be two S-schemes and $\mathscr{F}$ (resp. $\mathscr{G}$) be an $\mathscr{O}_X$-module (resp. $\mathscr{O}_Y$-module). Let p, q be the projections of $Z = X \times_S Y$, z be a point of Z, and put $x = p(z)$, $y = q(z)$. Then the stalk $(\mathscr{F} \otimes_S \mathscr{G})_z$ is isomorphic to $(\mathscr{F}_x \otimes_{\mathscr{O}_{X,x}} \mathscr{O}_{Z,z}) \otimes_{\mathscr{O}_{Z,z}} (\mathscr{G}_y \otimes_{\mathscr{O}_{Y,y}} \mathscr{O}_{Z,z}) = \mathscr{F}_x \otimes_{\mathscr{O}_{X,x}} \mathscr{O}_{Z,z} \otimes_{\mathscr{O}_{Y,y}} \mathscr{G}_y$.*

Proof. Since the question is local, we can reduce to the affine case, and the assertion follows from Corollary 8.1.10. □

Corollary 8.3.23. *With the notations in Proposition 8.3.22, if $\mathscr{F}$ and $\mathscr{G}$ are of finite type, then*

$$\mathrm{supp}(\mathscr{F} \otimes_S \mathscr{G}) = p^{-1}(\mathrm{supp}(\mathscr{F})) \cap q^{-1}(\mathrm{supp}(\mathscr{G})).$$

Proof. As $p^*(\mathscr{F})$ and $q^*(\mathscr{G})$ are of finite type over $\mathscr{O}_Z$, in view of Proposition 8.3.22 and **??**, we can reduce to the case $\mathscr{G} = \mathscr{O}_Y$, and the assertion then follows from the formula $\mathrm{supp}(p^{-1}(\mathscr{F})) = p^{-1}(\mathrm{supp}(\mathscr{F}))$. □

8.3.4 Scheme valued points

Let X be a scheme; for any scheme T, we denote by $X(T)$ the set $\mathrm{Hom}(T, X)$ of morphisms from T to X, and the elements of this set will be called **points of X with values in T**. The operation $T \mapsto X(T)$ is then a contravariant functor from the category of schemes to that of sets (in one word, we identify the scheme X with the induced functor h_X on **Sch**). Moreover, any morphism $g : X \to Y$ of schemes defines a natural transform $X(T) \to Y(T)$, which send $v \in X(T)$ to $g \circ v \in Y(T)$. The product of two S-schemes X and Y is then defined by the canonical isomorphism

$$(X \times_S Y)(T) \xrightarrow{\sim} X(T) \times_{S(T)} Y(T) \tag{8.3.3}$$

where the maps $X(T) \to S(T)$ and $Y(T) \to S(T)$ corresponds to the structural morphisms $X \to S$ and $Y \to S$.

If we are given a scheme S and consider S-schemes and S-morphisms, we denote by $X(T)_S$ the set $\mathrm{Hom}_S(T, X)$ of S-morphisms $T \to X$, and omit the index S if there is no risk of confusion. We also say the elements of $X(T)_S$ are the (S-)points of the S-scheme X with values in the S-scheme T. In particular, an S-section of X is none other than a point of X with values in S. The formula (8.3.3) is then written as

$$(X \times_S Y)(T)_S = X(T)_S \times Y(T)_S;$$

more generally, if Z is an S-scheme, X, Y, T are Z-schemes, we have

$$(X \times_Z Y)(T)_S = X(T)_S \times_{Z(T)_S} Y(T)_S.$$

We remark that for any morphism $S' \to S$, the set $X(S')_S = \mathrm{Hom}_S(S', X)$ is identified with the set $\mathrm{Hom}_{S'}(S', X')$, where $X' = X \times_S S'$, which is the set of S'-sections of X'.

In T (resp. S) is an affine scheme with ring B (resp. A), we replace T (resp. S) by B (resp. A) in the above notations, and we then refer points of X with values in the ring B, or points of the A-scheme X with values in the A-algebra B for the elements $X(B)$ of $X(B)_A$, respectively. We also call $X(T)_A$ the set of points of the A-scheme X with values in the A-scheme T.

Consider in particular the case where T is a local scheme $\mathrm{Spec}(A)$, where A is a local ring; the elements $X(A)$ corresponds to local homomorphisms $\mathscr{O}_{X,x} \to A$ for $x \in X$ (Corollary 8.2.15); we say the point x of the underlying topological space X is the **locality** of the point of X with values in A to which it corresponds (of course, several distinct points of X with values in A can have the same locality), or that the point of X with values in A which corresponds to x is **localizaed in x**.

Even more particularly, a point of X with values in a field K correspond to a point $x \in X$ and a field extension $\kappa(x) \to K$. If X is an S-scheme, saying that $S' = \mathrm{Spec}(K)$ is an S-scheme means K is an extension of the residue field $\kappa(s)$ for an point $s \in S$; an element of $X(K)_S$, which is called **a point of X lying over s with values in K**, corresponds then to a $\kappa(s)$-homomorphism $\kappa(x) \to K$, where x is a point of the topological space X lying over s (hence $\kappa(x)$ is an extension of $\kappa(s)$).

The points of X with values in an algebraically closed field K are called **geometric points** of the scheme X;[2] the field K is called the **value field** of the geometric point. If X is an S-scheme and s is an point of S, a **geometric point of X lying over s** is then a geometric point of X localizaed in a point of X lying over s. We then have a map $X(K) \to X$, which send a geometric point with values in K to the point it locates.

If $S = \mathrm{Spec}(k)$ is the spectrum of a field k and X is an S-scheme, the S-points of X with values in k is identified with the S-sections of X, or with the points x of X such that the canonical homomorphism $k \to \kappa(x)$ is an isomorphism since only at such a point there exists a homomorphism $\kappa(x) \to k$ such that the composition $k \to \kappa(x) \to k$ is the identity. Such points are called the **rational points** over k of the k-scheme X. Note that if k' is an extension of k, the points of X with values in k' correspond to the points of $X' = X_{(k')}$ rational over k' (cf. (8.3.2)).

The example $X = \mathrm{Spec}(K)$, where K is an nontrivial extension of k, shows that there do not necessarily exist in X rational points on k, even if X is nonempty. Still assuming that X is a k-scheme. For any point $x \in X$, there is always an extensions k' of k for which there is a point x' of $X' = X_{(k')}$ rational over k' and whose image by the canonical projection $X' \to X$ is x: it suffices to take for k' an extension of $\kappa(x)$, the k-monomorphism $\kappa(x) \to k'$ giving the sought point x'. When we thus passes from a point x to a rational point $x' \in X'$ over k' and above x, we say that we "make x rational."

Proposition 8.3.24. *Let $S = \mathrm{Spec}(k)$ be the specturm of a field k, and X be an S-scheme. Then any k-rational point of X is closed in X.*

Proof. In fact, it suffices to show that the point x is closed in any open affine open set containing x, so we may assume that $X = \mathrm{Spec}(A)$ is affine. In this case, since the composition homomorphism $k \to A \to \kappa(x)$ is an isomorphism (we know that $\kappa(x) = k$), we conclude in particular that $A/\mathfrak{p}_x \to k$ is an integral extension, which implies that $A/\mathfrak{p}_x$ is a field (**??**). □

Proposition 8.3.25. *Let $(X_i)_{1 \le i \le n}$ be S-schemes, s a point of X, and x_i a point of X_i lying over s for each i. For there to be a point y of the scheme $Y = X_1 \times_S \cdots \times_S X_n$ whose projections on X_i is x_i, it is necessary and sufficient that the x_i are over the same point s of S.*

[2] This terminology is also sometimes used when K is only separably closed, but at that time we will explicitly clarify which convention we adapt.

Proof. This condition is clearly necessary. Now let s be an element of S and x_i a point of X_i lying over s. Then there exist $\kappa(s)$-homomorphisms $\kappa(x_i) \to K$ where K is a common field. The composition $\kappa(s) \to \kappa(x_i) \to K$ are all identical, so the morphisms $\mathrm{Spec}(K) \to X_i$ corresponding to $\kappa(x_i) \to K$ are S-morphisms, and we conclude that they define a unique morphism $\mathrm{Spec}(K) \to Y$. If y is the corresponding point of Y, it is clear that its projection into each of the X_i is x_i. □

In other words, if we denote by (X) the set underlying X, we see that we have a canonical surjective map $(X \times_S Y) \to (X) \times_{(S)} (Y)$; we have already pointed that this maps is not injective in general; that is, there can be multiple points distinct in $X \times_S Y$ having same projections to X and Y.

Corollary 8.3.26. *Let $f : X \to Y$ be an S-morphism, $f_{(S')} : X_{(S')} \to Y_{(S')}$ the S'-morphism induced by a base change $S' \to S$. Let p (resp. q) be the projection $X_{(S')} \to X$ (resp. $Y_{(S')} \to Y$); for any subset V of X, we have*

$$q^{-1}(f(M)) = f_{(S')}(p^{-1}(M)).$$

Proof. By Corollary 8.3.10, $X_{(S')}$ is identified with the product $X \times_Y Y_{(S')}$ and we have the following commutative diagram

$$\begin{tikzcd}
X_{(S')} \arrow[r, "f_{(S')}"] \arrow[d, "p"'] & Y_{(S')} \arrow[d, "q"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

By Proposition 8.3.25, the relation $q(y') = f(x)$ for $x \in V$, $y' \in Y_{(S')}$ is equivalent to the existence of a point $x' \in X_{(S')}$ such that $p(x') = x$ and $f_{(S')}(x') = y'$, whence the corollary. □

Proposition 8.3.27. *Let X, Y be S-schemes and $x \in X$, $y \in Y$ two points lying over the same point $s \in S$. Then the set of points $X \times_S Y$ with projections x and y is in canonical correspondence with the set of types of the composition field extension of $\kappa(x)$ and $\kappa(y)$, considered as extensions of $\kappa(s)$.*

Proof. Let p (resp. q) be the projection of $X \times_S Y$ to X (resp. Y) and let E be the subspace $p^{-1}(x) \cap q^{-1}(y)$ of the underlying topological space of $X \times_S Y$. We first note that since x and y are lying over s, the morphisms $\mathrm{Spec}(\kappa(x)) \to S$ and $\mathrm{Spec}(\kappa(y)) \to S$ factor through $\mathrm{Spec}(\kappa(s))$:

$$\begin{tikzcd}
 & \mathrm{Spec}(\kappa(x)) \arrow[d] & \\
\mathrm{Spec}(\kappa(y)) \arrow[r] & \mathrm{Spec}(\kappa(s)) \arrow[r] & S
\end{tikzcd}$$

since $\mathrm{Spec}(\kappa(s)) \to S$ is a monomorphism by Corollary 8.2.18, it follows immediately that we have

$$P = \mathrm{Spec}(\kappa(x)) \times_S \mathrm{Spec}(\kappa(y)) = \mathrm{Spec}(\kappa(x)) \times_{\mathrm{Spec}(\kappa(s))} \mathrm{Spec}(\kappa(y)) = \mathrm{Spec}(\kappa(x) \otimes_{\kappa(s)} \kappa(y)).$$

Let $i : \mathrm{Spec}(\kappa(x)) \to X$ and $j : \mathrm{Spec}(\kappa(y)) \to Y$ be the canonical morphisms, we put $\alpha = i \times_S j : P \to E$ to be the map on the underlying topological space. On the other hand, any point $z \in E$ defines two $\kappa(s)$-homomorphisms $\kappa(x) \to \kappa(z)$ and $\kappa(y) \to \kappa(z)$, hence a $\kappa(s)$-homomorphism $\kappa(x) \otimes_{\kappa(s)} \kappa(y) \to \kappa(z)$, which corresponds to a morphism $\mathrm{Spec}(\kappa(z)) \to P$; we take $\beta(z)$ to be the image of this morphism, which defines a map $\beta : E \to P$.

To verify that $\alpha \circ \beta$ and $\beta \circ \alpha$ are identity maps, we need the following commutative diagram

$$\begin{tikzcd}
\mathrm{Spec}(\kappa(z)) \arrow[dr] \arrow[drr, bend left] \arrow[ddr, bend right] & & & \\
 & P \arrow[r] \arrow[d] \arrow[dr, "\alpha"] & \mathrm{Spec}(\kappa(y)) \arrow[dr, "j"] & \\
 & \mathrm{Spec}(\kappa(x)) \arrow[dr, "i"'] & E \arrow[r] \arrow[d] & Y \arrow[d] \\
 & & X \arrow[r] & S
\end{tikzcd}$$

for $z \in E$. By the uniqueness part of the universal property of fiber products, the canonical morphism $\mathrm{Spec}(\kappa(z)) \to E$ induced by $\mathrm{Spec}(\kappa(z)) \to X$ and $\mathrm{Spec}(\kappa(z)) \to Y$ is given by the composition

$\mathrm{Spec}(\kappa(z)) \to P \to E$, and also equal to the canonical morphism $\mathrm{Spec}(\kappa(z)) \to E$ of the scheme $X \times_S Y$ at z (Corollary 8.2.17). This means the image of $\beta(z)$ under α is exactly the image of the canonical morphism $\mathrm{Spec}(\kappa(z)) \to E$, which is just z; this shows $\alpha \circ \beta = 1_E$. As for $\beta \circ \alpha$, we just note that if $z = \alpha(\mathfrak{p})$ for some $\mathfrak{p} \in P$ (a prime ideal), then the morphism α induces a field extensins $\kappa(z) \to \kappa(\mathfrak{p})$, which corresponds to morphism $\mathrm{Spec}(\kappa(\mathfrak{p})) \to \mathrm{Spec}(\kappa(z))$. Again by the uniqueness part of the fiber product P, we conclude that the canonical morphism $\mathrm{Spec}(\kappa(\mathfrak{p})) \to P$ factors through $\mathrm{Spec}(\kappa(z))$, which means $\beta(z) = \mathfrak{p}$, so $\beta \circ \alpha = 1_P$. Finally, we recall that the set P corresponds to composition fields of $\kappa(x)$ and $\kappa(y)$ over $\kappa(s)$. □

8.3.5 Surjective morphisms

Let $\mathcal{P}$ be a property for morphisms of schemes. We consider the following conditions:

(i) If $f : X \to X'$ and $g : Y \to Y'$ are S-morphisms satisfying $\mathcal{P}$, then $f \times_S g$ also satisfies $\mathcal{P}$.

(ii) If $f : X \to Y$ is an S-morphism satisfying $\mathcal{P}$ and $S' \to S$ is a morphism, then $f_{(S')} : X_{(S')} \to Y_{(S')}$ also satisfies $\mathcal{P}$.

Since $f_{(S')} = f \times_S 1_{S'}$, we see if any identity morphism satisfies $\mathcal{P}$, then (i) implies (ii). On the other hand, since $f \times_S g$ is the following composition

$$X \times_S Y \xrightarrow{f \times 1_Y} X' \times_S Y \xrightarrow{1_{X'} \times g} X' \times_S Y'$$

it is clear that if the composition of two morphisms satisfying $\mathcal{P}$ still satisfies $\mathcal{P}$, then (ii) implies (i) (in this case we say $\mathcal{P}$ is stable under composition). In general, a property $\mathcal{P}$ is called **stable under base change** if it satisfies the condition (ii). For example Proposition 8.3.12 just says that being a monomorphism is stable under base change. On the other hand, if $\mathcal{P}$ is an arbitrary property of morphisms, we say a morphism $f : X \to S$ **satisfies** $\mathcal{P}$ **universally** (or is **universally** $\mathcal{P}$), if for any morphism $S' \to S$ the inverse image $f_{(S')}$ satisfies $\mathcal{P}$.

Our first application of the above definition is that surjectivity is stable under base change:

Proposition 8.3.28. *Surjective morphisms of schemes are stable under base change.*

Proof. Note that it is clear that surjectivity is stable under composition, in fact we have the both conditions (i) and (ii) described above. But condition (ii) follows from Corollary 8.3.26 by setting $V = X$. □

Proposition 8.3.29. *For a morphism $f : X \to Y$ of schemes to be surjective, it is necessary and sufficient that for any field K and any morphism $\mathrm{Spec}(K) \to Y$, there exists an extension K' of K and a morphism $\mathrm{Spec}(K') \to X$ fitting into the following diagram*

$$\begin{tikzcd} \mathrm{Spec}(K') \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \mathrm{Spec}(K) \arrow[r] & Y \end{tikzcd}$$

Proof. The condition is sufficient, since for any $y \in Y$, we can apply the canonical morphism $\mathrm{Spec}(\kappa(y)) \to Y$ to get a morphism $\mathrm{Spec}(K) \to X$, which gives a inverse image of y in X. Conversely, suppose that f is surjective, and let $y \in T$ be the image of $\mathrm{Spec}(K)$ in Y; there exists $x \in X$ such that $f(x) = y$. Consider the monomorphism $\kappa(y) \to \kappa(x)$ corresponding to f, and take an extension K' of $\kappa(y)$ containing $\kappa(x)$ and K; the morphism $\mathrm{Spec}(K') \to X$ corresponding to $\kappa(x) \to K'$ then satisfies the requirement. □

Corollary 8.3.30. *For a morphism $f : X \to Y$ to be surjective, it is necessary and sufficient that, for any field K, there exist an algebraically closed extension K' of K such that the map $X(K') \to Y(K')$ corresponding to f is surjective.*

Proof. In view of Proposition 8.3.29, this condition is sufficient. Conversely, suppose that f is surjective and let K be a field. If p is the characteristic of K, let us take for K' an algebraically closed extension of K having over the prime field P a transcendence basis of strictly larger cardinality to the cardinals of all the transcendence bases on P of the residual fields of X and Y having characteristic p. It then remains to see, with the same notations as in Proposition 8.3.29, that any monomorphism $u : \kappa(y) \to K'$ factors into

$$\kappa(y) \xrightarrow{w} \kappa(x) \xrightarrow{v} K'$$

where $w = f^x$. Now, let L a purely transcendental extension of P contained in $\kappa(y)$ and over which $\kappa(y)$ is algebraic; if B is a transcendence basis of L over P, we can complete $w(B)$ into a transcendence basis B' of $\kappa(x)$ on P, and then (due to the assumption made on the trancendence bases of K') define a monomorphism $v_1 : P(B') \to K'$ such that $v_1 \circ (w|_L)$ coincides with $u|_L$. There is also an isomorphism $v_2 = u \circ w^{-1}$ from $w(\kappa(y))$ to $u(\kappa(y))$ such that v_2 and v_1 coincide in $w(L)$; as $w(\kappa(y))$ and $P(B' - w(B))$ are linearly disjoint on $w(L)$, we can extend v_1 and v_2 into a monomorphism v_0 of $M = P(B')(w(\kappa(y)))$ in K'; as K' is algebraically closed and $\kappa(x)$ is algebraic over M, we can finally extend v_0 into the monomorphism $v : \kappa(x) \to K'$, which completes the proof. □

8.3.6 Radical morphisms

Proposition 8.3.31. *Let $f : X \to Y$ be a morphism of schemes. The following conditions are equivalent:*

(i) *f is universally injective.*

(ii) *The map f is injective and for any $x \in X$, the extension $f^x : \kappa(f(x)) \to \kappa(x)$ is purely inseparable.*

(iii) *For any field K, the map $X(K) \to Y(K)$ corresponding to f is injective.*

(iv) *For any field K, there exists an algebraically closed extension K' of K such that the map $X(K') \to X(K')$ corresponding to f is injective.*

(v) *The diagonal morphism $\Delta_f : X \to X \times_Y X$ is surjective.*

*The morphism f is called **radical** if it satisfies the above equivalent conditions.*

Proof. It is clear that (i) implies f is injective; on the other hand, if $\kappa(x)$ is not a purely inseparable extension of $\kappa(f(x))$, there exist two distinct $\kappa(f(x))$-monomorphisms $\kappa(x) \to K$ into an algebraically closed extension K of $\kappa(x)$; hence we get two distinct morphisms g_1, g_2 of $\mathrm{Spec}(K)$ to X, whose compositions $f \circ g_1$, $f \circ g_2$ equal to the same morphism $\mathrm{Spec}(K) \to Y$. If we set $Y' = \mathrm{Spec}(K)$, there then would be two distinct Y'-sections of $X_{(Y')}$; since K is a field, the Y'-sections of $X_{(Y')}$ correspond one-to-one to their images (the rational points of $X_{(Y')}$ over K), so $f_{(Y')} : X_{(Y')} \to Y'$ would not be injective, contrary to the assumption.

To show that (ii) implies (iii), we note that by Corollary 8.2.17, (iii) signifies that for any $y \in Y$ and a monomorphism $\kappa(y) \to K$ to a field K, there do not exist two distinct $\kappa(y)$-monomorphisms $\kappa(x_1) \to K$, $\kappa(x_2) \to K$, where x_1, x_2 are both lying over y. Now (ii) implis that if we have two such monomorphisms, they come from the same point x since f is injective; moreover, since $\kappa(x)$ is a purely inseparable extension of $\kappa(y)$, the two monomorphisms $\kappa(x) \to K$ are necessarily equal.

It is clear that (iii) implies (iv). Conversely, suppose that (iv) holds; let K be a field and K' be an algebraically closed extension of K; then the diagram

$$\begin{tikzcd} X(K) \arrow[r, "\alpha"] \arrow[d, "\varphi"] & Y(K) \arrow[d, "\varphi'"] \\ X(K') \arrow[r, "\alpha'"] & Y(K') \end{tikzcd} \tag{8.3.4}$$

is commutative. Since the homomorphism $K \to K'$ is injective, φ is injective by Corollary 8.2.17, and by hypothesis we can choose K' such that α' is also injective. Then α is injective, which shows (iii).

To see that (iv) and (v) are equivalent, we note that, for the morphism Δ_f to be surjective, it is necessary and sufficient that, in view of Corollary 8.3.30, for any field K, there exists an algebraically closed extension K' of K such that the diagonal map

$$X(K') \to (X \times_Y X)(K') = X(K') \times_{Y(K')} X(K')$$

corresponding to Δ_f is surjective. But by the definition of this fiber product, this signifies that the map $X(K') \to Y(K')$ is injective, whence our claim.

Finally, we prove that (iii) implies (i). If (iii) is satisfied, then for any base change $Y' \to Y$, the map

$$(X \times_Y Y')(K) \to Y'(K)$$

is still injective, as we immediately verify by noting that $(X \times_Y Y')(K) = X(K) \times_{Y(K)} Y'(K)$ and that $X(K) \to Y(K)$ is injective. Therefore, it suffices to prove that if $X(K) \to Y(K)$ is injective for any field

K, then f is injective. Now if x_1 and x_2 are two points of X such that $f(x_1) = f(x_2) = y$, there then exists a field extension K of $\kappa(y)$ and $\kappa(y)$-monomorphisms $\kappa(x_1) \to K$, $\kappa(x_2) \to K$; the corresponding morphisms u_1, u_2 of $\mathrm{Spec}(K)$ to X are then such that $f \circ u_1 = f \circ u_2$, and by hypothesis this implies $u_1 = u_2$, so $x_1 = x_2$. □

Remark 8.3.32. We then obtain examples of injective morphisms (and even bijective) of schemes but not universally injective: it suffices to take a morphism $\mathrm{Spec}(K) \to \mathrm{Spec}(k)$, where K is a separable extension of k distinct from k.

Corollary 8.3.33. *A monomorphism of schemes $f : X \to Y$ is radical. In particular, if A is a ring, S is a multiplicative subset of A, then the canonical morphism $\mathrm{Spec}(S^{-1}A) \to \mathrm{Spec}(A)$ is radical.*

Proof. The first assertion follows from Proposition 8.3.31(iii), and the second from the fact that $\mathrm{Spec}(S^{-1}A) \to \mathrm{Spec}(A)$ is a monomorphism. □

Corollary 8.3.34. *Let $f : X \to Y$ be a radical morphism, $g : Y' \to Y$ a morphism, and $X' = X_{(Y')}$. Then the radical morphism $f_{(Y')}$ is a bijection from the underlying space X' to $g^{-1}(f(X))$. Moreover, for any field K, the set $X'(K)$ is identified with the inverse image in $Y'(K)$ under the map $Y'(K) \to Y(K)$ (corresponding to g) of the subset $X(K)$ of $Y(K)$.*

Proof. The first assertion follows from Proposition 8.3.31(ii) and Corollary 8.3.26; the second one follows from the commutative diagram (8.3.4). □

Proposition 8.3.35. *Let $f : X \to Y$ and $g : Y \to Z$ be two morphism of schemes.*

(a) *If f and g are radical, so is $g \circ f$.*

(b) *Conversely, if $g \circ f$ is radical, so is f.*

Proof. It suffices to apply the functors X, Y, Z on any field K, and use the characterization of Proposition 8.3.31(iii); the verification boils down to set-theoretic issues, which are straightforward. □

Proposition 8.3.36. *Radical morphisms are stable under base changes. In particular, if $f : X \to X'$ and $g : y \to Y'$ are radical S-morphisms, then so is $f \times_S g$.*

Proof. Since radical is equivalently to universally injective, the first assertion is clear. The second one follows from the first one since radical morphisms are stable under composition by Proposition 8.3.35. □

8.3.7 Fibers of morphisms

Proposition 8.3.37. *Let $f : X \to Y$ be a morphism, y be a point of Y, and $\mathfrak{a}_y$ be an ideal of $\mathscr{O}_{Y,y}$. Put $Y' = \mathrm{Spec}(\mathscr{O}_{Y,y}/\mathfrak{a}_y)$, $X' = X \times_Y Y'$, and let $p : X' \to X$ be the canonical projection. Then p is a homeomorphism from X' onto the subspace $f^{-1}(Y')$ of X (where we identify Y' are a subspace of Y, cf. Corollary 8.2.12). Moreover, for any $x' \in X'$, the homomorphism $p^{\#}_{x'} : \mathscr{O}_{X,p(x')} \to \mathscr{O}_{X',x'}$ is surjective with kernel $\mathfrak{a}_y \mathscr{O}_{X,x}$.*

Proof. The morphism $\mathrm{Spec}(\mathscr{O}_{Y,y}/\mathfrak{a}_y) \to Y$ is radical (Proposition 8.3.31), so we conclude from Corollary 8.3.34 that p identifies the space $X' = X \times_Y \mathrm{Spec}(\mathscr{O}_{Y,y}/\mathfrak{a}_y)$ with $f^{-1}(Y')$. It remains to show that p is a homeomorphism and identify its morphism on stalks. Since this question is local, we may assume that $X = \mathrm{Spec}(B)$, $Y = \mathrm{Spec}(A)$, where B is an A-algebra. Then the morphism p corresponds to the homomorphism $1 \otimes \rho : B \to B \otimes_A A'$, where $\rho : A \to A' = A_{\mathfrak{p}_y}/\mathfrak{a}_y$ is the canonical homomorphism. Now any element of $B \otimes_A A'$ is of the form

$$\sum_i b_i \otimes \rho(a_i)/\rho(s) = \rho\Big(\sum_i a_i b_i \otimes 1\Big)(1 \otimes \rho(s))^{-1}$$

where $s \notin \mathfrak{p}_y$, so we can apply **??**. The assertion on the homomorphism $p^{\#}_{x'}$ also follows from the equality. □

We will mainly use Proposition 8.3.37 for the case $\mathfrak{a}_y = \mathfrak{m}_y$ is the maximal ideal of $\mathscr{O}_{Y,y}$. If there is no confusion, we denote by X_y the $\kappa(y)$-scheme obtained by transporting of scheme structure from $X' = X \otimes_S \kappa(s)$ to $f^{-1}(y)$ via the projection p, and this is always the scheme that will be involved when we speak of the **fiber** $f^{-1}(y)$ of the morphism f as a scheme.

Let X, Y be two S-schemes and $f : X \to Y$ an S-morphism. By the transitivity of base change, we have the canonical isomorphism

$$X_s = X \times_Y Y_s$$

for any $s \in S$; the morphism $f_s : X_s \to Y_s$ induced by f by the base change $Y_s \to Y$ is such that, for any $y \in Y_s$, the fiber $f_s^{-1}(y)$ is identified with the $\kappa(y)$-scheme $f^{-1}(y)$, since the residue field of Y_s at y is the same as that of Y at y, in view of Proposition 8.3.37

Proposition 8.3.38 (Transitivity of Fibers). *Let $f : X \to Y$, $g : Y' \to Y$ be two morphisms; put $X' = X_{(Y')}$ and $f' = f_{(Y')}$. For any $y' \in Y'$, if $y = g(y')$, then the scheme $X'_{y'}$ is canonically isomorphic to $X_y \otimes_{\kappa(y)} \kappa(y')$.*

Proof. In fact, by the transitivity of base change, we have canonical isomorphisms

$$(X \otimes_Y \kappa(y)) \otimes_{\kappa(y)} \kappa(y') \cong X \times_Y \mathrm{Spec}(\kappa(y')) \cong (X \times_Y Y') \otimes_{Y'} \kappa(y')$$

The left one is $X_y \otimes_{\kappa(y)} \kappa(y')$, and the right one is $X'_{y'} \otimes_{Y'} \kappa(y')$, so our assertion follows. □

Proposition 8.3.39. *Let $f : X \to Y$ be a monomorphism of schemes. Then for each $y \in Y$, the fiber X_y is a $\kappa(y)$-scheme which is either empty or $\kappa(y)$-isomorphic to $\mathrm{Spec}(\kappa(y))$.*

Proof. By Proposition 8.3.37, X_y is reduced to a point, and hence affine. By Proposition 8.3.12, the morphism $f_y : X_y \to \mathrm{Spec}(\kappa(y))$ induced by f under base change is still a monomorphism. If A is the ring of X_y, this signifies that the homomorphism $A \otimes_{\kappa(y)} A \to A$ which maps $a \times a'$ to aa' is bijective, and clearly implies that $A = \kappa(y)$, since otherwise there exist an element $a \in A$ not contained in $\kappa(y)$ and the image $a \otimes 1$ and $1 \otimes a$ are distinct, but both mapped to a. □

Proposition 8.3.40. *Let $f : X \to Y$ be an S-morphism of S-schemes, $g : S' \to S$ a surjective morphism, and $f' = f_{(S')} : X' = X_{(S')} \to Y' = Y_{(S')}$. Consider the following properties:*

(a) *surjective;*

(b) *injective;*

(c) *dominant;*

(d) *finite fiber (as sets);*

Then if $\mathcal{P}$ denotes one of the properties above and if f' satisfies $\mathcal{P}$, then so does f.

Proof. Since the projection $Y' \to Y$ is surjective by Proposition 8.3.28, we can, by virtue of Corollary 8.3.10, limiting ourselves to the case where $Y = S$, $Y' = S'$. For any $y' \in Y'$, let $y = g(y')$; we have the transitivity relation $X'_{y'} \cong X_y \otimes_{\kappa(y)} \kappa(y')$ (Proposition 8.3.38). Since the morphism $\mathrm{Spec}(\kappa(y')) \to \mathrm{Spec}(\kappa(y))$ is surjective, so is the projection $X'_{y'} \to X_y$ (Proposition 8.3.28). Thus, if $X'_{y'}$ is nonempty (resp. a singleton, resp. a finite set), the same holds for X_y. Since $S' \to S$ is surjective, this proves (a), (b), and (d). On the other hand, if f' is dominant, so is the composition $g \circ f' = f \circ g'$; but since $g' : X' \to X$ is surjective by Proposition 8.3.28, this implies f is dominant. □

8.3.8 Universally open and closed morphisms

Following the usual terminology, we say a morphism $f : X \to Y$ is **universally open** (resp. **universally closed**, resp. a **universal embedding**, resp. a **universal homeomorphism**) if for any base change $Y' \to Y$, the morphism $f_{(Y')} : X_{(Y')} \to Y'$ is open (resp. closed, resp. an embedding, resp. a homeomorphism).

Proposition 8.3.41.

(i) *The composition of two universally open morphisms (resp. universally closed morphisms, resp. two universal embeddings, resp. two universal homeomorphisms) is universally open (resp. universally closed, resp. a universal embedding, resp. a universal homeomorphism).*

(ii) *If $f : X \to X'$, $g : Y \to Y'$ are two universally open (resp. universally closed, resp. two universal embeddings, resp. two universal homeomorphisms) S-morphisms, so is the product $f \times_S g$.*

(iii) *If $f : X \to Y$ is a universally open (resp. universally closed, resp. a universal embedding, resp. a universal homeomorphism) S-morphism, so is $f_{(S')} : X_{(S')} \to Y_{(S')}$ for any base change $S' \to S$.*

(iv) *Let $f : X \to Y$, $g : Y \to Z$ be two morphism such that f is surjective; if $g \circ f$ is universally open (resp. universally closed, resp. a universal embedding, resp. a universal homeomorphism), so is g.*

(v) *Let (U_α) be an open cover of Y. For a morphism $f : X \to Y$ to be universally open (resp. universally closed, resp. a universal embedding, resp. a universal homeomorphism), it is necessary and sufficient that, for each α, the restriction $f_\alpha : f^{-1}(U_\alpha) \to U_\alpha$ is universally open (resp. closed).*

Proof. The assertion (i) follows from definiton, and so does (iii). We have already remareked that (i) and (iii) together imply (ii), since identity morphisms satisfies all the properties mentioned above. To prove (iv), we note that for any morphism $Z' \to Z$, the morphism $f_{(Z')} \to X_{(Z')}$ is surjective, so it suffices to prove that if $g \circ f$ is open (resp. closed, resp. an embedding, resp. a homeomorphism) and f is surjective, then so is g. For the case where $g \circ f$ is open or closed, the fact that g is open or closed result easily; for the other two cases, we can limit ourselves to the case $g(f(X)) = g(Y) = Z$, so that $g \circ f$ is a homeomorphism from X to Z. As f is surjective, g is necessarily bijective, and as it is open by the first two cases already shown, g is then a homeomorphism from Y to Z.

Finally, the necessity in (v) follows from (iii) and Corollary 8.3.2. Conversely, suppose the condition in (v) and let $g : Y' \to Y$ be a morphism; then $g^{-1}(U_\alpha) = U'_\alpha$ form an open cover of Y' and if $f' = f_{(Y')}$, the restriction $f'^{-1}(U'_\alpha) \to U'_\alpha$ of f' is none other than $(f_\alpha)_{(U'_\alpha)}$ (Corollary 8.3.2). We can then reduce to proving that f is open (resp. closed, resp. an embedding, resp. a homeomorphism) if each f_α is, which is immediate. □

We recall that openness of a map is a local property, i.e., a map $f : X \to Y$ is open if and only if it is open at every point of X. Simialrly, the property of universally open morphisms is also local. To justify this, we define a morphism $f : X \to Y$ is **universally open at a point** $x \in X$ if for any base change $Y' \to Y$ the morphism $f_{(Y')} : X_{(Y')} \to Y_{(Y')}$ is open at any point x' of X' lying over x.

Proposition 8.3.42. *Let $f : X \to Y$, $g : Y \to Z$ be two morphisms of schemes, x a point of X, and $y = f(x)$.*

(a) *If f is universally open at x and g is universally open at y, then $g \circ f$ is universally open at x. Conversely, if $g \circ f$ is universally open at x, then g is universally open at y.*

(b) *If $f : X \to Y$ is an S-morphism universally open at a point $x \in X$, then for any base change $S' \to S$, the morphism $f_{(S')} : X_{(S')} \to Y_{(S')}$ is universally open at any point of $X_{(S')}$ lying over x.*

Proof. Assertion (b) is an immediate concequence of the definition of universally openness at a point and transitivity of base change. Also, it follows from Corollary 8.3.26 that to prove (a), we may drop the "universally" condition and prove the assertion for openness, which then follows from ([**?**] new, 0_{I}, 2.8.2). □

Proposition 8.3.43. *Let X, Y be two schemes, $f : X \to Y$ be a morphism, and x be a point of X. Let (Y_i) be a locally finite covering of Y by closed subschemes, and suppose that for each i such that $f(x) \in Y_i$, the restriction $f_i : f^{-1}(Y_i) \to Y_i$ of f is an open morphism (resp. universally open) at the point x. Then f is open (resp. universally open) at the point x.*

Proof. The assertion about openness is immediate. For the universal part, consider a morphism $g : Y' \to Y$ and in Y' the closed subschemes $Y'_i = g^{-1}(Y_i)$ (Proposition 8.4.16), which underlying spaces form a locally finite covering of Y'. If $f' = f_{(Y')} : X_{(Y')} \to Y'$ is the base change of f, the restriction $f'_i : f'^{-1}(Y_i) \to Y'_i$ of f' equals to $(f_i)_{(Y')}$, so we can apply ([**?**] new, 0_{I}, 2.10.2 (ii)) to f'. □

Proposition 8.3.44. *Let $f : X \to Y$ be a morphism of schemes, $(X_i)_{1 \le i \le n}$ (resp. $(Y_i)_{1 \le i \le n}$) be a finite family of closed subschemes of X (resp. Y), and $j_i : X_i \to X$ (resp. $h_i : Y_i \to Y$) be the canonical injections. Suppose that $(X_i)_{1 \le i \le n}$ covers X and for each i there exists a morphism $f_i : X_i \to Y_i$ fitting into the following diagram*

$$\begin{tikzcd} X_i \arrow[r, "f_i"] \arrow[d, "j_i"'] & Y_i \arrow[d, "h_i"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

Then for f to be closed (resp. universally closed), it is necessary and sufficient that each f_i is.

Proof. If f is closed, then each f_i is closed since Y_i are closed in Y. Conversely, if the f_i are closed, for any closed subset F of X, we have $f_i(F \cap X_i) = f(F \cap X_i)$ and it closed in Y_i, hence in Y, and as $f(F)$ is the union of $f(F \cap X_i)$, it is therefore closed in Y.

For the case of universally closed morphisms, the condition is necessary because j_i is a closed immersion (hence universally closed, cf. Corollary 8.4.14), and if f is universally closed, so is $f \circ j_i = h_i \circ f_i$. But h_i is a closed immersion, hence separated (Proposition 8.5.26), it then follows that f_i is universally closed (Proposition 8.5.23).

Conversely, suppose that each f_i is universally closed, and consider the scheme Z that is the coproduct of that X_i. Let $u : Z \to X$ be the induced morphism by the j_i's. The restriction of $f \circ u$ to X_i is equal to $f \circ j_i = h_i \circ f$, hence universally closed (Corollary 8.4.14 and Proposition 8.3.41(i)); we then deduce from Corollary 8.3.3 that $f \circ u$ is universally closed. But since u is surjective by hypotheses, we conclude that f is universally closed (Proposition 8.3.41(iv)). □

Remark 8.3.45. If X only have finitely many irreducible components, then we deduce from Proposition 8.3.44 that, to verify a morphism $f : X \to Y$ is closed (resp. universally closed), we can reduce ourselves to doing it for dominant morphisms of integral schemes. In fact, let $(X_i)_{1 \le i \le n}$ be the reduced subschemes of X with underlying spaces the irreducible components of X (Proposition 8.4.44), which are then integral. Let Y_i be the unique reduced closed subscheme of Y with underlying space $\overline{f(X_i)}$ (Proposition 8.4.44), which iss irreducible (**??**). If $j_i : X_i \to X$ (resp. $h_i : Y_i \to Y$) is the canonical injection, there then exists a dominant morphism $f_i : X_i \to Y_i$ such that $f \circ g_i = h_i \circ f_i$ (Proposition 8.4.48); we are then in the case of Proposition 8.3.44, so f is closed (resp. universally closed) if and only if each f_i is.

Proposition 8.3.46. *Let X, Y be two S-schemes and $f : X \to Y$ be an S-morphism. For f to be universally closed, it suffices that, for any base change $S' \to S$ where $S' = S \otimes_{\mathbb{Z}} \mathbb{Z}[(T_\lambda)_{\lambda \in I}]$ (denoted by $S[(T_\lambda)_{\lambda \in I}]$), the morphism $f_{(S')} : X_{(S')} \to Y_{(S')}$ is closed.*

Proof. We note that if (S_α) is an open cover of S and Y_α is the inverse image of S_α in Y, it suffices to prove that the restricted morphism $f_\alpha : f^{-1}(Y_\alpha) \to Y_\alpha$ is closed (Proposition 8.3.41(v)). Now the inverse image of S_α in S' is $S_\alpha[(T_\lambda)_{\lambda \in I}] = S'_\alpha$ and $(f_\alpha)_{(S')}$ is the restriction $(f_{(S')})^{-1}(Y'_\alpha) \to Y'_\alpha$ of $f_{(S')}$, where Y'_α is the inverse image of S'_α in $Y_{(S')}$. If the proposition is proved for S_α and f_α, it is then true for S and f. We can then assume that S is affine.

Now let (U_β) be an open covering of Y; for f to be universally closed, it suffices to prove that $f_\beta : f^{-1}(U_\beta) \to U_\beta$ is universally closed for each β (Proposition 8.3.41(v)). Again, the morphism $(f_\beta)_{(S')}$ is the restriction $(f_{(S')})^{-1}(U'_\beta) \to U'_\beta$ of $f_{(S')}$, where $U'_\beta = U_\beta \times_S S'$ is the inverse image of U_β in $Y_{(S')}$. If the proposition is proved for U_β and f_β, it then holds for Y and f. Therefore, we can further assume that Y is affine.

Let us first show that if $f_{(S')}$ is closed for any base change $S' \to S$, then f is universally closed. In fact, any Y-scheme Y' can be considered as an S-scheme, and as the morphism $Y \to S$ is separated (recall that Y and S are assumed to be affine), $X \times_Y Y'$ (resp. $Y \times_Y Y' = Y'$) is identified with a closed subscheme of $X \times_S Y'$ (resp. $Y \times_S Y'$) (Proposition 8.4.13). In the following commutative diagram

$$\begin{tikzcd} X \times_Y Y' \arrow[r, "f \times 1_{Y'}"] \arrow[d] & Y \times_Y Y' = Y' \arrow[d] \\ X \times_S Y' \arrow[r, "f_{(Y')}"] & Y \times_S Y \end{tikzcd}$$

the vertical morphisms are closed immersions, so if $f_{(Y')}$ is a closed morphism, so is $f \times 1_{Y'}$.

It remains to prove that $f_{(S')}$ is closed for arbitrary base change $S' \to S$ if this is true for $S' = S[(T_\lambda)]$. Now by hypotheses S is affine, and if (S'_γ) is an open covering of S', we see in the same manner as before that for $f_{(S')}$ to be closed, it suffices to prove that $f_{(S'_\gamma)}$ is closed. We can then assume S' to be affine. If $S = \mathrm{Spec}(A)$, we have $S' = \mathrm{Spec}(A')$, where A' is an A-algebra. Let $(t_\lambda)_{\lambda \in I}$ be a generator for A', which means there is a surjective A-homomorphism $A[(T_\lambda)] \to A'$ identifying A' with $A[(T_\lambda)]/\mathfrak{b}$, where $\mathfrak{b}$ is an ideal. If $S'' = \mathrm{Spec}(A[(T_\lambda)])$, S' is then a closed subscheme of S'', and $X_{(S')}$ (resp. $Y_{(S')}$) is identified with a closed subscheme of $X_{(S'')}$ (resp. $Y_{(S'')}$). The morphism $f_{(S')}$ is the restriction of $f_{(S'')}$ on $X_{(S')}$, and since $f_{(S'')}$ is closed by hypotheses, we conclude that $f_{(S')}$ is closed. This completes the proof. □

8.4 Subschemes and immersions

8.4.1 Subschemes

Proposition 8.4.1. *Let X be a scheme and $\mathscr{I}$ be a quasi-coherent ideal of $\mathscr{O}_X$. The support Y of the sheaf $\mathscr{O}_X/\mathscr{I}$ is then closed, and if $\mathscr{O}_Y$ is the sheaf induced on Y by $\mathscr{O}_Y/\mathscr{I}$, $(Y,\mathscr{O}_Y)$ is a scheme.*

Proof. Since the problem is local, it suffices to consider the affine case and show that Y is closed and $(X,\mathscr{O}_Y)$ is an affine scheme. In fact, if $X=\operatorname{Spec}(A)$, we have $\mathscr{O}_X=\widetilde{A}$ and $\mathscr{I}=\widetilde{\mathfrak{a}}$, where $\mathfrak{a}$ is an ideal of A (Theorem 8.1.21); Y is then equal to the closed subset $V(\mathfrak{a})$ of X and is identified with the spectrum of $B=A/\mathfrak{a}$ (**??**). Moreover, if $\rho:A\to B=A/\mathfrak{a}$ is the canonical homomorphism, the direct image ${}^a\rho_*(\widetilde{B})$ is canonically identified with the sheaf $\widetilde{A}/\widetilde{\mathfrak{a}}=\mathscr{O}_X/\mathscr{I}$ (Corollary 8.1.6 and Proposition 8.1.12). These complete the proof. □

We say $(Y,\mathscr{O}_Y)$ is the **subscheme of** $(X,\mathscr{O}_X)$ **defined by the quasi-coherent ideal** $\mathscr{I}$. More generally, we say a locally ringed space $(Y,\mathscr{O}_Y)$ is a **subscheme** of a scheme $(X,\mathscr{O}_X)$ if Y is a locally closed subspace of X and if U denote the largest open subset of X containing Y such that Y is open in U (in other words, the complement of $\overline{Y}-Y$, so $U=(X-\overline{Y})\cup Y$), then $(Y,\mathscr{O}_Y)$ is a subscheme of $(U,\mathscr{O}_X|_U)$ defined by a quasi-coherent ideal of $\mathscr{O}_X|_U$. We say the subscheme $(Y,\mathscr{O}_Y)$ of $(X,\mathscr{O}_X)$ is **closed** if Y is closed in X (in this case $U=X$). It follows from this definition and Proposition 8.4.1 that closed subschemes of X are in one-to-one correspondence with quasi-coherent ideals of $\mathscr{O}_X$, since if two such ideals $\mathscr{I}$, $\mathscr{J}$ have the same support (closed) Y and the sheaf induced by $\mathscr{O}_X/\mathscr{I}$ and $\mathscr{O}_X/\mathscr{J}$ on Y are identical, then $\mathscr{I}=\mathscr{J}$.

Let $(Y,\mathscr{O}_Y)$ be a subscheme of X, U the largest open subset of X containing Y such that Y is closed in U, V an open subset of X contained in U; then $V\cap Y$ is closed in V. Moreover, if Y is defined by the quasi-coherent ideal $\mathscr{I}$ of $\mathscr{O}_X|_U$, then $\mathscr{I}|_V$ is a quasi-coherent ideal of $\mathscr{O}_X|_V$, and it is immediate that the scheme induced by Y over $Y\cap V$ is the closed subscheme of V defined by the ideal $\mathscr{I}|_V$.

In particular, the scheme induced by X over an open subset of X is a subscheme of X; such schemes are called **open subschemes** of X. One should note that a subscheme of X can have the underlying space being an open set U of X without being induced on this open subset by X: it is induced over U by X only if it is defined by the ideal 0 of $\mathscr{O}_X|_U$, and there are in general quasi-coherent ideals $\mathscr{I}$ of $\mathscr{O}_X|_U$ such that $(\mathscr{O}_X|_U)/\mathscr{I}$ have support U but is nonzero.

Proposition 8.4.2. *Let $(Y,\mathscr{O}_Y)$ be a locally ringed space such that Y is a subspace of X and there exists a covering (V_α) of Y by open sets of X such that for each α, $Y\cap V_\alpha$ is closed in V_α and the locally ringed space $(Y\cap V_\alpha,\mathscr{O}_Y|_{Y\cap V_\alpha})$ is a closed subscheme of the scheme induced over V_α by X. Then $(Y,\mathscr{O}_Y)$ is a subscheme of X.*

Proof. The hypothesis implies that Y is locally closed in X and the largest open set U containing Y and in which Y is closed contains the V_α. We are then reduced to the case $U=X$ and Y is closed in X. We define a quasi-coherent ideal $\mathscr{I}$ of $\mathscr{O}_X$ by letting $\mathscr{I}_{V_\alpha}$ to be the sheaf of ideal of $\mathscr{O}_X|_{V_\alpha}$ that defines the closed subscheme $(Y\cap V_\alpha,\mathscr{O}_Y|_{Y\cap V_\alpha})$ and for any open set W of X not meeting Y, $\mathscr{I}_W=\mathscr{O}_X|_W$. It is immediately verified that there exists a unique sheaf of ideals $\mathscr{I}$ satisfying these conditions and that it defines the closed subscheme $(Y,\mathscr{O}_Y)$. □

Proposition 8.4.3. *A (closed) subscheme of a (closed) subscheme of X is canonically identified with a (closed) subscheme of X.*

Proof. Since a locally closed subset of a locally closed subset of X is still locally closed in X, it is clear by Proposition 8.4.2 that the question is local and we may assume that X is affine. The proposition then follows from the identification $A/\mathfrak{b}$ and $(A/\mathfrak{a})/(\mathfrak{b}/\mathfrak{a})$ where $\mathfrak{a},\mathfrak{b}$ are ideals of the ring A such that $\mathfrak{a}\subseteq\mathfrak{b}$. □

Let Y be a subscheme of X and denote by $\iota:Y\to X$ the canonical injection of the underlying space; we know the inverse image $\iota^*(\mathscr{O}_X)$ is the restriction $\mathscr{O}_X|_Y$. If for any $y\in Y$, we denote by ι_y the canonical homomorphism $\mathscr{O}_{X,y}\to\mathscr{O}_{Y,y}$, these homomorphisms are then the restrictions to the stalks of $\mathscr{O}_X$ at the points of Y of a surjective homomorphism of sheaves of rings $\iota^\#:\mathscr{O}_X|_Y\to\mathscr{O}_Y$: it suffices indeed to check it locally on Y, so we can assume that X is affine and Y is a closed subscheme; in this case, if $\mathscr{I}$ is the quasi-coherent ideal of $\mathscr{O}_X$ which defines Y, the ι_y's are nothing but the restrictions to the stalks of the canonical homomorphism $\mathscr{O}_X|_Y\to(\mathscr{O}_X/\mathscr{I})|_Y$. We have therefore defined a monomorphism $j_Y=(\iota,\iota^\#)$

of locally ringed spaces, which is called the **canonical injection morphism**. If $f : X \to Z$ is another morphism of schemes, we say the composition

$$Y \xrightarrow{j_Y} X \xrightarrow{f} Z$$

is the **restriction** of f to the subscheme Y of X.

A subscheme Y of a scheme X is considered as an X-scheme via the canonical injection $j_Y : Y \to X$. Two subschemes Y, Z of X that are X-isomorphic are then necessary identical. In fact, if $u : Y \to Z$ is an X-isomorphism, the relation $j_Y = j_Z \circ u$ shows the underlying spaces of Y and Z are identical. Moreover if $U \supseteq Y$ is an open subset of X such that $Y = Z$ are closed in U, and $\mathscr{I}$, $\mathscr{J}$ are the ideals of U defining respectively Y and Z, for each $x \in Y$ we then have a commutative diagram

$$\begin{tikzcd}
\mathscr{O}_{X,x}/\mathscr{J}_x \arrow[rr, "u_x^\#"] & & \mathscr{O}_{X,x}/\mathscr{I}_x \\
& \mathscr{O}_{X,x} \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Since u is an isomorphism, this implies $\mathscr{I}_x = \mathscr{J}_x$, so $Y = Z$ and $u = 1_Y$.

According to the general definitions, we say a morphism $f : Z \to X$ is **dominated by the canonical injection** $j_Y : Y \to X$ of a subscheme Y of X, if f factors through j_Y:

$$Z \xrightarrow{g} Y \xrightarrow{j_Y} X$$

where g is a morphism of schemes. Since j_Y is a monomorphism, the morphism g is unique.

Proposition 8.4.4. *Let Y be a subscheme of a scheme X and $j : Y \to X$ be the cannical injection. For a morphism $f : Z \to X$ to be dominated by the injection j, it is necessary and sufficient that $f(Z) \subseteq Y$ and for each $z \in Z$, the homomorphism $f_z^\# : \mathscr{O}_{X,f(z)} \to \mathscr{O}_{Z,z}$ factors through $\mathscr{O}_{Y,f(z)}$ (or equivalently, the kernel of $\mathscr{O}_{X,f(z)} \to \mathscr{O}_{Z,z}$ is contained in that of $\mathscr{O}_{X,f(z)} \to \mathscr{O}_{Y,f(z)}$).*

Proof. The condition is cleraly necessary. For the sufficiency, we may assume that Y is a closed subscheme of X, and replace X by an open subset U such that Y is closed in U. Y is then defined by a quasi-coherent ideal $\mathscr{I}$ of $\mathscr{O}_X$. Let $\mathscr{J}$ be the kernel of the homomorphism $f^*(\mathscr{O}_X) \to \mathscr{O}_Z$. In view of the properties of the functor f^*, the hypothesis implies tht for each $z \in Z$ we have $(f^*(\mathscr{I}))_z \subseteq \mathscr{J}_z$, and concequently $f^*(\mathscr{I}) \subseteq \mathscr{J}$. Therefore the homomorphism $f^*(\mathscr{O}_X) \to \mathscr{O}_Z$ factors into

$$f^*(\mathscr{O}_X) \longrightarrow f^*(\mathscr{O}_X)/f^*(\mathscr{I}) = f^*(\mathscr{O}_X/\mathscr{I}) \xrightarrow{\theta_z} \mathscr{O}_Z$$

the first arrow being the canonical homomorphism. Let g be the continuous map of Z in Y coincide with f; it is clear that $g^*(\mathscr{O}_Y) = f^*(\mathscr{O}_X/\mathscr{I})$; on the other hand, for any $z \in Z$, θ_z is obviously a local homomorphism, so $(g, \theta) : Z \to Y$ is a morphism of schemes which satisfies $f = j \circ g$, whence the proposition. □

Corollary 8.4.5. *Let Y and Z be subschemes of X. For the canonical injection $Z \to X$ to be dominated by the injection $Y \to X$, it is necessary and sufficient that Z is a subscheme of Y.*

Due to this corollary, for two subschemes Y, Z of X we write $Y \preceq Z$ if Y is a subscheme of Z. It is clear that this defines an order relation on the set of subschemes of X, since two subschemes Y and Z are identical if $Y \preceq Z$ and $Z \preceq Y$.

8.4.2 Immersions of schemes

We say a morphism $f : Y \to X$ is an **immersion** (resp. a **closed immersion**, resp. an **open immersion**) if it is factorized into

$$Y \xrightarrow{g} Z \xrightarrow{j} X$$

where g is an isomorphism, Z is a subscheme (resp. a closed subscheme, resp. an open subscheme) of X, and j is the canonical injection. The subscheme Z and the isomorphism g are then uniquely determined since two X-isomorphic subschemes are identical. We say $f = i \circ g$ is the **canonical factorization** of the immersion f, and the subscheme Z and the isomorphism g is called **associated** with f. It is clera that an immersion is a monomorphism of schemes (since j is a monomorphism), and a fortiori a radical morphism (Corollary 8.3.33). Also, it is clear from Proposition 8.4.3 that the composition of two immersions (resp. two open immersions, resp. two closed immersions) is an immersion (resp. an open immersion, resp. a closed immersion).

Again, one should note that an immersion $f : Y \to X$ such that $f(Y)$ is a open subset of X, in other words which is an open morphism, is not necessarily an open immersion.

Example 8.4.6. Let X be an affine scheme. Then from the definition of closed subschemes, we see that for a morphism $f : Y \to X$ to be a closed immersion, it is necessary and sufficient that Y is an affine scheme and $\Gamma(f) : \Gamma(X, \mathscr{O}_X) \to \Gamma(Y, \mathscr{O}_Y)$ is surjective.

Lemma 8.4.7. *Let $f : Y \to X$ be a morphism of schemes such that $f(Y)$ is closed and f is a homeomorphism onto $f(Y)$. Then for each point $x \in X$, there exists an affine open neighborhood U of x such that $f^{-1}(U)$ is an affine open of Y.*

Proof. Since $f(Y)$ is closed in X, the lemma is trivial if $x \notin f(Y)$, since it suffices to choose an affine open neighborhood of x disjoint from $f(Y)$. If $x \in f(Y)$, there exists a unique point $y \in Y$ such that $f(y) = x$. Let W be an affine open neighborhood of x in X and V an affine open neighborhood of y in Y such that $f(V) \subseteq W$. By hypothesis $f(V)$ is an open neighborhood of x in $f(Y)$, so there exists an open neighborhood $U' \subseteq W$ of x such that $U' \cap f(Y) = f(V)$. Let U be an open neighborhood of x contained in U' and is of the form $D(s)$, where $s \in A = \Gamma(W, \mathscr{O}_X)$ (recall that W is chosen to be affine); in view of **??**(b), $f^{-1}(U) \subseteq V$ is of the form $D(t)$, where t is the image of s in $B = \Gamma(V, \mathscr{O}_Y|_V)$, hence proves the lemma. ☐

Lemma 8.4.8. *Let $f : Y \to X$ be a morphism of schemes and (U_λ) be an affine open covering of X such that for each λ, $f^{-1}(U_\lambda)$ is an affine open of Y. Then for any quasi-coherent $\mathscr{O}_Y$-module $\mathscr{F}$, the direct image $f_*(\mathscr{F})$ is a quasi-coherent $\mathscr{O}_X$-module.*

Proof. For each λ, put $V_\lambda = f^{-1}(U_\lambda)$, and let $f_\lambda : V_\lambda \to U_\lambda$ be the restriction of f to V_λ. Then the restriction $f_*(\mathscr{F})$ to U_λ is equal to $(f_\lambda)_*(\mathscr{F}_\lambda)$, where $\mathscr{F}_\lambda = \mathscr{F}|_{U_\lambda}$. But since U_λ and V_λ are affine by hypothesis, we see $(f_\lambda)_*(\mathscr{F}_\lambda)$ is quasi-coherent by Proposition 8.1.12. This proves the lemma. ☐

Proposition 8.4.9. *Let $f : Y \to X$ be a morphism of schemes.*

(a) *For f to be an open immersion, it is necessary and sufficient that f is a homeomorphism onto an open subset of X and for each $y \in Y$, the homomorphism $f_y^\# : \mathscr{O}_{X,f(y)} \to \mathscr{O}_{Y,y}$ is bijective.*

(b) *For f to be an immersion (resp. a closed immersion), it is necessary and sufficient that f is a homeomorphism onto a locally closed (resp. closed) subset of X and for each $y \in Y$, the homomorphism $f_y^\# : \mathscr{O}_{X,f(y)} \to \mathscr{O}_{Y,y}$ is surjective.*

Proof. In the two cases, the conditions are clearly necessary, so we only need to prove the sufficiency. If the conditions in (a) holds, it is clear that $f^\#$ induces an isomorphism of $\mathscr{O}_Y$ to $f^*(\mathscr{O}_X)$, and $f^*(\mathscr{O}_X)$ is the sheaf defined by the transport by structure by means of f^* from the induced sheaf $\mathscr{O}_X|_{f(Y)}$, hence the conclusion.

Suppose then the conditions in (b) holds. Let U_0 be the largest open set of X such that $Z = f(Y)$ is closed in U_0; by replacing X by the subscheme induced by X over U_0, we may assume that $Z = f(Y)$ is closed in X. By Lemma 8.4.7 and Lemma 8.4.8, the sheaf $f_*(\mathscr{O}_Y)$ is a quasi-coherent $\mathscr{O}_X$-module. We have an exact sequence

$$0 \longrightarrow \mathscr{I} \longrightarrow \mathscr{O}_X \xrightarrow{f^\#} f_*(\mathscr{O}_Y) \longrightarrow 0$$

where two terms are quasi-coherent $\mathscr{O}_X$-modules; we then deduce that (Corollary 8.2.23) $\mathscr{I}$ is a quasi-coherent ideal of $\mathscr{O}_X$, and $f^\#$ factors into

$$\mathscr{O}_X \longrightarrow \mathscr{O}_X/\mathscr{I} \xrightarrow{\theta} f_*(\mathscr{O}_Y)$$

where the first arrow is the canonical homomorphism and θ is an isomorphism. If $\mathscr{O}_Z = (\mathscr{O}_X/\mathscr{I})|_Z$, $(Z, \mathscr{O}_Z)$ is then a closed subscheme of $(X, \mathscr{O}_X)$ and f factors through the canonical injection $j_Z : Z \to X$. Since the corresponding morphism $Y \to Z$ is just (f_0, θ_0), where f_0 is the map f considered as a homeomorphism from Y to Z and θ_0 is the restriction of θ to $\mathscr{O}_Z$, it is clear that f is a closed immersion, which completes the proof. ☐

Remark 8.4.10. It may happen that $f : Y \to X$ is a closed immersion and for all $y \in Y$, $f_y^\# : \mathscr{O}_{X,f(y)} \to \mathscr{O}_{Y,y}$ is bijective, without f being an open immersion (that is, $f(Y)$ is not necessarily open in X). For example, let $X = \mathrm{Spec}(A)$ be an affine scheme and $x \in X$ be a closed point of X that is not isolated. Then if $Y = \mathrm{Spec}(A/\mathfrak{m}_x)$, the canonical morphism $Y \to X$ is a closed immersion satisfying the desired property, since the subspace $\{x\}$ is not open in X.

Corollary 8.4.11. *Let $f : Y \to X$ be a morphism of schemes.*

(a) *Let (V_λ) be a covering of $f(Y)$ by open subsets of X. Then for f to be an immersion (an open immersion), it is necessary and sufficient that for each λ, the restriction $f^{-1}(V_\lambda) \to V_\lambda$ of f is an immersion (an open immersion).*

(b) *Let (U_λ) be an open covering of X. Then for f to be a closed immersion, it is necessary and sufficient that for each λ, the restriction $f^{-1}(U_\lambda) \to U_\lambda$ of f is a closed immersion.*

Proof. In the case (a), $f_y^\#$ is surjective (resp. bijective) for every point $y \in Y$, and in case (b) it is surjective for every point $y \in Y$; it then suffices to verify that in case (a) f is a homeomorphism of Y onto a locally closed (resp. open) subset of X and in case (b), a homeomorphism onto a closed subset of X. Now, the hypothesis imply that f is clearly injective and maps each neighborhood of $y \in Y$ to a neighborhood of $f(y)$ in $f(Y)$. In case (a), $f(Y) \cap V_\lambda$ is locally closed (resp. open) in the union of the V_λ, and a fortiori in X; in case (b), $f(Y) \cap U_\lambda$ is closed in U_λ, hence closed in X since $X = \bigcup_\lambda U_\lambda$. ☐

Remark 8.4.12. We can generalize the notions of immersions to any ringed spaces. We define a **ringed subspace** of a ringed space $(X, \mathscr{O}_X)$ to be a ringed space of the form $(Y, (\mathscr{O}_U/\mathscr{I})|_Y)$, where U is an open of X, $\mathscr{I}$ an ideal of $\mathscr{O}_U$ and Y the support of the sheaf of rings $\mathscr{O}_U/\mathscr{I}$ (support which is no longer necessarily closed in U). We can define the canonical injection $Y \to X$, and the definitions and results of Proposition 8.4.4 are valid without modification. We then define the notion of immersion (resp. of closed immersion) in the same manner. The characterizations of open (resp. closed) immersions given in Proposition 8.4.9 still hold (observing that if $f(Y)$ is closed in X, $f_*(\mathscr{O}_Y)$ has support $f(Y)$). The result of Proposition 8.4.9 can therefore be stated by saying that if a scheme Y is a ringed subspace of a scheme X, then Y is a subscheme of X.

8.4.3 Inverse image of subschemes

Proposition 8.4.13. *Let $f : X' \to X$ and $g : Y' \to Y$ be two S-morphisms. Let p, q be the projections of $X \times_S Y$.*

(a) *If f and g are immersions (resp. open immersions, resp. closed immersions), then $f \times_S g$ is an immersion (resp. an open immersion, resp. a closed immersion).*

(b) *If X' (resp. Y') is identified with a subscheme of X (resp. Y) via the immersion f (resp. g), then $f \times_S g$ identifies the underlying space of $X' \times_S Y'$ with the subspace $p^{-1}(X') \cap q^{-1}(Y')$ of the underlying space of $X \times_S Y$.*

Proof. We can restrict ourselves to the case where X' and Y' are subschemas and f and g are the canonical injection morphisms. The proposition has already been established for the subschemes induced on the open sets (Corollary 8.3.2); as any subscheme is a closed subscheme of an open scheme, we are reduced in case X' and Y' are closed subschemes.

We can further assume that S is affine. In fact, let (S_λ) be an affine open cover of S, φ and ψ be the structural morphisms of X and Y, and let $X_\lambda = \varphi^{-1}(S_\lambda)$, $Y_\lambda = \psi^{-1}(S_\lambda)$. The restriction X'_λ (resp. Y'_λ) of X' (resp Y') to $X_\lambda \cap X'$ (resp. $Y_\lambda \cap Y'$) is a closed subscheme of X_λ (resp. Y_λ), the schemes X_λ, Y_λ, X'_λ, Y'_λ can then be considered as S_λ-schemes and the product $X_\lambda \times_S Y_\lambda$ and $X_\lambda \times_{S\lambda} Y_\lambda$ (resp. $X'_\lambda \times_S Y'_\lambda$ and $X'_\lambda \times_{S_\lambda} Y'_\lambda$) are identified (Corollary 8.3.2). If the proposition is true when S is affine, the restriction of $f \times_S g$ to the $X'_\lambda \times_S Y'_\lambda$ will therefore be an immersion. As the product $X'_\lambda \times_S Y'_\mu$ (resp. $X_\lambda \times_S Y_\mu$) is identified with $(X'_\lambda \cap X'_\mu) \times_S (Y'_\lambda \cap Y'_\mu)$ (resp. $(X_\lambda \cap X_\mu) \times_S (Y_\lambda \cap Y_\mu)$), the restriction of $f \times_S g$ to $X'_\lambda \times_S Y'_\mu$ is also an immersion; it follows from Corollary 8.4.11 that $f \times_S g$ is an immersion.

Secondly, let's prove that we can also assume that X and Y are affine. In fact, let (U_i) (resp. (V_j)) be an affine open cover of X (resp. Y), and let X'_i (resp. Y'_j) be the restriction of X' (resp. Y') to $X' \cap U_i$ (resp. $Y' \cap V_j$), which is a closed subscheme of U_i (resp. V_j). Then $U_i \times_S V_j$ is identified with the restriction of $X \times_S Y$ to $p^{-1}(U_i) \cap q^{-1}(V_j)$ by Corollary 8.3.2, and similarly, if $p' : X' \times_S Y' \to X'$ and

$q' : X' \times_S Y' \to Y'$ are the canonical projections, the product $X'_i \times_S Y'_j$ is identified with the restriction of $X' \times_S Y'$ to $p'^{-1}(X'_i) \cap q'^{-1}(Y'_j)$. Put $h = f \times_S g$, then since $X'_i = f^{-1}(U_i)$, $Y'_j = g^{-1}(V_j)$, we have

$$p'^{-1}(X'_i) \cap q'^{-1}(Y'_j) = h^{-1}(p^{-1}(U_i) \cap q^{-1}(V_j)) = h^{-1}(U_i \times_S V_j).$$

Again, by the same reasoning and using Corollary 8.4.11, we can show that h is an immersion.

Suppose then that X, Y, and S are affine, with rings B, C, and A, respectively. Then B and C are A-algebras, X' and Y' are affine subschemes with rings quotients B', C' of B and C, respectively. Moreover, f and g are induced by ring homomorphisms $\rho : B \to B'$ and $\sigma : C \to C'$. With these, we see $X \times_S Y$ (resp. $X' \times_S Y'$) is the affine scheme with ring $B \otimes_A C$ (resp. $B' \otimes_A C'$), and $f \times_S g$ corresponds to the ring homomorphism $\rho \otimes \sigma : B \otimes_A C \to B' \otimes_A C'$. Since this homomorphism is surjective, $f \times_S g$ is an immersion. Moreover, if $\mathfrak{b}$ (resp. $\mathfrak{c}$) is the kernel of ρ (resp. σ), the kernel of $\rho \otimes \sigma$ is $u(\mathfrak{b}) + v(\mathfrak{c})$, where u (resp. v) is the homorphism $b \mapsto b \otimes 1$ (resp. $c \mapsto 1 \otimes c$). As p corresponds to the ring homomorphism u and q corresponds to v, this kernel corresponds, in the spectrum $\mathrm{Spec}(B \otimes_A C)$, to the closed subset $p^{-1}(X') \cap q^{-1}(Y')$, which proves the demonstration. □

Corollary 8.4.14. *If $f : X \to Y$ is an immersion (resp. an open immersion, resp. a closed immersion) and an S-morphism, then $f_{(S')}$ is an immersion (resp. an open immersion, resp. a closed immersion) for any extension $S' \to S$ of base schemes.*

Proof. This follows from the observation that the identity morphism is an immersion (resp. an open immersion, resp. a closed immersion). □

Corollary 8.4.15. *An immersion (resp. a closed immersion, resp. an open immersion) is a universally embedding (resp. universally closed, resp. universally open).*

Proposition 8.4.16. *Let $f : X \to Y$ be a morphism, Y' a subscheme (resp. a closed subscheme, resp. an open subscheme) of Y, and $j : Y' \to Y$ the canonical injection.*

(a) *The projection $j' : X \times_Y Y' \to X$ is an immersion (resp. a closed immersion, resp. an open immersion), and the subscheme of X associated with j' has underlying space $f^{-1}(Y')$.*

$$\begin{tikzcd} X \times_Y Y' \arrow[r] \arrow[d, hook, "p"'] & Y' \arrow[d, hook, "j"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

Moreover, for a morphism $h : Z \to X$ to be such that $f \circ h : Z \to Y$ is dominated by j, it is necessary and sufficient that h is dominated by j'.

(b) *If Z is a closed subscheme defined by a quasi-coherent ideal $\mathscr{K}$ of $\mathscr{O}_Y$, the inverse image of Z by f is defined by the quasi-coherent ideal $f^*(\mathscr{K})\mathscr{O}_X$.*

Proof. As $p = 1_X \times_Y j$, the first assertion in (a) follows from Proposition 8.4.13. The second one is a special case of Corollary 8.3.34. Finally, if we have $f \circ h = j \circ h'$, where $h' : Z \to Y'$ is a morphism, it follows from the universal property of product that we have $h = p \circ u$, where $u : Z \to X \times_Y Y'$ is a morphism, whence assertion (a).

To prove (b), since the question is local on X and Y, we may assume that X and Y are affine. It then suffices to note that if A is an B-algebra and $\mathfrak{b}$ is an ideal of B, we have $A \otimes_B (B/\mathfrak{b}) = A/\mathfrak{b}A$, and apply Proposition 8.1.14. □

We say the subscheme of X thus defined is the **inverse image** of the subscheme Y' of Y by the morphism f. We say the morphism $f \times 1_{Y'} : f^{-1}(Y') \to Y'$ is the restriction of f to $f^{-1}(Y')$. When we speak of $f^{-1}(Y')$ as a subscheme of X, it is always this subscheme that will be involved.

Example 8.4.17. If the scheme $f^{-1}(Y')$ and X are identical, $j' : f^{-1}(Y') \to X$ is then the identity and any morphism $h : Z \to X$ is dominated by j'; hence the morphism $f : X \to Y$ factors into

$$X \xrightarrow{g} Y' \xrightarrow{j} Y$$

Example 8.4.18. If y is a closed point of Y and $Y' = \operatorname{Spec}(\kappa(y))$ is the smallest closed subscheme of Y having $\{y\}$ as underlying space, the closed subscheme $f^{-1}(Y')$ is then canonically isomorphic to the **fiber** $f^{-1}(y)$.

Corollary 8.4.19. *Retain the hypotheses of Proposition 8.4.16(b). Let X' be a closed subscheme of X defined by a quasi-coherent ideal $\mathscr{I}$ of $\mathscr{O}_X$ and $i : X' \to X$ be the canonical injection. For the restriction $f \circ i$ of f to X' is dominated by the injection $j : Y' \to Y$, it is necessary and sufficient that $f^*(\mathscr{K})\mathscr{O}_X \subseteq \mathscr{I}$.*

Proof. This follows from Proposition 8.4.16(b) and (a). □

Corollary 8.4.20. *Let $f : X \to Y$, $g : Y \to Z$ be two morphisms, and $h = g \circ f$ be their composition. For any subscheme Z' of Z, the subscheme $f^{-1}(g^{-1}(Z'))$ and $h^{-1}(Z')$ of X are identical.*

Proof. This follows from the transitivity of products and Proposition 8.4.16. □

Corollary 8.4.21. *Let X', X'' be two subschemes of X and $j' : X' \to X$, $j'' : X'' \to X$ be the canonical injections. Then $j'^{-1}(X'')$ and $j''^{-1}(X')$ are both equal to the infimum $\inf(X', X'')$ of X' and X'' for the ordered relation on subschemes, and is canonically isomorphic to $X' \times_S X''$.*

Proof. This follows from Proposition 8.4.16, Proposition 8.4.3, and the universal property of products. □

Corollary 8.4.22. *Let $f : X \to Y$ be a morphism and Y', Y'' be two subschemes of Y. Then we have $f^{-1}(\inf(Y', Y'')) = \inf(f^{-1}(Y'), f^{-1}(Y''))$.*

Proof. In fact, we have the canonical isomorphism of $(X \times_Y Y') \times_X (X \times_Y Y'')$ and $X \times_Y (Y' \times_Y Y'')$. □

8.4.4 Local immersions and local isomorphisms

Let $f : X \to Y$ be a morphism of schemes. We say f is a **local immersion** (resp. a **local isomorphism**) **at a point** $x \in X$ if there exists an open neighborhood U of x in X and an open neighborhood V of $f(x)$ in Y such that the restriction of f to U is a closed immersion (resp. an open immersion) into V. We say f is a local immersion (resp. a local isomorphism) if f is a local immersion (resp. a local isomorphism) at every point of X.

An immersion (resp. closed immersion) $f : X \to Y$ can then be characterized as a local immersion such that f is a homeomorphism onto a subset of Y (resp. a closed subset of Y). An open immersion f can be characterized as an injective local isomorphism.

Proposition 8.4.23. *Let X be an irreducible scheme, $f : X \to Y$ be a injective dominant morphism. If f is a local immersion, then it is an immersion and $f(X)$ is open in Y.*

Proof. In fact, let $x \in X$ and U be an open neighborhood of x in X, V an open neighborhood of $f(x)$ in Y such that $f|_Y$ is a closed immersion into V. As U is dense in X, $f(U)$ is also dense in Y by hypothesis, hence $f(U) = V$ and f is a homeomorphism from U to V. The hypothesis f is injective implies $f^{-1}(V) = U$, whence the proposition. □

Proposition 8.4.24. *Let $f : X \to X'$, $g : Y \to Y'$ be two S-schemes.*

(a) *The composition of two local immersions (resp. local isomorphisms) is a local immersion (resp. a local isomorphism).*

(b) *If f and g are local immersions (resp. local isomorphisms), so is the product $f \times_S g$.*

(c) *If f is a local immersion (resp. a local isomorphism), so is $f_{(S')}$ for any extension $S' \to S$ of base schemes.*

Proof. It suffices to prove (a) and (b). Now (a) follows from the transitivity of closed immersions (resp. open immersions) and the fact that if f is a homeomorphism of X to a closed subset Y, then for any open set $U \subseteq X$, $f(U)$ is open in $f(X)$, so there exists an open subset V of Y such that $f(U) = V \cap f(X)$, and $f(U)$ is therefore closed in V.

To prove (b), let p, q be the projections of $X \times_S Y$ and p', q' that of $X' \times_S Y'$. There exist by hypotheses open neighborhoods U, U', V, V' of $x = p(z)$, $x' = p'(z')$, $y = q(z)$, $y' = q'(z')$, respectively, such that $f|_U$ and $g|_V$ are closed immersions (resp. open immersions) onto U' and V', respectively. As the underlying space of $U \times_S V$ is $p^{-1}(U) \cap q^{-1}(V)$ and that of $U' \times_S V'$ is $p'^{-1}(U') \cap q'^{-1}(V')$, which are neighborhoods of z and z', respectively (Corollary 8.3.2), the claim follows by Corollary 8.4.14. □

Remark 8.4.25. A local isomorphism is clearly flat and universally open, and therefore is universally generalizing.

Proposition 8.4.26. *Let X be an irreducible scheme, Y an integral scheme, and $f : X \to Y$ be a morphism.*

(a) *If f is dominant, then for any $x \in X$, the homomorphism $f_x^\# : \mathscr{O}_{Y,f(x)} \to \mathscr{O}_{X,x}$ is injective.*

(b) *If f is dominant and a local immersion, then f is a local isomorphism (and therefore X is integral).*

Proof. Let ξ and η be the generic points of X and Y, respectively. If f is dominant, we then have $f(\xi) = \eta$; moreover $\mathscr{O}_{Y,\eta}$ is a field since Y is reduced, so $f_\xi^\# : \mathscr{O}_{Y,\eta} \to \mathscr{O}_{X,\xi}$ is a monomorphism. For any point $x \in X$, and any affine neighborhood U of $y = f(x)$, there exists an affine neighborhood V of x contained in $f^{-1}(U)$. The open set U (resp. V) contains η (resp. ξ), and the ring $\Gamma(U, \mathscr{O}_Y)$ is integral with fraction field $\mathscr{O}_{Y,\eta}$. If $\rho : \Gamma(U, \mathscr{O}_Y) \to \Gamma(V, \mathscr{O}_X)$ is the homomorphism corresponding to f, the composition

$$\Gamma(U, \mathscr{O}_Y) \xrightarrow{\rho} \Gamma(V, \mathscr{O}_X) \longrightarrow \mathscr{O}_{X,\xi}$$

is then the restriction of $f_\xi^\#$ to $\Gamma(U, \mathscr{O}_Y)$, so the homomorphism ρ. We now deduce that $f_x^\#$ is injective: in fact the hypotheses that X is irreducible implies that $\Gamma(V, \mathscr{O}_X)$ has a unique minimal $\mathfrak{n}$, which is its nilradical; the homomorphism $f_x^\#$ just send an element u/s (where $u, s \in \Gamma(U, \mathscr{O}_Y)$ and $s \neq 0$) to the element $\rho(u)/\rho(s) \in \mathscr{O}_{X,x}$, which is zero only if there exists $t \notin \mathfrak{p}_x$ such that $t\rho(u) = 0 \in \mathfrak{n}$. But as $t \notin \mathfrak{n}$, this then implies $\rho(u) \in \mathfrak{n}$, so $\rho(u)$ is nilpotent and since ρ is injective, this shows that u is nilpotent, which means $u = 0$ for $\Gamma(U, \mathscr{O}_Y)$ being integral.

To prove the second assertion, let f be dominant and a local immersion. We see $f(Y)$ is open in Y by Proposition 8.4.23. Since $f_x^\#$ is surjective for every point $x \in X$ (Proposition 8.4.9), it follows that $\mathfrak{f}_x^\#$ is an isomorphism by (a), and this shows f is a local isomorphism, again by Proposition 8.4.9 . □

Proposition 8.4.27. *Let Y be a reduced scheme such that the family of irreducible components of Y is locally finite. Let $j : X \to Y$ be an immersion. For j to be a local isomorphism at a point $x \in X$, it is necessary and sufficient that the homomorphism $f_x^\# : \mathscr{O}_{Y,j(x)} \to \mathscr{O}_{X,x}$ is flat.*

Proof. Since the question is local on Y, we may assume that Y is affine, that j is a closed immersion, and that all irreducible components of Y contain $j(x)$ (hence are finite in number), and we prove that j is an isomorphism in this case. If $Y = \mathrm{Spec}(A)$, and if $\mathfrak{p}$ is the prime ideal of A corresponding to $j(x)$, the morphism $\mathrm{Spec}(A_\mathfrak{p}) \to \mathrm{Spec}(A)$ is then dominant since $\mathfrak{p}$ contains the minimal ideals of A (it is containd in every irreducible component of $\mathrm{Spec}(A)$). As A is reduced, the homomorphism $A \to A_\mathfrak{p}$ is injective (**??**). If $\mathscr{O}$ is the ideal of $\mathscr{O}_Y$ defining the closed subscheme of Y associated with j, we have $\mathscr{I} = \widetilde{\mathfrak{a}}$, where $\mathfrak{a}$ is an ideal of A, and it is identified with a subset of $\mathfrak{a}_\mathfrak{p}$. If j is flat at x, then $(A/\mathfrak{a})_\mathfrak{p}$ is a flat $A_\mathfrak{p}$-module, and since the homomorphism $A_\mathfrak{p} \to (A/\mathfrak{a})_\mathfrak{p}$ is local, it is faithfully flat (**??**). By **??** this implies $\mathfrak{a}_\mathfrak{p} = 0$, hence $\mathfrak{a} = 0$ and the claim follows. □

8.4.5 Nilradical and associated reduced scheme

Proposition 8.4.28. *Let $(X, \mathscr{O}_X)$ be a scheme and $\mathscr{B}$ be a quasi-coherent $\mathscr{O}_X$-algebra. There exist a unique quasi-coherent ideal $\mathscr{N}$ of $\mathscr{B}$ such that for each point $x \in X$, the stalk $\mathscr{N}_x$ is the nilradical of the ring $\mathscr{B}_x$. If $X = \mathrm{Spec}(A)$ is affine and $\mathscr{B} = \widetilde{B}$, where B is an A-algebra, then $\mathscr{N} = \widetilde{\mathfrak{n}}$, where $\mathfrak{n}$ is the nilradical of B.*

Proof. Since the question is loca, we may assume that $X = \mathrm{Spec}(A)$ is affine and $\mathscr{B} = \widetilde{B}$. We know that $\widetilde{\mathfrak{n}}$ is a quasi-coherent $\mathscr{O}_X$-module and for each point $x \in X$, the stalk $\mathfrak{n}_x$ is an ideal of the fraction ring B_x. It suffices to show that the nilradical of B_x is contained in $\mathfrak{n}_x$, the opposite inclusion being evident. Now, let z/s be a nilpotent element in B_x, where $z \in B$ and $s \notin \mathfrak{p}_x$. By hypotheses, there exist $k \geq 0$ such that $(z/s)^k = 0$, which means there exists $t \notin \mathfrak{p}_x$ such that $tz^k = 0$. We then conclude that $(tz)^k = 0$, so $z/s = (tz)/(ts)$ is indeed in $\mathfrak{n}_x$. □

The quasi-coherent ideal $\mathscr{N}$ is called the **nilradical** of the $\mathscr{O}_X$-algebra $\mathscr{B}$. In particular, we denote by $\mathscr{N}_X$ the nilradical of $\mathscr{O}_X$.

Corollary 8.4.29. *Let X be a scheme. Then the closed subscheme of X defined by the quasi-coherent ideal $\mathscr{N}_X$ is the unique reduced subscheme of X with underlying space X. It is also the smallest subscheme of X having X as underlying space.*

Proof. Since the structural sheaf of the closed subscheme Y defined by $\mathscr{N}_X$ is $\mathscr{O}_X/\mathscr{N}_X$, it is immediate that Y is reduced and has X as underlying space, since $\mathscr{N}_x \neq \mathscr{O}_{X,x}$ for each $x \in X$. To prove the second claim, let Z be a subscheme of X with X as underlying space. Then Z is closed in X, so let $\mathscr{I}$ be the ideal defining it. We can assume that X is affine, so $\mathscr{I} = \tilde{\mathfrak{a}}$ where $\mathfrak{a}$ is an ideal of A. Then for each $x \in X$ we have $\mathfrak{a} \subseteq \mathfrak{p}_x$, so $\mathfrak{a} \subseteq \mathfrak{n}$, where $\mathfrak{n}$ is the nilradical of A. This shows Y is the smallest subscheme of X with underlying space X, and if Z is distinct from Y, we necessarily have $\mathscr{I}_x \neq \mathscr{N}_x$ for some $x \in X$, and concequently Z is not reduced. □

The reduced scheme defined by $\mathscr{N}_X$ on X is called the **reduced scheme associated with X**, and denoted by X_{red}. To say that a schema X is reduced therefore means that $X_{\mathrm{red}} = X$. Clearly, we have a canonical closed immersion $X_{\mathrm{red}} \to X$, which is also a universal homeomorphism.

Proposition 8.4.30. *For the spectrum of a ring A to be reduced (resp. integral), it is necessary and sufficient that A is reduced (resp. integral).*

Proof. In fact, the condition $\mathscr{N} = 0$ is necessary and sufficient for $\mathrm{Spec}(A)$ to be reduced, and the integral conditions follows from **??**. □

Proposition 8.4.31. *A scheme X is integral if and only if for each open subset U of X, the ring $\Gamma(U, \mathscr{O}_X)$ is integral.*

Proof. We first assume that $\Gamma(U, \mathscr{O}_X)$ is integral for any open set U. It is clear that X is reduced. If X is not irreducible, then one can find two nonempty disjoint open subsets U_1 and U_2. Then $\Gamma(U_1 \cup U_2, \mathscr{O}_X) = \Gamma(U_1, \mathscr{O}_X) \times \Gamma(U_2, \mathscr{O}_X)$, which is not an integral domain. Thus X is an integral scheme.

Conversely, suppose that X is reduced and irreducible. Let $U \subseteq X$ be an open subset, and suppose that there are elements $f, g \in \Gamma(U, \mathscr{O}_X)$ with $fg = 0$. Let

$$Y = \{x \in U : f_x \in \mathfrak{m}_x\}, \quad Z = \{x \in U : g_x \in \mathfrak{m}_x\};$$

then Y and Z are closed subsets, and $Y \cup Z = U$. Since X is irreducible, so U is irreducible, and one of Y or Z is equal to U, say $Y = U$. But then the restriction of f to any open affine subset of U will be contained in every point of that subset, hence nilpotent and thus zero. This shows that $\Gamma(U, \mathscr{O}_X)$ is integral. □

Proposition 8.4.32. *Let X be a scheme and x a point of X.*

(a) *For x to belong to a unique irreducible component of X, it is necessary and sufficient that the nilradical of $\mathscr{O}_{X,x}$ is prime.*

(b) *If the nilradical of $\mathscr{O}_{X,x}$ is prime and the family of irreducible components of X is locally finite, there exists an open neighborhood U of x that is irreducible.*

(c) *For X to be the coproduct of its irreducible components, it is necessary and sufficient that the family of irreducible components of X is locally finite and for each $x \in X$, the nilradical of $\mathscr{O}_{X,x}$ is prime.*

Proof. To check that whether x belongs to distinct irreducible components of X, we may assume that $X = \mathrm{Spec}(A)$ is affine (**??**). Then this signifies that $\mathfrak{p}_x$ contains two distinct minimal prime ideals of A, and equivalently $\mathfrak{m}_x$ contains two distinct minimal prime ideals of $\mathscr{O}_{X,x}$, which is a contradiction if and only if the nilradical of $\mathscr{O}_{X,x}$ is prime.

Now assume the conditions in (a). As the family of irreducible components of X is locally finite, the union of those of these components which do not contain x is closed, so its complement U is open and contained in the unique irreducible component of X containing x, and therefore irreducible (**??**). The assertion in (c) follows from (b) and **??**. □

Proposition 8.4.33. *For a scheme X to be locally integral, it is necessary and sufficient that the family of irreducible components is locally finite and for each point $x \in X$, the local ring $\mathscr{O}_{X,x}$ is integral. In this case, X is the coproduct of its irreducible components, which are integral subschemes.*

Proof. As the localizations of integral domains are integral, for a locally integral scheme X, its local rings $\mathscr{O}_{X,x}$ are inegral. The set of the irreducible components of X is locally finite since each $x \in X$ admits an irreducible open neighborhoods, and X is the coproduct of its irreducible components by Proposition 8.4.32. Conversely, if X satisfies these conditions, then X is the coproduct of its irreducible components, which are open and integral. It follows immediately that X is locally integral. □

Corollary 8.4.34. *Let X be a scheme whose set of irreducible components is locally finite (for example if X is locally Noetherian). Then for X to be integral, it is necessary and sufficient that it is connected and for each $x \in X$, the local ring $\mathscr{O}_{X,x}$ is integral. In particular, a locally integral and connected scheme is integral.*

Proposition 8.4.35. *Let X be a locally Noetherian scheme and let $x \in X$ be a point such that the nilradical $\mathscr{N}_x$ of $\mathscr{O}_{X,x}$ is prime (resp. such that $\mathscr{O}_{X,x}$ is reduced, resp. such that $\mathscr{O}_{X,x}$ is integral). Then there exist an open neighborhood U of x that is irreducible (resp. reduced, resp. integral).*

Proof. It suffices to consider the case where $\mathscr{N}_x$ is prime and where $\mathscr{N}_x = 0$, the third one is the conjunction of the first two cases. If $\mathscr{N}_X$ is prime, the claim follows from Proposition 8.4.32. If $\mathscr{N}_x = 0$, we then have $\mathscr{N}_y = 0$ for y in a neighborhood of x, since $\mathscr{N}$ is quasi-coherent, hence coherent since X is locally Noetherian and $\mathscr{N}$ is of finite type, and the conclusion follows from **??**. □

Proposition 8.4.36. *For a Noetherian scheme X, the nilradical $\mathscr{N}_X$ of $\mathscr{O}_X$ is nilpotent.*

Proof. Since X is quasi-compact, We can cover X by a finite number of affine open sets U_i, and it suffices to prove that there exist integers n_i such that $(\mathscr{N}_X|_{U_i})^{n_i} = 0$. If n is the largest of the n_i, we will then have $\mathscr{N}_X^n = 0$. We are therefore reduced to the case where $X = \mathrm{Spec}(A)$ is affine, A being a Noetherian ring. It then suffices to observe that the nilradical of A is nilpotent. □

Corollary 8.4.37. *A Noetherian sheme X is affine if and only if X_{red} is affine.*

Proof. If X is affine it is clear that X_{red} is affine, regardless of X being Noetherian. Conversely, assume that X_{red} is affine and X is Noetherian. Then by Proposition 8.4.36, the nilradical $\mathscr{N}$ of $\mathscr{O}_X$ is nilpotent. For any quasi-coherent sheaf $\mathscr{F}$ on X, consider the follows exact sequence (where $k \geq 0$)

$$0 \longrightarrow \mathscr{N}^k\mathscr{F} \longrightarrow \mathscr{N}^{k-1}\mathscr{F} \longrightarrow \mathscr{N}^{k-1}\mathscr{F}/\mathscr{N}^k\mathscr{F} \longrightarrow 0$$

Since $\mathscr{N}^{k-1}\mathscr{F}/\mathscr{N}^k\mathscr{F}$ is an $\mathscr{O}_X/\mathscr{N}$-module and $(X, \mathscr{O}_X/\mathscr{N})$ is affine, by Serre's criterion we have

$$H^1(X, \mathscr{N}^{k-1}\mathscr{F}/\mathscr{N}^k\mathscr{F}) = 0,$$

so $H^1(X, \mathscr{N}^k\mathscr{F}) = 0$ implies $H^1(X, \mathscr{N}^{k-1}\mathscr{F}) = 0$. Since $\mathscr{N}$ is nilpotent, this shows $H^1(X, \mathscr{F}) = 0$, so $(X, \mathscr{O}_X)$ is affine, by Serre's criterion again. □

Let $f : X \to Y$ be a morphism of schemes; let $i : X_{\mathrm{red}} \to X$ and $j : Y_{\mathrm{red}} \to Y$ be the canonical injections. The homomorphism $f_x^\# : \mathscr{O}_{Y,f(x)} \to \mathscr{O}_{X,x}$ maps nilpotent elements of $\mathscr{O}_{Y,f(x)}$ into nilpotent elements of $\mathscr{O}_{X,x}$, so $f^*(\mathscr{N}_Y)\mathscr{O}_X \subseteq \mathscr{N}_X$. It then follows from Proposition 8.4.4 that $f \circ i$ factors through j, so we get an induced morphism $f_{\mathrm{red}} : X_{\mathrm{red}} \to Y_{\mathrm{red}}$ and a commutative diagram

$$\begin{tikzcd} X_{\mathrm{red}} \arrow[r, "f_{\mathrm{red}}"] \arrow[d, "i"'] & Y_{\mathrm{red}} \arrow[d, "j"] \\ X \arrow[r, "f"] & Y \end{tikzcd} \tag{8.4.1}$$

In particular, if X is reduced, then the morphism f factors into

$$X \xrightarrow{f_{\mathrm{red}}} Y_{\mathrm{red}} \xrightarrow{j} Y$$

or in other words, f is dominated by the canonical injection j. We also conclude that Y_{red} satisfies the universal property that any morphism from a reduced scheme to Y factors through Y_{red}.

For two morphisms $f : X \to Y$, $g : Y \to Z$, it follows from the uniqueness of factorization that we have

$$(g \circ f)_{\mathrm{red}} = g_{\mathrm{red}} \circ f_{\mathrm{red}};$$

we can equivalently say that the operation $X \mapsto X_{\mathrm{red}}$ is a covariant functor on the category of schemes.

Proposition 8.4.38. *If X and Y are S-schemes, the schemes $X_{\mathrm{red}} \times_{S_{\mathrm{red}}} Y_{\mathrm{red}}$ and $X_{\mathrm{red}} \times_S Y_{\mathrm{red}}$ are identical, and are canonically identified with a subscheme of $X \times_S Y$ having the same underlying space as this product.*

Proof. The fact that $X_{\text{red}} \times_S Y_{\text{red}}$ is identified with a subscheme of $X \times_S Y$ follows from the fact that if $f : X' \to X$ and $g : Y' \to Y$ are surjective S-immersions, then $f \times_S g : X' \times_S Y' \to X \times_S Y$ is a surjective immersion (Proposition 8.4.13 and Proposition 8.3.28). On the other hand, if $\varphi : X_{\text{red}} \to S$ and $\psi : Y_{\text{red}} \to S$ are the structural morphisms, it is clear that they factors through S_{red}, and as $S_{\text{red}} \to S$ is a monomorphism, the second assertion follows. □

Corollary 8.4.39. *The schemes $(X \times_S Y)_{\text{red}}$ and $(X_{\text{red}} \times_{S_{\text{red}}} Y_{\text{red}})_{\text{red}}$ are canonically identified.*

We note that if X and Y are reduced S-schemes, it is not necessarily that case that $X \times_S Y$ is reduced, since the tensor product of two reduced algebras (even two fields) may not be reduced.

Corollary 8.4.40. *For any morphism $f : X \to Y$ of schemes, the diagram (8.4.1) factors into*

$$\begin{tikzcd} X_{\text{red}} = (X \times_Y Y_{\text{red}})_{\text{red}} \arrow[r] & X \times_Y Y_{\text{red}} \arrow[r] \arrow[d] & Y_{\text{red}} \arrow[d] \\ & X \arrow[r, "f"] & Y \end{tikzcd} \tag{8.4.2}$$

Proof. For this, we only need to note that $(X \times_Y Y_{\text{red}})_{\text{red}} = (X_{\text{red}} \times_{Y_{\text{red}}} Y_{\text{red}})_{\text{red}} = X_{\text{red}}$. □

Proposition 8.4.41. *Let X and Y be two schemes. Consider the following properties for a morphism $f : X \to Y$:*

(i) *surjective;*

(ii) *injective;*

(iii) *open at the point x (resp. open);*

(iv) *closed;*

(v) *a homeomorphism onto its image;*

(vi) *universally open at a point x (resp. universally open);*

(vii) *universally closed;*

(viii) *a universal embedding;*

(ix) *a universal homeomorphism;*

(x) *radical;*

(xi) *generalizing at a point x (resp. generalizing);*

(xii) *universally generalizing at a point x (resp. universally generalizing).*

Then, if $\mathcal{P}$ denote one of the above properties, for f to possess the property $\mathcal{P}$, it is necessary and sufficient that f_{red} possess $\mathcal{P}$.

Proof. The proposition is evident for the properties (i), (ii), (iii), (iv), (v), (xi), which only depend on the map of the underlying spaces. For (x), the proposition follows from the fact that the fibers of f and f_{red} at a point $y \in Y$ have the same underlying space and the residue field at a point of X (resp. Y) is the same for X_{red} and Y_{red}. For the properties (vi), (vii), (viii), (ix) and (xii), if f possesses one of these properties, the same is true of f_{red} due to Corollary 8.4.40 and that the morphism $X_{\text{red}} \to X \times_Y Y_{\text{red}}$ is a universal homeomorphism. Conversely, if f_{red} possesses one of these properties, it suffices to note that for any morphism $Y' \to Y$ we have $(X_{\text{red}} \times_{Y_{\text{red}}} Y'_{\text{red}})_{\text{red}} = (X \times_Y Y')_{\text{red}}$, so the morphism

$$(f_{(Y')})_{\text{red}} : (X \times_Y Y')_{\text{red}} \to Y'_{\text{red}}$$

possesses the "nonuniversal" version of the same property, and by what we have already seen, $f_{(Y')}$ then has the corresponding property. □

Proposition 8.4.42. *Let X and Y be two schemes and x be a point of X. Consider the following properties for a morphism $f : X \to Y$:*

(i) *a monomorphism;*

(ii) *an immersion;*

(iii) *an open immersion;*

(iv) *a closed immersion;*

(v) *a local immersion at the point x;*

(vi) *a local isomorphism at the point x;*

(vii) *birational.*

Then, if f possesses one of the above properties, f_{red} also possesses that property.

Proof. For the properties (ii) to (vii), the result follows from the observation that $(f_{\mathrm{red}})_x^\# : \mathscr{O}_{Y,f(x)} \to \mathscr{O}_{X,x}$ is surjective (resp. bijective) if $f_x^\# : \mathscr{O}_{Y,f(x)} \to \mathscr{O}_{X,x}$ is surjective (resp. bijective) (Proposition 8.4.9). For (i) it suffices to note that a monomorphism is universal (Proposition 8.3.12), the diagram (8.4.2), and the fact that $X_{\mathrm{red}} \to X \times_Y Y_{\mathrm{red}}$ is a closed immersion, hence a monomorphism. □

Note that if f_{red} is an immersion, it is not necessarily true that f is. For example, let $Y = \mathrm{Spec}(k)$ where k is a field and $X = \mathrm{Spec}(A)$, where $A = k[T]/(T^2)$. Then the canonical injection $\rho : k \to k[T]/(T^2)$ corresponds to a morphism $f : X \to Y$. It is clear that f is not an immersion (in fact any nonzero immersion into Y is automatically closed); but $A_{\mathrm{red}} = k$ so f_{red} is an isomorphism.

Remark 8.4.43. To say that an immersion $f : Y \to X$ is surjective means it is closed and that the subscheme of X associated with f is defined by an ideal $\mathscr{I}$ contained in the nilradical $\mathscr{N}_X$. In this case, we say f is a **nilimmersion**, and $\mathscr{I}$ is a nilideal of X; f is then a homeomorphism from Y to X, and f_{red} is an isomorphism from Y_{red} to X_{red}. We say the nilimmersion f is **nilpotent** (resp. **locally nilpotent**) if the ideal $\mathscr{I}$ is nilpotent (resp. locally nilpotent, i.e. that every $x \in X$ has an open neighborhood U such that $\mathscr{I}|_U$ is nilpotent). More precisely, we say f is **nilpotent of order** n if $\mathscr{I}^{n+1} = 0$. If Y is a subscheme of X and f is the canonical immersion, we say X is an **infinitesimal neighborhood** (resp. an **infinitesimal neighborhood of order** n) of Y if f is nilpotent (resp. nilpotent of order n).

8.4.6 Reduced scheme structure on closed subsets

Proposition 8.4.44. *For any locally closed subspace Y of the underlying space of a scheme X, there exists a unique reduced subscheme of X with underlying space Y.*

Proof. The uniqueness is immediate from Corollary 8.4.29, so we only need to construct a reduced scheme structure on Y. If X is affine with ring A and Y is closed in X, the proposition is immediate: $I(Y)$ is the largest ideal $\mathfrak{a} \subseteq A$ such that $V(\mathfrak{a}) = Y$, and is radical, hence the ring $A/I(Y)$ is reduced, and we can take the scheme structure $(Y, \widetilde{A/I(Y)})$ on Y.

In the general case, for any affine open $U \subseteq X$ such that $U \cap Y$ is closed in U, consider the closed subscheme Y_U of U defined by the quasi-coherent ideal associated with the ideal $I(U \cap Y)$ of $\Gamma(U, \mathscr{O}_X|_U)$, which is reduced. If V is an open affine of X contained in U, then Y_V is induced by Y_U on $V \cap Y$ since this induced scheme is a closed subscheme of V which is reduced and has $V \cap Y$ as underlying space; the uniqueness of Y_V therefore entails our assertion. □

Corollary 8.4.45. *Let X be a scheme and Y be a locally closed subset of X. Then any point $x \in Y$ admits a maximal generalization y (i.e. y has no further generalization in Y). In particular, if $Y \neq \varnothing$, there exist a maximal element $y \in Y$ under generalization.*

Proof. It suffices to give Y a subscheme structure of X and take y to be the generic point of the irreducible components of Y containing x. □

Example 8.4.46. Let X be a scheme and x be a closed point of X. Let U be an open neighborhood of x. Then $Z = (X - U) \cup \{x\}$ is a closed subset of X, so we can consider the reduced scheme structure on it. Let $\mathscr{I}$ be the corresponding quasi-coherent ideal of $\mathscr{O}_X$, we want to determine the stalk $\mathscr{I}_x$. For this, we can assume that $X = \mathrm{Spec}(A)$ is affine and $X - U = V(\mathfrak{a})$, where $\mathfrak{a}$ is an ideal of A. Then the point x

corresponds to a maximal ideal $\mathfrak{m}$ of A such that $\mathfrak{a} \not\subseteq \mathfrak{m}$, and $Z = V(\mathfrak{a}) \cup \{x\} = V(\mathfrak{a} \cap \mathfrak{m})$. By definition, $\mathscr{I}$ is the quasi-coherent ideal on X associated with $I(Z) = \sqrt{\mathfrak{a} \cap \mathfrak{m}}$, and therefore

$$\mathscr{I}_x = (\sqrt{\mathfrak{a} \cap \mathfrak{m}})_{\mathfrak{m}} = \sqrt{(\mathfrak{a} \cap \mathfrak{m})_{\mathfrak{m}}},$$

which is the intersection of prime ideals $\mathfrak{p}$ containing $\mathfrak{a} \cap \mathfrak{m}$ and contained in $\mathfrak{m}$. But if a prime ideal $\mathfrak{p} \subseteq \mathfrak{m}$ of A contains $\mathfrak{a} \cap \mathfrak{m}$, then by **??** we have $\mathfrak{p} \supset \mathfrak{a}$ or $\mathfrak{p} \supset \mathfrak{m}$. If $\mathfrak{p} \neq \mathfrak{m}$, then this implies $\mathfrak{p} \supset \mathfrak{a}$ and therefore $\mathfrak{m} \supset \mathfrak{p} \supset \mathfrak{a}$, which is a contradiction (since x is not contained in $X - U = V(\mathfrak{a})$). From this, we conclude that $\mathscr{I}_x = \mathfrak{m}_x$.

Example 8.4.47. Let X be a reduced locally Noetherian scheme and X' be a reduced closed subscheme of X with underlying space an irreducible component of X. Then X' is defined by a quasi-coherent ideal $\mathscr{I}$ of $\mathscr{O}_X$. Let x be the generic point of X'; we claim that $\mathscr{I}_x = 0$. For this, we can assume that $X = \mathrm{Spec}(A)$ is affine, where A is a reduced Noetherian ring, so $\mathscr{I} = \tilde{\mathfrak{p}}$ where $\mathfrak{p}$ is a minimal prime ideal of A. By definition the stalk of $\mathscr{I}$ at x is indeitified with $\mathfrak{p}A_{\mathfrak{p}}$, which is the maximal ideal of $A_{\mathfrak{p}} = \mathscr{O}_{X,x}$. But since A is reduced and $\mathfrak{p}$ is minimal, $A_{\mathfrak{p}}$ is a reduced Artinian local ring, whence a field, and we then conclude that $\mathfrak{p}A_p = 0$, so $\mathscr{I}_x = 0$.

Proposition 8.4.48. *Let X be a reduced scheme, $f : X \to Y$ be a morphism, and Z be a closed subscheme of Y containing $f(X)$. Then f factors into*

$$X \xrightarrow{g} Z \xrightarrow{j} Y$$

where j is the canonical injection.

Proof. The hypotheses implies that the closed subscheme $f^{-1}(Z)$ of X has underlying space X (Proposition 8.4.16). As X is reduced, this subscheme coincides with X by Corollary 8.4.29, and the proposition then follows from Proposition 8.4.16. □

Corollary 8.4.49. *Let X be a reduced subscheme of a scheme Y. If Z is the reduced closed subscheme of Y with underlying space $\overline{X}$, then X is an open subscheme of Z.*

Proof. Since X is locally closed, there is an open set U of Y such that $X = U \cap \overline{X}$. By Proposition 8.4.48, X is then a reduced subscheme of Z, with underlying space open in Z. Since the scheme structure induced by Z is also reduced, we conclude that X is induced by Z, in view of the uniqueness part of Proposition 8.4.44. □

Corollary 8.4.50. *Let $f : X \to Y$ be morphism and X' (resp. Y') be a closed subscheme of X (resp. Y) defined by a quasi-coherent ideal $\mathscr{I}$ (resp. $\mathscr{K}$) of $\mathscr{O}_X$ (resp. $\mathscr{O}_Y$). Suppose that X' is reduced and $f(X') \subseteq Y'$, then we have $f^*(\mathscr{K})\mathscr{O}_X \subseteq \mathscr{I}$.*

Proof. The restriction of f to X' factors into $X' \to Y' \to Y$ by Proposition 8.4.48, so it suffices to use Corollary 8.4.19. □

8.5 Separated schemes and morphisms

8.5.1 Diagonal and graph of a morphism

Let X be an S-scheme; recall that the diagonal morphism $X \to X \times_S X$, denoted by $\Delta_{X/S}$ or Δ_X, is the S-morphism $(1_X, 1_X)_S$, which means Δ_X is the unique S-morphism such that

$$p_1 \circ \Delta_X = p_2 \circ \Delta_X = 1_X,$$

where p_1, p_2 are the canonical projections of $X \times_S X$. If $f : T \to X$ and $g : T \to Y$ are two S-morphisms, we verify that

$$(f, g)_S = (f \times_S g) \circ \Delta_{T/S}.$$

If $\varphi : X \to S$ is the structural morphism of X, we also write Δ_φ for $\Delta_{X/S}$.

Proposition 8.5.1. *Let X, Y be S-schemes. If we identify $(X \times_S Y) \times_S (X \times_S Y)$ and $(X \times_S X) \times_S (Y \times_S Y)$, the morphism $\Delta_{X \times Y}$ is identified with $\Delta_X \times \Delta_Y$.*

Proof. In fact, if p_1, q_1 are the canonical projections $X \times_S X \to X$, $Y \times_S Y \to Y$, the projection $(X \times_S Y) \times_S (X \times_S Y) \to X \times_S Y$ is identified with $p_1 \times q_1$, and we have

$$(p_1 \times q_1) \circ (\Delta_X \times \Delta_Y) = (p_1 \circ \Delta_X) \times (q_1 \circ \Delta_Y) = 1_{X \times Y}$$

similar for the projection to the second factor. □

Corollary 8.5.2. *For any extension $S' \to S$ of base schemes, $\Delta_{X_{(S')}}$ is identified with $(\Delta_X)_{(S')}$.*

Proof. It suffices to remark that $(X \times_S X)_{(S')}$ is identified with $X_{(S')} \times_{S'} X_{(S')}$ canonically. □

Proposition 8.5.3. *Let X, Y be S-schemes and $S \to T$ be a morphism. Let $\varphi : X \to S$, $\psi : Y \to S$ be the structural morphisms, p, q the projection of $X \times_S Y$, and $\pi = \varphi \circ p = \psi \circ q$ the structural morphism $X \times_S Y \to S$. Then the diagram*

$$\begin{tikzcd}
X \times_S Y \arrow[r, "(p,q)_T"] \arrow[d, "\pi"'] & X \times_T Y \arrow[d, "\varphi \times_T \psi"] \\
S \arrow[r, "\Delta_{S/T}"] & S \times_T S
\end{tikzcd} \tag{8.5.1}$$

commutes and cartesian.

Proof. By the definition of products, we may prove the proposition in the category of sets, and replace X, Y, S by $X(Z)_T$, $Y(Z)_T$, $S(Z)_T$, where Z is an arbitrary T-scheme, and it is then immediate. □

Corollary 8.5.4. *If $f : X \to Y$ is an S-morphism, the diagram*

$$\begin{tikzcd}
X \arrow[r, "\Gamma_f"] \arrow[d, "f"'] & X \times_S Y \arrow[d, "f \times_S 1_Y"] \\
Y \arrow[r, "\Delta_Y"] & Y \times_S Y
\end{tikzcd} \tag{8.5.2}$$

is commutative and cartesian.

Proof. It suffcies to apply Proposition 8.5.3 with S replaced by Y and T by S, and note that $X \times_Y Y = X$. □

Proposition 8.5.5. *For a morphism $f : X \to Y$ of schemes to be a monomorphism, it is necessary and sufficient that $\Delta_{X/Y}$ is an isomorphism from X to $X \times_Y X$.*

Proof. In fact, f is monic means for any Y-scheme Z, the corresponding map $X(Z)_Y \to Y(Z)_Y$ is an injection, and as $Y(Z)_Y$ is reduced to a singleton, this means $X(Z)_Y$ is either empty or a singleton. But this is equivalent to saying that $X(Z)_Y \times X(Z)_Y$ is canonically isomorphic to $X(Z)_Y$ via the diagonal map, where the first set is $(X \times_Y X)(Z)_Y$, and this means $\Delta_{X/Y}$ is an isomorphism. □

Proposition 8.5.6. *The diagonal morphism is an immersion from X to $X \times_S X$, and the corresponding subscheme of $X \times_S X$ is called the **diagonal** of $X \times_S X$.*

Proof. Let p_1, p_2 be the projections of $X \times_S X$. As the continuous maps p_1 and Δ_X are such that $p_1 \circ \Delta_X = 1_X$, Δ_X is a homeomorphism from X onto $\Delta_X(X)$. Simialrly, the composition of the homomorphisms $\mathscr{O}_{X,x} \to \mathscr{O}_{\Delta_X(X),x} \to \mathscr{O}_{X,x}$ corresponding to p_1 and Δ_X is the identity, so the homomorphism corresponding to Δ_X on stalks are surjective. The proposition then follows from Proposition 8.4.9 . □

Corollary 8.5.7. *With the hypotheses of Proposition 8.5.3, the morphisms $(p, q)_T$ is an immersion.*

Proof. This follows from Proposition 8.5.3 and Corollary 8.4.14. □

Corollary 8.5.8. *Let X and Y be S-schemes and $f : X \to Y$ be an S-morphism. Then the graph morphism $\Gamma_f : X \to X \times_S Y$ of f is an immersion.*

Proof. This follows from the diagram (8.5.2) and Corollary 8.4.14. □

The subscheme of $X \times_S Y$ associated with the immersion Γ_f is called the **graph** of the morphism f; the subschemes of $X \times_S Y$ which are graphs of morphisms $X \to Y$ are characterized by the fact that the restriction of the projection $p_1 : X \times_S Y \to X$ to such a subscheme G is an isomorphism g from G to X: in fact, if this is the case, G is then the graph of the morphism $p_2 \circ g^{-1}$, where $p_2 : X \times_S Y \to Y$ is the second projection.

In particular, if $X = S$, the S-morphisms $S \to Y$, which are none other than the S-sections of Y, are equal to their graph morphisms; the subschemes of Y which are graphs of S-sections (in other words, those which are isomorphic to S by the restriction of the structural morphism $Y \to S$) are then called the **images of these sections**, or, by abuse of language, the S-sections of Y.

Proposition 8.5.9. *If $f : X \to Y$ is an S-morphism, the diagram*

$$\begin{tikzcd} X \arrow[r, "\Delta_X"] \arrow[d, "f"'] & X \times_S X \arrow[d, "f \times_S f"] \\ Y \arrow[r, "\Delta_Y"] & Y \times_S Y \end{tikzcd} \tag{8.5.3}$$

is commutative (in other words, Δ_X is a functorial morphism on the category of schemes).

Proof. The morphisms $\Delta_Y \circ f$ satisfies the condition that

$$p_1 \circ (\Delta_Y \circ f) = p_2 \circ (\Delta_Y \circ f) = f$$

where p_1, p_2 are the projections of $Y \times_S Y$. Similarly, if q_1, q_2 are the projections of $X \times_S X$,

$$\begin{aligned} p_1 \circ (f \times_S f) \circ \Delta_X &= f \circ q_1 \circ \Delta_X = f, \\ p_2 \circ (f \times_S f) \circ \Delta_X &= f \circ q_2 \circ \Delta_X = f. \end{aligned}$$

It then follows from the universal property of products that $\Delta_Y \circ f = (f \times_S f) \circ \Delta_X$. □

Corollary 8.5.10. *If X is a subscheme of Y, the diagonal $\Delta_X(X)$ is identified with a subscheme of $\Delta_Y(Y)$ whose the underlying space is identified with*

$$\Delta_Y(Y) \cap p_1^{-1}(X) = \Delta_Y(Y) \cap p_2^{-1}(X)$$

where p_1, p_2 are the projections of $Y \times_S Y$.

Proof. Apply Proposition 8.5.9 to the immersion $f : X \to Y$, we see $(f \times_S f)$ is an immersion which identifies $X \times_S X$ with the subspace $p_1^{-1}(X) \cap p_2^{-1}(X)$ of $Y \times_S Y$ (Proposition 8.4.13). Moreover, if $z \in \Delta_Y \cap p_1^{-1}(X)$, we have $z = \Delta_Y(y)$ and $y = p_1(z) \in X$, so $y = f(y)$, and $z = \Delta_Y(f(y))$ belongs to $\Delta_X(X)$ in view of the diagram (8.5.3). □

Proposition 8.5.11. *Let $u_1 : X \to Y$, $u_2 : X \to Y$ be two S-morphisms. Then the kernel $\ker(u_1, u_2)$ is canonically isomorphic to the inverse image in X of the diagonal $\Delta_Y(Y)$ under the morphism $(u_1, u_2)_S : X \to Y \times_S Y$.*

Proof. Let $Z \to X$ be the inverse image of $\Delta_Y(Y)$ under the morphism $(u_1, u_2)_S$. Then if $f : T \to X$ is an S-morphism such that $u_1 \circ f = u_2 \circ f$, then

$$\begin{aligned} p_1 \circ (u_1, u_2)_S \circ f &= u_1 \circ f = u_2 \circ f = p_1 \circ \Delta_Y \circ u_2 \circ f, \\ p_2 \circ (u_1, u_2)_S \circ f &= u_2 \circ f = p_2 \circ \Delta_Y \circ u_2 \circ f \end{aligned}$$

where p_1, p_2 are the projections of $Y \times_S Y$. We conclude that $(u_1, u_2)_S \circ f = \Delta_Y \circ u_2 \circ f$, and by the definition of Z, the morphism f factors uniquely through Z, which proves our claim. □

Corollary 8.5.12. *For a point $x \in X$ to belong to $\ker(u_1, u_2)$, it is necessary and sufficient that $u_1(x) = u_2(x) = y$ and the homomorphisms $\kappa(y) \to \kappa(x)$ corresponding to u_1 and u_2 are equal.*

Proof. In fact, if $j : Z \to X$ is the kernel of u_1 and u_2, to say $x \in Z$ signifies that the canonical morphism $h : \mathrm{Spec}(\kappa(x)) \to X$ factors into $h = j \circ g$, where g is a morphism from $\mathrm{Spec}(\kappa(x))$ to Z. This is equivalent to $u_1 \circ h = u_2 \circ h$, and by Corollary 8.2.17 to that $u_1(x) = u_2(x)$ and the field extensions $\kappa(y) \to \kappa(x)$ corresponding to u_1 and u_2 are equal. □

Proposition 8.5.13. *Let X and Y be S-schemes and $f : X \to Y$, $g : X \to Y$ be S-morphisms. Then we have the following commutative diagram*

$$\begin{tikzcd}
\ker(f,g) \arrow[r] \arrow[d] & X \arrow[r, "g"] \arrow[d, "\Gamma_g"] & Y \arrow[d, "\Delta_Y"] \\
X \arrow[r, "\Gamma_f"] \arrow[d, "f"'] & X \times_S Y \arrow[r, "g\times_S 1_Y"] \arrow[d, "f\times_S 1_Y"] & Y \times_S Y \\
Y \arrow[r, "\Delta_Y"] & Y \times_S Y &
\end{tikzcd}$$

where all squares are cartesian.

Proof. The fact that $\ker(f,g)$ is identified with the kernel of Γ_f and Γ_g can be deduced from by applying the projection $X \times_S Y \to Y$, or by Yoneda since this is clearly true for sets. The other two small squares are cartesian by Corollary 8.5.4, so the claim follows from the transitivity of products. □

Proposition 8.5.14. *Let $\mathcal{P}$ be a property for morphisms of schemes and consider the following conditions:*

(i) *If $j : X \to Y$ is an immersion and $g : Y \to Z$ is a morphism possessing the property $\mathcal{P}$, then $g \circ j$ possesses the property $\mathcal{P}$.*

(ii) *If $f : X \to X'$ and $g : y \to Y'$ are two S-morphisms possessing the property $\mathcal{P}$, then $f \times_S g$ possesses the property $\mathcal{P}$.*

(iii) *If the composition $g \circ f$ of two morphisms $f : X \to Y$, $g : Y \to Z$ possesses the property $\mathcal{P}$, then f possesses the property $\mathcal{P}$.*

Then (iii) *is a concequence of* (i) *and* (ii)*.*

Proof. The morphism f factors into

$$X \xrightarrow{\Gamma_f} X \times_Z Y \xrightarrow{p_2} Y$$

where p_2 is the projection. On the other hand, p_2 is idnetified with $(g \circ f) \times_Z 1_Y$, and by (ii) it possesses the property $\mathcal{P}$; as Γ_f is an immersion, it then follows from (i) that f possesses $\mathcal{P}$. □

Proposition 8.5.15. *Let $f : X \to Y$ and $g : Y \to Z$ be two morphisms. Consider the following properties for a morphism:*

(i) *a monomorphism;*

(ii) *an immersion;*

(iii) *a local immersion;*

(iv) *a universal embedding;*

(v) *radical.*

Then, if $g \circ f$ possesses one of these properties, so does f.

Proof. The properties (i) and (v) have only been put for memory, because for (i) this is a property valid for any category, and for (v), the proposition has already been proven in Proposition 8.3.31.

An immersion has each of these properties, and the composition of two morphisms having one (determined) of these properties also possesses it; moreover, all the above properties are stable under products. Thus the claim follows from Proposition 8.5.14. □

Corollary 8.5.16. *Let $j : X \to Y$ and $g : X \to Z$ be two S-morphisms. If j possesses one of the properties in Proposition 8.5.15, so does $(j,g)_S$.*

Proof. In fact, if $p : Y \times_S Z \to Y$ is the projection, we have $j = p \circ (j,g)_S$, and it suffices to apply Proposition 8.5.15. □

8.5.2 Separated morphisms and schemes

A morphism $f : X \to Y$ of schemes is called **separated** if the diagonal morphism $\Delta_f : X \to X \times_Y X$ is a closed immersion. In this case, X is said to be a **separated** Y**-scheme**, or **separated over** Y. A scheme X is called **separated** if it is separated over $\mathbb{Z}$. In view of Proposition 8.5.6, for X to be separated over Y, it is necessary and sufficient that the diagonal is a closed subscheme of $X \times_Y X$.

Proposition 8.5.17. *Any morphism of affine schemes is separated. In particular, any affine scheme is separated.*

Proof. If $f : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is a morphism of affine schemes, where B is an A-algebra, the diagonal morphism corresponds to the ring homomorphism $B \otimes_A B \to B$ given by $b \otimes b' \mapsto bb'$. Since this is surjective, we conclude that Δ_f is a closed immersion, so f is separated. □

Proposition 8.5.18. *Let X, Y be S-schemes and $S \to T$ be a separated morphism. Then the canonical immersion $X \times_S Y \to X \times_T Y$ in (8.5.1) is closed.*

Proof. In fact, in the diagram (8.5.1), the diagonal $\Delta_{S/T}$ is a closed immersion, so its base change $\varphi \times_T \psi : X \times_S Y \to X \times_T Y$ is also closed. □

Corollary 8.5.19. *Let X, Y be S-schemes and $f : X \to Y$ be an S-morphism. If Y is speareted over S, the graph morphism $\Gamma_f : X \to X \times_S Y$ is a closed immersion. In particular, any S-section of Y is a closed immersion.*

Proof. This follows from (8.5.2) and Corollary 8.4.14. □

Proposition 8.5.20. *Let Y be a separated S-scheme. Then for any S-morphisms $u_1 : X \to Y$, $u_2 : X \to Y$, the kernel of u_1 and u_2 is a closed subscheme of X.*

Proof. Recall that by Proposition 8.5.11 the kernel is the inverse image of $\Delta_Y(Y)$ under the morphism $(u_1, u_2)_S$. Since $\Delta_Y(Y)$ is closed, it follows from Corollary 8.4.14 that its inverse image is also closed. □

Corollary 8.5.21. *Let S be an integral scheme, η its generic point, and X a separated S-scheme. If two S-sections u, v of X satisfy $u(\eta) = v(\eta)$, then $u = v$.*

Proof. In fact, if $x = u(\eta) = v(\eta)$, the corresponding homomorphisms $\kappa(x) \to \kappa(\eta)$ are necessarily identical, since their composition with the homomorphism $\kappa(\eta) \to \kappa(x)$ corresponding to the structural morphism $X \to S$ is the idnetity on $\kappa(\eta)$. We then deduce from Corollary 8.5.12 that $\eta \in \ker(u_1, u_2)$, and by hypothesis $\ker(u_1, u_2)$ is a closed subscheme of S (Proposition 8.5.20). As S is reduced and η is its generic point, the unique closed subscheme of S containing η is S (Corollary 8.4.29), so $u = v$. □

Proposition 8.5.22. *Let $\mathcal{P}$ be a property of morphisms of schemes, and consider the following properties:*

(i) *If $j : X \to Y$ is a closed immersion and $g : Y \to Z$ is a morphism possessing the property $\mathcal{P}$, then $g \circ j$ possesses the property $\mathcal{P}$.*

(ii) *If $f : X \to X'$ and $g : Y \to Y'$ are two S-morphisms possessing the property $\mathcal{P}$, then $f \times_S g$ possesses the property $\mathcal{P}$.*

(iii) *If the composition $g \circ f$ of two morphisms $f : X \to Y$, $g : Y \to Z$ possesses the property $\mathcal{P}$ and if g is separated, then f possesses the property $\mathcal{P}$.*

(iv) *If $f : X \to Y$ possesses the property $\mathcal{P}$, so does f_{red}.*

Then, (iii) *and* (iv) *are concequences of* (i) *and* (ii)*.*

Proof. For the property (iii), the demonstration is similar to Proposition 8.5.14, with the fact that Γ_f is a closed immersion by Corollary 8.5.19. On the other hand, in the commutative diagram

$$\begin{tikzcd} X_{\mathrm{red}} \arrow[r, "f_{\mathrm{red}}"] \arrow[d] & Y_{\mathrm{red}} \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

the vertical morphisms are closed immersions, so we see that (iv) is a consequence of (i) and (iii), observing that a closed immersion is separated in view of the definition and Proposition 8.5.5. □

Proposition 8.5.23. *Let $f : X \to Y$ be a morphism and $g : Y \to Z$ be a separated morphism. Suppose that $g \circ f$ possesses one of the following properties:*

(i) *universally closed;*

(ii) *a closed immersion;*

Then f possesses the same property.

Proof. In fact, these properties satisfy the conditions (i) and (ii) in Proposition 8.5.22. □

Corollary 8.5.24. *Let Z be a separated S-scheme and $f : X \to Y$, $g : X \to Z$ be two S-morphisms. If f is universally closed (resp. a closed immersion), so is $(f, g)_S : X \to Y \times_S Z$.*

Proof. The morphism j factors into

$$X \xrightarrow{(f,g)_S} Y \times_S Z \xrightarrow{p} Y$$

and the projection $p : Y \times_S Z \to Y$ is a separate morphism by Proposition 8.5.26 (which do not use Corollary 8.5.24), so it suffices to apply Proposition 8.5.23. □

Remark 8.5.25. From the diagram in Proposition 8.5.13, we conclude that a morphism $Y \to S$ is separated if and only if the following equivalent conditions holds:

(i) The diagonal morphism $\Delta_{Y/S}$ is a closed immersion.

(ii) For every S-scheme X and for any two S-morphisms $f, g : X \to Y$, the kernel $\ker(f, g)$ is a closed subscheme of X.

(iii) For every S-scheme X and for any S-morphism $f : X \to Y$, the graph morphism Γ_f is a closed immersion.

Also, if the conclusion in Proposition 8.5.23 holds for the morphisms $\Delta_Y : Y \to Y \times_S Y$ and $p_2 : Y \times_S Y \to Y$, then Δ_Y is a closed immersion so Y is separated over S.

8.5.3 Criterion of separated morphisms

Proposition 8.5.26 (Propoerties of Seperated Morphisms).

(i) *Any radical morphism (and in particular any monomorphism, hence any immersion) is a separated morphism.*

(ii) *The composition of two separated morphisms is separated.*

(iii) *If $f : X \to X'$ and $g : Y \to Y'$ are two separated S-morphisms, $f \times_S g$ is separated.*

(iv) *If $f : X \to Y$ is a separated S-morphism, the S'-morphism $f_{(S')}$ is separated for any base change $S' \to S$.*

(v) *If the composition $g \circ f$ of two morphisms is separated, then f is separated.*

(vi) *For a morphism f to be separated, it is necessary and sufficient that f_{red} is separated.*

Proof. A radical morphism, its diagonal morphism is surjective (Proposition 8.3.31), so it is separated. If $f : X \to Y$, $g : Y \to Z$ are two morphisms, the diagram

$$\begin{tikzcd} X \arrow[rr, "\Delta_{X/Z}"] \arrow[dr, "\Delta_{X/Y}"'] & & X \times_Z X \\ & X \times_Y X \arrow[ur, "j"'] & \end{tikzcd}$$

where j is the canonical immersion in (8.5.1), is commutative. If f and g are separated, $\Delta_{X/Y}$ is a closed immersion and j is a closed immersion by Proposition 8.5.18, so $\Delta_{X/Z}$ is closed, which proves (ii). With (i) and (ii), conditions (iii) and (iv) are equivalent, and it suffices to prove (iv). Now by transitivity,

$X_{(S')} \times_{Y_{(S')}} X_{(S')}$ is identified with $(X \times_Y X) \times_Y Y_{(S')}$, and the diagonal morphism $\Delta_{X_{(S')}}$ is identified with $\Delta_X \times_Y 1_{Y_{(S')}}$. The assertion then follows from Corollary 8.4.14.

To prove (v), consider the factorization

$$X \xrightarrow{\Gamma_f} X \times_Z Y \xrightarrow{p_2} Y$$

of f, and note that $p_2 = (g \circ f) \times_Z 1_Y$; the hypothesis that $g \circ f$ is separated implies p_2 is separated by (iii), and as Γ_f is an immersion, Γ_f is separated by (i), hence f is separated by (ii). Finally, for (vi), we recall that the schemes $X_{\mathrm{red}} \times_{Y_{\mathrm{red}}} X_{\mathrm{red}}$ and $X_{\mathrm{red}} \times_Y X_{\mathrm{red}}$ is canonically identified (Proposition 8.4.38); if $j : X_{\mathrm{red}} \to X$ is the canonical injection, the diagram

$$\begin{tikzcd} X_{\mathrm{red}} \arrow[r, "\Delta_{X_{\mathrm{red}}}"] \arrow[d, "j"'] & X_{\mathrm{red}} \times_Y X_{\mathrm{red}} \arrow[d, "j \times_Y j"] \\ X \arrow[r, "\Delta_X"] & X \times_Y X \end{tikzcd}$$

is commutative, and the assertion follows from the fact that the vertical morphisms are homeomorphisms. ☐

Corollary 8.5.27. *If $f : X \to Y$ is separated, the restriction of f to any subscheme of X is separated.*

Proof. This follows from Proposition 8.5.26(i) and (iii). ☐

Corollary 8.5.28. *If X and Y are S-schemes and Y is separated over S, $X \times_S Y$ is separated over X.*

Proof. This is a particular case of Proposition 8.5.26(iv). ☐

Proposition 8.5.29. *Let X be a scheme and suppose that $(X_i)_{1 \le i \le n}$ is a finite covering of X by closed subsets. Let $f : X \to Y$ be a morphism and for each i let Y_i be a closed subset of Y such that $f(X_i) \subseteq Y_i$. Consider the reduced subscheme structures on each X_i and Y_i and let $f_i : X_i \to Y_i$ be the restriction of f on X_i. Then for f to be separated, it is necessary and sufficient that each f_i is separated.*

Proof. The necessity follows from Proposition 8.5.26(i), (ii) and (v). Conversely, if each f_i is separated, then the restriction $X_i \to Y$ of f is separated (Proposition 8.5.26). If p_1, p_2 are the projections of $X \times_Y X$, the subspace $\Delta_{X_i}(X_i)$ is identified with the subspace $\Delta_X(X) \cap p_1^{-1}(X_i)$ of $X \times_Y X$ (Corollary 8.5.10). This subspace is closed in $X \times_Y X$ by hypothesis, and their union is $\Delta_X(X)$, so Δ_X is closed and f is separated. ☐

Suppose in particular that X_i are the irreducible components of X; then we can suppose that each Y_i is a irreducible closed subset of Y (**??**); Proposition 8.5.29 then enable us to reduce the separation problem to integral schemes.

Proposition 8.5.30. *Let (Y_λ) be an open covering of a scheme Y. For a morphism $f : X \to Y$ to be separated, it is necessary and sufficient that each of the restriction $f_\lambda : f^{-1}(Y_\lambda) \to Y_\lambda$ is separated.*

Proof. If we set $X_\lambda = f^{-1}(Y_\lambda)$ and identify $X_\lambda \times_Y X_\lambda$ and $X_\lambda \times_{Y_\lambda} X_\lambda$, the $X_\lambda \times_Y X_\lambda$ form an open covering of $X \times_Y X$. If $Y_{\lambda\mu} = Y_\lambda \cap Y_\mu$ and $X_{\lambda\mu} = X_\lambda \cap X_\mu = f^{-1}(Y_{\lambda\mu})$, then $X_\lambda \times_Y X_\mu$ is identified with $X_{\lambda\mu} \times_{Y_{\lambda\mu}} X_{\lambda\mu}$ by Corollary 8.3.2, hence with $X_{\lambda\mu} \times_Y X_{\lambda\mu}$, and finally to an open subset of $X_\lambda \times_Y X_\lambda$, which establishes our assertion (Corollary 8.4.11). ☐

Proposition 8.5.30 allows, by taking a covering of Y byopen affines, to reduce the study of separated morphisms to separated morphisms with values in affine schemes.

Proposition 8.5.31. *Let Y be an affine scheme, X be a scheme, and (U_α) be an affine open covering of X. For a morphism $f : X \to Y$ to be separated, it ie necessary and sufficient that, for any couple of indices (α, β), the intersection $U_\alpha \cap U_\beta$ is affine, and the ring $\Gamma(U_\alpha \cap U_\beta, \mathscr{O}_X)$ is generated by the images of $\Gamma(U_\alpha, \mathscr{O}_X)$ and $\Gamma(U_\beta, \mathscr{O}_X)$.*

Proof. The open sets $U_\alpha \times_Y U_\beta$ form an open cover of $X \times_Y X$ (Corollary 8.3.2). Let p, q be the projections of $X \times_Y X$, we have

$$\Delta_X^{-1}(U_\alpha \times_Y U_\beta) = \Delta_X^{-1}(p^{-1}(U_\alpha) \cap q^{-1}(U_\beta)) = U_\alpha \cap U_\beta.$$

It therefore amounts to show that the restriction of Δ_X to $U_\alpha \cap U_\beta$ is a closed immersion into $U_\alpha \times_Y U_\beta$. But this restriction is just $(j_\alpha, j_\beta)_Y$, where j_α (resp. j_β) is the canonical injection of $U_\alpha \cap U_\beta$ to U_α (resp. to U_β). As $U_\alpha \times_Y U_\beta$ is an affine scheme with ring isomorphic to $\Gamma(U_\alpha, \mathscr{O}_X) \otimes_{\Gamma(Y,\mathscr{O}_Y)} \Gamma(U_\beta, \mathscr{O}_X)$, we see that $U_\alpha \cap U_\beta$ is a closed subscheme of $U_\alpha \times_Y U_\beta$ if and only if it is affine and the ring homomorphism

$$\Gamma(U_\alpha, \mathscr{O}_X) \otimes_{\Gamma(Y,\mathscr{O}_Y)} \Gamma(U_\beta, \mathscr{O}_X) \to \Gamma(U_\alpha \cap U_\beta, \mathscr{O}_X), \quad h_\alpha \otimes h_\beta \to h_\alpha h_\beta$$

is surjective (Example 8.4.6), which proves our assertion. □

Corollary 8.5.32. *Let Y be an affine scheme. For a morphism $f : X \to Y$ to be separated, it is necessary and sufficient that X is separated.*

Proof. In fact, the criterion in Proposition 8.5.31 does not depend on f. □

Corollary 8.5.33. *For a morphism $f : X \to Y$ to be separated, it is necessary that for any open affine subscheme U that is separated, the open subscheme $f^{-1}(U)$ is separated, and it suffices that this true for every affine open subset $U \subseteq Y$.*

Proof. The necessity follows from Proposition 8.5.29 and Proposition 8.5.26(ii). The sufficiency follows from Proposition 8.5.30 and Corollary 8.5.32. □

Proposition 8.5.34. *Let Y be a separated scheme and $f : X \to Y$ be a morphism. For any affine open U of X and any affine open V of Y, $U \cap f^{-1}(V)$ is affine.*

Proof. Let p_1, p_2 be the projections of $X \times_{\mathbb{Z}} Y$. Using the universal property of Γ_f, the subspace $U \cap f^{-1}(V)$ can be characterized by

$$\Gamma_f(U \cap f^{-1}(V)) = \Gamma_f(X) \cap p_1^{-1}(U) \cap p_2^{-1}(V)$$

Now $p_1^{-1}(U) \cap p_2^{-1}(V)$ is idnetified with the product $U \times_{\mathbb{Z}} V$, and therefore is affine; as $\Gamma_f(X)$ is closed in $X \times_{\mathbb{Z}} Y$ (Corollary 8.5.19), $\Gamma_f(X) \cap p_1^{-1}(U) \cap p_2^{-1}(V)$ is closed in $U \times_{\mathbb{Z}} V$, hence also affine. The assertion then follows from the fact that Γ_f is a closed immersion and Example 8.4.6. □

Example 8.5.35. The scheme in Example 8.2.10 is separated. In fact, for the covering (X_1, X_2) of X by affine opens, $X_1 \cap X_2 = U_{12}$ is affine and $\Gamma(U_{12}, \mathscr{O}_X)$, the fraction ring of the form $f(s)/s^m$ where $f \in K[s]$, is generated by $K[s]$ and $1/s$, so the conditions in Proposition 8.5.31 are satisfied.

With the same choice of X_1, X_2, U_{12} and U_{21} as in Example 8.2.10, take this time for u_{12} the isomorphism which sends $f(s)$ to $f(t)$; this time we obtain by gluing together a non-separated integral scheme X, because the first condition of Proposition 8.5.31 holds, but the second fails. It is immediate here that $\Gamma(X, \mathscr{O}_X) \to \Gamma(X_1, \mathscr{O}_X) = K[s]$ is an isomorphism; the inverse isomorphism defines a morphism $f : X \to \mathrm{Spec}(K[s])$ which is surjective, and for any $y \in \mathrm{Spec}(K[s])$ such that $\mathfrak{p}_y \neq (0)$, $f^{-1}(y)$ is reduced to a singleton, but for $\mathfrak{p}_y = (0)$, $f^{-1}(y)$ consists of two distinct points (we say that X is the "affine line on K, where the point 0 is doubled").

We can also give examples where neither of the two conditions of Proposition 8.5.31 does not hold. Note first that in the prime spectrum Y of the ring of polynomials $A = K[s,t]$ in two indeterminates over a field K, the open set $U = D(s) \cup D(t)$ is not an affine open set. Indeed, if z is a section of $\mathscr{O}_Y$ over U, there exist two integers $m, n \geq 0$ such that $s^m z$ and $t^n z$ are the restrictions to U of polynomials in s and t (Theorem 8.1.21), which is obviously only possible if the section z extends into a section over the entire space Y, identified with a polynomial in s and t. If U were affine, the canonical injection $U \to Y$ would therefore be an isomorphism by Theorem 8.1.17, which is absurd since $U \neq Y$.

This being so, let us take two affine schemes Y_1, Y_2, with rings $A_1 = K[x_1, t_1]$, $A_2 = K[s_2, t_2]$. Let $U_{12} = D(s_1) \cup D(t_1)$, $U_{21} = D(s_2) \cup D(t_2)$, and let u_{12} be the restriction to U_{21} of the isomorphism $Y_2 \to Y_1$ corresponding to the isomorphism of rints, which sends $f(s_1, t_1)$ to $f(s_2, t_2)$. We thus obtain an example where none of the conditions of Theorem 8.1.17 is satisfied (the integral scheme thus obtained is called "affine plane over K, where point 0 is doubled").

8.6 Finiteness conditions for morphisms

We study, in this section, various "finiteness conditions" for a morphism $f : X \to Y$ of schemes. There are basically two notions of "global finiteness" on X: quasi-compactness and quasi-separateness. On the other hand, there is two notions of "local finiteness" on X: locally of finite type and locally of finite presentation. By combining these local notions with the previous global notions, we obtain the notions of morphism of finite type and of morphism of finite presentation.

8.6.1 Quasi-compact and quasi-separated morphisms

We say a morphism $f : X \to Y$ is **quasi-compact** is for any quasi-compact open subset U of Y, the inverse image $f^{-1}(U)$ is quasi-compact. It is clear that this condition is purely topological, and if X is Noetherian, then any morphism $f : X \to Y$ is quasi-compact. We say a Y-scheme X is **quasi-compact over** Y if its structural morphism is quasi-compact.

If $\mathcal{B}$ is a base of Y formed by quasi-compact open sets (for example, affine opens), for a morphism f to be quasi-compact, it is necessary and sufficient that for any open set $V \in \mathcal{B}$, $f^{-1}(V)$, since any quasi-compact open set of Y is a finite union of open sets in $\mathcal{B}$.

If $f : X \to Y$ is a quasi-compact morphism, it is clear that for any open set V of Y, the restriction $f^{-1}(V) \to V$ of f is quasi-compact. Conversely, if (U_α) is an open covering of Y and $f : X \to Y$ is a morphism such that the restriction $f^{-1}(U_\alpha) \to U_\alpha$ is quasi-compact, then f is quasi-compact, since there exist a basis of quasi-compact open sets for Y, each set of which is contained in at least one of the U_α. We conclude that if $f : X \to Y$ is an S-morphism of S-schemes, and if there is an open covering (S_λ) of S such that the restrictions $\varphi^{-1}(S_\lambda) \to \psi^{-1}(S_\lambda)$ of f (where $\varphi : X \to S$ and $\psi : Y \to S$ are the structural morphisms) are quasi-compact morphisms, then f is quasi-compact.

Proposition 8.6.1. *Let Y be a separated scheme. Then for a morphism $f : X \to Y$ to be quasi-compact, it is necessary and sufficient that X is quasi-compact.*

Proof. If X is quasi-compact, it is a union of finitely many affine opens U_i, and for any affine open V of Y, $U_i \cap f^{-1}(V)$ is an affine open by Proposition 8.5.34, hence quasi-compact; therefore $f^{-1}(V)$ is quasi-compact. Conversely, if f is a quasi-compact morphism, then since Y is quasi-compact open in Y, we see $X = f^{-1}(Y)$ is also quasi-compact. □

Example 8.6.2. A closed immersion is quasi-compact since a closed subset of a quasi-compact set is again quasi-compact. However, open immersions are in general not quasi-compact: the standard example is the affine scheme $X = \mathrm{Spec}(k[x_1, x_2, \dots])$ and consider $U = X - \{0\}$, where 0 is the point of X corresponding to the maximal ideal $(x_1, x_2, \dots)$. The canonical injection $j : U \to X$ is not quasi-compact because U is not quasi-compact. To see this, consider the covering $(D(x_i))_{i\in\mathbb{N}}$ of U; for any finite subset J of $\mathbb{N}$, the family $(D(x_i))_{i\in J}$ can not cover U simply because the prime ideal $\mathfrak{p}_J$ generated by x_i with $i \in J$ is contained in U but not in the union of the $D(x_i)$ for $i \in J$.

We say a morphism $f : X \to Y$ of schemes is **quasi-separated** (of X is an Y**-scheme quasi-separated over** Y) if the diagonal morphism $\Delta_f : X \to X \times_Y X$ is quasi-compact. A scheme X is called **quasi-separated** if it is quasi-separated over $\mathbb{Z}$. Since a closed immersion is quasi-compact, we see any separated morphism is quasi-separated. In particular, any separated scheme is quasi-separated.

Proposition 8.6.3 (Properties of Quasi-Compact Morphisms).

(i) *An immersion $j : X \to Y$ is quasi-compact if it is closed, or Y is locally Noetherian, or X is Noetherian.*

(ii) *The composition of two quasi-compact morphisms is quasi-compact.*

(iii) *If $f : X \to Y$ is a quasi-compact S-morphism, $f_{(S')} : X_{(S')} \to Y_{(S')}$ is quasi-compact for any base change $g : S' \to S$.*

(iv) *If $f : X \to X'$ and $g : Y \to Y'$ is two quasi-compact S-morphisms, then $f \times_S g$ is quasi-compact.*

(v) *If the composition $g \circ f$ of two morphisms $f : X \to Y$, $g : Y \to Z$ is quasi-compact, and if g is quasi-separated or if X is locally Noetherian, then f is quasi-compact.*

(vi) *For a morphism f to be quasi-compact, it is necessary and sufficient that f_{red} is quasi-compact.*

Proof. Assertion (vi) is evident, since the quasi-compactness for a morphism only depends on the map of the underlying spaces. Also, (ii) follows from the definition of quasi-compactness.

The assertion in (i) is clear if j is closed; if j is an immersion and Y is locally Noetherian, any quasi-compact open V of Y is Noetherian, so $j^{-1}(V) = X \cap V \subseteq V$ is quasi-compact (here we identify X as a subscheme of Y). If X is Noetherian, then any morphism from X is quasi-compact.

To prove (iii), we can assume that $S = Y$ by the transitivity of products; put $f' = f_{(S')}$, and let U' be a quasi-compact open subset of S'. For any $s' \in U'$, let T be an affine open neighborhood of $g(s')$ in S, and let W be an affine open neighborhood of s' contained in $U' \cap g^{-1}(T)$; it suffices to show that $f'^{-1}(W)$ is quasi-compact, or in other words, we only need to show that if S and S' are affine, then $X \times_S S'$ is quasi-compact. This is true because by hypothesis X is a finite union of affine opens V_j, and $X \times_S S'$ is then the union of the affine schemes $V_j \times_S S'$, hence quasi-compact. With (ii) and (iii), asserion (iv) then follows.

We now prove (v) in the case where X is locally Noetherian. Put $h = g \circ f$ and let U be a quasi-compact open of Y; $g(U)$ is then quasi-compact in Z (not necessarily open), so it is contained in a finite union of quasi-compact opens V_j, and $f^{-1}(U)$ is contained in the union of the $h^{-1}(V_j)$, which are all quasi-compact by hypothesis. We then conclude that $f^{-1}(U)$ is a Noetherian space (**??**), and a fortiori quasi-compact.

To prove (v) in the case that g is quasi-separated, recall that f factors into

$$X \xrightarrow{\Gamma_f} X \times_Z Y \xrightarrow{p_2} Y$$

where p_2 is identified with $(g \circ f) \times_Z 1_Y$, and if $g \circ f$ is quasi-compact, so is p_2 by (iii). Finally, we have the following cartesian square (Corollary 8.5.4)

$$\begin{tikzcd} X \arrow[r, "\Gamma_f"] \arrow[d, "f"'] & X \times_Z Y \arrow[d, "f \times 1_Y"] \\ Y \arrow[r, "\Delta_g"] & Y \times_Z Y \end{tikzcd}$$

as by hypothesis Δ_g is quasi-compact, Γ_f is also quasi-compact, and by (ii) we conclude that f is quasi-compact. □

Proposition 8.6.4. *Let $f : X \to Y$, $g : Y \to Z$ be two morphisms. If $g \circ f$ is quasi-compact and f is surjective, then g is quasi-compact.*

Proof. If fact, if V is a quasi-compact open of Z, $f^{-1}(g^{-1}(V))$ is quasi-compact by hypothesis, and we have $g^{-1}(V) = f(f^{-1}(g^{-1}(V)))$ since f is surjective, so $g^{-1}(V)$ is quasi-compact. □

Proposition 8.6.5. *Let f be a quasi-compact morphism of schemes.*

(a) *The following conditions are equivalent:*

  (i) *f is dominant;*

  (ii) *for any maximal point $y \in Y$, $f^{-1}(y) \neq \varnothing$.*

  (iii) *for any maximal point $y \in Y$, $f^{-1}(y)$ contains a maximal point of X.*

(b) *If f is dominant, for any generalizing morphism $g : Y' \to Y$, the morphism $f_{(Y')} : X_{(Y')} \to Y'$ is quasi-compact and dominant.*

Proof. It is clear that (iii)⇒(ii)⇒(i). Suppose that f is dominant and consider an affine open neighborhood U of y; $f^{-1}(U)$ is quasi-compact, hence a union of finitely many affine opens V_i, and by hypothesis y belongs to the closure of $f(V_i)$ in U. We can evidently suppose that X and Y are reduced. As the closure in X of an irreducible component of V_i is an irreducible component of X (**??**), we can replace X by V_i, Y by the closed reduced subscheme $\overline{f(V_i)} \cap U$ of U, and we are thus reduced to proving the proposition when $X = \mathrm{Spec}(B)$ and $Y = \mathrm{Spec}(A)$ are affine and reduced. Since f is dominant, A is then a subring of B (**??**); the proposition then follows from the fact that any minimal prime ideal of A is the intersection with A of a minimal prime ideal of B (**??**).

If $f : X \to Y$ is quasi-compact and dominant, then $f' = f_{(Y')}$ is quasi-compact by Proposition 8.6.3. On the other hand, a maximal point y' of Y' is hypothesis lying over a maximal point y of Y ([**?**] new, 3.9.5); as $f^{-1}(y)$ is nonempty by (i), the same holds for $f'^{-1}(y')$ (Proposition 8.3.38), whence the conclusion. □

Proposition 8.6.6. *For a quasi-compact morphism $f : X \to Y$, the following conditions are equivalent:*

(i) *The morphism f is closed.*

(ii) *For any $x \in X$ and any specialization y' of $y = f(x)$ distinct from y, there exists a specialization x' of x such that $f(x') = y'$.*

In particular, if $f : X \to Y$ is a quasi-compact immersion, for f to be a closed immersion, it is necessary and sufficient that X (considered as a subspace of Y) contains any specializations (in Y) of its points.

Proof. The condition (ii) expresses as $f(\overline{\{x\}}) = \overline{\{y\}}$, and is therefore a concequence of (i). To show that (ii) implies (i), consider a closed subset of X' of X; let $Y' = \overline{f(X')}$ and we prove that $Y' = f(X')$. Endow X' and Y' the reduced subscheme structure, there then exists a morphism $f' : X' \to Y'$ such that the diagram

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d] & Y' \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

commutes. As f is quasi-compact, so is f' (Proposition 8.6.3(i) and (v)). We are then reduced to proving that if f is a quasi-compact dominant morphism, then $f(X) = Y$. Now let y' be a point of y and let y be the generic point of a irreducible component of Y containing y'; by (ii), it suffices to note that $f^{-1}(y)$ is nonempty, which follows from Proposition 8.6.5. □

Proposition 8.6.7 (Properties of Quasi-Separated Morphisms).

(i) *Any radical morphism $f : X \to Y$ (in particular, any monomorphism and any immersion) is quasi-separated.*

(ii) *If $f : X \to Y$ and $g : Y \to Z$ are two quasi-separated morphisms, $g \circ f$ is quasi-separated.*

(iii) *Let X, Y be two S-schemes and $f : X \to Y$ be a quasi-separated S-morphism. Then, for any base change $g : S' \to S$, the morphism $f_{(S')} : X_{(S')} \to Y_{(S')}$ is quasi-separated.*

(iv) *If $f : X \to X'$, $g : Y \to Y'$ are two quasi-separated S-morphisms, $f \times_S g$ is quasi-separated.*

(v) *If the composition $g \circ f$ of two morphisms $f : X \to Y$, $g : Y \to Z$ is quasi-separated, then f is quasi-separated; if moreover f is quasi-compact and surjective, g is also quasi-separated.*

(vi) *For a morphism $f : X \to Y$ to be quasi-separated, it is necessary and sufficient that f_{red} is quasi-separated.*

Proof. The assertion (i) follows from Proposition 8.5.26(i). To prove (iii), we may reduce to the case $Y = S$, and the assertion then follows from $\Delta_{f_{(S')}} = (\Delta_f)_{(S')}$ (Corollary 8.5.2) and Proposition 8.6.3.

For assertion (ii), consider the projections p, q of $X \times_Y X$; if $\pi : X \times_Y X \to Y$ is the structural morphism and $j = (p, q)_Z$, we have the following cartesian square (Eq. (8.5.1))

$$\begin{tikzcd} X \times_Y X \arrow[r, "j"] \arrow[d, "\pi"'] & X \times_Z X \arrow[d, "f \times_Z f"] \\ Y \arrow[r, "\Delta_g"] & Y \times_Z Y \end{tikzcd}$$

If g is quasi-separated then Δ_g is quasi-compact, so j is also quasi-compact by Proposition 8.6.3(iii). If f is quasi-separated, Δ_f is quasi-compact and so is $j \circ \Delta_f$, which equals to $\Delta_{g \circ f}$. With these, assertion (iv) then follows from (ii) and (iii).

Suppose now that $g \circ f$ is quasi-separated. Then with the preceding notations, $\Delta_{g \circ f} = j \circ \Delta_f$ is quasi-compact, so Δ_f is quasi-compact by Proposition 8.6.3(v) and f is then quasi-separated. If moreover f is quasi-compact and surjective, $f \times_{\mathbb{Z}} f$ is also quasi-compact by Proposition 8.6.3(iv), and we conclude that $\Delta_g \circ \pi \circ \Delta_f$ is quasi-compact. Since $\pi \circ \Delta_f = f$ is surjective, it follows from Proposition 8.6.3(v) that Δ_g is quasi-compact, so g is quasi-separated.

Finally, for a morphism $f : X \to Y$, consider the following diagram

$$\begin{tikzcd} X_{\text{red}} \arrow[r, "f_{\text{red}}"] \arrow[d, "j_X"'] & Y_{\text{red}} \arrow[d, "j_Y"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

where j_X and j_Y are surjective closed immersions, and so quasi-separated and quasi-compact. From the equality $f \circ j_X = j_Y \circ f_{\text{red}}$ and (v), we see f is quasi-separated if and only if f_{red} is quasi-separated. □

Corollary 8.6.8. *Let X and Y be schemes.*

(i) *If f is quasi-separated, any morphism $f : X \to Y$ is quasi-separated.*

(ii) *If Y is quasi-separated, for a morphism $f : X \to Y$ to be quasi-separated, it is necessary and sufficient that the scheme X is quasi-separated.*

(iii) *Let X be a quasi-compact and Y be quasi-separated. Then any morphism $f : X \to Y$ is quasi-compact.*

Proof. To show (i) we only need to note that any morphism $f : X \to Y$ is a $\mathbb{Z}$-morphism, and if X is quasi-separated, then for any morphism $f : X \to Y$ the composition $X \to Y \to \mathrm{Spec}(\mathbb{Z})$ is quasi-separated, so f is quasi-separated by Proposition 8.6.7(v). Similarly, assertion (ii) follows from Proposition 8.6.7(ii) and (v). Assertion (iii) follows from Proposition 8.6.3(v). □

Proposition 8.6.9. *Let $f : X \to Y$ be a morphism and (U_α) be a covering of Y by open subschemes that are quasi-separated. For f to be quasi-separated, it is necessary and sufficient that the open subschemes $f^{-1}(U_\alpha)$ is quasi-separated.*

Proof. The inverse image of U_α in $X \times_Y X$ is $X_\alpha \times_{U_\alpha} X_\alpha$, where $X_\alpha = f^{-1}(U_\alpha)$, and the restriction $X_\alpha \to X_\alpha \times_{U_\alpha} X_\alpha$ of Δ_f is just Δ_{f_α}, where f_α is the restriction $X_\alpha \to U_\alpha$ of f. Since quasi-compactness is local on target, we see f is quasi-separated if and only if each f_α is. But by hypothesis U_α is separated, so the conclusion follows from Corollary 8.6.8(ii). □

By Proposition 8.6.9, to verify a morphism is quasi-separated, it suffices to verify the quasi-separateness of some subschemes. This can be done by the following simple criteria:

Proposition 8.6.10. *Let X be a scheme and (U_α) be a covering of X formed by quasi-compact open subsets. Then the following conditions are equivalent:*

(i) *X is a quasi-separated scheme.*

(ii) *For any quasi-compact open subset U of X, the canonical injection $U \to X$ is quasi-compact (that is, U is retrocompact in X).*

(iii) *The intersection of two quasi-compact open subsets of X is quasi-compact.*

(iv) *For any couple of indices (α, β), the intersection $U_\alpha \cap U_\beta$ is quasi-compact.*

Proof. Properties (ii) and (iii) are equivalent by the definition of quasi-compactness. As a quasi-compact open is a finie union of affine open sets, for two quasi-compact open subsets U, V of X, $U \times_{\mathbb{Z}} V$ is a quasi-compact open subset of $X \times_{\mathbb{Z}} X$ (Corollary 8.3.2), with inverse image $U \cap V$ under Δ_X, hence (i) implies (iii). It is clear that (iii) implies (iv); finally, if (iv) holds, the $U_\alpha \times_{\mathbb{Z}} U_\beta$ form a covering of $X \times_{\mathbb{Z}} X$ by quasi-compact open sets and the inverse image of $U_\alpha \times_{\mathbb{Z}} U_\beta$ under Δ_X is $U_\alpha \cap U_\beta$, hence quasi-compact. It then follows that Δ_X is quasi-compact, so (iv) implies (i). □

Corollary 8.6.11. *Any locally Noetherian scheme X is quasi-separated, and any morphism $f : X \to Y$ is then quasi-separated.*

Proof. It suffices to note that any quasi-compact open subset of X is Noetherian, so X is quasi-separated by Proposition 8.6.10 and any morphism $f : X \to Y$ is quasi-separated by Proposition 8.6.9, since any open subscheme of X is again locally Noetherian. □

Proposition 8.6.12. *Let $f : X \to Y$ be a morphism and $g : Y' \to Y$ be a base change that is surjective and quasi-compact. Put $f' = f_{(Y')}$ and consider the following properties:*

(i) *quasi-compact;*

(ii) *quasi-separated.*

Then if $\mathcal{P}$ denotes one of these properties and f' possesses the property $\mathcal{P}$, then f possesses the property $\mathcal{P}$.

Proof. Let $g' : X' \to X$ be the cannical projection, which is surjective and quasi-compact (Proposition 8.3.28 and Proposition 8.6.3(iii)). If f' is quasi-compact, so is $g \circ f'$ and since $f \circ g' = g \circ f'$ we conclude that f is quasi-compact by Proposition 8.6.3(v).

Now assume that f' is quasi-separated. We have $X' \times_{Y'} X' = (X \times_Y X)_{(Y')}$ and $\Delta_{f'} = (\Delta_f)_{(Y')}$. The projection $X' \times_{Y'} X' \to X \times_Y X$ is quasi-compact and surjective by the same reasoning, and we can apply (i) to the morphism Δ_f. Since by hypothesis $\Delta_{f'}$ is quasi-compact, we conclude that Δ_f is quasi-compact, so f is quasi-separated. □

Proposition 8.6.13. *Let $f_i : X_i \to Y$ be a finite family of morphisms and $f : X \to Y$ be their coproduct. Then for f to be quasi-compact (resp. quasi-separated), it is necessary and sufficient that each f_i is quasi-compact (resp. quasi-separated).*

Proof. The assertion about quasi-compactness follows from definition. We also note that $X \times_Y X$ is the coproduct of that $X_i \times_Y X_j$, and Δ_f is the morphism that coincides with Δ_{f_i} on each X_i, so the assertion for quasi-separatedness also follows. □

Theorem 8.6.14. *Let X be a quasi-compact and quasi-separated scheme, $\mathscr{L}$ be an invertible $\mathscr{O}_X$-module, s be a section over X, X_s the open subset of $x \in X$ such that $s(x) \neq 0$, and $\mathscr{F}$ be a quasi-coherent $\mathscr{O}_X$-module.*

(a) *If $t \in \Gamma(X, \mathscr{F})$ is such that $t|_{X_s} = 0$, there exists $n > 0$ such that $t \otimes s^{\otimes n} = 0$.*

(b) *For any section $t \in \Gamma(X_s, \mathscr{F})$, there exists an integer $n > 0$ such that $t \otimes s^{\otimes n}$ can be extended to a section $t' \in \Gamma(X, \mathscr{F} \otimes \mathscr{L}^{\otimes n})$.*

Proof. As the space X is a finite union of affine opens U_i such that $\mathscr{L}|_{U_i}$ is isomorphic to $\mathscr{O}_X|_{U_i}$, we can assume that X is affine and $\mathscr{L} = \mathscr{O}_X$. The assertion (a) then follows from Theorem 8.1.21(iv).

Now let t be a sectin of $\mathscr{F}$ over X_s. Since $\mathscr{L}|_{U_i}$ is isomorphic to $\mathscr{O}_X|_{U_i}$, the restriction $(t \otimes s^{\otimes n})|_{U_i \cap X_s}$ is identified with $(t|_{U_i \cap X_s})(s|_{U_i \cap X_s})^n$ under the isomorphism $(\mathscr{F} \otimes \mathscr{L}^{\otimes n})|_{U_i} \cong \mathscr{F}|_{U_i}$. We then conclude from Theorem 8.1.21(iv) that there exists an integer $n \geq 0$ such that for each i, $(t \otimes s^{\otimes n})|_{U_i \cap X_s}$ extends to a section t_i of $\mathscr{F} \otimes \mathscr{L}^{\otimes n})$ over U_i. Let t_{ij} be the restriction of t_i to $U_i \cap U_j$; by definition we have $(t_{ij} - t_{ji})|_{X_s \cap U_i \cap X_j} = 0$. Since X is quasi-separated, $U_i \cap U_j$ is quasi-compact, so by Theorem 8.1.21(iv) there exists an integer $m \geq 0$ such that $(t_{ij} - t_{ji}) \otimes s^{\otimes m} = 0$. The sections $t_i \otimes s^{\otimes m}$ then glue together to give a section $t' \in \Gamma(X, \mathscr{F} \otimes \mathscr{L}^{\otimes (n+m)})$, which induces $t_i \otimes s^{\otimes m}$ over each U_i, and hence induces $t \otimes s^{\otimes (n+m)}$ over X_s. □

Corollary 8.6.15. *With the hypotheses of Theorem 8.6.14, consider the ring $A = \Gamma_*(\mathscr{L})$ and the graded A-module $M = \Gamma_*(\mathscr{F}; \mathscr{L})$ of type $\mathbb{Z}$. Then for each $s \in A_n$, there exists a canonical isomorphism $\Gamma(X_s, \mathscr{F}) \cong M_{(s)}$, where $M_{(s)} = (M_s)_0$ is the degree zero part of the localization M_s.*

Proof. With the notations of Theorem 8.6.14(b), we see that any element $t \in \Gamma(X_s, \mathscr{F})$ corresponds to an element t'/s^n in $M_{(s)}$, which is independent of the integer n and the chosen extension t', in view of Theorem 8.6.14(a). It is immediate that this defines a homomorphism, and is bijective. □

Corollary 8.6.16. *Let X be a quasi-compact and quasi-separated scheme and $\mathscr{F}$ a quasi-coherent $\mathscr{O}_X$-module. Then if $A = \Gamma(X, \mathscr{O}_X)$ and $M = \Gamma(X, \mathscr{F})$, the A_s-module $\Gamma(X_s, \mathscr{F})$ is canonically isomorphic to M_s.*

Proof. This is a special case of Corollary 8.6.15, by taking $\mathscr{L} = \mathscr{O}_X$. □

Proposition 8.6.17. *Let X be a quasi-compact scheme, $\mathscr{F}$ be a quasi-coherent $\mathscr{O}_X$-module of finite type, and $\mathscr{J}$ be a quasi-coherent ideal of $\mathscr{O}_X$ of finite type such that* $\mathrm{supp}(\mathscr{F})$ *is contained in* $\mathrm{supp}(\mathscr{O}_X/\mathscr{J})$. *Then there exists an integer $n \geq 0$ such that $\mathscr{J}^n \mathscr{F} = 0$.*

Proof. As X is a finite union of affine open subsets, we may assume that $X = \mathrm{Spec}(A)$ is affine. Then $\mathscr{F} = \widetilde{M}$ and $\mathscr{J} = \tilde{\mathfrak{a}}$, where M is a finitely generated A-module and $\mathfrak{a}$ is a finitely generated ideal of A, and

$$\mathrm{supp}(\mathscr{F}) = \mathrm{supp}(M) = V(\mathrm{Ann}(M)), \quad \mathrm{supp}(\mathscr{O}_X/\mathscr{J}) = \mathrm{supp}(A/\mathfrak{a}) = V(\mathfrak{a}).$$

By hypothesis we have $V(\mathrm{Ann}(M)) \subseteq V(\mathfrak{a})$, so $\mathfrak{a} \subseteq \sqrt{\mathrm{Ann}(M)}$. Since $\mathfrak{a}$ is finitely generated, there exists an integer $n \geq 0$ such that $\mathfrak{a}^n \subseteq \mathrm{Ann}(M)$, and therefore $\mathscr{J}^n \mathscr{F} = \widetilde{\mathfrak{a}^n M} = 0$. □

Corollary 8.6.18. *Under the hypothesis of Proposition 8.6.17, there exists a closed subscheme Y of X with underlying space* $\operatorname{supp}(\mathscr{O}_X/\mathscr{I})$ *such that, if $j: Y \to X$ is the canonical injection, we have $\mathscr{F} = j_*(j^*(\mathscr{F}))$.*

Proof. Note that the support of $\mathscr{O}_X/\mathscr{I}$ and $\mathscr{O}_X/\mathscr{I}^n$ are the same because if $\mathscr{I}_x = \mathscr{O}_{X,x}$, we also have $\mathscr{I}_x^n = \mathscr{O}_{X,x}$, and on the other hand we have $\mathscr{I}_n^x \subseteq \mathscr{I}_x$ for each $x \in X$. In view of Proposition 8.6.17, we may then suppose that $\mathscr{I}\mathscr{F} = 0$, so $\mathscr{F}$ is also an $(\mathscr{O}_X/\mathscr{I})$-module. If Y is the subscheme defined by $\mathscr{I}$, the conclusion is immediate. □

8.6.2 Morphisms of finite type and of finite presentation

Let $f: X \to Y$ be a morphism of schemes, x be a point of X, and $y = f(x)$. We say f is **of finite type** (resp. **of finite presentation**) **at the point x** if there exist an affine neighborhood V of y and an affine neighborhood U of x such that $f(U) \subseteq V$ and $\Gamma(U, \mathscr{O}_X)$ is a $\Gamma(V, \mathscr{O}_Y)$-algebra of finite type (resp. of finite presentation). We say f is **locally of finite type** (resp. **locally of finite presentation**) if it is of finite type (resp. of finite presentation) at every point of X. In this case, we say the Y-scheme X is locally of finite type (resp. locally of finite presentation) over Y.

Lemma 8.6.19. *Let $f: X \to Y$ be a morphism of schemes, x be a point of X, and $y = f(x)$. If there exist an affine neighborhood V of y and an affine neighborhood U of x such that $f(U) \subseteq V$ and $\Gamma(U, \mathscr{O}_X)$ is a $\Gamma(V, \mathscr{O}_Y)$-algebra of finite type (resp. of finite presentation), then for any affine open neighborhoods U' of x and V' of y, there exist affine open neighborhoods $U_1 \subseteq U \cap U'$ of x and $V_1 \subseteq V \cap V'$ of y, respectively of the form* $\operatorname{Spec}(\Gamma(U', \mathscr{O}_X)_{s'})$ *and* $\operatorname{Spec}(\Gamma(V', \mathscr{O}_Y)_{t'})$*, such that $f(U_1) \subseteq V_1$ and $\Gamma(U_1, \mathscr{O}_X)$ is a $\Gamma(V_1, \mathscr{O}_Y)$-algebra of finite type (resp. of finite presentation).*

Proof. Let $t' \in \Gamma(V', \mathscr{O}_Y)$ such that $V_1 = \operatorname{Spec}(\Gamma(V', \mathscr{O}_Y)_t)$ is an affine neighborhood of y contained in $V \cap V'$ and choose $s_0' \in \Gamma(U', \mathscr{O}_X)$ such that $U'' = \operatorname{Spec}(\Gamma(U', \mathscr{O}_X)_{s_0'})$ is a neighborhood of x contained in $U \cap U' \cap f^{-1}(V_1)$. There then exists $s \in \Gamma(U, \mathscr{O}_X)$ such that $U_1 = \operatorname{Spec}(\Gamma(U, \mathscr{O}_X)_s)$ is a neighborhood of x contained in U''. If s'' is the image of s in $\Gamma(U'', \mathscr{O}_X)$, we then have $U_1 = \operatorname{Spec}(\Gamma(U'', \mathscr{O}_X)_{s''})$, so there exists $s' \in \Gamma(U', \mathscr{O}_X)$ such that $U_1 = \operatorname{Spec}(\Gamma(U', \mathscr{O}_X)_{s'})$. Now $\Gamma(U_1, \mathscr{O}_X) = \Gamma(U, \mathscr{O}_X)[1/s]$, so it is a $\Gamma(U, \mathscr{O}_X)$-algebra of finite presentation, and a fortiori a $\Gamma(V, \mathscr{O}_Y)$-algebra of finite type (resp. of finite presentation). The homomorphism $\Gamma(V, \mathscr{O}_Y) \to \Gamma(U_1, \mathscr{O}_X)$ factors into

$$\Gamma(V, \mathscr{O}_X) \longrightarrow \Gamma(V_1, \mathscr{O}_Y) \longrightarrow \Gamma(U_1, \mathscr{O}_X)$$

If $\Gamma(U_1, \mathscr{O}_X)$ is identified with a quotient algebra $\Gamma(V, \mathscr{O}_Y)[T_1, \dots, T_n]/\mathfrak{a}$, then it is also identified with the quotient algebra $\Gamma(V_1, \mathscr{O}_Y)[T_1, \dots, T_n]/\mathfrak{b}$, where $\mathfrak{b}$ is the ideal generated by $\mathfrak{a}$. It then follows that $\Gamma(U_1, \mathscr{O}_X)$ is a $\Gamma(V_1, \mathscr{O}_Y)$-algebra of finite type (resp. finite presentation). □

Proposition 8.6.20. *If Y is locally Noetherian, then $f: X \to Y$ is locally of finite type if and only if it is locally of finite presentation. Moreover, if this holds, then X is also locally Noetherian.*

Proof. The first assertion is clear since we can take $\Gamma(V, \mathscr{O}_Y)$ to be Noetherian. The second one follows because $\Gamma(U, \mathscr{O}_X)$ is then also Noetherian. □

Proposition 8.6.21 (Properties of Morphisms Locally of Finite Type).

(i) *Any local immersion is locally of finite type.*

(ii) *If $f: X \to Y$ and $g: Y \to Z$ are two morphisms locally of finite type, then $g \circ f$ is locally of finite type.*

(iii) *If $f: X \to Y$ is an S-morphism locally of finite type, $f_{(S')}: X_{(S')} \to Y_{(S')}$ is locally of finite type for any base change $g: S' \to S$.*

(iv) *If $f: X \to X'$ and $g: Y \to Y'$ are two S-morphisms locally of finite type, $f \times_S g$ is locally of finite type.*

(v) *If the composition $g \circ f$ of two morphisms is locally of finite type, then f is locally of finite type.*

(vi) *If a morphism f is locally of finite type, so is f_{red}.*

Proof. Assertion (vi) follows from the fact that if a ring homomorphism $A \to B$ is of finite type, then so is $A/\mathfrak{n}(A) \to B/\mathfrak{n}(B)$. Now in view of Proposition 8.5.14, it suffices to prove (i), (ii) and (iii). If $j : X \to Y$ is a local immersion, for any $x \in X$ there exists an affine open neighborhood V of $j(x)$ in Y and an affine open neighborhood U of x such that the restriction $U \to V$ of j is a closed immersion. Then $\Gamma(U, \mathscr{O}_X)$ is a quotient ring of $\Gamma(V, \mathscr{O}_Y)$, and is therefore of finite type.

To establish (iii), we may assume that $Y = S$; let $p : X_{(S')} \to X$ and $q : X_{(S')} \to S$ be the canonical projections, x' be a point of $X_{(S')}$, and $x = p(x')$, $s' = q(x')$, $s = f(p(x')) = g(q(x'))$. Let V be an affine neighborhood of s in S and U be an affine neighborhood of x in X such that $f(U) \subseteq V$ and $\Gamma(U, \mathscr{O}_X)$ is a $\Gamma(V, \mathscr{O}_S)$-algebra of finite type. Let V' be an affine open neighborhood of s' in S' contained in $g^{-1}(V)$; then $p^{-1}(U) \cap q^{-1}(V)$ is an affine neighborhood of x' and is identified with $U \times_V V'$ (Corollary 8.3.2). This is an affine scheme with ring $\Gamma(U, \mathscr{O}_X) \otimes_{\Gamma(V, \mathscr{O}_S)} \Gamma(V', \mathscr{O}_{S'})$; as this is a $\Gamma(V', \mathscr{O}_{S'})$-algebra of finite type, we see (iii) follows.

Finally, to prove (ii), consider a point $x \in X$; there exists by hypothesis an affine open neighborhood W of $g(f(x))$ in Z and an affine open neighborhood V of $f(x)$ in Y such that $g(V) \subseteq W$ and $\Gamma(V, \mathscr{O}_Y)$ is a $\Gamma(W, \mathscr{O}_Z)$-algebra of finite type. By Lemma 8.6.19 there exists an affine open neighborhood $V' \subseteq V$ of $f(x)$ and an affine open neighborhood $U \subseteq f^{-1}(V')$ of x such that $\Gamma(V', \mathscr{O}_Y)$ is a $\Gamma(V, \mathscr{O}_Y)$-algebra of finite type and $\Gamma(U, \mathscr{O}_X)$ is a $\Gamma(V', \mathscr{O}_Y)$-algebra of finite type. We then conclude that $\Gamma(U, \mathscr{O}_X)$ is a $\Gamma(W, \mathscr{O}_Z)$-algebra of finite type, so (ii) follows. □

Corollary 8.6.22. *Let $f : X \to Y$ be a morphism locally of finite type. For any morphism $Y' \to Y$ such that Y' is locally Noetherian, $X \times_Y Y'$ is locally Noetherian.*

Proof. This follows from Proposition 8.6.20, since $f_{(Y')} : X \times_Y Y' \to Y'$ is locally of finite type by Proposition 8.6.21. □

Proposition 8.6.23. *Let $\rho : A \to B$ be a homomorphism of rings. For the corresponding morphism f : $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ to be locally of finite type, it is necessary and sufficient that B is an A-algebra of finite type.*

Proof. This condition is clearly sufficient. Conversely, assume that f is locally of finite type. Then by Lemma 8.6.19 there exists a finite cover of $\mathrm{Spec}(B)$ by open sets $D(g_i)$ (where $g_i \in B$) such that B_{g_i} is an A-algebra of finite type. Since the $D(g_i)$'s cover $\mathrm{Spec}(B)$, we see g_i generate the ring B, and it follows from **??** that B is of finite type over A. □

Proposition 8.6.24 (Properties of Morphisms Locally of Finite Presentation).

(i) *Any local isomorphism is locally of finite presentation.*

(ii) *If two morphisms $f : X \to Y$, $g : Y \to Z$ are locally of finite presentation, so is $g \circ f$.*

(iii) *If $f : X \to Y$ is an S-morphism locally of finite presentation, $f_{(S')} : X_{(S')} \to Y_{(S')}$ is locally of finite presentation for any base change $g : S' \to S$.*

(iv) *If $f : X \to X'$ and $g :\to Y'$ are two S-morphisms locally of finite presentation, $f \times_S g$ is locally of finite presentation.*

(v) *If the compositin $g \circ f$ of two morphisms $f : X \to Y$, $g : Y \to Z$ is locally of finite presentation and if g is locally of finite type, then f is locally of finite presentation.*

Proof. The first assertion is trivial, and to prove (ii), (iii), it suffices to replace the "algebra of finite type" in the proof of Proposition 8.6.21 by "algebra of finite presentation", and use Lemma 8.6.19. Again, assertion (iv) then follows from these. For (v), consider the cartesian square

$$\begin{tikzcd} X \arrow[r, "\Gamma_f"] \arrow[d, "f"'] & X \times_Z Y \arrow[d, "f \times_Z 1_Y"] \\ Y \arrow[r, "\Delta_g"] & Y \times_Z Y \end{tikzcd}$$

If we can show that Δ_g is locally of finite presentation, then it follows from (iii) that Γ_f is also locally of finite presentation. But f factors into

$$X \xrightarrow{\Gamma_f} X \times_Z Y \xrightarrow{p_2} Y$$

and we have $p_2 = (g \circ f) \times_Z 1_Y$, which is locally of finite presentation by (iv) since $g \circ f$ is. We then deduce that f is locally of finite presentation.

It then suffices to prove that is $g : Y \to Z$ is a morphism locally of finite type, then the diagonal morphism $\Delta_g : Y \to Y \times_Z Y$ is locally of finite presentation. To do this we may assume that $Z = \mathrm{Spec}(A)$, $Y = \mathrm{Spec}(B)$, and B is an A-algebra of finite type. The diagonal Δ_g corresponds to the homomorphism $\pi : B \otimes_A B \to B$ such that $\pi(x \otimes y) = xy$. Let $\mathfrak{I}$ be the kernel of π. We claim that $\mathfrak{I}$ is generated by the elements $1 \otimes s - s \otimes 1$, where s runs through a system of generators for the A-algebra B (this then proves the claim since B is of finite type over A). Now, it is clear that for any $x \in B$, we have $x \otimes 1 - 1 \otimes x \in \mathfrak{I}$; on the other hand, for $x, y \in B$, we have

$$x \otimes y = xy \otimes 1 + (x \otimes 1)(1 \otimes y - y \otimes 1)$$

If $\sum_i (x_i \otimes y_i) \in \mathfrak{I}$, we have by definition that $\sum_i x_i y_i = 0$, so

$$\sum_i (x_i \otimes y_i) = \sum_i (x_i \otimes 1)(1 \otimes y_i - y_i \otimes 1).$$

Moreover, if $x = st$, then

$$x \otimes 1 - 1 \otimes x = (s \otimes 1)(t \otimes 1 - 1 \otimes t) + (s \otimes 1 - 1 \otimes t)(1 \otimes t).$$

The claim then follows by indcution on the number of factors of a product in B. □

Corollary 8.6.25. *Let $g : Y \to Z$ be a morphism locally of finite type. Then the diagonal morphism $\Delta_g : Y \to Y \times_Z Y$ is locally of finite presentation.*

Proof. This is contained in the proof of Proposition 8.6.24. □

Proposition 8.6.26. *Let A be a ring, B be an A-algebra, $B' = A[T_1, \dots, T_n]$, and $\rho : B' \to B$ be a surjective homomorphism of A-algebras. Then for B to be an A-algebra of finite presentation, it is necessary and sufficient that the kernel $\mathfrak{a}$ of ρ is finitely generated in B'.*

Proof. The condition is sufficient by definition. Conversely, we note that the morphism $g : \mathrm{Spec}(B') \to \mathrm{Spec}(A)$ is locally of finite type; if B is an A-algebra of finite presentation, the morphism $f : \mathrm{Spec}(B) \to \mathrm{Spec}(B')$ corresponding to ρ and $g \circ f : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ are locally of finite presentation, so it follows from Proposition 8.6.24(v) that f is locally of finite presentation. Now it suffices to apply **??**. □

Corollary 8.6.27. *Let X, Y be two schemes, $j : X \to Y$ be an immersion, U an open subset of Y such that $j(X)$ is closed in U, and $\mathscr{I}$ the quasi-coherent ideal of $\mathscr{O}_U$ defining the closed subscheme of Y associated with j. For j to be locally of finite presentation, it is necessary and sufficient that $\mathscr{I}$ is a $\mathscr{O}_U$-module of finite type.*

Proof. Since the question is local, we can assume that X and Y are affine. The assertion then reduces to Proposition 8.6.26. □

Proposition 8.6.28. *Let $\rho : A \to B$ be a ring homomorphism. For the corresponding morphism $f : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ to be locally of finite presentation, it is necessary and sufficient that B is an A-algebra of finite presentation.*

Proof. The condition is clearly sufficient, so we only need to prove the necessity. If f is locally of finite presentation, then it follows from Proposition 8.6.23 that B is an A-algebra of finite type, so there exists a surjective homomorphism $\rho : C' = A[T_1, \dots, T_n] \to B$ of A-algebras. It then follows from Proposition 8.6.24(v) that the closed immersion $j : \mathrm{Spec}(B) \to \mathrm{Spec}(B')$ is locally of finite presentation, so, if $\mathfrak{b}$ is the kernel of ρ, the $\widetilde{B}'$-module $\tilde{\mathfrak{b}}$ is of finite type, and $\mathfrak{b}$ is therefore finitely generated in B' by Corollary 8.1.24. □

Proposition 8.6.29. *Let $\rho : A \to B$ be a homomorphism of rings such that B is a finite A-algebra. For B to be an A-algebra of finite presentation, it is necessary and sufficient that B is an A-module of finite presentation.*

Proof. There exists a finite A-algebra B' of finite presentation such that B' is a free A-module, and a surjective A-homomorphism of A-algebras $u : B' \to B$ (**??**); we have a surjective homomorphism $v : B'' = A[T_1, \dots, T_m] \to B'$ of A-algebras whose kernel is finitely generated. If $w = v \circ u : B'' \to B$ and $\mathfrak{b}$ (resp. $\mathfrak{a}$) is the kernel of w (resp. u), we have $\mathfrak{a} = v(\mathfrak{b})$ since v is surjective. If B is an A-algebra of finite presentation, $\mathfrak{b}$ is a finitely generated ideal of B'' by Proposition 8.6.26, so $\mathfrak{a}$ is a finitely generated ideal in B', hence a finitely generated A-module since B' is a finite A-algebra. As B' is a free A-module, B is then an A-module of finite presentation. The converse is proved in **??**. □

Proposition 8.6.30. *Let $f : X \to Y$ be a local immersion of finite type at a point of $y \in Y$. The following conditions are equivalent:*

(i) *f is an open map at y.*

(ii) *There exists an open neighborhood U of y in Y such that $f|_U$ is a nilimmersion over the open subscheme U.*

(iii) *There exists an open neighborhood U of y in Y such that $f|_U$ is a nilpotent immersion over the open subscheme U.*

Proof. It is clear that (iii) implies (ii) and (ii) implies (i). To show that (i) implies (iii), we can, by restricting f, suppose that f is a closed immersion from an affine open U of Y to an affine open V of X. Moreover, by choosing an irreducible component containing $f(y)$, we can further assume that V is irreducible. As f is a homeomorphism from U to $f(U)$, the hypothesis of (i) then implies that $f(U) = V$ since V is connected. If $V = \operatorname{Spec}(A)$, $U = \operatorname{Spec}(B)$, we have $B = A/\mathfrak{a}$, where $\mathfrak{a}$ is a nilideal of A (i.e. contained in the nilradical of A). On the other hand, in view of Lemma 8.6.19, we can, by replacing A with a fraction field A_s, suppose that B is an A-algebra of finite presentation. But then $\mathfrak{a}$ is a finitely generated ideal of A by Proposition 8.6.26, so it is nilpotent. □

Proposition 8.6.31. *Let $f : X \to Y$ be a morphism locally of finite type. Suppose that f admits a Y-section g, and for every $y \in Y$, the fiber $X_y = f^{-1}(y)$ is a $\kappa(y)$-scheme $\kappa(y)$-isomorphic to $\operatorname{Spec}(\kappa(y))$ (and necessarily has underlying space $\{g(y)\}$). Then f is an isomorphism.*

Proof. In fact, g is a nilimmersion of Y to X (Corollary 8.5.8), so the image $g(Y)$ has underlying space X and is defined by a nilideal $\mathscr{I}$ of $\mathscr{O}_X$. As $f \circ g = 1_Y$ and f is locally of finite type, g is locally of finite presentation by Proposition 8.6.24(v), so $\mathscr{I}$ is an ideal of finite type of $\mathscr{O}_X$ (Corollary 8.6.27). For any $y \in Y$, put $x = g(y)$, and consider the following exact sequence:

$$0 \longrightarrow \mathscr{I}_x \longrightarrow \mathscr{O}_{X,x} \underset{f_x^{\#}}{\overset{g_y^{\#}}{\rightleftarrows}} \mathscr{O}_{Y,y} \longrightarrow 0$$

The relation $f \circ g = 1_Y$ implies $f_x^{\#} \circ g_y^{\#} = 1$, so the above exact sequence splits. By tensoring with $\kappa(y) = \mathscr{O}_{Y,y}/\mathfrak{m}_y$, we then get an isomorphism $\mathscr{O}_{X,x}/\mathfrak{m}_y\mathscr{O}_{X,x} \cong \kappa(y) \oplus (\mathscr{I}_x/\mathfrak{m}_y\mathscr{I}_x)$. But the hypothesis on X_y implies that $\kappa(y)$-isomorphic to $\kappa(y)$, so we deduce that $\mathfrak{m}_y\mathscr{I}_x = \mathscr{I}_x$ and a fortiori $\mathfrak{m}_x\mathscr{I}_x = \mathscr{I}_x$. As $\mathscr{I}_x$ is a finitely generated ideal of $\mathscr{O}_{X,x}$, by Nakayama lemma we conclude $\mathscr{I}_x = 0$, so $\mathscr{I} = 0$ and f is an isomorphism. □

Corollary 8.6.32. *Let X, Y be two S-schemes and $f : X \to Y$ be an S-morphism. Suppose that X is locally of finite type over S. For each $s \in S$, let X_s, Y_s be the fiber of X and Y at the point s, and $f_s : X_s \to Y_s$ be the morphism induced by f under the base change $\operatorname{Spec}(\kappa(s)) \to S$. Then if for each $s \in S$, f_s is a monomorphism, f is a monomorphism.*

Proof. If f_s is a monomorphism, so is $f_y : X_y \to \operatorname{Spec}(\kappa(y))$. By hypothesis f is locally of finite type (Proposition 8.6.21(v)), so we can limit ourselves to the case $Y = S$. To see that f is a monomorphism, it suffices to prove that the first projection $p : X \times_S X \to X$ is an isomorphism (Proposition 8.5.5). Now the hypothesis on f_s implies that the projections $p_s : X_s \otimes_{\kappa(s)} X_s \to X_s$ are isomorphisms for all $s \in S$. Since p admits an S-section, namely the diagonal Δ_f, it follows from Proposition 8.6.31 that p is an isomorphism. □

We now come to the definition of *morphisms of finite type*, which can be seen as a global version of morphisms locally of finite type. Briefly speaking, the notion of finite type concerns the "global finiteness" of a morphism: we have the following definition and proposition.

Proposition 8.6.33. *Let $f : X \to Y$ be a morphism and (U_α) be a covering of Y by affine opens. The following conditions are equivalent:*

(i) *f is locally of finite type and quasi-compact.*

(ii) *For each α, $f^{-1}(U_\alpha)$ is a finite union of affine opens $V_{\alpha,i}$ such that the ring $\Gamma(V_{\alpha,i}, \mathscr{O}_X)$ is a $\Gamma(U_\alpha, \mathscr{O}_Y)$-algebra of finite type.*

(iii) *For any affine open U of Y, $f^{-1}(U)$ is a finite union of affine opens V_j such that $\Gamma(V_j, \mathscr{O}_X)$ is a $\Gamma(U, \mathscr{O}_X)$-algebra of finite type.*

*We say the morphism f is **of finite type** if it satisfies the above equivalent conditions. In this case, we say X is of finite type over Y.*

Proof. It is clear that (iii)⇒(ii)⇒(i). To prove that (i) implies (iii), we may assume that $Y = U$ is affine; then X is quasi-compact, hence is a finte union of affine opens V_j such that the restriction $V_j \to U$ of f is locally of finite type. By Proposition 8.6.23, we see $\Gamma(V_j, \mathscr{O}_X)$ is a $\Gamma(U, \mathscr{O}_X)$-algebra of finite type. □

Proposition 8.6.34. *Let $f : X \to Y$ be a morphism of finite type. If Y is Noetherian (resp. locally Noetherian), so is X.*

Proof. This follows from Proposition 8.6.20 and **??**. □

Proposition 8.6.35 (Properties of Morphisms of Finite Type).

(i) *Any quasi-compact immersion is of finite type.*

(ii) *The composition of two morphisms of finite type is of finite type.*

(iii) *If $f : X \to Y$ is an S-morphism of finite type, $f_{(S')} : X_{(S')} \to Y_{(S')}$ is of finite type for any base change $S' \to S$.*

(iv) *If $f : X \to X'$ and $g : Y \to Y'$ are two S-morphisms of finite type, $f \times_S g$ is of finite type.*

(v) *If the composition $g \circ f$ of two morphisms $f : X \to Y$, $g : Y \to Z$ is of finite type and if g is quasi-separated or X is Noetherian, then f is of finite type.*

(vi) *If a morphism f is of finite type, so is f_{red}.*

Proof. This follows directly from Proposition 8.6.21 and Proposition 8.6.3. □

Corollary 8.6.36. *Let $f : X \to Y$ be a morphism of finite type. For any morphism $Y' \to Y$ such that Y' is Noetherian, $X \times_Y Y'$ is Noetherian.*

Proof. This follows from Proposition 8.6.35(iii) and Proposition 8.6.34. □

Corollary 8.6.37. *Let X be a scheme of finite type over a locally Noetherian scheme S. Then any S-morphism $f : X \to Y$ is of finite type.*

Proof. The morphism f is locally of finite type by Proposition 8.6.21(v). To see it is quasi-compact, we can suppose that S is Noetherian. If $\varphi : X \to S$ and $\psi : Y \to S$ are the structural morphism, we have $\varphi = \psi \circ f$ and X is Noetherian by Proposition 8.6.34, so f is of finite type, by Proposition 8.6.35(v). □

Let X and Y be two schemes. We say a morphism $f : X \to Y$ is **of finite presentation** if it satisfies the following conditions:

(i) f is locally of presentation;

(ii) f is quasi-compact;

(iii) f is quasi-separated.

In this case, we say X is **of finite presentation over** Y, or is an Y**-scheme of finite presentation**. It is clear that condition (iii) is automatic if f is separated, or if X is locally Noetherian. If Y is locally Noetherian, then again, f is of finite type if and only if it is of finite presentation, and in this case X is also locally Noetherian.

Proposition 8.6.38 (Properties of Morphisms of Finite Presentation).

(i) *Any quasi-compact immersion that is locally of finite presentation (in particular any quasi-compact open immersion) is of finite presentation.*

(ii) *The composition of two morphisms of finite presentation is of finite presentation.*

(iii) *If $f : X \to Y$ be an S-morphism of finite presentation, $f_{(S')} : X_{(S')} \to Y_{(S')}$ is of finite presentation for any base change $S' \to S$.*

(iv) *If $f : X \to X'$ and $g : Y \to Y'$ are two S-morphisms of finite presentation, $f \times_S g$ is of finite presentation.*

(v) *If the composition $g \circ f$ of two morphisms $f : X \to Y$, $g : Y \to Z$ is of finite presentation and if g is quasi-separated and locally of finite type, then f is of finite presentation.*

Proof. This follows from Proposition 8.6.3, Proposition 8.6.7, and Proposition 8.6.24. □

It follows from Proposition 8.6.38(iii) that if f is a morphism of finite presentation and U is an open subset of Y, the restriction $f^{-1}(U) \to U$ of f is also of finite presentation. conversely, let (U_α) be a covering of Y by affine opens and suppose that the restriction $f^{-1}(U_\alpha) \to U_\alpha$ of f is a morphism of finite presentation. Then it follows that f is of finite presentation, since f is clearly of finite presentation and quasi-compact and it is quasi-separated by Proposition 8.6.9.

If X is a quasi-separated scheme, any morphism $f : X \to Y$ is quasi-separated by Corollary 8.6.8. Therefore, if f is quasi-compact and locally of finite presentation, it is of finite presentation.

Corollary 8.6.39. *Let $\rho : A \to B$ be a ring homomorphism. For the corresponding morphism $f : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ to be of finite type (resp. of finite presentation), it is necessary and sufficient that B is an A-algebra of finite type (resp. of finite presentation).*

Proof. Since any morphism of affine schemes is quasi-compact and separated, this follows from Proposition 8.6.23 and Proposition 8.6.28. □

Remark 8.6.40. In the definition of morphisms of finite presentation, the condition (iii) is not a concequence of the other two conditions. For example, let Y be a non-Noetherian affine scheme and let Y be a non-quasi-compact open subset of Y (an example for this is $Y = \mathrm{Spec}(k[x_1, x_2, \dots])$ and $U = Y - \{0\}$, cf. Example 8.6.2). Let X be the scheme obtained by glueing two schemes Y_1, Y_2 isomorphic to Y along the open sets U_1, U_2 corresponding to U, so that X is the union of two affine opens isomorphic to Y_1, Y_2, respectively, and $Y_1 \cap Y_2 = U$. Let $f : X \to Y$ be the morphism which coincides with the canonical isomorphism $Y_i \to Y$ on each Y_i. Then it is clearly locally of finite presentation, and is quasi-compact since the inverse image of a quasi-compact open of Y is the union of its images in Y_1 and Y_2; but as $Y_1 \cap Y_2 = U$ is not quasi-compact, it is not quasi-separated by Proposition 8.6.10 and Corollary 8.6.8(ii).

Proposition 8.6.41. *Let $f_i : X_i \to Y$ be a finite family of morphisms and $f : X \to Y$ be their coproduct, where $X = \coprod_i X_i$. Then for f to be of finite type (resp. finite presentation), it s necessary and sufficient that each f_i is.*

Proof. In view of Proposition 8.6.13, it suffices to note that the same assertion holds for morphisms of locally finite type and of finite presentation. □

8.6.3 Algebraic schemes

We say a K-scheme is **algebraic** (resp. **locally algebraic**) if it is of finite type over K (resp. locally of finite type over K). The field K is called the **base field** of X.

Proposition 8.6.42. *Let K be a field. A locally algebraic (resp. algebraic) K-scheme is locally Noetherian (Noetherian). Moreover X is a Jacobson scheme and a point $x \in X$ is closed if and only if $\kappa(x)$ is a finite extension of K.*

Proof. The first assertion is clear, and X is Jacobson by **??**. To characterize close points in X, we note that for a point $x \in X$ to be closed, it is necessary and sufficient that for an open covering (U_α) of X, s is closed in the U_α containing it. As there is a covering of X by affine opens U_α such that $\Gamma(U_\alpha, \mathscr{O}_X)$ is a K-algebra of finite type, we can then assume that $X = \mathrm{Spec}(A)$ where A is a K-algebra of finite type. The closed points of X are then maximal ideals of A; but then $A/\mathfrak{p}_x = \kappa(x)$ is a finite extension by **??**. Conversely, if $\kappa(x)$ is a finite K-algebra, so is the ring $A/\mathfrak{p}_x \subseteq \kappa(x)$, and as an integral K-algebra is also a field (**??**), we have $A/\mathfrak{p}_x = \kappa(x)$, so x is closed. □

Corollary 8.6.43. *Let K be an algebraically closed field and X be a locally algebraic K-scheme. Then the closed points of X are exactly the rational points of X over K, which are identified with the K-points of X with values in K.*

Proposition 8.6.44. *Let K be a field and X be a locally algebraic scheme over K. Then the following conditions are equivalent:*

(i) *X is Artinian.*

(ii) *The underlying space of X has only finitely many closed points.*

(iii) *The underlying space of X is finite.*

(iv) *X is isomorphic to* Spec(A) *where A is K-algebra of finite dimension.*

If X is algebraic over K, then these conditions are equivalent to the following:

(v) *The underlying space of X is discrete.*

(vi) *The points of X are all closed.*

Proof. We see (i) implies any other conditions, and (v) or (vi) implies (i) if X is Noetherian. Moreover, it is clear that (iv) implies (i), since a finite dimensional K-algebra is Artianian. In the condition of (ii), the set X_0 of closed points of X is then finite, closed and very dense in X, whence equal to X and X is therefore Artinian, since it is then Noetherian. □

Corollary 8.6.45. *Let K be a field, X be a locally algebraic K-scheme, and x be a point of X. The following conditions are equivalent:*

(i) *x is isolated in X;*

(ii) *x is closed in X and $\mathscr{O}_{X,x}$ is Artinian;*

(iii) *$\mathscr{O}_{X,x}$ is a finite dimensional K-algebra.*

Proof. If $\mathscr{O}_{X,x}$ is a finite dimensional K-algebra, so is the residue field $\kappa(x) = \mathscr{O}_{X,x}/\mathfrak{m}_x$, so (iii) implies (ii) in view of Proposition 8.6.42. The local ring $\mathscr{O}_{X,x}$ is Artinian signifies that x is a maximal point of X, since a Noetherian local ring is Artinian if and only if it has a unique primei deal. If x is moreover closed, the set $\{x\}$ is closed and stable under generalization, hence open ([?] new, 0_{I}, 2.1.5), and this proves x is isolated in X. Finally, if x is isolated in X, there exists an affine open neighborhood U of x such that $U = \{x\}$ and $\Gamma(U, \mathscr{O}_X)$ is a finite type K-algebra. But then $\Gamma(U, \mathscr{O}_X) = \mathscr{O}_{X,x}$, so (i) implies (iii). □

If the conditions in Proposition 8.6.44, we say X is a finite scheme over K, or a finite K-scheme. For such a scheme, we denote by $\dim_K(X)$ the dimension of the ring $\Gamma(X, \mathscr{O}_X)$ over K. If X and Y are two finite schemes over K, we have

$$\dim_K(X \amalg Y) = \dim_K(X) + \dim_K(Y), \quad \dim_K(X \times_K Y) = \dim_K(X)\dim_K(Y).$$

Corollary 8.6.46. *Let X be a finite scheme over a field K. For any extension K' of K, $X \otimes_K K'$ is a finite scheme over K', with $\dim_{K'}(X') = \dim_K(X)$.*

Proof. In fact, if $X = \operatorname{Spec}(A)$, we have $[A \otimes_K K' : K'] = [A : K]$, whence the claim. □

Corollary 8.6.47. *Let X be a finite scheme over a field K. We put*

$$n = \sum_{x \in X} [\kappa(x) : K]_s$$

Then, for any algebraically closed extension Ω of K, the underlying space of $X \otimes_K \Omega$ has exactly n points, which are identified with the Ω-valued points of X.

Proof. By Proposition 8.2.33, we can assume that $A = \Gamma(X, \mathscr{O}_X)$ is local; let $\mathfrak{m}$ be the maximal ideal of A, $L = A/\mathfrak{m}$ the residue field, which is a finite algebraic extension of K by Proposition 8.6.44. The Ω-points of X correspond bijectively to Ω-sections of $X \otimes_K \Omega$, and to the closed points of $X \otimes_K \Omega$ by Corollary 8.6.43, and finally to the points of this Artinian scheme (Proposition 8.2.33). They also correspond to K-homomorphisms of L into Ω, and the assertion then follows from the definition of separable degree. □

The number n defined in Corollary 8.6.47 is called the **separable rank** of A (or X) over K, or the **geometric number of points** of X. This is also the number of elements in $X(\Omega)_K$. It follows from this definition that for any extension K' of K, $X \otimes_K K'$ has the same geometric number of points as X. If we denote this number by $n(X)$, it is clear that, if X and Y are two finite schemes over K, we have

$$n(X \amalg Y) = n(X) + n(Y), \quad n(X \times_K Y) = n(X)n(Y).$$

Proposition 8.6.48. *Let $f : X \to Y$ be a morphism locally of finite type (resp. of finite type). Then, for any $y \in Y$, the fiber $X_y = f^{-1}(y)$ is a locally algebraic (resp. algebraic) scheme over $\kappa(y)$, and for each $x \in X_y$, $\kappa(x)$ is a extension of $\kappa(y)$ of finite type.*

Proof. As $X_y = X \otimes_Y \kappa(y)$, this follows from Proposition 8.6.21(iii) and Proposition 8.6.35(iii). □

Proposition 8.6.49. *Let $f : X \to Y$, $g : Y' \to Y$ be two morphisms, and put $X' = X \times_Y Y'$, $f' = f_{(Y')} : X' \to Y'$. Let $y' \in Y'$ and $y = g(y')$. If the fiber $X_y = f^{-1}(y)$ is a finite scheme over $\kappa(y)$, then the fiber $X'_{y'} = f'^{-1}(y')$ is a finite scheme over $\kappa(y')$, and we have*

$$\dim_{\kappa(y')}(X'_{y'}) = \dim_{\kappa(y)}(X_y), \quad n(X'_{y'}) = n(X_y).$$

Proof. This follows from the observation $X'_{y'} = X_y \otimes_{\kappa(y)} \kappa(y')$. □

Proposition 8.6.48 shows that the morphisms of finite type (resp. locally of finite type) correspond intuitively to "algebraic families of algebraic varieties (resp. locally algebraic)", where Y plays the role of "parameters." Because of this, these morphisms are of significant geometric interests. The morphisms which are not locally of finite type will intervene them by the process of "base change", for example by localization and completion.

8.6.4 Local determination of morphisms

Proposition 8.6.50. *Let X and Y be S-schemes, $x \in X$, $y \in Y$ be points lying over the same point $s \in S$.*

(a) *Suppose that Y is locally of finite type over S at y. Then if two S-morphisms f, g from X to Y are such that $f(x) = g(x) = y$ and the $\mathscr{O}_{S,s}$-homomorphisms $f_x^\#$ and $g_x^\#$ from $\mathscr{O}_{Y,y}$ to $\mathscr{O}_{X,x}$ coincide, then f and g coincide in an open neighborhood of x.*

(b) *Suppose that Y is locally of finite presentation over S at y. Then, for any $\mathscr{O}_{X,x}$-homomorphism $\varphi : \mathscr{O}_{Y,y} \to \mathscr{O}_{X,x}$ there exists an open neighborhood U of x in X and an S-morphism f from U to Y such that $f(x) = y$ and $f_x^\# = \varphi$.*

Proof. We first consider case (a). The question is local over S, X and Y, so we can suppose that S, X, Y are affine with rings A, B, C, respectively. The morphisms f and g then correspond to A-homomorphisms ρ, σ from C to B such that $\rho^{-1}(\mathfrak{p}_x) = \sigma^{-1}(\mathfrak{p}_x) = \mathfrak{p}_y$, and the homomorphisms ρ_x and σ_x from C_y to B_x, deduced by ρ and σ, coincide. We can morphism suppose that C is an A-algebra of finite type. Let $(c_i)_{1 \le i \le n}$ be generators of the A-algebra C, and put $b_i = \rho(c_i)$, $b'_i = \sigma(c_i)$. By hypothesis, we have $b_i/1 = b'_i/1$ in the ring B_x. This means there exist elements $s_i \in B - \mathfrak{p}_x$ such that $s_i(b_i - b'_i) = 0$ for each i, and we can evidently choose one $s \in B - \mathfrak{p}_x$ for all i. We then conclude that $b_i/1 = b'_i/1$ for each i in the ring B_s; if $i_s : B \to B_s$ is the canonical homomorphism, we then have $i_s \circ \rho = i_s \circ \sigma$, so the restriction of f and g on $D(s)$ are identical.

We now come to case (b). Again we can suppose that S, X, Y are affine with rings A, B, C. Put $\mathfrak{p} = \mathfrak{p}_x$, $\mathfrak{q} = \mathfrak{p}_y$, and let $\varphi : C_\mathfrak{q} \to B_\mathfrak{p}$ be an A-homomorphism. We then get an A-homomorphism $\rho : C \to C_\mathfrak{q} \xrightarrow{\varphi} B_\mathfrak{p}$. Since we can consider $B_\mathfrak{p}$ as an inductive limit of the filtered system of A-algebras B_s, where s runs through elements of $B - \mathfrak{p}$, and C is by hypothesis an A-algebra of finite presentation, we deduce from **??** that there exists $s \notin \mathfrak{p}$ and an A-homomorphism $\sigma : C \to B_s$ whose canonical image is ρ, that is, the following diagram is commutative

$$\begin{tikzcd} C \arrow[r, "\sigma"] \arrow[d] & B \arrow[d] \\ C_\mathfrak{q} \arrow[r, "\varphi"] & B_\mathfrak{p} \end{tikzcd} \tag{8.6.1}$$

It then suffices to take $U = D(s)$ and let f be the morphism induced by σ. □

Corollary 8.6.51. *Under the hypotheses of Proposition 8.6.50(ii), if moreover X is locally of finite type over S at the point of x, we can choose f to be of finite type.*

Proof. To see this, we can assume that S, X, Y are affine, so that the structural morphisms $X \to S$ and $Y \to S$ are respectively of finite type and of finite presentation; then the results follows from Lemma 8.6.19 and Proposition 8.6.21(iv). □

Corollary 8.6.52. *Retain the hypotheses of Proposition 8.6.50(ii) and suppose that Y is integral and $\varphi : \mathscr{O}_{Y,y} \to \mathscr{O}_{X,x}$ is injective. Then we can suppose that U is affine and f factors into*

$$U \xrightarrow{g} V \longrightarrow Y$$

where V is an affine open containing y and $g : U \to V$ is a morphism corresponding to a injective homomorphism $\Gamma(V, \mathscr{O}_Y) \to \Gamma(U, \mathscr{O}_X)$.

Proof. In fact, with the notations of Proposition 8.6.50(ii), C is integral and the canonical homomorphism $C \to C_g$ is then injective; the result then follows from the diagram (8.6.1), since σ is injective. □

Proposition 8.6.53. *Let $f : X \to Y$ be a morphism, x be a point of X and $y = f(x)$.*

(a) *Suppose that f is locally of finite type at the point x. For f to be a local immersion at the point x, it is necessary and sufficient that $f_x^\# : \mathscr{O}_{Y,y} \to \mathscr{O}_{X,x}$ is surjective.*

(b) *Suppose that f is locally of finite presentation at the point x. For f to be a local isomorphism at the point x, it is necessary and sufficient that $f_x^\# : \mathscr{O}_{Y,y} \to \mathscr{O}_{X,x}$ is an isomorphism.*

Proof. We only need to prove the sufficiency, and we first consider case (b). Then X is locally of finite presentation over y at the point x, so by Proposition 8.6.50(i) and (ii), there exists an open neighborhood V of y and a morphism $g : V \to X$ such that $g \circ f$ (resp. $f \circ g$) is defined and coincide with the identity on an open neighborhood W of x (resp. an open neighborhood T of y). Put $T' = T \cap g^{-1}(w)$ and $W' = f^{-1}(T')$, we then verify that $g(T') \subseteq W'$, $f(W') \subseteq T'$ and $(g \circ f)|_{W'} = 1_{W'}$, whence f is a local isomorphism.

For (a), we can assume that X and Y are affine, with ring A and B. Then f corresponds to a homomorphism $\varphi : B \to A$ of finite type; we have $\varphi^{-1}(\mathfrak{p}_x) = \mathfrak{p}_y$, and the homomorphism $\varphi_x : B_y \to A_x$ induced by φ is surjective. Let $(t_i)_{1 \le i \le n}$ be a system of generators of the B-algebra A. The hypothesis on φ_x then implies $t_i/1 = \varphi(b_i)/\varphi(c)$ in the ring A_x, where $b_i \in B$ and $c \in B - \mathfrak{p}_y$, so we can find $a \in A - \mathfrak{p}_x$ such that

$$a(t_i\varphi(c) - \varphi(b_i)) = 0.$$

If we put $g = a\varphi(c)$, then $t_i/1 = a\varphi(b_i)/g$ in the ring A_g. Now there exists by hypothesis a polynomial $Q(X_1, \dots, X_n)$ with coefficients in $\varphi(B)$ such that $a = Q(t_1, \dots, t_n)$; write $Q(X_1/T, \dots, X_n/T) = P(X_1, \dots, X_n, T)/T^m$, where P is a polynomial of degree m. In the ring A_g, we then have

$$\begin{aligned} a/1 &= Q(t_1/1, \dots, t_n/1) = Q(a\varphi(b_1)/g, \dots, a\varphi(b_n)/g) \\ &= a^m P(\varphi(b_1), \dots, \varphi(b_n), \varphi(c))/g^m = a^m \varphi(d)/g^m \end{aligned}$$

where $d \in B$. Since $g/1 = (a/1)(\varphi(c)/1)$ is invertible in A_g by definition, so is $a/1$ and $\varphi(c)/1$, and we can then write $a/1 = (\varphi(d)/1)(\varphi(c)/1)^{-m}$. We conclude that $\varphi(d)/1$ is also invertible in A_g. Put $h = cd$, as $\varphi(h)/1$ is invertible in A_g, the composed homomorphism $B \to A \to A_g$ factors into

$$\begin{tikzcd} B \arrow[r, "\varphi"] \arrow[dr] & A \arrow[r] & A_g \\ & B_h \arrow[ur, "\gamma"'] & \end{tikzcd}$$

We claim that γ is surjective. For this, it suffices to verify that the image of B_h in A_g contains $t_i/1$ and $1/g$. Now we have

$$1/g = (\varphi(c)/1)^{m-1}(\varphi(d)/1)^{-1} = \gamma(c^m/h)$$

and $a/1 = \gamma(d^{m+1}/h^m)$, so $(a\varphi(b_i))/1 = \gamma(b_i d^{m+1}/h^m)$, and as $t_i/1 = (a\varphi(b_i)/1)(g/1)^{-1}$, we conclude our assertion. The choice of h implies $f(D(g)) \subseteq D(h)$, and the restriction of f to $D(g)$ is induced by γ. Since γ is surjective, this restriction is a closed immersion from $D(g)$ to $D(h)$, so f is a local immersion at x. □

Corollary 8.6.54. *With the notations of Proposition 8.6.53, suppose that f is a local immersion at the point x and is locally of finite presentation at x. For f to be open at x, it is necessary and sufficient that the kernel of $f_x^\#$ is nilpotent.*

Proof. In view of Proposition 8.6.30, it suffices to prove the sufficiency of the condition. We can suppose that $Y = \mathrm{Spec}(A)$ and $X = \mathrm{Spec}(A/\mathfrak{n})$, where $\mathfrak{n}$ is a finitely generated ideal of A (Corollary 8.6.27), and by hypothesis $\mathfrak{n}_x$ is nilpotent. If $(s_i)_{1\leq i\leq n}$ is a system of generators of $\mathfrak{n}$, we then have $s_i^m/1 = 0$ in A_x for an integer m and all i. Then there exists $t \in A - \mathfrak{p}_x$ such that $ts_i^m = 0$ for all i, so $(s_i/1)^m = 0$ in the ring A_t. This shows $\mathfrak{n}_t$ is nilpotent, whence the conclusion. □

8.6.5 Direct image of quasi-coherent sheaves

Proposition 8.6.55. *Let X, Y be two schemes and $f : X \to Y$ be a quasi-compact and quasi-separated morphism. Then for any quasi-coherent $\mathscr{O}_X$-module, $f_*(\mathscr{F})$ is quasi-coherent.*

Proof. Since the question is local over Y, we can assume that Y is affine. If f is quasi-compact, X is then a union of finitely many open affines X_i, and in view of Proposition 8.6.7(ii), X is a quasi-separated scheme, hence the intersections $X_i \cap X_j$ are qausi-compact (Proposition 8.6.10).

We first assume that each intersection $X_i \cap X_j$ is affine. Put $\mathscr{F}_i = \mathscr{F}|_{X_i}$, $\mathscr{F}_{ij} = \mathscr{F}|_{X_i\cap X_j}$ and let $\mathscr{F}_i'$ and $\mathscr{F}_{ij}'$ be the inverse image of $\mathscr{F}_i$ and $\mathscr{F}_j$ under the restriction of f to X_i and to $X_{i\cap X_j}$. We see that $\mathscr{F}_i'$ and $\mathscr{F}_{ij}'$ are quasi-coherent (**??**). We define a homomorphism

$$u : \bigoplus_i \mathscr{F}_i' \to \bigoplus_{i,j} \mathscr{F}_{ij}'$$

such that $f_*(\mathscr{F})$ is the kernel of u, and this then implies $f_*(\mathscr{F})$ is quasi-coherent by Corollary 8.1.6. For this, it suffices to define u as a homomorphism of presheaves, so for each open subset $W \subseteq Y$, we need a homomorphism

$$u_W : \bigoplus_i \Gamma(f^{-1}(W) \cap X_i, \mathscr{F}) \to \bigoplus_{ij} \Gamma(f^{-1}(W) \cap X_i \cap X_j, \mathscr{F})$$

so as to satisfy the compatibility for the restrictions to a smaller open subset. If for any section s_i of $\mathscr{F}$ over $f^{-1}(W) \cap X_i$, we denote by s_{ij} its restriction to $f^{-1}(W) \cap X_i \cap X_j$, we set

$$u_W((s_i)) = (s_{ij} - s_{ji})$$

and the compatibility is evident. To identity the kernel $\mathscr{R}$ of u, we define a homomorphism $v : f_*(\mathscr{F}) \to \mathscr{R}$ which sends a section s of $\mathscr{F}$ over $f^{-1}(W)$ to the family (s_i), where s_i is the restriction of s to $f^{-1}(W) \cap X_i$. By the sheaf axoims of $\mathscr{F}$, it is clear that v is bijective, which proves the assertion in this case.

In the general case, the same reasoning can be applied if we can show that each $\mathscr{F}_{ij}'$ is quasi-coherent. But by hypotheses, $X_i \cap X_j$ is a union of finitely many affine opens X_{ijk}, and since each X_{ijk} are affine open subschemes of the affine scheme X_i, their intersections are again affine (affine schemes are separated), so we can apply the previous arguments to conclude that $\mathscr{F}_{ij}'$ is quasi-coherent, and the proof is then complete. □

Remark 8.6.56. We should note that even if X and Y are Noetherian schemes and $f : X \to Y$ is a morphism of finite type, the direct image $f_*(\mathscr{F})$ of a coherent $\mathscr{O}_X$-module $\mathscr{F}$ is in general not coherent. For example, let Y be the spectrum of a field K, $X = \mathrm{Spec}(K[T])$, and choose $\mathscr{F} = \mathscr{O}_X$.

Proposition 8.6.57. *Let $f : X \to Y$ be a quasi-compact and quasi-separated morphism. Let $(\mathscr{F}_\lambda)$ be a inductive system of quasi-coherent $\mathscr{O}_X$-modules and $\mathscr{F} = \varinjlim \mathscr{F}_\lambda$ be the inductive limit. Then $\varinjlim f_*(\mathscr{F}_\lambda) \cong f_*(\mathscr{F})$.*

Proof. For each affine open subset W of Y and any λ, we have a canonical homomorphism

$$u_{W,\lambda} : (f_*(\mathscr{F}_\lambda))|_W \to f_*(\mathscr{F})|_W$$

whence a canonical homomorphism

$$u_W : (\varinjlim f_*(\mathscr{F}_\lambda))|_W \to f_*(\mathscr{F})|_W$$

and this hommorphism is compatible with restrictions. Since f is quasi-compact and quasi-separated, by Proposition 8.6.55 $\varinjlim f_*(\mathscr{f}_\lambda)$ and $f_*(\mathscr{F})$ are quasi-coherent. Moreover, the homomorphism u_W corresponds by taking global section over W to the canonical homomorphism

$$\varphi_W : \Gamma(W, \varinjlim f_*(\mathscr{F}_\lambda)) \to \Gamma(f^{-1}(W), \mathscr{F}).$$

Since f is quasi-compact and quasi-separated, by (Stack Project. Lemma 6.29.1) we have

$$\Gamma(W, \varinjlim f_*(\mathscr{F}_\lambda)) = \varinjlim \Gamma(W, f_*(\mathscr{F}_\lambda)) = \varinjlim \Gamma(f^{-1}(W), \mathscr{F}_\lambda) = \Gamma(f^{-1}(W), \mathscr{F})$$

and φ_W is therefore the identity homomorphism. By Corollary 8.1.22, it then follows that u_W is an isomorphism for each W, and the assertion then follows. □

8.6.6 Extension of quasi-coherent sheaves

Let X be a topological space and $\mathscr{F}$ be a sheaf of sets (resp. of groups, of rings) over X. Let U be an open subset of X with $j : U \to X$ the canonical injection, and let $\mathscr{G}$ be a subsheaf of $\mathscr{F}|_U = j^{-1}(\mathscr{F})$. As the functor j_* is left exact, $j_*(\mathscr{G})$ is then a subsheaf of $j_*(j^{-1}(\mathscr{F}))$. Let $\rho_{\mathscr{F}} : \mathscr{F} \to j_*(j^{-1}(\mathscr{F}))$ be the canonical homomorphism associated with $\mathscr{F}$ and consider the subsheaf $\overline{\mathscr{G}} = \rho_{\mathscr{F}}^{-1}(j_*(\mathscr{G}))$ of $\mathscr{F}$. It follows immediately from definition that, for any open subset V of X, $\Gamma(V, \overline{\mathscr{G}})$ is formed by sections $s \in \Gamma(V, \mathscr{F})$ whose restriction on $V \cap U$ is a section of $\mathscr{G}$ over $V \cap U$. In particular, we have $\overline{\mathscr{G}}|_U = j^{-1}(\overline{\mathscr{G}}) = \mathscr{G}$, and $\overline{\mathscr{G}}$ is the largest subsheaf of $\mathscr{F}$ inducing $\mathscr{G}$ on U. We say the subsheaf $\overline{\mathscr{G}}$ is the **canonical extension** of the subsheaf $\mathscr{G}$ of $\mathscr{F}|_U$ to a subsheaf of $\mathscr{F}$.

Proposition 8.6.58. *Let X be a scheme and U be an open subset of X such that the canonical injection $j : U \to X$ is quasi-compact (in other words, U is retrocompact in X).*

(a) *For any quasi-coherent $(\mathscr{O}_X|_U)$-module $\mathscr{G}$, $j_*(\mathscr{G})$ is a quasi-coherent $\mathscr{O}_X$-module and we have $j_*(\mathscr{G})|_U = j^*(j_*(\mathscr{G})) = \mathscr{G}$.*

(b) *For any quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$ and any quasi-coherent sub-$(\mathscr{O}_X|_U)$-module $\mathscr{G}$ of $\mathscr{F}|_U$, the canonical extension $\overline{\mathscr{G}}$ of $\mathscr{G}$ is a quasi-coherent sub-$\mathscr{O}_X$-module of $\mathscr{F}$.*

Proof. Assertion (a) is a special case of Proposition 8.6.55 since j is quasi-separated by Proposition 8.6.7(i), and the relation $j_*(\mathscr{G})|_U = j^*(j_*(\mathscr{G})) = \mathscr{G}$ can be checked directly. By the same reasoning, $j_*(j^*(\mathscr{F}))$ is quasi-coherent, and as $\overline{G}$ is the inverse image of $j_*(\mathscr{G})$ under the homomorphism $\rho_{\mathscr{F}} : \mathscr{F} \to j_*(j^*(\mathscr{F}))$, assertion (b) follows from Corollary 8.2.23. □

Corollary 8.6.59. *Let X be a scheme and U be a quasi-compact open subset of X such that the canonical injection $j : U \to X$ is quasi-compact. Suppose that any quasi-coherent $\mathscr{O}_X$-module is the inductive limit of its quasi-coherent sub-$\mathscr{O}_X$-modules of finite type (this is true if X is an affine scheme). Let $\mathscr{F}$ be a quasi-coherent $\mathscr{O}_X$-module and $\mathscr{G}$ be a quasi-coherent sub-$(\mathscr{O}_X|_U)$-module of finite type of $\mathscr{F}|_U$. Then there exists a quasi-coherent sub-$\mathscr{O}_X$-module $\mathscr{G}'$ of finite type such that $\mathscr{G}'|_U = \mathscr{G}$.*

Proof. We have $\mathscr{G} = \overline{\mathscr{G}}|_U$, and $\overline{\mathscr{G}}$ is quasi-coherent by Proposition 8.6.58, hence is the inductive limit of its quasi-coherent sub-$\mathscr{O}_X$-modules $\mathscr{H}_\lambda$ of finite type. Then $\mathscr{G}$ is the inductive limit of the $\mathscr{H}_\lambda|_U$, hence equals to one of $\mathscr{H}_\lambda|_U$ since they are of finite type (**??**). □

Remark 8.6.60. Suppose that for any affine open $U \subseteq X$ the injection $U \to X$ is quasi-compact. Then if the conclusion of Corollary 8.6.59 holds for any affine open U and any quasi-coherent sub-$(\mathscr{O}_X|_U)$-module of finite type $\mathscr{G}$ of $\mathscr{F}|_U$, it follows that $\mathscr{F}$ is the inductive limit of its quasi-coherent sub-$\mathscr{O}_X$-modules of finite type. In fact, for any affine open $U \subseteq X$, we have $\mathscr{F}|_U = \widetilde{M}$, where M is a $\Gamma(U, \mathscr{O}_X)$-module, and as the latter is the inductive limit of its finitely generated sub-modules, $\mathscr{F}|_U$ is the inductive limit of its quasi-coherent sub-$(\mathscr{O}_X|_U)$-modules of finite type. Now, by hypotheses, such a submodule is induced over U by a quasi-coherent sub-$\mathscr{O}_X$-module of finite type $\mathscr{G}_{\lambda,U}$ of $\mathscr{F}$. The finite sums of $\mathscr{G}_{\lambda,U}$ are then quasi-coherent of finite type, since the question is local and we can assume that X is affine, where the conclusion is trivial. It then follows that $\mathscr{F}$ is the inductive limit of these finite sums, whence our assertion.

Corollary 8.6.61. *Under the hypotheses of Corollary 8.6.59, if $\mathscr{G}$ is a quasi-coherent $(\mathscr{O}_X|_U)$-module of finite type, there exists a quasi-coherent $\mathscr{O}_X$-module $\mathscr{G}'$ such that $\mathscr{G}'|_U = \mathscr{G}$.*

Proof. As $\mathscr{F} = j_*(\mathscr{G})$ is quasi-coherent (Proposition 8.6.58) and $\mathscr{F}|_U = \mathscr{G}$, it suffices to apply Corollary 8.6.59 to $\mathscr{F}$. □

Lemma 8.6.62. *Let X be a scheme, $(V_\lambda)_{\lambda \in L}$ be a covering of X by affine opens where L is well-ordered, and U be an open subset of X. For each $\lambda \in L$, let $W_\lambda = \bigcup_{\mu < \lambda} V_\mu$. Suppose that*

(i) *for any* $\lambda \in L$, $V_\lambda \cap W_\lambda$ *is quasi-compact;*

(ii) *the canonical injection* $j : U \to X$ *is quasi-compact.*

Then for any quasi-coherent $\mathscr{O}_X$*-module* $\mathscr{F}$ *and any quasi-coherent sub-*$(\mathscr{O}_X|_U)$*-module of finite type* $\mathscr{G}$ *of* $\mathscr{F}|_U$*, there exists a quasi-coherent sub-*$\mathscr{O}_X$*-module* $\mathscr{G}'$ *of* $\mathscr{F}$ *such that* $\mathscr{G}'|_U = \mathscr{G}$.

Theorem 8.6.63. *Let* X *be a scheme and* U *be an open subset of* X*. Suppose that one of the following conditions is satisfied:*

(a) X *is locally Noetherian;*

(b) X *is quasi-compact and quasi-separated and* U *is quasi-compact.*

Then for any quasi-coherent $\mathscr{O}_X$*-module* $\mathscr{F}$ *and any quasi-coherent sub-*$(\mathscr{O}_X|_U)$*-module of finite type* $\mathscr{G}$ *of* $\mathscr{F}|_U$*, there exists a quasi-coherent sub-*$\mathscr{O}_X$*-module* $\mathscr{G}'$ *of* $\mathscr{F}$ *such that* $\mathscr{G}'|_U = \mathscr{G}$.

Corollary 8.6.64. *With the conditions of Theorem 8.6.63, for any quasi-coherent* $(\mathscr{O}_X|_U)$*-module* $\mathscr{G}$ *of finite type, there exists a quasi-coherent* $\mathscr{O}_X$*-module* $\mathscr{G}'$ *of finite type such that* $\mathscr{G}'|_U = \mathscr{G}$.

Corollary 8.6.65. *Let* X *be a locally Noetherian scheme or a quasi-compact and quasi-separated scheme. Then any quasi-coherent* $\mathscr{O}_X$*-module is the inductive limit of its quasi-coherent sub-*$\mathscr{O}_X$*-modules of finite type.*

Proof. This follows from Theorem 8.6.63 and Remark 8.6.60. □

Corollary 8.6.66. *Under the hypotheses of Corollary 8.6.65, if a quasi-coherent* $\mathscr{O}_X$*-module* $\mathscr{F}$ *is such that any quasi-coherent sub-*$\mathscr{O}_X$*-module of finite type of* $\mathscr{F}$ *is generated by its global sections, then* $\mathscr{F}$ *is generated by its global sections.*

Proof. Let U be an affine neighborhood of a point $x \in X$, and let s be a section of $\mathscr{F}$ over U. The sub-$\mathscr{O}_X$-module $\mathscr{G}$ of $\mathscr{F}|_U$ generated by s is quasi-coherent and of finite type, hence there exists a quasi-coherent sub-$\mathscr{O}_X$-module of finite type $\mathscr{G}'$ of $\mathscr{F}$ such that $\mathscr{G}'|_U = \mathscr{G}$ (Theorem 8.6.63). By hypotheses, there is then a finite number of sections t_i of $\mathscr{G}'$ over X and sections a_i of $\mathscr{O}_X$ over a neighborhood $V \subseteq U$ of x such that $s|_V = \sum_i a_i \cdot (t_i|_V)$, which proves the corollary. □

8.6.7 Scheme-theoretic image

Let $f : X \to Y$ be a morphism of schemes. If there exists a smallest closed subscheme Y' of Y such that the canonical injection $j : Y' \to Y$ dominates f (or equivalently, the inverse image $f^{-1}(Y')$ is equal to X), we then say that Y' is the **scheme-theoretic image** of X under f, or the **scheme-theoretic image of** f. If X is a subscheme of Y, the scheme-theoretic image of the canonical injection $j : X \to Y$ is called the **scheme-theoretic closure** of X.

Proposition 8.6.67 (Transitivity). *Let* $f : X \to Y$ *and* $g : Y \to Z$ *be morphisms. Suppose that* Y' *is the scheme-theoretic image of* X *under* f*, and if* g' *is the restriction of* g *to* Y'*, the scheme-theoretic image of* Z' *under* g' *exists. Then the scheme-theoretic image of* X *under* $g \circ f$ *is equal to* Z'.

Proof. To say that a closed subscheme Z_1 of Z is such that $(g \circ f)^{-1}(Z_1) = X$ signifies that $f^{-1}(g^{-1}(Z_1)) = X$, or that f is dominated by the canonical injection $g^{-1}(Z_1) \to Y$. Now, in view of the existence of the scheme-theoretic image Y', for any closed subscheme Z_1 of Z having this property, $g^{-1}(Z_1)$ dominates Y', which, if $j : Y' \to Y$ is the canonical injection, amounts to saying that $j^{-1}(g^{-1}(Z_1)) = g'^{-1}(Z_1) = Y'$. We then conclude that Z' is the smallest closed subscheme Z_1 having this property, whence our claim. □

Corollary 8.6.68. *Let* $f : X \to Y$ *be an* S*-morphism such that* Y *is the scheme-theoretic image of* Y *under* f*. Let* Z *be a separated* S*-scheme; if two* S*-morphisms* $g_1, g_2 : Y \to Z$ *are such that* $g_1 \circ f = g_2 \circ f$*, then* $g_1 = g_2$.

Proof. Let $h = (g_1, g_2)_S : Y \to Z \times_S Z$. As the diagonal $T = \Delta_Z(Z)$ is a closed subscheme of $Z \times_S Z$, $Y' = h^{-1}(T)$ is a closed subscheme of Y. Put $u = g_1 \circ f = g_2 \circ f$; we then have $h \circ f = (u, u)_S = \Delta_Z \circ u$. As $\Delta_Z^{-1}(T) = Z$, we have $(h \circ f)^{-1}(T) = u^{-1}(Z) = X$, so $f^{-1}(Y') = X$. We then conclude that the canonical injection $Y' \to Y$ dominates f, so $Y' = Y$ by hypothesis. Then by Proposition 8.4.16, h factors into $\Delta_Z \circ v$ where v is a morphism $Y \to Z$, which implies $g_1 = g_2 = v$. □

Let $f : X \to Y$ be a morphism and suppose that the scheme-theoretic image Y' of f exists. Then Y' is defined by a quasi-coherent ideal $\mathscr{J}'$ of $\mathscr{O}_Y$, and by definition, $\mathscr{J}'$ is the largest quasi-coherent ideal such that the homomorphism $f^\# : \mathscr{O}_Y \to f_*(\mathscr{O}_X)$ factors into $\mathscr{O}_Y \to \mathscr{O}_Y/\mathscr{J}' \to f_*(\mathscr{O}_X)$. This implies that $\mathscr{J}' \subseteq \ker f^\# = \mathscr{J}$; we therefore obtain a case where Y' exists, the one where $\mathscr{J}$ is quasi-coherent, and where $\mathscr{J}' = \mathscr{J}$.

Proposition 8.6.69. *Let $f : X \to Y$ be a morphism. Then the scheme-theoretic image of X under f exists if one of the following conditions is satisfied:*

(a) *$f_*(\mathscr{O}_X)$ is quasi-coherent (which is the case if f is quasi-compact and quasi-separated).*

(b) *X is reduced.*

In this case, the underlying space of Y' is equal to $\overline{f(X)}$, and if f factors into

$$X \xrightarrow{f'} Y' \xrightarrow{Y} Y$$

where j is the canonical injection, f' is scheme-theoretic dominant. Moreover, if X is reduced (resp. integral), so is Y'.

Proof. The case (a) is immediate by our previous argument; moreover, as $\mathscr{O}_Y/\mathscr{J} \to f_*(\mathscr{O}_X)$ is then injective, this shows that f' is scheme-theoretic dominant. We still need to verify that the closed subscheme of Y defined by $\mathscr{J} = \ker f^\#$ has underlying space $\overline{f(X)}$. Since the support of $f_*(\mathscr{O}_X)$ is contained in $\overline{f(X)}$, we have $\mathscr{J}_y = \mathscr{O}_y$ for $y \notin \overline{f(X)}$, so the support pf $\mathscr{O}_Y/\mathscr{J}$ is contained in $\overline{f(X)}$. Moreover, this support is closed and contains $f(X)$: if $y \in f(X)$, the identity element of the ring $(f_*(\mathscr{O}_X))_y$ is nonzero, being the germ at y of the section $1 \in \Gamma(X, \mathscr{O}_X) = \Gamma(Y, f_*(\mathscr{O}_X))$; as this is the image of the identity element under $f^\#$, it is not contained in $\mathscr{J}_y$, so $\mathscr{O}_y/\mathscr{J}_y \neq 0$; this proves our first claim. The case (b) follows from Proposition 8.4.48, because there is a smallest closed subscheme Z with underlying space $\overline{f(X)}$ such that $f(X) \subseteq Z$. □

Proposition 8.6.70. *Suppose the conditions of Proposition 8.6.69, and let Y' be the scheme-theoretic image of X under f. For any open subset V of Y, let $f_V : f^{-1}(V) \to V$ be the restriction of f. Then the scheme-theoretic image of $f^{-1}(V)$ under f_V exists and is equal to the oepn subscheme $V \cap Y'$ of Y'.*

Proof. Put $X' = f^{-1}(V)$; as the direct image of $\mathscr{O}_{X'}$ is the restriction of $f_*(\mathscr{O}_X)$ to V, it is clear that the kernel of the homomorphism $\mathscr{O}_V \to (f_V)_*(\mathscr{O}_{X'})$ is the restriction of $\mathscr{J}$ to V, whence the assertion. □

Proposition 8.6.71. *Let Y be a subscheme of a scheme X, such that the canonical injection $j : Y \to X$ is quasi-compact. Then the scheme-theoretic closure of Y exists and has $\overline{Y}$ as underlying space.*

Proof. It suffices to apply Proposition 8.6.69 to the injection j, which is separated (Proposition 8.5.26) and quasi-compact by hypothesis. □

With these notations, let $\overline{Y}$ be the scheme-theoretic closure of Y in X. If the injection $\overline{Y} \to X$ is quasi-compact, and if $\mathscr{I}$ is the quasi-coherent ideal of $\mathscr{O}_X|_{\overline{Y}}$ defining the closed subscheme Y of $\overline{Y}$, then the quasi-coherent ideal of $\mathscr{O}_X$ defining $\overline{Y}$ is the canonical extension (Proposition 8.6.58) $\overline{\mathscr{I}}$ of $\mathscr{I}$, because it is evidently the largest quasi-coherent ideal of $\mathscr{O}_X$ inducing $\mathscr{I}$ over Y.

Corollary 8.6.72. *Under the hypothesis of Proposition 8.6.71, any section of $\mathscr{O}_{\overline{Y}}$ over an open subset V of $\overline{Y}$ that is zero on $V \cap Y$ is zero.*

Proof. In view of Proposition 8.6.70, we can assume that $V = \overline{Y}$. If we consider sections of $\mathscr{O}_{\overline{Y}}$ over $\overline{Y}$ as $\overline{Y}$-sections of $\overline{Y} \otimes_{\mathbb{Z}} \mathbb{Z}[T]$, which is separated over $\overline{Y}$, the assertion is then a particular case of Corollary 8.6.68. □

8.7 Rational maps over schemes

8.7.1 Rational maps and rational functions

Let X and Y be two schemes, U and V be open dense sets of X, and f (resp. g) be a morphism from U (resp. V) to Y. We say the morphisms f and g are equivalent if they coincide over an open subset dense in $U \cap V$. As the intersection of finitely many open dense subsets of X is an open dense subset of X, it is clear that this relation is an equivalence relation.

Given two schemes X and Y, a **rational map** from X to Y is defined to be an equivalent class of morphisms from an open dense subset of X to Y. If X and Y are S-schemes, this class is called an **rational S-map** is there exists an S-morphism in it. An rational S-map from S to X is called an **rational S-section** of X. The rational X-sections of the X-scheme $X \otimes_{\mathbb{Z}} \mathbb{Z}[T]$ (where T is an indeterminate) are called the **rational functions over** X (cf. Example 8.3.13). A rational map f from X to Y is usually denoted by $f : X \dashrightarrow Y$.

Let $f : X \dashrightarrow Y$ be a rational map and U be an open subset of X. If f_1, f_2 are morphisms belonging to the class f, defined respectively over the open dense sets V_1, V_2 of X, the restrictions $f_1|_{U\cap V_1}$ and $f_2|_{U\cap V_2}$ coincide on $U \cap V_1 \cap V_2$, which is dense in U; the class of morphisms f therefore defines a rational map $U \dashrightarrow Y$, called the **restriction** of f to U and denoted by $f|_U$.

It is clear that we have a canonical map from $\mathrm{Hom}_S(X, Y)$ to the set of rational S-maps from X to Y, which associates any S-morphism $f : X \to Y$ to the rational S-map it belongs to. If we denote by $\Gamma_{\mathrm{rat}}(X/Y)$ the set of rational Y-sections of X, we then have a canonical map $\Gamma(X/Y) \to \Gamma_{\mathrm{rat}}(X/Y)$. It is also clear that if X and Y are two S-schems, the set of rational S-maps from X to Y is canonically identified with $\Gamma_{\mathrm{rat}}((X \times_S Y)/X)$.

In view of Example 8.3.13, we see the rational functions over X are canonically identified with the equivalent classes of sections of the structural sheaf $\mathscr{O}_X$ over open dense sets of X, where two sections are equivalent if they coincide over an open dense subset of the intersection of their defining domain. In partiuclar, we see the rational functions over X form a ring $K(X)$.

If X is an irreducible scheme, any nonempty open subset of X is dense; we can also say that the non-empty open sets of X are the open neighborhoods of the generic point x of X. To say that two morphisms from nonempty open subsets of X to Y are equivalent therefore means in this case that they have the same germ at the point x. In other words, rational maps (resp. rational S-maps) $X \dashrightarrow Y$ are identified with the germs of morphisms (resp. of S-morphisms) of non-empty open subsets of X to Y at the generic point x of X. In particular:

Proposition 8.7.1. *If X is an irreducible scheme, the ring $K(X)$ of rational functions over X is canonically identified with the local ring $\mathscr{O}_{X,x}$ at the generic point x of X. This is a local ring of zero dimension, and therefore an Artinian local ring if X is Noetherian. It is a field if X is integral, and is identified with the fraction field of $\Gamma(X, \mathscr{O}_X)$ if X is moreover affine.*

Proof. Since we can identify rational functions with sections over X, the first assertion follows from the definiton of stalks. For the second one, we can assume that X is affine with ring A; then $\mathfrak{p}_x$ is the nilradical of A, and in particular $\mathscr{O}_{X,x}$ has zero dimension. If A is integral, $\mathfrak{p}_x = (0)$ and $\mathscr{O}_{X,x}$ is the fraction field of A. Finally, if A is Noetherian, then $\mathfrak{p}_x$ is nilpotent and $\mathscr{O}_{X,x} = A_x$ is Artinian. □

If X is integral, the ring $\mathscr{O}_{X,z}$ is integral for any $z \in X$. Any affine open U containing x must contain x as its generic point, and $\mathscr{O}_{X,z}$, equal to a fraction field of $\Gamma(U, \mathscr{O}_X)$, is identified with $K(X)$. We then conclude that $K(X)$ is identified with the fraction field of $\mathscr{O}_{X,z}$, and in this way, $\mathscr{O}_{X,z}$ is canonically identified with a subring of $K(X)$, so that a germ $s \in \mathscr{O}_{X,z}$ is canonically identified with a rational function over X.

Proposition 8.7.2. *Let X and Y be two S-schemes such that the family (X_λ) of irreducible components of X is locally finite. For each λ, let x_λ be the generic point of X_λ. If R_λ is the set of germs at x_λ of S-morphisms from open subsets of X to Y, the set of rational S-maps from X to Y is identified with the product of R_λ. In particular, the ring of rational functions over X is identified with the product of the local rings $\mathscr{O}_{X,x_\lambda}$.*

Proof. The set of the intersections $X_\lambda \cap X_\mu$ for $\lambda \neq \mu$ is then locally finite, so their union is closed and contains the maximal points of X. If we set $X'_\lambda = X_\lambda - \bigcup_{\mu\neq\lambda} X_\lambda \cap X_\mu$, then X'_λ is irreducible, with generic point equal to that of X_λ, and pairwise disjoint with union dense in X. For any open dense subset U of X, $U'_\lambda = U \cap X'_\lambda$ is a nonempty open dense subset of X'_λ, and U'_λ are pairwise disjoint with $U' = \bigcup_\lambda U'_\lambda$ closed in X. To give a morphism from U' to Y is then equivalent to giving (arbitrarily) a morphism from each of the U'_λ in Y, so the assertion follows. □

Corollary 8.7.3. *Let A be a Noetherian ring and $X = \mathrm{Spec}(A)$. The ring of rational function functions over X is identified with the total fraction ring $Q(A)$.*

Proof. Let S be the complement of the union of minimal prime ideals of A. Then by **??**, the ring of sections $\Gamma(D(f), \mathscr{O}_X)$ is identified with A_f, so $D(f)$ with $f \in S$ form a cofinal subset of the open dense sets of X, and the ring of rational functions over X is then identified with the inductive limit of A_f, $f \in S$, which is exactly $Q(A)$. □

Suppose that X is irreducible with generic point x. As any nonempty open set U of X contains x, and therefore contains any generalization $z \in X$, any morphism $U \to Y$ can be composed with the canonical morphism $\mathrm{Spec}(\mathscr{O}_{X,x}) \to X$ (Corollary 8.2.12). Two morphisms from nonempty open subsets of X to Y which coincide on a smaller open subset then give the same morphism $\mathrm{Spec}(\mathscr{O}_{X,x}) \to Y$. In other words, to any rational map X to Y, there corresponds a well-defined morphism $\mathrm{Spec}(\mathscr{O}_{X,x}) \to Y$.

Proposition 8.7.4. *Let X and Y be S-schemes. Suppose that X is irreducible with generic point x, and Y is of finite type over S. Then two rational S-maps X to Y corresponding to the same S-morphism $\mathrm{Spec}(\mathscr{O}_{X,x}) \to Y$ are identical. If moreover S is locally of finite presentation over S, then any S-morphism $\mathrm{Spec}(\mathscr{O}_{X,x}) \to Y$ corresponds to a rational S-map from X to Y.*

Proof. Given that every non-empty open subset of X is dense, this result follows immediately from Proposition 8.6.50. □

Corollary 8.7.5. *Suppose that X is irreducible with generic point x and Y is localy of finite presentation. Then the rational S-maps from X to Y are identified with the points of Y with values in the S-scheme $\mathrm{Spec}(\mathscr{O}_{X,x})$.*

Proof. This is just a reformulation of Proposition 8.7.4. □

Corollary 8.7.6. *Suppose that X is irreducible with generic point x and Y is localy of finite presentation. Let s be the image of x in S. Then giving a rational S-map from X to Y is equivalent to giving a point y of Y lying over s and a $\mathscr{O}_{S,s}$-homomorphism $\mathscr{O}_{Y,y} \to \mathscr{O}_{X,x} = K(X)$.*

Proof. This follows from Proposition 8.7.4 and Proposition 8.2.14. □

Corollary 8.7.7. *Suppose that X is irreducible with generic point x and Y is localy of finite presentation, then the rational S-maps from X to Y (with Y given) only depends on the S-scheme $\mathrm{Spec}(\mathscr{O}_{X,x})$ and in particular remain the same if we replace X by $\mathrm{Spec}(\mathscr{O}_{X,z})$, $z \in X$.*

Proof. In fact, if $z \in \overline{\{x\}}$ then x is the generic point of $Z = \mathrm{Spec}(\mathscr{O}_{X,z})$ and $\mathscr{O}_{X,x} = \mathscr{O}_{Z,x}$. □

Corollary 8.7.8. *Suppose that X is integral with generic point x and Y is localy of finite presentation. Let s be the image of x in S. Then following dates are equivalent:*

(i) *a rational S-map from X to Y;*

(ii) *a point of $Y \otimes_S \kappa(s)$ with values in the extension $K(X)$ of $\kappa(s)$;*

(iii) *a point $y \in Y$ over s and an $\kappa(s)$-homomorphism $\kappa(y) \to \kappa(x) = K(X)$.*

Proof. The points of Y over s belong to $Y \otimes_S \kappa(s)$ and the $\mathscr{O}_{S,s}$-homomorphisms $\mathscr{O}_{Y,y} \to K(X)$ are $\kappa(s)$-homomorphisms $\kappa(y) \to K(X)$, since $K(X)$ is a field. □

Corollary 8.7.9. *Let k be a field and X, Y be two schemes locally algebraic over k. Suppose that X is integral, then the rational k-maps from X to Y are identified with the points of Y with values in the extension $K(X)$ of k.*

8.7.2 Defining domain of a rational map

Let X and Y be schemes, f a rational map from X to Y. We say f is **defined at a point** $x \in X$ if there exists an open dense subset U containing x and a morphism $U \to Y$ representing f. The set of points $x \in X$ where f is defined is called the **defining domain** of the rational map f. It is clearly an open dense subset of X.

Proposition 8.7.10. *Let X and Y be two S-schemes such that X is reduced and Y is separated over S. Let f be a rational S-map from X to Y, and U_0 be its domain. There then exists a unique S-morphism $U_0 \to Y$ belonging to the class f.*

Proof. For any morphism $U \to Y$ belonging to the class f, we necessarily have $U \subseteq U_0$, so we only need to prove that if U_1, U_2 are two dense subsets of X and $f_i : U_i \to Y$ $(i = 1, 2)$ are two S-morphisms that coincide on an open subset $V \subseteq U_1 \cap U_2$, then f_1 and f_2 coincide on $U_1 \cap U_2$. For this, we can clearly assume that $X = U_1 = U_2$. As X (hence V) is reduced, X is smallest closed subscheme of X dominating V (Proposition 8.4.48). Let $g = (f_1, f_2)_S : X \to Y \times_S Y$; as by hypothesis the diagonal $T = \Delta_Y(Y)$ is a closed subscheme of $Y \times_S Y$, $Z = g^{-1}(T)$ is a closed subscheme of X. If $h : V \to Y$ is the restriction of f_1 and f_2 to V, the restriction of g to V is $\tilde{g} = (h, h)_S$, which factors into $\tilde{g} = \Delta_Y \circ h$. As $\Delta_Y^{-1}(T) = Y$, we have $\tilde{g}^{-1}(T) = V$, and Z is therefore a closed subscheme of X inducing the subscheme structure on V, hence dominates V, and this implies $Z = X$. From the relation $g^{-1}(T) = X$, we deduce that g factors into $\Delta_Y \circ f$, where f is a morphism $X \to Y$ (Proposition 8.4.16), and we have $f_1 = f_2 = f$ from the definition of the diagonal morphism. □

It is clear that the morphism $U_0 \to Y$ defined in Proposition 8.7.10 is the unique morphism in the class f that admits no further extension to open dense subsets of X containing U_0. Under the conditions of Proposition 8.7.10, we can then identify the rational maps from X to Y with the morphisms unextendable to open dense subsets of X to Y.

Corollary 8.7.11. *Let X and Y be two S-schemes such that X is reduced and Y is separated over S. Let U be an open dense subset of X, then there is a canonical bijective correspondence between the S-morphisms from U to Y and the rational S-maps from X to Y defined at each point of U.*

Proof. In view of Proposition 8.7.10, for any S-morphism $f : U \to Y$, there exists a rational S-map $\bar{f}$ from X to Y which extends f. □

Corollary 8.7.12. *Let S be a separated scheme, X be a reduced S-scheme, Y be an S-scheme, and $f : U \to Y$ be an S-morphism from an open dense subset U of X to Y. If $\bar{f}$ is the rational $\mathbb{Z}$-map from X to Y extending f, $\bar{f}$ is an S-morphism (and therefore the rational S-map from X to Y extending f).*

Proof. In fact, if $\varphi : X \to S$ and $\psi : Y \to S$ are the structural morphisms, U_0 is the defining domain of $\bar{f}$, and $j : U_0 \to X$ is the injection, it suffices to prove that $\psi \circ \bar{f} = \varphi \circ j$, which follows from the proof of Proposition 8.7.10, since f is an S-morphism. □

Corollary 8.7.13. *Let X and Y be S-schemes. Suppose that X is reduced and X, Y are spearated over S. Let $p : Y \to X$ be an S-morphism, U be an open dense subset of X, and f be a U-section of Y. Then the rational map $\bar{f}$ from X to Y extending f is a rational X-section of Y.*

Proof. We only need to prove that $p \circ \bar{f}$ is the identity on the defining domain of $\bar{f}$; since X is separated over S, this follows from the proof of Proposition 8.7.10. □

Corollary 8.7.14. *Let X be a reduced scheme and U be an open dense subset of X. There exists a canonical bijective correspondence between sections of $\mathscr{O}_X$ over U and rational functions f over X defined on each point of U.*

Proof. It suffices to remark that the X-scheme $X \otimes_{\mathbb{Z}} \mathbb{Z}[T]$ is separated over X by Proposition 8.5.26. □

Corollary 8.7.15. *Let Y be a reduced scheme, $f : X \to Y$ be a separated morphism, U be an open dense subset of Y, $g : U \to f^{-1}(U)$ be a U-section of $f^{-1}(U)$, and Z the reduced subscheme of X induced on $\overline{g(U)}$. For g to be the restriction of a Y-section of X, it is necessary and sufficient that the restriction of f to Z is an isomorphism from Z to Y.*

Proof. The restriction of f to $f^{-1}(U)$ is a separated morphism (Proposition 8.5.26(i)), so g is a closed immersion by Corollary 8.5.19, and therefore $g(U) = Z \cap f^{-1}(U)$ coincides with the subscheme induced by Z over the open subset $g(U)$ of Z. It is then clear that the given condition is sufficient, since if $f_Z : Z \to Y$ is an isomorphism and $\bar{g} : Y \to Z$ is the inverse morphism, then $\bar{g}$ extends g. Conversely, if g is the restriction to U of a Y-section h of X, h is then a closed immersion by Corollary 8.5.19, so $h(Y)$ is clossed, and as it contains $g(U)$ and we have (as h is continuous) $h(Y) = h(\overline{U}) \subseteq \overline{h(U)} = \overline{g(U)}$, we conclude that $h(Y) = Z$. It then follows from Proposition 8.4.44 that h is necessarily an isomorphism from Y to the closed subscheme Z of X, so $f|_Z$ is also an isomorphism. □

Let X and Y be two S-schemes, where X is reduced and Y is separated over S. Let f be a rational S-map from X to Y, and let x be a point of X. We can compose f with the canonical morphism $\mathrm{Spec}(\mathscr{O}_{X,x}) \to X$ provided that the trace on $\mathrm{Spec}(\mathscr{O}_{X,x})$ of the defining domain of f is dense in $\mathrm{Spec}(\mathscr{O}_{X,x})$ (identified with the set $z \in X$ such as $x \in \overline{\{z\}}$ (cf. Corollary 8.2.12). This happens if the family of irreducible components of X is *locally finite*:

Lemma 8.7.16. *Let X be a scheme such that the family of irreducible components of X is locally finite, and x be a point of X. The irreducible components of $\mathrm{Spec}(\mathscr{O}_{X,x})$ are then the traces over $\mathrm{Spec}(\mathscr{O}_{X,x})$ of the irreducible components of X containing x. For an open subset U of X to be such that $U \cap \mathrm{Spec}(\mathscr{O}_{X,x})$ is dense in $\mathrm{Spec}(\mathscr{O}_{X,x})$, it is necessary and sufficient that it meets the irreducible components of X containing x (and this is true in particular if U is dense in X).*

Proof. The second assertion clearly follows from the first one. As $\mathrm{Spec}(\mathscr{O}_{X,x})$ is contained in any affine open U containing x and the irreducible components of U containing x are the traces of the irreducible components of X containing x on U (**??**), we can suppose that X is affine with ring A. Then the prime ideals of A_x correspond to prime ideals of A contained in $\mathfrak{p}_x$, so the minimal prime ideals of A_x correspond to minimal prime ideals of A contained in $\mathfrak{p}_x$, and the lemma follows from **??**. □

Suppose that we are under the assumption of Lemma 8.7.16. If U is the defining domain of definition of the rational S-map f, denote by f' the rational map from $\mathrm{Spec}(\mathscr{O}_{X,x})$ to Y which coincides with f over $U \cap \mathrm{Spec}(\mathscr{O}_{X,x})$; we will say that this rational map is **induced** by f.

Proposition 8.7.17. *Let S be a scheme, X be a reduced S-scheme, and Y be a separted S-scheme that is locally of finite presentation over S. Suppose that the family of irreducible components of X is locally finite. Let f be a rational S-map from X to Y and x be a point of X. For f to be defined at the point x, it is necessary and sufficient that the rational map f' from $\mathrm{Spec}(\mathscr{O}_{X,x})$ to Y induced by f is a morphism.*

Proof. The conditions is clearly necessary since $\mathrm{Spec}(\mathscr{O}_{X,x})$ is contained in any open subset containing x. We now prove the sufficiency, so suppose that f' is a morphism. In view of Proposition 8.6.50, there exists an open neighborhood U of x in X and an S-morphism $g : U \to Y$ inducing f' on $\mathrm{Spec}(\mathscr{O}_{X,x})$. The point is that U is not necessarily dense in X, so we want to extend g to a morphism defined on an open dense subset of X. Now by Lemma 8.7.16, there are finitely many irreducible components X_i of X containing x, and we can assume that these are the only ones meeting U, by replacing U with a smaller open subset. As the generic points of X_i belong to the defining domain of f and to U, we see that f and g coincide over a non-empty open dense subset of each of the X_i (Proposition 8.6.50). Consider the morphism f_1 defined on an open dense subset of $U \cup (X - \overline{U})$ which equals to g over U and to f over the intersection of $X - \overline{U}$ and the defining domain of f (we also note that each X_i is contained in $\overline{U}$). As $U \cup (X - \overline{U})$ is dense in X, f_1 and f coincide on an open dense subset of X, and f is an extension of f_1. Since f_1 is defined at x, this shows f is defined at x. □

8.7.3 Sheaf of rational functions

Let X be a scheme. For each open subset U of X, we denote by $K(U)$ the ring of rational functions over U, which is an $\Gamma(U, \mathscr{O}_X)$-algebra. Moreover, if $V \subseteq U$ is a second open subset of X, any section of $\mathscr{O}_X$ over a dense subset of U restricts to a section over a dense subset of V, and if two sections coincide over an open dense subset of U, their restriction also coincide over a smaller open dense subset of V. We then define a homomorphism of algebras $\mathrm{Res}_V^U : K(U) \to K(V)$, and it is clear that for $U \supseteq V \supseteq W$ open in X we have $\mathrm{Res}_W^U = \mathrm{Res}_W^V \circ \mathrm{Res}_V^U$. Therefore, we get a presheaf of algebras over X. The associated sheaf of $\mathscr{O}_X$-algebras over X is then called the **sheaf of rational functions** over the scheme X, and denoted by $\mathscr{K}_X$. For any open subset U of X, it is clear that the restriction $\mathscr{K}_X|_U$ is equal to $\mathscr{K}_U$.

Proposition 8.7.18. *Let X be a scheme such that the family (X_λ) of irreducible components of X is locally finite. Then the $\mathscr{O}_X$-module $\mathscr{K}_X$ is quasi-coherent and for any open subset U of X, $K(U)$ is equal to $\Gamma(U, \mathscr{K}_X)$ and is identified with the product of the local rings of the generic points x_λ of the irreducible components X_λ such that $X_\lambda \cap U \neq \varnothing$.*

Proof. The fact that $K(U)$ is identified with the product follows from Proposition 8.7.2. We now show that the presheaf $U \mapsto K(U)$ is a sheaf. Consider an open subset U of X and an open covering (V_α) of U. If $s_\alpha \in K(V_\alpha)$ are such that s_α and s_β coincide over $V_\alpha \cap V_\beta$ for each pair of indices, we then conclude that for any index λ such that $U \cap X_\lambda \neq \varnothing$, the component in $K(X_\lambda)$ of all s_α such that $V_\alpha \cap X_\lambda \neq \varnothing$ are the same. Denoting by t_λ this component, it is clear that the element of $K(U)$ with component t_λ in $K(X_\lambda)$

has restriction s_α on each V_α. Finally, to see the sheaf $\mathscr{K}_X$ is quasi-coherent, we can limit ourselves to the case $X = \mathrm{Spec}(A)$ is affine with finitely many irreducible components; by taking for U the affine open sets of the form $D(f)$, where $f \in A$, it follows from the above argument that we have $\mathscr{K}_X = \widetilde{M}$, where M is the direct sum of the A-modules A_{x_λ}. □

Corollary 8.7.19. *Let X be a reduced scheme with irreducible components $(X_i)_{1\leq i\leq n}$, endowed with the reduced subscheme structures. If $\iota_i : X_i \to X$ is the canonical injection, $\mathscr{K}_X$ is the direct product of the $\mathscr{O}_X$-algebras $(\iota_i)_*(\mathscr{K}_{X_i})$.*

Proof. This is a particular case of Proposition 8.7.18, in view of the conditions in that proposition. □

Corollary 8.7.20. *If X is irreducible, any quasi-coherent $\mathscr{K}_X$-module $\mathscr{F}$ is a simple sheaf.*

Proof. It suffices to show that any $x \in X$ admits a neighborhood U such that $\mathscr{F}|_U$ is a simple sheaf, which means we can assume that X is affine. We can then suppose that $\mathscr{F}$ is the cokernel of a homomorphism $\mathscr{K}_X^{\oplus I} \to \mathscr{K}_X^{\oplus J}$, and it all boils down to seeing that $\mathscr{K}_X$ is a simple sheaf. But this is evident since $\Gamma(U, \mathscr{K}_X) = K(X)$ for any nonempty open subset U, since U contains the generic point of X. □

Corollary 8.7.21. *If X is irreducible, for any quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{K}_X$ is a simple sheaf. If moreover X is reduced (hence integral), $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{K}_X$ is isomorphic to a sheaf of the form $\mathscr{K}_X^{\oplus I}$.*

Proof. The first claim follows from Corollary 8.7.20, and the second one follows from the fact that if X is integral then $K(X)$ is a field. □

Proposition 8.7.22. *Let X be a scheme such that the family (X_λ) of irreducible components of X is locally finite. Then $\mathscr{K}_X$ is a quasi-coherent $\mathscr{O}_X$-algebra. If X is reduced, the canonical homomorphism $\mathscr{O}_X \to \mathscr{K}_X$ is injective.*

Proof. Since the question is local, the first claim follows from Proposition 8.7.18. The second one follows from Corollary 8.7.14. □

Let X and Y be integral schemes, so that $\mathscr{K}_X$ (resp. $\mathscr{K}_Y$) is a quasi-coherent $\mathscr{O}_X$-module (resp. $\mathscr{O}_Y$-module). Let $f : X \to Y$ be a dominant morphism; then there exists a canonical homomorphism of $\mathscr{O}_X$-modules:

$$\tau : f^*(\mathscr{K}_Y) \to \mathscr{K}_X.$$

Suppose that $X = \mathrm{Spec}(A)$ and $Y = \mathrm{Spec}(B)$ are affine with integral domains A and B, then f corresponds to the injective homomorphism $B \to A$ (**??**), which extens to a monomorphism $L \to K$ of fraction fields. The homomorphism τ then corresponds to the canonical homomorphism $L \otimes_B A \to K$.

In the general case, for any couple of affine opens $U \subseteq X$, $V \subseteq Y$ such that $f(U) \subseteq V$, we define similarly a homomorphism $\tau_{U,V}$ and note that if $U' \subseteq U$, $V' \subseteq V$ and $f(U') \subseteq V'$, then $\tau_{U,V}$ extends $\tau_{U',V'}$. If x and y are the generic points of X and Y, respectively, then $f(x) = y$ and

$$(f^*(\mathscr{K}_Y))_x = \mathscr{O}_{Y,y} \otimes_{\mathscr{O}_{Y,y}} \mathscr{O}_{X,x} = \mathscr{O}_{X,x}$$

and τ_x is therefore an isomorphism. However, the homomorphism τ is usually not an isomorphism: for example, if $B = L$ is a field containing the integral domain A and A is not a field, the canonical homomorphism $L \otimes_B A \to K$ is then the canonical homomorphism $A \to K$, which is not bijective.

8.7.4 Torsion sheaves and torsion-free sheaves

Let X be a reduced scheme whose family of irreducible components is locally finite. For any $\mathscr{O}_X$-module $\mathscr{F}$, the canonical homomorphism $\mathscr{O}_X \to \mathscr{K}_X$ is injective by Proposition 8.7.22, and defines a homomorphism $\mathscr{F} \to \mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{K}_X$ which on each stalk, is none other than the homomorphism $z \mapsto z \otimes 1$ from $\mathscr{F}_x$ to $\mathscr{F}_x \otimes_{\mathscr{O}_{X,x}} \mathscr{K}_{X,x}$. The kernel $\mathscr{T}$ (also denoted by $\mathscr{T}(\mathscr{F})$) of this homomorphism is a sub-$\mathscr{O}_X$-module $\mathscr{F}$ is called the **torsion sheaf** of $\mathscr{F}$, which is quasi-coherent if $\mathscr{F}$ is quasi-coherent (Proposition 8.7.18). The sheaf $\mathscr{F}$ is called **torsion-free** if $\mathscr{T} = 0$, and a **torsion sheaf** if $\mathscr{T} = \mathscr{F}$. For any $\mathscr{O}_X$-module $\mathscr{F}$, $\mathscr{F}/\mathscr{T}$ is torsion-free.

Proposition 8.7.23. *If X is an integral scheme, for a quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$ to be torsion-free, it is necessary and sufficient that it is isomorphic to a sub-$\mathscr{O}_X$-module $\mathscr{G}$ of a simple sheaf of the form $\mathscr{K}_X^{\oplus I}$, generated (as a $\mathscr{K}_X$-module) by $\mathscr{G}$.*

Proof. This follows from Corollary 8.7.21, since $\mathscr{F}$ is torsion-free if and only if the canonical homomorphism $\mathscr{F} \to \mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{K}_X$ is injective. □

The cardinality of the set I is called the **rank** of $\mathscr{F}$. For any affine open U of X, since the generic point of x is contained in U, the rank of $\mathscr{F}$ is also equal to the rank of $\Gamma(U, \mathscr{F})$ as a $\Gamma(U, \mathscr{O}_X)$-module.

Corollary 8.7.24. *Over an integral scheme X, any torsion-free quasi-coherent $\mathscr{O}_X$-module of rank 1 (and in particular any invertible $\mathscr{O}_X$-module) is isomorphic to a sub-$\mathscr{O}_X$-module of $\mathscr{K}_X$, and the converse is also true.*

Corollary 8.7.25. *Let X be an integral scheme, $\mathscr{L}$, $\mathscr{L}'$ be two torsion-free $\mathscr{O}_X$-module, s (resp. s') be two sections of $\mathscr{L}$ (resp. $\mathscr{L}'$) over X. For $s \otimes s' = 0$, it is necessary and sufficient that one of the sections s, s' is zero.*

Proof. Let x be the generic point of X. We have by hypothesis $(s \otimes s')_x = s_x \otimes s'_x = 0$. As $\mathscr{L}_x$ and $\mathscr{L}'_x$ are identified with sub-$\mathscr{O}_{X,x}$-modules of the field $\mathscr{O}_{X,x}$, the preceding relation implies $s_x = 0$ or $s'_x = 0$, and therefore $s = 0$ or $s' = 0$ since $\mathscr{L}$ and $\mathscr{L}'$ are torsion-free (Corollary 8.7.20). □

Proposition 8.7.26. *Let X and Y be two integral schemes and $f : X \to Y$ be a dominant morphism. For any torsion-free quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, $f_*(\mathscr{F})$ is a torsion-free $\mathscr{O}_Y$-module.*

Proof. As f_* is left exact, it suffices, in view of Proposition 8.7.23, to prove the proposition for $\mathscr{F} = \mathscr{K}_X^{\oplus I}$. Now any open subset U of Y contains the generic point of Y, hence $f^{-1}(U)$ contains the generic point of X, so we have $\Gamma(U, f_*(\mathscr{F})) = \Gamma(f^{-1}(U), \mathscr{F}) = K(X)^{\oplus I}$. Therefore $f_*(\mathscr{F})$ is the simple sheaf with stalk $K(X)^{\oplus I}$, considered as a $\mathscr{K}_Y$-module, and it is evidently torsion-free. □

Proposition 8.7.27. *Let X be reduced scheme whose family of irreducible components is locally finite. For any quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$ of finite type, the following conditions are equivalent:*

(i) *$\mathscr{F}$ is a torsion sheaf.*

(ii) *$\mathscr{F}_x = 0$ for every maximal point of x.*

(iii) supp$(\mathscr{F})$ *contains no irreducible component of X.*

Proof. Since the question is local, we may assume that X has finitely many irreducible components $(X_i)_{1 \le i \le n}$, with generic points x_i. Endow each X_i the reduced subscheme structure of X, and let $\iota_i : X_i \to X$ be the canonical injection. If we put $\mathscr{F} = \iota_i^*(\mathscr{F})$, we see immediately that (Corollary 8.7.19) $\mathscr{F}$ is torsion-free if and only if each $\mathscr{F}_i$ is torsion-free. As $\mathscr{F}_{x_i} = (\mathscr{F}_i)_{x_i}$, to establish the equivalence of (i) and (ii), we can assume that X is integral. But then if x is the generic point of X, the relation $\mathscr{F}_x = 0$ and $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{K}_X = 0$ are equivalent by Proposition 8.7.23 and Corollary 8.7.20. The equivalenct of (ii) and (iii) results from the fact that supp$(\mathscr{F})$ is closed in X (since $\mathscr{F}$ is quasi-coherent) and that the conditions supp$(\mathscr{F}) \cap X_i = \varnothing$ and $x_i \notin$ supp$(\mathscr{F})$ are then equivalent. □

8.7.5 Separation criterion for integral schemes

Let X be an integral scheme, K its function field, identified with the local ring at the generic point ξ of X. For any $x \in X$, we can identify $\mathscr{O}_{X,x}$ as a subring of K, formed by the rational functions defined at the point x. For any rational function $f \in K$, the defining domain $\delta(f)$ of f is then the open subset of $x \in X$ such that $f \in \mathscr{O}_{X,x}$, and in view of Corollary 8.7.14 we have, for each open subset $U \subseteq X$, that

$$\Gamma(U, \mathscr{O}_X) = \bigcap_{x \in U} \mathscr{O}_{X,x}. \tag{8.7.1}$$

Given a field K, for any subring A of K, we denote by $L(A)$ the set of localizations $A_{\mathfrak{p}}$, where $\mathfrak{p}$ runs through prime ideals of A; they are identified with local subrings of K containing A. As $\mathfrak{p} = (\mathfrak{p}A_{\mathfrak{p}}) \cap A$, the map $\mathfrak{p} \mapsto A_{\mathfrak{p}}$ from Spec(A) to $L(A)$ is bijective.

Lemma 8.7.28. *Let K be a field and A be a subring of K. For a local subring R to dominate a ring in $L(A)$, it is necessary and sufficient that $A \subseteq R$. In this case, the local ring $A_{\mathfrak{p}}$ dominated by R is then unique and corresponds to the prime ideal $\mathfrak{p} = \mathfrak{m}_R \cap A$, where $\mathfrak{m}_R$ is the maximal ideal of R.*

Proof. In fact, if R dominates $A_{\mathfrak{p}}$, then $\mathfrak{m}_R \cap A_{\mathfrak{p}} = \mathfrak{p}A_{\mathfrak{p}}$ by **??**, hence the uniqueness of $\mathfrak{p}$. On the other hand, if $A \subseteq R$, $\mathfrak{m}_R \cap A = \mathfrak{p}$ is a prime ideal of A, and as the elements of $A - \mathfrak{p}$ are then invertible in R, we have $A_{\mathfrak{p}} \subseteq R$, so $\mathfrak{p}A_{\mathfrak{p}} \subseteq \mathfrak{m}_R$ and R dominates $A_{\mathfrak{p}}$. □

Lemma 8.7.29. *Let K be a field, A, B be two local subrings of K, and C be the subring of K generated by $A \cup B$. The following conditions are equivalent:*

(i) *There exists a prime ideal $\mathfrak{r}$ of C such that $\mathfrak{m}_A = \mathfrak{r} \cap A$ and $\mathfrak{m}_B = \mathfrak{r} \cap B$.*

(ii) *The ideal $\mathfrak{c}$ generated in C by $\mathfrak{m}_A \cup \mathfrak{m}_B$ is proper.*

(iii) *There exists a local subring R of K dominating both A and B.*

Proof. It is clear that (i) implies (ii). Conversely, if $\mathfrak{c}$ is proper, it is contained in a maximal ideal $\mathfrak{n}$ of C, and $\mathfrak{n} \cap A$ contains $\mathfrak{m}_A$ and is proper, so $\mathfrak{n} \cap A = \mathfrak{m}_A$ and similarly $\mathfrak{n} \cap B = \mathfrak{m}_B$. Finally, it is clear that if R dominates A and B then $C \subseteq R$ and $\mathfrak{m}_A = \mathfrak{m}_R \cap A = (\mathfrak{m}_R \cap C) \cap A$, $\mathfrak{m}_B = \mathfrak{m}_R \cap B = (\mathfrak{m}_R \cap C) \cap B$, so (iii) implies (i). the converse is clear since we can take $R = C_{\mathfrak{r}}$. □

If the equivalent conditions in Lemma 8.7.29 hold, we say the two local subrings A and B are **related**.

Proposition 8.7.30. *Let A and B be subrings of a field K and C be the subring of K generated by $A \cup B$. The following conditions are equivalent:*

(i) *For any local ring R containing A and B, we have $A_{\mathfrak{p}} = B_{\mathfrak{q}}$, where $\mathfrak{p} = \mathfrak{m}_R \cap A$ and $\mathfrak{q} = \mathfrak{m}_R \cap B$.*

(ii) *For any prime ideal $\mathfrak{r}$ of C, we have $A_{\mathfrak{p}} = B_{\mathfrak{q}}$, where $\mathfrak{p} = \mathfrak{r} \cap A$ and $\mathfrak{q} = \mathfrak{r} \cap B$.*

(iii) *If $P \in L(A)$ and $Q \in L(B)$ are related, they are identical.*

(iv) *We have $L(A) \cap L(B) = L(C)$.*

Proof. It follows from Lemma 8.7.28 and Lemma 8.7.29 that (i) and (iii) are equivalent, and (i) implies (ii) by applying (i) to the ring $R = C_{\mathfrak{r}}$. Conversely, (ii) implies (i) because if R contains $A \cup B$, it contains C, and if $\mathfrak{r} = \mathfrak{m}_R \cap C$, we have $\mathfrak{p} = \mathfrak{r} \cap A$ and $\mathfrak{q} = \mathfrak{r} \cap B$, so $A_{\mathfrak{p}} = B_{\mathfrak{q}}$. We also see that (iv) implies (i), because if R contains $A \cup B$, it then dominates a local ring $C_{\mathfrak{r}} \in L(C)$ by Lemma 8.7.28; we have by hypothesis that $L(C) = L(A) \cap L(B)$, and as R dominates a unique ring in $L(A)$ (resp. $L(B)$), we conclude that $C_{\mathfrak{r}} = A_{\mathfrak{p}} = B_{\mathfrak{q}}$.

Finally, we show that (iii) implies (iv). Let $R \in L(C)$; R then dominates a ring $P \in L(A)$ and a ring $Q \in L(B)$ by Lemma 8.7.28, so P and Q are related, hence identical by hypothesis. As we then have $C \subseteq P$, P dominates a ring $R' \in L(C)$ (Lemma 8.7.28), so R dominates the ring R', and by Lemma 8.7.28 we necessarily have $R = R' = P$, so $R \in L(A) \cap L(B)$. Conversely, if $R \in L(A) \cap L(B)$, we have $C \subseteq R$, so R dominates a ring $R'' \in L(C)$ by Lemma 8.7.28. The two subrings R and R'' are clearly related, and as $L(C) \subseteq L(A) \cap L(B)$, we conclude from condition (iii) that $R = R''$, so $R \in L(C)$ and the proof is complete. □

Proposition 8.7.31. *Let X be an integral scheme and K be its field of rational functions. Then for X to be separated, it is necessary and sufficient that the relation "$\mathscr{O}_{X,x}$ and $\mathscr{O}_{X,y}$ are related" for two points $x, y \in X$ implies $x = y$.*

Proof. Suppose the given condition on X, we prove that X is separated. Let $U = \operatorname{Spec}(A)$ and $V = \operatorname{Spec}(B)$ be two distinct affine opens of X, with A, B identified as subrings of K. Then U (resp. V) is identified with the set $L(A)$ (resp. $L(B)$), and by Proposition 8.7.30 the hypothesis on X implies that, if C is the subring of K generated by $A \cup B$, $W = U \cap V$ is identified with $L(A) \cap L(B) = L(C)$. Moreover, we have seen from **??** that any subring R of K is equal to the intersection of the local rings belong to $L(R)$, so

$$C = \bigcap_{z \in W} \mathscr{O}_{X,z} = \Gamma(W, \mathscr{O}_X) \tag{8.7.2}$$

where we use formula (8.7.1). Consider then the subscheme induced by X over W. The identity homomorphism $\varphi : C \to \Gamma(W, \mathscr{O}_X)$ corresponds to a morphism $\psi : W \to \operatorname{Spec}(C)$. In view of (8.7.2) and the relation $L(C) = L(A) \cap L(B)$, any prime ideal $\mathfrak{r}$ of C is of the form $\mathfrak{r} = \mathfrak{m}_x \cap C$, where $x \in W$ is the point in $\operatorname{Spec}(C)$ corresponding to $\mathfrak{r}$, and the map ψ just sends x to $\mathfrak{r}$, so it is bijective. On the other hand, for any $x \in W$, $\psi_x^\#$ is the canonical injection $C_{\mathfrak{r}} \to \mathscr{O}_{X,x}$, where $\mathfrak{r} = \mathfrak{m}_x \cap C$. Now the local ring $\mathscr{O}_{X,x}$ dominates $A_{\mathfrak{p}}$, $B_{\mathfrak{q}}$ and $C_{\mathfrak{r}}$, where $\mathfrak{p} = \mathfrak{m}_x \cap A$ and $\mathfrak{q} = \mathfrak{m}_x \cap B$, and as $C_{\mathfrak{r}} \in L(C) = L(A) \cap L(B)$, by Lemma 8.7.28 we then conclude that $C_{\mathfrak{r}} = A_{\mathfrak{p}} = B_{\mathfrak{q}}$ (we already have seen this in the proof of Proposition 8.7.30). But the local rings $A_{\mathfrak{p}}$ and $B_{\mathfrak{q}}$ are both identified with the stalk $\mathscr{O}_{X,x}$, so we see that

$C_{\mathfrak{r}} = \mathscr{O}_{X,x}$ and $\psi_x^\#$ is bijective. It then remains to show that ψ is a homeomorphism, which amounts to show that, for any closed subset $F \subseteq W$, the image $\psi(F)$ is closed in $\mathrm{Spec}(C)$. Now F is the intersection with W of a closed subset E of the form $V(\mathfrak{a})$, where $\mathfrak{a}$ is an ideal of A. We claim that $\psi(F) = V(\mathfrak{a}C)$: in fact, the prime ideal of C containing $\mathfrak{a}C$ are the prime ideals of C containing $\mathfrak{a}$, hence the ideals of the form $\psi(x) = \mathfrak{m}_x \cap C$, where $\mathfrak{a} \subseteq \mathfrak{m}_x$ and $x \in W$. As $\mathfrak{a} \subseteq \mathfrak{m}_x$ is equivalent to $x \in V(\mathfrak{a}) = W \cap E$ for $x \in U$, we then get $\psi(F) = V(\mathfrak{a}C)$. In view of Proposition 8.5.31, we then conclude that X is separated, because $U \cap V$ is affine and the ring C is generated by $A \cup B$.

Conversely, suppse that X is separated, and let x, y be two points of X such that $\mathscr{O}_{X,x}$ and $\mathscr{O}_{X,y}$ are related. Let U (resp. V) be an open affine containing x (resp. y), with ring A (resp. B). We then see $U \cap V$ is affine and its ring C is generated by $A \cup B$ (Proposition 8.5.31). If $\mathfrak{p} = \mathfrak{m}_x \cap A$ and $\mathfrak{q} = \mathfrak{m}_x \cap B$, we have $A_{\mathfrak{p}} = \mathscr{O}_{X,x}$ and $B_{\mathfrak{q}} = \mathscr{O}_{X,y}$, so $A_{\mathfrak{p}}$ and $B_{\mathfrak{q}}$ are related. Then by Lemma 8.7.29 there exists a prime ideal $\mathfrak{r}$ of C such that $\mathfrak{p} = \mathfrak{r} \cap A$, $\mathfrak{q} = \mathfrak{r} \cap B$. But the prime ideal $\mathfrak{r}$ then corresponds to a point $z \in U \cap V$ since $U \cap V$ is affine, and we have $x = z$ and $y = z$, so $x = y$. ☐

Corollary 8.7.32. *Let X be a separated integral scheme and x, y be two points of X. For $x \in \overline{\{y\}}$, it is necessary and sufficient that $\mathscr{O}_{X,x} \subseteq \mathscr{O}_{Y,y}$, which means the rational functions defined at x are also defined at y.*

Proof. This condition is clearly necessary since the defining domain $\delta(f)$ of a rational function is open, hence stable under generalization. To see it is also sufficient, assume that $\mathscr{O}_{X,x} \subseteq \mathscr{O}_{X,y}$, so there exists a prime ideal $\mathfrak{p}$ of $\mathscr{O}_{X,x}$ such that $\mathscr{O}_{Y,y}$ dominates $(\mathscr{O}_{X,x})_{\mathfrak{p}}$ (Lemma 8.7.28). By Corollary 8.2.12, there exists $z \in X$ such that $x \in \overline{\{z\}}$ and $\mathscr{O}_{X,z} = (\mathscr{O}_{X,x})_{\mathfrak{p}}$; as $\mathscr{O}_{X,z}$ and $\mathscr{O}_{X,y}$ are then related, we have $z = y$ by Proposition 8.7.31, whence the corollary. ☐

Corollary 8.7.33. *If X is a separated integral scheme, the map $x \mapsto \mathscr{O}_{X,x}$ is injective. In other words, if x, y are two distinct points of X, there exists a rational function defined at only one of these points.*

Proof. This follows from Corollary 8.7.32 and the T_0-axiom. ☐

Corollary 8.7.34. *Let X be a Noetherian separated integral scheme. The sets $\delta(f)$ for $f \in K(X)$ form a subbasis the topology of X.*

Proof. In fact, any closed subset of X is then a finite union of irreducible closed subsets, which are of the form $\overline{\{y\}}$. Now if $x \notin \overline{\{y\}}$, there exists a rational function f defined at x but not at y (Corollary 8.7.33), which means $x \in \delta(f)$ and $\delta(f) \cap \overline{\{y\}} = \varnothing$. The complement of $\overline{\{y\}}$ is then a union of sets of the form $\delta(f)$, and in view of the previous remark, any open subset of X is a union of finite intersections of sets of the form $\delta(f)$. ☐

Proposition 8.7.35. *Let X, Y be two integral schemes with rational function fields K and L, respectively. Suppose that Y is separated and let $f : X \to Y$ be a dominant morphism. Then L is identified with a subfield of K, and for every point $x \in X$, $\mathscr{O}_{Y,f(x)}$ is the unique local ring of Y dominated by $\mathscr{O}_{X,x}$.*

Proof. Ths first assertion is already proved in Proposition 8.4.23. Now for every $x \in X$, the local homomorphism $f_x^\# : \mathscr{O}_{Y,f(x)} \to \mathscr{O}_{X,x}$ is injective by Proposition 8.4.23, so, if we identify L as a subfield of K, $\mathscr{O}_{Y,f(x)}$ is dominated by $\mathscr{O}_{X,x}$. As Y is separated, two local rings of Y can not be dominated by the same local ring (Proposition 8.7.31), so our assertion follows. ☐

Proposition 8.7.36. *Let X be an irreducible scheme and $f : X \to Y$ be a local immersion (resp. a local isomorphism). Suppose that f is separated, then it is an immersion (resp. an open immersion).*

Proof. It suffices to prove that f is a homeomorphism from X to $f(X)$ (Proposition 8.4.9). By replacing f with f_{red}, we may assume that X and Y are reduced. If Y' is the reduced subscheme of Y with underlying space $\overline{f(X)}$, f then factors into

$$X \xrightarrow{f'} Y' \xrightarrow{j} Y$$

where j is the canonical injection. Then f' is separated by Proposition 8.5.26(v) and is a local immersion by Proposition 8.5.15(iii), so we may reduce to the case that f is dominant. But then Y is irreducible by **??**, and by Proposition 8.4.23, we see f is in fact a local isomorphism, so for each $x \in X$ the homomorphism $f_x^\#$ is an isomorphism. By Corollary 8.7.33, this implies that f is injective, so f is in fact a homeomorphism. ☐

8.8 Formal schemes

8.8.1 Formal affine schemes and morphisms

Let A be a admissible topological ring, with a nilideal $\mathfrak{I}$ (recall that this means $\mathfrak{I}$ is open and $(\mathfrak{I}^n)$ tends to 0 in A). The specturm $\mathrm{Spec}(A/\mathfrak{I})$ is then a closed subscheme of $\mathrm{Spec}(A)$, which is the set of open prime ideals of A. This topological space does not depend on the nilideal of $\mathfrak{I}$, and we denote it by $\mathfrak{X}$. Let $(\mathfrak{I}_\lambda)$ be a system of fundamental neighborhood of 0 in A, formed by the nilideals of A, and for each λ, let $\mathscr{O}_\lambda$ be the structural sheaf of $\mathrm{Spec}(A/\mathfrak{I}_\lambda)$. This sheaf is induced over $\mathfrak{X}$ by $\widetilde{A}/\widetilde{\mathscr{I}}_\lambda$ (and is zero outside $\mathfrak{X}$). For $\mathfrak{I}_\mu \subseteq \mathfrak{I}_\lambda$, the canonical homomorphism $A/\mathfrak{I}_\mu \to A/\mathfrak{I}_\lambda$ defines a homomorphism $u_{\lambda\mu} : \mathscr{O}_\mu \to \mathscr{O}_\lambda$ of sheaves of rings, and $(\mathscr{O}_\lambda)$ is a projective system of sheaves of rings for these homomorphisms. As the topology of $\mathfrak{X}$ admits a basis formed by quasi-compact open subsets, if we view each $\mathscr{O}_\lambda$ as a sheaf of discrete rings, the $\mathscr{O}_\lambda$ then form a projective system of sheaves of topological rings, and we denote by $\mathscr{O}_\mathfrak{X}$ the limit of this system $(\mathscr{O}_\lambda)$. By ([**?**] 0_I 3.2.6), for any quasi-compact open subset U of $\mathfrak{X}$, $\Gamma(U, \mathscr{O}_\mathfrak{X})$ is then the limit topological ring of the discrete rings $\Gamma(U, \mathscr{O}_\lambda)$.

Given an admissible topological ring A, the closed subspace $\mathfrak{X}$ of $\mathrm{Spec}(A)$ formed by open prime ideals of A is called the **formal spectrum** of A and denoted by $\mathrm{Spf}(A)$. A topoloigcally ringed space is called a **formal affine scheme** if it is isomorphic to a formal spectrum $\mathrm{Spf}(A) = \mathfrak{X}$ endowed with the sheaf of topological rings $\mathscr{O}_\mathfrak{X}$, which is the limit of the sheaf of discrete rings $(\widetilde{A}/\widetilde{\mathfrak{I}}_\lambda)|_\mathfrak{X}$, where $\mathfrak{I}_\lambda$ runs through the filtered set of nilideals of A. When we speak of a formal spectrum $\mathfrak{X} = \mathrm{Spf}(A)$ as an formal affine scheme, it will always be understood that the topologically ringed space $(\mathfrak{X}, \mathscr{O}_\mathfrak{X})$ where $\mathscr{O}_\mathfrak{X}$ is defined as above. By an **adic** (resp. **Noetherian**) formal affine scheme, we mean a formal affine scheme which is isomorphic to a formal specturm $\mathrm{Spf}(A)$, where A is adic (resp. adic and Noetherian).

We note that any affine scheme $X = \mathrm{Spec}(A)$ can be considered as a formal affine scheme in a unique way: consider A as a discrete topological ring, the rings $\Gamma(U, \mathscr{O}_X)$ are then discrete if U is quasi-compact (but not true in general if U is any open set of X).

Proposition 8.8.1. *If $\mathfrak{X} = \mathrm{Spf}(A)$, where A is an admissible ring, then $\Gamma(X, \mathscr{O}_\mathfrak{X})$ is homeomorphic to A.*

Proof. In fact, as $\mathfrak{X}$ is closed in $\mathrm{Spec}(A)$, it is quasi-compact, and therefore $\Gamma(\mathfrak{X}, \mathscr{O}_\mathfrak{X})$ is homeomorphic to the limit of the discrete rings $\Gamma(\mathfrak{X}, \mathscr{O}_\lambda)$. But $\Gamma(\mathfrak{X}, \mathscr{O}_\lambda)$ is isomorphic to $A/\mathfrak{I}_\lambda$, and as A is separated and complete, this is homeomorphic to $\varinjlim A/\mathfrak{I}_\lambda$, whence the proposition. □

Proposition 8.8.2. *Let A be an admissible ring, $\mathfrak{X} = \mathrm{Spf}(A)$, and for $f \in A$, let $\mathfrak{D}(f) = D(f) \cap \mathfrak{X}$. Then the topologically ringed space $(\mathfrak{D}(f), \mathscr{O}_\mathfrak{X}|_{\mathfrak{D}(f)})$ is isomorphic to a formal affine spectrum $\mathrm{Spf}(A_{\{f\}})$.*

Proof. For any nilideal $\mathfrak{I}$ of A, the discrete ring $A_f/\mathfrak{I}_f$ is canonically identified with $A_{\{f\}}/\mathfrak{I}_{\{f\}}$, so the topological space $\mathrm{Spf}(A_{\{f\}})$ is canonically identified with $\mathfrak{D}(f)$. Moreover, for any quasi-compact open U of $\mathfrak{X}$ contained in $\mathfrak{D}(f)$, $\Gamma(U, \mathscr{O}_\lambda)$ is identified with the module of sections of the structural sheaf of $\mathrm{Spec}(A_f/\mathfrak{I}_\lambda)$ over U, so, if we put $\mathfrak{Y} = \mathrm{Spf}(A_{\{f\}})$, $\Gamma(U, \mathscr{O}_\mathfrak{X})$ is identified with $\Gamma(U, \mathscr{O}_\mathfrak{Y})$, whence the proposition. □

As a sheaf of rings, the stalk of the structural sheaf $\mathscr{O}_\mathfrak{X}$ of $\mathrm{Spf}(A)$ for any $x \in X$ is, by Proposition 8.8.2, identified with the inductive limit $\varinjlim A_{\{f\}}$ for $f \notin \mathfrak{p}_x$. Therefore, by **??** and **??**, we have the following:

Proposition 8.8.3. *For any $x \in \mathfrak{X} = \mathrm{Spf}(A)$, the stalk $\mathscr{O}_{\mathfrak{X},x}$ is a local ring whose residue field is isomorphic to $\kappa(x)$. If A is adic and Noetherian, then $\mathscr{O}_{\mathfrak{X},x}$ is a Noetherian ring.*

As the field $\kappa(x)$ is not reduced to 0, we conclude in particular that the support of $\mathscr{O}_\mathfrak{X}$ is equal to $\mathfrak{X}$, and $(\mathfrak{X}, \mathscr{O}_\mathfrak{X})$ is a locally topologically ringed space.

We now consider morphisms of formal affine schemes. Let A, B be admissible rings, and $\varphi : B \to A$ be a continuous homomorphism. The continuous map ${}^a\varphi : \mathrm{Spec}(A) \to \mathrm{Spec}(B)$ then maps $\mathfrak{X} = \mathrm{Spf}(A)$ into $\mathfrak{Y} = \mathrm{Spf}(B)$, because the inverse image of an open prime ideal of A is an open prime ideal of B. On the other hand, for any $g \in B$, φ defines a continuous homomorphism $\Gamma(\mathfrak{D}(g), \mathscr{O}_\mathfrak{Y}) \to \Gamma(\mathfrak{D}(\varphi(g), \mathscr{O}_\mathfrak{X}))$ in view of Proposition 8.8.1 and Proposition 8.8.2; as these homomorphisms are compatible with restrictions and $\mathfrak{D}(\varphi(g)) = ({}^a\varphi)^{-1}(\mathfrak{D}(g))$, we obtain a continuous homomorphism of sheaves of topological rings $\mathscr{O}_\mathfrak{Y} \to {}^a\varphi_*(\mathscr{O}_\mathfrak{X})$, which we denoted by $\tilde{\varphi}$. We then get a morphism $({}^a\varphi, \tilde{\varphi}) : (X, \mathscr{O}_\mathfrak{X}) \to (\mathfrak{Y}, \mathscr{O}_\mathfrak{Y})$ of topologically ringed spaces.

Proposition 8.8.4. *Let A, B be admissible topological rings, and $\mathfrak{X} = \mathrm{Spf}(A)$, $\mathfrak{Y} = \mathrm{Spf}(B)$. For a morphism $\psi : \mathfrak{X} \to \mathfrak{Y}$ of topologically ringed spaces to be of the form $({}^a\varphi, \tilde{\varphi}) : \mathfrak{X} \to \mathfrak{Y}$, it is necessary and sufficient that for each $x \in X$, $f_x^\# : \mathscr{O}_{\mathfrak{Y},\psi(x)} \to \mathscr{O}_{\mathfrak{X},x}$ is a local homomorphism.*

Proof. This conditions is necessary: in fact, let $\mathfrak{p} = \mathfrak{p}_x \in \mathrm{Spf}(A)$, and $\mathfrak{q} = \varphi^{-1}(\mathfrak{p}_x)$; if $g \notin \mathfrak{q}$, then $\varphi(g) \notin \mathfrak{p}$, and it is immediate that the homomorphism $B_{\{g\}} \to A_{\{\varphi(g)\}}$ induced from φ maps $\mathfrak{q}_{\{g\}}$ into $\mathfrak{p}_{\{\varphi(g)\}}$; by passing to inductive limit, we then see that $\tilde{\varphi}_x$ is a local homomorphism.

Conversely, let ψ be a morphism satisfying this condition. By Proposition 8.8.1, $\psi^{\#}$ defines a continuous homomorphism

$$\varphi = \Gamma(\psi^{\#}) : B = \Gamma(\mathfrak{Y}, \mathscr{O}_{\mathfrak{Y}}) \to \Gamma(\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}) = A.$$

By the hypothesis on $\psi^{\#}$, for the section $\varphi(g)$ of $\mathscr{O}_{\mathfrak{X}}$ over $\mathfrak{X}$ has invertible germ at a point x, it is necessary and sufficient that g has invertible germ at $\psi(x)$. But by **??**, the sections of $\mathscr{O}_{\mathfrak{X}}$ (resp. $\mathscr{O}_{\mathfrak{Y}}$) over $\mathfrak{X}$ (resp. $\mathfrak{Y}$) which have non-invertible germs at x (resp. $\psi(x)$) are exactly the elements of $\mathfrak{p}_x$ (resp $\mathfrak{p}_{\psi(x)}$), so we conclude that $\psi = {}^a\varphi$. Finally, for any $g \in B$, the diagram

$$\begin{tikzcd}
B = \Gamma(\mathfrak{Y}, \mathscr{O}_{\mathfrak{Y}}) \arrow[r, "\varphi"] \arrow[d] & \Gamma(\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}) = A \arrow[d] \\
B_{\{g\}} = \Gamma(\mathfrak{D}(g), \mathscr{O}_{\mathfrak{Y}}) \arrow[r, "\Gamma(\psi^{\#}_{\mathfrak{D}(g)})"] & \Gamma(\mathfrak{D}(\varphi(g), \mathscr{O}_{\mathfrak{X}})) = A_{\{\varphi(g)\}}
\end{tikzcd}$$

is commutative. By the universal property of localization of complete rings (**??**), we conclude that $\psi^{\#}_{\mathfrak{D}(g)}$ is equal to $\tilde{\varphi}_{\mathfrak{D}(g)}$ for $g \in B$, so we have $\psi^{\#} = \tilde{\varphi}$. $\square$

We say a morphism $\psi : \mathfrak{X} \to \mathfrak{Y}$ satisfying the condition in Proposition 8.8.4 is a **morphism of formal affine schemes**. Then by Proposition 8.8.4, the functor $A \mapsto \mathrm{Spf}(A)$ and $\mathfrak{X} \to \Gamma(\mathfrak{X}, \mathscr{O}_{\mathfrak{X}})$ define an equivalence from the category of admissible topological rings to the opposite of the category of formal affine schemes.

As a particular case of Proposition 8.8.4, note that for $f \in A$, the canonical injection of the formal affine scheme over $\mathfrak{D}(f)$ induced by $\mathfrak{X}$ corresponds to the canonical homomorphism $A \to A_{\{f\}}$. Under the hypothesis of Proposition 8.8.4, let h be an element of B and g be an element of A, which is a multiple of $\varphi(h)$. We then have $\psi(\mathfrak{D}(g)) \subseteq \mathfrak{D}(h)$; the restriction of ψ to $\mathfrak{D}(g)$, considered as a morphism $\mathfrak{D}(g) \to \mathfrak{D}(h)$, is the unique morphism η such that the diagram

$$\begin{tikzcd}
\mathfrak{D}(g) \arrow[r, "\eta"] \arrow[d] & \mathfrak{D}(h) \arrow[d] \\
\mathfrak{X} \arrow[r, "\psi"] & \mathfrak{Y}
\end{tikzcd}$$

This morphism corresponds to the unique continuous homomorphism $\tilde{\varphi} : B_{\{h\}} \to A_{\{g\}}$ such that the diagram

$$\begin{tikzcd}
B \arrow[r, "\varphi"] \arrow[d] & A \arrow[d] \\
B_{\{h\}} \arrow[r, "\tilde{\varphi}"] & A_{\{g\}}
\end{tikzcd}$$

is commutative.

Let A be an admissible ring, $\mathfrak{J}$ be an open ideal of A, and $\mathfrak{X}$ be the formal affine scheme $\mathrm{Spf}(A)$. Let $(\mathfrak{J}_\lambda)$ be the set of nilideals of A contained in $\mathfrak{J}$; then $\widetilde{\mathfrak{J}}/\widetilde{\mathfrak{J}}_\lambda$ is a sheaf of ideals of $\widetilde{A}/\widetilde{\mathfrak{J}}_\lambda$. We denote by $\mathfrak{J}^\Delta$ the projective limit of the sheaves induced by $\widetilde{\mathfrak{J}}/\widetilde{\mathfrak{J}}_\lambda$ over $\mathfrak{X}$, which is considered as an ideal of $\mathscr{O}_{\mathfrak{X}}$. For any $f \in A$, $\Gamma(\mathfrak{D}(f), \mathfrak{J}^\Delta)$ is the projective limit of $\mathfrak{J}_f/(\mathscr{J}_\lambda)_f$, which is identified with the open ideal $\mathfrak{J}_{\{f\}}$ of the ring $A_{\{f\}}$, and in particular $\Gamma(\mathfrak{X}, \mathfrak{J}^\Delta) = \mathfrak{J}$. We then conclude that (the $\mathfrak{D}(f)$ form a base of $\mathfrak{X}$) that we have

$$\mathfrak{J}^\Delta|_{\mathfrak{D}(f)} = (\mathfrak{J}_{\{f\}})^\Delta \tag{8.8.1}$$

With these notations, for $f \in A$ the canonical map of $A_{\{f\}} = \Gamma(\mathfrak{D}, \mathscr{O}_{\mathfrak{X}})$ in $\Gamma(\mathfrak{D}(f), (\widetilde{A}/\widetilde{\mathfrak{J}})|_{\mathfrak{X}}) = A_f/\mathfrak{J}_f$ is surjective with kernel $\Gamma(\mathfrak{D}(f), \mathfrak{J}^\Delta) = \mathfrak{J}_{\{f\}}$. These maps define a canonical continuous epimorphism from the sheaf $\mathscr{O}_{\mathfrak{X}}$ to the sheaf of discrete rings $(\widetilde{A}/\widetilde{\mathfrak{J}})|_{\mathfrak{X}}$, whose kernel is $\mathfrak{J}^\Delta$; this homomorphism is none other than the homomorphism $\tilde{\varphi}$, where φ is the canonical continuous homomorphism $A \to A/\mathfrak{J}$. The morphism $({}^a\varphi, \tilde{\varphi}) : \mathrm{Spec}(A/\mathfrak{J}) \to \mathfrak{X}$ of the formal affine schemes is then called the canonical morphism. We then have a canonical isomorphism

$$\mathscr{O}_{\mathfrak{X}}/\mathfrak{J}^{\Delta} \xrightarrow{\sim} (\widetilde{A}/\widetilde{\mathfrak{J}})|_{\mathfrak{X}}. \tag{8.8.2}$$

It is clear (in view of $\Gamma(X, \mathfrak{J}^{\Delta}) = \mathfrak{J}$) that the map $\mathfrak{J} \mapsto \mathfrak{J}^{\Delta}$ is strictly increasing: in fact, for $\mathfrak{J} \subseteq \mathfrak{J}'$, the sheaf $\mathfrak{J}'^{\Delta}/\mathfrak{J}^{\Delta}$ is canonically isomorphic to $\widetilde{\mathfrak{J}}'/\widetilde{\mathfrak{J}} = \widetilde{\mathfrak{J}'/\mathfrak{J}}$.

An ideal $\mathscr{J}$ of $\mathscr{O}_{\mathfrak{X}}$ is called a **nilideal** of $\mathfrak{X}$ if, for any $x \in \mathfrak{X}$, there exists an open neighborhood of x in $\mathfrak{X}$ of the form $\mathfrak{D}(f)$, where $f \in A$, such that $\mathscr{J}|_{\mathfrak{D}(f)}$ is of the form $\mathfrak{J}^{\Delta}$ for a nilideal $\mathfrak{J}$ of $A_{\{f\}}$. It is clear from our definiton that for any $f \in A$, any nilideal of $\mathfrak{X}$ induces a nilideal of $\mathfrak{D}(f)$.

Proposition 8.8.5. *If A is an admissible ring, any nilideal of $\mathfrak{X} = \mathrm{Spf}(A)$ is of the form $\mathfrak{J}^{\Delta}$, where $\mathfrak{J}$ is a nilideal of A.*

Proof. Let $\mathscr{J}$ be a nilideal of $\mathfrak{X}$; by hypothesis, and since $\mathfrak{X}$ is quasi-compact, there exist finitely many elements $f_i \in A$ such that $\mathfrak{D}(f)$ cover $\mathfrak{X}$ and such that $\mathscr{J}|_{\mathfrak{D}(f)} = \mathfrak{K}_i$, where $\mathfrak{K}_i$ is a nilideal of $A_{\{f_i\}}$. For any i, there then exists an open ideal $\mathfrak{N}_i$ of A such that $(\mathfrak{N}_i)_{\{f_i\}} = \mathfrak{K}_i$; let $\mathfrak{N}$ be a nilideal of A contained in each the $\mathfrak{N}_i$. The canonical image of $\mathscr{J}/\mathfrak{N}^{\Delta}$ in the structural sheaf $\widetilde{(A/\mathfrak{N})}$ of $\mathrm{Spec}(A/\mathfrak{N})$ is then such that its restriction to each $\mathfrak{D}(f_i)$ is equal to $\widetilde{\mathfrak{N}_i/\mathfrak{N}}$; we then conclude that this canonical image is a quasi-coherent ideal over $\mathrm{Spec}(A/\mathfrak{N})$, hence is of the form $\widetilde{\mathfrak{J}/\mathfrak{N}}$, where $\mathfrak{J}$ is an ideal of A containing $\mathfrak{N}$, and whence $\mathscr{J} = \mathfrak{J}^{\Delta}$ by (8.8.2). Moreover, as for each i there exists an integer n_i such that $\mathfrak{K}_i^{n_i} \subseteq \mathfrak{N}_{\{f_i\}}$, we have $(\mathscr{J}/\mathfrak{N}^{\Delta})^n = 0$ for n sufficiently large, and therefore $(\widetilde{\mathfrak{J}/\mathfrak{N}})^n = 0$, and finally $(\mathfrak{J}/\mathfrak{N})^n = 0$, which proves that $\mathfrak{J}$ is a nilideal of A. □

Proposition 8.8.6. *Let A be an adic ring, $\mathfrak{J}$ be a nilideal of A such that $\mathfrak{J}/\mathfrak{J}^2$ is an $A/\mathfrak{J}$ of finte type. For any integer $n > 0$, we then have $(\mathfrak{J}^{\Delta})^n = (\mathfrak{J}^n)^{\Delta}$.*

Proof. In fact, for any $f \in A$ we have (since $\mathfrak{J}^n$ is an open ideal)

$$(\Gamma(\mathfrak{D}(f), \mathfrak{J}^{\Delta}))^n = (\mathfrak{J}_{\{f\}})^n = (\mathfrak{J}^n)_{\{f\}} = \Gamma(\mathfrak{D}(f^n), (\mathfrak{J}^n)^{\Delta})$$

in view of (8.8.1) and **??**. As $(\mathfrak{J}^{\Delta})^n$ is associated with the presheaf $U \mapsto (\Gamma(U, \mathfrak{J}^{\Delta}))^n$, the corollary then follows since $\mathfrak{D}(f)$ form a basis for $\mathfrak{X}$. □

A family $(\mathscr{J}_\lambda)$ of nilideals of $\mathfrak{X}$ is called a **fundamental system of nilideals** if any nilideal of $\mathscr{J}$ contains at least one of these $\mathscr{J}_\lambda$. As $\mathscr{J}_\lambda = \mathfrak{J}_\lambda^{\Delta}$, this is equivalent to saying that the $\mathfrak{J}_\lambda$ form a fundamental neighborhood of 0 in A, where $\mathscr{J}_\lambda = \mathscr{J}_\lambda^{\Delta}$. Let (f_α) be a family of elements of A such that the $\mathfrak{D}(f_\alpha)$ cover $\mathfrak{X}$. If $(\mathscr{J}_\lambda)$ is a filtered decreasing family of ideals of $\mathscr{O}_{\mathfrak{X}}$ such that for any α, the family $(\mathscr{J}_\lambda|_{\mathfrak{D}(f_\alpha)})$ is a fundamental system of nilideals of $\mathfrak{D}(f_\alpha)$, then $(\mathscr{J}_\lambda)$ is a fundamental system of nilideals of $\mathfrak{X}$. In fact, for any nilideal of $\mathfrak{X}$, there exists a finite covering of $\mathfrak{X}$ by the $\mathfrak{D}(f_i)$ such that, for any i, $\mathscr{J}_{\lambda_i}|_{\mathfrak{D}(f_i)}$ is a nilideal of $\mathfrak{D}(f_i)$ contained in $\mathfrak{J}|_{\mathfrak{D}(f_i)}$. If μ is an index such that $\mathscr{J}_\mu \subseteq \mathscr{J}_{\lambda_i}$ for all i, then $\mathscr{J}_\mu$ is a nilideal of $\mathfrak{X}$ which is evidently contained in $\mathscr{J}$, whence the assertion.

8.8.2 Formal schemes and morphisms

Given a topologically ringed space $\mathfrak{X}$, we say an open subset $U \subseteq \mathfrak{X}$ is a **formal affine open** (resp. **an adic formal affine open**, resp. **a Noetherian formal affine open**) if the topologically ringed space $(U, \mathscr{O}_{\mathfrak{X}}|_U)$ is a formal affine scheme (resp. an adic formal affine scheme, resp. a Noetherian formal affine scheme). We way $(\mathfrak{X}, \mathscr{O}_{\mathfrak{X}})$ is a **formal scheme** (resp. **adic formal scheme**, resp. **locally Noetherian formal scheme**) if each of its point admits a formal affine open neighborhood (resp. an adic formal affine open, resp. a locally Noetherian formal affine open). We say that $\mathfrak{X}$ is **Noetherian** if it is locally Noetherian and the underlying space is quasi-compact (hence Noetherian). As any affine scheme can be considered as a formal affine scheme, any scheme can be considered as a formal scheme.

Proposition 8.8.7. *If $\mathfrak{X}$ is a formal scheme (resp. locally Noetherian formal scheme), the set of formal affine opens (resp. Noetherian formal affine opens) form a base for $\mathfrak{X}$.*

Proof. This follows from Proposition 8.8.2, and the fact that if A is a Noetherian adic ring, so is $A_{\{f\}}$ for any $f \in A$ (**??**). □

Corollary 8.8.8. *If $\mathfrak{X}$ is a formal scheme (resp. locally Noetherian formal scheme, resp. Noetherian formal scheme), the topological ringed space over any open subset of $\mathfrak{X}$ is a formal scheme (resp. locally Noetherian formal scheme, resp. Noetherian formal scheme).*

Given two formal schemes $\mathfrak{X}, \mathfrak{Y}$, we say that $f : \mathfrak{X} \to \mathfrak{Y}$ is a morphism of formal schemes if it is a morphism of the underlying locally ringed spaces. That is, if $(f, f^\#)$ is a morphism of ringed spaces and $f_x^\# : \mathscr{O}_{\mathfrak{Y}, f(x)} \to \mathscr{O}_{\mathfrak{X}, x}$ is a local homomorphism. The composition of two morphisms are defined as the same and clearly a morphism of formal schemes. The formal schemes then form a category, which we denoted by **Schf**, and we denote by $\mathrm{Hom}_{\mathbf{Schf}}(\mathfrak{X}, \mathfrak{Y})$ the set of morphisms of formal schemes $\mathfrak{X} \to \mathfrak{Y}$.

If U is an open subset of $\mathfrak{X}$, the canonical injection $U \to \mathfrak{X}$ is then a morphism of formal schemes, if we endow U the formal scheme structure induced by $\mathfrak{X}$. It is clear that this morphism is a monomorphism in the category **Schf**.

Proposition 8.8.9. *Let $\mathfrak{X}$ be a formal scheme, $\mathfrak{Y} = \mathrm{Spec}(A)$ be a formal affine scheme. Then there exists a canonical bijection*

$$\mathrm{Hom}_{\mathbf{Scf}}(\mathfrak{X}, \mathfrak{Y}) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{TopRing}}(A, \Gamma(\mathfrak{X}, \mathscr{O}_{\mathfrak{X}})).$$

Proof. We first note that, if $(\mathfrak{X}, \mathscr{O}_{\mathfrak{X}})$ and $(\mathfrak{Y}, \mathscr{O}_{\mathfrak{Y}})$ are two topologically ringed spaces, a morphism $f : (\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}) \to (\mathfrak{Y}, \mathscr{O}_{\mathfrak{Y}})$ defines canonically a continuous homomorphism of rings $\varphi : \Gamma(\mathfrak{Y}, \mathscr{O}_{\mathfrak{Y}}) \to \Gamma(\mathfrak{X}, \mathscr{O}_{\mathfrak{X}})$. In our case, we need to show that a continuous hommorphism $\varphi : A \to \Gamma(\mathfrak{X}, \mathscr{O}_{\mathfrak{X}})$ is of the form $\Gamma(\psi^\#)$ for a unique morphism $(\psi, \psi^\#)$. Now there exists by hypothesis a covering (V_α) of $\mathfrak{X}$ by formal affine opens; by composing φ with the restriction homomorphisms $\Gamma(\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}) \to \Gamma(V_\alpha, \mathscr{O}_{\mathfrak{X}}|_{V_\alpha})$, we obtain a continuous homomorphism $\varphi_\alpha : A \to \Gamma(V_\alpha, \mathscr{O}_{\mathfrak{X}}|_{V_\alpha})$, which corresponds to a unique morphism $\psi_\alpha : (V_\alpha, \mathscr{O}_{\mathfrak{X}}|_{V_\alpha}) \to (\mathfrak{Y}, \mathscr{O}_{\mathfrak{Y}})$, in view of Proposition 8.8.4. Moreover, for any couple (α, β) of indices, any point of $V_\alpha \cap V_\beta$ admits a formal affine open neighborhood W contained in $V_\alpha \cap V_\beta$ and it is clear that the compositions of φ_α and φ_β with the canonical restriction are the same continuous homomorphism $\Gamma(\mathfrak{Y}, \mathscr{O}_{\mathfrak{Y}}) \to \Gamma(W, \mathscr{O}_{\mathfrak{X}}|_W)$, so, in view of the relations $(\psi_\alpha^\#)_x = (\tilde{\varphi}_\alpha)_x$ for any $x \in V_\alpha$, the restrictions of ψ_α and ψ_β coincides on $V_\alpha \cap V_\beta$. We then conclude that there exists a unique morphism $\psi : (\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}) \to (\mathfrak{Y}, \mathscr{O}_{\mathfrak{Y}})$ whose restriction to V_α coincides with ψ_α, and it is clear that this is the unique morphism such that $\Gamma(\psi^\#) = \varphi$. □

Given a formal scheme $\mathfrak{S}$, a **formal $\mathfrak{S}$-scheme** is defined to be a formal scheme $\mathfrak{X}$ together with a morphism $\varphi : \mathfrak{X} \to \mathfrak{S}$, called the **structural morphism** of $\mathfrak{X}$. If $\mathfrak{S} = \mathrm{Spf}(A)$, where A is an admissible ring, we also say that the $\mathfrak{S}$-formal scheme $\mathfrak{X}$ is a formal A-scheme or a formal scheme over A. Any formal scheme can be clearly considered as a formal scheme over $\mathbb{Z}$ (endowed with the discrete topology).

If $\mathfrak{X}, \mathfrak{Y}$ are two formal $\mathfrak{S}$-schemes, we say a morphism $f : \mathfrak{X} \to \mathfrak{Y}$ is an **$\mathfrak{S}$-morphism** if the diagram

$$\begin{tikzcd}
\mathfrak{X} \arrow[rr, "f"] \arrow[dr] & & \mathfrak{Y} \arrow[dl] \\
& \mathfrak{S} &
\end{tikzcd}$$

where the vertical arrows are structural morphisms, is commutative. With this definition, the $\mathfrak{S}$-schemes form (for $\mathfrak{S}$ fixed) a category $\mathbf{Scf}_{\mathfrak{S}}$. We denote by $\mathrm{Hom}_{\mathfrak{S}}(\mathfrak{X}, \mathfrak{Y})$ the set of $\mathfrak{S}$-morphisms $\mathfrak{X} \to \mathfrak{Y}$. If $\mathfrak{S} = \mathrm{Spf}(A)$, we also say A-morphism for $\mathfrak{S}$-morphisms.

Let $\mathfrak{X}$ be a formal scheme; we say an ideal $\mathscr{I}$ of $\mathscr{O}_{\mathfrak{X}}$ is a nilideal of $\mathfrak{X}$ if any $x \in \mathfrak{X}$ admits a formal affine open neighborhood U such that $\mathscr{I}|_U$ is a nilideal of the formal scheme U induced by $\mathfrak{X}$. In view of Proposition 8.8.7, for any open $V \subseteq \mathfrak{X}$, $\mathscr{I}|_V$ is then a nilideal of the formal scheme induced over V.

A family $(\mathscr{I}_\lambda)$ of nilideals of $\mathfrak{X}$ is called a **fundamental system of nilideals** if there exists a covering (U_α) of $\mathfrak{X}$ by formal affine opens such that, for any α, the family $(\mathscr{I}_\lambda|_{U_\alpha})$ form a fundamental system of nilideals of U_α. For any open subset V of $\mathfrak{X}$, the family $(\mathscr{I}_\lambda|_V)$ then forms a fundamental system of nilideals for V, in view of (8.8.1). If $\mathfrak{X}$ is locally Noetherian, and $\mathscr{I}$ is a nilideal of $\mathfrak{X}$, it then follows from Proposition 8.8.6 that the powers of $\mathscr{I}^n$ form a fundamental system of nilideals of $\mathfrak{X}$.

Let $\mathfrak{X}$ be a formal scheme, $\mathscr{I}$ be a nilideal of $\mathfrak{X}$. Then the ringed space $(\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}/\mathscr{I})$ is a scheme, which is affine (resp. locally Noetherian, resp. Noetherian) if $\mathfrak{X}$ is a formal affine scheme (resp. a locally Noetherian formal scheme, resp. a Noetherian formal scheme). Moreover, if $\varphi : \mathscr{O}_{\mathfrak{X}} \to \mathscr{O}_{\mathfrak{X}}/\mathscr{I}$ is the canonical homomorphism, then $(1_{\mathfrak{X}}, \varphi)$ is a morphism (called canonical) of formal schemes $(\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}/\mathscr{I}) \to (\mathfrak{X}, \mathscr{O}_{\mathfrak{X}})$.

Proposition 8.8.10. *Let $\mathfrak{X}$ be a formal scheme, $(\mathscr{I}_\lambda)$ be a fundamental system of nilideals of $\mathfrak{X}$. Then the sheaf $\mathscr{O}_\mathfrak{X}$ is the projective limit of the sheaf of discrete rings $\mathscr{O}_\mathfrak{X}/\mathscr{I}_\lambda$.*

Proof. As $\mathfrak{X}$ admits a basis by quasi-compact open sets, we are reduced to the affine case, where the proposition follows from Proposition 8.8.5 and the definition of $\mathscr{O}_\mathfrak{X}$. □

Proposition 8.8.11. *Let $\mathfrak{X}$ be a locally Noetherian formal scheme. Then there exists a largest nilideal $\mathscr{T}$ of $\mathfrak{X}$, which is the unique nilideal $\mathscr{I}$ such that the scheme $(\mathfrak{X}, \mathscr{O}_\mathfrak{X}/\mathscr{I})$ is reduced. If $\mathscr{I}$ is a nilideal of $\mathfrak{X}$, then $\mathscr{T}$ is the inverse image of the nilradical of $\mathscr{O}_\mathfrak{X}/\mathscr{I}$ under the homomorphism $\mathscr{O}_\mathfrak{X} \to \mathscr{O}_\mathfrak{X}/\mathscr{I}$. The reduced (usual) scheme $(\mathfrak{X}, \mathscr{O}_\mathfrak{X}/\mathscr{T})$ is denoted by $\mathfrak{X}_{\mathrm{red}}$.*

Proof. Suppose first that $\mathfrak{X} = \mathrm{Spf}(A)$, where A is a Noetherian adic ring. The existence of $\mathscr{T}$ and its properties then follows from Proposition 8.8.5, in view of **??** about the largest nilideal of A. To prove the existence of $\mathscr{T}$ in the general case, it suffices to prove that if $V \subseteq U$ are two Noetherian formal affine opens of X, the largest nilideal $\mathscr{T}_U$ of U induces the largest nilideal $\mathscr{T}_V$ of V; but as $\Gamma(V, (\mathscr{O}_\mathfrak{X}|_V)/(\mathscr{T}_U|_V))$ is reduced, this is immediate. □

Corollary 8.8.12. *Let $\mathfrak{X}$ be a locally Noetherian formal scheme, $\mathscr{T}$ be the largest nilideal of $\mathfrak{X}$. Then for any open subset V of $\mathfrak{X}$, $\mathscr{T}|_V$ is the largest nilideal of V.*

Proof. This is already shown in the proof of Proposition 8.8.11. □

Proposition 8.8.13. *Let $\mathfrak{X}, \mathfrak{Y}$ be formal schemes, $\mathscr{I}$ (resp. $\mathscr{K}$) be the nilideal of $\mathfrak{X}$ (resp. $\mathfrak{Y}$), $f : \mathfrak{X} \to \mathfrak{Y}$ be a morphism of formal schemes.*

(i) *If $f^*(\mathscr{K})\mathscr{O}_\mathfrak{X} \subseteq \mathscr{I}$, there exists a unique morphism*

$$f_{\mathrm{red}} : (\mathfrak{X}, \mathscr{O}_\mathfrak{X}/\mathscr{I}) \to (\mathfrak{Y}, \mathscr{O}_\mathfrak{Y}/\mathscr{K})$$

of schemes such that the following diagram

$$\begin{tikzcd}
(\mathfrak{X}, \mathscr{O}_\mathfrak{X}/\mathscr{I}) \arrow[r, "f_{\mathrm{red}}"] \arrow[d] & (\mathfrak{Y}, \mathscr{O}_\mathfrak{Y}/\mathscr{K}) \arrow[d] \\
(\mathfrak{X}, \mathscr{O}_X) \arrow[r, "f"] & (\mathfrak{Y}, \mathscr{O}_\mathfrak{Y})
\end{tikzcd} \tag{8.8.3}$$

(ii) *Suppose that $\mathfrak{X} = \mathrm{Spf}(A)$, $\mathfrak{Y} = \mathrm{Spf}(B)$ are formal affine schemes, $\mathscr{I} = \mathfrak{I}^\Delta$, $\mathscr{K} = \mathfrak{K}^\Delta$, where $\mathfrak{I}$ (resp. $\mathfrak{K}$) is a nilideal of A (reps. B), and $f = ({}^a\varphi, \tilde{\varphi})$ where $\varphi : B \to A$ is a continuous homomorphism. For $f^*(\mathscr{K})\mathscr{O}_\mathfrak{X} \subseteq \mathscr{K}$, it is necessary and sufficient that $\varphi(\mathfrak{K}) \subseteq \mathfrak{I}$, and f_{red} is then the morphism $({}^a\varphi', \tilde{\varphi}')$, where $\varphi' : B/\mathfrak{K} \to A/\mathfrak{I}$ is the induced homomorphism by passing to quotients.*

Proof. In case (a), the hypothesis implies that the image of the ideal $f^{-1}(\mathscr{K})$ of $f^{-1}(\mathscr{O}_\mathfrak{Y})$ under $f^\# : f^{-1}(\mathscr{O}_\mathfrak{Y}) \to \mathscr{O}_\mathfrak{X}$ is contained in $\mathscr{I}$. By passing to quotients, we then deduce that $f^\#$ is a homomorphism

$$\omega : f^{-1}(\mathscr{O}_\mathfrak{Y}/\mathscr{K}) = f^{-1}(\mathscr{O}_\mathfrak{Y})/f^{-1}(\mathscr{K}) \to \mathscr{O}_\mathfrak{X}/\mathscr{I};$$

moreover, as for any $x \in \mathfrak{X}$, $f_x^\#$ is a local homomorphism, so is ω_x. The morphism $(f, \omega^\flat)$ is then the unique morphism of ringed spaces satisfying the requirements.

With the assumptions of (b), the canonical correspondence between morphisms of formal affine schemes and continuous homomorphisms of ringed spaces shows that the relation $f^*(\mathscr{K})\mathscr{O}_\mathfrak{X} \subseteq \mathscr{I}$ implies that $f' = ({}^a\varphi', \tilde{\varphi}')$, where $\varphi' : B/\mathfrak{K} \to A/\mathfrak{I}$ is the unique homomorphism such that the diagram

$$\begin{tikzcd}
B \arrow[r, "\varphi"] \arrow[d] & A \arrow[d] \\
B/\mathfrak{K} \arrow[r, "\varphi'"] & A/\mathfrak{I}
\end{tikzcd} \tag{8.8.4}$$

The existence of φ' implies then that $\varphi(\mathfrak{K}) \subseteq \mathfrak{I}$. Conversely, if this condition is verified, we have a canonical homomorphism $\varphi' : B/\mathfrak{K} \to A/\mathfrak{I}$, whence the induced morphism $f' = ({}^a\varphi', \tilde{\varphi}')$ satisfies the commutativity of (8.8.4). By considering the homomorphisms ${}^a\varphi^*(\mathscr{O}_\mathfrak{Y}) \to \mathscr{O}_\mathfrak{X}$ and ${}^a\varphi'^*(\mathscr{O}_\mathfrak{Y}/\mathscr{K}) \to \mathscr{O}_\mathfrak{X}/\mathscr{I}$, we then see that $f^*(\mathscr{K})\mathscr{O}_\mathfrak{X} \subseteq \mathscr{I}$. □

To conclude this paragraph, we discuss the fiber product of formal schemes. Let $\mathfrak{S}$ be a formal scheme. It turns out that the fiber product of formal schemes has the same construction as that of usual schemes, provided that we replace tensor products by completed tensor products.

Proposition 8.8.14. *Let $\mathfrak{X} = \mathrm{Spf}(B)$, $\mathfrak{Y} = \mathrm{Spf}(C)$ be two formal affine schemes over a formal affine scheme $\mathrm{Spf}(A)$. Let $\mathfrak{Z} = \mathrm{Spf}(B\widehat{\otimes}_A C)$ and p_1, p_2 be the $\mathfrak{S}$-morphisms corresponding to the A-homomorphisms $\rho_1 : B \to B\widehat{\otimes}_A C$ and $\rho_2 : C \to B\widehat{\otimes}_A C$. Then $(\mathfrak{Z}, p_1, p_2)$ forms a product in the category of the formal $\mathfrak{S}$-schemes $\mathfrak{X}$ and $\mathfrak{Y}$.*

Proof. In view of Proposition 8.8.4, we note that for any continuous A-homomorphism $\varphi : B\widehat{\otimes}_A C \to D$, where D is an admissible ring that is a topological A-algebra, we can associate the couple $(\varphi \circ \sigma_1, \varphi \circ \sigma_2)$, so that we define a bijection

$$\mathrm{Hom}_A(B\widehat{\otimes}_A C, D) \xrightarrow{\sim} \mathrm{Hom}_A(B, D) \times \mathrm{Hom}_A(C, D)$$

which follows from the universal property of the complete tensor product. □

Proposition 8.8.15. *For any formal $\mathfrak{S}$-schemes $\mathfrak{X}, \mathfrak{Y}$, the product $\mathfrak{X} \times_{\mathfrak{S}} \mathfrak{Y}$ exists.*

Proof. The demonstration is similar as the case for usual schemes, where we replace affine schemes by formal affine schemes and use Proposition 8.8.14. □

8.8.3 Inductive limits of schemes

Let $\mathfrak{X}$ be a formal scheme, $(\mathscr{I}_\lambda)$ be a fundamental system of nilideals of $\mathfrak{X}$; for each λ, let $f_\lambda : (\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}/\mathscr{I}_\lambda) \to \mathfrak{X}$. For $\mathscr{I}_\mu \subseteq \mathscr{I}_\lambda$, the canoincal homomorphism $\mathscr{O}_{\mathfrak{X}}/\mathscr{I}_\mu \to \mathscr{O}_{\mathfrak{X}}/\mathscr{I}_\lambda$ defines a canonical morphism

$$f_{\mu\lambda} : (\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}/\mathscr{I}_\lambda) \to (\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}/\mathscr{I}_\mu)$$

of (usual) schemes such that we have $f_\lambda = f_\mu \circ f_{\mu\lambda}$. The scheme $X_\lambda = (\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}/\mathscr{I}_\lambda)$ is the morphisms $f_{\mu\lambda}$ then constitutes an inductive system in the category of formal schemes.

Proposition 8.8.16. *The formal scheme and the morphisms f_λ constitute an inductive limit of the system $(X_\lambda, f_{\mu\lambda})$ in the cateogry of formal schemes.*

Proof. Let $\mathfrak{Y}$ be a formal scheme, and for each index λ, let

$$g_\lambda : X_\lambda \to \mathfrak{Y}$$

be a morphism such that $g_\lambda = g_\mu \circ f_{\mu\lambda}$ for $\mathscr{I}_\mu \subseteq \mathscr{I}_\lambda$. This last condition and the definition of X_λ then imply that the g_λ are equal to the same continuous map $g : \mathfrak{X} \to \mathfrak{Y}$ on the underlying space; moreover, the homomorphisms $g_\lambda^\# : g^{-1}(\mathscr{O}_{\mathfrak{Y}}) \to \mathscr{O}_{X_\lambda} = \mathscr{O}_{\mathfrak{X}}/\mathscr{I}_\lambda$ form a projective system of homomorphisms of sheaves of rings. By passing to projective limit, we then deduce a homomorphism $\omega : g^{-1}(\mathscr{O}_{\mathfrak{Y}}) \to \lim \mathscr{O}_{\mathfrak{X}}/\mathscr{I}_\lambda = \mathscr{O}_{\mathfrak{X}}$, and it is clear that (g, ω) is a morphism of ringed spaces such that the diagram

$$\begin{tikzcd} X_\lambda \arrow[rr, "g_\lambda"] \arrow[dr, "f_\lambda"'] & & \mathfrak{Y} \\ & \mathfrak{X} \arrow[ur, "g"'] & \end{tikzcd} \tag{8.8.5}$$

It remains to prove that g is a morphism of formal schemes; the question is local on $\mathfrak{X}$ and $\mathfrak{Y}$, we can assume that $\mathfrak{X} = \mathrm{Spf}(A)$, $\mathfrak{Y} = \mathrm{Spf}(B)$, where A, B are admissible rings, and $\mathscr{I}_\lambda = \mathfrak{I}^\Delta$ where $(\mathfrak{I}_\lambda)$ is a fundamental system of nilideals of A (Proposition 8.8.5). As $A = \varprojlim A/\mathfrak{I}_\lambda$, the existence of the morphism of formal affine schemes g fitting into the diagram (8.8.5) then follows from the one-to-one correspondence Proposition 8.8.4 between morphisms of formal affine schemes and continuous homomorphisms of rings, and of the definition of the projective limit. But the uniqueness of g as a morphism of ringed spaces shows that it coincides with the morphism at the beginning of the demonstration. □

Proposition 8.8.17. *Let $\mathfrak{X}$ be a topological space, $(\mathscr{O}_i, u_{ji})$ a projective system of sheaves of rings over $\mathfrak{X}$ indexed by $\mathbb{N}$. Let $\mathscr{I}_i$ be the kernel of $u_{0,i} : \mathscr{O}_i \to \mathscr{O}_0$ and suppose that*

(a) *For each i, the ringed space $X_i = (\mathfrak{X}, \mathscr{O}_i)$ is a scheme.*

(b) *For any $x \in \mathfrak{X}$ and any $i \in \mathbb{N}$, there exists an open neighborhood U_i of x in $\mathfrak{X}$ such that the restriction $\mathscr{I}_i|_{U_i}$ is nilpotent.*

(c) *The homomorphisms u_{ji} are surjective.*

Let $\mathscr{O}_{\mathfrak{X}}$ be the sheaf of topological rings which is the projective limit of the sheaf of discrete rings $\mathscr{O}_i$, and $u_i : \mathscr{O}_{\mathfrak{X}} \to \mathscr{O}_i$ be the canonical homomorphism. Then the topologically ringed space $(\mathfrak{X}, \mathscr{O}_{\mathfrak{X}})$ is a formal scheme and the homomorphisms u_i are surjective. If $\mathscr{I}^{(i+1)}$ is the kernel of u_i, then $(\mathscr{I}^{(i)})$ form a fundamental system of nilideals of $\mathfrak{X}$, and $\mathscr{I}^{(1)}$ is the projective limit of the sheaf of ideals $\mathscr{I}_i$.

Proof. We first note that at each stalk, u_{ji} is a surjective hommorphism and a fortiori a local homomorphism, so $v_{ij} = (1_{\mathfrak{X}}, u_{ji}) : X_j \to X_i$ is a morphism of schemes for $i \geq j$. Suppose that each X_i is an affine scheme with ring A_i. Then there exists a homomorphism $\varphi_{ji} : A_i \to A_j$ such that $u_{ji} = \tilde{\varphi}_{ji}$, so the sheaf $\mathscr{O}_j$ is a quasi-coherent $\mathscr{O}_i$-module over X_i, associated with A_j considered as an A_i-module via φ_{ji}. For each $f \in A_i$, let $f' = \varphi_{ji}(f)$; by hypothesis, the opens $D(f)$ and $D(f')$ are identical over $\mathfrak{X}$, and the homomorphism from $\Gamma(D(f), \mathscr{O}_i) = (A_i)_f$ to $\Gamma(D(f), \mathscr{O}_j) = (A_j)_{f'}$ corresponding to u_{ji} is none other than $(A_j)_{f'}$. But if we consider A_j as an A_i-module, $(A_j)_{f'}$ is the $(A_j)_f$-module $(A_j)_f$, so we have $u_{ji} = \tilde{\varphi}_{ji}$, if φ_{ji} is considered as a homomorphism of A_i-modules. Then, as u_{ji} is surjective, so is the φ_{ji} and if ji is the kernel of φ_{ji}, the kernel of u_{ji} is the quasi-coherent $\mathscr{O}_i$-module $\widetilde{\mathfrak{I}}_{ji}$. In particular, we have $\mathscr{I}_i = \widetilde{\mathfrak{I}}_i$, where $\mathfrak{I}_i$ is the kernel of $\varphi_{0,i} : A_i \to A_0$. The hypothesis (b) implies that $\mathscr{I}_i$ is nilpotent: in fact, as $\mathfrak{X}$ is quasi-compact, we can cover X by finitely many opens U_k such that $(\mathscr{I}_i|_{U_k})^{n_k} = 0$ and by choosing n to be the largest n_k, we have $\mathscr{I}_i^n = 0$; we then conclude that each $\mathfrak{I}_i$ is nilpotent. Then the ring $A = \varprojlim A_i$ is admissible by **??**, the canonical homomorphisms $\varphi_i : A \to A_i$ is surjective and its kernel $\mathfrak{I}^{(i+1)}$ is equal to the projective limit of the $\mathfrak{I}_{ik}$ for $k \geq i$; the $\mathfrak{I}^{(i+1)}$ form a fundamental system of neighborhoods of 0 in A. The assertion of the proposition then follows from the fact that $(\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}) = \mathrm{Spf}(A)$. We also note that if $f = (f_i)$ is an element in the projective limit $A = \varprojlim A_i$, the open subsets $D(f_i)$ (affine open in X_i) is then identified with $\mathfrak{D}(f)$, and the scheme induced by X_i over $\mathfrak{D}(f)$ is then identified with the affine scheme $\mathrm{Spec}((A_i)_{f_i})$.

In the general case, we remark that for any quasi-compact open U of $\mathfrak{X}$, the $\mathscr{I}_i|_U$ is nilpotent as we have seen. We claim that for any $x \in \mathfrak{X}$, there is an open neighborhood U of x in $\mathfrak{X}$ which is an affine open for any X_i. In fact, let U be an affine open for X_0, and observe that $\mathscr{O}_{X_0} = \mathscr{O}_X / \mathscr{I}_i$. As $\mathscr{I}_i|_U$ is nilpotent in view of the preceding arguments, U is also an affine open for X_i in view of Example 8.4.6. Now for any U satisfying this property, it follows from the same arguments that $(U, \mathscr{O}_{\mathfrak{X}}|_U)$ is a formal scheme such that the $\mathscr{I}^{(i)}|_U$ form a fundamental system of nilideals and $\mathscr{I}^{(1)}|_U$ is the projective limit of $\mathscr{I}_i|_U$, whence the conclusion. □

Corollary 8.8.18. *Suppose that for $i \geq j$, the kernel of u_{ji} is $\mathscr{I}_i^{j+1}$ and that $\mathscr{I}_1 / \mathscr{I}_1^2$ is of finite type over $\mathscr{O}_0 = \mathscr{O}_1 / \mathscr{I}_1$. Then $\mathfrak{X}$ is an adic formal scheme, and we have $\mathscr{I}^{(i)} = \mathscr{I}^{i+1}$ and $\mathscr{I} / \mathscr{I}^2$ is isomorphic to $\mathscr{I}_1$, where we put $\mathscr{I} = \mathscr{I}^{(1)}$. If moreover X_0 is locally Noetherian (resp. Noetherian), then $\mathfrak{X}$ is locally Noetherian (resp. Noetherian).*

Proof. As the underlying space of $\mathfrak{X}$ and X_0 are the same, the question is local and we can assume that each X_i is affine. In view of the relations $\mathscr{I}_{ji} = \widetilde{\mathfrak{I}}_{ji}$ (With the notations of Proposition 8.8.17), we are then reduced to the case of **??**, and note that $\mathfrak{I}_1 / \mathfrak{I}_1^2$ is then a finitely generated A_0-module (Corollary 8.1.24). □

In particular, any locally Noetherian formal scheme $\mathfrak{X}$ is the inductive limit of a sequence (X_n) of locally Noetherian (usual) schemes verifying the conditions of Proposition 8.8.17 and Corollary 8.8.18: it suffices to consider a nilideal $\mathscr{I}$ of $\mathfrak{X}$ (Proposition 8.8.11) and put $X_n = (\mathfrak{X}, \mathscr{O}_{\mathfrak{X}} / \mathscr{I}^{n+1})$ (Proposition 8.8.16). More generally, the same is true if $\mathfrak{X}$ is an adic formal scheme having a nilideal $\mathscr{I}$ such that $\mathscr{I} / \mathscr{I}^2$ is a finitely generated $(\mathscr{O}_{\mathfrak{X}} / \mathscr{I})$-module.

Corollary 8.8.19. *Let A be an admissible ring. For the formal affine scheme $\mathfrak{X} = \mathrm{Spf}(A)$ to be Noetherian, it is necessary and sufficient that A is adic and Noetherian.*

Proof. This condition is clearly sufficient. Conversely, suppose that $\mathfrak{X}$ is Noetherian, and let $\mathfrak{I}$ be a nilideal of A, $\mathscr{I} = \mathfrak{I}^{\Delta}$ the nilideal of $\mathfrak{X}$. The (usual) schemes $X_n = (\mathfrak{X}, \mathscr{O}_{\mathfrak{X}} / \mathscr{I}^{n+1})$ is then affine and Noetherian, so the ring $A_n = A / \mathscr{I}^{n+1}$ is Noetherian (Proposition 8.2.29), and we conclude that $\mathfrak{I} / \mathfrak{I}^2$ is a finitely generated $(A/\mathfrak{I})$-module. As the $\mathfrak{I}^n$ form a fundamental system of nilideals of $\mathfrak{X}$, we have $\mathscr{O}_{\mathfrak{X}} = \varprojlim (\mathscr{O}_{\mathfrak{X}} / \mathfrak{I}^n)$ (Proposition 8.8.10). □

Remark 8.8.20. With the notations of Proposition 8.8.17, let $\mathscr{F}_i$ be an $\mathscr{O}_i$-module, and suppose that for $i \geq j$ we are given a v_{ij}-morphism $\theta_{ji} : \mathscr{F}_i \to \mathscr{F}_j$, such that $\theta_{kj} \circ \theta_{ji} = \theta_{ki}$ for $k \leq j \leq i$. As the underlying continuous map of v_{ij} is the identity, θ_{ji} is a homomorphism of sheaves of abelian groups over $\mathfrak{X}$. Moreover, if $\mathscr{F}$ is the limit of the projective system $(\mathscr{F}_i)$ of sheaves of abelian groups, the fact that each θ_{ji} is a v_{ij}-morphism permits us to define over $\mathscr{F}$ an $\mathscr{O}_{\mathfrak{X}}$-module structure by passing to projective limits. With this, we say that $\mathscr{F}$ is the **projective limit** (for the θ_{ji}) of the system of $\mathscr{O}_i$-modules $(\mathscr{F}_i)$. In the particular case where $v_{ij}^*(\mathscr{F}_i) = \mathscr{F}_j$ and there θ_{ji} is the identity, we then say that $\mathscr{F}$ is the limit of the system $(\mathscr{F}_i)$ such that $v_{ij}^*(\mathscr{F}_i) = \mathscr{F}_j$ for $j \leq i$ (without mension of θ_{ji}).

Let $\mathfrak{X}, \mathfrak{Y}$ be two formal schemes, $\mathscr{J}$ (resp. $\mathscr{K}$) be a nilideal of $\mathfrak{X}$ (resp. $\mathfrak{Y}$), and $f : \mathfrak{X} \to \mathfrak{Y}$ be a morphism such that $f^*(\mathscr{K})\mathscr{O}_{\mathfrak{X}} \subseteq \mathscr{J}$. We then have for each integer $n > 0$ that $f^*(\mathscr{K}^n)\mathscr{O}_{\mathfrak{X}} = (f^*(\mathscr{K})\mathscr{O}_{\mathfrak{X}})^n \subseteq \mathscr{J}^n$, so Proposition 8.8.13 deduce a morphism $f_n : X_n \to Y_n$ of (usual) schemes such that the diagram

$$\begin{tikzcd} X_m \arrow[r, "f_m"] \arrow[d] & Y_m \arrow[d] \\ X_n \arrow[r, "f_n"] & Y_n \end{tikzcd} \tag{8.8.6}$$

is commutative for $m \leq n$; in other words, (f_n) is a inductive system of morphisms.

Conversely, let (X_n) (resp. (Y_n)) be a inductive system of schemes satisfying the conditions (b), (c) of Proposition 8.8.17, and let $\mathfrak{X}$ (resp. $\mathfrak{Y}$) be the inductive limits (whose existence is proved by Proposition 8.8.17). By the definition of inductive limits, any inductive system (f_n) of morphisms $f_n : X_n \to Y_n$ admits an inductive limit $f : \mathfrak{X} \to \mathfrak{Y}$, which is the unique morphism of formal schemes rendering the diagram

$$\begin{tikzcd} X_n \arrow[r, "f_n"] \arrow[d] & Y_n \arrow[d] \\ \mathfrak{X} \arrow[r, "f"] & \mathfrak{Y} \end{tikzcd}$$

Proposition 8.8.21. *Let $\mathfrak{X}, \mathfrak{Y}$ be adic formal schemes, $\mathscr{J}$ (resp. $\mathscr{K}$) be a nilideal of $\mathfrak{X}$ (resp. $\mathfrak{Y}$). The map $f \mapsto (f_n)$ is a bijection from the set of morphisms $f : \mathfrak{X} \to \mathfrak{Y}$ such that $f^*(\mathscr{K})\mathscr{O}_{\mathfrak{X}} \subseteq \mathscr{J}$ to the set of sequences (f_n) of morphisms rendering the diagram (8.8.6).*

Proof. If f is the inductive limit of this sequence, it is necessary to prove that $f^*(\mathscr{K})\mathscr{O}_{\mathfrak{X}} \subseteq \mathscr{J}$. This question is local over $\mathfrak{X}$ and $\mathfrak{Y}$, so we can assume that $\mathfrak{X} = \mathrm{Spf}(A)$, $\mathfrak{Y} = \mathrm{Spf}(B)$ are affine, where A, B are adic, $\mathscr{J} = \mathfrak{J}^\Delta$, $\mathscr{K} = \mathfrak{K}^\Delta$, where $\mathfrak{J}$ (resp. $\mathfrak{K}$) is a nilideal of A (resp. B). We then have $X_n = \mathrm{Spec}(A_n)$, $Y_n = \mathrm{Spec}(B_n)$, where $A_n = A/\mathfrak{J}^{n+1}$ and $B_n = B/\mathfrak{K}^{n+1}$, in view of Proposition 8.8.6. Then $f_n = ({}^a\varphi_n, \tilde{\varphi}_n)$, where $\varphi_n : B_n \to A_n$ is the homomorphism forming a projective system, so $f = ({}^a\varphi, \tilde{\varphi})$, where $\varphi = \varprojlim \varphi_n$. The commutative diagram (8.8.6) shows that $\varphi_n(\mathfrak{K}/\mathfrak{K}^{n+1}) \subseteq \mathfrak{J}/\mathfrak{J}^{n+1}$ for each n (by take $m = 0$), so by passing to limit, $\varphi(\mathfrak{K}) \subseteq \mathfrak{J}$, and this implies $f^*(\mathscr{K})\mathscr{O}_{\mathfrak{X}} \subseteq \mathscr{J}$ (Proposition 8.8.13). $\square$

Corollary 8.8.22. *Let $\mathfrak{X}, \mathfrak{Y}$ be locally Noetherian formal schemes, $\mathscr{T}$ be the largest nilideal of $\mathfrak{X}$.*

(i) *For any nilideal $\mathscr{K}$ of $\mathfrak{Y}$ and any morphism $f : \mathfrak{X} \to \mathfrak{Y}$, we have $f^*(\mathscr{K})\mathscr{O}_{\mathfrak{X}} \subseteq \mathscr{T}$.*

(ii) *There exists a bijective correspondence between* $\mathrm{Hom}(\mathfrak{X}, \mathfrak{Y})$ *and the set of sequences (f_n) of morphisms rendering the diagram (8.8.6), where $X_n = (\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}/\mathscr{T}^{n+1})$, $Y_n = (\mathfrak{Y}, \mathscr{O}_{\mathfrak{Y}}/\mathscr{K}^{n+1})$.*

Proof. It is clear that (ii) follows from (i) and Proposition 8.8.21. To prove (i), we can assume that $\mathfrak{X} = \mathrm{Spf}(A)$, $\mathfrak{Y} = \mathrm{Spf}(B)$, A, B being Noetherian and adic, $\mathscr{T} = \mathfrak{T}^\Delta$, $\mathscr{K} = \mathfrak{K}^\Delta$, where $\mathfrak{T}$ is the largest nilideal of A and $\mathfrak{K}$ is a nilideal of B. Let $f = ({}^a\varphi, \tilde{\varphi})$ where $\varphi : B \to A$ is a continuous homomorphism; as the elements of $\mathfrak{K}$ are topologically nilpotent, so are those of $\varphi(\mathfrak{K})$, so $\varphi(\mathfrak{K}) \subseteq \mathfrak{T}$ since $\mathfrak{T}$ is the set of topologically nilpotent elements of A (**??**). The conclusion then follows from Proposition 8.8.13(ii). $\square$

Corollary 8.8.23. *Let $f : \mathfrak{X} \to \mathfrak{S}$ and $g : \mathfrak{Y} \to \mathfrak{S}$ be morphisms of adic formal schemes. Let $\mathscr{J}$ (resp. $\mathscr{K}$, $\mathscr{L}$) be a nilideal of $\mathfrak{S}$ (resp. $\mathfrak{X}$, $\mathfrak{Y}$), and suppose that $f^*(\mathscr{J})\mathscr{O}_{\mathfrak{X}} \subseteq \mathscr{K}$, $g^*(\mathscr{J})\mathscr{O}_{\mathfrak{Y}} = \mathscr{L}$. Let $S_n = (\mathfrak{S}, \mathscr{O}_{\mathfrak{S}}/\mathscr{J}^{n+1})$, $X_n = (\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}/\mathscr{K}^{n+1})$, $Y_n = (\mathfrak{Y}, \mathscr{O}_{\mathfrak{Y}}/\mathscr{L}^{n+1})$. Then there exists a bijective correspondence between* $\mathrm{Hom}_{\mathfrak{S}}(\mathfrak{X}, \mathfrak{Y})$ *and the set of sequences (u_n) of S_n-morphisms $u_n : X_n \to Y_n$ rendering the diagram (8.8.6).*

Proof. For any $\mathfrak{S}$-morphism $u : \mathfrak{X} \to \mathfrak{Y}$, we have $f = g \circ u$ by definition, so

$$u^*(\mathscr{L})\mathscr{O}_{\mathfrak{X}} = u^*(g^*(\mathscr{I})\mathscr{O}_{\mathfrak{Y}})\mathscr{O}_{\mathfrak{X}} = f^*(\mathscr{I})\mathscr{O}_{\mathfrak{X}} \subseteq \mathscr{K},$$

so the corollary follows from Proposition 8.8.21. $\square$

We note that, for $m \leq n$, the datum of a morphism $f_n : X_n \to Y_n$ determines uniquely a morphism $f_m : X_m \to Y_m$ rendering the diagram (8.8.6), as we immediately see by reducing to the affine case; we have thus defined a map

$$\varphi_{mn} : \mathrm{Hom}_{S_n}(X_n, Y_n) \to \mathrm{hom}_{S_m}(X_m, Y_m)$$

and the $\mathrm{Hom}_{S_n}(X_n, Y_n)$ form for the φ_{mn} a projective system of sets; Corollary 8.8.23 then shows that there exists a canonical bijection

$$\mathrm{Hom}_{\mathfrak{S}}(\mathfrak{X}, \mathfrak{Y}) \xrightarrow{\sim} \varprojlim_n \mathrm{Hom}_{S_n}(X_n, Y_n).$$

Remark 8.8.24. Let $\mathfrak{S}$, $\mathfrak{X}, \mathfrak{Y}$ be formal schemes and $f : \mathfrak{X} \to \mathfrak{S}$, $g : \mathfrak{Y} \to \mathfrak{S}$ be morphisms. Suppose that there are fundamental system of nilideals $(\mathscr{I}_\lambda)$, $(\mathscr{K}_\lambda)$, $(\mathscr{L}_\lambda)$ in $\mathfrak{S}$, $\mathfrak{X}, \mathfrak{Y}$, respectively, with the same index set I, such that $f^*(\mathscr{I}_\lambda)\mathscr{O}_{\mathfrak{X}} \subseteq \mathscr{K}_\lambda$ and $g^*(\mathscr{I}_\lambda)\mathscr{O}_{\mathfrak{Y}} \subseteq \mathscr{L}_\lambda$ for any λ. Put $S_\lambda = (\mathfrak{S}, \mathscr{O}_{\mathfrak{S}}/\mathscr{I}_\lambda)$, $X_\lambda = (\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}/\mathscr{K}_\lambda)$, $Y_\lambda = (\mathfrak{Y}, \mathscr{O}_{\mathfrak{Y}}/\mathscr{L}_\lambda)$; for $\mathscr{I}_\mu \subseteq \mathscr{I}_\lambda$, $\mathscr{K}_\mu \subseteq \mathscr{K}_\lambda$, $\mathscr{L}_\mu \subseteq \mathscr{L}_\lambda$, note that S_λ (resp. X_λ, Y_λ) is a closed subscheme of S_μ (X_μ, Y_μ) with the same underlying space. As $S_\lambda \to S_\mu$ is a monomorphism of schemes, we then see that the products $X_\lambda \times_{S_\lambda} Y_\lambda$ and $X_\lambda \times_{S_\mu} Y_\lambda$ are identical (Corollary 8.3.2), because $X_\lambda \times_{S_\mu} Y_\lambda$ is identified with a closed subscheme of $X_\mu \times_{S_\mu} Y_\mu$ with the same underlying space. Now the product is the inductive limit of the schemes $X_\lambda \times_{S_\lambda} Y_\lambda$: in fact, we see as in Proposition 8.8.16, we can assume that $\mathfrak{S}$, $\mathfrak{X}$, $\mathfrak{Y}$ are formal affine schemes. In view of Proposition 8.8.13 and our hypotheses, immediately see that our assertion follows from the definition of the completed tensor product of two algebras.

Moreover, let $\mathfrak{Z}$ be a formal $\mathfrak{S}$-scheme, $(\mathscr{M}_\lambda)$ be a fundamental system of nilideals of $\mathfrak{Z}$ with index set I, $u : \mathfrak{Z} \to \mathfrak{X}$, $v : \mathfrak{Z} \to \mathfrak{Y}$ be morphisms such that $u^*(\mathscr{K}_\lambda)\mathscr{O}_{\mathfrak{Z}} \subseteq \mathscr{M}_\lambda$ and $v^*(\mathscr{L}_\lambda)\mathscr{O}_{\mathfrak{Z}} \subseteq \mathscr{M}_\lambda$. If we put $Z_\lambda = (\mathfrak{Z}, \mathscr{O}_{\mathfrak{Z}}/\mathscr{M}_\lambda)$, and if $u_\lambda : Z_\lambda \to X_\lambda$ and $v_\lambda : Z_\lambda \to Y_\lambda$ are the corresponding S_λ-morphisms, we then verify that $(u, v)_{\mathfrak{S}}$ is the inductive limits of the S_λ-morphisms $(u_\lambda, v_\lambda)_{S_\lambda}$.

8.8.4 Formal completion of schemes

Let X be a (usual) scheme, X' be a subscheme of X, U be an open subset of X containing X' and such that X' is a closed subscheme of U; then X' is defined by a quasi-coherent ideal $\mathscr{I}_U$ of $\mathscr{O}_U$. For any integer $n > 0$, and any quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, $(\mathscr{F}|_U) \otimes_{\mathscr{O}_U} (\mathscr{O}_U/\mathscr{I}_U^n)$ is then a quasi-coherent $\mathscr{O}_U$-module whose support is contained in X', which is therefore often identified with its restriction to X'. The family $\{(\mathscr{F}|_U) \otimes_{\mathscr{O}_U} (\mathscr{O}_U/\mathscr{I}_U^n)\}_{n \geq 1}$ then forms a projective system of sheaves of abelian groups. The limit $\varprojlim((\mathscr{F}|_U) \otimes_{\mathscr{O}_U} (\mathscr{O}_U/\mathscr{I}_U^n))$ is called the **completion of $\mathscr{F}$ along the subscheme X' of X**, and denoted by $\widehat{\mathscr{F}}_{/X'}$ of simply $\widehat{\mathscr{F}}$ (if there is no confusion). The sections of $\widehat{\mathscr{F}}$ over X' are called the **formal sections** of $\mathscr{F}$ along X'.

This definition is justified by the fact that it obviously does not depend on the choice open subset U, because at every point x of $U - X'$, there is a neighborhood of x in which $\mathscr{O}_U/\mathscr{I}_U^n$ is zero for any integer n. We can therefore limit ourselves to the case where X' is a closed subscheme of X, and we will always assume this henceforth. Also, it is clear that for any open subset $U \subseteq X$, we have $(\mathscr{F}|_U)_{/(U \cap X')} = (\mathscr{F}_{/X'})|_{U \cap X'}$.

By passing to projective limits, it is clear that $(\mathscr{O}_X)_{/X'}$ is a sheaf of rings, and that $\mathscr{F}_{/X'}$ can be considered as an $(\mathscr{O}_{/X'})$-module. Furthermore, as there existss a basis for X' formed by quasi-compact opens, we can consider $(\mathscr{O}_X)_{/X'}$ (resp. $\mathscr{F}_{/X'}$) as a sheaf of topological rings (resp. topological groups) which is the projective limit of the sheaf of discrete rings $\mathscr{O}_X/\mathscr{I}^n$ (resp. the sheaf of groups $\mathscr{F} \otimes_{\mathscr{O}_X} (\mathscr{O}_X/\mathscr{I}^n) = \mathscr{F}/\mathscr{I}^n\mathscr{F}$), and by passing to projective limits, $\mathscr{F}_{/X'}$ is then a topological $(\mathscr{O}_X)_{/X'}$-module. Note that for any quasi-compact open subset U of X, $\Gamma(U \cap X', (\mathscr{O}_X)_{/X'})$ (resp. $\Gamma(U \cap X', \mathscr{F}_{/X'})$) is then the projective limit of the discrete rings (resp. groups) $\Gamma(U, \mathscr{O}_X/\mathscr{I})$ (resp. $\Gamma(U, \mathscr{F}/\mathscr{I}\mathscr{F})$).

Now let $u : \mathscr{F} \to \mathscr{G}$ be a homomorphism of $\mathscr{O}_X$-modules, we then deduce a canonical homomorphism

$$u_{\mathscr{I}^n} : \mathscr{F} \otimes_{\mathscr{O}_X} (\mathscr{O}_X/\mathscr{I}^n) \to \mathscr{G} \otimes_{\mathscr{O}_X} (\mathscr{O}_X/\mathscr{I}^n)$$

for any $n \geq 1$, and these homomorphisms form a projective system. By passing to projective limits and restricting to X', we obtain a continuous $(\mathscr{O}_X)_{/X'}$-homomorphism $\mathscr{F}_{/X'} \to \mathscr{G}_{/X'}$, denoted by $u_{/X'}$ or $\hat{u}$, and is called the **completion of u along X'**. It is clear that if $v : \mathscr{G} \to \mathscr{H}$ is a second homomorphism of $\mathscr{O}_X$-modules, then $(v \circ u)_{/X'} = (v_{/X'}) \circ (u_{/X'})$, so $\mathscr{F}_{/X'}$ is a covariant additive functor on $\mathscr{F}$ from the category of quasi-coherent $\mathscr{O}_X$-modules, with values in the category of $(\mathscr{O}_X)_{/X'}$-modules.

Proposition 8.8.25. *Let X' be a closed subscheme of X defined by a quasi-coherent ideal $\mathscr{J}$ of finite type. Then the support of $(\mathscr{O}_X)_{/X'}$ is X', the topologically ringed space $(X', (\mathscr{O}_X)_{/X'})$ is an adic formal scheme, and $\mathscr{J}_{/X'}$ is a nilideal of this formal scheme. If $X = \operatorname{Spec}(A)$ is an affine scheme, $\mathscr{J} = \widetilde{\mathfrak{J}}$ where $\mathfrak{J}$ is an ideal of A, and $X' = V(\mathfrak{J})$, then $(X', (\mathscr{O}_X)_{/X'})$ is canonically identified with $\operatorname{Spf}(\widehat{A})$, where $\widehat{A}$ is the Hausdorff completion of A for the $\mathfrak{J}$-adic topology.*

Proof. We can evidently assume that $X = \operatorname{Spec}(A)$ is affine. By **??** the the Hausdorff completion $\widehat{\mathfrak{J}}$ of $\mathfrak{J}$ for the $\mathfrak{J}$-adic topology is identified with the ideal $\mathfrak{J}\widehat{A}$ of $\widehat{A}$, and that $\widehat{A}$ is a $\widehat{A}$-adic ring such that $\widehat{A}/\widehat{\mathfrak{J}}^n = A/\mathfrak{J}^n$. This last relation (for $n = 1$) proves that the open prime ideals of $\widehat{A}$ are the ideals $\widehat{\mathfrak{p}} = \mathfrak{p}\widehat{\mathfrak{J}}$, where $\mathfrak{p}$ is a prime ideal of A containing $\mathfrak{J}$, whence $\operatorname{Spf}(\widehat{A}) = X'$. As $\mathscr{O}_X/\mathscr{J}^n = \widetilde{A/\mathfrak{J}^n}$, the proposition then follows from the definition of $\operatorname{Spf}(\widehat{A})$. ☐

The formal scheme therefore defined is called the **completion** of X along X' and denoted by $X_{/X'}$ or $\widehat{X}$. If $X' = X$, we can set $\mathscr{J} = 0$, and then $X_{/X'} = X$. It is clear that if U is an open subscheme of X, then $U_{/(U\cap X')}$ is canonically identified with the formal subscheme of $X_{/X'}$ induced over the open subset $U \cap X'$ of X'.

Corollary 8.8.26. *Under the hypothesis of Proposition 8.8.25, assume that X is locally Noetherian. Then the (usual) scheme $\widehat{X}_{\text{red}}$ is the unique reduced subscheme of X with underlying space X' (Proposition 8.4.44). For $\widehat{X}$ to be Noetherian, it is necessary and sufficient that $\widehat{X}_{\text{red}}$ is Noetherian, and it is sufficient that X is Noetherian.*

Proof. The determination of $\widehat{X}_{\text{red}}$ is local (Proposition 8.8.11), so we can assume that X is affine; with the notations of Proposition 8.8.25, the ideal $\mathfrak{T}$ of topological nilpotent elements of $\widehat{A}$ is the inverse image of the nilradical of $A/\mathfrak{T}$ under the canonical map $\widehat{A} \to \widehat{A}/\widehat{\mathfrak{J}} = A/\mathfrak{J}$, so $\widehat{A}/\mathfrak{T}$ is isomorphic to $(A/\mathfrak{J})_{\text{red}}$. The first assertion then follows from Proposition 8.8.11 and Proposition 8.4.44. If $\widehat{X}_{\text{red}}$ is Noetherian, its underlying space X' is also Noetherian, so the $X'_n = \operatorname{Spec}(\mathscr{O}_X/\mathscr{J}^n)$ are Noetherian and so is $\widehat{X}$ (Corollary 8.8.18); the converse of this is immeidate. ☐

The canonical homomorphisms $\mathscr{O}_X \to \mathscr{O}_X/\mathscr{J}^n$ form a projective system and therefore gives, by passing to limit, a homomorphism of sheaves of rings $\theta : \mathscr{O}_X \to i_*((\mathscr{O}_X)_{/X'}) = \varprojlim(\mathscr{O}_X/\mathscr{J}^n)$, where $i : X' \to X$ is the canonical injection. We therefore obtain a morphism

$$(i, \theta) : X_{/X'} \to X$$

of ringed spaces, called the **canonical morphism**. By tensoring, for any coherent $\mathscr{O}_X$-module $\mathscr{F}$, the canonical homomorphisms $\mathscr{O}_X \to \mathscr{O}_X/\mathscr{J}^n$ gives homomorphisms $\mathscr{F} \to \mathscr{F} \otimes_{\mathscr{O}_X} (\mathscr{O}_X/\mathscr{J}^n)$ of $\mathscr{O}_X$-modules which form a projective system, and hence defines a canonical functorial homomorphism $\gamma : \mathscr{F} \to i_*(\mathscr{F}_{/X'})$ of $\mathscr{O}_X$-modules.

Example 8.8.27. Let X', X'' be closed subschemes of X, defined by quasi-coherent ideals $\mathscr{J}, \mathscr{J}'$ of $\mathscr{O}_X$. Suppose that for any affine open U of X, the ideals $\mathscr{J}|_U, \mathscr{J}'|_U$ of $\mathscr{O}_U$ are such that there exists integers $m, m' > 0$ such that $(\mathscr{J}|_U)^m \subseteq \mathscr{J}'|_U$ and $(\mathscr{J}'|_U)^{m'} \subseteq \mathscr{J}|_U$. It is clear that under this condition, for any quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, the sheaf of abelian groups $\varprojlim(\mathscr{F} \otimes_{\mathscr{O}_X} (\mathscr{O}_X/\mathscr{J}^n))$ and $\varprojlim(\mathscr{F} \otimes_{\mathscr{O}_X} (\mathscr{O}_X/\mathscr{J}'^n))$ are canonically isomorphic, which means $\mathscr{F}_{/X'} = \mathscr{F}_{/X''}$. Note that this condition on X' and X'' implies that these two closed subschemes have the same underlying space, but is not in general equivalent to the latter property.

However, if the quasi-coherent ideals $\mathscr{J}$ and $\mathscr{J}'$ are of finite type, then it follows from Proposition 8.6.17 that if the subschemes X' and X'' have the same underlying space, the above condition is satisfied. In particular, if X is locally Noetherian, so that any quasi-coherent ideal of $\mathscr{O}_X$ is of finite type, then for any closed subset (or locally closed) X' of X, we can define $\mathscr{F}_{/X'}$ to be equal to $\mathscr{F}_{/Y}$ for any subscheme Y of X with underlying space X'.

Proposition 8.8.28. *Suppose that X is locally Noetherian and let X' be a closed subset of X, $\mathscr{F}$ be a coherent $\mathscr{O}_X$-module.*

(i) *The functor $\mathscr{F}_{/X'}$ is exact on the category of coherent $\mathscr{O}_X$-modules.*

(ii) *The functorial homomorphism $\gamma^\sharp : i^*(\mathscr{F}) \to \mathscr{F}_{/X'}$ of $(\mathscr{O}_X)_{/X'}$-modules is an isomorphism.*

Proof. To prove (i), it suffices to prove that if $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ is an exact sequence of coherent $\mathscr{O}_X$-modules and U is an affine open of X with Noetherian ring $A = \Gamma(U, \mathscr{O}_X)$, the sequence

$$0 \longrightarrow \Gamma(U \cap X', \mathscr{F}'_{/X'}) \longrightarrow \Gamma(U \cap X', \mathscr{F}_{/X'}) \longrightarrow \Gamma(U \cap X', \mathscr{F}''_{/X'}) \to 0$$

is exact. Then $\mathscr{F}|_U = \widetilde{M}$, $\mathscr{F}'|_U = \widetilde{M}'$, $\mathscr{F}''|_U = \widetilde{M}''$, where M, M', M'' are A-modules of finite type such that the sequence $0 \to M' \to M \to M'' \to 0$ is exact. Let $\mathscr{I}$ be a quasi-coherent ideal of $\mathscr{O}_U$ defining a subscheme of U with underlying space $U \cap X'$, and $\mathfrak{I}$ be the ideal of A such that $\mathscr{I} = \widetilde{\mathfrak{I}}$. We have (Corollary 8.1.10)

$$\Gamma(U \cap X', \mathscr{F} \otimes_{\mathscr{O}_X} (\mathscr{O}_X/\mathscr{I}^n)) = M \otimes_A (A/\mathfrak{I}^n);$$

so by the definition of projective limit,

$$\Gamma(U \cap X', \mathscr{F}_{/X'}) = \varprojlim (M \otimes_A (A/\mathfrak{I}^n)) = \widehat{M}$$

where $\widehat{M}$ is the Huasdorff completion of M for the $\mathfrak{I}$-adic topology, and similarly

$$\Gamma(U \cap X', \mathscr{F}'_{/X'}) = \widehat{M}', \quad \Gamma(U \cap X', \mathscr{F}''_{/X'}) = \widehat{M}'';$$

our assertion then follows from the fact that A is Noetherian and the functor $\widehat{M}$ on M is exact on the category of finitely generated A-modules (**??**).

For assertion (i), the assertion is local, so we can assume that there exists an exact sequence $\mathscr{O}_X^m \to \mathscr{O}_X^n \to \mathscr{F} \to 0$; as $\gamma^\sharp$ is functorial, and the functors $i^*(\mathscr{F})$ and $\mathscr{F}_{/X'}$ are right exact, we have a commutative diagram

$$\begin{tikzcd}
i^*(\mathscr{O}_X^m) \arrow[r] \arrow[d, "\gamma^\sharp"] & i^*(\mathscr{O}_X^n) \arrow[r] \arrow[d, "\gamma^\sharp"] & i^*(\mathscr{F}) \arrow[r] \arrow[d, "\gamma^\sharp"] & 0 \\
(\mathscr{O}_X^m)_{/X'} \arrow[r] & (\mathscr{O}_X^n)_{/X'} \arrow[r] & \mathscr{F}_{/X'} \arrow[r] & 0
\end{tikzcd} \tag{8.8.7}$$

with exact rows. Moreover, the functors $i^*(\mathscr{F})$ and $\mathscr{F}_{/X'}$ commutes with finite sums, so we are reduced to the case where $\mathscr{F} = \mathscr{O}_X$. We then have $i^*(\mathscr{O}_X) = (\mathscr{O}_X)_{/X'} = \mathscr{O}_{\widehat{X}}$, and $\gamma^\sharp$ is a homomorphism of $\mathscr{O}_{\widehat{X}}$-modules; it then suffices to verify that $\gamma^\sharp$ maps the unit section of $\mathscr{O}_{\widehat{X}}$ over an open subset of X' to itself, which is immediate and also shows that $\gamma^\sharp$ is the identity. □

Corollary 8.8.29. *Under the hypotheses of Proposition 8.8.28, the morphism $i : \widehat{X} \to X$ is flat.*

Corollary 8.8.30. *Let X be a locally Noetherian scheme. If $\mathscr{F}$ and $\mathscr{G}$ are coherent $\mathscr{O}_X$-modules, there exists a canonical functorial isomorphism*

$$(\mathscr{F}_{/X'}) \otimes_{(\mathscr{O}_X)_{/X'}} (\mathscr{G}_{/X'}) \xrightarrow{\sim} (\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G})_{/X'}, \tag{8.8.8}$$

$$(\mathcal{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G}))_{/X'} \xrightarrow{\sim} \mathcal{H}om_{(\mathscr{O}_X)_{/X'}}(\mathscr{F}_{/X'}, \mathscr{G}_{/X'}). \tag{8.8.9}$$

Proof. This follows from the canonical identification of $i^*(\mathscr{F})$ and $\mathscr{F}_{/X'}$; the existence of the first isomorphism is clear for any morphism of ringed spaces and the second is the homomorphism of (**??**), which is an isomorphism for any flat morphism. □

Proposition 8.8.31. *Let X be a locally Noetherian scheme. For any coherent $\mathscr{O}_X$-module $\mathscr{F}$, the kernel of the canonical homomorphism $\Gamma(X, \mathscr{F}) \to \Gamma(X', \mathscr{F}_{/X'})$ induced from $\mathscr{F} \to \mathscr{F}_{/X'}$ is formed by the sections which are zero on an open neighborhood of X'.*

Proof. It follows from the definition of $\mathscr{F}_{/X'}$ that the canonical image of such a section is zero. Conversely, if the image of $s \in \Gamma(X, \mathscr{F})$ is zero in $\Gamma(X', \mathscr{F}_{/X'})$, it suffices to see that any $x \in X'$ admits an open neighborhood in X over which s is zero, and we can therefore assume that $X = \mathrm{Spec}(A)$ is affine, A is Noetherian, $X' = V(\mathfrak{I})$ where $\mathfrak{I}$ is an ideal of A, and $\mathscr{F} = \widetilde{M}$, where M is a finitely generated A-module. Then $\Gamma(X', \mathscr{F}_{/X'})$ is the Huasdorff completion $\widehat{M}$ of M for the $\mathfrak{I}$-topology, and the homomorphism $\Gamma(X, \mathscr{F}) \to \Gamma(X', \mathscr{F}_{/X'})$ is the canonical homomorphism $M \to \widehat{M}$. We have seen that the kernel of this homomorphism (**??**) consists of elements $z \in M$ which is annihilated by an element of $1 + \mathfrak{I}$. We then have $(1 + f)s = 0$ for $f \in \mathfrak{I}$, and for any $x \in X'$ we deduce that $(1_x + f_x)s_x = 0$; as $1_x + f_x$ is invertible in $\mathscr{O}_{X,x}$ ($\mathfrak{I}_x\mathscr{O}_{X,x}$ is contained in $\mathfrak{m}_x$), we then have $s_x = 0$, which proves the assertion. □

Corollary 8.8.32. *Let X be a locally Noetherian scheme and $u : \mathscr{F} \to \mathscr{G}$ be a homomorphism of coherent $\mathscr{O}_X$-modules. For the homomorphism $u_{/X'} : \mathscr{F}_{/X'} \to \mathscr{G}_{/X'}$ to be zero, it is necessary and sufficient that u is zero on an open neighborhood of X'.*

Proof. In fact, by Proposition 8.8.28, $u_{/X'}$ is identified with $i^*(u)$, so if we consider u as a section of $\mathscr{H} = \mathcal{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})$ over X, $u_{/X'}$ is the section of $i^*(\mathscr{H}) = \mathscr{H}_{/X'}$ over X'. It then suffices to apply Proposition 8.8.31 on $\mathscr{H}$. □

Corollary 8.8.33. *Let X be a locally Noetherian sceme and $u : \mathscr{F} \to \mathscr{G}$ be a homomorphism of coherent $\mathscr{O}_X$-modules. For the homomorphism $u_{/X'} : \mathscr{F}_{/X'} \to \mathscr{G}_{/X'}$ to be a monomorphism (resp. epimorphism), it is necessary and sufficient that u is a monomorphism (resp. epimorphism) on an open neighborhood of X'.*

Proof. Let $\mathscr{P}$ and $\mathscr{N}$ be the kernel and cokernel of u, so that we have an exact sequence

$$0 \longrightarrow \mathscr{P} \longrightarrow \mathscr{F} \xrightarrow{u} \mathscr{G} \longrightarrow \mathscr{N} \longrightarrow 0$$

By applying $(-)_{/X'}$, we then get an exact sequence

$$0 \longrightarrow \mathscr{P}_{/X'} \longrightarrow \mathscr{F}_{/X'} \xrightarrow{u_{/X'}} \mathscr{G}_{/X'} \longrightarrow \mathscr{N}_{/X'} \longrightarrow 0$$

That $u_{/X'}$ is a monomorphism (resp. epimorphism) is equivalent to $\mathscr{P}_{/X'} = 0$ (resp. $\mathscr{N}_{/X'} = 0$), so we can apply Proposition 8.8.31 to get the conclusion. □

Corollary 8.8.34. *Let X be a locally Noetherian scheme and $\mathscr{F}$ be a coherent $\mathscr{O}_X$-module. For $\mathscr{F}_{/X'}$ to be locally free (resp. locally free of rank n), it is necessary and sufficient that there exists an open neighborhood U of X' such that $\mathscr{F}|_X'$ is locally free (resp. locally free of rank n).*

Proof. To say that $\mathscr{F}_{/X'}$ is locally free signifies that any point $x \in X'$ admits an open neighborhood V in X such that there exists an isomorphism $v : (\mathscr{O}_X^n)_{/X'} \xrightarrow{\sim} \mathscr{F}_{/X'}|_{V \cap X'}$. We can evidently assume that $V = X$, and then it follows from (8.8.9) that v is of the form $u_{/Z}$, where u is a homomorphism $\mathscr{O}_X^n \to \mathscr{F}$; moreover, by replacing X with an open neigbourhood of Z, we can assume, in view of Corollary 8.8.33, that u is bijective, whence the corollary. □

We now consider the induced morphisms between formal completions. Let X, Y be schemes, $f : X \to Y$ be a morphism, X' (resp. Y') be a closed subscheme of X (resp. Y), and $i : X' \to X$, $j : Y' \to Y$ be the canonical injections. Suppose that the composition morphism $f \circ i$ dominates j, so that we have commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ X' \arrow[u, "i"] \arrow[r, "f'"] & Y' \arrow[u, "j"'] \end{tikzcd}$$

where f' is the restriction of f to X'. If $\mathscr{I}$ (resp. $\mathscr{K}$) is the quasi-coherent ideal defining X' (resp. Y'), then this means $f^*(\mathscr{K})\mathscr{O}_X \subseteq \mathscr{I}$ (Proposition 8.4.16). We then have, for each integer $n > 0$, $f^*(\mathscr{K}^n)\mathscr{O}_X \subseteq \mathscr{I}^n$, so if we put $X'_n = (X', \mathscr{O}_X/\mathscr{I}^{n+1})$, $Y'_n = (Y', \mathscr{O}_Y/\mathscr{K}^{n+1})$, the morphism f induces a morphism $f_n : X'_n \to Y'_n$, and it is immediate that the f_n form a inductive system. The inductive limit of this system (Proposition 8.8.21) is denote by $\hat{f} : X_{/X'} \to Y_{/Y'}$, and called the **completion of f along the subschemes X' and Y'**. It is clear from definition that the following diagram is commutative

$$\begin{tikzcd} X_{X'} \arrow[r, "\hat{f}"] \arrow[d, "i_X"'] & Y_{/Y'} \arrow[d, "i_Y"] \\ X \arrow[r, "f"] & Y \end{tikzcd} \tag{8.8.10}$$

where the vertical morphism are canonical morphism.

If $X = \mathrm{Spec}(B)$ and $Y = \mathrm{Spec}(A)$ are affine schemes, then $\mathscr{I} = \widetilde{\mathfrak{I}}$, $\mathscr{K} = \widetilde{\mathfrak{K}}$, where $\mathfrak{I}$ (resp. $\mathfrak{K}$) is an ideal of B (resp. A), and f corresponds to a homomorphism $\varphi : A \to B$ such that $\varphi(\mathfrak{K}) \subseteq \mathfrak{I}$. The morphism $\hat{f}$ then corresponds to the continuous homomorphism $\hat{\varphi} : \widehat{A} \to \widehat{B}$ (Proposition 8.8.9), where $\widehat{A}$ (resp. $\widehat{B}$) is the $\mathfrak{K}$-adic completion of A (resp. $\mathfrak{I}$-adic completion of B).

Let Z be a third scheme, $g : Y \to Z$ is a morphism, Z' is a closed subscheme of Z defined by a quasi-coherent ideal $\mathscr{R}$ of $\mathscr{O}_Z$, and suppose that we have $g^*(\mathscr{R})\mathscr{O}_Y \subseteq \mathscr{K}$. Then, if $\hat{g}$ is the completion of g along Y' and Z', then it follows from our definition that $\widehat{(g \circ f)} = \hat{g} \circ \hat{f}$.

Now suppose that X and Y are locally Noetherian schemes, X', Y' are closed subsets of X, Y, respectively, and $f : X \to Y$ is a morphism such that $f(X') \subseteq Y'$. Then there exists a coherent ideal $\mathscr{J}$ of $\mathscr{O}_X$ (resp. $\mathscr{K}$ of $\mathscr{O}_Y$) such that $\operatorname{supp}(\mathscr{O}_X/\mathscr{J}) = X'$ (resp. $\operatorname{supp}(\mathscr{O}_Y/\mathscr{K}) = Y'$) and that $f^*(\mathscr{K})\mathscr{O}_X \subseteq \mathscr{J}$: in fact, it suffices to choose $\mathscr{J}$ to be the ideal defining the reduced subscheme structure of X' and $\mathscr{K}$ to be the ideal defining that of Y'. The relation $f^*(\mathscr{K})\mathscr{O}_X \subseteq \mathscr{J}$ then follows from Corollary 8.4.50. We can then define the morphism $\hat{f} : X_{/X'} \to Y_{/Y'}$, and it follows from Example 8.8.27 that $\hat{f}$ does not depend on the choice of the idelas $\mathscr{J}$ and $\mathscr{K}$.

Proposition 8.8.35. *Let X and Y be locally Noetherian S-schemes and suppose that Y is of finite type over S. Let X', Y' be closed subsets of X, Y, respectively, and f, g be two S-morphisms from X to Y such that $f(X') \subseteq Y'$, $g(X') \subseteq Y'$. For that $\hat{f} = \hat{g}$, it is necessary and sufficient that f and g coincides in an open neighborhood of X'.*

Proof. This conditions is clearly sufficient (without the finiteness conditio on Y). To see that it is necessary, we first note that $\hat{f} = \hat{g}$ implies $f(x) = g(x)$ for any $x \in X'$. On the other hand, since the question is local, we can assume that $X = \operatorname{Spec}(B)$ and $Y = \operatorname{Spec}(A)$ are affine neighborhoods of x and $y = f(x) = g(x)$, respectively, with Noetherian rings, and that $S = \operatorname{Spec}(R)$ is affine. Then A is an R-algebra of finite type (Corollary 8.6.39), and f, g correspond to R-homomorphisms ρ, σ from A into B. By hypothesis, the induced homomorphisms $\hat{\rho}, \hat{\sigma}$ on completions are equal. We then conclude from Proposition 8.8.31 that for any section $s \in A$, the sections $\rho(s)$ and $\sigma(s)$ coincides in an open neighborhood of X' (dependent of s); as A is a finite type algebra over R, we then deduce that there exists an open neighborhood V of X' such that $\rho(s)$ and $\sigma(s)$ coincides over V for any section $s \in A$. If $h \in A$ is such that $D(h)$ is an open neighborhood of X' contained in V, we then conclude so f and g coincides on $D(h)$. □

Proposition 8.8.36. *Let X and Y be locally Noetherian schemes, $f : X \to Y$ be a morphism, X', Y' be closed subsets of X, Y, respectively, such that $f(X') \subseteq Y'$. Then, for any coherent $\mathscr{O}_Y$-module $\mathscr{G}$, there exists a canonical isomorphism of $(\mathscr{O}_X)_{/X'}$-modules*

$$(f^*(\mathscr{G}))_{/X'} \xrightarrow{\sim} \hat{f}^*(\mathscr{G}_{/Y'}). \tag{8.8.11}$$

Proof. If we identify canonically $(f^*(\mathscr{G}))_{/X'}$ with $i_X^*(f^*(\mathscr{G}))$ and $\hat{f}(\mathscr{G}_{/Y'})$ with $\hat{f}^*(i_Y^*(\mathscr{G}))$ (Proposition 8.8.28), the proposition then follows from the commutative diagram (8.8.10). □

Remark 8.8.37. Retain the hypotheses of Proposition 8.8.36, and let $\mathscr{F}$ be a coherent $\mathscr{O}_X$-module, $\mathscr{G}$ be a coherent $\mathscr{O}_Y$-module. If $u : \mathscr{G} \to \mathscr{F}$ is an f-morphism, then it correponds to a homomorphism $u^\sharp : f^*(\mathscr{G}) \to \mathscr{F}$, hence by completion a continuous $(\mathscr{O}_X)_{/X'}$-homomorphism

$$(u^\sharp)_{/X'} : (f^*(\mathscr{G}))_{/X'} \to \mathscr{F}_{/X'}.$$

In view of (8.8.11), there exists a unique $\hat{f}$-morphism $v : \mathscr{G}_{/Y'} \to \mathscr{F}_{/X'}$ such that $v^\sharp = (u^\sharp)_{/X'}$. If we consider the triple $(\mathscr{F}, X, X')$ (where $\mathscr{F}$ is quasi-coherent $\mathscr{O}_X$-module and X' is a closed subset of X) as a category $\mathcal{C}$, with morphisms $(\mathscr{F}, X, X') \to (\mathscr{G}, Y, Y')$ consisting of a morphism $f : X \to Y$ of schemes such that $f(X') \subseteq Y'$ and an f-morphism $u : \mathscr{G} \to \mathscr{F}$, we can then say that $(X_{/X'}, \mathscr{F}_{/X'})$ is a functor from $\mathcal{C}$ to the category of couples $(\mathfrak{Z}, \mathscr{H})$ formed by a locally Noetherian formal scheme $\mathfrak{Z}$ and an $\mathscr{O}_{\mathfrak{Z}}$-module $\mathscr{H}$, with morphisms given by morphisms g of formal schemes and g-morphisms of sheaves.

Proposition 8.8.38. *Let X and Y be S-schemes, S' be a closed subscheme of S and X', Y' be closed subschemes of X, Y, respectively, such that, if $\mathscr{J}, \mathscr{K}, \mathscr{L}$ are the nilideals of S', X', Y', then $\varphi^*(\mathscr{J})\mathscr{O}_X \subseteq \mathscr{K}$ and $\psi^*(\mathscr{J})\mathscr{O}_Y \subseteq \mathscr{L}$ (where $\varphi : X \to S$ and $\psi : Y \to S$ are the structural morphisms). Let $Z = X \times_S Y$ and $Z' = p^{-1}(X') \cap q^{-1}(Y')$, where p, q are the canonical projections.*

(a) *Then the completion $Z_{/Z'}$ is identified with the product of the formal $S_{S'}$-schemes $(X_{/X'}) \times_{S_{S'}} (Y_{/Y'})$, the structural morphisms with $\hat{\varphi}, \hat{\psi}$, and the projections with $\hat{p}, \hat{q}$.*

(b) *If T is an S-scheme, $u : T \to X$, $v : T \to Y$ are S-morphisms, and T' is a closed subscheme of T such that, if $\mathscr{M}$ is the ideal defining T', then $u^*(\mathscr{K})\mathscr{O}_T \subseteq \mathscr{M}$ and $v^*(\mathscr{L})\mathscr{O}_T \subseteq \mathscr{M}$. Then the completion of $(u, v)_S$ along T' and Z' is canonically identified with $(\hat{u}, \hat{v})_{S_{S'}}$.*

Proof. It is immediate that the question is local for S, X, Y, so we can assume that $S = \operatorname{Spec}(A)$, $X = \operatorname{Spec}(B)$, $Y = \operatorname{Spec}(C)$, $S' = V(\mathfrak{I})$, $X' = V(\mathfrak{K})$, $Y' = V(\mathfrak{N})$, where $\mathfrak{I}, \mathfrak{K}, \mathfrak{N}$ are ideals such that $\rho(\mathfrak{I}) \subseteq \mathfrak{K}$ and $\sigma(\mathfrak{I}) \subseteq \mathfrak{N}$, where $\sigma : A \to B$ and $\sigma : A \to C$ are the corresponding homomorphisms, Then we see that $Z = \operatorname{Spec}(B \otimes_A C)$ and that $Z' = V(\mathfrak{L})$, where $\mathfrak{L}$ is the ideal generated by $\operatorname{im}(\mathfrak{K} \otimes_A C) + \operatorname{im}(B \otimes_A \mathfrak{N})$. The conclusion then follows from Proposition 8.8.14 and the fact that the complete tensor product $\widehat{B}\widehat{\otimes}_{\widehat{A}}\widehat{C}$ is the completion of $B \otimes_A C$ for the $\mathfrak{L}$-adic topology. □

Corollary 8.8.39. *With the hypotheses of Proposition 8.8.38, for any S-morphism $f : X \to Y$ satisfying $f^*(\mathscr{L})\mathscr{O}_X \subseteq \mathscr{K}$, the graph diagram $\Gamma_{\hat{f}}$ is identified with the completion $\widehat{(\Gamma_f)}$ of the diagram morphism of f.*

Corollary 8.8.40. *Let X, Y be schemes, $f : X \to Y$ be a morphism, Y' be a closed subscheme of Y, and $X' = f^{-1}(Y')$. Then following commutative diagram is cartesian:*

$$\begin{tikzcd} X_{/X'} \arrow[r, "\hat{f}"] \arrow[d, "i_X"'] & Y_{/Y'} \arrow[d, "i_Y"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

Proof. It suffices to apply Proposition 8.8.38(a), where we take $X' = X$ and $S = S' = Y$. □

8.8.5 Coherent sheaves over formal schemes

In this paragraph, we discuss coherent sheaves over locally Noetherian formal schemes, and characterize them as projective limits of coherent modules over (usual) schemes. For this, we shall first consider the affine case, so let A be an adic ring and $\mathfrak{I}$ be a nilideal of A. Let $X = \operatorname{Spec}(A)$, $\mathfrak{X} = \operatorname{Spf}(A)$, whose underlying space is identified with the closed subset $V(\mathfrak{I})$ of $\operatorname{Spec}(A)$. If $X' = \operatorname{Spec}(A/\mathfrak{I})$ is the closed subscheme of X defined by $\widetilde{\mathfrak{I}}$, it then follows from definigion that $\mathfrak{X}$ is identified with $X_{/X'}$. For any A-module M, the sheaf $M^{\Delta} = (\widetilde{M})_{/X'}$ is then an $\mathscr{O}_{\mathfrak{X}}$-module. Moreover, if $u : M \to N$ is a homomorphism of A-modules, it then corresponds to a homomorphism $\tilde{u} : \widetilde{M} \to \widetilde{N}$, and hence to a continuous homomorphism $\tilde{u}_{/X'} : (\widetilde{M})_{/X'} \to (\widetilde{N})_{/X'}$, which we denote by u^{Δ}. It is evident that $(v \circ u)^{\Delta} = v^{\Delta} \circ u^{\Delta}$, so we obtain an additive covariant functor $M \mapsto M^{\Delta}$ from the category of A-module to the category of $\mathscr{O}_{\mathfrak{X}}$-modules.

As A is an adic ring, the ideals $\mathfrak{I}^n$ are open in A, hence separated and complete. The ideal $(\mathfrak{I}^n)^{\Delta}$ of $\mathscr{O}_{\mathfrak{X}}$, with the preceding definition, is then equal to the ideal defined in Eq. (8.8.1), and if we put $\mathscr{I} = \mathfrak{I}^{\Delta}$, then $(\mathfrak{I}^n)^{\Delta} = \mathscr{I}^n$ if $\mathfrak{I}/\mathfrak{I}^2$ is a finitely generated A-module (Proposition 8.8.6). Under this hypothesis, let $A_n = A/\mathfrak{I}^{n+1}$ and $X_n = \operatorname{Spec}(A_n) = (\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}/\mathscr{I}^{n+1})$. If $u_{nm} : X_m \to X_n$ is the canonical morphism induced by $A_n \to A_m$ for $m \leq n$, the formal scheme $\mathfrak{X}$ is then the inductive limit of X_n for the u_{nm} (Proposition 8.8.17).

Proposition 8.8.41. *Let A be an adic Noetherian ring. Then the functor $M \mapsto M^{\Delta}$ is exact on the category of finitely generated A-modules, and we have a canonical isomorphism*

$$\Gamma(\mathfrak{X}, M^{\Delta}) = M.$$

Proof. The exactness of the functor $M \mapsto M^{\Delta}$ follows from that of $M \mapsto \widetilde{M}$ (Proposition 8.1.5) and $\mathscr{F} \to \mathscr{F}_{/X'}$ (Proposition 8.8.28). By definition, $\Gamma(\mathfrak{X}, M^{\Delta})$ is the $\mathfrak{I}$-adic completion of the A-module $\Gamma(X, \widetilde{M}) = M$ ($\mathfrak{I}$ is a nilideal of A). But as A is complete and M is finitely generated, we see that M is complete and separated (**??**), and this proves the proposition. □

Proposition 8.8.42. *Let A be an adic Noetherian ring.*

(a) *If M and N are finitely generated A-modules, there exists a canonical isomorphism*

$$(M \otimes_A N)^{\Delta} \xrightarrow{\sim} M^{\Delta} \otimes_{\mathscr{O}_{\mathfrak{X}}} N^{\Delta}, \tag{8.8.12}$$

$$(\operatorname{Hom}_A(M, N))^{\Delta} \xrightarrow{\sim} \mathcal{H}om_{\mathscr{O}_{\mathfrak{X}}}(M^{\Delta}, N^{\Delta}). \tag{8.8.13}$$

(b) *The map $u \mapsto u^{\Delta}$ is a functorial isomorphism*

$$\operatorname{Hom}_A(M, N) \xrightarrow{\sim} \operatorname{Hom}_{\mathscr{O}_{\mathfrak{X}}}(M^{\Delta}, N^{\Delta}). \tag{8.8.14}$$

Proof. The first two isomorphisms follow from Corollary 8.1.10(a) and Corollary 8.8.30. Now as $\mathrm{Hom}_A(M,N)$ is a finitely generated A-module, we can apply Proposition 8.8.41 to identify $\Gamma(\mathfrak{X},(\mathrm{Hom}_A(M,N))^\Delta)$ with $\mathrm{Hom}_A(M,N)$, and by (8.8.13), we see that (8.8.14) is an isomorphism. □

Proposition 8.8.43. *If A is an adic Noetherian ring, $\mathcal{O}_\mathfrak{X}$ is a coherent sheaf of rings.*

Proof. If $f \in A$, we see that $A_{\{f\}}$ is an adic Noetherian ring (**??**) and as the question is local (Proposition 8.8.2), we are reduced to prove that the kernel of a homomorphism $v : \mathcal{O}_\mathfrak{X}^n \to \mathcal{O}_\mathfrak{X}$ is an $\mathcal{O}_\mathfrak{X}$-module of finite type. We then have $v = u^\Delta$, where $u : A^n \to A$ is a homomorphism (8.8.13). As A is Noetherian, the kernel of u is finitely generated, which means we have a homomorphism $w : A^m \to A^n$ such that the following sequence is exact:

$$A^m \xrightarrow{w} A^n \xrightarrow{u} A$$

We then conclude from Proposition 8.8.41 that the sequence

$$\mathcal{O}_\mathfrak{X}^m \xrightarrow{w^\Delta} \mathcal{O}_\mathfrak{X}^n \xrightarrow{v} \mathcal{O}_\mathfrak{X}$$

is exact, which means the kernel of v is of finite type. □

With the preceding notations, let $A_n = A/\mathfrak{I}^{n+1}$ and X_n be the affine scheme $\mathrm{Spec}(A_n) = (\mathfrak{X}, \mathcal{O}_\mathfrak{X}/\mathfrak{I}^{n+1})$, where $\mathscr{I} = \mathfrak{I}^\Delta$ is the nilideal of $\mathcal{O}_\mathfrak{X}$ corresponding to $\mathfrak{I}$. Let $u_{nm} : X_m \to X_n$ be the morphism induced by the ring homomorphism $A_n \to A_m$ for $m \leq n$. As we have remarked, $\mathfrak{X}$ is then the inductive limit of the X_n (Proposition 8.8.17).

Proposition 8.8.44. *Suppose that A is an adic Noetherian ring and let $\mathscr{F}$ be an $\mathcal{O}_\mathfrak{X}$-module. Then the following conditions are equivalent:*

(i) *$\mathscr{F}$ is a coherent $\mathcal{O}_\mathfrak{X}$-module;*

(ii) *there exists a finitely generated A-module M (uniquely determined up to isomorphism) such that $\mathscr{F}$ is isomorphic to M^Δ.*

(iii) *$\mathscr{F}$ is isomorphic to the projective limit of a sequence $(\mathscr{F}_n)$ of coherent $\mathcal{O}_{X_n}$-modules such that $u_{nm}^*(\mathscr{F}_n) = \mathscr{F}_m$ for $m \leq n$, and the projective system $(\mathscr{F}_n)$ is then isomorphic to the system $(\mathscr{F} \otimes_{\mathcal{O}_\mathfrak{X}} \mathcal{O}_{X_n})$*

Proof. By definition, we have $\mathcal{O}_\mathfrak{X} = A^\Delta$. If condition (ii) is satisfied, then M is the cokernel of a homomorphism $A^m \to A^n$, so it follows from Proposition 8.8.41 that M^Δ is the cokernel of a homomorphism $\mathcal{O}_\mathfrak{X}^m \to \mathcal{O}_\mathfrak{X}^n$. As the sheaf $\mathcal{O}_\mathfrak{X}$ is coherent (Proposition 8.8.43), so is M^Δ (**??**) and this proves (ii)⇒(i).

Now assume the conditions in (iii). Then since each X_n is an affine (usual) scheme, we have $\mathscr{F}_n = \widetilde{M}_n$, where M_n is a finitely generated A_n-module. Since $u_{mn}^*(\mathscr{F}_n) = \mathscr{F}_m$ for $m \leq n$, we have $M_m = M_n \otimes_{A_n} A_m$ (Proposition 8.1.14). The modules M_n then form a projective system for the canonical bi-homomorphisms $M_n \to M_m$, and it follows from the definition of A_n that this projective system satisfies the conditions of **??**, so its projective limit M is a finitely generated A-module such that $M_n = M \otimes_A A_n$ for each n. We then deduce that $\mathscr{F}_n$ is induced over X_n by $\widetilde{M} \otimes_{\mathcal{O}_\mathfrak{X}} (\mathcal{O}_\mathfrak{X}/\mathscr{I}^{n+1})$, so $\mathscr{F} = M^\Delta$ by definition. Conversely, if $\mathscr{F} = M^\Delta$ for a finitely generated A-module M, then by definition, $\mathscr{F}$ is the projective limit of the system $\widetilde{M}_n \otimes_{\mathcal{O}_\mathfrak{X}} (\mathcal{O}_\mathfrak{X}/\mathscr{I}^{n+1})$, and we have $u_{nm}^*(\mathscr{F}_n) = \mathscr{F}_m$ for $m \leq n$.

Finally, assume that $\mathscr{F}$ is a coherent $\mathcal{O}_\mathfrak{X}$-module. Considered as an $\mathcal{O}_\mathfrak{X}$-module, we have $\mathcal{O}_{X_n} = \mathcal{O}_\mathfrak{X}/\mathscr{I}^{n+1} = \widetilde{A}_n$, so $\mathscr{F}_n = \mathscr{F} \otimes_{\mathcal{O}_\mathfrak{X}} \mathcal{O}_{X_n}$ is a coherent $\mathcal{O}_\mathfrak{X}$-module (**??**), and since it is also a $\mathcal{O}_{X_n}$-module and $\mathscr{I}^{n+1}$ is coherent ($\mathfrak{I}$ is finitely generated), we conclude that $\mathscr{F}_n$ is a coherent $\mathcal{O}_{X_n}$-module (**??**), and it is immediate that $u_{mn}^*(\mathscr{F}_n) = \mathscr{F}_m$ for $m \leq n$. If $\mathscr{G} = \varprojlim \mathscr{F}_n$ is the projective limit, then it is a coherent $\mathcal{O}_\mathfrak{X}$-module in view of the equivalence of (ii) and (iii), and it remains to prove that $\mathscr{F}$ is isomorphic to $\mathscr{G}$. Now the canonical homomorphisms $\mathscr{F} \to \mathscr{F}_n$ form a projective system, hence induces a canonical homomorphism $w : \mathscr{F} \to \mathscr{G}$, so it suffices to prove that w is an isomorphism. Since this question is local, we can assume that $\mathscr{F}$ is the cokernel of a homomorphism $\mathcal{O}_\mathfrak{X}^m \to \mathcal{O}_\mathfrak{X}^n$, which is of the form v^Δ, where v is a homomorphism $v : A^m \to A^n$ of A-modules (Proposition 8.8.42), and $\mathscr{F}$ is then isomorphic to M^Δ, where $M = \operatorname{coker} v$. In view of Proposition 8.8.42, we then have

$$\mathscr{F}_n = M^\Delta \otimes_{\mathcal{O}_\mathfrak{X}} A_n^\Delta = (M \otimes_A A_n)^\Delta = (M \otimes_A A_n)^\Delta = \widetilde{M \otimes_A A_n}.$$

since the $\mathfrak{I}$-adic topology on $M \otimes_A A_n$ is discrete. This then implies $M^\Delta = \varprojlim \mathscr{F}_n = \mathscr{G}$, so w is an isomorphism. □

Theorem 8.8.45. *Let A be an adic Noetherian ring. Then the functor $M \mapsto M^{\Delta}$ is an equivalence from the category of finitely generated A-modules to the category of coherent $\mathscr{O}_{\mathfrak{X}}$-module.*

Proof. This follows from Proposition 8.8.41, Proposition 8.8.42 and Proposition 8.8.44. □

Now let A, B be two adic Noetherian rings and $\varphi : B \to A$ be a continuous homomorphism. We denote by $\mathfrak{J}$ (resp. $\mathfrak{K}$) the nilideal of A (resp. B), so that $\varphi(\mathfrak{K}) \subseteq \mathfrak{J}$, and we put $X = \operatorname{Spec}(A)$, $Y = \operatorname{Spec}(B)$, $\mathfrak{X} = \operatorname{Spf}(A)$, $Y = \operatorname{Spf}(B)$. Let $f : X \to Y$ be the corresponding morphism and $\hat{f} : \mathfrak{X} \to \mathfrak{Y}$ be the completion of f, which is also the morphism of formal schemes corresponding to φ. We then have a commutative diagram

$$\begin{tikzcd} \mathfrak{X} \arrow[r, "\hat{f}"] \arrow[d, "i_X"'] & \mathfrak{Y} \arrow[d, "i_Y"] \\ X \arrow[r, "f"] & Y \end{tikzcd} \tag{8.8.15}$$

Proposition 8.8.46. *For any finitely generated B-module N, there exists a canonical homomorphism*

$$\hat{f}^*(N^{\Delta}) \xrightarrow{\sim} (N \otimes_B A)^{\Delta}.$$

Proof. By Proposition 8.8.28, we have a canonical isomorphism $N^{\Delta} = i_Y^*(\widetilde{N})$, so by Proposition 8.1.14

$$(N \otimes_B A)^{\Delta} = i_X^*(\widetilde{N \otimes_B A}) = i_X^*(f^*(\widetilde{N})).$$

The proposition then follows from the commutative diagram Eq. (8.8.15). □

Corollary 8.8.47. *For any ideal $\mathfrak{b}$ of B, we have*

$$\hat{f}^*(\mathfrak{b}^{\Delta})\mathscr{O}_{\mathfrak{X}} = (\mathfrak{b}A)^{\Delta}.$$

Proof. Let $j : \mathfrak{b} \to B$ be the canonical injection, which corresponds to the canonical injection $j^{\Delta} : b^{\Delta} \to \mathscr{O}_{\mathfrak{Y}}$. By definition, $\hat{f}^*(\mathfrak{b}^{\Delta})\mathscr{O}_{\mathfrak{X}}$ is the image of the homomorphism $f^*(æ^{\Delta}) : \hat{f}^*(\mathfrak{b}^{\Delta}) \to \mathscr{O}_{\mathfrak{X}} = \hat{f}^*(\mathscr{O}_{\mathfrak{Y}})$. But this homomorphism is identified with $(j \otimes 1)^{\Delta} : (\mathfrak{b} \otimes_B A)^{\Delta} \to \mathscr{O}_{\mathfrak{X}} = (B \otimes_B A)^{\Delta}$ in view of Proposition 8.8.46. Since the image of $j \otimes 1$ is the ideal $\mathfrak{b}A$ of A, the image of $(j \otimes 1)^{\Delta}$ is equal to $(\mathfrak{b}A)^{\Delta}$, whence the corollary. □

Proposition 8.8.48. *If $\mathfrak{X}$ is a locally Noetherian formal scheme, the sheaf of rings $\mathscr{O}_{\mathfrak{X}}$ is coherent and any nilideal of $\mathfrak{X}$ is coherent.*

Proof. In fact, this question is local, so we can assume that $\mathfrak{X}$ is affine, and the proposition then follows from Proposition 8.8.43 and Proposition 8.8.44. □

Let $\mathfrak{X}$ be a locally Noetherian formal scheme, $\mathscr{J}$ be a nilideal of $\mathfrak{X}$, and X_n be the (usual) scheme $(\mathfrak{X}, \mathscr{O}_{\mathfrak{X}}/\mathscr{J}^{n+1})$, so that $\mathfrak{X}$ is the inductive limit of (X_n) for the morphisms $u_{mn} : X_m \to X_n$ (Proposition 8.8.17). With these notations, we have the following theorem:

Theorem 8.8.49. *For an $\mathscr{O}_{\mathfrak{X}}$-module $\mathscr{F}$ to be coherent, it is necessary and sufficient that it is isomorphic to the projective limit of a sequence $(\mathscr{F}_n)$ of coherent $\mathscr{O}_{X_n}$-modules such that $u_{nm}^*(\mathscr{F}_n) = \mathscr{F}_m$ for $m \leq n$. In this case, the projective system $(\mathscr{F}_n)$ is then isomorphic to the system $u_n^*(\mathscr{F}) = \mathscr{F} \otimes_{\mathscr{O}_{\mathfrak{X}}} \mathscr{O}_{X_n}$, where $u_n : X_n \to \mathfrak{X}$ is the canonical morphism.*

Proof. This question is local, so we can assume that $\mathfrak{X}$ is an affine formal scheme, and the theorem then follows from Proposition 8.8.44. □

In view of Theorem 8.8.49, we can then say that giving a coherent $\mathscr{O}_{\mathfrak{X}}$-module is equivalent to giving a projective system $(\mathscr{F}_n)$ of coherent $\mathscr{O}_{X_n}$-modules such that $u_{mn}^*(\mathscr{F}_n) = \mathscr{F}_m$ for $m \leq n$.

Corollary 8.8.50. *Under the hypotheses of Theorem 8.8.49, if $\mathscr{F}$ and $\mathscr{G}$ are coherent $\mathscr{O}_{\mathfrak{X}}$-modules, we have a canonical isomorphism*

$$\operatorname{Hom}_{\mathscr{O}_{\mathfrak{X}}}(\mathscr{F}, \mathscr{G}) \xrightarrow{\sim} \varprojlim \operatorname{Hom}_{\mathscr{O}_{X_n}}(\mathscr{F}_n, \mathscr{G}_n). \tag{8.8.16}$$

Proof. The transition homomorphism $\mathrm{Hom}_{\mathscr{O}_{X_n}}(\mathscr{F}_n,\mathscr{G}_n)\to\mathrm{Hom}_{\mathscr{O}_{X_m}}(\mathscr{F}_m,\mathscr{G}_m)$ is given by $\theta_n\mapsto u_{mn}^*(\theta_n)$ ($m\leq n$), and the homomorphism Eq. (8.8.16) sends an element $\theta\in\mathrm{Hom}_{\mathscr{O}_{\mathfrak{X}}}(\mathscr{F},\mathscr{G})$ to the sequence $(u_n^*(\theta))$. In view of Theorem 8.8.49, we see that the inverse homomorphism is given by sending a projective system $(\theta_n)\in\varprojlim\mathrm{Hom}_{\mathscr{O}_{X_n}}(\mathscr{F}_n,\mathscr{G}_n)$ to its projective limit. □

Corollary 8.8.51. *Under the hypotheses of Theorem 8.8.49, for a homomorphism $\theta:\mathscr{F}\to\mathscr{G}$ to be surjective, it is necessary and sufficient that the corresponding homomorphism $\theta_0=u_0^*(\theta):\mathscr{F}_0\to\mathscr{G}_0$ is surjective.*

Proof. The question is local, so we can assume that $\mathfrak{X}=\mathrm{Spf}(A)$, where A is an adic Noetherian ring, $\mathscr{F}=M^\Delta$, $\mathscr{G}=N^\Delta$, and $\theta=u^\Delta$, where M,N are finitely generated A-modules and $u:M\to N$ is a homomorphism. We then have $\theta_0=\tilde{u}_0$, where u_0 is the induced homomorphism

$$u\otimes 1:M\otimes_A A/\mathfrak{J}\to N\otimes_A A/\mathfrak{J}.$$

The conclusion then follows if we can prove that u is surjective if and only if u_0 is surjective. To this end, recall that $\mathfrak{J}$ is contained in the Jacobson radical of A (**??**(b)), so the assertion follows from Nakayama's lemma. □

Remark 8.8.52. In view of Theorem 8.8.49, we see that any coherent $\mathscr{O}_{\mathfrak{X}}$-module $\mathscr{F}$ is canonically endowed with a structure of topological $\mathscr{O}_{\mathfrak{X}}$-module, which is the projective limit of the sheaf of discrete groups $\mathscr{F}_n$. It then follows from Corollary 8.8.50 that any homomorphism $u:\mathscr{F}\to\mathscr{G}$ of coherent $\mathscr{O}_{\mathfrak{X}}$-modules is automatically continuous. Moreover, if $\mathscr{H}$ is a coherent sub-$\mathscr{O}_{\mathfrak{X}}$-module of a coherent $\mathscr{O}_{\mathfrak{X}}$-module $\mathscr{F}$, then for any open subset $U\subseteq\mathfrak{X}$, $\Gamma(U,\mathscr{H})$ is the kernel of the (continuous) homomorphism $\Gamma(U,\mathscr{F})\to\Gamma(U,\mathscr{F}/\mathscr{H})$ (because the functor Γ is left exact). Since $\Gamma(U,\mathscr{F}/\mathscr{H})$ is a separated topological group, we conclude that $\Gamma(U,\mathscr{H})$ is a *closed* subgroup of $\Gamma(U,\mathscr{F})$.

Proposition 8.8.53. *Let $\mathfrak{X}$ be a locally Noetherian formal scheme, $\mathscr{F}$ and $\mathscr{G}$ be coherent $\mathscr{O}_{\mathfrak{X}}$-modules. Then there are canonical isomorphism of topological $\mathscr{O}_{\mathfrak{X}}$-modules*

$$\mathscr{F}\otimes_{\mathscr{O}_{\mathfrak{X}}}\mathscr{G}\xrightarrow{\sim}\varprojlim(\mathscr{F}_n\otimes_{\mathscr{O}_{X_n}}\mathscr{G}_n),\tag{8.8.17}$$

$$\mathcal{H}om_{\mathscr{O}_{\mathfrak{X}}}(\mathscr{F},\mathscr{G})\xrightarrow{\sim}\mathcal{H}om_{\mathscr{O}_{X_n}}(\mathscr{F}_n,\mathscr{G}_n).\tag{8.8.18}$$

As $\mathrm{Hom}_{\mathscr{O}_{\mathfrak{X}}}(\mathscr{F},\mathscr{G})$ is the global section of the topological $\mathscr{O}_{\mathfrak{X}}$-module $\mathcal{H}om_{\mathscr{O}_{\mathfrak{X}}}(\mathscr{F},\mathscr{G})$, it is endowed with a canonical topology. If $\mathfrak{X}$ is Noetherian, then it follows from Eq. (8.8.18) that a fundamental system of neighborhoods of 0 is given by the subgroups $\mathrm{Hom}_{\mathscr{O}_{\mathfrak{X}}}(\mathscr{F},\mathscr{I}^n\mathscr{G})$ (cf. [**?**], 0, 7.8.2).

Proposition 8.8.54. *Let $\mathfrak{X}$ be a Noetherian formal scheme and $\mathscr{F},\mathscr{G}$ be coherent $\mathscr{O}_{\mathfrak{X}}$-modules. Then in the topological group $\mathrm{Hom}_{\mathscr{O}_{\mathfrak{X}}}(\mathscr{F},\mathscr{G})$, the set of surjective (resp. injective) homomorphisms is open.*

Proof. In view of Corollary 8.8.51, the set of surjective homomorphisms in $\mathrm{Hom}_{\mathscr{O}_{\mathfrak{X}}}(\mathscr{F},\mathscr{G})$ is the inverse image of the set of surjective homomorphisms in $\mathrm{Hom}_{\mathscr{O}_{X_0}}(\mathscr{F}_0,\mathscr{G}_0)$ under the continuous map $\mathrm{Hom}_{\mathscr{O}_{\mathfrak{X}}}(\mathscr{F},\mathscr{G})\to\mathrm{Hom}_{\mathscr{O}_{X_0}}(\mathscr{F}_0,\mathscr{G}_0)$. For the second assertion, we may cover $\mathfrak{X}$ by finitely many open Noetherian affine formal schemes U_i. For an element $\theta\in\mathrm{Hom}_{\mathscr{O}_{\mathfrak{X}}}(\mathscr{F},\mathscr{G})$, it is necessary and sufficient that its restriction in $\mathrm{Hom}_{\mathscr{O}_{\mathfrak{X}}|_{U_i}}(\mathscr{F}|_{U_i},\mathscr{G}|_{U_i})$ is injective for each i. Since there are only finitely many U_i, we can then reduce to the affine case, and the assertion follows from ([**?**], 0_{I}, 7.8.3). □

Chapter 9

Global properties of morphisms of schemes

9.1 Affine morphisms

9.1.1 Schemes affine over a scheme

Let S be a scheme and X be an S-scheme. If $f : X \to S$ is the structural morphism, then the direct image $f_*(\mathscr{O}_X)$ is an $\mathscr{O}_S$-algebra, which we denote by $\mathscr{A}(X)$ if there is no confusion. If U is an open subset of S, we have

$$\mathscr{A}(f^{-1}(U)) = \mathscr{A}(X)|_U.$$

Similarly, for any $\mathscr{O}_X$-module $\mathscr{F}$ (resp. any $\mathscr{O}_X$-algebra $\mathscr{B}$), we denote by $\mathscr{A}(\mathscr{F})$ (resp. $\mathscr{A}(\mathscr{B})$) the direct image $f_*(\mathscr{F})$ (resp. $f_*(\mathscr{B})$) which is an $\mathscr{A}(X)$-module (resp. $\mathscr{A}(X)$-algebra), and also an $\mathscr{O}_S$-module (resp. $\mathscr{O}_S$-algebra).

Let Y be another S-scheme with $g : Y \to S$ the structural morphism, and $h : X \to Y$ be an S-morphism. We then have a commutative diagram

$$\begin{tikzcd} X \arrow[rr, "h"] \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & S & \end{tikzcd}$$

By definition we have a homomorphism $h^\# : \mathscr{O}_Y \to h_*(\mathscr{O}_X)$ of sheaves of rings, and we deduce from this a homomorphism of $\mathscr{O}_S$-algebras $g_*(h^\#) : g_*(\mathscr{O}_Y) \to g_*(h_*(\mathscr{O}_X)) = f_*(\mathscr{O}_X)$, which means, a homomorphism $\mathscr{A}(Y) \to \mathscr{A}(X)$ of $\mathscr{O}_S$-algebras, and we denote it by $\mathscr{A}(h)$. If $h' : Y \to Z$ is another S-morphism, it is immediate that $\mathscr{A}(h' \circ h) = \mathscr{A}(h) \circ \mathscr{A}(h')$. Therefore we have defined a contravariant functor $\mathscr{A}(X)$ from the category of S-schemes to the category of $\mathscr{O}_S$-algebras.

Now let $\mathscr{F}$ be an $\mathscr{O}_X$-module, $\mathscr{G}$ be an $\mathscr{O}_Y$-module, and $u : \mathscr{G} \to \mathscr{F}$ be an h-morphism, wihch is a homomorphism $\mathscr{G} \to h_*(\mathscr{F})$ of $\mathscr{O}_Y$-modules. Then $g_*(u) : g_*(\mathscr{G}) \to g_*(h_*(\mathscr{F})) = f_*(\mathscr{F})$ is a homomorphism $\mathscr{A}(\mathscr{G}) \to \mathscr{A}(\mathscr{F})$ of $\mathscr{O}_S$-modules, which we denote by $\mathscr{A}(u)$. The couple $(\mathscr{A}(h), \mathscr{A}(u))$ is then a bi-homomorphism of $\mathscr{A}(Y)$-modules $\mathscr{A}(\mathscr{G})$ ot the $\mathscr{A}(X)$-module $\mathscr{A}(\mathscr{F})$. If we fix S and consider the couples $(X, \mathscr{F})$, where X is an S-scheme and $\mathscr{F}$ is an $\mathscr{O}_X$-module, we then see that $(\mathscr{A}(X), \mathscr{A}(\mathscr{F}))$ defines a contravariant functor from the category of these couples to the category of couples of $\mathscr{O}_S$-algebras and modules of this algebra.

Consider now an S-scheme X and let $f : X \to S$ be a structural morphism. We say that X is **affine over *S*** if there exists a covering (S_α) of S by affine opens such that, for each α, the open subscheme $f^{-1}(S_\alpha)$ of X is affine. If this is true, we also say that X is an **affine *S*-scheme**, or the structural morphism f is affine.

Example 9.1.1. Any closed subscheme of S is an affine S-scheme. In fact, if Y is a closed subscheme of S, then for any affine open U of S, the intersection $U \cap Y$ is a closed subscheme of U, whence affine.

Remark 9.1.2. One should note that an affine S-scheme X is not necessary an affine scheme (for example S is affine over S, but note that this is true if S itself is affine). On the other hand, if X is an S-scheme and is affine, it is not necessarily true that X is an affine S-scheme (we will see this later). However, if S is a separated scheme, then any affine scheme is affine over S by Proposition 8.5.34.

Proposition 9.1.3. *Any affine S-scheme is separated over S.*

Proof. Recall that separatedness is local on target (Proposition 8.5.30), and if $f^{-1}(S_\alpha)$ is affine, then the restriction of f to $f^{-1}(S_\alpha)$ is a morphism between affine schemes, so is separated. □

Proposition 9.1.4. *Let X be an affine S-scheme and $f : X \to S$ be the structural morphism. Then for any open subset $U \subseteq S$, $f^{-1}(U)$ is affine over U. In particular, if U is affine, so is $f^{-1}(U)$.*

Proof. In view of the definition, we can reduce to the case $S = \operatorname{Spec}(A)$ and $X = \operatorname{Spec}(B)$, so that f corresponds to a homomorphism $\rho : A \to B$. As the standard opens $D(g)$ with $g \in A$ form a basis for S, we only need to prove the assertion for $U = D(g)$. But recall that $f^{-1}(D(g)) = D(\rho(g))$, so our assertion follows. □

Corollary 9.1.5. *Let S be an affine scheme. Then for an S-scheme X to be affine over S, it is necessary and sufficient that X is an affine scheme.*

Proposition 9.1.6. *Let X be an affine S-scheme and $f : X \to S$ be the structural morphism. For any quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, $f_*(\mathscr{F})$ is a quasi-coherent $\mathscr{O}_S$-module. In particular, the $\mathscr{O}_S$-algebra $\mathscr{A}(X)$ is quasi-coherent.*

Proof. The morphism f is separted by Proposition 9.1.3 and quasi-compact by Proposition 9.1.4 (since any quasi-compact open subset is a finite union of affine opens), so we can apply Proposition 8.6.55. □

Proposition 9.1.7. *Let X ba an affine S-scheme. For any S-scheme Y, the map $h \mapsto \mathscr{A}(h)$ from $\operatorname{Hom}_S(Y, X)$ to $\operatorname{Hom}_{\mathscr{O}_S\text{-}\mathbf{alg}}(\mathscr{A}(X), \mathscr{A}(Y))$ is bijective.*

Proof. Let $f : X \to S$, $g : Y \to S$ be the structural morphisms. Suppose first that $S = \operatorname{Spec}(A)$ and $X = \operatorname{Spec}(B)$ are affine; we must shows that for any homomorphism $\omega : f_*(\mathscr{O}_X) \to g_*(\mathscr{O}_Y)$ of $\mathscr{O}_S$-algebras, there exists a unique S-morphism $h : Y \to X$ such that $\mathscr{A}(h) = \omega$. By definition, for any open subset $U \subseteq S$, ω defines a homomorphism $\omega_U : \Gamma(f^{-1}(U), \mathscr{O}_X) \to \Gamma(g^{-1}(U), \mathscr{O}_Y)$ of $\Gamma(U, \mathscr{O}_S)$-algebras. In particular, for $U = S$, this gives a homomorphism $\varphi : \Gamma(X, \mathscr{O}_X) \to \Gamma(Y, \mathscr{O}_Y)$, which by Proposition 8.2.4, since X is affine, corresponds to a morphism $h : Y \to X$. To see that $\mathscr{A}(h) = \omega$, we need to prove that for any open subset $U \subseteq S$, ω_U coincides with the algebra homomorphism φ_U, which corresponds to the S-morphism $h|_{g^{-1}(U)} : g^{-1}(U) \to f^{-1}(U)$. We may assume that $U = D(\lambda)$ where $\lambda \in A$; then, if $f : X \to S$ corresponds to the ring homomorphism $\rho : A \to B$, we have $f^{-1}(U) = D(\mu)$ where $\mu = \rho(\lambda)$, and $\Gamma(f^{-1}(U), \mathscr{O}_X)$ is the fraction ring B_μ. Now the following diagram commutes

$$\begin{tikzcd}
B = \Gamma(X, \mathscr{O}_X) \arrow[r, "\varphi"] \arrow[d] & \Gamma(Y, \mathscr{O}_Y) \arrow[d] \\
B_\mu = \Gamma(f^{-1}(U), \mathscr{O}_X) \arrow[r, bend left, "\varphi_U"] \arrow[r, bend right, "\omega_U"'] & \Gamma(g^{-1}(U), \mathscr{O}_Y)
\end{tikzcd}$$

By the universal property of localization, we then conclude that $\varphi_U = \omega_U$, whence the assertion in this case.

In the general case, let (S_α) be a covering of S by affine opens such that $f^{-1}(S_\alpha)$ are affine. Then any homomorphism $\omega : \mathscr{A}(X) \to \mathscr{A}(Y)$ of $\mathscr{O}_S$-algebras restricts to a family of homomorphisms

$$\omega_\alpha : \mathscr{A}(f^{-1}(S_\alpha)) \to \mathscr{A}(g^{-1}(S_\alpha))$$

of $\mathscr{O}_{S_\alpha}$-algebras, so there is a family of S_α-morphisms $h_\alpha : g^{-1}(S_\alpha) \to f^{-1}(S_\alpha)$ such that $\mathscr{A}(h_\alpha) = \omega_\alpha$. It all boils down to seeing that for any affine open U of the base $S_\alpha \cap S_\beta$, the restriction of h_α and h_β to $g^{-1}(U)$ coincide, which is immediate since these restrictions both correspond to the restriction homomorphism $\mathscr{A}(X)|_U \to \mathscr{A}(Y)|_U$ of ω. □

Corollary 9.1.8. *Let X and Y be affine S-schemes. For an S-morphism $h : Y \to X$ to be an isomorphism, it is necessary and sufficient that $\mathscr{A}(h) : \mathscr{A}(X) \to \mathscr{A}(Y)$ is an isomorphism.*

Proof. This follows from Proposition 9.1.7 and the functoriality of $\mathscr{A}(X)$. □

9.1.2 Affine S-scheme associated with an $\mathscr{O}_S$-algebra

Proposition 9.1.9. *Let S be a scheme. For any quasi-coherent $\mathscr{O}_S$-algebra $\mathscr{B}$, there exists an affine S-scheme X, defined up to S-isomorphisms, such that $\mathscr{A}(X) = \mathscr{B}$. The affine S-scheme X is said to be **associated with the $\mathscr{O}_S$-algebra** $\mathscr{B}$, and denoted by* $\mathrm{Spec}(\mathscr{B})$.

Proof. The uniqueness follows from Corollary 9.1.8, so we only need to construct the affine S-scheme X. For any affine open $U \subseteq S$, let X_U be the scheme $\mathrm{Spec}(\Gamma(U, \mathscr{B}))$; as $\Gamma(U, \mathscr{B})$ is an $\Gamma(U, \mathscr{O}_S)$-algebra, X_U is an S-scheme, and is affine over U since U and X_U are both affine. Moreover, as $\mathscr{B}$ is quasi-coherent, the $\mathscr{O}_S$-algebra $\mathscr{A}(X_U)$ is canonically idnetified with $\mathscr{B}|_U$ (Proposition 8.1.12). Let V be another affine open of S, and $X_{U,V}$ be the open subscheme of X_U over $\varphi_U^{-1}(U \cap V)$, where $\varphi_U : X_U \to S$ is the structural morphism. Then $X_{U,V}$ and $X_{V,U}$ are affine over $U \cap V$ (Proposition 9.1.4), and by definition $\mathscr{A}(X_{U,V})$ and $\mathscr{A}(X_{V,U})$ are canonically identified with $\mathscr{B}|_{U \cap V}$. There then exists (Corollary 9.1.8) a canonical S-isomorphism $\theta_{U,V} : X_{V,U} \to X_{U,V}$; furthermore, if W is a third affine open of S, and if $\theta'_{U,V}, \theta'_{V,W}, \theta'_{U,W}$ are the restrictions of $\theta_{U,V}, \theta_{V,W}, \theta_{U,W}$ over the inverse images of $U \cap V \cap W$ in X_V, X_W, X_W, then $\theta'_{U,V} \circ \theta'_{V,W} = \theta'_{U,W}$. By glueing the $X_{U,V}$, there then exists a scheme X and an affine open cover (T_U) of X such that for each U there is an isomorphism $\varphi_U : X_U \to T_U$ such that φ_U maps $\varphi_U^{-1}(U \cap V)$ to $T_U \cap T_V$ and we have $\theta_{U,V} = \varphi_U^{-1} \circ \varphi_V$. The morphism $g_U = \varphi_U \circ \varphi_U^{-1}$ then makes T_U an S-scheme, and the morphisms g_U and g_V coincide on $T_U \cap T_V$, so X is an S-scheme. It is clear by definition that X is affine over S and $\mathscr{A}(T_U) = \mathscr{B}|_U$, so $\mathscr{A}(X) = \mathscr{B}$. ☐

Corollary 9.1.10. *Let S be a scheme. The functor $\mathscr{A}(X)$ defines an equivalence of categories between the category of affine S-schemes and the category of quasi-coherent $\mathscr{O}_S$-algebras.*

Proof. By Proposition 9.1.7 we now that $\mathscr{A}(X)$ is fully faithful, and Proposition 9.1.9 proves that it is essentially surjective, whence the claim. ☐

Corollary 9.1.11. *Let S be a scheme. Then for any quasi-coherent $\mathscr{O}_S$-algebra $\mathscr{B}$, the contravariant functor*

$$Y \mapsto \mathrm{Hom}_{\mathscr{O}_S\text{-}\mathbf{alg}}(\mathscr{B}, \mathscr{A}(Y)) = \mathrm{Hom}_{\mathscr{O}_Y\text{-}\mathbf{alg}}(\mathscr{B} \otimes_{\mathscr{O}_S} \mathscr{O}_Y, \mathscr{O}_Y)$$

from the category of S-schemes to the category of sets, is represented by $\mathrm{Spec}(\mathscr{B})$.

Proof. Let $X = \mathrm{Spec}(\mathscr{B})$, then we know that $\mathscr{B} = \mathscr{A}(X)$, so the claim follows from Proposition 9.1.7. ☐

Corollary 9.1.12. *Let X be an affine S-scheme and $f : X \to S$ be the structural morphism. For any affine open $U \subseteq S$, the open subscheme $f^{-1}(U)$ of X is an affine scheme with ring $\Gamma(U, \mathscr{A}(X))$.*

Proof. We can suppose that X is associated with the $\mathscr{O}_S$-algebra $\mathscr{A}(X)$, the corollary then follows from the construction of X in Proposition 9.1.9. ☐

Example 9.1.13. Let S be the affine plane for a field K with the point 0 is doubled (Example 8.5.35). With the notations there, S is the union of two affine opens Y_1, Y_2. If f is the open immersion $Y_1 \to S$, then $f^{-1}(Y_2) = Y_1 \cap Y_2$ and we have already seen in Example 8.5.35 that this is not affine. So we obtain an example of an affine scheme not affine over a scheme S.

Remark 9.1.14. Let S be a scheme and $f : X \to S$ be a quasi-compact and quasi-separated morphism, so that $\mathscr{A}(X) = f_*(\mathscr{O}_X)$ is a quasi-coherent $\mathscr{O}_S$-algebra (Proposition 8.6.55). The affine S-scheme

$$X^0 = \mathrm{Aff}(X/S) = \mathrm{Spec}(f_*(\mathscr{O}_X)) = \mathrm{Spec}(\mathscr{A}(X))$$

is called the **affine envelope** of the S-scheme X. If $f^0 : X^0 \to S$ is the structural morphism, by Proposition 9.1.9 we then have

$$\mathscr{A}(X^0) = f^0_*(\mathscr{O}_{X^0}) = \mathscr{A}(X) = f_*(\mathscr{O}_X);$$

by Corollary 9.1.11, the identity homomorphism on $\mathscr{A}(X)$ therefore corresponds to a canonical S-morphism $\iota_X : X \to X^0$ such that f factors into

$$X \xrightarrow{\iota_X} X^0 \xrightarrow{f^0} S$$

This factorization for f is called the **Stein factorization** of f. For the morphism i_X to be an isomorphism, it is necessary and sufficient that the morphism f is affine. Moreover, for any S-scheme Y affine over S, the map $u \mapsto u \circ i_X$ is then a bijection

$$\mathrm{Hom}_S(X^0, Y) \xrightarrow{\sim} \mathrm{Hom}_S(X, Y). \tag{9.1.1}$$

which is functorial on Y: this follows from the canonical bijections

$$\mathrm{Hom}_S(X, Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{O}_S\text{-}\mathbf{alg}}(\mathscr{A}(Y), \mathscr{A}(X)) = \mathrm{Hom}_{\mathscr{O}_S\text{-}\mathbf{alg}}(\mathscr{A}(Y), \mathscr{A}(X^0)) \xrightarrow{\sim} \mathrm{Hom}_S(X^0, Y).$$

That is, the S-affine scheme X^0 satisfies the universal property that any S-morphism $f : X \to Y$ such that Y is affine over S must factors through X^0, or equivalently that X^0 represents the covariant functor $Y \mapsto \mathrm{Hom}_S(X, Y)$ on the category of S-affine schemes. We also deduce that for S fixed, $X \mapsto \mathrm{Aff}(X/S)$ is a covariant functor from the category of S-schemes that are quasi-compact and quasi-separted over S to the cateogry of S-schemes affine over S. Consider a commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\iota_X"'] & X' \arrow[d, "\iota_{X'}"] \\ X^0 \arrow[r, "f^0"] & X'^0 \end{tikzcd}$$

The relation (9.1.1) can then be interpreted as the following: the functor $X \mapsto \mathrm{Aff}(X/S)$ is the left adjoint of the forgetful functor from the category of S-schemes affine over S to the category of S-schemes. We then conclude that this functor commutes with inductive limits, hence finite sums.

Corollary 9.1.15. *Let X be an affine S-scheme and Y be an X-scheme. For Y to be affine over S, it is necessary and sufficient that Y is affine over X.*

Proof. We can assume that S is affine, and then X is also affine by Corollary 9.1.5. Then Y is affine over S if and only if it is affine over X, if and only if it is affine, so our claim follows. □

Let X be an affine S-scheme. Then by Corollary 9.1.15, to define of a scheme Y affine over X is equivalent to giving a scheme Y affine over S and an S-morphism $g : Y \to X$. In view of Proposition 9.1.9 and Proposition 9.1.7, this is equivalent to giving a quasi-coherent $\mathscr{O}_S$-algebra $\mathscr{B}$ and a homomorphism $\mathscr{A}(X) \to \mathscr{B}$ of $\mathscr{O}_S$-algebras (which dfines over $\mathscr{B}$ an $\mathscr{A}(X)$-algebra structure). If $f : X \to S$ is the structural morphism, we then have $\mathscr{B} = f_*(g_*(\mathscr{O}_Y))$.

Corollary 9.1.16. *Let X be an affine S-scheme. For X to be of finite type over S, it is necessary and sufficient that the quasi-coherent $\mathscr{O}_X$-algebra $\mathscr{A}(X)$ is of finite type.*

Proof. By definition, we can assume that S is affine. Then X is an affine scheme, hence quasi-compact; if $S = \mathrm{Spec}(A)$, $X = \mathrm{Spec}(B)$, $\mathscr{A}(X)$ is the $\mathscr{O}_S$-algebra $\widetilde{B}$. As $\Gamma(X, \tilde{B}) = B$, the corollary follows from Corollary 8.6.39. □

Corollary 9.1.17. *Let X be an affine S-scheme. For X to be reduced, it is necessary and sufficient that the quasi-coherent $\mathscr{O}_X$-algebra is reduced.*

Proof. The question is local on S so we can assume that S is affine, and the corollary then follows from Proposition 8.4.30. □

9.1.3 Quasi-coherent sheaves over affine S-schemes

Proposition 9.1.18. *Let X be an affine S-scheme, Y be an S-scheme, and $\mathscr{F}$ (resp. $\mathscr{G}$) be a quasi-coherent $\mathscr{O}_X$-module (resp. $\mathscr{O}_Y$-module). Then the map $(h, u) \mapsto (\mathscr{A}(h), \mathscr{A}(u))$ from the set of morphisms $(Y, \mathscr{G}) \to (X, \mathscr{F})$ to the set of bi-homomorphisms $(\mathscr{A}(X), \mathscr{A}(\mathscr{F})) \to (\mathscr{A}(Y), \mathscr{A}(\mathscr{G}))$ is bijective.*

Proof. The proof is the same as in Proposition 9.1.7, by using Proposition 8.2.6. □

Corollary 9.1.19. *Under the hypotheses of Proposition 9.1.18, suppose that Y is affine over S. Then for the couple (h, u) to be an isomorphism, it is necessary and sufficient that $(\mathscr{A}(h), \mathscr{A}(u))$ is a bi-isomorphism.*

Proposition 9.1.20. *For any couple $(\mathscr{B}, \mathscr{M})$ formed by a quasi-coherent $\mathscr{O}_S$-algebra $\mathscr{B}$ and a quasi-coherent $\mathscr{B}$-module $\mathscr{M}$, there exists a couple $(X, \mathscr{F})$ formed by an affine S-scheme and a quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$ such that $\mathscr{A}(X) = \mathscr{B}$ and $\mathscr{A}(\mathscr{F}) = \mathscr{M}$, and this couple is determined up to isomorphisms.*

Proof. The uniqueness part follows from Corollary 9.1.19. The existence for the scheme X follows from Proposition 9.1.9. To define $\mathscr{M}$, we can consider an affine open $U \subseteq S$ and set $\mathscr{F}|_{f^{-1}(U)} = \Gamma(\widetilde{U, \mathscr{M}})$, where $f : X \to S$ is the structural morphism. We will use $\widetilde{\mathscr{M}}$ to denote the quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$ associated with $\mathscr{M}$. □

Corollary 9.1.21. *In the category of quasi-coherent $\mathscr{B}$-modules, $\widetilde{\mathscr{M}}$ is an additive covariant functor which commutes with inductive limits and direct sums.*

Proof. We can in fact assume that S is affine, and the claim then reduces to the functor $\widetilde{M}$ for B-modules, where $B = \Gamma(S, \mathscr{B})$. □

Corollary 9.1.22. *Under the hypotheses of Proposition 9.1.20, assume that $\mathscr{B}$ is an $\mathscr{O}_X$-algebra of finite type. Then for $\widetilde{\mathscr{M}}$ to be an $\mathscr{O}_X$-module of finite type, it is necessary and sufficient that $\mathscr{M}$ is an $\mathscr{B}$-module of finite type.*

Proof. We can reduce to the case where $S = \operatorname{Spec}(A)$ is affine. Then $\mathscr{B} = \widetilde{B}$ where B is an A-algebra of finite type, and $\mathscr{M} = \widetilde{M}$ where M is an B-module. Over the scheme X, $\mathscr{O}_X$ is associated with the ring B and $\widetilde{\mathscr{M}}$ is associated with the B-module M. For $\widetilde{\mathscr{M}}$ to be of finite type, it is necessary and sufficient that M is of finite type, whence our claim. □

Proposition 9.1.23. *Let Y be an affine S-scheme and X, X' be two schemes affine over Y. Let $\mathscr{B} = \mathscr{A}(Y)$, $\mathscr{A} = \mathscr{A}(X)$, and $\mathscr{A}' = \mathscr{A}(X')$. Then $X \times_Y X'$ is affine over Y and $\mathscr{A}(X \times_Y X')$ is identified with $\mathscr{A} \otimes_{\mathscr{B}} \mathscr{A}'$.*

Proof. In fact, $\mathscr{A} \otimes_{\mathscr{B}} \mathscr{A}'$ is a quasi-coherent $\mathscr{B}$-algebra (Proposition 8.2.25), so is a quasi-coherent $\mathscr{O}_S$-algebra. Let Z be the spectral of $\mathscr{A} \otimes_{\mathscr{B}} \mathscr{A}'$. The canonical $\mathscr{B}$-homomorphisms $\mathscr{A} \to \mathscr{A} \otimes_{\mathscr{B}} \mathscr{A}'$ and $\mathscr{A}' \to \mathscr{A} \otimes_{\mathscr{B}} \mathscr{A}'$ corresponds to Y-morphisms $p : Z \to X$ and $p' : Z \to X'$ (Proposition 9.1.7). To see that the triple (Z, p, p') is a product $X \times_Y X'$, we can reduce to the case $S = \operatorname{Spec}(C)$ is affine. But then Y, X, X' are all affine schemes with rings B, A, A', which are C-algebras such that $\mathscr{B} = \widetilde{B}$, $\mathscr{A} = \widetilde{A}$, $\mathscr{A}' = \widetilde{A}$. We then see that $\mathscr{A} \otimes_{\mathscr{B}} \mathscr{A}'$ is identified with the $\mathscr{O}_S$-algebra $\widetilde{A \otimes_B A'}$ (Corollary 8.1.10), so the ring of Z is identified with $A \otimes_B A'$, and the morphisms p, p' correspond to the canonical homomorphisms $A \to A \otimes_B A'$ and $A' \to A \otimes_B A'$. The proposition then follows from Proposition 8.3.1. □

Corollary 9.1.24. *Let $\mathscr{F}$ (resp. $\mathscr{F}'$) be a quasi-coherent $\mathscr{O}_X$-module (resp. $\mathscr{O}_{X'}$-module). Then $\mathscr{A}(\mathscr{F} \otimes_Y \mathscr{F}')$ is canonically identified with $\mathscr{A}(\mathscr{F}) \otimes_{\mathscr{A}(Y)} \mathscr{A}(\mathscr{F}')$.*

Proof. The sheaf $\mathscr{F} \otimes_Y \mathscr{F}'$ is coherent over $X \times_Y X'$ by **??**. Let $g : Y \to S$, $f : X \to Y$, $f' : X' \to Y$ be the structural morphisms, so the structural morphism $h : Z \to S$ is equal to $g \circ f \circ p$ and to $g \circ f' \circ p'$. We define a canonical homomorphism

$$\mathscr{A}(\mathscr{F}) \otimes_{\mathscr{A}(Y)} \mathscr{A}(\mathscr{F}') \to \mathscr{A}$$

by the following: for any open subset $U \subseteq S$, we have canonical homomorphisms

$$\Gamma(f^{-1}(g^{-1}(U)), \mathscr{F}) \to \Gamma(h^{-1}(U), p^*(\mathscr{F})), \quad \Gamma(f'^{-1}(g^{-1}(U)), \mathscr{F}') \to \Gamma(h^{-1}(U), p'^*(\mathscr{F}')),$$

whence a canonical homomorphism

$$\begin{array}{c} \Gamma(f^{-1}(g^{-1}(U)), \mathscr{F}) \otimes_{\Gamma(g^{-1}(U), \mathscr{O}_Y)} \Gamma(f'^{-1}(g^{-1}(U)), \mathscr{F}') \\ \downarrow \\ \Gamma(h^{-1}(U), p^*(\mathscr{F})) \otimes_{\Gamma(h^{-1}(U), \mathscr{O}_Z)} \Gamma(h^{-1}(U), p'^*(\mathscr{F}')) \end{array}$$

To see this is an isomorphism of $\mathscr{A}(Z)$-modules, we can assume that S is affine, and $\mathscr{F} = \widetilde{M}$, $\mathscr{F}' = \widetilde{M}'$, where M (resp. M') is an A-module (resp. A'-module). Then $\mathscr{F} \otimes_Y \mathscr{F}'$ is identified with the sheaf over $X \times_Y X'$ associated with the $(A \otimes_B A')$-module $M \otimes_B M'$ and the corollary follows from the canonical indentification $\widetilde{M \otimes_B M'}$ with $\widetilde{M} \otimes_{\widetilde{B}} \widetilde{M}'$. □

Corollary 9.1.25. *Let X and Y be affine S-schemes and $f : Y \to X$ be an S-morphism. Let $\mathscr{F}$ be a quasi-coherent $\mathscr{O}_X$-module, then $\mathscr{A}(f^*(\mathscr{F}))$ is identified with $\mathscr{A}(\mathscr{F}) \otimes_{\mathscr{A}(X)} \mathscr{A}(Y)$.*

Proof. This is a special case of Corollary 9.1.24, by replacing X' with Y and Y with X. □

In particular, if $X = X' = Y$ (where X is an affine S-scheme), we see that if $\mathscr{F}, \mathscr{G}$ are two quasi-coherent $\mathscr{O}_X$-modules, then

$$\mathscr{A}(\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G}) = \mathscr{A}(\mathscr{F}) \otimes_{\mathscr{A}(X)} \mathscr{A}(\mathscr{G})$$

If moreover $\mathscr{F}$ is of finite presentation, then it follows from Proposition 8.1.12 and Corollary 8.1.10 that

$$\mathscr{A}(\mathcal{H}om_{\mathscr{O}_X}(\mathscr{F}, \mathscr{G})) = \mathcal{H}om_{\mathscr{A}(X)}(\mathscr{A}(\mathscr{F}), \mathscr{A}(\mathscr{G})).$$

Proposition 9.1.26. *If X and X' are two affine S-schemes with $\mathscr{B} = \mathscr{A}(X)$ and $\mathscr{B}' = \mathscr{A}(X')$. Then the coproduct $X \amalg X'$ is affine over S with $\mathscr{A}(X \amalg X') = \mathscr{B} \times \mathscr{B}'$.*

Proof. The coproduct is affine over S since the product of two affine schemes is affine, and the second assertion also follows from this, and the fact that $\operatorname{Spec}(A) \amalg \operatorname{Spec}(A') = \operatorname{Spec}(A \times A')$ for two rings A, A'. □

Proposition 9.1.27. *Let S be a scheme, $\mathscr{B}$ be a quasi-coherent $\mathscr{O}_S$-algebra, and $X = \operatorname{Spec}(\mathscr{B})$. For any quasi-coherent ideal $\mathscr{I}$ of $\mathscr{B}$, $\widetilde{\mathscr{I}}$ is a quasi-coherent ideal of $\mathscr{O}_X$, and the closed subschem of X defined by $\widetilde{\mathscr{I}}$ is canonically isomorphic to $\operatorname{Spec}(\mathscr{B}/\mathscr{I})$.*

Proof. In fact, it follows from Example 8.4.6 that Y is affine over S, and in view of Proposition 9.1.9, we can then assume that S is affine, and the proposition follows from the corresponding result in affine schemes. □

We can also express the result of Proposition 9.1.27 by saying that if $h : \mathscr{B} \to \mathscr{B}'$ is a surjective homomorphism of quasi-coherent $\mathscr{O}_S$-algebras, then $\mathscr{A}(h) : \operatorname{Spec}(\mathscr{B}') \to \operatorname{Spec}(\mathscr{B})$ is a closed immersion.

Proposition 9.1.28. *Let S be a scheme, $\mathscr{B}$ be a quasi-coherent $\mathscr{O}_S$-algebra, and $X = \operatorname{Spec}(\mathscr{B})$. For any quasi-coherent ideal $\mathscr{K}$ of $\mathscr{O}_S$, we have (where $f : X \to S$ is the structural morphism) $f^*(\mathscr{K})\mathscr{O}_X = \widetilde{\mathscr{K}\mathscr{B}}$.*

Proof. The question is local over S, so we can assume that $S = \operatorname{Spec}(A)$ is affine, and the proposition then follows Proposition 8.1.14. □

9.1.4 Base change of affine S-schemes

Proposition 9.1.29. *Let X be an affine S-scheme. For any extension $g : S' \to S$ of base scheme, $X' = X_{(S')}$ is affine over S'.*

Proof. If $f' : X' \to S'$ is the projection, it suffices to prove that $f'^{-1}(U')$ is an affine open for any affine open subset U' of S' such that $g(U')$ is contained in an affine open U of S. We can then assume that S and S' are affine, so X is affine. But then X' is affine, so the claim follows. □

Corollary 9.1.30. *Let $f : X \to S$ be the structural morphism, $f' : X' \to S'$, $g' : X' \to X$ the projections such that the diagram*

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$

is commutative. For any quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, there exists a canonical isomorphism $u : g^*(f_*(\mathscr{F})) \to f'_*(g'^*(\mathscr{F}))$ of $\mathscr{O}_{S'}$-modules. In particular, there exists a canonical isomorphism from $\mathscr{A}(X')$ to $g^*(\mathscr{A}(X))$.

Proof. To define u, it suffices to define a homomorphism

$$v : f_*(\mathscr{F}) \to g_*(f'^*(\mathscr{F})) = f_*(g'_*(g'^*(\mathscr{F})))$$

and let u be the homomorphism corresponding to v (via the adjointness). We set $v = f_*(\rho)$, where $\rho : \mathscr{F} \to g'_*(g'^*(\mathscr{F}))$ is the canonical homomorphism. To prove that u is an isomorphism, we can assume that S and S', hence X and X', are affine. Let A, A', B, B' be the ring of X, X', S, S', then $\mathscr{F} = \widetilde{M}$ where M is an B-module. We then see that $g^*(f_*(\mathscr{F}))$ and $f'_*(g'^*(\mathscr{F}))$ are equal to the $\mathscr{O}_{S'}$-module associated with the A'-module $A' \otimes_A M$, and u is the homomorphism associated with the identity. □

Corollary 9.1.31. *For any affine S-scheme X and $s \in S$, the fiber X_s is an affine scheme.*

Proof. It suffices to apply Proposition 9.1.29 on $\operatorname{Spec}(\kappa(s)) \to S$. □

Corollary 9.1.32. *Let X be an S-scheme and S' be an affine S-scheme. Then $X' = X_{(S')}$ is affine over X. Moreover, if $f : X \to S$ is the structural homomorphism, there exists a canonical isomorphism of $\mathscr{O}_X$-algebras $\mathscr{A}(X') \cong f^*(\mathscr{A}(S'))$, and for any quasi-coherent $\mathscr{A}(S')$-module $\mathscr{M}$, a canonical bi-isomorphism $f^*(\mathscr{M}) \cong \mathscr{A}(f'^*(\widetilde{\mathscr{M}}))$, where $f' = f_{(S')}$ is the structural morphism $X' \to S'$.*

Proof. It suffices to apply Proposition 9.1.29 and Corollary 9.1.30, with the role of X and S' exchanged. □

Let S, S' be two schemes, $q : S' \to S$ be a morphism, $\mathscr{B}$ (resp. $\mathscr{B}'$) be a quasi-coherent $\mathscr{O}_S$-algebra (resp. $\mathscr{O}_{S'}$-algebra), and $u : \mathscr{B} \to \mathscr{B}'$ be a q-morphism (which means a homomorphism $\mathscr{B} \to q_*(\mathscr{B}')$ of $\mathscr{O}_S$-algebras). If $X = \operatorname{Spec}(\mathscr{B})$ and $X' = \operatorname{Spec}(\mathscr{B}')$, we deduce a canonical morphism $v = \operatorname{Spec}(u) : X' \to X$ such that the diagram

$$\begin{tikzcd} X' \arrow[r, "v"] \arrow[d] & X \arrow[d] \\ S' \arrow[r, "q"] & S \end{tikzcd} \tag{9.1.2}$$

is commutatie. In fact, the homomorphism u corresponds to a homomorphism $u^\sharp : q^*(\mathscr{B}) \to \mathscr{B}'$ by adjointness, and there then exists a canonical S'-morphism

$$w : \operatorname{Spec}(\mathscr{B}') \to \operatorname{Spec}(q^*(\mathscr{B}))$$

such that $\mathscr{A}(w) = u^\sharp$ (Proposition 9.1.7). On the other hand, it follows from Corollary 9.1.30 that $\operatorname{Spec}(q^*(\mathscr{B}))$ is canonically identified with $X \times_S S'$; the morphism v is the defined to be the composition

$$X' \xrightarrow{w} X \times_S S' \xrightarrow{p_1} X$$

where p_1 is the projection, and the commutativity of (9.1.2) is easily verified. Let U (resp. U') be an affine open of S (resp. S') such that $q(U') \subseteq U$, $A = \Gamma(U, \mathscr{O}_S)$, $A' = \Gamma(U', \mathscr{O}_{S'})$, $B = \Gamma(U, \mathscr{B})$, $B' = \Gamma(U'\mathscr{B}')$. The restriction of u is a $(q|_{U'})$-morphism $u|_U : \mathscr{B}|_U \to \mathscr{B}'|_{U'}$ corresponding to a bi-homomorphism $B \to B'$ of algebras. If V, V' are the inverse images of U, U' in X, X', respectively, the morphism $V' \to V$, which is the restriction of v, corresponds to the preceding bi-homomorphism.

Now let $\mathscr{M}$ be a quasi-coherent $\mathscr{B}$-module. There then exists a canonical isomorphism of $\mathscr{O}_{X'}$-modules

$$v^*(\widetilde{\mathscr{M}}) \cong (q^*(\mathscr{M}) \otimes_{q^*(\mathscr{B})} \mathscr{B}')^\sim. \tag{9.1.3}$$

In fact, the canonical isomorphism of Corollary 9.1.30 provides a canonical isomorphism of $p_1^*(\widetilde{\mathscr{M}})$ with the sheaf over $\operatorname{Spec}(q^*(\mathscr{B}))$ associated with $q^*(\mathscr{B})$-module $q^*(\mathscr{M})$, and it suffices to apply Corollary 9.1.24.

Recall that we say a morphism $f : X \to Y$ is affine if X is an affine scheme over Y. The properties of affine S-schemes then translate into properties of affine morphisms.

Proposition 9.1.33 (Properties of Affine Morphisms).

(i) *A closed immersion is affine.*

(ii) *The composition of two affine morphisms is affine.*

(iii) *If $f : X \to Y$ is an affine S-morphism, then $f_{(S')}$ is affine for any base change $S' \to S$.*

(iv) *If $f : X \to Y$ and $f' : X' \to Y'$ are two affine S-morphisms, then $f \times_S f'$ is affine.*

(v) *If $f : X \to Y$ and $g : Y \to Z$ are two morphisms such that $g \circ f$ is affine and g is separated, then f is affine.*

(vi) *If f is affine, so is f_{red}.*

Proof. In view of Proposition 8.5.22, it suffices to prove (i), (ii), and (iii). Now (i) follows from Example 8.4.6, (ii) follows from Corollary 9.1.5, and (iii) follows from Proposition 9.1.29. □

Corollary 9.1.34. *If X is an affine scheme and Y is a separated scheme, any morphism $f : X \to Y$ is affine.*

Proof. This is a direct concequence of Proposition 9.1.33(v), since the canonical morphism $X \to \operatorname{Spec}(\mathbb{Z})$ is affine. □

Proposition 9.1.35. *Let Y be a locally Noetherian scheme and $f : X \to Y$ be a morphism of finite type. Then for f to be affine, it is necessary and sufficient that f_{red} is affine.*

Proof. It suffcies to prove that f is affine if f_{red} is affine. For this, we can assume that Y is affine and Noetherian, and show that X is affine. Now Y_{red} is affine, and X_{red} is therefore affine by hypothesis. Since X is Noetherian, the assertion follows from Corollary 8.4.37. □

9.1.5 Vector bundles

Let A be a ring and E be an A-module. Recall that the symmetric algebra over E is the A-algebra $\boldsymbol{S}(E)$ (or $\boldsymbol{S}_A(E)$) which is the quotient of $\boldsymbol{T}(E)$ by the ideal generated by elements $x \otimes y - y \otimes x$, where x, y belongs to E. The algebra $\boldsymbol{S}(E)$ is characterized by the universal property that if $\sigma : E \to \boldsymbol{S}(E)$ is the canonical map, any A-linear map $E \to B$, where B is a commutative algebra, factors through $\boldsymbol{S}(E)$ and gives a homomorphism $\boldsymbol{S}(E) \to B$ of A-algebras. We deduce from this property that for two A-modules E, F, we have

$$\boldsymbol{S}(E \oplus F) = \boldsymbol{S}(E) \otimes \boldsymbol{S}(F).$$

Moreover, $\boldsymbol{S}(E)$ is a covariant functor on E from the category of A-modules to that of commutative A-algebras. Finally, the preceding characterization shows that if $E = \varinjlim E_\lambda$, then $\boldsymbol{S}(E) = \varinjlim \boldsymbol{S}(E_\lambda)$. By abuse of language, a product $\sigma(x_1) \cdots \sigma(x_n)$, where $x_i \in E$, is usually written as $x_1 \cdots x_n$. The algebra $\boldsymbol{S}(E)$ is graded, where $\boldsymbol{S}_n(E)$ is the set of linear combinations of products of n elements of E. In particular, the algebra $\boldsymbol{S}(A)$ is canonically isomorphic to the polynomial algebra $A[T]$ over A, and the algebra $\boldsymbol{S}(A^n)$ is the polynomial algebra $A[T_1, \dots, T_n]$ over A. More particularly, if E is free of rank 1, then $\boldsymbol{S}_n(E)$ is isomorphic to the tensor algebra $\boldsymbol{T}(E) = \bigoplus_{n \geq 0} E^{\otimes n}$.

Let $\varphi : A \to B$ be a ring homomorphism. If F is an B-module, the canonical map $F \to \boldsymbol{S}(F)$ then gives a canonical map $F_{(\varphi)} \to \boldsymbol{S}(F)_{(\varphi)}$, which factors into $F_{(\varphi)} \to \boldsymbol{S}(F_{(\varphi)}) \to \boldsymbol{S}(F)_{(\varphi)}$. The canonical homomorphism $\boldsymbol{S}(F_{(\varphi)}) \to \boldsymbol{S}(F)_{(\varphi)}$ is surjective, but not necessarily bijective. If E is an A-module, any bi-homorphism $E \to F$ (which is an A-homomorphism $E \to F_{(\varphi)}$) then gives an A-homomorphism $\boldsymbol{S}(E) \to \boldsymbol{S}(F_{(\varphi)}) \to \boldsymbol{S}(F)_{(\varphi)}$ of algebras, which is a bi-homomorphism $\boldsymbol{S}(E) \to \boldsymbol{S}(F)$. Also, for any A-module E, $\boldsymbol{S}(E \otimes_A B)$ is canonically identified with the algebra $\boldsymbol{S}(E) \otimes_A B$, which follows from the universal property of $\boldsymbol{S}(E)$.

Let R be a multiplicative subset of A. Then apply the previous arguments for the ring $B = R^{-1}A$, and recall that $R^{-1}E = E \otimes_A R^{-1}A$, we see that $\boldsymbol{S}(R^{-1}E) = R^{-1}\boldsymbol{S}(E)$. Moreover, if $R' \supseteq R$ is another multiplicative subset of A, the diagram

$$\begin{tikzcd}
R^{-1}E \arrow[r] \arrow[d] & R'^{-1}E \arrow[d] \\
\boldsymbol{S}(R^{-1}E) \arrow[r] & \boldsymbol{S}(R'^{-1}E)
\end{tikzcd}$$

is commutative.

Now let $(S, \mathscr{A})$ be a ringed space and $\mathscr{E}$ be an $\mathscr{A}$-module over S. If for each open subset $U \subseteq S$, we associate the $\Gamma(U, \mathscr{A})$-module $\boldsymbol{S}(\Gamma(U, \mathscr{E}))$, we then define a presheaf of algebras. The associated sheaf is called the **symmetric $\mathscr{A}$-algebra** of the $\mathscr{A}$-module $\mathscr{E}$ and denoted by $\boldsymbol{S}(\mathscr{E})$. It follows immediately that $\boldsymbol{S}(\mathscr{E})$ satisfies the following universal property: any homomorphism $\mathscr{E} \to \mathscr{B}$ of $\mathscr{A}$-modules, where $\mathscr{B}$ is an $\mathscr{A}$-algebra, factors through $\boldsymbol{S}(\mathscr{E})$ to give a homomorphism $\boldsymbol{S}(\mathscr{E}) \to \mathscr{B}$ of $\mathscr{A}$-algebras. In particular, any homomorphism $u : \mathscr{E} \to \mathscr{F}$ of $\mathscr{A}$-modules defines a homomorphism $\boldsymbol{S}(u) : \boldsymbol{S}(\mathscr{E}) \to \boldsymbol{S}(\mathscr{F})$ of $\mathscr{A}$-algebras and $\boldsymbol{S}(\mathscr{E})$ is a covariant functor $\mathscr{E}$.

Now since the functor $\boldsymbol{S}$ commutes with inductive limits, we have $\boldsymbol{S}(\mathscr{E})_s = \boldsymbol{S}(\mathscr{E}_s)$ for any point $s \in S$. If $\mathscr{E}, \mathscr{F}$ are two $\mathscr{A}$-module, $\boldsymbol{S}(\mathscr{E} \oplus \mathscr{F})$ is canonically identified with $\boldsymbol{S}(\mathscr{E}) \otimes_{\mathscr{A}} \boldsymbol{S}(\mathscr{F})$, as we can check this for the corresponding presheaves.

We see that $\boldsymbol{S}(\mathscr{E})$ is a graded $\mathscr{A}$-algebra, and $\boldsymbol{S}_n(\mathscr{E})$ is the $\mathscr{A}$-module associated with the presheaf $U \mapsto \boldsymbol{S}_n(\Gamma(U,\mathscr{E}))$. In particular, the algebra $\boldsymbol{S}(\mathscr{A})$ is identified with $\mathscr{A}[T] = \mathscr{A} \otimes_{\mathbb{Z}} \mathbb{Z}[T]$, and if $\mathscr{E}$ is an invertible sheaf, then $\boldsymbol{S}(\mathscr{E})$ is isomorphic to the tensor algebra $\boldsymbol{T}(\mathscr{E}) = \bigoplus_{n\geq 0} \mathscr{E}^{\otimes n}$.

Proposition 9.1.36. *Let $f : (S,\mathscr{A}) \to (T,\mathscr{B})$ be a morphism of ringed spaces. If $\mathscr{F}$ is an $\mathscr{B}$-module, then $\boldsymbol{S}(f^*(\mathscr{F}))$ is canonically identified with $f^*(\boldsymbol{S}(\mathscr{F}))$*

Proof. To see this, we may make use the universal property of $\boldsymbol{S}$. By definition, $\boldsymbol{S}(f^*(\mathscr{F}))$ is defined to be the unique $\mathscr{A}$-algebra satisfying the following equality

$$\mathrm{Hom}_{\mathscr{A}\text{-}\mathbf{alg}}(\boldsymbol{S}(f^*(\mathscr{F})),\mathscr{C}) = \mathrm{Hom}_{\mathscr{A}}(f^*(\mathscr{F}),\mathscr{C})$$

for any $\mathscr{A}$-algebra $\mathscr{C}$. On the other hand, by the adjointness property of f_* and f^*, we have

$$\begin{aligned}\mathrm{Hom}_{\mathscr{A}\text{-}\mathbf{alg}}(f^*(\boldsymbol{S}(\mathscr{F})),\mathscr{C}) &= \mathrm{Hom}_{\mathscr{B}\text{-}\mathbf{alg}}(\boldsymbol{S}(\mathscr{F}),f_*(\mathscr{C}))\\ &= \mathrm{Hom}_{\mathscr{B}}(\mathscr{F},f_*(\mathscr{C})) = \mathrm{Hom}_{\mathscr{A}}(f^*(\mathscr{F}),\mathscr{C}).\end{aligned}$$

This implies the desired isomorphism. □

Proposition 9.1.37. *Let A be a ring, $S = \mathrm{Spec}(A)$ be the spectrum, and $\mathscr{E} = \widetilde{M}$ be the $\mathscr{O}_S$-module assocaited with an A-module M. Then the $\mathscr{O}_S$-algebra $\boldsymbol{S}(\mathscr{E})$ is associated with the A-algebra $\boldsymbol{S}(M)$.*

Proof. In fact, for any $f \in A$, $\boldsymbol{S}(M_f) = \boldsymbol{S}(M)_f$, so the proposition follows from the definition of $\widetilde{\boldsymbol{S}(M)}$. □

Corollary 9.1.38. *If S is a scheme and $\mathscr{E}$ is a quasi-coherent $\mathscr{O}_S$-module. Then the $\mathscr{O}_S$-algebra $\boldsymbol{S}(\mathscr{E})$ is quasi-coherent. If moreover $\mathscr{E}$ is of finite type (resp. of finite presentation), then each $\mathscr{O}_S$-module $\boldsymbol{S}_n(\mathscr{E})$ is of finite type (resp. finite presentation) and the $\mathscr{O}_S$-algebra $\boldsymbol{S}(\mathscr{E})$ is of finite type (resp. of finite presentation).*

Proof. The first assertion is immediate by Proposition 9.1.37. The second one follows from the fact that, if E is a finitely generated A-module, $\boldsymbol{S}_n(E)$ is also finitely generated. For the last assertion, we are reduced to the case $S = \mathrm{Spec}(A)$ and $\mathscr{E} = \widetilde{E}$ where E is an A-module of finite type (resp. of finite presentation). Now if we have an exact sequence

$$0 \longrightarrow N \longrightarrow A^n \longrightarrow E \longrightarrow 0$$

then we deduce an exact sequence

$$0 \longrightarrow \mathfrak{J} \longrightarrow \boldsymbol{S}(A^n) \longrightarrow \boldsymbol{S}(E) \longrightarrow 0$$

where $\mathfrak{J}$ is the ideal of $\boldsymbol{S}(A^n)$ generated by $N \subseteq \boldsymbol{S}_1(A^n)$, whence our conclusion. □

Let S be a scheme and $\mathscr{E}$ be a quasi-coherent $\mathscr{O}_S$-module. For any S-scheme T, with structural morphism $f : T \to S$, let $\mathscr{E}_{(T)} = f^*(\mathscr{E})$, which is a quasi-coherent $\mathscr{O}_T$-module. The map

$$T \mapsto F_{\mathscr{E}}(T) = \mathrm{Hom}_{\mathscr{O}_T}(\mathscr{E}_{(T)},\mathscr{O}_T) = \Gamma(T,\mathscr{E}_{(T)}^\vee)$$

then defines a contravariant functor from the category of S-schemes to that of sets if for any S-morphism $g : T' \to T$ we define $F_{\mathscr{E}}(g) : F_{\mathscr{E}}(T) \to F_{\mathscr{E}}(T')$ to be the map $g^* : u \mapsto g^*(u)$ (note that the structural morphism $T' \to S$ is $f \circ g$ and we have $\mathscr{E}_{(T')} = g^*(\mathscr{E}_{(T)})$ and $\mathscr{O}_{T'} = g^*(\mathscr{O}_T)$).

Proposition 9.1.39. *For any quasi-coherent $\mathscr{O}_S$-module $\mathscr{E}$, the contravariant functor $F_{\mathscr{E}}$ is represented by the couple formed by the affine S-scheme $\mathbb{V}(\mathscr{E}) = \mathrm{Spec}(\boldsymbol{S}(\mathscr{E}))$. The S-scheme $\mathbb{V}(\mathscr{E})$ is called the **vector bundle over S defined by** $\mathscr{E}$.*

Proof. This follows from the following canonical isomorphisms for any S-scheme T:

$$\begin{aligned}\mathrm{Hom}_S(T,\mathbb{V}(\mathscr{E})) &= \mathrm{Hom}_{\mathscr{O}_S\text{-}\mathbf{alg}}(\boldsymbol{S}(\mathscr{E}),\mathscr{A}(T)) = \mathrm{Hom}_{\mathscr{O}_S}(\mathscr{E},f_*(\mathscr{O}_T))\\ &= \mathrm{Hom}_{\mathscr{O}_T}(f^*(\mathscr{E}),\mathscr{O}_T) = \mathrm{Hom}_{\mathscr{O}_T}(\mathscr{E}_{(T)},\mathscr{O}_T).\end{aligned}$$

□

The canonical $\boldsymbol{S}(\mathscr{E})$-homomorphism $\mathscr{E}\otimes_{\mathscr{O}_S}\boldsymbol{S}(\mathscr{E})\to\boldsymbol{S}(\mathscr{E})$ induced by Proposition 9.1.7 an $\mathscr{O}_{\mathbb{V}(\mathscr{E})}$-homomorphism $\mathscr{E}_{(\mathbb{V}(\mathscr{E}))}\to\mathscr{O}_{\mathbb{V}(\mathscr{E})}$, which is a section over $\mathbb{V}(\mathscr{E})$ of dual sheaf $\mathscr{E}^*_{(\mathbb{V}(\mathscr{E}))}$ of $\mathscr{E}_{(V(\mathscr{E}))}$, called the **universal section** of this dual. If $U=\operatorname{Spec}(A)$ is an affine open of S, its inverse image in $\mathbb{V}(\mathscr{E})$ is identified with $\operatorname{Spec}(\boldsymbol{S}(M))$, if $\mathscr{E}|_U=\widetilde{M}$ where M is an A-module. Over the scheme $\operatorname{Spec}(\boldsymbol{S}(M))$, the universal section is identified with the homomorphism $m\otimes p\mapsto mp$ of $M\otimes_A\boldsymbol{S}(M)$ to $\boldsymbol{S}(M)$, where M is identified with the subset $\boldsymbol{S}_1(M)$ of $\boldsymbol{S}(M)$.

Consider in particular an open subset U of S. Then the S-morphisms $U\to\mathbb{V}(\mathscr{E})$ are the U-sections of the U-scheme induced by $\mathbb{V}(\mathscr{E})$ over $p^{-1}(U)$ (where $p:\mathbb{V}(\mathscr{E})\to S$ is the structural morphism). By the definition of $\mathbb{V}(\mathscr{E})$, these U-sections correspond bijectively to sections of the dual $\mathscr{E}^*$ of $\mathscr{E}$ over U. The functorial of $\mathbb{V}$ shows that this interpretation is compatible with the restriction to an open subset $U'\subseteq U$, so we can say that the dual $\mathscr{E}^*$ of $\mathscr{E}$ is canonically identified with the sheaf of germs of S-sections of $\mathbb{V}(\mathscr{E})$. In particular, if $T=S$, the zero homomorphism $\mathscr{E}\to\mathscr{O}_S$ corresponds to an S-section of $\mathbb{V}(\mathscr{E})$, called the **zero section**.

If now we choose T to be the spectrum $\{\xi\}$ of a field K, the structural morphism $f:T\to S$ corresponds to a monomorphism $\kappa(s)\to K$, where $s=f(\xi)$ (Corollary 8.2.17), and the S-morphisms $\{\xi\}\to\mathbb{V}(\mathscr{E})$ are none other than points of $\mathbb{V}(\mathscr{E})$ with values in the extension K of $\kappa(s)$, which all locate at some points of $p^{-1}(s)$. The set of these points, which is called the **rational fiber** of $\mathbb{V}(\mathscr{E})$ over K lying over the point s, is then identified (by the definition of $\mathbb{V}(\mathscr{E})$) with the dual of the K-vector space $\mathscr{E}_s\otimes_{\mathscr{O}_{S,s}}K=\mathscr{E}^s\otimes_{\kappa(s)}K$ where we set $\mathscr{E}^s=\mathscr{E}_s/\mathfrak{m}_s\mathscr{E}_s$. If $\mathscr{E}^s$ and K are of finite rank over $\kappa(s)$, this dual is identified with $(\mathscr{E}^s)^*\otimes_{\kappa(s)}K$, where $(\mathscr{E}^s)^*$ is the dual space of the $\kappa(s)$-vector space $\mathscr{E}^s$.

These properties justify the terminology of "vector bundle" introduced above, but note that the definition we obtained is dual to the classical definition, since one would expect to obtain the space $\mathscr{E}^s\otimes_{\kappa(s)}K$ for the fiber of $\mathbb{V}(\mathscr{E})$, rather than its dual. This distinction is imposed for the need of defining $\mathbb{V}(\mathscr{E})$ for any quasi-coherent $\mathscr{O}_S$-module $\mathscr{E}$, not only for locally free $\mathscr{O}_S$-modules of finite rank. We can indeed show that the functor $T\mapsto\Gamma(T,\mathscr{E}_{(T)})$ is only representable if $\mathscr{E}$ is locally free of finite rank.

Proposition 9.1.40. *Let S be a scheme.*

- (i) *$\mathbb{V}$ is a contravariant functor on $\mathscr{E}$ from the category of quasi-coherent $\mathscr{O}_S$-modules to the category of affine S-schemes.*
- (ii) *If $\mathscr{E}$ is of finite type (resp. of finite presentation), $\mathbb{V}(\mathscr{E})$ is of finite type (resp. of finite presentation) over S.*
- (iii) *If $\mathscr{E}$ and $\mathscr{F}$ are two quasi-coherent $\mathscr{O}_S$-modules, $\mathbb{V}(\mathscr{E}\oplus\mathscr{F})$ is canonically identified with $\mathbb{V}(\mathscr{E})\times_S\mathbb{V}(\mathscr{F})$.*
- (iv) *Let $g:S'\to S$ be a morphism. For any quasi-coherent $\mathscr{O}_S$-module $\mathscr{E}$, $\mathbb{V}(g^*(\mathscr{E}))$ is canonically identified with $\mathbb{V}(\mathscr{E})_{(S')}=\mathbb{V}(\mathscr{E})\times_S S'$.*
- (v) *A surjective homomorphism $\mathscr{E}\to\mathscr{F}$ of quasi-coherent $\mathscr{O}_S$-modules corresponds to a closed immersion $\mathbb{V}(\mathscr{F})\to\mathbb{V}(\mathscr{E})$.*

Proof. Assertion (i) follows from Proposition 9.1.7, since for any homomorphism $\mathscr{E}\to\mathscr{F}$ of $\mathscr{O}_S$-modules we have a homomorphism $\boldsymbol{S}(\mathscr{E})\to\boldsymbol{S}(\mathscr{F})$ of $\mathscr{O}_S$-algebras. Assertion (ii) follows immediately from Corollary 9.1.22 and Corollary 9.1.38. To prove (iii), it suffices to recall the canonical isomorphism $\boldsymbol{S}(\mathscr{E}\oplus\mathscr{F})\cong\boldsymbol{S}(\mathscr{E})\otimes_{\mathscr{O}_S}\boldsymbol{S}(\mathscr{F})$ and apply Proposition 9.1.23. Similarly, to prove (iv), it suffices to remark that if the homomorphism $\mathscr{E}\to\mathscr{F}$ is surjective, so is the corresponding homomorphism $\boldsymbol{S}(\mathscr{E})\to\boldsymbol{S}(\mathscr{F})$ of $\mathscr{O}_S$-algebras, and apply Proposition 9.1.27. □

Example 9.1.41. Consider in particular $\mathscr{E}=\mathscr{O}_S$. The scheme $\mathbb{V}(\mathscr{O}_S)$ is the specturm of the $\mathscr{O}_S$-algebra $\boldsymbol{S}(\mathscr{O}_S)$, which is identified with $\mathscr{O}_S[T]=\mathscr{O}_S\otimes_{\mathbb{Z}}\mathbb{Z}[T]$. This is evident if $S=\operatorname{Spec}(\mathbb{Z})$, in view of Proposition 9.1.37, and we pass from this to the general case by considering the structural morphism $S\to\operatorname{Spec}(\mathbb{Z})$ and using Proposition 9.1.40(iv). Because of this result, we again set $\mathbb{V}(\mathscr{O}_S)=S[T]$, and we obtain the identification of the sheaf of germs of S-sections of $S[T]$ over $\mathscr{O}_S$ as a particular case.

For any S-scheme X, by the definition of $\mathbb{V}(\mathscr{O}_S)$, the set $\operatorname{Hom}_S(X,S[T])$ is canonically identified with $\operatorname{Hom}_{\mathscr{O}_S}(\mathscr{O}_S,\mathscr{A}(X))$, which is canonically isomorphic to $\Gamma(S,\mathscr{A}(X))$ and therefore has a ring structure. Moreover, any S-morphism $h:X\to Y$ corresponds to a homomorphism $\Gamma(\mathscr{A}(h)):\Gamma(S,\mathscr{A}(Y))\to\Gamma(S,\mathscr{A}(X))$, so we obtain a contravariant functor $\operatorname{Hom}_S(X,S[T])$ from the category of S-schemes to the cateogry of rings. On the other hand, $\operatorname{Hom}_S(X,\mathbb{V}(\mathscr{E}))$ is identified similarly with $\operatorname{Hom}_{\mathscr{O}_S}(\mathscr{E},\mathscr{A}(X))$ (where $\mathscr{A}(X)$ is considered as an $\mathscr{O}_S$-module); we can then endow this set a $\operatorname{Hom}_S(X,S[T])$-module structure, and the couple

$$(\operatorname{Hom}_S(X,S[T]),\operatorname{Hom}_S(X,\mathbb{V}(\mathscr{E})))$$

is a contravariant functor on X with values in the category of couples (A, M) formed by a ring A and an A-module M, with morphisms being the bi-homomorphisms. In view of this, we say that $S[T]$ is the S-**scheme of ring** and $\mathbb{V}(\mathscr{E})$ is the S-**scheme of module** over the S-scheme of ring $S[T]$.

9.2 Homogeneous specturm of graded algebras

Let S be a graded ring and S_+ be the irrelevant ideal. We say a subset $\mathfrak{I}$ of S_+ is an **ideal of** S_+ if it is an ideal of S, and it is called a **graded prime ideal of** S_+ if it is the intersection with S_+ of a graded prime ideal of S not containing S_+ (in particular $\mathfrak{I} \neq S_+$, and this graded prime ideal of S is unique by **??**). If $\mathfrak{I}$ is an ideal of S_+, the radical of $\mathfrak{I}$ in S_+, denoted by $\mathfrak{r}_+(\mathfrak{I})$, is the set of elements of S_+ which have some power contained in $\mathfrak{I}$, or equivalently, $\mathfrak{r}_+(\mathfrak{I}) = \sqrt{\mathfrak{I}} \cap S_+$. In particularm the radical of 0 in S_+ is called the **nilradical** of S_+ and denoted by $\mathfrak{n}_+$: this is the subset of nilpotent elements of S_+. If $\mathfrak{I}$ is a graded ideal of S_+, its radical $\mathfrak{r}_+(\mathfrak{I})$ is also graded: by passing to $S/\mathfrak{I}$, we may assume that $\mathfrak{I} = 0$, and note that if $x = x_h + x_{h+1} + \cdots + x_k$ is nilpotent, then so is each $x_i \in S_i$; we can suppose that $x_k = 0$ and the top degree component of x^n is then x_k^n, so x_k is nilpotent, and we then preceed by induction on k. We say the graded ring S is essentially reduced if $\mathfrak{n}_+ = 0$, which means S_+ contains no nonzero nilpotent elements.

We note that in a graded ring S, if an element x is a zero divisor, so is its homogeneous component of top degree. We then say that the ring S is **essentially integral** if the ring S_+ (with the unit element) does not contain nonzero zero divisors; it suffices for this that a nonzero homogeneous element in S_+ is not divisor of 0 in this ring. It is clear that if $\mathfrak{p}$ is a graded prime ideal of S_+, $S/\mathfrak{p}$ is essentially integral. Let S be an essentially integral graded ring, and let $x_0 \in S_0$. It there exists a homogeneous element $f \neq 0$ in S_+ such that $x_0 f = 0$, we then have $x_0 S_+ = 0$, because $(x_0 g)f = (x_0 f)g = 0$ for any $g \in S_+$, and the hypothesis on S implies that $x_0 g = 0$. Therfore, for that S is integral, it is necessary and sufficient that S_0 is integral and the annihilator of S_+ in S_0 reduces to zero.

9.2.1 Localization of graded rings

Let S be a graded ring with positive degrees, f be a homogeneous element of S of degree $d > 0$. Then the fraction ring S_f is graded, where $(S_f)_n$ is the set of elements x/f^k, where $x \in S_{n+kd}$ with $k \geq 0$ (note that n can be an arbitrary integer). We denote by $S_{(f)}$ the subring $(S_f)_0$ of S_f formed by elements of degree 0.

If $f \in S_d$, the monomials $(f/1)^h$ in S_f (where h is an integer) form a lineraly independent system over the ring $S_{(f)}$, and the set of their linear combinations over $S_{(f)}$ is exactly the ring $(S^{(d)})_f$ (recall that $S^{(d)}$ is the direct sum of S_{nd}), and then we get an isomorphism

$$(S^{(d)})_f \cong S_{(f)}[T, T^{-1}] = S_{(f)} \otimes_{\mathbb{Z}} \mathbb{Z}[T, T^{-1}] \tag{9.2.1}$$

(where T is an indeterminate). In fact, if we have a relatin

$$\sum_{h=-a}^{a} z_h (f/1)^h = 0$$

where $z_h = x_h/f^m \in S_{(f)}$, then there exists an integer $k > -a$ such that

$$\sum_{h=-a}^{b} f^{h+k} x_h = 0,$$

and as the degrees of these terms are distinct, we have $f^{h+k} x_h = 0$ for all h, so $z_h = 0$ for all h. Similarly, if M is a graded S-module, the localization M_f is a graded S_f-module with $(M_f)_n$ being the set of elements z/f^k where $z \in M_{n+kd}$. We denote by $M_{(f)}$ the set of elements of degree 0 in M_f. It is immediate that $M_{(f)}$ is an $S_{(f)}$-module and we have $(M^{(d)})_f = M_{(f)} \otimes_{S_{(f)}} (S^{(d)})_f$.

Lemma 9.2.1. *Let $f \in S_d$ and $g \in S_e$ be two homogeneous elements of S with positive degrees. Then there exists a canonical isomorphism*

$$S_{(fg)} \cong (S_{(f)})_{g^d/f^e}.$$

If we identify these two rings, then for any S-module M, we have a canonical isomorphism

$$M_{(fg)} \cong (M_{(f)})_{g^d/f^e}.$$

Proof. Note that (fg) divides $f^e g^d$ and $f^e g^d$ divides $(fg)^{de}$, so the rings S_{fg} and $S_{f^e g^d}$ are canonically identified. On the other hand, $S_{f^e g^d}$ is also identified with $(S_{f^e})_{g^d/1}$, and as $f^e/1$ is invertible in S_{f^e}, $s_{f^e g^d}$ is also identified with $(S_{f^e})_{g^d/f^e}$. Now the element g^e/f^e is of degree zero in S_{f^e}, so we can conclude that the subring of $(S_{f^e})_{g^d/f^e}$ formed by elements of degree zero is $(S_{(f^e)})_{g^d/f^e}$, and as we have $S_{(f^e)} = S_{(f)}$, we see the assertion follows. $\square$

With the hypotheses of Lemma 9.2.1, it is clear that the canonical homomorphism $S_f \to S_{fg}$, which maps x/f^k to $xg^k/(fg)^k$, is of degree 0 so restricts to a canonical homomorphism $S_{(f)} \to S_{(fg)}$, such that the diagram

$$\begin{tikzcd} & S_{(f)} \arrow[dl] \arrow[dr] & \\ S_{(fg)} \arrow[rr, "\sim"] & & (S_{(f)})_{(g^d/f^e)} \end{tikzcd}$$

is commutative. We define similarly a canonical homomorphism $M_{(f)} \to M_{(fg)}$.

Lemma 9.2.2. *If f, g are two homogeneous elements of S_+, the ring $S_{(fg)}$ is generated by the union of the canonical images of $S_{(f)}$ and $S_{(g)}$.*

Proof. In view of Lemma 9.2.1, it suffices to show that $1/(g^d/f^e) = f^{d+e}/(fg)^d$ belongs to the canonical image of $S_{(g)}$ in $S_{(fg)}$, which is evident from the definition. $\square$

Proposition 9.2.3. *Let $f \in S_d$ be a homogeneous element of positive degree. Then there exists a canonical isomorphism $S_{(f)} \cong S^{(d)}/(f-1)S^{(d)}$ of rings. If we identify these two rings, then for any S-module M, there exists a canonical isomorphism of modules $M_{(f)} \cong M^{(d)}/(f-1)M^{(d)}$*

Proof. The first isomorphism is defined by sending the element x/f^n, where $x \in S_{nd}$, to the element $\bar{x}$, the class of x mod $(f-1)S^{(d)}$. This map is well-defined, because we have the congruence $f^h x \equiv x$ mod $(f-1)S^{(d)}$ for any $x \in S^{(d)}$, so if $f^h x = 0$ for some $h > 0$ then $\bar{x} = 0$. On the other hand, if $x \in S_{nd}$ is such that $x = (f-1)y$ with $y = y_{hd} + y_{(h+1)d} \cdots + y_{kd}$, where $y_{jd} \in S_{jd}$ and $y_{hd} \neq 0$, we have necessarily $h = n$ and $x = -y_{hd}$, as well as the relations $y_{(j+1)d} = f y_{jd}$ for $h \leq j \leq k-1$ and $f_{ykd} = 0$; in particular, this implies $f^{k-n}x = 0$. We therefore have an inverse homomorphism from $S^{(d)}/(f-1)S^{(d)}$ to $S_{(f)}$ by corresponding a class $\bar{x}$ mod $(f-1)S^{(d)}$ (where $x \in S_{nd}$) the element x/f^n of $S_{(f)}$, since the preceding remark shows that this map is well-defined. The first assertion is therefore proves, and the second one can be done similarly. $\square$

Corollary 9.2.4. *If S is Noetherian, so is $S_{(f)}$ for any homogeneous element f of positive degree.*

Proof. This follows from Proposition 9.2.3 and **??**. $\square$

Let T be a multiplicative subset of S_+ formed by homogeneous elements; $T_0 = T \cup \{1\}$ is then a multiplicative subset of S. As the elements of T_0 are homogeneous, the ring $T_0^{-1}S$ is graded in a natrual way, and we denote by $S_{(T)}$ the subring of $T_0^{-1}S$ formed by elements of degree 0. We know that $T_0^{-1}S$ is identified with the inductive limit of the rings S_f, where $f \in T$ (with the canonical homomorphisms $S_f \to S_{fg}$). As this identification preserves the degrees, it identified $S_{(T)}$ as the inductive limit of $S_{(f)}$, where $f \in T$. For any graded S-module M, we define similarly the module $M_{(T)}$ (over the ring $S_{(T)}$) formed by degree zero elements of $T_0^{-1}M$, and we conclude that $M_{(T)}$ is the inductive limit of $M_{(f)}$ for $f \in T$.

If $\mathfrak{p}$ is a graded prime ideal of S_+, we denote by $S_{(\mathfrak{p})}$ and $M_{(\mathfrak{p})}$ the ring $S_{(T)}$ and the module $M_{(T)}$ respectively, where T is the homogeneous elements of S_+ not contained in $\mathfrak{p}$.

9.2.2 The homogeneous specturm of a graded ring

Given a graded ring S with positive degrees, we denote by $\mathrm{Proj}(S)$ the **homogeneous specturm** of S, which is the set of graded prime ideals of S_+, or, equivalently, the set of graded prime ideals of S not containing S_+. We will define a scheme structure on $\mathrm{Proj}(S)$, just as we have done for $\mathrm{Spec}(A)$ for a ring A.

For a subset E of S, let $V_+(E)$ be the set of graded prime ideals of S containing E and not containing S_+, which is also the subset $V(E) \cap \mathrm{Proj}(S)$ of $\mathrm{Spec}(S)$. We have immediately the following equalities:

$$V_+(0) = \mathrm{Proj}(S), \quad V_+(S) = V_+(S_+) = \varnothing,$$
$$V_+\Big(\bigcup_\lambda E_\lambda\Big) = \bigcap_\lambda V_+(E_\lambda),$$
$$V_+(EF) = V_+(E) \cup V_+(F).$$

Again, the set $V_+(E)$ remain unchanged if we replace E by the graded ideal it generates; moreover, if $\mathfrak{I}$ is a graded ideal of S, we have

$$V_+(\mathfrak{I}) = V_+\Big(\bigcup_{i \ge n} (\mathfrak{I} \cap S_i)\Big) \tag{9.2.2}$$

for any $n > 0$: in fact, if $\mathfrak{p} \in \mathrm{Proj}(S)$ contains the homogeneous elements of $\mathfrak{I}$ with degrees $\ge n$, as by hypothesis there exists a homogeneous element $f \in S_d$ not contained in $\mathfrak{p}$, for any $m \ge 0$ and any $x \in S_m \cap \mathfrak{I}$, we have $f^r x \in \mathfrak{I} \cap S_{m+rd}$ for r sufficiently large, hence $f^r x \in \mathfrak{p} \cap S_{m+rd}$, which implies $x \in \mathfrak{p} \cap S_m$. Finally, for any graded ideal $\mathfrak{I}$ of S, we have

$$V_+(\mathfrak{I}) = V_+(\mathfrak{r}_+(\mathfrak{I}))$$

where $\mathfrak{r}_+(\mathfrak{I})$ is the radical of $\mathfrak{I}$ in S_+.

By definition, $V_+(E)$ is a closed subset of $X = \mathrm{Proj}(S)$ for the topology induced by $\mathrm{Spec}(S)$. For each element $f \in S$, we set

$$D_+(f) = D(f) \cap \mathrm{Proj}(S) = \mathrm{Proj}(S) \setminus V_+(f).$$

Then for two elements $f, g \in S$, $D_+(fg) = D_+(f) \cap D_+(g)$, and the subsets $D_+(f)$, with $f \in S_+$, form a basis for the topology of $X = \mathrm{Proj}(S)$.

Let f be a homogeneous element of S_+ with degree $d > 0$. For any prime ideal $\mathfrak{p}$ of S not containing f, we see the set of x/f^n, where $x \in \mathfrak{p}$ and $n \ge 0$, is a prime ideal of the fraction ring S_f. Its trace on $S_{(f)}$ is then a prime ideal of this ring, which we denote by $\psi_f(\mathfrak{p})$: this is the set of elements x/f^n, for $n \ge 0$, $x \in \mathfrak{p}_{nd}$. We have therefore defined a map

$$\psi_f : D_+(f) \to \mathrm{Spec}(S_{(f)});$$

moreover, if g is another homogeneous element of S_+ with degree $e > 0$, we have a commutative diagram

$$\begin{tikzcd}
D_+(fg) \arrow[r, "\psi_{fg}"] \arrow[d] & \mathrm{Spec}(S_{(fg)}) \arrow[d] \\
D_+(f) \arrow[r, "\psi_f"] & \mathrm{Spec}(S_{(f)})
\end{tikzcd} \tag{9.2.3}$$

where the left vertical maps are inclusions, and the right one is the map ${}^a\omega_{fg,f}$ induced from the canonical homomorphism $\omega_{fg,f} : S_{(f)} \to S_{(fg)}$. In fact, if $x/f^n \in \omega^{-1}(\psi_{fg}(\mathfrak{p}))$, where $fg \notin \mathfrak{p}$, we then have $g^n x/(fg)^n \in \psi_{fg}(\mathfrak{p})$, so $g^n x \in \mathfrak{p}$ and therefore $x \in \mathfrak{p}$, and the converse inclusion is evident.

Proposition 9.2.5. *The map ψ_f is a homeomorphism from $D_+(f)$ to $\mathrm{Spec}(S_{(f)})$.*

Proof. For $h \in S_{nd}$ is such that $h/f^n \in \psi_f(\mathfrak{p})$, by definition it is necessary and sufficient that $h \in \mathfrak{p}$, so $\psi^{-1}(D(h/f^n)) = D_+(fh) = D_+(h) \cap D_+(f)$ and the map ψ_f is therefore continuous. Moreover, the sets $D_+(hf)$, where h runs through the set S_{nd}, form a basis of the topology of $D_+(f)$, so the preceding argument proves, in view of the T_0-axiom for $D_+(f)$ and $\mathrm{Spec}(S_{(f)})$, that ψ_f is injective and the inverse map $\psi_f(D_+(f)) \to D_+(f)$ is continuous. Finally, to show that ψ_f is surjective, we remark that, if $\mathfrak{q}_0$ is a prime ideal of $S_{(f)}$ and if, for any $n > 0$, we denote by $\mathfrak{p}_n$ the set of elements $x \in_S n$ such that $x^d/f^n \in \mathfrak{q}_n$, the $\mathfrak{p}_n$ then verify the conditions **??**: if $x \in S_n$, $y \in S_n$ are such that $x^d/f^n \in \mathfrak{q}_0$ and $y^d/f^n \in \mathfrak{q}_0$, we have $(x+y)^{2d}/f^{2n} \in \mathfrak{q}_0$, whence $(x+y)^d/f^n \in \mathfrak{q}_0$ since $\mathfrak{q}_0$ is prime; this proves that $\mathfrak{p}_n$ is a subgroup of S_n, and the verification of other conditions of **??** is immediate. If $\mathfrak{p}$ is the corresponding graded ideal of S, then $\psi_f(\mathfrak{p}) = \mathfrak{q}_0$, since if $x \in S_{nd}$, the relations $x/f^n \in \mathfrak{q}_0$ and $x^d/f^{nd} \in \mathfrak{q}_0$ are equivalent. □

Corollary 9.2.6. *For $D_+(f) \neq \varnothing$, it is necessary and sufficient that f is nilpotent.*

Proof. For $\mathrm{Spec}(S_{(f)}) = \varnothing$, it is necessary and sufficient that $S_{(f)} = 0$, which means $1 = 0$ in S_f, and this is equivalent to that f is nilpotent. □

Corollary 9.2.7. *Let E be a subset of S_+. The following conditions are equivalent:*

(i) $V_+(E) = X = \mathrm{Proj}(S)$.

(ii) *Every element of E is nilpotent.*

(iii) *The homogeneous components of every element of E are nilpotent.*

Proof. It is clear that (iii)⇒(ii)⇒(i). If $\mathfrak{I}$ is the graded ideal of S generated by E, conditions (i) is equivalent to that $V_+(\mathfrak{I}) = X$, and a fortiori, (i) implies that any homogeneous element $f \in \mathfrak{I}$ is such that $V_+(f) = X$, so f is nilpotent by Corollary 9.2.6. □

Corollary 9.2.8. *If $\mathfrak{I}$ is a graded ideal of S_+, $\mathfrak{r}_+(\mathfrak{I})$ is the intersection of graded prime ideals in $V_+(\mathfrak{I})$.*

Proof. By considering the ring $S/\mathfrak{I}$, we may assume that $\mathfrak{I} = 0$. It then suffices to prove that if $f \in S_+$ is not nilpotent, then there exists a graded prime ideal of S not containing f. Now, since there exists at least homogeneous component of f that is not nilpotent, we may assume that f is homogeneous, the result then follows from Corollary 9.2.6. □

For any subset Y of $X = \mathrm{Proj}(S)$, we denote by $I_+(Y)$ the subset of $f \in S_+$ such that $Y \subseteq V_+(f)$, which is in other words $I(Y) \cap S_+$; the set $I_+(Y)$ is then a radical ideal of S_+.

Proposition 9.2.9. *Let E be a subset of S and Y be a subset of X.*

(a) *The ideal $I_+(V_+(E))$ is the radical in S_+ of the graded ideal of S_+ generated by E.*

(b) *The set $V_+(I_+(Y))$ is the closure of Y in X.*

Proof. If $\mathfrak{I}$ is the graded ideal of S_+ generated by E, we have $V_+(E) = V_+(\mathfrak{I})$ and the first assertion follows from Corollary 9.2.8. As for (b), since $V_+(\mathfrak{I}) = \bigcap_{f \in \mathfrak{I}} V_+(f)$, the relation $Y \subseteq V_+(\mathfrak{I})$ implies $Y \subseteq V_+(f)$ for any $f \in \mathfrak{I}$, and therefore $I_+(Y) \supseteq \mathfrak{I}$, so $V_+(I_+(Y)) \subseteq V_+(\mathfrak{I})$, which implies (b) by the definition of closure. □

Corollary 9.2.10. *The closed subsets Y of $X = \mathrm{Proj}(S)$ and the graded radical ideals of S_+ correspond bijectively via $Y \mapsto I_+(Y)$ and $\mathfrak{I} \mapsto V_+(\mathfrak{I})$. Also, the union $Y_1 \cup Y_2$ of two closed subsets of X corresponds to $I_+(Y_1) \cap I_+(Y_2)$, and the intersection of a family (Y_λ) of closed subsets corresponds to the radical of the sum of $I_+(Y_\lambda)$.*

Corollary 9.2.11. *Let (f_α) be a family of homogeneous elements of S_+ and f be an element of S_+. The following conditions are equivalent:*

(i) $D_+(f) \subseteq \bigcup_\alpha D_+(f_\alpha)$;

(ii) $V_+(f) \supseteq \bigcap_\alpha V_+(f_\alpha)$;

(ii) *a power of f is contained in the ideal generated by the f_α.*

In particular, if $\mathfrak{I}$ is a graded ideal of S_+, then $V_+(\mathfrak{I}) = \varnothing$ if and only if $\mathfrak{r}_+(\mathfrak{I}) = S_+$.

Corollary 9.2.12. *For $X = \mathrm{Proj}(S)$ to be empty, it is necessary and sufficient that every element of S_+ is nilpotent.*

Corollary 9.2.13. *The closed irreducible subset of $X = \mathrm{Proj}(S)$ correspond to graded prime ideals of S_+.*

Proof. In fact, if $Y = Y_1 \cup Y_2$, where Y_1, Y_2 are closed and distinct in Y, then

$$I_+(Y) = I_+(Y_1) \cap I_+(Y_2)$$

the ideals $I_+(Y_1)$ and $I_+(Y_2)$ are distinct from $I_+(Y)$, so $I_+(Y)$ can not be prime. Conversely, if $\mathfrak{I}$ is a graded non-prime ideal of S_+, there exist elements f, g of S_+ such that $fg \in \mathfrak{I}$ but $f, g \notin \mathfrak{I}$. Then $V_+(f) \nsubseteq V_+(\mathfrak{I})$, $V_+(g) \nsubseteq V_+(\mathfrak{I})$, but $V_+(\mathfrak{I}) \subseteq V_+(f) \cup V_+(g)$. We then conclude that $V_+(\mathfrak{I})$ is the union of the closed subsets $V_+(f) \cap V_+(\mathfrak{I})$ and $V_+(g) \cap V_+(\mathfrak{I})$, both are distinct from $V_+(\mathfrak{I})$. □

We now define the scheme structure on the homogeneous specturm $\mathrm{Proj}(S)$. Let f, g be two homogeneous elements of S_+ and consider the affine schemes $Y_f = \mathrm{Spec}(S_{(f)})$, $Y_g = \mathrm{Spec}(S_{(g)})$, and $Y_{fg} = \mathrm{Spec}(S_{(fg)})$. In view of Lemma 9.2.1, the morphism $w_{fg,f} : Y_{fg} \to Y_f$ corresponding to the canonical homomorphism $\omega_{fg,f} : S_{(f)} \to S_{(fg)}$, is an open immersion. By the homeomorphism $\psi_f : D_+(f) \to Y_f$ (Proposition 9.2.5), we can transport to $D_+(f)$ the affine scheme structure of Y_f; in view of the commutative diagram (9.2.3), the affine scheme $D_+(fg)$ is identified with the subscheme induced over the open subset $D_+(fg)$ by the affine scheme $D_+(f)$. It is then clear that $X = \mathrm{Proj}(S)$ is endowed with a unique scheme structure such that each $D_+(f)$ is an affine open subscheme of X. When we speak of the homogeneous spectrum $\mathrm{Proj}(S)$ as a scheme, it will always be the structure defined in this way.

Proposition 9.2.14. *The scheme* $\mathrm{Proj}(S)$ *is separated.*

Proof. By Proposition 8.5.31, it suffices to show that for any homogeneous elements f, g of S_+, $D_+(f) \cap D_+(g) = D_+(fg)$ is affine and the canonical images of the rings of $D_+(f)$ and $D_+(g)$ in $D_+(fg)$ generate the ring of $D_+(fg)$. The first one is clear by definition, and the second one follows from Lemma 9.2.2, □

Example 9.2.15. Let $S = K[T_1, T_2]$ where K is a field and T_1, T_2 are indeterminates. Then it follows from Corollary 9.2.11 that $\mathrm{Proj}(S)$ is the union of $D_+(T_1)$ and $D_+(T_2)$. We see that each of these affine subscheme is isomorphic to $K[T]$, and that $\mathrm{Proj}(S)$ is obtained by glueing these two schemes as described in Example 8.2.10.

Proposition 9.2.16. *Let S be a graded ring with positive degrees and $X = \mathrm{Proj}(S)$.*

(i) *If $\mathfrak{n}_+$ is the nilradical of S_+, the scheme X_{red} is canonically isomorphic to* $\mathrm{Proj}(S/\mathfrak{n}_+)$. *In particular, if S is essentially reduced, then* $\mathrm{Proj}(S)$ *is reduced.*

(ii) *Suppose that S is essentially reduced, then for X to be integral, it is necessary and sufficient that S is essentially integral.*

Proof. Let $\bar{S} = S/\mathfrak{n}_+$, and denote by $x \mapsto \bar{x}$ the canonical homomorphism $S \to \bar{S}$, of degree 0. For any $f \in S_d$ ($d > 0$), the canonical homomorphism $S_f \to \bar{S}$ is surjective and of degree 0, hence restricts to a surjection $S_{(f)} \to \bar{S}_{(\bar{f})}$. If we suppose that $f \notin \mathfrak{n}_+$, then $\bar{S}_{(\bar{f})}$ is reduced and the kenrel of the preceding homomorphism is the nilradical of $S_{(f)}$, whence $\bar{S}_{(\bar{f})} = (S_{(f)})_{\mathrm{red}}$. This homomorphism then corresponds to a closed immersion $D_+(\bar{f}) \to D_+(f)$ which identifies $D_+(\bar{f})$ with $(D_+(f))_{\mathrm{red}}$ (Corollary 8.4.29), and in particular is a homeomorphism of affine scheme. Further, if $g \notin \mathfrak{n}_+$ is another homogeneous element of S_+, the diagram

$$\begin{tikzcd} S_{(f)} \arrow[r] \arrow[d] & \bar{S}_{(\bar{f})} \arrow[d] \\ S_{(fg)} \arrow[r] & \bar{S}_{(\overline{fg})} \end{tikzcd}$$

is commutative. As the sets $D_+(f)$ for f homogeneous in S_+ and $f \notin \mathfrak{n}_+$ form a covering for $X = \mathrm{Proj}(S)$, we conclude that the morphisms $D_+(\bar{f}) \to D_+(f)$ glue together to a closed immersion $\mathrm{Proj}(\bar{S}) \to \mathrm{Proj}(S)$ which is a homeomorphism on the underlying spaces, whence the conclusion of (i) by Corollary 8.4.29.

Suppose now that S is essentially integral, which means (0) is a graded ideal of S_+ distinct from S_+. Then X is reduced by (a) and irreducible by Corollary 9.2.13. Conversely, if S is essentially reduced and X is integral, then for any $f \neq 0$ homogeneous in S_+, we have $D_+(f) \neq \varnothing$ by Corollary 9.2.6; the hypothesis that X is irreducible implies that $D_+(f) \cap D_+(g) \neq \varnothing$ for any f, g homogeneous and nonzero in S_+, so in particular $fg \neq 0$, and we then conclude that S_+ is integral. □

Proposition 9.2.17. *Suppose that S is a graded A-algebra where A is a ring. Then over $X = \mathrm{Proj}(S)$ the structural sheaf $\mathscr{O}_X$ is an A-algebra, which means X is a scheme over $\mathrm{Spec}(A)$.*

Proof. It suffices to note that for any f homogeneous in S_+, $S_{(f)}$ is an A-algebra and the homomorphism $S_{(f)} \to S_{(fg)}$ is an A-algebra homomorphism for any f, g homogeneous in S_+. □

Proposition 9.2.18. *Let S be a graded ring with positive degrees.*

(a) *For any integer $d > 0$, there exists a canonical isomorphism $\mathrm{Proj}(S) \cong \mathrm{Proj}(S^{(d)})$.*

(b) *Let S' be the graded ring such that $S'_0 = \mathbb{Z}$ and $S'_n = S_n$ for $n > 0$, then there exists a canonical isomorphism $\mathrm{Proj}(S) \cong \mathrm{Proj}(S')$.*

Proof. We have already seen in **??** that the map $\mathfrak{p} \mapsto \mathfrak{p} \cap S^{(d)}$ is a bijection from $\mathrm{Proj}(S)$ to $\mathrm{Proj}(S^{(d)})$. As for any f homogeneous in S_+, we have $V_+(f) = V_+(f^d)$, this bijection is a homeomorphism of topological spaces. Finally, with the same notations, $S_{(f)}$ and $S_{(f^d)}$ are canonically identified by Lemma 9.2.1, so $\mathrm{Proj}(S)$ and $\mathrm{Proj}(S^{(d)})$ are canonically identified as schemes.

If to any $\mathfrak{p} \in \mathrm{Proj}(S)$, we correspond the unique prime ideal $\mathfrak{p}' \in \mathrm{Proj}(S')$ such that $\mathfrak{p}' \cap S_n = \mathfrak{p} \cap S_n$ for $n > 0$, then it is clear that this defines a homeomorphism $\mathrm{Proj}(S) \cong \mathrm{Proj}(S')$ of the underlying spaces, since $V_+(f)$ is the same set for S and S' if f is a homogeneous element of S_+. We also note that $S_{(f)} = S'_{(f)}$: to see this it suffices to note that if $x/1$ is an element of $S_{(f)}$ with $x \in S_0$, then $x/1 = xf/f \in S'_{(f)}$; we then conclude that $\mathrm{Proj}(S)$ and $\mathrm{Proj}(S')$ are identified as schemes. □

Corollary 9.2.19. *Let S be a graded A-algebra and S_A be the graded A-algebra such that $(S_A)_0 = A$ and $(S_A)_n = S_n$ for $n > 0$, then there exists a canonical isomorphism $\mathrm{Proj}(S) \cong \mathrm{Proj}(S_A)$.*

Proof. In fact, these two schemes are isomorphic to $\mathrm{Proj}(S')$, where $S'_0 = \mathbb{Z}$ and $S'_n = S_n$ for $n > 0$, in view of Proposition 9.2.18. □

9.2.3 Sheaf associated with a graded module

Let M be a graded S-module. For any homogeneous element f of S_+, $M_{(f)}$ is an $S_{(f)}$-module, and it therefore corresponds to a quasi-coherent sheaf $\widetilde{M_{(f)}}$ over the affine $\mathrm{Spec}(S_{(f)})$, identified with $D_+(f)$.

Proposition 9.2.20. *There existss a unique quasi-coherent $\mathscr{O}_X$-module $\widetilde{M}$ such that for any homogeneous element $f \in S_+$, we have $\Gamma(D_+(f), \widetilde{M}) = M_{(f)}$, and the restriction homomorphism $\Gamma(D_+(f), \widetilde{M}) \to \Gamma(D_+(fg), \widetilde{M})$ for f, g homogeneous in S_+ corresponds to the canonical homomorphism $M_{(f)} \to M_{(fg)}$.*

Proof. Suppose that $f \in S_d$, $g \in S_e$. As $D_+(fg)$ is identified with the prime specturm $(S_{(f)})_{g^d/f^e}$ by Lemma 9.2.1, the restriction of $\widetilde{M_{(f)}}$ to $D_+(fg)$ is canonically identified with the sheaf associated with the module $(M_{(f)})_{(g^d/f^e)}$, hence to $\widetilde{M_{(fg)}}$ (Lemma 9.2.1). We then conclude that there is a canonical isomorphism

$$\theta_{g,f} : \widetilde{M_{(f)}}|_{D_+(fg)} \to \widetilde{M_{(g)}}|_{D_+(fg)}$$

such that, if g is another homogeneous element of S_+, we have $\theta_{f,h} = \theta_{f,g} \circ \theta_{g,h}$ over $D_+(fgh)$. By glueing, there then exists a quasi-coherent sheaf $\mathscr{F}$ over X such that for any homogeneous element $f \in S_+$, we have an isomorphism $\eta_f : \mathscr{F}|_{D_+(f)} \cong \widetilde{M_{(f)}}$ and $\theta_{g,f} = \eta_g \circ \eta_f^{-1}$. Since over $D_+(f)$ we have $\Gamma(D_+(f), \widetilde{M})$, $\mathscr{F}$ can be identified with the sheaf extended from the presheaf $D_+(f) \mapsto M_{(f)}$ over the basis of standard open sets of X, whence the assertions of the proposition. In particular, we have $\widetilde{S} = \mathscr{O}_X$. □

We say the quasi-coherent $\mathscr{O}_X$-module $\widetilde{M}$ is **associted** with the graded S-module M. Recall that the graded S-modules form a category whose morphisms are graded homomorphisms of degrees. With this convention:

Proposition 9.2.21. *The functor $\widetilde{M}$ is a covariant exact functor from the category of graded S-modules to the category of quasi-coherent $\mathscr{O}_X$-modules, which commutes with inductive limits and direct sums.*

Proof. Since the properties are local, it suffices to verify over the sheaf $\widetilde{M}|_{D_+(f)} = \widetilde{M_{(f)}}$. Now the functor M_f on M, the functor N_0 on N, and the functor $\widetilde{P}$ on P all satisfy the stated properties, whence the claim. □

We denote by $\tilde{u} : \widetilde{M} \to \widetilde{N}$ the homomorphism corresponding to a graded homomorphism $u : M \to N$ of degree 0. We also deduce from Proposition 9.2.21 that the results of Corollary 8.1.6 and Corollary 8.1.8 are also true for graded S-modules and homomorphism of degree 0, via the same demonstration.

Proposition 9.2.22. *For any $\mathfrak{p} \in X = \mathrm{Proj}(S)$, we have $\widetilde{M}_{\mathfrak{p}} = M_{(\mathfrak{p})}$.*

Proof. By definition we have $\widetilde{M}_{\mathfrak{p}} = \varinjlim \Gamma(D_+(f), \widetilde{M})$, where f runs through homogeneous elements $f \in S_+$ such that $f \notin \mathfrak{p}$. As $\Gamma(D_+(f), \widetilde{M}) = M_{(f)}$, the proposition follows from the definition of $M_{(\mathfrak{p})}$. □

In particular, the local ring $\mathscr{O}_{X,\mathfrak{p}}$ is just the ring $S_{(\mathfrak{p})}$, the set of elements x/f where f is homogeneous in S_+ and not contained in $\mathfrak{p}$, and x is homogeneous with the same degree as f. If moreover S is essentially integral, then $\mathrm{Proj}(S) = X$ is integral (Proposition 9.2.16), and if $\xi = (0)$ is the generic point of X, the rational function field $K(X) = \mathscr{O}_{X,\xi}$, is the field formed by f/g where f, g are homogeneous elements of S_+ and $g \neq 0$.

Proposition 9.2.23. *If, for any $z \in M$ and any homogeneous element $f \in S_+$, there exists a power of f annihilating z, then $\widetilde{M} = 0$. This condition is also necessary if $S = S_0[S_1]$.*

Proof. The condition $\widetilde{M} = 0$ is equivalent to $M_{(f)} = 0$ for any homogeneous element of S_+. Now if $f \in S_d$, the condition $M_{(f)} = 0$ signifies that for any $z \in M$ homogeneous whose degree is a multiple of d, there exists power f^n such that $f^n z = 0$; this implies the first claim. Conversely, if moreover S is generated by S_1, then condition then implies that $f^n z = 0$ for any $z \in M$ and any $f \in S_+$, since any element $f \in S_+$ is a finite linear combination of elements of S_1. □

Proposition 9.2.24. *Let $f \in S_d$ with $d > 0$. Then for any $n \in \mathbb{Z}$, the $(\mathscr{O}_X|_{D_+(f)})$-module $\widetilde{S(nd)}|_{D_+(f)}$ is canonically isomorphic to $\mathscr{O}_X|_{D_+(f)}$.*

Proof. The multiplication by the invertible element $(f/1)^n$ of S_f defines a bijection from $S_{(f)} = (S_f)_0$ to the ring
$$(S_f)_{nd} = (S_f(nd))_0 = (S(nd)_f)_0 = S(nd)_{(f)},$$
whence the assertion. □

Corollary 9.2.25. *Over the open subset $U = \bigcup_{f \in S_d} D_+(f)$, the restriction of the $\mathscr{O}_X$-module $\widetilde{S(nd)}$ is an invertible $(\mathscr{O}_X|_U)$-module.*

Corollary 9.2.26. *If the ideal S_+ of S is generated by S_1, then the $\mathscr{O}_X$-module $\widetilde{S(n)}$ is invertible for any $n \in \mathbb{Z}$.*

Proof. It suffices to note that under the hypothesis we have $X = \bigcup_{f \in S_1} D_+(f)$ by Corollary 9.2.11. □

The quasi-coherent $\mathscr{O}_X$-modules $\widetilde{S(n)}$ is of particular interest in the theory of projective schemes, so for each $n \in \mathbb{Z}$, we put $\mathscr{O}_X(n) = \widetilde{S(n)}$ and for an open subset U of X and any $(\mathscr{O}_X|_U)$-module $\mathscr{F}$, set
$$\mathscr{F}(n) = \mathscr{F} \otimes_{\mathscr{O}_X|_U} (\mathscr{O}_X(n)|_U).$$
If the ideal S_+ is generated by S_1, the functor $\mathscr{F}(n)$ is exact on $\mathscr{F}$ for any $n \in \mathbb{Z}$, since $\mathscr{O}_X(n)$ is then an invertible $\mathscr{O}_X$-module.

Example 9.2.27. Let k be a field and consider the graded algebra $S = k[x_0, \dots, x_n]$; let $X = \mathrm{Proj}(S)$. Let d be an inter and consider the twist sheaf $\mathscr{O}_X(d)$. We compute the global sections for $\mathscr{O}_X(d)$: by definition, for each $x_i \in S$, the section of $\mathscr{O}_X(d)$ over $U_i = D_+(x_i)$ is given by
$$\Gamma(U_i, \mathscr{O}_X(d)) = S(d)_{(x_i)} = (S_{(x_i)})_d = \{f/x_i^n : f \in S_{n+d}\} = \{x_i^d f : f \in S^{(i)}\},$$
where $S^{(i)} = k[x_0/x_i, \dots, x_n/x_i]$. Therefore a section of $\mathscr{O}_X(d)$ is a family of rational polynomials (f_i) with $f_i \in S^{(i)}$ such that $x_i^d f_i = x_j^d f_j$ for each $i \neq j$; let f be this common rational polynomial. Then by

construction, we have $f/x_i^d \in S^{(i)}$ for each i, which implies that f is a polynomial in S of degree d when $d \geq 0$. If on the other hand $d < 0$, then f can only have poles at x_i for each i, which is impossible, so there is no global sections for $\mathscr{O}_X(d)$ when $d < 0$.

Let M, N be graded S-modules. For any $f \in S_d$ we define a canonical homomorphism of $S_{(f)}$-modules

$$\lambda_f : M_{(f)} \otimes_{S_{(f)}} N_{(f)} \to (M \otimes_S N)_{(f)}$$

by composing the homomorphism $M_{(f)} \otimes_{S_{(f)}} N_{(f)} \to M_f \otimes_{S_f} N_f$ (induced from the canonical injections) with the canonical isomorphism $M_f \otimes_{S_f} N_f \cong (M \otimes_S N)_{(f)}$, and note that by the definition of the grading of tensor products, these isomorphisms preserve degrees. Unwinding the definitions, for $x \in M_{md}$, $y \in N_{nd}$, we have

$$\lambda_f((x/f^m) \otimes (y/f^n)) = (x \otimes y)/f^{m+n}.$$

It then follows that, if $g \in S_e$ is another homogeneous element, the diagram

$$\begin{tikzcd}
M_{(f)} \otimes_{S_{(f)}} N_{(f)} \arrow[r, "\lambda_f"] \arrow[d] & (M \otimes_S N)_{(f)} \arrow[d] \\
M_{(fg)} \otimes_{S_{(fg)}} N_{(fg)} \arrow[r, "\lambda_{fg}"] & (M \otimes_S N)_{(fg)}
\end{tikzcd}$$

(wher the vertical homomorphism are canonical) is commutative. We then deduce that λ is a canonical homomorphism of $\mathscr{O}_X$-modules

$$\lambda : \widetilde{M} \otimes_{\mathscr{O}_X} \widetilde{N} \to \widetilde{M \otimes_S N}.$$

Consider in particular two graded ideals $\mathfrak{J}, \mathfrak{K}$ of S. As $\widetilde{\mathfrak{J}}$ and $\widetilde{\mathfrak{K}}$ are two quasi-coherent ideals of $\mathscr{O}_X$, we have a canonical homomorphism $\widetilde{\mathfrak{J}} \otimes_{\mathscr{O}_X} \widetilde{\mathfrak{K}} \to \mathscr{O}_X$, and the diagram

$$\begin{tikzcd}
\widetilde{\mathfrak{J}} \otimes_{\mathscr{O}_X} \widetilde{\mathfrak{K}} \arrow[rr, "\lambda"] \arrow[dr] & & \widetilde{\mathfrak{J} \otimes_S \mathfrak{K}} \arrow[dl] \\
& \mathscr{O}_X &
\end{tikzcd} \tag{9.2.4}$$

is commutative. Finally, note that if M, N, P are graded S-modules, the diagram

$$\begin{tikzcd}
\widetilde{M} \otimes_{\mathscr{O}_X} \widetilde{N} \otimes_{\mathscr{O}_X} \widetilde{P} \arrow[r, "\lambda\otimes 1"] \arrow[d, "1\otimes\lambda"'] & \widetilde{M \otimes_S N} \otimes_{\mathscr{O}_X} \widetilde{P} \arrow[d, "\lambda"] \\
\widetilde{M} \otimes_{\mathscr{O}_X} \widetilde{N \otimes_S P} \arrow[r, "\lambda"] & (M \otimes_S N \otimes_S P)^{\sim}
\end{tikzcd} \tag{9.2.5}$$

is commutative. Simialrly, we define a canonical homomorphism of $S_{(f)}$-modules

$$\mu_f : \mathrm{Hom}_S(M, N)_{(f)} \to \mathrm{Hom}_{S_{(f)}}(M_{(f)}, N_{(f)})$$

which sends an element u/f^n, where u is a homomorphism of degre nd, the homomorphism $M_{(f)} \to N_{(f)}$ which sends x/f^m ($x \in M_{md}$) to $u(x)/f^{m+n}$. For $g \in S_e$, we have the following commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_S(M, N)_{(f)} \arrow[r, "\mu_f"] \arrow[d] & \mathrm{Hom}_{S_{(f)}}(M_{(f)}, N_{(f)}) \arrow[d] \\
\mathrm{Hom}_S(M, N)_{(fg)} \arrow[r, "\mu_{fg}"] & \mathrm{Hom}_{S_{(fg)}}(M_{(fg)}, N_{(fg)})
\end{tikzcd} \tag{9.2.6}$$

We then conclude that the μ_f define a canonical homomorphism

$$\mu : (\mathrm{Hom}_S(M, N))^{\sim} \to \mathcal{H}om_{\mathscr{O}_X}(\widetilde{M}, \widetilde{N}).$$

Proposition 9.2.28. *Suppose that the ideal S_+ is generated by S_1. Then the homomorphism λ is an isomorphism; this holds for μ if the graded S-module M is of finite presentation.*

Proof. As X is the union of $D_+(f)$ for $f \in S_1$, we are reduced to prove tht λ_f and μ_f are isomorphisms for f homogeneous of degree 1. We then define a $\mathbb{Z}$-linear map $M_n \times N_n \to M_{(f)} \otimes_{S_{(f)}} N_{(f)}$ that send a pair (x, y) to the element $(x/f^m) \otimes (y/f^n)$. This then defines a $\mathbb{Z}$-linear map $M \otimes_{\mathbb{Z}} N \to M_{(f)} \otimes_{S_{(f)}} N_{(f)}$, and if $s \in S_q$, this map send $(sx) \otimes y$ to $(s/f^q)((x/f^m) \otimes (y/f^n))$ (where $x \in M_m$, $y \in N_n$), so we get a bi-homomorphism $\gamma_f : M \otimes_S N \to M_{(f)} \otimes_{S_{(f)}} N_{(f)}$ relative to the canonical homomorphism $S \to S_{(f)}$ (sending $s \in S_q$ to s/f^q). Suppose that for an element $\sum_i (x_i \otimes y_i)$ of $M \otimes_S N$ (where x_i, y_i are homogeneous elements of degrees m_i, n_i, respectively) we have $f^r \sum_i (x_i \otimes y_i) = 0$, which means $\sum_i (f^r x_i \otimes y_i) = 0$. Then we deduce from the isomorphism $M_f \otimes_{S_f} N_f \cong (M \otimes_S N)_f$ that $\sum_i (f^r x^i / f^{m_i + r}) \otimes (y_i / f^{n_i}) = 0$, which means $\gamma_i(\sum_i (x_i \otimes y_i)) = 0$. Therefore γ_f factors through $(M \otimes_S N)_f$ and give a homomorphism $\tilde{\gamma}_f : (M \otimes_S N)_f \to M_{(f)} \otimes_{S_{(f)}} N_{(f)}$; if $\tilde{\lambda}_f$ is the restriction of $\tilde{\gamma}_f$ to $(M \otimes_S N)_{(f)}$, we then verify that λ_f and $\tilde{\lambda}_f$ are inverses of each other, so the first assertion follows.

To demonstrate the second assertion, we now assume that M is of finite presentation, so is the cokernel of a homomorphism $P \to Q$ of graded S-module, P, Q being direct sums of finitely many modules of the form $S(n)$. By using the left exactness of $\mathrm{Hom}_S(-, N)$ and the exactness of $M_{(f)}$ on M, we are reduced to prove that μ_f is an isomorphism in the case $M = S(n)$. Now for any homogeneous $z \in N$, let u_z be the homomorphism from $S(n)$ to N such that $u_z(1) = z$; we then see that $\eta : z \mapsto u_z$ is an isomorphism of degree 0 from $N(-n)$ to $\mathrm{Hom}_S(S(n), N)$. It thus corresponds to an isomorphism

$$\eta_f : N(-n)_{(f)} \to \mathrm{Hom}_S(S(n), N)_{(f)}.$$

On the other hand, let $\tilde{\eta}_f$ be the isomorphism $N_{(f)} \to \mathrm{Hom}_{S_{(f)}}(S(n)_{(f)}, N_{(f)})$ which send $z' \in N_{(f)}$ to the homomorphism $v_{z'}$ such that $v_{z'}(s/f^k) = sz'/f^{n+k}$ (for $s \in S_{n+k} = S(n)_k$). We consider the composition

$$N(-n)_{(f)} \xrightarrow{\eta_f} \mathrm{Hom}_S(S(n), N)_{(f)} \xrightarrow{\mu_f} \mathrm{Hom}_{S_{(f)}}(S(n)_{(f)}, N_{(f)}) \xrightarrow{\tilde{\eta}_f^{-1}} N_{(f)}$$

is the isomorphism $z/f^h \mapsto z/f^{h-n}$ from $N(-n)_{(f)} \to N_{(f)}$, so μ_f is an isomorphism. □

If the ideal S_+ is generated by S_1, we deduce from Proposition 9.2.28 that for any graded ideal $\mathfrak{I}$ of S and any graded S-module M, we have $\widetilde{\mathfrak{I}}\widetilde{M} = \widetilde{\mathfrak{I}M}$.

Corollary 9.2.29. *Suppose that the ideal S_+ is generated by S_1. Then for integers m, n, we have canonical isomorphisms*

$$\mathscr{O}_X(m) \otimes_{\mathscr{O}_X} \mathscr{O}_X(n) = \mathscr{O}_X(m+n), \quad \mathscr{O}_X(n) = (\mathscr{O}_X(1))^{\otimes n}.$$

Proof. The first formula follows from Proposition 9.2.28 and the existence of the canonical isomorphism $S(m) \otimes_S S(n) \cong S(m+n)$, which sends the element $1 \otimes 1 \in S(m)_{-m} \otimes S(n)_{-n}$ to the element $1 \in S(m+n)_{-(m+n)}$. It then suffices to demonstrate the second formula for $n = -1$, and in view of Proposition 9.2.28, this follows from the fact that $\mathrm{Hom}_S(S(1), S)$ is canonically isomorphic to $S(-1)$. □

Corollary 9.2.30. *Suppose that the ideal S_+ is generated by S_1. For any graded S-module M and $n \in \mathbb{Z}$, we have a canonical isomorphism $\widetilde{M(n)} = \widetilde{M}(n)$.*

Proof. This follows from Proposition 9.2.28 and the canonical isomorphism $M(n) \cong M \otimes_S S(n)$ which send $z \in M(n)_h = M_{n+h}$ to $z \otimes 1 \in M_{n+h} \otimes S(n)_{-n} \subseteq (M \otimes_S S(n))_h$. □

Example 9.2.31. Let S' be the graded ring such that $S'_0 = \mathbb{Z}$ and $S'_n = S_n$ for $n > 0$. Then if $f \in S_d$ $(d > 0)$, we have $S(n)_{(f)} = S'(n)_{(f)}$ for any $n \in \mathbb{Z}$, because an element of $S'(n)_{(f)}$ is of the form x/f^k where $x \in S'_{n+kd}$ $(k > 0)$, and we can always choose k such that $n + kd \neq 0$. As $X = \mathrm{Proj}(S)$ and $X' = \mathrm{Proj}(S')$ are canonically identified, we see that for any $n \in \mathbb{Z}$, $\mathscr{O}_X(n)$ and $\mathscr{O}_{X'}(n)$ are canonically isomorphic under this identification.

On the other hand, for any $d > 0$ and $n \in \mathbb{Z}$, we have

$$S^{(d)}(n)_h = S_{(n+h)d} = S(nd)_{hd};$$

so for any $f \in S_d$ we have $S^{(d)}(n)_{(f)} = S(nd)_{(f)}$. We have seen that the schemes $X = \operatorname{Proj}(S)$ and $X^{(d)} = \operatorname{Proj}(S^{(d)})$ are canonically identified, so under this identification, $\mathscr{O}_X(nd)$ and $\mathscr{O}_{X^{(d)}}(n)$ are canonically isomorphic, for any $n \in \mathbb{Z}$.

Proposition 9.2.32. *Let $d > 0$ be an integer and $U = \bigcup_{f \in S_d} D_+(f)$. Then the restriction to U of the canonical homomorphism $\mathscr{O}_X(nd) \otimes_{\mathscr{O}_X} \mathscr{O}_X(-nd) \to \mathscr{O}_X$ is an isomorphism for each $n \in \mathbb{Z}$.*

Proof. In view of Corollary 9.2.30, we can assume that $d = 1$, and the conclusion then follows the proof of Proposition 9.2.24. □

9.2.4 Graded S-module associated with a sheaf

In this paragraph, for simplicity, we always assume that the ideal S_+ is generated by S_1, which also means that $S = S_0[S_1]$ by **??**. The $\mathscr{O}_X$-module $\mathscr{O}_X(1)$ is then inveritble by Corollary 9.2.26; we then put, for any $\mathscr{O}_X$-module $\mathscr{F}$, that

$$\Gamma_*(\mathscr{F}) = \Gamma_*(\mathscr{O}_X(1), \mathscr{F}) = \bigoplus_{n \in \mathbb{Z}} \Gamma(X, \mathscr{F}(n))$$

Recall that $\Gamma_*(\mathscr{O}_X)$ is endowed with a graded ring structure and $\Gamma_*(\mathscr{F})$ is a graded module over $\Gamma_*(\mathscr{O}_X)$. Since $\mathscr{O}_X(n)$ is locally free, $\Gamma_*(\mathscr{F})$ is a covariant left-exact functor on $\mathscr{F}$; in particular, if $\mathscr{I}$ is a sheaf of ideals of $\mathscr{O}_X$, $\Gamma_*(\mathscr{F})$ is canonically a graded ideal of $\Gamma_*(\mathscr{O}_X)$.

Suppose that M is a graded S-module. For any $f \in S_d$ with $d > 0$, $x \mapsto x/1$ is a homomorphism of abelian groups $M_0 \mapsto M_{(f)}$, and as $M_{(f)}$ is canonically identified with $\Gamma(D_+(f), \widetilde{M})$, we obtain a homomorphism $\alpha_0^f : M_0 \to \Gamma(D_+(f), \widetilde{M})$ of abelian groups. It is clear that, for any $g \in S_e$ with $e > 0$, the diagram

$$\begin{tikzcd} & \Gamma(D_+(f), \widetilde{M}) \arrow[d] \\ M_0 \arrow[ur, "\alpha_0^f"] \arrow[dr, "\alpha_0^{fg}"'] & \\ & \Gamma(D_+(fg), \widetilde{M}) \end{tikzcd}$$

is commutative, and this signifies that for any $x \in M_0$, the sections $\alpha_0^f(x)$ and $\alpha_0^{fg}(x)$ of M coincide over $D_+(fg)$, and therefore there exists a unique section $\alpha_0(x) \in \Gamma(X, \widetilde{M})$ whose restriction on $D_+(f)$ is $\alpha_0^f(x)$. We then define (under the hypothesis that S_+ is generated by S_1) a homomorphism

$$\alpha_0 : M_0 \to \Gamma(X, \widetilde{M}).$$

By applying this result on each graded S-module $M(n)$ (where $n \in \mathbb{Z}$), we then obtain homomorphisms of abelian groups

$$\alpha_n : M_n = M(n)_0 \to \Gamma(X, \widetilde{M}(n))$$

and therefore a homomorphism of graded abelian groups

$$\alpha : M \to \Gamma_*(\widetilde{M})$$

(also denoted by α_M) such that α_M coincides with α_n on each M_n.

If we consider in particular $M = S$, then it is easy to see that (by the definition of the multiplication of $\Gamma_*(\mathscr{O}_X)$) $\alpha : S \to \Gamma_*(\mathscr{O}_X)$ is a homomorphism of graded rings, therefore, for any S-module M, α is a bi-homomorphism of graded modules.

Proposition 9.2.33. *For any $f \in S_d$ with $d > 0$, $D_+(f)$ is identified with the subset of $\mathfrak{p} \in X$ such that the section $\alpha_d(f)$ of $\mathscr{O}_X(d)$ is nonzero at $\mathfrak{p}$.*

Proof. As $X = \bigcup_{g \in S_1} D_+(g)$, it suffices to prove that for any $g \in S_d$, the set $\mathfrak{p} \in D_+(g)$ where $\alpha_d(f)$ is nonzero is identified with $D_+(fg)$. Now the restriction of $\alpha_d(f)$ to $D_+(g)$ is by defnition the section corresponding to the element $f/1$ of $S(d)_{(g)}$; by the canonical isomorphism $S(d)_{(g)} \cong S_{(g)}$, this section of $\mathscr{O}_X(d)$ over $D_+(g)$ is identified with the section of $\mathscr{O}_X$ over $D_+(g)$ corresponding to the element f/g^d of $S_{(g)}$. To see that this section is zero on $\mathfrak{p} \in D_+(g)$ then signifies that $f/g^d \in \mathfrak{q}$, where $\mathfrak{q}$ is the prime ideal of $S_{(g)}$ corresponding to $\mathfrak{p}$; by definition this means $f \in \mathfrak{p}$, whence the proposition. □

Now let $\mathscr{F}$ be an $\mathscr{O}_X$-module and put $M = \Gamma_*(\mathscr{F})$. In view of the homomorphism $\alpha : S \to \Gamma_*(\mathscr{O}_X)$ of graded rings, M can also be considered as a graded S-module. For any $f \in S_d$ ($d > 0$), it follows from Proposition 9.2.33 that the restriction of the section $\alpha_d(f)$ of $\mathscr{O}_X(d)$ to $D_+(f)$ is invertible, and so is the restriction to $D_+(f)$ of $\alpha_d(f^n)$ for any $n > 0$. Let $z \in M_{nd} = \Gamma(X, \mathscr{F}(nd))$, if there existss an integer $k \geq 0$ such that the restriction to $D_+(f)$ of $f^k z$, which is the section $(\alpha_d(f^k)z)|_{D_+(f)}$ of $\mathscr{F}((n+k)d)$, is zero, then we conclude that $z|_{D_+(f)} = 0$. This shows that we can define an $S_{(f)}$-homomorphism $\beta_f : M_{(f)} \to \Gamma(D_+(f), \mathscr{F})$ which corresponds the element $z/f^n \in M_{(f)}$ the section $(z|_{D_+(f)})(\alpha_d(f^n)|_{D_+(f)})^{-1}$ of $\mathscr{F}$ over $D_+(f)$. We also verify that for $g \in S_e$ ($e > 0$), the diagram

$$\begin{tikzcd}
M_{(f)} \arrow[r, "\beta_f"] \arrow[d] & \Gamma(D_+(f), \mathscr{F}) \arrow[d] \\
M_{(fg)} \arrow[r, "\beta_{fg}"] & \Gamma(D_+(fg), \mathscr{F})
\end{tikzcd}$$

is commutative. Since $M_{(f)}$ is canonically identified with $\Gamma(D_+(f), \widetilde{M})$ and the $D_+(f)$ form a basis for the topological space X, the homomorphisms β_f glue together to a unique canonical homomorphism of $\mathscr{O}_X$-modules

$$\beta : \widetilde{\Gamma_*(\mathscr{F})} \to \mathscr{F}$$

(also denoted by $\beta_{\mathscr{F}}$) which is evidently functorial.

Proposition 9.2.34. *Let M be a graded S-module and $\mathscr{F}$ be an $\mathscr{O}_X$-module. Then the composition homomorphisms*

$$\widetilde{M} \xrightarrow{\tilde{\alpha}} (\Gamma_*(\widetilde{M}))^{\sim} \xrightarrow{\beta} \widetilde{M} \tag{9.2.7}$$

$$\Gamma_*(\mathscr{F}) \xrightarrow{\alpha} \Gamma_*(\widetilde{\Gamma_*(\mathscr{F})}) \xrightarrow{\Gamma_*(\beta)} \Gamma_*(\mathscr{F}) \tag{9.2.8}$$

are isomorphisms.

Proof. The verification of (9.2.7) is local: on an open subset $D_+(f)$, this result follows from the definition and the fact that the action of β is determined by its action of sections over $D_+(f)$ (Corollary 8.1.3). The verification of (9.2.8) can be done at each degree: if we put $M = \Gamma_*(\mathscr{F})$, we have $M_n = \Gamma(X, \mathscr{F}(n))$ and $\Gamma_*(\widetilde{M})_n = \Gamma(X, \widetilde{M}(n)) = \Gamma(X, \widetilde{M(n)})$. If $f \in S_1$ and $z \in M_n$, $\alpha_n^f(z)$ is the element $z/1$ of $M(n)_{(f)}$, and equals to $(f/1)^n(z/f^n)$; it then corresponds by β_f to the section

$$(\alpha_1(f)^n|_{D_+(f)})(z|_{D_+(f)})(\alpha_1(f)^n|_{D_+(f)})^{-1}$$

over $D_+(f)$, which is the restriction of z to $D_+(f)$. □

In general, the homomorphisms α and β are not isomorphisms (for example, a graded S-module M can be nonzero with $\widetilde{M}$ being zero). To obtain some nice results about these two homomorphisms, we need to impose further conditions on the graded ring S and the graded S-module M.

Proposition 9.2.35. *Let S be a graded ring and A be a ring.*

(a) *If S is Noetherian, then $X = \mathrm{Proj}(S)$ is a Noetherian scheme.*

(b) *If S is a graded A-algebra of finite type, then $X = \mathrm{Proj}(S)$ is a scheme of finite type over $Y = \mathrm{Spec}(A)$.*

Proof. If S is Noetherian, the ideal S_+ is generated by finitely many homogeneous elements $(f_i)_{1 \leq i \leq p}$, so the space X is the union of $D_+(f_i) = \mathrm{Spec}(S_{(f_i)})$, and since each $S_{(f_i)}$ is Noetherian by Corollary 9.2.4, we see X is Noetherian.

Now assume that S is an A-algebra of finite type, then S_0 is an A-algebra of finite type and S is an S_0-algebra of finite type, so S_+ is a finitely generated ideal by **??**. We are then reduced to prove as in (a) that for any $f \in S_d$, $S_{(f)}$ is an A-algebra of finite type. In view of Proposition 9.2.3, it suffices to show that $S^{(d)}$ is an A-algebra of finite type, which follows from **??**. □

Let M be a graded S-module. We say M is **eventually null** if there exists an integer n such that $M_i = 0$ for $i \neq n$, and is **eventually finite** if there exists an integer n such that $\bigoplus_{i\geq n} M_i$ is a finitely generated S-module, or equivalently, that there exists a finitely generated graded sub-S-module M' of M such that M/M' is eventually null. We also note that if M is eventually null, then $M_{(f)} = 0$ for any homogeneous element f in S_+, so $\widetilde{M} = 0$.

Let M, N be two graded S-modules. We say a homomorphism $u : M \to N$ of degree 0 is **eventually injective** (resp. **eventually surjective**, **eventually bijective**) if there exists an integer n such that $u_i : M_i \to N_i$ is injective (resp. surjective, bijective) for $i \geq n$. Equivalently, the homomorphism u is eventually injective (resp. eventually surjective) if and only if $\ker u$ (resp. $\operatorname{coker} u$) is eventually null. If u is eventually bijective, we say it is an **eventual isomorphism**.

Proposition 9.2.36. *Let S be a graded ring such that S_+ is finitely generated and M be a graded S-module.*

(a) *If M is eventually finite, the $\mathscr{O}_X$-module $\widetilde{M}$ is of finite type.*

(b) *Suppose that M is eventually finite, then for $\widetilde{M} = 0$, it is necessary and sufficient that M is eventually null.*

Proof. If $M_n = 0$ for $n \geq n_0$ then $M_{(f)} = 0$ for any homogeneous element $f \in S_+$, so if M is eventually null, then $\widetilde{M} = 0$. On the other hand, if M is eventually finite, then for $n \gg 0$ the graded submodule $M' = \bigoplus_{k\geq n} M_k$ is finitely generated by hypothesis, and M/M' is eventually null, so $\widetilde{(M/M')} = 0$ and therefore $\widetilde{M} = \widetilde{M}'$ by the exactness of the functor $\widetilde{M}$ (Proposition 9.2.21). Therefore, to prove that $\widetilde{M}$ is of finite type, we may assume that M is finitely generated. Now since this question is local, we only need to show that $M_{(f)}$ is finitely genrated over $S_{(f)}$ for any homogeneous $f \in S_d$ with $d > 0$. But $M^{(d)}$ is a finitely generated $S^{(d)}$ module by **??**, and the assertion follows from Proposition 9.2.3.

Suppose now that M is eventually finite and $\widetilde{M} = 0$; then we have $\widetilde{M}' = 0$, so the condition that M' is eventually null is equivalent to that of M. We may therefore assume that M is finitely generated over S by homogeneous elements x_i $(1 \leq i \leq p)$; let $(f_j)_{1\leq j\leq q}$ be a system of generators of the ideal S_+. We have by hypothesis $M_{(f_j)} = 0$ for any j, so there exists an integer n such that $f_j^n x_i = 0$ for any i, j. Let $n_j = \deg(f_j)$ and m be the supremum of $\sum_j r_j n_j$ for any finite system of integers (r_j) such that $\sum_j r_j \leq nq$. It is then clear that if $k > m$, we have $S_k x_i = 0$ for any i; if d is the supremum of the degrees of x_i, we then conclude that $M_k = 0$ for $k > d + m$, which proves our assertion. □

Corollary 9.2.37. *Let S be a graded ring such that S_+ is finitely generated. For $X = \operatorname{Proj}(S) = \varnothing$, it is necessary and sufficient that S is eventually null.*

Proof. The condition $X = \varnothing$ is in fact equivalent to $\mathscr{O}_X = \widetilde{S} = 0$, and S is clearly a finite generated S-module. □

Example 9.2.38. To give a counterexample of Proposition 9.2.36, let k be a field and $S = k[X_1, \dots, X_n]$ be the polynomial ring with n variables. Consider the graded S-module M given by

$$M = k[X_1, X_2, \dots, X_n, \dots]/(X_1, X_2^2, \dots, X_n^n, \dots)$$

with the usual multiplication of polynomials. Then M is not finitely generated over S, and for every homogeneous polynomial f of degree 1, we see $f^N M = 0$ for sufficiently large N. Therefore $\widetilde{M} = 0$. However, note that $M_n \neq 0$ for every n.

Theorem 9.2.39. *Suppose that S is a graded ring such that the ideal S_+ is finitely generated by S_1, and let $X = \operatorname{Proj}(S)$. Then, for any quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, the canonical homomorphisms $\beta : \widetilde{\Gamma_*(\mathscr{F})} \to \mathscr{F}$ is an isomorphism.*

Proof. If S_+ is generated by finitely many $f_i \in S_1$, then X is the union of $\operatorname{Spec}(S_{(f_i)})$ which are quasi-compact, so X is quasi-compact. Also, $\mathscr{O}_X(n)$ is invertible for any $n \in \mathbb{Z}$ by Corollary 9.2.29, and since X is separated, by Corollary 8.6.15 and Proposition 9.2.33, we have for any $f \in S_d$ a canonical isomorphism $\Gamma_*(\mathscr{F})_{(\alpha_d(f))} \cong \Gamma(D_+(f), \mathscr{F})$ (the first module (considered as a $\Gamma_*(\mathscr{O}_X)$-module) is none other than $\Gamma_*(\mathscr{F})_{(f)}$ (considered as an S-module)). If we trace the definition of this isomorphism, we see that it coincides with β_f, whence our assertion. □

Corollary 9.2.40. *Under the hypotheses of Theorem 9.2.39, any quasi-coherent $\mathscr{O}_X$-module (of finite type) is isomorphic to an $\mathscr{O}_X$-module of the form $\widetilde{M}$, where M is a (finitely generated) graded S-module.*

Proof. Let $\mathscr{F}$ be a quasi-coherent $\mathscr{O}_X$-module, then $\mathscr{F} = \widetilde{M}$ for a graded S-module M by Theorem 9.2.39. Let $(f_\lambda)_{\lambda\in I}$ be a system of homogeneous generators of M; for each finite subset H of I, let M_H be the graded submodule of M generated by f_λ for $\lambda \in H$. It is clear that M is the inductive limit of the submodules M_H, so $\mathscr{F}$ is the inductive limit of the sub-$\mathscr{O}_X$-modules $\widetilde{M}_H$ (Proposition 9.2.21). If $\mathscr{F}$ is of finite type, we conclude from **??**. □

Corollary 9.2.41. *Under the hypotheses of Theorem 9.2.39, for any quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$ of finite type, there exists an integer n_0 such that for any $n \geq n_0$, $\mathscr{F}(n)$ is isomorphic to a quotient of $\mathscr{O}_X^r$ (where $r > 0$ depends on n), and therefore is generated by finitely many global sections.*

Proof. By Corollary 9.2.40, we can assume that $\mathscr{F} = \widetilde{M}$ where M is a quotient of a finite dierct sum of $S(m_i)$. By Proposition 9.2.21, we are therefore reduced to the case where $M = S(m)$, so $\mathscr{F}(n) = (S(m+n))^\sim = \mathscr{O}_X(m+n)$. It then suffices to prove that for each $n \geq 0$ there exists r and a surjective homomorphism $\mathscr{O}_X^r \to \mathscr{O}_X(n)$. For this, it suffcies to prove that, for a suitable r, there exists an eventually surjective homomorphism $u : S^r \to S(n)$ of degree zero. Now we have $S(n)_0 = S_n$, and by hypothesis $S_h = S_1^h$ for any $h > 0$, so $SS_n = \bigoplus_{h\geq n} S_h$. As S_n is a finitely generated S_0-module (**??**), consider a system $(a_i)_{1\leq i\leq r}$ of generators of this module, and let $u : S^r \to S(n)$ be the homomorphism that sends the i-th basis e_i of S^r to a_i. Then the image of u contains $\bigoplus_{h\geq 0} S(n)_h$, so u satisfies the requirement and the proof is complete. □

Corollary 9.2.42. *Under the hypotheses of Theorem 9.2.39, for any quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$ of finite type, there exists an integer n_0 such that for any $n \geq n_0$, $\mathscr{F}$ is isomorphic to a quotient of an $\mathscr{O}_X$-module of the form $\mathscr{O}_X(-n)^r$ (where $r > 0$ depends on n).*

Proof. This follows from Corollary 9.2.41 by tensoring with the invertible sheaf $\mathscr{O}_X(-n)$, which preserves the exactness. □

Proposition 9.2.43. *Assume the hypotheses of Theorem 9.2.39 and let M be a graded S-module.*

(a) *The cannical homomorphism $\tilde{\alpha} : \widetilde{M} \to (\Gamma_*(\widetilde{M}))^\sim$ is an isomorphism.*

(b) *Let $\mathscr{G}$ be a quasi-coherent sub-$\mathscr{O}_X$-module of $\widetilde{M}$ and let N be the graded sub-S-module of M which is the inverse image of $\Gamma_*(\mathscr{G})$ under α. Then we have $\widetilde{N} = \mathscr{G}$.*

Proof. As $\beta : (\Gamma_*(\widetilde{M}))^\sim \to \widetilde{M}$ is an isomorphism, $\tilde{\alpha}$ is its inverse isomorphism in view of (9.2.7), whence (a). Let P be the graded submodule $\alpha(M)$ of $\Gamma_*(\widetilde{M})$; as $\widetilde{M}$ is an exact functor, the image of $\widetilde{M}$ under $\tilde{\alpha}$ is equal to $\widetilde{P}$, so in view of (a), $\widetilde{P} = (\Gamma_*(\widetilde{M}))^\sim$. Put $Q = \Gamma_*(\mathscr{G}) \cap P$, so that $N = \alpha^{-1}(Q)$. Then by the preceding argument and Proposition 9.2.21, the image of $\widetilde{N}$ under $\tilde{\alpha}$ is $\widetilde{Q}$, and we have $\widetilde{Q} = \widetilde{\Gamma_*(\mathscr{G})}$. Since the image of $\widetilde{\Gamma_*(\mathscr{G})}$ under β is $\mathscr{G}$ and $\tilde{\alpha}$ is the inverse of β, we conclude that $\widetilde{N} = \mathscr{G}$. □

9.2.5 Functorial properties of Proj(S)

Let S, S' be two graded rings with positive degree and $\varphi : S' \to S$ be a homomorphisms of graded rings. We denote by $G(\varphi)$ the open subset of $X = \mathrm{Proj}(S)$ which is the complement of $V_+(\varphi(S'_+))$, or, the union of $D_+(\varphi(f'))$ where f' runs through homogeneous elements of S'_+. The restriction to $G(\varphi)$ of the continuous map ${}^a\varphi : \mathrm{Spec}(S') \to \mathrm{Spec}(S)$ is then a continuous map from $G(\varphi)$ to $\mathrm{Proj}(S')$, which is still denoted by ${}^a\varphi$. If $f' \in S'_+$ is homogeneous, we have

$$
{}^a\varphi^{-1}(D_+(f')) = D_+(\varphi(f')) \tag{9.2.9}
$$

since ${}^a\varphi$ maps $G(\varphi)$ into $\mathrm{Proj}(S')$. On the other hand, the homomorphism φ defines canonically a homomorphism of graded rings $S'_{f'} \to S_f$ of degree 0 (where $f = \varphi(f')$), whence a homomorphism $S'_{(f')} \to S_{(f)}$, which we denote by $\varphi_{(f)}$. It then corresponds to a morphism $({}^a\varphi_{(f)}, \tilde{\varphi}_{(f)}) : \mathrm{Spec}(S_{(f)}) \to \mathrm{Spec}(S'_{(f')})$ of affine schemes. If we identify $\mathrm{Spec}(S_{(f)})$ with the open subscheme $D_+(f)$ of $\mathrm{Proj}(S)$, we then obtain a morphism $\Phi_f : D_+(f) \to D_+(f')$ and ${}^a\varphi_{(f)}$ is identified with the restriction of ${}^a\varphi$ to $D_+(f)$. If g' is another homogeneous element of S'_+ and $g = \varphi(g')$, it is immediate that the diagram

$$\begin{tikzcd} D_+(fg) \arrow[r, "\Phi_{fg}"] \arrow[d] & D_+(f'g') \arrow[d] \\ D_+(f) \arrow[r, "\Phi_f"] & D_+(f') \end{tikzcd}$$

is commutative.

Proposition 9.2.44. *Let $\varphi : S' \to S$ be a homomorphism of graded rings. There exists a unique morphism $({}^a\varphi, \tilde{\varphi}) : G(\varphi) \to \operatorname{Proj}(S')$ (called the morphism associated with φ and denoted by $\operatorname{Proj}(\varphi)$), such that for any homogeneous element $f' \in S'_+$, the restriction of this morphism to $D_+(\varphi(f'))$ coincides with the morphism associated with the homomorphism $\varphi_{(f)} : S'_{(f')} \to S_{(\varphi(f'))}$.*

Proof. The morphism $({}^a\varphi, \tilde{\varphi})$ is obtained form glueing the morphisms Φ_f over $D_+(f)$, and the claim property is immediate. □

Corollary 9.2.45. *Let $\varphi : S' \to S$ be a homomorphism of graded rings.*

(a) *The morphism* $\operatorname{Proj}(\varphi)$ *is affine.*

(b) *If* $\ker\varphi$ *is nilpotent (and in particular if φ is injective), the morphism* $\operatorname{Proj}(\varphi)$ *is dominant.*

(c) *If φ is eventually surjective, then* $G(\varphi) = \operatorname{Proj}(S)$.

Proof. The first assertion follows from Corollary 9.2.45 and the relation ${}^a\varphi^{-1}(D_+(f')) = D_+(\varphi(f'))$. On the other hand, if $\ker\varphi$ is nilpotent, for any f' homogeneous in S'_+, we verify that $\ker\varphi_f$ is also nilpotent, and so is $\ker\varphi_{(f)}$. The conclusion then follows from **??**. Finally, if φ is eventually surjective, then every homogeneous element $f \in S_+$ has some power contained in the image of φ, so by Corollary 9.2.11 we conclude that $G(\varphi) = \bigcup_{f' \in S'_+} D_+(\varphi(f')) = \operatorname{Proj}(S)$, whence the claim. □

Remark 9.2.46. Note that there are in general morphisms from $\operatorname{Proj}(S)$ to $\operatorname{Proj}(S')$ which are not affine, and therefore do not come from graded ring homomorphisms $S' \to S$; an example is the structural morphism $\operatorname{Proj}(S) \to \operatorname{Spec}(A)$, where A is field ($\operatorname{Spec}(A)$ being identified with $\operatorname{Proj}(A[T])$ (cf. Corollary 9.3.5)).

Let $\varphi' : S'' \to S'$ be another homomorphism of graded rings, and put $\varphi'' = \varphi \circ \varphi'$. Then by the formula ${}^a\varphi'' = {}^a\varphi' \circ {}^a\varphi$ and $G(\varphi'') \subseteq G(\varphi)$, if Φ, Φ', and Φ'' are the associated morphisms of φ, φ' and φ'', we have $\Phi'' = \Phi' \circ (\Phi|_{G(\varphi'')})$.

Suppose that S (resp. S') is a graded A-algebra (resp. a graded A'-algebra), and let $\psi : A' \to A$ be a homomorphism of rings such that the diagram

$$\begin{tikzcd} A' \arrow[r, "\psi"] \arrow[d] & A \arrow[d] \\ S' \arrow[r, "\varphi"] & S \end{tikzcd}$$

is commutative. We can then consider $G(\varphi)$ and $\operatorname{Proj}(S')$ as schemes over $\operatorname{Spec}(A)$ and $\operatorname{Spec}(A')$, respectively. If Φ and Ψ are the associated morphisms of φ and ψ, respectively, the diagram

$$\begin{tikzcd} G(\varphi) \arrow[r, "\Phi"] \arrow[d] & \operatorname{Proj}(S') \arrow[d] \\ \operatorname{Spec}(A) \arrow[r, "\Psi"] & \operatorname{Spec}(A') \end{tikzcd}$$

is commutative.

Now let M be a graded S-module and consider the S'-module $\varphi^*(M)$, which is clearly graded. Let f' be a homogeneous element in S'_+, and set $f = \varphi(f')$. We then have a canonical isomorphism

$(\varphi^*(M))_{f'} \cong \varphi_f^*(M_f)$, and it is clear that this isomorphism preserves degrees, so induces an isomorphism $(\varphi^*(M))_{(f')} \cong \varphi_{(f)}^*(M_{(f)})$. There is then canonically an isomorphism of sheaves $\widetilde{\varphi^*(M)}|_{D_+(f')} \cong (\Phi_f)_*(\widetilde{M}|_{D_+(f)})$ (Proposition 8.1.12). Moreover, if g' is another homogeneous element of S'_+ and $g = \varphi(g')$, the diagram

$$\begin{CD} (\varphi^*(M))_{(f')} @>{\sim}>> (M_{(f)})_{(\varphi_{(f)})} \\ @VVV @VVV \\ (\varphi^*(M))_{(f'g')} @>{\sim}>> (M_{(fg)})_{(\varphi_{(fg)})} \end{CD}$$

is commutative, whence we conclude that the isomorphism

$$\widetilde{\varphi^*(M)}|_{D_+(f'g')} \cong (\Phi_{fg})_*(\widetilde{M}|_{D_+(fg)})$$

is the restriction to $D_+(f'g')$ of the isomorphism $\widetilde{\varphi^*(M)}|_{D_+(f')} \cong (\Phi_f)_*(\widetilde{M}|_{D_+(f)})$. As Φ_f is the restriction of Φ on $D_+(f)$, we then obtain the following result:

Proposition 9.2.47. *There exists a canonical isomorphism $\widetilde{\varphi^*(M)} \cong \Phi_*(\widetilde{M}|_{G(\varphi)})$ of $\mathscr{O}_X$-modules.*

We also deduce a canonical functorial map from the set of φ-homomorphisms $M' \to M$ from a graded S'-module to a graded S-module M, to the set of Φ-morphisms $\widetilde{M}' \to \widetilde{M}|_{G(\varphi)}$. If $\varphi' : S'' \to S'$ is another ring homomorphism and M'' is a graded S''-module, the composition of a φ-morphism $M' \to M$ and a φ'-morphism $M'' \to M'$ canonically corresponds to the composition of $\widetilde{M}'|_{G(\varphi')} \to \widetilde{M}|_{G(\varphi'')}$ and $\widetilde{M}'' \to \widetilde{M}'|_{G(\varphi')}$.

Proposition 9.2.48. *Let $\varphi : S' \to S$ be a homomorphism of graded rings and M' be a graded S'-module. Then there exists a canonical homomorphism $\nu : \Phi^*(\widetilde{M}') \to \widetilde{\varphi_*(M)}|_{G(\varphi)}$. If the ideal S'_+ is generated by S'_1, then ν is an isomorphism.*

Proof. For $f' \in S'_d$ with $d > 0$, we define a canonical homomorphism of $S_{(f)}$-modules (where $f = \varphi(f')$)

$$\nu_f : M'_{(f)} \otimes_{S'_{(f)}} S_{(f)} \to (M' \otimes_{S'} S)_{(f)}$$

by composing the homomorphism $M'_{(f)} \otimes_{S'_{(f)}} S_{(f)} \to M_{f'} \otimes_{S'_{f'}} S_f$ with the canonical homomorphism $M'_{f'} \otimes_{S'_{f'}} S_f \cong (M' \otimes_{S'} S)_f$. It is immediate to verify that compatibility of ν_f with the restriction homomorphisms $D_+(f)$ to $_+(fg)$ (for $g' \in S'_+$ and $g = \varphi(g')$), so we obtain a homomorphism

$$\nu : \Phi^*(\widetilde{M}') \to \widetilde{\varphi_*(M)}|_{G(\varphi)}.$$

For the second assertion, it suffices to prove that ν_f is an isomorphism for each $f' \in S'_1$, since $G(\varphi)$ is the union of $D_+(\varphi(f'))$. We first define a $\mathbb{Z}$-bilinear map $M'_m \times S_n \to M'_{(f')} \otimes_{S'_{f'}} S_{(f)}$ which sends (x', s) to the element $(x'/f'^m) \otimes (s/f^n)$. As in the proof of Proposition 9.2.28, this map then induces a bi-homomorphism

$$\eta_f : M' \otimes_{S'} S \to M'_{(f')} \otimes_{S'_{(f')}} S_{(f)}.$$

Moreover, if, for $r > 0$, we have $f^r \sum_i (x'_i \otimes s_i) = 0$, then $\sum_i (f'^r x'_i \otimes s_i) = 0$, so $\sum_i (f'^r x_i / f'^{m_i + r}) \otimes (s_i / f^{n_i}) = 0$, which means $\eta_f(\sum_i x_i \otimes y_i) = 0$; the homomorphism then factors through $(M' \otimes_{S'} S)_f$ and gives a homomorphism $\tilde{\eta}_f : (M' \otimes_{S'} S)_f \to M'_{(f')} \otimes_{S'_{(f')}} S_{(f)}$. It is easy to verify that $\tilde{\eta}_f$ is the inverse of ν_f, whence our assertion. □

In particular, since $\varphi_*(S'(n)) = S(n)$ for each $n \in \mathbb{Z}$, it follows from Proposition 9.2.48 that we have a canonical homomorphism $\Phi^*(\mathscr{O}_{X'}(n)) \to \mathscr{O}_X(n)|G(\varphi)$, and this an isomorphism if S'_+ is generated by S'_1.

Remark 9.2.49. We note that it follows from Proposition 9.2.18 that the morphism Φ is unchanged if we replace S by $S^{(d)}$, S' by $S'^{(d)}$, and φ by $\varphi^{(d)}$. Also, it is also unchanged if we replace S_0 and S'_0 by $\mathbb{Z}$ and φ_0 be the identity map.

Let A, A' be two rings and $\psi : A' \to A$ be a homomorphism of rings, which defines a morphism $\Psi : \mathrm{Spec}(A) \to \mathrm{Spec}(A')$. Let S' be an A'-algebra with positive degrees, and put $S = S' \otimes_{A'} A$, which is a graded A-algebra by setting $S_n = S'_n \otimes_{A'} A$. The map $s' \mapsto s' \otimes 1$ is then a homomorphism of graded rings and also a bi-homomorphism. Since S_+ is the A-module generated by $\varphi(S'_+)$, we have $G(\varphi) = \mathrm{Proj}(S) = X$, so, if we put $X' = \mathrm{Proj}(S')$, we have a commutative diagram

$$\begin{tikzcd} X \arrow[r, "\Phi"] \arrow[d, "p"'] & X' \arrow[d] \\ Y \arrow[r, "\Psi"] & Y' \end{tikzcd} \tag{9.2.10}$$

Now let M' be a graded S'-module, and set $M = M' \otimes_{A'} A = M' \otimes_{S'} S$.

Proposition 9.2.50. *The commutative diagram (9.2.10) is cartesian and the canonical homomorphism $\nu : \Phi^*(\widetilde{M}') \to \widetilde{M}$ in Proposition 9.2.48 is an isomorphism.*

Proof. The first assertion follows if we can prove that for any f' homogeneous in S'_+ and $f = \varphi(f')$, the restriction of Φ and p to $D_+(f)$ identify this scheme as $D_+(f') \times_{Y'} Y$; in other words, it suffices to prove that $S_{(f)}$ is canonically identified with $S'_{(f')} \otimes_{A'} A$, which is immedaite from the fact that the canonical isomorphism $S_f \cong S'_{f'} \otimes_{A'} A$ preserves degrees. The second assertion follows from the isomorphism $M'_{(f')} \otimes_{S'_{(f')}} S_{(f)} \cong M'_{(f')} \otimes_{A'} A$, and the later one is isomorphic to $M_{(f)}$ since M_f is canonically identified with $M'_{f'} \otimes_{A'} A$. $\square$

Corollary 9.2.51. *For any integer $n \in \mathbb{Z}$, $\widetilde{M}(n)$ is identified with $\Phi^*(\widetilde{M}'(n)) = \widetilde{M}'(n) \otimes_{Y'} \mathscr{O}_Y$. In particular, $\mathscr{O}_X(n) = \Phi^*(\mathscr{O}_{X'}(n)) = \mathscr{O}_{X'}(n) \otimes_{Y'} \mathscr{O}_Y$.*

Proof. This follows from Proposition 9.2.50 and Corollary 9.2.30. $\square$

Now let $\mathscr{F}'$ be an $\mathscr{O}_{X'}$-module and $\mathscr{F} = \Phi^*(\mathscr{F}')$. Then we have for each $n \in \mathbb{Z}$ that $\mathscr{F}(n) = \Phi^*(\mathscr{F}'(n))$ in view of Corollary 9.2.51. Therefore, by the definition of Φ^*, we have a canonical homomorphism

$$\Gamma(\rho) : \Gamma(X', \mathscr{F}'(n)) \to \Gamma(X, \mathscr{F}(n))$$

which then gives a canonical bi-homomorphism $\Gamma_*(\mathscr{F}') \to \Gamma_*(\mathscr{F})$ of graded modules.

Suppose that the ideal S'_+ is generated by S'_1 and $\mathscr{F}' = \widetilde{M}'$, so $\mathscr{F} = \widetilde{M}$ where $M = M' \otimes_{A'} A$. If f' is homogeneous in S'_+ and $f = \varphi(f')$, we have $M_{(f)} = M'_{(f')} \otimes_{A'} A$ and the diagram

$$\begin{tikzcd} M'_0 \arrow[r] \arrow[d] & M'_{(f')} = \Gamma(D_+(f), \widetilde{M}') \arrow[d] \\ M_0 \arrow[r] & M_{(f)} = \Gamma(D_+(f), \widetilde{M}) \end{tikzcd}$$

is commutative. We then conclude from the definition of the homomorphism $\alpha : M \to \Gamma_*(\widetilde{M})$ that the following diagram

$$\begin{tikzcd} M' \arrow[r, "\alpha_{M'}"] \arrow[d] & \Gamma_*(\widetilde{M}') \arrow[d] \\ M \arrow[r, "\alpha_M"] & \Gamma_*(\widetilde{M}) \end{tikzcd} \tag{9.2.11}$$

is commutative. Similarly, the diagram

$$\begin{tikzcd} \widetilde{\Gamma_*(\mathscr{F}')} \arrow[r, "\beta_{\mathscr{F}'}"] \arrow[d] & \mathscr{F}' \arrow[d] \\ \widetilde{\Gamma_*(\mathscr{F})} \arrow[r, "\beta_{\mathscr{F}}"] & \mathscr{F} \end{tikzcd} \tag{9.2.12}$$

is commutative (where the vertical is the canonical Φ-morphism $\mathscr{F}' \to \Phi^*(\mathscr{F}') = \mathscr{F}$).

Now let N' be another graded S'-module and $N = N' \otimes_{A'} A$. It is immediate that the canonical bi-homomorphisms $M' \to M$, $N' \to N$ give a bi-homomorphism $M' \otimes_{S'} N' \to M \otimes_S N$, and therefore an S-homomorphism $(M' \otimes_{S'} N') \otimes_{A'} A \to M \otimes_S N$ of degree 0, which then corresponds to an $\mathscr{O}_X$-homomorphism

$$\Phi^*((M' \otimes_{S'} N')^{\sim}) \to (M \otimes_S N)^{\sim}.$$

Moreover, it is immedaite to verify that the following diagram

$$\begin{tikzcd}
\Phi^*(\widetilde{M}' \otimes_{\mathscr{O}_{X'}} \widetilde{N}') \arrow[r, "\sim"] \arrow[d, "\Phi^*(\lambda)"] & \widetilde{M} \otimes_{\mathscr{O}_X} \widetilde{N} = \Phi^*(\widetilde{M}') \otimes_{\mathscr{O}_X} \Phi^*(\widetilde{N}') \arrow[d, "\lambda"] \\
\Phi^*((M' \otimes_{S'} N')^{\sim}) \arrow[r] & (M \otimes_S N)^{\sim}
\end{tikzcd} \tag{9.2.13}$$

is commutative (where the first row is an isomorphism by (**??**)). If the ideal S'_+ is generated by S'_1, it is clear that S_+ is generated by S_1, so the two vertical homomorphisms are isomorphisms, so the second row is also an isomorphism.

We have similarly a cannical bi-homomorphism $\mathrm{Hom}_{S'}(M', N') \to \mathrm{Hom}_S(M, N)$, which sends a homomorphism u' of degree k the homomorphism $u' \otimes 1$, which is also of degree k. We then deduce an S-homomorphism of degree 0:

$$\mathrm{Hom}_{S'}(M', N') \otimes_{A'} A \to \mathrm{Hom}_S(M, N)$$

which corresponds to a homomorphism of $\mathscr{O}_X$-modules:

$$\Phi^*((\mathrm{Hom}_{S'}(M', N'))^{\sim}) \to (\mathrm{Hom}_S(M, N))^{\sim}.$$

Similarly, the diagram

$$\begin{tikzcd}
\Phi^*((\mathrm{Hom}_{S'}(M', N'))^{\sim}) \arrow[r] \arrow[d, "\Phi^*(\mu)"] & (\mathrm{Hom}_S(M, N))^{\sim} \arrow[d, "\mu"] \\
\Phi^*(\mathcal{H}om_{\mathscr{O}_{X'}}(\widetilde{M}', \widetilde{N}')) \arrow[r] & \mathcal{H}om_{\mathscr{O}_X}(\widetilde{M}, \widetilde{N})
\end{tikzcd} \tag{9.2.14}$$

is commutative (where the second row is the canonical homomorphism of (**??**)).

9.2.6 Closed subschemes of $\mathrm{Proj}(S)$

Recall that if $\varphi : S \to S'$ is a homomorphism of graded rings, we say that φ is eventually surjective (resp. eventually injective, eventually bijective) if $\varphi_i : S_i \to S'_i$ is surjective (resp. injective, bijective) for sufficiently large i. It follows from Proposition 9.2.18 that the study of Φ can be reduced to the case where φ is surjective (resp. injectij, bijective). Instead of saying that φ is eventually bijective, we also say that it is then an eventual isomorphism.

Proposition 9.2.52. *Let S, S' be graded rings with positive degrees and set $X = \mathrm{Proj}(S)$, $X' = \mathrm{Proj}(S')$.*

(a) *If $\varphi : S \to S'$ is an eventually surjective homomorphism of graded rings, the corresponding morphism Φ is defined over $\mathrm{Proj}(S')$ and is a closed immersion. If $\mathfrak{I}$ is the kernel of φ, the closed subscheme of X associated with Φ is defined by the quasi-coherent ideal $\widetilde{\mathfrak{I}}$ of $\mathscr{O}_X$.*

(b) *Suppose moreover that the ideal S_+ is finitely generated by S_1. Let X' be a closed subscheme of X defined by a quasi-coherent ideal $\mathscr{J}$ of $\mathscr{O}_X$. Let $\mathfrak{I}$ be the graded ideal of S which is the inverse image of $\Gamma_*(\mathscr{J})$ under the canonical homomorphism $\alpha : S \to \Gamma_*(\mathscr{O}_X)$, and put $S' = S/\mathfrak{I}$. Then X' is the subscheme associated with the closed immersion $\mathrm{Proj}(S') \to X$ corresponding to the canonical homomorphism $S \to S'$ of graded rings.*

Proof. Let $\varphi : S \to S'$ be an eventually surjective homomorphism of graded rings. We can suppose that φ is surjective, so $\varphi(S_+)$ is generated by S'_+, we have $G(\varphi) = \mathrm{Proj}(S')$. Now the second assertion in (a) can be verified locally over X; let f be a homogeneous element of S_+ and put $f' = \varphi(f)$. As

φ is a surjective homomorphism of rings, $\varphi_{(f')}: S_{(f)} \to S'_{(f')}$ is surjective with kernel $\mathscr{I}_{(f)}$, so the corresponding morphism is closed.

We now consider the case of (b); in view of (a), we only need to verify that the homomorphism $\tilde{j}: \widetilde{\mathscr{I}} \to \mathscr{O}_X$ induced from the injection $j: \mathscr{I} \to S$ is an isomorphism from $\widetilde{\mathscr{I}}$ to $\mathscr{I}$, which follows from Proposition 9.2.43(b). □

Remark 9.2.53. Note that $\mathfrak{I}$ is the largest graded ideal $\mathfrak{I}'$ of S such that $\widetilde{\mathscr{I}}' = \mathscr{I}$ (where we identify $\widetilde{\mathscr{I}}'$ as a subsheaf of $\mathscr{O}_X$), since one immedaitely verify that this relation implies $\alpha(\mathfrak{I}') \subseteq \Gamma_*(\mathscr{I})$.

Corollary 9.2.54. *Assume the hypotheses of Proposition 9.2.52(a) and that S_+ is generated by S_1. Then $\Phi^*(\widetilde{S(n)})$ is canonically isomorphic to $\widetilde{S'(n)}$ for any $n \in \mathbb{Z}$, and therefore $\Phi^*(\mathscr{F}(n))$ is isomorphic to $\Phi^*(\mathscr{F})(n)$ for any $\mathscr{O}_X$-module $\mathscr{F}$.*

Proof. This is a partiuclar case of Proposition 9.2.48, in view of the definition of $\mathscr{F}(n)$ and Proposition 9.2.52(a). □

Corollary 9.2.55. *Assume the hypotheses of Proposition 9.2.52(a). Then for the closed subscheme X' of X to be integral, it is necessary and sufficient that the graded ideal $\mathfrak{I}$ is prime in S.*

Proof. As X' is isomorphic to $\mathrm{Proj}(S/\mathfrak{I})$, this condition is sufficient in view of Proposition 9.2.16. To see the necessity, assume that $\mathrm{Proj}(S')$ is integral and consider the exact sequence $0 \to \mathscr{I} \to \mathscr{O}_X \to \mathscr{O}_X/\mathscr{I} \to 0$, which gives an exact sequence

$$0 \longrightarrow \Gamma_*(\mathscr{I}) \longrightarrow \Gamma_*(\mathscr{O}_X) \longrightarrow \Gamma_*(\mathscr{O}_X/\mathscr{I})$$

by the left-exactness of the global section functor. In view of the canonical homomorphism $\alpha: S/\mathfrak{I} \to \Gamma_*(\mathscr{O}_X/\mathscr{I})$, it then suffices to prove that if $f \in S_m$, $g \in S_n$ are such that the image in $\Gamma_*(\mathscr{O}_X/\mathscr{I})$ of $\alpha_{n+m}(fg)$ is zero, then one of the images of $\alpha_m(f)$, $\alpha_n(g)$ is zero. Now by definition, these images are sectins of the invertible $(\mathscr{O}_X/\mathscr{I})$-modules $\mathscr{L} = (\mathscr{O}_X/\mathscr{I})(m)$ and $\mathscr{L}' = (\mathscr{O}_X/\mathscr{I})(n)$ over the integral scheme X'. The hypotheses implies that their product is zero in $\mathscr{L} \otimes \mathscr{L}'$ (Corollary 9.2.29), so one of them is zero by Corollary 8.7.25. □

Corollary 9.2.56. *Let A be a ring, M be an A-module, and S be a graded A-algebra generated by S_1. Let $u: M \to S_1$ be a surjective homomorphism of A-modules and $\bar{u}: \boldsymbol{S}(M) \to S$ be the unique homomorphism of A-algebras extending u. Then the morphism corresponding to $\bar{u}$ is a closed immersion from* $\mathrm{Proj}(S)$ *to* $\mathrm{Proj}(\boldsymbol{S}(M))$.

Proof. The homomorphism $\bar{u}$ is surjective by hypothesis, so it suffices to apply Proposition 9.2.52. □

9.3 Homogeneous spectrum of sheaves of graded algebras

Let Y be a scheme and $\mathscr{S}$ be an $\mathscr{O}_Y$-algebra. We say that $\mathscr{S}$ is **graded** if $\mathscr{S}$ is the direct sum of a family $(\mathscr{S}_n)$ of $\mathscr{O}_Y$-algebras such that $\mathscr{S}_m\mathscr{S}_n \subseteq \mathscr{S}_{m+n}$. If $\mathscr{S}$ is a graded $\mathscr{O}_Y$-algebra, by a **graded $\mathscr{S}$-module** $\mathscr{M}$ we mean an $\mathscr{S}$-module $\mathscr{M}$ which is the direct sum of a family $(\mathscr{M}_n)_{n\in\mathbb{Z}}$ such that $\mathscr{S}_m\mathscr{M}_n \subseteq \mathscr{M}_{m+n}$. We say $\mathscr{S}$ is a graded $\mathscr{O}_Y$-algebra with positive degrees if $\mathscr{S}_n = 0$ for $n < 0$, and $\mathscr{M}$ is a graded $\mathscr{S}$-module if $\mathscr{M}_n = 0$ for $n < 0$. In this section, without further specifications, we will only consider graded algebras with positive degree.

9.3.1 Homogeneous spectrum of a graded $\mathscr{O}_Y$-algebra

Let $\mathscr{S}$ be a graded $\mathscr{O}_Y$-algebra (with positive degrees) and $\mathscr{M}$ be a graded $\mathscr{S}$-module. If $\mathscr{S}$ is quasi-coherent, each homogeneous component $\mathscr{S}_n$ is also a quasi-coherent $\mathscr{O}_Y$-module, since it is the image of $\mathscr{S}$ under the projection of $\mathscr{S}$ onto $\mathscr{S}_n$. Similarly, if $\mathscr{M}$ is quasi-coherent as an $\mathscr{O}_Y$-module, so is each of its homogeneous components, and the converse also holds. If $d > 0$ is an integer, we denote by $\mathscr{S}^{(d)}$ the direct sum of the $\mathscr{O}_Y$-modules $\mathscr{S}_{nd}$, which is quasi-coherent if $\mathscr{S}$ is; for any integer k such that $0 \le k \le d-1$, we denote by $\mathscr{M}^{(d,k)}$ (or $\mathscr{M}^{(d)}$ if $k = 0$) the direct sum of $\mathscr{M}_{nd+k}$ (for $n \in \mathbb{Z}$). If $\mathscr{S}$ and $\mathscr{M}$ are quasi-coherent sheaves, $\mathscr{M}(n)$ is a quasi-coherent $\mathscr{S}$-module by Proposition 8.2.25.

We say that $\mathscr{M}$ is a graded $\mathscr{S}$-module **of finite type** (resp. **of finite presentation**) if for any $y \in Y$, there exists an open neighborhood U of y and integers n_i (resp. integers m_i and n_i) such that there exists a surjective homomorphism $\bigoplus_{i=1}^{r}(\mathscr{S}(n_i)|_U) \to \mathscr{M}|_U$ of degree 0 (resp. such that $\mathscr{M}|_U$ is isomorphic to the cokernel of a homomorphism $\bigoplus_{i=1}^{r}\mathscr{S}(m_i)|_U \to \bigoplus_{j=1}^{r}\mathscr{S}(n_j)|_U$ of degree 0).

Let U be an affine open of Y and $A = \Gamma(U, \mathscr{O}_Y)$ by its ring. By hypothesis, the graded $(\mathscr{O}_Y|_U)$-algebra $\mathscr{S}|_U$ is isomorphic to $\widetilde{S}$ where $S = \Gamma(U, \mathscr{S})$ is a graded A-algebra; we put $X_U = \operatorname{Proj}(\Gamma(U, \mathscr{S}))$. Let $U' \subseteq U$ be another affine open subset of Y and $j : U' \to U$ the canonical injection, which corresponds to a homomorphism $A \to A'$, we have $\mathscr{S}|_{U'} = j^*(\mathscr{S}|_U)$, and therefore $S' = \Gamma(U', \mathscr{S})$ is identified with $S \otimes_A A'$ by Proposition 8.1.14. We then conclude from Proposition 9.2.50 that $X_{U'}$ is canonically identified with $X_U \times_U U'$, and therefore with $p_U^{-1}(U')$, where p_U is the structural morphism $X_U \to U$. Let $\sigma_{U',U}$ be the canonical isomorphism $p_U^{-1}(U') \cong X_{U'}$ thus defined, and $\rho_{U',U}$ be the open immersion $X_{U'} \to X_U$ obtained by composing $\sigma_{U',U}^{-1}$ with the canonical injectin $p_U^{-1}(U') \to X_U$. It is immediate that if $U'' \subseteq U'$ is a third affine open of Y, we have $\rho_{U'',U} = \rho_{U'',U'} \circ \rho_{U',U}$.

Proposition 9.3.1. *Let Y be a scheme and $\mathscr{S}$ be a quasi-coherent graded $\mathscr{O}_Y$-algebra. Then there exists a unique scheme X over Y such that, if $p : X \to Y$ is the structural morphism, for any affine open U of Y, there exists an isomorphism $\eta_U : p^{-1}(U) \xrightarrow{\sim} X_U = \operatorname{Proj}(\Gamma(U, \mathscr{S}))$ such that, if V is another affine open of Y contained in U, the following diagram*

$$\begin{tikzcd} p^{-1}(V) \arrow[r, "\eta_V"] \arrow[d] & X_V \arrow[d, "\rho_{V,U}"] \\ p^{-1}(U) \arrow[r, "\eta_U"] & X_U \end{tikzcd}$$

is commutative. The scheme X is called the ***homogeneous spectrum*** *of the quasi-coherent $\mathscr{O}_Y$-algebra $\mathscr{S}$ and denoted by* $\operatorname{Proj}(\mathscr{S})$.

Proof. For two affine opens U, V of Y, let $X_{U,V}$ be the scheme induced over $p_U^{-1}(U \cap V)$ by X_U; we shall define a Y-isomorphism $\theta_{U,V} : X_{V,U} \to X_{U,V}$. For this, consider an affine open $W \subseteq U \cap V$; by composing the isomorphisms

$$p_U^{-1}(W) \xrightarrow{\sigma_{W,U}} X_W \xrightarrow{\sigma_{W,V}^{-1}} p_V^{-1}(W)$$

we obtain an isomorphism $\tau_W : p_U^{-1}(W) \to p_V^{-1}(W)$, and we can verify that if $W' \subseteq W$ is an affine open, $\tau_{W'}$ is the restriction of τ_W to $p_U^{-1}(W')$; the morphisms τ_W then glue together to a Y-isomorphism $\theta_{V,U}$, which is what we want. Moreover, if U, V, W are affine opens of Y and $\theta'_{U,V}$, $\theta'_{V,W}$, and $\theta'_{U,W}$ are restrictions of $\theta_{U,V}, \theta_{V,W}, \theta_{U,W}$ on the inverse images of $U \cap V \cap W$ in X_V, X_W, X_W, respectively, it follows from the preceding definition that $\theta'_{U,V} \circ \theta'_{V,W} = \theta'_{U,W}$. The existence of X then follows from glueing these schemes via the isomorphisms $\theta_{U,V}$, and the uniqueness is clear. □

It is clear that the Y-scheme $\operatorname{Proj}(\mathscr{S})$ is separated over Y since homogeneous specturms are separated. If $\mathscr{S}$ is an $\mathscr{O}_Y$-algebra of finite type, it follows from Proposition 9.2.35 and Proposition 8.6.33 that $\operatorname{Proj}(\mathscr{S})$ is of finite type over Y. If $p : X \to Y$ is the structural morphism, it is immediate that for any open subschem U of Y, $p^{-1}(U)$ is identified with the homogeneous specturm $\operatorname{Proj}(\mathscr{S}|_U)$.

Proposition 9.3.2. *Let $f \in \Gamma(Y, \mathscr{S}_d)$ with $d > 0$ and $X = \operatorname{Proj}(\mathscr{S})$. Then there exists an open subset X_f of X such that, for any affine open subset U of Y, we have $X_f \cap p^{-1}(U) = D_+(f|_U)$ in $X_U = \operatorname{Proj}(\Gamma(U, \mathscr{S}))$ (where $p : X \to Y$ is the structural morphism). Moreover, the Y-scheme induced over X_f by X is canonically isomorphic to* $\operatorname{Spec}(\mathscr{S}^{(d)}/(f-1)\mathscr{S}^{(d)})$.

Proof. For any affine open U, we have $f|_U \in \Gamma(U, \mathscr{S}_d) = \Gamma(U, \mathscr{S})_d$ since U is quasi-compact. If U, U' are two affine opens of Y such that $U' \subseteq U$, $f|_{U'}$ is the image of $f|_U$ by the restriction homomorphism $\Gamma(U, \mathscr{S}) \to \Gamma(U', \mathscr{S})$, so $D_+(f|_{U'})$ is equal to the open subscheme $\rho_{U',U}^{-1}(D_+(f|_U))$ of $X_{U'}$. The subset X_f can be then defined by glueing these subschemes, and the first assertion is then obvious. On the other hand, the open subscheme $D_+(f|_U)$ of X_U is canonically identified with $\operatorname{Spec}(\Gamma(U, \mathscr{S})_{(f|_U)})$, and this identification is clearly compatible with restrictions; the second assertion then follows from Proposition 9.2.3. □

Corollary 9.3.3. *If $f \in \Gamma(Y, \mathscr{S}_d)$ and $g \in \Gamma(Y, \mathscr{S}_e)$, we have $X_{fg} = X_f \cap X_g$.*

Proof. It suffices to consider the intersection of two members of $p^{-1}(U)$, where U is an affine open of Y, and the assertion follows from $D_+(fg) = D_+(f) \cap D_+(g)$ for a graded ring S. $\square$

Corollary 9.3.4. *Let (f_α) be a family of sections of $\mathscr{S}$ over Y such that $f_\alpha \in \Gamma(Y, \mathscr{S}_{d_\alpha})$. If the sheaf of ideals of $\mathscr{S}$ generated by this family containes all the $\mathscr{S}_n$ for sufficiently large n, then the underlying space X is the union of X_{f_α}.*

Proof. In fact, for any affine open U of Y, $p^{-1}(U)$ is the union of $X_{f_\alpha} \cap p^{-1}(U)$ by Corollary 9.2.11, so the claim follows from the construction of X_{f_α}. $\square$

Corollary 9.3.5. *Let $\mathscr{A}$ be a quasi-coherent $\mathscr{O}_Y$-algebra and put*

$$\mathscr{S} = \mathscr{A}[T] = \mathscr{A} \otimes_{\mathbb{Z}} \mathbb{Z}[T]$$

where T is an indeterminate. Then $X = \mathrm{Proj}(\mathscr{S})$ is canonically identified with $\mathrm{Spec}(\mathscr{A})$. In particular, $\mathrm{Proj}(\mathscr{O}_Y[T])$ is identified with Y.

Proof. By applying Corollary 9.3.4 to the unique section $f \in \Gamma(Y, \mathscr{S})$ equal to T on each point of Y, we see that $X_f = X$. Moreover, we have $f \in \mathscr{S}_1$, and $\mathscr{S}^{(1)}/(f-1)\mathscr{S}^{(1)} = \mathscr{S}/(f-1)\mathscr{S}$ is canonically isomorphic to $\mathscr{A}$, whence the corollary $\square$

Let $g \in \Gamma(Y, \mathscr{O}_Y)$; if we put $\mathscr{S} = \mathscr{O}_Y[T]$, then $g \in \Gamma(Y, \mathscr{S}_0)$; let

$$h = gT \in \Gamma(Y, \mathscr{S}_1).$$

If $X = \mathrm{Proj}(\mathscr{S})$, the canonical identification of Corollary 9.3.5 identifies X_h with the open subset Y_g of Y: in fact, we can assume that $Y = \mathrm{Spec}(A)$ is affine, and this then follows from the fact that the ring A_g is canonically identified with $A[T]/(gT-1)A[T]$.

Proposition 9.3.6. *Let $\mathscr{S}$ be a quasi-coherent graded $\mathscr{O}_Y$-algebra.*

(a) *For any $d > 0$, there exists a canonical Y-isomorphism from $\mathrm{Proj}(\mathscr{S})$ to $\mathrm{Proj}(\mathscr{S}^{(d)})$.*

(b) *Let $\mathscr{S}'$ be the graded $\mathscr{O}_Y$-algebra $\mathscr{O}_Y \oplus \bigoplus_{n>0} \mathscr{S}_n$, then the schemes $\mathrm{Proj}(\mathscr{S}')$ and $\mathrm{Proj}(\mathscr{S})$ are canonically Y-isomorphic.*

(c) *Let $\mathscr{L}$ be an invertible $\mathscr{O}_Y$-module and $\mathscr{S}_{(\mathscr{L})}$ be the graded $\mathscr{O}_Y$-algebra $\bigoplus_{d>0} \mathscr{S}_d \otimes \mathscr{L}^{\otimes d}$; then the schemes $\mathrm{Proj}(\mathscr{S})$ and $\mathrm{Proj}(\mathscr{S}_{(\mathscr{L})})$ are canonically Y-isomorphic.*

Proof. In all three cases, it suffices to define an isomorphism locally over Y and verifying the compatibility of restriction morphisms is immediate. We can then assume that Y is affine, and assertions (a) and (b) then follow from Proposition 9.2.18. As for (c), if the invertible sheaf $\mathscr{L}$ is just isomorphic to $\mathscr{O}_Y$ then the isomorphism between $\mathrm{Proj}(\mathscr{S})$ and $\mathrm{Proj}(\mathscr{S}_{(\mathscr{L})})$ is evident. To define a canonical isomorphism, let $Y = \mathrm{Spec}(A)$, $\mathscr{S} = \widetilde{S}$, where S is a graded A-algebra, and let c be a generator of the free A-module L such that $\mathscr{L} = \widetilde{L}$. Then for any $n > 0$, $x_n \mapsto x_n \otimes c^{\otimes n}$ is an A-isomorphism from S_n to $S_n \otimes L^{\otimes n}$, and these A-isomorphisms define an A-isomorphism of graded algebras

$$p_c : S \to S_{(L)} = \bigoplus_{n \geq 0} S_n \otimes L^{\otimes n}.$$

Let $f \in S_+$ be homogeneous of degree d; for any $x \in S_{nd}$, we have $(x \otimes c^{nd})/(f \otimes c^d)^n = (x \otimes (\varepsilon c)^{nd})/(f \otimes (\varepsilon c)^d)^n$ for any invertible element $\varepsilon \in A$, which implies that the isomorphism $S_{(f)} \to (S_{(L)})_{(f \otimes c^d)}$ induced by p_c is independent from the generator c of L, whence the assertion. $\square$

Recall that for the quasi-coherent graded $\mathscr{O}_Y$-algebra $\mathscr{L}$ to be *generated* by the $\mathscr{O}_Y$-module $\mathscr{S}_1$, it is necessary and sufficient that there exists a covering (U_α) of Y by affine opens such that the graded algebra $\Gamma(U_\alpha, \mathscr{S})$ over $\Gamma(U_\alpha, \mathscr{S}_0)$ is generated by $\Gamma(U_\alpha, \mathscr{S}_1)$. If this is true, then for any open subset V of Y, $\mathscr{S}|_V$ is then generated by $(\mathscr{O}_Y|_V)$-algebra $\mathscr{S}_1|_V$.

Proposition 9.3.7. *Suppose that there exists a finite affine open cover (U_i) of Y such that the graded algebra $\Gamma(U_i, \mathscr{S})$ is of finite type over $\Gamma(U_i, \mathscr{O}_Y)$. Then there exists $d > 0$ such that $\mathscr{S}^{(d)}$ is generated by $\mathscr{S}_d$, which is an $\mathscr{O}_Y$-module of finite type.*

Proof. In fact, it follows from **??** that for each i, there exist an integer m_i such that $\Gamma(U_i, \mathscr{S}_{nm_i}) = (\Gamma(U_i, \mathscr{S}_{m_i}))^n$ for all $n > 0$; it suffices to take d a common multiple of the m_i. ☐

Corollary 9.3.8. *Under the hypotheses of Proposition 9.3.7,* $\mathrm{Proj}(\mathscr{S})$ *is* Y*-isomorphic to a homogeneous specturm* $\mathrm{Proj}(\mathscr{S}')$*, where* $\mathscr{S}'$ *is a graded* $\mathscr{O}_Y$*-algebra generated by* $\mathscr{S}'_1$*, where* $\mathscr{S}'$ *is an* $\mathscr{O}_Y$*-algebra of finite type.*

Proof. It suffices to take $\mathscr{S}' = \mathscr{S}^{(d)}$, where d is determined by the properties of Proposition 9.3.7, and apply Proposition 9.3.6(a). ☐

If $\mathscr{S}$ is a quasi-coherent graded $\mathscr{O}_Y$-algebras, we have seen in Proposition 8.4.28 that its nilradical is a quasi-coherent $\mathscr{O}_Y$-module. We say that $\mathscr{N}_+ = \mathscr{N} \cap \mathscr{S}_+$ is the nilradical of $\mathscr{S}_+$, which is a quasi-coherent graded $\mathscr{S}_0$-module, since this is the case if Y is affine. For any $y \in Y$, $(\mathscr{N}_+)_y$ is then the nilradical of $(\mathscr{S}_+)_y = (\mathscr{S}_y)_+$. Similar to the local case, we say the graded $\mathscr{O}_Y$-algebra $\mathscr{S}$ is **essentially reduced** if $\mathscr{N}_+ = 0$, which means $\mathscr{S}_y$ is an essentially reduced graded $\mathscr{O}_{Y,y}$-algebra for any $y \in Y$. It is clear that for any quasi-coherent graded $\mathscr{O}_Y$-algebra $\mathscr{S}$, $\mathscr{S}/\mathscr{N}_+$ is essentially reduced. Finally, we say $\mathscr{S}$ is **integral** if $\mathscr{S}_y$ is an integral ring for each $y \in Y$ and if $(\mathscr{S}_+)_y = (\mathscr{S}_y))_+ \neq 0$ for all $y \in Y$.

Proposition 9.3.9. *Let* $\mathscr{S}$ *be a graded* $\mathscr{O}_Y$*-algebra. If* $X = \mathrm{Proj}(\mathscr{S})$*, the* Y*-scheme* X_{red} *is canonically isomorphic to* $\mathrm{Proj}(\mathscr{S}/\mathscr{N}_+)$*. In particular, if* $\mathscr{S}$ *is essentially reduced, then* X *is reduced.*

Proof. The fact that $X' = \mathrm{Proj}(\mathscr{S}/\mathscr{N}_+)$ is reduced follows from Proposition 9.2.16, since the question is local. Moreover, for any affine open $U \subseteq Y$, $p'^{-1}(U)$ is equal to $(p^{-1}(U))_{\mathrm{red}}$ (where p and p' are the structural morphisms $X \to Y$, $X' \to Y$, respectively); we also verify t hat the canonical U-morphisms $p'^{-1}(U) \to p^{-1}(U)$ is compatible with restrictions and define therefore a closed immersion $X' \to X$, which is a homeomorphism on underlying spaces. Our assertion then follows from Corollary 8.4.29. ☐

Proposition 9.3.10. *Let* Y *be an integral scheme and* $\mathscr{S}$ *be a quasi-coherent graded* $\mathscr{O}_Y$*-algebra such that* $\mathscr{S}_0 = \mathscr{O}_Y$*.*

(a) *If* $\mathscr{S}$ *is integral then* $X = \mathrm{Proj}(\mathscr{S})$ *is integral and the structural morphism* $p : X \to Y$ *is dominant.*

(b) *Suppose moreover that* $\mathscr{S}$ *is essentially reduced. Then, conversely, if* X *is integral and* p *is dominant, then* $\mathscr{S}$ *is integral.*

Proof. We first assume that $\mathscr{S}$ is integral. Then if (U_α) is a basis of Y formed by affine opens, it suffices to prove for Y being replaced by U_α and $\mathscr{S}$ by $\mathscr{S}|_{U_\alpha}$: in fact, is this is true, the underlying space $p^{-1}(U_\alpha)$ is an open irreducible subset of X such that $p^{-1}(U_\alpha) \cap p^{-1}(U_\beta) \neq \varnothing$ for any couple of indices α, β (since $U_\alpha \cap U_\beta$ contains an U_γ and $\mathscr{S}$ is integral), so X is irreducible by **??**; it is clear that X is reduced since $\mathscr{S}$ is reduced, so X is integral. It is clear that $p(X)$ is dense in Y since this holds for each U_α.

Suppose then that $Y = \mathrm{Spec}(A)$ where A is integral (Proposition 8.4.30) and $\mathscr{S} = \widetilde{S}$, where S is a graded A-algebra; the hypotheses on $\mathscr{S}$ is that for any $y \in Y$, $\widetilde{S}_y = S_y$ is an integral graded ring such that $(S_y)_+ \neq 0$. It then suffices to prove that S is an integral ring, since then $S_+ \neq 0$ and we can apply Proposition 9.2.16. Now, let f, g be two nonzero elements of S and suppose that $fg = 0$; for any $y \in Y$ we have $(f/1)(g/1) =$ in S_y, so $f/1 = 0$ or $g/1 = 0$ by hypothesis. Suppose for example that $f/1 = 0$ in S_y, so there exists $a \in A$ such that $a \notin \mathfrak{p}_y$ and $af = 0$. We then see that for each $z \in Y$, $(a/1)(f/1) = 0$ in the integral ring S_z, and as $a/1 \neq 0$ (since A is integral), $f/1 = 0$, which implies $f = 0$.

Now consider the hypothesis in (b) and assume that X is integral and p is dominant. Since the question is local over Y, we can assume that $Y = \mathrm{Spec}(A)$, where A is an integral ring, and $\mathscr{S} = \widetilde{S}$. By hypothesis for any $y \in Y$, $(S_y)_+$ is reduced, and so is $(S_0)_y = A_y$ by hypothesis, so S_y is a reduced ring and we conclude that S is reduced. The hypothesis that X is integral implies that S is essentially integral (Proposition 9.2.16). The proposition then boils down to see that the annihilator $\mathfrak{I}$ of S_+ over $A = S_0$ is reduced to zero. In the contrary case, we would have $(S_h)_+ = 0$ for an $h \neq 0$ in $\mathfrak{I}$, which implies $p^{-1}(D(h)) = \varnothing$ by Proposition 9.3.1, and $p(X)$ is then not dense in Y, contradicting the hypothesis (since $D(h) \neq \varnothing$, h is not nilpotent). We then see that the ring S is integral, which conclude our assertion. ☐

9.3.2 Sheaves associated with a graded $\mathscr{S}$-module

Let Y be a scheme, $\mathscr{S}$ be a quasi-coherent $\mathscr{O}_Y$-algebra, and $\mathscr{M}$ be a quasi-coherent graded $\mathscr{S}$-module over the ringed space $(Y, \mathscr{S})$. With the notations of Proposition 9.3.1, we denote by $\widetilde{\mathscr{M}}_U$ the quasi-coherent $\mathscr{O}_{X_U}$-module $\widetilde{\Gamma(U, \mathscr{M})}$. For $U' \subseteq U$, $\Gamma(U', \mathscr{M})$ is canonically identified with $\Gamma(U, \mathscr{M}) \otimes_A A'$ by Proposition 8.1.14, so $\widetilde{M}_{U'} = \rho^*_{U',U}(\widetilde{\mathscr{M}}_U)$ by Proposition 9.2.50.

Proposition 9.3.11. *There exists over* $\operatorname{Proj}(\mathscr{S}) = X$ *a unique quasi-coherent* $\mathscr{O}_X$*-module* $\widetilde{\mathscr{M}}$ *such that, for any affine open* U *of* Y*, we have* $\eta_U^*(\widetilde{\mathscr{M}}_U) = \widetilde{\mathscr{M}}|_{p^{-1}(U)}$ *, where* $p : X \to Y$ *is the structural morphism and* η_U *is the isomorphism* $p^{-1}(U) \cong \operatorname{Proj}(\Gamma(U, \mathscr{S}))$*. We say that* $\widetilde{\mathscr{M}}$ *is the* $\mathscr{O}_X$*-module associated with* $\mathscr{M}$*.*

Proof. As $\rho_{U',U}$ is identified with the injection morphism $p^{-1}(U') \to p^{-1}(U)$, the proposition follows from the relation $\widetilde{M}_{U'} = \rho_{U',U}^*(\widetilde{\mathscr{M}}_U)$ and glueing the $\widetilde{\mathscr{M}}_U$. □

Proposition 9.3.12. *Let* $\mathscr{M}$ *be a quasi-coherent* $\mathscr{S}$*-module and* $f \in \Gamma(Y, \mathscr{S}_d)$ *with* $d > 0$*. If* ξ_f *is the canonical isomorphism from* X_f *to the* Y*-scheme* $Z_f = \operatorname{Spec}(\mathscr{S}^{(d)}/(f-1)\mathscr{S}^{(d)})$ *in Proposition 9.3.2, then* $(\xi_f)_*(\widetilde{\mathscr{M}}|_{X_f})$ *is the* $\mathscr{O}_{Z_f}$*-module* $(\mathscr{M}^{(d)}/(f-1)\mathscr{M}^{(d)})^\sim$*.*

Proof. The question is local over Y we we are reduced to Proposition 9.2.3, and its compatibility with restrictions. □

Proposition 9.3.13. *The* $\mathscr{O}_X$*-module* $\widetilde{\mathscr{M}}$ *is an additive exact covariant functor from the category of quasi-coherent graded* $\mathscr{S}$*-modules to the category of quasi-coherent* $\mathscr{O}_X$*-modules, which commutes with direct sums and inductive limits.*

Proof. This follows from Corollary 8.1.6 and Proposition 9.2.34, since the question is local on Y. □

In particular, if $\mathscr{N}$ is a quasi-coherent graded sub-$\mathscr{S}$-module of $\mathscr{M}$, $\widetilde{\mathscr{N}}$ is cannically identified with a quasi-coherent sub-$\mathscr{O}_X$-module of $\widetilde{\mathscr{M}}$; if we take $\mathscr{M} = \mathscr{S}$, then for any quasi-coherent ideal $\mathscr{J}$ of $\mathscr{S}$, $\widetilde{\mathscr{J}}$ is a quasi-coherent ideal of $\mathscr{O}_X$.

If $\mathscr{M}$ is a quasi-coherent graded $\mathscr{S}$-module and $\mathscr{J}$ is a quasi-coherent ideal of $\mathscr{O}_Y$, then $\mathscr{J}\mathscr{M}$ is a quasi-coherent graded sub-$\mathscr{S}$-module of $\mathscr{M}$ and we have $\widetilde{\mathscr{J}\mathscr{M}} = \mathscr{J} \cdot \widetilde{\mathscr{M}}$: it suffices to verify this formula if $Y = \operatorname{Spec}(A)$, $\mathscr{S} = \widetilde{S}$, where S is a graded A-algebra, $\mathscr{M} = \widetilde{M}$ where M is a graded S-module, and $\mathscr{J} = \tilde{\mathfrak{a}}$, where $\mathfrak{a}$ is an ideal of A. For any homogeneous element f of S_+, the restriction to $D_+(f) = \operatorname{Spec}(S_{(f)})$ of $\widetilde{\mathscr{J}\mathscr{M}}$ is the assocaited sheaf of $(\mathfrak{a}M)_{(f)} = \mathfrak{a} \cdot M_{(f)}$, and the identification is compatible with restrictions.

Proposition 9.3.14. *Let* $f \in \Gamma(Y, \mathscr{S}_d)$ *with* $d > 0$*. Over the open subset* X_f*, the* $(\mathscr{O}_X|_{X_f})$*-module* $\widetilde{\mathscr{S}(nd)}|_{X_f}$ *is canonically isomorphic to* $\mathscr{O}_X|_{X_f}$ *for any* $n \in \mathbb{Z}$*. In particular, if the* $\mathscr{O}_Y$*-algebra* $\mathscr{S}$ *is generated by* $\mathscr{S}_1$*, the* $\mathscr{O}_X$*-module* $\widetilde{\mathscr{S}(n)}$ *is invertible for all* $n \in \mathbb{Z}$*.*

Proof. For any affine open U of Y, by Proposition 9.2.24 we have a canonical isomorphism $\widetilde{\mathscr{S}(nd)}|_{X_f \cap p^{-1}(U)} \cong \mathscr{O}_X|_{X_f \cap p^{-1}(U)}$, in view of Proposition 9.3.2 (where $p : X \to Y$ is the structural morphism). It is immediate that this isomorphism is compatible with restrictions, whence the first assertion. For the second one, if $\mathscr{S}$ is generated by $\mathscr{S}_1$, there exists an affine open cover (U_α) of Y such that $\Gamma(U_\alpha, \mathscr{S})$ is generated by $\Gamma(U_\alpha, \mathscr{S}_1) = \Gamma(U_\alpha, \mathscr{S})_1$, and we can then use Corollary 9.2.26. □

Again, for any integer $n \in \mathbb{Z}$ and any $\mathscr{O}_X$-module $\mathscr{F}$, we set

$$\mathscr{O}_X(n) = \widetilde{\mathscr{S}(n)}, \quad \mathscr{F}(n) = \mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{O}_X(n).$$

It follows from this definition that, for any open subset U of Y,

$$\widetilde{\mathscr{S}|_U(n)} = \mathscr{O}_X|_{p^{-1}(U)},$$

where $p : X \to Y$ is the structural morphism.

Proposition 9.3.15. *Let* $\mathscr{M}$ *and* $\mathscr{N}$ *be quasi-coherent graded* $\mathscr{S}$*-modules. Then there exists cannical homomorphisms*

$$\lambda : \widetilde{M} \otimes_{\mathscr{O}_X} \widetilde{\mathscr{N}} \to (\mathscr{M} \otimes_{\mathscr{S}} \mathscr{N})^\sim$$
$$\mu : (\mathcal{H}om_{\mathscr{S}}(\mathscr{M}, \mathscr{N}))^\sim \to \mathcal{H}om_{\mathscr{O}_X}(\widetilde{\mathscr{M}}, \widetilde{\mathscr{N}}).$$

If $\mathscr{S}$ *is generated by* $\mathscr{S}_1$*, then* λ *is an isomorphism; if moreover* $\mathscr{M}$ *is of finite presentation,* μ *is an isomorphism.*

Proof. The isomorphisms λ and μ are defined in the arguments before Proposition 9.2.28 if Y is affine, and this definition is local and then glue together to define global morphisms, in view of the diagrams (9.2.13) and (9.2.14). □

Corollary 9.3.16. *If $\mathscr{S}$ is generated by $\mathscr{S}_1$, for any integers $m,n\in\mathbb{Z}$, we have*

$$\mathscr{O}_X(m)\otimes_{\mathscr{O}_X}\mathscr{O}_X(n)=\mathscr{O}_X(m+n),\quad \mathscr{O}_X(n)=\mathscr{O}_X^{\otimes 1}.$$

Corollary 9.3.17. *If $\mathscr{S}$ is generated by $\mathscr{S}_1$, for any quasi-coherent graded $\mathscr{S}$-module $\mathscr{M}$ and $n\in\mathbb{Z}$, we have*

$$\widetilde{\mathscr{M}(n)}=\widetilde{\mathscr{M}}(n).$$

Remark 9.3.18. If $\mathscr{S}=\mathscr{A}[T]$ where $\mathscr{A}$ is a quasi-coherent $\mathscr{O}_Y$-algebra, we verify immediately that the invertible $\mathscr{O}_X$-module $\mathscr{O}_X(n)$ is canonically isomorphic to $\mathscr{O}_X$. Moreover, let $\mathscr{N}$ be a quasi-coherent $\mathscr{A}$-module, and put $\mathscr{M}=\mathscr{N}\otimes_{\mathscr{A}}\mathscr{A}[T]$. It follows from Proposition 9.3.12 and Corollary 9.3.5 that under the canonical isomorphism of $X=\mathrm{Proj}(\mathscr{A}[T])$ and $X'=\mathrm{Spec}(\mathscr{A})$, the $\mathscr{O}_X$-module $\widetilde{\mathscr{M}}$ is identified with the $\mathscr{O}_X$-module $\widetilde{\mathscr{N}}$.

Remark 9.3.19. Let $\mathscr{S}$ be a quasi-coherent graded $\mathscr{O}_Y$-algebra and define $\mathscr{S}'$ to be the $\mathscr{O}_Y$-algebra such that $\mathscr{S}'=\mathscr{O}_Y$ and $\mathscr{S}'_n=\mathscr{S}_n$ for $n>0$. Then the canonical isomorphism of $X=\mathrm{Proj}(\mathscr{S})$ and $\mathrm{Proj}(\mathscr{S}')$ identifies $\mathscr{O}_X(n)$ and $\mathscr{O}_{X'}(n)$ for any $n\in\mathbb{Z}$: this follows from the corresponding result in the affine case (Example 9.2.31) and the fact that this identification is compatible with restrictions. Similarly, let $X^{(d)}=\mathrm{Proj}(\mathscr{S}^{(d)})$; the canonical isomorphism of X and $X^{(d)}$ identifies $\mathscr{O}_X(nd)$ with $\mathscr{O}_{X^{(d)}}(n)$ for any $n\in\mathbb{Z}$.

Proposition 9.3.20. *Let $\mathscr{L}$ be an invertible $\mathscr{O}_Y$-module and ψ be the structural morphism from $X_{(\mathscr{L})}=\mathrm{Proj}(\mathscr{S}_{(\mathscr{L})})$ to $X=\mathrm{Proj}(\mathscr{S})$. Then for any integer $n\in\mathbb{Z}$, $\psi_*(\mathscr{O}_{X_{(\mathscr{L})}}(n))$ is canonically isomorphic to $\mathscr{O}_X(n)\otimes_Y\mathscr{L}^{\otimes n}$.*

Proof. Suppose first that Y is affine with ring A and $\mathscr{L}=\widetilde{L}$, where L is a free A-module of rank 1. With the notations of Proposition 9.3.6(c), we define for each $f\in S_d$ an isomorphism from $S(n)_{(f)}\otimes_A L^{\otimes n}$ to $S_{(L)}(n)_{(f\otimes c^d)}$ which sends $(x/f^k)\otimes c^n$, where $x\in S_{kd+n}$, to the element $(x\otimes c^{n+kd})/(f\otimes c^d)^k$. It is immedaite that this isomorphism is independent of the generator c of L, and is compatible with restrictions $D_+(f)\to D_+(fg)$. The general case then follows from glueing these isomorphisms. □

9.3.3 Graded $\mathscr{S}$-module assocaited with a sheaf

For simplicity, in the following discussion, we always assume that the quasi-coherent $\mathscr{O}_Y$-algebra $\mathscr{S}$ is generated by $\mathscr{S}_1$, which by Proposition 9.3.6 is not at all essential if we impose the finiteness conditions of Proposition 9.3.7 on Y. Let $p:X\to Y$ be the structural morphism where $X=\mathrm{Proj}(\mathscr{S})$, which is separated by Proposition 9.2.14. For any $\mathscr{O}_X$-module $\mathscr{F}$, we define

$$\Gamma_*(\mathscr{F})=\bigoplus_{n\in\mathbb{Z}}p_*(\mathscr{F}(n))$$

and in particular

$$\Gamma_*(\mathscr{O}_X)=\bigoplus_{n\in\mathbb{Z}}p_*(\mathscr{O}_X(n)).$$

We have seen in (**??**) that there exists a canonical homomorphism

$$p_*(\mathscr{F})\otimes_{\mathscr{O}_Y}p_*(\mathscr{G})\to p_*(\mathscr{F}\otimes_{\mathscr{O}_X}\mathscr{G})$$

for any $\mathscr{O}_X$-modules $\mathscr{F}$ and $\mathscr{G}$, so we deduce from Corollary 9.3.16 that $\Gamma_*(\mathscr{O}_X)$ is endowed with a graded $\mathscr{O}_Y$-algebra structure and $\Gamma_*(\mathscr{F})$ is a graded module over $\Gamma_*(\mathscr{O}_X)$.

In view of Proposition 9.3.14 and the left-exactness of the functor f_*, $\Gamma_*(\mathscr{F})$ is an additive left-exact covariant functor from the category of $\mathscr{O}_X$-modules to the category of graded $\mathscr{O}_Y$-modules. In particular, if $\mathscr{J}$ is a sheaf of ideals of $\mathscr{O}_X$, $\Gamma_*(\mathscr{J})$ is identified with a sheaf of graded ideals of $\Gamma_*(\mathscr{O}_X)$.

Now let $\mathscr{M}$ be a quasi-coherent graded $\mathscr{S}$-module. For any affine open U of Y, we have defined a homomorphism of abelian groups

$$\alpha_{0,U}:\Gamma(U,\mathscr{M}_0)\to\Gamma(p^{-1}(U),\widetilde{\mathscr{M}}).$$

It is immediate that these homomorphisms commutes with restrictions and define (which do not use the hypothesis that $\mathscr{S}$ is generated by $\mathscr{S}_1$) a homomorphism of sheaf of abelian groups

$$\alpha_0:\mathscr{M}_0\to\widetilde{\mathscr{M}}.$$

Apply this result to $\mathscr{M}_n = (\mathscr{M}(n))_0$ and use Corollary 9.3.17, we define a homomorphism of abelian groups

$$\alpha_n : \mathscr{M}_n \to p_*(\widetilde{\mathscr{M}}(n)) \tag{9.3.1}$$

for each $n \in \mathbb{Z}$, whence a functorial homomorphism of graded sheaves of abelian groups

$$\alpha : \mathscr{M} \to \Gamma_*(\widetilde{\mathscr{M}}) \tag{9.3.2}$$

(we also denote it by $\alpha_{\mathscr{M}}$). In the particular case $\mathscr{M} = \mathscr{S}$, we verify that $\alpha : \mathscr{S} \to \Gamma_*(\mathscr{O}_X)$ is a homomorphism of graded $\mathscr{O}_Y$-algebra and is a bi-homomorphism of graded modules, relative to this homomorphism of graded homomorphism of algebras.

We also remark that the homomorphism α_n corresponds to a canonical homomorphism of $\mathscr{O}_X$-modules

$$\alpha_n^\sharp : p^*(\mathscr{M}_n) \to \widetilde{\mathscr{M}}(n).$$

Moreover, it is easy to verify that this homomorphism is none other than the associated homomorphism (by Proposition 9.3.13) of the canonical homomorphism $\mathscr{M}_n \otimes_{\mathscr{O}_Y} \mathscr{S} \to \mathscr{M}(n)$ of $\mathscr{O}_Y$-modules, where the $\mathscr{O}_Y$-module $\mathscr{M}_n \otimes_{\mathscr{O}_Y} \mathscr{S}$ is given the natrual graduation. To see this, we can in fact assume that $Y = \mathrm{Spec}(A)$ is affine, $\mathscr{M} = \widetilde{M}$ and $\mathscr{S} = \widetilde{S}$, where S is a graded A-algebra generated by S_1. Returning to the definition of α, we see that the restriction to $D_+(f)$ of the homomorphism $\alpha_n^\sharp$ corresponds to the homomorphism $M_n \otimes_A S_{(f)} \to M(n)_{(f)}$, where $x \otimes 1$ is maped to $x/1$.

Proposition 9.3.21. *For any section $f \in \Gamma(Y, \mathscr{S}_d)$ with $d > 0$, X_f is identified with the set of points of X on which $\alpha_d(f)$ is nonzero.*

Proof. The element $\alpha_d(f)$ is a section of $p_*(\mathscr{O}_X(d))$ over Y, and by definition is then a section of $\mathscr{O}_X(d)$ over X. The definition of X_f (Proposition 9.3.2) proves our claim in the affine case, in view of Proposition 9.2.33. □

We shall henceforth suppose, in addition to the hypothesis at the beginning of this paragraph, that for every quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, $p_*(\mathscr{F}(n))$ is quasi-coherent over Y, and therefore $\Gamma_*(\mathscr{F}) = \bigoplus_{n \in \mathbb{Z}} p_*(\mathscr{F}(n))$ is also a quasi-coherent $\mathscr{O}_Y$-module; this circumstance will always occur if X is of finite type on Y (Proposition 8.6.55). We then conclude that $\widetilde{\Gamma_*(\mathscr{F})}$ is defined and is a quasi-coherent $\mathscr{O}_X$-module. For any affine open subset U of Y, we have (Corollary 8.1.6, Proposition 9.3.13, and note that U is quasi-compact)

$$\begin{aligned}
\Big(\Gamma(U, \bigoplus_{n \in \mathbb{Z}} p_*(\mathscr{F}(n)))\Big)^\sim &= \bigoplus_{n \in \mathbb{Z}} (\Gamma(U, p_*(\mathscr{F}(n))))^\sim = \bigoplus_{n \in \mathbb{Z}} (\Gamma(p^{-1}(U), \mathscr{F}(n)))^\sim \\
&= \Big(\bigoplus_{n \in \mathbb{Z}} \Gamma(p^{-1}(U), \mathscr{F}(n))\Big)^\sim = (\Gamma_*(\mathscr{F}|_{p^{-1}(U)}))^\sim
\end{aligned}$$

and therefore a canonical homomorphism

$$\beta_U : \Big(\Gamma(U, \bigoplus_{n \in \mathbb{Z}} p_*(\mathscr{F}(n)))\Big)^\sim \to \mathscr{F}|_{p^{-1}(U)}.$$

Moreover, the diagram (9.2.12) shows that these homomorphisms are compatible with restrictions on Y, so we deduce a canonical homomorphism

$$\beta : \widetilde{\Gamma_*(\mathscr{F})} \to \mathscr{F}$$

(also denoted by $\beta_{\mathscr{F}}$) for the quasi-coherent $\mathscr{O}_X$-modules.

Proposition 9.3.22. *Let $\mathscr{M}$ be a quasi-coherent $\mathscr{S}$-module and $\mathscr{F}$ be a quasi-coherent $\mathscr{O}_X$-module. Then the composition homomorphisms*

$$\widetilde{\mathscr{M}} \xrightarrow{\tilde{\alpha}} (\Gamma_*(\widetilde{\mathscr{M}}))^\sim \xrightarrow{\beta} \widetilde{\mathscr{M}} \tag{9.3.3}$$

$$\Gamma_*(\mathscr{F}) \xrightarrow{\alpha} \Gamma_*(\widetilde{\Gamma_*(\mathscr{F})}) \xrightarrow{\Gamma_*(\beta)} \Gamma_*(\mathscr{F}) \tag{9.3.4}$$

Proof. The question is local over Y, so we can apply Proposition 9.2.34. □

Again, the homomorphisms α and β are in general not isomorphisms, and further finiteness conditions must be imposed. We note also that the homomorphism β is not always defined, unlike the affine case. However, we shall see that if $\mathscr{S}$ is of finite type and generated by $\mathscr{S}_1$, the homomorphisms α and β are well defined and the corresponding results of the affine cases carry over without difficulties.

Proposition 9.3.23. *Let Y be a scheme and $\mathscr{S}$ be a quasi-coherent $\mathscr{O}_Y$-algebra generated by $\mathscr{S}_1$. Suppose that $\mathscr{S}_1$ is of finite type, then $X = \mathrm{Proj}(\mathscr{S})$ is of finite type over Y.*

Proof. Again we can assume that Y is affine with ring A, so $\mathscr{S} = \widetilde{S}$ where S is a graded A-algebra generated by S_1, and S_1 is a finitely generated A-module by hypothesis. Then S is an A-algebra of finite type, and the proposition follows from Proposition 9.2.35. □

Let $\mathscr{S}$ be a quasi-coherent graded $\mathscr{O}_Y$-algebra and $\mathscr{M}$ be a quasi-coherent $\mathscr{S}$-module. We say that $\mathscr{M}$ is **eventually null** if there exists an integer n such that $\mathscr{M}_k = 0$ for $k \geq n$, and is **eventually finite** if there exists an integer n such that the $\mathscr{S}$-module $\bigoplus_{k\geq n}\mathscr{M}_k$ is of finite type. If $\mathscr{M}$ is eventually null, it is clear that $\widetilde{\mathscr{M}} = 0$, as in the affine case.

Let $\mathscr{M}$ and $\mathscr{N}$ be quasi-coherent graded $\mathscr{S}$-modules. We say a homomorphism $u : \mathscr{M} \to \mathscr{N}$ of degree 0 is eventually injective (resp. eventually sujective, eventually bijective) if there exists an integer n such that $u_k : \mathscr{M}_k \to \mathscr{N}_k$ is injective (resp. surjective, bijective) for $k \geq n$. It is clear that in this case, $\tilde{u} : \widetilde{\mathscr{M}} \to \widetilde{\mathscr{N}}$ is injective (resp. surjective, bijective), since this can be checked locally over Y and we can apply Proposition 9.3.13. If u is eventually bijective, we also say that it is an eventual isomorphism.

Proposition 9.3.24. *Let Y be a scheme and $\mathscr{S}$ be a quasi-coherent $\mathscr{O}_Y$-algebra generated by $\mathscr{S}_1$ and such that $\mathscr{S}_1$ is of finite type. Let $\mathscr{M}$ be a quasi-coherent graded $\mathscr{S}$-module.*

(a) *If $\mathscr{M}$ is eventually finite, $\widetilde{\mathscr{M}}$ is of finite type.*

(b) *If $\mathscr{M}$ is eventually finite, for $\widetilde{\mathscr{M}} = 0$, it is necessary and sufficient that $\mathscr{M}$ is eventually null.*

Proof. The question is local over Y, so we can assume that $Y = \mathrm{Spec}(A)$, and the proposition then follows from Proposition 9.2.36. □

Theorem 9.3.25. *Let Y be a scheme and $\mathscr{S}$ be a quasi-coherent $\mathscr{O}_Y$-algebra generated by $\mathscr{S}_1$ and such that $\mathscr{S}_1$ is of finite type. Let $X = \mathrm{Proj}(\mathscr{S})$, then for any quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, the canonical homomorphism $\beta : \widetilde{\Gamma(\mathscr{F})} \to \mathscr{F}$ is an isomorphism.*

Proof. We first note that the homomorphism β is defined because of Proposition 9.3.23. To see that β is an isomorphism, we can assume $Y = \mathrm{Spec}(A)$ is affine, and then apply Theorem 9.2.39. □

Corollary 9.3.26. *Under the hypotheses of Theorem 9.3.25, any quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$ is isomorphic to an $\mathscr{O}_X$-module of the form $\widetilde{\mathscr{M}}$, where $\mathscr{M}$ is a quasi-coherent $\mathscr{S}$-module. If moreover $\mathscr{F}$ is of finite type, and if we suppose that Y is a quasi-compact scheme, then we can choose $\mathscr{M}$ to be of finite type.*

Proof. The first assertion follows from Theorem 9.3.25 by take $\mathscr{M} = \Gamma_*(\mathscr{F})$. For the second one, it suffices to prove that $\mathscr{M}$ is the inductive limit of graded sub-$\mathscr{S}$-modules of finite type $\mathscr{N}_\lambda$: in fact, it then follows that $\widetilde{\mathscr{M}}$ is the inductive limit of the $\widetilde{\mathscr{N}}_\lambda$ (Proposition 9.3.13), hene $\mathscr{F}$ is the inductive limit of the $\beta(\mathscr{N}_\lambda)$. As X is quasi-compact (Proposition 9.3.23) and $\mathscr{F}$ is of finite type, $\mathscr{F}$ then necessarily equal to one of the $\beta(\widetilde{\mathscr{N}}_\lambda)$ (**??**).

To define the $\mathscr{N}_\lambda$, it suffices to consider for each $n \in \mathbb{Z}$ the quasi-coherent $\mathscr{O}_Y$-module $\mathscr{M}_n$, which is the inductive limit of its sub-$\mathscr{O}_Y$-modules $\mathscr{M}_n^{(\mu_n)}$ of finite type (by Corollary 8.6.65). It is immediate that $\mathscr{P}_{\mu_n} = \mathscr{S} \cdot \mathscr{M}_n^{(\mu_n)}$ is a graded $\mathscr{S}_n$-module of finite type, and $\mathscr{M}$ is then the inductive limit of finite direct sums of these $\mathscr{S}$-modules. □

Corollary 9.3.27. *Suppose the hypotheses of Theorem 9.3.25 and that Y is quasi-compact. Let $\mathscr{F}$ be a quasi-coherent $\mathscr{O}_X$-module of finite type. There exists an integer n_0 such that for $n \geq n_0$, the canonical homomorphism $\sigma : p^*(p_*(\mathscr{F}(n))) \to \mathscr{F}(n)$ is surjective.*

Proof. For any $y \in Y$, let U be an affine open neighborhood of y in Y. There then exists an inter $n_0(U)$ such that, for $n \geq n_0(U)$, $\mathscr{F}(n)|_{p^{-1}(U)}$ is generated by finitely many sections over $p^{-1}(U)$ (Corollary 9.2.41); but these are canonical images of sections of $p^*(p_*(\mathscr{F}(n)))$ over $p^{-1}(U)$, so $\mathscr{F}(n)|_{p^{-1}(U)}$ is equal to the canonical image of $p^*(p_*(\mathscr{F}(n)))|_{p^{-1}(U)}$. Finally, as Y is quasi-compact, there is a finite affine open cover (U_i) of Y, and we can choose n_0 to be the largest of the $n_0(U_i)$. □

Remark 9.3.28. If $p : X \to Y$ is a morphism of ringed spaces and $\mathscr{F}$ is an $\mathscr{O}_X$-module, the fact that the canonical homomorphism $\sigma : p^*(p_*(\mathscr{F})) \to \mathscr{F}$ is surjective is explained as follows: for any $x \in X$ and any sections of $\mathscr{F}$ over an open neighborhood V of x, there exists an open neighborhood U of $p(x)$ in Y, finitely many sections $(t_i)_{1 \leq i \leq m}$ of $\mathscr{F}$ over $p^{-1}(U)$, a neighborhood $W \subseteq V \cap p^{-1}(U)$ of x and sections $(a_i)_{1 \leq i \leq m}$ of $\mathscr{O}_X$ over W such that

$$s|_W = \sum_i a_i \cdot (t_i|_W).$$

If Y is an affine scheme and $p_*(\mathscr{F})$ is *quasi-coherent*, this condition is equivalent to the fact that $\mathscr{F}$ is generated by its sections over X: in fact, if $Y = \operatorname{Spec}(A)$, we can suppose that $U = D(f)$ with $f \in A$. Since $p_*(\mathscr{F})$ is quasi-coherent, by Theorem 8.1.21 there exists an integer $n > 0$ and sections s_i of $\mathscr{F}$ over X such that $g^n t_i$ is the restriction of s_i (where $g = \rho(f)$, and $\rho : A \to \Gamma(X, \mathscr{O}_X)$ is the homomorphism corresponding to p by Proposition 8.2.4) to $p^{-1}(U)$. As g is invertible over $p^{-1}(U)$, we then have

$$s|_W = \sum_i b_i \cdot (s_i|_W)$$

where $b_i = a_i \cdot (g|_W)^{-n}$, whence our assertion. Therefore, if Y is affine, Corollary 9.3.27 then recovers Corollary 9.2.41, in view of **??**.

We finally conclude that if Y is a scheme, then the following three conditions are equivalent for a quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$ such that $p_*(\mathscr{F})$ is quasi-coherent:

(i) The canonical homomorphism $\sigma : p^*(p_*(\mathscr{F})) \to \mathscr{F}$ is surjective.

(ii) There is a quasi-coherent $\mathscr{O}_Y$-module $\mathscr{G}$ and a surjective homomorphism $p^*(\mathscr{G}) \to \mathscr{F}$.

(iii) For any affine open U of Y, $\mathscr{F}|_{p^{-1}(U)}$ is generated by its sections over $p^{-1}(U)$.

We have already established the equivalence of (i) and (iii), and (i) clearly implies (ii). conversely, any homomorphism $u : p^*(\mathscr{G}) \to \mathscr{F}$ factors into $p^*(\mathscr{G}) \to p^*(p_*(\mathscr{F})) \xrightarrow{\sigma} \mathscr{F}$ by (**??**), so if u is surjective, so is the canonical homomorphism $\sigma : p^*(p_*(\mathscr{F})) \to \mathscr{F}$.

Corollary 9.3.29. *Suppose the hypotheses of Theorem 9.3.25 and that Y is quasi-compact. Let $\mathscr{F}$ be a quasi-coherent $\mathscr{O}_X$-module of finite type. There then exists an integer n_0 such that for $n \geq n_0$, $\mathscr{F}$ is isomorphic to a quotient of an $\mathscr{O}_X$-module of the form $(p^*(\mathscr{G}))(-n)$, where $\mathscr{G}$ is a quasi-coherent $\mathscr{O}_Y$-module of finite type (dependent on n).*

Proof. As the structural morphism $X \to Y$ is separated and of finite type, $p_*(\mathscr{F}(n))$ is quasi-coherent by Proposition 8.6.55, and so is the inductive limits of its sub-$\mathscr{O}_Y$-modules of finite type, in view of Corollary 8.6.65. Since p^* commutes with inductive limits, we deduce from Corollary 9.3.27 and **??** that $\mathscr{F}(n)$ is the canonical image under $\sigma_{\mathscr{F}(n)}$ of an $\mathscr{O}_X$-module of the form $p^*(\mathscr{G})$, where $\mathscr{G}$ is a quasi-coherent sub-$\mathscr{O}_Y$-module of $p_*(\mathscr{F}(n))$ of finite type. The corollary then follows from Corollary 9.3.16 and Corollary 9.3.17. □

9.3.4 Functorial properties of $\operatorname{Proj}(\mathscr{S})$

Let Y be a scheme and $\mathscr{S}, \mathscr{S}'$ be two quasi-coherent graded $\mathscr{O}_Y$-algebras with positive degrees. Let $X = \operatorname{Proj}(\mathscr{S})$, $X' = \operatorname{Proj}(\mathscr{S}')$, and p, p' be the structural morphisms of X and X', respectively. Let $\varphi : \mathscr{S}' \to \mathscr{S}$ be an $\mathscr{O}_Y$-homomorphism of graded algebras. For any affine open U of Y, let $S_U = \Gamma(U, \mathscr{S})$, $S'_U = \Gamma(U, \mathscr{S}')$; the homomorphism φ defines a homomorphism $\varphi_U : S'_U \to S_U$ of graded A_U-algebras, where $A_U = \Gamma(U, \mathscr{O}_Y)$. It then corresponds to an open subset $G(\varphi_U)$ of $p^{-1}(U)$ and a morphism $\Phi_U : G(\varphi_U) \to p'^{-1}(U)$. Moreover, if $V \subseteq U$ is another affine open subset, the diagram

$$\begin{tikzcd} S'_U \arrow[r, "\varphi_U"] \arrow[d] & S_U \arrow[d] \\ S'_V \arrow[r, "\varphi_V"] & S_V \end{tikzcd} \tag{9.3.5}$$

is commutative, and we also verify, by the definition of $G(\varphi_U)$, that

$$G(\varphi_V) = G(\varphi_U) \cap p^{-1}(V)$$

and that Φ_V is the restriction of Φ_U to $G(\varphi_V)$. We thus define an open subset $G(\varphi)$ of X such that $G(\varphi) \cap p^{-1}(U) = G(\varphi_U)$ for any affine open $U \subseteq Y$, and an affine Y-morphism $\Phi : G(\varphi) \to X'$, which is called the morphism associated with φ and denoted by $\mathrm{Proj}(\varphi)$. If for any $y \in Y$, there exists an affine open neighborhood U of Y such that $\Gamma(U, \mathscr{O}_Y)$-module $\Gamma(U, \mathscr{S}_+)$ is generated by $\varphi(\Gamma(U, \mathscr{S}'_+))$, we then have $G(\varphi_U) = p^{-1}(U)$, and thus $G(\varphi) = X$.

Proposition 9.3.30. *Let $\mathscr{M}$ (resp. $\mathscr{M}'$) be a quasi-coherent graded $\mathscr{S}$-module (resp. $\mathscr{S}'$-module). Then there exist a canonical isomorphism $\widetilde{\varphi^*(\mathscr{M})} \xrightarrow{\sim} \Phi_*(\widetilde{\mathscr{M}}|_{G(\varphi)})$ of $\mathscr{O}_{X'}$-modules and a canonical homomorphism ν : $\Phi^*(\widetilde{\mathscr{M}'}) \to \widetilde{\varphi_*(\mathscr{M}')}|_{G(\varphi)}$. If $\mathscr{S}'$ is generated by $\mathscr{S}'_1$, ν is an isomorphism.*

Proof. The homomorphisms considered are in fact already defined locally over Y (see Proposition 9.2.47 and Proposition 9.2.48), and the general case then follows from their compatibility with restrictions, and diagram (9.3.5). □

In particular, for any $n \in \mathbb{Z}$, we have a canonical homomorphism $\Phi^*(\mathscr{O}_{X'}(n)) \to \mathscr{O}_X(n)|_{G(\varphi)}$, and this is a homomorphism if $\mathscr{S}'$ is generated by $\mathscr{S}'_1$.

Proposition 9.3.31. *Let Y, Y' be schemes, $\psi : Y' \to Y$ be a morphism, $\mathscr{S}$ be a quasi-coherent $\mathscr{O}_Y$-algebra, and put $\mathscr{S}' = \psi^*(\mathscr{S})$. Then the Y'-scheme $X' = \mathrm{Proj}(\mathscr{S}')$ is canonically identified with $\mathrm{Proj}(\mathscr{S}) \times_Y Y'$. Moreover, if $\mathscr{M}$ is a quasi-coherent graded $\mathscr{S}$-module, the $\mathscr{O}_{X'}$-module $\widetilde{\psi^*(\mathscr{M})}$ is identified with $\widetilde{\mathscr{M}} \otimes_Y \mathscr{O}_{Y'}$.*

Proof. We first note that $\psi^*(\mathscr{S})$ and $\psi^*(\mathscr{M})$ are quasi-coherent $\mathscr{O}_{Y'}$-modules. Let U be an affine open of Y, $U' \subseteq \psi^{-1}(U)$ an affine open of Y', and A, A' the ring of U, U', respectively. We then have $\mathscr{S}|_U = \widetilde{S}$ where S is a graded A-algebra, and $\mathscr{S}'|_{U'}$ is identified with $\widetilde{S \otimes_A A'}$ by Proposition 8.1.14. The first assertion then follows from Proposition 9.2.50 and Corollary 8.3.2, since we can easily verify that the projection $\mathrm{Proj}(\mathscr{S}'|_{U'}) \to \mathrm{Proj}(\mathscr{S}|_U)$ defined by this identification is compatible with restrictions over U and U' and therefore define a morphism $\mathrm{Proj}(\mathscr{S}') \to \mathrm{Proj}(\mathscr{S})$. Now let $p : \mathrm{Proj}(\mathscr{S}) \to Y$ and $p' : \mathrm{Proj}(\mathscr{S}') \to Y'$ be the structural morphisms; $p'^{-1}(U')$ is identified with $p^{-1}(U) \times_U U'$, and the two sheaves $\widetilde{\psi^*(\mathscr{M})}|_{p'^{-1}(U')}$ and $\widetilde{\mathscr{M}} \otimes_Y \mathscr{O}_{Y'}|_{p'^{-1}(U')}$ are then canonically identified to $\widetilde{M \otimes_A A'}$, where $M = \Gamma(U, \mathscr{M})$, in view of Proposition 9.2.50 and Proposition 8.1.14; whence the second assertion, since these identifications are compatible with restrictions. □

Corollary 9.3.32. *With the notations of Proposition 9.3.31, $\mathscr{O}_{X'}(n)$ is canonically identified with $\mathscr{O}_X(n) \otimes_Y \mathscr{O}_{Y'}$ for any $n \in \mathbb{Z}$.*

Proof. With the notations of Proposition 9.3.31, it is clear that $\psi^*(\mathscr{S}(n)) = \mathscr{S}'(n)$ for any $n \in \mathbb{Z}$, whence the corollary. □

Retain the notations in Proposition 9.3.31, denote by $\Psi : X' \to X$ the canonical projection, and put $\mathscr{M}' = \psi^*(\mathscr{M})$. We suppose that $\mathscr{S}$ is generated by $\mathscr{S}_1$ and that X is of finite type over Y (for example if $\mathscr{S}_1$ is of finite type, cf. Proposition 9.3.23). Then $\mathscr{S}'$ is generated by $\mathscr{S}'_1$ (as can be checked locally on affine opens of Y) and X' is of finite type over Y by Proposition 8.6.35. Let $\mathscr{F}$ be an $\mathscr{O}_X$-module and set $\mathscr{F}' = \Psi^*(\mathscr{F})$; it then follows from Corollary 9.3.32 that we have $\mathscr{F}'(n) = \Psi^*(\mathscr{F})$ for each $n \in \mathbb{Z}$. We define a canonical Ψ-homomorphism $\theta_n : p_*(\mathscr{F}(n)) \to p'_*(\mathscr{F}'(n))$ as follows: from the commutative diagram

$$\begin{tikzcd} X' \arrow[r, "\Psi"] \arrow[d, "p'"'] & X \arrow[d, "p"] \\ Y' \arrow[r, "\psi"] & Y \end{tikzcd}$$

we see that it suffices to define a homomorphism

$$p_*(\mathscr{F}(n)) \to \psi_*(q'_*(\Psi^*(\mathscr{F}(n)))) = q_*(\Psi_*(\Psi^*(\mathscr{F}(n)))),$$

and we can take $\theta_n = p_*(\rho_n)$, where ρ_n is the canonical homomorphism $\rho_n : \mathscr{F}(n) \to \Psi_*(\Psi^*(\mathscr{F}(n)))$. It is immediate that for any affine open U of Y and any affine open U' of Y such that $U' \subseteq \psi^{-1}(U)$, the homomorphism θ_n thus defined gives a canonical homomorphism $\Gamma(p^{-1}(U), \mathscr{F}(n)) \to \Gamma(p'^{-1}(U'), \mathscr{F}'(n))$, and the commutative diagram (9.2.12) shows that if $\mathscr{F}$ is quasi-coherent, the diagram

$$\begin{tikzcd}
\widetilde{\Gamma_*(\mathscr{F})} \arrow[r, "\tilde{\theta}"] \arrow[d, "\beta_{\mathscr{F}}"'] & \widetilde{\Gamma_*(\mathscr{F}')} \arrow[d, "\beta_{\mathscr{F}'}"] \\
\mathscr{F} \arrow[r, "\rho"] & \mathscr{F}'
\end{tikzcd}$$

is commutative (where the send row is the canonical Ψ-morphism $\mathscr{F} \to \Phi^*(\mathscr{F})$).

Similarly, the commutative diagram (9.2.11) shows that the diagram

$$\begin{tikzcd}
\mathscr{M} \arrow[r, "\rho"] \arrow[d, "\alpha_{\mathscr{M}}"] & \mathscr{M}' \arrow[d, "\alpha_{\mathscr{M}'}"] \\
\Gamma_*(\widetilde{\mathscr{M}}) \arrow[r, "\theta"] & \Gamma_*(\widetilde{\mathscr{M}'})
\end{tikzcd}$$

is commutative (where the first row is the canonical ψ-morphism $\mathscr{M} \to \psi^*(\mathscr{M})$).

Consider now a morphism $\psi : Y' \to Y$ of schemes, a quasi-coherent graded $\mathscr{O}_Y$-algebra (resp. $\mathscr{O}_{Y'}$-algebra) $\mathscr{S}$ (resp. $\mathscr{S}'$), and a ψ-morphism $u : \mathscr{S} \to \mathscr{S}'$ of graded algebras. This is equivalent to giving an $\mathscr{O}_{Y'}$-homomorphism of graded algebras $u^\sharp : \psi_*(\mathscr{S}) \to \mathscr{S}'$, and we deduce from $u^\sharp$ an Y'-morphism

$$w = \mathrm{Proj}(u^\sharp) : G(u^\sharp) \to \mathrm{Proj}(\psi^*(\mathscr{S})),$$

where $G(u^\sharp)$ is an open subset of $X' = \mathrm{Proj}(\mathscr{S}')$. On the other hand, $\mathrm{Proj}(\psi^*(\mathscr{S}))$ is canonically identified with $X \times_Y Y'$, where $X = \mathrm{Proj}(\mathscr{S})$ (Proposition 9.3.31). By composing the morphism $\mathrm{Proj}(u^\sharp)$ with the first projection $\pi : X \times_Y Y' \to X$, we then obtain a morphism $v = \mathrm{Proj}(u) : G(u^\sharp) \to X$ such that the diagram

$$\begin{tikzcd}
G(u^\sharp) \arrow[r, "v"] \arrow[d] & X \arrow[d] \\
Y' \arrow[r, "\psi"] & Y
\end{tikzcd} \tag{9.3.6}$$

is commutative. Moreover, for any quasi-coherent $\mathscr{O}_Y$-module $\mathscr{M}$, we have a canonical v-morphism

$$\nu : \widetilde{\mathscr{M}} \to (\psi^*(\mathscr{M}) \otimes_{\psi^*(\mathscr{S})} \mathscr{S}')^\sim|_{G(u^\sharp)} \tag{9.3.7}$$

such that $\nu^\sharp$ is the composition

$$v^*(\widetilde{\mathscr{M}}) = w^*(\pi^*(\widetilde{\mathscr{M}})) \xrightarrow{\sim} w^*(\widetilde{\psi^*(\mathscr{M})}) \xrightarrow{\nu} (\psi^*(\mathscr{M}) \otimes_{\psi^*(\mathscr{M})} \mathscr{S}')^\sim|_{G(u^\sharp)} \tag{9.3.8}$$

where the first arrow is the isomorphism in Proposition 9.3.31 and the second one is the homomorphism ν of Proposition 9.3.30. If $\mathscr{S}$ is generated by $\mathscr{S}_1$, then it follows from Proposition 9.3.30 that $\nu^\sharp$ is an isomorphism. As a particular case, for any $n \in \mathbb{Z}$ we have a canonical v-homomorphism

$$\nu : \mathscr{O}_X(n) \to \mathscr{O}_{X'}(n)|_{G(u^\sharp)}. \tag{9.3.9}$$

9.3.5 Closed subschemes of $\mathrm{Proj}(\mathscr{S})$

Let Y be a scheme and $\varphi : \mathscr{S} \to \mathscr{S}'$ be a homomorphism of quasi-coherent graded $\mathscr{O}_Y$-algebras of degree 0. We say that φ is **eventually injective** (resp. **eventually surjective**, **eventually bijective**) if there exists an integer n such that, for $k \geq n$, $\varphi_k : \mathscr{S}_k \to \mathscr{S}'_k$ is surjective (resp. injective, bijective). If this is the case, we can then reduce the study of the morphism $\Phi : \mathrm{Proj}(\mathscr{S}') \to \mathrm{Proj}(\mathscr{S})$ to the case where φ is surjective (resp. injective, bijective) (this follows from Proposition 9.3.6). If φ is eventually bijective, we also say that φ is an **eventual isomorphism**.

Proposition 9.3.33. *Let Y be a scheme, $\mathscr{S}$ be a quasi-coherent $\mathscr{O}_Y$-algebra, and $X = \mathrm{Proj}(\mathscr{S})$.*

(a) *If $\varphi : \mathscr{S} \to \mathscr{S}'$ is a eventually surjective homomorphism of graded $\mathscr{O}_Y$-algebra, the corresponding morphism $\Phi = \mathrm{Proj}(\varphi)$ is defined over $\mathrm{Proj}(\mathscr{S}')$ and is a closed immersion from $\mathrm{Proj}(\mathscr{S}')$ into X. If $\mathscr{I}$ is the kernel of φ, the closed subscheme of X associated with Φ is defined by the quasi-coherent ideal $\widetilde{\mathscr{I}}$ of $\mathscr{O}_X$.*

(b) *Suppose moreover that $\mathscr{S}_0 = \mathscr{O}_Y$ and $\mathscr{S}$ is generated by $\mathscr{S}_1$ where $\mathscr{S}_1$ is of finite type. Let X' be a closed subscheme of X defined by a quasi-coherent ideal $\mathscr{I}$ of $\mathscr{O}_X$. Let $\mathscr{J}$ be the inverse image of $\Gamma_*(\mathscr{I})$ under the canonical homomorphism $\alpha : \mathscr{S} \to \Gamma_*(\mathscr{O}_X)$ and put $\mathscr{S}' = \mathscr{S}/\mathscr{J}$. Then X' is the subscheme associated with the closed immersion $\mathrm{Proj}(\mathscr{S}') \to X$ corresponding to the canonical homomorphism $\mathscr{S} \to \mathscr{S}'$ of graded $\mathscr{O}_Y$-algebras.*

Proof. For the assertion of (a), we can assume that φ is surjective. Then for any affine open U of Y, $\Gamma(U, \mathscr{S}) \to \Gamma(U, \mathscr{S}')$ is surjective by Corollary 8.1.6, so we have $G(\varphi) = X$. We are immediately reduced to the case where Y is affine, and the assertion follows from Proposition 9.2.52(a).

As for case (b), we are reduced to prove that the homomorphism $\widetilde{\mathscr{J}} \to \mathscr{O}_X$ induced from the canonical injection $\mathscr{J} \to \mathscr{S}$ is an isomorphism from $\widetilde{\mathscr{J}}$ to $\mathscr{I}$. As the question is local, we can assume that Y is affine with ring A, which implies $\mathscr{S} = \widetilde{S}$, where S is a graded A-algebra generated by S_1 and S_1 is of finite type over A. It then suffices to apply Proposition 9.2.52(b). $\square$

Corollary 9.3.34. *Under the hypotheses of Proposition 9.3.33(a), suppose that $\mathscr{S}$ is generated by $\mathscr{S}_1$. Then $\Phi^*(\mathscr{O}_X(n))$ is canonically identified with $\mathscr{O}_{X'}(n)$ for any $n \in \mathbb{Z}$.*

Proof. We have defined such an isomorphism if Y is affine; in the general case, it suffices to verify that these isomorphisms are compatible with restrictions. $\square$

Corollary 9.3.35. *Let Y be a scheme, $\mathscr{S}$ be a quasi-coherent $\mathscr{O}_Y$-algebra generated by $\mathscr{S}_1$, $\mathscr{M}$ be a quasi-coherent $\mathscr{O}_Y$-module, and $u : \mathscr{M} \to \mathscr{S}_1$ be a surjective $\mathscr{O}_Y$-homomorphism. If $\bar{u} : \boldsymbol{S}(\mathscr{M}) \to \mathscr{S}$ is the canonical homomorphism of $\mathscr{O}_Y$-algebras extending u, then the morphism corresponding to $\bar{u}$ is a closed immersion from $\mathrm{Proj}(\mathscr{S})$ into $\mathrm{Proj}(\boldsymbol{S}(\mathscr{M}))$.*

Proof. In fact, $\bar{u}$ is surjective by hypothesis, so we can apply Proposition 9.3.33(a). $\square$

9.3.6 Morphisms into $\mathrm{Proj}(\mathscr{S})$

Let $q : X \to Y$ be a morphism of schemes, $\mathscr{L}$ be an invertible $\mathscr{O}_X$-module, and $\mathscr{S}$ be a quasi-coherent $\mathscr{O}_Y$-algebra with positive degrees. Then $q^*(\mathscr{S})$ is a quasi-coherent graded $\mathscr{O}_X$-algebra with positive degrees. Suppose that we are given a graded homomorphism of $\mathscr{O}_X$-algebras

$$\psi : q^*(\mathscr{S}) \to \boldsymbol{S}(\mathscr{L}) = \bigoplus_{n \geq 0} \mathscr{L}^{\otimes n}$$

(recall that for an invertible $\mathscr{O}_X$-module $\mathscr{L}$ we have $\boldsymbol{T}(\mathscr{L}) = \boldsymbol{S}(\mathscr{L})$) or equivalently, a $\mathscr{O}_Y$-homorphism of graded algebras

$$\psi^\flat : \mathscr{S} \to q_*(\boldsymbol{S}(\mathscr{L})).$$

Since $\mathscr{L}$ is invertible and $\boldsymbol{T}(\mathscr{O}_X) = \boldsymbol{S}(\mathscr{O}_X) = \mathscr{O}_X[T]$, by Proposition 9.3.6 and Corollary 9.3.5 we know that $\mathrm{Proj}(\boldsymbol{S}(\mathscr{L}))$ is canonically identified with X. We then conclude that the homomorphism ψ induces a Y-morphism

$$r_{\mathscr{L},\psi} : G(\psi) \to \mathrm{Proj}(\mathscr{S}) = P,$$

where $G(\psi)$ is an open subset of X. Recall that this morphism is by definition obtained by composing the first projection $\pi : \mathrm{Proj}(q^*(\mathscr{S})) = P \times_Y X \to P$ with the Y-morphism $\tau = \mathrm{Proj}(\psi) : G(\psi) \to \mathrm{Proj}(q^*(\mathscr{S}))$, which is shown in the following diagram:

$$\begin{tikzcd}
 & P \times_Y X \arrow[r, "\pi"] \arrow[d] & P \arrow[d, "p"] \\
G(\psi) \arrow[ur, "\tau=\mathrm{Proj}(\psi)"] \arrow[urr, bend left, "r_{\mathscr{L},\psi}"] \arrow[r] & X \arrow[r, "q"] & Y
\end{tikzcd}$$

Remark 9.3.36. Let us explain the morphism $r = r_{\mathscr{L},\psi}$ when $Y = \mathrm{Spec}(A)$ is affine, and therefore $\mathscr{S} = \widetilde{S}$, where S is a graded A-algebra with positive degrees. First suppose that $X = \mathrm{Spec}(B)$ is also affine and $\mathscr{L} = \widetilde{L}$, where L is a free B-module of rank 1. We then have $q^*(\mathscr{S}) = \widetilde{S \otimes_A B}$ by Proposition 8.1.14. If c is a generator of L, the homomorphism $\psi_n : q^*(\mathscr{S}_n) \to \mathscr{L}^{\otimes n}$ then corresponds to a B-homomorphism

$$w_n : S_n \otimes_A B \to L^{\otimes n}, \quad s \otimes b \mapsto b v_n(s) c^{\otimes n}, \tag{9.3.10}$$

where $v_n : S_n \to B$ is the n-th component of a homomorphism $v : S \to B$ of algebras. Let $f \in S_d$ with positive degree and put $g = v_d(f)$. We have $\pi^{-1}(D_+(f)) = D_+(f \otimes 1)$ in view of Proposition 9.2.50 and the identification of $D_+(f)$ with $\mathrm{Spec}(S_{(f)})$. On the other hand, the formula (9.2.9) and (9.3.10) shows that (using the canonical isomorphism of X and $\mathrm{Proj}(\boldsymbol{S}(\mathscr{L}))$)

$$\tau^{-1}(D_+(f \otimes 1)) = D(g)$$

whence $r^{-1}(D_+(f)) = D(g)$. Furthermore, the restriction of the morphism $\tau = \mathrm{Proj}(\psi)$ to $D(g)$ corresponds to the homomorphism $(S \otimes_A B)_{(f \otimes 1)} \to B_g$, which send $(s \otimes 1)/(f \otimes 1)^n$ (for $s \in S_{nd}$) to $v_{nd}(s)/g^n$, and the restriction of the projection π to $D_+(f \otimes 1)$ corresponds to the homomorphism $S_{(f)} \to (S \otimes_A B)_{(f \otimes 1)}$ given by $s/f^n \mapsto (s \otimes 1)/(f \otimes 1)^n$. We then conclude that the restriction of the morphism r to $D(g)$ corresponds to the homomorphisms $\omega : S_{(f)} \to B_g$ of A-algebras such that $\omega(s/f^n) = v_{nd}(s)/g^n$ for $s \in S_{nd}$.

Proposition 9.3.37. *Let $Y = \mathrm{Spec}(A)$ be affine and $\mathscr{S} = \widetilde{S}$, where S is a graded A-algebra. For any $f \in S_d = \Gamma(Y, \mathscr{S}_d)$, we have (where $\psi^\flat(f) \in \Gamma(X, \mathscr{L}^{\otimes d})$)*

$$r_{\mathscr{L},\psi}^{-1}(D_+(f)) = X_{\psi^\flat(f)}. \tag{9.3.11}$$

Moreover, under the canonical isomorphism of X and $\mathrm{Proj}(\boldsymbol{S}(\mathscr{L}))$, the restriction morphism $r_{\mathscr{L},\psi} : X_{\psi^\flat(f)} \to D_+(f) = \mathrm{Spec}(S_{(f)})$ corresponds to the homomorphism

$$\psi^\flat_{(f)} : S_{(f)} \to \Gamma(X_{\psi^\flat(f)}, \mathscr{O}_X)$$

such that, for any $s \in S_{nd} = \Gamma(Y, \mathscr{S}_{nd})$, we have

$$\psi^\flat_{(f)}(s/f^n) = (\psi^\flat(s)|_{X_{\psi^\flat(f)}})(\psi^\flat(f)|_{X_{\psi^\flat(f)}})^{-n}.$$

Proof. This follows from Remark 9.3.36 by passing to the general case. ☐

We say the morphism $r_{\mathscr{L},\psi}$ is **everywhere defined** if $G(\psi) = X$. For this to be the case, it is necessary and sufficient that $G(\psi) \cap q^{-1}(U) = q^{-1}(U)$ for any affine open $U \subseteq Y$, so this question is local over Y. If Y is affine, $G(\psi)$ is then the union of $r^{-1}(D_+(f))$ for $f \in S_+$, so by (9.3.11) the $X_{\psi^\flat(f)}$ then form a covering of X. In other words:

Corollary 9.3.38. *Under the hypotheses of Proposition 9.3.37, for the morphism $r_{\mathscr{L},\psi}$ to be everywhere defined, it is necessary and sufficient that for any $x \in X$, there exists an integer $n > 0$ and a section $s \in S_n$ such that $t = \psi^\flat(s) \in \Gamma(X, \mathscr{L}^{\otimes n})$ is nonzero at x. In particular, this is true if ψ is eventually surjective.*

Corollary 9.3.39. *Under the hypotheses of Proposition 9.3.37, for the morphism $r_{\mathscr{L},\psi}$ to be dominant, it is necessary and sufficient that for any integer $n > 0$, any section $s \in S_n$ such that $\psi^\flat(b) \in \Gamma(X, \mathscr{L}^{\otimes n})$ is locally nilpotent, is itself nilpotent.*

Proof. We must check that $r_{\mathscr{L},\psi}^{-1}(D_+(s))$ is nonempty if $D_+(s)$ is nonempty, and the corollary follows from (9.3.11) and Corollary 9.2.6. ☐

Proposition 9.3.40. *Let $q : X \to Y$ be a morphism, $\mathscr{L}$ be an invertible $\mathscr{O}_X$-module, and $\mathscr{S}$, $\mathscr{S}'$ be quasi-coherent graded $\mathscr{O}_Y$-algebras. Let $u : \mathscr{S}' \to \mathscr{S}$ be a homomorphism of graded algebras, $\psi : q^*(\mathscr{S}) \to \boldsymbol{S}(\mathscr{L})$ be a homomorphism of graded algebras, and $\psi' = \psi \circ q^*(u)$ be the composition.*

(i) *If $r_{\mathscr{L},\psi'}$ is everywhere defined, then $r_{\mathscr{L},\psi}$ is everywhere defined;*

(ii) *If u is eventually surjective and $r_{\mathscr{L},\psi'}$ is dominant, then $r_{\mathscr{L},\psi}$ is dominant;*

(iii) *If u is eventually injective and $r_{\mathscr{L},\psi}$ is dominant, then $r_{\mathscr{L},\psi}$ is dominant.*

Proof. We have $G(\psi') \subseteq G(\psi)$, whence the first assertion. If u is eventually surjective, $\mathrm{Proj}(u) : \mathrm{Proj}(S) \to \mathrm{Proj}(S')$ is everywhere defined and is a closed immersion; as $r_{\mathscr{L},\psi'}$ is the composition of $\mathrm{Proj}(u)$ and the restriction of $r_{\mathscr{L},\psi}$ to $G(\psi')$, we then conclude that if $r_{\mathscr{L},\psi'}$ is dominant, so is $r_{\mathscr{L},\psi}$. Finally, if u is eventually injective, then $\mathrm{Proj}(u)$ is a dominant morphism from $G(u)$ into $\mathrm{Proj}(\mathscr{S}')$ (Corollary 9.2.45); as $G(\psi')$ is the inverse image of $G(u)$ under $r_{\mathscr{L},\psi}$, we see that if $r_{\mathscr{L},\psi}$ is dominant, so is $r_{\mathscr{L},\psi'}$. □

Proposition 9.3.41. *Let Y be a quasi-compact scheme, $q : X \to Y$ be a quasi-compact morphism, $\mathscr{L}$ be an invertible $\mathscr{O}_X$-module, and $\mathscr{S}$ be a quasi-coherent $\mathscr{O}_Y$-algebra which is the filtered limit of a system $(\mathscr{S}^\lambda)$ of quasi-coherent $\mathscr{O}_Y$-algebras. Let $\varphi_\lambda : \mathscr{S}^\lambda \to \mathscr{S}$ be the canonical homomorphism, $\psi : q^*(\mathscr{S}) \to \boldsymbol{S}(\mathscr{L})$ be a homomorphism of graded algebras, and put $\psi_\lambda = \psi \circ q^*(\varphi_\lambda)$. Then for the morphism $r_{\mathscr{L},\psi}$ to be everywhere defined, it is necessary and sufficient that there exists an index λ such that $r_{\mathscr{L},\psi_\lambda}$ is everywhere defined; in this case, $r_{\mathscr{L},\psi_\mu}$ is everywhere defined for $\mu \geq \lambda$.*

Proof. The condition is sufficient in view of Proposition 9.3.40. Conversely, suppose that $r_{\mathscr{L},\psi}$ is everywhere defined; we can assume that Y is affine, because if for any affine open $U \subseteq Y$ there exists $\lambda(U)$ such that the restriction of $r_{\mathscr{L},\psi_{\lambda_{\lambda(U)}}}$ to $q^{-1}(U)$ is defined everywhere, it then suffices to cover Y by finitely many affine opens U_i (recall that Y is quasi-compact) and choose $\lambda \geq \lambda(U_i)$ for all i, by Proposition 9.3.40. If Y is affine (so $\mathscr{S} = \widetilde{S}$ where $S = \Gamma(Y,\mathscr{S})$) the hypotheses implies that for any $x \in X$, there exists a section $s^{(x)} \in S_n$ for some integer n such that, if $t^{(x)} = \psi^\flat(s^{(x)})$, then $t^{(x)}(x) \neq 0$ (where $t^{(x)}$ is a section of $\mathscr{L}^{\otimes n}$ over X), which implices $t^{(x)}(z) \neq 0$ for z in a neighborhood $V(x)$ of X. As the morphism $q : X \to Y$ is quasi-compact, we see X is quasi-compact, so we can cover X by finitely many $V(x_i)$ and let $s^{(i)}$ be the corresponding section of S. There is then an index λ such that $s^{(i)}$ is of the form $\varphi_\lambda(s_\lambda^{(i)})$, where $s_\lambda^{(i)} \in S^\lambda$ for all i, and it follows from Corollary 9.3.38 that $r_{\mathscr{L},\psi_\lambda}$ is everywhere defined. The second assertion is trival by Proposition 9.3.40. □

Corollary 9.3.42. *Under the hypotheses of Proposition 9.3.41, if the morphisms $r_{\mathscr{L},\psi_\lambda}$ are dominant, so is $r_{\mathscr{L},\psi}$. The converse is also true if the homomorphisms φ_λ are eventually injective.*

Proof. The second assertion is a particular case of Proposition 9.3.40. To show that $r_{\mathscr{L},\psi}$ is dominant if each $r_{\mathscr{L},\psi_\lambda}$ is, we can assume that Y is affine and thus $\mathscr{S} = \widetilde{S}$ where $S = \Gamma(Y,\mathscr{S})$. If $s \in S$ is such that $\psi^\flat(s)$ is locally nilpotent, as we can write $s = \varphi_\lambda(s_\lambda)$ for some λ, from the definition of ψ_λ and by Corollary 9.3.39, we conclude that s_λ is nilpotent, so s is nilpotent, and the assertion follows by applying Corollary 9.3.39. □

Remark 9.3.43. With the hypotheses and notations of Proposition 9.3.37, for each $n \in \mathbb{Z}$ we have a homomorphism

$$\nu : r_{\mathscr{L},\psi}^*(\mathscr{O}_P(n)) \to \mathscr{L}^{\otimes n}|_{G(\psi)} \tag{9.3.12}$$

which is in fact the homomorphism ν defined in (9.3.7) on $\mathscr{O}_P(n)$. We also see that under the hypotheses of Proposition 9.3.37, the restriction of ν to $X_{\psi^\flat(f)}$ corresponds to the homomorphism sending the element s/f^k (with $s \in S_{n+kd}$) to the section

$$(\psi^\flat(s)|_{X_{\psi^\flat(f)}})(\psi^\flat(f)|_{X_{\psi^\flat(f)}})^{-k} \in \Gamma(X_{\psi^\flat(f)}, \mathscr{L}^{\otimes n}),$$

where we also use the notations of Proposition 9.3.37.

Remark 9.3.44. Let $\mathscr{F}$ be a quasi-coherent $\mathscr{O}_X$-module and suppose that q is quasi-compact and quasi-separated, so for each $n \geq 0$, $q_*(\mathscr{F} \otimes \mathscr{L}^{\otimes n})$ is a quasi-coherent $\mathscr{O}_Y$-module (Proposition 8.6.55). Let $\mathscr{M}' = \bigoplus_{n\geq 0} \mathscr{F} \otimes \mathscr{L}^{\otimes n}$, which is a quasi-coherent $\mathscr{O}_Y$-module, and consider the image $\mathscr{M} = q_*(\mathscr{M}') = \bigoplus_{n\geq 0} q_*(\mathscr{F} \otimes \mathscr{L}^{\otimes n})$ (which is a quasi-coherent $\mathscr{S}$-module via the homomorphism $\psi^\flat$). We shall see that there is a canonical homomorphism of $\mathscr{O}_X$-modules

$$\xi : r_{\mathscr{L},\psi}^*(\widetilde{\mathscr{M}}) \to \mathscr{F}|_{G(\psi)}. \tag{9.3.13}$$

For this, recall that we have defined a canonical homomorphism (9.3.7):

$$\nu : r_{\mathscr{L},\psi}^*(\widetilde{\mathscr{M}}) \to (q^*(\mathscr{M}) \otimes_{q^*(\mathscr{S})} \boldsymbol{S}(\mathscr{L}))^\sim|_{G(\psi)},$$

where the right hand side is considered as a quasi-coherent sheaf over X. On the other hand, for any quasi-coherent graded $\boldsymbol{S}(\mathscr{L})$-module $\mathscr{M}'$, we have a canonical homomorphism

$$q^*(q_*(\mathscr{M}'))\otimes_{q^*(\mathscr{S})}\boldsymbol{S}(\mathscr{L})\to\mathscr{M}'$$

which, for any open subset U of X, any section t' of $q^*(q_*(\mathscr{M}'_h))$ over U and any section b' of $\mathscr{L}^{\otimes k}$ over U, sends the section $t'\otimes b'$ to the section $b'\sigma(t')$ of $\mathscr{M}'_{h+k}$, where σ is the canonical homomorphism $q^*(q_*(\mathscr{M}'))\to\mathscr{M}'$. We then conclude a canonical homomorphism

$$(q^*(q_*(\mathscr{M}'))\otimes_{q^*(\mathscr{S})}\boldsymbol{S}(\mathscr{L}))^{\sim}|_{G(\psi)}\to\widetilde{\mathscr{M}}'|_{G(\psi)}$$

and as $\widetilde{\mathscr{M}}'$ is canonically identified with $\mathscr{F}$ by Remark 9.3.18, we obtain the canonical homomorphism ξ.

Under the hypotheses and notations of Proposition 9.3.20, the restriction of this homomorphism to $X_{\psi^\flat(f)}$ is defined as follows: giving a section t_{nd} of $\mathscr{F}\otimes\mathscr{L}^{\otimes d}$ over X (which is also a section of $q_*(\mathscr{F}\otimes\mathscr{L}^{\otimes n})$ over Y), we send the element t_{nd}/f^n to the sectin $(t_{nd}|_{X_{\psi^\flat(f)}})(\psi^\flat(f)|_{X_{\psi^\flat(f)}})^{-n}$ of $\mathscr{F}$ over $X_{\psi^\flat(f)}$.

We now consider the important question that whether the induced morphism $r_{\mathscr{L},\psi}$ is an immersion (reps. an open immersion, a closed immersion). It is clear that this question is local over Y, and we shall give a criterion in this situation together with the condition that $r_{\mathscr{L},\psi}$ is defined everywhere.

Proposition 9.3.45. *Under the hypothesis and notations of Proposition 9.3.37, for the morphism $r_{\mathscr{L},\psi}$ be everywhere defined and an immersion, it is necessary and sufficient that there exists a family of sections $s_\alpha\in S_{n_\alpha}$ (with $n_\alpha>0$) such that, if $f_\alpha=\psi^\flat(s_\alpha)$, the following conditions are satisfied:*

(i) *The X_{f_α} form a covering of X.*

(ii) *The X_{f_α} are affine open subset of X.*

(iii) *For any index α and any section $t\in\Gamma(X_{f_\alpha},\mathscr{O}_X)$, there exists an integer $n>0$ and $s\in S_{mn_\alpha}$ such that $t=(\psi^\flat(s)|_{X_{f_\alpha}})(f_\alpha|_{X_{f_\alpha}})^{-m}$.*

For the morphism $r_{\mathscr{L},\psi}$ to be everywhere defined and an open immersion, it is necessary and sufficient that the family (s_α) satisfies the following additional condition:

(iv) *For any integer $m>0$ and any $s\in S_{mn_\alpha}$ such that $\psi^\flat(s)|_{X_{f_\alpha}}=0$, there exists an integer $n>0$ such that $s_\alpha^n s=0$.*

Similarly, for the morphism $r_{\mathscr{L},\psi}$ to be everywhere defined and a closed immersion, it is necessary and sufficient that the family (s_α) satisfies the following additional condition:

(v) *The $D_+(s_\alpha)$ form a covering of $P=\mathrm{Proj}(S)$.*

Proof. By Proposition 8.4.9, for the morphism $r=r_{\mathscr{L},\psi}$ to be an immersion (resp. a closed immersion), it is necessary and sufficent that there exists a covering of $r(G(\psi))$ (resp. of P) by the sets $D_+(s_\alpha)$ such that if $V_\alpha=r^{-1}(D_+(s_\alpha))$, the restriction of r on V_α is a closed immersion of V_α into $D_+(s_\alpha)$ (cf. Corollary 8.4.11). Now condition (i) just means that r is everywhere defined and that $D_+(s_\alpha)$ cover $r(X)$, by (9.3.11). As each $D_+(s_\alpha)$ is affine, condition (ii) and (iii) express that the restriction of r to X_{f_α} is a closed immersion into $D_+(s_\alpha)$ (Example 8.4.6). Finally, as (iii) and (iv) means the ring homomorphism $\psi^\flat_{(s_\alpha)}:S_{(s_\alpha)}\to\Gamma(X_{f_\alpha},\mathscr{O}_X)$ is an isomorphism, (ii), (iii), (iv) mean that the restriction of r to X_{f_α} is an isomorphism from X_{f_α} to $D_+(s_\alpha)$ for each α, so together with (i), they mean that r is an open immersion. □

Corollary 9.3.46. *Under the hypothesis and notations of Proposition 9.3.40, if $r_{\mathscr{L},\psi'}$ is everywhere defined and is an immersion, so is $r_{\mathscr{L},\psi}$. If we suppose that u is eventually surjective and if $r_{\mathscr{L},\psi'}$ is everywhere defined and is a closed immersion (resp. open), then so is $r_{\mathscr{L},\psi}$.*

Proof. We first suppose that $r_{\mathscr{L},\psi'}$ is everywhere defined and and is an immersion. Then by Proposition 9.3.45, there is a family $s'_\alpha \in S'_{n_\alpha}$ such that, if $f_\alpha = \psi'^\flat(s'_\alpha)$, the conditions (i), (ii), (iii) are satisfied. Set $s_\alpha = u(s'_\alpha)$, then $f_\alpha = \psi^\flat(s_\alpha)$, and we have a commutative diagram

$$\begin{tikzcd}
S'_{(s'_\alpha)} \arrow[r, "\psi'^\flat_{(s'_\alpha)}"] \arrow[d, "u_{(s'_\alpha)}"'] & \Gamma(X_{f_\alpha}, \mathscr{O}_X) \\
S_{(s_\alpha)} \arrow[ur, "\psi^\flat_{(s_\alpha)}"'] &
\end{tikzcd}$$

The hypothesis then implies that $\psi'^\flat_{(s'_\alpha)}$ is surjective, so the homomorphism $\psi^\flat_{(s_\alpha)}$ is also surjective. This shows that $r_{\mathscr{L},\psi}$ is everywhere defined and is an immersion, in view of Proposition 9.3.45. If $r_{\mathscr{L},\psi'}$ is moreover an open immersion, then the homomorphism $\psi'^\flat_{(s'_\alpha)}$ is also injective, and this implies the homomorphism $\psi^\flat_{(s_\alpha)}$ is injective if u is eventually surjective, since in this case the homomorphism $u_{(s'_\alpha)}$ is just surjective.

Finally, if $r_{\mathscr{L},\psi'}$ is a closed immersion, then condition (v) is satisfied for (s'_α), and hence satisfied for (s_α) if u is eventually surjective (since in this case $\mathrm{Proj}(u)$ is a closed immersion from $\mathrm{Proj}(S)$ to $\mathrm{Proj}(S')$); this implies $r_{\mathscr{L},\psi}$ is a closed immersion. □

Proposition 9.3.47. *Suppose the hypotheses of Proposition 9.3.41 and moreover that $q : X \to Y$ is a morphism of finite type. Then, for the morphism $r_{\mathscr{L},\psi}$ to be everywhere defined and an immersion, it is necessary and sufficient that there exists an index λ such that $r_{\mathscr{L},\psi_\lambda}$ is everywhere defined and is an immersion; in this case, $r_{\mathscr{L},\psi_\lambda}$ is everywhere defined and an immersion for $\mu \geq \lambda$.*

Proof. By Corollary 9.3.46, it suffices to prove that if $r_{\mathscr{L},\psi}$ is defined everywhere and is an immersion, then there exists an index λ such that $r_{\mathscr{L},\psi_\lambda}$ is defined everywhere and is an immersion. By the same reasoning of Proposition 9.3.41 and using the quasi-compactness of Y, we are reduced to the case where $Y = \mathrm{Spec}(A)$ is affine. As X is then quasi-compact, Proposition 9.3.45 shows that there exists a finite family (s_i) of elements of S ($s_i \in S_{n_i}$) satisfying the conditions (i), (ii), and (iii). The morphism $X_{f_i} \to Y$ (where $f_i = \psi^\flat(s_i)$) is of finite type since X_{f_i} is affine and the morphism $q : X \to Y$ is locally of finite type. The ring B_i of X_{f_i} is therefore an A-algebra of finite type by Corollary 8.6.39, and we choose (t_{ij}) to be a family of generators of this algebra. There are then elements $s_{ij} \in S_{m_{ij}n_i}$ such that

$$t_{ij} = (\psi^\flat(s_{ij})|_{X_{f_i}})(\psi^\flat(s_i)|_{X_{f_i}})^{-m_{ij}}$$

We can choose an index λ and elements $s_{i\lambda} \in S^\lambda_{n_i}$, $s_{ij\lambda} \in S^\lambda_{m_{ij}n_i}$ such that their images under φ_λ is s_i and s_{ij}, respectively. It is then clear that the family $(s_{i\lambda})$ satisfies the conditions (i), (ii), and (iii), so $r_{\mathscr{L},\psi}$ is everywhere defined and an immersion. □

Proposition 9.3.48. *Let Y be a quasi-compact scheme, $q : X \to Y$ be a morphism of finite type, $\mathscr{L}$ be an invertible $\mathscr{O}_X$-module, $\mathscr{S}$ be a quasi-coherent graded $\mathscr{O}_Y$-algebra, and $\psi : q^*(\mathscr{S}) \to \boldsymbol{S}(\mathscr{L})$ be a homomorphism of graded algebras. For the morphism $r_{\mathscr{L},\psi}$ to be defined everywhere and an immersion, it is necessary and sufficient that there exists an integer $n > 0$ and a quasi-coherent sub-$\mathscr{O}_Y$-module $\mathscr{E}$ of $\mathscr{S}_n$ of finite type such that:*

(a) *the homomorphism $\psi_n \circ q^*(j_n) : q^*(\mathscr{E}) \to \mathscr{L}^{\otimes n}$ (where $j_n : \mathscr{E} \to \mathscr{S}_n$ is the canonical injection) is surjective.*

(b) *if $\mathscr{S}'$ is the graded sub-$\mathscr{O}_Y$-algebra of $\mathscr{S}$ genrated by $\mathscr{E}$ and ψ' is the homomorphism $\psi \circ q^*(j')$, where $j' : \mathscr{S}' \to \mathscr{S}$ is the canonical injection, $r_{\mathscr{L},\psi'}$ is everywhere defined and an immersion.*

If these are true, any quasi-coherent sub-$\mathscr{O}_Y$-module $\mathscr{E}'$ of $\mathscr{S}_n$ containing $\mathscr{E}$ possesses the same properties, and so does the sub-$\mathscr{O}_Y$-module $\mathscr{S}'_{kn}$ of $\mathscr{S}_{kn}$ for any $k > 0$.

Proof. The sufficiency of these conditions is a particular case of Corollary 9.3.46 in view of the isomorphism between $\mathrm{Proj}(\mathscr{S})$ and $\mathrm{Proj}(\mathscr{S}^{(d)})$ (Proposition 9.3.6). We now prove the necessity, so let (U_i) be a finite affine open covering of Y and set $A_i = \Gamma(U_i, \mathscr{O}_Y)$. As $q^{-1}(U_i)$ is compact, the hypotheses that $r_{\mathscr{L},\psi}$ is an immersion defined on X implies by Proposition 9.3.45 the existence of a finite family (s_{ij}) of elements of $S^{(i)} = \Gamma(U_i, \mathscr{S})$ (where $s_{ij} \in S^{(i)}_{n_{ij}}$) satisfying conditions (i), (ii), and (iii). Since $q : X \to Y$ is of finite type, the restricted homomorphism $X_{f_{ij}} \to U_i$ is of finite type (where $f_{ij} = \psi^\flat(s_{ij})$), so the ring B_{ij} of $X_{f_{ij}}$ is an A_i-algebra of finite type, and we choose $(\psi^\flat(t_{ijk})|_{X_{f_{ij}}})(f_{ij}|_{X_{f_{ij}}})^{-m_{ijk}}$ to be a system of generators of B_{ij}, where $t_{ijk} \in S^{(i)}_{m_{ijk}n_{ij}}$. Let n be a common multiple of all the $m_{ijk}n_{ij}$ and put $s'_{ij} = s_{ij}^{h_{ij}} \in S^{(i)}_n$, where $h_{ij} = n/n_{ij}$. For any given piar (i,j,k), the element $t'_{ij} = s_{ij}^{h-m_{ijk}} t_{ijk}$ belongs to $S^{(i)}_n$, and it is clear that the $(\psi^\flat(t'_{ijk})|_{X_{f'_{ij}}})(f'_{ij}|_{X_{f'_{ij}}})^{-1}$ also generate B_{ij} (where $f'_{ij} = \psi^\flat(s'_{ij})$, and we note that $X_{f'_{ij}} = X_{f_{ij}}$). Let E_i be the sub-A_i-module of $S^{(i)}$ generated by these s'_{ij} and t'_{ijk}; then there exists a quasi-coherent sub-$\mathscr{O}_Y$-module $\mathscr{E}_i$ of $\mathscr{S}_n$ of finite type such that $\mathscr{E}_i|_{U_i} = \widetilde{E}_i$ (Theorem 8.6.63). It is then clear that the sub-$\mathscr{O}_Y$-module $\mathscr{E}$ of $\mathscr{E}$, which is the sum of the $\mathscr{E}_i$, satisfies the required properties. □

Remark 9.3.49. The point of Proposition 9.3.48 is that, for a scheme X of finite type over a quasi-compact scheme Y, if X can be embedded into $\mathrm{Proj}(\mathscr{S})$ via a morphism $r_{\mathscr{L},\psi}$, then we can choose $\mathscr{S}$ so that it is generated by $\mathscr{S}_1$ and $\mathscr{S}_1$ of finite type (we already know that in this case the twisted sheaves over $\mathrm{Proj}(\mathscr{S})$ have nice properties).

9.4 Projective bundles and ample sheaves

9.4.1 Projective bundles

Let Y be a scheme, $\mathscr{E}$ be a quasi-coherent $\mathscr{O}_Y$-module, and $\boldsymbol{S}(\mathscr{E})$ be the symmetric $\mathscr{O}_Y$-algebra of $\mathscr{E}$, which is quasi-coherent by Corollary 9.1.38. The **projective bundle** over Y associated with $\mathscr{E}$ is defined to be the Y-scheme $P = \mathrm{Proj}(\boldsymbol{S}(\mathscr{E}))$. The $\mathscr{O}_P$-module $\mathscr{O}_P(1)$ is called the **fundamental sheaf** of P.

If $\mathscr{E} = \mathscr{O}_Y^n$, we then denote by $\mathbb{P}_Y^{n-1}$ instead of $\mathbb{P}(\mathscr{E})$; if moreover Y is affine with ring A, we then denote this scheme by $\mathbb{P}_A^{n-1}$. As $\boldsymbol{S}(\mathscr{O}_Y)$ is canonically isomorphic to $\mathscr{O}_Y[T]$, we see $\mathbb{P}_Y^0$ is canonically identified with Y.

If $Y = \mathrm{Spec}(A)$ and $\mathscr{E} = \widetilde{E}$ where E is an A-module, we also denote by $\mathbb{P}(E)$ the projective bundle $\mathbb{P}(E)$. The simplest example is $\mathbb{P}(E)$ where E is a vector space over a field k. In this case, we see $\mathbb{P}(E)$ is isomorphic to $\mathbb{P}_k^{n-1}$, where n is the dimension of E.

Let $\mathscr{E}$, $\mathscr{F}$ be two qusi-coherent $\mathscr{O}_Y$-modules and $u : \mathscr{E} \to \mathscr{F}$ be an $\mathscr{O}_Y$-homomorphism. Then u corresponds canonically to a homomorphism $\boldsymbol{S}(u) : \boldsymbol{S}(\mathscr{E}) \to \boldsymbol{S}(\mathscr{F})$ of graded $\mathscr{O}_Y$-algebras. If u is surjective, so is $\boldsymbol{S}(u)$, and therefore $\mathrm{Proj}(\boldsymbol{S}(u))$ is a closed immersion $\mathbb{P}(\mathscr{F}) \to \mathbb{P}(\mathscr{E})$, which we then denoted by $\mathbb{P}(u)$. We can then say that $\mathbb{P}(\mathscr{E})$ is a contravariant functor on the category of quasi-coherent $\mathscr{O}_Y$-modules with *surjective homomorphisms*. Suppose that u is surjective and put $P = \mathbb{P}(\mathscr{E})$, $Q = \mathbb{P}(\mathscr{F})$, and $j = \mathbb{P}(u)$. We then have an isomorphism

$$j^*(\mathscr{O}_P(n)) = \mathscr{O}_Q(n)$$

by Corollary 9.3.34.

If $\psi : Y' \to Y$ is a morphism and $\mathscr{E}' = \psi^*(\mathscr{E})$, we then have $\boldsymbol{S}_{\mathscr{O}_{Y'}}(\mathscr{E}') = \psi^*(\boldsymbol{S}_{\mathscr{O}_Y}(\mathscr{E}))$ by Proposition 9.1.36, so from Proposition 9.3.31 we deduce that

$$\mathbb{P}(\psi^*(\mathscr{E})) = \mathbb{P}(\mathscr{E}) \times_Y Y'. \tag{9.4.1}$$

Moreover, it is clear that $\psi^*((\boldsymbol{S}_{\mathscr{O}_Y}(\mathscr{E}))(n)) = (\boldsymbol{S}_{\mathscr{O}_{Y'}}(\mathscr{E}'))(n)$ for each $n \in \mathbb{Z}$, so if $P = \mathbb{P}(\mathscr{E})$ and $P' = \mathbb{P}(\mathscr{E}')$, we have a canonical isomorphism

$$\mathscr{O}_{P'}(n) = \mathscr{O}_P(n) \otimes_Y \mathscr{O}_{Y'}. \tag{9.4.2}$$

Proposition 9.4.1. *Let $\mathscr{L}$ be an invertible $\mathscr{O}_Y$-module. For any quasi-coherent $\mathscr{O}_Y$-module, there exists a canonical Y-isomorphism $i_{\mathscr{L}} : Q = \mathbb{P}(\mathscr{E} \otimes \mathscr{L}) \xrightarrow{\sim} P = \mathbb{P}(\mathscr{E})$. Moreover, $(i_{\mathscr{L}})_*(\mathscr{O}_Q(n))$ is cannically isomorphic to $\mathscr{O}_P(n) \otimes_Y \mathscr{L}^{\otimes n}$ for any $n \in \mathbb{Z}$.*

Proof. If A is a ring, E is an A-module, L is an A-module free of rank 1, we can define a canonical homomorphism

$$\boldsymbol{S}_n(E \otimes L) \to \boldsymbol{S}_n(E) \otimes L^{\otimes n}$$

which maps an element $(x_1 \otimes y_1)\cdots(x_n \otimes y_n)$ to

$$(x_1 \cdots x_n) \otimes (y_1 \otimes \cdots \otimes y_n)$$

where $x_i \in E$ and $y_i \in L$. This is easily seen to be an isomorphism, so we get an isomorphism $\boldsymbol{S}(E \otimes L) \cong \bigoplus_{n \geq 0} \boldsymbol{S}_n(E) \otimes L^{\otimes n}$. In the situation of the proposition, the preceding remark allows us to define a canonical isomorphism of graded $\mathscr{O}_Y$-algebras

$$\boldsymbol{S}(\mathscr{E} \otimes_{\mathscr{O}_Y} \mathscr{L}) \xrightarrow{\sim} \bigoplus_{n \geq 0} \boldsymbol{S}_n(\mathscr{E}) \otimes_{\mathscr{O}_Y} \mathscr{L}^{\otimes n}$$

The proposition then follows from Proposition 9.3.6 and Proposition 9.3.20. □

Let $P = \mathbb{P}(\mathscr{E})$ and denote by $p : P \to Y$ the structural morphism. As by definition $\mathscr{E} = (\boldsymbol{S}(\mathscr{E}))_1$, we have a canonical homomorphism $\alpha_1 : \mathscr{E} \to p_*(\mathscr{O}_P(1))$, and therefore a canonical homomorphism

$$\alpha_1^\sharp : p^*(\mathscr{E}) \to \mathscr{O}_P(1).$$

Proposition 9.4.2. *The canonical homomorphism $\alpha_1^\sharp$ is surjective.*

Proof. We have seen that $\alpha_1^\sharp$ corresponds to the functorial homomorphism $\mathscr{E} \otimes_{\mathscr{O}_Y} \boldsymbol{S}(\mathscr{E}) \to (\boldsymbol{S}(\mathscr{E}))(1)$ (see the remark before Proposition 9.3.21). Since $\mathscr{E}$ generates $\boldsymbol{S}(\mathscr{E})$, this homomorphism is surjective, whence our assertion in view of Proposition 9.3.13. □

9.4.2 Morphisms into $\mathbb{P}(\mathscr{E})$

With the notations of the last subsection, we now let X be an Y-scheme, $q : X \to Y$ be the structural morphism, and $r : X \to P$ be an Y-morphism, which gives a commutative diagram

$$\begin{tikzcd} X \arrow[r, "r"] \arrow[dr, "q"'] & P \arrow[d, "p"] \\ & Y \end{tikzcd}$$

As the functor r^* is right-exact, we deduce from the surjective homomorphism $\alpha_1^\sharp$ in Proposition 9.4.2 a surjective homomorphism

$$r^*(\alpha_1^\sharp) : r^*(p^*(\mathscr{E})) \to r^*(\mathscr{O}_P(1)).$$

But $r^*(p^*(\mathscr{E})) = q^*(\mathscr{E})$ and $r^*(\mathscr{O}_P(1))$ is locally isomorphic to $r^*(\mathscr{O}_P) = \mathscr{O}_X$, which is then an invertible sheaf $\mathscr{L}_r$ over X, so we obtain a canonical surjective $\mathscr{O}_X$-homomorphism

$$\varphi_r : q^*(\mathscr{E}) \to \mathscr{L}_r.$$

If $Y = \operatorname{Spec}(A)$ is affine and $\mathscr{E} = \widetilde{E}$, we can explicitly explain this homomorphism: given $f \in E$, it follows from Proposition 9.2.33 that

$$r^{-1}(D_+(f)) = X_{\varphi_r^\flat(f)}.$$

Let V be an affine open of X contained in $r^{-1}(D_+(f))$, and let B be its ring, which is an A-algebra; put $S = \boldsymbol{S}_A(E)$. The restriction of r to V then corresponds to an A-homomorphism $\omega : S_{(f)} \to B$, and we have $q^*(\mathscr{E})|_V = \widetilde{E \otimes_A B}$ and $\mathscr{L}_r|_V = \widetilde{L}_r$, where by Proposition 8.1.14, $L_r = S(1)_{(f)} \otimes_{S_{(f)}} B$. In view of the definition of α_1, the restriction of φ_r to $q^*(\mathscr{E})|_V$ therefore corresponds to the B-homomorphism

$$u : E \otimes_A B \to L_r, \quad x \otimes 1 \mapsto (x/1) \otimes 1 = (f/1) \otimes \omega(x/f)$$

The canonical extension of φ_r to a homomorphism of $\mathscr{O}_X$-algebras (recall that $(\mathscr{O}_P(1))^{\otimes n} = \mathscr{O}_P(n)$ by Corollary 9.2.29)

$$\psi_r : q^*(\boldsymbol{S}(\mathscr{E})) = \boldsymbol{S}(q^*(\mathscr{E})) \to \boldsymbol{S}(\mathscr{L}_r) = \bigoplus_{n \geq 0} \mathscr{L}_r^{\otimes n} = \bigoplus_{n \geq 0} r^*(\mathscr{O}_P(n))$$

is then such that the restriction of ψ_r to $q^*(\boldsymbol{S}_n(\mathscr{E}))|_V$ corresponds to the B-homomorphism

$$\boldsymbol{S}_n(\mathscr{E}\otimes_A B)=\boldsymbol{S}_n(E)\otimes_A B\to L_r^{\otimes n}=(S(1)_{(f)})^{\otimes n}\otimes_{S_{(f)}}B$$

which send the element $s\otimes 1$ to $(f/1)^{\otimes n}\otimes\omega(s/f^n)$.

Conversely, given an invertible $\mathscr{O}_X$-module $\mathscr{L}$ and a quasi-coherent $\mathscr{O}_Y$-module $\mathscr{E}$, then any homomorphism $\varphi:q^*(\mathscr{E})\to\mathscr{L}$ corresponds to a canonical homomorphism of quasi-coherent $\mathscr{O}_X$-algebras

$$\psi:\boldsymbol{S}(q^*(\mathscr{E}))=q^*(\boldsymbol{S}(\mathscr{E}))\to\boldsymbol{S}(\mathscr{L})=\bigoplus_{n\geq 0}\mathscr{L}^{\otimes n}$$

which then gives an Y-morphism $r_{\mathscr{L},\psi}:G(\psi)\to\operatorname{Proj}(\boldsymbol{S}(\mathscr{E}))\to\mathbb{P}(\mathscr{E})$, which we also denoted by $r_{\mathscr{L},\varphi}$. If φ is surjective, then so is ψ and by Corollary 9.3.38 the morphism $r_{\mathscr{L},\psi}$ is everywhere defined.

Proposition 9.4.3. *Let $q:X\to Y$ be a morphism and $\mathscr{E}$ be a quasi-coherent $\mathscr{O}_Y$-module. Then the maps $r\mapsto(\mathscr{L}_r,\varphi_r)$ and $(\mathscr{L},\varphi)\mapsto r_{\mathscr{L},\varphi}$ form a bijective correspondence between the set of Y-morphisms $r:X\to\mathbb{P}(\mathscr{E})$ to the set of equivalence classes of couples $(\mathscr{L},\varphi)$ formed by an invertible $\mathscr{O}_X$-module and a surjective homomorphism $\varphi:q^*(\mathscr{E})\to\mathscr{L}$, where two couples $(\mathscr{L},\varphi)$, $(\mathscr{L}',\varphi')$ are equivalent if there exists an $\mathscr{O}_X$-isomorphism $\tau:\mathscr{L}\to\mathscr{L}'$ such that $\varphi'=\tau\circ\varphi$.*

Proof. Let us start with an Y-morphism $r:X\to\mathbb{P}(\mathscr{E})$, form $\mathscr{L}_r$ and φ_r, and put $r'=r_{\mathscr{L}_r,\varphi_r}$. To see the morphisms r and r' coincide, we may assume that $Y=\operatorname{Spec}(A)$ is affine, so $\mathscr{E}=\widetilde{E}$, and let $S=\boldsymbol{S}_A(E)$. Let $V=\operatorname{Spec}(B)$ be an affine open of X contained in $r^{-1}(D_+(f))$, where $f\in E$. Then as we have already seen, the restriction of r to V corresponds to an A-homomorphism $\omega:S_{(f)}\to B$, and the restriction of ψ_r to $q^*(\boldsymbol{S}_n(\mathscr{E}))|_V$ corresponds to the B-homomorphism $\boldsymbol{S}_n(E)\otimes_A B\to L_r^{\otimes n}$ which sends $s\otimes 1$ to $(f/1)^{\otimes n}\otimes\omega(s/f^n)$. The restriction of $\psi_r^\flat$ to $\boldsymbol{S}(\mathscr{E})|_V$ then corresponds to the homomorphism $\boldsymbol{S}_n(E)\to L_r^{\otimes n}$ which sends $s\in\boldsymbol{S}_n(E)$ to $(f/1)^{\otimes n}\otimes\omega(s/f^n)$, and by Proposition 9.3.37, the restriction of $r_{\mathscr{L}_r,\psi_r}$ to V corresponds to the homomorphism $(\psi_r^\flat)_{(f)}$, which send $s\in S_n$ to

$$(\psi_r^\flat(s))(\psi_r^\flat(f))^{-n}=[(f/1)^{\otimes n}\otimes\omega(s/f^n)][(f/1)\otimes 1]^{-n}=1\otimes\omega(s/f^n).$$

Therefore, under the identification of X with $\operatorname{Proj}(\boldsymbol{S}(\mathscr{L}))$, $r_{\mathscr{L},\psi_r}$ coincides with r over V, so they coincide on X.

Conversely. let $(\mathscr{L},\varphi)$ be a couple and form $r=r_{\mathscr{L},\varphi}$, $\mathscr{L}_r$, and φ_r. We show that there is a canonical isomorphism $\tau:\mathscr{L}_r\to\mathscr{L}$ such that $\varphi=\tau\circ\varphi_r$. For this, we can assume that $Y=\operatorname{Spec}(A)$ and $X=\operatorname{Spec}(B)$ are affine, and (with the notations of Remark 9.3.36) define τ be sending an element $(x/1)\otimes 1$ of L_r (where $x\in E$) to the element $v_1(x)c$ of L. It is easy to verify that τ is independent of the choice of the generator c of L. As v_1 is surjective, to show that τ is an isomorphism, it suffices to prove that if $x/1=0$ in $S(1)_{(f)}$, then $v_1(x)/1=0$ in B_g. But the first condition means that $f^nx=0$ in S_{n+1} for some n, and we then deduce that $v_{n+1}(f^nx)=g^nv_1(x)=0$ in B, whence the conclusion. Finally, it is immediate that for two equivalent couples $(\mathscr{L},\varphi)$, $(\mathscr{L}',\varphi')$, we have $r_{\mathscr{L},\varphi}=r_{\mathscr{L}',\varphi'}$. □

Theorem 9.4.4. *The set of Y-sections of $\mathbb{P}(\mathscr{E})$ is in bijective correspondence to the set of quasi-coherent sub-$\mathscr{O}_Y$-modules $\mathscr{F}$ of $\mathscr{E}$ such that $\mathscr{E}/\mathscr{F}$ is invertible.*

Proof. This is a particular case of Proposition 9.4.3 by taking $X=Y$ and note that if two pairs $(\varphi,\mathscr{L})$ and $(\varphi',\mathscr{L}')$ are equivalent, then $\ker\varphi$ and $\ker\varphi'$ are identical. □

Note that this property corresponds to the classical definition of the "projective space" as the set of hyperplanes of a vector space (the classical case corresponding to $Y=\operatorname{Spec}(k)$, where k is a field, and $\mathscr{E}=\widetilde{E}$, E being a finite dimensional k-vector. The sheaves $\mathscr{F}$ having the property stated in Theorem 9.4.4 corresponds then to the hyperplanes of E.

Remark 9.4.5. As there is a canonical correspondence between Y-morphisms from X to P and their graph morphisms, which are X-sections of $P\times_Y X$, we see conversely that Proposition 9.4.3 can be deduced from Theorem 9.4.4. Let $\operatorname{Hyp}_Y(X,\mathscr{E})$ be the set of quasi-coherent sub-$\mathscr{O}_X$-modules $\mathscr{F}$ of $\mathscr{E}\otimes_Y\mathscr{O}_X=q^*(\mathscr{E})$ such that $q^*(\mathscr{E})/\mathscr{F}$ is an invertible $\mathscr{O}_X$-module. If $g:X'\to X$ is an Y-morphism, then $g^*(q^*(\mathscr{E})/\mathscr{F})=g^*(q^*(\mathscr{E}))/g^*(\mathscr{F})$ by the right exactness of g^*, so the second sheaf is invertible, and therefore $\operatorname{Hyp}_Y(X,\mathscr{E})$ is a covariant functor over the category of Y-schemes. We can then interprete Theorem 9.4.4 by saying that the Y-scheme $\mathbb{P}(\mathscr{E})$ representes the functor $\operatorname{Hyp}_Y(-,\mathscr{E})$. This also provides a characterization of the projective bundle $P = \mathbb{P}(\mathscr{E})$ by the following universal property, more close to the geometric intuition that the constructions of $r_{\mathscr{L},\psi}$: for any morphism $q : X \to Y$ and any invertible $\mathscr{O}_X$-module $\mathscr{L}$ which is a quotient of $q^*(\mathscr{E})$, there exists a unique Y-morphism $r : X \to P$ such that $\mathscr{L}$ is isomorphic to $r^*(\mathscr{O}_P(1))$.

Corollary 9.4.6. *Suppose that any invertible $\mathscr{O}_Y$-module is trivial. Let $E = \mathrm{Hom}_{\mathscr{O}_Y}(\mathscr{E}, \mathscr{O}_Y)$ considered as a module over the ring $A = \Gamma(Y, \mathscr{O}_Y)$, and let $E^\times$ be the subset of E formed by surjective homomorphisms. Then the set of Y-sections of $\mathbb{P}(\mathscr{E})$ is canonically identified with $E^\times / A^\times$, where $A^\times$ is the group of units of A.*

Example 9.4.7. Let Y be a scheme, y be a point of Y, and $Y' = \mathrm{Spec}(\kappa(y))$. The fiber $p^{-1}(y)$ of $\mathbb{P}(\mathscr{E})$ is, in view of (9.4.1), identified with $\mathbb{P}(\mathscr{E}^y)$, where $\mathscr{E}^y = \mathscr{E}_y \otimes_{\mathscr{O}_{Y,y}} \kappa(y) = \mathscr{E}_y / \mathfrak{m}_y \mathscr{E}_y$ is considered as a vector space over $\kappa(y)$. More generally, if K is an extension of $\kappa(y)$, $p^{-1}(y) \otimes_{\kappa(y)} K$ is identified with $\mathbb{P}(\mathscr{E}^y \otimes_{\kappa(y)} K)$. Since any invertible sheaves over a local scheme is trivial, Corollary 9.4.6 shows that the points of $\mathbb{P}(\mathscr{E})$ lying over y with values in K, which are called the **rational geometric fibers** of $\mathbb{P}(\mathscr{E})$ over K lying over y, is identified with the projective space of the dual of the vector K-space $\mathscr{E}^y \otimes_{\kappa(y)} K$.

Example 9.4.8. Suppose now that Y is affine with ring A, and any invertible sheaf on Y is trivial; we put $\mathscr{E} = \mathscr{O}_Y^n$. Then with the notations of Corollary 9.4.6, E is identified with A^n by Corollary 8.1.3 and $E^\times$ is identified with the set of systems $(f_i)_{1 \le i \le n}$ of elements of A which generate the unit ideal of A. By Corollary 9.4.6, two such systems determine the same Y-section of $\mathbb{P}_Y^{n-1} = \mathbb{P}_A^{n-1}$, which means the same point of $\mathbb{P}_A^{n-1}$ with values in A, if and only if one is deduced from the other by multiplication by an invertible element of A.

Remark 9.4.9. Let $u : X' \to X$ be a morphism. If the Y-morphism $r : X \to P$ corresponds to the homomorphism $\varphi : q^*(\mathscr{E}) \to \mathscr{L}$, then by definition, the morphism $r \circ u$ corresponds to $u^*(\varphi) : u^*(q^*(\mathscr{E})) \to u^*(\mathscr{L})$. On the other hand, let $\mathscr{E}, \mathscr{F}$ be two quasi-coherent $\mathscr{O}_Y$-modules and $j = \mathbb{P}(v)$ be the closed immersion $\mathbb{P}(\mathscr{F}) \to \mathbb{P}(\mathscr{E})$ corresponding to a surjective homomorphism $v : \mathscr{E} \to \mathscr{F}$. If the Y-morphism $r : X \to \mathbb{P}(\mathscr{E})$ corresponds to the homomorphism $\varphi : q^*(\mathscr{E}) \to \mathscr{L}$, then $j \circ r$ corresponds to the composition

$$q^*(\mathscr{E}) \xrightarrow{q^*(v)} q^*(\mathscr{F}) \xrightarrow{\varphi} \mathscr{L}$$

Let $\psi : Y' \to Y$ be a morphism and put $\mathscr{E}' = \psi^*(\mathscr{E})$. If the Y-morphism $r : X \to P$ corresponds to the homomorphism $\varphi : q^*(\mathscr{E}) \to \mathscr{L}$, the Y'-morphism

$$r_{(Y')} : X_{(Y')} \to P' = \mathbb{P}(\mathscr{E}')$$

correspond to $\varphi_{(Y')} : q^*_{(Y')}(\mathscr{E}') = q^*(\mathscr{E}) \otimes_Y \mathscr{O}_{Y'} \to \mathscr{L} \otimes_Y \mathscr{O}_{Y'}$. In fact, by (9.4.1), we have the following commutative diagram

$$\begin{tikzcd}
X_{(Y')} \arrow[r, "r_{(Y')}"] \arrow[d, "v"] & P' = P_{(Y')} \arrow[r, "p_{(Y')}"] \arrow[d, "u"] & Y' \arrow[d, "\psi"] \\
X \arrow[r, "r"] & P \arrow[r, "p"] & Y
\end{tikzcd}$$

In view of (9.4.2), we have

$$(r_{(Y')})^*(\mathscr{O}_{P'}(1)) = (r_{(Y')})^*(u^*(\mathscr{O}_P(1))) = v^*(r^*(\mathscr{O}_P(1))) = v^*(\mathscr{L}) = \mathscr{L} \otimes_Y \mathscr{O}_{Y'}.$$

Also, $u^*(\alpha_1^\sharp)$ is equal to the canonical homomorphism $\alpha_1^\sharp : (p_{(Y')})^*(\mathscr{E}') \to \mathscr{O}_{P'}(1)$, in view of the definition of α_1, whence our assertion.

9.4.3 The Segre morphism

Let Y be a scheme and $\mathscr{E}, \mathscr{F}$ be two quasi-coherent $\mathscr{O}_Y$-modules. Put $P_1 = \mathbb{P}(\mathscr{E})$, $P_2 = \mathbb{P}(\mathscr{F})$, and denote by p_1, p_2 their morphisms; let $Q = P_1 \times_Y P_2$ and q_1, q_2 be the canonical projections. The $\mathscr{O}_Q$-module

$$\mathscr{L} = \mathscr{O}_{P_1}(1) \times_Y \mathscr{O}_{P_2}(1) = q_1^*(\mathscr{O}_{P_1}(1)) \times_{\mathscr{O}_Q} q_2^*(\mathscr{O}_{P_2}(1))$$

is invertible as a tensor product of invertible modules. On the other hand, if $r = p_1 \circ q_1 = p_2 \circ q_2$ is the structural morphism of Q, we have $r^*(\mathscr{E} \otimes_{\mathscr{O}_Y} \mathscr{F}) \otimes_{\mathscr{O}_Q} q_2^*(p_2^*(\mathscr{F}))$; the canonical surjective homomorphism $p_1^1(\mathscr{E}) \to \mathscr{O}_{P_1}(1)$ and $p_2^*(\mathscr{F}) \to \mathscr{O}_{P_2}(1)$ then give a canonical surjective homomorphism

$$s : r^*(\mathscr{E} \otimes_{\mathscr{O}_Y} \mathscr{F}) \to \mathscr{L} \tag{9.4.3}$$

we then deduce a canonical homomorphism, called the Segre morphism:

$$\zeta : \mathbb{P}(\mathscr{E}) \times_Y \mathbb{P}(\mathscr{F}) \to \mathbb{P}(\mathscr{E} \otimes_{\mathscr{O}_Y} \mathscr{F}). \tag{9.4.4}$$

To explain this morphism ζ, let us consider the case where $Y = \operatorname{Spec}(A)$ is affine, $\mathscr{E} = \widetilde{E}$, $\mathscr{F} = \widetilde{F}$, where E and F are two A-module; whence $\mathscr{E} \otimes_{\mathscr{O}_Y} \mathscr{F} = \widetilde{E \otimes_A F}$. Put $R = \boldsymbol{S}(E)$, $S = \boldsymbol{S}(F)$, and $T = \boldsymbol{S}(E \otimes_A F)$. Let $f \in E$, $g \in F$, and consider the affine open

$$D_+(f) \times_Y D_+(g) = \operatorname{Spec}(B)$$

of Q, where $B = R_{(f)} \otimes_A S_{(g)}$. The restriction of $\mathscr{L}$ on this affine open is $\widetilde{L}$, where

$$L = (R(1)_{(f)}) \otimes_A (S(1)_{(g)})$$

and the element $c = (f/1) \otimes (g/1)$ is a generator of L as a free B-module (Proposition 9.2.24). The homomorphism (9.4.3) then corresponds to the homomorphism

$$(x \otimes y) \otimes b \mapsto b((x/1) \otimes (y/1))$$

from $(E \otimes_A F) \otimes_A B$ to L. With the notations of Remark 9.3.36, we then have $v_1(x \otimes y) = (x/f) \otimes (y/g)$, so the restriction of the morphism ζ to $D_+(f) \times_Y D_+(g)$ is a morphism from this affine scheme to $D_+(f \otimes g)$, which corresponds to the ring homomorphism

$$\omega((x \otimes y)/(f \otimes g)) = (x/f) \otimes (y/g) \tag{9.4.5}$$

for $x \in E$ and $y \in F$.

From Proposition 9.4.3, there is a canonical isomorphism

$$\tau : \zeta^*(\mathscr{O}_P(1)) \xrightarrow{\sim} \mathscr{O}_{P_1}(1) \otimes_Y \mathscr{O}_{P_2}(1)$$

where we put $P = \mathbb{P}(\mathscr{E} \otimes_{\mathscr{O}_Y} \mathscr{F})$. Moreover, for $x \in \Gamma(Y, \mathscr{E})$ and $y \in \Gamma(Y, \mathscr{F})$, we have

$$\tau(\alpha_1(x \otimes y)) = \alpha_1(x) \otimes \alpha_1(y) \tag{9.4.6}$$

To see this, we can assume that Y is affine, so with the notations above and the definition of α_1, we have $\alpha_1^{f \otimes g}(x \otimes y) = (x \otimes y)/1$, $\alpha_1^f(x) = x/1$, and $\alpha_1^g(y) = y/1$. The definition of τ given in the proof of Proposition 9.4.3 says τ maps $(x/1) \otimes 1$ to $v_1(x)c$. Since we have seen that $v_1(x \otimes y) = (x/f) \otimes (y/g)$, this implies the assertion by a simple computation. From this, we then deduce the formula

$$\zeta^{-1}(P_{x \otimes y}) = (P_1)_x \times_Y (P_2)_y \tag{9.4.7}$$

where we need to use the following lemma:

Lemma 9.4.10. *Let B, B' be two A-algebras, and let $Y = \operatorname{Spec}(A)$, $Z = \operatorname{Spec}(B)$, $Z' = \operatorname{Spec}(B')$. Then for $t \in B$, $t' \in B'$, we have $D(t \otimes t') = D(t) \times_Y D(t')$.*

Proof. Let p, p' be the canonical projections of $Z \times_Y Z'$. Then it follows from **??** that $p^{-1}(D(t)) = D(t \otimes 1)$ and $p'^{-1}(D(t')) = D(1 \otimes t')$. Corollary 8.3.2 then implies the lemma, since $(t \otimes 1)(1 \otimes t') = t \otimes t'$. ☐

Proposition 9.4.11. *The Segre morphism is a closed immersion.*

Proof. Since the question is local on Y, we can assume that Y is affine. With the previous notations, the $D_+(f \otimes g)$ form a basis for P, since the elements $f \otimes g$ generate T for $f \in E$, $g \in F$. On the other hand, we have $\zeta^{-1}(D_+(f \otimes g)) = D_+(f) \times_Y D_+(g)$ in view of (9.4.7). It then suffices to use Corollary 8.4.11 to prove that the restriction of ζ to $D_+(f) \times_Y D_+(g)$ is a closed immersion into $D_+(f \otimes g)$. But this is a morphism between affine schemes whose corresponding ring homomorphism ω is surjective in view of the formula (9.4.5), so our assertion follows. ☐

The Segre morphism is functorial on $\mathscr{E}$ and $\mathscr{F}$ if we restrict ourselves to quasi-coherent $\mathscr{O}_Y$-modules with *surjective* homomorphisms. To see this, it suffices to consider a surjective $\mathscr{O}_Y$-homomorphism $\mathscr{E} \to \mathscr{E}'$ and prove that the following diagram is commutative

$$\begin{tikzcd} \mathbb{P}(\mathscr{E}') \times \mathbb{P}(\mathscr{F}) \arrow[r, "j\times 1"] \arrow[d, "\zeta"'] & \mathbb{P}(\mathscr{E}') \times \mathbb{P}(\mathscr{F}) \arrow[d, "\zeta"] \\ \mathbb{P}(\mathscr{E}' \otimes \mathscr{F}) \arrow[r] & \mathbb{P}(\mathscr{E} \otimes \mathscr{F}) \end{tikzcd}$$

where j is the canonical closed immersion $\mathbb{P}(\mathscr{E}') \to \mathbb{P}(\mathscr{E})$. Put $P_1' = \mathbb{P}(\mathscr{E}')$ and retain the previous notations; $j \times 1$ is a closed immersion by Proposition 9.3.33 and we have

$$(j \times 1)^*(\mathscr{O}_{P_1}(1) \otimes \mathscr{O}_{P_2}(1)) = j^*(\mathscr{O}_{P_1}(1)) \otimes \mathscr{O}_{P_2}(1) = \mathscr{O}_{P_1'}(1) \otimes \mathscr{O}_{P_2}(1)$$

in view of (9.4.2) and Corollary 8.3.16. The assertion then follows from Remark 9.4.9.

Proposition 9.4.12. *Let $\psi : Y \to Y'$ be a morphism and put $\mathscr{E}' = \psi^*(\mathscr{E})$, $\mathscr{F}' = \psi^*(\mathscr{F})$. Then the Segre morphism $\mathbb{P}(\mathscr{E}') \times \mathbb{P}(\mathscr{F}') \to \mathbb{P}(\mathscr{E}' \otimes \mathscr{F}')$ is identified with $\zeta_{(Y')}$.*

Proof. Put $P_1' = \mathbb{P}(\mathscr{E}')$ and $P_2' = \mathbb{P}(\mathscr{F}')$. Then by Remark 9.4.9, P_i' is identified with $(P_i)_{(Y')}$ for $i = 1, 2$, so the structural morphism $P_1' \times_{Y'} P_2' \to Y'$ is identified with $r_{(Y')}$, where r is the structural morphism of $P_1 \times_Y P_2$. On the other hand, $\mathscr{E}' \otimes \mathscr{F}'$ is identified with $\psi^*(\mathscr{E} \otimes \mathscr{F})$, so $\mathbb{P}(\mathscr{E}' \otimes \mathscr{F}')$ is identified with $\mathbb{P}(\mathscr{E} \otimes \mathscr{F})_{(Y')}$ by Proposition 9.3.31. Finally, $\mathscr{O}_{P_1'}(1) \otimes_{Y'} \mathscr{O}_{P_2'}(1) = \mathscr{L}$ is identified with $\mathscr{L} \otimes_Y \mathscr{O}_{Y'}$ in view of (9.4.2) and Proposition 8.3.15. The canonical homomorphism $(r_{(Y')})^*(\mathscr{E}' \otimes \mathscr{F}') \to \mathscr{L}'$ is then identified with $s_{(Y')}$, and our assertion follows from Proposition 9.4.3. □

Remark 9.4.13. The coproduct of $\mathbb{P}(\mathscr{E})$ and $\mathbb{P}(\mathscr{F})$ is similarly canonical isomorphic to a closed subscheme of $\mathbb{P}(\mathscr{E} \oplus \mathscr{F})$. In fact, the surjective homomorphisms $\mathscr{E} \oplus \mathscr{F} \to \mathscr{E}$, $\mathscr{E} \oplus \mathscr{F} \to \mathscr{F}$ correspond to closed immersions $\mathbb{P}(\mathscr{E}) \to \mathbb{P}(\mathscr{E} \oplus \mathscr{F})$, $\mathbb{P}(\mathscr{E}) \to \mathbb{P}(\mathscr{E} \oplus \mathscr{F})$; it then boils down to showing that the underlying spaces of these closed subschemes of $\mathbb{P}(\mathscr{E} \oplus \mathscr{F})$ thus obtained have no common point. The question being local on Y, we can assume that Y is affine adapt our previous notations. Now $\boldsymbol{S}_n(E)$ and $\boldsymbol{S}_n(F)$ are identified with submodules of $S_n(E \oplus F)$ with intersection reduced to 0, and if $\mathfrak{p}$ is a graded prime ideal of $\boldsymbol{S}(E)$ such that $\mathfrak{p} \cap S_n(E) \neq S_n(E)$ for all $n \geq 0$, then it corresponds to a unique graded prime ideal in $\boldsymbol{S}(E \oplus F)$ whose trace on $\boldsymbol{S}_n(E)$ is $\mathfrak{p} \cap S_n(E)$, but which contains $S_+(F)$. Therefore, two distinct points of $\mathrm{Proj}(\boldsymbol{S}(E))$ and $\mathrm{Proj}(\boldsymbol{S}(F))$ can not have same image in $\mathrm{Proj}(\boldsymbol{S}(E \oplus F))$.

9.4.4 Very ample sheaves

Proposition 9.4.14. *Let Y be a quasi-compact scheme, $q : X \to Y$ a morphism of finite type, and $\mathscr{L}$ be an invertible $\mathscr{O}_X$-module.*

(a) *Let $\mathscr{S}$ be a quasi-coherent graded $\mathscr{O}_Y$-algebra and $\psi : q^*(\mathscr{S}) \to \boldsymbol{S}(\mathscr{L})$ be a graded homomorphism of algebras. For the morphism $r_{\mathscr{L},\psi}$ to be everywhere defined and an immersion, it is necessary and sufficent that there exists an integer $n \geq 0$ and a quasi-coherent sub-$\mathscr{O}_Y$-module of finite type $\mathscr{E}$ of $\mathscr{S}_n$ such that the homomorphism $\varphi' = \psi_n \circ q^*(j) : q^*(\mathscr{E}) \to \boldsymbol{S}(\mathscr{L}) = \mathscr{L}'$ ($j : \mathscr{E} \to \mathscr{S}_n$ being the canonical injection) is surjective and the morphism $r_{\mathscr{L}',\varphi'} : X \to \mathbb{P}(\mathscr{E})$ is an immersion.*

(b) *Let $\mathscr{F}$ be a quasi-coherent $\mathscr{O}_Y$-module and $\varphi : q^*(\mathscr{F}) \to \mathscr{L}$ be a surjective homomorphism. For the morphism $r_{\mathscr{L},\varphi} : X \to \mathbb{P}(\mathscr{F})$ to be an immersion, it is necessary and sufficent that there exists a quasi-coherent sub-$\mathscr{O}_Y$-module of finite type $\mathscr{E}$ of $\mathscr{F}$ such that the homomorphism $\varphi' = \varphi \circ q(j) : q^*(\mathscr{E}) \to \mathscr{L}$ (where $j : \mathscr{E} \to \mathscr{F}$ is the canonical injection) is surjective and such that $r_{\mathscr{L},\varphi'} : X \to \mathbb{P}(\mathscr{E})$ is an immersion.*

Proof. We first consider case (a). The fact that $r_{\mathscr{L},\psi}$ is everywhere defined and an immersion is equivalent by Proposition 9.3.48 to the existence of an integer $n > 0$ and $\mathscr{E}$ such that, if $\mathscr{S}'$ is the subalgebra of $\mathscr{S}$ generated by $\mathscr{E}$, the homomorphism $q^*(\mathscr{E}) \to \mathscr{L}^{\otimes n}$ is surjective and $r_{\mathscr{L},\psi'} : X \to \mathrm{Proj}(\mathscr{S}')$ is everywhere defined and an immersion. We also have a closed immersion corresponding to the surjective homomorphism $\boldsymbol{S}(\mathscr{E}) \to \mathscr{S}'$, so these the morphism $X \to \mathbb{P}(\mathscr{E})$ is an immersion.

Now consider the situation of (b). As $\mathscr{F}$ is the inductive limit of its quasi-coherent submodules of finite type $\mathscr{E}_\lambda$ (Corollary 8.6.65), $\boldsymbol{S}(\mathscr{F})$ is the inductive limit of the $\boldsymbol{S}(\mathscr{E}_\lambda)$, so by Proposition 9.3.47 there exists λ such that $r_{\mathscr{L},\varphi_\mu}$ is everywhere defined and an immersion for $\mu \geq \lambda$. Also, since the functor f^* is left-adjoint, it commutes with inductive limits and therefore $q^*(\mathscr{F})$ is the inductive limits of the $q^*(\mathscr{E}_\lambda)$. Since $\mathscr{L}$ is an $\mathscr{O}_X$-module of finite type and $q^*(\mathscr{F}) \to \mathscr{L}$ is surjective, it follows from **??** that there exists λ' such that $q^*(\mathscr{E}_\mu) \to \mathscr{L}$ is surjective for $\mu \geq \lambda'$. It then suffices to choose $\mathscr{E} = \mathscr{E}_\mu$ for $\mu \geq \lambda$ and $\mu \geq \lambda'$. □

Let Y be a scheme and $q : X \to Y$ be a morphism. We say an invertible $\mathscr{O}_X$-module $\mathscr{L}$ is **very ample for q** (or **very ample relative to q**, or simple **very ample**) if there exists a quasi-coherent $\mathscr{O}_Y$-module $\mathscr{E}$ and a Y-immersion $r : X \to \mathbb{P}(\mathscr{E})$ such that $\mathscr{L}$ is isomorphic to $r^*(\mathscr{O}_P(1))$. In view of Proposition 9.4.3, this is equivalent to the existence of a quasi-coherent $\mathscr{O}_Y$-module $\mathscr{E}$ and a surjective homomorphism $\varphi : q^*(\mathscr{E}) \to \mathscr{L}$ such that the associated morphism $r_{\mathscr{L},\varphi} : X \to \mathbb{P}(\mathscr{E})$ is an immersion. We also note that the existence of a very ample $\mathscr{O}_X$-module relative to Y implies that q is separated (Proposition 9.2.14 and Proposition 8.5.26).

Corollary 9.4.15. *Suppose that there exists a quasi-coherent $\mathscr{O}_Y$-algebra $\mathscr{S}$ genrated by $\mathscr{S}_1$ and a Y-immersion $r : X \to P = \mathrm{Proj}(\mathscr{S})$ such that $\mathscr{L} \cong r^*(\mathscr{O}_P(1))$. Then $\mathscr{L}$ is very ample relative to q.*

Proof. If $\mathscr{F} = \mathscr{S}_1$, the canonical homomorphism $\boldsymbol{S}(\mathscr{F}) \to \mathscr{S}$ is surjective, so by composing r with the corresponding closed immersion $\mathrm{Proj}(\mathscr{S}) \to \mathbb{P}(\mathscr{F})$, we obtain an immersion $r' : X \to \mathbb{P}(\mathscr{F}) = P'$ such that $\mathscr{L}$ is isomorphic to $r'^*(\mathscr{O}_{P'}(1))$. □

Proposition 9.4.16. *Let $q : X \to Y$ be a quasi-compact morphism and $\mathscr{L}$ be an invertible $\mathscr{O}_X$-module. Then $\mathscr{L}$ is very ample relative to q if and only if $q_*(\mathscr{L})$ is quasi-coherent, the cannical homomorphism $\sigma : q^*(q_*(\mathscr{L})) \to \mathscr{L}$ is surjective, and the morphism $r_{\mathscr{L},\sigma} : X \to \mathbb{P}(q_*(\mathscr{L}))$ is an immersion.*

Proof. As q is quasi-compact, $q_*(\mathscr{L})$ is quasi-coherent if q is separated (Proposition 8.6.55). By Remark 9.3.28, the existence of a surjective homomorphism $\varphi : q^*(\mathscr{E}) \to \mathscr{L}$ ($\mathscr{E}$ being a quasi-coherent $\mathscr{O}_Y$-module) implies that σ is surjective. Moreover, the factorization $\varphi : q^*(\mathscr{E}) \to q^*(q_*(\mathscr{L})) \xrightarrow{\sigma} \mathscr{L}$ of (**??**) corresponds to a canonical factorization (recall that q^* commutes with $\boldsymbol{S}$)

$$q^*(\boldsymbol{S}(\mathscr{E})) \longrightarrow q^*(\boldsymbol{S}(q_*(\mathscr{L}))) \longrightarrow \boldsymbol{S}(\mathscr{L})$$

so by Corollary 9.3.46 the hypothesis that $r_{\mathscr{L},\varphi}$ is an immersion implies that $j = r_{\mathscr{L},\sigma}$ is an immersion. Moreover, by Proposition 9.4.3, $\mathscr{L}$ is isomorphic to $j^*(\mathscr{O}_{P'}(1))$ where $P' = \mathbb{P}(q_*(\mathscr{L}))$. The converse of this is clear by the definition of very ampleness. □

Corollary 9.4.17. *Let $q : X \to Y$ be a quasi-compact morphism and $\mathscr{L}$ be an invertible $\mathscr{O}_X$-module. For $\mathscr{L}$ to be very ample relative to Y, it is necessary and sufficent that there exists an open covering (U_α) of Y such that $\mathscr{L}|_{q^{-1}(U_\alpha)}$ is very ample relative to U_α for each α.*

Proof. This follows from the fact that the criterion of Proposition 9.4.16 is local over Y. □

Proposition 9.4.18. *Let Y be a quasi-compact scheme, $q : X \to Y$ a morphism of finite type, and $\mathscr{L}$ be an invertible $\mathscr{O}_X$-module. Then the following conditions are equivalent:*

(i) *$\mathscr{L}$ is very ample relative to Y.*

(ii) *There exists a quasi-coherent $\mathscr{O}_Y$-module $\mathscr{E}$ of finite type and a surjective homomorphism $\varphi : q^*(\mathscr{E}) \to \mathscr{L}$ such that $r_{\mathscr{L},\varphi}$ is an immersion.*

(iii) *There exists a coherent sub-$\mathscr{O}_Y$-module of finite type $\mathscr{E}$ of $q_*(\mathscr{L})$ and a surjective homomorphism φ : $q^*(\mathscr{E}) \to \mathscr{L}$ such that $r_{\mathscr{L},\varphi}$ is an immersion.*

Proof. It is clear that (ii) or (iii) implies (i); but (i) implies (ii) by Proposition 9.4.14, and similarly (i) implies (iii) in view of Proposition 9.4.16. □

Corollary 9.4.19. *Suppose that Y is a quasi-compact scheme. If $\mathscr{L}$ is very ample relative to Y, there exists a quasi-coherent graded $\mathscr{O}_Y$-algebra $\mathscr{S}$ generated by $\mathscr{S}_1$ and such that $\mathscr{S}_1$ is of finite type, and a dominant open Y-immersion $i : X \to P = \mathrm{Proj}(\mathscr{S})$ such that $\mathscr{L}$ is isomorphic to $i^*(\mathscr{O}_P(1))$.*

Proof. Since $\mathscr{L}$ is very ample, by Proposition 9.4.18 there exists a coherent sub-$\mathscr{O}_Y$-module of finite type $\mathscr{E}$ of $q_*(\mathscr{L})$ and a surjective homomorphism $\varphi : q^*(\mathscr{E}) \to \mathscr{L}$ such that $r_{\mathscr{L},\varphi}$ is an immersion. The structural morphism $p : P' = \mathbb{P}(\mathscr{E}) \to Y$ is then separated and of finite type (Proposition 9.3.23), so P' is a quasi-compact scheme if Y is quasi-compact. Let Z be scheme theoretic image of X in P', with underlying space $\overline{j(X)}$, where $j = r_{\mathscr{L},\varphi}$; then j factors through Z into a dominant open immersion $i : X \to Z$. But Z is identified with the scheme $\mathrm{Proj}(\mathscr{S})$, where $\mathscr{S}$ is the quotient graded $\mathscr{O}_Y$-algebra of $\boldsymbol{S}(\mathscr{E})$ by a quasi-coherent graded ideal (Proposition 9.3.33), and it is clear that $\mathscr{S}_1$ is generated by $\mathscr{S}$ (since $\boldsymbol{S}(\mathscr{E})$ satisfies this condition). Moreover, by Corollary 9.3.34, $\mathscr{O}_Z(1)$ is the inverse image of $\mathscr{O}_{P'}(1)$ under the canonical injection, so $\mathscr{L} = i^*(\mathscr{O}_Z(1))$. □

Proposition 9.4.20. *Let $q : X \to Y$ be a morphism, $\mathscr{L}$ be a very ample $\mathscr{O}_X$-module relative to q, and $\mathscr{L}'$ be an invertible $\mathscr{O}_X$-module such that there exists a quasi-coherent $\mathscr{O}_Y$-module $\mathscr{E}'$ and a surjective homomorphism $q^*(\mathscr{E}') \to \mathscr{L}'$. Then $\mathscr{L} \otimes_{\mathscr{O}_X} \mathscr{L}'$ is very ample relative to q.*

Proof. The hypothesis on $\mathscr{L}'$ implies the existence of an Y-morphism $r' : X \to P' = \mathbb{P}(\mathscr{E}')$ such that $\mathscr{L}'$ is isomorphic to $r'^*(\mathscr{O}_{P'}(1))$ (Proposition 9.4.3). There is by hypothesis a quasi-coherent $\mathscr{O}_Y$-module $\mathscr{E}$ and a Y-immersion $r : X \to P = \mathbb{P}(\mathscr{E})$ such that $\mathscr{L}$ is isomorphic to $r^*(\mathscr{O}_P(1))$. Consider the Segre morphism $\zeta : P \times_Y P' \to Q$ where $Q = \mathbb{P}(\mathscr{E} \otimes \mathscr{E}')$. As r is an immersions, so is the morphism $(r, r')_Y : X \to P \times_Y P'$ by Corollary 8.5.16, and therefore we get an immersion

$$r'' : X \stackrel{(r,r')_Y}{\to} P \times_Y P' \stackrel{\zeta}{\to} Q.$$

Since $\zeta^*(\mathscr{O}_Q(1))$ is isomorphic to $\mathscr{O}_P(1) \otimes_Y \mathscr{O}_{P'}(1)$, we conclude from Corollary 8.3.16 that $r''^*(\mathscr{O}_Q(1))$ is isomorphic to $\mathscr{L} \otimes \mathscr{L}'$, this proves the assertion. □

Remark 9.4.21. Note that $q^*(\mathscr{O}_Y^{\oplus I}) = \mathscr{O}_X^{\oplus I}$ and there exists a surjection $\mathscr{O}_X^{\oplus I} \to \mathscr{L}'$ if and only if $\mathscr{L}'$ is generated by global sections, so Proposition 9.4.20 is applicable if $\mathscr{L}'$ is generated by global sections.

Corollary 9.4.22. *Let $q : X \to Y$ be a morphism.*

(a) *Let $\mathscr{L}$ be an invertible $\mathscr{O}_X$-module and $\mathscr{K}$ be an invertible $\mathscr{O}_Y$-module. For $\mathscr{L}$ to be very ample relative to q, it is necessary and sufficent that $\mathscr{L} \otimes q^*(\mathscr{K})$ is very ample relative to q.*

(b) *If $\mathscr{L}$ and $\mathscr{L}'$ are two invertible $\mathscr{O}_X$-modules that are very ample relative to q, then so is $\mathscr{L} \otimes \mathscr{L}'$. In particular, $\mathscr{L}^{\otimes n}$ is very ample relative to q for any $n > 0$.*

Proof. The assertions in (b) is an immediate concequence of Proposition 9.4.20, so is the half implication of (a). Now assume that $\mathscr{L} \otimes q^*(\mathscr{K})$ is very ample; then so is $(\mathscr{L} \otimes q^*(\mathscr{K})) \otimes q^*(\mathscr{K}^{-1})$ by Proposition 9.4.20, which is isomorphic to $\mathscr{L}$. □

Proposition 9.4.23. *Let $f : X \to Y$ be a quasi-compact morphism where Y is quasi-compact, and $\mathscr{L}$ be a very ample $\mathscr{O}_X$-module relative to f. Let $\mathscr{F}$ be a quasi-coherent $\mathscr{O}_X$-module of finite type and for each $n \in \mathbb{Z}$, set $\mathscr{F}(n) = \mathscr{F} \otimes \mathscr{L}^{\otimes n}$. Then there exists an integer n_0 such that for $n \geq n_0$, the canonical homomorphism $\sigma : f^*(f_*(\mathscr{F}(n))) \to \mathscr{F}(n)$ is surjective.*

Proof. □

Proposition 9.4.24. *Let $f : X \to Y$ be a quasi-compact morphism where Y is quasi-compact, and $\mathscr{L}$ be a very ample $\mathscr{O}_X$-module relative to f. Let $\mathscr{F}$ be a quasi-coherent $\mathscr{O}_X$-module of finite type. Then there exists an integer n_0 such that, for $n \geq n_0$, $\mathscr{F}$ is isomorphic to a quotient of an $\mathscr{O}_X$-module $f^*(\mathscr{G}) \otimes_{\mathscr{O}_X} \mathscr{L}^{\otimes(-n)}$, where $\mathscr{G}$ is a quasi-coherent $\mathscr{O}_Y$-module of finite type (dependent on n).*

Proof. Since $\mathscr{L}$ is very ample, f is separated and by Proposition 9.4.23 the canonical homomorphism $\sigma : f^*(f_*(\mathscr{F}(n))) \to \mathscr{F}(n)$ is surjective for n sufficiently large. The proposition is then a generalization of Corollary 9.3.29, and can be proved similarly. □

Proposition 9.4.25 (Properties of Very Ample Sheaves).

(i) *For any scheme Y, any invertible $\mathscr{O}_Y$-module $\mathscr{L}$ is very ample relative to the identify morphism 1_Y.*

(ii) *Let $f : X \to Y$ be a morphism and $j : X' \to X$ be an immersion. If $\mathscr{L}$ is a very ample $\mathscr{O}_X$-module relative to f, then $j^*(\mathscr{L})$ is very ample relative to $f \circ j$.*

(iii) *Let $f : X \to Y$ be a morphism of finite type and $g : Y \to Z$ be a quasi-compact morphism where Z is quasi-compact. Let $\mathscr{L}$ a very ample $\mathscr{O}_X$-module relative to f and $\mathscr{K}$ be a very ample $\mathscr{O}_Y$-module relative to g. Then there exists an integer $n_0 > 0$ such that for any $n \geq n_0$, $\mathscr{L} \otimes f^*(\mathscr{K}^{\otimes n})$ is very ample relative to $g \circ f$.*

(iv) *Let $f : X \to Y$, $g : Y' \to Y$ be two morphisms, and put $X' = X_{(Y')}$. If $\mathscr{L}$ is a very ample $\mathscr{O}_X$-module relative to f, $\mathscr{L}' = \mathscr{L} \otimes_Y \mathscr{O}_{Y'}$ is very ample relative to $f_{(Y')}$.*

(v) *Let $f_i : X_i \to Y_i$ ($i = 1, 2$) be two S-morphisms. If $\mathscr{L}_i$ is a very ample $\mathscr{O}_{X_i}$-module relative to f_i, then $\mathscr{L}_1 \otimes_S \mathscr{L}_2$ is very ample relative to $f_1 \times_S f_2$.*

(vi) *Let $f : X \to Y$, $g : Y \to Z$ be two morphisms. If an $\mathscr{O}_X$-module $\mathscr{L}$ is very ample relative to $g \circ f$, then $\mathscr{L}$ is very ample relative to f.*

(vii) *Let $f : X \to Y$ be a morphism and $j : X_{\mathrm{red}} \to X$ be the canonical injection. If $\mathscr{L}$ is a very ample $\mathscr{O}_X$-module relative to f, then $j^*(\mathscr{L})$ is very ample relative to f_{red}.*

Proof. The property (ii) follows from the definition and it is immedaite that (vii) is deduced from (ii) and (vi). To prove (vi), consider the factorization

$$X \xrightarrow{\Gamma_f} X \times_Z Y \xrightarrow{p_2} Y$$

and note that $p_2 = (g \circ f) \times 1_Y$. It follows from the hypothesis and from (i) and (v) that $\mathscr{L} \otimes_{\mathscr{O}_Z} \mathscr{O}_Y$ is very ample relative to p_2. On the other hand, we have $\mathscr{L} = \Gamma_f^*(\mathscr{L} \otimes_{\mathscr{O}_Z} \mathscr{O}_Y)$ by Corollary 8.3.16, and Γ_f is an immersion (Corollary 8.5.8), so we can apply (ii). As for (i), we can apply the definition with $\mathscr{E} = \mathscr{L}$, and note that $\mathbb{P}(\mathscr{L})$ is identified with Y (Proposition 9.3.6).

We now prove (iv). Under the hypothesis of (iv), there exists a quasi-coherent $\mathscr{O}_Y$-module $\mathscr{E}$ and a Y-immersion $i : X \to P = \mathbb{P}(\mathscr{E})$ such that $\mathscr{L} = i^*(\mathscr{O}_P(1))$. Then, if $\mathscr{E}' = g^*(\mathscr{E})$, $\mathscr{E}'$ is a quasi-coherent $\mathscr{O}_Y$-module and we have $P' = \mathbb{P}(\mathscr{E}') = P_{(Y')}$, $i_{(Y')}$ is an immersion from $X_{(Y')}$ to P', and $\mathscr{L}'$ is isomorphic to $(i_{(Y')})^*(\mathscr{O}_{P'}(1))$ (Remark 9.4.9).

To prove (v), remark that there exists by hypothesis a Y_i-immersion $r_i : X_i \to P_i = \mathbb{P}(\mathscr{E}_i)$, where $\mathscr{E}_i$ is a quasi-coherent $\mathscr{O}_{Y_i}$-module, and $\mathscr{L}_i = r_i^*(\mathscr{O}_{P_i}(1))$. Then $r_1 \times_S r_2$ is an S-immersion of $X_1 \times_S X_2$ to $P_1 \times_S P_2$ (Proposition 8.4.13) and the inverse image of $\mathscr{O}_{P_1}(1) \otimes_S \mathscr{O}_{P_2}(1)$ by this immersion is $\mathscr{L}_1 \otimes_S \mathscr{L}_2$. On the other hand, put $T = Y_1 \times_S Y_2$, and let p_1, p_2 be the projection of T, respectively. If $P_i' = \mathbb{P}(p_i^*(\mathscr{E}_i))$, we have $P_i' = P_i \times_{Y_i} T$, whence

$$P_1' \times_T P_2' = (P_1 \times_{Y_1} T) \times_T (P_2 \times_{Y_2} T) = P_1 \times_{Y_1} (T \times_{Y_2} P_2) = P_1 \times_{Y_1} (Y_1 \times_S P_2) = P_1 \times_S P_2.$$

Similarly, we have $\mathscr{O}_{P_i'}(1) = \mathscr{O}_{P_i'}(1) \otimes_{Y_i} \mathscr{O}_T$, and $\mathscr{O}_{P_1'}(1) \otimes_T \mathscr{O}_{P_2'}(1)$ is identified with $\mathscr{O}_{P_1}(1) \otimes_S \mathscr{O}_{P_2}(1)$. We can then consider $r_1 \times_S r_2$ as an T-immersion from $X_1 \times_S X_2$ to $P_1' \times_T P_2'$, the inverse image of $\mathscr{O}_{P_1'}(1) \otimes_T \mathscr{O}_{P_2'}(1)$ by this immersion being $\mathscr{L}_1 \otimes_S \mathscr{L}_2$. We can then conclude as in Proposition 9.4.20 that $\mathscr{L}_1 \otimes_S \mathscr{L}_2$ is very ample.

It remains to prove (iii). We can first restrict to the case where Z is an affine scheme, since there exists a finite covering (U_i) of Z by affine opens; if the property is proved for $\mathscr{K}|_{g^{-1}(U_i)}$, $\mathscr{L}|_{f^{-1}(g^{-1}(U_i))}$ and an integer n_i, it suffices to choose n_0 to be the largest n_i to prove the property for $\mathscr{K}$ and $\mathscr{L}$ (Corollary 9.4.17). The hypotheses imply that f, g are spearated morphisms, so X and Y are quasi-compact schemes. Since $\mathscr{L}$ is very ample relative to f, there exists an immersion $r : X \to P = \mathbb{P}(\mathscr{E})$, where $\mathscr{E}$ is a quasi-coherent $\mathscr{O}_Y$-module of finite type and $\mathscr{L} = r^*(\mathscr{O}_P(1))$, in view of Proposition 9.4.18. We claim that there exists an integer m_0 such that for any $m \geq m_0$, there is a very ample $\mathscr{O}_P$-module $\mathscr{M}$ relative to the composition morphism $j : P \to Y \xrightarrow{g} Z$ such that $\mathscr{O}_P(1)$ is isomorphic to $\mathscr{M} \otimes_Y \mathscr{K}^{\otimes(-m)}$. For $n \geq m+1$, $\mathscr{O}_P(1) \otimes_Y \mathscr{K}^{\otimes n}$ will then be very ample relative to Z in view of the hypothesis and applying (v) to the morphism $j : P \to Z$ and 1_Z; as r is an immersion and $\mathscr{L} \otimes f^*(\mathscr{K}^{\otimes n}) = r^*(\mathscr{O}_P(1) \otimes_Y \mathscr{K}^{\otimes n})$, the conclusion then follows from (ii).

To estabilsh the claim, we can use Proposition 9.4.24 to obtain a closed immersion j_1 of P to $P_1 = \mathbb{P}(g^*(\mathscr{F}) \otimes \mathscr{K}^{\otimes(-m)})$ such that $\mathscr{O}_P(1)$ is isomorphic to $j_1^*(\mathscr{O}_{P_1}(1))$ (Proposition 9.3.33). On the other hand, there is an isomorphism from P_1 to $P_2 = \mathbb{P}(g^*(\mathscr{F}))$, identifying $\mathscr{O}_{P_1}(1)$ with $\mathscr{O}_{P_2}(1) \otimes_Y \mathscr{K}^{\otimes(-m)}$ (Proposition 9.4.1); we then have a closed immersion $j_2 : P \to P_2$ such that $\mathscr{O}_P(1)$ is isomorphic to $j_2^*(\mathscr{O}_{P_2}(1)) \otimes_Y \mathscr{K}^{\otimes(-m)}$. Finally, P_2 is identified with $P_3 \times_Z Y$ where $P_3 = \mathbb{P}(\mathscr{F})$, and $\mathscr{O}_{P_2}(1)$ is identified with $\mathscr{O}_{P_3}(1) \otimes_Z \mathscr{O}_Y$ (9.4.2). By definition, $\mathscr{O}_{P_3}(1)$ is very ample for Z, and so is $\mathscr{K}$, so it follows from (v) that the $\mathscr{O}_{P_2}$-module $\mathscr{O}_{P_3}(1) \otimes_Y \mathscr{K}$ is very ample for Z. In view of (ii), $\mathscr{M} = j_2^*(\mathscr{O}_{P_3}(1) \otimes_Y \mathscr{K})$ is then very ample for Z, and $\mathscr{O}_P(1)$ is isomorphic to $\mathscr{M} \otimes_Y \mathscr{K}^{\otimes(-m-1)}$, whence the demonstration. □

Proposition 9.4.26. *Let $f : X \to Y$, $f' : X' \to Y$ be two morphisms, $X'' = X \amalg X'$, and $f'' = f \amalg f'$. Let $\mathscr{L}$ (resp. $\mathscr{L}'$) be an invertible $\mathscr{O}_X$-module (resp. $\mathscr{O}_{X'}$-module), and let $\mathscr{L}''$ be the invertible $\mathscr{O}_{X''}$-module which coincides with $\mathscr{L}$ over X and with $\mathscr{L}'$ over X'. For $\mathscr{L}''$ to be very ample relative to f'', it is necessary and sufficent that $\mathscr{L}$ is very ample relative to f and $\mathscr{L}'$ is very ample relative to f'.*

Proof. We can assume that Y is affine. If $\mathscr{L}''$ is very ample then so is $\mathscr{L}$ and $\mathscr{L}'$ in view of Proposition 9.4.25(ii). Conversely, if $\mathscr{L}$ and $\mathscr{L}'$ are very ample, it follows from Remark 9.4.13 that $\mathscr{L}''$ is very ample. □

9.4.5 Ample sheaves

Let X be a scheme, $\mathscr{L}$ be an invertible $\mathscr{O}_X$-module, and $\mathscr{F}$ be an $\mathscr{O}_X$-module. For any $n \in \mathbb{Z}$, we set $\mathscr{F}(n) = \mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{L}^{\otimes n}$ (if there is no confusion), and $S = \bigoplus_{n \geq 0} \Gamma(X, \mathscr{L}^{\otimes n})$. If we consider X as a $\mathbb{Z}$-scheme and let $p : X \to Y = \operatorname{Spec}(\mathbb{Z})$ be the structural morphism, there are bijections

$$\operatorname{Hom}_{\mathsf{Qcoh}(X)}(p^*(\widetilde{S}), \boldsymbol{S}(\mathscr{L})) \xrightarrow{\sim} \operatorname{Hom}_{\mathsf{Qcoh}(Y)}(\widetilde{S}, p_*(\boldsymbol{S}(\mathscr{L}))) \xrightarrow{\sim} \operatorname{Hom}_{\mathbb{Z}}(S, \Gamma(X, \boldsymbol{S}(\mathscr{L})))$$

where we use Proposition 8.2.6. The homomorphism $\varepsilon : p^*(\widetilde{S}) \to \boldsymbol{S}(\mathscr{L})$ corresponding to the canonical injection of S into $\Gamma(X, \boldsymbol{S}(\mathscr{L}))$ is called the **canonical homomorphism associated with** $\mathscr{L}$. It then corresponds to a canonical morphism

$$r_{\mathscr{L},\varepsilon} : G(\varepsilon) \to \operatorname{Proj}(S). \tag{9.4.8}$$

Theorem 9.4.27. *Let X be a quasi-compact and quasi-separated scheme, $\mathscr{L}$ be an invertible $\mathscr{O}_X$-module and $S = \bigoplus_{n \geq 0} \Gamma(X, \mathscr{L}^{\otimes n})$. The following conditions are equivalent:*

- (i) *The subsets X_f, as f runs through the set of homogeneous elements of S_+, form a basis for X.*
- (i′) *The subsets X_f which are affine, as f runs through the set of homogeneous elements of S_+, form a basis for X.*
- (ii) *The canonical morphism $r_{\mathscr{L},\varepsilon} : G(\varepsilon) \to \operatorname{Proj}(S)$ is everywhere defined and a dominant open immersin.*
- (ii′) *The canonical morphism $r_{\mathscr{L},\varepsilon} : G(\varepsilon) \to \operatorname{Proj}(S)$ is everywhere defined and a homeomorphism from X onto a subspace of $\operatorname{Proj}(S)$.*
- (iii) *For any quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, if $\mathscr{F}_n$ is the sub-$\mathscr{O}_X$-module of $\mathscr{F}(n)$ generated by the sections of $\mathscr{F}(n)$ over X, then $\mathscr{F}$ is the direct sum of the sub-$\mathscr{O}_X$-modules $\mathscr{F}_n(-n)$ for $n > 0$.*
- (iii′) *Property* (iii) *holds for any quasi-coherent ideal of $\mathscr{O}_X$.*

Moreover, in this case, if (f_α) is a family of homogeneous elements of S_+ such that X_{f_α} is affine, then the restriction to $\bigcup_\alpha X_{f_\alpha}$ of the canonical morphism $r_{\mathscr{L},\varepsilon} : X \to \operatorname{Proj}(S)$ is an isomorphism from $\bigcup_\alpha X_{f_\alpha}$ to $\bigcup_\alpha (\operatorname{Proj}(S))_{f_\alpha}$.

Proof. It is clear that (ii) implies (ii′), and (ii′) implies (i) in view of the formula (9.3.11). Condition (i) implies (i′), because any $x \in X$ admits an affine neighborhood U such that $\mathscr{L}|_U$ is isomorphic to $\mathscr{O}_X|_U$; if $f \in \Gamma(X, \mathscr{L}^{\otimes n})$ is such that $x \in X_f \subseteq U$, X_f is also the set of $x' \in U$ such that $(f|_U)(x') \neq 0$, and this is an affine open subset of U (hence of X). To prove that (i′) implies (ii), it suffices to show the last assertion of the statement, hence to check that if $X = \bigcup_\alpha X_{f_\alpha}$, the condition (iv) of Proposition 9.3.45 is satisfied; this follows immediately from Theorem 8.6.14(a). To see that $r_{\mathscr{L},\varepsilon}$ is dominant, we note that for $f \in S_+$ homogeneous, X_f is the inverse image of $D_+(f)$ by $r_{\mathscr{L},\varepsilon}$ and by Corollary 8.6.15 we have $\Gamma(X_f, \mathscr{O}_X) = S_{(f)}$ is nonzero if f is not nilpotent, so X_f is nonempty if $D_+(f)$ is not empty.

To prove that (i′) implies (iii), note that if X_f is affine (where $f \in S_d$), $\mathscr{F}|_{X_f}$ is generated by its sections over X_f (Theorem 8.1.21); on the other hand, by Theorem 8.6.14 such a section s is of the form $(t|_{X_f}) \otimes (f|_{X_f})^{-m}$ where $t \in \Gamma(X, \mathscr{F}(md))$. By definition, t is also a section of $\mathscr{F}_{md}$, so s is a section of $\mathscr{F}_{md}(-md)$ over X_f, which proves (iii). It is clear that (iii) implies (iii′), and it rests to show that (iii′) implies (i). Now let U be an open neighborhood of $x \in X$, and let $\mathscr{J}$ be a quasi-coherent ideal of $\mathscr{O}_X$ defining a closed subscheme of X with underlying subspace $X - U$. The hypothesis of (iii′) implies that there exists an integer $n > 0$ and a section $\mathscr{J}(n)$ over X such that $f(x) \neq 0$. Then we have evidently $f \in S_n$ and $x \in X_f \subseteq U$, this proves (i). □

If X is a quasi-compact and quasi-separated scheme, the equivalent conditions of Theorem 9.4.27 implies that X is separated, since it is isomorphic to a subscheme of $\mathrm{Proj}(S)$. We say an invertible $\mathscr{O}_X$-module $\mathscr{L}$ is **ample** if X is a qausi-compact and quasi-separated scheme and the equivalent conditions of Theorem 9.4.27 are satisfied. It follows from Theorem 9.4.27(i) that if $\mathscr{L}$ is an ample $\mathscr{O}_X$-module, then for any open subset U of X, $\mathscr{L}|_U$ is an ample $(\mathscr{O}_X|_U)$-module.

Corollary 9.4.28. *Let $\mathscr{L}$ be an ample $\mathscr{O}_X$-module. For any finite subspace Z of X and any open neighborhood U of Z, there exists an integer $n > 0$ and a section $f \in \Gamma(X, \mathscr{L}^{\otimes n})$ such that X_f is an affine neighborhood of Z contained in U.*

Proof. In view of Theorem 9.4.27(ii), we can view X as a subscheme of $\mathrm{Proj}(S)$ and we only need to prove that for any finite subset Z' of $\mathrm{Proj}(S)$ and any open neighborhood U of Z', there exists a homogeneous element $f \in S_+$ such that $Z \subseteq D_+(f) \subseteq U$. Now by definition the closed set Y, which is the complement of U in $\mathrm{Proj}(S)$, is of the form $V_+(\mathfrak{I})$ where $\mathfrak{I}$ is a graded ideal of S, not containing S_+; on the other hand, the points of Z' are by definition graded prime ideals $\mathfrak{p}_i$ of S_+ not containing $\mathfrak{I}$. There then exists an element $f \in \mathfrak{I}$ not contained in each $\mathfrak{p}_i$ (**??**), and as the $\mathfrak{p}_i$ are graded, we can assume that f is homogeneous. This element f then satisfies the required. $\square$

Proposition 9.4.29. *Suppose that X is a quasi-compact and quasi-separated scheme. Then the conditions of Theorem 9.4.27 are equivalent to the following conditions:*

(iv) *For any quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$ of finite type, there exists an integer $n_0 > 0$ such that, for any $n \geq n_0$, $\mathscr{F}(n)$ is generated by its sections over X.*

(iv′) *For any quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$ of finite type, there exists integers $n > 0, k > 0$ such that $\mathscr{F}$ is isomorphic to a quotient of the $\mathscr{O}_X$-module $\mathscr{L}^{\otimes(-n)} \otimes \mathscr{O}_X^k$.*

(iv″) *Property (iv′) holds for any quasi-coherent ideal of $\mathscr{O}_X$ of finite type.*

Proof. As X is quasi-compact, if a quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$ of finite type is such that $\mathscr{F}(n)$ is generated by global sections, $\mathscr{F}(n)$ is then generated by finitely many global sections (**??**), so (iv) impliess (iv′) and it is clear that (iv′) implies (iv″). As any quasi-coherent $\mathscr{O}_X$-module $\mathscr{G}$ is the inductive limit of its sub-$\mathscr{O}_X$-modules of finte type (Corollary 8.6.65), to verify conditin (iii′) of Theorem 9.4.27, it suffices to verify that for a quasi-coherent ideal of $\mathscr{O}_X$ that is of finite type, and this is clear if condition (iv″) holds. It remains to prove that if $\mathscr{L}$ is an ample $\mathscr{O}_X$-module, then condition (iv) holds. Consider a finite affine open covering (X_{f_i}) of X with $f_i \in S_{n_i}$; by changing f_i by its power, we can assume that the integers n_i equal to the same integer m. The sheaf $\mathscr{F}|_{X_{f_i}}$, being of finite type by hypotheses, is generated by a finitely number of sections h_{ij} over X_{f_i} (Corollary 8.1.24). By Theorem 8.6.14, there then exists an integer k_0 such that the section $h_{ij} \otimes f_i^{k_0}$ extend to a section of $\mathscr{F}(k_0 m)$ over X for any couple (i, j). A fortiori the $h_{ij} \otimes f_i^k$ extend to sections of $\mathscr{F}(km)$ over X for $k \geq k_0$, and for such values of k, $\mathscr{F}(km)$ is then generated by global sections. For any integer p such that $0 < p < m$, $\mathscr{F}(p)$ is also of finite type, so there exist an integer k_p such that $\mathscr{F}(p)(km) = \mathscr{F}(p + km)$ is generated by global sections for $k \geq k_p$. Let n_0 be the largest of the $k_p m$ for $0 < p < m$; we then conclude that $\mathscr{F}(n)$ is generated by global sections for $n \geq n_0$. $\square$

Proposition 9.4.30. *Let X be a quasi-compact and quasi-separated scheme and $\mathscr{L}$ be an invertible $\mathscr{O}_X$-module.*

(a) *Let $n > 0$ be an integer. For $\mathscr{L}$ to be ample, it is necessary and sufficent that $\mathscr{L}^{\otimes n}$ is ample.*

(b) *Let $\mathscr{L}'$ be an inveritble $\mathscr{O}_X$-module such that, for any $x \in X$, there exists a section s' of $\mathscr{L}'^{\otimes n}$ over X such that $s'(x) \neq 0$. Then, if $\mathscr{L}$ is ample, so is $\mathscr{L} \otimes \mathscr{L}'$.*

Proof. Property (a) is a concequence of (i) of Theorem 9.4.27 since $X_{f^{\otimes n}} = X_f$. On the other hand, if $\mathscr{L}$ is ample, for any $x \in X$ and any neighborhood U of x, there exists $m > 0$ and $f \in \Gamma(X, f^{\otimes m})$ such that $x \in X_f \subseteq U$; if $f' \in \Gamma(X, \mathscr{L}'^{\otimes n})$ is such that $f'(x) \neq 0$, then $s(x) \neq 0$ for $s = f^{\otimes n} \otimes f'^{\otimes m} \in \Gamma(X, (\mathscr{L} \otimes \mathscr{L}')^{\otimes mn})$, so $x \in X_s \subseteq X_f \subseteq U$, and therefore $\mathscr{L} \otimes \mathscr{L}'$ is ample. $\square$

Corollary 9.4.31. *The tensor product of two ample $\mathscr{O}_X$-modules is ample.*

Proof. An ample $\mathscr{O}_X$-module satisfies the condition of Proposition 9.4.30(b). $\square$

Corollary 9.4.32. *Let $\mathscr{L}$ be an ample $\mathscr{O}_X$-module and $\mathscr{L}'$ be an invertible $\mathscr{O}_X$-module. There then exists an integer $n_0 > 0$ such that $\mathscr{L}^{\otimes n} \otimes \mathscr{L}'$ is ample and generated by global sections for $n \geq n_0$.*

Proof. It follows from Proposition 9.4.29 that there exists an integer m_0 such that for $m \geq m_0$, $\mathscr{L}^{\otimes m} \otimes \mathscr{L}'$ is generated by global sections, and therefore satisfies the condition of Proposition 9.4.30(b); we can then choose $n_0 = m_0 + 1$. □

Remark 9.4.33. Let $P = \mathrm{Pic}(X) = H^1(X, \mathscr{O}_X^\times)$ be the picard group of $\mathscr{O}_X$-modules, and let P^+ be the subset of P formed by ample sheaves. Suppose that P^+ is nonempty. Then it follows from Corollary 9.4.31 and Corollary 9.4.32 that we have

$$P^+ + P^+ \subseteq P^+, \quad P^+ - P^+ = P.$$

which means $P^+ \cup \{0\}$ is the set of positive elements of P for an order structure over P compatible with the group structure, which is archimedean in view of Corollary 9.4.32.

Proposition 9.4.34. *Let $q : X \to Y$ be a quasi-compact and quasi-separated morphism where Y is affine, and $\mathscr{L}$ be an inveritble $\mathscr{O}_X$-module.*

(a) *If $\mathscr{L}$ is very ample relative to q then $\mathscr{L}$ is ample.*

(b) *Suppose that q is of finite type. Then for $\mathscr{L}$ to be ample, it is necessary and sufficent that it satisfies the following equivalent conditions:*

  (v) *There exists $n_0 > 0$ such that for any integer $n \geq n_0$, $\mathscr{L}^{\otimes n}$ is very ample relative to q.*

  (v′) *There exists $n > 0$ such that $\mathscr{L}^{\otimes n}$ is very ample relative to q.*

Proof. The first assertion follows from the definition of very ample: if A is the ring of Y, there exists an A-module E and a surjective homomorphism

$$\psi : q^*(\widetilde{\boldsymbol{S}(E)}) \to \boldsymbol{S}(\mathscr{L})$$

such that $i = r_{\mathscr{L},\psi}$ is an immersion from X to $P = \mathbb{P}(\widetilde{E})$ such that $\mathscr{L} \cong i^*(\mathscr{O}_P(1))$. As the $D_+(f)$ for $f \in \boldsymbol{S}(E)_+$ homogeneous form a basis for P and $i^{-1}(D_+(f)) = X_{\psi^\flat(f)}$, we see that condition (i) of Theorem 9.4.27 holds, so $\mathscr{L}$ is ample.

Now assume that q is of finite type and $\mathscr{L}$ is ample. It follows from Theorem 9.4.27(ii) and Proposition 9.4.14(a) that there exists an integer $k_0 > 0$ such that $\mathscr{L}^{\otimes k_0}$ is very ample relative to q. On the other hand, in view of Proposition 9.4.29, there exists an integer m_0 such that, for $m \geq m_0$, $\mathscr{L}^{\otimes m}$ is generated by global sections. Put $n_0 = k_0 + m_0$; if $n \geq n_0$, we can write $n = k_0 + m$ where $m \geq m_0$, whence $\mathscr{L}^{\otimes n} = \mathscr{L}^{\otimes k_0} \otimes \mathscr{L}^{\otimes m}$. As $\mathscr{L}^{\otimes m}$ is generated by global sections, it follow from Proposition 9.4.20 and Remark 9.3.28 that $\mathscr{L}^{\otimes n}$ is very ample relative to q. Finally, it is clear that (v) implies (v′), and (v′) implies that $\mathscr{L}$ is ample in view of (a) and Proposition 9.4.30. □

Corollary 9.4.35. *Let $q : X \to Y$ be a quasi-compact and quasi-separated morphism of finite type where Y is affine, $\mathscr{L}$ be an ample $\mathscr{O}_X$-module, and $\mathscr{L}'$ be an invertible $\mathscr{O}_X$-module. Then there exists an integer $n_0 > 0$ such that for $n \geq n_0$, $\mathscr{L}^{\otimes n} \otimes \mathscr{L}'$ is very ample relative to q.*

Proof. In fact, there exists integer m_0 such that for $m \geq m_0$, $\mathscr{L}^{\otimes m} \otimes \mathscr{L}'$ is generated by global section (Corollary 9.4.32); on the other hand, there exists k_0 such that $\mathscr{L}^{\otimes k}$ is very ample relative to q for $k \geq k_0$. Thus $\mathscr{L}^{\otimes (k+m_0)} \otimes \mathscr{L}'$ is very ample for $k \geq k_0$ (Corollary 9.4.19). □

Proposition 9.4.36. *Let X be a quasi-compact scheme, Z be a closed subscheme of X defined by a qausi-coherent nilpotent ideal $\mathscr{J}$ of $\mathscr{O}_X$, and $j : Z \to X$ be the canonical injectin. For an inveritble $\mathscr{O}_X$-module $\mathscr{L}$ to be ample, it is necessary and sufficient that $\mathscr{L}' = j^*(\mathscr{L})$ is an ample $\mathscr{O}_Z$-module.*

Proof. This conditions is necessary. In fact, for any section f of $\mathscr{L}^{\otimes n}$ over X, let f' be the image $f \otimes 1$, which is a section of $\mathscr{L}'^{\otimes n} = \mathscr{L}^{\otimes n} \otimes_{\mathscr{O}_X} (\mathscr{O}_X / \mathscr{J})$ over the subspace Z (identified with X); it is clear that $Z_{f'} = X_f$, hence condition (i) of Theorem 9.4.27 shows that $\mathscr{L}'$ is ample.

To prove the sufficiency, note first that we can reduce to the case $\mathscr{J}^2 = 0$ by considering the finite sequence of schemes $X_k = (X, \mathscr{O}_X / \mathscr{J}^{k+1})$, which is a closed subscheme of the previous one and is defined by a square zero ideal. Now X is quasi-separated if X_{red} is quasi-separated (Proposition 8.6.7(vi)). Criterion (i) of Theorem 9.4.27 shows that it will suffice to prove that, if g is a section of $\mathscr{L}'^{\otimes n}$ over Z

such that Z_g is affine, then there exists $m > 0$ such that $g^{\otimes m}$ is the canonical image of a section f of $\mathscr{L}^{\otimes nm}$ over X. For this, we consider the exact sequence

$$0 \longrightarrow \mathscr{I}(n) \longrightarrow \mathscr{O}_X(n) = \mathscr{L}^{\otimes n} \longrightarrow \mathscr{O}_Z(n) = \mathscr{L}'^{\otimes n} \longrightarrow 0$$

since $\mathscr{F}(n)$ is an exact functor on $\mathscr{F}$; whence an exact sequence on cohomology:

$$0 \longrightarrow \Gamma(X, \mathscr{I}(n)) \longrightarrow \Gamma(X, \mathscr{L}^{\otimes n}) \longrightarrow \Gamma(X, \mathscr{L}'^{\otimes n}) \xrightarrow{\delta} H^1(X, \mathscr{I}(n))$$

which associates in particular g to an element $\delta g \in H^1(X, \mathscr{I}(n))$.

Note that since $\mathscr{I}^2 = 0$, $\mathscr{I}$ can be considered as a quasi-coherent $\mathscr{O}_Z$-module and we have, for any k, $\mathscr{L}'^{\otimes k} \otimes_{\mathscr{O}_Z} \mathscr{I}(n) = \mathscr{I}(n+k)$. For any section $s \in \Gamma(X, \mathscr{L}'^{\otimes k})$, tensoring by s is then a homomorphism $\mathscr{I}(n) \to \mathscr{I}(n+k)$ of $\mathscr{O}_Z$-modules, which gives a homomorphism $H^i(X, \mathscr{I}(n)) \to H^i(X, \mathscr{I}(n+k))$ of cohomology groups. We claim that

$$g^{\otimes m} \otimes \delta g = 0 \tag{9.4.9}$$

for $m > 0$ sufficiently large. In fact, Z_g is an affine open of Z and we have $H^1(Z_g, \mathscr{I}(n)) = 0$ where $\mathscr{I}(n)$ is considered as an $\mathscr{O}_Z$-module. In particular, if we put $g' = g|_{Z_g}$, and if we consider its image under $\delta : \Gamma(Z_g, \mathscr{L}'^{\otimes n}) \to H^1(Z_g, \mathscr{I}(n))$, we have $\delta g' = 0$. To explain this relation, observe that the first cohomology group of a sheaf coincides with the Cech cohomology; to form δg, it is necessary to consider an open covering (U_α) of X, which we can assume that is finite and formed by affine opens, and choose for each α a section $g_\alpha \in \Gamma(U_\alpha, \mathscr{L}^{\otimes n})$ whose image in $\Gamma(U_\alpha, \mathscr{L}'^{\otimes n})$ is $g|_{U_\alpha}$, and consider the class of cocycle $(g_{\alpha\beta} - g_{\beta\alpha})$, where $g_{\alpha\beta}$ is the restriction of g_α to $U_\alpha \cap U_\beta$ (a cocycle with values in $\mathscr{I}(n)$). We can moreover suppose that $\delta g'$ is in the same manner using the covering formed by $U_\alpha \cap Z_g$ and the restrictions $g_\alpha|_{U_\alpha \cap Z_g}$; the relation $\delta g' = 0$ signifies then that there exists for each α a section $h_\alpha \in \Gamma(U_\alpha \cap Z_g, \mathscr{I}(n))$ such that $(g_{\alpha\beta} - g_{\beta\alpha})|_{U_\cap \cap U_\beta \cap Z_g} = h_{\alpha\beta} - h_{\eta\alpha}$, where $h_{\alpha\beta}$ denotes the restriction of h_α to $U_\alpha \cap U_\beta \cap Z_g$. Now by Theorem 8.6.14 there exists an integer $m > 0$ such that $g^{\otimes m} \otimes h_\alpha$ is the restriction to $U_\alpha \cap Z_g$ of a section $t_\alpha \in \Gamma(U_\alpha, \mathscr{I}(n+nm))$ for each α; we then have $g^{\otimes m} \otimes (g_{\alpha\beta} - g_{\beta\alpha}) = t_{\alpha\beta} - t_{\beta\alpha}$ for any couple of indices, which proves $g^{\otimes m} \otimes \delta g = 0$.

We remark on the other hand that if $s \in \Gamma(X, \mathscr{O}_Z(p))$, $t \in \Gamma(X, \mathscr{O}_Z(q))$, we have, in the group $H^1(X, \mathscr{I}(p+q))$, that

$$\delta(s \otimes t) = (\delta s) \otimes t + s \otimes (\delta t). \tag{9.4.10}$$

For this, we can still consider an open cover (U_α) of X, and for each α a section $s_\alpha \in \Gamma(U_\alpha, \mathscr{O}_X(p))$ (resp. a section $t_\alpha \in \Gamma(U_\alpha, \mathscr{O}_X(q))$) whose canonical image in $\Gamma(U_\alpha, \mathscr{O}_Z(p))$ (resp. in $\Gamma(U_\alpha, \mathscr{O}_Z(q))$) is $s|_{U_\alpha}$; the relation (9.4.10) then follows from

$$(s_{\alpha\beta} \otimes t_{\alpha\beta}) - (s_{\beta\alpha} \otimes t_{\beta\alpha}) = (s_{\alpha\beta} - s_{\beta\alpha}) \otimes t_{\alpha\beta} + s_{\beta\alpha} \otimes (t_{\alpha\beta} - t_{\beta\alpha})$$

with the same notations before. By recurrence on k, we then have

$$\delta(g^{\otimes k}) = (kg^{\otimes(k-1)}) \otimes (\delta g) \tag{9.4.11}$$

and in view of (9.4.9) and (9.4.11), we have $\delta(g^{\otimes(m+1)}) = 0$, so $g^{\otimes(m+1)}$ is the canonical image of a section f of $\mathscr{L}^{\otimes n(m+1)}$ over X, whence our demonstration. □

Corollary 9.4.37. *Let X be a Noetherian scheme and $j : X_{\mathrm{red}} \to X$ be the canonical injection. For an inveritble $\mathscr{O}_X$-module $\mathscr{L}$ to be ample, it is necessary and sufficient that $j^*(\mathscr{L})$ is an ample $\mathscr{O}_{X_{\mathrm{red}}}$-module.*

Proof. The nilradical $\mathscr{N}_X$ is nilpotent and we can apply Proposition 9.4.36. □

9.4.6 Relatively ample sheaves

Let $f : X \to Y$ be a quasi-compact morphism and $\mathscr{L}$ be an invertible $\mathscr{O}_X$-module. We say that $\mathscr{L}$ is **ample relative to f**, or relative to Y, or **f-ample**, or **Y-ample**, if there exists an affine open covering (U_α) of Y such that if $X_\alpha = f^{-1}(U_\alpha)$, $\mathscr{L}|_{X_\alpha}$ is an ample $\mathscr{O}_{X_\alpha}$-module for each α. Again, the existence of an f-ample $\mathscr{O}_X$-module implies that X is separated, so f is necessarily separated by Proposition 8.5.26.

Proposition 9.4.38. *Let $f: X \to Y$ be a quasi-compact morphism and $\mathscr{L}$ be an invertible $\mathscr{O}_X$-module. If $\mathscr{L}$ is very ample relative to f, then it is ample relative to f.*

Proof. If $\mathscr{L}$ is very ample relative to f then the morphism f is separated, so by Proposition 9.4.34(a) the restriction $\mathscr{L}|_{f^{-1}(U)}$ for any affine open U of Y is very ample, hence ample. □

Let $f: X \to Y$ be a quasi-compact morphism and $\mathscr{L}$ be an invertible $\mathscr{O}_X$-module. We consider the graded $\mathscr{O}_Y$-algebra $\mathscr{S} = \bigoplus_{n \geq 0} f_*(\mathscr{L}^{\otimes n})$. Then the canonical homomorphisms $\sigma_n : f^*(f_*(\mathscr{L}^{\otimes n})) \to \mathscr{L}^{\otimes n}$ induce a canonical homomorphism

$$\sigma : f^*(\mathscr{S}) \to \bigoplus_{n \geq 0} \mathscr{L}^{\otimes n} = \boldsymbol{S}(\mathscr{L}).$$

On the other hand, it is easy to see that $\sigma^\flat$ is the canonical injection from $\mathscr{S}$ into $f_*(\boldsymbol{S}(\mathscr{L}))$. The homomorphism σ gives an Y-morphism

$$r_{\mathscr{L},\sigma} : G(\sigma) \to \mathrm{Proj}(\mathscr{S}) = P.$$

Proposition 9.4.39. *Let $f: X \to Y$ be a quasi-compact morphism and $\mathscr{L}$ be an invertible $\mathscr{O}_X$-module. Then the following conditions are equivalent:*

(i) *$\mathscr{L}$ is f-ample.*

(ii) *$\mathscr{S}$ is quasi-coherent and the Y-morphism $r_{\mathscr{L},\sigma}$ is everywhere defined and a dominant open immersion.*

(ii′) *The morphism f is separated, the Y-morphism $r_{\mathscr{L},\sigma}$ is everywhere defined and is a homeomorphism from X onto a subspace of* $\mathrm{Proj}(\mathscr{S})$.

Moreover, if these are satisfied, for any $n \in \mathbb{Z}$ the canonical homomorphism of (9.3.12)

$$\nu : r^*_{\mathscr{L},\sigma}(\mathscr{O}_P(n)) \to \mathscr{L}^{\otimes n} \tag{9.4.12}$$

is an isomorphism. Finally, for any quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, if we put $\mathscr{M} = \bigoplus_{n \geq 0} f_*(\mathscr{F} \otimes \mathscr{L}^{\otimes n})$, the canonical homomorphism of (9.3.13)

$$\xi : r^*_{\mathscr{L},\sigma}(\widetilde{\mathscr{M}}) \to \mathscr{F} \tag{9.4.13}$$

is an isomorphism.

Proof. We have remarked that (i) implies that f is separated, so $\mathscr{S}$ is quasi-coherent by Proposition 8.6.55. As the fact that $r_{\mathscr{L},\sigma}$ is an open immersion everywhere defined is local over Y, to shows that (i) implies (ii), we can assume that Y is affine and $\mathscr{L}$ is ample; the assertion then follows from Theorem 9.4.27. It is clear that (ii) implies (ii′); finally, to show that (ii′) implies (i), it suffices to consider an affine open cover (U_α) of Y and use Theorem 9.4.27(ii′) to $\mathscr{L}|_{X_\alpha}$.

To prove the last two assertions, we use the fact that $\sigma^\flat$ is the canonical injection of $\mathscr{S}$ to $f_*(\boldsymbol{S}(\mathscr{L}))$ and the expression of the morphisms ν and ξ in Remark 9.3.43 and Remark 9.3.44. It then follows that ν and ξ are injective. As for the surjectivity, we can assume that Y is affine, so $\mathscr{L}$ is ample; the criterion of Theorem 9.4.27(iii) then shows that ν and ξ are surjective, whence the assertion. □

Remark 9.4.40. From Proposition 9.4.39 and its proof, we conclude that if $\mathscr{L}$ is f-ample then $f_*(\boldsymbol{S}(\mathscr{L}))$ is equal to $\mathscr{S}$, so the homomorphism $\sigma^\flat$ is the identity on $\mathscr{S}$. This can also be seen from Proposition 8.6.57 since in this case f is separated.

Corollary 9.4.41. *Let (U_α) be an open covering of Y. For $\mathscr{L}$ to be Y-ample, it is necessary and sufficient that $\mathscr{L}|_{f^{-1}(U_\alpha)}$ is U_α-ample for each α.*

Proof. This is true since the condition (ii) of Proposition 9.4.39 is local over Y. □

Corollary 9.4.42. *Let $\mathscr{K}$ be an invertible $\mathscr{O}_Y$-module. For $\mathscr{L}$ to be Y-ample, it is necessary and sufficient that $\mathscr{L} \otimes f^*(\mathscr{K})$ is Y-ample.*

Proof. This is a concequence of Corollary 9.4.41 by taking U_α to be such that $\mathscr{K}|_{U_\alpha}$ is isomorphic to $\mathscr{O}_Y|_{U_\alpha}$ for each α. □

Corollary 9.4.43. *Suppose that Y is affine. For $\mathscr{L}$ to be Y-ample, it is necessary and sufficient that $\mathscr{L}$ is ample.*

Proof. This is immediate from the definition of Y-ample, and Proposition 9.4.39(ii) and Theorem 9.4.27(ii), since

$$\operatorname{Proj}(\mathscr{S}) = \operatorname{Proj}(\Gamma(Y,\mathscr{S})) = \operatorname{Proj}(\bigoplus_{n\geq 0}\Gamma(Y,f_*(\mathscr{L}^{\otimes n}))) = \operatorname{Proj}(S)$$

in this case (note that Y is quasi-compact). □

Corollary 9.4.44. *Let $f : X \to Y$ be a quasi-compact morphism. Suppose that there exists a quasi-coherent $\mathscr{O}_Y$-module $\mathscr{E}$ and a Y-morphism $i : X \to P = \mathbb{P}(\mathscr{E})$ which is a homeomorphism from X onto a subspace of P. Then $\mathscr{L} = i^*(\mathscr{O}_P(1))$ is Y-ample.*

Proof. We can assume that Y is affine, and the corollary then follows from the criterion (i) of Theorem 9.4.27 and the formula (9.3.11). □

Proposition 9.4.45. *Let X be a quasi-compact and quasi-separated scheme and $f : X \to Y$ be a quasi-compact and quasi-separated morphism. For an invertible $\mathscr{O}_X$-module $\mathscr{L}$ to be f-ample, it is necessary and sufficient that following equivalent conditions are satisfied:*

- (iii) *For any quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$ of finite type, there exists an integer $n_0 > 0$ such that, for any $n \geq n_0$, the canonical homomorphism $\sigma : f^*(f_*(\mathscr{F}\otimes\mathscr{L}^{\otimes n})) \to \mathscr{F}\otimes\mathscr{L}^{\otimes n}$ is surjective.*
- (iii′) *For any ideal $\mathscr{J}$ of $\mathscr{O}_X$ of finite type, there exist an integer $n > 0$ such that the canonical homomorphism $\sigma : f^*(f_*(\mathscr{J}\otimes\mathscr{L}^{\otimes n})) \to \mathscr{J}\otimes\mathscr{L}^{\otimes n}$ is surjective.*

Proof. □

Proposition 9.4.46. *Let $f : X \to Y$ be a quasi-compact morphism and $\mathscr{L}$ be an invertible $\mathscr{O}_X$-module.*

- (a) *Let $n > 0$ be an integer. For $\mathscr{L}$ to be f-ample, it is necessary and necessary that $\mathscr{L}^{\otimes n}$ is f-ample.*
- (b) *Let $\mathscr{L}'$ be an invertible $\mathscr{O}_X$-module, and suppose that there exists an integer $n > 0$ such that the canonical homomorphism $\sigma : f^*(f_*(\mathscr{L}'^{\otimes n})) \to \mathscr{L}'^{\otimes n}$ is surjective. Then, if $\mathscr{L}$ is f-ample, so is $\mathscr{L}\otimes\mathscr{L}'$.*

Corollary 9.4.47. *The tensor product of two f-ample $\mathscr{O}_X$-modules is f-ample.*

Proposition 9.4.48. *Let $f : X \to Y$ be a morphism of finite type where Y is quasi-compact, and $\mathscr{L}$ be an invertible $\mathscr{O}_X$-module. For $\mathscr{L}$ to be f-ample, it is necessary and sufficient that the following equivalent conditions are satisfied:*

- (iv) *There exists $n_0 > 0$ such that, for any integer $n \geq n_0$, $\mathscr{L}^{\otimes n}$ is very ample relative to f.*
- (iv′) *There exist $n > 0$ such that $\mathscr{L}^{\otimes n}$ is very ample relative to f.*

Lemma 9.4.49. *Let $u : Z \to S$ be a morphism, $\mathscr{L}$ be an invertible $\mathscr{O}_S$-module, s a section of $\mathscr{L}$ over S, and t be the inverse image of s under u. Then $Z_t = u^{-1}(S_s)$.*

Proof. □

Lemma 9.4.50. *Let Z, Z' be two S-schemes, p, p' be the projections of $T = Z\times_S Z'$, $\mathscr{L}$ (resp. $\mathscr{L}'$) be an invertible $\mathscr{O}_Z$-module (resp. $\mathscr{O}_{Z'}$-module), t (resp. t') be a section of $\mathscr{L}$ (resp. $\mathscr{L}'$) over Z (resp. Z'), s (resp. s') be the inverse image of t (resp. t') under p (resp. p'). Then we have $T_{s\otimes s'} = Z_t\times_S Z'_{t'}$.*

Proof. □

Proposition 9.4.51 (Properties of Relative Ample Sheaves).

- (i) *For any scheme Y, any invertible $\mathscr{O}_Y$-module $\mathscr{L}$ is relative ample relative to the identify morphism 1_Y.*
- (ii) *Let $f : X \to Y$ be a quasi-compact morphism and $j : X' \to X$ be a quasi-compact morphism that is a homeomorphism from X' onto a subspace of X. If $\mathscr{L}$ is an $\mathscr{O}_X$-module relative to f, then $j^*(\mathscr{L})$ is relative relative to $f\circ j$.*
- (iii) *Let $f : X \to Y$ and $g : Y \to Z$ be quasi-compact morphisms where Z is quasi-compact. Let $\mathscr{L}$ an ample $\mathscr{O}_X$-module relative to f and $\mathscr{K}$ be an ample $\mathscr{O}_Y$-module relative to g. Then there exists an integer $n_0 > 0$ such that for any $n \geq n_0$, $\mathscr{L}\otimes f^*(\mathscr{K}^{\otimes n})$ is ample relative to $g\circ f$.*

(iv) *Let $f : X \to Y$ be a quasi-compact morphism, $g : Y' \to Y$ be a morphism, and put $X' = X_{(Y')}$. If $\mathscr{L}$ is an ample $\mathscr{O}_X$-module relative to f, $\mathscr{L}' = \mathscr{L} \otimes_Y \mathscr{O}_{Y'}$ is ample relative to $f_{(Y')}$.*

(v) *Let $f_i : X_i \to Y_i$ $(i = 1, 2)$ be two quasi-compact S-morphisms. If $\mathscr{L}_i$ is an ample $\mathscr{O}_{X_i}$-module relative to f_i, then $\mathscr{L}_1 \otimes_S \mathscr{L}_2$ is ample relative to $f_1 \times_S f_2$.*

(vi) *Let $f : X \to Y$, $g : Y \to Z$ be two morphisms such that $g \circ f$ is quasi-compact. If an $\mathscr{O}_X$-module $\mathscr{L}$ is ample relative to $g \circ f$ and if g is separated or X is locally Noetherian, then $\mathscr{L}$ is ample relative to f.*

(vii) *Let $f : X \to Y$ be a quasi-compact morphism and $j : X_{\mathrm{red}} \to X$ be the canonical injection. If $\mathscr{L}$ is an ample $\mathscr{O}_X$-module relative to f, then $j^*(\mathscr{L})$ is ample relative to f_{red}.*

Proof. □

Proposition 9.4.52. *Let $f : X \to Y$ be a quasi-compact morphism, $\mathscr{I}$ be a locally nilpotent ideal of $\mathscr{O}_X$, Z the closed subscheme of X defined by $\mathscr{I}$, and $j : Z \to X$ be the cannical injection. For an invertible $\mathscr{O}_X$-module to be ample relative to f, it is necessary and sufficient that $j^*(\mathscr{L})$ is ample relative to $f \circ j$.*

Corollary 9.4.53. *Let X be a locally Noetherian scheme and $f : X \to Y$ be a quasi-compact morphism. For an invertible $\mathscr{O}_X$-module $\mathscr{L}$ to be ample relative to f, it is necessary and sufficient that its inverse image $\mathscr{L}'$ under the canonical injection $X_{\mathrm{red}} \to X$ is ample relative to f_{red}.*

Proposition 9.4.54. *Let $f : X \to Y$, $f' : X' \to Y$ be two morphisms, $X'' = X \amalg X'$, and $f'' = f \amalg f'$. Let $\mathscr{L}$ (resp. $\mathscr{L}'$) be an invertible $\mathscr{O}_X$-module (resp. $\mathscr{O}_{X'}$-module), and let $\mathscr{L}''$ be the invertible $\mathscr{O}_{X''}$-module which coincides with $\mathscr{L}$ over X and with $\mathscr{L}'$ over X'. For $\mathscr{L}''$ to be ample relative to f'', it is necessary and sufficent that $\mathscr{L}$ is ample relative to f and $\mathscr{L}'$ is ample relative to f'.*

Proposition 9.4.55. *Let Y be a quasi-compact scheme, $\mathscr{S}$ be a quasi-coherent graded $\mathscr{O}_Y$-algebra of finite type, $X = \mathrm{Proj}(\mathscr{S})$, and $f : X \to Y$ be the structural morphism. Then f is of finite type and there exists an integer $n > 0$ such that $\mathscr{O}_X(d)$ is invertible and f-ample.*

9.5 Projective morphisms and Chow's lemma

9.5.1 Quasi-affine morphisms

We say a scheme is **quasi-affine** if it is isomorphic to the subscheme induced over a quasi-compact open subset of an affine scheme. We say a morphism $f : X \to Y$ is quasi-affine, or that X is a quasi-affine Y-scheme, if there exists an affine open cover (U_α) of Y such that $f^{-1}(U_\alpha)$ is a quasi-affine scheme. Since any quasi-compact open subscheme of an affine scheme is separated, it is clear that quasi-affine morphisms are separated and quasi-compact, and any affine morphism is quasi-affine.

Recall that for any scheme X, if $A = \Gamma(X, \mathscr{O}_X)$, the identify homomorphism $A \to A$ induces a canonical morphism $q : X \to \mathrm{Spec}(A)$ by Proposition 8.2.4. This is also the morphism $r_{\mathscr{L},\varepsilon} : X \to \mathrm{Proj}(S)$ in (9.4.8) defined for $\mathscr{L} = \mathscr{O}_X$, since $\Gamma(X, -)$ commutes with taking tensor product with $\mathscr{O}_X$ and we have $S = \bigoplus_{n \geq 0} \Gamma(X, \mathscr{O}_X)^{\otimes n} = A[T]$, and $\mathrm{Proj}(A[T])$ is canonically identified with $\mathrm{Spec}(A)$.

Proposition 9.5.1. *Let X be a quasi-compact scheme and $A = \Gamma(X, \mathscr{O}_X)$. The following conditions are equivalent:*

(i) *X is a quasi-affine scheme.*

(ii) *The canonical morphism $q : X \to \mathrm{Spec}(A)$ is an open immersion.*

(ii') *The canonical morphism $q : X \to \mathrm{Spec}(A)$ is a homeomorphism from X onto a subspace of $\mathrm{Spec}(A)$.*

(iii) *The $\mathscr{O}_X$-module $\mathscr{O}_X$ is very ample relative to q.*

(iii') *The $\mathscr{O}_X$-module $\mathscr{O}_X$ is ample.*

(iv) *The subsets X_f, as f runs through A, form a basis for X.*

(iv') *The subsets X_f which are affine, as f runs through A, form a basis for X.*

(v) *Any quasi-coherent $\mathscr{O}_X$-module is generated by its global sections.*

(v′) *Any quasi-coherent ideal of finite type of $\mathscr{O}_X$ is generated by its global sections.*

Proof. It is clear that (ii)⇒(i) by definition, and (iii)⇒(iii′) by Proposition 9.4.34. Since the canonical morphisms $q : X \to \mathrm{Spec}(A)$ and $r_{\mathscr{O}_X,\varepsilon} : X \to \mathrm{Proj}(S)$ are identified, we see that (iii′)⇔(ii)⇔(ii′)⇔(iv)⇔(iv′) by Theorem 9.4.27. Also, (iii′)⇔(v)⇔(v′) in view of Proposition 9.4.29.

We also note that if X is quasi-affine, then it can be identified as an open subscheme of an affine scheme $Y = \mathrm{Spec}(B)$. Let $\varphi : B \to A$ be the correspond homomorphism (Proposition 8.2.4). Since the affine opens $D(g)$, with $g \in B$, form a basis of Y, and we have $X_f = D(g) \cap X$ where $f = \varphi(g)$, it follows that the subsets X_f which are affine, with $f \in A$, form a basis for X, which proves (i)⇒(iv′).

Finally, it remains to show that (i)⇒(iii). For this we first note that if X is quasi-affine then it is quasi-compact and separated, so by Corollary 8.6.15, for $f \in A$ we have $\Gamma(X_f, \mathscr{O}_X) = A_f$. Since we have $q^{-1}(D_+(f)) = X_f$, we conclude that the canonical morphism $q : X \to \mathrm{Spec}(A)$ is of finite type, and by Proposition 9.4.34, since $\mathscr{O}_X^{\otimes n}$ is isomorphic to $\mathscr{O}_X$ for any integer $n > 0$, $\mathscr{O}_X$ is very ample relative to q. This completes the proof. □

Remark 9.5.2. Let X be a quasi-affine scheme and $A = \Gamma(X, \mathscr{O}_X)$. By Proposition 9.5.1 we know that $\mathscr{O}_X$ is very ample relative to $q : X \to \mathrm{Spec}(A)$. Since X is separated and quasi-compact, $q_*(\mathscr{O}_X)$ is quasi-coherent by Proposition 8.6.55, and from $\Gamma(\mathrm{Spec}(A), q_*(\mathscr{O}_X)) = A$ we conclude that $q_*(\mathscr{O}_X) = \widetilde{A}$, so $q^*(q_*(\mathscr{O}_X)) = \mathscr{O}_X$ and the canonical homomorphism $\sigma : q^*(q_*(\mathscr{O}_X)) \to \mathscr{O}_X$ is identified with the identity on $\mathscr{O}_X$. This being so, the canonical morphism $r_{\mathscr{O}_X,\sigma} : X \to \mathbb{P}(q_*(\mathscr{O}_X))$ is then identified with $q : X \to \mathrm{Spec}(A)$, because we have

$$\mathbb{P}(q_*(\mathscr{O}_X)) = \mathrm{Proj}(\boldsymbol{S}(q_*(\mathscr{O}_X))) = \mathrm{Proj}(\boldsymbol{S}(A)) = \mathrm{Proj}(A[T]) = \mathrm{Spec}(A);$$

and we conclude from Proposition 9.5.1 that this is an open immersion, which justifies Proposition 9.4.16.

Corollary 9.5.3. *Let X be a quasi-compact scheme. If there exists a morphism $r : X \to Y$ from X into an affine scheme Y which is a homeomorphism onto an open subspace of Y, then X is quasi-affine.*

Proof. In fact, there then exists a family (g_α) of sections of $\mathscr{O}_Y$ over Y such that the $D(g_\alpha)$ form a basis for the topology of $r(X)$. If we put $f_\alpha = \theta(g_\alpha)$ where $\theta : \Gamma(Y, \mathscr{O}_Y) \to \Gamma(X, \mathscr{O}_X)$ is the corresponding ring homomorphism, then we have $X_{f_\alpha} = r^{-1}(D(g_\alpha))$, so the X_{f_α} form a basis for X, and by Proposition 9.5.1 X is then quasi-affine. □

Corollary 9.5.4. *If X is a quasi-affine scheme, any invertible $\mathscr{O}_X$-module is very ample (relative to the canonical morphism $q : X \to \mathrm{Spec}(A)$) and a fortiori ample.*

Proof. In fact any such module $\mathscr{L}$ is generated by its global sections (Proposition 9.5.1(v)), so $\mathscr{L} \otimes \mathscr{O}_X$ is very ample by Proposition 9.4.20. We also note that the morphism q is of finite type. □

Corollary 9.5.5. *Let X be a quasi-compact scheme. If there exists an invertible $\mathscr{O}_X$-module $\mathscr{L}$ such that $\mathscr{L}$ and $\mathscr{L}^{-1}$ are ample, then X is quasi-affine.*

Proof. In fact, $\mathscr{O}_X = \mathscr{L} \otimes \mathscr{L}^{-1}$ is then ample by Corollary 9.4.31. □

Proposition 9.5.6. *Let $f : x \to Y$ be a quasi-compact morphism. The following conditions are equivalent:*

(i) *f is quasi-affine.*

(ii) *The $\mathscr{O}_Y$-algebra $f_*(\mathscr{O}_X) = \mathscr{A}(X)$ is quasi-coherent and the canonical morphism $X \to \mathrm{Spec}(\mathscr{A}(X))$ corresponding to the identity homomorphism of $\mathscr{A}(X)$ is an open immersion.*

(ii′) *The $\mathscr{O}_Y$-algebra $\mathscr{A}(X)$ is quasi-coherent and the canonical morphism $X \to \mathrm{Spec}(\mathscr{A}(X))$ is a homeomorphism from X onto a subspace of $\mathrm{Spec}(\mathscr{A}(X))$.*

(iii) *The $\mathscr{O}_X$-module $\mathscr{O}_X$ is very ample relative to f.*

(iii′) *The $\mathscr{O}_X$-module $\mathscr{O}_X$ is ample relative to f.*

(iv) *The morphism f is separated and for any quasi-coherent $\mathscr{O}_X$-modle $\mathscr{F}$, the canonical homomorphism $\sigma : f^*(f_*(\mathscr{F})) \to \mathscr{F}$ is surjective.*

Morever, if f is quasi-affine, any invertible $\mathscr{O}_X$-module $\mathscr{L}$ is very ample relative to f.

Proof. The equivalence of these properties follows from the fact that they are all local over Y and the criteria of Proposition 9.5.1. Also, we note that $f_*(\mathscr{F})$ is quasi-coherent if f is separated (Proposition 8.6.55). The last assertion follows from Corollary 9.5.4. $\square$

Corollary 9.5.7. *Let Y be an affine scheme and $f : X \to Y$ be a quasi-compact morphism. For f to be quasi-affine, it is necessary and sufficient that X is quasi-affine scheme.*

Proof. This is an immediate concequence of Proposition 9.5.6 and Corollary 9.4.43. $\square$

Corollary 9.5.8. *Let Y be a quasi-compact and quasi-separated scheme, $f : X \to Y$ be a morphism of finite type. If f is quasi-affine, there exists a quasi-coherent sub-$\mathscr{O}_Y$-algebra $\mathscr{B}$ of $\mathscr{A}(X)$ of finite type such that the morphism $X \to \operatorname{Spec}(\mathscr{B})$ corresponding to the canonical injection $\mathscr{B} \to \mathscr{A}(X)$ is an immersion. Moreover, any quasi-coherent sub-$\mathscr{O}_Y$-algebra of finite type $\mathscr{B}'$ of $\mathscr{A}(X)$, containing $\mathscr{B}$, has the same property.*

Proof. In fact, $\mathscr{A}(X)$ is the inductive limit of its quasi-coherent sub-$\mathscr{O}_Y$-algebras of finite type (Corollary 8.6.65); the assertion is then a particular case of Proposition 9.3.47, in view of the identification of $\operatorname{Spec}(\mathscr{A}(X))$ and $\operatorname{Proj}(\mathscr{A}(X)[T])$ (Corollary 9.3.5) and the canonical morphisms from X into them (cf. Remark 9.5.2). $\square$

Proposition 9.5.9 (Properties of Quasi-affine Morphisms).

(i) *A quasi-compact morphism $f : X \to Y$ that is a homeomorphism from X onto a subspace of Y (and in particular a quasi-compact immersion) is quasi-affine.*

(ii) *The composition of two quasi-affine morphisms is quasi-affine.*

(iii) *If $f : x \to Y$ is a quasi-affine S-morphism, $f_{(S')} : X_{(S')} \to Y_{(S')}$ is a quasi-affine morphism for any base change $S' \to S$.*

(iv) *If $f : X \to Y$ and $g : X' \to Y'$ are two quasi-affine S-morphisms, so is $f \times_S g$.*

(v) *If $f : X \to Y$, $g : Y \to Z$ are two morphisms such that $g \circ f$ is quasi-affine and if g is separated or X is locally Noetherian, then f is quasi-affine.*

(vi) *Let $f : X \to Y$ be a quasi-compact morphism and $j : X_{\mathrm{red}} \to X$ the canonical injection. If an $\mathscr{O}_X$-module $\mathscr{L}$ is ample relative to f, then $j^*(\mathscr{L})$ is ample relative to f_{red}.*

Proof. In view of the criterion (iii') of Proposition 9.5.6, (i), (iii), (iv), (v) and (vi) are concequences of Proposition 9.4.51. To prove (ii), we can assume that Z is affine, and the assertion then follows from Proposition 9.4.51(iii), applied to $\mathscr{L} = \mathscr{O}_X$ and $\mathscr{K} = \mathscr{O}_Y$. $\square$

Remark 9.5.10. Let $f : X \to Y$, $g : Y \to Z$ be two morphisms such that $X \times_Z Y$ is locally Noetherian. Then the graph morphism $\Gamma_f : X \to X \times_Z Y$ is a quasi-compact immersion, hence quasi-affine, and the reasoning of Proposition 8.5.14 shows that the conclusion of (v) remains valid if we remove the hypothesis that g is separated.

Proposition 9.5.11. *Let $f : X \to Y$ be a quasi-compact morphism and $g : X' \to X$ be a quasi-affine morphism. If $\mathscr{L}$ is an f-ample $\mathscr{O}_X$-module, then $g^*(\mathscr{L})$ is ample relative to $f \circ g$.*

Proof. By hypothesis $\mathscr{O}_{X'}$ is very ample relative to f, and since the question is local over Y, it follows from Proposition 9.4.51(iii) that there exists (for Y affine) an integer n such that $g^*(\mathscr{L}^{\otimes n}) = (g^*(\mathscr{L}))^{\otimes n}$ is ample relative to $f \circ g$, whence $g^*(\mathscr{L})$ is ample relative to $f \circ g$. $\square$

9.5.2 Serre's criterion on affineness

Theorem 9.5.12 (Serre's criterion). *For a quasi-compact and quasi-separated scheme X, then the following conditions are equivalent:*

(i) *X is an affine scheme.*

(ii) *There exists a family (f_α) of elements of $A = \Gamma(X, \mathscr{O}_X)$ such that X_{f_α} are affine and the ideal generated by the f_α equals to A.*

(iii) *The functor $\Gamma(X, -)$ is exact on the category of quasi-coherent $\mathscr{O}_X$-modules.*

(iii') *For any exact sequence of quasi-coherent $\mathscr{O}_X$-modules*

$$0 \longrightarrow \mathscr{F}' \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}'' \longrightarrow 0$$

where $\mathscr{F}$ is isomorphic to a sub-$\mathscr{O}_X$-module of a finite product $\mathscr{O}_X^n$, the induced sequence

$$0 \longrightarrow \Gamma(X, \mathscr{F}') \longrightarrow \Gamma(X, \mathscr{F}) \longrightarrow \Gamma(X, \mathscr{F}'') \longrightarrow 0$$

is exact.

(iv) $H^1(X, \mathscr{F}) = 0$ *for any quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$.*

(iv') $H^1(X, \mathscr{I}) = 0$ *for any quasi-coherent ideal $\mathscr{I}$ of $\mathscr{O}_X$.*

Proof. It is clear that (i) implies (ii); (ii) implies on the oter hand that the X_{f_α} cover X, since by hypothesis the unit section 1 is a linear combination of f_α, and that the $D(f_\alpha)$ cover $\mathrm{Spec}(A)$. The last assertion of Theorem 9.4.27 then implies that $X \to \mathrm{Spec}(A)$ is an isomorphism.

We have seen that (i) implies (iii), and it is trivial that (iii) implies (iii'). On the other hand, (iii') implies that, for any closed point $x \in X$ and any open neighborhood U of x, there exists $f \in A$ such that $x \in X_f \subseteq X - U$. To see this, let $\mathscr{I}$ (resp. $\mathscr{I}'$) be the quasi-coherent ideal of $\mathscr{O}_X$ defining the reduced closed subscheme of X with underlying space $X - U$ (resp. $(X - U) \cup \{x\}$). It is clear that $\mathscr{I}' \subseteq \mathscr{I}$, and the quotient $\mathscr{I}'' = \mathscr{I}/\mathscr{I}'$ is a quasi-coherent $\mathscr{O}_X$-module. By hypothesis, the stalk of $\mathscr{O}_X/\mathscr{I}$ and $\mathscr{O}_X/\mathscr{I}'$ are zero at any point $x \in U - \{x\}$. Moreover, since $\{x\}$ is closed in X, the subscheme $X - U$ is *open and closed* in $(X - U) \cup \{x\}$, so we conclude that $(\mathscr{O}_X/\mathscr{I})_z = (\mathscr{O}_X/\mathscr{I}')_z$ for $z \in X - U$, and therefore $\mathscr{I}''_z = 0$. At the point x, we have $\mathscr{I}_x = \mathscr{O}_X$, while $\mathscr{I}'_x = \mathfrak{m}_x$ (cf. Example 8.4.46), so $\mathscr{I}''$ is supported at $\{x\}$ and $\mathscr{I}''_x = \kappa(x)$. The hypothesis of (iii') applied to the exact sequence $0 \to \mathscr{I}' \to \mathscr{I} \to \mathscr{I}'' \to 0$ shows that $\Gamma(X, \mathscr{I}) \to \Gamma(X, \mathscr{I}'')$ is surjective, so the section of $\mathscr{I}''$ whose germ at x equals to 1_x is the image of a section $f \in \Gamma(X, \mathscr{I}) \subseteq \Gamma(X, \mathscr{O}_X)$, and we have by definition $f(x) = 1_x$ and $f(y) = 0$ over $X - U$, which proves the assertion. Moreover, if U is affine, so is X_f, and the union X' of these affine opens X_f (with $f \in A$) is then an open subset of X containing any closed point of X. As X is a quasi-compact Kolmogoroff space, we then have $X' = X$ ([**?**] new, 0_{I}, 2.1.3). Since X is quasi-compact, there are finitely many elements $f_i \in A$ $(1 \leq i \leq n)$ such that (X_{f_i}) is an affine open cover of X. Consider the homomorphism $\mathscr{O}_X^n \to \mathscr{O}_X$ defined by the sections f_i; since for any $x \in X$ at least one of the $(f_i)_x$ is invertible, this homomorphism is surjective, and we then get an exact sequence

$$0 \longrightarrow \mathscr{R} \longrightarrow \mathscr{O}_X^n \longrightarrow \mathscr{O}_X \longrightarrow 0$$

where $\mathscr{R}$ is a quasi-coherent sub-$\mathscr{O}_X$-module of $\mathscr{O}_X$. It then follows from (iii') that the corresponding homomorphism $\Gamma(X, \mathscr{O}_X^n) \to \Gamma(X, \mathscr{O}_X)$ is surjective, which proves (ii).

Finally, (i) implies (iv) and (iv) implies (iv'). We show that (iv') implies (iii'). Now, if $\mathscr{F}'$ is a quasi-coherent sub-$\mathscr{O}_X$-module of $\mathscr{O}_X^n$, the filtration $0 \subseteq \mathscr{O}_X \subseteq \mathscr{O}_X^2 \cdots \subseteq \mathscr{O}_X^n$ defines over $\mathscr{F}'$ a filtration of the form $\mathscr{F}'_k = \mathscr{F} \cap \mathscr{O}_X^k$ $(0 \leq k \leq n)$, which are quasi-coherent $\mathscr{O}_X$-modules (Corollary 8.2.23(ii)), and $\mathscr{F}'_{k+1}/\mathscr{F}'_k$ is isomorphic to a quasi-coherent sub-$\mathscr{O}_X$-module of $\mathscr{O}_X^{k+1}/\mathscr{O}_X^k = \mathscr{O}_X$, which is thus a quasi-coherent ideal of $\mathscr{O}_X$. In the exact sequence

$$H^1(X, \mathscr{F}'_k) \longrightarrow H^1(X, \mathscr{F}'_{k+1}) \longrightarrow H^1(\mathscr{F}'_{k+1}/\mathscr{F}'_k) = 0$$

by hypothesis of (iv') we have $H^1(\mathscr{F}'_{k+1}/\mathscr{F}'_k) = 0$; since $H^0(X, \mathscr{F}'_0) = 0$, we conclude by recurrence on k that $H^1(X, \mathscr{F}'_k) = 0$ for each k, whence the claim. ☐

Remark 9.5.13. Note that if X is a covering of (X_{f_i}) with X_{f_i} being affine, then X is automatically quasi-separated, since for any couple (i, j) of indices we have $X_{f_i} \cap X_{f_j} = D_{X_{f_i}}(f_j|_{X_{f_i}})$, which is an affine open of X_{f_i} and hence quasi-compact (Proposition 8.6.10).

Remark 9.5.14. If X is a Noetherian scheme, then in conditions (iii') and (iv') we can replace "quasi-coherent" by "coherent." In fact, in the demonstration that (iii') implies (ii), $\mathscr{I}$ and $\mathscr{I}'$ are then coherent ideals, and moreover, any quasi-coherent submodule of a coherent module is coherent (Theorem 8.1.30), whence the assertion.

Corollary 9.5.15. *Let $f : X \to Y$ be a quasi-compact and quasi-separated morphism. Then the following conditions are equivalent:*

(i) *f is an affine morphism.*

(ii) *The functor f_* is exact on the cateogry of quasi-coherent $\mathscr{O}_X$-modules.*

(iii) *For any quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, we have $R^1 f_*(\mathscr{F}) = 0$.*

(iii') *For any quasi-coherent ideal $\mathscr{J}$ of $\mathscr{O}_X$, we have $R^1 f_*(\mathscr{J}) = 0$.*

Proof. Any of these conditions are local over Y, by the definition of $R^1 f_*(\mathscr{F})$ (that is, the sheaf associated with the presheaf $U \mapsto H^1(f^{-1}(U), \mathscr{F})$), so we may assume that Y is affine. If f is affine, X is then affine and (ii) follows from Corollary 8.1.13. Conversely, we prove that (ii) implies (i): for any quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, $f_*(\mathscr{F})$ is a quasi-coherent $\mathscr{O}_Y$-module by Proposition 8.6.55. By hypothesis the fonctor f_* is exact; $\Gamma(Y, -)$ is exact since Y is affine, so we conclude that $\Gamma(Y, f_*(-)) = \Gamma(X, -)$ is exact, which proves that X is affine in view of Theorem 9.5.12.

If f is affine, $f^{-1}(U)$ is affine for any affine open U of Y, so $H^1(f^{-1}(U), \mathscr{F}) = 0$ by Theorem 9.5.12, which means $R^1 f_*(\mathscr{F}) = 0$. Finally, suppose that (iii') is satisfied; the exact sequence of low-degree terms in the Leray spectral sequence gives

$$0 \longrightarrow H^1(Y, f_*(\mathscr{J})) \longrightarrow H^1(X, \mathscr{J}) \longrightarrow H^0(Y, R^1 f_*(\mathscr{J}))$$

As Y is affine and $f_*(\mathscr{J})$ is quasi-coherent (Proposition 8.6.55), we have $H^1(Y, f_*(\mathscr{J})) = 0$, so the hypothesis of (iii') imlies that $H^1(X, \mathscr{J}) = 0$, and we conclude from Theorem 9.5.12 that X is an affine scheme. □

Corollary 9.5.16. *If $f : X \to Y$ is an affine morphism then for any quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, the canonical homomorphism $H^1(Y, f_*(\mathscr{F})) \to H^1(X, \mathscr{F})$ is bijective.*

Proof. In fact, we have an exact sequence

$$0 \longrightarrow H^1(Y, f_*(\mathscr{F})) \longrightarrow H^1(X, \mathscr{F}) \longrightarrow H^0(Y, R^1 f_*(\mathscr{F}))$$

which comes from the lower terms of the Leray spectral sequence, and the conclusion follows from Corollary 9.5.15. □

9.5.3 Quasi-projective morphisms

We say a morphism $f : X \to Y$ is **quasi-projective**, of that X is **quasi-projective over** Y, or that X is a **quasi-projective** Y**-scheme**, if f is of finite type and there exists an invertible $\mathscr{O}_X$-module that is f-ample. It is clear that a quasi-projective morphism is necessarily separated. If Y is quasi-compact, it is also equivalent to say that f is of finite type and there exists a very ample $\mathscr{O}_X$-module relative to f (Proposition 9.4.38).

Remark 9.5.17. It should be noted that this definition is not local over Y. There exist examples where X and Y are nonsingular algebraic schemes over an algebraically closed field such that any point of Y admits an affine neighborhood U such that $f^{-1}(U)$ is quasi-projective over U, but f is not quasi-projective.

Proposition 9.5.18. *Let Y be a quasi-compact and quasi-separated scheme and X be a Y-scheme. Then the following conditions are equivalent:*

(i) *X is a quasi-projective Y-scheme.*

(ii) *X is of finite type over Y and there exists a quasi-coherent $\mathscr{O}_Y$-module $\mathscr{E}$ of finite type such that X is Y-isomorphic to a subscheme of $\mathbb{P}(\mathscr{E})$.*

(iii) *X is of finite type over Y and there exists a quasi-coherent graded $\mathscr{O}_Y$-algebra $\mathscr{S}$ that is generated by $\mathscr{S}_1$ and $\mathscr{S}_1$ is of finite type such that X is isomorphic to a dense open subscheme of* $\mathrm{Proj}(\mathscr{S})$.

Proof. This follows from Corollary 9.4.15, Proposition 9.4.18 and Corollary 9.4.19. □

Corollary 9.5.19. *Let Y be a quasi-compact and quasi-separated scheme such that there exists an ample $\mathscr{O}_Y$-module $\mathscr{L}$. For a Y-scheme X to be quasi-projective, it is necessary and sufficient that X is of finite type over Y and is isomorphic to a sub-Y-scheme of a projective bundle of the form $\mathbb{P}^r_Y$.*

Proof. By the hypothesis on Y, if $\mathscr{E}$ is a quasi-coherent $\mathscr{O}_Y$-module of finite type, $\mathscr{E}$ is isomorphic to a quotient of an $\mathscr{O}_Y$-module $\mathscr{L}^{\otimes(-n)} \otimes_{\mathscr{O}_Y} \mathscr{O}_Y^k$ (Proposition 9.4.29), so $\mathbb{P}(\mathscr{E})$ is isomorphic to a closed subscheme of $\mathbb{P}^{k-1}_Y$ (Proposition 9.4.1). □

Proposition 9.5.20 (Properties of Quasi-projective Morphisms).

(i) *A quasi-affine morphism of finite type (in particular a quasi-compact immersion or an affine morphism of finite type) is quasi-projective.*

(ii) *If $f : X \to Y$ and $g : Y \to Z$ are quasi-projective and if Z is quasi-compact, $g \circ f$ is quasi-projective.*

(iii) *If $f : X \to Y$ is a quasi-projective S-morphism, $f_{(S')} : X_{(S')} \to Y_{(S')}$ is quasi-projective for any base change $S' \to S$.*

(iv) *If $f : X \to Y$ and $g : X' \to Y'$ are two quasi-projective S-morphisms, $f \times_S g$ is quasi-projective.*

(v) *If $f : X \to Y$, $g : y \to Z$ are two morphisms such that $g \circ f$ is quasi-projective and if g is separated or X is locally Noetherian, then f is quasi-projective.*

(vi) *If f is quasi-projective, so is f_{red}.*

Proof. Property (i) follows from Proposition 9.5.6 and Proposition 9.4.51(i). The other parts follows from the definition of quasi-projective morphism and Proposition 9.4.51, with the corresponding properties of morphisms of finite type (Proposition 8.6.35). □

Remark 9.5.21. Note that it may happen that f_{red} is quasi-projective without f being so, even we assume that Y is the spectrum of a finite dimensional algebra over C and f is proper.

Corollary 9.5.22. *If X and X' are two quasi-projective Y-schemes, $X \amalg X'$ is quasi-projective over Y.*

Proof. This follows from Proposition 9.4.54. □

9.5.4 Universally closed and proper morphisms

As the terminology indicates, we say a morphism $f : X \to Y$ is **universally closed** if the projection $f \times_Y 1_{Y'} : X \times_Y Y' \to Y'$ is closed for any base change $Y' \to Y$. By Corollary 8.4.14, we know that a closed immersion is universally closed. We say a morphism $f : X \to Y$ is **proper** if it is separated, of finite type, and universally closed, and in this case X is said to be **proper over** Y, or a **proper** Y**-scheme**. It is clear that all these notations are local over Y. We also note that, to verify that the image of a closed subset Z of $X \times_Y Y'$ under the projection $q : X \times_Y Y' \to Y'$ is closed in Y', it suffices to shows that $q(Z) \cap U'$ is closed in U' for any affine open subset U' of Y'. As $q(Z) \cap U' = q(Z \cap q^{-1}(U'))$ and $q^{-1}(U')$ is identified with $X \times_Y U'$ (Corollary 8.3.2), we see that to verify the universally closedness of f, it suffices to limit the case where Y' is affine. We will see later that if Y is locally Noetherian, we can even assume that Y' is of finite type over Y.

Proposition 9.5.23 (Properties of Proper Morphisms).

(i) *A closed immersion is proper.*

(ii) *The composition of two proper morphisms is proper.*

(iii) *If $f : X \to Y$ is a proper S-morphism, then $f_{(S')} : X_{(S')} \to Y_{(S')}$ is proper for any base change $S' \to S$.*

(iv) *If $f : X \to Y$ and $g : X' \to Y'$ are two proper S-morphisms, then $f \times_S g$ is proper.*

Proof. It suffices to prove the first three properties. In view of Proposition 8.5.26 and Proposition 8.6.35, it suffices to verify the universally closedness in each cases. This is trivial in (i) since closed immersions are universal. For (ii), consider two proper morphisms $X \to Y$, $Y \to Z$, and a morphism $Z' \to Z$. We have $X \times_Z Z' = X \times_Y (Y \times_Z Z')$ and therefore the projection $X \times_Z Z' \to Z'$ factors into $X \times_Y (Y \times_Z Z') \to Y \times_Z Z' \to Z'$. By hypothesis, this is a composition of two closed morphisms, hence closed. Finally, in (iii), for any morphism $S' \to S$, $X_{(S')}$ is identified with $X \times_Y Y_{(S')}$; for any morphism $Z \to Y_{(S')}$, we have

$$X_{(S')} \times_{Y_{(S')}} Z = (X \times_Y Y_{(S')}) \times_{Y_{(S')}} Z = X \times_Y Z$$

and $X \times_Y Z \to Z$ is closed by hypothesis, so (iii) follows. □

Corollary 9.5.24. *Let $f : X \to Y$, $g : Y \to Z$ be two morphisms such that $g \circ f$ is proper.*

(a) *If g is separated, f is proper.*

(b) *If g is separated and of finite type and f is surjective, then g is proper.*

Proof. The first claim follows from Proposition 8.5.22. To prove (b), we only need to verify that g is universally closed. For any morphism $Z' \to Z$, the diagram

$$\begin{tikzcd} X \times_Z Z' \arrow[r, "f \times 1_{Z'}"] \arrow[dr, "p"'] & Y \times_Z Z' \arrow[d, "p'"] \\ & Z' \end{tikzcd}$$

(where p and p' are projections) is commutative. Moreover, $f \times 1_{Z'}$ is surjective if f is (Proposition 8.3.28), and p is a closed immersion by hypothesis. Any closed subset F of $Y \times_Z Z'$ is then the image under $f \times 1_{Z'}$ of a closed subset E of $X \times_Z Z'$, so $p'(F) = p(E)$ is closed in Z' by hypothesis, whence the corollary. □

Corollary 9.5.25. *If X is a proper scheme over Y and $\mathscr{S}$ is a quasi-coherent $\mathscr{O}_Y$-algebra, any Y-morphism $f : X \to \mathrm{Proj}(\mathscr{S})$ is proper (and a fortiori closed).*

Proof. In fact, the structural morphism $p : \mathrm{Proj}(\mathscr{S}) \to Y$ is separated, and $p \circ f$ is proper by hypothesis. □

Corollary 9.5.26. *Let $f : X \to Y$ be a separated morphism of finite type. Let $(X_i)_{1 \le i \le n}$ (resp. $(Y_i)_{1 \le i \le n}$) be a family of closed subscheme of X (resp. Y), $j_i : X_i \to X$ (resp. $h_i : Y_i \to Y$) be the canonical injection. Suppose that (X_i) forms a covering of X and for each i, let $f_i : X_i \to Y_i$ be a morphism such that the diagram*

$$\begin{tikzcd} X_i \arrow[r, "f_i"] \arrow[d, "j_i"'] & Y_i \arrow[d, "h_i"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

is commutative. Then, for f to be proper, it is necessary and sufficient that each f_i is proper.

Proof. If f is proper, so is each $f \circ j_i$, since j_i is a closed immersion; as each h_i is a closed immersion, hence separated, f_i is proper by Corollary 9.5.24. Suppose conversely that each f_i is proper, and consider the sum Z of X_i; let $u : Z \to X$ be the morphism that induces j_i on X_i. The restriction of $f \circ u$ to each X_i is equal to $f \circ j_i = h_i \circ f_i$, hence proper; it then follows that $f \circ u$ is proper. Since u is surjective by hypothesis, we conclude from Corollary 9.5.24 that f is proper. □

Corollary 9.5.27. *Let $f : X \to Y$ be a separated morphism of finite type. For f to be proper, it is necessary and sufficient that $f_{\mathrm{red}} : X_{\mathrm{red}} \to Y_{\mathrm{red}}$ is proper.*

Proof. This is a particular case where $n = 1$, $X_1 = X_{\mathrm{red}}$ and $Y_1 = Y_{\mathrm{red}}$. □

If X and Y and Noetherian schemes and $f : X \to Y$ is a separated morphism of finite type, to verify that f is proper, we can reduce to dominant morphisms of integral schemes. In fact, let X_i ($1 \le i \le n$) be the irreducible components of X and consider for each i the unique reduced closed subscheme structure on X_i. Let Y_i be the reduced closed subscheme with underlying space $\overline{f(X_i)}$. If $j_i : X_i \to X$ (resp. $h_i : Y_i \to Y$) are the canonical injections, we then have $f \circ j_i = h_i \circ f_i$, where f_i is a dominant morphism $f_i : X_i \to Y_i$. We then see that the conditions of Corollary 9.5.26 are satisfied, and for f to be proper, it is necessary and sufficient that each f_i is.

Corollary 9.5.28. *Let X and Y be separated S-schemes of finite type and $f : X \to Y$ be an S-morphism. For f to be proper, it is necessary and sufficient that for any S-scheme S', the morphism $f_{(S')} : X_{(S')} \to Y_{(S')}$ is closed.*

Proof. We note that if $\varphi : X \to S$ and $\psi : Y \to S$ are the structural morphisms, we have $\varphi = \psi \circ f$, so f is separated and of finite type (Proposition 8.5.26 and Proposition 8.6.35). If f is proper, so is $f_{(S')}$, and is a fortiori closed. Conversely, assume this condition and let Y' be a Y-scheme; Y' can be considered as an S-scheme, and the morphism $Y \to S$ is separated. In the commutative diagram

$$\begin{tikzcd} X \times_Y Y' \arrow[r, "f\times_Y 1_{Y'}"] \arrow[d] & Y \times_Y Y' = Y' \arrow[d] \\ X \times_S Y' \arrow[r, "f_{(Y')}"] & Y \times_S Y' \end{tikzcd}$$

the vertical morphisms are a closed immersion by Proposition 8.5.18. It follows from the assumption that $f_{(Y')}$ is closed, and so is $f \times_Y 1_{Y'}$. □

Let $f : X \to Y$ be a morphism of finit type. We say that a closed subset Z of X is **proper over** Y (or Y**-proper**, or $\boldsymbol{f}$**-proper**) if the restriction of f to a closed subscheme of X with underlying space Z is proper. As this restriction is then separated, it follows from Corollary 9.5.27 and Proposition 8.5.26(vi) the property that Z is proper over Y is independent of the closed subscheme structural chosen for Z.

Let Z be a proper subset of X for f and let $g : X' \to X$ be a proper morphism. Then $g^{-1}(Z)$ is then a proper subset of X': if T is a subscheme of X with underlying space Z, it suffices to note that the restriction of g to the closed subscheme $g^{-1}(T)$ of X' is a proper morphism $g^{-1}(T) \to T$ by Proposition 9.5.23(iii), and we can apply Proposition 9.5.23(ii) to conclude that $g^{-1}(T)$ is proper.

On the other hand, if X'' is a Y-scheme of finite type and $h : X \to X''$ is a Y-morphism, $h(Z)$ is also a proper subset of X'': in fact, for any reduced closed subscheme T of X with underlying space Z. The restriction of f to T is proper, and so is the restriction of h to T (Corollary 9.5.24(a)), so $h(Z)$ is closed in X''. Let T'' be a closed subscheme of X'' with underlying space $h(Z)$ so that the morphism $h|_T$ factors into (cf. Proposition 8.4.48)

$$T \xrightarrow{h|_T} T'' \xrightarrow{j} X''$$

where j is the canonical injection. Then $h|_T$ is proper by Corollary 9.5.26 and surjective. If $\psi : X'' \to Y$ is the structural morphism, $\psi|_{T''}$ is then separated of finite type (Proposition 8.5.26 and Proposition 8.6.35), and we have $f|_T = (\psi|_{T''}) \circ (h|_T)$; it then follows from Proposition 9.5.23(ii) that $\psi|_{T''}$ is proper, whence the assertion.

In particular, for a Y-proper subset of X, we have the following:

(a) For any closed subset X' of X, $Z \cap X'$ is a Y-proper subset of X'.

(b) If X is a subscheme of a Y-scheme of finite type X'', Z is also a Y-proper subset of X'' (and in particular is closed in X'').

9.5.5 Projective morphisms

Proposition 9.5.29. *Let X be a Y-scheme. The following conditins are equivalent:*

(a) *X is Y-isomorphic to a closed subscheme of a projective bundle $\mathbb{P}(\mathscr{E})$, where $\mathscr{E}$ is a quasi-coherent $\mathscr{O}_Y$-module of finite type.*

(b) *There exists a quasi-coherent graded $\mathscr{O}_Y$-algebra such that $\mathscr{S}_1$ is of finite type and generates $\mathscr{S}$, and that X is Y-isomorphic to a* $\mathrm{Proj}(\mathscr{S})$.

Proof. Condition (a) implies (b) by Proposition 9.3.33(b): if $\mathscr{I}$ is the quasi-coherent graded ideal of $\boldsymbol{S}(\mathscr{E})$, the quasi-coherent graded $\mathscr{O}_Y$-algebra $\mathscr{S} = \boldsymbol{S}(\mathscr{E})/\mathscr{I}$ is generateed by $\mathscr{S}_1$ and the later, which is the canonical image of $\mathscr{E}$, is an $\mathscr{O}_X$-module of finite type. Condition (b) implies (a) in view of Corollary 9.3.35 applied to the case where $\mathscr{M} \to \mathscr{S}_1$ is the identify homomorphism. □

We say a Y-scheme X is **projective over** Y or a projective Y**-scheme** if it satisfies the equivalent conditions of Proposition 9.5.29. We say a morphism $f : X \to Y$ is **projective** if X is a projective Y-scheme via this morphism. It is clear that if $f : X \to Y$ is projective, then there exists a very ample $\mathscr{O}_X$-module relative to f (Corollary 9.4.15).

Theorem 9.5.30. *Any projective morphism is quasi-projective and proper. Conversely, if Y is a quasi-compact and quasi-separated scheme, any quasi-projective and proper morphism $f : X \to Y$ is projective.*

Proof. It is clear that any projective morphism is of finite type and quasi-projective. On the other hand, it follows from Proposition 9.5.29(b) and Proposition 9.3.31 that if f is projective, so is $f \times_Y 1_{Y'} : X \times_Y Y' \to Y'$ for any morphism $Y' \to Y$. The proof that f is universally closed then boils down to show that a projective morphism f is closed. The question is local over Y, so we can suppose that $Y = \mathrm{Spec}(A)$ is affine, so by Proposition 9.5.29 $X = \mathrm{Proj}(S)$ where S is a graded A-algebra generated by finitely many elements of S_1. For any $y \in Y$, the fiber $f^{-1}(y)$ is identified with $\mathrm{Proj}(S) \times_Y \mathrm{Spec}(\kappa(y))$, hence to $\mathrm{Proj}(S \otimes_A \kappa(y))$ (Proposition 9.2.50). Therefore, $f^{-1}(y)$ is empty if and only if $S \otimes_A \kappa(y)$ is eventually zero, which meaens $S_n \otimes_A \kappa(y) = 0$ for n sufficiently large. Now as $(S_n)_y$ is an $\mathscr{O}_{Y,y}$-module of finite type, the preceding condition signifies that $(S_n)_y = 0$ for n sufficiently large, in view of Nakayama lemma. If $\mathfrak{a}_n$ is the annihilator in A of the A-module S_n, the preceding condition is then equivalent to that $\mathfrak{a}_n \subseteq \mathfrak{p}_y$ for n sufficiently large. Now as $S_n S_1 = S_{n+1}$ by hypothesis, we have $\mathfrak{a}_n \subseteq \mathfrak{a}_{n+1}$, and if $\mathfrak{a}$ is the sum of $\mathfrak{a}_n$, we then have $f(X) = V(\mathfrak{a})$, so $f(X)$ is closed in Y. If now X' is a closed subset of X, there exists a closed subscheme of X with underlying space X' and it is clear (by Proposition 9.5.29(a)) that the composition morphism $X' \to X \to Y$ is projective, so $f(X')$ is closed in Y.

Now conversely, assume that Y is quasi-compact and quasi-separated. The hypothesis that f is quasi-projective implies the existence of a quasi-coherent $\mathscr{O}_Y$-module of finite type $\mathscr{E}$ and a Y-immersion $j : X \to \mathbb{P}(\mathscr{E})$ (Proposition 9.5.18). Since f is proper and the structural morphism $\mathbb{P}(\mathscr{E}) \to Y$ is separated, j is proper (hence closed) by Corollary 9.5.24(a), so f is projective. ☐

Remark 9.5.31. Let $f : X \to Y$ be a morphism such that

(i) f is proper;

(ii) there exists a very ample $\mathscr{O}_X$-module $\mathscr{L}$ relative to f;

(iii) the quasi-coherent $\mathscr{O}_Y$-module $\mathscr{E} = f_*(\mathscr{L})$ is of finite type.

Then f is projective: there then exists a Y-immersion $j : X \to \mathbb{P}(\mathscr{E})$, and since f is proper, j is a closed immersion by Corollary 9.5.24(a). We will see that if Y is locally Noetherian, the last condition (iii) is a concequence of the others, and conditions (i) and (ii) characterize projective morphisms. If Y is Noetherian, we can further repalce in (ii) that there exists a *ample* $\mathscr{O}_X$-module relative to f (Proposition 9.4.48). We also note that there are proper morphisms that is not projective.

Remark 9.5.32. Let Y be a quasi-compact scheme such that there exists an ample $\mathscr{O}_Y$-module. For a Y-scheme X to be projective, it is necessary and sufficient that it is isomorphic to a closed subscheme of a projective bundle of the form $\mathbb{P}^r_Y$. This conditions is clearly sufficient. Conversely, if X is projective over Y, it is quasi-projective, so there exists a Y-immersion $j : X \to \mathbb{P}^r_Y$ by Corollary 9.5.19, which is closed by Corollary 9.5.24(a) and Theorem 9.5.30.

Remark 9.5.33. The reasoning of Theorem 9.5.30 shows that for any scheme Y, and any integer $r \geq 0$, the structural morphism $\mathbb{P}^r_Y \to Y$ is surjective, because if we put $\mathscr{S}_{\mathscr{O}_Y} = \boldsymbol{S}(\mathscr{O}_Y^{r+1})$, we have evidently $\mathscr{S}_y = \boldsymbol{S}_{\kappa(y)}(\kappa(y)^{r+1})$, so $(\mathscr{S}_n)_y \neq 0$ for any $y \in Y$ and $n \geq 0$.

Proposition 9.5.34 (Properties of Projective Morphisms).

(i) *A closed immersion is projective.*

(ii) *If $f : X \to Y$ and $g : Y \to Z$ are projective morphisms and if Z is quasi-compact and quasi-separated, then $g \circ f$ is projective.*

(iii) *If $f : X \to Y$ is a projective morphism, $f_{(S')} : X_{(S')} \to Y_{(S')}$ is projective for any base change $S' \to S$.*

(iv) *If $f : X \to Y$ and $g : X' \to Y'$ are projective S-morphisms, so is $f \times_S g$.*

(v) *If $g \circ f$ is a projective morphism and if g is separated, f is projective.*

(vi) *If f is projective, so is f_{red}.*

Proof. Property (i) follows from Corollary 9.3.5. It is necessary here to prove (iii) and (iv) separately, because of the restriction introduced on Z in (ii). To prove (iii), we can reduce to the case where $S = Y$ (Corollary 8.3.10) and the assertion then follows immediately from Proposition 9.5.29(b) and from Proposition 9.3.31. To prove (iv), we can assume that $X = \mathbb{P}(\mathscr{E})$, $X' = \mathbb{P}(\mathscr{E}')$, where $\mathscr{E}$ (resp. $\mathscr{E}'$) is a quasi-coherent $\mathscr{O}_Y$-module of finite type. Let p, p' be the projection of $T = Y \times_S Y'$ to Y and Y' respectively; by (9.4.1), we have $\mathbb{P}(p^*(\mathscr{E})) = \mathbb{P}(\mathscr{E}) \times_Y T$ and $\mathbb{P}(p'^*(\mathscr{E})) = \mathbb{P}(\mathscr{E}') \times_{Y'} T$, whence

$$\begin{aligned}\mathbb{P}(p^*(\mathscr{E})) \times_T \mathbb{P}(p'^*(\mathscr{E}')) &= (\mathbb{P}(\mathscr{E}) \times_Y T) \times_T (T \times_{Y'} \mathbb{P}(\mathscr{E}')) = \mathbb{P}(\mathscr{E}) \times_Y (T \times_{Y'} \mathbb{P}(\mathscr{E}')) \\ &= \mathbb{P}(\mathscr{E}) \times_Y ((Y \times_S Y') \times_{Y'} \mathbb{P}(\mathscr{E}')) = \mathbb{P}(\mathscr{E}) \times_S \mathbb{P}(\mathscr{E}').\end{aligned}$$

Now $p^*(\mathscr{E})$ and $p'^*(\mathscr{E}')$ are of finite type over T (**??**), and so is $p^*(\mathscr{E}) \otimes_{\mathscr{O}_T} p'^*(\mathscr{E}')$; as $\mathbb{P}(p^*(\mathscr{E})) \times_T \mathbb{P}(p'^*(\mathscr{E}'))$ is identified with a closed subscheme of $\mathbb{P}(p^*(\mathscr{E}) \otimes_{\mathscr{O}_T} p'^*(\mathscr{E}'))$ (Proposition 9.4.12), this proves (iv). For (v) and (vi), we can apply Proposition 8.5.22, since any closed subscheme of a projective Y-scheme is projective by Proposition 9.5.29(a). □

Proposition 9.5.35. *If X and X' are two projective Y-schemes, so is $X \amalg X'$.*

Proof. This is clear from Remark 9.4.13. □

Proposition 9.5.36. *Let X be a projective Y-scheme, $\mathscr{L}$ be a Y-ample $\mathscr{O}_Y$-module. For any section f of $\mathscr{L}$ over X, X_f is affine over Y.*

Proof. Since the question is local over Y, we can assume that $Y = \operatorname{Spec}(A)$ is affine; since $X_{f^{\otimes n}} = X_f$, by replacing $\mathscr{L}$ by $\mathscr{L}^{\otimes n}$ we can assume that $\mathscr{L}$ is very ample for the structural morphism $q : X \to Y$ (Proposition 9.4.34). The canonical homomorphism $\sigma : q^*(q_*(\mathscr{L})) \to \mathscr{L}$ is then surjective and the corresponding morphism

$$r = r_{\mathscr{L},\sigma} : X \to P = \mathbb{P}(q_*(\mathscr{L}))$$

is an immersion such that $\mathscr{L} \cong r^*(\mathscr{O}_P(1))$ (Proposition 9.4.16). Moreover, as X is proper over Y, the immersion r is closed by Corollary 9.5.24. By definition $f \in \Gamma(Y, q_*(\mathscr{L}))$ and $\sigma^\flat$ is the identity of $q_*(\mathscr{L})$; it then follows from the formula (9.3.11) that we have $X_f = r^{-1}(D_+(f))$. Then X_f is a closed subscheme of the affine scheme $D_+(f)$, and therefore is affine. □

Remark 9.5.37. If we take $Y = X$ in Proposition 9.5.36, we obtain that, for any scheme X and any invertible $\mathscr{O}_X$-module $\mathscr{L}$, the open subset X_f is affine over X.

Proposition 9.5.38. *Let X be a projective Y-scheme and $\mathscr{L}$ be a Y-ample $\mathscr{O}_Y$-module. Then X is Y-isomorphic to $\operatorname{Proj}(\mathscr{S})$, where $\mathscr{S} = \bigoplus_{n\geq 0} f_*(\mathscr{L}^{\otimes n})$.*

Proof. By Proposition 9.4.39 we see that the canonical morphism $r_{\mathscr{L},\sigma} : X \to \operatorname{Proj}(\mathscr{S})$ is a dominant open immersion. Since X is proper over Y and $\operatorname{Proj}(\mathscr{S})$ is separated over Y, the image of $r_{\mathscr{L},\sigma}$ is closed by Corollary 9.5.24, so $r_{\mathscr{L},\sigma} : X \to \operatorname{Proj}(\mathscr{S})$ is an isomorphism. □

9.5.6 Chow's lemma

Theorem 9.5.39 (Chow's lemma). *Let X be a separated S-scheme of finite type and suppose that one of the following conditions is satisfied:*

(a) *S is Noetherian.*

(b) *S is quasi-compact and X has finitely many irreducible components.*

Then there exists a quasi-projective S-scheme X' and a surjective projective S-morphism $f : X' \to X$ that induces an isomorphism an isomorphism $f^{-1}(U) \cong U$ for some open dense subset of X. If X is reduced (resp. irreducible), we can also choose X' to be reduced (resp. irreducible).

Proof. The proof is divided into several steps. First of all, we can assume that X is irreducible. To see this, we note that in both cases the scheme X has finitely many irreducible components X_i. If the theorem is demonstrated for each reduced subscheme X_i, and if $f_i : X_i' : X_i$ is the corresponding hommorphism which induces an isomorphism $f_i^{-1}(U_i) \cong U_i$ with $U_i \subseteq X_i$, the sum $X' = \coprod_i X_i'$ is then quasi-projective over S (Corollary 9.5.22 and Proposition 9.5.20) and the morphism $f : X' \to X$ whose restriction on X_i' equals to $j_i \circ f_i$ (where $j_i : X_i \to X$ is the canonical injection), is then surjective and projective (Proposition 9.5.35); it is immediate to see that X' is reduced if each X_i' is. We now choose U to be the union of $U_i \cap (\bigcup_{j \neq i} X_j)^c$; since U_i is dense in X_i and X_i is maximal irreducible, we conclude that each $U_i \cap (\bigcup_{j \neq i} X_j)$ is nonempty. The open subset U is then dense in X and f clearly induces an isomorphism $f^{-1}(U) \cong U$.

So suppose now that X is irreducible. As the structural morphism $\eta : X \to S$ is of finite type, there exists a finite covering (S_i) of S by affine opens, and for each i there is a finite covering (T_{ij}) of $\eta^{-1}(S_i)$ by affine opens, with the morphism $T_{ij} \to S_i$ being affine and of finite type, hence quasi-projective (Proposition 9.5.20(i)). As in both hypotheses the immersion $S_i \to S$ is quasi-compact, it is quasi-projective by Proposition 9.5.20(i), so the restriction of η to T_{ij} is quasi-projective (Proposition 9.5.20(ii)). We relabel the T_{ij} by U_k with $1 \leq k \leq n$. There exists, for each index k, an open immersion $\varphi_k : U_k \to P_k$, where P_k is projective over S (Proposition 9.5.18). Let $U = \bigcap_k U_k$; as X is irreducible and the U_k is nonempty, U is nonempty, and therefore is dense in X; the restrictions of φ_k to U together define a morphism

$$\varphi : U \to P = P_1 \times_S P_2 \times_S \cdots \times_S P_n$$

which fits into the diagram

$$\begin{tikzcd} U \arrow[r, "\varphi"] \arrow[d, "j_k"'] & P \arrow[d, "p_k"] \\ U_k \arrow[r, "\varphi_k"] & P_k \end{tikzcd} \tag{9.5.1}$$

where j_k is the canonical injection and p_k is the canonical projection. If $j : U \to X$ is the canonical injection, the morphism $\psi = (j, \varphi)_S : U \to X \times_S P$ is then an immersion by Corollary 8.5.16. Under the hypotheses of (a), $X \times_S P$ is locally Noetherian (Proposition 9.3.23 and Corollary 8.6.22), and under the hypotheses of (b), $X \times_S P$ is quasi-compact. In both cases the scheme-theoretic image X' of ψ in $X \times_S P$ exists (which is the closure of $\psi(U)$ in $X \times_S P$) and ψ factors into

$$\psi : U \xrightarrow{\psi'} X' \xrightarrow{h} X \times_S P$$

where ψ' is a dominant open immersion and h is a closed immersion. Let $q_1 : X \times_S P \to X$ and $q_2 : X \times_P \to P$ be the canonical projections; we put

$$f : X' \xrightarrow{h} X \times_S P \xrightarrow{q_1} X, \quad g : X' \xrightarrow{h} X \times_S P \xrightarrow{q_2} P. \tag{9.5.2}$$

We shall verify that the scheme X' and the morphism f satisfy the requirements. First we show that f is projective and surjective, and that the restriction of $U' = f^{-1}(U)$ is an isomorphism from U' to U. As the P_k are projective over S, so is P (Proposition 9.5.34(iv)), and $X \times_S P$ is projective over X by Proposition 9.5.34(iii); then X' is also projective over X, since it is a closed subscheme of $X \times_S P$. On the other hand, we have $f \circ \psi' = q_1 \circ (h \circ \psi') = q_1 \circ \psi = j$, so $f(X')$ contains the dense open subset U of X; but f is proper by Theorem 9.5.30, so $f(X') = X$. Now $q_1^{-1}(U) = U \times_S P$ is an open subscheme of $X \times_S P$, and the the immersion ψ factors into

$$\psi : U \xrightarrow{\Gamma_\varphi} U \times_S P \xrightarrow{j \times 1} X \times_S P.$$

By Proposition 8.6.70, $U' = h^{-1}(U \times_S P)$ is the scheme-theoretic image of $\psi^{-1}(U \times_S P) = U$ under $\psi_U : U \to U \times_S P$, and therefore the closure of the image of Γ_φ in $U \times_S P$. As P is separted over S, Γ_φ is a closed immersion (Corollary 8.5.19), so we conclude that $U' = \psi(U)$. As ψ is an immersion, the restriction of f to U' is then an isomorphism, with inverse ψ'. Finally, by definition, $U' = \psi(U) = \psi'(U)$ is open and dense in X'.

We now show that g is an immersion, which implies that X' is quasi-projective over S, since P is projective over S. Let $V_k = \varphi_k(U_k)$ be the image of U_k in P_k, $W_k = p_k^{-1}(V_k)$ be the inverse image in P, and put $U_k' = f^{-1}(U_k)$, $U_k'' = g^{-1}(W_k)$. Since the U_k cover X, it clear that the U_k' form an open covering of X'; we first shows that this is also true for the U_k'', by proving that $U_k' \subseteq U_k''$. For this, it suffices to estabilsh the commutativity of the diagram

$$\begin{tikzcd} U_k' \arrow[r, "g|_{U_k'}"] \arrow[d, "f|_{U_k'}"'] & P \arrow[d, "p_k"] \\ U_k \arrow[r, "\varphi_k"] & P_k \end{tikzcd} \tag{9.5.3}$$

Since $U'_k = h^{-1}(U_k \times_S P)$ and $\psi^{-1}(U_k \times_S P) = U$, by Proposition 8.6.70 U'_k is the scheme-theoretic image of U in $U_k \times_S P$ under the morphism $\psi_k : U \to U_k \times_S P$ induced by ψ. It then suffices to prove the commutativity of the diagram obtained by composing (9.5.3) with the morphism ψ_k (Corollary 8.6.68), and this comes from the commutative diagram (9.5.1).

The W_k then form an open covering of $g(X')$, so to show that g is an immersion, it suffices to show the restriction $g|_{U''_k}$ is an immersion into W_k (Corollary 8.4.11). For this, consider the morphism

$$u_k : W_k \xrightarrow{p_k} V_k \xrightarrow{\varphi_k^{-1}} U_k \to X$$

which fits into the following commutative diagram

$$\begin{tikzcd}
U' \arrow[r, hook] \arrow[dr] & U'_k \arrow[r, hook] \arrow[dr] & U''_k \arrow[r, hook, "h|_{U''_k}"] \arrow[dr, "f|_{U''_k}"'] \arrow[rr, bend left=50, "g|_{U''_k}"] & X \times_S W_k \arrow[r, "q_2"] \arrow[d, "q_1"] & W_k \arrow[l, bend right=40, "\Gamma_{u_k}"'] \arrow[dl, "u_k"'] \arrow[d, "p_k"] \\
& U \arrow[r, hook] & U_k \arrow[r, hook] \arrow[rr, bend right=30, "\cong", "\varphi_k"'] & X & V_k
\end{tikzcd}$$

By the definition of g (formula (9.5.2)), we have $U''_k = h^{-1}(X \times_S W_k) \subseteq X'$ and $\psi^{-1}(X \times_S W_k) = U$, so by Proposition 8.6.70, U''_k is the scheme-theoretic image of U under the morphism $U \to X \times_S W_k$ induced by ψ; since $U' = \psi(U)$, it is therefore dense in U''_k. On the other hand, as X is separted over S, the graph morphism $\Gamma_{u_k} : W_k \to X \times_S W_k$ is a closed immersion, so the graph $T_k = \Gamma_{u_k}(W_k)$ is a closed subscheme of $X \times_S W_k$. If we can prove that T_k dominates the canonical image of the open subscheme U' in $X \times_S W_k$, it will then dominates the subscheme U''_k. As the restriction of q_2 to T_k is an isomorphism onto W_k and h is a closed immersion, the restriction of g to X''_k will then be an immersion in W_k, and our assertion will be proved. For this, we let $v_k : U' \to X \times_S W_k$ be the canonical injection, and $w_k = q_2 \circ v_k$; then from the definition of Γ_{u_k} we have $v_k = \Gamma_{u_k} \circ w_k$, and the image of U' in $X \times_S W_k$ is therefore contained in T_k, verifying our claim.

It is clear that U, and therefore U', are irreducible, and so is X' by our construction, and that f is birational. If X is reduced, so is U', and X' is then reduced (Proposition 8.6.69). This completes the proof. □

Corollary 9.5.40. *Suppose the hypotheses of Theorem 9.5.39. For X to be proper over S, it is necessary and sufficient that there exists a projective S-scheme X' and a surjective S-morphism $f : X' \to X$ (which is projective by Proposition 9.5.34(v)). If this is the case, we can choose an open dense subset U of X such that f induces an isomorphism $f^{-1}(U) \cong U$ and that $f^{-1}(U)$ is dense in X'. If X is irreducible (resp. reduced), we can choose X' to be irreducible (resp. reduced). If X and X' are irreducible, f is then a birational morphism.*

Proof. The conditions is sufficient by Theorem 9.5.30 and Corollary 9.5.24(b). This is necessary because with the notations of Theorem 9.5.39, if X is proper over S, X' is then proper over S (since it is proper over X by Theorem 9.5.30), and our assertion follows from Proposition 9.5.23(ii). Moreover, as X' is quasi-projective over S, it is projective over S in view of Theorem 9.5.30. □

Corollary 9.5.41. *Let S be a locally Noetherian scheme, X be an S-scheme of finite type, and $\varphi : X \to S$ be the structural morphism. For X to be proper over S, it is necessary and sufficient that for any morphism of finite type $S' \to S$, the morphism $\varphi_{(S')} : X_{(S')} \to S'$ is closed. Moreover, it suffices to verify this condition for any S-scheme of the form $S' = S \otimes_{\mathbb{Z}} \mathbb{Z}[T_1, \dots, T_n]$.*

Proof. The conditions is clearly necessary, and we now prove the sufficiency. The question is local over S and S', so we may assume that S, S' are affine and Noetherian. By Chow's lemma, there exists a projective S-scheme P, an immersion $j : X' \to P$, and a projective surjective morphism $f : X' \to X$ such that the diagram

$$\begin{tikzcd}
X' \arrow[dr, "\psi"] \arrow[drr, bend left, "f"] \arrow[ddr, bend right, "j"'] & & \\
& X \times_S P \arrow[r, "q_1"] \arrow[d, "q_2"'] & X \arrow[d, "\varphi"] \\
& P \arrow[r, "r"] & S
\end{tikzcd}$$

is commutative; let $\psi = (f, j)_S$. As P is of finite type over S, the projection $q_2 : X \times_S P \to P$ is a closed morphism by hypotheses. On the other hand, since f is projective and the projection $q_1 : X \times_S P \to X$ is separated (since P is separated over S), we conclude from Proposition 9.5.34(v) that ψ is projective, hence closed. Since the immersion j is the composition of q_2 with ψ, it is therefore a closed immersion, whence proper. Moreover, the structural morphism $r : P \to S$ is projective, hence proper (Theorem 9.5.30), so $\varphi \circ f = r \circ j$ is proper. As f is surjective, we conclude by Corollary 9.5.24(b) that φ is proper.

To estabilsh the second assertion of the proposition, it suffices to prove that it implies that $\varphi_{(S')}$ is closed for any morphism $S' \to S$ of finite type. Now if S' is affine and of finite type over $S = \operatorname{Spec}(A)$, we have $S' = \operatorname{Spec}(A[x_1, \dots, x_n])$, and S' is then isomorphic to a closed subscheme of $S'' = \operatorname{Spec}(A[T_1, \dots, T_n])$ (where T_i are indeterminates). In the following commutative diagram

$$\begin{tikzcd}
X \times_S S' \arrow[r, "1_X \times j"] \arrow[d, "\varphi_{(S'')}"'] & X \times_S S'' \arrow[d, "\varphi_{(S'')}"] \\
S' \arrow[r, "j"] & S''
\end{tikzcd}$$

where j and $q_X \times j$ are closed immersions (Corollary 8.4.14) and $\varphi_{(S'')}$ is closed by hypothesis. We then conclude that $\varphi_{(S')}$ is closed, whence the claim. □

9.6 Integral morphisms and finite morphisms

9.6.1 Integral and finite morphisms

Let X be an S-scheme and $f : X \to S$ be the structural morphism. We say that X is **integral over S**, or that f is an **integral morphism**, if there exists an affine open covering (S_α) of S such that for each α, the open subscheme $f^{-1}(S_\alpha)$ of X is affine and its ring $B_\aleph$ is an integral algebra over the ring A_α of S_α. We say that X is **finite over** S, or that f is a **finite morphism**, if X is integral and of finite type over S. If S is affine with ring A, we also say that X is inegral or finite over A.

It is clear that any integral S-scheme is affine over S. Conversely, form the definition of integral morphisms, we see that for an affine S-scheme X to be integral over S (resp. finite), it is necessary and sufficient that the associated quasi-coherent $\mathscr{O}_S$-algebra $\mathscr{A}(X)$ is such that there exists an affine open covering (S_α) of S such that for each α, $\Gamma(S_\alpha, \mathscr{A}(X))$ is an integral algebra (resp. an integral algebra of finite type) over $\Gamma(S_\alpha, \mathscr{O}_S)$. A quasi-coherent $\mathscr{O}_S$-algebra satisfying this property is said to be **integral** (resp. **finite**) over $\mathscr{O}_S$. We note that a quasi-coherent $\mathscr{O}_S$-algebra $\mathscr{B}$ is finite if and only if it is an $\mathscr{O}_S$-module of finite type; it amounts to the same thing to say that $\mathscr{B}$ is an integral $\mathscr{O}_S$-algebra of finite type, because an integer algebra of finite type over a ring A is an A-module of finite type.

Proposition 9.6.1. *Let S be a locally Noetherian scheme. For an S-scheme X affine over S to be finite over S, it is necessary and sufficient that the $\mathscr{O}_S$-algebra $\mathscr{A}(X)$ is coherent.*

Proof. With the preceding remark, this follows from the fact that if S is locally Noetherian, then a quasi-coherent $\mathscr{O}_S$-module is of finite type if and only if it is coherent (Theorem 8.1.30). □

Proposition 9.6.2. *Let X be an integral (resp. finite) scheme over S with $f : X \to S$ the structural morphism. Then for any affine open subset $U \subseteq S$ with ring A, $f^{-1}(U)$ is affine and its ring B is an integral (resp. finite) algebra over A.*

Proof. To prove this proposition, we need **??**. We now that $f^{-1}(U)$ is affine by Proposition 9.1.4. If $\varphi : A \to B$ is the corresponding homomorphism, there exists a finite covering of U by open subsets $D(g_i)$ ($g_i \in A$) such that, if $h_i = \varphi(g_i)$, then B_{h_i} is an integral (resp. finite) algebra over A_{g_i}. In fact, by assumption, there is a covering of U by affine open subsets $V_\alpha \subseteq U$ such that if $A_\alpha = \Gamma(V_\alpha, \mathscr{O}_S)$ and $B_\alpha = \Gamma(f^{-1}(V_\alpha), \mathscr{O}_X)$, then B_α is an integral (resp. finite) algebra over A_α. Any $x \in U$ belongs to one V_α, so there exists $g \in A$ such that $x \in D(g) \subseteq V_\alpha$. If g_α is the image of g in A_α, we have $\Gamma(D(g), \mathscr{O}_S) = A_g = (A_\alpha)_{g_\alpha}$; let $h = \varphi(g)$, and let h_α be the image of g_α in B_α. We have

$$\Gamma(D(h), \mathscr{O}_S) = B_h = (B_\alpha)_{h_\alpha}$$

and as B_α is integral over A_α, $(B_\alpha)_{h_\alpha}$ is integral (resp. finite) over $(A_\alpha)_{g_\alpha}$. Since U is quasi-compact, we obtain a finite cover.

If we suppose first that each B_{h_i} is a finite algebra over A_{g_i}, then as an A_{g_i}-module, B_{h_i} is finitely generated, so **??** shows that B is a finitely generated A-module. Now assume that each B_{h_i} is integral over A_{g_i}; let $b \in B$, and let C be a sub-A-algebra of B generated by b. For each i, C_{h_i} is the A_{g_i}-algebra generated by $b/1$ over B_{h_i}. It then follows from the hypothesis that each C_{h_i} is finitely generated A_{g_i}-module, so by **??** C is a finitely generated A-module. This shows that B is integral over A. □

Proposition 9.6.3 (Properties of Integral and Finite morphisms).

(i) *A closed immersion is finite (and a fortiori integral).*

(ii) *The composition of two integral morphisms (resp. finite) is integral (resp. finite).*

(iii) *If $f : X \to Y$ is an integral (resp. finite) S-morphism, $f_{(S')} : X_{(S')} \to Y_{(S')}$ is integral (resp. finite) for any base change $S' \to S$.*

(iv) *If $f : X \to Y$ and $g : X' \to Y'$ are two integral (resp. finite) S-morphisms, so is $f \times_S g$.*

(v) *If $f : X \to Y$ and $g : Y \to Z$ are two morphisms such that $g \circ f$ is integral (resp. finite) and g is separated, then f is integral (resp. finite).*

(vi) *If $f : X \to Y$ is an integral (resp. finite) morphism, so is f_{red}.*

Proof. In view of Proposition 8.5.22, it suffices to prove (i), (ii), and (iii). To prove that a closed immersion $X \to S$ is finite, we can assume that $S = \mathrm{Spec}(A)$, and this then follows from the fact that a quotient ring $A/\mathfrak{a}$ is a finitely generated A-module. To prove the composition of two integral (resp. finite) morphisms $X \to Y$, $Y \to Z$ is integral (finite), we can assume that Z (and therefore X and Y) is affine, and the assertion is then equivalent to that if B is an integral (resp. finite) A-algebra and C is an integral (resp. finite) B-algebra, then C is an integral (resp. finite) A-algebra, which is immediate. Finally, to prove (iii), we can Similarly assume that $S = \mathrm{Spec}(A)$, $S' = \mathrm{Spec}(A')$; then X is affine with ring B (Proposition 9.6.2), $X_{(S')}$ is affine with ring $A' \otimes_A B$, and it suffices to note that if B is an integral (resp. finite) A-algebra, then $A' \otimes_A B$ is an integral (resp. finite) A'-algebra. □

Note also that if X and Y are two integral (resp. finite) S-schemes, the sum $X \amalg Y$ is an integral (resp. finite) over S, because a product of two integral (resp. finite) A-algebras is still integral (resp. finite).

Corollary 9.6.4. *If X is an integral (resp. finite) scheme over S, then for any open subset $U \subseteq S$, $f^{-1}(U)$ is integral (resp. finite) over U.*

Proof. This is a particular case of Proposition 9.6.3(iii). □

Corollary 9.6.5. *Let $f : X \to Y$ be a finite morphism. Then for any $y \in Y$, the fiber $f^{-1}(y)$ is a finite algebraic scheme over $\kappa(y)$, and a fortiori with discrete and finite underlying space.*

Proof. The $\kappa(y)$-scheme $f^{-1}(y)$ is identified with $X \times_Y \mathrm{Spec}(\kappa(y))$, so is finite over $\kappa(y)$ by Proposition 9.6.3(iii). This is then an affine scheme whose ring is a finite dimensional $\kappa(y)$-algebra, so is Artinian. The proposition then follows from Proposition 8.2.33. □

Corollary 9.6.6. *Let X and S be integral schemes and $f : X \to S$ be a dominant morphism. If f is integral (resp. finite), then the rational function field $K(X)$ of X is an algebraic (resp. finite) extension of $K(S)$.*

Proof. Let s be the generic point of S; the $\kappa(s)$-scheme $f^{-1}(s)$ is integral (resp. finite) over $\operatorname{Spec}(\kappa(s))$ by Proposition 9.6.3(iii), and contains by hypothesis the generic point of x of X; the local ring of $f^{-1}(s)$ at x, equal to $\kappa(x)$ (Proposition 8.3.37), is a localization of an integral (resp. finite) algebra over $\kappa(s)$, whence the corollary. □

Proposition 9.6.7. *Any integral morphism is universally closed.*

Proof. Let $f : X \to Y$ be an integral morphism. In view of Proposition 9.6.3(iii), it suffices to prove that f is closed. Let Z be a closed subset of X. In view of Proposition 9.6.3(vi), we can suppose that X and Y are reduced; moreover, if T is the reduced closed subscheme of Y with underlying space $\overline{f(X)}$, we see that f factors into

$$f : X \xrightarrow{g} T \xrightarrow{j} Y,$$

where $j : T \to Y$ is the canonical injection, and as j is separated, it follows from Proposition 9.6.3(v) that g is an integral morphism. We can then assume that $f(X)$ is dense in Y, and prove that $f(X) = Y$. Since the question is local over Y, we can assume that $Y = \operatorname{Spec}(A)$ is affine, so $X = \operatorname{Spec}(B)$ where B is an integral algebra over A (Proposition 9.6.2); moreover A is reduced and the hypothesis that $f(X)$ is dense in Y implies that the corresponding homomorphism $\varphi : A \to B$ is injective (**??**). The condition that $f(X) = Y$ then follows from **??**. □

Remark 9.6.8. The hypothesis that g is separated is essential for the validity of Proposition 9.6.3(v): in fact, if Y is not separated over Z, the identity 1_Y is the composition morphism

$$Y \xrightarrow{\Delta_Y} Y \times_Z Y \xrightarrow{p_1} Y$$

but Δ_Y is not integral, since it is not closed (Proposition 9.6.7).

Corollary 9.6.9. *Any finite morphism $f : X \to Y$ is projective.*

Proof. As f is affine, $\mathscr{O}_X$ is a very ample $\mathscr{O}_X$-module relative to f; moreover $f_*(\mathscr{O}_X)$ is a quasi-coherent $\mathscr{O}_Y$-algebra of finite type by hypothesis. Finally, f is separated, of finite type, and universally closed (Proposition 9.6.7), and we then have the conditions of Remark 9.5.31. □

Lemma 9.6.10. *Let Y be a scheme, $\mathscr{E}$ be a locally free $\mathscr{O}_Y$-module of rank r, and Z be a finite subset of Y contained in an affine open subset V. Then there exists an open neighborhood $U \subseteq V$ of Z such that $\mathscr{E}|_U$ is isomorphic to $\mathscr{O}_Y^r|_U$.*

Proof. □

Proposition 9.6.11. *Let $f : X' \to X$ be a finite morphism, and let $\mathscr{B} = f_*(\mathscr{O}_{X'})$ (which is a quasi-coherent finite $\mathscr{O}_X$-algebra). Let $\mathscr{F}'$ be a quasi-coherent $\mathscr{O}_{X'}$-module; for $\mathscr{F}'$ to be locally free of rank r, it is necessary and sufficient that $f_*(\mathscr{F}')$ is a locally free $\mathscr{B}$-module of rank r.*

Proof. It is clear that if $f_*(\mathscr{F}')|_U$ is isomorphic to $\mathscr{B}^r|_U$ (where U is open in X), $\mathscr{F}'|_{f^{-1}(U)}$ is isomorphic to $\mathscr{O}_{X'}^r|_{f^{-1}(U)}$ (Corollary 9.1.19). Conversely, suppose that $\mathscr{F}'$ is locally free of rank r and we prove that $f_*(\mathscr{F}')$ is locally isomorphic to $\mathscr{B}^r$ as $\mathscr{B}$-modules. Let x be a point of X; if U runs through affine neighborhoods of x, $f^{-1}(U)$ form a fundamental system of affine neighborhoods (Proposition 9.1.4) of the finite subset $f^{-1}(x)$, since f is closed (Proposition 9.6.7). The proposition then follows from Lemma 9.6.10. □

Proposition 9.6.12. *Let $g : X' \to X$ be an integral morphism of schemes, Y be a locally integral and normal scheme, f be a rational map from Y to X' such that $g \circ f$ is a everywhere defined rational map; then f is everywhere defined.*

Proof. Recall that we say a scheme X is normal if it is normal as a ringed space, which means the stalk $\mathscr{O}_{X,x}$ is an integrally closed domain for every $x \in X$. If f_1 and f_2 are two morphisms (densely defined from Y to X') in the class of f, it is clear that $g \circ f_1$ and $g \circ f_2$ are equivalent morphisms, which justifies the notation $g \circ f$ for their equivalent class. We recall also that if Y is locally Noetherian, then the hypothesis on Y implies that Y is locally integral (Proposition 8.4.35).

To prove the proposition, we first note that the question is local over Y and we can assume that there exists a morphism $h : Y \to X$ in the class of $g \circ f$. Consider the inverse image $Y' = X'_{(h)} = X'_{(Y)}$, and note that the morphism $g' = g_{(Y)} : Y' \to Y$ is integral by Proposition 9.6.3(iii). Via the correspondence of rational maps from Y to X' with rational Y-sections of Y', we see that we are reduced to the case $X = Y$. □

Corollary 9.6.13. *Let X be a locally integral and normal scheme, $g : X' \to X$ be an integral morphism, and f be a rational X-section of X'. Then f is everywhere defined.*

Corollary 9.6.14. *Let X be a normal and integral scheme, X' be an integral scheme, and $g : X' \to X$ be an integral morphism. If there exists a rational X-section f of X', g is an isomorphism.*

9.6.2 Quasi-finite morphisms

Proposition 9.6.15. *Let $f : X \to Y$ be a morphism locally of finite type, x be a point of X and $y = f(x)$. The following conditions are equivalent:*

(i) *The point x is isolated in the fiber X_y.*

(ii) *The point x is closed in X_y and there is no generalization of x in X_y.*

(iii) *The $\mathscr{O}_{Y,y}$-module $\mathscr{O}_{X,x}$ is quasi-finite.*

Proof. The question is local over X and Y, so we can suppose that $X = \mathrm{Spec}(B)$ and $Y = \mathrm{Spec}(A)$ are affine, where B is finite A-algebra. Moreover, replacing X by $X \times_Y \mathrm{Spec}(\mathscr{O}_{Y,y})$ does not change the fiber X_y and the local ring $\mathscr{O}_{X,x}$ (Proposition 8.3.37), so we can suppose that A is a local ring with maximal ideal $\mathfrak{m}$ (which equals to the local ring $\mathscr{O}_{Y,y}$). The fiber X_y is then the affine sheme of the ring $B/\mathfrak{m}B$, of finite type over $\kappa(y) = A/\mathfrak{m}$ (Proposition 8.6.48). Let $\mathfrak{P}$ be the prime ideal of B corresponding to x.

We note that the fiber $X_y = X \times_Y \mathrm{Spec}(\kappa(y)) = \mathrm{Spec}(B \otimes_A \kappa(y))$ is Jacobson. If (i) is satisfied, then $\{x\}$ is an open subset of X_y, hence contains a closed point (by the Jacobson property) which must be x, and x is therefore closed in X_y. Also, since $\{x\}$ is open in X_y, it is clear that there is no further generalization x' of x (which means $x \in \overline{\{x'\}}$) in X_y; this proves (i)⇒(ii).

We now consider the conditions in (ii) and (iii). Consider the ring $\bar{B} = B \otimes_A \kappa(y) = B/\mathfrak{m}B$ and let $\bar{\mathfrak{P}}$ be the prime ideal corresponding to $\mathfrak{P}$. If x is closed in X_y, we see that $\bar{\mathfrak{P}}$ is maximal in $\bar{B}$ and by condition (ii) there is no prime ideal of $\bar{B}$ contained in $\bar{\mathfrak{P}}$. This shows that $\bar{B}_{\bar{\mathfrak{P}}}$ is Artinian, so we conclude that the ring homomorphism $A \to B$ is quasi-finite at $\mathfrak{P}$; that is, the $\mathscr{O}_{Y,y}$-module $\mathscr{O}_{X,x}$ is quasi-finite. Conversely, if $\bar{B}_{\bar{\mathfrak{P}}}$ is Artinian, then $\bar{\mathfrak{P}}$ is maximal and there is no prime ideal of $\bar{B}$ contained in $\bar{\mathfrak{P}}$; we then conclude that x is closed in X_y and there is no generalization of x in X_y. This shows that (ii)⇔(iii).

We finally prove that (ii) implies (i). If (ii) is satisfied, then the prime ideal $\mathfrak{P}$ is both maximal and minimal in B since $\bar{B}_{\bar{\mathfrak{P}}}$ is Artinian. Let $\bar{\mathfrak{P}}_1 = \bar{\mathfrak{P}}, \bar{\mathfrak{P}}_2, \dots, \bar{\mathfrak{P}}_r$ be the minimal prime ideals of $\bar{B}$. Then the intersection $\bigcap_{i=1}^r \bar{\mathfrak{P}}_i$ is equal to the nilradical of $\bar{B}$, and $\bigcap_{i=2}^r \bar{\mathfrak{P}}_i$ is contained in this intersection in view of **??**, so there exists an element $\bar{b} \in \bigcap_{i=2}^r \bar{\mathfrak{P}}_i$ that is not nilpotent. The open subset $D_{\bar{B}}(\bar{b})$ of $X_y = \mathrm{Spec}(\bar{B})$ then reduces to $\{x\}$, which shows that x is isolated. □

Corollary 9.6.16. *Let $f : X \to Y$ be a morphism of finite type. Then the following conditions are equivalent:*

(i) *Any point $x \in X$ is isolated in the fiber $X_{f(x)}$ (that is, $X_{f(x)}$ is discrete).*

(ii) *For any $x \in X$, $X_{f(x)}$ is a finite $\kappa(f(x))$-scheme.*

(iii) *For any $x \in X$, the $\mathscr{O}_{Y,y}$-module $\mathscr{O}_{X,x}$ is quasi-finite.*

Proof. The equivalence of (i) and (iii) follows from Proposition 9.6.15. On the other hand, as $X_{f(x)}$ is an algebraic $\kappa(f(x))$-scheme, the equivalence of (i) and (ii) follows from Proposition 8.6.44. □

We say a morphism $f : X \to Y$ of finite type is **quasi-finite**, or X is **quasi-finite** over Y, if it satisfies the equivalent conditions of Corollary 9.6.16. We say a morphism $f : X \to Y$ is quasi-finite at a point $x \in X$ if there exists an affine open neighborhood V of $y = f(x)$ and an affine open neighborhood U of x such that $f(U) \subseteq V$ and the restriction $f|_U : U \to V$ is quasi-finite. We say that $f : X \to Y$ is locally quasi-finite if it is quasi-finite at every point of X. From Corollary 9.6.5, it is clear that any finite morphism is quasi-finite.

Proposition 9.6.17 (Properties of Quasi-finite Morphisms).

(i) *Any quasi-compact immersion (in particular any closed immersion) is quasi-finite.*

(ii) *If $f : X \to Y$ and $g : Y \to Z$ are quasi-finite morphisms, $g \circ f$ is quasi-finite.*

(iii) *If $f : X \to Y$ is a quasi-finite S-morphism, $f_{(S')} : X_{(S')} \to Y_{(S')}$ is quasi-finite for any base change $S' \to S$.*

(iv) *If $f : X \to Y$ and $g : X' \to Y'$ are quasi-finite S-morphisms, $f \times_S g$ is quasi-finite.*

(v) *Let $f : X \to Y$ and $g : Y \to Z$ be two morphisms such that $g \circ f$ is quasi-finite; if g is separated, or X is Noetherian, or $X \times_Z Y$ is locally Noetherian, then f is quasi-finite.*

(vi) *If f is quasi-finite, so is f_{red}.*

Proof. If $f : X \to Y$ is an immersion, any fiber is reduced to a singleton, so (i) follows from Proposition 8.6.35(i). For (ii), we note that $h = g \circ f$ is of finite type by Proposition 8.6.35(ii); if $z = h(x)$ and $y = f(x)$, y is isolated in $g^{-1}(z)$, so there exists an open neighborhood V of y in Y not containing other points of $g^{-1}(z)$; $f^{-1}(V)$ is then an open neighborhood of x not containing other points of $f^{-1}(y')$, where $y' \neq y$ is in $g^{-1}(z)$, and therefore not containing points $x' \neq x$ in $h^{-1}(z)$ that is not in $f^{-1}(y)$. As x is isolated in $f^{-1}(y)$ by hypothesis, it is then isolated in $h^{-1}(z) = f^{-1}(g^{-1}(z))$. As for (iii), we can limit ourselves to the case where $Y = S$ (Corollary 8.3.10); we first note that $f' = f_{(S')}$ is of finite type (Proposition 8.6.35(iii)). On the other hand, if $x' \in X' = X_{(S')}$ and $y' = f'(x')$, $X'_{y'}$ is identified with $X_y \otimes_{\kappa(y)} \kappa(y')$ by Proposition 8.3.38. As X_y is of finite dimension over $\kappa(y)$ by hypothesis, $X'_{y'}$ is of finite dimension over $\kappa(y')$, hence discrete. The assertions (iv), (v), (vi) then follows from the first three assertions in view of the general principle Proposition 8.5.22, where in (v) we assume that g is separated. The other cases, we first remark that if x is isolated in $X_{g(f(x))}$, it is also isolated in $X_{f(x)}$; the fact that f is of finite type follows from Proposition 8.6.35. □

Proposition 9.6.18. *Let A be a complete Noetherian local ring, $Y = \mathrm{Spec}(A)$, X be a separated Y-scheme locally of finite type, x be a point over the closed point y of Y, and suppose that x is isolated in the fiber X_y. Then $\mathscr{O}_{X,x}$ is a finitely generated A-module and X is Y-isomorphic to the sum of $X' = \mathrm{Spec}(\mathscr{O}_{X,x})$ (which is a finite Y-scheme) and an A-scheme X''.*

Proof. It follows from Proposition 9.6.15 that $\mathscr{O}_{X,x}$ is a quasi-finite A-module. As $\mathscr{O}_{X,x}$ is Noetherian (Proposition 8.6.20) and the homomorphism $A \to \mathscr{O}_{X,x}$ is local, the hypothesis that A is complete implies that $\mathscr{O}_{X,x}$ is a finitely generated A-module ([?] 0_{I}, 7.4.3). Let $X' = \mathrm{Spec}(\mathscr{O}_{X,x})$ be the local scheme of X at x and $g : X' \to X$ be the canonical morphism. The composition $f \circ g : X' \to Y$ is then finite, and since f is separated, g is finite by Proposition 9.6.3, so $g(X')$ is closed in X (Corollary 9.6.9). On the other hand, as g is of finite type and A is Noetherian, it is of finite presentation, and hence a local immersion at the closed point x' of X' (Proposition 8.6.53 and the definition of g). But X' is the only open neighborhood of x' in X', so it follows that $g(X')$ is open in X, which proves our assertion. □

Corollary 9.6.19. *Let A be a complete Noetherian local ring, $Y = \mathrm{Spec}(A)$, $f : X \to Y$ be a quasi-finite and separted morphism. Then X is Y-isomorphic to a sum $X' \amalg X''$, where X' is a finite Y-scheme and X'' is a quasi-finite Y-scheme such that, if y is the closed point of y, $X'' \cap f^{-1}(y) \neq \varnothing$.*

Proof. The fiber $f^{-1}(y)$ is finite and discrete by hypothesis, and the corollary then follows by recurrence on the number of points of $f^{-1}(y)$, using Proposition 9.6.18. □

9.6.3 Integral closure of a scheme

Proposition 9.6.20. *Let $(X, \mathscr{A})$ be a ringed space, $\mathscr{B}$ be an $\mathscr{A}$-algebra, and f be a section of $\mathscr{B}$ over X. The following properties are equivalent:*

(i) *The sub-$\mathscr{A}$-algebra of $\mathscr{B}$ generated by f is finite (that is, of finite type as an $\mathscr{A}$-module).*

(ii) *There exists a sub-$\mathscr{A}$-algebra $\mathscr{C}$ of $\mathscr{B}$, which is an $\mathscr{A}$-module of finite type, such that $f \in \Gamma(X, \mathscr{C})$.*

(iii) *For any $x \in X$, f_x is integral over the fiber $\mathscr{A}_x$.*

*If these equivalent conditions are satisfied, the section f is said to be **integral** over $\mathscr{A}$.*

Proof. As the sub-$\mathscr{A}$-module of $\mathscr{B}$ genearted by f^n is an $\mathscr{A}$-algebra, it is clear that (i) implies (ii). On the other hand, (ii) implies that for any $x \in X$, the $\mathscr{A}_x$-module $\mathscr{C}_x$ is of finite type, which implies that any element of the algebra $\mathscr{C}_x$, and in particular f_x, is integral over $\mathscr{A}_x$. Finally, if for any point $x \in X$, we have a relation

$$f_x^n + (a_1)_x f_x^{n-1} + \cdots + (a_n)_x = 0$$

where a_i are sections of $\mathscr{A}$ over an open neighborhood U of x, the section $f^n|_U + a_1 \cdot f^{n-1}|_U + \cdots + a_n$ is zero over an open neighborhood $V \subseteq U$ of x, so $f^k|_V$ (for $k \geq 0$) is a linear combination over $\Gamma(V, \mathscr{A})$ of $f^j|_V$ with $0 \leq j \leq n-1$. We then conclude that (iii) implies (i). □

Corollary 9.6.21. *Under the hypothesis of Proposition 9.6.20, there exists a (unique) sub-$\mathscr{A}$-algebra $\mathscr{A}'$ of $\mathscr{B}$ such that for any $x \in X$, $\mathscr{A}'_x$ is the set of germs $f_x \in \mathscr{B}_x$ that is integral over $\mathscr{A}_x$. For any open subset $U \subseteq X$, the sections of $\mathscr{A}'$ over U is the sections of $\Gamma(U, \mathscr{B})$ that is integral over $\mathscr{A}|_U$. We say that $\mathscr{A}'$ is the **integral closure** of $\mathscr{A}$ in $\mathscr{B}$.*

Proof. The existence of $\mathscr{A}'$ is immediate, by setting $\Gamma(U, \mathscr{A}')$ to be the set of $f \in \Gamma(U, \mathscr{B})$ such that f_x is integral over $\mathscr{A}_x$ for any $x \in U$. It is clear that $\mathscr{A}'$ is an algebra, and the second assertion follows from Proposition 9.6.20. □

Let $(X, \mathscr{A})$, $(Y, \mathscr{B})$ be two ringed space and $f : X \to Y$ be a morphism. Let $\mathscr{C}$ (resp. $\mathscr{D}$) be an $\mathscr{A}$-algebra (resp. $\mathscr{B}$-algebra) and let $u : \mathscr{D} \to \mathscr{C}$ be a f-morphism. Then, if $\mathscr{A}'$ (resp. $\mathscr{B}'$) is the integral closure of $\mathscr{A}$ (resp. $\mathscr{B}$) in $\mathscr{C}$ (resp. $\mathscr{D}$), the restriction of u to $\mathscr{B}'$ is then a f-morphism $u' : \mathscr{B}' \to \mathscr{A}'$. In fact, if j is the canonical injection $\mathscr{B}' \to \mathscr{D}$, it suffices to show that

$$v = u^\sharp \circ f^*(j) : f^*(\mathscr{B}') \to \mathscr{C}'$$

maps $f^*(\mathscr{B}')$ into $\mathscr{A}'$. Now an element of $(f^*(\mathscr{B}'))_x = \mathscr{B}'_{f(x)} \otimes_{\mathscr{B}_{(f(x))}} \mathscr{A}_x$ is integral over $\mathscr{A}_x$ by the definition of $\mathscr{B}'$, and hence so is its image under v_x, which proves our assertion.

Proposition 9.6.22. *Let X be a scheme and $\mathscr{A}$ be a quasi-coherent $\mathscr{O}_X$-algebra. The integral closure $\mathscr{O}'_X$ of $\mathscr{O}_X$ in $\mathscr{A}$ is then a quasi-coherent $\mathscr{O}_X$-algebra, and for any affine open U of X, $\Gamma(X, \mathscr{O}'_X)$ is the integral closure of $\Gamma(U, \mathscr{O}_X)$ in $\Gamma(U, \mathscr{A})$.*

Proof. We can assume that $X = \operatorname{Spec}(B)$ is affine and $\mathscr{A} = \widetilde{A}$, where A is an B-algebra. Let B' be the integral closure of B in A. It then boils down to seeing that for any $x \in X$, an element of A_x, integer over B_x, necessarily belongs to B'_x, which follows from the fact that taking integral closure commutes with localization (**??**). □

Under the hypothesis of Proposition 9.6.22, the X-scheme $X' = \operatorname{Spec}(\mathscr{O}'_X)$ is then called the **integral closure of X relative to** $\mathscr{A}$. We also deduce from Proposition 9.6.22 that if $f : X' \to X$ is the structural morphism, then for any open subset U of X, $f^{-1}(U)$ is the integral closure of the induced subscheme U by X, relative to $\mathscr{A}|_U$. In particular, we conclude that f is integral.

Let X and Y be schemes, $f : X \to Y$ be a morphism, $\mathscr{A}$ (resp. $\mathscr{B}$) be a quasi-coherent $\mathscr{O}_X$-algebra (resp. a $\mathscr{O}_Y$-algebra), and $u : \mathscr{B} \to \mathscr{A}$ be an f-morphism. We have seen that we have an induced f-morphism $u' : \mathscr{O}'_Y \to \mathscr{O}'_X$, where $\mathscr{O}'_X$ (resp. $\mathscr{O}'_Y$) is the integral closure of $\mathscr{O}_X$ (resp. $\mathscr{O}_Y$) relative to $\mathscr{A}$ (resp. $\mathscr{B}$), we deduce a canonical morphism $f' : \operatorname{Spec}(u') : X' \to Y'$ (Corollary 9.1.11) fitting into the following commutative diagram

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d] & Y' \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd} \tag{9.6.1}$$

Suppose that X has only finitely many irreducible components $(X_i)_{1 \leq i \leq r}$, with generic points $(\xi_i)_{1 \leq i \leq r}$, and consider in particular the integral closure of X relative to a quasi-coherent $\mathscr{K}_X$-algebra $\mathscr{A}$. By Corollary 8.7.19 and Corollary 8.7.20, $\mathscr{A}$ is the direct product of r quasi-coherent $\mathscr{O}_X$-algebras $\mathscr{A}_i$, the support of $\mathscr{A}_i$ being contained in X_i, and the induced sheaf of $\mathscr{A}_i$ over X_i is the constant sheaf whose fiber A_i is an algebra over $\mathscr{O}_{X,\xi_i}$. It is clear that the integral closure $\mathscr{O}'_X$ of $\mathscr{O}_X$ is the direct product of the integral closures $\mathscr{O}_X^{(i)}$ of $\mathscr{O}_X$ in each $\mathscr{A}_i$, and therefore the integral closure $X' = \operatorname{Spec}(\mathscr{O}'_X)$ of X relative to $\mathscr{A}$ is an X-scheme which is the sum of $\operatorname{Spec}(\mathscr{O}_X^{(i)}) = X'_i$.

Now suppose that the $\mathscr{O}_X$-algebra $\mathscr{A}$ is reduced, or equivalently, each algebra A_i is reduced, and therefore can be considered as an algebra over the field $\kappa(\xi_i)$ (equal to the rational function field of the reduced subscheme X_i of X); then the X'_i is a reduced X-scheme and X' is also the integral closure of X_{red}. Suppose moreover that the algebras A_i is a direct product of finitely many field K_{ij} ($1 \leq j \leq s_i$); if $\mathscr{K}_{ij}$ is the subalgebra of $\mathscr{A}_i$ corresponding to K_{ij}, it is clear that $\mathscr{O}_X^{(i)}$ is the direct product of integral closures $\mathscr{O}_X^{(ij)}$ of $\mathscr{O}_X$ in $\mathscr{K}_{ij}$. Therefore, X'_i is the sum of $X'_{ij} = \mathrm{Spec}(\mathscr{O}_X^{(ij)})$. Moreover, under this hypothesis, we have the following:

Proposition 9.6.23. *Each X'_{ij} is an integral and normal X-scheme, and its rational function field $K(X'_{ij})$ is canonically identified with the algebraic closure K'_{ij} of $\kappa(\xi_i)$ in K_{ij}.*

Proof. In view of the preceding remarks, we can assume that X is integral, so $r = 1$, $s_1 = 1$, so that the unique algebra A_1 is a field K; let ξ be the generic point of X, and let $f : X' \to X$ be the structural morphism. For any nonempty affine open U of X, $f^{-1}(U)$ is identified with the integral closure B'_U in the field K of the intagral ring $B_U = \Gamma(U, \mathscr{O}_X)$ (Proposition 9.6.22); as the ring B'_U is integrally closed, so is its localizations, and $f^{-1}(U)$ is by definition an integral and normal scheme. Moreover, as (0) is the unique prime ideal of B'_U lying over the prime ideal (0) of B_U, $f^{-1}(\xi)$ is reduced to a singleton ξ', and $\kappa(\xi')$ is the fraction field K' of B'_U, which is none other than the algebraic closure of $\kappa(\xi)$ in K. Finally, X' is irreducible, because if U runs through the nonempty affine open subsets of X, the $f^{-1}(U)$ constitute an open covering of X' formed by irreducible open subsets; moreover the intersection $f^{-1}(U \cap V)$ two two opens contains ξ', hence nonempty, and we conclude from **??** that X' is irreducible. □

Corollary 9.6.24. *Let X be a reduced scheme with finitely many irreducible components (X_i), and let ξ_i be the generic point of X_i. The integral closure X' of X relative to $\mathscr{K}_X$ is the sum of r separated X-schemes X'_i which are integral and normal. If $f : X' \to X$ is the structural morphism, $f^{-1}(\xi_i)$ is reduced to the generic point ξ'_i of X'_i and we have $\kappa(\xi'_i) = \kappa(\xi_i)$, which means f is birational.*

Proof. This is a particular case of Proposition 9.6.23 by taking $K'_{ij} = \kappa(\xi_i)$. The rational function field of X'_i (which is $\kappa(\xi'_i)$) is then equal to $\kappa(\xi_i)$, whence our claim. □

The integral closure X' of X relative to $\mathscr{K}_X$ is called that **normalization** of the reduced scheme X. We note that the morphism $f : X' \to X$, being birational and integral, is closed by Proposition 9.6.7, hence surjective (recall that a birational morphism is dominant). For $X' = X$, it is necessary and sufficient that X is normal. If X is an integral scheme, it follows from Corollary 9.6.24 that its normalization X' is integral.

Let X, Y be integral schemes, $f : X \to Y$ be a dominant morphism, $L = K(X)$, $K = K(Y)$ be the rational function field of X and Y. The morphism f corresponds to an injection $K \to L$, and if we identify K (resp. L) with the simple sheaf $\mathscr{K}_Y$ (resp. $\mathscr{K}_X$), this injection is an f-morphism. Let K_1 (resp. L_1) be an extension of K (resp. L) and suppose that we are given a monomorphism $K_1 \to L_1$ such that the diagram

$$\begin{tikzcd} K_1 \arrow[r] & L_1 \\ K \arrow[u] \arrow[r] & L \arrow[u] \end{tikzcd}$$

is commutative; if K_1 (resp. L_1) is considered as a simple sheaf over Y (resp. X), hence a $\mathscr{K}_Y$-algebra (resp. a $\mathscr{K}_X$-algebra), this signifies that $K_1 \to L_1$ is an f-morphism. Now if X' (resp. Y') is the integral closure of X (resp. Y) relative to L_1 (resp. K_1), X' (resp. Y') is a normal and integral scheme (Proposition 9.6.23) and its rational function field is canonically identified with the algebraic closure L' (resp. K') of L (resp. K) in L_1 (resp. K_1), and there exists a canonical morphism (necessarily dominant) $f' : X' \to Y'$ rendering the diagram (9.6.1). The important case is that $L_1 = L$, K_1 is an extension of K contained in L, and where we suppose that X is integral and normal, hence $X' = X$.

The preceding arguments then show that if X is a normal scheme and Y' is integrally closure of Y relative to a field $K_1 \subseteq L = K(X)$, any dominant morphism $f : X \to Y$ factors into

$$f : X \xrightarrow{f'} Y' \to Y$$

where f' is dominant; if the monomorphism $K_1 \to L$ is fixed, f' is necessarily unique (this can be verified when X and Y are both affine). We then say that given Y, L, and a K-monomorphism $K_1 \to L$, the integral closure Y' of Y relative to K_1 is a universal object.

Remark 9.6.25. Retain the hypothesis of Proposition 9.6.23 and suppose moreover that each algebra A_i is of finite dimension over $\kappa(\xi_i)$ (which implies that A_i is a direct product of finitely many fields); we can prove that the structural morphism $X' \to X$ is finite. For this, we can reduce to the case where X is reduced and affine with ring C, and that C has finitely many minimal prime ideals $\mathfrak{p}_i$ ($1 \leq i \leq r$) with $C_i = C/\mathfrak{p}_i$. Then by Proposition 9.6.22 X' is finite over X if the integral closure of each C_i in finite extension of its fraction field is a finitely generated C-module, or equivalently, if C_i is Japanese for each i. We know that this condition is true if C is an algebra of finite type over a field, or over $\mathbb{Z}$, or over a complete Noetherian local ring. We then conclude that $X' \to X$ is a finite morphism if X is a scheme of finite type over a field, or over $\mathbb{Z}$, or over a complete Noetherian local ring.

9.6.4 Determinant of an endomorphism of $\mathscr{O}_X$-modules

Let A be a ring, E be a free A-module of rank n, and $u : E \to E$ be an endomorphism of E; recall that in order to define the characteristic polynomial of u, we consider the endomorphism $u \otimes 1$ of free the $A[T]$-module $E \otimes_A A[T]$ (which is of rank n), and we put

$$P(u, T) = \det(T \cdot I - (u \otimes 1))$$

(I is the identity morphism on $E \otimes_A A[T]$). We have

$$P(u, T) = T^n - \sigma_1(u)T^{n-1} + \cdots + (-1)^n \sigma_n(u)$$

where $\sigma_i(u)$ is an element of A, equal to a homogeneous polynomial of degree i (with integer coefficients) with entries the elements of the matrix of u relative to any basis of E. We say that the $\sigma_i(u)$ are the **elementray symmetric functions** of u, and we have in particular $\sigma_1(u) = \operatorname{tr}(u)$ and $\sigma_n(u) = \det(u)$. By Hamilton-Cayley theorem, we have

$$P(u, u) = u^n - \sigma_1(u)u^{n-1} + \cdots + (-1)^n \sigma_n(u) = 0 \tag{9.6.2}$$

which can also be written as

$$(\det(u)) \cdot 1_E = uQ(u) \tag{9.6.3}$$

(1_E is the identity morphism on E), where

$$Q(u) = (-1)^{n+1}(u^{n-1} - \sigma_1(u)u^{n-2} + \cdots + (-1)^{n-1}\sigma_{n-1}(u)) \tag{9.6.4}$$

Let $\varphi : A \to B$ be a homomorphism of rings; consider the B-module $E_{(B)} = E \otimes_A B$ which is free of rank n, and the extension $u \otimes 1$ of u to an endomorphism on $E_{(B)}$. It is immediate that we have $\sigma_i(u \otimes 1) = \varphi(\sigma_i(u))$ for all i.

Suppose now that A is an integral domain, with fraction field K, and E is a finitely generated A-module (not necessarily free now). Let n be the rank of E, which equals to the dimension of $E \otimes_A K$ over K. Any endomorphism u of E corresponds canonically to the endomorphism $u \otimes 1$ of $E \otimes_A K$. By abuse of language, we call $P(u \otimes 1, T)$ the characteristic polynomial of u and denoted by $P(u, T)$, and the coefficients $\sigma_i(u \otimes 1)$ is called the elementray symmetric functions of u and denoted by $\sigma_i(u)$. In particular the determinant $\det(u) = \det(u \otimes 1)$ is defiend. With these notations, the formulas (9.6.2) and (9.6.3) are meaningful and still valid, if we interpret the u^i as the homomorphism $E \to E \otimes_A K$ whici is the composition of the endomorphism $u^j \otimes 1 = (u \otimes 1)^j$ of $E \otimes_A K$ and the canonical homomorphism $x \mapsto x \otimes 1$.

If F is the torsion module of R and $E_0 = E/F$, we have $u(F) \subseteq F$, hence, by taking quotient, u induces an endomorphism u_0 of E_0; moreover $E \otimes_A K$ is identified with $E_0 \otimes_A K$ and $u \otimes 1$ is identified with $u_0 \otimes 1$, hence $\sigma_i(u) = \sigma_i(u_0)$ for $1 \leq i \leq n$.

If E is torsion-free, E is identified with a sub-A-module of $E \otimes_A K$, and the relation $u \otimes 1 = 0$ is equivalent to $u = 0$. If E is a free A-module, the two definitions of $\sigma_i(u)$ given above coincide according to the preceding remarks, which justifies the notations adopted. We also note that if E is a torsion module then $E_0 = \{0\}$, the exterior algebra of E_0 is reduced to K and the determinant of the endomorphism u_0 of E_0 is equal to 1.

Proposition 9.6.26. *Let A be an integral domain, E be a finitely generated A-module, u be an endomorphism of u. Then the elementray symmetric functions $\sigma_i(u)$ of u (and in particular* $\det(u)$*) are integral elements of K over A.*

Proof. This is a particular case of **??**, where we set $B = K$ and note that condition (ii) is satisfied for $M = E$. □

Corollary 9.6.27. *Under the hypothesis of Proposition 9.6.26, if A is normal, the $\sigma_i(u)$ belong to A.*

Proposition 9.6.28. *Let A be an integral domain, E be a finitely generated A-module, of rank n, and u be an endomorphism of E such that the $\sigma_i(u)$ belong to A for each i. For u to be an automorphism of E, it is necessary that $\det(u)$ is invertible in A; this condition is sufficient if E is torsion free.*

Proof. This conditions is sufficient by (9.6.3) and (9.6.4), if E is torsion free, since E is then a sub-A-module of $E \otimes_A K$, and $(\det(u))^{-1}Q(u)$ is the inverse of u. Conversely, this is necessary, because if u is invertible, it follows from Proposition 9.6.26 that $\det(u^{-1})$ belongs to the integral closure A' of A in K, and is clearly the inverse of $\det(u)$ in A'. If $\det(u)$ is not invertible in A, then it belongs to a maximal ideal $\mathfrak{m}$ of A, which is the contraction of a maximal ideal of A' (**??**), contradiction. □

We note a generalization of the preceding results. Consider a reduced Noetherian ring A and let $\mathfrak{p}_i$ $(1 \leq i \leq r)$ be the minimal prime ideals of A, and K_i be the fractional field of $A_i = A/\mathfrak{p}_i$. Then the total fraction field K of A is the direct product of the fields K_i (**??**). Let E be a finitely generated A-module, and suppose that $E \otimes_A K$ is a K-module of dimension n. Then each K_i-vector space $E_i = E \otimes_A K_i$ is of dimension n. If u is an endomorphism of E, we put $P(u,T) = P(u \otimes 1, T)$ and $\sigma_j(u) = \sigma_j(u \otimes 1)$, and in particular $\det(u) = \det(u \otimes 1)$; the $\sigma_j(u)$ are then elements of K. It is immediate that $E \otimes_A K$ is a direct sum of E_i and each of them is stable under $u \otimes 1$. The restriction of $u \otimes 1$ to E_i is just the extension of u to E_i, and we conclude that $\sigma_j(u)$ is the element of K with component in K_i being $\sigma_j(u_i)$. As the integral closure of A in K is the direct product of that of A in K_i (**??**), the $\sigma_j(u)$ are integral over A.

Lemma 9.6.29. *The sub-A-algebra of K genrated by the elements $\sigma_j(u)$ $(1 \leq j \leq n)$ for $u \in \mathrm{Hom}_A(E,E)$, is a finitely generated A-module.*

Let $(X, \mathscr{A})$ be a ringed space, $\mathscr{E}$ be a locally free $\mathscr{A}$-module (of finite rank). There is then by hypothesis a basis $\mathfrak{B}$ of X such that for any $V \in \mathfrak{B}$, $\mathscr{E}|_V$ is isomorphic to $\mathscr{A}^n|_V$ (the integer n may vary with V). Let u be an endomorphism of $\mathscr{E}$; for any $V \in \mathfrak{B}$, u_V is then an endomorphism of the $\Gamma(V,\mathscr{A})$-module $\Gamma(V,\mathscr{E})$, which is free by hypothesis; the determinant of u_V is then defined and belongs to $\Gamma(V,\mathscr{A})$. Moreover, if $e_1,\dots,e_n$ is a basis of $\Gamma(V,\mathscr{E})$, their restriction to any open subset $W \subseteq V$ form a basis of $\Gamma(W,\mathscr{E})$ over $\Gamma(W,\mathscr{A})$, so $\det(u_W)$ is the restriction of $\det(u_V)$ to W. There then exists a unique section of $\mathscr{A}$ over X, which we denote by $\det(u)$ and call the **determinant** of u, such that the restriction of $\det(u)$ to any $V \in \mathfrak{B}$ is $\det(u_V)$. It is clear that for any $x \in X$, we have $\det(u)_x = \det(u_x)$; for two endomorphisms u, v of $\mathscr{E}$, we have

$$\det(u \circ v) = (\det(u))(\det(v)), \quad \det(1_{\mathscr{E}}) = 1_{\mathscr{A}}.$$

If $\mathscr{E}$ is of rank n (for example if X is connected), we have

$$\det(s \cdot u) = s^n \det(u)$$

for any $s \in \Gamma(X,\mathscr{A})$ (we note that $\det(0) = 0_{\mathscr{A}}$ if $n \geq 1$, but $\det(0) = 1_{\mathscr{A}}$ if $n = 0$). Moreover, for u to be an automorphism of $\mathscr{E}$, it is necessary and sufficient that $\det(u)$ is invertible in $\Gamma(X,\mathscr{A})$ (Proposition 9.6.28).

If $\mathscr{E}$ is of rank n, we can similarly define the elementary symmetric functions $\sigma_i(u)$ for u, which are elements of $\Gamma(X,\mathscr{A})$, and we also have the relations (9.6.3) and (9.6.4).

We have then define a homomorphism $\det : \mathrm{Hom}_{\mathscr{A}}(\mathscr{E},\mathscr{E}) \to \Gamma(X,\mathscr{A})$ of multiplicative monoids. Note that $\mathrm{Hom}_{\mathscr{A}}(\mathscr{E},\mathscr{E}) = \Gamma(X, \mathcal{H}om_{\mathscr{A}}(\mathscr{E},\mathscr{E}))$ by definition, so we can replace X by any open subset U in this definition of det, and therefore obtain a homomorphism $\det : \mathcal{H}om_{\mathscr{A}}(\mathscr{E},\mathscr{E}) \to \mathscr{A}$ of sheaves of multiplicative monoids. If $\mathscr{E}$ is of constant rank, we can similarly define the homomorphisms $\sigma_i : \mathcal{H}om_{\mathscr{A}}(\mathscr{E},\mathscr{E}) \to \mathscr{A}$ of sheaves of sets; for $i = 1$, the homomorphism $\sigma_1 = \mathrm{tr}$ is a homomorphism of $\mathscr{A}$-modules.

Let $(Y,\mathscr{B})$ be a second ringed space and $f : (X,\mathscr{A}) \to (Y,\mathscr{B})$ be a morphism of ringed spaces; if $\mathscr{F}$ is a locally free $\mathscr{B}$-module, $f^*(\mathscr{F})$ is a locally free $\mathscr{A}$-module (with the same rank of $\mathscr{F}$). For any endomorphism v of $\mathscr{F}$, $f^*(v)$ is then an endomorphism of $f^*(\mathscr{F})$, and it follows from these definitions that $\det(f^*(v))$ is the section of $\mathscr{A} = f^*(\mathscr{B})$ over X which corresponds canonically to $\det(v) \in \Gamma(Y,\mathscr{B})$. We can then say that the homomorphism $f^*(\det) : f^*(\mathcal{H}om_{\mathscr{B}}(\mathscr{F},\mathscr{F})) \to f^*(\mathscr{B}) = \mathscr{A}$ is the composition

$$f^*(\mathcal{H}om_{\mathscr{B}}(\mathscr{F},\mathscr{F})) \xrightarrow{\gamma^\sharp} \mathcal{H}om_{\mathscr{A}}(f^*(\mathscr{F}), f^*(\mathscr{F})) \xrightarrow{\det} \mathscr{A} \tag{9.6.5}$$

(formula (**??**)). We have a similar result for σ_i.

Suppose now that X is a locally integral scheme, so its sheaf of rational function $\mathscr{K}_X$ is locally simple over X (Corollary 8.7.19) and quasi-coherent as $\mathscr{O}_X$-module. If $\mathscr{E}$ is a qausi-coherent $\mathscr{O}_X$-module of finite type, $\mathscr{E}' = \mathscr{E} \otimes_{\mathscr{O}_X} \mathscr{K}_X$ is then a locally free $\mathscr{K}_X$-module (Corollary 8.7.21). For any endomorphism u of $\mathscr{E}$, $u \otimes 1_{\mathscr{K}_X}$ is then an endomorphism of $\mathscr{E}'$, and $\det(u \otimes 1)$ is a section of $\mathscr{K}_X$ over X, which is called the **determinant** of u and denoted by $\det(u)$. It follows from Proposition 9.6.26 that $\det(u)$ is a section of the integral closure of $\mathscr{O}_X$ in $\mathscr{K}_X$; if X is also normal, $\det(u)$ is then a section of $\mathscr{O}_X$ over X, and if we suppose moreover that $\mathscr{E}$ is torsion free, for u to be an automorphism of $\mathscr{E}$, it is necessary and sufficient that $\det(u)$ is invertible (Proposition 9.6.28). The formulae (9.6.3) and (9.6.4) are still valid; the homomorphism $u \mapsto \det(u)$ then defines a homomorphism $\det : \mathcal{H}om_{\mathscr{O}_X}(\mathscr{E},\mathscr{E}) \to \mathscr{K}_X$, which has values in $\mathscr{O}_X$ if X is normal. We have analogous results for the elementary symmetric function functions $\sigma_j(u)$, if $\mathscr{E}'$ has constant rank; if moreover X is normal, the $\sigma_j(u)$ are sections of $\mathscr{O}_X$ over X.

Finally, let X and Y be integral schemes, and $f : X \to Y$ be a dominant morphism. We see that there exists a canonical homomorphism $f^*(\mathscr{K}_Y) \to \mathscr{K}_X$, whence induces, for any quasi-coherent $\mathscr{O}_Y$-module $\mathscr{F}$ of finite type, a canonical homomorphism $\theta : f^*(\mathscr{F} \otimes_{\mathscr{O}_Y} \mathscr{K}_Y) \to f^*(\mathscr{F}) \otimes_{\mathscr{O}_X} \mathscr{K}_X$. If v is an endomorphism of $\mathscr{F}$, $f^*(v \otimes 1_{\mathscr{K}_Y})$ is an endomorphism of $f^*(\mathscr{F} \otimes_{\mathscr{O}_Y} \mathscr{K}_Y)$, and we have a commutative diagram

$$\begin{tikzcd}
f^*(\mathscr{F} \otimes_{\mathscr{O}_Y} \mathscr{K}_Y) \arrow[r, "f^*(v\otimes 1)"] \arrow[d, "\theta"'] & f^*(\mathscr{F} \otimes_{\mathscr{O}_Y} \mathscr{K}_Y) \arrow[d, "\theta"] \\
f^*(\mathscr{F}) \otimes_{\mathscr{O}_X} \mathscr{K}_X \arrow[r, "f^*(v)\otimes 1"] & f^*(\mathscr{F}) \otimes_{\mathscr{O}_X} \mathscr{K}_X
\end{tikzcd}$$

We then conclude that $\det(f^*(v))$ is the canonical image of the section $\det(v)$ of $\mathscr{K}_Y$ under the canonical homomorphism $f^*(\mathscr{K}_Y) \to \mathscr{K}_X$. In fact, it is immediate that we are reduced to the case where $X = \operatorname{Spec}(B)$ and $Y = \operatorname{Spec}(A)$ are affine, A, B being integral domains with fraction fields K, L respectively, the hommorphism $A \to B$ being injective and extends to a monomorphism $K \to L$. If $\mathscr{F} = \widetilde{M}$ where M is a finitely generated A-module, the dimension of $M \otimes_A K$ is equal to that of $(M \otimes_A B) \otimes_B L$ over L, and $\det((u \otimes 1) \otimes 1)$ is the image of $\det(u \otimes 1)$ in L for any endomorphism u of M, whence our conclusion.

Finally, suppose that X is a reduced locally Noetherian scheme, whose sheaf of rational functions $\mathscr{K}_X$ is quasi-coherent by Proposition 8.7.22. Let $\mathscr{E}$ be a coherent $\mathscr{O}_X$-module such that $\mathscr{E}' = \mathscr{E} \otimes_{\mathscr{O}_X} \mathscr{K}_X$ is locally free of rank n. We can then define for each endomorphism u of $\mathscr{E}$ the elementary symmetric functions $\sigma_j(u)$, which are sections of $\mathscr{K}_X$ over X.

9.6.5 Norm of invertible sheafs

Let $(X, \mathscr{A})$ be a ringed space and $\mathscr{B}$ be an $\mathscr{A}$-algebra. The $\mathscr{A}$-module $\mathscr{B}$ is canonically identified with a sub-$\mathscr{A}$-module of $\mathcal{H}om_{\mathscr{A}}(\mathscr{B},\mathscr{B})$, where a section f of $\mathscr{B}$ over an open subset U of X is identified with the multiplication by this section. Assume that $(X, \mathscr{A})$ and $\mathscr{B}$ satisfies the conditions given in the previous subsection, so that we can define $\det(f)$ (resp. $\sigma_j(f)$) to be a section of $\mathscr{K}_X$ over U, which is called the **norm** of f (resp. the elementary symmetric functions) of f and denoted by $N_{\mathscr{B}/\mathscr{A}}(f)$. We suppose that one of the following conditions is satisfied:

(α) $\mathscr{B}$ is a locally free $\mathscr{A}$-module of finite rank n.

(β) $(X, \mathscr{A})$ is a reduced locally Noetherian scheme, $\mathscr{B}$ is a coherent $\mathscr{A}$-module such that $\mathscr{B} \otimes_{\mathscr{A}} \mathscr{K}_X$ is a locally free $\mathscr{K}_X$-module of rank n, and for any section $f \in \Gamma(U, \mathscr{B})$ over an open subset $U \subseteq X$, $\sigma_j(f)$ $(1 \le j \le n)$ is a section of $\mathscr{A}$ over U (this is true for example if X is normal).

The hypothesis that $\mathscr{B} \otimes_{\mathscr{A}} \mathscr{K}_X$ is locally free can be expressed by the following: denote by X_i the reduced closed subschemes of X with underlying space the irreducible components of X, which are then locally Noetherian integral schemes. Any $x \in X$ belongs to finitely many X_i, and $\mathscr{B} \otimes_{\mathscr{A}} \mathscr{K}_{X_i}$ is a locally free $\mathscr{K}_{X_i}$-module of constant rank k_i (Corollary 8.7.20); to say that $\mathscr{B} \otimes_{\mathscr{A}} \mathscr{K}_X$ is locally free $\mathscr{K}_X$-module signifies that, for any $x \in X$, the ranks k_i such that $x \in X_i$ are all equal. This question is in fact local, and we can assume that $X = \operatorname{Spec}(A)$, where A is a reduced Noetherian ring, and $\mathscr{B} = \widetilde{B}$ where B is a finite A-algebra. If $\mathfrak{p}_i$ $(1 \le i \le r)$ are the minimal prime ideals of A, the total fraction ring K of A is then the direct product of K_i, where K_i is the fraction field of $A_i = A/\mathfrak{p}_i$, and $B \otimes_A K$ is then the direct sum of $B \otimes_A K_i$, whence our conclusion.

It is clear that under the hypotheses (α) or (β), we then define a homomorphism $N_{\mathscr{B}/\mathscr{A}} : \mathscr{B} \to \mathscr{A}$ of sheaves of multiplicative monoids, which is also denoted by N if there is no confusion, and called the norm homomorphism. For two sections f, g of $\mathscr{B}$ over an open subset U, we then have

$$N_{\mathscr{B}/\mathscr{A}}(fg) = N_{\mathscr{B}/\mathscr{A}}(f)N_{\mathscr{B}/\mathscr{A}}(g), \quad N_{\mathscr{B}/\mathscr{A}}(1_{\mathscr{B}}) = 1_{\mathscr{A}} \tag{9.6.6}$$

for the corresponding sections of $\mathscr{A}$ over U. Also, for any section s of $\mathscr{A}$ over U, we have

$$N_{\mathscr{B}/\mathscr{A}}(s \cdot 1_{\mathscr{B}}) = s^n. \tag{9.6.7}$$

In case (α), for any $f \in \Gamma(U, \mathscr{B})$ to be invertible, it is necessary and sufficient that $N(f) \in \Gamma(U, \mathscr{A})$ is invertiblel; in case (β), this condition is necessary, and is sufficient if $\mathscr{B}$ is a torsion free $\mathscr{A}$-module.

Suppose the one of the hypotheses (α), (β) is satisfied, and let $\mathscr{L}'$ be an invertible $\mathscr{B}$-module. We can canonically associate an invertible $\mathscr{A}$-module by the following. Denote by $\mathscr{A}^\times$ (resp. $\mathscr{B}^\times$) the subsheaf of $\mathscr{A}$ (resp. $\mathscr{B}$) such that $\Gamma(U, \mathscr{A}^\times)$ (resp. $\Gamma(U, \mathscr{B}^\times)$) is the set of invertible elements of $\Gamma(U, \mathscr{A})$ (resp. $\Gamma(U, \mathscr{B})$) for any open subset $U \subseteq X$; this is a sheaf of multiplicative groups, and $N_{\mathscr{B}/\mathscr{A}}$, restricted to $\mathscr{B}^\times$, is a homomorphism $\mathscr{B}^\times \to \mathscr{A}^\times$ of sheaves of groups. Let $\mathfrak{L}$ be the set of couples $(U_\lambda, \eta_\lambda)$, with the following property: U_λ is an open subset of X and $\eta_\lambda : \mathscr{L}'|_{U_\lambda} \xrightarrow{\sim} \mathscr{B}|_{U_\lambda}$ is an isomorphism of $(B|_{U_\lambda})$-modules. By hypothesis, the U_λ for an open covering og X; for two indices λ, μ, we put $\omega_{\lambda\mu} = (\eta_\lambda|_{U_\lambda \cap U_\mu}) \circ (\eta_\mu|_{U_\lambda \cap U_\mu})^{-1}$, which is an automorphism of $\mathscr{B}|_{U_\lambda \cap U_\mu}$, and canonically identified with a section of $\mathscr{B}^\times$ over $U_\lambda \cap U_\mu$, and $(\omega_{\lambda\mu})$ is a 1-cocycle over the covering $\mathfrak{U} = (U_\lambda)$ with values in $\mathscr{B}^\times$. The fact that $N_{\mathscr{B}/\mathscr{A}} : \mathscr{B}^\times \to \mathscr{A}^\times$ is a homomorphism implies that $(N_{\mathscr{B}/\mathscr{A}}(\omega_{\lambda\mu}))$ is a 1-cocycle of $\mathfrak{U}$ with values in $\mathscr{A}^\times$, which then corresponds (up to isomorphism) to an invertible $\mathscr{A}$-module. This invertible $\mathscr{A}$-module is denoted by $N_{\mathscr{B}/\mathscr{A}}(\mathscr{L}')$ and is called the norm of the invertible $\mathscr{B}$-module $\mathscr{L}'$.

Let $\mathfrak{M}$ be a subset of $\mathfrak{L}$ such that the U_λ form an open covering of X, and let $\mathfrak{B}$ be a covering of X. The restriction of the cocycle $(\omega_{\lambda\mu})$ to $\mathfrak{B}$ defines a 1-cocycle $(N_{\mathscr{B}/\mathscr{A}}(\omega_{\lambda\mu}))$, which is the restriction of the 1-cocycle $(N_{\mathscr{B}/\mathscr{A}}(\omega_{\lambda\mu}))$ to $\mathfrak{U}$; it is clear that there is a canonical isomorphism of the invertible $\mathscr{A}$-modules thus defined, and we can therefore define $N_{\mathscr{B}/\mathscr{A}}(\mathscr{L}')$ by a refinement of the covering $\mathfrak{U}$. This shows that, if $\mathscr{L}'$ and $\mathscr{K}'$ are two invertible $\mathscr{B}$-modules, by (9.6.6) we have

$$N(\mathscr{L}' \otimes_{\mathscr{B}} \mathscr{K}') = N(\mathscr{L}') \otimes_{\mathscr{A}} N(\mathscr{K}'), \quad N_{\mathscr{B}/\mathscr{A}}(\mathscr{B}) = \mathscr{A}, \tag{9.6.8}$$

and therefore $N(\mathscr{L}'^{-1}) = N(\mathscr{L}')^{-1}$. Also, it follows from (9.6.7) that if $\mathscr{L}$ is an invertible $\mathscr{A}$-module, we have

$$N_{\mathscr{B}/\mathscr{A}}(\mathscr{L} \otimes_{\mathscr{A}} \mathscr{B}) = \mathscr{L}^{\otimes n}. \tag{9.6.9}$$

We show that $N_{\mathscr{B}/\mathscr{A}}$ is a covariant functor on the category of invertible $\mathscr{B}$-modules. Let $u' : \mathscr{L}' \to \mathscr{K}'$ be a homomorphism of invertible $\mathscr{B}$-modules, and let $\mathfrak{B} = (U_\lambda)$ be an open covering of X such that for any λ, we have an isomorphism $\eta_\lambda : \mathscr{L}'|_{U_\lambda} \xrightarrow{\sim} \mathscr{B}|_{U_\lambda}$ and $\tau_\lambda : \mathscr{K}'|_{U_\lambda} \xrightarrow{\sim} \mathscr{B}|_{U_\lambda}$; there is then for each λ an endomorphism u'_λ of $\mathscr{B}|_{U_\lambda}$ such that $u'_\lambda \circ \eta_\lambda = \tau_\lambda \circ (u'|_{U_\lambda})$, and we can evidently identify u'_λ with a section of $\mathscr{B}$ over U_λ. Hence, for any couple (λ, μ) of indices, the restriction of $(\tau_\lambda)^{-1} \circ u'_\lambda \circ \eta_\lambda$ and $(\tau_\mu)^{-1} \circ u'_\mu \circ \eta_\mu$ to $U_\lambda \cap U_\mu$ coincide. We then deduce for the 1-cocycle $(\omega_{\lambda\mu})$ corresponding to $\mathscr{L}'$ and the 1-cocycle $(\gamma_{\lambda\mu})$ corresponding to $\mathscr{K}'$ the relation

$$\gamma_{\lambda\mu} u'_\mu = u'_\lambda \omega_{\lambda\mu}.$$

If we put $u_\lambda = N(u'_\lambda)$, we then have the analogous relation

$$N(\gamma_{\lambda\mu}) u_\mu = u_\lambda N(\omega_{\lambda\mu})$$

and therefore the u_λ define a homomorphism $N(\mathscr{L}') \to N(\mathscr{K}')$, which is denoted by $N_{\mathscr{B}/\mathscr{A}}(u)$ or $N(u)$. In view of Proposition 9.6.28, it is clear that under the hypothesis (α), u' is an isomorphism if and only if u is, and this is true under the hypothesis (β) if $\mathscr{B}$ is moreover torsion free. In particular, if consider the hommorphisms $\mathscr{B} \to \mathscr{L}'$, which correspond to global sections of $\mathscr{L}'$, since $N_{\mathscr{B}/\mathscr{A}}(\mathscr{B}) = \mathscr{A}$, we get a canonical homomorphism

$$N_{\mathscr{B}/\mathscr{A}} : \Gamma(X, \mathscr{L}') \to \Gamma(X, N_{\mathscr{B}/\mathscr{A}}(\mathscr{L}')).$$

It also follows from (9.6.6) that if $f' \in \Gamma(X, \mathscr{L}')$, $g' \in \Gamma(X, \mathscr{K}')$, we have

$$N(f' \otimes g') = N(f') \otimes N(g'). \tag{9.6.10}$$

Also, for any invertible $\mathscr{A}$-module $\mathscr{L}$ and any section $f \in \Gamma(X, \mathscr{L})$, we have

$$N_{\mathscr{B}/\mathscr{A}}(f \otimes 1_{\mathscr{B}}) = f^{\otimes n}. \tag{9.6.11}$$

Finally, for the homomorphism $\mathscr{B} \to \mathscr{L}'$ corresponding to a section f' of $\mathscr{L}'$ over X to be an isomorphism, it is necessary and sufficient that f'_x generates $\mathscr{L}'_x$ for any $x \in X$; under condition (α), this is equivalent to that $N(f')_x$ generates $(N(\mathscr{L}'))_x$ for any x, and this is true for condition (β) if $\mathscr{B}$ is torsion free.

Let $(X, \mathscr{A})$, $(X', \mathscr{A}')$ be two ringed spaces and $\varphi : X' \to X$ be a morphism, $\mathscr{B}$ be an $\mathscr{A}$-algebra, and $\mathscr{B}' = \varphi^*(\mathscr{B})$. Suppose that one of the following conditions is satisfied:

(i) $\mathscr{B}$ satisfies condition (α).

(ii) $(X, \mathscr{A})$ and $\mathscr{B}$ satisfy condition (β), $(X', \mathscr{A}')$ is a reduced locally Noehterian scheme, and if we denote by X_α and X'_β the reduced closed subschemes of X and X' with underlying space the irreducible components of these spaces, the restriction of φ to X'_β is a dominant morphism from X'_β to X_α.

Under these conditions, we claim that $\mathscr{B}'$ verifies the conditions (α) or (β); the first case is clear, and to prove the second one, it suffices to prove that for any $x' \in X'$, the ranks of $\mathscr{B}' \otimes_{\mathscr{O}_{X'}} \mathscr{K}_{X'_\beta}$ for the indices β such that $x' \in X'_\beta$ are the same. Now, if the restriction of φ to X'_β is a dominant morphism into X_α, the rank of $\mathscr{B}' \otimes_{\mathscr{O}_{X'}} \mathscr{K}_{X'_\beta}$ is equal to that of $\mathscr{B} \otimes_{\mathscr{O}_X} \mathscr{K}_{X_\alpha}$ (which can be seen from the affine case), whence our claim.

This being established, it follows that if f is a section of $\mathscr{B}$ over an open subset $U \subseteq X$, and f' is the inverse image of f under φ, $N_{\mathscr{B}'/\mathscr{A}'}(f')$ is the section of $\mathscr{A}'$ over $\varphi^{-1}(U)$ which is the inverse image of $N_{\mathscr{B}/\mathscr{A}}(f)$ under φ. If $\mathscr{L}$ is an invertible $\mathscr{B}$-module and if $\mathscr{L}' = \varphi^*(\mathscr{L})$ (which is an invertible $\mathscr{B}'$-module), we have

$$N_{\mathscr{B}'/\mathscr{A}'}(\mathscr{L}') = \varphi^*(N_{\mathscr{B}/\mathscr{A}}(\mathscr{L})). \tag{9.6.12}$$

Suppose now that $(X, \mathscr{A})$ is a scheme. Then giving a quasi-coherent finite $\mathscr{A}$-algebra $\mathscr{B}$ is equivalent to giving a finite morphism $\varphi : X' \to X$ such that $\varphi_*(\mathscr{O}_{X'}) = \mathscr{B}$, defined up to X-isomorphisms (Corollary 9.1.11), and in this case X' is isomorphic to the affine specturm $\mathrm{Spec}(\mathscr{B})$. Moreover, if this morphism $\varphi : X' \to X$ is fixed, then giving a quasi-coherent $\mathscr{O}_{X'}$-module $\mathscr{F}'$ is equivalent to giving a quasi-coherent $\mathscr{B}$-module such that $\varphi_*(\mathscr{F}') = \mathscr{F}$ (Proposition 9.1.20), and for $\mathscr{F}'$ to be invertible, it is necessary and sufficient that $\mathscr{F}$ is (Proposition 9.6.11). To utilize the preceding results for the finite morphism φ, it is then necessary to assume that $\mathscr{B} = \varphi_*(\mathscr{O}_{X'})$ satisfies condition (α) or (β). For any invertible $\mathscr{O}_{X'}$-module $\mathscr{L}'$, we then set

$$N_{X'/X}(\mathscr{L}') := N_{\varphi_*(\mathscr{O}_{X'})/\mathscr{O}_X}(\varphi_*(\mathscr{L}')) \tag{9.6.13}$$

which is called the **norm** (relative to φ) of $\mathscr{L}'$. Similarly, if $u' : \mathscr{L}' \to \mathscr{K}'$ is a homomorphism of invertible $\mathscr{O}_{X'}$-modules, we put

$$N_{X'/X}(u') = N_{\varphi_*(\mathscr{O}_X)/\mathscr{O}_X}(\varphi_*(u')) : N_{X'/X}(\mathscr{L}') \to N_{X'/X}(\mathscr{K}'). \tag{9.6.14}$$

In particular, if we consider homomorphisms $\mathscr{O}_{X'} \to \mathscr{L}'$, we obtain a canonical homomorphism

$$N_{X'/X} : \Gamma(X', \mathscr{L}') \to \Gamma(X, N_{X'/X}(\mathscr{L}')). \tag{9.6.15}$$

Proposition 9.6.30. *Let $\varphi : X' \to X$ be a finite morphism and suppose that condition* (i) *or* (ii) *is satisfied. For a homomorphism $u' : \mathscr{L}' \to \mathscr{K}'$ of invertible $\mathscr{O}_{X'}$-modules to be an isomorphism, it is necessary and sufficient that, in the first case, that $N_{X'/X}(u')$ is an isomorphism; in the second case, this condition is necessary, and is sufficient if $\varphi_*(\mathscr{O}_{X'})$ is torsion free.*

Proof. This is a particular case of our previous discussions, where we put $\mathscr{B} = \varphi_*(\mathscr{O}_{X'})$, which is a quasi-coherent finite $\mathscr{O}_X$-algebra. We also note that by Corollary 9.1.19, for $\varphi_*(u')$ to be an isomorphism, it is necessary and sufficient that u' is an isomorphism. □

Corollary 9.6.31. *Retain the hypothesis of Proposition 9.6.30 and suppose that $\varphi_*(\mathscr{O}_{X'})$ is torsion free. Let $\mathscr{L}'$ be an invertible $\mathscr{O}_{X'}$-module, f' be a section of $\mathscr{L}'$ over X', and $f = N_{X'/X}(f')$ the section of $\mathscr{L} = N_{X'/X}(\mathscr{L}')$ over X corresponding to f'. Then we have $\varphi(X' - X'_{f'}) = X - X_f$ and X_f is the largest open subset of X such that $\varphi^{-1}(U) \subseteq X'_{f'}$.*

Proof. In fact, $\varphi(X' - X'_{f'})$ is closed in X by Proposition 9.6.7, and it then suffices to prove the second assertion. Now the relation $U \subseteq X_f$ is equivalent to that the hommorphism $\mathscr{O}_X|_U \to \mathscr{L}|_U$ defined by $f|_U$ is an isomorphism. In view of Proposition 9.6.30, this is equivalent to that the homomorphism $\mathscr{O}_{X'}|_{\varphi^{-1}(U)} \to \mathscr{L}'|_{\varphi^{-1}(U)}$ defined by $f'|_{\varphi^{-1}(U)}$ is an isomorphism, which means $\varphi^{-1}(U) \subseteq X'_{f'}$. □

Proposition 9.6.32. *Let $\varphi : X' \to X$ be a finite morphism, $\psi : Y \to X$ be a morphism; let $Y' = X'_{(Y)}$, $\varphi' = \varphi_{(Y)}$, $\psi' = \psi_{(X')}$ such that the following diagram is commutative*

$$\begin{tikzcd} Y' \arrow[r, "\psi'"] \arrow[d, "\varphi'"'] & X' \arrow[d, "\varphi"] \\ Y \arrow[r, "\psi"] & X \end{tikzcd}$$

Assume the hypotheses of Proposition 9.6.30. Then for any invertible $\mathscr{O}_{X'}$-module $\mathscr{L}'$, we have

$$N_{Y'/Y}(\psi'^*(\mathscr{L}')) = \psi^*(N_{X'/X}(\mathscr{L}')).$$

Proof. Note that we have $\psi^*(\varphi_*(\mathscr{L}')) = \varphi'_*(\psi'^*(\mathscr{L}'))$ in view of Corollary 9.1.30, and in particular $\varphi'_*(\mathscr{O}_{Y'}) = \psi^*(\varphi_*(\mathscr{O}_{X'}))$; if $\varphi_*(\mathscr{O}_{X'})$ is locally free, so is $\varphi'_*(\mathscr{O}_{Y'})$. The conclusion then follows from the definition of $N_{X'/X}$, $N_{Y'/Y}$, and (9.6.12). □

9.6.6 A criterion for ample sheaves

Proposition 9.6.33. *Let $f : X \to Y$ be a quasi-compact morphism, $g : X' \to X$ be a finite and surjective morphism such that $(X, \mathscr{O}_X)$ and $g_*(\mathscr{O}_{X'})$ satisfy condition (β). Then, for an ample invertible $\mathscr{O}_{X'}$-module $\mathscr{L}'$ relative to $f \circ g$, $N_{X'/X}(\mathscr{L}') = \mathscr{L}$ is ample relative to f.*

Proof. We can suppose that Y is affine, and then, in view of Corollary 9.4.43, it suffices to prove that, if $\mathscr{L}'$ is ample, then $\mathscr{L} = N_{X'/X}(\mathscr{L}')$ is ample. For this, we can assume that $g_*(\mathscr{O}_{X'})$ is torsion free. In fact, let $\mathscr{T}$ be the kernel of the homomorphism $g_*(\mathscr{O}_{X'}) \to g_*(\mathscr{O}_{X'}) \otimes_{\mathscr{O}_X} \mathscr{K}_X$, which is a coherent ideal of $\mathscr{B} = g_*(\mathscr{O}_{X'})$ by hypothesis, and put $X'' = \operatorname{Spec}(\mathscr{B}/\mathscr{T})$; we then have a commutative diagram

$$\begin{tikzcd} X'' \arrow[rr, "j"] \arrow[dr, "g'"'] & & X' \arrow[dl, "g"] \\ & X & \end{tikzcd}$$

where j is a closed immersion (Proposition 9.1.27). Moreover, since $\mathscr{T}$ is a torsion sheaf, by Proposition 8.7.27 and **??** we see that the support of $\mathscr{T}$ is a closed subset that is rare in X, so for the generic point x of an irreducible component of X, there exists an affine open neighborhood U of x such that $\mathscr{B}|_U = (\mathscr{B}/\mathscr{T})|_U$. As g is by hypothesis surjective, we then conclude that $x \in g'(X'')$; g' is then dominant, and hence surjective by Proposition 9.6.7 since it is a finite morphism. By definition we have

$$g'_*(\mathscr{O}_{X''}) \otimes \mathscr{K}_X = (\mathscr{B}/\mathscr{T}) \otimes_{\mathscr{O}_X} \mathscr{K}_X = g_*(\mathscr{O}_{X'}) \otimes_{\mathscr{O}_X} \mathscr{K}_X,$$

so $(X, \mathscr{O}_X)$ and $g'_*(\mathscr{O}_{X''})$ satisfies condition (β), and $g'_*(\mathscr{O}_{X''})$ is torsion free. Finally, $j^*(\mathscr{L}') = \mathscr{L}''$ is an ample $\mathscr{O}_{X''}$-module (Proposition 9.4.51(ii)), and $N_{X''/X}(\mathscr{L}'') = N_{X'/X}(\mathscr{L}')$. To see this, we note that to define these two invertible $\mathscr{O}_X$-modules we can utilize an affine open covering (U_λ) of X such that $g_*(\mathscr{L}')$ and $g'_*(\mathscr{L}'')$ to U_λ are respectively isomorphic to $\mathscr{B}|_{U_\lambda}$ and $(\mathscr{B}/\mathscr{T})|_{U_\lambda}$. By Corollary 9.1.25, we immediately see that for any isomorphism $\eta_\lambda : g_*(\mathscr{L}')|_{U_\lambda} \to \mathscr{B}|_{U_\lambda}$ corresponds canonically to an isomorphism

$$\eta'_\lambda : g'_*(\mathscr{L}'')|_{U_\lambda} \to (\mathscr{B}/\mathscr{T})|_{U_\lambda}$$

so that, if $(\omega_{\lambda\mu})$ and $(\omega'_{\lambda\mu})$ are the 1-cocycles corresponding to the isomorphisms (η_λ) and (η'_λ), $\omega'_{\lambda\mu}$ is the canonical image of $\omega_{\lambda\mu} \in \Gamma(U_\lambda \cap U_\mu, \mathscr{B})$ to $\Gamma(U_\lambda \cap U_\mu, \mathscr{B}/\mathscr{T})$. In view of the definition of $\mathscr{T}$, we conclude that

$$N_{\mathscr{B}/\mathscr{A}}(\omega_{\lambda\mu}) = N_{(\mathscr{B}/\mathscr{T})/\mathscr{A}}(\omega'_{\lambda\mu})$$

(where $\mathscr{A} = \mathscr{O}_X$), whence the equality.

Suppose then that $g_*(\mathscr{O}_{X'})$ is torsion free. It then suffices to prove that if f runs through the sections of $\mathscr{L}^{\otimes n}$ ($n > 0$) over X, the X_f form a basis of X (Theorem 9.4.27). Now, let $x \in X$, and let U be an open neighborhood of x; as $g^{-1}(x)$ is finite by Corollary 9.6.5 and $\mathscr{L}'$ is ample, there exists an integer $n > 0$ and a section f' of $\mathscr{L}'^{\otimes n}$ over X' such that $X'_{f'}$ is an open neighborhood of $g^{-1}(x)$ contained in $g^{-1}(U)$. As we have $\mathscr{L}^{\otimes n} = N_{X'/X}(\mathscr{L}'^{\otimes n})$, it then suffices to choose $f = N_{X'/X}(f')$: in fact, we have $X - X_f = g(X' - X'_{f'})$ by Corollary 9.6.31, so $x \in X_f \subseteq U$. □

Corollary 9.6.34. *Under the hypotheses of Proposition 9.6.33, for an invertible $\mathscr{O}_X$-module $\mathscr{L}$ to be ample relative to f, it is necessary and sufficient that $\mathscr{L}' = g^*(\mathscr{L})$ is ample relative to $f \circ g$.*

Proof. This condition is necessary, since g is affine (Proposition 9.5.11). To see the sufficiency, we can assume that Y is affine, so X and X' are quasi-compact and $\mathscr{L}'$ is ample (Corollary 9.4.43). Now the set of points $x \in X$ such that there is a neighborhood of x over which $g_*(\mathscr{O}_{X'})$ (resp. $g_*(\mathscr{O}_{X'}) \otimes_{\mathscr{O}_X} \mathscr{K}_X$) is of rank n is open and closed in X by hypotheses, so X is a finite sum of such open subschemes (recall that X is quasi-compact), and we can therefore suppose that it is equal to one of them (Proposition 9.4.54). But we then have $N_{X'/X}(\mathscr{L}') = \mathscr{L}^{\otimes n}$, so $\mathscr{L}^{\otimes n}$ is ample in view of Proposition 9.6.33, and $\mathscr{L}$ is therefore ample. □

Corollary 9.6.35. *Under the hypotheses of Proposition 9.6.33, suppose moreover that $f : X \to Y$ is of finite type. Then, for f to be quasi-projective, it is necessary and sufficient that $f \circ g$ is quasi-projective. If we suppose that Y is quasi-compact and quasi-compact, then, for f to be projective, it is necessary and sufficient that $f \circ g$ is projective.*

Proof. The hypotheses implies that $f \circ g$ is of finite type. By the definition of quasi-projective morphisms, the first assertion then follows from Proposition 9.6.33 and Corollary 9.6.34. In view of this result and Theorem 9.5.30, it remains to prove that if f is quasi-projective, then for f to be proper, it is necessary and sufficient that $f \circ g$ is proper. But f is then separated and of fintie type, and as g is surjective, this follows from Corollary 9.5.24(ii). □

Corollary 9.6.36. *Let X be a scheme of finite type over a field K and K' be finite extension of K. For X to be projective (resp. quasi-projective) over K, it is necessary and sufficient that $X' \otimes_K K'$ is projective (resp. quasi-projective) over K'.*

Proof. This condition is necessary by Proposition 9.5.20(iii) and Proposition 9.5.34(iii). Conversely, suppose that X' is projective (resp. quasi-projective), and let $g : X' \to X$ be the canonical projective. Since K' is finite over K, it is clear that g is a finite morphism by Proposition 9.6.3 and is surjective (Proposition 8.3.28). Moreover, $g_*(\mathscr{O}_{X'})$ is a locally free $\mathscr{O}_X$-module, being isomorphic to $\mathscr{O}_X \otimes_K K'$ (Corollary 9.1.32). It then follows from the hypotheses and Corollary 9.6.9 and Proposition 9.5.34(ii) that X' is projective (resp. quasi-projective) over K. We then deduce from Corollary 9.6.35 that X is projective (resp. quasi-projective) over K. □

Remark 9.6.37. In fact, later we will see that the statement of Corollary 9.6.36 is valid for arbitrary extension K' of K.

The end of this subsection is devoted to the proof of the criterion in Proposition 9.6.42, which is a refinement of the techniques we have currently used.

Lemma 9.6.38. *Let X be a reduced Noetherian scheme and $\mathscr{E}$ be a coherent $\mathscr{O}_X$-module such that $\mathscr{E} \otimes_{\mathscr{O}_X} \mathscr{K}_X$ is locally free of rank n. Then there exists a reduced Noetherian scheme Z and a birational and finite morphism $h : Z \to X$ such that the homomorphisms $\sigma_i : \mathcal{H}om_{\mathscr{O}_X}(\mathscr{E}, \mathscr{E}) \to \mathscr{K}_X$ send $\mathcal{H}om_{\mathscr{O}_X}(\mathscr{E}, \mathscr{E})$ into the coherent $\mathscr{O}_X$-algebra $h_*(\mathscr{O}_Z)$.*

Corollary 9.6.39. *Under the hypotheses of Lemma 9.6.38, let W be an open subset of X such that for any $x \in W$, either X is normal at x or $\mathscr{E}_x$ is a free $\mathscr{O}_{X,x}$-module. Then we can choose h so that the restriction of h to $h^{-1}(W)$ is an isomorphism $h^{-1}(W) \cong W$.*

Proof. In fact, the hypotheses imply that if $U \subseteq W$ is an affine open subset, we have, in the notations of Lemma 9.6.38, that $(\sigma_i(u))_x \in A_x$ for any $x \in U$ (Proposition 9.6.26), hence $\sigma_i(u) \in A$, and the conclusion follows from the definition of h given in Lemma 9.6.38. □

Proof. Let X be a reduced Noetherian scheme, $g : X' \to X$ be a finite and surjective morphism such that $\mathscr{B} = g_*(\mathscr{O}_{X'})$ is a coherent $\mathscr{O}_X$-module. Suppose that $\mathscr{B} \otimes_{\mathscr{O}_X} \mathscr{K}_X$ is locally free of rank n. Then we can apply Lemma 9.6.38 on $\mathscr{E} = \mathscr{B}$, with the same notations, to obtain a homomorphism σ_n : $\mathcal{H}om_{\mathscr{O}_X}(\mathscr{B}, \mathscr{B}) \to h_*(\mathscr{O}_Z)$, and by composing with the canonical injection $\mathscr{B} \to \mathcal{H}om_{\mathscr{O}_X}(\mathscr{B}, \mathscr{B})$, we obtain a homomorphism of sheaves of multiplicative monoids:

$$\widetilde{\mathrm{N}} : \mathscr{B} = g_*(\mathscr{O}_{X'}) \to h_*(\mathscr{O}_Z) = \mathscr{C}. \tag{9.6.16}$$

For any invertible $\mathscr{O}_{X'}$-module $\mathscr{L}'$, $g_*(\mathscr{L}')$ is an invertible $\mathscr{B}$-module by Proposition 9.6.11, and using the same method, we can define an invertible $\mathscr{C}$-module $\widetilde{\mathrm{N}}(g_*(\mathscr{L}'))$, which is functorial on $\mathscr{L}'$. □

Lemma 9.6.40. *Let X be a reduced Noetherian scheme, $g : X' \to X$ be a finite and surjective morphism such that $g * (\mathscr{O}_{X'}) \otimes_{\mathscr{O}_X} \mathscr{K}_X$ is locally free of rank n. Then there exists a reduced Noetherian scheme Z and a finite and birational morphism $h : Z \to X$ such that for any ample $\mathscr{O}_X$-module $\mathscr{L}'$, the invertible $\mathscr{O}_Z$-module $\mathscr{M}$ such that $h_*(\mathscr{M}) = \widetilde{\mathrm{N}}(g_*(\mathscr{L}'))$ is ample.*

Corollary 9.6.41. *Under the hypotheses of Lemma 9.6.40, for any invertible $\mathscr{O}_X$-module $\mathscr{L}$ such that $g^*(\mathscr{L})$ is ample, $h^*(\mathscr{L})$ is ample.*

Proposition 9.6.42. *Let Y be an affine scheme, X be a reduced Noetherian scheme, $f : X \to Y$ be a quasi-compact morphism, and $g : X' \to X$ be a finite and surjective morphism. Let W be an open subset of X such that, for any $x \in W$, either X is normal at x, or there exists an open neighborhood $T \subseteq W$ of x such that $(g_*(\mathscr{O}_{X'}))|_T$ is a locally free $(\mathscr{O}_X|_T)$-module. Then there exists a reduced Y-scheme Z and a finite and birational Y-morphism $h : Z \to X$ such that the restriction of h to $h^{-1}(W)$ is an isomorphism $h^{-1}(W) \cong W$ and satisfies the following property: for any invertible $\mathscr{O}_X$-module such that $g^*(\mathscr{L})$ is ample relative to $f \circ g$, $h^*(\mathscr{L})$ is ample relative to $f \circ h$.*

Corollary 9.6.43. *If in Proposition 9.6.42 we have $W = X$, then for an invertible $\mathscr{O}_X$-module $\mathscr{L}$ to be ample relative to f, it is necessary and sufficient that $g^*(\mathscr{L})$ is ample relative to $f \circ g$.*

Remark 9.6.44. We shall see in Chapter 10 that if Y is Noetherian, f is of finite type, and if the restriction of f to the reduced closed subscheme of X having $X - W$ as underlying space is proper, then the conclusion of Corollary 9.6.43 is still valid. But we will also give examples of algebraic schemes X over a field K (the structural morphism $X \to \mathrm{Spec}(K)$ not being proper) whose normalize X' is quasi-affine, but which is not quasi-affine (so that $\mathscr{O}_X$ is not ample, although $\mathscr{O}_{X'}$ is, cf. Proposition 9.5.1, and that the morphism $g : X' \to X$ is finite and surjective (cf. Remark 9.6.25)). We will also see that this circumstance cannot occur when we replace "quasi-affine" by "affine" (by Chevalley's theorem).

9.6.7 Chevally's theorem

Lemma 9.6.45. *Let X, Y be integral Noetherian schemes, x (resp. y) be the generic point of X (resp. Y), and $f : X \to Y$ be a finite and surjective morphism. Let $\mathscr{L}$ be an invertible $\mathscr{O}_X$-module such that there exists an affine neighborhood U of y and a section $s \in \Gamma(X, \mathscr{L})$ such that $x \in X_s \subseteq f^{-1}(U)$. Then there exist integers $m, n > 0$, a homomorphism $u : \mathscr{O}_Y^m \to f_*(\mathscr{L}^{\otimes n})$ and an open neighborhood V of y such that the restriction $u|_V$ is an isomorphism $\mathscr{O}_Y^m|_V \xrightarrow{\sim} f_*(\mathscr{L}^{\otimes n})|_V$.*

Theorem 9.6.46 (Chevalley). *Let X be an affine scheme, Y be a Noetherian scheme, and $f : X \to Y$ be a finite and surjective morphism. Then Y is an affine scheme.*

Proof. It is clear that $f_{\mathrm{red}} : X_{\mathrm{red}} \to Y_{\mathrm{red}}$ is fiite (Proposition 9.6.3(vi)); as X_{red} is an affine scheme, and Y is affine if and only if Y_{red} is (Corollary 8.4.37), we can assume that X, Y are reduced. For any closed subset Y' of Y, there is then a unique reduced subscheme subscheme structure on Y' whose inverse image $f^{-1}(Y')$, canonically isomorphic to $X \times_Y Y'$, is affine as a closed subscheme of X, and the restriction of f to $f^{-1}(Y')$, which is identified with $f \times_Y 1_{Y'}$, is a finite and surjective morphism (Proposition 8.3.28 and Proposition 9.6.3(iv)). In view of the Noetherian induction principle (**??**), we are then (in view of Corollary 8.4.37) reduced to prove the theorem under the hypothesis that for any closed subset $Y' \neq Y$, any closed subscheme of Y with underlying space Y' is affine. With this hypothesis, we first note that, for any coherent $\mathscr{O}_Y$-module $\mathscr{F}$ whose support (closed) Z is distinct from Y, we have $H^1(Y,\mathscr{F})=0$. In fact, there exists a closed subscheme structure on Z such that, if $j:Z\to Y$ is the canonical injection, we have $\mathscr{F}=j_*(j^*(\mathscr{F}))$ (Corollary 8.6.18 and Theorem 8.1.30), and therefore $H^1(Y,\mathscr{F})=H^1(Z,j^*(\mathscr{F}))=0$ (Corollary 9.5.16, since j is affine).

Suppose first that Y is not irreducible, and let Y' be an irreducible component of Y; we endow Y' with the reduced subscheme structure, and let $j:Y'\to Y$ be the canonical injection. Let $\mathscr{F}$ be a coherent $\mathscr{O}_Y$-module, and consider the canonical homomorphism

$$\rho:\mathscr{F}\to\mathscr{F}'=j_*(j^*(\mathscr{F}));$$

Since j is proper and Y',Y are Noetherian schemes, $\mathscr{F}'$ is a coherent $\mathscr{O}_Y$-module by **??** (since we have $j_*(\mathscr{O}_{Y'})=\mathscr{O}_Y/\mathscr{J}$, where $\mathscr{J}$ is the quasi-coherent ideal of $\mathscr{O}_Y$ defining the subscheme Y'), so $\mathscr{G}=\ker\rho$ and $\mathscr{K}=\operatorname{im}\rho$ are coherent $\mathscr{O}_Y$-modules (**??**). On the other hand, by definition the fiber $\mathscr{F}'_y$ of $\mathscr{F}'$ at the generic point y of Y' is equal to $\mathscr{F}_y/\mathscr{J}_y\mathscr{F}_y$, and hence to $\mathscr{F}_y$ (Example 8.4.47), so y is not contained in the support of $\mathscr{G}$ and we conclude that $H^1(Y,\mathscr{G})=0$. Since the support of $\mathscr{F}'$ (and a fortiori that of $\mathscr{K}$) is contained in Y', it is distinct from Y, and we also conclude that $H^1(Y,\mathscr{K})=0$. From the exact sequence $0\to\mathscr{G}\to\mathscr{F}\to\mathscr{K}\to 0$ we then get $H^1(Y,\mathscr{F})=0$, so by Serre's criterion Y is then affine.

Suppose now that Y is irreducible, and therefore integral. We can also assume that X is integral: in fact, if we denote by X_i the reduced closed subschemes of X with underlying spaces the irreducible components of X and by $f_i:X_i\to Y$ the restriction of f to X_i, then one of f_i is dominant (Y is irreducible, so if its generic point is contained in the image of f_i, then f_i is dominant, and we note that f is surjective), and as there are finite morphisms (Proposition 9.6.3), it is surjective (Proposition 9.6.7); as X_i is an affine scheme, we see that we can replace X by X_i. In this case, we can apply Lemma 9.6.45 to $\mathscr{L}=\mathscr{O}_X$, since X is affine, to obtain a homomorphism $u:\mathscr{O}_Y^m\to f_*(\mathscr{L})=\mathscr{B}$ and an open neighborhood V of y such that $u|_V$ is an isomorphism. In view of Serre's criterion, it suffices to prove that for any coherent $\mathscr{O}_Y$-module $\mathscr{F}$ such that $\mathscr{F}\subseteq\mathscr{O}_Y$, we have $H^1(Y,\mathscr{F})=0$. We note that then $\mathscr{F}$ is torsion free since Y is integral, and we only need to show that $H^1(Y,\mathscr{F})=0$ for any torsion free coherent $\mathscr{O}_Y$-module $\mathscr{F}$. Now the homomorphism u defines a homomorphism

$$v:\mathscr{G}=\mathcal{H}om_{\mathscr{O}_Y}(\mathscr{B},\mathscr{F})\to\mathcal{H}om_{\mathscr{O}_Y}(\mathscr{O}_Y^m,\mathscr{F})=\mathscr{F}^m.$$

By hypotheses the support of $\mathscr{T}=\operatorname{coker}u$ does not meet V, so is a torsion $\mathscr{O}_Y$-module (Proposition 8.7.27). From the exact sequence $\mathscr{O}_Y^m\to\mathscr{B}\to\mathscr{T}\to 0$ induces, by the left exactness of $\mathcal{H}om_{\mathscr{O}_Y}$, an exact sequence

$$0\longrightarrow\mathcal{H}om_{\mathscr{O}_Y}(\mathscr{T},\mathscr{F})\longrightarrow\mathscr{G}\xrightarrow{\ v\ }\mathscr{F}^m$$

But as $\mathscr{F}$ is torsion free and $\mathscr{T}$ is torsion, we have $\mathcal{H}om_{\mathscr{O}_Y}(\mathscr{T},\mathscr{F})=0$, so v is injective. We then obtain an exact sequence

$$0\longrightarrow\mathscr{G}\longrightarrow\mathscr{F}^m\longrightarrow\operatorname{coker}v\longrightarrow 0$$

where $\mathscr{G}$ and $\operatorname{coker}v$ are coherent $\mathscr{O}_Y$-modules (**??**). In view of the exact sequence on cohomology, it suffices to prove that $H^1(Y,\mathscr{F})=H^1(Y,\operatorname{coker}v)=0$ since this implies $H^1(Y,\mathscr{F}^m)=(H^1(Y,\mathscr{F}))^m=0$, so $H^1(Y,\mathscr{F})=0$. Now the restriction $v|_V$ is an isomorphism, so the support of $\operatorname{coker}v$ is distinct from Y, and we have $H^1(Y,\operatorname{coker}v)=0$ by our hypothesis. On the other hand, $\mathscr{G}$ is a coherent $\mathscr{B}$-module by Corollary 8.2.31; as X is affine over Y, there exists a quasi-coherent $\mathscr{O}_X$-module $\mathscr{K}$ such that $\mathscr{G}$ is isomorphic to $f_*(\mathscr{K})$ (Proposition 9.1.20), and since $H^1(X,\mathscr{K})=0$ (X is affine), we then have $H^1(Y,\mathscr{G})=0$ by Corollary 8.6.18, which completes the proof. □

Corollary 9.6.47. *Let X be a Noetherian scheme and $(X_i)_{1\leq i\leq n}$ be a finite covering of X by closed subsets. Then for X to be affine, it is necessary and sufficient that for each i, there exists a closed subscheme of X that is affine and has underlying space X_i.*

Proof. Let X' be the sum of the X_i, then it is clear that X' is affine if each X_i is affine, and we have a surjective morphism $f:X'\to X$ whose restriction to X_i is the canonical injection. To apply Theorem 9.6.46, it remains to verifying that f is finite, and this follows from Proposition 9.6.3(i). □

9.7 Valuative criterion

In this section we give the valuative criterion of sparation and properness of a morphis, which are criteria which involve a auxiliary scheme $\operatorname{Spec}(A)$, where A is a valuation ring. With a convenient

"Noetherian" hypothesis, these criterion can be refined to the case where A is a discrete valuation ring, and this will probably be the only case that we will apply later.

9.7.1 Remainders for valuation rings

Among the vast properties that characterize valuation rings, we shall use the following one: a ring A is called a valuation ring if it is an integral domain which is not a field, and if in the set of proper local rinng contained in the fraction field K of A, A is maximal under the dominant relation. Recall that a valuation ring is integrally closed. If A is a valuation ring, then $A_{\mathfrak{p}}$ is also a valuation ring for any nonzero prime ideal $\mathfrak{p} \neq 0$.

Let K be a field, A be a proper local subring of K; then there exists a valuation ring of K dominating A (**??**). On the other hand, let B be a valuation ring, k be its residue field, and K be the fraction field, L be an extension of k. Then there exists a complete valuation ring C dominating B with residue field equals to L. In fat, L is an algebraic extension of a purely transcendental extension $L' = k(T_\mu)_{\mu\in M}$; we can extend the valuation of K corresponding to B to a valuation of $K' = K(T_\mu)_{\mu\in M}$ with residue field L'; replace B by this complete valuation ring C, we can assume then that B is complete that L is an algebraic closure of k. If $\overline{K}$ is an algebraic closure of K, we can then extend the defining valuation of B to $\overline{K}$, and the corresponding residue field is an algebraic closure of k, as can be seen by lifting the coefficients of a monic polynomial of $k[T]$ to $\overline{K}$. We are therefore finally reduced to the case where $L = k$ and it suffices then to take for C the completion of B to answer the question.

Let K be a field and A be a subring of K; the integral closure A' of A in K is the intersection of valuation rings of the fraction field of A containing A (**??**). The preceding argument then have the following geometric form:

Proposition 9.7.1. *Let Y be a scheme, $p : X \to Y$ be a morphism, x be a point of X, $y = p(x)$, and $y' \neq y$ be a specialization of y. Then there exists a local scheme Y', which is the specturm of a valuation ring, and a separated morphism $f : Y' \to Y$ such that, if a is the unique closed point of Y' and b is the generic point of Y', we have $f(a) = y'$ and $f(b) = y$. We can moreover suppose that one of the following additional conditions are satisfied:*

(i) *Y' is the specturm of a complete valuation ring whose residue field is algebraically closed, and there exists a $\kappa(y)$-homomorphism $\kappa(x) \to \kappa(b)$.*

(ii) *There exists a $\kappa(y)$-isomorphism $\kappa(x) \cong \kappa(b)$.*

Proof. Let Y_1 be the reduced closed subscheme of Y with $\overline{\{y\}}$ as underlying space, and X_1 be the closed subscheme $p^{-1}(Y_1)$; as $y' \in \overline{\{y\}}$ by hypothesis and $\kappa(x)$ is the same for X and X_1, by replacing Y with Y_1 and X with X_1, we can suppose that Y is integral with generic point y; $\mathscr{O}_{Y,y'}$ is then an integral local ring which is not a field, whose fraction field is $\mathscr{O}_{Y,y} = \kappa(y)$, and $\kappa(x)$ is an extension of $\kappa(y)$. To realize the conditions $f(a) = y'$ and $f(b) = y$ with the additional condition (i) (resp. (ii)), we choose $Y' = \mathrm{Spec}(A')$, where A' is a valuation ring dominating $\mathscr{O}_{Y,y'}$ and which is complete and with residue field an algebraically closed extension of $\kappa(x)$ (resp. a valuation ring dominating $\mathscr{O}_{Y,y'}$ with fraction field $\kappa(x)$); the existence of such rings are proved by the above remarks. □

Recall that a local ring $(A, \mathfrak{m})$ is of dimension if and only if any prime ideal of A distinct from $\mathfrak{m}$ is minimal; if A is integral, this means $\mathfrak{m}$ and (0) are the only prime ideals, and $\mathfrak{m} \neq (0)$; equivalent, $Y = \mathrm{Spec}(A)$ is reduced to two points a, b: a is the closed point, $\mathfrak{p}_a = \mathfrak{m}$, and $\kappa(a) = k$ is the residue field $A/\mathfrak{m}$; b is the generic point of Y, $\mathfrak{p}_b = (0)$, the set $\{b\}$ is the unique nontrivial open subset of Y, and $\kappa(b) = K$ is the fraction field of A. For an integral Noetherian local ring A of dimension 1, the following conditions are then equivalent:

(i) A is normal;

(ii) A is regular;

(iii) A is a valuation ring.

Moreover, if these are ture, A is then a discrete valuation ring.

Proposition 9.7.2. *Let A be a Noetherian local integral domain which is not a field, K be its fraction field, L be a extension of K of finite type. There then exists a discrete valuation ring of L dominating A.*

Proof. □

Corollary 9.7.3. *Let A be an integral Noetherian ring, K be its fraction field, and L be an extension of K of finite type. Then the integral closure of A in L is the intersection of discrete valuation rings of L containing A.*

Proposition 9.7.4. *Let Y be a locally Noetherian scheme, $p : X \to Y$ be a morphism locally of finite type, x be a point of X, $y = p(x)$, $y' \neq y$ be a specialization of y. Then there exists a local scheme Y', which is a specturm of a discrete valuation ring, and a separated morphism $f : Y' \to Y$ and rational Y-map $g : Y' \dashrightarrow X$ such that, if a is the closed point of Y' and b is the generic point, we have $f(a) = y'$, $f(b) = y$, $g(b) = x$, and in the following commutative diagram*

$$\begin{tikzcd} & \kappa(x) \arrow[dl, "\gamma"'] \\ \kappa(b) & \kappa(y) \arrow[l, "\varphi"] \arrow[u, "\pi"'] \end{tikzcd}$$

(whee π, φ, γ are the homomorphisms corresponding to p, f and g), γ is a bijection.

Proof. □

9.7.2 Valuative criterion of separation

Proposition 9.7.5 (Valuative Criterion of Separation). *Let Y be a scheme (resp. a locally Noetherian scheme), $f : X \to Y$ be a morphism (resp. a morphism locally of finite type). The following conditions are equivalent:*

(i) *f is separated.*

(ii) *f is quasi-separated and for any Y-scheme of the form $Y' = \mathrm{Spec}(A)$, where A is a valuation ring (resp. a discrete valuation ring), two Y-morphisms of Y' to X which coincide at the generic point of Y' are equal.*

(iii) *f is quasi-seaprated and for any Y-scheme of the form $Y' = \mathrm{Spec}(A)$, where A is a valuation ring (resp. a discrete valuation ring), two Y'-sections of $X' = X_{(Y')}$ which coincide at the generic point of Y' are equal.*

Proof. The equivalence of (ii) and (iii) follows from the correspondence of Y-morphisms $Y' \to X$ and Y'-sections of X'. If X is separated over Y, condition (ii) follows from the proof of Proposition 8.7.10, since Y' is integral. It then remains to prove that condition (ii) implies that the diagonal morphism $\Delta : X \to X \times_Y X$ is closed, and for this we can use Proposition 8.6.6. Now, let z be a point of the diagonal $\Delta(X)$, $z' \neq z$ be a specialization of z in $X \times_Y X$. Then there exists by Proposition 9.7.1 a valuation ring A and a morphism $g : Y' \to X \times_Y X$ such that $g(a) = z'$, $g(b) = z$ (with the notations of Proposition 9.7.1, a is the closed point of Y' and b is the generic point of Y'); this morphism makes Y' an $(X \times_Y X)$-scheme, and a fortiori a Y-scheme. If we compose g with the two projections of $X \times_Y X$, we obtain two Y-morphisms $g_1, g_2 : Y' \to X$, which by hypotheses send the point b to the same point in X; in view of (ii), these two morphisms coincide with a morphism $h : Y' \to X$, which signifies that g factors into $g = \Delta \circ h$, and therefore $z' \in \Delta(X)$. If we suppose that Y is locally Noetherian and f is of finite type, $X \times_Y X$ is locally Noetherian by Corollary 8.6.22, and we can therefore replace Proposition 8.7.10 by Proposition 9.7.4. □

The condition (ii) of Proposition 9.7.5 signifies that if $Y' = \mathrm{Spec}(A)$ and $X' = \mathrm{Spec}(K)$ where K is the fraction field of A, the canonical map

$$\mathrm{Hom}_Y(Y', X) \to \mathrm{Hom}_Y(X', X)$$

is injective. Equivalently, this means in the following diagram

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ Y' \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

the dashed morphism $Y' \to X$, if exists, is necessarily unique.

Remark 9.7.6. In the criterion (ii) of Proposition 9.7.5, we can restrict ourselves to valuation rings A which is complete whose residue field is algebraically closed; this follows from the additional condition (i) of Proposition 8.7.10.

9.7.3 Valuative criterion of properness

Proposition 9.7.7. *Let A be a valuation ring, $Y = \operatorname{Spec}(A)$, b be the generic point of Y. Let X be an integral and separated scheme and $f : X \to Y$ be a closed morphism such that $f^{-1}(b)$ is reduced to a point x and the corresponding homomorphism $\kappa(b) \to \kappa(x)$ is bijective. Then f is an isomorphism.*

Proof. As f is closed and dominant, we have $f(X) = Y$; it then suffices to prove that for any $y' \neq b$ in Y, there exists a unique point x' such that $f(x') = y'$ and the corresponding homomorphism $\mathscr{O}_{Y,y'} \to \mathscr{O}_{X,x'}$ is bijective, because f is then a homeomorphism. Now, if $f(x') = y'$, $\mathscr{O}_{X,x'}$ is a local ring contained in $K = \kappa(x) = \kappa(y)$ and dominates $\mathscr{O}_{Y,y'}$; the later is the local ring $A_{y'}$, which a valuation ring for the fraction field K of A. But $\mathscr{O}_{X,x'} \neq K$ since x' is not the generic point of X, and we then conclude that $\mathscr{O}_{X,x'} = \mathscr{O}_{Y,y'}$ by maximality. As X is an integral scheme, the relation $\mathscr{O}_{X,x'} = \mathscr{O}_{X,x''}$ implies $x' = x''$ by Proposition 8.7.31, which proves our claim. □

Let A be a valuation ring, $Y = \operatorname{Spec}(A)$, b the generic point of Y, so that $\mathscr{O}_{Y,b} = \kappa(b)$ is equal to the fraction field K of A. Let $f : X \to Y$ be a morphism. We have seen that the rational Y-sections of X correspond to the germs of Y-sections (defined over a neighborhood of b) at b, whence a canonical map

$$\Gamma_{\mathrm{rat}}(X/Y) \to \Gamma(f^{-1}(b)/\operatorname{Spec}(K)) \tag{9.7.1}$$

where the elements of $\Gamma(f^{-1}(b)/\operatorname{Spec}(K))$ are identified with the rational points of $f^{-1}(b) = X \otimes_A K$ over K. If f is separted, it then follows from Corollary 8.5.21 that the map of (9.7.1) is injective, since Y is integral.

Composing (9.7.1) with the canonical map $\Gamma(X/Y) \to \Gamma_{\mathrm{rat}}(X/Y)$, we then obtain a canonical map

$$\Gamma(X/Y) \to \Gamma(f^{-1}(b)/\operatorname{Spec}(K)). \tag{9.7.2}$$

If f is separated, this map is injective by Corollary 8.5.21.

Proposition 9.7.8. *Let A be a valuation ring with fraction field K, $Y = \operatorname{Spec}(A)$, b be the generic point of Y, and $f : X \to Y$ be a separated and closed morphism. Then the canonical map (9.7.2) is bijective.*

Proof. Let x be a rational point of $f^{-1}(b)$ over K. As f is separated, so is the morphism $f^{-1}(b) \to \operatorname{Spec}(K)$ corresponding to f (Proposition 8.5.26(iv)), since and any section of $f^{-1}(b)$ a closed immersion by Corollary 8.5.19, $\{x\}$ is closed in $f^{-1}(b)$. Consider the reduced closed subscheme X' of X with underlying space $\overline{\{x\}}$ of $\{x\}$ in X. It is clear that the restriction of f to X' satisfies the conditions of Proposition 9.7.7 (note that since x is rational over K, we have $\kappa(x) = K$), hence an isomorphism from X' to Y, whose inverse isomorphism is the Y-section of X we want. □

Recall that if F is a subset of the scheme Y, the codimension of F in Y is equal to the infimum of $\dim(\mathscr{O}_{Y,z})$ where $z \in F$ (this can be easily verified after reducing to affine case), and we denote this number by $\operatorname{codim}_Y(F)$.

Corollary 9.7.9. *Let Y be a reduced locally Noetherian scheme such that the subset N of $y \in Y$ where Y is not regular has codimension ≥ 2. Let $f : X \to Y$ be a separted and closed morphism of finite type and g be a rational Y-section of X. If Y' is the set of points of Y where g is not defined, then $\operatorname{codim}_Y(Y') \geq 2$.*

Proof. It suffices to prove that g is defined at any point $y \in Y$ such that $\dim(\mathscr{O}_{Y,y}) \leq 1$. If $\dim(\mathscr{O}_{Y,y}) = 0$, then y is the generic point of an irreducible component of Y. For any open dense subset V of Y, by restricting to an affine neighborhood of y and apply **??**, we conclude that V contains y. In particular, y belongs to the defining domain of g. Suppose now that $\dim(\mathscr{O}_{Y,y}) = 1$; then $\mathscr{O}_{Y,y}$ is a regular local ring, hence a discrete valuation ring. Let $Z = \operatorname{Spec}(\mathscr{O}_{Y,y})$; as $U = Y - Y'$ is open and dense, by Corollary 8.2.12 and our preceding arguments, $U \cap Z$ is nonempty (contains the generic of an irreducible component of Y containing y), so we can consider the rational map $g' : Z \dashrightarrow X$ induced by g. It then suffices to prove that g' is a morphism (Proposition 8.7.17). Now, g' can be considered as a rational Z-section of the Z-scheme $f^{-1}(Z) = X \times_Y Z$; it is clear that the morphism $f^{-1}(Z) \to Z$ corresponding to f is closed, and is separted by Proposition 8.5.26(i). We then conclude from Proposition 9.7.8 that g' is everywhere defined, and as Z is reduced and X is separated over Y, g' is a morphism (Proposition 8.7.10). □

Corollary 9.7.10. *Let S be a locally Noetherian scheme, X, Y be S-scheme, and assume that X is proper over X. Suppose that Y is reduced and the subset N of $y \in Y$ where Y is not regular has codimension ≥ 2. Let $f : Y \dashrightarrow X$ be a rational map and Y' be the set of points where f is not defined, then $\operatorname{codim}_Y(Y') \geq 2$.*

Proof. The rational S-maps $Y \dashrightarrow X$ correspond to rational Y-sections of $X \times_S Y$; as the structural morphism $X \times_S Y \to Y$ is closed by Proposition 9.5.23, we can apply Corollary 9.7.9, whence the corollary. □

Remark 9.7.11. The hypothesis on Y in Corollary 9.7.9 and Corollary 9.7.10 are satisfied in particular if Y is normal (by Serre's criterion for normality).

Theorem 9.7.12 (Valuative Criterion of Properness). *Let Y be a (resp. locally Noetherian) scheme and $f : X \to Y$ be a quasi-compact and separted morphism (resp. a quasi-compact morphism of finite type). The following conditions are equivalent:*

(i) *f is universally closed (resp. proper)*

(ii) *For any Y-scheme $Y' = \mathrm{Spec}(A)$ where A is a valuation ring (resp. a discrete valuation ring) with fraction field K, the canonical map (where $X' = \mathrm{Spec}(K)$)*

$$\mathrm{Hom}_Y(Y', X) \to \mathrm{Hom}_Y(\mathrm{Spec}(K), X)$$

corresponding to the cannical injection $A \to K$, is surjective (resp. bijective).

(iii) *For any Y-scheme $Y' = \mathrm{Spec}(A)$ where A is a valuation ring (resp. a discrete valuation ring) with fraction field K, the canonical map of (9.7.2) relative to the Y'-scheme $X' = X_{(Y')}$ is surjective (resp. bijective).*

Proof. The equivalence of (ii) and (iii) follows from the correspondence of Y-morphisms $Y' \to X$ and Y'-morphisms $Y' \to X'$. If f is universally closed then $f_{(Y')}$ is closed and separated, and it then suffices to apply Proposition 9.7.8. It remains to prove that (ii) implies (i). Consider first the case where Y is arbitrary, f is separated and quasi-compact. If the condition of (ii) is satisfied for f, it is also true for $f_{(Y'')} : X_{(Y'')} \to Y''$, where Y'' is an arbitrary Y-scheme, in view of the equivalent of (ii) and (iii), and the fact that $X_{(Y'')} \times_{Y''} Y' = X \times_Y Y'$ for any morphism $Y' \to Y''$; as $f_{(Y'')}$ is also quasi-compact and separated, we then conclude that we only need to prove (ii) implies f is closed, and for this we shall use Proposition 8.6.6. Let $x \in X$, $y' \neq y$ be a specialization of $y = f(x)$; in view of Proposition 9.7.1, there is a scheme $Y' = \mathrm{Spec}(A)$ where A is a valuation ring, and a separated morphism $g : Y' \to Y$ such that, if a is the closed point and b is the generic point of Y, we have $g(a) = y'$, $g(b) = y$, and there exists a $\kappa(y)$-hommorphism $\kappa(x) \to \kappa(b)$. This homomorphism corresponds to a canonical Y-morphism $\mathrm{Spec}(\kappa(b)) \to X$ (Corollary 8.2.17), and it then follows from condition (ii) that there exists a Y-morphism $h : Y' \to X$ which corresponds the previous morphism such that the following diagram commutes

$$\begin{tikzcd}
\mathrm{Spec}(\kappa(b)) \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
\mathrm{Spec}(A) \arrow[r, "g"] & Y
\end{tikzcd}$$

We then have $h(b) = x$, and if we put $h(a) = x'$, x' is then a specialization of x, and we have $f(x') = f(h(a)) = g(a) = y'$.

If now Y is locally Noetherian and f is a quasi-compact morphism of finite type, then condition (ii) implies that f is separated (Proposition 9.7.5), so the diagonal morphism $\Delta : X \to X \times_Y X$ is quasi-compact. Moreover, to verify that f is proper, it suffices to show that $f_{(Y'')} : X_{(Y'')} \to Y''$ is closed for any Y-scheme Y'' of finite type (Corollary 9.5.41). As then Y'' is locally Noetherian, we can resume the reasoning given in the first case by taking for Y' the specturm of discrete valuation ring, and applying Proposition 9.7.1 instead of Proposition 9.7.4. □

Remark 9.7.13. We deduce from the criterion (iii) of Theorem 9.7.12 a new proof of the fact that a projective morphism $X \to Y$ is closed, which is closer to classical methods. We can in fact assume that Y is affine, and X is therefore a closed subscheme of a projective bundle $\mathbb{P}^n_Y$ (Corollary 9.5.19). To show that $X \to Y$ is closed, it suffices to verify that the structural morphism $\mathbb{P}^n_Y \to Y$ is closed. The criterion (iii) of Theorem 9.7.12, together with (9.4.1), show that we are reduced to proving the following fact: if Y is the spectrum of a valuation ring A with fraction field K, every point of $\mathbb{P}^n_Y$ with values in K comes (by restriction to the generic point of Y) from a point of $\mathbb{P}^n_Y$ with values in A. Now, any invertible $\mathscr{O}_Y$-module is trivial, so it follows from Example 9.4.8 that a point of $\mathbb{P}^n_Y$ with values in K is identified with a class of elements $(\xi c_0, \xi c_1, \dots, \xi c_n)$ of K^{n+1}, where $\xi \neq 0$ and the c_i are elements of K which generate the unit ideal of K. By multiplying the c_i with an element of A, we can suppose that the c_i belong to A, and generate the unit ideal of A. But then (Example 9.4.8) the system $(c_0,\dots,c_n)$ defines a point of $\mathbb{P}^n_Y$ with values in A, whence our assertion.

Remark 9.7.14. The criteria Proposition 9.7.5 and Theorem 9.7.12 are especially convenient when we consider a Y-scheme X as a functor

$$X(Y') = \mathrm{Hom}_Y(Y', X)$$

where Y' is a Y-scheme. These criteria will allow us, for example, to prove that under certain conditions the "Picard schemas" are proper.

Corollary 9.7.15. *Let Y be a separated integral scheme (resp. a separated integral locally Noetherian scheme) and $f : X \to Y$ be a dominant morphism.*

(a) *If f is quasi-compact and universally closed, any valuation ring with fraction field the rational function field $K(X)$ and which dominates a local ring of Y, also dominates a local ring of X.*

(b) *Conversely, suppose that f is of finite type, and the property of (a) is satisfied for any valuation ring (resp. any discrete valuation ring) with fraction field $K(X)$. Then f is proper.*

Proof. Assume the hypotheses of (a) and let $K = K(Y)$, $L = K(X)$, y be a point of Y, A be a valuation ring with L the fraction field and dominate $\mathscr{O}_{Y,y}$. The injection $\mathscr{O}_{Y,y} \to A$ is local, so it defines a morphism

$$h : Y' = \mathrm{Spec}(A) \to \mathrm{Spec}(\mathscr{O}_{Y,y}) \to Y$$

(Proposition 8.2.14) such that $h(a) = y$, where a is the closed point of Y'. Moreover, since $K \subseteq L$, the morphism $\mathrm{Spec}(A) \to \mathrm{Spec}(\mathscr{O}_{Y,y})$ is dominant, so if η is the generic point of Y (which is also that of $\mathrm{Spec}(\mathscr{O}_{Y,y})$), we have $h(b) = \eta$, where b is the generic point of Y'. If ξ is the generic point of X, we have $\kappa(\xi) = \kappa(b) = L$ by hypothesis, so there is a Y-morphism $g : \mathrm{Spec}(L) \to X$ such that $g(b) = \xi$. In view of Theorem 9.7.12, we obtain a Y-morphism $g' : Y' \to X$ such that $g(b) = \xi$. If we set $x = g'(a)$, then A dominates $\mathscr{O}_{X,x}$.

We now prove (b); since the question is local over Y, we can assume that Y is affine (resp. affine and Noetherian). As f is of finite type, we can apply Chow's lemma, so there exists a projective morphism $p : P \to Y$, an immersion $j : X' \to P$, and a projective and surjective birational morphism $g : X' \to X$ (where X' is integral) such that the diagram

$$\begin{tikzcd} X' \arrow[r, "j"] \arrow[d, "g"'] & P \arrow[d, "\pi"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

is commutative. It suffices to prove that j is a closed immersion, because then $f \circ g = p \circ j$ is projective, hence proper, and as g is surjective we conclude that f is proper by Corollary 9.5.24. Let Z be a reduced closed subscheme of P with underlying space $\overline{j(X')}$; as X' is integral, j factors into

$$j : X' \xrightarrow{h} Z \xrightarrow{i} P$$

where $i : Z \to P$ is the canonical injection and $h : X' \to Z$ is a dominant open immersion. Since Z is integral and projective over Y by Proposition 9.5.34, we are then reduced to the case where P is integral, j is dominant and birational, and prove that j is surjective. Now let $z \in P$; $\mathscr{O}_{Z,z}$ is an integral local ring (resp. Noetherian integral) whose fraction field is

$$L = K(P) = K(X') = K(X).$$

We can assume that z is not the generic point of P (since the later is contained in $j(Z)$ as j is dominant), so $\mathscr{O}_{Z,z} \neq L$ and by **??** and Proposition 9.7.2, there exists a valuation ring (resp. a discrete valuation ring) A with fraction field L that dominates $\mathscr{O}_{Z,z}$. A fortiori A dominates the local ring $\mathscr{O}_{Y,y}$ where $y = p(z)$, and by hypotheses there is then a point $x \in X$ such that A dominates $\mathscr{O}_{X,x}$. As the morphism g is proper, it satisfies the conditions of (a), so our previous arguments then prove that A also dominates $\mathscr{O}_{X,x'}$, for some $x' \in X'$. Then the local rings $\mathscr{O}_{Z,z}$ and $\mathscr{O}_{Z,j(x')} = \mathscr{O}_{X,x'}$ are related, and by Proposition 8.7.31, as P is separated, we have $z = j(x')$, which completes the proof. $\square$

Corollary 9.7.16. *Let A be an integral domain, $Y = \mathrm{Spec}(A)$, and $f : X \to Y$ be a dominant morphism of integral schemes which is quasi-compact and universally closed. Then $\Gamma(X, \mathscr{O}_X)$ is canonically isomorphic to a subring of the integral closure of A in $K(X)$.*

Proof. Recall that by (8.7.1), $B = \Gamma(X, \mathscr{O}_X)$ is indeitified with the intersection of $\mathscr{O}_{X,x}$ for $x \in X$. If R is a valuation ring of $K(X)$ containing A, then it dominates the local ring $A_{\mathfrak{P}}$ where $\mathfrak{P} = \mathfrak{m}_R \cap B$, and therefore by Corollary 9.7.15 dominates a local ring of X. Then B is contained in R, and the conclusion follows from **??**. □

Remark 9.7.17. Under the hypothesis of Corollary 9.7.16, if we suppose that $K(X)$ is a finite extension of $K(Y)$, then we can in many cases conclude that $\Gamma(X, \mathscr{O}_X)$ is a finitely generated module over the ring $B = \Gamma(Y, \mathscr{O}_X)$. This is the case for example if B is a Japanese ring. In particular, if $X = \mathrm{Spec}(A)$ and $Y = \mathrm{Spec}(k)$ where k is a algebraically closed field, then the the corresponding homomorphism $k \to A$ is injective by **??** and since the integral closure of k in $K(X)$ is equal to k, we conclude that $\Gamma(X, \mathscr{O}_X) = k$.

9.7.4 Algebraic curves

Let k be a field. In this paragraph, all schemes are considered to be separated k-schemes of finite type, and any morphism are k-morphism.

Proposition 9.7.18. *Let X be a scheme of finite type over k; let x_i $(1 \leq i \leq n)$ be the generic points of the irreducible components X_i of X, and $K_i = \kappa(x_i)$. Then the following conditions are equivalent:*

(i) *For each i, the transcendence degree of K_i over k is equal to* 1.

(ii) *For any closed point x of X, the local ring $\mathscr{O}_{X,x}$ is of dimension* 1.

(iii) *The closed irreducible subsets of X distinct from the X_i are closed points of X.*

Proof. As X is quasi-compact, any irreducible closed subset F of X contains a closed point ([**?**] 0_{I}, 2.1.3). Let x be a closed point of X; in view of Corollary 8.2.12, there is a correspondence between prime ideals of $\mathscr{O}_{X,x}$ and the irreducible closed subsets of X containing x. The equivalence of (ii) and (iii) then follows. On the other hand, if $\mathfrak{p}_\alpha$ $(1 \leq \alpha \leq r)$ is the minimal prime ideals of the local Noetherian ring $\mathscr{O}_{X,x}$, the local ring $\mathscr{O}_{X,x}/\mathfrak{p}_\alpha$ are integral, whose fraction fields are the K_i such that $x \in X_i$. Moreover, the dimension of a k-algebra of finite type is equal to its transcendental degree over k. Finally, the dimension of $\mathscr{O}_{X,x}$ is the supremum of the $\mathscr{O}_{X,x}/\mathfrak{p}_\alpha$, and a k-algebra of finite type is catenary, so (i) and (ii) are equivalent. □

We note that under the conditions of Proposition 9.7.18, by Proposition 8.6.44 that set X is either empty of finite. We define an algebraic curve over k to be a nonempty scheme X over k satisfying the conditions of Proposition 9.7.18. Equivalently, we will see that this condition is equivalent to that the irreducible components of X has dimension 1. In particular, we note that if X is an algebraic curve over k, the reduced closed subschemes X_i of X with underlying spaces the irreducible components of X are also algebraic curves over k.

Corollary 9.7.19. *Let X be an irreducible algebraic curve. Then the only non-closed point of X is its generic point, the proper closed subsets of X are the finite subsets of X, which are also the non dense subsets of X.*

Proof. If a point $x \in X$ is not closed, then its closure is an irreducible closed subset of X, hence equal to X by Proposition 9.7.18, so x is the generic point of X. A proper closed subset F of X can not contain the generic point of X, so its points are all closed, hence T1, and by Proposition 8.2.33 we conclude that F is finite and discrete. The closure of an infinite subset of X is therefore necessarily equal to X, which proves the last assertion. □

If X is an arbitrary algebraic curve, then by applying Corollary 9.7.19, we conclude that the only non-closed points of X are the generic points of the irreducible components of X.

Corollary 9.7.20. *Let X and Y be irreducible algebraic curves over k and $f : X \to Y$ be a k-morphism. Then for f to be dominant, it is necessary and sufficient that $f^{-1}(y)$ is finite for any $y \in Y$.*

Proof. If f is not dominant, $f(X)$ is necessarily a finite subset of Y by Corollary 9.7.19, so it is not possible that $f^{-1}(y)$ is finite for any $y \in Y$ (since X is an infinite set). Conversely, if f is dominant, for any $y \in Y$ which is not the generic point η of Y, $f^{-1}(y)$ is closed in X since $\{y\}$ is closed in Y (Corollary 9.7.19); on the other hand, by hypotheses, $f^{-1}(y)$ does not contain the generic point of X, so is finite by Corollary 9.7.19. Finally, to see that $f^{-1}(\eta)$ is finite, we note that the morphism f is of finite type by Corollary 8.6.37, so the fiber $f^{-1}(\eta)$ is an irreducible scheme of finite type over $\kappa(\eta)$ with generic point ξ (Proposition 8.6.35). As $\kappa(\xi)$ and $\kappa(\eta)$ are extensions of k of finite type with transcendental degree 1, it follows that $\kappa(\xi)$ is a finite extension of $\kappa(\eta)$, so ξ is closed in $f^{-1}(\eta)$ by Corollary 9.6.5, and $f^{-1}(\eta)$ is therefore reduced to a point ξ. □

Remark 9.7.21. We will see later that a proper morphism $f : X \to Y$ of Noetherian schemes, such that $f^{-1}(y)$ is finite for any $y \in Y$, is necessarily finite. It then follows from Corollary 9.7.19 that such a dominant proper morphism of irreducible algebraic curves is finite.

Corollary 9.7.22. *Let X be an algebraic curve over k. For X to be regular, it is necessary and sufficient that X is normal, or the local ring of its closed points are discrete valuation rings.*

Proof. This comes from conditions (ii) of Proposition 9.7.18. □

Corollary 9.7.23. *Let X be a reduced algebraic curve, $\mathscr{A}$ be a reduced coherent $\mathscr{K}_X$-algebra. Then the integral closure X' of X relative to $\mathscr{A}$ is a normal algebraic curve, and the canonical morphism $X' \to X$ is finite.*

Proof. The fact that $X' \to X$ is finite follows from Remark 9.6.25, and X' is then a normal algebraic scheme over k. Moreover, we note that if X is irreducible with generic point ξ and its integral closure X' has generic point ξ', then $\kappa(\xi') = \kappa(\xi)$ by Corollary 9.6.24, so X' is also an algebraic curve over k. □

Corollary 9.7.24. *For a reduced algebraic curve X to be proper over k (which is called **complete**), it is necessary and sufficient that the normalization X' of X is proper over k.*

Proof. The canonical morphism $f : X' \to X$ is finite by Corollary 9.7.23, hence proper (Corollary 9.6.9) and surjective (Corollary 9.6.24). If $g : X \to \operatorname{Spec}(k)$ is the structural morphism, g and $g \circ f$ are then simultaneously proper, in view of Proposition 9.5.23 and Corollary 9.5.24. □

Proposition 9.7.25. *Let X be a normal algebraic curve over k and Y be a proper algebraic scheme over k. Then any rational k-map $f : X \dashrightarrow Y$ is everywhere defined, hence a morphism.*

Proof. It follows from Corollary 9.7.10 that the set of points $x \in X$ where this rational map is not defined, the dimnesion of $\mathscr{O}_{X,x}$ is ≥ 2, hence is empty. The assertion then follows from Proposition 8.7.10. □

Corollary 9.7.26. *A normal algebraic curve over k is quasi-projective over k.*

Proof. As X is the sum of finitely many integral and normal algerbaic curves (Corollary 9.6.24), we can assume that X is integral (Corollary 9.5.22). As X is quasi-compact, it can be covered by finitely many affine opens U_i ($1 \leq i \leq n$), and as each of this is of finite type over k, there exists an integer n_i and a k-immersion $f_i : U_i \to \mathbb{P}_k^{n_i}$ (Corollary 9.5.19 and Proposition 9.5.20(i)). As U_i is dense in X (recall that X is integral by our assumption), it follows from Proposition 9.7.25 that f_i extends to a k-morphism $g_i : X \to \mathbb{P}_k^{n_i}$, and we obtain a k-morphism $g = (g_1, \dots, g_n)_k$ from X into the product P of the $\mathbb{P}_k^{n_i}$ over k. Moreover, for each index i, as the restriction of g_i to U_i is an immersion, so is the restriction of g to U_i (Corollary 8.5.16). As the U_i cover X and g is separated by Proposition 8.5.26(v), g is an immersion from X into P by Proposition 8.7.36. The Segre morphism then provides from g an immersion of X into a projective bundle $\mathbb{P}_k^n$, so X is quasi-projective. □

Corollary 9.7.27. *A normal algebraic curve X is isomorphic to a dense open subscheme of a normal and complete algebraic curve $\widehat{X}$, determined up to isomorphisms.*

Proof. If X_1, X_2 are two normal and complete algebraic curves containing X as open dense subscheme, it follows from Proposition 9.7.25 there is an isomorphism of X_1 and X_2, whence the uniqueness of $\widehat{X}$. To prove the existence of $\widehat{X}$, it suffices to remark that we can consider X as a subscheme of a projective bundle $\mathbb{P}_k^n$ (Corollary 9.7.26). Let $\overline{X}$ be the scheme-theoretic closure of X in $\mathbb{P}_k^n$ (Proposition 8.6.71); as X is an open and dense subscheme of $\overline{X}$, the generic points x_i of the irreducible components of X are those of $\overline{X}$, and the residue fields $\kappa(x_i)$ are the same for both schemes, so $\overline{X}$ is an algebraic curve over k, which is reduced (Proposition 8.6.69) and projective over k (Proposition 9.2.52), hence complete by

Theorem 9.5.30. We then take $\widehat{X}$ to be the noramlization of $\overline{X}$, which is complete by Corollary 9.7.24. If $h : \widehat{X} \to \overline{X}$ is the canonical morphism, the restriction of h to $h^{-1}(X)$ is an isomorphism since X is normal (Proposition 9.6.22), and as $h^{-1}(X)$ contains the generic point of the irreducible components of $\widehat{X}$ (Corollary 9.6.24), it is therefore dense in $\widehat{X}$, which proves the assertion. □

Remark 9.7.28. We will later see that the conclusion of Corollary 9.7.27 is still valid without assuming the algebraic curve to be normal (or even reduced); we will also see that for an algebraic curve (reduced or not) to be affine, it is necessary and sufficient that its irreducible (reduced) components are not complete.

Corollary 9.7.29. *Let X be an irreducible normal algebraic curve with $L = K(X)$, Y be a complete and integral algebraic curve with $K = K(Y)$. Then there exists a canonical correspondence between dominant k-morphisms $X \to Y$ and k-monomorphisms $K \to L$.*

Proof. By Proposition 9.7.25, the rational k-maps $X \dashleftarrow Y$ are identified with k-morphisms $f : X \to Y$. The morphism f is dominante if and only if $f(x) = y$, where x and y are the generic points of X and Y, respectively. The corollary then follows from Corollary 8.7.6. □

Example 9.7.30. We can precise the result of Corollary 9.7.29 if Y is the projective line $\mathbb{P}^1_k = \mathrm{Proj}(k[T_0, T_1])$, where T_0 and T_1 are indeterminates. This is an integral and separated k-scheme, and the induced subscheme $D_+(T_0)$ of Y is isomorphic to $\mathrm{Spec}(k[T])$, so the generic point of Y is the ideal (0) of $k[T]$ and its rational function field is $k(T)$, which shows that Y is a complete algebraic cuver over k. Moreover, the only graded ideal of $S = k[T_0, T_1]$ containing T_0 and distinct from S_+ is the principal ideal (T_0), so the complement of $D_+(T_0)$ in Y is reduced to a closed point, called the "infinite point" and denoted by ∞.

Corollary 9.7.31. *Let X be an irreducible normal algebraic curve with $K = K(X)$. Then there exists a canonical correspondence between K and the set of morphisms $u : X \to \mathbb{P}^1_k$ which is distinct from the constant morphism with value ∞. For u to be dominant, it is necessary and sufficient that the corresponding element in K is transcendental over k.*

Proof. By Corollary 8.7.6 and Example 9.7.30, the rational maps $X \dashrightarrow \mathbb{P}^1_k$ (hence morphisms $X \to \mathbb{P}^1_k$, in view of Proposition 9.7.25) correspond to points of $\mathbb{P}^1_k$ with values in K. For any element $\xi \in K$, we have a induced homomorphism $k[T_0] \to K$ which maps T_0 to ξ, and therefore a morphism $\mathrm{Spec}(K) \to D_+(T_0)$. By composing with the canonical injection $D_+(T_0) \to \mathbb{P}^1_k$, we obtain a point of $\mathbb{P}^1_k$ with values in K which is not located at ∞, which is completely determined by ξ in view of Proposition 8.2.4. On the other hand, by Corollary 8.2.17, any constant morphism $u : \mathrm{Spec}(K) \to \mathbb{P}^1_k$ with value y factors into

$$u : \mathrm{Spec}(K) \to \mathrm{Spec}(\kappa(y)) \to \mathbb{P}^1_k.$$

Since $y \in D_+(T_0)$, the morphism $\mathrm{Spec}(\kappa(y)) \to \mathbb{P}^1_k$ factors through the canonical injection $D_+(T_0) \to \mathbb{P}^1_k$, and u therefore is obtained by morphism $\mathrm{Spec}(K) \to D_+(T_0)$, which corresponds to an element $\xi \in K$; this proves the first part of the corollary. For the morphism u to be dominant, it is necessary that the morphism $\mathrm{Spec}(K) \to \mathrm{Spec}(k[T_0]) \cong D_+(T_0)$ is dominant, which means the homomorphism $k[T] \to K$ is injective (**??**), and this is true if and only if ξ is transcendental. □

Remark 9.7.32. With the notations of Corollary 9.7.31, we now determine the image of the morphism $\mathrm{Spec}(K) \to \mathbb{P}^1_k$ induced by an algerbaic element $\xi \in K$ over k. By definition, if $y \in \mathbb{P}^1_k$ is this image, the morphism factors into

$$\mathrm{Spec}(K) \to \mathrm{Spec}(\kappa(y)) \to \mathrm{Spec}(k[T]) \to \mathbb{P}^1_k$$

where we write $\mathrm{Spec}(k[T])$ for $D_+(T_0)$. We first note that any prime ideal in $k[T]$ is maximal, so $\kappa(y) = k[T]/\mathfrak{p}_y$, and the morphism $\mathrm{Spec}(K) \to \mathrm{Spec}(\kappa(y))$ is identified with the canonical injection $k[T]/\mathfrak{p}_y \to K$. By definition, this homomorphism is induced by the homomorphism $k[T] \to K, T \mapsto \xi$, so if $f(T)$ is the irreducible polynomial of ξ over k, we have $\mathfrak{p}_y = (f)$, and y is therefore the point of $\mathrm{Spec}(k[T])$ corresponding to (f). In particular, if $\xi \in k$, then $\mathfrak{p}_y = (T - \xi)$, and if k is an algebraically closed field, we conclude that any element $\xi \in k$ corresponds to a morphism $\mathrm{Spec}(K) \to \mathbb{P}^1_k$ with image ξ (identified with its corresponding maximal ideal $(T - \xi)$), and any element $\xi \in K - k$ corresponds to a dominant morphism $\mathrm{Spec}(K) \to \mathbb{P}^1_k$.

Corollary 9.7.33. *Let X and Y be normal, complete and irreducible algebraic curves, with $K = K(Y)$, $L = K(X)$. Then there exist a bijective correspondense between the set of k-isomorphisms $X \xrightarrow{\sim} Y$ and the set of k-isomorphisms $K \xrightarrow{\sim} L$.*

Corollary 9.7.33 shows that a normal, complete and irreducible algerbaic curves over k is determined by its rational function field K up to an isomorphism. By definition, K is a field of finite type over k with transcendental degree 1 (which is called a algerbaic function field of one variable).

Proposition 9.7.34. *For any extension K of k of finite type and transcendental degree 1, there exists a normal, complete and irreducible algebraic curve X such that $K(X) = K$. The set of local rings of X is identified with the set formed by K and the valuation rings containing k with fraction field K.*

Proof. In fact, K is a finite extension of a purely transcendental extension $k(T)$ of k, which is identified with the rational function field of $Y = \mathbb{P}^1_k$. Let X be the inegral closure of Y relative to K; X is then a normal algebraic curve with field K (Proposition 9.6.23), and it is complete since the morphism $X \to Y$ is finite (Corollary 9.7.23). For $x \in X$, the local ring $\mathscr{O}_{X,x}$ is equals to K if x is the generic of X, and otherwise it is a discrete valuation ring of K. Conversely, let A be a discrete valuation ring with fraction field K; as the morphism $X \to \mathrm{Spec}(k)$ is proper and A dominates k, it also dominates a local ring $\mathscr{O}_{X,x}$ of X by Corollary 9.7.15, and therefore equals to $\mathscr{O}_{X,x}$. $\square$

Remark 9.7.35. It follows from Proposition 9.7.34 and Corollary 9.7.33 that giving a normal, complete and irreducible algebraic curve over k is essentially equivalent to giving a extension K of k of finite type and transcendental degree 1. We note that if k' is an extension of k, $X \otimes_k k'$ is also a complete algebraic curve over k' (Proposition 9.5.23(iii)), but in general it is neither reduced nor irreducible. However, this will be the case if K is a separable extension of k and if k is algebraically closed in K (which is expressed, in a classical terminology, that K is a "regular extension" of k). But even in this case, it may happen that $X \otimes_k k'$ is not normal.

9.8 Blow up of schemes, projective cones and closures

9.8.1 Blow up of schemes

Let Y be a scheme and $(\mathscr{I}_n)_{n\geq 0}$ be a decreasing sequence of quasi-coherent ideal of $\mathscr{O}_Y$. Suppose that the following conditions are satisfied:

$$\mathscr{I}_0 = \mathscr{O}_Y, \quad \mathscr{I}_n\mathscr{I}_n \subseteq \mathscr{I}_{m+n}$$

where m, n are integers. If this is true, we say that the sequence $(\mathscr{I}_n)_{n\geq 0}$ is **filtered**, or that $(\mathscr{I}_n)_{n\geq 0}$ is a **filtered sequence of quasi-coherent ideals of** $\mathscr{O}_Y$. We note that this hypothesis implies that $\mathscr{I}_1^n \subseteq \mathscr{I}_n$. Put

$$\mathscr{S} = \bigoplus_{n\geq 0} \mathscr{I}_n.$$

It follows the assumption that $\mathscr{S}$ is a quasi-coherent $\mathscr{O}_Y$-algebra, hence defines a Y-scheme $X = \mathrm{Proj}(\mathscr{S})$. If $\mathscr{J}$ is an invertible ideal of $\mathscr{O}_Y$, $\mathscr{I}_n \otimes_{\mathscr{O}_Y} \mathscr{J}^{\otimes n}$ is canonically idealtified with $\mathscr{I}_n\mathscr{J}^n$, and if we replace $\mathscr{I}_n$ by $\mathscr{I}_n\mathscr{J}^n$, then the obtained $\mathscr{O}_Y$-algebra $\mathscr{S}_{(\mathscr{J})}$ satisfies that $X_{(\mathscr{J})} = \mathrm{Proj}(\mathscr{S}_{(\mathscr{J})})$ is canonically isomorphic to X (Proposition 9.3.6).

Suppose that Y is locally integral, so that $\mathscr{K}_Y$ is a quasi-coherent $\mathscr{O}_Y$-algebra (Proposition 8.7.22). We say a sub-$\mathscr{O}_Y$-module $\mathscr{I}$ of $\mathscr{K}_Y$ is a **fractional ideal** of $\mathscr{K}_Y$ if it is of finite type. Given a filtered sequence $(\mathscr{I}_n)_{n\geq 0}$ of quasi-coherent fractional ideal of $\mathscr{K}_Y$, we can then define the quasi-coherent graded $\mathscr{O}_Y$-algebra $\mathscr{S}$ and the corresponding Y-scheme $X = \mathrm{Proj}(\mathscr{S})$. We then see that for an invertible fractional ideal $\mathscr{J}$ of $\mathscr{K}_Y$, there is a canonical isomorphism of X and $X_{(\mathscr{J})}$.

Let Y be a scheme (resp. a locally integral scheme), and $\mathscr{I}$ be a quasi-coherent ideal of $\mathscr{O}_Y$ (resp. a quasi-coherent fractional ideal of $\mathscr{K}_Y$); put $\mathscr{S} = \bigoplus_{n\geq 0}\mathscr{I}^n$. The Y-scheme $X = \mathrm{Proj}(\mathscr{S})$ is said to be the scheme obtained by **blowing up along the ideal** $\mathscr{I}$, or the **blow up** of Y relative to $\mathscr{I}$. If $\mathscr{I}$ is a quasi-coherent ideal of $\mathscr{O}_Y$ and Y' is the closed subscheme defined by $\mathscr{I}$, we also say that X is the Y-scheme obtained by blowing up Y'. By definition, $\mathscr{S}$ is generated by $\mathscr{S}_1 = \mathscr{I}$; if $\mathscr{I}$ is a $\mathscr{O}_Y$-module of finite type, X is then projective over Y. By the hypotheses on $\mathscr{I}$, the $\mathscr{O}_X$-module $\mathscr{O}_X(1)$ is invertible (Proposition 9.3.14) and very ample in view of Corollary 9.4.15 for the structural morphism $j : X \to Y$. We also note that the restriction of f to $f^{-1}(Y - Y')$ is an isomorphism if $\mathscr{I}$ is the quasi-coherent ideal of $\mathscr{O}_Y$ defining Y': in fact, this question is local over Y, so it suffice to suppose $\mathscr{I} = \mathscr{O}_Y$, and this then follows from Remark 9.3.18.

If we replace $\mathscr{I}$ by $\mathscr{I}^d$ for some $d > 0$, the blow up Y-scheme X is then replaced by a Y-scheme canonically isomorphic to X' (Proposition 9.3.6). Simialrly, for any invertible ideal (resp. fractional ideal) $\mathscr{J}$, the blow up scheme $X_{(\mathscr{J})}$ relative to $\mathscr{I}\mathscr{J}$ is canonically isomorphic to X. In particular, if $\mathscr{I}$ is an invertible ideal (resp. fractional ideal), the blow up Y-scheme relative to $\mathscr{I}$ is isomorphic to Y.

Proposition 9.8.1. *Let Y be an integral scheme.*

(i) *For any filtered sequence $(\mathscr{I}_n)$ of quasi-coherent fractional ideals of $\mathscr{K}_Y$, the Y-scheme $X = \mathrm{Proj}(\bigoplus_{n\geq 0}\mathscr{I}_n)$ is integral and the structural morphism $f : X \to Y$ is dominant.*

(ii) *Let $\mathscr{I}$ be a quasi-coherent fractional ideal of $\mathscr{K}_Y$ and X be the blow up Y-scheme relative to $\mathscr{I}$. If $\mathscr{I} \neq 0$, the structural morphism $f : X \to Y$ is surjective and birational.*

Proof. In case (i) the quasi-coherent graded $\mathscr{O}_Y$-algebra $\mathscr{S} = \bigoplus_{n\geq 0}\mathscr{I}_n$ is integral since for any $y \in Y$, $\mathscr{O}_{Y,y}$ is an integral domain, so the claim follows from Proposition 9.3.10. For (ii), we obtain from (i) that X is integral; if morever x and y are generic points of X and Y, we have $f(x) = y$, and it is necessary to prove that $\kappa(x) = \kappa(y)$. Now x is also the generic point of the fiber $f^{-1}(y)$; if $\psi : Z \to Y$ is the canonical morphism, where $Z = \mathrm{Spec}(\kappa(y))$, then the scheme $f^{-1}(y)$ is identified with $\mathrm{Proj}(\mathscr{S}')$, where $\mathscr{S}' = \psi^*(\mathscr{S})$ (Proposition 9.3.31). But it is clear that $\mathscr{S}' = \bigoplus_{n\geq 0}(\widetilde{\mathscr{I}_y})^n$, and as $\mathscr{I}$ is a nonzero quasi-coherent fractional ideal of $\mathscr{K}_Y$, $\mathscr{I}_y \neq 0$ (Corollary 8.7.21), so $\mathscr{I}_y = \kappa(y)$ (since y is the generic point of Y, $\mathscr{I}_y$ is a $\kappa(y)$-vector space). The scheme $\mathrm{Proj}(\mathscr{S}')$ is then identified with $\mathrm{Spec}(\kappa(y))$ (Remark 9.3.18), whence the assertion. □

Retain the notations of Proposition 9.8.1. By definition, the injection $\mathscr{I}_{n+1} \to \mathscr{I}_n$ defines for each $k \in \mathbb{Z}$ a injective homomorphism of degree 0 of graded $\mathscr{S}$-modules

$$u_k : \mathscr{S}_+(k+1) \to \mathscr{S}(k). \tag{9.8.1}$$

As $\mathscr{S}_+(k+1)$ and $\mathscr{S}(k+1)$ are eventually isomorphic $\mathscr{S}$-modules, the hommorphism u_k corresponds to a canonical injective homomorphism of $\mathscr{O}_X$-modules (Proposition 9.3.24):

$$\tilde{u}_k : \mathscr{O}_X(k+1) \to \mathscr{O}_X(k). \tag{9.8.2}$$

Recall on the other hand that we have defined a canonical homomorphism

$$\lambda : \mathscr{O}_X(d) \otimes_{\mathscr{O}_X} \mathscr{O}_X(k) \to \mathscr{O}_X(d+k)$$

and as the diagram

$$\begin{tikzcd}
\mathscr{S}(d) \otimes_{\mathscr{S}} \mathscr{S}(k) \otimes_{\mathscr{S}} \mathscr{S}(l) \arrow[r] \arrow[d] & \mathscr{S}(d+k) \otimes_{\mathscr{S}} \mathscr{S}(l) \arrow[d] \\
\mathscr{S}(d) \otimes_{\mathscr{S}} \mathscr{S}(k+l) \arrow[r] & \mathscr{S}(d+k+l)
\end{tikzcd}$$

is commutative, it follows from the functoriality of λ that the homomorphism λ define a quasi-coherent graded $\mathscr{O}_X$-algebra structure on $\mathscr{S}_X = \bigoplus_{n\in\mathbb{Z}}\mathscr{O}_X(n)$. Moreover, the diagram

$$\begin{tikzcd}
\mathscr{S}(d) \otimes_{\mathscr{S}} \mathscr{S}_+(k+1) \arrow[r] \arrow[d, "1\otimes u_k"'] & \mathscr{S}_+(d+k+1) \arrow[d, "u_{k+d}"] \\
\mathscr{S}(d) \otimes_{\mathscr{S}} \mathscr{S}(k) \arrow[r] & \mathscr{S}(d+k)
\end{tikzcd}$$

is commutative; the functoriality of λ shows that we have a commutative diagram

$$\begin{tikzcd}
\mathscr{O}_X(d) \otimes_{\mathscr{O}_X} \mathscr{O}_X(k+1) \arrow[r, "\lambda"] \arrow[d, "1\otimes\tilde{u}_k"'] & \mathscr{O}_X(d+k+1) \arrow[d, "\tilde{u}_{d+h}"] \\
\mathscr{O}_X(d) \otimes_{\mathscr{O}_X} \mathscr{O}_X(k) \arrow[r, "\lambda"] & \mathscr{O}_X(d+k)
\end{tikzcd} \tag{9.8.3}$$

where the horizontal arrows are the canonical homomorphisms. We can then say that the $\tilde{u}_k$ define an injective homomorphism (of degree 0) of graded $\mathscr{S}_X$-modules

$$\tilde{u} : \mathscr{S}_X(1) \to \mathscr{S}_X \tag{9.8.4}$$

Now we consider for each $n \geq 0$ the homomorphism $\tilde{v}_n = \tilde{u}_{n-1} \circ \cdots \tilde{u}_0$, which is an injective homomorphism $\mathscr{O}_X(n) \to \mathscr{O}_X$; we denote its image by $\mathscr{I}_{n,X}$, which is a quasi-coherent ideal $\mathscr{O}_X$ isomorphic to $\mathscr{O}_X(n)$. Moreover, the diagram

$$\begin{tikzcd} \mathscr{O}_X(m) \otimes_{\mathscr{O}_X} \mathscr{O}_X(n) \arrow[r, "\lambda"] \arrow[d, "\tilde{v}_m \otimes \tilde{v}_n"'] & \mathscr{O}_X(m+n) \arrow[d, "\tilde{v}_{m+n}"] \\ \mathscr{O}_X \arrow[r, "\mathrm{id}"] & \mathscr{O}_X \end{tikzcd}$$

is commutative for $m, n \geq 0$. We also conclude that $(\mathscr{I}_{n,X})_{n \geq 0}$ is a filtered sequence of quasi-coherent ideals of $\mathscr{O}_X$.

Proposition 9.8.2. *Let Y be a scheme, $\mathscr{I}$ be a quasi-coherent ideal of $\mathscr{O}_Y$, and X be the blow up Y-scheme relative to $\mathscr{I}$. Then for each $n > 0$ we have a canonical isomorphism*

$$\mathscr{O}_X(n) \xrightarrow{\sim} \mathscr{I}^n \mathscr{O}_X = \mathscr{I}_{n,X}$$

and therefore $\mathscr{I}^n \mathscr{O}_X$ is a very ample invertible $\mathscr{O}_X$-module relative to Y for $n > 0$.

Proof. The last assertion is immediate since $\mathscr{O}_X(1)$ is invertible (Proposition 9.3.14) and very ample relative to Y by definition. On the other hand, the image of the homomorphism $v_n : S_+(n) \to S$ is none other than $\mathscr{I}^n \mathscr{S}$, and the first assertion then follows from the exactness of the functor $\widetilde{\mathscr{M}}$ (Proposition 9.3.13) and the formula $\widetilde{\mathscr{I}\mathscr{M}} = \mathscr{I} \cdot \widetilde{\mathscr{M}}$. ☐

Corollary 9.8.3. *Under the hypotheses of Proposition 9.8.2, if $f : X \to Y$ is the structural morphism and Y' is the closed subscheme of Y defined by $\mathscr{I}$, the closed subscheme $X' = f^{-1}(Y')$ of X is defined by $\mathscr{I}\mathscr{O}_X$ (isomorphic to $\mathscr{O}_X(1)$), so we have an exact sequence*

$$0 \longrightarrow \mathscr{O}_X(1) \longrightarrow \mathscr{O}_X \longrightarrow \mathscr{O}_{X'} \longrightarrow 0$$

Proof. This follows from Proposition 9.8.2 and Proposition 8.4.16(b). ☐

Under the hypotheses of Proposition 9.8.2, we can specify that structure of $\mathscr{I}_{n,X}$. Note that that the homomorphism

$$\tilde{u}_{-1} : \mathscr{O}_X \to \mathscr{O}_X(-1)$$

corresponds canonically to a section s of $\mathscr{O}_X(-1)$ over X, which is called the canonical section (relative to $\mathscr{I}$). In the diagram (9.8.3), the horizontal homomorphisms are isomorphisms (Corollary 9.3.16), so by replacing d by k and k by -1 in that diagram, we obtain $\tilde{u}_k = 1_k \otimes \tilde{u}_{-1}$ (where 1_k is the identity of $\mathscr{O}_X(k)$), which means the homomorphism $\tilde{u}_k$ is none other than the tensor product by the canonical section k (for any $k \in \mathbb{Z}$). The homomorphism $\tilde{u}$ of (9.8.4) can be interpreted in the same manner, and we then deduce that, for any $n \geq 0$, the homomorphism $\tilde{v}_n : \mathscr{O}_X(n) \to \mathscr{O}_X$ is the tensor product by $s^{\otimes n}$.

Corollary 9.8.4. *With the notations of Corollary 9.8.3, the underlying of X' is the set of $x \in X$ such that $s(x) = 0$, where s is the canonical section of $\mathscr{O}_X(-1)$.*

Proof. In fact, if c_x is a generator for the fiber $(\mathscr{O}_X(1))_x$ at a point x, $s_x \otimes c_x$ is canonically identified with a generator for the fiber $\mathscr{I}_{1,X}$ at the point x, and is therefore invertible if and only if $s_x \notin \mathfrak{m}_x(\mathscr{O}_X(-1))_x$, which means $s(x) \neq 0$. ☐

Proposition 9.8.5. *Let Y be an integral scheme, $\mathscr{I}$ be a quasi-coherent fractional ideal of $\mathscr{K}_Y$, and X be the blow up Y-scheme relative to $\mathscr{I}$. Then there is an isomorphism $\mathscr{O}_X \xrightarrow{\sim} \mathscr{I}\mathscr{O}_X$, and in particular $\mathscr{I}\mathscr{O}_X$ is a very ample invertible $\mathscr{O}_X$-module relative to Y.*

Proof. The question is local over Y (Corollary 9.4.17), so we can assume that $Y = \mathrm{Spec}(A)$ is affine, where A is an integral domain with fraction field K and $\mathscr{I} = \tilde{\mathfrak{I}}$, where $\mathfrak{I}$ is a fractional ideal of K. Then there exists an element $a \neq 0$ such that $a\mathfrak{I} \subseteq A$. Put $S = \bigoplus_n \mathfrak{I}^n$; the map $x \mapsto ax$ is an A-isomorphism of $\mathfrak{I}^{n+1} = S(1)_n$ to $a\mathfrak{I}^{n+1} = a\mathfrak{I}S_n \subseteq \mathfrak{I}^n = S_n$, so defines a eventual isomorphism of degree 0 of graded S-modules $S_+(1) \to a\mathfrak{I}S$. But $x \mapsto a^{-1}x$ is an isomorphism of degree 0 of graded S-modules $a\mathfrak{I}S \xrightarrow{\sim} \mathfrak{I}S$, so we obtain an isomorphism $\mathscr{O}_X(1) \xrightarrow{\sim} \mathscr{I}\mathscr{O}_X$. As S is generated by $S_1 = \mathfrak{I}$, $\mathscr{O}_X(1)$ is invertible and very ample, whence the conclusion. ☐

Proposition 9.8.6. *Let Y be a locally Noetherian scheme, $\mathscr{I}$ ba a quasi-coherent ideal of $\mathscr{O}_Y$, and X be the blow up Y-scheme relative to $\mathscr{I}$. Let $f : X \to Y$ be the structural morphism. If $g : Z \to Y$ is any morphism such that $g^*(\mathscr{I})\mathscr{O}_Z$ is an invertible $\mathscr{O}_Z$-module, then there exists a unique morphism $\tilde{g} : Z \to X$ such that the following diagram is commutative:*

$$\begin{tikzcd} Z \arrow[r, "\tilde{g}"] \arrow[dr, "g"'] & X \arrow[d, "f"] \\ & Y \end{tikzcd}$$

Proof. The question is local over Y, so we can assume that $Y = \mathrm{Spec}(A)$ is affine, where A is Noetherian, and $\mathscr{I} = \widetilde{\mathfrak{I}}$ where $\mathfrak{I}$ is an ideal of A. Then $X = \mathrm{Proj}(S)$ where $S = \bigoplus_{n\geq 0}\mathfrak{I}^n$, and we note that since A is Noetherian, $\mathfrak{I}$ is finitely generated, so S is of finite type over A. Let $a_0, \dots, a_n \in \mathfrak{I}$ be a set of generators for $\mathfrak{I}$, so that we have a surjective homomorphism $\varphi : A[T_0, \dots, T_n] \to S$ which maps T_i to a_i, and this gives a closed immersion $i : X \to \mathbb{P}^n_A$, and we can identify X with its image. If $g : Z \to Y$ is a morphism such that $\mathscr{L} = g^*(\mathscr{I})\mathscr{O}_Z$ is invertible, then the inverse images of the a_i, which are global sections of $\mathscr{I}$, give global sections $s_0, \dots, s_n$ of $\mathscr{L}$ which generate $\mathscr{L}$. Then by Proposition 9.4.3 $r : Z \to P = \mathbb{P}^n_A$ such that $\mathscr{L} \cong r^*(\mathscr{O}_P(1))$ and $s_i = r^{-1}(T_i)$. We now claim that this morphism factors through the closed subscheme X of $\mathbb{P}^n_A$: this follows from the fact that if $F(T_0, \dots, T_n)$ is a homogeneous polynomial of degree d of $\ker\varphi$, then $F(a_0, \dots, a_n) = 0$ in A and so $F(s_0, \dots, s_n) = 0$ in $\Gamma(Z, \mathscr{L}^{\otimes d})$. This gives the desired morphism $\tilde{g} : Z \to X$, and for any such morphism we necessarily have

$$g^*(\mathscr{I})\mathscr{O}_Z = \tilde{g}^*(f^*(\mathscr{I})\mathscr{O}_X)\mathscr{O}_Z = \tilde{g}^*(\mathscr{O}_X(1))\mathscr{O}_Z$$

by Proposition 9.8.2, so we obtain a surjective hommorphism $\tilde{g}^*(\mathscr{O}_X(1)) \to g^*(\mathscr{I})\mathscr{O}_Z = \mathscr{L}$, hence an isomorphism by **??**. Clearly the sections s_i of $\mathscr{L}$ are the inverse images of the sections T_i of $\mathscr{O}_P(1)$ on $\mathbb{P}^n_A$, so the uniqueness of $\tilde{g}$ follows from Proposition 9.4.3. □

Corollary 9.8.7. *Let $q : Y' \to Y$ be a morphism of locally Noetherian schemes and $\mathscr{I}$ be a quasi-coherent ideal of $\mathscr{O}_Y$. Let X be the blow up Y-scheme relative to $\mathscr{I}$ and X' be the blow up Y'-scheme relative to $\mathscr{J} = q^*(\mathscr{I})\mathscr{O}_Y$. Then there exists a unique morphism $p : X' \to X$ such that the diagram*

$$\begin{tikzcd} X' \arrow[r, "p"] \arrow[d] & X \arrow[d] \\ Y' \arrow[r, "q"] & Y \end{tikzcd} \tag{9.8.5}$$

is commutative. Moreover, if q is a closed immersion, so is p.

Proof. The existence and uniqueness of q follows from Proposition 9.8.6 and Proposition 9.8.2. To show that p is a closed immersion if q is, we trace the definition of the blow up: $X = \mathrm{Proj}(\mathscr{S})$ where $\mathscr{S} = \bigoplus_{n\geq 0}\mathscr{I}^n$ and $X' = \mathrm{Proj}(\mathscr{S}')$ where $\mathscr{S}' = \bigoplus_{n\geq 0}\mathscr{J}^n$. Since Y' is a closed subscheme of Y, we can consider $\mathscr{S}'$ as a sheaf of graded algebras over Y. Then there exists a natrual surjective homomorphism $\mathscr{S} \to \mathscr{S}'$, which gives rise to the closed immersion p. □

In the situation of Corollary 9.8.7, if Y' is a closed subscheme of Y, we call the closed subscheme X' of X the **strict transform** of Y' under the blowing-up $f : X \to Y$.

Example 9.8.8. Let $Y = \mathbb{A}^n_k$ be the affine space over a field k and we consider the blow up of Y at the origin y of Y. Then $Y = \mathrm{Spec}(A)$ where $A = k[X_1, \dots, X_n]$, y corresponds to the ideal $\mathfrak{I} = (X_1, \dots, X_n)$, and $X = \mathrm{Proj}(S)$ where $S = \bigoplus_{n\geq 0}\mathfrak{I}^n$. We can define a surjective homomorphism

$$\varphi : A[Y_0, \dots, Y_n] \to S$$

of graded rings by sending Y_i to X_i as an element of degree 1 in S, which gives a closed immersion of X into $\mathbb{P}^{n-1}_A$. It is not hard to see that the kernel of φ is generated by the homogeneous polynomials $X_iY_j - X_jY_i$, where $i, j = 1, \dots, n$, so this definition is compatible with the definition of the blow up of the affine variety $\mathbb{A}^n_k$.

Now if Y' is a closed subscheme of Y passing through y, then the strict transform X' of Y' is a closed subscheme of X. Hence, provided that Y' is not reduced to the point y, we can recover X' as the closure of $f^{-1}(Y' - \{y\})$, where $f : f^{-1}(Y' - \{y\}) \to Y' - \{y\}$ is an isomorphism. Again this definition is compatible with the definition of blow up of closed subvarieties of $\mathbb{A}^n_k$.

Example 9.8.9. As an example of the general concept of blowing up a coherent sheaf of ideals, we show how to eliminate the points of indeterminacy of a rational map determined by an invertible sheaf. So let A be a ring, X be a Noetherian scheme over A, $\mathscr{L}$ be an invertible sheaf on X, and $s_0, \dots, s_n$ be global sections of $\mathscr{L}$. Let U be the open subset of X where the s_i generate the sheaf $\mathscr{L}$ (that is, the subset where the corresponding homomorphism $\mathscr{O}_X^{n+1} \to \mathscr{L}$ is surjective, cf. **??**(iii)). Then the invertible sheaf $\mathscr{L}|_U$ on U and the global sections $s_0, \dots, s_n$ determine an A-morphism $\varphi : U \to \mathbb{P}^n_A$, which is also a rational map $X \dashrightarrow \mathbb{P}^n_A$. We will now show how to blow up a certain sheaf of ideals $\mathscr{I}$ on X, whose corresponding closed subsheme Y has support equal to $X - U$ (i.e., the underlying topological space of Y is $X - U$), so that the morphism φ extends to a morphism $\tilde{\varphi} : \widetilde{X} \to \mathbb{P}^n_A$.

$$\begin{tikzcd}
\widetilde{X} \arrow[d, "\pi"'] \arrow[dr, dashed, "\tilde{\varphi}"] & \\
X & \mathbb{P}^n_A \\
U \arrow[u, hook] \arrow[ur, "\varphi"'] &
\end{tikzcd}$$

Let $\mathscr{F}$ be the quasi-coherent sub-$\mathscr{O}_X$-module of $\mathscr{L}$ generated by $s_0, \dots, s_n$. We define a coherent ideal $\mathscr{I}$ of $\mathscr{O}_X$ as follows: for any open subset $V \subseteq X$ and an isomorphism $\psi : \mathscr{L}|_V \cong \mathscr{O}_X|_V$, we take $\mathscr{I}|_V = \psi(\mathscr{F}|_V)$. It is clear that $\mathscr{I}_V$ is independent of the choice of ψ, so we get a well-defined coherent ideal $\mathscr{I}$ of $\mathscr{O}_X$. We also note that $\mathscr{I}_x = \mathscr{O}_{X,x}$ if and only if $x \in U$, so the corresponding closed subscheme Y has support $X - U$. Let $\pi : \widetilde{X} \to X$ be the corresponding blow up relative to $\mathscr{I}$. We claim that $\pi^*(\mathscr{I})$ is a coherent ideal of $\mathscr{O}_{\widetilde{X}}$, so is invertible by Proposition 9.8.2. This can be verified on each affine open X_{s_i}. Then the global sections $\pi^*(s_i)$ of $\pi^*(\mathscr{L})$ generate an invertible sub-$\mathscr{O}_{\widetilde{X}}$-module $\mathscr{L}'$ of $\pi^*(\mathscr{L})$. Now $\mathscr{L}'$ and the sections $\pi^*(s_i)$ define a morphism $\tilde{\varphi} : \widetilde{X} \to \mathbb{P}^n_A$ whose restriction on $\pi^{-1}(U)$ corresponds to the morphism φ under the isomorphism $\pi : \pi^{-1}(U) \xrightarrow{\sim} U$.

9.8.2 Homogenization of graded rings

Let S be a graded ring, which we do not suppose to be with positive degrees. We set

$$S^{\geqslant} = \bigoplus_{n \geq 0} S_n, \quad S^{\leqslant} = \bigoplus_{n \leq 0} S_n$$

which are subrings of S, with positive of negative degrees respectively. If f is a homogeneous element of degree d (positive of negative) of S, the localization $S_f = S'$ is endowed with a graded ring structure, where S'_n is the set of elements x/f^k, where $x \in S_{n+kd}$ $(k \geq 0)$; we also note that $S_{(f)} = S'_0$, and we denote $S_f^{\geqslant}$ and $S_f^{\geqslant}$ for $S'^{\geqslant}$ and $S'^{\leqslant}$, respectively. This notation is justified by the fact that if $d > 0$, then we have

$$(S^{\geqslant})_f = S_f; \tag{9.8.6}$$

in fact, if $x \in S_{n+kd}$ where $n + kd < 0$, we can write $x/f^k = xf^h/f^{h+k}$ so that $n + (h+k)d > 0$ for $h > 0$ large enough. We then conclude by definition that

$$(S^{\geqslant})_{(f)} = (S_f^{\geqslant})_0 = S_{(f)}. \tag{9.8.7}$$

If M is a graded S-module, we put similarly

$$M^{\geqslant} = \bigoplus_{n \geq 0} M_n, \quad M^{\leqslant} = \bigoplus_{n \leq 0} M_n$$

which are respectively $S^{\geqslant}$-module and $S^{\geqslant}$-module, with intersection the S_0-module M_0. If $f \in S_d$, we also define M_f as the graded S_f-module such that $(M_f)_n$ is the set of elements z/f^k, where $z \in M_{n+kd}$. We denote by $M_{(f)}$ the set of degree 0 elements in M_f, which is an $S_{(f)}$-module, and we write $M_f^{\geqslant}$ and $M_f^{\leqslant}$ for $(M_f)^{\geqslant}$ and $(M_f)^{\leqslant}$ respectively. If $d > 0$, we also have

$$(M^{\geqslant})_f = M_f, \quad (M^{\geqslant})_{(f)} = (M_f^{\geqslant})_0 = M_{(f)}. \tag{9.8.8}$$

Let z be an indeterminate, which is called the **homogenization variable**. If S is a graded ring, the polynomial algebra

$$\widehat{S} = S[z]$$

is a graded S-algebra, where for f homogeneous we put

$$\deg(fz^n) = n + \deg(f).$$

Lemma 9.8.10. *Let $f \in S_d$ with $d > 0$. We have canonical isomorphisms*

$$\widehat{S}_{(z)} \cong \widehat{S}/(z-1)\widehat{S} \cong S, \tag{9.8.9}$$

$$\widehat{S}_{(f)} \cong S_f^{\leqslant}. \tag{9.8.10}$$

Proof. The first isomorphism of (9.8.9) is already defined in Proposition 9.2.3 and the second one is trivial; the isomorphism $\widehat{S}_{(z)} \cong S$ thus defined then send an element xz^n/z^{n+k} (where $\deg(x) = k$ for $k \geq -n$) to the element x. The homomorphism (9.8.10) is defined by sending xz^n/f^k (where $\deg(x) = kd - n$) to the element x/f^k, of degree $-k$ in $S_f^{\leqslant}$, and it is easy to verify that this is an isomorphism. □

Let M be a graded S-module. It is clear that the S-module

$$\widehat{M} = M \otimes_S \widehat{S} = M \otimes_S S[z]$$

is the direct sum of the S-modules $M \otimes Sz^n$, whence the abelian groups $M_k \otimes Sz^n$. We define over $\widehat{M}$ an $\widehat{S}$-module structure by

$$\deg(x \otimes z^n) = n + \deg(x)$$

for x homogeneous in M.

Lemma 9.8.11. *Let $f \in S_d$ with $d > 0$. We have a canonical isomorphism*

$$\widehat{M}_{(z)} \cong \widehat{M}/(z-1)\widehat{M} \cong M, \tag{9.8.11}$$

$$\widehat{M}_{(f)} \cong M_f^{\leqslant}. \tag{9.8.12}$$

Proof. This can be proved as Lemma 9.8.10 by using the second part of Proposition 9.2.3. □

Let S be a graded ring with positive degrees. Then for each $n \geq 0$, we can consider $S(n) = \bigoplus_{m \geq n} S_m$ as a graded iedal of S (in particular $S(0) = S$ and $S(1) = S_+$). As it is clear that $S(m)S(n) \subseteq S(m+n)$, we can then define a graded ring

$$S^\natural = \bigoplus_{n \geq 0} S(n)$$

whence $S_n^\natural = S(n)$. Then $S_0^\natural$ is equal to S considered as a nongraded ring, and $S^\natural$ is therefore an S-algebra. For any homogeneous element $f \in S_d$ with $d > 0$, we denote by $f^\natural$ the element f considered as an element of $S(d) = S_d^\natural$.

Lemma 9.8.12. *Let S be a graded ring with positive degrees, f be a homogeneous element with $d > 0$. We have canonical isomorphisms:*

$$S_f \cong \bigoplus_{n \in \mathbb{Z}} S(n)_{(f)}, \tag{9.8.13}$$

$$(S_f^{\geqslant})_{f/1} \cong S_f, \tag{9.8.14}$$

$$S^\natural_{(f^\natural)} \cong S_f^{\geqslant}. \tag{9.8.15}$$

the first two of which are bi-isomorphisms of graded rings.

Proof. It is immediate that we have $(S_f)_n = (S(n)_f)_0 = S(n)_f$, whence the first isomorphism. On the other hand, as $f/1$ is invertible in S_f, there is a canonical isomorphism $S_f \cong S_f^{\geqslant} = (S_f)_{f/1}$ in view of (9.8.6) applide to S_f. Finally, if $x = \sum_{m \geq n} y_n$ is an element of $S(n)$, where $n = kd$, we can correspond the element $x/(f^\natural)^k$ to the element $\sum_m y_m/f^k$ of $S_f^{\geqslant}$, and we verify that this is an isomorphism. □

If M is a graded S-module, we can similarly put for $n \in \mathbb{Z}$

$$M^\natural = \bigoplus_{n\in\mathbb{Z}} M(n)$$

as $S(m)M(n) \subseteq M(m+n)$. Then $M^\natural$ is a graded $S^\natural$-module, and similarly we have the following:

Lemma 9.8.13. *Let $f \in S_d$ be a homogeneous element with $d > 0$. We have the following bi-homomorphisms*

$$M_f \cong \bigoplus_{n\in\mathbb{Z}} M(n)_{(f)}, \tag{9.8.16}$$

$$(M_f^{\geqslant})_{f/1} \cong M_f, \tag{9.8.17}$$

$$M^\natural_{(f^\natural)} \cong M_f^{\geqslant}. \tag{9.8.18}$$

the first two of which are bi-isomorphisms of graded modules.

Remark 9.8.14. We can think that $S^\natural$ is obtained from S by adding a "phantom" element $\boldsymbol{y}$ of degree -1. The component $S(n)$ can be then considered as the S-module $(S\boldsymbol{y}^n)_0$, which is the set of degree 0 elements in $S\boldsymbol{y}^n$. With this understanding, we can then relate the results of Lemma 9.8.10 and Lemma 9.8.12.

Lemma 9.8.15. *Let S be a graded ring with positive degrees.*

(i) *For $S^\natural$ to be an $S_0^\natural$-algebra of finite type (resp. Noetherian), it is necessary and sufficient that S is an S_0-algebra of finite type (resp. Noetherian).*

(ii) *For $S^\natural_{n+1} = S^\natural_1 S^\natural_n$ for $n \geq n_0$, it is necessary and sufficient that $S_{n+1} = S_1 S_n$ for $n \geq n_0$.*

(iii) *For $S^\natural_n = (S^\natural_1)^n$ for $n \geq n_0$, it is necessary and sufficient that $S_n = S_1^n$ for $n \geq n_0$.*

(iv) *If (f_α) is a set of homogeneous elements of S_+ such that the radical in S_+ of the ideal of S_+ generated by the f_α is equal to S_+, then $S_+^\natural$ is the radical in $S_+^\natural$ of the ideal of $S_+^\natural$ generated by the $f_\alpha^\natural$.*

Proof. If $S^\natural$ is an $S_0^\natural$-algebra of finite type, $S_+ = S_1^\natural$ is a finitely generated module over $S = S_0^\natural$ by **??**, so S is an S_0-algebra of finite type by **??**. If $S^\natural$ is a Noetherian ring, so is the ring $S_0^\natural = S$ by **??**. Conversely, if S is an S_0-algebra of finite type, then by **??** there exists $d > 0$ and $n_0 > 0$ such that $S_{n+d} = S_h S_n$ for any $n \geq n_0$; we can clearly suppose that $n_0 \geq d$. Moreover, the S_n are finitely generated S_0-modules (**??**(c)), so if $n \geq n_0 + d$, we have $S_n^\natural = S_n S^\natural_{n-d} = S^\natural_d S^\natural_{n-d}$; and if $n < n_0 + d$, we have

$$S_n^\natural = S_n + \cdots + S_{n_0+d-1} + S_d E + S_d^2 E + \cdots$$

where $E = S_{n_0} + \cdots + S_{n_0+d-1}$. For $1 \leq n \leq n_0$, let G_n be the union of the generators of the S_0-module S_i where $n \leq i \leq n_0 + d - 1$, considered as subsets of $S(n)$. For $n_0 + 1 \leq n \leq n_0 + d - 1$, similarly let G_n be the union of the generators of the S_0-module S_i where $n \leq i \leq n_0 + d - 1$ and of $S_d E$, considered as subsets of $S(n)$. Then it is clear that we have $S_n^\natural = S_0^\natural G_n$ for $1 \leq n \leq n_0 + d - 1$, and therefore the union G of the G_n for $1 \leq n \leq n_0 + d - 1$ is a system of generators of the $S_0^\natural$-algebra $S^\natural$. We then conclude that if $S = S_0^\natural$ is Noehterian, then so is $S^\natural$.

It is clear that if $S_{n+1} = S_1 S_n$ for $n \geq n_0$, then we have $S^\natural_{n+1} = S_1 S^\natural_n$, and a fortiori $S^\natural_{n+1} = S^\natural_1 S^\natural_n$ for $n \geq n_0$. Conversely, the latter relation means that

$$S_{n+1} + S_{n+2} + \cdots = (S_1 + S_2 + \cdots)(S_n + S_{n+1} + \cdots);$$

by comparing the degree $n+1$ component of both sides, we conclude that $S_{n+1} = S_1 S_n$. This proves the assertion (ii).

If $S_n = S_1^n$ for $n \geq n_0$, we have $S_n^\natural = \bigoplus_{m \geq n} S_1^m$, and as $S_1^\natural$ contains $\bigoplus_{m\geq 1} S_1^m$, we then have $S_n^\natural \subseteq (S_1^\natural)^n$ for $n \geq n_0$. Conversely, the degree n component of $(S_1^\natural)^n = (S_1 + S_2 + \cdots)^n$ considered as elements of S is equal to S_1^n; so the relation $S_n^\natural = (S_1^\natural)^n$ implies $S_n = S_1^n$.

Finally, to prove (iv), it suffices to show that if an element $g \in S_{k+d}$ is considered as an element of $S_k^\natural$ ($k > 0$, $d \geq 0$), then there exists an integer $n > 0$ such that in $S_{kn}^\natural$, g^n is a linear combination of the $f_\alpha^\natural$ with coefficients in $S^\natural$. By hypothesis, there exists an integer n_0 such that for $n \geq n_0$, we have $g^n = \sum_\alpha c_{\alpha n} f_\alpha$ in S, where the indices α appearing in this formula are independent of n. Moreover, we can evidently suppose that the $c_{\alpha n}$ are homogeneous, with

$$\deg(c_{\alpha n}) = n(k+d) - \deg(f_\alpha)$$

in S. Let n_0 be large enough such that we have $kn_0 > deg(f_\alpha)$ for the f_α appearing in the formula of g^n; for any α, let $c'_{\alpha n}$ be the element $c_{\alpha n}$ considered as an element of degree $kn - \deg(f_\alpha)$ in $S^\natural$. We then have $g^n = \sum_\alpha c'_{\alpha m} f_\alpha^\natural$ in $S^\natural$, which proves our assertion. □

Consider the graded S_0-algebra

$$S^\natural \otimes_S S_0 = S^\natural / S_+ S^\natural = \bigoplus_{n\geq 0} S(n)/S_+S(n).$$

As S_n is a quotient S_0-module of $S(n)/S_+S(n)$, we have a canonical hommorphism of graded S_0-algebras

$$S^\natural \otimes_S S_0 \to S \tag{9.8.19}$$

which is evidently surjective, and corresponds to a canonical closed immersion

$$\operatorname{Proj}(S) \to \operatorname{Proj}(S^\natural \otimes_S S_0) \tag{9.8.20}$$

Proposition 9.8.16. *The canonical morphism (9.8.20) is bijective. For the homomorphism (9.8.19) to be eventually bijective, it is necessary and sufficient that there exists an integer n_0 such that $S_{n+1} = S_1 S_n$ for $n \geq n_0$. If this is satisfied, then the morphism (9.8.20) is an isomorphism, and the converse of this is also true if S is Noetherian.*

Proof. To prove the first assertion, it suffices (Corollary 9.2.45) to prove that the kernel $\mathfrak{I}$ of the homomorphism (9.8.19) is formed by nilpotent elements. Now if $f \in S(n)$ is an element whose class mod $S_+S(n)$ belongs to this kernel, then $f \in S(n+1)$ (by the definition of (9.8.19)); the element f^{n+1}, considered as an element of $S(n(n+1))$, then belongs to $S_+S(n(n+1))$, if we write it as $f \cdot f^n$. Then the class of f^{n+1} mod $S_+S(n(n+1))$ is zero, which proves our assertion.

As the hypothesis $S_{n+1} = S_1 S_n$ for $n \geq n_0$ is equivalent to $S_{n+1}^\natural = S_1^\natural S_n^\natural$ for $n \geq n_0$ (Lemma 9.8.15(ii)), these hypotheses are equivalent to that (9.8.19) is eventually injective, hence eventually bijective, and then (9.8.20) is an isomorphism by Proposition 9.3.6(a). Conversely, assume that S is Noetherian, hence so is $S^\natural$ and $S^\natural \otimes_S S_0$ (Lemma 9.8.15(i)). If (9.8.20) is an isomorphism, the sheaf $\widetilde{\mathscr{F}}$ over $\operatorname{Proj}(S^\natural \otimes_S S_0)$ is zero (Proposition 9.3.33(a)); as $S^\natural \otimes_S S_0$ is Noetherian, we then conclude from Proposition 9.2.36(b) that $\mathfrak{I}$ is eventually null, so $S_{n+1}^\natural = S_1^\natural S_n^\natural$ for $n \geq n_0$. □

Consider now the canonical injection $(S_+)^n \to S(n)$, which defines an injective homomorphism of degree 0 of graded rings

$$\bigoplus_{n\geq 0} (S_+)^n \to S^\natural. \tag{9.8.21}$$

Proposition 9.8.17. *For the homomorphism (9.8.21) to be an eventual isomorphism, it is necessary and sufficient that there exists an integer n_0 such that $S_n = S_1^n$ for $n \geq n_0$. If this is the case, the corresponding morphism*

$$\operatorname{Proj}(S^\natural) \to \operatorname{Proj}(\bigoplus_{n\geq 0} (S_+)^n) \tag{9.8.22}$$

is everywhere defined and an isomorphism, and the converse is also true if S is Noetherian.

Proof. The first two conditions are equivalent in view of Lemma 9.8.15(iii). The third assertion follows from Lemma 9.8.15(i), (iii) and Lemma 9.8.18. □

Lemma 9.8.18. *Let S be a graded ring with positive degrees which is a S_0-algebra of finite type. If the morphism corresponding to the canonical injection $S' = \bigoplus_{n\geq 0} S_1^n \to S$ is everywhere defined and an isomorphism, then there exists $n_0 > 0$ such that $S_n = S_1^n$ for $n \geq n_0$.*

Proof. In fact, let f_i ($1 \leq i \leq r$) be a system of generators for the S_0-module S_1. Then the hypotheses implies that the $D_+(f_i)$ cover $\mathrm{Proj}(S)$. Let $(g_j)_{1\leq j\leq n}$ be a system of homogeneous elements of S_+, with $n_j = \deg(g_j)$, which together with the f_i form a system of generators of the ideal S_+, or a system of generators of S as an S_0-algebra. The elements $g_j/f_i^{n_j}$ of the ring $S_{(f_i)}$ then by hypotheses belong to the subring $S'_{(f_i)}$, so there exists an integer k such that $S_1^k g_j \subseteq (S_1)^{k+n_j}$ for any j. We then conclude by recurrence on r that $S_1^k g_j^r \subseteq S'$ for any $r \geq 1$, and by the choice of the g_j, we then have $S_1^k S \subseteq S'$. On the other hand there exists for any j an integer m_j such that $g_j^{m_j}$ belongs to the ideal of S generated by the f_i (Corollary 9.2.11), so $g_j^{m_j} \in S_1 S$, and $g_j^{m_j k} \in S_1^k S \subseteq S'$. Therefore there exists an integer $m_0 \geq k$ such that $g_j^m \in S_1^{mn_j}$ for $m \geq m_0$. Now if d is the largest of the n_j, the number $n_0 = dm_0 + k$ then satisfies the requirement. In fact, an element of S_n, for $n \geq n_0$, is a sum of elements of $S_1^\alpha u$, where u is a product of powers of g_j; if $\alpha \geq k$, it follows from the choice of k that $S_1^\alpha u \subseteq S_1^n$; in the contrary case, at least one of the exponent of g_j is $\geq m_0$, so $u \in S_1^\beta v$ where $\beta \geq m_0 \geq k$ and v is a product of powers of g_j, so we are reduced to the previous case, and $S_1^\alpha u \subseteq S_1^n$. This completes the proof. $\square$

Remark 9.8.19. The condition $S_n = S_1^n$ for $n \geq n_0$ clearly implies that $S_{n+1} = S_1 S_n$ for $n \geq n_0$, but the converse is not ture, even we assume that S is Noetherian. For example, let K be a field, $A = K[\boldsymbol{x}]$, $B = K[\boldsymbol{y}]/\boldsymbol{y}^2 K[\boldsymbol{y}]$, where $\boldsymbol{x}, \boldsymbol{y}$ are two indeterminates, with $\deg(\boldsymbol{x}) = 1$ and $\deg(\boldsymbol{y}) = 2$, and let $S = A \otimes_K B$, so that S is a graded algebra over K having a basis formed by the elements $\boldsymbol{x}^n$ and $\boldsymbol{x}^n\boldsymbol{y}$. It is immediate that $S_{n+1} = S_1 S_n$ for $n \geq 2$, but $S_1^n = K\boldsymbol{x}^n$ while $S_n = K\boldsymbol{x}^n + K\boldsymbol{x}^{n-2}\boldsymbol{y}$ for $n \geq 2$.

9.8.3 Projective cones

Let Y be a scheme; in this subsection, we only consider Y-schemes and Y-morphisms. Let $\mathscr{S}$ be a quasi-coherent graded $\mathscr{O}_Y$-algebra with positive degrees, which we suppose that $\mathscr{S}_0 = \mathscr{O}_Y$. According to the notations of the previous part, we put

$$\widehat{\mathscr{S}} = \mathscr{S}[\boldsymbol{z}] = \mathscr{S} \otimes_{\mathscr{O}_Y} \mathscr{O}_Y[\boldsymbol{z}] \tag{9.8.23}$$

which we consider as a graded $\mathscr{O}_Y$-algebra with positive degrees, so that for any affine open subset U of Y, we have

$$\Gamma(U, \mathscr{S}) = \Gamma(U, \mathscr{S})[\boldsymbol{z}].$$

In the following, we put

$$P = \mathrm{Proj}(\mathscr{S}), \quad C = \mathrm{Spec}(\mathscr{S}), \quad \widehat{C} = \mathrm{Proj}(\widehat{\mathscr{S}})$$

(where in the definition of C, $\mathscr{S}$ is considered as a nongraded $\mathscr{O}_Y$-algebra), and we say that C (resp. $\widehat{C}$) is the **affine cone** (resp. **projective cone**) defined by $\mathscr{S}$; we will also say "cone" instead of "cone affine." By abuse of language, we sat that C (resp. $\widehat{C}$) is the **affine projecting cone** of P (resp. the **projective projecting cone** of P), which P is understood to be given of the form $\mathrm{Proj}(\mathscr{S})$. Finally, we say that $\widehat{C}$ is the projective closure of C, where C is understood to be a scheme of the form $\mathrm{Spec}(\mathscr{S})$.

Lemma 9.8.20. *Let S be a graded ring with positive degrees, $X = \mathrm{Proj}(S)$, and f be a homogeneous element of S with degree $d > 0$. If f is not a divisor of zero in S, X is the smallest closed subscheme of X such that $X_f = D_+(f)$.*

Proof. This question is clearly local on X; for any homogeneous element $g \in S_h$ ($h > 0$), it suffices to prove that X_g is the smallest closed subscheme of X_g which dominates X_{fg}. It follows from the definition and **??** that this condition is equivalent to the fact that the canonical homomorphism $S_{(g)} \to S_{(fg)}$ is injective. Now this homomorphism is identified canonically with the homomorphism $S_{(g)} \to (S_{(g)})_{f^h/g^d}$ (Lemma 9.2.1). But as f^h is not a zero divisor of S, f^h/g^e is not a zero divisor in S_g (and a fortiori in $S_{(g)}$), because the relation $(f^h/g^d)(t/g^m) = 0$ with $t \in S$ and $m > 0$ implies the existence of an integer $n > 0$ such that $g^n f^h t = 0$, whence $g^n t = 0$, and therefore $t/g^m = 0$ in S_g. This proves the claim. $\square$

Proposition 9.8.21. *There is a commutative diagram*

$$\begin{tikzcd} & \widehat{C} \arrow[dd] & \\ P \arrow[ur, "j"] \arrow[dr] & & C \arrow[ul, "i"'] \arrow[dl] \\ & Y \arrow[ur, bend left, "\varepsilon"] & \end{tikzcd}$$

where ε and j are closed immersions and i is an affine morphism which is an open dominant immersion such that

$$i(C) = \widehat{C} - j(P). \tag{9.8.24}$$

Moreover $\widehat{C}$ is the smallest closed subscheme of $\widehat{C}$ dominating $i(C)$.

Proof. To define i, we consider the open subset $\widehat{C}_{\boldsymbol{z}} = \operatorname{Spec}(\widehat{\mathscr{S}}/(\boldsymbol{z}-1)\widehat{\mathscr{S}})$ of $\widehat{C}$ (by (9.8.9)), where $\boldsymbol{z}$ is canonically identified with a section of $\widehat{\mathscr{S}}$ over Y. The isomorphism $i : C \xrightarrow{\sim} \widehat{C}_{\boldsymbol{z}}$ corresponds then to the canonical isomorphism

$$\widehat{\mathscr{S}}/(\boldsymbol{z}-1)\widehat{\mathscr{S}} \cong \mathscr{S}$$

of (9.8.9). The morphism ε corresponds to the augmentation hommorphism $\mathscr{S} \to \mathscr{S}_0 = \mathscr{O}_Y$ with kernel $\mathscr{S}_+$, which is surjective so ε is a closed immersion (Proposition 9.1.27). Finally, j corresponds similar to the surjective homomorphism $\widehat{\mathscr{S}} \to \mathscr{S}$ of degree 0, which is the identity on $\mathscr{S}$ and zero on $\boldsymbol{z}\widehat{\mathscr{S}}$. By Proposition 9.3.33 it is clear that j is everywhere defined and a closed immersion.

To prove the other assertions of the proposition, we can evidently assume that $Y = \operatorname{Spec}(A)$ is affine, $\mathscr{S} = \widetilde{S}$ where S is a graded A-algebra, whence $\widehat{\mathscr{S}} = (\widehat{S})^{\sim}$); the homogeneous elements f of S_+ are then identified with the sections of $\widehat{\mathscr{S}}$ over Y, and the open subset $D_+(f)$ of $\widehat{C}$ is identified with $\widehat{C}_f$ (Proposition 9.2.5); similarly the open subset $D(f)$ of C is identified with C_f. It then follows from Corollary 9.2.11 and the definition of $\widehat{S}$ that the open subset $\widehat{C}_{\boldsymbol{z}} = i(C)$ together with $\widehat{C}_f$ (where f is homogeneous in S_+) constitute an open covering of $\widehat{C}$. Moreover, we have

$$i^{-1}(\widehat{C}_f) = C_f. \tag{9.8.25}$$

In fact, if we identify $i(C)$ with $\widehat{C}_{\boldsymbol{z}}$, then

$$\widehat{C}_f \cap i(C) = \widehat{C}_f \cap \widehat{C}_{\boldsymbol{z}} = \widehat{C}_{f\boldsymbol{z}} = \operatorname{Spec}(\widehat{S}_{(f\boldsymbol{z})}).$$

Now if $d = \deg(f)$, $\widehat{S}_{(f\boldsymbol{z})}$ is canonically isomorphic to $(\widehat{S}_{(\boldsymbol{z})})_{(f/\boldsymbol{z}^d)}$ (Lemma 9.2.1), and it follows from the definition of the isomorphism (9.8.9) that the image of $(\widehat{S}_{(\boldsymbol{z})})_{(f/\boldsymbol{z}^d)}$ under the corresponding isomorphism is exactly S_f. As $C_f = \operatorname{Spec}(S_f)$, this proves (9.8.25) and also shows that the morphism i is affine. Moreover, the restriction of i to C_f, considered as a morphism into $\widehat{C}_f$, corresponds (Proposition 8.2.4) to the canonical homomorphism $\widehat{S}_{(f)} \to \widehat{S}_{(f\boldsymbol{z})} \cong S_f$. We also note that under the isomorphism (9.8.10), $\widehat{C}_f$ is canonically idnetified with $\operatorname{Spec}(S_f^{\leqslant})$ and the morphism restriction $i|_{C_f} : C_f \to \widehat{C}_f$ corresponds to the canonical injection $S_f^{\leqslant} \to S_f$. The complement of $\widehat{C}_{\boldsymbol{z}}$ in $\widehat{C} = \operatorname{Proj}(\widehat{S})$ is, by definition, the set of graded prime ideals of $\widehat{S}$ containing $\boldsymbol{z}$, which is $j(P)$ from the defintion of j, whence (9.8.24).

To prove the final assertion, we can still assume that Y is affine. With the preceding notations, we note that $\boldsymbol{z}$ is not a zero divisor in $\widehat{S}$, so we can apply Lemma 9.8.20. □

We now identify the affine cone C with the open subscheme $i(C)$ of the projective cone $\widehat{C}$, which is dense in $\widehat{C}$. The closed subscheme of C associated with the closed immersion ε is called the **sommet scheme** of C. We also say that ε, which is a Y-section of C, is the **sommet section** or the **zero section** of C; we can then identify Y with the sommet scheme of C via the morphism ε. The composition $i \circ \varepsilon$ is a Y-section of $\widehat{C}$, which is also a closed immersion (Corollary 8.5.19), corresponding to the canonical surjective homomorphism $\widehat{\mathscr{S}} = \mathscr{S}[z] \to \mathscr{O}_Y[z]$ (cf. Remark 9.3.18), with kernel $\mathscr{S}_+[z] = \widehat{\mathscr{S}}_+$. The closed subscheme of $\widehat{C}$ associated with this closed immersion is called the **sommet scheme** of $\widehat{C}$, which can be identified with Y via $i \circ \varepsilon$, and $i \circ \varepsilon$ is called the **sommet section** of $\widehat{C}$. Finally, the closed subscheme of $\widehat{C}$ associated with j is called the **place of infinity** of C, which is identified with P via j.

The subscheme of C (resp. $\widehat{C}$) induced respectively over the open subsets

$$E = C - \varepsilon(Y), \quad \widehat{E} = \widehat{C} - i(\varepsilon(Y)) \tag{9.8.26}$$

are called respectively (by abuse of language) the **blunt affine cone** (resp. **blunt projective cone**) defined by $\mathscr{S}$. We note that with this terminology, E is not necessarily affine over Y, nor is it projective over Y (cf. Example 9.8.29). If we identify C with $i(C)$, we then have

$$C \cup \widehat{E} = \widehat{C}, \quad C \cap \widehat{E} = E. \tag{9.8.27}$$

so that $\widehat{C}$ can be considered as obtaining by gluing the open subschemes C and $\widehat{E}$ along E; moreover, in view of (9.8.24),

$$E = \widehat{E} - j(P). \tag{9.8.28}$$

If $Y = \mathrm{Spec}(A)$ is affine, we then have (with the notations of Proposition 9.8.21),

$$E = \bigcup C_f, \quad \widehat{E} = \bigcup \widehat{C}_f, \quad C_f = C \cap \widehat{C}_f \tag{9.8.29}$$

where f runs through homogeneous elements of S_+ (or a family of homogeneous elements of S_+ generating the ideal S_+). The glueing of C and the $\widehat{C}_f$ along the C_f is then determined by the injections $C_f \to C$, $C_f \to \widehat{C}_f$, which correspond to the cannical homomorphisms $S \to S_f$, $S_f^{\leqslant} \to S_f$. On the other hand, we note that $\bigcup \widehat{E}_f$ is the defining domain $G(\varphi)$ of the morphism associated with the canonical injection $\varphi : \mathscr{S} \to \widehat{\mathscr{S}} = \mathscr{S}[\boldsymbol{z}]$, so we obtain a morphism $p : \widehat{E} \to P$.

Proposition 9.8.22. *The associated morphism $p : \widehat{E} \to P$ is an affine and surjective morphism (called the **canonical retraction**) such that*

$$p^{-1}(P_f) = \widehat{C}_f \tag{9.8.30}$$

and we have $p \circ j = 1_P$. Moreover, if Y is affine and $f \in S_1$, then $\widehat{C}_f$ is isomorphic to $P_f \otimes_{\mathbb{Z}} \mathbb{Z}[T]$ (T is an indeterminate).

Proof. To prove the proposition we may assume that Y is affine, so $\mathscr{S} = \widetilde{S}$. For any $f \in S_+$ homogeneous, by (9.2.9) we have (9.8.30) and the restriction $p : \widehat{C}_f \to P_f$ corresponds to the canonical injection $S_{(f)} \to S_f^{\leqslant}$. The formula $p \circ j = 1_P$ and the fact that p is surjective follows from the fact that the composition $\mathscr{S} \to \widehat{\mathscr{S}} \to \mathscr{S}$ is the identity on $\mathscr{S}$. Finally, the last assertion follows from the fact that $S_f^{\leqslant}$ is isomorphic to $S_{(f)}[T]$ (cf. (9.2.1)). □

Corollary 9.8.23. *The restriction $\pi : E \to P$ of p to E is a surjective and affine morphism. If Y is affine and $f \in S_+$ is homogeneous, we have*

$$\pi^{-1}(P_f) = C_f \tag{9.8.31}$$

and the restriction of $\pi|_{C_f} : C_f \to P_f$ corresponds to the cannical injection $S_{(f)} \to S_f$. If $f \in S_1$, then C_f is isomorphic to $P_f \otimes_{\mathbb{Z}} \mathbb{Z}[T, T^{-1}]$ (T is an indeterminant).

Proof. The formula (9.8.31) follows from (9.8.30) and (9.8.25), which also proves the surjectivity of π. We also have seen that the canonical injection $C_f \to \widehat{C}_f$ corresponds to $S_{(f)} \to S_f$, whence the second assertion. Finally, the last assertion is a concequence of the fact that for $f \in S_1$, S_f is isomorphic to $S_{(f)}[T, T^{-1}]$ (cf. (9.2.1)). □

Remark 9.8.24. If Y is affine, the elements of the underlying space of E are the prime ideals $\mathfrak{p}$ (not necessarily graded) of S not containing S_+, in ivew of the definition of the immersion ε. For such a prime ideal $\mathfrak{p}$, the intersections $\mathfrak{p} \cap S_n$ satisfy the conditions of **??**, so there exists a graded prime ideal $\mathfrak{q}$ of S such that $\mathfrak{q} \cap S_n = \mathfrak{p} \cap S_n$ for any n. The map $\pi : E \to P$ on the underlying topological space is then interpreted by the relation

$$\pi(\mathfrak{p}) = \mathfrak{q}.$$

In fact, to verify this relation, it suffices to consider a homogeneous element f of S_+ such that $\mathfrak{p} \in D(f)$, and we then observe that $\mathfrak{q}_{(f)}$ is the inverse image of $\mathfrak{p}_f$ under the canonical injection $S_{(f)} \to S_f$.

Corollary 9.8.25. *If $\mathscr{S}$ is generated by $\mathscr{S}_1$, the morphism p and π are of finite type. Moreover, for any $x \in P$, the fiber $p^{-1}(x)$ is isomorphic to* $\mathrm{Spec}(\kappa(x)[T])$ *and $\pi^{-1}(x)$ is isomorphic to* $\mathrm{Spec}(\kappa(x)[T,T^{-1}])$.

Proof. This follows from Proposition 9.8.22 and Corollary 9.8.23, since if Y is affine and S is genrated by S_1, then the P_f for $f \in S_1$ form an open covering of P. □

Remark 9.8.26. The blunt affine cone E corresponding to the graded $\mathscr{O}_Y$-algebra $\mathscr{O}_Y[T]$ (where T is an indeterminate) is identified with $G_m = \mathrm{Spec}(\mathscr{O}_Y[T,T^{-1}])$, since it is none other than C_T as we have seen in Proposition 9.8.21. This scheme is canonically endowed with an abelian group Y-scheme structure.

Example 9.8.27. Let k be a field, $k[T_0,\dots,T_n]$ be the polynomial ring, and $\mathfrak{p}$ be a graded prime ideal of $k[T_0,\dots,T_n]$ not containing the irrelevant ideal. Consider the quotient graded ring $S = k[T_0,\dots,T_n]/\mathfrak{p}$, and set

$$P = \mathrm{Proj}(S), \quad C = \mathrm{Spec}(S), \quad \widehat{C} = \mathrm{Spec}(\widehat{S}).$$

In the language of varieties, if $V \subseteq \mathbb{P}_k^n$ is the variety defined by S, C can be viewed as the affine cone obtained by considering the lines connecting the origin with points of V. Moreover, $\widehat{C}$ is the closure of C in $\mathbb{P}_k^{n+1}$ if we embed $\mathbb{A}_k^{n+1}$ into $\mathbb{P}_k^{n+1}$ via the map $(x_0,\dots,x_n) \to [x_0 : \cdots : x_n : 1]$. Also, the morphism $j : P \to \widehat{C}$ corresponds to the injection $[x_0 : \cdots : x_n] \mapsto [x_0 : \cdots : x_n : 0]$. With these, the projective closure $\widehat{C}$ of C in $\mathbb{P}^{n+1}$ is given by the equivalent classes $[x_0 : \cdots : x_n : x_{n+1}]$ in $\mathbb{P}^{n+1}$ such that $[x_0 : \cdots : x_n] \in P$, and we can divide into two cases depending on whether $x_{n+1} \neq 0$:

(a) $(x_0,\dots,x_n) \in C$, $x_{n+1} \neq 0$;

(b) $[x_0 : \cdots : x_n] \in P$, $x_{n+1} = 0$.

Thus we see that the variety $\widehat{C}$ can be viewed as a union of P with C, which justifies the formula (9.8.24). Also, by definition the blunt affine cone E is the subvariety of C obtained by removing the origin of $\mathbb{A}^{n+1}$, and $\widehat{E}$ can be considered as a union of E and P, which is also the projective cone $\widehat{C}$ removing the point $[0 : \cdots : 0 : 1]$ in $\mathbb{P}^{n+1}$. We also remark that if we base change C through the structural morphism $P \to Y$, then the projection $C_{(P)} \to C$ can be viewed as the projection from $\mathbb{A}^{n+1} \times \mathbb{P}^n$ to $\mathbb{P}_k^n$ which maps (x,ξ) to ξ (where the class of $x \in \mathbb{A}_k^{n+1}$ is equal to ξ), and this is the blow up map of $\mathbb{A}_k^{n+1}$ at the origin.

Let Y be a scheme, $\mathscr{E}$ be a quasi-coherent $\mathscr{O}_Y$-module. If $\mathscr{S}$ is the graded $\mathscr{O}_Y$-algebra $\boldsymbol{S}_{\mathscr{O}_Y}(\mathscr{E})$, then $\widehat{\mathscr{S}}$ is identified with $\boldsymbol{S}_{\mathscr{O}_Y}(\mathscr{E} \oplus \mathscr{O}_Y)$. The affine cone $\mathrm{Spec}(\mathscr{S})$ defined by $\mathscr{S}$ is by definition the vector bundle $\boldsymbol{V}(\mathscr{E})$, and $\mathrm{Proj}(\mathscr{S})$ is by definition $\mathbb{P}(\mathscr{E})$, so we see that:

Proposition 9.8.28. *The projective closure of a vector bundle $\boldsymbol{V}(\mathscr{E})$ over Y is canonically isomorphic to $\mathbb{P}(\mathscr{E} \oplus \mathscr{O}_Y)$, and the place of infinity of this is canonically isomorphic to $\mathbb{P}(\mathscr{E})$.*

Example 9.8.29. Put for example $\mathscr{E} = \mathscr{O}_Y^r$ where $r \geq 2$. Then the blunt cones $E, \widehat{E}$ defined by $\mathscr{S}$ are neither affine nor projective over Y if $Y \neq \varnothing$. The second assertion is immediate, since $\widehat{C} = \mathbb{P}(\mathscr{O}_Y^{r+1})$ is projective over Y and the underlying spaces of E and $\widehat{E}$ are not closed in $\widehat{C}$, so the canonical immersions $E \to \widehat{C}$ and $\widehat{E} \to \widehat{C}$ are not projective (Theorem 9.5.30 and Proposition 9.5.34(v)). On the other hand, suppose that $Y = \mathrm{Spec}(A)$ is affine and for example $r = 2$; we have $C = \mathrm{Spec}(A[T_1,T_2])$ and E is the open subscheme $D(T_1) \cup D(T_2)$ of C, and we have seen that this is not affine (Example 8.5.35); a fortiori $\widehat{E}$ is not affine, since E is the open subset of $\widehat{E}$ where the section $\boldsymbol{z}$ is nonzero (9.8.27).

Proposition 9.8.30. *Let $\mathscr{L}$ be an invertible $\mathscr{O}_Y$-module, we have canonical isomorphisms for the blunt cones corresponding to $C = V(\mathscr{L})$:*

$$\mathrm{Spec}\Big(\bigoplus_{n\in\mathbb{Z}} \mathscr{L}^n\Big) \xrightarrow{\sim} E, \quad \boldsymbol{V}(\mathscr{L}^{-1}) \xrightarrow{\sim} \widehat{E}. \tag{9.8.32}$$

Moreover, there exists a canonical isomorphism from the projective closue of $V(\mathscr{L})$ to that of $V(\mathscr{L}^{-1})$, which transform the sommet scheme (resp. the place of infinity) of the first one to the place of infinity (resp. the sommet scheme) of the second one.

Proof. Here we have $\mathscr{S} = \bigoplus_{n\geq 0} \mathscr{L}^{\otimes n}$; the canonical injection $\mathscr{S} \to \bigoplus_{n\in\mathbb{Z}} \mathscr{L}^{\otimes n}$ defines a cannical dominant morphism

$$\mathrm{Spec}\Big(\bigoplus_{n\in\mathbb{Z}} \mathscr{L}^{\otimes n}\Big) \to \boldsymbol{V}(\mathscr{L}) = \mathrm{Spec}\Big(\bigoplus_{n\geq 0} \mathscr{L}^{\otimes n}\Big) \tag{9.8.33}$$

and it suffices to prove that this morphism is an isomorphism from $\mathrm{Spec}(\bigoplus_{n\in\mathbb{Z}}\mathscr{L}^{\otimes n})$ to E. The question is local over Y, so we can suppose that $Y = \mathrm{Spec}(A)$ is affine and $\mathscr{L} = \mathscr{O}_Y$, so $\mathscr{S} = \widetilde{A[T]}$ and $\bigoplus_{n\in\mathbb{Z}}\mathscr{L}^{\otimes n} = \widetilde{A[T,T^{-1}]}$. Now $A[T,T^{-1}]$ is the fraction ring $A[T]_T$ of $A[T]$, so (9.8.33) identify $\mathrm{Spec}(\bigoplus_{n\in\mathbb{Z}}\mathscr{L}^{\otimes n})$ as the open subscheme $D(T)$ of $C = \boldsymbol{V}(\mathscr{L})$, which by definition is E.

The isomorphism $\boldsymbol{V}(\mathscr{L}^{-1}) \cong \widehat{E}$ will on the other hand be a consequence of the last assertion, since $\boldsymbol{V}(\mathscr{L}^{-1})$ is the complement of the place of infinity of its integral closure and $\widehat{E}$ is the complement of the sommet scheme of projective closure of $C = \boldsymbol{V}(\mathscr{E})$. Now these projective closures are respectively $\mathbb{P}(\mathscr{L}^{-1}\oplus\mathscr{O}_Y)$ and $\mathbb{P}(\mathscr{L}\oplus\mathscr{O}_Y)$; but we have

$$\mathscr{L}\oplus\mathscr{O}_Y = \mathscr{L}\oplus(\mathscr{L}\otimes\mathscr{L}^{-1}) = \mathscr{L}\otimes(\mathscr{L}^{-1}\oplus\mathscr{O}_Y),$$

so the existence of the isomorphism follows from Proposition 9.4.1, and it remains to see that this isomorphism exchanges the sommet scheme and the place of infinity. For this we can assume that $Y = \mathrm{Spec}(A)$ is affine, $\mathscr{L} = \widetilde{L}$, with $L = Ac$, $L^{-1} = Ac'$, and the canonical isomorphism $L\otimes L^{-1}\to A$ sends $c\otimes c'$ to 1. Then

$$\boldsymbol{S}(L\oplus A) = A[\boldsymbol{z}]\otimes\bigoplus_{n\geq 0}Ac^{\otimes n},\quad \boldsymbol{S}(L^{-1}\oplus A) = A[\boldsymbol{z}]\otimes\bigoplus_{n\geq 0}Ac'^{\otimes n},$$

and the isomorphism defined in Proposition 9.4.1 sends $\boldsymbol{z}^h\otimes c'^{\otimes(n-h)}$ to the element $\boldsymbol{z}^{n-h}\otimes c^{\otimes h}$. Now, in $\mathbb{P}(\mathscr{L}^{-1}\oplus\mathscr{O}_Y)$ the place of infinity is the set of points where the section $\boldsymbol{z}$ vanishes, and the sommet section is the set of points where the section c' vanishes. As we have a similary description for $\mathbb{P}(\mathscr{L}\oplus\mathscr{O}_Y)$, our conclusion follows immediately from the preceding explanations. □

9.8.4 Functorial properties

Let Y, Y' be two schemes, $q : Y'\to Y$ be a morphism, $\mathscr{S}$ (resp. $\mathscr{S}'$) be a quasi-coherent $\mathscr{O}_Y$-algebra (resp. $\mathscr{O}_{Y'}$-algebra) with positive degrees. Consider a q-morphism of graded algebras

$$\varphi : \mathscr{S}\to\mathscr{S}'.$$

We have seen that this corresponds canonically to a morphism

$$\Phi = \mathrm{Spec}(\varphi) = \mathrm{Spec}(\mathscr{S}')\to\mathrm{Spec}(\mathscr{S})$$

such that the diagram

$$\begin{tikzcd} C' \arrow[r, "\Phi"] \arrow[d] & C \arrow[d] \\ Y' \arrow[r, "q"] & Y \end{tikzcd} \tag{9.8.34}$$

where $C = \mathrm{Spec}(\mathscr{S})$, $C' = \mathrm{Spec}(\mathscr{S}')$, is commutative. Suppose moreover that $\mathscr{S}_0 = \mathscr{O}_Y$ and $\mathscr{S}'_0 = \mathscr{O}_{Y'}$; let $\varepsilon : Y\to C$ and $\varepsilon' : Y'\to C'$ be the cannical immersions, we then have a commutative diagram

$$\begin{tikzcd} C' \arrow[r, "\Phi"] & C \\ Y' \arrow[r, "q"] \arrow[u, "\varepsilon'"] & Y \arrow[u, "\varepsilon"'] \end{tikzcd} \tag{9.8.35}$$

which corresponds to the diagram

$$\begin{tikzcd} \mathscr{S} \arrow[r, "\varphi"] \arrow[d] & \mathscr{S}' \arrow[d] \\ \mathscr{O}_Y \arrow[r] & \mathscr{O}_{Y'} \end{tikzcd}$$

where the vertical are augmentation homomorphisms, and the commutativity follows from the hypotheses that φ is a homomorphism of graded algebras.

Proposition 9.8.31. *If E (resp. E') is the blunt affine cone defined by $\mathscr{S}$ (resp. $\mathscr{S}'$), we have $\Phi^{-1}(E)\subseteq E'$. Moreover, the morphism* $\mathrm{Proj}(\varphi) : G(\varphi)\to\mathrm{Proj}(\mathscr{S})$ *is everywhere defined (in other words $G(\varphi) = \mathrm{Proj}(\mathscr{S}')$) if and only if $\Phi^{-1}(E) = E'$.*

Proof. The first assertion followsfrom (9.8.35), since $E = C - \varepsilon(Y)$ and $E' = C' - \varepsilon'(Y')$. To prove the second assertion, we can assume that $Y = \operatorname{Spec}(A)$ and $Y' = \operatorname{Spec}(A')$ are affine, $\mathscr{S} = \widetilde{S}$, $\mathscr{S}' = \widetilde{S}'$. For f homogeneous in S_+, if we put $f' = \varphi(f)$, we have $\Phi^{-1}(C_f) = C'_{f'}$ (9.2.9); to say that $G(\varphi) = \operatorname{Proj}(S')$ signifies that in S'_+ the radical of the ideal generated by the $f' = \varphi(f)$ is equal to S'_+ (Corollary 9.2.11), and this is equivalent to that the $C'_{f'}$ cover E' (9.8.29). □

The q-morphism φ extends canonically to a q-morphism of graded algebras

$$\hat{\varphi} : \widehat{\mathscr{S}} \to \widehat{\mathscr{S}}'$$

which satisfies $\hat{\varphi}(\boldsymbol{z}) = \boldsymbol{z}$. We then deduce a morphism

$$\hat{\Phi} = \operatorname{Proj}(\hat{\varphi}) : G(\hat{\varphi}) \to \widehat{C} = \operatorname{Proj}(\widehat{\mathscr{S}})$$

such that the diagram

$$\begin{tikzcd} G(\hat{\varphi}) \arrow[r, "\hat{\Phi}"] \arrow[d] & \widehat{C} \arrow[d] \\ Y' \arrow[r, "q"] & Y \end{tikzcd}$$

is commutative. It then follows from the definition that if we denote by $i : C \to \widehat{C}$ and $i' : C' \to \widehat{C}'$ are the canonical immersions, we have $i'(C') \subseteq G(\hat{\varphi})$ and the diagram

$$\begin{tikzcd} G(\hat{\varphi}) \arrow[r, "\hat{\Phi}"] & \widehat{C} \\ C' \arrow[u, "i'"] \arrow[r, "\Phi"] & C \arrow[u, "i"'] \end{tikzcd} \tag{9.8.36}$$

is commutative. Finally, if we put $P = \operatorname{Proj}(\mathscr{S})$, $P' = \operatorname{Proj}(\mathscr{S}')$, and if $j : P \to \widehat{C}$, $j' : P' \to \widehat{C}'$ are the canonical closed immersions, we have $j'(G(\varphi)) \subseteq G(\hat{\varphi})$ and the diagram

$$\begin{tikzcd} G(\hat{\varphi}) \arrow[r, "\hat{\Phi}"] & \widehat{C} \\ G(\varphi) \arrow[u, "j'"] \arrow[r, "\operatorname{Proj}(\varphi)"] & P \arrow[u, "j"'] \end{tikzcd} \tag{9.8.37}$$

is commutative.

Proposition 9.8.32. *If $\widehat{E}$ (resp. $\widehat{E}'$) is the blunt projective cone defined by $\mathscr{S}$ (resp. $\mathscr{S}'$), we have $\hat{\Phi}^{-1}(\widehat{E}) \subseteq \widehat{E}'$. Moreover, if $p : \widehat{E} \to P$ and $p' : \widehat{E}' \to P'$ are the canonical retractions, we have $p'(\hat{\Phi}^{-1}(\widehat{E})) \subseteq G(\varphi)$, and the diagram*

$$\begin{tikzcd} \hat{\Phi}^{-1}(\widehat{E}) \arrow[r, "\hat{\Phi}"] \arrow[d, "p'"'] & E \arrow[d, "p"] \\ G(\varphi) \arrow[r, "\operatorname{Proj}(\varphi)"] & P \end{tikzcd}$$

is commutative. If $\operatorname{Proj}(\varphi)$ is everywhere defined, so is $\hat{\Phi}$ and we have $\hat{\Phi}^{-1}(\widehat{E}) = \widehat{E}'$.

Proof. The first assertion follows from the commutative diagrams (9.8.34) and (9.8.36), and the next two follow from the definition of the canonical retraction, the definition of $\hat{\varphi}$, and the fact that $\widehat{E}$ is the defining domain of the morphism induced by the canonical injection $\mathscr{S} \to \widehat{\mathscr{S}}$. On the other hand, to see that $\hat{\Phi}$ is everywhere defined if $\operatorname{Proj}(\varphi)$ is, we can assume that $Y = \operatorname{Spec}(A)$, $Y' = \operatorname{Spec}(A')$ are affine, $\mathscr{S} = \widetilde{S}$, $\mathscr{S}' = \widetilde{S}'$; the hypothesis is that if f runs through homogeneous elements of S_+, the ideal in S'_+ generated by the $\varphi(f)$ has radical in S'_+ equal to S'_+. We then conclude that the radical of the ideal genrated by $\boldsymbol{z}$ and the $\varphi(f)$ in $(S'[\boldsymbol{z}])_+$ is equal to $(S'[\boldsymbol{z}])_+$, whence our assertion. This proves similarly that $\widehat{E}'$ is the union of the $\widehat{C}'_{(\varphi(f))}$, which is equal to $\hat{\Phi}^{-1}(\widehat{E})$. □

Corollary 9.8.33. *If Φ is everywhere defined, the inverse image under $\hat{\Phi}$ of underlying space of the place of infinity (resp. the sommet scheme) of $\widehat{C}'$ is the underlying space of the place of infinity (resp. the sommet scheme) of $\widehat{C}$.*

Proof. This follows from Proposition 9.8.32 and Proposition 9.8.31, in ivew of the relations (9.8.26) and (9.8.26). ☐

9.8.5 Blunt cones over a homogeneous specturm

Let Y be a scheme, $\mathscr{S}$ be a quasi-coherent $\mathscr{O}_Y$-algebra with positive degrees such that $\mathscr{S}_0 = \mathscr{O}_Y$, and $X = \mathrm{Proj}(\mathscr{S})$. We now apply the previous results to the structure morphism $q : X \to Y$. Let

$$\mathscr{S}_X = \bigoplus_{n \in \mathbb{Z}} \mathscr{O}_X(n) \tag{9.8.38}$$

which is a quasi-coherent $\mathscr{O}_X$-algebra, the multiplication γ being defined by the canonical homomorphisms

$$\mathscr{O}_X(m) \otimes \mathscr{O}_X(n) \to \mathscr{O}_X(m+n)$$

which satisfies the associativity in view of Proposition 9.3.15. Let $\mathscr{S}'$ be the quasi-coherent sub-algebra

$$\mathscr{S}_X^{\geqslant} = \bigoplus_{n \geq 0} \mathscr{O}_X(n)$$

of $\mathscr{S}_X$, with positive degrees. For each $n \in \mathbb{Z}$, we have a canonical q-morphisms $\alpha_n : \mathscr{S}_n \to \mathscr{O}_X(n)$ defined in (9.3.1), which together give a homomorphism

$$\alpha : \mathscr{S} \to \bigoplus_{n \geq 0} q_*(\mathscr{O}_X(n)).$$

By composing with the canonical homomorphism $\bigoplus_{n \geq 0} q_*(\mathscr{O}_X(n)) \to q_*(\mathscr{S}_X^{\geqslant})$, this gives a q-homomorphism $\mathscr{S} \to \mathscr{S}_X^{\geqslant}$, still denoted by α. We set

$$C_X = \mathrm{Spec}(\mathscr{S}_X^{\geqslant}), \quad \widehat{C}_X = \mathrm{Proj}(\mathscr{S}_X^{\geqslant}[z]), \quad P_X = \mathrm{Proj}(\mathscr{S}_X^{\geqslant})$$

and denote by E_X and $\widehat{E}_X$ the corresponding blunt cones. We then have the canonical morphisms

$$\begin{tikzcd}
& \widehat{C}_X \arrow[dd] & \\
P_X \arrow[ur, "j_X"] \arrow[dr] & & C_X \arrow[ul, "i_X"'] \arrow[dl] \\
& X \arrow[ur, bend left, "\varepsilon_X"] &
\end{tikzcd}$$

and $p_X : \widehat{E}_X \to P_X$, $\pi_X : E_X \to P_X$.

Proposition 9.8.34. *The structural morphism $\psi : P_X \to X$ is an isomorphism, and the morphism $\mathrm{Proj}(\alpha)$ is everywhere defined and equals to ψ. The morphism $\mathrm{Proj}(\hat{\alpha}) : \widehat{C}_X \to \widehat{C}$ is everywhere defined and its restriction to $\widehat{E}_X$ and E_X are isomorphisms into $\widehat{E}$ and E, respectively. Finally, if we identify P_X and X via ψ, the morphisms p_X and π_X are identified with the structural morphisms of the X-schemes $\widehat{E}_X$ and E_X.*

Proof. We can clearly assume that $Y = \mathrm{Spec}(A)$ is affine and $\mathscr{S} = \widetilde{S}$; then X is the union of the affine opens X_f, where $f \in S_+$ is homogeneous, the ring of X_f being $S_{(f)}$. It follows from the isomorphism (9.8.13) that

$$\Gamma(X_f, \mathscr{S}_X^{\geqslant}) = S_f^{\geqslant}. \tag{9.8.39}$$

We then have $\psi^{-1}(X_f) = \mathrm{Proj}(S_f^{\geqslant})$. But if $f \in S_d$ with $d > 0$, $\mathrm{Proj}(S_f^{\geqslant})$ is canonically isomorphic to $\mathrm{Proj}((S_f^{\geqslant})^{(d)})$ by Remark 9.3.19, and $(S_f^{\geqslant})^{(d)} = (S^{(d)})_f^{\geqslant}$ is identified with $S_{(f)}[T]$ by the map $T \mapsto$

$f/1$ (cf. (9.2.1)), so we conclude from Remark 9.3.18 that the structural morphism $\psi^{-1}(X_f) \to X_f$ is an isomorphism, whence the first assertion. To prove the second one, we first note that $\operatorname{Proj}(\alpha)$ is everywhere defined by Lemma 9.8.15. Since $\psi^{-1}(X_f) = (\psi^{-1}(X_f))_{f/1}$, it follows from (9.2.9) that the image of $\psi^{-1}(X_f)$ under $\operatorname{Proj}(\alpha)$ is contained in X_f, and the restriction of $\operatorname{Proj}(\alpha)$ to $\psi^{-1}(X_f)$, considered as a morphism into $X_f = \operatorname{Spec}(S_{(f)})$, is identified with ψ. Finally, the formula (9.8.30) and (9.8.10) show that $p_X^{-1}(\psi^{-1}(X_f)) = \operatorname{Spec}((S_f^{\geqslant})_{f/1}^{\leqslant})$, and this open subset is, by Proposition 9.8.32 and formula (9.8.30), the inverse image of $p^{-1}(X_f) = \operatorname{Spec}(S_f^{\leqslant})$ under $\operatorname{Proj}(\hat{\alpha})$. By the isomorphism (9.8.10), the restriction of $\operatorname{Proj}(\hat{\alpha})$ to $p_X^{-1}(\psi^{-1}(X_f))$ corresponds to the isomorphism $S_f^{\leqslant} \cong (S_f^{\geqslant})_{f/1}^{\leqslant}$, whence the third assertion. The last assertion is clear by definition. □

We note that by (9.8.36) the restriction of $\operatorname{Proj}(\hat{\alpha})$ to C_X is equal to $\operatorname{Spec}(\alpha)$.

Corollary 9.8.35. *Considered as X-schemes, $\widehat{E}_X$ is canonically isomorphic to $\operatorname{Spec}(\mathscr{S}_X^{\leqslant})$, E_X is canonically isomorphic to $\operatorname{Spec}(\mathscr{S}_X)$, and C_X is canonically isomorphic to $\operatorname{Spec}(\mathscr{S}_X^{\geqslant})$.*

Proof. As we have seen that p_X and π_X are affine, it suffices to verify the corollary if $Y = \operatorname{Spec}(A)$ is affine and $\mathscr{S} = \widetilde{S}$. The first assertion follows from the canonical isomorphism $(S_f^{\geqslant})_{f/1}^{\leqslant} \cong S_f^{\leqslant}$, which are compatible with passage from f to fg (f, g homogeneous in S_+). Similarly, the formula (9.8.31), applied to π_X, shows that $\pi_X^{-1}(\psi^{-1}(X_f)) = \operatorname{Spec}((S_f^{\geqslant})_{f/1})$ for f homogeneous in S_+, and the second assertion then follows from the canonical isomorphism $(S_f^{\geqslant})_{f/1} \cong S_f$. □

We can then say that $\widehat{C}_X$, considered as an X-scheme, is obtained by glueing the affine X-schemes $C_X = \operatorname{Spec}(\mathscr{S}_X^{\geqslant})$ and $\widehat{E}_X = \operatorname{Spec}(\mathscr{S}_X^{\leqslant})$ along their intersection $E_X = \operatorname{Spec}(\mathscr{S}_X)$.

Corollary 9.8.36. *Suppose that $\mathscr{O}_X(1)$ is an invertible $\mathscr{O}_X$-module and that $\mathscr{S}_X \cong \bigoplus_{n\in\mathbb{Z}}(\mathscr{O}_X(1))^{\otimes n}$ (for example if $\mathscr{S}$ is generated by $\mathscr{S}_1$). Then the blunt projective cone $\widehat{E}$ is identified with the rank one vector bundle $\mathbf{V}(\mathscr{O}_X(-1))$ over X, and the bulnk affine cone E is isomorphic to the open subscheme induced over the complement of the zero section in this vector bundle. With these indentifications, the canonical retraction $\widehat{E} \to X$ is identified with the structural morphism of the X-scheme $\mathbf{V}(\mathscr{O}_X(-1))$. Finally, there exists a canonical Y-morphism $\mathbf{V}(\mathscr{O}_X(1)) \to C$, whose restriction to the complement of the zero section of $\mathbf{V}(\mathscr{O}_X(1))$ is an isomorphism from this complement to the blunt affine cone E.*

Proof. In fact, if $\mathscr{L} = \mathscr{O}_X(1)$, then $\mathscr{S}_X^{\geqslant}$ is identified with $\mathbf{S}_{\mathscr{O}_X}(\mathscr{L})$ and $\mathscr{S}_X^{\leqslant}$ is identified with $\mathbf{S}_{\mathscr{O}_X}(\mathscr{L}^{-1})$, so $\widehat{E}_X$ is identified with $\mathbf{V}(\mathscr{L}^{-1})$ in view of Corollary 9.8.35 and C_X is identified with $\mathbf{V}(\mathscr{L})$. The morphism $\mathbf{V}(\mathscr{L}) \to C$ is the restriction of $\operatorname{Proj}(\hat{\alpha})$, and the assertion of the corollary is a particular case of Proposition 9.8.34. □

We note that the inverse image of the sommet scheme of C under the morphism $\mathbf{V}(\mathscr{O}_X(1)) \to C$ is the zero section of $\mathbf{V}(\mathscr{O}_X(1))$ (Corollary 9.8.33). But in general the corresponding subschemes of C and of $\mathbf{V}(\mathscr{O}_X(1))$ are not isomorphic.

9.8.6 Blow up of projective cones

With the notations of the previous subsection, we have a commutative diagram

$$\begin{tikzcd} \widehat{C}_X \arrow[r, "r"] & \widehat{C} \\ X \arrow[u, "i_X\circ\varepsilon_X"] \arrow[r, "q"] & Y \arrow[u, "i\circ\varepsilon"'] \end{tikzcd}$$

where $r = \operatorname{Proj}(\hat{\alpha})$. Moreover, the restriction of r to the complement $\widehat{C}_X - i_X(\varepsilon_X(X))$ of the sommet section is an isomorphism to $\widehat{C} - i(\varepsilon(Y))$ of the sommet section in view of Proposition 9.8.34. If we suppose for simplicity that Y is affine, $\mathscr{S}$ is of finite type and generated by $\mathscr{S}_1$, X is projective over Y and $\widehat{C}_X$ is projective over X, so $\widehat{C}_X$ is projective over Y (Proposition 9.5.34(ii)), and a fortiori over $\widehat{C}$ (Proposition 9.5.34(v)). We thus have a projective Y-morphism $r : \widehat{C}_X \to \widehat{C}$ (hence restricts to a projective Y-morphism $C_X \to C$) which contract X to Y and induces an isomorphism when restricted to the complement of X and of Y. We therefore have a relation between C_X and C, analogous to that which takes place between a blow up scheme and its initial scheme. We will effectively show that we can identify C_X with the homogeneous spectrum of a graded $\mathscr{O}_C$-algebra.

For each $n \geq 0$, we consider the quasi-coherent ideal

$$\mathscr{S}(n) = \bigoplus_{m \geq n} \mathscr{S}_m$$

of the graded $\mathscr{O}_Y$-algebra of $\mathscr{S}$. It is clear that $(\mathscr{S}(n))_{n\geq 0}$ is a filtered sequence of ideals of $\mathscr{S}$. Consider the $\mathscr{O}_C$-module associated with $\mathscr{S}(n)$, which is a quasi-coherent ideal of $\mathscr{O}_C = \widetilde{\mathscr{S}}$:

$$\mathscr{I}_n = \widetilde{\mathscr{S}(n)}.$$

Then $(\mathscr{I}_n)$ is also a filtered sequence of quasi-coherent $\mathscr{O}_C$-ideals, so we can consider the quasi-coherent graded $\mathscr{O}_C$-algebra

$$\mathscr{S}^\natural = \bigoplus_{n\geq 0} \mathscr{I}_n = \Big(\bigoplus_{n\geq 0} \mathscr{S}(n)\Big)^{\sim}.$$

Proposition 9.8.37. *There exists a canonical C-isomorphism*

$$h : C_X \to \operatorname{Proj}(\mathscr{S}^\natural). \tag{9.8.40}$$

Proof. Suppose first that $Y = \operatorname{Spec}(A)$ is affine, so $\mathscr{S} = \widetilde{S}$ where S is a graded A-algebra with positive degrees and $C = \operatorname{Spec}(S)$. We then have $\mathscr{S}^\natural = \widetilde{(S^\natural)}$. To define the morphism h, consider an element $f \in S_d$ $(d > 0)$ and the corresponding element $f^\natural \in S^\natural$; the S-isomorphism (9.8.15) defines a C-isomorphism

$$\operatorname{Spec}(S_f^{\geqslant}) \xrightarrow{\sim} \operatorname{Spec}(S^\natural_{(f^\natural)}). \tag{9.8.41}$$

But with the notations of Proposition 9.8.34, if $\varphi : C_X \to X$ is the structural morphism, it follows from (9.8.39) that $\varphi^{-1}(X_f) = \operatorname{Spec}(S_f^{\geqslant})$. We have on the other hand $D_+(f^\natural) = \operatorname{Spec}(S^\natural_{(f^\natural)})$, so that (9.8.41) define an isomorphism $v^{-1}(X_f) \to D_+(f^\natural)$. Moreover, if $g \in S_e$ with $e > 0$, the diagram

$$\begin{tikzcd}
\varphi^{-1}(X_{fg}) \arrow[r, "\sim"] \arrow[d] & D_+(f^\natural g^\natural) \arrow[d] \\
\varphi^{-1}(X_f) \arrow[r, "\sim"] & D_+(f^\natural)
\end{tikzcd}$$

is commutative, which is clear from the definition of (9.8.15). By definition S_+ is generated by these homogeneous elements F, so it follows from Lemma 9.8.15(iv) that the $D_+(f^\natural)$ form a covering of $\operatorname{Proj}(S^\natural)$ and the $\varphi^{-1}(X_f)$ form a covering of C_X, if X_f form a covering of X. These together gives a isomorphism $h : C_X \to \operatorname{Proj}(\mathscr{S}^\natural)$.

To prove the proposition in the general case, it suffices to see that if U, U' are two affine opens of Y such that $U' \subseteq U$, with rings A and A', and if $\mathscr{S}|_U = \widetilde{S}$, $\mathscr{S}|_{U'} = \widetilde{S}'$, the diagram

$$\begin{tikzcd}
C_{U'} \arrow[r] \arrow[d] & \operatorname{Proj}(S^\natural) \arrow[d] \\
C_U \arrow[r] & \operatorname{Proj}(S^\natural)
\end{tikzcd} \tag{9.8.42}$$

is commutative. But S' is canonically identified with $S \otimes_A A'$, so $S'^\natural$ is identified with $S^\natural \otimes_S S' = S^\natural \otimes_A A'$ and we then have $\operatorname{Proj}(S'^\natural) = \operatorname{Proj}(S^\natural) \times_U U'$ (Proposition 9.2.50). Similarly, if $X = \operatorname{Proj}(S)$ and $X' = \operatorname{Proj}(S')$, we have $X' = X \times_U U'$ and $\mathscr{S}_{X'} = \mathscr{S}_X \otimes_{\mathscr{O}_U} \mathscr{O}_{U'}$ (Corollary 9.3.32), which means $\mathscr{S}_{X'} = j^*(\mathscr{S}_X)$, where $j : X' \to X$ is the projection. By Corollary 9.1.30 we then have $C_{U'} = C_U \times_X X' = C_U \times_U U'$, and the commutativity of (9.8.42) is immediate. $\square$

Remark 9.8.38. The end of the reasoning of Proposition 9.8.37 is immediately generalized in the following way.: let $g : Y' \to Y$ be a morphism, $\mathscr{S}' = g^*(\mathscr{S})$, $X' = \mathrm{Proj}(\mathscr{S}')$; we then have a commutative diagram

$$\begin{tikzcd} C_{X'} \arrow[r] \arrow[d] & \mathrm{Proj}(\mathscr{S}'^{\natural}) \arrow[d] \\ C_X \arrow[r] & \mathrm{Proj}(\mathscr{S}^{\natural}) \end{tikzcd} \tag{9.8.43}$$

On the other hand, let $\varphi : \mathscr{S}'' \to \mathscr{S}$ be a homomorphism of graded $\mathscr{O}_Y$-algebras such that the induced morphism $\Phi = \mathrm{Proj}(\varphi) : X \to X''$ is everywhere defined, where $X'' = \mathrm{Proj}(\mathscr{S}'')$. We have an Y-morphism $v : C \to C''$ (where $C'' = \mathrm{Spec}(\mathscr{S}''))$ such that $\mathscr{A}(v) = \varphi$, and as φ is a graded homomorphism, we deduce from φ a v-morphism $\psi : \mathscr{S}''^{\natural} \to \mathscr{S}^{\natural}$ (Proposition 9.1.18). Moreover, it follows from Lemma 9.8.15(iv) and the hypotheses on φ that $\Psi = \mathrm{Proj}(\psi)$ is everywhere defined. Finally, in view of (9.3.9), we have a canonical Φ-morphism $\mathscr{S}_{X''} \to \mathscr{S}_X$, whence a morphism $w : C_{X''} \to C_X$. The diagram

$$\begin{tikzcd} C_{X''} \arrow[r, "\sim"] \arrow[d, "w"] & \mathrm{Proj}(\mathscr{S}^{\natural}) \arrow[d, "\Psi"] \\ C_X \arrow[r, "\sim"] & \mathrm{Proj}(\mathscr{S}^{\natural}) \end{tikzcd}$$

is commutative, as can be verified in the case where Y is affine.

Remark 9.8.39. Recall that $(\mathscr{I}_n)_{n\geq 0}$ is a filtered sequence where $\mathscr{I}_n = \mathscr{S}(n)$, so we have $\mathscr{I}_1^n \subseteq \mathscr{I}_n \subseteq \mathscr{I}_1$ for any $n > 0$. Now by definition, $\mathscr{I}_1 = \widetilde{\mathscr{S}_+}$, so $\mathscr{I}_1$ defines in C the closed subscheme $\varepsilon(Y)$ (Proposition 9.1.27 and Proposition 9.8.21). We then conclude that for any $n > 0$, the support of $\mathscr{O}_C/\mathscr{I}_n$ is contained in the underlying space of tge sommet scheme $\varepsilon(Y)$. In the inverse image of the blunt affine cone E, the structural morphism $\mathrm{Proj}(\mathscr{S}^{\natural}) \to C$ reduces to an isomorphism (as it follows from Proposition 9.8.37 and Proposition 9.8.34). Moreover, if we canonically identify C as a dense open subset of $\widehat{C}$, we can evidently extend the ideal $\mathscr{I}_n$ of $\mathscr{O}_C$ to an ideal $\mathscr{J}_n$ of $\mathscr{O}_{\widehat{C}}$, such that it coincides with $\mathscr{O}_{\widehat{C}}$ on the open subset $\widehat{E}$ of $\widehat{C}$. If we put $\mathscr{T} = \bigoplus_{n\geq 0} \mathscr{J}_n$, which is a graded $\mathscr{O}_{\widehat{C}}$-algebra, we can then extend the isomorphism (9.8.40) into a $\widehat{C}$-isomorphism

$$\widehat{C}_X \xrightarrow{\sim} \mathrm{Proj}(\mathscr{T}). \tag{9.8.44}$$

In fact, over $\widehat{E}$, it follows from the definition of $\mathscr{J}$ that $\mathrm{Proj}(\mathscr{T})$ is identified with $\widehat{E}$, and we therefore define the isomorphism (9.8.44) so that it coincides with the canonical isomorphism $\widehat{E}_X \to \widehat{E}$ on $\widehat{E}$ (Proposition 9.8.34); it is then clear that this isomorphism and (9.8.40) coincides over $\widehat{E}$.

Corollary 9.8.40. *Suppose that there exists $n_0 > 0$ such that $\mathscr{S}_{n+1} = \mathscr{S}_1\mathscr{S}_n$ for $n \geq n_0$. Then the sommet scheme of C_X (isomorphic to X) is the inverse image of the sommet scheme of C (isomorphic to Y) under the canonical morphism $r = \mathrm{Proj}(\alpha) : C_X \to C$. The converse of this is true if moreover Y is Noetherian and $\mathscr{S}$ is of finite type.*

Proof. The first assertion is local over Y, so we can assume that $Y = \mathrm{Spec}(A)$ is affine, $\mathscr{S} = \widetilde{S}$ where S is a graded A-algebra with positive degrees. This then follows from Proposition 9.8.16, because we have

$$\mathrm{Proj}(S^{\natural} \otimes_S S_0) = \mathrm{Proj}(S^{\natural} \otimes_S (S/S_+)) = C_X \times_C \varepsilon(Y)$$

(in view of the identification (9.8.40) and Proposition 9.2.50), which is also the inverse image of $\varepsilon(Y)$ in C_X under the morphism $r : C_X \to C$. The converse of this also follows from Proposition 9.8.16 if Y is Noetherian and affine and S is of finite type. If Y is Noetherian (not necessarily affine) and $\mathscr{S}$ is of finite type, there exists a finite convering of Y by Noetherian affine covers U_i, and we then deduce that for each i, there is an integer n_i such that $\mathscr{S}_{n+1}|_{U_i} = (\mathscr{S}_1|_U)(\mathscr{S}_n|_U)$ for $n \geq n_i$; the largest integer n_0 of the n_i then satisfies the requirement. □

We now consider the C-scheme $\widetilde{C}$ with is obtained by blowing up the affine cone C along the sommet scheme $\varepsilon(Y)$. By definiton this is $\mathrm{Proj}(\bigoplus_{n\geq 0}(\mathscr{S}_+)^n)$; the canonical injection

$$\iota : \bigoplus_{n\geq 0}(\mathscr{S}_+)^n \to \mathscr{S}^{\natural}$$

defines (by the identification of (9.8.40)) a canonical dominant C-morphism $\mathrm{Proj}(\iota) : G(\iota) \to \widetilde{C}$, where $G(\iota)$ is an open subset of C_X. We note that it is possible that $G(\iota) \neq C_X$; for example, $Y = \mathrm{Spec}(k)$ where k is a field, $\mathscr{S} = \widetilde{S}$ where $S = k[\boldsymbol{y}]$ and $\boldsymbol{y}$ is an indeterminate of degree 2. If R_n is the set $(S_+)^n$, considered as a subset of $S(n) = S_n^\natural$, then $S_+^\natural$ is not equal to the radical in $S_+^\natural$ of the ideal generated by the R_n.

Corollary 9.8.41. *Suppose that there exists $n_0 > 0$ such that $\mathscr{S}_n = \mathscr{S}_1^n$ for $n \geq n_0$. Then the canonical morphism* $\mathrm{Proj}(\iota) : G(\iota) \to \widetilde{C}$ *is everywhere defined and an isomorphism from C_X to $\widetilde{C}$. The converse of this is also true if moreover Y is Noetherian and $\mathscr{S}$ is of finite type.*

Proof. This assertion is local over Y, and therefore follows from Proposition 9.8.17. The converse of this is also true if Y is Noetherian and $\mathscr{S}$ is of finite type, as can be shown similarly to Corollary 9.8.40. $\square$

Remark 9.8.42. As the condition of Corollary 9.8.41 implies that of Corollary 9.8.40, we see that if this condition is verified, not only C_X is identified with the scheme obtained by blowing up C along the sommet scheme (isomorphic to Y), but also the sommet scheme of C_X (isomorphic to X) is identified with the inverse image of the sommet scheme of C in C_X. Moreover, the hypothesis of Corollary 9.8.41 implies that over $X = \mathrm{Proj}(\mathscr{S})$, the $\mathscr{O}_X$-modules $\mathscr{O}_X(n)$ are invertible (Proposition 9.3.14) and we have $\mathscr{O}_X(n) = \mathscr{L}^{\otimes n}$, where $\mathscr{L} = \mathscr{O}_X(1)$ (Corollary 9.3.16). By definition C_X is then the vector bundle $\boldsymbol{V}(\mathscr{L})$ over X, and the sommet scheme is the zero section of this vector bundle.

9.8.7 Ample sheaves and contractions

Let Y be a scheme, $f : X \to Y$ be a quasi-compact and quasi-separated morphism $\mathscr{L}$ be an ample inveritble $\mathscr{O}_X$-module realtive to f. Consider the graded $\mathscr{O}_Y$-algebra with positive degrees

$$\mathscr{S} = \mathscr{O}_Y \oplus \bigoplus_{n \geq 1} f_*(\mathscr{L}^{\otimes n})$$

which is quasi-coherent by Proposition 8.6.55. We have a canonical homomorphism of graded $\mathscr{O}_X$-algebras

$$\sigma : f^*(\mathscr{S}) \to \bigoplus_{n \geq 0} \mathscr{L}^{\otimes n}$$

which, for each $n \geq 1$, coincides with the canonical homomorphism $\sigma_n : f^*(f_*(\mathscr{L}^{\otimes n})) \to \mathscr{L}^{\otimes n}$, and for $n = 0$ is the identify on $\mathscr{O}_X$. The hypothesis that $\mathscr{L}$ is f-ample implies that (Proposition 9.4.39 and Proposition 9.3.24) the corresponding Y-morphism

$$r = r_{\mathscr{L},\sigma} : X \to P = \mathrm{Proj}(\mathscr{S})$$

is everywhere defined and a dominant open immersion, and we have $\mathscr{L}^{\otimes n} = r^*(\mathscr{O}_P(n))$ for $n \in \mathbb{Z}$.

Proposition 9.8.43. *Let $C = \mathrm{Spec}(\mathscr{S})$ the affine cone defined by $\mathscr{S}$. If $\mathscr{L}$ is f-ample, there exists a canonical Y-morphism*

$$g : V = \boldsymbol{V}(\mathscr{L}) \to C \tag{9.8.45}$$

such that the diagram

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[d, "f"'] & \boldsymbol{V}(\mathscr{L}) \arrow[r, "\pi"] \arrow[d, "g"] & X \arrow[d, "f"] \\ Y \arrow[r, "\varepsilon"] & C \arrow[r, "\psi"] & Y \end{tikzcd} \tag{9.8.46}$$

is commutative, where ψ and π are structural morphisms, j and ε are the canonical immersions which maps X and Y respectively to the zero section of $\boldsymbol{V}(\mathscr{L})$ and the sommet scheme of C. Moreover, the restriction of g to $\boldsymbol{V}(\mathscr{L}) - j(X)$ is an open immersion

$$\boldsymbol{V}(\mathscr{L}) - j(X) \to E = C - \varepsilon(Y)$$

into the blunt affine cone E corresponding to $\mathscr{S}$.

9.8.8 Quasi-coherent sheaves over the projective cone

Let Y be a scheme, $\mathscr{S}$ be a quasi-coherent graded $\mathscr{O}_Y$-algebra, $X = \operatorname{Proj}(\mathscr{S})$, $C = \operatorname{Spec}(\mathscr{S})$ and $\widehat{C} = \operatorname{Proj}(\widehat{\mathscr{S}})$. Let $\mathscr{M}$ be a quasi-coherent graded $\mathscr{S}$-module; to avoid any possible confusion, we denote by $\widetilde{\mathscr{M}}$ the quasi-coherent $\mathscr{O}_C$-module associated with $\mathscr{M}$ if $\mathscr{M}$ is considered as a *nongraded* $\mathscr{S}$-module, and by $\mathscr{P}roj_0(\mathscr{M})$ the quasi-coherent $\mathscr{O}_X$-module associated with $\mathscr{M}$, where $\mathscr{M}$ is considered as a graded $\mathscr{S}$-module. We also set

$$\mathscr{M}_X = \mathscr{P}roj(\mathscr{M}) = \bigoplus_{n\in\mathbb{Z}} \mathscr{P}roj_0(\mathscr{M}(n));$$

with the quasi-coherent $\mathscr{O}_X$-algebra being defined by (9.8.38), $\mathscr{P}roj(\mathscr{M})$ is endowed a quasi-coherent graded $\mathscr{S}_X$-module structure, via the canonical homomorphisms

$$\mathscr{O}_X(m)\otimes_{\mathscr{O}_X}\mathscr{P}roj_0(\mathscr{M}(n)) \to \mathscr{P}roj_0(\mathscr{S}(m)\otimes_{\mathscr{S}}\mathscr{M}(n)) \to \operatorname{Proj}_0(\mathscr{M}(m+n))$$

which satisfies the axioms of modules in view of the commutative diagram (9.2.5). If $Y = \operatorname{Spec}(A)$ is affine, $\mathscr{S} = \widetilde{S}$ and $\mathscr{M} = \widetilde{M}$, where S is a graded A-algebra and M is a graded S-module, then for any homogeneous element $f \in S_+$, we have

$$\Gamma(X_f, \mathscr{P}roj(\mathscr{M})) = M_f$$

in view of the definition and (9.8.16).

Now consider the quasi-coherent graded $\widehat{\mathscr{S}}$-module

$$\widehat{\mathscr{M}} = \mathscr{M}\otimes_{\mathscr{S}}\widehat{\mathscr{S}}$$

(where $\widehat{\mathscr{S}} = \mathscr{S}[T] = \mathscr{S}\otimes_{\mathbb{Z}}\mathbb{Z}[T]$); we then deduce a quasi-coherent graded $\mathscr{O}_{\widehat{C}}$-module $\mathscr{P}roj_0(\widehat{\mathscr{M}})$ (recall that $\widehat{C} = \operatorname{Proj}(\widehat{\mathscr{S}})$), which we also denote by

$$\mathscr{M}^{\square} = \mathscr{P}roj_0(\widehat{\mathscr{M}}).$$

It is clear that $\mathscr{M}^{\square}$ is an exact functor on $\mathscr{M}$ and commutes with inductive limits and direct sums.

Proposition 9.8.44. *With the notations of Proposition 9.8.21 and Proposition 9.8.22, we have canonical homomorphisms*

$$i^*(\mathscr{M}^{\square}) \xrightarrow{\sim} \widetilde{\mathscr{M}}, \tag{9.8.47}$$
$$j^*(\mathscr{M}^{\square}) \to \mathscr{P}roj_0(\mathscr{M}), \tag{9.8.48}$$
$$p^*(\mathscr{P}roj_0(\mathscr{M})) \to \mathscr{M}^{\square}|_{\widehat{E}} \tag{9.8.49}$$

Moreover, the homomorphism (9.8.48) is an isomorphism if $\mathscr{S}$ is generated by $\mathscr{S}_1$.

Proof. In fact, $i^*(\mathscr{M}^{\square})$ is canonically identified with $(\widehat{\mathscr{M}}/(z-1)\widehat{\mathscr{M}})^{\sim}$ over $\operatorname{Spec}(\widehat{\mathscr{S}}/(z-1)\widehat{\mathscr{S}})$ in view of Proposition 9.3.12 and the definition of i. The first isomorphism of (9.8.47) is then deduced from Proposition 9.1.18 and the canonical isomorphism $\widehat{\mathscr{M}}/(z-1)\widehat{\mathscr{M}} \cong \mathscr{M}$. On the other hand, the canonical immersion $j : X \to \widehat{C}$ corresponds to the canonical homomorphism $\widehat{\mathscr{S}} \to \mathscr{S}$ with kernel $z\widehat{\mathscr{S}}$, so the second isomorphism is a particular case of the canonical homomorphism of Proposition 9.3.30, using the fact that $\widehat{\mathscr{M}}\otimes_{\widehat{\mathscr{S}}}\mathscr{S} = \mathscr{M}$. Finally, the homomorphism of (9.8.49) is a particular case of the homomorphisms $\nu^{\sharp}$ defined in (9.3.8). If $Y = \operatorname{Spec}(A)$ is affine, $\mathscr{S} = \widetilde{S}$, $\mathscr{M} = \widetilde{M}$, then we see from Proposition 9.2.48 that the restriction of (9.8.49) to $p^{-1}(X_f) = \widehat{C}_f$ (for $f \in S_+$ homogeneous) corresponds to the canonical homomorphism

$$M_{(f)}\otimes_{S_{(f)}} S_f^{\leqslant} \to M_f^{\leqslant}$$

in view of (9.8.10) and (9.8.12). The last assertion also follows from Proposition 9.3.30. ☐

By abuse of language, we say that $\mathscr{M}^{\square}$ is the projective closure of the $\mathscr{O}_C$-module $\widetilde{\mathscr{M}}$, where $\mathscr{M}$ is understood to be a graded $\mathscr{S}$-module.

Let us consider a morphism $q : Y' \to Y$ and a q-homomorphism $\varphi : \mathscr{S} \to \mathscr{S}'$. It then follows from (9.1.3) that for any quasi-coherent graded $\mathscr{S}$-module $\mathscr{M}$, we have a canonical isomorphism

$$\Phi^*(\widetilde{\mathscr{M}}) \xrightarrow{\sim} (q^*(\mathscr{M})\otimes_{q^*(\mathscr{S})}\mathscr{S}')^{\sim}$$

of $\mathscr{O}_C$-modules, where $\Phi = \mathrm{Spec}(\varphi)$. On the other hand, if $w = \mathrm{Proj}(\varphi)$ and $\hat{\Phi} = \mathrm{Proj}(\hat{\varphi})$, (9.3.7) gives a canonical w-homomorphism

$$\mathscr{P}roj_0(\mathscr{M}) \to (\mathscr{P}roj_0(q^*(\mathscr{M}) \otimes_{q^*(\mathscr{S})} \mathscr{S}'))|_{G(\varphi)} \tag{9.8.50}$$

and also a canonical $\hat{\Phi}$-morphism

$$\mathscr{P}roj_0(\widehat{\mathscr{M}}) \to (\mathscr{P}roj_0(q^*(\widehat{\mathscr{M}}) \otimes_{q^*(\widehat{\mathscr{S}})} \widehat{\mathscr{S}}'))|_{G(\hat{\varphi})}. \tag{9.8.51}$$

Now we consider the situation of the structural morphism $q : X \to Y$, where $X = \mathrm{Proj}(\mathscr{S})$, with the canonical q-homomorphism $\alpha : \mathscr{S} \to \mathscr{S}_X^{\geqslant}$. We then have a canonical isomorphism

$$q^*(\mathscr{M}) \otimes_{q^*(\mathscr{S})} \mathscr{S}_X^{\geqslant} \xrightarrow{\sim} \mathscr{M}_X^{\geqslant} \tag{9.8.52}$$

where $\mathscr{M}_X^{\geqslant} = \bigoplus_{n\geq 0} \mathscr{P}roj_0(\mathscr{M}(n))$. To see this, we can assume that $Y = \mathrm{Spec}(A)$ is affine, $\mathscr{S} = \widetilde{S}$ and $\mathscr{M} = \widetilde{M}$, and define the isomorphism (9.8.52) in each affine open X_f (f is homogeneous in S_+), and verify the compatibility when passing to a homogeneous multiple of f. Now, the restriction of the left side of (9.8.52) to X_f is $\widetilde{M}' = ((M \otimes_A S_{(f)}) \otimes_{S\otimes_A S_{(f)}} S_f^{\geqslant})^{\sim}$ by (9.8.39). As we have a canonical isomorphism $M \otimes_A S_{(f)} \cong M \otimes_S (S \otimes_A S_{(f)})$, we conclude that $\widetilde{M}' \cong (M \otimes_S S_f^{\geqslant})^{\sim}$, and this is canonically isomorphic to the restriction of $\mathscr{M}_X^{\geqslant}$ by (9.8.13). The compatibility of this isomorphism with restrictions is clear.

By replace $\mathscr{M}$ by $\widehat{\mathscr{M}}$, $\mathscr{S}$ by $\widehat{\mathscr{S}}$ and $\mathscr{S}_X$ by $(\mathscr{S}_X^{\geqslant})^{\wedge}$ in the preceding arguments, we obtain similarly a canonical isomorphism

$$q^*(\widehat{\mathscr{M}}) \otimes_{q^*(\widehat{\mathscr{S}})} (\mathscr{S}_X^{\geqslant})^{\wedge} \xrightarrow{\sim} (\mathscr{M}_X^{\geqslant})^{\wedge} \tag{9.8.53}$$

If we recall Proposition 9.8.34 that the structural morphism $\psi : \mathrm{Proj}(\mathscr{S}_X^{\geqslant}) \to X$ is an isomorphism, we then deduce a canonical ψ-isomorphism

$$\mathscr{P}roj_0(\mathscr{M}) \xrightarrow{\sim} \mathscr{P}roj_0(\mathscr{M}_X^{\geqslant}) \tag{9.8.54}$$

as a particular case of (9.8.50). In fact, we observe that, in the notations of Proposition 9.8.34, that this reduces to the fact that canonical homomorphism $M_{(f)} \otimes_{S_{(f)}} (S_f^{\geqslant})^{(d)} \to (M_f^{\geqslant})^{(d)}$ is an isomorphism if $f \in S_d$ is homogeneous, which is immediate.

The isomorphism (9.8.53) permits us, by apply (9.8.51) to the canonical morphism $r = \mathrm{Proj}(\hat{\alpha}) : \widehat{C}_X \to \widehat{C}$, to obtain a canonical r-homomorphism

$$\mathscr{M}^{\square} \to (\mathscr{M}_X^{\geqslant})^{\square}. \tag{9.8.55}$$

Now recall that the restrictions of r to the blunt cones $\widehat{E}_X$ and E_X are isomorphisms onto $\widehat{E}$ and E, respectively.

Proposition 9.8.45. *The restriction of the canonical r-homomorphism (9.8.55) to $\widehat{E}_X$ and to E_X are isomorphisms*

$$\mathscr{M}^{\square}|_{\widehat{E}} \xrightarrow{\sim} (\mathscr{M}_X^{\geqslant})^{\square}|_{\widehat{E}_X}, \tag{9.8.56}$$

$$\mathscr{M}^{\square}|_{E} \xrightarrow{\sim} (\mathscr{M}_X^{\geqslant})^{\square}|_{E_X}. \tag{9.8.57}$$

Proof. We can assume that $Y = \mathrm{Spec}(A)$ is affine as in the proof of Proposition 9.8.34; with the notations there, we must show that the canonical homomorphism

$$\widehat{M}_{(f)} \otimes_{\widehat{S}_{(f)}} (S_f^{\geqslant})^{\frown}_{(f/1)} \to (M \otimes_S S_f^{\geqslant})^{\frown}_{(f/1)}$$

is an isomorphism. But in view of (9.8.10) and (9.8.12), the left side is canonically identified with $M_f^{\geqslant} \otimes_{S_f^{\geqslant}} (S_f^{\geqslant})_{f/1}^{\leqslant}$, hence with $M_f^{\geqslant}$ in view of (9.8.14); the right side is identified with $(M_f^{\geqslant})_{f/1}^{\leqslant}$, hence also to $M_f^{\geqslant}$ by (9.8.17), whence our assertion about (9.8.56). The isomorphism (9.8.57) then follows from (9.8.56) and (9.8.47). □

Corollary 9.8.46. *With the notations of Corollary 9.8.35, the restriction of* $(\mathscr{M}_X^{\geqslant})^{\square}$ *to* $\widehat{E}_X$ *is identified with* $\widetilde{\mathscr{M}_X^{\leqslant}}$ *and its restriction to* E_X *is identified with* $\widetilde{\mathscr{M}}_X$.

Proof. We can clearly reduce to the affine case, and this follows from the idnetification of $(M_f^{\geqslant})_{f/1}^{\leqslant}$ with $M_f^{\leqslant}$ and $(M_f^{\geqslant})_{f/1}$ with M_f (cf. (9.8.17)). $\square$

Proposition 9.8.47. *Under the hypotheses of Corollary 9.8.36, the canonical homomorphism (9.8.49) is an isomorphism.*

Proof. In view of the fact that the structural morphism $\operatorname{Proj}(\mathscr{S}_X^{\geqslant}) \to X$ is an isomorphism and the isomorphisms (9.8.54) and (9.8.56), we only need to prove that the canonical homomorphism $p_X^*(\mathscr{P}roj_0(\mathscr{M}_X^{\geqslant})) \to (\mathscr{M}_X^{\geqslant})^{\square}|_{E_X}$ is an isomorphism, which means that we can assume that $\mathscr{S}_1$ is an invertible $\mathscr{O}_Y$-module and $\mathscr{S}$ is generated by $\mathscr{S}_1$. With the notations of Proposition 9.8.44, we then have, for $f \in S_1$, $S_f^{\leqslant} = S_{(f)}[1/f]$ and the canonical isomorphism $M_{(f)} \otimes_{S_{(f)}} S_f^{\leqslant} \to M_f^{\leqslant}$ is an isomorphism by the definition of $M_f^{\geqslant}$. $\square$

We now consider the quasi-coherent $\mathscr{S}$-module $\mathscr{M}(n) = \bigoplus_{m \geq n} \mathscr{M}_m$ and the quasi-coherent graded $\mathscr{S}^{\natural}$-module

$$\mathscr{M}^{\natural} = \Big(\bigoplus_{n \geq 0} \mathscr{M}(n)\Big)^{\sim}.$$

By Proposition 9.8.37 we have a canonical C-isomorphism $h : C_X \xrightarrow{\sim} \operatorname{Proj}(\mathscr{S}^{\natural})$.

Proposition 9.8.48. *There exists a canonical h-isomorphism*

$$\mathscr{P}roj_0(\mathscr{M}^{\natural}) \xrightarrow{\sim} \widetilde{\mathscr{M}}_X. \tag{9.8.58}$$

Proof. This can be proved as Proposition 9.8.37, by using the bi-isomorphism (9.8.18) here instead of (9.8.15). $\square$

Chapter 10

Cohomology of coherent sheaves over schemes

10.1 Cohomology of affine schemes

10.1.1 Čech cohomology and Koszul complex

Let X be a quasi-compact and quasi-separated scheme and $\mathscr{F}$ be a quasi-coherent $\mathscr{O}_X$-module. Let $A = \Gamma(X, \mathscr{O}_X)$, $M = \Gamma(X, \mathscr{F})$, $\boldsymbol{f} = (f_i)_{1 \le i \le r}$ be a family of elements of A, and $U_i = X_{f_i}$ be the open subset of X. Let $U = \bigcup_{i=1}^{r} U_i$ and $\mathfrak{U} = (U_i)$ be the covering of U. For any sequence $(i_0, i_1, \dots, i_p) \in I^{p+1}$ with $I = \{1, \dots, r\}$, we set

$$U_{i_0 \dots i_p} = \bigcap_{j=0}^{p} U_{i_j} = X_{f_{i_0} \cdots f_{i_p}}.$$

By Corollary 8.6.15, we have $\Gamma(U_{i_0 \dots i_p}, \mathscr{F}) = M_{f_{i_0} \cdots f_{i_p}}$, where $M = \Gamma(X, \mathscr{F})$. Note that the localization module $M_{f_{i_0} \cdots f_{i_p}}$ is identified with the limit of the inductive system $\{M^n_{i_0 \dots i_p}\}_{n \ge 0}$, where $M^n_{i_0 \dots i_p} = M$ and the homomorphism $\varphi_{nm} : M^m_{i_0 \dots i_p} \to M^n_{i_0 \dots i_p}$ is given by multiplication by $(f_{i_0} \dots f_{i_p})^{n-m}$ for $m \le n$.

For any $n \ge 0$, let $C^{p+1}_n(M)$ be the set of alternating maps from I^{p+1} to M, and consider the inductive system formed by these A-modules and the homomorphisms induced by φ_{nm}. If $C^p(\mathfrak{U}, \mathscr{F})$ is the group of alternating Čech p-cochains relative to the covering $\mathfrak{U}$ with coefficients in $\mathscr{F}$, then we see that

$$C^p(\mathfrak{U}, \mathscr{F}) = \varinjlim_n C^{p+1}_n(M).$$

On the other hand, from the definition of $C^{p+1}_n(M)$ it is easy to see that it is canonically identified with the Koszul complex $K^{p+1}(\boldsymbol{f}^n, M)$, and the homomorphism φ_{nm} is identified with the map

$$\varphi_{\boldsymbol{f}^{n-m}} : K^\bullet(\boldsymbol{f}^n, M) \to K^\bullet(\boldsymbol{f}^m, M)$$

induced by the map $(x_1, \dots, x_r) \mapsto (f_1^{n-m} x_1, \dots, f_r^{n-m} x_r)$ on A^r. We then have, for any $p \ge 0$, a functorial isomorphism

$$C^p(\mathfrak{U}, \mathscr{F}) \xrightarrow{\sim} C^{p+1}(\mathfrak{f}, M), \tag{10.1.1}$$

where $\mathfrak{f}$ is the ideal generated by $\boldsymbol{f}$. Moreover, the definition of the differentials of $C^p(\mathfrak{U}, \mathscr{F})$ and $C^p_n(M)$ show that the isomorphism (10.1.1) is in fact a morphism of complexes.

Proposition 10.1.1. *If X is a quasi-compact and quasi-separated scheme, there exists a canonical functorial isomorphism*

$$H^p(\mathfrak{U}, \mathscr{F}) \xrightarrow{\sim} H^{p+1}(\mathfrak{f}, M) \quad \text{for} \quad p \ge 1, \tag{10.1.2}$$

where $\mathfrak{f}$ is the ideal generated by $\boldsymbol{f}$. Moreover, we have a functorial exact sequence

$$0 \longrightarrow H^0(\mathfrak{f}, M) \longrightarrow M \longrightarrow H^0(\mathfrak{U}, \mathscr{F}) \longrightarrow H^1(\mathfrak{f}, M) \longrightarrow 0 \tag{10.1.3}$$

Proof. The relation (10.1.2) is in fact a concequence of (10.1.1). On the other hand, we have $C^0(\mathfrak{U},\mathscr{F}) = C^1(\mathfrak{f},M)$, so $H^0(\mathfrak{U},\mathscr{F})$ is identified with a subgroup of 1-cocycles of $C^1(\mathfrak{f},M)$. As $C^0(\mathfrak{f},M) = M$, the exact sequence (10.1.3) follows from the definition of $H^0(\mathfrak{f},M)$ and $H^1(\mathfrak{f},M)$. □

Corollary 10.1.2. *Suppose that the X_{f_i} are quasi-compact and there exists $g_i \in \Gamma(U,\mathscr{F})$ such that $\sum_i g_i(f_i|_U) = 1|_U$. Then for any quasi-coherent $(\mathscr{O}_X|_U)$-module $\mathscr{G}$, we have $H^p(\mathfrak{U},\mathscr{G}) = 0$ for $p > 0$. If moreover $U = X$, then the canonical homomorphism $M \to H^0(\mathfrak{U},\mathscr{F})$ in (10.1.3) is bijective.*

Proof. By hypothesis $U_i = X_{f_i}$ is quasi-compact, so U is quasi-compact, and we can assume that $U = X$. Then the hypothesis implies that $\mathfrak{f} = A$, so by **??** we have $H^p(\mathfrak{f},M) = 0$ for $p \geq 1$, and the corollary follows from (10.1.2) and (10.1.3). □

Remark 10.1.3. Let X be an affine scheme, so that the $U_i = X_{f_i} = D(f_i)$ are affine opens, and so is each $U_{i_0\dots i_p}$ (but U is not necessarily affine). In this case, the functors $\Gamma(X,\mathscr{F})$ and $\Gamma(U_{i_0\dots i_p},\mathscr{F})$ are exact by Theorem 9.5.12. If we have an exact sequence $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ of quasi-coherent $\mathscr{O}_X$-modules, the sequence of complexes

$$0 \longrightarrow C^\bullet(\mathfrak{U},\mathscr{F}') \longrightarrow C^\bullet(\mathfrak{U},\mathscr{F}) \longrightarrow C^\bullet(\mathfrak{U},\mathscr{F}'') \longrightarrow 0$$

is exact, so we obtain a long exact sequence of coholomogy groups

$$\cdots \longrightarrow H^p(\mathfrak{U},\mathscr{F}') \longrightarrow H^p(\mathfrak{U},\mathscr{F}) \longrightarrow H^p(\mathfrak{U},\mathscr{F}'') \xrightarrow{\delta} H^{p+1}(\mathfrak{U},\mathscr{F}') \longrightarrow 0$$

On the other hand, if we put $M' = \Gamma(X,\mathscr{F}')$, $M'' = \Gamma(X,\mathscr{F}'')$, $M = \Gamma(X,\mathscr{F})$, then the sequence $0 \to M' \to M \to M' \to 0$ is exact; as $C^\bullet(\mathfrak{f},M)$ is an exact functor on M, we then get a long exact sequence on cohomology

$$\cdots \longrightarrow H^p(\mathfrak{f},M') \longrightarrow H^p(\mathfrak{f},M) \longrightarrow H^p(\mathfrak{f},M'') \xrightarrow{\delta} H^{p+1}(\mathfrak{f},M') \longrightarrow 0$$

Now as the diagram

$$\begin{tikzcd}
0 \arrow[r] & C^\bullet(\mathfrak{U},\mathscr{F}') \arrow[r] \arrow[d] & C^\bullet(\mathfrak{U},\mathscr{F}) \arrow[r] \arrow[d] & C^\bullet(\mathfrak{U},\mathscr{F}'') \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C^\bullet(\mathfrak{f},M') \arrow[r] & C^\bullet(\mathfrak{f},M) \arrow[r] & C^\bullet(\mathfrak{f},M'') \arrow[r] & 0
\end{tikzcd}$$

is commutative, we conclude that the diagram

$$\begin{tikzcd}
H^p(\mathfrak{U},\mathscr{F}'') \arrow[r, "\delta"] \arrow[d] & H^{p+1}(\mathfrak{U},\mathscr{F}') \arrow[d] \\
H^{p+1}(\mathfrak{f},M'') \arrow[r, "\delta"] & H^{p+2}(\mathfrak{f},M')
\end{tikzcd} \tag{10.1.4}$$

is commutative for any $p > 0$.

10.1.2 Cohomology of affine schemes

Theorem 10.1.4. *Let X be an affine scheme. For any quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, we have $H^p(X,\mathscr{F}) = 0$ for $p > 0$.*

Proof. Let $\mathfrak{U}$ be a finite covering of X by affine opens $X_{f_i} = D(f_i)$ $(1 \leq i \leq r)$; then the ideal generated by f_i is equal to $A = \Gamma(X,\mathscr{O}_X)$. We then conclude from Corollary 10.1.2 that we have $H^p(\mathfrak{U},\mathscr{F}) = 0$ for any $p > 0$. As affine opens for a basis for X, we then conclude from the definition of the Čech cohomology that $\check{H}^p(X,\mathscr{F}) = 0$ for any $p > 0$. But this is also applicable on X_f for $f \in A$, so $\check{H}^p(X_f,\mathscr{F}) = 0$ for $p > 0$; as $X_f \cap X_g = X_{fg}$, we then conclude from Leray's vanishing theorem that $H^p(X,\mathscr{F}) = 0$ for $p > 0$. □

Corollary 10.1.5. *Let $f : X \to Y$ be an affine morphism. Then for any quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, we have $R^p f_*(\mathscr{F}) = 0$ for $p > 0$.*

Proof. By definition $R^p f_*(\mathscr{F})$ is defined to be the sheaf associated with the presheaf $U \mapsto H^p(f^{-1}(U), \mathscr{F})$, where U runs through open subsets of Y. Now the affine opens U form a basis for Y, and for such U, $f^{-1}(U)$ is affine, so $H^p(f^{-1}(U), \mathscr{F}) = 0$ by Theorem 10.1.4, so we conclude that $R^p f_*(\mathscr{F}) = 0$. □

Corollary 10.1.6. *Let $f : X \to Y$ be an affine morphism. Then for any quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, the canonical homomorphism $H^p(Y, f_*(\mathscr{F})) \to H^p(X, \mathscr{F})$ is bijective for any p.*

Proof. Consider the Leray spectral sequence

$$E_2^{p,q} = (R^p\Gamma \circ R^q f_*)(\mathscr{F}) = H^p(Y, R^q f_*(\mathscr{F})),$$

it follows from Corollary 10.1.5 that $E_2^{p,q} = 0$ for $q > 0$, so this sequence collapses at E_2 page, whence our assertion. □

Corollary 10.1.7. *Let $f : X \to Y$ be an affine morphism and $g : Y \to Z$ be a morphism. Then for any quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, the canonical homomorphism $R^p g_*(f_*(\mathscr{F})) \to R^p(g \circ f)_*(\mathscr{F})$ is bijective for any p.*

Proof. It suffices to remark that, by Corollary 10.1.6, for any affine open W of Z, the canonical homomorphism $H^p(g^{-1}(W), f_*(\mathscr{F})) \to H^p(f^{-1}(g^{-1}(W)), \mathscr{F})$ is bijective; this homomorphism of presheaves then defines a canonical homomorphism $R^p g_*(f_*(\mathscr{F})) \to R^p(g \circ f)_*(\mathscr{F})$ which is bijective. □

10.1.3 Applications to cohomology of schemes

Proposition 10.1.8. *Let X be a separated scheme, $\mathfrak{U} = (U_\alpha)$ be a covering of X by affine opens. For any quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, the cohomology module $H^\bullet(X, \mathscr{F})$ and $H^\bullet(\mathfrak{U}, \mathscr{F})$ (over $\Gamma(X, \mathscr{O}_X)$) are canonically isomorphic.*

Proof. In fact, as X is separated, any finite intersection V of open sets in the covering $\mathfrak{U}$ is affine (Proposition 8.5.31), so $H^p(V, \mathscr{F}) = 0$ for $q > 0$ in view of Theorem 10.1.4. The proposition then follows from Leray's vanishing theorem. □

Remark 10.1.9. We note that the conclusion of Proposition 10.1.8 is still valid if the finite intersections of U_α are affine, even if X is not necessarily separated.

Corollary 10.1.10. *Let X be a quasi-compact and separated scheme, $A = \Gamma(X, \mathscr{O}_X)$, $\boldsymbol{f} = (f_i)_{1 \le i \le r}$ be a sequence of elements of A such that the X_{f_i} are affine. Then for any quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, we have a canonical functorial isomorphism*

$$H^p(U, \mathscr{F}) \xrightarrow{\sim} H^{p+1}(\mathfrak{f}, M) \quad \text{for} \quad p > 0 \tag{10.1.5}$$

where $\mathfrak{f}$ is the ideal generated by $\boldsymbol{f}$. Moreover, we have a functorial exact sequence

$$0 \longrightarrow H^0(\mathfrak{f}, M) \longrightarrow M \longrightarrow H^0(U, \mathscr{F}) \longrightarrow H^1(\mathfrak{f}, M) \longrightarrow 0 \tag{10.1.6}$$

If X is an affine scheme, it then follows from Remark 10.1.3 and Proposition 10.1.8 that for any $q \ge 0$, the diagram

$$\begin{tikzcd} H^p(U, \mathscr{F}'') \arrow[r, "\delta"] \arrow[d] & H^{p+1}(U, \mathscr{F}') \arrow[d] \\ H^{p+1}(\mathfrak{f}, M'') \arrow[r, "\delta"] & H^{p+2}(\mathfrak{f}, M') \end{tikzcd} \tag{10.1.7}$$

corresponding to an exact sequence $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ of quasi-coherent $\mathscr{O}_X$-modules, is commutative.

Proposition 10.1.11. *Let X be a quasi-compact and separated X scheme, $\mathscr{L}$ be an invertible $\mathscr{O}_X$-module, and consider the graded ring $A_* = \Gamma_*(\mathscr{L})$. Then $H^\bullet(\mathscr{F},\mathscr{L}) = \bigoplus_{n\in\mathbb{Z}} H^\bullet(X,\mathscr{F}\otimes\mathscr{L}^{\otimes n})$ is a graded A_*-module, and for any $f \in (A_*)_n$, we have a canonical isomorphism*

$$H^\bullet(X_f,\mathscr{F}) \xrightarrow{\sim} (H^\bullet(\mathscr{F},\mathscr{L}))_{(f)} \tag{10.1.8}$$

of $(A_*)_{(f)}$-modules.

Proof. As X is quasi-compact and separated, we can compute the cohomology of $\mathscr{O}_X$-module $\mathscr{F}\otimes\mathscr{L}^{\otimes n}$ using the same finite covering $\mathfrak{U} = (U_i)$ by open affine subsets such that $\mathscr{L}|_{U_i}$ is isomorphic to $\mathscr{O}_X|_{U_i}$ for each i (Proposition 10.1.8). Also, since each $U_i \cap X_f$ is open and affine, we can also compute $H^\bullet(X_f,\mathscr{F}\otimes\mathscr{L}^{\otimes n})$ using the covering $\mathfrak{U}|_{X_f} = (U_i\cap X_f)$. Now for any $f\in A_n$, it is immediate that the multiplication by f defines a homomorphism $C^\bullet(\mathfrak{U},\mathscr{F}\otimes\mathscr{L}^{\otimes m}) \to C^\bullet(\mathfrak{U},\mathscr{F}\otimes\mathscr{L}^{\otimes(m+n)})$, whence a homomorphism $H^\bullet(\mathfrak{U},\mathscr{F}\otimes\mathscr{L}^{\otimes m}) \to H^\bullet(\mathfrak{U},\mathscr{F}\otimes\mathscr{L}^{\otimes(m+n)})$. On the other hand, by Corollary 8.6.16 and Corollary 8.1.6 (ii), any $f\in A_n$ gives an isomorphism of $(A_*)_{(f)}$-modules

$$C^\bullet(\mathfrak{U}|_{X_f},\mathscr{F}) \xrightarrow{\sim} (C^\bullet(\mathfrak{U},\bigoplus_{n\in\mathbb{Z}}\mathscr{F}\otimes\mathscr{L}^{\otimes n}))_{(f)}.$$

By passing to cohomologies, we then deduce the desired isomorphism (10.1.8), using the fact that $M\mapsto M_{(f)}$ is an exact functor on the category of graded modules. □

Corollary 10.1.12. *If $A = \Gamma(X,\mathscr{O}_X)$, then for any $f\in A$ we have a canonical isomorphism $H^\bullet(X_f,\mathscr{F}) \xrightarrow{\sim} (H^\bullet(X,\mathscr{F}))_f$ of A_f-module.*

Corollary 10.1.13. *Let X be a quasi-compact and separated scheme and $f\in\Gamma(X,\mathscr{O}_X)$.*

(a) *If X_f is affine, then for any quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$ and any $\xi\in H^i(X,\mathscr{F})$ with $i>0$, there exists an integer $n>0$ such that $f^n\xi = 0$.*

(b) *Conversely, suppose that X_f is quasi-compact and for any quasi-coherent ideal $\mathscr{I}$ and any $\zeta\in H^1(X,\mathscr{I})$, there exists $n>0$ such that $f^n\zeta = 0$. Then X_f is affine.*

Proof. First, if X_f is affine then $H^i(X_f,\mathscr{F}) = 0$ for $i>0$, so (a) follows from the isomorphism of Corollary 10.1.12. Conversely, in case (b), in view of Serre's criterion, it suffices to prove that for any quasi-coherent ideal $\mathscr{K}$ of $\mathscr{O}_X|_{X_f}$, we have $H^1(X_f,\mathscr{K}) = 0$. Now as X_f is a quasi-compact open subset of the quasi-compact scheme X, by Theorem 8.6.63 there exists a quasi-coherent ideal $\mathscr{I}$ of $\mathscr{O}_X$ such that $\mathscr{K} = \mathscr{I}|_{X_f}$. By Corollary 10.1.12 we have $H^1(X_f,\mathscr{K}) = (H^1(X,\mathscr{I}))_f$, and the hypothesis then implies our claim. □

Lemma 10.1.14 (Induction principle). *Let X be a quasi-compact and quasi-separated scheme and $\mathcal{P}$ be a property for quasi-compact open subsets of X. Assume that the following conditions are satisfied:*

(a) *$\mathcal{P}$ holds for affine opens of X,*

(b) *if U is a quasi-compact open subset of X, V is an affine open of X, and $\mathcal{P}$ holds for U, V, $U\cap V$, then $\mathcal{P}$ holds for $U\cup V$.*

Then $\mathcal{P}$ holds for every quasi-compact open subset of X, and in particular holds for X.

Proof. We first prove that $\mathcal{P}$ holds for separated quasi-compact open subset $W\subseteq X$. For this, note that W can be written as a union $W = U_1\cup\cdots\cup U_n$ of affine opens and we can applying induction on n with $U = U_1\cup\cdots\cup U_n$ and $V = U_n$. This is allowed because $U\cap V = (U_1\cap U_n)\cup\cdots\cup(U_{n-1}\cap U_n)$ is again a union of $n-1$ affine open subschemes. Now for any quasi-compact open subset $W\subseteq X$, we can induct on the number of affine opens needed to cover W using the same trick as before and using that the quasi-compact open $U_i\cap U_n$ is separated as an open subscheme of the affine scheme U_n. □

Proposition 10.1.15. *Let $f : X\to Y$ be a quasi-compact and quasi-separated morphism. Then for any quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, $R^pf_*(\mathscr{F})$ is quasi-coherent for $p\geq 0$.*

Proof. Since taking restriction commutes with higher direct images, we may assume that Y is affine. Then X is quasi-compact and quasi-separated. For a quasi-compact open subset $U \subseteq X$ and $f_U = f|_U$, we let $\mathcal{P}(U)$ be the property that $R^p(f_U)_*(\mathscr{F})$ is quasi-coherent for all quasi-coherent modules $\mathscr{F}$ on U and $o \geq 0$. It then suffices to prove that the conditions of Lemma 10.1.14 hold. If U is affine, then the morphism f_U is affine, so by Corollary 9.5.15 we have $R^p(f_U)_*(\mathscr{F}) = 0$ for $p > 0$, and $f_*(\mathscr{F})$ is quasi-coherent by Proposition 8.6.55. Now let $U \subseteq X$ be a quasi-compact open subset, $V \subseteq X$ be an affine open subset, and assume that property $\mathcal{P}$ holds for U, V and $U \cap V$. Then for any quasi-coherent $\mathscr{O}_X|_{U \cup V}$-module $\mathscr{F}$, we have the relative Mayer-Vietoris sequence

$$0 \to (f_{U\cup V})_*(\mathscr{F}) \to (f_U)_*(\mathscr{F}|_U) \oplus (f_V)_*(\mathscr{F}|_V) \to (f_{U\cap V})_*(\mathscr{F}|_{U\cap V}) \to R^1(f_{U\cup V})_*(\mathscr{F}) \to \cdots$$

It then follows from our assumption and Corollary 8.1.6 that $R^p(f_{U\cup V})_*(\mathscr{F})$ is quasi-coherent for $p \geq 0$, so the assertion follows by applying Lemma 10.1.14. □

Corollary 10.1.16. *Let $f : X \to Y$ be a quasi-compact and quasi-separated morphism. Then for any quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$ and any affine open $V \subseteq Y$, the canonical homomorphism*

$$H^p(f^{-1}(V), \mathscr{F}) \to H^0(V, R^p f_*(\mathscr{F}))$$

is an isomorphism for $p \geq 0$.

Proof. Since this question is local, we may assume that Y is affine. Consider the Leray spectral sequence

$$E_2^{p,q} = H^p(Y, R^q f_*(\mathscr{F})) \Rightarrow H^{p+q}(X, \mathscr{F}).$$

By Proposition 10.1.15, we see that the $R^q f_*(\mathscr{F})$ are quasi-coherent $\mathscr{O}_Y$-module, so we have $H^p(Y, R^q f_*(\mathscr{F})) = 0$ for $p > 0$ (Theorem 9.5.12). The spectral sequence therefore collapses at E_2 page and we obtain an isomorphism $H^q(X, \mathscr{F}) \cong H^0(Y, R^q f_*(\mathscr{F}))$. □

Corollary 10.1.17. *Let $f : X \to Y$ be a quasi-compact and separated morphism, and suppose that Y is quasi-compact. Then there exists an integer $r > 0$ such that for any quasi-coherent $\mathscr{O}_X$-module, we have $R^p f_*(\mathscr{F}) = 0$ for $p > r$. If Y is affine, then we can choose r so that there exists a covering of X by r affine opens.*

Proof. Since Y is a union of affine opens, it suffices to prove the second assertion, in view of Corollary 10.1.16. Now if $\mathfrak{U}$ is a covering of X by r affine opens, then $H^p(\mathfrak{U}, \mathscr{F}) = 0$ for $p > r$, since the cochains can be taken to be alternating. The conclusion then follows from Proposition 10.1.8. □

Corollary 10.1.18. *Under the hypothesis of Proposition 10.1.15, suppose that $Y = \mathrm{Spec}(A)$ is affine. Then for any quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$ and any $f \in A$, we have*

$$\Gamma(Y_f, R^p f_*(\mathscr{F})) = (\Gamma(Y, R^p f_*(\mathscr{F})))_f.$$

Proof. This follows from Proposition 10.1.15 and Corollary 8.6.15. □

Proposition 10.1.19. *Let Y be a quasi-compact scheme, $f : X \to Y$ be a quasi-compact and quasi-separated morphism, $g : Y \to Z$ be an affine morphism. For any quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, the canonical homomorphism $R^p(g \circ f)_*(\mathscr{F}) \to g_*(R^p f_*(\mathscr{F}))$ is bijective for $p \geq 0$.*

Proof. Consider the Grothendieck spectral sequence

$$E_2^{p,q} = R^p g_*(R^q f_*(\mathscr{F})) \Rightarrow R^{p+q}(g \circ f)_*(\mathscr{F}).$$

Since each $R^q f_*(\mathscr{F})$ is quasi-coherent by Proposition 10.1.15, it suffices to prove that $R^p g_*(\mathscr{G}) = 0$ for any quasi-coherent $\mathscr{O}_Y$-module $\mathscr{G}$ and $p > 0$. Since this question is local, we may assume that Z is affine; but then Y is also affine so the assertion follows from Corollary 10.1.16. □

Proposition 10.1.20. *Let $f : X \to Y$ be a quasi-compact and separated morphism and $g : Y' \to Y$ be a morphism. Let $f' = f_{(Y')} : X' = X_{(Y')} \to Y'$, and consider the commutative diagram*

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "g'"'] & Y' \arrow[d, "g"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

Let $\mathscr{F}$ (resp. $\mathscr{G}$) be a quasi-coherent $\mathscr{O}_X$-module (resp. $\mathscr{O}_{Y'}$-module) and assume that $\mathscr{F}$ and $\mathscr{G}$ are tor-independent, i.e., that for any points $x \in X$, $y' \in Y'$ such that $g(y') = f(x) = y$, we have

$$\operatorname{Tor}_p^{\mathscr{O}_{Y,y}}(\mathscr{G}_{y'}, \mathscr{F}_x) = 0 \quad \textit{for} \quad p > 0.$$

Then there exists a natural isomorphism in the derived category $D(Y')$:

$$\mathscr{G} \otimes_Y^L Rf_*(\mathscr{F}) \xrightarrow{\sim} Rf'_*(\mathscr{G} \otimes_Y \mathscr{F}), \tag{10.1.9}$$

where $G \otimes_Y^L Rf_*(\mathscr{F}) := \mathscr{G} \otimes_{\mathscr{O}_{Y'}}^L Lg^*(Rf_*(\mathscr{F}))$.

When $Y = Y'$ (resp. $\mathscr{G} = \mathscr{O}_{Y'}$), the isomorphism (10.1.9) is called the **projection isomorphism** (resp. **base change isomorphism**). When $\mathscr{G} = \mathscr{O}_{Y'}$, we have $\mathscr{G} \otimes_Y \mathscr{F} = g'^*(\mathscr{F})$, so we deduce from (10.1.9) a canonical map

$$g^*(R^p f_*(\mathscr{F})) \to R^p f'_*(g'^*(\mathscr{F})) \tag{10.1.10}$$

This is the composition of the canonical map $g^*(R^p f_*(\mathscr{F})) \to H^p(Lg^*(Rf_*(\mathscr{F})))$ and the isomorphism $H^p(Lg^*(Rf_*(\mathscr{F}))) \xrightarrow{\sim} R^p f'_*(g'^*(\mathscr{F}))$ obtained from (10.1.9) by applying H^p. However, this is not an isomorphism in general.

Proof. First, we consider the case where $X = \operatorname{Spec}(B)$, $Y = \operatorname{Spec}(A)$, $Y' = \operatorname{Spec}(A')$ are affine, so that X' is affine with ring $B' = A' \otimes_A B$, and $\mathscr{F} = \widetilde{M}$, $\mathscr{G} = \widetilde{N}$ for some B-module M and A'-module N. Then $Rf_*(\mathscr{F})$ is represented by the underlying A-module $M_{[A]}$ of M, and $Rf'_*(\mathscr{G} \otimes_Y \mathscr{F})$ by the underlying A'-module $(N \otimes_A M)_{[A']}$ of $(N \otimes_{A'} B') \otimes_{B'} (B' \otimes_A M)$. On the other hand, the derived tensor product $\mathscr{G} \otimes_Y^L Rf_*(\mathscr{F})$ is represented by

$$N \otimes_A^L M_{[A]} := N \otimes_{A'}^L (A' \otimes_A^L M_{[A]}),$$

which can be calculated as $N \otimes_A P$, where P is a flat resolution of $M_{[A]}$. The tor-independence hypothesis implies that $\operatorname{Tor}_p^A(N, M_{[A]}) = 0$ for $p > 0$, i.e., the natural map

$$N \otimes_A^L M_{[A]} \to N \otimes_A M_{[A]} \tag{10.1.11}$$

is an isomorphism (in $D(A')$). The isomorphism (10.1.9) is then composition of (10.1.11) and the (trivial) isomorphism

$$N \otimes_A M_{[A]} \xrightarrow{\sim} (N \otimes_A M)_{[A']}.$$

In the general case, as f and Y are quasi-compact, X has a finite affine open covering $\mathfrak{U} = (U_i)$. Since f is separated, any finite intersection $U_{i_0 \dots i_p}$ is separated over Y, so it follows from Proposition 10.1.8 that $Rf_*(\mathscr{F}) = f_*(\check{\mathscr{C}}(\mathfrak{U}, \mathscr{F}))$, where $\check{\mathscr{C}}(\mathfrak{U}, \mathscr{F})$ is the alternating Čech complex of $\mathfrak{U}$ with values in $\mathscr{F}$. The preceding discussion, applied to affine open subsets of Y' above affine open subsets of Y, then shows that we have natural identifications

$$\mathscr{G} \otimes_Y^L Rf_*(\mathscr{F}) \xrightarrow{\sim} \mathscr{G} \otimes_{\mathscr{O}_{Y'}} g^*(f_*(\check{\mathscr{C}}(\mathfrak{U}, \mathscr{F}))) \xrightarrow{\sim} f'_*(\check{\mathscr{C}}(\mathfrak{U}', \mathscr{G} \otimes_Y \mathscr{F})) \xrightarrow{\sim} Rf'_*(\mathscr{G} \otimes_Y \mathscr{F}),$$

where $\mathfrak{U}'$ is the covering of X' formed by the inverse images of the U_i's. It is easy to check that the above composition does not depend on the choice of $\mathfrak{U}$, so we take this as the definition of the isomorphism (10.1.9). □

Corollary 10.1.21. *Let Y be a quasi-compact scheme, $f : X \to Y$ be a quasi-compact and separated morphism, and $g : Y' \to Y$ be a flat morphism. Then for any quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$, we have a base change isomorphism*

$$g^* Rf_*(\mathscr{F}) \xrightarrow{\sim} Rf'_*(g'^*(\mathscr{F})),$$

and the induced base change maps (10.1.10) are isomorphisms.

Proof. If g is flat, then we can take $\mathscr{G} = \mathscr{O}_{Y'}$ in Proposition 10.1.20, so that we obtain a base change isomorphism

$$g^* Rf_*(\mathscr{F}) \xrightarrow{\sim} Rf'_*(g'^*(\mathscr{F})).$$

Moreover, since g is flat, the canonical map $g^*(R^p f_*(\mathscr{F})) \to H^p(Lg^*(Rf_*(\mathscr{F})))$ is an isomorphism, so the induced base change maps (10.1.10) are isomorphisms. □

Corollary 10.1.22. *Let Y be a quasi-compact scheme and $f : X \to Y$ be a quasi-compact and separated morphism. Let $y \in Y$ be a point and X_y be the fiber of f at y. Then for any quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$ which is flat over Y, we have a natural isomorphism (in the derived category of $\kappa(y)$-vector spaces)*

$$\kappa(y) \otimes^L_{\mathscr{O}_Y} Rf_*(\mathscr{F}) \xrightarrow{\sim} R\Gamma(X_y, \mathscr{F}_y) \tag{10.1.12}$$

where $\mathscr{F}_y = \mathscr{O}_{X_y} \otimes_{\mathscr{O}_X} \mathscr{F}$.

Proof. By hypothesis $\mathscr{F}$ is tor-independent of $\kappa(y)$, so we can apply Proposition 10.1.20. □

Corollary 10.1.23. *Let A be a ring, X be an A-scheme of finite type, and B be a faithfully flat A-algebra. Then for X to be affine, it is necesary and sufficient that $X \otimes_A B$ is affine.*

Proof. The condition is clearly necessary. Conversely, as X is separated over A and the morphism $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is flat, it follows from Corollary 10.1.21 that we have

$$H^i(X \otimes_A B, \mathscr{F} \otimes_A B) = H^i(X, \mathscr{F}) \otimes_A B$$

for $i \geq 0$ and any quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$. As $X \otimes_A B$ is affine, we see that $H^i(X \otimes_A B, \mathscr{F} \otimes_A B) = 0$ for $i > 0$, so $H^i(X, \mathscr{F}) = 0$ for $i > 0$ since B is faithfully flat over A. As X is quasi-compact, the conclusion then follows from Serre's criterion. □

Remark 10.1.24. The projection isomorphism and the base change isomorphism together give a proof of the Künneth formula. To see this, let $f : X \to S$, $g : Y \to S$ be two quasi-compact and separated morphisms, and form the fiber product $X \times_S Y$:

$$\begin{tikzcd} X \times_S Y \arrow[r, "q"] \arrow[d, "p"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & S \end{tikzcd}$$

Let $\mathscr{F}$ (resp. $\mathscr{G}$) be a quasi-coherent $\mathscr{O}_X$-module (resp. $\mathscr{O}_Y$-module) and $h = f \circ p = g \circ q$. Then (assuming the projection isomorphism and the base change isomorphism) we have the following isomorphisms:

$$\begin{aligned} Rh_*(\mathscr{F} \otimes^L_S \mathscr{G}) &= Rf_*(Rp_*(Lp^*(\mathscr{F}) \otimes^L_S Lq^*(\mathscr{G}))) \cong Rf_*(\mathscr{F} \otimes^L_S Rp_*(Lq^*(\mathscr{G}))) \\ &\cong Rf_*(\mathscr{F} \otimes^L_S Lf_*(Rg_*(\mathscr{G}))) \cong Rf_*(\mathscr{F}) \otimes^L_S Rg_*(\mathscr{G}). \end{aligned}$$

10.2 Coholomogical properties of projective morphisms

10.2.1 Cohomology associated with an invertible sheaf

Let X be a scheme, $\mathscr{L}$ be an invertible $\mathscr{O}_X$-module, and consider the graded ring

$$S = \Gamma_*(X, \mathscr{L}) = \bigoplus_{n \in \mathbb{Z}} \Gamma(X, \mathscr{L}^{\otimes n}).$$

Let $\boldsymbol{f} = (f_i)_{1 \leq i \leq r}$ be a family of homogeneous elements of S, where $f_i \in S_{d_i}$. Put $U_i = X_{f_i}$, $U = \bigcup_i U_i$, and let $\mathfrak{U} = (U_i)$ be the covering of U. For any quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$ of X, we set

$$H^\bullet(\mathfrak{U}, \mathscr{F}; \mathscr{L}) := \bigoplus_{n \in \mathbb{Z}} H^\bullet(\mathfrak{U}, \mathscr{F} \otimes \mathscr{L}^{\otimes n}), \quad H^\bullet(U, \mathscr{F}; \mathscr{L}) := \bigoplus_{n \in \mathbb{Z}} H^\bullet(U, \mathscr{F} \otimes \mathscr{L}^{\otimes n}). \tag{10.2.1}$$

We note that the abelian groups in (10.2.1) are bigraded: for $m, n \in \mathbb{Z}$ we set

$$(H^\bullet(\mathfrak{U}, \mathscr{F}; \mathscr{L}))_{mn} = H^m(\mathfrak{U}, \mathscr{F} \otimes \mathscr{L}^n), \quad (H^\bullet(U, \mathscr{F}; \mathscr{L}))_{mn} = H^m(U, \mathscr{F} \otimes \mathscr{L}^n).$$

For any fixed $m \in \mathbb{Z}$, it is clear that $H^m(\mathfrak{U}, \mathscr{F}; \mathscr{L})$ and $H^m(U, \mathscr{F}; \mathscr{L})$ are graded S-modules. We now consider the graded S-module

$$M = \Gamma_*(\mathscr{F}; \mathscr{L}) = H^0(X, \mathscr{F}; \mathscr{L}) = \bigoplus_{n \in \mathbb{Z}} \Gamma(X, \mathscr{F} \otimes \mathscr{L}^{\otimes n}).$$

If X is quasi-compact and quasi-separated, then it follows from Theorem 8.6.14 that for any sequence $(i_0, i_1, \dots, i_p)$, we have a canonical isomorphism

$$\Gamma(U_{i_0\dots i_p}, \mathscr{F}; \mathscr{L}) = H^0(U_{i_0\dots i_p}, \mathscr{F}; \mathscr{L}) = M_{f_{i_0}\cdots f_{i_p}}.$$

Recall that $M_{f_{i_0}\cdots f_{i_p}}$ is idnetified with $\varinjlim_n M^n_{i_0\dots i_p}$, and this identification is an isomorphism of graded S-modules, if we define the degree of an element $z \in \varinjlim_n M^n_{i_0\dots i_p}$ to be the number $m - n(d_{i_0} + \cdots + d_{i_p})$ if z is the image of a homogeneous element $x \in M^n_{i_0\dots i_p} = M$ of degree m (from the definition of the transition homomorphism, it follows that this definition does not depend on the choice of x). Let $C^p_n(M)$ be the set of alternating maps $I^{p+1} \to M$ (for any n), then we can similarly define a graded S-module structure on $\varinjlim_n C^p_n(M)$. Now the canonical isomorphism

$$C^p(\mathfrak{U}, \mathscr{F}; \mathscr{L}) = \varinjlim_n C^p_n(M)$$

is then an isomorphism of graded S-modules. By Proposition 10.1.1, we have isomorphism of graded S-modules

$$C^p(\mathfrak{U}, \mathscr{F}; \mathscr{L}) = C^{p+1}(\mathfrak{f}, M) = \varinjlim_n K^{p+1}(\boldsymbol{f}^n, M)$$

where the degree of an element in $\varinjlim K^{p+1}(\boldsymbol{f}^n, M)$ is defined similarly. It is easy to see that the isomorphisms above are compatible with differential maps, so from Proposition 10.1.1 we conclude the following:

Proposition 10.2.1. *Let X be a quasi-compact and quasi-separated scheme. Then there exists a canonical isomorphism of graded S-modules*

$$H^p(\mathfrak{U}, \mathscr{F}; \mathscr{L}) \xrightarrow{\sim} H^{p+1}(\mathfrak{f}, M) \quad \textit{for} \quad p \geq 1, \tag{10.2.2}$$

where $\mathfrak{f}$ is the ideal generated by $\boldsymbol{f}$. Moreover, we have a functorial exact sequence

$$0 \longrightarrow H^0(\mathfrak{f}, M) \longrightarrow M \longrightarrow H^0(\mathfrak{U}, \mathscr{F}; \mathscr{L}) \longrightarrow H^1(\mathfrak{f}, M) \longrightarrow 0 \tag{10.2.3}$$

of homomorphisms of degree 0.

Corollary 10.2.2. *If X is quasi-compact and separated and $U_i = X_{f_i}$ are affine, then there exists a canonical isomorphism of degree* 0*:*

$$H^p(U, \mathscr{F}; \mathscr{L}) \xrightarrow{\sim} H^{p+1}(\mathfrak{f}, M) \quad \textit{for} \quad p > 0.$$

and we have a functorial exact sequence

$$0 \longrightarrow H^0(\mathfrak{f}, M) \longrightarrow M \longrightarrow H^0(\mathfrak{U}, \mathscr{F}; \mathscr{L}) \longrightarrow H^1(\mathfrak{f}, M) \longrightarrow 0$$

of homomorphisms of degree 0.

Corollary 10.2.3. *Let S be a graded ring with positive degrees, $(f_i)_{1\leq i\leq r}$ be homogeneous elements of S_+ with $f_i \in S_{d_i}$, and M be a graded S-module. Let $X = \mathrm{Proj}(S)$, $U_i = D_+(f_i)$, and $\mathscr{L} = \mathscr{O}_X(1)$, then there exists a canonical isomorphism of degree* 0*:*

$$H^p(U, \widetilde{M}; \mathscr{L}) \xrightarrow{\sim} H^{p+1}(\mathfrak{f}, M) \quad \textit{for} \quad p > 0.$$

and we have a functorial exact sequence

$$0 \longrightarrow H^0(\mathfrak{f}, M) \longrightarrow M \longrightarrow H^0(\mathfrak{U}, \widetilde{M}; \mathscr{L}) \longrightarrow H^1(\mathfrak{f}, M) \longrightarrow 0$$

of homomorphisms of degree 0.

Proof. In fact, we have $\Gamma(U_{i_0\dots i_p}, \widetilde{M(n)}) = (M_{f_{i_0}\cdots f_{i_p}})_n$, so $\Gamma(U_{i_0\dots i_p}, \widetilde{M}; \mathscr{L}) = M_{f_{i_0}\cdots f_{i_p}}$. The conclusion then follows from Corollary 10.2.2 since X is quasi-compact and separated. □

Remark 10.2.4. Corollary 10.2.3 is intersting if S is an A-algebra generated by S_1 where A is Noetherian. In fact, in this case any quasi-coherent $\mathscr{O}_X$-module $\mathscr{F}$ is of the form $\widetilde{M}$ by Theorem 9.2.39.

Remark 10.2.5. Under the hypothesis of Corollary 10.2.3, the functor $\Gamma(U_{i_0\cdots i_p}, \widetilde{M}; \mathscr{L}) = M_{f_{i_0}\cdots f_{i_p}}$ is exact on M, and as in Remark 10.1.3, if $0 \to M' \to M \to M'' \to 0$ is an exact sequence of graded S-modules, we have a commutative diagram for $p \geq 0$:

$$\begin{tikzcd}
H^p(U, \widetilde{M}''; \mathscr{L}) \arrow[r, "\partial"] \arrow[d] & H^{p+1}(U, \widetilde{M}'; \mathscr{L}) \arrow[d] \\
H^{p+1}(\mathfrak{f}, M'') \arrow[r, "\partial"] & H^{p+2}(\mathfrak{f}, M')
\end{tikzcd}$$

We now apply Corollary 10.2.3 to the polynomial ring $S = A[T_0, \dots, T_r]$, where A is a ring and T_i are indeterminantes. Let $M = S$ and $f_i = T_i$, we are then reduced to compute $H^\bullet(\mathfrak{m}, S)$, where $\boldsymbol{T} = (T_i)_{0\leq i\leq r}$ and $\mathfrak{m}$ is the maximal ideal of S generated by $T_0, \dots, T_r$.

Lemma 10.2.6. *If $S = A[T_0, \dots, T_r]$ and $\boldsymbol{T} = (T_i)_{0\leq i\leq r}$, then*

$$H^i(\boldsymbol{T}^n, S) = \begin{cases} 0 & i \neq r+1, \\ S/\mathfrak{m}^n & i = r+1. \end{cases} \tag{10.2.4}$$

Proof. This is an immediate from the fact that the sequence $\boldsymbol{T}$ is regular. □

By passing to inductive limits over n, we see that $H^i(\mathfrak{m}, S) = 0$ for $i \neq r+1$. If $i = r+1$, then the inductive system is formed by $S/\mathfrak{m}^n$ and $\varphi_{nm} : S/\mathfrak{m}^n \to S/\mathfrak{m}^m$ for $0 \leq n \leq m$ is the multiplication by $(T_0 \cdots T_r)^{n-m}$. For any sequence $\alpha = (\alpha_0, \dots, \alpha_r)$ and integer $n \geq \sup_i\{\alpha_i\}$, we define

$$\xi_\alpha^n = T_0^{n-\alpha_0} \cdots T_r^{n-\alpha_r} \mod \mathfrak{m}^n.$$

Then $\varphi_{nm}(\xi_\alpha^n) = \xi_\alpha^m$, so the ξ_α^n form an element ξ_α in the inductive limit $H^{r+1}(\mathfrak{m}, S)$.

Corollary 10.2.7. *With the notations of Lemma 10.2.6, we have $H^i(\mathfrak{m}, S) = 0$ for $i \neq r+1$, and $H^{r+1}(\mathfrak{m}, S)$ is a free A-module with basis formed by the elements ξ_α with $\alpha_i > 0$ for each i.*

Proof. In fact, for each $n \geq 0$, the elements ξ_α^n for $0 < \alpha_i \leq n$ form a basis of the A-module $S/\mathfrak{m}^n$, so the corollary follows from (10.2.4). □

Proposition 10.2.8. *Let A be a ring, $r > 0$ be an integer, and $X = \mathbb{P}_A^r$.*

(a) *We have $H^i(X, \mathscr{O}_X; \mathscr{O}_X(1)) = 0$ for $i \neq 0, r$.*

(b) *The canonical homomorphism $\alpha : S \to H^0(X, \mathscr{O}_X; \mathscr{O}_X(1))$ is an isomorphism.*

(c) *The A-module $H^r(X, \mathscr{O}_X; \mathscr{O}_X(1))$ is free with a basis formed by the elements ξ_α with $\alpha_i > 0$ for each i. Moreover, ξ_α is of degree $-|\alpha| = -(\alpha_0 + \cdots + \alpha_r)$ and $T_i \cdot \xi_\alpha = \xi_{\alpha_0,\dots,\alpha_i-1,\dots,\alpha_r}$.*

Proof. By Corollary 10.2.7 we have $H^i(\mathfrak{m}, S) = 0$ for $i \neq r+1$, so the assertion in (a) follows from (10.2.2). From the exact sequence (10.2.3), it is easy to see that $S \cong H^0(X, \mathscr{O}_X; \mathscr{O}_X(1))$, and the isomorphism is given by the canonical homomorphism α. The last assertion also follows from (10.2.2) and Corollary 10.2.7. □

Corollary 10.2.9. *The values (i, n) such that $H^i(X, \mathscr{O}_X(n)) \neq 0$ are the following: $i = 0$ and $n \geq 0$, or $i = r$ and $n \leq -(r+1)$.*

Proof. We note that if $A \neq 0$ then $H^i(X, \mathscr{O}_X(n)) \neq 0$ by the listed values of (i, n). □

Corollary 10.2.10. *The A-modules $H^i(X, \mathscr{O}_X(n))$ are free of finite rank. If $i > 0$, then they are zero for $n > 0$.*

Proposition 10.2.11. *Let Y be a scheme, $\mathscr{E}$ be a locally free $\mathscr{O}_Y$-module of rank $r+1$, and $X = \mathbb{P}(\mathscr{E})$ be the projective bundle defined by $\mathscr{E}$. Let $f : X \to Y$ be the structural morphism, then the values (n, i) such that $R^i f_*(\mathscr{O}_X(n)) \neq 0$ are the following: $i = 0$ and $n \geq 0$, or $i = r$ and $n \leq -(r+1)$. Moreover, the canonical homomorphism*

$$\alpha : \boldsymbol{S}_{\mathscr{O}_Y}(\mathscr{E}) \to \Gamma_*(\mathscr{O}_X) = \bigoplus_{n\in\mathbb{Z}} f_*(\mathscr{O}_X(n))$$

is an isomorphism.

Proof. This question is local over Y, so we can assume that $Y = \operatorname{Spec}(A)$ is affine and $\mathscr{E} = \widetilde{E}$, where $E = A^{r+1}$. We can then apply Proposition 10.2.8, in view of Corollary 10.1.16. □

10.2.2 The coherence theorem for projective morphisms

Proposition 10.2.12. *Let A be a Noetherian ring and S be a graded A-algebra with positive degrees that is generated by $r+1$ elements of S_1. Let $X = \operatorname{Proj}(S)$, and consider a coherent $\mathscr{O}_X$-module $\mathscr{F}$.*

(a) *The A-module $H^p(X, \mathscr{F})$ is finitely generated.*

(b) *We have $H^p(X, \mathscr{F}) = 0$ for $p > r$.*

(c) *There exists an integer n_0 such that for $n \geq n_0$, we have $H^p(X, \mathscr{F}(n))$ for $p > 0$.*

(d) *There exists an integer n_0 such that for $n \geq n_0$, $\mathscr{F}(n)$ is generated by global sections.*

Proof. Note that X can be identified with a closed subscheme of $P = \mathbb{P}^r_A$ (Proposition 9.2.52). Moreover, if $j : X \to P$ is the canonical injection, $j_*(\mathscr{F})$ is a coherent $\mathscr{O}_P$-module and we have $j_*(\mathscr{F}(n)) = (j_*(\mathscr{F}))(n)$ (Corollary 9.3.26 and Proposition 9.3.30). In view of (G, II, cor. du th.4.9.1), we only need to consider the case where $X = \mathbb{P}^r_A$ and $S = A[T_0, \dots, T_r]$. Now X can be covered by $r+1$ affine opens $D_+(T_i)$, so (b) follows from Corollary 10.1.16 and Corollary 10.1.17. We also note that (d) is proved in Corollary 9.3.27.

We now prove (a) and (c). By Proposition 10.2.8, these assertions hold for $\mathscr{F} = \mathscr{O}_X(m)$, hence for direct sums of finitely many $\mathscr{O}_X$-module of the form $\mathscr{O}_X(m_j)$. On the other hand, (a) and (c) are trivial for $p > r$ in view of (b). We now proceed by descendent induction on p. Since $\mathscr{F}$ is coherent, it is a quotient of direct sume of finitely many sheaves $\mathscr{O}_X(m_j)$ (Corollary 9.2.42). That is, we have an exact sequence $0 \to \mathscr{R} \to \mathscr{E} \to \mathscr{F} \to 0$ where $\mathscr{R}$ is coherent and $\mathscr{E}$ satisfies (a) and (c). Since tensoring with $\mathscr{O}_X(n)$ is exact, we obtain an exact sequence

$$H^{p-1}(X, \mathscr{E}(n)) \longrightarrow H^{p-1}(X, \mathscr{F}(n)) \longrightarrow H^p(X, \mathscr{R}(n)).$$

As $\mathscr{E}(n)$ is a direct sum of $\mathscr{O}_X(m_j + n)$, we see $H^{p-1}(X, \mathscr{E}(n))$ is finitely generated by Corollary 10.2.10, and so is $H^p(X, \mathscr{R}(n))$ by induction hypothesis. As A is Noetherian, we then conclude that $H^{p-1}(X, \mathscr{F}(n))$ is finitely generated for any $n \in \mathbb{Z}$, and in particular for $n = 0$. On the other hand, by induction hypothesis there exists an integer n_0 such that for $n \geq n_0$ we have $H^p(X, \mathscr{R}(n)) = 0$, and we can choose n_0 such that $H^{p-1}(X, \mathscr{E}(n)) = 0$ for $n \geq n_0$, since $\mathscr{E}$ satisfies (c). Therefore we see that $H^{p-1}(X, \mathscr{F}(n)) = 0$ for $n \geq n_0$, which completes the proof. □

Theorem 10.2.13 (Serre). *Let Y be a Noetherian scheme, $f : X \to Y$ be a projective morphism, and $\mathscr{L}$ be an invertible $\mathscr{O}_X$-module that is ample for f. For any coherent $\mathscr{O}_X$-module $\mathscr{F}$, set $\mathscr{F}(n) = \mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{L}^{\otimes n}$.*

(a) *The $\mathscr{O}_Y$-module $R^p f_*(\mathscr{F})$ is coherent for $p \geq 0$.*

(b) *There exists an integer n_0 such that for $n \geq n_0$, we have $R^p f_*(\mathscr{F}(n)) = 0$ for $p > 0$.*

(c) *There exists an integer n_0 such that for $n \geq n_0$, the canonical homomorphism $f^*(f_*(\mathscr{F}(n))) \to \mathscr{F}(n)$ is surjective.*

Proof. We note that the theorem is unchanged if we replace $\mathscr{L}$ by $\mathscr{L}^{\otimes d}$ for $d > 0$. In fact, we then have $\mathscr{F}(n) = (\mathscr{F} \otimes \mathscr{L}^{\otimes r}) \otimes \mathscr{L}^{\otimes kd}$ for $k > 0$ and $0 \leq r < d$, and by hypothesis for any r there is an integer n_r such that for $k \geq n_r$, the properties (b) and (c) holds for $\mathscr{F} \otimes \mathscr{L}^{\otimes r}$. Let n_0 be the supremum of dn_r, then assertion (b) and (c) hold for $n \geq n_0$. In view of Proposition 9.4.48, we may assume that $\mathscr{L}$ is very ample relative to f, so there exists a dominant open immersion $i : X \to P$, where $P = \operatorname{Proj}(\mathscr{S})$ for a quasi-coherent graded $\mathscr{O}_Y$-algebra $\mathscr{S}$ with positive degrees that is finitely generated by $\mathscr{S}_1$; moreover, $\mathscr{L} \cong i^*(\mathscr{O}_P(1))$. Since f is proper, the morphism i is also proper by Corollary 9.5.24, so it is an isomorphism $X \cong P$. We can therefore assme that $X = \operatorname{Proj}(\mathscr{S})$ and $\mathscr{L} = \mathscr{O}_X(1)$, and the theorem then follows from Proposition 10.2.12. □

Corollary 10.2.14. *Under the hypothesis of Theorem 10.2.13, let $0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{H} \to 0$ be an exact sequence of coherent $\mathscr{O}_X$-modules. Then there exists an integer n_0 such that for $n \geq n_0$, the sequence*

$$0 \longrightarrow f_*(\mathscr{F}(n)) \longrightarrow f_*(\mathscr{G}(n)) \longrightarrow f_*(\mathscr{H}(n))$$

is exact.

Proof. This follows from the long exact sequence of f_* and property (b) of Theorem 10.2.13. $\square$

Corollary 10.2.15. *Let Y be a Noetherian scheme, $f : X \to Y$ be a morphism of finite type, and $\mathscr{L}$ be an invertible $\mathscr{O}_X$-module that is ample for f. Let $0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{H} \to 0$ be an exact sequence of coherent $\mathscr{O}_X$-modules such that the supports of $\mathscr{F}$ and $\mathscr{H}$ are proper over Y. Then there exists an integer n_0 such that for $n \geq n_0$, the sequence*

$$0 \longrightarrow f_*(\mathscr{F}(n)) \longrightarrow f_*(\mathscr{G}(n)) \longrightarrow f_*(\mathscr{H}(n))$$

is exact.

Proof. The same reasoning of Theorem 10.2.13 show that we can assume that $\mathscr{L}$ is very ample relative to f, so we can identify X as an open subscheme of $Z = \mathrm{Proj}(\mathscr{S})$, where $\mathscr{S}$ is a quasi-coherent graded $\mathscr{O}_Y$-algebra with positive degrees, such that $\mathscr{S}$ is finitely generated by $\mathscr{S}_1$, and $\mathscr{L} = i^*(\mathscr{O}_Z(1))$, where $i : X \to P$ is the caonical immersion. Now as $\mathrm{supp}(\mathscr{G})$ is closed in X and contained in $\mathrm{supp}(\mathscr{F}) \cap \mathrm{supp}(\mathscr{H})$, it is proper over Y; the supports of $\mathscr{F}, \mathscr{G}, \mathscr{H}$ are then closed in Z. The sheaves $\mathscr{F}' = i_*(\mathscr{F})$, $\mathscr{G}' = i_*(\mathscr{G})$, and $\mathscr{H}' = i_*(\mathscr{H})$ are then coherent $\mathscr{O}_Z$-modules, and the sequence $0 \to \mathscr{F}' \to \mathscr{G}' \to \mathscr{H}' \to 0$ is exact. Moreover, if $g : Z \to Y$ is the structral morphism, then $f = g \circ i$, and it is clear that $\mathscr{F}'(n) = i_*(\mathscr{F}(n))$ and similarly for $\mathscr{G}', \mathscr{H}'$. The conclusion then follows from Corollary 10.2.14. $\square$

Remark 10.2.16. The assertion (a) of Theorem 10.2.13 is still valid if we only assume that Y is locally Noetherian. In fact, this property is local over Y, and the hypotheses inTheorem 10.2.13 imply that for any affine open $U \subseteq Y$, the restriction f to $f^{-1}(U)$ is projective and $\mathscr{L}|_{f^{-1}(U)}$ is ample for this restriction.

Remark 10.2.17. The assertion (a) of Theorem 10.2.13 is still valid, as we have seen, when we only assumes that X is a quasi-compact and quasi-separated scheme and $f : X \to Y$ is a quasi-compact and quasi-separated morphism (Proposition 9.4.45). However, note that assertion (b) is not true if we suppose that Y is the spectrum of a field k and that f is quasi-projective. For example, let $X' = \mathrm{Spec}(k[T_0, \dots, T_r])$ and X be the union of the affine opens $D(T_i)$ (so that X can be considered as the space $\mathbb{A}_k^r$ with the origin removed). As the immersion $X \to X'$ is quasi-compact, the structural morphism $f : X \to Y$ is quasi-affine, so $\mathscr{O}_X$ is very ample relative to f (Proposition 9.5.1). But the ring $\Gamma(X, \mathscr{O}_X)$ is identified with the intersection of $K[T_0, \dots, T_r]_{T_i}$ for $0 \leq i \leq r$ (Eq. (8.7.1)), which is $K[T_0, \dots, T_r]$, so it follows from (10.1.5) that we have $H^r(X, \mathscr{O}_X^{\otimes n}) = H^r(X, \mathscr{O}_X) = A \neq 0$ for any $n \in \mathbb{Z}$.

Theorem 10.2.18. *Let Y be a Noetherian scheme and $\mathscr{S}$ be a quasi-coherent $\mathscr{O}_Y$-algebra of finite type. Let $f : X \to Y$ be a projective morphism, $\mathscr{S}' = f^*(\mathscr{S})$, $\mathscr{M}$ be a quasi-coherent $\mathscr{S}'$-module of finite type.*

(a) *For any $p \geq 0$, $R^p f_*(\mathscr{M})$ is an $\mathscr{S}$-module of finite type.*

(b) *Let $\mathscr{L}$ be an f-ample $\mathscr{O}_X$-module, and put $\mathscr{M}(n) = \mathscr{M} \otimes \mathscr{L}^{\otimes n}$. Then there exists an integer n_0 such that for $n \geq n_0$, we have $R^p f_*(\mathscr{M}(n)) = 0$ for $p > 0$, and the canonical homomorphism $f^*(f_*(\mathscr{M}(n))) \to \mathscr{M}(n)$ is surjective.*

10.2.3 Applications to associated sheaves of graded modules

Theorem 10.2.19. *Let Y be a Noetherian scheme, $\mathscr{S}$ be a quasi-coherent $\mathscr{O}_Y$-algebra of finite type with positive degrees, $X = \mathrm{Proj}(\mathscr{S})$, and $q : X \to Y$ be the structural morphism. Let $\mathscr{M}$ be a quasi-coherent graded $\mathscr{S}$-module that is eventually finite. Then there exists an integer n_0 such that for $n \geq n_0$, the canonical homomorphism of ([**?**], 8.14.5.1)*

$$\alpha_n : \mathscr{M}_n \to q_*(\mathscr{P}roj_0(\mathscr{M}(n))) = q_*((\mathscr{P}roj(\mathscr{M}))_n)$$

is bijective. In other words, the canonical homomorphism $\alpha : \mathscr{M} \to \Gamma_*(\mathscr{P}roj(\mathscr{M}))$ is an eventual isomorphisms.

Proof. By Proposition 9.2.36, we can assume that $\mathscr{M}$ is an $\mathscr{S}$-module of finite type. As Y is quasi-compact, by Proposition 9.3.7 there exists an integer $d > 0$ such that $\mathscr{S}^{(d)}$ is generated by $\mathscr{S}_d$, which is of finite type. Now $\mathscr{M}$ is the direct sum of $\mathscr{M}^{(d,k)}$ for $0 \leq k < d$, and each $\mathscr{M}^{(d,k)}$ is quasi-coherent $\mathscr{S}^{(d)}$-module of finite type (**??**), so it suffices to prove that the canonical homomorphism $\alpha : \mathscr{M}^{(d,k)} \to \Gamma_*((\mathscr{P}roj(\mathscr{M}))^{(d,k)})$ is an eventual isomorphism. In view of ([**?**], 8.14.13) (and the diagram ([**?**], 8.14.13.4)), we can therefore assume that $\mathscr{S}$ is finitely generated by $\mathscr{S}_1$. As Y is Noetherian, the same reasoning as in Proposition 10.2.12 shows that we can further assume that $Y = \mathrm{Spec}(A)$, $\mathscr{S} = \widetilde{S}$, $\mathscr{M} = \widetilde{M}$, where A is a Noetherian ring, S_1 is a finitely generated A-module, and M is a finitely generated S-module.

We also note that it suffices to prove the assertion for $M = S$. In fact, in the general case, we have an exact sequence $R \to L \to M \to 0$, where L and R are direct sums of graded modules of the form $S(m)$. If the assertion is proved for $M = S$, then it also holds for $M = S(m)$, hence for L and R. Consider the commutative diagram

$$\begin{tikzcd} \widetilde{R}_n \arrow[r] \arrow[d, "\alpha_n"] & \widetilde{L}_n \arrow[r] \arrow[d, "\alpha_n"] & \widetilde{M}_n \arrow[r] \arrow[d, "\alpha_n"] & 0 \\ q_*(\widetilde{R}(n)) \arrow[r] & q_*(\widetilde{L}(n)) \arrow[r] & q_*(\widetilde{M}(n)) \arrow[r] & 0 \end{tikzcd}$$

The first and second vertical arrows are isomorphisms for $n \gg 0$, so by five lemma we conclude that the middle one is also an isomorphism for sufficiently large n, whence our assertion.

This being done, we are left to prove the theorem for $M = S$; for this, we first suppose that $S = A_0[T_0, \dots, T_r]$ (where T_i are indeterminates). In this case, the assertion follows from Proposition 10.2.8(b). In the general case, S is identified with a quotient of a ring $S' = A[T_0, \dots, T_n]$ by a graded ideal, so X is a closed subscheme of $X' = \mathbb{P}^r_A$. If $j : X \to X'$ is the canonical injection, then $j_*(\widetilde{S}(n))$ is equal to the $\mathscr{O}_{X'}$-module $(\mathscr{P}roj(\widetilde{S}))(n)$ where S is considered as a graded S'-module (Proposition 9.2.47). As $j_*(\widetilde{S}(n))$ is a eventually finite $\mathscr{O}_{X'}$-module, the canonical homomorphism $\alpha_n : S_n \to \Gamma(X', j_*(\widetilde{S}(n)))$ is bijective for $n \gg 0$, and this proves our assertion since $\Gamma(X', j_*(\widetilde{S}(n))) = \Gamma(X, \widetilde{S}(n))$. □

Corollary 10.2.20. *Under the hypotheses of Theorem 10.2.19, let $\mathscr{S}_X = \bigoplus_{n\in\mathbb{Z}} \mathscr{O}_X(n)$, and $\mathscr{F}$ be a quasi-coherent graded $\mathscr{S}_X$-module of finite type. Then $\Gamma_*(\mathscr{F})$ is eventually finite.*

Proof. We see that in the proof of Theorem 10.2.19, X is isomorphic to $\mathrm{Proj}(\mathscr{S}^{(d)})$ which is of finite type over Y. It then follows from ([**?**], 8.14.9) that $\mathscr{F}$ is isomorphic to a graded $\mathscr{S}_X$-module of the form $\mathscr{P}roj_0(\mathscr{M})$, where $\mathscr{M}$ is a quasi-coherent $\mathscr{S}$-module of finite type. In view of Theorem 10.2.19, we see $\Gamma_*(\mathscr{F})$ is eventually isomorphic to $\mathscr{M}$, so is eventually finite. □

Remark 10.2.21. Let Y be a Noetherian scheme, $\mathscr{S}$ be a graded $\mathscr{O}_Y$-alegbra satisfying the conditions of Theorem 10.2.19, and $X = \mathrm{Proj}(\mathscr{S})$. Let $\mathcal{K}_{\mathscr{S}}$ be the abelian category of quais-coherent graded $\mathscr{S}$-modules that are eventually finite, and $\mathcal{K}'_{\mathscr{S}}$ be the subcategory of $\mathcal{K}_{\mathscr{S}}$ consists of quais-coherent graded $\mathscr{S}$-modules that are eventually null. Finally, let $\mathcal{K}_X$ be the category of quasi-coherent graded $\mathscr{S}_X$-module of finite type (which amounts to saying, since $\mathscr{S}_X$ is periodic by ([**?**], 8.14.4) and ([**?**], 8.14.12), that each $\mathscr{F}_i$ is a coherent $\mathscr{P}_X$-module). Then in view of ([**?**], 8.14.8), ([**?**], 8.14.10) and Theorem 10.2.19, the functors $\mathscr{M} \mapsto \mathscr{P}roj(\mathscr{M})$ and $\mathscr{F} \mapsto \Gamma_*(\mathscr{F})$ define an equivalence from the quotient category $\mathcal{K}_{\mathscr{S}}/\mathcal{K}'_{\mathscr{S}}$ to the category $\mathcal{K}_X$. If $\mathscr{S}$ is generated by $\mathscr{S}_1$, we can also replace $\mathcal{K}_X$ by the category of coherent $\mathscr{O}_X$-modules.

Proposition 10.2.22. *Let Y be a Noetherian scheme.*

(a) *Let $\mathscr{S}$ be a quasi-coherent graded $\mathscr{O}_Y$-algebra of finite type with positive degrees, $X = \mathrm{Proj}(\mathscr{S})$, and $\mathscr{S}_X = \mathscr{P}roj(\mathscr{S}) = \bigoplus_{n\in\mathbb{Z}} \mathscr{O}_X(n)$. Then $\mathscr{S}_X$ is a periodic graded $\mathscr{O}_X$-algebra whose homogeneous components $(\mathscr{S}_X)_n = \mathscr{O}_X(n)$ are coherent $\mathscr{O}_X$-modules. If $d > 0$ is a period of $\mathscr{S}_X$, then $(\mathscr{S}_X)_d = \mathscr{O}_X(d)$ is an invertible $\mathscr{O}_X$-module that is Y-ample. Moreover, the canonical homomorphism $\alpha : \mathscr{S} \to \Gamma_*(\mathscr{S}_X)$ is a eventual isomorphism.*

(b) *Conversely, let $q : X \to Y$ be a projective morphism, and $\mathscr{S}'$ be a graded $\mathscr{O}_X$-algebra whose homogeneous components $\mathscr{S}'_n$ are coherent $\mathscr{O}_X$-modules, and that admits a period $d > 0$ such that $\mathscr{S}'_d$ is an invertible $\mathscr{O}_X$-module that is Y-ample. Then $\mathscr{S} = \bigoplus_{n\geq 0} q_*(\mathscr{S}'_n)$ is a quasi-coherent graded $\mathscr{O}_Y$-algebra of finite type with positive degrees, and there exists a Y-isomorphism $i : X \cong \mathrm{Proj}(\mathscr{S})$ such that $i^*(\mathscr{P}roj(\mathscr{S}))$ is isomorphic to $\mathscr{S}'$.*

Proposition 10.2.23. *Let Y be a Noetherian scheme, $q : X \to Y$ be a projective morphism, and $\mathscr{L}$ be an invertible $\mathscr{O}_X$-module that is very ample for q. Then $\mathscr{S} = \bigoplus_{n\geq 0} q_*(\mathscr{L}^{\otimes n})$ is a quasi-coherent $\mathscr{O}_Y$-algebra of finite type such that $\mathscr{S}_n = \mathscr{S}_1^n$ for $n \gg 0$, and there exists an Y-isomorphism $r : X \cong P = \mathrm{Proj}(\mathscr{S})$ such that $\mathscr{S} \cong r^*(\mathscr{O}_P(1))$.*

Proposition 10.2.24. *Let Y be a Noetherian integral scheme, X be an integral scheme, and $f : X \to Y$ be a projective birational morphism. Then there exists a coherent fractional ideal $\mathscr{I} \subseteq \mathscr{K}_Y$ such that X is Y-isomorphic to blow up Y-scheme relative to $\mathscr{I}$. Moreover, there exists an open subset $U \subseteq Y$ such that the restriction $f : f^{-1}(U) \to U$ is an isomorphic, and $\mathscr{I}|_U$ is invertible.*

Proof. By Proposition 9.5.18, there exists an invertible $\mathscr{O}_X$-module $\mathscr{L}$ that is very ample for f, so we can apply Proposition 10.2.23, hence identify X with $\mathrm{Proj}(\mathscr{S})$, where $\mathscr{S} = \bigoplus_{n\geq 0} f_*(\mathscr{L}^{\otimes n})$. We also see that each $f_*(\mathscr{L}^{\otimes n})$ is a torsion-free $\mathscr{O}_Y$-module (Proposition 8.7.26), so $\mathscr{S}$ is canonically identified with a sub-$\mathscr{O}_Y$-module of $\mathscr{S} \otimes_{\mathscr{O}_Y} \mathscr{K}_Y$. By Corollary 8.7.20 this sheaf is simple, so is completely determined when we know its restriction to any non-empty open set, for example to a open nonempty $U' \subseteq U$ (here $U \subseteq Y$ is an open subset such that f is an isomorphic on $f^{-1}(U)$) such that $\mathscr{L}|_{f^{-1}(U')}$ is isomorphic to $\mathscr{O}_X|_{f^{-1}(U')}$. As by hypotheses $f_*(\mathscr{L}^{\otimes n})|_{U'}$ is then isomorphic to $\mathscr{O}_Y|_{U'}$, we conclude that $\mathscr{S} \otimes \mathscr{K}_Y$ is an $\mathscr{K}_Y$-module isomorphic to $\mathscr{K}_Y[T]$, where T is an indeterminate, and $\mathscr{S}$ is eventually isomorphic to the sub-$\mathscr{O}_Y$-module generated by the canonical image of $f_*(\mathscr{L})$ in $\mathscr{S} \otimes_{\mathscr{O}_Y} \mathscr{K}_Y$ (Proposition 10.2.23). But if we identify $\mathscr{S} \otimes_{\mathscr{O}_Y} \mathscr{K}_Y$ with $\mathscr{K}_Y[T]$, then the image of $f_*(\mathscr{L})$ is identified with $\mathscr{J} \cdot T$, where $\mathscr{J}$ is a coherent sub-$\mathscr{O}_Y$-module of $\mathscr{K}_Y$ (Theorem 10.2.13), and its restriction to U' is isomorphic to $\mathscr{O}_Y|_{U'}$. We therefore conclude that $\mathscr{J}|_U$ is invertible and $\mathscr{S}$ is eventually isomorphic to $\bigoplus_{n\geq 0} \mathscr{J}^n$, whence our assertion. □

Corollary 10.2.25. *Under the hypothesis of Proposition 10.2.24, suppose that for any nontrivial coherent sub-$\mathscr{O}_Y$-module $\mathscr{J}$ of $\mathscr{K}_Y$, there exists an invertible $\mathscr{O}_Y$-module $\mathscr{L}$ such that*

$$\Gamma(Y, \mathscr{L} \otimes_{\mathscr{O}_Y} \mathcal{H}om_{\mathscr{O}_Y}(\mathscr{J}, \mathscr{O}_Y)) \neq 0.$$

Then, in the situation of Proposition 10.2.24, we can suppose that $\mathscr{J}$ is an ideal of $\mathscr{O}_Y$. This additional condition is always verified if there exists an ample $\mathscr{O}_Y$-module.

Proof. We first note that

$$\mathscr{L} \otimes_{\mathscr{O}_Y} \mathcal{H}om_{\mathscr{O}_Y}(\mathscr{J}, \mathscr{O}_Y) = \mathcal{H}om_{\mathscr{O}_Y}(\mathscr{L}^{-1}, \mathcal{H}om_{\mathscr{O}_Y}(\mathscr{J}, \mathscr{O}_Y)) = \mathcal{H}om_{\mathscr{O}_Y}(\mathscr{J} \otimes_{\mathscr{O}_Y} \mathscr{L}^{-1}, \mathscr{O}_Y)$$

so the hypothesis signifies that there is a nonzero homomorphism $u : \mathscr{J} \otimes \mathscr{L}^{-1} \to \mathscr{O}_Y$. Now for any $y \in Y$, $(\mathscr{J} \otimes \mathscr{L}^{-1})_y$ is identified with a sub-$\mathscr{O}_{Y,y}$-module of the fraction field $\mathscr{K}_{Y,y}$ of $\mathscr{O}_{Y,y}$, so u_y is necessarily injective, and u is therefore an isomorphism from $\mathscr{J} \otimes \mathscr{L}^{-1}$ onto an ideal $\mathscr{J}'$ of $\mathscr{O}_Y$. As the blow up Y-scheme relative to $\mathscr{J}$ and $\mathscr{J} \otimes \mathscr{L}^{-1}$ are isomorphic (Remark 9.3.19), this proves the corollary. The last remark is a direct concequence of Proposition 9.4.29. □

Corollary 10.2.26. *Let X and Y be integral schemes that are projective over a field k, and $f : X \to Y$ be a birational k-morphism. Then X is k-isomorphic to a blow up Y-scheme relative to a closed subscheme Y' (not necessarily reduced) of Y.*

Proof. In fact, f is projective by Proposition 9.5.34, and as Y is projective over k, the condition of Corollary 10.2.25 is satisfied. It then suffices to consider the closed subscheme of Y defined by the coherent ideal $\mathscr{J}$. □

10.2.4 Euler characteristic and Hilbert polynomial

Let A be an Artinian ring, X be a projective scheme over $Y = \mathrm{Spec}(A)$. For any coherent $\mathscr{O}_X$-module $\mathscr{F}$, the cohomology module $H^i(X, \mathscr{F})$ is finitely generated (Theorem 10.2.13), hence is of finite length. We also see that $H^i(X, \mathscr{F}) = 0$ for $i \gg 0$, so the integer

$$\chi_A(\mathscr{F}) = \sum_{i=0}^{\infty} (-1)^i \ell_A(H^i(X, \mathscr{F}))$$

is defined for any coherent $\mathscr{O}_X$-module. If A is local Artinian, then we say that $\chi_A(\mathscr{F})$ is the Euler characteristic of $\mathscr{F}$ (over the ring A). For $\mathscr{F} = \mathscr{O}_X$, the integer $\chi_A(\mathscr{O}_X)$ is called the **arithmetic genus** of X (over A).

It is clear that the map χ_A is additive on the category of coherent $\mathscr{O}_X$-modules. That is, for an exact sequence $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$, we have

$$\chi_A(\mathscr{F}) = \chi_A(\mathscr{F}') + \chi_A(\mathscr{F}'').$$

Theorem 10.2.27. *Let A be a local Artinian ring, X be a projective scheme over $Y = \mathrm{Spec}(A)$, $\mathscr{L}$ be a very ample $\mathscr{O}_X$-module relative to Y, and $\mathscr{F}$ be a coherent $\mathscr{O}_X$-module. For $n \in \mathbb{Z}$, we set $\mathscr{F}(n) = \mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{L}^{\otimes n}$.*

(a) *There exists a polynomial $P \in \mathbb{Q}[T]$ such that $\chi_A(\mathscr{F}(n)) = P(n)$ for $n \in \mathbb{Z}$ (the polynomial P is called the Hilbert polynomial of $\mathscr{F}$ over A).*

(b) *For $n \gg 0$, we have $\chi_A(\mathscr{F}(n)) = \ell_A(\Gamma(X, \mathscr{F}(n)))$.*

(c) *The leading coefficient of $\chi_A(\mathscr{F}(n))$ is positive.*

Example 10.2.28. Let k be a field, $r > 0$ be an integer, and $X = \mathbb{P}^r_k$. Then we have $\chi_A(\mathscr{O}_X(n)) = \binom{n+r}{r}$ for $n \in \mathbb{Z}$. To see this, we divide into three cases.

- For $n > 0$, we have $\chi_A(\mathscr{O}_X(n)) = \dim_k(H^0(X, \mathscr{O}_X(n)))$, which is the number of homogeneous polynomials of degree n and is equal to $\binom{n+r}{r}$.
- For $n < -r$, we have $\chi_A(\mathscr{O}_X(n)) = (-1)^r \dim_k(H^r(X, \mathscr{O}_X(n)))$. If $n = -r - d$, the dimension of $H^r(X, \mathscr{O}_X(n))$ over k is is the number of sequences of integers $(\alpha_0, \dots, \alpha_r)$ with $\alpha_i > 0$ and $|\alpha| = r + d$ (Proposition 10.2.8), which is equal to the number $\binom{d+r-1}{r} = (-1)^r\binom{n+r}{r}$.
- For $-r \leq n \leq 0$, we have $\binom{n+r}{r} = 0$ and also $H^i(X, \mathscr{O}_X(n)) = 0$ for $i \geq 0$.

Corollary 10.2.29. *Let A be a local Artinian ring, S be a graded A-algebra of finite type generated by S_1, M be a graded S-module, and $X = \mathrm{Proj}(S)$. Then we have $\chi_A(\widetilde{M}(n)) = \ell_A(M_n)$ for $n \gg 0$.*

Proof. This follows from Theorem 10.2.19, since $H^i(X, \widetilde{M}(n)) = 0$ for $i > 0$ if $n \gg 0$. □

10.2.5 Cohomological criterion for ampleness

Proposition 10.2.30. *Let Y be a Noetherian scheme, $f : X \to Y$ be a proper morphism, and $\mathscr{L}$ be an invertible $\mathscr{O}_X$-module. For any coherent $\mathscr{O}_X$-module $\mathscr{F}$, we set $\mathscr{F}(n) = \mathscr{F} \otimes \mathscr{L}^{\otimes n}$ for $n \in \mathbb{Z}$.*

(i) *$\mathscr{L}$ is ample for f.*

(ii) *For any coherent $\mathscr{O}_X$-module $\mathscr{F}$, there exists an integer n_0 such that for $n \geq n_0$, we have $R^p f_*(\mathscr{F}(n)) = 0$ for $p > 0$.*

(iii) *For any coherent ideal $\mathscr{J}$ of $\mathscr{O}_X$, there exists an integer n_0 such that for $n \geq n_0$, we have $R^1 f_*(\mathscr{J}(n)) = 0$.*

10.3 The finiteness theorem for proper morphisms

10.3.1 The dévissage lemma

Let $\mathcal{A}$ be an abelian category. We say that a subset $\mathfrak{E}$ of objects of $\mathcal{A}$ is exact if $0 \in \mathcal{E}$, and for any exact sequence $0 \to A' \to A \to A'' \to 0$ in $\mathcal{A}$ such that two of A, A', A'' are in $\mathfrak{E}$, then the third one is in $\mathfrak{E}$. The following technique, called the *dévissage lemma*, is introduced by Alexander Grothendieck to prove statements about coherent sheaves on Noetherian schemes. One can think this method as an adaption of Noetherian induction.

Theorem 10.3.1 (Dévissage Lemma). *Let X be a Noetherian scheme, $\mathfrak{E}$ be an exact subset of the category $\mathcal{A}$ of coherent $\mathscr{O}_X$-modules, and X' be a closed subset of X. Suppose that for any irreducible subset Y of X', with generic point y, there exists a coherent $\mathscr{O}_X$-module $\mathscr{G} \in \mathfrak{E}$ such that $\mathscr{G}_y$ is a $\kappa(y)$-vector space of dimension 1. Then any coherent $\mathscr{O}_X$-module with support in X' belongs to $\mathfrak{E}$.*

Proof. Consider the following property $\mathcal{P}(Y)$ on a closed subset Y of X: any coherent $\mathscr{O}_X$-module with support contained in Y belongs to $\mathfrak{E}$. In view of the Noetherian induction principle (**??**), we are reduced to prove the following: if Y is a closed subset of X' such that $\mathcal{P}(Y')$ is valid for any proper closed subset $Y' \subseteq Y$, then $\mathcal{P}(Y)$ is valid.

Now let $\mathscr{F}$ be a coherent $\mathscr{O}_X$-module with support contained in Y, we show that $\mathscr{F} \in \mathfrak{E}$. In this case, we endow Y with the reduced subscheme structure of X, and let $\mathscr{J}$ be the ideal of $\mathscr{O}_X$ defining it. By Proposition 8.6.17, we see there exists an integer $n > 0$ such that $\mathscr{J}^n\mathscr{F} = 0$; for $1 \leq k \leq n$, we then have an exact sequence

$$0 \longrightarrow \mathscr{J}^{k-1}\mathscr{F}/\mathscr{J}^k\mathscr{F} \longrightarrow \mathscr{F}/\mathscr{J}^k\mathscr{F} \longrightarrow \mathscr{F}/\mathscr{J}^{k-1}\mathscr{F} \longrightarrow 0$$

of coherent $\mathscr{O}_X$-modules. As $\mathfrak{E}$ is exact, by recurrence on k, it suffices to prove that $\mathscr{F}_k = \mathscr{J}^{k-1}\mathscr{F}/\mathscr{J}^k\mathscr{F}$ belongs to $\mathfrak{E}$; in other words, we may also assume that $\mathscr{J}\mathscr{F} = 0$, which means $\mathscr{F} = j_*(j^*(\mathscr{F}))$, where $j : Y \to X$ is the canonical injection.

First suppose that Y is reducible, and let $Y = Y' \cup Y''$, where Y', Y'' are proper closed subsets of Y. We endow Y', Y'' with the reduced subscheme structure, and let $\mathscr{J}', \mathscr{J}''$ be the defining ideals of $\mathscr{O}_X$. Put $\mathscr{F}' = \mathscr{F} \otimes_{\mathscr{O}_X} (\mathscr{O}_X/\mathscr{J}')$ and $\mathscr{F}'' = \mathscr{F} \otimes_{\mathscr{O}_X} (\mathscr{O}_X/\mathscr{J}'')$. The canonical homomorphisms $\mathscr{F} \to \mathscr{F}'$, $\mathscr{F} \to \mathscr{F}''$ then define a homomorphism $u : \mathscr{F} \to \mathscr{F} \oplus \mathscr{F}''$ such that any point $z \notin Y' \cap Y''$, the induced homomorphism $u_z : \mathscr{F} \to \mathscr{F}'_z \oplus \mathscr{F}''_z$ is bijective. In fact, we have $\mathscr{J}' \cap \mathscr{J}'' = \mathscr{J}$; if $z \notin Y''$ then $\mathscr{J}'_z = \mathscr{J}_z$, so $\mathscr{F}'_z = \mathscr{F}_z$ and $\mathscr{F}''_z = 0$; and similarly if $z \notin Y'$. The kernel and cokernel of u, which belong to $\mathcal{A}$, are then supported in $Y' \cap Y''$, and hence belong to $\mathfrak{C}$ by hypothesis. By the same reasoning, $\mathscr{F}'$ and $\mathscr{F}''$ are in $\mathfrak{C}$, hence so is $\mathscr{F}' \oplus \mathscr{F}''$. We now conclude from the following exact sequences

$$0 \longrightarrow \ker u \longrightarrow \mathscr{F} \longrightarrow \operatorname{im} u \longrightarrow 0$$

$$0 \longrightarrow \operatorname{im} u \longrightarrow \mathscr{F}' \oplus \mathscr{F}'' \longrightarrow \operatorname{coker} u \longrightarrow 0$$

that $\mathscr{F}$ belongs to $\mathfrak{C}$.

If on the other hand Y is irreducible, then the subscheme Y of X is integral. If y is the generic point of Y, we have $\mathscr{O}_{Y,y} = \kappa(y)$, and as $j^*(\mathscr{F})$ is a coherent $\mathscr{O}_Y$-module, $\mathscr{F}_y = (j^*(\mathscr{F}))_y$ is a $\kappa(y)$-vector space of finite dimension m. By hypothesis, there exists a coherent $\mathscr{O}_X$-module $\mathscr{G} \in \mathfrak{C}$ (necessarily supported in Y) such that $\mathscr{G}_y$ is a $\kappa(y)$-vector space of dimension 1, so there is an $\kappa(y)$-isomorphism $(\mathscr{G}_y)^m \cong \mathscr{F}_y$, which is also an $\mathscr{O}_Y$-isomorphism, and as $\mathscr{G}^m$ and $\mathscr{F}$ are coherent, there exists an open neighborhood W of y in X and an isomorphism $\mathscr{G}^m|_W \cong \mathscr{F}|_W$ (**??**). Let $\mathscr{H}$ be the graph of this isomorphism, which is a coherent sub-$(\mathscr{O}_X|_W)$-module of $(\mathscr{G}^m \oplus \mathscr{F})|_W$. Then there is a sub-$\mathscr{O}_X$-module $\mathscr{H}_0$ of $\mathscr{G}^m \oplus \mathscr{F}$, inducing $\mathscr{H}$ over W and zero over $X - Y$ (**??**). The restrictions $v : \mathscr{H}_0 \to \mathscr{G}^m$ and $w : \mathscr{H}_0 \to \mathscr{F}$ of the canonical projections of $\mathscr{G}^m \oplus \mathscr{F}$ are then homomorphisms of coherent $\mathscr{O}_X$-modules which are isomorphic over W and over $X - Y$. The kernel and cokernels of these homomorphisms are then supported in the proper closed subset $Y - (Y \cap W)$ of Y, so by hypotheses they belong to $\mathfrak{C}$. On the other hand, we have $\mathscr{G}^m \in \mathfrak{C}$ since $\mathscr{G} \in \mathfrak{C}$, so by the exactness of $\mathfrak{C}$ we conclude that $\mathscr{H}_0 \in \mathfrak{C}$, hence $\mathscr{F} \in \mathfrak{C}$. □

Corollary 10.3.2. *Suppose that the exact subset $\mathfrak{C}$ satisfies the additional property that any direct factor in $\mathcal{A}$ of a coherent $\mathscr{O}_X$-module $\mathscr{F} \in \mathfrak{C}$ belongs to $\mathfrak{C}$. Then the conclusion of Theorem 10.3.1 is still valid if we replace "$\mathscr{G}_y$ is a $\kappa(y)$-vector space of dimension 1" by the condition that $\mathscr{G}_y \neq 0$ (or equivalently $\operatorname{supp}(\mathscr{G}) = Y$).*

Proof. In fact, in this case the proof of Theorem 10.3.1 when Y is irreducible can be modified as follows: $\mathscr{G}_y$ is a $\kappa(y)$-vector space of dimension $q > 0$, and we then have an isomorphism $(\mathscr{G}_y)^m \cong (\mathscr{F}_y)^q$. By the same reasoning, we obtain that $\mathscr{F}^q \in \mathfrak{C}$, so $\mathscr{F} \in \mathfrak{C}$ by our additional assumption on $\mathfrak{C}$. □

10.3.2 The finiteness theorem for proper morphisms

Theorem 10.3.3. *Let Y be a locally Noetherian scheme and $f : X \to Y$ be a proper morphism. Then for any coherent $\mathscr{O}_X$-module $\mathscr{F}$, the $\mathscr{O}_Y$-modules $R^p f_*(\mathscr{F})$ are coherent for $p \geq 0$.*

Proof. Since this question is local over Y, we can suppose that Y is Noetherian, and hence X is Noetherian (Proposition 8.6.20). The coherent $\mathscr{O}_X$-module $\mathscr{F}$ satisfying the conclusion of Theorem 10.3.3 is easily seen to form an exact subset $\mathfrak{C}$ of the category $\mathcal{A}$ of coherent $\mathscr{O}_X$-modules. In fact, let $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ be an exact sequence of coherect $\mathscr{O}_X$-modules, and suppose for example that $\mathscr{F}'$, $\mathscr{F}$ belong to $\mathfrak{C}$. Then we have an exact sequence

$$R^{p-1} f_*(\mathscr{F}'') \xrightarrow{\partial} R^p f_*(\mathscr{F}) \longrightarrow R^p f_*(\mathscr{F}) \longrightarrow R^p f_*(\mathscr{F}'') \xrightarrow{\partial} R^{p+1} f_*(\mathscr{F}')$$

in which the four outer terms are coherent. It then follows from **??** that $R^p f_*(\mathscr{F})$ is coherent. We also note that any direct factor $\mathscr{F}' \in \mathcal{A}$ of a coherent $\mathscr{O}_X$-module $\mathscr{F} \in \mathfrak{C}$ belongs to $\mathfrak{C}$: in fact, $R^p f_*(\mathscr{F}')$ is then a direct factor of $R^p f_*(\mathscr{F})$ (G, II 4.4.4), so is of finite type, and since it is quasi-coherent (Proposition 10.1.15), it is coherent (note that Y is Noetherian). In view of Corollary 10.3.2, we only need to prove that if X is irreducible with generic point x, then there exists a coherent $\mathscr{O}_X$-module $\mathscr{F} \in \mathfrak{C}$ such that $\mathscr{F}_x \neq 0$. In fact, for any irreducible closed subscheme Y of X with canonical injection $j : Y \to X$, the composition $f \circ j$ is proper (Proposition 9.5.23), and if $\mathscr{G}$ is a coherent $\mathscr{O}_Y$-module supported in Y, then $j_*(\mathscr{G})$ is a coherent $\mathscr{O}_X$-module such that $R^p(f \circ j)_*(\mathscr{G}) = R^p f_*(j_*(\mathscr{G}))$ (G, II, 4.9.1), so we can apply our result on Y.

By Chow's lemma (Theorem 9.5.39), there exists an irreducible scheme X' and a surjective projective morphism $g : X' \to X$ such that $f \circ g : X' \to Y$ is projective. By Proposition 9.5.18, there exists a g-ample $\mathscr{O}_{X'}$-module $\mathscr{L}$, so by Theorem 10.2.13 we see that there exists an integer n_0 such that $\mathscr{F} = g_*(\mathscr{O}_{X'}(n))$

is a coherent $\mathscr{O}_X$-module and $R^pg_*(\mathscr{O}_{X'}(n)) = 0$ for $p > 0$ and $n \geq n_0$. Moreover, if x (resp. x') is the generic point of X (resp. X'), there exists an open subset U of x such that g is an isomorphism from $g^{-1}(U)$ onto U, so $\mathscr{F}_x \cong (\mathscr{O}_{X'}(n))_{x'} \neq 0$. On the other hand, since $f \circ g$ is projective and Y is Noetherian, the $R^p(f \circ g)(\mathscr{O}_{X'}(n))$ are coherent by Theorem 10.2.13. Consider the Grothendieck's spectral sequence:

$$E_2^{p,q} = R^pf_*(R^qg_*(\mathscr{O}_{X'}(n))) \Rightarrow R^*(f \circ g)_*(\mathscr{O}_{X'}(n)).$$

We have already remarked that for $n \gg 0$ we have $R^qg_*(\mathscr{O}_{X'}(n)) = 0$ for $q > 0$, so this sequence collapse at E_2 and we obtain an isomorphism $E_2^{p,0} = R^pf_*(\mathscr{F}) \cong R^p(f \circ g)_*(\mathscr{O}_{X'}(n))$, which implies $\mathscr{F} \in \mathfrak{C}$ and completes the proof. □

Corollary 10.3.4. *Let A be a Noetherian ring, X be a proper scheme over A. For any coherent $\mathscr{O}_X$-module $\mathscr{F}$, the $H^p(X, \mathscr{F})$ are finitely generated A-modules, and there exists an integer r such that $H^p(X, \mathscr{F}) = 0$ for any coherent $\mathscr{O}_X$-module $\mathscr{F}$ and $p > r$.*

Proof. The second assertion is proved in Proposition 10.2.12, and the first one follows from Theorem 10.3.3, in view of Corollary 10.1.16. □

Corollary 10.3.5. *Let Y be a locally Noetherian scheme, $f : X \to Y$ be a morphism of finite type. For any coherent $\mathscr{O}_X$-module with support proper over Y, the $\mathscr{O}_Y$-modules $R^pf_*(\mathscr{F})$ are coherent for $p \geq 0$.*

Proof. Since this question is local over Y, we may assume that Y is Noetherian, and then so is X. By hypothesis, any closed subscheme Z of X with underlying space $\mathrm{supp}(\mathscr{F})$ is proper over Y, so if $j : Y \to X$ is the canonical injection, $f \circ j$ is proper. Now we may choose Z such that $\mathscr{F} = j_*(\mathscr{G})$, where $\mathscr{G} = j^*(\mathscr{F})$ is a coherent $\mathscr{O}_Z$-module (Corollary 8.6.18). Since we have $R^pf_*(\mathscr{F}) = R^p(f \circ j)_*(\mathscr{G})$ by Corollary 10.1.7, the conclusion follows from Theorem 10.3.3. □

Proposition 10.3.6. *Let Y be a Noetherian scheme, $\mathscr{S}$ be a quasi-coherent graded $\mathscr{O}_Y$-algebra of finite type with positive degrees, $Y' = \mathrm{Proj}(\mathscr{S})$, and $g : Y' \to Y$ be the structural morphism. Let $f : X \to Y$ be a proper morphism, $\mathscr{S}' = f^*(\mathscr{S})$, and $\mathscr{M}$ be a quasi-coherent graded $\mathscr{S}'$-module of finite type. Then the $R^pf_*(\mathscr{M})$ are graded $\mathscr{S}$-modules of finite type for $p \geq 0$. Suppose moreover that $\mathscr{S}$ is generated by $\mathscr{S}_1$, then for any $p \in \mathbb{N}$, there exists an integer k_p such that for $k \geq k_p$ and $r \geq 0$, we have*

$$R^pf_*(\mathscr{M}_{k+r}) = \mathscr{S}_rR^pf_*(\mathscr{M}_k). \tag{10.3.1}$$

Proof. The first assertion follows from Theorem 10.2.18 since in its proof the condition on f is only used to derive the coherence of the $\mathscr{O}_{Y'}$-modules $R^pf'_*(\widetilde{\mathscr{M}})$. With the hypotheses of Proposition 10.3.6, f' is proper (Proposition 9.5.23(iii)), so we can now utilize Theorem 10.3.3 to complete the proof. □

Corollary 10.3.7. *Let A be a Noetherian ring, $\mathfrak{I}$ be an ideal of A, X be a proper A-scheme, and $\mathscr{F}$ be a coherent $\mathscr{O}_X$-module. Then for any $p \geq 0$, the direct sum $\bigoplus_{k\geq 0} H^p(X, \mathfrak{I}^k\mathscr{F})$ is a finitely generated module over $S = \bigoplus_{k\geq 0} \mathfrak{I}^k$. In particular, there exists an integer $k_p \geq 0$ such that for $k \geq k_p$, $r \geq 0$, we have*

$$H^p(X, \mathfrak{I}^{k+r}\mathscr{F}) = \mathfrak{I}^rH^p(X, \mathfrak{I}^k\mathscr{F}). \tag{10.3.2}$$

Proof. It suffices to apply 10.3.6 to $Y = \mathrm{Spec}(A)$, $\mathscr{S} = \widetilde{S}$, and $\mathscr{M}_k = \mathfrak{I}^k\mathscr{F}$. □

10.3.3 Generalization to formal schemes

10.4 Zariski's main theorem and applications

10.4.1 Grothendieck's comparison theorem

Let X, Y be Noetherian schemes, $f : X \to Y$ be a proper morohism, Y' be a closed subscheme of Y, and X' be the inverse image $f^{-1}(Y')$. We denote by $\widehat{X}$ and $\widehat{Y}$ the formal completion $X_{/X'}$ and $Y_{/Y'}$ of X and Y along X' and Y', respectively, and let $\hat{f} : \widehat{X} \to \widehat{Y}$ be the extension of f to completions. For any coherent $\mathscr{O}_X$-module $\mathscr{F}$, let $\widehat{\mathscr{F}}$ be the completion $\mathscr{F}_{/X'}$ of $\mathscr{F}$ along X', which is coherent by Proposition 8.8.28.

Let $\mathscr{J}$ be a coherent ideal of $\mathscr{O}_Y$ defining Y', then by Proposition 8.4.16(b) the coherent ideal $\mathscr{K} = f^*(\mathscr{J})\mathscr{O}_X$ defines the closed subscheme X' of X. For each $k \geq 0$, we consider the coherent $\mathscr{O}_X$-module

$$\mathscr{F}_k = \mathscr{F} \otimes_{\mathscr{O}_Y} (\mathscr{O}_Y/\mathscr{J}^{k+1}) = \mathscr{F}/\mathscr{K}^{k+1}\mathscr{F}.$$

The $\mathscr{O}_Y$-modules $R^pf_*(\mathscr{F})$ and $R^pf_*(\mathscr{F}_k)$ are coherent for $p \geq 0$ (Theorem 10.3.3). For any $k \geq 0$ and $p \geq 0$, the canonical homomorphism $\mathscr{F} \to \mathscr{F}_k$ defines by functoriality a homomorphism

$$R^pf_*(\mathscr{F}) \to R^pf_*(\mathscr{F}_k) \tag{10.4.1}$$

Moreover, as $\mathscr{F}_k$ is an $\mathscr{O}_X/\mathscr{K}^{k+1}$-module, $R^pf_*(\mathscr{F}_k)$ is an $\mathscr{O}_Y/\mathscr{J}^{k+1}$-module ([?] 0_{III}, 12.2.1) and we then deduce from (10.4.1) a homomorphism

$$R^pf_*(\mathscr{F}) \otimes_{\mathscr{O}_Y} (\mathscr{O}_Y/\mathscr{J}^{k+1}) \to R^pf_*(\mathscr{F}_k). \tag{10.4.2}$$

The two sides of (10.4.2) form two projective systems, and the projective limit of the left side is just the completion $(R^pf_*(\mathscr{F}))_{/Y'}$, which we also denote by $\widehat{R^pf_*(\mathscr{F})}$. Moreover, it is immediate that the homomorphisms in (10.4.2) form a projective system, so by passing to projective limit we obtain a canonocal homomorphism

$$\varphi_p : \widehat{R^pf_*(\mathscr{F})} \to \varprojlim_k R^pf_*(\mathscr{F}_k). \tag{10.4.3}$$

Since (10.4.2) is a homomorphism of $(\mathscr{O}_Y/\mathscr{J}^{k+1})$-modules, and can be considered as a continuous homomorphism of discrete $\mathscr{O}_{\widehat{Y}}$-modules, we see that the homomorphism φ_p is a continuous homomorphism of topological $\mathscr{O}_Y$-modules.

On the other hand, let $i_X : \widehat{X} \to X$ be the canonical morphism, which fits into the commutative diagram

$$\begin{tikzcd} X_k \arrow[r, "h_k"] \arrow[dr, "i_k"'] & \widehat{X} \arrow[d, "i_X"] \\ & X \end{tikzcd} \tag{10.4.4}$$

where X_k is the closed subscheme of X defined by the ideal $\mathscr{K}^{k+1}$, $i_k : X_k \to X$ is the canonical injection, and h_k is given the canonical homomorphism $\mathscr{O}_{\widehat{X}} \to \mathscr{O}_X/\mathscr{K}^{k+1}$. By Proposition 8.8.28, we have $\widehat{\mathscr{F}} = i_X^*(\mathscr{F})$. Since $\mathscr{F}_k = (i_k)_*(i_k^*(\mathscr{F}_k))$ (G, II, 4.9.1), we see that

$$H^p(X_k, i_k^*(\mathscr{F}_k)) = H^p(X, \mathscr{F}_k). \tag{10.4.5}$$

The canonical homomorphism $H^p(\widehat{X}, \widehat{\mathscr{F}}) \to H^p(X_k, h_k^*(\widehat{\mathscr{F}}))$ can then by identified as the following homomorphism:

$$H^p(\widehat{X}, \widehat{\mathscr{F}}) \to H^p(X, \mathscr{F}_k). \tag{10.4.6}$$

These homomorphisms evidently form a projective system, so by passing to projective limit we obtain a canonical homomorphism

$$\psi_{p,X} : H^p(\widehat{X}, \widehat{\mathscr{F}}) \to \varprojlim_k H^p(X, \mathscr{F}_k). \tag{10.4.7}$$

Now in view of Corollary 10.1.16, by replacing X with open subsets of the form $f^{-1}(V)$, where V is an affine open of Y, we also obtain homomorphisms

$$\psi_{p,V} : H^p(\widehat{X} \cap f^{-1}(V), \widehat{\mathscr{F}}) \to \varprojlim_k \Gamma(V, R^pf_*(\mathscr{F}_k)).$$

It is clear that the homomorphisms $\psi_{p,V}$ are compatible with restrictions, so by shefification, we obtain an induced canonical homomorphism

$$\psi_p : R^p\hat{f}_*(\widehat{\mathscr{F}}) \to \varprojlim_k R^pf_*(\mathscr{F}_k). \tag{10.4.8}$$

Finally, let $i_Y : \widehat{Y} \to Y$ be the canonical morphism; as $R^pf_*(\mathscr{F})$ is a coherent $\mathscr{O}_Y$-module by Theorem 10.3.3, we have $i_Y^*(R^pf_*(\mathscr{F})) = \widehat{R^pf_*(\mathscr{F})}$ (Proposition 8.8.28), and therefore a canonical homomorphism

$$\rho_p : \widehat{R^pf_*(\mathscr{F})} = i_Y^*(R^pf_*(\mathscr{F})) \to R^p\hat{f}_*(i_X^*(\mathscr{F})) = R^p\hat{f}_*(\widehat{\mathscr{F}}), \tag{10.4.9}$$

which is defined in the same way as the canonical homomorphism (**??**). From the commutative diagram (10.4.4), we then obtain a commutative diagram

$$\begin{tikzcd} \widehat{R^p f_*(\mathscr{F})} \arrow[rr, "\rho_p"] \arrow[dr, "\varphi_p"'] & & R^p \hat{f}_*(\widehat{\mathscr{F}}) \arrow[dl, "\psi_p"] \\ & \varprojlim_k R^p f_*(\mathscr{F}_k) & \end{tikzcd} \tag{10.4.10}$$

Theorem 10.4.1 (Grothendieck's Comparison Theorem). *Let $f : X \to Y$ be a proper morphism between Noetherian schemes, Y' be a closed subset of Y, and $X' = f^{-1}(Y')$. Then for any coherent $\mathscr{O}_X$-module $\mathscr{F}$, $R^p \hat{f}_*(\widehat{\mathscr{F}})$ is a coherent $\mathscr{O}_{\widehat{X}}$-module and the homomorphisms in (10.4.10) are homeomorphisms for $p \geq 0$.*

The fact that ρ_p is an isomorphism signifies that the formation of $R^p f_*$ commutes with that of completion, so Theorem 10.4.1 gives a comparision result between formal geometry and algebraic geometry. We being its proof by establishing the following affine case:

Corollary 10.4.2. *Under the hypotheses of Theorem 10.4.1, assume that $Y = \operatorname{Spec}(A)$, where A is Noetherian, and $\mathscr{J} = \widetilde{\mathfrak{I}}$ is an ideal of A, so that $\mathscr{F}_k = \mathscr{F}/\mathfrak{I}^{k+1}\mathscr{F}$. Then for each $p \geq 0$, the projective system $(H^p(X, \mathscr{F}_k))_{k \geq 0}$ satisfies the Mittag-Leffler condition, and the canonical homomorphism*

$$\psi_p : H^p(\widehat{X}, \widehat{\mathscr{F}}) \to \varprojlim_k H^p(X, \mathscr{F}_k) \tag{10.4.11}$$

is an isomorphism. Moreover, the filtration on $H^p(X, \mathscr{F})$ defined by the kernel of the canonical homomorphisms $H^p(X, \mathscr{F}) \to H^p(X, \mathscr{F}_k)$ is $\mathfrak{I}$-good and the canonical homomorphisms

$$\varphi_p : \widehat{H^p(X, \mathscr{F})} \to \varprojlim_k H^p(X, \mathscr{F}_k) \tag{10.4.12}$$

is a homeomorphism (where the left side is the $\mathfrak{I}$-adic completion of $H^p(X, \mathscr{F})$).

Proof. Fix an integer $p \geq 0$, and we simplfy the notation by setting

$$H = H^p(X, \mathscr{F}), \quad H_k = H^p(X, \mathscr{F}_k), \quad R_k = \ker(H \to H_k) \subseteq H.$$

The exact sequence on cohomology

$$H^p(X, \mathfrak{I}^{k+1}\mathscr{F}) \longrightarrow H^p(X, \mathscr{F}) \longrightarrow H^p(X, \mathscr{F}_k) \xrightarrow{\partial} H^{p+1}(X, \mathfrak{I}^{k+1}\mathscr{F}) \longrightarrow H^{+1}(X, \mathscr{F})$$

shows that R_k is also the image of the homomorphism $H^p(X, \mathfrak{I}^{k+1}\mathscr{F}) \to H^p(X, \mathscr{F})$; we set

$$Q_k = \ker(H^{p+1}(X, \mathfrak{I}^{k+1}\mathscr{F}) \to H^{p+1}(X, \mathscr{F})) = \operatorname{im}(H^p(X, \mathscr{F}_k) \to H^{p+1}(X, \mathfrak{I}^{k+1}\mathscr{F})) \tag{10.4.13}$$

so that there is an exact sequence

$$0 \longrightarrow R_k \longrightarrow H \longrightarrow H_k \longrightarrow Q_k \longrightarrow 0$$

Let x be an element of $\mathfrak{I}^m$ for $m \geq 0$; the multiplication by x on $\mathfrak{I}^k\mathscr{F}$ is a homomorphism $\mathfrak{I}^k\mathscr{F} \to \mathfrak{I}^{k+m}\mathscr{F}$, and therefore gives a homomorphism

$$\mu_{x,m} : H^p(X, \mathfrak{I}^k\mathscr{F}) \to H^p(X, \mathfrak{I}^{k+m}\mathscr{F}).$$

If we denote by S the graded A-algebra $\bigoplus_{k \geq 0} \mathfrak{I}^k$, then the multiplications $\mu_{x,m}$ define over $E = \bigoplus_{k \geq 0} H^p(X, \mathfrak{I}^k\mathscr{F})$ a finitely generated graded S-module structure (Corollary 10.3.7), which is Noetherian since S is Noetherian (**??**).

We begin by showing that the submodules (R_k) define a $\mathfrak{I}$-good filtration on H. First, for any $x \in \mathfrak{I}^m$, the diagram

$$\begin{tikzcd} \mathfrak{I}^{k+1}\mathscr{F} \arrow[r, "x"] \arrow[d] & \mathfrak{I}^{k+m+1}\mathscr{F} \arrow[d] \\ \mathscr{F} \arrow[r, "x"] & \mathscr{F} \end{tikzcd}$$

is commutative, so the correponding diagram

$$\begin{tikzcd} H^p(X,\mathfrak{I}^{k+1}\mathscr{F}) \arrow[r, "\mu_{x,m}"] \arrow[d] & H^p(X,\mathfrak{I}^{k+m+1}\mathscr{F}) \arrow[d] \\ H^p(X,\mathscr{F}) \arrow[r, "\mu_{x,0}"] & H^p(X,\mathscr{F}) \end{tikzcd} \tag{10.4.14}$$

is commutative, which proves, in view of the interpretation of R_k as the image of $H^p(X,\mathfrak{I}^{k+1}\mathscr{F})$ in $H^p(X,\mathfrak{F})$, that $\mathfrak{I}^m R_k \subseteq R_{k+m}$ and that the graded S-module $R = \bigoplus_{k\geq 0} R_k$ is a quotient of the sub-S-module $M = \bigoplus_{k\geq 0} H^p(X,\mathfrak{I}^{k+1}\mathscr{F})$ of E. Since M is also a finitely generated S-module by Corollary 10.3.7, the S-module R is then finitely generated, which is equivalent to the condition that (R_k) is $\mathfrak{I}$-good (**??**).

Consider now the graded S-module $N = \bigoplus_{k\geq 0} H^{p+1}(X,\mathfrak{I}^{k+1}\mathscr{F})$, which is a finitely generated S-module in view of Corollary 10.3.7. For each $k \geq 0$, by (10.4.13) we have $Q_k \subseteq N_k$, and by replacing p by $p+1$ in the diagram (10.4.14) we see that $S_m Q_k = \mathfrak{I}^m Q_k \subseteq Q_{k+m}$. In other words, $Q = \bigoplus_{k\geq 0} Q_k$ is a graded sub-S-module of N, and is therefore finitely generated. We denote by $\alpha_m : \mathfrak{I}^m \to A$ the canonical injection, which can also be written as $S_m \to S_0$. Since $\mathfrak{I}^{k+1}\mathscr{F}_k = 0$, the A-module $H^p(X,\mathscr{F}_k)$ is annihilated by $\mathfrak{I}^{k+1}$, so Q_k, as the image of the A-homomorphism $H^p(X,\mathscr{F}_k) \to H^{p+1}(X,\mathfrak{I}^{k+1}\mathscr{F})$, is also annihilated by $\mathfrak{I}^{k+1}$. This signifies that, in the S-module Q, we have

$$\alpha_{k+1}(S_{k+1})Q_k = 0. \tag{10.4.15}$$

As Q is a finitely generated S-module, there exists an integers k_0 and d such that $Q_{k+d} = S_d Q_k$ for $k \geq k_0$ (**??**); we then deduce from this relation and (10.4.15) that there exists an integer $r > 0$ such that

$$\alpha_r(S_r)Q = 0. \tag{10.4.16}$$

Now note that the the canonical injection $\mathfrak{I}^{k+m}\mathscr{F} \to \mathfrak{I}^k\mathscr{F}$ gives by passing to cohomology an A-homomorphism

$$\nu_m : H^{p+1}(X,\mathfrak{I}^{k+m}\mathscr{F}) \to H^{p+1}(X,\mathfrak{I}^k\mathscr{F})$$

and, for any $x \in \mathfrak{I}^m$, we have evidently a factorization

$$\mu_{x,0} : H^{p+1}(X,\mathfrak{I}^k\mathscr{F}) \xrightarrow{\mu_{x,m}} H^{p+1}(X,\mathfrak{I}^{k+m}\mathscr{F}) \xrightarrow{\nu_m} H^{p+1}(X,\mathfrak{I}^k\mathscr{F})$$

from which we conclude that, for any sub-A-module P of $H^{p+1}(X,\mathfrak{I}^k\mathscr{F})$, we have, in the S-module N,

$$\nu_m(S_m P) = \alpha_m(S_m)P. \tag{10.4.17}$$

If we choose $m \geq r$ to be a multiple of d, then as $Q_{k+m} = S_m Q_k$ for $k \geq k_0$ (by our choice of d), we derive from (10.4.17) and (10.4.16) that $\nu_m(Q_{k+m}) = \alpha_m(S_m)Q_k \subseteq \alpha_r(S_r)Q_k = 0$ for $k \geq k_0$.

Consider now the commutative diagram

$$\begin{tikzcd} H^p(X,\mathscr{F}) \arrow[r] \arrow[d, equal] & H^p(X,\mathscr{F}_{k+m}) \arrow[r, "\partial"] \arrow[d] & H^{p+1}(X,\mathfrak{I}^{k+1}\mathscr{F}) \arrow[r] \arrow[d] & H^{p+1}(X,\mathscr{F}) \arrow[d] \\ H^p(X,\mathscr{F}) \arrow[r] & H^p(X,\mathscr{F}_k) \arrow[r, "\partial"] & H^{p+1}(X,\mathfrak{I}^{k+1}\mathscr{F}) \arrow[r] & H^{p+1}(X,\mathscr{F}) \end{tikzcd}$$

induced from the homomorphisms $\mathfrak{I}^{k+m+1}\mathscr{F} \to \mathfrak{I}^k\mathscr{F}$ and $\mathscr{F}_{k+m} \to \mathscr{F}_k$. From this, we conclude the following commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & R_{k+m} \arrow[r] \arrow[d] & H \arrow[r] \arrow[d, equal] & H_{k+m} \arrow[r] \arrow[d] & Q_{k+m} \arrow[r] \arrow[d, "\nu_m"] & 0 \\ 0 \arrow[r] & R_k \arrow[r] & H \arrow[r] & H_k \arrow[r] & Q_k \arrow[r] & 0 \end{tikzcd}$$

whose rows are exact. As the last vertical arrow is zero for $k \geq k_0$, the image of H_{k+m} in H_k is contained in $\ker(H_k \to Q_k) = \operatorname{im}(H \to H_k)$, whence equals to $\operatorname{im}(H \to H_k)$ by the commutativity of the diagram.

This image then equals to the image of $H_{k'}$ in H_k for $k' \geq k+m$, so the projective system $(H_k)_{k\geq 0}$ satisfies the Mittag-Leffler condition. Moreover, for any affine open U of X, we have $H^i(U,\mathscr{F}_k) = 0$ for $i > 0$, and the map $H^0(U,\mathscr{F}_{k+m}) \to H^0(U,\mathscr{F}_k)$ is surjective for $m \geq 0$ (Proposition 8.1.5). We can then apply ([**?**] 0_{III}, 13.3.1) to conclude that the canonical homomorphism $H^p(\widehat{X},\widehat{\mathscr{F}}) \to \varprojlim_k H^p(X,\mathscr{F}_k)$ is bijective for each $p \geq 0$.

Now since the projective system (H/R_k) also satisfies the Mittag-Leffler condition, taking projective limit preserves the exactness of the sequence

$$0 \longrightarrow H/R_k \longrightarrow H_k \longrightarrow Q_k \longrightarrow 0$$

As $\nu_m(Q_{k+m}) = 0$, we have $\varprojlim_k Q_k = 0$, so we obtain an isomorphism $\varprojlim_k (H/R_k) \cong \varprojlim_k H_k$. But the filtration (R_k) of H is $\mathfrak{I}$-good, so it defines the $\mathfrak{I}$-adic topology on H, and $\varprojlim_k (H/R_k)$ is therefore the $\mathfrak{I}$-adic completion of H. □

Proof of Theorem 10.4.1. We now return to the proof of Theorem 10.4.1. For any affine open V of Y, since $R^pf_*(\mathscr{F})$ is coherent, we see that $\Gamma(V,\widehat{R^pf_*(\mathscr{F})})$ is equal to the $\mathfrak{I}$-adic completion of $\Gamma(V,R^pf_*(\mathscr{F}))$, and $\Gamma(V,\varprojlim_k R^pf_*(\mathscr{F}_k))$ is equal to $\varprojlim_k \Gamma(V,R^pf_*(\mathscr{F}_k))$ ([**?**] 0_{I}, 3.2.6). The fact that φ_p is a homeomorphism then follows from Corollary 10.4.2 and Corollary 10.1.16. Similarly, we see that each $\psi_{p,V}$ is an isomorphism, so ψ_p is an isomorphism by the definition of $R^p\hat{f}_*(\widehat{\mathscr{F}})$, and hence a homeomorphism by . □

Corollary 10.4.3. *Under the hypotheses of Theorem 10.4.1, for any affine open V of Y, the canonical homomorphism*

$$H^p(\widehat{X}\cap f^{-1}(V),\widehat{\mathscr{F}}) \to \Gamma(\widehat{Y}\cap V, R^p\hat{f}_*(\widehat{\mathscr{F}}))$$

is bijective.

Proof. This follows from the isomorphism ψ_p in Theorem 10.4.1 and Corollary 10.1.16. □

Remark 10.4.4. Let $f: X \to Y$ be a morphism of finite type between Noetherian schemes, and $\mathscr{F}$ be a coherent $\mathscr{O}_Y$-module with support proper over Y. Then we see from Corollary 10.3.5 that $R^pf_*(\mathscr{F})$ is coherent for $p \geq 0$. Moreover, we can also suppose that $\mathscr{F} = j_*(\mathscr{G})$, where $\mathscr{G} = j^*(\mathscr{F})$ is a coherent $\mathscr{O}_Z$-module, Z being a closed subscheme of X with underlying space $\mathrm{supp}(\mathscr{F})$, and $j: Z \to X$ is the canonical injection (Corollary 8.6.18). If we set $\mathscr{G}_k = \mathscr{G} \otimes_{\mathscr{O}_Y} (\mathscr{O}_Y/\mathscr{J}^{k+1})$, then

$$\mathscr{G}_k = j^*(\mathscr{F}_k),\quad R^pf_*(\mathscr{F}_k) = R^p(f\circ j)_*(\mathscr{G}_k),\quad R^pf_*(\mathscr{F}) = R^p(f\circ j)_*(\mathscr{G})$$

by Corollary 10.1.7. On the other hand, in view of Proposition 8.8.36, we also have

$$R^p\hat{f}(\widehat{\mathscr{F}}) = R^p\widehat{(f\circ j)}_*(\mathscr{G}).$$

We can then apply Theorem 10.4.1 to conclude that the result of Theorem 10.4.1 is also valid for $\mathscr{F}$ and f.

Proposition 10.4.5 (Formal Function Theorem). *Let Y be a locally Noetherian scheme, $f: X \to Y$ be a proper morphism, and $\mathscr{F}$ be a coherent $\mathscr{O}_X$-module. Then for any $y \in Y$ and $p \geq 0$, the $\mathscr{O}_{Y,y}$-module $(R^pf_*(\mathscr{F}))_y$ is finitely generated, separated for the $\mathfrak{m}_y$-adic topology, and we have a homeomorphism*

$$(\widehat{R^pf_*(\mathscr{F})})_y \xrightarrow{\sim} \varprojlim_k H^p(f^{-1}(y),\mathscr{F}\otimes_{\mathscr{O}_Y}(\mathscr{O}_{Y,y}/\mathfrak{m}_y^k))$$

where the left side is the $\mathfrak{m}_y$-adic completion of $(R^pf_*(\mathscr{F}))_y$ and $f^{-1}(y)$ is considered as the underlying space of $X\times_Y \mathrm{Spec}(\mathscr{O}_{Y,y}/\mathfrak{m}_y^k)$ (Proposition 8.4.16).

Proof. As $\mathscr{O}_{Y,y}$ is a Noetherian local ring and $(R^pf_*(\mathscr{F}))_y$ is finitely generated by Theorem 10.3.3, the $\mathfrak{m}_y$-adic topology on $(R^pf_*(\mathscr{F}))_y$ is separated (**??**). The assertions therefore result from Corollary 10.4.2 if Y is Noetherian and the poinr y is closed, since we can then replace Y be an affine neighborhood of y and put $Y' = \{y\}$, in view of (G, II, 4.9.1). In the general case, we set

$$Y_1 = \mathrm{Spec}(\mathscr{O}_{Y,y}),\quad X_1 = X\times_Y Y_1,\quad \mathscr{F}_1 = \mathscr{F}\otimes_{\mathscr{O}_Y}\mathscr{O}_{Y_1},\quad f_1 = f\times 1_{Y_1}: X_1 \to Y_1.$$

Then Y_1 is Noetherian, f_1 is proper, and $\mathscr{F}_1$ is coherent. Let y_1 be the unique closed point in Y_1; the proposition is then valid for f_1, $\mathscr{F}_1$ and y_1. We have $\mathscr{O}_{Y_1,y_1} = \mathscr{O}_{Y,y}$, $f_1^{-1}(y_1) = f^{-1}(y)$ (Proposition 8.3.37), and the schemes $X \times_Y \operatorname{Spec}(\mathscr{O}_{Y,y}/\mathfrak{m}_y^k)$ and $X_1 \times_{Y_1} \operatorname{Spec}(\mathscr{O}_{Y_1,y_1}/\mathfrak{m}_{y_1}^k)$ are canonically idnetified (Proposition 8.3.8). Moreover, $\mathscr{F}_1 \otimes_{\mathscr{O}_{Y_1}} (\mathscr{O}_{Y_1,y_1}/\mathfrak{m}_{y_1}^k)$ is identified with $\mathscr{F} \otimes_{\mathscr{O}_Y} (\mathscr{O}_{Y,y}/\mathfrak{m}_y^k)$ (Proposition 8.3.19). It then remains to see that $R^p(f_1)_*(\mathscr{F}_1)$ is canonically isomorphic to $R^p f_*(\mathscr{F}) \otimes_{\mathscr{O}_Y} \mathscr{O}_{Y_1}$, which follows from Corollary 10.1.21, since the local morphism $\operatorname{Spec}(\mathscr{O}_{Y,y}) \to Y$ is flat. □

Corollary 10.4.6. *Let Y be a locally Noetherian scheme. $f : X \to Y$ be a proper morphism, y be a point of Y, and r be the dimension of $f^{-1}(y)$. Then for any coherent $\mathscr{O}_X$-module $\mathscr{F}$, the sheaf $R^p f_*(\mathscr{F})$ vanishes in a neighborhood of y for $p > r$.*

Proof. In fact, if $p > r$, we then have $H^p(f^{-1}(y), \mathscr{F} \otimes (\mathscr{O}_{Y,y}/\mathfrak{m}_y^k)) = 0$ (by Leray's vanishing theorem) for any k, so the $\mathfrak{m}_y$-adic completion of $(R^p f_*(\mathscr{F}))_y$ is zero. As this topology is separated by Proposition 10.4.5, we then have $(R^p f_*(\mathscr{F}))_y = 0$, so the conclusion follows from **??**. □

Corollary 10.4.7. *Under the hypothesis of Proposition 10.4.5, we have a canonical homeomorphism*

$$\widehat{(f_*(\mathscr{F}))_y} \xrightarrow{\sim} \varprojlim_k \Gamma(f^{-1}(y), \mathscr{F}_y/\mathfrak{m}_y^k \mathscr{F}_y).$$

Remark 10.4.8. Many applications of Proposition 10.4.5 use only the case $p = 0$, in which case the right-hand side is equal to $\Gamma(\widehat{X}, \widehat{\mathscr{F}})$, where $\widehat{X}$ is the formal completion of X along X_y, and $\widehat{\mathscr{F}} = \mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{O}_{\widehat{X}}$. In particular, if $\mathscr{F} = \mathscr{O}_X$, we have $\Gamma(\widehat{X}, \mathscr{O}_{\widehat{X}})$, which is the ring of **formal-regular functions** (also called **holomorphic functions**) on X along X_y.

10.4.2 Zariski's connectedness theorem

Theorem 10.4.9 (Zariski's Connectednes Theorem). *Let Y be a locally Noetherian scheme, $f : X \to Y$ be a proper morphism, and*

$$X \xrightarrow{f'} Y' \xrightarrow{g} Y$$

be the Stein factorization of f. Then g is finite, f' is proper, $f'_*(\mathscr{O}_X)$ is isomorphic to $\mathscr{O}_{Y'}$, and the fibers $f'^{-1}(y')$ are nonempty and connected for any $y' \in Y'$.

Proof. Let $\theta : \mathscr{O}_{Y'} \to f'_*(\mathscr{O}_X)$ be the morphism induced by f'. Then since $g_*(\mathscr{O}_{Y'}) = \mathscr{A}(Y')$ and $f_*(\mathscr{O}_X) = \mathscr{A}(X)$, the homomorphism $g_*(\theta) : g_*(\mathscr{O}_{Y'}) \to g_*(f'_*(\mathscr{O}_X)) = f_*(\mathscr{O}_X)$ is an isomorphism, so θ is an isomorphism by Corollary 9.1.8. Since $\mathscr{A}(X)$ is coherent by Theorem 10.3.3, we see f' is finite, and it is proper by Proposition 9.5.23. It then remains to prove the following assertion: if $f : X \to Y$ is a proper morphism with $f_*(\mathscr{O}_X) \cong \mathscr{O}_Y$ and Y being locally Noetherian, then the fibers $f^{-1}(y)$ are nonempty and connected for any $y \in Y$. To this end, we note that the hypothesis $f_*(\mathscr{O}_X) \cong \mathscr{O}_Y$ implies $(f_*(\mathscr{O}_X))_y \cong \mathscr{O}_{Y,y} \neq 0$ for any $y \in Y$, so f is dominant and hence is surjective since f is closed. We may then, as in 10.4.5, reduce to the case where y is closed in Y. Then $f^{-1}(y)$ is a Noetherian space with finitely many connected components, and is equal to the underlying space of the completion $\widehat{X}$ of X along $f^{-1}(y)$. If $(Z_i)_{1\leq i\leq n}$ is its connected components, it is clear that $\Gamma(\widehat{X}, \mathscr{O}_{\widehat{X}})$ is the direct product of the rings $\Gamma(Z_i, \mathscr{O}_{\widehat{X}})$, and each of them is nonzero since the unit section is nozero at any point of $\widehat{X}$. Now if we apply Theorem 10.4.1 to $\mathscr{F} = \mathscr{O}_X$, whose completion along $f^{-1}(y)$ is $\mathscr{O}_{\widehat{X}}$, then we see that $\Gamma(\widehat{X}, \mathscr{O}_{\widehat{X}})$ is isomorphic to the $\mathfrak{m}_y$-adic completion $\widehat{\mathscr{O}_{Y,y}}$ of the local ring $\mathscr{O}_{\widehat{X}}$; this is a local ring which can not be a direct product of nonzero proper rings, since otherwise there are elements e_1, e_1 such that $e_1 + e_1 = 1$ and $e_1 e_2 = 0$. But then e_1, e_2 are nonunits so are contained in the maximal ideal, and their sum cannot be 1. We therefore have $n = 1$, which proves our assertion. □

Corollary 10.4.10. *Under the hypothesis of Theorem 10.4.9, if $f_*(\mathscr{O}_X) \cong \mathscr{O}_Y$, then the fibers $f^{-1}(y)$ are connected and nonempty for any $y \in Y$.*

Corollary 10.4.11. *Under the hypothesis of Theorem 10.4.9, for any $y \in Y$, the set of connected components of the fiber $f^{-1}(y)$ is in one-to-one correspondence to the fiber $g^{-1}(y)$ (in other words, the set of maximal ideals of $(f_*(\mathscr{O}_X))_y$).*

Proof. Since Y' is finite over Y, it has finite fiber at y (Corollary 9.6.5). As $f^{-1}(y) = f'^{-1}(g^{-1}(y))$, the corollary then follows from Theorem 10.4.9. $\square$

Remark 10.4.12. Let k be an extension field of $\kappa(y)$. If the scheme $f^{-1}(y) \otimes_{\kappa(y)} k = X \times_Y \operatorname{Spec}(k)$ is connected, so is $f^{-1}(y)$, since it is the image under a projection. For a morphism $f : X \to Y$ of scheme and a point $y \in Y$, we say that the fiber $f^{-1}(y)$ is **geometrically connected** if for any field extension k of $\kappa(y)$, the scheme $f^{-1}(y) \otimes_{\kappa(y)} k = X \times_Y \operatorname{Spec}(k)$ is connected. Under the hypothesis of Corollary 10.4.10, we can then conclude that the fibers $f^{-1}(y)$ are in fact geometrically connected. To see this, observe that for any extension k of $\kappa(y)$, there exists a Noetherian local ring A and a local hommorphism $\varphi : \mathscr{O}_{Y,y} \to A$ which is flat and such that the residue field of A is $\kappa(y)$-isomorphic to k ([**?**] 0_{III}, 10.3.1). Let $Y_1 = \operatorname{Spec}(A)$, $h : Y_1 \to Y$ be the local morphism corresponding to φ, sending the unique closed point y_1 of Y_1 to y, and put $X_1 = X \times_Y Y_1$ and $f_1 = f \times 1_{Y_1}$. Then f_1 is proper and $f_1^{-1}(y_1)$ is $\kappa(y_1)$-isomorphic to $X \times_Y \operatorname{Spec}(k)$. It then remains to show that $(f_1)_*(\mathscr{O}_{X_1}) = \mathscr{O}_{Y_1}$. Now since g is flat, we have $(f_1)_*(\mathscr{O}_{X_1}) = h^*(f_*(\mathscr{O}_X)) = h^*(\mathscr{O}_Y) = \mathscr{O}_Y$, in view of Corollary 10.1.21 applied to $p = 0$.

In the general case of Theorem 10.4.9, the same reasoning shows that we have $(f_1)_*(\mathscr{O}_{X_1}) = h^*(g_*(\mathscr{O}_{Y'}))$ (Corollary 9.1.30), and the Stein factorization $f_1 = g_1 \circ f_1'$ of f_1 is such that we have the Cartesian square

$$\begin{tikzcd}
X_1 \arrow[r, "f_1'"] \arrow[d] & Y_1' \arrow[r, "g_1"] \arrow[d] & Y_1 \arrow[d, "h"] \\
X \arrow[r, "f'"] & Y' \arrow[r, "g"] & Y
\end{tikzcd}$$

By the transitivity of fibers, we then see that the number of connected components of $f_1^{-1}(y_1)$ is, in view of Corollary 10.4.11, equal to the number of elements of $g_1^{-1}(y_1) = g^{-1}(y) \otimes_{\kappa(y)} k$. If we choose k to be an algebraic closure of $\kappa(y)$, this is also the number of geometric points of $g^{-1}(y)$, or the sum of the separable degrees $[\kappa(y_i') : \kappa(y)]_s$, where y_i' runs through the finite set $g^{-1}(y)$ (Corollary 8.6.47). Note that the $\kappa(y_i')$ are none other than the residual fields of the semi-local ring $(f_*(\mathscr{O}_X))_y$.

Proposition 10.4.13. *Let X and Y be two locally Noetherian integral schemes and $f : X \to Y$ be a proper and dominante morphism. For any $y \in Y$, the number of connected components of $f^{-1}(y)$ is also equal to the number of maximal ideals of the integral closure $\mathscr{O}_{Y,y}'$ of $\mathscr{O}_{Y,y}$ in the rational function field $K(X)$.*

Proof. Recall that for any open subset U of Y, $\Gamma(U, f_*(\mathscr{O}_X)) = \Gamma(f^{-1}(U), \mathscr{O}_X)$ is the intersection of the local rings $\mathscr{O}_{X,x}$ for $x \in f^{-1}(U)$ (formula (8.7.1)). We then conclude that the fiber $(f_*(\mathscr{O}_X))_y$ is a subring of $K(X)$ containing $\mathscr{O}_{Y,y}$. Moreover, as $f_*(\mathscr{O}_X)$ is a coherent $\mathscr{O}_X$-module, $(f_*(\mathscr{O}_X))_y$ is a finitely generated $\mathscr{O}_{Y,y}$-module, hence contained in $\mathscr{O}_{Y,y}'$. By **??**, any maximal ideal of the ring A is the intersection of A of a maximal ideal of $\mathscr{O}_{Y,y}'$, whence the proposition. $\square$

A local ring A is called **unibranch** if A_{red} is an integral ring and the integral closure of A_{red} is a local ring. We say a point y of an integral scheme Y is unibrach if the local ring $\mathscr{O}_{Y,y}$ is **unibranch** (this is the case if Y is normal at y). Let A be an integral local ring, and K be its fractional field. For A to be unibranch, it is then necessary and sufficient that any subring R of K, containing A and is a finite A-algebra, is a local ring. In fact, if A' is the integral closure A, then any such ring is containd in A' and any maximal ideal of R is trace of a maximal ideal of A' along R, so if A' is local, so is R. Conversely, A' is the inductive limit of the filtered family of such finite A-algebras A_α of A', and if each A_α is a local ring, then for $A_\alpha \subseteq A_\beta$, the maximal ideal of A_α is the trace over A_α of the maximal ideal of A_β, so A' is a local ring.

We also note that if the completion of a Noetherian local ring A is integral (in this case A is called **analytically integral**), then A is unibranch. In fact, let $\mathfrak{m}$ be the maximal ideal of A, K be its fraction field, and L be that of $\widehat{A}$; we then have $L = K \otimes_A \widehat{A}$. Let B be a finite sub-A-algebra of K, then the subring R of L generated by $\widehat{A}$ and B is isomorphic to $B \otimes_A \widehat{A}$; this is a finitely generated $\widehat{A}$-module, and is the $\mathfrak{m}$-adic completion of B. As B is a semi-local ring (**??**) and this completion is integral, we conclude from **??** that B has a unique maximal ideal, whence our assertion.

Corollary 10.4.14. *Under the hypotheses of 10.4.13, suppose that $[K(Y) : K(X)]_s = n$ and that $y \in Y$ is unibranch. Then the fiber $f^{-1}(y)$ has at most n connected components. In particular, if $K(X)$ is purely inseparable over $K(X)$, then $f^{-1}(y)$ is connected.*

Proof. Let $\mathscr{O}'_{Y,y}$ be the integral closure of $\mathscr{O}_{Y,y}$, then the integral closure R of $\mathscr{O}_{Y,y}$ in $K(X)$ is also that of $\mathscr{O}'_{Y,y}$. If $\mathscr{O}'_{Y,y}$ is a local ring, then R is a semi-local ring with at most n maximal ideals (**??**). □

Remark 10.4.15. Corollary 10.4.14 is essentially the "connectedness theorem" proved by Zariski for algebraic schemes. Note that in Corollary 10.4.14 if we suppose that Y is normal at y, then the fiber $f^{-1}(y)$ is geometrically connected, since (with the notation of Remark 10.4.12) $g^{-1}(y)$ then reduces to a point y' and $\kappa(y')$ is purely inseparable over $\kappa(y)$.

Corollary 10.4.16. *Under the hypothesis of Proposition 10.4.13, suppose moreover that $K(Y)$ is algebraically closed in $K(X)$, and let y be a normal point of Y. Then $f^{-1}(y)$ is geometrically connected, and there exists an open neighborhood U of y such that $f_*(\mathscr{O}_X|_{f^{-1}(U)})$ is isomorphic to $\mathscr{O}_Y|_U$. In particular, if we suppose that Y is normal (and $K(Y)$ is algebraically closed in $K(X)$), then $f_*(\mathscr{O}_X)$ is isomorphic to $\mathscr{O}_Y$.*

Proof. The first assertion concerning $f^{-1}(y)$ is a particular case of Remark 10.4.15. If $f : X \xrightarrow{f'} Y' \xrightarrow{g} Y$ is the Stein factorization, then $g^{-1}(y)$ is reduced to a single point y'. Moreover, we have $\mathscr{O}_{Y,y} \subseteq \mathscr{O}_{Y',y'} = (f_*(\mathscr{O}_X))_y \subseteq K(X)$, and as $\mathscr{O}_{Y',y'}$ is finite over $\mathscr{O}_{Y,y}$ (and a fortiori over $K(Y)$), it is contained in $K(Y)$ by our hypothesis; as y is normal, we necessarily have $\mathscr{O}_{Y',y'} = \mathscr{O}_{Y,y}$. We then conclude that g is a local isomorphism at the point y' (Proposition 8.6.53), which completes the proof of the first part of the corollary. There seconds one follows from the first, because the additional assumption entails that g is an isomorphism in the neighborhood of any point of Y', whence an isomorphism. □

Proposition 10.4.17. *Let A be a unibranch Noethreian local ring, $\mathfrak{a}$ be a defining ideal of A, $A_0 = A/\mathfrak{a}$, and $S = \mathrm{gr}_{\mathfrak{a}}(A)$. Then $\mathrm{Proj}(S)$ is a connected A_0-scheme.*

Proof. Let $\mathfrak{m}$ be the maximal ideal of A; $Y = \mathrm{Spec}(A)$ is an integral scheme with unique closed point y. By hypothesis, we have $\mathfrak{m}^k \subseteq \mathfrak{a} \subseteq \mathfrak{m}$ for an integer k, so $V(\mathfrak{a}) = \{\mathfrak{m}\}$. Let $S' = \bigoplus_{n\geq 0} \mathfrak{a}^n$, and $X = \mathrm{Proj}(S')$, which is the blow up Y-scheme relative to $\mathfrak{a}$. Then X is integral and the structural morphism $f : X \to Y$ is birational and projective (Proposition 9.8.1). Corollary 10.4.14 is then applicable and shows that $f^{-1}(y)$ is connected. But $f^{-1}(y)$ is the underlying space of $\mathrm{Proj}(S' \otimes_A A_0)$ (Proposition 9.2.50 and Proposition 8.3.37), and as $S' \otimes_A A_0 = S$ by definition, the proposition follows. □

10.4.3 Zariski's "Main Theorem"

Proposition 10.4.18. *Let Y be a locally Noetherian scheme and $f : X \to Y$ be a proper morphism. Let X' be the set of points $x \in X$ which is isolated in the fiber $f^{-1}(f(x))$. Then X' is open in X, and if $f = g \circ f'$ is the Stein factorization of f, the restriction of f' to X' is an isomorphism from X' onto an open subscheme U of Y', and we have $X' = f'^{-1}(U)$.*

Proof. As $g^{-1}(f(x))$ is finite and discrete (Corollary 9.6.5), for x to be isolated in $f^{-1}(f(x))$, it is necessary and sufficient that it is isolated in $f'^{-1}(f'(x))$. We may then assume that $f' = f$, so $f_*(\mathscr{O}_X) = \mathscr{O}_Y$. Then, if $x \in X'$, the fiber $f^{-1}(f(x))$ is connected by Corollary 10.4.10, and is necessarily reduced to the point x. As f is closed, for any open neighborhood V of x, $f(X - V)$ is closed in Y and does not contain $y = f(x)$, since $f^{-1}(y) = \{x\}$; if U is the complement of $f(X - V)$ in Y, then we have $f^{-1}(U) \subseteq V$, so we conclude that the inverse images under f of a fundamental system of open neighborhoods of y form a fundamental system of open neighborhoods of x. The hypothesis $f_*(\mathscr{O}_X) = \mathscr{O}_Y$ then implies that the homomorphism $f_y^\# : \mathscr{O}_{Y,y} \to \mathscr{O}_{X,x}$ is an isomorphism, so by Proposition 8.6.53 there exists an open neighborhood V of x and an open neighborhood U of y such that $f^{-1}(U) \subseteq V$ and the restriction $f|_V : V \to U$ is an isomorphism. Moreover, by the remarks above, we may also assume that $V = f^{-1}(U)$, so that $V \subseteq X'$, which completes the proof. □

Proposition 10.4.19. *Let Y be a locally Noetherian scheme, $f : X \to Y$ be a morphism. Then the following conditions are equivalent:*

(i) *f is finite.*

(ii) *f is affine and proper.*

(iii) *f is proper, and for any $y \in Y$, the fiber $f^{-1}(y)$ is a finite set.*

Proof. We see that (i)⇒(ii) by Corollary 9.6.9. If f is proper and affine, then so is the induced morphism $f^{-1}(y) \to \operatorname{Spec}(\kappa(y))$ (Proposition 9.5.23 and Proposition 9.1.33), and the finiteness theorem Theorem 10.3.3, applied to the structural sheaf of $f^{-1}(y)$, shows that $f^{-1}(y) = \operatorname{Spec}(A)$, where A is a finite dimensional $\kappa(y)$-algebra. Then $f^{-1}(y)$ is a finite $\kappa(y)$-scheme, so is a finite set (Proposition 8.6.44), and we see that (ii)⇒(iii). Finally, as $f^{-1}(y)$ is an algebraic scheme, the hypothesis that $f^{-1}(y)$ is a finite set implies that it is discrete (Proposition 8.6.44). With the notations of Proposition 10.4.18, we then have $X' = X$, and $f' : X' \to Y'$ is an isomorphism; as g is a finite morphism, we then conclude that (iii)⇒(i). □

Theorem 10.4.20 (Zariski's "Main Theorem"). *Let Y be a Noetherian scheme, $f : X \to Y$ be a quasi-projective morphism, and X' be the set of points $x \in X$ that is isolated in the fiber $f^{-1}(f(x))$. Then X' is an open subset of X, and the open subscheme X' of X is isomorphic via f to an open subscheme Y' of Y that is finite over Y.*

Proof. The hypotheses implies that there exists a projective Y-scheme P such that X is Y-isomorphic to an open subscheme of P. We then reduce to prove the theorem if f is a projective morphism, hence proper (Theorem 9.5.30), and this follows from Proposition 10.4.18. □

Remark 10.4.21. If X is reduced (resp. irreducible and X' is nonempty), we can suppose, in the situation of Theorem 10.4.20, that Y' is reduced (resp. irreducible). In fact, we can replace Y' by the scheme-theoretic closure $\overline{X'}$ of X' in Y', and it is reduced if X' is (Proposition 8.6.69). If X' is nonempty, then it is irreducible if X is, and then $\overline{X'}$ is also irreducible.

Corollary 10.4.22. *Let Y be a locally Noetherian and separated scheme, $f : X \to Y$ be morphism of finite type, and $x \in X$ be a point that is isolated in $f^{-1}(f(x))$. then there exists an open neighborhood U of x that is isomorphic to an open subscheme of Y that is finite over Y.*

Proof. Let $y = f(x)$, U be an affine open neighborhood of Y, V an affine open neighborhood of x contained in $f^{-1}(U)$. As Y is separated, the injection $U \to Y$ is affine (Corollary 9.1.34), and as V is affine over U (again by Corollary 9.1.34), the restriction of f to V is an affine morphism $V \to Y$ (Proposition 9.1.33(ii)); a fortiori, this restriction is quasi-projective since it is of finite type (Proposition 8.6.35(i) and Proposition 9.5.20). It then suffices to apply Theorem 10.4.20. □

Corollary 10.4.23. *Let A be a Noetherian ring, B be an A-algebra of finite type, $\mathfrak{P}$ be a prime ideal of B, $\mathfrak{p}$ be its contraction in A. Suppose that $\mathfrak{P}$ is both maximal and minimal among prime ideals of B with contraction $\mathfrak{p}$, then there exists $g \in B - \mathfrak{P}$, a finite A-algebra A' and an element $f' \in A'$ such that the A-algebra B_g and $A'_{f'}$ are isomorphic.*

Proof. It suffices to apply Corollary 10.4.22 to $Y = \operatorname{Spec}(A)$ and $X = \operatorname{Spec}(B)$, since the hypothesis on $\mathfrak{P}$ signified that it is isolated in the fiber. □

Corollary 10.4.24. *Let A be a Noetherian local ring, B be an A-algebra of finite type, $\mathfrak{M}$ be a maximal ideal of B whose contraction in A is the maximal ideal $\mathfrak{m}$ of A. Suppose that $\mathfrak{M}$ and minimal among prime ideals of B with contraction $\mathfrak{m}$. Then there exists a finite A-algebra A' and a maximal ideal $\mathfrak{m}'$ of A' (whose contraction is $\mathfrak{m}$) such that $B_{\mathfrak{M}}$ is isomorphic to the A-algebra $A'_{\mathfrak{m}'}$.*

Proof. This follows from Corollary 10.4.23 by taking stalks at $\mathfrak{m}$ and $\mathfrak{M}$. □

Corollary 10.4.25. *Under the hypothesis of Corollary 10.4.24, suppose that A and B are integral with the same fraction field K. Then, if A is integrally closed, we have $B = A$.*

Proof. By Remark 10.4.21 we can suppose that A' is integral with fraction field K. The hypothesis over A implies that $A' = A$, so $B_{\mathfrak{M}} = A$. As $A \subseteq B \subseteq B_{\mathfrak{M}}$, we conclude that $A = B$. □

Corollary 10.4.26. *Let Y be a locally Noetherian integral scheme, $f : X \to Y$ be a separated morphism that is of finite type and birational. Suppose that Y is normal and the fibers $f^{-1}(y)$ are finite for $y \in Y$. Then f is an open immersion; if moreover f is closed, then it is an isomorphic.*

Proof. Let $x \in X$ and put $y = f(x)$. As $f^{-1}(y)$ is an algebraic scheme over $\kappa(y)$, the hypothesis that it is finite implies that it is discrete (Proposition 8.6.44); moreover $\mathscr{O}_{Y,y}$ is integrally closed and $\mathscr{O}_{X,x}$ and $\mathscr{O}_{Y,y}$ have the same fraction field. We can then apply Corollary 10.4.25, and the homomorphism $f^\sharp_y : \mathscr{O}_{Y,y} \to \mathscr{O}_{X,x}$ is bijective. We then conclude that f is a local isomorphism. □

Proposition 10.4.27. *Let Y be a locally Noetherian scheme and $f : X \to Y$ be a morphism locally of finite type. Then the set X' of $x \in X$ isolated in the fiber $f^{-1}(f(x))$ is open in X.*

Proof. This question is local over X and Y, so we can assume that X and Y are affine Noetherian; f is then affine and of finite type, hence quasi-projective (Proposition 9.5.20(i)), and it suffices to apply Theorem 10.4.20. ☐

Corollary 10.4.28. *Let Y be a locally Noetherian scheme, $f : X \to Y$ be a proper morphism. Then the set U of points $y \in Y$ such that $f^{-1}(y)$ is discrete is open in Y, and the restriction morphism $f|_{f^{-1}(U)} : f^{-1}(U) \to U$ is finite. In particular, a proper and quasi-finite morphism is finite.*

Proof. In fact, the complement of U in Y is the image of $X - X'$ under f, which is closed in X by Proposition 10.4.27. As f is a closed map, U is then open in Y. Moreover, it follows from Proposition 8.6.44 that $f^{-1}(y)$ is finite for any $y \in U$; as the restriction morphism $f|_{f^{-1}(U)} : f^{-1}(U) \to U$ is proper (Proposition 9.5.23), it is finite in view of Proposition 10.4.19. ☐

10.5 Covariant functors on $\mathbf{Mod}(A)$ and base change

Let $f : X \to Y$ be a morphism of schemes and $\mathscr{F}$ be a quasi-coherent $\mathscr{O}_X$-module. In the study of the higher direct images $R^p f_*(\mathscr{F})$, the following problem is proposed: given a base change morphism $g : Y' \to Y$, we put $X' = X \times_Y Y'$, $f' = f_{(Y)}$ and $\mathscr{F}' = \mathscr{F} \otimes_Y \mathscr{O}_{Y'}$, so that the following diagram commutes:

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Y \end{array}$$

Our purpose is to understanding the higher direct images $R^p f'_*(\mathscr{F}')$ (assume giving the information of $R^p f_*(\mathscr{F})$), which (as we will see) can be reduced to considering the behaviour of $R^p f_*(\mathscr{F} \otimes_{\mathscr{O}_Y} \mathscr{G})$ for quasi-coherent $\mathscr{O}_Y$-modules $\mathscr{G}$, or that of the functor $R^p f_*(\mathscr{F} \otimes_{\mathscr{O}_Y} (-))$. If $\mathscr{F}$ is flat over Y, the functor $\mathscr{F} \otimes_{\mathscr{O}_Y} (-)$ is exact so the composition $R^\bullet f_*(\mathscr{F} \otimes_{\mathscr{O}_Y} (-))$ is a cohomological functor. But in general this is not true, and we want to replace $R^\bullet f_*(\mathscr{F} \otimes_{\mathscr{O}_Y} (-))$ by some good bahaved cohomological functor. We shall see in ([**?**], 6.10) that this new cohomological functor $\mathscr{G} \mapsto \mathscr{T}^\bullet(\mathscr{G})$ is defined locally (over Y) by $H^\bullet(\mathscr{L}^\bullet \otimes_Y \mathscr{G})$, where $\mathscr{L}^\bullet$ is a complex of locally free $\mathscr{O}_Y$-modules (uniquely determined up to homotopy). It is then interesting to forget about $\mathscr{T}^\bullet$ and consider the properties of functors of the form $H^\bullet(\mathscr{L}^\bullet \otimes_Y (-))$ (with appropriate finiteness conditions on $\mathscr{L}^\bullet$ of $H^i(\mathscr{L}^\bullet)$). The most important properties of these functors concern the exactness of the component $\mathscr{T}^i$ of $\mathscr{T}^\bullet$, and we will give various criterions for establishing such properties. As an application, we will obtain conditions allowing us to confirm that the functor $R^p f_*(\mathscr{F} \otimes_{\mathscr{O}_Y} (-))$ is exact (what we will express by saying that $\mathscr{F}$ is *cohomologically flat* on Y in dimension p). Another important property for the components $\mathscr{T}^i$ of $\mathscr{T}^\bullet$ is the semi-continuous property of the function $y \mapsto \dim_{\kappa(y)}(\mathscr{T}^i(\kappa(y)))$. If $\mathscr{T}^i$ is exact, this property is replaced by a continuity (hence locally constant) property, the converse being true according to Grauert when Y is reduced.

10.5.1 Functor on $\mathbf{Mod}(A)$

Let A be a ring (not necessarily commutative) and $\mathbf{Mod}(A)$ be the category of left A-modules. Consider an additive covariant functor $T : \mathbf{Mod}(A) \to \mathbf{Ab}$, and let M be a (A, A)-bimodule. Then the abelian group $T(M)$ is naturally endowed with a right A-module structure. In fact, for any $a \in A$, let $h_{a,M}$ be the endomorphism $x \mapsto xa$ induced on M. Then by hypothesis, $T(h_{a,M})$ is an endomorphism of $T(M)$. Moreover, since T is additive, for $a, b \in A$ we have

$$T(h_{ab}) = T(h_b \circ h_a) = T(h_b) \circ T(h_a), \quad T(h_{a+b}) = T(h_a) + T(h_b)$$

so the map $(a, y) \mapsto T(h_a)(y)$ is a right A-module structure on $T(M)$. In particular, $T(A)$ is a right A-module.

If A is a commutative ring, then for any A-module M, $T(M)$ is also an A-module. If $u : M \to N$ is a homomorphism of A-modules, we have, for $a \in A$, $u \circ h_a = h_{a,N} \circ u$, so $T(u) \circ T(h_{a,M}) = T(h_{a,N}) \circ T(u)$, which means $T(u) : T(M) \to T(N)$ is an A-homomorphism. We then say that T can be considered as a covariant endofunctor on $\mathbf{Mod}(A)$. More precisely, we define an equivalence from the category of additive covariant functors $\mathbf{Mod}(A) \to \mathbf{Ab}$ to the category of A-linear covariant functors $\mathbf{Mod}(A) \to \mathbf{Mod}(A)$, given by $T(h_{a,M}) = h_{a,T(M)}$ for $a \in A$. As the forgetful functor $\mathbf{Mod}(A) \to \mathbf{Ab}$ is exact and faithful, the exactness of these functors are therefore equivalent.

Assume that A is commutative and let B be an A-algebra (not necessarily commutative) and $\rho : A \to B$ be the ring homomorphism. Then we have an additive covariant functor $\rho^* : N \mapsto \rho^*(N)$ from the category of B-modules to that of A-modules. By composition with T, we then deduce a functor $T_{(B)} : \mathbf{Mod}(B) \to \mathbf{Ab}$, which is evidently additive and covariant, and called the **extension of scalar** of T by B. It is clear that the extension of scalars is functorial and additive on T. Moreover, if T commutes with inductive limits or direct sums (resp. is left exact, resp. is right exact), so is the functor $T_{(B)}$, since the functor ρ^* is exact and commutes with inductive limits and direct sums.

Suppose that A is commutative and T commutes with inductive limits. Then for any multiplicative subset S of A and any A-module M, we have a canonical isomorphism of A-modules

$$T(S^{-1}M) \xrightarrow{\sim} S^{-1}T(M). \tag{10.5.1}$$

In fact, suppose first that S is of the form f^n for $f \in A$. Then we see that $M_f = \varinjlim M_n$, where (M_n, φ_{nm}) is the inductive system of A-module $M_n = M$ with $\varphi_{nm} = h_{f^{n-m}}$, so the isomorphism (10.5.1) follows from the hypothesis on T. In the general case, we have $S^{-1}M = \varinjlim_{f \in S} M_f$, so we can similarly conclude the isomorphism. Moreover, the functoriality of (10.5.1) shows that it is an isomorphism of $S^{-1}A$-modules, and we can then write

$$T_{(S^{-1}A)}(S^{-1}M) = S^{-1}T(M) = T(S^{-1}M). \tag{10.5.2}$$

If $S = A - \mathfrak{p}$ is the complement of a prime ideal $\mathfrak{p}$ of A, we then write $T_{\mathfrak{p}}$ instead of $T_{(A_{\mathfrak{p}})}$.

Proposition 10.5.1. *Under the above hypothesis, if $T_{\mathfrak{m}}$ is left exact (resp. right exact) for any maximal ideal $\mathfrak{m}$ of A, then T is left exact (resp. right exact).*

Proof. This is a direct concequence of that fact that two submodules N, P of an A-module M are equal if and only if $N_{\mathfrak{m}} = P_{\mathfrak{m}}$ for any maximal ideal $\mathfrak{m}$ of A (**??**). □

10.5.2 Characterization of tensor product functor

Let A be a ring (not necessarily commutative) and $T : \mathbf{Mod}(A) \to \mathbf{Ab}$ be an additive covariant functor. For any left A-module M, we note that $\mathrm{Hom}_{\mathbb{Z}}(T(A), T(M))$ is canonically endowed with a left A-module structure such that $(a \cdot u)(y) = u(ya)$ for $y \in T(A)$, $a \in A$, $v \in \mathrm{Hom}_{\mathbb{Z}}(T(A), T(M))$. We define a homomorphism $\tilde{t}_M$ from M into $\mathrm{Hom}_{\mathbb{Z}}(T(A), T(M))$ to be the composition

$$M \xrightarrow{\sim} \mathrm{Hom}_A(A, M) \xrightarrow{T} \mathrm{Hom}_{\mathbb{Z}}(T(A), T(M))$$

where the first arrow is the canonical isomorphism $x \mapsto \theta_x$, given by $\theta_x(\xi) = \xi x$ for $\xi \in A$ and $x \in M$. Note that we have $\theta_{ax} = \theta_x \circ h_a$, so $T(\theta_{ax}) = T(\theta_x \circ h_a) = T(\theta_x) \circ T(h_a)$ and for $y \in T(A)$,

$$T(\theta_{ax})(y) = T(\theta_x)(T(h_a)(y)) = T(\theta_x)(ya).$$

We therefore conclude that $\tilde{t}_M$ induces a homomorphism

$$t_M : T(A) \otimes_A M \to T(M) \tag{10.5.3}$$

such that $t_M(a \otimes x) = T(\theta_x)(a)$ for $a \in T(A)$, $x \in M$. It is immediate to verify that t_M is functorial on M, which means for any homomorphism $u : M \to N$ of left A-modules, the diagram

$$\begin{tikzcd} T(A) \otimes_A M \arrow[r, "t_M"] \arrow[d, "1\otimes u"'] & T(M) \arrow[d, "T(u)"] \\ T(A) \otimes_A M \arrow[r, "t_N"] & T(N) \end{tikzcd} \tag{10.5.4}$$

is commutative.

More generally, if A is commutative, then for any A-module N we can define a canonical homomorphism

$$t_{N,M} : T(N) \otimes_A M \to T(N \otimes_A M). \tag{10.5.5}$$

For this, it suffices to replace θ_x by the A-module homomorphism $N \to N \otimes_A M$, which sends $y \in N$ to $y \otimes x$. It is clear that this functor is functorial on M and N. In particular, if B is an A-algebra (not necessarily commutative), we have a functorial homomorphism

$$T(M)_{(B)} = T(M) \otimes_A B \to T_{(B)}(M_{(B)}) \tag{10.5.6}$$

which is a homomorphism of B-modules. Moreover, the following diagram is commutative

$$\begin{tikzcd} T(A) \otimes_A M \arrow[r, "t_M"] \arrow[d] & T(M) \arrow[d] \\ T_{(B)}(B) \otimes_B M_{(B)} \arrow[r, "t_{M_{(B)}}"] & T_{(B)}(M_{(B)}) \end{tikzcd}$$

where the right vertical arrow is the composition

$$T(M) \to T(M) \otimes_A B \to T(M \otimes_A B) = T_{(B)}(M_{(B)})$$

and the left vertical arrow is the homomorphism

$$T(A) \otimes_A M \to T_{(B)}(B) \otimes_B (B \otimes_A M) = T_{(B)}(B) \otimes_A M,$$

where $T(A) \to T_{(B)}(B) = T(B)$ is the homomorphism induced by $A \to B$.

Lemma 10.5.2. *If $T : \mathbf{Mod}(A) \to \mathbf{Ab}$ is a covariant additive functor which commutes with direct sums, the canonical homomorphism t_L is an isomorphism for any free A-module L.*

Proof. In fact, we can write $L = \bigoplus_i L_i$, where L_i are isomorphic to A for each $i \in I$. The definition of t_L shows that $t_L = \bigoplus_i t_{L_i}$, since

$$T : \mathrm{Hom}_A(A, L) \to \mathrm{Hom}_{\mathbb{Z}}(T(A), T(L))$$

is the direct sum of the $\mathbb{Z}$-linear maps $T_i : \mathrm{Hom}_A(A, L_i) \to \mathrm{Hom}_{\mathbb{Z}}(T(A), T(L_i))$ in view of the hypothesis on T. We are then reduced to the case where $L = A$, and t_A is none other than the canonical isomorphism $T(A) \otimes_A A \cong T(A)$ for the A-module $T(A)$. $\square$

Lemma 10.5.3. *Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories, F, G be covariant additive functors from $\mathcal{A}$ to $\mathcal{B}$, and $\gamma : F \Rightarrow G$ be a morphism such that, for any object $A \in \mathcal{A}$, $\gamma_A : F(A) \to G(A)$ is an epimorphism. Then, if F is right exact and G is semi-exact, G is right exact.*

Proof. Since G is semi-exact, it suffices to prove that for any epimorphism $v : A \to B$ in $\mathcal{A}$, $G(v) : G(A) \to G(B)$ is an epimorphism. Now, we have a commutative diagram

$$\begin{tikzcd} F(A) \arrow[r, "F(v)"] \arrow[d, "\gamma_A"'] & F(B) \arrow[d, "\gamma_B"] \\ G(A) \arrow[r, "G(v)"] & G(B) \end{tikzcd}$$

in which $F(v)$, γ_A, γ_B are epimorphisms. It then follows from $G(v)$ is an epimorphism. $\square$

Proposition 10.5.4. *Let $T : \mathbf{Mod}(A) \to \mathbf{Ab}$ be a covariant additive functor which commutes with direct sums. Then the following conditions are equivalent:*

(i) *T is right exact;*

(ii) *the canonical homomorphism t_M is an isomorphism for any A-module M;*

(ii′) *T is semi-exact and the homomorphism t_M is surjective for any A-module M;*

(iii) *T is isomorphic to the functor $N \otimes_A (-)$, where N is an A-module.*

Proof. It is clear that (ii) implies (iii) and (iii) implies (i), since we can then take $N = T(A)$. We now prove that (i)⇒(ii); for this, put $F(M) = T(A) \otimes_A M$ for any A-module M. There exists an exact sequence $P \to L \to M \to 0$, where P and L are free A-modules. As T and F are right exact, we then have a commutative diagram

$$\begin{tikzcd} F(P) \arrow[r] \arrow[d, "t_P"] & F(L) \arrow[r] \arrow[d, "t_L"] & F(M) \arrow[r] \arrow[d, "t_M"] & 0 \\ T(P) \arrow[r] & T(L) \arrow[r] & T(M) \arrow[r] & 0 \end{tikzcd}$$

with exact rows. As t_P and t_L are isomorphism by Lemma 10.5.2, so is t_M by five lemma. Finally, it is clear that (ii) implies (ii'), and to see that (ii') implies (ii), it suffices to apply Lemma 10.5.3. □

Remark 10.5.5. For any right A-module N, let $T_N(M) = N \otimes_A M$, where M is a left A-module. Then T_N is a right exact covariant additive functor from $\mathbf{Mod}(A)$ to $\mathbf{Ab}$, and commutes with direct sums. If we canonically identify $T_N(A)$ with N, then the corresponding homomorphism (10.5.5) is the identity. We therefore conclude that the right A-module N in Proposition 10.5.4 is uniquely determined up to isomorphisms, and is isomorphic to $T(A)$. In other words, the morphisms $T \mapsto T(A)$ and $N \mapsto T_N$ define a equivalence from the category of covariant additive functors which are right exact and commute with direct sums to the category of right A-modules.

Proposition 10.5.6. *Let A be a left Artinian ring with Jacobson radical $\mathfrak{m}$ such that $A/\mathfrak{m}$ is a field k. Let $T : \mathbf{Mod}(A) \to \mathbf{Ab}$ be a covariant additive functor which commutes with direct sums. Then the conditions of Proposition 10.5.4 are equivalent to the following:*

(v) *T is semi-exact and the homomorphism $T(\varepsilon) : T(A) \to T(k)$ induced by the canonical homomorphism $\varepsilon : A \to k$ is surjective.*

10.5.3 Exactness of cohomological functors on $\mathbf{Mod}(A)$

Proposition 10.5.7. *Let A be a ring (not necessarily commutative) and $T^\bullet$ be a covariant cohomological functor from $\mathbf{Mod}(A)$ to $\mathbf{Ab}$ which commutes with direct sums. Let p be an integer such that T^p and T^{p-1} are defined. Then the following conditions are equivalent:*

(i) *T^p is right exact;*

(ii) *T^{p+1} is left exact;*

(iii) *for any left A-module M, the canonical homomorphism*

$$T^p(A) \otimes_A M \to T^p(M) \tag{10.5.7}$$

is an isomorphism.

(iv) *for any A-module M, the homomorphism (10.5.7) is an epimorphism;*

(v) *T^p is isomorphic to a functor $N \otimes_A (-)$, where N is a (uniqely determined) right A-module and isomorphic to $T^p(A)$.*

If the conditions of Proposition 10.5.6 are satisfied, then these conditions are equivalent to:

(vi) *the canonical homomorphism $T^p(\varepsilon) : T^p(A) \to T^p(k)$ is an epimorphism.*

Proof. By the definition of cohomological functors, T^i are semi-exact for any i such that T^i is defined. Moreover, for any exact sequence

$$0 \longrightarrow A \xrightarrow{u} B \xrightarrow{v} C \longrightarrow 0$$

we have $\ker(T^{p+1}(u)) = \operatorname{coker}(T^p(v))$, so it is clear that (i) and (ii) are equivalent. The rest of the proposition follows from Proposition 10.5.4 and Proposition 10.5.6. □

Corollary 10.5.8. *Let A be a commutative ring. With the notations of Proposition 10.5.7, suppose that T^p is right exact. If $f \in A$ does not belong to the annihilator of any nonzero element of an A-module M, then f doe not belong to the annihilator of any nonzero element of the A-module $T^{p+1}(M)$. In particular, if A is an integral domain, then the A-module $T^{p+1}(A)$ is torsion free.*

Proof. Let h_f be the homothety with ratio f on M. Then the hypothesis signifies that h_f is injective, and by Proposition 10.5.7, the homomorphism $T^{p+1}(h_f)$ is also injective, whence the corollary. □

Proposition 10.5.9. *Let A be a ring and $T^\bullet : \mathbf{Mod}(A) \to \mathbf{Ab}$ be a cohomological functor which commutes with direct sums. Let p be an integer such that T^{p-1}, T^p and T^{p+1} are defined. Then the following conditions are satisfied:*

(i) *T^p is exact;*

(ii) *T^{p-1} is right exact and T^{p+1} is left exact;*

(iii) *for any left A-module M, the canonical homomorphism*

$$T^i(A) \otimes_A M \to T^i(M) \tag{10.5.8}$$

is an isomorphism for $i = p, p-1$;

(iii') *for any left A-module M, the canonical homomorphism (10.5.8) is an epimorphism for $i = p, p-1$;*

(iv) *for any left A-module M, the canonical homomorphism (10.5.8) is an isomorphism for $i = p$ and $T^p(A)$ is a flat right A-module;*

(iv') *for any left A-module M, the canonical homomorphism (10.5.8) is an epimorphism for $i = p$ and $T^p(A)$ is a flat right A-module.*

Proof. The equivalence of (i) and (ii) follows from the equivalence of conditions (i) and (ii) of Proposition 10.5.7, and the equivalence of (ii), (iii) and (iii') follows from that of conditions (i), (iii) and (iv) of Proposition 10.5.7. Finally, to sya that $T^p(A)$ is flat signifies that the functor $T^p(A) \otimes_A M$ is left exact, so the equivalence of (i), (iv') and (iv) also follows from that of conditions (i), (iii) and (iv) of Proposition 10.5.7. □

Corollary 10.5.10. *Suppose that A is commutative, T^p is exact, and that $T^p(A)$ is an A-module of finite presentation. Then the function $x \mapsto \dim_{\kappa(x)}(T^p(\kappa(x)))$ is locally free over $X = \operatorname{Spec}(A)$, hence constant if $\operatorname{Spec}(A)$ is connected.*

Proof. As $T^p(A)$ is a flat A-module by Proposition 10.5.9, it is projective and finitely generated (**??**), so $\widetilde{T^p(A)}$ is a locally free $\mathscr{O}_X$-module. We have $T^p(\kappa(x)) = T^p(A) \otimes_A \kappa(x)$, so the rank function is locally constant by **??**. □

Proposition 10.5.11. *Let A be a left Artinian ring with Jacobson radical $\mathfrak{m}$ such that $A/\mathfrak{m}$ is a field k. Let $T^\bullet : \mathbf{Mod}(A) \to \mathbf{Ab}$ be a cohomological functor which commutes with direct sums. Then the conditions of Proposition 10.5.9 are equivalent to the following:*

(v) *the canonical homomorphism $T^i(\varepsilon) : T^i(A) \to T^i(k)$ is an epimorphism for $i = p, p+1$;*

(vi) *$T^p(\varepsilon)$ is an epimorphism and $T^p(A)$ is a flat right A-module.*

Suppose that A is commutative. Then the above conditions are equivalent to the following:

(vii) *for any A-module M of finite length, we have $\ell(T^p(M)) = \ell(T^p(k)) \cdot \ell(M)$;*

(viii) *$\ell(T^p(A)) = \ell(T^p(k)) \cdot \ell(A)$.*

Proof. The equivalence of (v), (vi) and the conditions of Proposition 10.5.9 can be deduced from Proposition 10.5.7. □

Proposition 10.5.12. *Let A be a ring and $T^\bullet : \mathbf{Mod}(A) \to \mathbf{Ab}$ be a cohomological functor which commutes with direct sums. Suppose that there exists an integer n_0 such that T^p is exact for $p \geq n_0$. Then, for an integer $n < n_0$, the following conditions are equivalent:*

(i) *T^p is exact for $p \geq n$;*

(ii) *$T^p(A)$ is a flat right A-module for $p \geq n$;*

(iii) *for any left A-module M, the canonical homomorphism $T^p(A) \otimes_A M \to T^p(M)$ is surjective for $p \geq n-1$.*

Proof. The equivalence of (i) and (ii) follows from Proposition 10.5.9 by induction, and (iii) follows from Proposition 10.5.9(iii'). □

If A is a commutative ring, B is an A-algebra (not necessarily) commutative, and $T^\bullet : \mathbf{Mod}(A) \to \mathbf{Ab}$ be a cohomological functor, then since the functor ρ^* is exact and faithful ($\rho : A \to B$ being the ring homomorphism), we see that the extension of scalar $T^\bullet_{(B)}$ is also a cohomological functor.

Corollary 10.5.13. *Suppose that $T^\bullet$ satisfies the conditions of Proposition 10.5.12 and commutes with inductive limits. Moreover, assume that A is an integral domain and the A-modules $T^p(A)$ are of finite presentation. Then for any integer $n < n_0$, there exists a nonzero element $f \in A$ such that the functor $T^p_{(A_f)} : \mathbf{Mod}(A_f) \to \mathbf{Ab}$ is exact for $p \geq n$. In particular, if there are finitely many indices p such that $T^p \neq 0$, then there exists a nonzero element $f \in A$ such that the functors $T^p_{(A_f)}$ are all exact.*

Proof. By hypothesis, T^p is exact for $p \geq n_0$, so $T^p(A)$ is flat for these values of p. In view of Proposition 10.5.12 and (10.5.1), it suffices to choose f such that $T^p_{(A_f)}(A_f) = T^p(A_f) = (T^p(A))_f$ is a free A_f-module for $n \leq p < n_0$. If x is the generic point of $\mathrm{Spec}(A)$, $(T^p(A))_x$ is a vector space of finite dimension over the fraction field of A. As each $T^p(A)$ is of finite presentation, there exists an element $f \in A$ satisfying the desired property (**??**). □

Corollary 10.5.14. *Suppose that $T^\bullet$ satisfies the conditions of Proposition 10.5.12 and commutes with inductive limits. Moreover, assume that A is Noetherian and the A-modules $T^p(A)$ are finitely generated. Then for any integer n, there exists an open dense subset U of $\mathrm{Spec}(A)$ such that, for any $p \geq n$, the function $x \mapsto \dim_{\kappa(x)}(T^p(\kappa(x)))$ is constant on U.*

Proof. Let $\mathfrak{p}$ be a minimal prime ideal of A. By hypothesis, the ring $B = A/\mathfrak{p}$ is integral and $\mathrm{Spec}(B)$ is identified with an irreducible component of $\mathrm{Spec}(A)$. We now prove by induction on the integer $p \geq n$ that, for each p, there exists $f_p \in B - \{0\}$ such that if we put $B' = B_{f_p}$, then $T^i_{(B')}$ is exact and the $T_i(B')$ are finitely generated B'-modules for $i \geq p$. By our hypothesis, this is true for $p \geq n_0$, since in this case T^p is exact and $T^p(B) \cong T^p(A)/T^p(\mathfrak{p})$, hence is a finitely generated A-module (and a fortiori B-module), so we can choose $f_p = 1$ (so that $B' = B = A/\mathfrak{p}$). We preceed by induction on p, so let f_p be such an element. Then f_p is the canonical image of an element $g_p \in A$, and if we put $A' = A_{g_p}$, we have $B' = A'/\mathfrak{p}'$ where $\mathfrak{p}'$ is the minimal prime ideal $\mathfrak{p}_{g_p}$ of A'. As $T^i(A_{g_p}) = (T_i(A))_{g_p}$, the $T^i(A')$ are finitely generated A'-modules, so $T^\bullet_{(A')}$ satisfies the same condition as $T^\bullet$, with n_0 replaced by p. We can then reduce the proof to the case where $A' = A$ and T^p is exact. The exact sequence $0 \to \mathfrak{p} \to A \to A/\mathfrak{p} \to 0$ then gives an exact sequence

$$T^{p-1}(A) \longrightarrow T^{p-1}(A/\mathfrak{p}) \xrightarrow{\delta} T^p(\mathfrak{p}) \longrightarrow T^p(A)$$

and as T^p is exact, the last arrow is injective, so $T^{p-1}(A/\mathfrak{p})$ is a quotient of $T^{p-1}(A)$ and therefore finitely generated. We note that in the proof of Corollary 10.5.13, we only use the fact that $T^p(A)$ are finitely generated for $p \geq n$, so we can apply it to the integral ring B and the functor $T^\bullet_{(B)}$, with $n = p-1$, which produces the desired element f_N. If the corollary is proved for the ring B_{f_N}, then there exists an open dense subset W of $\mathrm{Spec}(B_{f_N})$ such that the function $\dim_{\kappa(x)}(T^p(\kappa(x)))$ is constant, since $A_x = (B_{f_N})_x$ for any $x \in W$. By applying this reasoning to each irreducible component of $\mathrm{Spec}(A)$, the corollary is then proved. To summarize, the corollary is now reduced to the case where A is an integral domain, so by Corollary 10.5.13 there exists an element $f \in A - \{0\}$ such that the A_f-modules $T^p(A_f)$ are free of finite rank for $p \geq n$. The corollary then follows from Corollary 10.5.10. □

Proposition 10.5.15. *Let A be a commutative local ring, k be its residue field, $T^\bullet : \mathbf{Mod}(A) \to \mathbf{Ab}$ be a cohomological functor which commutes with direct sums. Suppose that there exists an integer n_0 such that T^p is exact for $i \geq n_0$, and that the $T^p(A)$ are finitely presented A-modules. Then the conditions of Proposition 10.5.12 imply the following conditions, and they are equivalent if A is reduced:*

(iv) *for any* $x \in \operatorname{Spec}(A)$, *we have*

$$\dim_{\kappa(x)}(T^p(\kappa(x))) = \operatorname{rank}_k(T^p(k)) \quad \textit{for} \quad p \geq n;$$

(iv') *for any generic point* x_j *of an irreducible component of* $\operatorname{Spec}(A)$, *we have*

$$\operatorname{rank}_{\kappa(x_j)}(T^p(\kappa(x_j))) = \operatorname{rank}_k(T^p(k)) \quad \textit{for} \quad p \geq n.$$

Proof. As $T^p(A)$ is a finitely presented A-module, condition (ii) of Proposition 10.5.12 is equivalent to that $T^p(A)$ is a free A-module for $p \geq n$ (**??**). Condition (iii) implies that $T^p(\kappa(x)) = T^p(A) \otimes_A \kappa(x)$ for $p \geq n$, so the conditions of Proposition 10.5.12 implies (iv), and it is clear that (iv) implies (iv'). Conversely, we need to show that (iv') implies (i) if A is reduced. We preceed by descending induction on the integer $p \geq n$, since we know that T^p is exact for $p \geq n_0$. Suppose that T^i is exact for $i \geq p > n$, we show that $T^{p-1}(A)$ is a free A-module. In view of the induction hypothesis, $T^{p-1}(A) \otimes_A M$ is isomorphic to $T^{p-1}(M)$ for any A-module M, by condition (iii) of Proposition 10.5.12 and Proposition 10.5.9. Applying this property to $M = \kappa(x_j)$ and $M = k$, we conclude that for each j, we have

$$\operatorname{rank}_{\kappa(x_j)}(T^{p-1}(A) \otimes_A \kappa(x_j)) = \operatorname{rank}_k(T^{p-1}(k)).$$

But this implies that $T^{p-1}(A)$ is free (**??**), whence the proposition. □

10.5.4 Exactness of the fonctor $H^\bullet(P^\bullet \otimes_A M)$

Let A be a ring (notn ecessarily commutative) and $P^\bullet$ be a complex of flat right A-modules. As the functor $P^i \otimes_A (-)$ is then exact for each i, the δ-functor

$$T^\bullet(M) = H^\bullet(P^\bullet \otimes_A M) \tag{10.5.9}$$

is a cohomological functor from $\mathbf{Mod}(A)$ to $\mathbf{Ab}$, which is A-linear if A is commutative and commutes with inductive limits.

If A is commutative and B is an A-algebra, then the cohomological functor $T^\bullet_{(B)}$ can be defined by

$$T^\bullet_{(B)} = H^\bullet(P^\bullet \otimes_A \rho^*(N))$$

where $\rho : A \to B$ is the ring homomorphism. As we can write $P^\bullet \otimes_A \rho^*(N) = P^\bullet \otimes_A \rho^*(B \otimes_B N) = (P^\bullet \otimes_A B) \otimes_B N$, we see that

$$T^\bullet_{(B)}(N) = H^\bullet(P^\bullet_{(B)} \otimes_B N) \tag{10.5.10}$$

where $P^\bullet_{(B)} = P^\bullet \otimes_A B$ is a complex of flat B-modules (**??**).

Proposition 10.5.16. *Let $T^\bullet$ be the cohomological functor defined by (10.5.9). Then for an integer p, the following conditions are equivalent:*

(i) *T^p is left exact (or equivalently, T^{p-1} is right exact);*

(ii) *$W^p(P^\bullet) = \operatorname{coker}(P^{p-1} \to P^p)$ is a flat right A-module;*

(iii) *there exists a complex $Q^\bullet$ of flat right A-modules such that $d^{p-1} : Q^{p-1} \to Q^p$ is zero and an isomorphism $H^\bullet(P^\bullet \otimes_A M) \xrightarrow{\sim} H^\bullet(Q^\bullet \otimes_A M)$ of cohomological functors.*

Proof. By definition, we have a canonical exact sequence

$$0 \longrightarrow T^p(M) \longrightarrow W^p(P^\bullet \otimes_A M) \longrightarrow P^{p+1} \otimes_A M$$

where $W^p(P^\bullet \otimes_A M) = \operatorname{coker}(P^{p-1} \otimes_A M \to P^p \otimes_A M) = W^p(P^\bullet) \otimes_A M$ in view of the flatness of $P^\bullet$. For any homomorphism $f : M \to N$, we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & T^p(M) \arrow[r] \arrow[d, "u"] & W^p(P^\bullet) \otimes_A M \arrow[r] \arrow[d, "v"] & P^{p+1} \otimes_A M \arrow[d, "w"] \\
0 \arrow[r] & T^p(N) \arrow[r] & W^p(P^\bullet) \otimes_A N \arrow[r] & P^{p+1} \otimes_A N
\end{tikzcd} \tag{10.5.11}$$

with exact rows. If f is a monomorphism, so is w since P^{p+1} is flat; if T^p is left exact, then u is also a monomorphism. We then conclude that v is a monomorphism, which implies that $W^p(P^\bullet)$ is flat. Conversely, if this is the case, then v is a monomorphism for any monomorphism $f : M \to N$, so the diagram (10.5.11) shows that u is a monomorphism, and therefore T^p is left exact. This proves (i) implies (ii), and it is immediate that (iii) implies (i), because if $P^\bullet$ has zero differentials and $0 \to M' \to M \to M'' \to 0$ is an exact sequence of A-modules, the connecting homomorphism of the sequence

$$H^{p-1}(P^\bullet \otimes_A M'') \xrightarrow{\delta} H^p(P^\bullet \otimes_A M') \longrightarrow H^p(P^\bullet \otimes_A M)$$

is zero by definition, so T^p is left exact. Conversely, we prove that (ii) implies (iii). If $Z^p(P^\bullet) = \ker(P^p \to P^{p+1})$, we have an exact sequence

$$0 \longrightarrow Z^{p-1}(P^\bullet) \longrightarrow P^{p-1} \longrightarrow P^p \longrightarrow W^p(P^\bullet) \longrightarrow 0$$

in which P^{p-1}, P^p and $W^p(P^\bullet)$ are flat, so $Z^{p-1}(P^\bullet)$ is flat. We can then define $Q^\bullet$ to be the following complex

$$\cdots \longrightarrow P^{p-2} \longrightarrow Z^{p-1}(P^\bullet) \xrightarrow{0} W^p(P^\bullet) \longrightarrow P^{p+1} \longrightarrow P^{p+2} \longrightarrow \cdots$$

As the P^i are flat, we have

$$W^i(P^\bullet \otimes_A M) = W^i(P^\bullet) \otimes_A M, \quad Z_i(P^\bullet \otimes_A M) = Z_i(P^\bullet) \otimes_A M, \quad B^i(P^\bullet \otimes_A M) = B^i(P^\bullet) \otimes_A M$$

from which we conclude that for any M the functorial homomorphisms $H^i(P^\bullet \otimes_A M) \xrightarrow{\sim} H^i(Q^\bullet \otimes_A M)$ for each i, which proves the assertion. □

Remark 10.5.17. We note that the conditions of Proposition 10.5.16 imply that each $B^p(P^\bullet)$ is flat, because we have an exact sequence

$$0 \longrightarrow B^p(P^\bullet) \longrightarrow P^p \longrightarrow W^p(P^\bullet) \longrightarrow 0$$

in which P^p and $W^p(P^\bullet)$ are flat.

Corollary 10.5.18. *Suppose that A is a regular Noetherian ring of dimension 1. Then, for the T^p to be left exact, it is necessary and sufficient that $T^p(A)$ is a flat A-module. For T^p to be exact, it is necessary and sufficient that $T^p(A)$ and $T^{p+1}(A)$ are flat A-modules.*

Proof. Recall that for a module M over a Dedekind ting, flatness is equivalent to torsion free, so under our hypothesis, any submodule of a flat A-module is flat. The second assertion follows from the first, since T^p is exact if and only if T^p and T^{p+1} are left exact. To prove the frist assertion, we consider the following exact sequence

$$0 \longrightarrow H^p(P^\bullet) \longrightarrow W^p(P^\bullet) \longrightarrow B^{p+1}(P^\bullet) \longrightarrow 0$$

where $B^{p+1}(P^\bullet)$ is a submodule of the flat module P^{p+1}, hence flat, so $H^p(P^\bullet)$ is flat if and only if $W^p(P^\bullet)$ is flat. □

The most important application of Proposition 10.5.16 is the following result:

Proposition 10.5.19. *Let A be a (commutative) Noetherian ring and $P^\bullet$ be a complex of flat A-modules. Suppose that the P^i are finitely generated A-modules, or that the $H^i(P^\bullet)$ are finitely generated and there exists an integer n_0 such that $H^i(P^\bullet) = 0$ for $i > n_0$. Let $T^\bullet$ be the cohomological functor defined by (10.5.9), then the set U of $y \in \mathrm{Spec}(A)$ such that $(T^p)_y$ is left exact (resp. right exact) is open in $\mathrm{Spec}(A)$.*

Proof. In any case, by ([?] 0_{III}, 11.9.3), we can replace $P^\bullet$ by a complex $Q^\bullet$ consists of finitely generated A-modules such that the functors $H^\bullet(P^\bullet \otimes_A (-))$ and $H^\bullet(Q^\bullet \otimes_A (-))$ are isomorphic; in particular, the $W^i(P^\bullet)$ are finitely generated. Moreover, in view of Proposition 10.5.7, we only need to prove the assertion concerning left exactness of T^p. Now let $x \in U$; as the functor $M \mapsto M_x$ is exact, we have $(W^p(P^\bullet))_x = W^p(P^\bullet_x)$, and in view of (10.5.10) and Proposition 10.5.16, the hypothesis implies that $(W^p(P^\bullet))_x$ is a flat A_x-module, hence free (**??**). We then conclude that there exists $f \in A$ such that $(W^p(P^\bullet))_f$ is a free A_f-module (**??**), and a fortiori $(W^p(P^\bullet))_y$ is free over A_y for any $y \in D(f)$. This then proves our assertion, since $D(f) \subseteq U$. □

Corollary 10.5.20. *Under the hypothesis of Proposition 10.5.19, assume that A is integral. Then the set U of* $\mathrm{Spec}(A)$ *such that $(T^p)_x$ is exact is nonempty in* $\mathrm{Spec}(A)$.

Proof. It suffices to note that $(T^p)_x$ is exact for the generic point x of $\mathrm{Spec}(A)$, since A_x is then the fraction field. □

Proposition 10.5.21. *Under the hypothesis of Proposition 10.5.19, the conditions of Proposition 10.5.16 are equivalent to the following:*

(iv) *there exsist an A-module Q and an isomorphism*

$$T^p(M) \xrightarrow{\sim} \mathrm{Hom}_A(Q, M). \tag{10.5.12}$$

Moreover, the A-module Q is uniquely determined up to isomorphisms, and is finitely generated.

Proof. The uniqueness of Q follows easily from Yoneda's lemma, and it is clear that $\mathrm{Hom}_A(Q, -)$ is left exact. Conversely, assume that T^p is left exact, we prove the existence of Q. By ([?] 0_{III}, 11.9.3), we can assume that each P^i is free of finite rank, hence projective. The dual $\check{P}^i$ of P^i is then also a finitely generated projective module, P^i is isomorphic to the dual of $\check{P}^i$, and the canonical homomorphism $P^i \otimes_A M \to \mathrm{Hom}_A(\check{P}^i, M)$ is bijective. We see on the other hand (Proposition 10.5.16) that we can suppose that $d^{p-1} : P^{p-1} \to P^p$ is zero, so we have an exact sequence

$$0 \longrightarrow T^p(M) \xrightarrow{u} P^p \otimes_A M \longrightarrow P^{p+1} \otimes_A M$$

where $v = d^p \otimes 1$. Put $Q' = \ker d^p$, so that we have an exact sequence

$$0 \longrightarrow Q' \xrightarrow{w} P^p \xrightarrow{d^p} P^{p+1}$$

Then by transposition, the sequence

$$\check{P}^{p+1} \xrightarrow{(d^p)^t} \check{P}^p \xrightarrow{w^t} \check{Q}' \longrightarrow 0$$

is exact. We then claim that the module $Q = \check{Q}' = \mathrm{coker}(d^p)^t$ satisfies our requirement. In fact, we have the exact sequence

$$0 \longrightarrow \mathrm{Hom}_A(Q, M) \longrightarrow \mathrm{Hom}_A(\check{P}^p, M) \xrightarrow{v'} \mathrm{Hom}_A(\check{P}^{p+1}, M)$$

where $v' = \mathrm{Hom}((d^p)^t, 1)$. If we canonically identify $P^i \otimes_A M$ with $\mathrm{Hom}(\check{P}^i, M)$, then v' is identified with $v = d^p \otimes 1$, and we therefore have a functorial isomorphism $T^p(M) \xrightarrow{\sim} \mathrm{Hom}_A(Q, M)$. Morever, Q, being a quotient of $\check{P}^p$, is finitely generated. □

Proposition 10.5.22. *Under the hypothesis of Proposition 10.5.19, for any finitely generated A-module M:*

(a) *the $T^i(M)$ are finitely generated A-modules;*

(b) *for any ideal $\mathfrak{I}$ of A, the canonical homomorphism*

$$\widehat{T^i(M)} \to \varprojlim_n T^i(M \otimes_A (A/\mathfrak{I}^{n+1})) \tag{10.5.13}$$

is bijective.

Proof. As in Proposition 10.5.19, we can assume that each P^i is finitely generated. Then every submodule of $P^i \otimes_A M$ is finitely generated, whence assertion (a). To prove (b), it suffices to show that, if $u : E \to F$ is a homomorphism of finitely generated A-modules and M is a finitely generated A-module, then, if we set $K(M) = \ker u \otimes 1_M$ and $C(M) = \mathrm{coker}\, u \otimes 1_M$, we have canonical isomorphisms

$$\widehat{K(M)} \to \varprojlim K(M_n), \quad \widehat{C(M)} = \varprojlim C(M_n).$$

To this end, we note that since $E \otimes_A M$ and $F \otimes_A M$ are finitely generated A-module, $\widehat{K(M)}$ and $\widehat{C(M)}$ are respectively the kernel and cokernel of the homomorphism $\widehat{u \otimes 1_M} : \widehat{E \otimes_A M} \to \widehat{F \otimes_A M}$. By the left exactness of $\varprojlim$, we then have $\widehat{K(M)} = \varprojlim K(M_n)$, and $\widehat{K(M)} = \varprojlim K(M_n)$; on the other hand, since $C(M_n) = C(M) \otimes_A (A/\mathfrak{I}^{n+1})$, we see from defintion that $\widehat{C(M)} = \varprojlim C(M_n)$. □

10.5.5 The case of Noetherian local rings

We now consider the case where A is a Noetherian local ring. Let $\mathfrak{I}$ be the maximal ideal of A. Let $T : \mathbf{Mod}(A) \to \mathbf{Ab}$ be a covariant additive functor; then the canonical homomorphisms (10.5.3)

$$T(M) \otimes_A (A/\mathfrak{m}^{n+1}) \to T(M \otimes_A (A/\mathfrak{m}^{n+1}))$$

form a projective system of A-homomorphisms, hence give a functorial homomorphism of $\widehat{A}$-modules

$$\widehat{T(M)} \to \varprojlim T(M_n) \tag{10.5.14}$$

where $M_n = M \otimes_A (A/\mathfrak{m}^{n+1})$ and $A_n = A/\mathfrak{m}^{n+1}$.

Proposition 10.5.23. *Let A be a Noetherian local ring with maximal ideal $\mathfrak{m}$, residue field k, $T : \mathbf{Mod}(A) \to \mathbf{Ab}$ be a covariant additive functor, which is semi-exact and commutes with inductive limits. Suppose that for every finitely generated A-module M, $T(M)$ is a finitely generated A-module and the canonical homomorhism (10.5.14) is an isomorphism. Under these conditions, the following properties are equivalent:*

(i) *T is right exact;*

(ii) *for each n, the functor $N \mapsto T(N)$ is right exact on the category of finitely generated A_n-modules (which means T is exact on the category of A-modules of finite length);*

(iii) *the canonical homomorphism $T(\varepsilon) : T(A) \to T(k)$ is surjective;*

(iv) *for sufficiently large n, the canonical homomorphism $T(A_n) \to T(k)$ is surjective.*

Proof. It is clear that (i) implies (ii); to see that (ii) implies (i), let $u : M \to N$ be an epimorphism of A-modules. As T commutes with inductive limits and the functor $\varinjlim$ is exact on the category of A-modules (indexed by filtered sets), we can also suppose that M and N are finitely generated, so by hypothesis $T(M)$ and $T(N)$ are also finitely generated. As A is a Noethertian local ring (hence a Zariski ring with the $\mathfrak{m}$-adic topology), it suffices to prove that $\widehat{T(u)} : \widehat{T(M)} \to \widehat{T(N)}$ is surjective (**??**). By hypothesis, $\widehat{T(M)}$ and $\widehat{T(N)}$ are isomorphic to $\varprojlim T(M_n)$ and $\varprojlim T(N_n)$, respectively, and $\widehat{T(u)}$ can be identified with the projective limit of the homomorphisms $T(u \otimes 1_{A_n}) : T(M_n) \to T(N_n)$, which are all surjective by condition (ii). We then conclude that $\widehat{T(u)}$ is surjective, so (ii) implies (i). It is clear that (i) implies (iii), and since $T(\varepsilon)$ can be written as $T(A) \to T(A_n) \to T(k)$, (iii) implies (iv). Finally, by Proposition 10.5.6 we know that (ii) is equivalent to (iv), since T is semi-exact on $\mathbf{Mod}(A_n)$. □

Corollary 10.5.24. *Under the hypotheses of Proposition 10.5.23, if $T(k) = 0$, then $T = 0$.*

Proof. As k is the only simple A-module, we deduce from (**??** 7.3.5.4) that $T(E) = 0$ for any A-module E of finite length. If M is a finitely generated A-module, then $\widehat{T(M)}$ is isomorphic to $\varprojlim T(M_n)$, and as M_n are of finite length, we have $\widehat{T(M)} = 0$, so $T(M) = 0$ since it is finitely generated (**??**). Finally, for any A-module M, $T(M)$ is the inductive limit of $T(N_\alpha)$, where N_α are the finitely generated submodules of M, whence the assertion. □

Proposition 10.5.25. *Let A be a Noetherian local ring with maximal ideal $\mathfrak{m}$, residue field k, $T^\bullet : \mathbf{Mod}(A) \to$ $\mathbf{Ab}$ be a cohomological functor which is semi-exact and commutes with inductive limits. Suppose that for any integer i and any finitely generated A-module M, $T^i(M)$ is finitely generated and the canonical homomorphism $\widehat{T^i(M)} \to \varprojlim T^i(M_n)$ is bijective. For an integer p, the following conditions are equivalent:*

(i) *T^p is exact;*

(ii) *T^p is right exact and $T^p(A)$ is a free A-module.*

(iii) *the canonical homomorphism $T^i(A) \to T^i(k)$ is surjective for $i = p, p-1$;*

(iv) *for each n, the canonical homomorphism $T^i(A_n) \to T^i(k)$ is surjective for $i = p, p-1$;*

(v) *for each n, the functor $N \mapsto T^p(N)$ is exact on the category of finitely generated A_n-modules.*

Proof. It follows from Proposition 10.5.7 that (i) is equivalent to that T^p and T^{p-1} are right exact; by the same reasoning, (v) is equivalent to say that T^p and T^{p-1} are right exact on the category of finitely generated A_n-modules. We then deduce form Proposition 10.5.23 that (i) is equivalent to (v); the equivalence of (i), (iii) and (iv) also follows from Proposition 10.5.23. Finally, we see that any finitely generated flat A-module is projective, the equivalence of (i) and (ii) follows from Proposition 10.5.9. □

Corollary 10.5.26. *Suppose the hypotheses of Proposition 10.5.25.*

(a) *If $T^p(k) = 0$, then $T^p = 0$, T^{p-1} is right exact and T^{p+1} is left exact.*

(b) *If $T^{p-1}(k) = T^{p+1}(k) = 0$, then T^p is exact, the canonical homomorphism*

$$T^p(A) \otimes_A M \to T^p(M)$$

is bijective and $T^p(A)$ is a free A-module.

Proof. Assertion (a) follows directly from Corollary 10.5.24 since T^p is semi-exact, and the first two assertions of (b) also follows from this. Finally, the last assertion follows from Proposition 10.5.23. □

Corollary 10.5.27. *Retain the the hypotheses of Proposition 10.5.25 and suppose that A is a DVR.*

(a) *For T^p to be right exact, it is necessary and sufficient that $T^{p+1}(A)$ is a free A-module.*

(b) *For T^p to be exact, it is necessary and sufficient that $T^p(A)$ and $T^{p+1}(A)$ are free A-modules.*

Proof. It is clear that (a) implies (b), in view of Proposition 10.5.7. To prove (a), let π be a uniformizer of A; for a finitely generated A-module M to be free (or equivalently, flat), it is necessary and sufficient that the homothety $h_\pi : x \mapsto \pi x$ of M is injective, because this means M is torsion free. Consider the exact sequence

$$0 \longrightarrow A \xrightarrow{h_f} A \longrightarrow k \longrightarrow 0$$

which induces an exact sequence

$$T^p(A) \longrightarrow T^p(k) \longrightarrow T^{p+1}(A) \xrightarrow{h_f} T^{p+1}(A)$$

We then see that $T^{p+1}(A)$ is free if and only if $T^p(A) \to T^p(k)$ is surjective, which proves our assertion (Proposition 10.5.23). □

10.5.6 Descent of exactness and the semi-continuity theorem

Proposition 10.5.28. *Under the hypothesis of Proposition 10.5.16, let B be a commutative A-algebra. If T^p is right exact (resp. left exact), then so is $T^p_{(B)}$. The converse is true if B is a fiathfully flat A-module.*

Proof. The first assertion is trivial since the forgetful functor is exact and faithful. Conversely, suppose first that B is a flat A-module. We then have, for any A-module M, $H^\bullet(P^\bullet \otimes_A (M \otimes_A B)) = (H^\bullet(P^\bullet \otimes_A M)) \otimes_A B$, wich means, for each p,

$$T^p(M) \otimes_A B = T^p_{(B)}(M_{(B)}). \tag{10.5.15}$$

Suppose that $T^p_{(B)}$ is right exact (resp. right exact), then as $M \to M_{(B)}$ is an exact functor, the first member of (10.5.15) is a right exact (resp. left exact) functor on M; if B is assumed to be faithfully flat, then we deduce that T^p has the same exactness property. □

Proposition 10.5.29. *Under the hypothesis of Proposition 10.5.16, suppose that A is a reduced Noetherian ring and the P^i are finitely generated A-modules. For T^p to be right exact (resp. left exact), it is necessary and sufficient that, for any A-algebra B which is a DVR, $T^p_{(B)}$ is right exact (resp. left exact).*

Proof. In view of Proposition 10.5.7, we only need to consider the left exactness, hence prove the sufficiency of the condition of the corollary. In view of Proposition 10.5.16, it suffices to prove that $W^p(P^\bullet)$ is a flat A-module; as P^p is finitely generated, $W^p(P^\bullet)$ is also finitely generated, so the criterion of ([?] 0_{III}, 10.2.8) shows that it suffices to show that $W^p(P^\bullet) \otimes_A B$ is a flat B-module for any A-algebra B which is a DVR. Now, as $P^\bullet$ is a complex of flat A-modules, we have

$$W^p(P^\bullet) \otimes_A B = W^p(P^\bullet \otimes_A B);$$

where $P^\bullet \otimes_A B$ is a complex of flat B-modules and for any B-module N, we have $H^\bullet(P^\bullet \otimes_A N) = H^\bullet((P^\bullet \otimes_A B) \otimes_B N)$, so $T^p_{(B)}(N) = H^p((P^\bullet \otimes_A B) \otimes_B N)$. Applying Proposition 10.5.16 to $T^p_{(B)}$, we see that the hypothesis that $T^p_{(B)}$ is left exact is equivalent to that $W^p(P^\bullet \otimes_A B)$ is a flat B-module, whence our assertion. □

The criterion of Proposition 10.5.29 reduces the exactness of T^p to the case of DVRs. In this case, we have the following useful result, which can be considered as a "universal coefficient theorem".

Proposition 10.5.30. *Under the hypothesis of Proposition 10.5.16, suppose that A is a one dimensional regular Noetherian ring (in other words, A is Noetherian and for any $x \in \operatorname{Spec}(A)$, A_x is a field or a DVR). Then for any integer p and A-module M, we have a canonical exact sequence*

$$0 \longrightarrow T^p(A) \otimes_A M \xrightarrow{t_M} T^p(M) \longrightarrow \operatorname{Tor}_1^A(T^{p+1}(A), M) \longrightarrow 0 \tag{10.5.16}$$

Proof. We simplify our notation by writing H^p, Z^p, B^p for $H^p(P^\bullet)$, $Z^p(P^\bullet)$, $B^p(P^\bullet)$. Then we have the following exact sequences

$$0 \longrightarrow H^p \longrightarrow W^p \longrightarrow B^{p+1} \longrightarrow 0$$
$$0 \longrightarrow B^{p+1} \longrightarrow Z^{p+1} \longrightarrow H^{p+1} \longrightarrow 0$$
$$0 \longrightarrow Z^p \longrightarrow P^p \longrightarrow B^{p+1} \longrightarrow 0$$

As P^p and P^{p+1} are flat, so are their submodules (since for any $x \in \operatorname{Spec}(A)$, a A_x-module is flat if and only if it is torsion free). By tensoring with M, we then obtain exact sequences

$$0 \longrightarrow H^p \otimes_A M \longrightarrow W^p \otimes_A M \xrightarrow{u} B^{p+1} \otimes_A M \longrightarrow 0 \tag{10.5.17}$$
$$0 \longrightarrow \operatorname{Tor}_1^A(H^{p+1}, M) \longrightarrow B^{p+1} \otimes_A M \xrightarrow{v} Z^{p+1} \otimes_A M \longrightarrow H^{p+1} \otimes_A M \longrightarrow 0 \tag{10.5.18}$$
$$0 \longrightarrow Z^p \otimes_A M \xrightarrow{w} P^p \otimes_A M \longrightarrow B^{p+1} \otimes_A M \longrightarrow 0 \tag{10.5.19}$$

By definition, $T^p(M) = \ker(d^p \otimes 1_M)/\operatorname{im}(d^{p-1} \otimes 1_M)$, which is the kernel of the homomorphism

$$(P^p \otimes_A M)/\operatorname{im}(d^{p-1} \otimes 1_M) \to P^{p+1} \otimes_A M$$

obtained from $d^p \otimes 1_M$ by passing to quotient. By definition we have $W^p = P^p/B^p$, so this homomorphism can also be written as $W^p \otimes_A M \to P^{p+1} \otimes_A M$, which is exactly the composition

$$W^p \otimes_A M \xrightarrow{u} B^{p+1} \otimes_A M \xrightarrow{v} Z^{p+1} \otimes_A M \xrightarrow{w} P^{p+1} \otimes_A M$$

By (10.5.19), the homomorphism w is injective, so we get an exact sequence

$$0 \longrightarrow \ker u \longrightarrow T^p(M) \longrightarrow \ker v \longrightarrow 0$$

This is exactly (10.5.16), since we have $H^p = T^p(A)$ by (10.5.17) and (10.5.18). □

Remark 10.5.31. Since $H^\bullet(P^\bullet \otimes_A M)$ is the cohomology of the double complex $P^\bullet \otimes_A M$ (where M is considered as a complex with only zero-th term), it is the limit of a regular homological spectral sequence with term E^2 given by

$$E^2_{p,q} = \operatorname{Tor}_p^A(H_q(P^\bullet), M) = \operatorname{Tor}_p^A(T^q(A), M).$$

The hypothesis of Proposition 10.5.30 implies that for any A-module E, F, we have $\mathrm{Tor}_p^A(E,F) = 0$ for $p \geq 2$ (**??**), so we have an exact sequence

$$0 \longrightarrow E^2_{0,q} \longrightarrow H^q(P^\bullet \otimes_A M) \longrightarrow E^2_{1,q-1}$$

which is none other than (10.5.16).

Corollary 10.5.32. *Under the hypothesis of Proposition 10.5.16, suppose that A is a DVR with fraction field K, residue field k, and that the $T^i(A)$ are finitely generated A-modules. Then*

$$\dim_k(T^p(k)) \geq \dim_k(T^p(A) \otimes_A k) \geq \mathrm{rank}_A(T^p(A)) = \dim_K(T^p(K)). \tag{10.5.20}$$

Moreover, for the equality hold, it is necessary and sufficient that T^p is exact, or that $T^p(A)$ and $T^{p+1}(A)$ are free A-modules.

Proof. By taking $M = k$ in the exact sequence (10.5.16), we see that

$$\dim_k(T^p(k)) = \dim_k(T^p(A) \otimes_A k) + \dim_k(\mathrm{Tor}_1^A(T^{p+1}(A), k)).$$

On the other hand, as $T^p(A)$ is a finitely generated module over the integral local ring A, we have (**??**)

$$\dim_k(T^p(A) \otimes_A k) \geq \mathrm{rank}_A(T^p(A)) = \mathrm{rank}_K(T^p(A) \otimes_A K)$$

and the equality holds if and only if $T^p(A)$ is a free A-module. On the other hand, since K is a flat A-module, we have $T^p(A) \otimes_A K = H^p(P^\bullet) \otimes_A K = H^p(P^\bullet \otimes_A K) = T^p(K)$, whence the inequalities in (10.5.20). Moreover, by our remarks, the equality holds if and only if $T^p(A)$ is free and $\mathrm{Tor}_1^A(T^{p+1}(A), k) = 0$, where ther later conditions means $T^{p+1}(A)$ is a free A-module. Finally, the last assertion follows from Corollary 10.5.27. □

Retain the hypothesis of Proposition 10.5.16, we now consider the function

$$d_p(x) = d_p^T(x) = \dim_{\kappa(x)}(T^p(\kappa(x)))$$

on the space $\mathrm{Spec}(A)$.

Lemma 10.5.33. *Let $\rho : A \to A'$ be a ring homomorphism and $f : \mathrm{Spec}(A') \to \mathrm{Spec}(A)$ be the corresponding morphism. If we set $T'^\bullet = T^\bullet_{(A')}$, then*

$$d_p^{T'} = d_p^T \circ f.$$

Proof. For any $x' \in \mathrm{Spec}(A')$, we have (setting $x = f(x')$)

$$H^\bullet(P^\bullet \otimes_A \kappa(x')) = H^\bullet((P^\bullet \otimes_A \kappa(x)) \otimes_{\kappa(x)} \kappa(x')) = H^\bullet((P^\bullet \otimes_A \kappa(x))) \otimes_{\kappa(x)} \kappa(x')$$

since $\kappa(x')$ is flat over $\kappa(x)$, whence the assertion. □

Lemma 10.5.34. *If A is a Noetherian ring and $P^\bullet$ is a complex of finitely generated flat A-modules. Then the function $d_p^T(x)$ is constructible.*

Proof. It suffices to prove that for any closed irreducible subset Y of $\mathrm{Spec}(A)$, there exists a nonempty open subset U of Y such that d_p is constant on U ([**?**] 0_{III}, 9.2.2). As $Y = \mathrm{Spec}(A/\mathfrak{a})$ for some radical ideal $\mathfrak{a}$, we can, in view of Lemma 10.5.34, reduce to the case where $Y = X$ and A is an integral Noetherian domain. But the conclusion then follows from Corollary 10.5.14. □

Theorem 10.5.35. *Let A be a Noetherian ring, $P^\bullet$ be a complex of finitely generated flat A-modules, and $T^\bullet(M) = H^\bullet(P^\bullet \otimes_A M)$ the cohomological functor defined by $P^\bullet$. For each $x \in \mathrm{Spec}(A)$, let $d_p(x) = \dim_{\kappa(x)}(T^p(\kappa(x)))$.*

(a) *The function d_p is constructible and upper semi-continuous on* $\mathrm{Spec}(A)$.

(b) *If T^p is exact, then d_p is continuous (hence locally constant) on* $\mathrm{Spec}(A)$. *The converse is true if A is reduced.*

Proof. The fisrt part of assertion (a) follows from Lemma 10.5.34. To prove the second one, it suffices ([?] 0_{III}, 9.3.4) to show that if $y \rightsquigarrow x$ is a generalization of $x \in \operatorname{Spec}(A)$, then $d_p(y) \leq d_p(x)$. Now, there exists a discrete valuation ring B and a morphism $f : \operatorname{Spec}(B) \to \operatorname{Spec}(A)$ such that, if a denote the closed point of $\operatorname{Spec}(B)$ and b is the genetric point, we have $f(a) = x$ and $f(b) = y$ (Proposition 9.7.1). In view of Lemma 10.5.34, we are then reduced to prove the inequality $d_p(a) \geq d_p(b)$ in $\operatorname{Spec}(B)$, which is none other than the inequality (10.5.20).

The first assertion of (b) follows similarly from Corollary 10.5.32 by passing to DVRs. For the converse, we use the criterion of Proposition 10.5.29. In view of Lemma 10.5.34, we can reduce to the case where A is a DVR. But as $\operatorname{Spec}(A)$ consists of two points, the hypothesis that d_p is continuous implies that T^p is exact, in view of Corollary 10.5.32. □

Corollary 10.5.36. *Let A be a Noetherian ring, $(\mathfrak{p}_i)_{1\leq i\leq r}$ be its minimal prime ideals, and k_i be the residue field of $\mathfrak{p}_i$.*

(a) *For any $x \in \operatorname{Spec}(A)$, there exists an index i such that $d_p(x) \geq \dim_{k_i}(T^p(k_i))$. In particular, if A is integral with fraction field K, we have $d_p(x) \geq \dim_K(T^p(K))$.*

(b) *Suppose that A is local and reduced, and let k be its residue field. Then, for T^p to be exact, it is necessary and sufficient that for each $1 \leq i \leq r$, we have*

$$\dim_k(T^p(k)) = \dim_{k_i}(T^p(k)). \tag{10.5.21}$$

Proof. Assertion (a) is immediate since any neighborhood of x contains one of the $\mathfrak{p}_i$, and it suffices to apply the definition of upper semi-continuity. On the other hand, if A is local, the only neighborhood in $\operatorname{Spec}(A)$ of the maximal ideal $\mathfrak{m}$ is $\operatorname{Spec}(A)$, so we have $d_p(x) \leq \dim_k(T^p(k))$ for any $x \in \operatorname{Spec}(A)$. Now, the relation (10.5.21) then implies that $d_p(x)$ is constant on $\operatorname{Spec}(A)$, and therefore T^p is exact in view of Theorem 10.5.35(b). The converse is also evident in view of this, since the specturm of a local ring is connected. □

10.5.7 Application to proper morphisms

10.5.7.1 The semicontinuity and exchange property Let $f : X \to Y$ be a quasi-compact and separated morphis of schemes, and let $\mathscr{P}^\bullet$ be a complex of quasi-coherent $\mathscr{O}_X$-modules. Suppose that each $\mathscr{P}^i$ is a Y-flat $\mathscr{O}_X$-module, so that we can consider the δ-functor defined by

$$\mathscr{T}^p(\mathscr{P}^\bullet, \mathscr{M}) = \mathscr{T}^p(\mathscr{P}^\bullet, \mathscr{M}) = R^p f_*(\mathscr{P}^\bullet \otimes_{\mathscr{O}_Y} \mathscr{M}) \tag{10.5.22}$$

where $\mathscr{M}$ is a quasi-coherent $\mathscr{O}_Y$-module. It follows from our hypothesis that $\mathscr{T}^\bullet$ is a cohomological functor on $\mathbf{Qcoh}(Y)$.

Now let $g : Y' \to Y$ be a base change morphism, and put $X' = X \times_Y Y'$, $f' = f_{(Y')} : X' \to Y'$, which is a quasi-compact and separated morphism. Let $\mathscr{P}'^\bullet = \mathscr{P}^\bullet \otimes_{\mathscr{O}_Y} \mathscr{O}_{Y'}$, which is a complex of Y'-flat $\mathscr{O}_{X'}$-modules. We can then define

$$\mathscr{T}^\bullet_{Y'} = Rf'_*(\mathscr{P}^\bullet \otimes_{\mathscr{O}_Y} \mathscr{M}')$$

as a cohomological functor on $\mathbf{Qcoh}(Y')$. If Y' is an affine scheme with ring A', then we also write $\mathscr{T}^\bullet_{(A')}$ for $\mathscr{T}^\bullet_{Y'}$. For any A'-module M', we then have $\mathscr{T}^\bullet_{A'}(\widetilde{M}') = \widetilde{T^\bullet_{A'}(M')}$, where we set $T^\bullet_{A'}(M') = \Gamma(Y', \mathscr{T}^\bullet_{A'}(\widetilde{M}'))$. Then $T^\bullet_{A'}$ is a cohomological functor on the category of A'-modules. We observe that if $Y = \operatorname{Spec}(A)$ is also affine, the functor $T^\bullet_{A'}$ is then the extension of scalars of the functor $T^\bullet_A$: in fact, let $g' : X' \to X$ be the bese change morphism of g; if $\mathfrak{U}$ is an affine open covering of X, $\mathfrak{U}' = g'^{-1}(\mathfrak{U})$ is then an affine open covering of X'. In view of ([?], 6.2.2), it boils down to show that

$$\mathscr{C}^\bullet(\mathfrak{U}, \mathscr{P}^\bullet \otimes_{\mathscr{O}_Y} g_*(\mathscr{M}')) = \mathscr{C}^\bullet(\mathfrak{U}', \mathscr{P}^\bullet \otimes_{\mathscr{O}_Y} \mathscr{M}),$$

and finally, that for any affine open U of X, if $U' = g^{-1}(U)$, then we have $\Gamma(U, \mathscr{P}^\bullet \otimes_{\mathscr{O}_Y} g_*(\mathscr{M}')) = \Gamma(U', \mathscr{P}^\bullet \otimes_{\mathscr{O}_Y} \mathscr{M})$, which is trivial. In particular, if U is an open subset of Y, we then have

$$\mathscr{T}^\bullet_U(\mathscr{M}|_U) = (\mathscr{T}^\bullet(\mathscr{M}))|_U.$$

For any quasi-coherent $\mathscr{O}_Y$-module $\mathscr{M}$, we have a canonical homomorphism

$$\mathscr{T}^p(\mathscr{O}_Y) \otimes_{\mathscr{O}_Y} \mathscr{M} \to \mathscr{T}^p(\mathscr{M}). \tag{10.5.23}$$

In fact, if Y is affine, this is the canonical homomorphism (10.5.3); this definition extends to the general case, since if U, V are two affine opens of Y such that $V \subseteq U$, the diagram

$$\begin{tikzcd}
(\mathscr{T}^p(\mathscr{O}_Y) \otimes_{\mathscr{O}_Y} \mathscr{M})|_U = \mathscr{T}^p_U(\mathscr{O}_Y|_U) \otimes_{\mathscr{O}_Y|_U} (\mathscr{M}|_U) \arrow[r] \arrow[d] & \mathscr{T}^p(\mathscr{M}|_U) = (\mathscr{T}^p(\mathscr{M}))|_U \arrow[d] \\
(\mathscr{T}^p(\mathscr{O}_Y) \otimes_{\mathscr{O}_Y} \mathscr{M})|_V = \mathscr{T}^p_V(\mathscr{O}_Y|_V) \otimes_{\mathscr{O}_Y|_V} (\mathscr{M}|_V) \arrow[r] & \mathscr{T}^p(\mathscr{M}|_V) = (\mathscr{T}^p(\mathscr{M}))|_V
\end{tikzcd}$$

is commutative by (10.5.4). Similarly, for any morphism $g : Y' \to Y$, we have a canonical homomorphism

$$\mathscr{T}^p(\mathscr{O}_Y) \otimes_{\mathscr{O}_Y} \to \mathscr{T}^p_{Y'}(\mathscr{O}_{Y'})$$

which is none other than the homomorphism (10.5.6).

If f is a proper morphism, $Y = \operatorname{Spec}(A)$ is a Noetherian affine scheme and $\mathscr{P}^\bullet$ is a *bounded above* complex of Y-flat coherent $\mathscr{O}_Y$-modules, we then have ([?], 6.10.5) an isomorphism $\mathscr{T}^p \xrightarrow{\sim} H^\bullet(\mathscr{L}^\bullet \otimes_{\mathscr{O}_Y} \mathscr{M})$, where $\mathscr{L}^\bullet = \widetilde{L}^\bullet$ and $L^\bullet$ is a bounded above complex of free A-modules of finite rank. The functor $\mathscr{T}^\bullet$ is then of the type we encountered in Section 10.5.4, so the corresponding results can be applied.

Theorem 10.5.37. *Let Y be a locally Noetherian scheme, $f : X \to Y$ be a proper morphism, and $\mathscr{P}^\bullet$ be a bounded above complex of Y-flat coherent $\mathscr{O}_Y$-modules. Then the functor $\mathscr{T}^\bullet$ defined by (10.5.22) satisfies the following conditions.*

(a) *(**Semi-continuity**) The function $y \mapsto d_p(y) = \dim_{\kappa(y)}(T^p_{(\kappa(y))}(\kappa(y)))$ is upper semi-continuous on Y.*

(b) *(**Exchange property**) For any integer p, the following conditions are equivalent:*

  (i) *$\mathscr{T}^p$ is right exact;*

  (i′) *$\mathscr{T}^p$ is isomorphic to the functor $\mathscr{M} \mapsto \mathscr{N} \otimes_{\mathscr{O}_Y} \mathscr{M}$, where $\mathscr{N}$ is isomorphic to $\mathscr{T}^p(\mathscr{O}_Y) = R^p f_*(\mathscr{P}^\bullet)$;*

  (i″) *the canonical homomorphism (10.5.23) is an isomorphism;*

  (ii) *$\mathscr{T}^{p+1}$ is left exact;*

  (ii′) *$\mathscr{T}^{p+1}$ is isomorphic to the functor $\mathscr{M} \mapsto \mathcal{H}om_{\mathscr{O}_Y}(\mathscr{Q}, \mathscr{M})$, where $\mathscr{M}$ is a coherent $\mathscr{O}_Y$-module uniquely determined up to isomorphism;*

  (iii) *for any affine open $U = \operatorname{Spec}(A)$ of Y, the functor T^p_A is right exact;*

  (iv) *for any morphism $g : Y \to Y'$, the canonical homomorphism*

$$\mathscr{T}^p(\mathscr{O}_Y) \otimes_{\mathscr{O}_Y} \to \mathscr{T}^p_{Y'}(\mathscr{O}_{Y'})$$

  is an isomorphism.

Proof. The semi-continuous property is local over Y, hence follows from Theorem 10.5.35. It is clear that (i″) implies (i′) and (i′) implies (i), and the equivalence of (i), (i″), (ii) and (ii′) is proved in Proposition 10.5.16 and Proposition 10.5.7 if Y is affine. To pass to general case, we first prove that (i)⇔(iii), which shows that property (i) is local over Y; our demonstration will also apply to prove that properties (i″) and (ii) are local over Y. Now it is clear that (iii) implies (i), so we only need to consider the converse. It evidently suffices to show that for any affine open U of Y and any exact sequence $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ of quasi-coherent $(\mathscr{O}_Y|_U)$-modules, there exists an exact sequence $\mathscr{O}_Y$-modules whose restriction to U is the given sequence. Now, this follows from the hypothesis that Y is locally Noetherian and (Proposition 8.6.58): we can extend $\mathscr{F}$ (resp. $\mathscr{F}'$) to a quasi-coherent $\mathscr{O}_X$-module $\mathscr{G}$ (resp. sub-$\mathscr{O}_X$-module $\mathscr{G}'$ of $\mathscr{G}$), and it suffices to put $\mathscr{G}'' = \mathscr{G}/\mathscr{G}'$.

To prove the equivalence of (ii) and (ii′) in the general case, we note that if Y is affine, the $\mathscr{O}_Y$-module $\mathscr{Q}$ is determined uniquely up to isomorphism; if U is an affine open of the affine scheme Y, then there is a functorial isomorphism

$$\mathscr{T}^{p+1}_U(\mathscr{M}|_U) \xrightarrow{\sim} \mathcal{H}om_{\mathscr{O}_Y|_U}(\mathscr{Q}|_U, \mathscr{M}|_U).$$

In the general case, for any affine open U of Y, there exists a coherent $(\mathscr{O}_Y|_U)$-module $\mathscr{Q}_U$ and a functorial isomorphism $\mathscr{T}^{p+1}_U(\mathscr{M}|_U) \xrightarrow{\sim} \mathcal{H}om_{\mathscr{O}_Y|_U}(\mathscr{Q}_U, \mathscr{M}|_U)$; the preceding remark allows us to glue the $\mathscr{Q}_U$ to produce a coherent $\mathscr{O}_Y$-module which satisfies (ii′).

Finally, we prove the equivalence of (i) and (iv); it is clear that (iv) is local over Y, and we have seen that so is (i); moreover, (iv) is also local over Y'. If $Y = \mathrm{Spec}(A)$ and $Y' = \mathrm{Spec}(A')$, then $T^\bullet_{A'}$ is the extension of scalars of $T^\bullet_A$, and it is clear that (i′) implies (iv). Conversely, suppose that $Y = \mathrm{Spec}(A)$ and let A' be the A-algebra $A \oplus M$, where M is a given A-module and the multiplication on A' is defined by $(a_1, m_1)(a_2, m_2) = (a_1, a_2, a_1 m_2 + a_2 m_1)$. Then we have

$$T^p_{A'}(A') = T^p(A \oplus M) = T^p(A) \oplus T^p(M)$$

and the hypothesis (iv) implies that the canonical homomorphism $T^p(A) \otimes_A M \to T^p(M)$ is also bijective, so (iv) implies (i″) and this completes the proof. □

Theorem 10.5.38. *Let Y be a locally Noetherian scheme, $f : X \to Y$ be a proper morphism, and $\mathscr{F}$ be a Y-flat coherent $\mathscr{O}_Y$-module. Then there exists a coherent $\mathscr{O}_Y$-module $\mathscr{Q}$ (uniquely determined up to isomorphism) such that there exists an isomorphism of functors on quasi-coherent $\mathscr{O}_Y$-modules $\mathscr{M}$:*

$$f_*(\mathscr{F} \otimes_{\mathscr{O}_Y} \mathscr{M}) \xrightarrow{\sim} \mathcal{H}om_{\mathscr{O}_Y}(\mathscr{Q}, \mathscr{M}).$$

Proof. In fact, the functor $\mathscr{T}^0 = f_*$ is left exact, so the assertion follows from Theorem 10.5.37. □

Corollary 10.5.39. *Let Y be a locally Noetherian scheme, $f : X \to Y$ be a proper morphism, $\mathscr{F}, \mathscr{F}'$ be Y-flat coherent $\mathscr{O}_Y$-modules, and $u : \mathscr{F} \to \mathscr{F}'$ be a homomorphism. Consider the following functors on quasi-coherent $\mathscr{O}_Y$-modules $\mathscr{M}$:*

$$\mathscr{T}(\mathscr{M}) = \ker(f_*(\mathscr{F} \otimes_{\mathscr{O}_Y} \mathscr{M}) \to f_*(\mathscr{F}' \otimes_{\mathscr{O}_Y} \mathscr{M})),$$
$$T(\mathscr{M}) = \Gamma(Y, \mathscr{T}(\mathscr{M})) = \ker(\Gamma(X, \mathscr{F} \otimes_{\mathscr{O}_Y} \mathscr{M}) \to \Gamma(X, \mathscr{F}' \otimes_{\mathscr{O}_Y} \mathscr{M})).$$

Then there exists a coherent $\mathscr{O}_Y$-module $\mathscr{R}$ (uniquely determined up to isomorphism) and isomorphisms of functors

$$\mathscr{T}(\mathscr{M}) \xrightarrow{\sim} \mathcal{H}om_{\mathscr{O}_Y}(\mathscr{R}, \mathscr{M}), \quad T(\mathscr{M}) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{O}_Y}(\mathscr{R}, \mathscr{M}). \tag{10.5.24}$$

Proof. It suffices to consider the functor $T(\mathscr{M})$, so by Theorem 10.5.38 there exists two coherent $\mathscr{O}_Y$-modules $\mathscr{Q}, \mathscr{Q}'$ defining functorial isomorphisms

$$\Gamma(X, \mathscr{F} \otimes_{\mathscr{O}_Y} \mathscr{M}) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{O}_Y}(\mathscr{Q}, \mathscr{M}), \quad \Gamma(X, \mathscr{F}' \otimes_{\mathscr{O}_Y} \mathscr{M}) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{O}_Y}(\mathscr{Q}', \mathscr{M}).$$

Now, $u : \mathscr{F} \to \mathscr{F}'$ defines a morphism of functors

$$\Gamma(X, \mathscr{F} \otimes_{\mathscr{O}_Y} \mathscr{M}) \to \Gamma(X, \mathscr{F}' \otimes_{\mathscr{O}_Y} \mathscr{M})$$

which corresponds to a unique homomorphism $v : \mathscr{Q}' \to \mathscr{Q}$ such that the diagram

$$\begin{tikzcd}
\Gamma(X, \mathscr{F} \otimes_{\mathscr{O}_Y} \mathscr{M}) \arrow[r] \arrow[d, "\sim"] & \Gamma(X, \mathscr{F}' \otimes_{\mathscr{O}_Y} \mathscr{M}) \arrow[d, "\sim"] \\
\mathrm{Hom}_{\mathscr{O}_Y}(\mathscr{Q}, \mathscr{M}) \arrow[r] & \mathrm{Hom}_{\mathscr{O}_Y}(\mathscr{Q}', \mathscr{M})
\end{tikzcd}$$

is commutative. As the covariant functor $\mathscr{N} \mapsto \mathrm{Hom}_{\mathscr{O}_Y}(\mathscr{N}, \mathscr{M})$ is left exact, it then suffices to put $\mathscr{R} = \mathrm{coker}\, v$. □

Corollary 10.5.40. *Let Y be a locally Noetherian scheme, $f : X \to Y$ be a proper morphism, $\mathscr{F}, \mathscr{G}$ be coherent $\mathscr{O}_X$-modules satisfying the following conditions:*

(a) *$\mathscr{F}$ is Y-flat;*

(b) *$\mathscr{G}$ is isomorphic to the cokernel of a homomorphism of locally free $\mathscr{O}_X$-modules $\mathscr{E}_1 \to \mathscr{E}_0$.*

Consider the following functors on quasi-coherent $\mathscr{O}_Y$-modules $\mathscr{M}$:

$$\mathscr{T}(\mathscr{M}) = f_*(\mathcal{H}om_{\mathscr{O}_Y}(\mathscr{G}, \mathscr{F} \otimes_{\mathscr{O}_Y} \mathscr{M})),$$
$$T(\mathscr{M}) = \Gamma(Y, \mathscr{T}(\mathscr{M})) = \mathrm{Hom}_{\mathscr{O}_X}(\mathscr{G}, \mathscr{F} \otimes_{\mathscr{O}_Y} \mathscr{M}).$$

Then there exsits a coherent $\mathscr{O}_Y$-module $\mathscr{N}$ (uniquely deteremined up to isomorphism) and isomorphisms of functors

$$\mathscr{T}(\mathscr{M}) \xrightarrow{\sim} \mathcal{H}om_{\mathscr{O}_Y}(\mathscr{N}, \mathscr{M}), \quad T(\mathscr{M}) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{O}_Y}(\mathscr{N}, \mathscr{M}).$$

Proof. We have functorial isomorphisms

$$\mathcal{H}om_{\mathcal{O}_X}(\mathscr{E}_i, \mathscr{F} \otimes_{\mathcal{O}_Y} \mathscr{M}) \xrightarrow{\sim} \check{\mathscr{E}}_i \otimes_{\mathcal{O}_X} (\mathscr{F} \otimes_{\mathcal{O}_Y} \mathscr{M}) \xrightarrow{\sim} (\check{\mathscr{E}}_i \otimes_{\mathcal{O}_X} \mathscr{F}) \otimes_{\mathcal{O}_Y} \mathscr{M} \xrightarrow{\sim} \mathcal{H}om_{\mathcal{O}_X}(\mathscr{E}_i, \mathscr{F}) \otimes_{\mathcal{O}_Y} \mathscr{M}$$

for $i = 0, 1$. Put $\mathscr{F}_i = \mathcal{H}om_{\mathcal{O}_X}(\mathscr{E}_i, \mathscr{F})$, which are Y-flat coherent $\mathcal{O}_X$-modules, and let $u : \mathcal{H}om(v, 1_{\mathscr{F}}) : \mathscr{F}_0 \to \mathscr{F}_1$. In view of the left exactness of the functor $\mathcal{H}om_{\mathcal{O}_X}(-, \mathscr{F} \otimes_{\mathcal{O}_Y} \mathscr{M})$, we have functorial isomorphisms

$$\begin{aligned}\mathcal{H}om_{\mathcal{O}_X}(\mathscr{G}, \mathscr{F} \otimes_{\mathcal{O}_Y} \mathscr{M}) &\xrightarrow{\sim} \ker(\mathcal{H}om_{\mathcal{O}_X}(\mathscr{E}_0, \mathscr{F} \otimes_{\mathcal{O}_Y} \mathscr{M}) \to \mathcal{H}om_{\mathcal{O}_X}(\mathscr{E}_1, \mathscr{F} \otimes_{\mathcal{O}_Y} \mathscr{M}))\\ &\xrightarrow{\sim} \ker(\mathscr{F} \otimes_{\mathcal{O}_Y} \mathscr{M} \to \mathscr{F}_1 \otimes_{\mathcal{O}_Y} \mathscr{M}).\end{aligned}$$

Since f_* is left exact, we then deduce functorial isomorphisms

$$f_*(\mathcal{H}om_{\mathcal{O}_X}(\mathscr{G}, \mathscr{F} \otimes_{\mathcal{O}_Y} \mathscr{M})) \xrightarrow{\sim} \ker(f_*(\mathscr{F}_0 \otimes_{\mathcal{O}_Y} \mathscr{M}) \to f_*(\mathscr{F}_1 \otimes_{\mathcal{O}_Y} \mathscr{M}))$$

and it suffices to apply Corollary 10.5.39. □

Remark 10.5.41. In Theorem 10.5.38, Corollary 10.5.39 and Corollary 10.5.40, the formation of the $\mathcal{O}_Y$-modules $\mathscr{Q}$, $\mathscr{R}$, $\mathscr{N}$ commutes with base changes. For example, in Theorem 10.5.38, let $g : Y' \to Y$ be a base change morphism. Then we have, for any quasi-coherent $\mathcal{O}_{Y'}$-module $\mathscr{M}'$, the isomorphisms

$$f'_*(\mathscr{F} \otimes_{\mathcal{O}_Y} \mathscr{M}) \xrightarrow{\sim} \mathcal{H}om_{\mathcal{O}_{Y'}}(g^*(\mathscr{Q}), \mathscr{M}')$$

because in view of the adjunction property, we have

$$\mathrm{Hom}_{\mathcal{O}_Y}(\mathscr{Q}, g_*(\mathscr{M}')) = \mathrm{Hom}_{\mathcal{O}_{Y'}}(g^*(\mathscr{Q}), \mathscr{M}').$$

Similarly, if in Corollary 10.5.39 we replace $Y, f, \mathscr{M}, \mathscr{F}, \mathscr{F}'$ by $Y', f', \mathscr{M}', \mathscr{F} \otimes_{\mathcal{O}_X} \mathcal{O}_{X'}, \mathscr{F}' \otimes_{\mathcal{O}_X} \mathcal{O}_{X'}$, then it is necessary to replace $\mathscr{R}$ by $g^*(\mathscr{R})$.

Remark 10.5.42. The condition (b) of Corollary 10.5.40 on $\mathscr{G}$ is satisfied for any $\mathscr{G}$ if there exists a Y-ample $\mathcal{O}_X$-module. In fact, it suffices to note that there exists a locally free $\mathcal{O}_X$-module $\mathscr{E}_0$ such that $\mathscr{G}$ is isomorphic to a quotient of $\mathscr{E}_0$ (Proposition 9.4.29). As $\mathscr{E}_0$ and $\mathscr{G}$ are coherent, the kernel $\mathscr{G}_1$ of $\mathscr{E}_0 \to \mathscr{G}$ is also coherent, and by applying the same reasoning to $\mathscr{G}_1$, we obtain an exact sequence $\mathscr{E}_1 \to \mathscr{E}_0 \to \mathscr{G} \to 0$, where $\mathscr{E}_0$ and $\mathscr{E}_1$ are locally free of finite type.

Proposition 10.5.43. *Under the hypothesis of Theorem 10.5.37, let y be a point of Y and p be an integer. Then the following conditions are equivalent:*

- (i) *the functor $T^p_{\mathcal{O}_{Y,y}}$ is right exact;*
- (ii) *the canonical homomorphism $T^p_{\mathcal{O}_{Y,y}}(\mathcal{O}_{Y,y}) \to T^p_{\mathcal{O}_{Y,y}}(\kappa(y))$ is surjective;*
- (iii) *for any integer n, the canonical homomorphism $T^p_{\mathcal{O}_{Y,y}}(\mathcal{O}_{Y,y}/\mathfrak{m}_y^{n+1}) \to T^p_{\mathcal{O}_{Y,y}}(\kappa(y))$ is surjective.*

Moreover, the set of $y \in Y$ verifying these conditions is exactly the open subset U of Y such that $\mathscr{T}^p_U$ is right exact.

Proof. The equivalence of (i), (ii), (iii) follows from Proposition 10.5.22 and Proposition 10.5.23. The fact that the set U where $T^p_{\mathcal{O}_{Y,y}}$ is right exact is open is a concequence of Proposition 10.5.19, and conversely if $\mathscr{T}^p_V$ is right exact, so is $T^p_{\mathcal{O}_{Y,y}}$ for any $y \in V$, by the condition (iii) of Theorem 10.5.37 and Proposition 10.5.28. □

Corollary 10.5.44. *If $\mathscr{T}^p$ is right exact (resp. left exact), then, for any morphism $g : Y' \to Y$, $\mathscr{T}^p_{Y'}$ is right exact (resp. left exact). The converse is ture if the morphism g is faithfully flat.*

Proof. The first assertion follows immediately from Proposition 10.5.28 and the fact that the question is local on Y and Y', in view of Theorem 10.5.37(ii) and (iii). To prove the second assertion, it suffices to show that for any $y \in Y$, $T^p_{\mathcal{O}_{Y,y}}$ is right exact (resp. left exact). But by our hypothesis, there exists $y' \in Y'$ such that $g(y) = y'$ and $\mathcal{O}_{Y',y'}$ is fiathfully flat over $\mathcal{O}_{Y,y}$, so the assertion follows from Proposition 10.5.28. □

10.5.7.2 Cohomological flatness Let X, Y be schemes, $f : X \to Y$ be a quasi-compact and separated morphism, $\mathscr{P}^\bullet$ be a complex of Y-flat quasi-coherent $\mathscr{O}_X$-modules, $\mathscr{T}^\bullet$ be the cohomological functor defined by $\mathscr{P}^\bullet$, and y be a point of Y. We say that $\mathscr{P}^\bullet$ is **cohomologically flat over** Y **at the point** y**, in dimension** p, if there exists an open neighborhood U of y in Y such that $\mathscr{T}_U^p$ is exact. We say $\mathscr{P}^\bullet$ is **cohomologically flat in dimension** p **over** Y if it is cohomologically flat over Y at every point $y \in Y$, in dimension p.

If $\mathscr{P}^\bullet$ is cohomologically flat over Y (resp. over Y at point y) for any dimension p, we simply say that $\mathscr{P}^\bullet$ is **cohomologically flat over** Y (resp. over Y at y). By definition, the notion of cohomologically flatness is local over Y. If Y is locally Noetherian, then for $\mathscr{P}^\bullet$ to be cohomologically flat over Y in dimension p, it is necessary and sufficient that the functor $\mathscr{T}^p$ is exact (for this, mimic the proof Theorem 10.5.37).

Proposition 10.5.45. *Let Y be a locally Noetherian scheme, $f : X \to Y$ be a proper morphism, $\mathscr{P}^\bullet$ be a bounded above complex of Y-flat coherent $\mathscr{O}_X$-modules, $\mathscr{T}^\bullet$ be the cohomological functor defined by $\mathscr{P}^\bullet$. For any $y \in Y$, the following conditions are equivalent:*

(i) *$\mathscr{P}^\bullet$ is cohomologically flat over Y at y in dimension p;*

(ii) *the functor $\mathscr{T}^p$ is exact;*

(iii) *there exists an integer n_0 such that for $n \geq n_0$, we have*

$$\ell_{\mathscr{O}_{Y,y}}(T^p_{\mathscr{O}_{Y,y}}(\mathscr{O}_{Y,y}/\mathfrak{m}_y^{n+1})) = \ell_{\mathscr{O}_{Y,y}}(T^p_{\mathscr{O}_{Y,y}}(\kappa(y))) \cdot \ell_{\mathscr{O}_{Y,y}}(\mathscr{O}_{Y,y}/\mathfrak{m}_y^{n+1}). \tag{10.5.25}$$

(iv) *there exists an open neighborhood U of y such that $R^p f_*(\mathscr{P}^\bullet)|_U$ is isomorphic to $(\mathscr{O}_Y|_U)^n$ and that, for any quasi-coherent $(\mathscr{O}_Y|_U)$-module $\mathscr{M}$, the canonical homomorphism*

$$((R^p f_*(\mathscr{P}^\bullet))|_U) \otimes_{\mathscr{O}_Y|_U} \mathscr{M} \to R^p f_*((\mathscr{P}^\bullet|_U) \otimes_{\mathscr{O}_Y|_U} \mathscr{M}) \tag{10.5.26}$$

is bijective.

If these conditions are verified, then we have the following:

(v) *There exists an open neighborhood U of y such that the function d_p of Theorem 10.5.35 is constant in U.*

Moreover, if Y is reduced at the point y, then (v) is equivalent to the above conditions.

Proof. In fact, condition (ii) is equivalent to say that $T^p_{\mathscr{O}_{Y,y}}$ and $T^{p-1}_{\mathscr{O}_{Y,y}}$ are right exact, so the equivalence of (i) and (ii) are clear. As $\mathscr{O}_{Y,y}/\mathfrak{m}_y^{n+1}$ is Artinian, and that $T^p_{\mathscr{O}_{Y,y}}(\mathscr{O}_{Y,y}/\mathfrak{m}_y^{n+1})$ and $T^p_{\mathscr{O}_{Y,y}}(\kappa(y))$ are finitely generated $(\mathscr{O}_{Y,y}/\mathfrak{m}_y^{n+1})$-modules, hence of finite length, so the equivalence of (ii) and (iii) follows from Proposition 10.5.43 and Proposition 10.5.11. The fact that (i) implies (v), and the equivalence if Y is reduced at y, all results from Theorem 10.5.35. Finally, (i) implies that $(R^p f_*(\mathscr{P}^\bullet))_y$ is a flat $\mathscr{O}_{Y,y}$-module (Proposition 10.5.9(iv')), hence free since it is finitely generated over $\mathscr{O}_{Y,y}$. Since $R^p f_*(\mathscr{P}^\bullet)$ is coherent, this implies $R^p f_*(\mathscr{P}^\bullet)$ is free over an open neighborhood of y (**??**). Conversely, it is clear that (iv) implies (i) by the definition of $\mathscr{T}_U^p$. □

Proposition 10.5.46. *Under the hypothesis of Proposition 10.5.45, the following conditions are equivalent:*

(i) *$\mathscr{P}^\bullet$ is cohomologically flat over Y in dimension $i \geq p$;*

(ii) *for $i \geq p-1$, the functors $\mathscr{T}^i$ are right exact;*

(iii) *for $i \geq p$, the $\mathscr{O}_X$-modules $R^i f_*(\mathscr{P}^\bullet)$ is locally free.*

Proof. The equivalence of (i) and (ii) is verified in Proposition 10.5.9 and (i) implies (iii) in view of Proposition 10.5.45(iv). Conversely, assume that (iii) is satisfied; since the question is local over Y, we may assume that $Y = \operatorname{Spec}(A)$ is affine. In this case, since $\mathscr{P}^\bullet$ is bounded above, we then have $\mathscr{T}^\bullet \xrightarrow{\sim} H^\bullet(\mathscr{L}^\bullet \otimes_{\mathscr{O}_Y} \mathscr{M})$, where $\mathscr{L}^\bullet = \widetilde{L}^\bullet$ and $L^\bullet$ is a bounded above complex of free A-modules of finite rank. Then the functor T^i is exact for $i \gg 0$, and by our hypothesis, $T^i(A)$ is a free A-module for $i \geq p$. We conclude from **??** that $\mathscr{T}^i = T^i$ is exact for $i \geq p$. □

We usually apply the criterions of cohomologically flatness to the case where the complex $\mathscr{P}^\bullet$ has a single nonzero term $\mathscr{F}$ at degree 0. In this case, we have $\mathscr{T}^p(\mathscr{M}) = T^p f_*(\mathscr{F} \otimes_{\mathscr{O}_Y} \mathscr{M})$.

Proposition 10.5.47. *Let Y be a locally Noetherian scheme, $f : X \to Y$ be a flat and proper morphism, y be a point of Y. Suppose that $\Gamma(X_y, \mathcal{O}_{X_y}) = R$ is a separated $\kappa(y)$-algebra. Then $\mathcal{O}_X$ is cohomologically flat over Y at point y in dimension 0.*

Proof. In view of Proposition 10.5.45, we may assume that Y is the spectrum of $A = \mathcal{O}_{Y,y}$. Since f is exact, we can choose $\mathscr{F} = \mathcal{O}_X$ to define the functor $T^\bullet$, and it is clear that T^0 is left exact. It then remains to prove that T^0 is right exact, to which we can reduce to the case where $A = \mathcal{O}_{Y,y}$ is an Artinian ring (Proposition 10.5.43(iii)). Let k' be a finite extension of $\kappa(y)$ so that $R \otimes_{\kappa(y)} k'$ is a direct product of finitely many fields isomorphic to k'. Since there exists a local homomorphism from A to a local ring A', which is a free A-algebra and such that the residue field of A' is isomorphic to k' ([?] 0_{III}, 10.3.2), we can, in view of Proposition 10.5.28, assume further that R is isomorphic to a direct product of m fields isomorphic to $\kappa(y)$. In this case, it is easy to see that the fiber X_y has exactly m connected components X_i', and $\Gamma(X_i', \mathcal{O}_{X_i'}) = \kappa(y)$ for each i. Since A is now a local Artinian ring, its spectrum is reduced to a singleton, so X and X_y have the same underlying space; in particular, X also has m connected components X_i, and $X_i' = X_i \times_Y \kappa(y)$. We can then reduce to the case where $R = \kappa(y)$, and in view of Proposition 10.5.43(ii), it suffices to prove that the canonical homomorphism $\Gamma(X, \mathcal{O}_X) \to \Gamma(X_y, \mathcal{O}_{X_y})$ is surjective. But this is trivial, since the composition

$$\Gamma(Y, \mathcal{O}_Y) = A \to \Gamma(X, \mathcal{O}_X) \to \Gamma(X_y, \mathcal{O}_{X_y}) = \kappa(y)$$

is surjective. □

Corollary 10.5.48. *Under the hypothesis of Proposition 10.5.47, there exists an open neighborhood U of y such that:*

(a) *$f_*(\mathcal{O}_X)|_U$ is isomorphic to $(\mathcal{O}_Y|_U)^m$;*

(b) *for any $z \in U$, the canonical homomorphism*

$$(f_*(\mathcal{O}_X))_z \otimes_{\mathcal{O}_{Y,z}} \kappa(z) \to \Gamma(X_z, \mathcal{O}_{X_z})$$

is bijective.

(c) *there exists a coherent $\mathcal{O}_U$-module $\mathcal{Q}$ and a functorial isomorphism*

$$R^1 f_*(f^*(\mathscr{M})) \xrightarrow{\sim} \mathcal{H}om_{\mathcal{O}_U}(\mathcal{Q}, \mathscr{M})$$

Proof. Assertion (a) follows from Proposition 10.5.47 and Proposition 10.5.45, and (b), (c) follows from the fact that $\mathscr{T}_U^0$ is exact and Theorem 10.5.37. □

Corollary 10.5.49. *Under the Proposition 10.5.47, assume that $\Gamma(X_y, \mathcal{O}_{Y,y}) = \kappa(y)$. Then there exists an open neighborhood U of y such that the canonical homomorphism $\mathcal{O}_Y|_U$.*

Proof. In fact, it follows from Corollary 10.5.48 that the integer m in (a) is necessarily equal to 1. □

Remark 10.5.50. Under the conditions of Proposition 10.5.47, consider the Stein factorization

$$X \xrightarrow{f'} Y' \xrightarrow{g} Y$$

where $Y' = \mathrm{Spec}(f_*(\mathcal{O}_X))$. The finite morphism g is then such that $g_*(\mathcal{O}_{Y'}) = f_*(\mathcal{O}_X)$ is locally free in an neighborhood of y, and its fiber at y is the specturm of a separated algebra over $\kappa(y)$ (Proposition 9.1.29). We shall see later that, there exists an open neighborhood U of y such that for any $z \in U$, the fiber $g^{-1}(z)$ is the spectrum of a separatble $\kappa(z)$-algebra (this is called an étale covering of U). Therefore, the hypothesis of Proposition 10.5.47 made on y is in fact valid in an open neighborhood of y.

10.5.7.3 Invariance of Euler characteristic and Hilbert function Let A be a ring, M be a finitely generated projective A-module (which means $\widetilde{M}$ is locally free on $X = \mathrm{Spec}(A)$). For any $\mathfrak{p} \in \mathrm{Spec}(A)$, the rank of M at $\mathfrak{p}$ is defined to be the rank of the $A_\mathfrak{p}$-module $M_\mathfrak{p}$ (or the rank of the locally free $\mathcal{O}_X$-module $\widetilde{M}$ at $\mathfrak{p}$). We then have

$$\mathrm{rank}_\mathfrak{p}(M) = \mathrm{rank}_\mathfrak{p}(M_\mathfrak{p}) = \dim_{\kappa(\mathfrak{p})}(M \otimes_A \kappa(\mathfrak{p})). \tag{10.5.27}$$

Proposition 10.5.51. *Let $P^\bullet$ be a bounded complex of finitely generated projective A-modules, and for any module M, let $T^\bullet(M) = H^\bullet(P^\bullet \otimes_A M)$. Then for any $\mathfrak{p} \in \operatorname{Spec}(A)$, we have*

$$\sum_i (-1)^i \dim_{\kappa(\mathfrak{p})}(T^i(\kappa(\mathfrak{p}))) = \sum_i (-1)^i \operatorname{rank}_{\mathfrak{p}}(P^i). \tag{10.5.28}$$

Proof. In fact, by definition we have $T^i(\kappa(\mathfrak{p})) = H^i(P^\bullet \otimes_A \kappa(\mathfrak{p}))$, and in view of 10.5.27, the formula (10.5.28) is none other than the invariance of Euler characteristic of a bounded complex of finite dimensional vector spaces when passing to homology. $\square$

Corollary 10.5.52. *The function*

$$\mathfrak{p} \mapsto \sum_i (-1)^i \dim_{\kappa(\mathfrak{p})}(T^i(\kappa(\mathfrak{p})))$$

is locally constant on $\operatorname{Spec}(A)$.

Proof. This follows from (10.5.28), since the rank of a finitely projective module P^i is locally constant (**??**). $\square$

Theorem 10.5.53. *Let Y be a locally Noetherian scheme, $f : X \to Y$ be a proper morphism, $\mathscr{P}^\bullet$ be a bounded complex of Y-flat coherent $\mathscr{O}_X$-modules. Let $T^\bullet$ be the functor defined by (10.5.22), then the function*

$$y \mapsto \sum_i (-1)^i \dim_{\kappa(y)}(T^i(\kappa(y))) \tag{10.5.29}$$

is locally constant on Y.

Proof. We may assume that $Y = \operatorname{Spec}(A)$ is affine with ring A Noetherian. Then there exists a complex $L^\bullet$ of finitely generated A-modules such that $\mathscr{T}^p(\mathscr{M}) \xrightarrow{\sim} H^p(\mathscr{L}^\bullet \otimes_{\mathscr{O}_Y} \mathscr{M})$, where $\mathscr{L}^\bullet = \widetilde{L}^\bullet$. As the complex $\mathscr{P}^\bullet$ is bounded (take a finite affine covering $\mathfrak{U}$ of X), the double complex $\mathscr{C}^\bullet(\mathfrak{U}, \mathscr{P}^\bullet \otimes_{\mathscr{O}_Y} \mathscr{M})$ is bounded. More precisely, there exists a finite subset E (independent of $\mathscr{M}$), such that for any $(i,j) \notin E$, we have $\mathscr{C}^i(\mathfrak{U}, \mathscr{P}^j \otimes_{\mathscr{O}_Y} \mathscr{M}) = 0$. From this, we see that there exists an integer i_0 such that $\mathscr{T}^i(\mathscr{M}) = 0$ for any quasi-coherent $\mathscr{O}_Y$-module $\mathscr{M}$ and $i \geq i_0$. In particular, for such values of i, $\mathscr{T}^i$ is trivially an exact functor, so by Proposition 10.5.16, $W^i(L^\bullet)$ is a finitely generated flat (hence projective) A-module for such i. Consider the complex $(Q^\bullet)$, where $Q^i = L^i$ for $i < i_0$, $Q^{i_0} = W^{i_0}(L^\bullet)$, and $Q^i = 0$ for $i > i_0$, and set $\mathscr{Q}^\bullet = \widetilde{Q}^\bullet$. It is clear that $H^i(\mathscr{Q}^\bullet \otimes_{\mathscr{O}_Y} \mathscr{M}) = H^i(\mathscr{L}^\bullet \otimes_{\mathscr{O}_Y} \mathscr{M})$ for $i < i_0 - 1$ and also for $i \geq i_0$ (the two members being both zero). On the other hand, as $\operatorname{im}(W^{i_0} \otimes_A M) = \operatorname{im}(L^{i_0} \otimes_A M)$ by defintion, we also have $H^i(\mathscr{Q}^\bullet \otimes_{\mathscr{O}_Y} \mathscr{M}) = H^i(\mathscr{L}^\bullet \otimes_{\mathscr{O}_Y} \mathscr{M})$ for $i = i_0 - 1$. We then see that we may assume that $\mathscr{L}^\bullet$ is a bounded complex, with the weaker assumption that each $\mathscr{L}^i$ is a locally free $\mathscr{O}_Y$-module. The theorem now follows from module bounded complex of projective Euler characteristic locally constant. $\square$

Under the condition of Theorem 10.5.53, the function (10.5.29) is constant if Y is connected. If Y is connected and nonempty, we denote its value by $\chi(f, \mathscr{P}^\bullet)$ or $\chi(Y, \mathscr{P}^\bullet)$, or simply $\chi(\mathscr{P}^\bullet)$ if there is no confusion, and call it the **Euler characteristic of $\mathscr{P}^\bullet$ relative to f**. In the general case, we denote by $\chi(f, \mathscr{P}^\bullet; y)$ or $\chi(Y, \mathscr{P}^\bullet; y)$, or simple $\chi(\mathscr{P}^\bullet; y)$ the integer in (10.5.29).

Under the hypothesis of Theorem 10.5.53, let

$$0 \longrightarrow \mathscr{P}'^\bullet \xrightarrow{u} \mathscr{P}^\bullet \xrightarrow{v} \mathscr{P}''^\bullet \longrightarrow 0$$

be an exact sequence of bounded complexes of Y-flat coherent $\mathscr{O}_X$-modules, where the homomorphisms u, v are of even degrees $2d, 2d'$. Then as $\mathscr{T}^\bullet$ is a cohomological functor, we have an exact sequence

$$\cdots \to \mathscr{T}^i(\mathscr{P}'^\bullet, \kappa(y)) \to \mathscr{T}^{i+2d}(\mathscr{P}'^\bullet, \kappa(y)) \to \mathscr{T}^{i+2d+2d'}(\mathscr{P}'^\bullet, \kappa(y)) \to \mathscr{T}^{i+1}(\mathscr{P}'^\bullet, \kappa(y)) \to \cdots$$

with only finitely many nonzero terms. Since the Euler characteristic of this complex is zero, we conclude that

$$\chi(\mathscr{P}^\bullet; y) = \chi(\mathscr{P}'^\bullet; y) + \chi(\mathscr{P}''^\bullet; y) \tag{10.5.30}$$

for any $y \in Y$. By an induction process, it is not hard to see that (by (10.5.30))

$$\chi(\mathscr{P}^\bullet; y) = \sum_i (-1)^i \chi(\mathscr{P}^i; y) \tag{10.5.31}$$

where for any coherent $\mathscr{O}_X$-module $\mathscr{F}$, flat over Y, we denote by $\chi(\mathscr{F};y)$ (or $\chi(f,\mathscr{F};y)$) the function $\chi(\mathscr{L}^\bullet;y)$ corresponding to the complex $\mathscr{L}^\bullet$ with only zero-th term $\mathscr{F}$. Therefore, the study of Euler characteristics is reduced to that of a single coherent $\mathscr{O}_X$-module.

Proposition 10.5.54. *Under the hypothesis of Theorem 10.5.53, let Y' be a locally Noetherian scheme, $g: Y' \to Y$ be a morphism, $X' = X \times_Y Y'$, $f' = f_{(Y')}: X' \to Y'$, $\mathscr{P}'^\bullet$ be the bounded complex $\mathscr{P}^\bullet \otimes_{\mathscr{O}_Y} \mathscr{O}_{Y'}$ of coherent Y'-flat $\mathscr{O}_{X'}$-modules. Then for $y' \in Y'$, we have*

$$\chi(\mathscr{P}'^\bullet;y') = \chi(\mathscr{P}^\bullet;g(y')). \tag{10.5.32}$$

Proof. The $\mathscr{O}_{X'}$-modules $\mathscr{P}'^i$ is the inverse image off $\mathscr{P}^i$ under the projection $X' \to X$, hence Y'-flat. On the other hand, we see that f' is proper, so the first member of (10.5.32) is well-defined. The formula (10.5.32) follows from ([?], 6.10.4.2) and Lemma 10.5.34, by reducing to the case where Y' and Y are affine. □

Proposition 10.5.55. *Under the hypothesis of Theorem 10.5.53, suppose that there exists an integer i_0 such that $T^i(\kappa(y)) = 0$ for $i \neq i_0$ and any $y \in Y$. Then $\mathscr{T}^{i_0}(\mathscr{O}_Y) = R^{i_0}f_*(\mathscr{P}^\bullet)$ is a locally free $\mathscr{O}_Y$-module, whose rank n is equal to $(-1)^{i_0}\chi(f,\mathscr{P}^\bullet;y)$.*

Proof. Note that the hypothesis of Proposition 10.5.19 is verified, so we can apply Proposition 10.5.22, and the hypothesis implies that $T^i_{\mathscr{O}_{Y,y}}$ is zero for $i \neq i_0$ in view of Corollary 10.5.24. By Proposition 10.5.9, $\mathscr{T}^{i_0}$ is then exact, so by ([?], 7.8.4), $R^{i_0}f_*(\mathscr{P}^\bullet)$ is locally free with rank at $y \in Y$ equal to

$$\dim_{\kappa(y)}(T^{i_0}(\kappa(y))) = (-1)^{i_0}\chi(f,\mathscr{P}^\bullet;y)$$

by definition, since $T^i(\kappa(y)) = 0$ for $i \neq i_0$. □

Corollary 10.5.56. *Let Y be a locally Noetherian scheme, $f: X \to Y$ be a proper morphism, $\mathscr{F}$ be a Y-flat coherent $\mathscr{O}_X$-module. Suppose that $R^if_*(\mathscr{F}) = 0$ for any $i > 0$. Then $f_*(\mathscr{F})$ is a locally free $\mathscr{O}_Y$-module, with rank equal to $\chi(f,\mathscr{F};y)$.*

Proof. It suffices to prove that $H^i(X_y,\mathscr{F} \otimes_{\mathscr{O}_Y} \kappa(y)) = 0$ for any $i > 0$ and $y \in Y$. For this, we may assume that $Y = \operatorname{Spec}(A)$ is affine. With the notations of Theorem 10.5.53, and $\mathscr{P}^\bullet$ being reduced to a single term $\mathscr{F}$, we have in fact $\mathscr{T}^p(\mathscr{O}_Y) = 0$ for $p > 0$ by hypothesis. We then conclude from Proposition 10.5.9 that $\mathscr{T}^p$ is exact for $p > 0$, and the assertion then follows from the equivalence of Theorem 10.5.37(i) and (iv). □

Proposition 10.5.57. *Under the hypothesis of Theorem 10.5.53, let $\mathscr{L}$ be a very ample $\mathscr{O}_X$-module relative to Y, and put $\mathscr{P}^\bullet(n) = \mathscr{P}^\bullet \otimes_{\mathscr{O}_X} \mathscr{L}^{\otimes n}$ for each $n \in \mathbb{Z}$. Then for any $y \in Y$, the function*

$$n \mapsto \chi(f,\mathscr{P}^\bullet(n);y)$$

is a polynomial with coefficients in $\mathbb{Q}$, and is locally constant on Y.

Proof. It is clear that $\mathscr{P}^\bullet(n)$ is a complex of Y-flat $\mathscr{O}_X$-modules. In view of (10.5.31), we can assume that $\mathscr{P}^\bullet$ has a single nonzero term $\mathscr{F}$ at degree 0. Moreover, as the question is local on Y, we may reduce to the case where Y is affine and f is projective. Put $X_y = f^{-1}(y)$ and let $\mathscr{L}_y = \mathscr{L} \otimes_{\mathscr{O}_Y} \kappa(y)$, which is a very ample $\mathscr{O}_{X_y}$-module (Proposition 9.4.25). For the functor $\mathscr{T}^\bullet$ defined by $\mathscr{P}^\bullet(n)$, we then have

$$T^i(\kappa(y)) = H^i(X_y,\mathscr{F}_y \otimes \mathscr{L}_y^{\otimes n})$$

where $\mathscr{F}_y = \mathscr{F} \otimes_{\mathscr{O}_Y} \kappa(y)$. In this case, $\chi(f,\mathscr{F}(n);y)$ is none other than the Euler characteristic $\chi_{\kappa(y)}(\mathscr{F}_y(n))$ defined in ([?], 2.5.1). The fact that $\chi(f,\mathscr{P}^\bullet(n);y)$ is a rational polynomial then follows from ([?], 2.5.3). Moreover, for each n, this number is locally constant by Corollary 10.5.52, whence our assertion. □

We denote by $P(f,\mathscr{P}^\bullet;y)$ or $P(\mathscr{P}^\bullet;y)$ the polynomial of Proposition 10.5.57, with rational coefficients, and call it the **Hilbert polynomial** at y relative to $\mathscr{P}^\bullet$, f and $\mathscr{L}$ (or simply the Hilbert polynomial of $\mathscr{P}^\bullet$ at y, or of f, if there is no confusion). From the properties of $\chi(f,\mathscr{P}^\bullet;y)$, we have

$$P(\mathscr{P}^\bullet;y) = P(\mathscr{P}'^\bullet;y) + P(\mathscr{P}''^\bullet;y) \tag{10.5.33}$$

for an exact sequence $0 \to \mathscr{P}^\bullet \to \mathscr{P}'^\bullet \to \mathscr{P}''^\bullet \to 0$, and in particular

$$P(\mathscr{P}^\bullet; y) = \sum_i (-1)^i P(\mathscr{P}^\bullet; y). \tag{10.5.34}$$

Similarly, with the hypotheses and notations of Proposition 10.5.54, we have

$$P(\mathscr{P}'^\bullet; y') = P(\mathscr{P}^\bullet; g(y')). \tag{10.5.35}$$

The formula (10.5.34) reduces the study of Hilber functions to that of a single Y-flat $\mathscr{O}_X$-module. In this case, this polynomial has an interpretation which does not depend on cohomology.

Corollary 10.5.58. *Let Y be a Noetherian scheme, $f : X \to Y$ be a projective morphism, $\mathscr{L}$ be a very ample $\mathscr{O}_X$-module relative to Y, $\mathscr{F}$ be a Y-flat coherent $\mathscr{O}_X$-module. Then there exists an integer n_0 such that for $n \geq n_0$, $f_*(\mathscr{F}(n))$ is a locally free $\mathscr{O}_Y$-module whose rank at $t \in Y$ is equal to $P(f, \mathscr{F}; y)(n)$.*

Proof. As the morphism f is projective, by Theorem 10.2.13 there exists n_0 such that for $n \geq n_0$, we have $R^i f_*(\mathscr{F}(n)) = 0$ for $i > 0$. The conclusion then follows from Corollary 10.5.56. □

Proposition 10.5.59. *Let Y be a Noetherian scheme, $f : X \to Y$ be a projective morphism, $\mathscr{L}$ be a ample $\mathscr{O}_X$-module relative to f, and put $\mathscr{F}(n) = \mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{L}^{\otimes n}$ for any $\mathscr{O}_X$-module $\mathscr{F}$ and $n \in \mathbb{Z}$. For a coherent $\mathscr{O}_X$-module $\mathscr{F}$ to be Y-flat, it is necessary and sufficient that there exists an integer n_0 such that, for $n \geq n_0$, $f_*(\mathscr{F}(n))$ is a locally free $\mathscr{O}_Y$-module.*

Proof. The necessarity of this condition is proved in Corollary 10.5.58. To prove the converse, we can assume that Y is affine with ring A. In view of the hypothesis and Theorem 10.2.13, the A-modules $\Gamma(X, \mathscr{F}(n))$ are finitely generated and projective (**??**). Let S be the graded ring $\Gamma_{n \geq 0}\Gamma(X, \mathscr{L}^{\otimes n})$; then X is canonically idetified with $\mathrm{Proj}(S)$ (Theorem 9.4.27(ii) and Corollary 9.5.25). Let $M = \bigoplus_{n \geq n_0} \Gamma(X, \mathscr{F}(n))$; by replacing $\mathscr{L}$ with a power $\mathscr{L}^{\otimes d}$, we may assume that S is generated by finitely many elements of degree 1 (Proposition 9.2.18), and it then follows from Theorem 9.2.39 and Proposition 9.2.36 that $\mathscr{F}$ is identified with $\widetilde{M}$. For a homogeneous element $g \in S$ of positive degere, we then have $\Gamma(X_g, \mathscr{F}) = M_{(g)}$. Now M, a sum of projective modules, is a flat A-module, hence so is M_g, and therefore also $M_{(g)}$, which is a direct factor of M_g. We then conclude that $\mathscr{F}$ is Y-flat at any point of X_g, and as the X_g cover X, the assertion is proved. □

Chapter 11

Local study of schemes and morphisms of schemes

11.1 Unramified morphisms, smooth morphisms and étale morphisms

In this section, we introduce the notions of unramified morphisms, smooth morphisms and étale morphisms between schemes. These three classes of morphisms are analogues of the following types of maps of manifolds in differential geometry.

- **Submersions** are maps inducing surjections of tangent spaces everywhere. They are useful in the notion of a fibration. (Perhaps a more relevant notion from differential geometry, allowing singularities, is "locally on the source a smooth fibration".)
- **Immersions** are maps inducing injections of tangent spaces. They can be thought as a generalized notion for submanifolds.
- **Local isomorphisms** are maps inducing isomorphisms of tangent spaces. They are be viewed as covering spaces of manifolds.

11.1.1 Formally unramification and formally smoothness

Let $f : X \to Y$ be a morphism of schemes. We say that f is **formally smooth** (resp. **formally unramified**) if for any affine scheme Y' over Y and any closed subscheme Y'_0 of Y' defined by a square zero ideal $\mathscr{I}$ of $\mathscr{O}_{Y'}$, the canonical map

$$\mathrm{Hom}_Y(Y', X) \to \mathrm{Hom}_Y(Y'_0, X) \tag{11.1.1}$$

is surjective (resp. injective). In this case, we say that X is formally smooth (resp. formally unramified) over Y. If X is both formally smooth and formally unramified over Y, then it is said to be **étale** over Y.

Example 11.1.1. Suppose that $Y = \mathrm{Spec}(A)$ and $X = \mathrm{Spec}(B)$ are affine schemes, so that the morphism f corresponds to a ring homomorphism $\varphi : A \to B$. Then X is formally smooth (resp. formally unramified) over Y if and only if B is a formally smooth (resp. formally unramified) A-algebra.

Example 11.1.2. In view of the proof of **??**, if a morphism $f : X \to Y$ is formally smooth (resp. formally unramified), then for any affine scheme Z over Y and any closed subscheme Z_0 of Z defined by a nilpotent ideal $\mathscr{I}$ of $\mathscr{O}_Z$, the canonical map

$$\mathrm{Hom}_Y(Z, X) \to \mathrm{Hom}_Y(Z_0, X)$$

is surjective (resp. injective), so we can also take this as the definition of formally unramification (resp. formally smoothness). We also note that if f is formally smooth (resp. formally étale), then for an *arbitrary* scheme Z over Y and any closed subscheme Z_0 of Z defined by a locally nilpotent ideal $\mathscr{I}$ of $\mathscr{O}_Z$, the canonical map

$$\mathrm{Hom}_Y(Z, X) \to \mathrm{Hom}_Y(Z_0, X)$$

is surjective (resp. bijective). To see this, let (U_α) be an open affine covering of Z such that the ideal $\mathscr{I}|_{U_\alpha}$ is nilpotent, and for each α, let U_α^0 be the inverse image of U_α in Z_0, which is the closed subscheme of U_α defined by $\mathscr{I}|_{U_\alpha}$. Let $f_0 : Z_0 \to X$ be a Y-morphism, then by the hypotheses and **??**, for any α, there is a (resp. unique) Y-morphism $f_\alpha : U_\alpha \to X$ whose restriction to U_α^0 equals to $f_0|_{U_\alpha^0}$. Since f_α and f_β then coincide on any affine open of $U_\alpha \cap U_\beta$, we conclude that there exists a (resp. unique) morpism $f : Z \to X$ whose restriction to Z_0 is equal to f_0.

Proposition 11.1.3 (Properties of Formally Unramified and Formally Smooth Morphisms).

(i) *A monomorphism is formally unramified, and an open immersion is formally étale.*

(ii) *The composition of two formally smooth (resp. formally unramified) morphisms is formally smooth (resp. formally unramified).*

(iii) *If $f : X \to Y$ is a formally smooth (resp. formally unramified) S-morphism, so is $f_{(S')} : X_{(S')} \to Y_{(S')}$ for any base change $S' \to S$.*

(iv) *If $f : X \to X'$ and $g : Y \to Y'$ are two formally smooth (resp. formally unramified) S-morphisms, $f \times_S g$ is formally smooth (resp. formally unramified).*

(v) *Let $f : X \to Y$ and $g : Y \to Z$ be two morphisms. If $g \circ f$ is formally unramified, so is f.*

(vi) *If $f : X \to Y$ is a formally unramified morphism, so is $f_{\mathrm{red}} : X_{\mathrm{red}} \to Y_{\mathrm{red}}$.*

Proof. In view of Proposition 8.5.14 and Proposition 8.5.22, it suffices to prove (i), (ii) and (iii). The two assertions of (i) are trivial by definition. To prove (ii), consider two morphisms $f : X \to Y$, $g : Y \to Z$, an affine scheme Z' over Z, a closed subscheme Z'_0 of Z' defined by a square zero ideal. Suppose that f and g are formally smooth, and consider a Z-morphism $u_0 : Z'_0 \to X$:

$$\begin{tikzcd} Z'_0 \arrow[r, "u_0"] \arrow[dd, "j"'] & X \arrow[d, "f"] \\ & Y \arrow[d, "g"] \\ Z' \arrow[r] \arrow[uur, dashed, "u"] \arrow[ur, dashed, "v"] & Z \end{tikzcd}$$

The hypothesis over g implies that there exists a Z-morphism $v : Z' \to Y$ such that $f \circ u_0 = v \circ j$ (where $j : Z'_0 \to Z'$ is the canonical injection), and the hypothesis over f implies that there exists a morphism $u : Z \to X$ such that $f \circ u = v$ and $u \circ j = u_0$, so $(g \circ f) \circ u$ is equal to the structural morphism $Z' \to Z$ and $u \circ j = u_0$, which proves that $g \circ f$ is formally smooth. By a smilar reasoning, we see that $f \circ g$ is formally unramified if both morphisms are formally unramified.

Finally, to prove (iii), put $X' = X_{(S')}$, $Y' = Y_{(S')}$, $f' = f_{(S')}$. Consider an affine scheme Y'' over Y' and a closed subscheme Y''_0 of Y'' defined by a square zero ideal. Then $\mathrm{Hom}_{Y'}(Y'', X')$ is canonically identified with $\mathrm{Hom}_Y(Y'', X)$, and $\mathrm{Hom}_{Y'}(Y''_0, X')$ with $\mathrm{Hom}_Y(Y'', X)$, so our conclusion follows from the definition. □

Proposition 11.1.4. *Let $f : X \to Y$ and $g : Y \to Z$ be morphisms and suppose that g is formally unramified. Then, if $g \circ f$ is formally smooth (resp. formally étale), so is f.*

Proof. Let Y' be an affine scheme, $h : Y' \to Y$ be a morphism, Y'_0 be a closed subscheme of Y' defined by a square zero ideal, and $u_0 : Y'_0 \to X$ be a Y-morphism.

$$\begin{tikzcd} Y'_0 \arrow[r, "u_0"] \arrow[d, "j"'] & X \arrow[d, "f"] & \\ Y' \arrow[r, "h"] \arrow[ur, dashed, "u"] & Y \arrow[r, "g"] & Z \end{tikzcd}$$

If $g \circ f$ is formally smooth, there exists a Z-morphism $u : Y' \to X$ such that $u \circ j = u_0$ (where $j : Y'_0 \to Y'$ is the canonical injection) and $(g \circ f) \circ u = g \circ h$. But this then implies that $f \circ u$ and h are Z-morphisms from Y' to Y such that $(f \circ u) \circ j = h \circ j$, so since g is formally unramified, we conclude that $f \circ u = h$, so u is a Y-morphism. In view of Proposition 11.1.3(v), the proposition is therefore proved. □

Corollary 11.1.5. *Suppose that g is formally étale. Then for the composition $g \circ f$ to be formally smooth (resp. formally unramified), it is necessary and sufficient that f is formally smooth (resp. formally unramified).*

Proof. This follows from Proposition 11.1.4 and Proposition 11.1.3(ii) and (v). □

Proposition 11.1.6. *Let $f : X \to Y$ be a morphism of schemes.*

(i) *Let (U_α) be an open covering of X and, for each α, let $i_\alpha : U_\alpha \to X$ be the canonical injection. For f to be formally smooth (resp. formally unramified), it is necessary and sufficient that each morphism $f \circ i_\alpha$ is formally smooth (resp. formally unramified).*

(ii) *Let (V_λ) be an open covering of Y. For f to be formally smooth (resp. formally unramified), it is necessary and sufficient that each restriction $f|_{f^{-1}(V_\lambda)} : f^{-1}(V_\lambda) \to V_\lambda$ is formally smooth (resp. formally unramified).*

Proof. We first note that (ii) is a concequence of (i): in fact, if $j_\lambda : V_\lambda \to Y$ and $i_\lambda : f^{-1}(V_\lambda) \to X$ are the canonical injections, the restriction $f_\lambda : f^{-1}(V_\lambda) \to V_\lambda$ satisfies $j_\lambda \circ f_\lambda = f \circ i_\lambda$; if f is formally smooth (resp. formally unramified), then so is $f \circ i_\lambda$ (Proposition 11.1.3); but as j_λ is formally étale, this implies that f_λ is formally smooth (resp. formally unramified) by Corollary 11.1.5. Conversely, if each restriction f_λ is formally smooth (resp. formally unramified), then so is $j_\lambda \circ f_\lambda$ (Proposition 11.1.3), and hence is f in view of (i). Now since each i_α is formally étale, it then boils down to prove that if the $f \circ i_\alpha$ is formally smooth (resp. formally unramified), then so is f.

Let Y' be an affine scheme, Y'_0 be a closed subscheme of Y' defined by a square zero ideal, and $g : Y' \to Y$ be a morphism. Let $u_0 : Y'_0 \to X$ be a morphism, and denote by W_α (resp. W_α^0) the open subscheme of Y' (resp. Y'_0) induced on $u_0^{-1}(U_\alpha)$ (note that Y' and Y'_0 have the same underlying space). First suppose that $f \circ i_\alpha$ is formally unramified. We show that if u_1 and u_2 are two Y-morphisms from Y' to X whose restriction on Y'_0 coincide, then $u_1 = u_2$. To see this, note that by Proposition 11.1.3 (iv), the hypothesis that $f \circ i_\alpha$ is formally unramified implies that $u_1|_{W_\alpha} = u_2|_{W_\alpha}$ for each α, so the assertion is true in this case.

Now suppose that $f \circ i_\alpha$ is formally smooth for each α, we prove that there exists a Y-morphism $u : Y' \to X$ whose restriction to Y'_0 is u_0. Now since Y' is affine, we can apply ([?], 16.5.17), whose conclusion precisely proves the existence of u. □

11.1.2 Differential properties and characterizations

Proposition 11.1.7. *For a morphism $f : X \to Y$ to be formally unramified, it is necessary and sufficient that $\Omega^1_{X/Y} = 0$.*

Proof. By Proposition 11.1.6, the question is local on source and target, so we can assume that $X = \mathrm{Spec}(B)$ and $Y = \mathrm{Spec}(A)$ are affine, hence reduce to proving that the A-algebra B is formally unramified if and only if $\Omega^1_{B/A} = 0$. For this, recall that $\mathrm{Hom}_B(\Omega^1_{B/A}, M)$ is isomorphic to $\mathrm{Der}_A(B, M)$ for any B-module M, so if $\Omega^1_{B/A} = 0$, **??** implies that B is formally unramified over A. Conversely, assume that B is formally unramified over A and consider the multiplication map $\mu : B \otimes_A B \to B$ of the A-algebra B. Let $\mathfrak{I}$ be the kernel of μ and set $C = (B \otimes_A B)/\mathfrak{I}^2$, $N = \mathfrak{I}/\mathfrak{I}^2$. Then N is a square zero ideal of C, and since B is formally unramified over A, we get $\mathrm{Der}_A(B, N) = 0$ by **??**. But N is by definition the differential module $\Omega^1_{B/A}$, so we conclude that $\mathrm{Hom}_B(\Omega^1_{B/A}, \Omega^1_{B/A}) = 0$, whence $\Omega^1_{B/A} = 0$. □

Corollary 11.1.8. *Let $f : X \to Y$ and $g : Y \to Z$ be two morphism. For f to be formally unramified, it is necessary and sufficient that the canonical homomorphism $f^*(\Omega^1_{Y/Z}) \to \Omega^1_{X/Z}$ is surjective.*

Proof. This is an immediate concequence of Proposition 11.1.7 and the exact sequence

$$f^*(\Omega^1_{Y/Z}) \longrightarrow \Omega^1_{X/Z} \longrightarrow \Omega^1_{X/Y} \longrightarrow 0$$

□

Proposition 11.1.9. *Let $f : X \to Y$ be a formally smooth morphism.*

(i) *The $\mathscr{O}_X$-module $\Omega^1_{X/Y}$ is locally projective. If f is locally of finite type, then $\Omega^1_{X/Y}$ is locally free of finite rank.*

(ii) *For any morphism $g : Y \to Z$, the sequence of $\mathscr{O}_X$-modules*

$$0 \longrightarrow f^*(\Omega^1_{Y/Z}) \longrightarrow \Omega^1_{X/Z} \longrightarrow \Omega^1_{X/Y} \longrightarrow 0 \tag{11.1.2}$$

is exact. Moreover, for any $x \in X$, there exists an open neighborhood U of x such that the restriction of these homomorphisms to U form a split exact sequence.

Proof. If f is locally of finite type, then the diagonal map $\Delta_{X/Y} : X \to X \times_Y X$ is of finite presentation (Corollary 8.6.25), so $\Omega^1_{X/Y}$ is an $\mathscr{O}_X$-module of finite type in view of Corollary 8.6.27. To see that $\Omega^1_{X/Y}$ is locally projective, it suffices to assume that $Y = \mathrm{Spec}(A)$ and $X = \mathrm{Spec}(B)$ are affine, and the conclusion then follows from **??**. Now to prove (ii), we can also assume that X, Y, Z are affine schemes, and in this case the conclusion then follows from the interpretation of the modules appearing in the sequence (11.1.2) and ([**?**] 0_{IV}, 20.5.7). □

Corollary 11.1.10. *If $f : X \to Y$ is a formally étale morphism, then for any morphism $g : Y \to Z$, the canonical homomorphism $f^*(\Omega^1_{Y/Z}) \to \Omega^1_{X/Z}$ is bijective.*

Proof. This follows from the exact sequence (11.1.2) and the fact that we have $\Omega^1_{X/Y} = 0$ (Proposition 11.1.7). □

Proposition 11.1.11. *Let $f : X \to Y$ be a morphism, X' be a closed subscheme of X such that the composition morphism $X' \to X \to Y$ is formally smooth. Then the sequence of $\mathscr{O}_{X'}$-modules*

$$0 \longrightarrow \mathscr{N}_{X'/X} \longrightarrow \Omega^1_{X/Y} \otimes_{\mathscr{O}_X} \mathscr{O}_{X'} \longrightarrow \Omega^1_{X'/Y} \longrightarrow 0 \tag{11.1.3}$$

is exact. Moreover, for any $x \in X$, there exists an open neighborhood U of x such that the restriction of these homomorphisms to U form a split exact sequence.

Proof. In view of Proposition 11.1.6, we can assume that $X = \mathrm{Spec}(B)$, $X' = \mathrm{Spec}(B/\mathfrak{I})$ and $Y = \mathrm{Spec}(A)$ are affine, where $\mathfrak{I}$ is an idela of B. Then the conormal sheaf $\mathscr{N}_{X'/X}$ corresponds to the B-module $\mathfrak{I}/\mathfrak{I}^2$, and the conclusion follows from **??**. □

Proposition 11.1.12. *Let $f : X \to Y$ be a morphism locally of finite type. Then the following conditions are equivalent:*

(i) *f is a monomorphism;*

(ii) *f is radical and formally unramified;*

(iii) *for any $y \in Y$, the fiber $f^{-1}(y)$ is either empty or $\kappa(y)$-isomorphic to $\mathrm{Spec}(\kappa(y))$.*

Proof. The fact that (i) implies (iii) follows from Proposition 8.3.39. It is clear that (iii) implies that f is radical, and we prove that $\Omega^1_{X/Y} = 0$ in this case. Note that the $\mathscr{O}_X$-module $\Omega^1_{X/Y}$ is quasi-coherent of finite type ([**?**], 16.3.9), so it follows from Corollary 8.3.23 that, for $(\Omega^1_{X/Y})_x = 0$, it is necessary and sufficient that if we put $Y_1 = \mathrm{Spec}(\kappa(y))$, $X_1 = f^{-1}(y) = X \times_Y Y_1$, then $(\Omega^1_{X_1/Y_1})_x = 0$. But as the morphism $f_1 : X_1 \to Y_1$ induced from f is formally unramified in view of the hypotheses of (iii) (Proposition 11.1.3), the conclusion follows from Proposition 11.1.7.

Finally, we show that (ii) implies (i). For this, consider the diagonal morphism $g = \Delta_f : X \to X \times_Y X$; since f is radical, g is surjective (Proposition 8.3.31). On the other hand, $\Omega^1_{X/Y}$ is defined by the conormal sheaf of the immersion g, and the hypothesis that f is unramified implies $\Omega^1_{X/Y} = 0$. Moerover, g is locally of finite presentation (Corollary 8.6.25), so it is an open immersion by ([**?**], 16.1.10). Being surjective, this immersion is then an isomorphism, so f is a monomorphism by Proposition 8.5.5. □

11.1.3 Unramified morphisms and smooth morphisms

We say that a morphism $f : X \to Y$ is **smooth** (resp. **unramified**) if it is locally of finite presentation and formally smooth (resp. formally unramified), and **étale** if it is both smooth and unramified.

Example 11.1.13. Let A be a ring and B be an A-algebra. Then B is a smooth (resp. unramified) algebra if the corresponding morphism $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is smooth (resp. unramified).

Proposition 11.1.14 (Properties of Unramified and Smooth Morphisms).

(i) *An open immersion is étale. For an immersion to be unramified, it is necessary and sufficient that it is locally of finite presentation.*

(ii) *The composition of two smooth (resp. unramified) morphisms is formally smooth (resp. formally unramified).*

(iii) *If $f : X \to Y$ is a smooth (resp. unramified) S-morphism, so is $f_{(S')} : X_{(S')} \to Y_{(S')}$ for any base change $S' \to S$.*

(iv) *If $f : X \to X'$ and $g : Y \to Y'$ are two smooth (resp. unramified) S-morphisms, $f \times_S g$ is smooth (resp. unramified).*

(v) *Let $f : X \to Y$ and $g : Y \to Z$ be two morphisms. If g is locally of finite type and $g \circ f$ is unramified, so is f.*

Proof. This follows from Proposition 8.6.38 and Proposition 11.1.3. □

Proposition 11.1.15. *Let $f : X \to Y$, $g : Y \to Z$ be morphisms, and suppose that g is unramified. Then, if $g \circ f$ is smooth (resp. étale), so is f.*

Proof. In fact, as g and $g \circ f$ are locally of finite presentation, so is f (Proposition 8.6.24(v)), and the conclusion then follows from Proposition 11.1.4. □

Corollary 11.1.16. *Suppose that g is étale, then for f to be smooth (resp. étale), it is necessary and sufficient that $g \circ f$ is smooth (resp. étale).*

Proof. This follows from Proposition 11.1.15 and Proposition 11.1.14(ii). □

Proposition 11.1.17. *Let $g : Y \to S$ and $h : X \to S$ be morphisms locally of finite presentation. For an S-morphism $f : X \to Y$ to be unramified, it is necessary and sufficient that the canonical homomorphism $f^*(\Omega^1_{Y/S}) \to \Omega^1_{X/S}$ is surjective.*

Proof. As f is then locally of finite presentation (Proposition 11.1.14(ii)), the proposition follows from Corollary 11.1.8. □

In view of Proposition 11.1.6 and the local nature of morphisms locally of finite presentation, we say a morphism $f : X \to Y$ is smooth (resp. unramified) at a point $x \in X$ if there exists an open neighborhood U of x in X such that the restriction $f|_U$ is a smooth (resp. unramified) morphism from U into Y. Then a morphism $f : X \to Y$ is smooth (resp. unramified) if and only if it is smooth (resp. unramified) at every point of X. Moreover, it is clear from this definition that the points of X where f is smooth (resp. unramified) is open in X.

Proposition 11.1.18. *For any scheme Y and any locally free $\mathscr{O}_X$-module $\mathscr{E}$ of finite rank, the vector bundle $\boldsymbol{V}(\mathscr{E})$ is a smooth Y-scheme.*

Proof. In fact, by Proposition 11.1.6 we can assume that $Y = \operatorname{Spec}(A)$ is affine and $\boldsymbol{V}(\mathscr{E}) = \operatorname{Spec}(A[T_1, \dots, T_n])$. As $A[T_1, \dots, T_n]$ is formally smooth over A and finitely presented, this proves the proposition. □

Corollary 11.1.19. *Under the hypothesis of Proposition 11.1.18, the projective bundle $\mathbb{P}(\mathscr{E})$ is a smooth Y-scheme.*

Proof. We can assume that $Y = \operatorname{Spec}(A)$ and $\mathbb{P}(\mathscr{E}) = \mathbb{P}^n_Y$. Then there is a finite open covering of $\mathbb{P}^n_A$ by the $D_+(T_i)$. But the ring of $D_+(T_i)$, being $A[T_1, \dots, T_n]_{(T_i)}$, is isomorphic to the polynomial ring $A[T_0, \dots, \widehat{T_i}, \dots, T_n]$, and hence smooth, so the corollary follows from Proposition 11.1.6. □

11.1.4 Characterization of unramification and smoothness

11.1.4.1 Characterization of unramified morphisms

Theorem 11.1.20. *Let $f : X \to Y$ be a morphism locally of finite presentation and x be a point of X. Then the following conditions are equivalent:*

(i) *f is unramified at x;*

(ii) *the diagonal morphism $\Delta_f : X \to X \times_Y X$ is a local isomorphism at x.*

(ii′) *if $Z = X \times_Y X$ and $z = \Delta_f(x)$, the morphism $(\Delta_f)^\#_x : \mathscr{O}_{Z,z} \to \mathscr{O}_{X,x}$ is bijective;*

(ii″) *for any morphism $g : Y' \to Y$ and any point $y' \in Y'$ lying over $y = f(x)$, any Y'-section s' of $X' = X \times_Y Y'$ with $x' = s'(y')$ lying over x is a local isomorphism at y';*

(iii) *$(\Omega^1_{X/Y})_x = 0$;*

(iv) *the $\kappa(y)$-scheme X_y is unramified over $\kappa(y)$ at x;*

(iv′) *the point x is isolated in X_y and the local ring $\mathscr{O}_{X_y,x}$ is a field and separable over $\kappa(y)$;*

(iv″) *$\mathscr{O}_{X_y,x} = \mathscr{O}_{X,x}/\mathfrak{m}_y\mathscr{O}_{X,x}$ is a field and a finite separable extension of $\kappa(y)$;*

(v) *$\mathscr{O}_{X,x}$ is a formally unramified $\mathscr{O}_{Y,y}$-algebra.*

Proof. As f is locally of finite type, the $\mathscr{O}_X$-module $\Omega^1_{X/Y}$ is of finite type ([?], 16.3.9), so $(\Omega^1_{X/Y})_x = 0$ if and only if there exists an open neighborhood U of x such that $\Omega^1_{X/Y}|_U = 0$. In view of Proposition 11.1.7, this proves the equivalence of (i) and (iii). On the other hand, if we put $A = \mathscr{O}_{Y,y}$, $B = \mathscr{O}_{X,x}$, then $(\Omega^1_{X/Y})_x = \Omega^1_{B/A}$ ([?], 16.4.15), so the equivalence of (iii) and (v) also follows from the affine case of Proposition 11.1.7.

By the very definition of unramified morphism at x, we see that (i) is equivalent to (iv). Also, as properties (iv) and (iv′) only involve the morphism $X_y \to \operatorname{Spec}(\kappa(y))$, this also implies the equivalence of (iv) and (iv′). On the other hand, (iv′) and (iv″) are equivalent, because it amounts to the same thing to say that $\mathscr{O}_{X_y,x}$ is a finite $\kappa(y)$-algebra or that x is an isolated point of X_y, since X_y is a $\kappa(y)$-scheme locally of type (Proposition 8.6.44).

We now prove the equivalence of (ii) and (ii′). We can limit ourselves to the case where $Y = \operatorname{Spec}(R)$, $X = \operatorname{Spec}(S)$ are affine and f is finitely presented. Then we have $Z = \operatorname{Spec}(S \otimes_R S)$ and Δ_f corresponds to the multiplication map $S \otimes_R S \to S$, whose kernel $\mathfrak{I}$ is a finitely generated ideal ([?] 0_{IV}, 20.4.4). If we put $\mathscr{I} = \widetilde{\mathfrak{I}}$, the $\mathscr{O}_Z$-module $\Delta_f^*(\mathscr{O}_X) = \mathscr{O}_Z/\mathscr{I}$ is then of finite presentation, and the hypothesis that the homomorphism $(\Delta_f)^\#_x : \mathscr{O}_{Z,z} \to \mathscr{O}_{X,x}$ is bijective implies that, by replacing X by an open neighborhood of x, the homomorphism $(\Delta_f)^\# : \mathscr{O}_Z \to \Delta_f^*(\mathscr{O}_X)$ is itself bijective (**??**). This then proves that (ii′) implies (ii), and the converse is evident.

On the other hand, the equivalence of (ii) and (ii″) follows from Corollary 8.5.4 even without the finiteness hypothesis on f: in fact, giving a Y'-section $s' : Y' \to X'$ is equivalent to giving a Y-morphism $h = g' \circ s' : Y' \to X$ (where $g' : Y' \to X$ is the canonical projection) such that $s' = \Gamma_h$, and the diagram

$$\begin{tikzcd} Y' \arrow[r, "s'"] \arrow[d, "h"'] & X' = Y' \times_Y X \arrow[d, "h \times_X 1_X"] \\ X \arrow[r, "\Delta_f"] & X \times_Y X \end{tikzcd}$$

is then cartesian. Therefore if Δ_f is a local isomorphism at x, then s' is a local isomorphism at y' (since $x = h(y')$), and this proves (ii)⇒(ii″). The converse is obtained by applying (ii″) to the case where $Y' = X$, $y' = x$, $g = f$ and $s' = \Delta_f$.

To finish the proof of the theorem, it then suffices to prove the following implications

$$\text{(iv″)} \Rightarrow \text{(iii)} \Rightarrow \text{(ii)} \Rightarrow \text{(iv″)}.$$

First, as $\Omega^1_{X/Y}$ is an $\mathscr{O}_X$-module of finite type, it follows from Nakayama lemma that condition (iii) is equivalent to $(\Omega^1_{X/Y})_x/\mathfrak{m}_y(\Omega^1_{X/Y})_x = 0$, which means $(\Omega^1_{X_y/\operatorname{Spec}(\kappa(y))})_x = 0$ ([?], 16.4.5). We are therefore reduced to the case where Y is the specturm of a field k and X is an algebraic k-scheme. The hypothesis that $\mathcal{O}_{X,x}$ is a finite field extension k' of k implies that x is closed in X (Corollary 8.6.45), and hence isolated in X. But then the hypothsis that k' is separable over k implies that $\Omega^1_{k'/k} = 0$ ([?] 0_{IV}, 20.6.20), which proves (iii). Moreover, in this case $\Omega^1_{X/Y}|_U = 0$ for an open neighborhood of x in X, so assertion (ii) follows from the definition of $\Omega^1_{X/Y}$.

Finally, to show that (ii)⇒(iv″), we can, by replacing X by an open neighborhood of x, suppose that Δ_f is an open immersion. If we denote by $f_y : X_y \to \mathrm{Spec}(\kappa(y))$ the morphism induced by f on fiber, then Δ_{f_y} is also an open immersion (Corollary 8.4.14), and as condition (iv″) only concerns the $\kappa(y)$-scheme X_y, we can therefore assume that $Y = \mathrm{Spec}(k)$ and $X = \mathrm{Spec}(A)$, where k is a field and A is a finite type k-algebra. Condition (iv″) is then established if we can show that A is a finite and separable k-algebra. If K is an algebraic closure of k, this amounts to saying that $A \otimes_k K$ is a finite and separable K-algebra ([?], 4.6.1), so we can further assume that k is algebraically closed. We first prove that A is a finite k-algebra. For this, it suffices to show that every closed point x of X is isolated, since then the set of such points is open in X and discrete, hence finite (X is Noetherian), and the assertion then follows from Proposition 8.6.44. Now for such a point x, we have $\kappa(x) = k$ since k is algebraically closed (Proposition 8.6.42), so by Corollary 8.6.43 there is a Y-section of X such that $s(Y) = \{x\}$, and in view of ([?], 17.4.1.1), $\{x\}$ is the inverse image of the diagonal $\Delta_X(X)$ under a morphism $X \to X \times_Y X$, hence is open in X in view of the hypothesis of (ii). The k-algebra A is therefore finite, hence is isomorphic to a direct product of finite local k-algebras. Since our question is local, we can then assume that A is a finite local k-algebra, so that $X = \mathrm{Spec}(A)$ is reduced to a singleton. The residue field of A, being a finite extension of k, is necessarily equal to k, so in view of Proposition 8.3.27, the product $X \times_k X$ is reduced to a singleton, and Δ_f is then an isomorphism. Since Δ_f corresponds to the multiplication map $\mu : A \otimes_k A \to A$, we conclude that μ is an isomorphism, which means $\Omega^1_{A/k} = 0$ and A is separable over k. □

Remark 11.1.21. If we onlt assume that f is locally of finite type, then $\Omega^1_{X/Y}$ is an $\mathcal{O}_X$-module of finite type ([?], 16.3.9), and Δ_f is a morphism locally of finite presentation (Corollary 8.6.25). The proof of Theorem 11.1.20 is still valid, provided that condition (i) is replaced by the following: the restriction of f to an open neighborhood of x is formally unramified. We also see that in this case the restriction of f to an open neighborhood of x is a locally quasi-finite morphism.

In fact, many authors require unramified morphisms to be (locally of finite type), rather than (locally) of finite presentation. The benifits of this definition is that any closed immersion is then unramified, which is reasonable under our intuition about unramification. The requirement of being locally of finite presentation, however, is necessary when we consider étale morphisms, so we add it to the definition of unramified morphism for consistency (following Grothendieck's definition). However, one should note that most of the proofs involving unramified morphisms work through finite type cases, and the restriction of being locally of finite presentation in fact unnecessary in most of the cases when we talk about unramified morphisms.

Corollary 11.1.22. *Let $f : X \to Y$ be a morphism locally of finite presentation. Then the following conditions are equivalent:*

(i) *f is unramified;*

(ii) *the diagonal morphism $\Delta_f : X \to X \times_Y X$ is an open immersion;*

(ii′) *for any morphism $Y' \to Y$, any Y'-section of $X' \times_Y Y'$ is an open immersion;*

(iii) $\Omega^1_{X/Y} = 0$;

(iv) *for any $y \in Y$, the $\kappa(y)$-scheme X_y is unramified over $\kappa(y)$;*

(iv′) *for any $y \in Y$, the $\kappa(y)$-scheme X_y is isomorphic to $\coprod_{\lambda \in I} \mathrm{Spec}(K_\lambda)$, where for each λ, K_λ is a finite separable extension of $\kappa(y)$;*

(v) *for any $x \in X$, the local ring $\mathcal{O}_{X,x}$ is a formally unramified $\mathcal{O}_{Y,f(x)}$-algebra.*

Proof. This follows from the observation that a morphism $f : X \to Y$ is unramified if and only if it is unramified at every point of X (we also note that in (ii′), an injective local isomorphism is an open immersion). □

Corollary 11.1.23. *If $f : X \to Y$ is unramified, then it is locally quasi-finite.*

Proof. By Theorem 11.1.20, for any point $x \in X$, the $\kappa(y)$-algebra $\mathscr{O}_{X_y,x}$ is finite, so x is isolated in X_y and the conclusion follows from ([?], 13.1.4). □

Proposition 11.1.24. *Let Y be a locally Noetherian scheme, $f : X \to Y$ be a morphism locally of finite type, x be a point of X. If $A = \mathscr{O}_{Y,y}$ and $\mathscr{O}_{X,x}$, then the conditions of Theorem 11.1.20 are equivalent to the following:*

(vi) *$\widehat{B} \otimes_{\widehat{A}} \kappa_A$ is a finite and separable field extension of κ_A (which implies that $\widehat{B}$ is a finite $\widehat{A}$-algebra);*

(vi′) *$\widehat{B}$ is a formally unramified $\widehat{A}$-algebra.*

Moreover, if $\kappa(x) = \kappa(y)$, or if k is separably closed, these conditions are also equivalent to:

(vi″) *the homomorphism $\widehat{A} \to \widehat{B}$ is surjective.*

Proof. Note that by the same reasoning of ([?] 0_{IV}, 19.3.6), the A-alebra B is formally unramified if and only if $\widehat{B}$ is formally unramified over $\widehat{A}$ for the adic topology. On the other hand, since f is locally of finite type, $\Omega^1_{B/A}$ is a finitely generated B-module ([?], 16.3.9), $\Omega^1_{B/A} = 0$ if and only if $\widehat{\Omega}_{B/A} = 0$ (**??**), so B is a formally unramified A-algebra if and only if it is formally unramified over A for the adic topology ([?] 0_{IV}, 20.7.4), and this proves the equivalence of (v) and (vi′).

Since $\kappa_A = A/\mathfrak{m}_A = \widehat{A}/\mathfrak{m}_A\widehat{A}$, we have $\widehat{B} \otimes_{\widehat{A}} \kappa_A = \widehat{B}/\mathfrak{m}_A\widehat{B} = \widehat{B} \otimes_B (B/\mathfrak{m}_A B)$, and $\widehat{B}/\mathfrak{m}_A\widehat{B}$ is therefore the completion of $B/\mathfrak{m}_A B = B \otimes_A \kappa_A$ for the $\mathfrak{m}_B$-adic topology (**??**). Since $\widehat{B}$ is a fiathfully flat B-module, we see that $B/\mathfrak{m}_A B$ is a field if and only if $\widehat{B}/\mathfrak{m}_A\widehat{B}$ is a field. Moreover, since completion does not change the residue field, in this case $B/\mathfrak{m}_A B$ is separable over κ_A if and only if $\widehat{B}/\mathfrak{m}_A\widehat{B}$ is separable over κ_A; this proves the equivalence of (iv″) and (vi).

Finally, if $\kappa(x) = \kappa(y)$ or κ_A is separably closed, condition (vi) implies that the homomorphism $\widehat{A}/\mathfrak{m}_A\widehat{A} \to \widehat{B}/\mathfrak{m}_A\widehat{B}$ is bijective and that $\widehat{B}$ is a quasi-finite $\widehat{A}$-algebra ([?] 0_{I}, 7.4.4), hence finite (since $\widehat{A}$ is complete, $\widehat{B}$ is separated for the $\mathfrak{m}_A$-adic topology, and $\mathfrak{m}_A\widehat{B}$ is a defining ideal of $\widehat{B}$ ([?] 0_{I}, 7.4.1)). The homomorphism $\widehat{A} \to \widehat{B}$ is then surjective by Nakayama lemma, so (vi) implies (vi″); the converse of this is evident. □

Given an S-scheme Y and two S-morphisms $f : X \to Y$, $g : X \to Y$, we define the **coincidence scheme** of f and g to be the inverse image of the diagonal $\Delta_{Y/S}$ under the S-morphism $(f,g)_S$. Moreover, by Proposition 8.5.11, this subscheme is canonically identified with the kernel $\ker(f,g)$.

Proposition 11.1.25. *Let $h : Y \to S$ be an unramified morphism and $f : X \to Y$, $g : X \to Y$ be two S-morphisms. Then the coincidence scheme of f and g is an open subscheme of X.*

Proof. In fact, since $\Delta_h : Y \to Y \times_S Y$ is an open immersion by Corollary 11.1.22, the inverse image of $\Delta_{Y/S}$ under $(f,g)_S$ is an open subscheme of X. □

Corollary 11.1.26. *Under the hypotheses of Proposition 11.1.25, let x be a point of X such that the following diagram commutes*

$$\mathrm{Spec}(\kappa(x)) \longrightarrow X \overset{f}{\underset{g}{\rightrightarrows}} Y$$

Then there exists an open neighborhood U of x such that $f|_U = g|_U$. If Y is also separated over S, then there also exists an open neighborhood Z of x such that $f|_Z = g|_Z$. In particular, if X is also connected, then $f = g$.

Proof. By Corollary 8.5.12, x belongs to the subscheme $\ker(f,g)$, and the assertion then follows from Proposition 11.1.25. □

Corollary 11.1.27. *Under the hypotheses of Proposition 11.1.25, suppose that the structural morphism $\varphi : X \to S$ is closed. Let s be a point of X and suppose that the compositions the following diagram commutes*

$$X_s \longrightarrow X \overset{f}{\underset{g}{\rightrightarrows}} Y$$

Then there exists an open neighborhood V of s in S such that $f|_{\varphi^{-1}(V)} = g|_{\varphi^{-1}(V)}$. If Y is also separated over S and φ is open, then we can choose V to be clopen. In particular, if S is also connected, then $f = g$.

Proof. It follows from Corollary 11.1.26 that the subscheme $C = \ker(f, g)$ is open in X and contains X_s. As φ is closed, there exists an open neighborhood V of s such that $\varphi^{-1}(V) \subseteq C$. If Y is also separated over S, then C is also closed, so $\varphi(X - C)$ is clopen in S, and its complement V in S is then a clopen neighborhood of s such that $\varphi^{-1}(V) \subseteq C$. □

Proposition 11.1.28. *Let Y be a connected scheme, $f : X \to Y$ be a unramified and separated morphism. Then any Y-section g of X is an isomorphism from Y onto an connected component of X, and the map $g \mapsto g(Y)$ is a bijection from $\Gamma(X/Y)$ to the set of connected components Z of X (necessarily open in X) such that the restriction of f to Z is an isomorphism from Z onto Y. In particular, if g_1 and g_2 and two Y-sections of X such that $g_1(y) = g_2(y)$ for a point $y \in Y$, then $g_1 = g_2$.*

Proof. It follows from Corollary 11.1.22 that any Y-section s of X is an open immersion, and as X is a separated Y-scheme, s is also a closed immersion (Corollary 8.5.19). Then s is an isomorphism from Y onto a clopen subscheme of X, and as $s(Y)$ is connected, this is necessarily a connected component of X. The rest of the proposition is then immediate. □

Example 11.1.29. Let k be a field and $X = \coprod_{n \in \mathbb{N}} \mathrm{Spec}(k)$ be an infinite direct sum of $\mathrm{Spec}(k)$. Then the morphism $f : X \to \mathrm{Spec}(k)$ is étale. This can be considered as a trivial covering space formed by infinitely many copies of the base space.

Example 11.1.30. Let k be a field with $\mathrm{char}(k) \neq 2$ and consider the normalization $f : \mathbb{A}^1_k \to C$ morphism, where $C = \mathrm{Spec}(k[T^2, T^3])$ and $\mathbb{A}^k_1 = \mathrm{Spec}(k[T])$. The morphism f corresponds to the ring homomorphism

$$\varphi : k[T^2, T^3] \to k[T], \quad (X, Y) \mapsto (T^2, T^3).$$

If we set $A = k[T^2, T^3]$ and $B = k[T]$, then it is easy to see that $B = A[X]/(X^2 - T^2)$, so the B-module $\Omega^1_{B/A}$ is generated by the symbol dX subject to the relation

$$0 = d(X^2 - T^2) = 2XdX - dT^2 = 2XdX.$$

and it is therefore isomorphic, as a B-module, to

$$A[X]/(2X, X^2 - T^2) = k[T^2, T^3, X]/(2X, X^2 - T^2) = k[T]/(2T) = k[T]/(T).$$

From this, we see that the support of $\Omega^1_{B/A}$ is equal to $\{(T)\}$, so by Corollary 11.1.22 the unramified locus of f is $\mathbb{A}^1_k - \{0\}$. This ramification can also be detected from the local homomorphism $\varphi_{\mathfrak{p}} : A_{\mathfrak{p}} \to B_{\mathfrak{P}}$, where $\mathfrak{p} = (T^2)$ and $\mathfrak{P} = (T)$. The induced homomorphism on residue fields is an isomorphism on $k(T)$, but the image of the maximal ideal of $\mathfrak{p}A_{\mathfrak{p}}$ to $\mathfrak{P}B_{\mathfrak{P}}$ is equal to $\mathfrak{P}^2 B_{\mathfrak{P}}$, so it is not unramified.

Geometrically, the ramification of f at origin is resulted from the fact that the tangent vectors (two different directions) of $\mathbb{A}^1_k$ are both bended to the same tangent vector (the same direction) of the origin of C. This justifies our intuition of unramification by unicity of tangent vector liftings.

Example 11.1.31. Let k be a field and consider the morphism $f : \mathbb{A}^k_1 \to \mathbb{A}^k_1$ corresponding to the ring homomorphism

$$\varphi : k[X] \to k[Y], \quad X \mapsto Y^2.$$

This can be considered as the projection of the parabola onto $\mathbb{A}^1_k$. Let $A = k[X]$ and $B = k[Y]$, then we have $B = A[T]/(T^2 - X)$, so the B-module $\Omega^1_{B/A}$ is isomorphic to $k[Y]/(2T) = k[Y]/(Y)$. Therefore the ramification locus of f is again $\mathbb{A}^1_k - \{0\}$. This is not surprising, since geometrically the tangent map of f at the origin is zero, so it does not satisfy the unicity of tangent vector liftings.

We also remark that the morphism f is flat. In fact, for any $\lambda \in k$, the fiber of f over the closed point $(X - \lambda)$ is $\mathrm{Spec}(k[Y]/(Y^2 - \lambda))$, which is the disjioint union of 2 points if $\lambda \neq 0$, and is the tangen vector $\mathrm{Spec}(k[\varepsilon]/(\varepsilon^2))$ if $\lambda = 0$. In all cases, we see that $\dim_k(k[Y]/(Y^2 - \lambda)) = 2$, so f is flat. Another way to see this is to use the miracle flatness: since f is a morphism between regular schemes of equal dimension, it must be flat.

11.1.4.2 Characterization of smooth morphisms

Theorem 11.1.32. *Let $f : X \to Y$ be a morphism locally of finite presentation, x be a point of X and $y = f(x)$. Then the following conditions are equivalent:*

(i) *f is smooth at x;*

(ii) *f is flat at x and the $\kappa(y)$-scheme X_y is smooth over $\kappa(y)$ at x;*

(ii') *f is flat and geometrically regular at x;*

(iii) *$\mathscr{O}_{X,x}$ is a formally flat $\mathscr{O}_{Y,y}$-algebra.*

Proof. By the hypothesis on f, we can assume that $Y = \operatorname{Spec}(A)$, $X = \operatorname{Spec}(C)$, where $C = B/\mathfrak{J}$, $B = A[T_1, \dots, T_n]$ being a polynomial algebra and $\mathfrak{J}$ a finitely generated ideal of B. The equivalence of (i) and (iii) then follows from **??**. On the other hand, apply this result to the morphism $f_y : X_y \to \operatorname{Spec}(\kappa(y))$ (which is locally of finite type), we see that the equivalence of (i), (ii) and (ii') follows from **??**. □

Corollary 11.1.33. *Let $f : X \to Y$ be a morphism locally of finite presentation. For f to be smooth, it is necessary and sufficient that f is flat and for any $y \in Y$, X_y is a geometrically regular $\kappa(y)$-scheme.*

Proof. This follows from the definition of flat morphisms and Theorem 11.1.32. □

Proposition 11.1.34. *Let Y be a locally Noetherian scheme, $f : X \to Y$ be a morphism locally of finite type, x be a point of X and $y = f(x)$. If $A = \mathscr{O}_{Y,y}$ and $\mathscr{O}_{X,x}$, then the conditions of Theorem 11.1.32 are equivalent to the following:*

(iv) *B is a formally smooth A-algebra;*

(iv') *$\widehat{B}$ is a formally unramified $\widehat{A}$-algebra.*

Moreover, if $\kappa(x) = \kappa(y)$, these conditions are also equivalent to:

(iv'') *$\widehat{B}$ is isomorphic to a power series $\widehat{A}$-algebra $\widehat{A}[[T_1, \dots, T_n]]$.*

Proof. The equivalence of condition (iii) of Theorem 11.1.32 and (iv) follows from Jacobian criterion (**??**), and that of (iv) and (iv') follows from ([**?**] 0_{IV} 19.3.6). On the other hand, (iv'') implies (iv') without the hypothesis on residue fields (**??**). Finally, if $\mathfrak{m}$ is the maximal ideal of $\widehat{A}$, condition (iv') implies that $\widehat{B}/\mathfrak{m}\widehat{B}$ is a formally smooth complete Noetherian local $\kappa(y)$-algebra. The hypothesis $\kappa(x) = \kappa(y)$ then shows that $\widehat{B}/\mathfrak{m}\widehat{B}$ is $\kappa(y)$-isomorphic to a formal series algebra $\kappa(y)[[T_1, \dots, T_n]]$ (**??**). On the other hand, as $\widehat{A}[[T_1, \dots, T_n]]$ is a flat $\widehat{A}$-module and a complete Noetherian local $\widehat{A}$-algebra, we conclude from ([**?**] 0_{IV}, 19.6.4) that it is isomorphic to $\widehat{B}$, so (iv') imples (iv''). □

Proposition 11.1.35. *Let Y be a locally Noetherian scheme, $f : X \to Y$ be a morphism locally of finite type, x be a point of X and $y = f(x)$. Suppose that Y is reduced at y. Then for f to be smooth at x, it is necessary and sufficient that f is universally open in an open neighborhood of x in X_y and that the $\kappa(y)$-scheme X_y is a geometrically regular at x.*

Proof. In view of, it amounts to show that if X_y is geometrically regular at x, then f is flat at x if and only if it is universally open in an open neighborhood of x in X_y. Now if f is flat at x, so is it in an open neighborhood of x in X ([**?**], 11.1.1) and hence universally open in this neighborhood ([**?**], 2.4.6). Conversely, the hypotheses that X_y is geometrically regular at x and f is universally open in an open neighborhood of x in X_y together imply that f is flat at x, since $\mathscr{O}_{Y,y}$ is reduced ([**?**], 15.2.2). □

Corollary 11.1.36. *Let Y be a locally Noetherian scheme, $f : X \to Y$ be a morphism locally of finite type, x be a point of X and $y = f(x)$. Suppose that Y is reduced and geometrically unibrach at y. Then for f to be smooth at x, it is necessary and sufficient that f is equidimensional at x and that the $\kappa(y)$-scheme X_y is a geometrically regular at x.*

Proof. Since the set of points where f is equidimensional is open ([**?**], 13.3.2), this follows from Proposition 11.1.35 and Chevalley's criterion ([**?**], 14.4.4). □

Proposition 11.1.37. *Let $f : X \to Y$ be a morphism locally of finite presentation which is smooth at a point $x \in X$, and $y = f(x)$. Then, for the local ring $\mathscr{O}_{X,x}$ to be reduced (resp. integrally closed, resp. geometrically unibrach), it is necessary and sufficient that $\mathscr{O}_{Y,y}$ is.*

Proposition 11.1.38. *Let Y be a locally Noetherian scheme, $f : X \to Y$ be a morphism locally of finite type which is smooth at a point $x \in X$. Put $y = f(x)$, then*

- $\dim(\mathscr{O}_{X,x}) = \dim(\mathscr{O}_{Y,y}) + \dim(\mathscr{O}_{X,x} \otimes_{\mathscr{O}_{Y,y}} \kappa(y))$;
- $coprof(\mathscr{O}_{X,x}) = coprof(\mathscr{O}_{Y,y})$;
- *for the local ring $\mathscr{O}_{X,x}$ to possesses property (S_n) or (R_n), it is necessary and sufficient that $\mathscr{O}_{Y,y}$ to possesses property (S_n) or (R_n). In particular, if $\mathscr{O}_{X,x}$ is regular (resp. normal), so is $\mathscr{O}_{Y,y}$.*

11.1.4.3 Characterization of étale morphisms

Theorem 11.1.39. *Let $f : X \to Y$ be a morphism locally of finite presentation, x be a point of X and $y = f(x)$. then the following conditions are equivalent:*

(i) *f is étale at x.*

(i′) *f is smooth and unramified at x;*

(ii) *f is smooth and quasi-finite at x;*

(iii) *f is flat and unramified at x;*

(iii′) *f is flat at x and the ring $\mathscr{O}_{X,x}/\mathfrak{m}_y\mathscr{O}_{X,x}$ is a finite and separable field extension of $\kappa(y)$;*

(iv) *$\mathscr{O}_{X,x}$ is a formally étale $\mathscr{O}_{Y,y}$-algebra.*

Proof. The equivalence of (i) and (i′) follows from definition, and that of (i) and (iv) follows from Theorem 11.1.20 and Theorem 11.1.32. The equivalence of (iii) and (iii′) also follows from Theorem 11.1.20. The fact that (i′)⇒(iii) follows from Theorem 11.1.32. Conversely, if (iii′) is satisfied, then f is geometrically regular at x by **??**, and hence smooth at x by Theorem 11.1.32. Also, the implication (i)⇒(ii) follows from Corollary 11.1.23.

It then remains to prove that (ii)⇒(iii), and as we have seen that f is flat at x (11.1.32), it suffices to show that X_y is unramified over $\kappa(y)$. In other words, we are reduced to the case where $Y = \operatorname{Spec}(k)$. As the question is local over X, we can also assume that $X = \operatorname{Spec}(A)$, where A is a finite local k-algebra ([**?**] 0_{I}, 7.4.1). In view of the hypothesis (ii), A is a formally smooth k-algebra, and hence geometrically regular (**??**). Since it is Artinian, we then conclude $\mathfrak{m}_A = 0$, so A a finite and separable extension of k. □

Corollary 11.1.40. *Let f be a morphism locally of finite presentation. Then the following conditions are equivalent:*

(i) *f is étale.*

(i′) *f is smooth and unramified;*

(ii) *f is smooth and locally quasi-finite;*

(iii) *f is flat and unramified;*

(iii′) *f is flat and for any $y \in Y$, the fiber X_y is a sum of specturms of finite and separable field extensions of $\kappa(y)$;*

(iii″) *f is flat and for any $y \in Y$ and any separably closed extension k' of $\kappa(y)$, the geometric fiber $X_y \otimes_{\kappa(y)} k'$ is a sum of specturms of fields isomorphic to k'.*

Proposition 11.1.41. *Let Y be a locally Noetherian scheme, $f : X \to Y$ be a morphism locally of finite presentation, x be a point of X and $y = f(x)$. If $A = \mathscr{O}_{Y,y}$ and $\mathscr{O}_{X,x}$, then the conditions of Theorem 11.1.39 are equivalent to the following:*

(iv) *$\widehat{B}$ is a formally étale $\widehat{A}$-algebra.*

(iv′) *$\widehat{B}$ is a free $\widehat{A}$-module and $\widehat{B} \otimes_{\widehat{A}} \kappa_A$ is a finite and separable field extension of κ_A (which implies that $\widehat{B}$ is a finite $\widehat{A}$-algebra).*

Moreover, if $\kappa(x) = \kappa(y)$ or $\kappa(y)$ is separably closed, these conditions are also equivalent to:

(iv″) *the canonical homomorphism $\widehat{A} \to \widehat{B}$ is bijective.*

Proposition 11.1.42. *Let Y be a locally Noetherian scheme, $f : X \to Y$ be a morphism locally of finite type, x be a point of X and $y = f(x)$. If f is étale at x, then $\dim(\mathscr{O}_{X,x}) = \dim(\mathscr{O}_{Y,y})$.*

Proof. This follows from Proposition 11.1.38 since x is isolated in the fiber X_y. □

11.2 Galois categories

11.2.1 The axioms of Galois theory

11.3 The étale fundamental group

11.3.1 Finite group quotients of schemes

Let G be a (fixed) finite group and S be a scheme. By a ***G*-scheme** over S, we mean an S-scheme X with a right action of G on X. By definition, this means we have a homomorphism $\rho : G \to \mathrm{Aut}_S(X)$ from G into the set of S-automorphisms of G. For any S-scheme Z, G has an induced left action on the set $\mathrm{Hom}_S(X, Z)$, so we can consider the set $\mathrm{Hom}_S(X, Z)^G$ of G-invariant S-morphisms. Since this set depends functorially by Z, we then obtain a functor $\mathrm{Hom}(X, -)^G$, for which we can ask the representability. By Yoneda lemma, this is equivalent to the existence a S-scheme Y and a G-invariant S-morphism $p : X \to Y$ such that, for any S-scheme Z, the corresponding map

$$\mathrm{Hom}_S(Y, Z) \to \mathrm{Hom}_S(X, Z)^G, \quad g \mapsto g \circ p$$

is bijective. In this case, we say that (Y, p) (or the S-morphism $p : X \to Y$) is a **quotient scheme** of X by G, and denote it by X/G. It is clear that the pair (Y, p) is determined up to isomorphism.

If the scheme X is affine, then the quotient scheme of X always exists and has a simple interpretation. In fact, in this case G has a left action on the ring A of X, and the invariant subring A^G then provides such a quotient.

Proposition 11.3.1. *Let R be a ring, A be an R-algebra with an R-linear action by G. Let $X = \mathrm{Spec}(A)$, $Y = \mathrm{Spec}(A^G)$, and $p : X \to Y$ be the canonical morphism.*

(a) *The morphism p is integral and surjective.*

(b) *The fibers of p are orbits of G, and p is a quotient map.*

(c) *Let $x \in X$, $y = p(x)$, and G_x be the stabilizer of x. Then $\kappa(x)$ is a normal extension of $\kappa(y)$ and the canonical map $G_x \to \mathrm{Gal}(\kappa(x)/\kappa(y))$ is surjective.*

(d) *(Y, p) is a quotient scheme of X by G.*

Proof. The assertions (a), (b) and (c) follows from **??**, **??** and the fact that an integral morphism is closed (Proposition 9.6.7). Finally, assertion (d) follows from Theorem 8.1.17 and the fact that for any ring B, we have a canonical bijection

$$\mathrm{Hom}_{\mathbf{Ring}}(B, A)^G \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{Ring}}(B, A^G).$$ □

Corollary 11.3.2. *Under the hypotheses of Proposition 11.3.1, the canonical homomorphism $\mathscr{O}_Y \to p_*(\mathscr{O}_X)^G$ is an isomorphism.*

Proof. By Proposition 8.1.12, the sheaf $p_*(\mathscr{O}_X)$ corresponds to the A^G-module A, so the corollary follows from the isomorphism $(S^{-1}A)^G = S^{-1}A^G$ of **??**. □

Proposition 11.3.3. *Let S be a scheme, X be a G-scheme over S and $p : X \to Y$ be a G-invariant affine S-morphism such that $\mathscr{O}_Y \cong p_*(\mathscr{O}_X)^G$. Then the conditions of Proposition 11.3.1 are satisfied.*

Proof. Since conditions (a), (b) and (c) are local on X and Y, we can assume that $Y = \mathrm{Spec}(B)$ and $X = \mathrm{Spec}(A)$ are both affine (since the morphism p is affine). Then the hypothesis implies that $B = A^G$, so it suffices to apply Proposition 11.3.1. As for assertion (d), it suffices to note that $p : X \to Y$ is a quotient map. □

Corollary 11.3.4. *Under the hypotheses of Proposition 11.3.3, for any open subset U of Y, the restriction $p^{-1}(U) \to U$ is a quotient U by G.*

Proof. This follows from the fact that the restriction $p|_{p^{-1}(U)} : p^{-1}(U) \to U$ also satisfies the hypotheses of Proposition 11.3.3. □

Corollary 11.3.5. *Retain the hypotheses of Proposition 11.3.3.*

(a) *For X to be affine (resp. separated) over S, it is necessary and sufficient that Y is affine (resp. separated) over S.*

(b) *If X is of finite type over S, then it is finite over Y. If S is also locally Noetherian, then Y is of finite type over S.*

Proof. As X is affine (and a fortiori separated) over Y, we see that Y is affine (resp. separated) over S if and only if X is (Proposition 8.5.26 and Proposition 9.1.33). Now if X is of finite type over Z, so is it over Y, and hence finite over Y (since $p : X \to Y$ is integral). If S is also locally Noetherian, then since Y is already quasi-compact over S (Proposition 8.6.3), it suffices to show that Y is locally of finite type over S, so we may assume that $X = \mathrm{Spec}(A)$, $Y = \mathrm{Spec}(B)$ and $S = \mathrm{Spec}(R)$ are affine. Then the ring A is of finite type over R and the conclusion follows from **??**. □

Let X be a G-scheme over a scheme S. If there exists a quotient scheme (Y, p) of X by G, we then say that X is **admissible** (or an **admissible G-scheme** over S). By Proposition 11.3.1, we see that any affine G-scheme over S is admissible. In the general case, we have the following characterization for admissible G-schemes.

Proposition 11.3.6. *Let X be a G-scheme over a scheme S. Then the following conditions are equivelent:*

(i) *X is admissible;*

(ii) *X is a union of affine open G-invariant subsets;*

(iii) *any orbit of G in X is contained in an affine open subset.*

Proof. It is clear that (ii)⇒(iii), and conversely, if an orbit T of G is contained in an affine open subset U, then the intersection $U' = \bigcap_{g\in G} g\cdot U$ is a G-invariant open subset containing T and contained in the affine open U. As in U, any finite subset has a fundamental system of open affine neighborhoods, there exists an open affine neighborhood V of T contained in U'. The transforms of V by G are therefore affine and contained in U', which is separate, so their intersection U'' is an affine open subset which is invariant under G and contains T. Since X is the union of orbits of G, we conclude that (iii)⇒(ii).

We note that condition (ii) is necessary for (i), since if $p : X \to Y$ is a quotient scheme of X and (V_α) is an affine open covering of Y, then $U_\alpha = p^{-1}(V_\alpha)$ is G-invariant and affine in X, and they cover X. Conversely, if (X_α) is a covering of X by G-invariant affine opens, then by Proposition 11.3.1, we can form the quotient $Y_\alpha = X_\alpha/G$; in each Y_i, the image of $X_i \cap X_j$ is an open subset Y_{ij}, which is identified with X_{ij}/G in view of Corollary 11.3.4. In particular, we deduce an isomorphism $Y_{ij} \cong Y_{ji}$, so we can glue Y_i to construct Y. □

Corollary 11.3.7. *If X is an admissible G-scheme, then it is admissible for any subgroup H of G.*

Corollary 11.3.8. *Let S be a scheme and suppose that X is a G-scheme over S which is affine over S. Then X is admissible; in fact, if X is defined by the quasi-coherent $\mathscr{O}_S$-algebra $\mathscr{A}$, then its quotient Y is defined by the quasi-coherent algebra $\mathscr{A}^G$.*

Proof. We may assume that $X = \mathrm{Spec}(A)$ and $S = \mathrm{Spec}(R)$ are affine, where A is an R-algebra. Then by Proposition 11.3.1 the quotient is defined by the R-algebra A^G, so the conclusion follows. □

Proposition 11.3.9. *Let X be an admissible G-scheme over S and $p : X \to Y$ be its quotient scheme. Consider a base change $S' \to S$ and put $X' = X \times_S S'$, $Y' = Y \times_S S'$, so that X' is a G-scheme over S' and the morphism $p' : X' \to Y'$ is G-invariant. If S' is flat over S, then (Y', p') is the quotient of X' by G. In other words, we have an isomorphism*

$$(X/G) \times_S S' \cong (X \times_S S')/G$$

Proof. We can evidently assume that S, X and Y are affine. It then suffices to prove that, if A is an R-algebra acted by G and $B = A^G$ is the invariant subring, then for any flat R-algebra R', the invaiant subring of $A' = A \times_R R'$ is identified with $B' = B \otimes_R R'$. To see this, note that the subring B is characterized by the exact sequence

$$0 \longrightarrow B \longrightarrow A \xrightarrow{\varphi_A} \prod_{g\in G} A \longrightarrow 0$$

where the homomorphism φ_A is defined by $x \mapsto (gx - x)_{g\in G}$. Since the induced homomorphism $\varphi_A \otimes 1_{R'} : A' \to A'$ is identified with $\varphi_{A'}$ and R' is flat over R, the conclusion follows. □

Remark 11.3.10. We note that the flatness assumption is essential for Proposition 11.3.9. For example, if Y' is a closed subscheme of Y and X' is its inverse image under p, then Y' is not necessarily isomorphic to the quotient X'/G. However, as we shall see, this is true if X is étale over Y.

11.3.2 Decomposition groups and inertia groups

Let G be a finite group and X be a G-scheme. For $x \in X$, the stabilizer subgroup of x is called the **decomposition group** of x, and denoted by $G^Z(x)$. This group has a canonical action on the residue field $\kappa(x)$, and the kernel of the canonical homomorphism $G^Z(x) \to \mathrm{Aut}(\kappa(x))$ is called the **inertia group** of x, and denoted by $G^T(x)$.

Let Ω be a separably closed field and $\xi : \mathrm{Spec}(\Omega) \to X$ be a geometric point of X which image x. Then G acts on the set of Ω-points $X(\Omega)$, and the inertia group of x is identified with the stabilizer of ξ:

$$G^T(x) = \{g \in G : g\xi = \xi\}.$$

Using this interpretation of the inertia subgroup, we can prove the following fact:

Proposition 11.3.11. *For any base change morphism $S' \to S$, let $X' = X \times_S S'$, x' be a point of X', and x be its image in X. Then we have $G^T(x') = G^T(x)$.*

Proof. It suffices to choose Ω large enough so that it is an extension of $\kappa(s')$ (where s' is the image of x' in S'). □

Proposition 11.3.12. *Let X be an admissible G-scheme, $p : X \to Y$ be its quotient, and suppose that Y is locally Noetherian and p is finite. Let H be a subgroup of G and consider $X' = X/H$. Let $x \in X$, x' be its image in X', and $y = p(x)$.*

(a) *If $H \supseteq G^T(x)$, then the homomorphism $p^\#_{x'} : \mathscr{O}_{Y,y} \to \mathscr{O}_{X',x'}$ is étale.*

(b) *If $H \supseteq G^Z(x)$, then the homomorphism $p^\#_{x'} : \mathscr{O}_{Y,y} \to \mathscr{O}_{X',x'}$ induces an isomorphism on completions.*

Proof. Since Y is locally Noetherian, the local scheme $Y' = \mathrm{Spec}(\widehat{\mathscr{O}_{Y,y}})$ is flat over Y, so by considering the base change $Y' \to Y$ and use Proposition 11.3.9, we may assume that Y is the spectrum of a complete Noetherian local ring and X is the specturm of a finite A-algebra B. If $H \supseteq G^Z(x)$, then by **??**(b), $\kappa(x')$ is identified with $\kappa(y)$ and $\mathfrak{m}_y$ generates the maximal ideal of $\mathfrak{m}_{x'}$, so the homomorphism $p^\#_{x'} : \mathscr{O}_{Y,y} \to \mathscr{O}_{X',x'}$ induces an isomorphism on completions. Assertion (a) then follows from (b) if we make a flat base change of Y so that $G^Z(x) = G^T(x)$ (Proposition 11.3.11). □

Corollary 11.3.13. *Under the conditions of Proposition 11.3.12, suppose that $G^T(x)$ is trivial. Then X is étale over Y at x.*

Corollary 11.3.14. *Suppose that X is connected and the action of G is faithful on X. For X to be étale over Y, it is necessary and sufficient that the inertia groups of X are trivial. In this case, G is identified with the group of Y-automorphisms of X.*

Proof. In view of Corollary 11.3.13, it suffices to suppose that X is étale over Y. Let $x \in X$ and $g \in G^T(x)$, then it follows from Corollary 11.1.26 that g acts trivially on X, and hence equals to the identity since G acts faithfully on X. □

Proposition 11.3.15. *Let S be a locally Noetherian scheme, X be a separated and étale scheme of finite type over S, and G be a finite group of S-automorphisms of X. Then the G-scheme X is admissible and the quotient scheme X/G is étale over S.*

Proof. Since X is separated and étale over S, it is quasi-projective over S ([**?**], 8.11.2), so the existence of X/G follows from Proposition 11.3.6(iii). To see that X/G is étale over S, we may assume that G acts transitively on the set of connected components of X, and by considering the stabilizer of a connected component, that X is connected. Finally, we can assume that G acts faithfully on X. But then the inertia groups of X are trivial, so it follows from Corollary 11.3.13 that $p : X \to X/G$ is étale. To see that X/G is then étale over S, we consider a point $x \in X$ and let $y = p(x)$, $s = \varphi(x)$ (where $\varphi : X \to S$ is the structural morphism). By taking a flat base change, we may assume that $\kappa(s)$ is separably closed, so by Proposition 11.1.41, the induced homomorphisms $\widehat{\mathscr{O}_{Y,y}} \to \widehat{\mathscr{O}_{X,x}}$ and $\widehat{\mathscr{O}_{S,s}} \to \widehat{\mathscr{O}_{X,x}}$ are bijective, so $\widehat{\mathscr{O}_{S,s}} \to \widehat{\mathscr{O}_{Y,y}}$ is bijective and X/S is étale over S by Proposition 11.1.41. □

Corollary 11.3.16. *If X is finite étale over S, then X/G is finite étale over S.*

Proof. By Corollary 11.3.8, if X is defined by the quasi-coherent $\mathscr{O}_S$-algebra $\mathscr{A}$, then X/G is defined by $\mathscr{A}^G$. By assumption $\mathscr{A}$ is a finite $\mathscr{O}_S$-algebra, so its subalgebra $\mathscr{A}^G$ is also finite over S, since S is locally Noetherian. □

11.3.3 The Galois category $\mathbf{FÉt}_S$

Let S be a locally Noetherian and connected scheme. We denote by $\mathbf{FÉt}_S$ the category of finite étale coverings of S, with morphisms given by S-morphisms of finite étale coverings of S. Any object X of $\mathbf{FÉt}_X$ is often call a finite étale cover of S (even if X is empty). It is clear that there is a canonical functor $p : \mathbf{FÉt}_S \to \mathbf{Sch}_S$.

Example 11.3.17. Let k be a separably closed field and $S = \mathrm{Spec}(k)$. In this case $\mathbf{FÉt}_S$ is equivalent to the category of finite sets: in fact, a scheme étale over k is the disjoint union of spectra of fields finite separable over k, whose underlying space is finite. Conversely, a finite set with n points is uniquely endowed with the structure of a scheme étale over k, that is, the specturm $\mathrm{Spec}(k^n)$.

Consider now a geometric point $\bar{s} : \mathrm{Spec}(\Omega) \to S$ of S, with values in an algebraically closed field Ω. For an étale covering X of S, we define $F_{\bar{s}}(X)$ to be the underlying set of the fiber $X_{\bar{s}} = X \times_S \bar{s}$. Since $X_{\bar{s}}$ is finite and étale over $\bar{s}$, it is a disjoint union of copies of $\bar{s}$ (Example 11.3.17), so $F_{\bar{s}}(X)$ can also be considered as the set of geometric points $\mathrm{Spec}(\Omega) \to X$ lying over $\bar{s}$, in other words,

$$F_{\bar{s}}(X) = \mathrm{Hom}_S(\bar{s}, X).$$

The assignment $X \mapsto F(X)$ is clearly functorial on X, hence defines a functor $F_{\bar{s}} : \mathbf{FÉt}_S \to \mathbf{FSet}$.

Proposition 11.3.18. *The pair $(\mathbf{FÉt}_S, F_{\bar{s}})$ form a Galois category.*

Proof. After identifying $\mathbf{FÉt}_{\bar{x}}$ with the category of finite sets (Example 11.3.17), we see that our functor $F_{\bar{s}}$ is nothing but the base change functor under the morphism $\bar{s} \to S$. Condition (G1) follows from Proposition 11.1.14, (G2) follows from (Corollary 11.3.16), (G3) from ([**?**], 5.3.5), and (G4) is trivial by definition. On the other hand, (G5) follows from ([**?**], 5.3.5) and the begining of Section 11.3.2, and (G6) is proved in ([**?**], 5.3.7). □

We can therefore apply the results proved in §11.2, which permits us in particular to define a pro-object P of $\mathbf{FÉt}_S$ representing $F_{\bar{s}}$, which is called the **universal covering** of S at s, and a topological group $\pi = \mathrm{Aut}(F_{\bar{s}}) = \mathrm{Aut}^0(P)$, called the **étale fundamental group** of S at s, denoted by $\pi_1(S, \bar{s})$. The functor F then defines an equivalence from $\mathbf{FÉt}_S$ to the category of finite π-sets, where $\pi = \pi_1(S, \bar{s})$. This equivalence allows us to interpret the operations of projective limits and finite inductive limits on coverings (products, fiber products, sums, passing to quotients, etc.) in terms of the analogues operations on π-$\mathbf{FSet}$, i.e. in terms of the obvious operations over finite π-sets. We also note that, since the topological connected components of an étale covering are also étale coverings, an object X in $\mathbf{FÉt}_S$ is connected in $\mathbf{FÉt}_S$ if and only if it is topological connected; in view of ([**?**], 5.5.3), this signifies that π acts transitively on $F_{\bar{s}}(X)$. Note that for an object X in $\mathbf{FÉt}_S$ to be faithfully flat and quasi-compact over S (it is already flat and quasi-compact over S), it is necessary and sufficient that the structural morphism $X \to S$ is surjective, i.e., is an epimorphism in $\mathbf{FÉt}_S$, or that $X \neq \varnothing$.

If $\bar{s}'$ is another geometric point of S (corresponding to an algebraically closed field Ω'), then it defines a fiber functor $F' = F_{\bar{s}'}$ over $\mathbf{FÉt}_S$, which is necessarily isomorphic to $F = F_{\bar{s}}$. Therefore, the fundamental group $\pi_1(S, s)$ is independent of $\bar{s}$ up to isomorphisms. If we denote by $\pi_1(S; s, s')$ the set of these isomorphisms (which can also be regareded as the set of isomorphisms $F_{\bar{s}} \to F_{\bar{s}'}$), then we obtain a groupoid whose objects are geometric points of S, and the fundamental group being the automorphism of these objects. The set $\pi_1(S; \bar{s}, \bar{s}')$ can also be considered as the set of path classes from $\bar{s}$ to $\bar{s}'$, which has an evident composition law. Finally, we can define a pro-group Π_1^S of $\mathbf{FÉt}_S$, called the **fundamental pro-group of S** or **local system of fundamental groups over S**, by the condition that we have a functorial isomorphism at the geometric point $\bar{s}$ of S:

$$F_{\bar{s}}(\Pi_1^S) = \pi_1(S, \bar{s}).$$

(cf. [**?**], remarique 5.5.10). In particular, if s is an ordinary point of S, the fiber of Π_1^S at s is a pro-group over $\kappa(s)$, which is the projective limit of finite étale groups over $\kappa(s)$. This pro-group is the called the

fundamental group of S at the point s, and denoted by $\pi_1(S,s)$. By definition, the points with values in an algebraically closed extension Ω of $\kappa(s)$ are exactly the elements of $\pi_1(S,\bar{s})$, where $\bar{s}$ is the geometric point of S defined by this extension. In particular, (by taking S to be the spectrum of a field) any field k is canonically associated with a pro-finite group over k, denoted by $\pi_1(k)$, which is the projective limit of finite étale groups over k, and whose points in an algebraically closed extension Ω of k is identified with the elements of the Galois group of $\bar{k}/k$, where $\bar{k}$ is the Galois closure of k in Ω.

Now let $f : S' \to S$ be a morphism of locally Noetherian connected schemes, $\bar{s}'$ be a geometric point of S', and $\bar{s} = f(\bar{s}')$ be its image in S. Then the inverse image functor induces a functor of categories:

$$f^* : \mathbf{F\acute{E}t}_S \to \mathbf{F\acute{E}t}_{S'}$$

and we have a functorial isomorphism $F_{\bar{s}} \xrightarrow{\sim} F_{\bar{s}'} \circ f^*$, so that f^* is an exact functor and we have a commutative diagram

$$\begin{tikzcd}
\mathbf{F\acute{E}t}_S \arrow[r, "\text{base change}"] \arrow[d, "F_{\bar{s}}"] & \mathbf{F\acute{E}t}_{S'} \arrow[d, "F_{\bar{s}'}"] \\
\pi(S,\bar{s})\text{-}\mathbf{FSet} \arrow[r, "f^*"] & \pi(S',\bar{s}')\text{-}\mathbf{FSet}
\end{tikzcd}$$

We have in particular a canonical homomorphism

$$f_* = \pi_1(f,\bar{s}') : \pi_1(S',\bar{s}') \to \pi_1(S,\bar{s})$$

which allows us to consider the inverse image functor as an operation of restriction of the group action. The properties of the function f^* are therefore expressed in a simple way by the properties of the homomorphism of the associated groups. If in particular S' is an étale covering of S, then the homomorphism f_* is an isomorphism from $\pi_1(S',\bar{s}')$ onto an open subgroup of $\pi_1(S,\bar{s})$ defining the étale covering S' of S (i.e. the stabilizer of $\bar{s}' \in F_{\bar{s}}(S')$ in $\pi_1(S,\bar{s})$).

Proposition 11.3.19. *Let S be the spectrum of a field k, and Ω be an algebraically closed extension of k, defining a geometric point $\bar{s}$ of S with values in Ω. Let k^{sep} be the separable closure of k in Ω. Then there exists a canonical isomorphism $\pi_1(S,\bar{s}) \xrightarrow{\sim} \mathrm{Gal}(k^{\mathrm{sep}}/k)$.*

Proof. Let $\bar{k}$ be the algebraic closure of k in Ω, which corresponds to a geometric point $\bar{s}'$ of S with values in k'. The natural homomorphism $F_{\bar{s}'} \to F_{\bar{s}}$ is evidently an isomorphism, because any scheme X étale over S we have

$$\mathrm{Hom}_S(\bar{s},Y) = \mathrm{Hom}_S(\bar{s}',Y)$$

In fact, X is then isomorphic to the spectrum of finite separable fields extensions of k in Ω, and any such extension necessarily lies in k^{sep}, hence in $\bar{k}$. On the other hand, the group $G = \mathrm{Gal}(\bar{k}/k) = \mathrm{Gal}(k^{\mathrm{sep}}/k)$ clearly acts on $F_{\bar{s}'}$, so we have a homomorphism

$$G \to \mathrm{Aut}(F_{\bar{s}'}) \xrightarrow{\sim} \mathrm{Aut}(F_{\bar{s}}) = \pi_1(S,\bar{s}).$$

We then obtain a continuous homomorphism $G \to \pi_1(S,\bar{s})$, and it remains to prove that this is an isomorphism. To this end, note that this homomorphism is injective because any element in its kernel is an autumorphism of $\bar{k}/k$ which induces the identity on any finite separated extension of k, hence trivial. It is surjective because if X is a connected étale covering of S, hence defined by a finite separable extension L/k, then G is transitive on the set of k-homorphisms of L into $\bar{k}$. □

Remark 11.3.20. We can in fact write out the explicit isomorphism of Proposition 11.3.19. Observe that $\mathrm{Gal}(k^{\mathrm{sep}}/k) = \mathrm{Gal}(\bar{k}/k)$ and $F_{\bar{s}}(X) = \mathrm{Hom}_S(\mathrm{Spec}(\bar{k}),Y)$, so we can consider the map

$$\mathrm{Gal}(\bar{k}/k) \times F_{\bar{s}}(X) \to F_{\bar{s}}(X), \quad (\sigma,\bar{x}) \mapsto \bar{x} \circ {}^a\sigma$$

where ${}^a\sigma$ is the induced automorphism on $\mathrm{Spec}(\bar{k})$. This defines an action of $\mathrm{Gal}(\bar{k}/k)$ on $F_{\bar{s}}$ since

$$\sigma\tau \cdot \bar{x} = \bar{x} \circ {}^a(\sigma\tau) = \bar{x} \circ {}^a\tau \circ {}^a\sigma = \sigma \cdot (\tau \cdot \bar{x}).$$

The action is clearly functorial on X, so we obtain the canonical homomorpism $\mathrm{Gal}(\bar{k}/k) \to \pi_1(S,\bar{s})$.

Proposition 11.3.21. *Let S be a connected, locally Noetherian and integral normal scheme, K be the function field of S, Ω be an algebraically closed extension of K corresponding to a geometric point $\bar{s}'$ of $S' = \mathrm{Spec}(K)$ and a geometric point $\bar{s}$ of S. Then the canonical homomorphism $\pi_1(S', \bar{s}') \to \pi_1(S, \bar{s})$ is surjective. If we identify $\pi_1(S', \bar{s}')$ with the absolute Galois group $\mathrm{Gal}(K^{\mathrm{sep}}/K)$ of K in Ω, then the kernel of this homomorphism corresponds to the Galois group of the subextension M/K, where M is the composite of finite extensions L of k in Ω, which are unramified over S.*

Proof. The first assertion signifies that the inverse image under S' of a connected étale covering X of S is connected, i.e. that X is integral; this follows from ([**?**] 1.10.1). The kernel of this homomorphism can be interpreted as the automorphisms of K^{sep}/K which induces identity on the set $F_{\bar{s}}(X)$, where we can assume that the étale covering X of S is connected. But this then signifies, in view of the fact that $X_{\bar{s}}$ is étale over $\bar{s}$, that these automotphisms induce identity on the finite subextensions of K^{sep}/K which are unramified over S, whence our assertion. ☐

Chapter 12

Group schemes

12.1 Algebraic structures

12.1.1 Algebraic structures on the category of presheaves

Given a kind of algebraic structure in the category of sets, we propose to extend it to the category $\mathcal{C}$. Let us first consider an example: the case of groups.

12.1.1.1 Group objects in $\widehat{\mathcal{C}}$ Let $G \in \widehat{\mathcal{C}}$, a **group structure on G** is defined to be the assignment of a group structure on the set $G(S)$ for any $S \in \mathrm{Ob}(\mathcal{C})$, so that for any morphism $f : S' \to S$ in $\mathcal{C}$, the map $G(f) : G(S) \to G(S')$ is a homomorphism of groups. If G and H are groups in $\widehat{\mathcal{C}}$, a **group homomorphism** from G to H is defined to be a morphism $\theta \in \mathrm{Hom}(G, H)$ such that for any object $S \in \mathrm{Ob}(\mathcal{C})$, the map $\theta(S) : G(S) \to H(S)$ is a homomorphism of groups. We denote by $\mathrm{Hom}_{\mathbf{Grp}}(G, H)$ the set of group homomorphisms from G to H, and by $\mathbf{Grp}_{\widehat{\mathcal{C}}}$ the category of groups in $\widehat{\mathcal{C}}$.

Example 12.1.1. Let $E \in \widehat{\mathcal{C}}$, then the object $\mathcal{A}ut(E)$ is endowed with a group structure. The final object e also possesses a unique group structure and is a final object in $\mathbf{Grp}_{\widehat{\mathcal{C}}}$.

Let G be a group in $\widehat{\mathcal{C}}$. For any $S \in \mathrm{Ob}(\mathcal{C})$, let $e_G(S)$ be the unit element in $G(S)$. The family $e_G(S)$ then defines an element $e_G \in \Gamma(G) = \mathrm{Hom}(e, G)$, which is a morphism of groups $e \to G$ and called the **unit section** of G. We also note that giving a group structure over G amounts to given a composition law over G, which is a morphism

$$\mu_G : G \times G \to G$$

such that for any $S \in \mathrm{Ob}(\mathcal{C})$, $\mu_G(S)$ is a group structure on $G(S)$. With the same manner, $f : G \to H$ is a group homomorphism is and only if the following diagram is commutative:

$$\begin{tikzcd} G \times G \arrow[r, "\mu_G"] \arrow[d, "{(f,f)}"'] & G \arrow[d, "f"] \\ H \times H \arrow[r, "\mu_H"] & H \end{tikzcd}$$

A sub-object H of G such that for any $S \in \mathrm{Ob}(\mathcal{C})$, $H(S)$ is a subgroup of $G(S)$ possessing evidently a group structure induced by that of G: that is, such that the monomorphism $H \to G$ is a morphism of groups. The group H endowed with this structure is called a **subgroup** of G.

If G and H are two groups in $\widehat{\mathcal{C}}$, the product $G \times H$ is endowed with a group structure such that for any $S \in \mathrm{Ob}(\mathcal{C})$, $G(S) \times H(S)$ is endowed with the product group structure. The group $G \times H$ endowed with this structure is called the product group of G and H (and this is also the product in the category $\mathbf{Grp}_{\widehat{\mathcal{C}}}$).

If G is a group in $\widehat{\mathcal{C}}$ then for any $S \in \mathrm{Ob}(\mathcal{C})$, G_S is also a group in $\widehat{\mathcal{C}_{/S}}$. If G and H are groups in $\widehat{\mathcal{C}}$, then we can define an object $\mathcal{H}om_{\mathbf{Grp}}(G, H)$ of $\widehat{\mathcal{C}}$ by

$$\mathcal{H}om_{\mathbf{Grp}}(G, H)(S) = \mathrm{Hom}_{\mathbf{Grp}}(G_S, H_S).$$

One should note that $\mathcal{H}om_{\mathbf{Grp}}(G, H)$ is in general not a group, nor a fortiori the object $\mathcal{H}om$ in the category $\mathbf{Grp}_{\widehat{\mathcal{C}}}$. We define similarly objects $\mathcal{I}so_{\mathbf{Grp}}(G, H)$, $\mathcal{E}nd_{\mathbf{Grp}}(G)$ and $\mathcal{A}ut_{\mathbf{Grp}}(G)$.

Definition 12.1.2. Let $G \in \mathrm{Ob}(\mathcal{C})$. A **group structure over** $\boldsymbol{G}$ is defined to be a group structure over $h_G \in \widehat{\mathcal{C}}$. If G and H are groups in $\mathcal{C}$, a group homomorphism from G to H is defined to be an element $f \in \mathrm{Hom}(G, H) \cong \mathrm{Hom}(h_G, h_H)$ which is a group homomorphism from h_G to h_H. We denote by $\mathbf{Grp}_{\mathcal{C}}$ the category of groups in $\mathcal{C}$. Note that there is a Cartesian square in **Cat**:

$$\begin{tikzcd} \mathbf{Grp}_{\mathcal{C}} \arrow[r] \arrow[d] & \mathbf{Grp}_{\widehat{\mathcal{C}}} \arrow[d] \\ \mathcal{C} \arrow[r, "h"] & \widehat{\mathcal{C}} \end{tikzcd}$$

The preceding definitions and constructions carries over to groups in $\mathcal{C}$, provided that the corresponding functors (products, $\mathcal{H}om$ objects, etc.) are representable in $\mathcal{C}$. They also applies to categories of the form $\mathcal{C}_{/S}$, and in this case, we denote by $\mathcal{H}om_{S\text{-}\mathbf{Grp}}$ for $\mathcal{H}om_{\mathbf{Grp}}$, etc.

More generally, if $\mathcal{T}$ is a kind of structure over n base sets defined by finite projective limits (for example, by the commutativity of some diagrams constructed from Cartesian products: monoid, group, action by group, module over a ring, Lie algebra over a ring, etc.), we can define the notion of $\mathcal{T}$ structure over n objects $F_1, \dots, F_n$ over $\widehat{\mathcal{C}}$: such a structure is the assignment of a $\mathcal{T}$ structure over the sets $F_1(S), \dots, F_n(S)$ for each $S \in \mathrm{Ob}(\mathcal{C})$, so that for any morphism $S' \to S$ in $\mathcal{C}$, the family of maps $(F_i(S) \to F_i(S'))$ is a poly-homomorphism for the $\mathcal{T}$ structure. We define in a similar way the morphisms of the $\mathcal{T}$ structure, whence a category of $\mathcal{T}$ objects in $\widehat{\mathcal{C}}$. The fully faithful functor h permits us to define the category of $\mathcal{T}$ objects in $\mathcal{C}$ as a fiber product in **Cat**.

Suppose now that in $\mathcal{C}$ the pullbacks exist, and let $\mathcal{T}$ be an algebraic structure defined by the data of certain morphisms between Cartesian products satisfying some axioms consisting of the commutativity of certain diagrams constructed by the previous arrows. A $\mathcal{T}$ structure on a family of objects of $\mathcal{C}$ will therefore be defined by certain morphisms between Cartesian products satisfying certain commutation conditions. It follows that if $\mathcal{C}$ and $\mathcal{C}'$ are two categories with products and $\lambda : \mathcal{C} \to \mathcal{C}'$ is a functor commuting with products, then for any family of objects (F_i) of $\mathcal{C}$ equipped with a $\mathcal{T}$ structure, the family $(f(F_i))$ of objects of $\mathcal{C}'$ will thereby be endowed with a $\mathcal{T}$ structure. For example, any group in $\mathcal{C}$ will be transformed into a group in $\mathcal{C}'$, any pair of a ring in $\mathcal{C}$ and a module over this ring will be transformed into an analogous pair in $\mathcal{C}'$, etc.

In particular, let $\mathcal{C}$ be a category, then the constant functor $E \mapsto E_S$ commutes with finite projective limits, and hence transforms groups into S-groups (i.e. groups in $\mathcal{C}_{/S}$), rings to S-rings, etc.

Remark 12.1.3. It is worth noting that the previous construction, applied to the category $\widehat{\mathcal{C}}$, restores the notions that have already been defined there. In others words, it amounts to the same thing to give oneself a $\mathcal{T}$ structure over an object of $\widehat{\mathcal{C}}$ when we consider this object as a functor on $\mathcal{C}$, or to give ourselves a $\mathcal{T}$ structure on the representable functor over $\mathcal{C}$ defined by this object. For example, let $G \in \widehat{\mathcal{C}}$; if the functor $F \mapsto \mathrm{Hom}_{\widehat{\mathcal{C}}}(F, G)$ is endowed with a group structure, then so is its restriction to $\mathcal{C}$. Conversely, if G is a group in $\widehat{\mathcal{C}}$, then the multiplication morphism $\mu_G : G \times G \to G$ induces for each $F \in \widehat{\mathcal{C}}$ a group structure over $\mathrm{Hom}_{\widehat{\mathcal{C}}}(F, G)$, which is functorial on F.

12.1.1.2 Group action in $\widehat{\mathcal{C}}$ Let $E \in \widehat{\mathcal{C}}$ and $G \in \mathbf{Grp}_{\widehat{\mathcal{C}}}$. A ***G*-object structure** over E is defined to be an assignment over $E(S)$, for each $S \in \mathrm{Ob}(\mathcal{C})$, a $G(S)$-set structure on $G(S)$, so that for any morphism $S' \to S$ in $\mathcal{C}$, the map $E(S) \to E(S')$ is compatible with the group homomorphism $G(S) \to G(S')$. As usual, this is equivalent to giving a morphism

$$\mu : G \times E \to E$$

which for each S endows $E(S)$ with a $G(S)$-set structure. On the other hand, since $\mathrm{Hom}(G \times E, E) \cong \mathrm{Hom}(G, \mathcal{E}nd(E))$, the morphism μ defines also a morphism $G \to \mathcal{E}nd(E)$ and it is immediate to see that this is a group homomorphism which sends G into $\mathcal{A}ut(E)$. Therefore, giving a G-object structure over E is equivalent to giving a group homomorphism

$$\rho : G \to \mathcal{A}ut(E).$$

In particular, any element $g \in G(S)$ defines an automorphism $\rho(g)$ of the functor E_S, that is, an automorphism of $E \times h_S$ which commutes with the projection $E \times h_S \to h_S$, and in particular an automorphism of $E(S')$ for any morphism $S' \to S$.

Definition 12.1.4. Let G be a group in $\widehat{\mathcal{C}}$ and E be a G-object. We denote by E^G the sub-object of E defined by

$$E^G(S) = \{x \in E(S) : x_{S'} \text{ is invariant under } G(S') \text{ for any morphism } S' \to S\}.$$

Here $x_{S'}$ is the image of x under $E(S) \to E(S')$. It is clear that E^G (called the **invariant sub-object** of E) is the largest sub-object of E on which G acts trivially. If F is a sub-object of E, we denote by $N_G(F)$ and $Z_G(F)$ the subgroups of G defined by

$$\begin{aligned} N_G(F)(S) &= \{g \in G(S) : \rho(g)F_S = F_S\} \\ &= \{g \in G(S) : \rho(S)F(S') = F(S') \text{ for any morphism } S' \to S\}, \\ Z_G(F)(S) &= \{g \in G(S) : \rho(g)|_{F_S} = \mathrm{id}\} \\ &= \{g \in G(S) : \rho(g)|_{F(S')} = \mathrm{id} \text{ for any morphism } S' \to S\}. \end{aligned}$$

In particular, let $x \in \Gamma(E)$, i.e. a collection of elements $x_S \in E(S)$, $S \in \mathrm{Ob}(\mathcal{C})$, such that for any morphism $f : S' \to S$, we have $E(f)(x_s) = x_{S'}$ (if $\mathcal{C}$ admits a final object S_0, then we have $\Gamma(E) = E(S_0)$). Then x can be considered as a sub-functor of E, also denoted by x, and we have $N_G(x) = Z_G(x)$. This common functor is also denoted by $\mathrm{Stab}_G(x)$ and called the **stabilizer** of x. For any $S \in \mathrm{Ob}(\mathcal{C})$, we then have

$$\mathrm{Stab}_G(x)(S) = \{g \in G(S) : \rho(g)x_S = x_S\}.$$

Suppose that fiber products exist in $\mathcal{C}$. If $G = h_G$ (resp. $E = h_E$), where G is a group in $\mathcal{C}$ (resp. $E \in \mathrm{Ob}(\mathcal{C})$), and if $\mathcal{C}$ possesses a final object S_0, so that x is a morphism $S_0 \to E$, then the stabilizer $\mathrm{Stab}_G(x)$ is represented by the fiber product $G \times_E S_0$, where $G \to E$ is the composition of $\mathrm{id}_G \times x : G = G \times S_0 \to G \times E$ and $\mu : G \times E \to E$.

Remark 12.1.5. The formation of E^G, $N_G(F)$ and $Z_G(F)$ commute with base changes, so for any $S \in \mathrm{Ob}(\mathcal{C})$, weh ave

$$(E^G)_S = (E_S)^{G_S}, \quad N_G(F)_S \cong N_{G_S}(F_S), \quad Z_G(F)_S \cong Z_{G_S}(F_S).$$

If G is a group in $\mathcal{C}$ and E is an object of $\widehat{\mathcal{C}}$ (resp. an object of $\mathcal{C}$), a G-object structure over E is defined to be an h_G-object structure over E (resp. h_E). With this definition, the above notations carries to $\mathcal{C}$, if the corresponding functors are representable. For example, if $N_{h_G}(h_F)$ is representable, then it is represented by a unique sub-object of G, which is then a subgroup of G and denoted by $N_G(F)$.

We say that the group G in $\widehat{\mathcal{C}}$ acts on a group H in $\widehat{\mathcal{C}}$ if H is endowed with a G-object structure such that, for any $g \in G(S)$, the automorphism of $H(S)$ defined by g is a group automorphism. This is the same to say that for any $g \in G(S)$, the automorphism $\rho(g)$ of H_S is an automorphism of groups in $\widehat{\mathcal{C}_{/S}}$, or that the morphism $G \to \mathcal{A}ut(H)$ sends G into $\mathcal{A}ut_{\mathbf{Grp}}(H)$.

In the above situation, there exists over $H \times G$ a unique group structure such that, for any $S \in \mathrm{Ob}(\mathcal{C})$, $(H \times G)(S)$ is the semi-direct product of the groups $H(S)$ and $G(S)$ relative to the given action of $G(S)$ on $H(S)$. This group is denoted by $H \rtimes G$ and called the **semi-direct product** of H by G. By definition, we then have

$$(H \rtimes G)(S) = H(S) \rtimes G(S).$$

Let G be a group in $\widehat{\mathcal{C}}$. For any morphism $S' \to S$ of $\mathcal{C}$ and any $g \in G(S)$, let $\mathrm{Inn}(g)$ be the automorphism of $G(S')$ defined by $\mathrm{Inn}(g)h = ghg^{-1}$. This definition extends to a morphism of groups in $\widehat{\mathcal{C}}$:

$$\mathrm{Inn} : G \to \mathcal{A}ut_{\mathbf{Grp}}(G) \subseteq \mathcal{A}ut(G).$$

The above definitions then apply to H and we have subgroups $N_G(E)$ and $Z_G(E)$ for any sub-object E of G.

Definition 12.1.6. We define the **center** of G and denote by $Z(G)$ the subgroup $Z_G(G)$ of G. We say that G is **abelian** if $Z_G(G) = G$ or, equivalently, if $G(S)$ is abelian for any $S \in \mathrm{Ob}(\mathcal{C})$. A subgroup H of G is called **invariant** in G if $N_G(H) = G$, or equivalently, if $H(S)$ is invariant in $G(S)$ for any S. Moreover, we say that H is **cental** in G if $Z_G(H) = G$, or equivalently, if $H(S)$ is cental in $G(S)$ for any S.

Definition 12.1.7. Let $f : G \to G'$ be a group homomorphism. The kernel of f is the subgroup of G defined by

$$(\ker f)(S) = \{x \in G(S) : f(S)x = 1\} = \ker f(S)$$

for any $S \in \mathrm{Ob}(\mathcal{C})$. This is an invariant subgroup of G. Note that if G and G' belongs to $\mathcal{C}$, $\mathcal{C}$ possesses a final object S_0 and fiber products exist in $\mathcal{C}$, then $\ker(f)$ is represented by $S_0 \times_{G'} G$.

Definition 12.1.8. Let $E \in \widehat{\mathcal{C}}$ and G be a group acting on E. We say that the action of G on E is faithful if the kernel of the morphism $G \to \mathcal{A}ut(E)$ is trivial, that is, if for any $S \in \mathrm{Ob}(\mathcal{C})$ and $g \in G(S)$, the condition $g_{S'} \cdot x = x$ for any morphism $S' \to S$ and $x \in E(S')$ implies $g = 1$.

Many definitions and propositions of elementary group theory are easily transported to the setting of groups in $\widehat{\mathcal{C}}$. Let us simply point out the following which will be useful to us:

Proposition 12.1.9. *Let $f : W \to G$ be a group homomorphism and put $H(S) = \ker f(S)$ for $S \in \mathrm{Ob}(\mathcal{C})$. Let $u : G \to W$ be a group homomorphism which is a section of f. Then W is identified with a semi-direct product of H by G for the action of G over H defined by $(g, h) \mapsto \mathrm{Inn}(u(g))h$ for $g \in G(S)$, $h \in H(S)$ and $S \in \mathrm{Ob}(\mathcal{C})$.*

All the definitions and propositions are transported as usual to $\mathcal{C}$. We define in particular the semi-product of two groups H and G in $\mathcal{C}$, with G acting on H, when the Cartesian product $H \times G$ exists in $\mathcal{C}$. We have the following analogue of Proposition 12.1.9:

Proposition 12.1.10. *Let $f : W \to G$ and $i : H \to W$ be group homomorphisms in $\mathcal{C}$ such that for any $S \in \mathrm{Ob}(\mathcal{C})$, $(H(S), i(S))$ is a kernel of $f(S) : W(S) \to G(S)$. Let $u : G \to W$ be a homomorphism of groups in $\mathcal{C}$ which is a section of f. Then W is identified with the semi-direct product of H by G for the action of G over H such that if $S \in \mathrm{Ob}(\mathcal{C})$, $g \in G(S)$ and $h \in H(S)$, we have $\mathrm{Inn}(u(g))i(h) = i(ghg^{-1})$.*

To end this paragraph, we breifly introduce the concept of modules over a ring in $\widehat{\mathcal{C}}$. So let A and M be objects of $\widehat{\mathcal{C}}$, we say that F is a **module over the ring A**, of simply an A-module, if for each $S \in \mathrm{Ob}(\mathcal{C})$ the et $A(S)$ is endowed with a ring structure and $M(S)$ with a module structure over this ring, so that for any morphism $S' \to S$, the map $A(S) \to A(S')$ is a ring homomorphism and $M(S) \to M(S')$ is a bi-homomorphism. If the ring A is fixed, we define as usual morphisms of A-modules M, M', whence the abelian group $\mathrm{Hom}_A(M, M')$, and the category of A-modules, which we denote by $\mathbf{Mod}(A)$.

Proposition 12.1.11. *The category $\mathbf{Mod}(A)$ is endowed with an abelian category structure defined "argument by argument". Moreover, $\mathbf{Mod}(A)$ is an (AB5) category, that is, arbitrary direct sums exist in $\mathbf{Mod}(A)$ and if M is an A-module, N is a submodule, and $(M_i)_{i\in I}$ is a filtrant family of increasing submodules of M, then*

$$\bigcup_{i\in I}(M_i \cap N) = \Big(\bigcup_{i\in I} M_i\Big) \cap N.$$

Proof. In fact, let $f : M \to M'$ be a morphism of A-modules. We define the A-modules $\ker f$ (resp. $\mathrm{im}\, f$ and $\mathrm{coker}\, f$) so that for any $S \in \mathrm{Ob}(\mathcal{C})$, $(\ker f)(S) = \ker f(S)$ (resp. $\cdots$). Then $\ker f$ (resp. $\mathrm{coker}\, f$) is a kernel (resp. cokernel) of f, and we have an isomorphism of A-modules $M/\ker f \cong \mathrm{im}\, f$. This proves that $\mathbf{Mod}(A)$ is an abelian category.

Arbitrary direct sums exist in $\mathbf{Mod}(A)$ and are defined "argument by argument". Finally, if M is an A-module, N is a submodule, and $(M_i)_{i\in I}$ is a filtrant family of increasing submodules of M, then the inclusion

$$\bigcup_{i\in I}(M_i \cap N) \subseteq \Big(\bigcup_{i\in I} M_i\Big) \cap N$$

is an equality: in fact, if $S \in \mathrm{Ob}(\mathcal{C})$ and $x \in N(S) \cap \bigcup_i M_i(S)$, then there exists $i \in I$ such that $x \in N(S) \cap M_i(S)$. ☐

Proposition 12.1.12. *If the category $\mathcal{C}$ is $\mathcal{U}$-small, then A is a generator for the category $\mathbf{Mod}(A)$. Concequently, $\mathbf{Mod}(A)$ is a Grothendieck category, hence possesses enough injectives.*

Proof. Let M be an A-module. For any $S \in \mathrm{Ob}(\mathcal{C})$, let $M_0(S)$ be a system of generators of the $A(S)$-module $M(S)$. Since, by hypothesis, $\mathcal{C}$ is small, we can consider the set $I = \coprod_{S\in\mathrm{Ob}(\mathcal{C})} M_0(S)$. We then have an epimorphism $A^{\oplus I} \to M$. This proves that A is a generator for $\mathbf{Mod}(A)$ (cf. [?] 1.9.1). As $\mathbf{Mod}(A)$ satisfies (AB5), it then follows from (cf. [?] 1.10.2) that $\mathbf{Mod}(A)$ has enough injectives. ☐

Remark 12.1.13. If we consider $\mathbb{Z}$ as a constant functor on $\mathcal{C}$, then the category of $\mathbb{Z}$-modules is isomorphic to the category of abelian groups.

Definition 12.1.14. If M is an A-module, then for any $S \in \mathrm{Ob}(\mathcal{C})$, M_S is an A_S-module, so we can define an abelian group $\mathcal{H}om_A(M, N)$ by

$$\mathcal{H}om_A(M, N)(S) = \mathrm{Hom}_{A_S}(M_S, N_S).$$

We define similarly objects $\mathcal{I}so_A(M, N)$, $\mathcal{E}nd_A(M)$ and $\mathcal{A}ut_A(M)$, which are groups in $\widehat{\mathcal{C}}$ endowed with the structure of composition.

Definition 12.1.15. Let A be a ring in $\widehat{\mathcal{C}}$, M be an A-module and G be a group in $\widehat{\mathcal{C}}$. We denote by $A[G]$ the group algebra in $\widehat{\mathcal{C}}$ of G over A, so that for any $S \in \mathrm{Ob}(\mathcal{C})$, we have

$$(A[G])(S) = A(S)[G(S)].$$

An $\boldsymbol{A[G]}$**-module structure** on M is defined to be a G-object structure such that for any $S \in \mathrm{Ob}(\mathcal{C})$ and $g \in G(S)$, the automorphim of $F(S)$ defined by g is an automorphism of $A(S)$-module. Equivalently, this means the group homomorphism

$$\rho : G \to \mathcal{A}ut(M)$$

sends G to the subgroup $\mathcal{A}ut_A(M)$ of $\mathcal{A}ut(M)$. Therefore, given an $A[G]$-module structure on M, we have a group homomorphism

$$\rho : G \to \mathcal{A}ut_A(M).$$

We define similarly the abelian group $\mathrm{Hom}_{A[G]}(M, N)$ for $A[G]$-modules M, N, whence an additive category $\mathbf{Mod}(A[G])$.

The constructions above are immediately specialized in the case where G (or A, or both) are representable by objects of $\mathcal{C}$ which are thereby endowed with corresponding algebraic structures.

12.1.2 Algebraic structures on the category of schemes

We now apply the constructions of the previous paragraph to the category of schemes **Sch**, and more generally to categories $\mathbf{Sch}_{/S}$. We will simplify the notations in the following way: a group in **Sch** will also be called a **group scheme**, and a group scheme in $\mathbf{Sch}_{/S}$ will be called a **group scheme over *S***, or an ***S*-group**, or A-group when S is the spectrum of a ring A.

12.1.2.1 Constant schemes The category of schemes admits direct sums and fiber products, while direct sums commute with base changes. We can then define the constant objects: for any set E, we have a scheme $E_{\mathbb{Z}}$ and for any scheme S, an S-scheme $E_S = (E_{\mathbb{Z}})_S$. Recall that for any S-scheme T, $\mathrm{Hom}_S(T, E_S)$ is the set of locally constant maps from the space T to E.

The functor $E \mapsto E_S$ commutes with finite projective limits. In particular, if G is a group, then G_S is a group scheme over S; if A is a ring, then A_S is a ring scheme over S, etc.

12.1.2.2 Affine *S*-groups Let T be an affine S-scheme, or an S-scheme that is affine over S. Then the $\mathscr{O}_S$-algebra $f_*(\mathscr{O}_T)$ (also denoted by $\mathscr{A}(T)$), where $f : T \to S$ is the structural morphism, is then quasi-coherent. Conversely, any quasi-coherent $\mathscr{O}_S$-algebra $\mathscr{A}$ corresponds to an affine S-scheme $\mathrm{Spec}(\mathscr{A})$, and the constructions $T \mapsto \mathscr{A}(T)$, $\mathscr{A} \mapsto \mathrm{Spec}(\mathscr{A})$ are quasi-inverses of each other. It follows that giving an algebraic structure on an affine S-scheme T is equivalent to giving the corresponding structure on $\mathscr{A}(T)$ in the opposite category to that of quasi-coherent $\mathscr{O}_S$-algebras. In particular, if G is an affine S-group over S, $\mathscr{A}(G)$ is endowed with an augmented $\mathscr{O}_S$-bialgebra structure, that is, we have the following homomorphisms of $\mathscr{O}_S$-algebras

$$\Delta : \mathscr{A}(G) \to \mathscr{A}(G) \otimes_{\mathscr{O}_S} \mathscr{A}(G), \quad \varepsilon : \mathscr{A}(G) \to \mathscr{O}_S, \quad \tau : \mathscr{A}(G) \to \mathscr{A}(G)$$

corresponding to the morphisms of S-schemes

$$\pi : G \times G \to G, \quad e_G : S \to G, \quad i : G \to G.$$

The maps Δ, ε and τ satisfy the following conditions (which express that G is an S-monoid):

(HA1) Δ is coassociative: the following diagram is commutative

$$\begin{tikzcd}
\mathscr{A}(G) \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & \mathscr{A}(G) \otimes_{\mathscr{O}_S} \mathscr{A}(G) \arrow[d, "\mathrm{id} \otimes \Delta"] \\
\mathscr{A}(G) \otimes_{\mathscr{O}_S} \mathscr{A}(G) \arrow[r, "\Delta\otimes\mathrm{id}"] & \mathscr{A}(G) \otimes_{\mathscr{O}_S} \mathscr{A}(G) \otimes_{\mathscr{O}_S} \mathscr{A}(G)
\end{tikzcd}$$

(HA2) Δ is compatible with ε: the following compositions are identities:

$$\mathscr{A}(G) \xrightarrow{\Delta} \mathscr{A}(G) \otimes_{\mathscr{O}_S} \mathscr{A}(G) \xrightarrow{\mathrm{id}\otimes\varepsilon} \mathscr{A}(G) \otimes_{\mathscr{O}_S} \mathscr{O}_S \xrightarrow{\sim} \mathscr{A}(G)$$
$$\mathscr{A}(G) \xrightarrow{\Delta} \mathscr{A}(G) \otimes_{\mathscr{O}_S} \mathscr{A}(G) \xrightarrow{\varepsilon\otimes\mathrm{id}} \mathscr{O}_S \otimes_{\mathscr{O}_S} \mathscr{A}(G) \xrightarrow{\sim} \mathscr{A}(G)$$

Also, in this case $(\mathscr{A}(G), \Delta, \varepsilon, \tau)$ is a Hopf algebra. Let us take advantage of the circumstance to notice once again that it follows from the definition of an S-group structure that in order to give such a structure on a S-scheme G affine over S, it is not necessary to verify anything on $\mathscr{A}(G)$, but simply endow each $G(S')$ for S' above S with a group structure functorial in S'. This remark applies mutatis mutandis to morphisms.

12.1.2.3 The groups $\mathbb{G}_a$ and $\mathbb{G}_m$ We consider the **additive group functor** $\mathbb{G}_a : \mathbf{Sch}^{\mathrm{op}} \to \mathbf{Set}$ defined by the formula

$$\mathbb{G}_a(S) = \Gamma(S, \mathscr{O}_S),$$

endowed with the group structure defined by the additive group structure of the ring $\Gamma(S, \mathscr{O}_S)$. This is represented by the affine scheme, which we denote also by $\mathbb{G}_a$, and which is then a group scheme

$$\mathbb{G}_a = \mathrm{Spec}(\mathbb{Z}[T]).$$

In fact, we have bijections

$$\mathrm{Hom}(S, \mathbb{G}_a) = \mathrm{Hom}_{\mathbf{Alg}}(\mathbb{Z}[T], \Gamma(S, \mathscr{O}_S)) \cong \Gamma(S, \mathscr{O}_S).$$

For any scheme S, we then have an affine S-group over S, which we denote by $\mathbb{G}_{a,S}$, and it corresponds to the $\mathscr{O}_S$-bigebra $\mathscr{O}_S[T]$ with the comultiplication and counit given by

$$\Delta(T) = T \otimes 1 + 1 \otimes T, \quad \varepsilon(T) = 0.$$

Let $\mathbb{G}_m : \mathbf{Sch}^{\mathrm{op}} \to \mathbf{Set}$ be the **multiplication group functor** defined by

$$\mathbb{G}_m(S) = \Gamma(S, \mathscr{O}_S)^{\times},$$

where $\Gamma(S, \mathscr{O}_S)^{\times}$ denotes the multiplication group of invertible elements in the ring $\Gamma(S, \mathscr{O}_S)$, endowed with the canonical group structure. This is represented by an affine group, which is still denoted by $\mathbb{G}_m$:

$$\mathbb{G}_m = \mathrm{Spec}(\mathbb{Z}[T, T^{-1}]) = \mathrm{Spec}(\mathbb{Z}[\mathbb{Z}])$$

where $\mathbb{Z}[\mathbb{Z}]$ is the group algebra of the additive group $\mathbb{Z}$ over the ring $\mathbb{Z}$. In fact,

$$\mathrm{Hom}(S, \mathrm{Spec}(\mathbb{Z}[T, T^{-1}])) = \mathrm{Hom}_{\mathbf{Alg}}(\mathbb{Z}[T, T^{-1}], \Gamma(S, \mathscr{O}_S)) \cong \Gamma(S, \mathscr{O}_S)^{\times}.$$

For any scheme S, we then have an affine S-group $\mathbb{G}_{m,S}$ over S, which corresponds to the $\mathscr{O}_S$-bigebra $\mathscr{O}_S[\mathbb{Z}]$, with the comultiplication and counit given by

$$\Delta(x) = x \otimes x, \quad \varepsilon(x) = 1 \quad \text{for } x \in \mathbb{Z}.$$

We also note that the set $\Gamma(S, \mathscr{O}_S)$ is a ring for each scheme S, so we can endow the functor $\mathbb{G}_a$ with a natural ring structure, which we denote by $\mathbb{O}$. The ring $\mathbb{O}$ is represented by the scheme $\mathrm{Spec}(\mathbb{Z}[T])$, which is also denoted by $\mathbb{O}$, which is then a ring scheme in $\widehat{\mathbf{Sch}}$. For any scheme S, $\mathbb{O}_S = S \times_{\mathrm{Spec}(\mathbb{Z})} \mathrm{Spec}(\mathbb{Z}[T]) = \mathrm{Spec}(\mathscr{O}_S[T])$ is then an affine ring scheme over S. Note that this ring is also denoted by $S[T]$.

For any object F in $\widehat{\mathbf{Sch}}$, the set $\mathbb{O}(F) := \mathrm{Hom}(F, \mathbb{O})$ is then endowed with a ring structure and is functorial on F. In particular, if X is a scheme and we are given morphisms $x : X \to F$ and $f : F \to \mathbb{O}$ (that is, $x \in F(X)$ and $f \in \mathbb{O}(F)$), then $f(x) := f \circ x$ is an element in $\mathbb{O}(X) = \Gamma(X, \mathscr{O}_X)$.

Definition 12.1.16. Let $\pi : M \to X$ be a morphism in $\widehat{\mathbf{Sch}}$, and $\mathbb{O}_X = \mathbb{O} \times X$. We say that M is an $\mathbb{O}_X$**-module** if for each X-scheme X', we are given an $\mathbb{O}(X')$-module structure on $\mathrm{Hom}_X(X', M)$, which is functorial on X'. Equivalently, this amounts to giving oneself an X-abelian group structure $\mu : M \times_X M \to M$ on M and an "external law"

$$\mathbb{O} \times M = \mathbb{O}_X \times_X M \to M, \quad (f, m) \mapsto f \cdot m$$

which is an X-morphism and for any $x \in X(S)$, endows $M(x) = \{m \in M(S) : \pi(m) = x\}$ an $\mathbb{O}(S)$-module structure. In this case, for any $Y \in \widehat{\mathbf{Sch}}_{/X}$ (not necessarily representable), the set $\mathrm{Hom}_X(Y, M) = \Gamma(M_Y/Y)$ is an $\mathbb{O}(Y)$-module, which is functorial on Y.

Example 12.1.17. Let k be a field of characteristic zero and A be a k-algebra. Then the set

$$\mathrm{Hom}_{\mathbf{Grp}}(\mathbb{G}_a, \mathbb{G}_m)(\mathrm{Spec}(A))$$

consists of nilpotent elements of A; more precisely, all group homomorphisms from $\mathbb{G}_{a,\mathrm{Spec}(A)}$ to $\mathbb{G}_{m,\mathrm{Spec}(A)}$ are of the form $x \mapsto e^{ax}$ with $a \in A$ nilpotent. To see that, note that the underlying schemes of $\mathbb{G}_{a,\mathrm{Spec}(A)}$ and $\mathbb{G}_{m,\mathrm{Spec}(A)}$ are $\mathrm{Spec}(A[X])$ and $\mathrm{Spec}(A[Y, Y^{-1}])$, so any group homomorphism is of the form $Y \mapsto \sum_i f_i X^i$ for some $f_i \in A$. The condition that this is a group homomorphism is that

$$\sum_i f_i(X_1 + X_2)^i = \Big(\sum_i f_i X_1^i\Big)\Big(\sum_j f_j X_2^j\Big).$$

Expanding this, we conclude that $f_{i+j}/(i+j)! = f_i/i! f_j/j!$, so every such homomorphism is of the form $f_i = a^i/i!$, and a must be nilpotent since the sum is finite.

Now, we conclude that the functor $\mathcal{H}om_{\mathbf{Grp}}(\mathbb{G}_a, \mathbb{G}_m)$ is not representable. For any positive integer n, let $A_n = k[t]/(t^n)$. Then the morphism $x \mapsto e^{tx}$ is in $\mathrm{Hom}_{\mathbf{Grp}}(\mathbb{G}_a, \mathbb{G}_m)(\mathrm{Spec}(A_n))$ for each n. However, if A is the inverse limit $k[\![t]\!]$, then there is no corresponding morphism in $\mathrm{Hom}_{\mathbf{Grp}}(\mathbb{G}_a, \mathbb{G}_m)(\mathrm{Spec}(A))$, so $\mathcal{H}om_{\mathbf{Grp}}(\mathbb{G}_a, \mathbb{G}_m)$ is not representable.

12.1.2.4 Diagonalizable groups The construction of $\mathbb{G}_m$ can be generalized in the following manner. Let M be an abelian group and $M_{\mathbb{Z}}$ be the constant group scheme associated with M. We then consider the functor $D(M) : \mathbf{Sch}^{\mathrm{op}} \to \mathbf{Set}$ defined by

$$D(M)(S) = \mathrm{Hom}_{\mathbf{Grp}}(M_{\mathbb{Z}}(S), \mathbb{G}_m(S)) \cong \mathrm{Hom}_{S\text{-}\mathbf{Grp}}(M_S, \mathbb{G}_{m,S}) \cong \mathcal{H}om_{\mathbf{Grp}}(M_{\mathbb{Z}}, \mathbb{G}_m)(S).$$

This is an abelian group in $\widehat{\mathbf{Sch}}$ and is represented by the group scheme $\mathrm{Spec}(\mathbb{Z}[M])$, which is still denoted by $D(M)$. In fact, for any scheme S, we have

$$\mathrm{Hom}(S, \mathrm{Spec}(\mathbb{Z}[M])) = \mathrm{Hom}_{\mathbf{Alg}}(\mathbb{Z}[M], \Gamma(S, \mathscr{O}_S)) \cong \mathrm{Hom}_{\mathbf{Grp}}(M, \Gamma(S, \mathscr{O}_S)^\times).$$

For any scheme S, we then obtain an affine group scheme over S:

$$D_S(M) = D(M)_S = \mathcal{H}om_{\mathbf{Grp}}(M_{\mathbb{Z}}, \mathbb{G}_m)_S = \mathcal{H}om_{\mathbf{Grp}}(M_S, \mathbb{G}_{m,S}).$$

This is associated with the $\mathscr{O}_S$-bigebra $\mathscr{O}_S[M]$, whose comultiplication and counit are defined by

$$\Delta(x) = x \otimes x, \quad \varepsilon(x) = 1 \quad \text{for} \quad x \in M.$$

If $f : M \to N$ is a homomorphism of abelian groups, we then have obtain a morphism of S-groups

$$D_S(f) : D_S(N) \to D_S(M),$$

whence a functor $D_S : M \mapsto D_S(M)$ from the category of abelian groups to the category of affine groups over S, which can also be described as the composition of the functor $M \mapsto M_S$ with the functor $M_S \mapsto \mathcal{H}om_{\mathbf{Grp}}(M_S, \mathbb{G}_{m,S})$. This functor clearly commutes with base changes. An S-group isomorphic to a group of the form $D_S(M)$ is called **diagonalizable**. We note that the elements of M can be interpreted as some characters of $D_S(M)$, that is, certain elements of $\mathrm{Hom}_{\mathbf{Grp}}(D_S(M), \mathbb{G}_{m,S})$ (in fact, this latter group is isomorphic to M_S, as we shall see).

Example 12.1.18. It is clear that we have $D(\mathbb{Z}) = \mathbb{G}_m$ and $D(\mathbb{Z}^n) = (\mathbb{G}_m)^n$. We now consider the group scheme

$$\boldsymbol{\mu}_n = D(\mathbb{Z}/n\mathbb{Z})$$

which is called the **group of n-th roots of unity**. In fact, we have

$$\boldsymbol{\mu}_n(S) = \mathrm{Hom}_{\mathbf{Grp}}(\mathbb{Z}/n\mathbb{Z}, \Gamma(S, \mathscr{O}_S)^\times) = \{f \in \Gamma(S, \mathscr{O}_S) : f^n = 1\}.$$

The S-group $\boldsymbol{\mu}_{n,S}$ corresponds to the $\mathscr{O}_S$-algebra $\mathscr{O}_S[T]/(T^n - 1)$. Suppose in particular that S is the spectrum of a field k of characteristic p. Then by putting $T - 1 = s$, we have

$$k[T]/(T^p - 1) = k[s]/(s^p),$$

which shows that the underlying space of $\boldsymbol{\mu}_{p,S}$ is reduced to a single point, and the local ring of this point is the Artinian k-algebra $k[s]/(s^p)$. By the same ideas, we see that the S-schemes $\mathbb{G}_{a,S}$, $\mathbb{G}_{m,S}$, $\mathbb{O}_S$ are smooth on S, that $D_S(M)$ is flat on S and that it is formally smooth (resp. smooth) on S if and only if the residual characteristic of S does not divide the torsion of M (resp. and if moreover M is finite type).

Example 12.1.19. The above procedure applies to "classical groups" (linear groups GL_n, symplectic groups Sp_n, orthogonal groups O_n, etc.). We define for example GL_n as representing the functor such that

$$\mathrm{GL}_n(S) = \mathrm{GL}(n, \Gamma(S, \mathscr{O}_S)) = \mathrm{Aut}_{\mathscr{O}_S}(\mathscr{O}_S^n).$$

We can construct it for example as the open set of $\mathrm{Spec}(\mathbb{Z}[T_{ij}])$ ($1 \leq i, j \leq n$) defined by the function $\det(T_{ij})$, which is $\mathrm{Spec}(\mathbb{Z}[T_{ij}, \det(T_{ij})^{-1}])$. Similarly, the special linear group SL_n is defined to be the kernel of the function $\det : \mathrm{GL}_n \to \mathbb{G}_m$, and it is the closed subscheme $\mathrm{Spec}(\mathbb{Z}[T_{ij}]/(\det(T_{ij} - 1)))$ of GL_n. The group structure of SL_n is thus the induced one by GL_n.

12.1.2.5 Module functors in the category of schemes We now associate with any $\mathscr{O}_S$-module over the schema S, an $\mathbb{O}_S$-module (where $\mathbb{O}_S$ denotes the ring functor introduced in 12.1.2.3). This can be done in two different ways, as we shall now define.

Definition 12.1.20. Let S be a scheme. For any $\mathscr{O}_S$-module $\mathscr{F}$, we denote by $\Gamma_{\mathscr{F}}$ and $\check{\Gamma}_{\mathscr{F}}$ the contravariant functors over $\mathbf{Sch}_{/S}$ defined by

$$\Gamma_{\mathscr{F}}(S') = \Gamma(S', \mathscr{F} \otimes_{\mathscr{O}_S} \mathscr{O}_{S'}), \quad \check{\Gamma}_{\mathscr{F}}(S') = \mathrm{Hom}_{\mathscr{O}_{S'}}(\mathscr{F} \otimes_{\mathscr{O}_S} \mathscr{O}_{S'}, \mathscr{O}_{S'}).$$

Then $\Gamma_{\mathscr{F}}$ and $\check{\Gamma}_{\mathscr{F}}$ are endowed with natural structures of $\mathbb{O}_S$-modules (note that $\mathbb{O}_S(S') = \Gamma(S', \mathscr{O}_{S'}) = \Gamma_{\mathscr{O}_S}(S')$), so that we obtain functors Γ and $\check{\Gamma}$ from the category of $\mathscr{O}_S$-modules to that of $\mathbb{O}_S$-modules, Γ being convariant and $\check{\Gamma}$ being contracovariant.

We often restrict ourselves to the category of quasi-coherent $\mathscr{O}_S$-modules, so that Γ and $\check{\Gamma}$ are considered as functors from $\mathbf{Qcoh}(\mathscr{O}_S)$ to the category of $\mathbb{O}_S$-modules:

$$\Gamma : \mathbf{Qcoh}(\mathscr{O}_S) \to \mathbf{Mod}(\mathbb{O}_S), \quad \check{\Gamma} : \mathbf{Qcoh}(\mathscr{O}_S)^{\mathrm{op}} \to \mathbf{Mod}(\mathbb{O}_S).$$

The reader should however note that most of the propositions in this paragraph do not rely on the quasi-coherence hypothesis.

Proposition 12.1.21. *Let S be a scheme.*

(a) *The functors Γ and $\check{\Gamma}$ commute with base changes: if $S' \to S$ is a morphism and $\mathscr{F}$ is a quasi-coherent $\mathscr{O}_S$-module, then $\Gamma_{\mathscr{F} \otimes \mathscr{O}_{S'}} \cong (\Gamma_{\mathscr{F}})_{S'}$ and $\check{\Gamma}_{\mathscr{F} \otimes \mathscr{O}_{S'}} \cong (\check{\Gamma}_{\mathscr{F}})_{S'}$.*

(b) *The functors Γ and $\check{\Gamma}$ are fully faithful: the canonical maps*

$$\mathrm{Hom}_{\mathscr{O}_S}(\mathscr{F}, \mathscr{F}') \to \mathrm{Hom}_{\mathbb{O}_S}(\Gamma_{\mathscr{F}}, \Gamma_{\mathscr{F}'}),$$
$$\mathrm{Hom}_{\mathscr{O}_S}(\mathscr{F}, \mathscr{F}') \to \mathrm{Hom}_{\mathbb{O}_S}(\check{\Gamma}_{\mathscr{F}'}, \check{\Gamma}_{\mathscr{F}})$$

are bijective.

(c) *The functors Γ and $\check{\Gamma}$ are additive: we have $\Gamma_{\mathscr{F} \oplus \mathscr{F}'} \cong \Gamma_{\mathscr{F}} \times_S \Gamma_{\mathscr{F}'}$ and $\check{\Gamma}_{\mathscr{F} \oplus \mathscr{F}'} \cong \check{\Gamma}_{\mathscr{F}} \times_S \check{\Gamma}_{\mathscr{F}'}$.*

Proof. Assertions (a) and (c) are clear from the definitions. As for (b), we note that by taking S' to be the open subsets of S, we can construct a homomorphism $u : \mathscr{F} \to \mathscr{F}'$ from an $\mathbb{O}_S$-homomorphism $f : \Gamma_{\mathscr{F}} \to \Gamma_{\mathscr{F}'}$, and it is immediate to verify that this gives an inverse of the canonical map $\mathrm{Hom}_{\mathscr{O}_S}(\mathscr{F}, \mathscr{F}') \to \mathrm{Hom}_{\mathbb{O}_S}(\Gamma_{\mathscr{F}}, \Gamma_{\mathscr{F}'})$. A similar argument shows that the canonical map $\mathrm{Hom}_{\mathscr{O}_S}(\mathscr{F}, \mathscr{F}') \to \mathrm{Hom}_{\mathbb{O}_S}(\check{\Gamma}_{\mathscr{F}'}, \check{\Gamma}_{\mathscr{F}})$ is also bijective. ☐

We recall that if F, F' are $\mathbb{O}_S$-modules, then $\mathcal{H}om_{\mathbb{O}_S}(F, F')$ denote that S-functor which associates any morphism $S' \to S$ with $\mathrm{Hom}_{\mathbb{O}_{S'}}(F_{S'}, F'_{S'})$.

Proposition 12.1.22. *We have the following canonical morphisms in $\mathbf{Mod}(\mathbb{O}_S)$:*

$$\begin{tikzcd}
\mathcal{H}om_{\mathbb{O}_S}(\Gamma_{\mathscr{F}}, \Gamma_{\mathscr{F}'}) \arrow[rr, "\sim"] & & \mathcal{H}om_{\mathbb{O}_S}(\check{\Gamma}_{\mathscr{F}'}, \check{\Gamma}_{\mathscr{F}}) \\
& \Gamma_{\mathcal{H}om_{\mathscr{O}_S}(\mathscr{F}, \mathscr{F}')} \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Proof. For each S-scheme S', we have a canonical homomorphism

$$\Gamma_{\mathcal{H}om_{\mathscr{O}_S}(\mathscr{F},\mathscr{F}')}(S') = \Gamma(\mathcal{H}om_{\mathscr{O}_S}(\mathscr{F},\mathscr{F}') \otimes \mathscr{O}_{S'}) \to \mathrm{Hom}_{\mathscr{O}_{S'}}(\mathscr{F} \otimes \mathscr{O}_{S'}, \mathscr{F}' \otimes \mathscr{O}_{S'}).$$

The proposition then follows from Proposition 12.1.21 (a) and (b). □

Remark 12.1.23. Let $\mathscr{F}$ be a quasi-coherent $\mathscr{O}_S$-module. Recall that the S-functor $\check{\Gamma}_{\mathscr{F}}$ is represented by an affine S-scheme which is denoted by $\mathbb{V}(\mathscr{F})$ and called the vector bundle defined by $\mathscr{F}$:

$$\mathbb{V}(\mathscr{F}) = \mathrm{Spec}(\boldsymbol{S}(\mathscr{F})),$$

where $\boldsymbol{S}(\mathscr{F})$ denotes the symmetric algebra over $\mathscr{F}$. On the other hand, the article ([**?**]) shows that if S is Noetherian and $\mathscr{F}$ is a coherent $\mathscr{O}_S$-module, then $\Gamma_{\mathscr{F}}$ is representable if and only if $\mathscr{F}$ is locally free, and in this case we have an isomorphism $\Gamma_{\mathscr{F}} \cong \check{\Gamma}_{\mathscr{F}}$.

Proposition 12.1.24. *Let $\mathscr{F}$ and $\mathscr{F}'$ be quasi-coherent $\mathscr{O}_S$-modules and $\mathscr{A}$ be a quasi-coherent $\mathscr{O}_S$-algebra. Then we have a functorial isomorphism*

$$\mathrm{Hom}_S(\mathrm{Spec}(\mathscr{A}), \mathcal{H}om_{\mathbb{O}_S}(\Gamma_{\mathscr{F}'}, \Gamma_{\mathscr{F}})) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{O}_S}(\mathscr{F}', \mathscr{F} \otimes_{\mathscr{O}_S} \mathscr{A}).$$

Proof. If we put $X = \mathrm{Spec}(\mathscr{A})$, then the LHS is canonically isomorphic to $\mathcal{H}om_{\mathbb{O}_S}(\Gamma_{\mathscr{F}'}, \Gamma_{\mathscr{F}})(X)$, which by Proposition 12.1.21 is given by

$$\begin{aligned}\mathcal{H}om_{\mathbb{O}_S}(\Gamma_{\mathscr{F}'}, \Gamma_{\mathscr{F}})(X) &\cong \mathrm{Hom}_{\mathbb{O}_X}(\Gamma_{\mathscr{F}' \otimes \mathscr{O}_X}, \Gamma_{\mathscr{F}\mathscr{O}_X}) \cong \mathrm{Hom}_{\mathscr{O}_X}(\mathscr{F}' \otimes \mathscr{O}_X, \mathscr{F} \otimes \mathscr{O}_X)\\ &\cong \mathrm{Hom}_{\mathscr{O}_S}(\mathscr{F}', \varphi_*(\varphi^*(\mathscr{F})))\end{aligned}$$

where $\varphi : X \to S$ is the structural morphism. On the other hand, by Corollary 9.1.24 we have $\varphi_*(\varphi^*(\mathscr{F})) \cong \mathscr{F} \otimes \mathscr{A}$, so the assertion follows. □

Corollary 12.1.25. *We have a canonical isomorphism* $\Gamma_{\mathscr{F} \otimes \mathscr{A}} \cong \mathcal{H}om_S(\mathrm{Spec}(\mathscr{A}), \Gamma_{\mathscr{F}})$.

Proof. Let $f : S' \to S$ be an S-scheme and $X' = X \times_S S'$, we then have a Cartesian diagram

$$\begin{tikzcd} X' \arrow[r, "\varphi'"] \arrow[d, "f'"'] & S' \arrow[d, "f"] \\ X \arrow[r, "\varphi"] & S \end{tikzcd}$$

By Proposition 9.1.29 and Corollary 9.1.30, X' is affine over S' and $\varphi'_*(\mathscr{O}_{X'}) = f^*(\mathscr{A})$, so

$$\mathcal{H}om_S(\mathrm{Spec}(\mathscr{A}), \Gamma_{\mathscr{F}})(S') = \mathrm{Hom}_{S'}(\mathrm{Spec}(f^*(\mathscr{A})), \Gamma_{f^*(\mathscr{F})})$$

and by Proposition 12.1.24 applied to $f^*(\mathscr{F})$, $\mathscr{F}' = \mathscr{O}_{S'}$ and $f^*(\mathscr{A})$, this is equal to

$$\Gamma(S', f^*(\mathscr{F}) \otimes f^*(\mathscr{A})) = \Gamma(S', f^*(\mathscr{F} \otimes \mathscr{A})) = \Gamma_{\mathscr{F} \otimes \mathscr{A}}(S').$$ □

Proposition 12.1.26. *If $\mathscr{F}$ and $\mathscr{F}'$ are locally free of finite type, then the morphisms in Proposition 12.1.22 are isomorphisms.*

Proof. In fact, for any morphism $S' \to S$, we then have

$$\Gamma_{\mathcal{H}om_{\mathscr{O}_S}(\mathscr{F},\mathscr{F}')}(S') = \Gamma(S', \mathcal{H}om_{\mathscr{O}_S}(\mathscr{F},\mathscr{F}') \otimes \mathscr{O}_{S'}) = \mathrm{Hom}_{\mathscr{O}_S}(\mathscr{F},\mathscr{F}').$$

But this is also isomorphic to $\mathcal{H}om_{\mathbb{O}_S}(\Gamma_{\mathscr{F}}, \Gamma_{\mathscr{F}'})(S')$ and to $\mathcal{H}om_{\mathbb{O}_S}(\Gamma_{\mathscr{F}}, \Gamma_{\mathscr{F}'})(S')$, in view of Proposition 12.1.21 (b). □

Corollary 12.1.27. *Let $\mathscr{F}$ be a locally free $\mathscr{O}_S$-module of finite type and put $\check{\mathscr{F}} = \mathcal{H}om_{\mathscr{O}_S}(\mathscr{F}, \mathscr{O}_S)$. Then we have canonical isomorphisms*

$$\Gamma_{\check{\mathscr{F}}} \cong \mathcal{H}om_{\mathbb{O}_S}(\Gamma_{\mathscr{F}}, \mathbb{O}_S) \cong \check{\Gamma}_{\mathscr{F}}, \quad \check{\Gamma}_{\check{\mathscr{F}}} \cong \mathcal{H}om_{\mathbb{O}_S}(\check{\Gamma}_{\mathscr{F}}, \mathbb{O}_S) \cong \Gamma_{\mathscr{F}},$$

Proof. This follows from Proposition 12.1.26 by taking $\mathscr{F}' = \mathscr{O}_S$ and note that $\Gamma_{\mathscr{O}_S} = \mathbb{O}_S$. □

Proposition 12.1.28. *If $u : \mathscr{F} \to \mathscr{F}'$ is a morphism of locally free $\mathscr{O}_S$-modules of finite rank, then for $\Gamma_u : \Gamma_{\mathscr{F}} \to \Gamma_{\mathscr{F}'}$ to be a monomorphism, it is necessary and sufficient that f identifies $\mathscr{F}$ locally as a direct factor of $\mathscr{F}'$.*

Proof. One direction follows essentially from **??**. Conversely, if $\mathscr{F}$ is a direct factor of $\mathscr{F}'$, then for any $f : S' \to S$, $f^*(\mathscr{F})$ is a submodule of $f^*(\mathscr{F}')$, so $\Gamma_{\mathscr{F}}(S') = \Gamma(S', f^*(\mathscr{F}))$ is a submodule of $\Gamma_{\mathscr{F}'}(S') = \Gamma(S', f^*(\mathscr{F}'))$. □

12.1.2.6 The category of $\mathscr{O}_S[G]$-modules Let G be an S-group and $\mathscr{F}$ be an $\mathscr{O}_S$-module. Then an $\mathscr{O}_S[G]$**-module structure** on $\mathscr{F}$ is defined to be an $\mathbb{O}_S[h_G]$-module structure on $\Gamma_{\mathscr{F}}$. A morphism of $\mathscr{O}_S[G]$-modules is by definition a morphism of the associated $\mathbb{O}_S[h_G]$-modules. We thus obtain a category $\mathbf{Mod}(\mathscr{O}_S[G])$ of $\mathscr{O}_S[G]$-modules and the full subcategory $\mathbf{Qcoh}(\mathscr{O}_S[G])$ formed by quasi-coherent $\mathscr{O}_S$-modules. By definition, giving an $\mathscr{O}_S[G]$-module structure on $\mathscr{F}$ is equivalent to giving a morphism of groups

$$\rho : h_G \to \mathcal{A}ut_{\mathbb{O}_S}(\Gamma_{\mathscr{F}}).$$

Remark 12.1.29. Since by Proposition 12.1.21 we have an anti-isomorphism

$$\mathcal{A}ut_{\mathbb{O}_S}(\Gamma_{\mathscr{F}}) \cong \mathcal{A}ut_{\mathbb{O}_S}(\check{\Gamma}_{\mathscr{F}}),$$

we see that an $\mathbb{O}_S[h_G]$-module structure on $\Gamma_{\mathscr{F}}$ is equivalent to an $\mathbb{O}_S[h_G]$-module structure on $\check{\Gamma}_{\mathscr{F}}$, and these two structures are connected by the operation $\rho(g) \mapsto \rho^*(g^{-1})$, where ρ^* denotes the image of $\rho : h_G \to \mathcal{A}ut_{\mathbb{O}_S}(\Gamma_{\mathscr{F}})$ under the above isomorphism.

Remark 12.1.30. The categories we have just constructed can also be defined by the following Cartesian squares:

$$\begin{tikzcd}
\mathbf{Qcoh}(\mathscr{O}_S[G]) \arrow[r, hook] \arrow[d] & \mathbf{Mod}(\mathscr{O}_S[G]) \arrow[r] \arrow[d] & \mathbf{Mod}(\mathbb{O}_S[h_G]) \arrow[d, "\text{forget}"] \\
\mathbf{Qcoh}(\mathscr{O}_S) \arrow[r, hook] & \mathbf{Mod}(\mathscr{O}_S) \arrow[r, "\Gamma"] & \mathbf{Mod}(\mathbb{O}_S)
\end{tikzcd}$$

The categories $\mathbf{Mod}(\mathscr{O}_S)$ and $\mathbf{Mod}(\mathbb{O}_S)$ are abelian, but one should be careful that in general the functor Γ is not exact, neither left nor right.

Remark 12.1.31. Let $\mathscr{F}$ be an $\mathscr{O}_S[G]$-module. The **subsheaf of invariants** $\mathscr{F}^G$ is defined as follows: for any open subset U of S,

$$\mathscr{F}^G(U) = \Gamma^G_{\mathscr{F}}(U) = \{x \in \mathscr{F}(U) : g \cdot x_{S'} = x_{S'} \text{ for any morphism } f : S' \to U \text{ and } g \in G(S')\}$$

where $x_{S'}$ denotes the image of x in $\Gamma(S', f^*(\mathscr{F})) = \Gamma(U, f_*(f^*(\mathscr{F})))$.

Be careful that the natural morphism $\Gamma_{\mathscr{F}^G} \to \Gamma^G_{\mathscr{F}}$ is not an isomorphism in general. For example, if $S = \mathrm{Spec}(\mathbb{Z})$ and G is the constant group $\mathbb{Z}/2\mathbb{Z} = \{1, \tau\}$ acting on $\mathscr{F} = \mathscr{O}_S$ via $\tau \cdot 1 = -1$, then we have $\mathscr{F}^G = 0$ since the ring $\Gamma(U, \mathscr{F})$ has characteristic zero for any standard open U of S. However, it is clear that $\Gamma^G_{\mathscr{F}}(\mathrm{Spec}(R)) = R$ for any $\mathbb{F}_2$-algebra R.

From now on, we restrict ourselves to the case where the group scheme G is affine over S. Then, in view of Proposition 12.1.24, giving a morphism of S-functors

$$\rho : h_G \to \mathcal{A}ut_{\mathbb{O}_S}(\Gamma_{\mathscr{F}})$$

is equivalent to giving a morphism of $\mathscr{O}_S$-modules

$$\mu : \mathscr{F} \to \mathscr{F} \otimes_{\mathscr{O}_S} \mathscr{A}(G).$$

The condition that ρ is a group homomorphism is then translated into the folllowing conditions on μ:

(CM1) the following diagram is commutative:

$$\begin{tikzcd}
\mathscr{F} \arrow[r, "\mu"] \arrow[d, "\mu"'] & \mathscr{F} \otimes_{\mathscr{O}_S} \mathscr{A}(G) \arrow[d, "\mathrm{id} \otimes \Delta"] \\
\mathscr{F} \otimes_{\mathscr{O}_S} \mathscr{A}(G) \arrow[r, "\mu \otimes \mathrm{id}"] & \mathscr{F} \otimes_{\mathscr{O}_S} \mathscr{A}(G) \otimes_{\mathscr{O}_S} \mathscr{A}(G)
\end{tikzcd}$$

(CM2) the following composition is the identity:

$$\mathscr{F} \xrightarrow{\mu} \mathscr{F} \otimes_{\mathscr{O}_S} \mathscr{A}(G) \xrightarrow{\mathrm{id} \otimes \varepsilon} \mathscr{F} \otimes \mathscr{O}_S \xrightarrow{\sim} \mathscr{F}$$

These two axioms then endow a *comodule structure* on $\mathscr{F}$ over the bigebra $\mathscr{A}(G)$.

Put $\mathscr{A} = \mathscr{A}(G)$. If $\mathscr{F}$ and $\mathscr{F}'$ are $\mathscr{A}$-comodules, a morphism $f : \mathscr{F} \to \mathscr{F}'$ of comodules is then defined to be a morphism of $\mathscr{O}_S$-modules such that the following diagram is commutative:

$$\begin{tikzcd} \mathscr{F} \arrow[r, "f"] \arrow[d, "\mu_{\mathscr{F}}"'] & \mathscr{F}' \arrow[d, "\mu_{\mathscr{F}'}"] \\ \mathscr{F} \otimes \mathscr{A} \arrow[r, "f \otimes \mathrm{id}"] & \mathscr{F}' \otimes \mathscr{A} \end{tikzcd}$$

We thus obtain a category $\mathbf{CoMod}(\mathscr{A})$ of comodules over $\mathscr{A}$, and we denote by $\mathbf{CoQcoh}(\mathscr{A})$ the full subcategory formed by quasi-coherent $\mathscr{O}_S$-modules. From the above remarks, it is also clear that we have the following:

Proposition 12.1.32. *Let G be an affine S-group. Then we have equivalences of categories:*

$$\mathbf{Mod}(\mathscr{O}_S[G]) \cong \mathbf{CoMod}(\mathscr{A}(G)), \quad \mathbf{Qcoh}(\mathscr{O}_S[G]) \cong \mathbf{CoQcoh}(\mathscr{A}(G)).$$

If moreover $S = \mathrm{Spec}(A)$ is affine and we put $A[G] = \Gamma(S, \mathscr{A}(G))$, then we have an equivalence of categories

$$\mathbf{CoQcoh}(\mathscr{A}(G)) \cong \mathbf{CoMod}(A[G]).$$

Proposition 12.1.33. *Suppose that G is affine and flat over S. Then the category $\mathbf{Mod}(\mathscr{O}_S[G])$ (resp. $\mathbf{Qcoh}(\mathscr{O}_S[G])$), being equivalent to the category of $\mathscr{A}(G)$-comodules (resp. quasi-coherent over $\mathscr{O}_S$), is abelian.*

Proof. Suppose that $\mathscr{A} = \mathscr{A}(G)$ is a flat $\mathscr{O}_S$-module. Let $\mathscr{E}$ be an $\mathscr{A}$-comodule and $\mathscr{F}$ be a sub-$\mathscr{O}_S$-module of $\mathscr{E}$. As $\mathscr{A}$ is flat over $\mathscr{O}_S$, we can identify $\mathscr{F} \otimes \mathscr{A}$ (resp. $\mathscr{F} \otimes \mathscr{A} \otimes \mathscr{A}$) as a sub-$\mathscr{O}_S$-module of $\mathscr{E}$ (resp. $\mathscr{E} \otimes \mathscr{A} \otimes \mathscr{A}$). Assume that $\mu_{\mathscr{E}}$ sends $\mathscr{F}$ into $\mathscr{F} \otimes \mathscr{A}$, then the restriction $\mu_{\mathscr{F}} : \mathscr{F} \to \mathscr{F} \otimes \mathscr{A}$ induces a comodule structure on $\mathscr{F}$, and we say that $\mathscr{F}$ is a sub-comodule of $\mathscr{E}$. By passing to quotient, $\mu_{\mathscr{E}}$ then defies a morphism of $\mathscr{O}_S$-modules $\mathscr{E}/\mathscr{F} \to \mathscr{E}/\mathscr{F} \otimes \mathscr{A}$, which endows $\mathscr{E}/\mathscr{F}$ with an $\mathscr{A}$-comodule structure.

Now if $f : \mathscr{E} \to \mathscr{E}'$ is a morphism of $\mathscr{A}$-comodules, then $\ker f$ (resp. $\mathrm{im}\, f$) is a sub-$\mathscr{A}$-comodule of $\mathscr{E}$ (resp. $\mathscr{E}'$), and f induces an isomorphism $\mathscr{E}/\ker f \xrightarrow{\sim} \mathrm{im}\, f$ of $\mathscr{A}$-comodules. Moreover, if $\mathscr{E}$ and $\mathscr{E}'$ are quasi-coherent $\mathscr{O}_S$-modules, then so are $\ker f$ and $\mathrm{im}\, f$. Therefore, we conclude that $\mathbf{CoMod}(\mathscr{A})$ and $\mathbf{CoQcoh}(\mathscr{A})$ are abelian categories. ☐

We now suppose further that G is a diagonalizable group, which means $\mathscr{A}(G)$ is the algebra of an abelian group M over the ring $\mathscr{O}_S$. If $\mathscr{F}$ is an $\mathscr{O}_S$-module, we then have

$$\mathscr{F} \otimes \mathscr{A}(G) = \bigoplus_{m \in M} \mathscr{F} \otimes m\mathscr{O}_S,$$

so giving a morphism $\mu : \mathscr{F} \to \mathscr{F} \otimes \mathscr{A}(G)$ is equivalent to giving a family of endomorphisms $(\mu_m)_{m \in M}$ of $\mathscr{F}$ such that for any section x of $\mathscr{F}$ over an open subset S, $(\mu_m(x))$ is a section of the direct sum $\bigoplus_{m \in M} \mathscr{F}$ (this means that over any sufficiently small open subset, there are only a finite number of restrictions of the $\mu_m(x)$ which are non-zero). For a morphism μ defined by

$$\mu(x) = \sum_{m \in M} \mu_m(x) \otimes m$$

to satisfy (CM1) and (CM2), it is necessary and sufficient that we have

$$\mu_m \circ \mu_n = \delta_{mn}\mu_m, \quad \sum_{m \in M} \mu_m = \mathrm{id}_{\mathscr{F}}$$

which signify that the μ_m are orthogonal projections adding up to the identity. We have therefore proved the following result:

Proposition 12.1.34. *If $G = D_S(M)$ is a diagonalizable group over S, then the category of $\mathscr{O}_S[G]$-modules (resp. quasi-coherent $\mathscr{O}_S[G]$-modules) is equivalent to the category of graded $\mathscr{O}_S$-modules (resp. quasi-coherent $\mathscr{O}_S[G]$-modules) of type M.*

More precisely, if $\mathscr{F}$ is an $\mathscr{O}_S[G]$-module and $\mathscr{F} = \bigoplus_{m \in M} \mathscr{F}_m$ is the corresponding graduation on $\mathscr{F}$, then for any S' over S and $g \in G(S')$, the action of g on a section x of $\mathscr{F} \otimes_{\mathscr{O}_S} \mathscr{O}_{S'}$ over an open subset S is given by

$$g \cdot x = \sum_m \phi(m) x_m$$

where $\phi : M \to \Gamma(S', \mathscr{O}_{S'})^\times$ is the morphism corresponding to g.

Corollary 12.1.35. *The functor $\mathscr{A} \mapsto \mathrm{Spec}(\mathscr{A})$ induces an equivalence from the category of graded quasi-coherent $\mathscr{O}_S$-algebras of type M to the opposite category of that of affine S-schemes acted by the group $G = D_S(M)$.*

Proof. If X is an affine scheme over S acted by the affine S-group $D_S(M)$, then $\mathscr{A}(S)$ is a quasi-coherent $\mathscr{O}_S$-algebra which is acted by G, whence a graded $\mathscr{O}_S$-algebra of type M. The converse of this is immediate. □

Proposition 12.1.36. *Let G be a diagonalizable group over S. If*

$$0 \longrightarrow \mathscr{F}_1 \longrightarrow \mathscr{F}_2 \longrightarrow \mathscr{F}_3 \longrightarrow 0$$

is an exact sequence of quasi-coherent $\mathscr{O}_S[G]$-modules which split as a sequence of $\mathscr{O}_S$-modules, then it splits as a sequence of $\mathscr{O}_S[G]$-modules..

Proof. If $G = D_S(M)$, then each $\mathscr{F}_i$ is graded by the $(\mathscr{F}_i)_m$ and for each $m \in M$ the sequence

$$0 \longrightarrow (\mathscr{F}_1)_m \longrightarrow (\mathscr{F}_2)_m \longrightarrow (\mathscr{F}_3)_m \longrightarrow 0$$

of $\mathscr{O}_S$-modules is splitting. The proposition then follows from Proposition 12.1.34, since the corresponding result for graded modules is true. □

12.1.3 Cohomology of groups

12.1.3.1 The standard complex Let $\mathcal{C}$ be a category, G be a group in $\widehat{\mathcal{C}}$, A be a ring and M be a $A[G]$-module. For $n \geq 0$, we put

$$C^n(G, M) = \mathrm{Hom}(G^n, M), \quad \mathcal{C}^n(G, M) = \mathcal{H}om(G^n, M),$$

where G^0 is the final object e of $\widehat{\mathcal{C}}$. Then $\mathcal{C}^n(G, M)$ (resp. $C^n(G, M)$) is endowed evidently with a structure of $\mathbb{O}$-module (resp. $\Gamma(\mathbb{O})$-module), and we have

$$C^n(G, M) \cong \Gamma(\mathcal{C}^n(G, M)), \quad \mathcal{C}^n(G, M)(S) = C^n(G_S, M_S).$$

Giving an element of $C^n(G, M)$ is then equivalent to giving for each $S \in \mathrm{Ob}(\mathcal{C})$ an n-cochain of $G(S)$ in $M(S)$, which is functorial on S. The boundary operator

$$d : C^n(G(S), M(S)) \to C^{n+1}(G(S), M(S)),$$

which is defined by the formula

$$\begin{aligned}(df)(g_1, \dots, g_{n+1}) = g_1 \cdot f(g_2, \dots, g_{n+1}) + \sum_{i=1}^n (-1)^i f(g_1, \dots, g_i g_{i+1}, \dots, g_{n+1}) \\ + (-1)^{n+1} f(g_1, \dots, g_n)\end{aligned}$$

is then functorial on S and hence defines a homomorphism

$$d : C^n(G, M) \to C^{n+1}(G, M)$$

such that $d \circ d = 0$. We then obtain a complex of abelian groups, which we denote by $C^\bullet(G, M)$. We define similarly a complex of A-modules $\mathcal{C}^n(G, M)$, and we have

$$C^\bullet(G, M) = \Gamma(\mathcal{C}^n(G, M)).$$

We denote by $H^n(G, M)$ (resp. $\mathcal{H}^n(G, M)$) the cohomology group of the complex $C^\bullet(G, M)$ (resp. $\mathcal{C}^\bullet(G, M)$). In particular, we have

$$\mathcal{H}^0(G, M) = M^G, \quad H^0(G, M) = \Gamma(M^G).$$

Remark 12.1.37. The set-theoretic definition of d is given to verify that $d \circ d = 0$. We can also define d in terms of the multiplication $m : G \times G \to G$ and the action $\mu : G \times M \to M$ as follows: for any $f \in C^n(G, M)$,

$$df = \mu \circ (\mathrm{id}_G \times f) + \sum_{i=1}^{n} (-1)^i f \circ (\mathrm{id}_{G^{i-1}} \times m \times \mathrm{id}_{G^{n-i}}) + (-1)^{n+1} f \circ \mathrm{pr}_{[1,n]},$$

where $\mathrm{pr}_{[1,n]}$ is the projection of $G^{n+1} = G^n \times G$ to G^n. Similarly, for any $S \in \mathrm{Ob}(\mathcal{C})$ and $f \in \mathrm{Ob}(\mathcal{C})^n(G, M)(S) = C^n(G_S, M_S)$, we have

$$df = \mu_S \circ (\mathrm{id}_G \times f) + \sum_{i=1}^{n} (-1)^i f \circ (\mathrm{id}_{G_S^{i-1}} \times m_S \times \mathrm{id}_{G_S^{n-i}}) + (-1)^{n+1} f \circ \mathrm{pr}_{[1,n]},$$

where m_S and μ_S are defined by base change.

We recall that $\mathbf{Mod}(A[G])$ is endowed with an abelian category structure, defined "argument by argument" (Proposition 12.1.11); therefore a sequence of $A[G]$-modules

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

is exact if and only the sequence of abelian groups

$$0 \longrightarrow M'(S) \longrightarrow M(S) \longrightarrow M''(S) \longrightarrow 0$$

is exact for any $S \in \mathrm{Ob}(\mathcal{C})$. If $\mathcal{C}$ is $\mathcal{U}$-small, then by Proposition 12.1.12, $\mathbf{Mod}(A[G])$ possesses enough injectives, so that the derived functors of the left exact functors $\mathcal{H}^0$ and H^0 can be defined. We now show that the functors $\mathcal{H}^n$ and H^n are isomorphic to the derived functors of $\mathcal{H}^0$ and H^0, respectively.

Definition 12.1.38. For any A-module P, we denote by $\mathrm{CoInd}(P)$ the object $\mathcal{H}om(G, P)$ of $\widehat{\mathcal{C}}$ endowed with the structure of an $A[G]$-module defined as follows: for any $S \in \mathrm{Ob}(\mathcal{C})$, we have $\mathcal{H}om(G, P)(S) = \mathrm{Hom}_S(G_S, P_S)$, and we act $g \in G(S)$ and $a \in A[S]$ on $\phi \in \mathrm{Hom}_S(G_S, P_S)$ by the formule

$$(g \cdot \phi)(h) = \phi(hg), \quad (a \cdot \phi)(h) = a\phi(h),$$

for any $h \in G(S')$ and $S' \to S$. Moreover, for any $\phi \in \mathrm{Hom}_S(G_S, P_S)$, we set

$$\varepsilon(\phi) = \phi(1) \in P(S)$$

where 1 denotes the unit element of $G(S)$. Then it is clear that the construction of $\mathrm{CoInd}(P)$ is functorial on P, and we have thus defined a functor $\mathrm{CoInd} : \mathbf{Mod}(A) \to \mathbf{Mod}(A[G])$ and a natural transform $\iota \circ \mathrm{CoInd} \to \mathrm{id}$, where ι denotes the forgetful functor.

Remark 12.1.39. Let G_1 and G_2 be two copies of G. Then the morphism

$$G_1 \times \mathrm{CoInd}(P) \to \mathrm{CoInd}(P), \quad (g_1, \phi) \mapsto (g_2 \mapsto \phi(g_2 g_1))$$

corresponds via the isomorphisms

$$\begin{aligned} \mathrm{Hom}(G_1 \times \mathrm{CoInd}(P), \mathrm{CoInd}(P)) &\cong \mathrm{Hom}(\mathrm{CoInd}(P), \mathcal{H}om(G_1, \mathcal{H}om(G_2, P))) \\ &\cong \mathrm{Hom}(\mathrm{CoInd}(P), \mathcal{H}om(G_2 \times G_1, P)) \end{aligned}$$

to the morphism $\phi \mapsto ((g_2, g_1) \mapsto \phi(g_2 g_1))$, i.e. to the morphism

$$\mathcal{H}om(G, P) \to \mathcal{H}om(G_2 \times G_1, P)$$

induced by the multiplication $\mu_G : G \times G \to G$, $(g_2, g_1) \mapsto g_2 g_1$.

Lemma 12.1.40. *The functor* CoInd *is right adjoint to the forgetful functor* $\iota : \mathbf{Mod}(A[G]) \to \mathbf{Mod}(A)$. *More precisely,* $\varepsilon : \iota \circ \mathrm{CoInd} \to \mathrm{id}$ *induces for any* $M \in \mathbf{Mod}(A[G])$ *and* $P \in \mathbf{Mod}(A)$ *a bijection*

$$\mathrm{Hom}_{A[G]}(M, \mathrm{CoInd}(P)) \xrightarrow{\sim} \mathrm{Hom}_A(M, P).$$

Therefore, if I *is an injective object of* $\mathbf{Mod}(A)$, *then* $\mathrm{CoInd}(I)$ *is an injective object of* $\mathbf{Mod}(A[G])$.

Proof. To any A-morphism $f : M \to P$, we associate an element $\phi_f \in \mathrm{Hom}_A(M, \mathrm{CoInd}(P))$ defined as follows: for $S \in \mathrm{Ob}(\mathcal{C})$ and $m \in M(S)$, $\phi_f(m)$ is the element of $\mathrm{Hom}_S(G_S, P_S)$ such that for any $g \in G(S')$, $S' \to S$,

$$\phi_f(m)(g) = f(gm) \in P(S').$$

Then for any $h \in G(S)$, we have $\phi_f(hm) = h \cdot f(m)$, i.e. $\phi_f \in \mathrm{Hom}_{A[G]}(M, \mathrm{CoInd}(P))$. Now if $\phi \in \mathrm{Hom}_{A[G]}(M, \mathrm{CoInd}(P))$ and we denote, for $m \in M(S)$, $f(m) = \phi(m)(1)$, then

$$\phi_f(m)(g) = f(gm) = \phi(gm)(1) = (g \cdot \phi(m)) = \phi(m)(g),$$

so $\phi_f = \phi$. Conversely, it is clear that $\phi_f(m)(1) = f(m)$, whence the first claim. The second claim then follows since the forgetful functor ι is exact. □

Definition 12.1.41. Let M be an $A[G]$-module; the identity map on M (considered as an A-module) corresponds by adjunction to a morphism of $A[G]$-modules

$$\eta_M : M \to \mathrm{CoInd}(M)$$

such that for $S \in \mathrm{Ob}(\mathcal{C})$ and $m \in M(S)$, $\eta_M(m)$ is the morpism $G_S \to M_S$ such that for any $S' \to S$ and $g \in G(S')$, $\eta_M(m)(g) = g \cdot m_{S'} \in M(S')$. Note that η_M is a monomorphism: in fact, $\varepsilon_M : \mathrm{CoInd}(M) \to M$ is a morphism of A-modules such that $\varepsilon_M \circ \eta_M = \mathrm{id}_M$. Therefore, M is a direct factor of the A-module $\mathrm{CoInd}(M)$.

Lemma 12.1.42. *For any $P \in \mathbf{Mod}(A)$, we have*

$$H^n(G, \mathcal{H}om(G, P)) = 0, \quad \mathcal{H}^n(G, \mathcal{H}om(G, P)) = 0 \quad \text{for} \quad n > 0.$$

Therefore, the functors $H^n(G, -)$ and $\mathcal{H}^n(G, -)$ are effacable for $n > 0$.

Proof. It suffices to prove that $\mathcal{C}^\bullet(G, \mathcal{H}om(G, P))$ and $C^\bullet(G, \mathcal{H}om(G, P))$ are null-homotopic at positive degrees. To this end, we only need to consider the second one, since the corresponding result can be derived via base changes. Now, we define for $n \geq 0$ a morphism

$$\sigma : C^{n+1}(G, \mathcal{H}om(G, P)) \to C^n(G, \mathcal{H}om(G, P)).$$

Let $f \in C^{n+1}(G, \mathcal{H}om(G, P))$; for any $S \in \mathrm{Ob}(\mathcal{C})$ and $g_1, \dots, g_n \in G(S)$, $\sigma(f)(g_1, \dots, g_n)$ is the element of $\mathrm{Hom}_S(G_S, P_S)$ such that for any $S' \to S$ and $x \in G(S')$,

$$\sigma(f)(g_1, \dots, g_n)(x) = f(x, g_1, \dots, g_n)(1) \in P(S'),$$

where 1 denotes the unit element of $G(S')$. Then σ is a null homotopy at positive degrees. In fact, for any $g_1, \dots, g_{n+1} \in G(S)$ and $x \in G(S')$, we have, on the one hand,

$$\begin{aligned} d\sigma(f)(g_1, \dots, g_{n+1})(x) &= f(xg_1, g_2, \dots, g_{n+1})(1) + \sum_{i=1}^{n} (-1)^i f(x, g_1, \dots, g_i g_{i+1}, \dots, g_{n+1})(1) \\ &\quad + (-1)^{n+1} f(x, g_1, \dots, g_n)(1), \end{aligned}$$

and on the other hand,

$$\begin{aligned} \sigma(df)(g_1, \dots, g_{n+1})(x) &= (xf(g_1, \dots, g_{n+1}))(1) - f(xg_1, g_2, \dots, g_{n+1})(1) \\ &\quad + \sum_{i=1}^{n} (-1)^{i+1} f(x, g_1, \dots, g_i g_{i+1}, g_{n+1}) + (-1)^{n+2} f(x, g_1, \dots, g_n)(1), \end{aligned}$$

whence

$$(d\sigma(f) + \sigma(df))(g_1, \dots, g_{n+1})(x) = (xf(g_1, \dots, g_{n+1}))(1) = f(g_1, \dots, g_{n+1})(x),$$

i.e. $d\sigma + \sigma d$ is the identity map on $C^{n+1}(G, \mathcal{H}om(G, P))$, for any $n \geq 0$. □

Proposition 12.1.43. *Suppose that $\mathcal{C}$ is $\mathcal{U}$-small, finite products exist in $\mathcal{C}$, and that G is representable. Then the functors $H^n(G, -)$ (resp. $\mathcal{H}^n(G, -)$) are the derived functors of $H^0(G, -)$ (resp. $\mathcal{H}^n(G, -)$) over the category of $A[G]$-modules.*

Proof. In view of ([?] 2.2.1 and 2.3), it suffices to show that the $H^n(G)$ (resp. $\mathcal{H}^n(G,-)$) form a cohomological functors, since they are effacable for $n>0$ in view of Lemma 12.1.42. Let

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

be an exact sequence of $A[G]$-modules, and let $S \in \mathrm{Ob}(\mathcal{C})$. By hypothesis, G is represented by an object $G \in \mathrm{Ob}(\mathcal{C})$, and finite products exist in $\mathcal{C}$. In particular, $\mathcal{C}$ possesses a final object e. For each $n \geq 0$, the product $G^n \times h_S$ is then represented by $G^n \times S$ (where $G^0 = e$), and the sequence

$$0 \longrightarrow M'(G^n \times S) \longrightarrow M(G^n \times S) \longrightarrow M''(G^n \times S) \longrightarrow 0$$

is exact. Therefore, the sequence of A-modules

$$0 \longrightarrow \mathcal{C}^n(h_G, M') \longrightarrow \mathcal{C}^n(h_G, M) \longrightarrow \mathcal{C}^n(h_G, M'') \longrightarrow 0$$

is exact, which means $\mathcal{C}^\bullet(G,-)$, considered as a functor from $\mathbf{Mod}(A[G])$ to the category of complexes of $\mathbf{Mod}(A)$, is exact. It then follows from the induced long exact sequence that $\mathcal{H}^n(G,-)$ form a cohomological functor. As the functor Γ is exact, the same holds for the functors $H^n(G,-)$. □

12.1.3.2 Cohomology of $\mathscr{O}_S[G]$-modules Let S be a scheme, G be an S-group and $\mathscr{F}$ be a quasi-coherent $\mathscr{O}_S[G]$-module. We define the cohomology groups of G with values in $\mathscr{F}$ by

$$H^n(G,\mathscr{F}) = H^n(h_G, \Gamma_{\mathscr{F}}).$$

Suppose that G is affine over S, then by Corollary 12.1.25, this cohomology can be calculated in the following way: $H^n(G,\mathscr{F})$ is the n-th cohomology group of the complex $C^\bullet(G,\mathscr{F})$ whose n-th term is

$$C^n(G,\mathscr{F}) = \Gamma(S, \mathscr{F} \otimes \underbrace{\mathscr{A}(G) \otimes \cdots \otimes \mathscr{A}(G)}_{n\text{-fold}}).$$

If f (resp. a_i) is a section of $\mathscr{F}$ (resp. $\mathscr{A}(G)$) over an open subset of S, we then have

$$\begin{aligned} d(f \otimes a_1 \otimes \cdots \otimes a_n) = \mu_{\mathscr{F}}(f) \otimes a_1 \otimes \cdots \otimes a_n + \sum_{i=1}^{n} (-1)^i f \otimes a_1 \cdots \otimes \Delta a_i \otimes \cdots \otimes a_n \\ + (-1)^{n+1} f \otimes a_1 \otimes \cdots \otimes a_n \otimes 1 \end{aligned}$$

where $\Delta : \mathscr{A}(G) \to \mathscr{A}(G) \otimes \mathscr{A}(G)$ and $\mu_{\mathscr{F}} : \mathscr{F} \to \mathscr{F} \otimes \mathscr{A}(G)$ are induced from the cogebrea structure of $\mathscr{A}(G)$ and the comodule structure on $\mathscr{F}$. Note in passing that the cohomology of G with values in $\mathscr{F}$ therefore depends only on the comodule structure of $\mathscr{F}$ and the monoid structure of G. In particular, we obtain a functor

$$H^0(G,\mathscr{F}) = \Gamma(S, \mathscr{F}^G)$$

where $\mathscr{F}^G$ is the invariant sheaf of $\mathscr{F}$ defined in Remark 12.1.31.

Theorem 12.1.44. *Let S be an affine scheme and G be an affine and flat group over S. Then the functors $H^n(G,-)$ are the derived functors of $H^0(G,-)$ over the category of quasi-coherent $\mathscr{O}_S[G]$-modules.*

If G is affine and flat over S, then by Proposition 12.1.33, the category $\mathbf{Qcoh}(\mathscr{O}_S[G])$ is equivalent to the category $\mathbf{CoQcoh}(\mathscr{A}(G))$ of quasi-coherent $\mathscr{A}(G)$-comodules over $\mathscr{O}_S$ and is abelian. On the other hand, $\mathscr{A}(G)$ being a flat $\mathscr{O}_S$-module, the functor $\mathscr{F} \mapsto \mathscr{F} \otimes_{\mathscr{O}_S} \mathscr{A}(G)^{\otimes n}$ is exact; as S is also affine, we conclude that $C^\bullet(G,-)$ is an exact functor over $\mathbf{Qcoh}(\mathscr{O}_S[G])$.

We denote by Δ (resp. η) the coultiplication (resp. counit) of $\mathscr{A}(G)$. For any quasi-coherent $\mathscr{O}_S$-module $\mathscr{P}$, we denote by $\mathrm{Ind}(\mathscr{P}) = \mathscr{P} \otimes_{\mathscr{O}_S} \mathscr{A}(G)$ endowed with the $\mathscr{A}(G)$-comodule structure defined by

$$\mathrm{id}_{\mathscr{P}} \otimes \Delta : \mathscr{P} \otimes_{\mathscr{O}_S} \mathscr{A}(G) \to \mathscr{P} \otimes_{\mathscr{O}_S} \mathscr{A}(G) \otimes_{\mathscr{O}_S} \mathscr{A}(G);$$

this defines a functor $\mathrm{Ind} : \mathbf{Qcoh}(\mathscr{O}_S) \to \mathbf{Qcoh}(\mathscr{O}_S[G])$. It follows from Corollary 12.1.25 that we have an isomorphism of $\mathbb{O}_S[G]$-modules

$$\Gamma_{\mathrm{Ind}(\mathscr{P})} \cong \mathrm{CoInd}(\Gamma_{\mathscr{P}}) = \mathcal{H}om(G, \Gamma_{\mathscr{P}}). \tag{12.1.1}$$

Via this identification, the morphism $\varepsilon : \mathrm{CoInd}(\Gamma_{\mathscr{P}}) \to \Gamma_{\mathscr{P}}$ then corresponds to the morphism $\mathrm{id}_{\mathscr{P}} \otimes \eta : \mathrm{Ind}(\mathscr{P}) \to \mathscr{P}$ of $\mathscr{O}_S$-modules, where we use Proposition 12.1.21. From Lemma 12.1.40, we then conclude the following corolalry:

Corollary 12.1.45. *Let S be a scheme and G be an affine group over S. Then the functor* Ind *is right adjoint to the forgetful functor $\iota : \mathbf{Qcoh}(\mathscr{O}_S[G]) \to \mathbf{Qcoh}(\mathscr{O}_S)$. More precisely, the map $\mathrm{id}_{\mathscr{P}} \otimes \eta : \mathrm{Ind}(\mathscr{P}) \to \mathscr{P}$ induces for any object $\mathscr{M}$ of $\mathbf{Qcoh}(\mathscr{O}_S[G])$ a bijection*

$$\mathrm{Hom}_{\mathscr{O}_S[G]}(\mathscr{M}, \mathrm{Ind}(\mathscr{P})) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{O}_S}(\mathscr{M}, \mathscr{P}).$$

Therefore, if $\mathscr{I}$ is an injective object in $\mathbf{Qcoh}(\mathscr{O}_S)$, then $\mathrm{Ind}(\mathscr{I})$ is an injective object in $\mathbf{Qcoh}(\mathscr{O}_S)$.

Let $\mathscr{F}$ be an $\mathscr{O}_S[G]$-module and $\mu_{\mathscr{F}} : \mathscr{F} \to \mathrm{Ind}(\mathscr{F})$ be the map defining the $\mathscr{A}(G)$-comodule structure. It follows from the axioms (CM1) and (CM2) that $\mu_{\mathscr{F}}$ is a morphism of $\mathscr{O}_S[G]$-modules, and that $(\mathrm{id}_{\mathscr{F}} \otimes \eta) \circ \mu_{\mathscr{F}} = \mathrm{id}_{\mathscr{F}}$, so that $\mathscr{F}$ is a direct factor of $\mathrm{Ind}(\mathscr{F})$ considered as $\mathscr{O}_S$-modules. In particular, $\mu_{\mathscr{F}}$ is a monomorphism. As we have, by (12.1.1) and Lemma 12.1.42,

$$H^n(G, \Gamma_{\mathrm{Ind}(\mathscr{F})}) \cong H^n(G, \mathcal{H}om_S(G, \Gamma_{\mathscr{F}})) = 0 \quad \text{for} \quad n > 0$$

we conclude that $H^n(G, -)$ is effacable for $n > 0$.

Finally, as S is affine, $\mathbf{Qcoh}(\mathscr{O}_S)$ possesses enough injectives. Let $\mathscr{F} \rightarrowtail \mathscr{I}$ be a monomorphism of $\mathscr{O}_S$-modules where $\mathscr{I}$ is injective object of $\mathbf{Qcoh}(\mathscr{O}_S)$; then, $\mathscr{A}(G)$ being flat over $\mathscr{O}_S$, $\mathrm{Ind}(\mathscr{F})$ is a sub-$\mathscr{O}_S[G]$-module of $\mathrm{Ind}(\mathscr{I})$, so we conclude that

Corollary 12.1.46. *Under the hypothesis of Theorem 12.1.44, the abelian category $\mathbf{Qcoh}(\mathscr{O}_S[G])$ possesses enough injectives.*

In view of ([?] 2.2.1 and 2.3), we then conclude that proof of Theorem 12.1.44.

Remark 12.1.47. We can also prove Corollary 12.1.45by the following calculation. To any morphism of $\mathscr{O}_S[G]$-modules $\phi : \mathscr{M} \to \mathscr{P} \otimes_{\mathscr{O}_S} \mathscr{A}(G)$, we associate the $\mathscr{O}_S$-morphism $(\mathrm{id}_{\mathscr{P}} \otimes \eta) \circ \phi : \mathscr{M} \to \mathscr{P}$. Conversely, to any $\mathscr{O}_S$-morphism $f : \mathscr{M} \to \mathscr{P}$ we associate the $\mathscr{O}_S[G]$-morphism $(f \otimes \mathrm{id}_{\mathscr{A}(G)}) \circ \mu_{\mathscr{M}} : \mathscr{M} \to \mathrm{Ind}(\mathscr{P})$. On the one hand, from axiom (CM2) we see that

$$(\mathrm{id}_{\mathscr{P}} \otimes \eta) \circ (f \circ \mathrm{id}_{\mathscr{A}(G)}) \circ \mu_{\mathscr{M}} = (f \circ \mathrm{id}_{\mathscr{O}_S}) \circ (\mathrm{id}_{\mathscr{P}} \otimes \eta) \circ \mu_{\mathscr{M}} = f.$$

On the other hand, for any ϕ the following diagram is commutative:

$$\begin{tikzcd} \mathscr{M} \arrow[r, "\phi"] \arrow[d, "\mu_{\mathscr{M}}"'] & \mathscr{P} \otimes_{\mathscr{O}_S} \mathscr{A}(G) \arrow[d, "\mathrm{id}_{\mathscr{P}} \otimes \Delta"] \\ \mathscr{M} \otimes_{\mathscr{O}_S} \mathscr{A}(G) \arrow[r, "\phi \otimes \mathrm{id}_{\mathscr{A}(G)}"] & \mathscr{P} \otimes_{\mathscr{O}_S} \mathscr{A}(G) \otimes_{\mathscr{O}_S} \mathscr{A}(G) \end{tikzcd}$$

so it follows that

$$\begin{aligned} (((\mathrm{id}_{\mathscr{P}} \otimes \eta) \circ \phi) \otimes \mathrm{id}_{\mathscr{A}(G)}) \circ \mu_{\mathscr{M}} &= (\mathrm{id}_{\mathscr{P}} \otimes \eta \otimes \mathrm{id}_{\mathscr{A}(G)}) \circ (\phi \otimes \mathrm{id}_{\mathscr{A}(G)}) \circ \mu_{\mathscr{M}} \\ &= (\mathrm{id}_{\mathscr{P}} \otimes \eta \otimes \mathrm{id}_{\mathscr{A}(G)}) \circ (\mathrm{id}_{\mathscr{P}} \otimes \Delta) \circ \phi = \phi. \end{aligned}$$

This proves the first claim of Corollary 12.1.45, and the second one then follows.

Let $\mathscr{F}$ be an $\mathscr{O}_S[G]$-module. We have seen that the axiom (CM2) shows that considered as $\mathscr{O}_S$-modules, $\mathscr{F}$ is a direct factor of $\mathrm{CoInd}(\mathscr{F})$. This implies the following proposition:

Proposition 12.1.48. *Let S be an affine scheme and G be an affine and flat group scheme over S. Suppose that for any exact sequence*

$$0 \longrightarrow \mathscr{F}_1 \longrightarrow \mathscr{F}_2 \longrightarrow \mathscr{F}_3 \longrightarrow 0$$

of quasi-coherent $\mathscr{O}_S[G]$-modules, which splits as a sequence of $\mathscr{O}_S$-modules, also split as $\mathscr{O}_S[G]$-modules. Then the functors $H^n(G, -)$ are zero for $n > 0$.

Proof. In fact, by the hypothesis, the sequence of $\mathscr{O}_S[G]$-modules

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathrm{CoInd}(\mathscr{F}) \longrightarrow \mathrm{CoInd}(\mathscr{F})/\mathscr{F} \longrightarrow 0$$

is splitting, so $\mathscr{F}$ is a direct factor of $\mathrm{CoInd}(\mathscr{F})$ as an $\mathscr{O}_S[G]$-module. Since $\mathrm{CoInd}(\mathscr{F})$ has trivial higher cohomology, so does $\mathscr{F}$. □

Theorem 12.1.49. *Let S be an affine scheme and G be a diagonalizable S-group. Then for any quasi-coherent $\mathscr{O}_S[G]$-module $\mathscr{F}$, we have $H^n(G, \mathscr{F}) = 0$ for $n > 0$.*

Proof. This follows from Proposition 12.1.48 and Proposition 12.1.36. □

12.1.4 G-equivariant objects and modules

Let $\mathcal{C}$ be a category with a final object e and such that fiber products exist in $\mathcal{C}$. Let G be a group in $\widehat{\mathcal{C}}$, $\pi : M \to X$ be a morphism in $\widehat{\mathcal{C}}$, and $\lambda = \lambda_X : G \times X \to X$ be an action of G on X. In this paragraph, we denote by $Y \times_f M$ the fiber product of $\pi : M \to X$ and an X-functor $f : Y \to X$.

For any $U \in \mathrm{Ob}(\mathcal{C})$ and $x \in X(U)$, the **fiber** of M at x is defined by $M_x = U \times_x M$, i.e. for any $\phi : U' \to U$, we have

$$M_x(U') = \{m \in M(U') : \pi(m) = x_{U'} = \phi^*(x)\}.$$

Finally, if $g \in G(U)$, we denote by $g(x)$ the element $\lambda(g, x)$ in $X(U)$.

Definition 12.1.50. We say that M is a ***G*-equivariant object over *X***, or a ***G*-equivariant *X*-object**, if we are given an action $\Lambda : G \times M \to M$ of G on M compatible with λ, i.e. such that the following diagram is commutative:

$$\begin{tikzcd} G \times M \arrow[r, "\Lambda"] \arrow[d, "\mathrm{id}_G \times \pi"] & M \arrow[d, "\pi"] \\ G \times X \arrow[r, "\lambda"] & X \end{tikzcd}$$

This is equivalent to saying that we are given, for any morphism $(g, x) : U \to G \times X$, morphisms

$$\Lambda_x^U(g) : M_x(U) \to M_{g(x)}(U), \quad m \mapsto g \cdot m$$

satisfying $1 \cdot m = m$ and $g \cdot (h \cdot m) = (gh) \cdot m$ and functorial on the $(G \times X)$-object U. Alternatively, this means we are given morphisms of U-objects

$$\Lambda_x(g) : M_x \to M_{g(x)}$$

such that $\Lambda_x(1) = \mathrm{id}$ and $\Lambda_{h(x)}(g) \circ \Lambda_x(h) = \Lambda_x(gh)$.

Now let A be a ring in $\widehat{\mathcal{C}}$ and $A_X = A \times X$. Under the condition described above, we say that M is a ***G*-equivariant A_X-module** if it is an A_X-module and the action Λ is compatible with the A_X-module structure on M, that is, if for any morphism $(g, x) : U \to G \times X$, the map $\Lambda_x(g) : M_x \to M_{g(x)}$ is a morphism of A_U-modules.

Remark 12.1.51. In the above definition for G-equivariant objects, the conditions $\Lambda_x(1) = \mathrm{id}$ and $\Lambda_{h(x)}(g) \circ \Lambda_x(h) = \Lambda_x(gh)$ implies that $\Lambda_x(g)$ is an isomorphism, with inverse $\Lambda_{g(x)}(g^{-1})$. Conversely, if we suppose that each $\Lambda_x(g)$ is an isomorphism, the condition $\Lambda_{h(x)}(g) \circ \Lambda_x(h) = \Lambda_x(gh)$, applied to $h = 1$, then implies that $\Lambda_x(1) = \mathrm{id}$.

Remark 12.1.52. If M is an A_X-module, then in view of the universal property of fiber products, giving a morphism $\Lambda : G \times M \to M$ which is compatible with λ is equivalent to giving a homomorphism of $A_{G \times X}$-modules

$$\theta : G \times M = (G \times X) \times_{\mathrm{pr}_X} M \to (G \times X) \times_\lambda M, \quad (g, x, m) \mapsto (g, g(x), g \cdot m),$$

and the morphisms $\Lambda_x(g) : M_x \to M_{g(x)}, m \mapsto g \cdot m$ are isomorphisms of A_U-modules if and only if θ is an isomorphism. As we have supposed that each $\Lambda_x(h)$ is an isomorphism, the equality $\Lambda_x(1) = \mathrm{id}$ follows from the equality $\Lambda_{h(x)}(g) \circ \Lambda_x(h) = \Lambda_x(gh)$. Therefore, Λ is an action of G over M if and only the following diagram of $(G \times G \times X)$-isomorphisms is commutative (where we denote by m the multiplication of G and $f^*(\theta)$ is the isomorphism induced from θ under a base change $f : G \times G \times X \to G \times X$)

$$\begin{tikzcd} (G \times G \times X) \times_{\mathrm{pr}_X \circ \mathrm{pr}_{23}} M \arrow[r, "\mathrm{pr}_{23}^*(\theta)", "\sim"'] \arrow[d, equal] & (G \times G \times X) \times_{\lambda \circ \mathrm{pr}_{23}} M \arrow[d, equal] \\ (G \times G \times X) \times_{\mathrm{pr}_X \circ (m \times \mathrm{id}_X)} M \arrow[d, "(m \times \mathrm{id}_X)^*(\theta)"', "\sim"] & (G \times G \times X) \times_{\mathrm{pr}_X \circ (\mathrm{id}_G \times \lambda)} M \arrow[d, "\sim"', "(\mathrm{id}_G \times \lambda)^*(\theta)"] \\ (G \times G \times X) \times_{\lambda \times (m \times \mathrm{id}_X)} M \arrow[r, equal] & (G \times G \times X) \times_{\lambda \circ (\mathrm{id}_G \times \lambda)} M \end{tikzcd}$$

Remark 12.1.53. The above definitions extend to the case where G is only a monoid. In this case, giving an action $\Lambda : G \times M \to M$ that is compatible with λ and such that each $\Lambda_x(g) : M_x \to M_{g(x)}$ is a morphism of A_U-modules is equivalent to giving a morphism

$$\theta : G \times M = (G \times X) \times_{\mathrm{pr}_X} M \to (G \times X) \times_\lambda M, \quad (g, x, m) \mapsto (g, g(x), g \cdot m),$$

such as the diagram in Remark 12.1.52 (without the signs $\sim$ under the arrows) is commutative, and such that $\mathrm{pr}_M \circ \theta \circ (\varepsilon_G \times \mathrm{id}_M) = \mathrm{id}_M$, where ε_G denotes the unit section of G and pr_M the projection on M (this is added since in this case the equality $\Lambda_x(1) = \mathrm{id}$ can not be derived).

Let Y be another object of $\widehat{\mathcal{C}}$ which is endowed with an action $\lambda_Y : G \times Y \to Y$ by G and N be a G-equivariant A_X-module. A morphism $f : Y \to X$ in $\widehat{\mathcal{C}}$ (resp. a homomorphism of A_X-modules $\phi : M \to X$) is called G-equivariant if it commutes with the action of G, i.e. if we have $f(g \cdot y) = g \cdot f(y)$ (resp. $\phi(g \cdot m) = g \cdot \phi(m)$), which is equivalent to $f \circ \lambda_Y = \lambda_X \circ \mathrm{id}_G \times f$ (resp. $\phi \circ \Lambda_M = \Lambda_N \circ (\mathrm{id}_G \times \phi)$). We then obtain the following lemma:

Lemma 12.1.54. *Let $f : Y \to X$ be a G-equivariant morphism and M be a G-equivariant A-module. Then the inverse image $f^*(M) = Y \times_f M$ is a G-equivariant A_Y-module.*

Proof. □

12.2 Tangent spaces and Lie algebras

In this section, we construct the tangent spaces and Lie algebras in scheme theory. It will be useful not to restrict oneself to the diagrams themselves, but to also be intersted to certain functors on the category of schemes which are not necessarily representable. The exposition we give here easily generalize beyond the theory of schemes. For example, it is valid for the theory of complex analytic spaces, with suitable modifications.

12.2.1 The tangent bundle and tangent space

12.2.1.1 The functor $\mathcal{H}om_{Z/S}(X, Y)$ Let $\mathcal{C}$ be a category and S be an object of $\mathcal{C}$. We consider objects X, Y, Z in $\widehat{\mathcal{C}}$ with X, Y lying over Z and Z lying over S:

$$\begin{tikzcd} X \arrow[dr, "p_X"'] & & Y \arrow[dl, "p_Y"] \\ & Z \arrow[d] & \\ & S & \end{tikzcd}$$

Definition 12.2.1. We define an object $\mathcal{H}om_{Z/S}(X, Y)$ in $\widehat{\mathcal{C}_{/S}}$ by the formula

$$\mathcal{H}om_{Z/S}(X, Y)(S') = \mathrm{Hom}_{Z_{S'}}(X_{S'}, Y_{S'}) = \mathrm{Hom}_Z(X \times_S S', Y),$$

where S' is an object of $\mathcal{C}_{/S}$. We see that $\mathcal{H}om_{Z/S}(X, Y)$ is none other than the sub-object of $\mathcal{H}om_S(X, Y)$ formed by morphisms compatible with p_X and p_Y, that is, it is the kernel of the morphisms

$$\mathcal{H}om_S(X, Y) \rightrightarrows \mathcal{H}om_S(X, Z)$$

where the first map is defined by composing with p_Y and the seond one is the constant map of p_X.

On the other hand, we see as in (**??**) that, for any object T of $\widehat{\mathcal{C}}$ over S, we have a natural bijection

$$\mathrm{Hom}_S(T, \mathcal{H}om_{Z/S}(X, Y)) \cong \mathrm{Hom}_Z(X \times_S T, Y).$$

Moreover, by (**??**), if E, F are objects of $\widehat{\mathcal{C}}$ lying over Z, then

$$\mathrm{Hom}_Z(E, \mathcal{H}om_Z(F, Y)) \cong \mathrm{Hom}_Z(E \times_Z F, Y) \cong \mathrm{Hom}_Z(F, \mathcal{H}om_Z(E, Y)).$$

Apply this to $E = X$ and $F = Z \times_S T$, we then obtain the following bijections for any object T of $\widehat{\mathcal{C}_{/S}}$:

$$\mathrm{Hom}_S(T, \mathcal{H}om_{Z/S}(X,Y)) \cong \mathrm{Hom}_Z(X \times_S T, Y) \cong \begin{cases} \mathrm{Hom}_Z(Z \times_S T, \mathcal{H}om_Z(X,Y)), \\ \mathrm{Hom}_Z(X, \mathcal{H}om_Z(Z \times_S T, Y)). \end{cases} \tag{12.2.1}$$

Since these bijections are functorial over T, we then obtain isomorphisms of S-functors

$$\begin{tikzcd} \mathcal{H}om_S(T, \mathcal{H}om_{Z/S}(X,Y)) \arrow[rr, "\sim"] \arrow[dr, "\sim"'] & & \mathcal{H}om_{Z/S}(X, \mathcal{H}om_Z(Z \times_S T, Y)) \\ & \mathcal{H}om_{Z/S}(X \times_S T, Y) \arrow[ur, "\sim"'] & \end{tikzcd} \tag{12.2.2}$$

12.2.1.2 The restriction functor $\mathrm{Res}_{Z/S} Y$ We now consider the special case where $X = Z$: we put

$$\mathrm{Res}_{Z/S} Y = \mathcal{H}om_{Z/S}(Z, Y).$$

By definition, we then have

$$\mathrm{Res}_{Z/S}(Y)(S') = \mathrm{Hom}_Z(Z \times_S S', Y) = \Gamma(Y_{S'}/Z_{S'}).$$

The functor $\mathrm{Res}_{Z/S} : \widehat{\mathcal{C}_{/Z}} \to \widehat{\mathcal{C}_{/S}}$ is a right adjoint of the base change functor from S to Z. In fact, for any S-functor U, by (12.2.1) we have

$$\mathrm{Hom}_S(U, \mathrm{Res}_{Z/S} Y) = \mathrm{Hom}_S(U, \mathcal{H}om_{Z/S}(Z,Y)) \cong \mathrm{Hom}_Z(U \times_S Z, Y).$$

(If $\mathcal{C} = \mathbf{Sch}$ and Z is an S-scheme, the functor $\mathrm{Res}_{Z/S}$ is called the **Weil restriction**.) We also ntoe that since for any $S' \in \mathrm{Ob}(\mathcal{C}_{/S})$ we have

$$\mathcal{H}om_{Z/S}(X,Y)(S') = \mathrm{Hom}_Z(X_{S'}, Y) \cong \mathrm{Hom}_X(X_{S'}, Y \times_Z X) = \mathcal{H}om_{X/S}(X, Y \times_Z X),$$

so we obtain an isomorphism

$$\mathcal{H}om_{Z/S}(X,Y) \cong \mathcal{H}om_{X/S}(X, Y \times_Z X) = \mathrm{Res}_{X/S}(Y \times_Z X),$$

which for $Z = S$ gives an isomorphism

$$\mathcal{H}om_S(X,Y) \cong \mathrm{Res}_{X/S} Y_X.$$

Remark 12.2.2. The functore $Y \mapsto \mathcal{H}om_{Z/S}(X,Y)$ commutes with products in the sense that we have a functorial isomorphism

$$\mathcal{H}om_{Z/S}(X, Y \times_Z Y') \cong \mathcal{H}om_{Z/S}(X,Y) \times_S \mathcal{H}om_{Z/S}(X,Y') \tag{12.2.3}$$

It follows that if Y is a Z-group (resp. Z-ring, etc.), then $\mathcal{H}om_{Z/S}(X,Y)$ is an S-group (resp. S-ring, etc.).

Remark 12.2.3. Let $\pi : M \to Y$ be a Y-functor in $\mathbb{O}_Y$-modules. Put $H = \mathcal{H}om_{Z/S}(X,Y)$, then the functor $\mathcal{H}om_{Z/S}(X,M)$ is endowed with a natural $\mathbb{O}_H$-module structur. In fact, denote by $m : M \times_Y M \to M$ and $\lambda : \mathbb{O}_Y \times_Y M \to M$ the defining morphisms of abelian group structure and module structure of M. Let H' be an S-scheme over H, that is, we are given a Z-morphism $f : X \times_S H' \to Y$, which makes $X \times_S H'$ a Y-object. Then $\mathrm{Hom}_H(H', \mathcal{H}om_{Z/S}(X,M))$ is the set of Z-morphisms $\phi : X \times_S H' \to M$ such that $\pi \circ \phi = f$, that is, the Y-morphisms $X \times_S H' \to M$.

Let ϕ, ψ be two such morphisms, we define $\phi + \psi$ as the composition of Y-morphisms

$$X \times_S H' \xrightarrow{\phi \times \psi} M \times_Y M \xrightarrow{m} M$$

and this endows $\mathcal{H}om_{Z/S}(X,M)$ an abelian group structure over $H = \mathcal{H}om_{Z/S}(X,Y)$.

Similarly, if a is an element of $\mathbb{O}(X \times_S H')$, i.e. an S-morphism $a : X \times_S H' \to \mathbb{O}_S$, we define $a\phi$ as the composition $\lambda \circ (a \times \phi)$, where $a \times \phi$ denotes the Y-morphism from $X \times_S H$ to $\mathbb{O}_Y \times_Y M \cong \mathbb{O}_S \times_S M$ with components a and ϕ. We verify that this endows $\mathrm{Hom}_H(H', \mathcal{H}om_{Z/S}(X,M))$ with an $\mathbb{O}(X \times_S H')$-module structure, which is functorial on H'.

12.2.1.3 The scheme $I_S(\mathscr{M})$

Definition 12.2.4. Let S be a scheme and $\mathscr{M}$ be a quasi-coherent $\mathscr{O}_S$-module. We denote by $\mathscr{D}_{\mathscr{O}_S}(\mathscr{M})$ the quasi-coherent algebra $\mathscr{O}_S \oplus \mathscr{M}$ (where $\mathscr{M}$ is considered as a square zero ideal). We denote by $I_S(\mathscr{M})$ the S-scheme $\operatorname{Spec}(\mathscr{D}_{\mathscr{O}_S}(\mathscr{M}))$. In particular, we have $\mathscr{D}_{\mathscr{O}_S} = \mathscr{D}_{\mathscr{O}_S}(\mathscr{O}_S)$, $I_S = I_S(\mathscr{O}_S)$, which are called the **algebra of dual numbers over S** and the **dual number scheme over S**.

We then obtain a contravariant functor $\mathscr{M} \mapsto I_S(\mathscr{M})$ from the category of quasi-coherent $\mathscr{O}_S$-modules to the category of S-schemes. In particular, the morphisms $0 \to \mathscr{M}$ and $\mathscr{M} \to 0$ define respectively the structural morphism $\rho : I_S(\mathscr{M}) \to I_S(0) = S$ and a section $\varepsilon_{\mathscr{M}} : S \to I_S(\mathscr{M})$, which is called the **zero section** of $I_S(\mathscr{M})$.

As $\mathscr{M} \mapsto I_S(\mathscr{M})$ is a contravariant functor, for any endomorphism $a \in \operatorname{End}_{\mathscr{O}_S}(\mathscr{M})$, we have an S-endomorphism a^* of $I_S(\mathscr{M})$, and

$$1^* = \mathrm{id}, \quad (ab)^* = b^* \circ a^*, \quad 0^* = \varepsilon_{\mathscr{M}} \circ \rho, \quad a^* \circ \varepsilon_{\mathscr{M}} = \varepsilon_{\mathscr{M}}.$$

Therefore, the S-scheme $I_S(\mathscr{M})$ is endowed with a right action of the (multiplicative) monoid $\operatorname{End}_{\mathscr{O}_S}(\mathscr{M})$, which commutes with S-morphisms $I_S(\mathscr{M}) \to I_S(\mathscr{M}')$ induced by homomorphisms $\mathscr{M} \to \mathscr{M}'$. In particular, the operations a^* preserves the zero section of $I_S(\mathscr{M})$.

For any endomorphism $a \in \operatorname{End}_{\mathscr{O}_S}(\mathscr{M})$, $f : S' \to S$ and $m \in I_S(\mathscr{M})(S')$, we write $m \cdot a = a^*(m)$. Then we have

$$m \cdot 1 = m, \quad (m \cdot a) \cdot b = m \cdot (ab), \quad m \cdot 0 = \varepsilon_{\mathscr{M}}(\rho(m)).$$

Moreover, if $m = \varepsilon_{\mathscr{M}}(f)$, then $m \cdot a = m$.

Remark 12.2.5. The formation of $I_S(\mathscr{M})$ commutes with base changes: we have a canonical isomorphism

$$I_S(\mathscr{M})_{S'} \cong I_{S'}(\mathscr{M} \otimes_{\mathscr{O}_S} \mathscr{O}_{S'}).$$

For simplicity, we shall write $I_{S'}(\mathscr{M})$ for $I_S(\mathscr{M})_{S'}$. More generally, if X is an S-functor (not necessarily representable), then we define $I_X(\mathscr{M}) := I_S(\mathscr{M}) \times_S X$.

Remark 12.2.6. By consider the homotheties on $\mathscr{M}$, we see that the multiplicative monoid $\mathbb{O}(S')$ acts on the S'-scheme $I_{S'}(\mathscr{M})$, which is functorial on $\mathscr{M}$, i.e. the S-scheme $I_S(\mathscr{M})$ is endowed with a structure of an $\mathbb{O}_S$-object, which is functorial on $\mathscr{M}$. We then have a morphism of S-schemes

$$\lambda : I_S(\mathscr{M}) \times_S \mathbb{O}_S \to I_S(\mathscr{M}),$$

which satisfies the evident conditions. For any S-functor X, we then obtain by base change a morphism of X-functors

$$\lambda_X : I_X(\mathscr{M}) \times_S \mathbb{O}_S \to I_X(\mathscr{M})$$

which defines an action of monoid $\mathbb{O}(X)$ on the S-functor $I_X(\mathscr{M})$: any element a of $\mathbb{O}(X) = \operatorname{Hom}_S(X, \mathbb{O}_S)$ defines an X-endomorphism a^* of $I_X(\mathscr{M})$. More precisely, if $x \in X(S')$ and $m \in I_S(\mathscr{M})(S') = I_{S'}(\mathscr{M})(S')$, then $a(x) = a \circ x$ belongs to $\mathbb{O}(S')$ and we have

$$(m, x) \cdot a = (m \cdot a(x), x).$$

This operation is functorial on $\mathscr{M}$ and preserves the zero section $\varepsilon_{\mathscr{M}} : X \to I_X(\mathscr{M})$, i.e. $a^* \circ \varepsilon_{\mathscr{M}} = \varepsilon_{\mathscr{M}}$ for any $a \in \mathbb{O}(X)$.

Even further, this operation is functorial on X in the following sense: if $\pi : Y \to X$ is a morphism of S-functors and $u : \mathbb{O}(X) \to \mathbb{O}(Y)$ is the corresponding ring homomorphism (i.e. $u(a) = a \circ \pi$ for $a \in \mathbb{O}(X)$), then the following diagram is commutative

$$\begin{tikzcd} I_Y(\mathscr{M}) \arrow[r, "u(a)^*"] \arrow[d, "\pi"'] & I_Y(\mathscr{M}) \arrow[d, "\pi"] \\ I_X(\mathscr{M}) \arrow[r, "a^*"] & I_X(\mathscr{M}) \end{tikzcd}$$

We now consider two quasi-coherent $\mathscr{O}_S$-modules $\mathscr{M}$ and $\mathscr{N}$. The commutative diagram of the direct sum

$$\begin{tikzcd} & \mathscr{M} \oplus \mathscr{N} \arrow[dl] \arrow[dr] & \\ \mathscr{M} \arrow[dr] & & \mathscr{N} \arrow[dl] \\ & 0 & \end{tikzcd}$$

then defines a commutative diagram of S-schemes

$$\begin{tikzcd} & I_S(\mathscr{M} \oplus \mathscr{N}) & \\ I_S(\mathscr{M}) \arrow[ur] & & I_S(\mathscr{N}) \arrow[ul] \\ & S \arrow[ul, "\varepsilon_{\mathscr{M}}"] \arrow[ur, "\varepsilon_{\mathscr{N}}"'] \arrow[uu, "\varepsilon_{\mathscr{M}\oplus\mathscr{N}}"'] & \end{tikzcd} \tag{12.2.4}$$

Proposition 12.2.7. *For any S-scheme X, the diagram of functors over S obtained by applying the functor $\mathcal{H}om_S(-, X)$ to (12.2.4) is Cartesian:*

$$\begin{tikzcd} \mathcal{H}om_S(I_S(\mathscr{M} \oplus \mathscr{N}), X) \arrow[r] \arrow[d] & \mathcal{H}om_S(I_S(\mathscr{N}), X) \arrow[d] \\ \mathcal{H}om_S(I_S(\mathscr{M}), X) \arrow[r] & \mathcal{H}om_S(S, X) = X \end{tikzcd}$$

Proof. It suffices to verify that for any $S' \to S$, the diagram obtained by applying the functors on S' is Cartesian. As the formation of $I_S(-)$ commutes with base change, it then suffices to prove this for $S' = S$, hence to verify that the following diagram is Cartesian:

$$\begin{tikzcd} X(I_S(\mathscr{M} \oplus \mathscr{N})) \arrow[r] \arrow[d] \arrow[dr, "X(\varepsilon_{\mathscr{M}\oplus\mathscr{N}})"] & X(I_S(\mathscr{N})) \arrow[d, "X(\varepsilon_{\mathscr{N}})"] \\ X(I_S(\mathscr{M})) \arrow[r, "X(\varepsilon_{\mathscr{M}})"] & X(S) \end{tikzcd}$$

Now if we consider $x \in X(S)$, it follows from ([?] III, 5.1) that the fiber $X(\varepsilon_{\mathscr{M}})^{-1}(x)$ of x is isomorphic to $\mathrm{Hom}_{\mathscr{O}_S}(x^*(\Omega^1_{X/S}), \mathscr{M})$. Since this latter functor clearly commutes with finite direct sums of $\mathscr{O}_S$-modules, our assertion follows. $\square$

Corollary 12.2.8. *Let X be an S-scheme and $\mathscr{M}$ be a free $\mathscr{O}_X$-module of finite type. Then $\mathcal{H}om_S(I_S(\mathscr{M}), X)$ is isomorphic to a finite product of copies of $\mathcal{H}om_S(I_S, X)$.*

Remark 12.2.9. It follows from the proof of Proposition 12.2.7 that $\mathcal{H}om_S(I_S, X)$ is isomorphic to the X-functor $\check{\Gamma}_{\Omega^1_{X/S}}$, and hence represented by the vector bundle $\mathbb{V}(\Omega^1_{X/S})$.

12.2.1.4 The tangent bundle and condition (E)

Definition 12.2.10. Let S be a scheme and $\mathscr{M}$ be a free $\mathscr{O}_S$-module of finite rank. Let X be a functor over S. The **tangent bundle of** X **over** S **relative to the** $\mathscr{O}_S$**-module** $\mathscr{M}$ is defined to be the S-functor

$$T_{X/S}(\mathscr{M}) = \mathcal{H}om_S(I_S(\mathscr{M}), X).$$

In particular, the **tangent bundle of** X **over** S is the functor

$$T_{X/S} = T_{X/S}(\mathscr{O}_S) = \mathcal{H}om_S(I_S, X).$$

The construction $\mathscr{M} \mapsto T_{X/S}(\mathscr{M})$ is then a covariant functor from the category of free $\mathscr{O}_S$-modules of finite type to the category of S-functors. In particular, the morphisms $\mathscr{M} \to 0$ and $0 \to \mathscr{M}$ define respectively an S-morphism $\pi_{\mathscr{M}} : T_{X/S}(\mathscr{M}) \to T_{X/S}(0) \cong X$ and a section $\tau : X \to T_{X/S}(\mathscr{M})$, called the **zero section**. Moreover, it follows from the preceding remarks that $\mathbb{O}(S)$ is a monoid acting on the X-functor $T_{X/S}(\mathscr{M})$, which is functorial on $\mathscr{M}$.

Remark 12.2.11. We note that the projection $\pi_{\mathscr{M}} : T_{X/S}(\mathscr{M}) \to X$ is induced by the zero section $\varepsilon_{\mathscr{M}} : S \to I_S(\mathscr{M})$, while the zero section $\tau : X \to T_{X/S}(\mathscr{M})$ is induced by the structural morphism $\rho : I_S(\mathscr{M}) \to S$. For any point $t \in T_{X/S}(\mathscr{M})(S')$ (resp. $x \in X(S')$), which corresponds to an S-morphism $f : I_{S'}(\mathscr{M}) \to X$ (resp. $g : S' \to X$), we have

$$\pi(t) = f \circ (\mathrm{id}_{S'} \times \varepsilon_{\mathscr{M}}), \quad (\text{resp. } \tau(x) = g \circ (\mathrm{id}_{S'} \times \rho)).$$

It follows from the above definition that $\mathscr{M} \mapsto T_{X/S}(\mathscr{M})$ is a covariant functor from the category of free $\mathscr{O}_X$-modules of finite rank to that of functors over X. In particular, $\mathbb{O}(S)$ is a monoid operating on the X-functor $T_{X/S}(\mathscr{M})$, which *respects the functoriality of* $\mathscr{M}$.

Remark 12.2.12. In particular, the above arguments motivates the following construction. For any S-morphism $X' \to X$, we put

$$\Sigma(X', \mathscr{M}) = \mathrm{Hom}_X(X', T_{X/S}(\mathscr{M})).$$

We have an action of the multiplicative monoid $\mathrm{End}_{\mathscr{O}_S}(\mathscr{M})$ over $\Sigma(X', \mathscr{M})$, denoted by $(\lambda, x) \mapsto \lambda * x$, such that

$$\lambda * (\mu * x) = (\lambda\mu) * x, \quad 1 * x = x, \quad 0 * x = \tau_0 * \phi \tag{12.2.5}$$

where τ_0 is the zero section $X \to T_{X/S}(\mathscr{M})$. We have similarly an action of $\mathrm{End}_{\mathscr{O}_S}(\mathscr{M} \oplus \mathscr{M})$ over $\Sigma(X', \mathscr{M} \oplus \mathscr{M})$.

Moreover, let $m : \mathscr{M} \oplus \mathscr{M} \to \mathscr{M}$ (resp. $\delta : \mathscr{M} \to \mathscr{M} \oplus \mathscr{M}$) the addition (resp. diagonal map) of $\mathscr{M}$, and put $m_{X'} : \Sigma(X', \mathscr{M} \oplus \mathscr{M}) \to \Sigma(X', \mathscr{M})$ and $\delta_{X'} : \Sigma(X', \mathscr{M}) \to \Sigma(X', \mathscr{M} \oplus \mathscr{M})$ be the induced morphisms. For $\lambda, \mu \in \mathbb{O}(S)$, let h_λ (resp. $h_{\lambda,\mu}$) be the multiplication by λ on $\mathscr{M}$ (resp. by (λ, μ) on $\mathscr{M} \oplus \mathscr{M}$). Since $m \circ h_{\lambda,\lambda} = h_\lambda \circ m$ and $m \circ h_{\lambda,\mu} = h_{\lambda+\mu}$, we have, for $z \in \Sigma(X', \mathscr{M} \oplus \mathscr{M})$ and $x \in \Sigma(X', \mathscr{M})$:

$$\lambda * m(z) = m((\lambda, \lambda) * z), \quad m((\lambda, \mu) * \delta(x)) = (\lambda + \mu) * x. \tag{12.2.6}$$

Definition 12.2.13. Let $x \in X(S) = \mathrm{Hom}_S(S, X) = \Gamma(X/S)$. We then define the tangent space of X over S at the point x relative to $\mathscr{M}$ to be the S-functor obtained from $T_{X/S}(\mathscr{M})$ by base change via the morphism $x : S \to X$:

$$\begin{tikzcd}
T_{X/S,x}(\mathscr{M}) \arrow[r] \arrow[d] & T_{X/S}(\mathscr{M}) \arrow[d, "\pi"] \\
S \arrow[r, "x"] & X
\end{tikzcd}$$

In particular, $T_{X/S,x}(\mathscr{O}_X)$ is denoted by $T_{X/S,x}$, which is called the **tangent space of X over S at the point x**.

Remark 12.2.14. It follows from Remark 12.2.11 that, for any $t : S' \to S$, $T_{X/S,x}(\mathscr{M})(S')$ is the set of S-morphisms $f : I_{S'}(\mathscr{M}) \to X$ such that $f \circ (\mathrm{id}_{S'} \times \varepsilon_{\mathscr{M}}) = x \circ t$, where $\varepsilon_{\mathscr{M}} : S \to I_S(\mathscr{M})$ is the zero section.

Proposition 12.2.15. *If X is representable, then $T_{X/S}(\mathscr{M})$ and $T_{X/S,x}(\mathscr{M})$ are representable. In particular, $T_{X/S}$ and $T_{X/S,x}$ are represented by the vector bundles $\mathbb{V}(\Omega^1_{X/S})$ and $\mathbb{V}(x^*(\Omega^1_{X/S}))$.*

Proof. It suffices to prove for $T_{X/S}(\mathscr{M})$, since the analogous result follows from base change. By Corollary 12.2.8, it suffices to consider $T_{X/S}$, which follows from Remark 12.2.9. ☐

Remark 12.2.16. By Proposition 12.2.15, we can give a simple description of the vector bundle representing $T_{X/S,x}$: if $x : S \to X$ is an S-morphism, then the image of x is locally closed in S by Corollary 8.5.8, hence defined by a quasi-coherent ideal $\mathscr{I}$ of an open subscheme of X. The quotient $\mathscr{I}/\mathscr{I}^2$ can then be considered as a quasi-coherent module over S, whose vector bundle $\mathbb{V}(\mathscr{I}/\mathscr{I}^2)$ is the desired representing scheme.

For example, let X be an algebraic scheme over a field X and x be a rational point of X over k. Let $\mathfrak{m}_x$ be the maximal ideal of the local ring $\mathscr{O}_{X,x}$, then we have $T_{X/k,x} = \mathbb{V}(\mathfrak{m}_x/\mathfrak{m}_x^2)$.

We now retun to the general situation. We first note that $T_{X/S,x}$ is a covariant functor from the category of free $\mathscr{O}_S$-modules of finite rank to that of functors over S. In particular, $\mathbb{O}_S$ is a set of perators of the functor $T_{X/S,x}(\mathscr{M})$, which respects the functoriality on $\mathscr{M}$.

Proposition 12.2.17. *The formulation of $T_{X/S}(\mathscr{M})$ and $T_{X/S,x}(\mathscr{M})$ commutes with base changes: for any S-scheme S', we have functorial isomorphisms*

$$T_{X_{S'}/S'}(\mathscr{M}\otimes\mathscr{O}_S)\xrightarrow{\sim}T_{X/S}(\mathscr{M})_{S'},$$
$$T_{X_{S'}/S',x'}(\mathscr{M}\otimes\mathscr{O}_S)\xrightarrow{\sim}T_{X/S,x}(\mathscr{M})_{S'}$$

where $x'=x_{S'}$.

Proof. This follows from the fact that $\mathcal{H}om$ commutes with base changes. □

Corollary 12.2.18. *The X-functor $T_{X/S}(\mathscr{M})$ (resp. the S-functor $T_{X/S,x}(\mathscr{M})$) is naturally endowed with an $\mathbb{O}_X$-object (resp. $\mathbb{O}_S$-object) structure, which is functorial on $\mathscr{M}$, and the isomorphism of Proposition 12.2.17 are isomorphism of $\mathbb{O}_{X_{S'}}$-objects (resp. $\mathbb{O}_{S'}$-objects).*

Proof. We first prove the case for $T_{X/S,x}(\mathscr{M})$. For any S' over S, $\mathbb{O}(S')$ acts on $\mathscr{M}\otimes\mathscr{O}_{S'}$, and hence on $T_{X_{S'}/S',x'}(\mathscr{M}\otimes\mathscr{O}_{S'})=T_{X/S,x}(\mathscr{M})_{S'}$. It is easy to verify that this operation is functorail on S', so $T_{X/S,x}(\mathscr{M})$ is endowed with an $\mathbb{O}_S$-object structure.

For $T_{X/S}(\mathscr{M})$ this is more complicated. For each X' over X, put $T_{X/S}(\mathscr{M})_{X'}=T_{X/S}(\mathscr{M})\times_X X'$; we need to endow $T_{X/S}(\mathscr{M})_{X'}(X')=\mathrm{Hom}_X(X',T_{X/S}(\mathscr{M}))$ with a structure of $\mathbb{O}(X')$-set which is functorial in X'. For this we construct the following diagram, where $X_{X'}=X\times_S X'$ and f' is the section of $X_{X'}$ over X' defined by $f:X'\to X$:

$$\begin{tikzcd}
 & T_{X_{X'}/X'}(\mathscr{M}) \arrow[dl] \arrow[dd] & \\
T_{X/S}(\mathscr{M}) \arrow[dd] & & T_{X/S}(\mathscr{M})_{X'} \arrow[ul] \arrow[ll] \arrow[dd] \\
 & X_{X'} \arrow[dl] & \\
X \arrow[dr] & & X' \arrow[ll, "f"'] \arrow[ul, "f'"'] \arrow[dl] \\
 & S &
\end{tikzcd}$$

This diagram, together with Remark 12.2.14, shows that $T_{X/S}(\mathscr{M})_{X'}(X')$ is identified with

$$T_{X_{X'}/X',f'}(\mathscr{M})(X')=\{X'\text{-morphisms }\psi:I_{X'}(\mathscr{M})\to X_{X'}\text{ such that }\psi\circ\varepsilon_{\mathscr{M}}=f'\},\tag{12.2.7}$$

over which any $a\in\mathbb{O}(X')$ operates via the action over $I_{X'}(\mathscr{M})$, i.e. with the notations of 12.2.1.3, we have $a\psi=\psi\circ a^*$, so for any $X''\to X'$ and $x\in I_{X'}(\mathscr{M})(X'')$, $(a\psi)(x)=\psi(x\cdot a)$. We then verify that this construction is functorial on X'. □

Remark 12.2.19. The operation of $\mathbb{O}_X$ over $T_{X/S}(\mathscr{M})$ can be simply defined as follows. For any $f:X'\to X$, by (12.2.7) we have[1]

$$\mathrm{Hom}_X(X',T_{X/S}(\mathscr{M}))=T_{X/S}(\mathscr{M})_{X'}(X')=\{\phi\in\mathrm{Hom}_S(I_{X'}(\mathscr{M}),X)\mid\phi\circ\varepsilon_{\mathscr{M}}=f\},$$

and we have seen in Remark 12.2.6 that $I_{X'}(\mathscr{M})$, considered as an S-functor, is endowed with an operation by the monoid $\mathbb{O}(X')$ which conserve the zero section $\varepsilon_{\mathscr{M}}:X'\to I_{X'}(\mathscr{M})$. Therefore, if we denote by a^* the endomorphism of $I_{X'}(\mathscr{M})$ defined by $a\in\mathbb{O}(X')$, then we have $a^*\phi=\phi\circ a$, which means for any $S'\to S$ and $(m,x')\in\mathrm{Hom}_S(S',I_S(\mathscr{M})\times_S X')$,

$$(a\phi)(m,x')=\phi(m\cdot a(x'),x')$$

(note that $a^*\circ\varepsilon_{\mathscr{M}}=\varepsilon_{\mathscr{M}}$, whence $(a\phi)\circ\varepsilon_{\mathscr{M}}=f$). Similarly, the operation of $\mathbb{O}_S$ over $T_{X/S,x}(\mathscr{M})$ can be described as follows. For any $t:S'\to S$, $T_{X/S,x}(\mathscr{M})(S')$ is the set of S-morphisms $\phi:I_{S'}(\mathscr{M})\to X$ such that $\phi\circ\varepsilon_{\mathscr{M}}=u\circ t$; for such a ϕ and $a\in\mathbb{O}(S')$, we have $a\phi=\phi\circ a^*$.

[1] If X' is representable, this equality can also be deduced from Remark 12.2.11 and the equivalence $\widehat{\mathbf{Sch}}_{/X}\xrightarrow{\sim}\widehat{\mathbf{Sch}_{/X}}$. In fact, the equivalence $\alpha:\widehat{\mathbf{Sch}}_{/X}\to\widehat{\mathbf{Sch}_{/X}}$ commutes with Yoneda embedding, so we have

$$\mathrm{Hom}_X(X',T_{X/S}(\mathscr{M}))\cong\mathrm{Hom}_X(X',\alpha(T_{X/S}(\mathscr{M})))=\alpha(T_{X/S}(\mathscr{M}))(X')=\{\phi\in\mathrm{Hom}_S(I_{X'}(\mathscr{M}),X):\pi_{\mathscr{M}}(\phi)=f\}.$$

and Remark 12.2.11 shows that $\pi_{\mathscr{M}}(\phi)=\phi\circ\varepsilon_{\mathscr{M}}$.

Let S be a scheme and X be an S-functor. We say that **X satisfies conditon (E) relative to S** if, for any $S' \to S$ and any free $\mathscr{O}_{S'}$-module $\mathscr{M}$ and $\mathscr{N}$ of finite rank, the diagram of sets

$$\begin{tikzcd} & X(I_{S'}(\mathscr{M} \oplus \mathscr{N})) \arrow[dl] \arrow[dr] & \\ X(I_{S'}(\mathscr{M})) \arrow[dr] & & X(I_{S'}(\mathscr{N})) \arrow[dl] \\ & X(S') & \end{tikzcd}$$

obtained by applying X to the diagram (12.2.4), is Cartesian. Equivalently, this means the functor $\mathscr{M} \mapsto T_{X/S}(\mathscr{M})$ transforms direct sums of free $\mathscr{O}_S$-modules of finite rank to products of X-functors. If this is the case, the same holds for the functor $\mathscr{M} \mapsto T_{X/S,x}(\mathscr{M}) = S \times_X T_{X/S}(\mathscr{M})$, for any $x \in \Gamma(X/S)$. By Proposition 12.2.7, we see that any representable functor satisfies condition (E).

We often say that "X/S satisfies condition (E)" to abbreviate that X satisfies condition (E) relative to S. In this case, the functor $\mathscr{M} \mapsto T_{X/S}(\mathscr{M})$ commutes with products, hence transforms groups to groups. In particular, $T_{X/S}(\mathscr{M})$ is an abelian X-group, and for the same reason $T_{X/S,x}(\mathscr{M})$ is an abelian S-group.

Proposition 12.2.20. *If X/S satisfies condition (E), the abelian group structure over $T_{X/S}(\mathscr{M})$ (resp. $T_{X/S,x}(\mathscr{M})$) and the operation of $\mathbb{O}_X$ (resp. $\mathbb{O}_S$) endow $T_{X/S}(\mathscr{M})$ (resp. $T_{X/S,x}(\mathscr{M})$) with the structure of an $\mathbb{O}_X$-module (resp. $\mathbb{O}_S$-module).*

Proof. The operation of $\mathbb{O}_X$ (resp. $\mathbb{O}_S$) is functorial on $\mathscr{M}$, so it respects the abelian group structure induced by the functoriality of $\mathscr{M}$. In fact, retain the notations of Remark 12.2.12. The structure of (abelian) X-group of $T_{X/S}(\mathscr{M})$ is deduced by the composition

$$T_{X/S}(\mathscr{M}) \times_X T_{X/S}(\mathscr{M}) \cong T_{X/S}(\mathscr{M} \oplus \mathscr{M}) \xrightarrow{m} T_{X/S}(\mathscr{M}),$$

and on the other hand the morphism

$$T_{X/S}(\mathscr{M}) \xrightarrow{\delta} T_{X/S}(\mathscr{M} \oplus \mathscr{M}) \cong T_{X/S}(\mathscr{M}) \times_X T_{X/S}(\mathscr{M})$$

is the diagonal morphism. We then deduce from the equality (12.2.6) and Remark 12.2.12 that

$$\lambda(x+y) = \lambda x + \lambda y, \quad (\lambda + \mu)x = \lambda x + \mu x,$$

for any $f : X' \to X$, $x, y \in \mathrm{Hom}_X(X', T_{X/S}(\mathscr{M}))$ and $\lambda, \mu \in \mathbb{O}(X')$. □

Remark 12.2.21. If X is representable, then it satisfies (E) and $T_{X/S}$ and $T_{X/S,x}$ are represented by vector bundles. The previous laws are the same as those which are deduced from the vector bundle structures.

Proposition 12.2.22. *If X/S satisfies condition (E), then $X_{S'}/S'$ satisfies condition (E) and the isomorphisms of Proposition 12.2.20 respects the $\mathbb{O}_{X_{S'}}$-module (resp. $\mathbb{O}_{S'}$-module) structure.*

Proof. The formulation of $I_S(\mathscr{M})$ commutes with base change, so the first assertion is immediate. The second one follows from the proof of Proposition 12.2.20. □

Proposition 12.2.23. *The functors $T_{X/S}(\mathscr{M})$ and $T_{X/S,x}(\mathscr{M})$ are functorial on X, which means if $f : X \to X'$ is an S-morphism, we have commutative diagrams*

$$\begin{tikzcd} T_{X/S}(\mathscr{M}) \arrow[r, "T(f)"] \arrow[d] & T_{X'/S}(\mathscr{M}) \arrow[d] \\ X \arrow[r] & X' \end{tikzcd} \qquad \begin{tikzcd} T_{X/S,x}(\mathscr{M}) \arrow[rr, "T_x(f)"] \arrow[dr] & & T_{X'/S,f\circ x}(\mathscr{M}) \arrow[dl] \\ & S & \end{tikzcd}$$

Moreover, if f is a monomorphism, so are $T(f)$ and $T_x(f)$.

Proof. The existence of $T(f)$ and $T_x(f)$, as well as the last assertion, follow immediately from definition. The commutativity of the diagrams then follows from the functoriality of these morphisms with respect to $\mathscr{M}$ and of the fact that $X = T_{X/S}(0)$. □

Remark 12.2.24. In the situation of Proposition 12.2.23, suppose that X and X' are representable and r is the rank of the free $\mathscr{O}_S$-module $\mathscr{M}$. Then by Corollary 12.2.8, $T_{X/S}(\mathscr{M})$ is isomorphic to the product over X of r copies of $\mathbb{V}(\Omega^1_{X/S})$, and similarly for $T_{X'/S}(\mathscr{M})$. Therefore, the square in Proposition 12.2.23 are Cartesian if f is an open immersion, of more generally if $f^*(\Omega^1_{X'/S}) = \Omega^1_{X/S}$ (for example if f is étale). In this case, we have an isomorphism of S-functors

$$T_{X/S,x}(\mathscr{M}) \xrightarrow{\sim} T_{X'/S,f\circ x}(\mathscr{M}).$$

More generally, the Cartesian square of Proposition 12.2.23 defines a morphism of X-functors

$$\begin{tikzcd} T_{X/S}(\mathscr{M}) \arrow[rr] \arrow[dr] & & T_{X'/S}(\mathscr{M}) \times_{X'} X \arrow[dl] \\ & X & \end{tikzcd}$$

Proposition 12.2.25. *Let $f : X \to X'$ be an S-morphism. If X and X' satisfy condition (E) relative to S, then*

$$T_{X/S}(\mathscr{M}) \xrightarrow{T(f)} T_{X'/S}(\mathscr{M})_X \qquad (\textit{resp.}\quad T_{X/S,x}(\mathscr{M}) \xrightarrow{T_x(f)} T_{X'/S,f\circ x}(\mathscr{M}))$$

is a morphism of $\mathbb{O}_X$-modules (resp. $\mathbb{O}_S$-modules).

Proof. This follows from Proposition 12.2.23 by the functoriality on $\mathscr{M}$. ☐

Proposition 12.2.26. *Let X and Y be functors over S. We have isomorphisms functorial on $\mathscr{M}$:*

$$T_{X/S}(\mathscr{M}) \times_S T_{Y/S}(\mathscr{M}) \xrightarrow{\sim} T_{(X\times_S Y)/S}(\mathscr{M}), \tag{12.2.8}$$
$$T_{X/S,x}(\mathscr{M}) \times_S T_{Y/S,y}(\mathscr{M}) \xrightarrow{\sim} T_{(X\times_S Y)/S,(x,y)}(\mathscr{M}), \tag{12.2.9}$$

Proof. The first isomorphism follows from (12.2.3), and the second one is deduced by base change via $(x,y) : S \to X \times_S Y$. ☐

Corollary 12.2.27. *If X/S is endowed with an algebraic structure defined by finite Cartesian products, then $T_{X/S}(\mathscr{M})$ is endowed with the same structure and the projection $T_{X/S}(\mathscr{M}) \to X$ is a morphism of that structure.*

Proposition 12.2.28. *If X/S and Y/S satisfy condition (E), then $(X \times_S Y)/S$ satisfies condition (E) and (12.2.8) (resp. (12.2.9)) is an isomorphism of $\mathbb{O}_{X\times_S Y}$-modules (resp. $\mathbb{O}_S$-modules).*

Proof. Suppose that X/S and Y/S satisfy condition (E). Then by (12.2.8), so does $(X \times_S Y)/S$. Let $(x,y) : Z \to X \times_S Y$ be an S-morphism. To see that (12.2.8) is a morphism of $\mathbb{O}_{X\times_S Y}$-modules, in view of Remark 12.2.19, it suffices to show that the map

$$\begin{aligned}\{\phi \in \mathrm{Hom}_S(I_Z(\mathscr{M}), X) : \phi \circ \varepsilon_{\mathscr{M}} = x\} \times \{\psi \in \mathrm{Hom}_S(I_Z(\mathscr{M}), Y) : \psi \circ \varepsilon_{\mathscr{M}} = y\}\\ \to \{\theta \in \mathrm{Hom}_S(I_Z(\mathscr{M}), X \times_S Y) : \theta \circ \varepsilon_{\mathscr{M}} = (X, y)\}\end{aligned}$$

which to (ϕ, ψ) associated $\phi \times \psi$, is a morphism of $\mathbb{O}(Z)$-modules. But this is immediate, since for $a \in \mathbb{O}(Z)$ we have $a \cdot (\phi, \psi) = (\phi \circ a^*, \psi \circ a^*)$, and

$$(\phi \circ a^*) \times (\psi \circ a^*) = (\phi \times \psi) \circ a^* = a \cdot (\phi \times \psi).$$

Similarly, by using Remark 12.2.14, we can show that (12.2.9) is a morphism of O_S-modules. ☐

If X is an S-group and $e : S \to X$ is the unit section, we define

$$\mathfrak{Lie}(X/S, \mathscr{M}) = T_{X/S,e}(\mathscr{M}),$$

that is, $\mathfrak{Lie}(X/S, \mathscr{M})$ is defined by the Cartesian square

$$\begin{tikzcd} \mathfrak{Lie}(X/S, \mathscr{M}) \arrow[r, "i"] \arrow[d] & T_{X/S}(\mathscr{M}) \arrow[d, "\pi"] \\ S \arrow[r, "e"] & X \end{tikzcd}$$

By Corollary 12.2.27, the projection $\pi : T_{X/S}(\mathscr{M}) \to X$ is a morphism of S-groups, and it then follows that $\mathfrak{Lie}(X/S, \mathscr{M})$ is endowed with an S-group structure, and is isomorphic via i to the kernel of π.

If, moreover, X/S satisfies condition (E), we shall see in Proposition 12.2.29 that the S-group structure of $\mathfrak{Lie}(X/S, \mathscr{M})$, induced by that of X, coincides with the abelian group structure induced by functoriality of $\mathscr{M}$. To this end we introduce the following terminology: an **H-set** is a set X endowed with a composition law with a two-sided unit, denoted by e_X or simply e. If $f : X \to Y$ is a morphism of H-sets, its kernel $\ker f$ is defined to be $f^{-1}(e_Y)$, which is a sub-H-set of X.

An H-object in a category $\mathcal{C}$ is defined by the usual manner: this is an object X of $\mathcal{C}$, endowed with a morphism $X \times X \to X$ such that there exists a section of X (over the final object) possessing the property of being a two-sided unit. Any $\mathcal{C}$-monoid, and in particular any $\mathcal{C}$-group is therefore an H-object. In particular, an H-object of the category of functors over a scheme S is called an ***S*-H-functor**. If X is an S-H-functor (for example, an S-group), and $e : S \to X$ is the unit section of X, we define

$$\mathfrak{Lie}(X/S, \mathscr{M}) = T_{X/S,e}(\mathscr{M}), \quad \mathfrak{Lie}(X/S) = \mathfrak{Lie}(X/S, \mathscr{O}_S).$$

By Corollary 12.2.27, we see that $T_{X/S}(\mathscr{M})$ and $\mathfrak{Lie}(X/S, \mathscr{M})$ are also S-H-functors, and we have morphisms of S-H-functors

$$\mathfrak{Lie}(X/S, \mathscr{M}) \xrightarrow{i} T_{X/S}(\mathscr{M}) \underset{\tau}{\overset{\pi}{\rightleftarrows}} X \tag{12.2.10}$$

where i is an isomorphism from $\mathfrak{Lie}(X/S, \mathscr{M})$ to $\ker \pi$ and τ is a section of π.

Proposition 12.2.29. *Let X be an S-H-object satisfying condition (E) relative to S. Then the S-H-object structure of $\mathfrak{Lie}(X/S, \mathscr{M})$ induced by that of X coincides with the S-group structure induced by functoriality on $\mathscr{M}$.*

Since X satisfies condition (E), we see that $\mathfrak{Lie}(X/S, \mathscr{M})$ is an H-object in the category of $\mathbb{O}_S$-modules. The proposition then follows from the following lemma:

Lemma 12.2.30. *Let $\mathcal{C}$ be a category. Let G be an H-object in the category of $\mathcal{C}$-H-objects (i.e. G is a $\mathcal{C}$-H-object endowed with a morphism of $\mathcal{C}$-H-objects $h : G \times G \to G$). Then h coincides with the composition law of G and is commutative.*

Proof. By taking the values of the functors on a variable argument, we are reduced to the case where $\mathcal{C}$ is the category of sets. We then have a set G and two maps $f, h : G \times G \to G$ such that

$$h(f(x,y), f(z,t)) = f(h(x,z), h(y,t)), \tag{12.2.11}$$

and we have two elements e, u of G such that $f(e,x) = f(X,e) = x$ and $h(u,x) = h(x,u) = x$. This is the famous Eckmann-Hilton argument[2], which we now provide a proof. We first note that by (12.2.11),

$$h(f(u,y), f(x,u)) = f(x,y) = h(f(x,u), f(u,y)). \tag{12.2.12}$$

In particular, for $y = e$ (resp. $x = e$), we obtain, respectively,

$$\begin{aligned} x = f(x,e) = h(f(u,e), f(x,u)) = h(u, f(x,u)) = f(x,u), \\ y = f(e,y) = h(f(e,u), f(u,y)) = h(u, f(u,y)) = f(u,y), \end{aligned}$$

whence the equality $h(y,x) = f(x,y) = h(x,y)$ in view of (12.2.12). This proves the lemma, whence Proposition 12.2.29. □

Remark 12.2.31. The assertion of Proposition 12.2.29 can also be interpreted as follows: if we endow $\mathfrak{Lie}(X/S, \mathscr{M})$ with the abelian group structure induced by functoriality on $\mathscr{M}$, then the morphism $i : \mathfrak{Lie}(X/S, \mathscr{M}) \to T_{X/S}(\mathscr{M})$ is a morphism of S-H-objects.

Corollary 12.2.32. *If X is an S-H-functor satisfying condition (E) relative to S, any element of $X(I_S(\mathscr{M}))$, which projects to the unit element of $X(S)$, is invertible.*

Proof. This follows from the sequence (12.2.10) and Proposition 12.2.29, since $\mathfrak{Lie}(X/S, \mathscr{M})$ is a group hence any element has an inverse. □

Corollary 12.2.33. *If X is an S-monoid satisfying condition (E) relative to S, an element of $X(I_S(\mathscr{M}))$ is invertible if and only if its image in $X(S)$ is invertible.*

[2]This argument is used to prove, for example, that higher homotopy groups are abelian.

Proof. One direction is immediate, so assume that $x \in X(I_S(\mathscr{M}))$ is an element whose projection s to $X(S)$ is invertible in $X(S)$. Let s^{-1} be the inverse of s in $X(S)$, then $y = x\tau(s^{-1}) = x\tau(s)^{-1}$ is projective to the unit element of $X(S)$, and hence is invertible in $X(I_S(\mathscr{M}))$. If y^{-1} is this inverse, we then have

$$x \cdot \tau(s)^{-1}y^{-1} = (x\tau(s)^{-1}) \cdot (x\tau(s)^{-1})^{-1} = e,$$

so x is right invertible. Similarly, by considering $y' = \tau(s^{-1})x = \tau(s)^{-1}x$, we see that x is also left invertible, so it is invertible in $X(I_S(\mathscr{M}))$. □

Corollary 12.2.34. *If X is an S-group satisfying condition (E) relative to S, the two S-group laws on $\mathfrak{Lie}(X/S, \mathscr{M})$ coincide.*

Corollary 12.2.35. *Let G be an S-group satisfying condition (E) relative to S. For $n \in \mathbb{Z}$, let $n_G : G \to G$ be the morphism of S-functors defined by $g \mapsto g^n$. Then the induced morphism $\mathfrak{Lie}(n_G) : \mathfrak{Lie}(G/S) \to \mathfrak{Lie}(G/S)$ is the multiplication by n, i.e. the map which to any $x \in \mathfrak{Lie}(G/S)(S')$ associates nx.*

Proof. We first note that n_G is in general not a morphism of groups, but it perverses the unit section $e : S \to G$, hence the induced morphism $\mathfrak{Lie}(n_G) = T_e(n_G)$ sends $\mathfrak{Lie}(G/S)$ into itself. If we denote by $i : \mathfrak{Lie}(G/S) \to T_{G/S}$ the inclusion, then $\mathfrak{Lie}(n_G)$ is defined by the equality $i(\mathfrak{Lie}(n_G)(x)) = i(x)^n$, for any $S' \to S$ and $x \in \mathfrak{Lie}(G/S)(S')$. Now by Remark 12.2.31 we have $i(x)^n = i(nx)$, whence $\mathfrak{Lie}(n_G)(x) = nx$. □

Before deducing other consequences from Proposition 12.2.29, let us prove another result of functoriality:

Proposition 12.2.36. *In the situation of 12.2.1.1, we have a functorial isomorphism on $\mathscr{M}$:*

$$T_{\mathcal{H}om_{Z/S}(X,Y)/S}(\mathscr{M}) \xrightarrow{\sim} \mathcal{H}om_{Z/S}(X, T_{Y/Z}(\mathscr{M})).$$

Proof. In fact, by definition we have

$$T_{\mathcal{H}om_{Z/S}(X,Y)/S}(\mathscr{M}) = \mathcal{H}om_S(I_S(\mathscr{M}), \mathcal{H}om_{Z/S}(X,Y)) \cong \mathcal{H}om_{Z/S}(X, \mathcal{H}om_Z(Z \times_S I_S(\mathscr{M}), Y)),$$

where we have used the isomorphism (12.2.1) with $T = I_S(\mathscr{M})$. In view of the isomorphism $Z \times_S I_S(\mathscr{M}) \cong I_Z(\mathscr{M})$, we then obtain

$$T_{\mathcal{H}om_{Z/S}(X,Y)/S}(\mathscr{M}) \cong \mathcal{H}om_{Z/S}(X, \mathcal{H}om_Z(I_Z(\mathscr{M}), Y)) = \mathcal{H}om_{Z/S}(X, T_{Y/Z}(\mathscr{M})).$$ □

Corollary 12.2.37. *If Y/Z satisfies condition (E), then $\mathcal{H}om_{Z/S}(X,Y)/S$ satisfies condition (E) and the isomorphism of Proposition 12.2.36 respects the $\mathbb{O}$-module structure over $\mathcal{H}om_{Z/S}(X,Y)$.*

Proof. Let $\mathscr{M}, \mathscr{N}$ be two free $\mathscr{O}_S$-modules of finite rank. If Y/Z satisfies condition (E), then

$$T_{Y/Z}(\mathscr{M} \oplus \mathscr{N}) \cong T_{Y/Z}(\mathscr{M}) \times_Y T_{Y/Z}(\mathscr{N}).$$

The right side is a sub-functor of $T_{Y/Z}(\mathscr{M}) \times_S T_{Y/Z}(\mathscr{N})$ and via the isomorphism (12.2.3), we obtain an isomorphism

$$\mathcal{H}om_{Z/S}(X, T_{Y/Z}(\mathscr{M} \oplus \mathscr{N})) \cong \mathcal{H}om_{Z/S}(X, T_{Y/Z}(\mathscr{M})) \times_{\mathcal{H}om_{Z/S}(X,Y)} \mathcal{H}om_{Z/S}(X, T_{Y/Z}(\mathscr{N})).$$

Combined with Proposition 12.2.36, this implies

$$T_{\mathcal{H}om_{Z/S}(X,Y)/S}(\mathscr{M} \oplus \mathscr{N}) \cong T_{\mathcal{H}om_{Z/S}(X,Y)/S}(\mathscr{M}) \times_{\mathcal{H}om_{Z/S}(X,Y)} T_{\mathcal{H}om_{Z/S}(X,Y)/S}(\mathscr{N}),$$

so $\mathcal{H}om_{Z/S}(X,Y)$ satisfies condition (E).

For the second assertion, let $H = \mathcal{H}om_{Z/S}(X,Y)$ and consider an S-morphism $\Delta : H' \to \mathcal{H}om_{Z/S}(X,Y)$, that is, an Z-morphism $\delta : H' \times_S X \to Y$, which makes $H' \times_S X$ a Y-object. We then have a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_H(H', \mathcal{H}om_{Z/S}(X, T_{Y/Z}(\mathscr{M}))) \arrow[r, hook] \arrow[d, equal] & \mathrm{Hom}_S(H', \mathcal{H}om_{Z/S}(X, T_{Y/Z}(\mathscr{M}))) \arrow[d, equal] \\
\mathrm{Hom}_Y(H' \times_S X, T_{Y/Z}(\mathscr{M})) \arrow[r, hook] \arrow[d, equal] & \mathrm{Hom}_Z(H' \times_S X, T_{Y/Z}(\mathscr{M})) \arrow[d, equal] \\
\{\psi \in \mathrm{Hom}_Z(I_{H' \times_S X}(\mathscr{M}), Y) : \psi \circ \varepsilon_{\mathscr{M}} = \delta\} \arrow[r, hook] & \mathrm{Hom}_Z(I_{H' \times_S X}(\mathscr{M}), Y).
\end{tikzcd}$$

By Remark 12.2.3, the action of $\alpha \in \mathbb{O}(H' \times_S X)$ over $\Psi \in \mathrm{Hom}_Y(H' \times_S X, T_{Y/Z}(\mathscr{M}))$ is given as follows: for any $U \to S$ and $(h,x) \in \mathrm{Hom}_S(U, H' \times_S X)$ (U is then an Y-object via $\delta \circ (h,x)$), we have

$$(\alpha\Psi)(h,x) = \alpha(h,x)\Psi(h,x),$$

where $\alpha(h,x) \in \mathbb{O}(U)$ acts on $\Psi(h,x) \in T_{Y/Z}(\mathscr{M})(U)$ via the $\mathbb{O}_Y$-module structure of $T_{Y/Z}(\mathscr{M})$. By Remark 12.2.19, the latter is given, via the identification

$$\mathrm{Hom}_Y(H' \times_S X, T_{Y/Z}(\mathscr{M})) = \{\psi \in \mathrm{Hom}_Z(I_{H' \times_S X}(\mathscr{M}), Y) : \psi \circ \varepsilon_{\mathscr{M}} = \delta\},$$

by the following: for any $(m,h,x) \in \mathrm{Hom}_S(U, I_S(\mathscr{M}) \times_S H' \times_S X)$,

$$(\alpha\psi)(m,h,x) = \psi(m \cdot \alpha(h,x), h, x). \tag{12.2.13}$$

On the other hand, consider the tangent space $T_{H/S}(\mathscr{M}) = \mathcal{H}om_S(I_S(\mathscr{M}), H)$; we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_H(H', T_{H/S}(\mathscr{M})) \arrow[r, hook] \arrow[d, equal] & \mathrm{Hom}_S(H', T_{H/S}(\mathscr{M})) \arrow[d, equal] \\
\{\Phi \in \mathrm{Hom}_S(I_{H'}(\mathscr{M}), H) : \Phi \circ \varepsilon_{\mathscr{M}} = \Delta\} \arrow[r, hook] \arrow[d, equal, "(*)"] & \mathrm{Hom}_S(I_{H'}(\mathscr{M}), H) \arrow[d, equal] \\
\{\phi \in \mathrm{Hom}_Z(I_{H' \times_S X}(\mathscr{M}), Y) : \phi \circ \varepsilon_{\mathscr{M}} = \delta\} \arrow[r, hook] & \mathrm{Hom}_Z(I_{H' \times_S X}(\mathscr{M}), Y)
\end{tikzcd}$$

where the bijection $(*)$ is given as follows: for any $U \to S$ and $(m,h,x) \in \mathrm{Hom}(U, I_S(\mathscr{M}) \times_S H' \times_S X)$ (so that U is over Z via $U \xrightarrow{x} X \to Z$), we have $\Phi(m,h) \in \mathrm{Hom}_Z(X \times_S U, Y)$ and

$$\phi(m,h,x) = \Phi(m,h) \circ (x \times \mathrm{id}_U) \in \mathrm{Hom}_Z(U, Y). \tag{12.2.14}$$

By Remark 12.2.19 (where we replace X by $\mathcal{H}om_{Z/S}(X,Y)$ and X' by H'), the action of $a \in \mathbb{O}(H')$ over $\Phi \in \mathrm{Hom}_S(I_{H'}(\mathscr{M}), H)$ is given by

$$(a\Phi)(m,h) = \Phi(m \cdot a(h), h)$$

where $U \to S$ and $(m,h) \in \mathrm{Hom}_S(U, I_S(\mathscr{M}) \times_S H')$. Therefore, if ϕ (resp. $a\phi$) is the element of $\mathrm{Hom}_Z(I_{H' \times_S X}(\mathscr{M}), Y)$ associated with Φ (resp $a\Phi$), we have, by (12.2.14),

$$(a\Phi)(m,h,x) = \Phi(m \cdot a(h), h) \circ (x \times \mathrm{id}_U) = \phi(m \cdot a(h), h, x). \tag{12.2.15}$$

Together with (12.2.13), this shows that the isomorphism $T_{H/S}(\mathscr{M}) \xrightarrow{\sim} \mathcal{H}om_{Z/S}(X, T_{Y/Z}(\mathscr{M}))$ of Proposition 12.2.36 is an isomorphism of $\mathbb{O}(H)$-modules. Moreover, for any $H' \to H$, the $\mathbb{O}(H')$-module structure of $\mathrm{Hom}_H(H', T_{H/S}(\mathscr{M}))$ extends, in a functorial way on H', to an $\mathbb{O}(H' \times_S X)$-module structure. □

In particular, for $Z = S$, we obtain the following corollary:

Corollary 12.2.38. *We have a functorial isomorphism on $\mathscr{M}$:*

$$T_{\mathcal{H}om_S(X,Y)/S}(\mathscr{M}) \xrightarrow{\sim} \mathcal{H}om_S(X, T_{Y/S}(\mathscr{M})).$$

Moreover, if Y/S satisfies condition (E), then $\mathcal{H}om_S(X,Y)/S$ satisfies condition (E) and the preceding isomorphism respects the $\mathbb{O}$-module structure over $\mathcal{H}om_S(X,Y)$.

Let $u : X \to Y$ be an S-morphism, which can be identified with a constant morphism $\boldsymbol{u} : S \to \mathcal{H}om_S(X,Y)$ such that $\boldsymbol{u}(f) = u_{S'}$ for any $f : S' \to S$. The fiber product of $\boldsymbol{u}$ and $\mathcal{H}om_S(X, T_{Y/S}(\mathscr{M})) \to \mathcal{H}om_S(X,Y)$ is then identified with $\mathcal{H}om_{Y/S}(X, T_{Y/S}(\mathscr{M}))$, where X is over Y via u. Therefore, we deduce from the definition of $T_{\mathcal{H}om_S(X,Y)/S,\boldsymbol{u}}(\mathscr{M})$ and Corollary 12.2.38 the following:

Corollary 12.2.39. *Let $u : X \to Y$ be an S-morphism. We have a functorial isomorphism on $\mathscr{M}$ (where X is over Y via u):*

$$T_{\mathcal{H}om_S(X,Y)/S,\boldsymbol{u}}(\mathscr{M}) \xrightarrow{\sim} \mathcal{H}om_{Y/S}(X, T_{Y/S}(\mathscr{M})).$$

This is an isomorphism of $\mathbb{O}_S$-modules if Y/S satisfies condition (E).

In particular, for $Y = X$, $\mathcal{E}nd_S(X)$ is an S-functor in monoids, hence a fortiori an S-H-functor. Since $\mathfrak{Lie}(\mathcal{E}nd_S(X)/S, \mathscr{M})$ is by definition $T_{\mathcal{E}nd_S(X)/S,e}(\mathscr{M})$, where e is the unit section, we obtain (recall that $\mathcal{H}om_{X/S}(X, T_{X/S}(\mathscr{M})) \cong \mathrm{Res}_{X/S}T_{X/S}(\mathscr{M})$):

Corollary 12.2.40. *We have a functorial isomorphism on $\mathscr{M}$:*

$$\mathfrak{Lie}(\mathcal{E}nd_S(X)/S, \mathscr{M}) \xrightarrow{\sim} \mathrm{Res}_{X/S}T_{X/S}(\mathscr{M}).$$

This is an isomorphism of $\mathbb{O}_S$-modules if X/S satisfies condition (E).

Remark 12.2.41. Suppose that X/S satisfies condition (E). Then the functor

$$\mathrm{Res}_{X/S}T_{X/S}(\mathscr{M}) = \mathcal{H}om_{X/S}(X, T_{X/S}(\mathscr{M}))$$

is endowed with a $\mathrm{Res}_{X/S}\mathbb{O}_X$-module structure, i.e. for any $S' \to S$,

$$\mathcal{H}om_{X/S}(X, T_{X/S}(\mathscr{M}))(S') = \{\psi \in \mathrm{Hom}_X(I_{S'}(\mathscr{M}) \times_S X, X) : \psi \circ (\varepsilon_{\mathscr{M}} \times \mathrm{id}_X) = \mathrm{pr}_X\}$$

is endowed with a $\mathbb{O}(X \times_S S')$-module structure, which is functorial on S'. This follows either from Proposition 12.2.20 and the properties of the functor $\mathrm{Res}_{X/S}$, or from the proof of Corollary 12.2.37.

We now give a geometric interpretation of the tangent bundle. Let U be an S-functor; by (**??**), we have isomorphism functorial on $\mathscr{M}$:

$$\begin{aligned} T_{X/S}(\mathscr{M})(U) &= \mathrm{Hom}_S(U, \mathcal{H}om_S(I_S(\mathscr{M}), X)) \cong \mathrm{Hom}_S(I_S(\mathscr{M}), \mathcal{H}om_S(U, X)) \\ &= \mathrm{Hom}_{I_S(\mathscr{M})}(U_{I_S(\mathscr{M})}, X_{I_S(\mathscr{M})}). \end{aligned}$$

In particular, the morphism $\mathscr{M} \to 0$ induces a commutative diagram

$$\begin{tikzcd} \mathrm{Hom}_S(U, T_{X/S}(\mathscr{M})) \arrow[r, "\sim"] \arrow[d, "\circ\pi_{\mathscr{M}}"] & \mathrm{Hom}_{I_S(\mathscr{M})}(U_{I_S(\mathscr{M})}, X_{I_S(\mathscr{M})}) \arrow[d] \\ \mathrm{Hom}_S(U, X) \arrow[r, equal] & \mathrm{Hom}_S(U, X) \end{tikzcd}$$

where the second vertical arrow is given by base change $\varepsilon_{\mathscr{M}} : S \to I_S(\mathscr{M})$. We therefore obtain the following proposition:

Proposition 12.2.42. *Let $h_0 : U \to X$ be an S-morphism. Then $\mathrm{Hom}_X(U, T_{X/S}(\mathscr{M}))$ is identified with the set of $I_S(\mathscr{M})$-morphisms $h : U_{I_S(\mathscr{M})} \to X_{I_S(\mathscr{M})}$ that extend h_0 (we view U (resp. X) as a sub-object of $U \times_S I_S(\mathscr{M})$ (resp. $X \times_S I_S(\mathscr{M})$) via $\mathrm{id}_U \times_S \varepsilon_{\mathscr{M}}$ (resp. $\mathrm{id}_X \times_S \varepsilon_{\mathscr{M}}$)).*

In particular, for $U = X$ and $h_0 = \mathrm{id}_X$, we obtain:

Corollary 12.2.43. *The set $\Gamma(T_{X/S}(\mathscr{M})/X)$ is identified with the set of $I_S(\mathscr{M})$-endomorphisms ϕ of $X_{I_S(\mathscr{M})}$ which induce identity on X, i.e. such that the following diagram is commutative:*

$$\begin{tikzcd} I_X(\mathscr{M}) \arrow[rr, "\phi"] & & I_X(\mathscr{M}) \\ & X \arrow[ul, "\varepsilon_{\mathscr{M}}"] \arrow[ur, "\varepsilon_{\mathscr{M}}"'] & \end{tikzcd}$$

On the other hand, by Corollary 12.2.39, $\Gamma(T_{X/S}(\mathscr{M})/X) \cong \mathfrak{Lie}(\mathcal{E}nd_S(X)/S, \mathscr{M})(S)$. If X/S satisfies condition (E), then $\mathcal{E}nd_S(X)/S$ satisfies condition (E) and $\mathfrak{Lie}(\mathcal{E}nd_S(X)/S, \mathscr{M})$ is then an $\mathbb{O}_S$-module (and in fact a $\mathrm{Res}_{X/S}\mathbb{O}_X$-module). Applying Proposition 12.2.29, we then deduce that

Proposition 12.2.44. *If X/S satisfies condition (E), the abelian group $\Gamma(T_{X/S}(\mathscr{M})/X)$ is identified with the set of $I_S(\mathscr{M})$-endomorphisms of $X_{I_S}(\mathscr{M})$ which induce identity on X. In particular, any $I_S(\mathscr{M})$-endomorphism of $X_{I_S}(\mathscr{M})$ which induces the identity on X is an automorphism.*

Corollary 12.2.45. *Let $u : X \to Y$ be an S-isomorphism with Y/S satisfying condition (E). Any $I_S(\mathscr{M})$-morphism of $X_{I_S(\mathscr{M})}$ to $Y_{I_S(\mathscr{M})}$ which extends u is an isomorphism.*

Proof. By Proposition 12.2.42 the considered set is identified with $\mathrm{Hom}_Y(X, T_{Y/S}(\mathscr{M}))$, which is isomorphic to $\Gamma(T_{Y/S}(\mathscr{M})/Y)$ by our hypothesis. $\square$

Corollary 12.2.46. *If Y/S satisfies condition (E), the monomorphism $\mathcal{I}so_S(X,Y) \to \mathcal{H}om_S(X,Y)$ induces, for any $u \in \mathrm{Iso}_S(X,Y)$, an isomorphism*

$$T_{\mathcal{I}so_S(X,Y)/S,u}(\mathscr{M}) \xrightarrow{\sim} T_{\mathcal{H}om_S(X,Y)/S,u}(\mathscr{M}).$$

Proof. It suffices to see that $T_{\mathcal{I}so_S(X,Y)/S,u}(\mathscr{M}) \xrightarrow{\sim} T_{\mathcal{H}om_S(X,Y)/S,u}(\mathscr{M})$ is a bijection, for any $S' \to S$. By base change (cf. Proposition 12.2.26), it suffices to consider $S' = S$. In this case, we note that $T_{\mathcal{H}om_S(X,Y)/S,u}(\mathscr{M})(S)$ (resp. $T_{\mathcal{I}so_S(X,Y)/S,u}(\mathscr{M})(S)$) is the set of $I_S(\mathscr{M})$-morphisms (resp. automorphims) $X_{I_S(\mathscr{M})} \to Y_{I_S(\mathscr{M})}$ which extends u, and we can apply Corollary 12.2.45. $\square$

Corollary 12.2.47. *If X/S satisfies (E), the monomorphism $\mathcal{A}ut_S(X) \to \mathcal{E}nd_S(X)$ induces, for any $u \in \mathcal{A}ut_S(X)$, an isomorphism $T_{\mathcal{A}ut_S(X)/S,u}(\mathscr{M}) \xrightarrow{\sim} T_{\mathcal{E}nd_S(X)/S,u}(\mathscr{M})$. In particular, we have*

$$\mathfrak{Lie}(\mathcal{A}ut_S(X)/S, \mathscr{M}) \xrightarrow{\sim} \mathfrak{Lie}(\mathcal{E}nd_S(X)/S, \mathscr{M}) \xrightarrow{\sim} \mathrm{Res}_{X/S} T_{X/S}(\mathscr{M})$$

so that $\mathfrak{Lie}(\mathcal{A}ut_S(X)/S, \mathscr{M})$ is endowed with a $\mathrm{Res}_{X/S}\mathbb{O}_X$-module structure.

Example 12.2.48. There exist functors possessing infinitesimal endomorphisms which are not automorphisms, and hence a fortiori do not satisfy condition (E). For any pointed set (E, x_0), let $M(E)$ be the free commutative monoid generated by E and $M_P(E, x_0)$ be the commutative monoid obtained by quotient $M(E)$ by the equivalence relation generated by $m \sim x_0 + m$. Then $(E, x_0) \to M_P(E, x_0)$ is the left adjoint of the forgetful functor from the category of commutative monoid to that of pointed sets. We say that $M_P(E, x_0)$ is the **free commutative monoid over the pointed set** (E, x_0).

Let X be the functor which associates any scheme S to the free commutative monoid over the set $\mathbb{O}(S)$, pointed by the zero element. A morphism $f : S \to I_{\mathbb{Z}} = \mathrm{Spec}(\mathbb{Z}[t])$ corresponds to a square zero element u_f of $\mathbb{O}(S)$, hence defines an endomorphism of $X(S)$ by $x \mapsto x + u_f$ (taken in $M_P(\mathbb{O}(S), 0)$). We thus obtain an endomorphism ϕ of $X_{I_{\mathbb{Z}}} = X \times_{\mathbb{Z}} I_{\mathbb{Z}}$, defined as follows. For any $f \in I_{\mathbb{Z}}(S)$ and $x \in X(S)$,

$$\phi(x, f) = (x + u_f, f).$$

If $f_0 : S \to I_{\mathbb{Z}}$ is the composition of the structural morphism $S \to \mathrm{Spec}(\mathbb{Z})$ and the zero section of $I_{\mathbb{Z}}$, the corresponding element $u_{f_0} = 0$, and hence $\phi(x, f_0) = (x, f_0)$ (since $x + 0 = x$ in $M_P(\mathbb{O}(S), 0)$). Since the map $X(S) \to X_{I_{\mathbb{Z}}}(S)$ is given by $x \mapsto (x, f_0)$, this shows that ϕ induces the identity on X, hence is an infinitesimal endomorphism of X which is evidently not an automorphism.

Suppose that X is representable. In this case, we have seen in Proposition 12.2.15 that the X-functor $T_{X/S}$ is represented by $\mathbb{V}(\Omega^1_{X/S})$, whence the bijections

$$\Gamma(T_{X/S}/X) \cong \mathrm{Hom}_X(\Omega^1_{X/S}, \mathscr{O}_S) \cong \mathrm{Der}_{\mathscr{O}_S}(\mathscr{O}_X). \tag{12.2.16}$$

This can also be deduced as follows. According to Proposition 12.2.44, $\Gamma(T_{X/S}/X)$ is identified with the set of **infinitesimal endomorphisms** of X (i.e. I_S-endomorphisms of X_{I_S} inducing the identity on X). Now X and X_{I_S} have the same underlying topological space, with structural sheaves being $\mathscr{O}_X$ and $\mathscr{D}_{\mathscr{O}_X} = \mathscr{O}_X \oplus \mathscr{M}$, where $\mathscr{M} = \mathscr{O}_X$ is considered as a square zero ideal. Let $\pi : \mathscr{D}_{\mathscr{O}_X} \to \mathscr{O}_X$ be the morphism of $\mathscr{O}_X$-algebras which is zero on $\mathscr{M}$, we then deduce that giving an infinitesimal endomorphism of X is equivalent to giving a morphism of $\mathscr{O}_S$-algebras $\phi : \mathscr{O}_X \to \mathscr{D}_{\mathscr{O}_X}$ such that $\pi \circ \phi = \mathrm{id}_{\mathscr{O}_X}$, which then amouts to giving an $\mathscr{O}_S$-derivation of the sheaf of rings $\mathscr{O}_X$.

Moreover, we see that if $D, D' \in \mathrm{Der}_{\mathscr{O}_S}(\mathscr{O}_X)$ and if we denote by ϕ_D the infinitesimal endomorphism correponding to D, then

$$\phi_{D+D'} = \phi_D \circ \phi_{D'}.$$

This shows that the identification

$$\{\text{infinitesimal endomorphisms of } X\} \cong \mathrm{Der}_{\mathscr{O}_S}(\mathscr{O}_X)$$

is an isomorphism of abelian groups. In view of Proposition 12.2.44 (and Remark 12.2.41), we have then isomorphism of abelian groups (as well as $\mathbb{O}(X)$-modules)

$$\Gamma(T_{X/S}/X) \xrightarrow{\sim} \mathrm{Der}_{\mathscr{O}_S}(\mathscr{O}_X)$$

which ressume the classical interpretation of tangent vectors in view of derivations of the structural sheaf. Recall also that $\Gamma(T_{X/S}/X)$ is equal to $H^0(X, \mathfrak{g}_{X/S})$, where $\mathfrak{g}_{X/S}$ is the dual of $\Omega^1_{X/S}$.

12.2.2 Tangent space of a group

Let G be a functor in groups over S. By Corollary 12.2.27, $T_{G/S}(\mathscr{M})$ and $\mathfrak{Lie}(G/S,\mathscr{M})$ are endowed with group structures over S and we have group morphisms

$$\mathfrak{Lie}(G/S,\mathscr{M}) \xrightarrow{i} T_{G/S}(\mathscr{M}) \underset{\tau}{\overset{\pi}{\rightleftarrows}} G \tag{12.2.17}$$

By definition i is an isomorphism from $\mathfrak{Lie}(G/S)(\mathscr{M})$ onto the kernel of π, and τ is a section of π. It then follows from Proposition 12.1.10 that we can identify $T_{G/S}(\mathscr{M})$ with a semi-direct product of G by $\mathfrak{Lie}(G/S,\mathscr{M})$.

Definition 12.2.49. The corresponding operation of G on $\mathfrak{Lie}(G/S,\mathscr{M})$ is denoted by

$$\mathrm{Ad}: G \to \mathcal{A}ut_{\mathbf{Grp}}(\mathfrak{Lie}(G/S,\mathscr{M}))$$

and called the adjoint representation (relative to $\mathscr{M}$) of G. For any $S' \to S$, we then have by definition, for $x \in G(S')$ and $X \in \mathfrak{Lie}(G/S,\mathscr{M})(S')$, that

$$\mathrm{Ad}(x)X = i^{-1}(\tau(x)i(x)\tau(x)^{-1}).$$

Definition 12.2.50. If G and H are two functors in groups over S and if $f: G \to H$ is a group morphism, then we have an induced morphism of exact sequences which is compatible with sections:

$$\begin{tikzcd}
1 \arrow[r] & \mathfrak{Lie}(G/S,\mathscr{M}) \arrow[r] \arrow[d, "\mathfrak{Lie}(f)"] & T_{G/S}(\mathscr{M}) \arrow[r] \arrow[d, "T(f)"] & G \arrow[r] \arrow[d, "f"] & 1 \\
1 \arrow[r] & \mathfrak{Lie}(H/S,\mathscr{M}) \arrow[r] & T_{H/S}(\mathscr{M}) \arrow[r] & H \arrow[r] & 1
\end{tikzcd}$$

The morphism $\mathfrak{Lie}(f) = T_e(f)$ is the derived morphism of f. If G/S and H/S satisfy condition (E), then $\mathfrak{Lie}(f)$ respects the $\mathbb{O}_S$-module structure induced by functoriality on $\mathscr{M}$ (cf. Proposition 12.2.25).

Proposition 12.2.51. *Let $g \in G(S)$, then $\mathrm{Ad}(g): \mathfrak{Lie}(G/S,\mathscr{M}) \to \mathfrak{Lie}(G/S,\mathscr{M})$ is the derived morphism of $\mathrm{Inn}(g): G \to G$.*

Proof. In fact, $\mathrm{Ad}(g)X = i^{-1}(\mathrm{Inn}(g)i(X))$, which is none other than $T(\mathrm{Inn}(g))X$ by the definition of the derived morphism. □

Suppose that G/S satisfies condition (E). Then, by Proposition 12.2.29, the group structure of $\mathfrak{Lie}(G/S,\mathscr{M})$ defined from G coincides with that induced by the $\mathbb{O}_S$-module structure of $\mathscr{M}$. We then deduce from the preceding proposition and the functoriality of the operation of $\mathbb{O}_S$ (Proposition 12.2.25) that:

Corollary 12.2.52. *Suppose that G/S satisfies condition (E). Then Ad sends G into the subgroup $\mathcal{A}ut_{\mathbb{O}_S}(\mathfrak{Lie}(G/S,\mathscr{M}))$ of $\mathcal{A}ut_{\mathbf{Grp}}(\mathfrak{Lie}(G/S),\mathscr{M})$, that is, for any $g \in G(S')$, $\mathrm{Ad}(g)$ respects the $\mathbb{O}(S')$-module structure of $\mathfrak{Lie}(G_{S'}/S',\mathscr{M})$. In other words, Ad is a linear representation of G on the $\mathbb{O}_S$-module $\mathfrak{Lie}(G/S,\mathscr{M})$.*

Remark 12.2.53. Suppose that G/S satisfies condition (E). Then the derived morphism of the group law $m: G \times_S G \to G$ is none other than the addition law of $\mathfrak{Lie}(G/S,\mathscr{M})$ (m is not a morphism of groups, but $m(e,e) = e$, so the derived morphism $\mathfrak{Lie}(m)$ sends $T_{(G\times_S G)/S,(e,e)}(\mathscr{M}) = \mathfrak{Lie}(G/S,\mathscr{M}) \times_S \mathfrak{Lie}(G/S,\mathscr{M})$ into $\mathfrak{Lie}(G/S,\mathscr{M})$). For any $n \in \mathbb{Z}$, we show similarly that if $n_G: G \to G$ is the morphism of S-functors defined by $g \mapsto g^n$, then the derived morphism $\mathfrak{Lie}(n_G)$ is the multiplication by n on $\mathfrak{Lie}(G/S)$, cf. Corollary 12.2.35.

Now consider the S-functor $\mathcal{H}om_{G/S}(G, T_{G/S}(\mathscr{M}))$; for any $S' \to S$, we have $T_{G/S}(\mathscr{M})_{S'} \cong T_{G_{S'}/S'}(\mathscr{M})$ and hence

$$\mathcal{H}om_{G/S}(G, T_{G/S}(\mathscr{M}))(S') \cong \mathrm{Hom}_{G_{S'}}(G_{S'}, T_{G_{S'}/S'}(\mathscr{M})) = \Gamma(T_{G_{S'}/S'}(\mathscr{M})/G_{S'}).$$

Note that we have an isomorphism, functorial on S',

$$\mathrm{Hom}_{S'}(G_{S'}, \mathfrak{Lie}(G_{S'}/S',\mathscr{M})) \xrightarrow{\sim} \Gamma(T_{G_{S'}/S'}(\mathscr{M})/G_{S'}) \tag{12.2.18}$$

which to any $f : G_{S'} \to \mathfrak{Lie}(G_{S'}/S', \mathscr{M})$ associates the section $s_f : G_{S'} \to T_{G_{S'}/S'}(\mathscr{M})$ such that, for any $S'' \to S'$ and $g \in G(S'')$,

$$s_f(g) = i(f(g))\tau(g).$$

Let h be an automorphism of the functor $G_{S'}$ over S' (not necessarily respects the group structure). To any section s of $T_{G_{S'}/S'}(\mathscr{M})$, we can associate $h(s)$ defined by transport the structure: this for example the only section of $T_{G_{S'}/S'}(\mathscr{M})$ fitting into the commutative diagram

$$\begin{tikzcd} G_{S'} \arrow[r, "s"] \arrow[d, "h"'] & T_{G_{S'}/S'}(\mathscr{M}) \arrow[d, "T(h)"] \\ G_{S'} \arrow[r, "h(s)"] & T_{G_{S'}/S'}(\mathscr{M}) \end{tikzcd}$$

In particular, we can take h to be the right translation t_x by an element x of $G(S')$, that is, $h(g) = t_x(g) = g \cdot x$, for any $g \in G(S'')$, $S'' \to S'$. We have immediately

$$t_x(s_f) = s_{t_x(f)},$$

where $t_x(f) : G_{S'} \to \mathfrak{Lie}(G_{S'}/S', \mathscr{M})$ is defined by

$$t_x(f)(g) = f(g \cdot x^{-1})$$

for any $g \in G(S'')$, $S'' \to S'$. It follows that if we operate G on $\mathcal{H}om_{G/S}(G, T_{G/S}(\mathscr{M}))$ and $\mathcal{H}om_S(G, \mathfrak{Lie}(G/S, \mathscr{M}))$ by right translation in the following way: for any $S' \to S$, $x \in G(S')$, $\sigma \in \Gamma(T_{G_{S'}/S'}(\mathscr{M}/G_{S'}))$ and $f \in \mathrm{Hom}_{S'}(G_{S'}, \mathfrak{Lie}(G_{S'}/S', \mathscr{M}))$,

$$(\sigma \cdot x)(g) = \sigma(g \cdot x^{-1}) \cdot \tau(x), \quad (f \cdot x)(g) = f(g \cdot x^{-1}),$$

for any $g \in G(S'')$, $S'' \to S'$, then the isomorphism (12.2.18) respects the action of G.

In particular, by this isomorphism, the elements of $\mathcal{H}om_{G/S}(G, T_{G/S}(\mathscr{M}))^G(S')$ (called **right invariant sections** of $T_{G_{S'}/S'}(\mathscr{M})$) corresponds to constant morphisms of $G_{S'}$ into $\mathfrak{Lie}(G_{S'}/S', \mathscr{M})$ (i.e. which factors through the projection $G_{S'} \to S'$), or to elements of $\mathfrak{Lie}(G_{S'}/S', \mathscr{M})(S') = \mathfrak{Lie}(G/S, \mathscr{M})(S')$. We then have the following proposition:

Proposition 12.2.54. *The map $\mathfrak{Lie}(G/S, \mathscr{M})(S) \to \Gamma(T_{G/S}(\mathscr{M})/G)$ which associates an element $X \in \mathfrak{Lie}(G/S, \mathscr{M})(S)$ the section $x \mapsto X(\pi(x))$ is a bijection from $\mathfrak{Lie}(G/S, \mathscr{M})(S)$ onto the set of right invariant sections of $\Gamma(T_{G/S}(\mathscr{M})/G)$.*

Similarly, we can act G on $\mathcal{E}nd_{I_S(\mathscr{M})/S}(G_{I_S(\mathscr{M})})$ as follows: for any $S' \to S$, $x \in G(S')$ and $u \in \mathcal{E}nd_{I_S(\mathscr{M})/S}(G_{I_S(\mathscr{M})})(S') = \mathrm{End}_{I_{S'}}(G_{I_{S'}(\mathscr{M})})$,

$$(u \cdot x)(g) = u(g \cdot x^{-1}) \cdot x,$$

for any $g \in G(S'')$, $S'' \to I_{S'}(\mathscr{M})$. Then the morphism of Corollary 12.2.43

$$\mathcal{H}om_{G/S}(G, T_{G/S}(\mathscr{M})) \to \mathcal{E}nd_{I_S(\mathscr{M})/S}(G_{I_S(\mathscr{M})})$$

respects the operation of G and induces for any $S' \to S$ a bijection from $\Gamma(T_{G_{S'}/S'}(\mathscr{M})/G_{S'})$ and the set of $I_{S'}(\mathscr{M})$-endomorphisms u of $G_{I_{S'}(\mathscr{M})}$ inducing the identity on G and are invariant under right translations, i.e. satisfies $u_{S''} \cdot x = u_{S''}$ for any $S'' \to S'$ and $x \in G(S'')$. By Proposition 12.2.44, we then conclude the following theorem:

Proposition 12.2.55. *There exists a bijection (functorail on G) from the set $\mathfrak{Lie}(G/S, \mathscr{M})(S)$ to the set of $I_S(\mathscr{M})$-endomorphisms of $G_{I_S(\mathscr{M})}$ inducing the identity on G and commutes with right translations of G, and this is a group isomorphism if G/S satisfies condition (E).*

By considering the case $\mathscr{M} = \mathscr{O}_S$, we thus obtain the classical definitions of the Lie algebra of a group.

Before going further, let us establish some new corollaries of Proposition 12.2.36. Let X, Y be over Z and Z be over S, as in 12.2.1.1. As we have seen in Proposition 12.2.36, the isomorphisms (12.2.2):

$$\begin{tikzcd} \mathcal{H}om_S(I_S(\mathscr{M}), \mathcal{H}om_{Z/S}(X,Y)) \arrow[rr, "\cong"] \arrow[dr, "\cong"'] & & \mathcal{H}om_{Z/S}(X, \mathcal{H}om_Z(I_Z(\mathscr{M}), Y)) \\ & \mathcal{H}om_{Z/S}(X \times_S I_S(\mathscr{M}), Y) \arrow[ur, "\cong"'] & \end{tikzcd} \tag{12.2.19}$$

induces the isomorphism θ below

$$\begin{tikzcd} T_{\mathcal{H}om_{Z/S}(X,Y)}(\mathscr{M}) \arrow[rr, "\cong", "\theta"'] \arrow[dr, "\cong"'] & & \mathcal{H}om_{Z/S}(X, T_{Y/Z}(\mathscr{M})) \\ & \mathcal{H}om_{Z/S}(X \times_S I_S(\mathscr{M}), Y) \arrow[ur, "\cong"'] & \end{tikzcd} \tag{12.2.20}$$

By Remark 12.2.2, if Y is a Z-group, so is $\mathcal{H}om_Z(V,Y)$ for any $V \to Z$ (in particular for $V = I_Z(\mathscr{M})$); explicitly, if $Z'' \to Z' \to Z$ and $\phi, \psi \in \mathrm{Hom}_Z(V_{Z'}, Y)$, then $\phi \cdot \psi$ is defined by

$$(\phi \cdot \psi)(v) = \phi(v)\psi(v)$$

for any $v \in V_{Z'}(Z'')$.

Definition 12.2.56. Suppose that X and Y are Z-groups. Let $\mathcal{H}om_{(Z/S)\text{-}\mathbf{Grp}}(X,Y)$ be the sub-functor of $\mathcal{H}om_{Z/S}(X,Y)$ defined as follows: for any $S' \to S$,

$$\mathcal{H}om_{(Z/S)\text{-}\mathbf{Grp}}(X,Y)(S') = \mathrm{Hom}_{Z_{S'}\text{-}\mathbf{Grp}}(X_{S'}, Y_{S'}). \tag{12.2.21}$$

This definition applies equally if we replace Y by the Z-group $T_{Y/Z}(\mathscr{M})$.

We then easily see that $T_{\mathcal{H}om_{(Z/S)\text{-}\mathbf{Grp}}(X,Y)/S}(\mathscr{M})(S')$ corresponds, under the isomorphisms of (12.2.20), to $Z_{S'}$-morphisms $\phi : X_{S'} \times_{S'} I_{S'}(\mathscr{M}) \to Y_{S'}$ which is multiplicative on X, that is, which satisfies $\phi(x_1x_2, m) = \phi(x_1, m)\phi(x_2, m)$, and these correspond to $Z_{S'}$-group morphisms $X_{S'} \to T_{Y/Z}(\mathscr{M})_{S'}$. We then obtain the following:

Proposition 12.2.57. *Let X, Y be Z-groups and Z be over S. We have an isomorphism of S-functors, functorial on $\mathscr{M}$:*

$$T_{\mathcal{H}om_{(Z/S)\text{-}\mathbf{Grp}}(X,Y)}(\mathscr{M}) \xrightarrow{\sim} \mathcal{H}om_{(Z/S)\text{-}\mathbf{Grp}}(X, T_{Y/Z}(\mathscr{M})).$$

In particular, for $Z = S$, we obtain the following corollary. Before stating it, we note that if Y is an abelian S-group, then so is $T_{Y/S}(\mathscr{M})$, and hence $H = \mathrm{Hom}_{S\text{-}\mathbf{Grp}}(X,Y)$ and $\mathrm{Hom}_{S\text{-}\mathbf{Grp}}(X, T_{Y/S}(M))$, and finally is $T_{H/S}(\mathscr{M})$.

Corollary 12.2.58. *Let X, Y be S-groups. We have an isomorphism of S-functors, functorial on $\mathscr{M}$:*

$$T_{\mathcal{H}om_{S\text{-}\mathbf{Grp}}(X,Y)/S}(\mathscr{M}) \xrightarrow{\sim} \mathcal{H}om_{S\text{-}\mathbf{Grp}}(X, T_{Y/S}(\mathscr{M})).$$

If Y is commutative, then this is an isomorphism of abelian S-groups.

If Y is an $\mathbb{O}_S$-module, the functor $T_{Y/S}(\mathscr{M})$ (resp. $\mathfrak{Lie}(Y/S, \mathscr{M})$) is endowed with an $\mathbb{O}_S$-module structure deduced by that of Y, which we denote by $T'_{Y/S}(\mathscr{M})$ (resp. $\mathfrak{Lie}'(Y/S, \mathscr{M})$). Therefore, if X, Y are $\mathbb{O}_S$-modules, then $T'_{Y/S}(\mathscr{M}) = \mathcal{H}om_S(I_S(\mathscr{M}), Y)$ and $H = \mathcal{H}om_{\mathbb{O}_S}(X,Y)$, and hence $\mathcal{H}om_{\mathbb{O}_S}(X, T'_{Y/S}(\mathscr{M}))$ and $T'_{H/S}(\mathscr{M})$, are endowed with $\mathbb{O}_S$-module structures, and we have:

Corollary 12.2.59. *If X, Y are $\mathbb{O}_S$-modules, we have an isomorphism of $\mathbb{O}_S$-modules, functorial on $\mathscr{M}$:*

$$T'_{\mathcal{H}om_{\mathbb{O}_S}(X,Y)/S}(\mathscr{M}) \xrightarrow{\sim} \mathcal{H}om_{\mathbb{O}_S}(X, T_{Y/S}(\mathscr{M})).$$

Definition 12.2.60. Let X, M be S-groups and X acts on M by groups automorphisms. We define the sub-functor $\mathcal{Z}^1_S(X,M)$ of $\mathcal{H}om_S(X,M)$ as follows: for any $S' \to S$, $\mathcal{Z}^1_S(X,M)(S')$ is the set

$$\{\phi \in \mathrm{Hom}_{S'}(X_{S'}, M_{S'}) : \phi(x_1x_2) = \phi(x_1)(x_1 \cdot \phi(x_2)) \text{ for any } x_1, x_2 \in X(S''), S'' \to S'\}.$$

The functor $\mathcal{Z}^1_S(X,M)$ is called the **functor of cross homomorphisms** from X to M.

Remark 12.2.61. If M is an $\mathbb{O}_S[X]$-module, then $\mathcal{Z}^1_S(X,M)$ coincides with the kernel of the differential

$$d : \mathcal{H}om_S(X,M) \to \mathcal{H}om_S(X^2,M)$$

defined in 12.1.3.1. In particular, $\mathcal{Z}^1_S(X,M)$ is an $\mathbb{O}_S$-module in this case.

Let $u : X \to Y$ be a morphism of S-groups. We have seen in Corollary 12.2.39 that we have an isomorphism of S-functors, functorial on $\mathscr{M}$:

$$T_{\mathcal{H}om_S(X,Y)/S,u}(\mathscr{M}) \xrightarrow{\sim} \mathcal{H}om_{Y/S}(X, T_{Y/S}(\mathscr{M})). \tag{12.2.22}$$

On the other hand, as Y is an S-group, we have $T_{Y/S}(\mathscr{M}) = \mathfrak{Lie}(Y/S,\mathscr{M}) \rtimes Y$, whence an isomorphism

$$\begin{aligned}\mathcal{H}om_{Y/S}(X, T_{Y/S}(\mathscr{M})) &\xrightarrow{\sim} \mathcal{H}om_{Y/S}(X, \mathfrak{Lie}(Y/S,\mathscr{M}) \rtimes Y)\\ &\xrightarrow{\sim} \mathcal{H}om_{Y/S}(X, \mathfrak{Lie}(Y,S,\mathscr{M})_Y)\\ &\xrightarrow{\sim} \mathcal{H}om_S(X, \mathfrak{Lie}(Y,S,\mathscr{M})).\end{aligned} \tag{12.2.23}$$

For any $S' \to S$, denote by $u' : X' \to Y'$ the morphism induced by u from base change. Consider the S-functor defined as follows:

$$\begin{aligned}\mathcal{H}om_{(Y/S)\text{-}\mathbf{Grp}}(X, \mathfrak{Lie}(Y/S,\mathscr{M}) \rtimes Y)(S') &= \mathrm{Hom}_{Y'\text{-}\mathbf{Grp}}(X', (\mathfrak{Lie}(Y/S,\mathscr{M}) \rtimes Y)_{S'})\\ &= \mathrm{Hom}_{Y'\text{-}\mathbf{Grp}}(X', \mathfrak{Lie}(Y'/S',\mathscr{M}) \rtimes Y').\end{aligned}$$

The isomorphism (12.2.22) then induces an isomorphism

$$T_{\mathcal{H}om_{S\text{-}\mathbf{Grp}}(X,Y)/S,u}(\mathscr{M}) \xrightarrow{\sim} \mathcal{H}om_{(Y/S)\text{-}\mathbf{Grp}}(X, \mathfrak{Lie}(Y/S,\mathscr{M}) \rtimes Y). \tag{12.2.24}$$

The isomorphism (12.2.23) can be made explicit as follows: If $\Phi \in \mathcal{H}om_{Y/S}(X, \mathfrak{Lie}(Y/S,\mathscr{M}) \rtimes Y)$, then for any $S'' \to S' \to S$ and $x \in X(S'')$, we can write

$$\Phi(S')(x) = \phi(S')(x) \cdot u'(x) \quad \text{where} \quad \phi(S')(x) \in \mathfrak{Lie}(Y'/S',\mathscr{M})(S''),$$

which determines an element ϕ of $\mathcal{H}om_S(X, \mathfrak{Lie}(Y/S,\mathscr{M}))$. On the other hand, the composition of the morphisms

$$X \xrightarrow{u} Y \xrightarrow{\mathrm{Ad}} \mathrm{Aut}_{S\text{-}\mathbf{Grp}}(\mathfrak{Lie}(Y/S,\mathscr{M}))$$

defines an operation of X on $L = \mathfrak{Lie}(Y/S,\mathscr{M})$ by group automorphisms, and we note that $\Phi(S')$ is a group morphism if and only if for any $x_1, x_2 \in X(S'')$, we have

$$\phi(S')(x_1x_2) = \phi(S')(x_1)(u(x_1)\phi(S')(x_2)u(x_1)^{-1}) = \phi(S')(x_1)(x_1 \cdot \phi(S')(x_2)),$$

that is, if and only if $\phi \in \mathcal{Z}^1_S(X, \mathfrak{Lie}(Y/S,\mathscr{M}))$. We therefore obtain the following result:

Proposition 12.2.62. *Let $u : X \to Y$ be a morphism of S-groups. We have an isomorphism of S-functors, functorial on $\mathscr{M}$:*

$$T_{\mathcal{H}om_{S\text{-}\mathbf{Grp}}(X,Y)/S,u}(\mathscr{M}) \xrightarrow{\sim} \mathcal{Z}^1_S(X, \mathfrak{Lie}(Y/S,\mathscr{M})).$$

Suppose moreover that Y/S satisfies condition (E). Then it folows from Corollary 12.2.58, by the same proof of Corollary 12.2.37, that $\mathcal{H}om_{S\text{-}\mathbf{Grp}}(X,Y)/S$ satisfies condition (E). We then have (this also follows from Proposition 12.2.62)

$$T_{\mathcal{H}om_{S\text{-}\mathbf{Grp}}(X,Y)/S,u}(\mathscr{M} \oplus \mathscr{N}) \cong T_{\mathcal{H}om_{S\text{-}\mathbf{Grp}}(X,Y)/S,u}(\mathscr{M}) \times_S T_{\mathcal{H}om_{S\text{-}\mathbf{Grp}}(X,Y)/S,u}(\mathscr{N}).$$

Therefore, $T_{\mathcal{H}om_{S\text{-}\mathbf{Grp}}(X,Y)/S,u}(\mathscr{M})$ is endowed, as $\mathcal{Z}^1_S(X, \mathfrak{Lie}(Y/S,\mathscr{M}))$, with an $\mathbb{O}_S$-module structure induced by functoriality on $\mathscr{M}$. We then deduce that the isomorphism Proposition 12.2.62 is an isomorphism of $\mathbb{O}_S$-modules in this case:

Proposition 12.2.63. *Let $u : X \to Y$ be a morphism of S-groups and suppose that Y/S satisfies condition (E). We have an isomorphism of $\mathbb{O}_S$-modules, functorial on $\mathscr{M}$:*

$$T_{\mathcal{H}om_{S\text{-}\mathbf{Grp}}(X,Y)/S,u}(\mathscr{M}) \xrightarrow{\sim} \mathcal{Z}^1_S(X, \mathfrak{Lie}(Y/S,\mathscr{M})).$$

Moreover, if Y/S satisfies condition (E), we deduce from Corollary 12.2.45, as the proof of Corollary 12.2.46, that for any $u \in \mathrm{Iso}_{S\text{-}\mathbf{Grp}}(X,Y)$ we have an isomorphism functorial on $\mathscr{M}$

$$T_{\mathcal{I}so_{S\text{-}\mathbf{Grp}}(X,Y)/S,u}(\mathscr{M}) \xrightarrow{\sim} T_{\mathcal{H}om_{S\text{-}\mathbf{Grp}}(X,Y)/S,u}(\mathscr{M}).$$

We then deduce the following corollaries:

Corollary 12.2.64. *Let $u : X \to Y$ be a morphism of S-groups. If Y/S satisfies condition (E), we have an isomorphism of $\mathbb{O}_S$-modules, functorial on $\mathscr{M}$:*

$$T_{\mathcal{I}so_{S\text{-}\mathbf{Grp}}(X,Y)/S,u}(\mathscr{M}) \xrightarrow{\sim} \mathcal{Z}^1_S(X, \mathfrak{Lie}(Y/S,\mathscr{M})).$$

Corollary 12.2.65. *Let X be an S-group. If X/S satisfies condition (E), we have an isomorphism of $\mathbb{O}_S$-modules, functorial on $\mathscr{M}$:*

$$\mathfrak{Lie}(\mathcal{A}ut_{S\text{-}\mathbf{Grp}}(X)/S,\mathscr{M}) \xrightarrow{\sim} \mathcal{Z}^1_S(X, \mathfrak{Lie}(X/S,\mathscr{M})).$$

If Y is abelian, then the adjoint representation of Y on $L = \mathfrak{Lie}(Y/S,\mathscr{M})$ is trivial, so we have $\mathcal{Z}^1_S(X,L) = \mathcal{H}om_{S\text{-}\mathbf{Grp}}(X,L)$. We thus have:

Corollary 12.2.66. *Let Y be an abelian S-group. We have an isomorphism of S-functors, functorial on $\mathscr{M}$:*

$$T_{\mathcal{H}om_{S\text{-}\mathbf{Grp}}(X,Y)/S,u}(\mathscr{M}) \xrightarrow{\sim} \mathcal{H}om_{S\text{-}\mathbf{Grp}}(X, \mathfrak{Lie}(Y/S,\mathscr{M})).$$

If Y/S satisfies condition (E), this is an isomorphism of $\mathbb{O}_S$-modules.

Consider now the case where X, Y are $\mathbb{O}_S$-modules. Recall that we denote by $T'_{Y/S}(\mathscr{M})$ (resp. $\mathfrak{Lie}'(Y/S,\mathscr{M})$) the functor $T_{Y/S}(\mathscr{M})$ (resp. $\mathfrak{Lie}(Y/S,\mathscr{M})$) endowed with the $\mathbb{O}_S$-module structure induced by that of Y. If Y/S satisfies condition (E), we always endow $\mathfrak{Lie}(Y/S,\mathscr{M})$ the $\mathbb{O}_S$-module structure defined by functoriality on $\mathscr{M}$. In this case, the abelian group structures of $\mathfrak{Lie}(Y/S,\mathscr{M})$ and $\mathfrak{Lie}'(Y/S,\mathscr{M})$ coincide (cf. Proposition 12.2.29), but this is in general not true for the module structures. For any $S' \to S$ and $a \in \mathbb{O}(S')$, we denote by $a \cdot' m$ (resp. $a \cdot m$) the action of a on $m \in \mathfrak{Lie}'(Y/S,\mathscr{M})(S')$ (resp. $m \in \mathfrak{Lie}(Y/S,\mathscr{M})(S')$), and similarly for the actions of a on $T'_{Y/S}(\mathscr{M})$ and $T_{Y/S}(\mathscr{M})$.

We have $T'_{Y/S}(\mathscr{M}) \cong \mathfrak{Lie}'(Y/S,\mathscr{M}) \oplus Y$ as $\mathbb{O}_S$-modules; therefore, we obtain, as in Corollary 12.2.66, that:

Proposition 12.2.67. *Let $u : X \to Y$ be a morphism of $\mathbb{O}_S$-modules. We have an isomorphism of S-functors, functorial on $\mathscr{M}$:*

$$T_{\mathcal{H}om_{\mathbb{O}_S}(X,Y)/S,u}(\mathscr{M}) \xrightarrow{\sim} \mathcal{H}om_{\mathbb{O}_S}(X, \mathfrak{Lie}'(Y/S,\mathscr{M})). \tag{12.2.25}$$

If Y/S satisfies condition (E), then $\mathcal{H}om_{\mathbb{O}_S}(X,Y)/S$ satisfies condition (E) and (12.2.25) is an isomorphism of $\mathbb{O}_S$-modules if we endow both sides the $\mathbb{O}_S$-module structure induced by functoriality on $\mathscr{M}$.

Remark 12.2.68. Let $u : X \to Y$ be a morphism of $\mathbb{O}_S$-modules. Denote by τ_u the map which associates to any morphism $\phi : X \to \mathfrak{Lie}'(Y/S,\mathscr{M})$ of $\mathbb{O}_S$-modules the morphism

$$\phi \oplus u : X \to T'_{Y/S}(\mathscr{M}) = \mathfrak{Lie}'(Y/S,\mathscr{M}) \oplus Y.$$

Then the isomorphism of Proposition 12.2.67 fits into the following diagram, functorial on $\mathscr{M}$:

$$\begin{tikzcd} T_{\mathcal{H}om_{\mathbb{O}_S}(X,Y)/S,u} \arrow[r, "\cong"] \arrow[d, hook] & \mathcal{H}om_{\mathbb{O}_S}(X, \mathfrak{Lie}'(Y/S,\mathscr{M})) \arrow[d, hook, "\tau_u"] \\ T_{\mathcal{H}om_{\mathbb{O}_S}(X,Y)/S}(\mathscr{M}) \arrow[r, "\cong"] & \mathcal{H}om_{\mathbb{O}_S}(X, T'_{Y/S}(\mathscr{M})) \end{tikzcd} \tag{12.2.26}$$

Moreover, if Y/S satisfies condition (E), we deduce from Corollary 12.2.45, as the proof of Corollary 12.2.46, that for any $u \in \mathrm{Iso}_{\mathbb{O}_S}(X,Y)$, we have

$$T_{\mathcal{I}so_{\mathbb{O}_S}(X,Y)/S}(\mathscr{M}) \cong T_{\mathcal{H}om_{\mathbb{O}_S}(X,Y)/S}(\mathscr{M}). \tag{12.2.27}$$

Corollary 12.2.69. *Let X be an $\mathbb{O}_S$-module satisfying condition (E) relative to S. We have an isomorphism, functorial on $\mathscr{M}$:*

$$\mathfrak{Lie}(\mathcal{A}ut_{\mathbb{O}_S}(X)/S,\mathscr{M}) \xrightarrow{\sim} \mathcal{H}om_{\mathbb{O}_S}(X,\mathfrak{Lie}'(X/S,\mathscr{M}))$$

which respects the $\mathbb{O}_S$-module structure induced by functoriality on $\mathscr{M}$. In particular, $\mathcal{A}ut_{\mathbb{O}_S}(X)/S$ satisfies condition (E).

Proof. The first assertion follows from (12.2.25) and (12.2.27). For the second one, as X/S satisfies condition (E), we have an isomorphism of $\mathbb{O}_S$-modules $\mathfrak{Lie}'(X/S,\mathscr{M}\oplus\mathscr{N}) \cong \mathfrak{Lie}'(X/S,\mathscr{M}) \times_S \mathfrak{Lie}'(X/S,\mathscr{N})$, and hence

$$\mathfrak{Lie}(\mathcal{A}ut_{\mathbb{O}_S}(X)/S,\mathscr{M}\oplus\mathscr{N}) \cong \mathfrak{Lie}(\mathcal{A}ut_{\mathbb{O}_S}(X)/S,\mathscr{M}) \times_S \mathfrak{Lie}(\mathcal{A}ut_{\mathbb{O}_S}(X)/S,\mathscr{N}).$$

In view of the sequence (12.2.17), this proves that $\mathcal{A}ut_{\mathbb{O}_S}(X)/S$ satisfies condition (E). □

Before going further towards this direction, let us take a closer look at the relations between Y, $\mathfrak{Lie}(Y/S)$ and $\mathfrak{Lie}'(Y/S)$. We first notice that (cf. Remark 12.2.14)

$$\mathfrak{Lie}(\mathbb{O}_S/S,\mathscr{M}) = \mathfrak{Lie}'(\mathbb{O}_S/S,\mathscr{M}) = \Gamma_{\mathscr{M}} \tag{12.2.28}$$

and that we have a canonical isomorphism

$$d:\mathbb{O}_S \xrightarrow{\sim} \mathfrak{Lie}(\mathbb{O}_S/S). \tag{12.2.29}$$

Now let F be an $\mathbb{O}_S$-module. For any $S_2 \to S_1 \to S$, we have a bihomomorphism

$$F(S_1) \to F(S_2), \quad \mathbb{O}(S_1) \to \mathbb{O}(S_2), \tag{12.2.30}$$

whence a morphism of $\mathbb{O}(S_2)$-modules

$$F(S_1) \otimes_{\mathbb{O}(S_1)} \mathbb{O}(S_2) \to F(S_2).$$

In particular, for $S_1 = S'$ and $S_2 = I_{S'}(\mathscr{M})$, we deduce a morphism of $\mathbb{O}(S')$-modules, functorial on $\mathscr{M}$

$$F(S') \otimes_{\mathbb{O}(S')} T_{\mathbb{O}_S/S}(\mathscr{M})(S') \to T'_{F/S}(\mathscr{M})(S').$$

With S' varies, we obtain morphisms of $\mathbb{O}_S$-modules, functorial on $\mathscr{M}$:

$$F \otimes_{\mathbb{O}_S} T_{\mathbb{O}_S/S}(\mathscr{M}) \to T'_{F/S}(\mathscr{M}). \tag{12.2.31}$$

These morphisms are functorial on $\mathscr{M}$, hence compatible with the projections of tangent bundles onto their bases; they then define morphisms of $\mathbb{O}_S$-modules

$$F \otimes_{\mathbb{O}_S} \mathfrak{Lie}(\mathbb{O}_S/S,\mathscr{M}) \to \mathfrak{Lie}'(F/S,\mathscr{M}) \tag{12.2.32}$$

such that the following diagram is commutative:

$$\begin{tikzcd}
0 \arrow[r] & F \otimes_{\mathbb{O}_S} \mathfrak{Lie}(\mathbb{O}_S/S,\mathscr{M}) \arrow[r] \arrow[d] & F \otimes_{\mathbb{O}_S} T_{\mathbb{O}_S/S}(\mathscr{M}) \arrow[r] \arrow[d] & F \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathfrak{Lie}'(F/S,\mathscr{M}) \arrow[r] & T'_{F/S}(\mathscr{M}) \arrow[r] & F \arrow[r] & 0
\end{tikzcd}$$

We can consider the morphisms (12.2.32) as morphisms of abelian S-groups:

$$F \otimes_{\mathbb{O}_S} \mathfrak{Lie}(\mathbb{O}_S/S,\mathscr{M}) \to \mathfrak{Lie}(F/S,\mathscr{M}). \tag{12.2.33}$$

By tensoring F with the isomorphism $d:\mathbb{O}_S \xrightarrow{\sim} \mathfrak{Lie}(\mathbb{O}_S/S)$, we then deduce (for $\mathscr{M} = \mathscr{O}_S$) a morphism of abelian S-groups

$$d: F \xrightarrow{\sim} F \otimes_{\mathbb{O}_S} \mathfrak{Lie}(\mathbb{O}_S/S) \to \mathfrak{Lie}(F/S). \tag{12.2.34}$$

Remark 12.2.70. If F/S satisfies condition (E), the morphisms (12.2.33) and (12.2.34) are not necessarily morphisms of $\mathbb{O}_S$-modules, if we endow both sides the module structure induced by functoriality on $\mathscr{M}$. For example, let k be a field with characteristic $p > 0$, $S = \mathrm{Spec}(k)$, and F be the $\mathbb{O}_S$-module which to any S-scheme T associates $F(T) = \Gamma(T, \mathscr{O}_T)$, endowed with the $\mathbb{O}(T)$-module structure obtained by acting a scalar via its p-th power, that is, $r \cdot f = r^p f$ for $r \in \mathbb{O}(T)$ and $f \in F(T)$. As an S-group, F is isomorphic to $\mathbb{G}_{a,S}$, so F satisfies condition (E) and $\mathfrak{Lie}(F/S)$ is identified with $\mathfrak{Lie}(\mathbb{G}_{a,S}/S) \cong \mathbb{O}_S$. Then, the morphism $d : F \to \mathfrak{Lie}(F/S)$ is, for any $T \to S$, the identity map $F(T) \to \mathbb{O}(T)$; it respects the abelian group structure, but not the $\mathbb{O}_S$-module structure.

Remark 12.2.71. We can explicit the morphisms (12.2.31) and (12.2.32) as follows. The morphism Θ : $F \otimes_{\mathbb{O}_S} T_{\mathbb{O}_S/S}(\mathscr{M}) \to T'_{F/S}(\mathscr{M}) = \mathcal{H}om_S(I_S(\mathscr{M}), F)$ is defined so that for any $S' \to S$, $\alpha \in \mathbb{O}(I_{S'}(\mathscr{M}))$, and $f : S' \to F$,

$$\Theta(f \otimes \alpha) = \alpha(\tau_0 \circ f) = \alpha \cdot (f \circ \rho)$$

where $\tau_0 : F \to T'_{F/S}(\mathscr{M})$ is the zero section and $\rho : I_{S'}(\mathscr{M}) \to S'$ is the structural morphism.

Definition 12.2.72. We say that F is a **good** $\mathbb{O}_S$**-module** if the morphisms $F \otimes_{\mathbb{O}_S} T_{\mathbb{O}_S/S}(\mathscr{M}) \to T_{F/S}(\mathscr{M})$ (or equivalently, the morphisms $F \otimes_{\mathbb{O}_S} \mathfrak{Lie}(\mathbb{O}_S/S, \mathscr{M}) \to \mathfrak{Lie}(F/S, \mathscr{M})$) are isomorphisms of abelian S-groups (so that F/S satisfies condition (E)) and if moreover they respect the $\mathbb{O}_S$-module structures induced by functoriality on $\mathscr{M}$.

Proposition 12.2.73. *Let F be an $\mathbb{O}_S$-module. Consider the following conditions:*

(i) *F is a good $\mathbb{O}_S$-module.*

(ii) *F/S satisfies condition (E) and $d : F \to \mathfrak{Lie}(F/S)$ is an isomorphism of $\mathbb{O}_S$-modules.*

(iii) *$\mathfrak{Lie}(F/S, \mathscr{M}) = \mathfrak{Lie}'(F/S, \mathscr{M})$.*

Then we have (i)$\Leftrightarrow$(ii)$\Rightarrow$(iii)*.*

Proof. The implication (i)$\Rightarrow$(ii) follows from definition. To see that (ii)$\Rightarrow$(ii), it suffices to show that the morphisms of abelian S-groups

$$F \otimes_{\mathbb{O}_S} \mathfrak{Lie}(\mathbb{O}_S/S, \mathscr{M}) \xrightarrow{\sim} \mathfrak{Lie}(F/S, \mathscr{M})$$

are isomorphisms of $\mathbb{O}_S$-modules. As F/S satisfies condition (E), the two members transform finite direct sums of copies of $\mathscr{O}_S$ into finite products of abelian S-groups. We are then reduced to the case where $\mathscr{M} = \mathscr{O}_S$, which follows by the hypothesis.

Finally, (i)$\Rightarrow$(iii) follows from the definition and the fact that the isomorphisms

$$F \otimes_{\mathbb{O}_S} \mathfrak{Lie}(\mathbb{O}_S/S, \mathscr{M}) \xrightarrow{\sim} \mathfrak{Lie}'(F/S, \mathscr{M})$$

of (12.2.32) is an isomorphism of $\mathbb{O}_S$-modules. ☐

Example 12.2.74. For any quasi-coherent $\mathscr{O}_S$-module $\mathscr{E}$, the $\mathbb{O}_S$-module $\Gamma_{\mathscr{E}}$ and $\check{\Gamma}_{\mathscr{E}}$ are good. In fact, for any $f : S' \to S$, the morphisms

$$\Gamma_{\mathscr{E}}(S') \otimes_{\mathbb{O}(S')} \mathbb{O}(I_{S'}(\mathscr{M})) \to T_{\Gamma_{\mathscr{E}}/S}(\mathscr{M})(S')$$
$$\check{\Gamma}_{\mathscr{E}}(S') \otimes_{\mathbb{O}(S')} \mathbb{O}(I_{S'}(\mathscr{M})) \to T_{\check{\Gamma}_{\mathscr{E}}/S}(\mathscr{M})(S')$$

correspond, respectively, to morphisms

$$\Gamma(S', f^*(\mathscr{E})) \otimes_{\mathbb{O}(S')} \Gamma(S', \mathscr{D}_{\mathscr{O}_{S'}}(\mathscr{M})) \to \Gamma(S', f^*(\mathscr{E}) \otimes_{\mathscr{O}_{S'}} \mathscr{D}_{\mathscr{O}_{S'}}(\mathscr{M})),$$
$$\mathrm{Hom}_{\mathscr{O}_{S'}}(f^*(\mathscr{E}), \mathscr{O}_{S'}) \otimes_{\mathbb{O}(S')} \Gamma(S', \mathscr{D}_{\mathscr{O}_{S'}}(\mathscr{M})) \to \mathrm{Hom}_{\mathscr{O}_{S'}}(f^*(\mathscr{E}), \mathscr{D}_{\mathscr{O}_{S'}}(\mathscr{M}));$$

which are both isomorphisms since $\mathscr{D}_{\mathscr{O}_{S'}}(\mathscr{M})$ is isomorphic, as $\mathscr{O}_{S'}$-module, to a finite direct sum of copies of $\mathscr{O}_{S'}$ (recall that $\mathscr{M}$ is assumed to be free).

Proposition 12.2.75. *Let F be a good $\mathbb{O}_S$-module. Then $\mathcal{A}ut_{\mathbb{O}_S}(F)/S$ satisfies condition (E) and we have a isomorphism (functorial on $\mathscr{M}$)*

$$\mathfrak{Lie}(\mathcal{A}ut_{\mathbb{O}_S}(F)/S, \mathscr{M}) \xrightarrow{\sim} \mathcal{H}om_{\mathbb{O}_S}(F, \mathfrak{Lie}(F/S, \mathscr{M}))$$

which respects the $\mathbb{O}_S$ induced by the functoriality on $\mathscr{M}$. In particular, we have an isomorphism of $\mathbb{O}_S$-modules

$$\mathfrak{Lie}(\mathcal{A}ut_{\mathbb{O}_S}(F)/S) \xrightarrow{\sim} \mathcal{E}nd_{\mathbb{O}_S}(F).$$

Morover, $\mathcal{E}nd_{\mathbb{O}_S}(F)$ is a good $\mathbb{O}_S$-module.

Proof. In fact, by Proposition 12.2.73, F/S satisfies condition (E) and

$$\mathfrak{Lie}(F/S, \mathscr{M}) = \mathfrak{Lie}'(F/S, \mathscr{M}) \cong F \otimes_{\mathbb{O}_S} \mathfrak{Lie}(\mathbb{O}_S/S, \mathscr{M}). \tag{12.2.35}$$

The first assertion then follows from Corollary 12.2.69. Put $E = \mathcal{E}nd_{\mathbb{O}_S}(F)$; by (12.2.35) and ([?] remarque 4.3.5), we have the following commutative diagram of abelian groups

$$\begin{tikzcd}
\mathcal{E}nd_{\mathbb{O}_S}(F) \otimes_{\mathbb{O}_S} \mathfrak{Lie}(\mathbb{O}_S/S, \mathscr{M}) \arrow[r, "d_E"] \arrow[d, equal] & \mathfrak{Lie}(\mathcal{E}nd_{\mathbb{O}_S}(F)/S, \mathscr{M}) \\
\mathcal{H}om_{\mathbb{O}_S}(F, F \otimes_{\mathbb{O}_S} \mathfrak{Lie}(\mathbb{O}_S/S, \mathscr{M})) \arrow[r, "d_F"] & \mathcal{H}om_{\mathbb{O}_S}(F, \mathfrak{Lie}(\mathcal{E}nd_{\mathbb{O}_S}(F)/S, \mathscr{M})) \arrow[u, "\cong"', "(*)"]
\end{tikzcd}$$

where d_F and $(*)$ are isomorphisms of $\mathbb{O}_S$-modules; therefore, so is d_E, and this proves the proposition. □

Remark 12.2.76. We can provide an explicit description of the isomorphism in Proposition 12.2.75. For this, put $\mathscr{O}_{I_S} = \mathscr{O}_S \oplus t\mathscr{O}_S$ (with $t^2 = 0$) and let F be a good $\mathbb{O}_S$-module. Then, for any $S' \to S$, the morphism[3] (which is the identity on $F(S')$)

$$F(S') \oplus tF(S') = F(S') \otimes_{\mathbb{O}(S')} \mathbb{O}(I_{S'}) \to F(I_{S'}) = F(S') \oplus \mathfrak{Lie}(F/S)(S')$$

induces an isomorphism of $\mathbb{O}(S')$-modules $tF(S') \cong \mathfrak{Lie}(F/S)(S')$. Since this is functorial over S', we then obtain an isomorphism $\mathfrak{Lie}(F/S) \cong tF$. For any $S' \to S$, we have, by Proposition 12.2.75, a commutative diagram

$$\begin{tikzcd}
\mathrm{End}_{\mathbb{O}_{S'}}(F_{S'}) \arrow[r, "\cong"] & \mathrm{Hom}_{\mathbb{O}_{S'}}(F_{S'}, tF_{S'}) \arrow[r, "\cong"] \arrow[d, hook] & \mathfrak{Lie}(\mathcal{A}ut_{\mathbb{O}_S}(F)/S)(S') \arrow[d, hook] \\
& \mathrm{Aut}_{\mathbb{O}(I_{S'})}(F_{I_{S'}}) \arrow[r, equal] & T_{\mathcal{A}ut_{\mathbb{O}_S}(F)/S}(S')
\end{tikzcd}$$

and we deduce from the commutative diagram (12.2.68) (take $u = \mathrm{id}$ in the diagram) that any $X \in \mathrm{End}_{\mathbb{O}_{S'}}(F_{S'})$ corresponds to the element $\mathrm{id} + tX$ of $\mathrm{Aut}_{\mathbb{O}_{I_{S'}}}(F_{I_{S'}})$.

We say that the S-group G is **good** if G/S satisfies condition (E) and $\mathfrak{Lie}(G/S)$ is a good $\mathbb{O}_S$-module. Note that if F is a good $\mathbb{O}_S$-module, it is also a good S-groups: in fact, F/S satisfies condition (E) and $\mathfrak{Lie}(F/S) \cong F$ (cf. Proposition 12.2.73 (ii)) is a good $\mathbb{O}_S$-module.

Example 12.2.77. If G is representable, then it is good. In fact, G/S satisfies condition (E) and $\mathfrak{Lie}(G/S)$ is of the form $\mathbb{V}(\mathscr{E})$ by Proposition 12.2.15, hence good by Example 12.2.74.

Lemma 12.2.78. *Let G be an S-group such that G/S satisfies condition (E), and $F = \mathfrak{Lie}(G/S)$. Then F/S satisfies condition (E) and the abelian group morphism $d : F \to \mathfrak{Lie}(F/S)$ respects the $\mathbb{O}_S$-module structure. Therefore, G is good if and only if $d : F \to \mathfrak{Lie}(F/S)$ is bijective.*

Proof. □

Theorem 12.2.79. *If F is a good $\mathbb{O}_S$-module, the S-group $\mathcal{A}ut_{\mathbb{O}_S}(F)$ is good.*

Proof. By Proposition 12.2.75, $\mathcal{A}ut_{\mathbb{O}_S}(F)/S$ satisfies condition (E) and $\mathfrak{Lie}(\mathcal{A}ut_{\mathbb{O}_S}(F)/S) \cong \mathcal{E}nd_{\mathbb{O}_S}(F)$ is a good $\mathbb{O}_S$-module. □

Example 12.2.80. Let F be the $\mathbb{O}_S$-module defined in Remark 12.2.70. Then, the canonical morphism $d : F \to \mathfrak{Lie}(F/S)$ is, for any $T \to S$, the identity morphism $F(T) \to \mathbb{O}(T)$. It respects the abelian group structure, but not the module structure, so F is not good.

[3] The equality on the right follows from the exact sequence (12.2.17).

Let G be an S-group and F be a good $\mathbb{O}_S$-module. Suppose that we are given a linear representation of G on F, that is, an S-group morphism

$$\rho : G \to \mathcal{A}ut_{\mathbb{O}_S}(F).$$

If G/S satisfies condition (E), we deduce from Proposition 12.2.75 and Proposition 12.2.25 a morphism of $\mathbb{O}_S$-modules

$$d\rho : \mathfrak{Lie}(G/S) \to \mathfrak{Lie}(\mathcal{A}ut_{\mathbb{O}_S}(F)/S) \cong \mathcal{E}nd_{\mathbb{O}_S}(F).$$

Moreover, put $\mathscr{O}_{I_S} = \mathscr{O}_S \oplus t\mathscr{O}_S$ (with $t^2 = 0$), we then deduce from Remark 12.2.76 that, for $S' \to S$ and $X \in \mathfrak{Lie}(G/S)(S') \subseteq T_{G/S}(S') = G(I_{S'})$, we have the following equality in $\mathrm{Aut}_{\mathbb{O}_{I_{S'}}}(F_{I_{S'}})$:

$$\rho(X) = \mathrm{id} + t\, d\rho(X), \tag{12.2.36}$$

i.e. for any $S'' \to I_{S'}$ and $f \in F(S')$, we have $\rho(X)(f) = f + td\rho(X)(f)$ in $F(S'')$.

Let G be a good S-group. Then $\mathfrak{Lie}(G/S)$ is a good $\mathbb{O}_S$-module, and we have a morphism of S-groups

$$\mathrm{Ad} : G \to \mathcal{A}ut_{\mathbb{O}_S}(\mathfrak{Lie}(G/S)).$$

We then deduce a morphism of $\mathbb{O}_S$-modules

$$\mathrm{ad} : \mathfrak{Lie}(G/S) \to \mathcal{E}nd_{\mathbb{O}_S}(\mathfrak{Lie}(G/S)),$$

or equivalently, an $\mathbb{O}_S$-bilinear morphism

$$\mathfrak{Lie}(G/S) \times_S \mathfrak{Lie}(G/S) \to \mathfrak{Lie}(G/S), \quad (x, y) \mapsto [x, y] := \mathrm{ad}(x) \cdot y$$

where x, y denote elements of $\mathfrak{Lie}(G/S)(S') = \mathfrak{Lie}(G_{S'}/S')(S')$. If G is commutative, then Ad is trivial, and we have $[x, y] = 0$.

Remark 12.2.81. We can give an equivalent definition of the bracket: it suffices to do this for $x, y \in \mathfrak{Lie}(G/S)(S)$. We then note that there is a canonical isomorphism $I_S \times_S I_S \cong I_{I_S}$; to avoid confusions, we denote by I and I' the two copies of I_S and put $\mathscr{O}_I = \mathscr{O}_S[t]$, $\mathscr{O}_{I'} = \mathscr{O}_S[t']$, where $t^2 = t'^2 = 0$. We then have a commutative diagram

$$\begin{tikzcd}
I \times I' \arrow[r] \arrow[d] & I' \arrow[d] \\
I \arrow[r] & S
\end{tikzcd}$$

(the two arrows from $I \times I'$ identifying it as the dual number scheme over I or over I'), which gives a commutative diagram of abelian groups ($L = \mathfrak{Lie}(G/S)$) by (12.2.17):

$$\begin{tikzcd}
 & & 1 \arrow[d] & 1 \arrow[d] & \\
 & & L(I) \arrow[r] \arrow[d] & L(S) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & L(I') \arrow[r] \arrow[d] & G(I \times I') \arrow[r] \arrow[d] & G(I') \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & L(S) \arrow[r] \arrow[d] & G(I) \arrow[r] \arrow[d] & G(S) \arrow[r] \arrow[d] & 1 \\
 & 1 & 1 & 1 &
\end{tikzcd} \tag{12.2.37}$$

The final piece of this diagram is none other than $\mathfrak{Lie}(L/S)(S)$. If G is good, this is isomorphic to $L(S)$ and we then have the following diagram, where the rows and columns are exact sequences of groups and in view of the identification $L(I) = L(S) \oplus tL(S)$ (resp. $L(I') = L(S) \oplus t'L(S)$), the injection $L(S) \hookrightarrow L(I)$ (resp. $L(S) \hookrightarrow L(I')$) is given by $u \mapsto tu$ (resp. $u \mapsto t'u$):

$$\begin{tikzcd} L(S) \arrow[r, "t"] \arrow[d, "t'"] & L(I) \arrow[r] \arrow[d] & L(S) \arrow[d] \\ L(I') \arrow[r] \arrow[d] & G(I \times I') \arrow[r] \arrow[d] & G(I') \arrow[d] \\ L(S) \arrow[r] & G(I) \arrow[r] & G(S) \end{tikzcd} \tag{12.2.38}$$

Now in this diagram, if we take two elements x and y in $L(S)$ and choose arbitrarily element $\tilde{x} \in L(I)$ (resp. $\tilde{y} \in L(I')$) which maps to x (resp. to y), then the commutator $\tilde{x}\tilde{y}\tilde{x}^{-1}\tilde{y}^{-1}$ in $G(I \times I')$ does not depend on the choice of $\tilde{x}$ and $\tilde{y}$, and it is the image of an element $z \in L(S)$. In fact, if we identify x with its image under the canonical section $L(S) \to L(I)$ (and similarly for y), then $\tilde{x} = xu$ and $\tilde{y} = yv$, with $u, v \in L(S) = L(I) \cap L(I')$, and since $L(I)$, $L(I')$ are abelian, we have

$$\tilde{x}\tilde{y}\tilde{x}^{-1}\tilde{y}^{-1} = xuyvu^{-1}x^{-1}v^{-1}y^{-1} = xuyu^{-1}vx^{-1}v^{-1}y^{-1} = xyx^{-1}y^{-1}.$$

Moreover, this element is send to the unit element of $G(I)$ and of $G(I')$, hence comes from an element $z \in L(S)$. Finally, consider y (resp. x) as element of $L(I')$ (resp. $L(S) \subseteq G(I')$), by (12.2.36) we have

$$xyx^{-1} = \mathrm{Ad}(x)(y) = (\mathrm{id} + t'\mathrm{ad}(x))(y) = y + t'[x, y],$$

so the element $xyx^{-1}y^{-1}$ of $L(I')$ is the iamge of $z = [x, y] \in L(S)$.

From the above construction, we see that the bracket has the following properties:

(i) The bracket is functorial on G: more precisely, $G \mapsto \mathfrak{Lie}(G/S)$ is a functor from the category of good S-groups to the category of good $\mathbb{O}_S$-modules endowed with an $\mathbb{O}_S$-bilinear composition law.

(ii) We have $[x, y] + [y, x] = 0$. In fact, the diagram is symmetric, and by exchanging x and y we are considering the element $\tilde{y}\tilde{x}\tilde{y}^{-1}\tilde{x}^{-1}$, which is the inverse of $\tilde{x}\tilde{y}\tilde{x}^{-1}\tilde{y}^{-1}$.

Proposition 12.2.82. *Let F be a good $\mathbb{O}_S$-module. Via the identification $\mathfrak{Lie}(\mathcal{A}ut_{\mathbb{O}_S}(F)/S) = \mathcal{E}nd_{\mathbb{O}_S}(F)$, we have*

$$\mathrm{Ad}(g) \cdot Y = g \circ Y \circ g^{-1}, \quad [X, Y] = X \circ Y - Y \circ X,$$

for any $S' \to S$, $g \in \mathrm{Aut}_{\mathbb{O}_S}(F_{S'})$ and $X, Y \in \mathfrak{Lie}(\mathcal{A}ut_{\mathbb{O}_S}(F)/S)(S') = \mathrm{End}_{\mathbb{O}_S}(F_{S'})$.

Proof. By base change, we can assume that $S' = S$, which makes it possible to simplify the notations. Put $I = I_S$ and $\mathscr{O}_I = \mathscr{O}_S[t]$ (with $t^2 = 0$). Recall that the inclusion $i : \mathrm{End}_{\mathbb{O}_S}(F) \hookrightarrow \mathcal{A}ut_{\mathbb{O}_I}(F_I)$ sends Y to $\mathrm{id} + tY$, so by the definition of $\mathrm{Ad}(g)$, we have

$$\mathrm{id} + t\mathrm{Ad}(g)(Y) = g \circ (\mathrm{id} + tY) \circ g^{-1} = \mathrm{id} + t(g \circ Y \circ g^{-1}),$$

whence $\mathrm{Ad}(g)(Y) = g \circ Y \circ g^{-1}$.

Let I' be a second copy of I_S, and put $\mathscr{O}_{I'} = \mathscr{O}_S[t']$ (with $t'^2 = 0$). Apply the result of Remark 12.2.81 to $G = \mathcal{A}ut_{\mathbb{O}_S}(F)$ and $L = \mathfrak{Lie}(G/S) = \mathcal{A}ut_{\mathbb{O}_S}(F)$, where we identify X with its image under the canonical section $L(S) \hookrightarrow L(I)$; its image in $G(I \times I')$ is then $\mathrm{id} + t'X$, hence the inverse is $\mathrm{id} - t'X$. Similarly, Y is send to $\mathrm{id} + tY$, so the inverse is $\mathrm{id} - tY$. Then the commutator

$$(\mathrm{id} + t'X) \circ (\mathrm{id} + tY) \circ (\mathrm{id} - t'X) \circ (\mathrm{id} - tY) = \mathrm{id} + tt'(X \circ Y - Y \circ X)$$

is the image of $Z = X \circ Y - Y \circ X$ in $G(I \times I')$ (in fact, Z is send to $tZ \in L(I)$, hence to $\mathrm{id} + tt'Z \in G(I \times I')$). By Remark 12.2.81, we conclude that $[X, Y] = X \circ Y - Y \circ X$. ☐

Corollary 12.2.83. *Let G be a good S-group and $x, y, z \in \mathfrak{Lie}(G/S)(S')$. We have*

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0.$$

Proof. In fact, as G is good, $\mathfrak{Lie}(G/S)$ is a good $\mathbb{O}_S$-module and hence, by Theorem 12.2.79, $\mathcal{A}ut_{\mathbb{O}_S}(\mathfrak{Lie}(G/S))$ is a good S-group. Then, the morphism of S-groups

$$\mathrm{Ad} : G \to \mathcal{A}ut_{\mathbb{O}_S}(\mathfrak{Lie}(G/S))$$

gives by functoriality $\mathrm{ad}([x,y]) = [\mathrm{ad}(x), \mathrm{ad}(y)]$. Combined with Proposition 12.2.82, this shows that

$$\mathrm{ad}([x,y]) = [\mathrm{ad}(x), \mathrm{ad}(y)] = \mathrm{ad}(x) \circ \mathrm{ad}(y) - \mathrm{ad}(y) \circ \mathrm{ad}(x),$$

which implies the Jacobi identity after applied to an element z. □

Corollary 12.2.84. *Let G be a good S-group linearly acted on a good $\mathbb{O}_S$-module F (i.e. F is an $\mathbb{O}_S[G]$-module, G and F being good). Then the linear map $d\rho : \mathfrak{Lie}(G/S) \to \mathcal{E}nd_{\mathbb{O}_S}(F)$ is a representation, that is, we have*

$$d\rho([x,y]) = d\rho(x) \circ d\rho(y) - d\rho(y) \circ d\rho(x).$$

Proof. This follows from the functoriality of bracket and Proposition 12.2.82. □

To any good S-group (for example representable), we have associated a good $\mathbb{O}_S$-module $\mathfrak{Lie}(G/S)$ endowed functorially a bilinear map verifying

$$[x,y] + [y,x] = 0, \quad [x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0.$$

We therefore say that $\mathfrak{Lie}(G/S)$, endowed with this structure, is the **quasi-Lie algebra** of G over S. For any linear representation of G over a good $\mathbb{O}_S$-module F, we can associate a representation of the quasi-Lie algebra $\mathfrak{Lie}(G/S)$. In particular, the adjoint representation of G is associated to the adjoint representation of the quasi-Lie algebra.

A group functor G over S is called **very good** if it is good and $\mathfrak{Lie}(G/S)$ is a Lie algebra over $\mathbb{O}_S$ (that is, if we have the identity $[x,x] = 0$). The following S-groups are very good: $\mathcal{A}ut_{\mathbb{O}_S}(F)$ for any good $\mathbb{O}_S$-module F (cf. Proposition 12.2.82 and Corollary 12.2.83), any representable group (see below), any good S-group admitting a monomorphism into a very good S-group (cf. Proposition 12.2.23), for example any good subgroup of a very good representable group, or any good S-group admitting a faithful representation over a good $\mathbb{O}_S$-module, for example any good S-group such that Ad is faithful.

Now suppose that G is a group scheme. By Proposition 12.2.55, $\mathfrak{Lie}(G/S)(S)$ is identified with right invariant infinitesimal automorphisms of G, hence by (12.2.16) with derivations of $\mathscr{O}_G$ over $\mathscr{O}_S$ invariant under right translations. Moreover, this identification respects the module structure and is an *anti-isomorphism* of Lie algebras: put $\mathscr{O}_I = \mathscr{O}_S[t]$ and $\mathscr{O}_{I'} = \mathscr{O}_S[t']$ and let $x \in L(I)$ and $y \in L(I')$[4]. The left translation λ_x (resp. λ_y) is an S-automorphism of $G_{I\times I'}$ which induces the identity on $G_{I'}$ (resp. G_I) and which corresponds to an $\mathscr{O}_S$-automorphism

$$u = \mathrm{id} + td_x, \qquad (\text{resp.} \quad v = \mathrm{id} + t'd_y)$$

of $\mathscr{O}_{G_{I\times I'}} = \mathscr{O}_G[t,t']/(t^2, t'^2)$, where d_x, d_y are $\mathscr{O}_S$-derivations of $\mathscr{O}_G$ invariant under right translations. As the correspondence of S-automorphisms of $G_{I\times I'}$ and $\mathscr{O}_S$-automorphisms of $\mathscr{O}_{G_{I\times I'}}$ is contravariant, $\lambda_x\lambda_y\lambda_x^{-1}\lambda_y^{-1}$ corresponds to $v^{-1}u^{-1}vu = \mathrm{id} + tt'(d_yd_x - d_xd_y)$. We then deduce from Remark 12.2.81 that the map $x \mapsto -d_x$ is an isomorphism of Lie algebras. The preceding argument is valid for $\mathfrak{Lie}(G/S)(S') = \mathfrak{Lie}(G_{S'}/S')(S')$ for any $S' \to S$, so we recover the following classical definition:

Proposition 12.2.85. *Via the isomorphism $x \mapsto -d_x$, $\mathfrak{Lie}(G/S)$ is identified with the functor which associates any $S' \to S$ to the Lie algebra of derivations of $G_{S'}$ over S' invariant under right translations.*

As we have seen in Example 12.2.77 that any representable group is good, we conclude the following corollary:

Corollary 12.2.86. *Any representable grop is very good.*

Let $e : S \to G$ be the unit section of G. Put $\omega^1_{G/S} = e^*(\Omega^1_{G/S})$ and recall that (cf. Proposition 12.2.15) $\mathfrak{Lie}(G/S)$ is represented by the vector bundle $\mathrm{Lie}(G/S) = \mathbb{V}(\omega^1_{G/S})$. We then have assocaited functorially to any S-group scheme G a vector bundle $\mathbb{V}(\omega^1_{G/S})$ over S, which represents the functor $\mathfrak{Lie}(G/S)$, hence is endowed with the structure of a Lie algebra S-scheme over $\mathbb{O}_S$. Moreover, this construction commutes with base change and finite products.

[4] As before, we write $L = \mathfrak{Lie}(G/S)$.

Remark 12.2.87. Let $\pi : G \to S$ be the structural morphism. The $\mathscr{O}_G$-module $\Omega^1_{G/S}$ is evidently $(G \times_S G)$-equivariant and hence, by ([?] I, 6.8.1), we have $\Omega^1_{G/S} \cong \pi^*(\omega^1_{G/S})$. It follows for example that $\Omega^1_{G/S}$ is locally free (resp. locally free of finite rank) if $\omega^1_{G/S}$ is, which is in particular the case if S is the spectrum of a field (resp. if S is the spectrum of a field and G is locally of finite type over S). Moreover, by ([?] I, 6.8.2), $\omega^1_{G/S}$ is endowed with a canonical $\mathbb{O}_S[G]$-module structure, which induces over $\mathbb{V}(\omega^1_{G/S}) = \mathrm{Lie}(G/S)$ the adjoint representation.

On the other hand, e is an immersion, and is a closed immersion if G is separated over S (cf. Corollary 8.5.4). Hence $\omega^1_{G/S}$ is identified with $\mathscr{I}/\mathscr{I}^2$, where $\mathscr{I}$ is the quasi-coherent ideal defining $e(S)$ in an open subset U of G in which $e(G)$ is closed (if G is separated over S, we can put $U = G$, and if $G = \mathrm{Spec}(\mathscr{A}(G))$ is affine over S, $\mathscr{I}$ is none other than the augmented ideal of $\mathscr{A}(G)$, i.e. the kernel of $e^\sharp : \mathscr{A}(G) \to \mathscr{O}_S$).

Remark 12.2.88. We deduce from the isomorphism $\Omega^1_{G/S} \cong \pi^*(\omega^1_{G/S})$ that the $\mathscr{O}_S$-module $\omega^1_{G/S}$ is identified with the sheaf $\pi^G_*(\Omega^1_{G/S})$ of right invariant differentials of G over S, that is, the sheaf whose sections over an open subset U of S are the sections of $\Omega^1_{G/S}$ over $\pi^{-1}(U)$ which are invariant under right translations (cf. ([?] I, 6.8.3)).

We denote by $\mathscr{L}ie(G/S)$ the sheaf of sections of the vector bundle $\mathrm{Lie}(G/S) \to S$, which is the $\mathscr{O}_S$-module $(\omega^1_{G/S})^\vee = \mathscr{H}om_{\mathscr{O}_S}(\omega^1_{G/S}, \mathscr{O}_S)$ dual to $\omega^1_{G/S}$ (cf. Proposition 9.1.39). It is endowed with a Lie algebra structure over $\mathscr{O}_S$. As this construction does not commute with base change (in general), the Lie algebra structure on $\mathscr{L}ie(G/S)$ does not allow us to reconstruct the S-scheme structure on the $\mathbb{O}_S$-Lie algebra $\mathrm{Lie}(G/S)$. However, we have:

Proposition 12.2.89. *Suppose that $\omega^1_{G/S}$ is locally free of finite type. Then $\mathscr{L}ie(G/S)^\vee \cong (\omega_{G/S})^{\vee\vee} \cong \omega^1_{G/S}$ and hence*

$$\mathrm{Lie}(G/S) = \mathbb{V}(\omega^1_{G/S}) = \mathbb{V}(\mathscr{L}ie(G/S)^\vee) = \Gamma_{\mathscr{L}ie(G/S)}.$$

Proof. In fact, $\omega^1_{G/S}$ is reflexive if it is locally free of finite type, and the assertion follows from Corollary 12.1.27. □

Finally, let $G \to H$ be a monomorphism of group functors. Then $\mathfrak{Lie}(G/S) \to \mathfrak{Lie}(H/S)$ is also a monomorphism (cf. Proposition 12.2.23). As any monomorphism of vector bundles is a closed immersion[5], we obtain:

Corollary 12.2.90. *Let $G \to H$ be a monomorphism of S-groups.*

(i) *$\mathrm{Lie}(G/S) \to \mathrm{Lie}(H/S)$ is a closed immersion and hence $\omega^1_{H/S} \to \omega^1_{G/S}$ is an epimorphism.*

(ii) *If $\omega^1_{G/S}$ is locally free of finite type, then the corresponding morphism $\mathscr{L}ie(G/S) \to \mathscr{L}ie(H/S)$ is an isomorphism from $\mathscr{L}ie(G/S)$ to a submodule of $\mathscr{L}ie(H/S)$ which is locally a direct factor.*

Example 12.2.91. Let $S = \mathrm{Spec}(k)$ with k a field of characteristic $p > 0$. Let $\boldsymbol{\alpha}_{p,S}$ be the S-functor which to any S-scheme T associates

$$\boldsymbol{\alpha}_{p,S}(T) = \{x \in \mathscr{O}(T) : x^p = 0\}.$$

Then $\boldsymbol{\alpha}_{p,S}$ is represented by $\mathrm{Spec}(\mathscr{O}_S[X]/(X^p))$, and hence is a very good S-group. It is also endowed with an $\mathbb{O}_S$-module structure, which is not very good, because the canonical morphism

$$\boldsymbol{\alpha}_{p,S} \to \mathfrak{Lie}(\boldsymbol{\alpha}_{p,S}/S) = \mathbb{G}_{a,S}$$

is not bijective[6].

Example 12.2.92. Let Nil be the $\mathbb{Z}$-functor defined as follows: for any scheme S, $\mathrm{Nil}(S)$ is the nilideal of $\mathscr{O}_S$:

$$\mathrm{Nil}(S) = \{x \in \mathscr{O}(S) : \text{there exists } n \in \mathbb{N} \text{ such that } x^n = 0\}.$$

Let Nil^2, $\mathbb{O}_{\mathrm{red}}$ and F be the $\mathbb{Z}$-functors in groups which associate to any scheme S, respectively, the ideal $\mathrm{Nil}(S)^2$ and

$$\mathbb{O}_{\mathrm{red}}(S) = \mathscr{O}(S)/\mathrm{Nil}(S), \quad F(S) = \mathscr{O}(S)/\mathrm{Nil}(S)^2.$$

[5] Let $f : \mathscr{M} \to \mathscr{N}$ be a morphism of $\mathscr{O}_S$-modules and $\mathscr{P} = \mathrm{coker}\, f$. If $\mathbb{V}(\mathscr{N}) \to \mathbb{V}(\mathscr{M})$ is a monomorphism, the surjective morphism $\boldsymbol{S}(\mathscr{N}) \to \boldsymbol{S}(\mathscr{P})$ factors through $\mathscr{O}_S$, hence $\mathscr{P} = 0$.

[6] This can be deduced from the exact sequence (12.2.17), or we can also note that $\omega^1_{G/k} = k[X]$.

It is easily seen that $\mathfrak{Lie}(\mathbb{O}_{\mathrm{red}}/\mathbb{Z}) = 0$, hence the $\mathbb{O}_{\mathbb{Z}}$-module $\mathbb{O}_{\mathrm{red}}$ is not good (although it is a good $\mathbb{Z}$-group). If M, N are free $\mathbb{Z}$-modules of finite rank, we have

$$\mathrm{Nil}^2(I_S(M \oplus N)) = \mathrm{Nil}^2(S) \oplus \mathrm{Nil}^2(S) \otimes_{\mathbb{Z}} M \oplus \mathrm{Nil}(S) \otimes_{\mathbb{Z}} N$$

and hence

$$F(I_S(M \oplus N)) = F(S) \oplus \mathbb{O}_{\mathrm{red}}(S) \otimes_{\mathbb{Z}} M \oplus \mathbb{O}_{\mathrm{red}}(S) \otimes_{\mathbb{Z}} N.$$

We then deduce, on the one hand, that the $\mathbb{Z}$-functor F satisfies condition (E) and, on the other hand, that $\mathfrak{Lie}(F/\mathbb{Z}) = \mathbb{O}_{\mathrm{red}}$ (cf. (12.2.17)); as the latter is not a good $\mathbb{O}_{\mathbb{Z}}$-module, this shows that F is a $\mathbb{Z}$-group which satisfies condition (E) but is not good.

12.2.3 Calculation of some Lie algebras

12.2.3.1 Lie algebras of diagonalizable groups Let $G = D_S(M)$ be a diagonalizable group over S (cf. 12.1.2.4). The formation of $\mathfrak{Lie}(G/S)$ commutes with base change, so it suffices to consider this construction for $G = D(M)$. We then have

$$G(I_S) = \mathrm{Hom}_{\mathbf{Grp}}(M, \Gamma(I_S, \mathscr{O}_{I_S})^{\times}) = \mathrm{Hom}_{\mathbf{Grp}}(M, \Gamma(S, \mathscr{D}_{\mathscr{O}_S})^{\times}).$$

Now the section $S \to I_S$ induces a split exact sequence

$$1 \longrightarrow \Gamma(S, \mathscr{O}_S) \longrightarrow \Gamma(S, \mathscr{D}_{\mathscr{O}_S})^{\times} \longrightarrow \Gamma(S, \mathscr{O}_S)^{\times} \longrightarrow 0$$

which implies that $\mathfrak{Lie}(G)(S)$ is identified with $\mathrm{Hom}_{\mathbf{Grp}}(M, \mathbb{O}_S)$, endowed with the evident $\mathbb{O}(S)$-module structure. We then obtain by base change the following:

Proposition 12.2.93. *We have isomorphisms*

$$\mathcal{H}om_{S\text{-}\mathbf{Grp}}(M_S, \mathbb{O}_S) \xrightarrow{\sim} \mathfrak{Lie}(D_S(M)/S), \quad \mathcal{H}om_{\mathbf{Grp}}(\widetilde{M}_S, \mathscr{O}_S) \xrightarrow{\sim} \mathscr{L}ie(D_S(M)/S),$$

where, in the second isomorhism, $\widetilde{M}_S$ is the sheaf of constant group over S defined by M, and $\mathcal{H}om_{\mathbf{Grp}}$ is the sheaf of homomorphisms of groups.

Corollary 12.2.94. *If M is free of finite rank, then*

$$\Gamma_{\mathscr{L}ie(D_S(M)/S)} \xrightarrow{\sim} \mathfrak{Lie}(D_S(M)/S), \quad M^{\vee} \otimes_{\mathbb{Z}} \mathscr{O}_S \xrightarrow{\sim} \mathscr{L}ie(D_S(M)/S).$$

In particular, $\mathbb{O}_S \cong \mathfrak{Lie}(\mathbb{G}_{m,S}/S)$ and $\mathscr{O}_S \cong \mathscr{L}ie(\mathbb{G}_{m,S}/S)$.

Proof. The second isomorphism follows from Proposition 12.2.93 the isomorphism

$$M^{\vee} \otimes_{\mathbb{Z}} \mathscr{O}_S = \mathrm{Hom}_{\mathbb{Z}}(\widetilde{M}_S, \mathscr{O}_S) = \mathrm{Hom}_{\mathbf{Grp}}(\widetilde{M}_S, \mathscr{O}_S),$$

which it implies that $\Gamma_{\mathscr{L}ie(D_S(M)/S)} = \mathcal{H}om_{S\text{-}\mathbf{Grp}}(M_S, \mathbb{O}_S)$, whence the first isomorphism. □

12.2.3.2 Inverse system of group schemes Let (G_i, φ_{ij}) be an inverse system of group schemes over S indexed by a finite set I. For each i, the sequence

$$1 \longrightarrow \mathfrak{Lie}(G_i) \longrightarrow T_{G_i/S} \longrightarrow G_i \longrightarrow 1$$

is exact, so by passing to inverse limit, we obtain an exact sequence

$$0 \longrightarrow \varprojlim(\mathfrak{Lie}(G_i)) \longrightarrow \varprojlim T_{G_i/S} \longrightarrow \varprojlim G_i \longrightarrow 0$$

so we conclude that

$$\varprojlim(\mathfrak{Lie}(G_i)) \xrightarrow{\sim} \mathfrak{Lie}(\varprojlim G_i).$$

For example, an exact sequence of groups $e \to G' \to G \to G''$ gives an exact sequence of Lie algebras

$$0 \longrightarrow \mathfrak{Lie}(G') \longrightarrow \mathfrak{Lie}(G) \longrightarrow \mathfrak{Lie}(G'')$$

and the functor $\mathfrak{Lie}$ commutes with fiber products:

$$\mathfrak{Lie}(H_1 \times_G H_2) \cong \mathfrak{Lie}(H_1) \times_{\mathfrak{Lie}(G)} \mathfrak{Lie}(H_2).$$

In particular, if H_1 and H_2 are subgroups of G, then $\mathfrak{Lie}(H_1)$ and $\mathfrak{Lie}(H_2)$ are sub-$\mathscr{O}_S$-modules of $\mathfrak{Lie}(G)$, and we have

$$\mathfrak{Lie}(H_1 \cap H_2) = \mathfrak{Lie}(H_1) \cap \mathfrak{Lie}(H_2).$$

Example 12.2.95. Consider for example the subgroups SL_2 and $\mathbb{G}_m$ in GL_2 over a field k of characteristic 2. Then we have $\mathrm{SL}_2 \cap \mathbb{G}_m = \boldsymbol{\mu}_2$, and

$$\mathfrak{Lie}(\mathrm{SL}_2) \cap \mathfrak{Lie}(\mathbb{G}_m) = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} : a \in k \right\} = \mathfrak{Lie}(\mu_2).$$

Proposition 12.2.96. *Let $H \subseteq G$ be algebraic groups over a field k such that $\mathfrak{Lie}(H) = \mathfrak{Lie}(G)$. If H is smooth and G is connected, then $H = G$.*

Proof. Let $\mathfrak{g} = \mathfrak{Lie}(G)$ and $\mathfrak{h} = \mathfrak{Lie}(H)$. We recall that $\dim(\mathfrak{g}) = \dim(G)$, with equality if and only if G is smooth. From the inequalities

$$\dim(H) = \dim(\mathfrak{h}) = \dim(\mathfrak{g}) \geq \dim(G) \geq \dim(H)$$

we conclude that $\dim(\mathfrak{g}) = \dim(G)$, so G is smooth and $\dim(G) = \dim(H)$, hence $G = H$. □

Corollary 12.2.97. *Let H_1 and H_2 be smooth connected subgroups of G such that $\mathfrak{Lie}(H_1) = \mathfrak{Lie}(H_2)$. If $H_1 \cap H_2$ is smooth, then $H_1 = H_2$.*

Proof. In fact, from $\mathfrak{Lie}(H_1 \cap H_2) = \mathfrak{Lie}(H_1) \cap \mathfrak{Lie}(H_2) = \mathfrak{Lie}(H_1)$ we conclude that $H_1 \cap H_2 = H_1$, and similarly it equals to H_2. □

12.2.3.3 Normalizers and centralizers We recall that a sequence $0 \to F' \to F \to F'' \to 0$ of $\mathbb{O}_S$-modules is called **exact** if for any $S' \to S$ the sequence

$$0 \to F'(S') \to F(S') \to F''(S') \to 0$$

is exact. Similarly, a sequence $1 \to G' \to G \to G'' \to 1$ of S-groups is exact if for any $S' \to S$ the sequence of groups

$$1 \to G'(S') \to G(S') \to G''(S') \to 1$$

is exact.

Lemma 12.2.98. *Let $1 \to G' \to G \to G'' \to 1$ be an exact sequence of S-groups.*

(i) *The sequences*

$$1 \to T_{G'/S}(\mathscr{M}) \to T_{G/S}(\mathscr{M}) \to T_{G''/S}(\mathscr{M}) \to 1$$
$$1 \to \mathfrak{Lie}(G'/S, \mathscr{M}) \to \mathfrak{Lie}(G/S, \mathscr{M}) \to \mathfrak{Lie}(G''/S, \mathscr{M}) \to 1$$

are exact.

(ii) *Let $1 \to H' \to H \to H'' \to 1$ be a second exact sequence of groups; it is exact if and only if the following sequence is exact:*

$$1 \longrightarrow G' \times_S H' \longrightarrow G \times_S H \longrightarrow G'' \times_S H'' \longrightarrow 1$$

(iii) *If two of the S-groups G', G, G'' satisfy condition (E), then the third one satisfies condition (E).*

(iv) *If $0 \to F' \to F \to F' \to 0$ is an exact sequence of $\mathbb{O}_S$-modules and two of the modules F', F, F'' are good, the third one is good.*

(v) *If two of the S-groups are good, the third one is good.*

Lemma 12.2.99. *Let G be an S-group, E, F be G-objects, M be an $\mathbb{O}_S[G]$-module.*

(a) *The canonical homomorphism $E^G \times_S F^G \to (E \times_S F)^G$ is an isomorphism.*

(b) *If M is a good $\mathbb{O}_S$-module, so is M^G.*

If E is an S-group and F is a sub-S-group of E, we denote by E/F the S-functor which to any $S' \to S$ associates the set $E(S')/F(S')$ of classes $\bar{x} = xF(S')$, $x \in E(S')$. If E is an abelian group over S, then E/F is endowed with an abelian group structure.

Now let G be an S-group and H be a sub-S-group of G; put $E = \mathfrak{Lie}(G/S, \mathscr{M})$ and $F = \mathfrak{Lie}(H/S, \mathscr{M})$. The adjoint action of H on E stablizes F, hence induces an action of H over the S-functor E/F. For any $S' \to S$, we then have

$$(E/F)^H(S') = \{\bar{x} \in E(S')/F(S') : x_{S''}^{-1}\mathrm{Ad}(h)x_{S''} \in F(S'') \text{ for } S'' \to S', h \in H(S'')\}$$

where $x_{S''}$ denotes the image of x in $E(S'')$.

Theorem 12.2.100. *Let G be an S-group, H be a sub-S-group of G, $N = N_G(H)$ and $Z = Z_G(H)$.*

(i) *If the group law of $\mathfrak{Lie}(G/S, \mathscr{M})$ is abelian, then*

$$\mathfrak{Lie}(N/S, \mathscr{M})/\mathfrak{Lie}(H/S, \mathscr{M}) = \big(\mathfrak{Lie}(G/S, \mathscr{M})/\mathfrak{Lie}(H/S, \mathscr{M})\big)^H.$$

(ii) *If the group law of $\mathfrak{Lie}(G/S, \mathscr{M})$ is abelian, then $\mathfrak{Lie}(Z/S, \mathscr{M}) = \mathfrak{Lie}(G/S, \mathscr{M})^H$.*

(iii) *If G satisfies condition (E) (resp. if G and H satisfy condition (E)), then Z satisfies condition (E) (resp. N satisfies condition (E)).*

(iv) *If G is good (resp. very good), then Z is good (resp. very good).*

(v) *If G and H are good, then N is good. If moreover G is very good, then N is very good.*

Corollary 12.2.101. *We have $\mathfrak{Lie}(Z(G)/S) = \mathfrak{Lie}(G/S)^G$ if the group law of $\mathfrak{Lie}(G/S)$ is abelian.*

Corollary 12.2.102. *If the group law of $\mathfrak{Lie}(G/S)$ is abelian and H is a normal subgroup of G, then*

$$\mathfrak{Lie}(G/S, \mathscr{M})/\mathfrak{Lie}(H/S, \mathscr{M}) = \big(\mathfrak{Lie}(G/S, \mathscr{M})/\mathfrak{Lie}(H/S, \mathscr{M})\big)^H.$$

Let G be a good S-group acting linearly on a good $\mathbb{O}_S$-module F via

$$\rho : G \to \mathcal{A}ut_{\mathbb{O}_S}(F).$$

We have defined a corresponding linear representation

$$d\rho : \mathfrak{Lie}(G/S) \to \mathcal{E}nd_{\mathbb{O}_S}(F).$$

The subgroups $N_G(E)$ and $Z_G(E)$ are defined for any subset E of F. Similarly, for any $S' \to S$, we define

$$\begin{aligned} N_{\mathfrak{Lie}(G/S)}(E)(S') &= \{X \in \mathfrak{Lie}(G/S) : d\rho(X)E_{S'} \subseteq E_{S'}\}, \\ Z_{\mathfrak{Lie}(G/S)}(E)(S') &= \{X \in \mathfrak{Lie}(G/S) : d\rho(X)E_{S'} = 0\}. \end{aligned}$$

called the **normalizer** and **centralizer**, respectively, of E in F.

Theorem 12.2.103. *Let G be a good S-group acting linearly on a good $\mathbb{O}_S$-module F, and E be a sub-$\mathbb{O}_S$-module of F.*

(a) *We have $\mathfrak{Lie}(Z_G(E)/S) = Z_{\mathfrak{Lie}(G/S)}(E)$ and $Z_G(E)$ is a good S-group; it is very good if G is.*

(b) *Suppose that E is a good $\mathbb{O}_S$-module. Then $\mathfrak{Lie}(N_G(E)/S) = N_{\mathfrak{Lie}(G/S)}(E)$ and $N_G(E)$ is a good S-group; it is very good if G is.*

Example 12.2.104. Let G be a good S-group. Then Theorem 12.2.103 can be applied to the adjoint representation of G. Let E be a good submodule of $\mathfrak{Lie}(G/S)$, for which we can associate the normalizer and centralizer. By Theorem 12.2.103, their Lie algebras are respectively the normalizer and centralizer of E in $\mathfrak{Lie}(G/S)$, given by the usual definition:

$$\begin{aligned} N_{\mathfrak{Lie}(G/S)}(E)(S') &= \{X \in \mathfrak{Lie}(G/S) : d\rho(X)E_{S'} \subseteq E_{S'}\}, \\ Z_{\mathfrak{Lie}(G/S)}(E)(S') &= \{X \in \mathfrak{Lie}(G/S) : d\rho(X)E_{S'} = 0\}. \end{aligned}$$

Example 12.2.105. Let H be a sub-S-group of G, then $\mathfrak{Lie}(H/S)$ is a sub-$\mathbb{O}_S$-module of $\mathfrak{Lie}(G/S)$. Suppose that $\mathfrak{Lie}(H/S)$ is a good $\mathbb{O}_S$-module; we evidently have

$$N_G(H) \subseteq N_G(\mathfrak{Lie}(H/S)), \quad Z_G(H) \subseteq Z_G(\mathfrak{Lie}(H/S))$$

whence, by Theorem 12.2.103, we obtain

$$\mathfrak{Lie}(N_G(H)/S) \subseteq N_{\mathfrak{Lie}(G/S)}(\mathfrak{Lie}(H/S)), \quad \mathfrak{Lie}(Z_G(H)/S) \subseteq Z_{\mathfrak{Lie}(G/S)}(\mathfrak{Lie}(H/S)),$$

but none of these four inclusions is a priori an identity. In particular, if H is a normal subgroup of G, then $\mathfrak{Lie}(H/S)$ is an ideal of $\mathfrak{Lie}(G/S)$.

Example 12.2.106. Let k be an algebraically closed field of characteristic $p > 0$ and G be the algebraic group over k such that for any k-scheme S', $G(S')$ consists of the matrices

$$A(a,b) = \begin{pmatrix} a & 0 & 0 \\ 0 & a^p & b \\ 0 & 0 & 1 \end{pmatrix}, \quad a \in \Gamma(S', \mathscr{O}_{S'})^\times, b \in \Gamma(S', \mathscr{O}_{S'}).$$

Defined regular functions on G by $X : A(a,b) \mapsto a$ and $Y : A(a,b) \mapsto b$, then $\mathscr{O}(G) = k[X, Y, X^{-1}]$, which is an integral domain, and so G is connected and smooth. We note that

$$\begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_1^p & b_1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_2 & 0 & 0 \\ 0 & a_2^p & b_2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_1^p & b_1 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} a_2 & 0 & 0 \\ 0 & a_2^p & b_1 - a_2^p b_1 + a_1^p b_2 \\ 0 & 0 & 1 \end{pmatrix}$$

so G is not commutative and its center consists of the elements $A(a,b)$ with $a^p = 1$ and $b = 0$. It then follows that

$$\mathscr{O}(Z(G)) = \mathscr{O}(G)/(X^p - 1, Y) \cong k[X]/(X^p - 1),$$

which is not reduced (in fact $Z(G) = \boldsymbol{\mu}_p$). On the other hand, the kernel of the morphism $G(k[t]) \to G(k)$ is given by

$$\left\{ \begin{pmatrix} 1 + at & 0 & 0 \\ 0 & 1 & bt \\ 0 & 0 & 1 \end{pmatrix} : a, b \in k \right\},$$

so the Lie algebra of G is equal to

$$\mathfrak{Lie}(G/k) = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix} : a, b \in k \right\},$$

which is obviously commutative. Moreover,

$$\begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_1^p & b_1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 + a_2 t & 0 & 0 \\ 0 & 1 & b_2 t \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_1^p & b_1 \\ 0 & 0 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 1 + a_2 t & 0 & 0 \\ 0 & 1 & a_1^p b_2 t \\ 0 & 0 & 1 \end{pmatrix},$$

so the kernel of the morphism $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ consists of the elements $A(a,b)$ with $a^p = 1$, hence

$$\ker \mathrm{Ad} = \mathrm{Spec}(\mathscr{O}(G)/(X^p - 1)) = \mathrm{Spec}(k[X, Y]/(X^p - 1)),$$

which is not reduced, and so Ad is not smooth. We also note that

$$\dim(\mathfrak{z}(\mathfrak{g})) = 2 > \dim(\ker \mathrm{Ad}) = 1 > \dim(Z(G)) = 0.$$

Example 12.2.107. Let S be a scheme, F be the good $\mathbb{O}_S$-module $\mathbb{O}_S^2$ endowed with the natural action of the good S-group $G = \mathrm{GL}_{2,S}$, and E be the sub-$\mathbb{O}_S$-module of F formed by couples (x_1, x_2) such that x_2 is nilpotent. Put $N = N_G(E)$, then $\mathfrak{Lie}(N/S) = \mathfrak{Lie}(G/S)$ while, for any $S' \to S$, we have

$$N_{\mathfrak{Lie}(G/S)}(E)(S') = \left\{ \begin{pmatrix} a & b \\ x & c \end{pmatrix} : a, b, c, x \in \mathscr{O}(S'), x \text{ nilpotent} \right\}$$

hence $\mathfrak{Lie}(N_G(E)/S) \neq N_{\mathfrak{Lie}(G/S)}(E)$.

By considering the semi-direct product $G' = F \rtimes G$, we obtain a similar counter-example where E is a sub-$\mathbb{O}_S$-module of $\mathfrak{Lie}(G'/S)$. We also note that with the notations above, $E = \mathfrak{Lie}(H/S)$ where H is the subgroup $\mathbb{O}_S \oplus \mathrm{Nil}^2$ of F (that is, for any $S' \to S$, $H(S')$ is formed by couples (x_1, x_2) where $x_2 \in \mathrm{Nil}(S')^2$).

12.3 Equivalence relations and passing to quotient

12.3.1 Universally effective equivalence relations

12.3.1.1 Equivalence relations

Definition 12.3.1. Let $\mathcal{C}$ be a category. A **$\mathcal{C}$-equivalence relation** over $X \in \mathrm{Ob}(\mathcal{C})$ is defined to be a representable sunfunctor R of $X \times X$, such that for any $S \in \mathrm{Ob}(\mathcal{C})$, $R(S)$ is the graph of an equivalence relation over $X(S)$.

This definition is applicable in particular to the category $\widehat{\mathcal{C}}$. If we consider X as an object of $\widehat{\mathcal{C}}$, then a $\widehat{\mathcal{C}}$-equivalance relation over X is none other than a subfunctor R of $X \times X$ (not necessarily representable in $\mathcal{C}$) such that $R(S)$ is the graph of an equivalence relation on $X(S)$ for any $S \in \mathrm{Ob}(\mathcal{C})$. In fact, this conditon is evidently necessary. Conversely, if for any $S \in \mathrm{Ob}(\mathcal{C})$, $R(S)$ is the graph of an equivalence relation, then this equivalence relation extends to $R(F)$ for any $F \in \mathrm{Ob}(\widehat{\mathcal{C}})$ by declearing two morphisms $\phi, \psi : F \to R$ to be equivalent if, for any $S \in \mathrm{Ob}(\mathcal{C})$ and $x \in F(S)$, $\phi(x)$ is equivalent to $\psi(x)$ in $X(S)$.

If R is a $\mathcal{C}$-equivalence relation on X, we denote by $p_i : R \to X$ the morphism induced by the projection $\mathrm{pr}_i : X \times X \to X$. We then have a diagram

$$p_1, p_2 : R \rightrightarrows X.$$

A morphism $u : X \to Z$ is called **compatible with** R if $up_1 = up_2$. The cokernel in $\mathcal{C}$ of the couple (p_1, p_2) is also called the **quotient object** of X by R, and denoted by X/R. We then have an exact diagram

$$R \overset{p_1}{\underset{p_2}{\rightrightarrows}} X \xrightarrow{p} X/R$$

and X/R represents the covariant functor

$$\mathrm{Hom}_{\mathcal{C}}(X/R, Z) = \{\text{morphisms } X \to Z \text{ compatible with } R\}.$$

Since the quotient objects have been chosen in $\mathcal{C}$, the quotient X/R is unique (when it exists).

These definitions immediately generalize to $\widehat{\mathcal{C}}$-equivalence relations on X, but note that the Yoneda embedding functor $\mathcal{C} \to \widehat{\mathcal{C}}$ does not commutes with the formation of quotients, so the quotient X/R of X by R in $\mathcal{C}$ if not a priori a quotient of X by R in $\widehat{\mathcal{C}}$. Therefore, we will be careful not to identity $\mathcal{C}$ indiscriminetly with its image in $\widehat{\mathcal{C}}$ when dealing with questions involving passages to the quotient. In the following, by "equivalence relation", we simply mean $\widehat{\mathcal{C}}$-equivalence relations.

If X is an object of $\mathcal{C}$ over S, an **equivalence relation on X over S** is defined to be an equivalence relation R over X such that the structural morphism $X \to S$ is compatible with R. In this case, the canonical morphism $R \to X \times X$ then factors through the monomorphism

$$X \times_S X \to X \times X$$

and defines an equivalence relation over the object $X \to S$ of $\mathcal{C}_{/S}$. If the quotient X/R exists, it is endowed with a canonical morphism to S and the corresponding object of $\mathcal{C}_{/S}$ is a quotient of $X \in \mathrm{Ob}(\mathcal{C}_{/S})$ by the preceding equivalence relation. Conversely, if S is a squarable object of $\mathcal{C}$ and $Y \to S$ is a quotient of X by this equivalence relation (in $\mathcal{C}_{/S}$), then Y is a quotient by R in $\mathcal{C}$.

Definition 12.3.2. If X (resp. X') is an object of $\mathcal{C}$ endowed with an equivalence relation R (resp. R'), a morphism $u : X \to X'$ is called compatible with R and R' if the following equivalence relations are satisfied:

(i) For any $S \in \mathrm{Ob}(\mathcal{C})$, two points of $X(S)$ congruent modulo $R(S)$ are transformed by u to two points of $X'(S)$ congruent modulo $R'(S)$

(ii) There exists a morphism $R \to R'$ (necessarily unique) fitting into the diagram

$$\begin{tikzcd} R \arrow[r] \arrow[d] & R' \arrow[d] \\ X \times X \arrow[r, "u\times u"] & X' \times X' \end{tikzcd}$$

By the universal property of X/R, there then exists (if the quotients X/R and X'/R' exists) a unique morphism v fitting into the commutative diagram

$$\begin{tikzcd} X \arrow[r, "p"] \arrow[d, "u"] & X/R \arrow[d, "v"] \\ X' \arrow[r, "p'"] & X'/R' \end{tikzcd}$$

Definition 12.3.3. A sub-object Y of X is called **stable** under the equivalence relation R if the following equivalent conditions are satisfied:

(i) For any $S \in \mathrm{Ob}(\mathcal{C})$, the subset $Y(S)$ of $X(S)$ is stable under $R(S)$.

(ii) The inverse images of Y under p_1 and p_2 are identical.

A particular important case is the following: the quotient X/R exists and Y is the inverse image of a sub-object of X/R in X.

Definition 12.3.4. Let R be an equivalence relation over X and $X' \to X$ ve a morphism. The equivalence relation R' over X obtained by the Cartesian diagram

$$\begin{tikzcd} R' \arrow[r] \arrow[d] & R \arrow[d] \\ X' \times X' \arrow[r] & X \times X \end{tikzcd}$$

is called the inverse image of R in X'. In particular, if X' is a sub-object of X, the corresponding equivalence relation is called the induced relation on X', and denoted by $R_{X'}$.

The morphism $X' \to X$ is compatible with R' and R; we then have, if the quotients exist, a morphism $X'/R' \to X/R$. If X' is a sub-object of X, we shall see that in certain case we can prove that $X'/R' \to X/R$ is a monomorphism, hence identifies X'/R' with a sub-object of X/R. If this is the case, the inverse image of this sub-object in X will be a sub-object of X containing X' and stable under R, called the **saturation** of X' for the equivalence relation R.

Proposition 12.3.5. *If the sub-object Y of X is stable under R, we have two Cartesian squares for $i = 1, 2$:*

$$\begin{tikzcd} R_Y \arrow[r] \arrow[d, "p_i"'] & R \arrow[d, "p_i"] \\ Y \arrow[r] & X \end{tikzcd}$$

Proof. This follows from the definition of R_Y and the stability of Y under R. □

12.3.1.2 Equivalence relation defiend by a free group action

Definition 12.3.6. Let X be an object of $\mathcal{C}$ and H be a $\mathcal{C}$-group acting on X. We say that H acts freely on X if the following conditions are satisfied:

(i) For any $S \in \mathrm{Ob}(\mathcal{C})$, the group $H(S)$ acts freely on $X(S)$.

(ii) The morphism of functors $H \times X \to X \times X$ defined by $(h, x) \mapsto (hx, x)$ is a monomorphism.

If H acts freely on X, the image of $H \times X$ by the morphism in (ii) is an equivalence relation on X, called the **equivalence relation defined by the action of $\boldsymbol{H}$ over $\boldsymbol{X}$**. The quotient of X by this equivalence relation, if exists, is denoted by $H\backslash X$. It represents the following covariant functor: if Z is an object of $\mathcal{C}$, we have

$$\mathrm{Hom}(H\backslash X, Z) = \{\text{morphisms } X \to Z \text{ invariant under } H\}$$

where a morphism $f : X \to Z$ is invariant under H if for any $S \in \mathrm{Ob}(\mathcal{C})$, the corresponding morphism $X(S) \to Z(S)$ is invariant under the group $H(S)$.

Lemma 12.3.7. *Let H be a group acting freely on X and Y be a sub-object of X. The following conditions are equivalent:*

(i) *Y is stable under the equivalence relation defined by H.*

(ii) *For any $S \in \mathrm{Ob}(\mathcal{C})$, the subset $Y(S)$ of $X(S)$ is stable under $H(S)$.*

(iii) *There exists a morphism f (necessarily unique) fitting into the commutative diagram*

$$\begin{tikzcd} H \times Y \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ H \times X \arrow[r] & X \end{tikzcd}$$

Under these conditions, f defines a morphism of $\widehat{\mathcal{C}}$-groups $H \to \mathcal{A}ut(Y)$ and the equivalence relation over Y defined by H is the induced one from X.

Proof. The proof is immediate, by the definition of stable objects and the equivalence relation induced by H. The operation of H on Y is called the induced action. □

Now consider the following situation: H and G are two $\mathcal{C}$-groups and we are given a group morphism $u : H \to G$. Then H acts on G by translations (we put $h \cdot g = u(h)g$) and it acts freely on G if and only if u is a monomorphism. In this case, the quotient of G by this action of H is denoted (if exists) by $H\backslash G$. Similarly, we can define a right action of H on G, and a quotient G/H. These quotients are functorial relative to the two groups. More precisely, we have the following lemma for right actions of H:

Lemma 12.3.8. *Let $u : H \to G$ and $u' : H' \to G'$ be two monomorphisms of $\mathcal{C}$-groups. Suppose that we are given a morphism of $\mathcal{C}$-groups $f : G \to G'$, then the following conditions are equivalent:*

(i) *f is compatible with the equivalence relations defined by H and H'.*

(ii) *For any $S \in \mathrm{Ob}(\mathcal{C})$, we have $f(u(H(S))) \subseteq u'(H(S))$.*

(iii) *There exists a morphism $g : H \to H'$ (necessarily unique and multiplicative) such that the following diagram is commutative:*

$$\begin{tikzcd} H \arrow[r, "g"] \arrow[d, "u"'] & H' \arrow[d, "u'"] \\ G \arrow[r, "f"] & G' \end{tikzcd}$$

Under these conditions, if the quotients G/H and G'/H' exist, there is a unique morphism $\bar{f}$ fitting into the commutative diagram

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[d, "p"'] & G' \arrow[d, "p'"] \\ G/H \arrow[r, "\bar{f}"] & G'/H' \end{tikzcd}$$

Proof. The first assertion can be verified element-wisely, and the second one then follows from (i). □

We can then translate the notions introduced above for general equivalence relations to the present situation. Let us simply point out the following lemma, whose proof is immediate by reduction to the set case:

Lemma 12.3.9. *Let $u : H \to G$ be a monomorphism of $\mathcal{C}$-groups and G' be a sub-group of G. For a sub-object G' of G to be stable under the equivalence relation defined by H, it is necessary and sufficient that u factors through the canonical monomorphism $G' \to G$. In this case, the induced action of H on G' is none other than that deduced by the monomorphism $H \to G'$ factorizing u.*

12.3.1.3 Universally effective equivalence relations

Definition 12.3.10. Let $f : X \to Y$ be a morphism. The image of the canonical monomorphism

$$X \times_Y X \to X \times X$$

then defines a $\widehat{\mathcal{C}}$-equivalence relation on X, called the **equivalence relation defined by f over X** and denoted by $R(f)$.

Definition 12.3.11. Let R be an equivalence relation over X. We say that R is effective if

(i) R is representable (i.e. is a $\mathcal{C}$-equivalence relation);

(ii) the quotient Y/R exists in $\mathcal{C}$;

(iii) the diagram

$$R \overset{p_1}{\underset{p_2}{\rightrightarrows}} X \xrightarrow{p} Y$$

makes R the fiber product of X over Y, that is, R is the equivalence relation defined by p.

If R is an effective equivalence relation over X, then p is an effective epimorphism. If $f : X \to Y$ is an effective epimorphism, then $R(f)$ is an effective equivalence relation over X whose quotient is Y. There then exists a correspondence bewteen effective equivalence relations over X and effective quotients of X.

Definition 12.3.12. An equivalence relation R over X is called **universally effective** if the quotient $Y = X/R$ exists and if, for any $Y' \to Y$, the fiber product $X' = X \times_Y Y'$ and $R' = R \times_Y Y'$ exist and R' is a fiber product of X' over Y'. Equivalently, this amouts to saying that R is effective and $p : X \to X/R$ is a universally effective epimorphism.

Remark 12.3.13. Suppose that $\mathcal{C}$ is the category of S-schemes and denote by $\mathbb{G}_{a,S}$ the additive group over S. Let $R \subseteq X \times_S X$ be a universally effective equivalence relation and $p : X \to Y$ be the quotient. Then, for any open subset U of Y, $\mathscr{O}(U) = \mathrm{Hom}_S(U, \mathbb{G}_{a,S})$ is the set of elements $\phi \in \mathscr{O}(p^{-1}(U)) = \mathrm{Hom}_S(p^{-1}(U), \mathbb{G}_{a,S})$ such that $\phi \circ p_1 = \phi \circ p_2$. In particular, if R is given by a free right action over X of a group H, then $\mathscr{O}(U)$ is the set of $\phi \in \mathscr{O}(p^{-1}(U))$ such that $\phi(xh) = \phi(x)$ for any $S' \to S$ and $x \in X(S')$, $h \in H(S')$.

Proposition 12.3.14. *Let R be a universally effective equivalence relation over X, $f : X \to Z$ be a morphism compatible with R, with a factorization $g : X/R \to Z$. The following conditions are equivalent:*

(i) *g is a monomorphism;*

(ii) *R is the equivalence relation defined by f.*

Proof. In fact, (i) clearly implies (ii), and the converse follows from **??**. □

Definition 12.3.15. Let H be a $\mathcal{C}$-group acting freely on X. We say that H acts **effectively** on X, or the action of H on X is **effective** (resp. **universally effective**), if the equivalence relation defined by H is effective (resp. universally effective).

In practive, it is often difficult to characterize universally effective epimorphisms. We often have, however, a certain number of morphisms of this type, for example, faithfully flat and quasi-compact morphisms of schemes. This leads to the following definition: Let $\mathcal{M}$ be a family of morphisms of $\mathcal{C}$ satisfying the following properties:

(M1) $\mathcal{M}$ is *stable under base change*, i.e. for any morphism $u : T \to S$ in $\mathcal{M}$ is squarable and for any $S' \to S$, $u' : T \times_S S' \to S'$ belongs to $\mathcal{M}$.

(M2) The composition of two morphisms in $\mathcal{M}$ belongs to $\mathcal{M}$.

(M3) Any isomorphism belongs to $\mathcal{M}$.

(M4) Any morphism in $\mathcal{M}$ is an effective epimorphism.

Note that (M1) and (M2) imply:

(M1′) The Cartesian product of two morphisms in $\mathcal{M}$ is in $\mathcal{M}$: Let $u : X \to Y$ and $u' : X' \to Y'$ be two S-morphisms belonging to $\mathcal{M}$. If $Y \times_S Y'$ exsits, then $X \times_S X'$ exists and $u \times_S u'$ belongs to $\mathcal{M}$.

This follows from the diagram

$$\begin{tikzcd}
 & Y' & X' \arrow[l, "u'"'] \\
X \arrow[d, "u"] & X \times_S Y' \arrow[l] \arrow[u] \arrow[d] & X \times_S X' \arrow[l] \arrow[u] \arrow[dl, "u \times_S u'"] \\
Y & Y \times_S Y' \arrow[l] &
\end{tikzcd}$$

Similarly, (M1) and (M4) imply:

(M4′) Any morphism of $\mathcal{M}$ is a universally effective epimorphism.

The family $\mathcal{M}_0$ of universally effective morphisms verifies the conditions (M1)–(M4). In fact, (M1), (M3) and (M4) follows by definition, (M2) follows from ([?] IV, 1.8). In the following, we suppose that $\mathcal{M}$ is a family of morphisms in $\mathcal{C}$ verifying the above conditions. In particular, our result is applicable to the family $\mathcal{M}_0$.

Definition 12.3.16. We say that an equivalence relation R over X is **of type** $\mathcal{M}$ if it is representable and if $p_1 \in \mathcal{M}$[7]. We say that R is $\mathcal{M}$**-effective** if it is effective and if the canonical morphism $X \to X/R$ belongs to $\mathcal{M}$. Finally, we say the quotient Y of X is $\mathcal{M}$**-effective** if the canonical morphism $X \to Y$ belongs to $\mathcal{M}$.

Proposition 12.3.17. *Let $\mathcal{M}$ be a family of morphisms in $\mathcal{C}$ as above.*

(a) *An $\mathcal{M}$-effective equivalence relation is of type $\mathcal{M}$ and universally effective.*

(b) *An $\mathcal{M}$-effective quotient is universally effective.*

(c) *The map $R \mapsto X/R$ and $p \mapsto R(p)$ is a bijective correspondence from the set of effective equivalence relations over X to the set of $\mathcal{M}$-effective quotients of X.*

(d) *$\mathcal{M}_0$-effectivity is equivalent to universally effectivity.*

Proof. Let R be $\mathcal{M}$-effective, so that we have a Cartesian square

$$\begin{tikzcd}
R \arrow[r, "p_2"] \arrow[d, "p_1"'] & X \arrow[d, "p"] \\
X \arrow[r, "p"] & X/R
\end{tikzcd}$$

and $p \in \mathcal{M}$. By (M1), p_1 and p_2 belong to $\mathcal{M}$, so R is of type $\mathcal{M}$.

Put $Y = X/R$ and let $Y' \to Y$ be a morphism. By (M1), the fiber products $X' = X \times_Y Y'$ and $R' = R \times_Y Y'$ exist and the morphisms $X' \to Y'$ and $p_i' : R' \to X'$ belong to $\mathcal{M}$. Finally, as $R = X \times_Y X$, we obtain, by associativity of fiber products:

$$R' = X \times_Y X \times_Y Y' = X' \times_{Y'} X'$$

so R' is $\mathcal{M}$-effective and in particular R is universally effective. This proves (a) and also (d). The assertions of (b) and (c) then follows from this and the definition. □

Example 12.3.18. Let H be an S-group whose structural morphism belongs to $\mathcal{M}$. If H acts freely on the S-object X, then it defines an equivalence relation of type $\mathcal{M}$. In fact, by (M1) the fiber product $H \times_S X$ exists and $p_2 : H \times_S X \to X$ belongs to $\mathcal{M}$. We say that the operation of H is $\mathcal{M}$**-effective** if the equivalence relation over X defined by H is $\mathcal{M}$-effective.

[7] This by (M2) and (M3) implies $p_2 \in \mathcal{M}$, since p_1 and p_2 are exchanged by an isomorphism of $X \times X$.

Proposition 12.3.19 (**$\mathcal{M}$-effectivity and Base Change).** *Let R be an $\mathcal{M}$-effective equivalence relation on X over S and put $Y = X/R$. Let $S' \to S$ be a base change morphism such that $Y' = Y \times_S S'$ exists. Then $X' = X \times_S S'$ exists, $R' = R \times_S S'$ exists and is an $\mathcal{M}$-effective equivalence relation on X' over S' and $X'/R' \cong (X/R)'$.*

Proof. In fact, the canonical morphisms $X \to Y$ and $R \to Y$ belong to $\mathcal{M}$, hence by (M1′), X' and R' are representable. By associativity of fiber products, R' is the equivalence relation defined by the canonical morphism $X' \to Y'$ which belongs to $\mathcal{M}$, whence the conclusion. □

Proposition 12.3.20 (**$\mathcal{M}$-effectivity and Cartesian Product).** *Let R (resp. R') be an $\mathcal{M}$-effective equivalence relation on X (resp. X') over S. If $(X/R) \times_S (X'/R')$ exists, then $X \times_S X'$ exists, $R \times_S R'$ is an $\mathcal{M}$-effective equivalence relation on $X \times_S X'$ over S and*

$$(X \times_S X')/(R \times_S R') \cong (X/R) \times_S (X'/R').$$

Proof. Put $Y = X/R$ and $Y' = X'/R'$. By (M1′), the fiber product $X \times_S X'$ exists and the canonical morphism $q : X \times_S X' \to Y \times_S Y'$ belongs to $\mathcal{M}$. Now the formula

$$(X \times X') \times_{Y \times Y'} (X \times X') \cong (X \times_Y X) \times (X' \times_{Y'} X')$$

(where the product without subscript is taken over S) shows that $R \times_S R'$ is the equivalence relation defined by q on $X \times_S X'$, whence the proposition. □

Suppose that $\mathcal{C}$ possesses a final object e and let $f : G \to G'$ be a morphism of $\mathcal{C}$-groups such that $f \in \mathcal{M}$. Then by (M1), the kernel $\ker f$ is representable by $e \times_{G'} G$, and the morphism $\ker f \to e$ belongs to $\mathcal{M}$. On the other hand, the equivalence relation defined by f is the same as that defined by the action of $\ker f$ (right, say) over G, that is, the image of the morphism $G \times \ker f \to G \times G$, defined by $(g, h) \mapsto (g, gh)$.

Corollary 12.3.21. *Suppose that $\mathcal{C}$ possesses a final object e and let $f : G \to G'$ be a morphism of $\mathcal{C}$-groups such that $f \in \mathcal{M}$. Then the action of* $\ker f$ *on G is $\mathcal{M}$-effective and G' is the the quotient $G/$* $\ker f$.

Proof. Since f is a universally effective epimorphism by (M4′), G' is identified with the quotient of G by the equivalence relation defined by f, that is, by the action of $\ker f$. Since $\ker \to e$ belongs to $\mathcal{M}$, this equivalence relation is therefore representable by (M1), and we conclude the corollary. □

12.3.1.4 Construction of quotients by descent

Definition 12.3.22. We say that a descent data over X' relative to $S' \to S$ is **effective** if X' endowed with the descent data is isomorphic to the inverse image over S' of an object X over S.

If $S' \to S$ is a descent morphism, then the X in the above definition is unique up to unique isomorphism. The morphism $S' \to S$ is an effective descent morphism if it is a descent morphism and any descent data relative to $S' \to S$ is effective.

Now consider an equivalence relation R over an object X over S. Let X' (resp. X'', resp. X''') be the inverse image of X over S', $S'' = S' \times_S S'$ and $S''' = S' \times_S S' \times_S S'$ and let R', R'', R''' be the induced equivalence relations of R by inverse image. Suppose that the equivalence relation R' on X' is $\mathcal{M}$-effective, and consider the quotient $Y' = X'/R'$ which is an object over S'. Its inverse images under the two projections from S'' are isomorphic to X''/R'' by Proposition 12.3.19, so the S'-object Y' is endowed with a canonical glueing data. Using the same uniqueness for X'''/R''', we see that this is a descent data (note that we have implicitly assumed have all these fiber products exist, for example if $S' \to S$ is squarable).

Proposition 12.3.23. *Let R be an equivalence relation on an object X over S, and $S' \to S$ be a universally effective epimorphism. Suppose that any S-morphism whose inverse image over S' belongs to $\mathcal{M}$ is itself in $\mathcal{M}$. Then the following conditions are equivalent:*

(i) *R is $\mathcal{M}$-effective on X;*

(ii) *R' is $\mathcal{M}$-effective and the canonical descent date over X'/R' is effective.*

Moreover, if this is the case, the descent object of X'/R' is canonically isomorphic to X/R.

Proof. The fact that (i) implies (ii) follows directly from the definition of $\mathcal{M}$-effectivity and Proposition 12.3.17 (a). If the converse is true, then the last assertion follows from the fact that a universally effective epimorphism is a descent morphism, so the descent object is unique (up to isomorphism).

We now prove that (ii)⇒(i). Let $Y' = X'/R'$ and Y be the descent object. As the canonical morphism $p' : X' \to X'/R' = Y'$ is compatible with the descent data (its inverse image over S'' coincides with the canonical morphism $X'' \to X''/R''$ by Proposition 12.3.19), it comes from an S-morphism $p : X \to Y$. As p' belongs to $\mathcal{M}$, it follows from the hypothesis made on the morphism $S' \to S$ that p also belongs to $\mathcal{M}$. As p' is compatible with the equivalence relation R', p is compatible with R, since a universally effective epimorphism is a descent morphism. We then have a morphism

$$R \to X \times_Y X.$$

To see that R is $\mathcal{M}$-effective and that Y is isomorphic to X/R, it suffices to prove that this morphism is an isomorphism. Now since R' is effective, this becomes an isomorphism after base change to S', and it is therefore an isomorphism for the same reason. □

We note that the hypothesis of Proposition 12.3.23 is verified if we take $\mathcal{M} = \mathcal{M}_0$ to be the family of universally effective epimorphisms and if $\mathcal{C}$ possesses fiber products (cf. [**?**], IV, Corollaire 1.10). We then deduce the following corollary:

Corollary 12.3.24. *Suppose that $\mathcal{C}$ possesses fiber products (over S). Let R be an equivalence relation on X over S and $S' \to S$ be a universally effective epimorphism. Then the following conditions are equivalent:*

(i) *R is universally effective on X;*

(ii) *R' is universally effective and the canonical descent date over X'/R' is effective.*

Moreover, if this is the case, the descent object of X'/R' is canonically isomorphic to X/R.

12.3.2 Equivalence relations in the category of sheaves

12.3.2.1 Equivalence relations in $\widetilde{\mathcal{C}}$ Let $\mathcal{C}$ be a site and $\widetilde{\mathcal{C}}$ be the category of sheaves over $\mathcal{C}$. Let $i : \mathcal{C} \to \widehat{\mathcal{C}}$ be the inclusion functor.

Proposition 12.3.25. *Any equivalence relation in $\widetilde{\mathcal{C}}$ is universally effective: let R be a $\widetilde{\mathcal{C}}$-equivalence relation on the sheaf X, then the sheaf associated with the separated presheaf*

$$i(X)/i(R) : S \mapsto X(S)/R(S)$$

is a universally effective quotient sheaf of X by R.

Proof. Let $X/R = (i(X)/i(R))^{\#}$ be the quotient sheaf of X by R, which exists by ([**?**], IV, 4.4.1(ii)). It is necessary to show that $X \to X/R$ is a universally effective epimorphism, and that the morphism $f : R \to X \times_{X/R} X$ is an isomorphism. The first assertion follows from the proof of ([**?**], IV, 4.4.3). As for f, it comes from the sheafification of the morphism $i(R) \to i(X) \times_{i(X/R)} i(X)$, or, as $i(X)/i(R)$ is separated ([**?**], IV, 4.4.5(ii)) so that $i(X)/i(R) \to i(X/R)$ is a monomorphism, from the canonical morphism $i(R) \to i(X) \times_{i(X)/i(R)} i(X)$.

We are therefore reduced to the same assertion for the category of presheaves. But $i(X)/i(R)$ is the presheaf $S \mapsto X(S)/R(S)$ and we are then reduced to the same assertion for the category of sets, which is immediate. □

Proposition 12.3.26. *Under the conditions of Proposition 12.3.25, let Y be a subsheaf of X. Denote by R_Y the equivalence relation induced on Y by R, then the canonical morphism $Y/R_Y \to X/R$ is a monomorphism: it identifies Y/R_Y with the subsheaf of X/R, which is the image sheaf of the composition morphism $Y \to X \to X/R$.*

Proof. The morphism of presheaves

$$i(Y)/i(R_Y) = i(Y)/i(R)_{i(Y)} \to i(X)/i(R)$$

is a monomorphism. As the functor # is left exact, it preserves monomorphisms and hence $Y/R_Y \to X/R$ is a monomorphism. The last assertion then follows from the commutative diagram

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & X \arrow[d] \\ Y/R_Y \arrow[r] & X/R \end{tikzcd}$$

and the fact that $Y \to Y/R_Y$ is covering. □

In view of 12.3.26, we can identify Y/R_Y with a subsheaf of X/R.

Proposition 12.3.27. *Let R be a $\widetilde{\mathcal{C}}$-equivalence relation on a sheaf X. For any subsheaf Y of X stable under R, denote by $Y' = Y/R_Y$ the quotient considered as a subsheaf of $X' = X/R$. Then $Y = Y' \times_{X'} X$ and the maps $Y \mapsto Y/R_Y$ and $Y' \mapsto Y' \times_{X'} X$ give a bijective correspoondence between the set of subsheaves Y of X stable under R and the set of subsheaves Y' of X'.*

Proof. If Y' is a subsheaf of X', then $Y' \times_{X'} X$ is a subsheaf of X stable under R, and we have $(Y' \times_{X'} X)/R = Y'$. If Y' is obtained by passing to quotient of a subsheaf Y of X, then Y is a subobject of $Y' \times_{X'} X$. It then suffices to show that if we have two subobjects Y_1 and Y_2 of X, stable under R and $Y_1 \subseteq Y_2$, and if the quotients Y_1/R_{Y_1} and Y_2/R_{Y_2} are identical, then $Y_1 = Y_2$. For this, we are evidently reduced to the same assertion in the case $Y_2 = X$. Denote then by P (resp. Q) the presheaf $i(X)/i(R)$ (resp. $i(Y)/i(R_Y)$), the diagram

$$\begin{tikzcd} Y \arrow[r] \arrow[d, hook] & Q \arrow[d, hook] \\ X \arrow[r] & P \end{tikzcd}$$

is Cartesian. As we have a commutative diagram

$$\begin{tikzcd} Q \arrow[r, hook] \arrow[d, hook] & Q^{\#} \arrow[d, equal] \\ P \arrow[r, hook] & P^{\#} \end{tikzcd}$$

and $Q \mapsto Q^{\#}$ is covering, the monomorphism $Q \hookrightarrow P$ is covering, so Q is a refinement of P. By base change, Y is then a refinement of X. As X and Y are both sheaves, we conclude that $Y = X$. □

In particular, if Y is a subsheaf of X and $Y' = Y/R_Y$, then the preceding correspondence ddefines a subsheaf $\overline{Y}$ of X, stable under R, containing Y and minimal with these properties; this subsheaf is called the saturation of Y for the equivalence relation R.

12.3.2.2 Description of the quotient of a sheaf by an equivalence relation

Now assume that the topology of $\mathcal{C}$ is subcanonical. In this case, we know that any covering sieve is universally effective epimorphic, and the canonical functor $i : \mathcal{C} \to \widehat{\mathcal{C}}$ factors through $\widetilde{\mathcal{C}}$.

Proposition 12.3.28. *Let R be a $\widetilde{\mathcal{C}}$-equivalence relation on a sheaf X. Let $F \in \mathrm{Ob}(\widehat{\mathcal{C}})$ be the presheaf defined as follows: for any $S \in \mathrm{Ob}(\mathcal{C})$[8],*

$$F(S) = \{\textit{sub-}S\textit{-sheaves } Z \textit{ of } X \times S \textit{ stable under } R \times S \textit{ whose quotient by } R_Z \textit{ is } S\}.$$

Then for any sheaf Y, $\mathrm{Hom}(Y, F)$ is identified with the set

$$\{\textit{sub-}Y\textit{-sheaves of } X \times Y \textit{ stable under } R \times Y \textit{ whose quotient is } Y\}.$$

In particular, the subsheaf R of $X \times X$ corresponds to a morphism $p : X \to F$ and the diagram

$$R \overset{p_1}{\underset{p_2}{\rightrightarrows}} X \xrightarrow{p} F$$

is exact, hence identifies F with the quotient sheaf X/R.

[8] $R \times S$ is the equivalence relation on $X \times S$ defined by $R \times S \subseteq X \times X \times S \times S$ (induced by the diagonal) and R_Z is the equivalence relation induced over Z.

Proof. Let $Q = X/R$. For any sheaf Y and any morphism $f : Y \to Q$ corresponding to a section $s : Y \to Q \times Y$, consider the diagram

$$\begin{tikzcd} & Z \arrow[r] \arrow[d, hook] & Y \arrow[d, hook, "s"] \\ R \times Y \arrow[r, shift left] \arrow[r, shift right] & X \times Y \arrow[r] & Q \times Y \end{tikzcd} \tag{12.3.1}$$

where the square is Cartesian. It is immediate from Proposition 12.3.27 that Z is a sub-Y-sheaf of $X \times Y$ stable under $R \times Y$ whose quotient is Y; conversely, any Z with these properties provides a unique section of $Q \times Y$ over Y. Taking Y to be representable or arbitrary, we obtain an isomorphism $Q \cong F$ and the desired form of $\mathrm{Hom}(Y, F)$. Finally, consider the canonical morphism $X \to Q$, we immdediately see that it corresponds to the sub-X-sheaf R of $X \times X$, which proves our assertion. □

Corollary 12.3.29. *Let G be a subfunctor of F such that $\mathrm{Hom}(X, G) \subseteq \mathrm{Hom}(X, F)$ contains R. Then the canonical morphism $p : X \to F$ factors through G. As p is covering, it follows that G is a refinement of F. In particular, any subsheaf G of F verifying the preceding condition is equal to F.*

Proof. By the identification of Proposition 12.3.28, the hypothesis implies that $p : X \to F$ belongs to the image of $\mathrm{Hom}(X, G)$, whence it factors through G. □

We now consider the case where X and R are representable. Let's first introduce some terminology. In addition to the conditions (M1)–(M4) introduced in 12.3.1.3, we will use other conditions on a family $\mathcal{M}$ of morphisms of $\mathcal{C}$ (for completeness, we recall conditions (M1)–(M3)):

(M1) $\mathcal{M}$ is stable under base change.

(M2) The composition of two elements of $\mathcal{M}$ belongs to $\mathcal{M}$.

(M3) Any isomorphism belongs to $\mathcal{M}$.

(M4$_\mathcal{T}$) Any element of $\mathcal{M}$ is covering.

(M5$_\mathcal{T}$) Let $f : X \to Y$ be a morphism in $\mathcal{C}$. If there exists a covering sieve $R \hookrightarrow Y$ such that for any $Y' \to R$, $X \times_Y Y' \to Y'$ belongs to $\mathcal{M}$, then f belongs to $\mathcal{M}$.

Recall that (M1) and (M2) implies

(M1′) The Cartesian product of two morphisms in $\mathcal{M}$ belongs to $\mathcal{M}$.

and (M1) and (M4$_\mathcal{T}$) implies (by [**?**], IV, 4.3.9):

(M4′) Any morphism in $\mathcal{M}$ is a universally effective epimorphism.

The preceding conditions are verified for the family of covering morphisms, denoted by $\mathcal{M}_\mathcal{T}$, if $\mathcal{C}$ possesses fiber products. The results we are going to establish for a family $\mathcal{M}$ satisfying these conditions will apply in particular to the family $\mathcal{M}_\mathcal{T}$. In particular, we can take for $\mathcal{T}$ the canonical topology and for $\mathcal{M}$ the family of universally effective epimorphisms.

Proposition 12.3.30. *Let $\mathcal{M}$ be a family of morphisms verifying conditions (M1)–(M5$_\mathcal{T}$). Let R be a $\widetilde{\mathcal{C}}$-equivalence relation on $X \in \mathrm{Ob}(\mathcal{C})$ of type $\mathcal{M}$, $\widetilde{X}$ be the sheaf associated with X, $\widetilde{R}$ the $\widetilde{\mathcal{C}}$-equivalence relation on $\widetilde{X}$ defined by R, and $\widetilde{X}/\widetilde{R}$ the quotient sheaf. For R to be $\mathcal{M}$-effective, it is necessary and sufficient that $\widetilde{X}/\widetilde{R}$ is representable, and in this case it is represented by the quotient X/R.*

Proof. Suppose that R is $\mathcal{M}$-effective and let $Y = X/R$. The canonical morphism $p : X \to Y$ belongs to $\mathcal{M}$, hence is covering by (M4$_\mathcal{T}$). The corresponding morphism

$$\tilde{p} : \widetilde{X} \to \widetilde{Y}$$

is then a universally effective epimorphism in $\widetilde{\mathcal{C}}$, hence identifies $\widetilde{Y}$ with the quotient of $\widetilde{X}$ by the equivalence relation R' defined by $\tilde{p}$. As the functor $\mathcal{C} \to \widetilde{\mathcal{C}}$ commutes with fiber products, R' is none other than $\widetilde{R}$, because R is the equivalence relation defined by R (since it is effective). We then conclude that $\widetilde{X}/\widetilde{R}$ is represented by Y.

Conversely, suppose that $\widetilde{X}/\widetilde{R}$ is represented by an object Y of $\mathcal{C}$. Let $p : X \to Y$ be the morphism induced by the canonical morphism $\widetilde{X} \to \widetilde{X}/\widetilde{R}$, which is a covering morphism by ([**?**], IV, 4.4.3). It is clear as before that R is the equivalence relation defined by p, so it remains to show that $p \in \mathcal{M}$. But the Cartesian square

$$\begin{tikzcd} R \arrow[r, "\cong"] & X \times_Y X \arrow[r, "p_1"] \arrow[d, "p_2"] & X \arrow[d, "p"] \\ & X \arrow[r, "p"] & Y \end{tikzcd}$$

shows that the base change of p by the covering morphism p belongs to $\mathcal{M}$ (since $p_2 \in \mathcal{M}$ by our hypothesis). We then conclude from (M1) and (M5$_\mathcal{T}$) that $p \in \mathcal{M}$. □

Corollary 12.3.31. *Let $\mathcal{M}$ be a family of morphisms verifying conditions (M1)–(M5$_\mathcal{T}$) and $f : G \to G'$ be a morphism of $\mathcal{C}$-groups belonging to $\mathcal{M}$. Suppose that* $\ker f$ *is representable (for example, if $\mathcal{C}$ has a final object e), then the equivalence relation on G defined by $H =$* $\ker f$ *is $\mathcal{M}$-effective and G' represents the quotient sheaf $\widetilde{G}/\widetilde{H}$ for the topology $\mathcal{T}$.*

Proof. This follows from Corollary 12.3.21 and Proposition 12.3.30. □

We are now in a position to state the main theroem of this paragraph. Before this, let recall the following result:

Proposition 12.3.32. *Let $\{S_i \to S\}$ be a covering family and Z be a sheaf over S. Suppose that for each i, the S_i-functor $Z \times_S S_i$ is represented by an object T_i. Then the family T_i is endowed with a canonical descent data relative to $\{S_i \to S\}$. For Z to be representable, it is necessary and sufficient that this descent data is effective, and in this case the descent object represents Z.*

Proof. By ([**?**], IV, 4.4.3), $\{S_i \to S\}$ is universally effective epimorphic in $\widetilde{\mathcal{C}}$, hence is a descent family in $\widetilde{\mathcal{C}}$. If Z is represented by an object T, the $T \times_S S_i$ (considered as sheaves) is isomorphic to $Z \times_S S_i$, hence the descent data over T_i is effective and the descent object (necessarily unique) is isomorphic to Z. Conversely, suppose that the canonical descent data over T_i is effective and let T be a descent object. As the family $\{S_i \to S\}$ is a descent family, there exsits an S-morphism $T \to Z$ whose base change to S_i is the canonical morphism $T_i \to Z \times_S S_i$. This morphism is therefore locally an isomorphism, and it follows from ([**?**], IV, 4.4.8) that it is an isomorphism. □

Theorem 12.3.33. *Let $\mathcal{M}$ be a family of morphisms verifying conditions (M1)–(M5$_\mathcal{T}$) and R be a $\mathcal{C}$-equivalence relation of type $\mathcal{M}$ on an object X of $\mathcal{C}$. Consider the functor $F \in \mathrm{Ob}(\widehat{\mathcal{C}})$ defined as follows:*

$$F(S) = \{\textit{sub-}S\textit{-sheaf } Z \textit{ of } X \times S \textit{ stable under } R \times S \textit{ whose quotient by } R_Z \textit{ is } S\}.$$

Let F_0 be the sub-functor of F such that $F_0(S)$ is formed by representable $Z \in F(S)$, that is,

$$F_0(S) = \left\{\begin{array}{l}\textit{sub-}\mathcal{C}_{/S}\textit{-objects } Z \textit{ of } X \times S \textit{ stable under } R \times S \textit{ such that } R_Z \textit{ is } \mathcal{M}\text{-} \\ \textit{effective and the quotient of } Z \textit{ by } R_Z \textit{ is } S\end{array}\right\}.$$

(a) *The morphism $p : X \to F$ defined by the sub-object R of $X \times X$ identifies F with the quotient sheaf of X by R.*

(b) *The following conditions are equivalent:*

  (i) *F is representable.*

  (ii) *F_0 is representable.*

  (iii) *R is $\mathcal{M}$-effective.*

  and under these conditions, we have $F = F_0 = X/R$.

(c) *Let $\mathcal{N}$ be a family of morphisms which is stable under base change and such that for any covering family $\{S_i \to S\}$ and any family $\{T_i \to S_i\}$ of morphisms in $\mathcal{N}$, any descent data on the T_i relative to $\{S_i \to S\}$ is effective. Suppose that X is squarable and the morphism $R \to X \times X$ belongs to $\mathcal{N}$, then $F_0 = F$.*

Proof. The proof of (i) follows from Proposition 12.3.28. As for (ii), we have seen the equivalence of (i) and (iii) as well as the equality $F = X/R$ (cf. Proposition 12.3.30). It remains to prove that (ii) or (iii) implies $F_0 = F$, but for this we first note that F_0 is indeed a sub-functor of F. In fact, for any $S \in \mathrm{Ob}(\mathcal{C})$ and $Z \in F_0(S)$, the morphism $Z \to S$ belongs to $\mathcal{M}$ and hence is squarable, so $Z \times_S S'$ belongs to $F_0(S')$ for any $S' \to S$. As $R \in F(X)$ belongs to $F_0(X)$, Corollary 12.3.29 shows that (ii) implies $F_0 = F$.

Now suppose that (iii) is satisfied and let Q be an object of $\mathcal{C}$ representing X/R. Then the morphism $X \to Q$ belongs to $\mathcal{M}$ and, for any $S \in \mathrm{Ob}(\mathcal{C})$ and any $Z \in F(S)$, the diagram (12.3.1) of Proposition 12.3.28 shows that $Z = S \times_{(Q\times S)} X \times S$ is representable, and $Z \to S$ belongs to $\mathcal{M}$, hence $Z \in F_0(S)$.

Finally, to prove (c), let $f : S \to F$ be a morphism corresponding to $Z \in F(S)$. We must show that f factors through F_0, which means Z is representable. For this, we first note that if f factors through X, then it is the image of an element $x_0 \in X(S)$, and the corresponding sheaf Z is defined by the Cartesian squares (since the morphism $p : X \to F$ corresponds to the subsheaf R of $X \times X$)

$$\begin{tikzcd}
Z \arrow[r] \arrow[d] & R_S \arrow[r] \arrow[d] & R \arrow[d] \\
X_S \arrow[r, "\mathrm{id}_{X_S}\times\tau_{x_0}"] & X_S \times_S X_S \arrow[r] & X \times X
\end{tikzcd}$$

where τ_{x_0} is the morphism $X_S \to X_S$ defined by $(x,s) \mapsto (x_0(s), s)$[9]. Moreover, as $R \to X \times X$ belongs to $\mathcal{N}$, so is $Z \to X_S$.

For the general case, as $X \to F$ is a covering morphism, there exists a covering family $\{S_i \to S\}$ and for each i a morphism $S_i \to X$ fitting into the diagram

$$\begin{tikzcd}
X \arrow[r] & F \\
S_i \arrow[r] \arrow[u] & S \arrow[u, "f"']
\end{tikzcd}$$

By the preceding arguments, the morphism $f_i : S_i \to F$ defined by this diagram belongs to $F_0(S_i)$ and corresponds to the subsheaf $Z \times_S S_i$ of X_{S_i}, and the morphism $Z \times_S S_i \to X_{S_i}$ belongs to $\mathcal{N}$. As the family $\{X_{S_i} \to X_S\}$ is covering, the descent data on $Z \times_S S_i$ provides a descent object over S by our hypothesis, which must represents Z in view of Proposition 12.3.32. ☐

Corollary 12.3.34. *Let R be an $\mathcal{M}$-effective equivalence relation on X. For any sheaf F, the map*

$$\mathrm{Hom}(X/R, F) \to \mathrm{Hom}(X, F)$$

identifies $\mathrm{Hom}(X/R, F)$ *with the subset formed by morphisms $X \to F$ compatible with R.*

Proof. By Theorem 12.3.33, X/R represents the quotient sheaf $\widetilde{X}/\widetilde{R}$, and the defining property of $\widetilde{X}/\widetilde{R}$ gives the assertion. ☐

Remark 12.3.35. In the hypothesis of Theorem 12.3.33 (iii), if we further suppose that the descent object T is such that the morphism $T \to S$ belongs to $\mathcal{N}$, then the inclusion morphism $Z \to X_S$ also belongs to $\mathcal{N}$, as it is obtained by the descent data on the morphisms $Z \times_S S_i \to X_{S_i}$, which are in $\mathcal{N}$.

Remark 12.3.36. We have proved the implications (iii)⇒(ii)⇒(i) and (iii)⇒$[F_0 = F = X/R]$ in Theorem 12.3.33 without resorting the "sufficient" part of Theorem 12.3.33, which is the only place we use condition (M5$_{\mathcal{T}}$). Therefore, they remain valid if $\mathcal{M}$ only satisfies conditions (M1)–(M4$_{\mathcal{T}}$). An example of such a family of that of squarable covering morphisms.

Corollary 12.3.37. *Under the conditions of Theorem 12.3.33 (ii), X/R is also the quotient sheaf of X by R for any intermediate topology between $\mathcal{T}$ and the canonical topology.*

Proof. If $\mathcal{T}'$ is an intermediate topology between $\mathcal{T}$ and the canonical topology, then $\mathcal{M}$ satisfies (M1)–(M4$_{\mathcal{T}'}$), so F_0 is identified with the quotient sheaf of X by R for $\mathcal{T}'$ (Remark 12.3.36), which is X/R. ☐

[9]Unwinding the definitions, we see that for any $S' \to S$, $Z(S)$ consists of elements $(x,s) \in X(S') \times S'$ such that $(x_0(s), x) \in R(S')$, so its quotient by R_Z is S.

Corollary 12.3.38. *Let R be a universally effective equivalence relation on X. Then the object X/R of $\mathcal{C}$ represents the quotient sheaf of X by R for the canonical topology. Moreover, $(X/R)(S)$ is the set of sub-$\mathcal{C}_{/S}$-objects Z of X_S stable under $R \times S$ such that R_Z is universally effective and the quotient of Z by R_Z is S.*

Corollary 12.3.39. *Let $\mathcal{M}$ be the family of squarable covering morphisms. If R is an $\mathcal{M}$-effective equivalence relation on X, then X/R of $\mathcal{C}$ represents the quotient sheaf of X by R and it also represents the functor F_0 of Theorem 12.3.33.*

While in questions involving exclusively projective limits (fiber products, algebraic structures, etc.) we can identify $\mathcal{C}$ indiscriminately with a full subcategory of $\widetilde{\mathcal{C}}$ or of $\widehat{\mathcal{C}}$, it is not the same in those which combine projective and inductive limits. In questions involving both projective limits and inductive limits (in particular passages to the quotient), we should consider the given category as embedded in the category of sheaves. Thus if R is a $\mathcal{C}$-equivalence relation on the object X of $\mathcal{C}$, X/R will denote the quotient sheaf of X by R (designated previously by $(i(X)/i(R))^{\#}$), so in the case where this sheaf is representable, the object representing it. The previous results show that in the most important cases, a quotient in $\mathcal{C}$ will also be a quotient in the category of sheaves.

We now give an example of the usage of effectivity criteria. As before, let $\mathcal{T}$ be a subcanonical topology on $\mathcal{C}$ and choose a family $\mathcal{M}$ of morphisms satisfying conditions (M1)–(M5$_{\mathcal{T}}$). We consider a family $\mathcal{N}$ of morphisms in $\mathcal{C}$ with the following properties:

(N1) $\mathcal{N}$ is stable under base change.

(N$_{\mathcal{T}}$) The morphisms of $\mathcal{N}$ have descent property for the given topology. That is, for any $S \in \mathrm{Ob}(\mathcal{C})$, any covering family $\{S_i \to S\}$ and any family $\{T_i \to S_i\}$ of morphisms in $\mathcal{N}$, any dascent data on T_i relative to $\{S_i \to S\}$ is effective, and if T is the descent object, the morphism $T \to S$ belongs to $\mathcal{N}$.

As any element of $\mathcal{M}$ is covering, (N$_{\mathcal{T}}$) implies the following property:

(N$_{\mathcal{M}}$) If $Y' \to X'$ belongs to $\mathcal{N}$ and $X' \to X$ belongs to $\mathcal{M}$, any descent data over Y' relative to $X' \to X$ is effective. If Y is the descent object, then $Y \to X$ belongs to $\mathcal{N}$.

A particular important example is the following: $\mathcal{C}$ is the category of schemes, $\mathcal{T}$ is the fpqc topology, $\mathcal{M}$ is the family of faithfully flat and quasi-compact morphisms, $\mathcal{N}$ is the family of closed immersions, or that of quasi-compact immersions.

By Theorem 12.3.33, we then have the following result (cf. Remark 12.3.35):

Proposition 12.3.40. *Let X be a squarable object in $\mathcal{C}$ and R be an equivalence relation on X of type $\mathcal{M}$ such that $R \to X \times X$ belongs to $\mathcal{N}$. Then the quotient sheaf X/R is defined by*

$$(X/R)(S) = \left\{\begin{array}{l}\text{sub-}\mathcal{C}_{/S}\text{-objects } Z \text{ of } X \times S \text{ stable under } R \times S \text{ such that } Z \to X_S \\ \text{belongs to } \mathcal{N}\text{, that } R_Z \text{ is } \mathcal{M}\text{-effective, and that the quotient of } Z \text{ by } R_Z \\ \text{is } S\end{array}\right\}.$$

Moreover, we have the following correspondence of stable subobjects of X and $\mathcal{M}$-effective equivalence relations.

Proposition 12.3.41. *Let $X \in \mathrm{Ob}(\mathcal{C})$ and R be an $\mathcal{M}$-effective equivalence relation on X.*

(a) *For any sub-object Y of X, stable under R and such that $Y \to X$ belongs to $\mathcal{N}$, the equivalence relation induced on Y by R is $\mathcal{M}$-effective and the quotient $Y/R_Y = Y'$ is a sub-object of $X' = X/R$ such that $Y' \to X'$.*

(b) *The map $Y \mapsto Y'$ is a bijection from the set of sub-objects Y of X stable under R such that $Y \to X$ belongs to $\mathcal{N}$ to the set of sub-objects Y' of X' such that $Y' \to X'$ belongs to $\mathcal{N}$. The inverse map is given by $Y' \to Y' \times_{X'} X$.*

Proof. As R is $\mathcal{M}$-effective, the morphism $X \to X'$ belongs to $\mathcal{M}$. Let Y' be a sub-object of X' such that the canonical morphism $Y' \to X'$ belongs to $\mathcal{N}$. Then, the sub-object $Y = Y' \times_{X'} X$ of X is stable under R, and the morphism $Y \to X$ (resp. $Y \to Y'$) belongs to $\mathcal{N}$ (resp. $\mathcal{M}$) since $\mathcal{N}$ and $\mathcal{M}$ are stable under base change. By Proposition 12.3.27, the quotient sheaf R/R_Y is represented by Y' and hence, by Proposition 12.3.30, R_Z is $\mathcal{M}$-effective.

Conversely, any sub-object Y of X, stable under R and such that the morphism $Y \to X$ belongs to $\mathcal{N}$, is obtained in this way. In fact, if Y is stable under R, its two inverse images in $R = X \times_{X'} X$ are identical and Y is endowed with a descent data relative to $X \to X'$; our assertion then follows from (N$_{\mathcal{M}}$). ∎

Corollary 12.3.42. *Let $X \in \mathrm{Ob}(\mathcal{C})$ and R be an $\mathcal{M}$-effective equivalence relation on X. Suppose that $R \to X \times X$ belongs to $\mathcal{N}$, then for any Y as in Proposition 12.3.41, $R_Y \to Y \times Y$ belongs to $\mathcal{N}$ and hence, by Proposition 12.3.40, we have*

$$(Y/R_Y)(S) = \left\{\begin{array}{l}\text{sub-}\mathcal{C}_{/S}\text{-objects } Z \text{ of } Y \times S \text{ stable under } R_Y \times S \text{ such that } Z \to Y_S \\ \text{belongs to } \mathcal{N}, \text{ that } R_Z \text{ is } \mathcal{M}\text{-effective, and that the quotient of } Z \text{ by } R_Z \\ \text{is } S\end{array}\right\}.$$

12.3.3 Passage to quotient and algebraic structures

12.3.3.1 Principal homogeneous bundles We recall that an object X in $\widehat{\mathcal{C}}$ with a (right) group action by a group functor H is called **formally principal homogeneous** under H if the canonical morphism

$$X \times H \to X \times X, \quad (x,h) \mapsto (x, xh)$$

is an isomorphism. Equivalently, this means for any $S \in \mathrm{Ob}(\mathcal{C})$, $X(S)$ is formally principal homogeneous under $H(S)$, which is therefore empty or principal homogeneous under $H(S)$. In particular, if we act H on itself by (right) translations, then H is formally principal homogeneous under itself. The H-object X is called trivial if it is isomorphic to H acted by right translations.

Proposition 12.3.43. *If X be formally principal homogeneous under H, we have an isomorphism*

$$\Gamma(X) \xrightarrow{\sim} \mathrm{Iso}_H(H, X)$$

of principal homogeneous sets under $\Gamma(H)$.

Proof. To any section x of X, we can associate the morphism $H \to X$ defined set-wise by $h \mapsto xh$. The assertion is then immediate. ☐

Corollary 12.3.44. *We have an isomorphism of H-objects*

$$X \xrightarrow{\sim} \mathcal{I}so_H(H, X).$$

Moreover, for X to be trivial, it is necessary and sufficient that X is formally principal homogeneous and possesses a global section.

Definition 12.3.45. Let $\mathcal{C}$ be a site. An S-object X with an action by H is called a **principal homogeneous bundle under** H if it is **locally trivial**, that is, if the following equivalent conditions are satisfied:

(i) The set of morphisms $T \to S$ such that (the functor) $X \times_S T$ is trivial under $H \times_S T$ is a refinement of S.

(ii) There exists a covering family $\{S_i \to S\}$ such that for each i, the S_i-functor $X \times_S S_i$ is trivial under $H \times_S S_i$.

Proposition 12.3.46. *Let $\mathcal{C}$ be a site and $\mathcal{M}$ be a family of morphisms in $\mathcal{C}$ satisfying conditions (M1)–(M5$_\mathcal{T}$) of 12.3.2.2. Let H be an S-group such that the structural morphism $H \to S$ belongs to $\mathcal{M}$ and P be an S-object acted by H. The following conditions are equivalent:*

(i) P is a principal homogeneous bundle under H.

(ii) P is formally principal homogeneous under H and the structural morphism $P \to S$ belongs to $\mathcal{M}$.

(iii) There exists a morphism $S' \to S$ in $\mathcal{M}$ such that the base change of P to S' is trivial, that is, $P \times_S S'$ is trivial under $H \times_S S'$.

(iv) H acts freely and $\mathcal{M}$-effectively on P and the quotient P/H is isomorphic to S.

Proof. We first note that (ii) and (iv) are equivalent, in view of the fact that, in either case, $P \to S$ belongs to $\mathcal{M}$, hence is squarable, which ensures the representability of $H \times_S P$ and $P \times_S P$. It is clear that (ii) implies (iii), because we can take $S' = P$, and the hypothesis that P is formally principal homogeneous implies that $P \times_S P$ is trivial under $H \times_S P$, since it has a section (the diagonal section $P \to P \times_S P$). On the other hand, (iii) implies (i), since $\{S' \to S\}$ is a covering family by condition (M4$_\mathcal{T}$). It then remains to show that (i)⇒(ii). In this case, the morphism of sheaves $P \times_S H \to P \times_S P$ is locally an isomorphism, hence an isomorphism ([?] IV, 4.5.8); P is then formally principal homogeneous. The structural morphism $P \to S$ is locally isomorphic to the structural morphism $H \to S$, which belongs to $\mathcal{M}$. It is then an element of $\mathcal{M}$ by (M1) and (M5$_\mathcal{T}$). ☐

We note that if H acts freely on an S-object X and $p : X \to Y = X/H$ is the quotient morphism, then we have an induced morphism

$$(H \times_S Y) \times_Y X = H \times_S X \to X.$$

Therefore, $H \times_S Y$ has an induced action on X over Y, and the quotient $X/H \times_S Y$ is Y. The equivalence of (i) and (iv) in Proposition 12.3.46 can therefore be generalized to the following proposition:

Proposition 12.3.47. *Under the same hypothesis of Proposition 12.3.46, assume that the topology $\mathcal{T}$ is subcanonical. Let H be an S-group and X be an S-object over which H acts (on right). Suppose that the structural morphism $H \to S$ belongs to $\mathcal{M}$, then the following conditions are equivalent:*

(i) *H acts freely and $\mathcal{M}$-effectively on X.*

(ii) *There exists an S-morphism $p : X \to Y$ compatible with the equivalence relation on X defined by H and such that the induced action of $H \times_S Y$ on X over Y makes X a principal homogeneous bundle under H_Y over Y.*

Under these conditions, p identifies Y with the quotient X/H.

Corollary 12.3.48. *Let $\mathcal{C}$ be a category possessing a final object, arbitrary fiber products, and endowed with a subcanonical topology $\mathcal{T}$. Let $f : G \to H$ be a morphism of $\mathcal{C}$-groups and $K = \ker f$, and suppose that f belongs to a family $\mathcal{M}$ satisfying conditions (M1)–(M5$_{\mathcal{T}}$). Then H represents the quotient sheaf G/K, and f is a K_H-torsor*[10]*.*

Proof. In fact, as f is covering, it is a universally effective epimorphism, so H is the quotient of G by the equivalence relation $R(f) = G \times_H G$, which is also the equivalence relation defined by K. On the other hand, the morphism $G \times K \to G \times_H G$, $(g, k) \mapsto (g, gk)$ is an isomorphism of $K_G = G \times_H K_H$-objects. Since the morphism $f : G \to H$ is covering, f is a K_H-torsor by Proposition 12.3.46 (ii). □

We can now specify Theorem 12.3.33 in the case of passage to quotient by a group action:

Proposition 12.3.49. *Under the hypothesis of Proposition 12.3.46, assume that the topology $\mathcal{T}$ is subcanonical and denote by F_0 the functor over S defined as follows: for any $S' \to S$, $F_0(S')$ is the set of representable sub-S'-functors Z of $X \times_S S'$, stable under $H \times_S S'$ and is a principal homogeneous bundle under the induced S'-group action.*

(a) *The following conditions are equivalent:*

  (i) *The action of H on X is $\mathcal{M}$-effective and free.*

  (ii) *F_0 is representable.*

  Under these conditions, we have $F_0 = X/H$.

(b) *Let $\mathcal{N}$ be a family of morphisms which is stable under base change and such that for any covering family $\{S_i \to S\}$ and any family $\{T_i \to S_i\}$ of morphisms in $\mathcal{N}$, any descent data on the T_i relative to $\{S_i \to S\}$ is effective. Suppose that X is squarable and the morphism $X \times_S H \to X \times_S X$ belongs to $\mathcal{N}$, then the morphism $p : X \to F_0$ corresponding to the sub-object $X \times_S H$ of $X \times_S X$ identifies F_0 with the quotient sheaf X/H.*

12.3.3.2 Group structure and passage to quotient

We are now intersted in the algebraic structure induced on the quotient G/H of a group by a subgroup. We first consider category of sheaves over $\mathcal{C}$ for an arbitrary topology. By taking the canonical topology and apply Remark 12.3.36, we then obtain results for the passage to universally effective quotients in $\mathcal{C}$.

Proposition 12.3.50. *Let $u : H \to G$ be a monomorphisms of sheaves of groups. Then there exists a unique G-object structure on the quotient sheaf G/H such that the canonical morphism*

$$p : G \to G/H$$

[10] We also say that G is a K-torsor over H.

is a morphism of G-objects. This structure is functorial relative to (G, H): if we have a commutative diagram

$$\begin{array}{ccc} H & \longrightarrow & G \\ \downarrow & & \downarrow \\ H' & \longrightarrow & G' \end{array}$$

Then the induced morphism $G/H \to G'/H'$ is compatible with $G \to G'$.

Proof. The sheaf G/H is the sheaf associated with the presheaf

$$i(G)/i(H) : S \mapsto G(S)/H(S);$$

as # is left exact, it transforms objects acted by groups into objects acted by groups. Since the presheaf $i(G)/i(H)$ is endowed with an action by $i(G)$, $G/H = (i(G)/i(H))^{\#}$ is endowed with an action by $(i(G))^{\#} = G$. This structure clearly has the required properties. □

Corollary 12.3.51. *Let $u : H \to G$ be a monomorphism of $\mathcal{C}$-groups. Suppose that the action of H on G is universally effective, then there exists a unique G-object structure on the quotient object G/H in $\mathcal{C}$ such that $p : G \to G/H$ is a morphism of G-objects. This structure is functorial relative to (G, H).*

Proposition 12.3.52. *Let $u : H \to G$ be a monomorphism of sheaves of groups which identifies H with a normal subsheaf of G. Then there exists a unique group structure on the quotient sheaf G/H such that the caonical morphism $p : G \to G/H$ is a group morphism. This structure is functorial relative to the couple (G, H) (H being normal).*

Proof. The proof is the same as Proposition 12.3.50. □

Corollary 12.3.53. *Let $u : H \to G$ be a monomorphism of $\mathcal{C}$-groups identifying H with a normal subgroup of G. Suppose that the action of H on G is universally effective, then there exists a group structure on the quotient object G/H in $\mathcal{C}$ such that $p : G \to G/H$ is a morphism of groups. This structure is functorial relative to (G, H) (H being normal and acts universally effectively).*

We can characterize the group structure of G/H in the following way:

Proposition 12.3.54. *Under the conditions of Proposition 12.3.52, let K be a $\mathcal{C}$-group and $f : G \to K$ be a morphism. The following conditions are equivalent:*

(i) *f is a morphism of groups compatible with the equivalence relation defined by H.*

(ii) *f is a morphism of groups inducing the trivial morphism $H \to K$.*

(iii) *f factors into a morphism of groups $G/H \to K$.*

Proof. The equivalence of (i) and (ii) is proved set-wisely. We evidently have (iii)⇒(ii). The equivalence of (iii) and (ii) then follows from the formula

$$\mathrm{Hom}(G/H, K) \cong \mathrm{Hom}(i(G)/i(H), K)$$

and the definition of the group structure of G/H. □

Remark 12.3.55. In the preceding situation, if the kernel of f is exactly H, then the morphism $G/H \to K$ which factors f is a monomorphism. This follows from Proposition 12.3.14.

In the case of sheaves of groups, we can precise Proposition 12.3.27 as following:

Proposition 12.3.56. *Let G be a sheaf of groups, H be a normal subsheaf of groups. For any subsheaf of groups K of G containing H, let K' be the quotient group K/H considered as a a subgroup of $G' = G/H$. Then we have $K = K' \times_{G'} G$, and the maps $K \mapsto K/H$, $K' \mapsto K' \times_{G'} G$ define a bijection between the set of subsheaves of groups of G containing H and the set of subsheaves of groups of G'. In this correspondence, normal subgroups of G corresponds to that of G'.*

Proof. The first assertion follows equally from Proposition 12.3.27 and Lemma 12.3.9. It remains to show that K is normal in G if and only if K' is normal in G'. If K is normal in G, then the presheaf $i(K)/i(H)$ is normal in $i(G)/i(H)$, and the same is true for the associated sheaves. Conversely, if K' is normal in G', then the fiber product $K' \times_{G'} G$ is normal in G, which is equal to K. □

Now if L is a subsheaf of groups of G, then there exists a smallest normal subsheaf of groups $\overline{L}$ of G containing L, called the saturation of L. In fact, we have $\overline{L} = L \cdot H$.

Proposition 12.3.57. *Under the preceding conditions, $L \cdot H$ is a subsheaf of groups of G containing H and the image of L in G/H is identified with*

$$(L \cdot H)/H \cong L/(H \cap L).$$

Proof. Denote by L' the image sheaf of L in G/H. This is a subsheaf of groups of G/H corresponding to $L \cdot H$ by Proposition 12.3.56. As the morphism $L \to L'$ is covering, hence a universally effective epimorphism of sheaves, it follows from Proposition 12.3.25 that L' is identified with the quotient of L by the kernel of $L \to L'$, which is evidently $H \cap L$. □

Finally, we consider the following case: we have a sheaf of groups G, a subsheaf of groups K of G and a subsheaf of groups H of K, which is normal in K. Let us first define a (right) action of the sheaf in groups $H \backslash K$ $(= K/H)$ on G/H. The group K operates by right translations on G. As H is normal in K, this operation is compatible with the equivalence relation defined by the action of H and thus defines an operation of K on G/H, that is, a morphism of the opposite group K^{op} to $\mathcal{A}ut(G/H)$. Since the latter is a sheaf (cf. [?], IV, 4.5.13) and that this morphism is trivial on H, it factors through K/H and defines the desired operation. Since the right and left operations of G on itself commute, the operations of G and K/H on G/H commute.

Proposition 12.3.58. *Under the preceding conditions, K/H acts freely on G/H (on the right) and we have a canonical isomorphism of sheaves operated by G:*

$$(G/H)/(K/H) \cong G/K.$$

If K is normal in G, then K/H is normal in G/H and this isomorphism is a group isomorphism.

Proof. We have an isomorphism of presheaves

$$i(G)/i(K) \xrightarrow{\sim} (i(G)/i(H))/(i(K)/i(H))$$

which respects the action of $i(G)$. The result then follows by applying # on both sides. □

Corollary 12.3.59. *Let G be a $\mathcal{C}$-group, K be a sub-$\mathcal{C}$-group of G, H be a normal sub-$\mathcal{C}$-group of K. Let $\mathcal{M}$ be a family of morphisms in $\mathcal{C}$ verifying the conditions (M1)–(M5$_{\mathcal{T}}$). Suppose that the right action of H on G (resp. K) is $\mathcal{M}$-effective, then K/H acts freely on G/H, and this action commutes with that of G. The following conditions are equivalent:*

(i) *The action of K on G is $\mathcal{M}$-effective.*

(ii) *The action of K/H on G/H is $\mathcal{M}$-effective.*

Under these conditions, we have an isomorphism of G-objects in $\mathcal{C}$:

$$(G/H)/(K/H) \cong G/K.$$

Proof. Since H acts $\mathcal{M}$-effectively on G and K, by Proposition 12.3.58 we have a diagram

$$\begin{tikzcd} H \arrow[r, hook] & K \arrow[r, hook] \arrow[d] & G \arrow[d] \\ & K/H \arrow[r, hook] & G/H \arrow[d] \\ & & G/K \end{tikzcd}$$

where the square is Cartesian. Since $\mathcal{M}$ is stable under composition, if K/H acts on G/H $\mathcal{M}$-effectively, then we conclude that $G \to G/K \in \mathcal{M}$, so K acts on G $\mathcal{M}$-effectively. Conversely, if $G \to G/K$ belongs to $\mathcal{M}$, then consider the Cartesian diagram

$$\begin{tikzcd} K \times G \arrow[r] \arrow[d] & G \arrow[d] \\ (K/H) \times (G/H) \arrow[r] & (G/H) \end{tikzcd}$$

By hypothesis, the morphism $K \times G \to G \times G$ belongs to $\mathcal{M}$ (Proposition 12.3.17), and $G \to G/H$ belongs to $\mathcal{M}$. We then conclude from (M1) and (M5$_\mathcal{T}$) that $(K/H) \times (G/H) \to G/H$ belongs to $\mathcal{M}$, so the equivalence relation on G/H defined by K/H is of type $\mathcal{M}$. Since the quotient of G/H by this equivalence relation is represented by G/K, we conclude from Theorem 12.3.33 that the action of K/H on G/H is $\mathcal{M}$-effective. □

Now let $\mathcal{N}$ be a family of morphisms in $\mathcal{C}$ verifying conditions (N1) and (N$_\mathcal{M}$) of **??** 12.3.2.2. By Proposition 12.3.56 and Proposition 12.3.41, we obtain:

Proposition 12.3.60. *Let G be a $\mathcal{C}$-group and H be a normal sub-$\mathcal{C}$-group of G whose action on G is $\mathcal{M}$-effective.*

(a) *For any sub-$\mathcal{C}$-group K of G containing H and such that the morphism $K \to G$ belongs to $\mathcal{N}$, H acts $\mathcal{M}$-effectively on K and the quotient $K/H \to K'$ is a sub-$\mathcal{C}$-group of $G' = G/H$ such that the morphism $K' \to G'$ belongs to $\mathcal{N}$.*

(b) *The map $K \mapsto K' = K/H$ is a bijection from the set of sub-$\mathcal{C}$-groups K of G containing H and such that the morphism $K \to G$ belongs to $\mathcal{N}$, H acts $\mathcal{M}$-effectively on K to the set of sub-$\mathcal{C}$-groups K' of G' such that the morphism $K' \to G'$ belongs to $\mathcal{N}$. Under this correspondence, the normal subgroups of G correspond to that of G'.*

Corollary 12.3.61. *If $H \to G$ belongs to $\mathcal{N}$, then $\mathcal{C}$ possesses a final object e and the unit section $e \to G/H$ belongs to $\mathcal{N}$.*

Proof. This follows form Proposition 12.3.60 by taking $K = H$. □

12.3.4 Applications to the category of schemes

Let **Sch** be the category of schemes, to which we can assocate the Zariski topology, that is, the topology generated by the family of morphisms $\{S_i \to S\}$, where each $S_i \to S$ is an open immersion and the union of images of S_i is equal to S. A sheaf over the Zariski topology is also called of local nature: this is a contravariant functor $F : \mathbf{Sch}^{\mathrm{op}} \to \mathbf{Set}$ such that for any scheme S and any covering $\{S_i \to S\}$, we have an exact diagram

$$F(S) \longrightarrow \prod_i F(S_i) \rightrightarrows \prod_{i,j} F(S_i \cap S_j)$$

In particular, a functor of local natura transforms direct sums to products. As any representable functor is a sheaf, this topology is coarser than the canonical topology.

To introduce (and handle) more topologies on **Sch**, we need a general criterion to identify the covering families of the topology generated by certain family of morphisms. This is contained in the following proposition.

Proposition 12.3.62. *Let $\mathcal{C}$ be a category and $\mathcal{C}'$ be a full subcategory. Let P be a set of families of morphisms of $\mathcal{C}$ with the same codomains, which is stable under composition and base change, and P' be a set of families of morphisms of $\mathcal{C}'$ containing the families of identity morphisms. We endow $\mathcal{C}$ with the topology generated by P and P' and suppose that the following conditions are satisfied:*

(a) *If $\{S_i \to S\} \in P'$ (hence $S_i, S \in \mathrm{Ob}(\mathcal{C}')$) and $T \to S$ is a morphism in $\mathcal{C}'$, then the fiber products $S_i \times_S T$ (in $\mathcal{C}$) exist and the family $\{S_i \times_S T \to T\}$ belongs to P'.*

(b) *For any $S \in \mathrm{Ob}(\mathcal{C})$, there exists $\{S_i \to S\} \in P$ with $S_i \in \mathrm{Ob}(\mathcal{C}')$ for each i.*

(c) *In the following situation*

$$S_{ijk} \xrightarrow{(P')} S_{ij} \xrightarrow{(P)} S_i \xrightarrow{(P')} S$$

where $S, S_i, S_{ij}, S_{ijk} \in \mathrm{Ob}(\mathcal{C}')$, $\{S_i \to S\} \in P'$, $\{S_{ij} \to S_i\} \in P$ for each i, $\{S_{ijk} \to S_{ij}\} \in P'$ for any i, j, there exists a family $\{T_n \to S\} \in P'$ and for each n a multi-index ijk and a commutative diagram

$$\begin{tikzcd} T_n \arrow[rr] \arrow[dr] & & S_{ijk} \\ & S \arrow[ur] & \end{tikzcd}$$

Then for a sieve R of $S \in \mathrm{Ob}(\mathcal{C})$ to be covering, it is necessary and sufficient that there exists a composite family

$$\begin{tikzcd} S_{ij} \arrow[r, dashed] \arrow[d, "(P')"'] & R \arrow[d] \\ S_i \arrow[r, "(P)"] & S \end{tikzcd} \tag{12.3.2}$$

where $S_i, S_{ij} \in \mathrm{Ob}(\mathcal{C}')$, $\{S_i \to S\} \in P$, $\{S_{ij} \to S_i\} \in P'$ for each i, and the morphisms $S_{ij} \to S$ factors through R.

Proof. Since the families in P and P' are covering, any family which is the composite of such families is again covering, so a sieve of the form indicated in the proposition is covering for $\mathcal{C}$, since it contains a covering sieve. Conversely, it suffices to prove that sieves of the form (12.3.2) form a topology, i,e, it suffices to verify the axioms (T1)–(T3).

To verify (T3), let $S \in \mathrm{Ob}(\mathcal{C})$. There exists by (b) a family $\{S_i \to S\} \in P$ with $S_i \in \mathrm{Ob}(\mathcal{C}')$. The families $\{\mathrm{id}_{S_i} : S_i \to S_i\}$ belong to P' by hypothesis, so the sieve S of S is of the following form:

$$\begin{tikzcd} S_i \arrow[r] \arrow[d, "(P')"'] & S \arrow[d, "\mathrm{id}_S"] \\ S_i \arrow[r, "(P)"] & S \end{tikzcd}$$

Now let R be a sieve of S with desired form (12.3.2) and R' be a sieve such that for any $T \to R$ in $\mathcal{C}$, the sieve $R' \times_T S$ is of the desired from. Then as the morphism $S_{ij} \to S$ factors through R, the sieve $R'_{ij} = R' \times_S S_{ij}$ of S_{ij} is of the desicred form by hypothesis:

$$\begin{tikzcd} R'_{ij} \arrow[r, hook] \arrow[dd] & S_{ij} \arrow[d, "(P')"'] \arrow[dr] & \\ & S_i \arrow[d, "(P)"'] & R \arrow[dl] \\ R' \arrow[r, hook] & S & \end{tikzcd}$$

so for each ij, we have a diagram of the form

$$\begin{tikzcd} S_{ijkl} \arrow[d, "(P')"'] \arrow[dr] & \\ S_{ijk} \arrow[d, "(P)"'] & R'_{ij} \arrow[dl] \\ S_{ij} & \end{tikzcd}$$

We have thus proved that there exists a composite family

$$S_{ijkl} \xrightarrow{(P')} S_{ijk} \xrightarrow{(P)} S_{ij} \xrightarrow{(P')} S_i \xrightarrow{(P)} S$$

belonging to $P \circ P' \circ P \circ P'$, which factors through R' and with all objects (except S) belong to $\mathcal{C}'$. Applying condition (c) to the family $\{S_{ijkl} \to S_i\}$, we then obtain for each i a family $\{T_{in} \to S_i\} \in P'$, such that $T_{in} \to S$ factors through one of the S_{ijkl}, hence through R':

$$\begin{tikzcd} T_{in} \arrow[r] \arrow[d, "(P')"'] & S_{ijkl} \arrow[dd] \\ S_i \arrow[d, "(P)"'] & \\ S & R' \arrow[l, hook'] \end{tikzcd}$$

The sieve R' of S is therefore of the desired form (12.3.2), which verifies axiom (T2).

Fianlly, as for axiom (T1), let R be a sieve of S of the desired form and $T \to S$ be a morphism in $\mathcal{C}$. Let $T_i = S_i \times_S T$; the family $\{T_i \to T\}$ then belongs to P, and applying condition (b), we obtain for each i a family $\{U_{ik} \to T_i\} \in P$, with $U_{ik} \in \mathrm{Ob}(\mathcal{C}')$. By the hypotesis on P, we have $\{U_{ik} \to T\} \in P$, so by condition (a), $U_{ik} \times_{S_i} S_{ij} = U_{ikj}$ is an object of $\mathcal{C}'$ and for each ik, $\{U_{ikj} \to U_{ik}\} \in P'$.

$$\begin{tikzcd}
U_{ikj} \arrow[rr] \arrow[d, "(P')"'] & & S_{ij} \arrow[d, "(P')"] \arrow[ddr, bend left] & \\
U_{ik} \arrow[r, "(P)"] \arrow[dr, "(P)"'] & T_i \arrow[r] \arrow[d, "(P)"] & S_i \arrow[d, "(P)"] & \\
& T \arrow[r] & S & R \arrow[l, hook']
\end{tikzcd}$$

We therefore conclude that the family $\{U_{ikj} \to T\}$ factors through the sieve $T \times_S R$ of T, which is hence of the desired form. This proves axiom (T1) and completes the proof. □

Corollary 12.3.63. *If $S \in \mathrm{Ob}(\mathcal{C}')$ and R is a sieve of S, then R is covering if and only if there exists a family $\{T_i \to S\} \in P'$ which factors through R.*

Proof. In fact, any such sieve is covering. Conversely, it suffices to apply (c) to the family $\{S_i \to S\}$ and the identity morphisms of S_i to deduce that any covering sieve is of the indicated form. □

Corollary 12.3.64. *For a presheaf $F \in \mathrm{PSh}(\mathcal{C})$ to be separated (resp. a sheaf), it is necessary and sufficient that the morphisms*

$$F(S) \longrightarrow \textstyle\prod_i F(S_i)$$

is injective (resp. that the diagram

$$F(S) \longrightarrow \textstyle\prod_i F(S_i) \rightrightarrows \textstyle\prod_{i,j} F(S_i \times_S S_j)$$

is exact) for $\{S_i \to S\} \in P$ and $\{S_i \to S\} \in P'$, respectively.

Proof. In fact, these conditions are necessary, because the families above are covering. Conversely, if R is the sieve of S of a family of morphisms $\{S_{ij} \overset{(P')}{\to} S_i \overset{(P)}{\to} S\}$, a diagram chasing shows that the above conditions imply that $\mathrm{Hom}(S, F) \to \mathrm{Hom}(R, F)$ is injective (resp. bijective). But any covering sieve R' of S contains a sieve generated by such a family and we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}(S,F) \arrow[rr, "f"] \arrow[dr, "g"'] & & \mathrm{Hom}(R,F) \arrow[dl, "h"] \\
& \mathrm{Hom}(R',F) &
\end{tikzcd}$$

If g is injective, then so is f, so in this case F is separated. In this case, since the morphism $R' \to R$ is covering, we see that h is also injective (cf. **??**). Therefore, if g is bijective, so is f, hence F is a sheaf. □

Remark 12.3.65. The condition (c) of Proposition 12.3.62 is satisfies if P' is stable under composition and if any family $\{S_i \to S\}$ of morphisms in P with $S_i, S \in \mathrm{Ob}(\mathcal{C}')$ has a subfamily belonging to P'.

We now let $\mathcal{C} = \mathbf{Sch}$ be the category of schemes, and $\mathcal{C}'$ be the full subcategory formed by affine schemes. We shall consider the following sets P':

P'_1: finite and surjective families, formed by flat morphisms;

P'_2: finite and surjective families, formed by flat morphisms of finite presentation;

P'_3: finite and surjective families, formed by étale morphisms;

P'_4: finite and surjective families, formed by finite étale morphisms;

For each of these sets P_i' (except P_4'), the conditions of Proposition 12.3.62 are satisfied (as for (c), note that an affine scheme is quasi-compact, so any family of morphisms of $\mathcal{C}'$, belonging to P, contains a finite subfamily which is equally in P, hence in P_i' for $i = 1,2,3$). The corresponding topology $\mathcal{T}_i$ generated by P and P_i' is denoted and called by the following manner:

- $\mathcal{T}_1$ is the faifufully flat and quasi-compact topology (fpqc);
- $\mathcal{T}_2$ is the faifufully flat and finite presented topology (fppf);
- $\mathcal{T}_3$ is the étale topology (ét);
- $\mathcal{T}_4$ is the finite étale topology (étf).

As $P_1' \supseteq P_2' \supseteq P_3' \supseteq P_4'$, we have

$$\text{fpqc} \geq \text{fppf} \geq \text{ét} \geq \text{étf} \geq \text{Zar}.$$

Proposition 12.3.66. *Let $\mathcal{T}_i$ ($i = 1,2,3,4$) be the topologies on* **Sch** *defined above.*

(a) *For a sieve R of S to be covering for $\mathcal{T}_i$ ($1 \leq i \leq 3$), it is necessary and sufficient that there exists a covering (S_α) of S by affine opens and for each α a family $\{S_{\alpha\beta} \to S_\alpha\} \in P_i'$, with $S_{\alpha\beta}$ affine, such that the the family $\{S_{\alpha\beta} \to S\}$ factors through R.*

(b) *For a presheaf F over* **Sch** *to be a sheaf for the fpqc topology (resp. fppf, étale, finite étale), it is necessary and sufficient that*

  (i) *F is a sheaf over the Zariski topology, i.e. a functor of local nature.*

  (ii) *For any faithfully flat morphism (resp. faithfully flat morphism of finite presentation, resp. surjective étale, resp. finite surjective étale) $T \to S$ of affine schemes, we have an exact diagram*

$$F(S) \longrightarrow F(T) \rightrightarrows F(T \times_S T)$$

(c) *The topologies $\mathcal{T}_i$ ($1 \leq i \leq 4$) are subcanonical.*

(d) *Any surjective family formed by open and flat morphisms (resp. flat and locally of finite presentation, resp. étale, resp. finite and étale) is covering for the fpqc topology (resp. fppf, resp. étale, resp. finite étale).*

(e) *Any finite and surjective family, formed by flat and quasi-compact morphisms, is covering for the fpqc topology.*

Proof. Assertion (a) follows from Proposition 12.3.62, and (b) follows from Corollary 12.3.64, since a sheaf for the Zariski topology transforms direct sums into products. Any representable functor is a sheaf for Zariski topology, and satisfies condition (ii) by ([**?**], VIII, 5.3), so $\mathcal{T}_1$ is subcanonical, which proves (c).

Let $\{S_i \to S\}$ be a family of morphisms as in (d). By considering a covering of S by affine opens, we are reduced to the case where S is affine. We first deal with the case where each $S_i \to S$ is flat and open (resp. étale). Let S_{ij} be a covering og S_i be affine opens. As the morphisms considered are open, the images T_{ij} of S_{ij} in S form an open covering of S. As S is affine, hence quasi-compact, there exists a finite subcover of T_{ij}, with i,j belongs to a finite set F. Then $S' = \coprod_F S_{ij}$ is affine, and the morphism $S' \to S$ belongs to P_1' (resp. P_3'), hence is covering. As this factors through the given family $\{S_i \to S\}$, the latter is also covering.

In the case of the finite étale topology, each S_i is finite over S, hence is affine; in the preceding argument, we can then take for $\{S_{ij}\}$ the covering $\{S_i\}$ of S_i, and we obtain a morphism $S' \to S$ belonging to P_4'.

Now consider the case where $f_i : S_i \to S$ are flat and locally of finite presentation. For any $s \in S$, there exists (by the proof of ([**?**], 17.16.2)) an affine subscheme $X(s)$ of one of the S_i such that $s \in f_i(X(s))$ and that the morphism $g_i : X(s) \to S$, restriction of f_i, is flat and quasi-finite. Then $g_i(X(s))$ is an open neighborhood $U(s)$ of s (([**?**], 2.4.6)), and as S is affine, it is covered by a finite number of such opens $U(s_j)$, $j = 1,\dots,n$. Therefore, $X' = \coprod_j X(s_j)$ is affine, and the morphism $X' \to S$ is surjective, flat, of finite presentation and quasi-finite, hence belongs to P_2', which completes the proof of (d).

Finally, let $\{S_i \to S\}$ be a finite and surjective family of flat and quasi-compact morphisms. Let T_j be a covering of S be affine opens. Then $S_{ij} = T_j \times_S S_i$ is quasi-compact and hence has a finite affine covering T_{ijk}. Each morphism $T_{ijk} \to T_j$ is flat, and the family $\{T_{ijk} \to T_j\}$ is finite and surjective, hence covering for $\mathcal{T}_1$. The family $\{T_{ijk} \to S\}$ is hence also, by composition, covering, and it factors through the given family:

$$\begin{tikzcd} T_{ijk} \arrow[r] \arrow[dr] & S_{ij} \arrow[r] \arrow[d] & S_i \arrow[d] \\ & T_j \arrow[r] & S \end{tikzcd}$$

so the given family $\{S_i \to S\}$ is also covering for $\mathcal{T}_1$. □

Proposition 12.3.67. *Let $\mathcal{M}_i$ be the family of the following morphisms:*

$\mathcal{M}_1$: *faithfully flat and quasi-compact morphisms.*

$\mathcal{M}_2$: *faithfully flat and locally of finite presentaion morphisms.*

$\mathcal{M}_3$: *surjective étale morphisms.*

$\mathcal{M}_4$: *finite surjective étale morphisms.*

Then the family $\mathcal{M}_i$ verifies conditions (M1)–(M$_{\mathcal{T}_i}$) of 12.3.2.2.

Proof. For (M1)–(M3), these are classical results. By Proposition 12.3.66 (d) and (e), we see that $\mathcal{M}_i$ satisfies (M4$_{\mathcal{T}_i}$), so it remains to verify (M5$_{\mathcal{T}_i}$). For this, it suffices to show that each $\mathcal{M}_i$ satisfies condition (M5$_{\mathcal{T}_1}$), since it implies the others. This follows from ([**?**], VIII, n4 and n5). □

Corollary 12.3.68. *If X is a scheme and R is an equivalence relation of type $\mathcal{M}_i$, then R is $\mathcal{M}_i$-effective if and only if the quotient sheaf X by R for $\mathcal{T}_i$ is representable and in this case it is represented by X/R.*

Proof. In fact, this is a concequence of Proposition 12.3.30. □

We also consider families $\mathcal{N}$ of morphisms verifying conditions (N1) and (N$_{\mathcal{T}_i}$). But note, as above, that condition (N$_{\mathcal{T}_1}$) implies the others.

Proposition 12.3.69. *The following families satisfy conditions (N1) and (N$_{\mathcal{T}_1}$) of 12.3.2.2, that is, have descent property for the fpqc topology:*

$\mathcal{N}$: *open immersions.*

$\mathcal{N}'$: *closed immersions.*

$\mathcal{N}''$: *quasi-compact immersions.*

Proof. In view of Proposition 12.3.66 (b), it suffices to consider the descent data relative to the Zariski topology and to a faithfully flat and quasi-compact morphism. The first assertion is clear; for the second one, the case for $\mathcal{N}$ and $\mathcal{N}'$ follows from ([**?**], VIII, 4.4) and ([**?**], VIII, 1.9). The case for $\mathcal{N}''$ can be deduce as in ([**?**], VIII, 5.5), by using the previous two results. □

We can therefore apply the results of the previous subsections to the families of morphisms given above. Let us give one as an example (Corollary 12.3.37 and Proposition 12.3.40):

Corollary 12.3.70. *Let X be a scheme and R be an equivalence relation on X. Suppose that $R \to X$ is faithfully flat and quasi-compact and $R \to X \times X$ is a closed immersion (resp. open immersion, resp. quasi-compact immersion). Then the quotient sheaf X/R is the same for the fpqc topology and the canonical topology, and for any scheme S, we have*

$$(X/R)(S) = \left\{ \begin{array}{l} \text{closed (resp. open, resp. quasi-compact) subschemes } Z \text{ of } X \times S \text{ stable} \\ \text{under } R \times S \text{ such that } Z \to X_S \text{ belongs to } \mathcal{N}\text{, that } R_Z \text{ is faithfully flat} \\ \text{and quasi-compact, and that diagram } R_Z \rightrightarrows Z \to S \text{ is exact} \end{array} \right\}.$$

12.3.4.1 Homogeneous spaces Let G be an S-group scheme, X an S-scheme acted (on right) by G, and

$$\Phi : G \times_S X \to X \times_S X$$

be the S-morphism defined setwise by $(g, x) \mapsto (gx, x)$. Recall that X is called formally principal homogeneous under G if the following equivalent conditions are satisfied[11]:

(i) For any $T \to S$, the set $X(T)$ is either empty of pincipal homogeneous under $G(T)$,

(ii) Φ is an isomorphism of S-functors,

(iii) Φ is an isomorphism of S-schemes.

The definition of **formally homogeneous spaces** (not necessarily principal) is obtained by demanding that Φ is an epimorphism in the category of sheaves for an appropriate topology $\mathcal{T}$. On the other hand, the condition that Φ is an epimorphism of S-functors is equivalent to that, for any $T \to S$, the set $X(T)$ is empty or homogeneous (not necessarily principal) under $G(T)$. But this condition is too restrictive, as shown in the following example.

Example 12.3.71. Let $S = \mathrm{Spec}(\mathbb{R})$, $G = \mathbb{G}_{m,\mathbb{R}}$ and $X = \mathbb{G}_{m,\mathbb{R}}$ over which G acts by $t \cdot x = t^2 x$. Then the morphism $\Phi : G \times_S X \to X \times_S X$ is étale, finite and surjective, hence an epimorphism in the category of sheaves for the finite étale topology (a fortiori, an epimorphism of S-schemes). However, the points 1 and -1 of $X(\mathbb{R})$ are not conjugate under $G(\mathbb{R})$, so that the morphism $G(\mathbb{R}) \times X(\mathbb{R}) \to X(\mathbb{R}) \times X(\mathbb{R})$ is not surjective[12].

Definition 12.3.72. Let G be an S-group, X be an S-group scheme acted by G and $\mathcal{T}$ be a subcanonical topology over $\mathbf{Sch}_{/S}$. We say that X is a **formally homogeneous space under** G (relative to the topology $\mathcal{T}$) if the following equivalent conditions are satisfied:

(i) the morphism $\Phi : G \times_S X \to X \times_S X$ is an epimorphism in the category of sheaves for the topology $\mathcal{T}$.

(ii) for any $T \to S$ and $x, y \in X(T)$, there exists a covering morphism $T' \to T$ for the topology $\mathcal{T}$ and $g \in G(T')$ such that $y_{T'} = g \cdot x_{T'}$.

Remark 12.3.73. Condition (i) implies that Φ is a universally effective epimorphism in $\mathbf{Sch}_{/S}$ (cf. [**?**], IV, 4.4.3). This implies, in particular, that Φ is a surjective morphism of schemes.

Proposition 12.3.74. *Let G be an S-group, X be an S-scheme acted by G, and $\mathcal{T}$ be a subcanonical topology over* $\mathbf{Sch}_{/S}$*. The following conditions are equivalent:*

(i) *X verifies the following conditions:*

  (a) *the morphism $\Phi : G \times_S X \to X \times_S X$ is covering, i.e. X is a formally homogeneous G-space.*

  (b) *the morphism $X \to S$ is covering, i.e. locally for the topology $\mathcal{T}$, it possesses a section*[13].

(ii) *X is locally isomorphic (as a G-object) to the quotient sheaf (for $\mathcal{T}$) of G by a subgroup scheme H, that is, there exists a covering family $\{S_i \to S\}$ such that each $X \times_S S_i$ represents the quotient sheaf of $G \times_S S_i$ by a subgroup scheme H_i.*

*Under these conditions, we say that X is a **homogeneous G-space** (relative to the topology $\mathcal{T}$).*

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[11]The equivalence of (i) and (ii) is clear, and (ii)⇔(iii) since $\mathcal{C}$ is a full subcategory of $\widehat{\mathcal{C}}$.

[12]Obviously, this difficulty comes from the fact that if $\mathcal{C}'$ is a full subcategory of $\widehat{\mathcal{C}}$ containing $\mathcal{C}$, for example, the category of sheaves on $\mathcal{C}$ for a subcanonical topology $\mathcal{T}$, and if $f : X \to Y$ is a morphism in $\mathcal{C}$, then the implications

$$f \text{ is an epimorphism in } \widehat{\mathcal{C}} \Rightarrow f \text{ is an epimorphism in } \mathcal{C}' \Rightarrow f \text{ is an epimorphism in } \mathcal{C}$$

are in general strict.

[13]Note that the morphism $X \to S$ is an epimorphism in $\widehat{\mathbf{Sch}_{/S}}$ if and only if for any $T \to S$, the morphism $X(T) \to S(T) = \{T \to S\}$ is surjective, that is, the morphism $T \to S$ factors through X. But since $T \to S \in S(T)$ is the image of the identity morphism $\mathrm{id}_S : S \to S$, this is true if and only if id_S factors through X, which means X admits a section.

Proof. Suppose that (ii) is satisfied. Put $G_i = G \times_S S_i$ and $X_i = X \times_S S_i$, then X_i possesses a section over S_i, namely the composition of the unit section $\varepsilon_i : S_i \to G_i$ and the projection $\pi_i : G_i \to X_i = G_i/H_i$. We then conclude that $X \to S$ is covering.

On the other hand, π_i is covering, so $\pi_i \times \pi_i$ is also covering, and we have a commutative diagram

$$\begin{tikzcd} G_i \times_{S_i} G_i \arrow[r, "\Psi_i"] \arrow[d, "\mathrm{id} \times \pi_i"] & G_i \times_{S_i} X_i \arrow[d, "\pi_i \times \pi_i"] \\ G_i \times_{S_i} X_i \arrow[r, "\Phi_i", "\cong"'] & G_i \times_{S_i} X_i \end{tikzcd}$$

where Φ_i is induced by Φ by base change $S_i \to S$ and Ψ_i is the isomorphism defined by $(g, g') \mapsto (gg', g)$. Then $(\pi_i \times \pi_i) \circ \Psi_i$ is covering, hence Φ_i is a covering. This shows that Φ is locally covering, hence is covering, whence (ii)⇒(i).

Conversely, suppose that (i) is satisfied, and moreover the structural morphism $X \to S$ possesses a section $\sigma \in X(S)$. By Corollary 8.5.8, σ is an immersion. Define $H = G \times_X S$ by the diagram below, where the two squares are Cartesian:

$$\begin{tikzcd} H \arrow[r] & G \arrow[r, "\mathrm{id}_G \boxtimes \sigma"] \arrow[d, "\pi"] & G \times_S X \arrow[d, "\Phi"] \\ S \arrow[r, "\sigma"] & X \arrow[r, "\mathrm{id}_X \boxtimes \sigma"] & X \times_S X \end{tikzcd}$$

where π, $\mathrm{id}_G \boxtimes \sigma$ and $\mathrm{id}_X \boxtimes \sigma$ denote the morphisms defined setwisely, for $T \to S$ and $g \in G(T)$, $x \in X(T)$, by

$$\pi(g) = g \cdot \sigma_T, \quad (\mathrm{id}_G \boxtimes \sigma)(g) = (g, \sigma_T), \quad (\mathrm{id}_X \boxtimes \sigma)(x) = (x, \sigma_T).$$

Then π is covering, and H is a subgroup scheme of G, represented by the stabilizer $\mathrm{Stab}_G(\sigma)$ of σ, that is, for any $T \to S$, we have

$$H(T) = \{g \in G(T) : g \cdot \sigma_T = \sigma_T\}.$$

Denote by G/H the presheaf $T \mapsto G(T)/H(T)$, and $(G/H)^\#$ the associated sheaf for the topology $\mathcal{T}$. From the above arguments, we obtain a commutative diagram of morphisms of presheaves acted by G:

$$\begin{tikzcd} G \arrow[r, "\pi"] \arrow[d] & X \\ G/H \arrow[ur, "\bar{\pi}"'] & \end{tikzcd}$$

where $\bar{\pi}$ is a monomorphism (cf. Proposition 12.3.14). As π is covering, $\bar{\pi}$ is also a covering, so $\bar{\pi}$ induces an isomorphism $(G/H)^\# \cong X$. Therefore, we have shown that: if X is a homogeneous G-space such that $X \to S$ admits a section σ, then X represents the quotient sheaf G/H, where $H = G \times_X S$ is the stabilizer of σ.

In the general case, by hypothesis there exists a covering family $\{S_i \to S\}$ such that each morphism $X_i = X \times_S S_i \to S_i$ possesses a section σ_i. Put $G_i = G \times_S S_i$, then the morphism $\Phi_i = G_i \times_{S_i} X_i \to X_i \times_{S_i} X_i$ deduced from Φ by base change $S_i \to S$ is covering, hence, by the preceding arguments, $X_i \cong G_i/H_i$ where $H_i = \mathrm{Stab}_{G_i}(\sigma_i)$. This proves the implication (i)⇒(ii). □

12.4 Construction of quotient schemes

12.4.1 $\mathcal{C}$-groupoids

Let $\mathcal{C}$ be a category which has finite products and coproducts. Recall that a diagram

$$\begin{tikzcd} X_1 \arrow[r, shift left, "d_1"] \arrow[r, shift right, "d_0"'] & X_0 \arrow[r, "p"] & Y \end{tikzcd}$$

in $\mathcal{C}$ is called **exact** if $pd_0 = pd_1$ and if, for any $T \in \mathcal{C}$, $T(p)$ is a bijection from $T(Y)$ to the subset of $T(X_0)$ formed by morphisms $f : X_0 \to T$ such that $fd_0 = fd_1$. We also say that (Y, p) is the cokernel of (d_0, d_1), and write

$$(Y, p) = \operatorname{coker}(d_0, d_1).$$

Let $\mathcal{C}$ be the category **Rsp** of ringed spaces. In this case, there always exists a cokernel (Y, p), of which we can give the following description: the underlying topological space Y is obtained from X_0 by identifying the points $d_0(x)$ and $d_1(x)$, endowed with the quotient topology. The canonical morphisms $\pi : X_0 \to Y$ and d_0, d_1 then induce a double arrow of sheaves of rings over Y:

$$\pi_*(\mathscr{O}_0) \underset{\delta_0}{\overset{\delta_1}{\rightrightarrows}} \pi_*((d_0)_*(\mathscr{O}_1)) = \pi_*((d_1)_*(\mathscr{O}_1))$$

where $\mathscr{O}_i$ is the structural sheaf of X_i. We choose $\mathscr{O}_Y$ to be the sheaf of rings over Y whose sections s are such that $\delta_0(s) = \delta_1(s)$. The morphism $p : X_0 \to Y$ is defined in the evident way.

Let $d_0, d_1 : X_1 \rightrightarrows X_0$ be a diagram in **Rsp** and (Y, p) be the cokernel. We say that an open subset U of X_0 is saturated if $d_0^{-1}(U) = d_1^{-1}(U)$, which is equivalent to that $U = p^{-1}(p(U))$. In this case, as Y is endowed with the quotient topology, $p(U)$ is an open subset of Y.

Lemma 12.4.1. *Let U be a saturated open subset of X and $V = p(U)$. If we denote by $U_1 = d_0^{-1}(U) = d_1^{-1}(U)$ the open subset of X_1, and $\tilde{d}_0$, $\tilde{d}_1$ and $\tilde{p}$ the restriction of d_0, d_1 to U_1 and p to U, then $(V, \tilde{p})$ is the cokernel of the following diagram in* **Rsp**:

$$U_1 \underset{\tilde{d}_0}{\overset{\tilde{d}_1}{\rightrightarrows}} U \xrightarrow{\tilde{p}} V$$

Proof. Since U is saturated, the morphisms d_0, d_1 and p restricts to give the desired diagram. The claim that $(V, \tilde{p})$ is the cokernel is an immediate verification. ∎

Remark 12.4.2. The result of Lemma 12.4.1 is not true in the category of schemes. For example, let $S = \operatorname{Spec}(\mathbb{C})$, $X_0 = \mathbb{A}_S^2 = \operatorname{Spec}(\mathbb{C}[x_1, x_2])$, $d_1 : \mathbb{G}_{m,S} \times_S \mathbb{A}_S^2 \to \mathbb{A}_S^2$ be the action of $\mathbb{G}_{m,S}$ on $\mathbb{A}_S^2$ by multiplication, and $d_0 : \mathbb{G}_{m,S} \times_S \mathbb{A}_S^2 \to \mathbb{A}_S^2$ the projection to the second factor. Let $U = \mathbb{A}_S^2 - \{\mathfrak{m}\}$, where $\mathfrak{m}$ is the point $(0, 0)$. Then the projective space $\mathbb{P}_S^1$ is the cokernel of $(\tilde{d}_0, \tilde{d}_1)$ in **Rsp** and $\mathbf{Sch}_{/S}$, and the cokernel Y of (d_0, d_1) in **Rsp** is the union of $\mathbb{P}_S^1$ and the point $y_0 = \{p(\mathfrak{m})\}$, with the unique open subset containing y_0 being Y and we have $\Gamma(Y, \mathscr{O}_Y) = \mathbb{C}$ (note that Y is not a scheme). If $f : \mathbb{A}_S^2 \to T$ is a morphism of S-schemes such that $fd_0 = fd_1$ and $\bar{f} : Y \to T$ is the induced morphism of ringed spaces, then for any affine open subset $V = \operatorname{Spec}(A)$ of T containing the point $t_0 = f(\mathfrak{m})$, we have $f^{-1}(V) = Y$, so the ring homomorphism $A \to \mathbb{C}[x_1, x_2]$ factors through $\mathbb{C}$. This shows that $S = \operatorname{Spec}(\mathbb{C})$ is the cokernel of (d_0, d_1) in the category $\mathbf{Sch}_{/S}$.

Lemma 12.4.3. *Let $d_0, d_1 : X_1 \rightrightarrows X_0$ be a diagram in* **Sch** *and (Y, p) be the cokernel in* **Rsp**.

(a) *If Y is a scheme and p is a morphism of schemes, then (Y, p) is a cokernel of (d_0, d_1) in* **Sch**.

(b) *Suppose that any point of X_0 has a saturated open neighborhood U such that, if $(\tilde{d}_0, \tilde{d}_1)$ is the induced diagram to $d_0^{-1}(U) = d_1^{-1}(U)$ and (Q, q) is the cokernel of $(\tilde{d}_0, \tilde{d}_1)$ in* **Rsp**, *then Q is a scheme and q is a morphism of schemes. Then (Y, p) is a cokernel of (d_0, d_1) in* **Sch**.

Proof. In the situation of (a), since (Y, p) is a cokernel of (d_0, d_1) in **Rsp**, every morphism $f : X_0 \to T$ of schemes such that $fd_0 = fd_1$ factors into a morphism $\bar{f} : Y \to T$ of ringed spaces. Now we know that $f = \bar{f}p$, and f, p are both morphisms of locally ringed spaces with p being surjective; it follows that $\bar{f}$ is also a morphism of locally ringed spaces, hence (Y, p) is a cokernel on (d_0, d_1) in **Sch**. Now (b) follows from (a) and Lemma 12.4.1 by glueing. ∎

In this section, we consider the existence of $\operatorname{coker}(d_0, d_1)$ if the two morphisms arise from a groupoid. More precisely, denote by $X_2 = X_1 \times_{d_1, d_0} X_1$ the fiber product and d_0', d_2' the two projections of X_2 to X_1, so that we have a Cartesian square:

$$\begin{tikzcd} X_2 \arrow[r, "d_0'"] \arrow[d, "d_2'"'] & X_1 \arrow[d, "d_1"] \\ X_1 \arrow[r, "d_0"] & X_0 \end{tikzcd} \tag{12.4.1}$$

Moreover, suppose that we are given a third morphism $d_1' : X_2 \to X_1$. We say that $(d_0, d_1 : X_1 \rightrightarrows X_0, d_1')$ is a $\mathcal{C}$**-groupoid** if for any object T of $\mathcal{C}$, $X_1(T)$ is the set of morphisms of a groupoid $X_*(T)$ whose set of objects is $X_0(T)$, with source map $d_1(T)$, target map $d_0(T)$, and whose composition map is $d_1'(T)$ (we identify $(X_1 \times_{d_1,d_0} X_1)(T)$ with $X_1(T) \times_{d_1(T),d_0(T)} X_1(T)$)[14].

If φ is a morphism of the groupid $X_*(T)$, the map $f \mapsto \varphi f$ is a bijection from the set of morphisms f whose target coincides with the source of φ to the set of morphisms with the same target as φ. We then conclude that there is a Cartesian square

$$\begin{tikzcd} X_2 \arrow[r, "d_1'"] \arrow[d, "d_0'"'] & X_1 \arrow[d, "d_0"] \\ X_1 \arrow[r, "d_0"] & X_0 \end{tikzcd} \tag{12.4.2}$$

Moreover, this square is also cocartesian: if $\varphi : X_1 \to Y$ is a morphism in $\mathcal{C}$ such that $\varphi d_1' = \varphi d_0'$, then for any $T \in \mathrm{Ob}(\mathcal{C})$, the value of the morphism $\varphi(T) : X_1(T) \to Y(T)$ on $f \in X_1(T)$ only depends on the target of f (if g is another morphism with the same target as f, then $f^{-1}g$ is in the image of d_1' and we have $\varphi(f) = \varphi d_1'(g, g^{-1}f) = \varphi d_0'(g, g^{-1}f) = \varphi(g)$), so it factors through $d_0(T)$.

Similarly, the map $g \mapsto g \circ \varphi$ is a bijection from the set of morphisms g whose source coincides with the target of φ to the set of morphisms with the same source as φ. We then conclude that there is a Cartesian square

$$\begin{tikzcd} X_2 \arrow[r, "d_1'"] \arrow[d, "d_2'"'] & X_1 \arrow[d, "d_1"] \\ X_1 \arrow[r, "d_1"] & X_0 \end{tikzcd} \tag{12.4.3}$$

which is also cocartesian.

On the other hand, let $s : X_0 \to X_1$ be the unique morphis in $\mathcal{C}$ such that, for any $T \in \mathrm{Ob}(\mathcal{C})$, $s(T) : X_0(T) \to X_1(T)$ associates to any object of $X_*(T)$ the identity morphism of this object. The morphism s satisfies the following equalities:

$$d_1 s = \mathrm{id}_{X_0}, \tag{12.4.4}$$
$$d_0 s = \mathrm{id}_{X_0}. \tag{12.4.5}$$

Finally, the associativity of the composition maps is expressed by the commutativity of the following diagram

$$\begin{tikzcd} X_1 \times_{d_1,d_0} X_1 \times_{d_1,d_0} X_1 \arrow[r, "d_1' \times \mathrm{id}_{X_1}"] \arrow[d, "\mathrm{id}_{X_1} \times d_1'"'] & X_1 \times_{d_1,d_0} X_1 \arrow[d, "d_1"] \\ X_1 \times_{d_1,d_0} X_1 \arrow[r, "d_1'"] & X_0 \end{tikzcd} \tag{12.4.6}$$

Conversely, the conditions (12.4.2), (12.4.3) and (12.4.6) and the existence of a morphism s satisfying (12.4.4) imply that $(d_0, d_1 : X_1 \rightrightarrows X_0, d_1')$ is a groupoid. In the rest of this section, we mainly use the squares (12.4.1), (12.4.2) and (12.4.3), which are summarized into the following diagram:

$$\begin{tikzcd} X_2 \arrow[r, shift left, "d_0'"] \arrow[r, shift right, "d_1'"'] \arrow[d, shift right, "d_2'"'] \arrow[d, shift left, "d_1'"] & X_1 \arrow[r, "d_0"] \arrow[d, "d_1"] & X_0 \\ X_1 \arrow[r, "d_0"] \arrow[d, "d_1"'] & X_0 & \\ X_0 & & \end{tikzcd} \tag{12.4.7}$$

where the square is Cartesian and the first row and first column are exact.

[14]Therefore, in this case, $X_2(T)$ is the set of pairs of composable morphisms (f_2, f_1), that is, such that $d_0(f_1) = d_1(f_2)$, and d_0', d_1' and d_2' send (f_2, f_1) to f_2, $f_2 \circ f_1$, f_1, respectively.

We only use associativity in an indirect way, for example to ensure the existence of a morphism s satisfying (12.4.4) and (12.4.5), or to ensure the existence of a morphism $\sigma : X_1 \to X_1$ such that

$$d_0\sigma = d_1, \quad d_1\sigma = d_0. \tag{12.4.8}$$

(We choose σ so that $\sigma(T) : X_1(T) \to X_1(T)$ sends a morphism of $X_*(T)$ to its inverse.)

By abusing of languages, a $\mathcal{C}$-groupoid is also defined to be a diagram

$$\begin{tikzcd} X_2 \arrow[r, shift left=2, "d'_0"] \arrow[r, "d'_1" description] \arrow[r, shift right=2, "d'_2"'] & X_1 \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] & X_0 \end{tikzcd}$$

such that (12.4.1), (12.4.2) and (12.4.3) are Cartesian, that (12.4.6) is commutative and that there exists s satisfying (12.4.4) and (12.4.5)[15].

Example 12.4.4. Let X be an object in $\mathcal{C}$ and G be a $\mathcal{C}$-group acting on X (on the left). We denote by $d_0 : G \times X \to X$ the morphism defining the action of G over X, by $d_1 : G \times X \to X$ the projection onto the second factor, by $\mu : G \times G \to G$ the multiplication of G, and finally by $\mathrm{pr}_{2,3}$ the projection of $G \times G \times X = G \times (G \times X)$ onto the second and third factors. Then

$$\begin{tikzcd} G \times G \times X \arrow[r, shift left=2, "\mathrm{pr}_{2,3}"] \arrow[r, "\mu\times\mathrm{id}_X" description] \arrow[r, shift right=2, "\mathrm{id}_G\times d_0"'] & G \times X \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] & X_0 \end{tikzcd}$$

is a groupoid in $\mathcal{C}$. For any $T \in \mathrm{Ob}(\mathcal{C})$, the groupoid $X_*(T)$ has object set $X(T)$ and morphisms (g, x), where $g \in G(T)$ and $x \in X(T)$. Moreover, $X_*(T)$ is a setoid if and only if for any $x \in G(T)$, the automorphism group $\mathrm{Aut}(x)$ is trivial, that is, if and only if $G(T)$ acts freely on $X(T)$.

Example 12.4.5. Let $d_0, d_1 : X_1 \to X_0$ be an **equivalence couple**, that is, if $d_0 \boxtimes d_1 : X_1 \to X_0 \times X_0$ is the morphism with components d_0, d_1, then for any $T \in \mathrm{Ob}(\mathcal{C})$, $(d_0 \boxtimes d_1)(T)$ is a bijection from $X_1(T)$ to the graph of an equivalence relation on $X_0(T)$. The set $X_1(T)$ is then identified with the set of couples (x, y) formed by elements of $X_1(T)$ such that $x \sim y$; similarly, the set $X_2(T) = (X_1 \times_{d_1,d_0} X_1)(T)$ is identified with the set of triples (x, y, z) of elements of $X_0(T)$ such that $x \sim y$ and $y \sim z$. There is then a unique morphism $d'_1 : X_2 \to X_1$ fitting into the squares (12.4.2) and (12.4.3): $d'_1(T)$ sends $(x, y, z) \in X_2(T)$ to $(x, z) \in X_1(T)$. For this choice of d'_1, $(d_0, d_1 : X_1 \rightrightarrows X_0, d'_1)$ is a $\mathcal{C}$-groupoid.

Conversely, consider a $\mathcal{C}$-groupoid X_* such that $d_0 \boxtimes d_1 : X_1 \to X_0 \times X_0$ is a monomorphism (in other words, for any $T \in \mathrm{Ob}(\mathcal{C})$ and $x, y \in X_0(T)$, there exists a unique morphism from x to y). Then (d_0, d_1) is an equivalence couple and X_* can be reconstructed from (d_0, d_1) as explained above[16].

Example 12.4.6. Let $p : X \to Y$ be a morphism in $\mathcal{C}$ and $\mathrm{pr}_1, \mathrm{pr}_2$ be two projections from $X \times_{p,p} X$ to X. Then $(\mathrm{pr}_1, \mathrm{pr}_2) : X \times_{p,p} X \rightrightarrows X$ is an equivalence couple. We say that p is an **effective epimorphism** if the diagram

$$\begin{tikzcd} X \times_{p,p} X \arrow[r, shift left, "\mathrm{pr}_1"] \arrow[r, shift right, "\mathrm{pr}_2"'] & X \arrow[r, "p"] & Y \end{tikzcd}$$

is exact, that is, if $(Y, p) = \mathrm{coker}(\mathrm{pr}_1, \mathrm{pr}_2)$.

For example, let S be a Noetherian scheme and $\mathcal{C}$ be the category of schemes finite over S. We show that an epimorphism in $\mathcal{C}$ is not necassarily effective: we choose $S = \mathrm{Spec}(k[T^3, T^5])$, where k is a field, $Y = S$ and $X = \mathrm{Spec}(k[T])$. If i denotes the inclusion of $B = k[T^3, T^5]$ to $A = k[T]$ and $p = \mathrm{Spec}(i)$, then $X \times_{p,p} X$ is identified with $\mathrm{Spec}(A \otimes A)$ and $\mathrm{coker}(\mathrm{pr}_1, \mathrm{pr}_2)$ is equal to $\mathrm{Spec}(B')$, where B' is the subring of A formed by elements a such that $a \otimes_B 1 = 1 \otimes_B a$. Now

$$T^7 \otimes_B 1 = (T^2T^5) \otimes_B 1 = T^2 \otimes_B T^5 = T^2 \otimes_B (T^3T^2) = T^5 \otimes_B T^2 = 1 \otimes_B T^7$$

hence $T^7 \in B'$, $T^7 \notin B$ and $\mathrm{Spec}(B) \neq \mathrm{Spec}(B')$, whence a couterexample[17].

---

[15] For a gropoid X_*, we often say that X_0 is the base of the groupoid and X_1 is the equvialence prerelation.

[16] In particular, if G is a $\mathcal{C}$-group acting on the left on an object X of $\mathcal{C}$ and X_* is the associated groupoid, then (d_0, d_1) is an equivalence couple if and only if G acts freely on X.

[17] The same arguments apply to $B = k[T^n, T^{n+r}]$ and the element $T^{n+2r} \otimes_B 1$, provided that $2r \nmid n$.

Consider a $\mathcal{C}$-groupoid

$$\begin{tikzcd} X_2 \arrow[r, shift left=2, "d'_0"] \arrow[r, "d'_1" description] \arrow[r, shift right=2, "d'_2"'] & X_1 \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] & X_0 \end{tikzcd}$$

and let $f_0 : Y_0 \to X_0$ be a morphism in $\mathcal{C}$. Then by base change to Y_0, we obtain a $\mathcal{C}$-groupoid

$$\begin{tikzcd} Y_2 \arrow[r, shift left=2, "e'_0"] \arrow[r, "e'_1" description] \arrow[r, shift right=2, "e'_2"'] & Y_1 \arrow[r, shift left, "e_0"] \arrow[r, shift right, "e_1"'] & Y_0 \end{tikzcd}$$

which is said to be **induced** by X_* and f_0. We also say that Y_* is the **inverse image** of X_* by the base change morphism $f_0 : Y_0 \to X_0$. More precisely, we choose for Y_1 the fiber product of the diagram

$$\begin{tikzcd} Y_1 \arrow[r, dashed, "f_1"] \arrow[d, dashed] & X_1 \arrow[d, "d_0 \boxtimes d_1"] \\ Y_0 \times Y_0 \arrow[r, "f_0 \times f_0"] & X_0 \times X_0 \end{tikzcd}$$

for e_0 and e_1 the composition of the canonical morphism $Y_1 \to Y_0 \times Y_0$ and the first and second projections of $Y_0 \times Y_0$. The morphism $Y_1 \to Y_0 \times Y_0$ is then $e_0 \boxtimes e_1$, and we have $f_0 \circ e_i = d_i \circ f_1$ for $i = 0, 1$, where we denote by f_1 the projection of Y_1 to X_1. We put $Y_2 = Y_1 \times_{e_0, e_1} Y_1$. We can say that the couple (e_0, e_1) is defined such that, for any $T \in \mathrm{Ob}(\mathcal{C})$, and any couple (y, x) of elements of $Y_0(T)$, there is a one-to-one correspondence $\psi \mapsto {}_y\psi_x$ between the set of morphisms ψ of $X_*(T)$ with source $f_0(x)$, target $f_0(y)$ and the arrows ${}_y\psi_x$ of $Y_*(T)$ with source x and target y. We therefore determine $e'_1 : Y_2 \to Y_1$ by defining for all $T \in \mathrm{Ob}(\mathcal{C})$ the composition of the morphism of $Y_*(T)$ using the formula

$$ {}_z\psi_y \circ {}_y\varphi_x = {}_z(\psi \circ \varphi)_x. $$

It is then clear that this makes $Y_*(T)$ a groupoid.

From the $\mathcal{C}$-groupoid X_* and the base change $f_0 : Y_0 \to X_0$, we can reconstruct the couple $(e_0, e_1) : Y_1 \rightrightarrows Y_0$ in another way: consider $Y_0 \times_{X_0} X_1$ and $\mathrm{pr}_1, \mathrm{pr}_2$ such that the square

$$\begin{tikzcd} Y_0 \times_{X_0} X_1 \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] & X_1 \arrow[d, "d_0"] \\ Y_0 \arrow[r, "f_0"] & X_0 \end{tikzcd} \tag{12.4.9}$$

is Cartesian. We then verify that we have a Cartesian square

$$\begin{tikzcd} Y_1 \arrow[r, "e_0 \boxtimes f_1"] \arrow[d, "e_1"'] & Y_0 \times_{X_0} X_1 \arrow[d, "d_1 \circ \mathrm{pr}_2"] \\ Y_0 \arrow[r, "f_0"] & X_0 \end{tikzcd} \tag{12.4.10}$$

where f_1 denotes the canonical projection from $Y_1 = (Y_0 \times Y_0) \times_{X_0 \times X_0} X_1$ to X_1.

Example 12.4.7. Let's take $Y_0 = X_1$, $f_0 = d_0$. For any object T of $\mathcal{C}$, $Y_1(T)$ is then identified with the set of diagrams of the form

$$\begin{tikzcd} b \arrow[r, "\varphi"] & d \\ a \arrow[u, "f"] & c \arrow[u, "g"'] \end{tikzcd}$$

of $X_*(T)$. The source of this diagram is the morphism f, the target is the morphism g, and the composition of two diagrams is clear (by taking the compositin of the horizontal morphisms).

Similarly, by choosing $Y'_0 = X_1$ and $f'_0 = d_1$, the set $Y'_1(T)$ is identified for any $T \in \mathrm{Ob}(\mathcal{C})$ with the set of diagrams of the form

$$\begin{tikzcd} b & d \\ a \arrow[u, "f"] \arrow[r, "\psi"] & c \arrow[u, "g"'] \end{tikzcd}$$

of the groupoid $X_*(T)$. The source of this diagram is the morphism f, the target is the morphism g, and the composition of two diagrams is evident.

Now since $X_*(T)$ is a groupoid for any $T \in \mathrm{Ob}(\mathcal{C})$, the identity map on $Y_0(T)$ and the map

$$\begin{tikzcd} b \arrow[r, "\varphi"] & d & & b & d \\ a \arrow[u, "f"] & c \arrow[u, "g"'] \arrow[r, phantom, "\mapsto"] & & a \arrow[u, "f"] \arrow[r, "g^{-1}\varphi f"] & c \arrow[u, "g"'] \end{tikzcd}$$

from $Y_1(T)$ to $Y'_1(T)$ define an isomorphism of groupoids $Y_*(T)$ and $Y'_*(T)$. Moreover, this isomorphism depends functorial on T, hence is an isomorphism of the $\mathcal{C}$-groupoids Y_* and Y'_*.

Proposition 12.4.8. *Let X_* be a $\mathcal{C}$-groupoid and suppose that $f_0 : Y_0 \to X_0$ is a universally effective epimorphism. Then* $\mathrm{coker}(d_0, d_1)$ *exists if and only if* $\mathrm{coker}(e_0, e_1)$ *exists. Moreover, in this case f_0 induces an isomorphism*

$$\mathrm{coker}(d_0, d_1) \xrightarrow{\sim} \mathrm{coker}(e_0, e_1).$$

Proof. We denote by $C(d_0, d_1)$ the covariant functor from $\mathcal{C}$ to the category of sets which associates to any $T \in \mathrm{Ob}(\mathcal{C})$ the kernel of the couple $T(d_0), T(d_1) : T(X_0) \rightrightarrows T(X_1)$ (in **Set**), and similarly for $C(e_0, e_1)$. For any $T \in \mathcal{C}$, we then have a commutative diagram

$$\begin{tikzcd} C(d_0,d_1)(T) \arrow[r] \arrow[d, "T(f)"] & T(X_0) \arrow[r, shift left, "T(d_1)"] \arrow[r, shift right, "T(d_0)"'] \arrow[d, "T(f_0)"] & T(X_1) \arrow[d, "T(f_1)"] \\ C(e_0,e_1)(T) \arrow[r] & T(Y_0) \arrow[r, shift left, "T(e_1)"] \arrow[r, shift right, "T(e_0)"'] & T(Y_1) \end{tikzcd}$$

where $T(f)$ is the injection induced by the injection $T(f_0)$. If we can show that $T(f)$ is a surjection for any T, then we obtain a functorial isomorphsim $f : C(d_0, d_1) \xrightarrow{\sim} C(e_0, e_1)$, which implies the proposition. For this, consider the diagram

$$\begin{tikzcd} & Y_1 \arrow[r, "f_1"] \arrow[d, shift right, "e_1"'] \arrow[d, shift left, "e_0"] & X_1 \arrow[d, shift right, "d_1"'] \arrow[d, shift left, "d_0"] \\ Y_0 \times_{X_0} Y_0 \arrow[ur, "\Delta"] \arrow[r, shift left, "\mathrm{pr}_2"] \arrow[r, shift right, "\mathrm{pr}_1"'] & Y_0 \arrow[r, "f_0"] \arrow[d, "g"] & X_0 \arrow[dl, dashed, "h"] \\ & T & \end{tikzcd}$$

where Δ is the section of $Y_1 \to Y_0 \times Y_0$ defined by the morphism $s \circ f_0 \circ \mathrm{pr}_1 : Y_0 \times Y_0 \to X_1$, the morphism $s : X_0 \to X_1$ satisfying the equalities (12.4.4) and (12.4.5). If a morphism $g : Y_0 \to T$ is such that $ge_0 = ge_1$, we then have $ge_0\Delta = ge_1\Delta$, hence $g\mathrm{pr}_1 = g\mathrm{pr}_2$. As f_0 is an effective epimorphism, g is then the composition of f_0 with a morphism $h : X_0 \to T$, that is, we have $g = T(f_0)(h)$. It remains to show that h belongs to $C(d_0, d_1)(T)$, which means $hd_0 = hd_1$; now we have

$$hd_0f_1 = hf_0e_0 = ge_0 = ge_1 = hf_0e_1 = hd_1f_1$$

whence the desired equality since f_1 is an epimorphism (because f_0 is a universally epimorphism). □

Consider now a scheme S and choose $\mathcal{C} = \mathbf{Sch}_{/S}$. Then a $\mathcal{C}$-groupoid

$$\begin{tikzcd} X_2 \arrow[r, shift left=2, "d'_0"] \arrow[r, "d'_1" description] \arrow[r, shift right=2, "d'_2"'] & X_1 \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] & X_0 \end{tikzcd}$$

permits us to define an equivalence relation on the underlying set $|X_0|$: if $x, y \in |X_0|$, we write $x \sim y$ if there exists $z \in |X_1|$ such that $x = d_1(z)$ and $y = d_0(z)$. The reflecxivity and symmetricity of this equation is evident[18]. As for the transtivity, if $x \sim y$ and $y \sim z$, then there exists $u, v \in |X_1|$ such that $x = d_1(u)$, $y = d_0(u)$, $y = d_1(v)$, $z = d_0(v)$. It then follows that (v, u) belongs to the set $|X_1| \times_{d_1, d_0} |X_1|$. As the canonical map

$$|X_1 \times_{d_1, d_0} X_1| \to |X_1| \times_{d_1, d_0} |X_1|$$

on underlying sets is surjective, (v, u) is the image of some $w \in |X_2|$. We then have $x = d_1 d_1'(w)$ and $z = d_0 d_1'(w)$, then $x \sim z$.

Now let $f_0 : Y_0 \to X_0$ be a base change morphism of schemes over S. If x, y are points of $|Y_0|$, we see that $x \sim y$ if and only if $f_0(x) \sim f_0(y)$. In fact, if $x \sim y$, then there exists $z \in |Y_1|$ such that $x = e_1(z)$ and $y = e_0(z)$. As $f_0 \circ e_i = d_i \circ f_1$ for $i = 0, 1$, we then have $f_0(x) = d_1 f_1(z)$ and $f_0(y) = d_0 f_1(z)$, whence $f_0(x) \sim f_0(y)$.

Conversely, if $f_0(x) \sim f_0(y)$ and $z \in |X_1|$ is such that $f_0(y) = d_1(z)$ and $f_0(x) = d_0(z)$, then by the square (12.4.9), there exists a point $t \in |Y_0 \times_{X_0} X_1|$ such that $\mathrm{pr}_1(t) = x$ and $\mathrm{pr}_2(t) = z$. Similarly, as $f_0(y) = d_1 \mathrm{pr}_2(t)$, there exists $s \in |Y_1|$ such that $y = e_1(s)$ and $(e_0 \boxtimes f_1)(s) = t$ (cf. the square (12.4.10)). We then have $e_0(s) = \mathrm{pr}_1(e_0 \boxtimes f_1)(s) = \mathrm{pr}_1(t) = x$, whence $x \sim y$.

12.4.2 Quotient for a finite locally free groupoid

Let S be a scheme and consider a $\mathbf{Sch}_{/S}$-grupoid

$$\begin{tikzcd} X_2 \arrow[r, shift left=2, "d_0'"] \arrow[r, "d_1'" description] \arrow[r, shift right=2, "d_2'"'] & X_1 \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] & X_0 \end{tikzcd}$$

In this subsection, we prove the existence of a quotient of X_* under the hypothesis that the strucutral morphism is finite and locally free. More precisely, we shall prove the following theorem:

Theorem 12.4.9. *Suppose that X_* satisfies the following conditions*[19]*:*

(i) *d_1 is locally free and finite.*

(ii) *For any $x \in X_0$, the set $d_0 d_1^{-1}(x)$ is contained in an affine open subset of X_0.*

Then we have the following:

(a) *There exists a cokernel (Y, p) of (d_0, d_1) in $\mathbf{Sch}_{/S}$. Moreover, such a pair (Y, p) is a cokernel of (d_0, d_1) in the category of ringed spaces.*

(b) *The morphism p is open and integral, and Y is affine if X_0 is affine.*

(c) *The morphism $X_1 \to X_0 \times_Y X_0$ with components d_0 and d_1 is surjective.*

(d) *If (d_0, d_1) is an equivalence couple, then the morphism $X_1 \to X_0 \times_Y X_0$ in (c) is an isomorphism and $p : X_0 \to Y$ is locally free and finite. Further, (Y, p) is a cokernel of (d_0, d_1) in the category of sheaves for the fppf topology and, for any base change $Y' \to Y$, Y' is the cokernel of the groupoid $X_* \times_Y Y'$ induced from X_* by base change.*

In particular, for any base change $S' \to S$, $Y' = Y \times_S S'$ is the cokernel of the S'-groupoid $X'_* = X_* \times_S S'$. Hence, in this case, the formation of quotient commutes with base change.

It follows from Theorem 12.4.9 (a) that the underlying topological space of Y is the quotient of that of X_0 by the equivalence relation defined by the groupoid X_*. The rest of this subsection is devoted to the proof of Theorem 12.4.9.

[18]The reflecxivity follows from the existence of $s : X_0 \to X_1$ which is a section of d_0 and d_1; the symmetricity follows from the existence of an involution σ of X_1 which exchanges d_0 and d_1.

[19]As d_0 and d_1 are exchanged by the involution σ, these conditions are symemtric on d_0 and d_1. Moreover, for any $x \in X_0$ we have $d_0 d_1^{-1}(x) = d_1 d_0^{-1}(x)$.

12.4.2.1 Quotient by a finite and locally free groupoid (affine case) We first prove the theorem under the asssumption that X_0 is affine and d_1 is locally free of constant rank n (then we shall see how to reduce the general case to this particular one). In this case, X_0, X_1 and X_2 are all affine, so we can suppose that $X_i = \mathrm{Spec}(A_i)$, $d_j = \mathrm{Spec}(\delta_j)$, $d'_k = \mathrm{Spec}(\delta'_k)$, where δ_j, δ'_k are homomorphism of rings. From (12.4.7), we then obtain a solid diagram

$$\begin{tikzcd}
A_2 & A_1 \arrow[l, shift right, "\delta'_1"'] \arrow[l, shift left, "\delta'_0"'] & A_0 \arrow[l, "\delta_0"'] \\
A_1 \arrow[u, "\delta'_2"] & A_0 \arrow[l, shift right, "\delta_1"'] \arrow[l, shift left, "\delta_0"'] \arrow[u, "\delta_1"] & B \arrow[l, "\delta"'] \arrow[u, dashed, "\delta"]
\end{tikzcd} \tag{12.4.11}$$

where the two squares are cocartesian and the first row is exact. We denote by B the subring of A_0 formed by $a \in A_0$ such that $\delta_0(a) = \delta_1(a)$, and let $\delta : B \to A_0$ be the canonical inclusion. If $a_0 \in A_0$, let

$$P_{\delta_1}(T, \delta_0(a)) = T^n - \sigma_1 T^{n-1} + \cdots + (-1)^n \sigma_n$$

be the characteristic polynomial of $\delta_0(a)$ if we consider A_1 as an A_0-algebra via the homomorphism δ_1 (cf. [?] II §5, exercice 9). As the two squares of (12.4.11) are cocartesian, we have

$$\delta_0(P_{\delta_1}(T, \delta_0(a))) = P_{\delta'_2}(T, \delta'_0\delta_0(a)), \quad \delta_1(P_{\delta_1}(T, \delta_0(a))) = P_{\delta'_2}(T, \delta'_1\delta_0(a)).$$

As $\delta'_0\delta_0 = \delta'_1\delta_0$, we then conclude that

$$\delta_0(P_{\delta_1}(T, \delta_0(a))) = \delta_1(P_{\delta_1}(T, \delta_0(a)))$$

that is, $\delta_0(\sigma_i) = \delta_1(\sigma_i)$ for each i. By Hamilton-Cayley theorem, we then have

$$\begin{aligned} 0 = \delta_1(P_{\delta_1}(T, \delta_0(a)))(\delta_0(a)) &= \delta_0(a)^n - \delta_1(\sigma_1)\delta_0(a)^{n-1} + \cdots + (-1)^n \delta_1(\sigma_n) \\ &= \delta_0(a)^n - \delta_0(\sigma_1)\delta_0(a)^{n-1} + \cdots + (-1)^n \delta_0(\sigma_n), \end{aligned}$$

whence

$$a^n - \sigma_1 a^{n-1} + \cdots + (-1)^n \sigma_n = 0$$

because δ_0 has a section $\tau : A_1 \to A_0$ such that $\tau\delta_0 = \mathrm{id}_{A_0}$, so it is injective. We then conclude that A_0 *is integral over* B.

Now consider two prime ideals $\mathfrak{p}, \mathfrak{q}$ of A_0; we show that the equality $\mathfrak{p} \cap B = \mathfrak{q} \cap B$ implies the existence of a prime ideal $\mathfrak{r}$ of A_1 such that $\mathfrak{p} = d_0(\mathfrak{r})$ and $\mathfrak{q} = d_1(\mathfrak{r})$. In fact, if the assertion was not true, $\mathfrak{p}$ would be distinct from $\delta_0^{-1}(\mathfrak{n})$ any prime ideal $\mathfrak{n}$ of A_1 such that $\delta_1^{-1}(\mathfrak{n}) = \mathfrak{q}$. For such an ideal $\mathfrak{n}$ we would have $\delta_0^{-1}(\mathfrak{n}) \cap B = \delta_1^{-1}(\mathfrak{n}) \cap B = \mathfrak{q} \cap B = \mathfrak{p} \cap B$, so by **??**, $\mathfrak{p}$ is not contained in $\delta_0^{-1}(\mathfrak{n})$. But there are finitely many prime ideals $\mathfrak{n}$ of A_1 such that $\delta_1^{-1}(\mathfrak{n}) = \mathfrak{q}$ (**??**), hence, by prime aviodence, there exists $a \in \mathfrak{p}$ which is not in any of the $\delta_0^{-1}(\mathfrak{n})$. Therefore, $\delta_0(a)$ is not conained in these ideals $\mathfrak{n}$, and hence, by the lemma bolow, the norm $N_{\delta_1}(\delta_0(a))$ does not belong to $B \cap \mathfrak{q}$ (the norm is calculated by considering A_1 as an A_0-algebra via the homomorphism δ_1, and we have $N_{\delta_1}(\delta_0(a)) = \sigma_n$ with the notations above). As $(-1)^{n-1}\sigma_n = a^n + \sum_{i=1}^{n-1}(-1)^i \sigma_i a^{n-i}$, this norm belongs to $B \cap \mathfrak{p} = B \cap \mathfrak{q}$, which is a contradiction.

Lemma 12.4.10. *Let $A \to A'$ be a ring homomorphism such that A' is a projective A-module of rank n. Let $\mathfrak{p}$ be a prime ideal of A and $\mathfrak{q}_1, \dots, \mathfrak{q}_r$ be the prime ideals of A' lying over $\mathfrak{p}$. Let $a \in A'$, then a belongs to $\mathfrak{q}_1 \cup \cdots \cup \mathfrak{q}_r$ if and only if its norm $N(a)$ belongs to $\mathfrak{p}$.*

Proof. By replacing A and A' by the localizations $A_{\mathfrak{p}}$ and $A'_{\mathfrak{p}}$, we may assume that $(A, \mathfrak{p})$ is local and A' is semi-local, with $\mathrm{Spec}(A') = \{\mathfrak{q}_1, \dots, \mathfrak{q}_r\}$. In this case, A' is a free A-module of rank n, and $N(a)$ is the determinant of the endomorphism $h_a : A' \to A'$ with ratio a. We then conclude that $N(a) \notin \mathfrak{p}$ if and only if $N(a)$ is invertible, if and only if h_a is invertible, and this is equivalent to that $a \notin \mathfrak{q}_1 \cup \cdots \cup \mathfrak{q}_r$. □

We now prove assertion (a) of Theorem 12.4.9 in this case. Let $Y = \mathrm{Spec}(B)$ and $p = \mathrm{Spec}(\delta)$, where $\delta : B \to A_0$ is the inclusion. By the preceding arguments, $p : X_0 \to Y$ is integral, hence surjective, and the underlying space of $\mathrm{Spec}(B)$ is obtained from that of X_0 by identifying points x, y such that thre exists $z \in X_1$ such that $d_1(z) = y$, $d_0(z) = x$. Moreover, as i is integral, p is closed (**??**) so that Y is endowed with the quotient topology of that of X_0. In particular, p is open: if U is an open subset of X_0, as d_1 is surjective and locally free and finite (hence faithfully flat and finitely presented), hence open, the saturation $U' = d_1 d_0^{-1}(U')$ of U' under the equivalence relation defined by X_* is open, so $p(U') = p(U)$ is open, since Y is endowed with the quotient topology.

Finally, it follows from the choice of B and the fact that p, d_0, d_1 are affine that the canonical sequence of sheaf of rings

$$\mathscr{O}_Y \longrightarrow p_*(\mathscr{O}_{X_0}) \underset{p_*(\delta_0)}{\overset{p_*(\delta_1)}{\rightrightarrows}} p_*((d_0)_*(\mathscr{O}_{X_1})) = p_*((d_1)_*(\mathscr{O}_{X_1}))$$

is exact. It remains to prove that (Y, p) is also the cokernel of (d_0, d_1) in the category of schemes (or more generally in the category of locally ringed spaces). Let $f : X_0 \to Z$ be a morphism of schemes such that $fd_0 = fd_1$. By the above arguments, there exists a unique morphism of ringed spaces $\tilde{f} : Y \to Z$ such that $f = \tilde{f}p$. Since the composition f and p are both local morphisms, we conclude that $\tilde{f}$ is a local morphism, hence a morphism of schemes.

Now the assertion (b) of Theorem 12.4.9 follows immediately. On the other hand, since $p : |X_0| \to |Y|$ is a quotient map, the following map

$$d_0 \boxtimes d_1 : |X_1| \to |X_0| \times_{|Y|} |X_0|$$

is surjective. This map factors into

$$|X_1| \xrightarrow{d_0 \boxtimes d_1} |X_0 \times_Y X_0| \xrightarrow{q} |X_0| \times_{|Y|} |X_0|$$

where q is the canonical map. We therefore conclude that the image of $d_0 \boxtimes d_1$ then contains points $v \in X_0 \times_Y X_0$ such that $\{v\} = q^{-1}(q(v))$, which is staisfied if v is a rational point over Y (that is, if the residue field $\kappa(v)$ is identified with $\kappa(w)$, where w is the image of v in Y). If $v \in X_0 \times_Y X_0$ is not rational over Y, let w be the image of v in Y. By ([?] 0_{III}, 10.3.1) there exists a local ring C and a flat local homomorphism $\rho : \mathscr{O}_w \to C$ such that $C/\mathfrak{m}_w C$ is isomorphic to $\kappa(v)$ as $\kappa(w)$-algebras. If we put $Y' = \mathrm{Spec}(C)$ and $\pi : Y' \to Y$ is the morphism induced by ρ, it is clear that the canonical projection of $(X_0 \times_Y X_0) \times_Y Y'$ onto $X_0 \times_Y X_0$ sends v to a point v' of $(X_0 \times_Y X_0) \times_Y Y'$ which is rational over Y'. As

$$(X_0 \times_Y X_0) \times_Y Y' \cong (X_0 \times_Y Y') \times_{Y'} (X_0 \times_Y Y'),$$

and as the hypothesis of Theorem 12.4.9 and the preceding results, in particular that of (b), is valid under base change $\pi : Y' \to Y$, we conclude that v' is the image of an element $u' \in X_1 \times_Y Y'$ under the morphism deduced from $d_0 \boxtimes d_1$ by base change. If u is the image of u' in X_1, we then have $v = (d_0 \boxtimes d_1)(u)$.

Finally, we prove assertion (d) of Theorem 12.4.9. By hypothesis, $X_0 = \mathrm{Spec}(A_0)$, $X_1 = \mathrm{Spec}(A_1)$, and for $i = 0, 1$, the morphism $\delta_i : A_0 \to A_1$ is finite; hence the morphism $A_0 \otimes_B A_0 \to A_1$ is finite. Since (d_0, d_1) is assumed to be an equivalence couple, we may further assume that $d_0 \boxtimes d_1 : X_1 \to X_0 \times_Y X_0$ is a monomorphism; then, by ([?], 18.12.6), $d_0 \boxtimes d_1$ is a closed immersion, so $A_0 \otimes_B A_0 \to A_1$ is surjective. We will show that this is an isomorphism (and also that $p : X_0 \to Y$ is finite and locally free). For this, it suffices to show that for any prime ideal $\mathfrak{p}$ of B, the homomorphism $(A_0)_\mathfrak{p} \otimes_{B_\mathfrak{p}} (A_0)_\mathfrak{p} \to (A_1)_\mathfrak{p}$ with components $(\delta_0)_\mathfrak{p}$ and $(\delta_1)_\mathfrak{p}$ is bijective. In other words, we may assume that B is local. It then follows from **??** that $(A_0)_\mathfrak{p}$ is semi-local. By applying ([?] 0_{III}, 10.3.1) to perform a faithfully flat base change, we may also assume that the residue field of B is infinite, so that we can use the following lemma:

Lemma 12.4.11. *Let B be a local ring with infinite residue field, A be a semi-local ring and $i : B \to A$ be a homomorphism wich sends the maximal ideal $\mathfrak{m}$ of B into the radical $\mathfrak{r}$ of A. Let M be a free A-module of rank n and N be a sub-B-module of M which generates M as an A-module. Then N contains a basis of M over A.*

Proof. We recall that, by Nakayama's lemma, a sequence $m_1, \dots, m_n$ of elements of M is an A-basis of M if and only if the canonical images of $m_1, \dots, m_n$ in $M/\mathfrak{r}M$ form a basis of $M/\mathfrak{r}M$ over $A/\mathfrak{r}$. We can then replace M by $M/\mathfrak{r}M$, N by $N/(N \cap \mathfrak{r}M)$, A by $A/\mathfrak{r}$ and B by $B/\mathfrak{m}$. In this case, we then have $A = K_1 \times \cdots \times K_r$, and M can be identified with $K_1^n \times \cdots \times K_r^n$. Now we choose elements $(x_{ij})_{1 \le i \le r, 1 \le j \le n}$ of N such that for each $1 \le i \le r$, the i-th component of $(x_{i,1}, \dots, x_{i,n})$ in K_i^n is linearly independent over K_i. We can consider the polynomial

$$f(a_{1,1}, \dots, a_{n,r}) = \prod_{i=1}^{r} \det_{K_i}\Big(\sum_{j=1}^{n} a_{j,1} x_{j,1}, \dots, \sum_{j=1}^{n} a_{j,r} x_{j,r}\Big)$$

where $\det_{K_i}(z_1, \ldots, z_n)$ denotes the determinant of the i-th components of $z_1, \ldots, z_n$ over K_i. As k is an infinite field and each polynomial $\det_{K_i}(\sum_{j=1}^n a_{j,1}x_{j,1}, \ldots, \sum_{j=1}^n a_{j,r}x_{j,r})$ with coefficient in A is nonzero (take $a_{i,1} = \cdots = a_{i,n} = 1$ and others to be zero), we conclude that there exists a family $(y_\ell)_{1\le\ell\le n}$ of k-linear combinations of the x_{ij} such that for any $1 \le i \le n$, the i-th component of $(y_\ell)_{1\le\ell\le n}$ is linearly independent over K_i. Then $(y_\ell)_{1\le\ell\le n}$ is easily seen to be a basis of M over A. □

We shall apply Lemma 12.4.11 for the ring homomorphism $B \to A_0$, $M = A_1$ (as an A_0-module via the homomorphism δ_1), and $N = \delta_0(A_0)$. In fact, as $d_0 \boxtimes d_1 : X_1 \to X_0 \times_Y X_0$ is a closed immersion, the homomorphism $A_0 \otimes_B A_0 \to A_1$ with components δ_0 and δ_1 is surjective; this signifies that $\delta_0(A_0)$ generates the A_0-module A_1.

Let $a_1, \ldots, a_n$ be elements of A_0 such that $\delta_0(a_1), \ldots, \delta_0(a_n)$ form a basis of A_1 over A_0. If we can show that $a_1, \ldots, a_n$ is a basis of A_0 over B, then the homomorphism $A_0 \otimes_B A_0 \to A_1$ sends the basis $(1 \otimes a_i)_{1\le i\le n}$ to the basis $(\delta_0(a_i))_{1\le i\le n}$, hence is bijective. Therefore, if $\varepsilon : \mathbb{Z}^n \to A_0$ is the homomorphism of abelian groups sending the natural basis of $\mathbb{Z}^n$ to $a_1, \ldots, a_n$, it suffices to prove that the map $B \otimes_{\mathbb{Z}} \mathbb{Z}^n \to A_0$ with components i and ε is bijective. Now the diagram (12.4.11) gives the following commutative diagram:

$$\begin{tikzcd}
A_2 & A_1 \arrow[l, shift left, "\delta_1'"'] \arrow[l, shift right, "\delta_0'"] & A_0 \arrow[l, "\delta_0"'] \\
A_1 \otimes_{\mathbb{Z}} \mathbb{Z}^n \arrow[u, "u_2"] & A_0 \otimes_{\mathbb{Z}} \mathbb{Z}^n \arrow[u, "u_1", "\cong"'] \arrow[l, shift left, "\delta_1\otimes 1"'] \arrow[l, shift right, "\delta_0\otimes 1"] & B \otimes_{\mathbb{Z}} \mathbb{Z}^n \arrow[u, "u_0"] \arrow[l, "\delta\otimes 1"']
\end{tikzcd}$$

where u_0, u_1 and u_2 have components δ and ε, δ_1 and $\delta_0\varepsilon$, δ_2' and $\delta_0'\delta_0\varepsilon$, respectively. We see that u_1 is an isomorphism. As the two squares in (12.4.11) are cocartesian, u_2 is then an isomorphism. Since the two horizontal rows of the above diagram are exact, we conclude that u_0 is bijective. This shows that A_0 is a locally free B-module of rank n, whence $\delta_0 \otimes \delta_1 : A_0 \otimes_B A_0 \to A_1$ is an isomorphism. This proves Theorem 12.4.9 in the particular case where X_0 is affine and d_1 is locally free of rank n.

12.4.2.2 Quotient by a finite and locally free groupoid (general case) Let U^n be the largest open subset of X_0 over which d_1 is finite and locally free of rank n. We know that X_0 is the direct sum of these U^n. On the other hand, it follows from the Cartesian squares

$$\begin{tikzcd}
X_2 \arrow[r, "d_0'"] \arrow[d, "d_2'"'] & X_1 \arrow[d, "d_1"] \\
X_1 \arrow[r, "d_0"] & X_0
\end{tikzcd}
\qquad
\begin{tikzcd}
X_2 \arrow[r, "d_1'"] \arrow[d, "d_2'"'] & X_1 \arrow[d, "d_1"] \\
X_1 \arrow[r, "d_1"] & X_0
\end{tikzcd}$$

that the inverse images of U^n under d_0 and d_1 both coincide with the lagest open subset of X_1 over which d_2' is locally free of rank n: in fact, as d_0 (resp. d_1) is surjective, finite and flat, hence faithfully flat and affine, d_2' is of rank n at a point x of X_1 if and only if d_1 is of rank n on a neighborhood of $d_0(x)$ (resp. $d_1(x)$). We then have $d_0^{-1}(U^n) = d_1^{-1}(U^n)$ so that the groupoid X_* is the direct sum of the groupoids X_*^n induced from X_* on the open and closed subsets U^n. It then suffices to prove Theorem 12.4.9 for each X_*^n, so we can assume that d_1 is locally free of finite rank n.

We now prove the theorem in the general case. By the above arguments, we can assume that d_1 is locally free of rank n. Let (Y, p) be a cokernel of (d_0, d_1) in the category of ringed spaces. Then as before, for (a) it suffices to prove that Y is a scheme and $p : X_0 \to Y$ is a morphism of schemes. By Lemma 12.4.3, this question is local over Y: let $y \in Y$ and $x_0 \in X_0$ such that $p(x) = y$; if x has a saturated affine open neighborhood U, then $p(U)$ is an affine open of Y by Theorem 12.4.9 (b) in the affine case and $p|_U$ is a morphism of schemes. It then suffices to prove that any $x \in X_0$ has a saturated affine open neighborhood U. Here is how we proceed (the proof is taken from [**?**], VIII, cor. 7.6):

$$d_1(d_0^{-1}(x)) \subseteq U = \widetilde{V}_f \subseteq V_f \subseteq \widetilde{V} \subseteq V \subseteq X_0.$$

By condition (ii) of Theorem 12.4.9, there exists an open affine subset V of X_0 containing $d_1(d_0^{-1}(X))$; if $F = X_0 - V$, $d_1(d_0^{-1}(F))$ is closed because d_1 is integral and $\widetilde{V} = X_0 - d_1(d_0^{-1}(F))$ is the largest saturated open subset containing V. As $\widetilde{V}$ is an neighborhood of the finite set $d_1(d_0^{-1}(x))$ (d_0 is also finite, hence has finite fiber), there exists a section f of the structural sheaf of V which vanishes over

$V - \widetilde{V}$ and such that $d_1(d_0^{-1}(x))$ is contained in the open subset $V_f \subseteq V$ formed by points where f is non-vanishing[20]. We then see that the largest saturated open subset $\widetilde{V}_f$ of V_f is affine: in fact, let $Z(f) = \widetilde{V} - V_f$. Then $d_0^{-1}(Z(f))$ is the set of points of $d_0^{-1}(\widetilde{V}) = d_1^{-1}(\widetilde{V})$ where the image $d_0^*(f)$ of f under the map induced by d_0 vanishes. On the other hand, as d_1 induces a locally free morphism of rank n from $d_0^{-1}(\widetilde{V}) = d_1^{-1}(\widetilde{V})$ to $\widetilde{V}$, by Lemma 12.4.10, $d_1(d_0^{-1}(Z(f)))$ is the set of points where the norm $N(d_0^*(f))$ relative to the morphism d_1 vanishes. It follows that $\widetilde{V}_f = \widetilde{V} - d_1(d_0(Z(f)))$ is the set of points of V_f where $N(d_0^*(f))$ does not vanish, so $\widetilde{V}_f$ is affine.

We therefore conclude assertion (a), and (b), (c), together with the first part of (d), are then clear from the affine case. It remains to see the other concequences in (d). By hypothesis, the groupoid X_* is given by an equivalence relation $R \to X_0 \times_S X_0$, and we have proved that R is effective and that $p : X_0 \to Y = X_0/R$ is surjective, finite and locally free, hence, in particular, faithfully flat and finitely presented. Therefore, if $\mathcal{M}$ is the family of faithfully flat morphisms locally of finite presentaion, then R is $\mathcal{M}$-effective. By Corollary 12.3.68, we conclude that (Y, p) represents the quotient sheaf of X_0 by R for the fppf topology, and the assertion concerning base change follows from Proposition 12.3.17.

Remark 12.4.12. With the hypothesis and notations of Theorem 12.4.9, suppose that S is locally Noetherian and $\pi_0 : X_0 \to S$ is quasi-projective. Let $\mathscr{A}$ be an ample $\mathscr{O}_{X_0}$-module relative to π_0. By Proposition 9.6.11, $p_*(\mathscr{A})$ is an invertible $p_*(\mathscr{O}_{X_0})$-module, so there exists a covering $(V_i)_{i \in I}$ of Y by affine opens, such that $\mathscr{A}$ is trivial over each saturated affine open subset $U_i = p^{-1}(V_i)$. For each $i \in I$, let $A_{i,0} = \mathscr{O}_{X_0}(U_i)$, $A_{i,1} = \mathscr{O}_{X_1}(d_0^{-1}(U_i)) = \mathscr{O}_{X_1}(d_1^{-1}(U_i))$, $\delta_{i,0}$ (resp. $\delta_{i,1}$) be the morphism $A_{i,0} \to A_{i,1}$ induced by d_0 (resp. d_1), and $B_i = \mathscr{O}_Y(V_i) = \{b \in A_{i,0} : \delta_{i,0}(b) = \delta_{i,1}(b)\}$. Following §§ 9.6.5, consider the invertible $\mathscr{O}_{X_0}$-module $N_{d_1}(d_0^*(\mathscr{A}))$, the norm of $d_0^*(\mathscr{A})$ relative to the finite and locally free morphism $d_1 : X_1 \to X_0$. If $\mathscr{A}$ is given, relative to the open covering $(U_i)_{i \in I}$, by the transition functions $c_{ij} \in \mathscr{O}_{X_0}(U_i \cap U_j)^\times$, then $N_{d_1}(d_0^*(\mathscr{A}))$ is given by the transition functions $N_{\delta_1}(\delta_0(c_{ij})) \in \mathscr{O}_{X_0}(U_i \cap U_j)^\times$. As, by the proof of 12.4.2.1, these elements belong to $\mathscr{O}_Y(U_i \cap U_j)^\times$, they define an invertible $\mathscr{O}_Y$-module $\mathscr{L}$, such that $p^*(\mathscr{L}) = N_{d_1}(d_0^*(\mathscr{A}))$. We also note that, for any $n \in \mathbb{N}$, we have $p^*(\mathscr{L}^n) = N_{d_1}(d_0^*(\mathscr{A}^n))$.

We now prove that $\mathscr{L}$ is ample for the morphism $\pi : Y \to S$ (the proof that $\pi : Y \to S$ is of finite type follows from that of Lemma 12.4.13 (b)). For this, by replacing S with an affine open, we may assume that S is affine. Let $y \in Y$, $x \in X_0$ such that $p(x) = y$, V be an open subset of Y containing y, and $U = p^{-1}(V)$. As $\mathscr{A}$ is π_0-ample, there exists an integer $n \geq 1$ and a section $s \in \Gamma(X_0, \mathscr{A}^n)$ such that the open subset $(X_0)_s$ satisfies $x \in (X_0)_s \subseteq U$ (cf. Theorem 9.4.27). With the preceding notations, s is given by the sections $a_i \in A_{i,0} = \mathscr{O}_{X_0}(U_i)$ such that $a_i = c_{ij}a_j$ over $U_i \cap U_j$, and $(X_0)_s$ is the union of $U_i' = \{\mathfrak{p} \in \operatorname{Spec}(A_{i,0}) : a_i \notin \mathfrak{p}\}$. For each $i \in I$, put $N(a_i) = N_{\delta_1}(\delta_0(a_i)) \in B_i$. By Theorem 12.4.9 (a) and Lemma 12.4.10, we have

$$p(U_i') = pd_1d_0^{-1}(U_i') = pd_1(\{\mathfrak{q} \in \operatorname{Spec}(A_{i,1}) : \delta_{i,0}(a_i) \notin \mathfrak{q}\})$$

and $d_1(\{\mathfrak{q} \in \operatorname{Spec}(A_{i,1}) : \delta_{i,0}(a_i) \notin \mathfrak{q}\}) = \{\mathfrak{p} \in \operatorname{Spec}(A_{i,0}) : N_{\delta_1}(\delta_{i,0}(a_i)) \notin \mathfrak{p}\}$, whence

$$p(U_i') = \{\mathfrak{p} \in \operatorname{Spec}(B_i) : N(a_i) \notin \mathfrak{p}\}.$$

It then follows that $p((X_0)_s) = Y_{N(s)}$, so if we denote by $N(s)$ the section of $\mathscr{L}^n$ over Y defined by the sections $N(a_i) \in \mathscr{O}_Y(V_i)$, we then have

$$y \in p((X_0)_s) = Y_{N(s)} \subseteq p(U) = V. \tag{12.4.12}$$

This proves that $\mathscr{L}$ is ample for $\pi : Y \to S$ (Theorem 9.4.27), so $\pi : Y \to S$ is quasi-projective.

12.4.3 Quasi-sections for a groupoid

We now prove a techniqucal lemma which will be used in the proof of the forecoming two theorems. Let S be a scheme and

$$X_2 \overset{d_0'}{\underset{d_2'}{\xrightarrow{\quad d_1' \quad}}} X_1 \underset{d_1}{\overset{d_0}{\rightrightarrows}} X_0$$

be a $\mathbf{Sch}_{/S}$-groupoid. A **quasi-section** of X_* is defined to be a subscheme U of X_0 such that

[20]This can be proved by applying the prime avoidence lemma, since V is assumed to be affine.

(a) The restriction of d_1 to $d_0^{-1}(U)$ is a finite, locally free and surjective morphism from $d_0^{-1}(U)$ to X_0.

(b) Any subset E of U formed by equivalent points for the equivalence relation defined by X_* is contained in an affine open subset of U[21].

If U is a quasi-section of X_*, the $\mathbf{Sch}_{/S}$-groupoid

$$\begin{tikzcd} U_2 \arrow[r, shift left=2, "u_0'"] \arrow[r, "u_1'" description] \arrow[r, shift right=2, "u_2'"'] & U_1 \arrow[r, shift left, "u_0"] \arrow[r, shift right, "u_1"'] & U \end{tikzcd}$$

induced from X_* and the inclusion $U \to X_0$ verifies the hypotheses of Theorem 12.4.9. In fact, put $V = d_0^{-1}(U)$ and let u, v be morphisms induced by d_0 and d_1:

$$X_0 \xleftarrow{v} V \xrightarrow{u} U$$

By (12.4.10), we then have a Cartesian square

$$\begin{tikzcd} U_1 \arrow[r] \arrow[d, "u_1"'] & V \arrow[d, "v"] \\ U \arrow[r, hook] & X_0 \end{tikzcd}$$

hence u_1 is surjective and finite locally free by (a). With (b), condition (a) then assures that the groupoid U_* satisfies the hypotheses of Theorem 12.4.9. In particular, $\mathrm{coker}(u_0, u_1)$ exists in $\mathbf{Sch}_{/S}$. Moreover, as d_0 possesses a section (the morphism $s : X_0 \to X_1$), u is a universally effective epimorphism by ([**?**], IV, 1.12); this ensures, by Proposition 12.4.8, that $\mathrm{coker}(u_0, u_1)$ coincides with the cokernel $\mathrm{coker}(v_0, v_1)$ of the groupoid V_*:

$$\begin{tikzcd} V_2 \arrow[r, shift left=2, "v_0'"] \arrow[r, "v_1'" description] \arrow[r, shift right=2, "v_2'"'] & V_1 \arrow[r, shift left, "v_0"] \arrow[r, shift right, "v_1"'] & V \end{tikzcd}$$

induced by U_* and the base change $u : V \to U$, which is also the inverse image of X_* under the base change

$$V \hookrightarrow X_1 \overset{d_0}{\to} X_0.$$

By Example 12.4.7, V_* is isomorphic to the groupoid V_*', the inverse image of X_* under the base change

$$V \hookrightarrow X_1 \overset{d_1}{\to} X_0$$

and hence V_*' admits a cokernel in $\mathbf{Sch}_{/S}$. Now, being flat, surjective and finite, $v : V \to X_0$ is faithfully flat and quasi-compact, hence a universally effective epimorphism by Proposition 12.3.67. Therefore by Example 12.4.7, the groupoid X_* also admits a cokernel $\mathrm{coker}(d_0, d_1)$ in $\mathbf{Sch}_{/S}$. We have therefore proved the first assertion of (a) in the following lemma:

Lemma 12.4.13. *Suppose that the $\mathbf{Sch}_{/S}$-groupoid X_* possesses a quasi-section. Then:*

(a) *There exists a cokernel (Y, p) of (d_0, d_1) in $\mathbf{Sch}_{/S}$. Moreover, such a couple (Y, p) is also a cokernel of (d_0, d_1) in the category of ringed spaces.*

(a') *p is surjective, and is open (resp. universally closed) if d_0 is.*

(b) *Suppose that S is locally Noetherian and X_0 is locally of finite type (resp. of finite type) over S. Then p and $Y \to S$ are locally of finite presentation (resp. of finite presentation).*

(c) *The morphism $X_1 \to X_0 \times_Y X_0$ with components d_0 and d_1 is surjective.*

[21] If $x, y \in E$, then there exists $z \in X_1$ such that $d_1(z) = x$, $d_0(z) = y$, that is, $z \in (d_1|_{d_0^{-1}(U)})^{-1}(x)$, which is a finite set by (a). Hence E is contained in the finite subset $d_0 d_1^{-1}(x) \cap U$.

(d) *If (d_0, d_1) is an equivalence couple, $X_1 \to X_0 \times_Y X_0$ is an isomorphism. Moreover, if $d_0 : X_1 \to X_0$ is flat, then p is faithfully flat.*

Proof. Before proving the second assertion of (a), let us first consider (a'), (b) and (c). We have seen that (Y, p) is identified with $\mathrm{coker}(v_0, v_1)$ and $\mathrm{coker}(u_0, u_1)$. Let q and r be the canonical epimorphisms of U and Y onto Y:

$$\begin{tikzcd} X_0 \arrow[dr, "p"'] & V \arrow[l, "v"'] \arrow[r, "u"] \arrow[d, "r"] & U \arrow[dl, "q"] \\ & Y & \end{tikzcd} \tag{12.4.13}$$

By hypothesis, v is surjective and finite locally free, hence open. On the other hand, if $d_0 : X_1 \to X_0$ is open (resp. universally closed), then u, which is induced from d_0 by restriction, is also open (resp. universally closed). As, by Theorem 12.4.9, q is surjective, integral and open, we conclude that r is surjective, and open (resp. universally closed) if d_0 is. The same assertion then holds for p, since v is surjective. This proves (a').

Now suppose that S is locally Noetherian and X_0 is locally of finite type over S, so that X_0 is locally Noetherian. Let $S' = \mathrm{Spec}(R)$ be an open affine subset of S, $Y' = \mathrm{Spec}(B)$ an open affine subset of Y which projections into S', and $U' = \mathrm{Spec}(A)$ be the inverse image of Y' in U. As R is Noetherian, it suffices to show that B is a finite type R-algebra, and this follows from the fact that R is Noetherian and A is integral over B (cf. (**??**)). Finally, as $X_0 \to S$ is locally of finite type, so is p (Proposition 8.6.21), hence p is locally of finite presentation since Y is locally Neotherian.

It remains to see that second assertion of (b). Suppose that X_0 is of finite type over S. Then, as p is surjective, Y is also quasi-compact over S, hence of finite type over S. As S is locally Noetherian, $X_0 \to S$ and $Y \to S$ are then finitely presented, and hence $p : X_0 \to Y$ is also finitele presented (Proposition 8.6.38).

Finally, as the groupoid V_* with base V is isomorphic to the inverse image of U_* by the base change u and to the inverse image of X_* by the base change v, we have Cartesian squares

$$\begin{tikzcd} X_1 \arrow[d, "d_0 \boxtimes d_1"] & V_1 \arrow[l] \arrow[r] \arrow[d, "v_0 \boxtimes v_1"] & U_1 \arrow[d, "u_0 \boxtimes u_1"] \\ X_0 \times_Y X_0 & V \times_Y V \arrow[l, "v \times v"'] \arrow[r, "u \times u"] & U \times_Y U \end{tikzcd} \tag{12.4.14}$$

As $u_0 \boxtimes u_1$ is surjective, so is $v_0 \boxtimes v_1$. Since $v \times v$ is surjective, the composition $V_1 \to X_0 \times_Y X_0$ is also surjective, whence so is $d_0 \boxtimes d_1$.

We now turn to the proof of the second assertion of (a). To see that (Y, p) is a cokernel of (d_0, d_1) in **Rsp**, we first show that Y is obtained from X_0 by identifying points x, y such that there exists $z \in X_1$ with $d_0(z) = x$ and $d_1(z) = y$. In fact, p is surjective and we have $pd_0 = pd_1$; conversely, if $p(x) = p(y)$, there exists a point z' of $X_0 \times_Y X_0$ whose first projection is x and second projection is y. If z is a point of X_1 such that $(d_0 \boxtimes d_1)(z) = z'$, then $d_0(z) = x$ and $d_1(z) = y$, whence our assertion. Also, if W is a saturated open subset of X_0, $W \cap U$ is saturated open in U, so by Theorem 12.4.9, $q(W \cap U)$ is open in Y. As $q(W \cap U) = p(W)$, we see that Y is endowed with the quotient topology of that of X_0.

It remains to show that the canonical sequence

$$\mathscr{O}_Y \longrightarrow p_*(\mathscr{O}_{X_0}) \rightrightarrows p_*((d_0)_*(\mathscr{O}_{X_1})) = p_*((d_1)_*(\mathscr{O}_{X_1}))$$

is exact. Let Y' be an open subset of Y and put $U' = q^{-1}(Y')$, $X'_0 = p^{-1}(Y')$, etc. Then, U' is a saturated open subset of U for the equivalence relation defined by the groupoid U_*, and it follows from Lemma 12.4.1 and Lemma 12.4.3 that Y' is its cokernel, in $\mathbf{Sch}_{/S}$ and in **Rsp**. Similarly, X'_0 is saturated open in X_0 for the equivalence relation defined by X_* and we have the following Cartesian squares

$$\begin{tikzcd} X'_0 \arrow[d] & V' = d_0^{-1}(U') \arrow[l, "\tilde{d}_1"'] \arrow[r, "\tilde{d}_0"] \arrow[d] & U' \arrow[d] \\ X_0 & V = d_0^{-1}(U) \arrow[l, "d_1"'] \arrow[r, "d_0"] & U' \end{tikzcd}$$

Hence $\tilde{d}_1$ is surjective and finite locally free. On the other hand, let $x \in U'$. As U is a quasi-section, the set $E = d_0 d_1^{-1}(x) \cap U$ is finite and contained in an affine open W of U. The intersection $E' = E \cap U'$ is then a finite subset contained in a quasi-affine open subset $W \cap U'$. Therefore, there exists an affine open W' of $W \cap V$ containing E', so U' is a quasi-section of the groupoid X'_* induced by X_* over X'_0. The first assertion of (a), applied to X'_* and U', shows that Y' is the cokernel of X'_* in $\mathbf{Sch}_{/S}$. In particular, for any S-scheme T, we have an exact sequence

$$T(Y') \xrightarrow{T(p|_{X'_0})} T(X'_0) \underset{T(d_0|_{X'_1})}{\overset{T(d_1|_{X'_1})}{\rightrightarrows}} T(X'_1)$$

Now if $T = \mathbb{G}_{a,S}$, this sequence is identified with

$$\Gamma(Y', \mathcal{O}_Y) \longrightarrow \Gamma(p^{-1}(Y'), \mathcal{O}_{X_0}) \underset{\delta_0}{\overset{\delta_1}{\rightrightarrows}} \Gamma(d_0^{-1}p^{-1}(Y'), \mathcal{O}_{X_1}) = \Gamma(d_1^{-1}p^{-1}(Y'), \mathcal{O}_{X_1})$$

which is then exact for any open subset Y' of Y. This conclude the proof of (a).

Finally, if (d_0, d_1) is an equivalence couple, so is (u_0, u_1) and the morphism $u_0 \boxtimes u_1 : U_1 \to U \times_Y U$ is an isomorphism (Theorem 12.4.9). As $v \times v$ is faithfully flat and quasi-compact, we conclude from (12.4.14) that $d_0 \boxtimes d_1$ is an isomorphism ([**?**], VIII, 5.4). Moreover, if d_0 is flat, u is also flat. Now q is flat by Theorem 12.4.9, so r is also flat from the diagram (12.4.13). As v is faithfully flat, p is then flat, and hence faithfully flat since it is surjective. □

12.4.4 Quotient for a flat proper groupoid

12.4.5 Quotient by a group scheme

We now consider the action of a group scheme G over S on an S-scheme X, and use the preceding results to construct the quotient $G\backslash X$. We first recall the following result:

Theorem 12.4.14. *Let S be a scheme and $f : X \to Y$ be an S-morphism. Suppose that one of the following conditions is satisfied:*

(α) *The morphism f is locally of finite presentation.*

(β) *The scheme S is locally Noetherian and X is locally of finite type over S.*

Then the following conditions are equivalent:

(i) *There exists an S-scheme X' and a factorization of f:*

$$f : X \xrightarrow{f'} X' \xrightarrow{i} Y$$

where f' is a faithfully flat S-morphism locally of finite presentation and i is a monomorphism.

(ii) *The first projection $\mathrm{pr}_1 : X \times_Y X \to X$ is a flat morphism.*

Moreover, under these conditions, (X', f') is a quotient of X by the equivalence relation induced by f (for the fppf topology), so that the factorization $f = i \circ f'$ is unique up to isomorphisms.

Proof. The implication (i)⇒(ii) is trivial. In fact, the first projection $\mathrm{pr}'_1 : X \times_{X'} X \to X$ factors through $X \times_Y X$:

$$\mathrm{pr}'_1 : X \times_{X'} X \xrightarrow{u} X \times_Y X \xrightarrow{\mathrm{pr}_1} X.$$

The morphism u is an isomorphism, since i is a monomorphism, and pr'_1 is flat since f' is flat, hence pr_1 is flat.

To prove that (ii)⇒(i), we first note that the assertions of Theorem 12.4.14 are local on Y (hence are local on S); they are also local on X, as it easily follows from the fact that a flat morphism locally of finite presentation is open ([**?**], 11.3.1).

The case where Y is locally Noetherian and X is of finite type over Y is treated in ([**?**], cor.2 du th.2). We now see how to reduce ourselves to this case. Under the hypothesis (α), we can therefore assume that X, Y are affine and f is finitely presented. By replacing S with Y, we may also assume that X and Y are finitely presented over S. We then reduce to the case where S is Noetherian thanks to ([**?**], 11.2.6).

Under the hypothesis (β), we can suppose that X, Y, S are affine, S is Noetherian and X is of finite type over S. Consider Y as a filtered projective limit of affine schemes Y_i which are of finite type over S. The schemes $X \times_{Y_i} X$ then form a descreasing filtered system of closed subschemes of $X \times_S X$, whose projective limit is $X \times_Y X$. As $X \times_S X$ is Noetherian, we have $X \times_{Y_i} X = X \times_Y X$ for i large enough, so that the composition

$$f_i : X \xrightarrow{f} Y \to Y_i$$

satisfies the hypothesis of (ii) if so does for f. As the equivalence relation defined by f over X coincides with that by f_i, it is clear that it suffices to prove (ii)⇒(i) for f_i, which means we can reduce to the case where Y is of finite type over S. □

Now let S be a scheme, G be a group scheme over S which is locally of finite presentation over S, and X be an S-scheme acted by G. If $X \to S$ possesses a section ξ, we recall that the stabilizer $\mathrm{Stab}_G(\xi)$ is represented by a group subscheme of G (in fact, by the group subscheme $G \times_X S$, cf. 12.1.1.2).

Theorem 12.4.15. *Let S be a scheme and G be a S-group scheme locally of finite presentation over S, which acts on an S-scheme X. Suppose that $X \to S$ possesses a section ξ such that the stabilizer H of ξ in G is flat over S. If one of the following conditions is satisfied:*

(a) *X is locally of finite type over S;*

(b) *S is locally Noetherian,*

then the fppf quotient sheaf G/H is representable by an S-scheme which is locally of finite presentation over S, and the S-morphism induced by ξ:

$$f : G = G \times_S S \to G \times_S X \to X, \quad g \mapsto g \cdot \xi$$

factors into

$$\begin{tikzcd} G \arrow[rr, "f"] \arrow[dr, "p"'] & & X \\ & G/H \arrow[ur, hook, "j"'] & \end{tikzcd}$$

where p is the canonical projection, which is a faithfully flat morphism locally of finite presentation, and j is a monomorphism.

Proof. The morphism f makes G an X-scheme, and the definition of the stabilizer of ξ, the morphism

$$G \times_S H \to G \times_X G, \quad (g, h) \mapsto (g, gh)$$

is an isomorphism. As H is flat over S, $G \times_S H$ is flat over G, hence the first projection $\mathrm{pr}_1 : G \times_S G \to G$ is flat. Therefore, if X is locally of finite type over S, then f is locally of finite presentation (Proposition 8.6.24), and otherwise S is supposed to be Noetherian. It then suffices to apply Theorem 12.4.14 to the morphism f. Also, it follows from ([**?**], 17.7.5) that G/H is locally of finite presentation over S. □

Corollary 12.4.16. *Let S be a scheme and $u : G \to H$ be a morphism of S-group schemes. Suppose that G is locally of finite presentation over S and that, either H is locally of finite type over S, or S is locally Noetherian. Then, if $K = \ker u$ is flat over S, the quotient group G/K is representable by an S-group scheme which is of finite presentation over S, and u factors into*

$$\begin{tikzcd} G \arrow[rr, "u"] \arrow[dr, "p"'] & & H \\ & G/K \arrow[ur, hook, "j"'] & \end{tikzcd}$$

where p is the canonical projection which is faithfully flat and locally of finite presentation, and j is a monomorphism.

Proof. We can apply Theorem 12.4.15 to $X = H$ and the unit section of H. □

12.5 Generalities on algebraic groups

In this section, we denote by A a local Artinian ring with residue field k. A group scheme G over $\mathrm{Spec}(A)$ is called simply an A-group. This A-group is called **locally of finite type** if the underlying scheme is locally of finite type over A, and it is called **algebraic** if the underlying scheme is of finite type over A.

12.5.1 Some preliminary remarks

Consider a group scheme G over an arbitrary scheme S. The structural morphism $\mu : G \times_S G \to G$ is called the multiplication morphism of G and the inversion morphism $c : G \to G$ is defined by $c(T)(x) = x^{-1}$ (T being a scheme over S and $x \in G(T)$). If U and V are subsets of G, we denote by $U \cdot V$ the image under the multiplication morphism of the subset of $G \times_S G$ formed by points whose first projection belongs to U and the second projection belongs to V. The notations U^{-1} or $c(U)$ are defined similarly.

Let $\mathrm{pr}_1 : G \times_S G \to G$ be the first projection and $\sigma : G \times_S G \to G \times_S G$ be the morphism with components pr_1 and μ. For any S-scheme T, $\sigma(T)$ is the map $(x, y) \mapsto (x, xy)$; so σ is an automorphism. The composition of this automorphism and the projection pr_2 is the multiplication μ. Therefore, if G is flat (resp. smooth, etc.) over S, then pr_2, and hence μ, are flat (resp. smooth, etc.).

We now suppose that S is the spectrum of a local Artinian ring A with residue field k. We denote by $(\mathbf{Sch}_{/k})_{\mathrm{red}}$ the category of reduced schemes over k. For any scheme X over A, the reduced scheme X_{red} is then an object of $(\mathbf{Sch}_{/k})_{\mathrm{red}}$, and the functor $X \mapsto X_{\mathrm{red}}$ is right adjoint to the inclusion of $(\mathbf{Sch}_{/k})_{\mathrm{red}}$ to $\mathbf{Sch}_{/A}$. This ensures that, for any A-group G, G_{red} is a group in the category $(\mathbf{Sch}_{/k})_{\mathrm{red}}$, that is, for any reduced k-scheme T, $G_{\mathrm{red}}(T)$ is endowed with a group structure, functorial on T. We note that G_{red} is not necessarily a k-group, because the multiplication is only a morphism $(G_{\mathrm{red}} \times_k G_{\mathrm{red}})_{\mathrm{red}} \to G_{\mathrm{red}}$.

Nevertheless, if k is a *perfect* field, the inclusion $(\mathbf{Sch}_{/k})_{\mathrm{red}}$ into $\mathbf{Sch}_{/k}$ commutas with products (in this case a product of reduced k-schemes is still reduced) so that the groups in $(\mathbf{Sch}_{/k})_{\mathrm{red}}$ are identified with k-groups whose underlying scheme is reduced. In this case, if G is a k-group, G_{red} is a subgroup scheme of G; but this subgroup is in genral not normal in G.

For example, if k is a field with characteristic 3, the constant group $(\mathbb{Z}/2\mathbb{Z})_k$ acts nontrivially on the diagonalisable $D_k(\mathbb{Z}/3\mathbb{Z})$. If G denotes the semi-direct prodcut of $D_k(\mathbb{Z}/3\mathbb{Z})$ by $(\mathbb{Z}/2\mathbb{Z})_k$ defined by this action, then G_{red} is identified with $(\mathbb{Z}/2\mathbb{Z})_k$ and is not normal in G.

Let k be an arbitrary field, $k^{\mathrm{perf}} = k^{p^{-\infty}}$ bethe perfect closure of k, and H be a group in the category $(\mathbf{Sch}_{/k})_{\mathrm{red}}$. Then $(H \otimes_k k^{\mathrm{perf}})_{\mathrm{red}}$ is a group scheme over k^{perf}. As $H \otimes_k k^{\mathrm{perf}}$ and $(H \otimes_k k^{\mathrm{perf}})$ have the same underlying topological space, we see that the groups in $(\mathbf{Sch}_{/k})_{\mathrm{red}}$ have in common with the k-groups certain topological properties invariant by extension of the base field: for example, any group of $(\mathbf{Sch}_{/k})_{\mathrm{red}}$ is separated.

An A-group G is always *separated*, becasue the unit section $e : \mathrm{Spec}(A) \to G$ is a closed immersion. In fact, let x be the unique point of $\mathrm{Spec}(A)$ and η be the structural morphism $G \to \mathrm{Spec}(A)$. As $\eta \circ e = \mathrm{id}_{\mathrm{Spec}(A)}$, for any affine open $U = \mathrm{Spec}(B)$ of G containing $e(x)$, the morphism $B \to A$ possesses a section, hence is surjective. It follows that e is a closed immersion (this argument is valid for any zero dimension local ring A, not necessarily Artinian). Now, the diagonal $G \times_A G$ is identified with the inverse image of the unit section:

$$\begin{tikzcd} G \arrow[r] \arrow[d, "\Delta_G"'] & \mathrm{Spec}(A) \arrow[d, "e"] \\ G \times_S G \arrow[r, "\varphi"] & G \end{tikzcd}$$

where $\varphi : G \times_S G$ is defined by $(x, y) \mapsto xy^{-1}$.

Let G be an A-scheme. We say that a point g of G is **strictly rational** over A if there exists an A-morphism $s : \mathrm{Spec}(A) \to G$ which sends the unique point of $\mathrm{Spec}(A)$ to g, i.e. if the morphism $A \to \mathscr{O}_{G,g}$ admits a retraction. We note that in this case $\kappa(g) = k$, and hence hg is a clsed point of G.

Suppose that G is an A-group, then such a morphism $s : \mathrm{Spec}(A) \to G$ defines an automorphism r_s of the scheme G over A, which is called the **right translation** by s: for any morphism $\pi : S \to \mathrm{Spec}(A)$, $r_s(\pi)$ is the automorphism of $G(S)$ defined by $x \mapsto x \cdot G(\pi)(s)$, for any $x \in G(S)$. Similarly, we denote by ℓ_s the left translation defined by s, which is the automorphism of G defined by $\ell_s(\pi)(x) = G(\pi)(s) \cdot x$, for any $x \in G(S)$.

As $G \otimes_A k$ and G have the same underlying topological space $|G|$, that $G \otimes_A k$ is a k-group and that $s \otimes_A k$ depends only on g and not on s, we see that the automorphism of $|G|$ induced by r_s and ℓ_s (or by $r_{s\otimes k}$ and $\ell_{s\otimes k}$) only depends on the point g and not on s. If P is a subset of $|G|$, we then denote by $r_g(P)$ or $P \cdot g$ (resp. $\ell_g(P)$ or $g \cdot P$) the subset $r_s(P)$ (Resp. $\ell_s(P)$).

Remark 12.5.1. If g is a strictly rational point of G and if $A \to A'$ is a morphism of local Artinian rings, then $G' = G \otimes_A A'$ possesses a unique point g' over g, and g' is strictly rational over A'. Moreover, if we denote by P' the inverse image of P in G', then $P' \cdot g'$ is the inverse image of $P \cdot g$ (cf. Corollary 8.3.26).

Proposition 12.5.2. *Let G be an A-group and U, V be open dense subsets in G. Then $U \cdot V$ (i.e. the image of $U \times_A V$ under the multiplication morphism) is equal to G.*

Proof. In fact, as G and $G \otimes k$ have the same underlying space, we may suppose, by replacing A by k and G by $G \otimes k$, that $A = k$. Let $g \in G$ and put $K = \kappa(g)$, then the left translation ℓ_g is an automorphism of G_K. As the projection $G_K \to G$ is open (cf. [?], 2.4.10), U_K and V_K are two open dense subsets of G_K, and so is the image of V_K by ℓ_g. There then exists $v \in V_K$ such that $u = \ell_g(v)$ belongs to U_K. Let L be an extension of K containing $\kappa(v)$, and hence $\kappa(u)$, and g_L, v_L be the L-points of G_L defined by g and v. Then $g_L \cdot v_L = u'$ is a point of G_L lying over u, and hence $g_L = u' \cdot v_L^{-1}$ is lying over $U \cdot V$, whence $g \in U \cdot V$. This proves the proposition. □

Corollary 12.5.3. *If G is an irreducible A-group, then G is quasi-compact.*

Proof. If U is a nonempty affine open subset of G, then U is dense in G, so by Proposition 12.5.2 the morphism $\mu : U \times_A U \to G$ is surjective, hence G is quasi-compact (since $U \times_A U$ is). □

Corollary 12.5.4. *Let G be an A-group and H be a sub-A-group of G. Then H is closed in G.*

Proof. Let k' be the perfert closure of k. As the underlying spaces of G and H are unchanged by base changing to k', we may suppose that $A = k$ is a perfect field. We can also suppose that G and H are reduced, hence geometrically reduced.

Let $\overline{H}$ be the closure of H, then $\mu^{-1}(\overline{H})$ is a closed subset of $G \times G$ containing $H \times H$. As the morphism $H \to \operatorname{Spec}(k)$ (resp. $\overline{H} \to \operatorname{Spec}(k)$) is universally open, and as H is dense in $\overline{H}$, we see that $H \times H$ is dense in $H \times \overline{H}$ and $H \times \overline{H}$ is dense in $\overline{H} \times \overline{H}$, hence $H \times H$ is dense in $\overline{H} \times \overline{H}$. We conclude that $\mu(\overline{H} \times \overline{H}) \subseteq \overline{H}$, and as $\overline{H} \times \overline{H}$ is reduced, μ induces a morphism $\mu' : \overline{H} \times \overline{H} \to \overline{H}$.

Let $g \in \overline{H}$ and put $K = \kappa(g)$. As the projection $\overline{H}_K \to \overline{K}$ is open, H_K and $\ell_g(H_K)$ are open dense subsets of $\overline{H}_K$, so there exists $u, v \in H_K$ such that $\ell_g(v) = u$. We then conclude, as the proof of Proposition 12.5.2, that g belongs to $H \cdot H = H$, so $\overline{H} = H$. □

12.5.2 Local properties for algebraic groups

Without further specifications, we now suppose that G is an A-group locally of finite type.

Proposition 12.5.5. *Let x be a point of a A-group G locally of finite type and flat over A. Then the local ring $\mathscr{O}_{G,x}$ is Cohen-Macaulay and there exists a regular sequence $a_1, \dots, a_n$ of $\mathscr{O}_{G,x}$ such that $\mathscr{O}_{G,x}/(a_1, \dots, a_n)$ is a finitely generated and flat A-module (hence free).*

Proof. We first suppose that $A = k$ is a field. It then suffices to prove that $\mathscr{O}_{G,x}$ is Cohen-Macaulay and we can limit ourselves to the case where x is a closed point (cf. [?], 0_{IV}, 16.5.13). By Lemma 12.5.6 below, G contains a closed point y such that $\mathscr{O}_{G,y}$ is Cohen-Macaulay. By ([?], 6.7.1), for any finite type extension K of k and any point $\bar{y}$ of $\overline{G} = G \otimes_k K$ over y, $\mathscr{O}_{\overline{G},\bar{y}}$ is then Cohen-Macaulay. If the extension K is chosen large enough, i.e. if K contains a normal extension of k containing the residue fields $\kappa(x)$ and $\kappa(y)$, then $\bar{y}$ is strictly rational over K and thus so is any point $\bar{x}$ of $\overline{G}$ lying over x[22]. As the automorphism $r_{\bar{x}} \circ r_{\bar{y}}^{-1}$ sends $\bar{y}$ to $\bar{x}$, we conclude that $\mathscr{O}_{\overline{G},\bar{x}}$, and hence $\mathscr{O}_{G,x}$, is Cohen-Macaulay ([?], 6.7.1). □

Lemma 12.5.6. *Any nonempty scheme X, locally of finite type over an Artinian ring A, contains a closed point x whose local ring is Cohen-Macaulay.*

[22] In fact, the hypothesis on K implies that, for any extension L of K, any k-morphism $\kappa(x) \to L$ (resp. $\kappa(y) \to L$) factors through K. Therefore, any point of $G \otimes_k K$ lying over x or y has residue field K, and hence is (strictly) rational over K.

Proof. We can evidently assume that X is affine with ring B and prove by induction on $\dim(X)$ (the assertion is trivial if X is discrete, since then any local ring is Artinian, hence Cohen-Macaulay, cf. **??**). As B is of finite type over A, if $\dim(B) > 0$, B contains a non-invertible element a which is not a zero-divisor[23]. The closed subscheme $X' = \mathrm{Spec}(B/(a))$ of X then has dimension strictly smaller than $\dim(X)$, and hence by induction hypothesis has a closed point x such that $\mathscr{O}_{X',x}$ is Cohen-Macaulay. As $\mathscr{O}_{X',x} = \mathscr{O}_{X,x}/(a)$ and a is non-invertible and not a zero-divisor in $\mathscr{O}_{X,x}$, we conclude that $\mathscr{O}_{X,x}$ is Cohen-Macaulay (**??**). □

Proposition 12.5.7. *Let A be a local Artinian ring, G be an A-group locally of finite type and flat over A, and x be a closed point of G. Then there exists a local A-algebra A', finite and free over A, such that any point x' of $G \otimes_A A'$ lying over x is strictly rational over A'.*

Proof. Let k_1 be a normal extension of finite type of k containing the residue field $\kappa(x)$ of x. By ([**?**] 0_{III}, 10.3.1), there exists a local A-algebra A_1 which is finite and free over A_1 with residue field k_1. In this case, the points $g_1, \dots, g_n$ of $G \otimes_A A_1$ lying over $x \in G$ has residue field k_1, so they are rational over A_1. Let $B_1, \dots, B_n$ be the local rings of $g_1, \dots, g_n$. By Proposition 12.5.5, $B_1, \dots, B_n$ possess quotients $B_1', \dots, B_n'$ which are Artinian and finite and free over A_1. Put $A' = B_1' \otimes_{A_1} \cdots \otimes_{A_1} B_n'$, then A' is local, finite and free over A_1, and for each $i = 1, \dots, n$, we have a surjective homomorphism

$$B_i \otimes_{A_1} A' \twoheadrightarrow B_i' \otimes_{A_1} A' \twoheadrightarrow A',$$

where the second one is induced by the multiplication map $B_i' \otimes_{A_1} B_i' \twoheadrightarrow B_i'$. Therefore, A' satisfies our requirements. □

Now let $e : \mathrm{Spec}(A) \to G$ be the unit element of G, which is the image of the unit section $\mathrm{Spec}(A) \to G$. By definition, e is strictly rational over A.

Proposition 12.5.8. *Let G be a group locally of finite type and flat over a local Artinian ring A and k^{perf} (resp. $\bar{k}$) be the perfect closure (resp. algebraic closure) of the residue field k of A.*

(a) *For any closed point x of $\overline{G} = G \otimes_k \bar{k}$, the local ring $\mathscr{O}_{\overline{G},e}$ and $\mathscr{O}_{\overline{G},x}$ are isomorphic. In particular, the tangent spaces $T_e\overline{G}$ and $T_x\overline{G}$ are isomorphic.*

(b) *The following assertions are equivalent:*

  (i) *$G \otimes_k k^{\mathrm{perf}}$ is reduced.*

  (ii) *$\mathscr{O}_{G,e} \otimes_A k^{\mathrm{perf}}$ is reduced.*

  (iii) *G is smooth over A.*

  (iv) *G is smooth over A at e.*

Proof. Let x be a closed point of $\overline{G}$, so that there exists a unique $\bar{k}$-morphism $s : \mathrm{Spec}(\bar{k}) \to \overline{G}$ with image x. The right translation r_s then induces an isomorphism from $\mathscr{O}_{\overline{G},e} = \mathscr{O}_{G,e} \otimes_k \bar{k}$ to $\mathscr{O}_{\overline{G},x}$, whence the assertion of (a).

To prove assertion (b), in view of Theorem 11.1.32, we may assume that $A = k$ is a field. The implications (i)⇒(ii), (iii)⇒(iv), (iii)⇒(i) and (iv)⇒(ii) are clear, so it suffices to show that (ii)⇒(iii). In this case, $\mathscr{O}_{G,e} \otimes_k \bar{k}$ is then reduced, so by (a), $\mathscr{O}_{\overline{G},x}$ is reduced for any closed point x of $\overline{G}$, so that $\overline{G}$ is reduced. As $\overline{G}$ is locally of finite type over k, there then exists a closed point y such that $\mathscr{O}_{\overline{G},y}$ is regular, and by (a), this implies that $\overline{G}$ is regular at any closed point, hence smooth over $\bar{k}$. It then follows from ([**?**], 17.7.1) that G is smooth over k. □

We can now give the examples below, indicated by M. Raynaud, of group schemes G over a non-perfect field k, such that G_{red} is not a k-group.

Example 12.5.9. Let k be a non-perfect field with characteristic $p > 0$, $t \in k - k^p$, $\bar{k}$ be an algebraic closure of k, and $\alpha \in \bar{k}$ such that $\alpha^p = t$.

[23] In fact, B is a Noetherian Jacobson ring. If any non-invertible element of B is a zero divisor, then by **??**, any prime ideal is an associated prime of B. In particular, B has only finitely many maximal ideals $\mathfrak{m}_1, \dots, \mathfrak{m}_r$ (cf. **??**). As B is Jacobson, the intersection of $\mathfrak{m}_i$ is the nilradical of B, and it follows from **??** that each $\mathfrak{m}_i$ is a minimal prime ideal of B, so $\dim(B) = 0$.

(a) Consider the additive group $\mathbb{G}_{a,k} = \operatorname{Spec}(k[X])$ and let G be the subscheme, finite over k, defined by the additive polynomial $X^{p^2} - tX^p$. Then we have

$$G_{\text{red}} = \operatorname{Spec}(k[X]/(X(X^{p(p-1)} - t)))$$

which is étale at the origin. If this was a group scheme over k, it would be smooth over k by Proposition 12.5.8; but G_{red} is not geometrically reduced, so this is not true.

(b) Consider $\mathbb{G}^4_{a,k} = \operatorname{Spec}(k[X,Y,U,V])$ and let G be the subgroup scheme defined by the ideal $\mathfrak{I}$ generated by the additive polynomials $P = X^p - tY^p$, $Q = U^p - tV^p$. Then G is of dimension 2 and is irreducible, because $(G_{\bar{k}})_{\text{red}} \cong \operatorname{Spec}(\bar{k}[Y,V])$ is irreducible.

Let $A = k[X,Y,U,V]$ and $\mathfrak{m}$ be the augmentation ideal (that is, the ideal generated by X,Y,U,V). Denote by x,y,u,v the images of dX, dY, dU, dV in $\Omega^1_{A/k} \otimes_A (A/\mathfrak{m})$, considered as linear forms over the tangent space $k^4 = T_0\mathbb{G}^4_{a,k}$. We now prove that the subspace $E = T_0 G_{\text{red}}$ is equal to k^4. Otherwise, there should exist a linear form $f = ax + by + cu + dv$, with $a,b,c,d \in k$ not all zero, which vanishes on E. Since the formation of $\Omega^1_{A/k}$ (and hence that of the tangent space) commutes with base change, we can identify f with its image in $(\bar{k}^4)^\times$. As $(G_{\bar{k}})_{\text{red}} \subseteq (G_{\text{red}})_{\bar{k}}$, f then vanishes over the subspace $T_0(G_{\bar{k}})_{\text{red}}$ of $\bar{k}^4$, which is defined by the equations $g_1 = x - \alpha y$ and $g_2 = u - \alpha v$, and hence $f = \lambda g_1 + \mu g_2$, with $\lambda, \mu \in \bar{k}$. Now $\lambda g_1 + \mu g_2$ does not belong to k^4 unless $\lambda = \mu = 0$, and this contradiction implies $E = k^4$, whence $T_0(G_{\text{red}})_{\bar{k}} = \bar{k}^4$.

On the other hand, $R = XV - YU$ belongs to $\sqrt{\mathfrak{I}}$ because $R^p = (X^p - tY^p)V^p - Y^p(U^p - tV^p)$, so the tangent space F at point $(\alpha, 1, \alpha, 1)$ of $(G_{\text{red}})_{\bar{k}}$ is contained in the hyperplane H of $\bar{k}^4$ defined by the equation $\alpha dV + dX - dU - \alpha dY = 0$, hence is of dimension ≤ 3[24]. By Proposition 12.5.8 (a), G_{red} is not a k-group scheme.

In fact, any k-group locally of finite type over a field k of characteristic zero is smooth. To see this, we need the following useful criterion for the smoothness of a general group scheme (we recall that a scheme S is called of **characteristic zero** if for any $s \in S$, the residue field $\kappa(s)$ has characteristic zero):

Proposition 12.5.10. *Let S be a scheme of characteristic zero and G be an S-group scheme locally of finite presentation over S at the unit section $e(S)$[25]. For G to be smooth at the unit section $e(S)$, it is necessary and sufficient that the $\mathscr{O}_S$-module $\omega_{G/S} = e^*(\Omega^1_{G/S})$ (called the **conormal module** of the unit section of G) is locally free.*

Proof. We first recall that, if $\pi : G \to S$ is the structural morphism, we have $\Omega^1_{G/S} = \pi^*(\omega_{G/S})$ (cf. Remark 12.2.87), so the $\mathscr{O}_S$-module $\omega_{G/S}$ is locally free if and only if the $\mathscr{O}_G$-module $\Omega^1_{G/S}$ is locally free, which is in turn the case if G is smooth over S.

Conversely, if $\omega_{G/S}$ is locally free, then so is $\Omega^1_{G/S}$, and as S has characteristic zero, it follows from ([**?**], 16.12.2) that G is differentially smooth over S. It then follows from ([**?**], 17.12.5) that G is smooth at the unit section. □

Corollary 12.5.11 (Cartier). *Let k be a field of characteristic zero. Then any k-group locally of finite type is smooth over k.*

Proof. In fact, in this case $\omega_{G/S}$ is always locally free, so by Proposition 12.5.10, G is smooth at the identity, hence smooth by Proposition 12.5.8. □

12.5.3 Connected components

Consider first an A-group G and let G' be the connected component of the identity e of G. This connected component is clearly closed, so that we can identify it with the reduced closed subscheme of G which has G' as underlying space.

Proposition 12.5.12. *For any field extension K of k, the underlying space of $G' \otimes_A K$ is the connected component of the identity of the K-group $G \otimes_A K$ (i.e. G' is geometrically connected).*

[24] In fact, $\sqrt{\mathfrak{I}}$ is generated by P, Q, R, so the tangent space of $(G_{\text{red}})_{\bar{k}}$ at a point (x_0, y_0, u_0, v_0) is given by the hyperplane defined by the equation $v_0 dX - u_0 dY + x_0 dV - y_0 dU = 0$.

[25] By this, we mean that G is locally of finite presentation over S at any point of $e(S)$.

Proof. Let $(G \otimes_A K)'$ be the connected component of the identity in $G \otimes_A K$. As the image of $(G \otimes_A K)'$ in G is connected and contains the identity element, it is contained in G', so $(G \otimes_A K)'$ is contained in the inverse image $G' \otimes_A K$ of G' in $G \otimes_A K$. The proposition then follows from the connectedness of $G' \otimes_A K$, which results from ([?], 4.5.8 and 4.5.14). □

It is clear that G' is a reduced k-scheme, and ([?] 4.5.8 and 4.5.14) shows that $G' \times_k G'$ is connected, so that $(G' \times_k G')_{\mathrm{red}}$ is the reduced subscheme of $G \times_A G$ whose underlying space is the connected component of the identity. In particular, the multiplication morphism $\mu : G \times_A G \to G$ induces a morphism $\mu' : (G' \times_k G')_{\mathrm{red}} \to G'$, which makes G' a group in $(\mathbf{Sch}_{/k})_{\mathrm{red}}$.

We recall that for a scheme P we denote by $|P|$ the underlyig topological space of P. Then, we define a sub-A-functor G^0 of G so that for any A-scheme S,

$$G^0(S) = \{u \in G(S) : u(|S|) \subseteq |G'|\}.$$

Let $c : G \to G$ be the inversion morphism; as $c(|G'|) = |G'|$, we have $c \circ u \in G^0(S)$ for any $u \in G^0(S)$. On the other hand, if $u, v \in G^0(S)$, then $u \boxtimes v$ sends $|S|$ into the subspace $|G \times_A G|$ formed by points whose two projections belong to $|G'|$. This subspace is identified with the underlying space of $G' \times_A G'$, which is connected by ([?] 4.5.8). Therefore, $\mu \circ (u \boxtimes v)$ sends $|S|$ into $|G'|$, and we conclude that G^0 is a sub-A-functor in groups of G.

If the connected component of e is an open subset of $|G|$, then the sub-functor G^0 is representable by this open subscheme of G, which is then a subgroup scheme of G, also denoted by G. In this case, we have $G' = (G^0)_{\mathrm{red}}$, and the underlying spaces $|G'|$ and $|G^0|$ coincide.

We now assume that G is locally of finite type over A, so that G is locally Noetherian, hence locally connected (**??**). In this case, any connected component of G is open. We then denote by G^0 the induced open subscheme of G over $|G'|$. By the arguments above, G^0 is a subgroup of G, called the **identity component** of G. For any A-scheme S, we then have

$$G^0(S) = \{u \in G(S) : u(|S|) \subseteq |G^0| = |G'|\}.$$

Let G^α be a connected component of G and $\nu^\alpha : G^\alpha \times_A G^0 \to G$ be the morphism induced by the equality

$$\nu^\alpha(S)(g, \gamma) = g\gamma g^{-1},$$

for any $S \in \mathbf{Sch}_{/A}$, $g \in G^\alpha(S)$, $\gamma \in G^0(S)$. If e is the identity of G, the restriction of ν^α to $G^\alpha \times_A \{e\}$ is trivial; as $G^\alpha \times_A G^0$ is connected by ([?] 4.5.8), we see that ν^α factors through G^0. Hence, for any A-scheme S, $G^0(S)$ is a normal subgroup of $G(S)$. We then obtain assertion (a) of the following proposition:

Proposition 12.5.13. *Let G be an A-group locally of finite type.*

(a) *G^0 is irreducible normal (in fact characteristic) subgroup of G and $G^0 \otimes_A k$ is geometrically irreducible over k.*

(b) *G^0 is quasi-compact, hence of finite type over A.*

Proof. As G^0 and $G^0 \otimes_A k$ have the same underlying topological space, it suffices to prove that $G^0 \otimes_A k$ is geometrically irreducible. Let $\bar{k}$ be an algebraic closure of k. Then $(G \otimes_A \bar{k})_{\mathrm{red}}$ is a $\bar{k}$-group locally of finite type and reduced, hence smooth over $\bar{k}$ (Proposition 12.5.8). A fortiori, the local rings $(G \otimes_A \bar{k})_{\mathrm{red}}$ are integral, hence, since $G \otimes_A \bar{k}$ is locally Noetherian, the connected components of $G \otimes_A \bar{k}$ are irreducible (cf. **??** and Proposition 8.4.32). In particular, the connected component $G^0 \times_A \bar{k}$ is irreducible. Finally, as G^0 is locally of finite type over A, it suffices to prove that G^0 is quasi-compact, which follows from Corollary 12.5.3 since G^0 is irreducible. □

Corollary 12.5.14. *Any connected component of G is irreducible, of finite type over A, and of the same dimension as G^0.*

Proof. We can suppose that $A = k$. Let C be a connected component of G, x be a closed point of C, $\kappa(x)$ be the residue field of x and k' be a finite normal extension of k containing $\kappa(x)$ over k ($\kappa(x)$ is a finite extension, cf. Proposition 8.6.42). The canonical projection $\pi : C \otimes_k k' \to C$ is open and closed (cf. Proposition 9.6.7 and [?] 2.4.10), therefore, if C' is the connected component of $C \otimes_k k'$, the projection $C' \to C$ is surjective, hence C' contains a point $y \in \pi^{-1}(x)$, and such a point is rational over k' (cf. the proof of Proposition 12.5.7). We then conclude that C' is the disjoint union of $G^0 \times_k k'$ under the translation r_y for $y \in \pi^{-1}(x)$. Now $G^0 \times_k k'$ is of finite type over k' by Proposition 12.5.13 and $\pi^{-1}(x)$ is finite (with cardinality $\leq [k' : k]$), so $C \otimes_k k'$ is of finite type over k', and hence C is of finite type over k (cf. [**?**], 17.7.4). On the other hand, as $G^0 \times_k k'$ is irreducible by Proposition 12.5.13, so is C', and hence is C, since the projection $C' \to C$ is surjective.

We therefore conclude from above that $C \otimes_k k'$ is the disjoint union of a finite number of translations of $G^0 \otimes_k k'$. As the dimension is invariant under base change of fields, it follows that C has the same dimension as G^0 (moreover, it follows from ([**?**], 5.2.1) that we have $\dim_g(G) = \dim(G^0)$ for any point $g \in G$). □

Example 12.5.15. One should note that a connected component (not containing the identity) may not be geometrically connected in general. For example, if $k = \mathbb{R}$, the group $\boldsymbol{\mu}_{3,\mathbb{R}}$, represented by $\mathbb{R}[X]/(X^3 - 1)$, has two connected components:

$$\{e\} = \mathrm{Spec}(\mathbb{R}), \quad C = \mathrm{Spec}(\mathbb{R}[X]/(X^2 + X + 1)),$$

and $C \otimes_{\mathbb{R}} \mathbb{C}$ has two components.

Before proceeding further, we establish the following simply lemma, which allows us to convert many problems about A-group schemes to those over the residue field k.

Lemma 12.5.16. *Let $(A, \mathfrak{m})$ be a local Artinian ring and $k = A/\mathfrak{m}$ be the residue field.*

(a) *Let X be an A-scheme such that $X \otimes_A k$ is locally of finite type (resp. of finite type), then so is X.*

(b) *Let $u : X \to Y$ be a morphism of A-schemes. If $u \otimes_A k$ is an immersion (resp. a closed immersion), then so is u.*

Proof. Suppose that $X \otimes_A k$ is locally of finite type over k. Let $U = \mathrm{Spec}(B)$ be an affine open of X. By the hypothesis of (a), there exists elements $x_1, \dots, x_n$ of B whose image generate $B/\mathfrak{m}B$ as a k-algebra, and it follows from Nakayama's lemma that x_i generate B as an A-algebra. This shows that X is locally of finite type over A. If $X \otimes_A k$ is also quasi-compact, then so is X (they have the same underlying topological space), and hence X is of finite type over A. This proves (a).

Now let $u : X \to Y$ be a morphism of A-schemes and suppose that $u \otimes_A k$ is an immersion (resp. a closed immersion). Then u is a homeomorphism from X to a locally closed (resp. closed) subset of Y and, for any $x \in X$, the ring homomorphism $\phi_x : \mathscr{O}_{Y,\phi(x)} \to \mathscr{O}_{X,x}$ is such that $\phi_x \otimes_A k$ is surjective. By Nakayama's lemma, it follows that ϕ_x is surjective, so u is an immersion (resp. a closed immersion) by Proposition 8.4.9 (b). □

Proposition 12.5.17. *Let A be a local Artinian ring with residue field k and let $u : G \to H$ be a quasi-compact morphism between A-group schemes locally of finite type.*

(a) *The set $u(G)$ is closed in H, whose connected components are irreducible and of the same dimension.*

(b) *We have* $\dim(G) = \dim(u(G)) + \dim(\ker u)$.

(c) *If u is a monomorphism, it is a closed immersion.*

Proof. By Lemma 12.5.16, it suffices to prove the proposition in the case where $A = k$. Moreover, as the considered properties are stable under fpqc descent, and as the dimension is stable under base change of fields, we may assume that k is algebraically closed.

Denote by C the reduced subscheme of H whose underlying space if $\overline{u(G)}$. As $u(G)$ is stable under the inversion morphism of H, so is C. On the other hand, $u : G \to C$ is quasi-compact and dominant, hence by ([**?**], 2.3.7), so is $u \times_k \mathrm{id}_G$ and $\mathrm{id}_H \times_k u$, hence is their composition $u \times_k u : G \times_k G \to C \times_k C$. Therefore, the multiplication of H sends $C \times_k C$ into C, and C is a subgroup of H.

Hence, by replacing H with C, we may assume that u is dominant. Since k is algebraically closed and G is of finite type over k, we see that $u(G(k))$ is dense in H, hence meets any connected component of H, and acts transitively on the set of connected components. It then suffices to show that $u(G)$ contains H^0. By replacing G with $u^{-1}(H^0)$, we can then suppose that $H = H^0$; in this case, by Proposition 12.5.13, H is irreducible and of finite type over k, hence Noetherian. On the other hand, u is locally of finite type by Proposition 8.6.21 and quasi-compact, hence of finite type. The constructibility theorem of Chevalley ([**?**], 1.8.5), $u(G)$ is a constructible subset (and dense) of $H = \overline{u(G)}$, hence contains an open dense subset U of H ([**?**] 0_{III}, 9.2.2). Then by Proposition 12.5.2, we have $H = U \cdot U \subseteq u(G)$, whence $u(G) = H$. In view of Corollary 12.5.14, this proves the assertion of (a).

To prove (b), recall first that the functor $\ker u$ is representable by $u^{-1}(e)$, where e is the identity element of H. As u is of finite type, $\ker u$ is of finite type over k. On the other hand, by replacing H with the reduced closed subscheme $u(G)$, we may assume that u is surjective. Denote by u^0 the restriction of u to G^0. As G and $\ker u$ are equidimensional and $(\ker u)^0 \subseteq \ker u^0$, we are then reduced to the case where G, and hence H, are irreducible.

By ([?], 9.2.6.2 et 10.6.1(ii)), the set of $y \in H$ such that $\dim(u^{-1}(y)) = \dim(G) - \dim(H)$ contains a nonempty open subset V. Since u is surjective, $U = u^{-1}(V)$ is then a nonempty open subset of G, hence contains a closed point x of G, since G is a Jacobson scheme ([?], 10.4.8). Then the right translation r_x is an isomorphism from $\ker u$ to $u^{-1}(u(x))$, whence

$$\dim(\ker u) = \dim(u^{-1}(u(x))) = \dim(G) - \dim(H).$$

Now suppose that u is a momomorphism. If C is a connected component of G, there exists a closed point $x \in G$ such that $C = r_x(G^0)$, and if we denote by u_C (resp. u^0) the restriction of u to C (resp. to G^0), we have $u_C = r_{u(x)} \circ u^0 \circ r_x^{-1}$, so it suffices to show that u^0 is a closed immersion. We can then suppose that $G = G^0$, so that G is irreducible and of finite type over k.

Let ξ be the generic point of G, then $\mathscr{O}_{G,\xi}$ if a local Artinian ring, with maximal ideal $\mathfrak{m}_\xi$. Let $h = u(\xi)$, $\mathfrak{m}_h$ be the maximal ideal of $\mathscr{O}_{H,h}$, and $A = \mathscr{O}_{G,\xi}/\mathfrak{m}_h\mathscr{O}_{G,\xi}$. As u is a monomorphism, so is the morphism $u_h : \operatorname{Spec}(A) \to \operatorname{Spec}(\kappa(h))$ induced by base change, hence the multiplication map $A \otimes_{\kappa(h)} A \to A$ is an isomorphism, and we conclude $A = \kappa(h)$. By Nakayama's lemma (since $\mathfrak{m}_h\mathscr{O}_{G,\xi}$ is contained in $\mathfrak{m}_\xi$, hence nilpotent), it follows that the morphism $\mathscr{O}_{H,h} \to \mathscr{O}_{G,\xi}$ is surjective.

Let V be an affine open subset of H containing h, U be a nonempty affine open subset of G contained in $u^{-1}(V)$, $\phi : \mathscr{O}_H(V) \to \mathscr{O}_G(U)$ be the induced morphism of k-algebras, $\mathfrak{p}$ be the prime ideal of $\mathscr{O}_G(U)$ corresponding to ξ, and $\mathfrak{q} = \phi^{-1}(\mathfrak{p})$. As G is of finite type over k, $\mathscr{O}_G(U)$ is generated by finitely many elements $a_1, \dots, a_n$ as a k-algebra. By the preceding arguments, there then exist elements $b_1, \dots, b_n$ and s in $\mathscr{O}_H(V)$ such that $s \notin \mathfrak{q}$ and we have the equalities $a_i/1 = \phi(b_i)/\phi(s)$ in $\mathscr{O}_G(U)_\mathfrak{p}$. By definition, this means there are elements $t_1, \dots, t_n$ of $\mathscr{O}_G(U) - \mathfrak{p}$ such that $t_i(a_i\phi(s) - \phi(b_i)) = 0$. Then, putting $t = t_1 \cdots t_n\phi(s) \in \mathscr{O}_G(U) - \mathfrak{p}$, the equalities $a_i/1 = \phi(b_i)/\phi(s)$ are valid in $\mathscr{O}_G(U)_t$ and, as $t \in \mathscr{O}_G(U) = k[a_1, \dots, a_n]$, there exists $b \in \mathscr{O}_H(V)$ such that $t/1 = \phi(b)/\phi(s)^r$ for an integer $r \in \mathbb{N}$. Therefore ϕ induces a surjection from $\mathscr{O}_H(V)_{sb}$ to $\mathscr{O}_G(U)_t$, and hence u is a local immersion at the generic point ξ.

The open subset W of G formed by points over which u is a local immersion is therefore nonempty. As G is a Jacobson scheme, W contains a closed point y and, to show that $W = G$, it suffices to show that any closed point of G belongs to W. But any closed point x is the image of y under the translation $r_x \circ r_y^{-1}$, and hence belongs to W. This proves that u is a local immersion.

As G is irreducible, it then follows that u is an immersion. In fact, for any $x \in G$, let U_x and V_x be open subsets of G and H, respectively, such that $x \in U_x$ and that u induces a closed immersion from U_x into V_x. As U_x is dense in G, $u(U_x)$ is in $u(G) \cap V_x$, and as $u(U_x)$ is closed in V_x, we then have $u(U_x) = V_x$. As u is also injective, we also have $U_x = u^{-1}(V_x)$, so u induces a closed immersion from G into the open subscheme of H formed by the union of the V_x; whence $u : G \to H$ is an immersion. Since we have proved that $u(G)$ is closed in H, it is a closed immersion. □

Lemma 12.5.18. *Let A be a local Artinian ring, k be its residue field, G be a flat A-group, X be an A-scheme endowed with a right action $\mu : G \times_A X \to X$ of G and a section $s_0 : \operatorname{Spec}(A) \to X$. Let ϕ be the morphism $\mu \circ (\mathrm{id}_G \times s_0)$ from $G = G \times_A A$ to X. If ϕ is flat at a point $g \in G$, then it is flat.*

Proof. As G is flat over A, by the fiber criterion of flatness ([?], 11.3.10.2), it suffices to show that $\phi \otimes_A k$ is flat, hence we can assume that $A = k$. In this case, s_0 can be considered as a k-point $x_0 \in X(k)$, and ϕ is the orbit morphism $h \mapsto h \cdot x_0$.

Let $h \in G$, we show that ϕ is flat at h. Let K be an extension of k containing $\kappa(g)$ and $\kappa(h)$; we have a Cartesian square

$$\begin{tikzcd} G_K \arrow[r, "\phi_K"] \arrow[d] & X_K \arrow[d] \\ G \arrow[r, "\phi"] & X \end{tikzcd}$$

where the vertical morphisms are faithfully flat. By ([?], V, Lemme 7.4), ϕ_K is flat at any point $g' \in G_K$ lying over g, and to show that ϕ is flat at h, it suffices to show that ϕ_K is flat at any point h' lying over

h. We are then reduced to the case where g and h are rational. Let $u = hg^{-1}$ and ℓ_u (resp. μ_u) be the left translation of G (resp. of X) defined by u. As $\phi \circ \ell_u = \mu_u \circ \phi$, we obtain a commutative square

$$\begin{tikzcd} \mathscr{O}_{X,h\cdot x_0} \arrow[r, "\cong"] \arrow[d] & \mathscr{O}_{X,g\cdot x_0} \arrow[d] \\ \mathscr{O}_{G,h} \arrow[r, "\cong"] & \mathscr{O}_{G,g} \end{tikzcd}$$

in which the horizontal morphisms are isomorphisms. As the morphism $\mathscr{O}_{X,g\cdot x_0} \to \mathscr{O}_{G,g}$ is flat by hypothesis, we conclude that $\mathscr{O}_{X,h\cdot x_0} \to \mathscr{O}_{G,h}$ is also flat. □

Example 12.5.19. Let k be a field of characteristic zero, G be the constant group $\mathbb{Z}_k$ over k and H be the additive group $\mathbb{G}_{a,k}$. Let $u : G \to H$ be a morphism of k-groups. If $u \neq 0$, then $u(G)$ is not closed in H. In fact, in this case $u(G)$ is an infinite subset of closed points of the underlying scheme $\mathbb{A}^1_k$ of H, which is not closed by Corollary 9.7.19.

We recall that if G is a Lie group, then a homogeneous space $X = G/H$ has a natural manifold structure and its dimension is given by $\dim(G) - \dim(H)$. In the case of algebraic groups, we still have the following analogous result:

Proposition 12.5.20. *Let A be a local Artinian ring, k be its residue field, G be an A-group locally of finite type, X be a nonempty A-scheme locally of finite type endowed with a left action by G. Suppose that the morphism $\phi : G \times_S X \to X \times_S X$ defined by $(g, x) \mapsto (gx, x)$ is surjective, then:*

(a) *The connected components of X are of finite type, irredcuible and have the same dimension.*

(b) *More precisely, let $\bar{k}$ be an algebraic closure of k and x be a closed point of $X \otimes_A \bar{k}$. Then the stabilizer $F = \mathrm{Stab}_{G\otimes_A \bar{k}}(x)$ is a closed subgroup of $G \otimes_A \bar{k}$, and the dimension of the irreducible components of X is $\dim(G) - \dim(F)$.*

Proof. In view of Lemma 12.5.16, we may suppose that $A = k$. We first consider the case where k is algebraically closed. Then G_{red} is a k-group locally of finite type and hence, by replacing G with G_{red} and X with X_{red}, we may assume that G and X are reduced.

As $G \times_k X$ is locally of finite type over k, ϕ is locally of finite type by Proposition 8.6.21, hence locally of finite presentation since $X \times_k X$ is locally Noetherian. Let x be a rational point of X, then the orbit map $\phi_x : G \to X$, induced from ϕ by base change along $\mathrm{id} \times x : X \to X \times_A X$, is surjective and locally of finite presentation. If η is a maximal point of X, then $\mathscr{O}_{X,\eta}$ is a field (since X is reduced), so ϕ_x is flat at any point of G lying over η. By Lemma 12.5.18, we then conclude that ϕ_x is flat. Now ϕ_x is faithfully flat and locally of finite presentation, hence open (cf. [?], 2.4.6). As G^0 is open in G, irreducible and quasi-compact (Proposition 12.5.13), each orbit $G^0 \cdot x = \phi_x(G^0)$, for x a rational point of X, is an open subset of X, irreducible and quasi-compact, hence of finite type over k (since X is locally of finite type over k).

As any nonempty open subset of X contains a rational point (equivalently, a closed point, since k is assumed to be algebraically closed), it then follows that X is covered by the orbits of G^0. Moreover, any two orbits are either disjoint or equal. In fact, if $\phi_x(G^0) \cap \phi_y(G^0)$ is nonempty, it contains a rational point z and there then exists $g, h \in G^0$ such that $g \cdot x = z = h \cdot y$, whence $x = g^{-1} \cdot z$ and $y = h^{-1} \cdot z$ and hence $\phi_x(G^0) = \phi_z(G^0) = \phi_y(G^0)$. Therefore, the orbits $\phi_x(G^0)$ are the irreducible components of X, and also the connected components of X.

Finally, let x, y be two rational points of X. As ϕ_x is surjective, there exists a point $g \in G$ such that $y = g \cdot x$ and, as G^0 is a normal subgroup of G, the orbit $G^0 \cdot y$ is then the image of G^0 under the left translation ℓ_g of X, so that $G^0 \cdot y$ and $G^0 \cdot x$ have the same dimension.

If x is a closed point of X, the stabilizer of x is represented by the closed subscheme F of G defined by the Cartesian square:

$$\begin{tikzcd} F \arrow[r] \arrow[d] & G \arrow[d, "\phi_x"] \\ \mathrm{Spec}(k) \arrow[r, "x"] & X \end{tikzcd}$$

Then F is a k-group locally of finite type, $F \cap G^0$ is a k-group of finite type containing F^0, and by Corollary 12.5.14, F and $F \cap G^0$ are equidimensional, with the same dimension as F^0. Let $C = \phi_x(G^0)$ be the irreducible component of X containing x. By the same arguments as Proposition 12.5.17, we can show that $\dim(C) = \dim(G^0) - \dim(F^0) = \dim(G) - \dim(F)$.

In the general case (i.e. k is an arbitrary field), let $\bar{k}$ be an algebraic closure of k. Let C be a connected component of X and C' be a connected component of $C \otimes_k \bar{k}$, then C' is a connected component of $X' = X \otimes_k \bar{k}$. The morphism $\pi : X' \to X$ is open by ([?], 2.4.10), and as it is integral, it is also closed, so $\pi(C') = C$. As C' is irreducible and quasi-compact, so is C, and hence C is of fnite type over k (because X is locally of finite type over k).

Finally, since the dimension is invariant under base change of fields ([?], 4.1.4), $\dim(C) = \dim(C')$, and as any irreducible components of X' are of the same dimension, the same is true for X. □

Proposition 12.5.21 (Closed Orbit Lemma). *Let A be a local Artinian ring, k be its residue field, G be an A-group locally of finite type and X be a nonempty A-scheme of finite type endowed with a left action by G. Then each orbit of a rational point of X is open in its closure, and the boundary of each orbit is a union of orbits of lower dimension. In particular, the orbits of minimal dimension are closed.*

Proof. As in Proposition 12.5.20, we may assume that $A = k$ is algebraically closed and G, X are reduced. In this case, if $x \in X(k)$ is a rational point of X and $M = \phi_x(G)$ is the orbit of x under G, then G is clearly stable under G. We note that any two rational points of M are conjugate: in fact, if $y \in M(k)$, then by definition there exists $g \in G$ such that $\phi_x(g) = y$. Since y is rational and ϕ_x is a k-morphism, it follows that g is rational over k, whence $\ell_g(x) = y$.

Now as X is of finite type over k, it is Noetherian, so by Chevalley's constructibility theorem ([?], 1.8.5), M is a constructible subset (and dense) of $\overline{M}$, hence contains an open dense subset U of $\overline{M}$ ([?] 0_{III}, 9.2.2). It is clear that U contains the closed point x of M, so by homogeneity, every closed point of $\overline{M}$ is contained in an open subset of $\overline{M}$ contained in M. We therefore conclude from ([?], 10.1.2) that M is open in $\overline{M}$, so $\overline{M} - M$ is closed and of lower dimension, as well as being G-stable; it is therefore a union of orbits of G of lower dimension. □

Remark 12.5.22. We note that if G is reduced, then the morphism $\phi_x : G \to X$ is faithfully flat, so $\mathscr{O}_M \to (\phi_x)_*(\mathscr{O}_G)$ is an injective homomorphism, and remains so after base change of fields. Therefore, if G is smooth, then M is geometrically reduced, and since its smooth locus is nonempty (hence open dense in M, cf. [?] 17.15.12), we see that M is smooth over k by homogeneity.

Example 12.5.23. Let k be an algebraically closed field and consider the action

$$\mathrm{SL}_2 \times \mathbb{A}_k^2 \to \mathbb{A}_k^2, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}.$$

There are two orbits of $\mathbb{A}_k^2$, namely $\{(0,0)\}$ and its complement. We see that the smaller one is closed, but the larger one is not locsed, and not even affine.

12.5.4 Morphisms of algebraic groups

Let A be a local Artinian ring, G and H be A-groups and $u : G \to H$ be a morphism of A-groups. Then u induces a morphism of groups $u(A) : G(A) \to H(A)$. As $H(A)$ acts on H by right translations, $u(A)$ defines an action of $G(A)$ on H, which is compatible with the morphism u and the action of $G(A)$ on G defined by left translations. As $G(A)$ acts transitively on strictly rational points, we see that these points "share the same properties with respect to u". For example, we have the following result:

Proposition 12.5.24. *Let A be a local Artinian ring with residue field k, G be an A-group locally of finite type and flat, $u : G \to H$ be a morphism of A-groups. The following conditions are equivalent:*

(i) *u is flat (resp. quasi-finite, resp. unramified, resp, smooth, resp. étale) at a point of G.*

(ii) *u is flat (resp. quasi-finite, resp. unramified, resp, smooth, resp. étale)*

For the proof of Proposition 12.5.24, we need the following lemma:

Lemma 12.5.25. *Let $A \to B \to C$ be ring homomorphisms and $\mathfrak{n}$ be a nilpotent ideal of A. Suppose that $C/\mathfrak{n}C$ is a $(B/\mathfrak{n}B)$-algebra of finite type.*

(a) *C is a B-algebra of finite type.*

(b) *If C is flat over A and $C/\mathfrak{n}C$ is a $(B/\mathfrak{n}B)$-algebra of finite presentation, then C is a B-algebra of finite presentation.*

Proof. Let $x_1, \dots, x_n$ be elements of C whose images in $C/\mathfrak{n}C$ generate it as a $(B/\mathfrak{n}B)$-algebra. By nilpotent Nakayama's lemma, the x_i generate C as a B-algebra, and this proves (a). Let $\phi : B[X_1, \dots, X_n] \to C$ be the surjective homomorphism thus obtained, and $\mathfrak{I} = \ker \phi$. Suppose that C is flat over A and $C/\mathfrak{n}C$ is of finite presentation over $B/\mathfrak{n}B$. Then $\mathfrak{I}/\mathfrak{n}\mathfrak{I}$, which is identified with the kernel of $\bar{\phi} = \phi \otimes_A (A/\mathfrak{n})$, is finitely generated by Proposition 8.6.26. Let $P_1, \dots, P_s$ be the polynomials whose images generate $\mathfrak{I}/\mathfrak{n}\mathfrak{I}$, then by nilpotent Nakayama's lemma, they generate $\mathfrak{I}$, which proves (b). □

Proof of Proposition 12.5.24. It suffices to prove that (i)⇒(ii), so let x be an arbitrary point of G. As G is flat over A, by the fiber criterion of flatness ([?], 11.3.10.2), u will be flat at x if $u \otimes_A k$ is. Similarly, by the preceding lemma, we see that u will be locally of finite type (resp. locally of finite presentation) if $u \otimes_A k$. As the other properties are verified over fibers, we may then assume that $A = k$.

Now let x be a point of G where condition (a) is satisfied. As the considered properties are perserved under fpqc descent (cf. [?], 2.5.1, 2.7.1 et [?], 17.7.1), we may assume that k is an algebraic closure of $\kappa(x)$, so that k is algebraically closed and $x \in G(k)$.

As G is a Jacobson scheme (cf. [?], 10.4.7) as as the set W of points of G where u is flat, quasi-finite, unramified, smooth or étale is open, it suffices to show that any point y of $G(k)$ belongs to W. Now, for such a point y, the translation $r_y \circ r_x^{-1}$ sends x to y, hence u possesses the same property at y, i.e. $y \in W$. □

Corollary 12.5.26. *Let A be a local Artinian ring with residue field k, G be a flat A-group. The following assertions are equivalent:*

(a) *G is locally quasi-finite (resp. unramified, resp, smooth, resp. étale) over A at a point.*

(b) *G is locally quasi-finite (resp. unramified, resp, smooth, resp. étale) over A*

Proof. In fact, if G satisties one of the conditions of (a) at a point x, there exists an open neighborhood U of x which is of finite type over A. Therefore, it suffices to apply Proposition 12.5.24 to the case where H is the trivial A-group, in view of Lemma 12.5.27 below. □

Lemma 12.5.27. *Let A be a local Artinian ring and G be an A-group. If there exists a nonempty open subset G of finite type over A, then G is locally of finite type over G.*

Proof. By Lemma 12.5.25, we can suppose that $A = k$ is a field. Moreover, by fpqc descent, we can assume that k is algebraically closed (cf. [?], 2.7.1). Let V be an open subset of G formed by points where G is of finite type over k; by hypothesis, $V \neq \varnothing$. As G is a Jacobson scheme, V contains a closed point x and, to show that $V = G$, it suffices to show that any closed point y of G belongs to V. Now for such a point y, the translation $r_y \circ r_x^{-1}$ sends x to y, so $y \in V$. □

Corollary 12.5.28. *Let A be a local Artinian ring and $u : G \to H$ be a morphism between A-groups locally of finite type. The following conditions are equivalent:*

(i) *u is universally open,*

(ii) *u is open;*

(iii) *u is open at a point of G;*

(iv) *the morphism $u^0 : H^0 \to H^0$ induced by u is dominant;*

(iv′) *u^0 is surjective;*

(v) *there exists a connected component C of G such that, if D denotes the connected component of H containing $u(C)$, the morphism $u' : C \to D$ induced by u is dominant.*

Proof. The implications (i)⇒(ii)⇒(iii) and (iv′)⇒(iv)⇒(v) are clear. As G^0 of finite type over A (Proposition 12.5.13), hence Noetherian, the induced morphism u^0 is quasi-compact so $u^0(G^0)$ is closed in H^0 by Proposition 12.5.17, whence (iv)⇒(iv′). On the other hand, since G^0 (resp. C) is open in G (Proposition 12.5.13) and H^0 (resp. D) is irreducible (Corollary 12.5.14), we see that (ii) implies (iv) (resp. that (iii) implies (v)). It then remains to show that (v) implies (i).

Let C and D be as in (v) and endowed them with the induced scheme structure; denote by $u' : C \to D$ the induced morphism. Let k be a residue field of A. As the base change $\mathrm{Spec}(k) \to \mathrm{Spec}(A)$ is a universally homeomorphism (Cf. **??**, 18.12.11), we can suppose that $A = k$. By hypothesis, u' is dominant and, since C is of finite type over k (Corollary 12.5.14), hence Noetherian, u' is quasi-compact. By ([**?**], 2.3.7), $u' \otimes_k \bar{k}$ is then quasi-compact and dominant, where $\bar{k}$ is an algebraic closure of k. Then, since $C \otimes_k \bar{k}$ is a union of connected components of $G \otimes_k \bar{k}$, the morphism $u \otimes_k \bar{k} : G \otimes_k \bar{k} \to H \otimes_k \bar{k}$ satisfies condition (v). We are then reduced to the case where $A = k$ is algebraically closed (cf. **??**, 2.6.4).

In this case, we can further replace u by u_{red}, and therefore assume that H is reduced. Let ξ (resp. η) be the generic point of C (resp. D). Since u' is quasi-compact and dominant, $u'(\xi) = \eta$ by Proposition 8.6.5. On the other hand, as H is reduced, the local ring $\mathscr{O}_{H,\eta}$ is a field, and hence u' is flat at ξ. Hence by Proposition 12.5.24, u is flat; moreover, since u is locally of finite type and H is locally Noetherian, u is locally of finite presentation, so by ([**?**], 2.4.6) it is universally open. □

Proposition 12.5.29. *Let A be a local Artinian ring and $u : G \to H$ be a quasi-compact morphism between A-groups locally of finite type. The following assertions are equivalent:*

(i) *u is proper;*

(ii) *there exists $h \in H$ such that the fiber $u^{-1}(h)$ is nonempty and proper over $\kappa(h)$;*

(ii) $\ker u$ *is proper over A.*

Proof. It is clear that (i) implies (iii), and (iii) implies (ii). On the other hand, it follows from the hypothesis that u is of finite type and, since G is separated, u is also separated. It then remains to show that condition (ii) implies that u is universally closed, for which we can assume that $A = k$. Let k' be an algebraic closure of $\kappa(h)$, $u' : G' \to H'$ be the morphism induced by base change, h' be a point of H' over h; then the fiber $u'^{-1}(h') = u^{-1}(h) \times_{\kappa(h)} k'$ is nonempty and proper, and it suffices to show that u' is proper ([**?**], 2.6.4). We can therefore assume that k is algebraically closed and $h \in H(k)$.

We have seen in Proposition 12.5.17 that $u(G)$ is the underlying space of a closed and reduced subscheme of H; any closed immersion being proper, we can then suppose that u is surjective, and that H is reduced. In this case, $G(k)$ acts simply and transitively on closed points of H; for any closed point $y \in H$, $u^{-1}(y)$ is therefore proper over $\kappa(y)$. By ([**?**], 9.6.1), the set of points $y \in H$ such that $u^{-1}(y)$ is not proper over $\kappa(y)$ is locally constructible; since it contains no closed point, it is therefore empty (cf. [**?**], 10.3.1 et 10.4.7).

Now consider the generic point η of H^0; by the preceding arguments, the fiber $u^{-1}(\eta) = G \times_H \mathrm{Spec}(\kappa(\eta))$ is proper over $\kappa(\eta)$. On the other hand, since H is reduced, $\kappa(\eta) = \mathscr{O}_{H,\eta}$. As $\mathscr{O}_{H,\eta}$ is the inductive limit of $\mathscr{O}_H(V)$, for V runs through nonempty open subsets of H^0, it follows from ([**?**], 8.1.2(a) et 8.10.5(xii)) that there exists a nonempty open subset V of such that the restriction $u|_{u^{-1}(V)} : u^{-1}(V) \to V$ is proper. It is then clear that the $g \cdot V$, for $g \in G(k)$, form an open covering of H; we then deduce that u is proper (cf. Corollary 9.5.26). □

Corollary 12.5.30. *Let A be a local Artinian ring and $u : G \to H$ be a morphism between A-groups locally of finite type. The following conditions are equivalent:*

(i) *u is locally quasi-finite;*

(ii) *u is quasi-finite at a point;*

(iii) $\ker u$ *is discrete;*

(iv) *the restriction of u to a connected component of G is finite.*

Finally, if u is quasi-compact, these conditions are equivalent to:

(v) *u is finite.*

Proof. It is clear that (iv) implies (iii), that (iii) implies (ii), and that in the case where u is quasi-compact, (v) is equivalent to (iv). We have seen in Proposition 12.5.24 that conditions (i) and (ii) are equivalent.

We now show that (i)⇒(iv). Let C be a connected component of G; since C is of finite type over A (**??**) and that G, H are separated, the restriction of u' of u to C is separated and of finite type. As the fibers of u' are discrete, it follows that u' is quasi-finite (Corollary 9.6.16). As any quasi-finite and proper morphism is finite (Proposition 10.4.19), it then suffices to show that u' is universally closed, for which we may suppose that $A = k$ is a field. Then by fpqc descent (cf. [?], 2.6.4), it suffices to show that $u' \otimes_k \bar{k}$ is universally closed, where $\bar{k}$ is an algebraic closure of k. Further, as C is of fnite type over k, $C \otimes_k \bar{k}$ is a finite sum of connected components $C'_1, \dots, C'_n$ of $G \otimes_k \bar{k}$, so it suffices to consider the claim for each C'_i; this means we can reduce to the case where k is algebraically closed.

Now let g be a closed point of G; if $u^0 : G^0 \to H$ is the restriction of u to G^0, we have $u' = r_{u(g)} \circ u^0 \circ r_{g^{-1}}$, so we only need to show that u^0 is universally closed. By hypothesis, u is locally quasi-finite, so the fiber $\ker u$ is discrete (and nonempty); we note that u^0 is of finite type since G^0 is of finite type over k (hence Noetherian), so the fiber $\ker u^0$ is finite, hence proper. It then follows from Proposition 12.5.29 that u^0 is proper, and a fortiori universally closed. □

Corollary 12.5.31. *Let A be a local Artinian ring and $u : G \to H$ be a quasi-compact morphism between A-groups locally of finite type. The following conditions are equivalent:*

(i) *u is a closed immersion;*

(ii) *u is a monomorphism;*

(iii) $\ker u$ *is trivial, ie,e, isomorphic to the trivial k-group.*

Proof. It is clear that (i) implies (ii), and (ii) and (iii) are equivalent. Finally, if $\ker u$ is trivial, then it is proper over k and nonempty, so u is a proper monomorphism by Proposition 12.5.29, and is of finite presentation because H is locally Noetherian. We then conclude from ([?], 8.11.5) that u is a closed immersion. □

Example 12.5.32. Let k be a field of characteristic zero, G be the constant k-group $\mathbb{Z}_k$ and H be the k-group $\mathbb{G}_{a,k}$. Let $u : G \to H$ be a morphism of k-groups. If $u \neq 0$, then $\ker u = 0$ and u is a monomorphism. But u is not a closed immersion (cf. Example 12.5.19).

12.5.5 Construction of the quotient $F\backslash G$

Let A be a local Artinian ring and $u : F \to G$ be a homomorphism of A-groups. If $\mu_F : F \times_A F \to F$ and $\mu_G : G \times_A G \to G$ denote the multiplication morphisms and λ is the composition morphism

$$F \times_A G \overset{u\times \mathrm{id}_G}{\to} G \times_A G \overset{\mu_G}{\to} G,$$

we define the **left quotient** $F\backslash G$ of G by F to be the cokernel of the following $\mathbf{Sch}_{/A}$-groupoid G_*:

$$\begin{tikzcd} F \times_A F \times_A G \arrow[r, shift left=2, "\mathrm{pr}_{2,3}"] \arrow[r, "\mu\times\mathrm{id}_G" description] \arrow[r, shift right=2, "\mathrm{id}_F \times \lambda"'] & F \times_A G \arrow[r, shift left, "\lambda"] \arrow[r, shift right, "\mathrm{pr}_2"'] & G \end{tikzcd}$$

(where pr_2 and $\mathrm{pr}_{2,3}$ are the projections of $F \times_A G$ and $F \times_A (F \times_A G)$.) We say that G_* is the groupoid of base G defined by u. As the unique A-morphism $F \to \mathrm{Spec}(A)$ is universally open ([?], 2.4.9), pr_2 is an open morphism, hence so is λ which is the composition of pr_2 and the automorphism σ of $F \times_A G$ defined by $\sigma(S)(x,y) = (x, u(S)(x) \cdot y)$, where S is an A-scheme and $x \in F(S)$, $y \in G(S)$. Similarly, we see that pr_2 and λ are flat if F is flat over A.

On the other hand, as We also note that any A-morphism $s : \mathrm{Spec}(A) \to G$ defines an automorphism of the groupoid G_* which induces over G, $F \times_A G$ and $F \times_A F \times_A G$ the automorphisms r_s, $\mathrm{id}_F \times r_s$ and $\mathrm{id}_F \times \mathrm{id}_F \times r_s$, respectively. We then denote this automorphism of G_* by r_s and call it the right translation of G_* defined by s.

Theorem 12.5.33. *Let F and G be groups flat and locally of finite type over a local Artinian ring A and $u : F \to G$ be a quasi-compact homomorphism of A-groups with kernel finite over A. Then*

(a) *The left quotient $F\backslash G$ of G by F exists in $\mathbf{Sch}_{/A}$ and $F\backslash G$ is a quotient of G_* in the category of ringed spaces.*

(b) *The canonical morphism $p : G \to F\backslash G$ is surjective and open, and $F\backslash G$ is locally of finite type over A.*

(b') *More precisely, $X = F\backslash G$ is endowed with a right action of G such that $p(e) \cdot g = p(g)$ for any $g \in G$. Therefore, the connected components of X are of finite type over A, irreducible and of the same dimension* $\dim(G) - \dim(F)$.

(c) *The canonical morphism* $(\lambda, \mathrm{pr}_2) : F \times_A G \to G \times_{(F\backslash G)} G$ *is surjective.*

(d) *If* u *is a monomorphism, then:*

(i) $F \times_A G \to G \times_{(F\backslash G)} G$ *is an isomorphism and* $G \to F\backslash G$ *is fiathfully flat and locally of finite presentation.*

(i′) $F\backslash G$ *represents the fppf sheaf quotient* $\widetilde{F\backslash G}$ *and* $G \to F\backslash G$ *is an* F*-torsor locally trivial for the fppf topology.*

(ii) $F\backslash G$ *is flat over* A*, and is smooth over* A *is* G *is.*

(iii) $u : F \to G$ *is a closed immersion and* $F\backslash G$ *is separated.*

(iv) *If* F *is a normal subgroup of* G*, there exists a unique* A*-group structure on* $F\backslash G$ *such that* $p : G \to F\backslash G$ *is a morphism of* A*-groups.*

In the proof of this theorem, we denote by A' a local A-algebra, finite and free over A. If $\mathcal{R}$ is a relation concerning A', we shall say that "$\mathcal{R}(A')$ is valid for A' large enough" if there exists a local algebra A_1, finite and free over A, such that the relation $\mathcal{R}(A')$ is verified for any local algebra A', finite and free over A_1.

12.5.5.1 Passage to the quotient $F\backslash G$ (case where F and G are of finite type over A) In this paragraph, we prove the theorem for the case where F and G are of finite type over A. For this, we first make the following additional hypothesis:

Every point of G has a saturated open neighborhood W such that the groupoid induced by G_* on W has a quasi-section. (12.5.1)

(cf. §§12.4.3). Then, by Lemma 12.4.13, we have the assertions (a), (b), (c) and (d)(i), and $F\backslash G$ is of finite type over k. Moreover, under the assumption of (d), since $G \to F\backslash G$ is faithfully flat and locally of finite presentation, assertion (d)(ii) follows from ([**?**], 17.7.4). On the other hand, (d)(i′) follows from (d)(i), by Proposition 12.3.17, Corollary 12.3.38 and Proposition 12.3.46.

We now prove the following assertion:

$$\text{Any finite subset of } F\backslash G \text{ is contained in an affine open subset.} \tag{12.5.2}$$

If U is a quasi-section of the groupoid induced by G_* over a saturated open subset W of G, then $U \otimes_A A'$ is a quasi-section of the $\mathbf{Sch}_{/A'}$-groupoid induced by $G_* \otimes_A A'$ over $W \otimes_A A'$. Further, if U_* is the $\mathbf{Sch}_{/A}$-groupoid induced by G_* over U, then $U_* \otimes_A A'$ is identified with the $\mathbf{Sch}_{/A'}$-groupoid induced by $G_* \otimes_A A'$ over $U \otimes_A A'$. It then follows from the proof of Lemma 12.4.13 that the construction of the quotient $X = F\backslash G$ commutes with any base change $A \to A'$, where A' is a local A-algebra finite and free over A[26].

Now let $x_1, \dots, x_n$ be elements of $X = F\backslash G$, which we can assume to be closed[27], and $g_1, \dots, g_n$ be closed points of G which project to $x_1, \dots, x_n$. Let V be an affine everywhere dense open subset of X[28], and let U be the inverse image of V in G. By Proposition 12.5.7, there exists a local A-algebra A', finite and free over A, such that the points $g'_1, \dots, g'_r$ of $G' = G \otimes_A A'$ lying over the $g_1, \dots, g_n$ are strictly rational over A'. As the morphisms $G' \to G$ and $G \to X$ are open, $U' = U \otimes_A A'$ is dense in G', hence the open subset $\bigcap_{i=1}^r (U')^{-1} \cdot g'_i$ is nonempty, hence contains a closed point x. Therefore, by Proposition 12.5.7 (and Remark 12.5.1), we can suppose, by enlarging A' if necessary, that x is strictly rational over A'. Then, as $x \in (U')^{-1} \cdot g'_i$, we have $g'_i \in U' \cdot x$.

Denote by V' the inverse image of V in $X' = X \otimes_A A'$; this is an affine open subset of X', and is also the image of U' under the projection $G' \to X'$. As the right translation r_x is an automorphism of the groupoid $G_* \otimes_A A'$, it induces an automorphism, still denoted by r_x, or the quotient X'. Therefore, the image $V' \cdot x = r_x(V')$ of $U' \cdot x$ in X' is an affine open subset of X' containing the images $x'_1, \dots, x'_r$

[26] For this, it amounts to note that the formation of the direct images under p, λ and pr_2 commute with flat base changes $A \to A'$: As F and G are of finite type and separated over an Artinian ring A, any morphism f considered in question is quasi-compact and separated, and the equality $f_*(\mathscr{O}_X) \otimes_A A' = f'_*(\mathscr{O}_{X'})$ (with the evident notations) follows from Corollary 10.1.21.

[27] In fact, let $y_1, \dots, y_n$ be arbitrary points of X; as X is of finite type over A, each y_i is in the closure of a closed point x_i of X, and any open subset containing x_i also contains y_i.

[28] Such an open subset exists, since X is of finite type over k: X has finitely many irreducible components $C_1, \dots, C_r$, and it suffices to choose for each i a nonempty open affine subset contained in $C_i - \bigcup_{j \neq i} C_i$ (however, by (b′) we know that these C_i are disjoint).

of $g'_1, \dots, g'_r$. Now consider the equivalence relation over $X' = X \otimes_A A'$ defined by the projection $X \otimes_A A' \to X$:

$$X \otimes_A A' \otimes_A A' \overset{d_1}{\underset{d_0}{\rightrightarrows}} X \otimes_A A' \longrightarrow X$$

where d_0 and d_1 are induced by the two canonical injections of A' into $A' \otimes_A A'$. As A' is a finite and free A-algebra, say of rank n, we see that d_0 and d_1 are finite and locally free of rank n. Therefore, we can apply the reasoning of 12.4.2.2 (the proof taken from **[?]**, VIII, 7.6) to obtain a saturated affine open subset $W' \subseteq V' \cdot x$ containing $x'_1, \dots, x'_r$. The image of W' in X then contains $x_1, \dots, x_n$ and is an affine open subset of X, by Theorem 12.4.9[29].

We return to the case where F and G are of finite type over A (i.e. we do note assume (12.5.1)). For any local algebra A', finite and free over A, we now denote by $U(A')$ the set of points of $G \otimes_A A'$ admitting a satusrated open subset W such that the groupoid induced by $G_* \otimes_A A'$ over W possesses a quasi-section. It is clear that $U(A')$ is saturated for the operation of $G(\mathrm{Spec}(A'))$ over $G \otimes_A A'$. We claim that, if A' is taken large enough, then $U(A') = G \otimes_A A'$.

Since $\ker u$ is assumed to be finite over A (and hence discrete), we see that the morphism $(\lambda, \mathrm{pr}_2) : F \times_A G \to G \times_A G$, which is the composition of $u \times \mathrm{id}_G : F \times_A G \to G \times_A G$ with the automorphism σ of $G \times_A G$ defined by $(x, y) \mapsto (xy, y)$, is quasi-finite (cf. Corollary 12.5.30). It then follows from (**[?]** V, theoreme 8.1) that $U(A)$ is not empty, hence contains a closed point y. We now prove our claim by induction on $\dim(G - U(A))$. Let $g_1, \dots, g_n$ be closed points belonging to distinct irreducible components of $G - U(A)$. By Proposition 12.5.7, there exists a local A-algebra A', finite and free over A, such that the points $g'_1, \dots, g'_r$ (resp. $x = x_1, \cdots, x_r$) of $G' = G \otimes_A A'$ lying over $g_1, \dots, g_n$ (resp. y) are strictly rational over A'. Then $U(A')$ contains $(U(A) \otimes_A A') \cdot x^{-1} g'_i$ for each i, so it contains $g'_1, \dots, g'_r$ and we have

$$\dim(G' - U(A')) < \dim(G - U(A)).$$

The induction hypothesis implies the existence of a local algebra A'', finite and free over A', such that $U(A'') = G' \otimes_{A'} A'' = G \otimes_A A''$.

With the assertion in (12.5.2), we are now able to prove the existence of the existence $F \backslash G$ for F, G of finite type over A. By the arguments above, we can take a local algebra A', finite and free over A, such that $U(A') = G \otimes_A A'$. We put $A'' = A' \otimes_A A'$ and, for any A-scheme X, we put $X' = X \otimes_A A'$ and $X'' = X \otimes_A A''$. By what we have already proven, the quotients $F' \backslash G'$ and $F'' \backslash G''$ exist and we have the following commutative diagram, where the first two rows and columns are exact:

$$\begin{tikzcd}
F'' \otimes_{A''} G'' \arrow[r, shift left, "\mathrm{pr}''_2"] \arrow[r, shift right, "\lambda''"'] \arrow[d, shift right, "w_1"'] \arrow[d, shift left, "w_2"] & G'' \arrow[r, "p''"] \arrow[d, shift right, "v_1"'] \arrow[d, shift left, "v_2"] & F'' \backslash G'' \arrow[d, shift right, "u_1"'] \arrow[d, shift left, "u_2"] \\
F' \times_{A'} G' \arrow[r, shift left, "\mathrm{pr}'_2"] \arrow[r, shift right, "\lambda'"'] \arrow[d, "h"] & G' \arrow[r, "p'"] \arrow[d, "g"] & F' \backslash G' \\
F \otimes_A G \arrow[r, shift left, "\mathrm{pr}_2"] \arrow[r, shift right, "\lambda"'] & G &
\end{tikzcd} \tag{12.5.3}$$

In this diagram, pr'_2 and λ' (resp. pr''_2 and λ'') are obtained from pr_2 and λ by base change; the morphisms g and h are induced by the canonical injection $A \to A'$. We denote by p' and p'' the canonical morphisms, and the morphisms v_1, v_2 and w_1, w_2 are induced by the two canonical injections from A' to A''. Finally, as the construction of the quotient $F' \backslash G'$ commutes with base changes $f_1, f_2 : \mathrm{Spec}(A'') \rightrightarrows \mathrm{Spec}(A')$, we have, with $\pi' : F' \backslash G' \to \mathrm{Spec}(A')$ being the structural morphism, the canonical isomorphisms for $i = 1, 2$:

$$\tau_i : F'' \backslash G'' \xrightarrow{\sim} (F' \backslash G') \times_{\pi', f_i} \mathrm{Spec}(A''),$$

and the morphism u_i is the composition of τ_i with the projection $(F' \backslash G') \times_{\pi', f_i} \mathrm{Spec}(A'') \to F' \backslash G''$.

[29] We note that $X \otimes_A A' \to X$ is a universally effective epimorphism since it is faithfully flat and locally of finite presentation, cf. Proposition 12.3.67.

If we have a diagram of the form (12.5.3), the first two rows and columns being exact, we then verify that $\operatorname{coker}(\mathrm{pr}_2, \lambda)$ exists if and only if $\operatorname{coker}(u_1, u_2)$ exists, and these two cokernels are identified. Now if follows from the compatibility of the formation of $F\backslash G$ with base extensions (as we have mentioned) that the composition morphism

$$(F'\backslash G') \times_{\pi', f_1} \operatorname{Spec}(A'') \xrightarrow{\tau_1^{-1}} F''\backslash G'' \xrightarrow{\tau_2} (F'\backslash G') \times_{\pi', f_2} \operatorname{Spec}(A'')$$

is a descent data over $F'\backslash G'$ relative to $f : \operatorname{Spec}(A') \to \operatorname{Spec}(A)$. By (12.5.2) and ([?] VIII, 7.6), this descent data is effective, which means $\operatorname{coker}(u_1, u_2)$ exists.

To complete the proof of assertions (a), (b), (c) and (d)(i) of Theorem 12.5.33 in the case where F and G are of finite type over A, it remains to study the quotient $F\backslash G$. According to Lemma 12.4.13, the assertions (b), (c) and (d)(i) become true after a suitable base change $f : \operatorname{Spec}(A') \to \operatorname{Spec}(A)$, so they were true before the base change (cf. [?], 2.6.1, 2.6.2 et 2.7.1). To see the second assertion of (a), i.e. that $F\backslash G$ is the cokernel of (pr_2, λ) in the category of ringed spaces, we can proceed as in 12.4.3.

Finally, we prove assertion (d)(iii) of Theorem 12.5.33, the proof of (b') and (d)(iv) will be postponed to 12.5.5.3. Denote by $X = F\backslash G$ and let $d : F \times_A G \to G \times_A G$ be the morphism with components λ and pr_2. As u is a closed immersion by Proposition 12.5.17, and as $d = \sigma \circ (u \times \mathrm{id}_G)$, where σ is the automorphism of $G \times_A G$ defined by $\sigma(x, y) = (xy, y)$, we see that d is also a closed immersion. On the other hand, by (d)(i), we have a Cartesian square

$$\begin{tikzcd}
F \times_A G \arrow[r, "d"] \arrow[d] & G \times_A G \arrow[d, "p \times p"] \\
X \arrow[r, "\Delta_X"] & X \times_A X
\end{tikzcd}$$

and p hence also $p \times p$ is faithfully flat and locally of finite presentation. By fppf descent, as d is a closed immersion, so is Δ_X (cf. Proposition 12.3.69), i.e. X is separted.

Remark 12.5.34. In Theorem 12.5.33, the hypothesis that G is flat can be removed if $u : F \to G$ is a monomorphism.

Corollary 12.5.35. *Let A be a local Artinian ring, G be an A-group locally of finite type, H be a closed subgroup of G, flat over A. Denote by p the morphism $g \to G/H$ and λ (resp. pr_1) the morphism $G \times H \to G$ defined by $\lambda(g, h) = gh$ (resp. the projection $G \times H \to G$). Then for any open subset U of G/H, we have*

$$\mathscr{O}(U) = \{\phi \in \mathscr{O}(p^{-1}(U)) : \phi \circ \lambda = \phi \circ \mathrm{pr}_1\}$$

i.e. $\mathscr{O}(U)$ is the set of $\phi \in \mathscr{O}(p^{-1}(U))$ such that $\phi(gh) = g$ for any A-scheme S and $g \in G(S)$, $h \in H(S)$.

Proof. In fact, as $p : G \to G/H$ is faithfully flat and locally of finite presentation, hence a universally effective epimorphism, this follows from Remark 12.3.13. □

12.5.5.2 Passage to the quotient $F\backslash G$ (general case) Now we consider the general case of Theorem 12.5.33, so F and G are not necessarily of finite type over A.

Let G^α be a connected component of G. We show that the saturation $\mathcal{S}(G^\alpha)$ of G^α for the equivalence relation defined by the groupoid G_* is closed in G. By definition, this saturation is the image of $F \times_A G^\alpha$ under λ, hence is open in G (recall that λ is open). If k is the residue field of A and $\bar{k}$ is an algebraic closure of k, then it suffices to show that the image of $(F \times_A G^\alpha) \otimes_A \bar{k}$ under $\lambda \otimes_A \bar{k}$ is closed in $G \otimes_A \bar{k}$ (cf. Proposition 12.3.69). As $G^\alpha \otimes_A \bar{k}$ is a union of connected components of $G \otimes_A \bar{k}$, we are then reduced to the case where $A = k$ is an algebraically closed field. In this case, $\mathcal{S}(G^\alpha)$ is the union of the images of G^α under the left translations $\ell_{u(x)}$, where x runs through closed points of F[30]. The assertion therefore follows from the fact that these images are connected components of G.

In particular, consider the identity component G^0 of G. Then $\mathcal{S}(G^0)$ contains evidently the image of F under u, which is none other than the equivalent class of the identity of G. On the other hand, if F^β is a connected component of F, then $F^\beta \times_A G^0$ is connected (Proposition 12.5.12) so that its image under

[30] If $x, y \in F$ are contained in the same connected component C of F, then $\ell_{u(x)}(G^\alpha)$ and $\ell_{u(y)}(G^\alpha)$ is contained in the image of $F \times G^\alpha$, which is connected (since we have assumed that k is algebraically closed). Therefore, we have $\ell_{u(x)}(G^\alpha) = \ell_{u(y)}(G^\alpha)$, and our assertion follows from the fact that any connected component contains a closed point.

λ is contained in the connected component of $u(F^\beta)$ in G. In other words, $\mathcal{S}(G^0)$ is the union of the connected components that meet the image of F.

We also remark that the open subscheme of G with $\mathcal{S}(G^0)$ as underlying topological space is a subgroup of G (still denoted by $\mathcal{S}(G^0)$): the inversion morphism of G preserves the image of F and permutes the connected components of G which meet this image. It then suffices to show that $\mu_G : G \times_A G \to G$ sends $S(G^0) \times_A \mathcal{S}(G^0)$ into $\mathcal{S}(G^0)$ and for this we can suppose that A is an algebraically closed field (in fact, $\mathcal{S}(G^0) \otimes_A \bar{k}$ is identified with the saturation of $(G \otimes_A \bar{k})^0$ for the equivalence relation defined by $u \otimes_A \bar{k}$). In this case, if G^γ and G^δ are connected components of $\mathcal{S}(G^0)$, then $G^\gamma \times_A G^\delta$ is connected and its image under μ_G meets the image of F; therefore, $u(G^\gamma \times_A G^\delta)$ is contained in a connected component of G meeting $u(F)$.

We therefore obtain that the groupoid G_* induced by u on G is a direct sum of groupoids $\mathcal{S}(G^\alpha_*)$ induced by G_* over the distinct clopen subsets of G of the form $\mathcal{S}(G^\alpha)$. The cokernel of G_* is then the direct sum of the cokernels of these groupoids $\mathcal{S}(G^\alpha_*)$, which we now study separately.

First consider the groupoid $\mathcal{S}(G^0_*)$ induced by G_* over $\mathcal{S}(G^0)$. It is clear that $\mathcal{S}(G^0_*)$ is the groupoid with base $\mathcal{S}(G^0)$ defined by the homomorphism from F to $\mathcal{S}(G^0)$ induced by u. The cokernel whose existence we want to prove is therefore identified with $F\backslash\mathcal{S}(G^0)$. On the other hand, consider the groupoid

$$G^0_2 \;\overset{\ell'_0}{\underset{\ell'_2}{\xrightarrow[\ell'_1]{}}}\; G^0_1 \;\overset{\ell_0}{\underset{\ell_1}{\rightrightarrows}}\; G^0_0 = G^0$$

induced by $\mathcal{S}(G^0)$ over G^0. If we recall the construction of the inverse image groupoid, the object denoted $Y_0 \times_{X_0} X_1$ is none other than $F \times_A G^0$, so that G^0_1 is the inverse image of G^0 under the morphism $F \times_A G^0 \to \mathcal{S}(G^0)$ induced by λ. We claim that this inverse image is $F_0 \times_A G^0$, where we note by F_0 the inverse image of G^0 under u. In fact, since G^0 is clopen in G, its inverse image is also clopen in F. If F^β is a connected component of F contained in F^0, $F^\beta \times_A G^0$ is connected (Proposition 12.5.12) and $\lambda(F^\beta \times_A G^0)$ is contained in G^0. Conversely, if F^β is a connected component of F not contained in F_0, then the image $\lambda(F^\beta \times_A G^0)$ is still connected and contains $u(F^\beta)$. Since $u(F^\beta)$ is not contained in G^0, which is a connected component in G, we conclude that $\lambda(F^\beta \times_A G^0)$ does not meet G^0.

It then follows that the groupoid G^0_* induced by G_* over G^0 is the one defined by the homomorphism $F_0 \to G^0$ induced by u. As G^0 (and hence F_0, since u is *quasi-compact*) is of finite type over A, by 12.5.5.1, G^0_* possesses a cokernel which is none other than $F_0\backslash G^0$. We claim that $F_0\backslash G^0$ is identified with $F\backslash\mathcal{S}(G^0)$. In fact, the proof is similar to that of Lemma 12.4.13 (a). Consider the diagram

$$\mathcal{S}(G^0) \xleftarrow{v} F \times_A G^0 \xrightarrow{\mathrm{pr}_2} G^0$$

where v is the morphism induced by λ. As pr_2 possesses a section, pr_2 is a universally effective epimorphism (cf. [?] IV, 1.12) so that $F_0\backslash G^0$ coincides with the cokernel $\mathrm{coker}(v_0, v_1)$, where

$$V_2 \;\overset{v'_0}{\underset{v'_2}{\xrightarrow[v'_1]{}}}\; V_1 \;\overset{v_0}{\underset{v_1}{\rightrightarrows}}\; V = F \times_A G^0$$

is the inverse image of the groupoid G^0_* under pr_2 (cf. Proposition 12.4.8), which is also the inverse image of $\mathcal{S}(G^0)$ under the composition morphism

$$F \times_A G^0 \hookrightarrow F \times_A \mathcal{S}(G^0) \overset{\mathrm{pr}_2}{\to} \mathcal{S}(G^0).$$

Similarly, as v is faithfully flat and quasi-compact (hence a universally effective epimorphism), $F\backslash\mathcal{S}(G^0)$ coincides with the cokernel of the inverse image of $\mathcal{S}(G^0_*)$ under v. Now this inverse image is isomorphic to V_* by Example 12.4.7, so the canonical inclusion G^0_* in $\mathcal{S}(G^0_*)$ induces an isomorphism $F_0\backslash G^0 \cong F\backslash\mathcal{S}(G^0)$.

Finally, we notice that the construction of $F\backslash\mathcal{S}(G^0)$ commutes with finite and locally free base changes, because this is true for $F_0\backslash G^0$.

It remains to construct the cokernel of the groupoid $\mathcal{S}(G^\alpha_*)$ for an arbitrary connected component of G. By Proposition 12.5.7, if we choose a large enough local A-algebra A', finite and free over A, then $G^\alpha \otimes_A A'$ is the union of finitely many connected components $C^1, \ldots, G^n$ of $G \otimes_A A'$, each of which has a strict rational point. For each i, there then exists a right translation r_i of $G \otimes_A A'$ which sends $G^0 \otimes_A A'$ to C^i. This translation induces an isomorphism of the groupoid $\mathcal{S}(G^0_*) \otimes_A A'$ to $\mathcal{S}(C^i_*)$, so that the latter has a cokernel. As $\mathcal{S}(G^\alpha_*) \otimes_A A'$ is the direct sum of a certain number of the $\mathcal{S}(C^i_*)$, it therefore possesses a cokernel. This cokernel is a direct sum of a certain number of copies of $(F_0 \otimes_A A') \backslash (G^0 \otimes_A A')$, so any finite subset of it is contained in an affine open subset; moreover, the construction of this cokernel commutes with finite and free base changes. We then conclude as in 12.5.5.1 that this cokernel is of the form $Y \otimes_A A'$, where Y is a cokernel of $\mathcal{S}(G^\alpha_*)$.

We have therefore constructed $F \backslash G$ and shown that it is a direct sum of schemes of finite type over A. The other assertions of Theorem 12.5.33 reduce directly to those concerning the groupoids $\mathcal{S}(G^\alpha_*)$. As in §§12.4.3, the second assertion of (a) follows from the first one and (b), (c), so it suffices to prove (b), (c) and (d)(i). As A' is a local, finite and free A-algebra, the morphism $A \to A'$ is faithfully flat and of finite presentation, so according to ([**?**] VIII 3.1, 4.6, 5.4), it suffices to check the corresponding assertions for the groupoid $\mathcal{S}(G^\alpha_*) \otimes_A A'$. But this is isomorphic to the direct sum of a finite number of copies of $\mathcal{S}(G^0_*) \otimes_A A'$, so we are reduced to the groupoid $\mathcal{S}(G^0_*)$. For this, we can still modify the proof established in §§12.4.3, as we have already done in this paragraph. We have therefore completed the proof of §§12.4.3. We have therefore completed the proof of Theorem 12.5.33

With the notations of ([**?**], V §9.a), we now consider a condition under which the formation of the cokernel of X_* in $\mathbf{Sch}_{/S}$ commutes with a given base extension $\pi : S' \to S$. As the cokernel of X_* and X'_* are identified with the cokernel of the groupoid U_* and U'_* induced by X_* and X'_* over the quasi-sections U and U', respectively, we are then reduced to the case where $\mathbf{Sch}_{/S}$-groupoids verifying the hypothesis of Theorem 12.4.9. If we denote by Y the cokernel of U_*, $Y' = Y \times_S S'$ and Y_1 the cokernel of U'_*, we have seen in ([**?**], V §9.a) that the canonical morphism $Y_1 \to Y'$ is a homeomorphism (and even a universal homeomorphism); we can therefore identify Y_1 and Y' as topological spaces. If $p : U \to Y$ is the canonical morphism and if $p' : U' \to Y'$ is that induced by base change, we want the sequence of $\mathscr{O}_{Y'}$-modules

$$\mathscr{O}_{Y'} \longrightarrow p'_*(\mathscr{O}_{U'}) \rightrightarrows p'_*(u'_1)_*(\mathscr{O}_{U'_1}) = p'_*(u'_0)_*(\mathscr{O}_{U'_1}) \tag{12.5.4}$$

to be exact. As we are reduced to the hypothesis of Theorem 12.4.9, u_0 and u_1 are finite and locally free; and by Theorem 12.4.9, p is integral. Then p and $p \circ u_i$ are affine, hence separated and quasi-compact.

Therefore, if S' is flat over S, it follows from Corollary 10.1.21 that (12.5.4) is identified with the inverse image of the sequence

$$\mathscr{O}_Y \longrightarrow p_*(\mathscr{O}_U) \rightrightarrows p_*(u_1)_*(\mathscr{O}_{U_1}) = p_*(u_0)_*(\mathscr{O}_{U_1}) \tag{12.5.5}$$

which is exact. An analogous reasoning is applicable if the groupoid X_* possesses "locally" a quasi-section (cf. the proof of ([**?**], V 7.1)), and we therefore obtain the following:

Proposition 12.5.36. *The construction of the cokernel of X_* commutes with any flat base change if X_* possesses a quasi-section.*

Now consider the case of the groupoid G_* of Theorem 12.5.33 where we suppose that F and G are of finite type over A. By the proof in 12.5.5.1, there exists a local algebra A', finite and free over A, such that the groupoid $G_* \otimes_A A'$ possesses quasi-sections. For any extension $T \to \operatorname{Spec}(A)$, the seqence

$$(F'' \backslash G'') \times_{\operatorname{Spec}(A)} T \rightrightarrows (F' \backslash G') \times_{\operatorname{Spec}(A)} T \to (F \backslash G) \times_{\operatorname{Spec}(A)} T$$

induced from the diagram (12.5.3) is exact. If we suppose that T is flat over $\operatorname{Spec}(A)$, then $(F'' \backslash G'') \times_{\operatorname{Spec}(A)} T$ and $(F' \backslash G') \times_{\operatorname{Spec}(A)} T$ are identified, respectively, with the cokernel of the groupoids $(G_* \otimes_A A') \times_{\operatorname{Spec}(A)} T$ and $(G_* \otimes_A A') \times_{\operatorname{Spec}(A)} T$. The diagram induced from (12.5.3) by base change $T \to \operatorname{Spec}(A)$ then show that $(F \backslash G) \times_{\operatorname{Spec}(A)} T$ is identified with the cokernel of $G_* \times_{\operatorname{Spec}(A)} T$. An analogous reasoning is applicable in the general case (i.e. if G and F are locally of finite type over A), hence we obtain:

Proposition 12.5.37. *Under the hypothesis of Theorem 12.5.33, for any flat A-scheme T, $(F \backslash G) \times_{\operatorname{Spec}(A)} T$ is identified with the quotient of $G \times_{\operatorname{Spec}(A)} T$ by $F \times_{\operatorname{Spec}(A)} T$.*

12.5.5.3 Connections with §12.3 and consequences We recall the notations and hypothesis of Theorem 12.5.33. We then have the following commutative diagram

$$\begin{tikzcd}
F \times_A G \times_A G \arrow[r, "\mathrm{id}_F \times \mu_G"] \arrow[d, shift right, "\mathrm{pr}_2 \times \mathrm{id}_G"'] \arrow[d, shift left, "\lambda \times \mathrm{id}_G"] & F \times_A G \arrow[d, shift right, "\mathrm{pr}_2"'] \arrow[d, shift left, "\lambda"] \\
G \times_A G \arrow[r, "\mu_G"] \arrow[d, "p \times \mathrm{id}_G"'] & G \arrow[d, "p"] \\
(F\backslash G) \times_A G \arrow[r, dashed, "\rho"] & F\backslash G
\end{tikzcd}$$

which satisfies the equalities $\mathrm{pr}_2 \circ (\mathrm{id}_F \times \mu_G) = \mu_G \circ (\mathrm{pr}_2 \times \mathrm{id}_G)$ and $\lambda \circ (\mathrm{id}_F \times \mu_G) = \mu_G \circ (\lambda \times \mathrm{id}_G)$. Moreover, as G is supposed to be flat over A, the left column is exact by Proposition 12.5.36, so that μ_G induces a morphism of A-schemes:

$$\rho : (F\backslash G) \times_A G \to F\backslash G$$

This morphism ρ induces a right action of G on $F\backslash G$ as we immediately verify; moreover, the canonical morphism $G \to F\backslash G$ commutes to the right actions of G on G and $F\backslash G$. This proves the first assertion of (b') of Theorem 12.5.33. By Proposition 12.5.20, we conclude that the connected components of $X = F\backslash G$ are of finite type, irreducible, and all of the same dimension. To evaluate this dimension, we can assume that $A = k$ is an algebraically closed field. By 12.1.1.2, the stabilizer of the k-point $p(e)$ is represented by the fiber $H = p^{-1}(p(e))$, and since $F\backslash G$ is the quotient of G by F in the category of ringed spaces, this fiber has underlying space $u(F)$, and since $\ker u$ is finite, we therefore have $\dim(H) = \dim(u(F)) = \dim(F)$. By Proposition 12.5.20, we have therefore obtained that $\dim(X) = \dim(G) - \dim(F)$. This proves (b') of Theorem 12.5.33.

If the homomorphism of A-groups $u : F \to G$ is a monomorphism, the above arguments can also be deduced from the results of §12.3. In fact, the canonical morphism $p : G \to F\backslash G$ is faithfully flat and open by Theorem 12.5.33 (b) and (d)(i); it is then covering for the fpqc topology (Proposition 12.3.66) and we can apply Corollary 12.3.51 and Corollary 12.3.53. In particular, if we suppose in Theorem 12.5.33 that F is a normal subgroup of G, then there exists a unique A-group structure over $F\backslash G$ such that $p : G \to F\backslash G$ is a homomorphism of A-groups. This proves assertion (d)(iv) of Theorem 12.5.33.

We now review the assertions of §12.3 in the present situation. The statements of Proposition 12.3.56 and Proposition 12.3.60 are translated as follows: Let F and G be two groups locally of finite type and flat over A, F being a closed normal subgroup of G. The maps $H \mapsto\to F\backslash H$ and $H' \mapsto H' \times_{(F\backslash G)} G$ define a bijection correspondence between flat A-subgroups of G containing F and flat A-subgroups of $F\backslash G$. Under this bijection, closed (resp. normal) subgroups of G containing F correspond to closed (resp. normal) subgroups of $F\backslash G$. Moreover, since $p : G \to F\backslash G$ is faithfully flat, an A-subgroup H of G is flat over A if and only if $F\backslash H$ is.

The result of Proposition 12.3.58 implies the following: Let F, H and G be groups locally of finite type and flat over A; suppose that $F \subseteq H \subseteq G$, with F closed in G and normal in H. Then $F\backslash H$ acts freely on the left on $F\backslash G$, the quotient scheme $(F\backslash H)\backslash(F\backslash G)$ exists and we have a canonical isomorphism of scheme acted by G:

$$(F\backslash H)\backslash(F\backslash G) = H\backslash G.$$

Finally, Proposition 12.3.57 implies the following: Let F, H and G are groups locally of finite type and flat over A; suppose that F is contained in G and closed, normal in G, that H is contained in G and that $F \cap H$ is flat over A. Let $F \rtimes_A H$ be the semi-product A-group of F by H, $u : F \cap H \to F \rtimes_A H$ be the monomorphism defined by $x \mapsto (x^{-1}, x)$, and $F \cdot H$ be the quotient $(F \cap H)\backslash(F \rtimes_A H)$. Then there is a canonical isomorphism

$$F\backslash(F \cdot H) = (F \cap H)\backslash H.$$

12.5.5.4 Factorization for a quasi-compact group homomorphism Let $u : G \to H$ be a quasi-compact morphism between A-groups locally of finite type such that the kernel N of u is flat over A. In this case, by Remark 12.5.34, the quotient A-group $C = N\backslash G$ exists and the morphism $p : G \to C$ is faithfully flat and locally of finite presentation. On the other hand, by Proposition 12.3.54, u induces a monomorphism $i : C \to H$, which is quasi-compact (because u is quasi-compact and p is surjective, cf. Proposition 8.6.4), hence it is a closed immersion by Proposition 12.5.17. We then obtain the following proposition:

Proposition 12.5.38. *Let $u : G \to H$ be a quasi-compact morphism between A-groups locally of finite type such that $N = \ker u$ is flat over A. Then we have a factorization*

$$\begin{tikzcd} G \arrow[rr, "u"] \arrow[dr, "p"'] & & H \\ & N\backslash G \arrow[ur, "i"'] & \end{tikzcd}$$

where p is faithfuly flat, locally of finite presentation, and i is a closed immersion.

Suppose moreover that G is flat over A. Then by Remark 12.5.34, $C = N\backslash G$ is flat over A and hence the quotient $X = C\backslash H$ exists in $\mathbf{Sch}_{/A}$ and represents the fppf sheaf quotient $\widetilde{C\backslash H}$, and $q : H \to X$ is a C-torsor. Therefore, denoting by $e : \mathrm{Spec}(A) \to G$ the unit section of G, i induces an isomorphism of fppf sheaves between $\widetilde{C}$ and the fiber product of q and $q \circ e : \mathrm{Spec}(A) \to X$, which is represented by a closed subscheme H. We then conclude that i is an isomorphism from C to a closed subgroup K of H (equal to the stabilizer of the A-point $q \circ e$ of X). (This provides another proof of the fact that a quasi-compact monomorphism between A-groups locally of finite type is a closed immersion.)

If C is a normal subgroup of H, then the A-group $\overline{H} = C\backslash H$ is a cokernel in the category of A-groups of the morphism $u : G \to H$, and K is the kernel of the morphism $H \to \overline{H}$. If G and H are abelian A-groups, then K is the image of u in the category of abelian A-groups and $C = N\backslash G$ is the coimage of u. In view of the isomorphism $C \cong K$, we obtain:

Theorem 12.5.39. *Let k be a field. The category of abelian k-algebraic groups is abelian.*

Proof. In fact, since k is a field, $\ker u$ is always flat over k. □

We note that the full subcategory of affine abelian algebraic k-groups is thick. In fact, consider an exact sequence of abelian algebraic k-groups:

$$1 \longrightarrow N \longrightarrow G \longrightarrow G/N \longrightarrow 1$$

If G is affine, it is clear that so is N, and G/N is also affine by Chevalley theorem (cf. [?], VI_B, 11.17). Conversely, if N and G/N are affine, then G is also affine by (cf. [?], VI_B, 9.2(viii)). We therefore conclude that:

Corollary 12.5.40. *Let k be a field. The category of affine abelian k-algebraic groups is abelian.*

Remark 12.5.41. The category of affine abelian groups (not necessarily of finite type) is also abelian; this is deduced in ([?], VI_B, 11.17 et 11.18.2) (cf. [?] §III.3, 7.4), see also ([?], VII_B) for a proof using formal groups.

12.5.5.5 Complements on the identity component

Let G be a group locally of finite type and flat over a local Artinian ring A. We have seen in Proposition 12.5.13 that the identity component G^0 is an open and normal subgroup of G, hence also flat over A. By Theorem 12.5.33, $G^0\backslash G$ is a flat A-group. Moreover, as each component G^α of G is saturated for the equivalence relation defined by G^0, $G^0\backslash G$ is the direct sum of these $G^0\backslash G^\alpha$[31]. In particular, the identity component of $G^0\backslash G$ is none other than $G^0\backslash G^0 \cong \mathrm{Spec}(A)$, and hence $G^0\backslash G \to \mathrm{Spec}(A)$ is a local isomorphism at the identity. Therefore, $G^0\backslash G$ is étale over $\mathrm{Spec}(A)$ by Proposition 12.5.24. If $A = k$ is an algebraically closed field, then $G^0\backslash G$ is a constant k-group, operating simply transitively on the set of components connected of G (in particular, if G is of finite type, then $G^0\backslash G$ is finite). The group $G^0\backslash G$ is also denoted by $\pi_0(G)$, together with the canonical morphism $\pi : G \to \pi_0(G)$. This notation is justified by the following characterization:

Proposition 12.5.42. *Let A be a local Artinian ring and G be a group locally of finite type and flat over a local Artinian ring A. Then $(\pi_0(G), \pi)$ is the satisfies the following universal property: for any étale k-group H and any homomorphism $u : G \to H$, there exists a unique homomorphism $\tilde{u} : G$ such that $u = \tilde{u} \circ \pi$. Moreover, the fibers of π are the connected components of G, and its kernel is G^0.*

[31] The saturation $\mathcal{S}(G^\alpha)$ under this equivalence relation is the image of $G^\alpha \times G^0$ under multiplication, which is equal to G^α since it is connected (G^0 is geometrically connected) and contains G^0.

Proof. The universal property for $\pi_0(G)$ follows from Proposition 12.3.54, since any étale k-group H is discrete. By Theorem 12.5.33, the morphism $\pi : G \to \pi_0(G)$ is identified with the quotient map on the underlying space defined by the equivalence relation induced by G^0, so the fibers of π are connected components of G. Its kernel is the fiber at the unit section, which is G^0. $\square$

If K is a field extension of k, then by Proposition 12.5.37, the formatin of π_0 commutes with base change to K, so we have

$$\pi_0(G \otimes_A K) \cong \pi_0(G) \otimes_A K.$$

And for another group H locally of finite type and flat over A, we have

$$\pi_0(G \times H) \cong \pi_0(G) \times \pi_0(H).$$

Finally, recall that (Corollary 12.5.14) any connected component of G is of finite type over A, so for G to be of finite type over A, it is necessary and sufficient that the A-group $\pi_0(G)$ is finite.

12.5.5.6 The quotient $G_{\mathrm{red}}\backslash G$ Now let k be a perfect field and G be a k-group locally of finite type. We have remarked in §§12.5.1 that G_{red} is then a subgroup of G. Moreover, the equivalence class of the identity of G for the left action of G_{red} on G is the whole space G. Therefore, by Theorem 12.5.33, we obtain:

Proposition 12.5.43. *Let k be a perfect field and G be a k-group locally of finite type. Then the k-scheme $G_{\mathrm{red}}\backslash G$ is the spectrum of a local Artinian k-algebra, with residue field k.*

Proof. In fact, $G_{\mathrm{red}}\backslash G$ reduces to a point, with residue field k, and is of finite type over k. It is therefore the spectrum of a local Artinian k-algebra (Proposition 8.6.44). $\square$

Proposition 12.5.44. *Let $u : F \to G$ be a morphism between groups locally of finite type over a perfect field k. Then the following assertions are equivalent:*

(i) *u is flat.*

(ii) *$u^0 : F^0 \to G^0$ is dominant and the morphism $\tilde{u} : F_{\mathrm{red}}\backslash F \to G_{\mathrm{red}}\backslash G$ induced by u is flat.*

Proof. Consider the following commutative diagram

$$\begin{tikzcd} F \arrow[r, "p"] \arrow[d, "u"'] & F_{\mathrm{red}}\backslash F \arrow[d, "\tilde{u}"] \\ G \arrow[r, "q"] & G_{\mathrm{red}}\backslash G \end{tikzcd}$$

where p and q are canonical projections. By Theorem 12.5.33 (d), p and q are faithfully flat; therefore, if u is flat, then $q \circ u = \tilde{u} \circ p$ is flat, hence so is $\tilde{u}$. On the other hand, since F^0 and G^0 are open and irreducible in F and G, respectively, the induced morphism $u^0 : F^0 \to G^0$ is flat, hence dominant by ([?], 2.3.5).

Conversely, suppose that $\tilde{u}$ is flat and u^0 is dominant. As F^0 and G^0 are irreducible, $\tilde{u}^0$ sends the generic point ξ of F^0 to the generic point η of G^0[32]. Let R be the local Artinian k-algebra such that $G_{\mathrm{red}}\backslash G = \mathrm{Spec}(R)$, and $\mathfrak{m}$ be its maximal ideal. We have a local homomorphism of local rings $R \to \mathscr{O}_{G,\eta} \to \mathscr{O}_{F,\xi}$. Note that we have a Cartesian square:

$$\begin{tikzcd} G_{\mathrm{red}} \arrow[r] \arrow[d] & G \arrow[d, "q"] \\ \mathrm{Spec}(R/\mathfrak{m}) \arrow[r] & \mathrm{Spec}(R) \end{tikzcd}$$

and hence $\mathscr{O}_{G,\eta}/\mathfrak{m}\mathscr{O}_{G,\eta} \cong \mathscr{O}_{G_{\mathrm{red}},\eta} = \kappa(\eta)$, so that $\mathscr{O}_{F,\xi}/\mathfrak{m}\mathscr{O}_{F,\xi}$ is flat over $\mathscr{O}_{G,\eta}/\mathfrak{m}\mathscr{O}_{G,\eta}$. On the other hand, as q and $\tilde{u} \circ p$ are flat, G and F are flat over R. Therefore, by the local criterion of flatness (cf. [?], 11.3.10.2), $\mathscr{O}_{F,\xi}$ is flat over $\mathscr{O}_{G,\eta}$, that is, u is flat at the point ξ, so it is flat by Lemma 12.5.18. $\square$

[32] Let $y = \tilde{u}^0(\xi)$ be the image of ξ and consider the generalization $\eta \rightsquigarrow y$. By ([?], 2.3.5), there exists a generalization $x \rightsquigarrow \xi$ such that $\tilde{u}^0(x) = \eta$. Since ξ is generic in F^0, we must have $x = \xi$, so $\eta = \tilde{u}^0(\xi)$.

12.5.5.7 Complements on k-groups not necessarily of finite type We now generalise the results of this subsection to groups not necessarily of finite type over a fixed field k. Before this, let point out the following result.

Lemma 12.5.45. *Let G be a k-group. For any $x \in G$, there exists a point $u \in G \times G$ such that $\mu(u) = x$ and the two projections $p_1(u)$ and $p_2(u)$ are maximal points of G.*

Proof. Let $K = \kappa(x)$. As the projection $G_K \to G$ is flat, it sends maximal points to maximal points (cf. [?], 2.3.5), we are reduced to the case where x is rational. Then the translation ℓ_x (resp. r_x) gives a morphism $G \to G \times G, g \mapsto (\ell_x(g^{-1}), g)$ (resp. $g \mapsto (g, r_x(g^{-1}))$) which induces an isomorphism from G to $\mu^{-1}(x)$. Then, if u is a maximal point of $\mu^{-1}(x)$, $p_1(u)$ and $p_2(u)$ are maximal points of G (since p_1 and p_2 are flat). □

Corollary 12.5.46. *Let $f : G \to H$ be a quasi-compact and dominant morphism of k-groups.*

(a) *f is surjective.*

(b) *If H is reduced, f is faithfully flat.*

Proof. We denote by μ_G (resp. μ_G) the multiplication of H (resp. G). Let $h \in H$, then by Lemma 12.5.45, there exists $u \in H \times H$ such that $\mu_H(u) = h$ and that $\alpha = p_1(u)$ and $\beta = p_2(u)$ are maximal points of H. As f is quasi-compact and dominant, $f^{-1}(\alpha)$ and $f^{-1}(\beta)$ are nonempty (Proposition 8.6.5), and hence there exists $v \in G \times G$ such that $(f \times f)(v) = u$ (Proposition 8.3.27). Then $g = \mu_G(v)$ satisfies $f(g) = h$, so f is surjective.

Suppose morover that H is reduced. Then $\mathscr{O}_{H,\alpha}$ is a field, and we have $f^{-1}(\alpha) \neq \varnothing$, so f is flat at a point ξ of $f^{-1}(\alpha)$, hence flal by Lemma 12.5.18. □

Recall that a morphism $f : X \to Y$ is called **scheme-theoretic dominant** if it satisfies the following condition: for any open subset U of Y, if Z is a closed subscheme of U such that the morphism $f^{-1}(U) \to U$ factors through Z, then $Z = U$. If f is quasi-compact and quasi-separated, this is equivalent to saying that the scheme-theoretic image of X under f is Y (cf. Proposition 8.6.69).

Proposition 12.5.47. *Let $f : H \to G$ be a quasi-compact morphism of k-groups. Then the scheme-theoretic image of f[33] is a closed subgroup H' of G, and f factors into*

$$\begin{tikzcd} H \arrow[rr, "f"] \arrow[dr, "f'"'] & & G \\ & H' \arrow[ur, "i"'] & \end{tikzcd}$$

where f' is scheme-theoretic dominant, quasi-compact and surjective.

Proof. Denote by c_G and μ_G (resp. c_H and μ_G) the inversion and multiplication of G (resp. H). Then $c_G \circ f = f \circ c_H$ factors through the closed subscheme $i(H')$, whence $H' \subseteq c_G(H')$ and hence $H' = c_G(H')$ (since $c_H^2 = \mathrm{id}_H$). Similarly, as $f \circ \mu_H = \mu_G \circ (f \times f)$ factors through H', $f \times f$ factors through the closed subscheme $\mu_G^{-1}(H')$ of $G \times G$. On the other hand, as the formation of scheme-theoretic image commutes with flat base change (Corollary 10.1.21 and Proposition 8.6.69), the scheme-theoretic image of $f \times \mathrm{id}_H$ (resp. $\mathrm{id}_{H'} \times f$) is $H' \times H$ (resp. $H' \times H'$). Therefore, by the transitivity of scheme-theoretic image (Proposition 8.6.67), the scheme-theoretic image of $f \times f$ is $H' \times H'$, which is then contained in $\mu_G^{-1}(H')$, i.e. the restriction of μ_G to $H' \times H'$ factors through H'. This shows that H' is a closed subgroup of G. We denote by $i : H' \hookrightarrow G$ the inclusion. Then $f = i \circ f'$, where $f' : H \to H'$ is scheme-theoretic dominant and quasi-compact (since f is quasi-compact and i is separated). By Corollary 12.5.46, f' is surjective. □

We can now state the following theorem due to D. Perrin.

Theorem 12.5.48 (D. Perrin). *Let G be a quasi-compact k-group, then*

(a) *G is the projective limit of a filtered system (G_i) of k-groups of finite type (where the transition morphisms $u_{ij} : G_j \to G_i$ are affine for i large enough) and the morphisms $G \to G_i$ are faithfully flat (and affine for i large enough).*

[33] This scheme-theoretic image exists by Proposition 8.6.69, since f is separated (recall that G and H are both separated).

(b) *Let H be a closed sub-k-group of G. Then the fpqc quotient sheaf $\widetilde{G/H}$ is a k-scheme in the following two cases:*

 (i) *The immersion $H \to G$ is of finite presentation; in this case, G/H is of finite type over k.*

 (ii) *H is normal in G.*

Corollary 12.5.49. *Let $f : G \to H$ be a quasi-compact morphism of k-groups. If f is scheme-theoretic dominant, it is faithfully flat (this is the case if H is affine and the morphism $f^{\#} : \mathscr{O}(H) \to \mathscr{O}(G)$ is surjective).*

Proof. □

Corollary 12.5.50. *Let $u : G \to H$ be a quasi-compact morphism of k-groups and $N = \ker u$.*

(a) *The fpqc quotient sheaf $\widetilde{G/N}$ is represented by a k-group G/N, and u factors into*

$$\begin{tikzcd} G \arrow[rr, "u"] \arrow[dr, "p"'] & & G \\ & G/N \arrow[ur, "i"'] & \end{tikzcd}$$

where p is faithfully flat and i is a closed immersion.

(b) *If u is a monomorphism, it is a closed immersion; if u is scheme-theoretic dominant, then it is faithfully flat.*

Corollary 12.5.51. *The category of abelian quasi-compact k-groups is abelian.*

Corollary 12.5.52. *If $\operatorname{char}(k) = 0$, any k-group G is geometrically reduced.*

Proof. □

Corollary 12.5.53. *Let G be a quasi-compact k-group and suppose that k is algebraically closed.*

(a) *Let $f : G \to H$ be a faithfully flat morphism of k-groups, then the induced map $G(k) \to H(k)$ is surjective.*

(b) *The set of rational points is dense in G.*

12.6 Generalities on group schemes

12.6.1 Open properties for group morphisms

In this subsection, S denotes an arbitrary scheme; an S-group scheme is simply called an S-group. Given an S-group G, we denote by $e : S \to G$ the unit section, $c : G \to G$ the inversion and μ the multiplication morphism $G \times_S G \to G$. For any S-scheme X, we denote by π or π_X the structural morphism $X \to S$.

Given a property $\mathcal{P}$ for a morphism of S-schemes $u : X \to Y$, we say that $\mathcal{P}$ is stable under base change if, for any u verifying $\mathcal{P}$, so is the morphism $u_{(Y')}$ for any S-morphism $Y' \to Y$. We say that $\mathcal{P}$ is **local for the topology** $\mathcal{T}$ if $\mathcal{P}$ verifies the following conditions:

(a) $\mathcal{P}$ is stable under base change,

(b) if $\{Y_i \to Y\}$ is a covering family of S-morphisms for the topology $\mathcal{T}$ and each $u_{(Y_i)}$ verifies $\mathcal{P}$, then u verifies $\mathcal{P}$.

Proposition 12.6.1. *Let $\mathcal{P}$ be a property for a morphism of S-schemes which is local for the fpqc topology. Let $u : G \to H$ be a morphism of S-groups, and suppose that G is flat and universally open over S. Let W be the largest open subset of H over which u verifies the property $\mathcal{P}$ and put $V = u^{-1}(W)$. Then $U = \pi_G(V)$ is an open subset of S and V is an open subgroup of $G|_U$ (we use $G|_U$ to denote the U-group $\pi_G^{-1}(U)$ induced by G).*

Proof. The existence of a largets open subset W of H over which u verifies $\mathcal{P}$ follows from that fact that $\mathcal{P}$ is local for the Zariski topology. Since π_G universally open, $\pi_G(V)$ is an open subset of S, and it suffices to show that V is a subgroup of $G|_U$. We can then assume that $U = S$.

Let $G' = G \times_S V$, $H' = H \times_S V$, $V' = V \times_S V$, $W' = W \times_S V$ and $u' = u_{(V)}$; let W_1' be the largest open subset of H' over which u' satisfies $\mathcal{P}$. Since V is flat and universally open over S, so is H' over H, and Lemma 12.6.2 below shows that $W_1' = W'$. Consider then the automorphism a (resp. b) of the V-scheme G' (resp. H') defined by

$$a(g, v) = (g \cdot v^{-1}, v), \quad (\text{resp. } b(h, v) = (h \cdot u(v^{-1}), v)),$$

where $g \in G(T)$, $v \in G(T)$ and $h \in H(T)$, for any $T \to S$. It is clear that $u' \circ a = b \circ u'$, so W' is stable under b and hence V' is stable under a. But this then implies that V is a subgroup of G. □

Lemma 12.6.2. *Let $\mathcal{P}$ be a property for a morphism of S-schemes which is local for the fpqc topology. Consider the following Cartesian diagram of S-schemes:*

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ \downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Y \end{array}$$

where g is flat and open. Let W (resp. W_1') be the largest open subset of Y (resp. Y') over which f (resp. $f' = f_{(Y')}$) verifies $\mathcal{P}(u)$. Then $W_1' = W \times_Y Y'$.

Proof. Put $W' = W \times_Y Y'$; since $\mathcal{P}$ is table under base change, we have, we have $W' \subseteq W_1'$. As g is open, $W_1 = g(W_1')$ is open in Y. Put $V_1 = f^{-1}(W_1)$ and $V_1' = V_1 \times_{W_1} W_1'$; it is clear that $V_1' = f'^{-1}(W_1')$. Since g is flat and open, the morphism $W_1' \to W_1 = g(W_1')$ induced by g is faithfully flat and open, hence covering for the fpqc topology (cf. Proposition 12.3.66). Since the morphism $V_1' \to W_1'$ induced by f' verifies $\mathcal{P}$, so does the morphism $V_1 \to W_1$ induced by f, and hence $W_1 \subseteq W$. We then conclude that $W_1' \subseteq g^{-1}(W_1) \subseteq g^{-1}(W) = W'$, so $W' = W_1'$. □

Example 12.6.3. The following properties for a morphism are local for the fpqc topology: flat, (universally) open, (locally) of finite type, of finite presentation, quasi-finite (cf. [?], 2.5.1, 2.6.1 et 2.7.1), unramified, smooth, étale (EGA [?], 17.7.3).

Remark 12.6.4. We note that the proof of Proposition 12.6.1 in fact only uses basis changes by flat morphisms, so the proposition applies to any property satisfying condition (b) for the fpqc topology and stable under base changes by flat morphisms (eg. quasi-compact and dominant).

Of course, we can state an analogous proposition concerning the properties local for a topology $\mathcal{T}$ finer than the Zariski topology, with the required condition on G being then that π_G is universally open and covering for the $\mathcal{T}$ topology. In particular, if G is flat and locally of finite presentation over S, we have an analogous statement for properties stable under base changes by flat morphisms locally of finite presentation and satisfying condition (b) with respect to the fpqc topology (for example, regular, reduced, Cohen-Macaulay, etc., cf. [?], 6.8).

Proposition 12.6.5. *Let G and H be S-groups and $u : G \to H$ be a morphism of S-groups.*

(a) *Suppose that G or H is flat over S and G or H is locally of finite presentation over S, and let V be the largest open subset of G such that the restriction of u to V is flat and locally of finite presentation (resp. smooth, resp. étale). Then $U = \pi_V(V)$ is an open subset of S and V is an open subgroup of $G|_U$.*

(b) *Suppose that G or H is universally open over S, and let V be the largest open subset of G such that the restriction of u to V is universally open*[34]*. Then $U = \pi_V(V)$ is an open subset of S and V is an open subgroup of $G|_U$.*

[34] We note that V is the largest open subset contained in the set E of points of G over which u is universally open, but E may not be open, as the following example shows: let k be a field, $H = S = \mathrm{Spec}(A)$, where $A = k[x]$, and G be the S-group $\mathrm{Spec}(A[y]/(xy))$; then E equals to the unit section of G, which is not open.

Proof. To prove (a), we fist note that the restriction π_V of π_G to V is flat and locally of finite presentation: if π_G is flat (resp. locally of finite presentation), then so is its restriction π_V; on the other hand, if π_H is flat (resp. locally of finite presentation), then so is π_V since we have $\pi_V = \pi_H \circ u$. Hence, in any case the morphism π_V is flat and locally of finite presentation, whence universally open ([**?**], 2.4.6). Let $U = \pi_V(V)$, which is then open in S. It suffices to show that V is an open subgroup of $G|_U$, and we may assume that $U = S$.

Put $G' = G \times_S V$, $H' = H \times_S V$, $V' = V \times_S V$ and $u' = u \times_S \mathrm{id}_V$. Then since V is flat and locally of finite presentation over S, so is H' over H. By ([**?**], 17.4), V' is then the largest open subset of G' such that the restriction of u' to V' is flat and locally of finite presentation (resp. smooth, resp. étale). Consider the automorphisms a and b defind in the proof of Proposition 12.6.1; then V' is stable under a, hence V is a subgroup of G.

Now consider (b). The restriction π_V of π_G to V is a universally open morphism, either because it is the restriction of π_G or is the composition of u and π_V. Again, let $U = \pi_V(V)$, it suffices to show that V is an open subgroup of $G|_U$, for which we can assume that $U = S$.

Define G', H', V', u' as before; then $\pi_V : V \to S$ is surjective and universally open, hence so is $G' \to G$. Therefore, by ([**?**], 13.3.4(i) et (ii)), V' is the largest open subset of G' over which u' is universally open. The same arguments then apply to show that V' is stable under a, so V is a subgroup of G. □

Corollary 12.6.6. *Let G be an S-group and V be the largest open subset of G which is flat and locally of finite presentation (resp. smooth, étale, universally open) over S. Then $U = \pi_V(V)$ is open in S and V is an open subgroup of $G|_U$.*

Proof. It suffices to apply Proposition 12.6.5 to the trivial S-group H and where u is the morphism $G \to H$, bacause then π_H is an isomorphism and $\pi_G = \pi_H \circ u$. □

Corollary 12.6.7. *Let G be an S-group, if there exists a neighborhood V of the unit section with one (or more) of the following properties: V is flat and locally of finite presentation (resp. smooth, étale, universally open) over S, then there exists an open subgroup of G with the same properties.*

Proof. It suffices to apply Corollary 12.6.6, since with the notations of Proposition 12.6.5 we have $e(S) \subseteq V$, so $U = S$. □

Proposition 12.6.8. *Let $u : G \to H$ be a morphism of S-groups.*

(a) *Suppose that G (resp. H) is flat and of finite presentation (resp. flat and of finite type) over S at the unit section*[35]*. Then the sets*

$$U_{\mathrm{flat}} \supset U_{\mathrm{smooth}} \supset U_{\mathrm{t}}$$

formed by points $s \in S$ such that u_s is flat (resp. smooth, étale), are open in S.

If G (resp. H) is flat and locally of finite presentation (resp. flat and of finite type) over S, then the set V_{flat} (resp. V_{smooth}, V_{t}) of points of G where u is flat (resp. smooth, étale) is the inverse image of U_{flat} (resp. U_{smooth}, U_{t}) under π_G.

(b) *Suppose that for any $s \in S$, the fiber G_s is locally of finite type over $\kappa(s)$, and that u is locally of finite type (resp. locally of finite presentation) at the unit section of G. Then the sets*

$$U_{\mathrm{lqf}} \supset U_{\mathrm{u.ram}}$$

formed by $s \in S$ such that u_s is locally quasi-finite (resp. unramified), are open in S.

If u is also locally of finite type (resp. locally of finite presentation), then the set V_{lqf} (resp. $V_{\mathrm{u.ram}}$) of points of G where u is quasi-finite (resp. unramified) is the inverse image of U_{lqf} (resp. $U_{\mathrm{u.ram}}$) under π_G.

Proof. □

Corollary 12.6.9. *Let $u : G \to H$ be a morphism of S-groups which is radiciel and suppose that G (resp. H) is flat and locally of finite presentation (resp. flat and locally of finite type) over S. Then the set U of $s \in S$ such that u_s is an open immersion is open in S, and the restriction of u to U is an open immersion.*

[35] The hypothesis here is that G is flat and of finite presentation over S at the unit section of G *or* H is flat and of finite presentation over S at the unit section of G; and the conclusion is that the sets $U_{\mathrm{flat}} \supset U_{\mathrm{smooth}} \supset U_{\mathrm{t}}$ are then open in S. The same interpretation is valid in (b).

Proof. By Proposition 12.6.8 (a), the set U' of points $s \in S$ such that u_s is étale is open in S. Since u is radiciel, so is u_s for any $s \in S$, and by ([**?**], 17.9.1) we have $U = U'$, which shows that U is open. Finally, Proposition 12.6.8 (a), the restriction of u to U is étale; since u is radiciel, this restriction is an open immersion by ([**?**], 17.9.1). $\square$

Proposition 12.6.10. *Let G be an S-group. The following conditions are equivalent:*

(i) *G is unramified at the unit section.*

(ii) *The unit section is an open immersion.*

(iii) *G is of finite presentation over S at the unit section, and G_s is unramified over $\kappa(s)$ for any $s \in S$.*

If G is locally of finite presentation over S, then the above conditions are equivalent to:

(iv) *G is unramified over S.*

Proof. The implication (i)⇒(ii) follows from Theorem 11.1.20, since an injective local isomorphism is an open immersion. Conversely, if $e : S \to G$ is an open immersion, then the restriction of π_G to $e(S)$ is an isomorphism, so G is unramified at the points of $e(S)$. We also note that by Lemma 12.5.27, either (i) or (iii) implies that for any $s \in S$, G_s is locally of finite presentation over $\kappa(s)$. Then by Theorem 11.1.20, condition (i) is equivalent to that for any $s \in S$, G_s is unramified over $\kappa(s)$ at e_s, the identity of G_s, or by Corollary 12.5.26 that G_s is unramified over $\kappa(s)$ for any $s \in S$, hence (i)⇔(iii). Finally, if G is locally of finite presentation over S, then by Theorem 11.1.20, condition (iv) is equivalent to that G_s is unramified over $\kappa(s)$ for any $s \in S$, whence to (iii). $\square$

Corollary 12.6.11. *Let $u : G \to H$ be a morphism of S-groups. Suppose that for any $s \in S$, G_s is locally of finite type over $\kappa(s)$.*

(a) *If u is locally of finite type, the following conditions are equivalent:*

  (i) *u is locally quasi-finite;*
  (ii) *for any $s \in S$, $u_s : G_s \to H_s$ is locally quasi-finite;*
  (iii) $\ker u$ *is locally quasi-finite over S;*
  (iv) *the fibers of* $\ker u$ *are discrete.*

(b) *If u is locally of finite presentation, the following conditions are equivalent:*

  (v) *u is unramified;*
  (vi) *for any $s \in S$, $u_s : G_s \to H_s$ is unramified;*
  (vii) $\ker u$ *is unramified;*
  (viii) *the unit section $S \to$* $\ker u$ *is an open immersion.*

Proof. The equivalences (i)⇔(ii) and (v)⇔(vi) follows from Proposition 12.6.8 (b); also, since $\ker u$ is the inverse image of $e_H(S)$ under u, we see that (i)⇒(iii) and (v)⇒(vii).

For any point $s \in S$, denote by e_s the identity element of H_s. Then (iii) (resp. (vii)) implies that, for any $s \in S$,

$$(\ker u)_s = \ker u_s = u_s^{-1}(e_s)$$

is locally quasi-finite (resp. unramified) over $\kappa(s) = \kappa(e_s)$, hence that u_s is quasi-finite (resp. unramified) at the identity of G_s. By Proposition 12.5.24, u_s is locally quasi-finite (resp. unramified), so (iii)⇒(ii) and (vii)⇒(vi). Finally, (ii)⇔(iv) by Corollary 12.5.30 and (vii)⇔(viii) by Proposition 12.6.10. $\square$

Lemma 12.6.12. *Let $\pi : X \to S$ be a morphism admitting an S-section $e : S \to X$.*

(a) *If π is injective, it is integral.*

(b) *If π is locally of finite type and if for any $s \in S$, π_s is an isomorphism, then π is an isomorphism.*

Proof. In the situation of (a), π is then a bijective, hence a homeomorphism (and in particular affine). Since e is a surjective immersion, $e(S)$ is defined by a nilideal $\mathscr{I}$ of $\mathscr{O}_X$. Since e is a section of π, we have $\mathscr{O}_X = \mathscr{O}_S \oplus \mathscr{I}$ as $\mathscr{O}_S$-modules. The ideal $\mathscr{I}$ is clearly integral over $\mathscr{O}_S$ (since it is nilpotent), so $\mathscr{O}_X$ is integral over $\mathscr{O}_S$, and π is integral.

Now assume the condition in (b), in particular, π is bijective. As $\pi \circ e = \mathrm{id}_S$, e is then locally of finite presentation (Proposition 8.6.24), so $\mathscr{I}$ is of fintie type over $\mathscr{O}_X$. For any $s \in S$, we have $\mathscr{O}_{X_s} = \kappa(s) \oplus (\mathscr{I} \otimes_{\mathscr{O}_S} \kappa(s))$, and since e_s is an isomorphism, $\mathscr{I} \otimes_{\mathscr{O}_S} \kappa(s) = 0$, hence a fortiori $\mathscr{I} \otimes_{\mathscr{O}_X} \kappa(x) = 0$. It then follows from Nakayama's lemma that $\mathscr{I} = 0$, so π is an isomorphism. □

Proposition 12.6.13. *Let G be an S-group locally of finite type. Suppose that for any $s \in S$, G_s is the trivial $\kappa(s)$-group. Then G is the trivial S-group.*

Proof. By hypothesis, for any $s \in S$, the morphism π_s is an isomorphism. Since π has a section e, we conclude from Lemma 12.6.12 that π is an isomorphism from G to S, so G is the trivial S-group. □

12.6.2 Identity component of a group scheme

Given a subset A (resp. B) of an S-scheme X (resp. Y), by abusing the notation, we denote by $A \times_S B$ the subset of $X \times_S Y$ formed by points whose first projection belongs to A and second one belongs to B. If A is a subset of an S-group G, we say that A is **stable under the group law of** G if we have $A^{-1} \subseteq A$ and $\mu(A \times_S A) \subseteq A$.

Let G be an S-functor in groups. We say that G is **pointwise reprentable** if for any $s \in S$, the functor $G_s = G \otimes_S \kappa(s)$ is representable. In this case, we denote by G_s^0 the identity component of the $\kappa(s)$-group G_s. We define an S-subgroup functor of G, called the **identity component** of G and denoted by G^0, as follows: for any $T \to S$,

$$G^0(T) = \{u \in G(T) : \text{for any } s \in S,\, u_s(|T_s|) \subseteq |G_s^0|\}.$$

We therefore construct a functor $G \mapsto G^0$.

Remark 12.6.14. If G is an S-functor in groups that is pointwise reprentable, then $\mathfrak{Lie}(G^0/S) = \mathfrak{Lie}(G/S)$. In fact, for any S-scheme T, if I_T is the dual number scheme over T and $\varepsilon : T \to I_T$ is the zero section, then by Remark 12.2.14 we have

$$\mathfrak{Lie}(G/S)(T) = \{u \in G(I_T) : u \circ \varepsilon = e\}$$

where $e : T \to G$ denotes the composition of $T \to S$ and the unit section $S \to G$; similarly,

$$\begin{aligned}\mathfrak{Lie}(G^0/S)(T) &= \{u \in G^0(I_T) : u \circ \varepsilon = e\} \\ &= \{u \in G(I_T) : u \circ \varepsilon = e \text{ and } u_s(|(I_T)_s|) \subseteq |G_s^0| \text{ for any } s \in S\}.\end{aligned}$$

But for any $s \in S$, T_s and $(I_T)_s = I_{T_s}$ have the same underlying space, so for $u \in \mathfrak{Lie}(G/S)(T)$ we have $u_s(I_{T_s}) = e_s$, where e_s denotes the unit section of G_s, whence $u \in \mathfrak{Lie}(G^0/S)(T)$. Since this argument is valid for any T, we conclude that $\mathfrak{Lie}(G^0/S) = \mathfrak{Lie}(G/S)$.

Remark 12.6.15. If G and H are pointwise reprentable S-functors in groups, then:

(a) if $G \subseteq H$, then $G^0 \subseteq H^0$.

(b) if $G \subseteq H$ amd $H^0 \subseteq G$, then $G^0 = H^0$.

(c) if for any $s \in S$, G_s is locally of finite type over $\kappa(s)$, then $(G^0)_s$ is represented by the identity component of G_s, so G^0 is pointwise representable and we have $(G^0)^0 = G^0$.

Proposition 12.6.16. *Let G be a pointwise representable S-functor in groups and S' be an S-scheme. Then $(G \times_S S')^0 = G^0 \times_S S'$, so the formation of identity components commutes with base change.*

Proof. It suffices to see that, for any $s' \in S'$ with projection $s \in S$, we have

$$((G \times_S S') \otimes_{S'} \kappa(s'))^0 = (G_s \otimes_{\kappa(s)} \kappa(s'))^0 = G_s^0 \otimes_{\kappa(s)} \kappa(s');$$

this follows from Proposition 12.5.12. Note (for later use) that we did not use the group structure of G_s, but only the fact that G_s^0 has a rational point, namely e_s, so it is geometrically connected ([**?**], 4.5.14). □

Example 12.6.17. Let G be an S-group and denote by $|G^0|$ the subset of G which is the union of $|G_s^0|$ for $s \in S$. Then $|G^0|$ is a subset of G stable under the group law of G, and for any $S' \to S$, we have

$$G^0(S') = \{u \in G(S') : u(|S'|) \subseteq |G^0|\}.$$

If $|G^0|$ is an open subset of G, then it is clear that G^0 is represented by the open subscheme induced over $|G^0|$.

Proposition 12.6.18. *Let S be a quasi-compact and quasi-separated scheme, G be an S-group whose fibers are locally of finite type. Then there exists a quasi-compact open subset U of G containing $|G^0|$.*

Proof. The unit section being an immersion, $e(S)$ is a quasi-compact subspace of G, hence there exists a quasi-compact open subset V of G containing $e(S)$. Since S is quasi-separated and V is quasi-compact, V is then quasi-compact over S (Corollary 8.6.8 (iii)), hence $V \times_S V$ is quasi-compact over S. Put $V_s = V \cap G_s$ and $V_s^0 = V \cap G_s^0$, then V_s^0 is an open subset of G_s^0, dense in G_s^0 since G_s^0 is irreducible (Proposition 12.5.13), hence $V_s^0 \cdot V_s^0 = G_s^0$ (Proposition 12.5.2). This shows that $V_s \cdot V_s \supset |G^0|$, so $V \cdot V \supset |G^0|$. Finally, since $V \cdot V$ is quasi-compact, there exists a quasi-compact open subset U of G containing $V \cdot V$, and a fortiori $|G^0|$. □

Corollary 12.6.19. *Let G be an S-group whose fibers are locally of finite type and connected. Then G is quasi-compact over S.*

Proof. Since the assertion is local over S, we may assume that S is affine. By Proposition 12.6.18, there then exists a quasi-compact open subset U of G containing $|G^0| = G$, so G is quasi-compact, hence quasi-compact over the affine scheme S (Proposition 8.6.3 (iv)). □

Proposition 12.6.20. *Let G be an S-group locally of finite presentation.*

(a) *$|G^0|$ is ind-constructible in G.*

(b) *If G is quasi-separated over S and S is quasi-compact and quasi-separated, then $|G^0|$ is constructible.*

(c) *If G is quasi-separated over S, then $|G^0|$ is locally constructible.*

Proof. We first consider the assertion (a). Since $\pi : G \to S$ is locally of finite presentation, for any $s \in S$, there exists an open subset U of G containing $e(s)$ and an open subset V of S containing s such that $\pi(U) \subseteq V$ and that the morphism $\pi' : U \to V$ induced by π is of finite presentation. Then $T = e^{-1}(U)$ is an open subset of S and if we denote by $W = \pi'^{-1}(T)$ and $\pi'' = \pi'|_W$, then $\pi'' : W \to T$ is of finite presentation, and admits a section $e'' : T \to W$ induced by e.

For any $t \in T$, as G_t^0 is irreducible by Proposition 12.5.13, $W \cap G_t^0$ is dense in G_t^0, hence irreducible, hence connected: it is then the connected component of $\pi''^{-1}(t)$ containing $e''(t)$. It then follows from ([**?**], 9.7.12) that the union W^0 of the $W \cap G_t^0$, for $t \in T$, is locally constructible in W. On the other hand, it follows from Proposition 12.5.2 that $|G^0| \cap \pi^{-1}(T) = W^0 \cdot W^0$, i.e. $|G^0| \cap \pi^{-1}(T)$ is the image of $W \times_T W$ under the morphism $\mu'' : W \times_T W \to \pi^{-1}(T)$ induced by μ. As $W \times_T W$ (resp. $\pi^{-1}(T)$) is of finite presentation (resp. locally of finite presentation) over T, μ'' is locally of finite presentation and quasi-separated by Proposition 8.6.24 and Proposition 8.6.21; if $\pi^{-1}(T)$ is also quasi-separated over T, then μ'' is quasi-compact by Proposition 8.6.3, hence of finite presentation. As $W^0 \times_T W^0$ is locally constructible in $W \times_T W$ (since W^0 is in W), it follows from Chevalley's constructible theorem ([**?**], 1.8.4 et 1.9.5(viii)) that $|G^0| \cap \pi^{-1}(T)$ is ind-constructible in $\pi^{-1}(T)$ and is locally constructible in $\pi^{-1}(T)$ if G (and hence $\pi^{-1}(T)$) is quasi-separated over T. This proves the assertions of (a) and (c).

Now suppose that G is quasi-separated over S and S is quasi-compact and quasi-separated. Then by Proposition 12.6.18 there exists a quasi-compact open subset U of G containing $|G^0|$. Since G is quasi-separated over S, G is quasi-separated, hence the open subset U is retrocompact (Proposition 8.6.10), and it suffices to show that $|G^0|$ is constructible in U ([**?**], 0_{III}, 9.1.8). Moreover, U being quasi-compact, hence quasi-compact over S (Corollary 8.6.8 (iii)), and quasi-separated over S, the restriction of π to U is of finite presentation, hence by ([**?**], 9.7.12), $|G^0|$ is locally constructible in U, hence constructible in U, since U is quasi-compact and quasi-separated ([**?**], 1.8.1). This proves (b), and it follows that for any quasi-compact and quasi-separated open subset T of S (for example, any affine open of S), $|G^0| \cap \pi^{-1}(T)$ is constructible. □

Corollary 12.6.21. *Let S_0 be a quasi-compact and quasi-separated scheme, I be a directed set, $(\mathscr{A}_i)_{i \in I}$ be an inductive system of quasi-coherent $\mathscr{O}_{S_0}$-algebras, $\mathscr{A} = \varinjlim \mathscr{A}_i$, $S_i = \operatorname{Spec}(\mathscr{A}_i)$ for $i \in I$, and $S = \operatorname{Spec}(\mathscr{A})$. Let G be an S_0-group locally of finite presentation, then the canonical map $\varinjlim G^0(S_i) \to G^0(S)$ is bijective.*

Proof. Since G is locally of finite presentation over S, the canonical map $\varinjlim G(S_i) \to G(S)$ is bijective by ([**?**], 8.14.2(c)). It then ensures that the canonical map $\varinjlim G^0(S_i) \to G^0(S)$ is injective. To see that it is surjective, let $g \in G^0(S) \subseteq G(S)$. There exists $i \in I$ such that g factors through some $g_i \in G(S_i)$; by hypothesis, $g^{-1}(|G^0|) = S$, but by Proposition 12.6.20, $|G^0|$ is ind-constructible in G, so $g_i^{-1}(|G^0|)$ is in S_i. It follows from ([**?**], 8.3.4) that there exists an index $j \geq i$ such that $g_j^{-1}(|G^0|) = S_j$, where g_j is the map induced from g_i by base change $S_j \to S_i$. This shows that $g_j \in G^0(S_j)$, and since g_j maps to g, this proves our assertion. □

Proposition 12.6.22. *Let G be an S-group locally of finite presentation. Suppose that G^0 is representable, then the canonical morphism $i : G^0 \to G$ is an open immersion. Moreover, G^0 is quasi-compact over S.*

Proof. Since G^0 is a subfunctor of G, the morphism i is a monomorphism, hence is radiciel. By the definition of G^0, we verify immediately that i is formally étale (and note that $|G^0|$ is the image of i). Finally, it follows from the characterization ([**?**], 8.14.2(c)) on S-schemes locally of finite presentation and from Corollary 12.6.21 that, since G is locally of finite presentation over S, so is G^0 (Proposition 8.6.24). Hence i is locally of finite presentation; it is then a radiciel and étale morphism, hence an open immersion ([**?**], 17.9.1). Finally, the last assertion follows from Corollary 12.6.19. □

Theorem 12.6.23. *Let G be an S-group. The following conditions are equivalent:*

(i) *G is smooth over S at the unit section.*

(ii) *G is flat and locally of finite presentation over S at the unit section, and for any $s \in S$, G_s is smooth over $\kappa(s)$.*

(iii) *There exists an open subgroup G' of G which is smooth over S.*

(iv) *G^0 is representable by an open subscheme of G which is smooth over S,*

Proof. By Proposition 12.5.24 and Corollary 12.6.7, it is clear that (iv)⇒(iii)⇒(i) and (i) implies (ii) and (iii). We first show that (iii)⇒(iv). In the situation of (iii), Lemma 12.6.24 below shows that G' contains $|G^0$, and that $|G^0| = |G'^0|$. It then suffices to prove that $|G^0|$ is open in G, because we have as in Example 12.6.17 that G^0 is representable by the smooth open subscheme induced over $|G'^0| = |G^0|$. We can therefore suppose that $G' = G$.

To show that $|G^0|$ is open, it suffices to prove that any $s \in S$ possesses a neighborhood T in S such that $|G^0| \cap \pi^{-1}(T)$ is open in $\pi^{-1}(T)$. Let $s \in S$; since $G = G'$, π is then locally of finite presentation, so we can construct, as in the proof of Proposition 12.6.20, an open subset T of S containing s and an open subset W of G containing $e(S)$ such that the morphism $\pi'' : W \to T$ induced by π is of finite presentation and admits a section $e'' : T \to W$, which is induced by e. For any $t \in T$, $W \cap G_t^0$ is then the connected component of $\pi''^{-1}(t)$ containing $e''(t)$. Since π is smooth, so is π'' which is then smooth of finite presentation. Then by ([**?**], 15.6.5), the union W^0 of the $W \cap G_t^0$ for $t \in T$ is open in W.

On the other hand, by Proposition 12.5.2, we have $W^0 \cdot W^0 = |G^0| \cap \pi^{-1}(T)$, and it is necessary to show that it is open in $\pi^{-1}(T)$. We are then reduced to the case where $T = S$, and it rests to demonstrate that $W^0 \cdot W^0$ is open in G. Since π is universally open, so is μ, and since W^0 is open in G, so is $W^0 \cdot W^0 = \mu(W^0 \times_S W^0)$. □

Lemma 12.6.24. *Let G be an S-group whose fibers are locally of finite type. Then any open subgroup H of G contains $|G^0|$ and we have $|G^0| = |H^0|$.*

Proof. Let $s \in S$ and put $G'_s = H_s \cap G_s^0$. Then G'_s is an open subset of G_s^0, which is dense in G_s^0 since G_s^0 is irreducible (Proposition 12.5.13), hence $G'_s \cdot G'_s = G_s^0$ (Proposition 12.5.2). This shows that $G_s^0 = G'_s \cdot G'_s \subseteq H_s \cdot H_s = H_s$, so we have $G_s^0 = H_s^0$ for any $s \in S$, whence $|G^0| \subseteq H$ and $|H^0| = |G^0|$. □

Proposition 12.6.25. *Let $u : G \to H$ be a morphism between S-groups locally of finite presentation. If u is flat, the map $u^0 : |G^0| \to |H^0|$ induced from u is surjective. The converse is true if G is flat over S and H has reduced fibers.*

Proof. If u is flat, then for any $s \in S$, u_s is flat and locally of finite presentation, hence open by ([**?**], 2.4.6), so the morphism $u_s^0 : G_s^0 \to H_s^0$ is surjective by Corollary 12.5.28. The morphism $u^0 : |G^0| \to |H^0|$ is then surjective.

Conversely, suppose that the map $u^0 : |G^0| \to |H^0|$ is surjective, G is flat over S and H has reduced fibers. Then for any $s \in S$, the morphism $u_s^0 : G_s^0 \to H_s^0$ is surjective, hence flat at any point lying over the generic point η of H_s^0 (since $\mathcal{O}_{H_s^0,\eta}$ is a field), and u is flat by Proposition 12.5.24. The morphism u is therefore flat by fiber criterion of flatness ([**?**], 11.3.11). $\square$

12.6.3 Dimension of fibers

Proposition 12.6.26. *Let G be an S-scheme locally of finite type, endowed with an S-section e and such that for any $s \in S$, we have $\dim(G_s) = \dim_{e(s)}(G_s)$ (this is the case if G is an S-group, cf. Corollary 12.5.14).*

(a) *The function $s \mapsto \dim(G_s)$ is upper semi-continuous on S.*

(b) *If G is locally of finite presentation over S, then this function is locally constructible.*

Proof. Let $\pi : G \to S$ be the structural morphism. By Chevalley's semicontinuous theorem ([**?**], 13.1.3), the function $x \mapsto \dim_x(\pi^{-1}(\pi(x)))$ is upper semi-continuous on G. Now for any $s \in S$, we have

$$\dim(G_s) = \dim(\pi^{-1}(s)) = \dim_{e(s)}(\pi^{-1}(\pi(e(s))));$$

and since the function $s \mapsto e(s)$ is continuous on S, the composition functor $s \mapsto \dim(G_s)$ is upper semi-continuous on G.

Suppose that G is locally of finite presentation over S. To show that the functor $s \mapsto \dim(G_s)$ is locally constructible, we see from the preceding arguments that it suffices to show that the function $x \mapsto \dim_x(\pi^{-1}(\pi(x)))$ is locally constructible on G, which follows from ([**?**], 9.9.1). $\square$

Proposition 12.6.27. *Let $\pi : G \to S$ be an S-scheme locally of finite presentation, endowed with an S-section e and satisfies the following conditions:*

(a) *For any $s \in S$ and any $x \in G_s$, we have $\dim(G_s) = \dim_x(G_s)$ (or equivalently, the irreducible components of G_s have the same dimension).*

(b) *For any $s \in S$, if G_s^0 is the connected component of G_s containing $e(s)$, then G_s^0 is gemetrically irreducible.*

Let $s \in S$, the following conditions are equivalent:

(i) *G is universally open over S at points of G_s^0.*

(ii) *G is universally open over S at any point of a neighborhood of $e(s)$ in G_s^0.*

(iii) *The function $t \mapsto \dim(G_t)$ is constant in a neighborhood of s in S.*

(iv) *$|G^0|$ is universally open over S at points of G_s^0, that is, for any $S' \to S$, $s' \in S'$ lying over s and V an open neighborhood of g in $G' = G_{(S')}$, $\pi(V \cap |G'^0|)$ is an open neighborhood of s' in S'.*

Proof. It is clear that (i)⇒(ii). By ([**?**], 14.3.3.1(ii)), the set of points of G_s^0 where π_G is universally open is closed in G_s^0. Hence, as G_s^0 is irreducible, we have (ii)⇒(i).

We now prove that (i)⇒(iii). Let T be the set of $t \in S$ such that $\dim(G_t) = \dim(G_s)$. By Proposition 12.6.26, T is locally constructible, hence by ([**?**], 1.10.1), to show that T is a neighborhood of s, it suffices to show that any generalization s' of s belongs to T. Let ξ be the generic point of G_s^0 and U be an open subset of G containing ξ. As π_G is universally open at ξ, by ([**?**], 14.3.13), for any generalization s' of s, we have $\dim(U \cap G_{s'}) \geq \dim_x(U \cap G_s)$. In view of the hypothesis (a), this then implies $\dim(G_{s'}) \geq \dim(G_s)$. Now the function $s \mapsto \dim(G_s)$ is upper semi-continuous by Proposition 12.6.26, so we also have $\dim(G_{s'}) \leq \dim(G_s)$, whence $s' \in T$. This show the implication (i)⇒(iii).

It is clear that (iv)⇒(i); we prove that (iii)⇒(iv). As the dimension of fibers is stable under base change of fields and as the formation of $|G^0|$ commutes with base change (cf. the proof of Proposition 12.6.16), we can suppose that $S' = S$ and $s' = s$. Moreover, as any open subset V of G meeting G_s^0 contains the generic point η of G_s^0, we can suppose that $g = \eta$.

We can also suppose that S is affine. Let U be an affine open neighborhood of $e(s)$, which is then of finite presentation over S. Replacing S by $e^{-1}(U)$ and then U by $U \cap \pi^{-1}(S)$, we are reduced to the case where $\pi : U \to S$ is of finite presentation and admits a section $e : S \to U$. Then by ([**?**], 9.7.12), $|G^0| \cap U$ is constructible in U, and by replacing V by an affine open subset in $V \cap U$, we obtain that $|G^0| \cap V$ is constructible in V. As $\pi : V \to S$ is of finite presentation, $\pi(|G^0| \cap V)$ is then locally constructible in S, by Chevalley's constructibility theorem (cf. [**?**], 1.8.4).

Hence, by (cf. [**?**], 1.10.1), to show that $\pi(|G^0| \cap V)$ is an open neighborhood of s, it suffices to show that for any generalization t of s, there exists a generalization ξ of η belonging to $|G^0|$ (and hence to

$|G^0| \cap V)$. Now the generic point ξ of G_t^0 is a generalization of η. In fact, $e(s)$ belongs to the closure X of $\{\xi\}$ in G, hence, by Chevalley's semi-continuous theorem (cf. [?], 13.1.3), we have $\dim_{e(s)}(X_s) \geq \dim_\xi(X_t)$; on the other hand, the hypothesis of (iii) implies that $\dim_\xi(G_t^0) = \dim_{e(s)}(G_s)$. It then follows that an irreducible components of X_s containing $e(s)$ is equal to G_s^0, whence $\eta \in X$. This proves (ii)⇒(iii). □

Corollary 12.6.28. *Let G be an S-group locally of finite presentation. Then the function $s \mapsto \dim(G_s)$ is locally constant over S.*

Proof. This follows immediately from Proposition 12.6.27, because any flat morphism locally of finite presentation is universally open ([?], 2.4.6). □

Corollary 12.6.29. *Let G be a flat S-group locally of finite presentatio over S at the unit section. Consider the following conditions:*

(i) *G is smooth over S at the unit section.*

(ii) *For any $s \in S$, G_s is smooth over $\kappa(s)$, and the function $s \mapsto \dim(G_s)$ is locally constant on S.*

(iii) *For any $s \in S$, G_s is smooth over $\kappa(s)$, and there exists an neighborhood V of the unit section such that $\pi_V : V \to S$ is universally open.*

(iv) *For any $s \in S$, G_s^0 is smooth over $\kappa(s)$, and G^0 is representable by an open subgroup of G, which is universally open over S.*

Then we have the implications (i)⇒(ii)⇔(iii)⇔(iv). If we further suppose that S is reduced, then the above conditions are equivalent, and they imply that G^0 is smooth over S.

Proof. We first show that (i)⇒(ii). For any $x \in G$, we have $\dim_x(\pi^{-1}(\pi(x))) = \dim(\pi^{-1}(\pi(x)))$ by Corollary 12.5.14, so by ([?], 17.10.2), the function

$$x \mapsto \dim_x(\pi^{-1}(\pi(x))) = \dim(\pi^{-1}(\pi(x)))$$

is continuous in a neighborhood of the unit section. Hence the function $s \mapsto \dim(G_s)$ is continuous on S, hence locally constant on S. On the other hand, for any $s \in S$, G_s is smooth over $\kappa(s)$ by Proposition 12.5.8.

Now assume the conditions in (ii), we prove (iv). It suffices to show that $|G^0|$ is open in G, because then, by Example 12.6.17, G^0 is representable by the open subscheme induced over $|G^0|$, and the properties of G^0 given in (iv) follow from Proposition 12.6.27. Given a point $s \in S$, we construct W, T, π'', e'' and W^0 as in Proposition 12.6.20. Then by ([?], 15.6.7), W^0 is open in W. On the other hand, under the hypothesis of (ii), it follows from Proposition 12.6.27 that π is universally open at any point of W^0, hence μ is universally open at any point of $W^0 \times_S W^0$, and this shows that $W^0 \cdot W^0$ is open in G. We can then conclude (iv) as in the proof of Theorem 12.6.23.

It is clear that (iv)⇒(iii), and (iii)⇒(ii) follows from scheme group uo at identity component of s iff applied to V. Finally, suppose that (ii)–(iv) are verified. To show that G is smooth at the unit section, in view of Theorem 12.6.23, we may assume that $G = G^0$. Then G is of finite presentation over S by Corollary 12.6.35, thus π_G is of finite presentation, with geometrically integral fibers, whose dimension are locally constant over S. By ([?], 15.6.7), the morphism $G \times_S S_{\mathrm{red}} \to S\mathrm{red}$ induced from π_G is flat, hence π_G is flat if S is reduced. In this case, $G = G^0$ is smooth over S by Theorem 12.6.23. □

Example 12.6.30. If k is a field and $S = \mathrm{Spec}(k[\delta])$ with $\delta^2 = 0$, then the trivial k-group $G = \mathrm{Spec}(k)$ is an S-group verifying (ii)–(iv) of Corollary 12.6.29. But it is not flat, hence not smooth, over S.

12.6.4 Separation of groups and homogeneous spaces

Proposition 12.6.31. *For an S-group G to be separated, it is necessary and sufficient that the unit section of G is a closed immersion.*

Proof. This condition is necessary by Corollary 8.5.19; it is sufficient in view of the following diagram:

$$\begin{tikzcd} G \arrow[r, "\pi"] \arrow[d, "\Delta_{G/S}"'] & S \arrow[d, "e"] \\ G \times_S G \arrow[r, "\mu\circ(\mathrm{id}_G \times c)"] & G \end{tikzcd}$$

□

Proposition 12.6.32. *If S is discrete, any S-group is separated.*

Proof. In fact, S is then equal to $\coprod_{s\in S}\operatorname{Spec}(\mathscr{O}_{S,s})$, and by Proposition 8.5.30, it suffices to show that for any $s\in S$, $G\otimes_S\operatorname{Spec}(\mathscr{O}_{S,s})$ is separated, which is true since $\mathscr{O}_{S,s}$ is a local ring of zero dimension, □

Theorem 12.6.33. *Let S be a scheme, G be an S-group locally of finite presentation and universally open over S at a neighborhood of the unit section, X be an S-scheme acted by G such that the morphism*

$$\Phi : G\times_S X\to X\times_S X,\quad (g,x)\mapsto(gx,x)$$

is surjective. Suppose that for any $s\in S$:

(i) *there exists an open subscheme U of X, separated over S, such that U_s is dense in X_s,*

(ii) *the fiber X_s is locally of finite type over $\kappa(s)$.*

Then X is seprated over S.

Before proving this theorem, let's see some of its concequences. First, we consider the following weaker form of Theorem 12.6.33:

Corollary 12.6.34. *Let S be a scheme, G be an S-group locally of finite presentation and universally open over S at a neighborhood of the unit section, X be an S-scheme acted by G such that the morphism*

$$\Phi : G\times_S X\to X\times_S X,\quad (g,x)\mapsto(gx,x)$$

is surjective. Suppose that X has connected and locally of finite type fibers, then:

(a) *X is separated over S.*

(b) *If there exists an open subset V of X, quasi-compact over S and meets each fiber X_s, then X is quasi-compact over S.*

Proof. In fact, let $s\in S$ such that $X_s\neq\varnothing$. As the morphism $\Phi_s : G_s\times_{\kappa(s)}X_s\to X_s\times_{\kappa(s)}X_s$ induced by Φ is surjective and as X_s is connected, by Proposition 12.5.20, X_s is irreducible. Hence, if U is an affine open subset of X such that U_s is nonempty, then U_s is dense in X_s, and Theorem 12.6.33 implies that X is separated over S.

To prove (b), we may assume that S is affine. Then V is quasi-compact and, by Proposition 12.6.18, there exists a quasi-compact open subset U of G containing $|G^0|$. Let $s\in S$ be such that $X_s\neq\varnothing$, then X_s is irreducible by ([?], VI$_A$, 2.6.6), and hence quasi-compact by ([?], VI$_A$, 2.6.4(i')). It then follows that X_s is of finite type over $\kappa(s)$, hence Noetherian. As U_s contains G_s^0, the morphism $U_s\times_{\kappa(s)}V_s\to X_s,(g,x)\mapsto gx$ is then surjective by ([?], VI$_A$, 2.6.4(ii)), as U_s is retrocompact in X_s. Therefore the morphism $U\times_S V\to X$ is surjective and hence X is quasi-compact (since U and V are quasi-compact); as S is affine, hence separated, this implies that X is quasi-compact over S (cf. Corollary 8.6.8). □

Corollary 12.6.35. *Let S be a scheme, G be an S-group locally of finite presentation, with connected fibers, and universally open over S. Then G is separated and of finite presentation over S.*

Proof. By Corollary 12.6.19 and Corollary 12.6.34, G is quasi-compact and separated over S, whence of finite presentation over S. □

12.6.5 Representability of sub-functors in groups

Let X be an S-functor, G be an S-functor in groups, and u,v be two S-morphisms from X to G. The **transporter** of u to v, denoted by $\operatorname{Trans}(u,v)$, is the sub-$S$-functor of G defined by

$$\begin{aligned}\operatorname{Trans}(u,v)(S') &= \{g\in G(S') : \operatorname{Inn}(g)\circ u_{S'}=v_{S'}\}\\ &= \{g\in G(S') : g_{S''}u_{S''}(x)g_{S''}^{-1}=v_{S''}(x)\text{ for any } x\in X(S''),\ S''\to S'\}.\end{aligned}$$

In particular, $\operatorname{Trans}(u,u)$ is the sub-S-functor in groups of G, called the **centralizer** of u and denoted by $\operatorname{Centr}(u)$.

If X and Y are two sub-S-functors of G, the **transporter** of X to Y, denoted by $\operatorname{Trans}_G(X,Y)$, is the sub-$S$-functor of G defined by

$$\operatorname{Trans}_G(X,Y)(S')=\{g\in G(S') : \operatorname{Inn}(g)(X_{S'})\subseteq Y_{S'}\}$$

$$= \{g \in G(S') : g_{S''} X(S'') g_{S''}^{-1} \subseteq Y(S'') \text{ for any } S'' \to S'\}.$$

We also define the strict transporter of X to Y by

$$\begin{aligned}\mathrm{STrans}_G(X,Y)(S') &= \{g \in G(S') : \mathrm{Inn}(g)(X_{S'}) = Y_{S'}\}\\ &= \{g \in G(S') : g_{S''} X(S'') g_{S''}^{-1} = Y(S'') \text{ for any } S'' \to S'\}.\end{aligned}$$

Note that we have

$$\mathrm{STrans}_G(X,Y) = \mathrm{Trans}_G(X,Y) \cap c(\mathrm{Trans}_G(Y,X)),$$

where c is the inversion morphism of G.

Now let H be a sub-S-functor of G and $i : H \to G$ be the canonical S-morphism; the cantralizer and normalizer of H in G is then the sub-S-functors

$$Z_G(H) = \mathrm{Centr}(i) = \mathrm{Trans}(i,i), \quad N_G(H) = \mathrm{STrans}_G(H,H).$$

Finally, the center of G is the S-functor $Z(G) = Z_G(G) = \mathrm{Centr}(\mathrm{id}_G)$. From our definition, it is clear that these functors are stable under base change.

Definition 12.6.36. Let $f : X \to Y$ be a morphism of schemes. We say that f is **essentially free**, or that X is **essentially free over** S, if there exists an affine open covering (S_i) of S, for each i an S_i-scheme S_i' which is affine and faithfully flat over S_i, and an affine open covering (X_{ij}') of $X_i' = X \times_S S_i'$ such that for any i, j, the ring of X_{ij}' is a free module over that of S_i'.

Proposition 12.6.37. *Let S be a schems.*

(a) *If X is essentially free over S, it is flat over S. The converse is true if S is Artinian.*

(b) *If S is the spectrum of a field, any S-scheme is essentially free over S.*

(b) *If X is essentially free over S, then $X' = X \times_S S'$ is essentially free over S' for any $S' \to S$. The converse is true if $S' \to S$ is faithfully flat and quasi-compact.*

Proof. The proof is immediate, by noting that a flat module over a local Artinian ring is free. □

Example 12.6.38. Let G be a diagonalizable group over S (or more generally, which become diagonalizable after a suitable faithfully flat and quasi-compact base extension to any affine open subset of S, i.e. G is "of multiplicative type"). Then H is essentially free over S. In fact, if G is diagonalizable, it is affine over S and defined by a group alebra, which is free over S.

The introduction of the notion of essentially freeness is justified by the following theorem:

Theorem 12.6.39. *Let S be a sheme, Z be an essentially free S-scheme, and Y be a closed subscheme of Z. Consider the following functor*

$$F = \mathrm{Res}_{Z/S} Y : \mathbf{Sch}_{/S}^{\mathrm{op}} \to \mathbf{Set}, \quad F(S') = \Gamma(Y_{S'}/Z_{S'}) = \begin{cases} \varnothing & \text{if } Z_{S'} \neq Y_{S'}, \\ \{\mathrm{id}_{Z_{S'}}\} & \text{if } Z_{S'} = Y_{S'}. \end{cases}$$

Then F is representable by a closed subscheme T of S. If $Y \to Z$ is of finite presentation, so is $T \to S$.

Proof. We first note that F is a sheaf for the fpqc topology: as $F(S') = \varnothing$ or $\{*\}$ for any S', it reduces to verify that if (S_i) is an open covering of S (resp. $S' \to S$ is a faithfully flat and quasi-compact morphism), and if each $Y_{S_i} \to Z_{S_i}$ (resp. if $Y_{S'} \to Z_{S'}$) is an isomorphism, then so is $Y \to Z$. But this is clear (resp. follows from [?] VIII, 5.4 ou [?], 2.7.1). Moreover, by ([?] VIII, 1.9), faithfully flat and quasi-compact morphisms are effective descent for the fibre category of closed immersions. This allows us to limit ourselves to the case where $S_i' = S$ (with the notations of Definition 12.6.36).

Let (Z_j) be an affine open covering of Z such that $\mathscr{O}(Z_j)$ is a free module over $A = \mathscr{O}(S)$, and let $Y_j = Y \cap Z_j$ and $F_j = \mathrm{Res}_{Z_j/S} Y_j : \mathbf{Sch}_{/S}^{\mathrm{op}} \to \mathbf{Set}$. Each F_j is a sub-functor of the final functor, and we have evidently $F = \bigcap_j F_j$, which allows us to reduce to proving that each F_j is representable by a closed subscheme T_j of S (bacause then F is representable by the intersection of the T_j). We can hence suppose that Z is equally affine, $Z = \mathrm{Spec}(B)$, where B is a free A-module. Let $(e_\lambda)_{\lambda \in \Lambda}$ be a basis of B over A, and $\varphi_\lambda : B \to A$ be the "coordinate forms" relative to this basis, that is, $\varphi_\lambda(\sum_\mu a_\mu e_\mu) = e_\lambda$ for any $\lambda \in \Lambda$.

If $\mathfrak{J}$ is an ideal of B defining the subscheme Y of Z, we let $\mathfrak{I}$ be the ideal in A generated by the coordinates $\varphi_\lambda(x)$, with $x \in \mathfrak{J}$. Then the subscheme $T = V(\mathfrak{I}) = \operatorname{Spec}(A/\mathfrak{I})$ satisfies the desired condition: in fact, for any A-algebra C, since B is flat over A, we see that the morphism $B \otimes_A C \to (B/\mathfrak{J}) \otimes_A C$ is an isomorphism if and only if the image of $x \otimes 1$ in $B \otimes_A C$ is zero for $x \in \mathfrak{J}$, which is equivalent to that the kernel of $A \to C$ contains the ideal $\mathfrak{I}$. □

Example 12.6.40. Let S be a scheme and X, Y, Z be schemes over S.

(a) Consider a morphism $q : X \to \mathcal{H}om_S(Y, Z)$, i.e. X acts on Y with value in Z, and a subscheme Z' of Z, whence a monomorphism

$$\mathcal{H}om_S(Y, Z') \to \mathcal{H}om_S(Y, Z).$$

Let X' be the inverse image of $\mathcal{H}om_S(Y, Z')$ in X under the morphism q, which is the sub-functor of X such that $X'(T)$ is the set of $x \in X(T)$ such that $q(x) : Y_T \to Z_T$ factors through Z'_T. The functor X' can also be described as follows: we put $P = X \times_S Y$, and let P' the inverse image of Z' under $r : P \to Z$, the corresponding morphism of q; then there is an isomorphism of X-functors

$$X' \cong \operatorname{Res}_{P/X} P'.$$

We then conclude that if Y is essentially free over S and Z' is closed in Z, the subfunctor X' of X is representable by a closed subscheme of X. If $Z' \to Z$ is of finite presentation, then so is $X' \to X$.

(b) Let q_1, q_2 be two actions of X over Y with value in Z, i.e. two morphisms

$$q_1, q_2 : X \rightrightarrows \mathcal{H}om_S(Y, Z),$$

and put $X' = \ker(q_1, q_2)$, which is the sub-functor of X such that $X'(T)$ is the set of $x \in X(T)$ such that the two morphisms $q_1(x), q_2(x) : Y_T \rightrightarrows Z_T$ coincide. Now giving the morphisms q_1, q_2 is equivalent to a morphism

$$q : X \to \mathcal{H}om_S(Y, Z \times_S Z),$$

and hence, a morphism $r : X \times_S Y \to Z \times_S Z$. If $U = Z \times_S Z$, and U' is the diagonal subscheme of $Z \times_S Z$, then X' is none other than the inverse image of the sub-functor $\mathcal{H}om_S(Y, U')$ of $\mathcal{H}om_S(Y, U)$ under q, so it is isomorphism to $\operatorname{Res}_{P/X} P'$, where $P = X \times_S Y$ and P' is the inverse image of the diagonal under r, i.e. the kernel of $r_1, r_2 : X \times_S Y \rightrightarrows Z$. We therefore conclude that if Y is essentially free over S and Z is separted over S, then the sub-functor X' of X is representable by a closed subscheme of X. If $Z \to S$ is locally of finite type, then $X' \to X$ is of finite presentation.

(c) Consider a morphism $q : X \to \mathcal{H}om_S(Y, Y)$, i.e. X acts over Y. Let X' be the kernel of this morphism, which is defined such that $X'(T)$ is the set of $x \in X(T)$ such that $q(X) : Y_T \to Y_T$ is the identity. This functor is a special case of (b), as we can always introduce the morphism

$$q' : X \to \mathcal{H}om_S(Y, Y)$$

so that X acts trivially on Y. Therefore, if Y is essentially free and separated over S, the subfunctor X' of X is representable by a closed subscheme of X. If $Y \to S$ is locally of finite type, then $X' \to X$ is of finite presentation.

(d) Under the conditions of (c), consider the sub-functor Y' of invariants of Y, that is, $Y'(T)$ is the set of $y \in Y(T)$ such that the corresponding morphism $\bar{q}(y) : X_T \to Y_T$ is the constant T-morphism with value y (where $\bar{q} : Y \to \mathcal{H}om_S(X, Y)$ is the morphism induced by q). If q' is as in (c), with the corresponding morphisms

$$\bar{q}, \bar{q}' : Y \rightrightarrows \mathcal{H}om_S(X, Y)$$

then $Y' = \ker(\bar{q}, \bar{q}')$. Therefore, if X is essentially free over S and Y is separted over S, then the sub-functor Y' of invariants of Y is representable by a closed subscheme of Y. If $Y \to S$ is locally of finite type, then $Y' \to Y$ is of finite presentation.

The constructions of the type explained in the previous examples frequently occurs in group theory. For example, if G is a group S-scheme acting on the S-schema X:

$$q : G \to \mathcal{A}ut_S(X),$$

the kernel of q (the subgroup of G acting trivially on X) is an open subscheme of G provided that X is essentially free and separated over S, and the sub-object X^G of invariants of X is a closed subscheme of X provided that G is essentially free over S and X is separted over S.

Let $u, v : X \to G$ be morphisms of schemes and consider the sub-functor $\mathrm{Trans}(u, v)$ of G whose value at an S-scheme T is the set of $g \in G(T)$ such that $\mathrm{Inn}(g)(u_T) = v_T$. If we consider the morphism $G \to \mathcal{H}om_S(X, G)$ defined by the composition of the inner automorphism of G and u and by v, then we are in the situation of Example 12.6.40 (b). Therefore, if X is essentially free over S and G is separated over S, then $\mathrm{Trans}(u, v)$ is a closed subscheme of G.

Let Y, Z be subschemes of X and consider the sub-functor $\mathrm{Trans}_G(Y, Z)$ of G, whose value at an S-scheme T is the set $g \in G(T)$ such that the corresponding automorphism of X_T satisfies $g(Y_T) \subseteq Z_T$, i.e. such that the induced morphism $Y_T \to X_T$ factors through Z_T. Therefore, by Example 12.6.40 (a), if Y is essentially free over S and Z is closed in X, then $\mathrm{Trans}_G(Y, Z)$ is a closed subscheme of G.

We can also consider the strict transporer of Y to Z, whose value at an S-scheme T is the set $g \in G(T)$ such that $g(Y_T) = Z_T$. Since we have $\mathrm{STrans}_G(Y, Z) = \mathrm{Trans}_G(Y, Z) \cap c(\mathrm{Trans}_G(Z, Y))$, we obtain the same result concerning $\mathrm{STrans}_G(Y, Z)$. That is, it is a closed subscheme of X if Y is essentially free over S and Z is closed in X.

A particular important case is $X = G$, with G act on its self by inner automorphisms. If H is a subscheme of G, the transporter of H to H is then the normalizer $N_G(H)$ of H. Hence, if H is a closed subgroup of G and essentially free over S, then $N_G(H)$ is representable by a closed subgroup of G.

Finally, let X be a subscheme of G; then the centralizer $Z_G(X)$ in G is the sun-functor in groups of G defined by the procedure of Example 12.6.40 (d), where we act Z on G by inner automorphisms. Therefore, if Z is essentially free over S and G is separated over S, $Z_G(X)$ is a closed subgroup of G. In particular, if G is essentially free and separated over S, then its center $Z(G)$ is a closed subgroup of G.

If S is the spectrum of a field, then any scheme over S is essentially free over S; also, any k-group is separted over k, so we obtain the following corollary:

Corollary 12.6.41. *Let G be a group scheme over a field k and Y, Y' be two subscheme of G. Then*

(a) *The centralizer of Y in G is a closed subgroup of G.*

(a′) *More generally, for any $u, v : X \to G$ morphisms of schemes, the transporter $\mathrm{Trans}(u, v)$ is representable by a closed subscheme of G.*

(b) *If Y is closed, the transporter $\mathrm{Trans}_G(Y', Y)$ is a closed subscheme of G. If Y' is also closed, we have the same conclusion for $\mathrm{STrans}_G(Y', Y)$.*

(c) *For any subgroup H of G, $N_G(H)$ is a closed subgroup of G.*

Corollary 12.6.42. *Let G be a group scheme over a field k and X be a separated scheme over S. Then the sub-functor X^G of invariants of X under G is represented by a closed subscheme of X. The morphism $X^G \to X$ is of finite presentation over S if X is of finite type over S.*

Proof. This follows from Example 12.6.40 (d) since any scheme is essentially free over k. □

Corollary 12.6.43. *Let k be a field and G be a connected algebraic k-group. Then $Z(G)$ is representable by a closed subscheme of G, and $G/Z(G)$ is an affine algebraic k-group.*

Proof. The first assertion is contained in Corollary 12.6.41, but we will see that it also follows from the proof of the second assertion. In fact, G acts by the adjoint representation on the finite dimensional k-vector space $V_n = \mathfrak{m}_e^n/\mathfrak{m}_e^{n+1}$ (where $\mathfrak{m}_e$ is the maximal ideal of $\mathscr{O}_{G,e}$), and denote by K_n the kernel of $\rho_n : G \to \mathrm{GL}(V_n)$. By Proposition 12.5.17, ρ_n induces a closed immersion $G/K_n \hookrightarrow \mathrm{GL}(V_n)$, hence each G/K_n is affine (a linear algebraic group). As G is Noetherian, the intersection K of K_n is equal to one of K_n, so G/K is affine.

On the other hand, let Z be the center of G; it is clear that $Z \subseteq K$. Let $\widehat{\mathscr{O}}_{G,e}$ be the completion of $\mathscr{O}_{G,e}$ for the $\mathfrak{m}_e$-adic topology and $\widehat{S}$ be the spectrum of $\widehat{\mathscr{O}}_{G,e}$ (resp. $S = \mathrm{Spec}(\mathscr{O}_{G,e})$). As $\widehat{S} \to S$ is faithfully flat and the two morphisms

$$K \times_k S \to S, \quad (g, x) \mapsto gxg^{-1} \quad (\text{resp.} \quad (g, x) \mapsto x)$$

coincide after base change $\widehat{S} \to S$, they coincide, i.e. K acts trivially on $\mathscr{O}_{G,e}$. Now by Example 12.6.40 (d), the sub-object G^K of invariants of G under K (which is none other than $Z_G(K)$) is a closed subscheme of G, hence is defined by a qausi-coherent ideal $\mathscr{I}$ of $\mathscr{O}_G$. As G^K dominates the subscheme $S = \mathrm{Spec}(\mathscr{O}_{G,e})$

and $\mathscr{I}$ is of finite type (G is Noetherian), there exists an open neighborhood U of e such that $\mathscr{I}|_U = 0$. Then the subgroup G^K contains U and hence also $U \cdot U$, which is equal to G since G is irreducible (Proposition 12.5.2). Therefore $Z_G(K) = G$, whence $K \subseteq Z$ and $Z = K$. □

12.6.6 Sheaf quotients

Definition 12.6.44. Given a monomorphism $u : H \to G$ of S-groups, we denote by G/H (resp. $H\backslash G$) the sheaf (for the fpqc topology) of G by the equivalence relation defined by the monomorphism

$$G \times_S H \xrightarrow{\delta\circ(\mathrm{id}_G \times u)} G \times_S G \quad (\text{resp.} \quad H \times_S G \xrightarrow{\gamma\circ(\mathrm{id}_G \times u)} G \times_S G)$$

where δ (resp. γ) is the automorphism of $G \times_S G$ defined by $(g,h) \mapsto (g,gh)$ (resp. $(h,g) \mapsto (hg,g)$) for $g,h \in G(T)$.

Proposition 12.6.45. *Let $u : H \to G$ be a monomorphism of S-groups. Suppose that G/H is represented by an S-scheme G', then:*

(i) *The canonical morphism $p : G \to G'$ is covering for the fpqc topology.*

(ii) *If we put $e' = p \circ e$ (this is called the unit section of G'), the following diagrams are Cartesian:*

$$\begin{tikzcd} G \times_S H \arrow[r, "\delta\circ(\mathrm{id}_G \times u)"] \arrow[d, "\mathrm{pr}_1"'] & G \arrow[d, "p"] \\ G \arrow[r, "p"] & G' \end{tikzcd} \qquad \begin{tikzcd} H \arrow[r, "u"] \arrow[d, "\pi_H"'] & G \arrow[d, "p"] \\ S \arrow[r, "e'"] & G' \end{tikzcd}$$

In particular, u is an immersion.

(iii) *There exists a unique S-scheme structure on G' acted by G such that p is a morphism of S-schemes acted by G.*

(iv) *If we suppose that H is normal in G, then there exists a unique S-group structure on G' such that p is a morphism of S-groups.*

(v) *Let S_0 be an S-scheme and put $G_0 = G \times_S S_0$, $H_0 = H \times_S S_0$. Then G_0/H_0 is representable by $G'_0 = G' \times_S S_0$.*

(vi) *Let $\mathcal{P}$ be a property for S-morphisms. Suppose that $\mathcal{P}$ is stable under base change, then if $p : G \to G'$ verifies $\mathcal{P}$, so does the structural morphism $\pi_H : H \to S$.*

(vii) *Let $\mathcal{P}$ be a property for S-morphisms. Suppose that $\mathcal{P}$ is local for the fpqc topology. Then, for the morphism $p : G \to G'$ to verify $\mathcal{P}$, it is necessary and sufficient that so deos $\pi_H : H \to S$.*

(viii) *Let $\mathcal{P}$ be a property for S-morphisms. Suppose that $\mathcal{P}$ is local for the fpqc topology and stable under composition. Then, if the structural morphisms $H \to S$ and $G' \to S$ verify $\mathcal{P}$, so does the structural morphism $G \to S$.*

(ix) *If G is reduced, then G' is reduced.*

(x) *For G' to be separated over S, it is necessary and sufficient that u is a closed immersion.*

(xi) *For H to be flat over S, it is necessary and sufficient that p is a flat morphism.*

(xii) *For H to be faithfully flat and locally of finite presentation over S, it is necessary and sufficient that p is faithfully flat and localy of finite presentation. In this case, for G' to be locally of finite presentation (resp. locally of finite type, of finite type, unramified, smooth, étale, locally quasi-finite, quasi-finite) over S, it is sufficient that so is G over S.*

(xiii) *Suppose that H is flat and of finite presentation over S. Then p is faithfully flat and of finite presentation. Moreover, for G to be of finite presentation over S, it is necessary and sufficient that so is G'.*

Remark 12.6.46. Under the general hypothesis of Proposition 12.6.45, if we suppose that H is flat and locally of finite presentation over S, then p is covering for the fppf topology by Proposition 12.6.45 (vii), so assertions (vii) and (viii) of Proposition 12.6.45 can be extended to properties local for the fppf topology.

Remark 12.6.47. The question that whether a quotient G/H is representable is often tricky. In general, to be able to affirm that the quotient G/H is representable, it is not sufficient to suppose that G and H are of finite presentation on S and H flat on S. In fact, suppose further that G is smooth with connected fibres. In this case, if G/H is a scheme, then it is separated by Corollary 12.6.34, and hence $H \hookrightarrow G$ is a closed immersion according to Proposition 12.6.45 (ix). Therefore, if H is not closed in G, then G/H is not representable.

To obtain a counter-example, we can choose S to be the spectrum of a DVR (so S consists of a generic point and a special point), and put $G = \mathbb{G}_{m,S}$. Consider on the other hand an integer $n > 1$ which is invertible over S; then $\boldsymbol{\mu}_n = \ker(G \xrightarrow{n} G)$ is a closed subgroup of G which is étale over S (cf. [?] VII_A, 8.4). Let H be the open subgroup of $\boldsymbol{\mu}_n$ obtaine by deleting from $\boldsymbol{\mu}_n$ the subset of the special fiber of $\boldsymbol{\mu}_n$ complementray to the origin. Then H is not closed in G, hence G/H is not representable.

12.6.7 Affine group schemes

12.7 Diagonalizable groups

12.7.1 Duality for group schemes

Let $\mathcal{C}$ be a category, which we identity as a full subcategory of $\widehat{\mathcal{C}} = \mathrm{PSh}(\mathcal{C})$. Let I be an abelina group functor over $\mathcal{C}$, i.e. an object of $\widehat{\mathcal{C}}$ endowed with an abelian group structure. For any $F \in \mathrm{Ob}(\widehat{\mathcal{C}})$, the object $\mathcal{H}om(F, I)$ is then endowed with an abelian group structure, induced by that of I. For any group G in $\widehat{\mathcal{C}}$, let $D(G) = \mathcal{H}om_{\mathbf{Grp}}(G, I)$ be the sub-object of $\mathcal{H}om(G, I)$ defined, for any $S \in \mathrm{Ob}(\mathcal{C})$, by:

$$D(G)(S) = \mathrm{Hom}_{S\text{-}\mathbf{Grp}}(G_S, I_S), \tag{12.7.1}$$

where $G_S = G \times S$ and $I_S = I \times S$ are considered as S-groups, i.e. as groups in $\widehat{\mathcal{C}}_{/S}$. Then $D(G)$ is a sub-$\widehat{\mathcal{C}}$-group of $\mathcal{H}om(G, I)$. In this way, we obtain a contravariant functor D from the category of $\widehat{\mathcal{C}}$-groups to the category of abelian $\widehat{\mathcal{C}}$-groups.

The right hand side of (12.7.1) can also be interpreted as the subset of $\mathrm{Hom}(G \times S, I)$ formed by morphisms $G \times S \to I$ which is "multiplicative relative to the first argument G". Moreover, the formula (12.7.1) also valid when S is any object of $\widehat{\mathcal{C}}$, not necessarily in $\mathcal{C}$.

If now we let S be a group in $\widehat{\mathcal{C}}$, which we denote by G', then in the first number of (12.7.1), we can distinguish the subset $\mathrm{Hom}_{\mathbf{Grp}}(G', D(G))$ formed by morphisms which respect the group structures of G' and $D(G)$. It then corresponds to the subset of $\mathrm{Hom}(G \times G', I)$ formed by morphisms which are multiplicative relative to the first and second arguments, which will then be called **bilinear morphisms** from $G \times G'$ to I, or pairings of G and G' with values in I. We thus obtain

$$\mathrm{Hom}_{\mathbf{Grp}}(G', D(G)) \xrightarrow{\sim} \mathrm{Bil}(G \times G', I), \tag{12.7.2}$$

which is functorial on the couple (G, G'). As the second member of (12.7.2) on G and G', we then deduce a functorial bijection

$$\mathrm{Hom}_{\mathbf{Grp}}(G', D(G)) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{Grp}}(G, D(G')). \tag{12.7.3}$$

In other words, a group morphism $G' \to D(G)$ is equivalent to a morphism $G' \to D(G)$, which are both equivalent to giving a bilinear form $G \times G' \to I$. Applying this to the special case where $G' = D(G)$ and to the identity morphism of $D(G)$, we thus obtain a canonical homomorphism

$$G \to D(D(G)). \tag{12.7.4}$$

We say that G is **reflexive** (relative to I) if this homomorphism is an isomorphism. We note that this implies in particular that G is abelian. We thus obtain:

Proposition 12.7.1. *The functor D induces an anti-equivalence on the category of reflexive $\widehat{\mathcal{C}}$-groups.*

In particular, if G, H are two reflexive groups, D induces an isomorphism

$$\mathrm{Hom}_{\mathbf{Grp}}(G, H) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{Grp}}(D(H), D(G))$$

(in fact it is sufficient that H is reflexive, as can be seen from the formula (12.7.3)).

As usual, we say that a $\mathcal{C}$-group is reflexive if it is reflexive as an $\widehat{\mathcal{C}}$-group. We thus obtain similarly an anti-equivalence on the category of reflexive $\mathcal{C}$-groups.

Remark 12.7.2. We note that the formation of $D(G)$ commutes with base changes, which therefore transforms reflexive groups into reflective groups.

We are interesting in particular in the case where $\mathcal{C} = \mathbf{Sch}_{/S}$, the category of schemes over S, and $I = \mathbb{G}_{m,S}$, the multiplicative grou over S. For any (ordinary) group M, we consider the constant S-group M_S. We also see that for any group scheme G over S, we have a canonical isomorphism (functorial on M and G and compatible with base changes):

$$\mathrm{Hom}_{S\text{-}\mathbf{Grp}}(M_S, G) = \mathrm{Hom}_{\mathbf{Grp}}(M, G(S))$$

Apply this to $G = I = \mathbb{G}_{m,S}$ and to a scheme S' over S, we then obtain a functorial isomorphism

$$D(M_S)(S') \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{Grp}}(M, \mathbb{G}_{m,S}(S')). \tag{12.7.5}$$

We thus recover the functor $D_S(M)$ considered in 12.1.2.4, which is representable for M abelian, and in this case,

$$D_S(M) = D(M_S) = \mathrm{Spec}(\mathscr{O}_S[M]),$$

where $\mathscr{O}_S[M]$ is the group algebra of M with coefficients in $\mathscr{O}_S$ (note that by definition $D(M)$ is unchanged if we replace M by its abelianization, so we can always assume that M is abelian).

Definition 12.7.3. A group scheme G over S is called **diagonalizable** if it is isomorphic to a scheme of the form $D_S(M) = D(M_S) = \mathcal{H}om_{S\text{-}\mathbf{Grp}}(M_S, \mathbb{G}_{m,S})$ for an abelian group M. We say that G is **locally diagonalizable** if any point of S admits an open neighborhood U such that $G|_U$ is doagonalizable.

Theorem 12.7.4. *Let Γ be a constant abelian group scheme over S, i.e. isomorphic to a group scheme of the form M_S, where M is an ordinary abelian group. Then Γ is reflexive, i.e. the canonical homomorphism*

$$\Gamma \to D(D(\Gamma))$$

is an isomorphism. The diagonalizable group $D(M_S)$ is hence also reflexive.

In view of the definitions, this theorem follows from the following result (applied to any scheme S' over S):

Corollary 12.7.5. *Let $G = D(M_S)$, then any homomorphism of S-groups*

$$\chi : G \to \mathbb{G}_{m,S}$$

is defined by a uniquely determined section of M_S over S, i.e. by a uniquely determined locally constant map from S to M.

Proof. We first recall that $\Gamma(M_S/S)$ is identified with the set of locally constant maps from S to M (cf. **[?]** I, 1.8). As by definition we have

$$\mathbb{G}_{m,S} = \mathrm{GL}(1)_S = \mathcal{A}ut_{\mathscr{O}_S}(\mathscr{O}_S),$$

we see that giving a group homomorphism $\chi : G \to \mathbb{G}_{m,S}$ is equivalent to giving a $\mathscr{O}_S[G]$-module structure over $\mathscr{O}_S$, which is compatible with the natural $\mathscr{O}_S$-module structure of $\mathscr{O}_S$. By Proposition 12.1.34, this amounts to giving a graduation of type M over $\mathscr{O}_S$, i.e. a decomposition of $\mathscr{O}_S$ into modules $\mathscr{L}_m$ ($m \in M$). But in view of **??**, a direct factor of a locally free module of finite type is locally free of finite type, hence each $\mathscr{L}_m$ is, in a neighborhood of each point of S, either zero of free of rank 1, and in this case is identified with $\mathscr{O}_S$ in this neighborhood. Let S_m be the open subset of S formed by points where $\mathscr{L}_m$ is isomorphic to $\mathscr{O}_S$. Since $\mathscr{O}_S$ is the direct sum of $\mathscr{L}_m$, we see that the union of S_m is equal to S, and that S_m are pairwise disjoint. Therefore, giving a group homomorphism $G \to \mathbb{G}_{m,S}$ is equivalent to giving a decomposition of S as union of disjoint open subsets S_m ($m \in M$), i.e. to giving a locally constant map from S to M. This proves Corollary 12.7.5 and hence Theorem 12.7.4. $\square$

Corollary 12.7.6. *Any locally diagonalizable group is reflexive. If M, N are two ordinary abelian groups, then the natural homomorphism*

$$\mathrm{Hom}_{S\text{-}\mathbf{Grp}}(M_S, N_S) \to \mathrm{Hom}_{S\text{-}\mathbf{Grp}}(D(N_S), D(M_S))$$

is bijective.

Proof. The statement for locally diagonalizable groups follows from the uniqueness part of Corollary 12.7.5, and the last assertion has alreay been remarked. □

Since the preceding isomorphism is compatible with base changes, we then deduce an isomorphism of S-groups

$$\mathcal{H}om_{S\text{-}\mathbf{Grp}}(M_S, N_S) \xrightarrow{\sim} \mathcal{H}om_{S\text{-}\mathbf{Grp}}(D(N_S), D(M_S)). \tag{12.7.6}$$

For any S-scheme T, by adjunction, we have

$$\mathcal{H}om_{S\text{-}\mathbf{Grp}}(M_S, N_S)(T) = \mathrm{Hom}_{\mathbf{Grp}}(M, \mathrm{Hom}_T(T, N_T)) = \mathrm{Hom}_{\mathbf{Grp}}(M, \Gamma(N_T/T)) \tag{12.7.7}$$

and $\Gamma(N_T/T)$ is the abelian group of locally constant maps $T \to N$ by ([?] I, 1.8). On the other hand, let $\mathrm{Hom}_{\mathbf{Grp}}(M, N)_S$ be the constant S-group associated with the ordinary abelian group $\mathrm{Hom}_{\mathbf{Grp}}(M, N)$. We then have an evident homomorphism of abelian S-functors in groups

$$\theta : \mathrm{Hom}_{\mathbf{Grp}}(M, N)_S \to \mathcal{H}om_{S\text{-}\mathbf{Grp}}(M_S, N_S). \tag{12.7.8}$$

which is always a monomorphism. Moreover, this is an isomorphism if M is finitely generated:

Proposition 12.7.7. *Let M, N be ordinary abelian groups and assume that M is finitely generated. Then (12.7.8) is an isomorphism.*

Proof. Let $F(M)$ and $G(M)$ be the left (resp. right) member of (12.7.8). If $M = M_1 \oplus M_2$, then we have a canonical isomorphism $F(M) = F(M_1) \oplus F(M_1)$, and similarly for G. Therefore, it suffices to prove that θ is an isomorphism if $M = \mathbb{Z}/r\mathbb{Z}$ for an integer $r \geq 0$. In this case, $F(M) = (N[r])(S)$, where $N[r]$ is the kernel of $r \cdot \mathrm{id}_N$, and for any $T \to S$, the homomorphism

$$F(M)(T) = \Gamma(N[r]_T/T) \to G(M)(T) = \Gamma(N_T/T)[r]$$

is easily seen to be bijective, whence the assertion. □

Corollary 12.7.8. *Let M, N be ordinary abelian groups and assume that M is finitely generated. Then we have an isomorphism*

$$\mathrm{Hom}_{\mathbf{Grp}}(M, N)_S \xrightarrow{\sim} \mathcal{H}om_{S\text{-}\mathbf{Grp}}(D(N_S), D(M_S));$$

therefore $\mathcal{H}om_{S\text{-}\mathbf{Grp}}(D(N_S), D(M_S))$ is representable.

Proof. This follows from Proposition 12.7.7 and Corollary 12.7.6. □

Corollary 12.7.9. *Under the conditions of Proposition 12.7.7, if S is connected, we have*

$$\mathrm{Hom}_{S\text{-}\mathbf{Grp}}(D_S(N), D_S(M)) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{Grp}}(M, S), \quad \mathrm{Iso}_{S\text{-}\mathbf{Grp}}(D_S(N), D_S(M)) \xrightarrow{\sim} \mathrm{Iso}_{\mathbf{Grp}}(M, S)$$

Proof. If S is connected, then the set $\mathrm{Hom}_{\mathbf{Grp}}(M, N)_S(S)$ is identified with $\mathrm{Hom}_{\mathbf{Grp}}(M, N)$, whence the first assertion by Corollary 12.7.8. The case for $\mathrm{Iso}_{\mathbf{Grp}}(D_S(N), D_S(M))$ can be deduced from this, using the functoriality of D_S. □

To extend the preceding results to locally diagonalizable groups, we need to use the technique of descents. Let S be a scheme and X, Y, T be S-schemes. If (T_i) is an open covering of T, and if we put $T_{ij} = T_i \cap T_j = T_i \times_T T_j$, then as a morphism of T-schemes $X_T \to Y_T$ cen be glued locally over T, we see that the following sequence is exact:

$$\mathrm{Hom}_T(X_T, Y_T) \to \prod_i \mathrm{Hom}_{T_i}(X_{T_i}, Y_{T_i}) \rightrightarrows \prod_{ij} \mathrm{Hom}_{T_{ij}}(X_{T_{ij}}, Y_{T_{ij}}) \tag{12.7.9}$$

i.e. the functor $\mathcal{H}om_S(X, Y)$ is a sheaf over $\mathbf{Sch}_{/S}$ for the Zariski topology. More generally, by ([?] IV, 4.5.13), it is a sheaf for any subcanonical topology over $\mathbf{Sch}_{/S}$.

Let G, H be S-groups, we then deduce that the S-functor $\mathcal{H}om_{S\text{-}\mathbf{Grp}}(X, Y)$ is a sheaf for the fpqc topology.

Lemma 12.7.10. *Let F be a Zariski sheaf over* $\mathbf{Sch}_{/S}$.

(a) *Suppose that there exists an open covering (S_i) of S such that the fiber product $F_i = F \times_S S_i$ is representable by an S_i-scheme X_i. Then F is representable by an S-scheme X.*

(b) *Suppose that F is a fpqc sheaf and there exists a faithfully flat and quasi-compact morphism $S' \to S$ such that $F' = F \times_S S'$ is representable by an S'-scheme X'. Then X' is endowed with a canonical descent data relative to $S' \to S$. If this descent data is effective (for example if X' is affine over S), then F is representable by an S-scheme X.*

Proof. We first consider the situation of (a). It follows from the hypothesis that $X_i \times_S S_j$ and $X_j \times_S S_i$ represent the two restriction of F to $S_{ij} = S_i \times_S S_j$, hence, by Yoneda lemma, there exists a unique isomorphism of S_{ij}-schemes

$$\phi_{ji} : X_i \times_S S_j \xrightarrow{\sim} X_j \times_S S_i;$$

we then have isomorphisms of schemes over $S_{ijk} = S_i \times_S S_j \times_S S_k$:

$$\begin{tikzcd}
X_i \times_S S_j \times_S S_k \arrow[r, "\phi_{ji}\times \mathrm{id}_{S_k}"] \arrow[d, equal] & X_j \times_S S_i \times_S S_k \arrow[r, equal] & X_j \times_S S_k \times_S S_i \arrow[d, "\phi_{kj}\times \mathrm{id}_{S_i}"] \\
X_i \times_S S_k \times_S S_j \arrow[r, "\phi_{ki}\times \mathrm{id}_{S_j}"] & X_k \times_S S_i \times_S S_j \arrow[r, equal] & X_k \times_S S_j \times_S S_i
\end{tikzcd}$$

and as these objects all represent the restriction of F to S_{ijk}, this diagram is commutative, i,e, the ϕ_{ji} is a descent data over X_i. Thus we can glue X_i to an S-scheme X such that $X \times_S S_i = X_i$ for each i. For any Y over S, the $Y_i = Y \times_S S_i$ form an open covering of Y; put $Y_{ij} = Y_i \times_S Y_j = Y \times_S S_{ij}$. As F (resp. h_X) is a Zariski sheaf by hypothesis, we have the following commutative diagram with exact rows:

$$\begin{tikzcd}
F(Y) \arrow[r] \arrow[d] & \prod_i F(Y_i) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, equal] & \prod_{ij} F(Y_{ij}) \arrow[d, equal] \\
h_X(Y) \arrow[r] & \prod_i h_X(Y_i) \arrow[r, shift left] \arrow[r, shift right] & \prod_{ij} h_X(Y_{ij})
\end{tikzcd}$$

It follows that X represents F, which proves (a).

As for (b), it follows from the hypotheses in (b) that $F_1'' = F \times_{S'} S_1''$ (where $S_1'' = S'' = S' \times_S S'$, considered as an S'-scheme via the first projection) is represented by $X_1'' = X' \times_{S'} S_1''$; similarly, $F_2'' = F' \times_{S'} S_2''$ is represented by $X_2'' = X' \times_{S'} S_2''$ (again, $S_2'' = S''$ and is considered as an S'-scheme via the second projection). Now $F_1'' = F \times_S S'' = F_2''$, so there exists a unique S''-isomorphism $\phi : X_1'' \to X_2''$.; then, if we denote by q_i (resp. pr_{ji}) the projections of $S''' = S' \times_S S' \times_S S'$ onto the i-th factor (resp. to the i-th and j-th factor), $X_i''' = X' \times_{S'} S_i'''$ (where $S_i''' = S'''$ is considered as an S'-scheme using q_i), and $\mathrm{pr}_{ji}^*(\phi) : X_i''' \xrightarrow{\sim} X_j'''$ the isomorphism of S'''-schemes induced from ϕ by base change, we obtain a diagram of isomorphisms of S'''-schemes

$$\begin{tikzcd}
X_1''' \arrow[r, "\mathrm{pr}_{21}^*(\phi)"] \arrow[dr, "\mathrm{pr}_{31}^*(\phi)"'] & X_2''' \arrow[d, "\mathrm{pr}_{32}^*(\phi)"] \\
 & X_3'''
\end{tikzcd}$$

and as these objects all represent the restriction of F to S''', this diagram is commutative, so ϕ is a descent data over X' relative to $S' \to S$. Suppose further that this descent data is effective, i.e. there exists an S-scheme X such that $X' \cong X \times_S S'$ (by [?] VIII 2.1, this is the case if X' is affine over S'). For any $Y \to S'$, put $Y' = Y \times_S S'$ and $Y'' = Y' \times_Y Y' = Y \times_S S''$, then $Y' \to Y$ is, as $S' \to S$, faithfully flat and quasi-compact, hence a universally effective epimorphism (Proposition 12.3.67). Therefore, the equivalence relation

$$Y' \times_Y Y' \rightrightarrows Y'$$

is $\mathcal{M}$-effective ($\mathcal{M}$ is the family of faithfully flat and quasi-compact morphisms) with quotient Y. As F (resp. h_X) is a fpqc sheaf by hypothesis, we again obtain a commutative diagram with exact rows:

$$\begin{tikzcd} F(Y) \arrow[r] \arrow[d] & F(Y') \arrow[r, shift left] \arrow[r, shift right] \arrow[d, equal] & F(Y' \times_Y Y') \arrow[d, equal] \\ h_X(Y) \arrow[r] & h_X(Y') \arrow[r, shift left] \arrow[r, shift right] & h_X(Y' \times_Y Y') \end{tikzcd}$$

It then follows that X represents F, whence assertion (b). □

Corollary 12.7.11. *Let F be a fpqc sheaf over $\mathbf{Sch}_{/S}$. Suppose that there exists an open covering (S_i) of S and for each i a faithfully flat and quasi-compact morphism $S_i' \to S_i$ such that $F_i' = F \times_S S_i'$ is representable by an S_i'-scheme X_i' which is affine over S_i'. Then F is representable by an S-scheme which is affine over S (such that $X \times_S S_i' = X_i'$ for each i). If moreover each $X_i' \to S_i'$ is a closed immersion (resp. a finite étale morphism), so is $X \to S$.*

Proof. The first assertion follows from Lemma 12.7.10. For the second one, it suffices to verify that each morphism $X \times_S S_i \to S_i$ is a closed immersion (resp. finite étale), which follows from ([**?**], 2.7.1) (resp. [**?**], 17.7.3). □

Corollary 12.7.12. *If G and H are locally diagonalizable groups over S with H of finite type over S, then $\mathcal{H}om_{S\text{-}\mathbf{Grp}}(G, H)$ is representable.*

Proof. By Corollary 12.7.8, there exists an open covering (S_i) of S such that $\mathcal{H}om_{S_i\text{-}\mathbf{Grp}}(G|_{S_i}, H_{S_i})$ is representable for each i. Since $\mathcal{H}om_{S\text{-}\mathbf{Grp}}$ is a Zariski sheaf, the corollary then follows form Lemma 12.7.10. □

12.7.2 Scheme-theoretic properties

Proposition 12.7.13. *Let S be a nonempty scheme, M be an ordinary abelian group, $G = D(M_S)$ the diagonalizable S-group defined by M.*

(a) *G is faithfully flat and affine over S (a fortiori quasi-compact over S).*

(b) *G is of fintie type over S if and only if M is finitely generated, and in this case G is of fintie presentation over S.*

(c) *G is finite over S if and only if M is finite, if and only if G is of finite type over S and annihilated by an integer $n > 0$. In this case, we have $\deg(G/S) = \mathrm{Card}(M)$.*

(c′) *G is integral over S if and only if M is a torsion group.*

(d) *G is the trivial S-group if and only if M is trivial.*

(e) *G is smooth over S if and only if M is finitely generated and the order of the torsion subgroup of M is coprime to the residue characteristic of S.*

Proof. Since $D_S(M)$ is obtained by base change $D_{\mathbb{Z}}(M)$ to S, the first assertion (a) is clear. As for (b), it suffices to reduce to affine case and note that for any ring A, the group algebra $A[M]$ is of finite type over A if and only if M is finitely generated, and in this case $\mathbb{Z} \to \mathbb{Z}[M]$ is finitely presented, hence $A \to A[M]$ is finitely presented for any ring A. This proves (b), and (c) follows from a similar argument, noting that for a finitely generated abelian group M is finite if and only if it is torsion. Finally, (c′) follows from the fact that for any ring A, the A-algebra $A[M]$ is integral over A if and only if M is torsion.

We now prove (e). If G is smooth over S, it is locally of finite presentation over S, so M is finitely generated and G is of finite presentation over S. Conversely, if G is of finite presentation over S (and flat over S by (a)), then it is smooth over S if and only if the geometric fibers are smooth, so we are reduced to the case where S is the spectrum of an algebraically closed field k. Write $M = T \oplus L$, where T is the torsion subgroup of M and L is a free abelian group, then we have $D(M) = D(T) \times D(L)$, where $D(L) \cong \mathbb{G}_m^r$ is smooth over k. Therefore, $G = D(M)$ is smooth over k if and only if $D(T)$ is. Now T is isomorphic to a direct sum of groups of the form $\mathbb{Z}/n_i\mathbb{Z}$, with n being the product of n_i, so $D(T)$ is a product of $D(\mathbb{Z}/n_i\mathbb{Z}) = \boldsymbol{\mu}_{n_i}$. Since the k-group is represented by $\mathrm{Spec}(k[T]/(T^{n_i} - 1))$, we then conclude that $D(T)$ is smooth if and only if each n_i is coprime to $p = \mathrm{char}(k)$, i.e. if and only if n is coprime to p. □

Theorem 12.7.14. *Let S be a scheme and*

$$0 \longrightarrow M' \xrightarrow{u} M \xrightarrow{v} M'' \longrightarrow 0$$

be an exact sequence of ordinary abelian groups. Consider the following sequence of S-schemes:

$$0 \longrightarrow D_S(M'') \xrightarrow{v^t} D_S(M) \xrightarrow{u^t} D_S(M'') \longrightarrow 0$$

(a) *v^t induces an isomorphism from $D_S(M'')$ to the kernel of u^t, and u^t is faithfully flat and quasi-compact.*

(b) *$D_S(M')$ represents the fpqc quotient $D_S(M)/D_S(M'')$.*

Proof. Let $\mathcal{M}$ be the family of faithfully flat and quasi-compact morphisms. First, (ii) follows from (i). In fact, the equivalence relation of $D_S(M)$ defined by u^t is the same as that defined by the subgroup $\ker(u^t) = D_S(M'')$; as $u^t \in \mathscr{M}$, this equivalence relation is $\mathcal{M}$-effective, and hence $D_S(M')$ represents the quotient sheaf for the fpqc topology (cf. Corollary 12.3.31).

The first assertion of (i) is a trivial concequence of the definition of $D_S(-)$. More generally, for any exact sequence

$$M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

the transposed sequence is exact:

$$0 \longrightarrow D_S(M'') \longrightarrow D_S(M) \longrightarrow D_S(M')$$

On the other hand, as $D_S(M)$ and $D_S(M')$ are affine over S, u^t is necessarily an affine morphism, a fortiori quasi-compact. The second assertion of (i) now follows from Corollary 12.7.15 (a) below. □

Corollary 12.7.15. *Let S be a nonempty scheme and $u : M' \to M$ be a homomorphism of ordinary abelian groups, $u^t : G \to G'$ be the transpose morphism.*

(a) *For u to be a monomorphism, it is necessary and sufficient that u^t is faithfully flat.*

(b) *For u to be an epimorphism, it is necessary and sufficient that u^t is a monomorphism (and in this case u^t is a closed immersion).*

Proof. To prove (a), we note that if u is a monomorphism, then $\mathscr{O}_S[M]$ is a module over $\mathscr{O}_S[M']$ admitting a nonempty basis (namely, the system of sections defined by any system of representatives of M modulo M'), a fortiori it is faithfully flat. Conversely, if this is the case, then $u^t : \mathscr{O}_S[M'] \to \mathscr{O}_S[M]$ is injective, which implies (for $S \neq \varnothing$) that $u : M' \to M$ is injective.

To prove (b), we note that if u is an epimorphism, then $\mathscr{O}_S[M'] \to \mathscr{O}_S[M]$ is surjective, hence u^t is a closed immersion and a fortiori a monomorphism. Conversely, if u^t is a monomorphism, then $\ker u^t$ is trivial. If we put $M'' = \operatorname{coker} u$, then we see that $\ker u^t = D_S(M'')$ by the left-exactness of D_S, so by Proposition 12.7.13 we have $M'' = 0$ hence u is an epimorphism. □

Corollary 12.7.16. *Let $M' \xrightarrow{u} M \xrightarrow{v} M''$ be an exact sequence of ordinary abelian groups, and consider the transpose sequence*

$$G'' \xrightarrow{v^t} G \xrightarrow{u^t} G'.$$

Then v^t induces a faithfully flat and quasi-compact morphism from G'' to $\ker u^t$, and the latter is a diagonalizable group isomorphic to $D_S(v(M)) = D_S(\operatorname{coker} u)$.

Proof. It suffices to consider the extended sequence $M' \xrightarrow{u} M \xrightarrow{v} v(M) = \operatorname{coker} u$ and apply Theorem 12.7.14. □

Corollary 12.7.17. *Let S be a scheme, $u : G \to H$ be a homomorphism of locally diagonalizable S-groups, with H of finite type over S. Put $G' = \ker u$, then:*

(a) *G' is locally diagonalizable, and is of finite type over S if G is.*

(b) *The quotient G/G' exists, or more precisely the equivalence relation defined by G' over G is $\mathcal{M}$-effective ($\mathcal{M}$ is the family of faithfully flat and quasi-compact morphisms). Moreover G/G' is locally diagonalizable, of finite type over S.*

(c) *The homomorphism $u : G \to H$ factors uniquely into*

$$G \xrightarrow{v} G/G' \xrightarrow{w} H$$

where v is the canonical homomorphism (hence faithfully flat and quasi-compact) and w is a closed immersion.

(d) *The quotient $H' = H/\operatorname{im} w = \operatorname{coker} w = \operatorname{coker} u$ exists. More precisely, the equivalence relation defined by G/G' over H is $\mathcal{M}$-effective, and H' is of finite type over S.*

Proof. The first assertion is a concequence of (b), by the definition of the quotient G'/G. To show that the fpqc quotient sheaf $\widetilde{G}/\widetilde{G}'$ is representable, since it is local over S, we can suppose that G and H are diagonalizable, of the form $D_S(M)$ and $D_S(N)$, and S is connected. As H is of finite type over S, N is finitely generated by Proposition 12.7.13, so by Corollary 12.7.8, u is defined by a homomorphism $u' : N \to M$. Then, in view of Theorem 12.7.14 and Corollary 12.7.15, G' is isomorphic to $D_S(\operatorname{coker} u')$, and $\widetilde{G}/\widetilde{G}'$ is represented by $D_S(\operatorname{im} u')$. Further, consider the exact sequence

$$0 \longrightarrow \ker u' \longrightarrow N \xrightarrow{w^t} \operatorname{im} u' \longrightarrow 0$$

we obtain that w is a closed immersion and the quotient $H' = H/\operatorname{im} w$ is represented by $D_S(\ker u')$; this is of finite type over S since N, and hence $\ker u'$, is of finite type. □

Corollary 12.7.18. *Let G be a diagonalizable group scheme over S, and $n \neq 0$ be an integer. Then the subgroup $G[n]$ of G, the kernel of the homomorphism $n \cdot \mathrm{id}_G : G \to G$, is integral over S, and finite over S if G is of finite type over S.*

Proof. If $G = D_S(M)$, then $G[n] = D_S(M/nM)$ in view of Theorem 12.7.14, and we conclude by Proposition 12.7.13. □

12.7.3 Torsors under a diagonalizable group

Let S be a scheme and $G = D_S(M)$ be a diagonalizable group over S. We consider **G-torsors** (or **principal homogeneous G-bundles**) for the fpqc topology. Recall that this is a scheme P over S, acted by G (on the right), such that any point of S admits an open neighborhood U and a faithfully flat and quasi-compact morphism $S' \to U$ such that $P' = P \times_S S'$ is isomorphic to $G' = G \times_S S'$. As G is affine over S, it follows from ([**?**], VIII 5.6) that such a P is necessarily affine over S. We also note that since G is itself faithfully flat and quasi-compact over S, a scheme P is principal homogeneous under G if and only if it is formally principal homogeneous, and it is moreover faithfully flat and quasi-compact over S (cf. Proposition 12.3.46 (iii)).

Recall on the other hand that (cf. Proposition 12.1.34) giveing an S-scheme P, affine over S, acted by the group $G = D_S(M)$ is equivalent to giving a quasi-coherent graded algebra $\mathscr{A}$ of type M over S, i.e. a quasi-coherent algebra $\mathscr{A}$ over S, endowed with a decomposition into direct sums (as $\mathscr{O}_S$-modules):

$$\mathscr{A} = \bigoplus_{m \in M} \mathscr{A}_m$$

where $\mathscr{A}_m \cdot \mathscr{A}_n \subseteq \mathscr{A}_{m+n}$ for any $m, n \in M$.

Proposition 12.7.19. *For a scheme P affine over S acted by $G = D_S(M)$ (defined by a quasi-coherent graded algebra $\mathscr{A}$ of type M) to be a G-torsor, it is necessary and sufficient that $\mathscr{A}$ satisfies the following conditions:*

(i) *For any $m \in M$, $\mathscr{A}_m$ is an invertible module over S.*

(ii) *For any $m, n \in M$, the homomorphism*

$$\mathscr{A}_m \otimes_{\mathscr{O}_S} \mathscr{A}_n \to \mathscr{A}_{m+n}$$

induced by multiplication of $\mathscr{A}$, is an isomorphism.

Proof. The necessity of these conditions are immediate by fpqc descent ([**?**], VIII, remarque 1.12), since they are verified for the trivial G-torsor, i.e. where $\mathscr{A} = \mathscr{O}_S[M]$. On the otherh and, we first note that (i) implies that P is faithfully flat over S, and is also quasi-compact over S since it is affine over S. Therefore, it remains to verify that P is formally homogeneous under G, i.e. that the homomorphism

$$P \times_S G \to P \times_S P$$

is an isomorphism. Now this morphism corresponds to the homomorphism

$$\mathscr{A} \otimes \mathscr{A} \to \mathscr{A} \otimes \mathscr{O}_S(M)$$

whose (m,n)-component (where $m, n \in M$) is given by (cf. the arguments before Proposition 12.1.34)

$$x_m \otimes y_n \mapsto x_m y_n \otimes e_n.$$

Therefore, we see that condition (ii) is equivalent to that P is formally principal homogeneous, and this proves the proposition. ☐

Corollary 12.7.20. *The conditions of Proposition 12.7.19 imply that the homomorphism $\mathscr{O}_S \to \mathscr{A}_0$ is an isomorphism.*

Proof. In fact, we have $\mathscr{A}_0 \otimes_{\mathscr{O}_S} \mathscr{A}_0 \cong \mathscr{A}_0$ and $\mathscr{A}_0$ is invertible. ☐

If for example $M = \mathbb{Z}$, then the conditions of Proposition 12.7.19 are equivalent to that $\mathscr{A}_1 = \mathscr{L}$ is invertible and

$$\mathscr{A} = \bigoplus_{n \in \mathbb{Z}} \mathscr{L}^{\otimes n}$$

(isomorphism of graded algebras). We therefore obtain the following special case:

Corollary 12.7.21. *There is an equivalence between the category of $\mathbb{G}_{m,S}$-torsors P over S to the category of invertible modules $\mathscr{L}$ over S, which associates an invertible module $\mathscr{L}$ over S to the spectrum of the graded algebra $\bigoplus_{n\in\mathbb{Z}} \mathscr{L}^{\otimes n}$.*

Corollary 12.7.22. *The group of isomorphism classes of $\mathbb{G}_{m,S}$-torsors over S is isomorphic to the Picard group* $\mathrm{Pic}(S)$, *i.e. to $H^1(S, \mathscr{O}_S^\times)$.*

Since $\mathbb{G}_{m,S}$ can also be considered as the scheme $\mathcal{A}ut_{\mathscr{O}S}(\mathscr{O}_S)$ of automorphisms of the module $\mathscr{O}_S$, we see that Corollary 12.7.22 is equivalent to the following statement, which is a variant of Hilbert's theorem 90:

Corollary 12.7.23. *Any $\mathbb{G}_{m,S}$-torsor over G is locally trivial (in the sense of the Zariski topology).*

Proof. For any S-scheme Y, we note that

$$\mathrm{Hom}_{\mathscr{O}_S\text{-}\mathbf{alg}}\Big(\bigoplus_{n\in\mathbb{Z}} \mathscr{L}^{\otimes n}, \mathscr{A}(Y)\Big) = \mathrm{Hom}_{\mathscr{O}_Y\text{-}\mathbf{alg}}\Big(\bigoplus_{n\in\mathbb{Z}} \mathscr{L}^{\otimes n} \otimes_{\mathscr{O}_S} \mathscr{O}_Y, \mathscr{O}_Y\Big)$$

is isomorphic to $\mathrm{Iso}_{\mathscr{O}_Y}(\mathscr{L} \otimes \mathscr{O}_S \mathscr{O}_Y, \mathscr{O}_Y)$, so the equivalence in Corollary 12.7.21 associates an invertible sheaf $\mathscr{L}$ with the $\mathbb{G}_{m,S}$-torsor $\mathcal{I}so_{\mathscr{O}_S}(\mathscr{L}, \mathscr{O}_S)$. The fact that this is an equivalence of categories means that every $\mathbb{G}_{m,S}$-torsor is of the form $\mathcal{I}so_{\mathscr{O}_S}(\mathscr{L}, \mathscr{O}_S)$, which is clearly Zariski trivial since $\mathscr{L}$ is invertible (this can also be proved using the fact that invertible sheaves have fpqc descent). ☐

Remark 12.7.24. We note that the preceding result is not valid for the group $\boldsymbol{\mu}_n$, or for a twisted form of $\mathbb{G}_m$. For example, let S^1 be the kernel of the norm morphism $N : \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_{m,\mathbb{C}} \to \mathbb{G}_{m,\mathbb{R}}$ (induced by the norm from $\mathbb{C}$ to $\mathbb{R}$), which is a $\mathbb{C}/\mathbb{R}$-form of $\mathbb{G}_{m,\mathbb{R}}$. The equation $N(z) = -1$ in $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_{m,\mathbb{C}}$ defines a S^1-torsor X over $\mathrm{Spec}(\mathbb{R})$, which is locally trivial for the étale topology, but nontrivial since $X(\mathbb{R}) = \varnothing$. We now prove that $H^1_{\text{ét}}(\mathbb{R}, S^1) = \mathbb{Z}/2\mathbb{Z}$, so that there is no other S^1-torsors over $\mathrm{Spec}(\mathbb{R})$. In fact, we have an exact sequence of smooth algebraic $\mathbb{R}$-groups

$$1 \longrightarrow S^1 \longrightarrow \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_{m,\mathbb{C}} \longrightarrow \mathbb{G}_{m,\mathbb{R}} \longrightarrow 1$$

so we obtain a long exact sequence on étale cohomology:

$$0 \longrightarrow S^1(\mathbb{R}) \longrightarrow \mathbb{C}^\times \xrightarrow{N} \mathbb{R}^\times \longrightarrow H^1_{\text{ét}}(\mathbb{R}, S^1) \longrightarrow H^1_{\text{ét}}(\mathbb{R}, \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_{m,\mathbb{C}})$$

But we have $H^1_{\text{ét}}(\mathbb{R}, \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_{m,\mathbb{C}}) \cong H^1_{\text{ét}}(\mathbb{C}, \mathbb{G}_{m,\mathbb{C}})$ (cf. [**?**] XXIV, 8.4), and this is trivial by Corollary 12.7.22. We then obtain an isomorphism $H^1_{\text{ét}}(\mathbb{R}, S^1) \cong \mathbb{R}^\times/N(\mathbb{C}^\times) \cong \mathbb{Z}/2\mathbb{Z}$.

Proposition 12.7.25. *Under the hypothesis in Proposition 12.7.19, the conditions* (i) *and* (ii) *are equivalent the following conditions:*

(i′) $\mathscr{O}_S \to \mathscr{A}_0$ *is an isomorphism.*

(ii′) *For any* $m \in M$*, we have* $\mathscr{A}_m \cdot \mathscr{A}_{-m} = \mathscr{A}_0$.

Proof. These conditions are necessary by Proposition 12.7.19. To prove the converse, we first consider the case $M = \mathbb{Z}$. That is, we show that if $A = \bigoplus_{n\in\mathbb{Z}} A_n$ is a graded ring such that $A_1 \cdot A_{-1} = A_0$, then each A_n is a projective A_0-module and

$$A_m \otimes_{A_0} A_n \to A_{m+n}, \quad m, n \in \mathbb{Z}$$

is an isomorphism. For this, we note that by hypothesis, there exists $f_i \in A_1$, $g_i \in A_{-1}$ such that

$$\sum_i f_i g_i = 1. \tag{12.7.10}$$

As the conclusion to be established is local over $\mathrm{Spec}(A_0)$ and as, by Eq. (12.7.10), $\mathrm{Spec}(A_0)$ is covered by the affine open subsets $D(f_i g_i)$, we may reduce ourselves to the case where there exists an element $f \in A_1$ which is invertible in A. Then for each $n \in \mathbb{Z}$, we obtain an isomorphism $h \mapsto f^n h$ from A_0 to A_n, whence0 an isomorphism $A_0[t, t^{-1}] \to A$ of A_0-algebras, from which our assertion is clear.

Now in the general case, we see that under the conditions of Proposition 12.7.25, each $\mathscr{A}_m$ ($m \in M$) is invertible. To prove the second condition of Proposition 12.7.19, we can suppose that $\mathscr{A}_m$ and $\mathscr{A}_n$ have basis f_m and f_n, with inverses $f_m^{-1} \in \Gamma(S, \mathscr{A}_{-m})$, $f_n^{-1} \in \Gamma(S, \mathscr{A}_{-n})$. Then the homothety with ratio $f_m^{-1} f_n^{-1} \in \Gamma(S, \mathscr{A}_{-m-n})$ defines an isomorphism $\mathscr{A}_{m+n} \to \mathscr{A}_0 \cong \mathscr{O}_S$, which sends the image of $f_m \otimes f_n$ in $\mathscr{A}_{m+n}$ to the unit section 1 of $\mathscr{O}_S$. In the diagram

$$\mathscr{A}_m \otimes \mathscr{A}_n \xrightarrow{u} \mathscr{A}_{m+n} \xrightarrow{v} \mathscr{A}_0 \cong \mathscr{O}_S$$

we see that w and v are epimorphisms of invertible sheaves, hence are isomorphisms, whence u is an isomorphism. □

12.7.4 Quotient by diagonalizable groups

In this subsection, we denote by $\mathcal{M}$ the family of faithfully flat and quasi-compact morphisms, and the torsors are considered for the fpqc topology. Our main result is the following:

Theorem 12.7.26. *Let S be a scheme, M be an ordinary abelian group, $G = D_S(M)$ be the diagonalizable group over S defined by M, P be an S-scheme affine over S acted freely by G on the right. Then the equivalence relation defined by G over P is $\mathcal{M}$-effective, i.e. the quotient sheaf $X = P/G$ exists and P is a $G_X = D_X(M)$-torsor over X. Further, P/G is affine over S; more precisely, if P is defined by a graded algebra $\mathscr{A}$ of type M, then P/G is isomorphic to* $\mathrm{Spec}(\mathscr{A}_0)$*, where $\mathscr{A}_0 = \mathscr{A}^G$ is the zero-th component of $\mathscr{A}$.*

Put $X = \mathrm{Spec}(\mathscr{A}_0)$, then we have a structural morphism $P \to X$ induced by $\mathscr{A}_0 \to \mathscr{A}$, which is invariant under the action of G. In this way, P is an X-scheme which is affine over X and acted by $G_X = D_X(M)$, and the hypothesis that G acts freely on P/S implies that G_X acts freely on P/X. It then remains to show that P is a G_X-torsor, using the fact that $\mathscr{B}_0 = \mathscr{O}_X$, where $\mathscr{B}$ is the graded $\mathscr{O}_X$-algebra of type M defining P/X. We can then suppose that $X = S$ and S is affine, hence P is affine, given by a graded ring A of type M whose homogeneous components is denoted by A_m, and $S = \mathrm{Spec}(A_0)$. In view of Proposition 12.7.25, we only need to prove that

$$A_m \cdot A_{-m} = A_0 \quad \text{for} \quad m \in M. \tag{12.7.11}$$

As in the proof of Proposition 12.7.19, we see that (12.7.11) is equivalent to that the morphism $P \times_S G \to P \times_S P$ is a closed immersion (not only a monomorphism), i.e. that the ring homomorphism

$$\theta : A \otimes_{A_0} A \to A[M], \quad b_n \otimes a_m \mapsto b_n a_m \otimes e_m$$

is surjective[36]. This is the case if we assume further that the equivalence relation defined by G over P is closed, but we will show that this follows from the assumption that G acts freely over P (this is in fact

[36] In fact, θ is surjective if and only if each piece $A_0 \otimes e_m$ is in the image of $A \otimes_{A_0} A$, which means $A_m \cdot A_{-m} \to A_0$ is surjective.

implicitly contained in the conclusion of Theorem 12.7.26, since $G \times_S P = G_X \times_X P$ is then isomorphic to $P \times_X P$, which is closed in $P \times_S P$ since X is affine (hence separated) over S).

Let $R = P \times_S G$. The hypothesis that G acts freely, i.e. that $R \to P \times_S P$ is a monomorphism, is equivalent to that the diagonal

$$R \to R' = R \times_{(P \times_S P)} R$$

is an isomorphism. We have $R = \operatorname{Spec}(A[M])$ and $R' = \operatorname{Spec}(A[M \times M]/\mathfrak{K})$, where $\mathfrak{K}$ is the ideal generated by the elements of the form

$$x_m(e_{m,0} - e_{0,m}), \quad m \in M, x_m \in A_m$$

Let $\phi : A[M \times M] \to A[M]$ be the surjective ring homomorphism defined by

$$xe_{m,n} \mapsto xe_{m+n}, \quad m, n \in M, x \in A$$

(where $e_m, e_{m,n} = e_m \otimes e_n$ are elements of the canonical basis of $A[M]$ and $A[M \times M]$). Then the diagonal morphism $R \to R'$ corresponds to the ring homomorphism

$$\bar{\phi} : A[M \times M]/\mathfrak{K} \to A[M]$$

obtained by passing to quotient. Now the kernel of $\bar{\phi}$ is the ideal $\mathfrak{K}'$ generated by the elements $d_m = e_{m,0} - e_{0,m}$. We have $\mathfrak{K} \subseteq \mathfrak{K}'$, and the hypothesis that G acts free on P, i.e. that $\bar{\phi}$ is an isomorphism, is equivalent to the equality $\mathfrak{K} = \mathfrak{K}'$, which can be expressed by the relations

$$d_m \in \mathfrak{K} = \sum_p A[M \times M] A_p d_p, \quad \text{for} \quad m \in M. \tag{12.7.12}$$

Using the natural tri-graduation over $A[M \times M]$, and the fact that the first degree of d_m is zero for any $m \in M$, this signifies that we can write d_m into the form

$$d_m = f e_{r,s}(e_{p,0} - e_{0,p}), \quad f \in A_{-p} \cdot A_p,$$

and using the fact that the total degree of d_m is m, we can reduce ourselves to the terms such that $r + s + p = m$. We then conclude that we have, for any $m \in M$, an expression

$$d_m = e_{m,0} - e_{0,m} = \sum_{r,s} \lambda_{r,s}(e_{m-s,s} - e_{r,m-r}) \tag{12.7.13}$$

where $\lambda_{r,s} \in \mathfrak{J}_p = A_p \cdot A_{-p} \subseteq A_0$, and $p = m - (r + s)$.

To deduce (12.7.11) from this, it suffices to establish the same relation module any maximal ideal of A_0. As the hypotheses are invariant under such a reduction, we may therefore assume that A_0 is a field.

Lemma 12.7.27. *Under the preceding conditions (with A_0 be a field), if $M \neq 0$, there exists $p \in M \setminus \{0\}$ such that $\mathfrak{J}_p = A_0$.*

Proof. If this is not the case, then each $\mathfrak{J}_p$ is zero in A_0 except $\mathfrak{J}_0$, so $\lambda_{r,s} = 0$ except $r + s = m$. By comparing the coefficients of $e_{m,0}$ in (12.7.13), we then obtain that $\lambda_{m,0} - \lambda_{m,0} = 1$, which is absurd. □

Lemma 12.7.28. *Under the preceding conditions (with A_0 be a field), for any proper subgroup N of M, there exists $p \in M - N$ such that $\mathfrak{J}_p = A_0$.*

Proof. Let $M' = M/N$ and consider the graded ring A' of fype M', whose underlying ring is A, and whose graduation is given by

$$A'_{m'} = \bigoplus_{m \in \pi^{-1}(m')} A_m$$

where $\pi : M \to M' = M/N$ is the canonical homomorphism. Geometrically, this construction means we consider the action of P by the subgroup $G' = D_S(M')$ induced by G, which is therefore free. That is, the couple (M', A') satisfies the hypothesis of Lemma 12.7.27. We thus conclude from Lemma 12.7.27 that there exists $f_i \in A_{m_i}$ and $g_i \in A_{-m_i}$, with $m_i \in M$ belonging to a common coset of M/N, such that $\sum_i f_i g_i = 1$. Now as A_0 is a field, we can choose $m_j \in M - N$ and write

$$1 = \Big(1 - \sum_{i \neq j} f_i g_i\Big)^{-1} \cdot f_j g_j$$

and this shows that $\mathfrak{J}_{m_j} = A_0$, whence the lemma. □

Now note that we have $\mathfrak{J}_p \cdot \mathfrak{J}_q \subseteq \mathfrak{J}_{p+q}$ and $\mathfrak{J}_p = \mathfrak{J}_{-p}$, so if N denotes the subset of M of $m \in M$ such that $\mathfrak{J}_p = A_0$, then N is a subgroup of M. Using Lemma 12.7.28, we see that $N = M$, and this completes the proof of Theorem 12.7.26.

Corollary 12.7.29. *Under the conditions of Theorem 12.7.26, the graph morphism*

$$P \times_S G \to P \times_S P$$

is a closed immersion. In particular, if σ is a section of P over S, then the morphism $g \mapsto \sigma \cdot g$ from G to P defined by σ is a closed immersion.

Corollary 12.7.30. *Let G, H be two S-groups, with G diagonalizable and H affine over S, and let $u : G \to H$ be a monomorphism of S-groups. Then u is a closed immersion, $H \backslash G = X$ exists and H is a G_X-torsor over X, with X affine over S.*

Proof. By Corollary 12.7.29, the induced morphism $H \times_S G \to G \times_S G$ by u is a closed immersion; since G is faithfully flat and quasi-compact over S, it follows that u is a closed immersion (cf. [**?**], VIII 4.8 et [**?**], 8.11). The other assertions are contained in Theorem 12.7.26. □

Corollary 12.7.31. *Under the conditions of Theorem 12.7.26, if G is of finite type over S and P is of finite type (resp. of finite presentation) over S, so is $X = P/G$.*

Proof. By hypothesis, G_X is then of finite type over X, hence of finite presentation over X by Proposition 12.7.13. As P is a G_X-torsor over X, we conclude that it is of finite presentation over X (cf. [**?**], 2.7.1). As it is also faithfully flat over X, we conclude the corollary from ([**?**], 17.7.5). □

12.8 Group schemes of multiplicative type

Definition 12.8.1. Let S be a scheme, G be a group scheme over S. We say that G is a group **of multiplicative type** if it is locally diagonalizable for the fpqc topology, i.e. if for any $s \in S$, there exists an open neighborhood U of s and a faithfully flat and quasi-compact morphism $S' \to U$ such that $G' = G \times_S S'$ is a diagonalizable S'-group.

We say that G is of **quasi-isotrivial** multiplicative type if it is locally diagonalizable for the étale topology, i.e. if in the above definition we can take $S' \to U$ to be surjective étale, or equivalently (by taking the direct sum of S') if there is an étale and surjective morphism $S' \to S$ such that $G' = G \times_S S'$ is a locally diagonalizable S'-group. If we can take $S' \to S$ to be finite and surjective étale, then we say that G is of **isotrivial** multiplicative type.

Finally, we say that G is of **locally trivial** multiplicative type (resp. **locally isotrivial**) if any $s \in S$ admits an open neighborhood U such that $G \times_S U$ is a diagonalizable U-group (resp. a group of isotrivial multiplicative type, i.e. there exists a finite étale surjective morphism $S' \to U$ such that $G \times_S S'$ is a diagonalizable S'-group).

Note that the previous notions all follow from that of diagonalizable group by a localization process, in the sense that we consider different topologies associated with **Sch**. It is generally agreed that when the word "locally" is not made explicit, it is for the Zariski topology. In the terminology introduced here, "of locally trivial multiplicative type" is equivalent to "locally diagonalizable", and similarly "trivial" is equaivalent to "diagonalizable". For the five concepts introduced above, we have the following implication diagram, which results from the corresponding relations between the topologies in play:

$$\begin{tikzcd}
 & \text{general multiplicative type} & \\
 & \text{quasi-isotrivial} \arrow[u, Rightarrow] & \\
 & \text{locally isotrivial} \arrow[u, Rightarrow] & \\
\text{isotrivial} \arrow[ur, Rightarrow] & & \text{locally trivial} \arrow[ul, Rightarrow] \\
 & \text{trivial} \arrow[ul, Rightarrow] \arrow[ur, Rightarrow] &
\end{tikzcd}$$

From a practical point of view, let us point out that all the groups of multiplicative type that we encounter will be quasi-isotrivial: we will see that if G is of finite type over S, then G is automatically quasi-isotrivial. But we will give examples where the group is not locally isotrivial. We will also see there that G can be locally trivial, without being isotrivial nor a fortiori trivial (which easily implies that the inclusions of the diagram above are strict).

On the other hand, we will also see that when S is locally Noetherian and normal (or more generally geometrically unibranch), any group of multiplicative finite type over S is necessarily isotrivial, and moreover trivial as soon as it is locally trivial. This explains that most of the groups of multiplicative type which we meet in practice will be without doubt isotrivial, especially since we will see later that the maximal tori of semi-simple group schemes are automatically isotrivial.

Definition 12.8.2. Let S be a scheme and G be an S-group. We say that G is a **torus** if it is locally (for the fpqc topology) isomorphic to a group of the form $\mathbb{G}_m^r$ (where $r \geq 0$ is an integer).

With the above notions, this means we can choose morphisms $S' \to U \hookrightarrow S$ such that G' is isomorphic to a group of the form $(\mathbb{G}_{m,S'})^r$. We note that the integer r depends on $s \in S$, which is equal to the dimension of the fiber $G_s = G \otimes_S \kappa(s)$. This is clearly a locally constant function, as one can see easily. This observation can be generalized:

Definition 12.8.3. Let G be a diagonalizable group scheme over a field k, hence G is isomorphic to $D_k(M)$ for some ordinary abelian group M, defined up to isomorphisms and $M \cong \mathrm{Hom}_{k\text{-}\mathbf{Grp}}(G, \mathbb{G}_{m,k})$. The isomorphic class of M is called the **type** of the diagonalizable group G, which is evidently invariant under base field change.

Now if G is a group scheme of multiplicative type over a scheme S, then for any $s \in S$, there exists an extension k of $\kappa(s)$ such that $G \times_S \mathrm{Spec}(k)$ is a diagonalizable k-group[37], whose type is then independent of the chosen extension, and called the **type of** G **at** s, or the **type of** G_s. In particular, if G itself is diagonalizable, then the type of G at s is equal to that of G.

In the general case, if G is a group of multiplicative type over S and M is an ordinary abelian group, we say that G **is of type** M if it is of type M at any point $s \in S$, i.e. if it is locally isomorphic to $D_S(M)$ for the fpqc topology.

Remark 12.8.4. We see immediately that for a group G of multiplicative type over S, the function $s \mapsto$ type of G_s is locally constant over S: in fact, with the notations above, if G' is of type M, then G is of type M over the neighborhood U of s. We thus obtain a canonical partition of S into subschemes S_i such that for each i, G_{S_i} is of type M_i, where M_i are non-isomorphic ordinary abelian groups.

In particular, if S is connected, then the type of fibers of G is constant, i.e. there exists an ordinary abelian group M such that G is of type M. Finally, if G is a torus, then the type of G at s is characterized by the integer $\dim(G_s)$ (in fact, G_s is of type $\mathbb{Z}^r$, where $r = \dim(G_s)$).

Remark 12.8.5. It is clear that the above definitions are stable under base change. Thus, if G is a group scheme over S and $S' \to S$ is a base change morphism, then if G is of multiplicative type (resp. isotrival, etc.), so is G'. If the morphism $S' \to S$ is also faithfully flat and quasi-compact, then G' is of multiplicative type (resp. a torus) if and only if G is. Similarly, if $S' \to S$ is étale (resp. finite étale), then G' is quasi-isotrivial (resp. isotrivial) if and only if G is. Finally, for a general base change morphism $S' \to S$, if $s' \in S'$ and $s = f(s')$, then the type of G' at s' is equal to that of G at s, since we have $G'_{s'} = G_s \otimes_{\kappa(s)} \kappa(s')$.

12.8.1 Extension of cartain properties to groups of multiplicative type

We denote by $\mathcal{M}$ the family of faithfully flat and quasi-compact morphisms. As we shall see, certain properties for diagonalizable groups extends immedaitely to groups of multiplicative type.

Proposition 12.8.6. *Let G be a group of multiplicative type over a scheme S. We have the following:*

(a) *G is faithfully flat over S and affine over S (a fortiori quasi-compact over S).*

(b) *G is of finite type over S if and only if for any $s \in S$, the type of G at s is a finitely generated abelian group. In this case, G is of finite presentation over S.*

[37] In fact, by hypothesis there exists an open neighborhood U of s and a faithfully flat and quasi-compact morphism $S' \to U$ such that $G_{S'}$ is diagonalizable. Then for any $s' \in S'$ lying over s, $\kappa(s')$ is an extension of $\kappa(s)$ and $G_{s'} = G \times_S \mathrm{Spec}(\kappa(s'))$ is diagonalizable.

(c) *G is finite over S if and only if for any $s \in S$, the type of G at s is given by a finite abelian group. If S is quasi-compact, this is equivalent to that G is of fintie type over S and annihilated by an integer $n > 0$.*

(c′) *G is integral over S if and only if for any $s \in S$, the type of G at s is given by a torsion abelian group.*

(d) *G is the trivial S-group if and only if for any $s \in S$, the type of G at s is the trivial groups.*

(e) *G is smooth over S if and only if for any $s \in S$, the type of G at s is given by an abelian group whose torsion subgroup has order coprime to the characteristic of $\kappa(s)$.*

Proof. These results are concequences of Proposition 12.7.13, since they have fpqc descent (cf. [?] VIII ou [?], §2). □

Proposition 12.8.7. *Let G be a group of multiplicative and fintie type over S. Then for any integer $n \neq 0$, the kernel $G[n]$ of $n \cdot \mathrm{id}_G$ is a subgroup of multiplicative type and finite over S.*

Proof. This follows from Corollary 12.7.18. □

Proposition 12.8.8. *Let G be a group of multiplicative and fintie type over S, acting freely (on the right) over an S-scheme X which is affine over S. Then:*

(a) *The equivalence relation defined by G on X is $\mathcal{M}$-effective, and $Y = X/G$ is affine over S.*

(b) *If X is of finite presentation (resp. of finite type) over S, so is Y.*

Proof. The first assertion follows from Theorem 12.7.26, dealing with the case where G is diagonalizable, and from Proposition 12.3.23, since faithfully flat and quasi-compact morphisms are effective descent for the fibre category of schemes with affine morphisms, i.e. for any Y' affine over S, endowed with a descent dat relative to $S' \to S$, this descent data is effective, i.e. Y' comes from a scheme Y which is affine over S (cf. [?] VIII 2.1).

For the second assertion, we are equally reduced to the diagonalizable case, which follows from Corollary 12.7.31, bacause the finite conditions can be descent by faithfully flat and quasi-compact morphisms (cf. [?] VIII 3.3 et 3.6). □

As in Corollary 12.7.29 and Corollary 12.7.30, we obtain from Proposition 12.8.8 the following corollaries (note that closed immersions are fpqc local, cf. [?], VIII 4.8 et [?], 8.11):

Corollary 12.8.9. *Under the conditions of Proposition 12.8.8, the graph morphism*

$$X \times_S G \to X \times_S X$$

is a closed immersion. In particular, for any section σ of X over S, the corresponding morphism $G \to X, g \mapsto \sigma \cdot g$ is a closed immersion.

Corollary 12.8.10. *Let $u : G \to H$ be a monomorphism of S-groups, with G of multiplicative type and H affine over S. Then u is a closed immersion, $H/G = Y$ exists and is affine over S. Finally, H is a G_Y-torsor over Y.*

Proof. If we act G on H via u, then the morphism corresponding to the unit section of H over S is exactly u, so by Corollary 12.8.9 it is a closed immersion. The other claims follow from Proposition 12.8.8. □

Remark 12.8.11. Let $u : G \to H$ be a monomorphism of S-groups, with G of multiplicative finite type over S and H of finite presentation and separated over S. Then by ([?] VIII 7.12 and 7.13 (a)), we can prove that u is a closed immersion.

Proposition 12.8.12. *Let $u : G \to H$ be a homomorphism of S-groups of multiplicative type, with H of finite type over S. Then:*

(a) *$G' = \ker u$ is an S-group of multiplicative type, and if of finite type if G is.*

(b) *The equivalence relation defined by G' over G is $\mathcal{M}$-effective, hence u factors into*

$$G \longrightarrow G/G' \longrightarrow H$$

where $G/G' \to H$ is a closed immersion of S-groups and $G \to G/G'$ is faithfully flat and quasi-compact. Moreover, G/G' is of multiplicative finite type over S.

(c) *The equivalence relation on H defined by $I = G/G'$ is $\mathcal{M}$-effective, $H' = H/I$ exists, and H' is of fintie type over S.*

Proof. As in Proposition 12.8.8, we may assume that G and H are diagonalizable, and then the proposition reduces to Corollary 12.7.17. □

Corollary 12.8.13. *Let S be a scheme.*

(a) *The category of S-groups of multiplicative finite type over S is abelian.*

(b) *Let $u : G \to H$ be a homomorphism of the category in (a); for u to be a monomorphism (resp. an epimorphism, resp. an isomorphism) in this category, it is necessary and sufficient that it is a monomorphism of schemes (resp. faithfully flat, resp. an isomorphism of schemes).*

Proof. The first assertion follows from Proposition 12.8.12, noting that the product of two S-groups of multiplicative type is again of multiplicative type, and of finite type over S if these two groups are. The rest are immediate, for example, $u : G \to H$ is a monomorphism if and only if $\ker u$ is trivial, if and only if it is a closed immersion in view of Proposition 12.8.12. We also note that since G and H are affine over S, any S-morphism between them is quasi-compact by Proposition 8.6.3 (v). □

Corollary 12.8.14. *Let $u : G \to H$ be a homomorphism of S-groups of multiplicative finite type over S. Let U be the set of points $s \in S$ such that $u_s : G_s \to H_s$ is a monomorphism (resp. faithfully flat, resp. an isomorphism). Then U is clopen and the induced morphism $u_U : G|_U \to H|_U$ is a monomorphism (resp. faithfully flat, resp. an isomorphism).*

Proof. Let P (resp. Q) be the kernel (resp. cokernel) of u. By Proposition 12.8.12, Q exists and P, Q are of multiplicative type, and their formation are stable under base change $S' \to S$; in particular, stable under taking fibers. On the other hand, u is a monomorphism (resp. faithfully flat, resp. an isomorphism) if and only if $P = 0$ (resp. $Q = 0$, resp. $P = Q = 0$). We are therefore reduced to verify that: if R is an S-group of multiplicative type, then the set U of points $s \in S$ such that $R_s = 0$ is clopen, and $R|_U = 0$. This is contained in Remark 12.8.4 and Proposition 12.8.6 (d). □

Corollary 12.8.15. *Let G be an S-group of multiplicative finite type over S, H, H' be subgroup of multiplicative type.*

(a) *$H'' = H \cap H'$ is a subgroup of multiplicative type of G and of finite type over S.*

(b) *Let U be the set of $s \in S$ such that $H_s \subseteq H'_s$ (resp. $H_s = H'_s$), then U is clopen and $H|_U \subseteq H'|_U$ (resp. $H|_U = H'|_U$).*

Proof. Of course, the notation $H \cap H'$ means the intersection in the sense of functors, i.e. $H \times_G H'$, which is a sub-S-group of G. By applying Proposition 12.8.12 to the inclusion $H \to G$, we see that H is of finite type; similarly H' is of finite type. It then follows that $H \cap H'$ is of multiplicative finite type over S (we note that the canonical functor from the category considered in Corollary 12.8.13 to $\mathbf{Sch}_{/S}$ commutes with finite products).

The formation of $H \cap H'$ commutes with base changes, and in particular with taking fibers. On the other hand, $H \subseteq H'$ (resp. $H = H'$) if and only if $H' = H$ (resp. $H'' = H$ and $H'' = H'$). Consider the canonical homomorphism $H'' \to H$ and $H'' \to H'$. Then U is the set $s \in S$ such that the induced homomorphism $H''_s \to H_s$ is an isomorphism (resp. $H''_s \to H_s$ and $H''_s \to H'_s$ are isomorphisms), so the conclusion follows from Corollary 12.8.14. □

Proposition 12.8.16. *Let S be a scheme, G be an S-group of multiplicative finite type over S, H be a subgroup of multiplicative type of G, and $K = G/H$ (which is a group of multiplicative type).*

(a) *Suppose that G is trivial (i.e. diagonalizable). Then there exists a partition (S_i) of S by clopen subsets such that for each i, $H|_{S_i}$ and $K|_{S_i}$ are diagonalizable. In particular, if S is connected, then H and K are diagonalizable.*

(b) *The same assertion is true as in (a) by replacing "diagonalizable" with "isotrivial", provided that S is connected or quasi-compact.*

(c) *Suppose that G is locally trivial (resp. locally isotrivial, resp. quasi-isotrivial), then so is K and H.*

Proof. By hypothesis of (a), we have $G = D_S(M)$, where M is a finitely generated abelian group. For any quotient group M_i of M, let $H_i = D_S(M_i)$ be the diagonalizable group corresponding to G. Let S_i be the set of $s \in S$ such that $H_s = (H_i)_s$; in view of Corollary 12.8.15, S_i is clopen in S, and we have $H|_{S_i} = (H_i)|_{S_i}$, hhence $H|_{S_i}$ is diagonalizable, and also $K|_{S_i}$ by Theorem 12.7.14. Evidently, the S_i are pairwise disjoint, and they cover S: this follows from the fact that for any $s \in S$, H_s is diagonalizable and a subgroup of G_s, which is also diagonalizable (cf. Corollary 12.7.15). Restricting to the nonempty S_i, we then obtain (a).

For (b), by hypothesis there exists $S' \to S$ being étale and surjective, such that $G_{S'}$ is diagonalizable. Then any point of S' has a neighborhood U', clopen in S', such that $H|_{U'}$ and $K|_{U'}$ are diagonalizable. The image U of U' in S is then clopen, and $S' \to S$ induces a finite and étale surjective morphism $U' \to U$, so that any point $s \in S$ has a clopen neighborhood U such that $H|_U$ and $K|_U$ are isotrivial. The assertion of (b) then follows, since we can take the direct sum of these morphisms $U' \to U \hookrightarrow S$ to obtain a finite étale surjective morphism. Finally, (c) follows from (a) and (b). □

12.8.2 Infinitesimal properties: lifting and conjugation

Theorem 12.8.17. *Let S be an affine scheme, G be a group of multiplicative type, $\mathscr{F}$ be a quasi-coherent sheaf over S over which G acts. Then we have*

$$H^i(G, \mathscr{F}) = 0 \quad \textit{for} \quad i > 0,$$

where H^i denotes the Hochschild cohomology of §§12.1.3.

In fact, by **??** 12.1.3.2, if $S = \mathrm{Spec}(A)$, $G = \mathrm{Spec}(B)$ and M is the A-module defining $\mathscr{F}$, then the Hochschild cohomology is calculated as the cohomology of the complex $C^\bullet(G, \mathscr{F})$, where

$$C^n(G, \mathscr{F}) = \Gamma(S, \mathscr{F} \otimes \underbrace{\mathscr{A}(G) \otimes \cdots \otimes \mathscr{A}(G)}_{n\text{-fold}}) = M \otimes \underbrace{B \otimes \cdots \otimes B}_{n\text{-fold}}.$$

If f (resp. a_i) is a section of $\mathscr{F}$ (resp. $\mathscr{A}(G)$) over S, we then have

$$\begin{aligned} d(f \otimes a_1 \otimes \cdots \otimes a_n) = \mu_M(f) \otimes a_1 \otimes \cdots \otimes a_n + \sum_{i=1}^{n} (-1)^i f \otimes a_1 \cdots \otimes \Delta a_i \otimes \cdots \otimes a_n \\ + (-1)^{n+1} f \otimes a_1 \otimes \cdots \otimes a_n \otimes 1 \end{aligned}$$

where $\Delta : B \to B \otimes B$ and $\mu_M : M \to M \otimes B$ are induced from the cogebrea structure of B and the comodule structure on $\mathscr{F}$. We also note that the formation of $C^\bullet(G, \mathscr{F})$ commutes with base change. Therefore, for any base change morphism $A \to A'$ with A' flat over A, we have

$$H^i(G', \mathscr{F}') = H^i(G, \mathscr{F}) \otimes_A A',$$

and therefore, if we suppose that A' is faithfully flat over A, to show that $H^i(G, \mathscr{F})$ vanishes, it suffices to consider the cohomology $H^i(G', \mathscr{F}')$. By the definition of multiplicative type, we are therefore reduced to the case where G is diagonalizable, which follows from Theorem 12.1.49.

Using the results of ([**?**] III), we then obtain from Theorem 12.8.17 the following concequences:

Theorem 12.8.18. *Let S be an affine scheme and S_0 be an affine subscheme of S defined by an ideal $\mathscr{J}$ such that $\mathscr{J}^2 = 0$. Let $u, v : H \to G$ be group homomorphisms over S, where H is of multiplicative type, and $u_0, v_0 : H_0 \to G_0$ be the morphisms induced by base change $S_0 \to S$.*

(a) *If $u_0 = v_0$, then there exists $g \in G(S)$ such that $v = \mathrm{Inn}(g) \circ u$ and $g_0 = e$.*

(b) *In particular, if H and H' are subgroups of G of multiplicative type such that $H_0 = H'_0$, then there exists $g \in G(S)$ such that $H' = \mathrm{Inn}(g)(H)$ and $g_0 = e$.*

Proof. The first assertion follows from ([**?**] III, 2.1(ii) et 3.1), and the second one follows by applying (a) to the morphisms $H \times_S H' \to H \hookrightarrow G$ and $H \times_S H' \to H' \hookrightarrow G$ (note that $H \times_S H'$ is of multiplicative type). □

Corollary 12.8.19. *Under the conditions of Theorem 12.8.18, suppose that G is smooth over S and $\mathscr{J}$ is nilpotent.*

(a) *If there exists $g_0 \in G_0(S_0)$ such that $v_0 = \mathrm{Inn}(g_0) \circ u_0$, then g_0 comes from an element $g \in G(S)$ such that $v = \mathrm{Inn}(g) \circ u$.*

(b) *In particular, if H and H' are subgroups of G of multiplicative type such that $H'_0 = \mathrm{Inn}(g_0)(H_0)$ with $g_0 \in G_0(S_0)$, then g_0 comes from an element $g \in G(S)$ such that $H' = \mathrm{Inn}(g)(H)$.*

Proof. To prove (a), by induction on the integer $n > 0$ such that $\mathscr{I}^n = 0$, we are reduced to the case where $\mathscr{I}$ is square zero. Moreover, as G is smooth over S, we can lift g_0 into an element g' of $G(S)$. By replacing v with $v' = \mathrm{Inn}(g') \circ u$, we are then reduced to the situation of Theorem 12.8.18. Finally, (b) follows from (a) by a similar argument as in Theorem 12.8.18. □

Corollary 12.8.20. *Under the conditions of Theorem 12.8.18, with $\mathscr{I}$ nilpotent, suppose that v is central. Then $u_0 = v_0$ implies $u = v$. In particular, if u_0 is the trivial homomorphism, then u is trivial.*

Proof. The reduction to the case where $\mathscr{I}$ is square zero is still immediate, and then it suffices to apply Theorem 12.8.18. □

Theorem 12.8.21. *Let S be an affine scheme and S_0 be an affine subscheme of S defined by a nilpotent ideal $\mathscr{I}$. Let H be an S-group of multiplicative type and G be a smooth S-group.*

(a) *Let $u_0 : H_0 \to G_0$ be a morphism of S_0-groups induced by base change $S_0 \to S$. Then there exists a morphism $u : H \to G$ lifting u_0 (and by Theorem 12.8.18, two such liftings u, u' are conjugate by an element of $G(S)$ which reduces to the identity in $G(S_0)$).*

(b) *In particular, if H_0 is a subgroup of G_0 of multiplicative type, there exists a subgroup H of G such that $H \times_S S_0 = H_0$, which is of multiplicative type and flat over S. Moreover, any two such subgroups H, H' are conjugate by an element of $g \in G(S)$ which reduces to the identity in $G(S_0)$.*

Proof. The first assertion follows from ([?] III 2.1, 2.3 et 3.1) (vanishing of H^2 implies the existence of a lifting of u_0, vanishing of H^1 implies the uniqueness).

Now let H_0 be a subgroup of G_0. We then have an closed immersion $u_0 : H_0 \to G_0$, which by (a) comes from a morphism $u : H \to G$. As H and H_0 (resp. G and G_0) have the same underlying topological space and as for any $h \in H$, the morphism $\mathscr{O}_{G,u(h)} \to \mathscr{O}_{H,h}$ is surjective (by the nilpotent version of Nakayama's lemma), u is equally a closed immersion. It then remains to note that for any lifting H of H_0 of a flat subgroup of G, H is necessarily of multiplicative type by Corollary 12.9.9[38]. □

Proposition 12.8.22. *Let S be an affine scheme and S_0 be an affine subscheme of S defined by a nilpotent ideal $\mathscr{I}$. Let G be an S-group of multiplicative type, X be an S-scheme locally of fintie type acted by G so that G_0 acts trivially on X_0. Then G acts trivially on X.*

Proof. The proof is that of ([?] III 2.4(b)), so we are reduced to the case where G is diagonalizable, which is the case considered in ([?] III 2.4(b)). □

Corollary 12.8.23. *Let S be an affine scheme and S_0 be an affine subscheme of S defined by a nilpotent ideal $\mathscr{I}$. Let $u : G \to H$ be a homomorphism of S-groups, with G of multiplicative type and H locally of finite presentation over S. Suppose that the homomorphism $u_0 : G_0 \to H_0$ induced by base change $S_0 \to S$ is central, then u is central.*

Proof. It suffices to apply Proposition 12.8.22 to the action of G on H defined by $(g, h) \mapsto \mathrm{Inn}(u(g)) \cdot h$. □

12.8.3 The density theorem of torsion subgroups

The dentity theorem (Theorem 12.8.30), together with the algebricity theorem (Theorem 12.8.50), will be the essential tools in the present and next two sections, to pass from the infinitesimal properties of the groups of multiplicative type, which we have just developed, to finite properties.

Definition 12.8.24. Let X be a scheme. We say that a family $(Z^i)_{i \in I}$ of subschemes of X is **schematically dense** if for any open subset U of X and any closed subscheme Z of U which dominates the $Z^i \cap U$, we have $Z = U$. We say that a subscheme Z of X is **schematically dense** in X if this is true for the family reduced to Z.

[38] We note that the proof of Corollary 12.9.9 only depends on Theorem 12.8.21 (a), so there is no circular reasoning.

We see immediately ([?] 11.10.1) that the above definition is equivalent to that for any open subset U of X, any section f of $\mathscr{O}_U$ which vanishes over the $Z^i \cap U$, is zero. This also signifies that the intersection of the kernels of the canonical homomorphisms

$$u_i : \mathscr{O}_X \to (v_i)_*(\mathscr{O}_{Z^i})$$

is zero, where $v_i : Z^i \to X$ is the canonical immersion. If X is over a scheme S, this amounts to saying that for any open subset U of X and any couple (u, v) of morphisms of U to a separated S-scheme Y which coincides over the $Z^i \cap U$, we have $u = v$[39]. With the terminology introduced in §§ 8.6.7, we see that the subscheme Z of X is schematically dense in X if and only if X coincides with the scheme-theoretic closure of Z in X.

Example 12.8.25. Let $X = \mathrm{Spec}(k[x,y]/(x^2, xy))$ (i.e. a line with an embedded point at the origin) and $U = D(y)$. Then U is of course dense in X as a topological space, but is not schematically dense, since $U \hookrightarrow X$ factors through $U \hookrightarrow \mathrm{Spec}(k[y]) \hookrightarrow \mathrm{Spec}(k[x,y]/(x^2, xy))$.

Lemma 12.8.26. *Let X be a flat S-scheme, $(Z^i)_{i\in I}$ be a family of subschemes of X, which are flat over S. Let S_0 be the subscheme of S defined by a nilpotent ideal $\mathscr{I}$, and suppose that the modules $\mathscr{I}^n/\mathscr{I}^{n+1}$ are locally free over S_0. Let X_0 and (Z_0^i) be the induced schemes by base change $S_0 \to S$. If the family (Z_0^i) is schematically dense in X_0, then $(Z^i)_{i\in I}$ is schematically dense in X.*

Proof. Suppose that $\mathscr{I}^{n+1} = 0$ (where $n \geq 0$), we prove by induction on n. The assertion is trivial for $n = 0$, so assume that $n > 0$ and denote by S_m, X_m, Z_m^i the schemes obtained by reduction modulo $\mathscr{I}^{m+1}$. The induction hypothesis implies that $(Z_{n-1}^i)_{i\in I}$ is schematically dense in X_{n-1}. By replacing X with an open subset, we are then reduced to proving that any section f of $\mathscr{O}_X$ which vanishes over the Z^i is itself zero.

Now the section f_{n-1} of $\mathscr{O}_{X_{n-1}} = \mathscr{O}_X/\mathscr{I}^n\mathscr{O}_X$ defined by f vanishes over the Z_{n-1}^i, so is itself zero, hence f is a section of $\mathscr{I}^n\mathscr{O}_X$. As X is flat over S, we have

$$\mathscr{I}^n\mathscr{O}_X \xrightarrow{\sim} \mathscr{E} \otimes_{\mathscr{O}_{S_0}} \mathscr{O}_{X_0}$$

where $\mathscr{E} = \mathscr{I}^n = \mathscr{I}^n/\mathscr{I}^{n+1}$. Similarly, as each Z^i is flat over S, the restriction f_i of f to Z^i can be regareded as a section of

$$\mathscr{I}^n\mathscr{O}_{Z^i} \xrightarrow{\sim} \mathscr{E} \otimes_{\mathscr{O}_{S_0}} \mathscr{O}_{Z_0^i}.$$

Now $\mathscr{E}$ is locally free by hypothesis, hence so is $\mathscr{F} = \mathscr{E} \otimes_{\mathscr{O}_{S_0}} \mathscr{O}_{X_0}$, and f is therefore a section of a locally free module $\mathscr{F}$ over X_0, such that for any i its restriction to Z_0^i is zero. As $(Z_0^i)_{i\in I}$ is schematically dense in X_0, it follows from ([?] 11.10.1) that f is itself zero. $\square$

Lemma 12.8.27. *Let X be a locally Noetherian and flat S-scheme, $(Z^i)_{i\in I}$ be a family of subschemes of X which are flat over S. Suppose that for any $s \in S$, the family $(Z_s^i)_{i\in I}$ of fibers at s is schematically dense in X_s, then the family $(Z^i)_{i\in I}$ is schematically dense in X. Further, in this case, if S is locally Noetherian, then $(Z^i)_{i\in I}$ is universally schematically dense in X.*

Lemma 12.8.28. *Let X be an S-scheme, $(Z^i)_{i\in I}$ be a family of subschemes of X, and $S' \to S$ be a faithfully flat morphism. If the family $(Z'^i)_{i\in I}$ is schematically dense in X', then $(Z^i)_{i\in I}$ is schematically dense in X.*

Proof. This follows from ([?] 11.10.6 (i) et 11.9.10 (i)). $\square$

Corollary 12.8.29. *Let X be a flat S-scheme and U be an open subset of X. Suppose that for any $s \in S$, U_s is schematically dense in X_s, and that X is locally Noetherian or locally of finite presentation over S. Then U is schematically dense in X.*

Theorem 12.8.30. *Let S be a scheme and G be a group of multiplicative finite type over S. For any integer $n > 0$, let $G[n]$ be the kernel of the morphism $n \cdot \mathrm{id}_G$. Then the family $(G[n])_{n>0}$ of subschemes of G is schematically dense in G.*

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[39] In fact, the relation $u = v$ is equivalent to the relation $Z = U$, where Z is the inverse image of the diagonal of $Y \times_S Y$ under the S-morphism $X \to Y \times_S Y$ defined by (u, v); this diagonal is a closed subscheme of $Y \times_S Y$, hence Z is a closed subscheme of U, which dominates $Z^i \cap U$ by the hypothesis on (u, v), hence if the family (Z^i) is schematically dense, we have $Z = U$, whence $u = v$. The converse implication is deduced by putting $Y = \mathrm{Spec}(\mathscr{O}_S[T])$.

Proof. We first consider the case where S is locally Noetherian. Then by Lemma 12.8.27, we are reduced to the case where S is the spectrum of a field k. By Lemma 12.8.28, we can also suppose that k is algebraically closed and G is diagonalizable, i.e. of the form $D_k(M)$, where M is a finitely generated abelian group. Then M is of the form $\Gamma \times \mathbb{Z}^r$, with Γ finite, hence G is isomorphic to $G_1 \times T$, where $G_1 = D(\Gamma)$ and $T = \mathbb{G}_m^r$. Therefore for n multiplicatively large (i.e. n is a multiple of the order of Γ), we have $G[n] = G_1 \times T[n]$ (since $G_1[n] = G_1$).

Applying again Lemma 12.8.27 to the projection $G \to G_1$, we are reduced to the case where $G = \mathbb{G}_m^r$. As G is then reduced, the familt $(G[n])_{n>0}$ is schematically dense in G if and only if the union of $G[n]$ is dense in G for the usual topology. As $G[n] = (\mathbb{G}_m[n])^r$, we are also reduced to the case $G = \mathbb{G}_m$, hence G is an irreducible algebraic curve. Now the theorem follows from the fact that the union of the $G[n]$ (equals to the set of roots of unity in k) is infinite (cf. Corollary 9.7.19). In the general case, for any point $s \in S$, there exists a neighborhood U of s and a faithfully flat and quasi-compact morphism $S' \to U$ such that $G' = G_{S'}$ is diagonalizable, i.e. of the form $D_{S'}(M)$ with M a finitely generated abelian group. By Lemma 12.8.28, we can then reduce to the case where G is diagonalizable, hence consider the absolute group $H = D_{\mathbb{Z}}(M)$ over $\mathrm{Spec}(\mathbb{Z})$. By the preceding arguments, for any $s \in \mathrm{Spec}(\mathbb{Z})$, the family $(H_s[n])_{n>0}$ is schematically dense in H; it then suffices to apply Lemma 12.8.27. □

Corollary 12.8.31. *Let G be an S-group of multiplicative finite type and H be a subgroup of G which dominates $G[n]$ for each $n > 0$. Then $G = H$.*

Proof. In view of Theorem 12.8.30, we are reduced to show that the subscheme H is closed, or equivalently $|H| = |G|$. This reduces us to the case where S is the spectrum of a field, but then any subgroup of G is closed (Proposition 12.5.17). □

Corollary 12.8.32. *Let S be a scheme and G be an S-group.*

(a) *Let $u, v : H \to G$ be homomorphisms of S-groups, with H of multiplicative finite type, and suppose that for any integer $n > 0$, the restriction of u, v to $H[n]$ are identical. Then $u = v$.*

(b) *Let H_1, H_2 be subgroups of multiplicative finite type of G, and suppose that for any integer $n > 0$, we have $H_1[n] = H_2[n]$. Then $H_1 = H_2$.*

Proof. The first assertion follows from the second, by considering the graph subgroups H_1 and H_2 of $H \times G$ induced by u and v. To prove (b), let $H = H_1 \cap H_2$, which is a subgroup of H_i ($i = 1, 2$). We must show that this is identical to H_i, but the hypothesis means that it dominates $H_i[n]$ for each $n > 0$, we are therefore reduced to the situation of Corollary 12.8.31. □

Remark 12.8.33. Under the conditions of Theorem 12.8.30, let $m > 0$ be an integer which have the following properties: for any $s \in S$, m is not a power of the characteristic of $\kappa(s)$, and if G_s is of type M, the prime divisors of the order of the torsion subgroup of M divides m (this second condition is verified if G is a torus). Then the proof of Theorem 12.8.30 implies that in the statement of Theorem 12.8.30 and Corollary 12.8.31 and Corollary 12.8.32, we can only consider the subgroups of the form $G[m^r]$, with $r > 0$.

12.8.4 Central homomorphisms of groups of multiplicative type

Lemma 12.8.34. *Let $(A, \mathfrak{m})$ be a local Noetherian ring, $S = \mathrm{Spec}(A)$, and H be a finite scheme over S, hence $H = \mathrm{Spec}(B)$ with B a finite A-algebra. Let K be a subscheme of H such that $K_n = H_n$ by reduction modulo $\mathfrak{m}^{n+1}$ for any n. Then $K = H$.*

Proof. Let s be the closed point of S. We first note that K is a closed subscheme of H. In fact, it is a priori a closed subscheme of an open subset U of H. But K, hence U, contains the fiber $K_s = H_s$; as the morphism $H \to S$ is finite, hence closed, this ensures that the complement of U is empty, i.e. $U = H$[40]. Therefore K is defined by an ideal $\mathfrak{I}$ of B. The hypothesis implies that $\mathfrak{I}$ is comtained in $\mathfrak{m}^n B$ for any n; as B is a finite A-module, it is separated for the $\mathfrak{m}$-adic topology, whence $\mathfrak{I} = 0$. □

Theorem 12.8.35. *Let $u, v : H \to G$ be homomorphisms of S-groups, with H of multiplicative finite type over S and S locally Noetherian or G of finite presentation over S. Let $s \in S$ be such that $u_s = v_s$, and suppose that v_s is central. Then there exists a neighborhood U of s such that $u|_U = v|_U$.*

[40]We note that the image of $H - K$ in S (if not empty) is a closed subset of S, whence is defined by an ideal $\mathfrak{a}$ of A. This ideal must be contained in $\mathfrak{m}$, so $\mathfrak{m}$ belongs to the image of $H - K$.

Proof. We first assume that S is locally Noetherian. Let $K = \ker(u, v)$ be the inverse image of the diagonal of $G \times_S G$ under the morphism (u, v); this is a subgroup of H, and we want to choose U such that $K|_U = H|_U$. Note that as S is locally Noetherian and H is of finite type over S, H is also locally Noetherian, so the immersion $K \hookrightarrow H$ is of finite type (Corollary 8.6.37). Then K is of finite type over S, hence of finite presentation over S (S is locally Noetherian). Therefore, by ([?] 8.8.2.4), to show that there exists an open neighborhood U of s such that $K|_U = H|_U$, it suffices to prove that $K_{S'} = H_{S'}$, where $S' = \mathrm{Spec}(A)$, $A = \mathrm{Spec}(\mathscr{O}_{S,s})$. We can then reduce to the case where S is local with closed point s.

In view of Corollary 12.8.20 (here we use the hypothesis that v_s is central), we have $u_n = v_n$ for any n, where as usual the index n indicates the reduction modulo $\mathfrak{m}^{n+1}$ ($\mathfrak{m}$ being the maximal ideal of A). For any integer $m > 0$, denote by $u[m]$, $v[m]$ the homomorphism $H[m] \to G$ induced by u, v, we then have $(u[m])_n = (v[m])_n$. As this is valid for any n and $H[n]$ is finite over S in view of Proposition 12.8.7, it follows from Lemma 12.8.34 that $u[m] = v[m]$. Since this is valid for any m, we conclude that $u = v$ from Theorem 12.8.30.

We now consider the case where G is of finite presentation. As H is also of finite presentation over S, by ([?] 8.8.2), we can suppose that S is local with closed point s, and show that $u = v$ in this case. If $f : S' \to S$ is a faithfully flat and quasi-compact morphism and we denote by f_H the morphism $H' \to H$ and by $u', v' : H' \to G'$ the morphisms induced by u, v, then the equality $u' = v'$ implies $u \circ f_H = v \circ f_H$, whence $u = v$ since f_H is an epimorphism. Therefore, by taking a base change by faithfully flat and quasi-compact morphisms, we can suppose that H is diagonalizable, hence of the form $D_S(M)$, with M a finitely generated abelian group.

Now introduce as in the proof of Corollary 12.8.29 the directed family of finite type sub-$\mathbb{Z}$-algebras A_i of $A = \mathscr{O}_{S,s}$, and $S_i = \mathrm{Spec}(A_i)$. Note that $H = D_S(M)$ provides, for any i, a diagonalizable group $H_i = D_{S_i}(M)$. As G is of finite presentation over S, by ([?] 8.8.2) (also see [?] VI_B, 10.2 et 10.3), there exists an index i, a group scheme G_i of finite presentation over S_i, and morphisms of S_i-groups $u_i, v_i : H_i \to G_i$ which become u, v under base change. Let s_i be the image of s in S_i and $\rho_i : H_i \times_{S_i} G_i \to G_i$ be the morphism of S_i-schemes defined by $\rho_i(h, g) = u_i(h) g u_i(h)^{-1}$. Then, as u_s is central, $\rho_s = \rho_i \times_{\kappa(s_i)} \kappa(s)$ is equal to the second projection, and hence so is ρ_i (since $\kappa(s_i) \to \kappa(s)$ is faithfully flat and quasi-compact), i.e. $(u_i)_{s_i}$ is central. Similarly, as $u_s = v_s$, we have $(u_i)_{s_i} = (v_i)_{s_i}$. We can then apply the locally Noetherian case of the theorem to the situation over S_i, which proves the conclusion. □

Corollary 12.8.36. *Under the hypothesis of Theorem 12.8.35, let $s \in S$ and suppose that u_s is trivial. Then there exists an open neighborhood U of s such that $u|_U$ is trivial.*

Proof. It suffices to apply Theorem 12.8.35 to u and the trivial homomorphism, which is central. □

Corollary 12.8.37. *Under the hypothesis of Theorem 12.8.35, assume that G is separated over S. Let U be the set of $s \in S$ such that u_s is trivial, then U is clopen in S and $u|_U : H_U \to G_U$ is trivial homomorphism.*

Proof. It remains to show that U is also closed. For this, let $K = \ker u$; as G is separated over S, this is a closed subscheme of H, and U is the set of $s \in S$ such that $K_s = H_s$. We then easily see that U is closed, for example by applying Theorem 12.6.39 (H is essentially free on S according to Example 12.6.38). In fact, we note that if $K_{S'} = H_{S'}$ for a morphism $S' \to S$, then for any $s' \in S'$ with image $s \in S$, the fiber $K_{s'} = H_{s'}$, and hence $K_s = H_s$ (cf. [?] VIII 5.4). This implies $s \in U$, so the morphism $S' \to S$ factors through $U \to S$. The converse of this is immediate, and we conclue that U represents the functor $\mathrm{Res}_{H/S} K$. □

Corollary 12.8.38. *Let S be a scheme, H and G be S-groups, with H of multiplicative finite type and G separated and of finite presentation over S. Let $\pi : S' \to S$ be a universally effective epimorphism with geometrically connected fibers. Let $u' : H' \to G'$ be a central homomorphism of S'-groups, then there exists a unique homomorphism $u : H \to G$ of S-groups such that $u \times_S S' = u'$. If S' admits a section g over S, then u is the morphism induced from u' by base change $g : S \to S'$.*

Proof. As π is a universally effective epimorphism, so is $H' \to H$, whence the uniqueness of u. If π admits a section g, then $u' = \pi^*(u)$ implies $u = g^*\pi^*(u) = g^*(u')$. For the existence of u, as π is effective, we are reduced to show that the two homomorphisms $u_1'', u_2'' : H'' \to G''$ of S''-groups induced from u' by base change $\mathrm{pr}_1, \mathrm{pr}_2 : S'' = S' \times_S S' \to S'$, are identical. Now these coincide over the diagonal of S'', since the inverse image of u_1'', u_2'' under the diagonal morphism $S' \to S''$ are identical (equal to u'). As u_1'' and u_2'' are central, we can apply Corollary 12.8.36 to the morphism $u_1''(u_2'')^{-1}$. There then exists a clopen subset U of S'', containing the diagonal of S'', such that u_1'' and u_2'' coincide over U. Since the fibers of S' over S are geometrically connected, so are those of S'' over S, which are a fortiori connected, whence U contains these fibers, and hence equals to S''. This completes the proof of the corollary. □

Corollary 12.8.39. *Let S be a scheme, K be an S-group locally of finite type with connected fibers, H be a normal subgroup of multiplicative finite type of K. Then H is a central subgroup of K.*

Proof. We first note that $\pi : K \to S$ has connected fibers, since for a group scheme locally of finite type over a field, connected implies geometrically connected (Proposition 12.5.13). We can then apply Corollary 12.8.38 to $G = H$, $S' = K$, and to the homomorphism of K-groups $u' : H_K \to H_K$ induced setwisely by $(h,k) \mapsto (khk^{-1}, k)$, which is central since H is commutative. The inverse image of u' under the unit section $e : S \to K$ is the identity homomorphism of H, so by Corollary 12.8.38 the same is true for $H_K \to H_K$, which implies that H is central in K. □

Remark 12.8.40. Let G be a group scheme over S, H_1, H_2 be subgroups of K of multiplicative finite type over S, and assume that S is locally Noetherian or G is of finite presentation over S. By applying the above results to the projections of $H_1 \times H_2$, we obtain similar results on the coincidence locus of H_1 and H_2. For example, if $s \in S$, and $(H_1)_s$ is central, then there exists an open neighborhood U of s such that $H_1|_U = H_2|_U$. Moreover, if G is separated over S, then the set U of $s \in S$ such that $(H_1)_s = (H_2)_s$ is clopen in S, and $(H_1)|_U = (H_2)|_U$.

Theorem 12.8.41. *Let S be a scheme, $u : H \to G$ be a homomorphism of S-groups, with H of multiplicative finite type and G of finite presentation over S.*

(a) *Suppose that G has connected fibers, and let $s \in S$ be such that $u_s : H_s \to G_s$ is a central homomorphism. Then there exists an open neighborhood U of s such that the homomorphism $u|_U : H|_U \to G|_U$ is central.*

(b) *Suppose that for any $s \in S$, $u_s : H_s \to G_s$ is central, then u is central.*

Proof. To prove (a), by reasoning as in Theorem 12.8.35, we can assume that S is local with closed point s, that $H = D_S(M)$ is diagonalizable, and that there eixsts a finite type sub-$\mathbb{Z}$-algebra A_i of $A = \mathscr{O}_{S,s}$ and a morphism $u_i : D_{S_i}(M) \to G_i$ such that u_{s_i} is central (s_i is the image of s in S_i) and induces u by base change. We must show that u is central in this case.

Let K be the subscheme $\ker(v,w)$, where $v, w : H \times_S G \rightrightarrows G$ are defined by

$$v(h,g) = g, \quad w(h,g) = \mathrm{Inn}(u(h)) \cdot g = u(h)gu(h)^{-1}.$$

Then K is a subgroup of the G-group $H_G = H \times_S G$, and we need to show that it equals to H_G. In view of Corollary 12.8.32 (b), it suffices to show that $H_G[m] = H[m] \times_S G$ for any integer $m > 0$, and we can then assume that $H = H[m]$, so H is finite over S.

Let e be the unit element of the fiber G_s, then $S' = \mathrm{Spec}(\mathscr{O}_{G,e})$ is a Noetherian local scheme (G is locally Noetherian since S is Noetherian); put $S'_n = S' \times_S S_n$, where $S_n = \mathrm{Spec}(A/\mathfrak{m}^{n+1})$. Then $K_{S'} = K \times_G S'$ is a subscheme of $H_{S'} = H \times_S S'$ and, by Corollary 12.8.23, we have $K_{S'_n} = H_{S'_n}$ for any n. As $H_{S'}$ is finite over S', we conclude from Lemma 12.8.34 that $K_{S'} = H_{S'}$.

On the other hand, as $H \times_S G$ is Noetherian (being of finite presentation over S), the immersion $K \hookrightarrow H \times_S G$ is of finite type by Corollary 8.6.37, so K is of finite type over G, and hence of finite presentation over G. Then the equality $K_{S'} = H_G \times_G S'$ implies, by ([?] 8.8.2.4), that there exists an open neighborhood W of e in G such that $K \times_G W = H_G \times_G W = H \times_S W$. Now K and $V = H \times_S W$ can both be considered as subschemes of the scheme G_H over K, and $K \times_G W = K \times_{G_H} V$; therefore, we conclude that K dominates the open neighborhood V of the unit section of G_H over H.

For any $t \in H$, the fiber G_t (being an algebraic $\kappa(t)$-group) is connected, hence irreducible (Proposition 12.5.13). By Proposition 8.6.71, the open subset V_t is then schematically dense in G_t, so V is schematically dense in G_H by Lemma 12.8.27. Moreover, as K is a sub-H-group of G_H, it induces on each fiber G_h a subgroup K_h, and as the latter dominates an open neighborhood of the identity element of G_h (which is dense in G_h), it follows that $K_h = G_h$ for each h, whence $|K| = |G_H|$. Now K is then a closed subscheme of G_H which dominates the schematically dense subscheme V, so $K = G_H$ and this proves (a), and (b) is a direct concequence of (a), since the formation of K commutes with restrictions. □

12.8.5 Canonical factorization of morphisms from groups of multiplicative type

Lemma 12.8.42. *Let S be a quasi-compact scheme, G be an abelian S-group of finite presentation and quasi-finite over S. Then there exists an integer $n > 0$ such that $n \cdot \mathrm{id}_G = 0$, i.e. $G = G[n]$.*

Proof. □

Corollary 12.8.43. *Let S be an S-scheme, H be an S-group of multiplicative finite type, and K be a closed subgroup of H which is of finite presentation over S. Let $s \in S$ be such that K_s is quasi-finite over $\kappa(s)$ at the identity. Then there exists an open neighborhood U of s such that $K|_U$ is contained in $H[n]|_U$ for an integer $n > 0$, and a fortiori such that $K|_U$ is finite over U.*

Proof. By Proposition 12.6.8, we see that there exists an neighborhood U of s such that $K|_U$ is locally quasi-finite over U. By shrinking U if necessary, we may assume that U is affine, hence quasi-compact, and $K|_U$ is quasi-finite and of finite presentation over U. As H is commutative (hence so is K), Lemma 12.8.42 then implies that $n \cdot \mathrm{id}_{K|_U} = 0$, so $K|_U$ is contained in $H[n]|_U$. □

From Corollary 12.8.43, we deduce the finiteness of certain closed subgroups of a multiplicative group. By applying Nakayama's lemma, we immediately deduce the following important result:

Proposition 12.8.44. *Let S be an S-scheme, H be an S-group of multiplicative finite type, and K be a closed subgroup of H which is of finite presentation over S. Assume that K_s is the trivial group, then there exists an open neighborhood U of s such that $K|_U$ is the trivial group.*

Proof. By Corollary 12.8.43, there exists an open neighborhood V of s such that $K|_V$ is finite over V. By taking an open affine neighborhood in V, we may assume that $V = \mathrm{Spec}(A)$ is affine, and then $K|_V = \mathrm{Spec}(B)$ for a finite A-algebra B. Let $\mathfrak{p}$ be the prime ideal of A corresponding to s, then by hypothesis, we have $A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} \cong B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}}$, so $A_{\mathfrak{p}} \cong B_{\mathfrak{p}}$ in view of Nakayama's lemma. As K is of finite presentation over S, so is $K|_V$ over V, and we conclude from ([?] 8.8.2.4) that there exists an open neighborhood U of s in V such that $K|_U \cong U$, i.e. $K|_U$ is the trivial group. □

Remark 12.8.45. We already know that (cf. Proposition 12.6.13) a group G locally of finite type over S is trivial if and only if its fibers G_s are trivial. Proposition 12.8.44 then shows that this property is "open" for certain closed subgroups of a multiplicative group.

Corollary 12.8.46. *Let $u : H \to G$ be a homomorphism of S-groups, with H of multiplicative finite type and G separated over S. Suppose that S is locally Noetherian or G is locally of finite type over S. Let $s \in S$ be such that $u_s : H_s \to G_s$ is a monomorphism, then there exists an open neighborhood U of s such that $u|_U : H|_U \to G|_U$ is a monomorphism.*

Proof. Let $K = \ker u$, the hypothesis implies that K_s is the trivial group. Now G being separated over S, K is a closed subgroup of H, and in the case where S is not locally Noetherian but G is of finite presentation over S, as $H \to S$ is of finite presentation, by Proposition 8.6.24 (v) we see that $H \to G$ is locally of finite presentation, hence so is $K \to S$, and K is of finite presentation over S since it is separated over S and quasi-compact over S (being closed in H, which is quasi-compact over S). We can then apply Proposition 12.8.44 to conclude the corollary. □

Remark 12.8.47. Under the hypothese of Corollary 12.8.46, we note that if G is affine over S (resp. of finite presentation over S), then $H|_U \to G|_U$ is in fact a closed immersion (Corollary 12.8.10 and Remark 12.8.11).

Corollary 12.8.48. *Let $u : H \to G$ be a homomorphism of S-groups, with H of multiplicative finite type and G separated over S. Suppose that S is locally Noetherian or G is locally of finite type over S. If for any $s \in S$, the homomorphism $u_s : H_s \to G_s$ is a monomorphism, then u is a monomorphism.*

Proof. The reasoning is the same as Corollary 12.8.46, the hypothesis that G is separated over S ensures that K is closed in H, since the hypothesis that u_s are monomorphisms imply that K is reduced setwisely to the unit section of G. □

Theorem 12.8.49. *Let $u : H \to G$ be a homomorphism of S-groups, with H of multiplicative finite type and G separated over S. Suppose that S is locally Noetherian or G is of finite presentation over S. Then $K = \ker u$ is a subgroup of multiplicative finite type over H, and u factors into*

$$H \xrightarrow{\bar{u}} H/K \xrightarrow{u'} G$$

where H/K is of multiplicative finite type, $\bar{u}$ is the canonical homomorphism (which is faithfully flat and affine), and u' is a monomorphism.

Proof. It suffices to prove that K is of multiplicative type, since the rest of the proposition then follows from Proposition 12.8.12 (c). We first suppose that G is of finite presentation over S. This hypothesis is stable under base change, so we may assume that H is diagonalizable, i.e. $H = D_S(M)$ with M a finitely generated abelian group, and prove that K is of multiplicative type. Let $s \in S$, then K_s is a closed subgroup of $H_s = D_{\kappa(s)}(M)$, hence is of the form $D_{\kappa(s)}(N)$ (Proposition 12.8.16), where N is a quotient group of M. Put $K' = D_S(N)$, then K' is a subgroup of multiplicative type of H; let $v' : K' \to G$ be the morphism induced by u. Then v'_s is the trivial homomorphism by our construction, so by Corollary 12.8.36 there exists an open neighborhood U of s such that $v'|_U : K'|_U \to G|_U$ is trivial. By replacing S with U, we may hence assume that v' is the trivial homomorphism, so u factors into

$$H \xrightarrow{\bar{u}} H/K' \xrightarrow{u'} G$$

Now since u'_s is induced from u_s by factorizing through $H_s \to H_s/(\ker u_s)$, we see that it is a monomorphism (Remark 12.3.55), hence in view of Corollary 12.8.46, there exists an open neighborhood V of s such that $u'|_V : (H/K')|_V \to G|_V$ is a monomorphism. By replacing S with V, we can therefore assume that u' is a monomorphism, whence $\ker u = \ker \bar{u} = K'$, and this proves that $\ker u$ is of multiplicative type by our construction.

The same proof is valid if we assume that S is locally Noetherian, at least in the case where H is diagonalizable. If we do not make this assumption on H, we must show that there exists a faithfully flat and quasi-compact morphism $S' \to S$, with S' locally Noetherian, which trivializes H. This is indeed what we will see in the next section (§§12.9.3). □

12.8.6 Algebricity of formal homomorphisms

Theorem 12.8.50. *Let A be a Noetherian ring and $\mathfrak{I}$ be an ideal of A such that A is complete for the $\mathfrak{I}$-adic topology. Let $S = \mathrm{Spec}(A)$, $S_n = \mathrm{Spec}(A/\mathfrak{I}^{n+1})$, and H, G be S-groups with H of isotrivial multiplicative type and G affine over S. Then the canonical map*

$$\theta : \mathrm{Hom}_{S\text{-}\mathbf{Grp}}(H, G) \to \varprojlim_n \mathrm{Hom}_{S_n\text{-}\mathbf{Grp}}(H_n, G_n) \tag{12.8.1}$$

is bijective (H_n and G_n are the S_n-groups induced by base change $S_n \to S$).

Corollary 12.8.51. *Under the hypotheses of Theorem 12.8.50, suppose that G is smooth over S, and let $u_0 : H_0 \to G_0$ be a homomorphism of S_0-groups. Then there exists a homomorphism of S-groups $u : H \to G$ lifting u_0, and any two such liftings u, u' are conjugate by an element $g \in G(S)$ which reduces to the identity in $G_0(S_0)$.*

Proof. The conjugation statement follows from Theorem 12.8.21. To construct u, we construct inductively for each n a morphism $u_n : H_n \to G_n$ such that u_n is deduced from u_{n+1} by reduction, which is possible by Theorem 12.8.21. By virtue of Theorem 12.8.50, the system (u_n) comes from a morphism $u : H \to G$. Given two liftings u and u', to construct g such that $u' = \mathrm{Inn}(g)u$ and $g_0 = 1$, we can construct g_n step by step, such that $g_0 = 1$, g_n is deduced from g_{n+1} by reduction, and $u'_n = \mathrm{Int}(g_n)u_n$; this is possible thanks to Theorem 12.8.21. As A is separate and complete, the g_n come from an element $g \in G(S)$ and to prove that $u' = \mathrm{Inn}(g)u$, it suffices to use the injectivity of Theorem 12.8.50. □

Remark 12.8.52. We recall that a similar result is proved in ([?] III 5.1), with the assumption that H is proper over S and G is separated and locally of finite type over S. The fact that Theorem 12.8.50 is true without any properness assumption is quite unexpected, and can be interpreted as a "rigidity" of the structure of a group of multiplicative type. The analogous statement with $G = H = \mathbb{G}_a$ (additive group) is false in general, as we see by taking A of characteristic $p > 0$ and defining the u_n to be the reduction mod $\mathfrak{I}_{n+1}$ of an additive formal series

$$u(T) = \sum_{n\geq 0} a_n T^{p^n}$$

where a_n are elements of A which tends zero for the $\mathfrak{I}$-adic topology, but not for the discrete topology (i.e. (a_n) is not eventually zero). The statement of Theorem 12.8.50 is also false if we drop the hypothesis that G is affine, even for $H = \mathbb{G}_m$. To see an example, consider a discrete valuation ring A and an elliptic curve over the fraction field K of A, which reduces (in the reduction theory of Néron-Kodaira, say) to the group $\mathbb{G}_m$ over the residual field k. We will then have a smooth commutative group scheme G over S, whose special fiber is $\mathbb{G}_{m,k}$ (which by Theorem 12.8.21 allows us to define a projective system of isomorphisms $u_n : H_n \xrightarrow{\sim} G_n$, where $H = \mathbb{G}_{m,A}$), but whose generic fiber is an abelian variety, so that there is no nontrivial homomorphism of S-groups $H \to G$.

12.8.7 Groups of multiplicative type over a field

Let k be a field and G be an affine k-group. We denote by $\mathscr{O}(G)$ the corresponding Hopf algebra of G and by $X(G) = \mathrm{Hom}_{k\text{-}\mathbf{Grp}}(G, \mathbb{G}_{m,k})$ the group of characters of G. By the lemma of independence of characters, we see that $X(G)$ is a free subset of $\mathscr{O}(G)$ over k, and it corresponds to the group of group-like elements of $\mathscr{O}(G)$, i.e. to elements $e \in \mathscr{O}(G)$ such that $\Delta(e) = e \otimes e$ and $\varepsilon(e) = 1$.

Lemma 12.8.53. *Let A be a Hopf algebra over a field k. Then the group-like elements in A are linearly independent over k. In particular, if G is an affine k-group, then distinct characters of G are linearly independent.*

Proof. If not, it will be possible to express one group-like element e as a linear combination of group-like elements

$$e = \sum_i a_i e_i, \quad a_i \in k.$$

We may even suppose that the e_i occurring in the sum are linearly independent. Now we have

$$\Delta(e) = e \otimes e = \sum_{ij} a_i a_j e_i \otimes e_j,$$
$$\Delta(e) = \sum_i a_i \Delta(e_i) = \sum_i a_i e_i \otimes e_i.$$

Since $e_i \otimes e_j$ are linearly independent, this implies that

$$\begin{cases} a_i a_i = a_i & \text{for all } i, \\ a_i a_j = 0 & \text{if } i \neq j. \end{cases}$$

We also know that $1 = \varepsilon(e) = \sum_i a_i \varepsilon(e_i) = \sum a_i$, so combining these statements, we see that the a_i form a complete set of orthogonal idempotents in the field k, and so one of them equals 1 and the remainder are zero, which contradicts our assumption that e is not equal to any of the e_i. $\square$

Proposition 12.8.54. *An affine k-group G is diagonalizable if and only if $\mathscr{O}(G)$ is generated by group-like elements.*

Proof. If $G = D_k(M)$ for an abelian group M, then $\mathscr{O}(G)$ is isomorphic to $k[M]$, and we know that $k[M]$ is generated by its group-like elements. Conversely, if $\mathscr{O}(G)$ is generated by its subgroup M of group-like elements, then by Lemma 12.8.53, M is a basis over k for $\mathscr{O}(G)$, so the induced homomorphism $k[M] \to \mathscr{O}(G)$ is an isomorphism of k-vector spaces. This isomorphism is compatible with the comultiplication maps because it is on the basis elements $m \in M$. $\square$

Proposition 12.8.55. *Let G be a diagonalizable group (resp. group of multiplicative type) over a field k. Then for any subgroup H of G, H and G/H is diagonalizable (resp. of multiplicative type).*

Proof. By taking an fpqc base change, we may assume that G is diagonalizable. Using ([**?**] VI_B 11.13) and Theorem 12.7.14, we can reduce to the case where G is of finite type over k. If H is a subgroup of G, then $H \hookrightarrow G$ is a closed immersion, so H is affine and the ring homomorphism $\mathscr{O}(G) \to \mathscr{O}(H)$ is surjective, and it sends group-like elements to group-like elements (being a homomorphism of Hopf algebras). As the group-like elements of $\mathscr{O}(G)$ span it, the same is true of $\mathscr{O}(H)$, so H is diagonalizable. It then follows from Theorem 12.7.14 that G/H is identified with $D_k(N)$. $\square$

Proposition 12.8.56. *Let k be a field, H, K be k-groups of multiplicative finite type, and G be a k-group such that we have an exact sequence*

$$1 \longrightarrow H \longrightarrow G \longrightarrow K \longrightarrow 1$$

(this implies that G is of finite type over k.)

(a) *If G is abelian or K is connected, then G is of multiplicative type.*

(b) *If K and H are diagonalizable and K is a torus, then G is diagonalizable.*

Proof. For the proof of (a), we refer to ([?] XVII 7.1.1), of which (a) is a particular case; the case of a field is treated in part (a) of the proof of ([?] XVII 7.1.1). Now suppose that K and H are diagonalizable:

$$K \cong D_k(M), \quad H \cong D_k(N)$$

and that G is of multiplicative type (this follows from (a), since K is a torus, hence connected). Then by ([?] X 1.4), G is isotrivial over k, i.e. there exists a finite separable extension k'/k such that $G' = G \times_k k'$ is diagonalizable, hence $G' = D_{k'}(E)$ for an abelian group E, and we have an exact sequence

$$0 \longrightarrow D_{k'}(N) \longrightarrow D_{k'}(E) \longrightarrow D_{k'}(M) \longrightarrow 0 \tag{12.8.2}$$

Hence, by Theorem 12.7.14 and Corollary 12.7.15, M is a subgroup of E and we have an exact sequence

$$0 \longrightarrow M \longrightarrow E \longrightarrow N \longrightarrow 0 \tag{12.8.3}$$

Now for a given extension E_0 of N by M, consider the diagonalizable k-group $G_0 = D_k(E_0)$; the k-group functor of automorphisms of the extension

$$1 \longrightarrow H \longrightarrow G_0 \longrightarrow K \longrightarrow 1 \tag{12.8.4}$$

i.e. the subfunctor in groups of $\mathcal{A}ut_{k\text{-}\mathbf{Grp}}(G_0)$ whose points over a k-scheme T are the $\phi \in \mathrm{Aut}_{T\text{-}\mathbf{Grp}}(G_T)$ inducing the identity over H_T and K_T, is identified with $L = \mathcal{H}om_{k\text{-}\mathbf{Grp}}(K, H)$, which is, by Corollary 12.7.8, the constant k-group associated with $\mathrm{Hom}_{\mathbf{Grp}}(N, M)$. We hence see that the classification of extensions G of K by H, which over a separable closure k^{sep} become isomorphic to the extension (12.8.4), is the same as that of k-torsors for the étale topology under the constant group L_k, which is classified by the Galois cohomology group (L being a trivial Γ-module)

$$H^1_{\text{ét}}(k, L_k) = H^1(\Gamma, L) = \mathrm{Hom}_{\mathbf{Cont.Grp}}(\Gamma, L)$$

If K is a torus, then M is torsion-free, hence so is L, whence $\mathrm{Hom}_{\mathbf{Cont.Grp}}(\Gamma, L) = 0$; it then follows that any extension of K by H is diagonalizable. $\square$

Example 12.8.57. An extension of diagonalizable groups may not be diagonalzable. For example, let G be the group of functor sending S' to the group of monomial matrices over $\mathscr{O}_{S'}$ (that is, a matrix with exactly one nonzero element on each row and each column). Let $D_n \cong D_(\mathbb{Z}^n)$ be the diagonal subgroup of GL_n, then it is not hard to see that we have $G = D_n \rtimes \mathfrak{S}_n$, where $\mathfrak{S}_n$ is the constant group associated with the permutation group of n elements. We thus obtain an exact sequence

$$e \longrightarrow D_n \longrightarrow G \longrightarrow \mathfrak{S}_n \longrightarrow e$$

Now consider the case $n = 2$; the constant group $\mathfrak{S}_2$ is isomorphic to $D(\mathbb{Z}/2\mathbb{Z})$, hence diagonalizable, and G is not diagonalizable since it is not abelian.

12.9 Characterization and classification of groups of multiplicative type

12.9.1 Classification of isotrivial groups of multiplicative type

We recall that a group of multiplicative type H over a scheme S is called **isotrivial** if there exists a finite surjective étale morphism $S' \to S$ such that $H' = H \times_S S'$ is diagonalizable. We note that any such scheme S' over S is dominated by a finite étale connected S-scheme S'_1, which is Galois, i.e. a homogeneous principal bundle over S under the group Γ_S, where Γ is an ordinary finite group (cf. [?] 18.2.9). We may thus suppose that S' is of this form, and we propose the determination of groups of multiplicative type over S which *splits* over S', i.e. such that $H' = H \times_S S'$ is diagonalizable. By descent theory ([?] VIII 2.1 et 5.4), the category of such H is equivalent to the category of diagonalizable groups H' over S', endowed with an action of Γ compatible with the action of Γ on S'. Now as S' is connected, the contravariant functor

$$M \mapsto D_{S'}(M)$$

is an anti-equivalence from the category of ordinary abelian groups to the category of diagonalizable groups over S', whose quasi-inverse is given by $H \mapsto \mathrm{Hom}_{S'\text{-}\mathbf{Grp}}(H, \mathbb{G}_{m,S'})$[41] (cf. Corollary 12.7.9). We have therefore proved the following proposition:

Proposition 12.9.1. *Let S be a connected scheme and S' be a Galois covering of S with group Γ. Then the category of groups of multiplicative type over S which split over S' is anti-equivalent to the category of Γ-modules, i.e. the ordinary abelian groups endowed with an action $\Gamma \to \mathrm{Aut}_{\mathbf{Grp}}(M)$.*

Now we want to generalize Proposition 12.9.1 to the category of any isotrivial multiplicative group over S, without fixing a trivialization. For this, we recall that the category of finite surjective étale morphisms $S' \to S$ is a Galois category, provided we choose a base point $\bar{s}$, i.e. a geometric point $\bar{s} : \mathrm{Spec}(\Omega) \to S$.

Corollary 12.9.2. *Let S be a connected scheme and $\bar{s} : \mathrm{Spec}(\Omega) \to S$ be a geometric point of S (where Ω is a separably closed field). Consider the fundamental group of S with base $\bar{s}$:*

$$\pi = \pi_1(S, \bar{s}).$$

Then the category of isotrivial multiplicative groups H over S is anti-equivalent to the category of Galois π-modules[42] *whose action comes from a finite quotient of π (i.e. such that the kernel $\pi \to \mathrm{Aut}_{\mathbf{Grp}}(M)$ is open).*

Proof. For any isotrivial multiplicative group H over S, we can associate the group

$$M = \mathrm{Hom}_{\Omega\text{-}\mathbf{Grp}}(H_{\bar{s}}, \mathbb{G}_{m,\Omega}),$$

where $H_{\bar{s}}$ is the geometric fiber of H at $\bar{s}$; this group is endowed in a natural way an action of $\pi_1(S, \bar{s})$, which factors through the Galois group of a trivilizing Galois cover of S. Conversely, if M is a Galois π-module whose action comes from a finite quotient Γ, then this corresponds to a Galois covering S' over S, and we can form the diagonalizable group $H' = D_{S'}(M)$, which is naturally endowed with an action of Γ. By Proposition 12.9.1, we then obtain a group H of multiplicative type over S which split over S'. By taking a section of $\mathrm{Spec}(\Omega)$ to $S'_{\bar{s}}$, which is isomorphic to $\coprod_{g\in\Gamma} \mathrm{Spec}(\Omega)$, we then conclude that $\mathcal{H}om_{\Omega\text{-}\mathbf{Grp}}(H_{\bar{s}}, \mathbb{G}_{m,\Omega})(S_{\bar{s}}) \cong M$.

Finally, if H is an isotrivial multiplicative group over S and $S' \to S$ is a Galois covering which trivializes H with Galois group Γ. If $\Gamma_1 = G/\ker(\pi \to \mathrm{Aut}(M))$ with $M = \mathrm{Hom}_{\Omega\text{-}\mathbf{Grp}}(H_{\bar{s}}, \mathbb{G}_{m,\Omega})$, then the action of π factors through Γ, so Γ_1 is a quotient of Γ and hence the Galois covering S'_1 corresponding to Γ_1 dominates S'. Now the diagonalizable group $D_{S'_1}(M)$, endowed with the descent data given by the action of Γ_1, has descent object $D_{S'}(M)$ over S' (since the action of Γ is induced from that of Γ_1). The corresponding descent objects of these descent datum are therefore isomorphic, which proves that the above two constructions are inverses of each other. □

We will see below (cf. **??**) that if S is normal, or more generally geometrically unibranch, then any group of multiplicative finite type over S is necessarily isotrivial, so classification principle Corollary 12.9.2 is applicable to groups of multiplicative finite type on S, which correspond to Galois π-modules which are finitely generated over $\mathbb{Z}$. In particular, we have the following result:

Proposition 12.9.3. *Let k be a field, H be a group of multiplicative finite type over k. Then H is isotrivial, i.e. there exists a finite separable extension k' over k which trivializes H. Concequently, if π is the absolute Galois group of k, then the category of groups of multiplicative finite type over k is anti-equivalent to the category of finitely generated*[43] *Galois π-modules*[44].

Proof. Since H is of finite type over k, this follows from the "principle of finite extension" (cf. [**?**] 9.1.4): By hypothesis, there exists a diagonalizable group G of finite type over k, a faithfully flat and quasi-compact morphism $S' \to S = \mathrm{Spec}(k)$, and an isomorphism of S'-groups $u : H_{S'} \cong G_{S'}$. By replacing S' with the residue field of a point of S', we may asume that S' is the spectrum of a field extension K of k. The latter is the inductive limit of subalgebras A_i of finite type, it follows from ([**?**] 8.8.2.4) that u comes

[41] Note that this is the set of characters of H *defined over* S'.

[42] By definition, a Galois π-module M is an abelian group endowed with a continuous action by π. Equivalently, this means the stabilizer of any point of M is open in π.

[43] A Galois π-module M is finitely generated if and only if it is finitely generated over $\mathbb{Z}$. In fact, if $m_1, \dots, m_n$ is a generating family of M over $\mathbb{Z}[\pi]$, then the stabilizer of each m_i has finite index in π, so its orbit under π is finite. Therefore, the m_i, together with their conjugates, form a finite subset of M which generates M over $\mathbb{Z}$.

[44] We note that if M is a finitely generated Galois π-module, then the kernel of $\pi \to \mathrm{Aut}_{\mathbf{Grp}}(M)$ is necessarily open.

from an A_i-isomorphism $u_i : H_{A_i} \cong G_{A_i}$, for i large enough. In view of Hilbers Nullstellensatz, there exists a quotient field k' of A_i which is a finite extension of k, this extension then trivilizes H. Now k' is a purely inseparable extension of a separable extension k'_s of k. In view of Corollary 12.8.38 (the extension $k' \to k'_s$ is purely inseparable, so the corresponding morphism has geometrically connected fibers), the isomorphism $u' : H_{k'} \cong G_{k'}$ comes from an isomorphism $H_{k'_s} \cong G_{k'_s}$, which completes the proof. □

Example 12.9.4. The correspondence of Corollary 12.9.2 gives in particular a characterization of isotrivial tori over S of relative dimension n: put $\pi = \pi_1(S, \bar{s})$, then the isomorphism classes of such tori correspond to representations $\pi \to \mathrm{GL}(n, \mathbb{Z})$, whose kernel is an open subgroup of π.

Example 12.9.5. Let k be a field and $\bar{k}$ be an algebraic closure of k. Then any finite separable extension of k can be embedded into $\bar{k}$, so in the correspondence of Proposition 12.9.3, if M is a finitely generated π-module, the corresponding multiplicative group H over k can be deduced by descenting $D_{\bar{k}}(M)$. In particular, for any extension K of k, we then have

$$H(\mathrm{Spec}(K)) = D_{\bar{k}}(M)^{\pi}(K) = D_{\bar{k}}(M)^{\pi_K}(K) = \mathrm{Hom}_{\mathbf{Grp}}(M, \bar{k}^{\times})^{\pi_K},$$

where π_K is the Galois group $\mathrm{Gal}(\bar{k}/K)$. In other words, the value of H over K is the set of group homomorphisms $M \to \bar{k}^{\times}$ commuting with the actions of π_K.

As a particular example, consider $k = \mathbb{R}$ and $\bar{k} = \mathbb{C}$, so $\pi = \mathbb{Z}/2\mathbb{Z}$, and let $M = \mathbb{Z}$. Then the automorphism group of M is isomorphic to $\mathbb{Z}/2\mathbb{Z}$, so there are two possible actions of π on M (denote by H the corresponding multiplicative group):

(a) The trivial action, and in this case $H(\mathbb{R}) = \mathbb{R}^{\times}$, and $H \cong \mathbb{G}_{m,\mathbb{R}}$.

(b) The action given by the identity on $\mathbb{Z}/2\mathbb{Z}$. Then $H(\mathbb{R}) = \mathrm{Hom}(\mathbb{Z}, \mathbb{C}^{\times})^{\pi}$, which consists of elements of $\mathbb{C}^{\times}$ fixed by the conjugate action of $\mathbb{Z}/2\mathbb{Z}$:

$$z \mapsto \bar{z}^{-1}.$$

Thus $H(\mathbb{R}) = \{z \in \mathbb{C}^{\times} : |z|^2 = 1\} = S^1$, which is compact.

Example 12.9.6. Even if S is an algebraic curve, there may exist over S tori (of relative dimension 2) which are not locally isotrivial (and a fortiori not isotrivial); there can also exist non-isotrivial locally trivial tori. (Note however that such phenomena can occur only if S is not normal, cf. [**?**] X, 5.16). For example, let S be an irreducible algebraic curve (over an algebraically closed field) having a double point a, S' be its normalization, and b, c be the points of S' lying over a. We then constructs a connected principal homogeneous bundle P over S, of structural group $\mathbb{Z}$, by linking infinite copies of S' according to the diagram

$$\begin{tikzcd}
b \arrow[dr] & & b \arrow[dr] & & b \arrow[dr] & \\
& c \arrow[ur] & & c \arrow[ur] & & c
\end{tikzcd}$$

(This is a principal bundle for the étale topology.) Now we have a homomorphism

$$\varphi : \mathbb{Z} \to \mathrm{GL}(2, \mathbb{Z}), \quad n \mapsto \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$$

which allows us to construct a torus T over S, of relative dimension 2, from the trivial torus $\mathbb{G}_m^2$ over P and the descent data induced from φ. (We note that the projection $P \to S$ is covering for the étale topology and a fortiori for the canonical topology of **Sch**, and that the considered descent data is necessarily effective, since $\mathbb{G}_{m,P}^2$ is affine over P). It is not difficult to prove that T is not isotrivial in the neighborhood of a (see [**?**] X, 7.3) (it is trivial in $S - \{a\}$).

12.9.2 Variations of the infinitesimal structure

In this subsection, we let S be a (fixed) scheme and S_0 be a subscheme of S with the same underlying space (i.e. such that the immersion $S_0 \to S$ is a homeomorphism, or equivalently, defined by a nilideal $\mathscr{I}$). We recall that by ([**?**] 18.1.2), the functor

$$X \mapsto X_0 = X \times_S S_0$$

is an equivalence between the category of étale schemes over S and the analogous category over S_0. We now consider the similar question for multiplicative groups over S and S_0:

Proposition 12.9.7. *The functor $H \mapsto H_0 = H \times_S S_0$ from the category of groups of multiplicative type over S to that over S_0 is fully faithful, and it induces an equivalence between the category of quasi-isotrivial multiplicative groups over S and the analogous category over S_0.*

Proof. We first prove the fully faithfulness, i.e. if H, G over S are of multiplicative type, then

$$\mathrm{Hom}_{S\text{-}\mathbf{Grp}}(H, G) \to \mathrm{Hom}_{S_0\text{-}\mathbf{Grp}}(H_0, G_0)$$

is bijective. This question is local over S, so we can assume that S is affine, and there then exists a faithfully flat and quasi-compact morphism $S' \to S$ which splits H and G. Let $S'' = S' \times_S S'$, and denote by H', G' (resp. H'', G'') the groups deduced from H, G by base change $S' \to S$ (resp. $S'' \to S$); similarly, we define S'_0 and S''_0, the latter being isomorphic to $S'_0 \times_{S_0} S'_0$. We then have a commutative diagram with exact rows:

$$\begin{tikzcd}
\mathrm{Hom}_{S\text{-}\mathbf{Grp}}(H,G) \arrow[r] \arrow[d, "u"] & \mathrm{Hom}_{S'\text{-}\mathbf{Grp}}(H',G') \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "u'"] & \mathrm{Hom}_{S''\text{-}\mathbf{Grp}}(H'',G'') \arrow[d, "u''"] \\
\mathrm{Hom}_{S_0\text{-}\mathbf{Grp}}(H_0,G_0) \arrow[r] & \mathrm{Hom}_{S'_0\text{-}\mathbf{Grp}}(H'_0,G'_0) \arrow[r, shift left] \arrow[r, shift right] & \mathrm{Hom}_{S''_0\text{-}\mathbf{Grp}}(H''_0,G''_0)
\end{tikzcd}$$

so to prove that u is bijective, it suffices to show that so are u' and u'', which then reduces us to the case where H, G are diagonalizable, i.e. $H = D_S(M)$ and $G = D_S(N)$, with M and N being ordinary abelian groups. We have similarly $H_0 = D_{S_0}(M)$, $G_0 = D_{S_0}(N)$, so there is a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{S\text{-}\mathbf{Grp}}(N_S, M_S) \arrow[r, "\sim"] \arrow[d] & \mathrm{Hom}_{S\text{-}\mathbf{Grp}}(D_S(M), D_S(N)) \arrow[d] \\
\mathrm{Hom}_{S_0\text{-}\mathbf{Grp}}(N_{S_0}, M_{S_0}) \arrow[r, "\sim"] & \mathrm{Hom}_{S_0\text{-}\mathbf{Grp}}(D_{S_0}(M), D_{S_0}(N))
\end{tikzcd}$$

where the horizontal arrows are isomorphisms in view of Corollary 12.7.6. We are then reduced to prove that the homomorphism

$$\mathrm{Hom}_{S\text{-}\mathbf{Grp}}(N_S, M_S) \to \mathrm{Hom}_{S_0\text{-}\mathbf{Grp}}(N_{S_0}, M_{S_0}) \tag{12.9.1}$$

is bijective, i.e. the functor $M_S \mapsto M_{S_0}$ is faithfully flat. Now (12.9.1) is also identified with the natural map

$$\mathrm{Hom}_{\mathbf{Grp}}(N, \Gamma(M_S)) \to \mathrm{Hom}_{\mathbf{Grp}}(N, \Gamma(M_{S_0}))$$

induced from $\Gamma(M_S) \to \Gamma(M_{S_0})$ (cf. (Eq. (12.7.7))), which is evidently bijective (because $\Gamma(M_S)$ is the set of locally constant maps from M to S, which only depends on the underlying topological space of S), whence the first assertion.

To prove the second assertion of Proposition 12.9.7, we must show that any quasi-isotrivial multiplicative group H_0 over S_0 comes from a quasi-isotrivial multiplicative group H over S. For this, let $S'_0 \to S_0$ be a étale surjective morphism trivilizing H_0. As we have remarked, there then exists an étale morphism $S' \to S$ and an S_0-isomorphism $S' \times_S S_0 \cong S'_0$. As H'_0 is diagonalizable, we immediately see that it comes from a diagonalizable group H' over S' by base change $S'_0 \to S'$ (if $H'_0 = D_{S'_0}(M)$, we can put $H' = D_{S'}(M)$). Define as usual $S'' = S' \times_S S'$, $S''' = S' \times_S S' \times_S S'$, and S''_0, S'''_0. Using the fully faithfulness for the case (S''', S'''_0) and (S''', S'''_0), we see that the natural descent data over H'_0 relative to $S'_0 \to S_0$ comes from a well-defined descent data over H' relative to $S' \to S$. This descent data is effective since H' is affine over S' ([?] VIII 2.1), so there exists an S-group H such that $H \times_S S' = H' = D_{S'}(M)$, and hence is quasi-isotrivial multiplicative. We then verify easily, using the fully faithfulness for (S', S'_0) and (S, S_0), that the given isomorphism between H'_0 and $H' \times_{S'} S'_0$ comes from an isomorphism between H_0 and $H \times_S S_0$. ☐

Corollary 12.9.8. *Let H be an S-group of multiplicative type, and $H_0 = H \times_S S_0$. For H to be quasi-isotrivial (resp. locally isotrivial, resp. isotrivial, resp. locally trivial, resp. trivial), it is necessary and sufficient that H_0 be so.*

Proof. The necessary direction is trivial, and the sufficient part is done for "quasi-isotrivial", because thanks to full fidelity, it suffices to know that any quasi-isotrivial group over S_0 can be lifted into a quasi-isotrivial group over S. The same argument works for "trivial". For the "isotrivial" case, we use the reasoning establishing the second assertion of Proposition 12.9.7, but taking $S_0' \to S_0$ to be finite étale surjective. The "locally isotrivial" and "locally trivial" cases result immediately from the "isotrivial" and "trivial" cases. □

Corollary 12.9.9. *Suppose that the ideal $\mathscr{I}$ defining S_0 is locally nilpotent. Let H be a flat S-group and $H_0 = H \times_S S_0$. For H to be of quasi-isotrivial multiplicative, it is necessary and sufficient that H_0 be so.*

Proof. Suppose that H_0 is quasi-isotrivial multiplicative, and we show that H is so. As the question is local for the étale topology and the category of étale schemes over S is equivalent to that over S_0 under the function $S' \mapsto S' \times_S S_0$, we are then reduced to the case where H_0 is diagonalizable, hence isomorphic to $D_{S_0}(M)$. Let $G = D_S(M)$, we then have an isomorphism $u_0 : H_0 \xrightarrow{\sim} G_0$; we claim that this comes from a unique homomorphism $u : H \to G$, which must be an isomorphism[45]. To see this, note that we have (cf. Eq. (12.7.3))

$$\mathrm{Hom}_{S\text{-}\mathbf{Grp}}(H, G) \cong \mathrm{Hom}_{S\text{-}\mathbf{Grp}}(M_S, \mathcal{H}om_{S\text{-}\mathbf{Grp}}(H, \mathbb{G}_{m,S}))$$

and the second member is identified with $\mathrm{Hom}_{\mathbf{Grp}}(M, \mathrm{Hom}_{S\text{-}\mathbf{Grp}}(H, \mathbb{G}_{m,S}))$, so the homomorphism

$$\mathrm{Hom}_{S\text{-}\mathbf{Grp}}(H, G) \to \mathrm{Hom}_{S_0\text{-}\mathbf{Grp}}(H_0, G_0) \tag{12.9.2}$$

is isomorphic to the homomorphism

$$\mathrm{Hom}_{\mathbf{Grp}}(M, \mathrm{Hom}_{S\text{-}\mathbf{Grp}}(H, \mathbb{G}_{m,S})) \to \mathrm{Hom}_{\mathbf{Grp}}(M, \mathrm{Hom}_{S_0\text{-}\mathbf{Grp}}(H_0, \mathbb{G}_{m,S_0})) \tag{12.9.3}$$

induced from the restriction homomorphism

$$\mathrm{Hom}_{S\text{-}\mathbf{Grp}}(H, \mathbb{G}_{m,S}) \to \mathrm{Hom}_{S_0\text{-}\mathbf{Grp}}(H_0, \mathbb{G}_{m,S_0}). \tag{12.9.4}$$

Therefore, to prove that (12.9.2) is bijective, it suffices to show that (12.9.4) is bijective. Since this question is local over S, we can assume that S is affine, Now H_0 is of multiplicative type and $\mathbb{G}_{m,S}$ is smooth over S and abelian, so the assertion follows from Theorem 12.8.21 (a), which completes the proof[46]. □

Corollary 12.9.10. *Let A be a local Artinian ring with residue field k, $S = \mathrm{Spec}(A)$, and $S_0 = \mathrm{Spec}(k)$.*

(a) *Let H be a flat S-group locally of finite type, such that $H_0 = H \times_S S_0$ is of multiplicative type. Then H is of isotrivial multiplicative finite type over S. In particular, any S-group H of multiplicative finite type is isotrivial.*

(b) *The functor $H \mapsto H_0$ is an equivalence between the category of groups of multiplicative finite type over A to the analogous category over k.*

Proof. Let H be as in (a), then H_0 is of multiplicative type and locally of finite type over S_0, hence of finite type, and isotrivial by Proposition 12.9.3. By Corollary 12.9.8 and Corollary 12.9.9, H is of multiplicative type (hence of finite type) and isotrivial, and assertion (b) follows from Proposition 12.9.7. □

Corollary 12.9.11. *Let $S' \to S$ be a faithfully flat, quasi-compact and radiciel morphism.*

(a) *The functor $H \mapsto H' = H \times_S S'$, from the category of multiplicative type over S to the analogous category over S', is fully faithful, and it induces an equivalence between the subcategory formed by quasi-isotrivial multiplicative groups.*

(b) *If H is of multiplicative type, for it to be quasi-isotrivial (resp. locally isotrivial, resp. isotrivial, resp. locally trivial, resp. trivial), it is necessary and sufficient that H_0 be so.*

Proof. Let $S'' = S' \times_S S'$ and $S''' = S' \times_S S' \times_S S'$, then the hypothesis on $S' \to S$ implies that the diagonal immersions $S' \to S''$ and $S' \to S'''$ are surjective, hence base change along these immersions is justified by Proposition 12.9.7 and Corollary 12.9.8. Given that $S' \to S$ is an effective descent morphism for the fiber category of groups of multiplicative type over schemes (because it is faithfully flat and quasi-compact), our assertion follows formally from Proposition 12.9.7 and Corollary 12.9.8. □

[45]Since u_0 is an isomorphism it suffices to see that for any $h \in H$, the morphism $\phi : \mathscr{O}_{G,u(h)} \to \mathscr{O}_{H,h}$ is an isomorphism. This is true after reduction modulo $\mathfrak{J} = \mathscr{I}_s$ (s being the image of h in S), so its cokernel C satisfies $C = \mathfrak{J}C$, whence $C = 0$ as $\mathfrak{J}$ is nilpotent. As $\mathscr{O}_{H,h}$ is flat over $\mathscr{O}_{S,s}$, the kernel of ϕ also verifies $K = \mathfrak{J}K$, whence $K = 0$.

[46]We recall that the proof of Theorem 12.8.21 (a) only depends on the vanishing of H^1 and H^2; in view of ([?] III 2.1), we only need H_0 to be of multiplicative type.

12.9.3 Variations of finite structure and the quasi-isotrivial theorem

12.9.3.1 Variations of finite structure: case of complete base In this paragraph, we consider a (fixed) Noetherian ring A endowed with an ideal $\mathfrak{I}$ such that A is separated and complete for the $\mathfrak{I}$-adic topology. Let $S = \mathrm{Spec}(A)$, $S_0 = \mathrm{Spec}(A/\mathfrak{I})$, we want to extend the results of the previous subsection to multiplicative groups over S and S_0.

Lemma 12.9.12. *Let G, H be S-groups with G of isotrivial multiplicative type, H affine over S and flat over S at the points of H_0, and H_0 of quasi-isotrivial multiplicative type. Then the following natural map is bijective:*

$$\mathrm{Hom}_{S\text{-}\mathbf{Grp}}(G, H) \to \mathrm{Hom}_{S_0\text{-}\mathbf{Grp}}(G_0, H_0).$$

Proof. For any integer $n \geq 0$, we put $S_n = \mathrm{Spec}(A/\mathfrak{I}^{n+1})$, and let G_n, H_n be the induced groups by base change $S_n \to S$. As G/S is of isotrivial multiplicative type and H is affine over S, by Theorem 12.8.50, the natural homomorphism

$$\mathrm{Hom}_{S\text{-}\mathbf{Grp}}(G, H) \to \varprojlim_n \mathrm{Hom}_{S_n\text{-}\mathbf{Grp}}(G_n, H_n)$$

is bijective. On the other hand, in view of Corollary 12.9.9, the H_n are of quasi-isotrivial multiplicative type, and in view of Proposition 12.9.7, the transition homomorphisms in the projective system $(\mathrm{Hom}_{S_n\text{-}\mathbf{Grp}}(G_n, H_n))_n$ are isomorphism, whence our assertion. □

Theorem 12.9.13. *Consider a Noetherian ring A endowed with an ideal $\mathfrak{I}$ such that A is separated and complete for the $\mathfrak{I}$-adic topology. Let $S = \mathrm{Spec}(A)$ and $S_0 = \mathrm{Spec}(A/\mathfrak{I})$.*

(a) *The functor $H \mapsto H_0 = H \times_S S_0$ is an equivalence between the category of isotrivial multiplicative groups over S to the analogous category over S_0.*

(b) *For an S-group H of multiplicative finite type, H is isotrivial if and only if H_0 is.*

Proof. For the first assertion, one can either repeat the proof of Proposition 12.9.7, or use Corollary 12.9.2, noting that in both cases the functor

$$S' \mapsto S'_0 = S' \times_S S_0$$

from the category of finite étale schemes over S to the category of finite étale schemes over S_0, is an equivalence ([?] I 8.4).

We now prove the second assertion, i.e. that if H_0 is isotrivial, so is H. In view of what we have proved, there exists an isotrivial multiplicative group G over S and an S_0-isomorphism

$$u_0 : G_0 \xrightarrow{\sim} H_0.$$

As H is of finite type, so are H_0 and G_0, so by Proposition 12.8.6 (b), the type of G at each point of S is a finitely generated abelian group, and hence G is of finite type over S. On the other hand, in view of Lemma 12.9.12, u_0 comes from a homomorphism of S-groups

$$u : G \to H.$$

Now G, H are of multiplicative finite type over S, and u_0 is an isomorphism, so by Corollary 12.8.14, u is an isomorphism (any neighborhood of S_0 in S is equal to S). □

Corollary 12.9.14. *Let A be a complete Noetherian local ring with residue field k.*

(a) *Any group of multiplicative finite type over A is isotrivial.*

(b) *The functor $H \mapsto H_0 = H \otimes_A k$ is an equivalence between the category of groups of multiplicative finite type over A and over k.*

Proof. The first assertion follows from Theorem 12.9.13 (b) and Proposition 12.9.3, and the second one follows from Theorem 12.9.13 (a), in view of the fact that H is of finite type if and only if H_0 is. □

Remark 12.9.15. We note that in view of Proposition 12.9.3, Corollary 12.9.14 gives a complete classification of groups of multiplicative fintie type ove A in terms of modules under the absolute Galois group of k.

Remark 12.9.16. Under the hypothesis of Theorem 12.9.13 (i.e. A is Noetherian, separated and complete for the $\mathfrak{I}$-adic topology), it follows from **??** that the functor $H \mapsto H_0$, form the category of groups of multiplicative finite type over S to that over S_0, is fully faithful (without the hypothesis of isotrivality).

However, it is not in general essentially surjective, in fact there can be an S_0-group H_0, of multiplicative finite type, locally trivial if we want (but not isotrivial), which does not come from a group of multiplicative type H over S by reduction. To see this, let us take as in Example 12.9.6 a non-isotrivial multiplicative group over a non-normal curve. We can obviously assume S to be affine, and suppose that it is embedded in the affine space of dimension 2, thus defined by an ideal $\mathfrak{I}$ in $k[X,Y]$. We will take for A the completion of this ring for the $\mathfrak{I}$-adic topology, so that A is a regular ring of dimension 2, a fortiori normal. We will see in ([?] X 5.16) that any group of multiplicative finite type over $S = \operatorname{Spec}(A)$ is isotrivial; hence H_0, which is non-isotrivial and of finite type over S_0, does not come from a group of multiplicative type H over S (because H would necessarily be of finite type, hence isotrivial).

Lemma 12.9.17. *Let H be an abelian algebraic group over a field k which admits an open subgroup G of multiplicative type. Then the family of subschemes $(H[n])_{n>0}$ of H is schematically dense, and in particular if $H[n] = G[n]$ for any $n > 0$, then $H = G$.*

Proof. Let $\bar{k}$ be an algebraic closure of k; it suffices to prove that the family $(H_{\bar{k}}[n])_{n>0}$ is schematically dense in $H_{\bar{k}}$, because then $(H[n])_{n>0}$ is schematically dense in H (Lemma 12.8.28). Hence, we may assume that k is algebraically closed, hence $G = D_k(M)$ for a finitely generated abelian group M. Let $M_0 = M/M_{\mathrm{tor}}$, where M_{tor} denotes the torsion subgroup of M, then $T = D_k(M_0)$ is the largest torus in G, and H/T is finite[47], so $H(k)/T(k)$ is annihilated by an integer $\nu > 0$. We can find finitely many elements $g_i \in H(k)$ such that $H = \coprod_i g_i G$, and we have $g_i^\nu \in T(k)$. As k is algebraically closed, $\nu \cdot \mathrm{id}_T$ is surjective in $T(k) \cong (k^\times)^d$ for some $d > 0$, hence by replacing the g_i by $g_i t_i^{-1}$, where $t_i \in T(k)$ is such that $t_i^\nu = g_i^\nu$, we can assume that $g_i^\nu = 1$. If n is a multiple of ν, we then have

$$H[n] \supseteq g_i \cdot G[n]$$

and as the family $G[n]$ is schematically dense in G in view of Theorem 12.8.30, we conclude that $H[n]$ is schematically dense. □

Theorem 12.9.18. *Let A be a Noetherian ring endowed with an ideal $\mathfrak{I}$ such that A is separated and complete for the $\mathfrak{I}$-adic topology. Let $S = \operatorname{Spec}(A)$, $S_0 = \operatorname{Spec}(A/\mathfrak{I})$, and H be an affine S-group of finite type and affine over S at the points of H_0, such that $H_0 = H \times_S S_0$ is of isotrivial multiplicative type. Then there exists an clopen subgroup G of H of isotrivial multiplicative finite type such that $G_0 = H_0$.*

Proof. In view of Theorem 12.9.13, there exists a group of isotrivial multiplicative type over G and an isomorphism

$$u_0 : G_0 \xrightarrow{\sim} H_0.$$

By Lemma 12.9.12, u_0 comes from a unique homomorphism of S-groups

$$u : G \to H.$$

Using Corollary 12.8.46, we see that u is a monomorphism (because if U is the set of $s \in S$ such that $u_s : G_s \to H_s$ is a monomorphism, then U is an open neighborhood of S_0 in S, and $G|_U \to H|_U$ is a monomorphism). In view of Corollary 12.8.10, u is then a closed immersion, so G is of finite type, hence of finite presentation over S. By Proposition 12.6.5, there exists an open neighborhood U of S_0, hence equals to S, such that $G|_U \to H|_U$ is étale. The morphism u is then étale, whence an open immersion (as it is a étale monomorphism, cf. [?] 17.9.1), and this proves the assertion. □

Corollary 12.9.19. *Under the hypotheses of Theorem 12.9.18, let H be an S-group of finite type, affine and flat over S. For H to be of isotrivial multiplicative type, it is necessary and sufficient that H_0 be so, and that H satisfies the following equivalent conditions:*

(i) *The fibers of H are of multiplicative type, and their types are constant over each connected component of S.*

(ii) *H is abelian and the subgroups $H[n]$ ($n > 0$) are finite over S.*

(iii) *The fibers of H are connected.*

[47] We note that $(H/T)/(G/T)$ is isomorphic to H/G, which is discrete and quasi-compact, hence finite (over k). As G/T is by construction also finite over k, we conclude that H/T is finite over k.

Proof. Of course, if H is of multiplicative type (and isotrivial), then conditions (i) and (ii) are verified, by Proposition 12.8.6 (c), so we only need to prove the sufficient part. We shall utilize the group G indicated in Theorem 12.9.18. If condition (iii) is satisfied, then we have evidently $H = G$. In the case (ii), we note that the open immersion $u : G \to H$ induces an open immersion

$$u[n] : G[n] \to H[n]$$

which induces an isomorphism $(G[n])_0 \xrightarrow{\sim} (H[n])_0$; as $H[n]$ is finite over S, this immediately implies that $u[n]$ is an isomorphism (this follows by Nakayama's lemma, cf. **??**). In view of Lemma 12.9.17, this ensures that the induced morphisms $u_s : G_s \to H_s$ on the fibers are isomorphisms, so u is surjective, whence an isomorphism.

Finally, in case (i), we can assume that S is connected, and then for any $s \in S$, $u_s : G_s \to H_s$ is a monomorphism of algebraic groups of multiplicative type of the same type over $\kappa(s)$[48]. We claim that this monomorphism is necessarily an isomorphism. In fact, we can suppose, by replacing the base field, that the two groups over $\kappa(s)$ are diagonalizable, and then this follows from Corollary 12.7.15 and the fact that a surjective homomorphism of isomorphic finitely generated $\mathbb{Z}$-modules is necessarily bijective. □

Corollary 12.9.20. *Under the hypotheses of Theorem 12.9.18, let H be an S-group of finite type, affine and flat over S, with connected fibers. Then for H to be an isotrivial torus, it is necessary and sufficient that H_0 be so.*

Proof. If H_0 is an isotrivial torus, then its type (determined by the relative dimension) is contant over each connected component of S. Therefore, it follows from Corollary 12.9.19 that H is of isotrivial multiplicative over S; since H_0 is a torus, we conclude that so is H. □

12.9.3.2 Case of arbitrary base. Theorem of quasi-isotrivial We recall that a local ring is called Henselian if any finite algebra B over A is a product of finite local algebras over A. If A is a Henselian local ring, k its residue field, $S = \mathrm{Spec}(A)$, $S_0 = \mathrm{Spec}(k)$, and ξ is a geometric point of S_0, then the functor

$$X \mapsto X_0 = X \times_S S_0$$

is an equivalence from the category of étale coverings over S to the analogous category over S_0 (cf. **[?]** §18.5). In particular, we have $\pi_1(S_0, \xi) = \pi_1(S, \xi)$.

Suppose that A is Noetherian and denote by A' its completion, $S' = \mathrm{Spec}(A')$. Then A' is a complete local Noetherian ring, hence Henselian (**[?]** 18.5.14), and the functor

$$X \mapsto X' = X \times_S S'$$

from the category of étale coverings over S to the analogous category over S', fits into the following commutative diagram:

$$\begin{tikzcd}
\mathbf{F\acute{E}t}_S \arrow[rr] \arrow[dr, "\sim"'] & & \mathbf{F\acute{E}t}_{S'} \arrow[dl, "\sim"] \\
& \mathbf{F\acute{E}t}_{S_0} &
\end{tikzcd}$$

hence is also an equivalence of categories, and $\pi_1(S_0, \xi) = \pi_1(S', \xi)$. We also note that as S is connected (A being local), it follows from Corollary 12.9.2 that (since $\pi_1(S, \xi) = \pi_1(S_0, \xi)$) the category of groups of isotrivial multiplicative type over S is equivalent to the analogous category over S_0 (and also over S' if A is Noetherian).

Lemma 12.9.21. *Let A be a local Henselian ring with residue field k, $S = \mathrm{Spec}(A)$, $S_0 = \mathrm{Spec}(k)$.*

(a) *The functor $H \mapsto H_0 = H \times_S S_0$ is an equivalence from the category of groups of multiplicative type and finite over S to the analogous category over S_0.*

(b) *If A is Noetherian, denote by A' its completion and $S' = \mathrm{Spec}(A')$, the functor $H \mapsto H' = H \times_S S'$ is an equivalence from the category of groups of multiplicative type and finite over S to the analogous category over S'.*

[48] Since G_s and H_s are of the same type for any $s \in S_0$, hence for any $s \in S$ (as A is complete and separated, the connected components of S and S_0 in bijection under $C \mapsto C \cap S_0$, cf. **[?]** 18.5.4 et 18.15.13).

Proof. The second assertion is a concequence of (a), and we see that the considered functor in (a) is essentially surjective, because any group of multiplicative type H_0 finite over $S_0 = \mathrm{Spec}(k)$ (hence of finite type over S_0) is isotrivial by Proposition 12.9.3, hence comes from a group of isotrivial multiplicative type over S (by the remarks above).

To prove the fully faithfulness, i.e. that for any groups G, H of multiplicative type and finite over S, the following map is bijective:

$$\mathrm{Hom}_{S\text{-}\mathbf{Grp}}(G, H) \to \mathrm{Hom}_{S_0\text{-}\mathbf{Grp}}(G_0, H_0)$$

or equivalently, denoting $F = \mathcal{H}om_{S\text{-}\mathbf{Grp}}(G, H)$, that the natural map

$$\mathrm{Hom}_S(S, F) \to \mathrm{Hom}_{S_0}(S_0, F_0)$$

induced by base change $S_0 \to S$, is bijective. For this, in view of the equivalence remarked above, we can utilize Lemma 12.9.22. □

Lemma 12.9.22. *Let G, H be groups of multiplicative type and finite over S. Then $F = \mathcal{H}om_{S\text{-}\mathbf{Grp}}(G, H)$ is representable by a finite étale over S.*

Proof. Let $f : S' \to S$ be a faithfully flat and quasi-compact morphism such that G' and H' are diagonalizable. It then suffices to show that $F_{S'}$ is representable by a scheme X' which is étale and finite (hence affine) over S', because the induced descent data over X' relative to f (cf. Lemma 12.7.10) is then effective ([?] VIII 2.1), whence the existence of a scheme X over S such that $X \times_S S' = X'$, which represents F, and is étale and finite over S (cf. [?] VIII 5.7 et [?] 17.7.3(ii)).

We can hence suppose that $G = D_S(M)$ and $H = D_S(N)$, where M, N are finite abelian groups (cf. Proposition 12.7.13 (c)). Then $K = \mathrm{Hom}_{\mathbf{Grp}}(N, M)$ is a finite abelian group, and by Corollary 12.7.8, we have an isomorphism

$$\mathcal{H}om_{S\text{-}\mathbf{Grp}}(G, H) \cong K_S,$$

which completes the proof of Lemma 12.9.22 (note that K_S is a finite direct sum of copies of S, whence finite étale). □

Lemma 12.9.23. *Let S be a local Henselian scheme, s be its closed point, X be a scheme locally of fintie type over S, x be an isolated point in its fiber X_s.*

(a) *$\mathcal{O}_{X,x}$ is finite over $\mathcal{O}_{S,s}$.*

(b) *If X is separated over S, then $X' = \mathrm{Spec}(\mathcal{O}_{X,x})$ is a clopen subscheme of X, i.e. we have a decomposition $X = X' \coprod X''$.*

Proof. By the local form of Zariski's main theorem ([] IV Th.1), x admits an affine open neighborhood $U = \mathrm{Spec}(B)$ of finite type and quasi-finite over $A = \mathcal{O}_{S,s}$, and there is an open immersion $U \hookrightarrow Y = \mathrm{Spec}(C)$, where C is a finite A-algebra. As A is Henselian, Y is the direct sum of local schemes $Y_1, \dots, Y_n$ (with $Y_i = \mathrm{Spec}(C_i)$, C_i being a finite local A-algebra), each is finite over S, and the points of Y lying over s are the closed points $y_1, \dots, y_n$. Therefore $x = y_i$ for certain index i, and $\mathcal{O}_{X,x} = \mathcal{O}_{U,x} = C_i$ is finite over A. Moreover, $X' = \mathrm{Spec}(C_i)$ is an open subscheme of U, hence of X.

Suppose that X is separated over S. Then as the morphism $X' \to S$ is finite (C_i being finite over A), so is the immersion $X' \to X$ (Proposition 9.6.3), and hence X' is also closed in X. □

Proposition 12.9.24. *Let A be a Noetherian local Henselian ring, A' be its completion, $S = \mathrm{Spec}(A)$, $S' = \mathrm{Spec}(A')$, and s be the closed point of S. Let G be a group of multiplicative finite type over S and H be a S-group locally of finite type and separated over S, such that H_s is of multiplicative type and H is flat over S at the points of H_s. Then the following natural map*

$$\mathrm{Hom}_{S\text{-}\mathbf{Grp}}(G, H) \xrightarrow{\sim} \mathrm{Hom}_{S'\text{-}\mathbf{Grp}}(G', H')$$

is bijective, where G', H' are induced by base change $S' \to S$.

Theorem 12.9.25. *Let S be a scheme, H be an affine S-group of finite presentation over S, and $s \in S$. Suppose that*

(α) *H is flat over S at the points of H_s.*

(β) *H_s is of multiplicative type.*

Then there exsits an étale morphism $S' \to S$, a point s' of S' lying over s such that $\kappa(s') = \kappa(s)$, and a clopen subgroup G' of $H' = H \times_S S'$ of isotrivial multiplicative finite type, such that $G'_{s'} = H'_{s'}$.

Corollary 12.9.26. *Let S be a scheme and H be an S-group of multiplicative finite type. Then H is quasi-isotrivial, i.e. is trivialized by a surjective étale morphism $S' \to S$.*

Proof. For any point $s \in S$, by Theorem 12.9.25, there exists an étale morphism $S' \to S$, a point $s' \in S'$ lying over s such that $\kappa(s) = \kappa(s')$, and a clopen subgroup G' of H' of isotrivial multiplicative finite type such that $G'_{s'} = H'_{s'}$. As G' and H' are of multiplicative finite type, by Corollary 12.8.14 there exists an open neighborhood U' of s' such that $G'|_{U'} = H'|_{U'}$. As étale morphisms are open, by replacing U' by a finite étale surjective morphism, we may assume that there exists an open neighborhood U of s such that $U' \to U$ is étale surjective, and $H \times_U U'$ is diagonalizable; we therefore conclude that H is quasi-isotrivial. □

Corollary 12.9.27. *Let A be a local Henselian ring, k be its residue field, and π be the Galois group of an algebraic closure of k.*

(a) *Any group of multiplicative finite type over $S = \mathrm{Spec}(A)$ is isotrivial.*

(b) *The category of groups of multiplicative finite type over S is equivalent to the analogous category over $S_0 = \mathrm{Spec}(k)$, and hence is anti-equivalent to the category of finitely generated Galois π-modules.*

Proof. With the notations of Theorem 12.9.25, we note that since $S' \to S$ is étale, hence locally quasi-finite, by Lemma 12.9.23 (a), $\mathscr{O}_{S',s'}$ is finite over $\mathscr{O}_{S,s}$. As we have $\kappa(s') = \kappa(s)$ and $\mathfrak{m}_{s'} = \mathfrak{m}_s$ ($S' \to S$ is unramified, cf. Theorem 11.1.20), we conclude from Nakayama's lemma that $\mathscr{O}_{S',s'} \cong \mathscr{O}_{S,s}$. As the image of $\mathrm{Spec}(\mathscr{O}_{S',s'})$ in S' is contained in U' and $S \cong \mathrm{Spec}(\mathscr{O}_{S,s})$, we conclude that there is a section $\sigma : S \to S'$ of S' over S such that the base change of G' along σ is isomorphic to H, whence the assertion in (a). The second assertion now follows from Proposition 12.9.3. □

Corollary 12.9.28. *Under the conditions of Theorem 12.9.25, suppose that one of the following conditions is satisfied:*

(i) *For any generalization t of s, H_t is of multiplicative type and of the same type as H_s.*

(ii) *H is abelian and $H[n]$ ($n > 0$) are finite over S in a neighborhood of s.*

(iii) *For any generalization t of s, the fiber H_t is connected.*

Then there exists an open neighborhood U of s such that $H|_U$ is of multiplicative type (and hence the above conditions are equivalent).

Proof. Since étale morphisms are open, we are reduced to prove that there exists (with the notations of Theorem 12.9.25) an open neighborhood U' of s' such that $G'|_{U'} = H'|_{U'}$. Let $S'' = \mathrm{Spec}(\mathscr{O}_{S',s'})$; as G' and H' are of finite presentation over S', it suffices to prove, according to ([?] 8.8.2), that $G'' = H''$. We can then assume that $S = S''$, and hence reduce to the case where S is local and s is its closed point. In this case, any point of S is a generalization of s, and any neighborhood of s is equal to S (cf. Proposition 8.2.11), so it suffices to apply Lemma 12.9.29 below. □

Lemma 12.9.29. *Let S be a local scheme, s be its closed point, H be an S-group of finite type, and G be an clopen subgroup of H of multiplicative type such that $G_s = H_s$. Suppose that one of the following conditions is satisfied:*

(i) *The fibers of H are of multiplicative type and of the same type as H_s.*

(ii) *H is abelian and the H_n ($n > 0$) are finite over S.*

(iii) *The fibers of H are connected.*

Then $G = H$ (and hence the above conditions are equivalent).

Proof. The proof is the same as Corollary 12.9.19. □

Corollary 12.9.30. *Let S be a scheme, H be an affine S-group, flat and of finite presentation over S, with multiplicative fibers. For H to be multiplicative, it is necessary and sufficient that the following equivalent conditions are satisfied:*

(i) *The type of H_s is a locally constant function on S.*

(ii) *H is abelian, and the H_n ($n > 0$) are finite over S.*

(iii) *The fibers of H are connected.*

Corollary 12.9.31. *Let S be a scheme, H be a flat group scheme of finite presentation over S. Suppose that H is affine over S and has connected fibers. If $s \in S$ is such that H_s is a torus, there exists an open neighborhood U of s such that $H|_U$ is a torus. In particular, if the fibers of H are tori, then H is a torus.*

12.9.4 Twisted constant groups

Let S be a scheme and R be a group scheme over S. We say that R is a **twisted constant group** over S if it is locally, in the sense of the fpqc topology, isomorphic to a constant group, i.e. of the form M_S, where M is an abelian group.

We say that a twisted constant group R over S is quasi-isotrivial (resp. isotrivial, resp. locally isotrivial, resp. locally trivial, resp. trivial) if in the above definition we can replace the fpqc topology by the étale topology (resp. global finite étale topology, resp. finite étale topology, resp. Zariski topology, resp. chaotic topology). For example, to say that R is quasi-isotrivial (resp. isotrivial) signifies that there exists a surjective étale (resp. and finite) morphism $S' \to S$ such that $R' = R \times_S S'$ is a constant group over S.

We define similarly the **type** of a twisted constant group R over S at a point $s \in S$; this is an isomorphism class of ordinary groups, which is locally constant over S, hence constant if S is connected. We say that R is of type M if the fibers of R are of type M. We note that R is quasi-compact over S only if it is *finite* over S, i.e. if its fiber at any $s \in S$ is a finite group.

The most interesting case for us is the one where R is abelian. We then say that R is "*finitely generated*" if its type at any point $s \in S$ is given by a finitely generated $\mathbb{Z}$-module (this should not be confused with the schematic notion "R of finite type over S").

Remark 12.9.32. We can also consider S-schemes X which are locally isomorphic (for the fpqc topology) to constant schemes. We then say that X is a twisted constant bundle over S, and the terminologies introduced above can be extended to these schemes. Of course, we shall pay attention that when X is endowed with a structure of S-group, the meaning of the expressions "twisted constant", "isotrivial" etc. changes accordingly if we take into account the group structure over S.

Proposition 12.9.33. *Let R be a twisted constant abelian group over S.*

(a) *The functor $H = D_S(R) = \mathcal{H}om_{S\text{-}\mathbf{Grp}}(G, \mathbb{G}_{m,S})$ (cf. §§12.7.1) is representable by a group of multiplicative type over S.*

(b) *For any $s \in S$, the type of R at s is equal to that of H at s.*

(c) *For R to be quasi-isotrivial (resp. global finite étale topology, resp. finite étale topology, resp. Zariski topology, resp. chaotic topology)*

Proof. As the covering familes for the fpqc topology are effective descent for the fibre category of affine groups schemes over S (cf. **[?]** VIII 2.1), we see that H is representable (and is of multiplicative type over S), since this is true if R is constant (cf. Lemma 12.7.10). The fact that H is of multiplicative type is clear by definition, and its type is easily seen to be the same as R at $s \in S$. Finally, since $H_{S'} = D_{S'}(R_{S'})$, the last assertion is reduced to the trivial case, i.e. to verify that R is trivial if and only if H is; this follows from the biduality theorem (Theorem 12.7.4). □

To specify the correspondence between twisted constant groups and groups of multiplicative type, it is necessary to start from a group of multiplicative type H, and study $R = D_S(H)$. If the latter is representable, it is obviously a twisted consant group, and we have $H \cong D_S(R)$. In other words, the functor $R \mapsto D_S(R)$ is an anti-equivalence between the category of twisted constant groups over S and that of groups of multiplicative type H over S such that $D_S(H)$ is representable. In general, it is hard to see that whether $D_S(H)$ is representable for a group of multiplicative type H, but we shall see that this is the case if H is quasi-isotrivial, and in particular if H is of finite type (cf. Corollary 12.9.26).

Lemma 12.9.34. *Let $S' \to S$ be a faithfully flat and locally presented morphism, and X' be a separated S'-scheme, locally of finite presentation and locally quasi-finite over S. Then any descent data over X' relative to $S' \to S$ is effective.*

Proof. If $X' \to S'$ is quasi-compact, hence of finite presentation and quasi-finite, then it is a quasi-affine morphism (cf. [?] 8.11.2), and the effectivity follows from ([?] VIII 7.9). In the general case, we can reduce to the case where S and S' are affine. Let (U_i') be an affine open covering of X' and V_i' be the saturation of U_i' under the equivalence relation of X' defined by the descent data, i.e. $V_i' = q_2(q_1^{-1}(U_i'))$, where q_1, q_2 are the projections of $X_1'' = X' \times_{S',\mathrm{pr}_2} S''$ over X' ($q_1 = \mathrm{pr}_1$, and q_2 is deduced from the first projection of $X_2'' = X' \times_{S',\mathrm{pr}_2} S''$, thanks to the descent data $X_1'' \cong X_2''$). As $S' \to S$ is faithfully flat, locally of finite presentation, and quasi-compact (recall that S' and S are affine), so is $p_1 : S'' = S' \times_S S' \to S'$, hence also q_1 and q_2, which are therefore quasi-compact open morphisms ([?] 2.4.6). Therefore, V_i' is an open subset of X'. From what we have already seen, the descent data induced on the V_i' are effective, from which it follows that the same is true for X' ([?] VIII 7.2). □

Corollary 12.9.35. *A faithfully flat and locally presented morphism $S' \to S$ is effective descent for the fibre category of twisted constant groups.*

Proof. In fact, this amounts to the effectiveness of a descent datum under the conditions of Lemma 12.9.34, if X' is a constant S'-scheme. □

Theorem 12.9.36. *Let S be a scheme, G, H be S-groups of quasi-isotrivial multiplicative type, with G of finite type over S. Then $\mathcal{H}om_{S\text{-}\mathbf{Grp}}(H, G)$ is representable by a quasi-isotrivial twisted constant group over S. For any $s \in S$, it the type of G_s (resp. H_s) is M (resp. N), then that of $\mathcal{H}om_{S\text{-}\mathbf{Grp}}(H, G)_s$ is $\mathrm{Hom}_{\mathbf{Grp}}(M, N)$.*

Proof. We proceed as in Lemma 12.9.22, utilizing the fact that the assertion has been established if G and H are trivial. The effectivity of the descent data is justified by Corollary 12.9.35 (in the case of an étale surjective morphism $S' \to S$[49]). □

Corollary 12.9.37. *Let S be a scheme and H be an S-group of multiplicative type.*

(a) *The S-group functor $D_S(H)$ is representable by a quasi-isotrivial twisted constant group over S.*

(b) *The functors $H \mapsto D_S(H)$ and $R \mapsto D_S(R)$ give an anti-equivalence between the category of quasi-isotrivial twisted constant groups over S and that of groups of quasi-isotrivial multiplicative type over S.*

Proof. This follows directly from Theorem 12.9.36 by taking $G = \mathbb{G}_{m,S}$, which is of finite type over S. □

Corollary 12.9.38. *Let S be a scheme and G, H be S-groups of multiplicative finite type. Then $\mathcal{H}om_{S\text{-}\mathbf{Grp}}(H, G)$ is representable by a finitely generated quasi-isotrivial twisted constant group over S.*

Proof. This follows from Theorem 12.9.36, as any group of multiplicative finite type over S is quasi-isotrivial (Corollary 12.9.26). □

We also note that in Proposition 12.9.33, R is finitely generated if and only if $H = D_S(R)$ is of finite type over S (cf. Proposition 12.8.6 (b)). In view of Corollary 12.9.26, H is then quasi-isotrivial, so R is quasi-isotrivial. We thus obtain the following corollaries:

Corollary 12.9.39. *The functors in Corollary 12.9.37 induce an anti-equivalence between the category of S-groups of multiplicative finite type and that of finitely generated twisted constant groups over S. Moreover, any such group R is quasi-isotrivial.*

Corollary 12.9.40. *Let H, G be S-groups of multiplicative finite type. Then $\mathcal{I}so_{S\text{-}\mathbf{Grp}}(H, G)$ is representable by a clopen subscheme of $\mathcal{H}om_{S\text{-}\mathbf{Grp}}(H, G)$, and it is a twisted constant S-scheme. In particular, $\mathcal{A}ut_{S\text{-}\mathbf{Grp}}(H, G)$ is representable by a twisted constant S-group (not abelian in general).*

Proof. This can be proves as in Theorem 12.9.36, using the fact that the assertions are valid if G and H are trivial. □

Proposition 12.9.41. *Let S be a scheme, R be a twisted constant abelian group over S, $H = D_S(R)$ the group of multiplicative type it defines. Consider the following conditions:*

(i) *H is isotrivial (i.e. R is isotrivial).*

(ii) *R is the union of clopen subschemes R_i, which are quasi-compact over S (and hence nesessarily finite over S).*

[49] From this, we see that the assertions of Theorem 12.9.36 are already valid if G and H are trivial for the fppf topology.

(iii) *The connected components of R are finite over S.*

Then we have (i)⇒(ii)⇒(iii)*,* (ii)⇔(iii) *if S is locally Noetherian, and* (i)⇔(ii) *if R is finitely generated (i.e. if H is of finite type over S) and S is quasi-compact or its connected components are open.*

Proof. The assertion in the parenthesis of (ii) follows from Lemma 12.9.42 below. By decomposing S into a sum of subschemes S_i over which H is of constant type, we are reduced to the case where H, hence R, is of constant type M. It is clear that (ii)⇒(iii), since the connected components of R are closed; they are clopen in R if R is locally Noetherian (cf. **??**, since R is of finite type over S, hence locally Noetherian), whence (iii)⇒(ii) in this case.

To prove that (i)⇒(ii), let $S' \to S$ be a finite surjective étale morphism which trivializes H, hence R, so that $R' \cong M_{S'} = \coprod_{m\in M} R'_m$, where R'_m are disjoint open subsets of R', S'-isomorphic to S'. Let $g : R' \to R$ be the projection morphism, which is finite surjective étale, hence an open and closed morphism. Then the subsets $R_m = g(R'_m)$ are clopen in R, and evidently quasi-compact over S since the R'_m are (g being surjective, Proposition 8.6.4).

Finally, suppose that H is of finite type over S, and we show that (ii)⇒(i). The case where the connected components of S are open is immediately reduced to the case where S is connected, so we can assume that S is quasi-compact or connected. Since M is finitely generated, we can write $M = \mathbb{Z}^r \times N$, where $r > 0$ is an integer and N a finite abelian group. Let $G = D_S(M)$ and consider the schemes (cf. Corollary 12.9.40)

$$P = \mathcal{I}so_{S\text{-}\mathbf{Grp}}(H, G) \subseteq \mathcal{H}om_{S\text{-}\mathbf{Grp}}(H, G) = Q.$$

We have isomorphisms

$$Q \cong \mathcal{H}om_{S\text{-}\mathbf{Grp}}(M_S, R) \cong \mathcal{H}om_{S\text{-}\mathbf{Grp}}(\mathbb{Z}^r_S, R) \times \mathcal{H}om_{S\text{-}\mathbf{Grp}}(N_S, R) \cong R^r \times E,$$

where $E = \mathcal{H}om_{S\text{-}\mathbf{Grp}}(N_S, R)$ is finite over S (because it is a twisted constant group of type $\mathrm{End}_{\mathbf{Grp}}(N)$). The hypothesis on R then implies that Q is the union of clopen subschemes Q_i which are finite over S, so P is the union of clopen subschemes $P_i = P \cap Q_i$, which are finite over S. As they are étale over S, their image in S are clopen subsets S_i of S, and cover S. If S is connected or quasi-compact, there then exists finitely many indices i such that the S_i cover S; let S' be the union of the corresponding P_i. Then $S' \to S$ is finite surjective étale, and putting $P' = P \times_S S' = \mathcal{I}so_{S'\text{-}\mathbf{Grp}}(H', G')$, we see that P' has a section over S', i.e. there exists an isomorphism of S'-groups

$$H' = H \times_S S' \xrightarrow{\sim} G' = G \times_S S' = D_{S'}(M),$$

which proves that S' trivializes H. □

Lemma 12.9.42. *Let S be a scheme and R be a twisted constant scheme over S. Then any closed subscheme Z of R which is quasi-compact over S is finite over S.*

Proof. In fact, by fpqc descent, we may assume that R is constant, hence of the form I_S, where I is a set. Then R is a directed union of J_S, where J runs through finite subsets of I. We can also suppose that S is affine, so Z is quasi-compact, hence contained in one of the J_S. As J_S is finite over S, so is Z. □

Let S be a locally Noetherian scheme, and $\widetilde{S}$ be the normalization of S_{red}. Recall that S is called *geometrically unibranch* (cf. **??** 0_{IV}, §23.2 et **[?]** §6.15) if the morphism $\widetilde{S} \to S$ is radiciel (and hence a universal homeomorphism). In particular, the connected components of S are irreducible (cf. Proposition 8.4.32 (c)).

Suppose then that S is connected, hence irreducible, let η be its generic point, and $f : P \to S$ be a flat and locally quasi-finite morphism. Let P_i be the irreducible components of P and ξ_i be the generic point of P_i. As P is flat over S, each ξ_i is lying over η (cf. **[?]** 2.3.4), and hence $(P_i)_\eta = P_i \cap P_\eta$ is the closure of ξ_i in P_η. Since the fiber P_η is discrete by hypothesis, we then have $(P_i)_\eta = \{\xi_i\}$. This remark applies in particular if f is étale; in this case, P is also locally Noetherian and geometrically unibranch (cf. **[?]** 17.5.7), hence its connected components are irreducible, and open.

Corollary 12.9.43. *Let S be a locally Noetherian and geometrically unibranch scheme, P be a quasi-isotrivial twisted constant scheme over S. Then the connected components of P are finite over S.*

Proof. We can evidently suppose that S is connected, hence irreducible; let η be its generic point. By the remark above, each connected component P_i of P is open and closed, and meets the fiber at a single point. Therefore Lemma 12.9.44 below applies and shows that each P_i is finite over S. □

Lemma 12.9.44. *Let S be a locally Noetherian and connected scheme, P be a quasi-isotrivial twisted constant S-scheme, and Z be a clopen subset of P such that there exists $s \in S$ such that Z_s is finite. Then Z is finite over S.*

Proof. Let us first consider the non-connected case of S. Let U be the set of $s \in S$ such that Z_s is finite, we shall prove that U is clopen and that $Z|_U$ is finite over U. This assertion is essentially equivalent to Lemma 12.9.44, but has the advantage that it is local for the étale topology, so we can reduce to the case where P is constant, i.e. of the form I_S, where I is a set[50].

We may then assume that S is connected, since the connected components of S are open (S being locally Noetherian). But then we must have $Z = J_S$ for a subset J of I, and hence $U = \varnothing$ or $U = S$, according to □

Corollary 12.9.45. *Let S be a locally Noetherian and geometrically unibranch scheme. Then any S-group H of multiplicative finite type is isotrivial.*

Proof. We can suppose that S is connected, hence H is of constant type M. We can then apply Corollary 12.9.43 to $P = R = D_S(H)$, and then utilize Proposition 12.9.41. □

12.9.5 Principal Galois bundles and enlarged fundamental group

Let S be a scheme, we consider the question of determine principal homogeneous bundles P over S (for the fpqc topology) with structure group of the form G_S, the constant group over S defined by an ordinary group G (not necessarily finite), which are also called **principal Galois bundles over** S **with group** G. We note that as G_S is étale and the structure morphism $G_S \to G$ is surjective, any such P is étale over S, and the structural morphism $P \to S$ is surjective (cf. [?] VIII 3.1 et [?] 17.7.3), hence P is covering for the étale topology, and P is also locally trivial for the étale topology (Proposition 12.3.46).

Since we often consider locally Noetherian schemes, we may suppose that S is a sum of connected schemes, i.e. its connected components are open, and hence assume that S is connected. Let ξ : $\mathrm{Spec}(\Omega) \to S$ be a geometric point of S, where Ω is a separably closed field. Then for any principal Galois bundle P over S with group G, P_ξ is a principal Galois bundle over Ω with group G, whence is trivial. We therefore specify the initial problem by proposing to determine the category of the principal Galois bundles over S **pointed over** ξ, i.e. endowed with an S-morphism $\xi \to P$, or equivalently a trivialization of P_ξ. For fixed G, the set of isomorphic classes of such bundles, up to isomorphisms fixing the base point ξ, is denoted by $\pi^1(S, \xi; G)$. Then the set $\pi^1(S; G)$ of isomorphism classes of principal Galois bundles over S with group G (without a specified base point) is isomorphic to the set of orbits of G in $\pi^1(S, \xi; G)$:

$$\pi^1(S; G) = \pi^1(S, \xi; G)/G,$$

where G acts naturally on P by definition. In fact, since any principal Galois bundle over S is trivialized over ξ, the map $\pi^1(S, \xi; G) \to \pi^1(S; G)$ is surjective. If two ξ-pointed principal Galois bundle $\sigma : \xi \to P$ and $\sigma' : \xi \to P'$ over S are identified in $\pi^1(S; G)$, then there exists a G-equivariant isomorphism ϕ : $P \to P'$, which then induces a map

$$\phi^* : P(\xi) \to P'(\xi).$$

If we denote by $g \in G$ the (unique) element such that $\sigma' \cdot g = \phi^*(\sigma) = \sigma \circ \phi$, then ϕ induces an isomorphism $P \cong P' \cdot g$ of ξ-pointed principal Galois bundles, whence our assertion.

For any morphism $S' \to S$ which is universally effective descent for the fibre category of twisted constant schemes over a variable base (for example fppf morphisms, cf. Corollary 12.9.35), we propose the determination of the subsets of the preceding sets, denoted by $\pi^1(S'/S, \xi; G)$ and $\pi^1(S'/S; G)$, formed by principal Galois bundles over S which are trivialized by S'. We determine in fact the fibre category of principal Galois bundles P over S which are trivialized by S'. Of course we have

$$\pi^1(S, \xi; G) = \varinjlim_{S'} \pi^1(S'/S, \xi; G)$$

where S' runs through a cofinal system of the set of covering morphisms $S' \to S$ for the étale topology (for example, if S is quasi-compact, we can take the set of S' over S which are quasi-compact and have étale surjective structural morphism). Similarly, the fibre category of principal Galois bundles over S is the inductive limit of its subcategories defined by the S' (formed by bundles trivialized over S').

[50]The locally Noetherian hypothesis is preserved under étale base change; this is where we use the quasi-isotriviality of P over S.

Thanks to the assumption made on $S' \to S$, the fiber category of principal Galois bundles over S trivialized by S' is equivalent to the fibre category of principal Galois bundles trivial over S' (hence of the form $G_{S'}$, where G acts by right translation), endowed with a descent data relative to $S' \to S$. The datum of a base point over a principal Galois bundle P over S trivialized by S' then translates, in terms of the trivial bundle P' over S' and its descent data, to the data of a trivialization of $P' \times_{S'} S'_\xi$ compatible with the induced descent data under base change, relative to $S'_\xi \to \xi$ (we put $S'_\xi = S' \times_S \xi$), i.e. a section σ of P'_ξ over S'_ξ compatible with the descent data. For a general morphism $S' \to S$ (not necessarily universally effective descent for the fibre category of twisted constant groups), we can then define $\pi^1(S'/S; G)$ and $\pi^1(S'/S, \xi; G)$ to be the set of isomorphism classes of trivial bundle P' over S' endowed with a descent data (and a trivialization of P'_ξ compatible with the induced descent data).

We can now state the most important result of this subsection, which gives the description of the functor $G \mapsto \pi^1(S, \xi; G)$ in terms of the simplicial set $S_\bullet$ induced by $S' \to S$, and identify the fiber category of principal Galois bundles over S:

Proposition 12.9.46. *Suppose that the connected components of S' and S'' are open.*

(a) *The functor $G \mapsto \pi^1(S'/S, \xi; G)$ from the category of groups to the category of sets, is representable by a group, denoted by $\pi_1(S'/S, \xi)$ and called the fundamental group of S at ξ relative to $S' \to S$. We then have a functorial isomorphism*

$$\pi^1(S'/S, \xi; G) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{Grp}}(\pi_1(S'/S, \xi), G).$$

(b) *The group $\pi_1(S'/S, \xi; G)$ admits a set of generators bijective to $\pi_0(S'')$, and is described in terms of these generators by relations bijective to elements of $\pi_0(S''')$. In particular, $\pi_1(S'/S, \xi)$ is finitely generated (resp. of finite presentation) if $\pi_0(S'')$ (resp. as well as $\pi_0(S''')$) is finite.*

(c) *The fibre category of principal Galois bundles over S trivialized by S', pointed over ξ, is equivalent to the category of ordinary groups G, endowed with a homomorphism $\pi_1(S'/S, \xi) \to G$.*

If S is connected and Noetherian, then any étale scheme S' over S is locally Noetherian, hence its connected components are open. We then conclude from the arguments above that the functor $G \mapsto \pi^1(S, \xi; G)$, from the category of groups to that of sets, is strictly pro-representable, i.e. there exists a projective system

$$\Pi = \Pi_1(S; \xi) = (\pi_i)_{i \in I}$$

of ordinary groups, with a filtered index set I, that is strict (i.e. the transition morphisms $\pi_j \to \pi_i$ are surjective), and an isomorphism of functors on G:

$$\pi^1(S, \xi; G) \xrightarrow{\sim} \varinjlim_i \mathrm{Hom}_{\mathbf{Grp}}(\pi_i, G).$$

The right member is also denoted by $\mathrm{Hom}_{\text{pro-}\mathbf{Grp}}(\Pi, G)$.

In the case where the projective limit $\pi = \varprojlim \pi_i$ is "large enough", i.e. if the canonical homomorphisms $\pi \to \pi_i$ are surjective for i sufficiently large, it is necessary to endow π with the projective limit topology of the discrete topologies of the π_i, and the isomorphism can also be written:

$$\pi^1(S, \xi; G) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Cont}\mathbf{Grp}}(\pi, G),$$

where the right member denotes the set of continous homomorphism of topological groups, where G is endowed with the discrete topology.

The hypothesis that we have just formulated on the projective system Π is verified, as it is well known, when the π_i are finite groups (cf. [?], III §7.4, Th.1). This last condition also means that any principal Galois bundle over S is isotrivial, i.e. is trivialized by a finite surjective étale morphism, which is the case when S is geometrically unibranched (for example normal) as it follows immediately from Corollary 12.9.45. In the case where the π_i are finite, the group π also coincides with the étale fundamental group $\pi_1(S, \xi)$ of S at ξ.

Also, in the prefered case ($\pi \to \pi_i$ being surjectives), we could also call π the extended group fundamental of S at ξ. In other cases, π itself does not present much interest, and the role of the usual fundamental group is played by the projective system itself, which we will call the **enlarged fundamental pro-group of S at ξ**.

Let us quickly indicate the calculation of $\pi_1(S'/S, \xi)$. Let S_i be the $(i+1)$-th fiber power of S' over S (i.e. $S_0 = S'$, $S_1 = S''$, etc.). We have obvious simplicial operations over the S_i, which make $(S_i)_{i \in \mathbb{N}}$ a simplicial object of $\mathbf{Sch}_{/S}$. Transforming this simplicial object by the functor of connected component

$$\pi_0 : \mathbf{Sch}_{/S} \to \mathbf{Set},$$

we then obtain a simplicial set $K_\bullet = (K_i)_{i \in \mathbb{N}}$, with $K_i = \pi_0(S_i)$. Similarly, the $(S_i)_\xi$ (the $(i+1)$-th fiber product of S'_ξ over ξ) form a simplicial object of $\mathbf{Sch}_{/\xi}$, hence of $\mathbf{Sch}_{/S}$. It is endowed with a natural homomorphism of simplicial objects to $(S_i)_{i \in \mathbb{N}}$ (induced by the morphism $\xi \to S$), so we have a simplicial set $k_\bullet$ (with $k_i = \pi_0((S_i)_\xi)$) and a canonical homomorphism $k_\bullet \to K_\bullet$. We can form a new simplicial set by taking the cone of this morphism (see **[?]** X 9.5.1):

$$\widetilde{K}_\bullet = \mathrm{Cone}(k_\bullet \to K_\bullet).$$

In this way, we obtain a pointed simplicial set $\widetilde{K}_\bullet$ (i.e. a simplicial set endowed with a homomorphism $\tilde{\xi} : e_\bullet \to \widetilde{K}_\bullet$, where $e_\bullet$ is the final simplicial set). We can then construct the well-known combinatorial invariants $\pi_0(\widetilde{K}_0, \tilde{\xi})$ and $\pi_1(\widetilde{K}_\bullet, \tilde{\xi})$, the construction of which only involves the components of degree ≤ 1 (resp. of degree ≤ 2) of $\widetilde{K}_\bullet$ (these invariants are defined without any restriction on S or S'). We then verifies without difficulty that, if the connected components of $S_0 = S'$ and $S_1 = S''$ are open (in fact, it is sufficient that $\widetilde{K}_\bullet$ is connected), then $\pi_1(\widetilde{K}_\bullet, \tilde{\xi})$ represents the functor i.e. we have:

$$\pi_1(S'/S, \xi) \cong \pi_1(\widetilde{K}_\bullet, \tilde{\xi}).$$

We also remark that if the morphism $S' \to S$ is universally submersive (cf. **[?]** IV 2.1), and the connected components of S' are open, then the simplicial set $K_\bullet$, and hence $\widetilde{K}_\bullet$, is connected.

Example 12.9.47.

Let S be a scheme, which we may assume to be locally Noetherian, so that certain schemes over S (namely schemes of finite type over S) are locally Noetherian, so their connected components are open. Using the enlarged fundamental group, we can now give a classification of twisted consant groups over S.

Proposition 12.9.48. *Any twisted constant scheme X over S which is locally trivial for the fppf topology is quasi-isotrivial.*

Proof. We can suppose that S is connected, hence X is of constant type I, where I is a set. Let $S' \to S$ be a faithfully flat morphism locally of finite presentation that trivializes X. Then $X' = X \times_S S'$ is isomorphic to $I_{S'}$, so $I_{S'}$ is endowed with a descent data relative to $S' \to S$, i.e. we have an isomorphism $I_{S''} \xrightarrow{\sim} I_{S''}$ satisfying the cocycle condition. Now $S'' = S' \times_S S'$ is locally Noetherian, so its connected components are open, and the automorphisms of $I_{S''}$ corresponding to sections of $G_{S''}$, where $G = \mathrm{Aut}(I)$ is the permutation group of I. In this way, we obtain a descent data over $G_{S'}$ (considered as the trivial Galois bundle) relative to $S' \to S$. In view of Lemma 12.9.34, this descent data is effective, so it corresponds to a principal Galois bundle P over S, with group G. By construction, it represents the functor $\mathcal{I}so_S(I_S, X)$ in the category of schemes over S which are locally Noetherian, and we easily see that $P \times_S X \cong I_S$, so the étale surjective base change $P \to S$ trivializes X, hence X is quasi-isotrivial. $\square$

The proof given above shows at the same time that the classification of twisted constant scheme X over S, quasi-isotrivial and of type I, is equivalent to that of principal Galois bundles over S, with group $G = \mathrm{Aut}(I)$. In fact, in this way, we obtain an equivalence of categories.

Using the enlarged fundamental group Π, we can make use the above correspondence to obtain an action of Π. For this, suppose that S is connected, and endowed with a geometric point ξ. Then the enlarged fundamental group $\Pi = \Pi_1(S, \xi)$ is defined. On the other hand, for any quasi-isotrivial twisted constant scheme X over S, let $I = X(\xi)$ be its setwise fiber at ξ, so X is of type I, and is associated as we have just seen with a principal Galois bundle $P = \mathrm{Iso}_S(I_S, X)$ over S, with group $G = \mathrm{Aut}(I)$. According to the definition of Π, we therefore obtain a canonical homomorphism Π to G, i.e. of one of the π_i to G. As G is the permutation group of $I = X(\xi)$, this means "G acts continuously on $I = X(\xi)$", being understood that the π_i (large i) act on I, in a compatible way with the transition morphisms of Π.

If $X \to Y$ is an S-morphism between quasi-isotrivial twisted constant schemes over S, then we obtain an induced map $X(\xi) \to Y(\xi)$, which is compatible with the operation of Π. We thus obtain an equivalence of categories:

Proposition 12.9.49. *Let S be a connected locally Noetherian scheme, ξ be a geometric point of S, $\Pi = \Pi_1(S, \xi)$ be the pro-fundamental group of S at ξ. Then the functor*

$$X \mapsto X(\xi)$$

is an equivalence between the category of quasi-isotrivial twisted constant schemes over S and the category of Π-sets.

The functor $X \mapsto X(\xi)$ is compatible with finite sums and finite projective limits. Therefore, it transforms quasi-isotrivial twisted constant groups (or rings, etc.) over S to ordinary groups (or rings, etc.) endowed with a continuous action of Π. In particular:

Corollary 12.9.50. *The category of twisted constant abelian groups over S is equivalent to the category of Π-modules.*

Using Corollary 12.9.26 and Corollary 12.9.37, we therefore conclude the following results:

Theorem 12.9.51. *Let S be a connected locally Noetherian scheme, ξ be a geometric point of S, $\Pi = \Pi_1(S, \xi)$ be the pro-fundamental group of S at ξ. Then the functor*

$$G \mapsto \mathrm{Hom}_{\kappa(\xi)\text{-}\mathbf{Grp}}(G_\xi, \mathbb{G}_{m,\xi})$$

induces an anti-equivalence from the category of groups of quasi-isotrivial multiplicative type over S to the category of Π-modules.

Corollary 12.9.52. *The preceding functor induces an anti-equivalence between the category of groups of multiplicative finite type over S and the category of π-modules which are finitely generated over $\mathbb{Z}$.*

Example 12.9.53. Let S be a complete rational curve over an algebraically closed field, with exactly a point with $n+1$ distinct branches. By Example 12.9.47, the enlarged fundamental group $\Pi(S, \xi)$ is a free group with n generators. Therefore, by Corollary 12.9.52, the classification of tori of relative dimension m over S is equivalent to that of systems of n endomorphisms $A_1, \dots, A_n$ of the $\mathbb{Z}$-module $\mathbb{Z}^m$, up to isomorphisms of $\mathbb{Z}^m$.

Remark 12.9.54. If we make no assumption on S, it remains true that for an ordinary finitely generated abelian group M, the category of groups of multiplicative type of type M over S is anti-equivalent to the category of principal Galois bundles over S, of group $G = \mathrm{Aut}_{\mathbf{Grp}}(M)$. This follows easily from Corollary 12.9.39 and Corollary 12.9.40.

12.10 Criterion of representability and applications to subgroups of multiplicative type

As we have already seen in §§12.9.3 and §§12.9.4, the representability of certain functors, in particular of functors of the form $\mathcal{H}om_S(X, Y)$, plays an important role in many questions concerning group schemes. Among the results particularly useful in this direction, let us point out (in addition to the questions of the representability of quotient) the question of the representability of functors of the form $\mathrm{Res}_{X/S} Y$ (Y a sub-object of X) studied in §§12.6.5. In this section, we shall give various variants of this results, which will provide us the representativeness of various centralizers, standardizers, and transporters.

12.10.1 Formally smooth functors

We now interpret the results stated in §§12.8.2, concerning the infinitesimal extensions of a homomorphism from a group of multiplicative type, into the language introduced above.

Proposition 12.10.1. *Let S be a scheme and G be a smooth group over S. Consider a homomorphism $u : H_1 \to H_2$ of groups of multiplicative type over S, whence a morphism of functors over S:*

$$M_{H_1} \to M_{H_2}, \quad M_{H_i} = \mathcal{H}om_{S\text{-}\mathbf{Grp}}(H_i, G),$$

defined by $w \mapsto w \circ u$. Then each M_{H_i} ($i = 1, 2$) is formally smooth, and the homomorphism $M_{H_1} \to M_{H_2}$ is formally smooth.

Proof. The first assertion follows from Theorem 12.8.21 (a). In view of the definition, the second one signifies the following: if S is affine, S_0 is a subscheme of S defined by a nilpotent ideal, $v : H_1 \to G$ is a homomorphism of S-groups, and $w_0 : (H_2)_{S_0} \to G_{S_0}$ is a homomorphism of S_0-groups such that $w_0 \circ u_{S_0} = v_{S_0}$, then there exists a homomorphism of S-groups

$$w : H_2 \to G$$

which extends w_0 and such that $w \circ u = v$. For this, we can first extend w_0 into a homomorphism of S-groups $w' : H_2 \to G$, which is possible by Theorem 12.8.21 (a). Then we consider $v' = w' \circ u : H_1 \to G$, which is such that $v'_{S'} = v_{S_0}$ by the hypothesis on w_0. In view of Theorem 12.8.21 (a), there then exists an element $g \in G(S)$, whose image in $G(S_0)$ is the identity, and such that $v = \mathrm{Inn}(g)v'$, whence $v = \mathrm{Inn}(g)w' \circ u$. It then suffices to take $w = \mathrm{Inn}(g)w'$. □

Corollary 12.10.2. *With the notations of Proposition 12.10.1, let $v_1, v_2 : H \to G$ be two morphisms of S-groups, and $\mathrm{Trans}(v_1, v_2)$ be the subfunctor of G formed by g such that $\mathrm{Inn}(g)v_1 = v_2$. Then this functor is formally smooth over S.*

Proof. By base change $S' \to S$, we can let S be affine and S_0 be a subscheme of S defined by a nilpotent ideal. For any $g_0 \in G(S_0)$ such that $\mathrm{Inn}(g_0)(v_1)_{S_0} = (v_2)_{S_0}$, we must extend g_0 to a section $g \in G(S)$ such that $\mathrm{Inn}(g)v_1 = v_2$. For this, we can first extend g_0 to a section g' of G over S, which is possible since G is smooth over S. Put $v'_2 = \mathrm{Inn}(g')v_1$, we then note that v_2 and v'_2 have the same restriction to S_0, so by Theorem 12.8.21 (a) there exists $g'' \in G(S)$, which induces the identity over S_0, such that $v_2 = \mathrm{Inn}(g'')v'_2$. Then $v_2 = \mathrm{Inn}(g'')\mathrm{Inn}(g')v_1 = \mathrm{Inn}(g''g')v_1$, so it suffices to choose $g = g''g'$. □

Corollary 12.10.3. *With the notations of Proposition 12.10.1, consider $M = M_H$ as a functor acted by G (by $(v, g) \mapsto \mathrm{Inn}(g)v$). Then the corresponding morphism*

$$\Phi : G \times_S M \to M \times_S M, \quad (g, v) \mapsto (\mathrm{Inn}(g)v, v)$$

is formally smooth.

Proof. By base change $S' \to S$, we may consider the absolute case for S. A section $M \times_S M$ over S is a couple (v_1, v_2), and its inverse image under Φ is given by the transporter $\mathrm{Trans}(v_1, v_2)$. Therefore Corollary 12.10.2 implies our assertion. □

Proposition 12.10.4. *Let S be a scheme, G be a smooth S-group over S, and consider the functor $M : \mathbf{Sch}_{/S}^{\mathrm{op}} \to$ **Set** such that*

$$M(S') = \{\text{the set of subgroups of multiplicative type of } G_{S'}\}.$$

Then M is formally smooth over S.

Proof. This is a reformulation of Theorem 12.8.21 (b). □

Corollary 12.10.5. *With the notations of Proposition 12.10.4, let $n > 0$ be an integer and consider the morphism of functors*

$$\varphi_n : M \to M, \quad \varphi_n(H) = H[n] = \ker(n \cdot \mathrm{id}_H).$$

Then φ_n is a formally smooth morphism. If for any integer $p > 0$, M_p denotes the subfunctor of M such that $M_p(S')$ is the set of subgroups of multiplicative type H of $G_{S'}$ such that $H[p] = H$. Then the morphism $M_{np} \to M_n$ induced by φ_n is formally smooth.

Proof. The second assertion is trivially contained in the first one. The proof of the first one is analogous to that of Proposition 12.10.1, by involking Theorem 12.8.21 (b). □

Corollary 12.10.6. *With the notations of Proposition 12.10.4, let H_1, H_2 be two subgroups of multiplicative type of G, and $\mathrm{Trans}_G(H_1, H_2)$ be the subfunctor of G formed by g such that $\mathrm{Inn}(g)(H_1) = H_2$. Then this functor is formally smooth over S. In particular, if $H_1 = H_2 = H$, then the subfunctor $N_G(H)$ is formally smooth.*

Proof. The proof is analogous to Corollary 12.10.2, by using Theorem 12.8.21 (b). □

Corollary 12.10.7. *With the notations of Proposition 12.10.4, consider M as a functor acted by G (by $(g, H) \mapsto \mathrm{Inn}(g)(H)$). Then the corresponding morphism*

$$\Phi : G \times_S M \to M \times_S M, \quad (g, H) \mapsto (\mathrm{Inn}(g)(H), H)$$

is formally smooth.

Proof. Again this follows from Corollary 12.10.6, since the inverse images under Φ are given by transporters. □

Proposition 12.10.8. *Let S be a scheme, G be a smooth S-group over S, H be an S-group of multiplicative type, and $u : H \to G$ be a homomorphism of S-groups. Let K be a subgroup of G of multiplicative type, and consider the functor $\mathrm{Trans}_G(u, K)$ formed by $g \in G(S')$ such that $\mathrm{Inn}(g)u_{S'} : H_{S'} \to G_{S'}$ factors through $K_{S'}$. Then this functor is formally smooth over S.*

Proof. By base change $S' \to S$, we can let S be affine and S_0 be a subscheme of S defined by a nilpotent ideal. Let $g_0 \in G(S_0)$ be such that $\mathrm{Inn}(g_0)u_{S_0}$ factors through K_{S_0}, then by Proposition 12.9.7, the induced morphism $v_0 : H_{S_0} \to K_{S_0}$ can be lifted to a morphism $v : H \to K$. If $j : K \to G$ is the inclusion morphism and g is an extension of g_0 to a section of G (exists since G is smooth over S), then jv and $\mathrm{Inn}(g)u$ have the same restriction on S_0, so by Theorem 12.8.21 (a), there exists $g' \in G(S)$, induceing the identity on S_0, such that $\mathrm{Inn}(g')\mathrm{Inn}(g)u = jv$. It then suffices to take $g'g \in G(S)$. □

12.10.2 Auxiliary results on representability

Proposition 12.10.9. *Let $F \to S$ a functor over a scheme S. The following conditions are equivalent:*

(i) *F is representable, and $F \to S$ is an open immersion.*

(ii) *F is a fpqc sheaf, commutes with inductive limits of rings, $F \to S$ is a monomorphism, and the followng conditions are verified: for any local scheme S' over S with residue field k, any S-morphism $\mathrm{Spec}(k) = S'_0 \to F$ can be extended to an S-morphism $S' \to S$.*

(iii) *(If S is locally Noetherian) F is a fpqc sheaf, commutes with inductive limits of rings and adic projective limits of rings, $F \to S$ is a monomorphism, and is infinitesimally étale.*

Before proving Proposition 12.10.9, let us explain some terminologies. We say that a contravariant functor F over S **commutes with inductive limits of rings** if for any filtered projective system $(S'_i)_{i\in I}$ of rings over an open affine subset of S, with ring A'_i, the natural homomorphism

$$\varinjlim \mathrm{Hom}_S(S'_i, F) \to \mathrm{Hom}_S(S', F), \quad \text{where } S' = \mathrm{Spec}(A'),\ A' = \varinjlim_i A'_i \tag{12.10.1}$$

is bijective. We note that the scheme S' is none other than the projective limit of (S'_i) in the category of schemes (and even in that of ringed spaces), so the considered condition is naturally considered as a "right-exactness" (commutation with certain inductive limits in $\mathbf{Sch}_{/S}$), like the condition of being a sheaf for some topology. Note that the considered condition is essentially relative, i.e. involves the morphism $F \to S$ and not only the functor $F : \mathbf{Sch}^{\mathrm{op}} \to \mathbf{Set}$ itself. Thus, if F is representable, the considered condition means that F is locally of finite presentation over S (cf. [?] 8.14.2).

On the other hand, we say that a functor F over S **commutes with adic projective limits of rings**, if for any S' over S which is the spectrum of a complete Noetherian local ring A' with maximal ideal $\mathfrak{m}$, putting $S'_n = \mathrm{Spec}(A'/\mathfrak{m}^{n+1})$, the natural map

$$\mathrm{Hom}_S(S', F) \to \varprojlim_n \mathrm{Hom}_S(S'_n, F) \tag{12.10.2}$$

is bijective. We note that this condition, which is naturally a "right-exactness" condition, is satisfied whenever F is representable. However, we easily seen that this condition only concerns the functor F as an object of $\widehat{\mathbf{Sch}}$, i.e. is independent of the morphism $F \to S$.

Remark 12.10.10. Let F be a functor over S which is a Zariski sheaf. Let (S_i) be a covering of S by open subsets, then we esily verify that (by a glueing method) F is representable if and only if each $F_i = F \times_S S_i$ is representable, which permits us for example to reduce to the case where S is affine. Suppose that F also commutes with inductive limits of rings, then for F to be representable, it is necessary and sufficient that its restriction to the category of schemes locally of finite presentation over S be representable. In fact, in this case, if X is a scheme locally of finite presentation over S and $X \to F$ is a morphism such that, for any S' locally of finite presentation over S, the induced morphism

$$\mathrm{Hom}_S(S', X) \to \mathrm{Hom}_S(S', F)$$

is bijective, then as X and F are Zariski sheaves which commute with the inductive limits of rings[51], $X \to F$ must be an isomorphism.

[51] Note that any algebra B over a ring A is the inductive limit of finitely presented sub-A-algebras.

We now prove Proposition 12.10.9. The implications (i)⇒(ii) and (i)⇒(iii) are evident, so it suffices to prove the converse. In the situation of (ii), let U be the set of $s \in S$ such that the canonical monomorphism $\operatorname{Spec}(\kappa(s)) \to S$ factors through F. In view of the last condition of (ii), we see that U is also the set of $s \in S$ such that the canonical monomorphism $\operatorname{Spec}(\mathscr{O}_{S,s}) \to U$ factors through F. Now $\mathscr{O}_{S,s}$ is the inductive limit of the rings of affine neighborhoods of s in S, so by our hypothesis on F, there exists an open neighborhood U_s such that the canonical immersion $U_s \to S$ factors through F. This implies $U_s \subseteq U$, so the set U is open in S. As $F \to S$ is a monomorphism and F is a Zariski sheaf, the S-morphism $U_s \to F$ can be glued over $U_s \cap U_{s'}$ ($s, s' \in U$), hence provides an S-morphism $U \to F$ (which is monomorphism, since $F \to S$ is a monomorphism). It remains to show that this is an isomorphism, or equivalently that any S-morphism $S' \to F$ factors uniquely through U (where S' is an S-scheme). As $F \to S$ and $U \to S$ are monomorphism, this is also equivalent to that the structural morphism $S' \to S$ factors through U, for which we can reduce to the case where S' is the spectrum of a field (as U is open in S, the morphism $S' \to S$ factors through U if and only if its image in S is contained in U, and this can be verified using the morphism $\operatorname{Spec}(\kappa(s'))$, $s' \in S'$). Let $s \in S$ be a point lying over the unique point s' of S', we claim that the S-morphism $S' \to S$ factors through $S_0 = \operatorname{Spec}(\kappa(s)) \to F$ (this implies $s \in U$ and we are done). For this, since $S' \to S_0$ is covering for the fpqc topology and F is a fpqc sheaf, it suffices to note that the two compositions

$$S'' = S' \times_{S_0} S' \rightrightarrows S' \to F$$

are equal, which follows from the fact that $F \to S$ is a monomorphism.

Now assume that S is locally Noetherian, and we prove that (iii)⇒(ii). For this, it suffices to show that the last condition of (iii), and it suffices to assume that $S' = \operatorname{Spec}(\mathscr{O}_{S,s})$, with $s \in S$ (this is all we have used to prove (i), which in turn implies (ii)). Let $A = \mathscr{O}_{S,s}$, $A_n = A/\mathfrak{m}^{n+1}$, $S_n = \operatorname{Spec}(A_n)$, then it follows from the hypothesis that $F \to S$ is infinitesimally smooth, so the given morphism $S_0 \to F$ extends to morphisms $S_n \to F$. As $F \to S$ is a monomorphism, we thus obtain an element of $\varprojlim_n F(S_n)$, and as F commutes with adic projective limits of rings, the morphisms $S_n \to F$ provides an S-morphism $\operatorname{Spec}(\widehat{A}) = \widehat{S}' \to F$. Again, as $F \to S$ is a monomorphism, F is a fpqc sheaf, and $\widehat{S}' \to S'$ is covering for this topology (S being Noetherian), the morphism $\widehat{S}' \to F$ factors through $S' \to F$, which completes the proof.

Proposition 12.10.11. *Let S be a locally Noetherian scheme, $F \to S$ be a functor over S, $(X_i, u_i)_{i \in I}$ be a family of S-morphisms $u_i : X_i \to F$, where each X_i is a scheme locally of finite type over S. Suppose that the following conditions are satisfied:*

(a) *F is a fpqc sheaf, commutes with inductive limits of rings and adic projective limits of rings.*

(b) *The $u_i : X_i \to F$ are monomorphisms, and are infinitesimally étale.*

(c) *The family (u_i) is "jointly surjective", i.e. any point of F with values in a field k comes from a point of X_i with values in k.*

Then F is representable by a scheme locally of finite type over S, the u_i are open immersions, and the family X_i covers F.

Proof. For each couple of indices (i, j), we write $X_{ij} = X_i \times_F X_j$, and consider the projections

$$v_{ij} : X_{ij} \to X_i, \quad w_{ij} : X_{ij} \to X_j.$$

We claim that these are represented by open immersions. To see this, we apply the criterion Proposition 12.10.9 (iii): X_{ij} satisfies the exactness conditions, because F, X_i, X_j all satisfy them, and the conditions are stable under finite projective limits (in particular fiber products). As $X_i \to F$ is a monomorphism, so is $v_{ij} : X_{ij} \to X_i$, which is induced by base change $X_j \to F$, and similarly v_{ij} is a monomorphism; finally, the infinitesimal étale condition is preserved by base change. This proves that we are under the conditions of Proposition 12.10.9 (iii).

We now use the X_i, X_{ij}, v_{ij} and w_{ij} to construct an S-scheme X such that X_i is identified with an open subset of X, and X_{ij} is identified with $X_i \cap X_j$, with v_{ij}, w_{ij} be canonical immersions. We note that X is also the quotient of $X' = \coprod_i X_i$ by the equivalence relation $R = \coprod_{ij} X_{ij}$ (the two projections $v, w : R \rightrightarrows Y$ being defined by the v_{ij} and w_{ij}). More precisely, F being a fpqc sheaf, the $u_i : X_i \to F$ provides a morphism $u' : X' \to F$, and R is none other than the equivalence relation defined by u'.

Finally, it follows from definition that the quotient $X = X'/R$ is also a quotient in the category of fpqc sheaves. Therefore, u' factors into a unique morphism

$$u : X \to F$$

which is a monomorphism. It remains to show that this is an isomorhism. As F is a Zariski sheaf, we can suppose that S is affine, and as F commutes with inductive limit of rings, it suffices to verify that for any T affine of finite presentation over S, any morphism $T \to F$ factors through X (cf. Remark 12.10.10). For this, consider $G = X \times_F T \to T$; as T is Noetherian, we see as above that it is an open immersion. Since $X \to F$ is jointly surjectie and this conditions is stable under base change, the morphism $G \to T$ is also jointly surjective, hence an isomorphism because it is an open immersion. □

Proposition 12.10.12. *Let S be a locally Noetherian scheme, I be a directed set, $(T_i)_{i\in I}$ a projective system of S-schemes locally of finite type, $T = \varprojlim_i T_i$ the projective limit functor, F a functor over S, and $u : F \to T$ an S-morphism. Suppose that the following conditions are satisfied:*

(a) *F is a fpqc sheaf, commutes with inductive limits of rings and adic projective limits of rings.*

(b) *The morphism $u : F \to T$ is a monomorphism.*

(b′) *The morphism $u : F \to T$ is infinitesimally étale.*

(c) *For any point ξ of F with values in a field k, denote by $\xi_i \in T_i(\mathrm{Spec}(k))$ its image and by t_i the corresponding point of T_i, then there exists an index $i \in I$ such that for any $j \geq i$ the transition morphism $p_{ij} : T_j \to T_i$ is étale at t_j.*

(d) *For any scheme X locally of finite type over S, and any S-morphism $X \to F$, the set of $x \in X$ over which this morphism is infinitesimally étale is open.*

Then F is representable by a scheme locally of finite type over S.

In practice, we will check conditions (c) and (d) of Proposition 12.10.12 via the following way:

Corollary 12.10.13. *With the conditions (a), (b), (b′) of Proposition 12.10.12, the conditions (c) and (d) of Proposition 12.10.12 are implied by the following:*

(c′) *The T_i are smooth over S, and the transition morphisms $p_{ij} : T_j \to T_i$ are smooth.*

(d′) *For any point ξ of F with values in a field k, let $t_i(\xi)$ be the point of T_i defined by ξ, $d_i(\xi)$ the relative dimension of T_i over S at $t_i(\xi)$, and $d(\xi) = \sup_i d_i(\xi)$. Then $d(\xi) < +\infty$ and for any scheme X locally of finite type over S, any S-monomorphism $v : X \to F$, the function $x \mapsto d(\xi_x)$ on X is locally constant (where for $x \in X$, we denote by ξ_x the point of F with values in $\kappa(x)$ induced by v).*

Corollary 12.10.14. *Under the hypothesis of Proposition 12.10.12, for any quasi-compact open subset U of F which is separated over S, there exists an index $i \in I$ such that $j \geq i$ the morphism $u_j|_U : U \to T_j$ is an open immersion. In particular, if the T_i are quasi-affine over S, then any open subset U of F which is quasi-compact over S, i.e. of finite type over S, is quasi-affine over S.*

Proposition 12.10.15. *Let S be a scheme and G be an affine group scheme over S.*

(a) *Let $F : \mathbf{Sch}_{/S}^{\mathrm{op}} \to \mathbf{Set}$ such that, for any S' over S,*

$$F(T) = \{\textit{set of subgroups of multiplicative type of } G_{S'} \textit{ which is finite over } S'\}.$$

Suppose that S is locally Noetherian or G is of finite presentation over S. Then F is representable and affine over S. If G is of finite presentation over S, then F is locally of finite presentation over S.

(b) *Let H be a group of multiplicative type over S and finite over S. Then $\mathcal{H}om_{S\text{-}\mathbf{Grp}}(H, G)$ is representable and affine over S. If G is of finite type (resp. of finite presentation) over S, so is $\mathcal{H}om_{S\text{-}\mathbf{Grp}}(H, G)$.*

Remark 12.10.16. Except for the assertion that $\mathrm{Hom}_{S\text{-}\mathbf{Grp}}$ is affine, and in the case where G is of finite presentation over S (which will suffice), Proposition 12.10.15 is an immediate consequence from Hilbert's Schema Theory ([**?**]); it even sufficies that G be quasi-projective over S. In case (a), we can also represent the larger functor

$$F'(S') = \{\text{set of subgroups of } G_{S'}, \text{ flat proper and of finite presentation over } S'\},$$

(the canonical monomorphism $F \to F'$ is an open immersion, as it follows from Proposition 12.10.9 and Corollary 12.9.19), so that the representability of F' implies that of F. In case (b), we can confine ourselves to assuming that H is projective and of finite presentation over S. In both cases, we obtain a functor locally of finite presentation over S. In the section, Proposition 12.10.15 is only a technique lemma to prove a key result at the next subsection, so we will sketch an easy direct proof, without using Hilbert schemes.

12.10.3 The functor of subgroups of multiplicative type

The main result of this subsection is the following theorem, which gives the representability of the functor of subgroups of multiplicative type for an affine smooth group scheme.

Theorem 12.10.17. *Let S be a scheme, G be an affine smooth S-group, and consider the functor $F : \mathbf{Sch}_{/S}^{\mathrm{op}} \to \mathbf{Set}$ defined by*

$$F(S') = \{\text{set of subgroups of multiplicative type of } G_{S'}\}.$$

Then F is representable, and is smooth and separated over S.

Corollary 12.10.18. *Let G, H be S-groups, with G smooth and affine over S, H of multiplicative finite type over S. Then $\mathcal{H}om_{S\text{-}\mathbf{Grp}}(H, G)$ is representable, and is smooth and separated over S.*

Remark 12.10.19. We have deduced Corollary 12.10.18 from Theorem 12.10.17, which is immediate if H is also smooth over S. For the deduction of the general case, the representability result of Theorem 12.10.17 would have to be established without supposing G smooth over S, but only affine of finite presentation over S (of course, then F will no longer be smooth over S in general!). There is a little doubt that whether Theorem 12.10.17 remains true under these more general assumptions, but the proof seems to have to be more delicate. Note, however, that when G is a closed subgroup of a smooth affine group G' over S, then the functor F representing the subgroups of multiplicative type of G is represented by a closed subscheme of the scheme representing the analogous functor F' for G', as can easily be seen by applying Theorem 12.6.39. This also raises the following question: for a group scheme G over an affine scheme S, which is affine and of presentation finite over S, is it isomorphic to a group subscheme of a suitable $\mathrm{GL}_{n,S}$? This is true when S is the spectrum of a field (cf. [?] VI_B 11.11), but unfortunately false in general, even for tori. Finally, note that we can also directly prove Corollary 12.10.18 by exactly the same method as Theorem 12.10.17.

Corollary 12.10.20. *Under the conditions of Theorem 12.10.17, let $u_1, u_2 : H \rightrightarrows G$ be two homomorphisms of S-groups. Then the subfunctor $\mathrm{Trans}(u_1, u_2)$ of G is representable by a closed subscheme of G, which is smooth over S.*

Proof. By Corollary 12.10.2, it remains to show that $\mathrm{Trans}(u_1, u_2) \to G$ is a closed immersion, which follows from the fact that it is the kernel of the morphisms $G \to M$, defined by $g \mapsto \mathrm{Inn}(g) \circ u_1$ and the constant morphism $g \mapsto u_2$, and that M is separated over S. □

Corollary 12.10.21. *Under the conditions of Theorem 12.10.17, let $M = \mathcal{H}om_{S\text{-}\mathbf{Grp}}(H, G)$, which is a smooth and separated scheme over S. Let G acts on M via $(g, u) \mapsto \mathrm{Inn}(g) \circ u$, then the canonical morphism*

$$\Phi : G \times_S M \to M \times_S M \tag{12.10.3}$$

is smooth.

Proof. This follows from Corollary 12.10.20. In fact, by Corollary 12.10.3, the morphism Φ is formally smooth, it is locally of finite presentation because $G \times_S M$ and $M \times_S M$ are (cf. Proposition 8.6.24). □

Corollary 12.10.22. *Under the conditions of Theorem 12.10.17, let $u : H \to G$ be a morphism of S-groups. Then $\mathrm{Centr}_G(u) = \mathrm{Trans}(u, u)$ is represented by a closed subgroup of G, which is smooth over S. Further, $C/\mathrm{Centr}_G(u)$ is representable by an open subscheme of M.*

Proof. The morphism $g \mapsto \mathrm{Inn}(g) \circ u$ from G to M is smooth of finit etype in view of Corollary 12.10.21, and hence an open morphism ([?] 2.4.6). If U denotes its image, with the induced scheme structure by M, then the induced morphism $G \to U$ is smooth, surjective, of finite type (cf. Proposition 8.6.35), hence covering for the fpqc topology. Moreover, it is evident that the morphism $G \to M$ makes G a formally principal homogeneous sheaf under $\mathrm{Centr}_G(u)_M$ (with action defined by right translations), which implies that the sheaf $G/\mathrm{Centr}_G(u)$ is representable by U (cf. Proposition 12.3.47). □

Proposition 12.10.23. *Under the conditions of Theorem 12.10.17, let $u_1, u_2 : H \to G$ be two homomorphism of S-groups. Then the following conditions are equivalent:*

(i) *u_1 and u_2 are locally conjugate for the étale topology.*

(i′) *u_1 and u_2 are locally conjugate for the fpqc topology.*

(ii) *For any $s \in S$, denote by $\bar{s}$ the spectrum of an algebraic closure of $\kappa(s)$, then the morphisms $(u_1)_{\bar{s}}$ and $(u_2)_{\bar{s}}$ are conjugate by an element of $G(\kappa(\bar{s}))$.*

(ii′) *The structural morphism $\mathrm{Trans}_G(u_1, u_2) \to S$ is surjective.*

(iii) *$\mathrm{Trans}_G(u_1, u_2)$ is a torsor under the action of the smooth S-group $\mathrm{Centr}_G(u_1)$.*

Proof. The implications (i)⇒(i′) and (ii)⇒(ii′) are trivial, and (i′)⇒(ii) follows from the principle of finite extension (cf. [?] 9.1.4). On the other hand, (ii′)⇒(iii) because $\mathrm{Trans}_G(u_1, u_2)$ is smooth and of finite type over S, hence flat and quasi-compact over S; so it is faithfully flat and quasi-compact over S if and only if the structural morphism is surjective (it is easy to see that $\mathrm{Trans}_G(u_1, u_2)$ is a formally principal homogeneous bundle over S under the action of $\mathrm{Centr}_G(u_1)$). Finally, (iii)⇒(i) follows from the fact that the morphism $\mathrm{Trans}_G(u_1, u_2) \to S$ is smooth and surjective (cf. **??** 17.16.3). □

Remark 12.10.24. For $u_1 = u$ fixed, the functor $\mathbf{Sch}^{\mathrm{op}}_{/S} \to \mathbf{Set}$ which associates a scheme S' over S with the set of homomorphisms of S'-groups $u_2 : H_{S'} \to G_{S'}$ which are locally conjugate to $u_{S'} : H_{S'} \to G_{S'}$ for the étale topology (or equivalently, the fpqc topology), is represented by the open subset $M \cong G/\mathrm{Centr}_G(u)$ of G, considered in Corollary 12.10.22.

Let us now sketch the variants of the previous results, obtained by applying Theorem 12.10.17 instead of Corollary 12.10.18. Let G be a smooth and affine group scheme over S, and denote by M the smooth and separated S-scheme which represents the functor considered in Theorem 12.10.17. We still have an action of G on M:

$$G \times_S M \to M, \quad (g, H) \mapsto \mathrm{Inn}(g)(H),$$

whence as above a morphism

$$\Phi : G \times_S M \to M \times_S M. \tag{12.10.4}$$

By Theorem 12.10.17 and Corollary 12.10.7, this morphism is smooth, and we can also prove the following corollary:

Corollary 12.10.25. *Let H_1, H_2 be subgroups of multiplicative type of G. Then the subfunctor $\mathrm{Trans}_G(H_1, H_2)$ of G are representable by a closed subscheme of G, which is smooth over S.*

Proof. This follows from Corollary 12.10.6, and $\mathrm{Trans}_G(H_1, H_2)$ is closed in G for the same reason as in Corollary 12.10.20. □

Corollary 12.10.26. *Let H be a subgroup of multiplicative type of G. Then the subfunctor $N_G(H)$ of G is representable by a closed subgroup of G which is smooth over S. Moreover, the quotient $G/N_G(H)$ is representable by an open subscheme of M.*

Corollary 12.10.27. *Let H_1, H_2 be subgroups of multiplicative type of G. Then the following conditions are equivalent:*

(i) *H_1 and H_2 are locally conjugate for the étale topology.*

(i′) *H_1 and H_2 are locally conjugate for the fpqc topology.*

(ii) *For any $s \in S$, denote by $\bar{s}$ the spectrum of an algebraic closure of $\kappa(s)$, then $(H_1)_{\bar{s}}$ and $(H_2)_{\bar{s}}$ are conjugate by an element of $G(\kappa(\bar{s}))$.*

(ii′) *The structural morphism $\mathrm{Trans}_G(H_1, H_2) \to S$ is surjective.*

(iii) *$\mathrm{Trans}_G(H_1, H_2)$ is a torsor under the action of the smooth S-group $N_G(H_1)$.*

Remark 12.10.28. Similar to Remark 12.10.24, for a subgroup of multiplicative type H of G, the functor $\mathbf{Sch}^{\mathrm{op}}_{/S} \to S$ which assocaites a scheme S' over S with the set of subgroups H' of multiplicative type of $G_{S'}$ which are locally conjugate to $H_{S'}$ for the étale topology, is representable by the open subset $C/Z_G(H)$ of G.

Remark 12.10.29. As the morphism (12.10.3) (resp. (12.10.4)) is smooth hence open, its image is open in $M \times_S M$. We let R be this image, endowed with the induced scheme structure of $M \times_S M$. We then easily see that R is an equivalence relation over M, whose definition is none other than the condition considered in Proposition 12.10.23 (resp. Corollary 12.10.27). It is therefore interesting that whether the quotient sheaf M/R (which is formally étale over S, cf. Theorem 12.8.21) is representable (it is then representable by a étale scheme over S); this is true for example if S is the spectrum of a field. We also note that it may happen that R is not closed in $M \times_S M$, which signifies that (if M/R is representable) M/R may not be separated over S.

Theorem 12.10.30. *Let S be a scheme, G be a smooth and affine group over S, $s \in S$.*

(a) *For any subgroup of multiplicative type H_0 of G_s, there exists an étale morphism $S' \to S$, a point $s' \in S'$ lying over s such that $\kappa(s') = \kappa(s)$, and a subgroup of multiplicative type H' of $G' = G \times_S S'$ such that $H'_{S'} = H_0 \otimes_{\kappa(s)} \kappa(s')$.*

(b) *For any group homomorphsim $u_0 : H_s \to H_s$, where H is an S-group of multiplicative finite type, there exists an étale morphism $S' \to S$, a point $s' \in S$ lying over s such that $\kappa(s') = \kappa(s)$, and a morphism $u' : H' \to G'$ such that $u'_{s'} = u_0 \otimes_{\kappa(s)} \kappa(s')$.*

Proof. This follows from Theorem 12.10.17 and Corollary 12.10.18. For example, by Theorem 12.10.17 we know that the functor M considered in it is representable and smooth over S. Now a subgroup of multiplicative type H_0 of G_s corresponds to a morphism $\kappa(s) \to M$, located at $m \in M$. Then m is a rational point of M over S, and by ([?] 17.16.3 (i)), there exists a neighborhood U of s and a subscheme M' of M (contained in the inverse image of U in M) such that $m \in M'$ and the induced morphism $M' \to U$ is étale. We then obtain a factorization $\kappa(s) \to M'$, and it suffices to take $S' = M'$, $s' = m \in M'$. ☐

Proposition 12.10.31. *Let S be a scheme, G be a smooth and affine group over S, H be a subgroup of multiplicative finite type of G. Then $Z_G(H)$ is a clopen subgroup of $N_G(H)$, and the quotient sheaf*

$$W_G(H) = N_G(H)/Z_G(H)$$

is representable by an open subgroup of $\mathcal{A}ut_{S\text{-}\mathbf{Grp}}(H)$, which is hence quasi-finite, étale and separted over S.

Proof. Consider the evident homomorphism

$$\theta : N_G(H) \to \mathcal{A}ut_{S\text{-}\mathbf{Grp}}(H),$$

whose kernel is by definition $Z_G(H)$. As $\mathcal{A}ut_{S\text{-}\mathbf{Grp}}(H)$ is representable by a étale and separated group over S (cf. Corollary 12.9.40), its unit section is an open and closed immersion, hence its inverse image under θ is an open and closed subgroup of $N_G(H)$. We also claim that θ is a smooth morphism: the formally smoothness follows formally from the definition, and the fact that $N_G(H)$ is smooth over S and $\mathcal{A}ut_{S\text{-}\mathbf{Grp}}(H)$ is étale over S. We then conclude as in Corollary 12.10.22 that the image of θ is an open subset U of $\mathcal{A}ut_{S\text{-}\mathbf{Grp}}(H)$, and represents the quotient sheaf $N_G(H)/Z_G(H)$ (note that $N_G(H)$ is affine over S, as a closed subscheme of G). This scheme is hence étale and separated over S, since $\mathcal{A}ut_{S\text{-}\mathbf{Grp}}(H)$ is, and it is quasi-finite over S because it is quasi-compact as the image of $N_G(H)$. ☐

Corollary 12.10.32. *Under the hypothesis of Proposition 12.10.31, for any $s \in S$, let*

$$w(s) = \mathrm{rank}_{\kappa(s)}(N_{G_s}(H_s)/Z_{G_s}(H_s))$$

(which is also the index of $Z_{G(k)}(H(k))$ in $N_{G(k)}(H(k))$, where k is an algebraic closure of $\kappa(s)$). Then the function $s \mapsto w(s)$ is lower semi-continuous. For it to be constant in a neighborhood of s, it is necessary and sufficient that $W_G(H)$ be finite over S in a neighborhood of s.

Proof. In fact, for any S-scheme W which is étale, of finite type and separated over S, the function $s \mapsto w(s) = \mathrm{rank}_{\kappa(s)}(W_s)$ is lower semi-continuous, and it is constant in a neighborhood of s if and only if W is finite over S at a neighborhood of s. To see this, we note that we can reduce to the case where S is locally Noetherian: we can clearly assume that $S = \mathrm{Spec}(A)$ is affine, and then S is the inductive limit of $S_i = \mathrm{Spec}(A_i)$, where A_i are finitely generated sub-$\mathbb{Z}$-algebras of A. By ([?] 8.8.2 et 8.10.5), there exists an index i, a scheme W_i over S_i which is étale, of finite type and separated over S_i, such that W is isomorphic to $W_i \times_{S_i} S$. We recall that the rank function of fibers is invariant under base change (cf. Corollary 8.6.47), so it suffices to prove the assertion for W_i and S_i, which in turn follows from ([?] 15.5.1, [?] 2.4.6, et Proposition 10.4.19). ☐

12.11 Maximal tori, Weyl group, Cartan subgroups and reductive center of affine smooth group schemes

12.11.1 Maximal tori, Cartan subgroups and Weyl group

12.11.1.1 Maximal tori Let G be an algebraic group over an algebraically closed field k. We say that an algebraic subgroup T of G is a **maximal torus** if T is a torus (since k is algebraically closed, this means T is isomorphic to a group of the form $\mathbb{G}_m^r$), and is maximal for this property. Note that, k being perfect, G_{red} is a subgroup of G which is smooth over k, and any reduced subgroup of G is automatically a subgroup of G_{red}. Therefore, the maximal tori of G coincide with that of G_{red}. If G is affine, hence G_{red} is affine, a fundamental theorem of Borel tells us that any two maximal tori of G are conjugate under an element of $G(k) = G_{\text{red}}(k)$ ([**?**] 6, th.4 (c)), and in particular have the same dimension. This common dimension is called the **reductive rank** of G. Note also that the restriction that G is affine is harmless, as it follows from a theorem known from Chevalley that any smooth connected algebraic group over k is a extension of an abelian variety by an affine group. In this subsection, we will most often limit ourselves to group schemes affine over the base.

Let G be a smooth algebraic group over k and T be a maximal torus of G; the centralizer $C = Z_G(T)$ of T in G is called the **Cartan subgroup** of G associated with T, which is a subgroup of G thanks to Corollary 12.6.41. Note that G as smooth over k, C is also smooth over k in view of Corollary 12.10.26 and Proposition 12.10.31, so in this case C is the unique subgroup of G smooth over k such that $C(k)$ is the centralizer of $T(k)$ in $G(k)$. By the conjugation theorem cited above, the Cartan subgroups of different maximal tori are conjugate to each other, so they have the same dimension, which is called the **nilpotent rank** of G, and is equal to that of G_{red}. Let $\rho_r(G)$ and $\rho_n(G)$ be the reductive rank and nilpotent rank of G, respectively. Then we have the inequality:

$$\rho_r(G) \leq \rho_n(G),$$

and their difference

$$\rho_u(G) = \rho_n(G) - \rho_r(G) = \dim(C/T)$$

can be calld, if G is affine, the **unipotent rank** of G. If G is smooth, affine and connected, then C is a nilpotent and connected algebraic group ([**?**] 6, th.6 (a) et (c)), hence isomorphic to a product $C_s \times C_u$, where $C_s = T$ is the maximal torus and C_u is a smooth unipotent subgroup, i.e. a successive extension of groups isomorphic to $\mathbb{G}_a$ ([**?**] 6, th.1 cor.1 et 7, th.4). In this case, we also have

$$\rho_u(G) = \dim(C_u).$$

Remark 12.11.1. In addition to the notions of rank that we have just specified for an affine algebraic group, there are two others which are useful, namely the semisimple rank $\rho_s(G)$, which is by definition the reductive rank of the quotient G/R, where R is the radical of G, and the infinitesimal rank $\rho_i(G)$, which is defined to be the nilpotent rank of the Lie algebra of G (this will be defined and studied later). We only remark that there are inequalities

$$\rho_s \leq \rho_r \leq \rho_n \leq \rho_i.$$

For a general group scheme over S, the notion of a maximal torus is defined using the fibers over residue fields. For this to work, we must take an algebraic closure of each residue field $\kappa(s)$, $s \in S$. The validity of taking such a field extension is justified by the following lemma:

Lemma 12.11.2. *Let G be an algebraic group over an algebraically closed field k, T be an algebraic subgroup of G, k' be an algebraically closed extension of k, G' and T' be the induced group under base change. For T to be a maximal torus, it is necessary and sufficient that T' is a maximal torus of G'.*

Proof. Since k and k' are algebraically closed, by our arguments above, we can replace G with G_{red} and hence assume that G is smooth over k. In this case, we know from ([**?**]) that T is a maximal torus in G if and only if $Z_G(T)/T$ is unipotent, and this holds for T if and only if it holds for T' ([**?**]). □

Definition 12.11.3. Let S be a scheme, G be an S-group of finite type, T be a subgroup of G. We say that T is a maximal torus of G if

(a) T is a torus, i.e. is locally isomorphic to $\mathbb{G}_m^r$ for the fpqc topology;

(b) for any $s \in S$, denote by $\bar{s}$ the spectrum of an algebraic closure of $\kappa(s)$, $T_{\bar{s}}$ is a maximal torus in $G_{\bar{s}}$.

It follows from Lemma 12.11.2 that if S is the spectrum of a an algebraically closed field, then we recover the usual definition of maximal tori, and that the above definition is stable under base change. We also note that a maximal torus in the sense of Definition 12.11.3 is maximal in the set of sub-torus of G (this follows easily from Corollary 12.8.14). But the converse of this question is more tricky, that is, whether G admits effectively a maximal torus in the sense of Definition 12.11.3, which is in general not true even if G is semi-simple. However, we will see that this is true if S is artinian, or if S is a local scheme and G is "reductive": in this case, every torus of G is contained in a maximal torus.

Definition 12.11.4. Let G be an algebraic group over a field k. We define the **reductive rank** (resp. **nilpotent rank**, resp. **unipotent rank**, etc.) of G to be that of $G_{\bar{k}}$, where $\bar{k}$ is an algebraic closure of k.

We see in view of Lemma 12.11.2 and the commutation of $Z_G(T)$ with base change that the notion of various ranks of G is stable under base field extensions. On the other hand, if k is algebraically closed, then these coincide with the already defined ones.

Remark 12.11.5. It is not hard to construct a affine group scheme smooth over the spectrum S of a DVR, whose generic fiber is isomorphic to $\mathbb{G}_m$ and the special fiber is isomorphic to $\mathbb{G}_a$. For example, let $R = k[\![\pi]\!]$ be a complete DVR and consider the affine scheme $G = \operatorname{Spec}(R[t, (1-\pi t)^{-1}])$. Then G is a group scheme under the multiplication and inversion:

$$\mu(t_1, t_2) = t_1 + t_2 - \pi t_1 t_2, \quad c(t) = -t(1-\pi t)^{-1}.$$

On the level of algebras, these maps correspond to the comultiplication and antipode of $R[t, (1-\pi t)^{-1}]$, given by

$$\Delta(t) = 1 \otimes t + t \otimes 1 - \pi t \otimes t, \quad S(t) = -t(1-\pi t)^{-1}.$$

When π is invertible, then the k-linear homomorphism

$$\varphi : k[\pi, \pi^{-1}, t, (1-\pi t)^{-1}] \to k[\pi, \pi^{-1}, s, s^{-1}], \quad t \mapsto \pi^{-1}(1-s)$$

gives an isomorphism from G to the multiplicative group scheme $\mathbb{G}_{m,S}$ (and, in fact, that's how the above formulas for multiplication and inversion are easiest to obtain). When π is zero, it is clear that G reduces to the additive group.

For such a group G, it does not contain any torus except the trivial one T (reduced to the identity), which is evidently not a maximal torus in the sense of Definition 12.11.3. More precisely, in the special fiber $G_0 = \mathbb{G}_{a,k}$, T_0 is of course a maximal torus, but in the generic fiber $G_1 = \mathbb{G}_{m,K}$, T_1 is not maximal (k is the residue field, K is the fraction field). We also see that in this example the reductive rank of G_s ($s \in S$) is not a continuous function on S.

Despite the counter-example in Remark 12.11.5, we have the following result:

Theorem 12.11.6. *Let G be an affine smooth group over S. For any $s \in S$, consider $\rho_r(s) = \rho_r(G_s)$ and $\rho_n(s) = \rho_n(G_s)$, the reductive rank and nilpotent rank of G_s. With these notations, we have:*

(a) *The function ρ_r is lower semi-continuous on S, and ρ_n is upper semi-continous on S. Hence $\rho_u = \rho_n - \rho_r$ is upper semi-continuous on S.*

(b) *The following conditions (stable under arbitrary base change) are equivalent:*

  (i) *The function ρ_r is locally constant on S (in this case, we say that G is of locally constant reductive rank).*

  (ii) *There exists, locally for the étale topology, a maximal torus in G.*

  (iii) *There exists, locally for the fpqc topology, a maximal torus in G.*

(c) *Let T_1, T_2 be maximal tori of G, then T_1, T_2 are conjugate locally for the étale topology, i.e. there exists a surjective étale morphism $S' \to S$ such that the subgroups $(T_1)_{S'}$ and $(T_2)_{S'}$ of $G_{S'}$ are conjugate by a section of $G_{S'}$ over S'.*

(d) *If ρ_r is locally constant, so is ρ_n (hence also ρ_u).*

Proof. Note that for any morphism $S' \to S$, if $G' = G \times_S S'$, the functions ρ'_n, ρ'_r and ρ'_u over S' defined in terms of G', are obtained by composing ρ_n, ρ_r and ρ_u with the given morphism $S' \to S$. If $S' \to S$ is faithfully flat and quasi-compact, then ρ' is upper (resp. lower) semi-continuous if and only if ρ is, because the topology of S is obtained by a quotient of S ([**?**] VIII 4.3). Therefore, the assertions of (a) are local for the fpqc topology. Let $s \in S$, we can show that the set U of $t \in S$ such that $\rho_r(t) \geq \rho_r(s)$ (resp. $\rho_n(t) \leq \rho_n(s)$) is an open neighborhood of s. By taking an algebraic closure of $\kappa(s)$, we see that we are reduced to the case where G_s has a maximal torus T_s. Moreover, thanks to Theorem 12.10.30 (a), by replacing S with an étale scheme S' over S endowed with a point s' over s, we can suppose that T_s is the fiber of a torus T of G. Then for any $t \in S$, we have

$$\rho_r(t) \geq \rho_r(G_t) \geq \dim(T_t) = \dim(T_s) = \rho_r(G_s) = \rho_r(s),$$

which proves that ρ_r is lower semi-continuous. On the other hand, in view of Proposition 12.10.31, the functor $C = Z_G(T)$ is representable by a closed subgroup of G which is smooth over S. Hence by ([**?**] 17.10.2) there exists an open neighborhood U of s such that $t \in U$ implies $\dim(C_t) = \dim(C_s) = \rho_n(s)$. The upper semi-continuity of ρ_n then follows from the relation

$$\rho_n(t) \leq \dim(C_t) \quad \text{for} \quad t \in S,$$

which is contained in the following observation: if G is an affine smooth algebraic group over a field k and T is a torus in G, C its centralizer, then we have $\rho_n(G) \leq \dim(G)$. In fact, we can suppose that k is algebraically closed and choose a maximal torus T' containing T. Then the centralizer C' of T' is contained in C, hence $\dim(C') \leq \dim(C)$.

If ρ_r is locally constant, then for any torus T in G and any $s \in S$, if T_s is a maximal torus in G_s, then there exists an open neighborhood U of s such that $T|_U$ is a maximal torus in $G|_U$. Now using the reasoning of (a), we see that (i)⇒(iii). On the other hand, (iii)⇒(i), because if G admits a maximal torus T, then it is evident that $\rho_r(s) = \dim(T_s)$ is a locally constant function on s, but we note that the question of the continuity of ρ_r is local for the fpqc topology. It remains to show that (i)⇒(ii).

For this, we introduce the functor F of Theorem 12.10.17, which is a scheme smooth and separated over S, and consider the sub-functor $\mathscr{T}$ of F whose value over $S' \to S$ is the set of maximal tori in $G_{S'}$. We claim that $\mathscr{T}$ is representable by an open subscheme of F, hence is smooth and separated over S. To see this, we shall use Proposition 12.10.9. We first note that, since tori and their maximality are bith defined fpqc locally and the functor F is a fpqc sheaf, it is easy to see that $\mathscr{T}$ is a fpqc sheaf. Now let U be the set of $x \in F$ such that the canonical monomorphism $\mathrm{Spec}(\kappa(x)) \to F$ factors through $\mathscr{T}$. Then any point $x \in U$ (lying over $s \in S$) corresponds to a maximal torus $T_{\kappa(x)}$ of $G_{\kappa(x)}$, which comes from a torus T of $G_{\mathrm{Spec}(\mathscr{O}_{F,x})}$[52]. Since $T_{\kappa(x)}$ is a maximal torus of $G_{\kappa(x)}$, we see that T_s is a torus in G_s, so there exists an open neighborhood U of s such that $T|_U$ is a maximal torus in $G|_U$. But by Proposition 8.2.11, the inverse image of U in F then contains the image of $\mathrm{Spec}(\mathscr{O}_{F,x})$, so T is in fact a maximal torus of $G_{\mathrm{Spec}(\mathscr{O}_{F,x})}$. In other words, we have proved that the canonical morphism $\mathrm{Spec}(\mathscr{O}_{F,x}) \to F$ factors through $\mathscr{T}$. Now since $\mathscr{O}_{F,x}$ is the inductive limit of the rings of affine neighborhoods of x in F, we see that the torus T of $G_{\mathrm{Spec}(\mathscr{O}_{F,x})}$ in fact comes from a subgroup T' of multiplicative type of $G|_{U'}$, where U' is an affine open neighborhood of x. By possibly shrinking U', we may also assume that T' is a torus in $G|_{U'}$. Again, since T' is a maximal torus at x (hence the image of $\mathrm{Spec}(\mathscr{O}_{F,x})$), we may further assume that T' is a maximal torus of $G|_{U'}$. In this case, for any point $x' \in U'$, the canonical morphism $\mathrm{Spec}(\kappa(x')) \to F$ then factors through T', so we have $U' \subseteq U$, and hence U is open in F. Now following the proof of (ii)⇒(i), we conclude that the open subset U represents $\mathscr{T}$, and hence $\mathscr{T}$ is smooth and separated over S. As the structural morphism $\mathscr{T} \to S$ is evidently surjective, it admits locally for the étale topology a section over X in view of ([**?**] 17.16.3), and this proves (i)⇒(ii).

As for (c), this is an immediate concequence of Corollary 12.10.25 and ([**?**] 17.16.3), in view of Borel's conjugation theorem[53]. Finally, in view of the remarks in the proof of (a), if ρ_r is locally constant, we can assume that there is a maximal torus T in G. If C is its centralizer, then C is representable and is smooth over S by Proposition 12.10.31, so the function $s \mapsto \rho_n(s) = \dim(C_s)$ is indeed locally constant. □

[52]Since the canonical monomorphism $\mathrm{Spec}(\kappa(x)) \to F$ factors through $\mathrm{Spec}(\mathscr{O}_{F,x})$, $T_{\kappa(x)}$ comes from a subgroup T of $G_{\mathrm{Spec}(\mathscr{O}_{F,x})}$ of multiplicative type. Now since the image of $\mathrm{Spec}(\mathscr{O}_{F,x})$ in S is contained in any neighborhood of s (cf. Proposition 8.2.11), we see that T is in fact a torus of $G_{\mathrm{Spec}(\mathscr{O}_{F,x})}$.

[53]The transporter of two maximal tori of G is representable by a smooth and separated scheme over S, so we can proceed as in the proof of Theorem 12.10.30.

Corollary 12.11.7. *Let G be as in Theorem 12.11.6 and let $s \in S$ be such that $\rho_u(s) = 0$, i.e. $\rho_r(s) = \rho_n(s)$ (that is, the maximal tori in G_k are central, where k is an algebraic closure of $\kappa(s)$). Then there exists an open neighborhood U of s such that ρ_r and ρ_n are constant over U, and in particular, for any $t \in U$ the unipotent rank $\rho_u(t)$ of G_t is zero.*

Proof. This follows immediately from Theorem 12.11.6 and the inequality $\rho_r(t) \leq \rho_n(t)$ for any $t \in S$. □

In the proof of Theorem 12.11.6, we have also proved the following result:

Corollary 12.11.8. *Let G be as in Theorem 12.11.6 and suppose that G is of locally constant reductive rank. Consider the functor*

$$\mathscr{T} : \mathbf{Sch}_S^{\mathrm{op}} \to \mathbf{Set}$$

such that for any $S' \to S$, we have

$$\mathscr{T}(S') = \{\textit{the set of maximal torus in } G_{S'}\}.$$

Then $\mathscr{T}$ is representable by a scheme which is smooth, separated and of finite type over S.

Proof. It remains to show that $\mathscr{T}$ is of finite type over S, for which we can assume that G admits a maxila torus T. By Corollary 12.10.26, $N_G(T)$ and $G/N_G(T)$ are representable by schemes, and $\mathscr{T}$ is isomorphic to $G/N_G(T)$. The morphism $G \to \mathscr{T}$ defined by $g \mapsto \mathrm{Inn}(g)(T)$ being surjective, and G quasi-compact over S, we see that $\mathscr{T}$ is also quasi-compact over S, whence the corollary. □

The functor $\mathscr{T}$ of Corollary 12.11.8 is called the **scheme of maximal tori of** G. We will see in 12.11.2.1 that it is in fact affine over S.

Remark 12.11.9. As the functor F in Theorem 12.10.17 is representable by a scheme M over S, by definition, the identity morphism on M corresponds to a subgroup $\widetilde{H}$ of G_M, which is called the **universal subgroup of multiplicative type** of G_M. Its name is justified by the following observation: for any subgroup H of multiplicative type of G, which corresponds to an S-section $\sigma : S \to M$, consider the following commutative diagram

$$\begin{tikzcd} F(S) \arrow[r] \arrow[d, "\sim"'] & F(M) \arrow[d, "\sim"] \\ M(S) \arrow[r] & M(M) \end{tikzcd}$$

As the image of $\sigma \in M(S)$ in $M(M)$ is the identity (it is a section), we conclude that the image of $H \in F(S)$ in $F(M)$ is equal to $\widetilde{H}$, i.e. the inverse image of H under the morphism $G_M \to G$ equals to $\widetilde{H}$. On the other hand, since σ is a section of M over S, it induces a section G_M over G, and we then see that the inverse image of $\widetilde{H}$ under this section is equal to H, that is, *any subgroup of H of multiplicative type can be obtained from $\widetilde{H}$.*

More generally, for $S' \to S$, we see that a section $\sigma : S' \to M$ (which corresponds to a subgroup H of multiplicative type of $G_{S'}$) induces a commutative diagram with Cartesian squares:

$$\begin{tikzcd} H \arrow[r] & \widetilde{H} \arrow[d, hook] & \\ G_{S'} \arrow[r] & G_M \arrow[r] & G \arrow[d] \\ S' \arrow[r] & M \arrow[r] & S \end{tikzcd}$$

If the section σ factors through $\widetilde{H}$, i.e. σ comes from an element of $\widetilde{H}(S')$, then we conclude from the above diagram that we obtain a section of H over S'. Therefore, the subscheme $\widetilde{H} \subseteq G_M$ represents the following functor:

$$\widetilde{H}(S') = \begin{Bmatrix} \text{set of couples } (H,\sigma)\text{, where } H \text{ is a subgroup of multiplicative type} \\ \text{of } G_{S'}\text{, and } \sigma \text{ is a section of } H \text{ over } S' \end{Bmatrix}.$$

Now assume that G is of constant reductive rank r, so the functor $\mathscr{T}$ is representable (by an open subscheme of G). Let H be a subgroup of multiplicative type of G and $s \in S$, such that H_s is a torus of G_s of relative dimension r (a maximal torus). Then the fiber H_s corresponds to a morphism $\operatorname{Spec}(\kappa(s)) \to M$, and we have a commutative diagram

$$\begin{tikzcd} F(S) \arrow[r] \arrow[d, "\sim"] & F(M) \arrow[r] \arrow[d, "\sim"] & F(\operatorname{Spec}(\kappa(s))) \arrow[d, "\sim"] \\ M(S) \arrow[r] & M(M) \arrow[r] & M(\kappa(s)) \end{tikzcd}$$

so the fiber of $\widetilde{H}$ along this morphism is equal to H_s. In other words, the underlying subspace of $\mathscr{T}$ consists of points $x \in M$ such that $\widetilde{H}_x$ is a torus of $(G_M)_x$ of relative dimension r.

We now conclude this paragraph by giving some examples where maximal tori exsit.

Proposition 12.11.10. *Let G be an S-group of multiplicative finite type over S. Then G admits a unique maximal torus, and any torus of G is contained in this maximal torus.*

Proof. The uniqueness follows clearly from the last assertion, which characterizes the maximal torus as the largest torus of G. From the uniqueness, we see that the question of existence is local for the fpqc topology, which allows us to suppose that G diagonalizable, i.e. of the form $D_S(M)$, M a finitely generated abelian group finite type. Let M_0 be the quotient of M by its torsion subgroup, then the torus $T = D_S(M_0)$ in G is maximal and the greatest sub-torus. Indeed, a sub-torus T' of G is locally diagonalizable for the fpqc topology, so to prove that $T' \subseteq T$, we can suppose that T' is diagonalizable, therefore of the form $D_S(N)$, where N is a free quotient of M, hence N is a quotient of M_0. Since the construction of T as $D_S(M_0)$ is compatible with base change, this shows at the same time that T is a maximal torus of G, and completes the proof. ☐

In the case where G is smooth over S, we can generalize Proposition 12.11.10:

Proposition 12.11.11. *Let G be an S-group of finite presentation over S. Suppose that G admits fpqc locally a central maximal torus, then it admits (globally) a unique maximal torus, and this is the largest torus of G.*

Proof. The uniqueness follows from the last assertion, so the question is fpqc local and we can assume that G admits a central maximal torus T. Then any torus R of G is contained in T, in view of Lemma 12.11.12 below. ☐

Lemma 12.11.12. *Let G be an S-group of finite presentation over S and T be a maximal torus of G. If R is a sub-torus of G and R commutes with T, then $R \subseteq T$.*

Proof. As R commutes with T, the morphism $R \times_S T \to G$ defined by $(r, t) \mapsto rt$ is a homomorphism of groups, hence it admits an image subgroup T' in G (Theorem 12.8.49), which is a quotient group of multiplicative type of $R \times_S T$, hence a torus, and contains T. As T is a maximal torus, we then have $T' = T$, hence $R \subseteq T$. ☐

Corollary 12.11.13. *Let G be an abelian S-group, smooth and affine over S with locally constant reductive rank. Then G admits a unique maximal torus, and it contains any sub-torus of G.*

Proof. This follows from Proposition 12.11.11 and Theorem 12.11.6 (b). ☐

Corollary 12.11.14. *Let G be a smooth and affine group over S. Suppose that for any $s \in S$, denote by $\bar{s}$ the spectrum of an algebraic closure k of $\kappa(s)$, the geometric fiber $G_{\bar{s}}$ is a connected and nilpotent algebraic group. Suppose further that the reductive rank of G is locally constant, then G admits a unique maximal torus T, and T is central and is the largest sub-torus of G.*

Proof. In view of Theorem 12.11.6 (b), G admits fpqc locally a maximal torus, so by Proposition 12.11.11 we are reduced to prove that any maximal torus of G is central. In view of Theorem 12.8.41 (b), we can assume that S is the spectrum of an algebraically closed field. Then $T(k)$ is in the center of $G(k)$ by ([**?**] 6 th.2), which implies that T is in the center of G, bacause then $Z_G(T)$ is a closed subscheme of G which contains the points of $G(k)$, and hence by Nullstellensatz is equal to G. ☐

12.11.1.2 The Weyl group We first consider an algebraically closed field k and let G be an algebraic group over k, smooth and affine over k. If T is a maximal torus of G, C is its centralizer and N the normalizer, then in view of Corollary 12.10.26 and Proposition 12.10.31, these are closed smooth subgroups of G, and C is an open subgroup of N, so that $W = N/C$ is a finite étale group over k, hence determined by the group $W(k)$ of closed points of k (in fact, $W = W(k)_k$, the constant k-group associated with $W(k)$). The finite (ordinary) group $W(k)$ is called the **geometric Weyl group** (or simply the **Weyl group**) **of** G **relative to** T. In view of Borel's conjugation theorem, the Weyl groups relative to different maximal tori are all isomorphic, so we can say that Weyl group of G without specifying the maximal torus. As the formation of C, N and N/C commutes with arbitrary base extension, we see that if k' is algebraically closed field containing k, the geometric Weyl group of $G_{k'}$ relative to $T_{k'}$ is canonically isomorphic to that of G relative to T; therefore, the geometric Weyl group of G coincides with that of $G_{k'}$.

This allows us to define the geometric Weyl group of an algebraic group G, smooth and affine over an arbitrary field k, to be that of $G_{k'}$ for k' an algebraic closure of k. If G admits a maximal torus T, then we can evidently form $C = Z_G(T)$, $N = N_G(T)$, and $W = N/C$, which is a finite étale group over k, called the **geometric Weyl group relative to** T. This is then nothing other than the group of points of W with values in an arbitrary algebraically closed extension k' of k. However, the knowledge of the geometric Weyl group $W(k')$ is obviously not sufficient in general, to reconstruct the algebraic group W: it is also necessary to know the operation of the Galois group $\mathrm{Gal}(k'/k)$ on $W(k')$.

If finally G is a group scheme over an arbitrary base S, G is smooth and affine over S, and if T is a maximal torus of G, then Proposition 12.10.31 allows us to define the group

$$W(T) = N_G(T)/Z_G(T)$$

which is an étale S-group, separated and quasi-finite over S. Its geometric fiber (relative to an algebraic closure of the residue field $\kappa(x)$, $s \in S$) is then the geometric Weyl groups of the fiber G_s. As a result, Corollary 12.10.32 gives us information on the variation of these groups with $s \in S$. We can specify and generalize this information as follows:

Theorem 12.11.15. *Let S be a scheme, G be an S-group, smooth and affine over S. For any $s \in S$, let $w(s)$ be the geometric Weyl group of G_s, which is an isomorphism class of finite groups. In the set E of isomorphism classes of finite groups, we introduce the following order: $w \leq w'$ if w and w' are represented by finite groups W and W', respectively, such that W is isomorphic to a quotient of W'.*

(a) *The function $s \mapsto w(s)$ from S to E is lower semi-continuous.*

(b) *Suppose that the reductive rank of G is locally constant. Then the following conditions are equivalent:*

  (i) *The function $s \mapsto w(s)$ is locally constant.*

  (ii) *The function $s \mapsto \mathrm{Card}(w(s))$ is locally constant.*

  (iii) *There exists, locally for the étale topology, a maximal torus T of G such that $W(T)$ is finite over S.*

  (iv) *For any $S' \to S$ and any maximal torus T of $G_{S'}$, the Weyl group $W(T)$ is finite over S'.*

Preceding as in Theorem 12.11.6 (a), we are reduced to prove that for any $s \in S$, there exists an open neighborhood U of s such that $t \in U$ implies $w(t) \geq w(s)$, provided that threre exists a torus R in G such that R_s is a maximal torus of G_s. Let $W(R) = N_G(R)/Z_G(R)$ be as in Proposition 12.10.31, which is a étale group scheme, separated and quasi-finite over S. For any $t \in S$, let $w'(t) \in E$ be the geometic fiber of $W(R)$ at t. As R_s is a maximal torus in G_s and the formation of N_G, Z_G and N_G/Z_G are compatible with base change, we see that

$$w(s) = w'(s);$$

We further assert that $w(t) \geq w'(t)$ and $w'(t) \geq w'(s)$ for t in a neighborhood of s, and this proves (a). These two inequalities are contained in the following two lemmas.

Lemma 12.11.16. *Let S be a scheme, W be an S-group which is étale, separated and quasi-finite over S. For any $s \in S$, let $f(s)$ be the class of the geometric fiber of W at s, which is an element of the ordered set E of isomorphism classes of finite groups. Then the function $f : S \to E$ is lower semi-continuous. For it to be constant in a neighborhood of $s \in S$, it is necessary and sufficient that so is the function $s \mapsto \mathrm{Card}(f(s))$, and for this it is necessary and sufficient that W is finite over S on an open neighborhood U of s.*

Proof. This result is a refinement of the fact invoked in the proof of Corollary 12.10.32, we confine ourselves to a sketch of the demonstration, see also ([**?**] 15.5.1) and ([**?**], 18.10.7). We can again assume that S is affine and Noetherian, and hence the function f is constructible ([**?**] 0_{III}, 9.3.1 et 9.3.2). In view of ([**?**] 0_{III}, 9.3.4), we only need to prove that if t is a generalization of s, then $f(t) \geq f(s)$. Then thanks to Proposition 9.7.1, we can reduced to the case where S is the spectrum of a discrete valuation ring, which we can assume to be complete with algebraically closed residue field. But then, as G is étale and separated over S, it contains a clopen subscheme G', finite over S, such that $G'_s = G_s$ (Corollary 9.6.19), and we see immediately that G' is a subgroup of G in this case. Moreover, as G' is étale and finite over $S = \operatorname{Spec}(R)$, with R complete with algebraically closed residue field, it follows from ([**?**] 18.8.1) that G is a constant group, hence of the form A_S, where $A = G'(\kappa(s)) = G(\kappa(s))$ has class $f(s)$. If B is the geometric fiber of G at the generic point t of S, we then have a canonical monomorphism $A \to B$, and this proves $f(s) \leq f(t)$. The fact that f is constant near s (or equivalently, $\operatorname{Card}(f)$ is constant near s) if and only if W is finite over S in a neighborhood of s follows from Proposition 10.4.19 and ([**?**] 15.5.1). □

Lemma 12.11.17. *Let G be an affine smooth algebraic group over an algebraically closed field k, $R \subseteq T$ be two sub-torus of G, $W(R)$ and $W(T)$ be the finite groups relative to R and T. Then $W(R)$ is isomorphic to a quotient of $W(T)$.*

Proof. We consider the following diagram:

$$\begin{tikzcd}
T \arrow[r, hook] & C(T) \arrow[r, hook] \arrow[d, hook] & C(R) \arrow[d, hook] \\
& N(T) \cap N(R) \arrow[r, hook] \arrow[d, hook] & N(R) \\
& N(T) &
\end{tikzcd}$$

Then $(N(T) \cap N(R))/C(T)$ is a subgroup of $W(T) = N(T)/C(T)$, and we have an evident homomorphism

$$(N(T) \cap N(R))/C(T) \to W(R) = N(R)/C(R),$$

so it remains to show that this is surjective, i.e. for any point g of $N(R)$ with values in k, there exists a point c of $C(R)$ with values in k such that cg normalizes T, that is, such that

$$\operatorname{Inn}(c)(\operatorname{Inn}(g)T) = T.$$

For this, it suffices to note that $\operatorname{Inn}(g)T$ is a torus of $N(R)$, hence of $C(R)$ (which is an open subgroup). Then T and $\operatorname{Inn}(g)T$ are two maximal tori of $C(R)$, since they are maximal in G, and we conclude by Borel's conjugation theorem. □

We therefore conclude assertion (a) of Theorem 12.11.15. As for (b), we have already pointed out that (i) and (ii) are trivially equivalent, and they imply (iv) according to the converse of Lemma 12.11.16. On the other hand, (iv)⇒(iii) thanks to Theorem 12.11.6 (b) and Lemma 12.11.16. Finally, (iii)⇒(ii), because we have seen in Theorem 12.11.6 (a) that condition (ii) is local for fpqc topology, which allows us to assume that G admits a maximal torus T such that $W(T)$ is finite over S, and we conclude again using Lemma 12.11.16. This completes the proof of Theorem 12.11.15.

12.11.1.3 Cartan subgroups

Definition 12.11.18. Let G be a group scheme smooth and of finite type over a scheme S. A subgroup C of G is called a **Cartan subgroup** of G if it is smooth over S and such that for any $s \in S$, if $\bar{s}$ denotes the spectrum of an algebraic closure of $\kappa(s)$, then $C_{\bar{s}}$ is a Cartan subgroup of $G_{\bar{s}}$.

It is immediate that if C is a Cartan subgroup of G, then for any S' over S, $C_{S'}$ is also a Cartan subgroup of $G_{S'}$. We also note that the fact for a subgroup C of G to be a Cartan subgroup is local for the fpqc topology.

Theorem 12.11.19. *Let G be smooth group scheme, affine and of finite type over S, with locally constant reductive rank. Then the map*

$$T \mapsto Z_G(T)$$

induces a bijection from the set of maxmal tori of G to that of Cartan subgroups of G[54]*. If C corresponds to T, then T is the unique maximal torus of C.*

Proof. Let T be a maximal torus of G, the functor $Z_G(T)$ is representable by a closed and smooth subscheme of G (Proposition 12.10.31), $C = Z_G(T)$, and it follows by definition that C is a Cartan subgroup of G. Moreover, T is evidently a maximal torus in C, and being central in C, it is the unique maximal torus of C (Proposition 12.11.11). Hence the map $T \mapsto Z_G(T)$ is injective.

To see it is surjective, let C be a Cartan subgroup of G. It suffices to find a maximal torus T of C, because then T is a maximal torus of G (for any $s \in S$, C_s and G_s have the same reductive rank), and in view of ([?] IX 5.6(b)), T is in the center of C, hence $C \subseteq C' = Z_G(T)$. But C is a smooth subgroup of the smooth group C' over S and coincides with C' fiber by fiber, whence $C = C'$. Now since G is locally of constant reductive rank, so is C, and hence C admits locally a maximal torus (for the fpqc topology), in view of Theorem 12.11.6 (b). As this torus is central by the preceding arguments, it follows from Proposition 12.11.11 that C admits a maximal torus, whence the assertion. □

Corollary 12.11.20. *Let G be a group scheme smooth and affine over S with locally constant reductive rank. Consider the functor $\mathscr{C} : \mathbf{Sch}_{/S}^{\mathrm{op}} \to \mathbf{Set}$ defined by*

$$\mathscr{C}(S') = \{\textit{the set of Cartan subgroups of } G_{S'}\},$$

then $\mathscr{C}$ is isomorphic to the functor $\mathscr{T}$ of Corollary 12.11.8, hence is representable by a scheme which is smooth, separated and of finite type over S.

Corollary 12.11.21. *Under the conditions of Theorem 12.11.19, if $C = Z_G(T)$, we have*

$$N_G(C) = N_G(T).$$

Proof. In fact, for $S' \to S$ and $g \in G(S')$, if $\mathrm{Inn}(g)(C) \subseteq C$, then $\mathrm{Inn}(g)$ fixes the unique maximal torus of $C_{S'}$, which is $T_{S'}$, so we have $g \in N_G(T)(S')$. The converse follows from definition: if $g \in N_G(T)(S')$, then $\mathrm{Inn}(g)(T) = T$, and for any $t \in T(S')$, $c \in C(S')$, we have

$$\mathrm{Inn}(g)(c) \cdot \mathrm{Inn}(g)(t) = \mathrm{Inn}(g)(ct) = \mathrm{Inn}(g)(tc) = \mathrm{Inn}(g)(t) \cdot \mathrm{Inn}(g)(c)$$

so $\mathrm{Inn}(g)(C) \subseteq C$, and thus $g \in N_G(C)$. □

12.11.2 The reductive center

Definition 12.11.22. Let S be a scheme, G be an S-group of finite presentation over S with affine fibers, and Z be a subgroup of G. We say that Z is a **reductive center** of G if

(i) Z is central in G and of multiplicative finite type over S,

(ii) for any morphism $S' \to S$ and any central homomorphism $u : H \to G_{S'}$, where H is a group of multiplicative finite type over S, u factors through $Z_{S'}$.

We note that such a subgroup Z is uniquely determined as the largest central subgroup of G of multiplicative fintie type over S. It is easy to give examples where G (smooth and affine over S) admits a largest central subgroup Z of multiplicative finite type over S, but where Z is not a reductive center (cf. Remark 12.11.5). In fact, it follows from Theorem 12.8.49 that a subgroup Z of G is a reductive center if and only if it is a largest central subgroup of multiplicative type and preserves this property under any base change.

It is evident that if Z is a reductive center of G, then for any base change $S' \to S$, $Z_{S'}$ is the reductive center of $G_{S'}$. By fpqc descent ([?] VIII) and the uniqueness of reductive center, we see that the existence of the reductive center of G is local for the fpqc topology.

[54]The proof given here in fact only proves the theorem for closed Cartan subgroups. However, ([?] XII 7.1(a)) proves the theorem in the given form and shows that any Cartan subgroup of G is in fact closed.

Proposition 12.11.23. *Let G be an S-group of finite presentation with affine fibers, Z be a subgroup of G. For Z to be a reductive center of G, it is necessary and sufficient that it is of multiplicative type and for any $s \in S$, Z_s is a reductive center of G_s.*

Proof. The condition is clearly necessary. Conversely, let Z be such that Z_s is a reductive center of G_s for any $s \in S$. It then follows from Theorem 12.8.41 that Z is central in this case. As the considered properties are invariant under base change, it remains to show that any central homomorphism $u : H \to G$, with H of multiplicative and fintie type, factors through Z. Now since Z is central, the canonical immersion $Z \to G$ and u define a group homomorphism

$$w : H \times_S Z \to G.$$

In view of Theorem 12.8.49, this admits a image group K, which is a subgroup of multiplicative type of G, and we must show that $K = Z$. But this can be justified fiberwise using the hypothesis on Z, and then apply Remark 12.8.40. □

We also note that in Proposition 12.11.23, the hypothesis that for any $s \in S$, Z_s is a reductive center of G_s is in fact purely geometric, i.e. it suffices to verify this over the algebraic closure of $\kappa(s)$ (as it is local for the fpqc topology).

Theorem 12.11.24. *Let G be an affine algebraic group over a field k. Then G admits a reductive center.*

Proof. As the center of G is represented by a closed subgroup of G (cf. Corollary 12.6.43), we are easily reduced to the case where G is abelian. Moreover, since the considered property is fpqc local, we may asume that k is algebraically closed. □

Lemma 12.11.25. *Let H be an S-group of multiplicative type, then any group homomorphism $u : H \to \mathbb{G}_a$ is trivial.*

Proof. As a faithfully flat and quasi-compact morphism is epimorphic, it suffices to consider the case where $H = D_S(M)$ is diagonalizable. Consider the module $E = \mathscr{O}_S^2$, as an extension of $E' = \mathscr{O}_S$ by $E'' = \mathscr{O}_S$. Then $\mathbb{G}_a$ is identified with the scheme of automorphisms of this extension, hence a homomorphism $u : H \to \mathbb{G}_a$ is identified with an H-module structure over E respecting this extension, i.e. such that E' is stable under the action of H and that the operation induced by H on E'' is trivial. In view of Proposition 12.1.34, if $E = \bigoplus_m E_m$ is the corresponding graduation of E, this implies that $E_m = E_m \cap E'$ for $m \neq 0$, so H acts trivially on E, hence u is trivial. □

Remark 12.11.26. If G is a nontrivial abelian variety over k, then G does not admits reductive center in the sense of Definition 12.11.22, where we omit the "affine" restriction. In fact, for n coprime to the characteristic of k, $G[n]$ is étale over k with order coprime to $\mathrm{char}(k)$, hence is of multiplicative type. But the family $(G[n])$ is schematically dense in G, so if there exists a reductive center, then it must be equal to G, which is absurd.

Lemma 12.11.27. *Let S be a scheme, G be an affine smooth group over S, with connected fibers. Let T be a maximal torus of G and $u : H \to G$ be a central homomorphism, with H a group of multiplicative finite type over S. Then u factors through T.*

Proof. Let C be the centralizer of T, which is a closed subscheme of G and smooth over S (Proposition 12.10.31), hence affine over S. As T is contained in the center of C, it is normal, and we can consider the quotient group $C/T = U$, which is representable (Corollary 12.8.10). Now as u is central, it factors through C (its image commutes with T), and it remains to prove that the composition homomorphism $H \to C \to U = C/T$ is trivial. In view of Theorem 12.8.41, we may reduce to the case where S is the spectrum of a field, which we may assume to be algebraically closed. In this case, by ([**?**] 6, th.2), U is a connected unipotent algebraic group (smooth over k), which signifies that it admits a decomposition series with quotients isomorphic to $\mathbb{G}_a$. Therefore any homomorphism from a group of multiplicative finite type H to U is trival, whence our assertion. □

Corollary 12.11.28. *Let G be an affine smooth group over S with connected fibers. If G admits a reductive center, then it is contained in any maximal torus of G.*

Proof. This follows from Lemma 12.11.27 and condition (b) in Definition 12.11.22. □

Theorem 12.11.29. *Let S be a scheme, G be an affine smooth group over S with connected fibers.*

(a) *For any $s \in S$, let $z(s)$ be the type of the reductive center of G_s (exists by Theorem 12.11.24). Let E be the ordered set of isomorphic classes of finitely generated $\mathbb{Z}$-modules, which $M \leq M'$ if and only if M is isomorphic to a quotient of M'. Then the map $S \to E, s \mapsto z(s)$ is lower semi-continuous.*

(b) *For G to admit a reductive center Z, it is necessary and sufficient that the function $z : S \to E$ in (a) is locally constant. In this case, G/Z is representable (cf. Proposition 12.8.8), and G/Z admits the trivial subgroup as its reductive center.*

(c) *Suppose that the reductive rank of G is locally constant. Then G admits a reductive center Z, and the maximal tori T of G/Z (resp. Cartan subgroups C of G) correspond bijectively to the maximal tori T' of $G' = G/Z$ (resp. Cartan subgroups C' of G) via $T' = T/Z$ (resp. $C' = C/Z$).*

(d) *Let T be a maximal torus of G, $\mathfrak{g} = \mathfrak{Lie}(G)$ be the Lie algebra of G, and consider the homomorphism*

$$\theta : T \to \mathrm{GL}(\mathfrak{g})$$

induced by the adjoint representation of G. Then the kernel of θ is a reductive center of G.

Let us give a useful translation of (d), in the case where T is diagonalizable, hence of the form $D_S(M)$ where M is a free $\mathbb{Z}$-module of finite rank. Then by Proposition 12.1.34, the T-module $\mathfrak{g}$ admits a graduation of type M:

$$\mathfrak{g} = \bigoplus_{m \in M} \mathfrak{g}_m$$

where $\mathfrak{g}_m$ are sub-T-modules of $\mathfrak{g}$ (which are necessarily locally free, since G is smooth over S). Suppose that for any $m \in M$, the rank of $\mathfrak{g}_m$ is constant, then the set R of elements $m \in M$ such that $\mathfrak{g}_m \neq 0$ (roots of T) is finite. We then conclude the following corollary:

Corollary 12.11.30. *Under the conditions of Corollary 12.11.28 (d), the reductive center of G is the intersection of kernels of roots $m \in R$. We then have an isomorphism*

$$Z \cong D_S(N)$$

where N is the quotient of M by the subgroup generated by R.

Corollary 12.11.31. *Let G be an algebraic group over an algebraically closed field k. Suppose that G is smooth, connected, affine, with reductive center reduced to the identity, and that the Lie algebra $\mathfrak{g}$ of G is nilpotent. Then G is unipotent, i.e. admits a decomposition series with quotients isomorphic to $\mathbb{G}_a$.*

Proof. In view of ([**?**] 6, th.4, cor.3), it suffices to show that any maximal torus T of G is reduced to the trivial group, or equivalently, that the Lie algebra $\mathfrak{t}$ of T is reduced to 0. But this follows from the fact that the T-module $\mathfrak{g}$ decomposes according to roots α of T, that for any $t \in \mathfrak{t}$, the operator $\mathrm{ad}_{\mathfrak{g}}(t)$ on $\mathfrak{g}$ is semisimple (T is semisimple). As $\mathfrak{g}$ is nilpotent and $\mathfrak{t} \subseteq \mathfrak{g}$, this then implies that $\mathrm{ad}_{\mathfrak{g}}(t) = 0$. Now in view of Theorem 12.11.29 (d), as the reductive center of G reduces to the neutral element, the homomorphism $T \to \mathrm{GL}(\mathfrak{g})$ is a monomorphism, and therefore induces an injective map on Lie algebras, which implies that (Proposition 12.2.75) for $t \in \mathfrak{t}$, the relation $\mathrm{ad}_{\mathfrak{g}}(t) = 0$ implies $t = 0$. This proves that $\mathfrak{t} = 0$ and completes the demonstration. □

Proposition 12.11.32. *Let G be an affine smooth connected algebraic group over an algebraically closed field k. Then the reductive center of G is the intersection of maximal tori of G.*

Proof. Of course, this is the intersection in the schematic sense (or equivalently, in the sense of subfunctors of G), i.e. the largest closed subscheme of G dominated by the maximal tori of G. As G is Noetherian, this is also the intersection of a suitable finite set of maximal tori of G.

Let Z be this intersection, which is a closed subgroup of a torus, hence is of multiplicative type (cf. Proposition 12.8.55). By Lemma 12.11.27 (d), Z contains the reductive center of G; to prove the equality, it remains to prove that it is central. Since G is connected, it suffices in view of Corollary 12.8.39 to prove that Z is normal. But by construction Z is invariant under the $\mathrm{Int}(g)$, with $g \in G(k)$, hence the normalizer $N = N_G(Z)$ is a closed subgroup of G which contains the rational points of G. As G is reduced, we then have $N = G$, which completes the proof. □

Proposition 12.11.33. *Let S be a scheme and G be an affine smooth group over S with connected fibers and of zero unipotent rank. Let $\mathscr{T}$ be the scheme of maximal tori of G, which is smooth, separated and of finite type over S (cf. Theorem 12.11.6 and Corollary 12.11.8). Let G acts on $\mathscr{T}$ by inner automorphisms, so we have a homomorphism of group functors*

$$u : G \to \mathcal{A}ut_S(\mathscr{T}).$$

Then the following subfunctors of G are identical:

(i) *The reductive center Z of G.*

(ii) *The center $Z(G)$ of G.*

(iii) *The kernel Z' of the homomorphism u.*

In particular, the center of G is representable by a subgroup of multiplicative type of G.

Proof. It is clear that we have $Z \subseteq Z(G) \subseteq Z'$, so it remains to show that $Z' \subseteq Z$, which amounts to saying that (the hypothesis being stable under base change) any section $g \in G(S)$ acting trivially on $\mathscr{T}$ is a section of Z. Putting $G' = G/Z$ and utilizing Theorem 12.11.29 (b) and (c), which imply in particular that the scheme $\mathscr{T}'$ of maximal tori of G' is canonically isomorphic to $\mathscr{T}$, we are reduced to the case where $G = G'$, i.e. where the reductive center of G is trivial (note that in view of Theorem 12.11.29 (c), the unipotent rank of G' is equal to that of G, so is zero). It is then necessary to prove in this case that g is the unit section of G. The usual procedure of passing to limit reduces us to the case where S is Noetherian, and similarly to the case where S is local Artinian. In this case, the kernel Z' of u is representable (Proposition 12.6.37 (a) and Example 12.6.40 (c)), so to show that Z' reduces to the identity, it suffices in view of Nakayama's lemma to prove this for the fiber Z'_0. This then reduces us to the case where S is the spectrum of a field k, which we can evidently assume to be algebraically closed. Now Z' is contained in the stabilizer of any point of $\mathscr{T}(k)$, i.e. in the normalizer $N_G(T)$ of any maximal torus T of G. As the unipotent rank of G is zero, by Proposition 12.10.31, $T = Z_G(T)$ is an open subgroup of $N_G(T)$, hence the Lie algebra of $N_G(T)$ is identified with that of T, and the Lie algebra of Z' is contained in T. On the other hand, it follows from Proposition 12.11.32 that the intersection of the Lie algebras of maximal tori T of G is none other than the Lie algebra of the reductive center Z, so it is zero, since we have supposed $Z = 0$. Therefore, the Lie algebra of Z' is zero, i.e. Z' is étale over k. Furthermore, Z' is evidently normal in G, and as G is connected, it follows easily that Z' is contained in the center of G, and hence in $T = Z_G(T)$ for any maximal torus T. We then conclude that Z' is contained in the intersection of maximal tori of G, which is zero by Proposition 12.11.32, and this completes the proof. □

12.11.2.1 Application to subgroups of multiplicative type

Theorem 12.11.34. *Let S be a scheme, G be an affine smooth S-group, M be the scheme of subgroups of multiplicative type of G (cf. Theorem 12.10.17). For any integer $n > 0$, let T_n be the sub-functor of M such that $T_n(S')$ is the set of subgroups of multiplicative type H of $G_{S'}$ such that $n \cdot \mathrm{id}_H = 0$ (which is representable and affine over S, cf. Proposition 12.10.15). Let $u_n : M \to T_n$ be the morphism defined by $u_n(H) = H[n]$, where $H[n] = \ker(n \cdot \mathrm{id}_H)$.*

(a) *Any subscheme U of M, of finite type over S, is contained in a closed subscheme of finite type over S, and any closed subscheme Z of M, of finite type over S, is affine over S.*

(b) *Suppose that S is quasi-compact, and let Z be a closed subscheme of M of finite type over S. Then there exists an integer $n > 0$ such that for any multiple m of n, the morphism*

$$u_m|_Z : Z \to T_m$$

is a closed immersion.

Corollary 12.11.35. *With the notations of Theorem 12.11.34, let U be a clopen subset of M, of finite type over S. Then U is affine over S for the induced scheme structure by M, and if S is quasi-compact, there exists $n > 0$ such that for any multiple m of n, the induced morphism $u_m|_U : U \to T_m$ is an open and closed immersion.*

Proof. The first assertion follows from Theorem 12.11.34 (a), and the second one from Theorem 12.11.34 (b), as the morphism $u_m : M \to T_m$ is smooth (Corollary 12.10.5) and that a smooth immersion (i.e. étale immersion) is an open immersion. □

Corollary 12.11.36. *Let S be a scheme, G be an affine smooth S-group, of locally constant reductive rank, $\mathscr{T}$ be the scheme of maximal tori of G (cf. Corollary 12.11.8). Then $\mathscr{T}$ is smooth and affine over S. If T is a maximal torus of G, $N_G(T)$ its normalizer, then $G/N_G(T)$ is affine over S (cf. Corollary 12.10.26). The same is true for $G/Z_G(T)$ provided that $W(T) = N_G(T)/Z_G(T)$ is finite over S (cf. Theorem 12.11.15 (b)).*

Proof. The second asertion is contained in the first one, because by the conjugation theorem, $G/N_G(T)$ is isomorphic to $\mathscr{T}$. Note that by construction, $\mathscr{T}$ is isomorphic to a clopen subscheme of M. To see this, we can suppose that the reductive rank of G is constant and equals to r; then $\mathscr{T}$ is the subscheme of M which corresponds to sub-tori of relative dimension r, i.e. the largest subscheme of M over which the universal subgroup of multiplicative type $\widetilde{H} \subseteq G_M$ is of type $\mathbb{Z}^r$ (cf. Remark 12.11.9), which is clopen in M. We can then apply Corollary 12.11.35. Finally, for the last assertion, we note that $G/Z_G(T)$ is finite over $G/N_G(T) \cong (G/Z_G(T))/W(T)$[55], hence is affine if $G/N_G(T)$ is. □

Corollary 12.11.37. *Let G be an affine smooth algebraic group over a field k. Then the scheme M of subgroups of multiplicative type of G is a direct sum of affine schemes over S. For any subgroup H of multiplicative type of G, if $C = Z_G(T)$ and $N = N_G(T)$, the quotients G/C and G/N are affine.*

Proof. Using Corollary 12.10.7, we see that the saturation of any finite closed subset of M under the action of G is open: in fact, we are reduced to the case where k is algebraically closed, hence to the case of the orbit of a rational point x over k. But then by Corollary 12.10.7, the morphism $g \mapsto g \cdot x$ from G to M is smooth[56], so its image is open. Let U be the union of orbits of closed points of M under the equivalence relation defined by the action of G. Then U is open and contains any closed point of M, hence by Nullstellensatz is equal to M. Thus M is a disjoint union of open subsets, which are necessarily closed, hence M is a sum of clopen subschemes M_i, where each M_i is an orbit under G of a closed point, hence is quasi-compact and of finite type (as an image of G). In view of Corollary 12.11.35, each M_i is then affine. If H is a subgroup of multiplicative type of G, it corresponds to a rational point of M over k, and G/C is identified with the orbit of x under G (Remark 12.10.28), hence is affine by the arguments above. As C is an open subgroup of N (Proposition 12.10.31), G/C is finite over G/N, and hence is affine if G/C is (Proposition 9.6.3). □

Corollary 12.11.38. *Under the conditions of Corollary 12.11.37 on G and H, the subscheme U of M of subgroups of multiplicative type of G which is locally conjugate to H (cf. Remark 12.10.28) is a clopen subscheme of G.*

Proof. In the proof of Corollary 12.11.37 we have seen that any orbit in M under H is clopen. □

12.12 Regular elements of algebraic groups and Lie algebras

12.12.1 An auxilary lemma for schemes with operators

Let S be a scheme, G be an S-group acting (on the left) on an S-scheme V, W be a closed subscheme of V and $N = \mathrm{Trans}_G(W)$ be its stabilizer in G. We endow $\mathbf{Sch}_{/S}$ with the fpqc topology, and identify G, V, M with the corresponding sheaves over $\mathbf{Sch}_{/S}$. Consider the quotient sheaf G/N, it is easy to see that this is isomorphic to the functor which for any S' over S associates the set of subsheaves W' of $V_{S'}$ which is locally conjugate with $W_{S'}$ under G. Let X be the **twisted product of** G **and** V **over** N, i.e. the subsheaf of $(G/N) \times_S V$ whose value, for any S' over S, is the set of couples (W', v), where W' is as above and $v \in W'(S')$. Let Z be the inverse image of X in $G \times_S V$, so that we have a Cartesian diagram

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & X \arrow[d, hook, "i"] \\ G \times_S V \arrow[r] & (G/N) \times_S V \end{tikzcd} \tag{12.12.1}$$

where i is the canonical inclusion and the bottom arrow is induced by the canonical morphism $G \to G/N$ sending g to $g \cdot W_{S'}$. We then see that for S' over S, $Z(S')$ is the set of couples $(g, v) \in G(S') \times V(S')$ such that $v \in g \cdot W_{S'}(S')$. Therefore, Z is isomorphic to the sheaf $G \times_S W$ via the isomorphism

$$G \times_S W \xrightarrow{\sim} Z, \quad (g, w) \mapsto (g, g \cdot w).$$

[55] Recall that $G/Z_G(T) \to (G/Z_G(T))/W(T)$ is a $W(T)$-torsor, so we can apply ([?] VIII 5.7).
[56] This is the base change of the morphism Φ in Corollary 12.10.7 by the morphism $M \to M \times_S M, y \mapsto (y, x)$

The preceding Cartesian diagram then gives a Cartesian diagram

$$\begin{tikzcd} G\times_S W \arrow[r, "q"] \arrow[d, "\lambda"'] & X \arrow[d, "i"] \\ G\times_S V \arrow[r] & (G/N)\times_S V \end{tikzcd} \tag{12.12.2}$$

where $\lambda(g,w)=(g,g\cdot w)$, hence $q(g,w)=(\bar{g},g\cdot w)$, where $\bar{g}$ denotes the image of g under the canonical map $G(S')\to(G/N)(S')$. Finally, we see from the diagram (12.12.1) that $Z\to X$ is a principal bundle under the group N, acted by $(g,v)\cdot n=(gn,v)$, so that in (12.12.2), the morphism $q:G\times_S W\to X$ is a principal bundle under the right action of N defined by

$$(g,w)\cdot n=(gn,n^{-1}\cdot w).$$

We summarize the previous morphisms in the following diagram:

$$\begin{tikzcd} & G\times_S W \arrow[d, "q"] \arrow[dr, "\varphi"] & \\ G/N & X \arrow[l, "p"'] \arrow[r, "\psi"] \arrow[d, hook, "i"] & V \\ & (G/N)\times_S V \arrow[ul, dashed, "\mathrm{pr}_1"] \arrow[ur, dashed, "\mathrm{pr}_2"'] & \end{tikzcd} \tag{12.12.3}$$

where $p=\mathrm{pr}_1\circ i$, $\psi=\mathrm{pr}_2\circ i$ and $\varphi=\psi\circ q$, i.e. $\varphi(g,w)=g\cdot w$. If v is a section of V over S, the subsheaf $X_v\subseteq X$ of the inverse image of v under ψ is then given by $X_v(S')$ the set of subsheaves W' of $V_{S'}$ which is locally conjugate to $W_{S'}$ under G and contains the section $v_{S'}$ of $V_{S'}$. On the other hand, the inverse image of v under φ is isomorphic to the subsheaf M_v of G given by $M_v(S')$ is the set of $g\in G(S')$ such that $v_{S'}\in g\cdot W(S')$, i.e. such that $g^{-1}v_{S'}\in W(S')$. If v is a section of W and not only of V, then M_v obviously contains N.

Now suppose that N is representable and faithfully flat and quasi-compact over S, and that G/N is representable (this is true if S is the spectrum of a field k and G is of finite type over k, cf. Corollary 12.6.41). Then using the theory of fpqc descent and the fact that $Z\to G\times_S V$ is a closed immersion (recall that Z is isomorphic to $G\times_S W$), we see from the Cartesian diagram (12.12.2) that X is representable (it is obtained by descent of the closed subscheme Z from $G\times_S V$ by the faithfully flat and quasi-compact morphism $G\times_S V\to(G/N)\times_S V$), so (12.12.3) is a diagram of morphisms of schemes over S.

We suppose thereafter that S is the spectrum of a field k, and that G, V, W are of finite type over k. Let $\mathfrak{n}$ be the Lie algebra of N, so we have $\dim(N)\leq\dim_k(\mathfrak{n})$, with the equality if and only if N is smooth over k. Let $a\in W(k)$, and consider the subscheme $M_a=\varphi^{-1}(a)$ of G defined as above, which contains N. Let $\mathfrak{m}_a$ the tangent space of M_a at the identity element e of N, so that we have

$$\mathfrak{n}\subseteq\mathfrak{m}_a,\quad \dim(N)\leq\dim_k(\mathfrak{n})\leq\dim_k(\mathfrak{m}_a). \tag{12.12.4}$$

Lemma 12.12.1. *Consider the following conditions:*

(i) *$n=\mathfrak{m}_a$ and N is smooth over k.*

(i′) $\dim(N)=\dim_k(\mathfrak{m}_a)$.

(ii) *The morphism $\psi:X\to V$ is unramified at $(\bar{e},a)$.*

(iii) *M_a and N coincide in a neighborhood of e.*

Then we have the implications (i)⇔(i′)⇒(ii)⇔(iii). Further, suppose that $\varphi:G\times_S W\to V$ is smooth at (e,a), then M_a is smooth over k at e, and ψ is smooth at $(\bar{e},a)$.

Proof. The equivalence of (i) and (i') follows from (12.12.4), and the fact that N is smooth if and only if $\dim(N) = \dim_k(\mathfrak{n})$. On the other hand, consider the inclusion morphism $N \to M_a$, by ([?] 17.11.1 (d)), if N is smooth over k at e and the tangent map at e is surjective, then $N \to M_a$ is smooth at e, hence (being an immersion) is an isomorphism at e; this proves (i)⇒(iii). To prove the equivalence of (ii) and (iii), consider $X_a = \psi^{-1}(a)$; using the isomorphism $M_a \cong \varphi^{-1}(a) = q^{-1}(X_a)$, we obtain a morphism $p_a : M_a \to X_a$ which makes M_a a principal fiber bundle over X_a with group N_{X_a}. Consider the following diagram:

$$\begin{tikzcd} N \arrow[r, "j_a"] \arrow[d] & M_a \arrow[d, "p_a"] \\ S = \operatorname{Spec}(k) \arrow[r, "j'_a"] & X_a \end{tikzcd}$$

where $j'_a : \operatorname{Spec}(k) \to X_a$ is defined by the point $(\bar{e}, a)$ of X, and $j_a : N \to M_a$ is the canonical immersion. To say that ψ is unramified at $(\bar{e}, a)$ signifies that j'_a is an open immersion (Theorem 11.1.20), or that it induces an isomorphism $S \xrightarrow{\sim} \operatorname{Spec}(\mathscr{O}_{X_a,a})$. As p_a is flat, this is equivalent to say that the morphism induced from the preceding morphism by base change $\operatorname{Spec}(\mathscr{O}_{M_a,e}) \to \operatorname{Spec}(\mathscr{O}_{X_a,b})$ is an isomorphism; but this induced morphism is none other than the morphism $\operatorname{Spec}(\mathscr{O}_{N,e}) \to \operatorname{Spec}(\mathscr{O}_{M_a,e})$, so we obtain the equivalence of (ii) and (iii). Finally, as $M \cong \varphi^{-1}(a)$, the second assertion follows from the fact that q is flat and $q(e, a) = (\bar{e}, a)$. □

12.12.2 Regular elements of algebraic groups and density theorem

12.12.2.1 The case of an algebraic group over a field We now apply the construction and notations of the previous subsection to the case where G is a smooth connected algebraic group over k, $V = G$ over which G acts by inner automorphisms, and $W = H$ is a smooth connected subgroup of G. We denote by $\mathfrak{g}, \mathfrak{h}$ the Lie algebras of G and H, respectively, by N the normalizer of H in G, and $\mathfrak{n}$ be its Lie algebra. If $a \in G(k)$, we denote by M_a the subgroup of G isomorphic to $\varphi^{-1}(a)$, so that if $a \in H(k)$, we have $N \subseteq M_a$. In this case, we denote by $\mathfrak{m}_a$ the tangent space of M_a at the identity element of G. Note that for $a \in H(k)$, we have

$$\mathfrak{h} \subseteq \mathfrak{n} \subseteq \mathfrak{m}_a \subseteq \mathfrak{g}.$$

We shall utilize the following lemma:

Lemma 12.12.2. *For that $H = N^0$, it is necessary and sufficient that we have $(\mathfrak{g}/\mathfrak{h})^H = 0$ (where $\mathfrak{g}/\mathfrak{h}$ denotes the subspace of invariants under the adjoint action of H). If this condition is satisfied, N is smooth and we have* $\dim(X) = \dim(G)$. *In any case,* $\dim(X) \leq \dim(G)$, *and the equality holds if and only if H has finite index in N.*

Proof. In view of Theorem 12.2.100 (i), $\mathfrak{n}$ is equal to the inverse image of $(\mathfrak{g}/\mathfrak{h})^H$ under the quotient map $\mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$, hence $(\mathfrak{g}/\mathfrak{h})^H = 0$ if and only if $\mathfrak{n} = \mathfrak{h}$, which is equivalent to (H being a smooth connected subgroup of N) that $H = N^0$ (cf. Remark 12.6.14 and Proposition 12.2.96). This implies evidently that N is smooth. On the other hand, we have

$$\dim(X) = \dim(G) - \dim(N) + \dim(H) = \dim(G) - (\dim(N) - \dim(H)),$$

hence $\dim(X) \leq \dim(G)$, with the equality holds if and only if $\dim(H) = \dim(N)$, i.e. if and only if H has finite index in N. □

Theorem 12.12.3. *Let G be a smooth connected algebraic group over an algebraically closed field k, H be a smooth connected subgroup of G, N be its normalizer, and $X = G \times^N H$ be the twisted product of G and H over N. Let $\psi : X \to G$ be the canonical morphism, defined by $(\bar{g}, h) \mapsto \operatorname{Inn}(g)h = ghg^{-1}$. Then the following conditions are equivalent:*

(i) *H contains a Cartan subgroup C of G.*

(i') *The reductive rank and nilpotent rank H equal to those of G.*

(ii) *H contains a maximal torus T of G, and $(\mathfrak{g}/\mathfrak{h})^T = 0$.*

(iii) *The set of conjugates of H containing a given maximal torus is nonempty and finite, and H has finite index in N.*

(iv) *There exists $a \in H(k)$ which is contained in finitely many conjugates of H (or such that $\psi^{-1}(a)$ has an isolated point), and H has finite index in N.*

(iv′) *The morphism $\psi : X \to G$ is generically quasi-finite (i.e. there exists an open dense subset of X over which ψ is quasi-finite), and H has finite index in N.*

(v) *There exists an open dense subset U of G such that for any $x \in U(k)$, the set of conjugates of H containing x is nonempty and finite, i.e. $\psi : X \to G$ is dominant and generically quasi-finite.*

(vi) *There exists an open dense subset U of G such that any $x \in U(k)$ is contained in a conjugate of H, i.e. $\psi : X \to G$ is dominant.*

(vii) *There exists $a \in H(k)$ such that $(\mathfrak{g}/\mathfrak{h})^{\mathrm{Ad}(a)} = 0$.*

Furthermore, these conditions imply that $H = N^0$, i.e. N is smooth and $\dim(H) = \dim(N)$, *and that $\psi : X \to G$ is generically étale.*

By Lemma 12.12.2, we have $\dim(X) \le \dim(G)$, with the equality holds if and only if $\dim(H) = \dim(N)$, i.e. H has finite index in N. Now the inequality $\dim(X) \le \dim(G)$ implies that $\psi : X \to G$ is dominant if and only if it is domimant and generically quasi-finite, or equivalantly if ψ is generically quasi-finite and $\dim(X) = \dim(G)$. By Lemma 12.12.2 again, we then conclude the equivalence of (iv′), (v) and (vi). The equivalence of (iv) and (iv′) is immdiaite by Corollary 10.4.22, as X is irreducible (being the image of $G \times_S H$).

The equivalence of (i) and (i′) is immediate from definition. On the other hand, if H contains a Cartan subgroup C of G, it contains the maximal torus T of C (cf. Proposition 12.11.11), which is a maximal torus of G. As C is the centralizer of T, its Lie algebra $\mathfrak{c}$ is given by

$$\mathfrak{c} = \mathfrak{g}^T$$

(cf. Theorem 12.2.100 (ii)). Hence as $H \supseteq C$, whence $\mathfrak{h} \supseteq \mathfrak{c}$, it ensures that $\mathfrak{h} \supseteq \mathfrak{g}^T$, which in view of the root decomposition (Proposition 12.1.34) implies that

$$(g/\mathfrak{h})^T = 0. \tag{12.12.5}$$

Conversely, if H contains a maximal torus T and the preceding relation is verified, i.e. $\mathfrak{h} \supseteq \mathfrak{c} = \mathfrak{g}^T$, we claim that H contains the centralizer C of T (whence the equivalence (i) and (ii)). This follows form the following lemma:

Lemma 12.12.4. *Let G be a smooth algebraic group over a field k, T be a subgroup of multiplicative type of G, C be its connected centralizer (i.e. the connected component of $Z_G(T)$*[57]*), H be a smooth subgroup of G containing T. For H to contain C, it is necessary and sufficient that its Lie algebra $\mathfrak{h}$ contains the Lie algebra $\mathfrak{c}$ of C.*

Proof. We have seen in Proposition 12.10.31 that $Z_G(H)$ is smooth over k, hence C is smooth over k; similarly $Z_H(T)$ is smooth over k. Now the intersection $Z_H(T) = Z_G(T) \cap H$ has Lie algebra $\mathfrak{h} \cap \mathfrak{c}$, so the hypothesis $\mathfrak{h} \supseteq \mathfrak{c}$ implies that the smooth subgroup $Z_H(T)$ of the smooth group $Z_G(T)$ have the same Lie algebra, so it contains the connected component C of $Z_G(T)$, which means H contains C. The reverse implication is immediate. □

To prove the equivalence of (i) and (iii), it suffices to prove that if H contains the maximal torus T of a Cartan subalgebra C of G, then the condition $H \supseteq C$ (which is equivalent to (12.12.5), as we have already seen above) is equivalent to that H has finite index in N and the set of conjugates of H containing T is finite. If H contains C, hence $(\mathfrak{g}/\mathfrak{h})^T = 0$, then a fortiori

$$(\mathfrak{g}/\mathfrak{h})^H = 0, \tag{12.12.6}$$

now recall that $\mathfrak{n}$ is the inverse image of $(\mathfrak{g}/\mathfrak{h})^H$ under the canonical morphism $\mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$ (Theorem 12.2.100 (i)), so the preceding relation is equivalent to that $H = N^0$ by Lemma 12.12.2, a fortiori H

[57] This is equal to $Z_G(T)$ if G is connected and T is a torus, cf. ([?] XII 6.6 (b)).

has finite index in N. Now consider the diagraom of subgroups

$$\begin{tikzcd} T \arrow[r] & N(T)\cap H \arrow[r] \arrow[d] & H \arrow[d] \\ & N(T)\cap N(H) \arrow[r] \arrow[d] & N(H) \\ & N(T) & \end{tikzcd}$$

Using the conjugation theorem of maximal tori in H (**[?]** 6.6 (a)), we see that any conjugate of H containing T is conjugate to H by an element of $N(T)(k)$, hence the set of conjugates of H containing T is in bijection to the set of points of $N(T)/(N(T)\cap N(H))$. Now as $H\supseteq C$, we have $N(T)\cap H\supseteq C$, hence the preceding set is a quotient of $(N(T)/C)(k)$ (which is a finite set), hence is finite. Conversely, suppose that $N(T)/(N(T)\cap N(H))$ are $N(H)/H$ are finite. Using the conjugation theorem in H, we see that the homomorphism

$$(N(T)\cap N(H))/(N(T)\cap H)\to N(H)/H$$

induced by the preceding diagram is bijective on k-values points (in fact, it is an isomorphism), so as the second member is finite, so is the first one. It then follows that $N(T)\cap H$ has finite index in $N(T)$, so it contains $C=N(T)^0$, whence $H\supseteq C$. By now, we have proved that conditions (i), (i'), (ii) and (iii) are equivalent.

Now consider the implication (ii)⇒(vii). We easily see that the conditions (ii) and (vii) are each invariant under base extension $k\to k'$, with k' algebraically closed, so we can suppose that k is of infinite transcendendal degree over its prime field. Then it is well known (and we immediately verify) that there exists an element $a\in T(k)$ such that the subgroup of $T(k)$ it generates is dense in T for the Zariski topology. We then conclude that $(\mathfrak{g}/\mathfrak{h})^T=(\mathfrak{g}/\mathfrak{h})^{\mathrm{Ad}(a)}$, and as by hypothesis the first member is zero, we then conclude (vii).

Now as G is irreducible and the smooth locus of ψ is open (cf. **[?]** 17.11.4), the proof of (vii)⇒(vi) is contained in the following corollary, which refines Theorem 12.12.3:

Corollary 12.12.5. *Let G be a smooth algebraic group over a field k, H be a smooth subgroup of G, N be its normalizer in G, and $\varphi: G\times_S H\to G$ be the morphism defined by $\varphi(g,h)=\mathrm{Inn}(g)h=ghg^{-1}$. Let $\psi: X=G\times^N H\to G$ be the morphism induced from φ by passing to quotient, and $a\in H(k)$. Then the following conditions are equivalent:*

(i) *φ is smooth at (e,a).*

(ii) *ψ is étale at $(\bar{e},a)$ and N is smooth over k.*

(iii) *$(\mathfrak{g}/\mathfrak{h})^{\mathrm{Ad}(a)}=0$ (where $\mathfrak{g},\mathfrak{h}$ are the Lie algebras of G, H, respectively).*

These conditions imply $H^0=N^0$.

Proof. We see that the smoothness of φ (which is a morphism of smooth k-schemes) at a rational point over k is equivalent to the surjectivity of the tangent map at this point. Now an immedaite calculus shows that this tangent map is given by

$$d\varphi_{(e,a)}(\xi,\eta)=(\mathrm{Ad}(a^{-1})-\mathrm{id})(\xi)+\eta,$$

consider as a map from $\mathfrak{g}\times\mathfrak{h}\to\mathfrak{g}$ (as usual, we identity the tangent spaces of G and H with their Lie algebras, using left invariant vector fields)[58]. Its surjectivity is then equivalent to the surjectivity of $\mathrm{id}-\mathrm{Ad}(a^{-1})=\mathrm{id}-(\mathrm{Ad}(a))^{-1}$ on $\mathfrak{g}/\mathfrak{h}$, i.e. to the condition in (iii). Now (iii) implies a fortiori $(\mathfrak{g}/\mathfrak{h})^H=0$, i.e. $H^0=N^0$ (cf. Lemma 12.12.2). We deduce that N is also smooth, and $\dim(H)=\dim(N)$, whenc $\dim(X)=\dim(G)$. Now as $q: G\times_S H\to X$ is flat and $\psi=\varphi\circ q$, this implies that ψ is smooth at $q(e,a)=(\bar{e},a)$, hence étale at this point by dimension consideration. We have therefore proved (i)⇔(iii)⇒(ii), but (ii)⇒(i) because the smoothness of N implies that of q. □

[58] As the tangent map can be computed componentwise (i.e. by computing the partial derivatives), it amounts to consider the morphism $G\to G, g\mapsto gag^{-1}$. This morphism sends e to a, and to compute the tangent map on $\mathfrak{g}$, we may compose it with the left multiplication $\ell_{a^{-1}}$, but then note that $a^{-1}gag^{-1}=\mu(C_{a^{-1}}(g),g^{-1})$, where μ is the multiplication morphism of G.

Finally, we prove that (vi)⇒(i), which, together with the already established implications, prove the theorem. Suppose first that G is affine, and let U be the nonempty open subset of G such that $x \in U(k)$ implies that x is contained in a conjugate of H. Let C be a Cartan subgroup of G. Using the implication (i)⇒(v) for C replacing H (this is the density theorem of Borel), we see that we can find a conjugate of C which meets U, hence we can suppose that $U \cap C \neq \varnothing$, i.e. there exists a nonempty open subset V of C such that for any $x \in V(k)$, x is contained in a conjugate of H. Now write C as a product

$$C = T \cdot C_u$$

where T is the maximal torus of C (also a maximal torus of G) and C_u is the unipotent part of C, T being the center of C (**[?]** 6 th.2). We can still assume that k has infinite transcendental degree over its prime field, which allows us to find an element $t \in T(k)$ which belongs to the projection of V onto T (which is a non-empty open set of T), i.e. $(t \cdot C_u) \cap V \neq \varnothing$, and such that t "generates" T. As any algebraic subgroup of G which contains a product $t \cdot u$ (with $t \in T(k)$, $u \in C_u(k)$) contains both factors (**[?]** 4 th. 3), it follows, with the previous choice of t, and taking $t \cdot u \in V(k)$, that there exists a conjugate of H containing t, hence T. Therefore, by taking conjugate of C (and T), we can assume that $T \subseteq H$.

If $W \subseteq C_u$ is the inverse image of V under the morphism $u \mapsto t \cdot u$, then we see that for any element $x \in (T \cdot W)(k)$, there exists a conjugate of H which contains T and x. As we have already remarked (**[?]** 6.6 (a)), such a conjugate is of the form $\mathrm{Inn}(g) \cdot H$, where $g \in N(T)(k)$. Consider then the morphism

$$f : N(T) \times_S H \to G, \quad (g, h) \mapsto \mathrm{Inn}(g)h = ghg^{-1},$$

then the image of f contains $T \cdot W$, hence, as $N(T)$ is a finite union of translates $C \cdot g_i$ with $g_i \in N(T)(k)$ (since C has finite index in $N(T)$), there exists an open dense subset V' of $C = T \cdot C_u$ which is contained in the image of $(C \cdot g_i) \times_S H$ under f. By replacing H with $\mathrm{Inn}(g_i) \cdot H$, we can suppose that $g_i = e$, i.e. $f(C \times H) \supseteq V'$. Hence for any $u \in V'(k)$, there exists $v \in C(k)$ and $h \in H(k)$ such that $v^{-1}hv = u$, whence $vuv^{-1} \in H(k)$. Putting $C' = C \cap H = Z_H(T)$, we then get $vuv^{-1} \in C'$, whence $u \in \mathrm{Inn}(v) \cdot C'$. This proves that the union of conjugates of C' in C (under the elements of $C(k)$) is dense, which implies (in view of the implication (vi)⇒(iv)) that C' has finite index in its normalizer in C. By (**[?]** 7 lemma du n1), this implies that $C' = C$, hence $C = H$, and this proves (vi)⇒(i) if G is affine.

In the general case, we prceed by induction on $n = \dim(G)$, the assertion being trivial if $n = 0$. Let Z be the center of G, and distinguish two cases: if $\dim(Z \cap H) > 0$, then putting $G' = G/(Z \cap H)$, we have $\dim(G') < n$; on the other hand, the hypothesis (vi) on H implies the same condition for the image H' of H in G', so H' contains a Cartan subgroup C' of G', and then H contains its inverse image C of G', which is a Cartan subgroup in view of (**[?]** XII 6.6 (e)).

In the case where $\dim(Z \cap H) = 0$, the canonical morphism $H \to G/Z$ is a finite morphism, and as G/Z is affine in view of (**[?]** XII 6.1), it then follows that H is affinem hence any homomorphism from H into an abelian variety is zero: this follows from the fact that a connected affine smooth algebraic group over an algebraically closed field is a rational variaty, or simply that it is the union of its Borel subgroups (**[?]** 6 th.5 (b)). Using Chevalley's structure theorem, we see that G is an extension of an abelian veriety A by a smooth affine group. Then the image of H in A is zero, H being affine; but it is identical to A because the union of its conjugates in A must be dense, so $A = 0$ and G is affine. We are reduced to the affine case. This completes the proof of Theorem 12.12.3.

Corollary 12.12.6. *Suppose that the equivalent conditions of Corollary 12.12.5 are verified.*

(a) *Let $k(X)$ (resp. $k(G)$) be the rational function field of X (resp. G), then $k(X)$ is a separable finite extension of $k(G)$ (let d be its degree).*

(b) *Let T be a maximal torus of G contained in H (which exists by Theorem 12.12.3 (ii)) and C be the corresponding Cartan subgroup of G; then $C \subseteq H$. On the other hand, $N_G(T)$ is a smooth subgroup of G and $N_G(T) \cap N_G(H) = N_{N_G(H)}(T)$ is a smooth subgroup of finite index d in $N_G(T)$. The number of conjugates of H containing a given maximal torus or a Cartan subgroup of G is equal to d.*

(c) *Let U be the largest open subset of G such that $\psi : X \to G$ induces a finite étale morphism $\psi^{-1}(U) \to U$. Then U is an open dense subset of G and for $g \in G(k)$, we have $g \in U(k)$ if and only if there exists exactly d conjugates of H containing g, or equivalently there exists d distinct cojugates H_i of H containing g such that for each i, $(\mathfrak{g}/\mathfrak{h}_i)^{\mathrm{Ad}(g)} = 0$ (where $\mathfrak{h}_i = \mathfrak{Lie}(H_i)$).*

Proof. Assertion (a) follows from the fact that ψ is generically étale; this also implies that the open subset u introduces in (c) is nonempty, i.e. dense, and also the two characterization given for the elements of

$U(k)$ (ψ is separated, X is integral and G is integral normal, so this follows from ([?] I 10.11) and the fact that $(\mathfrak{g}/\mathfrak{h}_i)^{\mathrm{Ad}(g)} = 0$ signifies that ψ is étale at the point x_i of $\psi^{-1}(g)$ corresponding to H_i). If H contains a maximal torus T of G, then the centralizers of T in H and G have the same dimension, and are smooth and connected ([?] XII 6.6 (b)), hence are equal, which proves that $C \subseteq H$. Moreover, we know that the normalizer of T in a smooth group containing it is smooth (Corollary 12.10.26), so $N_G(T)$ and $N_{N_G(H)}(T)$ are smooth (we note that $N = N_G(H)$ is smooth by the last assertion of Theorem 12.12.3), moreover $N_{N_G(H)}(T)$ contains C, which has finite index in $N_G(T)$, so it also has finite index in $N_G(T)$. Using the conjugation theorem for maximal tori of H, we see that this index is equal to the number of conjugates of H which contain T, or equivalently, those contain C. Now as the union of conjugates of C in G is dense (in view of Theorem 12.12.3 (vi) applied to C), and the open subset U defined in (c) is evidently stable under inner automorphisms, we see that $C \cap U \neq \varnothing$. Proceed as in the proof of (vi)⇒(i) of Theorem 12.12.3, we conclude that (by a base change of field) that there exists $g \in (C \cap U)(k)$ such that any conjugate H containing g also contains T, and therefore also contains C (cf. the proof of (i)⇔(iii) of Theorem 12.12.3). Hence the conjugates of H containing C are those containing g, and as $g \in U(k)$, the number of such conjugates is equal to d, which proves the assertions of (b). We have in fact established that the set of conjugates of H containing T is a homogeneous set under the group of rational points of

$$W_G(T) = N_G(T)/Z_G(T)$$

which proves in particular that d is smaller or equal to the order of the Weyl group of G. □

Corollary 12.12.7. *With the notations of Theorem 12.12.3, the following conditions are equivalent:*

(i) *$\psi : X \to G$ is a birational morphism.*

(ii) *There exists a unique conjugate of H containing a given Cartan subgroup of G.*

(iii) *H contains a Cartan subgroup C of G, and $N_G(H) \supseteq N_G(C)$.*

(iv) *There exists a nonempty open subset V of G such that $g \in V(k)$ implies that g is contained in exactly one conjugate of H.*

Proof. This follows from Corollary 12.12.6 and Theorem 12.12.3, since in this case ψ is birational if and only if $d = 1$, which is equivalent by Corollary 12.12.6 to that $N_G(T) \cap N_G(H) = N_G(T)$ (recall that $N_G(T) = N_G(C)$ by [?]). □

Corollary 12.12.8. *Suppose that the conditions of Corollary 12.12.7 are satisfied, and let $g \in G(k)$. Then the following conditions are equivalent:*

(i) *$g \in U(k)$, where U is defined as in Corollary 12.12.6, i.e. g is contained in a unique conjugate of H.*

(ii) *The set of conjugates of H containing g is finite nonempty.*

(iii) *The scheme $\psi^{-1}(g)$ of conjugates of H containing g contains an isolated point.*

(iv) *There exists a conjugate H' of H containing g, and we have $(\mathfrak{g}/\mathfrak{h}')^{\mathrm{Ad}(g)} = 0$, where $\mathfrak{h}' = \mathfrak{Lie}(H')$.*

Finally, U is also the largest open subset of G such that ψ induces an isomorphism $\psi^{-1}(U) \xrightarrow{\sim} U$.

Proof. The equivalence of (i), (ii) and (iii), as well as the last assertion, follow from the Zariski's Main Theorem applied to the birational morphism $\psi : X \to G$, given that G is normal. The equivalence of these conditions to (iv) follows immediately from the last assertion of Theorem 12.12.3, which characterizes the set of elements of X over which ψ is étale. □

Theorem 12.12.9. *Let G be a connected smooth algebraic group over an algebraically closed field k, C be a Cartan subgroup, with maximal torus T, $N = N_G(C) = N_G(T)$ (cf. [?]), Let $X = G \times^N C$ be the twisted product of G and C over N, where N acts over C by inner automorphisms, and $\psi : X \to G$ be the canonical morphism.*

(a) *The morphism ψ is birational.*

(b) *Let U be the largest open subset of G such that ψ induces an isomorphism $\psi^{-1}(U) \xrightarrow{\sim} U$, and*

$$\rho = \rho_n(G) = \dim(C)$$

be the nilpotent rank of G. Then for any $g \in G(k)$, the multiplicity of the eigenvalue 1 of $\mathrm{Ad}(g)$ over $\mathfrak{g}$ is $\geq \rho$, and for it to be equal to ρ, it is necessary and sufficient that we have $g \in U(k)$.

Proof. As the conditions of Theorem 12.12.3, we can apply Corollary 12.12.7, which implies (a). In ([**?**] 7) (in the case where G is affine), the points of $U(k)$ are called regular points of $G(k)$, and we will follow this terminology by calling U the **open subset of the regular points of** G. To prove (b), we introduce for any $g \in G(k)$ the characteristic polynomial

$$P(\mathrm{Ad}(g), t) = t^n + c_1(g)t^{n-1} + \cdots + c_n(g).$$

By replacing k with an arbitrary k-algebra R, we easily see that the $c_i(g)$ comes from well-defined sections $c_i \in \Gamma(G, \mathscr{O}_G)$. If $g \in G(k)$ is an element contained in a Cartan subgroup (for example a regular element), which we can assume to be C, then by Corollary 12.12.8 (iv), we see that we have $(\mathfrak{g}/\mathfrak{c})^{\mathrm{Ad}(g)} = 0$ if and only if $\mathfrak{g}$ is regular (where $\mathfrak{c}$ denotes the Lie algebra of C). On the other hand, as C is nilpotent, we see immediately that $\mathrm{Ad}_{\mathfrak{c}}(g)$ only has eigenvalue 1, which proves that the multiplicity of the eigenvalue 1 for $\mathrm{Ad}_{\mathfrak{g}}(g)$ is $\geq \rho$, and exactly equals to $\dim(C) = \rho$ if and only if g is regular. In particular, the polynomial above is divisible by $(t-1)^\rho$. Like the relation of divisibility by $(t-1)^\rho$ is expressed by linear relations (with integer coefficients) between the coefficients of the polynomial, and that these relations hold for $g \in U(k)$, U being a dense open set, it follows (G being reduced) that they hold for any g, so that we have a relation

$$t^n + c_1 t^{n-1} + \cdots + c_0 = (t-1)^\rho (t^{n-\rho} + b_1 t_1^{n-\rho-1} + \cdots + b_{n-\rho}) \tag{12.12.7}$$

in the ring of polynomials over $\Gamma(G, \mathscr{O}_G)$. In particular, for any $g \in G(k)$, the multiplicity of 1 of $\mathrm{Ad}(g)$ is $\geq \rho$. Moreover, we see that we have the equality if g is regular. For the converse, suppose that G is affine, and write g as a product

$$g = g_s g_u$$

of semi-simple and unipotent part ([**?**] 4 n4), then

$$\mathrm{Ad}(g) = \mathrm{Ad}(g_s)\mathrm{Ad}(g_u)$$

is the analogous decomposition for $\mathrm{Ad}(g)$ ([**?**] 4 n4 cor au th.3), and therefore $\mathrm{Ad}(g)$ and $\mathrm{Ad}(g_s)$ have the same eigenvalue (considering multilicities), in particular the eigenvalue 1 has the same multiplicity in $\mathrm{Ad}(g)$ and $\mathrm{Ad}(g_s)$.

On the other hand, in view of ([**?**] 7 th.2 cor.1), g is regular if and only if g_s is. Hence to prove (b), we can suppose that $g = g_s$ is semi-simple, hence contained in a maximal torus in view of ([**?**] 7 th.2 cor.1), and a fortiori in a Cartan subgroup of G, which is the case we have already treated. This proves (b) in the case where G is affine. In the general case, let $Z = Z(G)_{\mathrm{red}}$, then in view of ([**?**] XII 6.6 (e)), the Cartan subgroups of G are the inverse images of that of $G' = G/Z$, hence g is regular in G if and only if its image g' in G' is regular in G'. On the other hand, as Z is smooth, the Lie algebra $\mathfrak{g}'$ of G' is equal to $\mathfrak{g}/\mathfrak{z}$, where $\mathfrak{z} = \mathfrak{Lie}(Z)$, and $\mathrm{Ad}(g')$ is equal to $\mathrm{Ad}_{\mathfrak{g}/\mathfrak{z}}(g)$, hence the multiplicity of eigenvalue 1 in $\mathrm{Ad}(g')$ is equal to $d = \dim(Z)$ plus the multiplicity of eigenvalue 1 in $\mathrm{Ad}(g')$, so the former is equal to the reductive rank of G if and only if the latter is equal to that of G'. We are therefore reduced to the case of G', but G' is affine in view of ([**?**] XII 6.1), so this completes the proof. □

Corollary 12.12.10. *With the notations of the preceding proof, let*

$$b = 1 + b_1 + \cdots + b_{n-\rho} \in \Gamma(G, \mathscr{O}_G).$$

Then the open subset of regular elements of G is given by

$$U = G_b$$

(set of points of G where b is invertible), in particular, U is an affine open subset if G is affine.

Proof. By the arguments above, g is a regular element if and only if the polynomial $t^{n-\rho} + b_1 t^{n-\rho-1} + \cdots + b_{n-\rho}$ does not divide $t-1$, which means, as k is algebraically closed, that 1 is not a root of it, whence the claim. □

Remark 12.12.11. We note that the equivalence condition of Theorem 12.12.9 (b) of regularity points of G is independent of C (it only concerns the action of g on $\mathfrak{g}$). Therefore, we conclude that any regular point $g \in G(k)$ is contained in contained in a Cartan subgroup of G (necessarily unique, in view of Corollary 12.12.8, since the Cartan subgroups of G are conjugate). Conversely, if a point $g \in G(k)$ is contained in a unique Cartan subgroup C of G, then it is regular in view of Corollary 12.12.8

Corollary 12.12.12. *Let H be a connected smooth algebraic subgroup of G containing a Cartan subgroup of G.*

(a) *Let C be an algebraic subgroup of H. For C to be a Cartan subgroup of H, it is necessary and sufficient that it be a Cartan subgroup of G.*

(b) *Let $g \in G(k)$ and d be the degree of ψ (cf. Corollary 12.12.6). For g to be a regular point of G, it is necessary and sufficient that there exists exactly d conjugates H_i of H containing g, and that for each i, g is a regular element of H_i (or equivalently, is regular in one of them). If this is the case, and if C is the unique Cartan subgroup containing g (cf. Remark 12.12.11), then the conjugates of H containing g are the conjugates of H containing C.*

(c) *Let $g \in H(k)$, for g to be regular in G, it is necessary and sufficient that it be rgular in H, and we have $(\mathfrak{g}/\mathfrak{h})^{\mathrm{Ad}(g)} = 0$.*

Proof. Under one of the hypothesis of (a), the unique maximal torus T of C is a maximal torus of G and of G (H having the same reductive rank as G, cf. Theorem 12.12.3), so as $Z_H(T) \subseteq Z_G(T)$ are smooth connected groups with the same dimension, they are equal, so it is equivalent to saying that C is equal to one of these subgroups, whence the assertion of (a).

Now consider the assertions of (b). Suppose first that g is regular in G, let C be the unique Cartan subgroup of G containing g, then in view of Corollary 12.12.6 (b), there exists exactly d conjugates H_i of H containing C. As $(\mathfrak{g}/\mathfrak{c})^{\mathrm{Ad}(g)} = 0$, i.e. $\mathrm{Ad}(g)$ has no eigenvalue 1 on $\mathfrak{g}/\mathfrak{c}$, we have a fortiori $(\mathfrak{g}/\mathfrak{h}_i)^{\mathrm{Ad}(g)} = 0$, hence in view of Corollary 12.12.6 (c), there exists exactly d conjugates of H containing g, namely the H_i. For such an H_i, a Cartan subgrou of H_i containing g is a Cartan subgroup of G containing g in view of (a), hence is equal to C; this proves that g is regular in H_i (Remark 12.12.11). Conversely, suppose that there exists at most d conjugates H_i of H containing g, and that g is regular in one of them, which can be assumed to be equal to H. Let us prove that g is regular in G. As g is regular in H, it is contained in a unique Cartan subgroup C of H, which by (a) is a Cartan subgroup of G. Let C' be a Cartan subgroup of G containing g, let us prove $C' = C$ (which implies that g is regular in C). Indeed, by virtue of **??** (b), there exist exactly d conjugates of H containing C', and since they all contain g, they are necessarily the H_i, therefore the H_i, and in particular H, contain C'. Now C and C' are two Cartan subgroups of H (in view of (a)), which contains the same regular element g of H, hence they are equal.

Finally, let $g \in H(k)$. Denote by $\nu(u)$ the dimension of the kernel of $\mathrm{id} - u$, for an endomorphism of a finite-dimensional vector space, we have

$$\nu(\mathrm{Ad}_{\mathfrak{g}}(g)) = \nu(\mathrm{Ad}_{\mathfrak{h}}(g)) + \nu(\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}(g))$$

now the two members on the right are respectively $\geq \rho_n(H) = \rho_n(G)$ and ≥ 0, so we have $\nu(\mathrm{Ad}_{\mathfrak{g}}(g)) = \rho$ if and only if $\nu(\mathrm{Ad}_{\mathfrak{h}}(g)) = \rho$ if and only if $\nu(\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}(g)) = 0$, i.e. g is regular in G if and only if it is regular in H and $\mathrm{Ad}_{\mathfrak{g}/\mathfrak{h}}(g)$ has no nontrivial invariants. □

Example 12.12.13. In the statement of Theorem 12.12.3, one cannot weaken the condition (iii) by assuming only that H contains a maximal torus and has finite index in its normalizer, even if we require that this normalizer be smooth, i.e. that we have $H = N^0$, and even when G is affine solvable. As an example, consider the group G of GL_3 of the form

$$G(k) = \left\{ \begin{pmatrix} t & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} : t \in k^\times, a, b, c \in k \right\},$$

and the subgroup H of matrices of the preceding form, with $b = c = 0$. The maximal torus T of G is given by matrices g with $a = b = c = 0$, and its centralizer C and normalizer N are given by matrices of the form

$$C(k) = N(k) = \left\{ \begin{pmatrix} t & 0 & 0 \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} : t \in k^\times, b \in k \right\}.$$

On the other hand, the normalizer of H is equal to H.

Remark 12.12.14. Let G be a smooth algebraic group over k, H be a smooth subgroup of G, but not suppose that H and G are connected. Suppose that H^0 contains a Cartan subgroup of G^0, then $(\mathfrak{g}/\mathfrak{h})^H \subseteq$

$(\mathfrak{g}/\mathfrak{h})^{H^0} = 0$ by Theorem 12.12.3, so $H^0 = N^0$ (N being the normalizer of H), in particular N is smooth. However, one can easily construct examples, with connected G, where H has a connected component H_i such that for no $h \in H_i(k)$, we have $(\mathfrak{g}/\mathfrak{h})^{\mathrm{Ad}(h)} = 0$, i.e. the morphism

$$G \times H_i \to G, \quad (g, h) \mapsto \mathrm{Inn}(g)h = ghg^{-1}$$

is not étale (nor even quasi-finite) at any point (e, h), $h \in H_i(k)$. Similarly, even if the image H' of H in the finite group $G' = G/G^0$ is equal to G', it is not necessarily true that the union of conjugates of H in G is dense in G (for example, let G be the semi-direct product $\mathbb{Z}/2\mathbb{Z} \rtimes T$, where T is a maximal torus of G^0, and $H = G^0$). On the other hand, if we do not suppose a priori that H^0 contains a Cartan subgroup of G^0, but the union of the conjugates of H in G is dense, then H^0 necessarily contains a Cartan subgroup of G^0: to see this, we can obviously suppose that G is connected, and it suffices to apply the proof of (vi)⇒(i) of Theorem 12.12.3, which is valid without assuming H to be connected.

12.12.2.2 The case of a group over arbitrary base Suppose first that we are over a base field k, not necessarily algebraically closed. As the conditions (i'), (iv'), (v) and (vi) of Theorem 12.12.3 are invariant under base field change, we see by passing to the algebraic closure $\bar{k}$ of k that they are equivalent to each other, and equivalent to that $H_{\bar{k}}$ contains a Cartan subgroup of $G_{\bar{k}}$. If these conditions are satisfied, then (with the notations of Corollary 12.12.6) the rational function field $k(X)$ is a finite separable extension of $k(G)$, of degree d, which is independent of base field change. If U is the largest open subset of G such that ψ induces a morphism $\psi^{-1}(U) \xrightarrow{\sim} U$ which is finite and étale, then the formation of U commutes with base field change (cf. Lemma 12.6.2). If ψ is birational, then U is also the largest open subset of G such that ψ induces an isomorphism $\psi^{-1}(U) \xrightarrow{\sim} U$, and then for $g \in U(k)$, there exists a unique subgroup H' of G, conjugate to H over the algebraic closure $\bar{k}$, such that $g \in H'$.

A point $g \in G(k)$ is called **regular** if it is regular as an element $G(\bar{k}) = G_{\bar{k}}(\bar{k})$. More generally, the construction of Corollary 12.12.10 gives us an open subset of G, whose formation commutes with any base field change, so is called the **open subset of regular points of** G. This is also characterized by the fact that for any algebraically closed field extension K of k and any point $g \in G(K)$, g is a regular point of G_K if and only if $g \in U(K)$. If $g \in U(K)$, we shall see that g is contained in a unique Cartan subgroup of G.

Let G be a smooth and separated scheme over S, of finite type over S and has connected fibers. Consider the functor $\mathscr{C} : \mathbf{Sch}_{/S}^{\mathrm{op}} \to \mathbf{Set}$ defined by

$$\mathscr{C}(S') = \{\text{set of Cartan subgroups of } G_{S'}\}$$

Suppose that this functor is representable by a smooth scheme over S (we will give in ([?] XV) an equivalent condition for this to be so, but we already know that this hypothesis is satisfied if G is affine over S of locally constant reductive rank (Corollary 12.11.20), or more generally if G admits locally for the fpqc topology a maximal torus ([?] 7.1 (a)), for example if S is the spectrum of a field). Let X be the "universal Cartan subgroup of the $\mathscr{C}$-group $G_{\mathscr{C}}$". As a scheme over S, X then represents the following functor (cf. Remark 12.11.9)

$$X(S') = \{\text{set of couples } (C, g), C \text{ being a Cartan subgroup of } G_{S'}, \text{and } g \text{ be a section of } C \text{ over } S'\}.$$

Consider the canonical projection $\psi : X \to G$, $(C, g) \mapsto g$. We then have the following theorem:

Theorem 12.12.15. *Under the preceding conditions on G and notations, let U be the set of $g \in G$ such that g is a regular element in the fiber G_s. Then U is open, and it is also the largest open subset U of G such that ψ induces an isomorphism $\psi^{-1}(U) \xrightarrow{\sim} U$.*

Proof. We first prove that U is open. By the hypothesis of the representability of $\mathscr{C}$ as a smooth scheme over S. as its structural morphism is evidently surjective, we conclude that G admits locally for the étale topology a Cartan subgroup (cf. **??** 17.16.3), and that the nilpotent rank of the fibers of G is locally constant. The same is true for the dimension of fibers of G (as G is smooth over S), and by taking localization of S, we can suppose that they are both constant, say ρ and n. Consider the **Kill polynomial**

$$P_G(t) = t^n + c_1 t^{n-1} + \cdots + c_n \in A[t], \quad A = \Gamma(G, \mathscr{O}_G)$$

(the polynomial in the proof of Theorem 12.12.9.) The restriction of this polynomial to fibers G_s of G, and in particular to the maximal points of S, is divisible by $(t-1)^\rho$, which is expressed by the fact that certain linear combinations with integer coefficients of the c_i are zero on the fibers G_s. If S is reduced (which we can assume to establish that U is open), it follows that they are themselves zero, so that the Killing polynomial itself is divisible by $(t-1)^\rho$, say

$$P_G(t) = (t-1)^\rho(t^{n-\rho} + b_1 t^{n-r-1} + \cdots + b_{n-\rho}).$$

Let b be the sum of these coefficients $b_0 = 1, b_1, \dots, b_{n-\rho}$, then in view of Corollary 12.12.10 applied to the fibers of G, we see that $U = G_b$, which proves that U is open.

To show that $\psi^{-1}(U) \to U$ is an isomorphism, we are reduced by ([?] I 5.7) to verify this fiber by fiber, hence to the case where the base is a field, which we may assume to be algebraically closed. Then there exists a Cartan subgroup C of G, and if N is its normalizer, then $\mathscr{C}$ is identified, by the conjugation theorem ([?] XII 7.1 (a) et (b)), with G/N, and the morphism $\psi : X \to G$ is none other than that defined in 12.12.2.1. We then conclude by Theorem 12.12.9 (b). The same reasoning shows that U is the largest open subset of G such that ψ induces an isomorphism $\psi^{-1}(U) \to U$. □

Corollary 12.12.16. *Under the conditions of Theorem 12.12.15, let g be a regular section of G, i.e. such that for any $s \in S$, $g(s)$ is a regular point of G_s. Then there exists a unique Cartan subgroup C of G such that g is a section of C.*

Proof. The hypothesis on g signifies that g is a section of U, and the conclusion that there exists a unique section of X which dominates g is none other an equivalent expression that $\psi^{-1}(U) \to U$ is an isomorphism. □

Note that the open subset $\psi^{-1}(U)$ of the Cartan subgroup X of $G_{\mathscr{C}}$ is none other than the open subset of X formed by points of X which are regular in $G_{\mathscr{C}}$ (regular in the fibers). We thus obtain a natural "fibration" of the open dense subset U of regular points of G over the scheme $\mathscr{C}$, whose fiber at a point x of $\mathscr{C}$ (corresponding to a Cartan subgroup C of G) equals to the set of sections of $C_{\kappa(x)}$ over $\kappa(x)$. We conclude for example the following corollary:

Corollary 12.12.17. *Let G be a connected smooth algebraic group over the field k, $\mathscr{T}$ be the scheme of maximal tori of G (isomorphic to the scheme of Cartan subgroups of G). Then the function field $k(G)$ of G is isomorphic to the function field of a connected nilpotent affine smooth algebraic group C over the function field $k(\mathscr{T})$ of $\mathscr{T}$ (called the **generic Cartan subgroup** of G). If G is affine with zero unipotent rank, i.e. if the Cartan subgroups of $G_{\bar{k}}$ are tori, then $k(G)$ is a unirational extension of $k(\mathscr{T})$.*

Of course, the generic Cartan subgroup is just the Cartan subgroup of $G_{k(\mathscr{T})}$ correponding to the generic fiber of X over $\mathscr{T}$. It only remains to prove the last assertion of Corollary 12.12.17, which is contained in the following well-known result (due to Chevalley):

Lemma 12.12.18. *Let k be a field, T be a torus over k, $k(T)$ be the rational function field of T. Then $k(T)$ is a unirational extension of k, i.e. is contained in a purely transcendental extension of k.*

Proof. Let k' be a finite separable extension of k which split X (Proposition 12.9.3), then $T \otimes_k k'$ is a rational variaty, i.e. admits an open dense subset isomorphic to an open dense subset of the affine space $\mathbb{A}^n_k$, hence $T' = \mathrm{Res}_{\mathrm{Spec}(k')/\mathrm{Spec}(k)} T_{k'}$ is a rational variaty (it admits an open dense subset isomorphic to an open dense subset $\mathrm{Res}_{\mathrm{Spec}(k')/\mathrm{Spec}(k)} \mathbb{A}^n_{k'}$, which is isomorphic to $\mathbb{A}^{mn}_k$, where $m = [k' : k]$). Consider the norm hommorphism $T' \to T$ (defined for abelian algebraic groups over k); then the composition $T \to T' \to T$ is the m-th power on T, hence dominant, so $T' \to T$ is dominant, which proves that T is unirational. □

Now let us return to the conditions of Theorem 12.12.15, and suppose that G admits locally for the fpqc topology a maximal torus (cf. [?] XII 7.1). Let T be the maximal torus of the Cartan subgroup C of G, so that the morphism $j : C \to G$ induces a morphism $T \to G$ whose image is formed setwisely of semi-simple elements fibers of G ([?] XII 8). Finally, it follows from Theorem 12.12.15 that the restriction of j to the open set T^{reg} of regular points of T induces a closed immersion

$$T^{\mathrm{reg}} \to U = G^{\mathrm{reg}}.$$

Explaining the meaning of T^{reg} as a functor over S, we find that:

Corollary 12.12.19. *Let G be a smooth and separated S-group of finite type over S with connected fibers, admitting locally for the fpqc topology a maximal torus. Let $Z : \mathbf{Sch}_{/S} \to \mathbf{Set}$ be the functor defined by*

$$Z(S') = \{\text{set of regular sections of } G_{S'} \text{ over } S' \text{ which is contained in a maximal torus of } G_{S'}\}.$$

Then Z is representable by a closed scheme of the open subset $U = G^{\mathrm{reg}}$ of G, and is smooth over S.

Proof. In fact, the functor is represented by the union of translates of T^{reg} under the action of G, so it is a closed subscheme of U. □

Corollary 12.12.20. *Under the conditions of Corollary 12.12.19, let C be a Cartan subgroup of G, and consider the morphism*

$$\varphi : Z \times_S C \to G, \quad (g,h) \mapsto \mathrm{Inn}(g)h = ghg^{-1}.$$

Then φ is dominant.

Proof. It suffices to prove this fiber by fiber, so we are reduced to the case where S is the spectrum of an algebraically closed field. Let T be the maximal torus of C, t_0 be an element of $T(k)$ regular in G, c_0 be a point of $C(k)$ regular in G, consider $\varphi^{-1}(\varphi(t_0,c_0))$, whose rational points over k are the couples (t,c), with $t \in Z(k)$, $c \in C(k)$, and such that $\mathrm{Inn}(t)c = \mathrm{Inn}(t_0)c_0$, i.e. $c = \mathrm{Inn}(t^{-1}t_0)c_0$. Its points therefore correspond to $t \in Z(k)$ such that $\mathrm{Inn}(t^{-1}t_0)c_0 \in C$, or equivalently, as c_0 is regular, such that $tt_0^{-1} \in N$ (normalizer of C), i.e. $t \in N$. By considering the the points $t \in Z(k)$ such that $t \in C(k)$, we obtain a clopen subset of this fiber (cf. Theorem 12.12.3), so there is a connected component of $\varphi^{-1}(\varphi(t_0,c_0))$ isomorphic to T. The generic fiber of φ therefore has dimension $\leq \dim(T)$, and

$$\dim(\mathrm{im}\,\varphi) \geq \dim(Z \times_S C) - \dim(T) = \dim(Z) + \dim(C) - \dim(T),$$

but we have

$$\dim(Z) = \dim(T^{\mathrm{reg}}) = \dim(G) - \dim(C) + \dim(T),$$

whence finally $\dim(\mathrm{im}\,\varphi) \geq \dim(G)$, so φ is dominant. □

Remark 12.12.21. Note that the reasoning also shows that the connected component connected of the fiber $\varphi^{-1}(\varphi(t_0,c_0))$ at (t_0,c_0) is isomorphic to T; in particular, it is smooth over k, and has the same dimension as the generic fiber, which implies that φ is in fact smooth at (t_0,c_0) (which we should also be able to verify by calculating the tangent map). It follows that under the conditions of Corollary 12.12.20, the induced morphism $Z \times_S C^{\mathrm{reg}} \to G^{\mathrm{reg}}$ (where we set $C^{\mathrm{reg}} = C \cap G^{\mathrm{reg}}$) is a smooth morphism. Similarly, we see that the analogous morphism $Z \times T^{\mathrm{reg}} \to Z$ (where T is a maximal torus of G) is smooth. More generally, for any connected smooth normal subgroup H of C containing a regular element c_0 of $G(k)$, the image of $Z \times_S H \to G$ is dense in that of $G \times_S H \to G$.

12.12.3 Cartan subalgebras and regular elements of Lie algebras

Let $\mathfrak{g}$ be a Lie algebra over a ring k. For any $x \in \mathfrak{g}$, recall that we denote by $\mathrm{ad}(x)$ the endomorphism

$$\mathrm{ad}(x)(y) = [x,y]$$

of $\mathfrak{g}$, which is a derivation of the Lie algebra $\mathfrak{g}$. For any derivation D on $\mathfrak{g}$, the nilspace of D, i.e. the union of the kernel of the powers of D, which is a Lie subalgebra of $\mathfrak{g}$, as we see from the Leibniz formula

$$D^n([x,y]) = \sum_{p=0}^{n} \binom{n}{p} [D^p x, D^{n-p} y].$$

As in §6.1, for $a \in \mathfrak{g}$, we denote by $\mathfrak{g}^0(a)$ the nilspace of $\mathrm{ad}(a)$, which is the union of kernel of $\mathfrak{a}(a)^n$.

Proposition 12.12.22. *For any $a \in \mathfrak{g}$, its nilspace $\mathfrak{g}^0(a)$ is a Lie subalgebra of $\mathfrak{g}$, and is self-normalizing.*

Proof. It remains to prove that it is self-normalizing, i.e. that any element of $\mathfrak{g}/\mathfrak{g}^0(a)$ annihilated by the adjoint representation of $\mathfrak{g}^0(a)$ on $\mathfrak{g}/\mathfrak{g}^0(a)$ is zero, but this is trivial (because any element in this quotient annihilated by $\mathrm{ad}(x)$ is zero). □

In this subsection, we suppose that k is a field (not necessarily of characteristic zero), and $\mathfrak{g}$ be finite dimension over k. Following **[?]**, we denote by $W(\mathfrak{g})$ the scheme over k defined by $\mathfrak{g}$, whose points on a k-algebra A are the elements of $\mathfrak{g} \otimes_k A$. In other words, we have

$$\mathrm{Hom}_{\mathrm{Spec}(k)}(\mathrm{Spec}(A), W(\mathfrak{g})) = \mathrm{Hom}_k(k, \mathfrak{g} \otimes_k A) = \mathrm{Hom}_{k\text{-}\mathbf{Alg}}(\boldsymbol{S}_k(\mathfrak{g}^\vee), A)$$

so $W(\mathfrak{g})$ is represented by the k-algebra $\boldsymbol{S}_k(\mathfrak{g}^\vee)$, the symetric algebra over $\mathfrak{g}^\vee$ ($W(\mathfrak{g})$ is the functor $\Gamma_{\mathfrak{g}}$ defined in 12.1.2.5, cf. Remark 12.1.23 and Corollary 12.1.27). If $x \in \mathfrak{g}$, the characteristic polynomial of $\mathrm{ad}(a)$ is also called the **characteristic polynomial** or **Killing polynomial** of a in $\mathfrak{g}$, say

$$P_{\mathfrak{g}}(x,t) = t^n + c_1(a)t^{n-1} + \cdots + c_n(a)$$

where $n = \mathrm{rank}_k(\mathfrak{g})$ and $c_i(a) \in k$. By considering these polynomials for $a \in \mathfrak{g} \otimes_k A$, where A is an arbitrary k-algebra, we see that the $c_i(a)$ comes from well-defined sections of the structural sheaf of $W(\mathfrak{g})$, i.e. from elements of $\boldsymbol{S}_k(\mathfrak{g}^\vee)$ (if k is an infinite field, the c_i are determined by the corresponding polynomial functions $g \to k$, but this is no longer true if k is finite). Let r be the largest integer such that the Killing polynomial

$$P_{\mathfrak{g}}(t) = t^n + c_1 t^{n-1} + \cdots + c_n$$

is divisible by r, i.e. we have

$$P_{\mathfrak{g}}(t) = t^n + c_1 t^{n-1} + \cdots + c_{n-r}t^r, \quad c_{n-r} \neq 0.$$

The integer r is called the **nilpotent rank** of the Lie algebra $\mathfrak{g}$ (this coincides with the rank of $\mathfrak{g}$ defined in §6.1, if k has characteristic zero), and denoted by $\rho_n(\mathfrak{g})$. It is clearly invariant under base field extension.

Proposition 12.12.23. *Let r be the nilpotent rank of $\mathfrak{g}$, and $x \in \mathfrak{g}$. Then we have*

$$\dim_k(\mathfrak{g}^0(a)) \geq r,$$

with equality if and only if we have $c_{n-r}(a) \neq 0$. In this case, $\mathfrak{g}^0(a)$ is a nilpotent subalgebra of $\mathfrak{g}$.

Proof. The first assertion is trivial from our definition, since $\dim_k(\mathfrak{g}^0(a))$ is the multiplicity of eigenvalue 0 for $\mathrm{ad}(a)$. Now assume that $c_{n-r}(a) \neq 0$, we need to show that $\mathfrak{g}^0(a)$ is nilpotent, which signifies that for any $x \in \mathfrak{g}^0(a)$, $\mathrm{ad}_{\mathfrak{g}^0(a)}(x)$ is a nilpotent endomorphism (cf. Corollary 1.4.13). For this, we may assume that k is algebraically closed, then as $\mathrm{ad}(a)$ is injective on $\mathfrak{g}/\mathfrak{g}^0(a)$, by Nullstellensatz there exists a nonempty open subset U of $W(\mathfrak{g}^0(a))$ such that for any $x \in U(k)$, $\mathrm{ad}_{\mathfrak{g}/\mathfrak{g}^0(a)}(x)$ is injective[59], hence $\mathfrak{g}^0(x) \subseteq \mathfrak{g}^0(a)$. We can also suppose that U is contained in the open subset of points where c_{n-r} do not vanish, i.e. the regular elements of $\mathfrak{g}$ (since this open subset is nonempty in view of $c_{n-r}(a) \neq 0$). Then $g^0(x)$ has the same dimension as $\mathfrak{g}^0(a)$ (both equal to the nilpotent rank of $\mathfrak{g}$), so they are equal, i.e. for any $x \in U(k)$, $\mathrm{ad}_{\mathfrak{g}^0(a)}(x)$ is nilpotent. Since the nilpotent condition for $x \in \mathfrak{g}^0(a)$ can be detected by the vanishing of the Killing polynomial of $\mathfrak{g}^0(a)$ and U is a dense subset of $W(\mathfrak{g}^0(a))$ ($W(\mathfrak{g}^0(a))$ being irreducible, since $\boldsymbol{S}_k(\mathfrak{g}^0(a)^\vee)$ is an integral domain), we conclude that it is equal to $W(\mathfrak{g}^0(a))$, i.e. $\mathrm{ad}_{\mathfrak{g}^0(a)}(x)$ is nilpotent for $x \in \mathfrak{g}^0(a)$, so $\mathfrak{g}^0(a)$ is nilpotent. □

We say that an element $a \in \mathfrak{g}$ is regular if $c_{n-r}(a) \neq 0$, i.e. if $\dim_k(\mathfrak{g}^0(a)) = r$. If k is infinite, this also signifies that $\dim_k(\mathfrak{g}^0(a))$ is as small as possible[60] (for a runs through $\mathfrak{g}$). In any case, the notion of regularity is invariant under base field change, and the set of points of $W(\mathfrak{g})$ which are regular (i.e. those come from the regular points of $W(\mathfrak{g})$ with values in a suitable extension field of k) is open, because it is identified with $W(\mathfrak{g})_{c_{n-r}}$ (the subset of $W(\mathfrak{g})$ where c_{n-r} is invertible).

Corollary 12.12.24. *Let $a \in \mathfrak{g}$ be a regular element and $\mathfrak{h}$ be a subalgebra of $\mathfrak{g}$ containing a. Then $\mathfrak{h}$ is nilpotent if and only if $\mathfrak{h} \subseteq \mathfrak{g}^0(a)$. In particular, $\mathfrak{g}^0(a)$ is a maximal nilpotent subalgebra of $\mathfrak{g}$.*

Proof. As $\mathfrak{g}^0(a)$ is nilpotent, the relation $\mathfrak{h} \subseteq \mathfrak{g}^0(a)$ implies that $\mathfrak{h}$ is nilpotent; conversely, if $\mathfrak{h}$ is nilpotent, then $\mathrm{ad}_{\mathfrak{h}}(a)$ is nilpotent, so $\mathfrak{h}$ is contained in the nilspace of its element a, i.e. $\mathfrak{h} \subseteq \mathfrak{g}^0(a)$. □

[59] We note that $W(\mathfrak{g}^0(a))(k) = \mathfrak{g}^0(a)$, so to define the open subset U, it suffices by Nullstellensatz to consider the subset of $x \in \mathfrak{g}^0(a)$ such that $\mathrm{ad}_{\mathfrak{g}/\mathfrak{g}^0(a)}(x)$ is injective, and prove that it is open. But this is immediate: this is the complement of the kernel of the adjoint representation $\mathrm{ad} : \mathfrak{g}^0(a) \to \mathrm{GL}(\mathfrak{g}/\mathfrak{g}^0(a))$.

[60] If k is finite, it may happen that $c_{n-r} \neq 0$ in $\boldsymbol{S}_k(\mathfrak{g}^\vee)$, but $c_{n-r}(a) = 0$ for any $a \in \mathfrak{g}$. In this case, for any $a \in \mathfrak{g}$, we will have $\dim_k(\mathfrak{g}^0(a)) > r$.

Proposition 12.12.25. *Suppose that k is infinite and let $\mathfrak{d}$ be a subalgebra of $\mathfrak{g}$. Consider the following conditions:*

(i) *$\mathfrak{d}$ is maximal nilpotent and contains a regular element of $\mathfrak{g}$.*

(i') *$\mathfrak{d}$ is of the form $\mathfrak{g}^0(a)$, where a is a regular element of $\mathfrak{g}$.*

(ii) *$\mathfrak{d}$ is nilpotent and of the form $\mathfrak{g}^0(a)$, where $a \in \mathfrak{g}$.*

(ii') *$\mathfrak{d}$ is nilpotent, and there exists $a \in \mathfrak{d}$ such that $\mathrm{ad}_{\mathfrak{g}/\mathfrak{d}}(a)$ is injective.*

(iii) *$\mathfrak{d}$ is nilpotent and self-normalizing.*

We have the implications (i)⇔(i')⇒(ii)⇔(ii')⇔(iii)[61].

Proof. The equivalence of (i) and (i') is trivial by Corollary 12.12.24, and these conditions imply trivially (ii). The equivalence of (ii) and (ii') is equally trivial, as well as (ii')⇒(iii) (cf. Proposition 12.12.22). It remains to prove (iii)⇒(ii'), which follows from **??** below (this is the point where we use the fact that k is infinite). □

Lemma 12.12.26. *Let $\mathfrak{d}$ be a nilpotent Lie algebra over an infinite field k, acting on a finite dimensional vector space V. Suppose that for any $x \in \mathfrak{d}$, the endomorphism $u(x)$, then there exists a nonzero element $v \in V$ annihilated by $\mathfrak{d}$.*

Proof. We can suppose that k is algebraically closed and $\mathfrak{d}$ is of finite dimension. We then see that V is a direct sum of finitely many nonzero subspaces V_i stable under $\mathfrak{d}$, such that for any i, and any $x \in \mathfrak{d}$, $u(x)|_{V_i}$ has a unique eigenvalue $\lambda_i(x)$ (cf. **[?]** I, §4, Exercice 22). Let $c_i(x)$ be the constant term of the characteristic polynomial of $u(x)|_{V_i}$, so that $\lambda_i(x) = 0$ if and only if $c_i(x) = 0$. Then c_i is a polynomial function over $\mathfrak{d}$, and the hypothesis signifies that the union of zero sets of c_i is equal to $\mathfrak{d}$. Hence one of the c_i is zero[62], which reduces us to the case (by replacing V with V_i) where V is such that $u(x)$ ($x \in \mathfrak{d}$) are nilpotent. But then Engel's theorem (Theorem 1.4.12) implies that there exists nonzero $v \in V$ such that $x \cdot v = 0$ for $x \in \mathfrak{d}$. □

We easily see that (k being finite) the equivalent conditions (i) and (i') of Proposition 12.12.25 are invariant under base field extension. If they are satisfied, we then say that $\mathfrak{d}$ is a **Cartan subalgebra** of $\mathfrak{g}$. In the general case (where k is not necessarily infinite), we say that $\mathfrak{d}$ is a Cartan subalgebra of $\mathfrak{g}$, if it becomes a Cartan subalgebra of $\mathfrak{g}$ after a (hence any) base field extension $k \to k'$, with k' infinite. This then implies that $\mathfrak{d}$ is nilpotent and self-normalizing.

Proposition 12.12.27. *Let $\mathfrak{g}$ be a Lie algebra over a field k.*

(a) *If $a \in \mathfrak{g}$ is a regular element, it is contained in a unique Cartan subalgebra of $\mathfrak{g}$.*

(b) *Let $\mathfrak{d}$ be a Cartan subgroup of $\mathfrak{g}$ and $a \in \mathfrak{d}$, then a is regular in $\mathfrak{g}$ if and only if $\mathrm{ad}_{\mathfrak{g}/\mathfrak{d}}(a)$ is injective.*

Proof. In fact, for (a) we note that if a is regular then $\mathfrak{g}^0(a)$ is a Cartan subalgebra of $\mathfrak{g}$ (because this is true over an infinite field k' over k), and it then follows from Corollary 12.12.24 that any Cartan subalgebra of $\mathfrak{g}$ containing a is equal to $\mathfrak{g}^0(a)$. As for (b), we note that the dimension of $\mathfrak{g}^0(a)$ is equal to the sum of $\mathfrak{d}^0(a)$ and $(\mathfrak{g}/\mathfrak{d})^0(a)$, and as the first is equal to r, this sum is equal to r if and only if $\mathrm{ad}_{\mathfrak{g}/\mathfrak{d}}(a)$ is injective. □

Corollary 12.12.28. *Let a be a regular element of $\mathfrak{g}$, $\mathfrak{d}$ be a Cartan subalgebra of $\mathfrak{g}$ containing a, A be a k-algebra, $\mathfrak{g}_A$ and $\mathfrak{d}_A$ be the Lie algebras over A induced by base change, and a_A be the image of a in $\mathfrak{g}_A$. Let u be an automorphism of $\mathfrak{g}$, then for that $u(\mathfrak{d}_A) = \mathfrak{d}_A$, it is necessary and sufficient that $u(a_A) \in \mathfrak{d}_A$.*

Proof. The condition is clearly necessary, to show that it is also sufficient, note that if $u(a_A) \in \mathfrak{d}_A$, then $\mathfrak{d}' = u(\mathfrak{d}_A)$ is a subalgebra of $\mathfrak{g}$ containing a_A, and any element $b \in \mathfrak{d}'$ is such that $\mathrm{ad}_{\mathfrak{d}'}(b)$ is nilpotent (because $\mathfrak{d}'$ is isomorphic to $\mathfrak{d}_A$, which has this property). Putting $b = a_A$, we then see that the nilspace $\mathfrak{d}_A = \mathfrak{g}_A^0(a_A)$ contains $\mathfrak{d}'$. As $\mathfrak{d}'$ is locally a direct factor of the module $\mathfrak{g}_A$ ($\mathfrak{d}$ being so), and hence of $\mathfrak{d}_A$, and it is a projective module of the same rank r as $\mathfrak{g}_A$, we conclude that $\mathfrak{d}' = \mathfrak{d}_A$. □

Proposition 12.12.29. *Let $\mathfrak{h}$ be a Lie subalgebra of $\mathfrak{g}$.*

[61] We shall see in **??** that if $\mathfrak{g}$ is the Lie algebra of a smooth algebraic group, then these conditions are all equivalent.

[62] Recall that over an infinite field, a vector space can not be written as a finite union of its proper subspaces.

(a) *The following conditions are equivalent if k is infinite:*

(i) $\mathfrak{h}$ *contains a Cartan subalgebra* $\mathfrak{d}$ *of* $\mathfrak{g}$.

(ii) $\mathfrak{h}$ *contains a regular element a of* $\mathfrak{g}$, *and an element b such that* $\mathrm{ad}_{\mathfrak{g}/\mathfrak{h}}(b)$ *is injective.*

(iii) $\mathfrak{h}$ *has the same nilpotent rank as* $\mathfrak{g}$, *and contains a regular element a of* $\mathfrak{g}$.

These conditions are invariant under base field change.

(b) *Suppose that the conditions in (a) are varified over an infinite field k' over k. Let $a \in \mathfrak{h}$, then a is regular in $\mathfrak{g}$ if and only if it is regular in $\mathfrak{h}$ are $\mathrm{ad}_{\mathfrak{g}/\mathfrak{h}}(a)$ is injective, i.e. if and only if $\mathfrak{h}^0(a) = \mathfrak{g}^0(a)$ and if this is a Cartan subalgebra of $\mathfrak{h}$.*

(c) *Under the conditions of (b), let $\mathfrak{d}$ be a subalgebra of $\mathfrak{h}$. For that this is a Cartan subalgebra of $\mathfrak{h}$, it is sufficient that it is a Cartan subalgebra of $\mathfrak{g}$.*

Proof. We see immediately that the conditions (ii) and (iii) of (a) are invariant under base field extension of k (assumed to be infinite), and that in statements (b) and (c), we can assume that k is infinite. If $\mathfrak{h}$ contains the Cartan subalgebra $\mathfrak{d} = \mathfrak{g}^0(a)$, then a is a regular element of $\mathfrak{g}$ such that $\mathrm{ad}_{\mathfrak{g}/\mathfrak{h}}(a)$ is injective, so (i)⇒(ii). Conversely, if (ii) holds, then for a "generic" element a of $\mathfrak{h}$, a simultaneously satisfies the two conditions considered in (ii), so $\mathfrak{g}^0(a)$ is a Cartan subalgebra of $\mathfrak{g}$ and is contained in $\mathfrak{h}$, whence (i). Now we have seen that (i) and (ii) are equivalent. Suppose that they are satisfied, and let a be a variable element of $\mathfrak{h}$, then

$$\dim_k(\mathfrak{g}^0(a)) = \dim_k(\mathfrak{h}^0(a)) + \dim_k((\mathfrak{g}/\mathfrak{h})^0(a)), \tag{12.12.8}$$

On the other hand, the two terms on the right are respectively $\geq r' = \rho_n(\mathfrak{h})$ and ≥ 0, and the equalities is attained for a "generic" element of $\mathfrak{h}$. Moreover, we also have $\dim_k(\mathfrak{g}^0(a)) \geq r = \rho_n(\mathfrak{g})$, the equality being attained if and only if a is regular in $\mathfrak{g}$. We then conclude that we have $r = r'$, and that a is regular in $\mathfrak{g}$ if and only if the two terms on the right of (12.12.8) are equal to r' and 0, respectively, i.e. if and only if a is regular in $\mathfrak{h}$ and $\mathrm{ad}_{\mathfrak{g}/\mathfrak{h}}(a)$ is injective; this proves (b), and (c) then follows trivially by consider a regular element of $\mathfrak{g}$ in $\mathfrak{d}$, so that $\mathfrak{g}^0(a) = \mathfrak{d}$. Moreover, the preceding result shows that (i)⇒(iii), and finally (iii)⇒(i) because with (iii), a generic element a of $\mathfrak{h}$ is regular in $\mathfrak{h}$ and is in $\mathfrak{g}$, hence $\mathfrak{h}^0(a) \subseteq \mathfrak{g}^0(a)$ are respectively Cartan subalgebras of $\mathfrak{h}$ and $\mathfrak{g}$. As $\mathfrak{g}$ and $\mathfrak{h}$ have the same nilpotent rank over k, the two subalgebras are therefore equal, which proves (i). □